# The Four Block Distance Problem in $H^{\infty}$ Control.

by

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## ABSTRACT

The aim of this thesis is to characterize all solutions to the four block general distance problem which arises in $H^\infty$- optimal control. The procedure is to embed the original problem in an all-pass matrix which is constructed. With this complete, one can demonstrate that part of this all-pass matrix acts as a generator of all solutions. Special attention is given to the characterization of the optimal solutions using a generalized state-space framework. As an application, a representation formula is derived for all solutions to $H^\infty$-optimal control problems of the third kind.

It is shown that all controllers can be parameterized in terms of the solutions to two n-dimensional algebraic Riccati equations and the state space realization of the original problem; "n" being the dimension of the problem. If P and S are the positive semi-definite stabilizing solutions to these equations, then necessary and sufficient conditions for the existence of a solution such that $||F_l(P,K)||_\infty \leq \gamma$ are given in terms of the inertia of S, P and SP; in which $\gamma$ is the target $H^\infty$-norm. Special attention is given to the cases when $D_{11} \neq 0$ and/or $D_{22} \neq 0$ by suitably modifying the problem using loop shifting arguments into one where $D_{11}=0$ and $D_{22}=0$. This approach to solving $H^\infty$ control problems is shown to be easily applicable to problems of the first and second kinds.

As an example of the latter case, the problem of mixed-sensitivity is investigated. Mixed-sensitivity optimization is a two block problem which normally requires an iterative solution. An explicit closed formula for the optimal cost is given in the cases of the unweighted mixed sensitivity and parametric mixed sensitivity problems. It is shown that all the controllers which satisfy an arbitrary $H^\infty$-norm bound may be parameterized in terms of the solutions to the standard $H^2$ Riccati equations.

As an engineering application, the $H^\infty$ optimal control methods developed above are used to improve the performance and robustness properties of a synchronous turbo-generator connected to the infinite bus-bar. An $H^\infty$ "piggy-back" controller is used to supplement the existing control loops to achieve the required goal. The results compare favorably with those obtained by other multivariable design methodologies.

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## 0. Notation and Preliminaries

The aim of this short section is to summarize the notation we intend to use. Most of it is standard, but we wish to draw particular attention to the state-space notation which will be used extensively.

| | |
|---|---|
| $\mathbb{R}, \mathbb{R}_+, \mathbb{C}$ | real, non-negative and complex numbers |
| $\mathbb{R}(s)$ | field of rational functions in s with real coefficients |
| $\mathbb{C}_+, \mathbb{C}_-$ | open right (resp. left) half plane |
| $\mathbb{F}^{p\times m}$ | set of $p\times m$ matrices with elements in $\mathbb{F}$ (= $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$ etc) |
| $\lambda(A), \lambda_{max}(A)$ | spectrum of a square matrix A, largest eigenvalue of A |
| $A^*$ | complex conjugate transpose of $A\in\mathbb{C}^{p\times m}$(transpose if $A\in\mathbb{R}^{p\times m}$) |
| $A\geq 0$, $A>0$ | A is positive semidefinite (resp. positive definite) |
| $A\leq 0$, $A<0$ | A is negative semidefinite (resp. negative definite) |
| $\mathcal{L}^{\infty,(p\times m)}$ | space of $p\times m$ matrices with entries which are bounded on the $j\omega$ axis (including the point at $\infty$) |
| $\|\cdot\|_\infty$ | $\mathcal{L}^\infty$-norm of matrices in $\mathcal{L}^\infty$ |
| $\mathcal{H}_+^{\infty,(p\times m)}$ | subspace of $\mathcal{L}^\infty$; $p\times m$ matrices which are analytic and bounded in $\mathbb{C}_+$ |
| $\mathcal{H}_-^{\infty,(p\times m)}$ | subspace of $\mathcal{L}^\infty$; $p\times m$ matrices which are analytic and bounded in $\mathbb{C}_-$ |
| $\mathcal{H}_\pm^{\infty,(p\times m)}(k)$ | set of pxm matrix functions in $\mathcal{L}^\infty$ with no more than k poles in $\mathbb{C}_\pm$ |
| $\mathfrak{B}\mathcal{H}_-^\infty, \mathfrak{B}\mathcal{H}_+^\infty$ | unit balls in $\mathcal{H}^\infty$: $\{f\in\mathcal{H}_-^\infty : \|f\|_\infty\leq 1\}$ , $\{f\in\mathcal{H}_+^\infty : \|f\|_\infty\leq 1\}$ |
| $\mathfrak{R}\mathcal{L}^{\infty,(p\times m)}$ | same as $\mathcal{L}^{\infty,(p\times m)}$ except elements are taken from $\mathbb{R}^{(p\times m)}(s)$ |
| $\mathfrak{R}\mathcal{H}_\pm^{\infty,(p\times m)}(k)$ | same as $\mathcal{H}_\pm^{\infty,(p\times m)}(k)$ except elements are taken from $\mathbb{R}^{(p\times m)}(s)$ |
| $\Gamma_G$ | Hankel operator with symbol $G(s)\in\mathcal{H}_+^\infty$ |
| $G^*(s)$ | $= G(-\bar{s})^*$, the para-hermitian conjugate of G(s) |
| $\|A\|_2$ | = the spectral norm of A |
| $In(A)$ | $=(\pi(A),\nu(A),\delta(A))$; $\pi(A)$=number of eigenvalues of A in the open right half plane, $\nu(A)$=number of eigenvalues of A in the open left half plane, $\delta(A)$=number of eigenvalues of A on the imaginary axis. |

Associated with a transfer function matrix $G(s)\in\mathbb{R}(s)^{p\times m}$ of McMillan degree n, is a state-space realization

$$G(s) = D+C(sI-A)^{-1}B \tag{0.1}$$

Throughout this thesis G(s) is assumed to be a system of linear time-invarient ordinary differential equations where $A \in \mathbb{C}^{nXn}$, $B \in \mathbb{C}^{nXm}$, $C \in \mathbb{C}^{pXn}$ and $D \in \mathbb{C}^{pXm}$. We will use the alternative notation $G(s) \stackrel{s}{=} (A,B,C,D)$ or

$$G(s) \stackrel{s}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{0.2}$$

for realizations of G(s). Generalized state-space models will be represented as

$$G(s) \stackrel{s}{=} \begin{bmatrix} E & A & B \\ 0 & C & D \end{bmatrix} \tag{0.3}$$

so that $G(s)=D+C(sE-A)^{-1}B$.

In the above notation, we have $G^*(s) \stackrel{s}{=} (-A^*,C^*,-B^*,D^*)$ and in the case that D is non-singular, we have $G^{-1}(s) \stackrel{s}{=} (A-BD^{-1}C,BD^{-1},-D^{-1}C,D^{-1})$. If G(s) is defined by (0.2), the system matrix of the given realization is defined as

$$G(s) = \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}$$

The system zeros of G(s) are defined to be the points at which the system matrix loses normal rank and are given by $\{\lambda(A-BD^{-1}C)\} \supseteq \{\text{McMillan zeros of } G(s)\}$; these sets are equal if the realization is minimal. If $G^{-1}(s) = G^*(s)$ , then G(s) is all-pass. G(s) is called stable if all it's poles are in $\mathbb{C}_-$.

We will talk about basis changes T in the state-space of G(s); we will take this to mean $G(s) \stackrel{s}{=} (A,B,C,D) \stackrel{T}{\rightarrow} G(s) \stackrel{s}{=} (TAT^{-1},TB,CT^{-1},D)$. For generalized state-space models basis changes are represented by

$$G(s) \stackrel{s}{=} \begin{bmatrix} UEV & UAV & UB \\ 0 & CV & D \end{bmatrix} \tag{0.4}$$

in which U and V are arbitrary nonsingular matrices.

We shall also make use of the lower and upper linear fractional transformations which are defined by

$$F_l\left\{ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, U \right\} = H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21} \tag{0.5}$$

where U is of dimension $l$xm if $H_{22}$ has dimension m$xl$. In the same way

$$F_u\left\{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, U\right\} = H_{22} + H_{21}U(I - H_{11}U)^{-1}H_{12} \tag{0.6}$$

in which U is of dimension $l$xm if $H_{11}$ has dimension m$xl$.

If $P(s) \in \mathbb{R}^{(p+q)\mathrm{x}(m+r)}(s)$ and $K(s) \in \mathbb{R}^{r\mathrm{X}q}(s)$ have state-space realizations

$$P(s) \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad K(s) \overset{s}{=} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$

Then if the well-posedness condition $\det(I-D_{22}\hat{D}) \neq 0$ is satisfied,

$$F_l(P,K) \overset{s}{=} \left[\begin{array}{cc|c} A+B_2\hat{D}L_1C_2 & B_2L_2\hat{C} & B_1+B_2\hat{D}L_1D_{21} \\ \hat{B}L_1C_2 & \hat{A}+\hat{B}L_1D_{22}\hat{C} & \hat{B}L_1D_{21} \\ \hline C_1+D_{12}L_2\hat{D}C_2 & D_{12}L_2\hat{C} & D_{11}+D_{12}\hat{D}L_1D_{21} \end{array}\right] \tag{0.7}$$

where

$$L_1=(I-D_{22}\hat{D})^{-1} \quad ; \quad L_2=(I-\hat{D}D_{22})^{-1} \tag{0.8}$$

This result follows by routine computation. □

# Chapter one
# Introduction

Over the past ten years, a substantial portion of the control litterature has been devoted to $H^{\infty}$ control theory. The importance of this theory stems from its ability to address engineering issues such as robust stability in the presence of uncertainty, accurate tracking or performance, and noise and disturbance rejection. Uncertainties or perturbations arise when accounting for model changes, uncertainty in operating conditions, unmodeled dynamics and non-linearities. Performance objectives such as good command response and small errors in estimation and regulation are also easily dealt with in an $H^{\infty}$ framework. Once formulated, the $H^{\infty}$ optimization problem is habitually converted into a so called *general distance* or *model matching problem.* The distance problem considered here is one in which a completely unstable transfer function R(s) of MacMillan degree "n" is approximated (in the $\mathcal{L}_{\infty}$ norm sense) by a stable transfer function Q(s) of degree no more than "n". In [11,12], Glover addresses the one block problem and gives a complete characterization of all approximations that minimize $||R(s)-Q(s)||_{\infty}$. In this thesis, particular attention will be given to the solution of the *four block general distance problem* which shall be described next.

The *four block general distance problem* has it's genesis in $H^{\infty}$-optimal control problems of the third kind [9,26,31]. In typical $H^{\infty}$ design situations, a nominal plant model is known, and the design engineer has the task of selecting various frequency dependent weights. The plant model and weights are then combined into a single matrix: P(s); where P(s) is a matrix transfer function which contains the structure between the plant model, weights and the controller. P(s) may be partitioned such that

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}(s) \tag{1.1}$$

Figure 1.1 shows the generalized regulator set-up in which K(s) is the controller transfer function matrix to be designed; any linear combination of inputs, outputs, commands, and a controller can be rearranged to match this diagram. The variable $v$ consists of all external inputs, $z$ are the error signals to be regulated, $u$ are the control inputs and $y$ the measurements. Throughout this thesis attention will be restricted to linear time invariant finite dimensional systems.

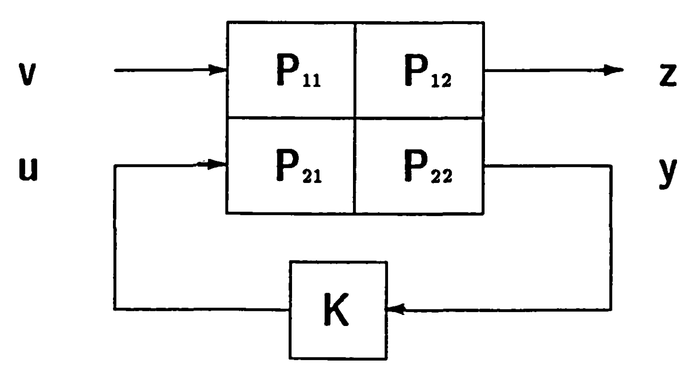


Figure 1.1 – Generalized regulator set-up

$$z=F_l(P,K)v=(P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21})v$$

The $H^{\infty}$ control problem is then concerned with minimizing $\|F_l(P, K)\|_{\infty}$ as K(s) ranges over the set of stabilizing controllers **S**. In other words, we seek to minimize the $L_2$ norm of $z$ over the set of allowable inputs $y$ such that $\|y\|_2=1$. Rather than tackling this (nonlinear) problem directly, it is widely believed prudent to convert it into another problem which is linear in a free parameter. This is done by the "Youla parameterization" of all stabilizing controllers, this is obtained by calculating coprime factorizations of P(s) over $\mathcal{H}_+^{\infty}$ and solving a double Bezout identity to obtain the coefficients of $K_o(s)$; the generator of all stabilizers.

$$\inf_{K\in S}\|F_l(P,K)\|_{\infty}=\inf_{Q\in\mathcal{H}_+^{\infty}}\|F_l(P,F_l(K_o,Q))\|_{\infty}=\inf_{Q\in\mathcal{H}_+^{\infty}}\|(T_{11}-T_{12}QT_{21})(s)\|_{\infty} \tag{1.2}$$

in which $T_{11}(s)$, $T_{12}(s)$ and $T_{21}(s)$ may always be chosen stable with $T_{12}(s)$ and $T_{21}(s)$ parts of inner matrices [9,26,31]. We change nothing by re-writing (1.2) as

$$\inf_{Q\in\mathcal{H}_+^{\infty}}\|(T_{11}-[T_{12}\ T_{\perp}]\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{21} \\ \tilde{T}_{\perp} \end{bmatrix})(s)\|_{\infty} \tag{1.3}$$

in which $T_{\perp}(s)$ and $\tilde{T}_{\perp}(s)$ are chosen to make $[T_{12}\ T_{\perp}](s)$ and $[T_{21}^{*}\ \tilde{T}_{\perp}^{*}]^{*}(s)$ inner. This, too, is always possible [9,26]. Finally, by invoking the norm preserving property of inner matrices, we see that (1.3) is equivalent to

$$\inf_{Q\in\mathcal{H}_+^{\infty}}\left\|\begin{bmatrix} R_{11}-Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s)\right\|_{\infty}\leq\gamma \tag{1.4}$$

where

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s)=\begin{bmatrix} T_{12}^{*} \\ T_{\perp}^{*} \end{bmatrix}T_{11}[T_{21}^{*}\ \tilde{T}_{\perp}^{*}](s)\in\mathcal{R}\mathcal{H}_-^{\infty} \tag{1.5}$$

and $\gamma$ is the target norm; (1.4) is the four block general distance problem which we will study in this thesis.

Five years ago, Doyle [31] suggested a solution to this problem which is based on the work of Davis, Kahan and Weinberger [6]. The technique proposed consisted of breaking the four block problem into two consecutive two block problems and finally a one block problem, which was subsequently solved. The desired controller is then recovered by a somewhat protracted back substitution. This approach has proved quite adequate for theoretical work, but when implemented on a computer, suffers from serious degree inflation problems. Computationally, each iteration requires the solutions to three factorization problems, furthermore, one has to make heavy use of model reduction routines to suppress any state inflation.

In an attempt to understand these inflation phenomena (in the context of $H^\infty$-control), several researchers undertook cancellation analyses for the one and two block problems [17,18,19]. Limebeer and Hung [19] investigated the one block problem arising in $H^\infty$ control problems of the first kind or single target problems. These problems are characterized by the assumption that $P_{12}(s)$ and $P_{21}(s)$ in (1.1) are square (but not necessarily of the same size). Using approximation theory [11], they carry out a detailed analysis of the cancellations which arise in such problems yielding a MacMillan degree bound for all $H^\infty$ optimal and suboptimal stabilizing controllers. Their main result states, "*If deg(P(s))=n, then there exist optimal stabilizing controllers of degree n-1 which achieve the target norm*". The same result has been derived by Limebeer and Anderson in [17], and Kimura in [58] using vector interpolation theory. In [18], Limebeer and Halikias use approximation theory again to extend the cancellation analysis of [19] to deal with problems of the second kind. That is, $P_{12}(s)$ has more rows than columns and $P_{21}(s)$ is square, alternatively, $P_{21}(s)$ has more columns than rows and $P_{12}(s)$ is square. They show that the controller degree bound derived for problems of the first kind also holds for problems of the second kind. In addition, their analysis shows that significant improvements may be made to existing computer software, which is backed up by computation trials which reveal that the new code is approximately eight times faster. Juang and Jonckheere [33] have also derived such a controller degree bound using their linear quadratic approximation technique [34]. At this point, it is natural to ask whether the controller degree bound for problems of the first and second kinds carries over to those of the third kind. To this end, a controller degree conjecture was made in [18] for the case of $H^\infty$ control problems of the third kind which was subsequently confirmed by Hung [15] using interpolation theory. The outcome of this work was most encouraging, in that problems with deg(R)=n, in (1.4), always had solutions with deg(Q)=n. In the same way, it was demonstrated that if deg(P)=n, in (1.1), then controllers with degree n always exist. These observations lead to the expectation that Doyle's original algorithm could be greatly improved from the computational point of view.

Over the past fifteen months there has been a flurry of activity among $H^\infty$ control researchers; the goal: a new state-space based approach to solving the four block general distance problem posed in (1.4). Limebeer et al [34] solved this problem by embedding (1.4) into a larger all-pass matrix, with this complete they demonstrate that all optimal and suboptimal solutions are generated by part of this all-pass embedding. Hung [15] also solves the problem using interpolation theory, he breaks down the model matching problem in (1.2) into two interpolation problems for which he characterizes all optimal and suboptimal error systems. In both these cases [15,34] explicit formulae for the solutions are given.

Clearly, the next step is to apply these solution techniques to the four block general distance problem associated with $H^\infty$ control. To this end, a number of closed form representation formulae have been derived for the generator of all $H^\infty$ controllers. Limebeer et al [34] replace Q(s) in the middle term of (1.2) with the generator of all extensions to yield a generator for all stabilizing controllers. The derivation of such formulae has been obtained using a variety of other techniques;

Glover and Doyle [14] have also used the all-pass embedding technique independently to derive a similar representation formula, however unlike [15,34], they do not treat the optimal case. Doyle et al [10] use the direct state space approach based on separation principle arguments which are reminicent of LQG ($H_2$ theory). In addition to obtaining controller representation formulae in which the controller degree bound is implicit, [10,14,20,34] show that all $H^\infty$ control problems may be solved with only two indefinite algebraic Riccati equations, further, each also give necessary and sufficient conditions for the existence of stabilizing controllers meeting the target norm. Hung [13,16] has also derived one such set of formulae based on the solutions to indefinite Riccati equations, again his approach was via matrix interpolation theory. Khargonekar et al [35] have shown that if all the states are accessible, the infimum of the norm of the closed loop transfer function can be achieved with a constant static feedback, their solution involves solving an indefinite Riccati equation, furthermore, they give similar necessary and sufficient conditions for the existence of a static feedback meeting the target norm. In addition to these techniques there exists the J-factorization approach of Glover et al [36] based on the canonical factorization theory of Bart et al [37], this method also leads to a representation formula for a generator of all stabilizing controllers. Verma has a closed formula for problems of the first kind [29]. Finally, a time domain approach using the maximum principle is due to Tadmor [28]. Limebeer et al [39] solve the finite horizon and time varying cases using linear quadratic differential game theory.

Another problem which has recieved much attention is that of *entropy minimization.* Arov and Krien [1] have studied a class of suboptimal extensions which they have termed *minimum entropy extensions.* The entropy of (1.4), which is defined at a point, is given by

$$I(R,s_o) = \frac{-1}{2\pi}\int_{-\infty}^{+\infty} \ln|\det(I-R^*(j\omega)R(j\omega))|\,\frac{Re(s_o)}{|j\omega-s_o|^2}\,d\omega \qquad Re(s_o)>0$$

Limebeer and Hung [19] studied these extensions in the case of one block $H^\infty$ control problems, where the problem is posed as: among all stable closed loops: $R(s)$ achieving a certain target norm $\gamma$ ($\gamma>\gamma_o$), obtain the one which minimizes the entropy integral. In [34], Limebeer et al extend these results to the general distance four block problem using the all-pass embedding technique. Mustafa and Glover [40] also employ the all-pass embedding approach to derive the entropy minimizing extension which they then apply to the four block $H^\infty$ distance problem. In both cases special attention is given to minimizing entropy at infinity, [40] goes further, discussing the interesting links with risk sensitive LQG problems, and they also give formulae for the entropy minimizing controller.

Other related problems of interest include the super-optimal Nehari extension of general rational matrices. In [69], Limebeer et al sequentially minimize each frequency dependent singular value of the error system until the problem is reduced to one of rank one, yielding the unique super-optimal extension. Postlethwaite et al [70] also develop one such algorithm for computing the super-optimal Nehari extensions. Jaimoukha [71], extends the results of [69,70] to the four block distance problem.

The recent breakthroughs in the $H^\infty$ scene have meant that theoretically cleaner and more robust algorithms exist for the computing of $H^\infty$ optimal controllers. Nevertheless, most engineering problems still require a computationally expensive $\gamma$-iteration. Ideally, one would like to remove the $\gamma$-iteration completely. Although this ideal seems some way off, some progress has been made in this direction. Limebeer and Hung [19] give a number of closed formulae for the optimal cost in the cases of unweighted single target problems, a cost formula is also derived for the weighted robustness problem. Glover and MacFarlane tackle a two block optimal robustness problem [41]. They derive a formula for $\gamma_o$ in terms of the Hankel norm of the normalized coprime factors associated with the plant to be controlled, in addition a representation formula for the generator of all sub-optimal controllers is given. Foias and Tannenbaum have a determinental formula for the (essential) norm of the four block operator associated with $H^\infty$ control [42]. Kasenally and Limebeer [21], use the new $H^\infty$ solution technique to derive a formula for $\gamma_o$ in the cases of unweighted mixed-sensitivity and parametric mixed-sensitivity.

As with any new engineering theory the success of feedback design with an $H^\infty$ optimization criterion can only be judged by its performance once applied to practical engineering problems. Currently researchers are having some degree of success [43,44,45,46,47] to name but a few. The aim of the final chapter is to perform a design for a turbogenerator with a view to improving the overall transient and dynamic stability. To date, LQG regulators have been considered [48,49,50,51] these have modeled uncertainty as white noise and performance in terms of the $L_2$ norm of the error. Unfortunately few physical systems are adequately modeled with additive white noise as the only uncertainty, hence the control of such systems display reasonable performance but poor robustness characteristics.

## 1.1 Thesis contributions:

In this thesis we have studied various important aspects of the $H^\infty$ control theory associated with $H^\infty$-control problems of the third kind. To start with, we address a general problem termed the four block general distance problem. The solution proposed here is to embed the four block problem into a larger all-pass matrix, transforming it into a one block problem, with this complete it is demonstrated that part of this all-pass matrix acts as a generator of all suboptimal solutions. It is established that this same generator captures a class of the inertia optimum solutions (that it captures all the optimal solutions is not proved). Furthermore, the loss in the effective dimension of the free contraction is also demonstrated.

In the next stage the all-pass embedding technique is employed to solve the four block problem associated with $H^\infty$control problems of the third kind. This approach yields a controller degree bound for the generator of all controllers. Combining the embedding and parameterization Riccati equations leads to a new way of solving $H^\infty$ control problems which requires the solution of only two indefinite Riccati equations of fixed dimension.

A representation formula for the generator of all controllers is derived, what is interesting is that this formula is expressed entirely in terms of P(s) and the solutions to the afore mentioned indefinite Riccati equations. Interpolation theory is used to derive necessary and sufficient conditions for the existence of stabilizing controllers achieving the target norm. It is known [10,14,34,36], that the $H^{\infty}$ formulae become cumbersome in the event that $D_{11}\neq 0$ and $D_{22}\neq 0$[1]. Glover and Doyle have already shown that a modest simplification is possible if $D_{22}=0$, and that this assumption may be made without loss of generality. It is demonstrated by a loop transformation argument that $D_{11}=0$ may also be assumed without loss of generality [34,81]. This observation brings about a substantial simplification in statement of the formulae.

Using only standard real matrix operations, it is possible to compute the $H^{\infty}$ optimal controller. An iterative procedure associated with this calculation is presented.

Unweighted mixed sensitivity is a two block problem which generally requires an iterative solution. For this special class of problem it is possible to obtain a closed formula for the optimal cost. A cost formula for the parametric mixed sensitivity problem is also derived. Both formulae and their controller representation equations are expressed in terms of the solutions to the standard $H_2$ algebraic equations and the state-space of P(s).

Software has been developed (using Fortran and PC Matlab) based on the results presented in chapters 2 and 3 that uses standard matrix operations and linear algebra techniques. As an engineering application this new approach to solving $H^{\infty}$ control problems is employed to enhance the stability of a synchronous machine connected to an infinite busbar. A so called supplementary controller is designed to feed existing control loops with signals during fault and transient periods. In the first instance a single inertia model is considered and finally, the effects shaft torsional vibrations are included.

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[1] $D_{ij}$ is the $(i\text{-}j)$th partition of the D-matrix in a realization of P(s). P(s) is referred to as equation number (1.1).

# Chapter two
# A characterization of all solutions to the four block general distance problem.

## 2.1 Introduction

In this chapter we characterize all the solutions to the four block general distance problem. The procedure is to embed the original problem in an all-pass matrix which we construct. With this complete, we demonstrate that part of this all-pass matrix acts as a generator of all solutions. Special attention is given to the characterization of all optimal solutions.

The four block general distance problem has it's genesis in $H^\infty$-optimal control problems of the third kind [9,26,31]. In typical $H^\infty$ design situations, a nominal plant model is known, and the design engineer has the task of selecting various frequency dependent weights. The plant model and weights are then combined into a single matrix

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}(s) \tag{2.1}$$

and one seeks to minimize $\|F_l(P, K)\|_\infty$ as K(s) ranges over the set of stabilizing controllers. Rather than tackling this (nonlinear) problem directly, it is widely believed prudent to convert it into another problem which is linear in a free parameter. That is

$$\inf_{Q \in \mathcal{H}_+^\infty} \|(T_{11}+T_{12}QT_{21})(s)\|_\infty \tag{2.2}$$

in which $T_{11}(s)$, $T_{12}(s)$ and $T_{21}(s)$ may always be chosen stable with $T_{12}(s)$ and $T_{21}(s)$ parts of inner matrices[9,26]. We change nothing by re-writing (2.2) as

$$\inf_{Q \in \mathcal{H}_+^\infty} \left\|\left(T_{11}+[T_{12}\ T_\perp]\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}\right)(s)\right\|_\infty \tag{2.3}$$

in which $T_\perp(s)$ and $\tilde{T}_\perp(s)$ are chosen to make $[T_{12}\ T_\perp](s)$ and $[T_{21}^*\ \tilde{T}_\perp^*]^*(s)$ inner. This, too, is always possible [2,9,11,26]. Finally, by invoking the norm preserving property of inner matrices, we see that (2.3) is equivalent to

$$\inf_{Q \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s) \right\|_\infty \tag{2.4}$$

where

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s) = \begin{bmatrix} T_{12}^* \\ T_\perp^* \end{bmatrix} T_{11} [T_{21}^*\ \tilde{T}_\perp^*](s) \in \mathcal{RH}_-^\infty \tag{2.5}$$

(2.4) is the four block general distance problem which we will study in this chapter.

Five years ago, Doyle [31] suggested a solution to this problem which is based on the work of Davis, Kahan and Weinberger[6]. This approach has proved quite adequate for theoretical work, but when implemented on a computer, suffers from serious degree inflation problems. The purpose of this chapter is to present a new solution to the four block general distance problem which is theoretically cleaner, and is also a great improvement over existing methods from an implementation viewpoint.

This chapter has been divided into several sub-sections, in Section 2.2 we construct an all-pass embedding for the generator of all solutions to the four block problem. In Section 2.3 we derive a representation formula for all sub-optimal solutions, the optimal cases are treated in section 2.4. We summarize the key findings in the conclusions (Section 2.5).

## 2.2. The construction of an all-pass embedding.

In this section we will show how to construct an all-pass embedding for R(s). With this initial step complete, we go on to prove that part of the resulting all-pass matrix acts as a generator for all solutions to the four block infinity norm problem. In this section we will only deal with suboptimal problems; the optimal cases will be addressed in the next section.

We begin by considering the following one block distance problem

$$\inf_{\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}(s)\in\mathcal{H}_{+}^{\infty}} \left\| \begin{bmatrix} R_{11}+Q_{11} & R_{12}+Q_{12} \\ R_{21}+Q_{21} & R_{22}+Q_{22} \end{bmatrix}(s) \right\|_{\infty} \tag{2.6}$$

in which $R(s)\in\mathcal{H}_{-}^{\infty}$. From standard Nehari extension theory, we know that this problem always has a solution such that

$$\begin{bmatrix} R_{11}+Q_{11} & R_{12}+Q_{12} \\ R_{21}+Q_{21} & R_{22}+Q_{22} \end{bmatrix}(s)\begin{bmatrix} R_{11}+Q_{11} & R_{12}+Q_{12} \\ R_{21}+Q_{21} & R_{22}+Q_{22} \end{bmatrix}^{*}(s)=\gamma_0^2 I \tag{2.7}$$

where $\gamma_0=\left\|\Gamma_{R^*}\right\|$. Lets now suppose that R(s) is special in that its second block row and column are all-pass in character. That is

$$\begin{bmatrix} R_{21} \; R_{22} \end{bmatrix}(s)\begin{bmatrix} R_{21}^{*} \\ R_{22}^{*} \end{bmatrix}(s) = \gamma_0^2 I \tag{2.8}$$

and

$$\begin{bmatrix} R_{12}^{*} \; R_{22}^{*} \end{bmatrix}(s)\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}(s) = \gamma_0^2 I \tag{2.9}$$

Comparing (2.8) and (2.9) with (2.7), one observes that there is a solution to the problem in (2.6) which is of the form

$$Q(s) = \begin{bmatrix} Q_{11}(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{2.10}$$

This looks like the solution to a four block problem in (2.4). In the usual event that (2.8) and (2.9) are not satisfied, we simply force the issue by introducing the spectral factors L(s) and N(s) such that

$$\begin{bmatrix} L & R_{21} & R_{22} \end{bmatrix}(s) \begin{bmatrix} L^* \\ R_{21}^* \\ R_{22}^* \end{bmatrix}(s) = \gamma^2 I \tag{2.11}$$

and

$$\begin{bmatrix} N^* & R_{12}^* & R_{22}^* \end{bmatrix}(s) \begin{bmatrix} N \\ R_{12} \\ R_{22} \end{bmatrix}(s) = \gamma^2 I \tag{2.12}$$

which is always possible provided of course that $\gamma \geq \max\left\{ \left\| \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}(s) \right\|_\infty , \left\| \begin{bmatrix} R_{21} & R_{22} \end{bmatrix}(s) \right\|_\infty \right\}$[1]. This gives the augmented problem

$$\inf_{Q \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} * & * & N \\ * & R_{11} & R_{12} \\ L & R_{21} & R_{22} \end{bmatrix}(s) + \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{10} & Q_{11} & Q_{12} \\ Q_{20} & Q_{21} & Q_{22} \end{bmatrix}(s) \right\|_\infty \tag{2.13}$$

which will have a solution of the form

$$Q(s) = \begin{bmatrix} Q_{00} & Q_{01} & 0 \\ Q_{10} & Q_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}(s) \tag{2.14}$$

provided $\gamma$ is chosen big enough. Finally, one may discard the augmentation in the knowledge that this will not increase the norm. So much for the intuition.

We start our formal analysis by assuming that $R(s) \in \mathcal{RH}_-^{\infty,(p+q)\times(m+l)}$ and that there exists some $Q(s) \in \mathcal{RH}_+^{\infty,p\times m}$ such that

[1] Incidentally, exactly the same construction may be used to prove Parrott's theorem which is a result we shall call upon again[22,23, pg.4].

$$\left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\infty} \leq \gamma_o. \tag{2.15}$$

We will also assume that

$$\left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(\infty) \right\|_2 = \left\| \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \right\|_2 = \max\{\| D_{21}\ D_{22} \|_2, \left\| \begin{matrix} D_{12} \\ D_{22} \end{matrix} \right\|_2 \} = \gamma_P < \gamma_o \tag{2.16}$$

The second equality in (2.16) follows from Parrott's theorem[22,23,Thm 1.2, 17], and the fact that we have complete freedom in our choice of $D_{11}$[2].

Suppose next that

$$\begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s) \stackrel{s}{=} \left[ \begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & A_Q & B_{Q3} & 0 \\ \hline C_1 & C_{Q3} & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & D_{22} \end{array} \right] \tag{2.17}$$

is minimal in which

$$R(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad \text{and} \quad Q(s) \stackrel{s}{=} \left[ \begin{array}{c|c} A_Q & B_{Q3} \\ \hline C_{Q3} & 0 \end{array} \right] \tag{2.18}$$

Invoking a standard result from dilation theory [11], we can write

$$\mathcal{A}(s) = \begin{array}{c} \\ m+l \\ p \\ q \end{array} \!\!\begin{array}{c} \begin{array}{ccc} p+q & m & l \end{array} \\ \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & R_{11}+Q & R_{12} \\ \mathcal{A}_{31} & R_{21} & R_{22} \end{bmatrix} \end{array} (s) \tag{2.19}$$

in which it is always possible to choose the first $m+l$ rows and $p+q$ columns of $\mathcal{A}(s)$ so that $\mathcal{A}(s)\mathcal{A}^*(s)=\gamma^2 I$ where we select $\gamma>\gamma_o$ (the case of $\gamma\geq\gamma_o$ will be addressed later). As a consequence of the all-pass character of $\mathcal{A}(s)$, we must have

$$\mathcal{A}_{31}(s)\mathcal{A}_{31}^*(s) = \gamma^2 I - \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} R_{21}^* \\ R_{22}^* \end{bmatrix}(s) \tag{2.20}$$

and

$$\mathcal{A}_{13}^*(s)\mathcal{A}_{13}(s) = \gamma^2 I - \begin{bmatrix} R_{12}^* & R_{22}^* \end{bmatrix} \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}(s) \tag{2.21}$$

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[2]The strict inequality in (2.16) may be relaxed, but the additional intricacies which this involves will unduly cloud the main theme of the construction. $\gamma_P$ is defined to be the Parrott lower bound.

which are spectral factorization problems. Suppose next, and this is always possible, that we choose

$$\mathcal{A}_{31}(s) = \begin{array}{cc} \begin{array}{cc} p & q \end{array} & \\ \begin{bmatrix} 0 & X_{42}(s) \end{bmatrix} & q \end{array} \tag{2.22}$$

and

$$\mathcal{A}_{13}(s) = \begin{array}{cc} l & \\ \begin{bmatrix} 0 \\ X_{24}(s) \end{bmatrix} & \begin{array}{c} m \\ l \end{array} \end{array} \tag{2.23}$$

It is important to notice that the dilation in (2.19) may be accomplished without changing the A–matrix in (2.17) [11, Thm 5.2]. Further, the spectral factorization compressions in (2.22) and (2.23) require no change of A–matrix either. To see this, one needs only study all the possible solutions to the bounded real equations [2]. Translating these observations into the state-space, gives the following special structure which we will soon analyse in greater detail:

$$\mathcal{A}(s) = \begin{array}{cc} \begin{array}{cccc} p & q & m & l \end{array} & \\ \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 \\ Q_{21} & X_{22}+Q_{22} & X_{23}+Q_{23} & X_{24} \\ Q_{31} & X_{32}+Q_{32} & R_{11}+Q & R_{12} \\ 0 & X_{42} & R_{21} & R_{22} \end{bmatrix} & \begin{array}{c} m \\ l \\ p \\ q \end{array} \end{array} \tag{2.24a}$$

$$\overset{s}{=} \left[\begin{array}{cc|cccc} A & 0 & 0 & B_O & B_1 & B_2 \\ 0 & A_Q & B_{Q1} & B_{Q2} & B_{Q3} & 0 \\ \hline 0 & C_{Q1} & J_{11} & J_{12} & J_{13} & 0 \\ C_O & C_{Q2} & J_{21} & J_{22} & J_{23} & J_{24} \\ C_1 & C_{Q3} & J_{31} & J_{32} & D_{11} & D_{12} \\ C_2 & 0 & 0 & J_{42} & D_{21} & D_{22} \end{array}\right] \tag{2.24b}$$

in which

$$X_{42}(s) \overset{s}{=} \begin{bmatrix} A & B_O \\ C_2 & J_{42} \end{bmatrix} \tag{2.25}$$

and

$$X_{24}(s) \overset{s}{=} \begin{bmatrix} A & B_2 \\ C_O & J_{24} \end{bmatrix} \tag{2.26}$$

Abbreviating (2.24b) to

$$\mathcal{A}(s) \overset{s}{=} \begin{bmatrix} A_{\mathcal{A}} & B_{\mathcal{A}} \\ C_{\mathcal{A}} & J \end{bmatrix} \tag{2.27}$$

and invoking the standard state-space characterization of all-pass matrices yields

Lemma 2.1:

$\mathcal{A}(s)$ is all pass if there exist Hermitian matrices $P_{\mathcal{A}}$, $Q_{\mathcal{A}}$ and an orthogonal J matrix such that

$$A_{\mathcal{A}}P_{\mathcal{A}} + P_{\mathcal{A}}A^*_{\mathcal{A}} + B_{\mathcal{A}}B^*_{\mathcal{A}} = 0 \tag{2.28}$$

$$A^*_{\mathcal{A}}Q_{\mathcal{A}} + Q_{\mathcal{A}}A_{\mathcal{A}} + C^*_{\mathcal{A}}C_{\mathcal{A}} = 0 \tag{2.29}$$

$$JB^*_{\mathcal{A}} + C_{\mathcal{A}}P_{\mathcal{A}} = 0 \tag{2.30a}$$

$$J^*C_{\mathcal{A}} + B^*_{\mathcal{A}}Q_{\mathcal{A}} = 0 \tag{2.30b}$$

$$P_{\mathcal{A}}Q_{\mathcal{A}}=\gamma^2 I \tag{2.31}$$

$$JJ^* = J^*J = \gamma^2 I \tag{2.32}$$

are satisfied. $P_{\mathcal{A}}$ and $Q_{\mathcal{A}}$ defined by

$$P_{\mathcal{A}} = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} \text{and } Q_{\mathcal{A}} = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix} \tag{2.33}$$

Proof: [9,11]

The proof proceeds by direct calculation. □

In the next phase of our development, we analyse several consequences of equations (2.28) to (2.33). Firstly,

The (4,4) block of (2.32)$\Rightarrow$ $\Phi = \gamma^2 I - D_{21}D_{21}^* - D_{22}D_{22}^* = J_{42}J_{42}^*$ (2.34)

which may be solved as a Cholesky factorization problem. $J_{42}$ is nonsingular by (2.16). Also,

The (4,4) block of (2.32) $\Rightarrow$ $\Theta = \gamma^2 I - D_{12}^*D_{12} - D_{22}^*D_{22} = J_{24}^*J_{24}$ (2.35)

in which $J_{24}$ is nonsingular for the same reason. Continuing further,

(2.30a)$\Rightarrow$ $J_{42}B_O^* + D_{21}B_1^* + D_{22}B_2^* + C_2P_1 = 0$ (2.36)

which gives

$$B_O = -[B_1D_{21}^* + B_2D_{22}^* + P_1C_2^*]\, J_{42}^{-*} \tag{2.37}$$

(2.28)$\Rightarrow$ $AP_1 + P_1A^* + B_OB_O^* + B_1B_1^* + B_2B_2^* = 0$ (2.38)

Substituting (2.37) into (2.38) gives

$$(A+(B_1D_{21}^*+B_2D_{22}^*)\Phi^{-1}C_2)P_1 + P_1(A+(B_1D_{21}^*+B_2D_{22}^*)\Phi^{-1}C_2)^* + P_1C_2^*\Phi^{-1}C_2P_1+$$

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix}\left\{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}+\begin{bmatrix} D_{21}^* \\ D_{22}^* \end{bmatrix}\Phi^{-1}\begin{bmatrix} D_{21} & D_{22} \end{bmatrix}\right\}\begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} = 0 \tag{2.39}$$

which will be recognized as the Riccati equation associated with the spectral factorization problem (2.20). At this point we remind the reader that every solution to (2.39) is symmetric and negative definite as it satisfies the controllability grammian equation in (2.38). The 'biggest' solution is the minimum phase spectral factor in (2.25) [1], the minimum phase spectral factor will be required later when characterizing all solution. A dual sequence of arguments based on (2.29) and (2.30b) leads to

$$C_O= -J_{24}^{-*}(D_{12}^*C_1 + D_{22}^*C_2 + B_2^*Q_1) \tag{2.40}$$

in which $Q_1=Q_1^*<0$ satisfies

$$A^*Q_1+Q_1A+C_O^*C_O+C_1^*C_1+C_2^*C_2=0 \tag{2.41}$$

and is the observability grammian of the unstable part of $\mathcal{A}(s)$. Substituting (2.35) and (2.40) into (2.41) gives

$$(A+B_2\Theta^{-1}(D_{12}^*C_1+D_{22}^*C_2))^*Q_1 + Q_1(A+B_2\Theta^{-1}(D_{12}^*C_1+D_{22}^*C_2)) + Q_1B_2\Theta^{-1}B_2^*Q_1+$$

$$\begin{bmatrix} C_1^* & C_2^* \end{bmatrix}\left\{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}+\begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix}\Theta^{-1}\begin{bmatrix} D_{12}^* & D_{22}^* \end{bmatrix}\right\}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0 \tag{2.42}$$

which may be used in the solution of the spectral factorization problem (2.21). As with (2.39), all the solutions to (2.42) are negative definite since they have to satisfy the observability equation in (2.41); further, we require the 'biggest' solution to (2.42) which corresponds to a minimum phase spectral factor.

With the aid of Lemma 2.1, we now set about the task of constructing a suitable stable matrix to replace

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q \end{bmatrix}(s)$$

in (2.24a). Such an exchange may be undertaken freely, because at no time have we made any use of Q(s) save assuming its existence. The all-pass embedding may now be completed with the aid of a construction due to Glover [12]. To begin with, we assemble an orthogonal J matrix. Following that, $A_Q$, $B_Q$ and $C_Q$ will be constructed to satisfy the equations in Lemma 2.1.

From (2.24b) we have

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & 0 \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & D_{11} & D_{12} \\ 0 & J_{42} & D_{21} & D_{22} \end{bmatrix} \tag{2.43}$$

as the direct feed-through matrix of the error system. In order to solve the problem at infinity, we are required to replace $D_{11}$ with $\mathfrak{D}_{11}=-D_{12}(\gamma_p^2 I-D_{22}^*D_{22})^{-1}D_{22}^*D_{21}$ in which $\gamma_p$ is the Parrott lower bound given by $\gamma_p= \max\left\{\|D_{21}\ \ D_{22}\|_2\ ;\ \|D_{12}^*\ \ D_{22}^*\|_2\right\}$ [22,23,30]. This is achieved by setting $Q(\infty)=-(D_{11}+D_{12}(\gamma_p^2 I-D_{22}^*D_{22})^{-1}D_{22}^*D_{21})$. The key point is that (2.32) together with (2.43) lead to a set of equations which may always be solved to yield an orthogonal J-matrix. The proof of this claim proceeds by direct calculation from (2.32):

$$J_{32}=-(\mathfrak{D}_{11}D_{21}^*+D_{12}D_{22}^*)J_{42}^{-*} \tag{2.44a}$$

$$J_{23}=-J_{24}^{-*}(D_{12}^*\mathfrak{D}_{11}+D_{22}^*D_{21}) \tag{2.44b}$$

and also

$$J_{31}J_{31}^*+J_{32}J_{32}^*+\mathfrak{D}_{11}\mathfrak{D}_{11}^*+D_{12}D_{12}^*=\gamma^2 I \tag{2.44c}$$

(2.34), (2.44a)$\Rightarrow$

$$J_{31}J_{31}^*=\gamma^2 I-\mathfrak{D}_{11}\mathfrak{D}_{11}^*-D_{12}D_{12}^*-(\mathfrak{D}_{11}D_{21}^*+D_{12}D_{22}^*)(\gamma^2 I-D_{21}D_{21}^*-D_{22}D_{22}^*)^{-1}(\mathfrak{D}_{11}D_{21}^*+D_{12}D_{22}^*)^*$$

As a consequence of (2.16), we can write

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^* & \Xi_{22} \end{bmatrix} = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} \mathfrak{D}_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \mathfrak{D}_{11}^* & D_{21}^* \\ D_{12}^* & D_{22}^* \end{bmatrix} > 0$$

which gives

$$J_{31}J_{31}^* = (\Xi_{11}-\Xi_{12}\Xi_{22}^{-1}\Xi_{12}^*) > 0$$

by a Schur complement argument. This shows that the Cholesky factor $J_{31}$ is also nonsingular. Continuing;

$$J_{13}^*J_{13}=\gamma^2 I-D_{11}^*D_{11}-D_{21}^*D_{21}-J_{23}^*J_{23}$$

(2.35) and (2.44)$\Rightarrow$

$$J_{13}^*J_{13}=\gamma^2 I-\mathfrak{D}_{11}^*\mathfrak{D}_{11}-D_{21}^*D_{21}-(D_{12}^*\mathfrak{D}_{11}+D_{22}^*D_{21})^*(\gamma^2 I-D_{12}^*D_{12}-D_{22}^*D_{22})^{-1}(D_{12}^*\mathfrak{D}_{11}+D_{22}^*D_{21})$$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^* & \Lambda_{22} \end{bmatrix} = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} \mathfrak{D}_{11}^* & D_{21}^* \\ D_{12}^* & D_{22}^* \end{bmatrix} \begin{bmatrix} \mathfrak{D}_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} > 0$$

we find that

$$J_{13}^* J_{13} = (\Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{12}^*) > 0 \tag{2.44d}$$

and (2.44d) $\Rightarrow$ $J_{13}$ is also nonsingular. Finally $J_{11}$, $J_{12}$, $J_{21}$ and $J_{22}$ can be evaluated from:

$$J_{22} = -J_{24}^{-*}(D_{12}^* J_{32} + D_{22}^* J_{42}) \tag{2.44e}$$

$$J_{21} = -J_{24}^{-*} D_{12}^* J_{31} \tag{2.44f}$$

$$J_{11} = -J_{13}^{-*}(J_{23}^* J_{21} + \mathfrak{D}_{11}^* J_{31}) \tag{2.44g}$$

$$J_{12} = -J_{13} D_{21}^* J_{42}^{-*} \tag{2.44h}$$

in which the indicated inverses have been proved to exist.

An alternative approach is to construct the J-matrix 4 blocks at a time by abbreviating (2.43) to

$$J = \begin{bmatrix} J_1 & J_2 \\ J_3 & D \end{bmatrix} \tag{2.45a}$$

in which

$$J_1 = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},\ J_2 = \begin{bmatrix} J_{13} & 0 \\ J_{23} & J_{24} \end{bmatrix},\ J_3 = \begin{bmatrix} J_{31} & J_{32} \\ 0 & J_{42} \end{bmatrix} \text{and } D = \begin{bmatrix} \mathfrak{D}_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{2.45b}$$

Next, we solve the two Cholesky factorization problems

$$J_3 J_3^* = \gamma^2 I - DD^* \tag{2.46a}$$

and

$$J_2^* J_2 = \gamma^2 I - D^* D \tag{2.46b}$$

As a consequence of the assumption in (2.16), $J_{13}$, $J_{24}$, $J_{31}$ and $J_{42}$ are all non-singular. Also, (2.46a) and (2.46b) are completely consistent with equations (2.34) and (2.35) which have already been used in the specification of $J_{42}$ and $J_{24}$. We may now use [23,page314] to complete the construction. That is

$$J_1 = -J_2 D^* J_3^{-*} \tag{2.47}$$

Following Glover [12], we define

$$\begin{array}{cccc} p & q & m & l \\ \end{array}$$
$$[\, B_{Q1}\ B_{Q2}\ B_{Q3}\ 0\,] = R^{-*}(Q_1[0\ B_O\ B_1\ B_2]+[0\ C_O^*\ C_1^*\ C_2^*]J) \tag{2.48}$$

in which

$$R=P_1Q_1-\gamma^2 I \tag{2.49}$$

The fact that the last '$l$' columns in (2.48) are zero follows from the (4,1) block of the all-pass equation (2.30b) and is crucial to our development. In this section we assume that $\det(R)\neq 0$; $\det(R)=0$ may also arise, but we defer the treatment of this case until the next section. Similarly we define

$$\begin{array}{c} \begin{bmatrix} C_{Q1} \\ C_{Q2} \\ C_{Q3} \\ 0 \end{bmatrix} \end{array} \begin{array}{c} m \\ l \\ p \\ q \end{array} = -\begin{bmatrix} 0 \\ C_O \\ C_1 \\ C_2 \end{bmatrix} P_1 - J \begin{bmatrix} 0 \\ B_O^* \\ B_1^* \\ B_2^* \end{bmatrix} \tag{2.50}$$

The zero block at the bottom of (2.50) follows from the (4,1) block of the all-pass equation (2.30a). Next suppose that

$$A_Q=-A^*-[B_{Q1}\ B_{Q2}\ B_{Q3}\ 0][0\ B_O\ B_1\ B_2]^* \tag{2.51}$$

A routine calculation, which is essentially the same as that in [12], will verify that (2.28) and (2.29) are satisfied by

$$\begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} = \begin{bmatrix} P_1 & I \\ I & Q_1R^{-1} \end{bmatrix} \tag{2.52}$$

and

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix} = \begin{bmatrix} Q_1 & -R^* \\ -R & RP_1 \end{bmatrix} \tag{2.53}$$

Also, it is not hard to show that (2.48) to (2.53) satisfy (2.28) to (2.32), and so $\mathcal{A}(s)$ in (2.24) will satisfy $\mathcal{A}(s)\mathcal{A}^*(s)=\gamma^2 I$ as required. The (2,2) block of (2.29) gives

$$A_Q^*RP_1+RP_1A_Q+C_{Q1}^*C_{Q1}+C_{Q2}^*C_{Q2}+C_{Q3}^*C_{Q3}=0 \tag{2.54}$$

in which $P_1$ is the controllability grammian of the augmented R(s). Since R(s) is assumed to be minimal and completely unstable, $P_1<0$. If $\gamma$ is chosen large enough such that $(Q_1P_1-\gamma^2 I)<0$, it is immediate from standard inertia arguments that $\mathrm{Re}(\lambda(A_Q))\leq 0$ [11]. As we will now demonstrate, $\mathrm{Re}(\lambda(A))>0 \Rightarrow \mathrm{Re}(\lambda(A_Q))<0$. Suppose that there exists $x\neq 0$ such that $A_Qx=\lambda x$ with $\lambda+\bar{\lambda}=0$.

