

Coherent Chaos Interest Rate Models and the Wick Calculus in Finance

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by

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Declaration of originality

I, Stala Hadjipetri, declare that the work presented in this thesis is my own. When material from other authors has been used, these have been duly acknowledged.

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Abstract

This thesis develops new tools in stochastic analysis with applications to finance. The first part presents novel developments in the Wiener chaos approach to the modelling, calibration, and pricing of interest rate derivatives. To price financial instruments it suffices to specify the pricing kernel, which in Brownian models can be represented as the conditional variance of a square-integrable random variable which serves as the “generator” of the pricing kernel. The coefficients of the chaos expansion of the generator act as the parameters of a generic interest-rate model. A special class of generators, arising from “coherent” chaos expansions, is considered, and the resulting interest rate models are investigated. Coherent representations are important since a kernel generator can be expressed as a linear superposition of coherent generators. This property is exploited to derive general expressions for the pricing kernel, along with the associated discount bond and short rate processes. Pricing formulae for bond options and swaptions are obtained in closed form. The pricing kernel of a generic incoherent model is then obtained by use of the underlying coherent elements. Finite-dimensional representations of coherent chaos models are investigated, and used to construct a class of tractable models having the feature that discount bond prices are piecewise-flat processes. In the second part of the thesis, a general theory of the Wick calculus is developed. Novel results concerning the Wick orders of random variables are derived. In the case where the underlying process is a Brownian motion the Wick calculus reduces to the Ito calculus, but the former is not restricted to the Gaussian class, and is applicable to other cases, such as Lévy processes. With financial applications in mind, the Wick calculus is extended to a wider class of stochastic processes. The thesis concludes with a change of measure analysis for Wick exponentials of Lévy processes, indicating that the Wick calculus can be used as a tool for modelling the dynamics of asset prices.

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Part I

Coherent Chaos Interest Rate Models

Chapter 1

Introduction to Part I

1.1 Background

1.1.1 Overview of interest rate models

Interest rate modelling has been developed for more than four decades now, leading to a range of approaches embodying a number of different points of emphasis (see, e.g., James & Webber 2000, Cairns 2004, Hunt & Kennedy 2004, Musiela & Rutkowski 2006, Brigo & Mercurio 2007, Filipović 2009, Björk 2009, Carmona & Tehranchi 2010). This area of research is of fundamental importance because interest rate levels affect all other asset prices (including those in other asset classes). One of the first attempts to model interest rate term structures was via short rate models, in particular one-factor short rate models. In this case, the term structure of interest rates is driven by the short rate alone. Once a short rate process is specified under the pricing measure, expressions for prices of discount bonds and interest rate derivatives can be calculated. The model due to Vasicek (1977) was one of the first such short rate models to be developed, and captures the mean reverting nature of interest rates by use of an Ornstein-Uhlenbeck process with constant coefficients. The model provides closed-form expressions for the prices of discount bonds and associated derivatives, but has the shortcoming of allowing the nominal interest rates to become negative. This issue is removed in another important short-rate model, due to Cox, Ingersoll & Ross (1985), wherein interest rate positivity is ensured. Unlike the Vasicek model, the diffusion term in the CIR model depends on the square root of the short rate and this state

dependence ensures that interest rates remain positive. However, the CIR model is less tractable than the Vasicek model in terms of implementation. Furthermore, both of these models suffer from the fact that they cannot be calibrated against the initial term structure. Other works on short rate models include, among many, Brennan & Schwartz (1979), Black, Derman & Toy (1990), Black & Karasinski (1991), Longstaff & Schwartz (1992), and Hull & White (1990).

Forward rate models were thereafter introduced, where one specifies the arbitrage-free evolution of instantaneous forward rates by choosing the volatility structure of those forward rates as well as the initial term structure. Ho & Lee (1986) developed the idea initially in a discrete-time setting, while Heath, Jarrow & Morton (1992) extended the approach to continuous time, and in doing so generated a relatively general framework (the HJM framework) for the modelling of interest rate derivatives. The result was an arbitrage-free framework for the stochastic evolution of the entire yield curve. A restriction of the HJM framework is the requirement that the drift and volatility structure of the forward rates must be related in a specific way in order to ensure the existence of an equivalent martingale measure. There is, however, no apparent economic intuition for the choice of forward rate volatility. Furthermore, the condition on the forward rate volatility process required to ensure interest rate positivity, obtained in Brody & Hughston (2002), is difficult to implement.

A popular class of models that ensued are the so called “market models” which specify observable market rates, for example LIBOR or swap rates. This eliminates the need to model instantaneous rates and with properly matched volatilities, these models recover cap and swaption prices given by the Black (1976) model, which is the standard model used in the industry. However, different models are required for each instrument type and they seem to be incompatible with one another, which would make it difficult to aggregate risk. For further information on market models we refer to Miltersen *et al.* (1997), Sandmann & Sondermann (1997), Brace, Gatarek & Musiela (1997), Glasserman & Merener (2003), Jamshidian (1997), Rebonato (2002), Brigo & Mercurio (2007) and Musiela & Rutkowski (2006). Important as they are in implementations, the theory of such models is rather disconnected from the underlying economics, and as a consequence researchers have been motivated in recent years to pursue new directions in interest rate theory.

1.1.2 The pricing kernel approach

One of the more recent approaches that has attracted attention is based, in effect, on the specification of the pricing kernel from which the dynamics of interest rates are deduced as a by-product. An important advantage of the pricing kernel method is that it allows the treatment and pricing of a wide range of financial contracts, across a variety of different asset classes, to be consistent and relatively transparent (see, for instance, Cochrane 2005). Perhaps the earliest attempt to directly model interest rates from this point of view is that of Constantinides (1992). Another early example in this spirit is that of Flesaker & Hughston (1996, 1997, 1998), who introduced a framework that incorporates interest rate positivity in a canonical way. Further significant developments of the theory of interest rates from a pricing kernel perspective appear in the work of Rutkowski (1997), Rogers (1997, 2006), Glasserman & Jin (2001) and Hunt & Kennedy (2004). In particular Hunt & Kennedy (2004) present a rather detailed analysis of the relation between the Flesaker-Hughston class of models and other interest-rate models.

Hughston & Rafailidis (2005) made the observation that, in the context of a Brownian filtration, the class of positive supermartingales appropriate for the characterisation of the pricing kernel admits a representation in terms of the conditional variance of a certain random variable. They proposed the use of the Wiener-Ito chaos expansion to model and calibrate the specification of this random variable. The so-called chaotic approach to interest-rate modelling was extended further in Brody & Hughston (2004), where the most general form for the arbitrage-free dynamics of a positive-interest term structure, again in the context of a Brownian filtration, is obtained by exploiting the calculus on function spaces. See Rafailidis (2005), Grasselli & Hurd (2005), Tsujimoto (2010) and Grasselli & Tsujimoto (2011) for further important contributions to the ‘chaotic’ approach to interest rate modelling.

From the year 2007 onward an additional element of complexity entered the interest rate modelling scene with the observation that markets were pricing different types of interest rate instruments in ways that were apparently inconsistent with the elementary application of no-arbitrage rules. The resulting attempts to understand the situation better have led to the development of so-called multiple-curve models. The debate continues among practitioners and in the literature as to the root of such phenomena, and how to

model in a multiple-curve setting. See for example Ametrano & Bianchetti (2009), Kijima, Tanaka & Wong (2009), Moreni & Morini (2010), Moreni & Pallavicini (2010), Piterbarg (2010), Crépey, Grbac & Nguyen (2011), Piterbarg (2012), Crépey, Grbac & Nguyen (2012), Moreni & Pallavicini (2013), Filipović & Trolle (2013), Brigo & Pallavicini (2013), Henrard (2014), and references cited therein, for some of the relevant work in the area. Generally speaking, the risks associated with inter-bank transactions appear to be part of the story, together with a variety of credit and liquidity related issues, along with the consideration of collateral costs. It is perhaps reasonable to suggest that some form of the usual rules of pricing continue in fact to hold, in such scenarios, but only when all of the relevant cash flows have been identified, and appropriate allowances have been made for liquidity risk. The fact is that, as matters stand, there is as of yet still no comprehensive theory of arbitrage-free pricing in the absence of perfectly liquid markets, so it remains to be seen how these developments will work out. In what follows we shall operate mainly within the context of a "pure" theory of interest rates, leaving such issues related to interbank lending, liquidity, collateral costs, and so on, to one side. Nevertheless, it may be that some form of "multiple-pricing kernel" approach could be applicable to the problem, in the context of which some of the results that we establish could be carried forward and applied in a multiple-curve setting. But that is a matter for the future.

1.2 Outline of Part I

The purpose of the first part of this thesis is to develop a new class of interest-rate models, called ‘coherent interest-rate models’, within the pricing kernel formalism. Coherent interest rate models emerge from the Wiener-Ito chaos expansion for the pricing kernel.

We extend these chaos-based models for the pricing kernel in two distinct ways: (a) by working out certain general representations for chaos models for each chaos order; and (b) by introducing finite-dimensional realisations of chaos models based on function spaces. With these objectives in mind, the first part of the thesis is organised as follows. In Sections 2.1–2.3 we review some of the background material on the theory of chaos models. More specifically, in Section 2.1 we give an overview of the pricing kernel and its role in financial modelling, based on a version of the axiomatic framework of Hughston & Rafailidis (2005)

and we go on to summarise the argument leading to the conditional variance representation of Hughston & Rafailidis (2005). Section 2.2 introduces a version of the Wiener-Ito chaos expansion that we use to model a square integrable random variable. In Section 2.3 we explain the definition of the coherent chaos representation introduced in Brody & Hughston (2004), and its role in interest-rate modelling.

In Section 2.4 we introduce the notion of an n^{th} -order coherent chaos model, and derive the general representation for the pricing kernel, the short rate, the bond price, and the risk premium in this model. Explicit examples of derivative pricing formulae are then obtained in Sections 3.1-3.3 for coherent interest rate models. The examples covered are bond option prices and swaption prices, which are obtained in closed form. Coherent chaos models are important because they act as building blocks for general interest rate models. That is, a generic interest rate model can be expressed in the form of a linear superposition of the underlying coherent components (in the sense explained in Section 2.3), and thus are incoherent. By exploiting the linearity structure, we are able to derive a generic expression in Section 4 for the pricing kernel for incoherent chaos models.

In Section 5 we investigate finite-dimensional realisations of coherent chaos models. A chaos coefficient is given by an element of an infinite-dimensional Hilbert space of square-integrable functions. When this coefficient is replaced by the square-root of a *finite* number of Dirac delta functions, the resulting system can in effect be treated in a finite-dimensional Hilbert space. The corresponding interest rate models are ‘simple’ in the sense that the pricing kernel and the bond price processes are given by piecewise step functions. The idea of a model based on finite-dimensional realisations is that it can be used to fit a finite number of initial bond prices, without relying on interpolation methods. We conclude the first part of the thesis in Section 6 with a brief discussion and further comments.

Chapter 2

The Wiener-Ito chaos expansion in interest rate modelling

2.1 The pricing kernel approach to interest rate modelling

The pricing kernel approach to valuation and risk management provides perhaps the most direct route towards deriving various familiar results in financial modelling. In addition, it provides an insight into the relation between the risk, risk aversion, and return arising from risky investments when asset prices can jump—an idea that is perhaps difficult to grasp from other approaches currently available (see Brody, Hughston & Mackie 2012b, Brody & Hughston 2013). In the pricing kernel approach, one is dealing in the physical measure as opposed to the risk neutral measure. Therefore, it is more suitable as a basis for a unified framework for pricing and hedging but also for asset risk management and asset allocation related issues. There has been growing interest in re-examining the importance of the physical measure, which in effect is equivalent to the pricing kernel method (see, e.g., Carr & Yu 2012; Hull, Sokol & White 2014). Here we shall adapt the pricing kernel approach to interest rate modelling. We shall begin in this section with a brief review of the idea of a pricing kernel. In particular, we find it convenient to follow a version of the axiomatic scheme introduced in Hughston & Rafailidis (2005).

The axioms listed below are neither minimum nor unique (see also Brody & Hughston 2004, Rogers 2006, Hughston & Macrina 2008, 2010, Hughston & Mina 2012 for

further developments of this axiomatic scheme); however, we find that the approach taken in Hughston & Rafailidis (2005) leads to the desired representation of the pricing kernel needed for our purposes in the most expedient manner, and thus we shall take this as our starting point.

2.1.1 The axiomatic formalism

We begin by modelling the economy by the specification of a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the ‘physical’ probability measure. We equip this space with the standard augmented filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ generated by a system of one or more independent Brownian motions $\{W_t^\alpha\}_{0 \leq t < \infty}$, $\alpha = 1, \dots, k$. We assume that asset prices are continuous semimartingales on $(\Omega, \mathcal{F}, \mathbb{P})$, which will enable us to use various standard results of stochastic calculus. The expectation under $\mathbb{P}$ will be denoted by $\mathbb{E}[-]$ and the conditional expectation with respect to $\mathcal{F}_t$ by $\mathbb{E}_t[-]$. Other notations will be introduced as needed. In what follows, all processes are $\mathcal{F}_t$ -adapted unless otherwise stated. With this setup in mind, Hughston & Rafailidis (2005) characterise the absence of arbitrage in the economy with no transaction costs by assuming the existence of a pricing kernel, a strictly positive Ito process $\{\pi_t\}_{t \geq 0}$ (also known as a state-price deflator, stochastic discount factor, or state-price density), such that the following set of axioms hold:

- (a) There exists a money-market account asset with a strictly-increasing and absolutely continuous price process $\{B_t\}_{t \geq 0}$.
- (b) Given any asset with price process $\{S_t\}_{t \geq 0}$ and $\{\mathcal{F}_t\}$ -adapted dividend rate process $\{D_t\}_{t \geq 0}$, the process $\{M_t\}_{t \geq 0}$ defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s D_s ds \quad (2.1)$$

is a $\mathbb{P}$ -martingale.

- (c) There exists an asset—a floating rate note—that offers a continuous dividend rate such that its value remains constant over time.
- (d) There exists a system of discount bonds $\{P_{tT}\}_{0 \leq t \leq T \leq \infty}$ such that for each value of t

it holds that the following asymptotic property is satisfied:

$$\lim_{T \rightarrow \infty} P_{tT} = 0. \quad (2.2)$$

We remark that the assumption of the existence of a pricing kernel is sufficient to ensure the absence of arbitrage, without the requirement of market completeness (see Duffie 1996). Hence in what follows market completeness will not be assumed. Let us now examine some of the consequences that follow on from the axioms stated above. From axiom (a), we deduce the existence of an $\{\mathcal{F}_t\}$ -adapted short rate process $\{r_t\}_{t \geq 0}$ such that $r_t > 0$ for all $t \geq 0$ and that

$$dB_t = r_t B_t dt. \quad (2.3)$$

Since the money market account is an asset that pays no dividend, it follows from axiom (b) that the process $\{\zeta_t\}_{t \geq 0}$ defined by

$$\zeta_t = \pi_t B_t \quad (2.4)$$

must be a martingale. We note that at most one process B_t satisfying (a) and (b) can exist. Due to the fact that $\pi_t > 0$ and $B_t > 0$, the martingale ζ_t is strictly positive for all $t \geq 0$. Since $\{\zeta_t\}$ is a positive martingale, on account of the martingale representation theorem and the fact that $\mathcal{F}_t$ is the Brownian filtration, we can write

$$d\zeta_t = -\zeta_t \lambda_t dW_t \quad (2.5)$$

for some $\{\mathcal{F}_t\}$ -adapted vector-valued process $\{\lambda_t\}_{t \geq 0}$. Here, for simplicity, we write $\lambda_t dW_t$ to denote the vector inner product

$$\lambda_t dW_t = \sum_{\alpha=1}^k \lambda_t^\alpha dW_t^\alpha. \quad (2.6)$$

As a consequence of (2.3), (2.4), and (2.5), a short calculation making use of Ito's quotient rule shows that the dynamical equation satisfied by the pricing kernel takes the form

$$d\pi_t = -r_t\pi_t dt - \lambda_t\pi_t dW_t. \quad (2.7)$$

Equivalently, we can write

$$\pi_t = \exp \left(- \int_0^t r_s ds - \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right). \quad (2.8)$$

We observe from the representation in (2.8) that the pricing kernel is a product of a martingale and a strictly decreasing process. It therefore follows that $\{\pi_t\}$ is a supermartingale.

Consider now a system of default-free discount (zero coupon) bonds. We assume, in accordance with axiom (d), that the economy supports the existence of such a system of bonds over all time horizons. We write P_{tT} for the value at time t of a T -maturity bond that pays one unit of currency at maturity T .

We deduce from axiom (b) that for each maturity T the process $\{\pi_t P_{tT}\}_{0 \leq t \leq T}$ is a martingale, which we denote by $\{L_{tT}\}_{0 \leq t \leq T}$. It follows from equation (2.4) that

$$P_{tT} = \frac{L_{tT} B_t}{\rho_t}. \quad (2.9)$$

Since $\{L_{tT}\}$ is a parametric family of martingales, there exists a family of vector-valued processes $\{\sigma_{tT}\}_{0 \leq t \leq T}$, which are differentiable in T , such that

$$dL_{tT} = L_{tT}(\sigma_{tT} - \lambda_t) dW_t. \quad (2.10)$$

By use of Ito's product and quotient rules, it follows on from equations (2.3), (2.5), (2.9) and (2.10) that the dynamical equation satisfied by the discount-bond system is given by

$$dP_{tT} = P_{tT}(r_t + \lambda_t \sigma_{tT}) dt + P_{tT} \sigma_{tT} dW_t, \quad (2.11)$$

which implies that P_{tT} can be written in the form

$$P_{tT} = P_{0T} B_t \exp \left(\int_0^t \sigma_{sT} (dW_s + \lambda_s ds) - \frac{1}{2} \int_0^t \sigma_{sT}^2 ds \right). \quad (2.12)$$

We recognise the process $\{\sigma_{tT}\}_{0 \leq t \leq T}$ for fixed T as the T -maturity discount bond volatility, while the vector process $\{\lambda_t\}_{t \geq 0}$ is the associated market price of risk. On account of the condition that $\lim_{t \rightarrow T} P_{tT} = 1$, we find that

$$B_t = \frac{1}{P_{0t}} \exp \left(- \int_0^t \sigma_{st} (dW_s + \lambda_s ds) + \frac{1}{2} \int_0^t |\sigma_{st}|^2 ds \right) \quad (2.13)$$

which shows that the short rate process can be expressed in terms of the market price of risk, the initial term structure, and the discount bond volatility. By taking logarithms in (2.13), differentiating with respect to t , and using the relation $\sigma_{tt} = 0$, we obtain

$$r_t = -\partial_t \ln P_{0t} + \int_0^t \sigma_{st} \partial_t \sigma_{st} ds - \int_0^t \partial_t \sigma_{st} (dW_s + \lambda_s ds). \quad (2.14)$$

We now turn to the pricing kernel method, which provides the basis for an effective technique for the valuation of contingent claims. Thus, for example, if V_T is the payout of a derivative at time T , then on account of axiom (b) we find that the value of the derivative at time t is given by

$$V_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T V_T]. \quad (2.15)$$

In particular, for a unit cash flow $V_T = 1$ we obtain the bond pricing formula

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T]. \quad (2.16)$$

Remark 2.1.1 The conditions

$$0 < P_{tT} \leq 1 \quad \text{and} \quad \frac{\partial}{\partial T} P_{tT} < 0 \quad (2.17)$$

for $0 \leq t < T < \infty$ imply that bond prices are decreasing in T and from the definition

$$r_t = - \lim_{T \rightarrow t} \left(\frac{\partial}{\partial T} P_{tT} \right), \quad (2.18)$$

the short rate is guaranteed to remain positive at all times. The first inequality in (2.17) can be seen from the fact that $\{\pi_t\}$ is a supermartingale and is positive. The second inequality follows from

$$\lim_{\varepsilon \rightarrow 0} \frac{P_{t,T+\varepsilon} - P_{t,T}}{\varepsilon} = - \frac{1}{\pi_t} \mathbb{E}_t[r_T \pi_T]. \quad (2.19)$$

Since $\{\pi_t\}$ is a supermartingale, it follows that $\pi_T \geq \mathbb{E}_T[\pi_U]$ for $T \leq U$, and hence that $\mathbb{E}_t[\pi_T] \geq \mathbb{E}_t[\pi_U]$ by the tower property of conditional expectation. Thus $\mathbb{E}_t[\pi_T]$ is monotonically decreasing as a function of T , and we deduce that the limit $\lim_{T \rightarrow \infty} \mathbb{E}_t[\pi_T]$ exists. It follows then by use of (2.2) that

$$\lim_{T \rightarrow \infty} \mathbb{E}_t[\pi_T] = 0, \quad (2.20)$$

for all $t \geq 0$, and hence $\lim_{T \rightarrow \infty} \mathbb{E}[\pi_T] = 0$. This shows, under the axioms assumed, that the pricing kernel is a potential, i.e. it is a right-continuous positive supermartingale whose expectation vanishes as $T \rightarrow \infty$. See, for example, Karatzas & Shreve (1998), Rogers & Williams (2000) and Protter (2004) for more details on potentials.

The specification of the pricing kernel therefore leads on the one hand to the bond price dynamics as well as the associated short rate process, while on the other hand to the pricing of general contingent claims. It is for these reasons that we prefer to model the pricing kernel directly.

2.1.2 The conditional variance representation

In this section we follow the observation made in Hughston & Rafailidis (2005) that the pricing kernel can be expressed in the form of the conditional variance of an $\mathcal{F}_\infty$ -measurable square-integrable random variable. The idea is as follows. We begin by writing (2.7) in the

integral form

$$\pi_T - \pi_t = - \int_t^T r_s \pi_s ds - \int_t^T \lambda_s \pi_s dW_s, \quad (2.21)$$

and taking the $\mathcal{F}_t$ -conditional expectation on each side of (2.21) to obtain

$$\pi_t = \mathbb{E}_t[\pi_T] + \mathbb{E}_t \left[\int_t^T r_s \pi_s ds \right]. \quad (2.22)$$

Now taking the limit $T \rightarrow \infty$ we deduce the following implicit relation satisfied by the pricing kernel:

$$\pi_t = \mathbb{E}_t \left[\int_t^\infty r_s \pi_s ds \right]. \quad (2.23)$$

Note that since $\{\pi_t\}$ and $\{r_t\}$ are both $\{\mathcal{F}_t\}$ -adapted, the vector valued process $\{\eta_t\}_{t \geq 0}$ defined by the relation

$$|\eta_t|^2 = \sum_{\alpha=1}^k \eta_t^\alpha \eta_t^\alpha = r_t \pi_t \quad (2.24)$$

is also $\{\mathcal{F}_t\}$ -adapted where the vector η_t , for each $t \geq 0$, is unique only up to $\text{SO}(k)$ -rotational degrees of freedom. For any given representative element $\{\eta_t\}$ of this equivalence class, we define an $\{\mathcal{F}_t\}$ -martingale $\{X_t\}$ by setting

$$X_t = \int_0^t \eta_s dW_s. \quad (2.25)$$

Then we have

$$X_\infty - X_t = \int_t^\infty \eta_s dW_s, \quad (2.26)$$

from which, on account of the conditional Wiener-Ito isometry, we deduce that

$$\begin{aligned}
 \pi_t &= \mathbb{E}_t [(X_\infty - X_t)^2] \\
 &= \mathbb{E}_t \left[\left(\int_t^\infty \eta_s dW_s \right)^2 \right] \\
 &= \mathbb{E}_t \left[\int_t^\infty \eta_s^2 ds \right] \\
 &= \mathbb{E}_t \left[\int_t^\infty r_s \pi_s ds \right], \tag{2.27}
 \end{aligned}$$

and hence that

$$\pi_t = \mathbb{E}_t [(X_\infty - \mathbb{E}_t[X_\infty])^2]. \tag{2.28}$$

We shall refer to this identity as the conditional-variance representation for the pricing kernel. Also, we shall refer to the random variable, X_∞ as the ‘generator’ of the pricing kernel.

2.1.3 The term structure density representation for the pricing kernel

A term structure process that satisfies the positive interest conditions (2.17) and equation (2.2) is called an *admissible term structure* in accordance with the terminology of Brody & Hughston (2001). Thus for every admissible term structure, the corresponding pricing kernel is a potential. Here we shall give a brief overview on the term-structure density representation.

In what follows, it will be useful to introduce the so-called Musiela parameterisation and write $x = T - t$ for the tenor variable, and $P_{t,t+x}$ for the discount bond. With this notation, we introduce the *term structure density process* $\{\rho_t(x)\}_{x \geq 0}$, defined by

$$\rho_t(x) = -\frac{\partial P_{t,t+x}}{\partial x}. \tag{2.29}$$

From equation (2.2) and the positive interest rate conditions (2.17), we see that this process has the characteristics of a probability density function (see Brody & Hughston 2001),

namely,

$$\rho_t(x) > 0 \quad \text{and} \quad \int_0^\infty \rho_t(x) dx = 1 \quad (2.30)$$

hold, for all $x \geq 0$, $t \leq T$. Setting $t = 0$ in (2.29), we obtain

$$\rho(x) = \rho_0(x) = -\frac{\partial P_{0x}}{\partial x} \quad (2.31)$$

which we call the *initial term structure density function*.

It turns out, by use of the Flesaker-Hughston representation, that the pricing kernel approach is also closely related to the term structure density approach of Brody & Hughston (2001). Flesaker & Hughston (1996, 1998) showed that an arbitrage free system of discount bond prices can always be expressed in the form

$$P_{tT} = \frac{\int_t^\infty \rho(s) M_t(s) ds}{\int_t^\infty \rho(s) M_t(s) ds}, \quad \forall 0 \leq t \leq T < \infty, \quad (2.32)$$

where $\{M_t(s)\}_{0 \leq t \leq s < \infty}$ is a family of strictly positive martingales indexed by s , and $\rho(s)$ is the positive deterministic function defined in (2.31). This family of martingales plays an important role in such a setup because it is the only source of randomness for the modelling of interest rates. Additionally, for every admissible term structure model, the pricing kernel admits the following representation in terms of an integral of the term-structure density:

$$\pi_t = \int_t^\infty U_t(s) ds \quad \text{where} \quad U_t(s) = \rho(s) M_t(s), \quad \forall 0 \leq t < \infty. \quad (2.33)$$

This is known as the Flesaker-Hughston representation for the pricing kernel and it shows that the pricing kernel can be represented as a Riemann integral of a one parameter family of weighted martingales $\{U_t(s)\}_{0 \leq t \leq s < \infty}$.

We can express the pricing kernel in terms of the term structure density by substituting expression (2.32) into definition (2.29) and rearranging the terms, to give

$$\pi_t = \frac{\rho_0(t+x) M_t(t+x)}{\rho_t(x)}. \quad (2.34)$$

By comparing expressions (2.23) and (2.33), we connect the Flesaker-Hughston approach with the conditional variance representation and conclude that $U_t(s) = \mathbb{E}_t[r_s \pi_s]$.

2.2 The Wiener-Ito chaos expansion

The foregoing analysis shows that to model the pricing kernel, it suffices to model the generating random variable X_∞ , that is to say, an element of the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ of square integrable random variables on $(\Omega, \mathcal{F}_\infty, \mathbb{P})$ where $\mathcal{F}_\infty$ is the σ -algebra generated by $\{W_t\}$ over the interval $[0, \infty)$. The proposal of Hughston & Rafailidis (2005) is to employ the Wiener-Ito chaos expansion to ‘parameterise’ the generator X_∞ , work out $\{\pi_t\}$, and use the result to obtain pricing formulae for various derivatives; which in turn allows one to calibrate the functional parameters in the chaos expansion (see also Björk 2007).

As a preliminary step to introducing the basic idea of the chaos expansion, we consider the subspace $S_n \subset \mathbb{R}_+^n$ defined by the triangular region

$$S_n = \{(s_1, s_2, \dots, s_n) \in \mathbb{R}_+^n \mid 0 \leq s_n \leq \dots \leq s_2 \leq s_1 \leq \infty\}, \quad (2.35)$$

and denote the space of deterministic square-integrable functions on S_n by $\mathcal{L}^2(S_n)$. Now, suppose $\phi_n \in \mathcal{L}^2(S_n)$ so that for $n \geq 1$,

$$\int_0^T \int_0^{s_1} \dots \int_0^{s_{n-1}} \phi_n^2(s_1, s_2, \dots, s_n) ds_n \dots ds_2 ds_1 < \infty. \quad (2.36)$$

Then we define the n -fold iterated Ito integral according to

$$I_n(\Phi_n)(T) = \int_0^T \int_0^{s_1} \dots \int_0^{s_{n-1}} \phi_n(s_1, s_2, \dots, s_n) dW_{s_n} \dots dW_{s_2} dW_{s_1}, \quad (2.37)$$

where $I_0(\phi_0)$ is constant. Since $I_n(\Phi_n)$ is an Ito integral on S_n , it is also a martingale. With this notation at hand, we state a form of the Wiener-Ito chaos expansion.

