

UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY

DEPARTMENT OF PHYSICS

APPLIED OPTICS SECTION

CONTRIBUTIONS TO THE STATISTICAL

CALCULATION AND MEASUREMENT

OF

SPECKLE PATTERNS

ALAIN SCRIBOT

THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY OF THE UNIVERSITY OF LONDON, 1975.

ERRATA SHEET
FOR
CONTRIBUTIONS TO THE STATISTICAL CALCULATION
AND MEASUREMENT OF SPECKLE PATTERNS.

The word "antocorrelation" should be replaced by the word "autocorrelation" wherever it occurs as for example on P. 35 lines 1 and 9

37	9
40	1 and 6
50	13 .

P.2 line 17, replace the comma "," by "and".

P.12 line 3, should read "...coherent light of wavelength λ ".

line 21, replace "as properly noted" by "a property noted".

N.B. The author wishes to insist upon the fact that the appearance of the speckle pattern is independent of the roughness for a strong scatterer only in the far field.

P.17 in equation (9) replace " $(E' - \langle E' \rangle^2)$ " by " $(E' - \langle E' \rangle)^2$ ".
line 17, replace "This suggests..." by "RICE suggests...".

P.20 line 4, replace "(ERDELYI 1953)" by "(ERDELYI 1954)".

P.24 in equation (46) replace "K" by " K^2 ".

P.37 in equation (8) replace " $\phi_n(x) dx$ " by " $\phi_n(x') dx'$ ",
in equation (11) replace " $\epsilon_m^* \phi_m^*(x)$ " by " $\epsilon_m^* \phi_m^*(x')$ ",
line 20, replace " $\phi_n(x)$ " by " $\phi_n(x')$ ".

P.41 line 2, replace "coefficients as" by "coefficients a_n as",
replace equation (24) by:

$$P(\{a_n\}) = \prod_n \frac{1}{\sqrt{\pi} \epsilon_n} \exp\left(-\sum_n \frac{|a_n|^2}{\epsilon_n}\right) \quad (24)$$

P.43 line 6, replace "(fig.(56))" by "(fig.(5b))".

P.48 line 13, replace "integrati" by "integrated".

P.50 in equation (60) replace " $\text{psf}(x-x'-s)$ " by " $\text{psf}^*(x-x'-s)$ ".

PP.74 to 79 incl., replace the words "relative apertures" in the captions by the words "space-bandwidth product".

P.82 line 3, replace "We observe is closely related..." by "We observe c is closely related...".

P.91 line 12, the sentence inside the brackets is "(i.e. the collecting pupil is fixed and the scanning aperture tends to a point)".

/ERRATA (contd.)

P.152 line 4, replace "diffusing pupil" by "collecting pupil".

P.161 line 6, replace "When $E' \rightarrow \infty$ " by $P_{\text{joint}}(E') \rightarrow \infty$ " by
"When $E' \rightarrow \infty$ " $P_{\text{joint}}(E') \rightarrow 0$ ".
P.161 line 6, replace "When $E' \rightarrow \infty$ " by $P_{\text{joint}}(E') \rightarrow \infty$ " by
"When $E' \rightarrow \infty$ " $P_{\text{joint}}(E') \rightarrow 0$ ".

P.162 line 11 replace "coefficiant" by "coefficient",

line 14 replace "scalling" by "scaling".

P.177 line 8, replace "fig.(64)" by "fig.(56)",

line 9, replace "P verus" by "P versus σ ".

P.188, replace "BLACMAN" by "BLACKMAN".

P.192, between "PARRY, G. 1974 a)..." and "RUSHFORTH, C.K. and HARRIS,
R.W. 1968..." insert,

"PARRY, G. 1974 b) Some effects of surface roughness on the
appearance of speckle in polychromatic light, Opt. Comm.
12, 75."

ABSTRACT

We first review the classical approach to the approximate first order statistics of speckle patterns expressed by the Gamma distribution in terms of the average integrated intensity and the number of independent correlation cells the scanning aperture can be divided into. We then examine in detail the properties of the exact distribution obtained with the eigenvalues of a homogeneous Fredholm integral equation whose kernel is the amplitude autocorrelation function of the scanned speckle pattern. A theorem by Mercer allows us to derive two important expressions for the sum of the eigenvalues and the sum of the squares of the eigenvalues. The first of these relations shows us the number of independent correlation cells for the approximate Gamma distribution is equal to the cut off index of the eigenvalues. The exact and approximate distributions are compared in detail and numerous computational results are given. The validity of the exact distribution is tested experimentally at a point where the difference between exact and approximate distributions is critical, good agreement is found with the exact distribution. Finally the statistics of depolarized speckle patterns are considered, experimental results show that the depolarization is a function of the thickness of the diffusing medium, the variance of size of the scatterers in the diffusing medium and of the wavelength used.

<u>0. INTRODUCTION</u>	5
<u>1. STATISTICS OF DETERMINISTICALLY POLARIZED SPECKLE PATTERNS IN MONOCHROMATIC COHERENT LIGHT</u>	10
<u>1. THE PHYSICAL DEVICE AND SOME BASIC ASSUMPTIONS</u>	12
<u>2. THE APPROXIMATE GAMMA DISTRIBUTION</u>	14
<u>2.1. Derivation - Direct Method</u>	14
<u>2.2. The Characteristic function Method</u>	18
<u>2.3. Evaluation of the number N of independent correlation cells and second order statistics.</u>	22
<u>2.4. Validity of the Gamma distribution</u>	25
<u>2.5. Statistics of speckle patterns with added illumination fields.</u>	27
<u>3. THE EXACT DISTRIBUTION</u>	34
<u>3.1. Introduction</u>	34
<u>3.2. The autocorrelation function, the Fredholm homogeneous integral equation, the corresponding sets of eigenvalues and eigenfunctions and some of their properties</u>	35
<u>3.3. The Exact Probability density Function in Terms of Eigenvalues</u>	41
<u>3.4. On solving the homogeneous Fredholm integral equation</u>	52
<u>3.5. Dependence of the Exact probability density function on the precision of the eigenvalues</u>	56
<u>4. APPLICATIONS</u>	58
<u>4.1. Introduction</u>	58
<u>4.2. 1) A slit (one dimensional) diffusing pupil with a lorentzian intensity transmittance</u>	61

profile and a slit scanning aperture with a rectangular intensity transmittance profile .	61
2) A slit (one dimensional) diffusing pupil with a rectangular intensity transmittance profile and a slit scanning aperture with a rectangular intensity transmittance profile. .	80
3) A rectangular (two dimensional) diffusing pupil with a rectangular intensity transmittance profile and a rectangular scanning aperture with a rectangular intensity transmittance profile .	111
4) A circular (two dimensional) diffusing pupil with a "Top hat" intensity transmittance profile and a circular scanning aperture with a "top hat" intensity transmittance profile.	122
<u>4.3. Experimental results</u>	152
<u>11. RANDOMLY POLARIZED SPECKLE PATTERNS</u>	157
<u>1. INTRODUCTION</u>	158
<u>2. JOINT STATISTICS OF TWO INDEPENDENT POLARIZATION COMPONENTS</u>	160
<u>3. EXPERIMENTAL OBSERVATIONS</u>	166
<u>3.1. Measuring Techniques</u>	166
<u>3.2 Experimental Results</u>	170
APPENDICES	178
<u>BIBLIOGRAPHY</u>	188
<u>ACKNOWLEDGMENTS</u>	195

0. INTRODUCTION

A speckle pattern is a random diffraction phenomenon occurring when coherent light falls on a diffusing medium. In 1877 EXNER observed Fraunhofer rings - a related phenomenon - with radially fibrous structure when lighting with a candle a glass plate upon which he had previously breathed. The radial structure was due to the polychromaticity of the candle's light. VON LAUE in 1914, 1916, 1917 studied the statistical properties of Fraunhofer rings produced by a plate covered with lycopodium powder. True monochromatic speckle patterns obtained in coherent light were first observed with the invention of the LASER. These early observations are related by ALLEN and JONES 1963 LANGMUIR 1963 , OLIVER 1963 and GOULD et al 1964. Since then speckle patterns have been the centre of a flourishing interest as further properties and applications are constantly being found.

The first part of this thesis consists of a brief section describing the apparatus used to observe and measure speckle statistics in monochromatic coherent light. In the second section, the arguments leading to the first order statistics of speckle patterns observed through non punctual finite scanning apertures are reviewed. These statistics are adapted from the mathematically related field of electronic noise and are based on a theory due to RICE 1945 . This theory was used in photo-counting statistics by MANDEL 1959 and applied to speckle patterns by GOODMAN 1965 . Here we retrace Rice 's arguments in the speckle context but we also give another derivation of the first order statistics^{by} a method based on characteristic functions which are the Fourier transforms of probability density functions. The approximate first order probability density function obtained by either method is a gamma distribution and depends only on two parameters, the average

integrated intensity and the number of independent correlation cells into which the scanning aperture can be divided. Although the average integrated intensity is easily determined, finding the number of independent correlation cells is not an easy matter and requires some complicated second order statistical manipulations. The validity of the Gamma distribution is examined by the use of k^{th} order cumulants which are convenient functions of the k^{th} order moments. At the end of this second section is given a review of the statistics of speckle patterns with coherently, incoherently and partially coherently added illumination.

In the third section we explain the theory leading to the exact statistics of speckle patterns. This theory was developed by KAC and SIEGERT 1947 in the context of electronic noise with square law detectors, here the noise is the speckle complex amplitude and the square law detector is the photomultiplier behind the scanning aperture. The theory here adapted to speckle patterns is based on Karhunen Loève expansions of statistical functions which are generalized Fourier expansions whose coefficients are uncorrelated. The eigenfunctions and eigenvalues of these expansions are given as the solutions of a homogeneous Fredholm integral equation whose kernel is the amplitude autocorrelation function of the scanned speckle pattern (i.e. the Fourier transform of the amplitude power spectrum). The eigenvalues obtained from the homogeneous Fredholm integral equation are interpreted physically as the average integrated intensity of several incoherently added speckle patterns and the characteristic function of the required exact first order probability density function is expressed in terms of them. The properties of the eigenvalues are examined closely and a theorem by MERCER 1909 which is a generalization of another theorem by SCHMIDT 1907 allows us to

determine two important relations between the eigenvalues and the amplitude autocorrelation function. The first of these relations shows us that the number of independent correlation cells (one of the two parameters defining the gamma distribution) the scanning aperture can be divided into is equal to the cut off index of the eigenvalues and this allows us to find easily the number of independent correlations and to compare easily the exact and approximate distributions. The second relation gives the sum of the squared eigenvalues as an integral function of the squared autocorrelation function. At the end of this third section, two methods are considered to solve the homogeneous Fredholm integral equation and an explanation is also given of how the precision of the exact distribution is a function of the precision with which the eigenvalues are known. A relative precision of 10^{-3} on the eigenvalues is generally found to be adequate.

In the fourth section four applications of the theory leading to the exact statistics are examined. These applications provide a good means of comparison between the approximate and exact distribution. The first of these cases concerns a slit diffusing pupil with a lorentzian intensity transmittance and a slit scanning aperture with rectangular intensity transmittance. It is perhaps more interesting from the mathematical point of view than from the physical point of view as it illustrates clearly one of the two methods considered as solving the homogeneous Fredholm integral equation. The two following cases concern slit and rectangular apertures with rectangular intensity transmittances. The fourth case concerns circular apertures with "top hat" intensity transmittances. This last case is the most interesting because of its relevance to the general geometrical configuration of most optical instruments and also because we provide experimental verification at a point where the difference between the exact and approximate

distributions is the largest. The agreement between experiment and exact distribution is found to be good and better than with the approximate distribution. For all the four cases the difference between the exact and approximate distributions is characterized by plots of the integrated mean square difference versus the relative aperture which is proportional to the product of the size of the collecting pupil by the size of the scanning aperture.

In a second part of this thesis is a short description of the statistics of randomly polarized speckle patterns. We mainly state our experimental observations concerning the random polarization of speckle patterns emerging from thick diffusing media.

In this thesis we do not consider the elimination of or the applications of speckle patterns. Authors who may be quoted in connexion with speckle elimination are GERRITSON et al 1968 , GABOR 1970 and DAINTY and WELFORD 1971 . Dainty and Welford's method is based on a theorem by WELFORD 1971 stating that the pupil of a coherently illuminated optical system varying in time gives a time averaged image equivalent to that of an optical system with a certain fixed pupil and partially coherent illumination. Speckle in partially coherent light is well known to be less contrasted. Mc-KECHNIE 1975 gives a detailed analysis of the change in contrast obtained by the application of this method.

Applications of speckle patterns to the study of vibrations have been considered by ARCHBOLD et al 1963 , FERNELIUS and TOME 1971 and ELIASSON and MOTTIER 1971 . If we consider a vibrating rough surface illuminated in coherent light the speckle at the nodes of the surface is more

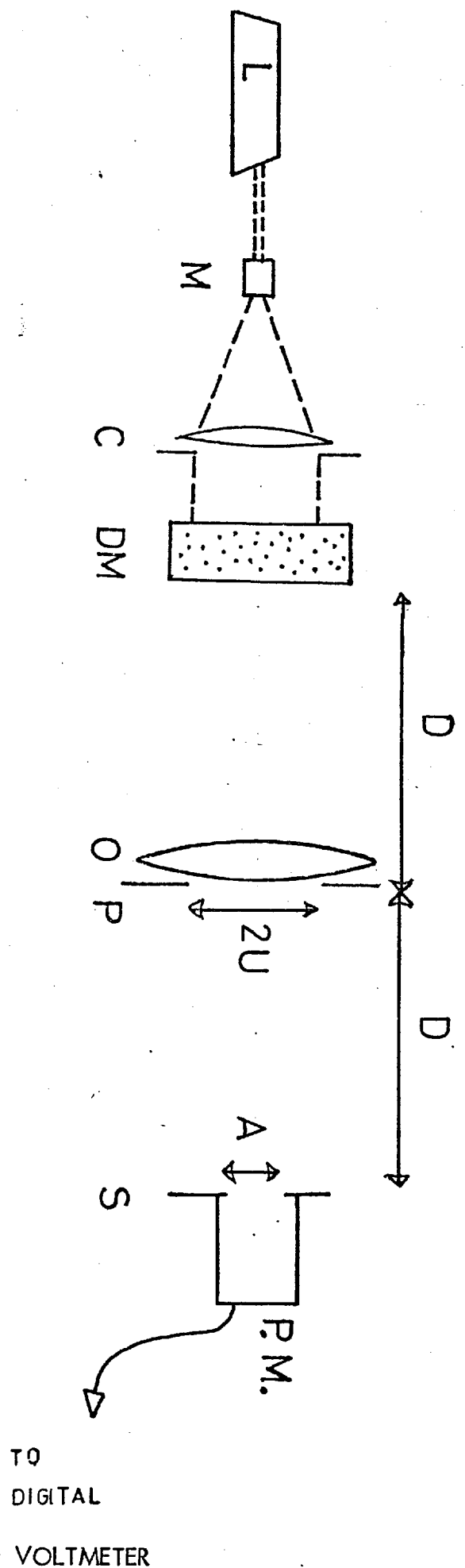
contrasted than at the vibration maxima . This can perhaps also be interpreted in the light of Welford's theorem that if the vibrating surface is considered as a secondary source the nodes behave as coherent light sources and the vibration maxima behave as partially coherent sources. LEENDERTZ 1970 and BUTTERS and LEENDERTZ 1971 also apply speckle patterns to vibration analysis, although their method is based on the superposition of two speckle patterns recorded at different times. Another application of speckle patterns concern the analysis of surface roughness. This is studied with polychromatic laser light, the statistics of the resulting speckle patterns are observed to be a function of the roughness of the diffusing surface. A way of characterizing this is to plot the standard deviation of the intensity fluctuations versus the standard deviation of the height fluctuations of the surface. Authors to be quoted in connexion with this work are SPRAGUE 1972 , TRIBILLON 1974 , PARRY 1974 a) and PARRY 1974 b) Other applications include attainment of diffraction limited imaging in telescopes (LABEYRIE 1970 , DAINTY 1972) and speckle interferometry (GEZARI et al 1972 and BURCH 1970 .

1. STATISTICS OF DETERMINISTICALLY
 POLARIZED SPECKLE PATTERNS
 IN
 MONOCHROMATIC COHERENT LIGHT

OPTICAL SET-UP FOR
OBSERVING SPECKLE PATTERNS IN
COHERENT LIGHT.

- L: LASER with wavelength λ
M: MICROSCOPE OBJECTIVE
C: COLLIMATOR AND DIAPHRAGM
DM: DIFFUSING MEDIUM
O: OPTICAL SYSTEM
P: COLLECTING PUPIL
S: SCANNING APERTURE DEFINED
OVER THE DOMAIN A
P.M: PHOTO MULTIPLIER

FIG. 1



1. THE PHYSICAL DEVICE AND SOME BASIC ASSUMPTIONS

Consider the apparatus described in fig. (1) for observing speckle patterns. It consists of a laser L emitting monochromatic coherent light of wavelength λ , a microscope objective M to expand the beam, a collimating lens and diaphragm C , a diffusing medium DM , an optical system O , a distance D away from the diffuser with a pupil P of size $2U$ we call the collecting pupil mounted close to it to limit the size of the scattered beam and lastly a scanning aperture S defined over the domain (A) . The collecting pupil and the scanning aperture are at a distance D apart and are considered in the Fourier space of each other, (i.e. the diffuser and scanning aperture are in the focal planes of the optical system).

This apparatus can be treated in analogy with electronic devices as a linear filter followed by a square law detector (see e.g. WHALEN 1971) the linear filtering device being the scanning aperture S and the square law device being the photomultiplier. The quantity that is linearly filtered is the complex amplitude of the speckle pattern and the photomultiplier detects the intensity which is the squared modulus of the complex amplitude.

The diffusing medium is assumed to give r.m.s. phase variations greater than the wavelength (strong scatterer). In this case the appearance of the speckle pattern is independent of the roughness of the scattering surface, as properly noted by RIGDEN and GORDON 1962, and we assume that the scattered light can be treated as stationary white noise (DAINTY 1972 d) i.e. as a random process with constant spectral density (DAVENPORT and ROOT 1958). The scattered light is assumed to be deterministically polarized

(e.g. plane polarized). This last assumption is only true for surface-like scatterers, i.e. with their scattering centres all lying approximately in the same plane. Later we shall see how thick scatterers tend to depolarize light. However we can study the deterministically polarized speckle statistics from a thick scatterer by inserting a polarising filter in front of the scanning aperture.

2. THE APPROXIMATE GAMMA DISTRIBUTION

2.1. DERIVATION - DIRECT METHOD

Given the optical apparatus and basic assumptions in the above section, the probability density function of speckle intensity at any given point is well known to be a negative exponential (GOODMAN 1963 , GOODMAN 1965 , BURCH 1970 , DAINY 1970). To show this, one assumes that the real and imaginary parts of the complex amplitude obey the central limit theorem, (see e.g. MIDDLETON 1960). This theorem states that the sum of a large number of random variables tends to be normally (Gaussianly) distributed even though the summands may not be. Here the summands are the real and imaginary amplitudes from each scatterer. The joint probability distribution $P(V_r, V_i)$ of the complex amplitude is then a Gaussian (equ. (1)). By simple probability transformations, one derives successively the amplitude and phase distribution $P(|V|, \phi)$, which is a Rayleigh distribution (equ. (2)) and the intensity and phase distribution $P(I, \phi)$, which is a negative exponential (equ. (3)).

$$P(V_r, V_i) = \frac{1}{2\pi\sigma_{|V|}^2} \exp\left(-\frac{V_r^2 + V_i^2}{2\sigma_{|V|}^2}\right) \quad (1)$$

$$P(|V|, \phi) = \frac{|V|}{2\pi\sigma_{|V|}^2} \exp\left(-\frac{|V|^2}{2\sigma_{|V|}^2}\right) \quad (2)$$

$$P(I, \phi) = \frac{1}{2\pi\sigma_{|V|}^2} \exp\left(-\frac{I}{2\sigma_{|V|}^2}\right) \quad (3)$$

with $V \geq 0$, $I \geq 0$, $0 \leq \phi \leq 2\pi$

$\sigma_{|V|}$ is the r.m.s. of the amplitude. Renormalizing equ. (3) and introducing the r.m.s. of the intensity σ_I rather than keeping the r.m.s. of the amplitudes, the final expressions for $P(I)$ and $P(\phi)$ are

$$P(I) = \frac{1}{\sigma_I} \exp\left(-\frac{I}{\sigma_I}\right) \quad (4)$$

with the average intensity $\langle I \rangle$ such that

$$\langle I \rangle = \sigma_I \quad (4a)$$

and

$$P(\phi) = \frac{1}{2\pi} \quad (5)$$

However the r.m.s. is measured through a finite aperture which is not a point and we must therefore find the distribution of the integrated intensity applicable when measuring with any finite aperture. The following derivation is due to RICE 1945, here, we apply his arguments to speckle patterns.

Let us first assume we are measuring σ_I with an aperture sufficiently small such that the intensity does not change appreciably over its surface. The integrated intensity E detected is proportional to the intensity at a point and to the size of the surface S of the scanning aperture, so that we may write,

$$P(E) = \frac{1}{\langle E \rangle} \exp\left(-\frac{E}{\langle E \rangle}\right) \quad (6)$$

with r.m.s. integrated intensity fluctuations,

$$\sigma_E = \langle E \rangle = \langle I \rangle \times S \quad (7)$$

This function is illustrated for various average integrated intensities in fig. 2.

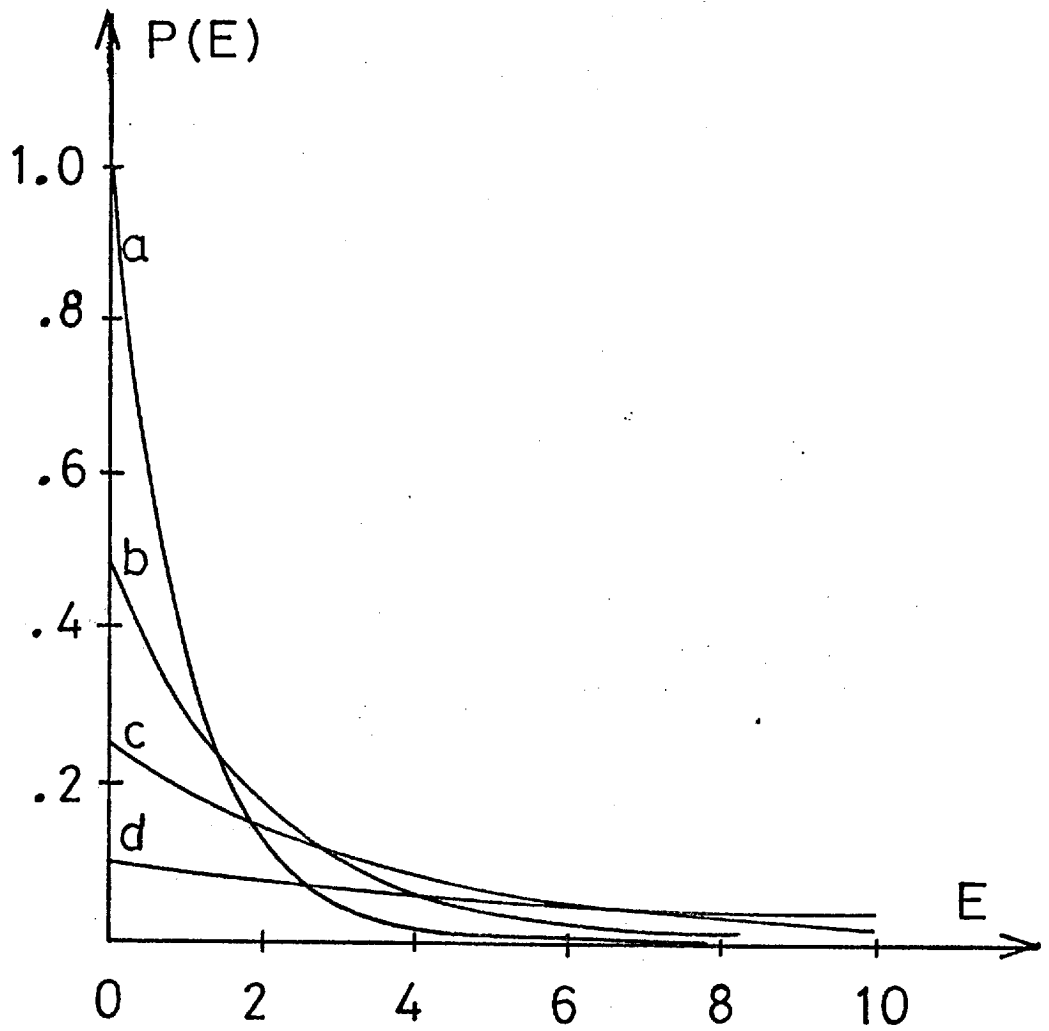


FIG. 2

GRAPH OF THE PROBABILITY DENSITY IN EQN. 2.1.(6)

FOR VARIOUS VALUES OF $\langle E \rangle$,

a) $\langle E \rangle = 1$

b) $\langle E \rangle = 2$

c) $\langle E \rangle = 4$

d) $\langle E \rangle = 10$

$$P(E) = \frac{1}{\langle E \rangle} e^{-\frac{E}{\langle E \rangle}}, \quad \sigma_E = \langle E \rangle$$

When the scanning aperture becomes very large, we may divide it into a number of smaller surfaces of constant integrated intensity, say N , each of area $s = S/N$, let E_r be the contribution of the r^{th} surface, the integrated intensity E of the entire area is then,

$$E' = E_1 + E_2 + E_3 + \dots + E_N \quad (8)$$

The integrated intensity incident on the r^{th} surface is taken to obey a negative exponential. If the sub-areas are large enough, the E_r 's are substantially independent random variables; if in addition N is large enough, by application of the central limit theorem, E' is approximately normally distributed, if all the E_r 's have same average and variance as in equ. (6), we obtain

$$P(E') = \frac{1}{\sqrt{2\pi} \sigma_{E'}} \exp \left(- \frac{(E' - \langle E' \rangle)^2}{\sigma_{E'}^2} \right) \quad (9)$$

with the average and variance given by

$$\langle E' \rangle = \frac{S}{s} \quad \text{and} \quad \sigma_{E'}^2 = \frac{S}{s} \sigma_E^2 \quad (10)$$

When we look at eqs. (6) and (9) we observe that a Gamma variate of the form

$$P(E') = \frac{\alpha^N E'^{N-1} e^{-\alpha E'}}{\Gamma(N)} \quad (11)$$

becomes a negative exponential for $N = 1$ and a normal law if we let N become large. This suggests equ. (11) is an appropriate approximate distribution for E' .

The average and variance of the integrated intensity are,

$$\alpha = \frac{\langle E' \rangle}{\sigma_{E'}^2} \quad \text{and} \quad N = \frac{\langle E' \rangle^2}{\sigma_{E'}^2} \quad (12)$$

giving,

$$a = \frac{\langle E' \rangle}{\sigma_{E'}^2} \quad \text{and} \quad N = \frac{\langle E' \rangle^2}{\sigma_{E'}^2} \quad (13)$$

These parameters are chosen in such a way that the approximate distribution and the experimental distribution have the same average and variance. This Gamma distribution was first used in optics in the photoelectron counting context by MANDEL 1959 and GOODMAN 1965 ; DAINITY 1971 applied it to speckle patterns.

Inserting equs. (12) and (13) in equ. (11) gives,

$$P(E') = \left(\frac{N}{\langle E' \rangle} \right)^N \frac{E'^{N-1} e^{-\frac{E'N}{\langle E' \rangle}}}{\Gamma(N)} \quad (14)$$

which is the classical first order probability density function scaled to average integrated intensity used for deterministically polarized speckle patterns. The behaviour of this function is illustrated in figure (3). To be precise, the above reasoning applies only for integer values of N ; however the Gamma distribution will often provide a closer fit to experimental data if non-integer values of N are used. (CONDIE 1966 , Mc-KECHNIE 1974). This is why the parameters N and a are chosen using experimental values of $\langle E' \rangle$ and $\sigma_{E'}^2$.

2.2. THE CHARACTERISTIC FUNCTION METHOD

The characteristic function of a probability density function is defined as the Fourier transform of the probability density function under consideration (CRAMER 1946 , LOEVE 1962 , PAPOULIS 1966). We explain the characteristic function and its essential properties more fully in Appendix A.

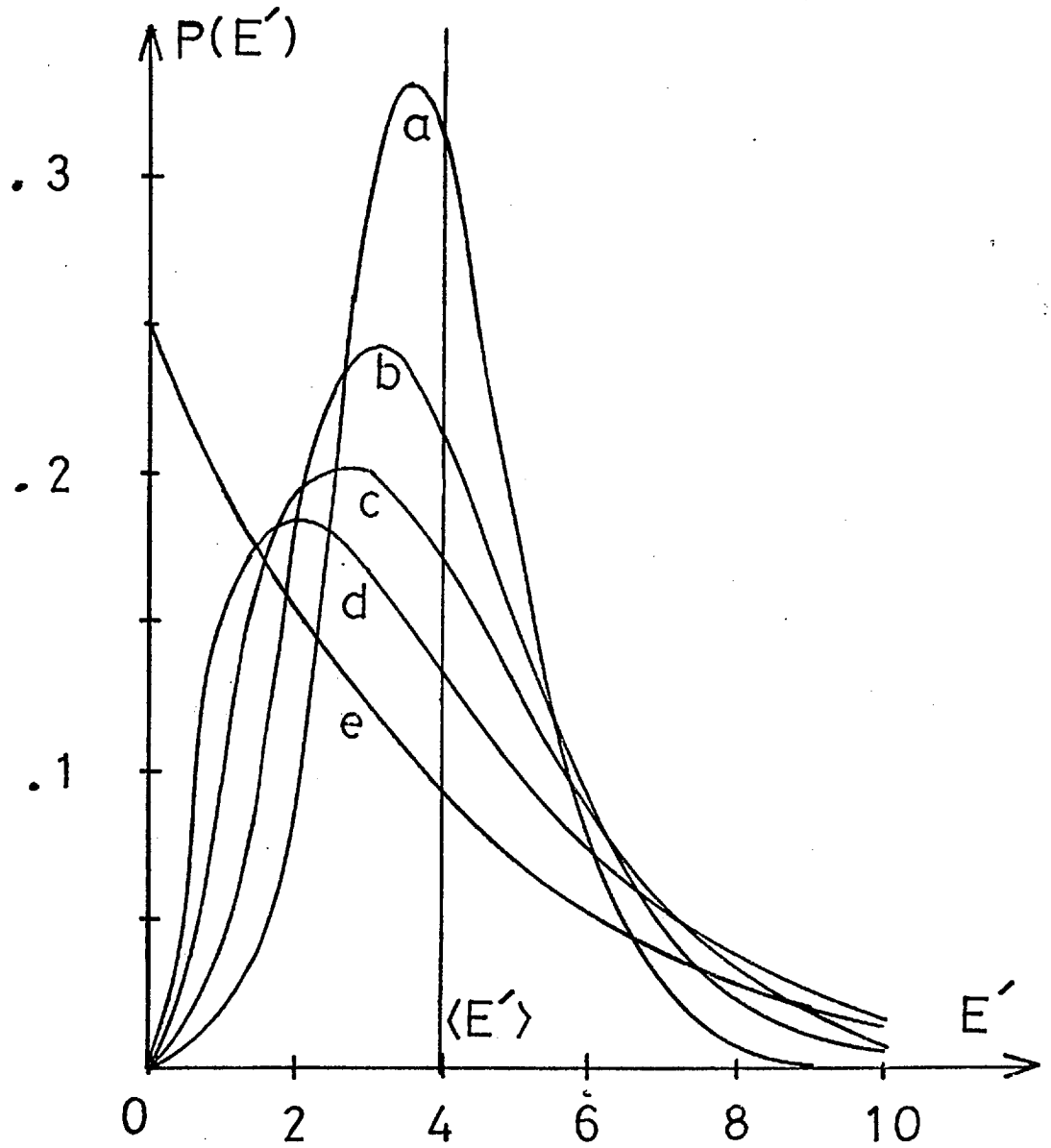


FIG. 3

GRAPH OF THE PROBABILITY DENSITY IN EQN. 2.1.(14)

a) $N = 10$

b) $N = 5$

c) $N = 3$

d) $N = 2$

e) $N = 1$

$$P(E') = \frac{\left(\frac{N}{\langle E' \rangle}\right)^N E'^{N-1} e^{-\frac{E'}{\langle E' \rangle}}}{\Gamma(N)}$$

Considering again an area of the speckle pattern small enough for the intensity to stay constant over its surface, the probability density of the detected integrated intensity is, according to equ. (6) a negative exponential and its characteristic function (i.e. its Fourier transform) is, (ERDELYI 1953)

$$C_E(z) = (1 - iz \langle E \rangle)^{-1} \quad (15)$$

The modulus of this function is illustrated in fig. 4 for various values of the average integrated intensity. For a large aperture, we again perform division into N sub-areas such that the total intensity received through that area is the sum of independent random variables E_r from each sub-area, each taken to obey a negative exponential. From Appendix A, the characteristic function of such a sum $C_{E'}(Z)$ is the product of the characteristic function of each variable,

$$C_{E'}(z) = \prod_{r=1}^N C_{E_r}(z) \quad (16)$$

If the quantities $C_{E_r}(Z)$ are empirically assumed all equal and given by equ. (15), then we have,

$$C_{E'}(z) = (1 - iz \langle E \rangle)^{-N} \quad (17)$$

The inverse Fourier transform of this yields the required distribution (ERDELYI 1953)

$$P(E') = \left(\frac{1}{\langle E \rangle} \right)^N \frac{E'^{N-1} e^{-\frac{E'}{\langle E \rangle}}}{\Gamma(N)} \quad (18)$$

The average integrated intensity and variance are,

$$\langle E' \rangle = N \langle E \rangle \quad (19)$$

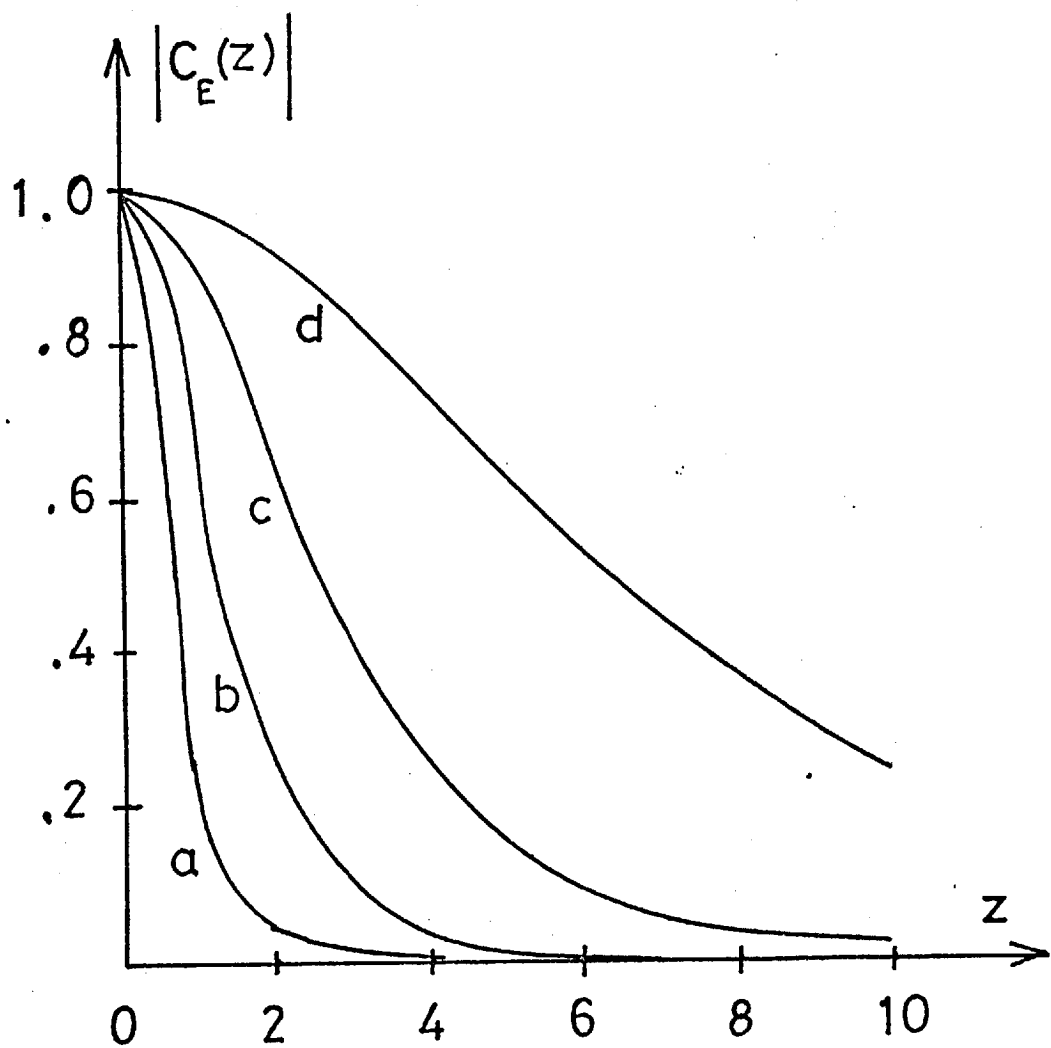


FIG. 4

GRAPH OF THE MODULUS OF THE CHARACTERISTIC FUNCTION IN
EQN. 2.2. (15)

a) $\langle E \rangle = 1$

b) $\langle E \rangle = 2$

c) $\langle E \rangle = 4$

d) $\langle E \rangle = 10$

$$C_E(z) = \left(1 - iZ \langle E \rangle \right)^{-1}$$

and

$$\sigma_{E'}^2 = N \sigma_E^2 = N \langle E \rangle^2 \quad (20)$$

Using these two equations we find again the Gamma distribution scaled to average integrated intensity

$$P(E') = \left(\frac{N}{\langle E' \rangle} \right)^N \frac{E'^{N-1} e^{-\frac{E' N}{\langle E' \rangle}}}{\Gamma(N)} \quad (21)$$

2.3. Evaluation of the number N of independent correlation cells and second order statistics

In the language of statisticians N defines the number of degrees of freedom of the Gamma distribution (see e.g. CRAMER 1946), in our language N defines the number of independent correlation cells into which our scanned speckle pattern can be divided. The Gamma distribution in equ. (21) depends only on the average integrated intensity and this parameter N . Physically, for a given collecting pupil P and scanning aperture S it is easy to choose or set the average integrated intensity as we please. But to determine N a priori we need an analytical expression. Equation (13) gave us N as a function of the variance $\sigma_{E'}^2$ and the average $\langle E' \rangle$, since the quantity $\langle E' \rangle$ is given, we now need an analytical expression for the quantity $\sigma_{E'}^2$. This can be found from second order statistics, (GOODMAN 1965, DAINTY 1970). The results used in this section entail considerable algebra, they are quoted without proof and the reader is referred to the required authors whenever appropriate.

Let $\Pi_{E',S}(u)$ be the integrated intensity power spectrum of the scanned speckle pattern, in one dimensional notation, without loss of generality (u) defines a point in Fourier space. (For a definition of power spectra and their

properties see e.g. BLACKMAN and TUKEY 1959 , see also GOLDFISCHER 1965 who considers power spectra of speckle patterns). The variance of the integrated intensity $\sigma_{E'}^2$, scanned with a finite aperture and the power spectrum are related by (DAINTY 1970),

$$\sigma_{E'}^2 = \int_{-\infty}^{+\infty} \pi_{E,S}(u) |s(u)|^2 du \quad (22)$$

where $s(u)$ is the Fourier transform of the scanning aperture. This equation expresses the only way in which the approximate Gamma distribution depends on the frequency spectrum. RICE 1945 , has shown that the intensity and amplitude power spectra $\pi_{E'}(u)$ and $\pi_{|V|}(u)$ are simply related by a correlation equation.

$$\pi_{E'}(u) = \pi_{|V|}(u) \otimes \pi_{|V|}(u) \quad (23)$$

where $\otimes$ denotes correlation. We must therefore find the amplitude power spectrum. Let us suppose $V_{DM}(x)$, the complex amplitude emerging from the diffuser is a stationary stochastic process with an amplitude power spectrum

$\pi_{|V|,DM}(u)$. Following DAINTY 1970 , the pupil P and scanning aperture S are associated to a linear filtering device with input and output power spectra respectively $\pi_{|V|,DM}(u)$ and $\pi_{|V|,S}(u)$. These power spectra are simply related (see e.g. PAPOULIS 1965) by,

$$\pi_{|V|,S}(u) = |P(u)|^2 \pi_{|V|,DM}(u) \quad (24)$$

where $P(u)$ is the pupil function of P (see fig. (1)). By using equations (23) and (24) we have the scanned intensity power spectrum,

$$\pi_{E',S}(u) = |P(u)|^2 \pi_{|V|,DM}(u) \otimes |P(u)|^2 \pi_{|V|,DM}(u) \quad (25)$$

If the complex amplitude emerging from the diffuser is treated as white noise,

$\pi_{|V|,DM}(u)$ is a constant K and equation (25) reduces to,

$$\pi_{E',S}(u) = K |P(u)|^2 \otimes |P(u)|^2 \quad (26)$$

It can be seen by integrating equation (24) that K is proportional to the average integrated intensity:

$$\langle E' \rangle = K \int_{-\infty}^{+\infty} |P(u)|^2 du \quad (27)$$

Substituting this result in equation (26) into equation (22) finally gives

$$\sigma_{E'}^2 = \langle E' \rangle^2 \frac{\int_{-\infty}^{+\infty} |P(u)|^2 \otimes |P(u)|^2 |s(u)| du}{\left\{ \int_{-\infty}^{+\infty} |P(u)|^2 du \right\}^2} \quad (28)$$

The quantity N is now seen to be the inverse of the normalized power spectrum of the integrated intensity,

$$N = \frac{\langle E' \rangle^2}{\sigma_{E'}^2} = \frac{\left\{ \int_{-\infty}^{+\infty} |P(u)|^2 du \right\}^2}{\int_{-\infty}^{+\infty} |P(u)|^2 \otimes |P(u)|^2 du} \quad (29)$$

Physically this means that as long as the scattered light is treated as stationary white noise the number of independent autocorrelation cells depends only (through this last equation) on the geometrical configuration of the optical set-up used to measure the statistics of the speckle pattern. As an example DAINTY

1970 gives the result of equation (29) for a circular configuration (i.e. a circular pupil P and a circular scanning aperture S with "Top Hat" intensity transmittances), this is

$$N = \frac{\pi^3 k^2 (0.305)^2}{4} \left\{ \int_0^1 \left[\cos^{-1}(u) - u \sqrt{1-u^2} \right] \frac{J_1[4\pi k u (0.305)]}{u} du \right\}^{-1} \quad (30)$$

where u is a radial coordinate in Fourier space and k is a measure of relative aperture defined by

$$k = \frac{\text{cut off frequency of power spectrum}}{\text{Some reference frequency on the aperture transfer function, usually the first zero of that function}} \quad (31)$$

As a note it should be added that the number N of independent correlation cells is often loosely referred to as the number of speckles in the scanned field. This has the advantage of giving a "visual significance" to N , ELIASSON & MOTTIER 1971 have elaborated this notion and define the size of a 3D speckle.

2.4. VALIDITY OF THE GAMMA DISTRIBUTION

In the preceding sections we have seen that the Gamma distribution for first order speckle statistics represents in fact an approximation of the true statistical distribution. MANDEL 1959 gives some consideration to the validity of the Gamma distribution in the photon counting context. His arguments are applied to speckle statistics by CONDIE - 1966. Their results are based on the properties of the characteristic function from which an error for the cumulant is evaluated. (For characteristic functions and cumulants, see appendix A). Their arguments are as follows.

The characteristic function of the Gamma distribution with respect to integrated intensity is (see section 2.2),

$$C_{E'}(z) = \left(1 - \frac{iz}{\alpha}\right)^{-N} \quad (32)$$

$$\text{where } \alpha = \frac{\langle E' \rangle}{\sigma_{E'}^2} \quad (33)$$

From equ. (A14) the expression for the k^{th} cumulant is

$$q_k = \frac{N}{a^k} (k-1)! \left(1 - \frac{iz}{a}\right)^{-k} \Big|_{z=0} = \frac{N(k-1)}{a^k} \quad (34)$$

The characteristic function $C_{E'}(z)$ can be regarded (as in section 2.2) as the product of N characteristic functions $C_{E_r}(z)$ such that,

$$C_{E_r}(z) = \left(1 - \frac{iz}{a_r}\right)^{-1} \quad (35)$$

With all the a_r 's equal to a , the k^{th} cumulant of $C_{E_r}(z)$ is from equ.

(A14)

$$q_{r,k} = \frac{(k-1)}{a_r^k} \quad (36)$$

If we apply the rule of composition of cumulants given by equ. (A20) we obtain

$$q_k = (k-1)! \sum_{r=1}^N \frac{1}{a_r^k} \quad (37)$$

If all the a_r 's are equal to a this equation simplifies and gives

$$q_k = (k-1)! \frac{N}{a^k} \quad (38)$$

On the other hand, if they are not all equal, the k^{th} cumulant will be found to be in error by a factor F_k such that

$$F_k = \frac{N}{a^k} \frac{1}{\sum_{r=1}^N \frac{1}{a_r^k}} \quad (39)$$

setting F_1 equal to unity gives

$$\frac{N}{a} = \sum_{r=1}^N \frac{1}{a_r} \quad (40)$$

Introducing this equation in equ. (39) and remembering from equ. (6) that for a negative exponential α_r is equal to $1/\langle E_r \rangle$, we obtain

$$F_k = \frac{\left(\sum_{r=1}^N \langle E_r \rangle \right)^k}{N^{k-1} \sum_{r=1}^N \langle E_r \rangle^k} \quad (41)$$

Physically, this is equivalent to performing two sets of measures to evaluate two sets of cumulants, first with a large aperture having N correlation cells as given by equ. (29), calculating a first set of cumulants from that, then with N quasi-punctual apertures (having only one correlation cell each), calculating a set of joint cumulants and comparing both sets. CONDIE 1966 has done this experimentally for an aperture of $N = 3$ independent correlation cells. His measured r.m.s. fluctuations are $\sigma_1 = .9$, $\sigma_2 = 1.0$, $\sigma_3 = .9$ giving,

$$F_1 = 1. , F_2 = .998 , F_3 = .994 , F_4 = .984$$

Condie states that this kind of error is quite insignificant, but he does not mention anything for scanning apertures with larger numbers of independent correlation cells (e.g. $N = 50$). This method is impractical and the exactitude of the Gamma distribution is not easy to evaluate.

2.5. Statistics of speckle patterns with added illumination fields

The above sections give a classical treatment of the statistics of ^amonochromatic speckle pattern on its own. However, it is possible to add various fields to the speckle pattern that modify these statistics. These may be classified as follows,

- Incoherent addition of one or more speckle patterns
- Coherent addition of one or more speckle patterns
- Addition of one or more speckle patterns that are partially coherent between each other
- Coherent addition of a uniform background
(i.e. reference wave)
- Incoherent addition of a uniform background

All these cases have been extensively treated in the literature by various authors, Mc-KECHNIE 1974 gives a good review of all of them.

Incoherent addition of one or more speckle patterns

BURCH 1970 shows that when two speckle patterns are added together incoherently the resulting probability density function is given by the convolution of the probability density functions of each speckle taken separately. For N speckle patterns added together incoherently it is easily seen that the resulting probability density will be an N -fold convolution of N probability densities of the speckle patterns taken separately. Rather than perform the convolutions, it is more convenient to reason on characteristic functions (i.e. the Fourier transforms of the probability densities) and say that the joint characteristic function of the resulting speckle pattern will be the product of the characteristic function of each speckle pattern taken separately. Thus, if we are scanning the incoherently added speckle patterns with a pinhole aperture, each speckle will have a characteristic function of the type

$$C_k(z) = \left(1 - iz \langle E'_k \rangle \right)^{-1} \quad (42)$$

where k is the index denoting the k^{th} patterns and $\langle E'_k \rangle$ is its average integrated intensity. The resulting characteristic function will be,

$$C(z) = \prod_{k=1}^N C_k(z) = \prod_{k=1}^N (1 - iz \langle E'_k \rangle)^{-1} \quad (43)$$

This product can be expanded into partial fractions with coefficients B_k ,

$$B_k = \prod_{\substack{l=1 \\ l \neq k}}^N \frac{\langle E'_k \rangle}{\langle E'_k \rangle - \langle E'_l \rangle} \quad (44)$$

and the joint characteristic function now is,

$$C(z) = \sum_{k=1}^N \frac{B_k}{(1 - iz \langle E'_k \rangle)} \quad (45)$$

The Fourier transform of this equation gives the required joint probability density function,

$$P(E') = \sum_{k=1}^N \frac{B_k}{\langle E'_k \rangle} e^{-\frac{E'}{\langle E'_k \rangle}} \quad (46)$$

With only two speckle patterns added incoherently of respective average integrated intensity $\langle E'_1 \rangle$ and $\langle E'_2 \rangle$, the probability density function is,

$$P(E') = \frac{e^{-\frac{E'}{\langle E'_1 \rangle}} - e^{-\frac{E'}{\langle E'_2 \rangle}}}{\langle E'_1 \rangle - \langle E'_2 \rangle} \quad (47)$$

this formula is given by BURCH - 1970.

If the quantities $\langle E'_k \rangle$ are all equal to, say, $\langle E' \rangle$, the joint characteristic function for N incoherently added speckly patterns is expressed by

$$C(z) = (1 - iz \langle E' \rangle)^{-N} \quad (48)$$

The Fourier transform of which is, of course, a Gamma distribution,

$$P(E') = \left(\frac{1}{\langle E' \rangle} \right)^N \frac{E'^{N-1} e^{-\frac{E'}{\langle E' \rangle}}}{\Gamma(N)} \quad (49)$$

We have the interesting result that the approximate statistics of one speckle pattern observed through a finite scanning aperture such that there are N independent correlation cells within it are the same as the statistics of N incoherently superposed speckle patterns with the same average intensity scanned through a pinhole aperture (i.e. scanning aperture with one correlation cell only).

If the N incoherently superposed speckle patterns are observed through a finite scanning aperture such that there are J correlation cells within it, using the Fourier transform of the approximate Gamma distribution with J degrees of freedom for each speckle patterns of average intensity $\langle E'_k \rangle$ gives the joint characteristic function as

$$C(z) = \left\{ \prod_{k=1}^N (1 - iz \langle E'_k \rangle) \right\}^{-J} \quad (50)$$

If all the $\langle E'_k \rangle$'s are equal to $\langle E' \rangle$, for N speckle patterns we have

$$C(z) = (1 - iz \langle E' \rangle)^{-JN} \quad (51)$$

The probability density function is given by the Fourier transform of this expression as,

$$P(E') = \left(\frac{1}{\langle E' \rangle} \right)^{JN} \frac{E'^{JN-1} e^{-\frac{E'}{\langle E' \rangle}}}{\Gamma(JN)} \quad (52)$$

This approximate distribution is identical to that of a single speckle pattern scanned with a finite aperture having $M = JN$ correlation cells or to that of an incoherent superposition of M speckle patterns scanned with a pinhole aperture.

Coherent addition of one or more speckle patterns

The case of the coherent addition of two speckle patterns is mentioned by ARCHBOLD et al 1970 but is considered more fully by DAINITY 1972. The general result is that the coherent superposition of two speckle patterns is another speckle pattern with similar statistics. The argument is based on the Gaussian statistical properties of the complex amplitudes. If the probability density of the complex amplitude of each speckle pattern is a Gaussian, the probability density of the complex amplitude of the resulting speckle pattern will be the convolution of the two Gaussians the result of which is another Gaussian, hence the statistics of the intensity are similar.

Addition of partially coherent speckle patterns

The case of the superposition of two partially coherent speckle patterns is treated by BARAKAT 1973 in the "coherent optics terminology", that is when the two speckle patterns are partially coherent in time (as opposed to space). Barakat derives an integral expression for their joint probability density

function of the intensity scanned with a pinhole. BURCH 1974 shows this integral can be analytically solved, but these results apply only to the case of two speckle patterns. GOODMAN 1975 develops a general formalism for the sum of N partially correlated speckly patterns. His method is based on techniques developed by WOLF 1959 in the theory of partial polarization. The coherence between each speckle pattern is expressed by a set of coefficients μ_{nm} where n and m denote the n^{th} and m^{th} partially correlated speckle pattern, these coefficients are arranged in matrix form, as the matrix is hermitian, it is diagonalizable. A set of eigen-values $\{ \lambda_n \}$ is obtained as the result of the diagonalization and ^{these} are physically interpreted as representing the average intensities of N uncorrelated or incoherent speckle patterns whose superposition would give the same statistics. One is then brought back to evaluating the probability density of N incoherently added speckle patterns.

Coherent addition of a uniform background

This problem is treated by many authors, GOODMAN 1967, ARCHBOLD et al 1970, BURCH 1970, VILKOMERSON 1971, DAINTY 1972 and McKECHNIE 1974.

The probability density function at a point is given by

$$P(E') = \frac{1}{\langle E' \rangle} e^{\frac{E' + E_b}{\langle E' \rangle}} I_0 \left(\frac{2 \sqrt{E_b E'}}{\langle E' \rangle} \right) \quad (53)$$

Where E_b is the background integrated intensity, $\langle E' \rangle$ is the average integrated intensity of the speckle pattern and I_0 is the modified bessel function of order zero (see e.g. ABRAMOWITZ & STEGUN 1970).

Incoherent addition of a uniform background

The probability density function at a point for this case is the convolution of a delta function and a negative exponential giving,

$$P(E') = \frac{1}{\langle E' \rangle} e^{-\frac{E' - E_b}{\langle E' \rangle}} \quad (54)$$

where E_b is the constant background integrated intensity.

3. THE EXACT DISTRIBUTION

3.1. Introduction

In the preceding section we presented the approximate Gamma distribution and its main properties. We now introduce a method to derive an exact distribution for the integrated intensity of speckle patterns observed through non-point apertures. The method is based on finding the eigenvalues and eigenfunctions of a Fredholm homogeneous integral equation (FREDHOLM 1903) whose kernel is the amplitude autocorrelation function of the speckle pattern. This method is due to KAC and SIEGERT 1947 who developed it in the context of noise in radio receivers with square law detectors. In the speckle context, CONDIE 1966 states some properties of the eigenvalues, DAINTY 1971 using expressions in terms of eigenvalues found in THOMAS 1969 gives an integral expression of the probability density function, but Thomas's expressions concern real arguments and as the speckle amplitude is a complex variable DAINTY's integral is wrong and cannot be solved. Here we give a full account of the method as applicable to speckle patterns and give the exact probability density function of integrated intensity in terms of the eigenvalues. The Gamma distribution is found to be a special case due to a particular set of eigenvalues.

3.2. The autocorrelation function, the homogeneous Fredholm integral equation, the corresponding set of eigenvalues and eigenfunctions and some of their properties.

Let the amplitude autocorrelation function of the scanned speckle pattern be defined by,

$$R(x, x') = \langle V_S^*(x') V_S(x) \rangle \quad (1)$$

where the brackets denote average, the asterisk denotes complex conjugate and $V_S(\cdot)$ represents the scanned complex amplitude and x' and x are two points within the scanning aperture. It can easily be verified that this autocorrelation function is hermitian and square summable, that is

$$R(x, x') = R^*(x', x) \quad (2)$$

$$\text{and } \int_{(A)} \int_{(A)} |R(x, x')|^2 dx' dx < \infty \quad (3)$$

Where (A) is the domain of integration defined by the scanning aperture function.

This autocorrelation function represents a transformation R of the complex amplitude which is characteristic of the scattered light and detector system.

$$f'(x) = \int_{(A)} R(x, x') f(x) dx \quad (3a)$$

Where $f(\cdot)$ is an arbitrary function and $f'(\cdot)$ is the transformed arbitrary function. Let us show some properties of this autocorrelation function.