Then the (2,2) block of (2.29) $\Rightarrow$ $x^*(\lambda+\bar{\lambda})RP_1x+x^*C_Q^*C_Qx=0$ $\Rightarrow$ $C_Qx=0$. As a consequence the (2,1) block of (2.29) $\Rightarrow$ $x^*RA=-\bar{\lambda}x^*R$. Since $x^*R\neq 0$, $-\bar{\lambda}$ must be an eigenvalue of A. This is impossible, however, since A was assumed to have all its eigenvalues in the open right half plane. We therefore conclude that $Re(\lambda(A_Q))<0$, as required.

## 2.3 Characterization of all solutions

In the last section we proved that

$$Q(s) \stackrel{s}{=} \begin{bmatrix} A_Q & B_{Q3} \\ C_{Q3} & \mathcal{D}_{11}-D_{11} \end{bmatrix} \in \mathcal{RH}_+^{\infty,pXm} \tag{2.55}$$

is one sub-optimal solution to the four block infinity norm problem. This follows from the fact that removing the first two rows and columns of (2.24) is not norm increasing. The aim of this section is to demonstrate that it is possible to parameterize all the sub-optimal solutions in terms of a stable contraction and certain blocks of (2.24a), and this is the main result of this section. To proceed further, we will require a series of lemmas to help us in the proof of theorem 2.5. Firstly, we need to pin down the locations of the zeros of $Q_{13}(s)$ and $Q_{31}(s)$.

Lemma 2.2

(i) $z_i$ is a system zero of $X_{42}(s)$ if and only if $-\bar{z}_i$ is a system zero of $Q_{13}(s)$.

(ii) $z_i$ is a system zero of $X_{24}(s)$ if and only if $-\bar{z}_i$ is a system zero of $Q_{31}(s)$.

Proof:

The zeros of $Q_{13}(s)$ are given by the eigenvalues of $(A_Q-B_{Q3}J_{13}^{-1}C_{Q1})$. By direct calculation:

$$A_Q-B_{Q3}J_{13}^{-1}C_{Q1} = -A^*-B_{Q2}B_0^*-B_{Q3}B_1^*+B_{Q3}J_{13}^{-1}(J_{12}B_0^*+J_{13}B_1^*)$$

by (2.47) and(2.51)

$$= -A^*-B_{Q2}B_0^*+B_{Q3}J_{13}^{-1}J_{12}B_0^*$$

(2.44h) $\Rightarrow$

$$= -A^*-B_{Q2}B_0^*-B_{Q3}D_{21}^*J_{42}^{-*}B_0^*$$

$$=-A^*-(B_{Q2}J_{42}^*+B_{Q3}D_{21}^*)J_{42}^{-*}B_0^*$$

The (3,2) block of (2.30) $\Rightarrow$

$$=-A^*+C_2^*J_{42}^{-*}B_0^*$$

$$=-(A-B_0J_{42}^{-1}C_2)^* \tag{2.56}$$

Noting that the zeros of $X_{42}(s)$ are given by the eigenvalues of $(A-B_0J_{42}^{-1}C_2)$ concludes the proof of (i).

The zeros of $Q_{31}(s)$ are given by the eigenvalues of $(A_Q-B_{Q1}J_{31}^{-1}C_{Q3})$. By direct calculation from the (1,2) block of (2.29), (2.53) $\Rightarrow$

$$
\begin{aligned}
& A_Q-B_{Q1}J_{31}^{-1}C_{Q3} && =R^{-*}\left\{-A^*R^*+C_o^*C_{Q2}+C_1^*C_{Q3}-R^*B_{Q1}J_{31}^{-1}C_{Q3}\right\} \\
&(2.48) \Rightarrow && =R^{-*}\left\{-A^*R^*+C_o^*C_{Q2}+C_1^*C_{Q3}-(C_o^*J_{21}+C_1^*J_{31})J_{31}^{-1}C_{Q3}\right\} \\
&(2.44f) \Rightarrow && =R^{-*}\left\{-A^*R^*+C_o^*C_{Q2}+C_o^*J_{24}^{-*}D_{12}^*C_{Q3}\right\} \\
&(4,2) \text{ block of } (2.31) \Rightarrow && =R^{-*}\left\{-A^*R^*+C_o^*J_{24}^{-*}B_2^*R^*\right\} \\
& && =-R^{-*}(A-B_2J_{24}^{-1}C_o)^*R^* \qquad (2.57)
\end{aligned}
$$

noting that the zeros of $X_{24}(s)$ are given by the eigenvalues of $(A-B_2J_{24}^{-1}C_o)$ proves the second part.

□

As a consequence of Lemma 2.1, we are able to locate the zeros of $Q_{13}(s)$ and $Q_{31}(s)$ in the open left half plane. This follows from the fact that the solutions to (2.39) and (2.42), corresponding to minimum phase spectral factors, will locate all the zeros of $X_{24}(s)$ and $X_{42}(s)$ in the open right half plane. From now on we assume that these solutions have been selected.

Before continuing we need two standard results. The first bounds the norm of $F_u\{H,U\}(s)$ and the McMillan degree of the unstable part, while the second result pins down the location of the non- minimal modes in a general linear fractional map.

Lemma 2.3 [13,24]

Suppose that $H(s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s)$ where $H(s) \in \mathcal{H}_+^{\infty,(p+m)\mathrm{x}(m+p)}(k)$ and consider a $U(s) \in \mathcal{H}_+^{\infty,m\mathrm{x}p}(\ell)$ such that $\| H_{11}U \|_\infty < 1$ then

(a) $\|H\|_\infty \le \gamma$ and $\| U \|_\infty \le \gamma^{-1} \Rightarrow \| F_u(H,U)\|_\infty \le \gamma$

(b) $F_u(H,U) \in \mathcal{H}_+^{\infty,m\mathrm{x}p}(k+\ell)$ □

Proof:

(a) We begin by assuming that H(s) is all-pass, $\|H(s)\|_\infty = \gamma$, using this property it is not hard to verify that

$$
\begin{aligned}
\gamma^2 I - \left\{F_u(H,U)\right\}^*\left\{F_u(H,U)\right\} &= H_{12}^*(I-U^*H_{11}^*)^{-1}\{I-U^*U\gamma^2\}(I-H_{11}U)^{-1}H_{12} \qquad (2.58) \\
&\ge 0 \qquad \text{if } \|U(s)\|_\infty \le \gamma^{-1}.
\end{aligned}
$$

Next we consider $\|H(s)\|_\infty \le \gamma$, then it is always possible to embed H(s) in a larger matrix such that

$$H_a(s) = \begin{bmatrix} H_{00} & H_{01} & H_{02} \\ H_{10} & H_{11} & H_{12} \\ H_{20} & H_{21} & H_{22} \end{bmatrix}(s) = \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & H_{22} \end{bmatrix}(s) \tag{2.59}$$

where $H_a(s)$ is all-pass and $||H_a(s)||_\infty = \gamma$; then as in the previous case, it is easily verified that

$$\begin{aligned} \gamma^2 I - \left\{F_u(H_a,\bar{U})\right\}^* \left\{F_u(H_a,\bar{U})\right\} &= \bar{H}_{12}^*(I-\bar{U}^*\bar{H}_{11}^*)^{-1}\{I-\bar{U}^*\bar{U}\gamma^2\}(I-\bar{H}_{11}\bar{U})^{-1}\bar{H}_{12} \\ &\geq 0 \qquad \text{if } ||U(s)||_\infty \leq \gamma^{-1} \text{ and } \bar{U}(s) = \begin{bmatrix} 0 & 0 \\ 0 & U \end{bmatrix}(s). \end{aligned} \tag{2.60}$$

This completes the proof of the first part.

(b) The proof of this part is obtained by considering the poles of $F_u\left\{H,\alpha U\right\}$ where $\alpha \in [0,1]$. See Glover et al [13] for the proof.

□

Lemma 2.4 [17,19]

Suppose that

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2.61}$$

in which $H_{12}(s) \in \mathbb{R}^{p_1 X m_2}$ with $m_2 \geq p_1$ and $H_{21}(s) \in \mathbb{R}^{p_2 X m_1}$ with $p_2 \geq m_1$. Suppose also that

$$U(s) \overset{s}{=} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \tag{2.62}$$

is a minimal realization and that the well posedness condition $\det(I-D_{11}\hat{D}) \neq 0$ is satisfied. Then in the closed loop realization, $F_u(H,U)(s)$,

(a) every uncontrollable mode is a Smith zero of

$$\begin{bmatrix} sI-A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \tag{2.63}$$

(b) every unobservable mode is a Smith zero of

$$\begin{bmatrix} sI-A & -B_1 \\ C_2 & D_{21} \end{bmatrix} \tag{2.64}$$

□

Proof:

a) Suppose that $s_o$ is an uncontrollable mode of the closed loop realization of $F_u(H,U)(s)$, then there exists $[\omega_1\ \omega_2]\neq 0$ such that

$$[\omega_1\ \omega_2]\begin{bmatrix} s_oI-A-B_1\hat{D}MC_1 & -B_1[I+\hat{D}MD_{11}]\hat{C} & -B_1\hat{D}MD_{12}-B_2 \\ -\hat{B}MC_1 & s_oI-\hat{A}-\hat{B}MD_{11}\hat{C} & -\hat{B}MD_{12} \end{bmatrix}=0 \tag{2.65}$$

where

$$M = (I-D_{11}\hat{D})^{-1}$$

define

$$z_2=-\omega_1B_1\hat{D}M-\omega_2\hat{B}M \tag{2.66}$$

$$\Rightarrow \qquad [\omega_1\ z_2]\begin{bmatrix} s_oI-A & -B_2 \\ C_1 & D_{12} \end{bmatrix}=0 \tag{2.67}$$

It remains to verify that $[\omega_1\ \omega_2]\neq 0 \Rightarrow [\omega_1\ z_2]\neq 0$. This we we do by supposing that $[\omega_1\ z_2]=0$

$$(2.66) \Rightarrow \qquad -\omega_2\hat{B}=0 \tag{2.68}$$

$$(2.65) \Rightarrow \qquad \omega_2(s_oI-\hat{A})=0 \tag{2.69}$$

which contradicts the minimality assumption of U(s) in (2.62), hence $[\omega_1\ z_2]\neq 0$, this completes the proof of part (a).

The proof to part (b) proceeds in a similar way with a dual set of arguments.

□

We are now ready to show that

$$\bar{Q}(s) = F_u\left\{\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix},U\right\}(s)=\left\{Q+Q_{31}U(I-Q_{11}U)^{-1}Q_{13}\right\}(s) \qquad \gamma U(s)\in\mathcal{BH}_+^{\infty,p\times m} \tag{2.70}$$

captures all the sub-optimal solutions to the infinity norm problem.

Theorem 2.5

The linear fractional map given in (2.70) generates every $Q(s)\in\mathcal{RH}_+^{\infty,p\times m}$ which satisfies (2.15).

Proof:

We begin by showing that (2.70) generates a class of solutions. Since $\det(Q_{13}(j\omega))\neq 0$ and

$\det(Q_{31}(j\omega))\neq 0$ for all real $\omega$ by lemma 2.2,

$\|Q_{11}\|_\infty<\gamma$ and $\|Q_{11}U\|_\infty<1$

$$\Rightarrow \|F_u(\mathcal{A}(s),\begin{bmatrix} U(s) & 0 \\ 0 & 0 \end{bmatrix})\|_\infty = \|\begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}+\begin{bmatrix} Q_{31} & X_{32}+Q_{32} \\ 0 & X_{42} \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\left\{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}-\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & X_{22}+Q_{22} \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}\right\}^{-1}\begin{bmatrix} Q_{13} & 0 \\ X_{23}+Q_{23} & X_{24} \end{bmatrix}\|_\infty$$

$$=\|\begin{bmatrix} R_{11}+\bar{Q} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\|_\infty\leq\gamma$$

in which $\bar{Q}(s)=(Q+Q_{31}U(I-Q_{11}U)^{-1}Q_{13})(s)$, for any stable U(s) such that $\|U\|_\infty\leq\frac{1}{\gamma}$ and $\mathcal{A}(s)$ given in (2.24). This follows from Lemma 2.3(a).

Since $U(s)\in\mathcal{H}_+^\infty$ and $\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix}(s)\in\mathcal{RH}_+^\infty$, $\bar{Q}(s)$ is in $\mathcal{H}_+^\infty$ also (by lemma 2.3(b)).

Establishing that (2.70) captures all the solutions is a little more tricky. Suppose that

$$\|\begin{bmatrix} R_{11}+\tilde{Q} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\|_\infty\leq\gamma \tag{2.71}$$

for some $\tilde{Q}(s)\in\mathcal{RH}_+^{\infty,pXm}$ we need to show that $\tilde{Q}(s)$ is generated via the linear fractional map $F_u\{\mathcal{A}(s),U(s)\}$ in which U(s) is stable such that $\|U\|_\infty\leq\gamma$ and furthermore, U(s) has to have structure $\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$. Next consider that $\mathcal{A}(s)$ in (2.24) is partitioned as

$$\left[\begin{array}{cc|cc} Q_{11} & Q_{12} & Q_{13} & 0 \\ Q_{21} & X_{22}+Q_{22} & X_{23}+Q_{23} & X_{24} \\ \hline Q_{31} & X_{32}+Q_{32} & R_{11}+Q & R_{12} \\ 0 & X_{42} & R_{21} & R_{22} \end{array}\right](s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s) \tag{2.72}$$

in which

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^*(s)\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s) =\gamma^2 I \tag{2.73}$$

$$\widetilde{U}(s)= F_u\left\{\gamma^{-2}\begin{bmatrix} H_{22}^* & H_{12}^* \\ H_{21}^* & H_{11}^* \end{bmatrix}(s), Z(s)\right\}, \tag{2.75}$$

it is easily verified (see Appendix A) using (2.73) that

$$Z(s)=F_u\left\{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s), \widetilde{U}(s)\right\} \tag{2.76}$$

Since $||Z||_\infty \leq \gamma$ by assumption, and $||\gamma^{-2}H_{22}^*||_\infty < \frac{1}{\gamma}$, the map given in (2.75) is well posed and further $||\widetilde{U}||_\infty \leq \frac{1}{\gamma}$ (by lemma 2.3). It now remains for us to show that $\widetilde{U}(s) \in \mathcal{RH}_+^\infty$ and is of the form $\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$. The bordering zeros allow one to ignore the second row and column of $\mathcal{A}(s)$ in (2.24) or alternatively (2.72). From (2.71) and (2.73) we obtain

$$H_{22}+\Delta\bar{Q} \ = H_{22}+H_{21}\widetilde{U}(I-H_{11}\widetilde{U})^{-1}H_{12} \tag{2.77}$$

in which

$$\Delta\bar{Q} = \begin{bmatrix} \Delta Q & 0 \\ 0 & 0 \end{bmatrix}$$

This gives

$$\widetilde{U} = H_{21}^{-1}\Delta\bar{Q}(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})^{-1} \tag{2.78}$$

Since

$$H_{21}^{-1}\Delta\bar{Q} = \begin{bmatrix} Q_{31}^{-1}\Delta Q & 0 \\ 0 & 0 \end{bmatrix} \tag{2.79}$$

and

$$H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q} \ = \begin{bmatrix} Q_{11}Q_{31}^{-1}\Delta Q+Q_{13} & 0 \\ * & X_{24} \end{bmatrix} \tag{2.80}$$

it is clear that

$$H_{21}^{-1}\Delta\bar{Q}(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})^{-1} = \begin{bmatrix} Q_{31}^{-1}\Delta Q(Q_{11}Q_{31}^{-1}\Delta Q+Q_{13})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.81}$$

establishes that $\widetilde{U}(s)$ has the required form.

The fact that $\widetilde{U}(s) \in \mathcal{RH}_+^\infty$ follows from a simple cancellation argument which is based on lemma 2.4. If $\widetilde{U}(s)$ had a right half plane pole,

$$\widetilde{Q}(s)=F_u\left\{\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix},\widetilde{U}_{11}\right\}(s) \tag{2.82}$$

would have a right half plane eigenvalue in its A-matrix (by lemma 2.3(b)). Since $\widetilde{Q}(s)\in\mathcal{RH}_+^\infty$ by assumption, a right half plane cancellation would have to occur. This is impossible, however, because all the system zeros of $Q_{13}(s)$ and $Q_{31}(s)$ are in the open left half plane and these are the only possible cancellation locations by lemma 2.4. The proof is complete. □

## 2.4. A characterization of all optimal solutions.

In this section it is our purpose to lift the assumption of sub-optimality which has been made so far. We begin our discussion with a simple example which illustrates several features of the general case. If

$$\begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s)=\begin{bmatrix} r_1+q & 0 \\ 0 & r_2 \end{bmatrix}(s) \tag{2.83}$$

where $R(s)\in\mathcal{RH}_-^\infty$ and $Q(s)\in\mathcal{RH}_+^\infty$, then there are two possible situations which require separate consideration. These are

$$(1) \qquad \left|\Gamma_{r_1}\right| < \|r_2\|_\infty \tag{2.84a}$$

and

$$(2) \qquad \left|\Gamma_{r_1}\right| \geq \|r_2\|_\infty \tag{2.84b}$$

In the first case, it is clear that $\gamma_o=\|r_2\|_\infty$ and any $q(s)\in\mathcal{H}_+^\infty$ which satisfies $\|r_1+q\|_\infty\leq\gamma_o$ will be an optimal solution; a continuum of such solutions exist. In the second case, however, the (unique) optimal Nehari extension of $r_1(s)$ is the only solution and $\gamma_o=\left|\Gamma_{r_1}\right|$.

In the general case there are also two possible forms of optimality. To see this, we note that

$$\gamma_o \geq \mu=\max\left\{ \left\|\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}\right\|_\infty, \left\|\begin{bmatrix} R_{21} & R_{22} \end{bmatrix}\right\|_\infty \right\} \tag{2.85}$$

by a result due to Parrott [22,23,Theorem 2.1,30 ]. If Q(s) were free to range over $\mathcal{L}^\infty$ (rather than just $\mathcal{H}_+^\infty$), one could always attain the lower bound of $\mu$. Since Q(s) is constrained to be an $\mathcal{H}_+^\infty$ function, however, $\gamma_o\geq\mu$ is clearly possible.

### 2.4.1 Optimality of the Parrott type.

We will take optimality of the Parrott type to mean $\gamma_o=\mu$, and we will also assume in this sub-section that $\lambda_{max}(P_1Q_1)<\gamma_o^2$; this second condition simply allows us to postpone the treatment of $R=P_1Q_1-\gamma_o^2I$ being singular until a little later.

The essential difficulty with optimality of the Parrott type are axis phenomena. Firstly, it becomes necessary to use special algorithms to solve the spectral factorization problems associated with (2.39) and (2.42). The presence of imaginary axis eigenvalues in the spectrum (spectra) of the Hamiltonean matrices associated with these equations will cause standard eigenvalue sorting algorithms to fail [4]. As a direct consequence of Lemma 2.2, $Q_{13}(s)$ and/or $Q_{31}(s)$ in (2.70) may also have zeros which lie on the imaginary axis. This follows from the fact that the hamiltonian eigenvalues are the zeros of the spectral factors. In such cases, the linear fractional map in (2.70) may be ill-posed for certain $\gamma U(s)$'s$\in\mathcal{BH}_+^\infty$, H(s) is all-pass and $Q_{13}(s)$ and/or $Q_{31}(s)$ have axis zeros therefore $||H_{11}(s)||_\infty=\gamma$. Further, the proof that (2.70) captures <u>all</u> the solutions breaks down; the reader will recall that we assumed that the system zeros of $Q_{13}(s)$ and $Q_{31}(s)$ lie in the <u>open</u> left half plane. One might consider a U(s) no longer in $\mathcal{H}_+^\infty$ which has imaginary axis poles, and that the map in (2.70) is still stable because of axis cancellations. To avoid these degeneracies, one should regard $\mu$ as an infimum which need not necessarily be achievable.

In the next example, we show that optimality of the Parrott type need not necessarily lead to troublesome axis phenomena although axis eigenvalues are present in the Hamiltonian matrix associated with the spectral factorization. In this case the axis system zeros of $Q_{13}(s)$ and $Q_{31}(s)$ may be removed with a non-minimal mode in the state-space realization of $\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix}(s)$.

<u>Example 4.1</u>

Suppose
$$R(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix};\ A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ C=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $||\Gamma||_\infty=\frac{1}{2}$; $\gamma_o = ||\frac{1}{s-1}||_\infty = 1$ and direct substitution gives

$$E_Q=\begin{bmatrix} -\frac{3}{4} & 0 \\ 0 & 0 \end{bmatrix};\ A_Q=\begin{bmatrix} \frac{5}{4} & 0 \\ 0 & 0 \end{bmatrix};\ B_Q=\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_Q=\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } J=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

removing the non-minimal mode and reverting to standard state-space yields:

$$\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix}(s) \stackrel{s}{=} \begin{bmatrix} -\frac{5}{3} & \frac{4}{3} & -\frac{2}{3} \\ 1 & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

$F_u\left\{\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix}, U \in \mathcal{B}\mathcal{H}_+^\infty\right\}$ generates all the extensions of $R_{11}(s)=\frac{1}{s-1}$ which have infinity norm $\leq 1$.

The system zeros of both $Q_{13}(s)$ and $Q_{31}(s)$ are at -1. The purpose of working in a generalized state space framework will become apparent in the next section. □

2.4.2 Optimality due to inertia

It follows from standard inertia arguments [11], that $\gamma_0^2 = \lambda_{max}(P_1Q_1) \geq \mu^2$ is the least $\gamma$ for which $Q(s) \in \mathcal{H}_+^\infty$. Unlike the one block problem, $\gamma_0$ has to be found by iteration. The so-called $\gamma$-iteration procedure which achieves this, is described in several places [9,18,26,31] to mention a few. At this limiting value of $\gamma$, the R-matrix which is given in (2.49) will be singular, and as a consequence $[\ B_{Q1}\ B_{Q2}\ B_{Q3}\ 0\ ]$ as represented in (2.48) becomes undefined. To remedy this, we simply switch to a generalized state-space representation thereby removing both the potentially troublesome $R^{-*}$ terms [27]. This gives

$$R^* \dot{x} = R^* A_Q\, x + R^* [\ B_{Q1}\ B_{Q2}\ B_{Q3}\ 0]\ u \tag{2.86}$$

$$y = \begin{bmatrix} C_{Q1} \\ C_{Q2} \\ C_{Q3} \\ 0 \end{bmatrix} x + J\ u \tag{2.87}$$

which we denote by

$$\mathcal{Q}(s) = \left[\begin{array}{c|ccc} Q_{11} & Q_{12} & Q_{13} & 0 \\ \hline Q_{21} & Q_{22} & Q_{23} & 0 \\ Q_{31} & Q_{32} & Q & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \stackrel{s}{=} \begin{bmatrix} R^* & \widetilde{A}_Q & \widetilde{B}_Q \\ 0 & \widetilde{C}_Q & J \end{bmatrix} \tag{2.88}$$

where

$$J = \begin{array}{c} \\ m \\ l+p+q \end{array}\!\!\begin{array}{c} \begin{array}{cc} p & l+m+q \end{array} \\ \begin{bmatrix} J_{11} & \widetilde{J}_{12} \\ \widetilde{J}_{21} & \widetilde{J}_{22} \end{bmatrix} \end{array} = \left[\begin{array}{c|ccc} J_{11} & J_{12} & J_{13} & 0 \\ \hline J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & \mathcal{D}_{11} & D_{12} \\ 0 & J_{42} & D_{21} & D_{22} \end{array}\right] \tag{2.89}$$

$$\widetilde{B}_Q = \begin{array}{c} \\ \end{array}[\ \overset{p}{\widetilde{B}_{Q1}}\ \overset{l+m+q}{\widetilde{B}_{Q2}}\ ] = Q_1\,[\ 0 \mid B_0\ B_1\ B_2\ ] + [\ 0 \mid C_0^*\ C_1^*\ C_2^*\ ]\begin{bmatrix} J_{11} & \widetilde{J}_{12} \\ \widetilde{J}_{21} & \widetilde{J}_{22} \end{bmatrix} \tag{2.90}$$

$$\widetilde{C}_Q = \begin{array}{c} m \\ l+p+q \end{array}\begin{bmatrix} \widetilde{C}_{Q1} \\ \widetilde{C}_{Q2} \end{bmatrix} = -\begin{bmatrix} 0 \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} P_1 - \begin{bmatrix} J_{11} & \widetilde{J}_{12} \\ \widetilde{J}_{21} & \widetilde{J}_{22} \end{bmatrix}\begin{bmatrix} 0 \\ B_0^* \\ B_1^* \\ B_2^* \end{bmatrix} \tag{2.91}$$

$$\widetilde{A}_Q = \gamma_0^2 A^* + Q_1 A P_1 - [\ C_0^*\ C_1^*\ C_2^*\ ]\,\widetilde{J}_{22}\begin{bmatrix} B_0^* \\ B_1^* \\ B_2^* \end{bmatrix} \tag{2.92}$$

Next, we find orthogonal matrices U and V such that $U^*R^*V=\begin{bmatrix} \bar{R} & 0 \\ 0 & 0 \end{bmatrix}$ in which $\bar{R}$ is nonsingular. This gives

$$Q(s) \overset{s}{=} \begin{bmatrix} U^*R^*V & U^*\widetilde{A}_Q V & U^*\widetilde{B}_Q \\ 0 & \widetilde{C}_Q V & J \end{bmatrix} \tag{2.93}$$

which if conformably partitioned with $U^*R^*V=\begin{bmatrix} \bar{R} & 0 \\ 0 & 0 \end{bmatrix}$ yields

$$Q(s) \overset{s}{=} \begin{bmatrix} \bar{R} & 0 & \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ 0 & 0 & \bar{A}_{21} & \bar{A}_{22} & \bar{B}_2 \\ 0 & 0 & \bar{C}_1 & \bar{C}_2 & J \end{bmatrix} \tag{2.94}$$

Under the assumption that $\bar{A}_{22}^{-1}$ exists, one obtains

$$Q(s) \overset{s}{=} \begin{bmatrix} \bar{R}^{-1}(\bar{A}_{11}-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}) & \bar{R}^{-1}(\bar{B}_1-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2) \\ \bar{C}_1-\bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21} & J-\bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2 \end{bmatrix} \tag{2.95}$$

by standard singular perturbation methods[3] [27]. The case of $\bar{A}_{22}$ singular is a little more delicate but may be treated via generalized inverses [27].

(2.95) is a standard state space model for Q(s), and is the basis of a representation formula for all solutions. What is not clear, and this is the subject of the next section, is the way in which the <u>effective</u> dimension of U(s) in (2.70) reduces at optimality. The next example illustrates this phenomenon.

Example 4.2

Take

$$R(s)=\begin{bmatrix}\frac{2}{s-1} & 0\\ 0 & \frac{1}{s-1}\end{bmatrix}$$

Clearly $\gamma_0=\|\frac{1}{s-1}\|_\infty = \|\Gamma_{2/(s-1)}\| = 1$. By direct computation, the descriptor form of the representation formula (2.70) is given by:

$$E_Q=\begin{bmatrix}0&0\\0&0\end{bmatrix};\ A_Q=\begin{bmatrix}2&0\\0&0\end{bmatrix};\ B_Q=\begin{bmatrix}1&0&-1&0\\0&0&0&0\end{bmatrix}$$

$$C_Q=\begin{bmatrix}-2&0\\0&0\\2&0\\0&0\end{bmatrix}\text{ and } J=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

removing the non-minimal mode

$$\begin{bmatrix}Q_{11}\ Q_{13}\\ Q_{31}\ \ Q\end{bmatrix}\overset{s}{=}\begin{bmatrix}0&2&-\sqrt{2}&\sqrt{2}\\0&\sqrt{2}&0&1\\0&-\sqrt{2}&1&0\end{bmatrix}$$

---

[3] As we will demonstrate, the process of calculating a singular perturbation is an operation of strict system equivalence [25, pg.53]. As a consequence, we will preserve the transfer function and all the finite system poles and zeros; a non-minimal mode(s) at infinity will be removed with the deletion of the row and column corresponding to $A_{22}$. In the case that $A_{22}^{-1}$ exists we may write.

$$\left[\begin{array}{cc|cc}I&-A_{12}A_{22}^{-1}&0&0\\0&I&0&0\\\hline 0&C_{12}A_{22}^{-1}&I&0\\0&C_{22}A_{22}^{-1}&0&I\end{array}\right]\left[\begin{array}{cc|cc}sE-A_{11}&-A_{12}&-B_{11}&-B_{12}\\-A_{21}&-A_{22}&-B_{21}&-B_{22}\\\hline C_{11}&C_{12}&D_{11}&D_{12}\\C_{21}&C_{22}&D_{21}&D_{22}\end{array}\right]\left[\begin{array}{cc|cc}I&0&0&0\\-A_{22}^{-1}A_{21}&I&-A_{22}^{-1}B_{21}&-A_{22}^{-1}B_{22}\\\hline 0&0&I&0\\0&0&0&I\end{array}\right]$$

$$=\left[\begin{array}{cc|cc}sE-A_{11}+A_{12}A_{22}^{-1}A_{21}&0&-B_{11}+A_{12}A_{22}^{-1}B_{21}&-B_{12}+A_{12}A_{22}^{-1}\\0&-A_{22}&0&0\\\hline C_{11}-C_{12}A_{22}^{-1}A_{21}&0&D_{11}-C_{12}A_{22}^{-1}B_{21}&D_{12}-C_{12}A_{22}^{-1}B_{22}\\C_{21}-C_{22}A_{22}^{-1}A_{21}&0&D_{21}-C_{22}A_{22}^{-1}B_{21}&D_{22}-C_{22}A_{22}^{-1}B_{22}\end{array}\right]$$

After a singular perturbation which removes the infinite pole we obtain the following constant matrix.

$$\begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $Q_{13}=0$ and $Q_{31}=0$ , $Q=1$ is the only solution. □

### 2.4.3 Reduction in the effective dimension of the free parameter (at optimality due to inertia)

In this section we study the reduction in effective dimension of the free parameter U(s) in the representation formula (2.70), which occurs with optimality due to inertia. In the case of one block problems, it is already well known that the effective dimension of the free parameter reduces at $\gamma=\gamma_o$. For classical scalar extension problems, this dimension reduction makes the difference between an infinite number of suboptimal solutions parameterized by a scalar u(s), and a unique optimal solution.

As it turns out, a detailed study of optimality in the four block case is significantly simplified by the assumption $R(\infty)=0$. Since the same basic method of approach will solve the general $R(\infty)\neq 0$ case, we will study the simple case first, and defer the more complex analysis of the $R(\infty)\neq 0$ case to the next subsection.

*Case1 $R(\infty)=0$*

If $\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}=0$, it is easy to show that the J-matrix in (2.89) is given by

$$J = \begin{bmatrix} 0 & 0 & \gamma I & 0 \\ 0 & 0 & 0 & \gamma I \\ \gamma I & 0 & 0 & 0 \\ 0 & \gamma I & 0 & 0 \end{bmatrix} \tag{2.96}$$

Simple computations based on (2.96), and (2.90) to (2.92) reveal that

$$\tilde{B}_Q = [\ \gamma C_1^* \mid Q_1B_o+\gamma C_2^* \mid Q_1B_1 \mid (Q_1B_2+\gamma C_o^*)=0\ ] \tag{2.97}$$

$$\tilde{C}_Q = -\begin{bmatrix} \gamma B_1^* \\ C_oP_1+\gamma B_2^* \\ C_1P_1 \\ (C_2P_1+\gamma B_o^*)=0 \end{bmatrix} \tag{2.98}$$

$$\widetilde{A}_Q = \gamma^2 A^* + Q_1 A P_1 - \gamma C_2^* B_0^* - \gamma C_0^* B_2^* \tag{2.99}$$

and finally

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q \end{bmatrix} = \left[\begin{array}{cc|ccc} R^* & \gamma^2 A^* + Q_1 A P_1 - \gamma C_2^* B_0^* - \gamma C_0^* B_2^* & \gamma C_1^* & Q_1 B_0 + \gamma C_2^* & Q_1 B_1 \\ \hline 0 & -\gamma B_1^* & 0 & 0 & \gamma I \\ 0 & -C_0 P_1 - \gamma B_2^* & 0 & 0 & 0 \\ 0 & -C_1 P_1 & \gamma I & 0 & 0 \end{array}\right] \tag{2.100}$$

To continue further, we select a basis in the state-space of (2.17) so that $P_1$ and $Q_1$ are given by :

$$P_1 = \begin{bmatrix} P & 0 \\ 0 & -\gamma_0 \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} Q & 0 \\ 0 & -\gamma_0 \end{bmatrix} \tag{2.101}$$

This is always possible by standard balancing arguments. To simplify the presentation, we have decided to assume that $\lambda_{max}(P_1 Q_1)$ has multiplicity one. In these new co-ordinates, we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & -\gamma_0 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & -\gamma_0 \end{bmatrix} \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \begin{bmatrix} \bar{B}_1^* & \bar{B}_2^* \end{bmatrix} = 0 \tag{2.102}$$

in which

$$[\, 0 \ B_0 \ B_1 \ B_2 \,] = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & B_{11} & B_{12} & B_{13} \\ 0 & B_{21} & B_{22} & B_{23} \end{bmatrix} \tag{2.103}$$

and

$$\begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & -\gamma_0 \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & -\gamma_0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} \bar{C}_1^* \\ \bar{C}_2^* \end{bmatrix} \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} = 0 \tag{2.104}$$

where

$$\begin{bmatrix} 0 \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = [\, \bar{C}_1 \ \bar{C}_2 \,] = \begin{bmatrix} 0 & 0 \\ C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \tag{2.105}$$

From the (2,2) blocks of (2.102) and (2.104) we see that

$$\bar{B}_2\bar{B}_2^* = \bar{C}_2^*\bar{C}_2 \tag{2.106}$$

The (1,4) block of (2.97) becomes

$$\begin{bmatrix} Q & 0 \\ 0 & -\gamma_o \end{bmatrix}\begin{bmatrix} B_{13} \\ B_{23} \end{bmatrix} + \gamma_o\begin{bmatrix} C_{11}^* \\ C_{12}^* \end{bmatrix} = 0 \tag{2.107}$$

which shows that

$$B_{23} = C_{12}^* \tag{2.108}$$

In the same way, the (4,1) block of (2.98) gives

$$B_{21}^* = C_{32} \tag{2.109}$$

Substituting (2.108) and (2.109) into (2.106) yields

$$B_{22}B_{22}^* = C_{22}^*C_{22} \tag{2.110}$$

which shows there exists an orthogonal U such that

$$UB_{22}^* = C_{22} \tag{2.111}$$

(2.111) may be regarded as a counterpart to [11, eqn(6.23)].

Since $\gamma_o = \lambda_{max}^{1/2}(P_1Q_1)$, (2.102), (2104), (2.108) and (2.109) verify that a generalized state-space model for the transfer function in (2.100) is given by

$$\begin{bmatrix} QP-\gamma_o^2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \gamma_o^2A_{11}^*+QA_{11}P-\gamma_oC_{31}^*B_{11}^*-\gamma_oC_{11}^*B_{13}^* & \gamma_oC_{21}^*C_{22} \\ \gamma_oB_{22}B_{12}^* & \gamma_oB_{22}B_{22}^* \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+\begin{bmatrix} \gamma_oC_{21}^* & QB_{11}+\gamma C_{31}^* & QB_{12} \\ \gamma_oC_{22}^* & -\gamma B_{21}+\gamma C_{32}^* & -\gamma_oB_{22} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{2.112}$$

(2.113)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\gamma_oB_{12}^* & -\gamma_oB_{22}^* \\ -C_{11}P-\gamma B_{13}^* & \gamma C_{12}-\gamma B_{23}^* \\ -C_{21}P & \gamma_oC_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \gamma_oI \\ 0 & 0 & 0 \\ \gamma_oI & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

In the notation of (2.103) and (2.105). The second row of (2.112) gives

$$x_2 = -(\gamma_o B_{22}B_{22}^*)^{-1}\left\{\gamma_o B_{22}B_{12}^* x_1 + \gamma_o C_{22}^* u_1 - \gamma_o B_{22}u_2\right\} \tag{2.114}$$

If we denote the reduced order model corresponding to (2.100) (ie after removal of $x_2$) by

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q \end{bmatrix} = \left[\begin{array}{cc|ccc} \bar{R} & A_p & B_{1p} & B_{2p} & B_{3p} \\ \hline 0 & C_{1p} & D_{11p} & D_{12p} & D_{13p} \\ 0 & C_{2p} & D_{21p} & D_{22p} & D_{23p} \\ 0 & C_{3p} & D_{31p} & D_{32p} & D_{33p} \end{array}\right] \tag{2.115}$$

then

$$\begin{aligned} B_{1p} &= \gamma_o C_{21}^* - \gamma_o C_{21}^* C_{22}(\gamma_o B_{22}B_{22}^*)^{-1}\gamma_o C_{22}^* \\ &= \gamma_o(B_{22}B_{22}^*)^{-1}C_{21}^*\left\{C_{22}^*C_{22}I_p - C_{22}C_{22}^*\right\} \end{aligned} \tag{2.116}$$

by (2.110), (2.112) and (2.114). In the same way

$$\begin{aligned} D_{21p} &= \gamma_o I - \gamma_o C_{22}(\gamma_o B_{22}B_{22}^*)^{-1}\gamma_o C_{22}^* \\ &= \gamma_o(B_{22}B_{22}^*)^{-1}\left\{C_{22}^*C_{22}I_p - C_{22}C_{22}^*\right\} \end{aligned} \tag{2.117}$$

which when combined with (2.116) becomes

$$\begin{bmatrix} B_{1p} \\ D_{21p} \end{bmatrix} = \gamma_o(B_{22}B_{22}^*)^{-1}\begin{bmatrix} C_{21}^* \\ I \end{bmatrix}\left\{C_{22}^*C_{22}I_p - C_{22}C_{22}^*\right\} \tag{2.118}$$

Substituting (2.114) into (2.113) gives

$$[C_{1p}\ D_{12p}] = \left\{B_{22}^*B_{22} - I_m B_{22}B_{22}^*\right\}[B_{12}^* \ \ -I]\gamma_o(B_{22}B_{22}^*)^{-1} \tag{2.119}$$

and finally we can also show that

$$D_{11p} = \gamma_o B_{22}^*(B_{22}B_{22}^*)^{-1}C_{22}^* \tag{2.120}$$

It is clear that (2.118) and (2.119) are rank deficient[4]. In order to make the reduction in the effective dimension explicit, we suppose that $B_{22}$ and $C_{22}$ have been scaled to unit length, and then define the orthogonal matrices

[4] $C_{22}$ and $B_{22}$ respectively verify this claim.

$$T_l := \begin{bmatrix} B_{22} \\ \{B_{22}\}_{\perp} \end{bmatrix} \text{and } T_r := \begin{bmatrix} C_{22} & \{C_{22}\}_{\perp} \end{bmatrix} \tag{2.121}$$

where $\{\cdot\}_{\perp}$ denotes the orthogonal completion of $\{\cdot\}$. With the aid of (2.118), (2.119), and (2.120), we see that

$$\begin{bmatrix} T_l & 0 \\ 0 & I_{l+p} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q \end{bmatrix}(s) \begin{bmatrix} T_r & 0 \\ 0 & I_{q+m} \end{bmatrix} = \begin{bmatrix} \gamma_o & 0\cdots0 & 0\cdots0 & 0\cdots0 \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{21} & Q_{22} & Q_{23} \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{31} & Q_{32} & Q \end{bmatrix} \tag{2.122a}$$

or

$$\begin{bmatrix} T_l & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q \end{bmatrix}(s) \begin{bmatrix} T_r & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} \gamma_o & 0\cdots0 & 0\cdots0 \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{11} & \hat{Q}_{13} \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{31} & Q \end{bmatrix} \tag{2.122b}$$

and note also that (2.70) may be written as

$$T_l \bar{Q} T_r = F_u\left\{ \begin{bmatrix} T_l Q_{11} T_r & T_l Q_{13} \\ Q_{31} T_r & Q \end{bmatrix}, T_r^* U T_l^* = \bar{U} \right\} \tag{2.123}$$

Next, we substitute the right hand side of (2.122b) into (2.123) and then describe this linear fractional map by the system of equations

$$y_1 = \gamma_o u_1 \tag{2.124a}$$

$$y_2 = \hat{Q}_{11} u_2 + \hat{Q}_{13} u_3 \tag{2.124b}$$

$$y_3 = \hat{Q}_{31} u_2 + Q u_3 \tag{2.124c}$$

and

$$u_1 = \bar{u}_{11} y_1 + \bar{u}_{12} y_2 \tag{2.125a}$$

$$u_2 = \bar{u}_{21} y_1 + \bar{U}_{22} y_2 \tag{2.125b}$$

where

$$\bar{U} = \begin{bmatrix} \bar{u}_{11} & \bar{u}_{12} \\ \bar{u}_{21} & \bar{U}_{22} \end{bmatrix} \tag{2.126}$$

The partitioning in (2.126) is conformable with that on the right hand side of (2.122). It is now easy to eliminate $u_1$ from (2.124a) and (2.125) to obtain

$$u_2 = [\bar{U}_{22} + \bar{u}_{21}\gamma_o(1-\bar{u}_{11}\gamma_o)^{-1}\bar{u}_{12}]\, y_2 \tag{2.127}$$

(2.124b), (2.124c) and (2.127) may be combined as

$$\bar{Q} = F_u\left\{ \begin{bmatrix} \gamma_o & 0\cdots 0 & 0\cdots 0 \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{11} & \hat{Q}_{13} \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{31} & Q \end{bmatrix}, \begin{bmatrix} 1/\gamma_o & 0\cdots 0 \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & F_u(\bar{U},\, \gamma_o) \end{bmatrix} \right\}$$

Removing the redundant equation in $u_1$ and $y_1$ yields

$$= F_u\left\{ \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{13} \\ \hat{Q}_{31} & Q \end{bmatrix}, F_u\{\bar{U},\, \gamma_o\} \right\} \tag{2.128}$$

and we see that (2.126) may always be replaced by $F_u\{\bar{U},\gamma_o\}$ which is a contraction of lower dimension. Further, $T_r^* U T_l^* = \bar{U}$ may be written out in full as

$$U(s) = \begin{bmatrix} C_{22} & \{C_{22}\}_\perp \end{bmatrix} \begin{bmatrix} 1/\gamma_o & 0\cdots 0 \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & F_u(\bar{U},\gamma_o) \end{bmatrix} \begin{bmatrix} B_{22} \\ \{B_{22}\}_\perp \end{bmatrix} \tag{2.129}$$

and $(2.129)B_{22}^*$ yields

$$\gamma_o U(s) B_{22}^* = C_{22} \tag{2.130}$$

which is a constraining equation on U(s) which is of the same form as that derived by Glover in [11, eqn(8.69)]. Finally, using (2.112) and (2.113) we see that the optimal solution is given by

$$\begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} \\ \hat{Q}_{21} & Q_{22} & Q_{23} \\ \hat{Q}_{31} & Q_{32} & Q \end{bmatrix}(s) \overset{s}{=}$$

$$\left[\begin{array}{c|cccc} \widetilde{A} & R^{-*}\gamma_o C_{21}^*\{C_{22}\}_\perp & -\gamma_o^{-1} C_{31}^* & R^{-*}\{QB_{12}+\gamma C_{21}^* C_{22} B_{22}\} \\ \hline -\gamma_o\{B_{22}\}_\perp B_{12}^* & 0 & 0 & \gamma\{B_{22}\}_\perp \\ -\gamma_o^{-1} B_{13}^* R^* & 0 & 0 & 0 \\ -(C_{21}P+\gamma C_{22} B_{22} B_{12}^*) & \gamma\{C_{22}\}_\perp & 0 & 0 \end{array}\right]$$

in which

$$\widetilde{A} = R^{-*}\left\{\gamma_o^2 A_{11}^* + QA_{11}P - \gamma_o C_{31}^* B_{11}^* - \gamma_o C_{11}^* B_{13}^* - \gamma_o C_{21}^* C_{22} B_{22} B_{12}\right\}$$

Remark 2.1

The transformations which perform the compression in (2.122), need not necessarily be calculated by balancing as shown in (2.121). All that is required is the singular perturbation as described in Section 2.2, followed by two singular value decompositions based on (2.118) and (2.119). If

$$\begin{bmatrix} B_{1p} \\ D_{21p} \end{bmatrix} = Y\begin{bmatrix} 0 & \\ \vdots & * \\ 0 & \end{bmatrix}U^* \quad \text{and} \quad [C_{1p}\ D_{12p}] = Q\begin{bmatrix} 0 \cdots 0 \\ * \end{bmatrix}Z^* \tag{2.131}$$

then we may simply set $T_l = Q^*$ and $T_r = U$. □

*Case2 $R(\infty) \neq 0$*

The aim of this section is to generalize the analysis to the case $R(\infty) \neq 0$. We will take equations (2.86) to (2.92) given in the previous section as our starting point. It will also be assumed that $P_1$ and $Q_1$ have the form given in (2.101). In these coordinates, (2.90) and (2.91) become

$$\begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} & 0 \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} & 0 \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & -\gamma_o \end{bmatrix}\begin{bmatrix} 0 & B_{11} & B_{12} & B_{13} \\ 0 & B_{21} & B_{22} & B_{23} \end{bmatrix} + \begin{bmatrix} 0 & C_{11}^* & C_{21}^* & C_{31}^* \\ 0 & C_{12}^* & C_{22}^* & C_{32}^* \end{bmatrix}\begin{bmatrix} J_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix} \tag{2.132}$$

where the J-matrix is defined in (2.89) and

$$\begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \\ \hat{C}_{31} & \hat{C}_{32} \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & -\gamma_o \end{bmatrix} - \begin{bmatrix} J_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ B_{11}^* & B_{21}^* \\ B_{12}^* & B_{22}^* \\ B_{13}^* & B_{23}^* \end{bmatrix} \tag{2.133}$$

where the notation of (2.103) and (2.105) has been used. Using $(2.90)J^*$ to eliminate $[\,0\ C_0^*\ C_1^*\ C_2^*\,]^*$ from (2.91) yields:

$$\gamma_o^2\begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \\ \hat{C}_{31} & \hat{C}_{32} \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} J_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}\begin{bmatrix} \hat{B}_{11}^* & \hat{B}_{21}^* \\ \hat{B}_{12}^* & \hat{B}_{22}^* \\ \hat{B}_{13}^* & \hat{B}_{23}^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & -\gamma_o \end{bmatrix} + \begin{bmatrix} * & 0 \\ * & 0 \\ * & 0 \\ * & 0 \end{bmatrix} \tag{2.134}$$

Then the (4,2) block of (2.134) $\Rightarrow$

$$J_{42}\hat{B}_{22}^* + D_{21}\hat{B}_{23}^* = 0 \tag{2.135}$$

and the (3,2) block of this same equation gives

$$\gamma_o J_{21}J_{31}^{-1}\hat{C}_{32} = J_{21}\hat{B}_{21}^* + J_{21}J_{31}^{-1}J_{32}\hat{B}_{22}^* + J_{21}J_{31}^{-1}\mathfrak{D}_{11}\hat{B}_{23}^* \tag{2.136}$$

Now

$$(2.44f) \Rightarrow J_{21}J_{31}^{-1} = -J_{24}^{-*}D_{12}^* \tag{2.137}$$

$$(2.44b) \text{ and } (2.137) \Rightarrow J_{23} + J_{24}^{-*}D_{22}^*D_{21} = J_{21}J_{31}^{-1}\mathfrak{D}_{11} \tag{2.138}$$

$$(2.44e) \text{ and } (2.137) \Rightarrow J_{22} + J_{24}^{-*}D_{22}^*J_{42} = J_{21}J_{31}^{-1}J_{32} \tag{2.139}$$

Substituting (2.137), (2.138) and (2.139) into (2.136) gives

$$\begin{aligned}\gamma_o J_{21}J_{31}^{-1}\hat{C}_{32} &= J_{21}\hat{B}_{21}^* + J_{22}\hat{B}_{22}^* + J_{23}\hat{B}_{23}^* + J_{24}^{-*}D_{22}^*(J_{42}\hat{B}_{22}^* + D_{21}\hat{B}_{23}^*) \\ &= \gamma_o\hat{C}_{22}\end{aligned}$$

by (2.135) and (2.134). Whence

$$J_{21}J_{31}^{-1}\hat{C}_{32} = \hat{C}_{22} \tag{2.140}$$

Next, we derive a dual to (2.134) by combining (2.90) and (2.91) as

$$\begin{aligned}\gamma_o^2\begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} & 0 \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} & 0 \end{bmatrix} &= \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} Q & 0 \\ 0 & -\gamma_o \end{bmatrix}\begin{bmatrix} \hat{C}_{11}^* & \hat{C}_{21}^* & \hat{C}_{31}^* & 0 \\ \hat{C}_{12}^* & \hat{C}_{22}^* & \hat{C}_{32}^* & 0 \end{bmatrix}\begin{bmatrix} J_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}\end{aligned} \tag{2.141}$$

