



# Funding, repo and credit inclusive valuation as modified option pricing



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## ABSTRACT

We take the holistic approach of computing an OTC claim value that incorporates credit and funding liquidity risks and their interplays, instead of forcing individual price adjustments: CVA, DVA, FVA, KVA. The resulting nonlinear mathematical problem features semilinear PDEs and FBSDEs. We show that for the benchmark vulnerable claim there is an analytical solution, and we express it in terms of the Black–Scholes formula with dividends. This allows for a detailed valuation analysis, stress testing and risk analysis via sensitivities.

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## 1. Introduction

Prior to the financial crisis of 2007–2008, institutions tended to ignore the credit risk of highly-rated counterparties in valuing and hedging contingent claims traded over-the-counter (OTC), claims which are in fact bilateral contracts negotiated between two default-risky entities. Then, in just the short span of one month of 2008 (Sep 7 to Oct 8), eight mainstream financial institutions experienced critical credit events in a painful reminder of the default-riskiness of even large names (the eight were Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landsbanki, Glitnir and Kaupthing, to which we could also add Merrill Lynch).

One of the explosive manifestations of this crisis was the sudden divergence between the rate of overnight indexed swaps (OISs) and the LIBOR rate, pointing to the credit and liquidity risk existing in the interbank market. This forced dealers and financial institutions to reassess the valuation of OTC claims, leading to various adjustments to their book value.

It is difficult to do justice to the entire literature on such valuation adjustments, which intertwines two strands that have been developed in parallel by academics and practitioners. For a full introduction to valuation adjustments and all related references

we refer to the first chapter of either Brigo et al. [15] or Crépey et al. [17].

All such adjustments may concern both over the counter (OTC) derivatives trades and derivatives trades done through central clearing houses (CCP), see for example Brigo and Pallavicini [16] for a comparison of the two cases where the full mathematical structure of the problem of valuation under possibly asymmetric initial and variation margins, funding costs, liquidation delay and credit gap risk is explored, resulting in BSDEs and semilinear PDEs. It is worth pointing out that the size of such derivatives markets remains quite relevant even post-crisis. At end of 2012, the market value of outstanding OTC derivative contracts was reported to be \$24.7 trillion with \$632.6 trillion in notional value (BIS 2013). Even if many deals are now collateralized in an attempt to avoid CVA altogether, contagion and gap risk may still result in important residual CVA, as was shown for the case of credit default swap trades in Brigo et al. [10].

As we mentioned above, the rigorous theory of valuation in presence of all such effects can be quite challenging, leading to models that are based on advanced mathematical tools such as semilinear PDEs or BSDEs, which make numerical analysis difficult and slow. See for example El Karoui et al. [18] for an example of how asymmetric interest rates, even in absence of credit risk, lead to BSDEs. The papers Brigo et al. [11] and Bichuch et al. [1] deal with the mathematical analysis of valuation equations in presence of all the abovementioned effects and risks, except KVA, for which we refer instead to Brigo et al. [12] for an indifference pricing

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approach. Biffis et al. [7] analyze such effects in the area of life insurance contracts, and longevity swaps in particular.

Isolating and computing each individual adjustment is difficult because there is a marked interplay between them in pricing. Therefore, the causes of these adjustments are accounted for at the level of the contract payoffs and the resulting all-inclusive price is written as a solution to an advanced mathematical problem of the type mentioned above. Is there a case, even for a simple contract, where this all-inclusive price of an uncollateralized contract can be calculated analytically? We present here an answer in the affirmative.

More specifically, we show that for standard benchmark products the above mathematically challenging structures can be solved analytically under a few simplifying assumptions. The solution is expressed in terms of the same explicit formula used for standard derivatives in the absence of these adjustments, namely the Black–Scholes pricing formula for vulnerable options on dividend-paying assets. This leads to a closed-form solution for the all-inclusive price of a benchmark product, the vulnerable call option, which then enables an analysis of all such effects that is more approachable from a numerical point of view. This link with the all-familiar Black–Scholes formula may be a way to reach out to a large portion of market participants and traders that are often discouraged by the full mathematical complexity of nonlinear valuation and the related nonlinearity valuation adjustment (NVA), see Brigo et al. [13]. We finally mention the credit and later funding adjustment works by Brigo and Masetti [14], Bo and Capponi [8], Bielecki et al. [2], Bielecki and Rutkowski [6], Brigo et al. [9] and the general credit risk references by Bielecki et al. [5] and Rutkowski [21], since they either deal with aspects we do not address here or are related to techniques we use in our paper.

