GROMOV–WITTEN INVARIANTS OF LOCAL $\mathbb{P}^2$ AND MODULAR FORMS

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Abstract. We construct a sheaf of Fock spaces over the moduli space of elliptic curves E_y with $\Gamma_1(3)$-level structure, arising from geometric quantization of $H^1(E_y)$, and a global section of this Fock sheaf. The global section coincides, near appropriate limit points, with the Gromov–Witten potentials of local $\mathbb{P}^2$ and of the orbifold $[\mathbb{C}^3/\mu_3]$. This proves that the Gromov–Witten potentials of local $\mathbb{P}^2$ are quasi-modular functions for the group $\Gamma_1(3)$, as predicted by Aganagic–Bouchard–Klemm, and proves the Crepant Resolution Conjecture for $[\mathbb{C}^3/\mu_3]$ in all genera.

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1. Introduction

Let Y be the total space $K_{\mathbb{P}^2}$ of the canonical line bundle of $\mathbb{P}^2$, and let $\mathcal{X}$ denote the orbifold $\big[\mathbb{C}^3/\mu_3\big]$, where the group μ_3 of third roots of unity acts with weights $(1,1,1)$. Let

$$F_Y^g = -\frac{t^3}{18}\delta_{g,0} - \frac{t}{12}\delta_{g,1} + \sum_{d=0}^{\infty} n_{g,d} e^{dt}, \qquad F_{\mathcal{X}}^g = \sum_{k=1}^{\infty} n_{g,k}^{\mathrm{orb}} \frac{\mathfrak{t}^k}{k!}$$

denote the genus-g Gromov–Witten potentials of Y and $\mathcal{X}$ respectively. Here $n_{g,d}$ is the genus-g, degree-d Gromov–Witten invariant of Y, and $n_{g,k}^{\mathrm{orb}}$ is the genus-g Gromov–Witten invariant of $\mathcal{X}$ with k insertions of the age-1 orbifold class. We regard F_Y^g as a function of $t \in H^2(Y)$, and $F_{\mathcal{X}}^g$ as a function of $\mathfrak{t} \in H^2_{\mathrm{orb}}(\mathcal{X})$. The main result of this paper is:

Theorem A (see Corollary 10.3.5, Theorem 10.5.3, Theorem 10.3.9 for precise statements)**.** *Introduce modular parameters τ, τ_{orb} by*

$$\begin{aligned} \tau &= -\frac{1}{2} - \frac{3}{2\pi\mathtt{i}}\frac{\partial^2 F_Y^0}{\partial t^2} & \tau_{\mathrm{orb}} &= 3\frac{\partial^2 F_{\mathcal{X}}^0}{\partial \mathfrak{t}^2} \\ &= -\frac{1}{2} + \frac{t}{2\pi\mathtt{i}} + O(e^t) & &= \mathfrak{t} + O(\mathfrak{t}^4) \end{aligned}$$

Then:

1. *when regarded as a function of τ, F_Y^g extends to a holomorphic function on the upper half-plane $\mathbb{H}$;*
2. *when regarded as a function of τ_{orb}, $F_{\mathcal{X}}^g$ extends to a holomorphic function on the disc $|\tau_{\mathrm{orb}}| < r$, where $r = \Gamma(\frac{1}{3})^3/\Gamma(\frac{2}{3})^3$;*
3. *for $g \geq 2$, F_Y^g is a quasi-modular function with respect to the congruence subgroup:*
$$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1,\ c \equiv 0 \bmod 3 \right\};$$
4. (Crepant Resolution Conjecture) *$\{F_Y^g\}$ and $\{F_{\mathcal{X}}^g\}$ are related by an explicit Feynman diagram expansion, which takes the following form for $g \geq 2$:*
$$F_{\mathcal{X}}^g = F_Y^g + (\textit{polynomial expressions in } \{\partial_t^k F_Y^h : 0 \leq h < g, 1 \leq k \leq 3g-3\})$$
where $\tau_{\mathrm{orb}} = r \cdot \frac{3\tau+1-\xi}{3\tau+1-\xi^2}$.

This proves conjectures of Aganagic–Bouchard–Klemm [2].

1.1. Geometric Quantization and the Fock Sheaf.

Aganagic–Bouchard–Klemm's prediction was based on Witten's discovery [81] that a topological string partition function

$$Z = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} F^g\right) \tag{1}$$

can be understood as a 'wave function' of a quantum-mechanical system that arises from geometric quantization of the state space of the theory. In the present setting, the state space is given by $H^1(E_y)$, where

$$E_y = \text{compactification of } \left\{(x_1,x_2) \in (\mathbb{C}^\times)^2 : x_1 + x_2 + \tfrac{y}{x_1 x_2} + 1 = 0\right\} \tag{2}$$

is the family of elliptic curves, parametrised by $y \in \overline{\mathbb{H}/\Gamma_1(3)} \cong \mathbb{P}(3,1)$, that corresponds to Y under mirror symmetry. Quasi-modularity follows by 'quantizing' monodromies of this family. The aim of the present paper is to verify this physics picture for local $\mathbb{P}^2$ mathematically.

Let us begin with the genus-zero part of the story. The graph of dF_Y^0 defines a Lagrangian submanifold $\mathcal{L}$ in the cotangent bundle $T^*H^2(Y) \cong H^2(Y) \oplus H^4(Y)$ of $H^2(Y)$, where we regard $H^4(Y) = H^4_c(Y)$ as the dual of $H^2(Y)$ via the intersection pairing:

$$\mathcal{L} = \Gamma(dF_Y^0) = \left\{ (t,p) \in H^2(Y) \oplus H^4(Y) : p = \frac{\partial F_Y^0}{\partial t} \right\}.$$

The Crepant Resolution Conjecture at genus zero – proved in this case by [17, 27] – says that the graphs of dF_Y^0 and $dF_{\mathcal{X}}^0$ coincide under an affine symplectic transformation $U\colon T^*H^2(Y) \to T^*H^2_{\mathrm{orb}}(\mathcal{X})$: see Figure 1. The family $\{T_t\mathcal{L}\}$ of tangent spaces to $\mathcal{L}$ defines a *variation of Hodge structure* (VHS) of weight 1; under mirror symmetry, this is identified with the VHS on $H^1(E_y)$ of the mirror family. The mirror curve E_y is parameterized by $y \in \mathbb{P}(3,1) \cong \overline{\mathbb{H}/\Gamma_1(3)}$ and the Gromov–Witten potentials F_Y^0, $F_{\mathcal{X}}^0$ describe the local behaviour of the VHS near $y = 0$ (the large-radius limit) and $y = \infty$ (the orbifold point) respectively. An important observation here is that the directions of the 'y-axes' $H^4(Y)$ and $H^4_{\mathrm{orb}}(\mathcal{X})$ do not coincide. In higher genus, these y-axes play the role of a polarization in geometric quantization.

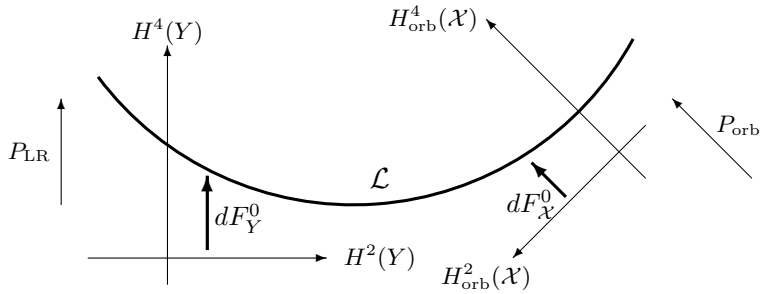


FIGURE 1. Genus-zero Crepant Resolution Conjecture for $\mathcal{X} = [\mathbb{C}^3/\mu_3]$

Geometric quantization (see [60]) associates to a symplectic vector space H a Hilbert space, called the Fock space, which is an irreducible representation of the Heisenberg algebra associated with H. To construct such a representation, we need the data of a *polarization*, that is, a Lagrangian subspace P of H. Given a polarization P, the Fock space is a space $\mathrm{Fun}(H/P)$ of functions on H/P of an appropriate class (C^∞, L^2, Schwartz, etc). It carries an action of the Heisenberg algebra given by 'canonical quantization'. For instance, if H is a 2-dimensional symplectic vector space with Darboux co-ordinates (p,x), and if we choose P to be the subspace $\langle \partial/\partial p \rangle$, then the corresponding Fock space is a space of functions of x, and the Heisenberg algebra acts by operators x and $\hbar\partial/\partial x$. If we have two different polarizations P_1, P_2, the corresponding Fock spaces are canonically isomorphic as projective representations of the Heisenberg algebra:

$$T(P_1,P_2)\colon \mathrm{Fun}(H/P_1) \xrightarrow{\cong} \mathrm{Fun}(H/P_2)$$

by the Stone–von Neumann theorem. Such an isomorphism is given by an integral transformation of Fourier type.

We are interested in Fock space elements of the form (1), which can be viewed as asymptotic series in $\hbar$. Agangagic–Bouchard–Klemm [2] described the isomorphism $T(P_1,P_2)$ for such asymptotic functions using a sum over Feynman diagrams. Using their Feynman rule, we construct in §7 a sheaf $\mathfrak{Fock}_{\mathrm{CY}}$ of Fock spaces[1] over the base $\mathbb{P}(3,1)$ of the mirror family.

[1]This is a sheaf of sets, not of vector spaces, as functions of the form (1) are not closed under addition.

Note that we need to construct a Fock *sheaf* here, instead of a Fock *space*, because there is no globally defined, single-valued, flat polarization over the moduli space $\mathbb{P}(3,1) \cong \overline{\mathbb{H}/\Gamma_1(3)}$, because the mirror family has non-trivial monodromies. Roughly speaking, we:

(a) choose an open covering $\{U_\alpha\}$ of $\mathbb{P}(3,1)$ by sufficiently small open sets U_α;
(b) choose a Gauss–Manin flat polarization $P_\alpha \subset H^2(E_y)$ such that $P_\alpha \pitchfork \mathcal{L}$ over U_α, i.e. $P_\alpha \oplus H^{1,0}(E_y) = H^1(E_y)$ for each $y \in U_\alpha$;
(c) define $\Gamma(U_\alpha, \mathfrak{Fock}_{\mathrm{CY}})$ to be the space of asymptotic series $\exp(\sum_{g=1}^{\infty} \hbar^{g-1} F^g)$, where F^g is a holomorphic function on U_α;

and then patch local Fock spaces $\Gamma(U_\alpha, \mathfrak{Fock}_{\mathrm{CY}})$ over overlaps $U_\alpha \cap U_\beta$ using the Feynman rule of Aganagic–Bouchard–Klemm. Theorem A above is a consequence of the following more fundamental result:

Theorem B (Theorem 8.6.1, Theorem 9.0.3)**.** *There exists a global section $\mathscr{C}_{\mathrm{CY}}$ of the Fock sheaf $\mathfrak{Fock}_{\mathrm{CY}}$ such that:*

(1) *in a neighbourhood of the large-radius limit point $y = 0$ and with respect to the polarization $P_{\mathrm{LR}} = H^4(Y)$, $\mathscr{C}_{\mathrm{CY}}$ is represented by the Gromov–Witten potentials of Y;*
(2) *in a neighbourhood of the orbifold point $y = \infty$ and with respect to the polarization $P_{\mathrm{orb}} = H^4_{\mathrm{orb}}(\mathcal{X})$, $\mathscr{C}_{\mathrm{CY}}$ is represented by the Gromov–Witten potentials of $\mathcal{X}$;*
(3) *in a neighbourhood of the conifold point $y = -\frac{1}{27}$ and with respect to a polarization P_{con}, $\mathscr{C}_{\mathrm{CY}}$ is represented by a collection $\{F^g_{\mathrm{con}}\}$ of functions such that F^g_{con} has poles of order $2g-2$ at $y_1 = -\frac{1}{27}$ for $g \geq 2$.*

There are various possible choices of polarization, which are summarized in Table 1. The VHS of the mirror family has singularities at the large-radius ($y = 0$), conifold ($y = -\frac{1}{27}$), and orbifold ($y = \infty$) points. Near these points, there are unique flat polarizations P_{LR}, P_{con}, P_{orb} characterized by invariance under local monodromy (Notation 7.2.7, Proposition 10.3.2). These polarizations become multi-valued when they are analytically continued.

On the other hand, we can also consider polarizations which are *not* Gauss-Manin flat, but are *single-valued*. Expressing the global section $\mathscr{C}_{\mathrm{CY}}$ with respect to a single-valued polarization yields $\exp(\sum_{g=1}^{\infty} \hbar^{g-1} F^g)$, where the correlation functions F^g are single-valued on $\mathbb{P}(3,1)$. The polarization P_{cc} defined by $H^{0,1}(E_y) = \overline{H^{1,0}(E_y)}$, which we call the complex-conjugate polarization, is single-valued, and coincides with P_{LR} at $y = 0$, P_{con} at $y = -\frac{1}{27}$, and P_{orb} at $y = \infty$. It varies non-holomorphically along $\mathbb{P}(3,1)$, and correlation functions for $\mathscr{C}_{\mathrm{CY}}$ with respect to P_{cc} satisfy the Bershadsky–Cecotti–Ooguri–Vafa holomorphic anomaly equation [9] (Proposition 10.6.3). We can also obtain single-valued polarizations by considering the algebraic structure of the bundle $H^1(E_y)$ over $\mathbb{P}(3,1)$: $\bigcup_y H^1(E_y) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$. The algebraic polarization P_{alg} is a single-valued holomorphically-varying non-flat polarization, corresponding to $\mathcal{O}(-1)$ over $\mathbb{P}(3,1)$; correlation functions for P_{alg} can be thought of as the 'holomorphic ambiguity' in the holomorphic anomaly equation. Correlation functions for P_{alg} are rational functions, and it follows that the Gromov–Witten potentials F^g_Y, $F^g_{\mathcal{X}}$ belong to certain polynomial rings (see Theorem 10.7.3).

Remark 1.1.1. Polarizations are called 'opposite line bundles' in the main body of the text.

Remark 1.1.2. Lho–Pandharipande [65, 66] also proved a similar finite generation result, and a version of the Crepant Resolution Conjecture for Y and $\mathcal{X}$. We give a proof of their version of Crepant Resolution Conjecture using our method below (Theorem 10.7.3) but we learned its elegant formulation from them.

polarization	flat/curved	global behaviour	correlation functions
P_{LR}	flat	multi-valued	F_Y^g, quasi-modular
P_{orb}	flat	multi-valued	$F_{\mathcal{X}}^g$
P_{con}	flat	multi-valued	quasi-modular
P_{cc}	curved	single-valued	almost-holomorphic modular
P_{alg}	curved	single-valued	holomorphic modular (rational functions)

TABLE 1. Various Polarizations

Remark 1.1.3. It was conjectured by Huang–Klemm [52, 53] that the correlation function F_{con}^g with respect to P_{con} should satisfy a certain 'gap condition' – see (127). We do not have a proof of this conjecture, but verify it up to genus $g = 7$. See §10.8.

1.2. **Summary of the Argument.** In outline: we pass from Y and $\mathcal{X}$ to their toric compactifications $\overline{Y} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(-3) \oplus \mathcal{O})$ and $\overline{\mathcal{X}} = \mathbb{P}(1,1,1,3)$. These have generically semisimple quantum cohomology, which is not true for Y or $\mathcal{X}$. We determine the Gromov–Witten potentials of $\overline{Y}$ and $\overline{\mathcal{X}}$ using the Givental–Teleman formula; this requires semisimplicity. We relate the two potentials via mirror symmetry for $\overline{Y}$ and $\overline{\mathcal{X}}$. The Gromov–Witten potentials of $\overline{Y}$ and $\overline{\mathcal{X}}$ glue together to give a single-valued section $\mathscr{C}_{\mathrm{B}}$ of an infinite-dimensional version of the Fock sheaf, which we constructed in [25]; this is a higher-genus version of the Crepant Resolution Conjecture for $\overline{Y}$ and $\overline{\mathcal{X}}$. The finite-dimensional version of the Fock sheaf, and the global section $\mathscr{C}_{\mathrm{CY}}$, emerge from their infinite-dimensional counterparts by taking a certain 'conformal limit' or 'local limit'. In this limit, the volume of the fiber of $\overline{Y} \to \mathbb{P}^2$ becomes infinitely large, and the Gromov–Witten theory of $\overline{Y}$ reduces to that of Y.

Let us explain some more details. We consider the Landau–Ginzburg model that is mirror to the small quantum cohomology of $\overline{Y}$. This is given by

$$W_{y_1,y_2} = \left(x_1 + x_2 + \frac{y_1}{x_1 x_2} + 1\right) x_3 + \frac{y_2}{x_3} \qquad (x_1, x_2, x_3) \in \left(\mathbb{C}^\times\right)^3 \tag{3}$$

where $(y_1, y_2) \in (\mathbb{C}^\times)^2$. This family of Laurent polynomials extends over a partial compactification $\mathcal{M}_{\mathrm{B}}$ of $(\mathbb{C}^\times)^2$, where the limits

$$(y_1, y_2) \to (0,0) \qquad\qquad (y_1^{-1/3}, y_1^{1/3} y_2) \to (0,0)$$

correspond respectively to the large-radius limit for $\overline{Y}$ and the large-radius limit for $\overline{\mathcal{X}}$: see Figure 2.

The Landau–Ginzburg mirror determines an infinite-dimensional symplectic vector bundle over $\mathcal{M}_{\mathrm{B}}$, with fiber over $(y_1, y_2) \in (\mathbb{C}^\times)^2$ equal to

$$H^3\big(\Omega^\bullet_{(\mathbb{C}^\times)^3}(\!(z)\!), zd + dW_{y_1,y_2} \wedge \big)$$

This vector bundle carries a flat Gauss–Manin connection, which has logarithmic singularities along $y_1 = 0$, $y_2 = 0$, and $y_1 = -\frac{1}{27}$, and has a Lagrangian subbundle

$$\mathsf{F}_{\mathrm{B}} = H^3\big(\Omega^\bullet_{(\mathbb{C}^\times)^3}[\![z]\!], zd + dW_{y_1,y_2} \wedge \big)$$

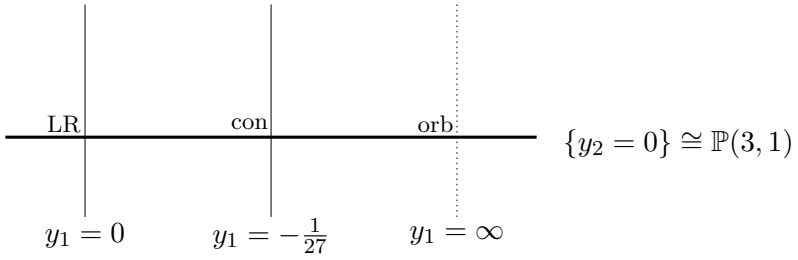


FIGURE 2. The B-model moduli space $\mathcal{M}_{\mathrm{B}}$: this is the base space of the family W_{y_1,y_2} of Landau–Ginzburg potentials.

Such structures have been studied by K. Saito [74] in the context of singularity theory. By transporting the Lagrangian subspaces F_{B} in the fibers to a fixed fiber[2], using the Gauss–Manin connection, we obtain a moving family of semi-infinite Lagrangian subspaces. This is an example of a *variation of semi-infinite Hodge structure* (VSHS) [5].

On the other side of mirror symmetry, we consider the descendant potentials $\mathcal{F}^g_{\overline{Y}}$ and $\mathcal{F}^g_{\overline{\mathcal{X}}}$, which are generating functions for genus-g Gromov–Witten invariants of $\overline{Y}$ and $\overline{\mathcal{X}}$ with descendants. $\mathcal{F}^g_{\overline{Y}}$ and $\mathcal{F}^g_{\overline{\mathcal{X}}}$ are functions on an infinite-dimensional space, and the graph of the differential $d\mathcal{F}^0_{\overline{Y}}$ defines a Lagrangian submanifold

$$\mathfrak{L}_{\overline{Y}} \subset H^\bullet(\overline{Y}) \otimes \mathbb{C}(\!(z^{-1})\!)$$

in Givental's symplectic space $H^\bullet(\overline{Y}) \otimes \mathbb{C}(\!(z^{-1})\!)$. Under mirror symmetry, the tangent spaces to the Givental cone $\mathfrak{L}_{\overline{Y}}$ are identified with the VSHS determined by the Landau–Ginzburg mirror, near the large-radius limit point for $\overline{Y}$. Analogous statements hold for $\overline{\mathcal{X}}$.

The Landau–Ginzburg model (3) is a mirror to the *small* quantum cohomology of $\overline{Y}$, rather than the full big quantum cohomology, and therefore the VSHS that it determines is not miniversal. In §5 we construct a miniversal unfolding of this semi-infinite variation, over a six-dimensional base $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$, that is a mirror to the big quantum cohomology of $\overline{Y}$. The base $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ is a thickening of $\mathcal{M}_{\mathrm{B}} \setminus \{y_1 = -\frac{1}{27}\}$. It carries an infinite-dimensional version $\mathfrak{Fock}_{\mathrm{B}}$ of the Fock sheaf, which we constructed in [25] and review in §6 below, and furthermore there is a distinguished global section $\mathscr{C}_{\mathrm{B}}$ of this Fock sheaf. The global section $\mathscr{C}_{\mathrm{B}}$ is constructed using Givental's formula for higher-genus potentials [41]; see [25, §7.2]. It coincides under mirror symmetry, near the large-radius limit points for $\overline{Y}$ and $\overline{\mathcal{X}}$, with the total descendant potentials $\mathcal{Z}_{\overline{Y}} = \exp(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}^g_{\overline{Y}})$ of $\overline{Y}$ and $\mathcal{Z}_{\overline{\mathcal{X}}}$ of $\overline{\mathcal{X}}$. This follows from Teleman's theorem [80] [25, Theorem 7.15].

Now we take the local limit. Observe that the Landau–Ginzburg potential W_{y_1,y_2} with y_2 set to zero defines the family (2) of elliptic curves E_{y_1}; thus the divisor $(y_2 = 0)$ in $\mathcal{M}_{\mathrm{B}}$ can be identified with the base $\mathbb{P}(3,1) \cong \overline{\mathbb{H}/\Gamma_1(3)}$ of the mirror family of Y. A key step in the argument is the construction of a "restriction map" from the infinite-dimensional Fock sheaf $\mathfrak{Fock}_{\mathrm{B}}$ over $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ to the finite-dimensional Fock sheaf $\mathfrak{Fock}_{\mathrm{CY}}$ over $\mathbb{P}(3,1)$. This requires care, as the VSHS associated with W_{y_1,y_2} has logarithmic singularities along $y_2 = 0$. We also need a result comparing polarizations for the VSHS with polarizations for the VHS.

[2]We are hiding some technical details here. To obtain a moving subspace realization, we need to analytify F_{B} in the z-direction. The analytification is denoted by $\mathcal{F}_{\mathrm{B}}$ in the main body of the text.

Theorem C (see Propositions 4.3.1, 5.1.5, 5.1.6 and 5.1.7, Notation 7.2.7, Theorem 8.0.8)**.** *Let $y_1 \in \mathbb{P}(3,1)$. There is a one-to-one correspondence between:*

(a) *flat polarizations near y_1 for the VSHS associated with the Landau–Ginzburg mirror that are compatible with the Deligne extension;*
(b) *flat polarizations near y_1 for the VHS associated with the mirror family $\{E_y\}$ of elliptic curves.*

Let $\mathfrak{Fock}_{\mathrm{B}}$ denote the infinite-dimensional Fock sheaf over $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ and let $\mathfrak{Fock}_{\mathrm{CY}}$ denote the finite-dimensional Fock sheaf over $\mathbb{P}(3,1)$. Write $\mathbb{P}(3,1)^\circ = \mathbb{P}(3,1) \setminus \{-\frac{1}{27}\}$ for the complement of the conifold point, and let $i\colon \mathbb{P}(3,1)^\circ \hookrightarrow \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ denote the inclusion map. There is a restriction map:

$$i^{-1}\mathfrak{Fock}_{\mathrm{B}} \longrightarrow \mathfrak{Fock}_{\mathrm{CY}}\big|_{\mathbb{P}(3,1)^\circ}$$

By applying the restriction map to the global section $\mathscr{C}_{\mathrm{B}}$ of $\mathfrak{Fock}_{\mathrm{B}}$, we obtain a section $\mathscr{C}_{\mathrm{CY}}$ of $\mathfrak{Fock}_{\mathrm{CY}}$ over $\mathbb{P}(3,1)^\circ$. It is then easy to check that $\mathscr{C}_{\mathrm{CY}}$ corresponds to the Gromov–Witten potentials of Y and $\mathcal{X}$, respectively, near $y_1 = 0$ and $y_1 = \infty$. In §9, we show that the genus-g potential of $\mathscr{C}_{\mathrm{CY}}$ has poles of order $2g-2$ at the conifold point $y_1 = -\frac{1}{27}$, by analysing the pole order of the ingredients in Givental's formula for higher-genus potentials. This proves Theorem B.

Remark 1.2.1. In the main body of the text, we consider various versions of VSHS but do not use the term 'VSHS' itself, instead using the equivalent notions of TEP structures, log-TEP structures, and log-cTEP structures.

Remark 1.2.2 (Related work)**.** Higher-genus Gromov–Witten invariants of local $\mathbb{P}^2$ and $\left[\mathbb{C}^3/\mu_3\right]$ have been studied by many authors. In string theory, Alim–Scheidegger–Yau–Zhou [4], Huang–Klemm [52], and Huang–Klemm–Quackenbush [53] have emphasized the importance of special geometry and the holomorphic anomaly equations. On the mathematics side, Bouchard–Cavalieri have computed Gromov–Witten invariants of $\left[\mathbb{C}^3/\mu_3\right]$ at genus 2 and 3 using Hodge and Hurwitz–Hodge integrals [13]. Lho–Pandharipande have recently established the holomorphic anomaly equation for local $\mathbb{P}^2$, in the precise form predicted by physicists, and used this to prove a higher-genus Crepant Resolution Conjecture for $\left[\mathbb{C}^3/\mu_3\right]$ [65, 66]. Another approach goes via the Remodelling Conjecture of Bouchard–Klemm–Mariño–Pasquetti [14] and Eynard–Orantin recursion [33]. Fang–Liu–Zong [34] have established the Remodelling Conjecture for all toric Calabi–Yau 3-orbifolds, and this should lead to a proof of modularity and the holomorphic anomaly equation in our setting.

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List of Notation

$V_{\mathrm{A},X} \subset H_X$	open subset of the form (8).
$\mathcal{M}^\times_{\mathrm{A},X} = \frac{V_{\mathrm{A},X}}{2\pi\mathrm{i}H^2(X,\mathbb{Z})}$	the base of the A-model TEP structure; see Example 2.7.5.
$(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X})$	the base of the A-model log-TEP structure; $\mathcal{M}^\times_{\mathrm{A},X} = \mathcal{M}_{\mathrm{A},X} \setminus D_{\mathrm{A},X}$; see Example 2.7.6, Theorem 5.4.1.
$(\widetilde{\mathcal{F}}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$	see Example 2.7.5.
$(\mathcal{F}^\times_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$	A-model TEP structure; see Example 2.7.5, Theorems 3.3.1, 3.3.2.
$(\mathcal{F}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$	A-model log-TEP structure; see Theorem 5.4.1.
$(\mathsf{F}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$	A-model log-cTEP structure; see Example 6.1.5.
$\mathbf{P}_{\mathrm{A}}$	canonical opposite module for the A-model TEP structure; see Example 2.8.6.
P_{A}	canonical opposite module for the A-model log-cTEP structure; see Example 6.7.5.
$\mathfrak{Fock}_{\mathrm{A},X}$	A-model Fock sheaf; see Definition 6.8.10.
$(\mathcal{M}_{\mathrm{B}}, D)$	the base of the B-model log-TEP structure; see (23) and Proposition-Definition 3.5.4.
$\mathcal{M}^\times_{\mathrm{B}}$	$\mathcal{M}_{\mathrm{B}} \setminus D$; the base of the B-model TEP structure; see (23), Definition 3.2.2.
$\mathcal{M}^\circ_{\mathrm{B}}$	$\mathcal{M}_{\mathrm{B}} \setminus \{y_1 = -1/27\}$; see Theorem 5.0.3.
$(\mathcal{M}^{\mathrm{big}}_{\mathrm{B}}, D^{\mathrm{big}})$	the base of the big B-model log-TEP structure; see Theorem 5.0.3; this contains $\mathcal{M}^\circ_{\mathrm{B}}$ but not $\mathcal{M}_{\mathrm{B}}$.
$(\mathcal{M}_{\mathrm{CY}}, D_{\mathrm{CY}})$	$\big(\mathbb{P}(3,1), \{0, -\frac{1}{27}\}\big)$; the base of $\mathcal{F}_{\mathrm{CY}}$, F_{CY} (§4.1) and $\mathfrak{Fock}_{\mathrm{CY}}$ (§7.3).
$\mathcal{M}^\circ_{\mathrm{CY}}$	$\mathbb{P}(3,1) \setminus \{-\frac{1}{27}\}$; the base of $\mathfrak{Fock}^\circ_{\mathrm{CY}}$ (§8).
$(\mathcal{F}^\times_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$	the B-model TEP structure (Definition 3.2.2) with base $\mathcal{M}_{\mathrm{B}}$.
$(\mathcal{F}^\times_{\mathrm{GKZ}}, \nabla^{\mathrm{GKZ}})$	the GKZ system, isomorphic to $(\mathcal{F}^\times_{\mathrm{B}}, \nabla^{\mathrm{B}})$; see §3.4.
$(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$	the B-model log-TEP structure with base $(\mathcal{M}_{\mathrm{B}}, D)$; see Proposition-Definition 3.5.4.
$(\mathcal{F}^{\mathrm{big}}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$	the big B-model log-TEP structure; see Theorem 5.0.3.
$(\mathsf{F}^{\mathrm{big}}_{\mathrm{B}}, \nabla, (\cdot,\cdot))$	the big B-model log-cTEP structure; see Example 6.1.6.
$(\mathcal{F}_{\mathrm{CY}}, \nabla, (\cdot,\cdot))$	the restriction of $\mathcal{F}_{\mathrm{B}}$ to $\mathcal{M}_{\mathrm{CY}}$; see §4.1.
H, $\overline{H}$, H_{vec}	vector bundles (of rank 6, 3, 2) on $\mathcal{M}_{\mathrm{CY}}$ obtained from $\mathcal{F}_{\mathrm{CY}}$; see §4.1–4.2.
$\mathbf{P}_{\mathrm{LR}}$, $\mathbf{P}_{\mathrm{con}}$, $\mathbf{P}_{\mathrm{orb}}$	unique Deligne-extension-compatible opposite modules for $\mathcal{F}_{\mathrm{B}}$ near $y = 0$, $y = -\frac{1}{27}$, and $y = \infty$ respectively; see Proposition 5.1.7.
$\mathfrak{Fock}_{\mathrm{B}}$	the B-model Fock sheaf over $\mathcal{M}^{\mathrm{big}}_{\mathrm{B}}$; see Definition 6.9.2.
$\mathfrak{Fock}_{\mathrm{CY}}$	the finite-dimensional Fock sheaf over $\mathcal{M}_{\mathrm{CY}} = \mathbb{P}(3,1)$; see Definition 7.3.9.
$\mathcal{F}^*$	restriction of a sheaf $\mathcal{F}$ over $\mathcal{M} \times \mathbb{C}$ to $\mathcal{M} \times \mathbb{C}^\times$; see Notation 2.8.3.