The (2,4) block of (2.141) gives

$$\hat{C}_{22}^*J_{24} + \hat{C}_{32}^*D_{12} = 0 \tag{2.142}$$

while the (2,3) block implies

$$\gamma_o\hat{B}_{23}J_{13}^{-1}J_{12} = \hat{C}_{12}^*J_{12} + \hat{C}_{22}^*J_{23}J_{13}^{-1}J_{12} + \hat{C}_{32}^*\mathfrak{D}_{11}J_{13}^{-1}J_{12} \tag{2.143}$$

$$\gamma_o J_{21}J_{31}^{-1}\hat{C}_{32} = J_{21}\hat{B}_{21}^* + J_{21}J_{31}^{-1}J_{32}\hat{B}_{22}^* + J_{21}J_{31}^{-1}\mathfrak{D}_{11}\hat{B}_{23}^* \tag{2.136}$$

Now

$$(2.44f) \Rightarrow J_{21}J_{31}^{-1} = -J_{24}^{-*}D_{12}^* \tag{2.137}$$

$$(2.44b) \text{ and } (2.137) \Rightarrow J_{23} + J_{24}^{-*}D_{22}^*D_{21} = J_{21}J_{31}^{-1}\mathfrak{D}_{11} \tag{2.138}$$

$$(2.44e) \text{ and } (2.137) \Rightarrow J_{22} + J_{24}^{-*}D_{22}^*J_{42} = J_{21}J_{31}^{-1}J_{32} \tag{2.139}$$

Substituting (2.137), (2.138) and (2.139) into (2.136) gives

$$\begin{aligned}\gamma_o J_{21}J_{31}^{-1}\hat{C}_{32} &= J_{21}\hat{B}_{21}^* + J_{22}\hat{B}_{22}^* + J_{23}\hat{B}_{23}^* + J_{24}^{-*}D_{22}^*(J_{42}\hat{B}_{22}^* + D_{21}\hat{B}_{23}^*) \\ &= \gamma_o\hat{C}_{22}\end{aligned}$$

by (2.135) and (2.134). Whence

$$J_{21}J_{31}^{-1}\hat{C}_{32} = \hat{C}_{22} \tag{2.140}$$

Next, we derive a dual to (2.134) by combining (2.90) and (2.91) as

$$\begin{aligned}\gamma_o^2\begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} & 0 \\ \hat{B}_{21} & \hat{B}_{22} & \hat{B}_{23} & 0 \end{bmatrix} &= \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} Q & 0 \\ 0 & -\gamma_o \end{bmatrix}\begin{bmatrix} \hat{C}_{11}^* & \hat{C}_{21}^* & \hat{C}_{31}^* & 0 \\ \hat{C}_{12}^* & \hat{C}_{22}^* & \hat{C}_{32}^* & 0 \end{bmatrix}\begin{bmatrix} J_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}\end{aligned} \tag{2.141}$$

The (2,4) block of (2.141) gives

$$\hat{C}_{22}^*J_{24} + \hat{C}_{32}^*D_{12} = 0 \tag{2.142}$$

while the (2,3) block implies

$$\gamma_o\hat{B}_{23}J_{13}^{-1}J_{12} = \hat{C}_{12}^*J_{12} + \hat{C}_{22}^*J_{23}J_{13}^{-1}J_{12} + \hat{C}_{32}^*\mathfrak{D}_{11}J_{13}^{-1}J_{12} \tag{2.143}$$

(2.44h) $\Rightarrow$ $\quad J_{13}^{-1}J_{12} = -D_{21}^*J_{42}^{-*}$ (2.144a)

the (2,4) block of $JJ^* \Rightarrow$ $\quad J_{22}+J_{24}D_{22}^*J_{42}^{-*} = -J_{23}D_{21}^*J_{42}^{-*} = J_{23}J_{13}^{-1}J_{12}$ (2.144b)

the (3,4) block of $JJ^* \Rightarrow$ $\quad J_{32}+D_{12}D_{22}^*J_{42}^{-*} = -\mathfrak{D}_{11}D_{21}^*J_{42}^{-*} = \mathfrak{D}_{11}J_{13}^{-1}J_{12}$ (2.144c)

| | | |
|---|---|---|
| (2.44h) $\Rightarrow$ | $J_{13}^{-1}J_{12}=-D_{21}^{*}J_{42}^{-*}$ | (2.144a) |
| the (2,4) block of $JJ^{*}$ $\Rightarrow$ | $J_{22}+J_{24}D_{22}^{*}J_{42}^{-*}=-J_{23}D_{21}^{*}J_{42}^{-*}=J_{23}J_{13}^{-1}J_{12}$ | (2.144b) |
| the (3,4) block of $JJ^{*}$ $\Rightarrow$ | $J_{32}+D_{12}D_{22}^{*}J_{42}^{-*}=-\mathfrak{D}_{11}D_{21}^{*}J_{42}^{-*}=\mathfrak{D}_{11}J_{13}^{-1}J_{12}$ | (2.144c) |

(2.143) may be rearranged as

$$\begin{aligned}\gamma_o\hat{B}_{23}J_{13}^{-1}J_{12} &= \hat{C}_{12}^{*}J_{12}+\hat{C}_{22}^{*}J_{22}+\hat{C}_{32}^{*}J_{32}+(\hat{C}_{22}^{*}J_{24}+\hat{C}_{32}^{*}D_{12})D_{22}^{*}J_{42}^{-*}\\ &= \gamma_o\hat{B}_{22}\end{aligned} \tag{2.145}$$

by (2.142), (2.144a), (2.144b) and (2.144c). Thus

$$\hat{B}_{23}J_{13}^{-1}J_{12}=\hat{B}_{22} \tag{2.146}$$

Substituting (2.102) to (2.105), and (2.132) and (2.133) into (2.86) and (2.87) yields

$$\begin{aligned}\begin{bmatrix}QP-\gamma_o^2I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}&=\begin{bmatrix}-(QP-\gamma_o^2I)A_{11}^{*}-(Q\mathfrak{B}_1+\mathfrak{C}_1^{*}\tilde{J}_{22})\mathfrak{B}_1^{*} & \gamma_o\mathfrak{C}_1^{*}\mathfrak{C}_2-\mathfrak{C}_1^{*}\tilde{J}_{22}\mathfrak{B}_2^{*}\\ \gamma_o\mathfrak{B}_2\mathfrak{B}_1^{*}-\mathfrak{C}_2^{*}\tilde{J}_{22}\mathfrak{B}_1^{*} & \gamma_o\mathfrak{B}_2\mathfrak{B}_2^{*}-\mathfrak{C}_2^{*}\tilde{J}_{22}\mathfrak{B}_2^{*}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\\ &\quad+\begin{bmatrix}\mathfrak{C}_1^{*}\tilde{J}_{21} & Q\mathfrak{B}_1+\mathfrak{C}_1^{*}\tilde{J}_{22}\\ \mathfrak{C}_2^{*}\tilde{J}_{21} & \mathfrak{C}_2^{*}\tilde{J}_{22}-\gamma_o\mathfrak{B}_2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}\end{aligned} \tag{2.147}$$

and

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=-\begin{bmatrix}\tilde{J}_{12}\mathfrak{B}_1^{*} & \tilde{J}_{12}\mathfrak{B}_2^{*}\\ \mathfrak{C}_1P+\tilde{J}_{22}\mathfrak{B}_1^{*} & \tilde{J}_{22}\mathfrak{B}_2^{*}-\gamma_o\mathfrak{C}_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}J_{11} & \tilde{J}_{12}\\ \tilde{J}_{21} & \tilde{J}_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{2.148}$$

in which

$$\begin{bmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{bmatrix}=\begin{bmatrix}B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\end{bmatrix} \tag{2.149}$$

and

$$\begin{bmatrix}\mathfrak{C}_1 & \mathfrak{C}_2\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\\ C_{31} & C_{32}\end{bmatrix} \tag{2.150}$$

The second row of (2.147) implies that

$$x_2=-(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}\left\{(\gamma_o\mathfrak{B}_2\mathfrak{B}_1^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_1^*)x_1+\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{21}u_1+(\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}-\gamma_o\mathfrak{B}_2)u_2\right\} \tag{2.151}$$

At this stage we may eliminate $x_2$ from (2.147) and (2.148) to obtain a reduced order model for $\mathfrak{Q}(s)$ in (2.88), which has the form

$$\mathfrak{Q}(s)=\left[\begin{array}{c|cc} A_p & B_{1p} & B_{2p} \\ \hline C_{1p} & D_{11p} & D_{12p} \\ C_{2p} & D_{21p} & D_{22p} \end{array}\right] \tag{2.152}$$

In order to demonstrate that $\mathfrak{Q}(s)$ may be transformed into the form given in (2.122), we need explicit expressions for:

$$\begin{bmatrix} B_{1p} \\ D_{21p} \end{bmatrix}, \begin{bmatrix} C_{1p} & D_{12p} \end{bmatrix} \text{ and } D_{11p}.$$

Substituting (2.151) into (2.147) and making use of (2.106) ($\mathfrak{B}_2\mathfrak{B}_2^*=\mathfrak{C}_2^*\mathfrak{C}_2$) gives

$$B_{1p}=\mathfrak{C}_1^*\tilde{\mathfrak{J}}_{21}-(\gamma_o\mathfrak{C}_1^*\mathfrak{C}_2-\mathfrak{C}_1^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{21}$$

$$=(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}\mathfrak{C}_1^*\left\{\mathfrak{C}_2^*(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)I_{l+p+q}-(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)\mathfrak{C}_2^*\right\}\tilde{\mathfrak{J}}_{21} \tag{2.153}$$

In the same way we obtain

$$D_{21p}=(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}\left\{\mathfrak{C}_2^*(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)I_{l+p+q}-(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)\mathfrak{C}_2^*\right\}\tilde{\mathfrak{J}}_{21} \tag{2.154}$$

Combining (2.153) and (2.154) gives

$$\begin{bmatrix} B_{1p} \\ D_{21p} \end{bmatrix}=(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}\begin{bmatrix} \mathfrak{C}_1^* \\ I \end{bmatrix}\left\{\mathfrak{C}_2^*(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)I_{l+p+q}-(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)\mathfrak{C}_2^*\right\}\tilde{\mathfrak{J}}_{21} \tag{2.155}$$

and a similar verification shows that:

$$[C_{1p}\ D_{12p}]=\tilde{\mathfrak{J}}_{12}\left\{\mathfrak{B}_2^*(\gamma_o\mathfrak{B}_2-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22})-I_{l+m+q}(\gamma_o\mathfrak{B}_2-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22})\mathfrak{B}_2^*\right\}[\mathfrak{B}_1\ -I](\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1} \tag{2.156}$$

One can also write down

$$D_{11p}=J_{11}+\tilde{\mathfrak{J}}_{12}\mathfrak{B}_2^*(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{21} \tag{2.157}$$

and

$$D_{22p}=\tilde{\mathfrak{J}}_{22}+(\gamma_o\mathfrak{C}_2-\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}\mathfrak{B}_2^*)^{-1}(\gamma_o\mathfrak{B}_2-\mathfrak{C}_2^*\tilde{\mathfrak{J}}_{22}) \tag{2.158}$$

from (2.147), (2.148) and (2.151).

The transformations which compress (2.152) into the form shown on the right hand side of (2.122), may now be constructed. We begin by observing from (2.132) and (2.133) that

$$-\gamma_o\mathfrak{B}_2+\mathfrak{C}_2^*\widetilde{J}_{22} = [\ \hat{B}_{22}\ \hat{B}_{23}\ 0\ ] = \hat{B}_{23}J_{13}^{-1}\widetilde{J}_{12} \tag{2.159}$$

$$\gamma_o\mathfrak{C}_2-\widetilde{J}_{22}\mathfrak{B}_2^* = \begin{bmatrix}\hat{C}_{22}\\ \hat{C}_{32}\\ 0\end{bmatrix} = \widetilde{J}_{21}J_{31}^{-1}\hat{C}_{32} \tag{2.160}$$

This gives

$$\begin{bmatrix}B_{1p}\\ D_{21p}\end{bmatrix}J_{31}^{-1}\hat{C}_{32} = 0 \text{ and } \hat{B}_{23}J_{13}^{-1}\begin{bmatrix}C_{1p} & D_{12p}\end{bmatrix} = 0 \tag{2.161}$$

With this motivation, we define

$$T_r = [\ J_{31}^{-1}\hat{C}_{32}\ \ \{J_{31}^{-1}\hat{C}_{32}\}_\perp\ ] \text{ and } T_l = \begin{bmatrix}\hat{B}_{23}J_{13}^{-1}\\ \{\hat{B}_{23}J_{13}^{-1}\}_\perp\end{bmatrix} \tag{2.162}$$

which reduce to (2.121) if $\begin{bmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{bmatrix} = 0$; in (2.162) we assume that $J_{31}^{-1}\hat{C}_{32}$ have been scaled to unit length. It is immediate from (2.156) and (2.157) that

$$\begin{bmatrix}D_{11p} & D_{12p}\\ D_{21p} & D_{22p}\end{bmatrix}\begin{bmatrix}0\\ \mathfrak{B}_2^*\end{bmatrix} = \gamma_o\begin{bmatrix}0\\ \mathfrak{C}_2\end{bmatrix} \tag{2.163}$$

What is less obvious is that (2.163) may be replaced by

$$\begin{bmatrix}D_{11p} & D_{12p}\\ D_{21p} & D_{22p}\end{bmatrix}\begin{bmatrix}J_{31}^{-1}\hat{C}_{32}\\ \mathfrak{B}_2^*\end{bmatrix} = \gamma_o\begin{bmatrix}-J_{13}^{-*}\hat{B}_{23}^*\\ \mathfrak{C}_2\end{bmatrix} \tag{2.164}$$

Before proving this, we note that $\mathfrak{B}_2\mathfrak{B}_2^* = \mathfrak{C}_2^*\mathfrak{C}_2 \Rightarrow \|J_{31}^{-1}\hat{C}_{32}\| = \|J_{13}^{-1}\hat{B}_{23}^*\|$ since the matrix on the right of (2.164) is orthogonal. This shows that there exists an orthogonal matrix U such that $UJ_{13}^{-1}\hat{B}_{23}^*=J_{31}^{-1}\hat{C}_{32}$. This equation is a replacement for (2.111) and is a counterpart to [11,equ(6.23)]. (2.164) also establishes that

$$T_lD_{11p}T_r = \begin{bmatrix}\gamma_o & 0\cdots0\\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & *\end{bmatrix} \tag{2.165}$$

This together with (2.169) shows that $T_r$ and $T_l$ given in (2.162) will indeed transform (2.152) to the form given in (2.122). Since $D_{21p}J_{31}^{-1}\hat{C}_{32}=$, (2.164) is established by proving that $D_{11p}J_{31}^{-1}\hat{C}_{32}=-\gamma_o J_{13}^{-*}\hat{B}_{23}^{*}$. From (2.157) we obtain

$$\begin{aligned}
D_{11p}J_{31}^{-1}\hat{C}_{32} &=\left\{J_{11}+\tilde{J}_{12}\mathfrak{B}_2^*(\gamma_o\mathfrak{B}_2\mathfrak{B}_2^*-\mathfrak{C}_2^*\tilde{J}_{22}\mathfrak{B}_2^*)^{-1}\mathfrak{C}_2^*\tilde{J}_{21}\right\}J_{31}^{-1}\hat{C}_{32}\\
&=J_{11}J_{31}^{-1}\hat{C}_{32}+\tilde{J}_{12}\mathfrak{B}_2^*\\
&=J_{11}J_{31}^{-1}\left\{\gamma_o C_{22}-J_{32}B_{21}^*-\mathfrak{D}_{11}B_{22}^*-D_{12}B_{23}^*\right\}+J_{12}B_{21}^*+J_{13}B_{22}^*
\end{aligned}\tag{2.166}$$

on substituting the (3,2) block of (2.132). Substituting from (2.133) yields

$$\begin{aligned}
&= \gamma_o J_{11}J_{31}^{-1}C_{22}+\tfrac{1}{\gamma_o}(J_{12}-J_{11}J_{31}^{-1}J_{32})\left\{C_{12}^*J_{22}+C_{22}^*J_{32}+C_{32}^*J_{42}-\hat{B}_{22}\right\}^*\\
&\quad+\tfrac{1}{\gamma_o}(J_{13}-J_{11}J_{31}^{-1}\mathfrak{D}_{11})\left\{C_{12}^*J_{23}+C_{22}^*\mathfrak{D}_{11}+C_{32}^*D_{21}-\hat{B}_{23}\right\}^*\\
&\quad-\tfrac{1}{\gamma_o}J_{11}J_{31}^{-1}D_{12}\left\{C_{12}^*J_{24}+C_{22}^*D_{12}+C_{32}^*D_{22}\right\}^*
\end{aligned}\tag{2.167}$$

$$\begin{aligned}
&= \tfrac{1}{\gamma_o}\left\{\gamma_o^2J_{11}J_{31}^{-1}+(J_{12}-J_{11}J_{31}^{-1}J_{32})J_{32}^*+(J_{13}-J_{11}J_{31}^{-1}\mathfrak{D}_{11})\mathfrak{D}_{11}^*-J_{11}J_{31}^{-1}D_{12}D_{12}^*\right\}C_{22}\\
&\quad+\tfrac{1}{\gamma_o}\left\{(J_{12}-J_{11}J_{31}^{-1}J_{32})J_{22}^*+(J_{13}-J_{11}J_{31}^{-1}\mathfrak{D}_{11})J_{23}^*-J_{11}J_{31}^{-1}D_{12}J_{24}^*\right\}C_{12}\\
&\quad+\tfrac{1}{\gamma_o}\left\{(J_{12}-J_{11}J_{31}^{-1}J_{32})J_{42}^*+(J_{13}-J_{11}J_{31}^{-1}\mathfrak{D}_{11})D_{21}^*-J_{11}J_{31}^{-1}D_{12}D_{22}^*\right\}C_{32}\\
&\quad+\tfrac{1}{\gamma_o}\left\{(J_{12}-J_{11}J_{31}^{-1}J_{32})J_{12}^*J_{13}^{-*}-(J_{13}-J_{11}J_{31}^{-1}\mathfrak{D}_{11})\right\}\hat{B}_{23}^*
\end{aligned}\tag{2.168}$$

Invoking various equations corresponding to the orthogonal character of the J-matrix given in (2.24b), enables one to show that

(1) The coefficient of $C_{22}$ is zero.
(2) The coefficient of $C_{12}$ is zero.
(3) The coefficient of $C_{32}$ is zero.
(4) The coefficient of $\hat{B}_{23}$ is $-\gamma_o J_{13}^{-*}$.

as required, and (2.164) is verified. We have thus proved that

$$\begin{bmatrix} T_l & 0\\ 0 & I_{l+p+q}\end{bmatrix}\mathcal{Q}(s)\begin{bmatrix} T_r & 0\\ 0 & I_{l+m+q}\end{bmatrix}=\begin{bmatrix}\gamma_o & 0\cdots0 & 0\cdots0 & 0\cdots0 & \\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & 0\\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{21} & Q_{22} & Q_{23} & 0\\ \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \hat{Q}_{31} & Q_{32} & Q & 0\\ & 0 & 0 & 0 & 0\end{bmatrix}\tag{2.169}$$

Arguments which are identical to those given in section 2.2.3, establish that U(s) must satisfy the constraining equation

$$\gamma_o U(s) J_{13}^{-*} \hat{B}_{23}^* = J_{31}^{-1} \hat{C}_{32} \tag{2.170}$$

and therefore undergoes an effective dimension reduction in the general $R(\infty)\neq 0$ case also. (2.170) generalizes (2.130) and is the four block counterpart to [11, eqn(8.69)]. □

Remark 2.2

At optimality,

the (2,2) block of (2.73) $\Rightarrow H_{12}^* H_{12} + H_{22}^* H_{22} = \gamma_o^2 I$ (2.171)

$$(2.169) \Rightarrow \qquad H_{12} = \begin{bmatrix} 0\ldots0 & 0\ldots0 \\ \hat{Q}_{13} & 0 \\ * & * \end{bmatrix} \tag{2.172}$$

Removing the redundant equation form (2.172) $\Rightarrow H_{12}^* H_{12}(s)$ looses rank all along the axis (outer product) $\Rightarrow ||H_{22}||_\infty = \gamma_o$; furthermore, $\bar{\sigma}(H_{22}(j\omega)) = \gamma_o \ \forall \omega \in \mathbb{R}$.

The (1,1) block of (2.73) $\Rightarrow H_{11}^* H_{11} + H_{21}^* H_{21} = \gamma_o^2 I$ (2.173)

$$(2.169) \Rightarrow \qquad H_{21} = \begin{bmatrix} \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix} & \hat{Q}_{31} & * \\ \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix} & 0 & * \end{bmatrix} \tag{2.174a}$$

and

$$H_{11} = \begin{bmatrix} \gamma_o & 0\ldots0 & 0\ldots0 \\ \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix} & \hat{Q}_{11} & * \\ \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix} & * & * \end{bmatrix} \tag{2.174b}$$

Removing the redundant equations from (2.174a) and (2.174b) $\Rightarrow H_{21}^* H_{21}$ is full rank along the axis (inner product) $\Rightarrow ||H_{11}||_\infty < \gamma_o$. The linear fractional map in (2.70) remains well posed at optimality.

□

## 2.5. Conclusions

The purpose of this chapter was to derive a representation formula for all $H_+^\infty$ solutions to the four block general distance problem. This formula is given in (2.70) and was constructed by all-pass embedding. There are a number of interesting comparisons which one can make with the well known one block, or Nehari extension problem. These are:

1] All solutions may be characterized by a representation formula in which the dimension of the free contraction is the same as that of Q(s) in (2.15). Furthermore, in problems where deg(R)=n, the solution also has degree n.

2] As opposed to the one block case, there are two types of optimality. We have called these optimality of the Parrott type (section 2.1) and optimality due to inertia (section 2.2).

3] In the case of optimality of the Parrott type, one needs to take special care of axis phenomena. Here a class of solutions is captured, however the proof of capturing <u>all</u> the solutions breaks down. (This problem has now been solved in a revised version of [34])

4] In the case of optimality due to inertia, the degree of the solution drops to n−1 using singular perturbation, in addition the effective dimension of the free contraction always reduces. The latter may be characterized by a constraining equation (2.170) which is very similar to that in Glover [11]. (In the event that the rank de~~f~~fect of $(QP-\gamma^2 I)=r$, where $r \geq 1$, the degree of the solution drops to n−r. Revised version of [34])

5] To avoid numerical difficulties, it seems prudent to invoke the generalized state-space ideas first given in [27]. Also, the generalized state-space approach gives a clear and easily understood treatment of optimality due to inertia.

# Chapter three
# State-space solutions to the $H^{\infty}$-control problem.

## 3.1 Introduction

In this chapter we present a cancellation analysis of $H^{\infty}$-control of the third kind. Youla parameterization theory [9,26,52] is used to parameterize the class of all stabilizing controllers, the all-pass embedding ideas discussed in chapter 2 and [34] will select those controllers which minimize the closed loop $\mathcal{L}^{\infty}$-norm. Furthermore, this approach yields a closed form representation formula for $H^{\infty}$-controllers (optimal and sub-optimal) in terms of P(s) in (3.2) and the solutions to two indefinite algebraic Riccati equations of the game theory type, not unlike LQG theory, in addition necessary and sufficient conditions for the existence of a solution to the $H^{\infty}$-control problem are also given in terms of these solutions to the indefinite equations.

A solution to the four block general distance problem in the $H^{\infty}$-control context suggested by Doyle [9,31] is based on the work of Davis, Kahan and Weinberger [6]. Although adequate for theoretical purposes, when implemented on a computer suffers from serious degree inflation problems. Cancelling the non-minimal modes after each step of the calculation is computationally intensive and hence time consuming, furthermore, the so called *almost* non-minimal modes (ie those modes corresponding to small Hankel singular values) are often removed introducing rounding-off errors and thereby obscuring cancellations that occur at latter stages of the algorithm. In an attempt to understand these inflation phenomena, several researchers undertook cancellation analyses for the one, two and four block cases [15,17,18,19]. The outcome of this work is most encouraging, in problems with $\deg(P(s))=n$, the controller always has degree n also. This result has lead to the expectation that Doyle's original algorithm could be improved from a computational point of view .

The second section of this chapter reviews $H^{\infty}$-optimization theory using Youla parameterization [9,26,52] followed by solving the four block distance problem a' la Doyle [9,26,31]. The third section presents a new solution to the four block $H^{\infty}$-control problem based on the all-pass embedding construction presented in chapter 2, the results are a theoretically cleaner solution and a great improvement over existing methods from an implementation view point. Necessary and sufficient conditions for the existence of $H^{\infty}$ optimal controllers are given in section four. Sections five and six deal with the cases of $D_{11}\neq 0$ and/or $D_{22}\neq 0$, and X and/or Y singular respectively.

## 3.2 Review of $H^{\infty}$-optimization theory

In this section we discuss Youla parameterization [9,26,52] which is used to characterize the class of all stabilizing compensators : S(P) and the corresponding closed loop transfer functions $\Re(s)$ in figure 1.1. Following that we identify those controllers in S(P) that minimize the $\mathcal{L}^{\infty}$–norm of the closed loop transfer function

$$\inf_{K_{opt}(s)\in S(P)} \| F_l(P,-K_{opt})\|_\infty \quad \leq \quad \inf_{K(s)\in S(P)} \| F_l(P,-K)\|_\infty \tag{3.1}$$

In the second part of this section we describe the solution to the four block $H^\infty$-control problem as suggested by Doyle [9,31].

### 3.2.1 Parameterization of all stabilizing controllers

Let P(s) in figure1.1be given by

$$P(s) = \begin{array}{c} \\ p \\ q \end{array}\!\!\begin{array}{c} \begin{array}{cc} m & l \end{array} \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{array}\!(s) \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \in \mathcal{L}_\infty^{(p+q)X(m+l)} \tag{3.2}$$

and suppose that $(A,B_2,C_2)$ is stabilizable and detectable [53], this guarantees that the set of internally stabilizing controllers, $S$, is not empty. Under these conditions K(s) stabilizes the feedback system in figure 1.1 if and only if it stabilizes $P_{22}(s)$ [26,53]. Then

$$S(P)=S(P_{22}) \ =\left\{ K(s)\in\mathcal{L}_\infty^{lXq}\text{: } (P_{22},K) \text{ is internally stable}\right\} \tag{3.3a}$$

or equivalently

$$=\left\{ K(s)\in\mathcal{L}_\infty^{lXq}\text{: } \begin{bmatrix} I & K \\ -P_{22} & I \end{bmatrix}^{-1} \in \mathcal{RH}_+^\infty \right\} \tag{3.3b}$$

Two additional assumptions will be made: (i) $D_{12}$ and $D_{21}$ have full column and row rank respectively. (ii) $(A-B_2(D_{12}^*D_{12})^{-1}D_{12}^*C_1)$ and $(A-B_1D_{21}^*(D_{21}D_{21}^*)^{-1}C_2)$ have no eigenvalues on the imaginary axis. Furthermore, we assume that $p\geq l$ and $m\geq q$, the cases when $p<l$ and/or $m<q$ are not treated.

In order to reduce the number of terms appearing in later calculations, we introduce scaling matrices $S_1$ and $S_2$ as has been described elsewhere [26]. If $S_1$ and $S_2$ orthogonalize the columns of $D_{12}$ and rows $D_{21}$, then P(s) in (3.1) becomes

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2S_1 \\ \hline C_1 & D_{11} & D_{12} \\ S_2C_2 & D_{21} & S_2D_{22}S_1 \end{array}\right] \tag{3.4}$$

in which $D_{12}$ and $D_{21}$ are part of orthogonal matrices such that $[D_{12}\ D_\perp]$ and $[D_{21}^*\ \tilde{D}_\perp^*]$ are orthogonal [26]. This assumption does not result in any loss of generality because any pre-scaling may be reversed by scaling K(s) with $S_1KS_2$ at the end of the design calculation [26]. From now we will assume that P(s) has already been scaled in this way.

Theorem 3.1:

Let

$$P_{22}(s) = N(s)M^{-1}(s) = \widetilde{M}^{-1}(s)\widetilde{N}(s) \tag{3.5}$$

be the right and left coprime factorizations of $P_{22}(s)$ and

$$\begin{bmatrix} \widetilde{V} & \widetilde{U} \\ -\widetilde{N} & \widetilde{M} \end{bmatrix}(s)\begin{bmatrix} M & -U \\ N & V \end{bmatrix}(s) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.6}$$

the corresponding Bezout identities. All the matrices in (3.6) belong to $\mathcal{RH}_+^\infty$. Furthermore we define

$$K_o(s) = \begin{array}{c} l \\ q \end{array}\begin{bmatrix} UV^{-1} & \widetilde{V}^{-1} \\ V^{-1} & V^{-1}N \end{bmatrix}(s) \in \mathcal{L}_\infty^{(l+q)X(q+l)} \tag{3.7}$$

Then:

i) $\mathbf{S}(P_{22})=\left\{K(s)=F_l(K_o,Q)(s) : \forall\ Q(s)\in\mathcal{RH}_\infty^{+(lXq)} \right\}$ (3.8)

or

ii) $\mathbf{S}(P_{22})=\left\{K(s)=(U+MQ)(V-NQ)^{-1}(s) : \forall\ Q(s)\in\mathcal{RH}_\infty^{+(lXq)} \text{ and } \det(V-NQ)(s)\neq 0 \right\}$ (3.9)

Proof:

$F_l(K_o,Q)(s)=(U+MQ)(V-NQ)^{-1}(s)$ is verified by direct calculation using (3.6) and (3.7). We now show that the linear fractional representation of $K(s)$ is internally stabilizing, using (3.6) we have

$$\begin{bmatrix} I & K \\ -P_{22} & I \end{bmatrix}^{-1}(s) = \begin{bmatrix} M & 0 \\ 0 & V-NQ \end{bmatrix}(s)\begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}(s)\begin{bmatrix} \widetilde{V} & -\widetilde{U} \\ \widetilde{N} & \widetilde{M} \end{bmatrix}(s) \tag{3.10}$$

hence $K(s)\in\mathbf{S}(P_{22})$ since all the matrices on the right hand side of (3.10) belong to $\mathcal{RH}_+^\infty$.

Next we need to show that all stabilizing controllers can be parameterized in terms of a linear fractional map. Suppose that $K(s)$ is a stabilizing controller with a right coprime factorization $K(s)=U_1V_1^{-1}(s)$ together with matrices $X(s)$ and $Y(s)$ $\in\mathcal{RH}_+^\infty$ such that $XU_1(s)+YV_1(s)=I$. (dropping s-dependence)

$$\begin{bmatrix} I & K \\ -P_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} M & 0 \\ 0 & V_1 \end{bmatrix}\begin{bmatrix} M & U_1 \\ -N & V_1 \end{bmatrix}^{-1} \in\mathcal{RH}_+^\infty \text{ by assumption.} \tag{3.11}$$

However

$$\begin{bmatrix} 0 & -\tilde{U} \\ X & 0 \end{bmatrix}\begin{bmatrix} M & U_1 \\ -N & V_1 \end{bmatrix}+\begin{bmatrix} \tilde{V} & \tilde{U} \\ -X & Y \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & V_1 \end{bmatrix} = I \tag{3.12}$$

$\begin{bmatrix} M & 0 \\ 0 & V_1 \end{bmatrix}(s)$ and $\begin{bmatrix} M & U_1 \\ -N & V_1 \end{bmatrix}(s)$ are therefore right coprime, hence $\begin{bmatrix} M & U_1 \\ -N & V_1 \end{bmatrix}^{-1}(s) \in \mathcal{RH}_+^\infty$. (3. 13)

From (3.6) define

$$\begin{bmatrix} \tilde{V} & \tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & -U_1 \\ N & V_1 \end{bmatrix} = \begin{bmatrix} I & -\tilde{Q} \\ 0 & \Delta \end{bmatrix} \in \mathcal{RH}_+^\infty \tag{3.14}$$

where $\tilde{Q}(s)=(\tilde{V}U_1-\tilde{U}V_1)(s)$ and $\Delta(s)=(\tilde{N}U_1+\tilde{M}V_1)(s)$, using (3.6) we can write

$$\begin{bmatrix} M & -U_1 \\ N & V_1 \end{bmatrix}^{-1}\begin{bmatrix} M & -U \\ N & V \end{bmatrix} = \begin{bmatrix} I & \tilde{Q}\Delta^{-1} \\ 0 & \Delta^{-1} \end{bmatrix} \in \mathcal{RH}_+^\infty, \tag{3.15}$$

follows from (3.6) and (3.13), continuing further (3.14) gives

$$\begin{bmatrix} -U_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} M & -U \\ N & V \end{bmatrix}\begin{bmatrix} -\tilde{Q} \\ \Delta \end{bmatrix} = \begin{bmatrix} -(U+MQ)\Delta \\ (V-NQ)\Delta \end{bmatrix} \tag{3.16}$$

where $Q(s)=\tilde{Q}\Delta^{-1}(s)\in\mathcal{RH}_+^\infty$ from (3.15).

Hence $K(s)=U_1V_1^{-1}(s)=(U+MQ)(V-NQ)^{-1}(s)$; the family of all stabilizing controllers can be parameterized in terms of a linear fractional transformation. This completes the proof. □

In the next stage of our development, the nonlinear optimization problem of (3.1) is converted to another problem which is linear in a free parameter. A routine calculation based on (3.6) will establish that

$$\begin{aligned} \Re(s) &= F_l(P,-K)(s) && (3.17)\\ &= [P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21}](s) && (3.18)\\ &= [(P_{11} - P_{12}U\tilde{M}P_{21}) - (P_{12}M)Q(\tilde{M}P_{21})](s) && (3.19)\\ &= [T_{11} - T_{12}QT_{21}](s) && (3.20) \end{aligned}$$

where $T_{ij}$ are defined in the obvious way. (3.20) shows that $\Re(s)$ is parameterized linearly in $Q(s)$.

Since $(A,B_2)$ is stabilizable there exists a state feedback matrix F such that $A-B_2F$ is stable. Similarly, since $(C_2,A)$ is detectable there exists an output injection matrix H such $A-HC_2$ is stable. Given any such pair of stabilizing matrices F and H, the right and left coprime factorizations of $P_{22}$ together with the solutions of the Bezout identities are given by [9,26]

$$\begin{bmatrix} M & -U \\ N & V \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{c|cc} A-B_2F & B_2 & H \\ \hline -F & I & 0 \\ C_2-D_{22}F & D_{22} & I \end{array}\right] \tag{3.21}$$

and

$$\begin{bmatrix} \tilde{V} & \tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{c|cc} A-HC_2 & B_2-HD_{22} & H \\ \hline F & I & 0 \\ -C_2 & -D_{22} & I \end{array}\right] \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.7) yields after simplification

$$K_o(s) \overset{s}{=} \left[\begin{array}{c|cc} A-B_2F-HC_2+HD_{22}F & -H & B_2-HD_{22} \\ \hline -F & 0 & I \\ C_2-D_{22}F & I & D_{22} \end{array}\right] \tag{3.23}$$

We will make the following specific choices of the stabilizing matrices F and H for reasons that will become apparent in the next subsection:

$$F = D_{12}^*C_1 + B_2^*X \tag{3.24}$$

where X is the unique stabilizing positive semi-definite solution of the algebraic Riccati equation

$$X(A-B_2D_{12}^*C_1) + (A-B_2D_{12}^*C_1)^*X - XB_2B_2^*X + C_1^*D_\perp D_\perp^* C_1 = 0 \tag{3.25}$$

Similarly

$$H = B_1D_{21}^* + YC_2^* \tag{3.26}$$

where Y is the unique stabilizing positive semi-definite solution to the algebraic Riccati equation

$$Y(A-B_1D_{21}^*C_2)^* + (A-B_1D_{21}^*C_2)Y - YC_2^*C_2Y + B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^* = 0 \tag{3.27}$$

A straight forward calculation will establish that the $T_{ij}$ 's of (3.20) are give by

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \\ \widetilde{T}_{\perp} & 0 & 0 \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|ccc} A-B_2F & B_2F & B_1 & B_2 & -X^{\dagger}C_1^*D_{\perp} \\ 0 & A-HC_2 & B_1-HD_{21} & 0 & 0 \\ \hline C_1-D_{12}F & D_{12}F & D_{11} & D_{12} & D_{\perp} \\ 0 & C_2 & D_{21} & 0 & 0 \\ 0 & -\widetilde{D}_{\perp}B_1^*Y^{\dagger} & \widetilde{D}_{\perp} & 0 & 0 \end{array}\right] \tag{3.28}$$

where $X^{\dagger}$ and $Y^{\dagger}$ are the Moore-Penrose generalized inverses of X and Y respectively. With this particular choice of F and H, $[T_{12} \mid T_{\perp}]$ and $[T_{21}^* \mid \widetilde{T}^*_{\perp}]$ are inner [9,26]. We call $T_{\perp}(s)$ and $\widetilde{T}_{\perp}(s)$ all-pass extensions of $T_{12}$ and $T_{21}$ respectively.

3.2.2 The $\gamma$- iteration (minimizing the cost)

In this section we review an algorithm [9,26,31] which identifies those compensators in $S(P_{22})$ which minimize the $\mathcal{L}^{\infty}$-norm of the closed loop transfer function

$$\inf_{Q(s)\in\mathfrak{R}\mathcal{H}_+^{\infty}} \| (T_{11}-T_{12}QT_{21})(s) \|_{\infty} = \gamma_o \tag{3.29}$$

by reducing the problem to finding an upper bound to the Nehari problem [11]. The approach will be to generate a sequence $\gamma_i = \| \mathfrak{R}_i(s) \|_{\infty} = \| T_{11}-T_{12}Q_iT_{21} \|_{\infty}$ which converges on $\gamma_o$ (from above). In applications the calculation of this sequence is terminated when the upper bound $\alpha$ for $\gamma_o$ has been found such that $(\alpha - \gamma_o) < \epsilon$ for some sufficiently small $\epsilon\in\mathfrak{R}_+$ .

Since $[T_{12} \mid T_{\perp}](s)$ and $\begin{bmatrix} T_{21} \\ \widetilde{T}_{\perp} \end{bmatrix}(s)$ are inner and thus norm preserving we can write: (dropping s-dependence)

$$\inf_{Q\in\mathfrak{R}\mathcal{H}_+^{\infty}} \left\| [T_{11} - T_{12}QT_{21}][T_{21}^* \mid \widetilde{T}^*_{\perp}] \right\|_{\infty} = \gamma_o \tag{3.30}$$

$$\inf_{Q\in\mathfrak{R}\mathcal{H}_+^{\infty}} \left\| [T_{11}T_{21}^* - T_{12}Q \mid T_{11}\widetilde{T}^*_{\perp}] \right\|_{\infty} = \gamma_o \tag{3.31}$$

$$\inf_{Q\in\mathfrak{R}\mathcal{H}_+^{\infty}} \left\| M_1(T_{11}T_{21}^* - T_{12}Q) \right\|_{\infty} = 1 \tag{3.32}$$

$$M_1(s) = (\gamma_o^2 I - T_{11}\widetilde{T}^*_{\perp}(T_{11}\widetilde{T}^*_{\perp})^*)^{-1/2} \qquad \text{(first spectral factorization)} \tag{3.33}$$

$$M_1T_{12} = \begin{bmatrix} \theta & \theta_{\perp} \end{bmatrix} \begin{bmatrix} M_2^{-1} \\ 0 \end{bmatrix} \qquad \text{(inner-outer factorization)} \tag{3.34a}$$

where

$$\begin{bmatrix} \theta & \theta_{\perp} \end{bmatrix}(s) \text{ and } \begin{bmatrix} M_2^{-1} \\ 0 \end{bmatrix}(s) \tag{3.34b}$$

are the inner and outer factors respectively.

$$\inf_{Q\in\mathcal{RH}_+^\infty}\left\| M_1T_{11}T_{21}^* - \begin{bmatrix} \theta & \theta_\perp \end{bmatrix}\begin{bmatrix} M_2^{-1} \\ 0 \end{bmatrix} Q \right\|_\infty = 1 \tag{3.35}$$

$$\inf_{Q\in\mathcal{RH}_+^\infty}\left\| \begin{bmatrix} \theta^* \\ \theta_\perp^* \end{bmatrix} [M_1T_{11}T_{21}^* - \begin{bmatrix} \theta & \theta_\perp \end{bmatrix}\begin{bmatrix} M_2^{-1} \\ 0 \end{bmatrix} Q] \right\|_\infty = 1 \tag{3.36}$$

$$\inf_{Q\in\mathcal{RH}_+^\infty}\left\| \begin{bmatrix} \theta^* M_1T_{11}T_{21}^* - M_2^{-1}Q \\ \theta_\perp^* M_1T_{11}T_{21}^* \end{bmatrix} \right\| = 1 \tag{3.37}$$

$$\inf_{Q\in\mathcal{RH}_+^\infty}\left\| \theta^* M_1T_{11}T_{21}^*M_3 - M_2^{-1}QM_3 \right\|_\infty = 1 \tag{3.38}$$

$$M_3 = \left( I - (\theta_\perp^* M_1T_{11}T_{21}^*)^*(\theta_\perp^* M_1T_{11}T_{21}^*)\right)^{-1/2} \text{ (second spectral factorization)}$$

decomposing $(\theta^* M_1T_{11}T_{21}^*M_3)(s)$ into stable and unstable projections.

$$\inf_{Q\in\mathcal{RH}_+^\infty}\left\| (\theta^* M_1T_{11}T_{21}^*M_3)_+ - [M_2^{-1}QM_3 - (\theta^* M_1T_{11}T_{21}^*M_3)_-\,] \right\|_\infty = 1 \tag{3.39}$$

define

$$\tilde{Q}(s) = [M_2^{-1}QM_3 - (\theta^* M_1T_{11}T_{21}^*M_3)_-\,](s) \tag{3.40}$$

We observe that (3.39) is an optimal approximation problem [11] in which we solve for $\tilde{Q}(s)$ defined in (3.40) and backsubstitute for Q(s) using (3.41).

$$Q(s) = M_2\{\, \tilde{Q} + (\theta^* M_1T_{11}T_{21}^*M_3)_-\, \}M_3^{-1}(s) \tag{3.41}$$

Continuing from (3.21) we have that $[T_{12}\ |\ T_\perp](s)$ and $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}(s)$ are all pass therefore:

$$\|\Re(s)\|_\infty = \left\| T_{11}(s) - [T_{12}\ |\ T_\perp]\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}(s) \right\|_\infty \tag{3.42}$$

$$= \left\| \begin{bmatrix} T_{12}^*T_{11}T_{21}^* - Q & T_{12}^*T_{11}\tilde{T}_\perp^* \\ T_\perp^*T_{11}T_{21}^* & T_\perp^*T_{11}\tilde{T}_\perp^* \end{bmatrix}(s) \right\|_\infty \tag{3.43}$$

$$= \left\| \begin{bmatrix} (R_{11} - Q)(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix} \right\|_\infty \tag{3.44}$$

where

$$R(s) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s) = \begin{bmatrix} T_{12}^{*} \\ T_{\perp}^{*} \end{bmatrix} T_{11}\,[T_{21}^{*} \mid \widetilde{T}_{\perp}^{*}](s) \tag{3.45}$$

The algorithm is as follows: Firstly, we find lower and upper bounds for $\gamma_o$ . By setting Q(s)=0 in (3.44) we see that

$$\gamma_o \leq \|R(s)\|_{\infty} \leq 2\sum_{i=1}^{n} \sigma_i(\, R(-s)\,) + \bar{\sigma}(D_R) = \gamma_{upper} \tag{3.46}$$

in which we have taken deg(R)=$n$ and $D_R$ is the D matrix of R(s) [11]. Assuming for argument's sake that R(s) and Q(s) have the same dimension, then we get from Nehari's theorem [11] that

$$\gamma_{low} = \left\| \Gamma_{R(-s)} \right\| \leq \gamma_o \tag{3.47}$$

thus $\gamma_{low} \leq \gamma_o \leq \gamma_{upper}$

In the second step, perform a binary search on the interval $[\gamma_{low}\,,\,\gamma_{upper}]$

Let

$$\gamma_1 = (\gamma_{low} + \gamma_{upper})/2 \tag{3.48}$$

and check the existence of a spectral factor $M_1(s)$ in

$$\left\{\gamma_1^2 I - T_{11}\widetilde{T}_{\perp}^{*}(T_{11}\widetilde{T}_{\perp}^{*})^{*}\right\}(s) \;=\; M_1 M_1^{*}(s) \tag{3.49}$$

should the spectral factor exist compute the inner-outer factorization:

$$M_1 T_{12}(s) = \begin{bmatrix} \theta & \theta_{\perp} \end{bmatrix} \begin{bmatrix} M_2^{-1} \\ 0 \end{bmatrix}(s) \tag{3.50}$$

and check the existence of the spectral factor in $M_3(s)$ in :

$$I - \;(\theta_{\perp}^{*} M_1 T_{11} T_{21}^{*})^{*}(\theta_{\perp}^{*} M_1 T_{11} T_{21}^{*})(s) \;=\; M_3^{*} M_3(s) \tag{3.51}$$

Upon the existence of such a spectral factor, check that

$$\left\| \Gamma_{(\theta^{*} M_1 T_{11} T_{21}^{*} M_3)_+} \right\| \leq 1 \tag{3.52}$$

If (3.52) is satisfied $\gamma_1$ is reduced to $\gamma_2 = (\gamma_{low} + \gamma_1)/2$. If either the spectral factors in (3.49) and (3.51) do not exist or the condition of (3.52) is not satisfied, gamma is increased to : $\gamma_2 = (\gamma_{upper} + \gamma_1)/2$. By successively bisecting the search interval in this way, the procedure will generate a sequence of upper and lower bounds which will converge to $\gamma_o$. Finally $\widetilde{Q}(s)$ is evaluated by solving the Nehari approximation problem [11] when (3.52) $\simeq 1$. Back-substituting via (3.41) and (3.8) yields the controller. Details concerning the computations at each stage of the algorithm outlined above are given in [26,31].

### 3.3 $H^{\infty}$-control four block distance problem

Rather than solving the optimization problem in (3.29) with the algorithm outlined in the previous section, we can easily convert it into a four block distance problem with the same free parameter $Q(s)\in\mathcal{RH}_{+}^{\infty}$ as shown in (3.44). Using the solution to the four block general distance problem presented in the chapter 2, it is possible to isolate all those Q(s) of interest. To this end we shall adapt the development of section 2.2 to the case of $H^{\infty}$-optimal control problems of the third kind and use the generator of all Q(s)'s to generate the class of all optimal and suboptimal controllers. Furthermore this approach allows us to derive a closed form representation formula for all controllers in terms of P(s) and the solutions to two indefinite Riccati equations. The results are a simpler $\gamma$-iteration with only two Riccati equations of fixed order and no degree inflation problems.

#### 3.3.1 The construction of an all-pass embedding

In this sub-section we will follow the development presented in section 2.2 for the special case of $H^{\infty}$-control problems of the third kind. Consider R(s) defined in (3.45), then it may be verified using (3.28) that R(s) has the realization [54]

$$R(s)\overset{s}{=}\left[\begin{array}{cc|cc} -(A-B_2F)^* & \Psi\Theta & \Psi D_{21}^* & \Psi\tilde{D}_{\perp}^* \\ 0 & -(A-HC_2)^* & -C_2^* & Y^{\dagger}B_1\tilde{D}_{\perp}^* \\ \hline -B_2^* & D_{12}^*D_{11}\Theta+FY & D_{12}^*D_{11}D_{21}^* & D_{12}^*D_{11}\tilde{D}_{\perp}^* \\ D_{\perp}^*C_1X^{\dagger} & D_{\perp}^*D_{11}\Theta & D_{\perp}^*D_{11}D_{21}^* & D_{\perp}^*D_{11}\tilde{D}_{\perp}^* \end{array}\right] \tag{3.53}$$

where we define $\Psi=\left\{(C_1-D_{12}F)^*D_{11}+XB_1\right\}$ and $\Theta=(B_1-HD_{21})^*$. It is clear that R(s) has all its poles in $\mathbb{C}_+$, $R(s)\in\mathcal{RH}_{-}^{\infty,(p+q)\times(m+l)}$ and we seek $Q(s)\in\mathcal{RH}_{+}^{\infty,(p\times m)}$ such that

$$\left\|\begin{bmatrix} (R_{11}-Q)(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix}\right\|_{\infty}\leq\gamma_o. \tag{3.54}$$

We will also assume that

$$\|\mathcal{E}(\infty)\|_2 = \left\| \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & \bar{D}_{22} \end{bmatrix} \right\|_2 = \max\{ \|\bar{D}_{21}\ \bar{D}_{22}\|_2, \left\|\begin{matrix}\bar{D}_{12}\\ \bar{D}_{22}\end{matrix}\right\|_2 \} < \gamma_o \tag{3.55}$$

so as to avoid the case of Parrott type optimality occurring at infinity.