Proposition 2.2.1. *The Wiener-Ito chaos expansion. Let ξ be an $\mathcal{F}_T$ -measurable random variable in $\mathcal{L}^2(\mathbb{P})$. Then there exists a unique sequence $\{f_n\}_{n=0}^\infty$ of functions $f_n \in \mathcal{L}^2(S_n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \quad (2.38)$$

where the convergence is in $\mathcal{L}^2(\mathbb{P})$. Moreover, we have the isometry

$$\|\xi\|_{\mathcal{L}^2(\mathbb{P})}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{L}^2(S_n)}^2. \quad (2.39)$$

We give a sketch proof of the Wiener-Ito chaos expansion for an $\mathcal{F}_T$ -measurable random variable X_T such that $\mathbb{E}[X_T^2] < \infty$. Then by the Ito representation theorem, there exists a $\{\mathcal{F}_t\}$ -adapted process $\theta_1(s_1)$, $0 \leq s_1 \leq T$, satisfying

$$\mathbb{E} \left[\int_0^T \theta_1^2(s_1) ds_1 \right] \leq \mathbb{E} [X_T^2] < \infty, \quad (2.40)$$

such that X_T can be expressed in the form

$$X_T = \mathbb{E}[X_T] + \int_0^T \theta_1(s_1) dW_{s_1}. \quad (2.41)$$

Applying the representation theorem once again to the process $\theta_1(s_1)$, implies the existence of a square-integrable process $\theta_2(s_1, s_2)$, $0 \leq s_2 \leq s_1$, satisfying

$$\mathbb{E} \left[\int_0^{s_1} \theta_2^2(s_1, s_2) ds_2 \right] \leq \mathbb{E} [\theta_1^2(s_1)] < \infty, \quad (2.42)$$

such that

$$\theta_1(s_1) = \mathbb{E}[\theta_1(s_1)] + \int_0^{s_1} \theta_2(s_1, s_2) dW_{s_2}. \quad (2.43)$$

Another application of the representation theorem then expresses $\theta_2(s_1, s_2)$ in terms of the process $\theta_3(s_1, s_2, s_3)$, and so on. By iteration we thus find that, if we write

$$\phi(s_1, s_2, \dots, s_n) = \mathbb{E} [\theta_n(s_1, s_2, \dots, s_n)], \quad (2.44)$$

then the random variable X_T can be expanded as a series in the form

$$X_T = \phi + \int_0^T \phi(s_1) dW_{s_1} + \int_0^T \int_0^{s_1} \phi(s_1, s_2) dW_{s_2} dW_{s_1} + \dots, \quad (2.45)$$

where $\phi = \mathbb{E}[X_T]$. For a full proof we refer to Di Nunno, Øksendal, Proske (2009).

If we fix T at infinity, then by the Proposition 2.2.1, we can express any $\mathcal{F}_\infty$ -measurable random variable $X_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ in the form

$$X_\infty = \sum_{n=0}^{\infty} I_n(\phi_n), \quad (2.46)$$

where the deterministic functions $\phi_n \in \mathcal{L}^2(S_n)$ are uniquely determined by X_∞ . The generator X_∞ , therefore admits a unique expansion of the form

$$X_\infty = \phi_0 + \int_0^\infty \phi_1(s_1) dW_{s_1} + \int_0^\infty \int_0^{s_1} \phi_2(s_1, s_2) dW_{s_2} dW_{s_1} + \dots, \quad (2.47)$$

where $\phi_0 = \mathbb{E}[X_\infty]$, $\phi_1(s) \in \mathcal{L}^2(S_1)$ is a square-integrable function of one variable, $\phi_2(s, s') \in \mathcal{L}^2(S_1) \otimes \mathcal{L}^2(S_1)$ is a square-integrable function of two variables, and so on.

Each chaos coefficient $\phi_n(s_1, \dots, s_n)$ belongs to the space of a symmetric tensor product of n copies of a Hilbert space of square integrable functions restricted to the subdiagonal domain S_n as defined in (2.35), e.g. $\phi_3(s_1, s_2, s_3) \in \mathcal{L}^2(S_3)$. The direct sum of these linear inner product spaces, denoted here by $\mathfrak{F}(\mathcal{L}^2(S_1))$, where

$$\begin{aligned} \mathfrak{F}(\mathcal{L}^2(S_1)) = & S_1 \oplus \mathcal{L}^2(S_1) \oplus (\mathcal{L}^2(S_1) \otimes \mathcal{L}^2(S_1)) \\ & \oplus (\mathcal{L}^2(S_1) \otimes \mathcal{L}^2(S_1) \otimes \mathcal{L}^2(S_1)) \oplus \dots, \end{aligned} \quad (2.48)$$

is spanned by the chaos coefficients $(\phi_0, \phi_1(s_1), \phi_2(s_1, s_2), \dots)$. The space $\mathfrak{F}(\mathcal{L}^2(S_1))$ is known as the *Fock space* over $\mathcal{L}^2(S_1)$.

For interest-rate modelling, we see that by substitution of (2.47) into (2.28), the pricing kernel is very naturally parameterised by a vector Φ of a set of deterministic quantities:

$$\Phi = (\phi_0, \phi_1(s), \phi_2(s, s_1), \phi_3(s, s_1, s_2), \dots), \quad (2.49)$$

where Φ can be regarded as an element of a Fock space $\mathfrak{F}$ of the direct sum of the Hilbert spaces of square-integrable functions. The components $\{\phi_i\}_{i=0,1,\dots}$ of Φ are called the *Wiener-Ito chaos coefficients*. These coefficients fully characterise the information in X_∞ . Since X_∞ determines the pricing kernel $\{\pi_t\}$, which in turn generates the bond price pro-

cess $\{P_{tT}\}$, each interest rate model can be viewed as depending on the specification of its Wiener-Ito chaos coefficients.

The idea of a chaos expansion was introduced by Wiener in his seminal paper entitled “Homogeneous Chaos” (Wiener 1938); the specific representation (2.47) in terms of the stochastic integral appears in a remarkable paper by Ito (1951). We refer to Nualart (2006), Janson (1997), Malliavin (1997), Øksendal (1997), and Di Nunno *et al.* (2009) for further discussion on the chaos expansion technique and its role in stochastic analysis.

Hughston & Rafailidis (2005) classify models whose chaos coefficients are nonzero up to order n as n^{th} order chaos models and examine first and second order chaos models in detail. They also look at a factorisable second chaos model where $\phi_2(s_1, s_2)$ factors into a product of two single-variable functions $\alpha(s_1)\beta(s_2)$.

2.3 The coherent chaos expansion

There is a special class of vectors in $\mathfrak{F}$ called ‘coherent vectors’. Such vectors admit a number of desirable characteristics. Let us consider the symmetric tensor product

$$\mathcal{H}^{(n)} = \mathcal{L}^2(S_1) \otimes \mathcal{L}^2(S_1) \otimes \cdots \otimes \mathcal{L}^2(S_1) \quad (2.50)$$

of n copies of Hilbert spaces. A generic element of $\mathcal{H}^{(n)}$ may be written in the form $\phi_n(s_1, s_2, \dots, s_n)$; whereas a coherent vector of $\mathfrak{F}$ is generated by a map from an element of $\mathcal{H}^{(1)}$ to an element of $\mathcal{H}^{(n)}$. Specifically, given $\phi(s) \in \mathcal{H}^{(1)} = \mathcal{L}^2(S_1)$, we consider an element of $\mathcal{H}^{(n)}$ of the ‘degenerate’ form

$$\phi_n(s_1, s_2, \dots, s_n) = \phi(s_1)\phi(s_2) \dots \phi(s_n). \quad (2.51)$$

The importance of coherent vectors is that the totality of such vectors in $\mathcal{H}^{(n)}$ constitutes a resolution of the identity, i.e. these vectors are in general not orthogonal but nevertheless are complete. Therefore, an arbitrary element of $\mathcal{H}^{(n)}$ can be expressed in the form of a linear combination of (possibly uncountably many) coherent vectors.

More generally, given an element $\phi(s) \in \mathcal{H}^{(1)}$ we can generate a coherent Fock vector

of the form

$$\Phi_\phi = (1, \phi(s), \phi(s_1)\phi(s_2), \dots, \phi(s_1)\phi(s_2) \dots \phi(s_n)). \quad (2.52)$$

Then an arbitrary element of $\mathfrak{F}$ can likewise be expressed as a linear combination of coherent vectors. The significance of the completeness of the system of coherent vectors for the chaos expansion, noted in Brody & Hughston (2004), is as follows. First we observe that if Φ_ϕ is coherent, the associated generating random variable X_∞^ϕ arising from the chaos expansion (2.47) takes the form

$$X_\infty^\phi = \sum_{n=0}^{\infty} \left(\int_0^\infty \int_0^{s_1} \dots \int_0^{s_{n-1}} \phi(s_1)\phi(s_2) \dots \phi(s_n) dW_{s_n} \dots dW_{s_2} dW_{s_1} \right) \quad (2.53)$$

where $\phi(s_i) \in \mathcal{L}^2(\mathbb{R}_+)$, $i = 1, \dots, n$, is a deterministic square integrable function of a single variable s_i . Here the $n = 0$ term is assumed to be unity. We now make use of the following well known identity introduced by Ito (1951), which can also be found in Di Nunno *et al.* (2009):

$$\int_0^T \int_0^{s_1} \dots \int_0^{s_{n-1}} \phi(s_1)\phi(s_2) \dots \phi(s_n) dW_{s_n} \dots dW_{s_1} = \frac{Q_T^{n/2}}{n!} H_n \left(\frac{R_T}{Q_T^{1/2}} \right), \quad (2.54)$$

where

$$R_t = \int_0^t \phi(u) dW_u \quad \text{and} \quad Q_t = \int_0^t \phi^2(u) du, \quad (2.55)$$

and where

$$H_n(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k x^{n-2k}}{k! (n-2k)! 2^k} \quad (2.56)$$

denotes the n^{th} Hermite polynomial, determined by the generating function

$$\exp \left(tx - \frac{1}{2} t^2 \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \quad (2.57)$$

where $\lfloor n/2 \rfloor$ is the floor function defined by

$$\lfloor n/2 \rfloor = \begin{cases} n/2, & n \text{ even} \\ n/2 - 1/2, & n \text{ odd.} \end{cases} \quad (2.58)$$

The role of the Hermite polynomials in relation to Gaussian random variables is well known (see, e.g., Schoutens 2000). We remark that if Y and Z are standard normal random variables, then

$$\mathbb{E}[H_n(Y)H_m(Z)] = (\mathbb{E}[YZ])^n \delta_{mn}, \quad (2.59)$$

which follows from the fact that Hermite polynomials are orthogonal with respect to the standard normal density function. By substituting the iterated integral in equation (2.54) into (2.53) we get

$$X_\infty^\phi = \sum_{n=0}^{\infty} \frac{Q_\infty^{n/2}}{n!} H_n \left(\frac{R_\infty}{Q_\infty^{1/2}} \right), \quad (2.60)$$

and comparing this with (2.57) we deduce that

$$X_\infty^\phi = \exp \left(\int_0^\infty \phi(s) dW_s - \frac{1}{2} \int_0^\infty \phi^2(s) ds \right). \quad (2.61)$$

On account of the completeness of the coherent vectors and linearity, it follows that an arbitrary $\mathcal{F}_\infty$ -measurable square-integrable random variable X_∞ admits a representation of the form

$$X_\infty = \sum_j c_j \exp \left(\int_0^\infty \phi_j(s) dW_s - \frac{1}{2} \int_0^\infty \phi_j^2(s) ds \right), \quad (2.62)$$

where $\{c_j\}$ are constants satisfying $\sum_j c_j^2 < \infty$, where $\phi_j(s) \in \mathcal{L}^2(\mathbb{R}_+)$ for each j is a deterministic square-integrable function, and where the summation in (2.62) is formal and may be replaced by an appropriate integration in the uncountable case. For example, both the expansion coefficient $c(\theta)$ and the square-integrable function $\phi(s, \theta)$ may depend on a parameter θ , in which case the summation over the index j in (2.62) is replaced by an

integration over the parameter θ . It should be evident that an analogous result holds more generally for an arbitrary $\mathcal{F}_t$ -measurable square-integrable random variable—that is, any such random variable can be expressed in terms of a linear combination of log-normally distributed random variables. Putting the matter differently, log-normal random variables are dense in the space of square-integrable random variables. This fact was applied in Brody & Hughston (2004) to identify the general expressions for the pricing kernel and other quantities (such as the bond price, market price of risk, and various interest rates) in the Brownian-based setting, owing to the fact that the conditional variance of X_∞ in (2.62) is easily calculated in closed form.

2.4 Coherent chaos interest-rate models

In this section, we extend the existing investigation by examining the n^{th} -order chaos models for each n . That is to say, we are concerned with interest-rate models that arise from the chaos expansion of the form

$$\int_0^\infty \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi(s_1, s_2, \dots, s_n) dW_{s_n} \cdots dW_{s_2} dW_{s_1} \quad (2.63)$$

for each n , where $\phi(s_1, s_2, \dots, s_n) \in \mathcal{H}^{(n)}$ is a symmetric function of n variables. As indicated above, any element of $\mathcal{H}^{(n)}$ can be expressed as a linear combination of a (possibly uncountable) collection of coherent functions of the form $\phi(s_1)\phi(s_2)\cdots\phi(s_n)$. That is to say, we have

$$\phi(s_1, s_2, \dots, s_n) = \sum_j c_j \phi_j(s_1)\phi_j(s_2)\cdots\phi_j(s_n) \quad (2.64)$$

for some choice of coefficients $\{c_j\}$ and functions $\{\phi_j(s)\}$. Let us therefore introduce:

Definition 2.4.1 An n^{th} -order coherent chaos model is generated by a random variable of the form

$$X_\infty^{[n]} = \int_0^\infty \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi(s_1)\phi(s_2)\cdots\phi(s_n) dW_{s_n} \cdots dW_{s_2} dW_{s_1}, \quad n \geq 1, \quad (2.65)$$

where $X_\infty^{[n]} = X_\infty^{[n]}(\phi)$, $\phi \in \mathcal{L}_2(\mathbb{R}_+)$ is determined by the choice of ϕ but for simplicity of notation, we write $X_\infty^{[n]}$.

Our strategy is first to work out the interest rate model arising from such a coherent element of $\mathcal{H}^{(n)}$, i.e. an n^{th} -order coherent chaos model, and then to consider linear combinations of such elements in order to obtain more general n^{th} -order chaos models for the pricing kernel.

To proceed, we substitute (2.54) into expression (2.65), from which it follows that for a coherent element we have

$$X_\infty^{[n]} = \frac{Q_\infty^{n/2}}{n!} H_n \left(\frac{R_\infty}{Q_\infty^{1/2}} \right) \quad (2.66)$$

and

$$X_t^{[n]} = \frac{Q_t^{n/2}}{n!} H_n \left(\frac{R_t}{Q_t^{1/2}} \right), \quad (2.67)$$

where R and Q are defined as in (2.55) and where we have used

$$X_t^{[n]} = \mathbb{E}_t [X_\infty^{[n]}]. \quad (2.68)$$

Note that the argument $R_t/Q_t^{1/2}$ appearing in the Hermite polynomial is, for each t , a standard normally distributed random variable. Thus on account of (2.59) we find that the martingales $\{X_t^{[n]}\}_{0 \leq t \leq \infty}$ for different values of n are mutually orthogonal. This property will be exploited in a calculation below when we analyse the more general ‘incoherent’ models. Let us turn to the determination of the pricing kernel $\{\pi_t^{[n]}\}$, defined by

$$\pi_t^{[n]} = \mathbb{E}_t \left[(X_\infty^{[n]})^2 \right] - (X_t^{[n]})^2. \quad (2.69)$$

Proposition 2.4.1. *The pricing kernel $\{\pi_t^{[n]}\}$ associated with an n^{th} -order coherent chaos*

generator $X_\infty^{[n]}$ of the form (2.66) is given by

$$\pi_t^{[n]} = \sum_{k=0}^n \frac{[2(n-k)]! (1 - Q_t^k) X_t^{[2n-2k]}}{k! [(n-k)!]^2}. \quad (2.70)$$

Proof. To analyse the expressions in (2.69) for each $n \in \mathbb{N}$, we make use of the following product identity for a pair of Hermite polynomials of different order (see, for example, Janson 1997, p. 28):

$$H_n(x)H_m(x) = \sum_{k=0}^{m \wedge n} \binom{m}{k} \binom{n}{k} k! H_{m+n-2k}(x), \quad (2.71)$$

where $m \wedge n = \min(m, n)$. Setting $m = n$ we obtain

$$H_n^2(x) = \sum_{k=0}^n \binom{n}{k} \binom{n}{k} k! H_{2n-2k}(x). \quad (2.72)$$

Since, from (2.56), we have

$$X_t^{[n]} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k R_t^{n-2k} Q_t^k}{k! (n-2k)! 2^k}, \quad (2.73)$$

then by squaring $X_t^{[n]}$, it follows from (2.67) that

$$\left(X_t^{[n]}\right)^2 = \frac{Q_t^n}{(n!)^2} H_n^2\left(\frac{R_t}{\sqrt{Q_t}}\right). \quad (2.74)$$

Now substituting the expression (2.72) in (2.74), we obtain

$$\left(X_t^{[n]}\right)^2 = \frac{Q_t^n}{(n!)^2} \sum_{k=0}^n \binom{n}{k} \binom{n}{k} k! H_{2n-2k}(x), \quad (2.75)$$

and we use (2.67) again to deduce that

$$\left(X_t^{[n]}\right)^2 = \sum_{k=0}^n \frac{Q_t^k [2(n-k)]! X_t^{[2n-2k]}}{k! [(n-k)!]^2}. \quad (2.76)$$

Taking the limit $t \rightarrow \infty$ in (2.76) and substituting the result in (2.69), we obtain (2.70), after unit-normalising the function $\phi(s)$ such that $Q_\infty = 1$. $\square$

We observe from (2.70) and (2.73) that $\{\pi_t^{[n]}\}$ is given by a polynomial of order $2n - 2$ of the Gaussian process $\{R_t\}$ defined in (2.55). Having obtained an expression for the pricing kernel for each $n \in \mathbb{N}$, we are in a position to determine representations for the various quantities of interest. To this end, let us first derive expressions for the short rate and market price of risk processes.

Proposition 2.4.2. *Let $\{r_t^{[n]}\}$ be the short rate process in the coherent chaos model of order n and $\{\lambda_t^{[n]}\}$ be the associated market price of risk process. Then*

$$r_t^{[n]} = \frac{\phi^2(t)}{\pi_t^{[n]}} \sum_{k=0}^n \frac{[2(n-k)]! Q_t^{k-1} X_t^{[2n-2k]}}{(k-1)! [(n-k)!]^2} \quad (2.77)$$

and

$$\lambda_t^{[n]} = -\frac{\phi(t)}{\pi_t^{[n]}} \sum_{k=0}^n \frac{(1 - Q_t^k) [2(n-k)]!}{k! [(n-k)!]^2} X_t^{[2n-2k-1]}. \quad (2.78)$$

As a preliminary result which we need for the proof of Proposition 2.4.2, we have the following lemma.

Lemma 2.4.3. *The n^{th} -order coherent chaos generator $X_\infty^{[n]}$ satisfies the following differential equation:*

$$dX_t^{[n]} = X_t^{[n-1]} \phi(t) dW_t. \quad (2.79)$$

Proof. Let $Z_t = R_t/\sqrt{Q_t}$. Then by Ito's quotient rule, we have

$$\begin{aligned} dZ_t &= Z_t \left[\frac{dR_t}{R_t} - \frac{d\sqrt{Q_t}}{\sqrt{Q_t}} \right] \\ &= \frac{\phi_t}{\sqrt{Q_t}} dW_t - \frac{1}{2} \frac{\phi_t^2 Z_t}{Q_t} dt, \end{aligned} \quad (2.80)$$

and

$$(dZ_t)^2 = \frac{\phi_t^2}{Q_t} dt. \quad (2.81)$$

On the other hand, by Ito's lemma we can write:

$$dH_m(Z_t) = H'_m(Z_t) dZ_t + \frac{1}{2} H''_m(Z_t) (dZ_t)^2. \quad (2.82)$$

Now differentiating the well-known relationship

$$H_{m+1}(Z_t) = Z_t H_m(Z_t) - H'_m(Z_t) \quad (2.83)$$

for Hermite polynomials and rearranging, we obtain

$$H''_m(Z_t) = H_m(Z_t) + Z_t H'_m(Z_t) - H'_{m+1}(Z_t). \quad (2.84)$$

Using (2.83) to replace $H'_m(Z_t)$ and $H'_{m+1}(Z_t)$ in (2.84), we find that

$$H''_m(Z_t) = (1 + Z_t^2) H_m(Z_t) - 2Z_t H_{m+1}(Z_t) + H_{m+2}(Z_t). \quad (2.85)$$

Now using (2.83) and (2.84) in (2.82) we write (2.82) as follows:

$$\begin{aligned} dH_m(Z_t) &= \frac{\phi_t}{\sqrt{Q_t}} [Z_t H_m(Z_t) - H_{m+1}(Z_t)] dW_t \\ &\quad + \frac{1}{2} \frac{\phi_t^2}{Q_t} [H_{m+2}(Z_t) - 2Z_t H_{m+1}(Z_t)] dt. \end{aligned} \quad (2.86)$$

We use another well-known recursion relation for the Hermite polynomials

$$H_{m+2}(Z_t) = Z_t H_{m+1}(Z_t) - (m+1)H_m(Z_t), \quad (2.87)$$

which we can use to replace $H_{m+2}(Z_t)$ in (2.86). Hence we find that

$$dH_m(Z_t) = \frac{\phi_t}{\sqrt{Q_t}} [Z_t H_m(Z_t) - H_{m+1}(Z_t)] dW_t - \frac{m}{2} \frac{\phi_t^2}{Q_t} H_m(Z_t) dt, \quad (2.88)$$

and by (2.67), we obtain

$$\begin{aligned} dX_t^{[m]} &= \frac{1}{m!} \left(Q_t^{m/2} dH_m(Z_t) + H_m(Z_t) dQ_t^{m/2} \right) \\ &= \frac{\phi_t Q_t^{m/2-1/2}}{m!} [Z_t H_m(Z_t) - H_{m+1}(Z_t)] dW_t. \end{aligned} \quad (2.89)$$

From (2.87) we observe that the term above in square brackets is just $mH_{m-1}(Z_t)$, and we deduce that

$$dX_t^{[m]} = X_t^{[m-1]} \phi(t) dW_t. \quad (2.90)$$

□

Proof of Proposition 2.4.2. On account of (2.7) we observe that the processes are given by the drift and the volatility of the pricing kernel. If the pricing kernel takes the form (2.70), then an application of the Ito calculus along with the relation

$$dX_t^{[2n-2k]} = X_t^{[2n-2k-1]} \phi(t) dW_t \quad (2.91)$$

which is shown for $m = 2n - 2k$ in Lemma 2.4.3, gives us the desired expressions at once.

As for the discount bonds, it follows from (2.16) that the bond price process takes the form of a ratio of polynomials of the Gaussian process $\{R_t\}$:

Proposition 2.4.4. *Let $\{P_{tT}^{[n]}\}_{0 \leq t \leq T}$ be the discount bond process for the coherent chaos*

model of order n . Then

$$P_{tT}^{[n]} = \frac{\sum_{k=0}^n A_{k,n} (1 - Q_T^k) X_t^{[2n-2k]}}{\sum_{k=0}^n A_{k,n} (1 - Q_t^k) X_t^{[2n-2k]}}, \quad (2.92)$$

where

$$A_{k,n} = \frac{[2(n-k)]!}{k![(n-k)!]^2}. \quad (2.93)$$

The initial term structure is given by the discount function

$$P_{0T}^{[n]} = 1 - Q_T^n. \quad (2.94)$$

Proof. Substituting (2.70) into (2.16) and making use of the fact that $\{X_t^{[k]}\}$ is a martingale for every k , we deduce (2.92) at once. The expression in (2.94) can be seen by first using the representation in (2.73) to write

$$X_t^{[2n-2k]} = \sum_{j=0}^{n-k} \frac{(-1)^j R_t^{2n-2k-2j} Q_t^j}{j! (2n-2k-2j)! 2^j}, \quad (2.95)$$

we can see that at time $t = 0$ the expression above is only non-zero when $j = 0$ and $j = n - k$ i.e. when $n = k$. In fact, we find that $X_0^{[2n-2k]} = \delta_{nk}$. $\square$

By substituting (2.94) into (2.31), we obtain the expression

$$\rho^{[n]}(x) = n Q_x^{n-1} \phi^2(x) \quad (2.96)$$

for the initial term structure density, where n denotes the chaos order. Now, taking the representations (2.69) for the pricing kernel and (2.66) for $X_\infty^{[n]}$ we can write

$$\pi_t^{[n]} = \int_t^\infty \phi^2(s_1) \mathbb{E}_t \left[\left(\int_0^{s_1} \dots \int_0^{s_{n-1}} \phi(s_2) \dots \phi(s_n) dW_{s_n} \dots dW_{s_2} \right)^2 \right] ds_1. \quad (2.97)$$

We notice that the integral inside the expectation of equation (2.97) is equal to $\left(X_{s_1}^{[n-1]} \right)^2$,

and so from (2.76) we get

$$\pi_t^{[n]} = \int_t^\infty \phi^2(s_1) \mathbb{E}_t \left[\sum_{k=1}^n \frac{Q_t^{k-1} [2(n-k)]! X_{s_1}^{[2n-2k]}}{(k-1)! [(n-k)!]^2} \right] ds_1. \quad (2.98)$$

We compare (2.98) with (2.33) to obtain

$$U_t^{[n]}(x) = M_t^{[n]}(x) \rho_n(x) = \phi^2(x) \sum_{k=1}^n \frac{[2(n-k)]! Q_x^{k-1} X_t^{[2n-2k]}}{(k-1)! [(n-k)!]^2} \quad (2.99)$$

for each n , where we have used the fact that $\{X_t^{[n]}\}$ is a martingale for each n . Since

$$U_t^{[n]}(x) = M_t^{[n]}(x) \rho^{[n]}(x), \quad (2.100)$$

we use (2.96) and divide (2.99) by $\rho^{[n]}(x)$ to obtain

$$M_t^{[n]}(x) = \frac{1}{n} \sum_{k=1}^n \frac{[2(n-k)]! Q_x^{k-n} X_t^{[2n-2k]}}{(k-1)! [(n-k)!]^2}, \quad (2.101)$$

which is a martingale for all x , and for all n .

Each added chaos order increases the stochastic complexity of the model in a controlled manner and this can provide a well defined way to tailor the model to the data available in the market (see Grasselli & Tsuimoto, 2011, for the calibration of factorisable chaos models). The pricing kernel, short rate process and market price of risk process are all polynomials in a Gaussian random variable and so for higher orders of n these are just higher order polynomials. We can create more complex models using the coherent structure as building blocks as we will see in Chapter 4.

The initial term structure (2.94) shows, in particular, that in the case of a single-factor setup, coherent chaos models are fully characterised by the initial term structure. Such a strong constraint, of course, is expected in the case of a single factor model for which the only model ‘parameter’ is a deterministic scalar function $\phi(t)$, and it is indeed natural that these functional degrees of freedom should be fixed unambiguously by the initial yield curve. On the other hand, the general cases are obtained simply by taking linear superpositions of the coherent chaos models in an appropriate manner. Before we examine these

general cases, let us first work out a number of derivative-pricing formulae in the case of single-factor coherent chaos models.

Chapter 3

Bond and Swaption pricing

3.1 Bond option pricing in second-order coherent chaos models

The purpose of this section is to derive explicit formulae for the prices of options on zero-coupon bonds in a second order coherent chaos model. i.e. when $n = 2$ and the model depends on the deterministic function ϕ . In this case, it follows from equations (2.70) and (2.73) that the pricing kernel is given by a quadratic function of a single Gaussian state variable R_t :

$$\pi_t^{[2]} = (1 - Q_t)(R_t^2 - Q_t) + \frac{1}{2}(1 - Q_t^2), \quad (3.1)$$

where we recall that Q and R are given in (2.55). On account of (2.16), the bond price process can then be written as a rational function of the state variable R_t :

$$P_{tT}^{[2]} = \frac{(1 - Q_T)(R_t^2 - Q_t) + \frac{1}{2}(1 - Q_T^2)}{(1 - Q_t)(R_t^2 - Q_t) + \frac{1}{2}(1 - Q_t^2)}. \quad (3.2)$$

We shall make use of the pricing formula (2.15) and the expressions in (3.1) and (3.2) to find the pricing formula for a European-style call option on a discount bond. In particular, let t be the option maturity, let $T \geq t$ be the bond maturity and let K be the option strike. Then bearing in mind that $\pi_0^{[2]} = \frac{1}{2}$ under the convention $Q_\infty = 1$ that we have chosen

here, we find that the initial price of the option is given by

$$\begin{aligned}
 C_0^{[2]}(t, T, K) &= 2 \mathbb{E} \left[\pi_t^{[2]} \left(P_{tT}^{[2]} - K \right)^+ \right] \\
 &= 2 \mathbb{E} \left[\left(R_t^2 [(1 - Q_T) - K(1 - Q_t)] - Q_t(1 - Q_T) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2}(1 - Q_T^2) - \frac{1}{2}K(1 - Q_t)^2 \right)^+ \right] \\
 &= 2 \mathbb{E} \left[(AZ_t^2 + B)^+ \right], \tag{3.3}
 \end{aligned}$$

where $Z_t = R_t/\sqrt{Q_t}$ is a standard normal random variable, and where

$$A = Q_t [(1 - Q_T) - K(1 - Q_t)] \tag{3.4}$$

and

$$B = \frac{1}{2} [(1 - Q_T^2) - K(1 - Q_t^2)] - Q_t [(1 - Q_T) - K(1 - Q_t)]. \tag{3.5}$$

The problem of pricing the call option in this model thus reduces to finding the roots of a quadratic equation $Az^2 + B = 0$. Letting z_1, z_2 denote the roots

$$z_1 = -\sqrt{-\frac{B}{A}} \quad \text{and} \quad z_2 = +\sqrt{-\frac{B}{A}}, \tag{3.6}$$

we consider the different scenarios depending on the signatures of the polynomial coefficients A and B . We shall proceed to examine this case by case.