Consider the inequality,

$$\left\langle \left| \int_{(A)} f(x) V(x) dx \right|^2 \right\rangle \geq 0 \quad (4)$$

Where $f(\cdot)$ is an arbitrary function. This inequality simply expresses that the average of a positive quantity is positive. The left hand side may be written

$$\int_{(A)} \int_{(A)} f^*(x') f(x) \langle V^*(x') V(x) \rangle dx' dx \quad (5)$$

$$= \int_{(A)} \int_{(A)} f^*(x') f(x) R(x, x') dx' dx$$

and therefore

$$\int_{(A)} \int_{(A)} f^*(x') f(x) R(x, x') dx' dx \geq 0 \quad (6)$$

A function $R(x, x')$ which satisfies such a relation is said to be positive semidefinite. To such a function we can apply the following two theorems, the first one is due to SCHMIDT 1907 and the second one is due to MERCER 1909

Theorem 1

"Every function $R(x, x')$ which is hermitian and square summable can be developed, in the sense of convergence in the mean, into the series

$$R(x, x') = \sum_{n=0}^{\infty} \epsilon_n \phi_n^*(x') \phi_n(x) \quad (7)$$

where $\{\phi_n(x)\}$ denotes the orthonormal sequence of eigenfunctions and $\{\epsilon_n\}$ the sequence of corresponding eigenvalues of the transformation R generated by the kernel $R(x, x')$."

Theorem 2

"If the transformation R generated by the kernel $R(x, x')$ is positive, that is if relation (6) holds for all f , or equivalently, if all the eigenvalues $\varepsilon_n \neq 0$ are positive, the expansion (7) is uniformly convergent."

The complete sequences of eigenfunctions $\{\phi_n(x)\}$ and eigenvalues $\{\varepsilon_n\}$ are found as solutions of the homogeneous Fredholm equation

$$\varepsilon_n \phi_n(x) = \int_{(A)} R(x, x') \phi_n(x) dx \quad (8)$$

where the autocorrelation function $R(x, x')$ is taken as the kernel of the integral equation. The eigenfunctions ϕ_n verify the orthonormality condition.

$$\int_{(A)} \phi_n(x) \phi_m(x) dx = \delta_{nm} \quad (9)$$

where δ_{nm} is the Kronecker symbol equal to unity when $n = m$ and zero otherwise.

Multiplying both sides of equ. (8) by $\phi_m^*(x)$ and integrating over (A), we obtain, using orthonormality relation (9)

$$\varepsilon_n \delta_{nm} = \int \int_{(A)(A)} \phi_n(x') R(x, x') \phi_m^*(x) dx' dx \quad (10)$$

By inverting x and x' in equ. (8) and taking the complex conjugate of both sides and using the hermitian relation (2) we have,

$$\varepsilon_m^* \phi_m^*(x) = \int_{(A)} R(x, x') \phi_m(x) dx \quad (11)$$

We now multiply this last relation by $\phi_n(x')$ and integrate thereby finding

$$\varepsilon_m^* \delta_{nm} = \int \int_{(A)(A)} \phi_n(x') R(x, x') \phi_m^*(x) dx' dx \quad (12)$$

The right hand sides of equs. (10) and (12) are identical and by subtracting equ. (12) from equ. (10) we find

$$(\varepsilon_n - \varepsilon_m^*) \delta_{nm} = 0 \quad (13)$$

when $n = m$ we have $\varepsilon_n = \varepsilon_m^*$, this means the eigenvalues we have obtained from the Fredholm integral equ. (8) are real. If in equ. (10) we set n equal to m , according to equ. (6) we find that all the eigenvalues different from zero are positive in accordance with the theorem 2.

This property allows us to sort them as follows,

$$\varepsilon_0 \geq \varepsilon_1 \geq \varepsilon_2 \dots \geq \varepsilon_n \geq \dots \geq 0 \quad (14)$$

Two important relations can be derived from equ. (7). The first one is found by setting n equal to m and integrating, with the orthonormality condition in equ. (9) this gives

$$\int_{(A)} R(x, x' = x) dx = \sum_{n=0}^{\infty} \varepsilon_n \quad (15)$$

The second is found by integrating the squared modulus of the autocorrelation function and again using equ. (9),

$$\int_{(A)} \int_{(A)} |R(x, x')|^2 dx dx' = \sum_{n=0}^{\infty} \varepsilon_n^2 \quad (16)$$

These last two equations will be of some importance in evaluating statistical properties of speckle patterns in the following sections.

As the eigenfunctions ϕ_n are complete and orthonormal any other function and particularly the complex amplitudes can be expanded in terms of

them , for example,

$$V_S(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \quad (17)$$

where the expansion coefficients are given by

$$a_n = \int_{(A)} \phi_n^*(x) V_S(x) dx \quad (18)$$

The advantage of using these particular eigenfunctions to write expansions resides in the fact the expansion coefficients a_n are uncorrelated. From equs. (1) and (17) we may write the autocorrelation function

$$R(x, x') = \left\langle \sum_n a_n^* \phi_n^*(x') \sum_m a_m \phi_m(x) \right\rangle \quad (19)$$

which after rearranging the terms gives

$$R(x, x') = \sum_n \sum_m \langle a_n^* a_m \rangle \phi_n^*(x') \phi_m(x) \quad (20)$$

Substituting this expansion in equ. (7) yields

$$\langle a_n^* a_m \rangle = \varepsilon_n \delta_{nm} \quad (21)$$

This equation shows the expansion coefficients a_n are uncorrelated and also gives a physical interpretation of the eigenvalues as quantities equivalent to average integrated intensities. Expansions of the type given by equ. (17) where the expansion coefficients are uncorrelated are called generalized Fourier expansions or Karhunen-Loève expansions (DAVENPORT and ROOT 1958 , Loève 1963).

One can, of course, write expansions similar to equ. (17) in terms of any complete orthonormal set of functions but the expansion coefficients are then correlated and do not lead to an expression of the type of equ. (21). This

argument tends to show that for a given autocorrelation function the set of eigenvalues and eigenfunctions are unique. A further remark is that nothing is implied about the stationarity of the speckle field, all that is required is that the autocorrelation function must be hermitian and square integrable. However, if the scanned speckle pattern is stationary, equ. (15) takes on an interesting form; the autocorrelation function of a stationary random variable depends only on the difference of its arguments (see e.g. THOMAS 1969), we have,

$$R(x, x') = R(x - x') \quad (22)$$

Inserting this expression in equ. (15), gives,

$$\sum_{n=0}^{\infty} \epsilon_n = R(0) A. \quad (23)$$

where A is the area of the scanning aperture.

Let us now examine how we can express the exact probability density function with the above arguments in view.

3.3. The Exact Probability Density Function in Terms of Eigenvalues.

In section 2.1 we make the hypothesis that the complex amplitude is Gaussianly distributed (equ. 2.1 (1)), and this is supported by the central limit theorem. In equ. (17) we write an expansion of the scanned complex amplitude in terms of a special set of eigenfunctions obtained by the homogeneous Fredholm integral equation (equ. (11)). We have shown that the corresponding expansion coefficients are uncorrelated.

Since these coefficients are linear functions of $V_S(x)$ (equ. (18)) they are distributed according to a multivariate Gaussian distribution (MEHTA 1970). From this and equ. (21) we can write the joint probability density function of the coefficients as,

$$P(\{a_n\}) = \prod_n \frac{1}{\pi \epsilon_n} e^{-\frac{|a_n|^2}{\epsilon_n}} \quad (24)$$

The intensity E' integrated over the whole scanning aperture can be derived from equ. (17) and the orthonormality condition expressed in equ. (9),

$$E' = \int_{(A)} I(x) dx = \int_{(A)} |V_S(x)|^2 dx = \sum_{n=0}^{\infty} |a_n|^2 \quad (25)$$

We can now proceed to evaluate the characteristic functions (i.e. the Fourier Transform) of the integrated intensity. From eqs. (A2) and (24) we may write

$$C_{E'}(z) = \langle e^{izE'} \rangle = \int_{(A)} [e^{iz \sum_n |a_n|^2} \prod_l \frac{1}{\pi \epsilon_l} e^{-\frac{|a_l|^2}{\epsilon_l}} \times \prod_j d^2 a_j] \quad (26)$$

By $d^2 a_j$ we mean $d a_j^* d a_j$ taking account of the fact that the correlation coefficients are complex. Rearranging equ. (26) gives us

$$C_{E'}(z) = \prod_n \frac{1}{\pi \varepsilon_n} \int_{-\infty}^{+\infty} \exp \left[- \sum_l \left\{ |a_l|^2 \left(\frac{1}{\varepsilon_l} - iz \right) \right\} \right] d^2 a_l \quad (27)$$

Integration is performed over the whole infinite range and yields,

$$C_{E'}(z) = \prod_n \frac{1}{\pi \varepsilon_n} \prod_l \frac{\pi}{\left(\frac{1}{\varepsilon_l} - iz \right)} \quad (28)$$

more neatly,

$$C_{E'}(z) = \prod_n \left(1 - iz \varepsilon_n \right)^{-1} \quad (29)$$

We must observe that because the coefficients a_n are complex and not real the characteristic function expressed by the above equation is not

$$C_{E'}(z) = \prod_n \left(1 - iz \varepsilon_n \right)^{-1/2} \quad (30)$$

as in DAINTY 1971 ; this point has also been mentioned by BARAKAT 1973 .

The required probability density function is the inverse Fourier transform of equ. (29)

$$P(E') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-izE'}}{\prod_n (1 - iz \varepsilon_n)} dz \quad (31)$$

This Fourier transform represents the second essential step in the determination of the probability density functions, the first one being the resolution of the Fredholm integral equation to find the eigenvalues ε_n . To perform the integration in equ. (31) we must generally expand the integrand into a series of partial fractions. This could not be readily done with Dainty's expression

(equ. (30)) because of the square root in the denominator. With the eigenvalues sorted as in equ. (14) there are basically three types of sets of eigenvalues. These are illustrated in fig. 5. In the first case (fig. (5a)) we have an infinite denumerable set of eigenvalues that are all different and tend to zero with increasing index fairly rapidly such that for large indices they can be neglected. In the second case, (fig. (5b)), we also have a denumerable infinity of eigenvalues but the first one is degenerate, i.e. several eigenvalues are equal to the first one and they then drop off smoothly to negligible values. In the third case all the eigenvalues are identical to the first one up to a cut off index and zero afterwards (fig. 5(c)). This case is unlikely as it can be shown (see e.g. RIESZ & NAGY 1955) that the eigenvalues given by an integral equation of the Fredholm type always come in a denumerable infinity. However it is a most interesting case because it leads to the Gamma distribution. As in the two other cases the eigenvalues tend to zero with increasing index this case can be considered a good approximation to the two others. We shall call this special set the "equivalent set". It will be defined by the following rule

$$n_0 \epsilon_0 = \sum_{n=0}^{\infty} \epsilon_n \quad (32)$$

Where the series on the right hand of this equation is made from the true sequence of eigenvalues generated by the Fredholm integral equation, ϵ_0 is the first value of that sequence and n_0 defines the cut off index of the equivalent set. As we will show in the next pages the equivalent set leads to the Gamma distribution and the true set leads to the exact distribution with equ. (32) we thus have a convenient way of comparing the two distributions. We now give the probability density function obtained with each case.

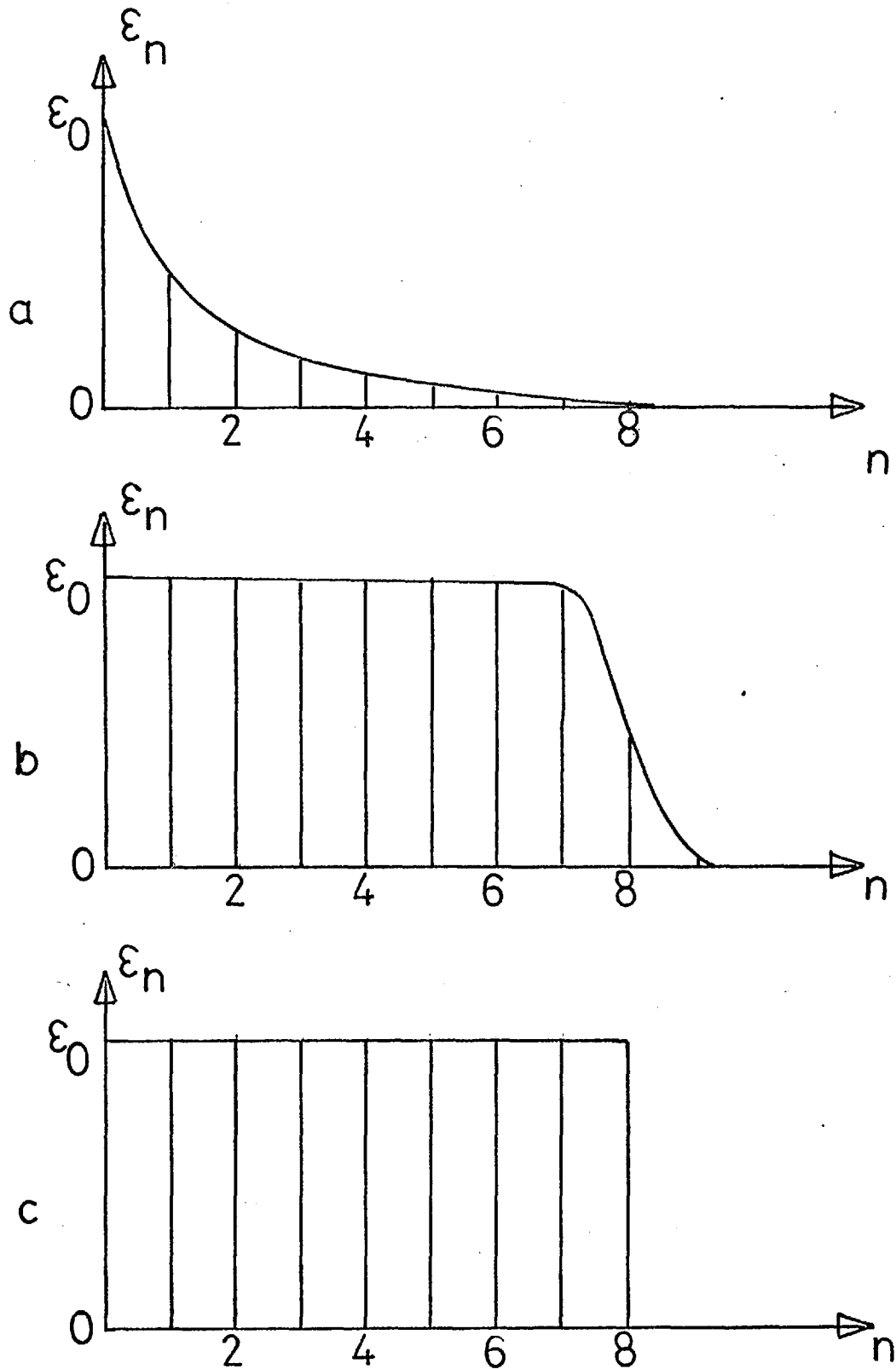


FIG.5 DIFFERENT EIGENVALUE BEHAVIOURS, a) LEADS TO EQN. 3.3. (35)

b) LEADS TO EQN. 3.3. (45),

c) LEADS TO THE GAMMA DISTRIBUTION IN EQN. 3.3. (51).

INDICATES THE ENVELOPE OF THE EIGENVALUES

INDICATES THE FIRST EIGENVALUE, n IS THE INDEX

Probability density function when all the eigenvalues are different (fig. 5a)

The characteristic function in equ. (29) is expanded into partial fractions with coefficients G_n such that

$$G_n = \prod_{m \neq n} \frac{\varepsilon_n}{\varepsilon_n - \varepsilon_m} \quad (33)$$

and becomes

$$C_{\varepsilon'}(z) = \sum_n \frac{G_n}{(1 - iz \varepsilon_n)} \quad (34)$$

Taking the Fourier transform of this expression (which is equivalent to solving equ. (31)) gives

$$P(\varepsilon') = \sum_{n=0}^{\infty} \frac{G_n}{\varepsilon_n} e^{-\frac{\varepsilon'}{\varepsilon_n}} \quad (35)$$

which is the required exact probability density function of integrated intensity.

In general the eigenvalues will all be different as on fig. 5a) and this formula will be the one most frequently used for the exact distribution. The average and variance of this distribution are found by using eqs. A(14) and A(17) of appendix A and are

$$\langle \varepsilon' \rangle = \sum_{n=0}^{\infty} \varepsilon_n \quad (36)$$

and

$$\sigma_{\varepsilon'}^2 = \sum_{n=0}^{\infty} \varepsilon_n^2 \quad (37)$$

Probability density function when the first eigenvalue is degenerate and the others drop off to zero (fig. 5b)

The condition for the behaviour of the eigenvalues can be written

$$\epsilon_n = \epsilon_0 \quad \text{for} \quad n \leq n_0$$

and

$$\epsilon_n > \epsilon_{n+1} > \epsilon_{n+2} \dots > 0$$

for $n > n_0$

In view of this we can rewrite the characteristic function in equ. (29) as

$$C_{E'}(z) = \frac{1}{(1 - iz \epsilon_0)^{n_0}} \times \frac{1}{\prod_{n > n_0} (1 - iz \epsilon_n)} \quad (38)$$

We can expand the right-most part of this expression in partial fractions, this gives,

$$C_{E'}(z) = \frac{1}{(1 - iz \epsilon_0)^{n_0}} \sum_{n > n_0}^{\infty} \frac{G_n}{(1 - iz \epsilon_n)} \quad (39)$$

with the expansion coefficients given by

$$G_n = \prod_{\substack{m \neq n, \\ m > n_0}} \frac{\epsilon_n}{\epsilon_n - \epsilon_m} \quad (40)$$

This equation is essentially the same as equ. (33) but with a different set of indices. The Fourier transform integral of equ. (39) is

$$P(E') = \sum_{n > n_0}^{\infty} \frac{G_n}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iz E'}}{(1 - iz \epsilon_0)^{n_0} (1 - iz \epsilon_n)} dz \quad (41)$$

The solution of this is found as (ERDELYI 1953)

$$P(E') = \sum_{n > n_0}^{\infty} \left\{ G_n \left(\frac{1}{\varepsilon_0} \right)^{n_0} \frac{1}{\varepsilon_n} \frac{E'^{n_0} e^{-\frac{E'}{\varepsilon_0}}}{{}_1F_1 \left[1; (n_0 + 1); \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_n} \right) E' \right]} \right\} \quad (42)$$

where ${}_1F_1$ is one of the hypergeometric functions, defined by (ABRAMOWITZ and STEGUN 1970),

$${}_1F_1 [a; b; z] = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \quad (43)$$

with

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (44)$$

Inserting these definitions in equ. (42) gives the final expression for the probability density function,

$$P(E') = \left(\frac{1}{\varepsilon_0} \right)^{n_0} E'^{n_0} e^{-\frac{E'}{\varepsilon_0}} \times \sum_{n > n_0}^{\infty} \frac{G_n}{\varepsilon_n} \sum_{k=0}^{\infty} \frac{\left[\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_n} \right) E' \right]^k}{\Gamma(n_0 + k + 1)} \quad (45)$$

The mean and variance are (from equs. A(14) and A(17) of appendix A)

$$\langle E' \rangle = n_0 \varepsilon_0 + \sum_{n > n_0}^{\infty} \varepsilon_n \quad (46)$$

and

$$\sigma_{E'}^2 = n_0 \varepsilon_0^2 + \sum_{n > n_0}^{\infty} \varepsilon_n^2 \quad (47)$$

BARAKAT 1973 also considers a similar set of eigenvalues and states the following

"It is left as an exercise to the reader to prove that the appropriate expression for

$P(E')$ is now"

$$P(E') = \sum_{n=0}^{\infty} \frac{1}{\Gamma(N_n)} \prod_{n=0}^{\infty} \left[\frac{(-1)^{N_n}}{(2\varepsilon_n)^{N_{n-1}}} \right] \times \frac{d^{N_{n-1}}}{dt^{N_{n-1}}} G_t \Big|_{t = (i2\varepsilon_n)} \quad (48)$$

where $N_0 = N, N_1 = N_2 = \dots = 1$, and

$$G_t = \exp(-iE't) \prod_{l=1}^{\infty} \left(t - \frac{i}{2\varepsilon_l} \right)^{-N_l} \quad (49)$$

Barakat continues by saying "This expression is not nearly as formidable as it looks; however we leave it ...".

Probability density function for the equivalent set of eigenvalues (fig. 5c)

All the eigenvalues are identical to the first one ε_0 up to the cut off index given by equ. (32) and zero afterwards. In this case the characteristic function in equ. (29) can be written straight-forwardly as

$$C_{E'}(z) = (1 - iz\varepsilon_0)^{-n_0} \quad (50)$$

This is the same as equ. 2.2 (17) and confirms the interpretation of equ. (21) of the eigenvalues as average integrated intensities. The Fourier transform of equ. (50) accordingly gives the Gamma distribution,

$$P(E') = \left(\frac{1}{\varepsilon_0} \right)^{n_0} \frac{E'^{n_0-1} e^{-\frac{E'}{\varepsilon_0}}}{\Gamma(n_0)} \quad (51)$$

with average integrated intensity and variance

$$\langle E' \rangle = n_0 \varepsilon_0 \quad (52)$$

and

$$\sigma_{E'}^2 = n_0 \varepsilon_0^2 \quad (53)$$

Scaling equ. (51) to the average integrated intensity gives

$$P(E') = \left(\frac{n_0}{\langle E' \rangle} \right)^{n_0} \frac{E'^{n_0-1} e^{-\frac{E' n_0}{\langle E' \rangle}}}{\Gamma(n_0)} \quad (54)$$

which clearly is identical to the distribution in equ. 2.2.(21). Although we will find no existing set of eigenvalues leading to this distribution from the Fredholm integral equation, it has been shown experimentally that the Gamma distribution is a good approximation (CONDIE 1966 , McKECHNIE 1974) and we can therefore induce the eigenvalues we will find will very nearly behave like the equivalent set.

The advantage of this derivation of the Gamma distribution is that we can now express its properties with the aid of eigenvalues, this is made easy by using the two equs. (15) and (16) derived from the theorems by SCHMIDT and MERCER and by using the equivalent set of eigenvalues (equ. (32)). Let us derive an expression for n_0 the cut off index and also the number of independent correlation cells.

From equs. (15) and (32) we find the first order statistical expression,

$$\int_{(A)} R(x, x) dx = n_0 \varepsilon_0 \quad (55)$$

From equs. (16) and (32) we find the second order statistical expression,

$$\int_{(A)} \int_{(A)} |R(x, x')|^2 dx dx' = n_0 \varepsilon_0^2 = \sigma_{E'}^2 \quad (56)$$

We immediately find from these two equations

$$n_0 = \frac{\langle E' \rangle}{\sigma_{E'}^2} = \frac{\left\{ \int_{(A)} R(x, x) dx \right\}^2}{\int_{(A)} \int_{(A)} |R(x, x')|^2 dx' dx} \quad (57)$$

This result is also given in the photon counting context by BURES et al 1972 .

If the speckle pattern under consideration is stationary, from equ. (23) we find

$$n_0 = \frac{\langle E' \rangle}{\sigma_{E'}^2} = \frac{R(0)^2 A^2}{\int_{(A)} \int_{(A)} |R(x, x')|^2 dx' dx} \quad (58)$$

From the Wiener-Kintchine theorem (see e.g. LAWSON and UHLENBECK 1950) the amplitude autocorrelation function is the Fourier transform of the amplitude power spectrum. With the white noise assumption, from equ. 2.3.(24) the scanned amplitude power spectrum is

$$\Pi_{|V|, S} = \text{cst} \times |P(u)|^2 \quad (59)$$

where $P(u)$ is the pupil function of P (see fig. 1). If $\text{psf}(x)$ (point spread function) is the Fourier transform of $P(u)$, by taking the Fourier transform of equ. (59) we find the expression of the autocorrelation function

$$R(x - x') = \text{cst} \times \int_{-\infty}^{+\infty} \text{psf}(x - x' - s) \text{psf}(s) ds \quad (60)$$

where $\text{psf}(x)$ is an even function. Introducing this equation in equ. (58) gives

$$n_0 = \frac{\left\{ \int_{-\infty}^{+\infty} \text{psf}^2(s) ds \right\}^2 A^2}{\int_{(A)} \int_{(A)} \left\{ \int_{-\infty}^{+\infty} \text{psf}(x - x' - s) \text{psf}(s) ds \right\}^2 dx' dx} \quad (61)$$

Again we see that if the scattered light is treated as stationary white noise the number of independent correlation cells depends only on the geometrical configuration of the optical set-up used to measure the statistics of the speckle pattern.

This equation for the number of independent correlation cells is expressed with integrals in the "hardware" space whereas equation 2.3 (29) is expressed with integrals in the Fourier space. This is perhaps the essential difference between the two relations.

The quantity n_0 is generally calculated in a much easier fashion, from eqs. (32) and (15) we have

$$n_0 = \frac{\sum_n \epsilon_n}{\epsilon_0} = \frac{\int_{(A)} R(x, x) dx}{\epsilon_0} \quad (61a)$$

For stationary kernels (equ. (23)) this simply is

$$n_0 = \frac{R(0) A}{\epsilon_0} \quad (61b)$$

This formula, which is much simpler than either equ. (61) or equ. 2.3 (29), we shall generally use to calculate the number of independent correlation cells.

ACKASU 1971 has shown it is equivalent to reason directly on Fourier space or "hardware" space in order to evaluate noise statistics. As an example for slit aperture geometry equ. (61) gives

$$n_0 = \frac{A^2}{\int_{-\frac{A}{2}}^{+\frac{A}{2}} \int_{-\frac{A}{2}}^{+\frac{A}{2}} \left\{ \int_{-\infty}^{+\infty} \frac{\sin\left(\frac{\pi U}{\lambda D} (x - x' - s)\right) \sin\left(\frac{\pi U}{\lambda D} s\right)}{\frac{\pi U}{\lambda D} (x - x' - s) \frac{\pi U}{\lambda D} s} ds \right\}^2 dx dx'} \quad (62)$$

where the scanning aperture is a slit of width A and the pupil P is a slit of width $2U$.

3.4. On Solving the Homogeneous Fredholm Integral Equation

In the preceding section we have seen the general method to evaluate the first order probability density function $P(E')$ of speckle patterns contains two essential steps. The first was to find the eigenvalues defined by the homogeneous Fredholm integral equation.

$$\epsilon_n \phi_n(x) = \int_{(\Lambda)} R(x, x') \phi_n(x') dx' \quad (63)$$

The second step was to insert these eigenvalues into the Fourier transform integral

$$P(E') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-izE'}}{\prod_n (1 - iz\epsilon_n)} dz \quad (64)$$

and to perform integration. A word on how to solve the Fredholm homogeneous integral equation is therefore in order.

We consider essentially two methods. The first consists in finding the differential equation associated to the Fredholm integral equation and then solving this differential equation. It is generally easier to find an integral equation from a given differential equation. Doing the contrary is generally difficult except for some obvious cases. The advantage of this technique is that exact or closed expressions can be found for the eigenfunctions and a great accuracy in the eigenvalues can be obtained for fairly little numerical manipulation and computing. We illustrate this method completely for the simple case of the slit collecting pupil with a Lorentzian intensity transmittance profile and a slit scanning aperture with a rectangular intensity transmittance profile in section 4. SLEPIAN 1961 also uses this method to calculate the Prolate Spheroidal Wave Functions and their associated eigenvalues. In section 4 we also use SLEPIAN'S results in the case of the slit and rectangular unit intensity transmittance apertures. Good accounts

of the relations between differential equations and integral equations can be found in COURANT and HILBERT 1953 and FRÖBERG 1970 .

The second method we consider to solve the integral equation consists in writing the integral in the Riemann sense as the limit of a series of N terms with N tending to infinity, (see e.g. RIESZ and NAGY 1955). This leads to a matrix formulation of the problem and as the kernel of the integral equation is Hermitian, all we have to do to find the eigenvalues is diagonalize a Hermitian matrix version of the kernel. As shown in paragraph 3.3, the eigenvalues will be real positive. This technique is conceptually much simpler but requires much more computing. However, as large computing facilities are available (a C.D.C. 6400 series computer) this method is worth using. We shall use it notably to solve the case of a two dimensional circular unit transmittance collecting pupil in section 4.

A brief account of this method is as follows, (a more detailed account will be given for the circular case in section 4.2.4). As the limit of a sum of N terms with N tending to infinity the Fredholm homogeneous integral equation is written as

$$\varepsilon \phi(x) = \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} R(x, nh) \phi(nh) h \quad (65)$$

where h is the elementary summing interval, n is the dummy index, as the integral in the Fredholm equation is defined over the domain (A) we must have a boundary relation, for example, when $(A) = \left[-\frac{A}{2}, +\frac{A}{2} \right]$, this is,

$$Nh = \frac{A}{2} \quad (66)$$

Writing the Fredholm equation as in equ. (65) is physically equivalent to filling

the scanning aperture with $2N$ points a distance h apart, each point being indexed n . That means we are effectively sampling the speckle pattern at $2N$ points. Mathematically we need an infinity of such points but physically if we take only a finite though large number of sampling points we will obtain satisfactory results. That is equivalent to truncating the series in equ. (65) at N large. Clearly the precision of the eigenvalues obtained by this method will depend on the value of N (see section 3.5). In practice a relative precision of 10^{-3} is found to yield satisfactory results and N is chosen accordingly.

In equ. (65) we observe that the value of the quantity ε is independent of the argument n . We can therefore set it freely to some value $m h$ such that $-N \leq m \leq +N$. We can now write our integral equation in the form

$$\varepsilon \phi(mh) = \sum_{n=-N}^{+N} R(h, m, n) \phi(nh) \quad (67)$$

Matrix formulation of this equation is now evident, with obvious notation we have

$$[\varepsilon \delta_{nm}] [\phi(n)] = h [R_{n,m}] [\phi(n)] \quad (68)$$

where δ_{nm} is the KRONECKER delta. $[\varepsilon \delta_{nm}]$ and $[R_{n,m}]$ are two $2N \times 2N$ square matrices and $[\phi(n)]$ is a $2N$ column matrix. The eigenvalues are now seen to be the solution of $2N$ independent linear equations and must satisfy the condition

$$\det [h R_{n,m} - \varepsilon \delta_{n,m}] = 0 \quad (69)$$

The eigenfunctions are then found at $2N$ points at the rate of one eigenfunction per eigenvalue by solving the set of equations

$$\begin{aligned}
 [h R_{n,m} - \varepsilon_0 \delta_{nm}] [\phi_{0,n}] &= 0 \\
 [h R_{n,m} - \varepsilon_1 \delta_{nm}] [\phi_{1,n}] &= 0 \\
 &\vdots \\
 [h R_{n,m} - \varepsilon_{2N} \delta_{nm}] [\phi_{2N,n}] &= 0
 \end{aligned}
 \tag{70}$$

where the indices $0, 1 \dots 2N$ correspond to the first, the second the two N^{th} eigenvalue and eigenfunction at point nh , with the notation,

$$\phi_{j,n} = \phi_j(x = nh) \tag{71}$$

3.5. Dependence of the Exact Probability Density on the Precision of the Eigenvalues

The exact probability density functions given by equs. (35) and (45) contain summations ranging to infinity. In numerical applications we cannot sum ad infinitum. Furthermore the eigenvalues are determined numerically from equ. (69) to a certain precision. The validity of the exact distribution will depend largely on this precision. It is therefore the purpose of this section to examine how the accuracy of the eigenvalues will affect the accuracy of the exact distribution and when to truncate the summations. We will limit these considerations to the exact formula in equ. (35) as this is the most widely used exact distribution.

The exact distribution is given by, (see equs. (33) and (35)),

$$P(E') = \sum_{n=0}^{\infty} \frac{G_n}{E_n} e^{-\frac{E'}{E_n}} = \sum_{n=0}^{\infty} P_n \quad (72)$$

where each of the summands is termed P_n for convenience. If we denote by ΔP and $\Delta |P_n|$ the absolute errors on P and P_n , we have

$$\Delta P = \sum_{n=0}^{\infty} \Delta |P_n| \quad (73)$$

In turn we can evaluate the $\Delta |P_n|$'s by logarithmic differentiation,

$$\frac{\Delta |P_n|}{|P_n|} = \Delta \log |P_n| \quad (74)$$

after differentiating and optimizing the partial errors we obtain a final expression for the complete absolute error on the probability density,

$$\Delta P(E') = \sum_{n=0}^{\infty} \text{err}(n) P_n \Delta E_n \quad (75)$$

with

$$\text{err}(n) = \frac{E'}{\epsilon_n^2} + \frac{1}{\epsilon_n} + \sum_{\substack{m \neq n \\ m=0}}^{\infty} \frac{\epsilon_m}{\epsilon_n^2 - \epsilon_n \epsilon_m} \quad (76)$$

The relative error on $P(E')$ is then given by

$$\frac{\Delta P(E')}{P(E')} = \frac{\sum_{n=0}^{\infty} \text{err}(n) P_n \Delta \epsilon_n}{\sum_{n=0}^{\infty} P_n} \quad (77)$$

Where $\Delta \epsilon_n$ is the absolute error made upon calculating the n^{th} eigenvalue these expressions are calculated by introducing an arbitrary $\Delta \epsilon_n$. Once a tolerable $\Delta \epsilon_n$ is found, the eigenvalues are calculated to the corresponding required relative precision. From previous consideration (equ. 14) and from experience (see section 4) we know the eigenvalues always decrease to zero, the summations are therefore truncated when the eigenvalues are less than the absolute error found to be tolerable.

4. APPLICATIONS

4.1 INTRODUCTION

In this section we examine the application of the theory in section 3 using an eigen-value integral equation and a Fourier transform integral to find some first order probability density functions of monochromatic coherent speckle patterns scanned with finite apertures. Such speckle patterns are known to be stationary (DAINTY 1972) and the Gamma distribution is known to give a good description, (McKECHNIE 1974). Such speckle patterns will provide a good testing ground for the theory.

We will examine four cases in all defined by the shapes of the diffusing pupil and scanning aperture.

- 1) A slit (one dimensional) collecting pupil with a Lorentzian intensity transmittance profile and a slit scanning aperture with a rectangular intensity transmittance profile.
- 2) A slit (one dimensional) collecting pupil with a rectangular intensity transmittance profile and a slit scanning aperture with a rectangular intensity transmittance profile.
- 3) A Rectangular (two dimensional) collecting pupil with a rectangular intensity transmittance profile and a rectangular scanning aperture with a rectangular intensity transmittance profile.
- 4) A Circular (two dimensional) collecting pupil with a "top hat" intensity transmittance profile and a circular scanning aperture with a "top hat" intensity transmittance profile.

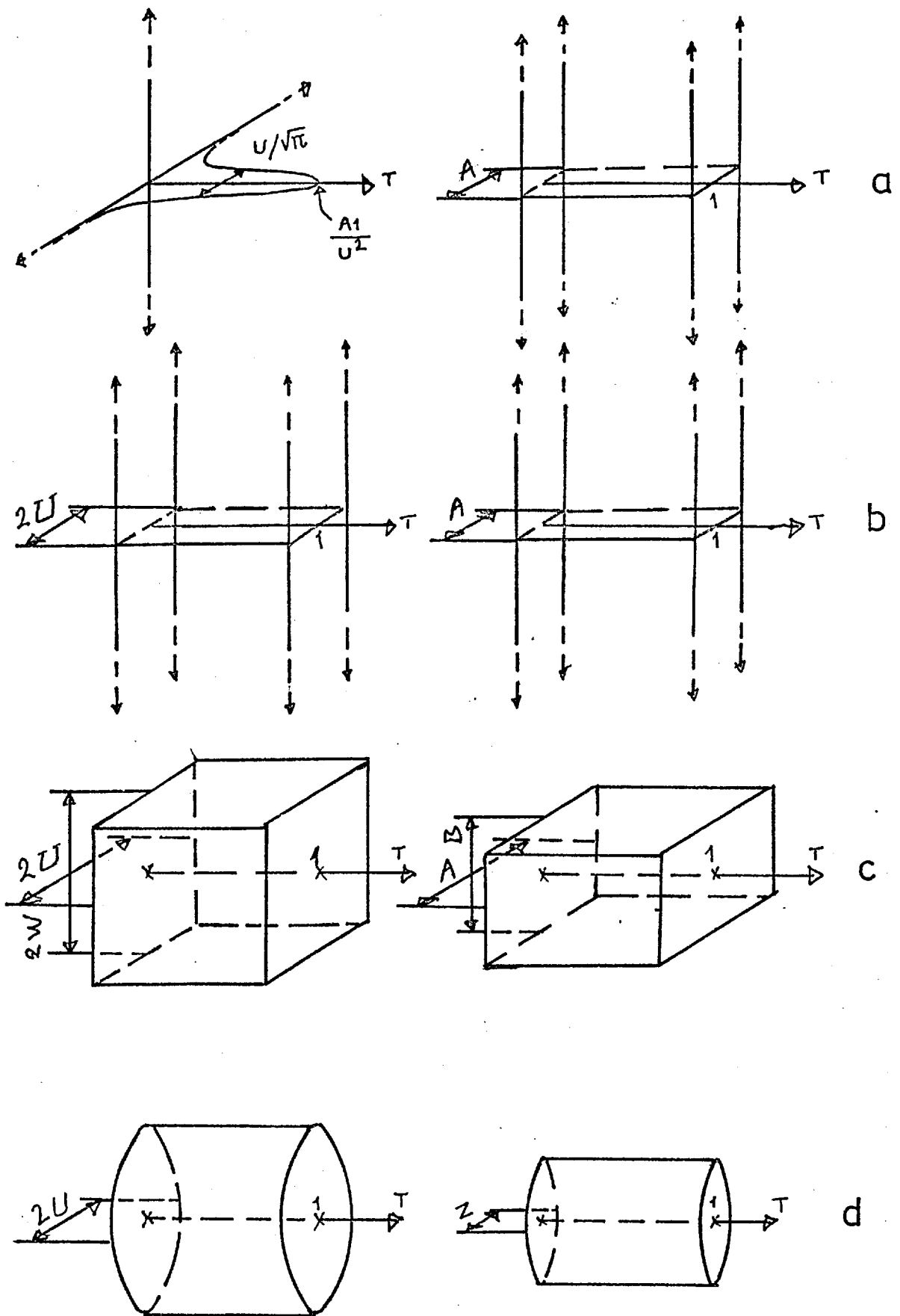
These four cases are illustrated for more clarity in fig. 6.

For most cases we give tables of the eigen-values, comparative plots of the exact distribution and approximate Gamma distribution found with the equivalent set of eigen-values defined by equ 3.2 (32), and also a plot of the integrated mean square difference between the exact and approximate distributions as a function of relative aperture (the relative aperture is proportional to the size of the collecting pupil times the size of the scanning aperture.)

The first case is essentially to illustrate the differential equation approach to solving the Fredholm integral equation. Mathematically it is quite simple and the mechanisms involved are quite clear. The second case is physically the simplest conceivable. The corresponding integral equation has been extensively studied in the literature. The eigen-values are found also by the equivalent differential equation approach, but the analysis is much more complicated and we use scaled versions of the eigenvalues tabulated by SLEPIAN and SONNENBLICK 1966 . The third case is a straightforward extension of the second.

The fourth case is perhaps the most important as it deals with circular systems and these are by far the most significant in optics as most optical instruments have circular apertures. The eigenvalues are found by the direct matrix approach.

Finally we give the results of experimental measurements made for the fourth case and compare the experimentally obtained distribution with the approximate Gamma distribution and the exact distribution.



T is transmittance

FIG 6

4.2 1) A slit (one dimensional) collecting pupil with a lorentzian intensity transmittance profile and a slit scanning aperture with a rectangular transmittance profile.

The collecting pupil P is a slit with a lorentzian intensity transmittance profile of half width $\frac{U}{2\sqrt{\pi}}$ (see fig. 6 a) and the scanning aperture is a slit of width A. Similar problems have been treated in communication theory in the noise context by SLEPIAN 1958 and in photon counting statistics by BEDARD 1966 , BEDARD et al 1967 JAISWAL and MEHTA 1969 and MEHTA 1970 .

The Autocorrelation function and the Fredholm integral equation.

The kernel of the Fredholm integral equation which is the autocorrelation function of the scanned speckle pattern is found either directly from equ. 3.3. (60) or by making use of the Wiener-Kintchine theorem (see e.g. LAWSON and UHLEHBECK 1950) stating that autocorrelation functions and power spectra are Fourier pairs. The latter approach is perhaps the easiest and the one we are using. The constant in equ. 3.3. (59) is found from equ. 2.3 (27), this gives the amplitude power spectrum as

$$\pi_{|V|,S} = \frac{|P(u)|^2 \langle E' \rangle}{\int_{-\infty}^{+\infty} |P(u)|^2 du} \quad (1)$$

where $|P(u)|^2$ is the lorentzian intensity transmittance function defined by

$$|P(u)|^2 = \frac{A_1}{4\pi u^2 + U^2} \quad (2)$$

The maximum transmittance is found for $u = 0$, and is A_1 / U^2 where A_1 is a constant and U is a parameter characterizing the spread of the function. Taking the Fourier transform of $\Pi_{|u|,S}$ with $|P(u)|^2$ defined as in equ. (2) gives us the autocorrelation function as

$$R(x - x') = \langle E' \rangle e^{-\frac{U}{\lambda D} |x - x'|} \quad (3)$$

The required eigenvalues that allow us to compute our probability density function are found by solving the integral equation,

$$\varepsilon_n \phi_n(x) = \int_{(A)} \langle E' \rangle e^{-\frac{U}{\lambda D} |x - x'|} \phi_n(x') dx' \quad (4)$$

where (A) is the domain of definition of the slit scanning aperture, i.e.

$(A) = \left[-\frac{A}{2}, +\frac{A}{2} \right]$. We now proceed to show how to solve this integral equation by the equivalent differential equation approach.

Solution of the integral equation.

First let us scale equ. (4). The kernel is an even function and we therefore consider only half the integration domain (A) . Let us introduce the dimensionless variable X and X' such that

$$X = \frac{A}{2x} \quad \text{and} \quad X' = \frac{A}{2x'} \quad (5)$$

The scaled integral equation we now consider is

$$\Lambda_n \phi_n(X) = \int_0^1 e^{-c|X - X'|} \phi_n(X') dX' \quad (6)$$

which will be easier to manipulate. The quantities Λ_n are scaled versions of

the eigenvalues ε_n such that,

$$\varepsilon_n = \langle E' \rangle \Lambda_n \quad (7)$$

From equ. (6), the $\{\Lambda_n\}$ are seen to be dimensionless quantities. The parameter c defines a measure of the relative aperture and is the analogue of the space-bandwidth product in electronic communication theory,

$$c = \frac{UA}{\lambda D} \quad (8)$$

where λ is the wavelength and D is the distance between the pupil P and the scanning aperture S as on fig. (1).

Let us differentiate equ. (6) twice with respect to X and use the relations

$$\frac{d}{dX} |X - X'| = \varepsilon(X - X') \quad (9)$$

and

$$\frac{d}{dX} \varepsilon(X - X') = 2 \delta(X - X') \quad (10)$$

where $\varepsilon(X)$ is $+1$ if X is positive and -1 if X

is negative. This quantity is not to be confused with the set of eigenvalues

$\{\varepsilon_n\}$. The quantity $\delta(X)$ is the usual Dirac delta function. This gives us the differential equation

$$\frac{d^2 \phi(X)}{dX^2} + \omega^2 \phi(X) = 0 \quad (11)$$

$$\text{where } \omega^2 = \left(2c/\Lambda\right) - c^2 \quad (12)$$

From equ. (11), the eigenfunctions must be of the form

$$\phi(X) = c_1 e^{+i\omega X} + c_2 e^{-i\omega X} \quad (13)$$

where C_1 and C_2 are two integration constants. Inserting equ. (12) and (13) in equ. (4) and evaluating the integral, we find that

$$e^{-c(X-1)} \left[\frac{C_1 e^{+i\omega}}{c - i\omega} + \frac{C_2 e^{-i\omega}}{c + i\omega} \right] + e^{-cX} \left[\frac{C_1}{c + i\omega} + \frac{C_2}{c - i\omega} \right] \quad (14)$$

This equation is verified if the terms in both brackets are nil,

$$\frac{C_1 e^{+i\omega}}{c - i\omega} + \frac{C_2 e^{-i\omega}}{c + i\omega} = 0 \quad (15)$$

$$\frac{C_1}{c + i\omega} + \frac{C_2}{c - i\omega} = 0 \quad (16)$$

By eliminating C_1 and C_2 from these equations we find ω is a set of solutions $\{\omega_k\}$ given by the roots of the transcendental equation

$$\frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right) \sin \omega - \cos \omega = 0 \quad (17)$$

Figures (7) and (8) show graphic solutions of this equation. The eigenvalues are then found from equ. (12) and are given by,

$$\Lambda_k = \frac{2c}{\omega_k^2 + c^2} \quad (18)$$

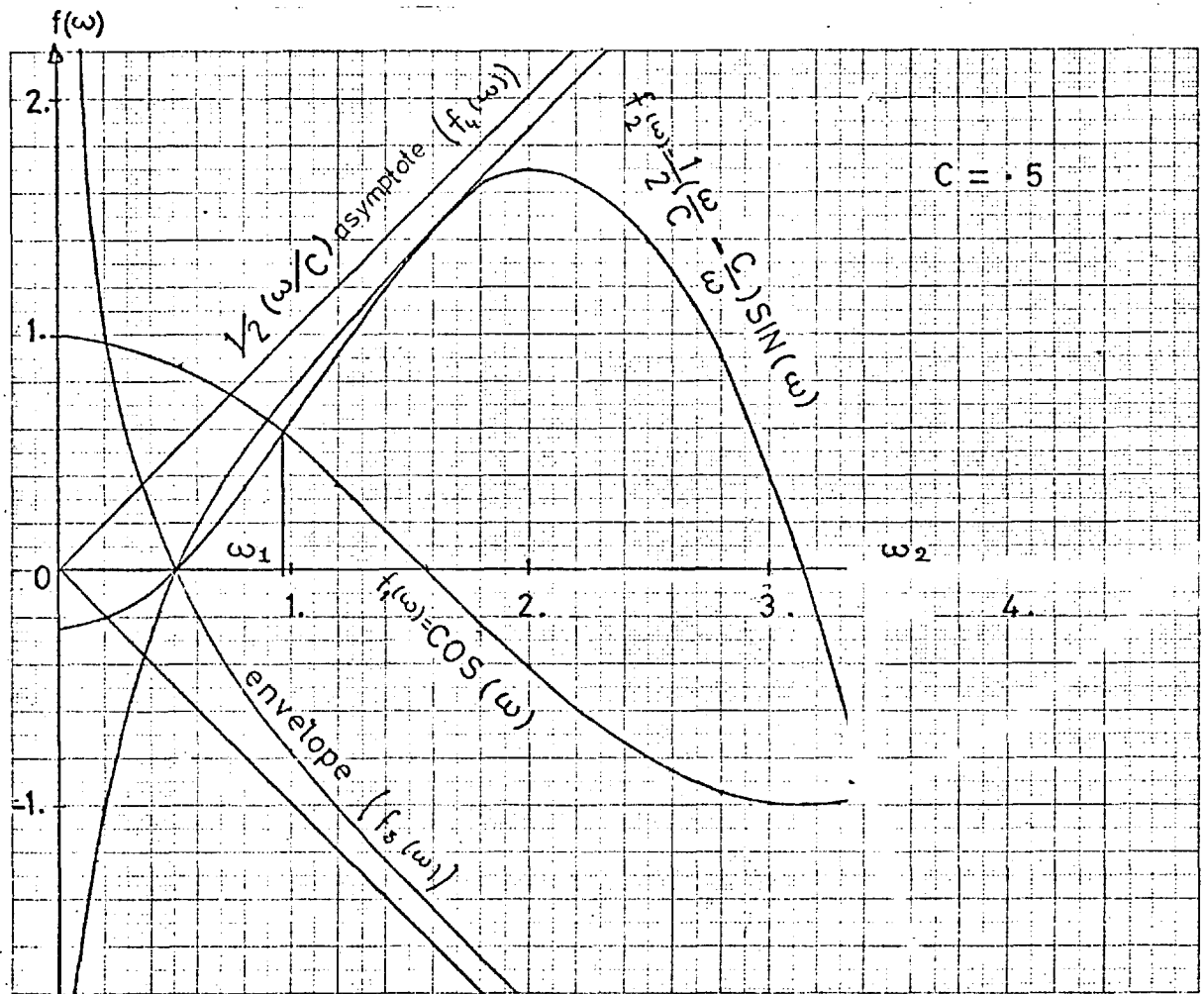


FIG. 7 GRAPHICAL SOLUTION OF THE TRANSCENDENTAL EQUATION

$$\frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right) \sin \omega - \cos \omega = 0 \quad \text{FOR } c = 0.5$$

THE QUANTITIES ω_k ARE FOUND AT THE INTERSECTION OF THE TWO CURVES OF EQUATIONS

$$f_1(\omega) = \cos \omega \quad \text{and} \quad f_2(\omega) = \frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right) \sin \omega$$

THE LATTER OSCILLATES INSIDE THE ENVELOPE

$$f_3(\omega) = \pm \frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right)$$

OF ASYMPTOTE

$$f_4(\omega) = \pm \frac{1}{2} \left(\frac{\omega}{c} \right)$$

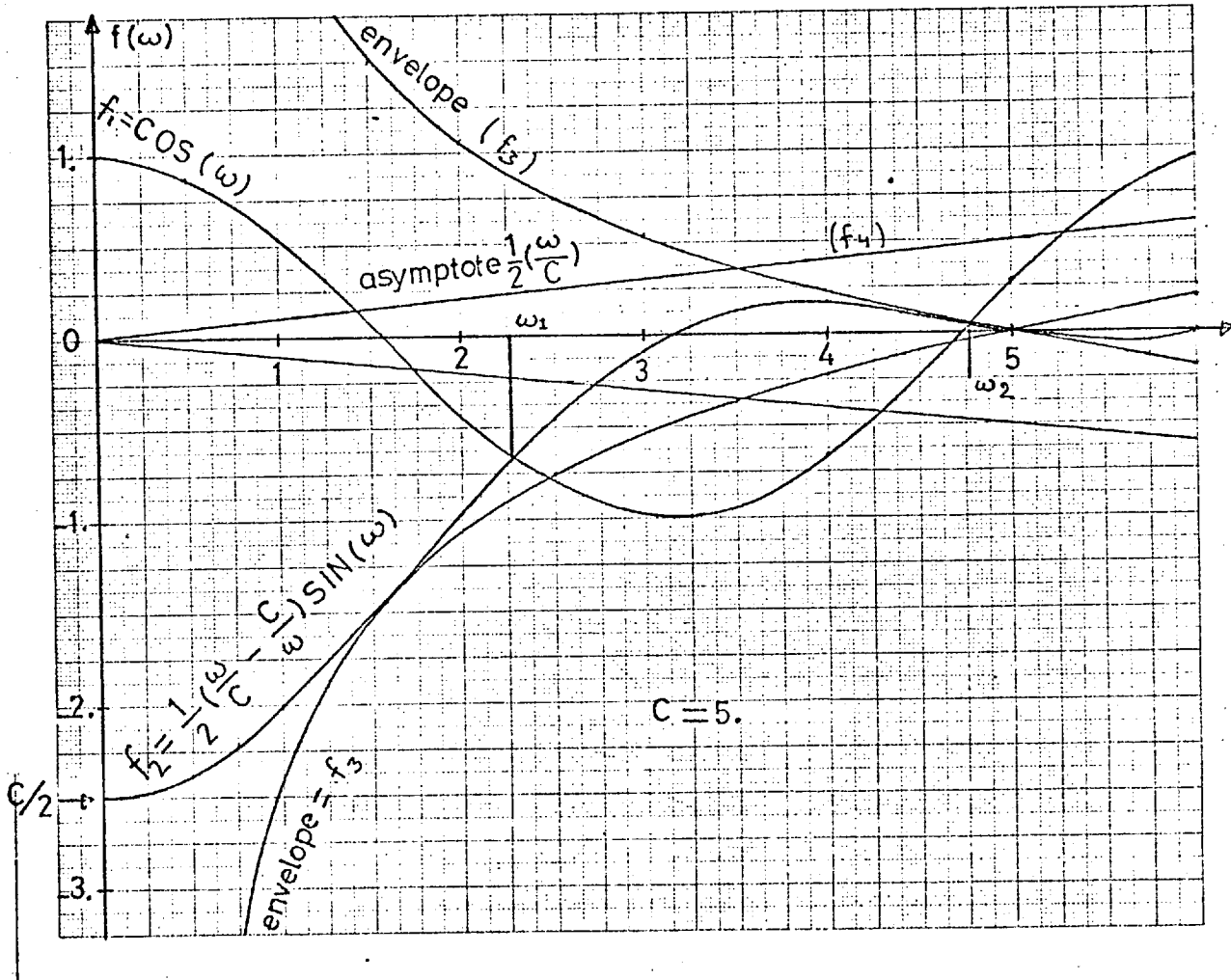


FIG. 8 GRAPHICAL SOLUTION OF THE TRANSCENDENTAL EQUATION

$$\frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right) \sin \omega - \cos \omega = 0 \quad \text{FOR } c = 5.$$

THE QUANTITIES ω_k ARE FOUND AT THE INTERSECTION OF THE TWO CURVES OF EQUATIONS

$$f_1(\omega) = \cos \omega \quad \text{and} \quad f_2(\omega) = \frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right) \sin \omega$$

THE LATTER OSCILLATES INSIDE THE ENVELOPE

$$f_3(\omega) = \pm \frac{1}{2} \left(\frac{\omega}{c} - \frac{c}{\omega} \right)$$

OF ASYMPTOTE

$$f_4(\omega) = \pm \frac{1}{2} \left(\frac{\omega}{c} \right)$$

It is seen from equ. (17) that the values of ω_k rapidly tend to the value $k\pi$ with increasing k and an asymptotic expression of Λ_k for large k will be

$$\Lambda_k = \frac{2c}{k^2\pi^2 + c^2} \quad (19)$$

Numerical Results

Table 1 shows these eigenvalues for various values of the relative aperture OMEGA is the value of ω_k found by solving equ. (17), EIGENVALUE is the value of Λ_k from equ. (18) and LIMIT-VALUE is the quantity found from equ. (19). For all c 's the eigenvalues are listed up to that which is just less than the thousandth of the first eigenvalue. All the given eigenvalues are given with a relative precision of 10^{-6} . These eigenvalues are all positive and decrease steadily to zero. They are non degenerate and obey the equation derived from Mercer's theorem for stationary kernels (equ. 3.2. (23)). The FORTRAN program called LOEIVA used to compute these eigenvalues can be found in appendix B. Table 2 shows numerical values for some essential functions of c given in Column 1, column 2 gives the first eigenvalue, column 3 gives the index N at which the ratio Λ_N / Λ_0 is less than a thousandth is reached, column 4 gives the sum of the eigenvalues up to that index with the reminder that the sum of the whole infinite range as calculated from equ. 3.2 (15) and 3.2 (23) is unity, column 5 gives the cut off index n_0 for the "equivalent set" of eigenvalues as defined by equ. 3.3. (32), column 6 gives the cut off index calculated with the first N eigenvalues from eqs. 3.3 (36), 3.3. (37) and 3.3 (58), finally column 7 gives the integrated mean square difference between the exact distribution and the corresponding Gamma distribution with n_0 taken from column 5.

LORENTZIAN EIGENVALUES.