Suppose next that

$$\mathcal{E}(s) = \begin{bmatrix} R_{11}-Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s) \overset{s}{=} \left[\begin{array}{cc|cc} A_R & 0 & B_{R1} & B_{R2} \\ 0 & A_Q & B_{Q3} & 0 \\ \hline C_{R1} & -C_{Q3} & \bar{D}_{11} & \bar{D}_{12} \\ C_{R2} & 0 & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] \tag{3.56}$$

is minimal in which

$$R(s) \overset{s}{=} \left[\begin{array}{c|cc} A_R & B_{R1} & B_{R2} \\ \hline C_{R1} & \bar{D}_{11} & \bar{D}_{12} \\ C_{R2} & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] \quad \text{and} \quad Q(s) \overset{s}{=} \left[\begin{array}{c|c} A_Q & B_{Q3} \\ \hline C_{Q3} & 0 \end{array}\right] \tag{3.57}$$

Invoking a standard result from dilation theory [11], we can write

$$\mathcal{S}(s) = \begin{array}{c} \\ m+l \\ p \\ q \end{array} \begin{array}{c} \begin{array}{ccc} p+q & m & l \end{array} \\ \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{13} \\ \mathcal{S}_{21} & R_{11}-Q & R_{12} \\ \mathcal{S}_{31} & R_{21} & R_{22} \end{bmatrix} \end{array} (s) \tag{3.58}$$

in which it is always possible to choose the first $m+l$ rows and $p+q$ columns of $\mathcal{S}(s)$ so that $\mathcal{S}(s)\mathcal{S}^*(s)=\gamma^2 I$ where we select $\gamma \geq \gamma_o$. As a consequence of the all-pass character of $\mathcal{S}(s)$, we must have

$$\mathcal{S}_{31}(s)\mathcal{S}_{31}^*(s) = \gamma^2 I - \begin{bmatrix} R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} R_{21}^* \\ R_{22}^* \end{bmatrix}(s) = \gamma^2 I - (T_\perp^* T_{11})(T_\perp^* T_{11})^*(s) \tag{3.59}$$

and

$$\mathcal{S}_{13}^*(s)\mathcal{S}_{13}(s) = \gamma^2 I - \begin{bmatrix} R_{12}^* & R_{22}^* \end{bmatrix} \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix}(s) = \gamma^2 I - (T_{11}\tilde{T}_\perp^*)^*(T_{11}\tilde{T}_\perp^*)(s) \tag{3.60}$$

which are spectral factorization problems. Suppose next, and this is always possible, that we choose

$$\mathcal{S}_{31}(s) = \begin{array}{c c} \scriptstyle p & \scriptstyle q \\ \left[\begin{array}{c c} 0 & N(s) \end{array}\right. & \left.\right]_{q} \end{array} \tag{3.61}$$

and

$$\mathcal{S}_{13}(s) = \begin{array}{c c} \scriptstyle l & \\ \left[\begin{array}{c} 0 \\ M(s) \end{array}\right] & \begin{array}{c} \scriptstyle m \\ \scriptstyle l \end{array} \end{array} \tag{3.62}$$

Building this structure into $\mathcal{S}(s)$ will give the following representation

$$\mathcal{S}(s) = \begin{array}{c c} \begin{array}{cccc} \scriptstyle p & \scriptstyle q & \scriptstyle m & \scriptstyle l \end{array} & \\ \left[\begin{array}{cccc} Q_{11} & Q_{12} & Q_{13} & 0 \\ Q_{21} & R_{00}-Q_{22} & R_{01}-Q_{23} & M \\ Q_{31} & R_{10}-Q_{32} & R_{11}-Q & R_{12} \\ 0 & N & R_{21} & R_{22} \end{array}\right] & \begin{array}{c} \scriptstyle m \\ \scriptstyle l \\ \scriptstyle p \\ \scriptstyle q \end{array} \end{array} \tag{3.63a}$$

$$\stackrel{s}{=} \left[\begin{array}{cc|cccc} A_R & 0 & 0 & B_o & B_{R1} & B_{R2} \\ 0 & A_Q & B_{Q1} & B_{Q2} & B_{Q3} & 0 \\ \hline 0 & -C_{Q1} & X_1 & X_2 & X_3 & 0 \\ C_o & -C_{Q2} & X_4 & X_5 & X_6 & D_m \\ C_{R1} & -C_{Q3} & X_7 & X_8 & \bar{D}_{11} & \bar{D}_{12} \\ C_{R2} & 0 & 0 & D_n & \bar{D}_{21} & \bar{D}_{22} \end{array}\right] \tag{3.63b}$$

in which

$$N(s) \stackrel{s}{=} \left[\begin{array}{cc} A_R & B_o \\ C_{R2} & D_n \end{array}\right] \tag{3.64}$$

and

$$M(s) \stackrel{s}{=} \left[\begin{array}{cc} A_R & B_{R2} \\ C_o & D_m \end{array}\right] \tag{3.65}$$

Abbreviating (3.63b) to

$$\mathcal{S}(s) \stackrel{s}{=} \left[\begin{array}{cc} A_{\mathcal{S}} & B_{\mathcal{S}} \\ C_{\mathcal{S}} & \bar{D} \end{array}\right] \tag{3.66}$$

and invoking the standard state-space characterization of all-pass matrices yields

$$A_{\mathcal{S}}P_{\mathcal{S}} + P_{\mathcal{S}}A_{\mathcal{S}}^* + B_{\mathcal{S}}B_{\mathcal{S}}^* = 0 \tag{3.67}$$

$$A_{\mathcal{S}}^*Q_{\mathcal{S}} + Q_{\mathcal{S}}A_{\mathcal{S}} + C_{\mathcal{S}}^*C_{\mathcal{S}} = 0 \tag{3.68}$$

$$\bar{D}B_{\mathcal{S}}^* + C_{\mathcal{S}}P_{\mathcal{S}} = 0 \tag{3.69}$$

$$\bar{D}^*C_g + B_g^*Q_g = 0 \tag{3.70}$$
$$\bar{D}\bar{D}^* = \bar{D}^*\bar{D} = \gamma^2 I \tag{3.71}$$

in which we define

$$P_g = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} \text{and } Q_g = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix} \tag{3.72}$$

In the next phase of our development, we write down the equations describing the spectral factorization problems

$$M^*M(s)=\gamma_2 I - (T_{11}\widetilde{T}^*_{\perp})^*(T_{11}\widetilde{T}^*_{\perp})(s) \tag{3.73}$$

and

$$NN^*(s)=\gamma_2 I - (T^*_{\perp}T_{11})(T^*_{\perp}T_{11})^*(s) \tag{3.74}$$

It is easily verified using (3.28) that $(T_{11}\widetilde{T}^*_{\perp})(s)$ has a realization given by :

$$(T_{11}\widetilde{T}^*_{\perp})(s) = [\hat{A}, \hat{B}, \hat{C}, \hat{D}] = \left[\begin{array}{c|c} -(A-HC_2)^* & Y^{\dagger}B_1\widetilde{D}^*_{\perp} \\ \hline D_{11}(B_1-HD_{21})^*+C_1Y & D_{11}\widetilde{D}^*_{\perp} \end{array}\right] \tag{3.75}$$

Then the spectral factor M(s) is [2]

$$M(s) = [\hat{A}, \hat{B}, C_m, D_m] = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{R}^{-1/2}(\hat{B}^*\hat{X} - \hat{D}^*\hat{C}) & \hat{R}^{1/2} \end{array}\right] \tag{3.76}$$

where

$$\hat{R}=\gamma^2 I - \hat{D}^*\hat{D} \tag{3.77}$$

and $\hat{X}$ satisfies the algebraic Riccati equation [2]

$$\hat{X}(\hat{A} + \hat{B}\hat{R}^{-1}\hat{D}^*\hat{C}) + (\hat{A} + \hat{B}\hat{R}^{-1}\hat{D}^*\hat{C})^*\hat{X} - \hat{X}\hat{B}\hat{R}^{-1}\hat{B}^*\hat{X} - \hat{C}^*(I + \hat{D}\hat{R}^{-1}\hat{D}^*)\hat{C}=0 \tag{3.78}$$

The minimum phase spectral factor is required in theorem 2.5 so we shall select the minimum solution to (3.78) [1](ie the least positive of all the solutions). A dual set of arguments for (3.74) will yield:

$$(T_{\perp}^{*}T_{11})(s) = [\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}] = \left[\begin{array}{c|c} -(A-B_2F)^{*} & (C_1-D_{12}F)^{*}D_{11}+XB_1 \\ \hline D_{\perp}^{*}C_1X^{\dagger} & D_{\perp}^{*}D_{11} \end{array}\right] \tag{3.79}$$

Then the spectral factor N(s) is[2]

$$N(s) = [\widetilde{A}, B_n, \widetilde{C}, D_n] = \left[\begin{array}{c|c} \widetilde{A} & (\widetilde{X}\widetilde{C}^{*} - \widetilde{B}\widetilde{D}^{*})\,\widetilde{R}^{-*/2} \\ \hline \widetilde{C} & \widetilde{R}^{*/2} \end{array}\right] \tag{3.80}$$

where

$$\widetilde{R}=\gamma^2 I - \widetilde{D}\widetilde{D}^{*} \tag{3.81}$$

and $\widetilde{X}$ satisfies the algebraic Riccati equation [2]

$$\widetilde{X}(\widetilde{A} + \widetilde{B}\widetilde{D}^{*}\widetilde{R}^{-1}\widetilde{C})^{*} + (\widetilde{A} + \widetilde{B}\widetilde{D}^{*}\widetilde{R}^{-1}\widetilde{C})\widetilde{X} - \widetilde{X}\widetilde{C}^{*}\widetilde{R}^{-1}\widetilde{C}\widetilde{X} - \widetilde{B}(I + \widetilde{D}^{*}\widetilde{R}^{-1}\widetilde{D})\widetilde{B}^{*}=0 \tag{3.82}$$

In this case we shall also select the minimal solution so that the spectral factor has the required minimum phase properties.

<u>Remark 3.1</u>

Observe that the two algebraic Riccati equations (3.78) and (3.82) correspond to (2.39) and (2.42), furthermore they have dimension '$n$' whereas (2.39) and (2.42) are '$2n$' equations, '$2n$' being the state dimension of R(s).

We now turn our attention to the construction of a $\bar{D}$-matrix in (3.63b)

$$\bar{D} = \begin{bmatrix} X_1 & X_2 & X_3 & 0 \\ X_4 & X_5 & X_6 & D_m \\ X_7 & X_8 & \bar{D}_{11} & \bar{D}_{12} \\ 0 & D_n & \bar{D}_{21} & \bar{D}_{22} \end{bmatrix} \tag{3.83}$$

in which $D_m$ and $D_n$ have already been specified by (3.76) and (3.80) respectively. To continue further, we need to solve the problem at infinity by replacing $\bar{D}_{11}$ with $\mathfrak{D}_{11}=-\bar{D}_{12}(\gamma_p^2 I+\bar{D}_{22}^{*}\bar{D}_{22})^{-1}\bar{D}_{22}^{*}\bar{D}_{21}$ in which $\gamma_p$ is the Parrott lower bound (see section 2.2). (3.83) together with (3.71) lead to a set of equations which may always be solved to yield an orthogonal $\bar{D}$-matrix. The proof of this claim proceeds by direct calculation in precisely the same way as the construction of a J-matrix in section (2.2), for completeness $X_1$, $X_2$, $X_3$, $X_4$, $X_5$, $X_6$, $X_7$ and $X_8$ can be evaluated from:

$$X_8 = -\,(\mathcal{D}_{11}\bar{D}_{21}^{*} + \bar{D}_{12}\bar{D}_{22}^{*})\,D_n^{-*} \tag{3.84}$$

$$X_6 = -\,D_m^{-*}(\bar{D}_{22}^{*}\bar{D}_{21} + \bar{D}_{12}^{*}\mathcal{D}_{11}) \tag{3.85}$$

$$X_7 = \left\{\,\gamma^2 I - \mathcal{D}_{11}\mathcal{D}_{11}^{*} - \bar{D}_{12}\bar{D}_{12}^{*} - X_8X_8^{*}\,\right\}^{1/2} \tag{3.86}$$

$$X_3 = \left\{\,\gamma^2 I - \mathcal{D}_{11}^{*}\mathcal{D}_{11} - \bar{D}_{21}^{*}\bar{D}_{21} - X_6^{*}X_6\,\right\}^{1/2} \tag{3.87}$$

$$X_5 = -\,D_m^{-*}(\bar{D}_{12}^{*}X_8 + \bar{D}_{22}^{*}D_n) \tag{3.88}$$

$$X_4 = -\,D_m^{-*}\bar{D}_{12}^{*}X_7 \tag{3.89}$$

$$X_1 = -\,X_3^{-*}(X_6^{*}X_4 + \mathcal{D}_{11}^{*}X_7) \tag{3.90}$$

$$X_2 = -\,X_3\bar{D}_{21}^{*}D_n^{-*} \tag{3.91}$$

it follows from (3.55) that the indicated inverses of $D_n$ and $D_m$ exist, and Schur complement argument shown in section (2.2) will establish that $X_3$ and $X_7$ are nonsingular. A tedious calculation will demonstrate that (3.84) to (3.91) are consistent with the remaining equations in (3.71).

The all-pass embedding may now be completed with the aid of a construction due to Glover [12]. The idea is to find a $\bar{Q}(s)$, which may be thought of as replacement for $Q(s)$ in (3.54), which gives $\mathcal{S}(s)$ the required all-pass properties.

$$[\,\overset{p}{B}_{Q1}\;\; \overset{q}{B}_{Q2}\;\; \overset{m}{B}_{Q3}\;\; \overset{l}{0}\,] = R^{-*}(Q_1[0\ B_o\ B_1\ B_2]+[0\ C_o^{*}\ C_1^{*}\ C_2^{*}]\bar{D}) \tag{3.92}$$

in which

$$R=P_1Q_1-\gamma^2 I \tag{3.93}$$

The fact that the last '$l$' columns in (3.92) are zero follows from the all-pass equation (3.70) and is crucial to our development. In this section we assume that $\det(R)\neq 0$; $\det(R)=0$ may also arise, but we defer the treatment of this case until latter. Similarly we define

$$\begin{array}{c@{}l} \begin{bmatrix} C_{Q1} \\ C_{Q2} \\ C_{Q3} \\ 0 \end{bmatrix} & \begin{array}{l} m \\ l \\ p \\ q \end{array} \end{array} = \begin{bmatrix} 0 \\ C_o \\ C_1 \\ C_2 \end{bmatrix} P_1 + \bar{D} \begin{bmatrix} 0 \\ B_o^{*} \\ B_1^{*} \\ B_2^{*} \end{bmatrix} \tag{3.94}$$

The zero block at the bottom of (3.94) follows from the all-pass equation (3.69). Finally suppose that

$$A_Q=-A^*-[B_{Q1}\ B_{Q2}\ B_{Q3}\ 0][0\ B_o\ B_1\ B_2]^* \tag{3.95}$$

A routine calculation, which is essentially the same as that in [12], will verify that (3.67) and (3.68) are satisfied by

$$\begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} = \begin{bmatrix} P_1 & I \\ I & Q_1R^{-1} \end{bmatrix} \tag{3.96}$$

and

$$\begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix} = \begin{bmatrix} Q_1 & -R^* \\ -R & RP_1 \end{bmatrix} \tag{3.97}$$

Also, it is not hard to show that (3.92) to (3.97) satisfy (3.67) to (3.71), and so $\mathcal{E}(s)$ in (3.63) will satisfy $\mathcal{E}(s)\mathcal{E}^*(s)=\gamma^2 I$ as required. The (2,2) block of (3.68) gives

$$A_Q^*RP_1+RP_1A_Q+C_{Q1}^*C_{Q1}+C_{Q2}^*C_{Q2}+C_{Q3}^*C_{Q3}=0 \tag{3.98}$$

In this equation we note that $RP_1>0$ by virtue of (3.54) and Nehari's Theorem, and consequently, we have $\mathrm{Re}(\lambda(A_Q))<0$ as required.

*Temporary assumption 1:* For simplicity we shall assume that X and Y are non-singular, if this is not the case the state dimension of R(s) may be deflated by balancing (3.25) and (3.27) against each other [18,19] and Cancelling the unobservable and uncontrollable modes.

Lemma 3.2

Under the assumption that X and Y are nonsingular the controllability and observability grammians $P_1$ and $Q_1$ of $R_a(s)$ are given by :

$$P_1 = \begin{bmatrix} -\tilde{X} & 0 \\ 0 & -Y^{-1} \end{bmatrix} \tag{3.99}$$

and

$$Q_1 = \begin{bmatrix} -X^{-1} & Y \\ Y & -(\hat{X}+YXY) \end{bmatrix} \tag{3.100}$$

where $R_a(s)$ is defined as the augmented R(s) in (3.63) and X, Y, $\tilde{X}$ and $\hat{X}$ satisfy the algebraic Riccati equations given in (3.25), (3.27), (3.78), and (3.82) respectively. □

Proof:

It follows from (3.63b), (3.76), (3.80) and (3.83) that $R_a(s)$ has a state-space realization

$$R_a(s) \stackrel{s}{=} [\, A_{R_a}, B_{R_a}, C_{R_a}, D_{R_a} \,]; \tag{3.101a}$$

$$R_a(s) \stackrel{s}{=} \left[\begin{array}{cc|ccc} -(A-B_2F)^* & \Psi\Theta & B_n & \Psi D_{21}^* & \Psi\widetilde{D}^*_{\perp} \\ 0 & -(A-HC_2)^* & 0 & -C_2^* & Y^{-1}B_1\widetilde{D}^*_{\perp} \\ \hline 0 & C_m & * & * & D_m \\ -B_2^* & D_{12}^*D_{11}\Theta+FY & * & D_{12}^*D_{11}D_{21}^* & D_{12}^*D_{11}\widetilde{D}^*_{\perp} \\ D^*_{\perp}C_1X^{-1} & D^*_{\perp}D_{11}\Theta & D_n & D^*_{\perp}D_{11}D_{21}^* & D^*_{\perp}D_{11}\widetilde{D}^*_{\perp} \end{array}\right] \tag{3.101b}$$

where the entries $(*)$ indicate don't care, $\Psi$ and $\Theta$ are defined in (3.53). The equation defining the controllability grammian of $R_a(s)$ is

$$A_{R_a}P_1 + P_1A^*_{R_a} + B_{R_a}B^*_{R_a} = 0 \tag{3.102}$$

Block (2,2) of (3.102) $\Rightarrow$

$$-(A - HC_2)^*P_{22} - P_{22}(A - HC_2) + C_2^*C_2 + Y^{-1}B_1\widetilde{D}^*_{\perp}\widetilde{D}_{\perp}B_1^*Y^{-1} = 0 \tag{3.103}$$

comparing Y(3.103)Y with (3.27) reveals that

$$P_{22} = -Y^{-1} \tag{3.104}$$

Block (1,2) of (3.102) $\Rightarrow$

$$-(A - B_2F)^*P_{12} - \Psi\Theta Y^{-1} - P_{12}(A - HC_2) - \Psi D_{21}^*C_2 + \Psi\widetilde{D}^*_{\perp}\widetilde{D}_{\perp}B_1^*Y^{-1} = 0 \tag{3.105}$$

Substituting for $\Theta$ from (3.53) and combining the $\Psi$-terms gives

$$-(A - B_2F)^*P_{12} - P_{12}(A - HC_2) = 0 \tag{3.106}$$

Hence

$$P_{12} = 0 \tag{3.107}$$

Block (1,1) of (3.102) in the notation of (3.79) and (3.80) gives

$$\widetilde{A}P_{11} + P_{11}\widetilde{A}^* + (\widetilde{X}\widetilde{C}^* - \widetilde{B}\widetilde{D}^*)\widetilde{R}^{-1}(\widetilde{X}\widetilde{C}^* - \widetilde{B}\widetilde{D}^*)^* + \widetilde{B}\widetilde{B}^* = 0 \tag{3.108}$$

comparing (3.108) with (3.82) shows that it is satisfied by

$$P_{11} = -\widetilde{X} \tag{3.109}$$

Hence the controllability grammian of $R_a(s)$ is

$$P_1 = \begin{bmatrix} -\widetilde{X} & 0 \\ 0 & -Y^{-1} \end{bmatrix} \tag{3.110}$$

verifies the first part of the claim. Continuing further, the equation defining the observability grammian of $R_a(s)$ is

$$A^*_{R_a}Q_1 + Q_1A_{R_a} + C^*_{R_a}C_{R_a} = 0 \tag{3.111}$$

The (1,1) block of (3.111) $\Rightarrow$

$$-(A - B_2F)Q_{11} - Q_{11}(A - B_2F)^* + B_2B_2^* + X^{-1}C_1^*D_\perp D_\perp^* C_1X^{-1} = 0 \tag{3.112}$$

comparing X(3.112)X with (3.25) reveals that

$$Q_{11} = -X^{-1} \tag{3.113}$$

The (1,2) block of (3.111) $\Rightarrow$

$$\begin{aligned} &-(A-B_2D_{12}^*C_1-B_2B_2^*X)Q_{12}-\left\{X^{-1}(C_1-D_{12}F)^*D_{11}+B_1\right\}(B_1-HD_{21})^*-Q_{12}(A-HC_2)^* \\ &\quad -B_2\left\{D_{12}^*D_{11}(B_1-HD_{21})^*+D_{12}C_1Y+B_2^*XY\right\}+X^{-1}C_1^*(I-D_{12}D_{12}^*)D_{11}(B_1-HD_{21})^*=0 \end{aligned} \tag{3.114}$$

substituting $Q_{12} = Y$, allows (3.114) to simplify to

$$AY + B_1(B_1 - HD_{21})^* + Y(A - HC_2)^* = 0 \tag{3.115}$$

which may be rewritten

$$Y(A-B_1D_{21}^*C_2)^* + (A-B_1D_{21}^*C_2)Y - YC_2^*C_2Y + B_1\widetilde{D}_\perp^*\widetilde{D}_\perp^*B_1^* = 0 \tag{3.116}$$

$$\Rightarrow \qquad Q_{12} = Y \tag{3.117}$$

satisfies (3.114).The (2,2) block of (3.111) $\Rightarrow$

$$(B_1-HD_{21})\Big\{D_{11}^*(C_1-D_{12}F)+B_1^*X\Big\}Y+Y\Big\{(C_1-D_{12}F)^*D_{11}+XB_1\Big\}(B_1-HD_{21})^*-(A-HC_2)Q_{22}$$
$$-Q_{22}(A-HC_2)^*+\{D_{12}^*D_{11}(B_1-HD_{21})^*+FY\}^*\{D_{12}^*D_{11}(B_1-HD_{21})^*+FY\}$$
$$+(B_1-HD_{21})D_{11}^*D_{\perp}D_{\perp}^*D_{11}(B_1-HD_{21})^* + (\hat{B}^*\hat{X}-\hat{D}^*\hat{C})^*\hat{R}^{-1}(\hat{B}^*\hat{X}-\hat{D}^*\hat{C}) = 0 \qquad (3.118a)$$

Substituting $Q_{22}=-(\hat{X}+YXY)$ and (3.78) will simplify (3.118) to

$$(A-HC_2)YXY+YXY(A-HC_2)^*+(B_1-HD_{21})B_1^*XY+YXB_1(B_1-HD_{21})^*+YF^*FY-YC_1^*C_1Y=0 \quad (3.118b)$$

substituting (3.24) and (3.25) will simplify (3.118b) to

$$YX(3.27) + (3.27)XY = 0. \qquad (3.119)$$

Hence the observability grammian of $R_a(s)$ is given by

$$Q_1 = \begin{bmatrix} -X^{-1} & Y \\ Y & -(\hat{X}+YXY) \end{bmatrix} \qquad (3.120)$$

establishes the second part of the claim. □

3.3.2 Characterizing all controllers

This section will be devoted to deriving a closed form representation formula for the generator of all $H^\infty$ controllers which satisfy (3.1); in addition we will show that for problems where $\deg(P(s))=n$, the controllers also have degree $n$.

Theorem 2.5 shows that

$$\bar{Q}(s)=F_l\left\{\begin{bmatrix} Q & Q_{31} \\ Q_{13} & Q_{11} \end{bmatrix};U\right\}(s)=\Big\{Q+Q_{31}U(I-\ Q_{11}U)Q_{13}\Big\}(s) \qquad \gamma U(s)\in\mathcal{BH}_+^{\infty,pXm} \quad (3.121)$$

generates all solutions satisfying (3.54). From theorem 3.1, the class of all stabilizing controllers which minimize (3.1) is generated by

$$-\bar{K}(s) = F_l\Big\{K_o;\bar{Q}\Big\}(s) \qquad (3.122)$$

To proceed further we need to determine the realization of $\begin{bmatrix} Q & Q_{31} \\ Q_{13} & Q_{11} \end{bmatrix}(s)$. For later convenience, a generalized state-space representation is adopted from here on.

(3.92) gives

$$(P_1Q_1-\gamma^2 I)^*[\,B_{Q1}\; B_{Q2}\; B_{Q3}\; 0] =$$

$$\begin{bmatrix} -X^{-1} & Y \\ Y & -(\hat{X}+YXY) \end{bmatrix}\begin{bmatrix} 0 & B_n & B_{12} & B_{13} \\ 0 & 0 & B_{22} & B_{23} \end{bmatrix}+\begin{bmatrix} 0 & 0 & C_{21}^* & C_{31}^* \\ 0 & C_m^* & C_{22}^* & C_{32}^* \end{bmatrix}\begin{bmatrix} X_1 & X_2 & X_3 & 0 \\ X_4 & X_5 & X_6 & D_m \\ X_7 & X_8 & \mathcal{D}_{11} & \bar{D}_{12} \\ 0 & D_n & \bar{D}_{21} & \bar{D}_{22} \end{bmatrix}$$

where (3.123)

$$[\,0\; B_O\; B_1\; B_2\,] = \begin{bmatrix} 0 & B_n & B_{12} & B_{13} \\ 0 & 0 & B_{22} & B_{23} \end{bmatrix} \tag{3.124a}$$

and

$$\begin{bmatrix} 0 \\ C_O \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & C_m \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \tag{3.124b}$$

The (1,1) block of (3.123) gives

$$(P_1Q_1-\gamma^2 I)^* B_{Q1} = \begin{bmatrix} C_{21}^* X_7 \\ C_m^* X_4 + C_{22}^* X_7 \end{bmatrix}, \tag{3.125}$$

substituting from (3.53) into the (1,3) block of (3.123) yields

$$(P_1Q_1-\gamma^2 I)^* B_{Q3} = \begin{bmatrix} -H \\ YB_{12}-(\hat{X}+YXY)B_{22}+C_m^* X_6+C_{22}^*\mathcal{D}_{11}+C_{32}^*\bar{D}_{21} \end{bmatrix} \tag{3.126}$$

continuing, (3.94 ) gives

$$\begin{bmatrix} C_{Q1} \\ C_{Q2} \\ C_{Q3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & C_m \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}\begin{bmatrix} -\widetilde{X} & 0 \\ 0 & -Y^{-1} \end{bmatrix}+\begin{bmatrix} X_1 & X_2 & X_3 & 0 \\ X_4 & X_5 & X_6 & D_m \\ X_7 & X_8 & \mathcal{D}_{11} & \bar{D}_{12} \\ 0 & D_n & \bar{D}_{21} & \bar{D}_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ B_n^* & 0 \\ B_{12}^* & B_{22}^* \\ B_{13}^* & B_{23}^* \end{bmatrix} \tag{3.127}$$

The (1,1) block of (3.127) gives

$$C_{Q1} = [\ X_2B_{11}^* + X_3B_{12}^* \mid X_3B_{22}^*\ ] \tag{3.128}$$

Substituting from (3.53) into the (3,1) of (3.127) block yields

$$C_{Q3} = [\ -C_{21}\widetilde{X} + X_8B_{11}^* + \mathfrak{D}_{11}B_{12}^* + \bar{D}_{12}B_{13}^* \mid -F] \tag{3.129}$$

$(P_1Q_1-\gamma^2I)^*$

Partitioning $A_Q$ defined in (3.95)

$$\begin{bmatrix} A_Q^{11} & A_Q^{12} \\ A_Q^{21} & A_Q^{22} \end{bmatrix} = (P_1Q_1-\gamma^2I)^* \begin{bmatrix} (A-B_2F) & 0 \\ -(B_1-HD_{21})\left\{(C_1-D_{12}F)D_{11}+XB_1\right\}^* & (A-HC_2) \end{bmatrix}$$

$$-(P_1Q_1-\gamma^2I)^*[\ B_{Q1}\ B_{Q2}\ B_{Q3}\ 0] \begin{bmatrix} 0 & 0 \\ B_{11}^* & 0 \\ B_{12}^* & B_{22}^* \\ B_{13}^* & B_{23}^* \end{bmatrix} \tag{3.130}$$

The next part of this development is concerned with calculating a closed form representation formula for all controllers. We will establish this with a series of intricate manipulations using the state-space representation of (3.122).

$$-\bar{K}(s) = F_l\left\{K_0;\ F_l\left\{\begin{bmatrix} Q & Q_{31} \\ Q_{13} & Q_{11} \end{bmatrix};U\right\}\right\}(s) = F_l\left\{\begin{bmatrix} E_k & A_k & B_k \\ 0 & C_k & D_k \end{bmatrix};U\right\} \quad \text{where } \gamma U(s)\in\mathfrak{B}\mathcal{H}_+^\infty \tag{3.131}$$

$A_k$ is partitioned into nine nxn sub-blocks, then following the construction given in Appendix B we have

$$A_k = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where

$$A_{11} = A-B_2F-HC_2+HD_{22}F \tag{3.132}$$

$$\begin{bmatrix} A_{12} & A_{13} \end{bmatrix} = (B_2-HD_{22})[\ -C_{21}\widetilde{X}+X_8B_{11}^*+\mathfrak{D}_{11}B_{12}^*+\bar{D}_{12}B_{13}^* \mid -F] \tag{3.133}$$

$$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} -H \\ YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathfrak{D}_{11}+C_{32}^*\bar{D}_{21} \end{bmatrix}(C_2-D_{22}F) \tag{3.134}$$

and

$$\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_Q^{11} & A_Q^{12} \\ A_Q^{21} & A_Q^{22} \end{bmatrix} + \begin{bmatrix} -H \\ YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathfrak{D}_{11}+C_{32}^*\bar{D}_{21} \end{bmatrix} \cdot$$
$$D_{22}[\, -C_{21}\widetilde{X} + X_8B_n^* + \mathfrak{D}_{11}B_{12}^* + \bar{D}_{12}B_{13}^* \mid -F] \tag{3.135}$$

$$E_k = \begin{bmatrix} I & 0 & 0 \\ 0 & X^{-1}\widetilde{X}-\gamma^2 I & -I \\ 0 & -Y\widetilde{X} & \hat{X}Y^{-1}+YX-\gamma^2 I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & R_1^* & R_3^* \\ 0 & R_2^* & R_4^* \end{bmatrix} \tag{3.136}$$

$$B_k = \left[\begin{array}{c|c} -H & -(B_2-HD_{22})X_7 \\ -H & -(B_2-HD_{22})X_7 \\ YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathfrak{D}_{11}+C_{32}^*\bar{D}_{21} & C_m^*X_4+C_{22}^*X_7-\left\{YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathfrak{D}_{11}+C_{32}^*\bar{D}_{21}\right\}D_{22}X_7 \end{array}\right] \tag{3.137}$$

$$C_k = \left[\begin{array}{c|c|c} -F & -C_{21}\widetilde{X} + X_8B_n^* + \mathfrak{D}_{11}B_{12}^* + \bar{D}_{12}B_{13}^* & -F \\ -X_3(C_2-D_{22}F) & X_2B_n^* + X_3B_{12}^* - X_3D_{22}(-C_{21}\widetilde{X}+X_8B_n^*+\mathfrak{D}_{11}B_{12}^*+\bar{D}_{12}B_{13}^*) & -X_3(C_2-D_{22}F) \end{array}\right] \tag{3.138}$$

and finally

$$D_k = \begin{bmatrix} 0 & -X_7 \\ -X_3 & X_3D_{22}X_7-X_1 \end{bmatrix} \tag{3.139}$$

$[E_k\ A_k\ B_k\ C_k\ D_k]$ given by (3.132) to (3.139) is a 3-$n$ state space representation formula for all $K(s)\in\mathbf{S}(P_{22})$, however as will be shown this representation can be reduced to an $n$-state formula.

The first basis change in the state-space of (3.132) to (3.139) is given by

$$T_l = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.140}$$

and

$$T_r = \begin{bmatrix} I & R_1^* & R_3^* \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{3.141}$$

where the subscripts '$l$' and '$r$' correspond to the left and right basis transformations respectively.

$$T_l B_k = \left[\begin{array}{c|c} 0 & 0 \\ -H & -(B_2 - HD_{22})X_7 \\ YB_{12} - (\hat{X}+YXY)B_{22} + C_m^* X_6 + C_{22}^* \mathfrak{D}_{11} + C_{32}^* \bar{D}_{21} & C_m^* X_4 + C_{22}^* X_7 - \left\{YB_{12} - (\hat{X}+YXY)B_{22} + C_m^* X_6 + C_{22}^* \mathfrak{D}_{11} + C_{32}^* \bar{D}_{21}\right\} D_{22} X_7 \end{array}\right] \tag{3.142}$$

$$T_l E_k T_r = \begin{bmatrix} I & 0 & 0 \\ 0 & R_1^* & R_3^* \\ 0 & R_2^* & R_4^* \end{bmatrix} \tag{3.143}$$

$$T_l A_k T_r = \begin{bmatrix} A_{11} - A_{21} & (A_{11} - A_{21})R_1^* + (A_{12} - A_{22}) & (A_{11} - A_{21})R_3^* + (A_{13} - A_{23}) \\ A_{21} & A_{21}R_1^* + A_{22} & A_{21}R_3^* + A_{23} \\ A_{31} & A_{31}R_1^* + A_{32} & A_{31}R_3^* + A_{33} \end{bmatrix} \tag{3.144}$$

$$C_k T_r = \left[\begin{array}{c|c|c} -F & -FR_1^* - C_{21}\widetilde{X} + X_8 B_n^* + \mathfrak{D}_{11} B_{12}^* + \bar{D}_{12} B_{13}^* & 0 \\ -X_3(C_2 - D_{22}F) & -X_3(C_2 - D_{22}F)R_1^* + X_2 B_n^* + X_3 B_{12}^* - X_3 D_{22}(-C_{21}\widetilde{X} + X_8 B_n^* + \mathfrak{D}_{11} B_{12}^* + \bar{D}_{12} B_{13}^*) & 0 \end{array}\right] \tag{3.145}$$

<u>Lemma</u> <u>3.3</u>

$$(A_{11} - A_{21})R_3^* + (A_{13} - A_{23}) = 0 \tag{3.146a}$$

and

$$(A_{11} - A_{21})R_1^* + (A_{12} - A_{22}) = 0 \tag{3.146b}$$

<u>Proof:</u>

Substituting from (3.132) to (3.136) we get

$$
\begin{aligned}
(A_{11}-A_{21})R_3^*+(A_{13}-A_{23}) &= -(A-HC_2-B_2F+HD_{22}F)-H(C_2-D_{22}F)-(B_2-HD_{22})F \\
&\qquad +(A-HC_2)+HC_2-HD_{22}F \qquad (3.147) \\
&= 0
\end{aligned}
$$

proves the first part.

Substituting (3.132) to (3.136) we get

$$
\begin{aligned}
(A_{11}-A_{21})R_1^*+(A_{12}-A_{22}) = {} & (A-B_2F)X^{-1}\widetilde{X}+B_2\left\{B_2^*\widetilde{X}+D_{12}^*D_{11}\widetilde{B}\right\}-X^{-1}\widetilde{X}(A-B_2F)-B_1\widetilde{B}^* \\
& -X^{-1}(\widetilde{X}\widetilde{C}^*-\widetilde{B}\widetilde{D}^*)\widetilde{R}^{-1}(\widetilde{X}\widetilde{C}^*-\widetilde{B}\widetilde{D}^*)^*+X^{-1}C_1^*D_{\perp}(\widetilde{X}\widetilde{C}^*-\widetilde{B}\widetilde{D}^*)^* \qquad (3.148)
\end{aligned}
$$

substituting from (3.25) gives

$$
\begin{aligned}
(A_{11}-A_{21})R_1^*+(A_{12}-A_{22}) &= X^{-1}\left\{\widetilde{A}\widetilde{X}+\widetilde{X}\widetilde{A}^*-\widetilde{B}\widetilde{B}^*-(\widetilde{X}\widetilde{C}^*-\widetilde{B}\widetilde{D}^*)\widetilde{R}^{-1}(\widetilde{X}\widetilde{C}^*-\widetilde{B}\widetilde{D}^*)^*\right\} \qquad (3.149) \\
&= 0
\end{aligned}
$$

by (3.82) proves the second part. □

Lemma 3.3 and (3.142) establish that there are n uncontrollable modes in the realization of $\bar{K}(s)$ given by (3.132) to (3.139), therefore in the new basis $\bar{K}(s)\stackrel{S}{=}[E_kA_kB_kC_kD_k]$ is given by

$$
E_k = \begin{bmatrix} R_1^* & R_3^* \\ R_2^* & R_4^* \end{bmatrix} \qquad (3.150)
$$

$$
A_k = \begin{bmatrix} A_{21}R_1^*+A_{22} & A_{21}R_3^*+A_{23} \\ A_{31}R_1^*+A_{32} & A_{31}R_3^*+A_{33} \end{bmatrix} \qquad (3.151)
$$

$$
B_k = \left[\begin{array}{c|c} -H & -(B_2-HD_{22})X_7 \\ YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathfrak{D}_{11}+C_{32}^*\bar{D}_{21} & C_m^*X_4+C_{22}^*X_7-\left\{YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathfrak{D}_{11}+C_{32}^*\bar{D}_{21}\right\}D_{22}X_7 \end{array}\right] \qquad (3.152)
$$

$$
C_kT_r = \left[\begin{array}{c|c} -FR_1^*-C_{21}\widetilde{X}+X_8B_n^*+\mathfrak{D}_{11}B_{12}^*+\bar{D}_{12}B_{13}^* & 0 \\ -X_3(C_2-D_{22}F)R_1^*+X_2B_n^*+X_3B_{12}^*-X_3D_{22}(-C_{21}\widetilde{X}+X_8B_n^*+\mathfrak{D}_{11}B_{12}^*+\bar{D}_{12}B_{13}^*) & 0 \end{array}\right] \qquad (3.153)
$$

$D_k$ is unchanged and is given in (3.139).

Introducing a second basis change

$$T_l = \begin{bmatrix} I & -R_3^*R_4^{-*} \\ 0 & I \end{bmatrix} \tag{3.154}$$

where we assume that $R_4$ is non-singular

$$T_lE_k = \begin{bmatrix} R_1^*-R_3^*R_4^{-*}R_2^* & 0 \\ R_2^* & R_4^* \end{bmatrix} \tag{3.155}$$

$$T_lB_k = \begin{bmatrix} -H-R_3^*R_4^{-*}\Delta & -(B_2-HD_{22})X_7-R_3^*R_4^{-*}\{C_m^*X_4+C_{22}^*X_7-\Delta D_{22}X_7\} \\ \Delta & C_m^*X_4+C_{22}^*X_7-\Delta D_{22}X_7 \end{bmatrix} \tag{3.156}$$

where

$$\Delta=YB_{12}-(\hat{X}+YXY)B_{22}+C_m^*X_6+C_{22}^*\mathcal{D}_{11}+C_{32}^*\bar{D}_{21} \tag{3.157}$$

$$T_lA_k= \begin{bmatrix} A_{21}R_1^*+A_{22}-R_3^*R_4^{-*}(A_{31}R_1^*+A_{32}) & A_{21}R_3^*+A_{23}-R_3^*R_4^{-*}(A_{31}R_3^*+A_{33}) \\ A_{31}R_1^*+A_{32} & A_{31}R_3^*+A_{33} \end{bmatrix} \tag{3.158}$$

<u>Lemma 3.4</u>

$$A_{21}R_3^*+A_{23}-R_3^*R_4^{-*}(A_{31}R_3^*+A_{33})=0 \tag{3.159}$$

<u>Proof:</u>

By direct calculation we have

$$\begin{aligned} A_{21}R_3^*+A_{23}-R_3^*R_4^{-*}(A_{31}R_3^*+A_{33}) &= H(C_2-D_{22}F)+R_3^*(A-HC_2)-HC_2+HD_{22}F+R_4^{-*}\left\{R_4^*(A-HC_2)+\Delta C_2\right\} \\ &\quad -R_4^{-*}\Delta D_{22}F-R_4^{-*}\Delta(C_2-D_{22}F) \qquad (3.160) \\ &= 0 \end{aligned}$$

completes the proof. □

Lemma 3.4 and (3.153) establish that the realization of $\bar{K}(s)$ given in (3.150) to (3.153) has n unobservable modes, therefore in the new basis $\bar{K}(s)=[E_kA_kB_kC_kD_k]$ becomes

$$E_k = R_4^*R_1^*+R_2^* \tag{3.161a}$$

$$A_k = R_4^*A_{21}R_1^*+R_4^*A_{22}+A_{31}R_1^*+A_{32} \tag{3.161b}$$

$$B_k = [-R_4^*H+\Delta \qquad -R_4^*(B_2-HD_{22})X_7+\{C_m^*X_4+C_{22}^*X_7-\Delta D_{22}X_7\}] \tag{3.161c}$$

$$C_k = \begin{bmatrix} -FR_1^* - C_{21}\widetilde{X} + X_8B_n^* + \mathfrak{D}_{11}B_{12}^* + \bar{D}_{12}B_{13}^* \\ -X_3(C_2-D_{22})R_1^* + X_2B_n^* + X_3B_{12}^* - X_3D_{22}(-C_{21}\widetilde{X} + X_8B_n^* + \mathfrak{D}_{11}B_{12}^* + \bar{D}_{12}B_{13}^*) \end{bmatrix} \tag{3.161d}$$

$$D_k = \begin{bmatrix} 0 & -X_7 \\ -X_3 & X_3D_{22}X_7 - X_1 \end{bmatrix} \tag{3.161e}$$

after substituting for $R_3^*$ and a left transformation by $R_4^*$; this last transformation avoids the use of $R_4^{-*}$ in the representation formula.

Theorem 3.5

For any $H^\infty$-optimal control problem of the third kind, every $H^\infty$-optimal controller is generated by

$$-K(s) = F_l\left\{\begin{bmatrix} E_k & A_k & B_k \\ 0 & C_k & D_k \end{bmatrix}; U\right\}(s) \qquad \text{where } \gamma U(s) \in \mathfrak{B}\mathcal{H}_\infty^+ \tag{3.162}$$

and satisfies

1) $\deg(K) \leq n - r + \deg(U)$

in the case of inertia type optimality, where 'r' is the multiplicity of the rank defect in $E_k$.

2) $\deg(K) \leq n + \deg(U)$

in the cases of Parrott type optimality and sub-optimal problems. □

It is clear that these degree bounds also hold for problems of the first and second kind, such results have already been established in [17,18,19,32].

The purpose of using a descriptor representation formula is the same as that described in section 2.4. If $\gamma$ is optimal due to inertia, a singular perturbation proceedure may be used to remove the infinite pole(s) from K(s), and will have associated with it the reduction in the effective dimension of the free parameter U(s). Even in the sub-optimal case $E_k^{-1}$ may be badly conditioned and is consequently best avoided.

At this stage let us summarize the findings of this chapter to this point. The four block $H^\infty$ control problem has been solved by embedding it in a larger all-pass matrix. The subsequent generator of all solutions is substituted into (3.122) to yield the generator of all stabilizing controllers; furthermore, the McMillan degree of the central solution is no more than n. Given a P(s), one may compute the $H^\infty$ optimal controller in the following way; solve for X and Y in (3.25) and (3.27) respectively and calculate $\gamma_{upper}$ and $\gamma_{low}$ from (3.46) and (3.47). Select a $\gamma$ between the bounds and solve the two spectral factorization problems in (3.59) and (3.60). Successively bisecting the search interval will lead to either R in (3.93) becoming singular or one/or both the spectral factorization problems not having a solution. These have been termed inertia and Parrott type optimalities respectively. Finally generate all solutions to this problem using (3.161). Although this algorithm is a significant improvement over the one presented in section 3.2.2, it turns out that the $H^\infty$ control problem has still further structure which shall be exploited. What is interesting, is that the two parameterization Riccati equations (3.25) and (3.27) may be combined with the two spectral factorization equations in (3.59) and (3.60) to yield two indefinite equations. Further, it can be shown that the generator of all controllers in (3.162) is expressed in terms of the state-space of P(s) and the solutions to these new Riccati equations.

*Temporary assumption 2:* As it turns out a detailed study of the structure in this problem is significantly simplified by assuming that $D_{11}=0$ and $D_{22}=0$ in (3.2). Since the same basic approach will solve the general case, we will study the simple case first and defer the more complicated calculations to later.

In the next stage of our development, equations (3.25) and (3.27) are combined with (3.82) and (3.78) respectively to yield two indefinite algebraic Riccati equations.

$\gamma^2$(3.25)-(3.82) can be rewritten as

$$\begin{aligned}&(\gamma^2X+\tilde{X})(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*(\gamma^2X+\tilde{X})+\gamma^{-2}(\gamma^2I-\tilde{X}X^{-1})C_1^*D_\perp D_\perp^*C_1(\gamma^2I-\tilde{X}X^{-1})\\&\quad+\tilde{X}X^{-1}C_1^*D_\perp D_\perp^*C_1+C_1^*D_\perp D_\perp^*C_1X^{-1}\tilde{X}-\tilde{X}B_2B_2^*X-\gamma^2XB_2B_2^*X-XB_2B_2^*\tilde{X}+XB_1B_1^*X=0\end{aligned}$$

Substituting $\tilde{X}X^{-1}$(3.25) and (3.25)$X^{-1}\tilde{X}$ yields

$$\begin{aligned}&X(A-B_2D_{12}^*C_1)(\gamma^2I-X^{-1}\tilde{X})+(\gamma^2I-\tilde{X}X^{-1})(A-B_2D_{12}^*C_1)^*X+\gamma^{-2}(\gamma^2I-\tilde{X}X^{-1})C_1^*D_\perp D_\perp^*C_1(\gamma^2I-X^{-1}\tilde{X})\\&\quad-X(\gamma^2B_2B_2^*-B_1B_1^*)X=0\end{aligned}$$

which may be rewritten as

$$P(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*P-P(\gamma^2B_2B_2^*-B_1B_1^*)P+C_1^*D_\perp D_\perp^*C_1\gamma^{-2}=0 \tag{3.163}$$

where $P=(\gamma^2I-\tilde{X}X^{-1})^{-1}X$ in which $(\gamma^2I-\tilde{X}X^{-1})$ is assumed non-singular. A dual set of arguments using (3.27) and (3.78) yields

$$S(A-B_1D_{21}^*C_2)^*+(A-B_1D_{21}^*C_2)S-S(\gamma^2C_2^*C_2-C_1^*C_1)S+B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*\gamma^{-2}=0 \tag{3.164}$$

where $S=(\gamma^2I-\hat{X}Y^{-1})^{-1}Y$ in which $(\gamma^2I-\hat{X}Y^{-1})$ is assumed non-singular.

Next we shall derive a closed form representation formula for the generator of all stabilizing controllers under the assumptions 1 and 2. Using (3.136), $E_k$ becomes

$$E_k=\hat{X}Y^{-1}X^{-1}\widetilde{X}-\hat{X}Y^{-1}\gamma^2-YX\gamma^2-X^{-1}\widetilde{X}\gamma^2+\gamma^4I$$

two consecutive basis changes from the left and right $T_l=SY^{-1}$ and $T_r=X^{-1}P$ yield

$$E_k=I-\gamma^2SP \tag{3.165}$$

Define $B_k=[\ B_{k1}\ B_{k2}\ ]$; substituting from (3.157), (3.136) and (3.101) gives

$$\begin{aligned}B_{k1} &=-(\hat{X}Y^{-1}+YX-\gamma^2I)B_1D_{21}^*+YXB_1D_{21}^*+(\hat{X}+YXY)C_2^*-(\hat{X}Y^{-1}+YX-\gamma^2I)YC_2^*\\ &=(\gamma^2I-\hat{X}Y^{-1})B_1D_{21}^*+\gamma^2YC_2^*\end{aligned}$$

the left transformation $T_l$ yields

$$B_{k1}=B_1D_{21}^*+\gamma^2SC_2^* \tag{3.166a}$$

Similarly,

$$\begin{aligned}B_{k2} &=-(\hat{X}Y^{-1}+YX-\gamma^2I)B_2\gamma+Y(D_{12}^*C_1+B_2^*X)^*\gamma\\ &=\gamma(B_2+SC_1^*D_{12})\end{aligned} \tag{3.166b}$$

after the basis transformation with $T_l$.