- (i) If $A = 0$ then $B = \frac{1}{2}(1 - Q_T)(Q_T - Q_t) > 0$ since $Q_T > Q_t$. Therefore the option is always in the money, and we have

$$C_0^{[2]} = P_{0T}^{[2]} - KP_{0t}^{[2]} = 2B. \tag{3.7}$$

- (ii) If $A > 0$ then $B > 0$. This follows from the fact that setting $A > 0$ in equation (3.4) implies that

$$K < (1 - Q_T)/(1 - Q_t). \tag{3.8}$$

Since $B = \frac{1}{2}(1 - Q_T^2) - Q_t(1 - Q_T) - \frac{1}{2}K(1 - Q_t)^2$, then the bound (3.8) on K implies that $B > \frac{1}{2}(1 - Q_T)(Q_T - Q_t)$. But since $1 > Q_T > Q_t$, we have $B > 0$. In this case, $Az^2 + B$ is always positive, so again the call option will always expire in the money. The price of such an option is thus

$$\begin{aligned} C_0^{[2]} &= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Az^2 + B) e^{-\frac{1}{2}z^2} dz \\ &= P_{0T}^{[2]} - KP_{0t}^{[2]} \\ &= 2(A + B). \end{aligned} \quad (3.9)$$

- (iii) If $A < 0$ and $B \leq 0$, then the payoff is never positive, resulting in an option whose value is zero.
- (iv) The nontrivial case is when we have $A < 0$ and $B > 0$. Then the quadratic polynomial is positive over the interval (z_1, z_2) , and we have

$$\begin{aligned} C_0^{[2]} &= \frac{2}{\sqrt{2\pi}} \int_{z_1}^{z_2} (Az^2 + B) e^{-\frac{1}{2}z^2} dz \\ &= 2(A + B) [1 - 2N(z_1)] + 4Az_1\psi(z_1), \end{aligned} \quad (3.10)$$

where $N(z)$ denotes the standard cumulative normal distribution function, and $\psi(z)$ the associated density function. It is easy to see that the option delta that gives the position on the underlying bond required to hedge the option, in cases (i) and (ii) is one and in case (iii) is zero. For case (iv) the option delta is given by

$$\Delta^{[2]} = 1 - 2N(z_1) \quad (3.11)$$

where we have used the relations

$$P_{0T}^{[2]} - KP_{0t}^{[2]} = 2(A + B) \quad (3.12)$$

and

$$z_1^2 - 1 = -\frac{A + B}{A}. \quad (3.13)$$

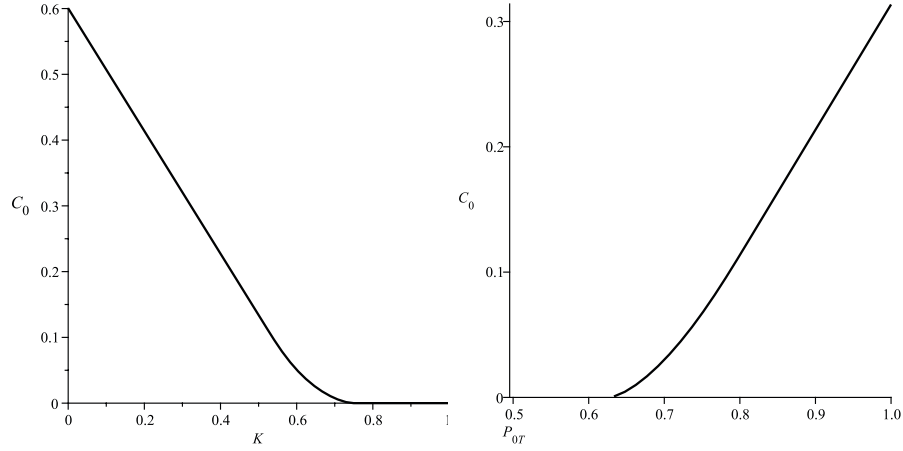


Figure 3.1: Left panel: the call price as a function of the strike for fixed bond maturity ($T = 10$) and fixed option maturity ($t = 3$) in the second-order coherent chaos model with $\phi^2(s) = (0.1)^{-1} \exp(-0.1s)$. Right panel: the call price as a function of the initial bond price, where the bond price is varied by changing its maturity T from 100 to zero. Other parameters are set as $t = 5$, $K = 0.7$, and $\phi^2(s) = (0.03)^{-1} \exp(-0.03s)$.

We see that in a coherent chaos model of the second order, the pricing of call options on discount bonds is entirely tractable and so is the determination of the associated hedge portfolio. As an illustrative example, let us consider the case in which we choose

$$\phi^2(s) = \lambda^{-1} \exp(-\lambda s) \quad (3.14)$$

for some constant $\lambda > 0$. As indicated above, in a coherent chaos model, this is all we need in order to calculate both bond prices and call option prices. In Figure 3.1, the left panel shows the relation between the call price and its strike. Here, the bond maturity is fixed at $T = 10$ and the option maturity at $t = 3$, and we set $\lambda = 0.1$. On the right-side of Figure 3.1 we show how the call price changes with respect to the bond values for different maturities. Here, the option maturity has been chosen to be $t = 5$, the bond maturities vary from 100 to zero, the strike has been fixed at 0.7, and we have set $\lambda = 0.03$.

3.2 Swaption pricing in second-order coherent chaos models

It turns out that swaptions can also be valued in closed form, in a manner analogous to the factorisable second-order chaos models of Hughston & Rafailidis (2005). In this case, the payout of the contract can be expressed in the form:

$$H_t^{[2]} = \left(1 - P_{tT_n}^{[2]} - K \sum_{i=1}^n P_{tT_i}^{[2]} \right)^+, \quad (3.15)$$

where T_i , $i = 1, \dots, n$, are specified future payment dates. The initial value of the swaption is then given by the expectation

$$\begin{aligned} H_0^{[2]} &= 2 \mathbb{E} \left[\pi_t^{[2]} \left(1 - P_{tT_n}^{[2]} - K \sum_{i=1}^n P_{tT_i}^{[2]} \right)^+ \right] \\ &= 2 \mathbb{E} \left[(AZ_t^2 + B)^+ \right], \end{aligned} \quad (3.16)$$

where $Z_t = R_t / \sqrt{Q_t}$, as before. In the present case, however, we have

$$A = Q_t [(Q_{T_n} - Q_t) - K \sum_{i=1}^n (1 - Q_{T_i})] \quad (3.17)$$

and

$$B = \frac{1}{2} (Q_{T_n} - Q_t)^2 + K Q_t \sum_{i=1}^n (1 - Q_{T_i}) - \frac{K}{2} \sum_{i=1}^n (1 - Q_{T_i}^2). \quad (3.18)$$

In other words, the valuation of a swaption proceeds in the same way as that of a call option, except that the coefficients defined as A and B are a little more complicated. Writing z_1 and z_2 for the roots of $Az^2 + B = 0$, we thus have the following:

(i) When $A < 0$ and $B \leq 0$, the swaption has value zero.

(ii) When $A > 0$ and $B \geq 0$, we have

$$H_0^{[2]} = P_{0t}^{[2]} - P_{0T_n}^{[2]} - K \sum_{i=1}^n P_{0T_i}^{[2]}. \quad (3.19)$$

(iii) When $A < 0$ and $B > 0$ so that $-B/A > 0$, we have

$$\begin{aligned} H_0^{[2]} &= 2(A + B) [N(z_2) - N(z_1)] + 2A [z_1\psi(z_1) - z_2\psi(z_2)] \\ &= \left(P_{0t}^{[2]} - P_{0T_n}^{[2]} - K \sum_{i=1}^n P_{0T_i}^{[2]} \right) [1 - 2N(z_1)] + 4Az_1\psi(z_1), \end{aligned} \quad (3.20)$$

where $\psi(z)$ is the standard normal density function as used in Section 3.1.

(iv) When $A > 0$ and $B < 0$ so that $-B/A > 0$, we have

$$H_0^{[2]} = 2N(z_1) \left(P_{0t}^{[2]} - P_{0T_n}^{[2]} - K \sum_{i=1}^n P_{0T_i}^{[2]} \right) - 4Az_1\psi(z_1). \quad (3.21)$$

3.3 Bond option pricing in third-order coherent chaos models

The pricing kernel for a third-order pure coherent chaos model can be expressed in the form

$$\pi_t^{[3]} = 6(1 - Q_t)X_t^{[4]} + (1 - Q_t^2)X_t^{[2]} + \frac{1}{6}(1 - Q_t^3). \quad (3.22)$$

The bond price process is consequently given by

$$P_{tT}^{[3]} = \frac{36(1 - Q_T)X_t^{[4]} + 6(1 - Q_T^2)X_t^{[2]} + (1 - Q_T^3)}{36(1 - Q_t)X_t^{[4]} + 6(1 - Q_t^2)X_t^{[2]} + (1 - Q_t^3)}. \quad (3.23)$$

We can therefore express the value at time zero of a European-style call option on a discount bond in this model as

$$\begin{aligned} C_0^{[3]} &= \mathbb{E} \left[\frac{\pi_t^{[3]}}{\pi_0^{[3]}} \left(P_{tT}^{[3]} - K \right)^+ \right] \\ &= 3! \mathbb{E} \left[(AZ_t^4 + BZ_t^2 + C)^+ \right], \end{aligned} \quad (3.24)$$

where

$$A = \frac{1}{4}Q_t^2 [(1 - Q_T) - K(1 - Q_t)], \quad (3.25)$$

$$B = \frac{1}{2}Q_t [(1 - Q_T^2) - K(1 - Q_t^2)] - \frac{3}{2}Q_t^2 [(1 - Q_T) - K(1 - Q_t)], \quad (3.26)$$

and

$$\begin{aligned} C = & \frac{1}{6} [(1 - Q_T^3) - K(1 - Q_t^3)] - \frac{1}{2}Q_t [(1 - Q_T^2) - K(1 - Q_t^2)] \\ & + \frac{3}{4}Q_t^2 [(1 - Q_T) - K(1 - Q_t)]. \end{aligned} \quad (3.27)$$

The roots of the quartic polynomial $Az^4 + Bz^2 + C$ are given by

$$z_1 = -\sqrt{\frac{-B - \sqrt{\delta}}{2A}}, \quad z_2 = -\sqrt{\frac{-B + \sqrt{\delta}}{2A}}, \quad z_3 = -z_2, \quad z_4 = -z_1, \quad (3.28)$$

where $\delta = B^2 - 4AC$. Thus, we have the following:

(i) If $A = 0$ then $B = \frac{1}{2}Q_t(1 - Q_T)(Q_T - Q_t)$ and

$$C = \frac{1}{6}(1 - Q_T)(Q_T - Q_t)(Q_T - Q_t + 1 - Q_t). \quad (3.29)$$

Since both B and C are positive, the option always expires in the money and we have

$$\begin{aligned} C_0^{[3]} &= \frac{6}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Bz^2 + C)e^{-\frac{1}{2}z^2} dz \\ &= (1 - Q_T)(Q_T - Q_t)(1 + Q_t + Q_T). \end{aligned} \quad (3.30)$$

(ii) If $A > 0$ and $-B - \sqrt{\delta} \geq 0$ then this implies the model at $B \leq -\sqrt{\delta}$. Since there are four roots, the call option in this case is

$$\begin{aligned} C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \left(\int_{-\infty}^{z_2} + \int_{z_1}^{z_4} + \int_{z_3}^{\infty} \right) (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\ &= 6(3A + B + C)(2N(z_2) - 2N(z_1) + 1) \\ &\quad + 12(3A + B)(z_1\psi(z_1) - z_2\psi(z_2)) \\ &\quad + 12A(z_1^3\psi(z_1) - z_2^3\psi(z_2)). \end{aligned} \quad (3.31)$$

(iii) If $A > 0$ and $-B - \sqrt{\delta} < 0$ but $-B + \sqrt{\delta} \geq 0$ then if $-\sqrt{\delta} < B \leq 0$, the initial

value of the call option is

$$\begin{aligned} C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \left(\int_{-\infty}^{z_2} + \int_{z_3}^{\infty} \right) (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\ &= 12(3A + B + C)N(z_2) - 12(Az_2^2 + 3A + B)z_2\psi(z_2), \end{aligned} \quad (3.32)$$

where $\psi(z)$ is the standard normal density function as used in Section 3.1.

If $0 < B \leq \sqrt{\delta}$, then

$$\begin{aligned} C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \int_{z_2}^{z_3} (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\ &= 6(3A + B + C)(1 - 2N(z_2)) + 12(Az_2^2 + 3A + B)z_2\psi(z_2). \end{aligned} \quad (3.33)$$

(iv) If $A > 0$ and $-B + \sqrt{\delta} < 0$ then the value of the option is zero.

(v) If $A < 0$ and $-B - \sqrt{\delta} > 0$ then

$$\begin{aligned} C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\ &= 6(3A + B + C). \end{aligned} \quad (3.34)$$

(vi) If $A < 0$ and $-B + \sqrt{\delta} > 0$ but $-B - \sqrt{\delta} \leq 0$ then if $-\sqrt{\delta} < B \leq 0$,

$$\begin{aligned} C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \left(\int_{-\infty}^{z_1} + \int_{z_4}^{\infty} \right) (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\ &= 12(3A + B + C)N(z_1) - 12(Az_1^2 + 3A + B)z_1\psi(z_1). \end{aligned} \quad (3.35)$$

If $0 < B \leq \sqrt{\delta}$,

$$\begin{aligned} C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \int_{z_1}^{z_4} (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\ &= 6(3A + B + C)(1 - 2N(z_1)) + 12(Az_1^2 + 3A + B)z_1\psi(z_1). \end{aligned} \quad (3.36)$$

(vii) Finally, if $A < 0$ and $-B + \sqrt{\delta} \leq 0$ then

$$\begin{aligned}
 C_0^{[3]} &= \frac{3!}{\sqrt{2\pi}} \left(\int_{z_1}^{z_2} + \int_{z_3}^{z_4} \right) (Az^4 + Bz^2 + C) e^{-\frac{1}{2}z^2} dz \\
 &= 12(3A + B + C)(N(z_2) - N(z_1)) + 12(Az_1^2 + 3A + B)z_1\psi(z_1) \\
 &\quad - 12(Az_2^2 + 3A + B)z_2\psi(z_2).
 \end{aligned} \tag{3.37}$$

We have shown that for orders two and three, call option prices can be found analytically. Going beyond the second-order and third-order chaos models, we observe that for a general coherent chaos model of order n the pricing kernel is a polynomial of order $2n - 2$ in the single Gaussian process $\{R_t\}$. It follows that for the valuation of an option or a swaption, the relevant calculation reduces to that of taking the expectation of the positive part of a polynomial of the same order in the standard normal random variable $Z_t = R_t/\sqrt{Q_t}$. In other words, the problem reduces to the identification of the roots of a polynomial of order $2n - 2$. Since such a root-finder can be carried out numerically, we find that semi-analytic expressions for both option and swaption prices are available in the case of an n^{th} -order coherent chaos model.

In the coherent setting, the dynamics are fully determined by a single function of one variable, $\phi(t)$, with possibly a number of parameters. Since there is only one functional degree of freedom, the initial yield curve can be determined but the model cannot be fit to other initial data that might be available and this is why an incoherent model is needed. This is described in the next section.

Grasselli & Tsujimoto (2011) calibrate a factorisable third order chaos model to option data and find that it outperforms the Hull and White and rational lognormal models as well as being comparable to LIBOR market models. Further details are discussed in the next section.

Chapter 4

Incoherent chaos models

We have examined the coherent chaos interest-rate models in some detail with a view to generalise these models to more generic cases, which we call *incoherent chaos models*. As indicated above, our key observation is that the $\mathcal{F}_\infty$ -measurable square-integrable generator $X_\infty^{[n]}$ associated with an n^{th} -order chaos expansion can be expressed as a linear combination of ‘coherent’ log-normal random variables $X_\infty^{[n]}(\phi_i)$ for different structure functions $\phi_i(s)$. More specifically, if the chaos coefficients are given by an incoherent element of the form

$$\phi_n(s_1, s_2, \dots, s_n) = \sum_i c_i \phi_i(s_1) \phi_i(s_2) \dots \phi_i(s_n) \quad (4.1)$$

and we have the representation

$$X_\infty^{[n]} = \sum_i c_i X_\infty^{[n]}(\phi_i), \quad (4.2)$$

where for clarity we have written for each ϕ_i

$$X_\infty^{[n]}(\phi_i) = \int_0^\infty \int_0^{s_1} \dots \int_0^{s_{n-1}} \phi_i(s_1) \phi_i(s_2) \dots \phi_i(s_n) dW_{s_n} \dots dW_{s_2} dW_{s_1} \quad (4.3)$$

for the coherent random variables. Then we obtain the following result for the pricing kernel in an n^{th} order incoherent chaos model:

Proposition 4.0.1. *The pricing kernel in a generic n^{th} -order incoherent model can be ex-*

pressed in the form

$$\pi_t^{[n]} = \sum_{i,j} c_i c_j \sum_{k=1}^n \langle \phi_i, \phi_j \rangle_t^{(k)} X_t^{[n-k]}(\phi_i) X_t^{[n-k]}(\phi_j), \quad (4.4)$$

where

$$\langle \phi_i, \phi_j \rangle_t^{(k)} = \int_t^\infty \int_t^{s_1} \cdots \int_t^{s_{k-1}} \phi_i(s_1) \phi_j(s_1) \cdots \phi_i(s_k) \phi_j(s_k) ds_k \cdots ds_1, \quad (4.5)$$

with $\langle \phi_i, \phi_j \rangle_t^{(0)} = 1$.

Proof. The pricing kernel associated with a general n^{th} -order chaos model can be obtained by working out the following expectation:

$$\pi_t^{[n]} = \mathbb{E}_t \left[\left(\sum_i c_i X_\infty^{[n]}(\phi_i) \right)^2 \right] - \left(\mathbb{E}_t \left[\sum_i c_i X_\infty^{[n]}(\phi_i) \right] \right)^2. \quad (4.6)$$

The second term on the right side of (4.6) can be evaluated on account of the martingale relation

$$\mathbb{E}_t[X_\infty^{[n]}(\phi_i)] = X_t^{[n]}(\phi_i). \quad (4.7)$$

Calculating the first term on the right side of (4.6) is evidently a little more complicated. We proceed as follows. First, observe from (4.3) that the random variable $X_\infty^{[n]}(\phi_i)$ can be written in the recursive form

$$X_\infty^{[n]}(\phi_i) = \int_0^\infty \phi_i(t) X_t^{[n-1]}(\phi_i) dW_t. \quad (4.8)$$

It follows, by use of the conditional form of the Wiener-Ito isometry, that

$$\begin{aligned} \mathbb{E}_t [X_\infty^{[n]}(\phi_i) X_\infty^{[n]}(\phi_j)] &= X_t^{[n]}(\phi_i) X_t^{[n]}(\phi_j) \\ &+ \int_t^\infty \phi_i(u) \phi_j(u) \mathbb{E}_t [X_u^{[n-1]}(\phi_i) X_u^{[n-1]}(\phi_j)] du. \end{aligned} \quad (4.9)$$

An application of the recursion relation (4.8) on $X_u^{[n-1]}$ together with another use of the conditional form of the Wiener-Ito isometry then shows that what remains in the conditional expectation on the right side of (4.9) is reduced to order $n - 2$. By iteration, we are then able to find that

$$\mathbb{E}_t [X_\infty^{[n]}(\phi_i) X_\infty^{[n]}(\phi_j)] = \sum_{k=0}^n \langle \phi_i, \phi_j \rangle_t^{(k)} X_t^{[n-k]}(\phi_i) X_t^{[n-k]}(\phi_j), \quad (4.10)$$

where $\langle \phi_i, \phi_j \rangle^{(k)}$ is defined as in (4.5). Putting these relations together, we obtain

$$\pi_t^{[n]} = \sum_{i,j} c_i c_j \left[\sum_{k=0}^n \langle \phi_i, \phi_j \rangle_t^{(k)} X_t^{[n-k]}(\phi_i) X_t^{[n-k]}(\phi_j) - X_t^{[n]}(\phi_i) X_t^{[n]}(\phi_j) \right], \quad (4.11)$$

from which we conclude that (4.4) holds. $\square$

Example 4.0.1 As an illustration, consider the case in which $n = 2$, and suppose that the chaos expansion involves the superposition of a pair of functions. Thus, we have

$$\Phi(s_1, s_2) = \sum_{i=1}^2 c_i \phi_i(s_1) \phi_i(s_2). \quad (4.12)$$

In this case, after some calculations, we obtain the following expression for the pricing kernel:

$$\begin{aligned} \pi_t^{[2]} &= 2c_1^2 (R_t^2(\phi_1)(1 - Q_t(\phi_1)) + \frac{1}{2} (1 - Q_t(\phi_1))^2) \\ &\quad + 2c_2^2 (R_t^2(\phi_2)(1 - Q_t(\phi_2)) + \frac{1}{2} (1 - Q_t(\phi_2))^2) \\ &\quad + 2c_1 c_2 R_t(\phi_1) R_t(\phi_2) \int_t^\infty \phi_1(s) \phi_2(s) ds \\ &\quad + c_1 c_2 \left(\int_t^\infty \phi_1(s) \phi_2(s) ds \right)^2. \end{aligned} \quad (4.13)$$

Here, we have written

$$R_t(\phi_i) = \int_0^t \phi_i(s) dW_s, \quad (4.14)$$

for $i = 1, 2$, for the two Gaussian state-variable processes. We see that the pricing kernel in this case is a simple quadratic polynomial of the two state variables. Analogous results hold for higher n and for a larger number of terms in the expansion.

Example 4.0.2 More generally, consider an ‘incoherent’ model that consists of a combination of coherent terms of different chaos orders. As an example, suppose that X_∞ is given by a combination of a first order and an n^{th} order coherent chaos element:

$$X_\infty = X_\infty^{[1]}(\phi_1) + X_\infty^{[n]}(\phi_2). \quad (4.15)$$

Taking the conditional variance and making use of (4.10), we find that the associated pricing kernel takes the form

$$\pi_t^{[n]} = 1 - Q_t(\phi_1) + \sum_{k=1}^n \frac{[X_t^{[n-k]}(\phi_2)]^2}{(k!)^2} + 2X_t^{[n-1]}(\phi_2) \int_t^\infty \phi_1(s) \phi_2(s) ds, \quad (4.16)$$

from which the corresponding term-structure dynamics can easily be derived.

Chapter 5

Finite dimensional realisations of coherent chaos models

In this section we consider the case in which the Hilbert space of square-integrable functions is approximated by (or replaced with) a finite-dimensional Hilbert space. To facilitate the calculation, we make use of the Dirac delta function so that the function $\phi(x)$ is given by the square root of a weighted sum of a finite number of delta functions. The idea of finite-dimensional realisations is to generate a set of models that can be calibrated purely in terms of a finite number of available market data, without evoking the assumption of hypothetical initial bond prices across all maturities.

To proceed, we recall first that we have made the normalisation convention such that

$$Q_\infty = \int_0^\infty \phi^2(s) ds = 1. \quad (5.1)$$

Hence, $\phi^2(s)$ can be interpreted as defining a probability density function, except that now in effect in a finite-dimensional space. More precisely, we continue to work in the infinite-dimensional Hilbert space, but by facilitating a finite number of distributions rather than functions, the analysis effectively reduces to that based on a finite-dimensional Hilbert space. We therefore choose ϕ^2 to take the following form:

$$\phi^2(s) = \sum_{i=1}^{N+1} p_i \delta(s - T_i), \quad (5.2)$$

where $\delta(s)$ denotes the Dirac delta function, and the $\{T_i\}_{i=1,\dots,N}$ represent N different maturity dates of bonds for which prices are available in the market. We remark that in (5.2) and (5.3) we have artificially introduced T_{N+1} , where $T_{N+1} > T_N$, such that the bond price is assumed to become zero at this point. Note that T_{N+1} is not an expiry date, but rather an arbitrary time that is beyond the maturity date of the bond in the market with the longest lifetime. The choice of T_{N+1} will not affect the valuation of contracts whose maturities are $\leq T_N$. Thus the analysis below will not be affected by the arbitrariness of the choice of T_{N+1} . The reason for introducing T_{N+1} is to fulfill the asymptotic condition (2.2) of axiom (d) in the finite-dimensional setup. Conversely, had we not introduced T_{N+1} , the discussion below would not be affected, but strictly speaking axiom (d) would be violated and hence the term-structure density formalism (which is not considered here) would not be directly applicable.

The coefficients $\{p_i\}$ are probability weights so that $0 \leq p_i \leq 1$ and that $\sum_{i=1}^{N+1} p_i = 1$. For this choice of $\phi(s)$, the integral Q_t then takes the form of a piecewise step function, given as follows:

$$Q_t = \sum_{i=1}^{N+1} p_i \mathbb{1}_{\{T_i \leq t\}} = \sum_{i=1}^{N+1} \mathbb{1}_{\{T_i \leq t < T_{i+1}\}} \sum_{j=1}^i p_j. \quad (5.3)$$

Let us now consider the n^{th} -order coherent chaos model associated with the positive square-root of (5.2). A short calculation using expression (5.3) in result (2.94) for $P_{0t}^{[n]}$ shows that the corresponding initial bond prices with maturity t admit the following representation:

$$P_{0t}^{[n]} = 1 - \sum_{i=1}^{N+1} \mathbb{1}_{\{T_i \leq t < T_{i+1}\}} \left(\sum_{j=1}^i p_j \right)^n. \quad (5.4)$$

To illustrate the initial term structure in the present model we sketch in Figure 5.1 an example of the initial bond prices as a function of the maturity time for $n = 2$. Three maturities have been used here: $T_1 = 1$, $T_2 = 4$ and the ‘artificially’ chosen $T_3 = 9$. According to (5.4) the initial value P_{0T_1} of a bond expiring at time T_1 (where the first jump occurs) is $P_{0T_1} = 1 - p_1^n$, the price at time T_2 is $P_{0T_2} = 1 - (p_1 + p_2)^n$, and so on. Hence in the second-order coherent chaos model with only two data points specified, we have the

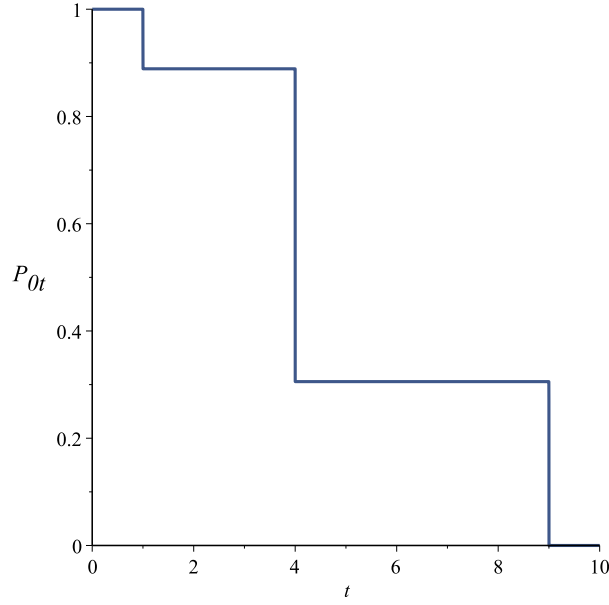


Figure 5.1: Initial bond prices $P_{0t}^{[n]}$ as a function of maturity time for $n = 2$. Two bond prices are assumed given at times $T_1 = 1$ and $T_2 = 4$; while at time $T_3 = 9$ the bond price is assumed to jump to zero. The parameter values chosen here are given by $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{2}$, and $p_3 = \frac{1}{3}$.

values $P_{0T_1} = 1 - p_1^2$, $P_{0T_2} = 1 - (p_1 + p_2)^2$, and $P_{0T_3} = 1 - (p_1 + p_2 + p_3)^2 = 0$.

Since it is assumed that the maturities $\{T_i\}_{i=1,\dots,N}$ are the points where we have market data for the bond prices, we see that the probability weights $\{p_i\}$ can be calibrated from the initial market prices of the discount bonds. This in turn determines the function $\phi(s)$, which in turn determines the subsequent dynamics for the term structure. Let us proceed to analyse the term structure dynamics in the finite-dimensional models. To begin, recall that $R_t \sim \mathcal{N}(0, Q_t)$, so in order to determine the Gaussian process $\{R_t\}$ whose probabilistic characteristics are identical to that of $\int_0^t \phi(s) dW_s$, where $\phi(s)$ is of the form (5.2), let us consider a family of independent Gaussian random variables

$$n_i \sim \mathcal{N}\left(0, \sum_{j=1}^i p_j\right), \quad (5.5)$$

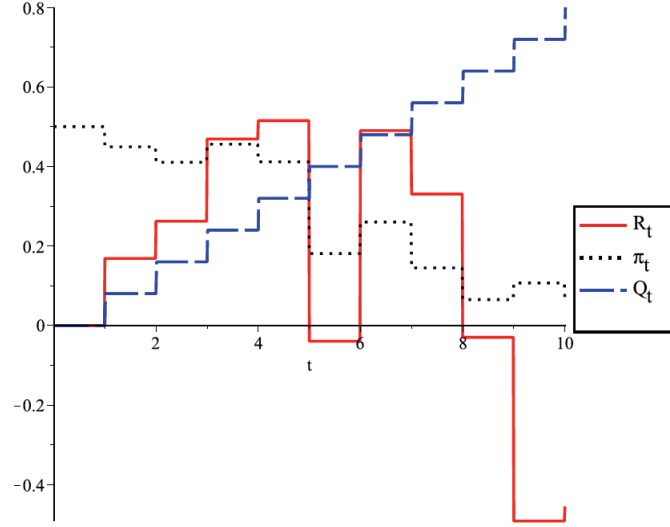


Figure 5.2: Sample path simulations for the Gaussian driver $\{R_t\}$, its quadratic variation $\{Q_t\}$, and the resulting pricing kernel $\{\pi_t\}$, in the case of a finite-dimensional coherent chaos model with $n = 2$. The parameters chosen here are $p_i = 0.08$ and $T_i = i$ for $i = 1 \dots 10$, and $p_{11} = 0.2$ with $T_{11} > 10$ arbitrary.

for $i = 1, \dots, N + 1$. Then we can represent the Gaussian process $\{R_t\}$ according to

$$R_t = \sum_{i=1}^{N+1} \mathbb{1}_{\{T_i \leq t < T_{i+1}\}} n_i, \quad (5.6)$$

where equality here holds in probability. By making use of (2.70) and (2.73), the corresponding pricing kernel can be identified in the finite-dimensional models. In figure 5.2 we illustrate typical sample paths of $\{Q_t\}$, $\{R_t\}$, and $\{\pi_t^{[2]}\}$ for $n = 2$ and $N = 10$. For simplicity we have chosen a uniform probability and equal spacing: $p_i = 0.08$ and $T_i = i$ for $i = 1, \dots, 10$. Similarly, in figure 5.3, we show a sample path of a corresponding bond price process, where we have again chosen $p_i = 0.08$ for $i = 1, \dots, 10$ with bond maturity being $T = 10$.