2. Notation and Preliminaries

2.1. **Bases for Cohomology and Orbifold Cohomology.** Let X denote one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y and $\overline{Y}$. We fix bases $\{\phi_0, \phi_1, \dots, \phi_N\}$ for the (orbifold) cohomology H_X of X such that

- ϕ_0 is the identity class
- writing r for the dimension of the (untwisted) degree two cohomology group $H^2(X)$ – so that $r = 0$, 1, 1, 2 respectively for $X = \mathcal{X}$, $\overline{\mathcal{X}}$, Y and $\overline{Y}$ – the classes $\phi_1, \dots, \phi_r$ form a nef integral basis of $H^2(X)$;
- if X is compact, $\{\phi^0, \dots, \phi^N\}$ is a basis dual to $\{\phi_0, \dots, \phi_N\}$ with respect to the (orbifold) Poincaré pairing.

More specifically we choose the following explicit bases. Let $H_{\mathcal{X}}$ denote the Chen–Ruan orbifold cohomology $H^\bullet_{\mathrm{orb}}(\mathcal{X}; \mathbb{C})$. We fix the basis:

$$\phi_0 = \mathbf{1}_0 \qquad \phi_1 = \mathbf{1}_{\frac{1}{3}} \qquad \phi_2 = \mathbf{1}_{\frac{2}{3}}$$

for $H_{\mathcal{X}}$, where $\mathbf{1}_k$, $k \in \{0, \frac{1}{3}, \frac{2}{3}\}$, denotes the fundamental class of the component of the inertia stack $\mathcal{I}(\mathcal{X})$ corresponding to the element $\exp(2\pi \mathtt{i} k) \in \mu_3$. The age of $\mathbf{1}_k$ is $3k$.

Let $H_{\overline{\mathcal{X}}}$ denote the Chen–Ruan orbifold cohomology $H^\bullet_{\mathrm{orb}}(\overline{\mathcal{X}}; \mathbb{C})$. Let $h \in H^2(\overline{\mathcal{X}}; \mathbb{C})$ denote the first Chern class of the line bundle $\mathcal{O}(1) \to \overline{\mathcal{X}}$, and regard elements of $H^\bullet(\overline{\mathcal{X}}; \mathbb{C})$ as orbifold cohomology classes via the canonical inclusion of $\overline{\mathcal{X}}$ into the inertia stack $\mathcal{I}(\overline{\mathcal{X}})$. We fix the basis:

$$\phi_0 = \mathbf{1}_0 \qquad \phi_1 = h \qquad \phi_2 = h^2 \qquad \phi_3 = h^3 \qquad \phi_4 = \mathbf{1}_{\frac{1}{3}} \qquad \phi_5 = \mathbf{1}_{\frac{2}{3}}$$

for $H_{\overline{\mathcal{X}}}$, where $\mathbf{1}_0$, $\mathbf{1}_{\frac{1}{3}}$, $\mathbf{1}_{\frac{2}{3}}$ denote the fundamental classes of the components of the inertia stack $\mathcal{I}(\overline{\mathcal{X}})$, ordered so that the age of $\mathbf{1}_k$ is $3k$. The orbifold Poincaré pairing on $\overline{\mathcal{X}}$ satisfies

$$(1, h^3) = (h, h^2) = \left(\mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}\right) = \frac{1}{3}$$

and that all other pairings among basis elements are zero.

Let $H_Y = H^\bullet(Y; \mathbb{C})$. Let $\mathbf{1} \in H_Y$ denote the unit class, let $\pi\colon Y \to \mathbb{P}^2$ denote the projection, and let $h \in H^2(Y; \mathbb{C})$ denote the first Chern class of the line bundle $\pi^\star \mathcal{O}(1) \to Y$. We fix the basis:

$$\phi_0 = \mathbf{1} \qquad \phi_1 = h \qquad \phi_2 = h^2$$

for H_Y.

Let $H_{\overline{Y}} = H^\bullet(\overline{Y}; \mathbb{C})$. Let $h_1, h_2 \in H^2(\overline{Y})$ be such that, regarding $\overline{Y}$ as the projective compactification of the line bundle $\mathcal{O}(-3) \to \mathbb{P}^2$, the zero section is Poincaré dual to $h_2 - 3h_1$, the infinity section is dual to h_2 and the fiber is dual to h_1. With these conventions, h_1 and h_2 are rays of the Kähler cone for $\overline{Y}$. We fix the basis:

$$\phi_0 = 1 \qquad \phi_1 = h_1 \qquad \phi_2 = h_2 \qquad \phi_3 = h_1^2 \qquad \phi_4 = h_1(h_2 - 3h_1) \qquad \phi_5 = h_1^2 h_2$$

for $H_{\overline{Y}}$.

2.2. **Gromov–Witten Invariants.** Let X denote one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y, $\overline{Y}$. Let $X_{g,n,d}$ denote the moduli space of n-pointed genus-g stable maps to X of degree $d \in H_2(X; \mathbb{Q})$. If X is a smooth algebraic variety (so $X = Y$ or $X = \overline{Y}$) then there are evaluation maps:

$$\mathrm{ev}_k\colon X_{g,n,d} \to X \qquad\qquad k \in \{1, 2, \dots, n\}$$

If X is an orbifold (so $X = \mathcal{X}$ or $X = \overline{\mathcal{X}}$) then there are evaluation maps to the rigidified cyclotomic inertia stack:

$$\mathrm{ev}_k \colon X_{g,n,d} \to \overline{\mathcal{I}}(X) \qquad\qquad k \in \{1, 2, \ldots, n\}$$

and a canonical isomorphism $H^\bullet(\overline{\mathcal{I}}X; \mathbb{Q}) \cong H^\bullet(\mathcal{I}X; \mathbb{Q})$, so we get cohomological pullbacks

$$\mathrm{ev}_k^\star \colon H_X \to H^\bullet(X_{g,n,d}; \mathbb{C}) \qquad\qquad k \in \{1, 2, \ldots, n\}$$

that behave like the pullbacks via evaluation maps: see [1] or [21, §2.2.2]. Write:

$$\langle \alpha_1, \ldots, \alpha_n \rangle^X_{g,n,d} = \int_{[X_{g,n,d}]^{\mathrm{vir}}} \prod_{k=1}^{k=n} \mathrm{ev}_k^\star(\alpha_k) \tag{4}$$

where $\alpha_1, \ldots, \alpha_n \in H_X$; the integral denotes cap product with the virtual fundamental class [7, 68] followed by push-forward (in homology) along the map from $X_{g,n,d}$ to a point; if X is non-compact (i.e. $X = \mathcal{X}$ or $X = Y$), we require that $d \neq 0$ or that at least one of the classes $\alpha_1, \ldots, \alpha_n$ has a compact support, so that the integral (4) is well-defined[3]. The right-hand side of (4) is a rational number when $\alpha_1, \ldots, \alpha_n$ are rational, called a *Gromov–Witten invariant* of X.

Let $\psi_1, \ldots, \psi_n \in H^2\big(X_{g,n,d}; \mathbb{Q}\big)$ denote the universal cotangent line classes [1, §8.3]. Write:

$$\Big\langle \alpha_1 \psi_1^{i_1}, \ldots, \alpha_n \psi_n^{i_n} \Big\rangle^X_{g,n,d} = \int_{[X_{g,n,d}]^{\mathrm{vir}}} \prod_{k=1}^{k=n} \mathrm{ev}_k^\star(\alpha_k) \cup \psi_k^{i_k} \tag{5}$$

where $\alpha_1, \ldots, \alpha_n \in H_X$; $i_1, \ldots, i_n$ are non-negative integers; the integral denotes cap product with the virtual fundamental class followed by push-forward to a point; and as before we insist that $d \neq 0$ or that one of $\alpha_1, \ldots, \alpha_n$ has compact support. The right-hand side of (5) is a rational number when $\alpha_1, \ldots, \alpha_n$ are rational, called a *gravitational descendant* of X.

Consider now the morphism $p_m \colon X_{g,m+n,d} \to \overline{\mathcal{M}}_{g,m}$ that forgets the map, forgets the last n marked points, forgets any stack structure at the marked points (if X is an orbifold), and then stabilises the resulting prestable curve. Let $\psi_{m|i} \in H^2(X_{g,n+m,d}; \mathbb{Q})$ denote the pullback along p_m of the ith universal cotangent line class on $\overline{\mathcal{M}}_{g,m}$. Write:

$$\begin{aligned} \Big\langle \alpha_1 \bar\psi_1^{i_1}, \ldots, \alpha_m \bar\psi^{i_m} : \beta_1, \ldots, \beta_n \Big\rangle^X_{g,m+n,d} \\ = \int_{[X_{g,m+n,d}]^{\mathrm{vir}}} \prod_{k=1}^{k=m} \Big(\mathrm{ev}_k^\star(\alpha_k) \cup \psi_{m|k}^{i_k} \Big) \cdot \prod_{l=m+1}^{l=m+n} \mathrm{ev}_l^\star(\beta_{l-m}) \end{aligned} \tag{6}$$

where $\alpha_1, \ldots, \alpha_m \in H_X$; $\beta_1, \ldots, \beta_n \in H_X$; $i_1, \ldots, i_m$ are non-negative integers; the integral denotes cap product with the virtual fundamental class followed by push-forward to a point; and as before we insist that $d \neq 0$ or that one of $\alpha_1, \ldots, \alpha_n$ has compact support. We insist also that $m \geq 3$ if $g = 0$ and that $m \geq 1$ if $g = 1$, so that the map p_m is well-defined. The right-hand side of (6) is a rational number, called an *ancestor invariant* of X.

[3]Here we use the property that the evaluation maps for $\mathcal{X}$ and Y are proper; this will also appear in §2.4.

2.3. **Gromov–Witten Potentials.** Let X denote one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y, $\overline{Y}$. Let r be the rank of $H_2(X)$. In §2.1 we fixed a basis $\phi_0, \dots, \phi_N$ for H_X such that $\phi_1, \dots, \phi_r$ is a nef basis for $H^2(X;\mathbb{C}) \subset H_X$. For $d \in H_2(X;\mathbb{Q})$, write:

$$Q^d = Q_1^{d_1} \cdots Q_r^{d_r}$$

where $d_i = d \cdot \phi_i$. Let $t^0, \dots, t^N$ be the co-ordinates on H_X defined by the basis $\phi_0, \dots, \phi_N$, so that $t \in H_X$ satisfies $t = t^0\phi_0 + \ldots + t^N\phi_N$. The *genus-$g$ Gromov–Witten potential* is:

$$F_X^g = \sum_{d \in \mathrm{NE}(X)} \sum_{n=0}^{\infty} \frac{Q^d}{n!} \left\langle t, \dots, t \right\rangle_{g,n,d}^X \tag{7}$$

where the first sum is over the set $\mathrm{NE}(X)$ of degrees of effective curves in X. This is a generating function for genus-g Gromov–Witten invariants. The genus-g Gromov–Witten potential is *a priori* a formal power series in variables Q_i and t^j:

$$F_X^g \in \mathbb{C}[\![Q_1, \dots, Q_r]\!][\![t^0, \dots, t^N]\!]$$

but the Divisor Equation [1, Theorem 8.3.1] implies that:

$$F_X^g \in \mathbb{C}[\![t^0, Q_1 e^{t^1}, \dots, Q_r e^{t^r}, t^{r+1} \dots, t^N]\!]$$

It thus makes sense to set $Q_1 = \cdots = Q_r = 1$, obtaining an element:

$$F_X^g\big|_{Q_1=\cdots=Q_r=1} \in \mathbb{C}[\![t^0, e^{t^1}, \dots, e^{t^r}, t^{r+1} \dots, t^N]\!]$$

There is an open region $V_{\mathrm{A},X} \subset H_X$ of the form:

$$\begin{cases} |t^i| < \epsilon_i & i = 0 \text{ or } r < i \le N \\ \Re t^i < -M_i & 1 \le i \le r \end{cases} \tag{8}$$

such that all of the power series $F_X^g\big|_{Q_1=\cdots=Q_r=1}$, $g \ge 0$, converge on $V_{\mathrm{A},X}$ [23]. In the rest of this paper we will write F_X^g for the analytic function $F_X^g\big|_{Q_1=\cdots=Q_r=1}$ defined on $V_{\mathrm{A},X}$, so that:

$$F_X^g(t) = \sum_{d \in \mathrm{NE}(X)} \sum_{n=0}^{\infty} \frac{e^{d \cdot t^{(2)}}}{n!} \left\langle t', \dots, t' \right\rangle_{g,n,d}^X \qquad\qquad t \in V_{\mathrm{A},X}$$

where we write $t^{(2)} = \sum_{i=1}^r t^i \phi_i$ for the degree two part of t and $t' = t - t^{(2)}$. We refer to the limit point:

$$\begin{cases} t^i = 0 & i = 0 \text{ or } r < i \le N \\ \Re t^i \to -\infty & 1 \le i \le r \end{cases}$$

as the *large-radius limit point* for X.

2.4. **Quantum Cohomology.** Let X be one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y, $\overline{Y}$. When X is compact, i.e. X is either $\overline{Y}$ or $\overline{\mathcal{X}}$, we define the quantum product $*$ on H_X by the formula:

$$(\phi_i * \phi_j, \phi_k)_X = \frac{\partial^3 F_X^0}{\partial t^i \partial t^j \partial t^k}(t)\bigg|_{Q_1=\cdots=Q_r=1} \tag{9}$$

where the pairing $(\cdot,\cdot)_X$ on the left-hand side is the (orbifold) Poincaré pairing. The product $*$ defines a family of commutative ring structures on H_X parameterized by $t \in V_{\mathrm{A},X}$, called

the *quantum cohomology* of X. When X is not compact, i.e. X is either $\mathcal{X}$ or Y, we define the quantum product by using the push-forward by the last marked point

$$\phi_i * \phi_j = \sum_{d\in \mathrm{NE}(X)} \sum_{n=0}^{\infty} \frac{e^{d\cdot t^{(2)}}}{n!} (\mathrm{ev}_{n+3})_\star \left(\mathrm{ev}_1^\star(\phi_i)\, \mathrm{ev}_2^\star(\phi_j) \prod_{k=1}^{n} \mathrm{ev}_{k+2}^\star(t') \cap [X_{0,n+3,d}]^{\mathrm{vir}} \right)$$

Here we write $t^{(2)} = \sum_{i=1}^r t^i \phi_i$ for the degree two part of t and $t' = t - t^{(2)}$. This makes sense because the evaluation map ev_{n+3} is proper. The quantum products for $\mathcal{X}$ and Y can be obtained as the limits of the quantum products for $\overline{\mathcal{X}}$ and $\overline{Y}$ respectively. We have

$$\lim_{\Re(t^1)\to-\infty} \iota^\star(\phi_i *_t^{\overline{\mathcal{X}}} \phi_j) = \iota^\star(\phi_i) *^{\mathcal{X}}_{\iota^\star(t)} \iota^\star(\phi_j)$$

$$\lim_{\Re(t^2)\to-\infty} \iota^\star(\phi_i *_t^{\overline{Y}} \phi_j) = \iota^\star(\phi_i) *^{Y}_{\iota^\star(t)} \iota^\star(\phi_j)$$

where ι denotes the natural inclusion of $\mathcal{X}$ into $\overline{\mathcal{X}}$ or Y into $\overline{Y}$ and $*_t^X$ denotes the quantum product of X at the parameter t. In particular, the quantum products for $\mathcal{X}$ and Y are also convergent on regions of the form (8).

2.5. **Dubrovin Connection, Fundamental Solution and J-Function.** Let X be one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y, $\overline{Y}$. Write $c_1(X) = \rho^1\phi_1 + \cdots + \rho^r\phi_r$. Define the *Euler vector field* E on H_X by:

$$E = t^0 \frac{\partial}{\partial t^0} + \sum_{i=1}^{r} \rho^i \frac{\partial}{\partial t^i} + \sum_{i=r+1}^{N} \big(1 - \tfrac{1}{2}\deg\phi_i\big) t^i \tfrac{\partial}{\partial t^i} \tag{10}$$

and the *grading operator* $\mu\colon H_X \to H_X$ by:

$$\mu(\phi_i) = \big(\tfrac{1}{2}\deg\phi_i - \tfrac{1}{2}\dim_{\mathbb{C}} X\big)\,\phi_i$$

Let $\pi\colon V_{\mathrm{A},X}\times\mathbb{C} \to V_{\mathrm{A},X}$ denote projection to the first factor. The *Dubrovin connection*[4] is a meromorphic flat connection ∇ on $\pi^\star\big(TV_{\mathrm{A},X}\big) \cong H_X \times (V_{\mathrm{A},X}\times\mathbb{C})$, defined by:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t^i}} &= \frac{\partial}{\partial t^i} - \frac{1}{z}\big(\phi_i *\big) && 0 \le i \le N \\ \nabla_{z\frac{\partial}{\partial z}} &= z\frac{\partial}{\partial z} + \frac{1}{z}\big(E*\big) + \mu && \text{where } z \text{ is the co-ordinate on } \mathbb{C} \end{aligned} \tag{11}$$

The Dubrovin connection defines the A-model TEP structure in Example 2.7.5 below.

The Dubrovin connection admits the following fundamental solution $L(t,-z)$ [38, Corollary 6.2; 54, Proposition 2.4]. Suppose that X is compact, i.e. X is either $\overline{\mathcal{X}}$ or $\overline{Y}$. Then the fundamental solution is an $\mathrm{End}(H_X)$-valued function of $(t,z) \in V_{\mathrm{A},X}\times\mathbb{C}^\times$ defined by

$$L(t,-z)\alpha = e^{t^{(2)}/z}\alpha + \sum_{\substack{d\in\mathrm{NE}(X), n\ge 0\\ (n,d)\neq(0,0)}} \sum_{i=0}^{N} \frac{e^{d\cdot t^{(2)}}}{n!} \left\langle \phi^i, t', \dots, t', \frac{e^{t^{(2)}/z}\alpha}{z-\psi} \right\rangle_{0,n+2,d} \phi_i \tag{12}$$

which satisfies the differential equation:

$$\nabla_{\frac{\partial}{\partial t^i}} \big(L(t,-z)\alpha\big) = 0 \qquad i = 0,\dots,N \tag{13}$$

and preserves the (orbifold) Poincaré pairing

$$(L(t,-z)\alpha, L(t,z)\beta)_X = (\alpha,\beta)_X \qquad \text{for all } \alpha,\beta \in H_X. \tag{14}$$

[4]The sign of z is often flipped in the literature: see e.g. [54].

Givental's J-function is defined to be

$$\begin{aligned} J(t,-z) &= L(t,-z)^{-1}\mathbf{1} \\ &= e^{-t^{(2)}/z}\left(\mathbf{1} - \frac{t'}{z} + \sum_{\substack{d\in \mathrm{NE}(X), n\geq 0 \\ (n,d)\neq(0,0),(1,0)}} \sum_{i=0}^{N} \frac{e^{d\cdot t^{(2)}}}{n!} \left\langle t', \dots, t', \frac{\phi^i}{z(z+\psi)} \right\rangle_{0,n+1,d} \phi_i \right). \end{aligned} \tag{15}$$

When X is non-compact, the fundamental solution $L(t,z)$ and the J-function $J(t,z)$ are defined similarly, replacing $\{\phi^i\}$ above with the dual basis of $\{\phi_i\}$ in the compactly-supported cohomology group. See [55, §2.5] for more details.

2.6. **Descendant Potentials and Ancestor Potentials.** Let X be one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y, $\overline{Y}$. Let $\phi_0, \dots, \phi_N$ be the basis for H_X defined in §2.1. Let $(t_0, t_1, t_2, \dots)$ be an infinite sequence of elements of H_X, and write $t_n = t_n^0\phi_0 + \dots + t_n^N\phi_N$. Set $\mathbf{t}(z) = \sum_{n=0}^{\infty} t_n z^n \in H_X[\![z]\!]$. The *genus-$g$ descendant potential* of X is:

$$\mathcal{F}_X^g = \sum_{d\in \mathrm{NE}(X)} \sum_{n=0}^{\infty} \frac{Q^d}{n!} \langle \mathbf{t}(\psi_1), \dots, \mathbf{t}(\psi_n) \rangle_{g,n,d}^X . \tag{16}$$

This is a formal power series[5] in variables Q_i, $1 \leq i \leq r$, and t_n^j, $0 \leq j \leq N$, $0 \leq n < \infty$; it is a generating function for genus-g gravitational descendants of X. The *total descendant potential* is:

$$\mathcal{Z}_X = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}_X^g \right).$$

This is a formal power series[5] in variables $\hbar$, $\hbar^{-1}$, Q_i, $1 \leq i \leq r$, and t_n^j, $0 \leq j \leq N$, $0 \leq n < \infty$; it is a generating function for all gravitational descendants of X.

Let $t \in H_X$, let $(a_0, a_1, a_2, \dots)$ be an infinite sequence of elements of H_X, and write:

$$t = t^0\phi_0 + \dots + t^N\phi_N \qquad a_n = a_n^0\phi_0 + \dots + a_n^N\phi_N \qquad \mathbf{a}(z) = \sum_{i=0}^{\infty} a_n z^n \in H_X[\![z]\!]$$

The *genus-g ancestor potential* of X is:

$$\bar{\mathcal{F}}_X^g = \sum_{d\in \mathrm{NE}(X)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{Q^d}{n!m!} \left\langle \mathbf{a}(\bar{\psi}_1), \dots, \mathbf{a}(\bar{\psi}_m) : \overbrace{t, \dots, t}^{n} \right\rangle_{g,m+n,d}^X \tag{17}$$

This is a formal power series[5] in variables Q_i, $1 \leq i \leq r$; t^j, $0 \leq j \leq N$; and a_n^k, $0 \leq k \leq N$, $0 \leq n < \infty$. It is a generating function for genus-g ancestor invariants of X. The *total ancestor potential* is:

$$\mathcal{A}_X = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} \bar{\mathcal{F}}_X^g \right)$$

This is a formal power series[5] in variables $\hbar$; $\hbar^{-1}$; Q_i, $1 \leq i \leq r$; t^j, $0 \leq j \leq N$; and a_n^k, $0 \leq k \leq N$, $0 \leq n < \infty$. It is a generating function for all ancestor invariants of X.

[5] See [23, §2.5] for a precise statement.

2.7. **TEP Structures and** log**-TEP Structures.**

Definition 2.7.1. Let $\mathcal{M}$ be a complex manifold. Let z denote the standard co-ordinate on $\mathbb{C}$, let $(-)\colon \mathcal{M}\times\mathbb{C}\to\mathcal{M}\times\mathbb{C}$ be the map sending (t,z) to $(t,-z)$, and let $\pi\colon \mathcal{M}\times\mathbb{C}\to\mathcal{M}$ be the projection. A *TEP structure* $\big(\mathcal{F},\nabla,(\cdot,\cdot)_{\mathcal{F}}\big)$ with base $\mathcal{M}$ consists of a locally free $\mathcal{O}_{\mathcal{M}\times\mathbb{C}}$-module $\mathcal{F}$ of rank $N+1$, a meromorphic flat connection:

$$\nabla\colon \mathcal{F}\to\big(\pi^\star\Omega^1_{\mathcal{M}}\oplus\mathcal{O}_{\mathcal{M}\times\mathbb{C}}z^{-1}dz\big)\otimes_{\mathcal{O}_{\mathcal{M}}}\mathcal{F}(\mathcal{M}\times\{0\})$$

and a non-degenerate pairing:

$$(\cdot,\cdot)_{\mathcal{F}}\colon (-)^\star\mathcal{F}\otimes_{\mathcal{O}_{\mathcal{M}\times\mathbb{C}}}\mathcal{F}\to\mathcal{O}_{\mathcal{M}\times\mathbb{C}}$$

which satisfies:

$$\begin{aligned}\big((-)^\star s_1,s_2\big)_{\mathcal{F}}&=(-)^\star\big((-)^\star s_2,s_1\big)_{\mathcal{F}}\\ d\big((-)^\star s_1,s_2\big)_{\mathcal{F}}&=\big((-)^\star\nabla s_1,s_2\big)_{\mathcal{F}}+\big((-)^\star s_1,\nabla s_2\big)_{\mathcal{F}}\end{aligned}\tag{18}$$

for local sections $s_1\in\mathcal{F}((-)^\star V)$, $s_2\in\mathcal{F}(V)$, where $V\subset\mathcal{M}\times\mathbb{C}$ is an open subset. Here $\mathcal{F}(\mathcal{M}\times\{0\})$ denotes the sheaf of sections of $\mathcal{F}$ with poles of order at most 1 along the divisor $\mathcal{M}\times\{0\}\subset\mathcal{M}\times\mathbb{C}$.

Definition 2.7.2. Let $D\subset\mathcal{M}$ be a normal crossing divisor. A log-*TEP structure* with base $(\mathcal{M},D)$ is a tuple $(\mathcal{F},\nabla,(\cdot,\cdot)_{\mathcal{F}})$ consisting of a locally free sheaf $\mathcal{F}$ of rank $N+1$ over $\mathcal{M}$, a meromorphic flat connection ∇

$$\nabla\colon \mathcal{F}\to\Omega^1_{\mathcal{M}\times\mathbb{C}}(\log Z)\otimes_{\mathcal{O}_{\mathcal{M}\times\mathbb{C}}}\mathcal{F}(\mathcal{M}\times\{0\})$$

where $Z=(\mathcal{M}\times\{0\})\cup(D\times\mathbb{C})$ is a normal crossing divisor in $\mathcal{M}\times\mathbb{C}$, and a non-degenerate pairing

$$(\cdot,\cdot)_{\mathcal{F}}\colon (-)^\star\mathcal{F}\otimes_{\mathcal{O}_{\mathcal{M}\times\mathbb{C}}}\mathcal{F}\to\mathcal{O}_{\mathcal{M}\times\mathbb{C}}$$

which satisfies the same properties (18) as TEP structure. Here $\Omega^1_{\mathcal{M}\times\mathbb{C}}(\log Z)$ denotes the sheaf of differential forms with logarithmic singularities along Z.

Remark 2.7.3. The notion of TEP structure is due to Hertling [47]: 'TEP' stands for Twister, Extension, and Pairing. This gives us a co-ordinate-free language in which to discuss mirror symmetry. More precisely, a TEP structure in our sense is what Hertling would call a TEP(0)-structure; for us all TEP structures have weight zero. A log-TEP structure is a TEP structure with logarithmic singularities; cf. Reichelt's notion of log-trTLEP structure [71, Definition 1.8]. When $D=\varnothing$, a log-TEP structure is the same thing as a TEP structure.

Definition 2.7.4. A log-TEP structure $(\mathcal{F},\nabla,(\cdot,\cdot))$ with base $(\mathcal{M},D)$ is said to be *miniversal* if for every point $x\in\mathcal{M}$, there exists a section ξ of $\mathcal{F}|_{z=0}$ on a neighbourhood U_x of x such that the map

$$\begin{aligned}\Theta_{\mathcal{M}}(\log D)&\longrightarrow\mathcal{F}|_{z=0}\\ X&\longmapsto z\nabla_X\xi\end{aligned}$$

is an isomorphism over U_x. Here $\Theta_{\mathcal{M}}(\log D)$ denotes the sheaf of logarithmic vector fields, that is, the subsheaf of $\Theta_{\mathcal{M}}$ consisting of vector fields tangent to the divisor D. (When $\mathcal{M}$ has an orbifold singularity at x, we take U_x above to be a uniformizing chart near x.) By taking $D=\varnothing$, this also defines miniversality for TEP structures.

Example 2.7.5 (A-model TEP structure)**.** An important class of examples of miniversal TEP structures is provided by the quantum cohomology of a smooth algebraic variety or orbifold X. We will need this only when X is one of $\mathcal{X}$, $\overline{\mathcal{X}}$, Y, $\overline{Y}$, but the definition here makes sense whenever the genus-zero Gromov–Witten potential F_X^0 defines an analytic function on a region $V_{\mathrm{A},X} \subset H_X$ of the form (8). The Dubrovin connection (11) defines a TEP structure $(\widetilde{\mathcal{F}}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$ with base $V_{\mathrm{A},X}$, where

- $\widetilde{\mathcal{F}}_{\mathrm{A},X}$ is the locally free sheaf corresponding to the trivial H_X-bundle over $V_{\mathrm{A},X} \times \mathbb{C}$;
- $\nabla^{\mathrm{A},X}$ is the Dubrovin connection;
- $(\cdot,\cdot)_{\mathrm{A},X}$ is the pairing induced by the orbifold Poincaré pairing.