C = 0.4

INDEX	OMEGA	EIGENVALUE	LIMIT VALUE
0	.865674E+00	.879709E+00	.500000E+01
1	.337736E+01	.691647E-01	.797638E-01
2	.640787E+01	.194077E-01	.201824E-01
3	.950879E+01	.883226E-02	.899013E-02
4	.126297E+02	.501039E-02	.506093E-02
5	.157587E+02	.321936E-02	.324018E-02
6	.188919E+02	.224050E-02	.225057E-02
7	.220275E+02	.164823E-02	.165386E-02
8	.251645E+02	.126300E-02	.126691E-02
9	.283026E+02	.998506E-03	.100050E-02
10	.314414E+02	.809127E-03	.810438E-03

C = 0.7

0	.111845E+01	.804171E+00	.285714E+01
1	.353281E+01	.107935E+00	.135140E+00
2	.649781E+01	.327780E-01	.350276E-01
3	.957080E+01	.152025E-01	.156746E-01
4	.126766E+02	.868561E-02	.883818E-02
5	.157965E+02	.559958E-02	.566274E-02
6	.189235E+02	.390420E-02	.393484E-02
7	.220546E+02	.287536E-02	.289196E-02
8	.251883E+02	.220493E-02	.221468E-02
9	.283237E+02	.174406E-02	.175016E-02
10	.314604E+02	.141379E-02	.141779E-02
11	.345980E+02	.116909E-02	.117183E-02
12	.377362E+02	.982793E-03	.984727E-03
13	.480749E+02	.838696E-03	.839100E-03
14	.440141E+02	.722495E-03	.723539E-03

C = 1.0

0	.130694E+01	.738846E+00	.200000E+01
1	.367318E+01	.138005E+00	.183999E+00
2	.658462E+01	.450885E-01	.494090E-01
3	.963168E+01	.213289E-01	.222652E-01
4	.127232E+02	.122789E-01	.125854E-01
5	.158341E+02	.794537E-02	.807297E-02

C = 1.0 (contd.)

INDEX	OMEGA	EIGENVALUE	LIMIT VALUE
6	.189549E+02	.555111E-02	.561316E-02
7	.220816E+02	.409335E-02	.412702E-02
8	.252120E+02	.314147E-02	.316128E-02
9	.283448E+02	.248623E-02	.249863E-02
10	.314794E+02	.201622E-02	.202437E-02
11	.346153E+02	.166776E-02	.167333E-02
12	.346153E+02	.140231E-02	.140625E-02
13	.408896E+02	.119549E-02	.119835E-02
14	.440277E+02	.103123E-02	.103336E-02
15	.471663E+02	.898611E-03	.900227E-03
16	.503052E+02	.790009E-03	.791258E-03
17	.534445E+02	.699958E-03	.700939E-03

C = 7.0

0	.246454E+01	.254204E+00	.285724E+00
1	.503591E+01	.188272E+00	.237814E+00
2	.775213E+01	.128328E+00	.158231E+00
3	.105927E+02	.868460E-01	.101577E+00
4	.135218E+02	.603871E-01	.676611E-01
5	.165100E+02	.435351E-01	.473389E-01
6	.195376E+02	.325038E-01	.346273E-01
7	.225921E+02	.250267E-01	.262856E-01
8	.256653E+02	.197822E-01	.205684E-01
9	.287520E+02	.159877E-01	.165009E-01
10	.318486E+02	.131661E-01	.135140E-01
11	.349528E+02	.110175E-01	.112611E-01
12	.380629E+02	.934716E-02	.952236E-02
13	.411775E+02	.802483E-02	.815393E-02
14	.442958E+02	.696131E-02	.705843E-02
15	.474170E+02	.609391E-02	.616832E-02
16	.505407E+02	.537765E-02	.543559E-02
17	.536665E+02	.477964E-02	.482540E-02
18	.567939E+02	.427540E-02	.431200E-02
19	.599228E+02	.384642E-02	.387605E-02
20	.630529E+02	.347855E-02	.350276E-02
21	.661841E+02	.316074E-02	.318074E-02

table 1 (contd.)

C = 7.0 (contd.)

INDEX	OMEGA	EIGENVALUE	LIMIT VALUE
22	.693163E+02	.288437E-02	.290102E-02
23	.724492E+02	.264256E-02	.265654E-02
24	.755829E+02	.242981E-02	.244162E-02
25	.787172E+02	.224165E-02	.225171E-02
26	.818520E+02	.207446E-02	.208307E-02
27	.849873E+02	.192523E-02	.193265E-02
28	.881231E+02	.179150E-02	.179792E-02
29	.912593E+02	.167119E-02	.167678E-02
30	.943958E+02	.156258E-02	.156746E-02
31	.975326E+02	.146419E-02	.146848E-02
32	.100670E+03	.137479E-02	.137857E-02
33	.103807E+03	.129331E-02	.129666E-02
34	.106945E+03	.121885E-02	.122183E-02
35	.110083E+03	.115063E-02	.115328E-02
36	.113221E+03	.108797E-02	.109034E-02
37	.116359E+03	.103029E-02	.103241E-02
38	.119498E+03	.977063E-03	.978972E-03
39	.122636E+03	.927851E-03	.929573E-03
40	.125775E+03	.882261E-03	.883818E-03
41	.128914E+03	.839945E-03	.841356E-03
42	.132053E+03	.800597E-03	.801879E-03
43	.135192E+03	.763948E-03	.765115E-03
44	.138331E+03	.729755E-03	.730820E-03
45	.141471E+03	.697805E-03	.698779E-03
46	.144610E+03	.667905E-03	.668798E-03
47	.147750E+03	.639885E-03	.640704E-03
48	.150889E+03	.613590E-03	.614343E-03
49	.154029E+03	.588881E-03	.589575E-03
50	.157169E+03	.565634E-03	.566274E-03
51	.160308E+03	.543735E-03	.544327E-03
52	.163448E+03	.523083E-03	.523630E-03
53	.166588E+03	.503585E-03	.504092E-03
54	.169728E+03	.485155E-03	.485626E-03
55	.172869E+03	.467719E-03	.468156E-03
56	.176009E+03	.451205E-03	.451612E-03

table 1 (contd.)

C = 7.0 (contd.)

INDEX	OMEGA	EIGENVALUE	LIMIT VALUE
57	.179149E+03	.435549E-03	.435929E-03
58	.182289E+03	.420694E-03	.421048E-03
59	.185429E+03	.406586E-03	.406916E-03
60	.188570E+03	.393175E-03	.393484E-03
61	.191710E+03	.380416E-03	.380706E-03
62	.194851E+03	.368268E-03	.368540E-03
63	.197991E+03	.356693E-03	.356947E-03
64	.201132E+03	.345654E-03	.345893E-03
65	.204272E+03	.335120E-03	.335345E-03
66	.207413E+03	.325060E-03	.325271E-03
67	.210553E+03	.315446E-03	.315645E-03
68	.213694E+03	.306252E-03	.306439E-03
69	.216834E+03	.297454E-03	.297630E-03
70	.219975E+03	.289029E-03	.289196E-03
71	.223116E+03	.280957E-03	.281115E-03
72	.226257E+03	.273219E-03	.273368E-03
73	.229397E+03	.265795E-03	.265937E-03
74	.232538E+03	.258670E-03	.258804E-03
75	.235679E+03	.251828E-03	.251955E-03

table 1 (end)

c	λ_0	N	$\sum_{n=0}^N \lambda_n$	$n_0 = \frac{1}{\lambda_0}$	$n_0 = \frac{N \sum_{n=0}^N \lambda_n^2}{(\sum_{n=0}^N \lambda_n)^2}$	integrated mean sq. diff.
.4	.8797	11	.9923	1.1368	1.1280	.000860
.6	.8281	14	.9910	1.2076	1.1967	.001138
.7	.8042	15	.9902	1.2435	1.2314	.001238
.8	.7813	16	.9896	1.2799	1.3554	.001157
.9	.7596	17	.9890	1.3165	1.3020	.001391
1.	.7388	18	.9885	1.3535	1.3379	.001451
2.	.5747	28	.9853	1.7400	1.7145	.001764
3.	.4649	38	.9838	2.1510	2.1162	.001900
4.	.3876	47	.9826	2.5799	2.5349	.001988
7.	.2542	76	.9812	3.9339	3.8600	.002181

SOME ESSENTIAL FUNCTIONS OF C THE RELATIVE APERTURE FOR LORENTZIAN EIGENVALUES, λ_0 IS THE CORRESPONDING FIRST EIGENVALUE, N IS THE INDEX FOR WHICH THE RATIO $\frac{\lambda_N}{\lambda_0}$ IS JUST LESS THAN 10^{-3} . THE SUM $\sum_{n=0}^N \lambda_n$ INDICATES HOW WELL WE ARE IN AGREEMENT WITH THE SUM PREDICTED BY THE USE OF MERCER'S THEOREM (WHICH IN THIS CASE IS UNITY), n_0 IS THE CUT-OFF INDEX OF THE "EQUIVALENT SET" OF EIGENVALUES CALCULATED WITH THE USE OF MERCER'S THEOREM, THE SUM $\frac{(\sum_{n=0}^N \lambda_n)^2}{\sum_{n=0}^N \lambda_n^2}$ IS ALSO EQUAL TO n_0 (SEE TEXT) AND SHOWS THE VALUE OF THIS QUANTITY WHEN ONLY N EIGENVALUES ARE USED. THE INTEGRATED MEANSQUARE DIFFERENCE IS PLOTTED ON FIG.14.

table 2

Figure (9) shows a plot of n_0 the cut off index of the eigenvalues for the "equivalent set" as defined in equ. 3.3. (32) versus the relative aperture c given by equ. (8). From the approximate Gamma distribution viewpoint n_0 is the number of independent correlation cells within the relative aperture. This is seen to tend towards the value $c/2$. From equ. (17) when we let c tend to infinity, we can write

$$-\frac{c}{2\omega} = \tan \omega \quad (20)$$

From fig (8) for $c = 5$, we see the first solution for ω is between $\pi/2$ and π this will also be true for greater values of c , the solution of equ. (20) in the domain $(\omega) = [\pi/2, \pi]$ when c tends to infinity is then $\omega_0 = \pi$. Inserting this value in equ. (18) gives

$$\Lambda_0 = \frac{2c}{\pi^2 + c^2} \quad (21)$$

which in turn gives for the cut off index

$$n_0 = \frac{1}{\Lambda_0} \rightarrow \frac{c}{2} \text{ as } c \rightarrow \infty \quad (22)$$

This justifies the asymptotic line in fig. (9).

Figures (10) to (13) show comparative plots between the exact distribution as given by equ. 3.3. (35) and the approximate Gamma distribution as given by equ. 3.3. (54) with the same n_0 as plotted on fig. (9). These distributions are plotted for an average integrated intensity equal to 4. Figure (14) shows a plot of the integrated mean square difference between both distributions. For a point aperture ($c \rightarrow 0$) both distributions are seen to be equivalent and tend to a negative exponential: for $c = .5$ the integrated mean square difference is about 10^{-3} . When c gets larger as far as we have calculated, the

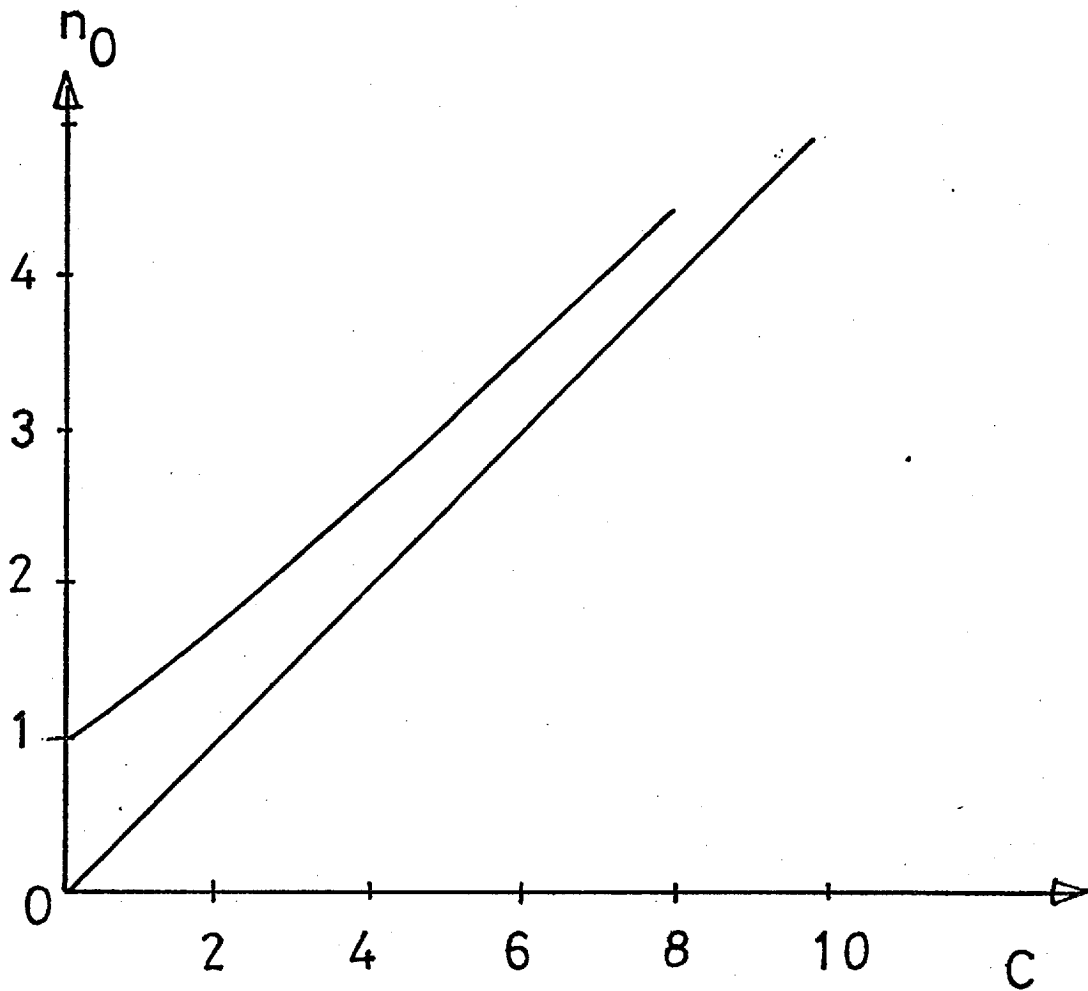


FIG. 9

CUT-OFF INDEX OF THE LORENTZIAN EIGENVALUES VS. c , THE RELATIVE APERTURE, n_0 ALSO DEFINES THE NUMBER OF INDEPENDENT CORRELATION THE SCANNING APERTURE CAN BE DEVIDED INTO.

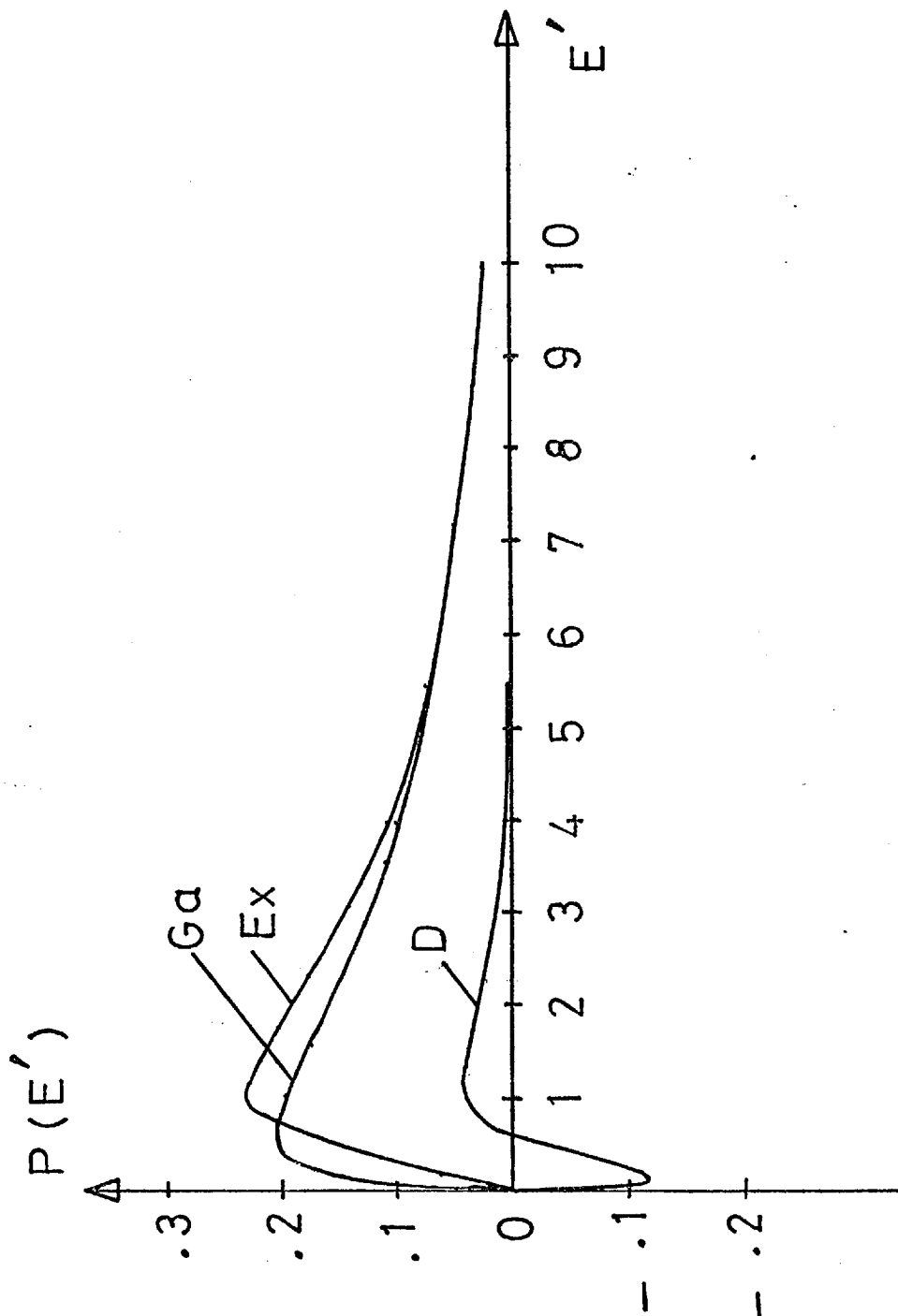


FIG.10 COMPARATIVE PLOTS OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR LORENTZIAN CASE, RELATIVE APERTURE $C = 0.4$,
 Ga = GAMMA DISTRIBUTION,
 Ex = EXACT DISTRIBUTION,
 D = DIFFERENCE.

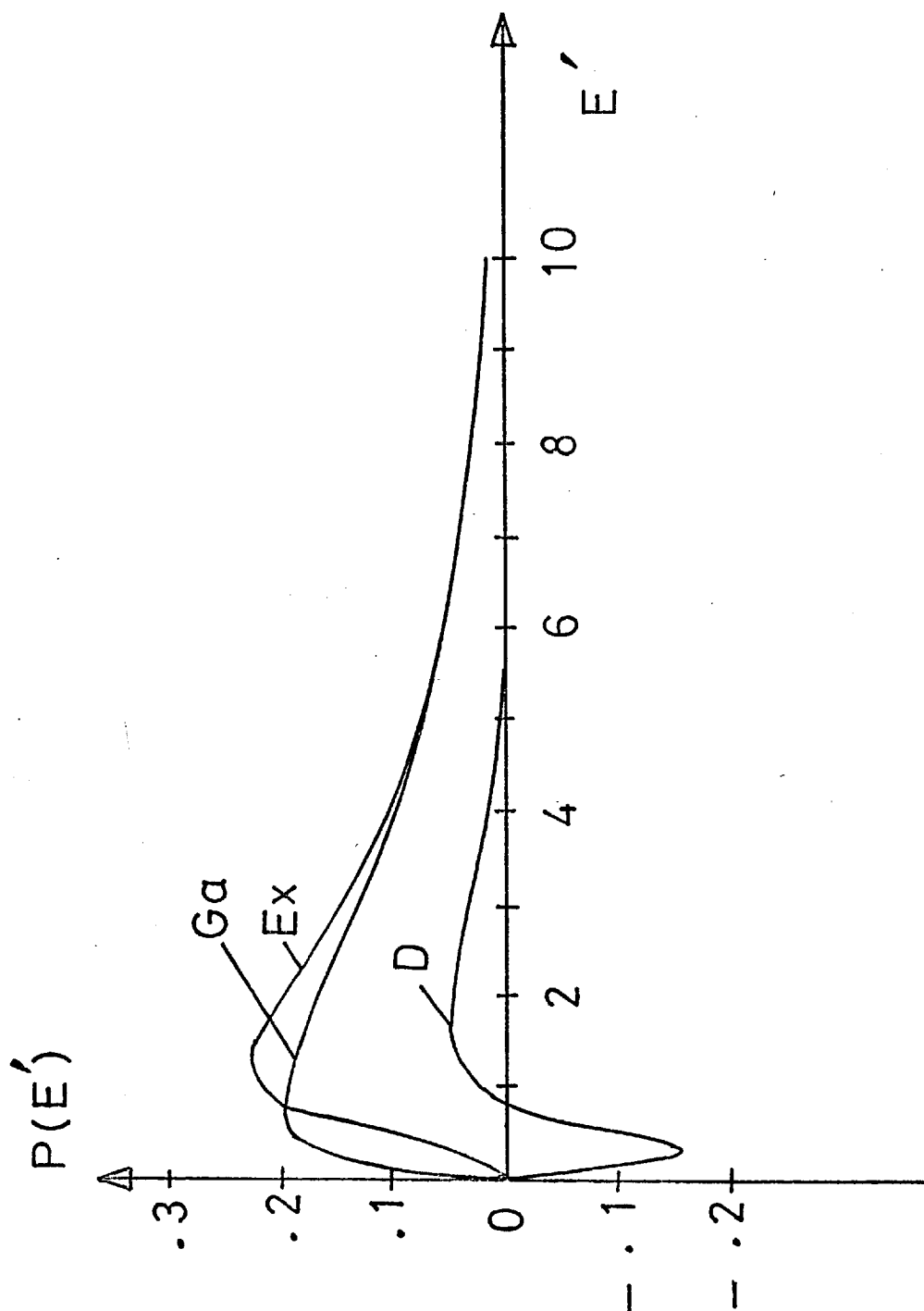


FIG. 11 COMPARATIVE PLOTS OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR LORENTZIAN CASE, RELATIVE APERTURE $C = 0.7$,
 Ga = GAMMA DISTRIBUTION,
 Ex = EXACT DISTRIBUTION,
 D = DIFFERENCE.

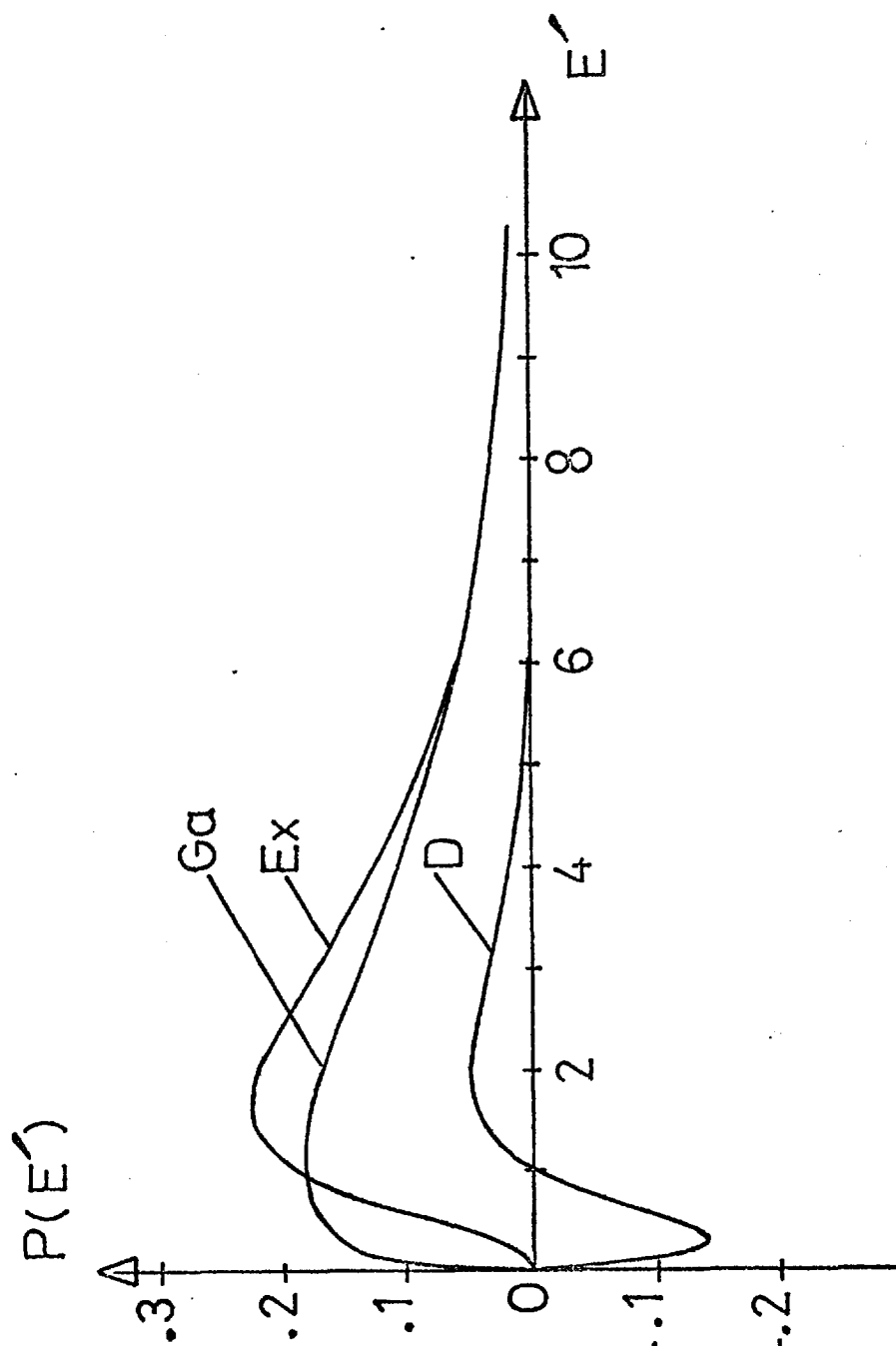


FIG. 12 COMPARATIVE PLOTS OF EXACT AND APPROXIMATE PROBABILITY

DENSITIES FOR LORENTZIAN CASE, RELATIVE APERTURE $C = 1.0$

Ga = GAMMA DISTRIBUTION,

Ex = EXACT DISTRIBUTION,

D = DIFFERENCE.

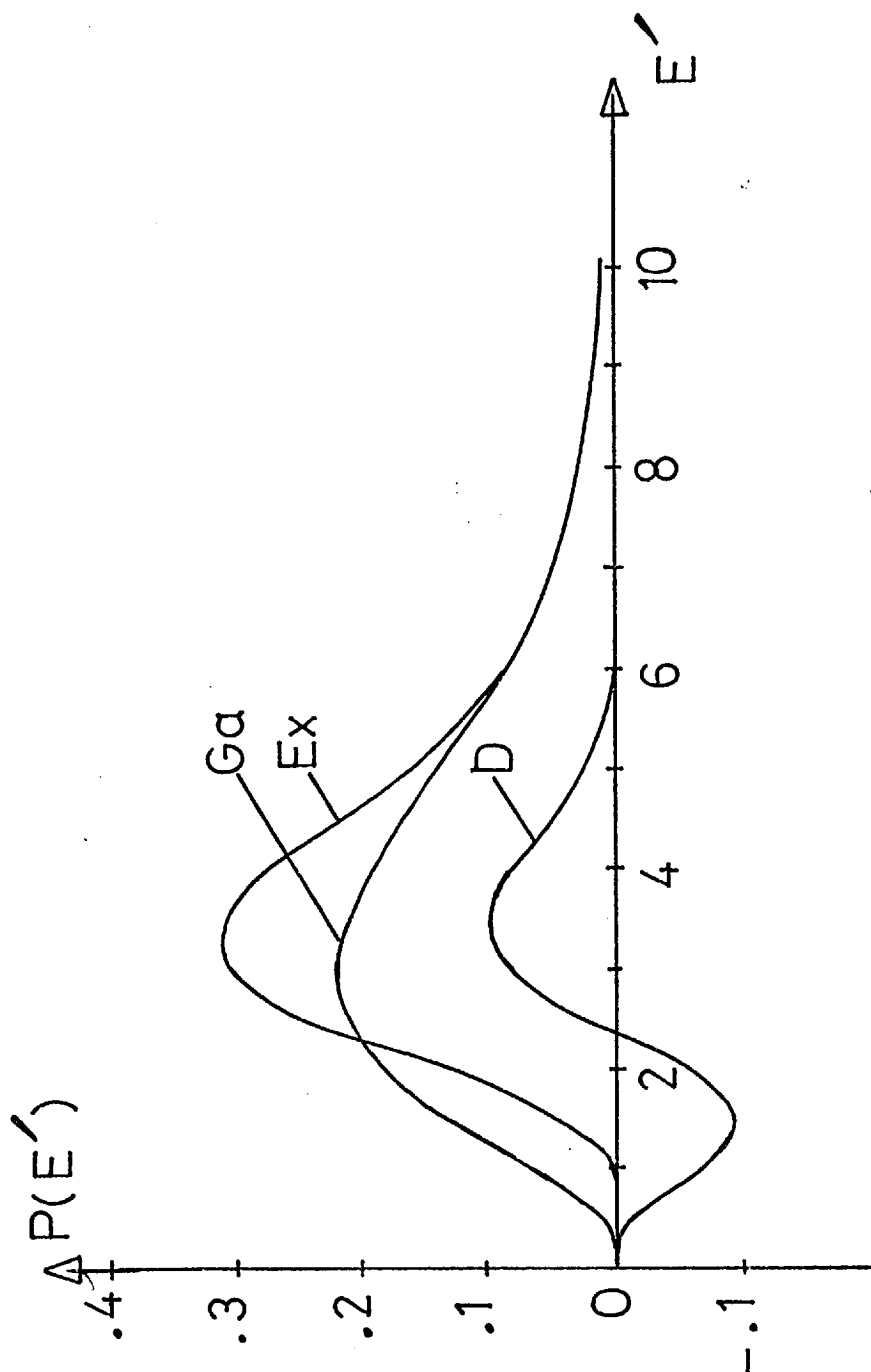


FIG. 13 COMPARATIVE PLOTS OF EXACT AND APPROXIMATE PROBABILITY
 DENSITIES FOR LORENTZIAN CASE? RELATIVE APERTURE $C = 7.0$,
 Ga = GAMMA DISTRIBUTION,
 Ex = EXACT DISTRIBUTION,
 D = DIFFERENCE.

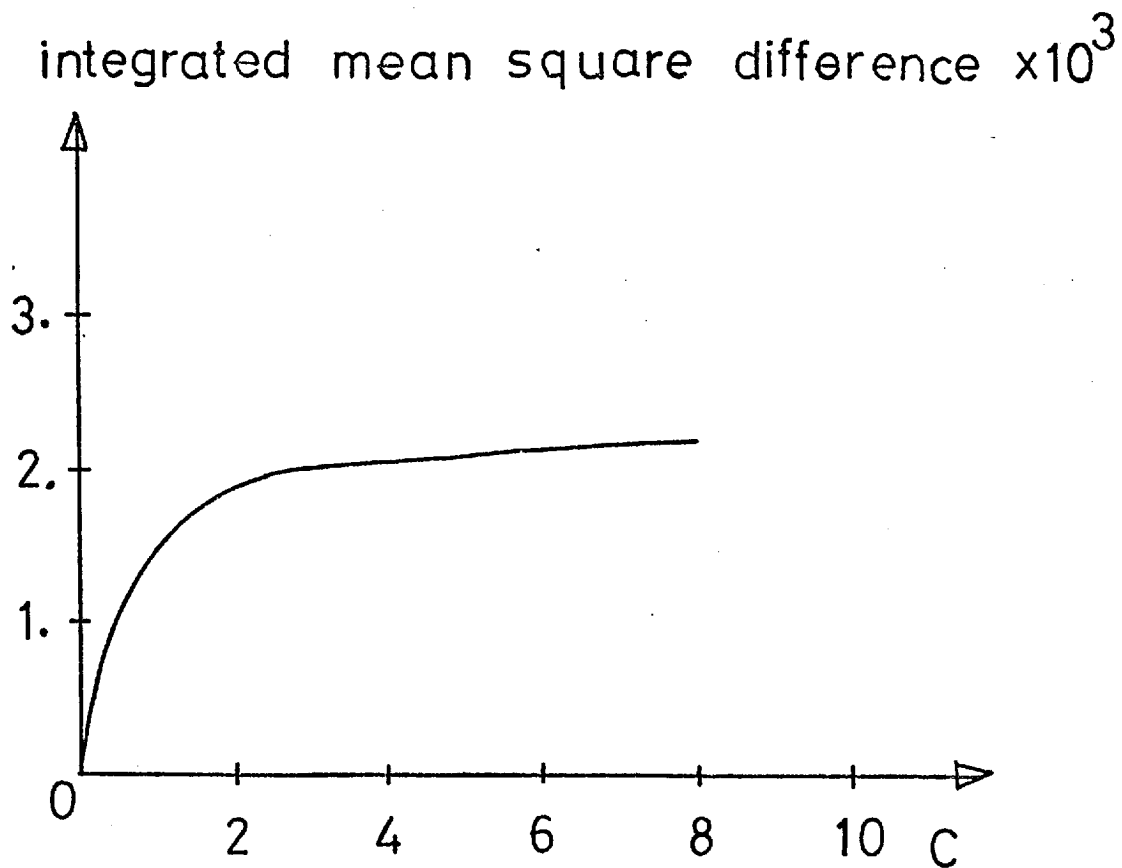


FIG. 14 THE DIFFERENCE BETWEEN THE APPROXIMATE AND EXACT FORMULAE FOR THE LORENTZIAN CASE IS CHARACTERISED BY THE GRAPH OF THE INTEGRATED MEAN SQUARE DIFFERENCE VERSUS THE RELATIVE APERTURE C .

integrated mean square difference levels off just above the value 2×10^{-3}

This could be due to the fact the lorentzian scattering spot is not bounded.

2) A slit (one dimensional) collecting pupil with a rectangular intensity transmittance profile and a slit scanning aperture with a rectangular intensity transmittance profile.

The collecting pupil P is a slit with a rectangular intensity transmittance profile of half width U and the scanning aperture is a unit intensity transmittance slit of width A. The advantage of slit geometry is that the problem can be treated one dimensionally. The autocorrelation function for this case is a sinc function. SLEPIAN and associated workers (SLEPIAN and POLLAK 1961, LANDAU and POLLACK 1961, LANDAU and POLLAK 1962, LANDAU 1966 and LANDAU 1967) have studied a scaled version of the corresponding integral equation and its properties by observing the equivalent differential equation was that of the prolate spheroidal wave functions studied previously notably by FLAMMER 1957.

The Autocorrelation Function and the Fredholm integral equation

Again the amplitude autocorrelation function is found either directly from equ. 3.3. (60) or from the amplitude power spectrum by using the Wiener-Kintchine theorem.

The power spectrum is given by (see equ. (1))

$$\pi_{|v|,s} = \frac{|P(v)|^2 \langle E' \rangle}{\int_{-\infty}^{+\infty} |P(v)|^2 dv} \quad (23)$$

If $P(v)$, the pupil function is a binary function, that is with unit transmittance

over all its transmitting portion and zero elsewhere such that,

$$P(u) = \text{rect} \frac{u}{2U} = 1 \quad \text{if } |u| \leq U \quad (24)$$

and zero elsewhere

(This function is real positive and even).

We observe that the squared modulus of this function is the function itself and we can therefore write

$$\Pi_{|v|,s} = \frac{P(u) \langle E' \rangle}{\int_{-\infty}^{+\infty} P(u) du} \quad (25)$$

The Fourier transform of this equation gives the autocorrelation function

$$R(x - x') = \langle E' \rangle \frac{\sin(2\pi U(x - x')/\lambda D)}{2\pi U(x - x')/\lambda D} \quad (26)$$

where U , λ and D are defined as on figs. (1) and (6b), x and x' are two points in hardware space. The eigenvalues and eigenfunctions we need are the solutions of the integral equation,

$$\epsilon_n \phi_n(x) = \int_{-\frac{A}{2}}^{+\frac{A}{2}} \frac{\langle E' \rangle \sin(2\pi U(x - x')/\lambda D)}{2\pi U(x - x')/\lambda D} \phi_n(x') dx' \quad (27)$$

Let us consider the scaled version of this equation as studied by Slepian and others mentioned above,

$$\Lambda_n \phi_n(x) = \int_{-1}^{+1} \frac{\sin(c(x - x'))}{\pi(x - x')} \phi_n(x') dx' \quad (28)$$

where c defines the relative aperture (or space-bandwidth product) and is

given by

$$c = \frac{\pi U A}{\lambda D} \quad (29)$$

We observe c is closely related to the measure of relative apertures as defined by DAINTY 1972 . This definition (equ. 2.3 (31)) can be interpreted as

$$k = \frac{\text{scanning aperture diameter or width}}{\text{Rayleigh resolution limit of pupil P}} = \frac{A}{\mathcal{R}_P} \quad (30)$$

where $\mathcal{R}_P$ denotes the Rayleigh resolution limit of the collecting pupil P.

For the slit geometry under consideration $\mathcal{R}_P$ is defined by the relation (see e.g. BORN and WOLF 1970),

$$\mathcal{R}_P = \frac{\lambda D}{2 U} \quad (31)$$

Inserting this in equ. (27) gives Dainty's relative aperture in k units as

$$k = \frac{2 U A}{\lambda D} \quad (32)$$

From this and equ. (26) we obtain the simple relation

$$k = \frac{2 c}{\pi} \quad (33)$$

Henceforth we shall use c units rather than k units to define relative aperture.

As the integral equ. (28) is a scaled version of the integral equ. (27) the set of eigenvalues $\{\Lambda_n\}$ will be a scaled version of the set $\{\varepsilon_n\}$. We know from previous arguments (equ. 3.3. (36)) that the average integrated intensity is related to the eigenvalues through their sum. If the scaling constant

between the $\{\varepsilon_n\}$ and the $\{\Lambda_n\}$ is α such that

$$\varepsilon_n = \alpha \Lambda_n \quad (34)$$

for all values of n , from equ. 3.3. (36) we can write

$$\langle E' \rangle = \sum_n \varepsilon_n = \alpha \sum_n \Lambda_n \quad (35)$$

Applying the first relation derived from Mercer's theorem expressed for stationary kernels (equ. 3.2. (23)) to the integral equ. (28), we find

$$\sum_n \Lambda_n = \frac{2c}{\pi} \quad (36)$$

Inserting this relation in equ. (35) gives the scaling constant

$$\alpha = \frac{\langle E' \rangle \pi}{2c} \quad (37)$$

Solution of the integral equation.

The integral equation

$$\Lambda_n \phi_n(x) = \int_{-1}^{+1} \frac{\sin(c(x-x'))}{\pi(x-x')} \phi_n(x') dx' \quad (38)$$

is the first iterate of the integral equation

$$s_n \phi_n(x) = \int_{-1}^{+1} e^{icxx'} \phi_n(x') dx' \quad (39)$$

$$\text{where } \Lambda_n = \frac{2c}{\pi} s_n^2 \quad (40)$$

Slepian (see e.g. SLEPIAN and POLLAK 1961) has shown the functions $\phi_n(x)$

are scaled versions of some of the angular prolate spheroidal equations,

(see BOUKWAMP 1947, FLAMMER 1957). Here we draw

freely from SLEPIAN and POLLAK 1961.

When c is real, the differential equation

$$(1 - x^2) \psi'' - 2x\psi' + (\chi - c^2 x^2) \psi = 0 \quad (41)$$

has continuous solutions in the closed interval $(x) = [-1, +1]$ only for some discrete real positive values χ_n of the quantity χ called eigenvalues.

To each eigenvalue χ_n ($n = 0, 1, 2, \dots$) corresponds a unique solution

$S_{0n}(c, x)$ (in Flammer's notation) such that $S_{0n}(c, 0) = P_n(0)$ where $P_n(x)$ is the n^{th} Legendre polynomial of order zero. The functions $S_{0n}(c, x)$ are called the angular prolate spheroidal wave functions.

They are real for x real and continuous functions of c , for $c \geq 0$.

They are orthonormal in $[-1, +1]$ and are square integrable in that interval.

$S_{0n}(c, x)$ has exactly n zeros in $[-1, +1]$ and tends to $P_n(x)$ uniformly in that interval as $c \rightarrow 0$, and is even or odd if n is respectively even or odd. The eigenvalues χ_n are continuous functions of c and for $c = 0$, $\chi_n(0) = n(n+1)$.

A second set of solutions $R_{0n}^{(1)}(c, x)$ called the radial prolate spheroidal wave functions exist which differ only from the angular functions by a scaling constant,

$$S_{0n}(c, x) = \kappa_n(c) R_{0n}^{(1)}(c, x) \quad (42)$$

the scaling constant is defined by (FLAMMER 1957)

$$\kappa_n(c) = \frac{n! \sum_{k=0}^{\infty} d_k^n(c)}{2^n d_0^n(c) \left[\left(\frac{n}{2} \right)! \right]^2} \quad (43)$$

for n even and

$$\kappa_n(c) = \frac{(3n+1)! \sum_{k=1}^{\infty} 'd_k^n(c)}{2^n d_1^n(c) c \left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} \quad (44)$$

for n odd, the prime denotes summation on even or odd terms accordingly.

The differential equation (41) is observed to be closely related to the differential equation defining Legendre polynomials P_n (ABRAMOWITZ and STEGUN 1970) and the d_k^n are expansion coefficients of the functions S_{0n} into these,

$$S_{0n}(c, x) = \sum_{k=0,1}^{\infty} 'd_k^n(c) P_{n+k}(x) \quad (44a)$$

The radial functions $R_{0n}^{(1)}(c, x)$ are normalized such that

$$R_{0n}^{(1)}(c, x) \rightarrow \frac{1}{cx} \cos \left[cx - \frac{(n+1)\pi}{2} \right] \quad (45)$$

as $x \rightarrow \infty$

From these angular and radial prolate spheroidal functions, the integral equs.

(37) and (38) are written

$$\frac{2c}{\pi} \left[R_{0n}^{(1)}(c, 1) \right]^2 S_{0n}(c, x) = \int_{-1}^{+1} \frac{\sin(c(x-x'))}{\pi(x-x')} (c, x') dx' \quad (46)$$

and

$$2i^n R_{0n}^{(1)}(c, 1) S_{0n}(c, x) = \int_{-1}^{+1} e^{icxx'} S_{0n}(c, x') dx' \quad (47)$$

and the corresponding eigenvalues Λ_n are seen to be given by,

$$\Lambda_n(c) = \frac{2c}{\pi} \left[R_{0n}^{(1)}(c, 1) \right]^2, \quad n = 0, 1, 2, \dots \quad (48)$$

These eigenvalues are extensively tabulated by SLEPIAN and SONNENBLICK 1966 and STUCKEY and LAYTON 1964.

Other fundamental properties of the prolate spheroidal wave functions are that they are orthonormal both over finite $[-1, +1]$ and infinite intervals and they are their own Fourier transforms. They are therefore ideally suited for studying band-limited functions and their transforms.

Prolate spheroidal wave functions have been applied to optics by a number of workers, des CLOIZEAUX and MEHTA, 1972, MEHTA and des CLOIZEAUX, 1972, MEHTA and MEHTA 1973 in the photo-electron counting context, BARNES 1966, FRIEDEN 1971, RUSHFORTH and HARRIS 1968 and TORALDO di FRANCIA 1969 a) and b) in the image restoration and super resolution context.

Numerical Results

Figure (15) shows plots of the eigenvalues Λ_n as defined by equs. (38) and (48) for various values of c . We see from these graphs that for smaller values of c (i.e. c typically less than 10.) the eigenvalues are all different and the first order probability density is calculated from equ. 3.3. (35). When c gets larger, the first eigenvalues are constantly equal to unity and the first order probability density function can therefore be calculated from equ. 3.3. (45). When c gets even larger the similarity of the envelope of the eigenvalues with a step function is clearly seen.

By application of Mercer's theorem for stationary kernels expressed in equ. 3.2 (23) we can write

$$\frac{\sin(c(x-x'))}{\pi(x-x')} = \sum_{n=0}^{\infty} \Lambda_n \phi_n^*(x') \phi_n(x) \quad (49)$$

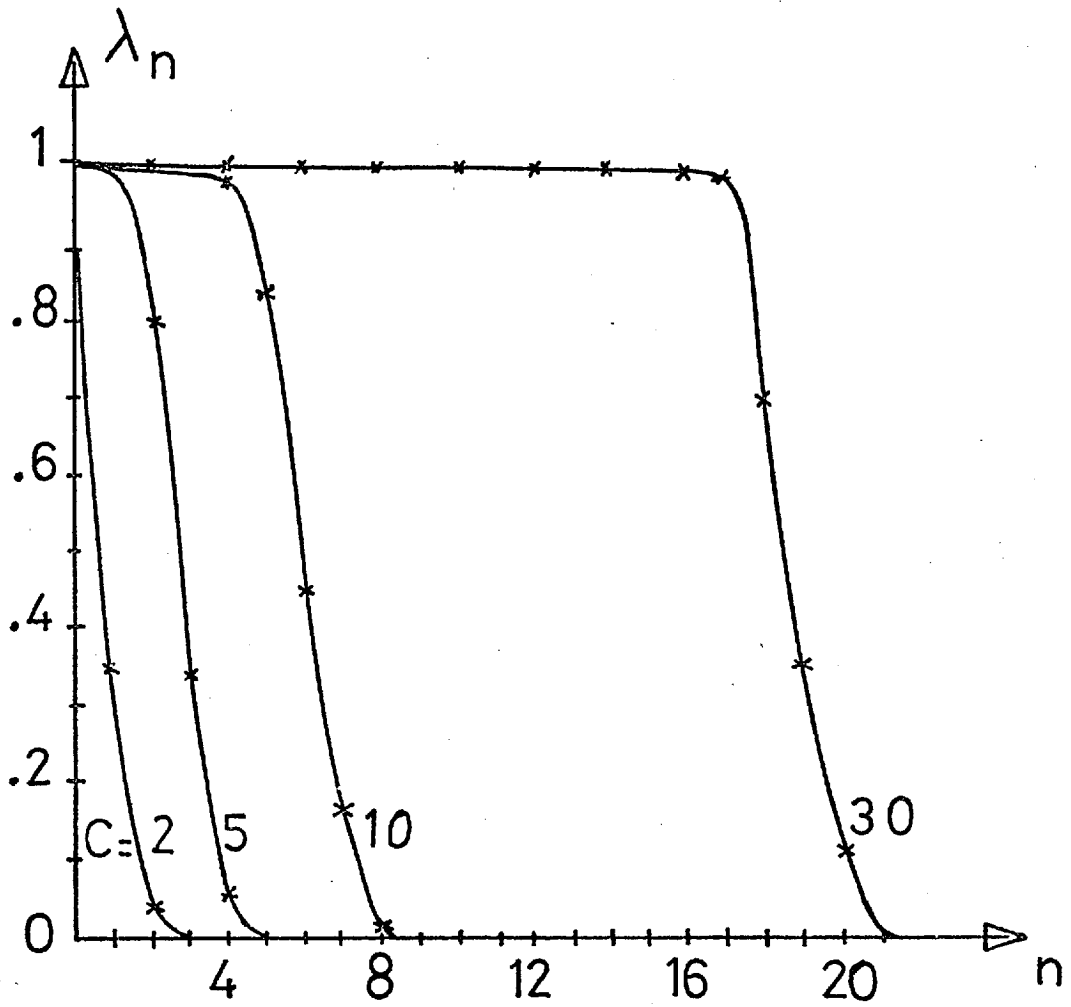


FIG. 15

PLOTS OF THE EIGENVALUES AND THEIR ENVELOPES
 GENERATED BY EQN. 4.2. (38) FOR THE UNIT INTENSITY
 TRANSMITTANCE SLIT DIFFUSING PUPIL FOR VARIOUS VALUES
 OF THE RELATIVE APERTURE C .

x EIGENVALUES,
 — ENVELOPES.

where the asterisk denotes complex conjugate. By setting x equal to x^* and integrating over the interval $[-1, +1]$ as the eigenfunctions are orthonormal we find

$$\frac{2c}{\pi} = \sum_{n=0}^{\infty} \Lambda_n \quad (50)$$

The "equivalent set" of eigenvalues is given by (equ. 3.3. (32))

$$n_0 \Lambda_0 = \sum_{n=0}^{\infty} \Lambda_n \quad (51)$$

inserting this definition in equ. (50) gives

$$n_0 = \frac{2c}{\pi \Lambda_0} \quad (52)$$

SLEPIAN 1965 finds a similar result for large values of c and large values of n by use of asymptotic expansion of the formulae giving the eigenvalues.

This result is

$$n_0 = \frac{2c}{\pi} \quad (53)$$

This formula differs from ours in that it does not take account of Λ_0 the first eigenvalue. Slepian's asymptotic expansions show that for c increasingly large, Λ_0 tends to unity, his asymptotic formula is (for c large)

$$1 - \Lambda_n = \frac{2^{3n+2} \sqrt{\pi} c^{n+\frac{1}{2}} e^{-2c}}{n!} \times \left[1 - \frac{6n^2 - 2n + 3}{32c} + O\left(\frac{1}{c^2}\right) \right] \quad (54)$$

setting $n = 0$ this equation gives

$$\Lambda_0 = 1 - \sqrt{\pi c} e^{-2c} \quad (55)$$

which indeed tends to unity as c tends to infinity and therefore shows that our result agrees with his. However our result derived by the use of Mercer's theorem is valid for all c and is therefore more general.

We can check the validity of our result by observing that as c decreases to zero n_0 must tend to unity. Physically, this means that as our scanning aperture tends to a pinhole (and therefore so does our relative aperture as the size of the scattering spot is maintained fixed) we are left with one correlation cell. For fixed n and small c SLEPIAN 1965, gives the asymptotic formula

$$\Lambda_n = \frac{2}{\pi} \left[\frac{2^{2n} (n!)^3}{(2n)! (2n+1)!} \right]^2 c^{2n+1} \times \left[1 - \frac{(2n+1)c^2}{(2n-1)^2 (2n+3)^2} + O(c^4) \right] \quad (56)$$

As c tends to zero, we see from this equation that we have

$$\Lambda_0 \gg \Lambda_1 \gg \Lambda_2 \dots$$

and

$$\Lambda_0 = \frac{2c}{\pi} \quad (57)$$

Inserting this value of Λ_0 in equ. (52) gives $n_0 = 1$ which is the required condition.

With the knowledge of Λ_0 and n_0 we may write the approximate first order probability density function as in equ. 3.3. (54)

$$P(E') = \left(\frac{1}{\alpha \Lambda_0} \right)^{n_0} \frac{E'^{n_0-1} e^{-\frac{E'}{\alpha \Lambda_0}}}{\Gamma(n_0)} \quad (58)$$

Where α is the scaling factor between the required set of eigenvalues $\{\epsilon_n\}$ and the computed set $\{\Lambda_n\}$ such that

$$\epsilon_n = \alpha \Lambda_n \quad (59)$$

the mean and variance found from equ. (58) are

$$\langle \epsilon' \rangle = n_0 \alpha \Lambda_0 \quad (60)$$

and

$$\sigma_{\epsilon'}^2 = n_0 \alpha^2 \Lambda_0^2 \quad (61)$$

From equ. (52) and (60) we find the scaling constant is

$$\alpha = \frac{\langle \epsilon' \rangle \pi c}{2 U A} \quad (62)$$

which is the same as equ. (37) showing α can be computed directly from the true set of eigenvalues or from the equivalent set of eigenvalues. Inserting the value of c as defined in equ. (29) in equ. (61) gives

$$\alpha = \frac{\langle \epsilon' \rangle \lambda D}{2 U A} \quad (63)$$

Inserting this value in equ. (58) gives

$$P(\epsilon') = \left(\frac{2 U A}{\langle \epsilon' \rangle \lambda D \Lambda_0} \right)^{n_0} \frac{\epsilon'^{n_0-1} e^{-\frac{2 U A \epsilon'}{\langle \epsilon' \rangle \lambda D \Lambda_0}}}{\Gamma(n_0)} \quad (64)$$

When all the eigenvalues are different (c less than 10) the exact first order probability density function is given by equ. 3.3. (35), with the scaling factor given by equ. (63) which holds for any eigenvalue (equ. (37)) we have

$$P(\epsilon') = \frac{2 U A}{\langle \epsilon' \rangle \lambda D} \sum_{n=0}^{\infty} \frac{G_n}{\Lambda_n} e^{-\frac{2 U A \epsilon'}{\langle \epsilon' \rangle \lambda D \Lambda_n}} \quad (65)$$

with the partial fraction coefficient given by

$$G_n = \prod_{m \neq n} \frac{\Lambda_n}{\Lambda_n - \Lambda_m} \quad (66)$$

When the relative aperture c tends to zero, as shown above from Slepian's asymptotic formula we have $\Lambda_0 \gg \Lambda_1 \gg \Lambda_2 \dots$

Neglecting the terms containing eigenvalues smaller than Λ_1 the probability density function is written (expressed with c rather than U, A, λ and D for convenience)

$$P(E') = \frac{2c}{\pi \langle E' \rangle} \left(\frac{\Lambda_0}{\Lambda_0 - \Lambda_1} \frac{1}{\Lambda_0} e^{-\frac{2c E'}{\pi \langle E' \rangle \Lambda_0}} + \frac{\Lambda_1}{\Lambda_1 - \Lambda_0} \frac{1}{\Lambda_1} e^{-\frac{2c E'}{\pi \langle E' \rangle \Lambda_1}} \right) \quad (67)$$

And now neglecting Λ_1 before Λ_0 and inserting the value of Λ_0 given by equ. (57) give

$$P(E') = \frac{1}{\langle E' \rangle} e^{-\frac{E'}{\langle E' \rangle}} \quad (68)$$

proving the exact distribution tends to a negative exponential when the relative aperture tends to zero (i.e. the scanning aperture tends to a point).

For c greater than 10 when several of the eigenvalues are equal to the first, Λ_0 up to the index n_0 and different for $n > n_0$ the exact first order probability density function is given by equ. 3.3. (45) with the scaling factor in equ. (63) we have

$$P(E') = \left(\frac{2UA}{\langle E' \rangle \lambda D \Lambda_0} \right)^{n_0+1} E'^{n_0} e^{-\frac{2UA}{\langle E' \rangle \lambda D \Lambda_0} E'} \times \sum_{n > n_0} \frac{G_n}{\Lambda_n} \left\{ \sum_{k=0}^{\infty} \frac{\left[\frac{2UA}{\langle E' \rangle \lambda D} \left(\frac{1}{\Lambda_0} - \frac{1}{\Lambda_n} \right) E' \right]^k}{\Gamma(n_0 + k + 1)} \right\} \quad (69)$$

with the partial fraction coefficients G_n given by

$$G_n = \prod_{\substack{m \neq n \\ m > n_0}} \frac{\Lambda_n}{\Lambda_n - \Lambda_m} \quad (70)$$

Table 3 shows some of the eigenvalues Λ_n . The ones for small c are calculated from equ. (56), the others are reproduced from SLEPIAN and SONNENBLICK 1966.

Table 4 shows some numerical values for some essential functions of C . Figures (16) to (28) show comparative plots of the exact and approximate distributions as given by eqs. 3.3 (35) and 3.3 (54). They are plotted for an average integrated intensity equal to 4. Figure (29) shows a plot of the integrated mean square difference as a function of the relative aperture C . This difference, calculated numerically, reaches a peak around $C = 2$. and the value of this peak is about 6×10^{-4} giving a mean absolute difference (the square root of the integrated mean square difference) of about 2.5×10^{-2} . For $C = 8$, the integrated mean square difference is less than 10^{-4} giving an absolute difference of less than 10^{-2} . These results show that for values of C around 2, it is perhaps more advisable to use the exact distribution but for values of C less than 1, or more than 8, the gamma distribution is quite suitable. Figure (30) shows a plot of n_0 (the cut off index of the eigenvalues and also the number of independent correlations cells contained in the relative aperture) as a function of C , as shown earlier n_0 tends to unity as C decreases to zero and asymptotically towards the value $\frac{2C}{\pi}$ as C gets larger (some of the results given here are stated in SCRIBOT 1974).