Let us now specify the probabilistic setting used in what follows. We fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\mathbb{P}$  is meant to represent the physical probability measure. Denote by  $\mathbb{F}$  the default-free filtration, namely the filtration generated by default-free financial quantities, typically driven by Brownian shocks. In general this may include for example information generated by default-free interest rates, the pre-default intensity of a name, stock prices of companies that are assumed default-free, and so on. Assume that the default time  $\tau$  of the entity of interest (e.g. the issuer of a bond and in whose name there is a traded CDS, or the issuer of an option) is a positive random variable on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The default time generates a filtration  $\mathbb{H} = (\mathcal{H}_t)$  where  $\mathcal{H}_t := \sigma(\mathbb{1}_{\{\tau \leq u\}}, u \leq t)$ , which is used to progressively enlarge  $\mathbb{F}$  in order to obtain the full filtration  $\mathbb{G} = (\mathcal{G}_t)$  where  $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ . We work under the assumption that  $F_t := \mathbb{P}(\tau \leq t | \mathcal{F}_t)$  is a continuous, increasing function and  $F_t < 1$  for any  $t$ . Note that this assumption on the default time has already appeared in Elliott et al. [19] in conjunction with the hypothesis (H).

## 2. Vulnerable call option pricing by replication

After a detailed analysis of valuation of the zero-recovery defaultable bond, we will now address a more advanced problem of valuation of vulnerable options on some risky asset. Once again, our goal is to compare various approaches and to identify the underlying assumptions, which are frequently ignored in the existing literature.

Here  $\mathcal{F}_t := \sigma(S_u, u \leq t)$  will be the natural filtration generated by the price process of a traded asset (stock), as all other typical  $\mathbb{F} = (\mathcal{F}_t)$  processes will be taken to be deterministic. Let the maturity date  $T$  be fixed and let  $X$  be an  $\mathcal{F}_T$ -measurable integrable random variable.

Let  $A$  be a **contract** (vulnerable call option) that has the payoff to its holder at maturity time  $T$  given by

$$X = \mathbb{1}_{\{\tau > T\}}(S_T - K)^+$$

where  $\tau$  is interpreted as the default time of the issuer of the option. We wish to find the positive price  $P_0$  of this contract, which will be computed by the buyer of the option, who pays to the issuer the amount  $P_0$  at time 0, and subsequently hedges his long position in the option by establishing a replicating portfolio for the payoff  $-X$ , using financial instruments available in the market and starting from the initial wealth  $-P_0$ . More generally, we will search for the price  $P_t$  at any date  $t \in [0, T]$  using analogous replication arguments.

From now on, we consider a market with the following **primary assets**  $(A^1, A^2, A^3, A^4)$ :

- (i) an unsecured funding account with the interest rate  $f$ ;
- (ii) a stock (the underlying asset of the contract);
- (iii) a repo agreement on the stock with the repo rate  $h$ ;
- (iv) a zero-recovery defaultable bond with the rate of return  $r^C$  issued by the counterparty.

At time  $t$ , the price  $P_t^i$  of the asset  $A^i$  is given by

$$P_t^1 = B_t^f, \quad P_t^2 = S_t, \quad P_t^3 = 0, \quad P_t^4 = B_t$$

and the gains process since inception of  $A^i$  is denoted by  $G_t^i$  with  $G_0^i = 0$  for all  $i$ .

As a preliminary step, we specify the model inputs: the treasury rate  $f$ , the repo rate  $h$  and the bond rate of return  $r^C$ . Note that the rates  $f$ ,  $h$  and  $r^C$  are postulated to be constant (or, at least, deterministic) and they are known. We assume also that the process  $S$  is continuous (obviously,  $B^f$  is continuous as well). We will later assume, in addition, that the stock price volatility  $\sigma$  is known as well. Hence we seek for the pricing formula in terms of the model parameters  $f$ ,  $h$ ,  $r^C$  and  $\sigma$  and the option data:  $T$  and  $K$ .