Summing up the section, we see that using the finite-dimensional approach, a class of elementary, highly tractable, and easily calibrated models can be constructed. These models have the characteristic feature that the discount bond process is piecewise flat and

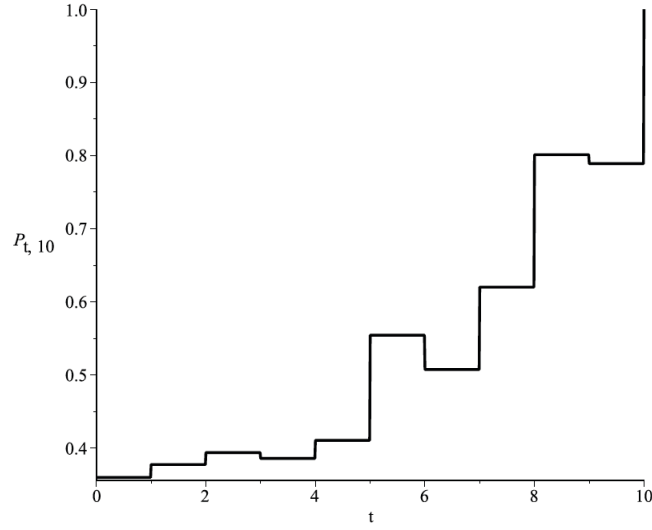


Figure 5.3: A sample path simulation for the discount bond price process $\{P_{tT}\}$ in the coherent chaos model for order $n = 2$. The parameters chosen here as $p_i = 0.08$ for $i = 1, \dots, 10$, and the bond maturity is set as $T = 10$. The bond price fluctuates up and down, but eventually converges to its terminal value one. The prices jump at the points at which the initial bond prices are assumed given in accordance with (5.4).

that the distribution of the bond price at any time is determined by a ratio of polynomials of standard normally distributed random variables. The bond price processes emerging in these models can alternatively be viewed as representing ‘simple’ approximations to more sophisticated continuous processes. To illustrate how a typical bond price in these models behaves, in Figure 6.1 we show an example of the bond price dynamics where the ‘grid size’ is made ten times finer than the one sketched in Figure 5.3. It is interesting to note in this connection that although the Gaussian process $R_t = \int_0^t \phi(s) dW_s$ formally appears to be continuous, owing to the appearance of distributions in (5.2) the resulting process contains jumps, as is evident in figure 5.2.

By using initial bond price data, we can calibrate the probabilities p_i via the expression given in (5.4). This will give us all the fundamental expressions needed in a coherent chaos model. However, an interesting open problem would be to consider an incoherent counterpart which would allow one to calibrate against additional data.

Chapter 6

Conclusion to Part I

The purpose of the first part of this thesis has been to offer an in-depth analysis of the coherent chaos interest-rate models of each order, with the view that these form the basis for generic (incoherent) interest-rate models. We have found that for a n^{th} order coherent chaos model, the pricing kernel is given by a polynomial of order $2n - 2$ in a Gaussian state variable process. This leads to tractable expressions for the bond price, market price of risk, and the short rate processes. Additionally, we have shown that in all these models it is possible to derive either analytic or semianalytic formulae for the pricing of both bond options and swaptions, involving at most the numerical determination of the roots of basic polynomials.

Single-factor coherent chaos models, although themselves considerably richer than elementary Gaussian interest rate models, are perhaps too restrictive on account of the fact that each coherent chaos model depends on a single functional degree of freedom. For more realistic models, it suffices to take linear superpositions of coherent chaos elements in the sense described in Section 2.3. The resulting expression for the pricing kernel, in the most generic case, is tractable albeit a shade cumbersome. For practical purposes it seems sufficient to consider a small number (two or three) of coherent vectors to generate rich and flexible interest-rate models. We hope that the results presented here will form a foundation for further investigations in this direction.

By way of comparison, we draw attention to the implementation of the Wiener-Ito chaos models that has been carried out in the work of Grasselli & Tsujimoto (2011) in the

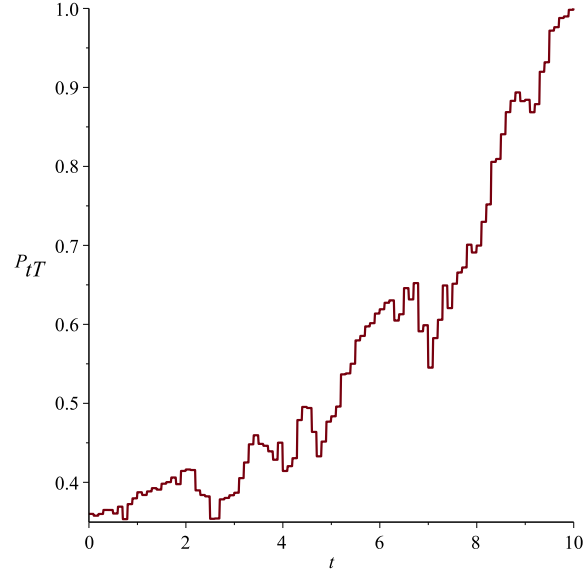


Figure 6.1: A sample path simulation for the discount bond price process $\{P_{tT}\}$ in the coherent chaos model for order $n = 2$. The parameters chosen here are $p_i = 0.008$ for $i = 1, \dots, 100$, and the bond maturity is set as $T = 10$. The bond price fluctuates up and down, but eventually converges to its terminal value one. The prices jump at the points at which the initial bond prices are assumed given in accordance with (5.4).

case of a third-order chaos model, with the choices $\phi(s_1) = \alpha(s_1)$, $\phi(s_1, s_2) = \beta(s_1)$, and $\phi(s_1, s_2, s_3) = \gamma(s_1)$. Their model has been shown to fit the forward curve more closely, and with less parameters, than various currently preferred models employed in industry. An interesting extension would therefore be to consider the implementation of the following generalisation: $\phi(s_1) = \alpha(s_1)$, $\phi(s_1, s_2) = \beta(s_1)\beta(s_2)$, and $\phi(s_1, s_2, s_3) = \gamma(s_1)\gamma(s_2)\gamma(s_3)$ in line with the foregoing discussion, and examine how well the generalised model fits both the forward and volatility curves.

In relation to the analysis on finite-dimensional models, it is worth making the following observation. For simplicity of discussion, if we set $N = 1$, then formally we are led to an expression of the form

$$R_t = \int_0^t \sqrt{\delta(s - T)} dW_s \quad (6.1)$$

for the Gaussian process $\{R_t\}$, the meaning of which *a priori* is not easily interpreted. Hanzon (1983) attempts to give a formal definition of the square root of delta functions

by how it acts on a test function. In the present analysis we have circumvented the direct handling of processes of the form (6.1) and its generalisations by means of identifying for each t , an alternative random variable whose probability law is identical to that of R_t . More specifically, we have used an idea similar to Ito's isometry to represent R_t in (6.1). We then proceeded to use this alternative representation to characterise interest-rate dynamics. For a more direct analysis of stochastic integrals of the form (6.1), calculus on the multiplication of distributions due to Colombeau (1990) might prove useful. Alternatively, the analysis of Hanzon (1983) on the square-root of the Dirac delta function, or that of Judge (1966) on the eigenfunction of the coordinate multiplication operator, might become relevant in this respect. We defer such an analysis to another occasion.

Part II

The Wick Calculus in Finance

Chapter 7

Introduction to Part II

In the first part of this thesis we have investigated the properties of a class of interest rate models that arise from mapping each admissible term structure to a point in the function space—the direct sum of the tensor products of Hilbert spaces of square-integrable functions—and made use of properties of this function space to draw conclusions about properties of interest rate models. The method and technique of the use of a function space extends to a wide range of applications in stochastic analysis, and is ultimately linked to various well-known ideas such as Malliavin calculus (see e.g. Nualart 2006), which allows one to compute derivatives of random variables. The idea of performing calculus on random variables, however, goes back much further, in part motivated by the requirement in theoretical physics for constructing nonlinear functions of random variables (e.g. Segal 1960). In the physics context the motivation for such investigations comes from the desire to quantise nonlinear classical systems, where classical variables (deterministic variables) are replaced by their quantum counterparts (random variables). There are, as a consequence, numerous approaches being proposed to deal with the calculus of random variables, with differing degrees of overlap. In particular, the formulation that is now commonly referred to as the ‘Wick calculus’ is possibly the most significant of all the approaches considered in the physics literature. Inspired by these developments, in the second part of the thesis we shall develop the Wick calculus of random variables in some detail as we attempt to fill in the gaps in the literature, with the view towards potential applications in financial modelling.

We begin by remarking that there are three main strands of independent research that are directly relevant to the content in the second part of this thesis. The first strand begins in 1880, with Paul Emile Appell, who studied sequences of polynomials $p_n(x)$ satisfying the differential equation $p'_n = np_{n-1}$ (Appell, 1880). Polynomials satisfying this equation are now called Appell polynomials. Subsequently, Sheffer (1939) further extended the class of Appell polynomials, which he called ‘polynomials of type zero’ but which are now more commonly referred to as Sheffer polynomials. The relevance of these polynomials in Wick calculus, as we shall see, is that the Wick powers of random variables are given by Appell polynomials evaluated at the random variable.

Another strand of research begins as early as the 17th century with the so-called ‘umbral calculus’, which remained relatively dormant until the 19th century when it received considerable attention by various authors, most notably in the work of Sylvester (1851) who invented the name. It was widely used but due to the fact that it is a highly symbolic language, it was frowned upon by some mathematicians owing to its lack of mathematical rigour. Early attempts, by for example Bell (1938, 1940), to axiomatise it were unsuccessful but it continued to be used extensively. It was finally in the 1970s that Gian-Carlo Rota was able to construct a rigorous foundation of the theory by using operator methods (see, for example, Mullin & Rota 1970, Rota 1975, Roman & Rota 1978). Umbral calculus is about an isomorphism between formal power series, linear functionals, and a certain class of (shift invariant) linear operators on polynomials. In particular, the isomorphism between finite operator calculus and formal power series allows one to express everything that can be done with operators in terms of power series.

More generally, umbral calculus studies the analogies between various sequences of polynomials p_n and the power sequence x^n . The line of umbral calculus that precedes the work in this thesis is the ‘modern classical umbral calculus’ as so named by Roman (1984). Roman (1984) describes it as a systematic study of the class of Sheffer sequences, made by employing the simplest techniques of modern algebra. Applications of umbral calculus include Lagrange inversion, symmetric functions, combinatorial counting and recurrences, graph theory, combinatorics, approximation theory, signal processing, coalgebras, statistics, probability theory, topology, analysis, physics and invariant theory. See Di Bucchianico & Loeb (2000) for a complete survey of research papers relating to umbral calculus up

to the year 2000.

The third strand begins with ‘normal ordering’, which was introduced by Houriet & Kind (1949) for bosonic field operators in quantum field theory. It was systemised and extended to mixed species of fields (in quantum physics, particles are represented by ‘fields’ that, in the language of probability theory, can be viewed as random variables) by Gian-Carlo Wick (1950) who called it the ‘S-product’ and used it as a tool to renormalise certain divergent series in quantum field theory. It was later called Wick ordering or Wick renormalisation and in 1974, Barry Simon defined the Wick power and Wick product for random variables with finite moments in a more systematic manner.

The majority of the second part of this thesis consists of the development of a calculus based on Wick powers and Wick products, which is often referred to as the Wick calculus. Our study begins with the representation of the Wick power and Wick product of random variables as presented in Simon (1974) but we modify these slightly so that a consistent calculus can be developed in a more systematic way.

As regards related references, a general account of the traditions of both mathematical physics and probability theory, regarding various aspects of the Wick calculus, is given in Dobrushin & Minlos (1977). In Meyer & Yan (1989), the work in Hida & Ikeda (1967) was extended to cover Wick products of Hida distributions. It should be remarked that the notion of Wick calculus (and in particular the Wick product) that we develop here is closely related to but *distinct* from what is also referred to as the ‘Wick product’ in stochastic analysis of white noise introduced by Hida & Ikeda (1967). The Wick product used in stochastic analysis is defined for Hida distributions and can be extended to general random variables via a limiting argument. However, the result depends on the topology used and different choices for the topology lead to different definitions. See Hu & Yan (2009), Hida, Kuo, Pothoff & Streit (1993) for more details as well as the survey in Gjessing *et al.* (1993) on this topic. The alternative Wick product has been used in various overlapping areas of stochastic analysis including fractional Brownian motion, Malliavin calculus and for solving stochastic differential equations.

Motivated by the existence of a rich and diverse wealth of closely related ideas, which perhaps have not been sufficiently systematically developed and some of which may prove particularly useful in certain applications to finance, our aim is to develop the calculus of

random variables initiated by Wick in as much detail as possible. Needless to say in the case of Malliavin calculus, the relevant ideas have already been proven to be useful in the development of various financial applications. Since the literature is rather diverse and yet lacks any consistent and systematic treatment of the subject, we focus on the development of the rules of the calculus itself as a preliminary step towards the possibility of various potential applications in finance.

In particular, we introduce Wick ordering, with a view to potential applications in finance. By means of an illustration we shall also derive some basic results in financial mathematics that can be derived using the Ito calculus. One key motivation behind the study embarked upon here is due to the fact that the applicability of the Wick calculus is by no means restricted to Gaussian random variables. Indeed, provided that the moments of the random variables are finite, the formalism is unchanged, irrespective of the distributional properties of the random variable. The advantage of this observation is that, when applied to financial modelling, we need not make distinction as regards whether the driving process is a Brownian motion or some other Lévy process. This is in sharp contrast to the use of Ito calculus, which in itself is only applicable to the Brownian case.

It is worth drawing attention to the fact that the Wick order has been studied quite extensively in the context of Gaussian random variables, especially in the field of probability theory. See, for example, Simon (1974) and Surgailis & Vaičiulis (1999). As regards financial application, Henry-Labordère (2008) uses Wick calculus for standard normal random variables and applies the theory to convexity adjustments, sensitivities and local volatility of stochastic volatility models. Here we are able to extend some of the basic results of asset pricing based on geometric Brownian motion model to a more general class of a geometric Lévy models. Specifically, the second part of the thesis is organised as follows. We begin in Chapter 8 by introducing an extended version of the definition of the Wick power as presented in Simon (1974). We proceed to use this definition to prove various results involving the Wick power, perhaps the most useful being Proposition 8.2.1 where we show that the Wick power of a random variable can be expressed as a polynomial of that random variable. Chapter 9 covers the Wick product which is a generalisation of the Wick power for multiple variables. In Chapter 10, we prove results analogous to those in Ito calculus and show that equations become simpler and are similar to deterministic calcu-

lus. For example, Proposition [10.1.2](#) shows an Ito's lemma type result where the quadratic variation term usually seen, is not present. Chapter [11](#) has some change of measure results for Gaussian processes as well as for general Lévy processes. We conclude the second part of the thesis with Chapter 12 by highlighting the main results and giving potential areas of development and further research.

Chapter 8

The Wick power

The Wick order was introduced by Wick (1950) in the context of quantum field theory. In this section, we begin by extending the Wick power as presented in Simon (1974). We then outline the link to the Appell polynomials as well as presenting various different representations and expansions. All measure valued quantities in what follows will be with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ unless otherwise stated.

In classical deterministic integration theory, we can formally differentiate a formal power series in the following way

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n. \quad (8.1)$$

We would like to construct something analogous for random variables. We therefore begin by considering the formal power series in a fixed random variable X and define a formal derivation with respect to X , as in Simon (1974), by

$$\frac{d}{dX} \left(\sum_{n=0}^{\infty} a_n X^n \right) = \sum_{n=0}^{\infty} (n+1) a_{n+1} X^n. \quad (8.2)$$

Note that the differential operator used in (8.2) has no definition in terms of limits but acts as an operator such that (8.2) holds true. In this way, we avoid dealing with the usual barriers when trying to use ordinary calculus on random variables. This derivative operator will be used in the remainder of the chapter and is defined in the same formal way.

The results that follow apply to a general random variable X with the only restricting condition being that the moments of X are finite, i.e. $\mathbb{E}[|X^m|] < \infty$, for every $m \geq 1$.

Definition 8.0.1 Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite moments. Then the *Wick power* $:X^n:$, $n = 0, 1, 2, \dots$ of X is defined recursively by the following axioms:

(a) The null power:

$$:X^0: = 1 \quad (8.3)$$

(b) The differentiation:

$$\frac{d}{dX} :X^n: = n :X^{n-1}:, \quad n = 1, 2, \dots \quad (8.4)$$

(c) The expectation:

$$\mathbb{E}[:X^n:] = 0, \quad n = 1, 2, \dots \quad (8.5)$$

(d) The linearity:

$$:\alpha X^m + \beta X^n: = \alpha :X^m: + \beta :X^n:, \quad n, m = 0, 1, 2, \dots \quad (8.6)$$

for any $\alpha, \beta \in \mathbb{R}$.

Remark 8.0.1 Note that our definition of the Wick power differs to that of Simon (1974) by the added linearity condition (d), which is required for the Wick power to be unambiguously defined, as we shall later see (see Section 9.2 for more details on this). This version of the Wick power can also be defined through its generating function.

Remark 8.0.2 The Wick power has also been defined without the coefficient of n as in (8.4). See for example Di Bucchianico (1997, page 41), who writes condition (b) in Definition 8.0.1 as follows:

$$Dp_n = p_{n-1} \quad (8.7)$$

for what he defines as the Wick polynomial sequence $\{p_n\}_{n \in \mathbb{N}}$ associated to X and where D denotes the differentiation operator. A translation of (8.7) to our notation is

$$\frac{d}{dX} :X^n: = :X^{n-1}:, \quad n = 1, 2, \dots \quad (8.8)$$

We now use Definition 8.0.1 to express the Wick power in terms of integrals. In particular, we find the antiderivative $:X^n:$ by integrating the expression in condition (b).

Proposition 8.0.2. *Let X be a random variable with finite moments. Then for $n \geq 1$,*

$$:X^n: = n \left(\int :X^{n-1}: dX - \mathbb{E} \left[\int :X^{n-1}: dX \right] \right). \quad (8.9)$$

Remark 8.0.3 Note that the integral in the proposition above is the inverse of the formal differential operator in (8.4).

Proof. We begin by integrating the expression in (8.4). We perform the integration as we would on a deterministic expression, since our definition allows us to use ordinary calculus on a Wick power. We therefore obtain

$$:X^n: = n \int :X^{n-1}: dX + c, \quad (8.10)$$

where c is the constant of integration. In order to determine c , we take the expectation of the expression in (8.10) and use (8.5) to obtain

$$c = -n \mathbb{E} \left[\int :X^{n-1}: dX \right]. \quad (8.11)$$

By substituting this into (8.10), we arrive at the expression in (8.9). $\square$

We can find the Wick power for each n by using Proposition 8.0.2. By setting $n = 1$ in equation (8.9), we can write,

$$:X: = X - \mathbb{E}[X]. \quad (8.12)$$

For $n = 2$, we can represent the second Wick power as

$$\begin{aligned} :X^2: &= 2 \int :X: dX - 2\mathbb{E} \left[\int :X: dX \right] \\ &= 2 \int X - \mathbb{E}[X] dX - 2\mathbb{E} \left[\int X - \mathbb{E}[X] dX \right] \\ &= X^2 - 2X\mathbb{E}[X] - \mathbb{E}[X^2] + 2(\mathbb{E}[X])^2. \end{aligned} \quad (8.13)$$

Finally, for $n = 3$, we obtain

$$\begin{aligned} :X^3: &= 3 \int :X^2: dX - 3\mathbb{E} \left[\int :X^2: dX \right] \\ &= X^3 - 3X^2\mathbb{E}[X] - 3X\mathbb{E}[X^2] + 6X(\mathbb{E}[X])^2 - \mathbb{E}[X^3] \\ &\quad + 6\mathbb{E}[X]\mathbb{E}[X^2] - 6(\mathbb{E}[X])^3. \end{aligned} \quad (8.14)$$

We observe from these first few examples that the Wick powers constitute a sequence of polynomials (one for each n). This leads us to another related area of mathematics.

8.1 Appell polynomials

There is a branch of mathematics that focuses on a class of polynomials called Appell polynomials, named after their originator Paul Emile Appell (1880). The Appell polynomials were originally defined as polynomials $A_n(x)$ satisfying

$$A'_n(x) = nA_{n-1}(x), \quad \text{for } n \geq 1. \quad (8.15)$$

Notice that this relation is equivalent to condition (b). The Wick power is a class of Appell polynomials that also satisfies conditions (a), (c) and (d) for a single variable, as accord-

ing to Giraitis & Surgailis (1986). However, the following definition is also used in, for example, Avram & Taqqu (1987) for the Appell polynomials.

Definition 8.1.1 The *Appell polynomials*, $\{A_n(x)\}$, is a sequence of polynomials of order $n = 0, 1, \dots$ with leading coefficient 1, such that

1. $A_0(x) = 1$,
2. $A'_n(x) = nA_{n-1}(x)$, for $n \geq 1$,
3. $\mathbb{E}[A_n(X)] = 0$ for $n \geq 1$.

We observe that this definition is analogous to the definition we have presented for the Wick power if we make the identification $:X^n: = A_n(X)$.

It is known in the theory of Appell polynomials (see, e.g., Avram & Taqqu 1987) that the following recursion relation

$$A_{n+1}(x) = xA_n(x) - \sum_{k=0}^n \binom{n}{k} \kappa_{n-k+1} A_k(x) \quad (8.16)$$

holds, where κ_{n-k+1} are the cumulants of X :

$$\kappa_{n-k+1} = \mathbb{E}[X^{n-k+1}] - \sum_{m=1}^{n-k} \binom{n-k}{m-1} \kappa_m \mathbb{E}[X^{n-k+1-m}]. \quad (8.17)$$

Rewriting (8.16) in terms of Wick powers, we have

$$:X^{n+1}: = X :X^n: - \sum_{k=0}^n \binom{n}{k} \kappa_{n-k+1} :X^k:. \quad (8.18)$$

Although we do not make explicit use of equation (8.18), it is interesting to compare with expressions that we derive in the next section for the Wick power.

8.2 Polynomial representation

While calculating the Wick powers recursively using (8.9), we observed that the following recursive formulation also holds:

Proposition 8.2.1. *Let X be a random variable with finite moments such that Definition 8.0.1 holds. Then the Wick power $:X^n:$, $n = 0, 1, \dots$ of X can be represented by the following recursive polynomial:*

$$:X^n: = \sum_{k=0}^n \binom{n}{k} c_k X^{n-k}, \quad (8.19)$$

where

$$c_k = - \sum_{j=0}^{k-1} \binom{k}{j} c_j \mathbb{E}[X^{k-j}], \quad k \geq 1, \quad c_0 = 1. \quad (8.20)$$

Remark 8.2.1 The real numbers $\{c_k\}$ are related to the cumulants $\{\kappa_k\}$ of X in a nontrivial way. In particular, by comparing equations (8.17) and (8.20), we obtain the relation

$$\sum_{j=1}^n \binom{n-1}{j-1} \kappa_j \mathbb{E}[X^{n-j}] = - \sum_{j=1}^n \binom{n}{j} c_j \mathbb{E}[X^{n-j}]. \quad (8.21)$$

Remark 8.2.2 A similar result to the one in Proposition 8.2.1 is given in Giraitis & Surgailis (1986). In particular, they have

$$:X^n: = \sum_{k=0}^n \binom{n}{k} c_k X^{n-k}, \quad (8.22)$$

where

$$c_k = \sum \binom{k}{j_1, \dots, j_r} \frac{(-1)^r}{r!} \kappa_{j_1} \dots \kappa_{j_r}. \quad (8.23)$$

Here, the sum is taken over all $r \geq 1$ and all integers $j_1 \geq 1, \dots, j_r \geq 1$ such that $j_1 + \dots + j_r = k$, while κ_j is the j^{th} cumulant of X , defined by

$$\kappa_j(X) = \frac{\partial^j}{\partial a^j} \log \mathbb{E} [e^{aX}] \Big|_{a=0}. \quad (8.24)$$

Expression (8.23) is more difficult to work with than (8.20), and so we continue with the representation in Proposition 8.2.1.

Proof of Proposition 8.2.1. We prove the proposition by induction. Setting $n = 0$ in equation (8.19) we obtain

$$:X^0: = 1. \quad (8.25)$$

Now, we assume that the proposition holds for $n - 1$ and proceed to show that it holds for n . By substituting

$$:X^{n-1}: = \sum_{k=0}^{n-1} \binom{n-1}{k} c_k X^{n-1-k} \quad (8.26)$$

into equation (8.9), we obtain

$$:X^n: = n \sum_{k=0}^{n-1} \binom{n-1}{k} c_k \left[\int X^{n-1-k} dX - \mathbb{E} \left[\int X^{n-1-k} dX \right] \right], \quad (8.27)$$

Since integrating a polynomial boils down to integrating each term, we can write

$$\begin{aligned} :X^n: &= n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{c_k}{n-k} [X^{n-k} - \mathbb{E} [X^{n-k}]] \\ &= \sum_{k=0}^n \binom{n}{k} c_k X^{n-k} - c_n - \sum_{k=0}^{n-1} \binom{n}{k} c_k \mathbb{E} [X^{n-k}]. \end{aligned} \quad (8.28)$$

We observe that the last term on the right hand side of (8.28) is exactly c_n and so the last two terms cancel to give the desired result. $\square$

Before we proceed further, let us consider a number of examples.

Example 8.2.1 If $n = 1$, equation (8.19) gives the following for the first Wick power of X

$$:X: = c_0 X + c_1 \quad (8.29)$$

and from (8.20), we have that

$$c_0 = 1, \quad c_1 = -\mathbb{E}[X]. \quad (8.30)$$

Putting (8.29) and (8.30) together, we obtain the known result for the first Wick power of X . Namely, $:X: = X - \mathbb{E}[X]$.

Example 8.2.2 Consider the case when $n = 2$. By (8.19), the second Wick power can then be written as

$$:X^2: = c_0 X^2 + 2c_1 X + c_2, \quad (8.31)$$

and from (8.20) we have

$$c_2 = -\mathbb{E}[X^2] + 2(\mathbb{E}[X])^2. \quad (8.32)$$

Substituting (8.30) and (8.32) into (8.31), we obtain

$$:X^2: = X^2 - 2X\mathbb{E}[X] - \mathbb{E}[X^2] + 2(\mathbb{E}[X])^2. \quad (8.33)$$

Example 8.2.3 Using Proposition 8.2.1, once again, we obtain for $n = 3$,

$$:X^3: = c_0 X^3 + 3c_1 X^2 + 3c_2 X + c_3, \quad (8.34)$$

with

$$c_3 = -\mathbb{E}[X^3] + 3\mathbb{E}[X]\mathbb{E}[X^2] - 6(\mathbb{E}[X])^3 + 3\mathbb{E}[X]\mathbb{E}[X^2]. \quad (8.35)$$

From (8.30), (8.32) and (8.35), we therefore find that

$$\begin{aligned} :X^3: &= X^3 - 3X^2\mathbb{E}[X] - 3X\mathbb{E}[X^2] + 6X(\mathbb{E}[X])^2 - \mathbb{E}[X^3] \\ &\quad + 6\mathbb{E}[X]\mathbb{E}[X^2] - 6(\mathbb{E}[X])^3. \end{aligned} \quad (8.36)$$

Finally, following the same line of reasoning for $n = 4$, we obtain

$$\begin{aligned} :X^4: &= X^4 - 4X^3\mathbb{E}[X] + 12X^2(\mathbb{E}[X])^2 - 6X^2\mathbb{E}[X^2] - 4X\mathbb{E}[X^3] \\ &\quad + 24X\mathbb{E}[X]\mathbb{E}[X^2] - 24X(\mathbb{E}[X])^3 - \mathbb{E}[X^4] + 6(\mathbb{E}[X^2])^2 \\ &\quad + 24(\mathbb{E}[X])^4 + 8\mathbb{E}[X]\mathbb{E}[X^3] - 36(\mathbb{E}[X])^2\mathbb{E}[X^2]. \end{aligned} \quad (8.37)$$

8.3 Properties of the Wick power

In this section, we provide various different results involving the Wick power. Specifically, we find a range of differential, integral and recursive relations satisfied by the Wick powers. The following result is well known in the literature for the Appell polynomials and the zero mean version (when $\mathbb{E}[X] = 0$) can be found in Avram & Taqqu (1987). We find, as expected, that for each k , we recover the k^{th} Wick power.