Define $C_k=[\ C_{k1}^T\ C_{k2}^T\ ]^T$; substituting from (3.136) and (3.101) gives

$$\begin{aligned}C_{k1} &=-(D_{12}^*C_1+B_2^*X)X^{-1}\widetilde{X}+B_2^*\widetilde{X}+(D_{12}^*C_1+B_2^*X)\gamma^2\\ &=D_{12}^*C_1(\gamma^2I-X^{-1}\widetilde{X})+B_2^*X\gamma^{-2}\end{aligned}$$

applying the right basis transformation $T_r=X^{-1}P$ yields

$$C_{k1}=D_{12}^*C_1+B_2^*P\gamma^2 \tag{3.167a}$$

Similarly,

$$\begin{aligned}C_{k2} &=\gamma C_2(\gamma^2I-X^{-1}\widetilde{X})+\gamma D_{21}B_1^*X\\ &=\gamma(C_2+D_{21}B_1^*P)\end{aligned} \tag{3.167b}$$

after the basis transformation with $T_r$. Clearly (3.161e) gives

$$D_k=\begin{bmatrix}D_{k11} & D_{k12}\\ D_{k21} & D_{k22}\end{bmatrix}=\begin{bmatrix}0 & -\gamma I\\ -\gamma I & 0\end{bmatrix} \tag{3.168}$$

Finally we evaluate the controller A-matrix. Substituting (3.134) to (3.136) into (3.161b) yields

$$A_k=-(\hat{X}Y^{-1}+YX-\gamma^2I)HC_2(X^{-1}\widetilde{X}-\gamma^2I)+\left\{YB_{12}-(\hat{X}+YXY)B_{22}\right\}C_2(X^{-1}\widetilde{X}-\gamma^2I)-Y\widetilde{X}(A-B_2F)$$
$$+(\hat{X}Y^{-1}+YX-\gamma^2I)\left\{(X^{-1}\widetilde{X}-\gamma^2I)(A-B_2F)+X^{-1}B_nB_n^*-C_{31}^*D_nB_n^*+HB_{12}^*\right\}-YB_nB_n^*$$
$$-(YB_{12}-(\hat{X}+YXY)B_{22})B_{12}^* \tag{3.169}$$

Substituting from (3.80) and (3.101) one can rewrite (3.169)

$$A_k=-(\hat{X}Y^{-1}-\gamma^2I)B_1D_{21}^*C_2(X^{-1}\widetilde{X}-\gamma^2I)+\gamma^2YC_2^*C_2(X^{-1}\widetilde{X}-\gamma^2I)-\gamma^2YC_2^*D_{21}B_1^*X$$
$$+(\hat{X}Y^{-1}-\gamma^2I)B_1D_{21}^*D_{21}B_1^*X+(\hat{X}Y^{-1}-\gamma^2I)X^{-1}\widetilde{X}(A-B_2F)-YC_1^*D_\perp D_\perp^*C_1X^{-1}\widetilde{X}$$
$$+\gamma^{-2}(\hat{X}Y^{-1}-\gamma^2I)(X^{-1}\widetilde{X}X^{-1}-X^{-1}\gamma^2)C_1^*D_\perp D_\perp^*C_1X^{-1}\widetilde{X}-\gamma^2(\hat{X}Y^{-1}+YX-\gamma^2I)(A-B_2F)$$

Substituting the $(A-B_2F)$ terms from (3.25) and grouping terms yields

$$A_k=-(\hat{X}Y^{-1}-\gamma^2I)B_1D_{21}^*C_2(X^{-1}\widetilde{X}-\gamma^2I)-(\hat{X}Y^{-1}-\gamma^2I)(X^{-1}\widetilde{X}X^{-1}-X^{-1}\gamma^2)(A-B_2D_{12}^*C_1)^*X$$
$$+YC_1^*D_\perp D_\perp^*C_1(\gamma^2I-X^{-1}\widetilde{X})+\gamma^2YC_2^*C_2(X^{-1}\widetilde{X}-\gamma^2I)+\gamma^2Y(A-B_2D_{12}^*C_1)^*X-\gamma^2YC_2^*D_{21}B_1^*X$$
$$+\gamma^{-2}(\hat{X}Y^{-1}-\gamma^2I)(X^{-1}\widetilde{X}X^{-1}-X^{-1}\gamma^2)C_1^*D_\perp D_\perp^*C_1(X^{-1}\widetilde{X}-\gamma^2I)+(\hat{X}Y^{-1}-\gamma^2I)B_1D_{21}^*D_{21}B_1^*X$$

Applying the right and left basis changes $T_l$ and $T_r$ gives

$$A_k=-B_1D_{21}^*C_2-P^{-1}(A-B_2D_{12}^*C_1)^*P+SC_1^*D_\perp D_\perp^*C_1-SC_2^*C_2\gamma^2+\gamma^2S(A-B_2D_{12}^*C_1)^*P$$
$$-\gamma^2SC_2^*D_{21}B_1^*P-P^{-1}C_1^*D_\perp D_\perp^*C_1\gamma^{-2}-B_1D_{21}^*D_{21}B_1^*P$$

substituting $P^{-1}$(3.163) and $\gamma^2S$(3.163) gives

$$A_k=(A-B_2D_{12}^*C_1)-(\gamma^2B_2B_2^*-B_1B_1^*)P-B_1D_{21}^*C_2-SP(A-B_2D_{12}^*C_1)\gamma^2+SP(\gamma^2B_2B_2^*-B_1B_1^*)P\gamma^2$$
$$-\gamma^2SC_2^*C_2-\gamma^2SC_2^*D_{21}B_1^*P-B_1D_{21}^*D_{21}B_1^*P$$

which may be rewritten as

$$A_k=E_k(A-B_2C_{k1}+B_1B_1^*P)-\gamma^{-1}B_{k1}C_{k2} \tag{3.170a}$$

or

$$A_k=(A-B_2D_{12}^*C_1-B_1D_{21}^*C_2)+S\gamma^2(A-B_2D_{12}^*C_1-B_1D_{21}^*C_2)^*P$$
$$-(\gamma^2B_2B_2^*-B_1\widetilde{D}_\perp^*\widetilde{D}_\perp B_1^*)P-S(\gamma^2C_2^*C_2-C_1^*D_\perp D_\perp^*C_1) \tag{3.170b}$$

<u>Remark</u> <u>3.2:</u>

In the case that $(\gamma^2I-\widetilde{X}X^{-1})$ and/or $(\gamma^2I-\hat{X}Y^{-1})$ are singular P and/or S become unbounded. It is possible to avoid this difficulty by expressing the representation formula in terms of the stable deflating subspaces associated with Hamiltonians which go with (3.163) and (3.164). These details have been worked out by Hung [16] in a related context.

One may write

$$P=P_2P_1^{-1}$$

and

$$S=S_2S_1^{-1} \Rightarrow S=S_1^{-*}S_2^* \quad (\text{since } S=S^*)$$

Introducing the basis change

(i) $S_1^*(sE_k-A_k)P_1$

(ii) $S_1^*B_{ki}$ $\qquad i=1,2$

(iii) $C_{ki}P_1$ $\qquad i=1,2$

yields

$$\begin{aligned}
E_k&=S_1^*P_1-\gamma^2S_2^*P_2\\
A_k&=S_1^*\widetilde{A}+\gamma^2S_2^*\widetilde{A}P_2-S_1^*(B_2B_2^*\gamma^2-B_1\widetilde{D}^*_\perp\widetilde{D}_\perp B_1^*)P_2-S_2^*(C_2^*C_2\gamma^2-C_1^*D_\perp D^*_\perp C_1)P_1\\
B_{k1}&=S_1^*B_1D_{21}^*+S_2^*C_2^*\gamma^2\\
B_{k2}&=(S_1^*B_2+S_2^*C_1^*D_{12})\gamma\\
C_{k1}&=D_{12}^*C_1P_1+B_2^*P_2\gamma^2\\
C_{k2}&=(C_2P_1+D_{21}B_1^*P_2)\gamma
\end{aligned}$$

which is free from terms in $P_1^{-1}$ and $S_1^{-1}$. These manipulations have not been proved rigorous in the cases when $P_1$ and/or $S_1$ are identically singular. □

Remark 3.3

Equations (3.163) and (3.164) form an interesting link with LQG theory. It is not difficult to show that $\lim_{\gamma\uparrow\infty}\gamma^2(3.163)=(3.25)$ and $\lim_{\gamma\uparrow\infty}\gamma^2(3.164)=(3.27)$. In the case of the central solution:

$$\lim_{\gamma\uparrow\infty} K(s) \stackrel{s}{=} \left[\begin{array}{cc} A-B_2F-HC_2 & H \\ -F & 0 \end{array}\right]$$

which is reminiscent of LQG. □

## 3.4 Necessary and sufficient conditions for the existence of $H^\infty$-optimal controllers.

Section 3.3.2 has shown that it is possible to parameterize all controllers in terms of two n-dimensional Riccati equations (3.163) and (3.164). If P and S are the solutions to these equations, then it will be shown that (1) $P\geq0$, (2) $S\geq0$ and (3) $\lambda_{max}(PS)\leq\gamma^{-2}$ are necessary and sufficient conditions for the existence of a solution such that $||F_l(P,K)||_\infty\leq\gamma$. The aim of this section is to prove this result using vector interpolation theory; this proof addresses the optimal cases in which P and S are bounded. We will prove that the four block general distance problem has an associated

Pick matrix which is congruent to

$$\Pi(\gamma) = \begin{bmatrix} P^{-1} & \gamma I \\ \gamma I & S^{-1} \end{bmatrix}$$

where the indicated inverses are assumed to exist. It will be shown that the necessary and sufficient condition for a solution to exist is $\Pi(\gamma)\geq 0$, which by a Schur complement argument, is equivalent to $P>0$, $S>0$ and $\lambda_{max}(PS)\leq\gamma^{-2}$. If either $P^{-1}$ or $S^{-1}$ do not exist, a routine balancing argument will reduce the dimension of the interpolation problem; $P^{-1}$ and $S^{-1}$ will always exist for the smaller problem. Once (P,S) with the desired properties have been found, they may be substituted into the representation formula which parameterizes all the solutions (3.165) to (3.170).

In section 3.3.1, we have shown that the four block problem given in (3.54) may be solved by embedding it in a larger all-pass matrix. Lets suppose that (3.54) is embedded in the all-pass matrix

$$\mathcal{S}(s) = \begin{array}{c} \begin{array}{ccc} p+q & m & l \end{array} \\ \begin{bmatrix} R_{00}-Q_{00} & R_{01}-Q_{01} & R_{02} \\ R_{10}-Q_{10} & R_{11}-Q_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix} \end{array} \begin{array}{c} m+l \\ p \\ q \end{array} (s) \tag{3.171}$$

in which, we choose

$$R_{02}(s) = \begin{array}{c} l \\ \begin{bmatrix} 0 \\ M(s) \end{bmatrix} \end{array} \begin{array}{c} m \\ l \end{array}$$

and

$$R_{20}(s) = \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \begin{bmatrix} 0 & N(s) \end{bmatrix} \end{array} q$$

$$\inf_{Q\in H_+^\infty} \left\| \begin{bmatrix} I & 0 & 0 \\ 0 & T_{12} & T_\perp \end{bmatrix} (3.171) \begin{bmatrix} I & 0 \\ 0 & T_{21} \\ 0 & \tilde{T}_\perp \end{bmatrix} (s) \right\|_\infty \tag{3.172}$$

$$= \inf_{Q\in \mathcal{H}_+^\infty} \| ( \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & T_{12} & T_\perp \end{bmatrix} \begin{bmatrix} Q_{00} & Q_{01} & 0 \\ Q_{10} & Q_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T_{21} \\ 0 & \tilde{T}_\perp \end{bmatrix} )(s) \|_\infty \tag{3.173}$$

which is a dilated version of the original problem posed in (3.42). The necessary and sufficient conditions are established by studying the vector interpolation problem associated with (3.173). Use will be made of the interpolation theory described in [17]. In particular, we show that the Pick matrix associated the problem in (3.173) is congruent[1] to

$$\Pi(\gamma) = \begin{bmatrix} P^{-1} & \gamma I \\ \gamma I & S^{-1} \end{bmatrix} \tag{3.174}$$

if X and Y are nonsingular (assumption 1). Otherwise, a smaller Pick matrix with dimension Rank(X)+Rank(Y) must be constructed.

3.4.1 Interpolation theory

We will prove the necessary and sufficient conditions in two steps. Firstly, we will establish that a solution to the $H^{\infty}$ control problem exists if and only if the Pick matrix associated with (3.173) is positive semi-definite. Secondly, we will show by state-space calculation that the Pick matrix has the particular form given in (3.174).

Lemma 3.6

A stabilizing controller K(s) such that $\|F_l(P,K)(s)\|_{\infty} \le \gamma$ exist if and only if $\Pi(\gamma) \ge 0$.

Proof:

Necessity

Suppose there exists a stabilizing controller K(s) such that $\|F_l(P(s),K(s))\|_{\infty} \le \gamma$; P(s) is the design problem matrix given in (3.2). This assumption implies that there exists a $Q(s) \in \mathcal{H}_+^{\infty}$ such that

$$\|(T_{11} - [T_{12}\ T_{\perp}] \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{21} \\ \tilde{T}_{\perp} \end{bmatrix})(s)\|_{\infty} \le \gamma \tag{3.175}$$

by standard parameterization theory arguments[9,26]. Since $[T_{12}\ T_{\perp}](s)$ and $[T_{21}^{*}\ \tilde{T}_{\perp}^{*}]^{*}(s)$ may be chosen inner, we have

$$\left\| \begin{bmatrix} R_{11}-Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix}(s) \right\|_{\infty} \le \gamma \tag{3.176}$$

By standard dilation theory [11], one may construct the all-pass matrix

[1]We say that two square matrices of the same dimension are congruent, if they have the same inertia.

$$\left\| \begin{bmatrix} R_{00}-Q_{00} & R_{01}-Q_{01} & R_{02} \\ R_{10}-Q_{10} & R_{11}-Q_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix}(s) \right\|_{\infty} = \gamma \tag{3.177}$$

in such a way that

$$[\, R_{20}\; R_{21}\; R_{22}](s) \tag{3.178a}$$

and

$$\begin{bmatrix} R_{02} \\ R_{12} \\ R_{22} \end{bmatrix}(s) \tag{3.178b}$$

are parts of inner matrices. Finally, $\begin{bmatrix} I & 0 & 0 \\ 0 & T_{12} & T_{\perp} \end{bmatrix}$(3.177)$\begin{bmatrix} I & 0 \\ 0 & T_{21} \\ 0 & \tilde{T}_{\perp} \end{bmatrix}$(s) gives

$$\| \mathcal{R}(s) = \left( \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & T_{12} & T_{\perp} \end{bmatrix} \begin{bmatrix} Q_{00} & Q_{01} & 0 \\ Q_{10} & Q_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T_{21} \\ 0 & \tilde{T}_{\perp} \end{bmatrix} \right)(s) \|_{\infty} = \gamma \tag{3.179}$$

Since $\mathcal{R}(s)$ is an interpolating matrix function, it is necessarily the case that the Pick matrix $\Pi(\gamma)$ is positive semi-definite [17].

Sufficiency

Suppose $\Pi(\gamma) \geq 0$. By standard interpolation theory there exists an interpolating matrix function $\mathcal{R}(s)$ such that

$$\| \mathcal{R}(s) = \left( \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & T_{12} & T_{\perp} \end{bmatrix} \begin{bmatrix} Q(s) \in \mathcal{H}_{+}^{\infty} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T_{21} \\ 0 & \tilde{T}_{\perp} \end{bmatrix} \right)(s) \|_{\infty} = \gamma \tag{3.180}$$

Invoking the norm preserving properties of all-pass matrices gives

$$\left\| \begin{bmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix}(s) - \begin{bmatrix} Q(s) \in \mathcal{H}_{+}^{\infty} \end{bmatrix} \right\|_{\infty} = \gamma$$

What remains to be shown is that Q(s) may always be chosen with the form

$$\begin{bmatrix} Q_{00} & Q_{01} & 0 \\ Q_{10} & Q_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}(s) \tag{3.181}$$

To demonstrate that this is the case, we make use of the following known fact[55,56,57],

$$\Pi(\gamma)\geq 0 \Leftrightarrow \left\| \begin{bmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix}(s) \right\|_H \leq \gamma \tag{3.182}$$

We complete the proof of sufficiency by noting that (3.182) together with (3.178) always admits the construction of a solution to (3.180) of the form indicated in (3.181) completing the proof. □

3.4.2 The state-space calculations.

In this section we will derive state-space expressions for the left and right interpolation constraints which are associated with (3.173). Following that, we go on to verify that the Pick matrix associated with (3.173), is (3.174). The case of interpolation constraints with multiplicities is not treated here, nor is the case of left and right constraints at the same point addressed.

It follows from the first standing assumption, (X and Y are non-singular) that $\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}(s)$ will have n right half plane zeros

$$s_i \qquad\qquad i= 1,2,\ldots,n \tag{3.183a}$$

together with a sequence of vectors $a_i$ which satisfy

$$\begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix}(s_i)a_{i2}=0^2 \qquad\qquad i=1,2,\ldots,n \tag{3.183b}$$

Further, we define

$$b_i=\begin{bmatrix} T_{01} \\ T_{11} \end{bmatrix}(s_i)a_{i2} \qquad\qquad i=1,2,\ldots,n \tag{3.183c}$$

and any interpolating matrix function associated with (3.173) must satisfy[17]:

2

In order that all the dimensions match, we should write $\begin{bmatrix} I & 0 \\ 0 & T_{21} \\ 0 & \tilde{T}_\perp \end{bmatrix}(s_i)\begin{bmatrix} a_{i1} \\ a_{i2} \end{bmatrix}=0$. This clearly requires that $a_{i1}=0$ $i=1,\ldots n$.

A similar remark applies in the case of the left vector constraints.

$$\mathcal{R}(s_i)a_i = b_i \qquad i=1,2,\ldots,n \tag{3.183d}$$

By duality, $[\,T_{12}\ T_\perp](s)$ will have n right half plane zeros

$$s_i \qquad i=n+1,\ldots,2n \tag{3.184a}$$

together with a sequence of vectors $a_{i2}$ which satisfy

$$a_{i2}^*[\,T_{12}\ T_\perp](s_i)=0 \qquad i=n+1,\ldots,2n \tag{3.184b}$$

If we define

$$b_i^* = a_{i2}^*[\,T_{10}\ T_{11}](s_i) \qquad i=n+1,\ldots,2n \tag{3.184c}$$

then it is necessarily the case that

$$a_i^*\mathcal{R}(s_i) = b_i^* \qquad i=n+1,\ldots,2n \tag{3.184d}$$

if $\mathcal{R}(s)$ is interpolating.

The Pick matrix associated with (3.183) and (3.184) is given by [17]

$$\Pi(\gamma) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}(\gamma) \tag{3.185}$$

in which

$$\Pi_{11} = \left\{\frac{\gamma^2 a_i^* a_k - b_i^* b_k}{\bar{s}_i + s_k}\right\}_{k=1,\ldots,n}^{i=1,\ldots,n} \qquad \Pi_{12} = \left\{\frac{\gamma a_i^* b_k - \gamma b_i^* a_k}{\bar{s}_i - \bar{s}_k}\right\}_{k=n+1,\ldots,2n}^{i=1,\ldots,n}$$

and

$$\Pi_{12}^* = \left\{\frac{\gamma b_i^* a_k - \gamma a_i^* b_k}{s_k - s_i}\right\}_{k=1,\ldots,2n}^{i=n+1,\ldots,2n} \qquad \Pi_{22} = \left\{\frac{\gamma^2 a_i^* a_k - b_i^* b_k}{\bar{s}_k + s_i}\right\}_{k=n+1,\ldots,2n}^{i=n+1,\ldots,2n}$$

We will now find explicit expressions for the $s_i$'s, $a_{i2}$'s and $b_i$'s given in (3.183) and (3.184). To begin with, we note that the $s_i$'s associated with the right constraints are given by $-\bar{s}_i = \lambda_i(A-HC_2)$ which follows from the all-pass property of $\begin{bmatrix} \tilde{T}_{21} \\ \tilde{T}_\perp \end{bmatrix}(s)$ and (3.28). To obtain state-space models for the $a_i$'s in (3.183b), we study the solutions to the equation

$$\left[\begin{array}{c|c} s_iI-A+(B_1D_{21}^*+YC_2^*)C_2 & -B_1+(B_1D_{21}^*+YC_2^*)D_{21} \\ \hline C_2 & D_{21} \\ -\widetilde{D}_\perp B_1^*Y^{-1} & \widetilde{D}_\perp \end{array}\right]\begin{bmatrix} Yx_i \\ a_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{0} \\ 0 \end{bmatrix} \tag{3.186}$$

The (2,1) and (3,1) blocks $\Rightarrow$

$$a_{i2}=[D_{21}^*\ \widetilde{D}_\perp^*]\begin{bmatrix} -C_2Y \\ \widetilde{D}_\perp B_1^* \end{bmatrix}x_i \qquad i=1,2,\ldots,n \tag{3.187}$$

Substituting this into the (1,1) partition of (3.186) yields

$$(s_iY-AY+(B_1D_{21}^*+YC_2^*)C_2Y+[(B_1D_{21}^*+YC_2^*)D_{21}-B_1]\,[D_{21}^*\ \widetilde{D}_\perp^*]\begin{bmatrix} -C_2Y \\ \widetilde{D}_\perp B_1^* \end{bmatrix})x_i=0 \tag{3.188a}$$

which simplifies to

$$(s_iI+A^*-C_2^*H^*)x_i=0 \qquad i=1,2,\ldots,n \tag{3.188b}$$

after substituting from (3.27)

A dual sequence of arguments for the left constraints gives

$$[x_i^*X\ \ a_{i2}^*]\left[\begin{array}{c|cc} s_iI-A+B_2F & B_2 & -X^{-1}C_1^*D_\perp \\ \hline -(C_1-D_{12}F) & D_{12} & D_\perp \end{array}\right] = [\,0\mid 0\ 0\,] \tag{3.189}$$

The (1,2) and (1,3) blocks $\Rightarrow$

$$a_{i2}^*=x_i^*[-XB_2\ \ C_1^*D_\perp]\begin{bmatrix} D_{12}^* \\ D_\perp^* \end{bmatrix} \qquad i=n+1,\ldots,2n \tag{3.190}$$

and the (1,1) partition of (3.189) gives

$$x_i^*X(s_iI-A+B_2F)+x_i^*[-XB_2D_{12}^*+C_1^*D_\perp D_\perp^*][-C_1+D_{12}F]=0 \tag{3.191a}$$

which simplifies to

$$x_i^*(s_iI+A^*-F^*B_2^*)=0 \qquad i=n+1,\ldots,2n \tag{3.191b}$$

after substituting from (3.25).

In the next stage, the $b_i$'s for the left and right constraints have to calculated. First however we require the realization of

$$\begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01}T_{21}+R_{02}\tilde{T}_{\perp} \\ T_{12}R_{10}+T_{\perp}R_{21} & T_{11} \end{bmatrix} \tag{3.192}$$

The realizations of $T_{00}$ and $T_{11}$ follow directly from (3.101b) and (3.28) respectively. Continuing further,

$$T_{01} = [R_{01}\ R_{02}] \begin{bmatrix} T_{21} \\ \tilde{T}_{\perp} \end{bmatrix} \tag{3.193a}$$

$$= \left[\begin{array}{c|c} A-HC_2 & B_1-HD_{21} \\ \hline -\gamma^{-1}\tilde{D}_{\perp}B_1^*S^{-1} & \gamma\tilde{D}_{\perp} \end{array}\right] \tag{3.193b}$$

after canceling n non-minimal modes. Similarly,

$$T_{10} = [T_{12}\ T_{\perp}] \begin{bmatrix} R_{10} \\ R_{20} \end{bmatrix} \tag{3.194a}$$

$$= \left[\begin{array}{c|c} A-B_2F & -\gamma^{-1}P^{-1}C_1^*D_{\perp} \\ \hline C_1-D_{12}F & \gamma D_{\perp} \end{array}\right] \tag{3.194b}$$

Now, we turn our attention to evaluating the $b_i$'s in (3.183c). A direct calculation using (3.187), (3.188) and (3.193) shows that

$$\left[\begin{array}{cc|c} s_iI-A+B_2F & -B_2F & -B_1 \\ 0 & s_iI-A+HC_2 & HD_{21}-B_1 \\ \hline 0 & -\gamma^{-1}\tilde{D}_{\perp}B_1^*S^{-1} & \gamma\tilde{D}_{\perp} \\ C_1-D_{12}F & D_{12}F & 0 \end{array}\right] \begin{bmatrix} Yx_i \\ Yx_i \\ a_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_{i1} \\ b_{i2} \end{bmatrix} \qquad i=1,2,\dots,n \tag{3.195}$$

which gives

$$b_i = \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix} = \begin{bmatrix} \gamma^{-1}\tilde{D}_{\perp}B_1^*(\gamma^2Y^{-1}-S^{-1})Y \\ C_1Y \end{bmatrix} x_i \qquad i=1,2,\dots,n \tag{3.196}$$

the (1,1) and (2,1) blocks of (3.195) are easily verified using (3.187). In the same way it is not difficult to calculate the $b_i$'s associated with the left constraints; substituting (3.190) and (3.191) into (3.184) yields

$$[x_i^*X \quad 0 \quad a_{i2}^*]\left[\begin{array}{cc|cc} s_iI-A+B_2F & -B_2F & -P^{-1}C_1^*D_\perp\gamma^{-1} & B_1 \\ 0 & s_iI-A+HC_2 & 0 & B_1-HD_{21} \\ \hline -(C_1-D_{12}F) & -D_{12}F & \gamma D_\perp & 0 \end{array}\right] = [0 \quad 0 \quad b_{i1}^* \; b_{i2}^*] \qquad i=n+1,\ldots,2n \tag{3.197}$$

in which

$$b_i^*=[b_{i1}^* \quad b_{i2}^*] = x_i^*[\;\gamma^{-1}X(\gamma^2X^{-1}-P^{-1})C_1^*D_\perp \qquad XB_1] \qquad i=n+1,\ldots,2n \tag{3.198}$$

In the last phase of our development, we substitute the state-space expressions for the interpolation constraints into (3.185). We begin with the (1,1) block

$$\Pi_{11}(\gamma) = \left\{\frac{\gamma^2a_i^*a_k - b_i^*b_k}{\bar{s}_i + s_k}\right\}_{k=1,\ldots,n}^{i=1,\ldots,n} \tag{3.199}$$

Substituting from (3.187) and (3.196) gives

$$\Pi_{11}(\gamma)=x_i^*\Big\{\gamma^2[B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*+YC_2^*C_2Y] -YC_1^*C_1 \\ -\gamma^{-2}[\gamma^2I-YS^{-1}]B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*[\gamma^2I-YS^{-1}]\Big\}x_k\Big/(\bar{s}_i+s_k) \tag{3.200}$$

Eliminating terms and substituting

$$YS^{-1}(A-B_1D_{21}^*C_2)Y+Y(A-B_1D_{21}^*C_2)^*S^{-1}Y-Y(\gamma^2C_2^*C_2-C_1^*C_1)Y+\gamma^{-2}YS^{-1}B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*S^{-1}Y=0 \tag{3.201}$$

yields:

$$\Pi_{11}(\gamma)=x_i^*Y\Big\{S^{-1}[A-B_1D_{21}^*C_2+B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*Y^{-1}]+ \\ [A-B_1D_{21}^*C_2+B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*Y^{-1}]^*S^{-1}\Big\}Yx_k\Big/(\bar{s}_i+s_k) \tag{3.202}$$

Substituting from (3.27)$Y^{-1}$ and $Y^{-1}$(3.27) allows us to write

$$\Pi_{11}(\gamma)=-x_i^*Y\Big\{S^{-1}Y[A-HC_2]^*Y^{-1}+Y^{-1}[A-HC_2]YS^{-1}\Big\}Yx_k\Big/(\bar{s}_i+s_k) \tag{3.203}$$

Finally we invoke (3.188b) to obtain

$$\Pi_{11}(\gamma)=\left\{x_i^*YS^{-1}Yx_k\right\}_{k=1,\dots,n}^{i=1,\dots,n} \tag{3.204}$$

Next, we consider the (1,2) term

$$\Pi_{12}(\gamma)=\left\{\frac{\gamma a_i^*b_k - \gamma b_i^*a_k}{\bar{s}_i - \bar{s}_k}\right\}_{k=n+1,\dots,2n}^{i=1,\dots,n} \tag{3.205}$$

which gives

$$\Pi_{12}(\gamma)=\frac{\gamma x_i^*Y\left\{[Y^{-1}B_1\tilde{D}_\perp^*\tilde{D}_\perp-C_2^*D_{21}]B_1^*-C_1^*[D_\perp D_\perp^*C_1X^{-1}-D_{12}B_2^*]\right\}Xx_k}{\bar{s}_i - \bar{s}_k} \tag{3.206}$$

Adding ($A^*$-$A^*$) and substituting $Y^{-1}$(3.27) and (3.25)$X^{-1}$ yields

$$\Pi_{12}(\gamma)=\frac{\gamma x_i^*Y\left\{X[A-B_2F]X^{-1}-Y^{-1}[A-HC_2]Y\right\}Xx_k}{\bar{s}_i - \bar{s}_k} \tag{3.207}$$

As a result of employing (3.188b) and (3.191b) one obtains

$$\Pi_{12}(\gamma)=\left\{\gamma x_i^*YXx_k\right\}_{k=n+1,\dots,2n}^{i=1,\dots,n} \tag{2.208}$$

A simple symmetry argument yields

$$\Pi_{21}(\gamma)=\left\{\gamma x_i^*XYx_k\right\}_{k=1,\dots,n}^{i=n+1,\dots,2n} \tag{3.209}$$

Finally for the the (2,2) block of (3.185)

$$x_i^*\left\{\gamma^2[XB_2B_2^*X+C_1^*D_\perp D_\perp^*C_1]-\gamma^{-2}(\gamma^2I-XP^{-1})C_1^*D_\perp D_\perp^*C_1(\gamma^2I-P^{-1}X)-XB_1B_1^*X\right\}x_i\Big/(\bar{s}_k-s_i)$$

Eliminating and substituting

$$X(A-B_2D_{12}^*C_1)P^{-1}X+XP^{-1}(A-B_2D_{12}^*C_1)^*X-X(\gamma^2B_2B_2^*-B_1B_1^*)X+XP^{-1}C_1^*D_\perp D_\perp^*C_1\gamma^{-2}P^{-1}X=0$$

yields

$$x_i^* X\left\{(A-B_2D_{12}^*C_1+X^{-1}C_1^*D_\perp D_\perp^* C_1)P^{-1}+P^{-1}(A-B_2D_{12}^*C_1+X^{-1}C_1^*D_\perp D_\perp^* C_1)^*\right\}Xx_k\Big/(\bar{s}_k+s_i)$$

Substituting from $X^{-1}$(3.25) and (3.25)$X^{-1}$ gives

$$x_i^* X\left\{-X^{-1}(A-B_2F)^*P^{-1}-P^{-1}(A-B_2F)X^{-1}\right\}Xx_k\Big/(\bar{s}_k+s_i)$$

$$\Pi_{22}(\gamma)=\left\{x_i^* XP^{-1}Xx_k\right\}_{k=n+1,\dots,2n}^{i=n+1,\dots,2n} \tag{3.210}$$

Since the $x_i$'s are linearly independent,

$$\mathrm{In}(\Pi(\gamma))=\mathrm{In}(\begin{bmatrix} YS^{-1}Y & YX\gamma \\ XY\gamma & XP^{-1}X \end{bmatrix})=\mathrm{In}(\begin{bmatrix} S^{-1} & \gamma I \\ \gamma I & P^{-1} \end{bmatrix}) \tag{3.211}$$

Lemma 3.6 and assumption 1 imply $S^{-1}>0$ and $P^{-1}>0$, a Schur complement argument will show that $(I-\gamma^2SP)\geq 0 \Rightarrow \lambda_{max}(SP)\leq\gamma^{-2}$, establishing the claim made at the begining of section 3.4.

Remark 3.4

Sufficiency may also be proved by a routine but somewhat protracted calculation which involves the use of a generalized state space version of the bounded real equations [2]. Again this approach treats the optimal case corresponding to $E_k$ in (3.165) being singular. Since sufficiency has already been established via interpolation an outline of the bounded real calculations has been relegated to appendix C.

Theorem 3.7

Internally stabilizing controllers exist such that $||F_l(P,K)||_\infty\leq\gamma$, if and only if there exist solution to the Riccati equations (3.163) and (3.164) such that $S>0$, $P>0$ and $\lambda_{max}(SP)\leq\gamma^{-2}$. If such solutions exist, all the controllers are given by the representation formula

$$-K(s)=F_l(\mathcal{K};U)(s) \qquad\qquad U(s)\in\tfrac{1}{\gamma}BH_+^\infty$$

where

$$\mathcal{K}(s) \stackrel{s}{=} \begin{bmatrix} E_k & A_k & B_{k1} & B_{k2} \\ 0 & C_{k1} & 0 & D_{k12} \\ 0 & C_{k2} & D_{k21} & 0 \end{bmatrix} \tag{3.212}$$

and is defined in (3.165)-(3.168) and (3.170). □

Corollary 3.8

$P=(\gamma^2 I-\tilde{X}X^{-1})^{-1}X$ and $S=(\gamma^2 I -\hat{X}Y^{-1})^{-1}Y$ are the stabilizing solutions to (3.163) and (3.164) respectively.

Proof:

$$\mathrm{In}\left\{(A-B_2D_{12}^*C_1)-(\gamma^2B_2B_2^*-B_1B_1^*)X(\gamma^2I-X^{-1}\tilde{X})^{-1}\right\}$$

$$=\mathrm{In}\left\{[\![(A-B_2D_{12}^*C_1)(\gamma^2I-X^{-1}\tilde{X})-(\gamma^2B_2B_2^*-B_1B_1^*)X]\!](\gamma^2I-X^{-1}\tilde{X})^{-1}\right\}$$

substituting $X^{-1}$(3.82)

$$=\mathrm{In}\Big\{[\![(\gamma^2-X^{-1}\tilde{X})(A-B_2F)-(A-B_2D_{12}^*C_1)X^{-1}\tilde{X}-X^{-1}(A-B_2F)^*\tilde{X}$$
$$-X^{-1}\tilde{X}X^{-1}C_1^*D_\perp D_\perp^*C_1\gamma^{-2}X^{-1}\tilde{X}]\!](\gamma^2I-X^{-1}\tilde{X})^{-1}\Big\}$$

substituting $X^{-1}(3.25)X^{-1}\tilde{X}$

$$=\mathrm{In}\left\{(\gamma^2-X^{-1}\tilde{X})(A-B_2F+X^{-1}C_1^*D_\perp D_\perp^*C_1\gamma^{-2}X^{-1}\tilde{X})(\gamma^2I-X^{-1}\tilde{X})^{-1}\right\}=(0,n,0)$$

which follows from the fact that $\tilde{X}$ is the destabilizing solution to (3.82), which verifies the P is indeed the stabilizing solution to (3.163).

A dual set of arguments using (3.164), (3.78) and (3.27) establish that S is the stabilizing solution to (3.164). □

## 3.5 The removal of temporary assumption 2

We now study the case when $D_{11}\neq 0$ and $D_{22}\neq 0$ in (3.2). It is shown that when this situation arises, the problem can be transformed into an equivalent one in which $D_{11}=0$ and $D_{22}=0$. To do this consider the loop transformations illustrated in Fig.1.

To begin with, select a constant matrix $F_\infty$ so that the $D_{11}$-matrix of the modified problem is reduced to the Parrott lower bound of $\max\left\{\|D_\perp^* D_{11}\|\ ,\ \|D_{11}\tilde{D}_\perp^*\|\right\}$. If P(s) is as given in (3.2), then figure 1 gives

$$\dot{x}=Ax+B_1y_2+B_2u_3$$
$$u_2=C_1x+D_{11}y_2+D_{12}u_3$$
$$y_3=C_2x+D_{21}y_2+D_{22}u_3$$
$$u_3=u'_3-F_\infty y_3$$

substituting for $u_3$ gives

$$y_3=C_2x+D_{21}y_2+D_{22}(u'_3-F_\infty y_3)$$

$$\Rightarrow \qquad y_3=(I+D_{22}F_\infty)^{-1}\left\{C_2x+D_{21}y_2+D_{22}u'_3\right\}$$

similarly

$$\dot{x}=Ax+B_1y_2+B_2u'_3-B_2F_\infty(I+D_{22}F_\infty)^{-1}\left\{C_2x+D_{21}y_2+D_{22}u'_3\right\}$$

$$\Rightarrow \qquad \dot{x}=(A-B_2F_\infty(I+D_{22}F_\infty)^{-1}C_2)x+(B_1-B_2F_\infty(I+D_{22}F_\infty)^{-1}D_{21})y_2 \\ +B_2(I-F_\infty(I+D_{22}F_\infty)^{-1}D_{22})u'_3$$

$$u_2=C_1x+D_{11}y_2+D_{12}u'_3-D_{12}F_\infty(I+D_{22}F_\infty)^{-1}\left\{C_2x+D_{21}y_2+D_{22}u'_3\right\}$$

$$\Rightarrow \qquad u_2=(C_1-D_{12}F_\infty(I+D_{22}F_\infty)^{-1}C_2)x+(D_{11}-D_{12}F_\infty(I+D_{22}F_\infty)^{-1}D_{21})y_2 \\ +D_{12}(I-F_\infty(I+D_{22}F_\infty)^{-1}D_{22})u'_3$$

combining x, $y_2$ and $u'_3$ yields

$$\begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} =$$

$$\left[\begin{array}{c|cc} A-B_2F_\infty(I+D_{22}F_\infty)^{-1}C_2 & B_1-B_2F_\infty(I+D_{22}F_\infty)^{-1}D_{21} & B_2(I+F_\infty D_{22})^{-1} \\ \hline C_1-D_{12}F_\infty(I+D_{22}F_\infty)^{-1}C_2 & D_{11}-D_{12}F_\infty(I+D_{22}F_\infty)^{-1}D_{21} & D_{12}(I+F_\infty D_{22})^{-1} \\ (I+D_{22}F_\infty)^{-1}C_2 & (I+D_{22}F_\infty)^{-1}D_{21} & D_{22}(I+F_\infty D_{22})^{-1} \end{array}\right] \tag{3.213}$$

The aim is first to find $F_\infty$ to minimize $\tilde{D}_{11}$. Next, we suppose that $Q_\infty(s)$ solves the Parrott problem

$$\inf_{Q_\infty} \left\| \begin{bmatrix} D_{12}^*D_{11}D_{21}^*-Q_\infty & D_{12}^*D_{11}\tilde{D}_\perp^* \\ D_\perp^*D_{11}D_{21}^* & D_\perp^*D_{11}\tilde{D}_\perp^* \end{bmatrix} \right\|_2 \tag{3.214}$$

in which one solution is [22,23,30]

$$Q_\infty=D_{12}^*(D_{11}+D_{11}\tilde{D}_\perp^*(\gamma_p^2 I-(D_\perp^*D_{11}\tilde{D}_\perp^*)^*(D_\perp^*D_{11}\tilde{D}_\perp^*))^{-1}(D_\perp^*D_{11}\tilde{D}_\perp^*)^*D_\perp^*D_{11}\Big)D_{21}^* \tag{3.215}$$

where

$$\gamma_p=\max\left\{||D_\perp^*D_{11}||\ ,\ ||D_{11}\tilde{D}_\perp^*||\right\} \tag{3.216}$$

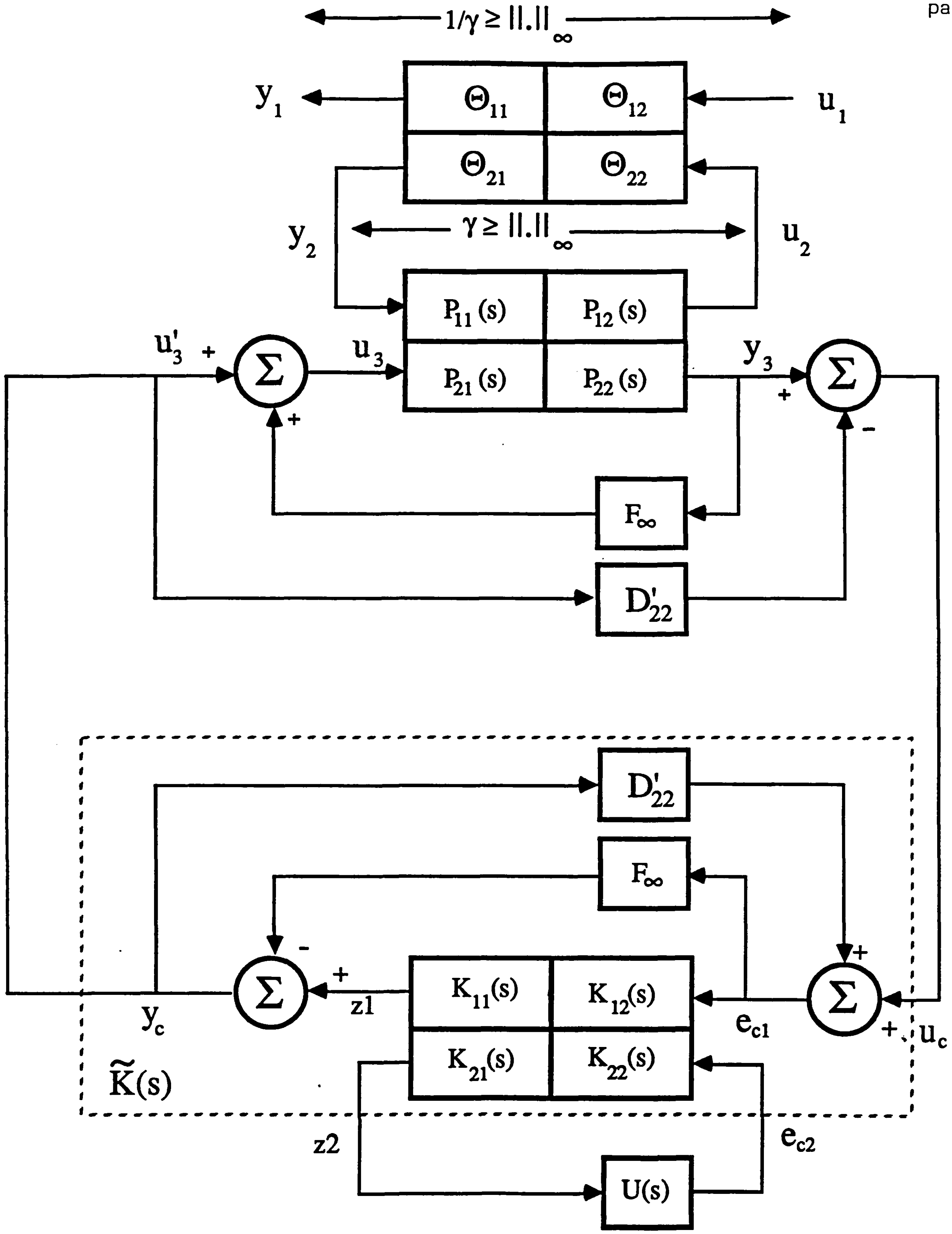



Fig.1 Loop transformations.

$F_\infty$ is now found by setting $Q_\infty := F_\infty(I+D_{22}F_\infty)^{-1}$, which after a small rearrangement yields

$$F_\infty=(I-Q_\infty D_{22})^{-1}Q_\infty \tag{3.217}$$

(well posedness is assumed). All we have done in this step is solve the original $H^\infty$ design problem at infinity. After applying the state feedback $F_\infty$, one obtains the new problem for which the following invariance properties are established:

(i) $(A, B_2, C_2)$ is stabilizable and detectable $\Leftrightarrow$ $(\tilde{A}, \tilde{B}_2, \tilde{C}_2)$ is stabilizable and detectable.

(ii) $\mathrm{rank}\left(\begin{bmatrix} j\omega I-A & -B_2 \\ C_1 & D_{12} \end{bmatrix}\right)=n+m \ \forall\omega\in\mathbf{R} \Leftrightarrow \mathrm{rank}\left(\begin{bmatrix} j\omega I-\tilde{A} & -\tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix}\right)=n+m \ \forall\omega\in\mathbf{R}$

(iii) $\mathrm{rank}\left(\begin{bmatrix} j\omega I-A & B_1 \\ -C_2 & D_{21} \end{bmatrix}\right)=(n+q) \ \forall\omega\in\mathbb{R} \Leftrightarrow \mathrm{rank}\left(\begin{bmatrix} j\omega I-\tilde{A} & \tilde{B}_1 \\ -\tilde{C}_2 & \tilde{D}_{21} \end{bmatrix}\right)=(n+q) \ \forall\omega\in\mathbb{R}$

<u>Proof:</u>

(i) follows from the identities

$$[sI-\tilde{A} \quad \tilde{B}_2] = [sI-A \quad B_2]\begin{bmatrix} I & 0 \\ F_\infty(I+D_{22}F_\infty)^{-1}C_2 & (I+F_\infty D_{22})^{-1} \end{bmatrix}$$

$$\begin{bmatrix} sI-\tilde{A} \\ \tilde{C}_2 \end{bmatrix} = \begin{bmatrix} I & B_2F_\infty(I+D_{22}F_\infty)^{-1} \\ 0 & (I+D_{22}F_\infty)^{-1} \end{bmatrix}\begin{bmatrix} sI-A \\ C_2 \end{bmatrix}$$

(ii) and (iii) follow from the relationships

$$\text{(a)} \begin{bmatrix} sI-\tilde{A} & -\tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix} = \begin{bmatrix} sI-A & -B_2 \\ C_1 & D_{12} \end{bmatrix}\begin{bmatrix} I & 0 \\ -F_\infty(I+D_{22}F_\infty)^{-1}C_2 & (I+F_\infty D_{22})^{-1} \end{bmatrix}$$

$$\text{(b)} \begin{bmatrix} sI-\tilde{A} & \tilde{B}_1 \\ -\tilde{C}_2 & \tilde{D}_{21} \end{bmatrix} = \begin{bmatrix} I & -B_2F_\infty(I+D_{22}F_\infty)^{-1} \\ 0 & (I+D_{22}F_\infty)^{-1} \end{bmatrix}\begin{bmatrix} sI-A & B_1 \\ -C_2 & D_{21} \end{bmatrix}$$

Having reduced the norm of $\tilde{D}_{11}$, the next step is to replace the original problem by an equivalent problem for which $D_{11}=0$. Define the Julia type matrix [30] by

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \gamma^{-2} \begin{bmatrix} \widetilde{D}_{11} & (\gamma^2 I - \widetilde{D}_{11}\widetilde{D}_{11}^{*})^{1/2} \\ -(\gamma^2 I - \widetilde{D}_{11}^{*}\widetilde{D}_{11})^{1/2} & \widetilde{D}_{11}^{*} \end{bmatrix} \tag{3.218}$$

which is easily seen to satisfy $\Theta\Theta^{*}=\gamma^{-2}I$ for all $\gamma>\gamma_p$. Next, construct $P'(s)$ given in figure 1

$$\begin{aligned}
&y_1=\Theta_{11}u_1+\Theta_{12}u_2 \\
&y_2=\Theta_{21}u_1+\Theta_{22}u_2 \\
&\dot{x}=\widetilde{A}x+\widetilde{B}_1 y_2+\widetilde{B}_2 u'_3 \\
&u_2=\widetilde{C}_1 x+\widetilde{D}_{11}y_2+\widetilde{D}_{12}u'_3 \\
&y_3=\widetilde{C}_2 x+\widetilde{D}_{21}y_2+\widetilde{D}_{22}u'_3
\end{aligned}$$

cancelling the variables $y_2$ and $u_2$; substituting for $y_2$ yields

$$\begin{aligned}
&u_2=\widetilde{C}_2 x+\widetilde{D}_{11}(\Theta_{21}u_1+\Theta_{22}u_2)+\widetilde{D}_{12}u'_3 \\
&u_2=(I-\widetilde{D}_{11}\Theta_{22})^{-1}\left\{\widetilde{C}_1 x+\widetilde{D}_{11}\Theta_{21}u_1+\widetilde{D}_{12}u'_3\right\} \\
&\dot{x}=\widetilde{A}x+\widetilde{B}_1(\Theta_{21}u_1+\Theta_{22}u_2)+\widetilde{B}_2 u'_3 \\
&\dot{x}=\left\{\widetilde{A}+\widetilde{B}_1\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{C}_1\right\}x+\widetilde{B}_1\left\{\Theta_{21}+\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{11}\Theta_{21}\right\}u_1 \\
&\qquad\qquad+\left\{\widetilde{B}_2+\widetilde{B}_1\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{12}\right\}u'_3
\end{aligned}$$

$$\begin{aligned}
&y_1=\Theta_{11}u_1+\Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\left\{\widetilde{C}_1 x+\widetilde{D}_{11}\Theta_{21}u_1+\widetilde{D}_{12}u'_3\right\} \\
&y_1=\Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{C}_1 x+\left\{\Theta_{11}+\Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{11}\Theta_{21}\right\}u_1 \\
&\qquad\qquad+\Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{12}u'_3
\end{aligned}$$

$$\begin{aligned}
&y_3=\widetilde{C}_2 x+\widetilde{D}_{21}(\Theta_{21}u_1+\Theta_{22}u_2)+\widetilde{D}_{22}u'_3 \\
&y_3=\left\{\widetilde{C}_2+\widetilde{D}_{21}\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{C}_1\right\}x+\widetilde{D}_{21}\left\{\Theta_{21}+\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{11}\Theta_{21}\right\}u_1 \\
&\qquad\qquad+\left\{\widetilde{D}_{22}+\widetilde{D}_{21}\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{12}\right\}u'_3
\end{aligned}$$

Then the combined input output map between $(u_1;u'_3)$ and $(y_1;y_3)$ has a state space realization given by

$$P'(s)\stackrel{s}{=}\left[\begin{array}{c|cc}
\widetilde{A}+\widetilde{B}_1\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{C}_1 & \widetilde{B}_1(I-\Theta_{22}\widetilde{D}_{11})^{-1}\Theta_{21} & \widetilde{B}_2+\widetilde{B}_1\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{12} \\ \hline
\Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{C}_1 & \Theta_{11}+\Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{11}\Theta_{21} & \Theta_{12}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{12} \\
\widetilde{C}_2+\widetilde{D}_{21}\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{C}_1 & \widetilde{D}_{21}(I-\Theta_{22}\widetilde{D}_{11})^{-1}\Theta_{21} & \widetilde{D}_{22}+\widetilde{D}_{21}\Theta_{22}(I-\widetilde{D}_{11}\Theta_{22})^{-1}\widetilde{D}_{12}
\end{array}\right] \tag{3.219}$$

In connection with this step, there are also a number of invariance properties:

(i)$'$ Suppose that $\mathrm{rank}\left(\begin{bmatrix} j\omega I-A & -B_2 \\ C_1 & D_{12} \end{bmatrix}\right)=n+m\ \forall\omega\in\mathbb{R}$ together with $\mathrm{rank}\left(\begin{bmatrix} j\omega I-A & B_1 \\ -C_2 & D_{21} \end{bmatrix}\right)=n+q$ $\forall\omega\in\mathbb{R}$. Then$(A,\ B_2,\ C_2)$ is stabilizable and detectable $\Leftrightarrow$ $(A',\ B'_2,\ C'_2)$ is stabilizable and detectable.

(ii)$'$ $\mathrm{rank}\left(\begin{bmatrix} j\omega I-A & B_2 \\ -C_1 & D_{12} \end{bmatrix}\right)=n+m\ \forall\omega\in\mathbb{R} \Leftrightarrow \mathrm{rank}\left(\begin{bmatrix} j\omega I-A' & B'_2 \\ -C'_1 & D'_{12} \end{bmatrix}\right)=n+m\ \forall\omega\in\mathbb{R}$

(iii)$'$ $\mathrm{rank}\left(\begin{bmatrix} j\omega I-A & -B_1 \\ C_2 & D_{21} \end{bmatrix}\right)=n+q\ \forall\omega\in\mathbb{R} \Leftrightarrow \mathrm{rank}\left(\begin{bmatrix} j\omega I-A' & -B'_1 \\ C'_2 & D'_{21} \end{bmatrix}\right)=n+q\ \forall\omega\in\mathbb{R}$

Proof:

(i)$'$ is a consequence (i) as follows. The rank assumptions together with the assumed stabilizability and detectability of $(A,B_2,C_2)$ imply that there exists a controller such that $\|F_l(\bar{P},K)\|_\infty=\gamma$. Since $\|\Theta_{22}\|_\infty<\gamma^{-1}$, it follows from a small gain argument and the rank assumptions that $F_l(P',K)$ is internally stable. As K is an internally stabilizing controller for $P'(s)$, it follows that $(A',B'_2,C'_2)$ must be stabilizable and detectable. The opposite implication may be argued in the same way.