Note that, in principle, all these quantities are observed in the market, provided that the volatility is understood as the implied volatility. By contrast, we do not need to assume that the CDS on the counterparty is traded, although this postulate would not change our derivation of the option pricing formula, and the knowledge of the CDS spread  $r^{CDS}$  is immaterial. In fact, we know from the preceding section that, for a fixed level of the treasury rate  $f$ , there is one-to-one correspondence between  $r^C$  and  $r^{CDS}$ .

Let us now determine the gains processes. Buying one repo contract amounts to selling the shares of stock against cash, under the agreement of repurchasing them back at the higher price that includes the interest payments corresponding to the repo rate. (Selling the repo results in the opposite cash flows.) Any appreciation (or depreciation) in the stock price is part of the positive (or negative) gains, while the outgoing repo interest payments are negative gains:  $dG_t^3 = dS_t - hS_t dt$ .

Under the standing assumption that the pre-default rate of return  $r^C$  on the counterparty's bond is deterministic, we obtain

$$B_t = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T r_u^C du} = (1 - J_t) e^{-\int_t^T r_u^C du}$$

where  $J_t := \mathbb{1}_{\{\tau \leq t\}}$  is the point process that models the jump to default of the counterparty. The gains have negative terms for outgoing cash flows corresponding to the drop in the bond value at the time of default. To summarize, the gains of primary assets are given by

$$\begin{aligned} dG_t^1 &= fB_t^f dt, \quad dG_t^2 = dS_t, \quad dG_t^3 = dS_t - hS_t dt, \\ dG_t^4 &= r_t^C B_t dt - B_{t-} dJ_t. \end{aligned} \quad (1)$$

A **trading strategy**  $\varphi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4)$  gives the number of units of each primary asset purchased to build a portfolio. Let  $\beta \in [0, 1]$  be a constant. A trading strategy  $\varphi$  is **admissible** if at any date  $t$  the investor can only use the repo market for a fraction  $\beta$  of the stock amount required and the rest has to be obtained in the stock market with funding from the treasury. The **wealth** at time  $t \in [0, T]$  of the portfolio resulted from an admissible strategy  $\varphi$  is

denoted by  $V_t^\varphi$  and equals

$$V_t^\varphi = \sum_{i=1}^4 \varphi_t^i P_t^i$$

and the gains process associated with this strategy satisfies  $G_0^\varphi = 0$  and

$$dG_t^\varphi := \sum_{i=1}^4 \varphi_t^i dG_t^i. \quad (2)$$

We then say that a strategy  $\varphi$  is **self-financing** if for all  $t \in [0, T]$

$$V_t^\varphi = V_0^\varphi + G_t^\varphi. \quad (3)$$

An admissible trading strategy  $\varphi$  **replicates** the payoff  $-X$  if  $V_T^\varphi = -X$ . We define the time  $t$  price of the contract as the negative of the wealth  $V_t^\varphi$  of the portfolio which replicates  $-X$ , that is,

$$P_t := -V_t^\varphi. \quad (4)$$

Since the market model under study is in fact linear, it is equivalent to postulate that  $\varphi$  replicates the payoff  $X$  and to set  $P_t = V_t^\varphi$ . Note, however, that this step would not be possible if trading in primary assets was nonlinear (for instance, in models with differing lending and borrowing treasury rates). The existence of the specific primary assets in our market ensures that any claim is **attainable**. In fact, the present market model is complete and no-arbitrage arguments show that the price of any contract is unique.

At time  $t$  before default, the investor builds a replicating portfolio for  $-X$  knowing that the assumptions on  $\tau$  imply that default may occur between  $t$  and  $t + dt$  for an arbitrarily small  $dt$ . To replicate the contract the investor:

1. buys  $\beta \Delta_t$  repos, borrows  $\beta \Delta_t S_t$  from treasury to buy and deliver  $\beta \Delta_t$  shares, and receives  $\beta \Delta_t S_t$  cash which is paid back to treasury;
2. borrows  $(1 - \beta) \Delta_t S_t$  from treasury and buys  $(1 - \beta) \Delta_t$  shares;
3. buys  $P_t/B_t$  units of the issuer's bond in order to match the value of this portfolio and the option payoff.