Proposition 8.3.1. *Let X be a random variable with finite moments. Then*

$$X^n = \sum_{k=0}^n \binom{n}{k} \mathbb{E}[X^{n-k}] :X^k: . \quad (8.38)$$

Proof. We prove the proposition by induction. Setting $n = 1$ in (8.38), we obtain $X = \mathbb{E}[X] + :X:$. We assume that the proposition holds for $n - 1$. Since we are able to formally differentiate with respect to X , we can write

$$\begin{aligned} \frac{d}{dX} X^n &= nX^{n-1} \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^{n-1-k}] :X^k: \end{aligned} \quad (8.39)$$

and by integration, we find that

$$X^n = n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[X^{n-1-k}] \int :X^k: dX + c, \quad (8.40)$$

where, by (8.5) and (8.9),

$$c = \mathbb{E}[X^n] - n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E} \left[\int :X^k: dX \right]. \quad (8.41)$$

By setting $j = k + 1$, we obtain the desired result. $\square$

The above result shows that there is a certain symmetry between $:X^n:$ and X^n , in the sense that we can either regard $:X^n:$ as a polynomial of X , or regard X^n as a polynomial in the Wick powers.

The following proposition shows that when we (formally) differentiate the first Wick power raised to the power n , the result is analogous to performing the operation on an ordinary monomial.

Proposition 8.3.2. *Let X be a random variable with finite moments. Then*

$$\frac{d}{dX} :X:^n = n :X:^{n-1}. \quad (8.42)$$

Proof. By expression (8.17) of the first Wick power, we use the binomial sum to write

$$\frac{d}{dX} :X:^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (\mathbb{E}[X])^{n-k} \frac{d}{dX} X^k. \quad (8.43)$$

Now since we can formally differentiate with respect to X , we can write the left hand side of (8.42) as

$$\frac{d}{dX} :X:^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (\mathbb{E}[X])^{n-k} k X^{k-1}. \quad (8.44)$$

After making the replacement $j = k - 1$, we obtain

$$\frac{d}{dX} :X:^n = n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1-j} (\mathbb{E}[X])^{n-1-j} X^j, \quad (8.45)$$

which concludes our proof. $\square$

Using Proposition 8.3.2, we can obtain the following integral relation:

Proposition 8.3.3. *Let X be a random variable with finite moments. Then*

$$:X:^n = \mathbb{E}[:X:^n] + n \left[\int :X:^{n-1} dX - \mathbb{E} \left[\int :X:^{n-1} dX \right] \right]. \quad (8.46)$$

Proof. Integrating (8.42), we can write the left hand side of (8.46) as

$$:X:^n = n \int :X:^{n-1} dX + c. \quad (8.47)$$

Hence upon taking the expectation of (8.47), we find that

$$c = \mathbb{E}[:X:^n] - n \mathbb{E} \left[\int :X:^{n-1} dX \right]. \quad (8.48)$$

Putting (8.47) and (8.48) together, we obtain the result. $\square$

The following result is similar to the polynomial representation in Proposition 8.2.1 with the difference that instead of the Wick power of a random variable being expressed as a polynomial in X , we show that it can be expressed as a polynomial of the first Wick power $:X:$.

Proposition 8.3.4. *Let X be a random variable with finite moments. Then the Wick power $:X^n:$, $n = 0, 1, \dots$ of X can be written as follows:*

$$:X^n: = \sum_{k=0}^n \binom{n}{k} g_k :X:^{n-k}, \quad (8.49)$$

where

$$g_k = - \sum_{j=0}^{k-1} \binom{k}{j} g_j \mathbb{E}[:X:]^{k-j}, \quad k \geq 1, \quad g_0 = 1. \quad (8.50)$$

Remark 8.3.1 For $n = 1$, this result is trivial and we find that $:X: = :X:$. For $n = 2$, we have $:X^2: = :X:]^2 - \mathbb{E}[:X:]^2$, while for $n = 3$ we have $:X^3: = :X:]^3 - 3 :X: \mathbb{E}[:X:]^2 - \mathbb{E}[:X:]^3$, and so on. Expanding these expressions for $:X^2:$ and $:X^3:$, we find that they give results (8.33) and (8.36).

Proof. We have seen in the remark above that (8.49) holds for $n = 1$. We prove the proposition by induction. Assuming that the proposition holds for $n - 1$, by the definition of the Wick power we have

$$\begin{aligned} \frac{d}{dX} :X^n: &= n :X^{n-1}: \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} g_k :X:]^{n-1-k}. \end{aligned} \quad (8.51)$$

Integrating (8.51), we obtain

$$:X^n: = n \sum_{k=0}^{n-1} \binom{n-1}{k} g_k \int :X:]^{n-1-k} dX + c, \quad (8.52)$$

where

$$c = -n \sum_{k=0}^{n-1} \binom{n-1}{k} g_k \mathbb{E} \left[\int :X:]^{n-1-k} dX \right]. \quad (8.53)$$

Here c has been found in the usual way, by taking the expectation of (8.53) and using (8.5). It follows from Proposition 8.3.3 that

$$:X^n: = n \sum_{k=0}^{n-1} \binom{n-1}{k} g_k \left(\frac{:X:]^{n-k} - \mathbb{E}[:X:]^{n-k}}{n-k} \right). \quad (8.54)$$

Extending the upper limit to n , we recover (8.49). $\square$

We also have the following interesting formulation involving Wick powers.

Proposition 8.3.5. *Let X be a random variable with finite moments. Then,*

$$:X:^n = \sum_{k=0}^n \binom{n}{k} :X^{n-k}: \mathbb{E}[:X:^k]. \quad (8.55)$$

Remark 8.3.2 For $n = 1$, we trivially obtain $:X: = :X:$. Setting $n = 2$, we have $:X:^2 = :X^2: + \mathbb{E}[:X:^2]$. Again, we find that by rearranging and expanding the Wick powers, we obtain the known expressions for the Wick powers.

Proof. Once again, we prove this proposition by induction. In the remark above, we have seen that it holds for $n = 1$ and we assume that it holds for $n - 1$. We then use (8.42) to obtain

$$\frac{d}{dX} :X:^n = n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[:X:^k] :X^{n-1-k}:. \quad (8.56)$$

Integrating (8.56), we find that

$$:X:^n = n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[:X:^k] \left[\int :X^{n-1-k}: dX - \mathbb{E} \left[\int :X^{n-1-k}: dX \right] \right] \quad (8.57)$$

and by equation (8.9), we have

$$:X:^n = n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathbb{E}[:X:^k] \frac{:X^{n-k}:}{n-k}. \quad (8.58)$$

Writing this sum to n , we obtain the desired result. $\square$

8.4 The Wick exponential

As mentioned at the beginning of this chapter, the Wick power can also be defined by its generating function. The so-called *Wick exponential* (or Wick ordered exponential) is the exponential generating function for the Wick powers and is defined in terms of formal power series by

$$:e^{\alpha X}: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :X^n:, \quad \alpha \in \mathbb{R}. \quad (8.59)$$

We would like to derive a representation of the Wick exponential that does not involve the Wick ordering, thus can be used more effectively as a generating function for the moments of the random variable. We proceed as follows. First, taking the formal derivative of $:e^{\alpha X}:$ with respect to X , we obtain, on account of (8.4), the following differential equation:

$$\frac{d}{dX} :e^{\alpha X}: = \alpha :e^{\alpha X}:, \quad (8.60)$$

whose solution is

$$:e^{\alpha X}: = k e^{\alpha X}, \quad (8.61)$$

where k is some constant. Then by applying (8.5), we find that

$$:e^{\alpha X}: = \frac{e^{\alpha X}}{\mathbb{E}[e^{\alpha X}]}, \quad (8.62)$$

which holds in the sense of formal power series. By differentiating (8.62) with respect to α k times, we can express the Wick power as follows:

$$:X^k: = \frac{\partial^k}{\partial \alpha^k} \left(\frac{e^{\alpha X}}{\mathbb{E}[e^{\alpha X}]} \right) \Big|_{\alpha=0}. \quad (8.63)$$

Remark 8.4.1 The expression on the right hand side of (8.62) is widely referred to as the Esscher transform. It has been used for decades in insurance mathematics and is also commonly used to change probability measures as we will see in Chapter 11. See Gerber

& Shiu (1994) for more details.

Remark 8.4.2 Both (8.69) and (8.63) are used as definitions of the Wick power.

Remark 8.4.3 In the foregoing discussion we have assumed that

$$\mathbb{E}[e^{\alpha X}] < \infty \quad (8.64)$$

for $\alpha \in A$, where A is a connected interval containing the origin.

8.5 Examples

Below we consider some examples of the Wick powers and Wick-ordered exponentials for a number of specific random variables.

Example 8.5.1 *Hermite polynomials.* If X is a centred Gaussian random variable with variance σ^2 , the formal power series in (8.62) converges, and we can use the moment generating function of X

$$\mathbb{E}[e^{\alpha X}] = e^{\frac{1}{2}\alpha^2\sigma^2}, \quad \alpha \in \mathbb{R}. \quad (8.65)$$

By substituting (8.65) into (8.62), we can express the Wick exponential as

$$:e^{\alpha X}: = e^{\alpha X - \frac{1}{2}\alpha^2\sigma^2}, \quad (8.66)$$

and by multiplying the series for $e^{\alpha X}$ and $e^{-\frac{1}{2}\alpha^2\sigma^2}$ and substituting into (8.66) we obtain

$$\sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{j=0}^{\infty} \frac{\alpha^j X^j}{j!} \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^{2k} \sigma^{2k}}{2^k k!}. \quad (8.67)$$

If we change the summation index from j to $n = j + 2k$, then we obtain

$$\sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{n=0}^{\infty} \frac{\alpha^n X^{n-2k}}{(n-2k)!} \sum_{k=0}^{\infty} \frac{(-1)^k \sigma^{2k}}{2^k k!}, \quad (8.68)$$

and hence we conclude that

$$:X^n: = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n! X^{n-2k} \sigma^{2k}}{k! (n-2k)! 2^k} = \sigma^n H_n \left(\frac{X}{\sigma} \right). \quad (8.69)$$

Here $H_n(\cdot)$ are the n^{th} order Hermite polynomials as defined in (2.56).

We can connect this with the first part of the thesis by first taking the following relation into account:

$$\begin{aligned} n! \left(\int_0^T \int_0^{s_1} \cdots \int_0^{s_{n-1}} \phi(s_1) \phi(s_2) \cdots \phi(s_n) dW_{s_n} \cdots dW_{s_2} dW_{s_1} \right) \\ = \left(\int_0^T \int_0^T \cdots \int_0^T \phi(s_1) \phi(s_2) \cdots \phi(s_n) dW_{s_n} \cdots dW_{s_2} dW_{s_1} \right). \end{aligned} \quad (8.70)$$

Then from (2.53) we can write

$$\begin{aligned} X_\infty^\phi &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^\infty \int_0^\infty \cdots \int_0^\infty \phi(s_1) \phi(s_2) \cdots \phi(s_n) dW_{s_n} \cdots dW_{s_2} dW_{s_1} \right) \\ &= \sum_{n=0}^{\infty} \frac{Q_\infty^{n/2}}{n!} H_n \left(\frac{R_\infty}{\sqrt{Q_\infty}} \right) \\ &= \sum_{n=0}^{\infty} \frac{:R_\infty^n:}{n!} \end{aligned} \quad (8.71)$$

Therefore, we see from (8.69) that in the case where X is a standard normal variable, the Wick power reduces to the Hermite polynomial of X and we verify that the Wiener chaos expansion is a sum of Hermite polynomials which are represented by the Wick power in equation (8.71). Expression (8.69) is a generalisation of a result that has been known for some time (McKean, 1969), namely, that

$$\left(\int_0^T dW_s \right)^n = T^{n/2} H_n \left(\frac{1}{\sqrt{T}} \int_0^T dW_s \right). \quad (8.72)$$

Example 8.5.2 *Bernoulli polynomials.* The Bernoulli polynomials are of Appell type for

random variables with moment generating function

$$\mathbb{E}[e^{\alpha X}] = \frac{\alpha}{e^\alpha - 1}. \quad (8.73)$$

We can show this by following the method used in the previous example. We begin with the expression for the Wick exponential.

$$:e^{\alpha X}: = \frac{1}{\alpha}(e^{\alpha(X+1)} - e^{\alpha X}). \quad (8.74)$$

Rearranging and writing the exponential functions in terms of formal power series, we find

$$\alpha \sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} [(X+1)^k - X^k]. \quad (8.75)$$

Now relabelling the summation index according to $k \rightarrow n+1$, we find that

$$\sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{n!} :X^n: = \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{(n+1)!} [(X+1)^{n+1} - X^{n+1}]. \quad (8.76)$$

Comparing the coefficients of α^j and setting $j = n+1$, we conclude that

$$:X^n: = \frac{1}{n+1} [(X+1)^{n+1} - X^{n+1}]. \quad (8.77)$$

Example 8.5.3 *Euler polynomials.* The Euler polynomials are of Appell type for random variables with moment generating function

$$\mathbb{E}[e^{\alpha X}] = \frac{2}{e^\alpha + 1}. \quad (8.78)$$

We begin again with the expression for the Wick exponential:

$$:e^{\alpha X}: = \frac{1}{2}(e^{\alpha(X+1)} + e^{\alpha X}). \quad (8.79)$$

Writing the exponential functions in terms of formal power series, we obtain

$$2 \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :X^n: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} [(X+1)^n + X^n]. \quad (8.80)$$

Comparing the coefficients of α^n we find that

$$:X^n: = \frac{1}{2} [(X+1)^n + X^n]. \quad (8.81)$$

Example 8.5.4 *Exponential random variable.* Let X be an exponentially distributed random variable with probability density function

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (8.82)$$

The moment generating function of $X \sim \text{Exp}(1)$ is

$$\mathbb{E}[e^{\alpha X}] = \frac{1}{1-\alpha}, \quad (8.83)$$

and from (8.62) we can write the Wick exponential as

$$:e^{\alpha X}: = e^{\alpha X} (1-\alpha), \quad \alpha \in \mathbb{R}_+. \quad (8.84)$$

Expanding in terms of formal power series, we obtain

$$\sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{n=0}^{\infty} \frac{\alpha^n X^n}{n!} - \sum_{j=0}^{\infty} \frac{\alpha^{j+1} X^j}{j!}. \quad (8.85)$$

Changing from j to $n = j + 1$ in the second sum of the right side of (8.85), we find

$$\sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{n=0}^{\infty} \frac{\alpha^n X^n}{n!} - \sum_{n=0}^{\infty} \frac{\alpha^n X^{n-1}}{(n-1)!}. \quad (8.86)$$

Hence the Wick power can be expressed as follows:

$$:X^n: = X^n - nX^{n-1}, \quad (8.87)$$

Example 8.5.5 *Gamma random variable.* Let X be a Gamma distributed random variable with shape parameter k and scale parameter θ . Then the probability density function of X , for $\theta = 1$, is

$$f(x) = \frac{1}{\Gamma(k)} x^{k-1} e^{-x}, \quad x \geq 0, \quad k > 0, \quad (8.88)$$

and its moment generating function is

$$\mathbb{E}[e^{\alpha X}] = (1 - \alpha)^{-k}, \quad \alpha < 1. \quad (8.89)$$

Hence by (8.62), we write the Wick exponential for the random variable $X \sim \Gamma(k, 1)$ as

$$:e^{\alpha X}: = e^{\alpha X} (1 - \alpha)^k. \quad (8.90)$$

Writing this in terms of formal power series, we obtain

$$\sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{i=0}^{\infty} \frac{\alpha^i X^i}{i!} \sum_{j=0}^{\infty} \binom{k}{j} (-1)^j \alpha^j. \quad (8.91)$$

Setting $j = n - i$ in the second sum on the right hand side of (8.91), we find

$$\sum_{n=0}^{\infty} \frac{\alpha^n :X^n:}{n!} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{i=0}^n \binom{k}{n-i} \frac{n! (-1)^{n-i} X^i}{i!} \quad (8.92)$$

and so we can write the Wick power of a gamma distributed random variable as follows:

$$:X^n: = n! \sum_{i=0}^n \binom{k}{n-i} \frac{(-1)^{n-i} X^i}{i!} = (-1)^n n! L_n^{(k-n)}(X). \quad (8.93)$$

Here $L_n^{(k)}(X)$ are the n^{th} order *generalised Laguerre polynomials* or *associated Laguerre*

polynomials. The Rodrigues formula for these is

$$L_n^{(k)}(z) = \frac{z^{-k}e^z}{n!} \frac{d^n}{dz^n} (z^{n+k}e^{-z}), \quad n \in \mathbb{N}, k \in \mathbb{R}. \quad (8.94)$$

Applying Leibniz's theorem for differentiation of a product to (8.94), we obtain

$$L_n^{(k)}(x) = \sum_{j=0}^n (-1)^j \binom{n+k}{n-j} \frac{x^j}{j!}. \quad (8.95)$$

The first few generalised Laguerre polynomials are:

$$\begin{aligned} L_0^{(k)}(x) &= 1, \\ L_1^{(k)}(x) &= -x + k + 1, \\ L_2^{(k)}(x) &= \frac{1}{2}x^2 - (k+2)x + \frac{(k+2)(k+1)}{2}, \\ L_3^{(k)}(x) &= \frac{-x^3}{6} + \frac{(k+3)x^2}{2} - \frac{(k+2)(k+3)x}{2} + \frac{(k+1)(k+2)(k+3)}{6}. \end{aligned} \quad (8.96)$$

We conclude this section by remarking that in general, expressions for the Wick power are either recursive or involve differentials. However, we can obtain explicit expressions under certain specific distributions. For more examples of Appell polynomials together with the associated distributions we refer to Bretti & Ricci (2004).

8.6 Multiple Wick powers

By a multiple Wick power we mean a repeated application of the Wick order on a Wick power. Thus, for example, we may take the Wick order of X^2 , which is a quadratic polynomial of X , upon which we may apply another Wick ordering. The result will be another quadratic polynomial of X , on which the Wick ordering can be applied again, and so on. If Wick ordering is applied on X^n m times, then we denote the result by $q_m(X^n)$. With this notation, the result is summarised in the following proposition:

Proposition 8.6.1. *Let X be a random variable with finite moments. Taking the Wick order*

of X^n repeatedly m times, we can express the multiple Wick power as follows:

$$q_0(X^n) = X^n \quad (8.97)$$

and for $m \geq 1$,

$$q_m(X^n) = \sum_{k_1=0}^n \binom{n}{k_1} c_{n-k_1} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} c_{k_1-k_2} \cdots \sum_{k_m=0}^{k_{m-1}} \binom{k_{m-1}}{k_m} c_{k_{m-1}-k_m} X^{k_m}, \quad (8.98)$$

where c_k are given in (8.20).

Proof. We prove the proposition by induction. Setting $m = 1$, then from equation (8.98) we find that

$$q_1(X^n) = \sum_{k=0}^n \binom{n}{k} c_{n-k} X^k. \quad (8.99)$$

If we assume that the proposition holds for m , then by taking the Wick order of $q_m(X^n)$, we obtain

$$\begin{aligned} q_{m+1}(X^n) &= \sum_{k_1=0}^n \binom{n}{k_1} c_{n-k_1} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} c_{k_1-k_2} \cdots \sum_{k_m=0}^{k_{m-1}} \binom{k_{m-1}}{k_m} c_{k_{m-1}-k_m} :X^{k_m}: \\ &= \sum_{k_1=0}^n \binom{n}{k_1} c_{n-k_1} \sum_{k_2=0}^{k_1} \binom{k_1}{k_2} c_{k_1-k_2} \\ &\quad \cdots \sum_{k_m=0}^{k_{m-1}} \binom{k_{m-1}}{k_m} c_{k_{m-1}-k_m} X^{k_m} \sum_{k_{m+1}=0}^{k_m} \binom{k_m}{k_{m+1}} c_{k_m-k_{m+1}} X^{k_{m+1}} \end{aligned} \quad (8.100)$$

□

Equation (8.98) above is quite general and may overshadow the simplicity and potential use of the result. We therefore look at some examples to show that taking multiple Wick orders of small values of n may be of interest.

Example 8.6.1 Taking the Wick order of the first Wick power of a random variable X , we

find that

$$q_2(X) = :X - \mathbb{E}[X]: = X - 2\mathbb{E}[X]. \quad (8.101)$$

Repeating this, we have for m repeated Wick orders

$$q_m(X) = X - m\mathbb{E}[X]. \quad (8.102)$$

Example 8.6.2 We now consider the case for $n = 2$ and find that taking the Wick order of X^2 m times we get

$$\begin{aligned} q_m(X^2) &= X^2 - 2X\mathbb{E}[X] + m(2(\mathbb{E}[X])^2 - \mathbb{E}[X^2]) \\ &\quad + 2(m+1)((\mathbb{E}[X])^2 - X\mathbb{E}[X]). \end{aligned} \quad (8.103)$$

8.7 Summary

We began this chapter with a version of the definition of the Wick power given by Simon (1974). We then used an integral equation to find each Wick power instead of the usual method of using (8.4) for each n and then integrating each time (see for example Simon 1974, Gjessing *et al.* 1993). Then, after linking the Wick power to the Appell polynomials, we derive a novel representation for the Wick power (Proposition 8.2.1). Section 8.3 consists of various (new) representations involving the Wick power. Using the well known Wick exponential (Esscher transform), we provide examples for the Wick power under various different distributions. Finally, we introduce the concept of multiple Wick powers.

Chapter 9

The Wick product

In the previous chapter we have presented a way of performing calculus involving a single random variable under the rules that we have referred to as the Wick calculus. In many applications, we are required to deal simultaneously with a number of random variables. Hence the calculus of a single random variable needs to be extended to that involving multiple random variables, which is referred to as the Wick product. This extension of course reduces to the analysis of Wick powers presented in the previous chapter when restricted to a single random variable. As in the case of the Wick powers we shall begin with an extended version of the definition of the Wick product introduced in Simon (1974).

Definition 9.0.1 Let $X_1, \dots, X_k$ be (real-valued) random variables, with finite joint moments. The *Wick product* $:X_1^{n_1} \cdots X_k^{n_k}:$ is defined recursively as a polynomial in $X_1, \dots, X_k$ of total degree $n = n_1 + \cdots + n_k$ satisfying

(a) The null power:

$$:X_1^0 \cdots X_k^0: = 1. \quad (9.1)$$

(b) The partial derivative:

$$\frac{\partial}{\partial X_i} :X_1^{n_1} \cdots X_k^{n_k}: = n_i :X_1^{n_1} \cdots X_i^{n_i-1} \cdots X_k^{n_k}:, \quad n_1, \dots, n_k = 1, 2, \dots \quad (9.2)$$

(c) The expectation:

$$\mathbb{E}[:X_1^{n_1} \cdots X_k^{n_k}:] = 0, \quad n_1 + \cdots + n_k \neq 0, \quad n_1, \dots, n_k = 1, 2, \dots \quad (9.3)$$

(d) The linearity:

$$\begin{aligned} :\alpha(X_1^{n_1} \cdots X_k^{n_k}) + \beta(Y_1^{m_1} \cdots Y_j^{m_j}): &= \alpha :X_1^{n_1} \cdots X_k^{n_k}: + \beta :Y_1^{m_1} \cdots Y_j^{m_j}:, \\ n_1, \dots, n_k, m_1, \dots, m_j &= 0, 1, 2, \dots \end{aligned} \quad (9.4)$$

As in the case of the Wick power, we can recursively find explicit expressions for the Wick products using the above definition.

Example 9.0.1 Let us first look at the Wick product $:XY:$. By (9.2) we have

$$\frac{\partial}{\partial X} :XY: = :Y: \quad \text{and} \quad \frac{\partial}{\partial Y} :XY: = :X:. \quad (9.5)$$

Integrating the equations in (9.5), we obtain

$$:XY: = :X: :Y: + \xi_1(Y) \quad \text{and} \quad :XY: = :X: :Y: + \xi_2(X) \quad (9.6)$$

and we observe that $\xi_1 = \xi_2 = c$ is required for the equations in (9.6) to agree. We find c in the usual way, by taking the expectation of (9.6), to obtain

$$\begin{aligned} :XY: &= :X: :Y: - \mathbb{E}[:X: :Y:] \\ &= XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] - \mathbb{E}[XY] + 2\mathbb{E}[X]\mathbb{E}[Y]. \end{aligned} \quad (9.7)$$

Example 9.0.2 Let us now consider the product $:XYZ:$ of a triplet of random variables.

By (9.2) we have

$$\begin{aligned}\frac{\partial}{\partial X} :XYZ: &= :YZ: = YZ - :Y: \mathbb{E}[Z] - :Z: \mathbb{E}[Y] - \mathbb{E}[YZ], \\ \frac{\partial}{\partial Y} :XYZ: &= :XZ: = XZ - :X: \mathbb{E}[Z] - :Z: \mathbb{E}[X] - \mathbb{E}[XZ], \\ \frac{\partial}{\partial Z} :XYZ: &= :XY: = XY - :X: \mathbb{E}[Y] - :Y: \mathbb{E}[X] - \mathbb{E}[XY].\end{aligned}\tag{9.8}$$

Integrating these equations, we get

$$\begin{aligned}:XYZ: &= :X: (YZ - :Y: \mathbb{E}[Z] - :Z: \mathbb{E}[Y] - \mathbb{E}[YZ]) + \xi_1 \\ :XYZ: &= :Y: (XZ - :X: \mathbb{E}[Z] - :Z: \mathbb{E}[X] - \mathbb{E}[XZ]) + \xi_2, \\ :XYZ: &= :Z: (XY - :X: \mathbb{E}[Y] - :Y: \mathbb{E}[X] - \mathbb{E}[XY]) + \xi_3,\end{aligned}\tag{9.9}$$

where $\xi_1 = \xi_1(Y, Z)$, $\xi_2 = \xi_2(X, Z)$ and $\xi_3 = \xi_3(X, Y)$ are integration “constants”. Now, in order for the three equations in (9.9) to agree, we set

$$\begin{aligned}\xi_1 &= \mathbb{E}[X] (Y\mathbb{E}[Z] + Z\mathbb{E}[Y]) - :Y: \mathbb{E}[XZ] - :Z: \mathbb{E}[XY] + c, \\ \xi_2 &= \mathbb{E}[Y] (X\mathbb{E}[Z] + Z\mathbb{E}[X]) - :X: \mathbb{E}[YZ] - :Z: \mathbb{E}[XY] + c, \\ \xi_3 &= \mathbb{E}[Z] (X\mathbb{E}[Y] + Y\mathbb{E}[X]) - :X: \mathbb{E}[YZ] - :Y: \mathbb{E}[XZ] + c,\end{aligned}\tag{9.10}$$

for some constant c . To find c , we recall that $\mathbb{E}[:XYZ:] = 0$, by the definition of the Wick product, and so

$$c = -\mathbb{E}[XYZ] + \mathbb{E}[X]\mathbb{E}[YZ] + \mathbb{E}[Y]\mathbb{E}[XZ] + \mathbb{E}[Z]\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y]\mathbb{E}[Z].\tag{9.11}$$

We can then substitute (9.10) and (9.11) into (9.9) to obtain

$$\begin{aligned}:XYZ: &= XYZ - :X: \mathbb{E}[YZ] - :Y: \mathbb{E}[XZ] - :Z: \mathbb{E}[XY] - :XY: \mathbb{E}[Z] \\ &\quad - :XZ: \mathbb{E}[Y] - :YZ: \mathbb{E}[X] - \mathbb{E}[XYZ].\end{aligned}\tag{9.12}$$

By expanding the Wick orders on the right hand side of (9.12), we obtain

$$\begin{aligned}
:XYZ: &= XYZ - X\mathbb{E}[YZ] - Y\mathbb{E}[XZ] - Z\mathbb{E}[XY] - XY\mathbb{E}[Z] \\
&\quad - XZ\mathbb{E}[Y] - YZ\mathbb{E}[X] + 2X\mathbb{E}[Y]\mathbb{E}[Z] + 2Y\mathbb{E}[X]\mathbb{E}[Z] \\
&\quad + 2Z\mathbb{E}[X]\mathbb{E}[Y] + 2\mathbb{E}[X]\mathbb{E}[YZ] + 2\mathbb{E}[Y]\mathbb{E}[XZ] \\
&\quad + 2\mathbb{E}[Z]\mathbb{E}[XY] - \mathbb{E}[XYZ] - 6\mathbb{E}[X]\mathbb{E}[Y]\mathbb{E}[Z].
\end{aligned} \tag{9.13}$$

The calculation for the Wick product becomes rather complicated even in the case of a trilinear function of three random variables. It is cumbersome to calculate the Wick order of more variables. We therefore introduce a recursion formula for the Wick product which is considerably easier to work with.

9.1 Polynomial representation

In what follows, it will be useful to employ the multivariable polynomial representation for the Wick order of a product of random variables. To this end, we begin with the following result:

Proposition 9.1.1. *Let $X_1, \dots, X_k$ be random variables such that their joint moments are finite. Then the Wick product $:X_1^{n_1} \dots X_k^{n_k}:$, can be expressed as a multivariable polynomial of the following form:*

$$:X_1^{n_1} \dots X_k^{n_k}: = \sum_{j_1=0}^{n_1} \dots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \dots \binom{n_k}{j_k} c_{j_1, \dots, j_k} X_1^{n_1-j_1} \dots X_k^{n_k-j_k} \tag{9.14}$$

where the coefficients $\{c_{j_1, \dots, j_k}\}$ are determined recursively by the relation

$$\sum_{i_1=0}^{j_1} \dots \sum_{i_k=0}^{j_k} \binom{j_1}{i_1} \dots \binom{j_k}{i_k} c_{i_1, \dots, i_k} \mathbb{E}[X_1^{j_1-i_1} X_2^{j_2-i_2} \dots X_k^{j_k-i_k}] = 0 \tag{9.15}$$

and where $c_{0, \dots, 0} = 1$.

Remark 9.1.1 A special case of this result under the Gaussian setting can be found in

Dobrushin & Minlos (1977).

Before we proceed to construct the proof of Proposition 9.1.1 in the general case, let us first prove the proposition for a pair of random variables X and Y , to gain some intuition on the logic of the general proof.