PROLATE SPHEROIDAL EIGENVALUES

INDEX	EIGENVALUE	INDEX	EIGENVALUE
C = 0.1		C = 4.0	
0	.635913E-01	0	.995885E+00
1	.706588E-04	1	.912107E+00
2	.125738E-07	2	.519055E+00
C = 0.5		3	.110211E+00
0	.309590E+00	4	.882788E-02
1	.858063E-02	5	.381292E-03
2	.391863E-04	C = 5.0	
C = 0.8		0	.999352E+00
0	.474337E+00	1	.979865E+00
1	.335393E-01	2	.799922E+00
2	.409086E-03	3	.343562E+00
C = 1.0		4	.560156E-01
0	.572582E+00	5	.418209E-02
1	.627913E-01	6	.193308E-03
2	.123748E-02	C = 6.0	
3	.920098E-05	0	.999902E+00
C = 2.0		1	.996062E+00
0	.880560E+00	2	.940173E+00
1	.355641E+00	3	.646792E+00
2	.358677E-01	4	.207349E+00
3	.115223E-02	5	.273872E-01
4	.188815E-04	6	.195500E-02
C = 3.0		7	.948488E-04
0	.975829E+00	C = 7.0	
1	.709963E+00	0	.999985E+00
2	.205139E+00	1	.999292E+00
3	.182038E-01	2	.985708E+00
4	.708147E-03	3	.864566E+00
		4	.477053E+00
		5	.115729E+00
		6	.130560E-01
		7	.906573E-03
		8	.456239E-04

table 3

PROLATE SPHEROIDAL EIGENVALUES (SUITE)

INDEX	EIGENVALUE
-------	------------

C = 8.0

0	.999998E+00
1	.999879E+00
2	.997005E+00
3	.960546E+00
4	.747903E+00
5	.320277E+00
6	.607844E-01
7	.612629E-02
8	.418252E-03
9	.216631E-04

C = 9.0

0	1.000000E+00
1	.999980E+00
2	.999419E+00
3	.990396E+00
4	.910133E+00
5	.599096E+00
6	.196940E+00
7	.305651E-01
8	.284661E-02
9	.192308E-03

table 3 (end)

c	λ_0	N	$\frac{2C}{\pi}$	$\sum_{n=0}^N \lambda_n$	$n_0 = \frac{2C}{\pi \lambda_0}$	integrated mean sq. diff.
.1	.6359E-01	3	.6366E-01	.6366E-01	1.0011	.000000
.5	.3095E+00	3	.3183E+00	.3182E+00	1.0282	.000077
.8	.4743E+00	3	.5093E+00	.5083E+00	1.0737	.000289
1.	.5725E+00	4	.6366E+00	.6366E+00	1.1118	.000405
2.	.8805E+00	5	1.273E+00	1.273E+00	1.4459	.000508
3.	.9758E+00	5	1.910E+00	=	1.9572	.000317
4.	.9958E+00	6	2.547E+00	=	2.5570	.000191
5.	.9993E+00	7	3.183E+00	=	3.1852	.000126
6.	.9990E+00	8	3.820E+00	=	3.8235	.000089
7.	.9999E+00	9	4.456E+00	=	4.4564	.000068
8.	.9999E+00	10	5.093E+00	=	5.0930	.000054
9.	1.000E+00	11	5.730E+00	=	5.7296	.000043

table 4

SOME ESSENTIAL FUNCTIONS OF C THE RELATIVE APERTURE FOR PRO-
 LATE SPHEROIDAL EIGENVALUES, λ_0 IS THE CORRESPONDING FIRST EIGEN-
 VALUE, N IS THE INDEX FOR WHICH THE RATIO $\frac{\lambda_N}{\lambda_0}$ IS JUST LESS
 THAN 10^{-3} . $\frac{2C}{\pi}$ IS THE SUM OF THE EIGENVALUES ACCORDING
 TO THE MERCER THEOREM, $\sum_{n=0}^N \lambda_n$ INDICATES HOW WELL THE TRUE SUM
 OF THE EIGENVALUES AGREES WITH MERCER'S THEOREM, n_0 IS THE CUT-
 OFF INDEX OF THE "EQUIVALENT SET" OF EIGENVALUES, THE INTEGRATED
 MEAN SQUARE DIFFERENCE IS PLOTTED ON FIG. 29.

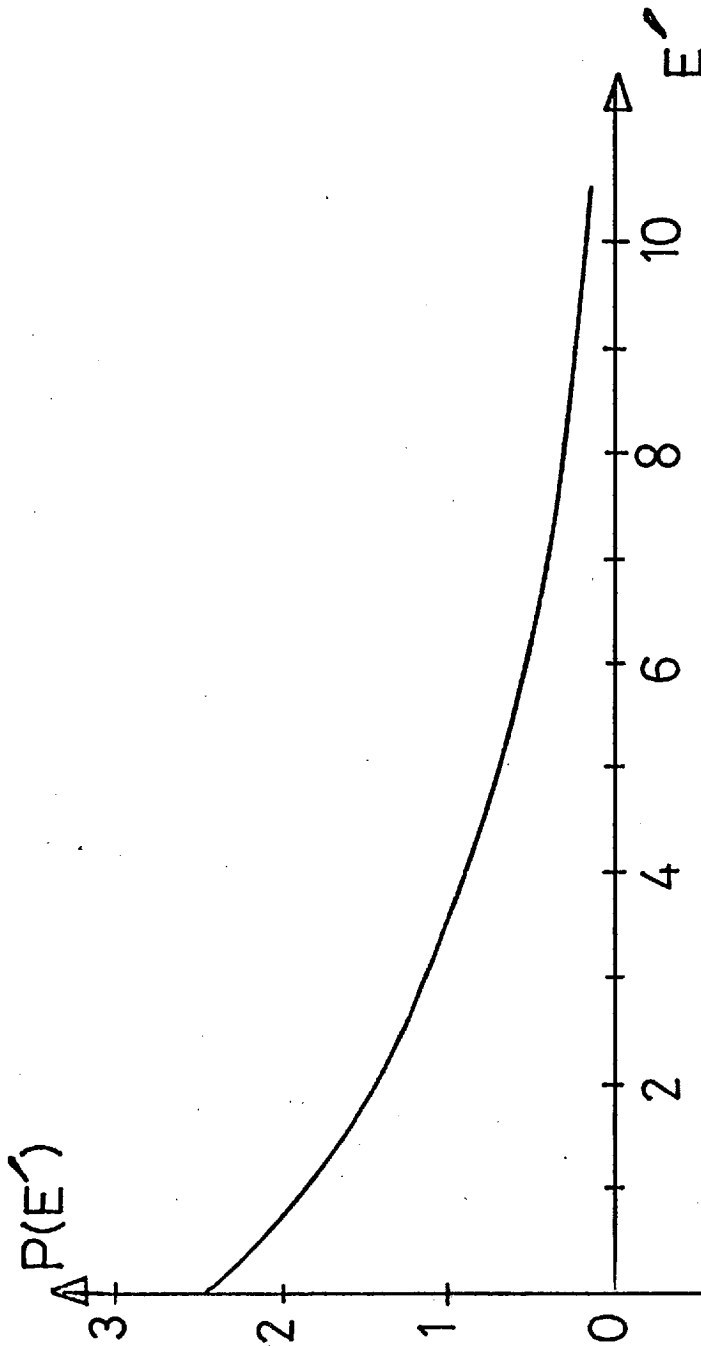


FIG. 16 COMPARATIVE PLOT FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR RELATIVE APERTURE $C = 0.1$, FOR THIS VALUE, BOTH CURVES ARE IDENTICAL.

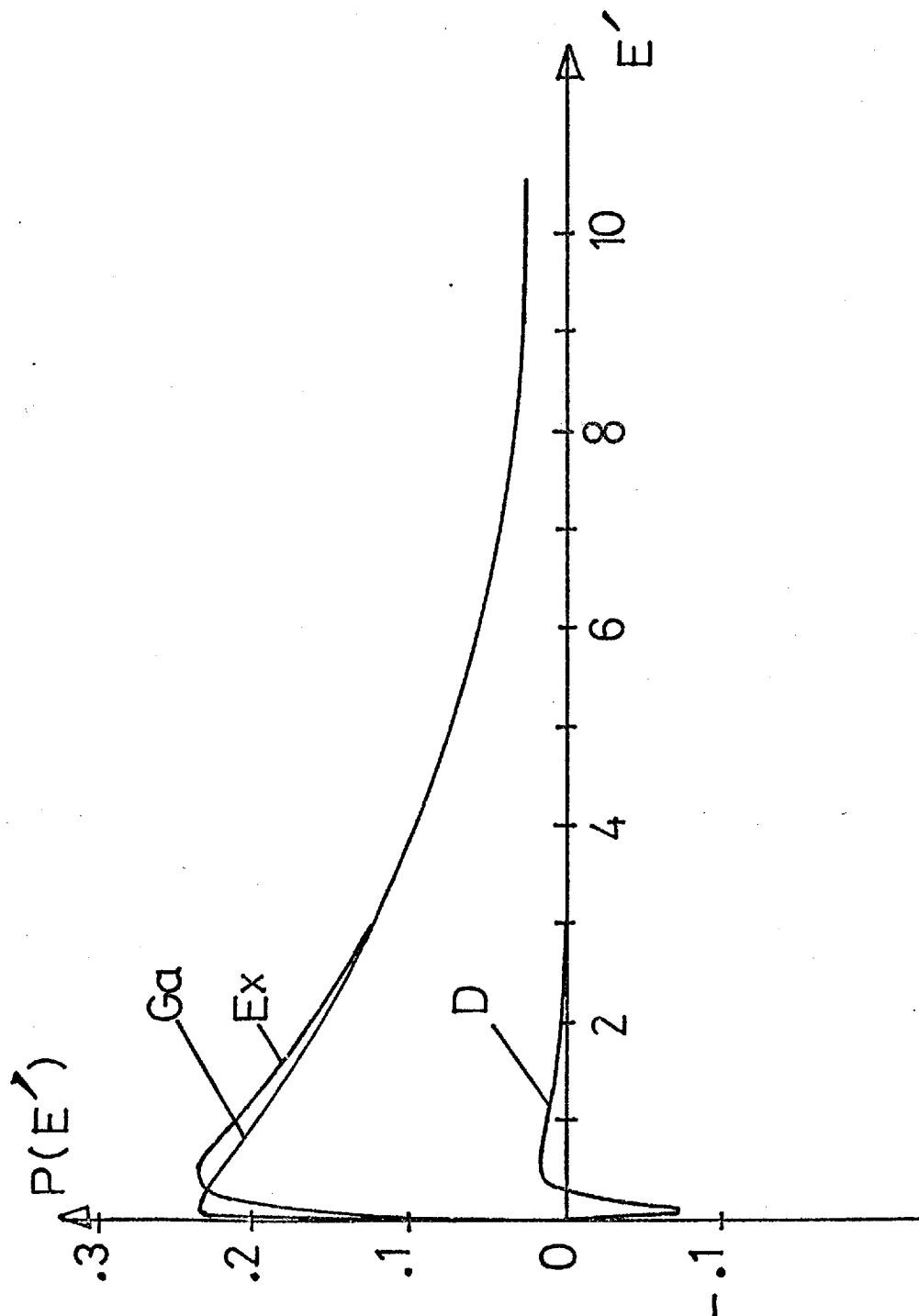


FIG. 17 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 0.5$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

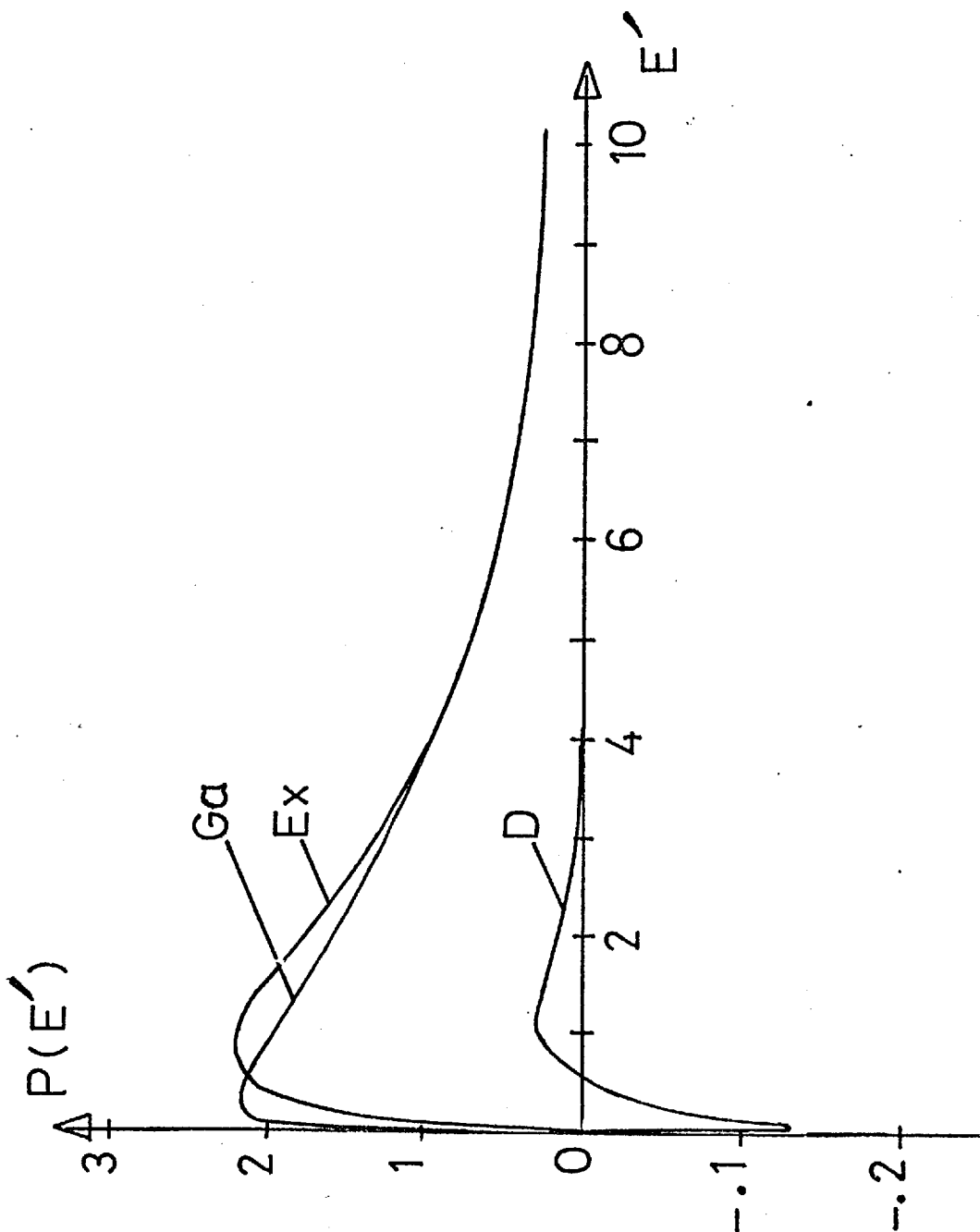


FIG. 18 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 0.8$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

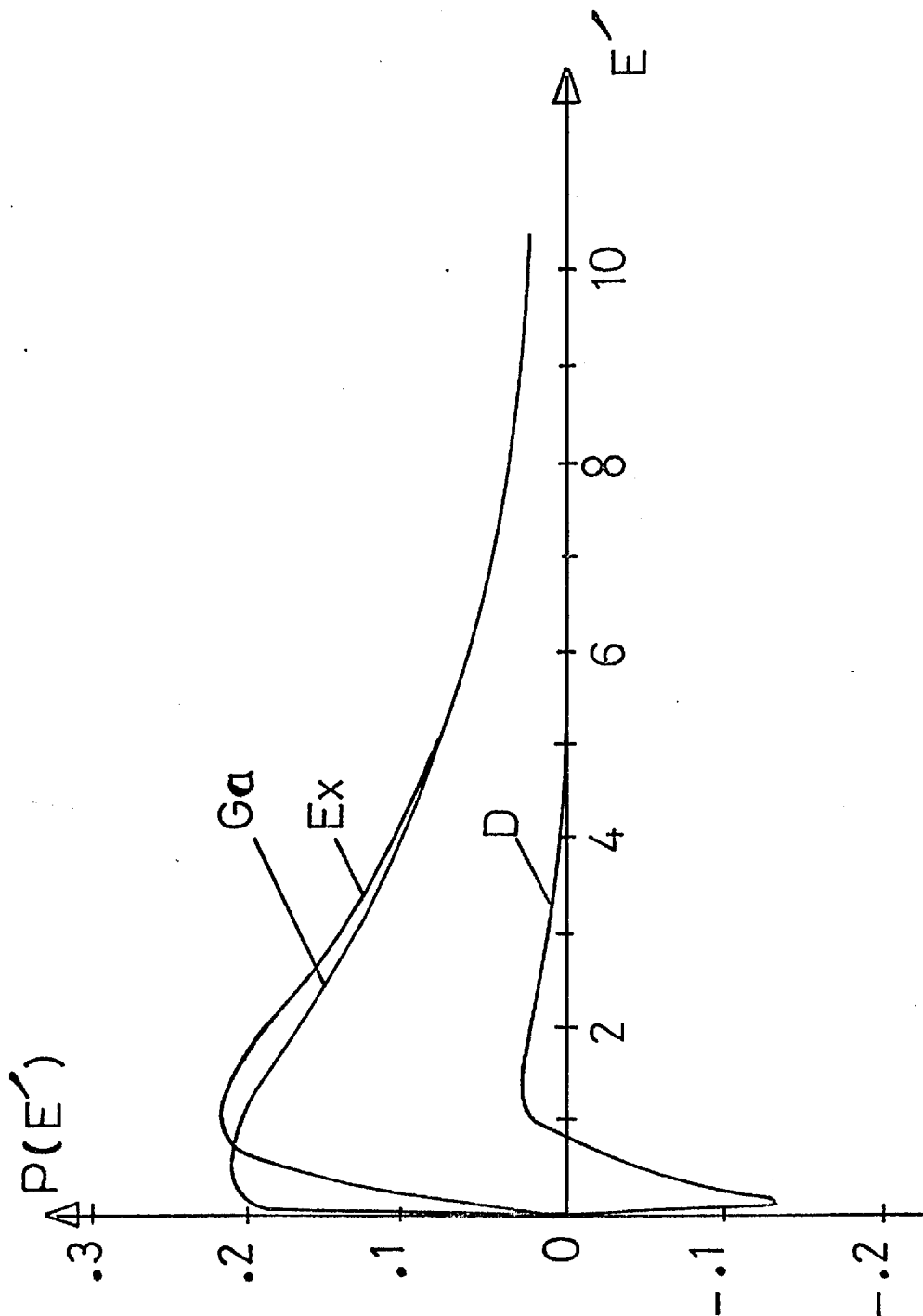


FIG. 19 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 1.0$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

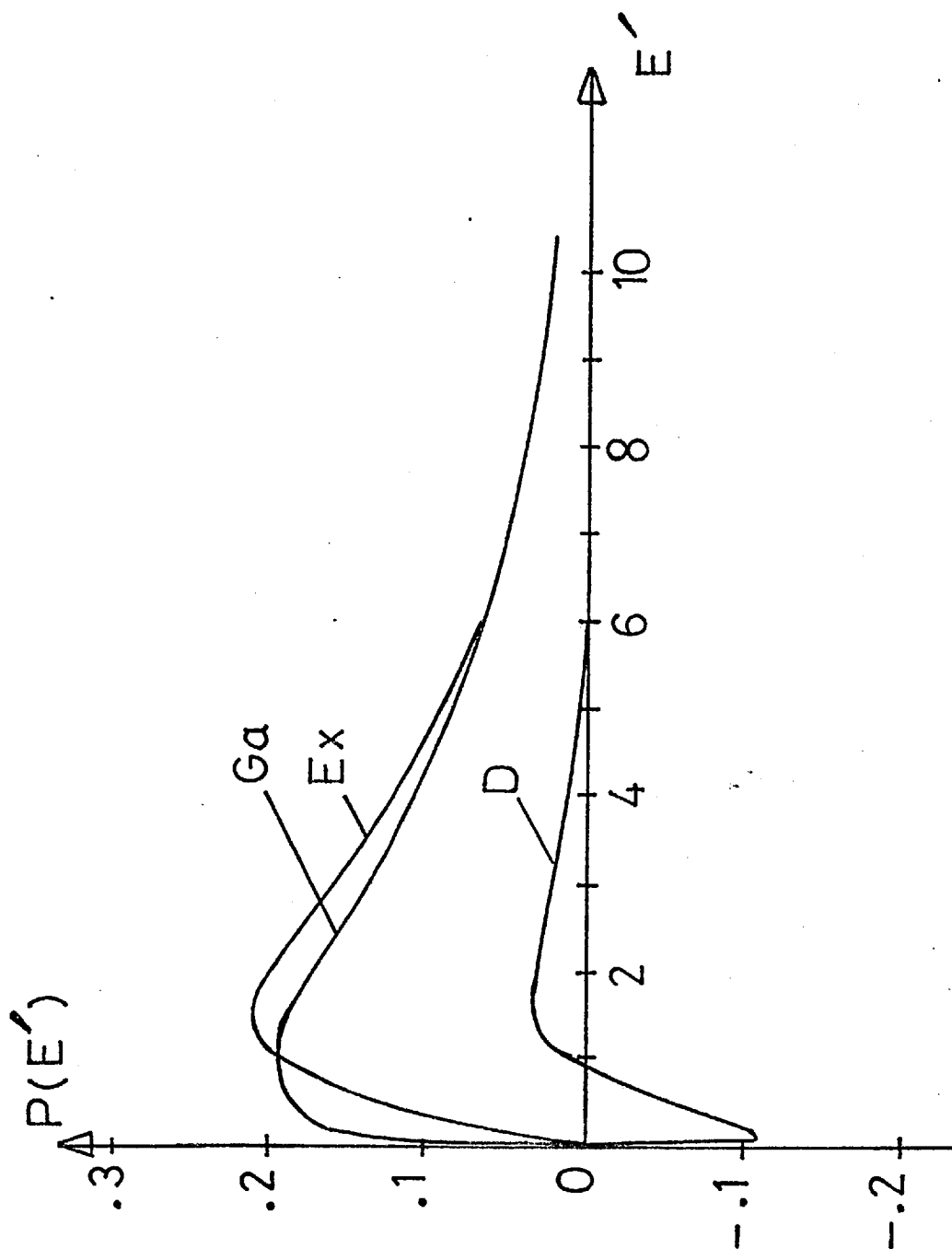


FIG. 20 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 1.5$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

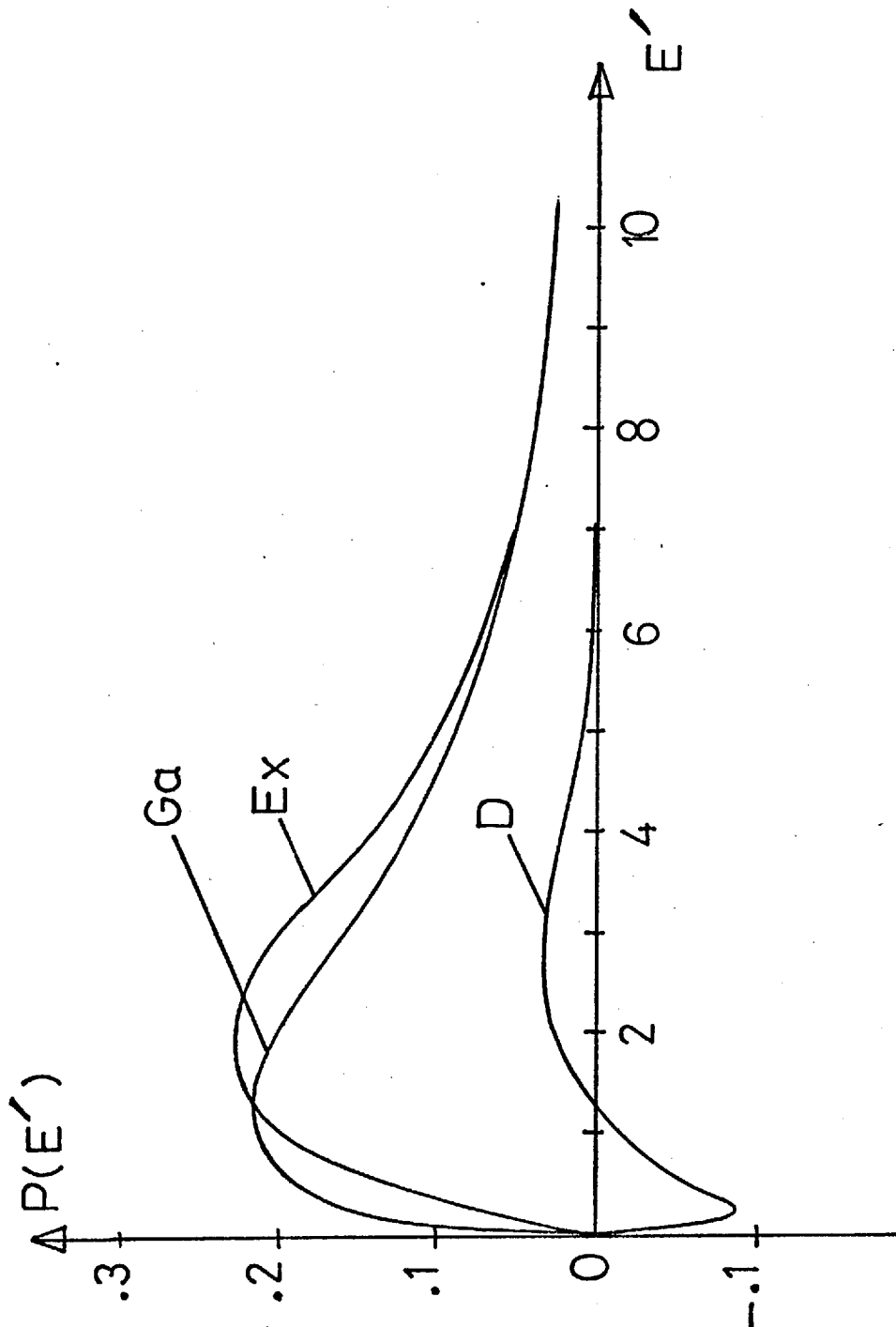


FIG. 21 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 2.0$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

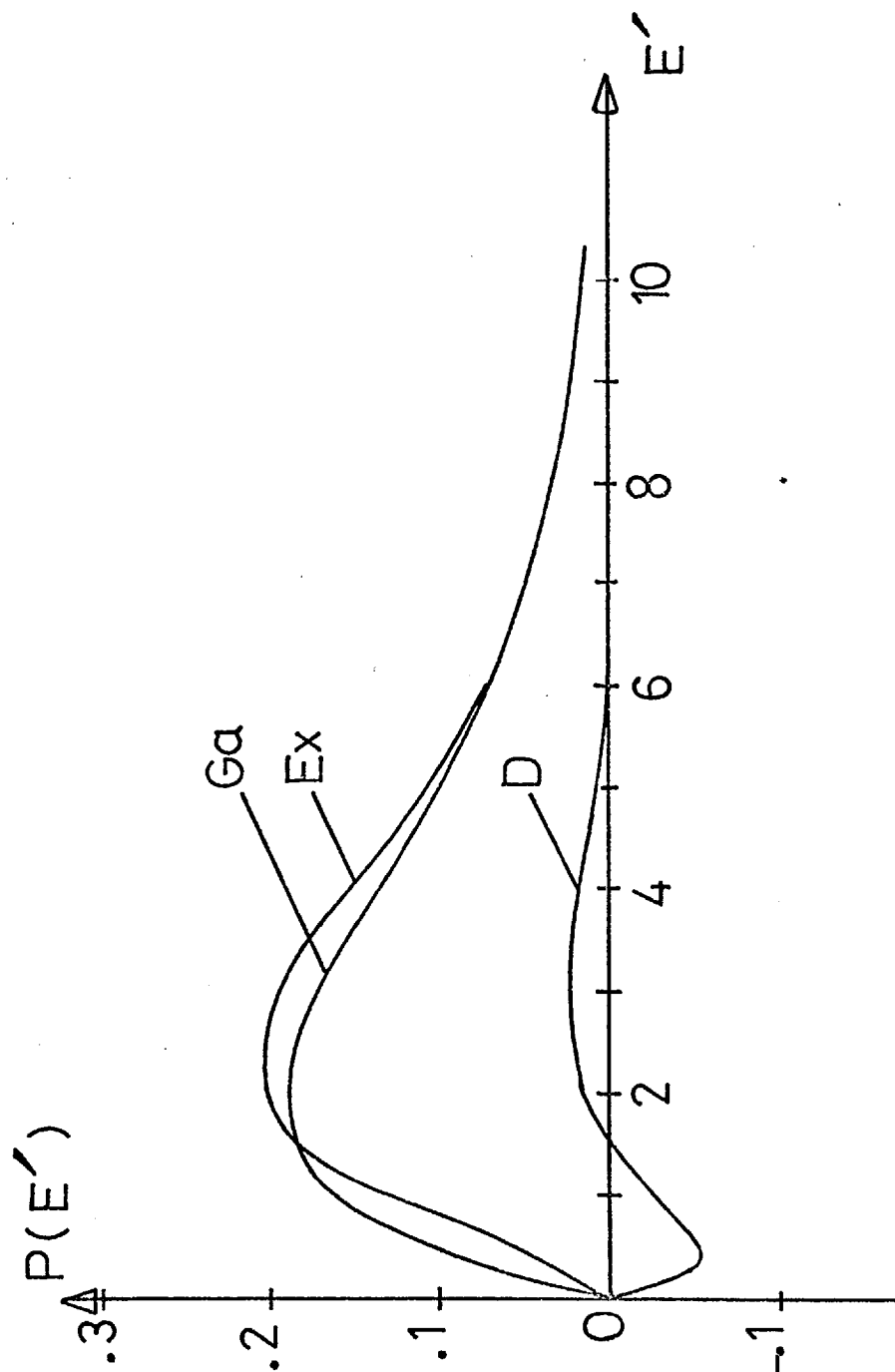


FIG. 22 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 3$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

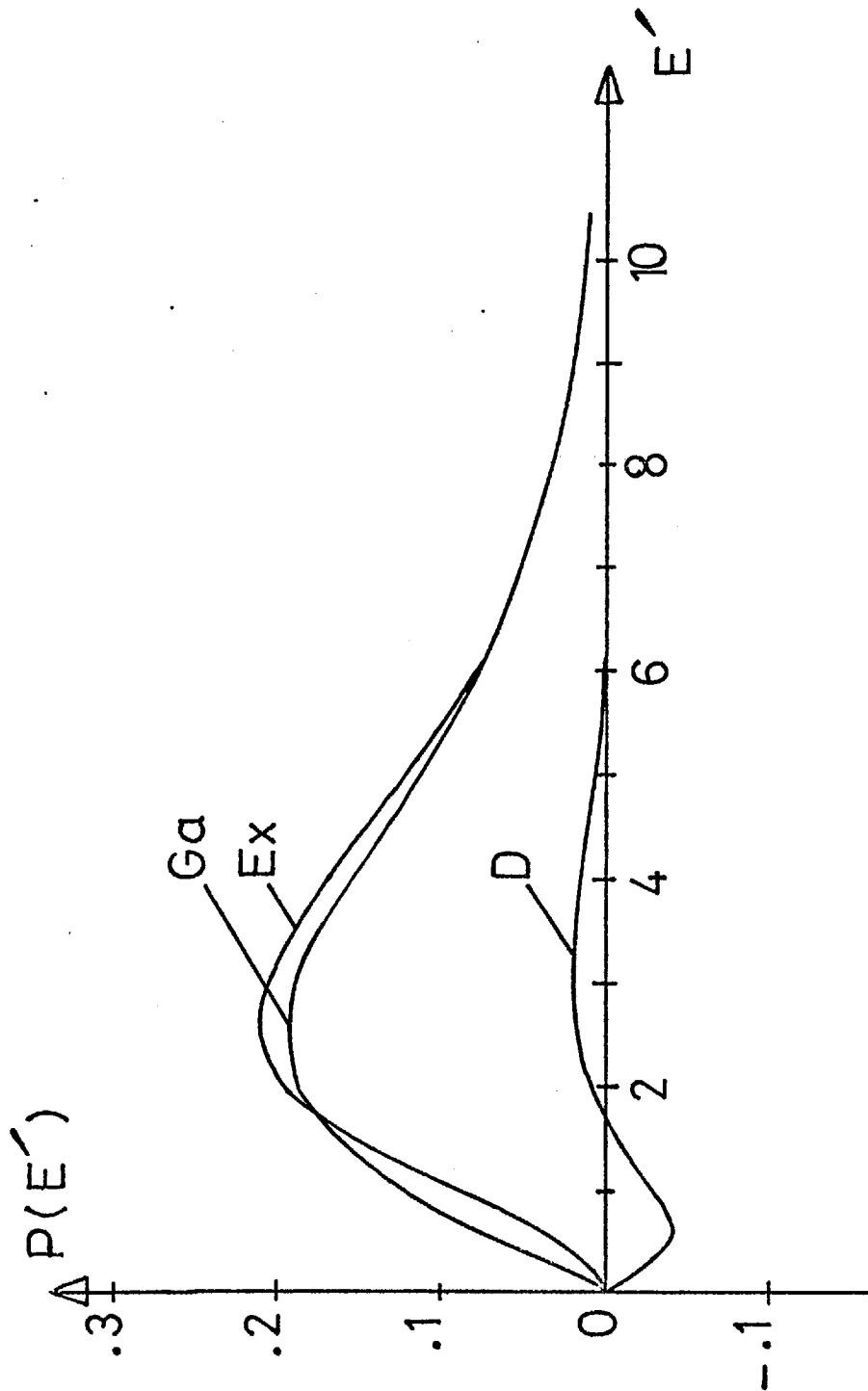


FIG. 23 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 4$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

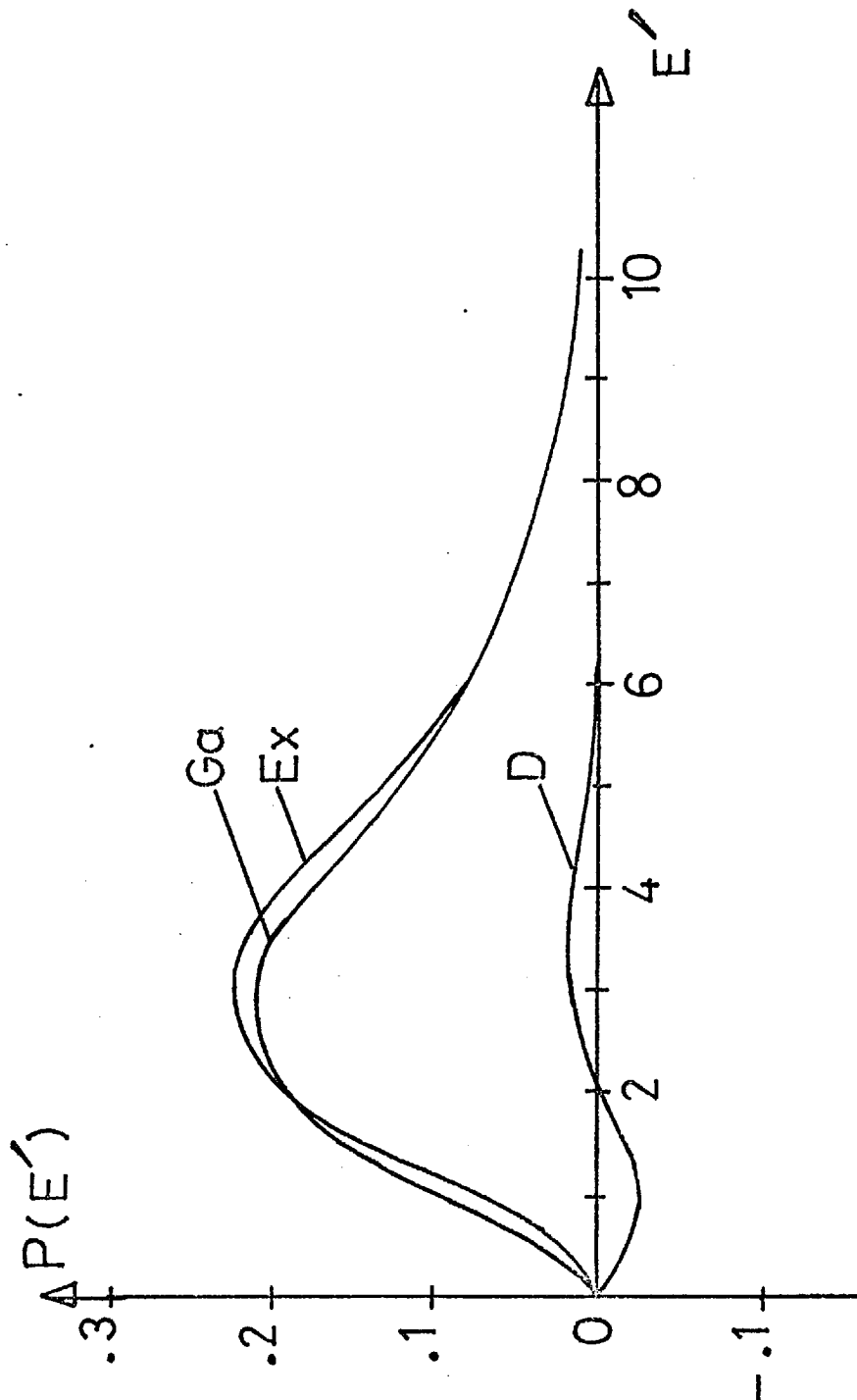


FIG. 24 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 5$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

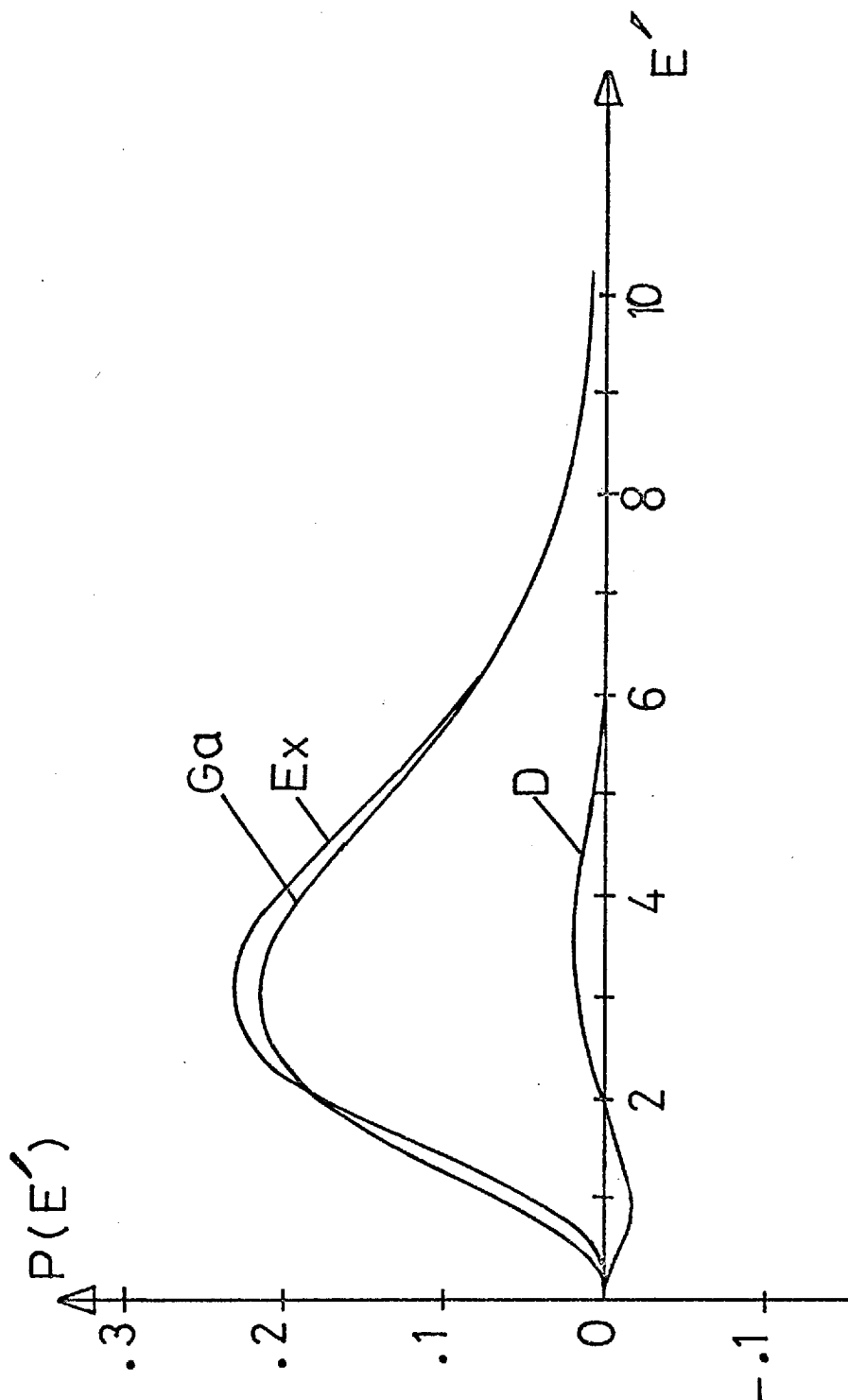


FIG. 25 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 6$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

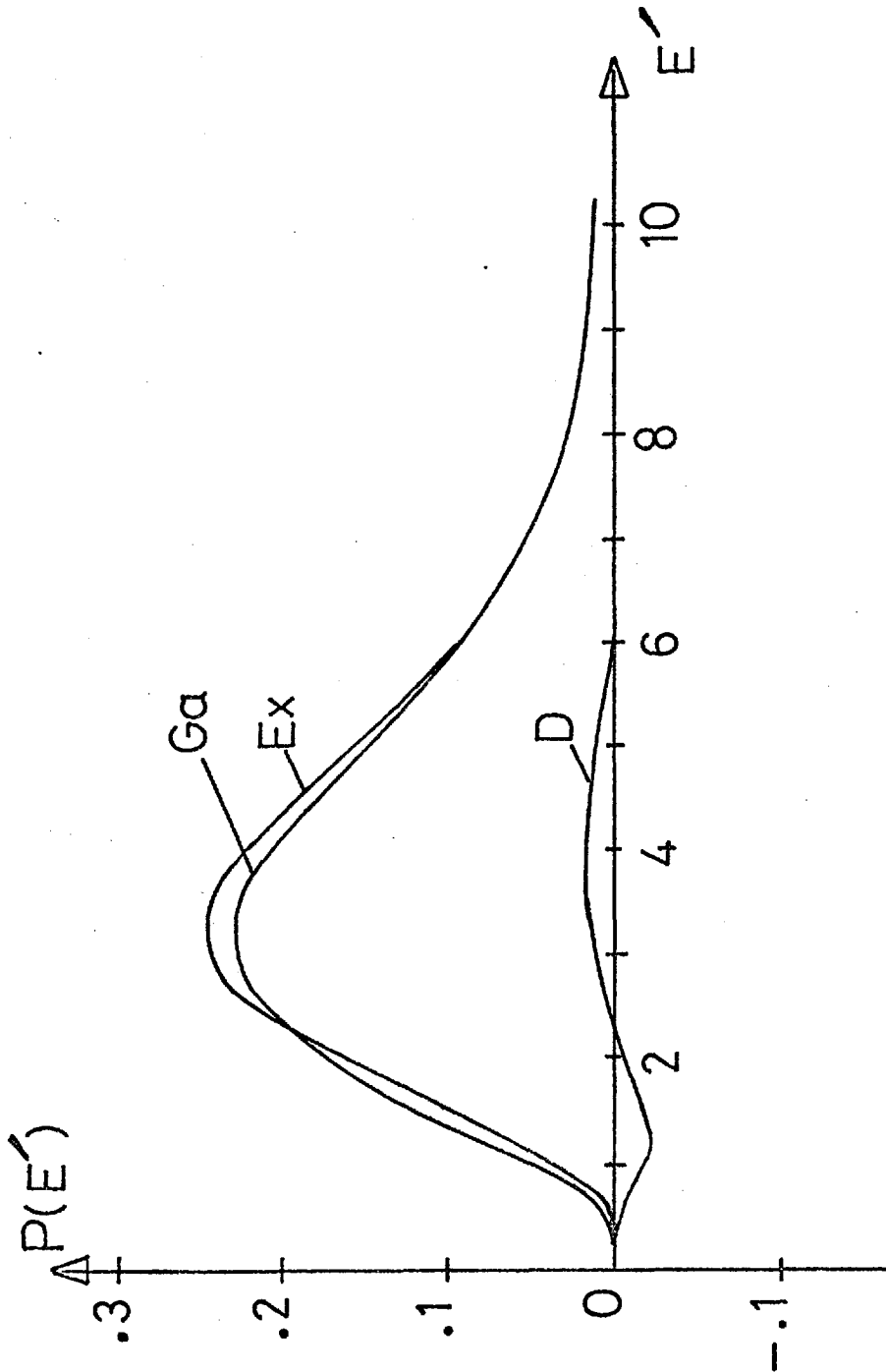


FIG. 26 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 7$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

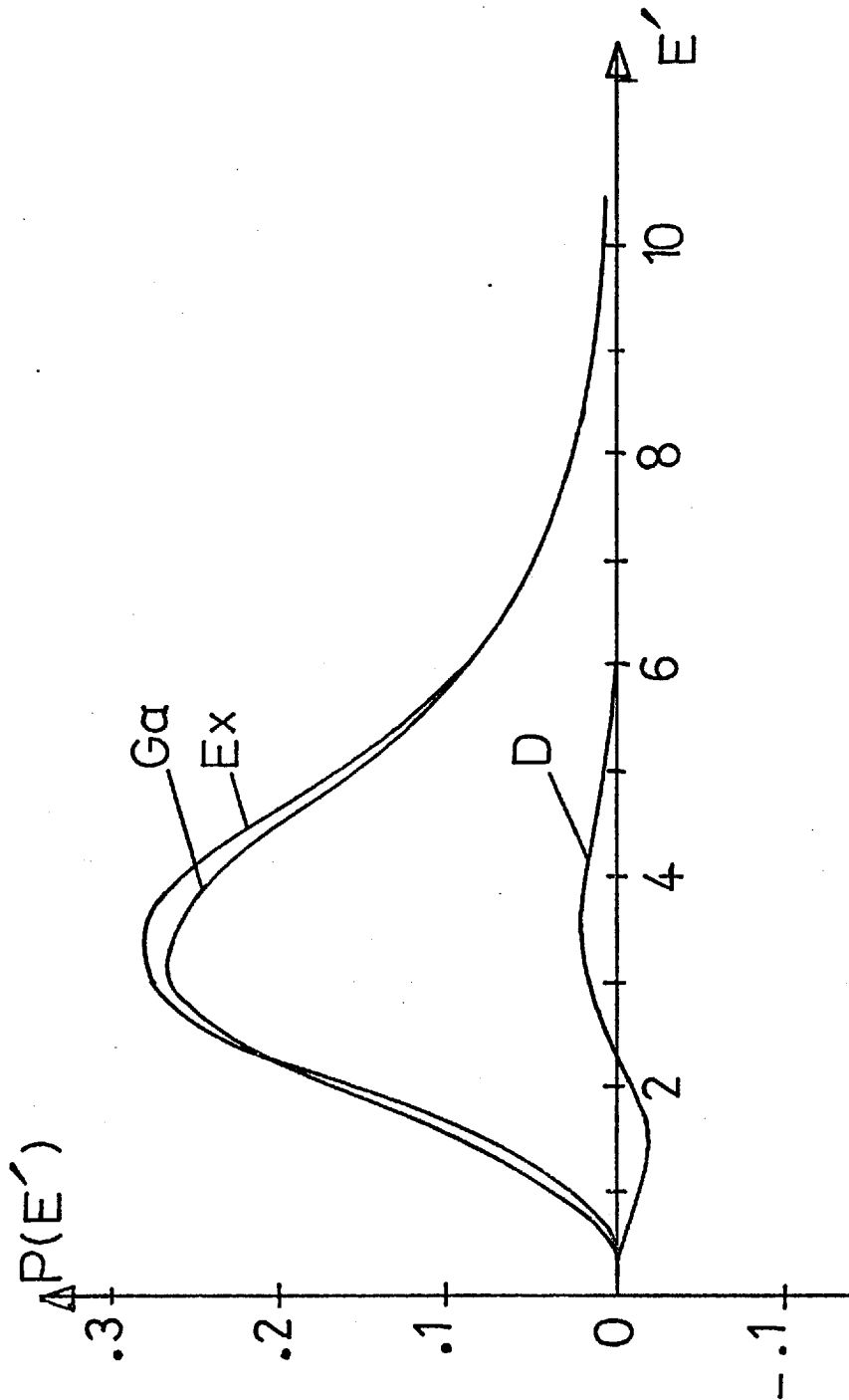


FIG. 27 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 8$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

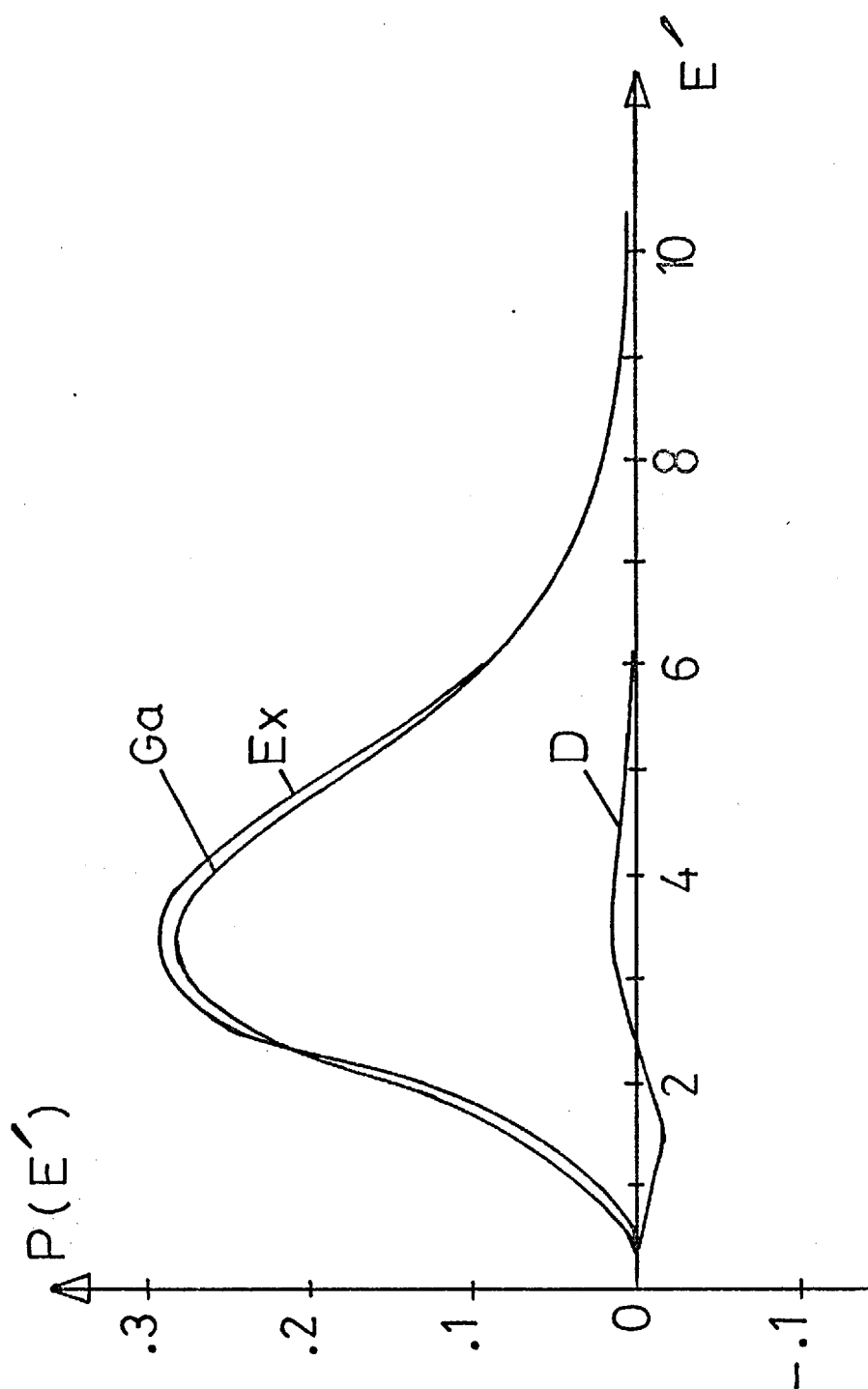


FIG. 28 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE SLIT CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE $C = 9$. Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

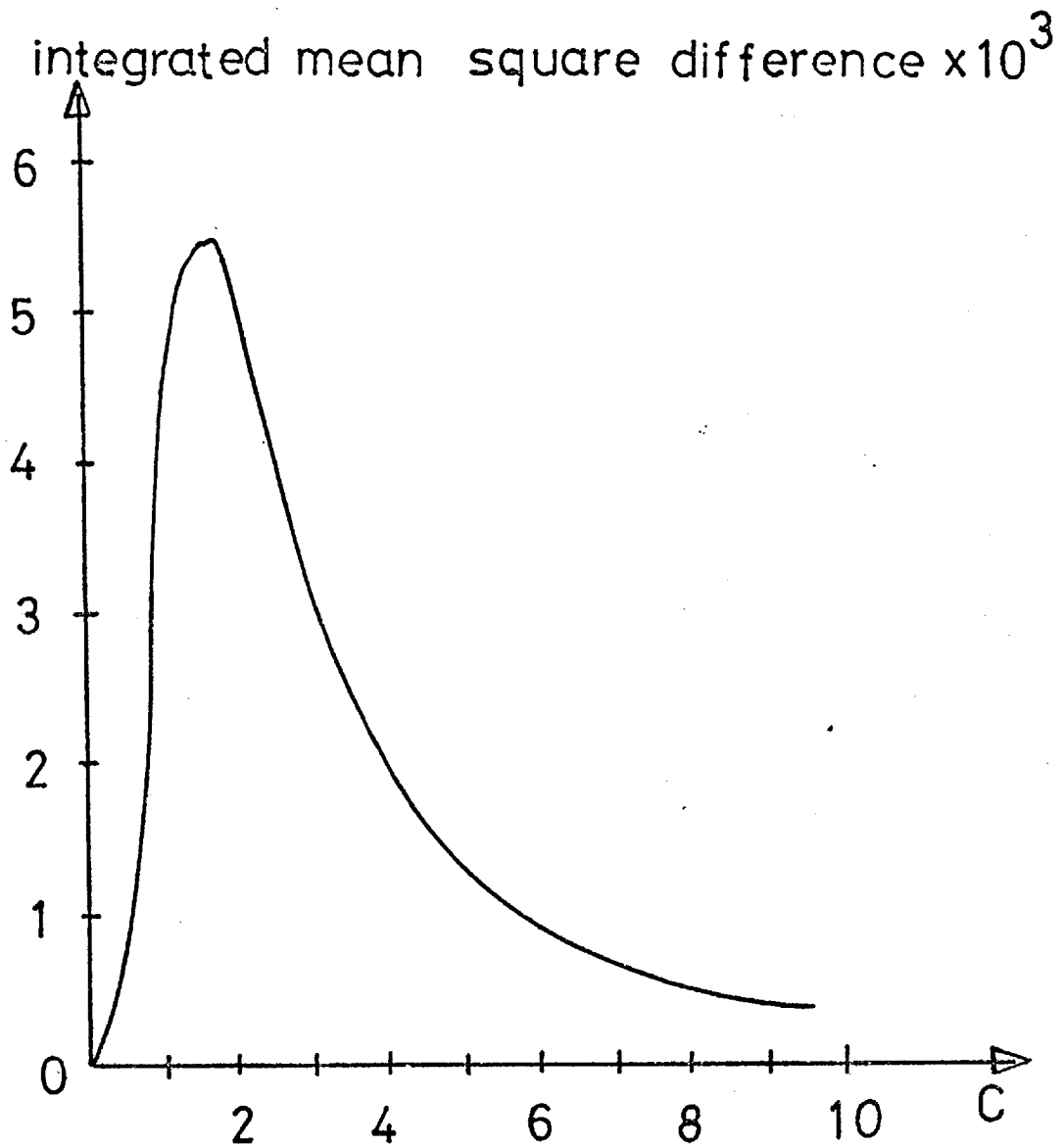


FIG. 29 THE DIFFERENCE BETWEEN THE APPROXIMATE AND EXACT DISTRIBUTION FOR SLIT APERTURES WITH UNIT INTENSITY TRANSMITTANCES IS CHARACTERISED BY THE GRAPH OF THE INTEGRATED MEAN SQUARE DIFFERENCE VERSUS THE RELATIVE APERTURE C .