When X is a smooth variety, the Divisor Equation implies that the Dubrovin connection descends to $\mathcal{M}_{\mathrm{A},X}^\times \times \mathbb{C}$, where

$$\mathcal{M}_{\mathrm{A},X}^\times := V_{\mathrm{A},X}/2\pi\mathtt{i}H^2(X,\mathbb{Z}).$$

and $2\pi\mathtt{i}H^2(X,\mathbb{Z})$ acts on $V_{\mathrm{A},X}$ by translation. When X is an orbifold, and we interpret $H^2(X,\mathbb{Z})$ as the sheaf cohomology[6] of the topological stack X, we again have that the Dubrovin connection descends to $\mathcal{M}_{\mathrm{A},X}^\times \times \mathbb{C}$. In this case, $2\pi\mathtt{i}H^2(X,\mathbb{Z})$ acts on the vector bundle $H_X \times (V_{\mathrm{A},X} \times \mathbb{C}) \to (V_{\mathrm{A},X} \times \mathbb{C})$ by the so-called *Galois action*, which is also nontrivial in the fibre direction. We refer the reader to [54, Proposition 2.3] for details; see also Example 2.7.6. The TEP structure $(\widetilde{\mathcal{F}}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$ described above descends, via the Galois action, to a TEP structure $(\mathcal{F}_{\mathrm{A},X}^\times, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$ with base $\mathcal{M}_{\mathrm{A},X}^\times$. This is the *A-model TEP structure*.

Example 2.7.6 (A-model log-TEP structure)**.** The quotient space $\mathcal{M}_{\mathrm{A},X}^\times$ has a natural partial compactification defined by our choice of nef basis for $H^2(X)$; this compactification, which we denote by $\mathcal{M}_{\mathrm{A},X}$, adds a normal crossing divisor $D_{\mathrm{A},X}$ at infinity. The A-model TEP structure extends to the partial compactification to give a miniversal log-TEP structure

$$\big(\mathcal{F}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X}\big)$$

with base $\big(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X}\big)$, called the *A-model* log*-TEP structure*: see [55, §2.2]. Concretely, this amounts to the following. Suppose first that X is a smooth variety. Recall that we have fixed a basis $\phi_0, \dots, \phi_N$ for H_X such that $\phi_0 \in H_X^0$ is the unit class and that $\phi_1, \dots, \phi_r$ is a nef basis for $H^2(X)$ in §2.1. This defines co-ordinates $t^0, \dots, t^N$ on H_X. Set $q_i = e^{t^i}$, $1 \le i \le r$, and consider $\mathbb{C}^{N+1} = \mathbb{C} \times \mathbb{C}^r \times \mathbb{C}^{N-r}$ with co-ordinates $(t^0, q_1, \dots, q_r, t^{r+1}, \dots, t^N)$. The partial compactification $\mathcal{M}_{\mathrm{A},X}$ is a neighbourhood of the origin in $\mathbb{C}^{N+1}$. The locally free sheaf $\mathcal{F}_{\mathrm{A},X}$ is given by the trivial H_X-bundle over $\mathcal{M}_{\mathrm{A},X} \times \mathbb{C}$. The divisor $D_{\mathrm{A},X}$ is the locus $q_1 q_2 \cdots q_r = 0$, the pairing is as in Example 2.7.5, and the meromorphic flat connection is:

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t^i}} &= \tfrac{\partial}{\partial t^i} - \tfrac{1}{z}\big(\phi_i *\big) && i = 0 \text{ or } r < i \le N \\
\nabla_{q_i \frac{\partial}{\partial q_i}} &= q_i \tfrac{\partial}{\partial q_i} - \tfrac{1}{z}\big(\phi_i *\big) && 1 \le i \le r \\
\nabla_{z\frac{\partial}{\partial z}} &= z\tfrac{\partial}{\partial z} + \tfrac{1}{z}\big(E *\big) + \mu
\end{aligned} \tag{19}$$

[6] An element of $H^2(X,\mathbb{Z})$ corresponds to an isomorphism class of a topological orbi-line bundle on X.

where (as before) z is the standard co-ordinate on $\mathbb{C}$ and E is the Euler vector field:

$$E = t^0 \frac{\partial}{\partial t^0} + \sum_{i=1}^{r} \rho^i q_i \frac{\partial}{\partial q_i} + \sum_{i=r+1}^{N} \left(1 - \tfrac{1}{2}\deg \phi_i\right) t^i \frac{\partial}{\partial t^i}$$

When X is an orbifold, $\mathcal{M}_{\mathrm{A},X}$ has orbifold singularities along the divisor $D_{\mathrm{A},X}$ and $\mathcal{F}_{\mathrm{A},X}$ is defined as an orbi-sheaf over $\mathcal{M}_{\mathrm{A},X}$. We shall describe the structure explicitly for $X = \overline{\mathcal{X}} = \mathbb{P}(1,1,1,3)$ (and this is the only case we need). In this case we have co-ordinates $t^0, \dots, t^5$ on H_X dual to the basis $\mathbf{1}_0, h, h^2, h^3, \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}$ from §2.1. Set $q = e^{t^1}$ and consider the space $\mathbb{C}^6$ with co-ordinates $(t^0, \sqrt[3]{q} = e^{t^1/3}, t^2, t^3, t^4, t^5)$. By the Divisor Equation, the Dubrovin connection (11) for $X = \overline{\mathcal{X}}$ induces a meromorphic flat connection of the form (19) on the trivial H_X-bundle over $V \times \mathbb{C}$, where V is a small open neighbourhood of the origin in the $\mathbb{C}^6$. Let μ_3 act[7] on the trivial bundle $H_X \times (V \times \mathbb{C}) \to (V \times \mathbb{C})$ by

$$\xi \cdot \left(\alpha, (t^0, \sqrt[3]{q}, t^2, t^3, t^4, t^5), z\right) = \left(G(\xi)\alpha, (t^0, \xi^{-1}\sqrt[3]{q}, t^2, t^3, \xi t^4, \xi^{-1} t^5), z\right)$$

where $G(\xi)$ is the endomorphism of H_X represented by the matrix

$$\left(\begin{array}{c|cc} I & & \\ \hline & \xi & 0 \\ & 0 & \xi^{-1} \end{array}\right)$$

in the basis $\mathbf{1}_0, h, h^2, h^3, \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}$ and I is the identity matrix of size 4. The μ_3-action here preserves the Dubrovin connection and the orbifold Poincaré pairing. The base of the A-model log-TEP structure is given by:

$$(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X}) = \left([V/\mu_3], [\{\sqrt[3]{q} = 0\}/\mu_3]\right)$$

$\mathcal{F}_{\mathrm{A},X}$ is the orbi-sheaf corresponding to the orbi-vector bundle:

$$[(H_X \times (V \times \mathbb{C}))/\mu_3] \to [V/\mu_3] \times \mathbb{C}$$

$\nabla^{\mathrm{A},X}$ is the meromorphic flat connection induced by the Dubrovin connection, and $(\cdot,\cdot)_{\mathrm{A},X}$ is the pairing on $\mathcal{F}_{\mathrm{A},X}$ induced by the orbifold Poincaré pairing.

Notation 2.7.7. As found in the notation $\mathcal{M}^{\times}_{\mathrm{A},X} = \mathcal{M}_{\mathrm{A},X} \setminus D_{\mathrm{A},X}$, $\mathcal{F}^{\times}_{\mathrm{A},X}$, we often put a cross "$\times$" to denote spaces (or sheaves) obtained by deleting normal crossing divisors from other spaces (or by restricting to the complement of these divisors).

2.8. **From TEP Structures to trTLEP Structures via Opposite Modules.** Hertling has defined the notion of a trTLEP structure with base $\mathcal{M}$. This consists of a TEP structure $\left(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\right)$ with base $\mathcal{M}$ together with certain extension data for $\mathcal{F}$, ∇, and $(\cdot,\cdot)_{\mathcal{F}}$ across $\mathcal{M} \times \{\infty\} \subset \mathcal{M} \times \mathbb{P}^1$. We encode these extension data using a subsheaf of $\pi_\star(\mathcal{F}|_{\mathcal{M}\times\mathbb{C}^\times})$ of semi-infinite rank called an *opposite module* (Definition 2.8.5). A TEP structure equipped with an opposite module is equivalent to a trTLEP structure, so the reader who prefers sheaves of finite rank can translate statements about opposite modules into statements about trTLEP structures. We will use both languages since opposite modules fit well with Givental quantization.

Definition 2.8.1 (Hertling [47, §5.2])**.** Let $\mathcal{M}$ be a complex manifold and let $(-)\colon \mathcal{M}\times\mathbb{P}^1 \to \mathcal{M}\times\mathbb{P}^1$ be the map sending (t,z) to $(t,-z)$. A trTLEP structure $\left(\mathcal{E}, \nabla, (\cdot,\cdot)_{\mathcal{E}}\right)$ with base $\mathcal{M}$ consists of:

[7] The group μ_3 here arises as $H^2(X,\mathbb{Z})/H^2(|X|,\mathbb{Z})$, where $|X|$ denotes the coarse moduli space of $X = \overline{\mathcal{X}}$.

- a locally free sheaf $\mathcal{E}$ on $\mathcal{M} \times \mathbb{P}^1$ such that $\mathcal{E}|_{\{y\}\times\mathbb{P}^1}$ is a free $\mathcal{O}_{\mathbb{P}^1}$-module for each $y \in \mathcal{M}$;
- a meromorphic flat connection ∇ on $\mathcal{E}$ with poles along $Z = \mathcal{M} \times \{0\} \cup \mathcal{M} \times \{\infty\}$:
$$\nabla\colon \mathcal{E} \to \Omega^1_{\mathcal{M}\times\mathbb{P}^1}(\log Z) \otimes \mathcal{E}(\mathcal{M} \times \{0\});$$
- a non-degenerate pairing:
$$(\cdot,\cdot)_{\mathcal{E}}\colon (-)^\star\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{M}\times\mathbb{P}^1}} \mathcal{E} \to \mathcal{O}_{\mathcal{M}\times\mathbb{P}^1}$$
which satisfies:
$$\big((-)^\star s_1, s_2\big)_{\mathcal{E}} = (-)^\star\big((-)^\star s_2, s_1\big)_{\mathcal{E}}$$
$$d\big((-)^\star s_1, s_2\big)_{\mathcal{E}} = \big((-)^\star \nabla s_1, s_2\big)_{\mathcal{E}} + \big((-)^\star s_1, \nabla s_2\big)_{\mathcal{E}}$$
for $s_1, s_2 \in \mathcal{E}$.

Note that ∇ has logarithmic singularities along $\mathcal{M}\times\{\infty\}$, and that the restriction of a trTLEP structure $\big(\mathcal{E}, \nabla, (\cdot,\cdot)_{\mathcal{E}}\big)$ to $\mathcal{M} \times \mathbb{C}$ is a TEP structure.

Remark 2.8.2. The 'L' in 'trTLEP structure' stands for logarithmic (along $\mathcal{M} \times \{\infty\}$) and the 'tr' stands for trivial (along $\{y\} \times \mathbb{P}^1$). Our trTLEP structure is what Hertling would call a trTLEP(0) structure: for us all trTLEP structures are of weight zero.

Notation 2.8.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{M} \times \mathbb{C}$. We write $\mathcal{F}^*$ for the restriction $\mathcal{F}\big|_{\mathcal{M}\times\mathbb{C}^\times}$.

Definition 2.8.4. Let $\big(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\big)$ be a TEP structure with base $\mathcal{M}$. The pairing $(\cdot,\cdot)_{\mathcal{F}}$ induces a symplectic pairing:
$$\begin{aligned}\Omega : \pi_\star\mathcal{F}^* \otimes_{\mathcal{O}_\mathcal{M}} \pi_\star\mathcal{F}^* &\longrightarrow \mathcal{O}_\mathcal{M}\\ s_1 \otimes s_2 &\longmapsto \operatorname{Res}_{z=0}\big((-)^\star s_1, s_2\big)_{\mathcal{F}}\, dz\end{aligned}$$
The connection ∇ induces an operator
$$\nabla\colon \pi_\star\mathcal{F}^* \to (\Omega^1_\mathcal{M} \oplus \mathcal{O}_\mathcal{M} dz) \otimes_{\mathcal{O}_\mathcal{M}} \pi_\star\mathcal{F}^*$$
which preserves the symplectic pairing Ω.

Definition 2.8.5. Let $\big(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\big)$ be a TEP structure with base $\mathcal{M}$. Recall that $\pi_\star\mathcal{F}^*$ is a $\pi_\star(\mathcal{O}_{\mathcal{M}\times\mathbb{C}^\times})$-module. This contains a locally free $\pi_\star(\mathcal{O}_{\mathcal{M}\times\mathbb{C}})$-module $\mathbf{F} := \pi_\star\mathcal{F}$ as a subsheaf. Let $\mathbf{P}$ be a locally free $\pi_\star(\mathcal{O}_{\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})})$-submodule of $\pi_\star\mathcal{F}^*$. We say that:

(1) $\mathbf{P}$ is opposite to $\mathbf{F}$ if $\pi_\star\mathcal{F}^* = \mathbf{F} \oplus \mathbf{P}$;
(2) $\mathbf{P}$ is isotropic if $\Omega(s_1, s_2) = 0$ for all $s_1, s_2 \in \mathbf{P}$;
(3) $\mathbf{P}$ is parallel if $\nabla_X\mathbf{P} \subset \mathbf{P}$ for all $X \in T\mathcal{M}$;
(4) $\mathbf{P}$ is homogeneous if $\nabla_{z\partial_z}\mathbf{P} \subset \mathbf{P}$.

An *opposite module* for $\big(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\big)$ is a locally free $\pi_\star(\mathcal{O}_{\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})})$-submodule $\mathbf{P}$ of $\pi_\star\mathcal{F}^*$ such that $\mathbf{P}$ is opposite to $\mathbf{F}$, isotropic, parallel, and homogeneous.

Example 2.8.6 (the A-model trTLEP structure and canonical opposite module)**.** Recall that the A-model TEP structure $\mathcal{F}^\times_{\mathrm{A},X}$ with base $\mathcal{M}^\times_{\mathrm{A},X}$ is given as the quotient of the trivial H_X-bundle over $V_{\mathrm{A},X} \times \mathbb{C}$ by the Galois action (see Example 2.7.5). The Dubrovin connection on the trivial H_X-bundle over $V_{\mathrm{A},X} \times \mathbb{C}$ extends to the trivial H_X-bundle over $V_{\mathrm{A},X} \times \mathbb{P}^1$ with only logarithmic poles along $V_{\mathrm{A},X} \times \{\infty\}$, and yields a trTLEP structure with base $V_{\mathrm{A},X}$. This trTLEP structure descends, via the Galois action, to give a trTLEP structure with base $\mathcal{M}^\times_{\mathrm{A},X}$ called the *A-model trTLEP structure*. This is an extension of the A-model TEP structure.

The corresponding opposite module can be described as follows. Consider the sheaf

$$\widetilde{\mathbf{P}}_{\mathrm{A}} = z^{-1}H_X \otimes \pi_\star(\mathcal{O}_{V_{\mathrm{A},X}\times(\mathbb{P}^1\setminus\{0\})}) \subset H_X \otimes \pi_\star(\mathcal{O}_{V_{\mathrm{A},X}\times\mathbb{C}^\times})$$

over $V_{\mathrm{A},X}$, where $\pi\colon V_{\mathrm{A},X}\times\mathbb{P}^1 \to V_{\mathrm{A},X}$ is the projection. The sheaf $\widetilde{\mathbf{P}}_{\mathrm{A}}$ gives an opposite module for the TEP structure $(\widetilde{\mathcal{F}}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X})$ introduced in Example 2.7.5. It descends to an opposite module $\mathbf{P}_{\mathrm{A}}$ of the A-model TEP structure via the Galois action. We call $\mathbf{P}_{\mathrm{A}}$ the *canonical opposite module* of the A-model TEP structure. Alternatively, $z\mathbf{P}_{\mathrm{A}}$ can be described as the push-forward along π of the restriction of the A-model trTLEP structure to $\mathcal{M}^\times_{\mathrm{A},X}\times(\mathbb{P}^1\setminus\{0\})$.

Remark 2.8.7. The subsheaf $\mathbf{F} = \pi_\star\mathcal{F}$ of $\pi_\star\mathcal{F}^*$ in the above definition gives a *variation of semi-infinite Hodge structure* (VSHS) in the sense of Barannikov [5]. It is maximally isotropic with respect to Ω and satisfies the Griffiths transversality condition $\nabla_X\mathbf{F} \subset z^{-1}\mathbf{F}$ for $X \in T\mathcal{M}$. It also satisfies $\nabla_{z^2\partial_z}\mathbf{F} \subset \mathbf{F}$. See [27, 55] for an exposition.

We now recall how an opposite module $\mathbf{P}$ for a TEP structure $\big(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\big)$ with base $\mathcal{M}$ determines a trTLEP structure with base $\mathcal{M}$. To give an extension of the locally free sheaf $\mathcal{F}^*$ on $\mathcal{M}\times\mathbb{C}^\times$ to a locally free sheaf on $\mathcal{M}\times\mathbb{C}$ is the same thing as to give a locally free $\pi_\star(\mathcal{O}_{\mathcal{M}\times\mathbb{C}})$-submodule $\mathbf{F}$ of $\pi_\star\mathcal{F}^*$ such that $\pi_\star\mathcal{F}^* = \mathbf{F}\otimes_{\pi_\star(\mathcal{O}_{\mathcal{M}\times\mathbb{C}})}\pi_\star(\mathcal{O}_{\mathcal{M}\times\mathbb{C}^\times})$. The submodule $\mathbf{F}$ consists of those sections which extend holomorphically to $z=0$; in the situation at hand the extension is given by the TEP structure $\mathcal{F}$ itself, so $\mathbf{F} = \pi_\star\mathcal{F}$. To give an extension of $\mathcal{F}^*$ to a locally free sheaf over $\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})$ is the same thing as to give a locally free $\pi_\star(\mathcal{O}_{\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})})$-submodule $\mathbf{F}'$ of $\pi_\star\mathcal{F}^*$ such that $\pi_\star\mathcal{F}^* = \mathbf{F}'\otimes_{\pi_\star(\mathcal{O}_{\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})})}\pi_\star\mathcal{O}_{\mathcal{M}\times\mathbb{C}^\times}$. The submodule $\mathbf{F}'$ consists of those sections which extend holomorphically to $z=\infty$; in the situation at hand we take $\mathbf{F}' = z\mathbf{P}$. Thus the opposite module $\mathbf{P}$ determines an extension of the locally free sheaf $\mathcal{F}$ on $\mathcal{M}\times\mathbb{C}$ to a locally free sheaf $\mathcal{E}$ on $\mathcal{M}\times\mathbb{P}^1$. The restriction $\mathcal{E}|_{\{y\}\times\mathbb{P}^1}$ is a free $\mathcal{O}_{\mathbb{P}^1}$-module because $\mathbf{P}_y$ is opposite to $\mathbf{F}_y$: the space of global sections of $\mathcal{E}|_{\{y\}\times\mathbb{P}^1}$ is $z\mathbf{P}_y\cap\mathbf{F}_y$, and the projection $z\mathbf{P}_y\cap\mathbf{F}_y \xrightarrow{\sim} z\mathbf{P}_y/\mathbf{P}_y$ gives a trivialization of $\mathcal{E}|_{\{y\}\times\mathbb{P}^1}$ (see [55, Lemma 3.8]). The pairing $(\cdot,\cdot)_{\mathcal{F}}$ on $\mathcal{F}$ extends holomorphically and non-degenerately across $z=\infty$ to a pairing on $\mathcal{E}$ because $\mathbf{P}$ is isotropic. The connection ∇ on $\mathcal{F}$ induces a connection on $\mathcal{E}$ with logarithmic singularity along $z=\infty$ because $\mathbf{P}$ is homogeneous and parallel. Thus an opposite module $\mathbf{P}$ for the TEP structure $\big(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}}\big)$ determines a trTLEP structure $(\mathcal{E}, \nabla, (\cdot,\cdot)_{\mathcal{E}})$. Conversely, a trTLEP structure $(\mathcal{E}, \nabla, (\cdot,\cdot)_{\mathcal{E}})$ determines an opposite module $\mathbf{P} = z^{-1}\pi_\star(\mathcal{E}|_{\mathcal{M}\times(\mathbb{P}^1\setminus\{0\})})$ of the underlying TEP structure. We have thus proved:

Proposition 2.8.8. *There is a bijective correspondence between opposite modules for a TEP structure and trTLEP structures which extend that TEP structure.*

Let $\mathbf{P}$ be an opposite module for a TEP structure $(\mathcal{F}, \nabla, (\cdot,\cdot)_{\mathcal{F}})$ with base $\mathcal{M}$. It defines a locally free sheaf $z\mathbf{P}/\mathbf{P}$ of rank $N+1 = \operatorname{rank}\mathcal{F}$ on $\mathcal{M}$. This is identified with the restriction to $z=\infty$ of the corresponding trTLEP structure $\mathcal{E}$, and is equipped with a flat connection

$$\nabla\colon z\mathbf{P}/\mathbf{P} \to \Omega^1_{\mathcal{M}}\otimes(z\mathbf{P}/\mathbf{P})$$

since ∇_X with $X\in T\mathcal{M}$ preserves $\mathbf{P}$. Therefore $z\mathbf{P}/\mathbf{P}$ defines a flat vector bundle over $\mathcal{M}$. The trivialization $\mathcal{E}|_{\{y\}\times\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}\otimes(z\mathbf{P}_y/\mathbf{P}_y)$ discussed before Proposition 2.8.8 yields an isomorphism:

$$\mathcal{F} \cong \pi^\star(z\mathbf{P}/\mathbf{P}) \tag{20}$$

Definition 2.8.9. We call the isomorphism (20) the *flat trivialization* associated to the opposite module $\mathbf{P}$. Over a simply-connected base, we can take a flat frame of $z\mathbf{P}/\mathbf{P}$ that yields a trivialization of $\mathcal{F}$. This is also called a *flat trivialization*.

Remark 2.8.10. The flat trivialization gives rise to a Frobenius-type structure. See Hertling [47, Theorem 5.7] and Coates–Iritani–Tseng [27, Proposition 2.11].

Example 2.8.11. The flat trivialization associated to the canonical opposite module $\mathbf{P}_{\mathrm{A}}$ in Example 2.8.6 corresponds to the standard trivialization of $\widetilde{\mathcal{F}}_{\mathrm{A},X}$ in Example 2.7.5.

3. The Mirror Landau–Ginzburg Model for $\overline{Y}$ and $\overline{\mathcal{X}}$

Mirror symmetry associates to each toric variety a *Landau–Ginzburg model* [39, 48]. In this context, a Landau–Ginzburg model consists of:

- a holomorphic family $\pi\colon Z \to \mathcal{M}_{\mathrm{B}}^{\times}$ of algebraic tori;
- a function $W\colon Z \to \mathbb{C}$, called the *superpotential*;
- a section ω of the relative canonical sheaf $K_{Z/\mathcal{M}_{\mathrm{B}}^{\times}}$ which gives a holomorphic volume form ω_q on each fibre $Z_q = \pi^{-1}(q)$.

The base space $\mathcal{M}_{\mathrm{B}}^{\times}$ of the family is called the *B-model moduli space*. In this section we define the Landau–Ginzburg model that corresponds to $\overline{Y}$ under mirror symmetry (§3.1) and use it to construct a TEP structure, called the *B-model TEP structure* (§3.2). We formulate mirror symmetry for $\overline{Y}$ as an equivalence of TEP structures (§3.3) between the A-model TEP structure – or rather its restriction to the small quantum cohomology locus $H^2(\overline{Y}) \subset H^\bullet(\overline{Y})$ – and the B-model TEP structure defined from the Landau–Ginzburg model. We then give an alternative construction of the B-model TEP structure, in terms of the so-called *GKZ system*, which is useful in computations (§3.4). The B-model TEP structure is defined over a non-compact base $\mathcal{M}_{\mathrm{B}}^{\times}$, but computations with the GKZ system allow us to define an extension of the B-model TEP structure over a toric partial compactification $\mathcal{M}_{\mathrm{B}}$ of $\mathcal{M}_{\mathrm{B}}^{\times}$, such that the extension has logarithmic singularities along the partially-compactifying divisor (§3.5).

Remark 3.0.12. The Landau–Ginzburg model that we consider in this section provides a mirror to the *small* quantum cohomology of $\overline{Y}$: an open subset in the base $\mathcal{M}_{\mathrm{B}}^{\times}$ corresponds to a relatively open subset in the small quantum cohomology locus $H^2(\overline{Y}) \subset H_{\overline{Y}}$. We will construct a mirror to *big* quantum cohomology, over a larger base $\mathcal{M}_{\mathrm{B}}$, in §5 below.

3.1. **The Mirror Landau–Ginzburg Model.** The toric variety $\overline{Y}$ is the GIT quotient of $\mathbb{C}^5$ by $(\mathbb{C}^\times)^2$ where $(\mathbb{C}^\times)^2$ acts via the inclusion

$$(\mathbb{C}^\times)^2 \hookrightarrow (\mathbb{C}^\times)^5, \quad (s,t) \mapsto (s,s,s,s^{-3}t,t). \tag{21}$$

Consider the map π given by restricting the dual of this inclusion

$$\begin{aligned} \pi\colon (\mathbb{C}^\times)^5 &\longrightarrow (\mathbb{C}^\times)^2 \\ (w_1,\dots,w_5) &\longmapsto (w_1w_2w_3w_4^{-3}, w_4w_5) \end{aligned}$$

to the following open subset of $(\mathbb{C}^\times)^2$:

$$\left\{(y_1,y_2) \in (\mathbb{C}^\times)^2 : y_1 \neq -\tfrac{1}{27}\right\} \tag{22}$$

The superpotential W is:

$$W = w_1 + w_2 + w_3 + w_4 + w_5$$

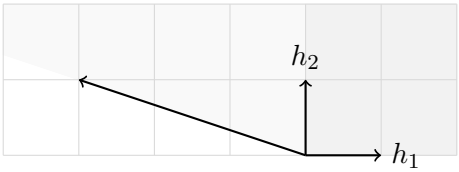


FIGURE 3. The secondary fan for $\overline{Y}$.

and the holomorphic volume form ω_y on the fibre $Z_y = \pi^{-1}(y_1, y_2)$ is:

$$\omega_y = \frac{d\log w_1 \wedge \cdots \wedge d\log w_5}{d\log y_1 \wedge d\log y_2}$$

We delete the locus $y_1 = -\frac{1}{27}$ in (22) because critical points of $W|_{Z_y}$ escape to infinity there: see [27, §3.1].

We now consider a partial compactification $\mathcal{M}_{\mathrm{B}}^{\times}$ of the open subset (22) and extend the Landau–Ginzburg model considered above to a Landau–Ginzburg model over this larger base. Consider the secondary fan (Figure 3) for the toric variety $\overline{Y}$; this records the weight data (21) defining the toric variety $\overline{Y}$. The toric orbifold $\mathcal{M}_{\mathrm{B}}$ associated to the secondary fan gives a partial compactification of the open set (22). The two cones in the secondary fan define toric co-ordinate patches on $\mathcal{M}_{\mathrm{B}}$. Let y_1, y_2 be the co-ordinates dual respectively to h_1 and to h_2, and let $\mathfrak{y}_1$ and $\mathfrak{y}_2$ be the co-ordinates dual respectively to $h_2 - 3h_1$ and to h_2. The two co-ordinate systems are related by:

$$\begin{aligned} \mathfrak{y}_1 &= y_1^{-1/3} & y_1 &= \mathfrak{y}_1^{-3} \\ \mathfrak{y}_2 &= y_1^{1/3} y_2 & y_2 &= \mathfrak{y}_1 \mathfrak{y}_2 \end{aligned} \tag{23}$$

Note that $\mathcal{M}_{\mathrm{B}}$ is an orbifold with a $\mathbb{Z}/3\mathbb{Z}$ quotient singularity at $(\mathfrak{y}_1, \mathfrak{y}_2) = 0$, and $(\mathfrak{y}_1, \mathfrak{y}_2)$ is a uniformizing system near this orbifold point.

We define the base $\mathcal{M}_{\mathrm{B}}^{\times}$ of our new Landau–Ginzburg model to be

$$\mathcal{M}_{\mathrm{B}}^{\times} := \mathcal{M}_{\mathrm{B}} \setminus \overline{\left\{(y_1, y_2) \in \mathbb{C}^2 : y_1 y_2 = 0 \text{ or } y_1 = -\tfrac{1}{27}\right\}}$$

Taking w_1, w_2, w_5 as co-ordinates on the fibre $Z_y \subset Z$, we see that:

$$\begin{aligned} W_y &= w_1 + w_2 + \frac{y_1 y_2^3}{w_1 w_2 w_5^3} + \frac{y_2}{w_5} + w_5 \\ &= w_1 + w_2 + \frac{\mathfrak{y}_2^3}{w_1 w_2 w_5^3} + \frac{\mathfrak{y}_1 \mathfrak{y}_2}{w_5} + w_5 \\ \omega_y &= d\log w_1 \wedge d\log w_2 \wedge d\log w_5 \end{aligned} \tag{24}$$

We can therefore extend the family of tori π, the superpotential W_y, and the section ω across the locus $\{\mathfrak{y}_1 = 0\}$. These extensions define a new Landau–Ginzburg model with base $\mathcal{M}_{\mathrm{B}}^{\times}$.

Notation 3.1.1. We refer to the point $(y_1, y_2) = (0, 0)$ as the *large-radius limit point* and to the point $(\mathfrak{y}_1, \mathfrak{y}_2) = (0, 0)$ as the *orbifold point*. We refer to the locus $y_1 = -\frac{1}{27}$ as the *conifold locus*.

Remark 3.1.2. The right-hand cone in Figure 3 is canonically identified with the Kähler cone of $\overline{Y}$, and under this identification the cohomology classes h_1, h_2 defined in §2.1 are as pictured.

Remark 3.1.3. The Landau–Ginzburg model described here is discussed in more detail in [27, §§2–3].