Of course, at this moment the option price  $P_t$  is yet unknown, but it will be found from the matching condition (4) combined with the terminal payoff  $-X$ . This replicating portfolio produces the following admissible strategy

$$\theta_t := \left( -\frac{(1 - \beta) \Delta_t S_t}{B_t^f}, (1 - \beta) \Delta_t, \beta \Delta_t, \frac{P_t}{B_t} \right). \quad (5)$$

At time  $t + dt$  the investor:

4. receives  $\beta \Delta_t$  shares from repo and sells them for  $\beta \Delta_t S_{t+dt}$ ;
5. borrows from treasury  $\beta \Delta_t S_t (1 + hdt)$  to close the repo;
6. sells  $(1 - \beta) \Delta_t$  shares for  $(1 - \beta) \Delta_t S_{t+dt}$ ;
7. sells the counterparty's bond for  $P_t B_{t+dt}/B_t$ ;
8. pays back to the treasury the amount  $(1 - \beta) \Delta_t S_t (1 + fdt)$ .

The change in the wealth of the replicating position resulting from these transactions equals

$$\begin{aligned} V_{t+dt}^\theta - V_t^\theta &= \beta \Delta_t S_{t+dt} - \beta \Delta_t S_t (1 + hdt) + (1 - \beta) \Delta_t S_{t+dt} \\ &\quad + \frac{P_t}{B_t} dB_t - (1 - \beta) \Delta_t S_t (1 + fdt) \\ &= \beta \Delta_t dS_t - \beta h \Delta_t S_t dt + (1 - \beta) \Delta_t dS_t \\ &\quad + \frac{P_t}{B_t} dB_t - (1 - \beta) f \Delta_t S_t dt \\ &= \Delta_t dS_t - ((1 - \beta) f + \beta h) \Delta_t S_t dt + P_t (r^C dt - dj_t). \end{aligned}$$

This can be derived formally by using (1) and computing the gains process (2) associated with the portfolio  $\theta$  given by (5)

$$\begin{aligned} dG_t^\theta &= -\frac{(1 - \beta) \Delta_t S_t}{B_t^f} f B_t^f dt + (1 - \beta) \Delta_t dS_t \\ &\quad + \beta \Delta_t (dS_t - h S_t dt) + \frac{P_t}{B_t} (r_t^C B_t dt - B_t - dj_t) \\ &= \Delta_t dS_t - ((1 - \beta) f + \beta h) \Delta_t S_t dt + P_t (r_t^C dt - dj_t) \end{aligned}$$

where we used the equality  $B_{t-} = B_t$ , which holds before default. Note also that the wealth of  $\theta$  at default equals zero, which is consistent with the option payoff at default. Consequently, we may and do set  $\theta_t = (0, 0, 0, 0)$  for  $t > \tau$ .

Let us now focus on the pricing problem before default. Since  $dV_t^\theta = dG_t^\theta$  (from (3)) and  $dP_t = dV_t^\theta$  (from (4)), we have

$$dP_t = \Delta_t dS_t - ((1 - \beta) f + \beta h) \Delta_t S_t dt + P_t (r_t^C dt - dj_t). \quad (6)$$

To derive the pre-default pricing PDE, we assume that under the statistical probability  $\mathbb{P}$  the stock price is governed by

$$dS_t = \mu_t S_t dt + \sigma S_t dW_t$$

and the price  $P_t$  can be expressed as

$$P_t = \mathbb{1}_{\{\tau > t\}} \tilde{P}_t = \mathbb{1}_{\{\tau > t\}} v(t, S_t) = (1 - J_t) v(t, S_t)$$

for some function  $v(t, s)$  of class  $C^{1,2}$ . Then the Ito formula yields

$$\begin{aligned} dP_t &= (1 - J_t) dv(t, S_t) + v(t, S_t) d(1 - J_t) \\ &= (1 - J_t) dv(t, S_t) - v(t, S_t) dj_t \end{aligned}$$

and

$$\begin{aligned} dP_t &= (1 - J_t) \left( v_t(t, S_t) + \frac{\sigma^2 S_t^2}{2} v_{ss}(t, S_t) \right) dt \\ &\quad + (1 - J_t) v_s(t, S_t) dS_t - v(t, S_t) dj_t. \end{aligned} \quad (7)$$