(ii)$'$ and (iii)$'$ follow from (ii) and (iii) and the identities

$$\text{(a)}\begin{bmatrix} sI-A' & B'_2 \\ -C'_1 & D'_{12} \end{bmatrix}=\begin{bmatrix} I & \bar{B}_1\Theta_{22}(I-\bar{D}_{11}\Theta_{22})^{-1} \\ 0 & \Theta_{12}(I-\bar{D}_{11}\Theta_{22})^{-1} \end{bmatrix}\begin{bmatrix} sI-\bar{A} & \bar{B}_2 \\ -\bar{C}_1 & \bar{D}_{12} \end{bmatrix}$$

$$\text{(b)}\begin{bmatrix} sI-A' & -B'_1 \\ C'_2 & D'_{21} \end{bmatrix}=\begin{bmatrix} sI-\bar{A} & -\bar{B}_1 \\ \bar{C}_2 & \bar{D}_{21} \end{bmatrix}\begin{bmatrix} I & 0 \\ \Theta_{22}(I-\bar{D}_{11}\Theta_{22})^{-1}\bar{C}_1 & (I-D_{22}\Theta_{22})^{-1}\Theta_{21} \end{bmatrix}$$

we have shown that the original problem has a solution $F_l\left\{P;K\right\}(s)\le\gamma$ if and only if the modified problem $F_l\left\{P';\widetilde{K}\right\}\le\gamma^{-1}$; furthermore, $P'_{11}(\infty)=F_l\left\{\Theta,\widetilde{D}_{11}\right\}=0$ by a small computation. The equivalence between the two problems follows from a specialization of Redheffer's theorem provided $\gamma>\gamma_p$ [13,lemma 4.1]. The next step is to eliminate $P'_{22}(\infty)=D'_{22}$ by a standard loop shifting argument; which has already been suggested by Glover and Doyle [14]. The result of these three transformations has been to change the original problem given in (3.2) to an equivalent one in which $D_{11}=0$ and $D_{22}=0$. Following this, one may compute $\widetilde{K}(s)$ using equations (3.163)-(3.170). Finally singularly perturb the controller and reverse the effects of $D'_{22}$ and $F_\infty$ to obtain a representation formula for all controllers. Define the perturbed model of $\widetilde{K}(s)$ by

$$\dot{x}=A_{kn}x+B_{k1n}u_c+B_{k2n}e_{c2} \tag{3.220a}$$

$$y_c = C_{k1n}x + D_{k11n}u_c + D_{k12n}e_{c2} \tag{3.220b}$$
$$z_2 = C_{k2n}x + D_{k21n}u_c + D_{k22n}e_{c2} \tag{3.220c}$$

the connecting equations are given by

$$e_{c1} = u_c + D'_{22}y_c \tag{3.221a}$$
$$y_c = z_1 + F_\infty e_{c1} \tag{3.221b}$$

(3.220b) and (3.221a) $\Rightarrow$ $$z_1 = C_{k1n}x + D_{k11n}u_c + D_{k12n}e_{c2} - F_\infty e_{c1} \tag{3.222a}$$

(3.220b) and (3.221a) $\Rightarrow$ $$\begin{aligned} u_c &= e_{c1} - D'_{22}\left\{C_{k1n}x + D_{k11n}u_c + D_{k12n}e_{c2}\right\} \\ &= (I + D'_{22}D_{k11n})^{-1}\left\{e_{c1} - D'_{22}C_{k1n}x - D'_{22}D_{k12n}e_{c2}\right\} \end{aligned} \tag{3.222b}$$

substituting (3.222b) into (3.220a), (3.220c) and (3.222a) to cancel $u_c$ yields the following realization for K(s)

$$A_k = A_{kn} - B_{k1n}(I + D'_{22}D_{k11n})^{-1}D'_{22}C_{k1n} \tag{3.223a}$$
$$B_{k1} = B_{k1n}(I + D'_{22}D_{k11n})^{-1} \tag{3.223b}$$
$$B_{k2} = B_{k2n} - B_{k1n}(I + D'_{22}D_{k11n})^{-1}D'_{22}D_{k12n} \tag{3.223c}$$
$$C_{k1} = C_{k1n} - D_{k11n}(I + D'_{22}D_{k11n})^{-1}D'_{22}C_{k1n} \tag{3.223d}$$
$$C_{k2} = C_{k2n} - D_{k21n}(I + D'_{22}D_{k11n})^{-1}D'_{22}C_{k1n} \tag{3.223e}$$
$$D_{k11} = -F_\infty + D_{k11n}(I + D'_{22}D_{k11n})^{-1} \tag{3.223f}$$
$$D_{k12} = D_{k12n} - D_{k11n}(I + D'_{22}D_{k11n})^{-1}D'_{22}D_{k12n} \tag{3.223g}$$
$$D_{k21} = D_{k21n}(I + D'_{22}D_{k11n})^{-1} \tag{3.223h}$$
$$D_{k22} = D_{k22n} - D_{k21n}(I + D'_{22}D_{k11n})^{-1}D'_{22}D_{k12n} \tag{3.223i}$$

A routine calculation shown in appendix D will verify that

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = F_l\left\{ \left[\begin{array}{cc|c} -F_\infty & \widetilde{K}_{12} & -I \\ \widetilde{K}_{21} & \widetilde{K}_{22} - \widetilde{K}_{21}D'_{22}\widetilde{K}_{12} & \widetilde{K}_{21}D'_{22} \\ \hline -I & D'_{22}\widetilde{K}_{12} & -D'_{22} \end{array}\right], \widetilde{K}_{11} \right\} \tag{3.224}$$

in which the s-dependence has been suppressed.

<u>Algorithm 1:</u>

The procedure presented above can be summarized in the form of a series of steps to facilitate implementation. Starting with a P(s) given in (3.2)

<u>Step 1</u> Scale $D_{12}$ and $D_{21}$ so that they are parts of orthogonal matrices; furthermore, obtain $D_\perp$ and $\widetilde{D}_\perp$ such that $[D_{12}\ D_\perp]$ and $[D_{21}^*\ \widetilde{D}_\perp^*]^*$ are orthogonal matrices. [9,26]

<u>Step 2</u> Calculate $Q_\infty$ and $F_\infty$ using (3.215)-(3.217) and construct $F_l\left\{P(s);-F_\infty\right\}$ given in (3.213b).

Step 3 Compute the bounds for the problem : $\gamma_{upper}$ and $\gamma_{low}$ using (3.47) and (3.46) respectively. (This step requires one to solve the parameterization Riccati equations)

Step 4 Select $\gamma$ between the bounds and calculate $\Theta$ in (3.218); construct $P'(s)$ in (3.219b) which is the new problem.

Step 5 Scale $P'_{12}(\infty)$ and $P'_{21}(\infty)$ as in step 1.

Step 6 Compute P and S : the solutions to (3.163) and (3.164). The minimization process takes place over $\frac{1}{\gamma}$ in this case, care must be taken to modify (3.163) and (3.164) accordingly. We seek $\gamma$ such that $\lambda_{min}(I\text{-}SP\gamma^{-2})=0$ or the minimum value of $\gamma$ for which $\lambda_{min}(S)>0$ and $\lambda_{min}(P)>0$.

Step 7 Assemble $\widetilde{K}(s)$ using (3.165)-(3.170), singularly perturb it in the case of inertia optimality and reverse the scaling carried out in step 5.

Step 8 Reverse the effects of $D'_{22}$ and $F_\infty$ using (3.223) and finally undo the scaling done in step 1.

Sucessively bisecting the search interval will give a sequence of $\gamma$'s which converge to $\gamma_o$. For each of these $\gamma$, steps 4, 5 and 6 are executed until $\gamma_o$ is attained. The convergence of such a sequence has been studied in a related context [15] where the $\gamma$-iteration is shown to be monotonic and continuous.

It is possibly the case that the transformation given in (2.218) is poorly conditioned from a numerical point of view. In this event, it is suggested that the loop shifts given by $F_\infty$ and $D'_{22}$ be left in service, but that the transformation be removed. This will require formulae corresponding to $D_{11}\neq 0$ which may be derived in a similar way to (3.163)-(3.170). The calculations involve long intricate manipulations of the terms in (3.161), for this reason they have not been included here. One such set of formulae is given in (3.225) to (3.235) below:

$$\begin{aligned}&\left\{A + (B_1D_{11}^*D_\perp D_\perp^* - \gamma^2B_2D_{12}^*)(\gamma^2I - D_{11}D_{11}^*D_\perp D_\perp^*)^{-1}C_1\right\}^*P \\ &\quad + P\left\{A + (B_1D_{11}^*D_\perp D_\perp^* - \gamma^2B_2D_{12}^*)(\gamma^2I - D_{11}D_{11}^*D_\perp D_\perp^*)^{-1}C_1\right\} \\ &\quad - P\left\{\gamma^2B_2B_2^* - (B_1 - B_2D_{12}^*D_{11})\gamma^2(\gamma^2I - D_{11}^*D_\perp D_\perp^*D_{11})^{-1}(B_1 - B_2D_{12}^*D_{11})^*\right\}P \\ &\quad + C_1^*D_\perp(\gamma^2I\text{-}D_\perp^*D_{11}D_{11}^*D_\perp)^{-1}D_\perp^*C_1=0\end{aligned} \tag{3.225}$$

$$\begin{aligned}&\left\{A + B_1(\gamma^2I - \widetilde{D}_\perp^*\widetilde{D}_\perp D_{11}^*D_{11})^{-1}(\widetilde{D}_\perp^*\widetilde{D}_\perp D_{11}^*C_1 - \gamma^2D_{21}^*C_2)\right\}S \\ &\quad + S\left\{A + B_1(\gamma^2I - \widetilde{D}_\perp^*\widetilde{D}_\perp D_{11}^*D_{11})^{-1}(\widetilde{D}_\perp^*\widetilde{D}_\perp D_{11}^*C_1 - \gamma^2D_{21}^*C_2)\right\}^* \\ &\quad - S\left\{\gamma^2C_2^*C_2 - (C_1 - D_{11}D_{21}^*C_2)^*\gamma^2(\gamma^2I - D_{11}\widetilde{D}_\perp^*\widetilde{D}_\perp D_{11}^*)^{-1}(C_1 - D_{11}D_{21}^*C_2)\right\}S \\ &\quad + B_1\widetilde{D}_\perp^*(\gamma^2I\text{-}\widetilde{D}_\perp D_{11}^*D_{11}\widetilde{D}_\perp^*)^{-1}\widetilde{D}_\perp B_1^*=0\end{aligned} \tag{3.226}$$

If solutions to (3.225) and (3.226) exist such that $S>0$, $P>0$ and $SP\leq\gamma^{-2}$, then all the stabilizing controllers which achieve $||F_l(P,K)||_\infty \leq \gamma$ are given by the representation formula:

$$-K(s)=F_l\{\mathcal{K};U\}(s) \qquad\qquad \text{where } \tfrac{1}{\gamma}U(s)\in\mathcal{BH}_\infty^+$$

where

$$\mathcal{K}(s) \stackrel{s}{=} \left[\begin{array}{cccc} E_k & A_k & B_{k1} & B_{k2} \\ 0 & C_{k1} & 0 & D_{k12} \\ 0 & C_{k2} & D_{k21} & D_{k22} \end{array}\right].$$

in which

$$E_k \;= I - SP\gamma^2 \tag{3.227}$$

$$B_{k1} \;=-\gamma^2\Big\{B_1+SC_1^*D_{11}+SC_2^*D_{21}(\gamma^2I-D_{11}^*D_{11})\Big\}(\gamma^2I-\tilde{D}_\perp^*\tilde{D}_\perp D_{11}^*D_{11})^{-1}D_{21}^* \tag{3.228}$$

$$C_{k1} \;=-\gamma^2D_{12}^*(\gamma^2I-D_{11}D_{11}^*D_\perp D_\perp^*)^{-1}\Big\{C_1+D_{11}B_1^*P+(\gamma^2I-D_{11}D_{11}^*)D_{12}B_2^*P\Big\} \tag{3.229}$$

$$D_{k12} =-\Big\{I-D_{12}^*D_{11}(\gamma^2I-D_{11}^*D_\perp D_\perp^*D_{11})^{-1}D_{11}^*D_{12}\Big\}^{1/2} \tag{3.230}$$

$$D_{k21} =-\Big\{I-D_{21}D_{11}^*(\gamma^2I-D_{11}\tilde{D}_\perp^*\tilde{D}_\perp D_{11}^*)^{-1}D_{11}D_{21}^*\Big\}^{1/2} \tag{3.231}$$

$$D_{k22} =D_{k21}^{-*}D_{21}D_{11}^*(\gamma^2I-D_{11}\tilde{D}_\perp^*\tilde{D}_\perp D_{11}^*)^{-1}D_{12}D_{k12} \tag{3.232}$$

$$\begin{aligned} A_k \;=E_k\Big\{A+B_2C_{k1}\Big\}+B_{k1}C_2+\Big\{E_kB_1+B_{k1}D_{21}\Big\}(\gamma^2I-D_{11}^*D_\perp D_\perp^*D_{11})^{-1} \\ \Big\{D_{11}^*D_\perp D_\perp^*C_1+B_1^*P\gamma^2-D_{11}^*D_{12}B_2^*P\gamma^2\Big\} \end{aligned} \tag{3.233}$$

$$B_{k2}=\Big\{B_2+(B_1\tilde{D}_\perp^*\tilde{D}_\perp D_{11}^*+\gamma^2SC_1^*-\gamma^2SC_2^*D_{21}D_{11}^*)(\gamma^2I-D_{11}\tilde{D}_\perp^*\tilde{D}_\perp D_{11}^*)^{-1}D_{12}\Big\}D_{k12} \tag{2.234}$$

$$C_{k2}=D_{k21}\Big\{C_2+D_{21}(\gamma^2I-D_{11}^*D_\perp D_\perp^*D_{11})^{-1}(D_{11}^*D_\perp D_\perp^*C_1+B_1^*P\gamma^2-D_{11}^*D_{12}B_2^*P\gamma^2)\Big\} \tag{2.235}$$

<u>Remark 3.5</u>

With the transformation in (3.218) disabled, the minimization procedure takes place over $\gamma$, (3.225)-(3.235) can be implemented as stated.

<u>3.6 Removal of temporary assumption 1</u>

In this section we shall lift the assumptions on the invertibility of X and Y. It will be shown that Theorem 3.7 are still valid in the general case when S and P are no longer definite but may also be semi-definite.

To begin with, we will consider balancing the Riccati equations (3.25) and (3.27); this technique has already been established in [18,19] and consists of obtaining a basis transformation in the original P(s) such that X and Y are diagonal.

*Balancing the Riccati equations.*

Consider the basis change T in the state-space of P(s) in (3.2) [18,19], then the algebraic Riccati equation in (3.25) becomes

$$X(TAT^{-1}-TB_2D_{12}^*C_1T^{-1})+(TAT^{-1}-TB_2D_{12}^*C_1T^{-1})^*X-XTB_2B_2^*T^*X+T^{-*}C_1^*D_\perp D_\perp^*C_1T^{-1}=0 \quad (3.236a)$$

pre- and post-multiply (3.236a) with $T^*$ and T respectively gives

$$T^*XT(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*T^*XT-T^*XTB_2B_2^*T^*XT+C_1^*D_\perp D_\perp^*C_1=0 \quad (3.236b)$$

which shows that X undergoes a congruence transformation ($T^{-*}XT^{-1}\rightarrow X$). Similarly in the new basis (3.27) becomes [18,19]

$$T^{-1}YT^{-*}(A-B_1D_{21}^*C_2)^*+(A-B_1D_{21}^*C_2)T^{-1}YT^{-*}-T^{-1}YT^{-*}C_2^*C_2T^{-1}YT^{-*}+B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^*=0 \quad (3.237)$$

and hence ($TYT^*\rightarrow Y$). Using the construction given in [11, appendix B], one can select T so that

$$X=\begin{bmatrix}\tilde{\Sigma}_1 & & & \\ & \tilde{\Sigma}_2 & & \\ & & 0 & \\ & & & 0\end{bmatrix}=\begin{bmatrix}\Sigma_1 & \\ & 0\end{bmatrix} \quad (3.238)$$

and

$$Y=\begin{bmatrix}\tilde{\Sigma}_1 & & & \\ & 0 & & \\ & & \tilde{\Sigma}_3 & \\ & & & 0\end{bmatrix}$$

introducing a permutation matrix J so that

$$JYJ^*=\begin{bmatrix}\tilde{\Sigma}_1 & & & \\ & \tilde{\Sigma}_3 & & \\ & & 0 & \\ & & & 0\end{bmatrix}=\begin{bmatrix}\Sigma_2 & \\ & 0\end{bmatrix} \quad (3.239)$$

where $\Sigma_1>0$ and $\Sigma_2>0$.

Assuming that the realization in (3.2) has been put into a basis which balances the Riccati equations. Continuing, define

$$W=J(A-B_1D_{21}^*C_2)J^* \tag{3.240}$$
$$Z=(A-B_2D_{12}^*C_1) \tag{3.241}$$

Introduce a partitioning on

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_{11}\ C_{12} \\ C_{21}\ C_{22} \end{bmatrix} \tag{3.242}$$

and

$$[B_1\ B_2] = \begin{bmatrix} B_{11}\ B_{12} \\ B_{21}\ B_{22} \end{bmatrix} \tag{3.243}$$

conformable with (3.238); similarly, partition Z and substitute into (3.25)

$$\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}+\begin{bmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$-\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}\begin{bmatrix} B_{12}^*\ B_{22}^* \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} C_{11}^* \\ C_{12}^* \end{bmatrix}D_\perp D_\perp^*[C_{11}\ C_{12}] \tag{3.244}$$

The (2,2) block of (3.244) yields

$$C_{12}^*D_\perp D_\perp^*C_{12} = 0 \quad \Rightarrow \quad D_\perp^*C_{12} = 0 \tag{3.245}$$

Consequently, the (1,2) block of (3.244) gives

$$\Sigma_1Z_{12} = 0 \quad \Rightarrow \quad Z_{12} = 0 \tag{3.246}$$

finally, the (1,1) block gives

$$\Sigma_1Z_{11}+Z_{11}^*\Sigma_1-\Sigma_1B_{12}B_{12}^*\Sigma_1+C_{11}^*D_\perp D_\perp^*C_{11}=0$$

which is a deflated Riccati equation with a positive definite solution. It is not difficult to verify that

$$A-B_2F = Z-B_2B_2^*X = \begin{bmatrix} Z_{11}-B_{12}B_{12}^*\Sigma_1 & 0 \\ Z_{12}-B_{22}B_{12}^*\Sigma_1 & Z_{22} \end{bmatrix} \tag{3.247}$$

from which one can infer that $Z_{22}$ is asymptotically stable since $(A-B_2F)$ is; as noted in [18], the eigenvalues of $Z_{22}$ are the stable modes of $(A-B_2D_{12}^*C_1)$ which are undetectable through $D^*_{\perp}C_1$. Partitioning

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} J^* = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \tag{3.248}$$

and

$$J\,[B_1\ B_2] = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} \tag{3.249}$$

conformably with (3.239), similarly for W. Substituting into (3.27), then it is not difficult to verify that

(2,2) block $\Rightarrow$ $\hat{B}_{21}\widetilde{D}^*_{\perp}=0$ (3.250)

(2,1) block $\Rightarrow$ $W_{21}=0$ (3.251)

and

$$J(A-HC_2)J^* = \begin{bmatrix} W_{11}-\Sigma_2\hat{C}_{21}^*\hat{C}_{21} & W_{12}-\Sigma_2\hat{C}_{21}^*\hat{C}_{22} \\ 0 & W_{22} \end{bmatrix} \tag{3.252}$$

$W_{22}$ is asymptotically stable since $(A-HC_2)$ is, further, the eigenvalues of $W_{22}$ are the stable modes of $(A-B_1D_{21}^*C_2)$ which are undetectable through $B_1\widetilde{D}^*_{\perp}$.

Lemma 3.9

(i) $(I-XX^{\dagger})\widetilde{X}=0$

and

(ii) $(I-YY^{\dagger})\hat{X}=0$

where X and Y, and $\widetilde{X}$ and $\hat{X}$ are the unique positive semi-definite stabilizing solutions to (3.25), (3.27), (3.78) and (3.82) respectively.

Proof:

Suppose that (3.25) is in balanced coordinates, then X is given by (3.238). Let $\widetilde{X}_{11}$ be the stabilizing solution to

$$\widetilde{X}_{11}(Z_{11}-B_{12}B_{12}^*\Sigma_1)+(Z_{11}-B_{12}B_{12}^*\Sigma_1)^*\widetilde{X}_{11}+\widetilde{X}_{11}\Sigma_1^{-1}C_{11}^*D_{\perp}D^*_{\perp}C_{11}\Sigma_1^{-1}\widetilde{X}_{11}\gamma^{-2}+\Sigma_1B_{11}B_{11}^*\Sigma_1=0$$

where the terms are defined in (3.242), (3.243) and (3.247). Set $\widetilde{X} = \begin{bmatrix} \widetilde{X}_{11} & 0 \\ 0 & 0 \end{bmatrix}$; then it is easy to verify that as such $\widetilde{X}$ satisfies (3.82) and is the unique stabilizing solution.

This structure on $\widetilde{X}$ allows one to deduce (i). The second part is established via a parallel set of arguments and setting $\hat{X} = \begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & 0 \end{bmatrix}$. □

Lifting the invertibility assumptions on X and Y allows one to rewrite the factorization Riccati equations (3.78) and (3.82)

$$\hat{X}(A-HC_2)^*+(A-HC_2)\hat{X}+\hat{X}Y^{\dagger}B_1\widetilde{D}_{\perp}^*\widetilde{D}_{\perp}B_1^*Y^{\dagger}\hat{X}\gamma^{-2}+YC_1^*C_1Y=0 \tag{3.253a}$$

and

$$\widetilde{X}(A-B_2F)+(A-B_2F)^*\widetilde{X}+\widetilde{X}X^{\dagger}C_1^*D_{\perp}D_{\perp}^*C_1X^{\dagger}\widetilde{X}\gamma^{-2}+XB_1B_1^*X=0 \tag{3.253b}$$

$\gamma^2$(3.25)+(3.253b) can be rewritten as

$$(\gamma^2X+\widetilde{X})(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*(\gamma^2X+\widetilde{X})+(\gamma^2I-\widetilde{X}X^{\dagger})C_1^*D_{\perp}D_{\perp}^*C_1(\gamma^2I-X^{\dagger}\widetilde{X})\gamma^{-2}$$
$$+\widetilde{X}X^{\dagger}C_1^*D_{\perp}D_{\perp}^*C_1+C_1^*D_{\perp}D_{\perp}^*C_1X^{\dagger}\widetilde{X}-\widetilde{X}B_2B_2^*X-\gamma^2XB_2B_2^*X-XB_2B_2^*\widetilde{X}+XB_1B_1^*X=0$$

substituting $\widetilde{X}X^{\dagger}$(3.25) and (3.25)$X^{\dagger}\widetilde{X}$ yields

$$X(A-B_2D_{12}^*C_1)(\gamma^2I-X^{\dagger}\widetilde{X})+(\gamma^2I-\widetilde{X}X^{\dagger})(A-B_2D_{12}^*C_1)^*X+(\gamma^2I-\widetilde{X}X^{\dagger})C_1^*D_{\perp}D_{\perp}^*C_1(\gamma^2I-X^{\dagger}\widetilde{X})\gamma^{-2}$$
$$-\gamma^2XB_2B_2^*X+XB_1B_1^*X+(A-\ BF)^*(I-XX^{\dagger})\widetilde{X}+\widetilde{X}(I-X^{\dagger}X)(A-B_2F)=0$$

Using Lemma 3.9 (i) gives

$$P(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*P-P(\gamma^2B_2B_2^*-B_1B_1^*)P+C_1^*D_{\perp}D_{\perp}^*C_1\gamma^{-2}=0 \tag{3.254}$$

where $P=(\gamma^2I-\widetilde{X}X^{\dagger})^{-1}X$ in which $(\gamma^2I-\widetilde{X}X^{\dagger})$ is assumed non-singular; Rank(P)=Rank(X). A dual set of arguments using (3.27), (3.253a) and lemma 3.9 (ii) yields

$$S(A-B_1D_{21}^*C_2)^*+(A-B_1D_{21}^*C_2)S-S(\gamma^2C_2^*C_2-C_1^*C_1)S+B_1\widetilde{D}_{\perp}^*\widetilde{D}_{\perp}B_1^*\gamma^{-2}=0 \tag{3.255}$$

where $S=(\gamma^2I-\hat{X}Y^{\dagger})^{-1}Y$ in which $(\gamma^2I-\hat{X}Y^{\dagger})$ is assumed non-singular; Rank(S)=Rank(Y).

In this balanced framework the realizations in (3.28) and (3.101b) may be replaced with others of dimension Rank(X)+Rank(Y). The analysis of section (3.4) may be repeated to give necessary and sufficient conditions for the existence of stabilizing controllers in terms of a smaller Pick matrix.

Using (3.238)-(3.252) one can derive a minimal realization for T(s) in (3.28)

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \\ \widetilde{T}_{\perp} & 0 & 0 \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|ccc} Z_{11}-B_{12}B_{12}^{*}\Sigma_1 & B_{12}\hat{F}_1 & B_{11} & B_{12} & -\Sigma_1^{-1}C_{11}^{*}D_{\perp} \\ 0 & W_{11}-\Sigma_2\hat{C}_{21}^{*}\hat{C}_{21} & \hat{B}_{11}\widetilde{D}_{\perp}^{*}\widetilde{D}_{\perp}-\Sigma_2\hat{C}_{21}^{*}D_{21} & 0 & 0 \\ \hline D_{\perp}D_{\perp}^{*}C_{11}-D_{12}B_{12}^{*}\Sigma_1 & D_{12}\hat{F}_1 & 0 & D_{12} & D_{\perp} \\ 0 & \hat{C}_{21} & D_{21} & 0 & 0 \\ 0 & -\widetilde{D}_{\perp}\hat{B}_{11}^{*}\Sigma_2^{-1} & \widetilde{D}_{\perp} & 0 & 0 \end{array}\right]$$

(3.256)

The minimality of T(s) is assured from the fact that the poles of this realization are located in the left half plane, furthermore, one can pinpoint the system zeroes of (3.256) in the open right half plane by showing that the A-matrix of its inverse is congruent to

$$\begin{bmatrix} -(Z_{11}-B_{12}B_{12}^{*}\Sigma_1)^{*} & * \\ 0 & -(W_{11}-\Sigma_2\hat{C}_{12}^{*}\hat{C}_{12})^{*} \end{bmatrix}$$

thus barring any pole-zero cancellations. ( $*$ indicates a don't care entry) As a result of removing the non-minimal modes in T(s) one can obtain a reduced order model for $R_a(s)$ in (3.101b).

$$R_a(s)=\begin{bmatrix} R_{00} & R_{01} & R_{02} \\ R_{10} & R_{11} & R_{12} \\ R_{20} & R_{21} & R_{22} \end{bmatrix}(s)\stackrel{s}{=}$$

$$\left[\begin{array}{cc|cccc} -(Z_{11}-B_{12}B_{12}^{*}\Sigma_1)^{*} & \Sigma_1B_{11}(\widetilde{D}_{\perp}^{*}D_{\perp}\hat{B}_{11}^{*}-D_{21}^{*}\hat{C}_{22}\Sigma_2) & 0 & \gamma^{-1}\widetilde{X}_{11}B_{r1} & \Sigma_1B_{11}D_{21}^{*} & \Sigma_1B_{11}\widetilde{D}_{\perp}^{*} \\ 0 & -(W_{11}-\Sigma_2\hat{C}_{21}^{*}\hat{C}_{21})^{*} & 0 & 0 & -\hat{C}_{21}^{*} & \Sigma_2^{-1}\hat{B}_{11}\widetilde{D}_{\perp}^{*} \\ \hline 0 & 0 & 0 & 0 & \gamma I & 0 \\ 0 & \gamma^{-1}\widetilde{D}_{\perp}\hat{B}_{11}^{*}\Sigma_2^{-1}\hat{X}_{11} & 0 & 0 & 0 & \gamma I \\ -B_{12}^{*} & \hat{F}_1\Sigma_2 & \gamma I & 0 & 0 & 0 \\ D_{\perp}^{*}C_{11}\Sigma_1^{-1} & 0 & 0 & \gamma I & 0 & 0 \end{array}\right]$$

where (3.257)

$B_{r1}=\Sigma_1^{-1}C_{11}^{*}D_{\perp}$ and $FJ=[\hat{F}_1\ \hat{F}_2]$

Following the steps in section 3.4

$$a_{i2}=[D_{21}^{*}\ \widetilde{D}_{\perp}^{*}]\begin{bmatrix} -\hat{C}_{21}\Sigma_2 \\ \widetilde{D}_{\perp}\hat{B}_{11}^{*} \end{bmatrix}x_i \tag{3.258a}$$

$$[\,s_iI+(W_{11}-\Sigma_2\hat{C}_{21}^*\hat{C}_{21})^*]\,x_i=0 \tag{3.258b}$$

$$a_{i2}^*=x_i^*\,[\,-\Sigma_1B_{12}\quad C_{11}^*D_\perp]\begin{bmatrix} D_{12}^* \\ D_\perp^* \end{bmatrix} \tag{3.258c}$$

$$x_i^*[s_iI+(Z_{11}-B_{12}B_{12}^*\Sigma_1)^*]=0 \tag{3.258d}$$

In the case that Rank(X)>Rank(Y)

$$\begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix}=\begin{bmatrix} \gamma^{-1}\widetilde{D}_\perp\hat{B}_{11}^*(\gamma^2\Sigma_2^{-1}-\hat{S}^{-1})\Sigma_2 \\ \hat{C}_{11}\Sigma_2 \end{bmatrix} \tag{3.259a}$$

$$[b_{i1}^*\quad b_{i2}^*]\;=\;[\,\gamma^{-1}\Sigma_1(\Sigma_1^{-1}\gamma^2-\hat{P}^{-1})C_{11}^*D_\perp\mid\Sigma_1B_{11}] \tag{3.258b}$$

in which $\hat{S}$ and $\hat{P}$ are the non-singular parts of S and P respectively. (a simple calculation shows that balancing X and Y also balances P and S).

The Pick matrix in this case is congruent to the expression on the right in (3.260):

$$\mathrm{In}(\Pi(\gamma))=\mathrm{In}(\begin{bmatrix} \Sigma_2^{-1} & \gamma I \;\begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ \begin{matrix}\gamma I \\ 0\cdots 0\end{matrix} & \Sigma_1^{-1} \end{bmatrix}) \tag{3.260}$$

In the case that Rank(X)<Rank(Y) the the subsequent inertia properties of the Pick matix are given by:

$$\mathrm{In}(\Pi(\gamma))=\mathrm{In}(\begin{bmatrix} \Sigma_2^{-1} & \begin{matrix}\gamma I \\ 0\cdots 0\end{matrix} \\ \gamma I \;\begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \Sigma_1^{-1} \end{bmatrix}) \tag{3.261}$$

Finally in the case that Rank(X)=Rank(Y), there is no need for bordering zeros:

$$\mathrm{In}(\Pi(\gamma))=\mathrm{In}(\begin{bmatrix} \Sigma_2^{-1} & \gamma I \\ \gamma I & \Sigma_1^{-1} \end{bmatrix}) \tag{3.262}$$

The findings of this section may be summarized in the following theorem

Theorem 3.8

Necessary and sufficient conditions for the existence of a stabilizing controller such that $||F_l(K,P)||_\infty \leq \gamma$ are:

i) $S \geq 0$

ii) $P \geq 0$

iii) $\lambda_{max}(SP) \leq \gamma^{-2}$ □

## 3.7 Problems of the first and second kind.

The aim of this section is to demonstrate how the controller representation formula and the indefinite Riccati equations can be specialized to solve problems of the first and second kinds.

*Problems of the first kind:* If $P_{12}(s)$ and $P_{21}(s)$ in (3.2) are square and not necessarily of the same size, we call the associated problem a problem of the first kind. To solve such problems it suffices to set $D_\perp=0$ and $\tilde{D}_\perp=0$ in the formulae, furthermore, $D_{12}$ and $D_{21}$ can be scaled so that $D_{12}=I$ and $D_{21}=I$. Algorithm 1 can be applied directly and $\gamma_{low}$ will be the optimal cost by Nehari's theorem. The drawback to this approach is that unless $\gamma_{low}$ is chosen a $\gamma$-iteration is required. However, the solution technique presented in [19] is computationally cheaper and requires no $\gamma$-iteration. In the case of $D_{11}=0$ and $D_{22}=0$, the Riccati equations and representation formula degenerate to

$$P(A-B_2C_1)+(A-B_2C_1)^*P-P(\gamma^2B_2B_2^*-B_1B_1^*)P=0$$
$$S(A-B_1C_2)^*+(A-B_1C_2)S-S(\gamma^2C_2^*C_2-C_1^*C_1)S=0$$

$$E_k=I-\gamma^2SP$$
$$A_k=(A-B_2C_1-B_1C_2)+S\gamma^2(A-B_2C_1-B_1C_2)^*P-\gamma^2B_2B_2^*P-S\gamma^2C_2^*C_2$$
$$B_{k1}=B_1+\gamma^2SC_2^*$$
$$B_{k2}=\gamma(B_2+SC_1^*)$$
$$C_{k1}=C_1+B_2^*P\gamma^2$$
$$C_{k2}=(C_2+B_1^*P)\gamma$$
$$D_k=\begin{bmatrix} D_{k11} & D_{k12} \\ D_{k21} & D_{k22} \end{bmatrix}=\begin{bmatrix} 0 & -\gamma I \\ -\gamma I & 0 \end{bmatrix}$$

*Problems of the second kind:* These problems are characterized by having either $P_{12}(s)$ or $P_{21}(s)$ non-square. Consider the case when $P_{21}(s)$ is square, (for problems in which $P_{12}(s)$ is square, one can transpose P(s), calculate the associated controller which when transposed solves the original problem) then Algorithm 1 carries over by setting $\tilde{D}_\perp=0$ and $D_{21}=I$. For this type of problem the solution presented here is superior from a computational point of view to the one in [18]. In this case one has to solve two n-dimensional Riccati equations (3.254) and (3.255) which require approximately $n^3$ floating point operations each. The old approach [9,26,31] requires one to solve a 2n-spectral factorization problem [18,eqn 4.34] which is performed in approximately $8n^3$ *flops*. The advantages are clear. In the case of $D_{11}=0$ and $D_{22}=0$, the Riccati equations and the controller representation formula degenerate to

$$P(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*P-P(\gamma^2B_2B_2^*-B_1B_1^*)P+C_1^*D_\perp D_\perp^*C_1\gamma^{-2}=0$$
$$S(A-B_1C_2)^*+(A-B_1C_2)S-S(\gamma^2C_2^*C_2-C_1^*C_1)S=0$$
$$E_k=I-\gamma^2SP$$
$$A_k=(A-B_2D_{12}^*C_1-B_1C_2)+S\gamma^2(A-B_2D_{12}^*C_1-B_1C_2)^*P-\gamma^2B_2B_2^*P-S(\gamma^2C_2^*C_2-C_1^*D_\perp D_\perp^*C_1)$$
$$B_{k1}=B_1+\gamma^2SC_2^*$$
$$B_{k2}=\gamma(B_2+SC_1^*D_{12})$$
$$C_{k1}=D_{12}^*C_1+B_2^*P\gamma^2$$
$$C_{k2}=(C_2+B_1^*P)\gamma$$
$$D_k=\begin{bmatrix} D_{k11} & D_{k12} \\ D_{k21} & D_{k22}\end{bmatrix}=\begin{bmatrix} 0 & -\gamma I \\ -\gamma I & 0\end{bmatrix}$$

Remark 3.6

Section 2.4.3 shows that for inertia optimality the generator of all Q(s) under-goes an effective reduction in dimension. This phenomenon carries over to the generator of all controllers and is the object of this remark. At inertia optimality the controller generator is given by

$$F_l\left\{\begin{bmatrix} K_o^{11} & K_o^{12} \\ K_o^{21} & K_o^{22}\end{bmatrix};\begin{bmatrix} Q & \tilde{Q}_{31} \\ \tilde{Q}_{13} & \tilde{Q}_{11}\end{bmatrix}\right\}\begin{bmatrix} u_1 \\ u_2\end{bmatrix}=\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \tag{3.264a}$$

$$F_l\left\{\begin{bmatrix} K_o^{11} & K_o^{12} \\ K_o^{21} & K_o^{22}\end{bmatrix};\begin{bmatrix} Q & Q_{31} & \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix} \\ Q_{13} & Q_{11} & \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix} \\ 0\cdots0 & 0\cdots0 & \gamma\end{bmatrix}\right\}\begin{bmatrix} u_1 \\ u_2\end{bmatrix}=\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \tag{3.264b}$$

expanding the $\{\cdot\}$ term in (3.264a) together with simple manipulations yields

$$y_1=K_o^{11}u_1+K_o^{12}(I-QK_o^{22})^{-1}QK_o^{21}u_1+K_o^{12}(I-QK_o^{22})^{-1}\tilde{Q}_{31}u_2$$

and

$$y_2=\tilde{Q}_{13}(I-K_o^{22}Q)^{-1}K_o^{21}u_1+\tilde{Q}_{13}(I-K_o^{22}Q)^{-1}K_o^{22}\tilde{Q}_{31}u_2+\tilde{Q}_{11}u_2$$

substituting from (3.264b) gives

$$\begin{bmatrix} \hat{K}_o^{11} & \hat{K}_o^{12} & 0 \\ \hat{K}_o^{21} & \hat{K}_o^{22} & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

illustrating the loss of effective dimension in the generator of all controllers. □

## 3.8 Computation time trials.

This section presents some data to validate the claim that this approach to solving $H^\infty$ control problems is better than that originally proposed by Doyle [9,31]. A number of computation trials are performed on both the original software [9,29,31] and the approach presented in this chapter. In each case a single iteration is timed (Unix - dtime). The results are given in the table below, showing the state-dimension of P(s) and cpu execution time (in seconds) for one iteration of the original and new approaches. Apart from a marked reduction in execution time, the improved program is more robust and able to tackle problems with higher state-dimension than its predecessor.

| N$^o$ of states | old (sec) | new (sec) |
|---|---|---|
| 1 | 2.46 | 0.26 |
| 2 | 15.8 | 0.58 |
| 3 | 22.8 | 0.93 |
| 4 | 90.6 | 1.58 |
| 5 | 171.5 | 2.03 |
| 6 | 244.5 | 3.50 |
| 7 | 303.1 | 6.30 |
| 8 | 463.5 | 10.3 |
| 9 | 509.2 | 13.4 |
| 10 | 1228 | 16.3 |

## 3.9 Conclusions.

The purpose of this chapter has been to derive a representation formula for all $H^\infty$ controllers. The generator of all Q(s) in (2.70) is used to generate all the solutions to the four block $H^\infty$ control problem, this generator is then collapsed into the generator of all stabilizers to yield the generator of all stabilizing controllers satisfying (3.1). A number of important points have been established in this chapter, these are:

1] The generator of all controllers has a McMillan degree no greater than that of the original problem setup in P(s). All the solutions may be characterized by a representation formula in which the dimension of the free contraction is the same as $P_{22}(s)$.

2] The generator of all controllers is expressed in terms of the stabilizing solutions to two indefinite Riccati equations: S and P and the state-space realization of P(s).

3] To avoid numerical difficulties, it is prudent to continue using the generalized state-space ideas introduced in chapter 2. In the case of inertia optimality on can characterize all the optimal controllers by invoking singular perturbations on $\mathcal{K}(s)$ in (3.212).

4] Necessary and sufficient conditions for the existence of a stabilizing controller such that $||F_l(P,K)||_\infty \leq \gamma$ are given in terms of the inertia of S, P and $(I-SP\gamma^2)$.

5] Problems with $D_{11} \neq 0$ and $D_{22} \neq 0$ result in cumbersome $H^\infty$ formulae, however these can always be modified using loop shifting arguments into problems where $D_{11}=0$ and $D_{22}=0$. One can therefore always assume that $D_{11}=0$ and $D_{22}=0$ without loss of generality.

6] For the cases when X and/or Y are singular, the controller representation formula is still valid. Necessary and sufficient conditions for the existence of stabilizing controllers are given in terms of the inertia of S, P and $(I-SP\gamma^2)$.

7] This approach can easily be degenerated into one and two block problems [17,18,19] where the controller degree bounds are known.

8] Experience has shown that typical engineering problems often result in a P(s) of high state dimension, the solution is time consuming, especially when several weight selections are required. The work presented in this chapter shows that considerable gains can be made vis-à-vis the approach in [26,31]. This solution technique has no state inflation problems and therefore there is no need for computationally demanding model reduction routines in the computer code. Although the level of benefit varies from problem to problem, a calculation time reduction of between 10 and 80 is easy to achieve, the higher the state dimension of P(s) the greater the benefit.

At this point it is appropriate to mention that several researchers have indepedently obtained similar results to those presented in this chapter. Glover and Doyle have also used the embedding idea independently to derive on such set of formulae [14], Doyle et al have used classical $L_2$ type arguments to obtain a representation formula based on the solution of two indefinite Riccati equations [10]. In contrast to the Doyle-Glover approach we are able to treat the optimal case. Hung [15] has also obtained one such set of formulae base on Riccati equations using interpolation theory, this work also treats the optimal case. Glover et al [36] have employed J-factorization, and Verma has similar results pertaining to problems of the first kind [29].

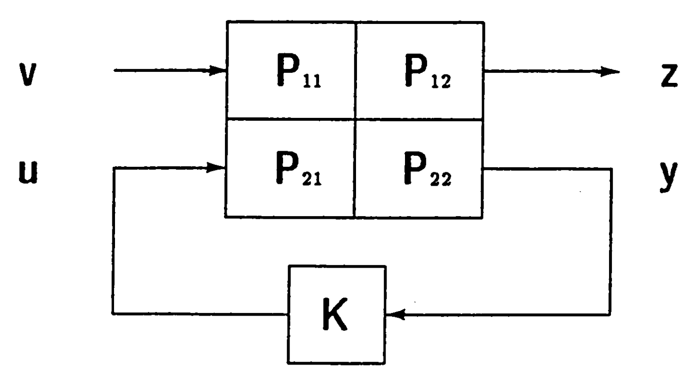



Figure 3.2 – Generalized regulator set-up

# Chapter four
# Closed formula for a parametric mixed sensitivity problem.

## 4.1 Introduction

The all-pass embedding technique presented in chapter 3 has reduced the computational burden associated with $H^{\infty}$-optimal control design to the solution of two n-dimensional Riccati equations provided an upper bound for the $H^{\infty}$-norm is known. If an upper bound for the norm is not known, or the optimal norm ($\gamma_o$) is required, a computationally expensive $\gamma$-iteration is still necessary. Ideally, one would like to remove the $\gamma$-iteration completely. Although this ideal still seems some way off, there has been some progress in this direction. Jonckheere and Juang have obtained an approximation to $\gamma_o$ in the case of the mixed sensitivity problem by converting it into a quadratic approximation problem [32]. Glover and McFarlane have obtained a closed formula for $\gamma_o$ in the case of a two block optimal robustness problem [41]. Foias and Tannenbaum have a determinantal formula for the (essential) norm of the four block operator associated with $H^{\infty}$ control [42].

In this chapter we will derive a closed formula for $\gamma_o$ in the case of a special mixed sensitivity problem which is concerned with minimizing the combined effects of plant output disturbance and noise on the measurement feedback signal; mixed sensitivity optimization has already been the subject of considerable scrutiny [33,38,65,66,67]. The mixed sensitivity problem we consider is

$$\gamma_o = \inf_{K \in S} \left\| \begin{matrix} (1-\epsilon)GK(I+GK)^{-1}(s) \\ \epsilon(I+GK)^{-1}(s) \end{matrix} \right\|_{\infty} \qquad \epsilon \in [0,1] \tag{4.1}$$

in which $S$ is the set of all stabilizing compensators as defined in (3.8) and (3.9). As we will show, this particular two block problem allows one to derive a closed formula for $\gamma_o$ in terms of $\lambda_{max}(YX)$ where X and Y are the two standard $H^2$ Riccati equation solutions (3.25) and (3.27). We will also show that the two indefinite Riccati equations associated with the representation formulae for all solutions (3.163) and (3.164), are soluble in terms of X and Y. This means that once $\gamma_o$ is found, a formula for all solutions corresponding to any $\gamma \geq \gamma_o$ is available for free. Each time a new value of $\epsilon$ is chosen, only one of the two $H^2$ Riccati equations need be re-solved. When deriving these results, we make extensive use of the formulae derived in chapter 3 and especially the conditions for optimality. Similar results have been found for a number of one block problems [19]. As an example, the unweighted sensitivity problem has $\gamma_o = \sqrt{1+\lambda_{max}(YX)}$ provided of course that X and Y are not both zero. In the degenerate case that X and Y are both zero, the plant is stable and minimum phase and the plant may be cancelled by setting $K(s)=cG^{-1}(s)$ for some constant c. In this case the problem is essentially non-dynamic and $\gamma_o$ can be made arbitrarily small.

We will analyse

$$\gamma_o = \inf_{K\in S} \left\| \begin{matrix} GK(I+GK)^{-1}(s) \\ (I+GK)^{-1}(s) \end{matrix} \right\|_\infty \tag{4.2}$$

in the next section. The solution to the problem in (4.1) follows by analogy with (4.2), but is considerably more complicated.

## 4.2. Uweighted mixed-sensitivity.

We will begin by analysing the special case of (4.1) given in (4.2). Firstly, we claim that

$$F_l(P(s),K(s)) = \begin{bmatrix} GK(I+GK)^{-1} \\ (I+GK)^{-1} \end{bmatrix}(s) \tag{4.3}$$

where

$$P(s) = \left[\begin{array}{c|c} 0 & -G \\ I & G \\ \hline -I & -G \end{array}\right](s) \stackrel{s}{=} \left[\begin{array}{c|cc} A & 0 & B \\ \hline -C & 0 & -D \\ C & I & D \\ -C & -I & -D \end{array}\right] \tag{4.4}$$

This fact is easily verified by calculation. Next, we perform a preliminary scaling in order that $D_{12}$ is part of an orthogonal matrix [9,26]. In this step we assume that $D^{-1}$ exists. If this is not the case, one may weight the plant by setting $G_w(s)=G(s)W(s)$ in which we choose $W(s)$ so that $G_w(\infty)$ is nonsingular. This is always possible and one may think of $G_w(s)$ as a replacement for $G(s)$. Multiplying the second column of $P(s)$ by $\frac{D^{-1}}{\sqrt{2}}$ gives

$$P(s) = \left[\begin{array}{c|cc} A & 0 & \frac{BD^{-1}}{\sqrt{2}} \\ \hline -C & 0 & \frac{-1}{\sqrt{2}}I \\ C & I & \frac{1}{\sqrt{2}}I \\ -C & -I & \frac{-1}{\sqrt{2}}I \end{array}\right] \tag{4.5}$$

with

$$[D_{12}\ D_\perp] = \begin{bmatrix} -I/\sqrt{2} & I/\sqrt{2} \\ I/\sqrt{2} & I/\sqrt{2} \end{bmatrix} \tag{4.6}$$

Substituting (4.5) and (4.6) into the two parameterization Riccati equations (3.25) and (3.27) gives

$$X(A - BD^{-1}C) + (A - BD^{-1}C)^*X - \frac{XBD^{-1}D^{-*}B^*X}{2} = 0 \tag{4.7a}$$

and

$$YA^* + AY - YC^*CY=0 \tag{4.7b}$$

respectively.

In the same way, one may substitute the state-space realization given in (4.5), into (3.225) and (3.226). This yields

$$P(A - BD^{-1}C) + (A - BD^{-1}C)^*P - \frac{PBD^{-1}D^{-*}B^*P}{2} \cdot \frac{\gamma^2(\gamma^2-1)}{(\gamma^2-\frac{1}{2})} = 0 \tag{4.8a}$$

and

$$SA^* + AS - SC^*CS(\gamma^2-1) =0 \tag{4.8b}$$

after some simplification.

Comparing (4.7a) and (4.8a) shows that

$$\boxed{P = \frac{(\gamma^2-\frac{1}{2})}{\gamma^2(\gamma^2-1)}X \qquad \gamma\geq 1} \tag{4.9}$$

while a comparison between (4.7b) and (4.8b) yields

$$\boxed{S = \frac{1}{\gamma^2-1}Y \qquad \gamma\geq 1} \tag{4.10}$$

As noted above, P and S will be stabilizing and positive semi-definite provided $(\gamma\geq 1)$.[1] By virtue of Theorem 3.6, and (4.9) and (4.10), $\gamma_o\geq 1$ and this is in agreement with the one block problems given in [19].

### 4.2.1 Parrott optimality.

When deriving a closed formula for $\gamma_o$, our approach will be to seek the value of $\gamma$ for which $(I - \gamma^2 SP)$ is singular. Before doing this calculation however, we need to guarantee that the optimality will always be of the inertia type (as opposed to the Parrott type) . We do this by studying the two block distance problem associated with (4.2) :

$$\inf_{Q\in\mathcal{H}_+^\infty} \left\| \begin{bmatrix} R_1-Q \\ R_2 \end{bmatrix}(s) \right\|_\infty \tag{4.11}$$

[1] Equations (4.9) and (4.10) form an interesting link with LQG theory along the line indicated in [14]. It is not hard to show that $\lim_{\gamma\uparrow\infty} \gamma^2 P=X$ and $\lim_{\gamma\uparrow\infty} \gamma^2 S=Y$ for general four block problems. (4.9) and (4.10) are a special case of this.