3.2. **The B-Model TEP structure.** We now use the Landau–Ginzburg model $\big(\pi : Z \to \mathcal{M}_{\mathrm{B}}^{\times}, W, \omega\big)$ to define a TEP structure, called the B-model TEP structure. This is almost the same as the discussions in [27, §2.5.1; 54, §3.3], with the main difference[8] being that there the (equivalent) language of variations of semi-infinite Hodge structure is used. Consider the locally free sheaf $\mathcal{R}$ over $\mathcal{M}_{\mathrm{B}}^{\times} \times \mathbb{C}^{\times}$ with fibre over (y,z) equal to the relative cohomology group $H^3\big(Z_y, \{x \in Z_y : \Re(W_y(x)/z) \gg 0\}\big)$. This sheaf carries a flat Gauss–Manin connection ∇^{GM}, and there is a distinguished global section of $\mathcal{R}$ given by:

$$(y,z) \longmapsto \exp\big(-W_y/z\big)\,\omega_y.$$

Let $\mathcal{O}_{Z\times\mathbb{C}}$ denote the analytic structure sheaf. Consider the $\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\times}\times\mathbb{C}}$-module $\mathcal{F}^{\times}$ consisting of sections of $\mathcal{R}$ of the form:

$$\Big[f(x,z)\exp\big(-W(x)/z\big)\,\omega\Big] \qquad\qquad \text{where } f(x,z) \in (\pi\times\mathrm{id})_{\star}\mathcal{O}_{Z\times\mathbb{C}}$$

such that, for each $z \in \mathbb{C}$, the function $x \mapsto f(x,z)$ is algebraic on each fibre Z_y. The sheaf $\mathcal{F}^{\times}$ is a locally free extension of $\mathcal{R}$ to $\mathcal{M}_{\mathrm{B}}^{\times} \times \mathbb{C}$ [54, Proposition 3.14]. The B-model TEP structure will, roughly speaking, be the twist of $\mathcal{F}^{\times}$ by a factor of $z^{-3/2}$: this twist will ensure that the pairing on the B-model TEP structure behaves correctly.

Lemma 3.2.1 (see [27, Lemma 2.19])**.** *The intersection pairing:*

$$I : H^3\big(Z_y, \{x \in Z_y : \Re(W_y(x)/z) \ll 0\}\big) \otimes H^3\big(Z_y, \{x \in Z_y : \Re(W_y(x)/z) \gg 0\}\big) \to \mathbb{C}$$

induces a pairing:

$$I : (-)^{\star}\mathcal{F}^{\times} \otimes \mathcal{F}^{\times} \to (2\pi\mathtt{i}z)^3\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\times}\times\mathbb{C}}$$

Proof. Observe that, on the one hand:

$$I\bigg(\Big[f(x,-z)e^{W(x)/z}\,\omega\Big], \Big[g(x,z)e^{-W(x)/z}\,\omega\Big]\bigg) \in \mathcal{O}_{\mathcal{M}\times\mathbb{C}^{\times}}$$

and on the other hand:

$$I\bigg(\Big[f(x,-z)e^{W(x)/z}\,\omega\Big], \Big[g(x,z)e^{-W(x)/z}\,\omega\Big]\bigg) = \sum_{\text{critical points } \sigma} \bigg(\int_{\Gamma_-(\sigma)} f(x,-z)e^{W(x)/z}\,\omega\bigg) \cdot \bigg(\int_{\Gamma_+(\sigma)} g(x,z)e^{-W(x)/z}\,\omega\bigg)$$

where

$$\begin{aligned} \Gamma_+(\sigma) &\in H_3\big(Z_y, \{x \in Z_y : \Re(W_y(x)/z) \gg 0\}\big) \\ \Gamma_-(\sigma) &\in H_3\big(Z_y, \{x \in Z_y : \Re(W_y(x)/z) \ll 0\}\big) \end{aligned} \tag{25}$$

are the Lefschetz thimbles given by upward (for Γ_+) and downward (for Γ_-) gradient flow of the function $x \mapsto \Re\big(\frac{W(x)}{z}\big)$ from the critical point $\sigma \in Z_y$ of $W|_{Z_y}$. Stationary phase

[8] A minor difference is that the sign of z is flipped compared to [27, 54].

approximation gives that, as $z \to 0$ in some angular sector:

$$I\left(\left[f(x,-z)e^{W(x)/z}\,\omega\right],\left[g(x,z)e^{-W(x)/z}\,\omega\right]\right) \sim \tag{26}$$
$$\sum_{\text{critical points } \sigma} (-2\pi z)^{3/2}\left(\frac{f(\sigma)}{\sqrt{\mathrm{Hess}_\sigma(W)}} + O(z)\right)\cdot(2\pi z)^{3/2}\left(\frac{g(\sigma)}{\sqrt{\mathrm{Hess}_\sigma(W)}} + O(z)\right)$$

Thus the function

$$I\left(\left[f(x,-z)e^{W(x)/z}\,\omega\right],\left[g(x,z)e^{-W(x)/z}\,\omega\right]\right) \in \mathcal{O}_{\mathcal{M}\times\mathbb{C}^\times}$$

is in fact regular at $z=0$ and lies in $(2\pi\mathtt{i}z)^3\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times\times\mathbb{C}}$. □

Definition 3.2.2. The *B-model TEP structure* $\left(\mathcal{F}_{\mathrm{B}}^\times, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\right)$ consists of:

- the locally free $\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times\times\mathbb{C}}$-module $\mathcal{F}_{\mathrm{B}}^\times := \mathcal{F}^\times$;
- the flat connection:
$$\nabla^{\mathrm{B}}\colon \mathcal{F}_{\mathrm{B}}^\times \to \left(\pi^\star\Omega^1_{\mathcal{M}_{\mathrm{B}}^\times} \oplus \mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times\times\mathbb{C}} z^{-1}dz\right) \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times\times\mathbb{C}}} \mathcal{F}_{\mathrm{B}}^\times(\mathcal{M}_{\mathrm{B}}^\times \times \{0\})$$
defined by:
$$\nabla^{\mathrm{B}} := \nabla^{\mathrm{GM}} - \frac{3}{2}\frac{dz}{z}$$
- the pairing:
$$(\cdot,\cdot)_{\mathrm{B}} := \frac{1}{(2\pi\mathtt{i}z)^3} I(\cdot,\cdot)$$

It is proven in [54, §3.3] that the B-model TEP structure is, in fact, a TEP structure.

Remark 3.2.3. The connection ∇^{B} is compatible with the pairing $(\cdot,\cdot)_{\mathrm{B}}$, whereas the connection ∇^{GM} is compatible with the pairing $I(\cdot,\cdot)$.

3.3. **Mirror Symmetry as an Isomorphism of TEP Structures.** Let X denote $\overline{\mathcal{X}}$ or $\overline{Y}$. Let $\left(\mathcal{F}_{\mathrm{A},X}^\times, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X}\right)$ be the A-model TEP structure for X, as defined in Example 2.7.5. This is a TEP structure with base $\mathcal{M}_{\mathrm{A},X}^\times = V_{\mathrm{A},X}/2\pi\mathtt{i}H^2(X,\mathbb{Z})$, where $V_{\mathrm{A},X} \subset H_X$ is an open subset of the form (8). Recall also that $\mathcal{M}_{\mathrm{A},X}^\times = \mathcal{M}_{\mathrm{A},X} \setminus D_{\mathrm{A},X}$, where $(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X})$ is the base of the A-model log-TEP structure in Example 2.7.6. With notation as in Example 2.7.6, we have

$$\mathcal{M}_{\mathrm{A},\overline{Y}} = \left\{(t^0,q_1,q_2,t^3,t^4,t^5) \in \mathbb{C}^6 : |t^i| < \epsilon_i, |q_i| < \epsilon_i\right\} \qquad D_{\mathrm{A},\overline{Y}} = \{q_1q_2 = 0\}$$

for $X = \overline{Y}$ and

$$\mathcal{M}_{\mathrm{A},\overline{\mathcal{X}}} = \left[\left\{(t^0, \sqrt[3]{q}, t^2, t^3, t^4, t^5) \in \mathbb{C}^6 : |t^i| < \epsilon_i, |\sqrt[3]{q}| < \epsilon_1\right\}/\mu_3\right] \qquad D_{\mathrm{A},\overline{\mathcal{X}}} = \{q = 0\}$$

for $X = \overline{\mathcal{X}}$.

Theorem 3.3.1 (Mirror Symmetry for $\overline{Y}$)**.** *Let (y_1,y_2) be the co-ordinates defined in §3.1. Let $h_1,h_2 \in H^2(\overline{Y})$ be as in §2.1. There are real numbers $\epsilon_1,\epsilon_2 > 0$ such that if:*

$$U^\times = \left\{(y_1,y_2) \in \mathcal{M}_{\mathrm{B}}^\times : |y_i| < \epsilon_i\right\}$$

and the map $\mathrm{mir}_{\overline{Y}}\colon U^\times \to \mathcal{M}^\times_{\mathrm{A},X}$ *is*

$$\mathrm{mir}_{\overline{Y}}(y_1,y_2) = (0, y_1 e^{-3g(y_1)}, y_2 e^{g(y_1)}, 0, 0, 0), \qquad g(y_1) = \sum_{d=1}^{\infty} \frac{(3d-1)!}{(d!)^3}(-1)^{d+1} y_1^d$$

then there is an isomorphism of TEP structures

$$\left(\mathcal{F}^\times_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\right)\Big|_{U^\times\times\mathbb{C}} \cong \mathrm{mir}^\star_{\overline{Y}}\left(\mathcal{F}_{\mathrm{A},\overline{Y}}, \nabla^{\mathrm{A},\overline{Y}}, (\cdot,\cdot)_{\mathrm{A},\overline{Y}}\right)$$

where on the left we have the B-model TEP structure and on the right we have the A-model TEP structure.

Proof. This is an example of [27, Conjecture 2.21], and is proven in [27, §3.2]. □

Theorem 3.3.2 (Mirror symmetry for $\overline{\mathcal{X}}$)**.** *Let* $(\mathfrak{y}_1,\mathfrak{y}_2)$ *be the co-ordinates defined in §3.1. Let* $h \in H^2(\overline{\mathcal{X}})$ *be the first Chern class of* $\mathcal{O}(1)$ *as in §2.1. There are real numbers* $\epsilon_1,\epsilon_2 > 0$ *such that if:*

$$U^\times = \left\{(\mathfrak{y}_1,\mathfrak{y}_2) \in \mathcal{M}^\times_{\mathrm{B}} : |\mathfrak{y}_i| < \epsilon_i\right\}$$

and the map $\mathrm{mir}_{\overline{\mathcal{X}}}\colon U^\times \to \mathcal{M}^\times_{\mathrm{A},\overline{\mathcal{X}}}$ *is*

$$\mathrm{mir}_{\overline{\mathcal{X}}}(\mathfrak{y}_1,\mathfrak{y}_2) = (0,\mathfrak{y}_2,0,0,\mathfrak{t}(\mathfrak{y}_1),0), \qquad \mathfrak{t}(\mathfrak{y}_1) = \sum_{n=0}^{\infty}(-1)^n \frac{\prod_{j=0}^{n-1}(\frac{1}{3}+j)^3}{(3n+1)!}\mathfrak{y}_1^{3n+1}$$

then there is an isomorphism of TEP structures:

$$\left(\mathcal{F}^\times_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\right)\Big|_{U^\times\times\mathbb{C}} \cong \mathrm{mir}^\star_{\overline{\mathcal{X}}}\left(\mathcal{F}_{\mathrm{A},\overline{\mathcal{X}}}, \nabla^{\mathrm{A},\overline{\mathcal{X}}}, (\cdot,\cdot)_{\mathrm{A},\overline{\mathcal{X}}}\right)$$

where on the left we have the B-model TEP structure and on the right we have the A-model TEP structure.

Proof. This is an example of [27, Conjecture 2.21], and is proven in [27, §3.4] along $\mathfrak{y}_1 = 0$. A proof including the $\mathfrak{y}_1$-direction is given in [54, Proposition 4.8]. □

Remark 3.3.3. The map $\mathrm{mir}_{\overline{Y}}(y_1,y_2) = (0,q_1,q_2,0,0,0)$ is determined by the I-function [39] of $\overline{Y}$ via the asymptotics $I_{\overline{Y}}(y_1,y_2,z) = 1 + (h_1\log q_1 + h_2 \log q_2)/z + o(z^{-1})$: see (35). On the other hand, $\mathrm{mir}_{\overline{\mathcal{X}}}(\mathfrak{y}_1,\mathfrak{y}_2)$ is determined by the extended I-function [19] of $\overline{\mathcal{X}}$

$$I_{\overline{\mathcal{X}}}(\mathfrak{y}_1,\mathfrak{y}_2,z) = \sum_{k_1,k_2\geq 0} \mathfrak{y}_1^{k_1}\mathfrak{y}_2^{k_2+3h/z} \frac{\prod_{c\leq 0, \langle c\rangle = \left\langle\frac{k_2-k_1}{3}\right\rangle}(h+cz)^3}{\prod_{c\leq \frac{k_2-k_1}{3}, \langle c\rangle = \left\langle\frac{k_2-k_1}{3}\right\rangle}(h+cz)^3} \frac{1}{k_1! z^{k_1}\prod_{c=1}^{k_2}(3h+cz)} \mathbf{1}_{\left\langle\frac{k_1-k_2}{3}\right\rangle}$$

via the expansion $I_{\overline{\mathcal{X}}}(\mathfrak{y}_1,\mathfrak{y}_2,z) = 1 + (\mathfrak{t}\mathbf{1}_{\frac{1}{3}} + (\log q)h)/z + o(1/z)$.

3.4. **The GKZ System and the B-Model TEP Structure.** We now give an alternative construction of the B-model TEP structure, which is very convenient for calculations. This construction is in terms of the so-called GKZ system, due to Gelfand–Kapranov–Zelevinsky [35].

Definition 3.4.1. Let $\mathcal{D}^z \subset \mathrm{End}_{\mathbb{C}}(\mathcal{O}_{\mathcal{M}^\times_{\mathrm{B}}\times\mathbb{C}})$ denote the subsheaf of the sheaf of differential operators on $\mathcal{M}^\times_{\mathrm{B}}\times\mathbb{C}$ generated, as a sheaf of rings, by $\mathcal{O}_{\mathcal{M}^\times_{\mathrm{B}}\times\mathbb{C}}$ and $\{zX : X \text{ is a vector field on } \mathcal{M}^\times_{\mathrm{B}}\}$, where z is the standard co-ordinate on $\mathbb{C}$.

Remark 3.4.2. Let $\mathcal{E}$ be a $\mathcal{D}^z$-module. The action of $(1/z)\cdot zX$ defines a map

$$\nabla_X \colon \mathcal{E} \to z^{-1}\mathcal{E} = \mathcal{E}(\mathcal{M}_{\mathrm{B}}^\times \times \{0\})$$

When $\mathcal{E}$ is a coherent $\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times \times \mathbb{C}}$-module, one may view ∇_X as a flat connection in the direction of $\mathcal{M}_{\mathrm{B}}^\times$ with poles along $z=0$.

Definition 3.4.3. By *the GKZ system* we mean the $\mathcal{D}^z$-module $\mathcal{F}_{\mathrm{GKZ}}^\times$ on $\mathcal{M}_{\mathrm{B}}^\times \times \mathbb{C}$ defined as follows. Recall from (23) that $\mathcal{M}_{\mathrm{B}}^\times$ is covered by two co-ordinate patches (y_1, y_2) and $(\mathfrak{y}_1, \mathfrak{y}_2)$ related by:

$$\mathfrak{y}_1 = y_1^{-1/3} \qquad y_1 = \mathfrak{y}_1^{-3}$$
$$\mathfrak{y}_2 = y_1^{1/3} y_2 \qquad y_2 = \mathfrak{y}_1 \mathfrak{y}_2$$

Define charts $U_{\mathrm{LR}}^\times$ and $U_{\mathrm{orb}}^\times$ on $\mathcal{M}_{\mathrm{B}}^\times$ by:

$$U_{\mathrm{LR}}^\times = \left\{(y_1,y_2) \in (\mathbb{C}^\times)^2 : y_1 \neq -\tfrac{1}{27}\right\} \qquad U_{\mathrm{orb}}^\times = \left\{(\mathfrak{y}_1,\mathfrak{y}_2) \in \mathbb{C}\times\mathbb{C}^\times : \mathfrak{y}_1^3 \neq -27\right\} \tag{27}$$

Let z denote the standard co-ordinate on $\mathbb{C}$. Consider the left ideal $\mathcal{I}_{\mathrm{LR}} \subset \mathcal{D}^z|_{U_{\mathrm{LR}}^\times \times \mathbb{C}}$ generated by:

$$\begin{aligned} &D_2(D_2 - 3D_1) - y_2, \\ &D_1^3 - y_1(D_2-3D_1)(D_2-3D_1+z)(D_2-3D_1+2z) \end{aligned} \tag{28}$$

where $D_1 = -zy_1\partial_{y_1}$, $D_2 = -zy_2\partial_{y_2}$. Consider the left ideal $\mathcal{I}_{\mathrm{orb}} \subset \mathcal{D}^z|_{U_{\mathrm{orb}}^\times \times \mathbb{C}}$ generated by:

$$\begin{aligned} &\mathfrak{D}_2\mathfrak{d}_1 - \mathfrak{y}_2, \\ &(\mathfrak{D}_2 - \mathfrak{y}_1\mathfrak{d}_1)^3 - 27(\mathfrak{d}_1)^3 \end{aligned} \tag{29}$$

where $\mathfrak{d}_1 = -z\partial_{\mathfrak{y}_1}$, $\mathfrak{D}_2 = -z\mathfrak{y}_2\partial_{\mathfrak{y}_2}$. The ideals $\mathcal{I}_{\mathrm{LR}}$ and $\mathcal{I}_{\mathrm{orb}}$ coincide on the overlap $(U_{\mathrm{LR}}^\times \cap U_{\mathrm{orb}}^\times)\times\mathbb{C}$ and define a left ideal $\mathcal{I} \subset \mathcal{D}^z$ over $\mathcal{M}_{\mathrm{B}}^\times \times \mathbb{C}$. The GKZ system $\mathcal{F}_{\mathrm{GKZ}}^\times$ is defined to be the $\mathcal{D}^z$-module $\mathcal{D}^z/\mathcal{I}$.

Definition 3.4.4 (Grading operator)**.** Define the *Euler vector field* E on $\mathcal{M}_{\mathrm{B}}$ by:

$$E = 2y_2\partial_{y_2} = 2\mathfrak{y}_2\partial_{\mathfrak{y}_2} \tag{30}$$

This matches up with the Euler vector field (10) on the A-model under the mirror maps $\mathrm{mir}_{\overline{\mathcal{X}}}$, $\mathrm{mir}_{\overline{Y}}$. Consider the endomorphism $\mathrm{Gr} \in \mathrm{End}_{\mathbb{C}}(\mathcal{D}^z)$ defined by the commutator:

$$\mathrm{Gr}(P) = [z\partial_z + E, P] \tag{31}$$

This preserves the GKZ ideal $\mathcal{I}$ and induces an endomorphism $\mathrm{Gr} \in \mathrm{End}_{\mathbb{C}}(\mathcal{F}_{\mathrm{GKZ}}^\times)$ of the GKZ system, called the *grading operator*.

Setting:

$$\begin{aligned} \nabla_{z\partial_z} = \mathrm{Gr} - E - \frac{3}{2} &= \mathrm{Gr} + 2z^{-1}D_2 - \frac{3}{2} \\ &= \mathrm{Gr} + 2z^{-1}\mathfrak{D}_2 - \frac{3}{2} \end{aligned} \tag{32}$$

defines a meromorphic connection on $\mathcal{F}_{\mathrm{GKZ}}^\times$ in the direction of z. Combining this with the connection defined in Remark 3.4.2, we obtain a meromorphic flat connection on $\mathcal{F}_{\mathrm{GKZ}}^\times$:

$$\nabla \colon \mathcal{F}_{\mathrm{GKZ}}^\times \to \left(\pi^\star\Omega^1_{\mathcal{M}_{\mathrm{B}}^\times} \oplus \mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times \times \mathbb{C}}\tfrac{dz}{z}\right) \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\times}} \mathcal{F}_{\mathrm{GKZ}}^\times\big(\mathcal{M}_{\mathrm{B}}^\times \times \{0\}\big). \tag{33}$$

Remark 3.4.5. The GKZ system is a version of what is sometimes referred to as the *Horn system*, homogenized by including the variable z.

Remark 3.4.6. Recall from Definition 2.7.1 that we consider TEP structures in the category of complex manifolds and holomorphic maps. The A-model TEP structure is naturally a holomorphic object, as the structure constants of quantum cohomology are transcendental rather than algebraic functions. The GKZ system and the B-model TEP structure can most naturally be defined in the algebraic category but, for simplicity of exposition, in this paper we will regard them as holomorphic objects.

3.4.1. *The GKZ System is Isomorphic to the B-Model TEP Structure.* The B-model TEP structure $\big(\mathcal{F}_{\mathrm{B}}^{\times}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\big)$ defines another $\mathcal{D}^z$-module $\big(\mathcal{F}_{\mathrm{B}}^{\times}, \nabla^{\mathrm{B}}\big)$ on $\mathcal{M}_{\mathrm{B}}^{\times} \times \mathbb{C}$, which we call the *B-model $\mathcal{D}^z$-module*. Recall that there is a distinguished global section of $\mathcal{F}_{\mathrm{B}}^{\times}$:

$$(y,z) \longmapsto \exp\big(-W_y/z\big)\,\omega_y. \tag{34}$$

Oscillating integrals:

$$\int_{\Gamma_+(\sigma)} e^{-W_y/z}\,\omega_y$$

over the Gauss–Manin-flat cycles (Lefschetz thimbles) $\Gamma_+(\sigma)$ defined in (25) are annihilated by the differential operators (28), (29), where we take:

$$D_1 = -z\nabla^{\mathrm{B}}_{y_1\partial_{y_1}} \qquad \mathfrak{d}_1 = -z\nabla^{\mathrm{B}}_{\partial_{\mathfrak{y}_1}}$$
$$D_2 = -z\nabla^{\mathrm{B}}_{y_2\partial_{y_2}} \qquad \mathfrak{D}_2 = -z\nabla^{\mathrm{B}}_{\mathfrak{y}_2\partial_{\mathfrak{y}_2}}$$

It is proven in [54, §4] that we have a $\mathcal{D}^z$-module isomorphism:

$$\varphi\colon (\mathcal{F}_{\mathrm{GKZ}}^{\times}, \nabla^{\mathrm{GKZ}}) \xrightarrow{\cong} (\mathcal{F}_{\mathrm{B}}^{\times}, \nabla^{\mathrm{B}})$$

defined by sending the distinguished section $1 \in \mathcal{F}_{\mathrm{GKZ}}^{\times}$ to the distinguished section (34) of $\mathcal{F}_{\mathrm{B}}^{\times}$.

3.4.2. *The Pairing on the GKZ System.* We can use the $\mathcal{D}^z$-module isomorphism between the GKZ system and the B-model $\mathcal{D}^z$-module to define a pairing on the GKZ system:

$$(\cdot,\cdot)_{\mathrm{GKZ}}\colon (-)^{\star}\mathcal{F}_{\mathrm{GKZ}}^{\times} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\times}\times\mathbb{C}}} \mathcal{F}_{\mathrm{GKZ}}^{\times} \to \mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\times}\times\mathbb{C}}$$

by pulling back the pairing $(\cdot,\cdot)_{\mathrm{B}}$ on the B-model $\mathcal{D}^z$-module along the isomorphism φ. This pairing can be computed using mirror symmetry: the isomorphisms in Theorem 3.3.1 and Theorem 3.3.2 intertwine the pairings $(\cdot,\cdot)_{\mathrm{B}}$ and $(\cdot,\cdot)_{\mathrm{A},-}$; moreover the pairing $(\cdot,\cdot)_{\mathrm{A},-}$ can be computed through Givental's I-function. For example if $f(z,y_1,y_2,D_1,D_2)$ and $g(z,y_1,y_2,D_1,D_2)$ are elements of the GKZ system defined near $(y_1,y_2)=(0,0)$, then their pairing can be written in terms of the A-model pairing:

$$\Big((-)^{\star} f(z,y_1,y_2,D_1,D_2), g(z,y_1,y_2,D_1,D_2)\Big)_{\mathrm{GKZ}}$$
$$= \Big(f(-z,y_1,y_2,z\nabla_1,z\nabla_2)\mathbf{1}, g(z,y_1,y_2,-z\nabla_1,-z\nabla_2)\mathbf{1}\Big)_{\mathrm{A},\overline{Y}}$$

where $\nabla_i = (\mathrm{mir}_{\overline{Y}}^{\star}\nabla^{\mathrm{A},\overline{Y}})_{y_i\partial_{y_i}}$ is the Dubrovin connection pulled back by the mirror map $\mathrm{mir}_{\overline{Y}}$. By applying the inverse $L(t,-z)^{-1}$ of the fundamental solution (12) to the sections of the

A-model TEP structure in the right-hand side and using the properties (13), (14) of $L(t,-z)$ and the definition (15) of the J-function, we find that the pairing equals

$$\Big(f(-z,y_1,y_2,zy_1\partial_{y_1},zy_2\partial_{y_2})J(\mathrm{mir}_{\overline{Y}}(y_1,y_2),z),\\ g(z,y_1,y_2,-zy_1\partial_{y_1},-zy_2\partial_{y_2})J(\mathrm{mir}_{\overline{Y}}(y_1,y_2),-z)\Big)_{\overline{Y}}$$

The mirror theorem of Givental [39] says that $J(\mathrm{mir}_{\overline{Y}}(y_1,y_2),-z)$ equals the cohomology-valued power series $I_{\overline{Y}}(y_1,y_2,-z)$:

$$I_{\overline{Y}}(y_1,y_2,-z)=\sum_{d_1,d_2\geq 0}\frac{y_1^{d_1-h_1/z}y_2^{d_2-h_2/z}}{\prod_{m=1}^{d_1}(h_1-mz)^3\prod_{m=1}^{d_2}(h_2-mz)}\frac{\prod_{m=-\infty}^{0}(h_2-3h_1-mz)}{\prod_{m=-\infty}^{d_2-3d_1}(h_2-3h_1-mz)} \tag{35}$$

Here we expand the right-hand side as a Taylor series in the (nilpotent) cohomology classes h_1, h_2 from §2.1; note that all but finitely many terms in the infinite products on the right-hand side cancel. Hence we obtain

$$\Big((-)^\star f(z,y_1,y_2,D_1,D_2),g(z,y_1,y_2,D_1,D_2)\Big)_{\mathrm{GKZ}} \tag{36}$$
$$=\Big(f(-z,y_1,y_2,zy_1\partial_{y_1},zy_2\partial_{y_2})I_{\overline{Y}}(y_1,y_2,z),g(z,y_1,y_2,-zy_1\partial_{y_1},-zy_2\partial_{y_2})I_{\overline{Y}}(y_1,y_2,-z)\Big)_{\overline{Y}}$$

Equations (36) and (35) together make clear that the pairing:

$$((-)^\star f(z,y_1,y_2,D_1,D_2),g(z,y_1,y_2,D_1,D_2))_{\mathrm{GKZ}}$$

extends holomorphically across the locus $y_1y_2=0$ if f and g depend polynomially on (y_1,y_2).

3.5. **The B-Model** log**-TEP Structure.** Recall that the B-model TEP structure has base $\mathcal{M}_{\mathrm{B}}^\times$, which is the open subset of the toric variety $\mathcal{M}_{\mathrm{B}}$ obtained by deleting the divisor $D=\overline{(y_1y_2=0)}\cup(y_1=-\frac{1}{27})$ from $\mathcal{M}_{\mathrm{B}}$. Here we construct a logarithmic extension of the B-model TEP structure across D, which we call the *B-model* log*-TEP structure*. This is a log-TEP structure in the sense of Definition 2.7.2.