By equating the  $dS_t$ ,  $dt$  and  $dJ_t$  terms in (6) and (7), we obtain the following equalities in which the variables  $(t, S_t)$  were suppressed

$$\begin{aligned} \Delta_t &= (1 - J_t) v_s, \\ (1 - J_t) \left( v_t + \frac{\sigma^2 S_t^2}{2} v_{ss} + ((1 - \beta) f + \beta h) S_t v_s \right) \\ &\quad - (1 - J_t) r_t^C v = 0, \\ -P_t dj_t &= -v dj_t. \end{aligned} \quad (8)$$

The pre-default pricing PDE for the function  $v(t, s)$  is now obtained from (8) as

$$v_t + ((1 - \beta) f + \beta h) s \frac{\partial v}{\partial s} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 v}{\partial s^2} - r_t^C v = 0 \quad (9)$$

with terminal condition  $v(T, s) = (s - K)^+$ . One recognizes (9) as the Black–Scholes PDE when the underlying stock pays dividends. To see this, it suffices to take the discount rate to be the return on the defaultable bond  $r := r^C$  and the instantaneous dividend yield to be the bond spread over the effective funding rate:  $q := r^C - f^\beta$  where the **effective funding rate** is defined as the weighted average:  $f^\beta := (1 - \beta) f + \beta h$ . We conclude that the following result is valid.

**Proposition 2.1.** *The time  $t$  price of the vulnerable call option obtained by replication equals*

$$P_t = \mathbb{1}_{\{\tau > t\}} \left( S_t e^{-q(T-t)} N(d_1^q) - K e^{-r^C(T-t)} N(d_2^q) \right) \quad (10)$$

with  $q = r^C - f^\beta$  and

$$d_1^q = \frac{\log \frac{S_t}{K} + (r^C - q + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}, \quad d_2^q = d_1^q - \sigma \sqrt{T - t}.$$

It is worth noting that (10) may also be derived from (6) through probabilistic means without resorting to the pricing PDE. From (6), we obtain the following equation for the pre-default price  $\tilde{P}$

$$d\tilde{P}_t = -f^\beta \Delta_t S_t dt + \Delta_t dS_t + r_t^C \tilde{P}_t dt. \quad (11)$$

Let now  $\mathbb{Q}^\beta$  be the probability measure, which is equivalent to  $\mathbb{P}$ , and such that the drift of the risky asset  $S$  under  $\mathbb{Q}^\beta$  is equal to the effective funding rate  $f^\beta$ . Then the process  $\tilde{P}$  is governed under  $\mathbb{Q}^\beta$  by

$$d\tilde{P}_t - r_t^C \tilde{P}_t dt = \Delta_t \sigma S_t dW_t^\beta$$

with terminal condition  $\tilde{P}_T = (S_T - K)^+$  where  $W^\beta$  is the Brownian motion under  $\mathbb{Q}^\beta$ . This leads to the following probabilistic representation for  $\tilde{P}_t$

$$\begin{aligned} \tilde{P}_t &= e^{-r^C(T-t)} \mathbb{E}^\beta[(S_T - K)^+ | \mathcal{F}_t] \\ &= e^{-(r^C - f^\beta)(T-t)} \mathbb{E}^\beta[e^{-f^\beta(T-t)}(S_T - K)^+ | \mathcal{F}_t], \end{aligned}$$

which in turn yields (10) through either standard computations of conditional expectation or by simply noting that it is given by the Black–Scholes formula with the interest rate  $f^\beta$  and no dividends.