Lemma 4.1

$$||R_2(s)||_\infty < 1$$

and

Parrott lower bound $= \frac{1}{\sqrt{2}}$

Proof:

Direct substitution of (4.5) and (4.6) into [18,eqn(2.26)] yields

$$R_2(s) = \left[\begin{array}{c|c} -(A-YC^*C)^* & C^* \\ \hline CY/\sqrt{2} & I/\sqrt{2} \end{array}\right] \tag{4.12}$$

after cancelling 'n' unobservable modes.

$$R_2^*R_2(s) \stackrel{s}{=} \left[\begin{array}{cc|c} (A-YC^*C) & (YC^*CY)/2 & YC^*/2 \\ 0 & -(A-YC^*C)^* & C^* \\ \hline -C & CY/2 & I/2 \end{array}\right]$$

introducing a basis transformation

$$T = \begin{bmatrix} I & -Y/2 \\ 0 & I \end{bmatrix}$$

gives

$$R_2^*R_2 = \tfrac{1}{2}I,$$

hence $R_2(s)$ is all-pass at a height of $\frac{1}{\sqrt{2}}$. This gives $||R_2(s)||_\infty < 1$ and furthermore, the Parrott lower bound $= \frac{1}{\sqrt{2}}$, completing the proof. □

Lemma 4.1 establishes that optimality will necessarily be accompanied by a loss of rank in $(I-\gamma^2SP)$.

Remark 4.1:

In the case that G(s) is stable and minimum phase, X=0 and Y=0 ($\Rightarrow$P=0 and S=0). Direct substitution of (4.5) and (4.6) into [18, eq(2.26)] yields

$$R(s) = \begin{bmatrix} D_{12}^*D_{11} \\ D_\perp^*D_{11} \end{bmatrix}$$

after cancelling 2n non-minimal modes. This leads to a non dynamic problem where the optimal cost is the Parrott lower bound. $\gamma_o=\|D^*_\perp D_{11}\|=\frac{1}{\sqrt{2}}$. In this case the solution to (4.11) is $Q=\frac{1}{\sqrt{2}}I$ and the central Parrott controller is given by: $-F_l(K_O,-\frac{1}{\sqrt{2}})$, where $K_O(s)$ is defined in (3.23)

$$-K(s) = \left[\begin{array}{cc} A-BD^{-1}C & BD^{-1} \\ \sqrt{2}\,C & -\sqrt{2}\,I \end{array}\right]$$

We also note from (4.9) and (4.10) that it is also possible for P and S to be unbounded at $\gamma_o=1$. To see how this might arise, one can consider the quadratic equation $ax^2+bx+c=0$ which has roots given by $(r_1,r_2)=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and $\lim_{|a|\downarrow 0}(r_1,r_2)=(\frac{-c}{b},\infty)$. $\gamma_o=1$ in the case that G(s) is either stable or minimum phase but not both. In this case $\gamma_o$ can be considered as an infimum which can be arbitrarily approached but not reached.

### 4.2.2 Inertia optimality.

Now that we have dealt with the case of that Parrott optimality, we may investigate the problem of

$$\lambda_{min}(I-\gamma^2 SP) = 0$$

Theorem 4.2

The optimal cost $\gamma_o$ for the unweighted mixed-sensitivity problem is given by

$$\gamma_o=\left\{1+\tfrac{1}{2}\left\{\lambda_{max}(YX)+\sqrt{\lambda_{max}(YX)\big(2+\lambda_{max}(YX)\big)}\right\}\right\}^{\frac{1}{2}}$$

where X and Y are the unique positive semi-definite solutions to (4.7a) and (4.7b).

Proof:

Eqns (4.9) and (4.10) establish a relation between P and S, and X and Y. At optimality

$$\lambda_{min}(I-\gamma^2 SP) = 0$$

Substituting from (4.9) and (4.10), we obtain

$$\lambda_{min}(I-\gamma^2 SP) = \lambda_{min}\Big(I-\frac{(\gamma^2-\frac{1}{2})}{(\gamma^2-1)^2}YX\Big) \qquad (\gamma\geq 1)$$

$$= 1-\frac{(\gamma^2-\frac{1}{2})}{(\gamma^2-1)^2}\cdot\lambda_{max}(YX) \qquad (4.14)$$

This gives

$$\lambda_{min}(\mathrm{I}-\gamma^2\mathrm{SP}) = 0$$

$$\Leftrightarrow \qquad (\gamma^2-1)^2-(\gamma^2-\tfrac{1}{2})\cdot\lambda_{max}(\mathrm{YX}) = 0$$

$$\Leftrightarrow \qquad \gamma^4 - \Big(2 + \lambda_{max}(\mathrm{YX})\Big)\gamma^2 + \tfrac{1}{2}\Big(2 + \lambda_{max}(\mathrm{YX})\Big) = 0$$

and therefore

$$\gamma_o^2 = \frac{2+\lambda_{max}(\mathrm{YX})+\sqrt{\Big(2 + \lambda_{max}(\mathrm{YX})\Big)^2-2\,\Big(2 + \lambda_{max}(\mathrm{YX})\Big)}}{2}$$

$$= 1+\tfrac{1}{2}\Big\{\lambda_{max}(\mathrm{YX})+\sqrt{\lambda_{max}(\mathrm{YX})\Big(2 + \lambda_{max}(\mathrm{YX})\Big)}\;\Big\}$$

So finally:

$$\gamma_o = \left\{ 1 + \tfrac{1}{2}\Big\{\lambda_{max}(\mathrm{YX}) + \sqrt{\lambda_{max}(\mathrm{YX})\Big(2 + \lambda_{max}(\mathrm{YX})\Big)}\;\Big\}\right\}^{\frac{1}{2}} \tag{4.15}$$

which completes the proof. □

Once X and Y have been calculated, one may obtain $\gamma_o$ by (4.15), P and S for any value of $\gamma \geq \gamma_o$ by (4.9) and (4.10). The controller for the unweighted mixed sensitivity is then given by:

$$\mathrm{E}_k=\mathrm{I} - \frac{(\gamma^2-\frac{1}{2})}{(\gamma^2-1)^2}\mathrm{YX} \tag{4.16a}$$

$$\mathrm{B}_{k1}=\frac{\gamma^2}{(\gamma^2-1)}\mathrm{YC}^* \tag{4.16b}$$

$$\mathrm{C}_{k1}=\frac{-1}{\sqrt{2}}\Big\{2\mathrm{C}+\mathrm{D}^{-*}\mathrm{B}^*\mathrm{X}\Big\} \tag{4.16c}$$

$$\mathrm{D}_{k12}=-\sqrt{\frac{\gamma^2(\gamma^2-1)}{(\gamma^2-\frac{1}{2})}}\;\cdot \mathrm{I} \tag{4.16d}$$

$$\mathrm{D}_{k21}=-\sqrt{\gamma^2-1}\;\cdot \mathrm{I} \tag{4.16e}$$

$$\mathrm{D}_{k22}=-\frac{\gamma^3}{\sqrt{2\gamma^2-1}}\;\cdot \mathrm{I} \tag{4.16f}$$

$$\mathrm{B}_{k2}=\frac{1}{\sqrt{2}}\Big\{\mathrm{BD}^{-1}-\mathrm{YC}^*\Big\}\mathrm{D}_{k12} \tag{4.16g}$$

$$C_{k2}=\frac{\gamma^2}{2}D_{k21}^{-1}D^{-*}B^*X \tag{4.16h}$$

$$A_k=E_k\left\{A+\frac{BD^{-1}C_{k1}}{\sqrt{2}}\right\}+B_{k1}D_{k21}^{-1}C_{k2} \tag{4.16i}$$

The standard singular perturbation arguments discussed in section 2.4.2 will remove the non-minimal modes at infinity for the case $\gamma=\gamma_0$.

Remark 4.2

If G(s) is either stable or minimum phase, it may be shown that $\gamma_0=1$ and that the Riccati equations in (4.8) become linear as $\gamma\rightarrow 1$. This causes one of the Riccati solutions to become unbounded. If we suppose that G(s) is stable, then S=0 is the required S-solution, if G(s) has a right half plane zero, however, the required P-solution will be unbounded. If G(s) is minimum phase and has a right half plane pole, the properties of S and P are reversed.

It is possible to avoid these difficulties by expressing the controller representation formula in terms of the stable deflating subspaces associated with the Hamiltonians which go with (4.8). One such representation formula is given in Remark 3.2 which is easily specialized to this class of problem. □

## 4.3 Unweighted parametric mixed sensitivity.

In this section we will address the problem posed in (4.1). The approach is analogous to that used to solve the simpler problem of the last section. In the case of the parametric mixed sensitivity problem we have

$$P(s)=\left[\begin{array}{c|c} 0 & -(1-\epsilon)G \\ \epsilon I & \epsilon G \\ \hline -I & -G \end{array}\right](s)\overset{s}{=}\left[\begin{array}{c|cc} A & 0 & B \\ \hline -(1-\epsilon)C & 0 & -(1-\epsilon)D \\ \epsilon C & \epsilon I & \epsilon D \\ -C & -I & -D \end{array}\right] \qquad \epsilon\in[0,\ 1] \tag{4.17}$$

After rescaling, under the assumption that D is non-singular, (4.1) becomes

$$P(s)=\left[\begin{array}{c|cc} A & 0 & BD^{-1}\mu \\ \hline -(1-\epsilon)C & 0 & -(1-\epsilon)\mu I \\ \epsilon C & \epsilon I & \epsilon\mu I \\ -C & -I & -\mu I \end{array}\right] \tag{4.18}$$

in which

$$\mu=(2\epsilon^2-2\epsilon+1)^{-\frac{1}{2}} \tag{4.19}$$

and

$$[D_{12}\ D_{\perp}] = \begin{bmatrix} -(1-\epsilon)\mu I & \epsilon\mu I \\ \epsilon\mu I & (1-\epsilon)\mu I \end{bmatrix} \tag{4.20}$$

Simple substitutions which we leave to the reader, verify that the $H^2$ Riccati equations in (3.25) and (3.27) are given by

$$X(A - BD^{-1}C) + (A - BD^{-1}C)^*X - \mu^2 XBD^{-1}D^{-*}B^*X = 0 \tag{4.21a}$$

and

$$YA^* + AY - YC^*CY = 0 \tag{4.21b}$$

respectively. Substituting (4.18) and (4.19) into (3.225) and (3.226) yields

$$P(A - BD^{-1}C) + (A - BD^{-1}C)^*P - \frac{\mu^2\gamma^2(\gamma^2-\epsilon^2)}{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2} PBD^{-1}D^{-*}B^*P= 0 \tag{4.22a}$$

and

$$SA^* + AS -(\gamma^2-(1-\epsilon)^2)\ SC^*CS =0 \tag{4.22b}$$

The manipulations required in this step are tedious. As before, we may compare (4.21) and (4.22) to yield the positive semi-definite solutions

$$\boxed{P = \frac{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2}{\gamma^2(\gamma^2-\epsilon^2)}X \qquad \gamma\geq\epsilon} \tag{4.23}$$

and

$$\boxed{S = \frac{1}{\gamma^2-(1-\epsilon)^2}Y \qquad \gamma\geq(1-\epsilon)} \tag{4.24}$$

to the indefinite $H^\infty$ Riccati equations. Theorems 3.6 and 3.7 mean that $\gamma_o\geq\max(\epsilon,1-\epsilon)\geq\frac{1}{2}$.

4.3.1 Parrott optimality.

We seek $\gamma_o$ for which $(I-SP\gamma^2)$ is singular, however, before doing this calculation we need to consider Parrott type optimality.

Lemma 4.3

$$\text{Parrott lower bound} = \mu\epsilon(1-\epsilon)\leq\frac{1}{2\sqrt{2}} \tag{4.25a}$$

and

$$||R_2(s)||_\infty < \frac{1}{2} \tag{4.25b}$$

Proof:

The proof proceeds in the same way as for Lemma 4.1; $R_2(s)$ is all-pass at $\mu\epsilon(1-\epsilon)$. The Parrott lower bound is $\mu\epsilon(1-\epsilon)$. The inequality follows by calculating the maximizing $\epsilon$ ($\epsilon=\frac{1}{2}$) and the maximum Parrott bound ($=\frac{1}{2\sqrt{2}}$). The second part is verified from

$$||R_2(s)||_\infty = \frac{1}{2\sqrt{2}} < \min_{\epsilon} \left\{ \max(\epsilon, 1-\epsilon) \right\} = \frac{1}{2} \tag{4.26}$$

□

Lemma 4.3 establishes that optimality will always be of the inertia type except for the cases when G(s) is stable and minimum phase.

Remark 4.2

In the case that G(s) is stable and minimum phase X=0 and Y=0 ($\Rightarrow$P=0 and S=0) leads to a non-dynamic problem with

$$R(s) \stackrel{s}{=} \begin{bmatrix} D_{12}^* D_{11} \\ D_\perp^* D_{11} \end{bmatrix}$$

where $\gamma_o$=Parrott lower bound$=||D^*_\perp D_{11}||=\mu\epsilon(1-\epsilon)$ and $Q=\epsilon^2\mu$ with the unique Parrott controller given by

$$-K(s) = \begin{bmatrix} A-BD^{-1}C & BD^{-1}\epsilon^2/(1-\epsilon)^2 \\ C/\mu & -\epsilon^2/\mu(1-\epsilon)^2 I \end{bmatrix}$$

4.3.2 Inertia optimality.

Theorem 4.4

The optimal cost $\gamma_o$ for the unweighted parametric mixed sensitivity problem is given by

$$\gamma_o = \left\{ \frac{1}{2}(\lambda_{max}(YX)+\mu^{-2}) + \frac{1}{2}\sqrt{\mu^2(\mu^{-2}+\lambda_{max}(YX))((2\epsilon-1)^2+\mu^{-2}\lambda_{max}(YX))} \right\}^{\frac{1}{2}} \tag{4.27}$$

Proof:

Inertia type optimality is accompanied with a loss of rank in $(I-SP\gamma^2)$

$$\Rightarrow \qquad \lambda_{min}(\text{I-SP}\gamma^2)=\lambda_{min}\left\{\text{I} - \frac{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2}{(\gamma^2-(1-\epsilon)^2)(\gamma^2-\epsilon^2)}\text{YX}\right\}=0$$

$$\Rightarrow \qquad (\gamma^2-\epsilon^2)(\gamma^2-(1-\epsilon)^2)=\left\{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2\right\}\lambda_{max}(\text{YX})$$

$$\Rightarrow \qquad \gamma^4-\gamma^2(\lambda_{max}(\text{YX})+\mu^{-2})+\epsilon^2(1-\epsilon)^2(1+\mu^2\lambda_{max}(\text{YX}))=0$$

$$\Rightarrow \qquad \gamma^2=\tfrac{1}{2}(\lambda_{max}(\text{YX})+\mu^{-2})+\tfrac{1}{2}\sqrt{(\lambda_{max}(\text{YX})+\mu^{-2})^2-4\epsilon^2(1-\epsilon)^2(1+\mu^2\lambda_{max}(\text{YX}))}$$

$$\Rightarrow \qquad \gamma_o=\left\{\tfrac{1}{2}(\lambda_{max}(\text{YX})+\mu^{-2})+\tfrac{1}{2}\sqrt{\mu^2(\mu^{-2}+\lambda_{max}(\text{YX}))((2\epsilon-1)^2+\mu^{-2}\lambda_{max}(\text{YX}))}\right\}^{\frac{1}{2}}$$

completes the proof. □

Once X and Y have been calculated, $\gamma_o$ is obtained from (4.27). Then the optimal and suboptimal controllers are given by

$$\text{E}_k=\text{I} - \frac{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2}{(\gamma^2-\epsilon^2)(\gamma^2-(1-\epsilon)^2)}\text{YX} \tag{4.28a}$$

$$\text{B}_{k1}=\frac{\gamma^2}{\gamma^2-(1-\epsilon)^2}\text{YC}^* \tag{4.28b}$$

$$\text{C}_{k1}=\frac{-1}{\mu}\left\{\text{C}+\mu^2\text{D}^{-*}\text{B}^*\text{X}\right\} \tag{4.28c}$$

$$\text{D}_{k12}=-\sqrt{\frac{\gamma^2(\gamma^2-\epsilon^2)}{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2}}\cdot\text{I} \tag{4.28d}$$

$$\text{D}_{k21}=-\sqrt{\gamma^2-\epsilon^2}\cdot\text{I} \tag{4.28e}$$

$$\text{D}_{k22}=-\frac{\mu\gamma^3}{\sqrt{\gamma^2-\epsilon^2\mu^2(1-\epsilon)^2}}\cdot\text{I} \tag{4.28f}$$

$$\text{B}_{k2}=\mu\left\{\text{BD}^{-1}-\text{YC}^*\right\}\text{D}_{k12} \tag{4.28g}$$

$$\text{C}_{k2}=\mu^2\gamma^2\text{D}_{k21}^{-1}\text{D}^{-*}\text{B}^*\text{X} \tag{4.28h}$$

$$\text{A}_k=\text{E}_k\left\{\text{A}+\mu\text{BD}^{-1}\text{C}_{k1}\right\}+\text{B}_{k1}\text{D}_{k21}^{-1}\text{C}_{k2} \tag{4.28i}$$

which specialized down to (4.16) in the non-parametric case. In the case of optimality a standard singular perturbation argument will remove the infinite poles.

## 4.4 Conclusions

The aim of this chapter is to derive a closed formula for the optimal, cost in the case of the parametric mixed sensitivity problem given in (4.1). It has been demonstrated that the optimal cost and a representation formula for all solutions may be evaluated in terms of the $H^2$ Riccati equation solutions and the state space realization of G(s) only. If one requires to change the target value of $\gamma \geq \gamma_o$, then this can be done without re-solving either of the indefinite $H^\infty$ Riccati equations. In the case of a change in the value of $\epsilon$, only one of the $H^2$ Riccati equations need be re-solved; review (4.21).

An alternative cost formula may be derived using the all-pass embedding techique, its draw-back is that $\gamma_o$ is the solution to a 2n eigen value problem which makes the approach presented in this chapter more attractive especially when solving problems with high state dimension.

The ease with which these results were obtained is most encouraging. Hopefully, the results recently reported in [10,14,16,34,36] and in chapter 3 will lead to a larger class of problems for which non-iterative solutions are possible. Recently, two more closed formulae for optimal $\gamma$ have been reported [80].

# Chapter five
# $H^{\infty}$-optimal control of synchronous turbo-generators.

## 5.1 Introduction.

Most modern power systems display complex and nonlinear dynamic behavior. Sudden changes in the system are followed by transients, if the changes in energy levels in the system are large enough, the system becomes unstable. Stability can be maintained however if sufficient damping is provided by manipulating the power flows between inputs and outputs. Over recent years, the trend has been towards transmitting more power at higher voltages across longer transmission lines. For economic reasons, large generating units are commonly located at some distance form the load centres. Power is transmitted via high voltage transmission lines causing adverse effects on the stability of the system. Another factor affecting stability is the low inertia of new generating units. During system faults these tend to accelerate faster than the older and smaller units, and are therefore less stable.

It has long been recognized that feedback control systems which are traditionally made up on an automatic voltage regulator (AVR) and a speed governor improve both the steady-state and transient performance of synchronous turbo-generators [59,72-75]. Early studies used one loop at a time Nyquist type methods for both analysis and design purposes. In the 1970's, power engineers began using truly multivariable design techniques such as LQG [48-51,76,77] and the Nyquist array method [71]. These are however, by no means the only approaches, optimal aiming strategies [63], bang-bang [61] and self tuning control [62] have also been used with varying degrees of success.

The aim of this chapter is to investigate the use of $H^{\infty}$-optimal control methods in improving the performance of synchronous machines. Initially this study will be concerned with the traditional single inertia model of a generator connected to an infinite busbar. The approach is to use an $H^{\infty}$ controller in addition to the AVR and governor systems which are fitted as standard equipment to most large turbogenerator sets. The controller feeds supplementary signals to both loops on the basis of a few measurements taken from the system. The objective of this strategy is to increase the synchronizing and damping torques at the zero mode frequency of this machine.

In the final stage we will consider the full mechanical dynamics of the turbine shaft. In this case the $H^{\infty}$-optimal controller is required to provide additional synchronizing and damping torques so as to suppress the shaft torsional vibrations.

## 5.2 Mathematical modeling of synchronous generators.

To obtain a complete model of a single machine infinite busbar power system, the generator, excitation system and governor are modeled separately, combining them gives an overall model of the system. To limit the complexity of the non-linear model the following simplifying assumptions are made: i) Constant values of machine reactance, time varying saturation is neglected: ie the magnetic circuits are taken to be linear. ii) The field flux harmonics are neglected: the wave is assumed to have sinusoidal distribution over the air gap. iii) The machine operating frequency $\omega_0$ equals that of the infinite bus and their respective phases match. iv) The motoring convention and per-unit system of [60] is employed. v) The transmission line capacitance is assumed to be nil, and the line is taken as forming part of a modified generator where the line and transformer reactances and impedances are lumped together with those of the machine.

*The generator model [60]*

The well known electromagnetic dynamics of a synchronous generator may be accurately modeled by the standard two axis theory of machines expressed in the rotor reference frame [60]. The synchronous machine together with the damper circuits can be represented by three electric circuits on the d-axis and two on the q-axis. Together with the second order equation of mechanical motion, the system is described by a total of seven first order non-linear differential equations.

With the above assumptions the well-known generator equations [60], expressed in terms of flux linkages and a rotor reference frame, can be summarized by the following matrix equation

$$\left[\omega_0\dot{\psi}\right] = -\left[\ \mathrm{R}\ \right]\left[\ \mathrm{B}\ \right]\left[\omega_0\psi\right] + \begin{bmatrix} \omega_0 \mathrm{V}_{bd} - \omega_0^2\psi_q + \omega_0\psi_q\dot{\theta} \\ \omega_0 \mathrm{V}_f \\ 0 \\ \omega_0 \mathrm{V}_{bq} + \omega_0^2\psi_d - \omega_0\psi_d\dot{\theta} \\ 0 \end{bmatrix} \tag{5.1}$$

where

$$[\omega_0\psi] = [\omega_0\psi_d\ ,\ \omega_0\psi_f\ ,\ \omega_0\psi_{kd}\ ,\ \omega_0\psi_q\ ,\ \omega_0\psi_{kq}]^{\mathrm{T}} \tag{5.2}$$

and

$$[\mathrm{R}] = \mathrm{diag}[\ \mathrm{R}_a+\mathrm{R}_e+\mathrm{R}_t\ ,\ \mathrm{R}_f\ ,\ \mathrm{R}_{kd}\ ,\ \mathrm{R}_a+\mathrm{R}_e+\mathrm{R}_t\ ,\ \mathrm{R}_{kq}\ ] \tag{5.3}$$

$\mathrm{R}_a$ is the generator armature resistance,

$\mathrm{R}_e$ is the transmission line resistance,

$\mathrm{R}_t$ is the transformer resistance,

$\mathrm{R}_f$ is the field resistance,

$\mathrm{R}_{kd}$ is the d-axis damper winding resistance,

$\mathrm{R}_{kq}$ is the q-axis damper winding resistance.

$$[\mathrm{B}]^{-1} = \begin{bmatrix} \mathrm{X}_{md}+\mathrm{X}_a+\mathrm{X}_e+\mathrm{X}_t & \mathrm{X}_{md} & \mathrm{X}_{md} & & \\ \mathrm{X}_{md} & \mathrm{X}_{md}+\mathrm{X}_f & \mathrm{X}_{md} & & \\ \mathrm{X}_{md} & \mathrm{X}_{md} & \mathrm{X}_{md}+\mathrm{X}_{kd} & & \\ & & & \mathrm{X}_{mq}+\mathrm{X}_a+\mathrm{X}_e+\mathrm{X}_t & \mathrm{X}_{mq} \\ & & & \mathrm{X}_{mq} & \mathrm{X}_{mq}+\mathrm{X}_{kq} \end{bmatrix} \tag{5.4}$$

where

$X_{md}$ = d-axis magnetizing reactance

$X_{mq}$ = q-axis magnetizing reactance.

$X_a$ = armature leakage reactance.

$X_f$ = field leakage reactance.

$X_e$ = line leakage reactance.

$X_t$ = transformer leakage reactance.

$X_{kd}$ = d-axis damper leakage reactance.

$X_{kq}$ = q-axis damper leakage reactance.

The electromagnetic torque is given by

$$M_e = \frac{\omega_0}{2}\left\{\psi_d\begin{bmatrix} B_{44} & B_{45}\end{bmatrix}\begin{bmatrix}\omega_0\psi_q \\ \omega_0\psi_{kq}\end{bmatrix} - \psi_d\begin{bmatrix} B_{11} & B_{12} & B_{13}\end{bmatrix}\begin{bmatrix}\omega_0\psi_d \\ \omega_0\psi_f \\ \omega_0\psi_{kd}\end{bmatrix}\right\} \tag{5.5}$$

where $B_{ij}$ are the components of [B]. Assuming the shaft linking the turbine and generator is rigid, the equation of motion for the rotor is:

$$J\ddot{\theta} + D\dot{\theta} + M_t = M_e. \tag{5.6}$$

Where J is the inertia of the rotor, D is the damping arising from the bearings and windage, $\theta$ is the rotor angle relative to the infinite bus, $M_t$ the turbine shaft torque, termed the prime mover torque.

Combining (5.2) and (5.6) yields a seven state non-linear model for the generator.

*The transmission line model*

Here an uncompensated line is assumed so that the transmission line and generator transformer may be represented by a series resistance and inductance in each of the stator phases; figure 5.1. The phase voltages are given by :

$$V_t = V_b - V_r - V_x \tag{5.7}$$

The maximum terminal voltage $V_{mt}$ is givet by $V_{mt}=(V_{td}^2+V_{tq}^2)^{1/2}$ where $V_{td}$ and $V_{tq}$ are related to $V_{bd}$ and $V_{bq}$ via the the voltage drop in the transmission circuit. The d and q axis components of the terminal voltage are obtained by using Park's transformation [60] resulting in:

$$V_{td} = V_{bd} - (R_t + R_e)i_d - \frac{(X_e + X_t)}{\omega_0}\dot{i}_d + \frac{\omega}{\omega_0}(X_e + X_t)i_q \tag{5.8}$$

and

$$V_{tq} = V_{bq} - (R_t + R_e)i_q + \frac{(X_e + X_t)}{\omega_0}\dot{i}_q - \frac{\omega}{\omega_0}(X_e + X_t)i_d \tag{5.9}$$

where $\omega=\omega_0-\dot{\theta}$, $(R_e+R_t)$ and $(X_e+X_t)$ are the total resistance and reactance of the transformer and transmissions line between the generator and the system busbar.

*Automatic voltage regulator/ exciter system [64]*

The AVR (automatic voltage regulator) loop controls the magnitude of the terminal voltage $V_t$. However $V_t$ is an alternating voltage which has to be rectified and smoothed to give $V_{trms}$. Any difference between $V_{trms}$ and the reference input $V_{ref}$ serves as an input to the exciter after amplification. The exciter in turn delivers a voltage $V_f$ to the generator field winding. The AVR/excitation system is represented as the cascade of two first order systems figure 5.1. It is assumed that positive as well as negative field forcing is allowed, but negative field current has been banned. The saturation levels of the exciter voltage $V_e$ and field voltage $V_f$ are chosen to be $\pm 3$ time the rated load value. The following equations describe the system:

$$\dot{V}_e = (V_{ref} + U_1 - V_{trms})G_a/T_a - V_e/T_a \tag{5.10a}$$

and

$$\dot{V}_f = (V_e G_e - V_f)/T_e \tag{5.10b}$$

where $U_1$ is the supplementary signal to the AVR system from the "piggy back" $H^\infty$ controller. This model has been identified as type AC4 in the IEEE models for excitation systems [64].

*Governor/turbine system [48-50]*

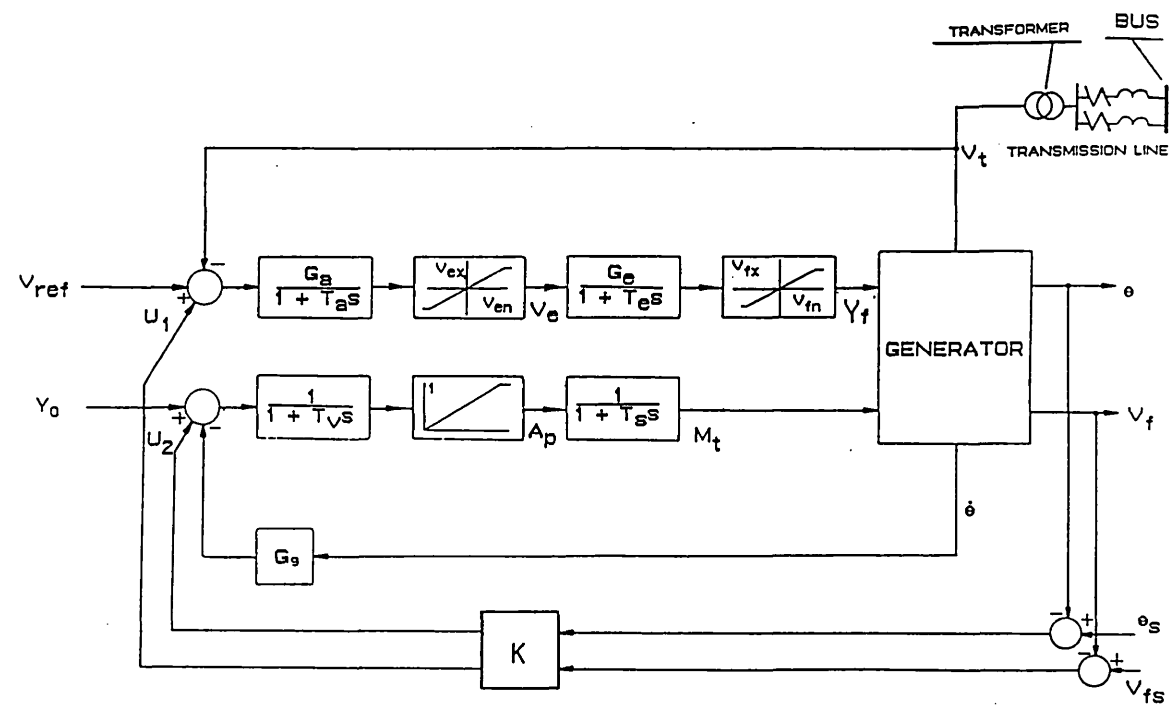
The speed governor regulates the power output and frequency of the generator, figure 5.1. The output speed $\dot{\theta}$ is continuously measured and compared with a reference input $Y_0$. The error activates a servo-valve which gauges the amount of steam entering the turbine. The energy contained in the high pressure steam is converted to angular momentum of the shaft: hence by regulating the aperture of the servo-valve one can achieve a steady state operating frequency. A simplified governor/turbine [48-50] with each stage given by first order equations are is given by the following:

$$\dot{A}_p = (-A_p - G_g\dot{\theta} + Y_0\,U_2)/T_v \tag{5.11a}$$

and

$$\dot{M}_t = (A_p - M_t)/T_s \tag{5.11b}$$

where $U_2$ is the supplementary control signal and $A_p$ is the valve position with a constraint $0 \leq A_p \leq 1$. Although the control loops described here are very simple, several researchers [48-50] have




SINGLE MACHINE INFINITE BUS SYSTEM

figure 5.1

shown them to be entirely adequate for the purposes of simulation and control system design.

Combining (5.1), (5.6), (5.10) and (5.11) leads to a non-linear equation of the form

$$\dot{x} = Ax + Bu + N(x) \tag{5.12}$$

where the state vector is

$$(\theta \,,\, \dot{\theta} \,,\, \omega_0\psi_d \,,\, \omega_0\psi_f \,,\, \omega_0\psi_{kd} \,,\, \omega_0\psi_q \,,\, \omega_0\psi_{kq} \,,\, V_e \,,\, V_f \,,\, A_P \,,\, M_t)^T \tag{5.13}$$

details of matrices A, B, and N(x) are in appendix A of [48]. The non-linear model is used to predict transient performance whereas the linearized approximation forms a basis for $H^\infty$ control system design. Expanding (5.12) in a Taylor series about an operating point leads to a linear approximation if the second and higher order terms are neglected.

$$\Delta\dot{x} = A_l\Delta x + B_l\Delta u \tag{5.14}$$

where $\Delta x$ is the deviation of the state variables from the steady state as $\Delta u$ changes.

*Shaft dynamics:*

Since two shaft failures at Mohave in 1970 and 1971 [79], the importance of the mechanical dynamics are well recognized by engineers concerned with subsynchonous resonance. Ignoring the spring like characteristics of the shaft during the design gives rise to unstable oscillations in closed loop control. This type of instability was first encountered while dealing with the problem of inter-area oscillations; power system stabilizers were subsequently introduced to deal with the problem, however, these had the effect of exciting the torsional vibrations in the generator shaft.

A real turbo-generator set consists of a high pressure, intermediate pressure, and three low pressure turbines as well as the generator and exciter. These stages are coupled together through shafts of finite stiffness which behave like springs when the system is disturbed. The shaft can be described by a set of second order differential equations

$$J\ddot{\theta} + D\dot{\theta} + K\theta + T = 0 \tag{5.15}$$

where

$$K = \begin{bmatrix} K_{12} & -K_{12} & & \\ -K_{12} & K_{12}+K_{23} & -K_{23} & \\ & & & \\ & & & K_{67} \end{bmatrix}$$

$J=\mathrm{diag}[J_1, J_2, ..., J_7]$; $D=\mathrm{diag}[D_1, D_2, ..., D_7]$; $T=[T_{m1}, T_{m2}, ..., T_{m5}, -T_e, 0]$ and $\theta=[\theta_1, \theta_2, ..., \theta_7]$ in which $J_i$ is the $i$th inertia, $D_i$ is the $i$th damping coefficient, $K_{ij}$ is the shaft stiffness between the $i$th and $j$th inertia, T is the the forcing torque and $\theta$ is the angular position. A complete description of the shaft used in this chapter is given in [48,49].

## 5.3 Problem formulation.

The objectives of the $H^\infty$ turbo-generator control system are to improve the performance and robustness characteristics with respect to existing LQG regulators [48-51]. This regulator has to enhance the stability of a single machine infinite bus system following a symmetrical three phase short circuit by:

a) Increasing the synchronizing torque in order to decrease thefirst load angle excursion.

b) Improve the damping torque in order to reduce subsequent oscillations.

c) Maintain a rapid terminal voltage recovery profile.

The problem is formulated so that the objectives are encompassed in a single performance index:

a) Robustness to additive perturbations around the plant: $K(I+GK)^{-1}$

b) Plant output disturbance rejection (sensitivity): $(I+GK)^{-1}$

$$\inf_{K(s)\in S} \left\| \begin{matrix} W_1K(I+GK)^{-1} \\ W_2(I+GK)^{-1} \end{matrix} \right\|_\infty \tag{5.16}$$

in which $W_1(s)$ and $W_2(s)$ are frequency dependent weighting functions which are used to shape the robustness and sensitivity functions so as to meet the design specifications.

*Robustness to additive perturbations:*

The reasons for building a robustness criterion into our performance index are numerous. Until now [48-51] the optimal controllers for single machine infinite bus systems have been based on well known LQG theory which does not account for the adverse effects of parameter variations, uncertainties and non-linearities which arise for the following reasons:

i) Linearization of the generator differential equations is carried out about a rated operating point. The system must be robust to variations in operating point caused by fluctuations in power demand.

ii) Modeling errors are incurred when optimal Hankel norm approximation is performed on both models.

iii) For large disturbances, the non-linearities in the control loops saturate and may lead to instability if not accounted for.

iv) For the fault duration the system model changes since the transmission line is short circuited.

v) The $H^\infty$ controller is discretized and the closed loop is simulated as though the controller were implemented on a microprocessor.

These uncertainties are usually associated with high frequency component errors which is well suited to an additive robustness performance index, we are therefore required to design a compensator which stabilizes the nominal plant as well as a class of perturbed plants $\left\{G(s)+\Delta(s),\ ||\Delta(s)||_{\infty}<\epsilon\right\}$ where $\epsilon$ is the as yet unspecified level of uncertainty. The term $K(I+GK)^{-1}(s)$ is used to improve the system robustness to additive perturbations [53] while reducing the amplitude of the control signals being applied to the turbo-generator inputs.

*Plant output disturbance rejection: Sensitivity:*

As the name suggests, the purpose of having a sensitivity measure in the design performance is used to suppress any large (but bounded) disturbances on the plant output or to obtain a good tracking performance of command inputs. To achieve this the transfer function between the reference and the measurement signals should be made as small as possible under the constraint of internal stability. The sensitivity term $(I+GK)^{-1}(s)$ is used to reduce the low frequency load angle deviations and speed up the terminal voltage recovery after a system disturbance.

## 5.4 The two block design:

The object of this design is to minimize (5.16) over the class of all stabilizing controllers. It is not difficult to show that (5.16) can be cast as a generalized regulator problem as shown in figure 3.2, we seek to minimize the transfer function between "v" and "z" subject to internal stability:

$$\min_{K(s)\,\in\mathbf{S}} ||F_l(P,K)||_{\infty}\leq\gamma$$

where $\gamma$ is the target norm, $\mathbf{S}$ is the class of all stabilizing controllers and

$$P(s)=\begin{bmatrix} 0 & W_1 \\ W_2 & W_2G \\ -I & -G \end{bmatrix}(s) \tag{5.17}$$

We now turn our attention to the central problem in all $H^{\infty}$ control system design which is weight selection. One advantage of working with the index in (5.16) is that, the robustness and performance requirements are specified in different parts of the frequency range. The key feature of this design methodology is that the largest frequency response singular value of the transfer function between "v" and "z" is frequency independent (i.e flat) and furthermore, it is as small as possible. This means that frequency responses of the robustness and sensitivity functions will be in inverse proportion to the magnitude of the weights if they are chosen to have disjoint frequency ranges.

Most of the uncertainty associated with this problem has a high frequency error component.

Designing for robustness requires that $K(I+GK)^{-1}(s)$ be made as small as possible (in the $\mathcal{L}\infty$ norm sens) over high frequencies where the uncertainty is chronic. To meet this objective $W_1(s)$ is chosen as a diagonal high pass filter where the larger gain over high frequencies places emphasis on minimizing the robustness function in this part of the spectrum.

$$W_1(s)=\mathrm{diag}\left\{ 2.2\,\frac{(s+5)^2}{(s+30)^2}\,;\,8.1\,\frac{(s+7)^2}{(s+40)^2} \right\}$$

with a relative penalty ratio of 36 and 32.7 on each loop. The robustness weight for the excitation loop has a narrow bandwidth since the AVR and exciter have fast dynamics and are more likely to saturate. The slower dynamics of the governor loop allows for a wider bandwidth in its associated weight.

To limit the controller bandwidth, the plant is post-multiplied by a strictly proper filter $F(s)$; $\widetilde{G}(s)=G(s)F(s)$. Once the design is completed on $\widetilde{G}(s)$, the desired compensator is recovered by simply scaling; $K(s)=F(s)\widetilde{K}(s)$. $F(s)$ is chosen to have a bandwidth approximately twenty times larger than that of the plant, so as not to interfere with the design requirements.

$$F(s)=\mathrm{diag}\left\{ \frac{1}{s/200+1}\,;\,\frac{1}{s/200+1} \right\}$$

The presence of $F(s)$ in the design dose not affect the subsequent sensitivity function:

$$\begin{aligned} S_{ens}(s) &= (I+\widetilde{G}\widetilde{K})^{-1}(s) \\ &= (I+GK)^{-1}(s) \end{aligned}$$

The robustness function on the other hand is forced to become strictly proper so as to limit the controller bandwidth and hence the amplitude of the control signals being applied to the turbogenerator inputs.

$$\begin{aligned} & R_{ob}(s) = \widetilde{K}(I+\widetilde{G}\widetilde{K})^{-1}(s) \\ \Rightarrow\quad & FR_{ob}(s) = K(I+GK)^{-1}(s) \end{aligned}$$

Alternatively one can absorb $F^{-1}(s)$ into $W_1(s)$ to give a non-proper weighting function, the effect of which is to weight the robustness function increasingly with frequency.

The turbogenerator model considered here has a bandwidth of 10.2 rad/s and a resonance peak at 9.4 rad/s, the latter is also the frequency of oscillation of the uncontrolled load angle (see figure 5.2a). To ensure adequate sensitivity over low frequencies, the $W_2(s)$ weighting function is chosen to be a strictly proper low pass filter. $W_2(\infty)=0$ is used since there are no high frequency sensitivity requirements, this ensures that there is no conflict between the cost functions over high frequencies. The plant output disturbance has a fundamental frequency component at 9.4 rad/s, to desensitize the system at this frequency, $W_2(s)$ is designed to have a resonance peak at this same frequency. The low bandwidth of the steam valve makes it impractical to widen the active frequency range of $W_2(s)$.

$$W_2(s)=\left\{1.9\ \frac{1990}{s^2+2.3s+93.65}\ ;\ 8.1\ \frac{1990}{s^2+2.3s+93.65}\right\}$$

(which has infinite penalty ratio since it is strictly proper). The two loops are weighted with the same filter to emphasize the control of the rotor angle through the governor by providing adequate synchronizing torque and via the AVR to give the necessary damping.

The weighting functions $W_1(s)$ and $W_2(s)$ together with plant model G(s) are substituted into (5.17) to yield a 16-state P(s). Algorithm 1 described in section 3.5 is used to calculate K(s) which is subsequently model reduced to remove any non-minimal modes. The final weight selection described above was arrived at after numerous designs, all the design calculations were done in the state space using computer software based on the equations in chapter 3 [14,34].

Figure 5.3a displays $\bar{\sigma}\left\{K(I+GK)^{-1}(s)\right\}$ which shows that the robustness has been increased over frequencies beyond the system bandwidth. The largest singular value response of the sensitivity function: $\bar{\sigma}\left\{(I+GK)^{-1}(s)\right\}$ is plotted in figure 5.3b, the information that this graph conveys, shows that sensitivity is quite poor even over low frequencies.

The $11^{th}$ order non-linear system described in section 5.2 is simulated and subjected to a symmetrical three phase short circuit of 100ms at the high voltage terminals of the generator transformer. The transmission line impedance after the clearance of the fault is assumed to remain unchanged. The load angle to the infinite bus-bar and field voltage are assumed measurable and are fed into the controller. The steady state conditions of the system corresponding to an operating point of P=0.8pu, Q=0.0pu and $V_t$=1.025pu represent the pre-fault conditions in all the simulations.

Figure 5.2a shows the response with and without the controller following the fault. The damping is quite substantial with the machine back to steady state in under 1 second after the fault recovery. The terminal voltage recovery does not differ much from that of the uncontrolled machine. However, there is no significant reduction in the first rotor swing which is reflected in the rather inactive behavior of the valve.

The AVR supplementary signal orients the field voltage and the exciter action towards good damping of the rotor angle and the terminal voltage. The exciter loop makes no contribution in the reduction of the first rotor excurtion which is due to its sharp drop of field voltage from its positive ceiling value to the negative one causing the terminal voltage not to increase sharply after the fault recovery. As a result there is less transmission power to the infinite bus and an increased acceleration of the machine.

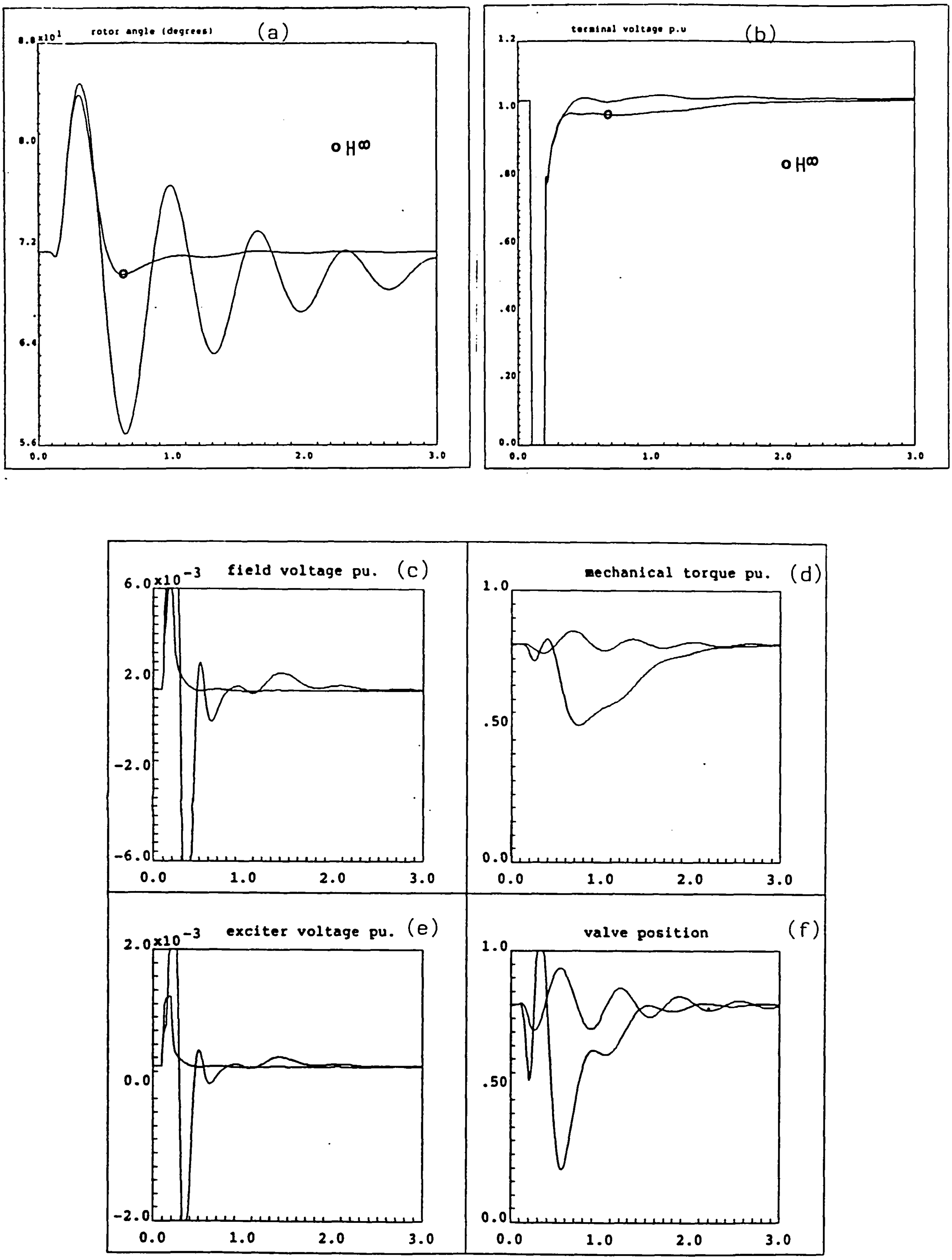

rotor angle (degrees)
(a)
o H∞
terminal voltage p.u
(b)
o H∞
field voltage pu.
(c)
6.0x10^-3
2.0
-2.0
-6.0
mechanical torque pu.
(d)
1.0
.50
0.0
exciter voltage pu.
(e)
2.0x10^-3
0.0
-2.0
valve position
(f)
1.0
.50
0.0
0.0
1.0
2.0
3.0

figure 5.2

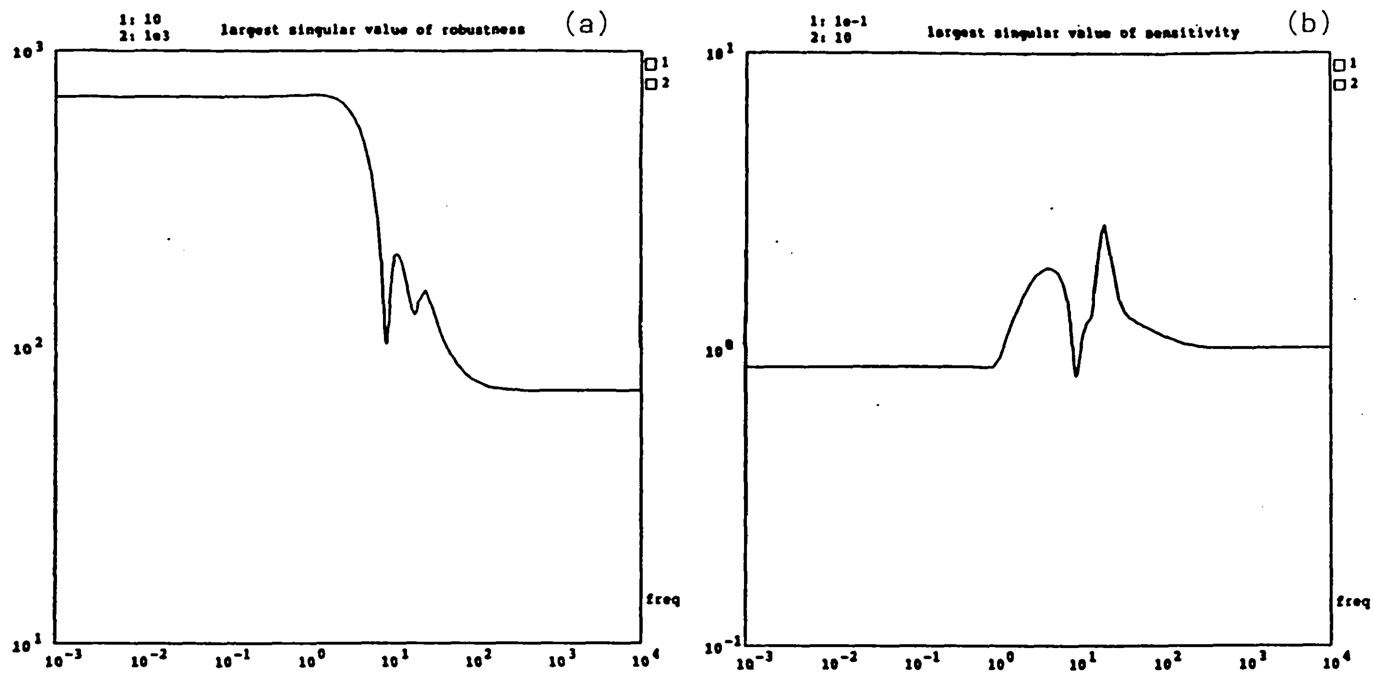



figure 5.3

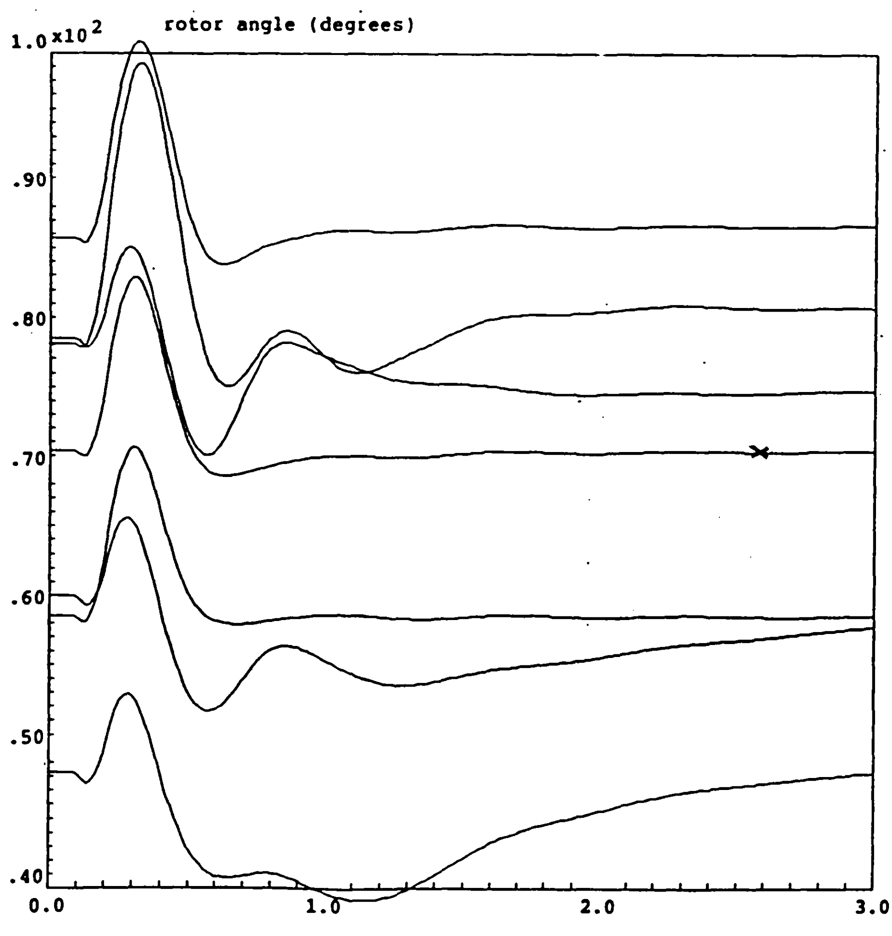



figure 5.4

Figure 5.4 shows the rotor angle responses to 100ms faults while the generator runs at different operating points. The system is well behaved for all of these changes in operation conditions, however, it is known that better robust performance can be achieved in particular with an LQG regulator [48,49].