Proposition 3.5.1. *The flat connection and the pairing of the GKZ system are described explicitly as follows.*

(a) *In the chart near $(y_1,y_2)=(0,0)$ with $y_1\neq-\frac{1}{27}$, writing $D_1=-zy_1\partial_{y_1}$, $D_2=-zy_2\partial_{y_2}$, the GKZ system has basis:*

$$1,D_2,D_2^2,D_2^3,D_1,(1+27y_1)D_1^2 \tag{37}$$

With respect to this basis, we have:

$$D_1 = \begin{pmatrix} 0 & -\frac{1}{3}y_2 & \frac{1}{3}zy_2 & 18y_2^2\big(y_1 - \frac{1}{54}\big) & 0 & 6zy_1y_2 \\ 0 & 0 & -\frac{1}{3}y_2 & 9zy_1y_2 & 0 & 2z^2y_1 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3}y_2(1-27y_1) & 0 & -3zy_1 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & y_1 \\ 1 & 0 & 0 & -zy_2(1+27y_1) & 0 & -3y_1(3y_2+2z^2) \\ 0 & 0 & 0 & 3y_2 & \frac{1}{1+27y_1} & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & 0 & 0 & y_2(54y_1y_2 - y_2 + z^2) & -\frac{1}{3}y_2 & \frac{1}{9}zy_2(1+27y_1) \\ 1 & 0 & 0 & -zy_2(2-27y_1) & 0 & -\frac{1}{9}y_2(1+27y_1) \\ 0 & 1 & 0 & y_2(2-27y_1) & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{9}(1+27y_1) \\ 0 & 0 & 0 & -3zy_2(1+27y_1) & 0 & -\frac{1}{3}y_2(1+27y_1) \\ 0 & 0 & 0 & 9y_2 & 0 & 0 \end{pmatrix}$$

and the Gram matrix of the pairing is:

$$\begin{pmatrix} 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & (1+27y_1) \\ 0 & 9 & 0 & 27y_2 & 3 & 0 \\ 9 & 0 & 27y_2 & 0 & 0 & y_2(1+27y_1) \\ 0 & 0 & 3 & 0 & 0 & 9y_1 \\ 0 & 1+27y_1 & 0 & y_2(1+27y_1) & 9y_1 & 0 \end{pmatrix}$$

(b) *In the chart near $(y_1, y_2) = (-\frac{1}{27}, 0)$, writing $t = y_1 + \frac{1}{27}$ and $D_t = zt\partial_t$, the following relations define the GKZ system:*

$$\begin{aligned} 9ty &= D_2(9tD_2 - (27t-1)D_t) \\ 729t^2D_t^3 &= \big[9tD_2 - (27t-1)(D_t+2z)\big] \\ &\qquad \times \big[9t(D_2+z) - (27t-1)(D_t+z)\big]\big[9t(D_2+2z) - (27t-1)D_t\big] \end{aligned}$$

and the GKZ system has basis:

$$1, D_2, D_2^2, D_2^3, \big(1 - \tfrac{1}{27t}\big)D_t, \tfrac{1}{27t}\big((27t-1)^2D_t^2 + (27t-1)D_t\big) \tag{38}$$

(This is the same basis as Part (a).) With respect to this basis, we have:

$$D_t = \begin{pmatrix} 0 & \frac{9ty_2}{1-27t} & \frac{-9tzy_2}{1-27t} & \frac{27t(18t-1)y_2^2}{27t-1} & 0 & 6tzy_2 \\ 0 & 0 & \frac{9ty_2}{1-27t} & 9tzy_2 & 0 & 2tz^2 \\ 0 & \frac{9t}{27t-1} & 0 & \frac{9(2-27t)ty_2}{27t-1} & 0 & -3tz \\ 0 & 0 & \frac{9t}{27t-1} & 0 & 0 & t \\ \frac{27t}{27t-1} & 0 & 0 & \frac{-729t^2zy_2}{27t-1} & 0 & -3t\left(2z^2+3y_2\right) \\ 0 & 0 & 0 & \frac{81ty_2}{27t-1} & \frac{1}{27t-1} & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & 0 & 0 & y_2(54ty_2 - 3y_2 + z^2) & -\frac{1}{3}y_2 & 3zy_2t \\ 1 & 0 & 0 & -zy_2(3-27t) & 0 & -3y_2t \\ 0 & 1 & 0 & y_2(3-27t) & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 3t \\ 0 & 0 & 0 & -81zy_2t & 0 & -9y_2t \\ 0 & 0 & 0 & 9y_2 & 0 & 0 \end{pmatrix}$$

and the Gram matrix of the pairing is:

$$\begin{pmatrix} 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 27t \\ 0 & 9 & 0 & 27y_2 & 3 & 0 \\ 9 & 0 & 27y_2 & 0 & 0 & 27ty_2 \\ 0 & 0 & 3 & 0 & 0 & 9t - \frac{1}{3} \\ 0 & 27t & 0 & 27ty_2 & 9t - \frac{1}{3} & 0 \end{pmatrix}$$

(c) *In the chart near* $(\mathfrak{y}_1, \mathfrak{y}_2) = (0, 0)$ *with* $\mathfrak{y}_1^3 \neq -27$*, writing* $\mathfrak{d}_1 = -z\partial_{\mathfrak{y}_1}$, $\mathfrak{D}_2 = -z\mathfrak{y}_2\partial_{\mathfrak{y}_2}$*, the GKZ system has basis:*

$$1, \mathfrak{D}_2, \mathfrak{D}_2^2, \mathfrak{D}_2^3, \mathfrak{d}_1, \mathfrak{d}_1^2 \tag{39}$$

With respect to this basis, we have:

$$\mathfrak{d}_1 = \begin{pmatrix} 0 & \mathfrak{y}_2 & -z\mathfrak{y}_2 & z^2\mathfrak{y}_2 & 0 & -\frac{3z\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1^3+27} \\ 0 & 0 & \mathfrak{y}_2 & -2z\mathfrak{y}_2 & 0 & -\frac{3\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1^3+27} \\ 0 & 0 & 0 & \mathfrak{y}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mathfrak{y}_1^3+27} \\ 1 & 0 & 0 & 0 & 0 & -\frac{z^2\mathfrak{y}_1}{\mathfrak{y}_1^3+27} \\ 0 & 0 & 0 & 0 & 1 & \frac{3z\mathfrak{y}_1}{\mathfrak{y}_1^3+27} \end{pmatrix}$$

$$\mathfrak{D}_2 = \begin{pmatrix} 0 & 0 & 0 & -2z^2\mathfrak{y}_1\mathfrak{y}_2 & \mathfrak{y}_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3\mathfrak{y}_1\mathfrak{y}_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3z\mathfrak{y}_1^2\mathfrak{y}_2 & 0 & \mathfrak{y}_2 \\ 0 & 0 & 0 & \mathfrak{y}_2(\mathfrak{y}_1^3+27) & 0 & 0 \end{pmatrix}$$

and the Gram matrix of the pairing is:

$$\begin{pmatrix} 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 27\mathfrak{y}_1\mathfrak{y}_2 & 0 & 0 \\ 9 & 0 & 27\mathfrak{y}_1\mathfrak{y}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{27+\mathfrak{y}_1^3} \\ 0 & 0 & 0 & 0 & \frac{9}{27+\mathfrak{y}_1^3} & 0 \end{pmatrix}$$

Proof. We will prove only part (a). Part (b) follows trivially from part (a), and part (c) is very similar. Consider first the subsheaf of the GKZ system spanned, over $\mathcal{O}_{U_{\mathrm{LR}}^\times \times \mathbb{C}}$, by:

$$1, D_2, D_2^2, D_2^3, D_1, (1 + 27y_1)D_1^2$$

This subsheaf is locally free over $U_{\mathrm{LR}}^\times \times \mathbb{C}$; to see this, it suffices to show that the Gram matrix of the pairing is as claimed, for this matrix is invertible for all y_1, y_2.

To compute the Gram matrix, observe that the pairing is homogeneous of degree -3 with respect to the grading (31) and that, in view of the discussion immediately above, only non-negative powers of y_1, y_2, and z can occur. Thus the Gram matrix takes the form:

$$\begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & *z & 0 & * \\ 0 & * & *z & *z^2 + *y_2 & * & *z \\ * & *z & *z^2 + *y_2 & *z^3 + *zy_2 & *z & *z^2 + *y_2 \\ 0 & 0 & * & *z & 0 & * \\ 0 & * & *z & *z^2 + *y_2 & * & *z \end{pmatrix}$$

where each asterisk denotes an unknown function of y_1. Consider now the matrix entry $((-)^\star D_2, D_1^2)_{\mathrm{GKZ}}$. Combining equation (36) with the equality:

$$I_{\overline{Y}}(y_1, y_2, -z)\Big|_{y_2=0} = e^{-h_1/z} e^{-h_2/z} \left(1 + \sum_{k>0} y_1^k \frac{\prod_{-3k<m\leq 0}(h_2 - 3h_1 - mz)}{\prod_{1\leq m\leq k}(h_1 - mz)^3}\right)$$

yields:

$$((-)^\star D_2, D_1^2)_{\mathrm{GKZ}} = \Big(h_2 + \textstyle\sum_{k>0} y_1^k h_2 \frac{\prod_{-3k<m\leq 0}(h_2-3h_1+mz)}{\prod_{1\leq m\leq k}(h_1+mz)^3}, h_1^2 + \sum_{l>0} y_1^l (h_1 - lz)^2 \frac{\prod_{-3l<m\leq 0}(h_2-3h_1-mz)}{\prod_{1\leq m\leq l}(h_1-mz)^3}\Big)_{\mathrm{A},\overline{Y}}$$

and hence:

$$\begin{aligned} &((-)^\star D_2, D_1^2)_{\mathrm{GKZ}} \\ &= \int_{\overline{Y}} \Big(h_2 + \textstyle\sum_{k>0} y_1^k h_2 \frac{\prod_{-3k<m\leq 0}(h_2-3h_1+mz)}{\prod_{1\leq m\leq k}(h_1+mz)^3}\Big) \Big(h_1^2 + \sum_{l>0} y_1^l (h_1 - lz)^2 \frac{\prod_{-3l<m\leq 0}(h_2-3h_1-mz)}{\prod_{1\leq m\leq l}(h_1-mz)^3}\Big) \\ &= \int_{\overline{Y}} h_2 h_1^2 = 1 \end{aligned}$$

where for the second equality we used the relation $h_2(h_2 - 3h_1) = 0$ in $H^\bullet(Y)$. Thus:

$$\big((-)^\star D_2, (1 + 27y_1) D_1^2\big)_{\mathrm{GKZ}} = 1 + 27y_1$$

The same reasoning allows us to fill in almost all terms in the Gram matrix that are not divisible by y_2:

$$\begin{pmatrix} 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 1 + 27y_1 \\ 0 & 9 & 0 & *y_2 & 3 & 0 \\ 9 & 0 & *y_2 & *zy_2 & 0 & *y_2 \\ 0 & 0 & 3 & 0 & 0 & * \\ 0 & 1 + 27y_1 & 0 & *y_2 & * & 0 \end{pmatrix}$$

Furthermore the symmetry:

$$\big((-)^\star s_1, s_2\big)_{\mathrm{B}} = (-)^\star \big((-)^\star s_2, s_1\big)_{\mathrm{B}}$$

gives a corresponding symmetry of the GKZ pairing, which in particular implies that

$$((-)^\star D_2^3, D_2^3)_{\mathrm{GKZ}} = 0.$$

All remaining terms in the Gram matrix are therefore independent of z. These can be calculated using the principal term of the stationary phase approximation (26), where we see the

residue pairing:

$$\left(\left[f(x,-z)e^{W(x)/z}\,\omega\right],\left[g(x,z)e^{-W(x)/z}\,\omega\right]\right)_{\mathrm{B}} = \sum_{\text{critical points } \sigma} \frac{f(\sigma,0)g(\sigma,0)}{\mathrm{Hess}_\sigma(W)} + O(z)$$

Thus:

$$\begin{aligned}\big((-)^\star D_2^2, D_2^3\big)_{\mathrm{GKZ}} &= \sum_{\text{critical points } \sigma} \frac{\big(y_2\frac{\partial W}{\partial y_2}(\sigma)\big)^2\big(y_2\frac{\partial W}{\partial y_2}(\sigma)\big)^3}{\mathrm{Hess}_\sigma(W)} + O(z)\\ &= \sum_{\text{critical points } \sigma} \frac{\big(\frac{3y_1y_2^3}{w_1w_2w_5^3} + \frac{y_2}{w_5}\big)^5}{w_1^2w^5} + O(z)\\ &= 27y_2 + O(z)\end{aligned}$$

where we use co-ordinates (w_1, w_2, w_5) on the fibre of the Landau–Ginzburg model as in (24), and at the last step we used the critical point equations:

$$w_1 - \frac{y_1y_2^3}{w_1w_2w_5^3} = 0 \qquad w_2 - \frac{y_1y_2^3}{w_1w_2w_5^3} = 0 \qquad w_5 - \frac{3y_1y_2^3}{w_1w_2w_5^3} - \frac{y_2}{w_5} = 0$$

On the other hand we know that $\big((-)^\star D_2^2, D_2^3\big)_{\mathrm{GKZ}}$ is independent of z, so:

$$\big((-)^\star D_2^2, D_2^3\big)_{\mathrm{GKZ}} = 27y_2$$

The same reasoning yields $(D_2^3, D_2^2)_{\mathrm{GKZ}} = 27y_2$ and:

$$\begin{aligned}\big((-)^\star D_2^3, (1+27y_1)D_1^2\big)_{\mathrm{GKZ}} &= y_2(1+27y_1) \qquad \big((-)^\star D_1, (1+27y_1)D_1^2\big)_{\mathrm{GKZ}} = 9y_1\\ \big((-)^\star (1+27y_1)D_1^2, D_2^3\big)_{\mathrm{GKZ}} &= y_2(1+27y_1) \qquad \big((-)^\star (1+27y_1)D_1^2, D_1\big)_{\mathrm{GKZ}} = 9y_1\end{aligned}$$

This completes the calculation of the Gram matrix.

We now compute the connection matrices, i.e. the matrices for the action of D_1 and D_2 on the elements:

$$1, D_2, D_2^2, D_2^3, D_1, (1+27y_1)D_1^2$$

This is routine, involving repeated application of the equations (28); one can do this systematically using Gröbner basis methods as in [44]. In particular we discover that the subsheaf of the GKZ system spanned over $\mathcal{O}_{U_{\mathrm{LR}}^\times \times \mathbb{C}}$ by the above elements is closed under the action of D_1 and D_2. It follows that this subsheaf is in fact the entire GKZ system over $U_{\mathrm{LR}}^\times \times \mathbb{C}$, and hence that (37) is a basis for the GKZ system over $U_{\mathrm{LR}}^\times \times \mathbb{C}$, as claimed. □

With these explicit connection matrices in hand, we now construct a logarithmic extension of the B-model TEP structure to all of $\mathcal{M}_{\mathrm{B}}$.

Definition 3.5.2 ([30, Proposition 5.2])**.** Let $(\mathcal{G}^\times, \nabla)$ be a locally free sheaf with flat connection on $\mathcal{M} \setminus D$, where D is a normal crossing divisor in $\mathcal{M}$. Let $\mathcal{G}$ be a locally free extension of $\mathcal{G}^\times$ to $\mathcal{M}$ such that ∇ is extended to a meromorphic flat connection on $\mathcal{G}^\times$ with logarithmic singularities along D. We say that $\big(\mathcal{G}, \nabla\big)$ is *the Deligne extension* of $(\mathcal{G}^\times, \nabla)$ across D if the residue endomorphisms of ∇ along D are nilpotent. Let $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$ be a TEP structure with base $\mathcal{M} \setminus D$. We say that a log-TEP structure $(\mathcal{F}, \nabla, (\cdot,\cdot))$ with base $(\mathcal{M}, D)$ is *the Deligne extension* of $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$ if $(\mathcal{F}, \nabla, (\cdot,\cdot))$ restricts to $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$ over $\mathcal{M} \setminus D$ and for each $z \in \mathbb{C}^\times$, $(\mathcal{F}, \nabla)|_{\mathcal{M}\times\{z\}}$ is the Deligne extension of $(\mathcal{F}^\times, \nabla)|_{(\mathcal{M}\setminus D)\times\{z\}}$

Remark 3.5.3. Deligne [30] called this logarithmic extension "prolongement canonique". The Deligne extension of a flat connection on $\mathcal{M} \setminus D$ across D exists if and only if the local monodromy around D is unipotent, and is unique if it exists. When the local monodromy around D is not unipotent, a logarithmic extension is given by the choice of a determination of logarithm, i.e. a section of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ [30, Proposition 5.4].

Proposition-Definition 3.5.4. *Recall from §3.1 that the toric variety $\mathcal{M}_{\mathrm{B}}$ is covered by two toric co-ordinate patches, with co-ordinate systems (y_1, y_2) and $(\mathfrak{y}_1, \mathfrak{y}_2)$. Let U_{LR} and U_{orb} denote the following co-ordinate patches of $\mathcal{M}_{\mathrm{B}}$ (see equation (27) for $U_{\mathrm{LR}}^\times$ and $U_{\mathrm{orb}}^\times$)*

$$U_{\mathrm{LR}} = \{(y_1, y_2) \in \mathbb{C}^2\} \qquad U_{\mathrm{orb}} = \{(\mathfrak{y}_1, \mathfrak{y}_2) \in \mathbb{C}^2 : \mathfrak{y}_1^3 \neq -27\}$$

Specifying that the following generators of $\mathcal{F}_{\mathrm{B}}^\times = \mathcal{F}_{\mathrm{GKZ}}^\times$:

$$1, D_2, D_2^2, D_2^3, D_1, (1+27y_1)D_1^2 \qquad \text{over } U_{\mathrm{LR}}^\times \times \mathbb{C}, \text{ as in (37)}$$
$$1, \mathfrak{D}_2, \mathfrak{D}_2^2, \mathfrak{D}_2^3, \mathfrak{d}_1, \mathfrak{d}_1^2 \qquad \text{over } U_{\mathrm{orb}}^\times \times \mathbb{C}, \text{ as in (39)}$$

form locally free bases for $\mathcal{F}_{\mathrm{B}}$ over (respectively) $U_{\mathrm{LR}} \times \mathbb{C}$ and $U_{\mathrm{orb}} \times \mathbb{C}$ defines a locally free sheaf $\mathcal{F}_{\mathrm{B}}$ over $\mathcal{M}_{\mathrm{B}} \times \mathbb{C}$. The sheaf $\mathcal{F}_{\mathrm{B}}$ carries a meromorphic flat connection ∇^{B} and a pairing $(\cdot,\cdot)_{\mathrm{B}}$ and the triple $(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ forms a log*-TEP structure with base $(\mathcal{M}_{\mathrm{B}}, D)$ in the sense of Definition 2.7.2, where*

$$D = \overline{(y_1 y_2 = 0)} \cup (y_1 = -1/27). \tag{40}$$

We call the triple $(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ the B-model log-TEP structure*. The restriction of the B-model* log*-TEP structure to $\mathcal{M}_{\mathrm{B}}^\times \times \mathbb{C}$ is canonically isomorphic to the B-model TEP structure, and the B-model* log*-TEP structure is the Deligne extension of the B-model TEP structure.*

Proof. We need to check that the generators specified give locally free bases for $\mathcal{F}_{\mathrm{B}}^\times$ over (respectively) $U_{\mathrm{LR}}^\times \times \mathbb{C}$ and $U_{\mathrm{orb}}^\times \times \mathbb{C}$, that the connection matrices with respect to these bases have logarithmic singularities along the divisor $D \times \mathbb{C}$, that the residue endomorphisms of the connection along D are nilpotent, and that the pairing extends holomorphically across D. These statements follow easily from Proposition 3.5.1. □

Remark 3.5.5. The locally free sheaf $\mathcal{F}_{\mathrm{B}}$ should be understood as an orbi-vector bundle on the orbifold chart, cf. Example 2.7.6. In other words, on the chart U_{orb}, $\mathcal{F}_{\mathrm{B}}$ is a μ_3-equivariant sheaf equipped with μ_3-invariant connection and pairing. The μ_3-action is given on the frame by $(1, \mathfrak{D}_2, \mathfrak{D}_2^2, \mathfrak{D}_2^3, \mathfrak{d}_1, \mathfrak{d}_2^2) \mapsto (1, \mathfrak{D}_2, \mathfrak{D}_2^2, \mathfrak{D}_2^3, e^{2\pi\mathtt{i}/3}\mathfrak{d}_1, e^{4\pi\mathtt{i}/3}\mathfrak{d}_2^2)$.

4. The Conformal Limit

Let $\mathcal{M}_{\mathrm{CY}} = \mathbb{P}(3,1)$, and let D_{CY} be the divisor $\{0, -\frac{1}{27}\} \subset \mathcal{M}_{\mathrm{CY}}$. A key ingredient in Aganagic–Bouchard–Klemm's modularity argument is the family of elliptic curves:

$$\big\{[X:Y:Z] \in \mathbb{P}^2 : X^3 + Y^3 + Z^3 + y^{-1/3}XYZ = 0\big\} \tag{41}$$

parametrized by $y \in \mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$, and the corresponding variation of Hodge structure. This variation of Hodge structure is a two-dimensional vector bundle over $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ equipped with a flat connection and a Hodge filtration. We will see in this section how this finite-dimensional variation of Hodge structure arises from the B-model TEP structure, by taking the *conformal limit* $y_2 \to 0$ of the Deligne extension $\mathcal{F}_{\mathrm{B}}$.

4.1. **A Vector Bundle of Rank 6 on $\mathcal{M}_{\mathrm{CY}}$ with a Logarithmic Connection.** The closure of the locus $\{y_2 = 0\}$ in $\mathcal{M}_{\mathrm{B}}$ is a copy of $\mathcal{M}_{\mathrm{CY}}$. Consider the restriction

$$\mathcal{F}_{\mathrm{CY}} := \mathcal{F}_{\mathrm{B}}|_{\mathcal{M}_{\mathrm{CY}}\times\mathbb{C}}$$

of the B-model log-TEP structure $\mathcal{F}_{\mathrm{B}}$ (Proposition-Definition 3.5.4) to $\mathcal{M}_{\mathrm{CY}} \times \mathbb{C} \subset \mathcal{M}_{\mathrm{B}} \times \mathbb{C}$. The sheaf $\mathcal{F}_{\mathrm{CY}}$ has the structure of a log-TEP structure with base $(\mathcal{M}_{\mathrm{CY}}, D_{\mathrm{CY}})$ together with the endomorphism $N\colon \mathcal{F}_{\mathrm{CY}} \to z^{-1}\mathcal{F}_{\mathrm{CY}}$ defined as the residue of ∇^{B} along the divisor $\mathcal{M}_{\mathrm{CY}} \times \mathbb{C} \subset \mathcal{M}_{\mathrm{B}} \times \mathbb{C}$ and the grading operator Gr. More precisely we have:

- a meromorphic flat connection with poles along $Z = (D_{\mathrm{CY}} \times \mathbb{C}) \cup (\mathcal{M}_{\mathrm{CY}} \times \{0\})$
$$\nabla\colon \mathcal{F}_{\mathrm{CY}} \to \Omega^1_{\mathcal{M}_{\mathrm{CY}}\times\mathbb{C}}(\log Z) \otimes \mathcal{F}_{\mathrm{CY}}(\mathcal{M}_{\mathrm{CY}} \times \{0\})$$
defined by
$$\begin{aligned}\nabla\left(s|_{y_2=0}\right) &= \left(\nabla^{\mathrm{B}}_{y_1\frac{\partial}{\partial y_1}-\frac{1}{3}y_2\frac{\partial}{\partial y_2}} s\right)\Big|_{y_2=0} \frac{dy_1}{y_1} + \left(\nabla^{\mathrm{B}}_{z\frac{\partial}{\partial z}} s\right)\Big|_{y_2=0} \frac{dz}{z} \\ &= \left(\nabla^{\mathrm{B}}_{\mathfrak{y}_1\frac{\partial}{\partial \mathfrak{y}_1}} s\right)\Big|_{y_2=0} \frac{d\mathfrak{y}_1}{\mathfrak{y}_1} + \left(\nabla^{\mathrm{B}}_{z\frac{\partial}{\partial z}} s\right)\Big|_{y_2=0} \frac{dz}{z}\end{aligned}$$
for a local section s of $\mathcal{F}_{\mathrm{B}}$;
- a flat non-degenerate pairing
$$(\cdot,\cdot) : (-)^\star\mathcal{F}_{\mathrm{CY}} \otimes \mathcal{F}_{\mathrm{CY}} \to \mathcal{O}_{\mathcal{M}_{\mathrm{CY}}\times\mathbb{C}}$$
induced by $(\cdot,\cdot)_{\mathrm{B}}$;
- the residue endomorphism $N\colon \mathcal{F}_{\mathrm{CY}} \to z^{-1}\mathcal{F}_{\mathrm{CY}}$:
$$N = \nabla^{\mathrm{B}}_{y_2\partial_{y_2}}\big|_{y_2=0} = \nabla^{\mathrm{B}}_{\mathfrak{y}_2\partial_{\mathfrak{y}_2}}\big|_{\mathfrak{y}_2=0} = -z^{-1}D_2 = -z^{-1}\mathfrak{D}_2$$
which is flat for ∇ and satisfies $((-)^\star N s_1, s_2) = -((-)^\star s_1, N s_2)$ for $s_1, s_2 \in \mathcal{F}_{\mathrm{CY}}$;
- the grading operator $\mathrm{Gr}\colon \mathcal{F}_{\mathrm{CY}} \to \mathcal{F}_{\mathrm{CY}}$ induced from the grading operator (31) of the GKZ system: this is related to $\nabla_{z\partial_z}$ by

$$\nabla_{z\partial_z} = \mathrm{Gr} - 2N - \frac{3}{2} \qquad \text{(cf. equation (32)).} \tag{42}$$

Remark 4.1.1. Let ∇ be a flat connection on $\mathcal{M}$ with logarithmic singularities along a smooth divisor $D \subset \mathcal{M}$. In order to obtain a flat connection along D from ∇, we need to choose a splitting of the sequence $0 \to \Omega^1_D \to \Omega^1_{\mathcal{M}}(\log D)|_D \xrightarrow{\mathrm{Res}} \mathcal{O}_D \to 0$ (see Example 8.1.1 below) otherwise the induced connection along D is defined only 'up to the residue endomorphism'. This choice is not canonical in general, and we chose a particular splitting when defining the connection ∇ on $\mathcal{F}_{\mathrm{CY}}$. The splitting does not play an important role in this section, but will appear again in §§8.1–8.2 and will be important there.

The triple $(\mathcal{F}_{\mathrm{CY}}, \nabla, (\cdot,\cdot))$ is a log-TEP structure with base $(\mathcal{M}_{\mathrm{CY}}, D_{\mathrm{CY}})$ in the sense of Definition 2.7.2. The grading operator Gr on $\mathcal{F}_{\mathrm{CY}}$ is $\pi^{-1}\mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$-linear since the variable y_1 of the base is of degree zero. Thus it serves as another connection in the z-direction. The GKZ description also passes to $\mathcal{F}_{\mathrm{CY}}$: on the chart $\mathcal{M}_{\mathrm{CY}} \setminus \{0, -\frac{1}{27}, \infty\}$, it is defined by the relations

$$\begin{aligned}&D_2(D_2 - 3D_1) = 0, \\ &D_1^3 - y_1(D_2 - 3D_1)(D_2 - 3D_1 + z)(D_2 - 3D_1 + 2z) = 0\end{aligned}$$

where $D_1 = -zy_1\partial y_1$ is as before and $D_2 = [-zy_2\partial_{y_2}]_{y_2=0} = -zN$ is now an $\mathcal{O}_{\mathcal{M}_{\mathrm{CY}}\times\mathbb{C}}$-linear endomorphism commuting with D_1. It is extended across the three points $\{0, -\frac{1}{27}, \infty\}$ by the bases specified in Proposition-Definition 3.5.4.

4.1.1. *The Rank 6 Vector Bundle H.* Consider now the push-forward $\pi_\star(\mathcal{F}^*_{\mathrm{CY}})$ where $\pi\colon \mathcal{M}_{\mathrm{CY}} \times \mathbb{C}^\times \to \mathcal{M}_{\mathrm{CY}}$ is the projection; see Notation 2.8.3 for the notation here. Consider the subsheaf of $\pi_\star(\mathcal{F}^*_{\mathrm{CY}})$ consisting of homogeneous sections of degree 1 with respect to Gr; this subsheaf is locally free of rank 6 over $\mathcal{M}_{\mathrm{CY}}$, and thus defines a rank-6 vector bundle $H \to \mathcal{M}_{\mathrm{CY}}$. The vector bundle H carries the following structures:

- a logarithmic flat connection $\nabla\colon \mathcal{O}(H) \to \Omega^1_{\mathcal{M}_{\mathrm{CY}}}(\log D_{\mathrm{CY}}) \otimes \mathcal{O}(H)$, induced from the meromorphic flat connection on $\mathcal{F}_{\mathrm{CY}}$;
- a ∇-flat endomorphism $N \in \mathrm{End}(H)$ of vector bundles, induced by the residue endomorphism $N\colon \mathcal{F}_{\mathrm{CY}} \to z^{-1}\mathcal{F}_{\mathrm{CY}}$;
- an $\mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$-bilinear symplectic pairing $\Omega\colon \mathcal{O}(H) \otimes \mathcal{O}(H) \to \mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$, induced by the pairing $(\cdot,\cdot)$ on $\mathcal{F}_{\mathrm{CY}}$.

The pairing $(\cdot,\cdot)$ induces a symplectic pairing Ω on $\mathcal{O}(H)$ because, when restricted to $\mathcal{O}(H)$, $(\cdot,\cdot)$ takes values in $z^{-1}\mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$; we set:

$$\Omega(s_1, s_2) = \mathrm{Res}_{z=0}\big((-)^\star s_1, s_2\big)\, dz \qquad \text{for } s_1,\, s_2 \in \mathcal{O}(H).$$

The connection ∇ on H preserves the symplectic form, and $N : H \to H$ is infinitesimally symplectic, i.e. $\Omega(Nv, w) + \Omega(v, Nw) = 0$. In view of Proposition-Definition 3.5.4, local frames of H over the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$ and the orbifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = 0\}$ are given respectively by:

$$\begin{array}{ll} -z,\ D_2,\ z^{-1}D_2^2,\ z^{-2}D_2^3,\ D_1 - \frac{1}{3}D_2,\ z^{-1}(1+27y_1)(D_1 - \frac{1}{3}D_2)^2 & \text{on } \mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\} \\ -z,\ \mathfrak{D}_2,\ z^{-1}\mathfrak{D}_2^2,\ z^{-2}\mathfrak{D}_2^3,\ \mathfrak{d}_1,\ z^{-1}(1 + \frac{1}{27}\mathfrak{y}_1^3)\mathfrak{d}_1^2 & \text{on } \mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = 0\} \end{array} \tag{43}$$

These two bases are related by the transition matrix

$$\left(\begin{array}{c|cc} I & & \\ \hline & -\frac{1}{3}\mathfrak{y}_1 & -\frac{1}{9}\mathfrak{y}_1(1+27\mathfrak{y}_1^{-3}) \\ & 0 & 3\mathfrak{y}_1^{-1} \end{array}\right)$$

where I is the identity matrix of rank 4. This implies that $\mathcal{O}(H) \cong \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1)$ as a bundle on $\mathcal{M}_{\mathrm{CY}} = \mathbb{P}(3,1)$.

4.1.2. *The Hodge Filtration.* The vector bundle H carries a 'Hodge filtration' given by pole order at $z = 0$:

$$F^p = \left[\pi_\star\Big(z^{p-2}\mathcal{F}_{\mathrm{CY}}\Big)\right]_{\deg 1}$$

where $\pi\colon \mathcal{M}_{\mathrm{CY}} \times \mathbb{C} \to \mathcal{M}_{\mathrm{CY}}$ is the projection and the subscript indicates that we take the subsheaf consisting of homogeneous elements of degree 1. This is a decreasing filtration by subbundles:

$$0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H$$

such that one has:

$$\nabla_v F^p \subset F^{p-1} \qquad NF^p \subset F^{p-1} \qquad \Omega(F^p, F^{4-p}) = 0$$

for any vector field $v \in \Theta_{\mathcal{M}_{\mathrm{CY}}}(\log D_{\mathrm{CY}}\})$. Explicit bases of the subbundles F^p on the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$ are given by:

$$\begin{aligned} F^3 &: && -z \\ F^2 &: && -z, D_2, D_1 - \tfrac{1}{3}D_2 \\ F^1 &: && -z, D_2, D_1 - \tfrac{1}{3}D_2, z^{-1}D_2^2, z^{-1}(1+27y_1)(D_1 - \tfrac{1}{3}D_2)^2 \\ F^0 &: && -z, D_2, D_1 - \tfrac{1}{3}D_2, z^{-1}D_2^2, z^{-1}(1+27y_1)(D_1 - \tfrac{1}{3}D_2)^2, z^{-2}D_2^3 \end{aligned}$$

There is a 'primitive section' $\zeta \in F^3$ of H, represented by $-z$ in the GKZ system. This satisfies $N^3\zeta \neq 0$, and $N^3\zeta$ is flat.