Proposition 9.1.2. *Let X, Y be random variables such that their joint moments exist. Then the Wick product $:X^m Y^n:$ can be written as follows:*

$$:X^m Y^n: = \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1, j_2} X^{m-j_1} Y^{n-j_2}, \quad (9.16)$$

where the coefficients $\{c_{j_1, j_2}\}$ are determined recursively by the relation

$$\sum_{i_1=0}^{j_1} \sum_{i_2=0}^{j_2} \binom{j_1}{i_1} \binom{j_2}{i_2} c_{i_1, i_2} \mathbb{E}[X^{j_1-i_1} Y^{j_2-i_2}] = 0 \quad (9.17)$$

and where $c_{0,0} = 1$.

Proof. We prove the proposition by induction. Setting $m = n = 0$ in equation (9.16), we find that

$$:X^0 Y^0: = 1. \quad (9.18)$$

Now, from Definition 9.0.1 of the Wick product, we have the following system of equations

$$\begin{aligned} \frac{\partial}{\partial X} :X^m Y^n: &= m :X^{m-1} Y^n:, \\ \frac{\partial}{\partial Y} :X^m Y^n: &= n :X^m Y^{n-1}:. \end{aligned} \quad (9.19)$$

Upon integration, we obtain

$$\begin{aligned} :X^m Y^n: &= m \int :X^{m-1} Y^n: dX + \xi_1(Y), \\ :X^m Y^n: &= n \int :X^m Y^{n-1}: dY + \xi_2(X). \end{aligned} \quad (9.20)$$

Assuming that the proposition holds for $:X^m Y^{n-1}:$ and $:X^{m-1} Y^n:$, we can write the expressions in (9.20) in the following way

$$\begin{aligned} :X^m Y^n: &= m \sum_{j_1=0}^{m-1} \sum_{j_2=0}^n \binom{m-1}{j_1} \binom{n}{j_2} c_{j_1, j_2} \frac{X^{m-j_1} Y^{n-j_2}}{m-j_1} + \xi_1(Y), \\ :X^m Y^n: &= n \sum_{j_1=0}^m \sum_{j_2=0}^{n-1} \binom{m}{j_1} \binom{n-1}{j_2} c_{j_1, j_2} \frac{X^{m-j_1} Y^{n-j_2}}{n-j_2} + \xi_2(X), \end{aligned} \quad (9.21)$$

and we can rewrite these as

$$\begin{aligned} :X^m Y^n: &= \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1, j_2} X^{m-j_1} Y^{n-j_2} \\ &\quad - \sum_{j_2=0}^n \binom{n}{j_2} c_{m, j_2} Y^{n-j_2} + \xi_1(Y), \\ :X^m Y^n: &= \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1, j_2} X^{m-j_1} Y^{n-j_2} \\ &\quad - \sum_{j_1=0}^m \binom{m}{j_1} c_{j_1, n} X^{m-j_1} + \xi_2(X). \end{aligned} \quad (9.22)$$

The equations in (9.22) agree if

$$- \sum_{j_2=0}^n \binom{n}{j_2} c_{m, j_2} Y^{n-j_2} + \xi_1(Y) = - \sum_{j_1=0}^m \binom{m}{j_1} c_{j_1, n} X^{m-j_1} + \xi_2(X) \quad (9.23)$$

holds. Since the left hand side is a function of Y and the right hand side a function of X , it must hold that

$$\begin{aligned} \xi_1(Y) - \sum_{j_2=0}^n \binom{n}{j_2} c_{m, j_2} Y^{n-j_2} &= c, \\ \xi_2(X) - \sum_{j_1=0}^m \binom{m}{j_1} c_{j_1, n} X^{m-j_1} &= c. \end{aligned} \quad (9.24)$$

Therefore, the equations in (9.22) become

$$:X^m Y^n: = \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1, j_2} X^{m-j_1} Y^{n-j_2} + c. \quad (9.25)$$

Now, we can determine the constant c by taking the expectation of the equation in (9.25) and using condition (c) of Definition 9.0.1 to obtain

$$c = - \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1, j_2} \mathbb{E} [X^{m-j_1} Y^{n-j_2}]. \quad (9.26)$$

But from (9.17) we find that the equation in (9.26) is in fact equal to zero. This completes our proof. $\square$

Example 9.1.1 Using the proposition above, we obtain for $:XY:$ the expression already derived in Example 9.0.1:

$$\begin{aligned} :XY: &= \sum_{j_1=0}^1 \sum_{j_2=0}^1 \binom{1}{j_1} \binom{1}{j_2} c_{j_1, j_2} X^{1-j_1} Y^{1-j_2} \\ &= XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + 2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[XY]. \end{aligned} \quad (9.27)$$

Example 9.1.2 The Wick product $:XY^2:$ can also be calculated with relative ease. From (9.16), we can write

$$\begin{aligned} :XY^2: &= \sum_{j_1=0}^1 \sum_{j_2=0}^2 \binom{1}{j_1} \binom{2}{j_2} c_{j_1, j_2} X^{1-j_1} Y^{2-j_2} \\ &= XY^2 + 2c_{0,1}XY + c_{0,2}X + c_{1,0}Y^2 + 2c_{1,1}Y + c_{1,2}, \end{aligned} \quad (9.28)$$

and we use (9.17) to find the coefficients:

$$\begin{aligned} c_{0,1} &= -\mathbb{E}[Y], & c_{0,2} &= -\mathbb{E}[Y^2] + 2(\mathbb{E}[Y])^2, \\ c_{1,0} &= -\mathbb{E}[X], & c_{1,1} &= -\mathbb{E}[XY] + 2\mathbb{E}[X]\mathbb{E}[Y], \\ c_{1,2} &= -\mathbb{E}[XY^2] + 4\mathbb{E}[Y]\mathbb{E}[XY] + 2\mathbb{E}[X]\mathbb{E}[Y^2] - 6\mathbb{E}[X](\mathbb{E}[Y])^2. \end{aligned} \quad (9.29)$$

Substituting the results of (9.29) into (9.28), we thus obtain:

$$\begin{aligned}
 :XY^2: &= XY^2 - 2XY\mathbb{E}[Y] - X\mathbb{E}[Y^2] + 2X(\mathbb{E}[Y])^2 - \mathbb{E}[X]Y^2 \\
 &\quad - 2\mathbb{E}[XY]Y + 4\mathbb{E}[X]Y\mathbb{E}[Y] - \mathbb{E}[XY^2] \\
 &\quad + 4\mathbb{E}[XY]\mathbb{E}[Y] + 2\mathbb{E}[X]\mathbb{E}[Y^2] - 6\mathbb{E}[X](\mathbb{E}[Y])^2.
 \end{aligned} \tag{9.30}$$

We provide one more preliminary step before the full multivariable proposition in order to make the transition smoother and clearer.

Proposition 9.1.3. *Let X, Y, Z be random variables such that their joint moments are finite. Then the Wick product $:X^l Y^m Z^n:$ can be written in the form*

$$:X^l Y^m Z^n: = \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} X^{l-j_1} Y^{m-j_2} Z^{n-j_3}, \tag{9.31}$$

where the coefficients $\{c_{j_1, j_2, j_3}\}$ are determined recursively by the relation

$$\sum_{i_1=0}^{j_1} \sum_{i_2=0}^{j_2} \sum_{i_3=0}^{j_3} \binom{j_1}{i_1} \binom{j_2}{i_2} \binom{j_3}{i_3} c_{i_1, i_2, i_3} \mathbb{E}[X^{j_1-i_1} Y^{j_2-i_2} Z^{j_3-i_3}] = 0 \tag{9.32}$$

and where $c_{0,0,0} = 1$.

Proof. We prove the proposition by induction. Setting $l = m = n = 0$ in equation (9.31), gives

$$:X^0 Y^0 Z^0: = 1. \tag{9.33}$$

From Definition 9.0.1 of the Wick product, we have the following system of equations

$$\begin{aligned}
 \frac{\partial}{\partial X} :X^l Y^m Z^n: &= l :X^{l-1} Y^m Z^n:, \\
 \frac{\partial}{\partial Y} :X^l Y^m Z^n: &= m :X^l Y^{m-1} Z^n:, \\
 \frac{\partial}{\partial Z} :X^l Y^m Z^n: &= n :X^l Y^m Z^{n-1}:.
 \end{aligned} \tag{9.34}$$

Upon integration, we obtain

$$\begin{aligned}
: X^l Y^m Z^n : &= l \int : X^{l-1} Y^m Z^n : dX + \xi_1(Y, Z), \\
: X^l Y^m Z^n : &= m \int : X^l Y^{m-1} Z^n : dY + \xi_2(X, Z), \\
: X^l Y^m Z^n : &= n \int : X^l Y^m Z^{n-1} : dZ + \xi_3(X, Y).
\end{aligned} \tag{9.35}$$

Assuming that the proposition holds for $: X^{l-1} Y^m Z^n :$, $: X^l Y^{m-1} Z^n :$ and $: X^l Y^m Z^{n-1} :$ and integrating, we can then write the expressions in (9.20) in the following way

$$\begin{aligned}
: X^l Y^m Z^n : &= l \sum_{j_1=0}^{l-1} \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l-1}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} \frac{X^{l-j_1} Y^{m-j_2} Z^{n-j_3}}{l-j_1} \\
&\quad + \xi_1(Y, Z), \\
: X^l Y^m Z^n : &= m \sum_{j_1=0}^l \sum_{j_2=0}^{m-1} \sum_{j_3=0}^n \binom{l}{j_1} \binom{m-1}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} \frac{X^{l-j_1} Y^{m-j_2} Z^{n-j_3}}{m-j_2} \\
&\quad + \xi_2(X, Z), \\
: X^l Y^m Z^n : &= n \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^{n-1} \binom{l}{j_1} \binom{m}{j_2} \binom{n-1}{j_3} c_{j_1, j_2, j_3} \frac{X^{l-j_1} Y^{m-j_2} Z^{n-j_3}}{n-j_3} \\
&\quad + \xi_3(X, Y).
\end{aligned} \tag{9.36}$$

By shifting the summation variable, we can express these expressions alternatively as

$$\begin{aligned}
: X^l Y^m Z^n : &= \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} X^{l-j_1} Y^{m-j_2} Z^{n-j_3} \\
&\quad - \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{m}{j_2} \binom{n}{j_3} c_{l, j_2, j_3} Y^{m-j_2} Z^{n-j_3} + \xi_1(Y, Z), \\
: X^l Y^m Z^n : &= \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} X^{l-j_1} Y^{m-j_2} Z^{n-j_3} \\
&\quad - \sum_{j_1=0}^l \sum_{j_3=0}^n \binom{l}{j_1} \binom{n}{j_3} c_{j_1, m, j_3} X^{l-j_1} Z^{n-j_3} + \xi_2(X, Z),
\end{aligned}$$

$$\begin{aligned}
: X^l Y^m Z^n : &= \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} X^{l-j_1} Y^{m-j_2} Z^{n-j_3} \\
&\quad - \sum_{j_1=0}^l \sum_{j_2=0}^m \binom{l}{j_1} \binom{m}{j_2} c_{j_1, j_2, n} X^{l-j_1} Y^{m-j_2} + \xi_2(X, Y). \quad (9.37)
\end{aligned}$$

The equations in (9.37) agree if

$$\begin{aligned}
\xi_1(Y, Z) - \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{m}{j_2} \binom{n}{j_3} c_{l, j_2, j_3} Y^{m-j_2} Z^{n-j_3} &= c, \\
\xi_2(X, Z) - \sum_{j_1=0}^l \sum_{j_3=0}^n \binom{l}{j_1} \binom{n}{j_3} c_{j_1, m, j_3} X^{l-j_1} Z^{n-j_3} &= c, \\
\xi_2(X, Y) - \sum_{j_1=0}^l \sum_{j_2=0}^m \binom{l}{j_1} \binom{m}{j_2} c_{j_1, j_2, n} X^{l-j_1} Y^{m-j_2} &= c. \quad (9.38)
\end{aligned}$$

Therefore, the equations in (9.37) become

$$: X^l Y^m Z^n : = \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} X^{l-j_1} Y^{m-j_2} Z^{n-j_3} + c. \quad (9.39)$$

We can determine the constant c by taking the expectation of the equation in (9.39) and using condition (c) of Definition 9.0.1. This gives us

$$c = - \sum_{j_1=0}^l \sum_{j_2=0}^m \sum_{j_3=0}^n \binom{l}{j_1} \binom{m}{j_2} \binom{n}{j_3} c_{j_1, j_2, j_3} \mathbb{E} [X^{l-j_1} Y^{m-j_2} Z^{n-j_3}]. \quad (9.40)$$

But by (9.32) the right hand side of (9.40) is equal to zero. This completes our proof. $\square$

We are now ready to prove Proposition 9.1.1 for a general Wick product of k random variables.

Proof of Proposition 9.1.1. We prove the proposition by induction. For $n_1 = n_2 = \dots = n_k = 0$, we have by definition that $: X_1^0 X_2^0 \dots X_k^0 : = 1$. By (9.2) we have for each i ,

$$\frac{\partial}{\partial X_m} : X_1^{n_1} \dots X_m^{n_m+1} \dots X_k^{n_k} : = n_m : X_1^{n_1} \dots X_m^{n_m} \dots X_k^{n_k} :. \quad (9.41)$$

Integrating (9.41), we obtain k equations of the form

$$\begin{aligned} :X_1^{n_1} \dots X_m^{n_m} \dots X_k^{n_k}: &= n_m \int :X_1^{n_1} \dots X_i^{n_m-1} \dots X_k^{n_k}: dX_m \\ &+ \xi_m(X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_k). \end{aligned} \quad (9.42)$$

Now assuming that the proposition holds for $n_1, \dots, n_m - 1, \dots, n_k$. Then we can write (9.42) as

$$\begin{aligned} &:X_1^{n_1} \dots X_m^{n_m} \dots X_k^{n_k}: \\ &= n_m \sum_{j_1=0}^{n_1} \dots \sum_{j_m=0}^{n_m-1} \dots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \dots \binom{n_m-1}{j_m} \dots \binom{n_k}{j_k} \\ &\quad \times c_{j_1, \dots, j_k} X_1^{n_1-j_1} \dots X_{m-1}^{n_{m-1}-j_{m-1}} X_{m+1}^{n_{m+1}-j_{m+1}} \dots X_k^{n_k-j_k} \\ &\quad \times \int X_m^{n_m-j_m-1} dX_m + \xi_m(X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_k). \end{aligned} \quad (9.43)$$

Integrating and writing the n_m^{th} sum to n_m , we obtain

$$\begin{aligned} &:X_1^{n_1} \dots X_m^{n_m} \dots X_k^{n_k}: \\ &= \sum_{j_1=0}^{n_1} \dots \sum_{j_m=0}^{n_m} \dots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \dots \binom{n_m}{j_m} \dots \binom{n_k}{j_k} c_{j_1, \dots, j_k} \\ &\quad \times X_1^{n_1-j_1} \dots X_{m-1}^{n_{m-1}-j_{m-1}} X_m^{n_m-j_m} X_{m+1}^{n_{m+1}-j_{m+1}} \dots X_k^{n_k-j_k} \\ &\quad - \sum_{j_1=0}^{n_1} \dots \sum_{j_{m-1}=0}^{n_{m-1}} \sum_{j_{m+1}=0}^{n_{m+1}} \dots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \dots \binom{n_{m-1}}{j_{m-1}} \binom{n_{m+1}}{j_{m+1}} \dots \binom{n_k}{j_k} \\ &\quad \times c_{j_1, \dots, j_{m-1}, n_m, j_{m+1}, \dots, j_k} X_1^{n_1-j_1} \dots X_{m-1}^{n_{m-1}-j_{m-1}} X_{m+1}^{n_{m+1}-j_{m+1}} \dots X_k^{n_k-j_k} \\ &\quad + \xi_m(X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_k). \end{aligned} \quad (9.44)$$

In order for all k equations to agree, we require that

$$\begin{aligned} c &= \xi_m(X_1, \dots, X_{m-1}, X_{m+1}, \dots, X_k) \\ &\quad - \sum_{j_1=0}^{n_1} \dots \sum_{j_{m-1}=0}^{n_{m-1}} \sum_{j_{m+1}=0}^{n_{m+1}} \dots \sum_{j_k=0}^{n_k} \binom{n_1}{j_1} \dots \binom{n_{m-1}}{j_{m-1}} \binom{n_{m+1}}{j_{m+1}} \dots \binom{n_k}{j_k} \\ &\quad \times c_{j_1, \dots, j_{m-1}, n_m+1, j_{m+1}, \dots, j_k} X_1^{n_1-j_1} \dots X_{m-1}^{n_{m-1}-j_{m-1}} X_{m+1}^{n_{m+1}-j_{m+1}} \dots X_k^{n_k-j_k}. \end{aligned} \quad (9.45)$$

Substituting (9.45) into (9.44) and then taking the expectation of (9.44) to find c , we find by equation (9.15) that we obtain the relation in equation (9.16). $\square$

Although the proof is somewhat intricate, the resulting expression proves to provide a quicker way of calculating the Wick products than the usual method described at the beginning of Chapter 9.

We proceed by investigating what happens to the Wick product when the random variables considered are mutually independent. We begin by observing that the Wick product of X and Y can also be expressed in the form

$$:XY: = :X: :Y: - \text{Cov}(X, Y), \quad (9.46)$$

and for X, Y independent, this simplifies to

$$:XY: = :X: :Y:. \quad (9.47)$$

More generally, when X and Y are independent, the Wick product of the two random variables factors in to the product of their Wick powers.

Proposition 9.1.4. *Let X and Y be independent random variables with finite moments. Then*

$$:X^m Y^n: = :X^m: :Y^n:. \quad (9.48)$$

Proof. From Proposition 9.1.2, we can write the Wick product as follows:

$$:X^m Y^n: = \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1, j_2} X^{m-j_1} Y^{n-j_2}. \quad (9.49)$$

When X and Y are independent, the expectation in equation (9.17) factors into a product

of expectations and so we can write

$$:X^m Y^n: = \sum_{j_1=0}^m \sum_{j_2=0}^n \binom{m}{j_1} \binom{n}{j_2} c_{j_1} c_{j_2} X^{m-j_1} Y^{n-j_2} = :X^m: :Y^n:. \quad (9.50)$$

□

Remark 9.1.2 The result in the previous proposition can be extended to k independent variables. In other words, we can write:

$$:\prod_{i=1}^k X_i^{n_i}: = \prod_{i=1}^k :X_i^{n_i}:. \quad (9.51)$$

See Avram & Taqqu (1987) for a proof of this “factorisation lemma”.

A still more general result than Proposition (9.1.4) is:

Proposition 9.1.5. *Let X and Y be independent random variables with finite moments. Then for analytic functions $f(X) = \sum_{n=0}^{\infty} \alpha_n X^n$, $g(Y) = \sum_{m=0}^{\infty} \beta_m Y^m$, we have*

$$:f(X)g(Y): = :f(X): :g(Y):. \quad (9.52)$$

Proof. Writing functions f and g in terms of formal power series,

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n, \quad g(y) = \sum_{m=0}^{\infty} \beta_m y^m, \quad (9.53)$$

we can take the Wick order of these functions to obtain

$$:f(X): = \sum_{n=0}^{\infty} \alpha_n :X^n:, \quad :g(Y): = \sum_{m=0}^{\infty} \beta_m :Y^m:. \quad (9.54)$$

On the other hand, multiplying f and g together and then taking the Wick order, we find

$$\begin{aligned}
 :f(X)g(Y): &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_n \beta_m :X^n Y^m: \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_n \beta_m :X^n : :Y^m: \\
 &= :f(X): :g(Y):,
 \end{aligned} \tag{9.55}$$

where the second line is due to the result in Proposition (9.1.4). $\square$

Remark 9.1.3 Proposition 9.1.5 can also be extended to k variables so that if we have independent random variables $X_i, i = 1, \dots, k$, then it holds true that

$$: \prod_{i=1}^N f_i(X_i) : = \prod_{i=1}^N : f_i(X_i) : . \tag{9.56}$$

We can see this by expanding the functions f_i in terms of power series.

9.2 Treatment of the Wick order

Several authors have commented on the lax definition of the Wick order (more specifically, the Wick power and the Wick product). Simon (1974), for instance, warns that not all algebraic properties are preserved by the Wick product and gives the following example: If $Y = X^{-1}$ and $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, then $0 = :XY:|_{Y=X^{-1}} \neq :XX^{-1}: = :X^0: = 1$. There are two issues here. The first is that there are two different methods being used to evaluate the Wick product which correspond to different parts of the definition. By taking $:XX^{-1}:$ and evaluating the quantity before taking the Wick order, we obtain a Wick power of order zero which corresponds to the first part of the definitions of the Wick power or Wick product, namely equations (8.3) or (9.1). On the other hand, with $:XY:|_{Y=X^{-1}}$, one calculates the Wick product of two random variables and thereafter makes the substitution. In order to arrive at the Wick product of two random variables, one uses the second part of the definition which holds for powers greater than zero. Consequently, if these two methods gave the same quantity, this would contradict the definitions of both the Wick power and

the Wick product. The second issue is exactly the fact that finding the expression of a Wick product and then performing a substitution is in general not permissible. This is due to the fact that in the polynomial expansion of the Wick power and Wick product the coefficients are moment dependent and so they will change with the number of variables.

An analogous point has also been raised in Gjessing *et al.* (1993) who point out that $:XY:|_{Y=X^2} \neq :X^3:$. The apparent contradiction arises because $:XY:$ consists of two random variables and so uses the Wick product of two random variables, whereas $:X^3:$ uses the Wick product of three random variables. Therefore, as stated above, substitutions of this type in general do not work. An exception, on the other hand, given by Møller (2013), shows that expressing the Wick product $:X_1^{k_1} \dots X_n^{k_n}:$ as another polynomial of a different set of random variables $Y_1 \dots Y_n$ does not give rise to a different Wick ordered random variable provided that the new random variables Y_j are linear combinations of the X_i 's. To make this point clearer, we state the following:

$$:XY:|_{Y=f(X)} \neq :Xf(X): \quad (9.57)$$

for a nonlinear function f .

Gjessing *et al.* (1993), also argue that the Wick product is not useful due to its lack of algebraic properties. In their argument, they first treat the number 1 as a random variable and show $:X1: \neq :X:$ by using the Wick product of two random variables, and thereafter perform a substitution to obtain an expression for $:X1:$. Condition (d) of our definition for the Wick power, together with the commutativity of the Wick product, implies that since 1 is a constant it should be taken out of the Wick order, i.e. $:X1: = 1 :X: = :X:$. In general, constants within any Wick order cannot be treated in the same way as random variables. Conversely, if they were to be treated the same way, (8.3) and (9.1) would be redundant.

All these observations are reminders that the Wick order should be treated with care, and substitutions should in general occur before taking the Wick order or else it leads to apparent contradictions.

9.3 Properties of the Wick product

In what follows we shall show that the Wick order is both commutative and distributive, but fails to satisfy a particular form of associativity condition, according to our formulation.

Proposition 9.3.1. *Let X, Y, Z be random variables such that their joint moments are finite. Then we have the following properties satisfied by the Wick product:*

1. *The product is commutative:*

$$:XY: = :YX: \quad (9.58)$$

2. *The product is not associative in the following sense:*

$$::XY:Z: \neq :X:YZ: \quad (9.59)$$

3. *The product is distributive:*

$$:X(Y + Z): = :XY: + :XZ: \quad (9.60)$$

Proof. The proof of these statements goes as follows.

1. For the commutativity, we have,

$$\begin{aligned} :XY: &= XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] - \mathbb{E}[XY] + 2\mathbb{E}[X]\mathbb{E}[Y] \\ &= YX - Y\mathbb{E}[X] - X\mathbb{E}[Y] - \mathbb{E}[YX] + 2\mathbb{E}[Y]\mathbb{E}[X] \\ &= :YX: \end{aligned} \quad (9.61)$$

2. As for the associativity, we observe that the left hand side of (9.59), when expanded gives

$$::XY:Z: = :XYZ: - \mathbb{E}[Y]:XZ: - \mathbb{E}[X]:YZ: + (2\mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[XY]):Z: \quad (9.62)$$

and similarly, we have

$$:X:YZ: = :XYZ: - \mathbb{E}[Z]:XY: - \mathbb{E}[Y]:XZ: + (2\mathbb{E}[Y]\mathbb{E}[Z] - \mathbb{E}[YZ]):X: \quad (9.63)$$

for the right hand side.

We provide a counterexample. Let $X \sim N(0, 1)$, $Y \sim N(0, s)$ and $Z \sim N(0, t)$, then (9.62) simplifies to

$$:XY:Z: = :XYZ: - \mathbb{E}[XY]Z \quad (9.64)$$

and similarly, (9.68) becomes

$$:X:YZ: = :XYZ: - \mathbb{E}[YZ]X. \quad (9.65)$$

The Wick product therefore fails to be associative since (9.64) and (9.65) cannot be equal.

3. Since $Y + Z$ is linear, we can use the work of Møller (2013) mentioned above, together with equation (9.27) for the Wick product of two random variables. In particular, regarding the first variable as X and the second as $Y + Z$, we can write

$$\begin{aligned} & :X(Y + Z): \\ &= X(Y + Z) - X\mathbb{E}[Y + Z] - (Y + Z)\mathbb{E}[X] + 2\mathbb{E}[X]\mathbb{E}[Y + Z] - \mathbb{E}[X(Y + Z)] \\ &= XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] - \mathbb{E}[XY] + 2\mathbb{E}[X]\mathbb{E}[Y] + XZ - X\mathbb{E}[Z] \\ &\quad - Z\mathbb{E}[X] - \mathbb{E}[XZ] + 2\mathbb{E}[X]\mathbb{E}[Z] \\ &= :XY: + :XZ:, \end{aligned} \quad (9.66)$$

which establishes the claim. $\square$

Before we conclude this chapter on the Wick product, let us briefly comment on the effect of the application of iterated Wick ordering on a product of random variables.

9.4 Multiple Wick products

In Section 8.6, we introduced the idea of taking repeated Wick orders of Wick powers. With the same line of thought, we take Wick orders of Wick products. The result can be summarised as follows.

Proposition 9.4.1. *Let $X_1, \dots, X_k$ be random variables with finite joint moments. Taking the Wick order of $X_1^{n_1}, \dots, X_k^{n_k}$ repeatedly, m times, we can express the multiple Wick product as follows:*

$$q_0(X_1^{n_1} \dots X_k^{n_k}) = X_1^{n_1} \dots X_k^{n_k} \quad (9.67)$$

for $m = 0$, and for $m \geq 1$,

$$\begin{aligned} q_m(X_1^{n_1} \dots X_k^{n_k}) &= \sum_{j_{1,1}=0}^{n_1} \dots \sum_{j_{1,k}=0}^{n_k} \binom{n_1}{j_{1,1}} \dots \binom{n_k}{j_{1,k}} c_{n_1-j_{1,1}, \dots, n_k-j_{1,k}} \times \\ &\quad \sum_{j_{2,1}=0}^{j_{1,1}} \dots \sum_{j_{2,k}=0}^{j_{1,k}} \binom{j_{1,1}}{j_{2,1}} \dots \binom{j_{1,k}}{j_{2,k}} c_{j_{1,1}-j_{2,1}, \dots, j_{1,k}-j_{2,k}} \times \dots \\ &\quad \times \sum_{j_{m,1}=0}^{j_{m-1,1}} \dots \sum_{j_{m,k}=0}^{j_{m-1,k}} \binom{j_{m-1,1}}{j_{m,1}} \dots \binom{j_{m-1,k}}{j_{m,k}} \times \\ &\quad c_{j_{m-1,1}-j_{m,1}, \dots, j_{m-1,k}-j_{m,k}} X_1^{j_{m,1}} \dots X_k^{j_{m,k}}. \end{aligned} \quad (9.68)$$

The notation being used here is analogous to that being used in Section 8.6 so that, for example, $q_1(X_1^{n_1} \dots X_k^{n_k}) = :X_1^{n_1} \dots X_k^{n_k}:$, $q_2(X_1^{n_1} \dots X_k^{n_k}) = (:X_1^{n_1} \dots X_k^{n_k}:)$, and so on.