5.5 The four block design:

In this section we will repeat the previous design, however with a new performance index:

$$\inf_{K(s)\in \mathbf{S}} \left\| \begin{bmatrix} W_1K(I+GK)^{-1}W_2 & W_1K(I+GK)^{-1}W_4 \\ W_3(I+GK)^{-1}W_2 & W_3(I+GK)^{-1}W_4 \end{bmatrix}(s) \right\|_\infty \tag{5.18}$$

where the weighting functions $W_1(s)$, $W_2(s)$, $W_3(s)$ and $W_4(s)$ are to be chosen. The weight selection in (5.16) was constrained on one side of the performance index with the shared weighting function forced to be the identity so as to avoid having conflicting requirements on the same index. Equation (5.18) lifts this constraint by allowing both side to have weights, however, there are cross terms which must be taken into consideration. (5.18) is equivalent to finding $K(s)\in\mathbf{S}$ which minimizes (under the constraints of internal stability) the the transfer function between "v" and "z" in the generalized regulator configuration:

$$P(s) = \begin{bmatrix} 0 & 0 & W_1 \\ W_3W_2 & W_3W_4 & W_3G \\ -W_2 & -W_4 & -G \end{bmatrix}(s) \tag{5.19}$$

To ensure adequate sensitivity over low frequencies the weighting functions $W_4(s)$ and $W_3(s)$ are both chosen as strictly proper low pass filters. The input to the sensitivity function is weighted by $W_4(s)$ which is :

$$W_4(s)=\text{diag}\left\{\frac{96.8}{s^2(1.07\cdot10^{-2})+s(4.63\cdot10^{-2})+1}\,;\,\frac{173.8}{(s/27+1)(s^2(1.11\cdot10^{-2})+s(2.22\cdot10^{-2})+1)}\right\}$$

where each entry of $W_4(s)$ has a resonance at 9.4 rad/sec because the plant output disturbance has a fundamental frequency component equal to the generator's natural frequency. The (2,2) entry of $W_4(s)$ has additional pole at -27, its effect on the rotor angle performance is to cancel low amplitude ripples and to improve the decay rate. Intuitively this can be considered as concentrating finite resources over a finite frequency range. The output of the sensitivity function has weights placing emphasis on the attenuation of low frequency signals and reduction of steady state error :

$$W_3(s) = \left\{\frac{59}{5s+1}\,;\,\frac{4.8}{s/17+1}\right\}.$$

The output of the robustness function is weighted by $W_1(s)$ which is a proper high pass filter, placing importance on the robustness to high frequency perturbations.

$$W_1(s) = \text{diag}\left\{\frac{(s/6+1)^2}{(s/45+1)^2}\, 6.66{*}10^{-3}\ ;\ \frac{(s/7+1)^2}{(s/33+1)^2}\, 6.3{*}10^{-3}\right\}$$

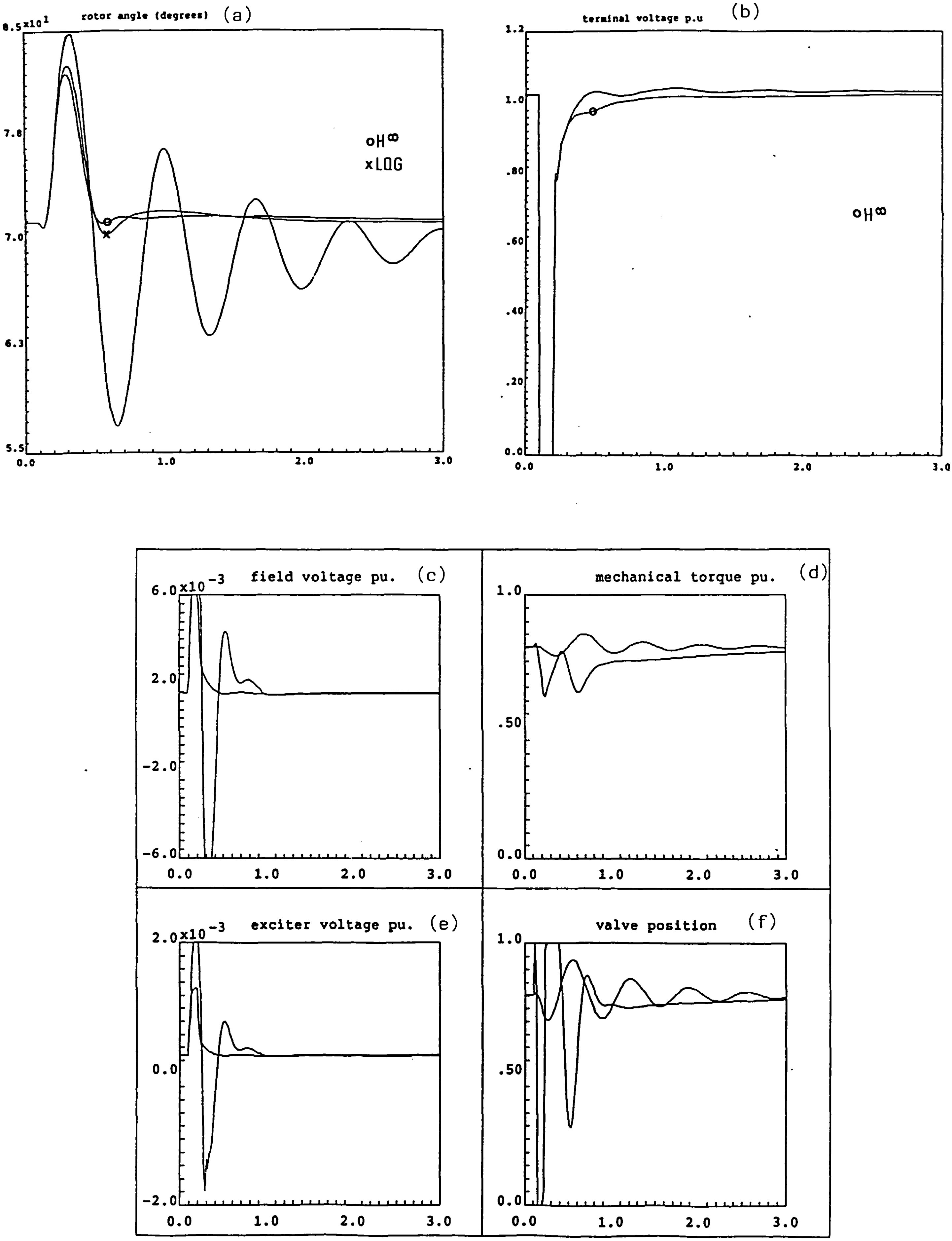
with a relative penalty ratio of 56.2 and 22.2 on each of the loops. The input to $K(I + GK)^{-1}$ is weighted by a constant scaling matrix $W_2 = \text{diag}\{0.7\ ,\ 1.0\}$. In addition to robustness, as explained in the previous section, the controller bandwidth is limited by introducing the filter F(s) between the plant and controller.

The weighting functions $W_1(s)$, $W_2(s)$, $W_3(s)$ and $W_4(s)$, together with the modified plant $\widetilde{G}(s)$ are substituted into (5.19) to yield P(s). $\widetilde{K}(s)$ is computed using Algorithm 1 and the desired controller is recovered by scaling: $K(s)=F(s)\widetilde{K}(s)$. Figure 5.6a is a plot of $\bar{\sigma}\left\{K(I+GK)^{-1}(s)\right\}$, the closed loop display an increased robustness around the machine's natural frequency, further, the presence of F(s) in the design forces $\bar{\sigma}\left\{K(I+GK)^{-1}(s)\right\}$ to go to zero over high frequencies (a desirable characteristic). $\bar{\sigma}\left\{(I+GK)^{-1}(s)\right\}$ is plotted in figure 5.6b showing that the weighting functions $W_4(s)$ and $W_3(s)$ have been effective in increasing the closed loop sensitivity up to 3rad/sec.

Figure 5.5a and figure 5.5b show the time responses of the generator load angle and terminal voltage recovery profile following a 100ms 3-phase symmetrical short circuit. There is a considerable reduction in the first and second rotor excursions. The excitation loop behaves much as it did in the first design, however, the valve activity has noticeably increased. The almost instant closing of the valve accounts for the reduction in the first swing.

Compared with the LQG regulator of [48,49], one sees that the $H^\infty$ controlled system is able to produce more synchronizing and damping torques, this is reflected in it's the nominally better performance. The terminal voltage recovery profile is very rapid with no ripples as in the case of the uncontrolled system.

In practice the generator steady state conditions drift over the safe operating region of the P-Q plane depending on the load fluctuations. Having designed the controller at a fixed operating point it is important to investigate the systems transient behavior for a variety of other points on the operating chart. Figure 5.7 shows the rotor angle responses to 100ms faults while the generator runs under different operating conditions. In general the closed loop system is able to maintain a good performance with higher synchronizing and damping torques than the LQG regulator [48,49] (not shown here). Trace 'c' is slightly different from the others, here the operating point is (1.0,0.0) where the generator is running at maximum load, the steam valve is completely open making it impossible to increase the power after the first swing. Nevertheless, under this unfair operating condition the $H^\infty$ controller returns the system to its pre-fault steady-state condition in under 3 seconds.



rotor angle (degrees)
(a)
8.5x10^1
7.8
7.0
6.3
5.5
0.0
1.0
2.0
3.0
oH∞
xLQG
terminal voltage p.u
(b)
1.2
1.0
.80
.60
.40
.20
0.0
oH∞
field voltage pu.
(c)
6.0x10^-3
2.0
-2.0
-6.0
mechanical torque pu.
(d)
1.0
.50
0.0
exciter voltage pu.
(e)
2.0x10^-3
0.0
-2.0
valve position
(f)


figure 5.5

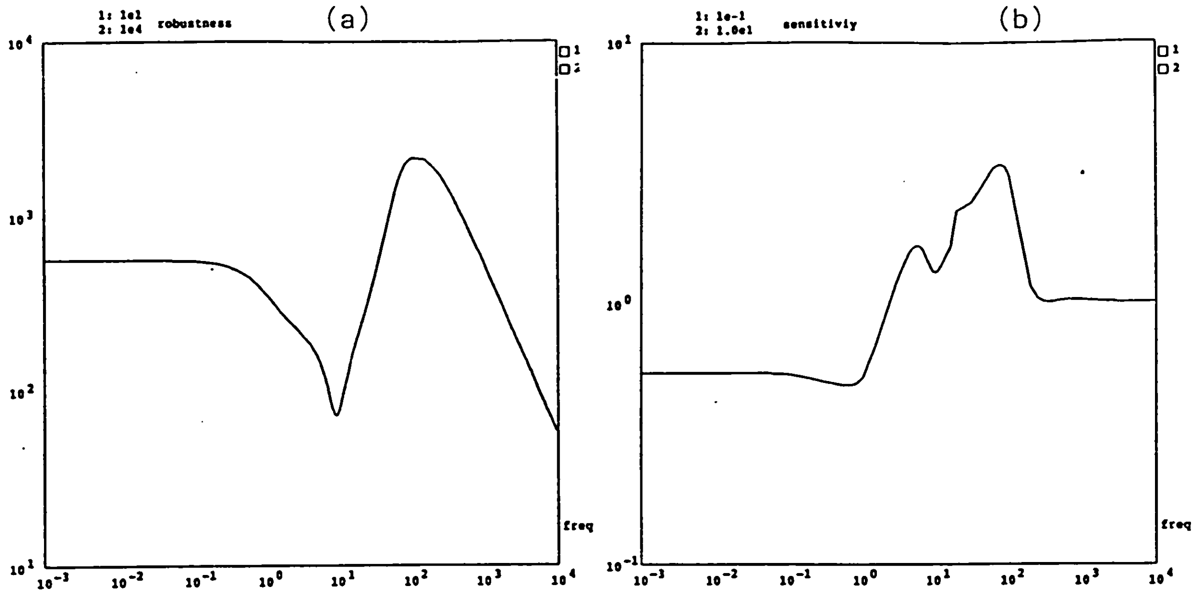



figure 5.6

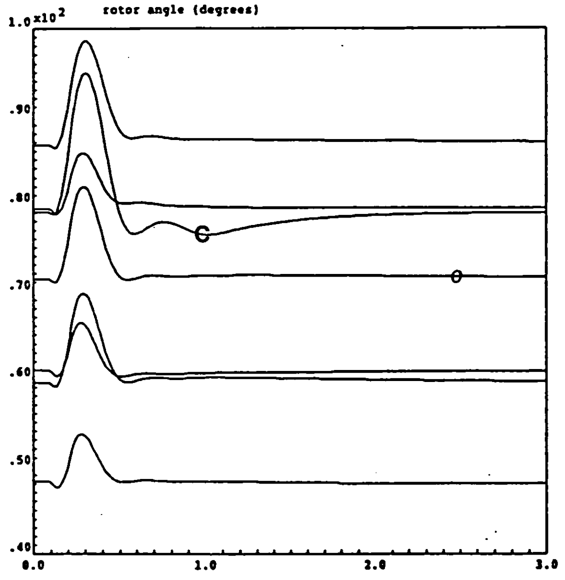



figure 5.7

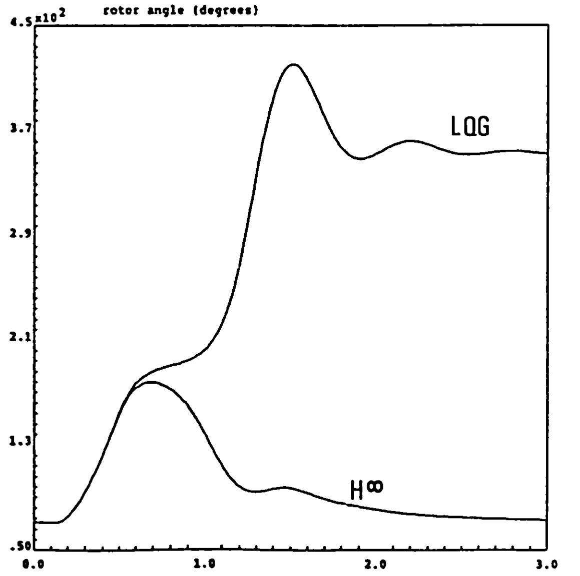



figure 5.8

By gradually increasing the fault clearing time until the controlled system loses synchronism, it is possible to test the robustness of both LQG and $H^\infty$ regulators to parameter variations which take place during the short circuit. Figure 5.8 shows that by providing more synchronizing torque and reducing the first swing the $H^\infty$ regulator is more robust than its LQG counter part. The fault was applied in each case for a period of 410ms to the high voltage side of the stepup transformer. While this fault duration period is much longer than typical fault clearing times, it was nevertheless selected as it constitutes a more severe test of the robustness of the $H^\infty$ controller. Further increases in the fault clearance time results in the $H^\infty$ controlled machine eventually losing synchronism for a fault duration of 480ms. Even though the LQG quadratic performance index overlooks the robustness issue it does have some inherent robustness characteristics namely infinite gain margin and up to $\pm$ $60^0$ phase margin which explains the good properties of this benchmark regulator. [48,49]

## 5.6 An $H^\infty$ multi mode controller.

A real turbine set has several inertia, 4 or 5 turbine stages, the generator and the exciter which are connected together though shafts of finite stiffness. This combination of inertia and springs introduces important dynamics into the behavior of the machine, if ignored these might have adverse effects on the generator stability [48,49].

In this section the full mechanical model of the generator shaft is added to the model used in sections 5.4 and 5.5 yielding a $23^{rd}$ order non-linear model. The design objectives are robust stabilization and disturbance rejection to improve the synchronizing and damping torques as well as maintain a rapid terminal voltage recovery. We will use the four block performance index employed in section 5.5.

To ensure adequate sensitivity $W_4(s)$ and $W_3(s)$ are both chosen as strictly proper low pass filters.

$$W_4(s)=\text{diag}\left\{ \frac{35.2}{(s/17+1)^2} ; \frac{154}{s^2/96.5+s/51.3+1} \right\}$$

in which the (2,2) entry has a resonance peak at the machine's natural frequency of 9.7rad/sec. The (1,1) entry has a wider bandwidth to enable the excitation loop to dampen out the first mode of vibration. The output of the sensitivity function has weights placing emphasis on the attenuation of low frequency disturbances and reduction of the steady state error:

$$W_3(s)=\text{diag}\left\{ \frac{200}{10s+1} ; \frac{2.18}{s/11+1} \right\}$$

The output of the robustness function is weighted with $W_1(s)$, placing importance on robustness to high frequency perturbations:

$$W_1(s)=\mathrm{diag}\left\{\ 5.6*10^{-3}\frac{(s/6+1)^2}{(s/45+1)^2}\ ;\ 1.9*10^{-2}\frac{(s/7+1)^2}{(s/33+1)^2}\ \right\}$$

The input to $K(I+GK)^{-1}(s)$ is weighted by a constant scaling matrix $W_2(s)=\mathrm{diag}\{0.25;0.38\}$.

Figure 5.9a is the plot of $\bar{\sigma}\left\{(I+GK)^{-1}(s)\right\}$, the closed loop displays a significant sensitivity to frequencies around the machine's natural frequency. Figures 5.10a and 10b show the time responses of the generator load angle and the terminal voltage, the $H^{\infty}$ controller produces a significant improvement in the load angle transient behavior, the good sensitivity to the machines natural frequency is reflected in the cancellation in the zero mode of oscillation. The subsequent ripples in the load angle are a result of the inadequate sensitivity over low frequencies. The wide bandwidth of the exciter weighting function enables the damping of the first mode of vibration, interaction with higher order modes is unlikely since the shaft is quite stiff and the energy required to produce appreciable oscillations is less likely to be available. The control loop responses are shown in figure 5.11, an initial period of 0.5 second is spent on the control of the zero mode, after this is achieved the controller concentrates on the cancellation of the first mode which reflected in the high frequency oscillations experienced in both exciter and governor loops. Figure 5.9b is a plot of $\bar{\sigma}\left\{K(I+GK)^{-1}(s)\right\}$, the closed loop shows a relative increase in robustness around the machines natural frequency, further the presence of F(s) in the design limits the bandwidth of the controller making it implementable. Figure 5.11 shows the effects of operating the generator at various points on the P-Q plane other than the nominal conditions. In each case the controller is able to cancel the zero mode effectively and reduce the first mode of torsional vibration in the shaft.

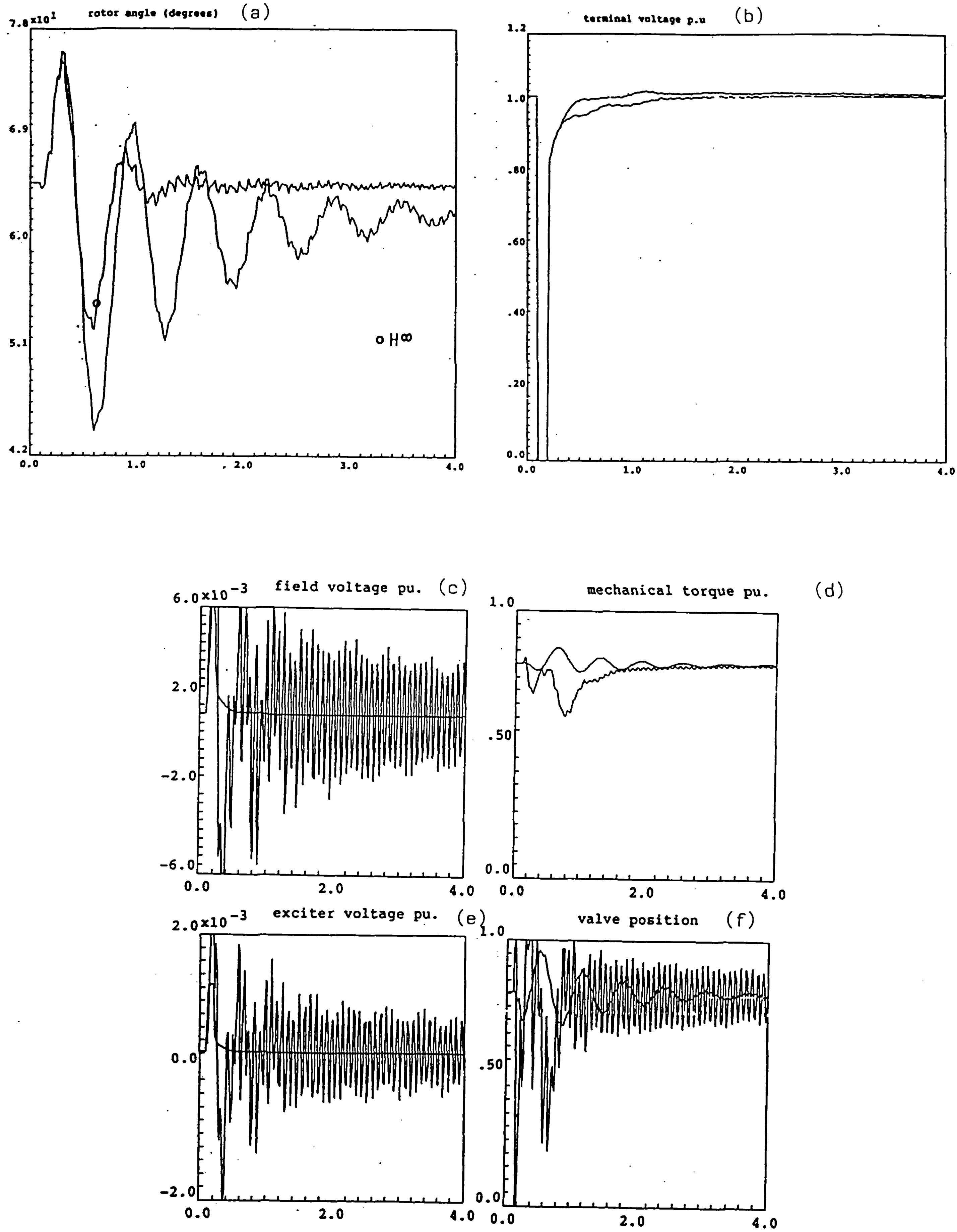

rotor angle (degrees) (a)
7.8x10^1
6.9
6.0
5.1
4.2
0.0 1.0 2.0 3.0 4.0
o H∞
terminal voltage p.u (b)
1.2
1.0
.80
.60
.40
.20
0.0
0.0 1.0 2.0 3.0 4.0
field voltage pu. (c)
6.0x10^-3
2.0
-2.0
-6.0
0.0 2.0 4.0
mechanical torque pu. (d)
1.0
.50
0.0
0.0 2.0 4.0
exciter voltage pu. (e)
2.0x10^-3
0.0
-2.0
0.0 2.0 4.0
valve position (f)
1.0
.50
0.0
0.0 2.0 4.0

figure 5.9

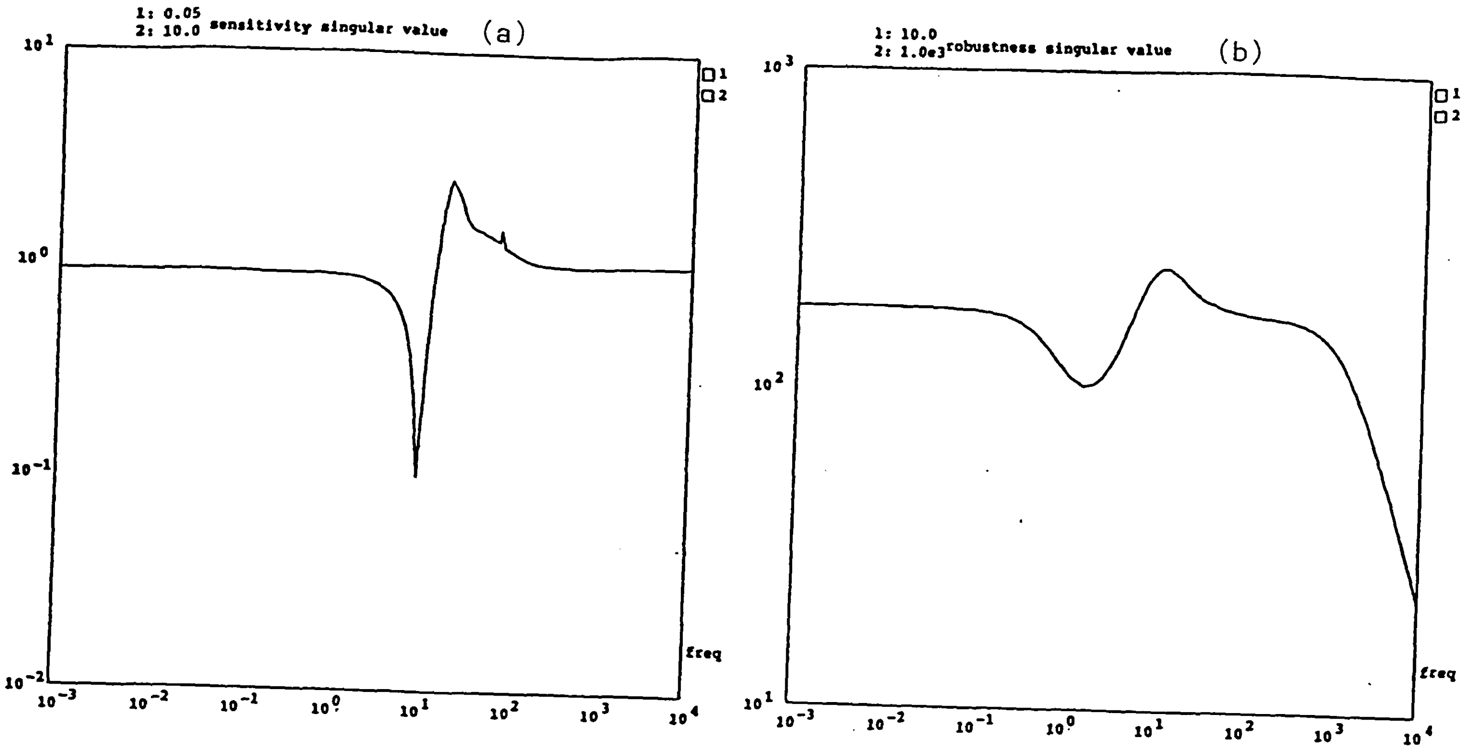



figure 5.10

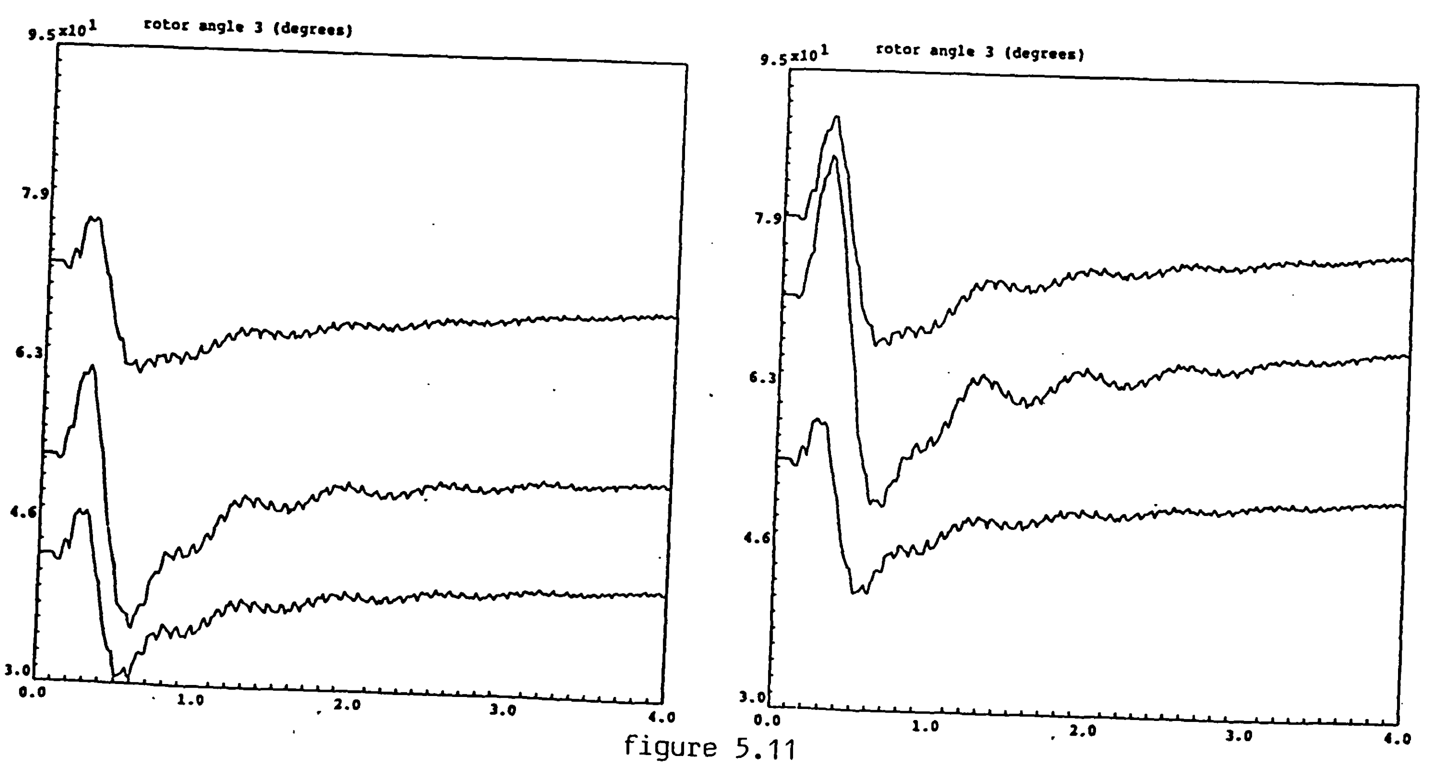



figure 5.11

## 5.7 Conclusions.

The aim of this chapter is to validate recently developed $H^\infty$ optimal control techniques [14,34] on a turbogenerator system. We have shown that with field voltage and rotor angle measurements it is possible to design an $H^\infty$ optimal controller giving robust performance. This problem has been particularly challenging in that there many sources of uncertainty which have to be taken into account at the design stage. Our experience is that $H^\infty$ optimal control techniques are still hard to use, having said that it is possible to arrive at a satisfactory design after several design iterations. With hindsight our selection of weighting functions is quite reasonable, the link between them and the achievable closed loop performance is clear. It is often the case that theoretical developments precede non-trivial applications "know how", $H^\infty$ optimal control is no exception.

## Appendix A

Using (2.74) we now verify that

$$Z(s)= F_u\left\{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}(s);\ \widetilde{U}(s)\right\}$$

where H(s) and $\widetilde{U}(s)$ are defined in (2.72) and (2.75) respectively. (dropping the s-deppendence we have)

$$\begin{aligned}
& \widetilde{U} = \gamma^{-2}H_{11}^{*}+\gamma^{-4}H_{21}^{*}Z(I-\gamma^{-2}H_{22}^{*}Z)^{-1}H_{12}^{*} \\
(2.71)\Rightarrow\quad & = \gamma^{-2}H_{11}^{*}+\gamma^{-4}(H_{21}^{*}H_{22}+H_{21}^{*}\Delta\bar{Q})(I-\gamma^{-2}H_{22}^{*}H_{22}-\gamma^{-2}H_{22}^{*}\Delta\bar{Q})^{-1}H_{12}^{*} \\
(2.70)\Rightarrow\quad & = \gamma^{-2}H_{11}^{*}+\gamma^{-4}(H_{21}^{*}H_{22}+H_{21}^{*}\Delta\bar{Q})(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})^{-1} \\
& = \gamma^{-2}\left\{H_{11}^{*}H_{12}+H_{11}^{*}H_{11}H_{21}^{-1}\Delta\bar{Q}+H_{21}^{*}H_{22}+H_{21}^{*}\Delta\bar{Q}\right\}(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})^{-1} \\
(2.70)\Rightarrow\quad & = H_{21}^{-1}\Delta\bar{Q}(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})^{-1}
\end{aligned} \tag{A1}$$

$$\begin{aligned}
(A1)\Rightarrow\qquad I-H_{11}\widetilde{U} &= I-H_{11}H_{21}^{-1}\Delta\bar{Q}(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})^{-1} \\
H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q} &= (I-H_{11}\widetilde{U})^{-1}(H_{12}+H_{11}H_{21}^{-1}\Delta\bar{Q})-(I-H_{11}\widetilde{U})^{-1}H_{11}H_{21}^{-1}\Delta\bar{Q} \\
H_{21}^{-1}\Delta\bar{Q} &= \widetilde{U}(I-H_{11}\widetilde{U})^{-1}H_{12} \\
H_{22}+\Delta\bar{Q} &= H_{21}\widetilde{U}(I-H_{11}\widetilde{U})^{-1}H_{12}+H_{22} \\
&= F_u\left\{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix};\ \widetilde{U}\right\}
\end{aligned}$$

verifies the claim. □

## Appendix B

A straight forward calculation will establish that

$$F_l\Big\{G_1;F_l\{G_2;U\}\Big\}(s)$$

has the following state-space realization

$$\left\{F_l\left[\begin{array}{cc|cc|cc} I & 0 & A & B_2\widetilde{C}_1 & B_1 & B_2\widetilde{D}_{12} \\ 0 & E & \widetilde{B}_1C_2 & \widetilde{A}+\widetilde{B}_1D_{22}\widetilde{C}_1 & \widetilde{B}_1 & \widetilde{B}_2+\widetilde{B}_1D_{22}\widetilde{D}_{12} \\ \hline 0 & 0 & C_1 & \widetilde{C}_1 & 0 & \widetilde{D}_{12} \\ 0 & 0 & \widetilde{D}_{21}C_2 & \widetilde{C}_2+\widetilde{D}_{21}D_{22}\widetilde{C}_1 & \widetilde{D}_{21} & \widetilde{D}_{21}D_{22}\widetilde{D}_{12}+\widetilde{D}_{22} \end{array}\right];U\right\}$$

where

$$G_1(s) \stackrel{s}{=} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & I \\ C_2 & I & D_{22} \end{bmatrix} \text{ and } G_2(s) \stackrel{s}{=} \begin{bmatrix} E & \widetilde{A} & \widetilde{B}_1 & \widetilde{B}_2 \\ 0 & \widetilde{C}_1 & 0 & \widetilde{D}_{12} \\ 0 & \widetilde{C}_2 & \widetilde{D}_{21} & \widetilde{D}_{22} \end{bmatrix}$$

□

# Appendix C

Lemma C1

Suppose $G(s) = D+C(sE-A)^{-1}B$ in which we assume that $\det(sE-A)\neq 0$. Then there exist matrices T, L and $W_o$ such that

$$AT+T^*A^*+BB^*=-LL^* \qquad (C1)$$
$$I-DD^*=W_oW_o^* \qquad (C2)$$
$$ET=T^*E^* \qquad (C3)$$
$$CT+DB^*=-W_oL^* \qquad (C4)$$

We have $||G||_\infty \leq 1$. If L and $W_o$ are both zero, then $GG^*=I$.

Proof:

The proof presented here is analogous to that presented in [2].

$$I-GG^* = I-\left\{D+C(sE-A)^{-1}B\right\}\left\{D^*+B^*(-sE^*-A^*)^{-1}C^*\right\}$$
$$=I-DD^*-DB^*(-sE^*-A^*)^{-1}C^*-C(sE-A)^{-1}BD^*-C(sE-A)^{-1}BB^*(-sE^*-A^*)^{-1}C^*$$

substituting (C4)

$$=I-DD^*+(CT+W_oL^*)(-sE^*-A^*)^{-1}C^*+C(sE-A)^{-1}(T^*C^*+LW_o^*)$$
$$-C(sE-A)^{-1}BB^*(-sE^*-A^*)^{-1}C^*$$

substituting from (C1) and (C3)

$$=W_oW_o^*+W_oL^*(-sE^*-A^*)^{-1}C^*+C(sE-A)^{-1}LW_o^*+C(sE-A)^{-1}LL^*(-sE^*-A^*)^{-1}C^*$$
$$=\left\{W_o+C(sE-A)^{-1}L\right\}\left\{W_o+C(sE-A)^{-1}L\right\}^*\geq 0$$

$\Rightarrow ||G||_\infty \leq 1$. (G(s) is a contraction)

Next we establish that although S and P may be singular, a stabilizing controller exists if $S\geq 0$, $P\geq 0$ and the spectral radius of (SP) is less than $\gamma^{-2}$; $\lambda_{max}(SP)\leq\gamma^{-2}$.

Using the constrution given in [11, appendixB], it is possible to select a basis change in the state-space of P(s) such that

$$S = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \text{and } P = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

define $Z=(A-B_2D_{12}^*C_1)$ and $W=(A-B_1D_{21}^*C_2)$

partitioning conformally with S and P

$$[B_1\ B_2] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix} \qquad (C5a)$$

and

$$\begin{bmatrix} C_1 C_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \end{bmatrix} \tag{C5b}$$

Finally partition Z into 4×4 submatrices conformally with S and P, and together with (C5), substitute into (3.254), then

(1,1) block $\Rightarrow Z_{11}^*\Sigma_1+\Sigma_1 Z_{11}-\Sigma_1(\gamma^2 B_{21}B_{21}^*-B_{11}B_{11}^*)\Sigma_1+\gamma^{-2}C_{11}^*D_\perp D_\perp^* C_{11}=0$

(3,3) block $\Rightarrow Z_{33}^*\Sigma_3+\Sigma_3 Z_{33}-\Sigma_3(\gamma^2 B_{23}B_{23}^*-B_{13}B_{13}^*)\Sigma_3+\gamma^{-2}C_{13}^*D_\perp D_\perp^* C_{13}=0$

(2,2) block $\Rightarrow C_{12}^*D_\perp=0$

(1,1) block $\Rightarrow C_{14}^*D_\perp=0$

column 2 $\Rightarrow$ $Z_{12}=0$ and $Z_{32}=0$

column 4 $\Rightarrow$ $Z_{14}=0$ and $Z_{34}=0$.

Similarly partioning W into 4×4 submatrices, and together with (C5), substitute into (3.225), then

(1,1) block $\Rightarrow$ $W_{11}\Sigma_1+\Sigma_1 W_{11}^*-\Sigma_1(\gamma^2 C_{21}^*C_{21}-C_{11}^*C_{11})\Sigma_1+\gamma^{-2}B_{11}\widetilde{D}_\perp^*\widetilde{D}_\perp B_{11}^*=0$

(2,2) block $\Rightarrow$ $W_{22}\Sigma_2+\Sigma_2 W_{22}^*-\Sigma_2(\gamma^2 C_{22}^*C_{22}-C_{12}^*C_{12})\Sigma_2+\gamma^{-2}B_{12}\widetilde{D}_\perp^*\widetilde{D}_\perp B_{12}^*=0$

(3,3) block $\Rightarrow$ $\widetilde{D}_\perp B_{13}^*=0$

(4,4) block $\Rightarrow$ $\widetilde{D}_\perp B_{14}^*=0$

column 3 $\Rightarrow$ $W_{13}=0$ and $W_{32}=0$

column 4 $\Rightarrow$ $W_{41}=0$ and $W_{42}=0$.

Using the construction in Appendix B

$$\begin{bmatrix} E_{cl} & A_{cl} & B_{cl} \\ 0 & C_{cl} & D_{cl} \end{bmatrix} = F_l\left\{P(s);\begin{bmatrix} E_k & A_k & B_k \\ 0 & -C_k & -D_k \end{bmatrix}\right\}$$

where P(s) and K(s) are given in (3.2), and, (3.164)-(3.168) and (3.170) respectively. Following the cancellation of uncontrollable and unobservable modes, one gets

$$B_{cl} = \begin{bmatrix} B_{11} & \gamma B_{21} \\ B_{13} & \gamma B_{23} \\ -B_{11}\widetilde{D}_\perp^*\widetilde{D}_\perp+\Sigma_1 C_{21}^*\gamma^2 D_{21} & \Sigma_1 C_{11}^* D_{12}\gamma \\ -B_{12}\widetilde{D}_\perp^*\widetilde{D}_\perp+\Sigma_2 C_{22}^*\gamma^2 D_{21} & \Sigma_2 C_{12}^* D_{12}\gamma \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} D_\perp D_\perp^* C_{11}-D_{12}\gamma^2 B_{21}^*\Sigma_1 & D_\perp D_\perp^* C_{13}-D_{12}\gamma^2 B_{23}^*\Sigma_3 & -D_{12}D_{12}^*C_{11}-\gamma^2 D_{12}B_{21}^*\Sigma_1 & -D_{12}D_{12}^*C_{12} \\ -\gamma D_{21}B_{11}^*\Sigma_1 & -\gamma D_{21}B_{13}^*\Sigma_3 & -\gamma(C_{21}+D_{21}B_{11}^*\Sigma_1) & \gamma C_{22} \end{bmatrix}$$

$$D_{cl} = \begin{bmatrix} 0 & \gamma D_{12} \\ \gamma D_{21} & 0 \end{bmatrix}$$

$$E_{cl}=\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -\gamma^2\Sigma_1^2 & 0 & I-\gamma^2\Sigma_1^2 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

Substituting into (C4):

$$C_{cl}T+D_{cl}B_{cl}^{*}= -\begin{bmatrix}\gamma D_{\perp} \\ 0\end{bmatrix}[\, L_{11}\ L_{12}\ L_{13}\ L_{14}\,]$$

the bottom row gives

$$T=\begin{bmatrix} \Sigma_1^{-1} & 0 & -\gamma^2\Sigma_1 & 0 \\ 0 & \Sigma_3^{-1} & 0 & 0 \\ 0 & 0 & \gamma^2\Sigma_1 & 0 \\ 0 & 0 & 0 & \gamma^2\Sigma_2 \end{bmatrix}$$

back substituting with T, then the top row gives

$$L_{11}=-\gamma^{-1}\Sigma_1^{-1}C_{11}^{*}D_{\perp}$$
$$L_{12}=-\gamma^{-1}\Sigma_3^{-1}C_{13}^{*}D_{\perp}$$
$$L_{13}=\gamma\Sigma_1C_{11}^{*}D_{\perp}$$
$$L_{14}=0$$

The $A_{cl}$ matrix has not been stated, we leave this to the reader to work out. A tedious calculation will verify that this L defined above satisfies (C1). Finally for stability $TE\geq 0$ is requiered.

$$ET=\begin{bmatrix} \Sigma_1^{-1} & 0 & -\gamma^2\Sigma_1 & 0 \\ 0 & \Sigma_3^{-1} & 0 & 0 \\ -\gamma^2\Sigma_1 & 0 & \gamma^2\Sigma_1 & 0 \\ 0 & 0 & 0 & \gamma^2\Sigma_2 \end{bmatrix}$$

which is congruent to

$$\begin{bmatrix} \Sigma_1^{-1} & 0 & 0 & 0 \\ 0 & \Sigma_3^{-1} & 0 & 0 \\ 0 & 0 & \gamma^2\Sigma_1(I-\gamma^2\Sigma_1^2) & 0 \\ 0 & 0 & 0 & \gamma^2\Sigma_2 \end{bmatrix}$$

$\Rightarrow$ for stability $\Sigma_1>0$, $\Sigma_2>0$, $\Sigma_3>0$ and $\lambda_{max}(\Sigma_1)\leq\gamma^{-1}$.

Therefore, a stabilizing controller K(s) exists such that $||F_l(P,K)||_\infty\leq\gamma$ if $S\geq 0$, $P\geq 0$ and $\lambda_{max}(SP)\leq\gamma^{-2}$. $\square$

# Appendix D

$\widetilde{K}(s)$ is described by

$$y_c=(K_{11}+F_\infty)(u_c+D'_{22}y_c)+K_{12}e_{c2} \tag{D1a}$$

$$z_2=K_{21}(u_c+D'_{22}y_c)+K_{22}e_{c2} \tag{D1b}$$

rearranging terms this gives

$$[I-(K_{11}+F_\infty)D'_{22}]^{-1}(K_{11}+F_\infty)u_c+[I-(K_{11}+F_\infty)D'_{22}]^{-1}K_{12}e_{c2}=y_c \tag{D2a}$$

$$K_{21}[I+D'_{22}[I-(K_{11}+F_\infty)D'_{22}]^{-1}(K_{11}+F_\infty)]u_c+[K_{22}+K_{21}D'_{22}[I-(K_{11}+F_\infty)D'_{22}]^{-1}K_{12}]e_{c2}=z_2 \quad \text{(D2a)}$$

$$\Rightarrow \quad \widetilde{K}_{11}=[I-(K_{11}+F_\infty)D'_{22}]^{-1}(K_{11}+F_\infty) \tag{D3a}$$

$$\widetilde{K}_{12}=[I-(K_{11}+F_\infty)D'_{22}]^{-1}K_{12} \tag{D3b}$$

$$\widetilde{K}_{21}=K_{21}[I+D'_{22}[I-(K_{11}+F_\infty)D'_{22}]^{-1}(K_{11}+F_\infty) \tag{D3c}$$

$$\widetilde{K}_{22}=K_{22}+K_{21}D'_{22}[I-(K_{11}+F_\infty)D'_{22}]^{-1}K_{12} \tag{D3d}$$

Form (D3a) $\Rightarrow K_{11}=-F_\infty+\widetilde{K}_{11}(I+D'_{22}\widetilde{K}_{11})^{-1}$ (D4a)

substituting (D4a) into (D3b) gives

$$K_{12}=[I-\widetilde{K}_{11}(I+D'_{22}\widetilde{K}_{11})^{-1}D'_{22}]\widetilde{K}_{12} \tag{D4b}$$

substituting (D4a) into (D3c) gives

$$K_{21}=\widetilde{K}_{21}[I-D'_{22}\widetilde{K}_{11}(I+D'_{22}\widetilde{K}_{11})^{-1}] \tag{D4c}$$

finally, substituting (Da4) into (D3d) yields

$$K_{22}=\widetilde{K}_{22}-\widetilde{K}_{21}[I-D'_{22}\widetilde{K}_{11}(I+D'_{22}\widetilde{K}_{11})^{-1}]D'_{22}\widetilde{K}_{12} \tag{D4d}$$

Expanding the $F_l\{\cdot\}$ term in (3.224) and comparing in with (D4) reveals that

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = F_l\left\{ \left[\begin{array}{cc|c} -F_\infty & \widetilde{K}_{12} & -I \\ \widetilde{K}_{21} & \widetilde{K}_{22}-\widetilde{K}_{21}D'_{22}\widetilde{K}_{12} & \widetilde{K}_{21}D'_{22} \\ \hline -I & D'_{22}\widetilde{K}_{12} & -D'_{22} \end{array}\right], \widetilde{K}_{11} \right\} \tag{D5}$$

is true. □

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