**Remarks 2.1.** (i) If we model the pre-default rate of the defaultable bond as  $r = r_t^C = r^{CDS} + f$ , where the CDS spread  $r^{CDS}$  (rather than the bond return  $r^C$ ) is taken as a model's input, then the pricing equation (10) holds with  $q := r^{CDS} - \beta(h - f)$ . In other words, the option pricing formula (10) is still valid when the defaultable bond is replaced by the counterparty's CDS in our trading model.

(ii) PDE (9) is in fact equivalent to PDE (32) obtained in [4, Eq. 4.4] using the martingale approach. To see this, it suffices to rewrite (9) with the dynamics of the primary assets

$$\begin{aligned} dB_t^f &= fB_t^f dt, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= B_{t-}(\mu_3 dt - dM_t) = B_{t-}((\mu_3 + \xi_t)dt - dJ_t), \end{aligned}$$

where  $\mu_3 = f$  and  $M_t := J_t + \log G_{t \wedge \tau} = J_t - \xi_t$  where  $G_t := \mathbb{P}(\tau > t | \mathcal{F}_t)$ . The process  $M$  is commonly known as the compensated  $\mathbb{G}$ -martingale of the default process  $J$ .

(iii) Though the stock  $S$  was assumed to pay no dividends, the present framework can be easily extended to the dividend-paying case. As a result, the effective funding rate  $f^\beta$  should be replaced by  $f^\beta - \delta$ .

### 3. Vulnerable call option pricing by adjusted cash flows approach

Let us consider again the problem of pricing the same vulnerable option, but this time using the **adjusted cash flows approach** originated in Pallavicini et al. [20], derived rigorously in Brigo et al. [11] and presented in a wider context in Brigo and Pallavicini [16]. We do not make here an attempt to justify their approach, but we start instead with the pricing Eq. (11) of Brigo and Pallavicini [16] and adapt it to the present context of a vulnerable call option, which is an uncollateralized contract. Note that the variation margin is  $M$ , while  $N^C$  and  $N^I$  are the initial margin accounts for the two counterparties, resulting in the total collateral account  $C = M + N^C + N^I$ . In our case, this means that all terms appearing in the last two lines of Eq. (11) in Brigo and Pallavicini [16] vanish. Moreover, the cash flow at default equals zero (due to zero recovery convention for the vulnerable option) and the promised cash flow over time period  $(t, t + dt)$  is

$$\Pi(t, t + dt) = (S_T - K)^+ \mathbb{1}_{\{T \in (t, t+dt)\}}.$$

Hence the pricing Eq. (11) in Pallavicini et al. [20] reduces to

$$V_t = E^h[\mathbb{1}_{\{\tau > T\}} D(t, T; f)(S_T - K)^+ | \mathcal{G}_t] \quad (12)$$

where  $\mathbb{E}^h$  is the expectation with respect to the probability measure  $\mathbb{Q}^h$  that makes the drift of the risky asset equal to  $h$ , meaning that

$$dS_t = hS_t dt + \sigma S_t dW_t^h$$

where  $W^h$  is a Brownian motion under  $\mathbb{Q}^h$ . Furthermore,  $\mathbb{G} = (\mathcal{G}_t)$  is the full filtration that includes the information on default times and the discount factor  $D(s, t; f)$  equals

$$D(s, t; f) := \exp\left(-\int_s^t f_u du\right).$$

We henceforth assume a constant treasury rate  $f$  and we use the pre-default intensity  $\lambda$  under  $\mathbb{Q}^h$  of the counterparty, which is defined in [16, (40)] by

$$\mathbb{1}_{\{\tau > t\}} \lambda dt := \mathbb{Q}^h(\tau \in dt | \tau > t, \mathcal{F}_t),$$

to obtain the survival probability  $G_t^h = e^{-\lambda t}$  where  $G_t^h := \mathbb{Q}^h(\tau > t | \mathcal{F}_t)$ . Note that this is consistent with the assumptions on  $\tau$  in the replication approach of Section 2. Using (12) and Cor. 3.1.1 of Bielecki et al. [3], we obtain

$$V_t = \mathbb{1}_{\{\tau > t\}} (G_t^h)^{-1} \mathbb{E}^h[D(t, T; f)(S_T - K)^+ | \mathcal{F}_t].$$