4.1.3. *The Kernel and the Image of N.* The endomorphism N is flat for ∇, and therefore the kernel and image of N are preserved by ∇. By examining the action of $N = -z^{-1}D_2$ on the basis (43), we know that both $\operatorname{Ker} N$ and $\operatorname{Im} N$ are of rank 3 and have the following explicit bases (on the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$):

$$\begin{aligned} \operatorname{Ker} N &: && z^{-2}D_2^3,\ D_1 - \tfrac{1}{3}D_2,\ z^{-1}(1+27y_1)(D_1 - \tfrac{1}{3}D_2)^2 \\ \operatorname{Im} N &: && D_2,\ z^{-1}D_2^2,\ z^{-2}D_2^3 \end{aligned}$$

4.2. **A Vector Bundle of Rank 2 on $\mathcal{M}_{\mathrm{CY}}$ with a Logarithmic Connection.** We now pass from H, which is a six-dimensional symplectic vector bundle over $\mathcal{M}_{\mathrm{CY}}$, to a two-dimensional symplectic vector bundle H_{vec} over $\mathcal{M}_{\mathrm{CY}}$. The bundle H_{vec} is obtained from H via the infinitesimally symplectic endomorphism N. A similar construction appears in the work of Konishi–Minabe [63, §8] in the A-model.

4.2.1. *The Rank 3 Vector Bundle $\overline{H} = \operatorname{Cok} N$ and Quantum D-Module of $K_{\mathbb{P}^2}$.* Consider the cokernel $\overline{H}$ of the map $N : H \to H$. This carries a flat connection ∇ with logarithmic poles along D_{CY} induced by ∇ on H. Write $\theta = \nabla_{y_1\partial_{y_1}} = -z^{-1}D_1$ for the operator[9] acting on $\mathcal{O}(\overline{H})$. Local frames for $\overline{H}$ on the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$ and the orbifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = 0\}$ are given respectively by:

$$\zeta = [-z], \quad \theta\zeta = [D_1 - \tfrac{1}{3}D_2], \quad -(1+27y_1)\theta^2\zeta = [z^{-1}(1+27y_1)(D_1 - \tfrac{1}{3}D_2)^2] \tag{44}$$

and

$$\zeta = [-z], \quad -3\mathfrak{y}_1^{-1}\theta\zeta = [\mathfrak{d}_1], \quad \tfrac{1}{3}\mathfrak{y}_1(1+27\mathfrak{y}_1^{-3})\theta(\theta+\tfrac{1}{3})\zeta = [-z^{-1}(1+\tfrac{1}{27}\mathfrak{y}_1^3)\mathfrak{d}_1^2]$$

We have $\mathcal{O}(\overline{H}) \cong \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1)$. The differential operator

$$\theta^3 - y_1(-3\theta)(-3\theta-1)(-3\theta-2) \tag{45}$$

annihilates the primitive section $\zeta \in \mathcal{O}(\overline{H})$. Hence the D-module $(\mathcal{O}(\overline{H}), \nabla)$ is isomorphic to the quantum D-module for $Y = K_{\mathbb{P}^2}$; equation (45) is the Picard–Fuchs equation for the family of elliptic curves (41) mirror to $K_{\mathbb{P}^2}$. With respect to the frame $\{\zeta, \theta\zeta, (1+27y_1)\theta^2\zeta\}$ (44) in the manifold chart, the action of θ is represented by the matrix:

$$\theta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -6y_1 \\ 0 & \frac{1}{1+27y_1} & 0 \end{pmatrix}$$

[9] As $z^{-1}D_2$ acts trivially on $\mathcal{O}(\overline{H})$, we have $\theta = -z^{-1}(D_1 - \frac{1}{3}D_2) = -\frac{1}{3}\mathfrak{y}_1\mathfrak{d}_1$.

4.2.2. *Affine Subbundle H_{aff} of $\overline{H}$.* Any local function $\psi(y_1)$ annihilated by the differential operator (45) gives a D-module homomorphism $\psi^\sharp\colon \mathcal{O}(H) \to \mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$ sending ζ to the function $\psi(y_1)$. The constant function 1 is a solution to the equation (45) and thus defines a homomorphism $1^\sharp\colon \mathcal{O}(H) \to \mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$. Consider the slice (affine subbundle) H_{aff} of $\overline{H}$ given by:

$$H_{\mathrm{aff}} = \{v \in \overline{H} : 1^\sharp(v) = 1\}$$

(cf. the dilaton shift in equation (66)). Elements of H_{aff} take the form:

$$\zeta + x \cdot \theta\zeta - p \cdot (1 + 27y_1)\theta^2\zeta \qquad\qquad x,\ p \in \mathbb{C}; \tag{46}$$

on the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$. As we see in §4.2.3 below, each fibre of the affine bundle H_{aff} is naturally equipped with an affine symplectic structure. The affine bundle H_{aff} is preserved by the connection ∇ on $\overline{H}$.

4.2.3. *Rank 2 Vector Subbundle H_{vec} of $\overline{H}$ Parallel to H_{aff}.* Consider the canonical projection $\operatorname{Ker} N \to \operatorname{Cok} N = \overline{H}$. This induces an embedding of vector bundles $\operatorname{Ker} N/(\operatorname{Im} N \cap \operatorname{Ker} N) \to \overline{H}$. Let H_{vec} denote the image of $\operatorname{Ker} N/(\operatorname{Im} N \cap \operatorname{Ker} N)$ in $\overline{H}$. From the description of $H_{\mathrm{aff}} \subset \overline{H}$ in §4.2.2 and $\operatorname{Ker} N$ in §4.1.3, there is a canonical identification between the tangent space to the affine space $H_{\mathrm{aff}}|_y$ and the fibre $H_{\mathrm{vec}}|_y$. In other words, H_{vec} is a vector subbundle of $\overline{H}$ parallel to H_{aff}. The bundle H_{vec} carries a flat connection ∇ with logarithmic poles along D_{CY} and one has $\mathcal{O}(H_{\mathrm{vec}}) \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

The symplectic structure on H descends to a symplectic structure on H_{vec}. Given a finite-dimensional symplectic vector space (H, Ω) and an infinitesimal symplectic transformation $N \in \mathfrak{sp}(H)$, the symplectic orthogonal $(\operatorname{Ker} N)^\perp$ coincides with $\operatorname{Im} N$: since $\Omega(Nv, w) = -\Omega(v, Nw)$ we have that $\operatorname{Im} N \subset (\operatorname{Ker} N)^\perp$, and the two spaces have the same dimension. The symplectic pairing Ω thus induces a symplectic pairing on the quotient space $\operatorname{Ker} N/(\operatorname{Im} N \cap \operatorname{Ker} N)$. Applying this construction to the (six-dimensional, flat) symplectic vector bundle H and the bundle map $N : H \to H$ yields a (two-dimensional, flat) symplectic vector bundle $H_{\mathrm{vec}} = \operatorname{Ker} N/(\operatorname{Im} N \cap \operatorname{Ker} N)$. The symplectic pairing is given by:

$$\Omega\big(\theta\zeta, -(1 + 27y_1)\theta^2\zeta\big) = -\frac{1}{3} = \Omega\big([\mathfrak{d}_1], [-z^{-1}(1 + \tfrac{1}{27}\mathfrak{y}_1^3)\mathfrak{d}_1^2]\big) \tag{47}$$

and therefore the symplectic form on H_{aff} is given by $\frac{1}{3} dp \wedge dx$ in the co-ordinates (46).

4.3. **Opposite Filtrations on H, $\overline{H}$, and H_{vec}.** The Hodge filtration $F^\bullet$ on H induces a filtration:

$$0 \subset \overline{F}^3 \subset \overline{F}^2 \subset \overline{F}^1 = \overline{H}$$

on $\overline{H}$, where $\overline{F}^k := F^k/(\operatorname{Im} N \cap F^k)$. They are spanned by the bases

$$\begin{aligned}
&\overline{F}^3 : \quad \zeta = [-z] \\
&\overline{F}^2 : \quad \zeta = [-z], \theta\zeta = [D_1 - \tfrac{1}{3}D_2] \\
&\overline{F}^1 : \quad \zeta = [-z], \theta\zeta = [D_1 - \tfrac{1}{3}D_2], -(1 + 27y_1)\theta^2\zeta = [z^{-1}(1 + 27y_1)(D_1 - \tfrac{1}{3}D_2)^2]
\end{aligned}$$

on the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$. This restricts to a filtration:

$$0 = F^3_{\mathrm{vec}} \subset F^2_{\mathrm{vec}} \subset F^1_{\mathrm{vec}} = H_{\mathrm{vec}}$$

on H_{vec}, where $F^k_{\mathrm{vec}} = H_{\mathrm{vec}} \cap \overline{F}^k$.

4.3.1. *Opposite Filtration.* Opposite filtrations are decreasing filtrations which are complementary to the Hodge filtration. We study a well-behaved class of opposite filtrations which yield trivializations of $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ (i.e. extensions of $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\mathbb{P}^1}$-module) with good properties. See [46, §7; 75, §3] for a closely related discussion.

Proposition 4.3.1. *Let $y \in \mathcal{M}_{\mathrm{CY}}$. Let z denote the standard co-ordinate on $\{y\}\times\mathbb{P}^1 \cong \mathbb{P}^1$. There is a one-to-one correspondence between:*

(A) *subspaces P of $H_{\mathrm{vec}}|_y$ such that $F^2_{\mathrm{vec}} \oplus P = H_{\mathrm{vec}}|_y$;*

(B) *filtrations:*
$$0 = \overline{U}_0 \subset \overline{U}_1 \subset \overline{U}_2 \subset \overline{U}_3 = \overline{H}|_y$$
satisfying $\overline{F}^p \oplus \overline{U}_{p-1} = \overline{H}|_y$ and $\overline{U}_2 = H_{\mathrm{vec}}|_y$.

(C) *filtrations:*
$$0 \subset U_0 \subset U_1 \subset U_2 \subset U_3 = H|_y$$
satisfying $F^p \oplus U_{p-1} = H|_y$, $N(U_p) \subset U_{p-1}$, $U_0^\perp = U_2$, and $U_1^\perp = U_1$;

(D) *extensions of $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}} = \mathcal{F}_{\mathrm{B}}\big|_{\{y\}\times\mathbb{C}}$ to a locally free sheaf $\mathcal{E}$ on $\{y\}\times\mathbb{P}^1$ such that:*
- *the corresponding holomorphic vector bundle on $\{y\}\times\mathbb{P}^1$ is trivial;*
- *the pairing $(\cdot,\cdot)_{\mathrm{B}}$ extends holomorphically across $z=\infty$ and is non-degenerate there;*
- *the connection ∇ has a logarithmic pole at $z=\infty$;*
- *the map N defined in §4.1 extends holomorphically across $z=\infty$ and vanishes there.*

This correspondence satisfies:
$$\overline{U}_k = U_k/(\operatorname{Im} N \cap U_k)$$
$$P = \overline{U}_1 = U_1/(\operatorname{Im} N \cap U_1)$$

Proof.

(A $\iff$ B). To give a subspace P as in (A) is exactly the same as to give a filtration $\overline{U}_\bullet$ as in (B) such that $\overline{U}_1 = P$.

(B $\implies$ C). Suppose that $\overline{U}_\bullet$ is a filtration as in (B). Set:
$$U_0 = \langle N^3\zeta\rangle$$
$$U_1 = \left\{s \in \operatorname{Ker} N : s + \operatorname{Im} N \in \overline{U}_1\right\} + \langle N^2\zeta\rangle$$
$$U_2 = \operatorname{Ker} N + \operatorname{Im} N$$
where recall that $\zeta = -z$ and $N = -z^{-1}D_2$. It is clear that $F^1 \oplus U_0 = H$, that $F^3 \oplus U_2 = H$, that $N(U_p) \subset U_{p-1}$, that $U_0^\perp = U_2$, and that $\overline{U}_k = U_k/(\operatorname{Im} N \cap U_k)$. It remains to show that $F^2 \oplus U_1 = H$ and that $U_1^\perp = U_1$. The space U_1 is certainly isotropic, and:
$$\dim U_1 = \dim \overline{U}_1 + \dim(\operatorname{Ker} N \cap \operatorname{Im} N) + 1 = 3$$
so U_1 is maximal isotropic: $U_1^\perp = U_1$. Both F^2 and U_1 have dimension 3, so to show that $F^2 \oplus U_1 = H$ it suffices to show that $F^2 + U_1 = H$. Let $v \in H$ be arbitrary, and let $\overline{v}$ denote the equivalence class of v in $\overline{H}$. Since $\overline{F}^2 \oplus \overline{U_1} = \overline{H}$, there exist $\overline{f} \in \overline{F}^2$ and $\overline{u} \in \overline{U}_1$ such that $\overline{v} = \overline{f} + \overline{u}$. Let $f \in F^2$ and $u \in U_1$ be lifts of $\overline{u}$ and $\overline{f}$ respectively. Then $v - f - u \in \operatorname{Im} N = \langle N\zeta, N^2\zeta, N^3\zeta\rangle$. Since $N\zeta \in F^2$ and $N^2\zeta,\ N^3\zeta \in U_1$, it follows that $v \in F^2 + U_1$. Thus if $\overline{U}_\bullet$ is a filtration as in (B), we can define $U_\bullet$ as above to obtain a filtration as in (C) which satisfies $\overline{U}_k = U_k/(\operatorname{Im} N \cap U_k)$.

(C $\implies$ B). Suppose that we are given a filtration $U_\bullet$ as in (C). The filtration $U_\bullet$ is opposite to $F^\bullet$, and counting dimensions gives:

$$\dim U_0 = 1 \qquad \dim U_1 = 3 \qquad \dim U_2 = 5$$

The elements $\zeta \in U_3$, $N\zeta \in U_2$, $N^2\zeta \in U_1$, and $N^3\zeta \in U_0$ are non-zero and linearly independent; in particular $U_0 = \langle N^3\zeta \rangle$. We have $U_1 = \langle N^3\zeta, N^2\zeta, e_1 \rangle$ for some e_1. Since $Ne_1 \in U_0$ is a scalar multiple of $N^3\zeta$ we may, by replacing e_1 with a linear combination of e_1 and $N^2\zeta$, without loss of generality assume that $Ne_1 = 0$. We have $U_2 = \langle N^3\zeta, N^2\zeta, e_1, N\zeta, e_2 \rangle$ for some e_2. Replacing e_2 with a linear combination of e_2, $N\zeta$, and $N^2\zeta$ we may without loss of generality assume that $Ne_2 \in U_1$ is a scalar multiple of e_1. Thus, with respect to the basis $N^3\zeta$, $N^2\zeta$, e_1, $N\zeta$, e_2, ζ for $\mathcal{O}(H)$, the matrix of N has the form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We know that the kernel of N is three-dimensional (§4.1.3), so $*$ must be zero and $Ne_2 = 0$.

Set $\overline{U}_k = U_k/(\operatorname{Im} N \cap U_k)$. We find:

$$\overline{U}_0 = 0 \qquad \overline{U}_1 = \langle [e_1] \rangle \qquad \overline{U}_2 = \langle [e_1], [e_2] \rangle \qquad \overline{U}_3 = \overline{U}$$

Now $Ne_1 = Ne_2 = 0$, so $\overline{U}_2 \subset H_{\mathrm{vec}}$, and both spaces are two-dimensional, so $\overline{U}_2 = H_{\mathrm{vec}}$. Since $F^p \oplus U_{p-1} = H$, it follows that $\overline{F}^p + \overline{U}_{p-1} = \overline{H}$. For dimensional reasons we have $\overline{F}^p \oplus \overline{U}_{p-1} = \overline{H}$. Thus given a filtration $U_\bullet$ as in (C), setting $\overline{U}_k = U_k/(\operatorname{Im} N \cap U_k)$ determines a filtration as in (B).

(C $\implies$ D). We construct the extension (D) using an appropriate opposite module, as in §2.8 but taking the base $\mathcal{M}$ there to be the point $\{y\}$. Let $\pi : \{y\} \times \mathbb{P}^1 \to \{y\}$ be the projection map and note, for comparison with §2.8, that:

$$\pi_\star(\mathcal{O}_{\{y\}\times(\mathbb{P}^1\setminus\{0\})}) = \mathcal{O}_{\mathbb{P}^1}\big(\mathbb{P}^1 \setminus \{0\}\big) \qquad \text{and} \qquad \pi_\star(\mathcal{O}_{\{y\}\times\mathbb{C}}) = \mathcal{O}_{\mathbb{P}^1}(\mathbb{C})$$

To match with §2.8, write:

$$\mathcal{F}^* = \mathcal{F}_{\mathrm{CY}}\big|_{\{y\}\times\mathbb{C}^\times} \qquad \text{and} \qquad \mathbf{F} = \pi_\star\big(\mathcal{F}_{\mathrm{CY}}\big|_{\{y\}\times\mathbb{C}}\big)$$

We construct the opposite module using the Rees construction.

Recall that $\mathcal{O}(H)$ is the submodule of $\pi_\star\mathcal{F}^*$ consisting of degree one sections. Define $\mathbf{P}$ to be the $\mathcal{O}_{\mathbb{P}^1}\big(\mathbb{P}^1 \setminus \{0\}\big)$-submodule of $\pi_\star\mathcal{F}^*$ spanned by:

$$z^{-2}U_3 + z^{-1}U_2 + U_1 + zU_0$$

The submodule $\mathbf{P}$ is homogeneous. Recall that $\mathbf{F}$ is the $\mathcal{O}_{\mathbb{P}^1}(\mathbb{C})$-submodule of $\pi_\star\mathcal{F}^*$ spanned by:

$$z^{-1}F^3 + F^2 + zF^1 + z^2F^0$$

The fact that $F^p \oplus U_{p-1} = H|_y$ implies that $\pi_\star\mathcal{F}^* = \mathbf{F} \oplus \mathbf{P}$. The facts that $U_0^\perp = U_2$ and $U_1^\perp = U_1$ imply that $\mathbf{P}$ is isotropic. Thus $\mathbf{P}$ is an opposite module. The discussion in §2.8 produces from $\mathbf{P}$ an extension of $\mathcal{F}_{\mathrm{CY}}\big|_{\{y\}\times\mathbb{C}}$ to a locally free sheaf $\mathcal{E}$ on $\{y\} \times \mathbb{P}^1$ such that:

- the corresponding holomorphic vector bundle on $\{y\} \times \mathbb{P}^1$ is trivial;
- the pairing $(\cdot,\cdot)_{\mathrm{B}}$ extends holomorphically across $z = \infty$ and is non-degenerate there;

- the connection ∇ has a logarithmic pole at $z = \infty$.

Recall that $z\mathbf{P}$ consists of sections of $\mathcal{E}$ over $\mathbb{P}^1 \setminus \{0\}$. The fact that $N(U_p) \subset U_{p-1}$ implies that the map N extends holomorphically across $z = \infty$ and vanishes there. Thus a filtration $U_\bullet$ as in (C) determines an extension as in (D).

(D $\implies$ C). Consider again the discussion in §2.8 with the base $\mathcal{M}$ there taken to be the point $\{y\}$. An extension $\mathcal{E}$ as in (D) determines an opposite module $\mathbf{P} = z^{-1}\pi_\star\big(\mathcal{E}|_{\{y\}\times(\mathbb{P}^1\setminus\{0\})}\big)$. Set:

$$U_p = z^{p-1}\mathbf{P} \cap H$$

This defines an increasing filtration. Recall that we have

$$F^p = z^{p-2}\mathbf{F} \cap H$$

The grading operator Gr preserves $z\mathbf{P} \cap \mathbf{F} \cong \mathbf{F}/z\mathbf{F}$ and is semisimple there, and therefore $\mathbf{P}$ is generated by homogeneous elements over $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \setminus \{0\})$. Thus the decomposition $z^{p-2}\mathbf{F} \oplus z^{p-2}\mathbf{P} = \pi_\star\mathcal{F}^*$ restricted to degree one part implies $U_{p-1} \oplus F^p = H$. The fact that $\mathbf{P}$ is isotropic implies that $\Omega(U_p, U_{2-p}) = 0$ and thus one has $U_p = U_{2-p}^\perp$ for dimension reasons. Furthermore $N(U_p) \subset U_{p-1}$ follows from the fact that N extends across $z = \infty$ and vanishes there. Thus an extension as in (D) determines a filtration $U_\bullet$ as in (C). □

4.3.2. *Opposite Filtration at the Cusps $y = 0, -\frac{1}{27}, \infty$.* At the large-radius point $y = 0$, the conifold point $y = -\frac{1}{27}$ and the orbifold point $y = \infty$, we have distinguished free extensions of $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ to $\{y\} \times \mathbb{P}^1$ characterized by local monodromy around them. By Proposition 4.3.1, each of them corresponds to a line P in the fibre of H_{vec} at y.

Proposition 4.3.2. *Let y be one of the three points $\{0, -\frac{1}{27}, \infty\}$ in $\mathcal{M}_{\mathrm{CY}} = \mathbb{P}(3,1)$. There exists a unique extension of $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ to a locally free $\mathcal{O}_{\{y\}\times\mathbb{P}^1}$-module $\mathcal{E}$ such that the condition (D) of Proposition 4.3.1 holds and that, in addition:*

- *when y is the large radius limit point or the conifold point, the residue endomorphism $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}} \to z^{-1}\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ of the connection ∇ at y extends regularly across $z = \infty$ and vanishes there;*
- *when y is the orbifold point, the action of $\mathrm{Aut}(y) = \mu_3$ on $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ extends across $z = \infty$. Here μ_3 acts trivially on the base $\{y\} \times \mathbb{C}$.*

The free extensions of $\mathcal{F}_{\mathrm{CY}}|_{\{y\}\times\mathbb{C}}$ to $\{y\} \times \mathbb{P}^1$ are given explicitly by the following bases:

$1,\ D_2,\ D_2^2,\ D_2^3,\ D_1,\ D_1^2$	*(large radius limit point $y = 0$)*
$1,\ D_2,\ D_2^2,\ D_2^3,\ D_1,\ (1+27y_1)D_1^2$	*(conifold point $y = -\frac{1}{27}$)*
$1,\ \mathfrak{D}_2,\ \mathfrak{D}_2^2,\ \mathfrak{D}_2^3,\ \mathfrak{d}_1,\ \mathfrak{d}_1^2$	*(orbifold point $y = \infty$)*

Let P_{LR}, P_{con}, P_{orb} denote the corresponding subspace P of $H_{\mathrm{vec}}|_y$ at the large radius, conifold and orbifold points under the correspondence between (A) and (D) in Proposition 4.3.1. They are given by:

$$P_{\mathrm{LR}} = \langle\theta^2\zeta\rangle, \qquad P_{\mathrm{con}} = \langle(1+27y_1)\theta^2\zeta\rangle \qquad P_{\mathrm{orb}} = \langle z^{-1}\mathfrak{d}_1^2\rangle = \langle\mathfrak{y}_1^{-2}\theta(\theta+\tfrac{1}{3})\zeta\rangle$$

Proof. We discuss the three cases $y = 0, -\frac{1}{27}, \infty$ separately.

($y = 0$, existence) Take the frame of $\mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}}$ described in the proposition. Recall that $\mathcal{F}_{\mathrm{CY}}$ is the restriction of the B-model log-TEP structure $\mathcal{F}_{\mathrm{B}}$ to $\mathcal{M}_{\mathrm{CY}}$. The connection ∇^{B} defines two residue endomorphisms $N_i\colon \mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}} \to z^{-1}\mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}}$ about the divisors $y_i = 0$,

$i = 1, 2$. The map N_2 equals N in §4.1. By Proposition 3.5.1, N_i are represented by the matrices:

$$-\frac{1}{z}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad -\frac{1}{z}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

respectively for $i = 1$ and $i = 2$. These are regular at $z = \infty$ and vanish there. The connection $\nabla_{z\partial_z}$ equals $\mathrm{Gr} - 2N_2 - \frac{3}{2}$ along $\{0\} \times \mathbb{C}$ (see equation (32)). This is regular at $z = \infty$ since the frame is homogeneous. The Gram matrix of the pairing $(\cdot, \cdot)_{\mathrm{B}}$ along $\{0\} \times \mathbb{C}$ is independent of z by Proposition 3.5.1. The frame thus gives an extension of $\mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}}$ to $\{0\} \times \mathbb{P}^1$ satisfying the conditions in the proposition.

($y = 0$, uniqueness) Suppose that we have an extension of $\mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{0\}\times\mathbb{P}^1}$-module $\mathcal{E}$ satisfying the conditions in the proposition. Set

$$V := \Gamma(\mathbb{P}^1, \mathcal{E}) \subset \Gamma(\mathbb{C}, \mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}})$$

Recall that Gr acts on $\mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}}$. It preserves the space V since $\mathrm{Gr} = \nabla_{z\partial_z} + 2N_2 + \frac{3}{2}$ is regular at $z = \infty$. Therefore V is graded. We have the graded isomorphism $\mathcal{F}_{\mathrm{CY}}|_{(0,0)} \cong V$. Under this isomorphism $1 \in \mathcal{F}_{\mathrm{CY}}|_{(0,0)}$ corresponds to a degree-zero global section of $\mathcal{F}_{\mathrm{CY}}|_{\{0\}\times\mathbb{C}}$ which restricts to 1 at $z = 0$, but 1 is the only such global section and therefore $1 \in V$. The operators zN_1, zN_2 are regular at both $z = 0$ and $z = \infty$ and thus they act on V. Therefore $\mathbb{C}[zN_1, zN_2] \cdot 1 \subset V$. On the other hand, $-zN_i$ is given by the multiplication by D_i in the GKZ system, and thus $\mathbb{C}[zN_1, zN_2] \cdot 1$ contains a 6-dimensional subspace spanned by $1, D_2, D_2^2, D_2^3, D_1, D_2^2$. Hence $V = \mathbb{C}[zN_1, zN_2] \cdot 1$. The conclusion follows.

($y = -\frac{1}{27}$, existence) This is essentially identical to ($y = 0$, existence). We use Proposition 3.5.1 again.

($y = -\frac{1}{27}$, uniqueness) Suppose that we have an extension of $\mathcal{F}_{\mathrm{CY}}|_{\{-\frac{1}{27}\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{E}$ satisfying the conditions in the proposition. Set $V = \Gamma(\mathbb{P}^1, \mathcal{E}) \subset \Gamma(\mathbb{C}, \mathcal{F}_{\mathrm{CY}}|_{\{-\frac{1}{27}\}\times\mathbb{C}})$ as before. For the same reason as in ($y = 0$, uniqueness), V is graded and is preserved by the operators zN_2, zN_t, where $N = N_2, N_t$ are the residue endomorphisms along the divisors $y_2 = 0$ and $t = y_1 + \frac{1}{27} = 0$ respectively. Therefore, under the graded isomorphism $\mathcal{F}_{\mathrm{CY}}|_{(-\frac{1}{27},0)} \cong V$, the homogeneous basis $1, D_2, D_2^2, D_2^3, D_1, (1 + 27y_1)D_1^2$ of $\mathcal{F}_{\mathrm{CY}}|_{(-\frac{1}{27},0)}$ lifts to a basis of V of the form:

$$1,\ D_2,\ D_2^2,\ D_2^3,\ D_1 + \alpha z 1,\ zN_t(D_1 + \alpha z 1)$$

for some α, where we used the fact that $zN_t(D_1 + \alpha z 1) = (1 + 27y_1)D_1^2$. We have $-zN_2(D_1 + \alpha z 1) = \frac{1}{3}D_2^2 + \alpha z D_2$ by Proposition 3.5.1 and it has to lie in V. Therefore $\alpha = 0$. The result follows.

($y = \infty$, existence) This is essentially identical to ($y = 0$, existence). We use Proposition 3.5.1 again.

($y = \infty$, uniqueness) Once again, suppose that we have an extension of $\mathcal{F}_{\mathrm{CY}}|_{\{\infty\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{\infty\}\times\mathbb{P}^1}$-module $\mathcal{E}$ satisfying the conditions in the proposition. Set $V = \Gamma(\mathbb{P}^1, \mathcal{E}) \subset \Gamma(\mathbb{C}, \mathcal{F}_{\mathrm{CY}}|_{\{\infty\}\times\mathbb{C}})$. As before V is graded and preserved by the residue endomorphism $zN = \mathfrak{D}_2$. Therefore a homogeneous basis $1, \mathfrak{D}_2, \mathfrak{D}_2^2, \mathfrak{D}_2^3, \mathfrak{d}_1, \mathfrak{d}_1^2$ of $\mathcal{F}_{\mathrm{CY}}|_{(\infty,0)}$ lifts to a basis of V of the form:

$$1,\ \mathfrak{D}_2,\ \mathfrak{D}_2^2,\ \mathfrak{D}_2^3,\ \mathfrak{d}_1 + \alpha' z 1,\ \mathfrak{d}_1^2 + \beta' z \mathfrak{d}_1 + \gamma' z \mathfrak{D}_2 + \delta' z^2 1$$

for appropriate scalars α', β', γ', δ'. The space V is invariant under the $\mathbb{Z}/3Z$-action; $\mathbb{Z}/3Z$ acts by $\mathfrak{d}_1 \mapsto e^{2\pi \mathrm{i}/3}\mathfrak{d}_1$, $\mathfrak{D}_2 \mapsto \mathfrak{D}_2$. Thus $\alpha' = 0$, as otherwise V contains both 1 and z, contradicting the fact that $V \cong \mathcal{F}_{\mathrm{CY}}|_{(\infty,0)}$. Similarly $\beta' = \gamma' = \delta' = 0$. This completes the proof. □

5. Enlarging the Base of the B-Model log-TEP Structure

In this section we enlarge the base of the B-model log-TEP structure (see §3.5) in such a way that the enlarged log-TEP structure, which we call the *big B-model* log-*TEP structure*, is miniversal (Definition 2.7.4). The process of enlarging the base, described below, should be an example of a universal unfolding of log-TEP structure. We prove:

Theorem 5.0.3. *Let $(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ be the B-model* log-*TEP structure with base $(\mathcal{M}_{\mathrm{B}}, D)$ in §3.5. Let $\mathcal{M}_{\mathrm{B}}^{\circ} := \mathcal{M}_{\mathrm{B}} \setminus \{y_1 = -1/27\}$ be the complement of the conifold locus. We have*

- *a 6-dimensional complex manifold $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$*
- *a closed embedding $\iota\colon \mathcal{M}_{\mathrm{B}}^{\circ} \to \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$*
- *a divisor D^{big} in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ such that $\iota^{-1}D^{\mathrm{big}} = D \cap \mathcal{M}_{\mathrm{B}}^{\circ}$;*
- *a miniversal* log-*TEP structure $(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ with base $(\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}, D^{\mathrm{big}})$ such that $\iota^{\star}(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ is isomorphic to $(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})|_{\mathcal{M}_{\mathrm{B}}^{\circ}\times\mathbb{C}}$.*

We call the triple $(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ the big B-model log-TEP structure.