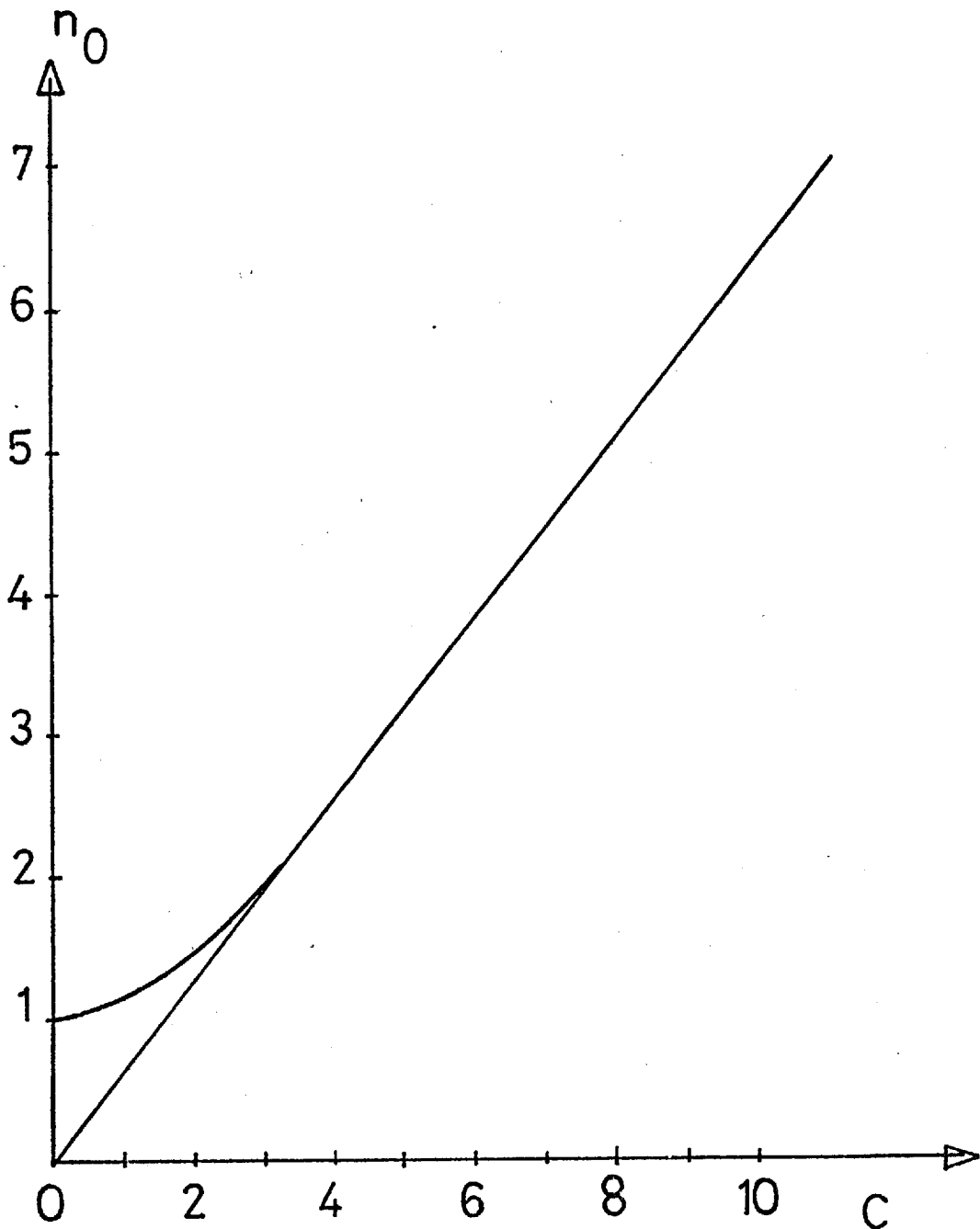


FIG. 30 CUT-OFF INDEX OF THE UNIT INTENSITY TRANSMITTANCE SLIT

APERTURES EIGENVALUES (PROLATE SPHEROIDAL EIGENVALUES) VERSUS

c THE RELATIVE APERTURE, n_0 ALSO DEFINES THE NUMBER OF INDEPENDENT

AUTOCORRELATION CELLS THE SCANNING APERTURE CAN BE DEVIDED INTO.

3) A Rectangular (two dimensional) collecting pupil with a rectangular intensity transmittance profile and a rectangular scanning aperture with a rectangular intensity transmittance profile. (see fig. 6 c)

This case is a generalization of the preceding one. We make the standard assumption that in rectangular geometry functions of two variables are separable, this in hardware space and Fourier space. Let (x, y) denote a point in hardware space of coordinates x and y and let $f(\cdot)$ be any function in hardware space. Similarly let $F(\cdot)$ be any function in Fourier space and (u, v) a point of coordinates u and v in Fourier space. Our assumption is then formulated by

$$f(x, y) = f_1(x) f_2(y) \quad (71)$$

$$F(u, v) = F_1(u) F_2(v) \quad (72)$$

It is this assumption that makes this case a straightforward generalization from the preceding case and we therefore give it a short treatment. The formulae are adapted with obvious notations from the above case and the corresponding equations are indicated in the first brackets after each new equation.

The collecting pupil P is now a rectangle $2U \times 2W$ defined by the function,

$$\begin{aligned} P(u, v) &= P_1(u) P_2(v) = \text{rect} \frac{u}{U} \text{rect} \frac{v}{W} \quad (24) \quad (73) \\ &= 1 \text{ if } |u| \leq U \text{ and } |v| \leq W \\ &= 0 \text{ otherwise} \end{aligned}$$

and the scanning aperture is a rectangle of size $A \times B$. The relative aperture is defined by

$$c = c_1 \times c_2 = \frac{\pi U A}{\lambda D} \frac{\pi V B}{\lambda D} = \frac{\pi^2 U V A B}{\lambda^2 D^2} \quad (29) \quad (74)$$

The amplitude power spectrum of the scanned speckle pattern is,

$$\Pi_{|V|,S} = \frac{P_1(u) P_2(v) \langle E' \rangle}{\int_{-\infty}^{+\infty} P_1(u) du \int_{-\infty}^{+\infty} P_2(v) dv} \quad (25) \quad (75)$$

The amplitude autocorrelation function is found from the Fourier transform of this equation and is,

$$\begin{aligned} R(x-x', y-y') &= R_1(x-x') R_2(y-y') \\ &= \langle E' \rangle \frac{\sin(2\pi U(x-x')/\lambda D)}{2\pi U(x-x')/\lambda D} \\ &\quad \times \frac{\sin(2\pi W(y-y')/\lambda D)}{2\pi W(y-y')/\lambda D} \end{aligned} \quad (26) \quad (76)$$

The integral equation leading to the required eigenvalues is given by,

$$\epsilon_n \epsilon_m \phi_n(x) \phi_m(y) = \int_{-\frac{A}{2}}^{+\frac{A}{2}} R_1(x-x') \phi_n(x') dx' \int_{-\frac{B}{2}}^{+\frac{B}{2}} R_2(y-y') \phi_m(y') dy' \quad (27) \quad (77)$$

Where the quantities ϵ_n and ϵ_m are the eigenvalues corresponding to each integral and are scaled versions of the eigenvalues Λ_n given in equs. (28) and (48). As we are interested only in the eigenvalues to evaluate the first order probability density functions we perform a re-indexing which allows us to express our formulae more simply. This is defined by

$$\gamma_k = \epsilon_n \epsilon_m \quad (78)$$

where to each index n and to each index m we associate a single index k .

Let the set γ be sorted in decreasing order such that

$$\gamma_0 \gg \gamma_1 \gg \gamma_2 \dots \gg 0$$

the corresponding eigenfunctions will be

$$\phi_k(x, y) = \phi_n(x) \phi_m(y) \quad (79)$$

From Mercer's theorem we can write

$$\frac{\sin(c_1(x-x'))}{\pi(x-x')} \frac{\sin(c_2(y-y'))}{\pi(y-y')} = \sum_k \gamma_k \phi_k^*(x',y') \phi_k(x,y) \quad (49) \quad (80)$$

Integration of this equation over the intervals $(x) = [-1, +1]$

and $(y) = [-1, +1]$ gives

$$\frac{2c_1c_2}{\pi^2} = \frac{2c}{\pi^2} = \sum_{k=0}^{\infty} \gamma_k \quad (50) \quad (81)$$

the "equivalent set" of eigenvalues defined by equ 3.3 (32) gives

$$n_0 \gamma_0 = \sum_{k=0}^{\infty} \gamma_k \quad (51) \quad (81)$$

inserting this equation in equ. (80) gives

$$n_0 = \frac{2c_1c_2}{\pi^2 \gamma_0} \quad (52) \quad (82)$$

with similar asymptotic conditions as in the slit case. The first order probability density functions will be given by,

- For the approximate Gamma distribution -

$$P(E') = \left(\frac{2UWAB}{\langle E' \rangle \lambda^2 D^2 \gamma_0} \right)^{n_0} \frac{e^{-\frac{2UWAB}{\lambda^2 D^2 \langle E' \rangle \gamma_0}}}{\Gamma(n_0)} \quad (64) \quad (83)$$

- For the exact distribution with all eigenvalues different from each other

$$P(E') = \frac{2UWAB}{\langle E' \rangle \lambda^2 D^2} \sum_{k=0}^{\infty} \frac{G_k}{\gamma_k} e^{-\frac{2UWAB}{\langle E' \rangle \lambda^2 D^2 \gamma_k}} \quad (65) \quad (84)$$

with the partial fraction coefficients given by

$$G_k = \prod_{l \neq k} \frac{\gamma_k}{\gamma_k - \gamma_l} \quad (66) \quad (85)$$

- For the exact distribution with n_0 eigenvalues equal to the first and the others different from the first.

$$P(E') = \left(\frac{2 U W A B}{\langle E' \rangle \lambda^2 D^2 \gamma_0} \right)^{n_0+1} E'^{n_0} e^{-\frac{2 U A W B}{\langle E' \rangle \lambda^2 D^2 \gamma_0}}$$

$$\times \sum_{k=0}^{\infty} \frac{G_k}{\gamma_k} \left\{ \sum_{p=0}^{\infty} \frac{\left[\frac{2 U W A B}{\langle E' \rangle \lambda^2 D^2} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_k} \right) E' \right]^p}{\Gamma(n_0 + p + 1)} \right\}$$

(69) (86)

with the partial fraction coefficients G_k given by

$$G_k = \prod_{\substack{l \neq k \\ l > n_0}} \frac{\gamma_k}{\gamma_k - \gamma_l} \quad (70) \quad (87)$$

One special case is of interest; when we are dealing with square apertures, i.e.

when $U = V$ and $A = B$ then we have

$$c = c_1^2 = \frac{\pi^2 U^2 A^2}{\lambda^2 D^2} \quad (88)$$

the cut off index of the "equivalent set" of eigenvalues or also the number of independent correlation cells is given by

$$n_0 = \frac{4 c_1^2}{\pi^2 \gamma_0^2} \quad (89)$$

As the two sets of eigenvalues corresponding respectively to each perpendicular dimension are identical, some of the γ_k 's in equ. (77) are two-fold degenerate and none of the exact first order probability density formula given so far applies.

This two-fold degeneracy introduces a computing difficulty which may be overcome as follows:

Consider the sequence of eigenvalues for square apertures obtained for $c_1 = c_2 = 2$.

$$\begin{array}{lll} \gamma_0 & = & .77539 \quad E + 00 \\ \gamma_1 & = & .31316 \quad E + 00 \\ \gamma_2 & = & .31316 \quad E + 00 \\ \gamma_3 & = & .12648 \quad E + 00 \\ \vdots & & \vdots \end{array} \quad \left. \begin{array}{l} \uparrow \\ \downarrow \end{array} \right\} \text{degeneracy}$$

where γ_1 and γ_2 show two-fold degeneracy. These eigenvalues are given with a relative precision of 10^{-5} . From experience we know we do not need such a precision to calculate our probability densities and that generally a relative precision of 10^{-3} on the eigenvalues is sufficient. Without inconvenience we can therefore add a 6th decimal to our eigenvalues that can be easily handled by the computer and has the effect of lifting the degeneracy.

For example our eigenvalues can be written for this case

$$\begin{aligned}\gamma_0 &= .775390 \text{ E} + 00 \\ \gamma_1 &= .313161 \text{ E} + 00 \\ \gamma_2 &= .313160 \text{ E} + 00 \\ \gamma_3 &= .126480 \text{ E} + 00 \\ &\vdots\end{aligned}$$

The exact first order probability density function is now given for this case by

$$P(E') = \frac{2 U^2 A^2}{\langle E' \rangle \lambda^2 D^2} \sum_{k=0}^{\infty} \frac{G_k}{\gamma_k} e^{-\frac{2 U^2 A^2}{\langle E' \rangle \lambda^2 D^2 \gamma_k}} \quad (90)$$

For rectangular and square apertures we limit our results to four cases with

$c_2 = 2$ fixed and c_1 respectively equal to 1, 2, 4 and 8. These cases are

illustrated in figs. (31) to (34). Figure (35) is a plot of the integrated mean

square difference as a function of c , fig. (36) shows a plot of the cut off index

in equ. (82) as a function of $c = c_1 \times c_2$. We do not give tables of eigen-

values as these are easily calculated from the slit eigenvalues.

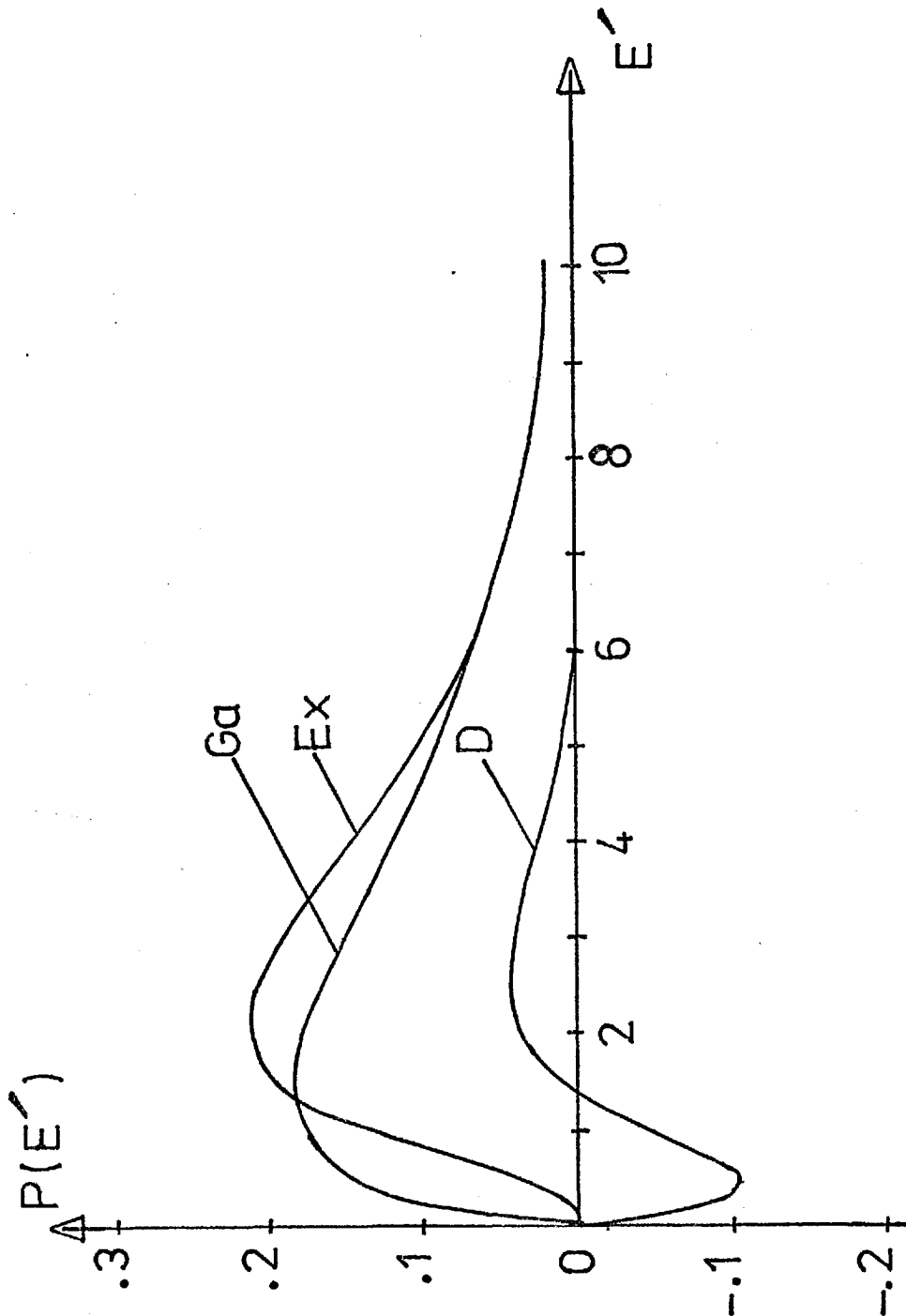


FIG. 31 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE
 RECTANGULAR APERTURES OF EXACT AND APPROXIMATE PROBABILITY
 DENSITIES FOR ARELATIVE APERTURE $C = 1 \times 2 = 2$. Ga = GAMMA DISTRI-
 BUTION, Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

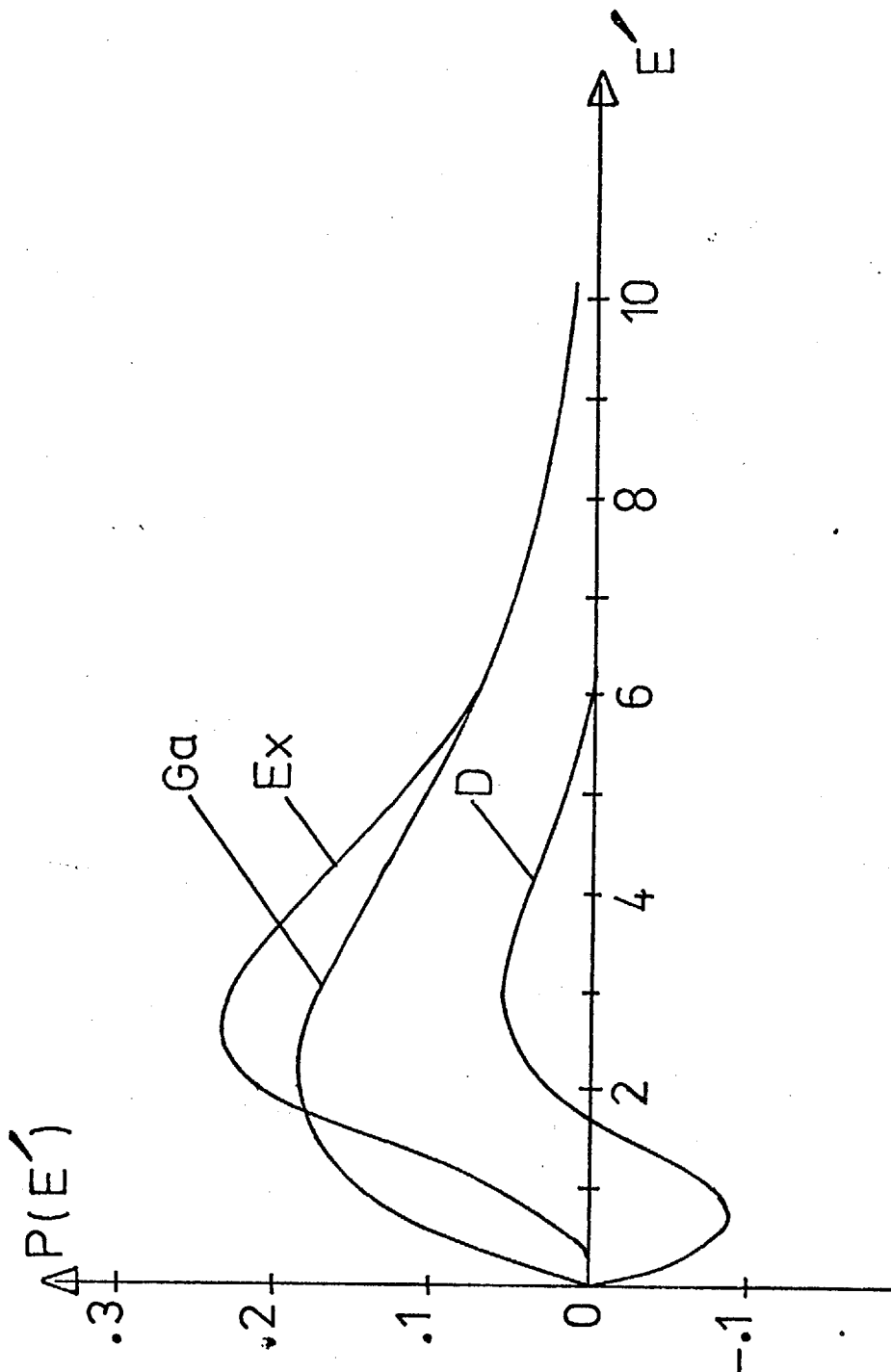


FIG. 32 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE
 RECTANGULAR APERTURES OF EXACT AND APPROXIMATE PROBABILITY DEN-
 SITIES FOR A RELATIVE APERTURE $C = 2 \times 2 = 4$. Ga = GAMMA DISTRIBUTION,
 Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

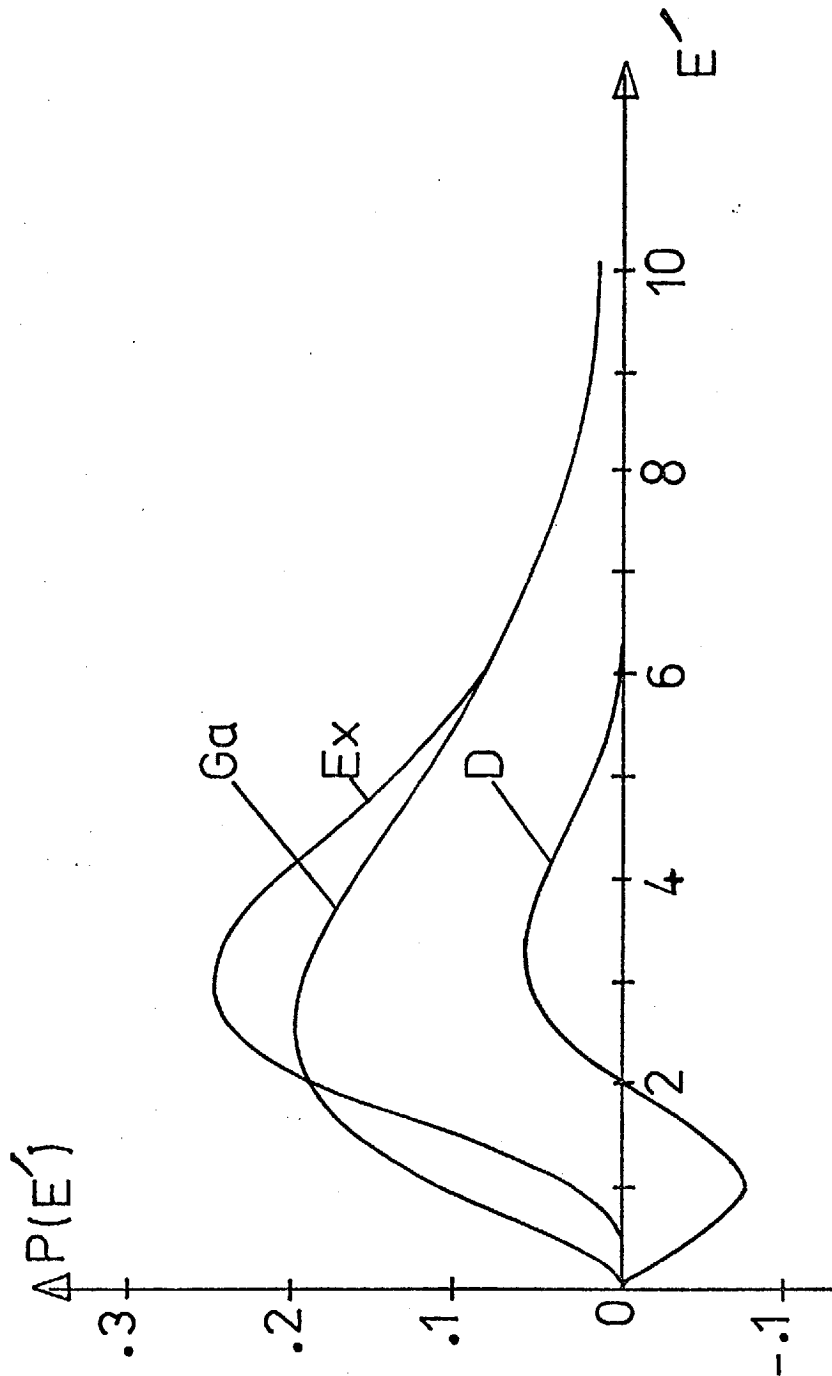


FIG. 33 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE
RECTANGULAR APERTURES OF EXACT AND APPROXIMATE PROBABILITY
DENSITIES FOR A RELATIVE APERTURE $C = 3 \times 2 = 6$. Ga = GAMMA DISTRI-
BUTION, Ex = EXACT DISTRIBUTION, D= DIFFERENCE.

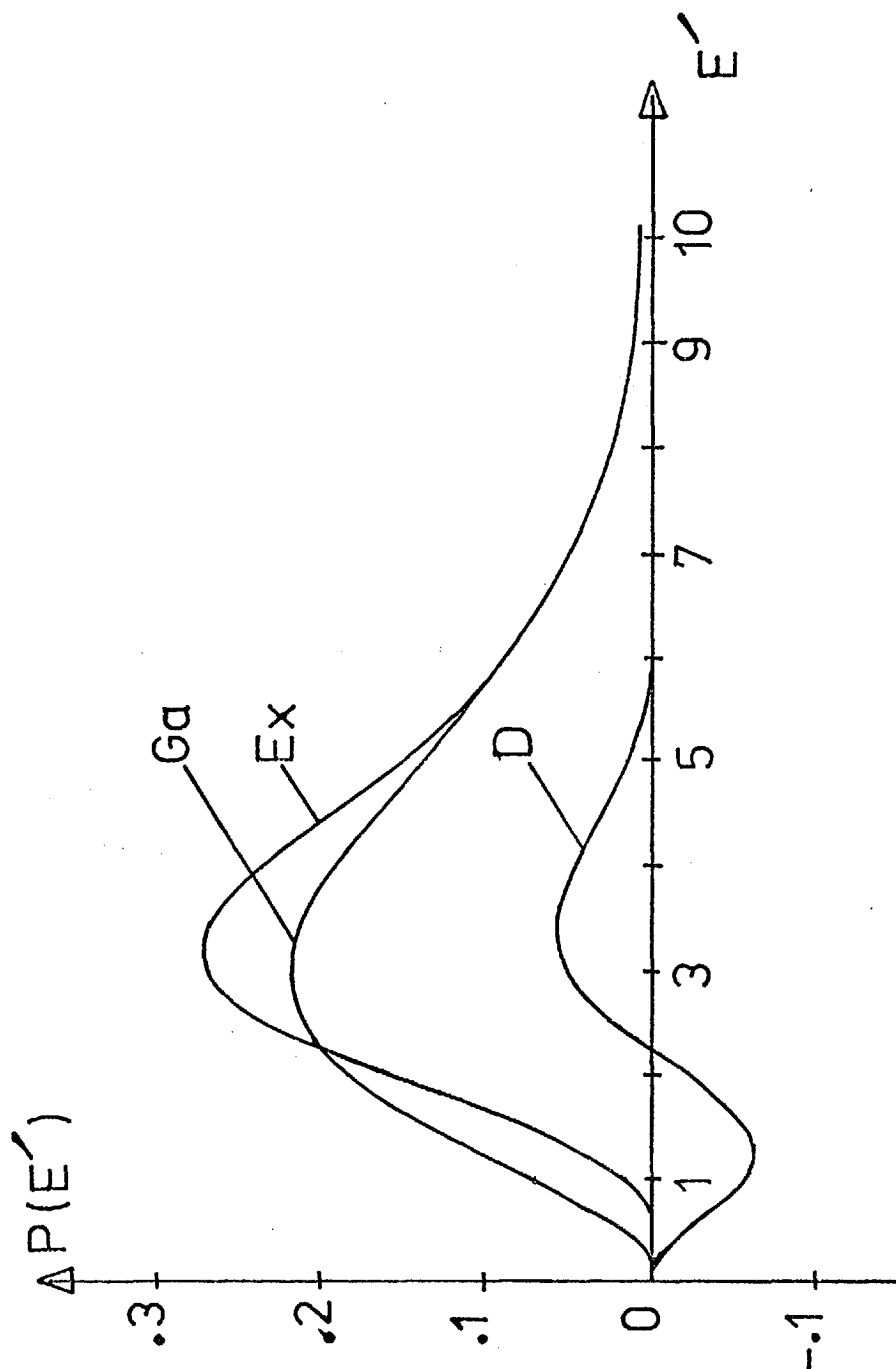


FIG. 34 COMPARATIVE PLOTS FOR UNIT INTENSITY TRANSMITTANCE
 RECTANGULAR APERTURES OF EXACT AND APPROXIMATE PROBABILITY
 DENSITIES FOR A RELATIVE APERTURE $C = 4 \times 2 = 8$. Ga = GAMMA DISTRI-
 BUTION , Ex = EXACT DISTRIBUTION, D = DIFFERENCE.

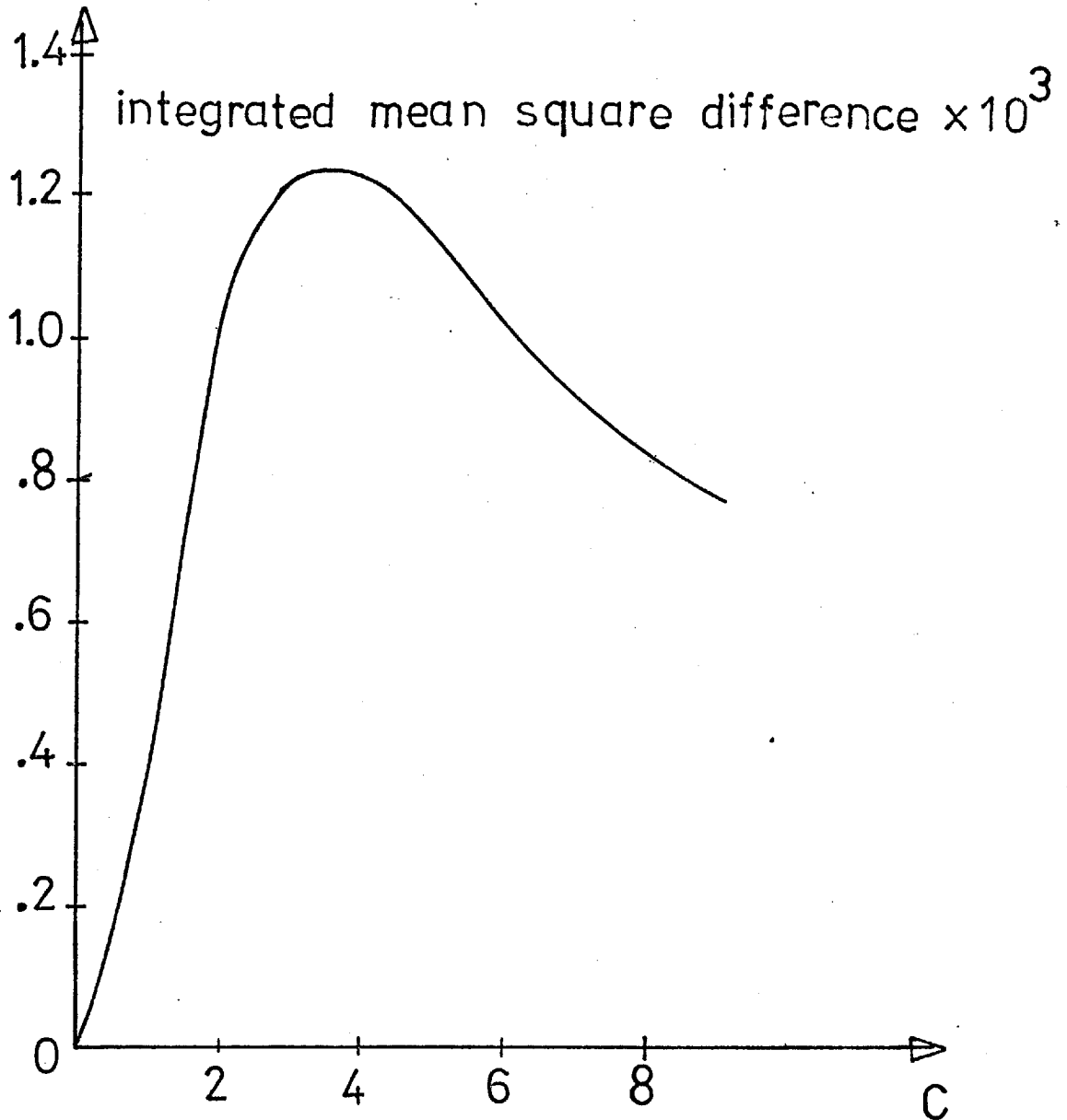


FIG. 35 THE DIFFERENCE BETWEEN THE APPROXIMATE AND EXACT DISTRIBUTION FOR UNIT TRANSMITTANCE RECTANGULAR APERTURES IS CHARACTERIZED BY THE GRAPH OF THE INTEGRATED MEAN SQUARE DIFFERENCE VERSUS THE RELATIVE APERTURE C . HERE WE HAVE $C = C_1 \times C_2$ (SEE TEXT) WITH C_1 CONSTANT AND SET TO 2.

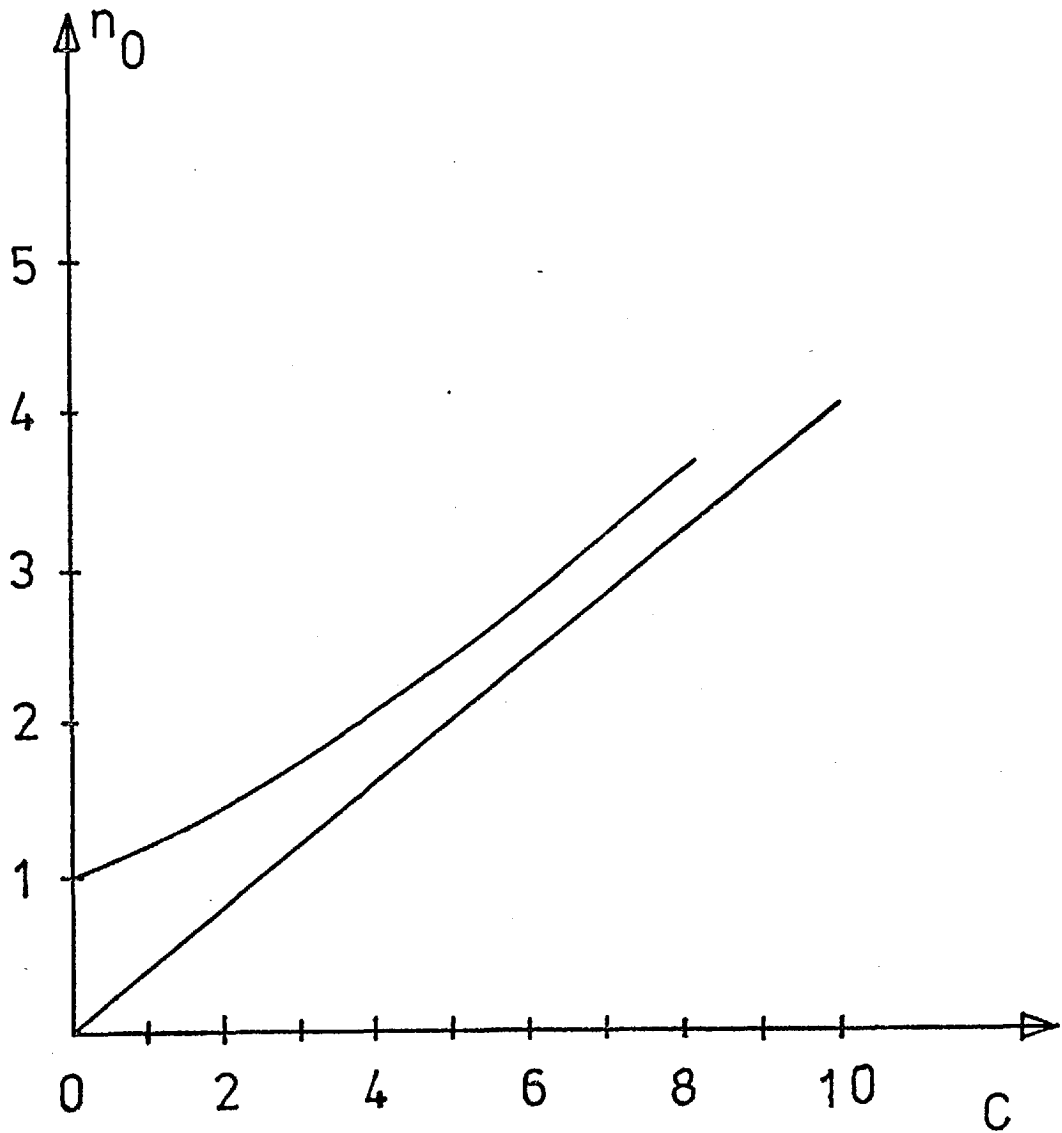


FIG. 36 CUT-OFF INDEX OF THE UNIT INTENSITY TRANSMITTANCE

RECTANGULAR APERTURES VERSUS C THE RELATIVE APERTURE, n_0 ALSO DEFINES THE NUMBER OF INDEPENDENT AUTOCORRELATION CELLS THE SCANNING APERTURE CAN BE DEVIDED INTO.

4) A Circular (two dimensional) collecting pupil with a "top hat" intensity transmittance profile and a circular scanning aperture with a "top hat" intensity transmittance profile.

This is perhaps the most important case to be considered as practically all optical instruments contain circular geometry. The collecting pupil and the scanning aperture are both circular holes in opaque screens and have unit intensity transmittance over their transmitting portions, see fig. 6 d. The radius of the collecting pupil is U and the radius of the scanning aperture is $Z/2$. The eigenvalues obtained for circular geometry will yield the correct statistics to naturally occurring speckle patterns viewed through conventional instruments.

For this case we find the eigenvalues by matrix diagonalisation as mentioned in section 3.

Autocorrelation function and integral equation.

The amplitude autocorrelation function is found as for the other cases by taking the Fourier transform of the amplitude power spectrum, although we now use two dimensional polar coordinates, (u, α) for the Fourier space and (r, θ) for the hardware space. With polar coordinates, the amplitude power spectrum is expressed by,

$$\pi_{|V|,S} = \frac{|P(u, \alpha)|^2 \langle E' \rangle}{\int_0^\infty \int_0^{2\pi} |P(u, \alpha)|^2 u \, du \, d\alpha} \quad (91)$$

where $P(u, \alpha)$ is the pupil function of unit intensity transmittance defined by

$$\begin{aligned} P(u, \alpha) &= 1 & \text{if} & & 0 \leq u \leq U \\ & & \text{and} & & 0 \leq \alpha \leq 2\pi \\ P(u, \alpha) &= 0 & \text{otherwise} & & \end{aligned} \quad (92)$$

This function is real positive with circular symmetry. As for the slit and rectangular cases its squared modulus is equal to itself and the amplitude power spectrum can therefore be written

$$\Pi_{A,S} = \frac{P(u, \alpha) \langle E' \rangle}{\int_0^\infty \int_0^{2\pi} P(u, \alpha) u du d\alpha} \quad (93)$$

Because the pupil function has circular symmetry, the Fourier transform and Hankel transform of the amplitude power spectrum are identical (BRACEWELL 1965).

This transformation gives us the required autocorrelation function

$$R(\rho) = \frac{\langle E' \rangle}{\int_0^\infty \int_0^{2\pi} P(u, \alpha) u du d\alpha} \int_0^U J_0 \left(\frac{2\pi u \rho}{\lambda D} \right) u du \quad (94)$$

Where $J_0(\cdot)$ is the zero order Bessel function (ABRAMOWITZ & STEGUN

1970) and the quantity ρ defines the distance between two points in hardware space expressed in polar coordinates (r, θ) and (r', θ') ,

$$\rho = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \quad (95)$$

With the well known recurrence relation (BORN and WOLF 1970)

$$\frac{d}{d\rho} \left\{ \rho^{n+1} J_{n+1}(\rho) \right\} = \rho^{n+1} J_n(\rho) \quad (96)$$

the expression in equ (94) becomes

$$R(\rho) = \frac{2 J_1 \left(\frac{2\pi U \rho}{\lambda D} \right)}{\frac{2\pi U \rho}{\lambda D}} \langle E' \rangle \quad (97)$$

where $J_1(\cdot)$ is the first order Bessel function. The eigenvalues leading to

the exact first order probability density functions are given by the integral

equation,

$$\varepsilon \phi(r, \theta) = \langle E' \rangle \int_0^{Z/2} \int_0^{2\pi} \frac{2 J_1 \left(\frac{2\pi U \rho}{\lambda D} \right)}{\frac{2\pi U \rho}{\lambda D}} \phi(r', \theta') r' dr' d\theta' \quad (98)$$

Solution of the integral equation.

Let us consider a scaled version of the above integral equation that yields scaled versions of the required eigenvalues but that is more practical to handle ,

$$\Lambda \phi(r, \theta) = \int_0^1 \int_0^{2\pi} \frac{2 J_1(c \rho)}{c \rho} \phi(r', \theta') r' dr' d\theta' \quad (99)$$

Where c defines as in the previous slit and rectangle cases the relative aperture (or space-bandwidth product) ,

$$c = \frac{2 \pi U Z}{\lambda D} \quad (100)$$

In equ (99) the variables r, r', θ and θ' are now dimensionless.

Up to now there does not seem to exist any differential equation approach to the solution of integral equations of the type in equ. (99). A related integral equation defining Generalized prolate spheroidal wave Functions invented by SLEPIAN 1964 and HEURTLEY 1964 and HEURTLEY and STREIFER 1965 ,

$$\Lambda_n \psi_n(r) = \int_0^1 \frac{c J_1(c r r')}{2 \pi^2 r r'} \psi_n(r') r' dr' \quad (101)$$

was considered by the author in early investigations. This equation is claimed by SLEPIAN 1964 to be the two dimensional analogue in circular coordinates of the integral equation used for slits defining ordinary prolate spheroidal wave functions (see equ. (37)). However the integral equ. (101) has two disadvantages. First, it only deals with the radial coordinates of the two points (r, θ) and (r', θ') under consideration, secondly, it does not obey the first equation (equ. 3.2 (15)) derived from Mercer's theorem, i.e. the integrated kernel with $r = r'$ is not equal to the sum of the eigenvalues and is less than this sum by a changing factor,

$$\int_0^1 \frac{c J_1(c r^2)}{2 \pi^2 r^2} dr < \sum_{n=0}^{\infty} \Lambda_n \quad (102)$$

These are the reasons that lead us to find the eigenvalues of equ. (99) by the technique of matrix diagonalization mentioned in section 3.

Let us formulate the integral in equ. (99) as the limit of an infinite sum of terms, to r' we associate the quantity nh and to θ' we associate the quantity ka ,

$$\Lambda \phi(r, \theta) = \lim_{\substack{N \rightarrow \infty \\ K \rightarrow \infty}} \sum_{n=0}^N \sum_{k=0}^K \frac{2 J_1(c \rho(r, n, \theta, k))}{c \rho(r, n, \theta, k)} \phi(n, k) nh^2 a \quad (103)$$

where

$$\phi(n, k) = \phi(nh, ka) \quad (104)$$

and

$$\rho(r, n, \theta, k) = \sqrt{r^2 + n^2 h^2 - 2 r n h \cos(\theta - ka)} \quad (105)$$

The quantities N, h, K and a verify the boundary relations,

$$Nh = 1 \quad \text{and} \quad Ka = 2\pi \quad (106)$$

or similarly

$$0 \leq nh \leq Nh = 1 \quad \text{and} \quad 0 \leq ka \leq 2\pi \quad (106a)$$

The eigenvalue Λ is independent of the parameters r and θ , therefore these can be chosen,

$$r = mh \quad \text{and} \quad \theta = la \quad (107)$$

with

$$0 \leq m \leq N \quad \text{and} \quad 0 \leq l \leq K \quad (107a)$$

In view of this, equ. (103) becomes,

$$\Delta \phi(m, l) = \lim_{\substack{N \rightarrow \infty \\ K \rightarrow \infty}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{2 J_1(c \rho(m, n, l, k))}{c \rho(m, n, l, k)} \phi(n, k) n h^2 a \quad (108)$$

with

$$\rho(m, n, l, k) = h \sqrt{m^2 + n^2 - 2 m n \cos((l-k) a)} \quad (109)$$

The quantities n, k, h and a define in fact a mesh over which to perform summation, h and a are the radial and angular increments and represent the lattice parameters of the mesh and n and k are the indices corresponding to the nodal points of the mesh. Let us choose this mesh such that each nodal point corresponds to a constant differential surface element expressed in polar coordinates,

$$r' dr' d\theta' = cst. \quad (110)$$

with the definitives in equ. (106a) this becomes

$$n h^2 a = cst. \quad (111)$$

For $n = 1$, let us choose the angular increment $a = \pi/2$, for a constant radial increment we then have the relation,

$$n h^2 a = \frac{h^2 \pi}{2} \quad \text{giving} \quad a = \frac{\pi}{2n} \quad (112)$$

From the boundary relation for K expressed in equ. (106) we find

$$\frac{K \pi}{2n} = 2\pi \quad \text{giving} \quad K = 4n \quad (113)$$

This lattice is illustrated in fig. (37). So defined the angular quantities k and a depend on the radial index n and the boundary conditions are now written

$$1 \leq k \leq 4n \quad \text{and} \quad 1 \leq n \leq N \quad (114)$$

In view of these considerations equ. (108) becomes

$$\Delta \phi(m, l) = \lim_{N \rightarrow \infty} \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{4n} \frac{2 J_1(c \rho(m, n, l, k))}{c \rho(m, n, l, k)} \phi(n, k) \right\} \frac{h^2 \pi}{2} \quad (115)$$

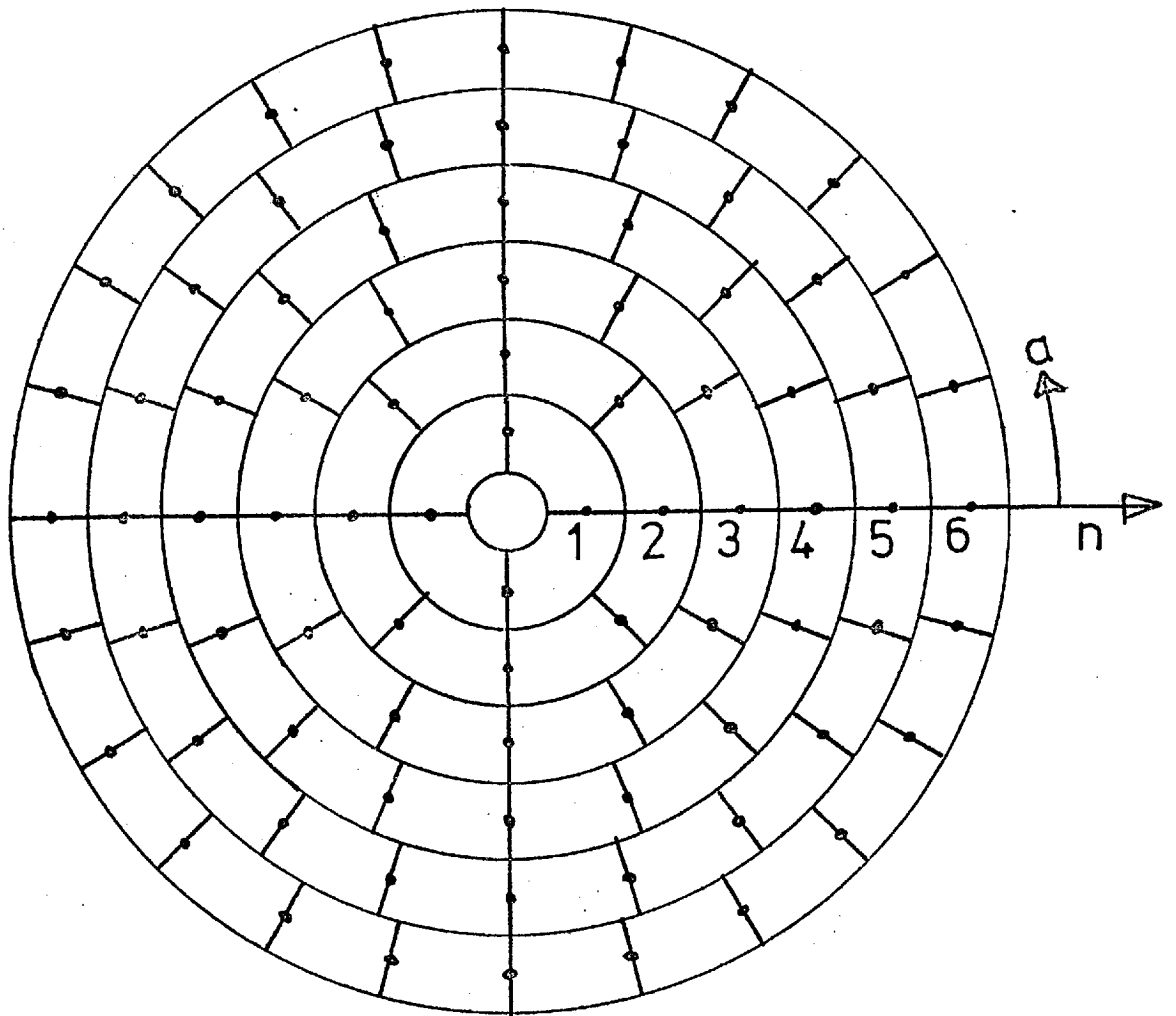


FIG.37

INTEGRATION MESH FOR CIRCULAR APERTURES AS DEFINED BY EQN. 4.2 (111)
 THE QUANTITIES h , n , a ARE RESPECTIVELY, THE RADIAL INTEGRATION
 INCREMENT, THE RADIAL INDEX AND THE ANGULAR INTEGRATION INCREMENT.

$$\text{with } 1 \leq l \leq 4m \quad \text{and} \quad 1 \leq m \leq N \quad (116)$$

Written in this form equ. (115) would yield the eigenvalues Λ by diagonalization of the four-dimensional matrix,

$$M(m, n, l, k) = \left[\frac{h^2 \pi}{2} \frac{2 J_1(c \rho(m, n, l, k))}{c \rho(m, n, l, k)} \right] \quad (117)$$

This is not practical and uselessly difficult. Each pair of indices (n, k) represents a single point of the mesh to which a single index can be associated provided some sort of order is used. Therefore to each pair (n, k) we associate the single number (j) and to each pair (m, l) we associate the single number (i) .

Equation (115) now becomes

$$\Lambda \phi(i) = \lim_{I \rightarrow \infty} \sum_{j=1}^I \frac{2 J_1[c \rho(i, j)]}{c \rho(i, j)} \phi(j) \frac{h^2 \pi}{2} \quad (118)$$

For the above defined mesh we find to a given number N corresponds points, this gives the boundary condition for i and j ,

$$1 \leq i, j \leq I = 2(N^2 + N) \quad (119)$$

An example of the order we use to associate the index j to the pair of indices (n, k) is illustrated below, in fact any other order is suitable

to	$1 \leq n \leq N$	and	$1 \leq k \leq 4n$	corresponds	j
	1		1		1
	1		2		2
	1		3		3
	1		4		4
	2		1		5
	2		2		6
	2		3		7
	⋮		⋮		⋮
	2		8		12
	⋮		⋮		⋮
	N		$4N$		$2(N^2 + N)$

In practice to compute the eigenvalues N does not tend to infinity but is fixed to a value large enough that the desired precision in the eigenvalues is obtained. The parameter h is a function of the value of N and can be determined by the following arguments.

The first equation derived from Mercer's theorem (equ. 3.2 (15)) applied to equ. (99) gives

$$\int_0^1 \int_0^{2\pi} \frac{2 J_1(0)}{(0)} r dr d\theta = \sum_f \Lambda_f = \pi \quad (120)$$

the same equation applied to equ. (118) gives

$$\sum_{j=1}^{2(N^2+N)} \frac{2 J_1(0)}{(0)} \frac{h^2 \pi}{2} = \sum_f \Lambda_f = h^2 \pi (N^2 + N) \quad (121)$$

Comparison of eqs (120) and (121) yields the required value

$$h = \sqrt{\frac{1}{N^2 + N}} \quad (122)$$

The eigenvalues are then found by solving the analogous equation to equ. 3.4 (69)

$$\det \left[\Lambda - \frac{1}{N^2 + N} \frac{\pi}{2} \frac{2 J_1(c p(i,j))}{c p(i,j)} \right] = 0 \quad (123)$$

which is equivalent to diagonalizing the matrix

$$M(i,j) = \sqrt{\frac{1}{N^2 + N}} \frac{\pi}{2} \left[\frac{2 J_1(c p(i,j))}{c p(i,j)} \right] \quad (124)$$

A computer program devised to do this is listed in appendix B.