Proof. We prove the proposition by induction. Setting $m = 1$, then from equation (9.68),

we find that

$$q_1(X_1^{n_1} \cdots X_k^{n_k}) = \sum_{j_{1,1}=0}^{n_1} \cdots \sum_{j_{1,k}=0}^{n_k} \binom{n_1}{j_{1,1}} \cdots \binom{n_k}{j_{1,k}} c_{n_1-j_{1,1}, \dots, n_k-j_{1,k}} \times X_1^{j_{m,1}} \cdots X_k^{j_{m,k}}. \quad (9.69)$$

Assuming that the proposition holds for m , by taking the Wick order of $q_m(X_1^{n_1} \cdots X_k^{n_k})$ we obtain

$$\begin{aligned} q_m(X_1^{n_1} \cdots X_k^{n_k}) &= \sum_{j_{1,1}=0}^{n_1} \cdots \sum_{j_{1,k}=0}^{n_k} \binom{n_1}{j_{1,1}} \cdots \binom{n_k}{j_{1,k}} c_{n_1-j_{1,1}, \dots, n_k-j_{1,k}} \times \\ &\quad \sum_{j_{2,1}=0}^{j_{1,1}} \cdots \sum_{j_{2,k}=0}^{j_{1,k}} \binom{j_{1,1}}{j_{2,1}} \cdots \binom{j_{1,k}}{j_{2,k}} c_{j_{1,1}-j_{2,1}, \dots, j_{1,k}-j_{2,k}} \times \cdots \\ &\quad \times \sum_{j_{m,1}=0}^{j_{m-1,1}} \cdots \sum_{j_{m,k}=0}^{j_{m-1,k}} \binom{j_{m-1,1}}{j_{m,1}} \cdots \binom{j_{m-1,k}}{j_{m,k}} \times \\ &\quad c_{j_{m-1,1}-j_{m,1}, \dots, j_{m-1,k}-j_{m,k}} : X_1^{j_{m,1}} \cdots X_k^{j_{m,k}} : \\ &= \sum_{j_{1,1}=0}^{n_1} \cdots \sum_{j_{1,k}=0}^{n_k} \binom{n_1}{j_{1,1}} \cdots \binom{n_k}{j_{1,k}} c_{n_1-j_{1,1}, \dots, n_k-j_{1,k}} \times \\ &\quad \sum_{j_{2,1}=0}^{j_{1,1}} \cdots \sum_{j_{2,k}=0}^{j_{1,k}} \binom{j_{1,1}}{j_{2,1}} \cdots \binom{j_{1,k}}{j_{2,k}} c_{j_{1,1}-j_{2,1}, \dots, j_{1,k}-j_{2,k}} \times \cdots \\ &\quad \times \sum_{j_{m,1}=0}^{j_{m-1,1}} \cdots \sum_{j_{m,k}=0}^{j_{m-1,k}} \binom{j_{m-1,1}}{j_{m,1}} \cdots \binom{j_{m-1,k}}{j_{m,k}} \\ &\quad \times \sum_{j_{m+1,1}=0}^{j_{m,1}} \cdots \sum_{j_{m+1,k}=0}^{j_{m,k}} \binom{j_{m,1}}{j_{m+1,1}} \cdots \binom{j_{m,k}}{j_{m+1,k}} \\ &\quad \times c_{j_{m,1}-j_{m+1,1}, \dots, j_{m,k}-j_{m+1,k}} X_1^{j_{m+1,1}} \cdots X_k^{j_{m+1,k}}. \end{aligned} \quad (9.70)$$

□

The expression in equation (9.68) appears somewhat cluttered, however, if the number

m and the number k of the random variables involved are relatively small, then in fact a rather simple expression emerges, as shown in the example below.

Example 9.4.1 For $m = 2$ and two random variables, we find

$$\begin{aligned}
 q_2(XY) &= \sum_{j_1=0}^1 \sum_{j_2=0}^1 \binom{1}{j_1} \binom{1}{j_2} c_{1-j_1, 1-j_2} :X^{j_1} Y^{j_2}: \\
 &= c_{1,1} + c_{1,0} :Y: + c_{0,1} :X: + c_{0,0} :XY: \\
 &= XY - 2Y\mathbb{E}[X] - 2X\mathbb{E}[Y] - 2\mathbb{E}[XY] + 6\mathbb{E}[X]\mathbb{E}[Y]. \quad (9.71)
 \end{aligned}$$

9.5 Summary

This chapter began with a version of the definition of the Wick product as defined by Simon (1974). We then introduced a polynomial representation for the Wick product (a multivariable version of the one found in Section 8.3). We went on to show some results for independent random variables, some of which are already present in the literature (see Remark 9.1.2). After commenting on the treatment of the Wick order and going through some properties, we introduced the idea of multiple Wick products (the multivariable version of multiple Wick powers introduced in Section 8.6).

Chapter 10

The Wick order of stochastic processes

10.1 The Wick power and stochastic processes

In the previous two chapters we have developed, in some detail, the algebraic operations of ‘fixed’ random variables under the rules of the Wick calculus. The purpose of this section is to extend the results of Chapters 8 and 9 by replacing the underlying random variables with a parametric family of random variables, i.e. stochastic processes, with the view towards potential financial applications.

We begin by noting that all stochastic processes considered in what follows are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an augmented filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ generated by the process $\{X_t\}_{t \geq 0}$ under consideration. As before, the results that follow require the existence of the moments of the process: $\mathbb{E}[|X_t^m|] < \infty, \forall m \in \mathbb{Z}_+$ and $\forall t \geq 0$. That is, the processes are assumed to have finite moments for each time t . Before we define the Wick power of a stochastic process, we define the formal partial derivatives of a formal power series in two variables as follows:

$$\frac{\partial}{\partial t} \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{mn} t^m x^n \right) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \gamma_{mn} m t^{m-1} x^n, \quad (10.1)$$

and

$$\frac{\partial}{\partial x} \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{mn} t^m x^n \right) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \gamma_{mn} n t^m x^{n-1}. \quad (10.2)$$

Let $\{P_n\}_{n \geq 1}$ be a sequence of polynomials in x with time-dependent coefficients. We write

$$P_n(t, x) = \sum_{k=0}^n p_k^{(n)}(t) x^k \quad (10.3)$$

for some n , where each $p_k^{(n)}(t)$ is a polynomial in t .

Definition 10.1.1 The *Wick power of a stochastic process* $\{X_t\}$, whose moments are all finite, is a two variable polynomial $P_n(t, X_t) \equiv :X_t^n:$ on $[0, \infty) \times \mathbb{R}$ such that the following axioms hold:

(a) The null power:

$$:X_t^0: = P_0(t, X_t) = 1 \quad (10.4)$$

(b) The differentiation:

$$\frac{\partial}{\partial x} P_n(t, x) = n P_{n-1}(t, x), \quad n = 1, 2, \dots \quad (10.5)$$

(c) The expectation:

$$\mathbb{E}[:X_t^n:] = \mathbb{E}[P_n(t, X_t)] = 0, \quad n = 1, 2, \dots \quad (10.6)$$

(d) The initial condition:

$$:X_0^n: = P_n(0, X_0) = X_0^n, \quad n = 1, 2, \dots \quad (10.7)$$

(e) The ‘monic’ property:

$$p_n^{(n)}(t) = 1 \quad n = 1, 2, \dots \quad (10.8)$$

(f) The linearity:

$$:\alpha X_t^m + \beta X_t^n: = \alpha :X_t^m: + \beta :X_t^n:, \quad n, m = 0, 1, 2, \dots \quad (10.9)$$

for any $\alpha, \beta \in \mathbb{R}$.

Remark 10.1.1 There is an analogous presentation of the axioms in the above definition in Sengupta & Goswami (2008), in the context of time-space harmonic polynomials for continuous-time Lévy processes. A time-space harmonic polynomial for a stochastic process $\{X_t\}$, defined in Sengupta (2000), is a polynomial $P(t, x)$ in x with time-dependent coefficients, such that $\{P(t, X_t)\}$ is a martingale with respect to the natural filtration associated with $\{X_t\}$. It was also discussed in Goswami & Sengupta (1995) where this concept was introduced in a discrete time setting.

Note that since the Wick power depends on the distributional property of the underlying random variable, in the case of a Wick power of a stochastic process, the resulting expression itself will become dynamic, i.e. time-dependent. With this in mind, let us rephrase Proposition 8.2.1 when the random variable X is replaced by a stochastic process $\{X_t\}$.

Proposition 10.1.1. *Let $\{X_t\}$ be a stochastic process with finite moments and let t be a parameter representing time. Then the Wick power $:X_t^n:$, $n = 0, 1, \dots$ of X_t can be written in the following way:*

$$:X_t^n: = P_n(t, X_t) = \sum_{k=0}^n \binom{n}{k} c_k(t) X_t^{n-k}, \quad (10.10)$$

where the coefficients $\{c_k(t)\}$ are determined recursively according to: $c_0(t) = 1$ for all t , and for $k \geq 1$,

$$c_k(t) = \begin{cases} -\sum_{j=0}^{k-1} \binom{k}{j} c_j(t) \mathbb{E}[X_t^{k-j}], & t \neq 0 \\ 0, & t = 0. \end{cases} \quad (10.11)$$

Remark 10.1.2 We remark here that the summation in equation (10.11) can be expressed more concisely as follows:

$$\sum_{j=0}^k \binom{k}{j} c_j(t) \mathbb{E}[X_t^{k-j}] = 0. \quad (10.12)$$

The recursion relation (10.11) can be used to obtain explicit expressions for each n for the Wick power of a process. For example, by setting $n = 1$ in equation (10.10), we obtain $c_0(t) = 1$, $c_1(t) = -\mathbb{E}[X_t]$, and hence

$$:X_t: = X_t - \mathbb{E}[X_t]. \quad (10.13)$$

Analogously, for $n = 2$, we can represent the second Wick power as

$$:X_t^2: = X_t^2 - 2X_t\mathbb{E}[X_t] - \mathbb{E}[X_t^2] + 2(\mathbb{E}[X_t])^2, \quad (10.14)$$

whereas for $n = 3$ we obtain

$$\begin{aligned} :X_t^3: &= X_t^3 - 3X_t^2\mathbb{E}[X_t] - 3X_t\mathbb{E}[X_t^2] + 6X_t(\mathbb{E}[X_t])^2 - \mathbb{E}[X_t^3] \\ &\quad + 6\mathbb{E}[X_t]\mathbb{E}[X_t^2] - 6(\mathbb{E}[X_t])^3. \end{aligned} \quad (10.15)$$

We observe that these expressions for the Wick powers of a stochastic process are analogous to (8.17), (8.18) and (8.14) for random variables.

Proof. We begin by showing that all the axioms of the Wick power definition (Definition 10.1.1) are satisfied.

(a) Setting $n = 0$ in equation (10.10) we obtain

$$:X_t^0: = c_0(t) = 1. \quad (10.16)$$

(b) For $X_t = x$, we take the derivative of the expression in equation (10.10) with respect

to x , and find that

$$\begin{aligned}
 \frac{\partial}{\partial x} P_n(t, x) &= \frac{\partial}{\partial x} \left(\sum_{k=0}^n \binom{n}{k} c_k(t) x^{n-k} \right) \\
 &= n \sum_{k=0}^{n-1} \binom{n-1}{k} c_k(t) x^{n-1-k} \\
 &= n P_{n-1}(t, x).
 \end{aligned} \tag{10.17}$$

(c) Taking the expectation of the expression in equation (10.10), we obtain, by (10.11),

$$\mathbb{E}[:X_t^n:] = \sum_{k=0}^n \binom{n}{k} c_k(t) \mathbb{E}[X_t^{n-k}] = 0, \quad n = 1, 2, \dots, \quad t \neq 0. \tag{10.18}$$

(d) Setting $t = 0$ in (10.10), we find that since $c_k(0) = 0$,

$$:X_0^n: = X_0^n, \quad n = 1, 2, \dots \tag{10.19}$$

(e) Comparing equations (eq:10.2) and (10.10), we observe that

$$p_k^{(n)}(t) = \binom{n}{k} c_{n-k}(t), \tag{10.20}$$

and setting $k = n$, we obtain

$$p_n^{(n)}(t) = c_{n-n}(t) = 1 \quad n = 1, 2, \dots \tag{10.21}$$

We now observe that the the representation for the Wick power is unique, since all coefficients $\{c_k(t)\}$ are fixed by the condition (10.6). But we have just shown that the representation (10.10) fulfills all the conditions. On account of uniqueness, therefore, we have established the claim. $\square$

Example 10.1.1 *Brownian motion.* If $X_t = W_t$, where $\{W_t\}_{t \geq 0}$ is a standard Brownian

motion, then for any fixed time t we have

$$:W_t^2: = :W_t:^2 + (\mathbb{E}[W_t])^2 - \mathbb{E}[W_t^2] = W_t^2 - t, \quad (10.22)$$

which of course is an $\{\mathcal{F}_t\}$ -martingale.

Example 10.1.2 *Gamma process.* If $X_t = \gamma_t$, where $\{\gamma_t\}_{t \geq 0}$ is an $(mt, 1)$ -Gamma process, then for a fixed time t we have

$$\begin{aligned} :\gamma_t^2: &= :\gamma_t:^2 + (\mathbb{E}[\gamma_t])^2 - \mathbb{E}[\gamma_t^2] \\ &= (\gamma_t - mt)^2 + m^2 t^2 - (mt + m^2 t^2) \\ &= \gamma_t^2 - 2mt\gamma_t + m^2 t^2 - mt. \end{aligned} \quad (10.23)$$

Note again that this is an $\{\mathcal{F}_t\}$ -martingale.

Remark 10.1.3 As we will show later in Proposition 10.2.3, the Wick orders in the above examples turn out to be martingales due to their shared property of independent increments.

In what follows, we shall use the notations f_x for the first derivative $\partial f / \partial x$, and f_{xx} for the second derivative $\partial^2 f / \partial x^2$.

Proposition 10.1.2. *Let $f(x)$ be a function for which the partial derivatives $f_x(x)$ and $f_{xx}(x)$ exist and are continuous and let $\{W_t\}$ be a Brownian motion with drift such that $X_t = \sigma W_t + \mu t$, for $\sigma, \mu \in \mathbb{R}$. Then*

$$:f(X_T): - :f(X_t): = \int_t^T :f_x(X_s): dX_s + \frac{1}{2} \sigma^2 \int_t^T f_{xx}(X_s) ds \quad (10.24)$$

Proof. From Ito's lemma, we have

$$f(X_T) - f(X_t) = \int_t^T f_x(X_s) dX_s + \frac{1}{2} \sigma^2 \int_t^T f_{xx}(X_s) ds \quad (10.25)$$

where

$$[X, X]_t = X_t^2 - 2 \int_0^t X_u dX_u \quad (10.26)$$

is the quadratic variation of $\{X_t\}$. Taking the Wick order of the expression in (10.25), we obtain

$$:f(X_T): - :f(X_t): = : \int_t^T f_x(X_s) dX_s : + \frac{1}{2} \sigma^2 : \int_t^T f_{xx}(X_s) d[X, X]_s : . \quad (10.27)$$

Now the first expression on the right hand side of (10.27) can be written as follows:

$$: \int_t^T f_x(X_s) dX_s : = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_x(X_{t_{i-1}}) (X_{t_i} - X_{t_{i-1}}) :, \quad (10.28)$$

where $\Delta = \{t_0, t_1, \dots, t_n\}$ is a partition on $[0, T]$ for $0 = t_0 < t_1 < \dots < t_n = T$ and $\|\Delta\| = \max_i(t_i - t_{i-1})$. Since $f_x(X_{t_{i-1}})$ and $X_{t_i} - X_{t_{i-1}}$ are independent, and since $:XY: = :X: :Y:$ for a pair of independent random variables, we can write

$$\begin{aligned} : \int_t^T f_x(X_s) dX_s : &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_x(X_{t_{i-1}}) : (X_{t_i} - X_{t_{i-1}}) : \\ &= \int_t^T : f_x(X_s) : dX_s . \end{aligned} \quad (10.29)$$

The second integral on the right hand side of (10.27) can be written as follows:

$$\begin{aligned} : \int_t^T f_{xx}(X_s) d[X, X]_s : &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_{xx}(X_{t_{i-1}}) ([X, X]_{t_i} - [X, X]_{t_{i-1}}) : \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_{xx}(X_{t_{i-1}}) : ([X, X]_{t_i} - [X, X]_{t_{i-1}}) : . \end{aligned} \quad (10.30)$$

Taking the Wick order of the expression in equation (10.26) we obtain

$$\begin{aligned} & : \int_t^T f_{xx}(X_s) d[X, X]_s : \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_{xx}(X_{t_{i-1}}) : \left(: W_{t_i}^2 : - 2 \int_0^{t_i} : X_u : d : X_u : - : W_{t_{i-1}}^2 : + 2 \int_0^{t_{i-1}} : X_u : d : X_u : \right). \end{aligned} \quad (10.31)$$

When $X_t = \sigma W_t + \mu t$, we find that

$$:[X, X]_t : = : X_t^2 : - 2 \int_0^t : X_u : d : X_u : = 0, \quad (10.32)$$

and so the expression in (10.31) becomes zero and we are left with the desired result. $\square$

We proceed to consider some examples where $X_t = W_t$, i.e. when $\mu = 0$ and $\sigma = 1$. We first make the observation that

$$: W_t : = W_t - \mathbb{E}[W_t] = W_t, \quad (10.33)$$

and hence $d : W_t : = dW_t$. It follows, on account of Proposition 10.1.2 for $f(W_t) = W_t^{n+1}$, that

$$\int_t^T : W_s^n : dW_s = \frac{1}{n+1} [: W_T^{n+1} : - : W_t^{n+1} :]. \quad (10.34)$$

Example 10.1.3 Setting $n = 1$ in (10.34), we obtain

$$\int_t^T : W_s : dW_s = \frac{1}{2}(W_T^2 - T) - \frac{1}{2}(W_t^2 - t), \quad (10.35)$$

which is in agreement with the Ito integration:

$$\int_0^t W_s dW_s \Big|_{\text{Ito}} = \int_0^t : W_s : dW_s \Big|_{\text{Wick}}. \quad (10.36)$$

We therefore observe that stochastic integration based on the use of Wick calculus reproduces that based on the Ito calculus, albeit that the Wick integration is somewhat simpler

since the familiar rules of differentiation are applicable in the latter case with the difference that all terms are contained within the Wick order.

Remark 10.1.4 We note that the integration used here differs from the similar concept of the Stratonovich integral in the following way. When dealing with the Stratonovich integral, we integrate as we would deterministically as follows:

$$\int_t^T W_s \circ dW_s = \frac{1}{2} (W_T^2 - W_t^2), \quad (10.37)$$

whereas in Wick integration, we have

$$\int_t^T W_s dW_s = \frac{1}{2} (:W_T^2: - :W_t^2:). \quad (10.38)$$

Example 10.1.4 Setting $n = 2$ in (10.34), we obtain

$$\int_t^T :W_s^2: dW_s = \frac{1}{3} [:W_T^3: - :W_t^3:]. \quad (10.39)$$

Expanding the Wick powers on both sides of (10.39), we obtain

$$\int_t^T W_s^2 dW_s = \int_t^T s dW_s + \frac{1}{3} [W_T^3 - 3TW_T - W_t^3 + 3tW_t]. \quad (10.40)$$

For simplicity, if we set $t = 0$ and $T = t$ in (10.40) we obtain

$$\int_0^t W_s^2 dW_s = \frac{1}{3} W_t^3 + \int_0^t s dW_s - tW_t. \quad (10.41)$$

On the other hand, an application of Ito's lemma on the cube of the Brownian motion, W_t^3 , gives the relation

$$\int_0^t W_s^2 dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s ds. \quad (10.42)$$

At first there seems to be a discrepancy between the Wick representation (10.41) and the Ito representation (10.42). However, on closer inspection one finds that if we let $f(t, x) = tx$,

and consider the case where $x = W_t$, then by using Ito's lemma again, we find that

$$tW_t = \int_0^t W_s ds + \int_0^t s dW_s. \quad (10.43)$$

Therefore, although it is not immediately apparent at first sight, (10.41) and (10.42) are equivalent. Note that (10.43) can also be shown using Wick calculus and a proof of this will be presented towards the end of this section.

Example 10.1.5 Let $f(x) = e^{\alpha x}$ and consider the Brownian motion W_t . Then from Proposition 10.1.2 we obtain

$$\int_t^T : e^{\alpha W_s} : dW_s = \frac{1}{\alpha} [: e^{\alpha W_T} : - : e^{\alpha W_t} :]. \quad (10.44)$$

Remark 10.1.5 In his study of orthogonal polynomials, Schoutens (2000) considers the Hermite polynomials with respect to the Brownian motion and obtains an analogous result to the one in Example 10.1.5 with $\alpha = 1$ (but uses a different notation to the one adopted here).

In the following proposition we use the notation $f_s = \partial f / \partial s$.

Proposition 10.1.3. *Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ exist and are continuous and let $\{X_t\}$ be a scaled Brownian motion with drift such that $X_t = \sigma W_t + \mu t$, for $\sigma, \mu \in \mathbb{R}$. Then*

$$: f(T, X_T) : - : f(t, X_t) : = \int_t^T : f_s(s, X_s) : ds + \int_t^T : f_x(s, X_s) : d : X_s : . \quad (10.45)$$

Remark 10.1.6 We remark again on the relationship between the results derived using the

Wick calculus and the Stratonovich integral. Using Stratonovich integration, we have

$$f(T, W_T) - f(t, W_t) = \int_t^T f_s(s, W_s) ds + \int_t^T f_x(s, W_s) \circ dW_s, \quad (10.46)$$

whereas from Proposition 10.1.3 we can write

$$:f(T, W_T): - :f(t, W_t): = \int_t^T :f_s(s, W_s): ds + \int_t^T :f_x(s, W_s): dW_s. \quad (10.47)$$

Proof. From Ito's lemma, we have

$$\begin{aligned} f(T, X_T) - f(t, X_t) &= \int_t^T f_s(s, X_s) ds + \int_t^T f_x(s, X_s) dX_s \\ &\quad + \frac{1}{2} \sigma^2 \int_t^T f_{xx}(s, X_s) d[X, X]_s, \end{aligned} \quad (10.48)$$

where the quadratic variation is as in (10.26). Taking the Wick order of the expression in (10.48), we obtain

$$\begin{aligned} :f(T, X_T): - :f(t, X_t): &= : \int_t^T f_s(s, X_s) ds : + : \int_t^T f_x(s, X_s) dX_s : \\ &\quad + \frac{1}{2} \sigma^2 : \int_t^T f_{xx}(s, X_s) d[X, X]_s :. \end{aligned} \quad (10.49)$$

Now the first expression on the right hand side of (10.49) simplifies to

$$: \int_t^T f_s(s, X_s) ds : = \int_t^T :f_s(s, X_s): ds \quad (10.50)$$

while the middle integral can be written as follows

$$: \int_t^T f_x(s, X_s) dX_s : = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n :f_x(t_{i-1}, X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}):. \quad (10.51)$$

Since $f_x(X_{t_{i-1}})$ and $X_{t_i} - X_{t_{i-1}}$ are independent, we can write

$$\begin{aligned} : \int_t^T f_x(s, X_s) dX_s : &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_x(t_{i-1}, X_{t_{i-1}}) : (X_{t_i} - X_{t_{i-1}}) : \\ &= \int_t^T : f_x(s, X_s) : d : X_s : . \end{aligned} \quad (10.52)$$

The final integral on the right hand side of (10.49) can be written as follows:

$$\begin{aligned} : \int_t^T f_{xx}(s, X_s) d[X, X]_s : &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_{xx}(t_{i-1}, X_{t_{i-1}}) : ([X, X]_{t_i} - [X, X]_{t_{i-1}}) : \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n : f_{xx}(t_{i-1}, X_{t_{i-1}}) : : ([X, X]_{t_i} - [X, X]_{t_{i-1}}) : . \end{aligned} \quad (10.53)$$

As shown in the proof of Proposition 10.1.2, by taking the Wick order of the quadratic variation, equation (10.53) becomes zero. The result follows. $\square$

We proceed with the following result, which is related to asset pricing in the geometric Brownian motion model, and uses Proposition 10.2.4.

Example 10.1.6 Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ exist and are continuous and let $\{W_t\}$ be a Brownian motion. Then by the above proposition, we have

$$d : f(t, W_t) : = : f_t(t, W_t) : dt + : f_x(t, W_t) : dW_t. \quad (10.54)$$

If f takes the form

$$f(t, x) = f(0, W_0) e^{\sigma x + \mu t}, \quad (10.55)$$

then the differentials of f are

$$f_t(t, x) = \mu f(t, x) \quad (10.56)$$

and

$$f_x(t, x) = \sigma f(t, x). \quad (10.57)$$

Substituting these into (10.63), we obtain the expression for the Wick differential

$$d : f(t, W_t) : = \mu : f(t, W_t) : dt + \sigma : f(t, W_t) : dW_t. \quad (10.58)$$

Now, the Wick order of the function f in (10.63) is

$$: f(t, W_t) : = f(0, W_0) : e^{\sigma W_t + \mu t} :, \quad (10.59)$$

and if we set $S(t, W_t) = : f(t, W_t) :$, we find that

$$S(t, W_t) = S(0, W_0) e^{\sigma W_t + \mu t - \frac{1}{2} \sigma^2 t} \quad (10.60)$$

satisfies the stochastic differential equation

$$dS(t, W_t) = \mu S(t, W_t) dt + \sigma S(t, W_t) dW_t. \quad (10.61)$$

This shows that the Wick ordering of $e^{\sigma W_t + \mu t}$ gives rise to the geometric Brownian motion model used for example in the Black-Scholes theory of option pricing. An analogous construction for other Lévy processes then gives the geometric Lévy models for asset pricing, as discussed in the next section.

Remark 10.1.7 We wish to show that we can derive equation (10.41) using the results that we have presented in this thesis. By Proposition 10.1.3 we have for $f(t, x) = tx$ and $x = W_t$, that

$$: tW_t : = \int_0^t s dW_s + \int_0^t W_s ds \quad (10.62)$$

Now the Wick order on the left hand side of (10.62) simplifies to

$$:tW_t: = t :W_t: = tW_t \quad (10.63)$$

where in the first equality the deterministic t is taken out of the Wick order. We can see that in general, a function $\alpha(t)$ can be factored out of the Wick ordering in the same way as a constant since the expectation operator in (10.11) treats them in the same way. In the second equality of equation (10.63), we have used the fact that $:W_t: = W_t$. Therefore, (10.62) can be written as follows:

$$\int_t^T s dW_s = TW_T - tW_t - \int_t^T W_s ds. \quad (10.64)$$

Substituting the integral in (10.40) with the expression in (10.64) we arrive at the same expression as we would have obtained via Ito's lemma. Needless to say, the result is in agreement with (10.41) obtained in the example earlier.

Example 10.1.7 Let $f(t, x) = tx + t^2x^2$. Then for $x = W_t$ we find

$$:tW_t + t^2W_t^2: = \int_0^t :W_s + 2sW_s^2: ds + \int_0^t :s + 2W_s s^2: dW_s. \quad (10.65)$$

Expanding the Wick orders and rearranging, we find that we recover the same result as via Ito's lemma. Namely,

$$tW_t + t^2W_t^2 = \frac{t^3}{3} + tW_t + 2 \int_0^t sW_s^2 ds + 2 \int_0^t s^2W_s ds. \quad (10.66)$$

10.2 The Wick power and Lévy processes

In this section we look at how Lévy processes behave under the Wick order. We will show that for a Lévy process $\{L_t\}$, its Wick power $:L_t^n:$ is a martingale. Thus the Wick power of a Lévy process falls in the class of time-space harmonic polynomials. Let us begin with a brief reminder of the key properties of a Lévy process, before we proceed to show that the Wick exponential of a Lévy process is also a martingale. We then derive the Wick

counterpart of the Ito-Lévy formula.

Definition 10.2.1 A stochastic process $\{L_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a *Lévy process* if it satisfies the following properties:

1. Initial value: $\mathbb{P}(L_0 = 0) = 1$. i.e. $L_0 = 0$ almost surely.
2. Independent increments: For any $0 \leq t_1 \leq t_2 \leq \dots \leq t_n < \infty$, $L_{t_2} - L_{t_1}, L_{t_3} - L_{t_2}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent.
3. Stationary increments: $L_t - L_s$ is equal in distribution to L_{t-s} , for any $0 \leq s \leq t$.
4. Stochastic continuity: For all $\varepsilon > 0$ and $t \geq 0$ it holds that

$$\lim_{h \rightarrow 0} \mathbb{P}(|L_{t+h} - L_t| > \varepsilon) = 0. \quad (10.67)$$

That is, the path of $\{L_t\}$ is $\mathbb{P}$ -a.s. right continuous with left limits.

To be able to apply the rules of calculus on stochastic processes with various applications, we need to extend the definition of the Wick power to include differentiation with respect to the time variable, since in many applications various quantities of interest depend not only on the value of the process but also on time. A version of the following result can also be found in Sengupta (2000), in the context of time-space harmonic polynomials.

Proposition 10.2.1. *Let $\{L_t\}$ be a Lévy process with finite moments satisfying*

$$\mathbb{E}[e^{\alpha L_t}] < \infty, \quad \forall t \geq 0, \quad (10.68)$$

for some $\alpha \in A$ where A is a (connected) interval containing the origin. Let $P_n(t, x)$, $n = 0, 1, \dots$ be a sequence of polynomials in t and in x such that $L_t^n := P_n(t, L_t)$. Then there exists a sequence $\{\psi_n\}_{n \geq 1}$ of real numbers such that P_n satisfies the following differential equation

$$\frac{\partial}{\partial t} P_n(t, x) = \sum_{k=0}^n \binom{n}{k} \psi_{n-k} P_k(t, x), \quad n = 1, 2, \dots \quad (10.69)$$

for each n .

Proof. Due to the stationary and independent increment properties of Lévy processes, there exists some function $\psi(\alpha)$, called the Laplace exponent, such that the moment generating function of L_t takes the following form:

$$\mathbb{E}[e^{\alpha L_t}] = e^{t\psi(\alpha)}, \quad \text{where} \quad \psi(\alpha) = \frac{1}{t} \ln \mathbb{E}[e^{\alpha L_t}]. \quad (10.70)$$

That is to say, it is a key property of a Lévy process that the function $\psi(\alpha)$ is independent of t . The Wick exponential can then be written in the form

$$:e^{\alpha L_t}: = e^{\alpha L_t - t\psi(\alpha)}. \quad (10.71)$$

By differentiating the explicit time-dependent part of the expression

$$e^{\alpha L_t - t\psi(\alpha)} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :L_t^n: \quad (10.72)$$

with respect to t (that is, by regarding the right side of (10.72) as a function of t and L_t , and by differentiating this function with respect to t), we obtain

$$-\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sum_{k=0}^n \binom{n}{k} \psi_{n-k} :L_t^k: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(\frac{\partial}{\partial t} :L_t^n: \right), \quad (10.73)$$

where

$$\psi(\alpha) = -\sum_{j=1}^{\infty} \frac{\psi_j}{j!} \alpha^j \quad (10.74)$$

and where we have used the following short-hand notation

$$\frac{\partial}{\partial t} :L_t^n: \equiv \frac{\partial}{\partial t} P_n(t, x) \Big|_{x=L_t}. \quad (10.75)$$

By equating the coefficients of α_n in equation (10.73), we arrive at the desired result. $\square$

We find that the Wick exponential for a Lévy process, defined as in (10.72), is a mar-

tingale with respect to the natural filtration associated with L_t .