If  $\tilde{V}$  denotes the  $\mathbb{F}$ -adapted pre-default price process such that for all  $t \in [0, T]$

$$\mathbb{1}_{\{\tau > t\}} V_t = \mathbb{1}_{\{\tau > t\}} \tilde{V}_t,$$

then from the above equation we immediately obtain

$$\tilde{V}_t = (G_t^h)^{-1} \mathbb{E}^h[D(t, T; f)(S_T - K)^+ | \mathcal{F}_t].$$

Since  $G^h$  is deterministic, for a constant treasury rate  $f$ , the pre-default price can be written as

$$\tilde{V}_t = e^{-(\lambda+f)(T-t)} \mathbb{E}^h[(S_T - K)^+ | \mathcal{F}_t]$$

or, equivalently,

$$\tilde{V}_t = e^{-(\lambda+f-h)(T-t)} \mathbb{E}^h[e^{-h(T-t)}(S_T - K)^+ | \mathcal{F}_t].$$

The last expectation can be computed yielding the usual Black–Scholes formula when the drift of the stock equals  $h$

$$\mathbb{E}^h[e^{-h(T-t)}(S_T - K)^+ | \mathcal{F}_t] = S_t N(d_1) - Ke^{-h(T-t)} N(d_2)$$

where

$$d_1 = \frac{\log \frac{S_t}{K} + (h + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t}.$$

We conclude that the pre-default price process satisfies

$$\tilde{V}_t = e^{-(\lambda+f-h)(T-t)} (S_t N(d_1) - Ke^{-h(T-t)} N(d_2))$$

and thus

$$V_t = \mathbb{1}_{\{\tau > t\}} (S_t e^{-(\lambda+f-h)(T-t)} N(d_1) - Ke^{-(\lambda+f)(T-t)} N(d_2)). \quad (13)$$

Upon setting  $\lambda + f - h = q$  and  $\lambda + f = r$ , we deduce that (13) coincides with the pricing formula (10) obtained by replication for  $\beta = 1$ . It is also not difficult to show that  $\lambda = r^{CDS}$ . This shows that the adjusted cash flow method and the replication approach lead to the same price for the vulnerable call option.

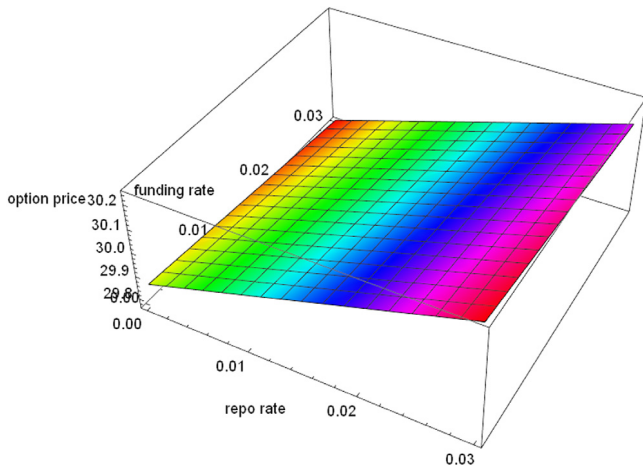


Fig. 4.1. Option price is increasing in repo rate  $h$  and decreasing in funding rate  $f$ .

#### 4. Sensitivity analysis

In the final step, we perform the sensitivity analysis for the vulnerable call option by focusing on the impact of the rates  $f$  and  $h$ . We leave aside the parameter  $r^C$ , since in our model the investor has the freedom to choose a particular combination of funding sources when purchasing shares, as formally represented by the parameter  $\beta \in [0, 1]$ , but the spread  $r^C$  is given by the market and thus it is natural to assume that it is fixed.

**Example 4.1.** Fig. 4.1 shows the dependence of the pre-default price (13) of a vulnerable call on the treasury rate  $f$  and repo rate  $h$  for  $\beta = 1$  when  $S_t = 80$ ,  $K = 100$ ,  $\sigma = 0.3$ ,  $T - t = 0.1$ ,  $r^{CDS} = 0.05$ . The pre-default price of the vulnerable call is decreasing in the treasury rate  $f$  and increasing in the repo rate  $h$ .