We construct the enlarged base for the B-model TEP structure using Reichelt's universal unfolding for log-trTLEP structures. The argument is in three steps, as follows. In the first step we construct, for each $y \in \mathcal{M}_{\mathrm{B}}$, a log-trTLEP structure on a neighbourhood U_y of y. In the second step we delete the conifold locus $y_1 = -\frac{1}{27}$ (because Reichelt's generation condition fails there) and apply Reichelt's unfolding result to construct a miniversal log-trTLEP structure on $U_y \times V_y$, where V_y is a neighbourhood of the origin in $\mathbb{C}^4$, such that the restriction to $U_y \times \{0\}$ is the log-trTLEP structure constructed in the first step. In the third step we show that the log-TEP structures that underly the log-trTLEP structures constructed in step two are compatible on chart overlaps, and thus assemble to give a global miniversal log-TEP structure over a six-dimensional base $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. (The log-trTLEP structures themselves are in general not compatible with each other on chart overlaps.) The six-dimensional base $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ contains $\mathcal{M}_{\mathrm{B}}^{\circ}$ as a subset.

5.1. **Step 1: Constructing** log**-trTLEP Structure Locally.** We begin with a general method to construct a log-trTLEP structure near a unipotent monodromy point of a log-TEP structure. As we discussed in §2.8, an opposite module for a TEP structure gives rise to a trTLEP structure and a flat trivialization (Definition 2.8.9). Suppose that a log-TEP structure with base $(\mathcal{M}, D)$ is the Deligne extension of a TEP structure with base $\mathcal{M} \setminus D$ (Definition 3.5.2). In this case, a flat trivialization of the TEP structure given by an opposite module does not necessarily extend to the log-TEP structure. We introduce below the notion of "compatibility with Deligne extension" for an opposite module. This describes a certain special situation where the flat trivialization extends to a trivialization of the log-TEP structure and yields a log-trTLEP structure. The resulting log-trTLEP structure is very special: the residue endomorphisms are nilpotent and vanish at $z = \infty$. We then show that opposite modules near p for the log-TEP structure which is compatible with the Deligne extension is uniquely determined by a trivialization of the log-TEP structure over $\{p\} \times \mathbb{C}$ satisfying certain conditions. Finally we apply this method to the B-model log-TEP structure and construct a log-trTLEP structure locally on $\mathcal{M}_{\mathrm{B}}$.

Definition 5.1.1. Let $\mathcal{M}$ be a complex manifold with normal crossing divisor D. Let $(\mathcal{F}, \nabla, (\cdot,\cdot))$ be a log-TEP structure with base $(\mathcal{M}, D)$ which is the Deligne extension of a TEP structure $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$ with base $\mathcal{M} \setminus D$ (Definition 3.5.2). Let p be a point in $\mathcal{M}$ and let U_p be a contractible open neighbourhood of p such that every (nonempty) irreducible component of $D \cap U_p$ contains p. An opposite module $\mathbf{P}$ for $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$ defined over $U_p \setminus D$ is said to be *compatible with the Deligne extension* $(\mathcal{F}, \nabla, (\cdot,\cdot))$ if the following conditions are satisfied:

(1) the flat trivialization of $\mathcal{F}^\times|_{(U_p\setminus D)\times\mathbb{C}}$ associated to $\mathbf{P}$ (Definition 2.8.9) has no monodromy around D and thus defines a locally free extension $\mathcal{E}$ of $\mathcal{F}$ to $U_p \times \mathbb{P}^1$ such that the corresponding vector bundle over $U_p \times \mathbb{P}^1$ is trivial;

(2) the connection ∇ defines a meromorphic flat connection on $\mathcal{E}$ with:
$$\nabla \colon \mathcal{E} \to \Omega^1_{U_p\times\mathbb{P}^1}(\log Z) \otimes \mathcal{E}(U_p \times \{0\})$$
where $Z = (D \times \mathbb{P}^1) \cup (U_p \times \{0\}) \cup (U_p \times \{\infty\})$;

(3) the pairing $(\cdot,\cdot)$ extends holomorphically across $(U_p \times \{\infty\}) \cup ((D \cap U_p) \times \mathbb{P}^1)$ and is non-degenerate there;

(4) the residue endomorphisms of ∇ along $(D \cap U_p) \times (\mathbb{P}^1 \setminus \{0\})$ are nilpotent and vanish at $(D \cap U_p) \times \{\infty\}$.

Condition (4) implies that $(\mathcal{E}, \nabla, (\cdot,\cdot))$ coincides with the Deligne extension $(\mathcal{F}, \nabla, (\cdot,\cdot))$ over $U_p \times \mathbb{C}$, because the Deligne extension is the unique logarithmic extension such that the residue endomorphisms are nilpotent.

Remark 5.1.2. When the base $\mathcal{M}$ has an orbifold singularity at p and the Deligne extension $\mathcal{F}$ is an orbi-sheaf (e.g. the B-model log-TEP structure, see Remark 3.5.5), we define the compatibility with the Deligne extension near p by replacing U_p with the uniformizing chart and requiring the same conditions (1)–(4) in Definition 5.1.1 over the uniformizing chart. The locally free sheaf $\mathcal{E}$ on $U_p \times \mathbb{P}^1$ in (1) becomes $\mathrm{Aut}(p)$-equivariant, where $\mathrm{Aut}(p)$ is the finite automorphism group at p which acts on U_p. The connection ∇ and the pairing $(\cdot,\cdot)$ on $\mathcal{E}$ are invariant under the $\mathrm{Aut}(p)$-action.

Remark 5.1.3. Compatibility with the Deligne extension has been discussed in the context of the Crepant Resolution Conjecture and mirror symmetry: see [27, Theorem 3.5] and [55, §3.5] where a characterization of the A-model opposite module is given at certain cusps in the B-model moduli space.

It is convenient to rephrase the above conditions (1)–(4) in Definition 5.1.1 in terms of an explicit trivialization. Choose local co-ordinates $(x_1, \dots, x_r, y_1, \dots, y_s)$ of $\mathcal{M}$ centred at $p \in \mathcal{M}$ such that the divisor $D \cap U_p$ can be written as $x_1 x_2 \cdots x_r = 0$. (We set $r = 0$ if $p \notin D$.) Then an opposite module $\mathbf{P}$ compatible with the Deligne extension yields a trivialization of $\mathcal{F}|_{U_p\times\mathbb{C}}$ with the following properties:

- the connection in the trivialization takes the form:
$$d + \frac{1}{z}\left(\sum_{i=1}^{r} A_i(x,y)\frac{dx_i}{x_i} + \sum_{i=1}^{s} B_i(x,y)dy_i + (C_0(x,y) + zC_1(x,y))\frac{dz}{z}\right) \tag{48}$$
where A_i, B_i, C_0, C_1 are matrix-valued holomorphic functions on U_p such that the residue endomorphisms $A_i|_{x_i=0}$ are nilpotent;
- the Gram matrix of the pairing $(\cdot,\cdot)$ is constant with respect to the trivialization.

This trivialization extends the flat trivialization of $\mathcal{F}^\times|_{(U_p\setminus D)\times\mathbb{C}}$ associated to $\mathbf{P}$, and we refer to it as a *flat trivialization* of $\mathcal{F}$ associated to $\mathbf{P}$. Conditions (1)–(3) in Definition 5.1.1 imply

that an opposite module $\mathbf{P}$ compatible with the Deligne extension yields a log-trTLEP structure with base U_p in the sense of Reichelt [71, Definition 1.8]. Note however that Reichelt's notion of log-trTLEP structure is more general than our notion of 'compatibility with the Deligne extension': the connection ∇ of a log-trTLEP structure has a form similar to (48) but the term A_i there can depend linearly on z, i.e. $A_i = A_{i0}(x,y) + A_{i1}(x,y)z$.

Remark 5.1.4. In view of the proof of Proposition A.0.4, slightly more is true about the connection (48): $A_i|_{x_i=0}$ is independent of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s$ and $C_1(x,y)$ is independent of x and y. These follow automatically from the flatness of the connection.

The existence of an opposite module over $U_p \setminus D$ which is compatible with the Deligne extension is reduced to the existence of a trivialization of $\mathcal{F}$ over $\{p\} \times \mathbb{C}$ (or equivalently, an extension of $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{p\}\times\mathbb{P}^1}$-module) satisfying certain properties.

Proposition 5.1.5. *Let D be a normal crossing divisor in $\mathcal{M}$ and let $(\mathcal{F}, \nabla, (\cdot,\cdot))$ be a* log*-TEP structure with base $(\mathcal{M}, D)$ which is the Deligne extension of a TEP structure $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$ with base $\mathcal{M}\setminus D$. Let p be a point in $\mathcal{M}$ and take local co-ordinates $(x_1, \dots, x_r, y_1, \dots, y_s)$ centred at p such that D can be written as $x_1 x_2 \cdots x_r = 0$ near p. (We take $r = 0$ when $p \notin D$.) There is a one-to-one correspondence between the following:*

(a) *an extension of $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{p\}\times\mathbb{P}^1}$-module such that*
   - *the residue endomorphisms $\nabla_{x_i\partial_{x_i}}|_p \colon \mathcal{F}|_{\{p\}\times\mathbb{C}} \to z^{-1}\mathcal{F}|_{\{p\}\times\mathbb{C}}$, $i = 1, \dots, r$ extend regularly across $z = \infty$ and vanish there;*
   - *the connection ∇ on $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ has a logarithmic pole at $z = \infty$, i.e. $\nabla_{z\partial_z}$ is regular at $z = \infty$;*
   - *the pairing $(\cdot,\cdot)$ on $\{p\}\times\mathbb{C}$ extends regularly across $z = \infty$ and is non-degenerate there;*
   - *when p is an orbifold point, the $\mathrm{Aut}(p)$-action on $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ extends across $z = \infty$.*

(b) *an opposite module $\mathbf{P}$ for $(\mathcal{F}^\times, \nabla, (\cdot,\cdot))$, defined near p, which is compatible with the Deligne extension $(\mathcal{F}, \nabla, (\cdot,\cdot))$.*

Proof. Let U_p be a contractible open neighbourhood of p in $\mathcal{M}$ such that every irreducible component of $D \cap U_p$ contains p. (If p is an orbifold point, we take U_p to be a uniformizing chart.) In view of the discussion after Definition 5.1.1, an opposite module $\mathbf{P}$, defined over $U_p \setminus D$, which is compatible with the Deligne extension yields a flat trivialization of $\mathcal{F}$ over $U_p \times \mathbb{C}$ such that

(i) the connection ∇ in the trivialization takes the form:

$$d + \frac{1}{z}\left(\sum_{i=1}^{r} A_i(x,y)\frac{dx_i}{x_i} + \sum_{i=1}^{s} B_i(x,y)dy_i + (C_0(x,y) + zC_1(x,y))\frac{dz}{z}\right) \tag{49}$$

where A_i, B_i, C_0, C_1 are matrix-valued holomorphic functions on U_p and $A_i|_{x_i=0}$ is nilpotent;

(ii) the pairing $(\cdot,\cdot)$ is constant with respect to the trivialization.

Restricting the trivialization to $\{p\}\times\mathbb{C}$, we obtain an extension of $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{p\}\times\mathbb{P}^1}$-module satisfying the conditions in (a). When p is an orbifold point, recall from Remark 5.1.2 that $\mathcal{F}|_{U_p\times\mathbb{C}}$ extends, via the trivialization, to an $\mathrm{Aut}(p)$-equivariant free $\mathcal{O}_{U_p\times\mathbb{P}^1}$-module $\mathcal{E}$.

Conversely, suppose that we have an extension of $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{p\}\times\mathbb{P}^1}$-module satisfying the conditions in (a). We take a trivialization of $\mathcal{F}|_{\{p\}\times\mathbb{C}}$ which yields the free extension. We shall show that there exists a unique trivialization of $\mathcal{F}$ over $U_p \times \mathbb{C}$ extending the trivialization over $\{p\}\times\mathbb{C}$ and satisfying the properties (i)–(ii) listed above.

To see the existence of a trivialization over $U_p \times \mathbb{C}$, we first extend the given trivialization of $\mathcal{F}$ along $\{p\} \times \mathbb{C}$ to $U_p \times \mathbb{C}$ in an arbitrary way (shrinking U_p if necessary). The connection ∇ in the trivialization takes the form

$$d + \frac{1}{z}\left(\sum_{i=1}^{r} A_i(x,y,z)\frac{dx_i}{x_i} + \sum_{i=1}^{s} B_i(x,y,z)dy_i + C(x,y,z)\frac{dz}{z}\right).$$

where A_i, B_i, C are matrix-valued holomorphic functions on $U_p \times \mathbb{C}$, $A_i(0,0,z)$, $1 \le i \le r$ are nilpotent and independent of z and $C(0,0,z)$ depends linearly on z, i.e. $C(0,0,z) = C_0(0,0) + zC_1(0,0)$. By Propositions A.0.2 and A.0.4, after shrinking U_p if necessary, there exists a gauge transformation L_+ defined on $U_p \times \mathbb{C}$ such that $L_+|_{\{p\}\times\mathbb{C}} = \mathrm{id}$ and that this connection is transformed to a connection of the form (49) by L_+. By Proposition A.0.5, the Gram matrix of the pairing $(\cdot,\cdot)_{\mathcal{F}}$ is constant over $U_p \times \mathbb{C}$ after the gauge transformation.

Next we show the uniqueness of such a trivialization. Suppose we have a gauge transformation G such that $G|_{\{p\}\times\mathbb{C}} = \mathrm{id}$ and that G transforms the connection (49) to a connection of the same form:

$$d + \frac{1}{z}\left(\sum_{i=1}^{r} A_i'(x,y)\frac{dx_i}{x_i} + \sum_{i=1}^{s} B_i'(x,y)dy_i + (C_0'(x,y) + zC_1'(x,y))\frac{dz}{z}\right) \tag{50}$$

where $A_i'(0,y)$, $1 \le i \le r$ are nilpotent. By Proposition A.0.2, the connections (49) and (50) admit respectively unique "fundamental solutions in the U-direction" of the form:

$$\tilde{L}(x,y,z)e^{\sum_{i=1}^{r} A_i(0,0)\log x_i/z}, \quad \tilde{L}'(x,y,z)e^{\sum_{i=1}^{4} A_i'(0,0)\log x_i/z}$$

satisfying the initial conditions $\tilde{L}(0,0,z) = \tilde{L}'(0,0,z) = \mathrm{id}$. Then we have

$$\tilde{L}'(x,y,z)e^{-\sum_{i=1}^{r} A_i'(0,0)\log x_i/z} = G(x,y,z)\tilde{L}(x,y,z)e^{-\sum_{i=1}^{r} A_i(0,0)\log x_i/z}$$

Since the trivialization along $\{p\} \times \mathbb{C}$ is fixed, the residue endomorphisms are the same $A_i(0,0) = A_i'(0,0)$. Since the connections (49), (50) in the U-direction are trivial along $z = \infty$, $\tilde{L}$ and $\tilde{L}'$ are regular on $U_p \times (\mathbb{P}^1 \setminus \{0\})$ and $\tilde{L}|_{z=\infty} = \tilde{L}'|_{z=\infty} = \mathrm{id}$. Therefore G has to be the identity on $U_p \times \mathbb{C}$.

When p is an orbifold point, we additionally need to check that the opposite module corresponding to the trivialization of $\mathcal{F}|_{U_p\times\mathbb{C}}$ is well-defined on the quotient $(U_p \setminus D)/\operatorname{Aut}(p)$. (The trivialization itself may not descend to the quotient.) It suffices to show that each $g \in \operatorname{Aut}(p)$ acts on the trivializing frame by a constant matrix (independent of z). This follows from the uniqueness statement: let $s_0, \dots, s_N$ be the trivializing frame of $\mathcal{F}|_{U_p\times\mathbb{C}}$ and define a matrix-valued function M on $U_p \times \mathbb{C}$ by $[g \cdot s_0, \dots, g \cdot s_N] = [s_0, \dots, s_N]M$. By the last condition in (a), $M_p := M|_{\{p\}\times\mathbb{C}}$ is a constant matrix independent of z. The frame $[s_0, \dots, s_N]M_p$ yields a trivialization of $\mathcal{F}|_{U_p\times\mathbb{C}}$ satisfying the properties (i)–(ii) above since M_p is constant. On the other hand, since ∇ and $(\cdot,\cdot)$ are $\operatorname{Aut}(p)$-invariant, the connection matrices and Gram matrix of the pairing $(\cdot,\cdot)$ do not change under the gauge transformation by M, and hence the trivialization given by the frame $[s_0, \dots, s_N]M$ also satisfies the properties (i)–(ii) above. The uniqueness argument shows that the two trivializing frames are the same, i.e. $M = M_p$ is a constant matrix. □

We now apply the above general method to the B-model log-TEP structure $(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$. Recall from §3.5 that $(\mathcal{F}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ is the Deligne extension of the B-model TEP structure $(\mathcal{F}_{\mathrm{B}}^{\times}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ with logarithmic singularities along

$$D = \overline{\{y_1y_2 = 0\}} \cup \{y_1 = -1/27\}$$

For each point y in $\mathcal{M}_{\mathrm{B}}$, we shall construct an opposite module $\mathbf{P}$ for the B-model TEP structure over $U_y \setminus D$ for a sufficiently small neighbourhood U_y of y, which is compatible with the Deligne extension. This yields a log-trTLEP structure with base $(U_y, U_y \cap D)$ which underlies the B-model log-TEP structure.

5.1.1. *Step 1, Case 1: $y \in \mathcal{M}_{\mathrm{CY}} \setminus (D_{\mathrm{CY}} \cup \{\infty\})$.* Let y be of the form $y = (y_1, y_2)$ such that $y_2 = 0$ and $y_1 \neq 0, -\frac{1}{27}, \infty$. In other words, we take $y \in \mathcal{M}_{\mathrm{CY}} \setminus \big(D_{\mathrm{CY}} \cup \{\infty\}\big)$. In this case there are many possible choices for $\mathbf{P}$:

Proposition 5.1.6. *Let $y \in \mathcal{M}_{\mathrm{CY}} \setminus \big(D_{\mathrm{CY}} \cup \{\infty\}\big)$. The following are equivalent:*

(A) *a subspace P of $H_{\mathrm{vec}}|_y$ such that $F^2_{\mathrm{vec}} \oplus P = H_{\mathrm{vec}}|_y$;*
(B) *an opposite module $\mathbf{P}$ defined on $U_y \setminus D$, where U_y is a neighbourhood of y in $\mathcal{M}_{\mathrm{B}}$, such that $\mathbf{P}$ is compatible with the Deligne extension.*

Proof. By Proposition 4.3.1, a subspace P of $H_{\mathrm{vec}}|_y$ such that $F^2_{\mathrm{vec}} \oplus P = H_{\mathrm{vec}}|_y$ is equivalent to an extension of $\mathcal{F}_{\mathrm{B}}|_{\{y\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{y\}\times\mathbb{P}^1}$-module satisfying the condition (a) of Proposition 5.1.5. The conclusion follows from Proposition 5.1.5. □

5.1.2. *Step 1, Case 2: the Large-Radius, Conifold, and Orbifold Points.* We next consider the large-radius, conifold, and orbifold points. In each case there is a unique choice for $\mathbf{P}$. For the large-radius and the orbifold points, the uniqueness has been shown in [27, Theorem 3.5] for the case at hand, and in [55, Theorem 3.13] for a more general target.

Proposition 5.1.7. *We have the following:*

(1) *Suppose that y is the large-radius limit point $y = (y_1, y_2) = (0, 0)$. There is a unique opposite module $\mathbf{P}_{\mathrm{LR}}$, defined near y, which is compatible with the Deligne extension. The corresponding flat trivialization of $\mathcal{F}_{\mathrm{B}}$ along $\{y\} \times \mathbb{C}$ is given by the frame:*
$$1,\ D_2,\ D_2^2,\ D_2^3,\ D_1,\ D_1^2$$
(2) *Suppose that y is the conifold point $y = (t, y_2) = (0, 0)$. There is a unique opposite module $\mathbf{P}_{\mathrm{con}}$, defined near y, which is compatible with the Deligne extension. The corresponding flat trivialization of $\mathcal{F}_{\mathrm{B}}$ along $\{y\} \times \mathbb{C}$ is given by the frame:*
$$1,\ D_2,\ D_2^2,\ D_2^3,\ D_1,\ (1 + 27y_1)D_1^2$$
(3) *Suppose that y is the orbifold point $y = (\mathfrak{y}_1, \mathfrak{y}_2) = (0, 0)$. There is a unique opposite module $\mathbf{P}_{\mathrm{orb}}$, defined near y, which is compatible with the Deligne extension. The corresponding flat trivialization of $\mathcal{F}_{\mathrm{B}}$ along $\{y\} \times \mathbb{C}$ is given by the frame:*
$$1,\ \mathfrak{D}_2,\ \mathfrak{D}_2^2,\ \mathfrak{D}_2^3,\ \mathfrak{d}_1,\ \mathfrak{d}_1^2$$

Proof. In all three cases, in view of Proposition 5.1.5, it suffices to check that there exists a unique extension of $\mathcal{F}_{\mathrm{B}}|_{\{y\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{y\}\times\mathbb{P}^1}$-module satisfying the condition (a) of Proposition 5.1.5 and that it is defined by the frame given in the proposition. This has been proved in Proposition 4.3.2. □

5.1.3. *Step 1, Case 3: $y \notin \mathcal{M}_{\mathrm{CY}}$.* We now turn to the remaining case, where $y \notin \mathcal{M}_{\mathrm{CY}}$. This means either that $y = (y_1, y_2)$ with $y_2 \neq 0$, or that $y = (\mathfrak{y}_1, \mathfrak{y}_2)$ with $\mathfrak{y}_1 = 0$ and $\mathfrak{y}_2 \neq 0$. We will use the fact that any connection ∇ as in Definition 5.1.1 defined on $U \times \mathbb{C}$ extends canonically to a connection on $V \times \mathbb{C}$, where V is the orbit of U under the flow of the Euler field: see e.g. Kim–Sabbah [59, Example 1.3]. In the case at hand, the Euler field is $2y_2\partial_{y_2} = 2\mathfrak{y}_1\partial_{\mathfrak{y}_2}$. The opposite submodules constructed in Step 1, Cases 1 and 2, are defined on neighbourhoods $\{U_y\}$ that together cover the locus $\mathcal{M}_{\mathrm{CY}} \subset \mathcal{M}_{\mathrm{B}}$ where $y_2 = 0$ or $\mathfrak{y}_2 = 0$, and so the orbits

of these neighbourhoods under the Euler flow cover all of $\mathcal{M}_{\mathrm{B}}$. Thus we construct, for any $y \in \mathcal{M}_{\mathrm{B}}$ with $y \notin \mathcal{M}_{\mathrm{CY}}$, a neighbourhood U_y of y in $\mathcal{M}_{\mathrm{B}}$ and an opposite module $\mathbf{P}$ over $U_y \setminus D$ which is compatible with the Deligne extension.

More precisely, we have the following statement:

Proposition 5.1.8. *Let $p\colon \mathcal{M}_{\mathrm{B}} \to \mathcal{M}_{\mathrm{CY}}$ be the map that sends $(y_1, y_2) \in \mathcal{M}_{\mathrm{B}}$ to the point $(y_1, 0) \in \mathcal{M}_{\mathrm{CY}} \subset \mathcal{M}_{\mathrm{B}}$, and which sends $(\mathfrak{y}_1, \mathfrak{y}_2) \in \mathcal{M}_{\mathrm{B}}$ such that $\mathfrak{y}_1 = 0$ to the orbifold point $(\mathfrak{y}_1, \mathfrak{y}_2) = (0, 0) \in \mathcal{M}_{\mathrm{CY}} \subset \mathcal{M}_{\mathrm{B}}$. Let $y \in \mathcal{M}_{\mathrm{CY}}$. For a sufficiently small open neighbourhood U_y of y, an opposite module defined over $U_y \setminus D$ which is compatible with the Deligne extension extends to an opposite module over $p^{-1}(p(U_y)) \setminus D$.*

Proof. Suppose for simplicity that $y \in \mathcal{M}_{\mathrm{CY}}$ is neither the large radius limit point, nor the conifold point, nor the orbifold point. (The argument in these three cases is essentially identical.) Then $p(x)$ is the limit as $t \to -\infty$ of the image of x under the time-t flow of the Euler field. With respect to the flat trivialization defined by $\mathbf{P}$, we have:

$$\begin{aligned}
\nabla_{z\partial_z} &= z\partial_z - 2z^{-1}B(y_1, y_2) + C(y_1, y_2) \\
\nabla_{\partial_{y_1}} &= \partial_{y_1} + z^{-1}A(y_1, y_2) \\
\nabla_{y_2\partial_{y_2}} &= y_2\partial_{y_2} + z^{-1}B(y_1, y_2)
\end{aligned}$$

for (y_1, y_2) in U_y, for some regular endomorphism-valued functions A, B, C on U_y. Flatness of ∇ gives that C is independent of y_1 and y_2 (see Remark 5.1.4) and yields the following differential equations:

$$\begin{aligned}
2y_2\partial_{y_2}B &= B - [C, B] \\
2y_2\partial_{y_2}A = 2\partial_{y_1}B &= A - [C, A]
\end{aligned}$$

These differential equations imply:

$$\begin{aligned}
B(y_1, y_2t^2) &= t \cdot t^{-C}B(y_1, y_2)t^C \\
A(y_1, y_2t^2) &= t \cdot t^{-C}A(y_1, y_2)t^C
\end{aligned} \qquad\qquad t \in \mathbb{C}^\times$$

The right-hand side defines an analytic continuation of $B(y_1, y_2)$, $A(y_1, y_2)$ – which are originally defined only near $y_2 = 0$ – to all of $V_y = p^{-1}(p(U_y))$. By the discussion after Definition 5.1.1 this yields an opposite module over $V_y \setminus D$ which is compatible with the Deligne extension. □

Remark 5.1.9. This completes Step 1: we have constructed, for each $y \in \mathcal{M}_{\mathrm{B}}$, a neighbourhood U_y of y in $\mathcal{M}_{\mathrm{B}}$ and an opposite module $\mathbf{P}$ over $U_y \cap \mathcal{M}_{\mathrm{B}}^\times$ which is compatible with the Deligne extension. In particular, $\mathbf{P}$ determines a log-trTLEP structure with base U_y.

5.2. **Step 2: Unfolding the** log**-trTLEP Structures Locally.** We now delete the conifold locus, $y_1 = -\frac{1}{27}$, from $\mathcal{M}_{\mathrm{B}}$, setting:

$$\mathcal{M}_{\mathrm{B}}^\circ := \left\{(y_1, y_2) \in \mathcal{M}_{\mathrm{B}} : y_1 \neq -\tfrac{1}{27}\right\}$$

Consider $y \in \mathcal{M}_{\mathrm{B}}^\circ$, a neighbourhood U_y of y in $\mathcal{M}_{\mathrm{B}}^\circ$, and an opposite module $\mathbf{P}$ over $U_y \setminus D$ such that $\mathbf{P}$ is compatible with the Deligne extension, as constructed in Step 1. The choice of U_y and $\mathbf{P}$ defines a log-trTLEP structure with base U_y such that the underlying TEP structure coincides with the B-model log-TEP structure. The section ξ of $\mathcal{F}_{\mathrm{B}}$ corresponding to the element $1 \in \mathcal{F}_{\mathrm{GKZ}}^\times$ satisfies the conditions (IC), (GC), (EC), and flatness in [71, Theorem 1.12]. We therefore consider Reichelt's universal unfolding of our log-trTLEP structure. This is a log-trTLEP structure with base $(U_y \times V_y, (D \cap U_y) \times V_y)$, where V_y is a neighbourhood of the

origin in $\mathbb{C}^4$, such that the restriction to $U_y \times \{0\}$ coincides with the log-trTLEP structure with base $(U_y, D \cap U_y)$ defined by $\mathbf{P}$. The underlying log-TEP structure is miniversal in the sense of Definition 2.7.4.

Remark 5.2.1. We delete the conifold locus $y_1 = -\frac{1}{27}$ because Reichelt's generation condition (GC) fails there.

5.3. **Step 3: A Global Miniversal TEP Structure.** Now that we have completed Steps 1 and 2, we are in the following situation. Given a sufficiently small open subset U of $\mathcal{M}_{\mathrm{B}}^{\circ}$, there exists an opposite module $\mathbf{P}$ over $U \setminus D$ that is compatible with the Deligne extension. Thus there exists a log-trTLEP structure with base $(U \times V, (U \cap D) \times V)$, where V is an open neighbourhood of the origin in $\mathbb{C}^4$; this log-trTLEP structure is constructed as a universal unfolding of the log-trTLEP structure with base $(U, U \cap D)$ defined by $\mathbf{P}$. The log-trTLEP structure with base $(U \times V, (U \cap D) \times V)$ determines a log-TEP structure with the same base, and we now show that all these log-TEP structures glue together, after shrinking the base $U \times V$ if necessary, to give a global, miniversal log-TEP structure, defined on a six-dimensional complex manifold $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ that contains $\mathcal{M}_{\mathrm{B}}^{\circ}$ as a closed submanifold. This global log-TEP structure is the *big B-model* log-*TEP structure*.

5.3.1. *The Gluing Map.* To simplify the notation, when there is no risk of confusion, we denote a log-TEP (or log-trTLEP) structure simply by the corresponding locally free sheaf $\mathcal{F}$, omitting the flat connection ∇ and the pairing $(\cdot,\cdot)_{\mathcal{F}}$.