Numerical Results

Table 5 shows the eigenvalues obtained by the above method. For a given relative aperture c , not all the eigenvalues are different, some are two-fold

EIGENVALUES FOR CIRCULAR TOP HAT APERTURES AS CALCULATED FROM
EQN. 4.2 (118)

m	EIGENVALUE	n	EIGENVALUE
	C = 0.1		C = 1.5
0	.313767 E+01	0	.239725 E+01
1	.196135 E-02	1	.345947 E+00
2	.196135 E-02	2	.345946 E+00
	C = 0.2	3	.221150 E-01
0	.312593 E+01	4	.221029 E-01
1	.781973 E-02	5	.643941 E-02
2	.781972 E-02	6	.780311 E-03
3	.855189 E-05	7	.780309 E-03
	C = 0.5		C = 2.0
0	.304532 E+01	0	.197410 E+01
1	.477654 E-01	1	.509667 E+00
2	.477653 E-01	2	.509666 E+00
3	.527723 E-03	3	.595566 E-01
4	.327572 E-03	4	.595186 E-01
	C = 1.0	5	.200456 E-01
0	.277822 E+01	6	.382782 E-02
1	.176068 E+00	7	.382781 E-02
2	.176068 E+00	8	.518811 E-03
3	.489677 E-02	9	.518810 E-03
4	.489437 E-02		
5	.127553 E-02		

table 5

n	EIGENVALUE	n	EIGENVALUE
C = 2.5		C = 3.5	
0	.157178 E+01	0	.958375 E+00
1	.627983 E+00	1	.674272 E+00
2	.627982 E+00	2	.674271 E+00
3	.118346 E+00	3	.263169 E+00
4	.118254 E+00	4	.262833 E+00
5	.470458 E-01	5	.146905 E+00
6	.122539 E-01	6	.576033 E-01
7	.122539 E-01	7	.576032 E-01
8	.195025 E-02	8	.137260 E-01
9	.195025 E-02	9	.137260 E-01
10	.788978 E-03	10	.775948 E-02
11	.788468 E-03	11	.775244 E-02
C = 3.0		12	.812654 E-03
0	.122908 E+01	13	.804227 E-03
1	.681853 E+00	14	.697832 E-03
2	.681852 E+00	15	.697831 E-03
3	.190874 E+00	16	.393703 E-03
4	.190688 E+00	C = 4.0	
5	.902431 E-01	0	.753780 E+00
6	.295013 E-01	1	.623184 E+00
7	.295013 E-01	2	.623183 E+00
8	.567933 E-02	3	.320531 E+00
9	.567932 E-02	4	.319976 E+00
10	.281863 E-02	5	.207455 E+00
11	.281650 E-02	6	.954378 E-01
12	.245440 E-03	7	.954376 E-01
13	.243096 E-03	8	.285254 E-01
14	.181050 E-03	9	.285254 E-01
15	.181049 E-03	10	.174745 E-01
		11	.174550 E-01
		12	.226019 E-02
		13	.223457 E-02
		14	.212097 E-02
		15	.212097 E-02

table 5 (cond.)

n	EIGENVALUE	m	EIGENVALUE
C = 4.0 (contd)		C = 5.0	
16	.115128 E-02	0	.489043 E+00
17	.182550 E-03	1	.474471 E+00
18	.181786 E-03	2	.474470 E+00
19	.130199 E-03	3	.359689 E+00
20	.130198 E-03	4	.358486 E+00
		5	.290132 E+00
		6	.178711 E+00
	C = 4.5	7	.178711 E+00
0	.601887 E+00	8	.833068 E-01
1	.550677 E+00	9	.833067 E-01
2	.550676 E+00	10	.558805 E-01
3	.353287 E+00	11	.557765 E-01
4	.352442 E+00	12	.117226 E-01
5	.258715 E+00	13	.115618 E-01
6	.138209 E+00	14	.114613 E-01
7	.138208 E+00	15	.114613 E-01
8	.518787 E-01	16	.669351 E-02
9	.518786 E-01	17	.164552 E-02
10	.334520 E-01	18	.163619 E-02
11	.334044 E-01	19	.107139 E-02
12	.546584 E-02	20	.107139 E-02
13	.539781 E-02	21	.356026 E-03
14	.533324 E-02	22	.356026 E-03
15	.533322 E-02	23	.172312 E-03
16	.294060 E-02	24	.172311 E-03
17	.599045 E-03	25	.762469 E-04
18	.596154 E-03	26	.740597 E-04
19	.399153 E-03		
20	.399153 E-03		
21	.122894 E-03		
22	.122894 E-03		

table 5 (contd)

n	EIGENVALUE	n	EIGENVALUE
	C = 5.5		C = 6.0
0	.404738 E+00	0	.345286 E+00
1	.404737 E+00	1	.345286 E+00
2	.404315 E+00	2	.339902 E+00
3	.344870 E+00	3	.317199 E+00
4	.343260 E+00	4	.315161 E+00
5	.298319 E+00	5	.286959 E+00
6	.209888 E+00	6	.227367 E+00
7	.209888 E+00	7	.227366 E+00
8	.118904 E+00	8	.152109 E+00
9	.118904 E+00	9	.152109 E+00
10	.830143 E-01	10	.111297 E+00
11	.828070 E-01	11	.110915 E+00
12	.225161 E-01	12	.388726 E-01
13	.221750 E-01	13	.382181 E-01
14	.215402 E-01	14	.360060 E-01
15	.215401 E-01	15	.360059 E-01
16	.137124 E-01	16	.253486 E-01
17	.389012 E-02	17	.807671 E-02
18	.386369 E-02	18	.800975 E-02
19	.257053 E-02	19	.558221 E-02
20	.257052 E-02	20	.558220 E-02
21	.927240 E-03	21	.219999 E-02
22	.927238 E-03	22	.219999 E-02
23	.508336 E-03	23	.129974 E-02
24	.508333 E-03	24	.129974 E-02
25	.222119 E-03	25	.581236 E-03
26	.215075 E-03	26	.560754 E-03
27	.523532 E-04	27	.164469 E-03
28	.488238 E-04	28	.152361 E-03
29	.460394 E-04	29	.132148 E-03
30	.430818 E-04	30	.123278 E-03
		31	.733533 E-04
		32	.457008 E-04
		33	.457007 E-04

table 5 (contd)

n	EIGENVALUE	n	EIGENVALUE
	C = 6.5		C = 7.0
0	.296236 E+00	0	.256209 E+00
1	.296236 E+00	1	.256209 E+00
2	.290566 E+00	2	.252374 E+00
3	.284527 E+00	3	.252180 E+00
4	.282070 E+00	4	.249335 E+00
5	.263184 E+00	5	.234206 E+00
6	.230670 E+00	6	.222611 E+00
7	.230669 E+00	7	.222611 E+00
8	.176449 E+00	8	.188422 E+00
9	.176449 E+00	9	.188422 E+00
10	.136351 E+00	10	.154445 E+00
11	.135695 E+00	11	.153389 E+00
12	.603929 E-01	12	.846121 E-01
13	.592500 E-01	13	.827801 E-01
14	.542297 E-01	14	.743603 E-01
15	.542296 E-01	15	.743701 E-01
16	.422244 E-01	16	.632614 E-01
17	.149533 E-01	17	.249796 E-01
18	.147990 E-01	18	.246527 E-01
19	.110397 E-01	19	.199087 E-01
20	.110397 E-01	20	.199087 E-01
21	.478879 E-02	21	.957908 E-02
22	.478878 E-02	22	.957907 E-02
23	.293213 E-02	23	.591639 E-02
24	.293212 E-02	24	.591636 E-02
25	.138522 E-02	25	.303290 E-02
26	.133078 E-02	26	.289966 E-02
27	.451277 E-03	27	.109891 E-02
28	.414687 E-03	28	.999905 E-03
29	.347525 E-03	29	.845349 E-03
30	.323149 E-03	30	.783459 E-03
31	.198346 E-03	31	.497004 E-03
32	.127871 E-03	32	.326854 E-03
33	.127870 E-03	33	.326852 E-03
34	.498920 E-04	34	.144708 E-03
35	.498913 E-04	35	.144706 E-03
		36	.509621 E-04
		37	.509619 E-04

table 5 (contd.)

n	EIGENVALUE	n	EIGENVALUE
	C = 7.0 (contd)		C = 8.0
38	.339421 E-04	0	.197345 E+00
39	.290503 E-04	1	.196756 E+00
40	.257137 E-04	2	.196755 E+00
	C = 7.5	3	.195610 E+00
0	.223534 E+00	4	.193857 E+00
1	.223533 E+00	5	.189378 E+00
2	.222834 E+00	6	.189378 E+00
3	.221519 E+00	7	.181255 E+00
4	.219645 E+00	8	.179947 E+00
5	.207507 E+00	9	.179946 E+00
6	.207506 E+00	10	.164482 E+00
7	.205926 E+00	11	.162237 E+00
8	.188508 E+00	12	.124854 E+00
9	.188507 E+00	13	.121077 E+00
10	.163705 E+00	14	.109516 E+00
11	.162113 E+00	15	.109516 E+00
12	.107460 E+00	16	.104116 E+00
13	.104739 E+00	17	.531030 E-01
14	.937257 E-01	18	.519238 E-01
15	.937355 E-01	19	.488085 E-01
16	.853196 E-01	20	.488084 E-01
17	.380092 E-01	21	.293562 E-01
18	.373668 E-01	22	.293562 E-01
19	.326927 E-01	23	.179465 E-01
20	.326926 E-01	24	.179464 E-01
21	.175611 E-01	25	.114189 E-01
22	.175611 E-01	26	.107906 E-01
23	.107919 E-01	27	.477696 E-02
24	.107918 E-01	28	.423121 E-02
25	.612606 E-02	29	.402960 E-02
26	.582500 E-02	30	.371067 E-02
27	.240482 E-02	31	.253358 E-02
28	.216193 E-02	32	.168778 E-02
29	.191239 E-02	33	.168778 E-02
30	.176659 E-02	34	.873220 E-03
31	.116096 E-02	35	.873208 E-03
		36	.331338 E-03

table 5 (contd)

n	EIGENVALUE	n	EIGENVALUE
	C = 7.5 (contd)		C = 8.0 (contd)
32	.770812 E-03	37	.331337 E-03
33	.770809 E-03	38	.263880 E-03
34	.374298 E-03	39	.213361 E-03
35	.374293 E-03	40	.172690 E-03
36	.134363 E-03	41	.146049 E-03
37	.134363 E-03	42	.146048 E-03
38	.985134 E-04	43	.249887 E-04
39	.830932 E-04		
40	.693226 E-04		
41	.562515 E-04		
42	.562514 E-04		

table 5 (contd)

n	EIGENVALUE	n	EIGENVALUE
	C = 10.0		C = 10.0 (contd)
0	.129957 E+00	28	.326257 E-01
1	.129088 E+00	29	.325960 E-01
2	.129088 E+00	30	.367450 E-01
3	.127501 E+00	31	.260090 E-01
4	.125612 E+00	32	.187462 E-01
5	.125515 E+00	33	.187462 E-01
6	.125515 E+00	34	.107955 E-01
7	.124701 E+00	35	.107954 E-01
8	.123749 E+00	36	.656681 E-02
9	.123749 E+00	37	.656679 E-02
10	.123369 E+00	38	.565336 E-02
11	.121783 E+00	39	.449827 E-02
12	.121626 E+00	40	.370932 E-02
13	.117407 E+00	41	.370932 E-02
14	.117406 E+00	42	.336500 E-02
15	.116151 E+00	43	.887338 E-03
16	.105697 E+00	44	.618949 E-03
17	.100035 E+00	45	.618942 E-03
18	.997398 E-01	46	.613524 E-03
19	.997396 E-01	47	.289741 E-03
20	.927645 E-01	48	.215134 E-03
21	.874444 E-01	49	.154835 E-03
22	.874442 E-01	50	.150818 E-03
23	.617909 E-01	51	.570218 E-04
24	.617906 E-01	52	.570214 E-04
25	.588095 E-01		
26	.533844 E-01		
27	.359146 E-01		

table 5 (end)

degenerate and some very nearly so.

For example, for $c = 1$

n	Λ_n
0	.277822 E + 01
1	.176068 E + 00
2	.176068 E + 00
3	.489677 E - 02
4	.489437 E - 02
5	.127553 E - 02
6	.754525 E - 04
7	.754523 E - 04
8	.821901 E - 05
9	.821900 E - 05
10	.733948 E - 06
:	:
:	:

This degeneracy is due to the circular symmetry of the system.

A similar kind of degeneracy was encountered for the eigenvalues of square diffusing pupils with square scanning apertures, and was also due to the symmetry of the system. In general systems with symmetry will give some kind of degeneracy in the eigenvalues (MEHTA 1964). None of the eigenvalues is more than two fold degenerate and the computing difficulty this introduces is lifted as for square systems by changing a sixth or seventh order decimal in the eigenvalues that does not affect the precision of the results.

The first order probability density function from these eigenvalues is found from equ. 3.3 (35), with the scaling factor α between the computed set and the required set of eigenvalues such that

$$\epsilon_n = \alpha \Lambda_n \quad (125)$$

$$P(E') = \sum_{n=0}^{\infty} \frac{G_n}{\alpha \Lambda_n} e^{-\frac{E'}{\alpha \Lambda_n}} \quad (126)$$

The average integrated intensity and variance are

$$\langle E' \rangle = \alpha \sum_n \Lambda_n \quad (127)$$

and

$$\sigma_{E'}^2 = \alpha^2 \sum_n \Lambda_n^2 \quad (128)$$

From eqs. (120) and (127) we find the scaling constant

$$\alpha = \langle E' \rangle / \pi \quad (129)$$

The "equivalent set" of eigenvalues as defined by equ. 3.3 (32) gives the cut off index (number of independent correlation cells) as

$$n_0 = \pi / \Lambda_0 \quad (130)$$

and the approximate Gamma distribution is,

$$P(E') = \left(\frac{\pi}{\langle E' \rangle \Lambda_0} \right)^{n_0} \frac{E'^{n_0-1} e^{-\frac{E' \pi}{\langle E' \rangle \Lambda_0}}}{\Gamma(n_0)} \quad (131)$$

Table 6 gives numerical values for some essential functions of c .

Figures (38) to (44) give some comparative plots of the exact and approximate distributions. Figure (45) gives a plot of the integrated mean square difference between the exact and approximate distribution as a function of the relative aperture c . Figure (46) gives a plot of the number of independent correlation cells as a function of c . From this graph we observe that when c tends to zero, the quantity n_0 tends to unity. Physically this means that when the scanning aperture tends to a pinhole with a fixed diffusing pupil we are left with one correlation cell only, which is what should be expected. Setting n_0 to unity in equ. (131) shows that the first eigenvalue Λ_0 tends to π as c tends to zero.

A plot of the square root of n_0 as a function of c (fig. (47)) shows

c	λ_0	N	$\sum_{n=0}^N \lambda_n$	$n_0 = \frac{\pi}{\lambda_0}$	integrated mean sq. diff.
.2	.3126E+01	4	3.1416	1.0050	.000001
.5	.3045E+01	5	3.1417	1.0316	.000135
1.	.2778E+01	6	3.1414	1.1308	.000779
2.	.1974E+01	10	3.1412	1.5914	.001226
3.	.1229E+01	13	3.1414	2.5560	.000997
4.	.7538E+00	19	3.1415	4.1678	.000726
6.	.3453E+00	28	3.1415	9.098	.000459

SOME ESSENTIAL FUNCTIONS OF C, THE RELATIVE APERTURE FOR THE EIGENVALUES OBTAINED IN THE CIRCULAR CONFIGURATION (FOURTH CASE) , λ_0 IS THE CORRESPONDING FIRST EIGENVALUE, N IS THE INDEX FOR WHICH THE RATIO $\frac{\lambda_N}{\lambda_0}$ IS JUST LESS THAN 10^{-3} . THE SUM $\sum_{n=0}^N \lambda_n$ INDICATES HOW WELL WE ARE IN AGREEMENT WITH THE SUM PREDICTED BY THE USE OF MERCER'S THEOREM (WHICH IN THIS CASE IS π), n_0 IS THE CUT-OFF INDEX OF THE "EQUIVALENT SET" OF EIGENVALUES, THE INTEGRATED MEAN SQUARE DIFFERENCE IS PLOTTED ON FIG. 45.

table 6

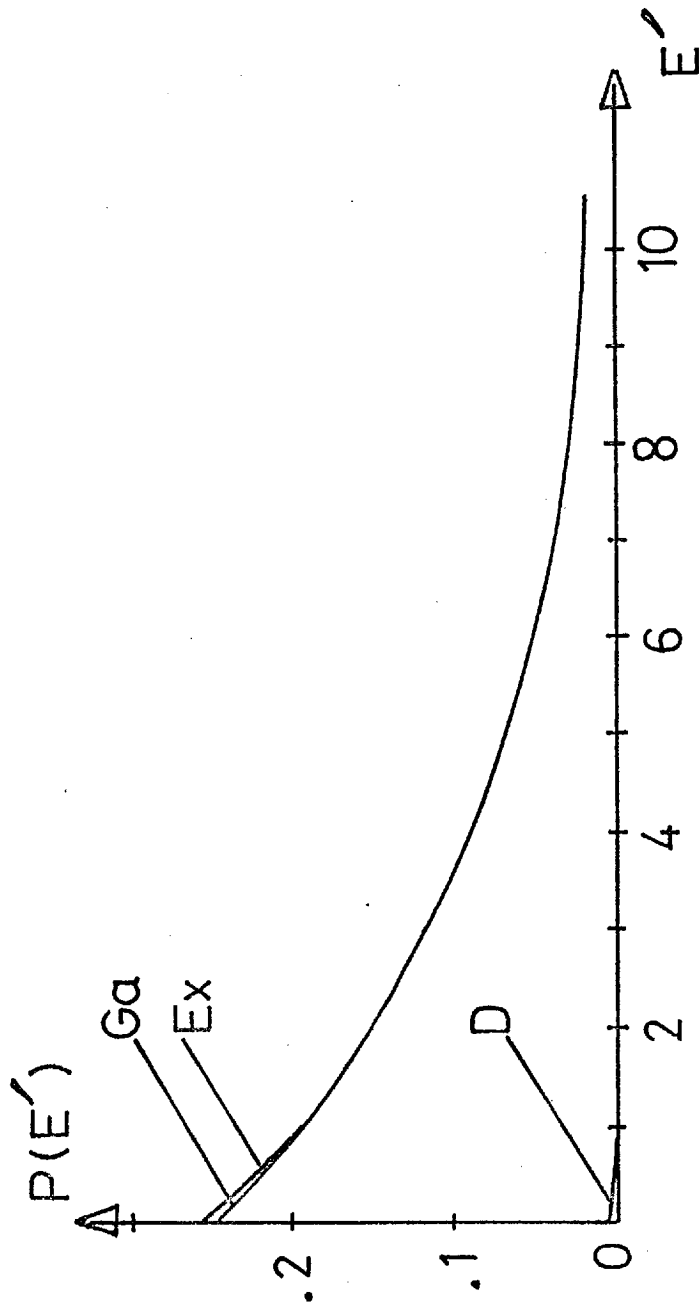


FIG. 38 COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 0.2$.

Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION,

D = DIFFERENCE.

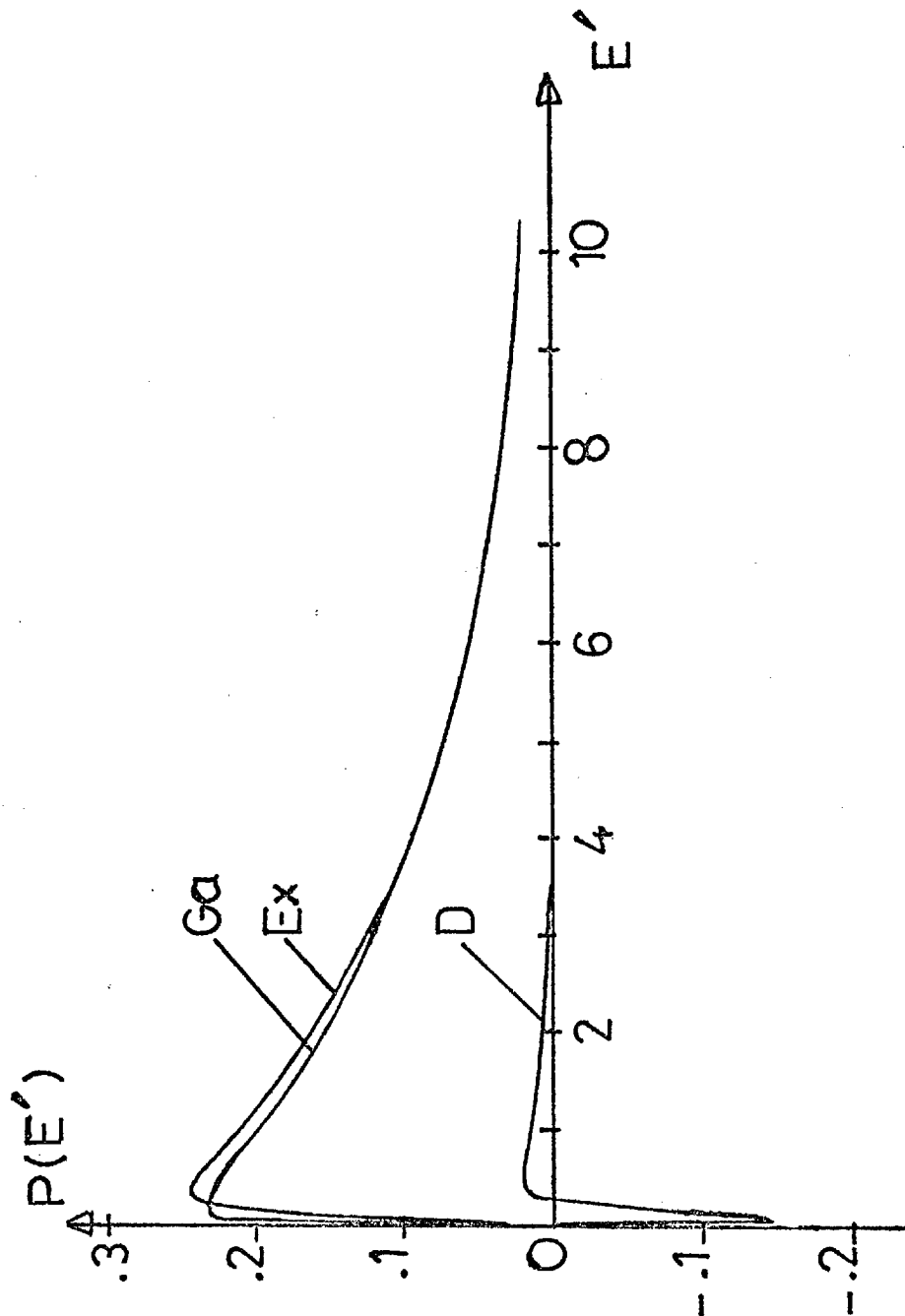


FIG. 39 COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 0.5$.

Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION,

D = DIFFERENCE.

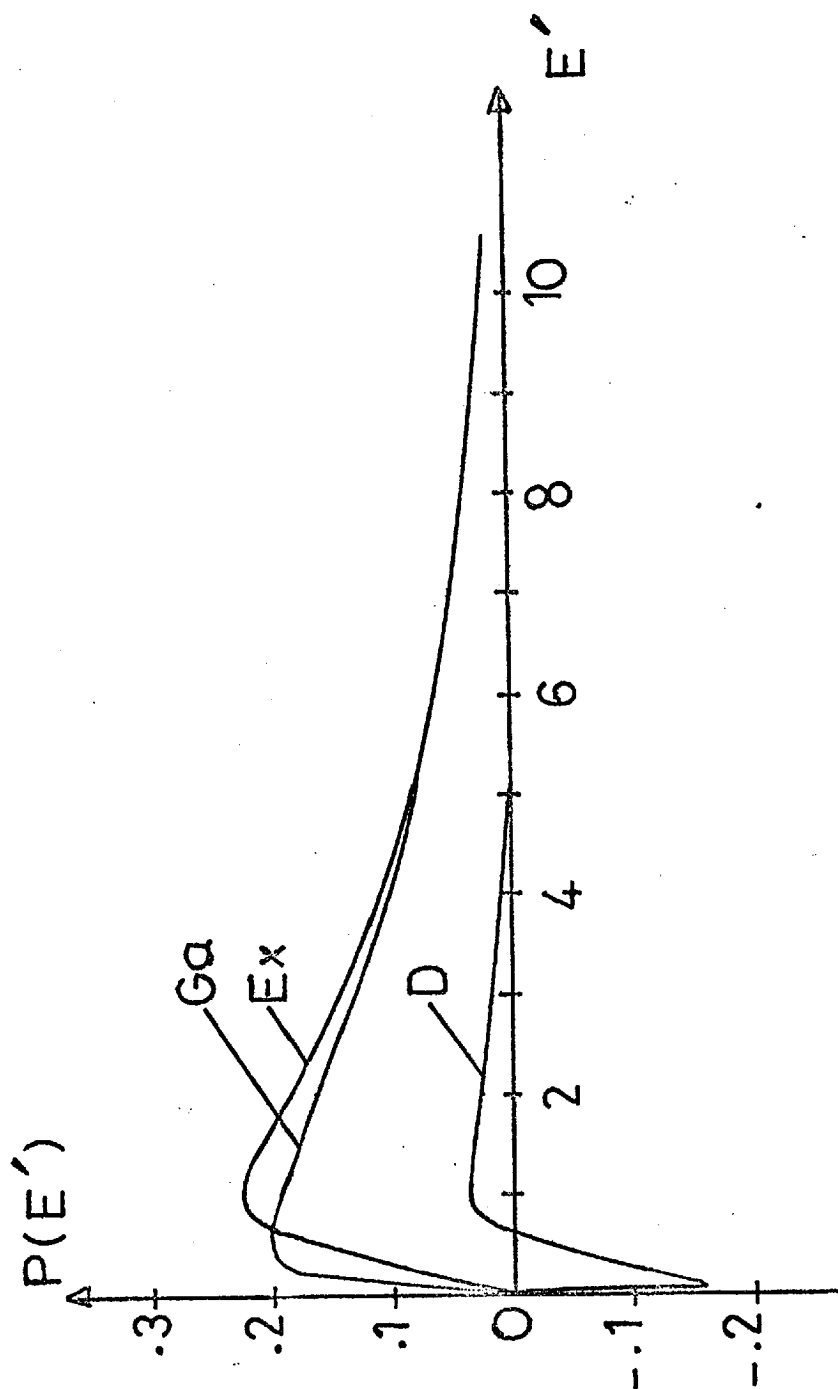


FIG. 40 COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 1$.

Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION,

D = DIFFERENCE.

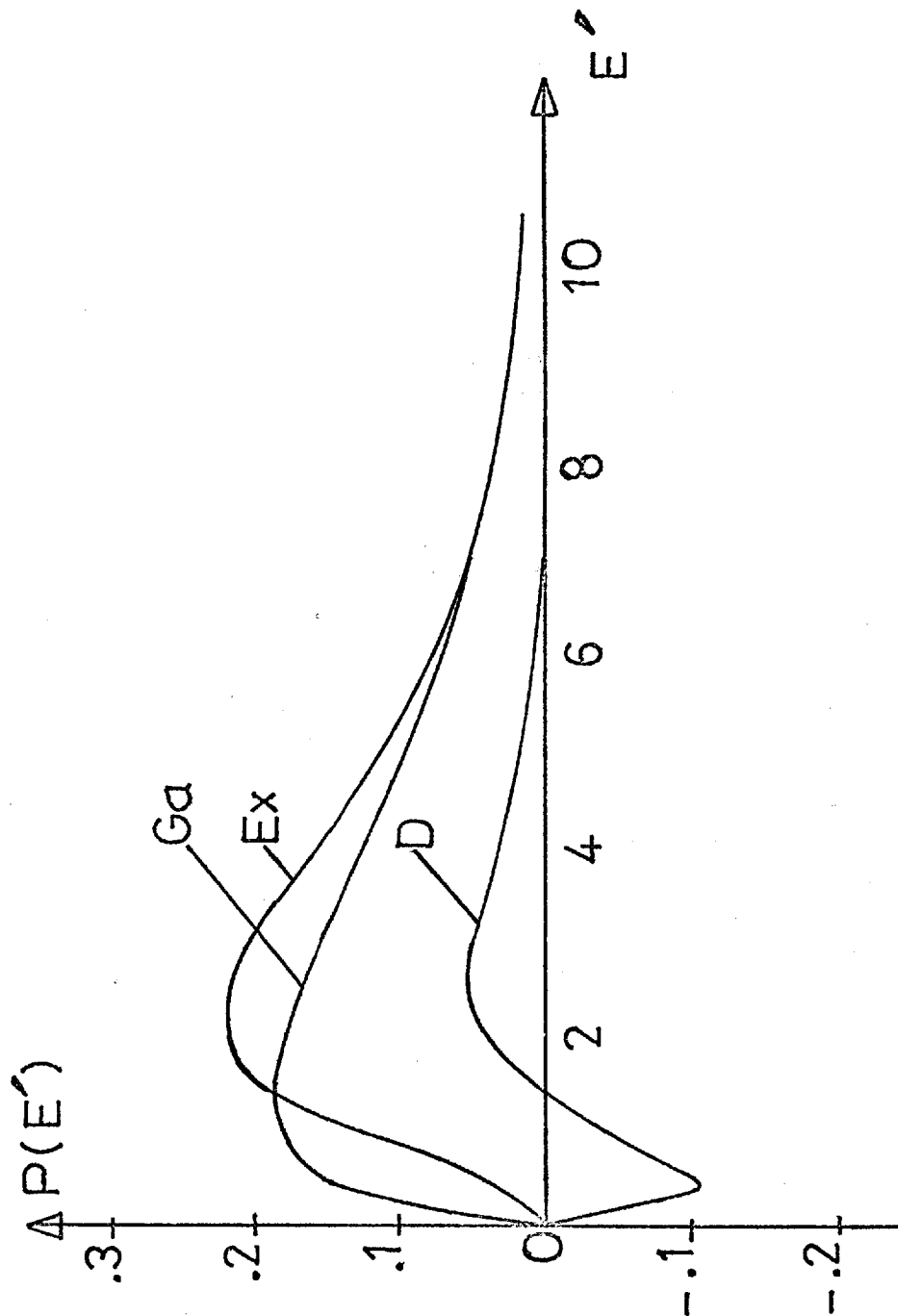


FIG. 4I COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 2$.

Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION,

D = DIFFERENCE.

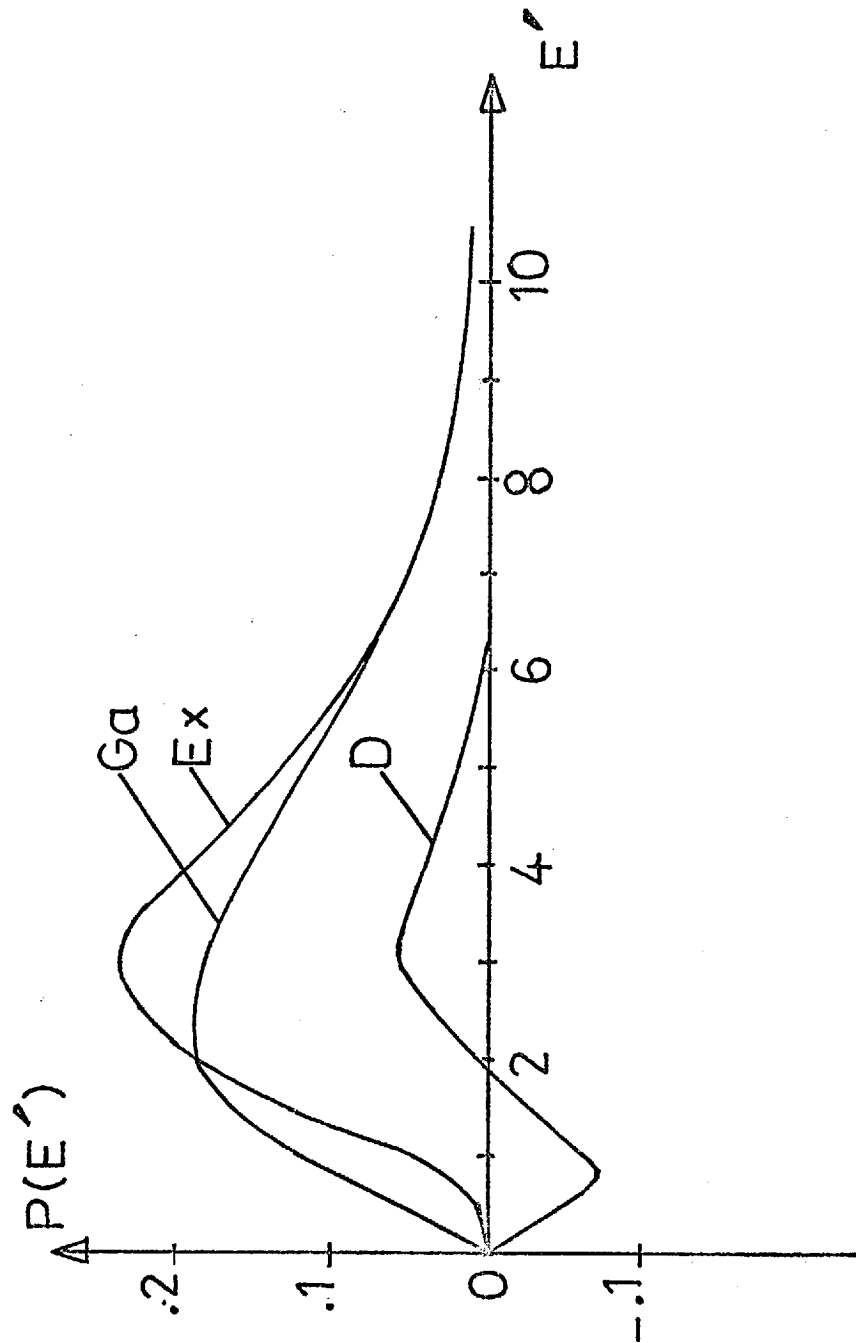


FIG. 42 COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE

PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 3$.

Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION,

D = DIFFERENCE.

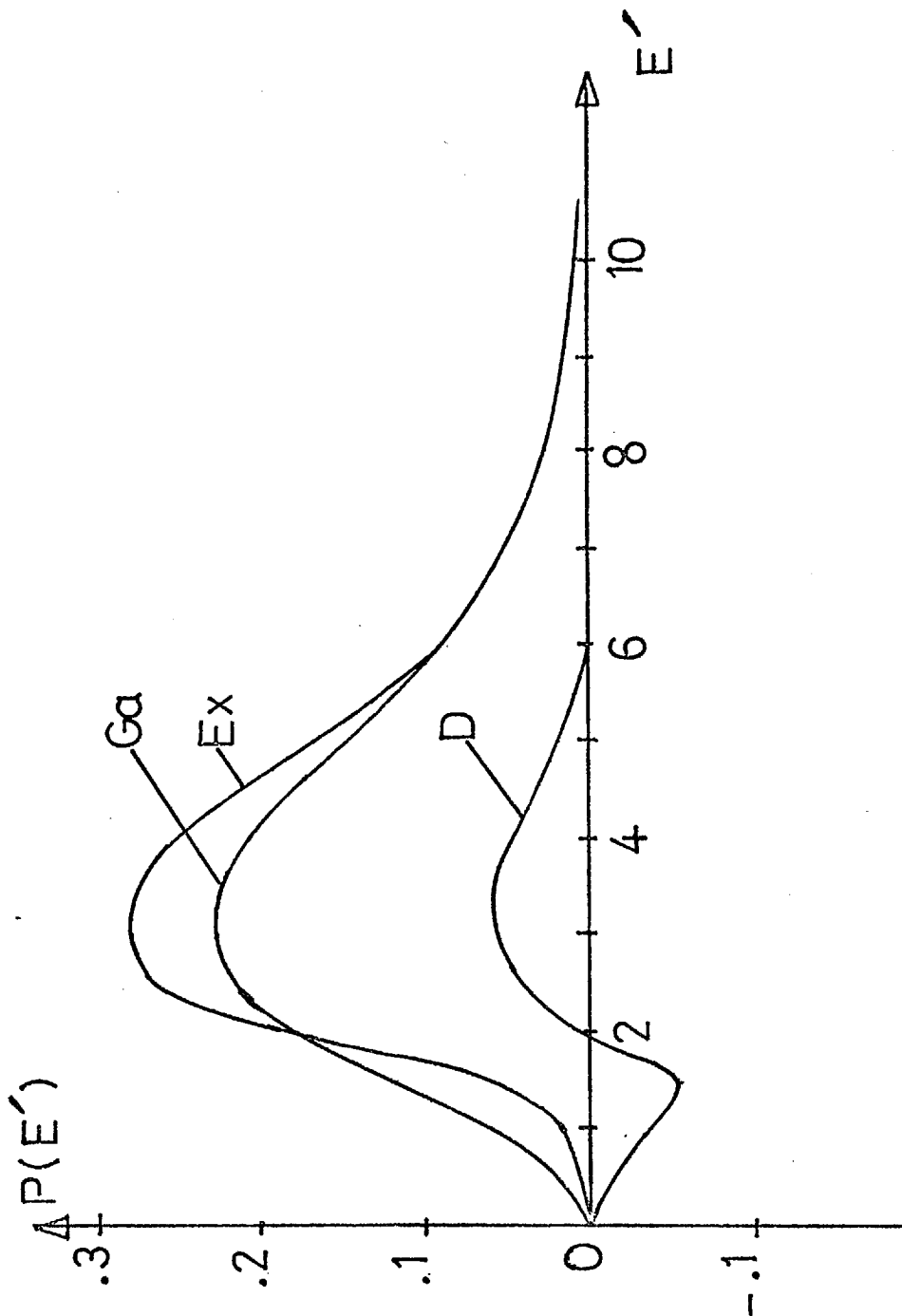


FIG. 43 COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 4$.

Ga = GAMMA DISTRIBUTION, Ex = EXACT DISTRIBUTION,
 D = DIFFERENCE.

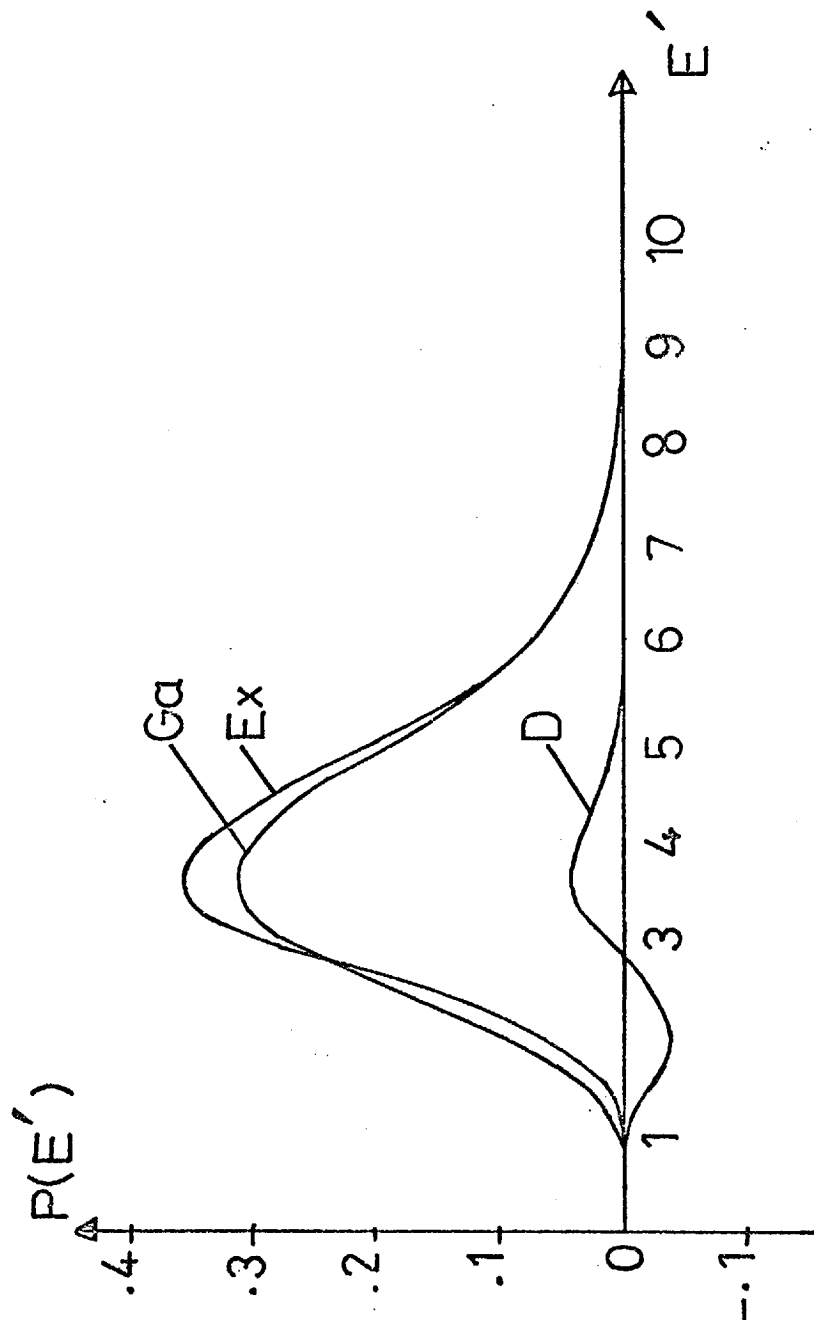


FIG. 44 COMPARATIVE PLOTS FOR CIRCULAR CASE OF EXACT AND APPROXIMATE PROBABILITY DENSITIES FOR A RELATIVE APERTURE OF $C = 6$.

D = DIFFERENCE.

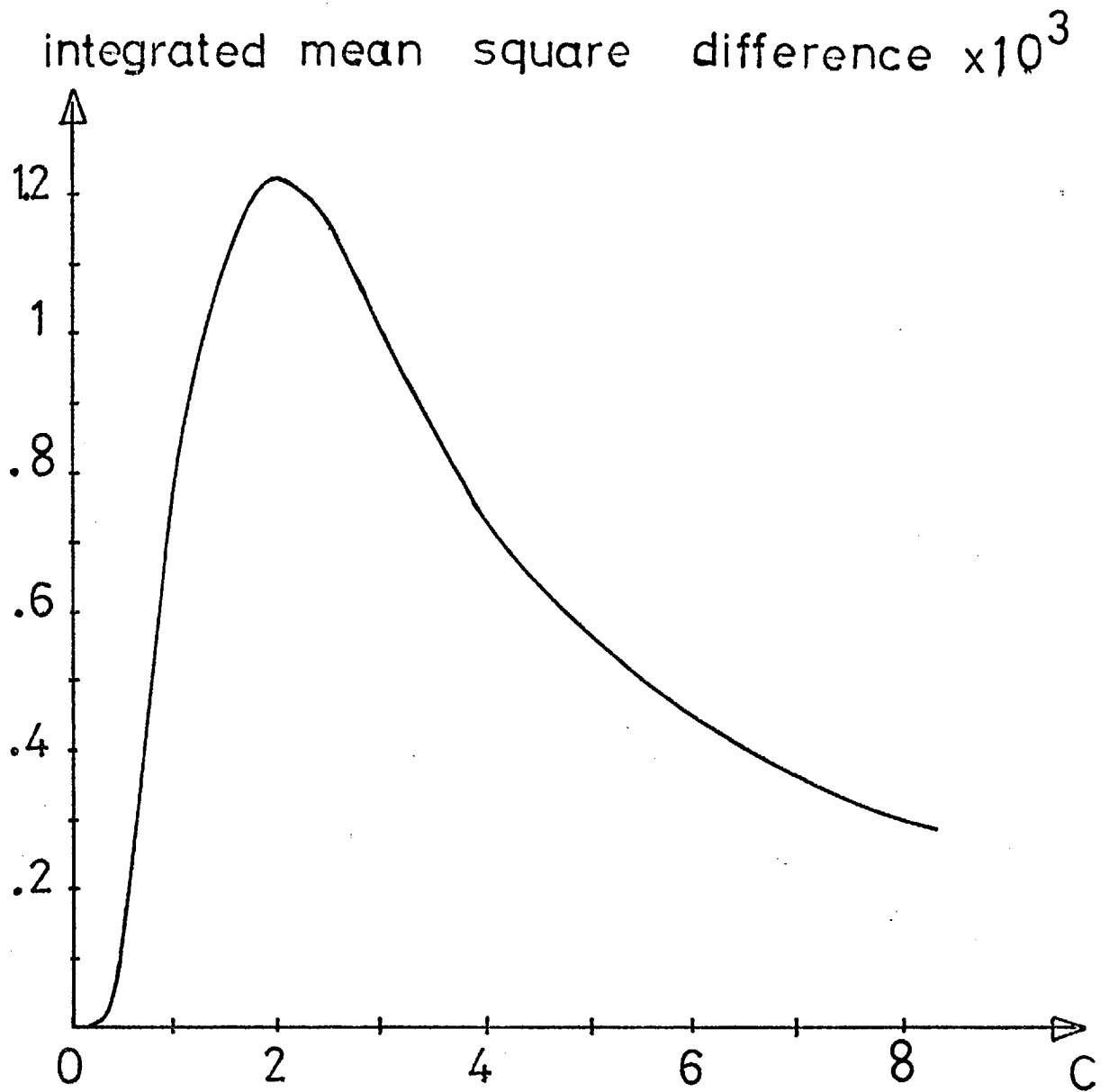


FIG. 45 THE DIFFERENCE BETWEEN THE APPROXIMATE AND THE EXACT FORMULAE FOR THE CASE OF CIRCULAR APERTURES IS CHARACTERISED BY THE GRAPH OF THE INTEGRATED MEAN SQUARE DIFFERENCE VERSUS THE RELATIVE APERTURE c .

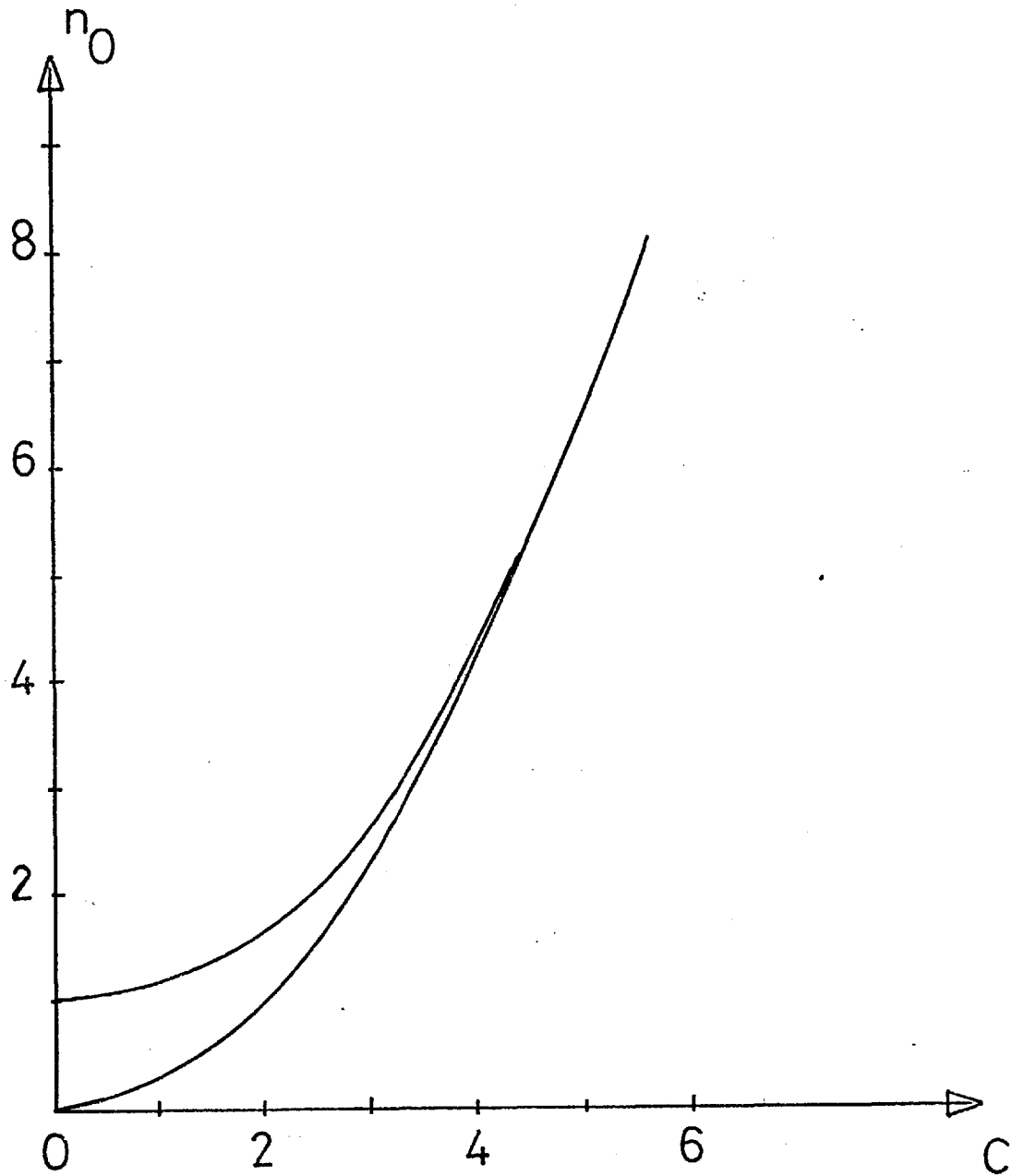


FIG. 46

CUT-OFF INDEX OF THE CIRCULAR EIGENVALUES VS. c THE RELATIVE APERTURE, n_0 ALSO DEFINES THE NUMBER OF INDEPENDENT CORRELATION CELLS THE SCANNING APERTURE CAN BE DEVIDED INTO. THE ASYMPTOTE IS THE PARABOLA OF EQN. $c^2/4$.

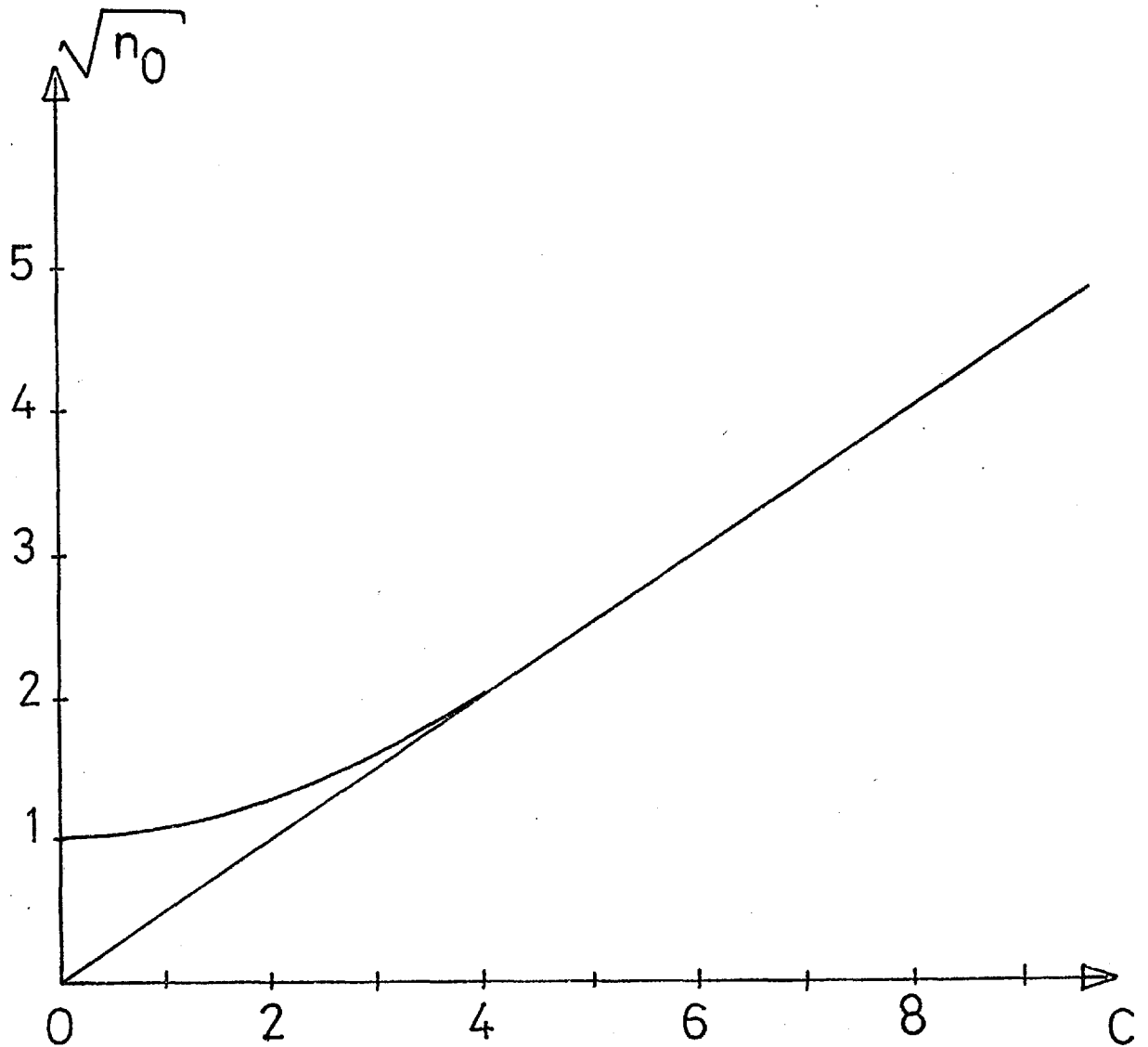


FIG. 47 PLOT OF THE SQUARE ROOT OF THE CUT-OFF INDEX USED TO FIND
THE ASYMPTOTIC LINE OF EQN. $c/2$.

that as c gets larger, $\sqrt{n_0}$ tends to the straight line of slope $c/2$, therefore n_0 tends to the parabola $c^2/4$ (plotted on fig. (46)).

Substituting this value in equ. (130) gives the asymptotic result

$$\Lambda_0 \rightarrow \frac{4\pi}{c^2} \quad \text{as } c \rightarrow \infty \quad (132)$$

for the first eigenvalue.

4.3. Experimental Results

The experiment described here is a fundamental test of the eigenvalue method to describe statistical properties of speckle patterns. This experiment is performed for the circular configuration of diffusing pupil and scanning aperture also as a test to the matrix diagonalization approach to the eigenvalues. The object of the experiment is a real non theoretical plot of the first order probability density function of integrated intensity made with some 8000 readings of the integrated intensity at different points.

The apparatus used is as described on fig. (1). The collecting Pupil P has a diameter of 1.15 cm. the scanning aperture S in front of the photomultiplier has a diameter of $50\ \mu$. The wavelength is $4880\ \text{\AA}$ and the distance D between the collecting pupil and scanning aperture is 1.50 m. This gives a relative aperture in c units (equ. (100)) of 2.45, (the absolute error on this value is .2 it is determined by the limited precision with which the diffusing pupil diameter, the scanning aperture diameter and the distance D are known. From fig (46) (the plot of n_0 vs. c) this corresponds to between 1.7 and 2.25 independent correlation cells.) We are thus operating near the maximum of the curve of the integrated mean square difference as a function of c. The scanning aperture with the photomultiplier is moved across the speckle field by a stepping motor controlled by logic pulses. The steps taken are multiples of the micron. Here we choose the stepping distance between the readings of the order of half the Rayleigh resolution limit of the collecting pupil. This is in accordance with the SHANNON-WEAVER 1962 sampling theorem (see also BLACKMAN and TUKEY 1959 and GOODMAN 1968). This theorem states that a function to be determined experimentally should be measured at least at every interval

corresponding to twice the maximum frequency, this being to avoid errors such as aliasing. Here the maximum frequency present is determined by the Rayleigh resolution limit,

$$\mathcal{R}_p = \frac{.61 \lambda D}{U} \neq 75 \mu \quad (133)$$

The chosen sampling interval is 35μ .

The output voltage from the photomultiplier is read on a digital voltmeter and the data for each reading between each step is converted into coded paper punched tape. (The code used is American Standard Code for Information Interchange, well known as the ASCII code) The paper tape is then taken to the computer (a C.D.C. 6400) where it is decoded and the data is stored on a disc file in array form. Stored in this fashion the data is easily accessible by any number of independent programmers.

For our own use a program is written that calculates the average integrated intensity then distributes each reading in bins such that those readings equal to zero are distributed in the first bin and those readings equal to average integrated intensity are distributed in the tenth bin.

The value of the integrated intensity E' is then simply related to the index of the bin j by

$$E' = \frac{\langle E' \rangle}{10} \times j \quad (134)$$

The height of the bins are normalized such that the sum of all the readings is unity. This is equivalent to normalising a probability density function to unity. Thus scaled the height of the bins represent the experimental first order probability density function of integrated intensity for a circular diffusing pupil and a circular scanning aperture with unit intensity transmittances of given size.