To explain the dependence observed in Example 4.1 and perform a general sensitivity analysis, we first compute “funding Greeks” when  $\beta = 1$ , that is, all shares are purchased at repo. We obtain the following expressions

$$\partial_f \tilde{V}_t = \partial_{r^C} \tilde{V}_t = -(T - t) \tilde{V}_t < 0, \quad (14)$$

$$\partial_h \tilde{V}_t = e^{-(h-f-r^{CDS})(T-t)}(T-t)S_t N(d_1^q) > 0, \quad (15)$$

which means that the pre-default call price decreases in both the treasury rate  $f$  and the bond return  $r^C$ , but increases in the repo rate associated with the risky asset. Furthermore, the relative sensitivity to funding  $\frac{\partial_f \tilde{V}_t}{\tilde{V}_t} = -(T - t)$  appears to be smaller in absolute value than the relative sensitivity to the repo rate  $\frac{\partial_h \tilde{V}_t}{\tilde{V}_t} > T - t$ . This simple benchmark case highlights that the repo rate may have an important impact on the contract value, often more significant than the treasury rate or the credit spread.

Let us now consider the price obtained in Section 2 where the additional parameter  $\beta \in [0, 1]$  dictates the structure of the funding arrangements for the investor. In view of (10) and Remark 2.1(i), we obtain the following funding Greeks:

$$\begin{aligned} \partial_f \tilde{V}_t &= -\beta(T - t) \tilde{V}_t \\ &\quad + (1 - \beta)(T - t)e^{(\beta(h-f)-r^{CDS})(T-t)}KN(d_2^q), \end{aligned} \quad (16)$$

$$\partial_h \tilde{V}_t = \beta e^{(\beta(h-f)-r^{CDS})(T-t)}(T - t)S_t N(d_1^q) \geq 0,$$

where the last inequality is strict when  $\beta > 0$ . In particular, for  $\beta = 1$  we recover (14)–(15) and for  $\beta = 0$  (pure treasury funding), we get

$$\partial_f \tilde{V}_t = (T - t)e^{(f-r^C)(T-t)}KN(d_2^q) > 0,$$

$$\partial_h \tilde{V}_t = 0,$$

where  $f - r^C = -r^{CDS} < 0$ . In general, it is hard to determine the sign of the sensitivity  $\partial_f \tilde{V}_t$  given by (16), though it is clear that it changes from a positive value for  $\beta = 0$  to a negative one for  $\beta = 1$ .

To give an interpretation of funding Greeks, we observe that the contract's payoff can be written as  $X = B_T(S_T - K)^+$ , so it can be seen as a hybrid contract which combines the call option on the stock with the long position in the counterparty bond. For any  $0 < \beta \leq 1$  the price  $\tilde{V}_t$  increases in  $h$ , since the cost of hedging the option component  $(S_T - K)^+$  is manifestly increasing with  $h$ .

The price dependence on  $f$  is a bit harder to analyze. Indeed, from representation (5) of the hedging portfolio, we see that for  $0 < \beta < 1$  the dependence on  $f$  is rather complex: the investor needs to borrow cash from  $B^f$  (which grows at the rate  $f$ ) and thus the cost of hedging increases in  $f$ , but he simultaneously invests in the bond  $B$  (with the rate of return  $r^C = f + r^{CDS}$  where  $r^{CDS}$  is constant) so that the cost of hedging decreases in  $f$ . The net impact of both legs may be negative, in the sense that the price of the option decreases when  $f$  increases. This is rather clear for  $\beta = 1$ , since in that case the investor does not use  $B^f$  for his hedging purposes (take  $\beta = 1$  in (5)) and we see that the cost of hedging the component  $B_T$  in the payoff  $X$  falls when  $f$  increases. By contrast, when  $\beta = 0$  the value of  $h$  is immaterial, and the increase of  $f$  makes the option more expensive. Finally, when only the CDS spread  $r^{CDS}$  increases and  $f$ ,  $h$  are kept fixed, then the cost of hedging decreases as well, since the bond  $B$  becomes cheaper.

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