Remark 10.2.1 In the first part of the thesis we observed that the theory of Hermite polynomials plays an important role in the characterisation of properties of Brownian motions. It is by now well known, more generally, that there is a close relation between orthogonal polynomials and Lévy processes. The reason, in a nutshell, is simple: Orthogonal polynomials are defined with respect to a measure. Therefore, in particular, if, for example, a Lévy process $\{L_t\}$ admits a density function $l_t(x)$, and if we can find a set of polynomials $\{P_n(x)\}$ orthogonal with respect to $l_t(x)$, then we have the relation $\mathbb{E}[P_n(L_t)P_m(L_t)] \propto \delta_{nm}$. Note that such a set of polynomials need not exist for a given measure, but for a number of standard Lévy processes it does exist, for example, a version of the associated Laguerre polynomials are orthogonal with respect to a Gamma process (see Schoutens, 2000). Furthermore, various martingales can be constructed by use of these polynomials via the generating-function technique.

Proposition 10.2.2. *Let $\{L_t\}_{t \geq 0}$ be a Lévy process such that for a fixed time t , L_t has finite moments. Then $:e^{\alpha L_t}: is a unit-initialised $(\mathcal{F}_t, \mathbb{P})$ -martingale.$*

Proof. By the definition of the Wick exponential, we have

$$\mathbb{E} [:e^{\alpha L_t} :] = \mathbb{E} \left[\left| \frac{e^{\alpha L_t}}{\mathbb{E} [e^{\alpha L_t}]} \right| \right] = \mathbb{E} \left[\frac{e^{\alpha L_t}}{\mathbb{E} [e^{\alpha L_t}]} \right] = 1 < \infty, \quad (10.76)$$

which satisfies the first part of the martingale definition. The next part of the proof is to show that $\mathbb{E}_s [:e^{\alpha L_t} :] = :e^{\alpha L_s} :$. This can be seen as follows:

$$\begin{aligned} \mathbb{E}_s [:e^{\alpha L_t} :] &= \mathbb{E}_s \left[\frac{e^{\alpha L_t}}{\mathbb{E} [e^{\alpha L_t}]} \right] \\ &= \frac{1}{\mathbb{E} [e^{\alpha L_t}]} \mathbb{E}_s [e^{\alpha L_t}] \\ &= \frac{e^{\alpha L_s} \mathbb{E}_s [e^{\alpha(L_t - L_s)}]}{\mathbb{E} [e^{\alpha(L_t - L_s)}] \mathbb{E} [e^{\alpha L_s}]}, \end{aligned} \quad (10.77)$$

where in the last equality we have used the independent increments property of Lévy processes. Finally, due to the independence and stationarity of the increments of the Lévy

process, we can write

$$\begin{aligned}
 \mathbb{E}_s [:e^{\alpha L_t} :] &= \frac{e^{\alpha L_s} \mathbb{E} [e^{\alpha(L_t - L_s)}]}{\mathbb{E} [e^{\alpha(L_t - L_s)}] \mathbb{E} [e^{\alpha L_s}]} \\
 &= \frac{e^{\alpha L_s}}{\mathbb{E} [e^{\alpha L_s}]} \\
 &= :e^{\alpha L_s} : .
 \end{aligned} \tag{10.78}$$

□

Remark 10.2.2 We note that the Wick exponential of a Lévy process is nothing but the Esscher martingale associated with that Lévy process. This is just the dynamic analogue of the Esscher transform we saw earlier in (8.62). For the reader familiar with the theory of Esscher transforms (see, e.g., Gerber & Shiu 1994), the result in the proposition above is in fact obvious. It follows also that the Wick exponential of a Lévy process can be used for the purpose of measure change—this will be examined in the chapter that follows.

The following proposition shows that the Wick power of a Lévy process is also a martingale.

Proposition 10.2.3. *Let $\{L_t\}_{t \geq 0}$ be a Lévy process with finite moments. Then $:L_t^n:$ for each value of $n = 0, 1, 2, \dots$ is an $(\mathcal{F}_t, \mathbb{P})$ -martingale.*

Proof. Analogous to (8.59), which holds for random variables, $:e^{\alpha L_t}:$ can be expanded in the form

$$:e^{\alpha L_t}: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :L_t^n: . \tag{10.79}$$

Since the definition of the Wick power is for stochastic processes with finite moments, it follows that

$$\mathbb{E}[|:L_t^n:|] < \infty. \tag{10.80}$$

Now, taking the conditional expectation of the equation above, the left side becomes

$$\mathbb{E}_s[:e^{\alpha L_t}:] = :e^{\alpha L_s}: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :L_s^n:, \quad (10.81)$$

since $:e^{\alpha L_t}:$ is a martingale (from the previous proposition). Equating this to the conditional expectation of the right side, we obtain

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} :L_s^n: = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \mathbb{E}_s[:L_t^n:], \quad (10.82)$$

and by comparing coefficients of α^n we see that $\mathbb{E}_s[:L_t^n:] = :L_s^n:.$ Therefore, $:L_t^n:$ is an $(\mathcal{F}_t, \mathbb{P})$ -martingale. $\square$

Di Nardo & Oliva (2013) study time-space harmonic polynomials for Lévy processes and emphasise that the usefulness of these polynomials is that they are martingales, even though the underlying process need not be. We will see below that it is particularly useful that the first Wick power is a compensated process.

Proposition 10.2.4. *Let $f(x)$ be a function for which the partial derivatives $f_x(t, x)$ and $f_{xx}(t, x)$ exist and are continuous, and let $\{L_t\}$ be a Lévy process with Lévy triplet (σ^2, ν, γ) such that its moments are finite. Then, for every $t \geq 0$, we have*

$$\begin{aligned} :f(L_t): &= f(L_0) + \int_0^t :f_x(L_s): dL_s: \\ &+ : \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(L_{s-} + \Delta L_s) - f(L_{s-}) - \Delta L_s f'(L_{s-}):. \end{aligned} \quad (10.83)$$

Proof. We recall first the well-known result on the generalisation of Ito's lemma for a scalar Lévy process (for details, see Cont & Tankov 2003), which asserts that

$$\begin{aligned} f(L_t) &= f(L_0) + \int_0^t \frac{\sigma^2}{2} f_{xx}(L_s) ds + \int_0^t f_x(L_s) dL_s \\ &+ \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(L_{s-} + \Delta L_s) - f(L_{s-}) - \Delta L_s f'(L_{s-}) \end{aligned} \quad (10.84)$$

where the first integral in (10.84) comes from

$$\int_0^t \frac{1}{2} f_{xx}(L_s) d[\sigma W_t + \mu t, \sigma W_t + \mu t]. \quad (10.85)$$

Now, if $L_t = \sigma W_t + \mu t + \Delta L_t$ we can replace $\sigma W_t + \mu t$ with $L_t - \Delta L_t$ to obtain

$$\begin{aligned} f(L_t) &= f(L_0) + \int_0^t \frac{1}{2} f_{xx}(L_s) d[L - \Delta L, L - \Delta L]_s + \int_0^t f_x(L_s) dL_s \\ &\quad + \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(L_{s-} + \Delta L_s) - f(L_{s-}) - \Delta L_s f'(L_{s-}). \end{aligned} \quad (10.86)$$

Taking the Wick order of the expression in equation (10.86), we find that

$$\begin{aligned} :f(L_t): &= f(L_0) + : \int_0^t \frac{1}{2} f_{xx}(L_s) d[L - \Delta L, L - \Delta L]_s : + : \int_0^t f_x(L_s) dL_s : \\ &\quad + : \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(L_{s-} + \Delta L_s) - f(L_{s-}) - \Delta L_s f'(L_{s-}) :. \end{aligned} \quad (10.87)$$

Since $L_t - \Delta L_t = \sigma W_t + \mu t = X_t$, by use of (10.32) we obtain

$$:[L - \Delta L, L - \Delta L]_t: = 0. \quad (10.88)$$

As a result, the first integral on the right hand side of (10.87) disappears. Due to the independent increments property of Lévy processes, the second integral on the right hand side of (10.87) can be written as follows:

$$: \int_0^t f_x(L_s) dL_s : = \int_0^t : f_x(L_s) : d : L_s :. \quad (10.89)$$

We are therefore left with the required result. $\square$

Example 10.2.1 We consider the case where L_t is a Poisson process N_t with unit intensity,

and let $f(x) = x^2$. Equation (10.83) simplifies to

$$\begin{aligned} :N_t^2: &= N_0^2 + 2 \int_0^t :N_s: dN_s: \\ &\quad + : \sum_{i=1}^{N_t} (N_{s-} + 1)^2 - N_{s-}^2 - 2N_{s-} : . \end{aligned} \quad (10.90)$$

Expanding the Wick order on the left hand side, we find

$$:N_t^2: = N_t^2 - 2tN_t - t + t^2. \quad (10.91)$$

Turning to the right hand side, we first expand the Wick orders to obtain

$$2 \int_0^t N_{s-} dN_s - 2 \int_0^t s dN_s - 2 \int_0^t (N_s - s) ds + : \sum_{i=1}^{N_t} 1 : . \quad (10.92)$$

We define τ_i to be the time of the i^{th} jump in the Poisson process $\{N_t\}_{t \geq 0}$ where we set $\tau_0 = 0$. The following observation will be used in our calculations below.

$$N_{\tau_{i-}} = \lim_{\substack{s \rightarrow \tau_i \\ s < \tau_i}} N_s = i - 1, \quad i \geq 1. \quad (10.93)$$

So we have

$$\begin{aligned} &2 \int_0^t N_{s-} dN_s - 2 \sum_{i=1}^{N_t} \tau_i - 2 \sum_{i=1}^{N_t} \int_{(\tau_{i-1}, \tau_i]} (i - 1 - s) ds - 2 \int_{(\tau_{N_t}, t]} (N_t - s) ds + N_t - t \\ &= 2 \int_0^t N_{s-} dN_s - 2 \sum_{i=1}^{N_t} \left[\tau_i + (i - 1)(\tau_i - \tau_{i-1}) - \frac{1}{2}(\tau_i^2 - \tau_{i-1}^2) \right] \\ &\quad - 2N_t(t - \tau_{N_t}) + (t^2 - \tau_{N_t}^2) + N_t - t \\ &= 2 \int_0^t N_{s-} dN_s - 2tN_t + t^2 + N_t - t. \end{aligned} \quad (10.94)$$

Now equating the expressions in (10.90) and (10.94) we obtain the known result

$$N_t^2 - 2 \int_0^t N_{s-} dN_s = N_t. \quad (10.95)$$

Example 10.2.2 Let $f(x) = e^{\alpha x}$ and consider the Poisson process N_t with unit intensity. Then from Proposition 10.2.4 we find that

$$:e^{\alpha N_t}: = 1 + \alpha \int_0^t :e^{\alpha N_{s-}}: d:N_s: + : \sum_{\substack{0 \leq s \leq t \\ \Delta N_s \neq 0}} e^{\alpha(N_{s-}+1)} - e^{\alpha N_{s-}} - \alpha e^{\alpha N_{s-}} :. \quad (10.96)$$

Rearranging and simplifying,

$$\begin{aligned} \int_0^t :e^{\alpha N_{s-}}: d:N_s: &= \frac{1}{\alpha} \left(:e^{\alpha N_t}: - 1 + (1 + \alpha - e^\alpha) : \sum_{i=1}^{N_t} e^{\alpha(i-1)} : \right) \\ &= \frac{(:e^{\alpha N_t}: - 1 + (1 + \alpha - e^\alpha)(1 - :e^{\alpha N_t}:))}{\alpha(1 - e^\alpha)} \\ &= \frac{:e^{\alpha N_t}: - 1}{e^\alpha - 1}. \end{aligned} \quad (10.97)$$

Schoutens (2000) also has an analogous result in his analysis of stochastic integration with respect to compensated stochastic processes, which in our setting is represented by the Wick power.

We refer to Sengupta (2008) for further examples of time-space harmonic polynomials for specific Lévy processes.

Proposition 10.2.5. *Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ exist and are continuous, and let $\{L_t\}$ be a Lévy process such that its moments are finite. Then for every $t \geq 0$ we have*

$$\begin{aligned} :f(t, L_t): &= f(0, L_0) + \int_0^t :f_s(s, L_s): ds + \int_0^t :f_x(s, L_{s-}): d:L_s: \\ &\quad + : \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(s, L_{s-} + \Delta L_s) - f(s, L_{s-}) - \Delta L_s f'(s, L_{s-}):. \end{aligned} \quad (10.98)$$

Proof. Ito's lemma for a scalar Lévy process (for details, see Cont & Tankov 2003) can be

written as follows:

$$\begin{aligned} f(t, L_t) = & f(0, L_0) + \int_0^t f_s(s, L_s) ds + \int_0^t \frac{\sigma^2}{2} f_{xx}(s, L_s) ds + \int_0^t f_x(s, L_{s-}) dL_s \\ & + \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(s, L_{s-} + \Delta L_s) - f(s, L_{s-}) - \Delta L_s f'(s, L_{s-}). \end{aligned} \quad (10.99)$$

Taking the Wick order of the expression in equation (10.99), we obtain:

$$\begin{aligned} :f(t, L_t): = & f(0, L_0) + : \int_0^t \frac{\sigma^2}{2} f_{xx}(s, L_s) d[L, L]_s : + : \int_0^t f_x(s, L_s) dL_s : \\ & + : \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(s, L_{s-} + \Delta L_s) - f(s, L_{s-}) - \Delta L_s f'(s, L_{s-}) : \end{aligned} \quad (10.100)$$

As shown in the proof of Proposition 10.2.4, the Wick order of the quadratic variation of the Lévy process $\{L_t\}$ is zero. As a result, the first integral on the right hand side of (10.100) disappears. Due to the independent increments property of Lévy processes, the second integral on the right hand side of (10.100) can be written as follows:

$$: \int_0^t f_x(L_s) dL_s : = \int_0^t : f_x(L_s) : d : L_s : . \quad (10.101)$$

We are therefore left with the required result. $\square$

Thus far we have considered the Wick power of a stochastic process and looked more closely at the Brownian motion as well as more general Lévy processes. We now take a look at examples of the Wick product of Lévy processes.

Remark 10.2.3 We begin with the Wick product $:X_t Y_t:$ and show that it can be expressed in terms of the first Wick powers, $:X_t:$, $:Y_t:$, and the covariance of X_t and Y_t :

$$\begin{aligned} :X_t Y_t: &= :X_t: :Y_t: + \mathbb{E}[X_t] \mathbb{E}[Y_t] - \mathbb{E}[X_t Y_t] \\ &= :X_t: :Y_t: - \text{Cov}(X_t, Y_t). \end{aligned} \quad (10.102)$$

Remark 10.2.4 If $X_t = Y_t$ in (10.102), we have

$$\begin{aligned} :X_t^2: &= :X_t: ^2 + (\mathbb{E}[X_t])^2 - \mathbb{E}[X_t^2] \\ &= :X_t: ^2 - \text{Var}(X_t). \end{aligned} \quad (10.103)$$

Janson (1997) introduces the Wick product via Feynman diagrams (a rather sophisticated version of the cumulant expansion used in quantum field theory) and includes the following example using a slightly different approach. Here we reproduce the result using our formulation of the Wick calculus.

Example 10.2.3 Let

$$X_\infty = \int_0^\infty f(s) dW_s \quad \text{and} \quad Y_\infty = \int_0^\infty g(s) dW_s, \quad (10.104)$$

where f, g are deterministic functions. The Wick order of these integrals is just the integrals themselves since the mean of the Brownian motion is zero and the integrands are deterministic. If we let $f(x, y) = xy$, then the product of the integrals in (10.104) can be represented as follows:

$$:X_\infty Y_\infty: = \int_0^\infty X_t dY_t + \int_0^\infty Y_t dX_t. \quad (10.105)$$

Note that we can also write

$$:X_\infty Y_\infty: = X_\infty Y_\infty - \mathbb{E}[X_\infty Y_\infty] \quad (10.106)$$

by the definition of the Wick product (9.7), since $\mathbb{E}[X_\infty] = \mathbb{E}[Y_\infty] = 0$. Putting (10.105) and (10.106) together, we obtain a rather curious result concerning the product of a Gaussian random variables:

$$X_\infty Y_\infty = \int_0^\infty X_t dY_t + \int_0^\infty Y_t dX_t + \mathbb{E}[X_\infty Y_\infty]. \quad (10.107)$$

10.3 Summary

We began this chapter by creating a definition for the Wick power of a stochastic process by using elements from both Simon (1974) and Goswami & Sengupta (2008). We proceeded by proving a polynomial representation for the Wick power of a general stochastic process. Novel results analogous to Ito's lemma are derived for a scaled Brownian motion with drift as well as for a scalar Lévy process.

Chapter 11

Change of measure

As indicated in the previous chapter, the Wick exponential of a Lévy process is just the associated Esscher martingale whose expectation is unity. As such, one can use it to effect a change of probability measure:

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = :e^{\alpha X_t} :_{\mathbb{P}} . \quad (11.1)$$

Under this class of measure change, if the process $\{X_t\}$ is a Lévy process, then it remains a Lévy process under the measure $\mathbb{Q}$ (see Kyprianou 2006, page 78). In fact, and more interestingly, the effect of the measure change in (11.1) varies greatly between different Lévy processes. For example, in the case of a Brownian motion (with drift), it is well known that its drift is modified under such a measure change. For a Poisson process, on the other hand, such a measure change modifies the jump rate, and for a gamma process it is the scale of the process that changes, and so on. In what follows we shall comment briefly on the measure change induced by the Wick exponential of a Lévy process, and use this to derive the form of the excess rate of return associated with a risky investment when the underlying price process is modelled by a geometric Lévy process.

11.1 Gaussian change of measure

We begin by examining the Brownian case. The purpose is not merely to show the well-known result concerning the ‘change of drift’ associated with a Brownian motion, but rather

to highlight the use of the Wick calculus in deriving the measure change on the Wick powers of a Brownian motion. We assume that the measurable space $(\Omega, \mathcal{F})$ supports two measures $\mathbb{Q}$ and $\mathbb{P}$ so that $\{W_t\}$ is a Gaussian process with respect to both $\mathbb{Q}$ and $\mathbb{P}$. More specifically, we consider the case where $\{W_t\}$ is a Brownian motion with respect to the measure $\mathbb{P}$. We can then form the Wick powers of W_t under the two measures, $\mathbb{Q}$ and $\mathbb{P}$, $:W_t^n:_\mathbb{Q}$ and $:W_t^n:_\mathbb{P}$, and find transformation laws from one to the other.

To begin, since $\{W_t\}$ is a Brownian motion under $\mathbb{P}$, we can write

$$:e^{\alpha W_t}:_\mathbb{P} = e^{\alpha W_t - \frac{1}{2}\alpha^2 t} \quad (11.2)$$

for the Wick exponential of W_t under $\mathbb{P}$. On the other hand, since W_t is Gaussian under $\mathbb{Q}$, we can also write

$$:e^{\alpha W_t}:_\mathbb{Q} = e^{\alpha(W_t - \mathbb{E}^\mathbb{Q}[W_t]) - \frac{1}{2}\alpha^2 \sigma_\mathbb{Q}^2}, \quad (11.3)$$

where $\sigma_\mathbb{Q}^2$ is the variance of W_t under $\mathbb{Q}$. Combining equations (11.2) and (11.3), we obtain the following relation:

$$e^{\alpha \mathbb{E}^\mathbb{Q}[W_t]} :e^{\alpha W_t}:_\mathbb{Q} = :e^{\alpha W_t}:_\mathbb{P} e^{\frac{1}{2}\alpha^2 (t - \sigma_\mathbb{Q}^2)}. \quad (11.4)$$

By writing the exponentials in terms of their formal power series, we find that

$$:(W_t + \mathbb{E}^\mathbb{Q}[W_t])^n:_\mathbb{Q} = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{n! (t - \mathbb{E}^\mathbb{Q}[W_t^2] + \mathbb{E}^\mathbb{Q}[W_t]^2)^m}{2^m m! (n - 2m)!} :W_t^{n-2m}:_\mathbb{P}, \quad (11.5)$$

where for clarity we have replaced $\sigma_\mathbb{Q}^2$ with $\mathbb{E}^\mathbb{Q}[W_t^2] - \mathbb{E}^\mathbb{Q}[W_t]^2$. This gives a general result concerning the effect of a measure change on the Wick power for a Brownian motion.

Let us now set $n = 1$ in (11.5). Then we have

$$:W_t + \mathbb{E}^\mathbb{Q}[W_t]:_\mathbb{Q} = :W_t:_\mathbb{P}. \quad (11.6)$$

If we further let $:e^{-\lambda W_t}:_{\mathbb{P}}$ for some real λ be the change of measure process, then

$$\begin{aligned}
 \mathbb{E}^{\mathbb{Q}}[W_t] &= \mathbb{E}^{\mathbb{P}}[W_t :e^{-\lambda W_t}:_{\mathbb{P}}] \\
 &= e^{-\frac{1}{2}\lambda^2 t} \mathbb{E}^{\mathbb{P}} \left[W_t \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k W_t^k}{k!} \right] \\
 &= e^{-\frac{1}{2}\lambda^2 t} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k!} \mathbb{E}^{\mathbb{P}} [W_t^{k+1}].
 \end{aligned} \tag{11.7}$$

Now the m^{th} moment of the Brownian motion is

$$\mathbb{E}^{\mathbb{P}}[W_t^m] = \begin{cases} \frac{t^j (2j)!}{j! 2^j}, & 2j = m \\ 0, & 2j + 1 = m \end{cases}. \tag{11.8}$$

We therefore conclude that

$$\begin{aligned}
 \mathbb{E}^{\mathbb{Q}}[W_t] &= e^{-\frac{1}{2}\lambda^2 t} \sum_{j=1}^{\infty} \frac{(-1)^{2j-1} \lambda^{2j-1}}{(2j-1)!} \mathbb{E}^{\mathbb{P}} [W_t^{2j}] \\
 &= -\frac{2}{\lambda} e^{-\frac{1}{2}\lambda^2 t} \sum_{k=0}^{\infty} \left(\frac{\lambda^2 t}{2} \right)^{k+1} \frac{1}{k!} \\
 &= -\lambda t
 \end{aligned} \tag{11.9}$$

and substituting this into (11.6), we obtain

$$:W_t:_{\mathbb{Q}} = :W_t:_{\mathbb{P}} + \lambda t, \tag{11.10}$$

which is the well-known result obtained from Girsanov's theorem. We also find from (11.10) that $\sigma_t^{\mathbb{Q}} = t$. Analogous results can be obtained for higher powers of the Brownian motion.

11.2 Lévy-Wick change of measure

Let $\{L_t\}$ be a Lévy process with the associated $\mathbb{P}$ -Laplace exponent $\psi(t)$, given by (10.70). We can write the Wick exponential of this process at time t under some measure $\mathbb{Q}$ as

follows:

$$:e^{\sigma L_t}:_{\mathbb{Q}} = e^{\sigma L_t - t\psi^*(\sigma)}, \quad \text{where} \quad \psi^*(\sigma) = \frac{1}{t} \ln \left(\mathbb{E}^{\mathbb{Q}} [e^{\sigma L_t}] \right), \quad (11.11)$$

where $:e^{\sigma L_t}:_{\mathbb{Q}}$ is one of many possible measure changes. We now observe that

$$\begin{aligned} e^{t\psi^*(\sigma)} &= \mathbb{E}^{\mathbb{Q}} [e^{\sigma L_t}] \\ &= \mathbb{E}^{\mathbb{P}} [e^{\sigma L_t} : e^{-\lambda L_t} :_{\mathbb{P}}] \\ &= \mathbb{E}^{\mathbb{P}} [e^{\sigma L_t} e^{-\lambda L_t - t\psi(-\lambda)}] \\ &= \mathbb{E}^{\mathbb{P}} [e^{(\sigma-\lambda)L_t - t\psi(\sigma-\lambda)}] e^{t\psi(\sigma-\lambda) - t\psi(-\lambda)} \\ &= e^{t\psi(\sigma-\lambda) - t\psi(-\lambda)}, \end{aligned} \quad (11.12)$$

and so we can write the Laplace exponent of $\{L_t\}$ under $\mathbb{Q}$ in terms of the Laplace exponent under $\mathbb{P}$ as follows:

$$\psi^*(\sigma) = \psi(\sigma - \lambda) - \psi(-\lambda). \quad (11.13)$$

We can therefore express the Wick exponential under $\mathbb{Q}$ in terms of Laplace exponents under the original $\mathbb{P}$ measure alone:

$$:e^{\sigma L_t}:_{\mathbb{Q}} = e^{\sigma L_t - t\psi(\sigma-\lambda) + t\psi(-\lambda)}. \quad (11.14)$$

It follows that under the Esscher-type measure change, the transformation of the Wick power of a Lévy process can be calculated according to the following scheme:

$$:L_t^n:_{\mathbb{Q}} = \left. \frac{\partial^n}{\partial \sigma^n} e^{\sigma L_t - t\psi(\sigma-\lambda) + t\psi(-\lambda)} \right|_{\sigma=0}. \quad (11.15)$$

Now rearranging (11.14), substituting the result into (10.70) and then using (11.12), we can also relate the two Wick exponentials under the two measures and obtain

$$:e^{\sigma L_t}:_{\mathbb{P}} = :e^{\sigma L_t}:_{\mathbb{Q}} e^{t\psi(\sigma-\lambda) - t\psi(-\lambda) - t\psi(\sigma)}. \quad (11.16)$$

Comparing this equation with (11.3), we see that when L_t is a Brownian motion under $\mathbb{P}$, (11.16) reduces to exactly the expression in (11.3). Equation (11.16) can be used for changing measure from $\mathbb{P}$ to $\mathbb{Q}$ or vice versa.

Example 11.2.1 As an elementary application of the above results, let us consider the modelling of financial asset prices using geometric Lévy models. In this context we assume that $\mathbb{P}$ denotes the ‘physical’ measure, whereas $\mathbb{Q}$ refers to the ‘risk-neutral’ or ‘pricing’ measure. Specifically, our objective here is to derive an expression for the excess rate of return above the short rate arising from a risky investment, when the underlying price process is modelled by a geometric Lévy model. This result has in fact recently been obtained by Brody, Hughston & Mackie (2012a). Here, however, we shall derive the result using the Wick calculus; the result of which is in agreement with the known expression. In this example we begin by noting that since $:e^{\sigma L_t}:_{\mathbb{Q}}$ is a $\mathbb{Q}$ martingale, we can express these discounted asset prices as a multiple of $:e^{\sigma L_t}:_{\mathbb{Q}}$ as follows:

$$e^{-rt} S_t = S_0 :e^{\sigma L_t}:_{\mathbb{Q}}. \quad (11.17)$$

Rearranging and using relation (11.16), we thus obtain

$$S_t = S_0 e^{rt} :e^{\sigma L_t}:_{\mathbb{P}} e^{R(\lambda, \sigma)t}, \quad (11.18)$$

where

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda) \quad (11.19)$$

is the excess rate of return above the short rate, which is given by a function of the risk measure σ and the risk aversion measure λ . This is identical to the result in Brody, Hughston & Mackie (2012a), which states that the price process of a risky asset in a geometric Lévy model under the $\mathbb{P}$ measure is given by the expression

$$S_t = S_0 e^{rt} e^{\sigma L_t - t\psi(\sigma)} e^{R(\lambda, \sigma)t}. \quad (11.20)$$

11.3 Summary

In this chapter we showed that the Wick exponential can be used as a change of measure process, which in itself is not new since it is equivalent to the Esscher transform. However, we do obtain some interesting results in the process. For example, equation (11.5) may have some interesting applications for $n > 1$. We also showed that the result in (11.20) obtained in Brody, Hughston & Mackie (2012a) can be derived using the Wick exponential of a Lévy process.

Chapter 12

Conclusion to Part II

In the second part of this thesis, we have presented various results for the Wick power of a random variable as well as the more general Wick product of multiple random variables. When compared to the more familiar Ito calculus, the setting here is a considerably general one where the random variables are only required to have finite moments. We then apply the theory, with some extensions, to stochastic processes. It follows, in particular, that the rules of the calculus are not restricted to the case of a Brownian motion. With this in mind, we have derived a Wick version of Ito's lemma for a function $f(t, x)$ of time t and a scaled and drifted Brownian motion $\{X_t\}$:

$$:f(T, X_T): - :f(t, X_t): = \int_t^T :f_s(s, X_s): ds + \int_t^T :f_x(s, X_s): d :X_s: \quad (12.1)$$

Note that the result given by the quadratic variation term in Ito's lemma is embedded within the Wick ordering.

Now more generally, when extending the theory to consider a Lévy process $\{L_t\}$ with finite moments and a smooth function $f(t, x)$ of two variables, we obtain the representation

$$\begin{aligned} :f(t, L_t): &= f(0, L_0) + \int_0^t :f_s(s, L_s): ds + \int_0^t :f_x(s, L_{s-}): d :L_s: \\ &+ : \sum_{\substack{0 \leq s \leq t \\ \Delta L_s \neq 0}} f(s, L_{s-} + \Delta L_s) - f(s, L_{s-}) - \Delta L_s f'(s, L_{s-}): . \end{aligned} \quad (12.2)$$

The results obtained have a number of desirable properties due to the fact that the pro-

cesses considered have independent increments, which makes the Wick exponential as well as the Wick power both martingales. However, the class of processes that have independent increments are still considerably rich.

As indicated earlier, our objective is to systematically develop the rules and properties of the Wick calculus. Having done so, we are in a position to explore its applications in various areas of applied stochastic analysis such as financial modelling. The analysis considered at the end of the thesis on change of measure techniques shows promising signs that the Wick calculus could be developed further in this direction.

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