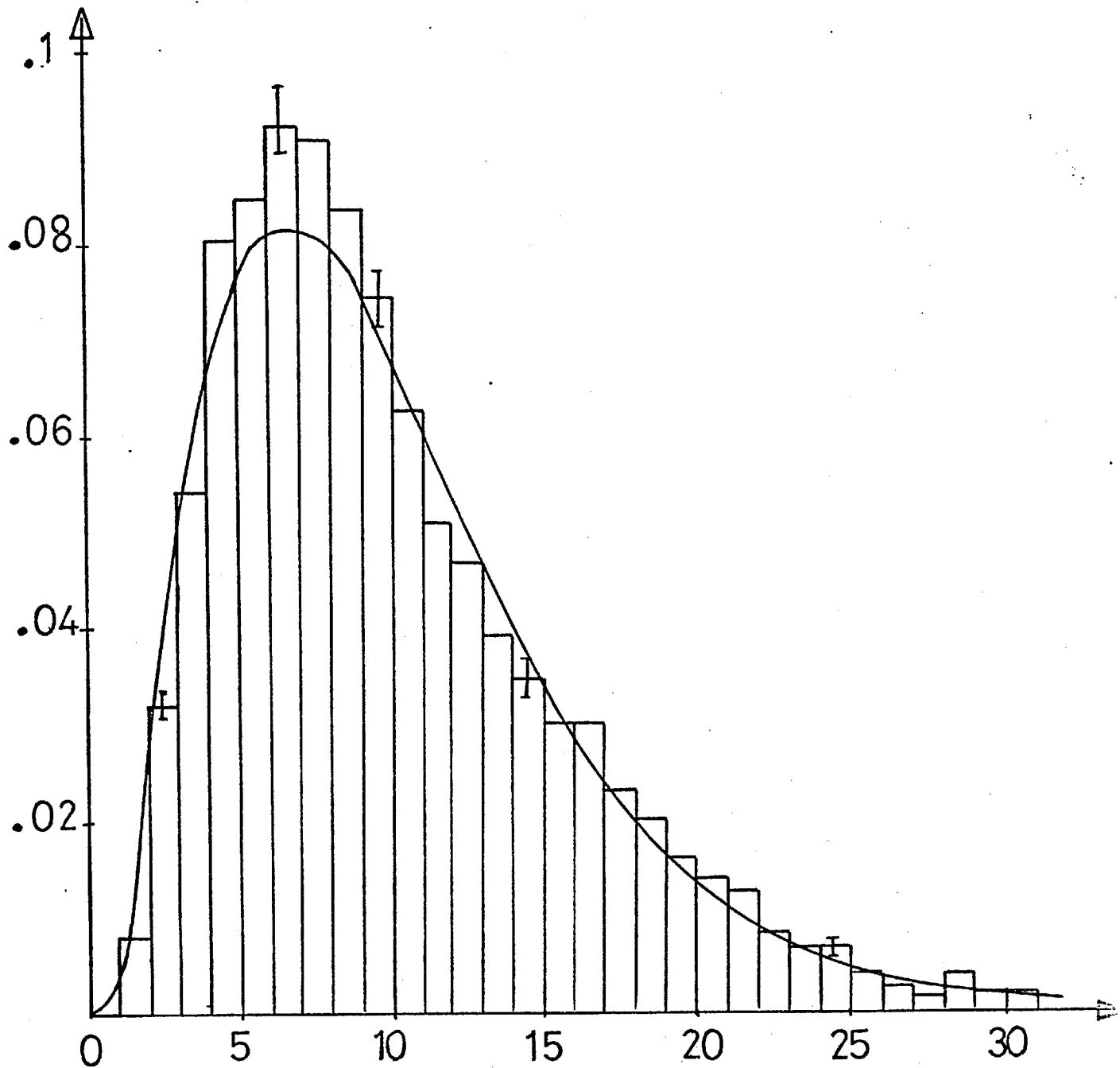


FIG. 48 EXPERIMENTAL FIRST ORDER PROBABILITY DENSITY FUNCTION PLOTTED FROM 8000 READINGS TAKEN WITH THE APARATUS ILLUSTRARED ON FIG. 1 AND MOUNTED WITH TOP HAT DIFFUSING PUPIL AND SCANNING APERTURE, THE RELATIVE APERTURE IN C UNITS (EQN. 4.2. (100)) IS 2.5 . THE READINGS ARE DISTRIBUTED IN BINS AND THEIR SUM IS NORMALISED TO UNITY. THE AVERAGE INTEGRATED INTENSITY IS 1.0 . THERE ARE TEN BINS TO THE AVERAGE. THE CORRESPONDING GAMMA DISTRIBUTION IS PLOTTED AS THE SMOOTH LINE.

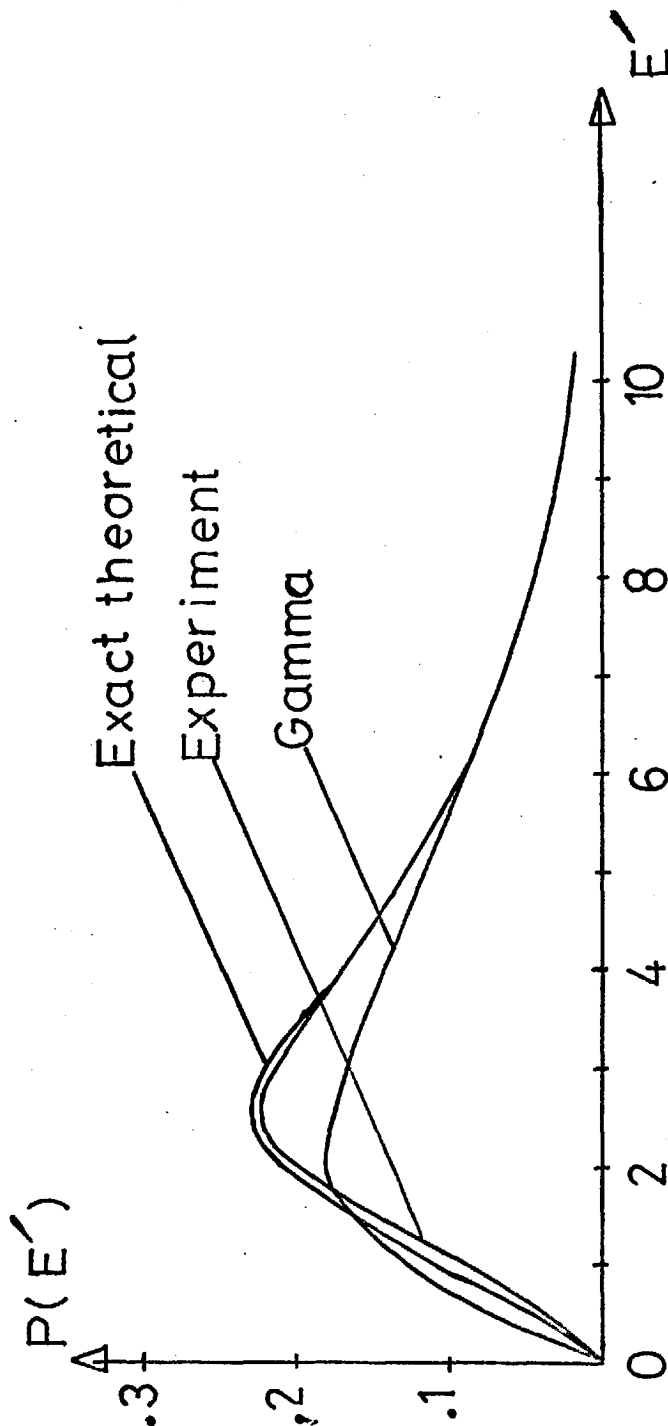


FIG. 49, PLOTS OF EXPERIMENTAL, EXACT THEORETICAL AND GAMMA

DISTRIBUTIONS OBTAINED WITH TOP HAT DIFFUSING PUPIL AND APERTURE, THE EXPERIMENTAL CURVE WAS OBTAINED WITH SOME 8000 READINGS. AS ON FIG. 48. THE RELATIVE APERTURE IS $c = 2.5$, THE AVERAGE INTEGRATED INTENSITY IS $\langle E' \rangle = 4$, THE NUMBER OF INDEPENDENT CORRELATION CELLS IS

$$n_0 = 1.97$$

This bin technique is illustrated in fig. (48). The error bars represent the absolute fluctuations of the number of readings in each bin. These bars are proportional to $\sqrt{n_j}$ where n_j is the number of readings in the j th bin. The processed data returns a number of independent correlation cells computed from the ratio of the squared average integrated intensity to the variance of the latter equal to 1.97. Figure (46) shows the corresponding value of c as just under 2.5 which means our settings are better than expected.

The experimentally obtained first order probability density function is plotted on the same graph as the exact theoretical distribution obtained with the eigenvalues computed for the given value of c , also on the same graph is a plot of the approximate Gamma distribution. These plots are shown on fig. (49). The closeness of the fit between the experimental and exact theoretical distribution is clearly seen, so is the difference with the approximate Gamma distribution.

II

RANDOMLY POLARIZED SPECKLE PATTERNS

1. INTRODUCTION

All the preceding results concern deterministically polarized speckle patterns. Experimentally we have observed only surface like scatterers, i.e. scatterers whose diffusing centres lie in the same plane, do not appreciably depolarize a polarized incident laser beam. This is the case for example, with a single layer of ground glass. On the other hand diffusing media with three dimensional structure, i.e. thick diffusing media, depolarize light (SCRIBOT 1975), this is the case, to take examples, with paper, opal glass and stacks of ground glass. Statistics of polarized speckle therefore represent an ideal case which is, however, easily obtained by inserting a polarizer between the collecting pupil and the scanning aperture.

If observed closely the speckle pattern is seen to be locally deterministically polarized over areas about the size of a speckle (i.e. areas of the order of the Rayleigh resolution limit of the collecting pupil), but each speckle is polarized in a random manner, hence the term "randomly polarized" speckle.

There is little to be found in the literature on the state of polarization of speckle patterns. GOODMAN 1965 states depolarized speckle can be decomposed into two perpendicular independent components and their probability density functions are then convolved to give the joint probability density function. BURCH 1970 gives the result of the convolution of two probability density functions of two incoherently added speckle patterns observed through a point aperture and McKECHNIE 1974 applies this to evaluate the depolarization from a single layer of ground glass and finds the ratio of the average intensity of the two components i.e. of the order of one thousandth. BOURRET 1974 gives

an analytical account of depolarization of electromagnetic radiation in random media but a fundamental assumption of his treatment is that backscatter is negligible and the resulting expression for his polarization factor is a negative exponential which does not agree with our experimental results. In the thick medium we consider, it is intuitively clear that there must be a large amount of backscatter present.

Here we introduce some general expressions for the joint first order probability density functions of two perpendicular polarization components. These depend on the average integrated intensity of each component, but a priori we know nothing about the value of these quantities or their behaviour. It is therefore the purpose of our experiments to try and find something about this. We give an explanation of the measuring technique involved and then state our experimental results. The polarization factor P introduced is seen to depend on the "roughness" of the medium (r.m.s. fluctuations of the size of the scattering centres), the thickness of the medium and also on the wavelength used.

2. JOINT STATISTICS OF TWO INDEPENDENT POLARIZATION COMPONENTS

Let us first examine the statistics of randomly polarized speckle patterns observed through a point aperture. Let $\langle E'_{\parallel} \rangle$ and $\langle E'_{\perp} \rangle$ be the average integrated intensity of the two components and $P_{\parallel}(E')$ and $P_{\perp}(E')$ be their respective first order probability densities, these are given by (see equ. 1.2.1. (6)),

$$P_{\parallel}(E') = \frac{1}{\langle E'_{\parallel} \rangle} e^{-\frac{E'}{\langle E'_{\parallel} \rangle}} \quad (1)$$

and

$$P_{\perp}(E') = \frac{1}{\langle E'_{\perp} \rangle} e^{-\frac{E'}{\langle E'_{\perp} \rangle}} \quad (2)$$

The Convolution of these two functions gives the joint first order probability density function of a depolarized speckle pattern observed through a pinhole.

This convolution can be evaluated from the product of the characteristic function of each distribution (see appendix A). The respective characteristic functions are

$$C_{\parallel}(z) = (1 - iz \langle E'_{\parallel} \rangle)^{-1} \quad (3)$$

and

$$C_{\perp}(z) = (1 - iz \langle E'_{\perp} \rangle)^{-1} \quad (4)$$

Their product is the characteristic function of the joint probability density function

$$C(z) = C_{\parallel}(z) C_{\perp}(z) = (1 - iz \langle E'_{\parallel} \rangle)^{-1} (1 - iz \langle E'_{\perp} \rangle)^{-1} \quad (5)$$

which can be expanded in the sum of two partial fractions,

$$C(z) = \frac{\langle E'_{\parallel} \rangle}{\langle E'_{\parallel} \rangle - \langle E'_{\perp} \rangle} \frac{1}{1 - iz \langle E'_{\parallel} \rangle} + \frac{\langle E'_{\perp} \rangle}{\langle E'_{\perp} \rangle - \langle E'_{\parallel} \rangle} \frac{1}{1 - iz \langle E'_{\perp} \rangle} \quad (6)$$

The Fourier transform of this yields

$$P_{\text{joint}}(E') = \frac{e^{-\frac{E'}{\langle E'_{\parallel} \rangle}} - e^{-\frac{E'}{\langle E'_{\perp} \rangle}}}{\langle E'_{\parallel} \rangle - \langle E'_{\perp} \rangle} \quad (7)$$

which is the required joint distribution. We observe the following properties for $P_{\text{joint}}(E')$

$$\text{When } E' \rightarrow 0 \quad P_{\text{joint}}(E') \rightarrow 0$$

$$\text{When } E' \rightarrow \infty \quad P_{\text{joint}}(E') \rightarrow \infty$$

When $\langle E'_{\perp} \rangle \rightarrow 0$ (i.e. when the observed speckle pattern is completely polarized in a single direction, the $\parallel$ one for example)

$$P_{\text{joint}}(E') = \frac{1}{\langle E' \rangle} e^{-\frac{E'}{\langle E' \rangle}} \quad (8)$$

Where $\langle E' \rangle = \langle E'_{\parallel} \rangle$, this distribution is, as expected, the same as in equ. 1.2.2 (6). Equation (7) is given by BURCH 1970 in connexion with the statistics of two unequally weighted speckle patterns added incoherently.

If the two speckle patterns are equally divided into the two perpendicular polarization components, we have $\langle E'_{\parallel} \rangle = \langle E'_{\perp} \rangle = \langle E' \rangle$ and the probability density given in equ. (7) is not applicable, we must derive the correct joint probability density function from the Fourier transform of

$$C(z) = (1 - iz \langle E' \rangle)^{-2} \quad (9)$$

which gives

$$P_{\text{joint}}(E') = \frac{E' e^{-\frac{E'}{\langle E' \rangle}}}{\langle E' \rangle^2} \quad (10)$$

which is instantly recognized as the Gamma distribution with two degrees of freedom.

Let us now consider the statistics of depolarized speckle patterns observed through non-punctual finite scanning apertures. The general exact probability density function expressed in equ. 1.3.3 (35) applied to the two polarization components gives

$$P_{\parallel}(E') = \sum_n \frac{G_{n\parallel}}{\varepsilon_{n\parallel}} e^{-\frac{E'}{\varepsilon_{n\parallel}}} \quad (11)$$

and

$$P_{\perp}(E') = \sum_n \frac{G_{n\perp}}{\varepsilon_{n\perp}} e^{-\frac{E'}{\varepsilon_{n\perp}}} \quad (12)$$

where $\{\varepsilon_{n\parallel}\}$ is the set of eigenvalues corresponding to the $\parallel$ polarization component and $\{\varepsilon_{n\perp}\}$ is the set of eigenvalues corresponding to the $\perp$ polarization component, $G_{n\parallel}$ and $G_{n\perp}$ being the respective partial fraction coefficient given by equ. 1.3.3 (33).

First we observe from the results in section 1.4, that the eigenvalues depend only on the geometrical configuration of the diffusing pupil and scanning aperture and on the mean intensity through a scaling constant. Since the geometrical configuration is the same for both polarization components we may write

$$\varepsilon_{n\parallel} = \alpha_{\parallel} \Lambda_n \quad (13)$$

$$\text{and } \varepsilon_{n\perp} = \alpha_{\perp} \Lambda_n \quad (14)$$

Where the set of eigenvalues $\{\Lambda_n\}$ depends only on the geometrical configuration. The scaling constants $\alpha_{\parallel}$ and $\alpha_{\perp}$ can be easily obtained from the relations between the average integrated intensities of both distributions,

$$\langle E'_{\parallel} \rangle = \sum_n \varepsilon_{n\parallel} = \alpha_{\parallel} \sum_n \Lambda_n \quad (15)$$

$$\text{and } \langle E'_{\perp} \rangle = \sum_n \varepsilon_{n\perp} = \alpha_{\perp} \sum_n \Lambda_n \quad (16)$$

The variances are given by

$$\sigma_{E'_{\parallel}}^2 = \alpha_{\parallel}^2 \sum_n \Lambda_n^2 \quad (17)$$

$$\text{and } \sigma_{E'_{\perp}}^2 = \alpha_{\perp}^2 \sum_n \Lambda_n^2 \quad (18)$$

From equs. (13), (14) and 1.3.3 (33) the partial fraction coefficients $G_{n\parallel}$ and $G_{n\perp}$ are seen to be identical and the two components of the probability density function can be rewritten

$$P_{\parallel}(E') = \sum_n \frac{G_n}{\alpha_{\parallel} \Lambda_n} e^{-\frac{E'}{\alpha_{\parallel} \Lambda_n}} \quad (19)$$

and

$$P_{\perp}(E') = \sum_n \frac{G_n}{\alpha_{\perp} \Lambda_n} e^{-\frac{E'}{\alpha_{\perp} \Lambda_n}} \quad (20)$$

The joint probability density function is again evaluated through the two corresponding characteristic functions

$$C_{\parallel}(z) = \sum_n G_n \frac{1}{1 - iz \alpha_{\parallel} \Lambda_n} \quad (21)$$

and

$$C_{\perp}(z) = \sum_n G_n \frac{1}{1 - iz \alpha_{\perp} \Lambda_n} \quad (22)$$

The product of these characteristic function after further expansion into partial fractions is ,

$$C(z) = \sum_m \sum_n G_m G_n \left(\frac{\alpha_{\parallel}}{(\alpha_{\parallel} - \alpha_{\perp})} \frac{1}{(1 - iz \alpha_{\parallel} \Lambda_m)} + \frac{\alpha_{\perp}}{(\alpha_{\perp} - \alpha_{\parallel})} \frac{1}{(1 - iz \alpha_{\perp} \Lambda_n)} \right) \quad (23)$$

The Fourier transform of this expression gives the required joint distribution ,

$$P_{\text{joint}}(E') = \sum_m \sum_n G_m G_n \left[\frac{e^{-\frac{E'}{\alpha_{\parallel} \Lambda_m}}}{(\alpha_{\parallel} - \alpha_{\perp}) \Lambda_m} - \frac{e^{-\frac{E'}{\alpha_{\perp} \Lambda_n}}}{(\alpha_{\parallel} - \alpha_{\perp}) \Lambda_n} \right] \quad (24)$$

If instead of starting from exact distributions for each component (eqs. (11) and (12)) we start from approximate Gamma distributions we obtain the following results.

The two components have respective probability densities (equ. 1.3.3. (54)),

$$P_{\parallel}(E') = \left(\frac{n_0}{\langle E'_{\parallel} \rangle} \right)^{n_0} \frac{E'^{n_0-1} e^{-\frac{n_0 E'}{\langle E'_{\parallel} \rangle}}}{\Gamma(n_0)} \quad (25)$$

and

$$P_{\perp}(E') = \left(\frac{n_0}{\langle E'_{\perp} \rangle} \right)^{n_0} \frac{E'^{n_0-1} e^{-\frac{n_0 E'}{\langle E'_{\perp} \rangle}}}{\Gamma(n_0)} \quad (26)$$

with the number of independent correlation cells given by (equ. 3.2 (32)),

$$n_0 = \frac{\sum_n \Lambda_n}{\Lambda_0} \quad (27)$$

This is the same for both components and again depends only on the geometrical configuration of the diffusing pupil and scanning aperture. The characteristic functions of these probability densities are ,

$$C_{\parallel}(z) = \frac{1}{\left(1 - i z \frac{\langle E'_{\parallel} \rangle}{n_0} \right)^{n_0}} \quad (28)$$

and

$$C_{\perp}(z) = \frac{1}{\left(1 - i z \frac{\langle E'_{\perp} \rangle}{n_0} \right)^{n_0}} \quad (29)$$

The product of these characteristic functions is the characteristic function of the joint probability density and is,

$$C(z) = \frac{1}{\left(1 - iz \frac{\langle E'_{\parallel} \rangle}{n_0}\right)^{n_0} \left(1 - iz \frac{\langle E'_{\perp} \rangle}{n_0}\right)^{n_0}} \quad (30)$$

The Fourier transform of this expression gives,

$$P_{\text{joint}}(E') = \left(\frac{n_0^2}{\langle E'_{\parallel} \rangle \langle E'_{\perp} \rangle} \right)^{n_0} \frac{E'^{2n_0-1} e^{-\frac{n_0 E'}{\langle E'_{\parallel} \rangle}}}{\Gamma(2n_0)} \\ \times {}_1F_1 \left[n_0; 2n_0; n_0 \left(\frac{1}{\langle E'_{\parallel} \rangle} - \frac{1}{\langle E'_{\perp} \rangle} \right) E' \right] \quad (31)$$

where, ${}_1F_1$ is the first order hypergeometric function already encountered in equ. 1.3.3 (42). With the definitions given in eqs. 1.3.3. (43) and 1.3.3. (44) we finally have

$$P_{\text{joint}}(E') = \left(\frac{n_0^2}{\langle E'_{\parallel} \rangle \langle E'_{\perp} \rangle} \right)^{n_0} \frac{E'^{2n_0-1} e^{-\frac{n_0 E'}{\langle E'_{\parallel} \rangle}}}{\Gamma(n_0)} \\ \times \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(n_0+k)}{\Gamma(2n_0+k)} \left(n_0 \frac{\langle E'_{\perp} \rangle - \langle E'_{\parallel} \rangle}{\langle E'_{\parallel} \rangle \langle E'_{\perp} \rangle} E' \right)^k \frac{1}{k!} \right\} \quad (32)$$

This equation represents the approximate first order probability density function of randomly polarized speckle patterns observed through a finite scanning aperture.

3. EXPERIMENTAL OBSERVATIONS.

The purpose of the experiments described here is to find something about the behaviour of the quantities $\langle E'_{\parallel} \rangle$ and $\langle E'_{\perp} \rangle$ as a function of the diffusing medium.

3.1. Measuring technique.

The thick diffusing medium we study in this section is particularly easy to make and to handle, being made from layers of ground glass and each layer being ground on one side only. The randomly polarized speckle pattern arising from this medium is observed through a large aperture so as to measure the average depolarization.

The measurable quantities we use in our experiment are called Stokes parameters and defined the column Stokes' vector (see e.g. SHURCLIFF 1962 or CLARKE and GRAINGER 1971)

$$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} \quad (33)$$

where each parameter has the dimension of integrated intensity. I is the total integrated intensity, M is the amount of horizontal plane polarized intensity, C is the amount of 45° plane polarized intensity and S is the amount of right circularly polarized intensity. These quantities are in fact time averaged quantities and the two components, φ_x and φ_y are the corresponding phases at $z = 0$ with λ the wavelength and ω the angular frequency.

The Stokes parameters are defined by

$$\begin{aligned}
 I &= \langle E_{x0}^2 + E_{y0}^2 \rangle \\
 M &= \langle E_{x0}^2 - E_{y0}^2 \rangle \\
 C &= \langle 2 E_{x0} E_{y0} \cos(\varphi_y - \varphi_x) \rangle \\
 S &= \langle 2 E_{x0} E_{y0} \sin(\varphi_y - \varphi_x) \rangle
 \end{aligned}
 \tag{36}$$

for a completely polarized beam of light. From these relations we see that if the light is completely polarized we have

$$I = \sqrt{M^2 + C^2 + S^2} \tag{37}$$

On the other hand if the light is partially polarized we have

$$I > \sqrt{M^2 + C^2 + S^2} \tag{38}$$

We now define the normalized polarization coefficient P which expresses the amount of light which is not depolarized,

$$P = \frac{\sqrt{M^2 + C^2 + S^2}}{I} \tag{39}$$

This quantity is of course positive and equal or less than unity. If in the depolarized beam there are no C and S components then the polarization factor becomes

$$P = \frac{M}{I} = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}} \tag{40}$$

$$\text{where } I_{\parallel} = \langle E_{x0}^2 \rangle \tag{41}$$

$$\text{and } I_{\perp} = \langle E_{y0}^2 \rangle \tag{42}$$

The quantity M is measured with a filter blind to vertical plane polarized light, C is measured with a filter which to -45° plane polarized light and S is measured with a filter blind to left circularly polarized light. If I_1, I_2

and I_3 are the intensities respectively measured with each filter, the following relations hold,

$$\begin{aligned} M &= 2 I_1 - I \\ C &= 2 I_2 - I \\ S &= 2 I_3 - I \end{aligned} \quad (43)$$

The filters are made from plastic sheet polarizer for the plane polarization filters (to measure I_1 and I_2) and a quarter wave-plate followed by a plate of sheet polarizer is used to measure the right-hand circular polarization component I_3 .

In order to evaluate how the polarization factor is affected by the roughness of the diffusing medium, this has to be evaluated beforehand. The diffusing medium is made of layers of ground glass and we take the r. m. s. of height fluctuations of the ground surfaces as a measure ~~of the roughness~~ of the size of the scattering centers. The height fluctuations are evaluated by a mechanical method which consists of a stylus that runs across a section of the surface and traces a record of its movements. The machine that does this is called a Talysurf and the device has some analogy with a record player although the output is not sound but a profile of the surface under test. An example of the kind of record this machine gives is shown in fig. (50). A quantity of interest given by the Talysurf is the Centre Line Average noted CLA. We can define the CLA by considering a zero mean stochastic process, x : (here the height of the ground glass), the CLA is defined as the average of the absolute value of x . If the stochastic process under consideration has a Gaussian distribution, we can relate easily the r. m. s. of the ^{fluctuations} process to the CLA. We have the relation

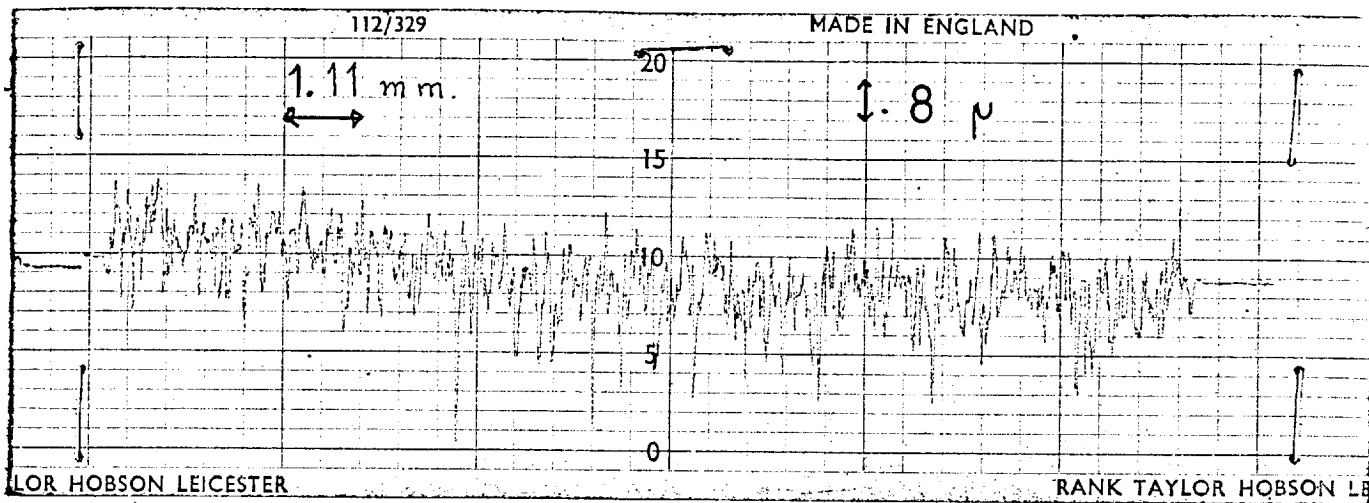


FIG. 50 EXAMPLE OF RECORD GIVEN BY THE TALYSURF SURFACE ROUGHNESS MEASURING MACHINE.

$$CLA = \langle |x| \rangle = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{+\infty} |x| e^{-\frac{x^2}{2\sigma_x^2}} dx \quad (44)$$

which gives after integration

$$CLA = \frac{2 \sigma_x}{\sqrt{2\pi}} \quad (45)$$

The Talysurf gives a direct reading of the CLA. Numerically from equ (45) we obtain

$$\sigma_x = 1.253 \quad CLA \quad (46)$$

The optical apparatus we use is the same as illustrated on fig. (1) except for the insertion of a horizontal plane polarizer before the diffuser and the presence of the analyzing filters required in front of the photomultiplier.

3.2. Experimental results

(Figure (51) shows a plot of the polarization factor P versus N the number of layers of ground glass of r.m.s. height fluctuations $\sigma_1 = .430 \mu$. For this roughness at least seven layers of ground glass are required to give an appreciable depolarization. For ten layers P is approximately 6.6. Fig. (52) illustrates the difference this introduces in the statistics of the speckle pattern scanned through a point aperture. Figure (53) shows a plot of P versus N for r.m.s. height fluctuations $\sigma_2 = 3.8 \mu$. Here three layers are enough to make a noticeable difference in the state of polarization of the speckle pattern. For ten layers of this ground glass P is approximately 3.7. Fig. (54) illustrates the difference this makes in the speckle statistics scanned through a point aperture. Figure (55) shows a plot of P versus σ the

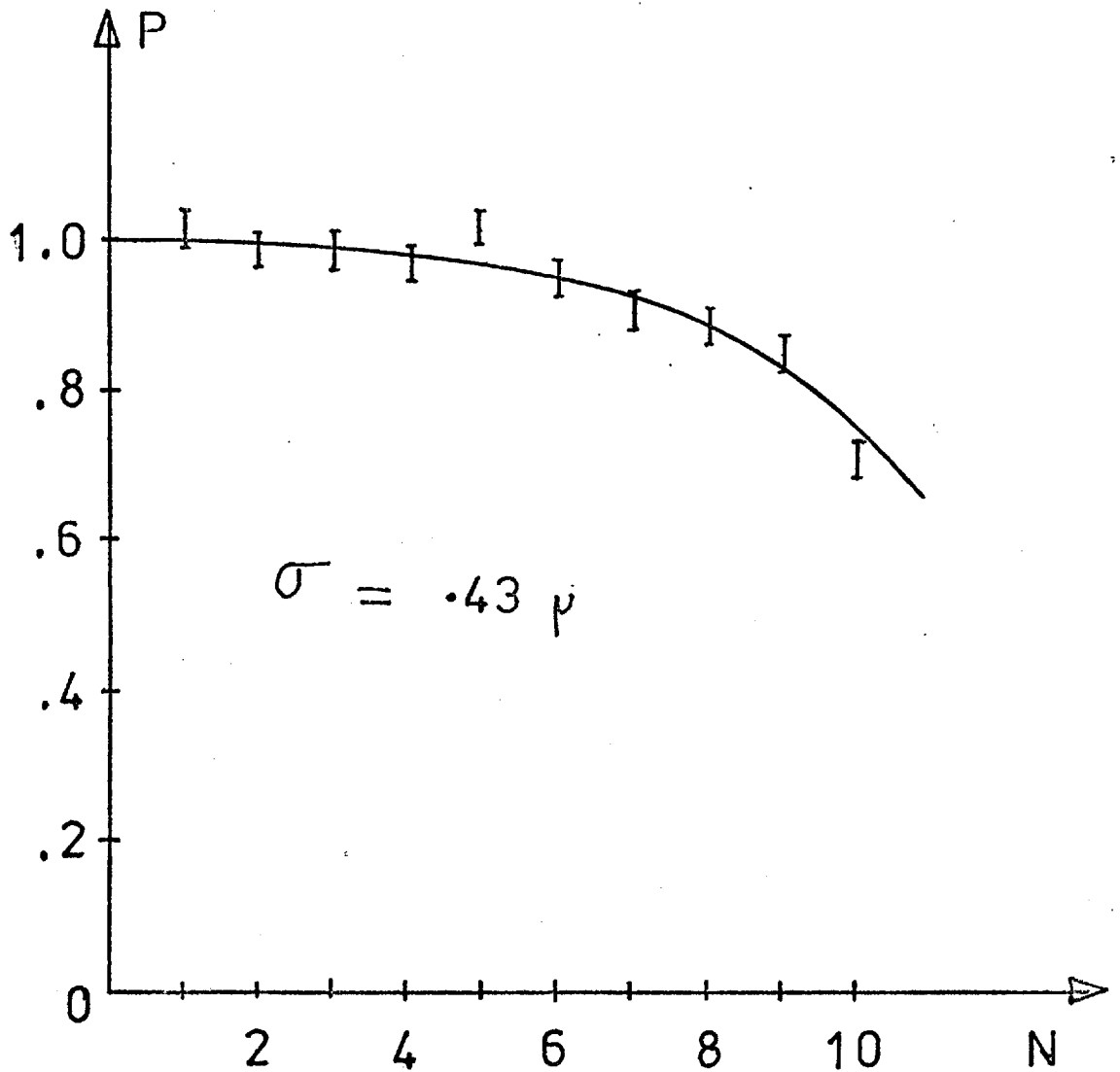


FIG. 51

GRAPH OF POLARIZATION COEFFICIENT P VERSUS N , THE NUMBER OF LAYERS OF GROUND GLASS (OR THICKNESS OF DIFFUSING MEDIUM) FOR A STANDARD DEVIATION OF THE ROUGHNESS OF $\sigma = 0.43 \mu$.

($\lambda = 4765 \text{ \AA}$ -blue-, $D = 1.18 \text{ m}$, U = collecting pupil radius = 1.5 cm , A = scanning aperture diameter = 0.8 cm).

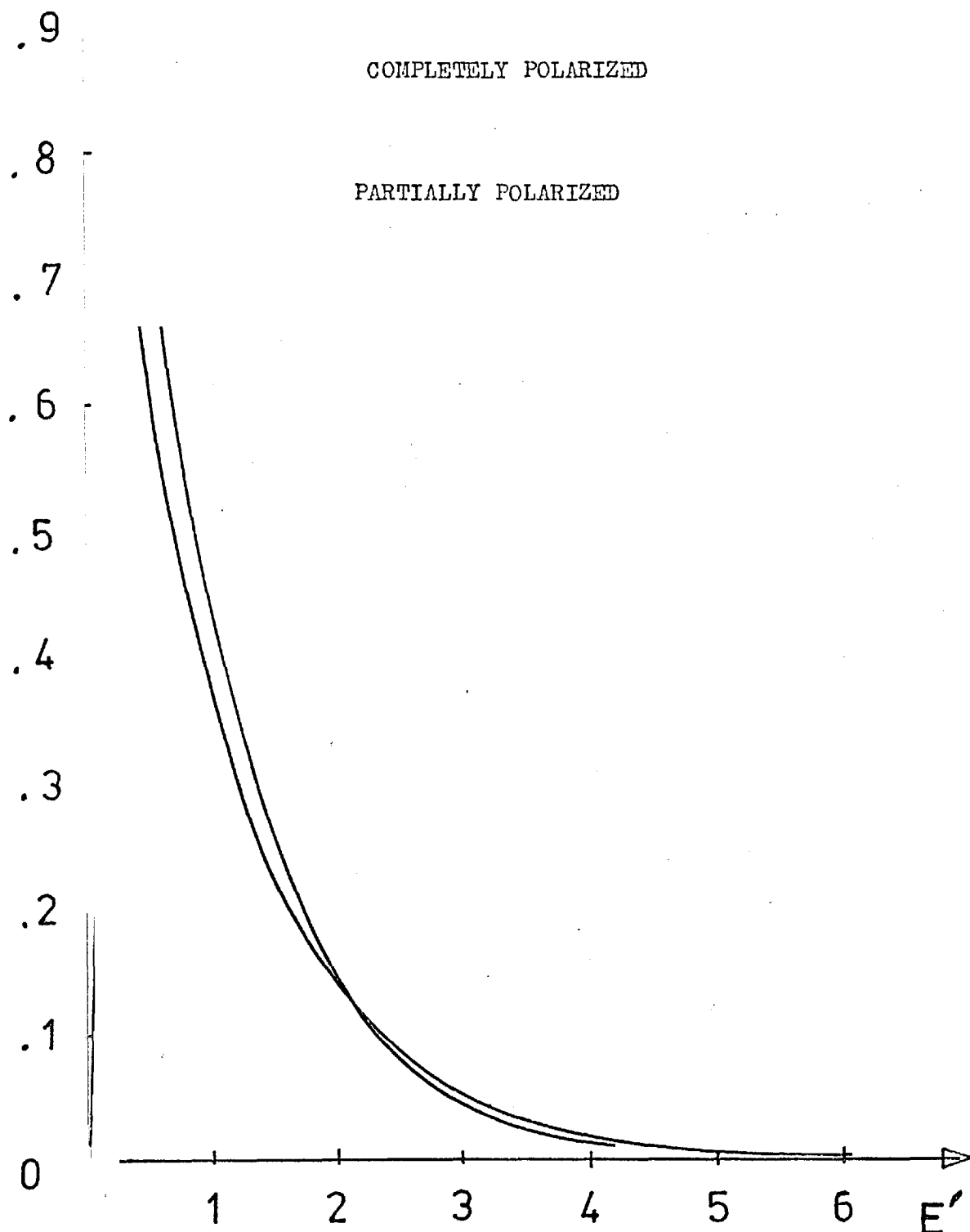


FIG. 52

FIRST ORDER PROBABILITY DENSITY FUNCTION OF INTENSITY FOR
PARTIALLY POLARIZED SPECKLE PATTERNS OBSERVED THROUGH POINT
APERTURES AS EXPRESSED BY EQN.2 (7). THE AVERAGE TOTAL INTENSITY
IS UNITY AND THE POLARIZATION FACTOR IS (EQN. 3 (40)) 0.76.

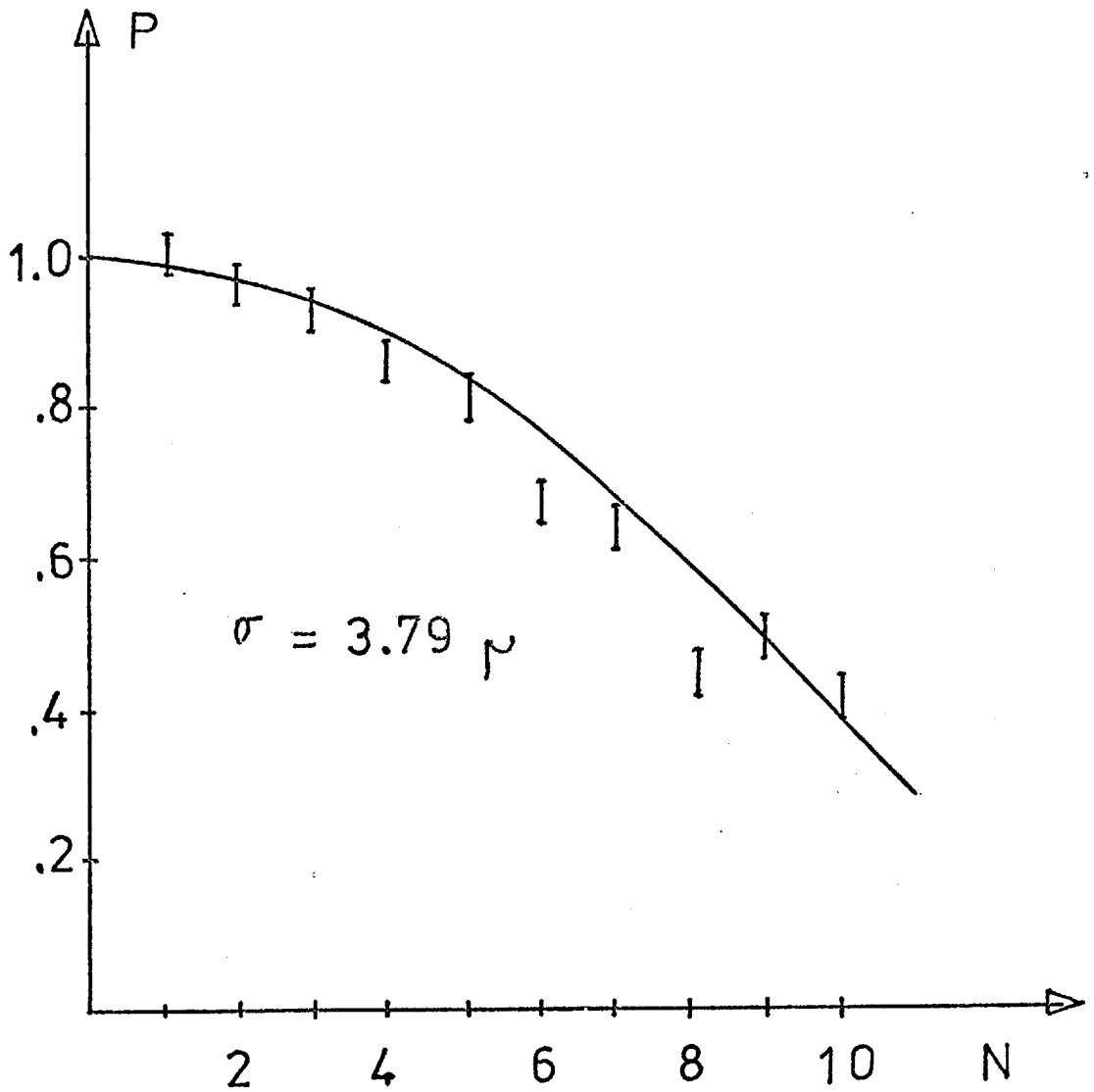


FIG. 53

GRAPH OF POLARIZATION COEFFICIENT P VERSUS N , THE NUMBER OF LAYERS OF GROUND GLASS (OR THICKNESS OF THE DIFFUSING MEDIUM) FOR A STANDARD DEVIATION OF THE ROUGHNESS OF $\sigma = 3.79 \mu$.

($\lambda = 4765 \text{ \AA}$ -blue-, $D = 1.18 \text{ m}$, U = collecting pupil radius = 1.5 cm , A = scanning aperture diameter = 0.8 cm).

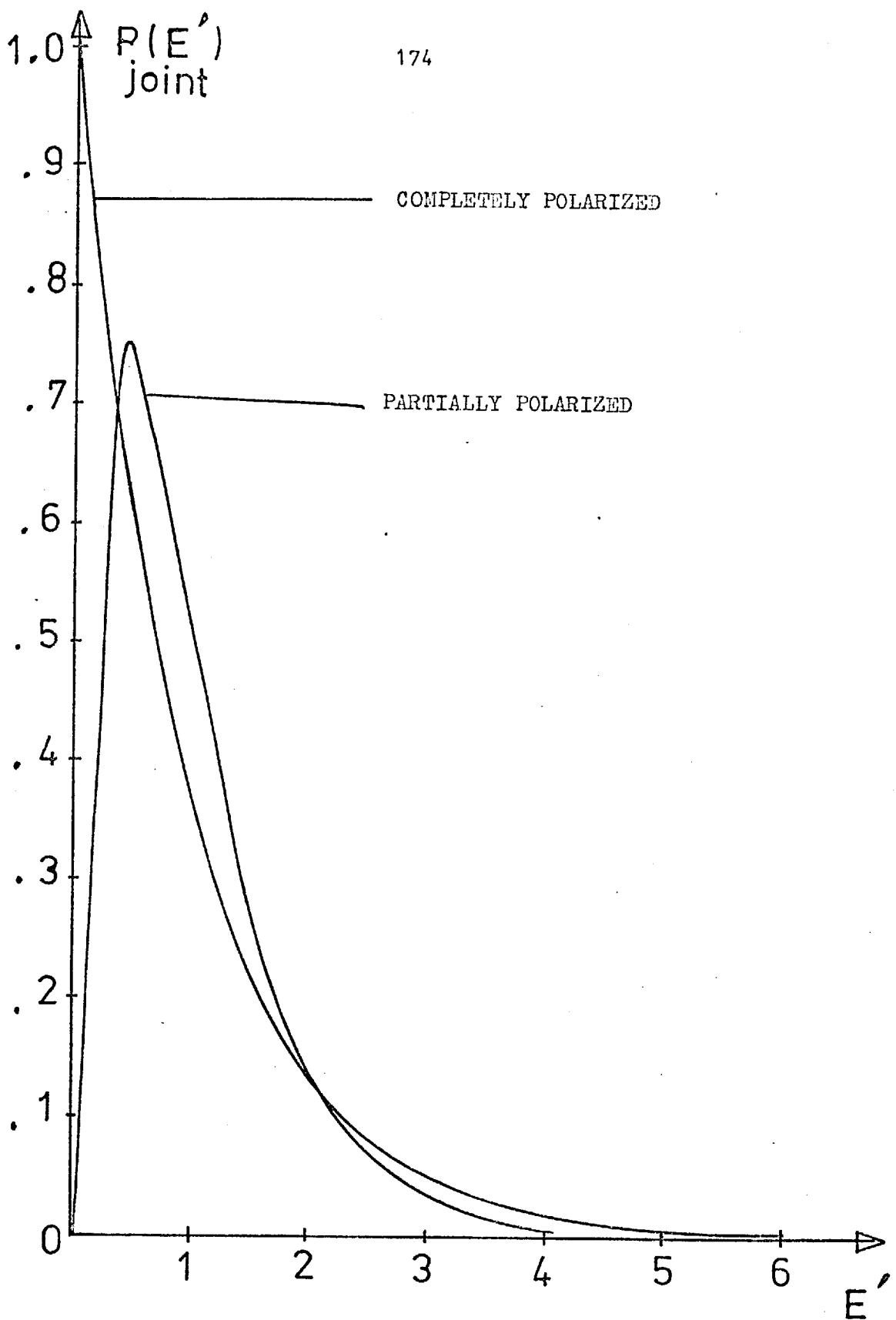


FIG. 54

FIRST ORDER PROBABILITY DENSITY FUNCTION OF INTENSITY FOR PARTIALLY POLARIZED SPECKLE PATTERNS OBSERVED THROUGH POINT APERTURES AS EXPRESSED BY EQN. 2 (7). THE AVERAGE TOTAL INTENSITY IS UNITY AND THE POLARIZATION FACTOR IS (EQN. 3 (40)) 0.37.

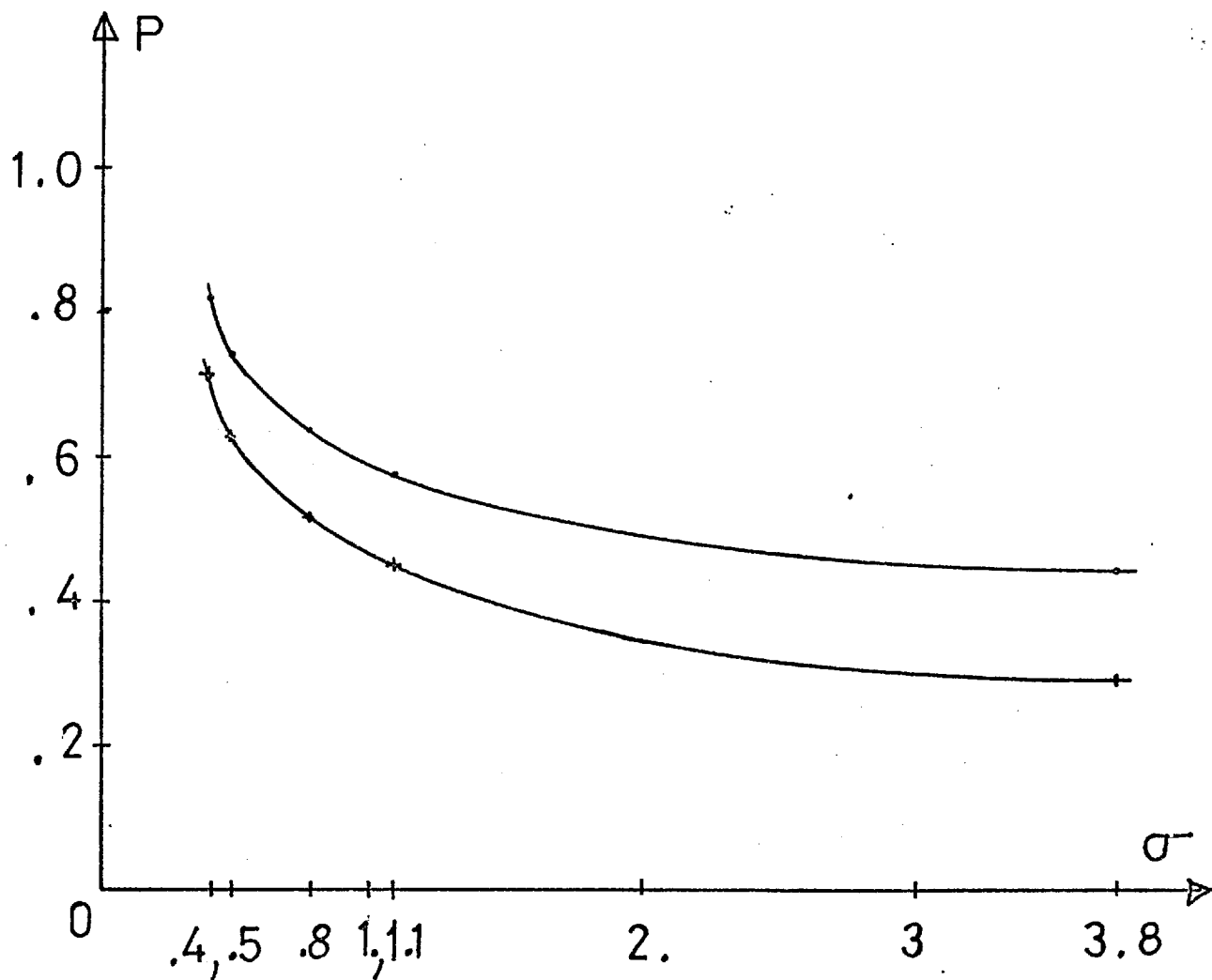


FIG. 55 GRAPH OF POLARIZATION COEFFICIENT P VERSUS σ , THE STANDARD DEVIATION OF THE ROUGHNESS, USING STACKS OF TEN GROUND GLASS PLATES. THIS GRAPH IS PLOTTED FOR TWO WAVELENGTHS,

$\lambda_1 = 6328 \text{ \AA}$ -RED-, $\bullet$, $\lambda_2 = 4765 \text{ \AA}$ -BLUE-, $+$, ($D = 1.18 \text{ m}$, $U = \text{collecting pupil radius} = 1.5 \text{ cm}$, $A = \text{scanning aperture diameter} = 0.8 \text{ cm}$).

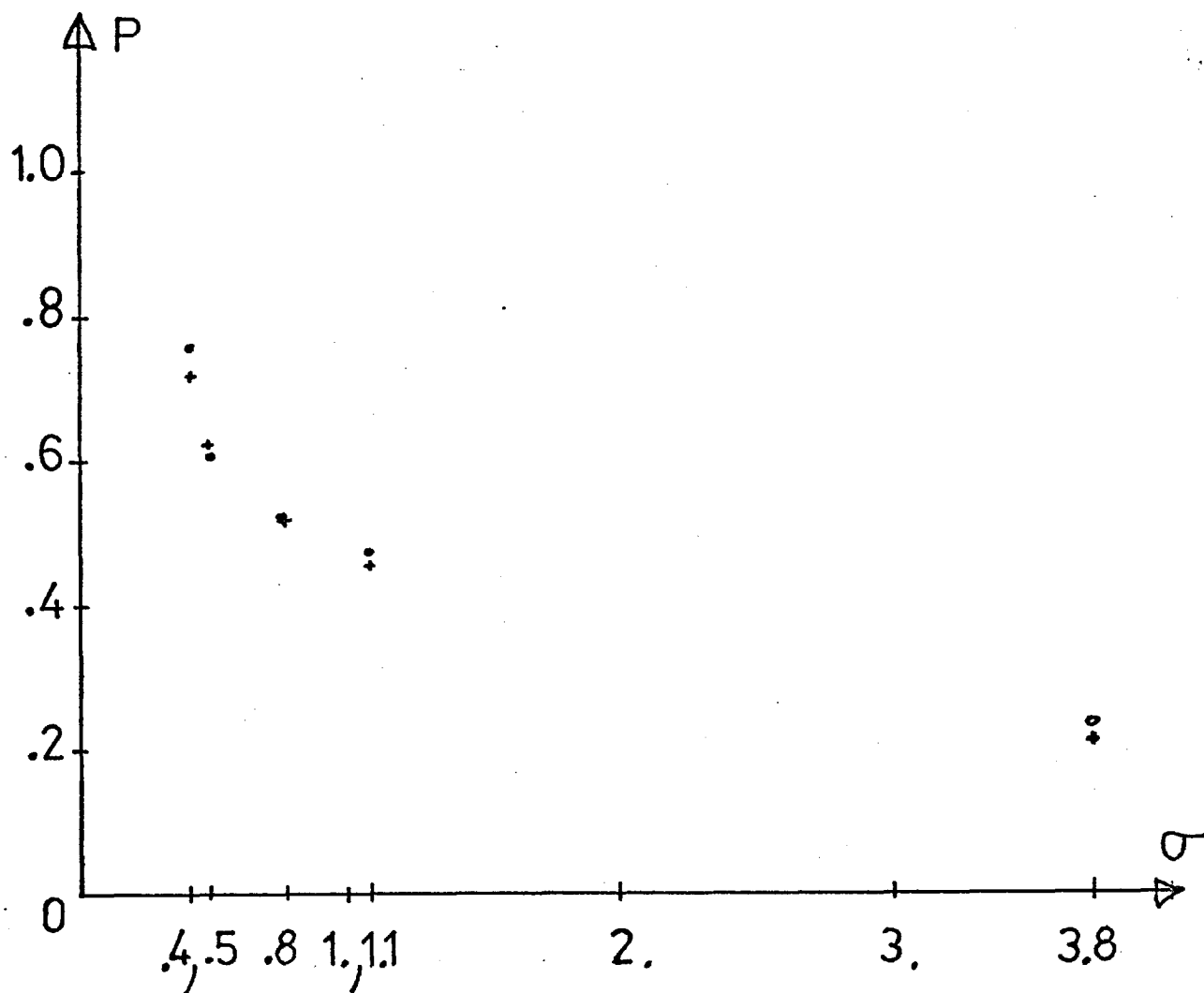


FIG. 56 COMPARATIVE PLOT OF P VERSUS σ MEASURED AS IN EQN.

3.1 (39) $\circ$, AS IN EQN. 3.1 (40) $+$. ($\lambda = 4765 \text{ \AA}$ -blue-,
 $D = 1.18 \text{ m}$, $U = 1.5 \text{ cm}$, $A = 0.8 \text{ cm}$).

roughness r.m.s. height fluctuation. This graph was obtained by using stacks of ten ground glass plates, the r.m.s. height fluctuation being constant within each stack. Two wavelengths were used, $\lambda_1 = 6328 \text{ \AA}$ red and $\lambda_2 = 4765 \text{ \AA}$ blue.

Throughout all the measurements, the coefficients C and S for $\pm 45^\circ$ plane polarization and right hand circular polarization remained negligibly small. The polarization coefficient P can therefore be expressed as in equ. (40) rather than as in equ. (39). This is illustrated by the graph in fig. (64) which represents a comparative plot of P versus θ for P defined as in equs. (39) and (40).

These experiments represent some of the very first investigations of randomly polarized speckle. It is felt by the author that the statistics of randomly polarized speckle can be studied in connexion with the properties of matter, for example, surface roughness, where opaque surfaces can serve as moulds to transparent resins (CHANDLEY and WELFORD 1975) When dried these resins can be built up in stacks to constitute a thick diffusing medium as in the above experiments.

APPENDIX A

CHARACTERISTIC FUNCTIONS AND CUMULANTS

Here we define characteristic functions and cumulants. We show some of their fundamental properties and their relevance to the sum of many independent variables, (CRAMER 1946 , LOEVE 1962 , PAPOULIS 1965)

CHARACTERISTIC FUNCTIONS

Let $P(x)$ be a normalized probability distribution of the real random variable x ,

$$\int_{-\infty}^{+\infty} P(x) dx = 1 \quad (A1)$$

The characteristic function $C_x(z)$ is defined as the mean value of e^{izx} or as the Fourier transform of $P(x)$,

$$C_x(z) = \langle e^{izx} \rangle = \int_{-\infty}^{+\infty} e^{izx} P(x) dx \quad (A2)$$

In general $C_x(z)$ is a complex valued function of z . Elementary properties are,

$$C_x(0) = 1 \quad (A3)$$

$$|C_x(z)| \leq \int_{-\infty}^{+\infty} P(x) dx = 1 \quad (A4)$$

$$C_x(-z) = C_x^*(z) \quad (A5)$$

*denotes complex conjugate.

A Mac-Laurin expansion of e^{izx} gives,

$$\langle e^{izx} \rangle = 1 + \sum_{k=1}^{\infty} \frac{i^k z^k \langle x^k \rangle}{k!} \quad (A6)$$

The moments of order k , $\langle x^k \rangle$ are easily obtained by differentiation,

$$\langle x^k \rangle = \frac{1}{i^k} \frac{d^k}{dz^k} C_x(z) \Big|_{z=0} \quad (A7)$$

CHARACTERISTIC FUNCTION OF TWO OR MORE INDEPENDENT RANDOM VARIABLES

Let x_1 and x_2 be two independent random variables with known distributions, the characteristic function of $x = x_1 + x_2$ will be from equ. (A2)

$$C_x(z) = \langle e^{iz(x_1 + x_2)} \rangle \quad (A8)$$

which can be written, because the variables are independent, as

$$C_x(z) = \prod_{i=1}^n C_{x_i}(z) \quad (A9)$$

For n independent variables,

(A10)

The characteristic function of the sum of n independent random variables is equal to the product of the characteristic function of each variable.

Unfortunately the k^{th} moment is not easily obtained, from eqs. (A7) and (A10),

$$\langle x^k \rangle = \frac{1}{i^k} \frac{d^k}{dz^k} \prod_{i=1}^n C_{x_i}(z) \quad (A11)$$

The problem of evaluating the $\langle x^k \rangle$'s is the chief reason for introducing cumulants.

CUMULANTS q_k OF ORDER K

The Mac-Laurin expansion of $\text{Log}(1+z)$ is,

$$\text{Log}(1+z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k} \quad (\text{A12})$$

Substituting $1+z$ by the Mac-Laurin expansion in equ. (A6), after rearranging the terms, we have

$$\text{Log } C_x(z) = \sum_{k=1}^{\infty} \frac{q_k}{k!} i^k z^k \quad (\text{A13})$$

The coefficients q_k are called cumulants and were introduced by Thiele (1903).

Differentiation gives a relation similar to equ. (A7),

$$q_k = \frac{1}{i^k} \frac{d^k}{dz^k} \text{Log } C_x(z) \Big|_{z=0} \quad (\text{A14})$$

To obtain the relations between the x^k and q_k we write the identities,

$$\text{Log } C_x(z) = \text{Log} \left(1 + \sum_{k=1}^{\infty} \frac{\langle x^k \rangle i^k z^k}{k!} \right) = \sum_{k=1}^{\infty} \frac{q_k i^k z^k}{k!} \quad (\text{A15})$$

or

$$C_x(z) = 1 + \sum_{k=1}^{\infty} \frac{\langle x^k \rangle i^k z^k}{k!} = \exp \left(\sum_{k=1}^{\infty} \frac{q_k i^k z^k}{k!} \right) \quad (\text{A16})$$

Formaly, this gives,

$$\begin{aligned} q_1 &= \langle x \rangle \\ q_2 &= \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2 \\ q_3 &= \langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle - 2\langle x^3 \rangle \\ q_4 &= \langle x^4 \rangle - 3\langle x^2 \rangle^2 - 4\langle x \rangle \langle x^3 \rangle + 12\langle x \rangle^2 \langle x^2 \rangle - 6\langle x^4 \rangle \end{aligned} \quad (\text{A17})$$

and conversely,

$$\begin{aligned}
 \langle x \rangle &= q_1 \\
 \langle x^2 \rangle &= q_2 + q_1^2 \\
 \langle x^3 \rangle &= q_3 + 3 q_1 q_2 + q_1^3 \\
 \langle x^4 \rangle &= q_4 + 3 q_1^2 q_2 + 4 q_1 q_3 + 6 q_1^2 q_2 + q_1^4 \\
 &\text{-----}
 \end{aligned}
 \tag{A18}$$

APPLICATION OF CUMULANTS TO n INDEPENDENT RANDOM VARIABLES

From equs. (A10) and (A14) we have,

$$q_k = \sum_{j=1}^n \frac{1}{i^k} \frac{d^k}{dz^k} \text{Log } C_{x_j}(z) \tag{A19}$$

or more simply

$$q_k = \sum_{j=1}^n q_{jk} \tag{A20}$$

The cumulant of order k of a sum of independent random variables is the sum of the individual cumulants of order k of each variable. By using relations (A17) and (A18) the $\langle x^k \rangle$ can be evaluated from the $\langle x_j^k \rangle$.

APPENDIX B COMPUTER PROGRAMMES

```

PROGRAM LOEIVA (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
COMMON C,PI
EXTERNAL C
PI=3.141593
E1=.0001
E2=.0001
7 READ(5,1)C
1 FORMAT(F10.0)
IF(C.LT.0.0) GOTO 8
WRITE(6,6) C
6 FORMAT(///*) C=*,F0.2)
WRITE(6,100)
100 FORMAT(///*) K OMEGA EIGENVALUE LIMIT VALUE*/
*)
AVE=1.0
A1=2.0*AVE
SUM=0.0
DO 2 K=1,101
I=K-1
A1=FLCAY(I)
ALP=A1*PI
IF(K.EQ.1) ALP=0.00001
UP=(A1+1.)*PI
IFAIL=1
CALL C05AAF (ALL,UP,E1,E2,G,XI,1,IFAIL)
IF (IFAIL.EQ.0) GOTO 4
WRITE (6,3)
3 FORMAT (IX, 41HFUNCTION HAS SAME SIGN AT INTERVAL LIMITS)
STOP 100
4 CONTINUE
AL=A1/(XI*XI+C*C)
IF(J.EQ.0) FIRST=AL
ALIM=A1/(AJ*AJ*PI-PI+C*C)
SUM=SUM+AL
WRITE(6,5) I,XI,AL,ALIM
5 FORMAT (/*,15,3E15.6/)
RATIO=AL/FIRST
IF(RATIO.LT.0.001) GOTO 11
2 CONTINUE
11 WRITE(6,10) SUM
10 FORMAT(/*) SUM OF EIG.VALS =*,E15.6/)
GOTO 7
8 STOP 1
END

```

THIS PROGRAMME COMPUTES THE EIGENVALUES FOR THE LORENTZIAN
INTENSITY TRANSMITTANCE PROFILE (FIRST CASE) SEE (TABLE 1)

THIS PROGRAMME COMARES BY PLOTTING THEM, THE APPROXIMATE AND EXACT FIRST ORDER PROBABILITY DENSITY FUNCTIONS OF SPECKLE PATTERNS SCANNED THROUGH FINITE APERTURES.

```

C      PROGRAM PROBONE (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C      THIS PROGRAM COMPUTES AND PLOTS THE EXACT AND APPROXIMATE
C      FIRST ORDER PROBABILITY DENSITY FUNCTION OF INTEGRATED
C      SPECKLE INTENSITY.
C      COMMON PE(101),CP(101),PA(101)
C      REAL NO
C      DIMENSION X(303),Y(303)
C      DIMENSION XL(100)
C      EQUIVALENCE (Y(1),PE(1))
C      DO 888 K=1,10
C      READ(5,3) N
C      3  FORMAT(I3)
C      IF(M.LT.0) STOP 2
C      -----
C      N IS NUMBER OF RELEVANT EIGENVALUES
C      XL(N) IS SET OF EIGENVALUES
C      XMI IS AVERAGE INTEGRATED INTENSITY
C      -----
C      READ(5,1) (XL(N),N=1,M)
C      WRITE(6,110) (XL(NU),NU=1,M)
C      110  FORMAT(15,6)
C      READ(5,2) SW
C      2    FORMAT(F10,6)
C      WRITE(6,120) SW
C      120  FORMAT(///* SW=*,F10,6///)
C      -----
C      SW=SPACE-BANDWIDTH PRODUCT
C      SW=0
C      -----
C      READ(5,6) XMI
C      6    FORMAT(F10,6)
C      NO=0
C      DO 7 I1=1,N
C      7    NO=NO+XL(I1)
C      NO=NO/XL(1)
C      ALPHA=XMI/(NO*XL(1))
C      XO=NO
C      CALL GMMMA(XO,GXO,IER)
C      WRITE(6,1001) IER
C      1001  FORMAT(///* IER=*,I10///)

```

CONTINUED ON NEXT PAGE...

CONTINUATION OF PROGRAMME PROBONE

```

C-----
C      IER = 0 , NO ERROR
C      IER = 1,  X0 IS WITHIN 10**(-6) OF BEING A
C                  NEGATIVE INTEGER.
C-----
      DO 100 I=1,101
      X(I)=X(I+101)=X(I+202)=FLOAT(I-1)/10.
      S=0.
      DO 200 N=1,M
      CALL BN(N,XL,*,*)
200  S=S+1*EXP(-X(I)/(ALPHA*XL(N)))/(ALPHA*XL(N))
      PE(I)=S
      IF (I.EQ.1) GOTO 103
      PA(I)=(NO/X**I)**NO*X(I)**(NO-1.)*EXP(-NO*X(I)/XMI)/GX0
103  PA(1)=1.
100  OP(I)=PE(I)-PA(I)
      CALL GRAFIT(X,Y,303,202,101,50)
C-----
C      X IS SYMBOOL FOR APPROXIMATE FORMULA
C      * IS SYMBOOL FOR EXACT FORMULA AND FOR DIFFERENCE.
C-----
      WRITE(6,777) NO
777  FORMAT(* NO=*,F10.6//)
      AIA=XMI
      SIG=ALPHA*XL(1)*SQRT(NO)
      RATIO=SIG/AIA
      RATIO2=AIA/(SIG*SIG)
      PARAM=0.
      DO 10 L=1,101
10  PARAM=PARAM+OP(L)*DP(L)
      PARAM=PARAM/101
      WRITE(6,4) AIA,SIG,RATIO
4  FORMAT(* MEAN=*,F20.6,* SIG=*,F20.6,* SIG/MEAN=*,F20.6/)
      WRITE(6,5) RATIO2,PARAM
5  FORMAT(* MEAN/SIGSQ=*,F20.6,* PARAM=*,F20.6/)
888  CONTINUE
      STOP 1
      END
      SUBROUTINE BN(N,XL,B,M)
      DIMENSION XL(M)
      A=1.
      DO 1 J=1,M
      IF (J.EQ.N) GOTO 2
      A=A*XL(N)/(XL(N)-XL(J))
2  CONTINUE
1  CONTINUE
      B=A
      RETURN
      END

```

THIS PROGRAMME COMPUTES THE EIGENVALUES FOR THE CIRCULAR CONFIGURATION
(FOURTH CASE) BY MATRIX DIAGONALISATION, (SEE TABLE 5).

```

PROGRAM GYROVA (INPUT,OUTPUT,TAP5=INPUT,TAP6=OUTPUT)
C THIS PROGRAM COMPUTES EIGENVALUES FOR CIRCULAR TOP HAT CASE
COMMON P(60,2),A(60,60),R(60),E(60)
C P IS ARRAY OF COORDINATES OF POINTS IN MESH
C A IS AUTO-CORRELATION KERNEL IN MATRIX FORM
C R IS EIGENVALUE MATRIX
C E IS WORK-SPACE MATRIX
LIM1=5
C LIM1 DEFINES RADIAL SIZE OF MESH
LIN=2*(LIM1*LIM1+LIM1)
C LIN DEFINES NUMBER OF POINTS IN MESH
ALIM1=FLOAT(LIM1)
PI=3.14159
READ(5,*) C
C C IS RELATIVE APERTURE, C=2.*PI*U*/(WAVELENGTH*D)
H=SQRT(1./(ALIM1*LIM1+ALIM1))
C H IS RADIAL SUMMATION ELEMENT
WRITE(6,10) C,H,LIM1,LIN
10 FORMAT(* C=*,F15.6,* H=*,F15.6,* LIM1=*,I5,* LIN=*,I5)
DO 1 N=1,LIN
C THIS DO LOOP FILLS IN THE COORDINATES OF THE MESH INTO
C THE ARRAY P
N0=2*(N-1)-1
I1=N0+1
N2=2*(N+N0)
AN=FLOAT(N)
ALPHA=PI/(2.*AN)
C ALPHA IS ANGULAR SUMMATION ELEMENT
DO 2 I=N1,N2
P(I,1)=AN*H
I4=4*N
AK=FLOAT(I-N1+1)
P(I,2)=AK*ALPHA
CONTINUE
2
1 CONTINUE
DO 100 I=1,LIN
DO 200 J=1,LIN
C THESE TWO DO LOOPS DEFINE THE KERNEL IN A RE-INDEXED
C MATRIX FORM
IF(J.EQ.I) GOTO 400
ARGSQ=P(I,1)*P(I,1)+P(J,1)*P(J,1)-2.*P(I,1)*P(J,1)*COS(P(I,2)-P(J,
*2))
ARG=0.5*SQRT(ARGSQ)
BJ=S17ADF(ARG)
C THIS S17ADF IS A LIBRARY FUNCTION FOR BESSEL FUNCTIONS
A(I,J)=2.*BJ/ARG
GOTO 200
400 A(I,J)=1.
A(J,I)=A(I,J)
CONTINUE
JFAIL=1
CALL FC2AAF (A,60,60,R,E,IFAIL)
C THIS LIBRARY STORED SUBROUTINE PERFORMS MATRIX DIAGONALISATION
WRITE(6,301)
301 FORMAT(1H3,FHINDEX,15H EIGENVALUE)
SUM=0.
DO 300 I=1,LIN
J=1-1
XL=H*H*PI*R(I)/2.
SUM=SUM+XL
WRITE(6,302) J,XL
302 FORMAT(1H3,I5,F15.6)
300 CONTINUE
SUM1=SUM/(C*0)
C THIS EQUATION VERIFIES HERCER-S THEOREM.
WRITE(6,20) SUM,SUM1
20 FORMAT(* SUM OF THE EIGENVALUES IS*,F15.6,* NORMALIZED SUM IS*,
*15.6)
STOP 1
END

```

PROGRAMME JUPITER

```
PROGRAM JUPITER (INPUT,OUTPUT,TAPE20,TAPE5=INPUT,TAPE6=OUTPUT,TAPE
*1=TAPE20)
```

```
COMMON B(50),P(50),A(4000)
```

```
SIN FUNCTION X(100),Y(100)
```

```
EQUIVALENCE (Y(1),P(1))
```

```
ASOMEAN=0.
```

```
AHEAD=0.
```

```
TOTAL=0.
```

```
READ(5,1) N
```

```
FORMAT(15)
```

```
READ(5,13) WL,D,RO,AP
```

```
FORMAT(4F5.1)
```

```
WL IS WAVELENGTH IN ANGSTROMS
```

```
D IS FOCUS LENGTH IN METERS
```

```
RO IS PUPIL RADIUS IN CENTIMETERS
```

```
AP IS SCANNING APERTURE DIAMETER IN MICRONS
```

```
READ(5,5) NPTA
```

```
FORMAT(13)
```

```
DO 2 J=1,N
```

```
READ(1,3) IA,IB,IS
```

```
FORMAT(15,I2,I1)
```

```
A(1)=FLOAT((-IA)*10.** (IS-1))
```

```
AHEAD=AHEAD+A(1)
```

```
ASOMEAN=ASOMEAN+A(1)*A(1)
```

```
AHEAD=AHEAD/FLOAT(N)
```

```
ASOMEAN=ASOMEAN/FLOAT(N)
```

```
AHEADSQ=AHEAD*AHEAD
```

```
SIGSQ=ASOMEAN-AHEADSQ
```

```
SIG=SQRT(SIGSQ)
```

```
RATIO1=SIGSQ/AHEADSQ
```

```
RATIO2=SIG/AHEAD
```

```
AA=AHEADSQ/(SIGSQ*NPTA)
```

```
XNO=AHEADSQ/SIGSQ
```

```
XN=XNO-1.
```

```
CALL GAMMA(XNO,GX,ITP)
```

```
B(1)=0.
```

```
P(1)=0.
```

```
X(1)=X(51)=1.
```

```
DO 4 I1=2,50
```

```
X(I1)=X(I1+50)=FLOAT(I1)
```

```
P(I1)=0.
```

```
P(I1)=AA**XNO*X(I1)**XN*EXP(-AA*X(I1))/GX
```

```
DO 6 N1=1,N
```

```
Y=IFIX(FLOAT(NPTA)*A(N1)/AHEAD)+1
```

```
IF (N.GT.50) GOTO 21
```

```
B(N)=B(N)+1.
```

```
GOTO 6
```

```
WRITE(6,21) N
```

```
FORMAT(//* N IS OVER 50 AND EQUAL TO*,I5//)
```

```
CONTINUE
```

```
WRITE(6,12)
```

```
FORMAT(//* BIN NO CONTENTS OF BIN*/)
```

```
WRITE(6,9) (X(L),P(L),L=1,50)
```

```
FORMAT(1XF5.2,F20.6)
```

```
DO 16 K=1,50
```

```
TOTAL=TOTAL+B(K)
```

```
P(K)=P(K)/FLOAT(N)
```

```
WRITE(6,1000)
```

```
FORMAT(//* INDEX GAMMA DIST. EXPERIMENT*/)
```

```
WRITE(6,1001) (X(K1),P(K1),B(K1),K1=1,50)
```

```
FORMAT(1XF5.1,F20.6,E20.6)
```

```
CALL GRAFIT(X,Y,100,50,0,50)
```

```
RAYLIN=.61*WL*D/RO*10.**(-5)
```

```
RAYLIN IS RAYLEIGH RESOLUTION LIMIT (CIRCULAR) IN NM
```

CONTINUED ON NEXT PAGE...

CONTINUATION OF PROGRAMME JUPITER

```

      XK=AP/2*Wavelength**(-3)
      C=3.14159*XK/2.
      LW=IFIX(WL)
      WRITE (6,14) LW,C,PO,AP
14  FORMAT(////* Wavelength =*,I5,* ANGSTROMS, SINCH LENG.=*,F5.2,* INCH, PU
      *PIL RADIUS=*,F5.2,* CM, SCAN. AP. DIA.=*,F5.2,* MICRONS*///)
      WRITE (6,15) Wavelength,XK,C
15  FORMAT(////* Wavelength=*,F10.6,* MM REL.AP. IN K UNITS=*,F10.6,* K
      *=K,PI/2=*,F10.6*///)
      WRITE (6,16) N
16  FORMAT(////* NUMBER OF SAMPLES USED=*,I10*///)
      IT=IFIX(TOTAL)
      WRITE (6,17) IT

17  FORMAT(////* TOTAL NUMBER OF SAMPLE DATA ACCOUNTED FOR=*,I10*///)
      WRITE (6,11) NDATA
11  FORMAT(////* NUMBER OF BINS TO AVERAGE=*,I5*///)
      WRITE (6,7)
7  FORMAT(////*      A MEAN      ASQ MEAN      MEANSQ      SIGSQ      SIGSQ/
      *MEANSQ      SIG/MEAN      AA      XNO*//)
      WRITE (6,8) ASQ MEAN,ASQ MEAN,MEANSQ,SIGSQ,RATIO1,RATIO2,AA,XNO
8  FORMAT(1X*E15.6*///)
      STOP 1
      END

```

THIS PROGRAMME PLOTS THE EXPERIMENTAL FIRST ORDER PROBABILITY DENSITY FUNCTION OF INTEGRATED INTENSITY OF SPECKLE PATTERNS SCANNED THROUGH FINITE APERTURES. IT ALSO PLOTS ON THE SAME GRAPH, FOR PURPOSE OF COMPARISON, THE APPROXIMATE (GAMMA) DISTRIBUTION, (SEE I.4.3 AND FIGS. 48 AND 49).

- ABRAMOWITZ, M. and STEGUN, I. 1970, Handbook of mathematical functions, Dover press N.Y.
- ACKASU, A.Z. 1961, Measurement of noise power spectra by Fourier analysis, J. Appl. Phys., 32, 565.
- ALLEN, L. and JONES, D.G.C. 1963. An analysis of the granularity of scattered optical maser light, Phys. Lett., 7, 321.
- ARCHBOLD E., BURCH, J.M., ENNOS A.E. and TAYLOR, P.A., 1969, Visual observation of surface vibration nodal patterns, Nature, 222, 263.
- ARCHBOLD, E., ENNOS, A.E., and TAYLOR, P.A., 1970, A laser speckle interferometer for the detection of surface movements and vibration, in "Optical Instruments and Techniques", J. Home Dickson ed., Oriel Press.
- BARAKAT, R. 1973, a) First order probability densities of laser speckle patterns observed through finite-size scanning apertures. Optica Acta 20 , 729.
- BARAKAT, R. 1973, b) The brightness distribution of the sum of two correlated speckle patterns. Opt. Comm. 8, 14.
- BARNES, C.W., 1966, object restoration in a diffraction limited imaging system. J.O.S.A. 56, 575
- BEDARD, G. 1966, Photon Counting Statistics of Gaussian light Phys. Rev. 151 1038
- BEDARD G. JANE C. CHANG and MANDEL, L., 1967. Approximate Formulas for photoelectric counting distributions Phys. Rev. 160, 1496.
- BLACMAN, R.B., and TUKEY, J.W., 1959. The measurement of power spectra, Dover press N.Y.
- BORN, M., and WOLF, E., 1970, Principles of Optics, Pergamon Press London.
- BOUKWAMP, C.J., 1947. On spheroidal wave functions of zero order, J. Math. Phys. 26, 79.
- BOURRET, R., 1974. Depolarization of electromagnetic radiation, Optica Acta 21, 721.
- BRACEWELL, R., 1965, The Fourier transform and its Applications, McGraw Hill Book Co. N.Y.

- BURCH, J.M., 1970. Interferometry with scattered light, in Optical Instrumentation and Techniques, J. Home Dickson Ed. Oriel Press, Newcastle upon Tyne.
- BURCH, J.M., 1975. Acknowledged in GOODMAN 1975.
- BURES, J., DELISLE, C., and ZARDECKI, A., 1972. Determination de la surface de coherence a partir d'une experience de photocomptage, Canadian Journal of Physics 50, 760.
- BUTTERS, J.N., and LEENDERTZ, J.A., 1971. Speckle pattern and holographic techniques in engineering metrology, Opt. and Laser Techn., 3, 26
- CHANDLEY, P.J. and WELFORD, W.T. ¹⁹⁷⁵ Surface Roughness by light scattering Private communication.
- CLARKE, D., and GRAINGER, J.F., 1971. Polarized light and Optical Measurement, Pergamon Press, Oxford.
- des CLOIZEAUX, J. and MEHTA M.L., 1972. Some Asymptotic expressions for prolate spheroidal functions and for the eigenvalues of differential and integral equations of which they are solutions, J. Math. Phys. 13, 1945.
- CONDIE, M.A. 1966. An experimental investigation of the statistics of diffusely reflected coherent light, Thesis, Stanford University.
- COURANT R. and HILBERT D., 1953. Methods of mathematical physics Interscience Publishers Ltd. London
- CRAMER, H., 1946. Mathematical methods of statistics, Princeton University Press.
- DAINTY, J.C. 1970. Some statistical properties of random speckle patterns in coherent and partially coherent illumination, Optica Acta, 17, 761.
- DAINTY, J.C., 1971. Detection of images immersed in speckle noise, Optica Acta 18, 327.
- DAINTY, J.C., 1972. a) Coherent addition of a uniform beam to a speckle pattern, J.O.S A. 62 p.
- DAINTY, J.C. 1972. b) Speckle statistics and the detection of images in in hologram reconstructions of bubble chamber tracks and other objects Thesis, London University.

- DAINTY, J.C., 1973. Diffraction limited imaging of stellar objects using telescopes of low optical quality, *Opt. Comm.* 7, 129.
- DAINTY, J.C., and WELFORD, W.T., 1971, reduction, of speckle in image plane hologram reconstruction by moving pupils, *Opt. Comm.*, 3, 289.
- DAVENPORT, W.B., and ROOT, W.L. 1958. An introduction to the theory of random signals and noise, Mc-Graw, Hill Book Co. N.Y.
- ELIASSON, B., and ROTTIER, F.M., 1971. Determination of the granular radiance distribution of a diffuser and its use for vibration analysis. *J.O.S.A.*, 61, 559.
- ERDELYI, A., 1954. Table of integral transforms. McGraw Hill Book Co. N.Y.
- EXNER, K. 1877, "Über die Fraunhofer'schen Ringe, die Quetelet'schen Streifen und verwandte Erscheinung, Sitzungsber, Kaiserl. Akad. wiss. (Wien), 76, 522.
- FERNELLIUS, N., and TOME, C. 1971. Vibration analysis studies using changes of laser speckle, *J.O.S.A.* 61, 566.
- FLAMMER, C. 1954. Spheroidal wave functions. Stanford University Press, Stanford, Cal.
- FREDHOLM, I. 1903, Sur une classe d'equations fonctionnelles, *Acta Math.* 27, 365.
- FRIEDEN, B.R. 1971. Evaluation, design and extrapolation methods for optical signals based on use of the prolate functions in *Progress in Optics* vol 9, p.311, E. Wolf edition.
- FROBERG, C.E. 1870. Introduction to numerical analysis, Addison-Wesley Publishing Co. London.
- GABOR, D. 1970, Laser speckle and its elimination, *I.B.M.J. Res. Develop.* 11, 509.
- GERRITSON, H.J. HANNAN, W.J., and RAMBERG, E.G., 1968, elimination of speckle noise in holograms with redundancy, *Appl. Opt.*, 7, 2301.
- GOLDFISCHER, L.I. 1865. Autocorrelation function and power spectral density of laser produced speckle patterns, *J.O.S.A.* 55, 247.
- GOODMAN, J.W. 1963. Statistical properties of laser sparkle patterns, Stanford University Electronics Labs. Tech. Rep. SEL - 63 - 140 (TR 2303 - 1)

- GOODMAN, J.W. 1965. Some effects of target induced scintillation on optical radar performance, Proc. I.E.E.E. 53 1688.
- GOODMAN, J.W. 1967. Film Grain noise in Wave front reconstruction Imaging. J.O.S.A. 57, 493.
- GOODMAN J.W. 1968. Introduction to Fourier Optics, McGraw Hill Book Co., N.Y.
- GOODMAN, J.W. 1975. Probability density function of the sum of N partially correlated speckle patterns, Opt. Comm. 13, 244.
- GOULD, G., JACOBS, S.F., LATOURETTE, J.T. NEWSTEIN M. and RABINOWITZ, P. 1964. Coherent detection of light scattered from a diffusely reflecting surface, Appl. Opt., 3, 648 (L).
- HEURTLEY, J.C. 1964, in Proceedings of the symposium on Quasi Optics, Polytechnic Press, Brooklyn, N.Y. 1964.
- HEURTLEY, J.C. and STREIFER W. 1965. Optical resonator modes - circular reflectors of spherical curvature. J.O.S.A. 55, 1472.
- KAC, M., and SIEGERT A.J.F. 1947, on the theory of noise in radio receivers with square law detectors, Jo. Appl. Phys. 18, 383.
- LANDAU, H.J. 1965. The eigenvalue behaviour of certain convolution equations. Trans. Amer. Math. Soc. 115, 242.
- LANDAU, H.J. 1967. Necessary density conditions for sampling and interpolation of certain entire functions, Acta Mathematica 117, 37.
- LANDAU, H.J., and POLLAK H.O. 1962. Prolate Spheroidal wave-functions, Fourier analysis and uncertainty - II, Bell Syst. Tech. Jo. 40, 65.
- LANDAU, H.J. and POLLAK, H.O. 1962. Prolate spheroidal wave functions, Fourier analysis and uncertainty - III, Bell Syst. Tech. Jo. 41, 1295.
- LANGMUIR, R.V. 1963. Scattering of laser light Appl. Phys. Lett. 2, 29.
- Von LAUE, M. 1914. Die Beugungserscheinungen an vielen unregelmässig verstreuten Teilchen, Sitzungs. Akad. Wiss. (Berlin) 44, 1144.
- Von LAUE M. 1916. Mathematische Betrachtungen über die Beugungerscheinungen auf vielen unregelmässig verstreuten Teilchen, M.H. Physik. G (Zurich) 18, 90.

- Von LAUE, M. 1917. Ein Versagen der klassischen Optik, Verhandl. Dent. Phys. Ges. 19, 19.
- LAWSON, J., and UHLENBECK G.E. 1950. Threshold noise signals, Mc.Graw Hill Book Co. N.Y.
- LEENDERTZ, J.A. 1970. Measurement of surface displacement by interference of speckle patterns, in Optical Instruments and Techniques, J. Home Dickson ed., Oriel press. p.256.
- LOEVE, M., 1963. Probability theory, Van Nostrand, Rheinhold Co. London.
- MANDEL, L. 1959. Fluctuations of photon Beams: the distribution of the photoelectrons, Proc. Phys. Soc. 74, 233.
- MEHTA, C.L. 1970. Theory of photoelectron counting, in Progress in Optics vol. VIII, E. WOLF ed. North Holland Publishing Co. Amsterdam.
- MEHTA, M.L. 1967. Random Matrices, Academic Press, London.
- MEHTA, M.L. and Des CLOIZEAUX, J. 1972. The probabilities of several consecutive eigenvalues of a random matrix.
- MEHTA, C.L., and MEHTA, M.L. 1973. Statistics of photoelectron counting with a rectangular spectral profile, J.O.S.A. 63, 826
- MERCER, T. 1909. Functions of positive and negative type and their connection with the theory of integral equations. Trans. London Phi. Soc. (A) 209 415.
- MIDDLETON, D. 1960. An introduction to statistical communication theory, McGraw Hill Book Co. N.Y.
- OLIVER, B.M. 1963. Sparkling spots and random diffraction, Proc. I.E.E.E. 51 220 (L)
- PAPOULIS, A. 1965. Probability, random variables and stochastic processes, McGraw Hill Book Co., N.Y.
- PARRY, G. 1974. a) Some effects of temporal coherence on the first order statistics of speckle, Optica Acta, 21, 763.
- RUSHFORTH, C.K. and FARRIS R.W. 1968. Restoration, resolution and noise, J.O.S.A. 58, 539.
- SHANNON C.E. and WEAVER, W. 1962. The mathematical theory of communication, The University of Illinois Press, Urbana.
- SCRIBOT, A.A. 1974. a) First Order probability density functions of speckle measured with a finite aperture. Opt.Comm. 4, 238.

- SCRIBOT, A.A. 1974. b) Depolarization of coherent laser light by a thick diffusing medium, *Opt. Comm.* 13, 81.
- SLEPIAN, D. 1958. Fluctuations of random noise power, *Bell syst. Tech. Jo.* 37, 163
- SLEPIAN, D. 1964. Prolate spheroidal wave functions, Fourier analysis and uncertainty, IV, extension to many dimensions, Generalised prolate spheroidal functions. *Bell Syst. Tech. Jo.* 43, 3009.
- SLEPIAN, D. 1965. Some asymptotic expansions for prolate spheroidal wave functions, *Jo. Math. Phys.* 44, 99.
- SLEPIAN, D., and POLLAK, H.O. 1961. Prolate spheroidal wave functions, Fourier analysis and uncertainty - I, *Bell Syst. Tech. Jo.* 40, 43.
- SLEPIAN, D., and SONNENBLICK, E. 1965. Eigenvalues associated with prolate spheroidal wave functions of zero order, *Bell Syst. Tech. Jo.* 44, 1745.
- SPRAGUE, R.A. 1972. Surface Roughness measurement using white light Speckle. *Appl. Optics* 11, 2811.
- STUCKEY, M.M., and LAYTON, L.L. 1964. Numerical determination of spheroidal wave functions, eigenvalues and expansion coefficients, A.M.L. report 164, U.S. Naval Research Lab. Washington D.C.
- THOMAS J.B. 1969. An introduction to statistical communication theory, Wiley, N.Y.
- THIELE, T.N. 1903. Theory of observations, London. Quoted by CRAMER 1946. as the inventor of cummulants.
- TORALDO di FRANCIA, G. 1969. a) Degrees of Freedom of an image, *J.O.S.A.* 59, 799.
- TORALDO di FRANCIA, G. 1969. b) Some recent progress in classical optics, *Rivista del Nuov. Cim. No Speciale* p.460.
- TRIBILLON, G. 1974. Correlation entre deux speckle obtenus avec deux longueurs d'onde - Application a la mesure de la rugosite moyenne. *Opt. Comm.* 11, 172.
- VILKOMERSON, D. 1971. Statistical distribution of irradiance in the creation of a hologram. *J.O.S.A.* 61, 929.
- WELFORD W.T. 1971. Time-averaged images produced by optical systems with time-varying pupils, *Opt. Comm.* 4, 275.

WHALEN A.D. 1971. Detection of signals in noise. Academic Press, N.Y.

WOLF, E. 1959. Coherence properties of partially polarized electromagnetic radiation. Nuov. Cim. 13, 1165.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor W.T. Welford and my colleagues for many useful discussions. It is also a pleasure to thank Mr. C.V. Cooper who prepared all the glass plates for the diffusing media. This work was performed under an exchange grant between the Centre National de la Recherche Scientifique (C.N.R.S. Paris) and the British Council.

FIRST-ORDER PROBABILITY DENSITY FUNCTIONS OF SPECKLE MEASURED WITH A FINITE APERTURE

A.A. SCRIBOT

*Physics Department, Imperial College,
London SW7 2BZ, UK*

Received 28 December 1973

Revised manuscript received 4 April 1974

An exact and an approximate first-order probability density function are computed in terms of the eigenvalues of the autocorrelation function of the speckle pattern obtained in coherent light. We examine the case of a sinc autocorrelation function obtained with a rectangular intensity transmitting pupil. The integrated mean square difference between both formulae is plotted.

1. Introduction

In this article, we compare an exact and an approximate probability density formula for first order speckle statistics obtained in coherent light. The approximate formula is a gamma distribution, the exact formula is an expansion in terms of eigenvalues of the autocorrelation function of the speckle pattern. Rice [1] was the first to use the gamma distribution in the context of electronic noise statistics. Mandel [2] used it in optics in photoelectron counting and Goodman [3] applied it to speckle patterns. Condie [4] first discussed the properties of an exact formula in speckle statistics using a lorentzian intensity distribution for the scatterer. Mehta [5] compared the moduli of the characteristic functions of the approximate and the exact distributions in photoelectron counting. He also used a lorentzian for the spectral profile and showed good agreement in the case of small and large width of the spectral profile. Mehta and Mehta [6], by using Slepian's results [7, 8] found similar results in the case of a rectangular profile.

Here we compare the approximate and exact probability densities in the speckle context using a rectangular intensity distribution for the scatterer. The difference is characterized by the integrated mean square difference.

2. The exact formula

The probability density of speckle intensity at a point is well known to be a negative exponential [9], but speckles are observed through non-point finite apertures and one must compute the probability of integrated intensity $P(I)$; this is given by the integral [10, 11]

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-izI} dz}{\prod_n (1 - iz\epsilon_n)}, \quad (1)$$

where the ϵ_n are eigenvalues given by the homogeneous Fredholm equation

$$\epsilon_n \phi(x, y) = \iint_A r(x-x', y-y') \phi_n(x', y') dx' dy', \quad (2)$$

A = scanning aperture with $\epsilon_0 \geq \epsilon_1 \geq \epsilon_2$ where $r(x, y)$ is the complex amplitude autocorrelation function of the scanned speckle pattern, (x, y) , (x', y') are two points of the speckle pattern, ϕ_n are a set of eigenfunctions orthonormal over A . If all the eigenvalues are different, the integration of eq. (1) gives the exact formula [6, 10]

$$P(I) = \sum_n (C_n/\epsilon_n) \exp(-I/\epsilon_n),$$

with

$$C_n = \prod_{m \neq n} \epsilon_n (\epsilon_n - \epsilon_m)^{-1}. \quad (3)$$

The mean and variance are

$$\langle I \rangle = \sum_n C_n \epsilon_n, \quad (4)$$

$$\sigma^2 = 2 \sum_n C_n \epsilon_n^2 - \left(\sum_n C_n \epsilon_n \right)^2. \quad (4a)$$

3. The gamma distribution

If, as suggested by Condie [4] the eigenvalues are taken as values of a step function such that

$$\epsilon_n = \epsilon_0 \quad \text{for } n \leq n_0,$$

and zero elsewhere (the envelope of these eigenvalues is shown in fig. 1 in dotted lines for $n_0 = 6$), the integral (1) now becomes

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-izI} dz}{(1 - iz\epsilon_0)^{n_0}}, \quad (5)$$

which gives

$$P(I) = \left(\frac{1}{\epsilon_0} \right)^{n_0} \frac{I^{n_0-1} e^{-I/\epsilon_0}}{\Gamma(n_0)}, \quad (6)$$

which is the gamma distribution. The mean and vari-

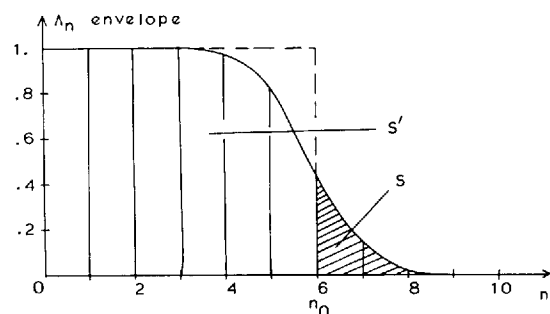


Fig. 1. An example of the behaviour of the eigenvalues in eq. (10) for $c = 10$. The ratio of the surfaces S and S' is a measure of the error made by replacing the eigenvalues by a step function.

ance are

$$\langle I \rangle = \epsilon_0 n_0, \quad (7)$$

$$\sigma^2 = \epsilon_0^2 n_0. \quad (7a)$$

This suggests eq. (6) be written as

$$P(I) = \left(\frac{n_0}{\langle I \rangle} \right)^{n_0} \frac{I^{n_0-1} e^{-I n_0 / \langle I \rangle}}{\Gamma(n_0)}, \quad (8)$$

which is the probability function scaled to mean intensity. The quantity n_0 is interpreted as the number of independent correlation cells into which the aperture is divided.

4. Case where the autocorrelation function $r(x, y)$ is a sinc function

We now examine the case where the autocorrelation function is a sinc function. Physically this corresponds to considering the speckle as white noise taken through a slit pupil (one dimensional) of uniform transmittance. Generalisation to a two-dimensional rectangular aperture is straightforward. The autocorrelation of the speckle in the far field is equal to the amplitude diffraction function of the pupil [11]. The eigenvalues are scaled versions of the eigenvalues Λ_n of the prolate spheroidal wave functions and have been extensively studied and tabulated by Slepian [7, 8]. They are defined by

$$\Lambda_n \phi_n(x) = \int_{-1}^{+1} \frac{\sin[c(x-x')]}{\pi(x-x')} \phi_n(x') dx' \quad (9)$$

with $1 \geq \Lambda_0 \geq \Lambda_1 \geq \Lambda_2 \dots$. The required eigenvalues are given by

$$\epsilon_n = \alpha \Lambda_n^i, \quad (10)$$

where α is a proportionality constant. The envelope of these eigenvalues is the continuous line in fig. 1 for $c = 10$, where c is the space bandwidth product of the system. The similarity between the two curves is clearly seen. The ratio of the crossed-hatched area S beyond n_0 to the area S' under the step function

$$S/S' = \sum_{n > n_0} \Lambda_n / n_0 \Lambda_0 \quad (11)$$

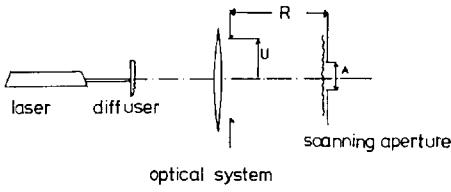


Fig. 2. Optical set-up for observing speckle patterns in coherent light.

is a measure of the error made on the eigenvalues by replacing them by a step function.

5. Computation of the cut-off index of the eigenvalues

By the Mercer theorem (see e.g. ref. [12]) we can write the kernel of eq. (9)

$$\frac{\sin[c(x-x')]}{\pi(x-x')} = \sum_n \Lambda_n \phi_n^*(x') \phi_n(x), \quad (12)$$

where the star denotes complex conjugate. By setting x' equal to x and integrating over $[-1, 1]$ we find

$$2c/\pi = \sum_n \Lambda_n. \quad (13)$$

If the eigenvalues are taken as a step function we can write

$$\sum_n \Lambda_n = n_0 \Lambda_0 \quad (14)$$

and

$$n_0 = 2c/\pi \Lambda_0; \quad (15)$$

c is the space bandwidth product [8,9], which in our case is

$$c = \pi UA/\lambda D \quad (16)$$

and

$$n_0 = 2UA/\lambda D \Lambda_0, \quad (17)$$

where λ is the wavelength, U , A and D are defined in fig. 2. This formula for n_0 is valid for both the step

Fig. 3. Comparative plots of the exact and approximate probability densities: a) $c = 0.5$, $n_0 = 1.03$, $n_0/\langle I \rangle = 0.26$; b) $c = 1.0$, $n_0 = 1.12$, $n_0/\langle I \rangle = 0.28$; c) $c = 2.0$, $n_0 = 1.45$, $n_0/\langle I \rangle = 0.36$; d) $c = 8.0$, $n_0 = 5.09$, $n_0/\langle I \rangle = 1.27$.

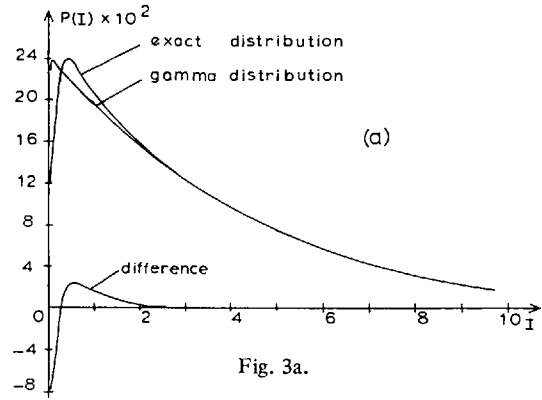


Fig. 3a.

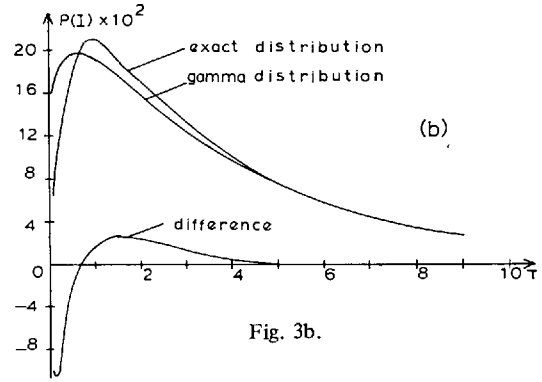


Fig. 3b.

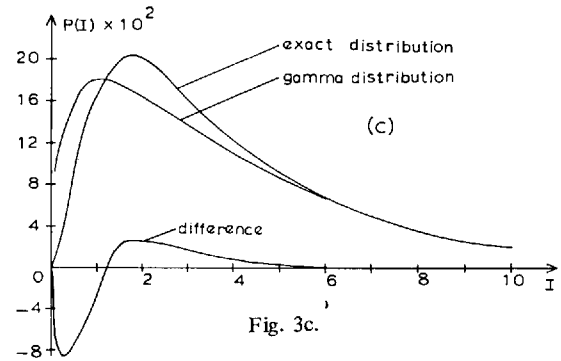


Fig. 3c.

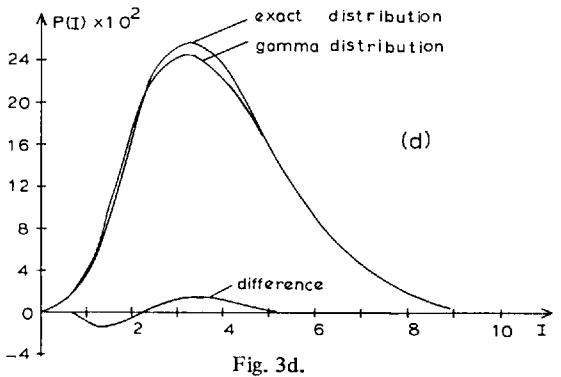


Fig. 3d.

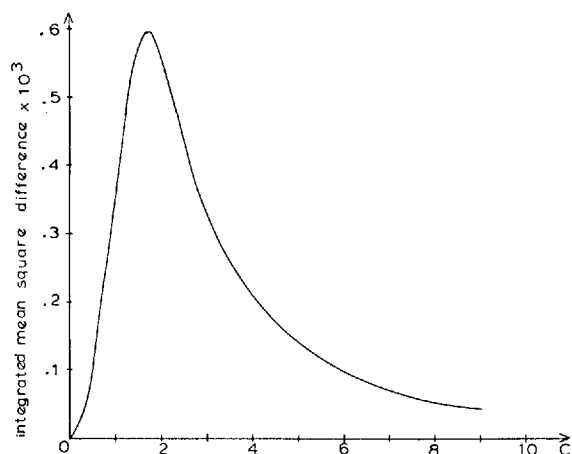


Fig. 4. The difference between the approximate and exact formulae is characterised by the graph of the integrated mean square difference versus the space bandwidth product c .

and the proper set of eigenvalues. This result differs from that of Barakat [10] who gives n_0 to be inversely proportional to A .

From eqs. (7), (10) and (17) we evaluate α ,

$$\alpha = \langle I \rangle \lambda D / 2UA. \quad (18)$$

6. Comparison of exact and approximate formulae

The exact probability density was calculated from eq. (3), using Slepian's eigenvalues, and the gamma distribution [eq. (8)] was computed with (non-integral) values of n_0 given by eq. (17), to give the best fit between the two.

Figs. 3 and 4 illustrate some of the differences between the approximate and exact formulae, fig. 3 shows some comparative plots for values of c equal

to 0.5, 1.0, 2.0 and 8.0. $\langle I \rangle$ is chosen equal to 4. Fig. 4 shows a graph, computed numerically, of the integrated mean square difference between the two formulae as a function of c , it reaches a maximum around $c = 2$ and goes to zero for both small and large c . When c is small, both formulae tend to a negative exponential. For c less than unity, eigenvalues were computed from an asymptotic formula given by Slepian [8].

Acknowledgements

It is a pleasure to thank Professor W.T. Welford, Dr. R.W. Smith and Dr. J.C. Dainty for discussing this work which performed under an exchange grant between the Centre National de la Recherche Scientifique (France) and the University of London.

References

- [1] S.O. Rice, Bell Syst. Tech. Jo. 24 (1945) 46.
- [2] L. Mandel, Proc. Phys. Soc. 74 (1959) 233.
- [3] J.W. Goodman, Proc. I.E.E.E. 53 (1965) 1688.
- [4] M.A. Condie, Thesis, Stanford University, Dept. of Electrical Engineering (1966).
- [5] C.L. Mehta, Progress in Optics VIII (1973) p. 390.
- [6] M.L. Mehta and C.L. Mehta, J. Opt. Soc. Am. 63 (1973) 826.
- [7] D. Slepian and E. Sonnenblick, Bell Syst. Tech. Jo. 44 (1966) 1745.
- [8] D. Slepian, Jo. Math. Phys. 44 (1965) 99.
- [9] J.C. Dainty, Optica Acta 17 (1970) 761.
- [10] R. Barakat, Optica Acta 20 (1973) 792.
- [11] J.C. Dainty, Optica Acta 18 (1971) 372.
- [12] W.B. Davenport and W.L. Root, An Introduction to the Theory of Random Signals and Noise (McGraw-Hill Book Co., Tokyo, 1958).

Reprinted from:

OPTICS COMMUNICATIONS

Volume 13, No. 1, January 1975

DEPOLARIZATION OF COHERENT LASER LIGHT BY A THICK DIFFUSING MEDIUM

A.A. SCRIBOT

Physics Department, Imperial College, London S.W.7 2BZ, UK

pp. 81–83



NORTH-HOLLAND PUBLISHING COMPANY-AMSTERDAM

DEPOLARIZATION OF COHERENT LASER LIGHT BY A THICK DIFFUSING MEDIUM

A.A. SCRIBOT

Physics Department, Imperial College, London S.W.7 2BZ, UK

Received 21 October 1974

We study experimentally the depolarization of a coherent laser beam in the far field of a thick diffusing medium made from stacks of ground glass. We show it is a function of the wavelength used, the thickness of the medium and the variance of the surface roughness.

1. Introduction

Many authors have considered intensity statistics in speckle patterns arising in coherent light [1–3] but little literature can be found on the state of polarization of the speckle field [4]. This is because the speckle pattern is usually obtained from a surface diffusing medium and this gives little depolarization. Here we look experimentally at the amount of depolarisation in a speckle pattern obtained from a thick diffusing medium. The thick medium used is particularly easy to make and to handle; it is made from one to ten layers of ground glass. Each layer is ground on one side only. The roughness is characterized by the variance of the surface profile which is estimated mechanically by a stylus method [5]. This can be associated with the variance of the size of the diffusing particles in other media. The polarisation state of the light field is characterized by a coefficient P derived from Stoke's vector [6, 7], which is measured over areas large compared with the average speckle size [8]. The depolarization effect observed is characteristic of the whole of the speckle field, each speckle being fully polarized but in a random manner.

The experimental results show that P is a function of wavelength, the number of layers of ground glass the light traverses (i.e. thickness of the medium) and the variance of the roughness.

2. Measuring technique

The initial laser light is horizontally plane polarized. The amount of depolarized light is estimated from a measure of Stoke's vector which is defined by four quantities I , M , C and S . I is the total light intensity across the scanning aperture, M is the amount of horizontal plane polarized intensity, C is the amount of $+45^\circ$ plane polarized intensity, S is the amount of right circular polarized intensity. If the light is completely polarized, we have the following relation,

$$I = \sqrt{M^2 + C^2 + S^2}. \quad (1)$$

If the light is partially depolarized then,

$$I > \sqrt{M^2 + C^2 + S^2}. \quad (2)$$

The amount of light which is not depolarized is expressed by the normalized coefficient P ,

$$P = \sqrt{M^2 + C^2 + S^2}/I. \quad (3)$$

P is of course positive and equal or less than unity. I is measured directly with a photomultiplier, M is measured with a filter blind to vertical plane polarized light, C is measured with a filter blind to -45° plane polarized light and S is measured with a filter blind to left circularly polarized light. Let I_1 , I_2 and I_3 be the intensities respectively measured with each filter, then we have the following relations

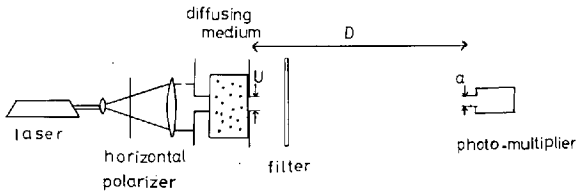


Fig. 1. Diagram of apparatus used for measuring depolarization; U = diffusing aperture diameter, a = scanning aperture diameter.

$$M = 2I_1 - I,$$

$$C = 2I_2 - I, \quad (4)$$

$$S = 2I_3 - I.$$

These measures are compensated for any losses. The apparatus used is as on fig. 1.

3. Experimental results

Fig. 2 shows a plot of P versus N , the number of layers of ground glass (i.e. thickness of the diffusing medium) with a roughness variance of $\sigma_1 = 0.430 \mu$. Fig. 3 shows the same kind of plot using a variance of $\sigma_2 = 3.791 \mu$. The difference in the steepness of the curves in figs. 2 and 3 is clearly seen. Fig. 4 shows a plot of P versus σ the roughness variance. This graph was obtained by using stacks of ten ground glass plates, the variance being constant within each stack.

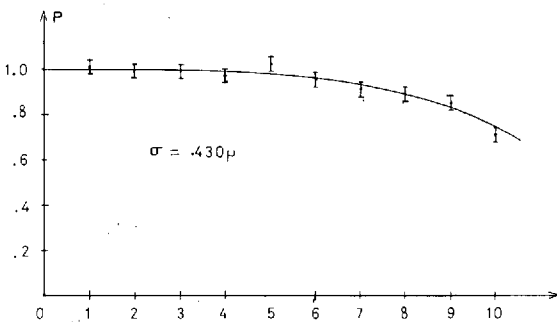


Fig. 2. Graph of polarization coefficient P versus N , number of layers of ground glass (or thickness of diffusing medium) for a roughness variance of $\sigma_1 = 0.430 \mu$ ($\lambda = 4765 \text{ \AA}$ -blue-, $D = 1.18 \text{ m}$, $U = 1.5 \text{ cm}$, $a = 0.8 \text{ cm}$).

Two wavelengths were used, $\lambda = 6328 \text{ \AA}$ -red- and $\lambda_2 = 4765 \text{ \AA}$ -blue-.

4. Conclusions

Throughout all the measurements the coefficients C and S for $+45^\circ$ plane polarization and right-hand circular polarization remained negligibly small. There is therefore, over the whole aperture, only degradation of the initial horizontal plane polarization and no acquisition of any other form of polarization. In this case it is easy to show [6, 7] the polarization coefficient P is equal to the ratio of the difference of the intensities measured with a parallel and perpendicular polarizer to the sum of the same intensities, thus

$$P = \frac{M}{I} = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}}. \quad (5)$$

We have checked this experimentally in every case considered and fig. 5 shows a comparative plot of P versus variance measured by both methods. Finally we would like to suggest depolarization measurements as a possible method for evaluating surface roughness when the surfaces are transparent and available in stacks. We hope to give a theoretical interpretation of depolarisation in thick media in terms of the mean free path between the diffusing elements in a forthcoming paper.

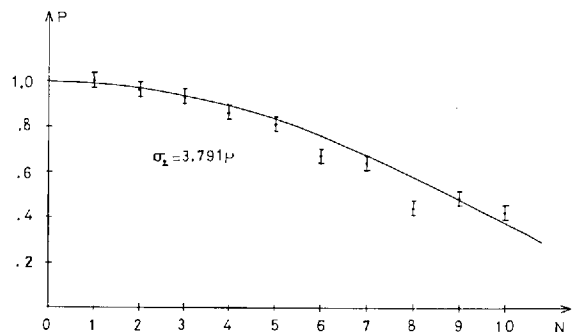


Fig. 3. Graph of polarization coefficient P versus N , number of layers of ground glass (or thickness of diffusing medium) for a roughness variance of $\sigma_2 = 3.791 \mu$ ($\lambda = 4765 \text{ \AA}$ -blue-, $D = 1.18 \text{ m}$, $U = 1.5 \text{ cm}$, $a = 0.8 \text{ cm}$).

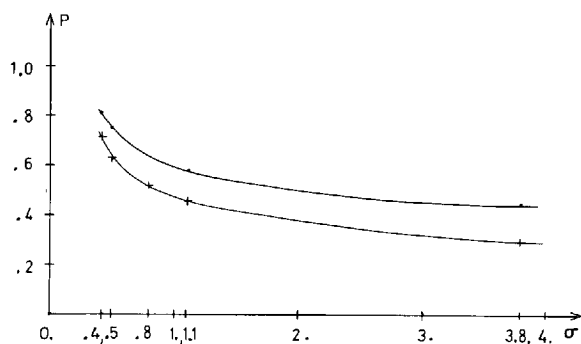


Fig. 4. Graph of polarization coefficient P versus σ , roughness variance, using stacks of ten ground glass plates. This graph is plotted for two wavelengths $\lambda_1 = 6328 \text{ \AA}$ -red-, $\bullet$, $\lambda_2 = 4765 \text{ \AA}$ -blue-, $+$, ($D = 1.18 \text{ m}$, $U = 1.5 \text{ cm}$, $a = 0.8 \text{ cm}$).

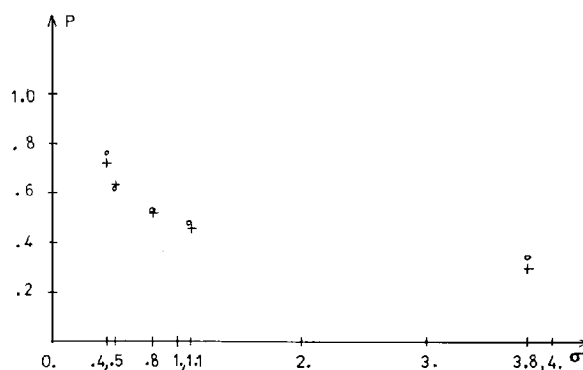


Fig. 5. Comparative plot of P versus σ measured as in eq. (3), $\circ$, as in eq. (5), $+$, ($\lambda = 4765 \text{ \AA}$ -blue-, $D = 1.18 \text{ m}$, $U = 1.5 \text{ cm}$, $a = 0.8 \text{ cm}$).

Acknowledgements

I would like to thank Professor W.T. Welford for useful discussion. This work was performed under an exchange grant between the "Centre National de la Recherche Scientifique" (C.N.R.S., Paris) and the British Council.

References

[1] J.W. Goodman, Proc. IEEE 53 (1965) 1688.

- [2] M.A. Condie, Thesis, Stanford University, Dept. of Electrical Engineering (1966).
- [3] J.C. Dainty, Opt. Acta 17 (1970) 761.
- [4] R. Bourret, Opt. Acta 21 (1974) 721.
- [5] E. Green, Proc. Instn. Mech. Engrs. 182 Pt 3K (1967-8) 330.
- [6] W.A. Shurcliff, Polarized Light (Oxford University Press, London 1962).
- [7] D. Clarke and J.F. Grainger, Polarized Light and Optical Measurement (Pergamon Press, Oxford 1971).
- [8] B. Eliasson and F.M. Mottier, J. Opt. Soc. Am. 61 (1971) 559.