Lemma 5.3.1. *Let U be an open set of $\mathcal{M}_{\mathrm{B}}^{\circ}$. Suppose that we have opposite modules $\mathbf{P}$, $\mathbf{P}'$ for the B-model TEP structure $\mathcal{F}_{\mathrm{B}}^{\times}$ over $U \setminus D$ that are compatible with the Deligne extension. These opposite modules define the* log-*trTLEP structures underlying the B-model* log-*TEP structure $\mathcal{F}_{\mathrm{B}}$. Suppose that U is sufficiently small so that the* log-*trTLEP structures admit the following universal unfolding as in Step 2:*

$$\textit{miniversal } \log\textit{-trTLEP structure } \mathcal{E}_{\mathbf{P}} \textit{ with base } (U \times V, (U \cap D) \times V)$$
$$\textit{miniversal } \log\textit{-trTLEP structure } \mathcal{E}_{\mathbf{P}'} \textit{ with base } (U \times V', (U \cap D) \times V')$$

where V, V' are open neighbourhoods of the origin in $\mathbb{C}^4$. We write $\mathcal{F}_{\mathbf{P}} = \mathcal{E}_{\mathbf{P}}|_{(U\times V)\times\mathbb{C}}$ and $\mathcal{F}_{\mathbf{P}'} = \mathcal{E}_{\mathbf{P}'}|_{(U\times V')\times\mathbb{C}}$ for the underlying log-*TEP structures. Let $\theta_{\mathbf{P}\mathbf{P}'}$ denote the canonical isomorphism of* log-*TEP structures*

$$\theta_{\mathbf{P}\mathbf{P}'} \colon \mathcal{F}_{\mathbf{P}}|_{U\times\{0\}\times\mathbb{C}} \cong \mathcal{F}_{\mathrm{B}}|_{U\times\mathbb{C}} \cong \mathcal{F}_{\mathbf{P}'}|_{(U\times\{0\})\times\mathbb{C}}$$

given by the construction. There exist open sets $O_{\mathbf{P}\mathbf{P}'} \subset U \times V$, $O_{\mathbf{P}'\mathbf{P}} \subset U \times V'$ and a biholomorphic map $\varphi_{\mathbf{P}\mathbf{P}'} \colon O_{\mathbf{P}\mathbf{P}'} \to O_{\mathbf{P}'\mathbf{P}}$ such that:

- *$U \times \{0\} \subset O_{\mathbf{P}\mathbf{P}'}$ and $U \times \{0\} \subset O_{\mathbf{P}'\mathbf{P}}$;*
- *$\varphi_{\mathbf{P}\mathbf{P}'}|_{U\times\{0\}}$ is the identity map;*
- *$\varphi_{\mathbf{P}\mathbf{P}'}$ maps the divisor $((U \cap D) \times V) \cap O_{\mathbf{P}\mathbf{P}'}$ onto $((U \cap D) \times V') \cap O_{\mathbf{P}'\mathbf{P}}$;*
- *there is an isomorphism $\Theta_{\mathbf{P}\mathbf{P}'} \colon \mathcal{F}_{\mathbf{P}}|_{O_{\mathbf{P}\mathbf{P}'}\times\mathbb{C}} \to (\varphi_{\mathbf{P}\mathbf{P}'} \times \mathrm{id})^{\star}(\mathcal{F}_{\mathbf{P}'}|_{O_{\mathbf{P}'\mathbf{P}}\times\mathbb{C}})$ of* log-*TEP structures which restricts to $\theta_{\mathbf{P}\mathbf{P}'}$ over $(U \times \{0\}) \times \mathbb{C}$.*

Moreover, the map $\varphi_{\mathbf{P}\mathbf{P}'}$ and the isomorphism $\Theta_{\mathbf{P}\mathbf{P}'}$ are unique as germs.

Proof. By construction, the log-TEP structures $\mathcal{F}_{\mathbf{P}}$ and $\mathcal{F}_{\mathbf{P}'}$ are equipped with natural opposite modules $\mathbf{P}$ and $\mathbf{P}'$ that are compatible with Deligne extensions and give rise to the log-trTLEP structures $\mathcal{E}_{\mathbf{P}}$ and $\mathcal{E}_{\mathbf{P}'}$. Recall from Proposition 5.1.5 that a Deligne-extension-compatible opposite module for $\mathcal{F}_{\mathbf{P}'}$ near $p \in U \times \{0\}$ corresponds bijectively to an extension of $\mathcal{F}_{\mathbf{P}'}|_{\{p\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{p\}\times\mathbb{P}^1}$-module satisfying certain conditions. By the isomorphism

$\theta_{\mathbf{P}\mathbf{P}'}\colon \mathcal{F}_{\mathbf{P}}|_{U\times\{0\}\times\mathbb{C}} \cong \mathcal{F}_{\mathbf{P}'}|_{U\times\{0\}\times\mathbb{C}}$, one can shift the opposite module $\mathbf{P}$ for $\mathcal{F}_{\mathbf{P}}|_{U\times\{0\}\times\mathbb{P}^1}$ to an opposite module $\mathbf{P}''$ for $\mathcal{F}_{\mathbf{P}'}|_{U\times\{0\}\times\mathbb{C}}$. For every point $p \in U\times\{0\}$, $\mathbf{P}''$ gives rise to an extension of $\mathcal{F}_{\mathbf{P}'}|_{\{p\}\times\mathbb{C}}$ to a free $\mathcal{O}_{\{p\}\times\mathbb{P}^1}$-module satisfying the conditions of Proposition 5.1.5, (a). Therefore, the opposite module $\mathbf{P}''$ for $\mathcal{F}_{\mathbf{P}'}|_{U\times\{0\}\times\mathbb{C}}$ extends to a Deligne-extension-compatible opposite module (which we denote by $\mathbf{P}''$ again) for $\mathcal{F}_{\mathbf{P}'}$ over an open neighbourhood O of $U\times\{0\}$ in $U\times V'$. This gives rise to a log-trTLEP structure $\mathcal{E}_{\mathbf{P}''}$ over $O\times\mathbb{P}^1$ underlain by the log-TEP structure $\mathcal{F}_{\mathbf{P}'}$ and one has an isomorphism of log-trTLEP structures:

$$\theta_{\mathbf{P}\mathbf{P}''}\colon \mathcal{E}_{\mathbf{P}}|_{(U\times\{0\})\times\mathbb{P}^1} \cong \mathcal{E}_{\mathbf{P}''}|_{(U\times\{0\})\times\mathbb{P}^1}$$

(The isomorphism $\theta_{\mathbf{P}\mathbf{P}''}$ is induced from $\theta_{\mathbf{P}\mathbf{P}'}$.) The universal property of Reichelt's unfolding implies that there exist a biholomorphic map $\varphi_{\mathbf{P}\mathbf{P}'}\colon O_{\mathbf{P}\mathbf{P}'} \to O_{\mathbf{P}'\mathbf{P}}$ between an open neighbourhood $O_{\mathbf{P}\mathbf{P}'}$ of $U\times\{0\}$ in $U\times V$ and an open neighbourhood $O_{\mathbf{P}'\mathbf{P}}$ of $U\times\{0\}$ in $O\subset U\times V'$ such that $\varphi_{\mathbf{P}\mathbf{P}'}$ satisfies the properties listed in the statement and that $\theta_{\mathbf{P}\mathbf{P}''}$ extends to an isomorphism of log-trTLEP structures:

$$\Theta_{\mathbf{P}\mathbf{P}''}\colon \mathcal{E}_{\mathbf{P}}|_{O_{\mathbf{P}\mathbf{P}'}\times\mathbb{P}^1} \cong (\varphi_{\mathbf{P}\mathbf{P}'}\times\mathrm{id})^\star(\mathcal{E}_{\mathbf{P}''}|_{O_{\mathbf{P}'\mathbf{P}}\times\mathbb{P}^1})$$

The map $\varphi_{\mathbf{P}\mathbf{P}'}$ and isomorphism $\Theta_{\mathbf{P}\mathbf{P}''}$ are unique as germs. Restricting $\Theta_{\mathbf{P}\mathbf{P}''}$ to $O_{\mathbf{P}\mathbf{P}'}\times\mathbb{C}$, we obtain the desired isomorphism $\Theta_{\mathbf{P}\mathbf{P}'}$ between $\mathcal{F}_{\mathbf{P}}$ and $(\varphi_{\mathbf{P}\mathbf{P}'}\times\mathrm{id})^\star(\mathcal{F}_{\mathbf{P}'})$.

We show the uniqueness of $\varphi_{\mathbf{P}\mathbf{P}'}$ and $\Theta_{\mathbf{P}\mathbf{P}'}$. Suppose we have $\varphi_{\mathbf{P}\mathbf{P}'}$ and $\Theta_{\mathbf{P}\mathbf{P}'}$ satisfying the conditions in the statement. Then the isomorphism $\Theta_{\mathbf{P}\mathbf{P}'}\colon \mathcal{F}_{\mathbf{P}} \cong (\varphi_{\mathbf{P}\mathbf{P}'}\times\mathrm{id})^\star\mathcal{F}_{\mathbf{P}'}$ of log-TEP structures induces a log-trTLEP structure $\mathcal{E}_{\mathbf{P}''}$ underlain by the log-TEP structure $\mathcal{F}_{\mathbf{P}'}$ which is isomorphic to $\mathcal{E}_{\mathbf{P}}$ as a log-trTLEP structure. By the uniqueness of Reichelt's universal unfolding, $\varphi_{\mathbf{P}\mathbf{P}'}$ and $\Theta_{\mathbf{P}\mathbf{P}'}$ should be the same (as germs) as what we constructed above. □

5.3.2. *The Big B-model* log*-TEP Structure.* The above Lemma 5.3.1 says that the underlying log-TEP structures of the miniversal log-trTLEP structures constructed locally in Step 2 do not depend on the choice of opposite modules. Therefore, they are glued together to give a global miniversal log-TEP structure over a 6-dimensional base $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. At first sight, the gluing construction looks obvious: however it is not so straightforward to show that the glued space is Hausdorff. We leave this technical (but elementary) problem to a separate paper [24] and adapt the result there to our setting.

We take an open covering $\{U_i\}_{i\in I}$ of $\mathcal{M}_{\mathrm{B}}^\circ$ such that for each $i\in I$ there exists an opposite module $\mathbf{P}_i$ for $\mathcal{F}_{\mathrm{B}}^\times$ over $U_i\setminus D$ which is compatible with the Deligne extension and that the log-trTLEP structure associated to $\mathbf{P}_i$ admits Reichelt's universal unfolding $\mathcal{E}_i$ with base $(U_i\times V_i, (U_i\cap D)\times V_i)$ for an open neighbourhood V_i of the origin in $\mathbb{C}^4$. We write $\mathcal{F}_i = \mathcal{E}_i|_{(U_i\times V_i)\times\mathbb{C}}$ for the log-TEP structure underlying $\mathcal{E}_i$. We glue the local charts $U_i\times V_i$ first and then glue the local log-TEP structures $\mathcal{F}_i$.

First we construct an ambient space $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ containing $\mathcal{M}_{\mathrm{B}}^\circ$. Write $\iota\colon U_i \cong U_i\times\{0\} \to U_i\times V_i$ for the inclusion map and define the sheaf of algebras over U_i by $\mathcal{A}_i := \iota^{-1}\mathcal{O}_{U_i\times V_i}$. For $i,j\in I$, the sheaves $\mathcal{A}_i$ and $\mathcal{A}_j$ are canonically isomorphic along $U_i\cap U_j$ by the map $\varphi_{\mathbf{P}_i\mathbf{P}_j}$ in Lemma 5.3.1. The gluing maps $\varphi_{\mathbf{P}_i\mathbf{P}_j}$ satisfy the cocycle condition by their uniqueness. Therefore $\mathcal{A}_i$ for all $i\in I$ are glued together to give a global sheaf $\mathcal{A}$ of algebras over $\mathcal{M}_{\mathrm{B}}^\circ$. The sheaf $\mathcal{A}$ is naturally equipped with a surjection $\mathcal{A}\to\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\circ}$. By [24, Theorem 1], there exists a global 6-dimensional complex manifold $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ together with a closed embedding $\iota\colon \mathcal{M}_{\mathrm{B}}^\circ \to \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ such that we have an isomorphism $\mathcal{A}\cong\iota^{-1}\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}}$ which commutes with the natural surjections to $\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^\circ}$. The space $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ is unique in the sense explained in *loc. cit.*

Next we construct a log-TEP structure on $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. Consider the inclusion $\iota \times \mathrm{id}\colon \mathcal{M}_{\mathrm{B}}^{\circ} \times \mathbb{C} \to \mathcal{M}_{\mathrm{B}}^{\mathrm{big}} \times \mathbb{C}$ and set $\widetilde{\mathcal{A}} = (\iota \times \mathrm{id})^{-1}\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\mathrm{big}} \times \mathbb{C}}$. This is the sheaf of algebras over $\mathcal{M}_{\mathrm{B}}^{\circ} \times \mathbb{C}$. Consider the pull-back $(\iota \times \mathrm{id})^{-1}\mathcal{F}_i$. This is a locally free $\widetilde{\mathcal{A}}|_{U_i \times \mathbb{C}}$-module of rank 6. Recall that the gluing maps $\varphi_{\mathbf{P}_i\mathbf{P}_j}$ are determined so that the log-TEP structures $\mathcal{F}_i$ and $(\varphi_{\mathbf{P}_i\mathbf{P}_j} \times \mathrm{id})^{\star}\mathcal{F}_j$ are isomorphic. In view of the construction of $\mathcal{A}$, this means that $(\iota \times \mathrm{id})^{-1}\mathcal{F}_i|_{(U_i \cap U_j) \times \mathbb{C}}$ and $(\iota \times \mathrm{id})^{-1}\mathcal{F}_j|_{(U_i \cap U_j) \times \mathbb{C}}$ are canonically isomorphic as $\widetilde{\mathcal{A}}|_{(U_i \cap U_j) \times \mathbb{C}}$-modules for each $i, j \in I$. Therefore the sheaves $(\iota \times \mathrm{id})^{-1}\mathcal{F}_i$ are glued together to a locally free $\widetilde{\mathcal{A}}$-module $\mathcal{B}$ of rank 6. By [24, Theorem 2, Remark 4], there exists a locally free sheaf $\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}$ of rank 6 over an open neighbourhood of $\mathcal{M}_{\mathrm{B}}^{\circ} \times \mathbb{C}$ in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}} \times \mathbb{C}$ such that $(\iota \times \mathrm{id})^{-1}\mathcal{F}_{\mathrm{B}}^{\mathrm{big}} \cong \mathcal{B}$. Similarly, we can glue the divisors $(U_i \cap D) \times V_i$ on local charts to construct a global divisor D^{big} in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ by regarding them as a coherent $\widetilde{\mathcal{A}}$-module and applying [24, Theorem 2]. The flat connection and the pairing on the local charts are glued to give *germs* of connections and pairings:

$$\nabla^{\mathrm{B}}\colon (\iota \times \mathrm{id})^{-1}\mathcal{F}_{\mathrm{B}}^{\mathrm{big}} \to (\iota \times \mathrm{id})^{-1}\left(\Omega^1_{\mathcal{M}_{\mathrm{B}}^{\mathrm{big}} \times \mathbb{C}}(\log \widehat{Z}) \otimes \mathcal{F}_{\mathrm{B}}^{\mathrm{big}}(\mathcal{M} \times \{0\})\right)$$

$$(\cdot,\cdot)_{\mathrm{B}}\colon (\iota \times \mathrm{id})^{-1}\left((-)^{\star}\mathcal{F}_{\mathrm{B}}^{\mathrm{big}} \otimes \mathcal{F}_{\mathrm{B}}^{\mathrm{big}}\right) \to (\iota \times \mathrm{id})^{-1}\mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\mathrm{big}} \times \mathbb{C}}$$

where $\widehat{Z} = \mathcal{M}_{\mathrm{B}}^{\mathrm{big}} \times \{0\} \cup D^{\mathrm{big}} \times \mathbb{C}$. These germs extend to an actual open neighbourhood of $\mathcal{M}_{\mathrm{B}}^{\circ} \times \mathbb{C}$ and satisfy the properties of a miniversal log-TEP structure. Because of the flat connection ∇, the structure $(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ extends automatically to an open set of the form $O \times \mathbb{C}$, where O is an open neighbourhood of $\mathcal{M}_{\mathrm{B}}^{\circ}$ in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. The proof of Theorem 5.0.3 is now complete.

5.4. **A Mirror Theorem for Big Quantum Cohomology.** The opposite module $\mathbf{P}$ in Proposition 5.1.7(1) coincides under mirror symmetry (Theorem 3.3.1) with the canonical opposite module for Gromov–Witten theory defined in Example 2.8.6: this is [27, Theorem 3.5]. Thus in a neighbourhood of the large-radius limit point $(y_1, y_2) = (0, 0)$, the A-model log-TEP structure (Example 2.7.6) is isomorphic to the big B-model log-TEP structure. Since the universal unfolding of a log-TEP structure is unique as an analytic germ, this proves:

Theorem 5.4.1. *Let $(\mathcal{M}_{\mathrm{A},\overline{Y}}, D_{\mathrm{A},\overline{Y}})$ denote the base of the A-model* log*-TEP structure for $\overline{Y}$, as described in Example 2.7.6. Let D^{big} be the divisor in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ as above. Consider:*

- *the A-model* log*-TEP structure $\left(\mathcal{F}_{\mathrm{A},\overline{Y}}, \nabla^{\mathrm{A},\overline{Y}}, (\cdot,\cdot)_{\mathrm{A},\overline{Y}}\right)$ for $\overline{Y}$; this is a* log*-TEP structure with base $(\mathcal{M}_{\mathrm{A},\overline{Y}}, D_{\mathrm{A},\overline{Y}})$.*
- *the big B-model* log*-TEP structure $\left(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\right)$; this is a* log*-TEP structure with base $(\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}, D^{\mathrm{big}})$.*

There exist:

- *an open neighbourhood U^{big} of the large-radius limit point in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$;*
- *an open embedding of pairs* $\mathrm{Mir}\colon (U^{\mathrm{big}}, D^{\mathrm{big}} \cap U^{\mathrm{big}}) \to (\mathcal{M}_{\mathrm{A},\overline{Y}}, D_{\mathrm{A},\overline{Y}})$*; and*
- *an isomorphism of* log*-TEP structures*

$$\left(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\right)\Big|_{U^{\mathrm{big}} \times \mathbb{C}} \cong \mathrm{Mir}^{\star}\left(\mathcal{F}_{\mathrm{A},\overline{Y}}, \nabla^{\mathrm{A},\overline{Y}}, (\cdot,\cdot)_{\mathrm{A},\overline{Y}}\right). \tag{51}$$

The map Mir *is called the mirror map; it sends the large-radius limit point in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ to the origin in $\mathcal{M}_{\mathrm{A},\overline{Y}}$, and coincides with the map $\mathrm{mir}_{\overline{Y}}$ in Theorem 3.3.1 when restricted to the small parameter space $U^{\times} \subset U^{\mathrm{big}}$ there. The isomorphism* (51) *intertwines the opposite*

module $\mathbf{P}_{\mathrm{LR}}$ *for* $\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}$ *defined in Proposition 5.1.7(1) with the canonical opposite module* $\mathbf{P}_{\mathrm{A}}$ *from Example 2.8.6.*

Remark 5.4.2. An analogous mirror symmetry statement for log-trTLEP structures was proved by Reichelt–Sevenheck [72].

Remark 5.4.3. A similar statement holds for the big quantum cohomology of the orbifold $\overline{\mathcal{X}}$: cf. [27, proof of Theorem 3.12].

6. Quantization Formalism and Fock Sheaf

As discussed in the Introduction, Givental's quantization formalism has been an essential ingredient in much recent work in Gromov–Witten theory. Givental's formulation of his quantization rules depends on a choice of flat co-ordinate system and so, in the context of mirror symmetry, is applicable only over certain patches of the moduli space $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. In previous work, we constructed a global and co-ordinate-free version of Givental's quantization, associating to a miniversal cTEP structure a *Fock sheaf* on (an open subset of) the total space of that cTEP structure [25]. Furthermore we showed that whenever the cTEP structure is semisimple, such as is the case for the A-model cTEP structure associated to a target space X with semisimple quantum cohomology, there is a canonically defined global section of this Fock sheaf. (In the A-model case, this global section coincides with the total descendant potential $\mathcal{Z}_X$.) In this section, we review the construction of the Fock sheaf.

6.1. cTEP Structures and log-cTEP Structures.

We will need the notions of cTEP structure and log-cTEP structure. One can think of these as being obtained from the notions of TEP structure (Definition 2.7.1) and log-TEP structure (Definition 2.7.2) by taking the formal completion along the divisor $z = 0$.

Definition 6.1.1. Let $\widehat{\mathbb{A}}^1 = \operatorname{Spf} \mathbb{C}[\![z]\!]$ denote the formal neighbourhood of zero in $\mathbb{C}$. Recall that a sheaf of modules over $\mathcal{M} \times \widehat{\mathbb{A}}^1$ is the same thing as a sheaf of $\mathcal{O}_{\mathcal{M}}[\![z]\!]$-modules. Let $(-)\colon \mathcal{M} \times \widehat{\mathbb{A}}^1 \to \mathcal{M} \times \widehat{\mathbb{A}}^1$ denote the map sending (t, z) to $(t, -z)$; this is consistent with our previous definition of $(-)$, in Definition 2.7.1. For an $\mathcal{O}_{\mathcal{M}}[\![z]\!]$-module F, we give the pull-back $(-)^\star\mathsf{F}$ the structure of an $\mathcal{O}_{\mathcal{M}}[\![z]\!]$-module by setting:

$$f(z)(-)^\star\alpha = (-)^\star f(-z)\alpha \qquad \text{for all } f(z) \in \mathcal{O}_{\mathcal{M}}[\![z]\!] \text{ and } \alpha \in \mathsf{F}.$$

Write $\mathsf{F}[z^{-1}]$ for the locally free $\mathcal{O}_{\mathcal{M}}(\!(z)\!)$-module $\mathsf{F} \otimes_{\mathcal{O}_{\mathcal{M}}[\![z]\!]} \mathcal{O}_{\mathcal{M}}(\!(z)\!)$, and F_0 for the quotient $\mathsf{F}/z\mathsf{F}$.

Notation 6.1.2. We will use sans serif font (F, G, etc.) to denote sheaves and similar structures over $\widehat{\mathbb{A}}^1$ or $\mathcal{M} \times \widehat{\mathbb{A}}^1$.

Definition 6.1.3 (cf. Definition 2.7.1)**.** Let $\mathcal{M}$ be a complex manifold. A *cTEP structure* $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ with base $\mathcal{M}$ consists of a locally free $\mathcal{O}_{\mathcal{M}}[\![z]\!]$-module F of rank $N+1$, a meromorphic flat connection:

$$\nabla\colon \mathsf{F} \to (\Omega^1_{\mathcal{M}} \oplus \mathcal{O}_{\mathcal{M}} z^{-1} dz) \otimes_{\mathcal{O}_{\mathcal{M}}} z^{-1}\mathsf{F}$$

and a non-degenerate pairing:

$$(\cdot,\cdot)_{\mathsf{F}}\colon (-)^\star\mathsf{F} \otimes_{\mathcal{O}_{\mathcal{M}}[\![z]\!]} \mathsf{F} \to \mathcal{O}_{\mathcal{M}}[\![z]\!]$$

which satisfies:

$$\begin{aligned} ((-)^\star s_1, s_2)_{\mathsf{F}} &= (-)^\star((-)^\star s_2, s_1)_{\mathsf{F}} \\ d((-)^\star s_1, s_2)_{\mathsf{F}} &= ((-)^\star \nabla s_1, s_2)_{\mathsf{F}} + ((-)^\star s_1, \nabla s_2)_{\mathsf{F}} \end{aligned}$$

for $s_1, s_2 \in \mathsf{F}$. Here we regard $z^{-1}\mathsf{F}$ as a subsheaf of $\mathsf{F}[z^{-1}]$; non-degeneracy of the pairing $(\cdot,\cdot)_{\mathsf{F}}$ means that the induced pairing on $\mathsf{F}_0 = \mathsf{F}/z\mathsf{F}$

$$(\cdot,\cdot)_{\mathsf{F}_0} \colon \mathsf{F}_0 \otimes_{\mathcal{O}_{\mathcal{M}}} \mathsf{F}_0 \to \mathcal{O}_{\mathcal{M}}$$

is non-degenerate.

Definition 6.1.4 (cf. Definition 2.7.2)**.** Let D be a divisor with normal crossings in a complex manifold $\mathcal{M}$. A log-*cTEP structure* $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ with base $(\mathcal{M}, D)$ consists of a locally free $\mathcal{O}_{\mathcal{M}}[\![z]\!]$-module F of rank $N+1$, a meromorphic flat connection:

$$\nabla \colon \mathsf{F} \to \left(\Omega^1_{\mathcal{M}}(\log D) \oplus \mathcal{O}_{\mathcal{M}} z^{-1} dz\right) \otimes_{\mathcal{O}_{\mathcal{M}}} z^{-1}\mathsf{F}$$

and a pairing:

$$(\cdot,\cdot)_{\mathsf{F}} \colon (-)^\star \mathsf{F} \otimes_{\mathcal{O}_{\mathcal{M}}[\![z]\!]} \mathsf{F} \to \mathcal{O}_{\mathcal{M}}[\![z]\!]$$

which satisfies the properties listed in Definition 6.1.3 and is non-degenerate in the same sense.

Example 6.1.5. Our first key example is the *A-model* log-*cTEP structure*, which is the formalization at $z = 0$ of the A-model log-TEP structure (Example 2.7.6). Write $\mathcal{F}_{\mathrm{A},X}$ for the sheaf underlying the A-model log-TEP structure and $(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X})$ for its base. The A-model log-cTEP structure $\left(\mathsf{F}_{\mathrm{A},X}, \nabla^{\mathrm{A},X}, (\cdot,\cdot)_{\mathrm{A},X}\right)$ has base $(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X})$ and:

$$\mathsf{F}_{\mathrm{A},X} := \mathcal{F}_{\mathrm{A},X} \otimes_{\mathcal{O}_{\mathcal{M}_{\mathrm{A},X}\times\mathbb{C}}} \mathcal{O}_{\mathcal{M}_{\mathrm{A},X}}[\![z]\!]$$

with meromorphic flat connection ∇ and pairing $(\cdot,\cdot)$ induced by the meromorphic flat connection and pairing on the A-model log-TEP structure. We refer to the restriction to $\mathcal{M}_{\mathrm{A},X} \backslash D_{\mathrm{A},X}$ of the A-model log-cTEP structure as the *A-model cTEP structure*.

Example 6.1.6. Our second key example is the *big B-model* log-*cTEP structure*, which is the formalization at $z = 0$ of the big B-model log-TEP structure from Theorem 5.0.3. This is a log-cTEP structure with base $(\mathcal{M}^{\mathrm{big}}_{\mathrm{B}}, D^{\mathrm{big}})$:

$$\mathsf{F}^{\mathrm{big}}_{\mathrm{B}} := \mathcal{F}^{\mathrm{big}}_{\mathrm{B}} \otimes_{\mathcal{O}_{\mathcal{M}^{\mathrm{big}}_{\mathrm{B}}\times\mathbb{C}}} \mathcal{O}_{\mathcal{M}^{\mathrm{big}}_{\mathrm{B}}}[\![z]\!]$$

with meromorphic flat connection ∇ and pairing $(\cdot,\cdot)$ induced by the meromorphic flat connection and pairing on the big B-model log-TEP structure.

Remark 6.1.7. A cTEP structure with base $\mathcal{M}$ is the same thing as a log-cTEP structure with base $(\mathcal{M}, D)$ where $D = \varnothing$. Thus the definitions of symplectic pairing, miniversality, etc. for log-cTEP structures given below also define the corresponding notions for cTEP structures.

Definition 6.1.8. Let $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ be a log-cTEP structure. The pairing $(\cdot,\cdot)_{\mathsf{F}}$ induces a symplectic pairing:

$$\Omega \colon \mathsf{F}[z^{-1}] \otimes_{\mathcal{O}_{\mathcal{M}}} \mathsf{F}[z^{-1}] \to \mathcal{O}_{\mathcal{M}},$$

defined by:

$$\Omega(s_1, s_2) = \mathrm{Res}_{z=0}((-)^\star s_1, s_2)_{\mathsf{F}}\, dz \tag{52}$$

Definition 6.1.9. Let $n \in \mathbb{Z}$ and let $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ be a log-cTEP structure. We set:

$$\begin{aligned} (z^n\mathsf{F})^\vee &:= \varinjlim_l \mathscr{H}om_{\mathcal{O}_{\mathcal{M}}}(z^n\mathsf{F}/z^l\mathsf{F}, \mathcal{O}_{\mathcal{M}}), \\ \mathsf{F}[z^{-1}]^\vee &:= \varprojlim_n \varinjlim_l \mathscr{H}om_{\mathcal{O}_{\mathcal{M}}}(z^{-n}\mathsf{F}/z^l\mathsf{F}, \mathcal{O}_{\mathcal{M}}). \end{aligned} \tag{53}$$

There are natural surjections:

$$\mathsf{F}[z^{-1}]^\vee \twoheadrightarrow \cdots \twoheadrightarrow (z^{-2}\mathsf{F})^\vee \twoheadrightarrow (z^{-1}\mathsf{F})^\vee \twoheadrightarrow \mathsf{F}^\vee \twoheadrightarrow (z\mathsf{F})^\vee \twoheadrightarrow \cdots .$$

The dual $(z^n\mathsf{F})^\vee$ has the structure of an $\mathcal{O}_{\mathcal{M}}[\![z]\!]$-module such that the action of z is nilpotent; it is locally isomorphic to $(\mathcal{O}_{\mathcal{M}}(\!(z)\!)/\mathcal{O}_{\mathcal{M}}[\![z]\!])^{N+1}$ as an $\mathcal{O}_{\mathcal{M}}(\!(z)\!)$-module. Also $\mathsf{F}[z^{-1}]^\vee$ is locally free as an $\mathcal{O}_{\mathcal{M}}(\!(z)\!)$-module. The symplectic pairing gives an isomorphism

$$\mathsf{F}[z^{-1}] \cong \mathsf{F}[z^{-1}]^\vee, \quad s \mapsto \iota_s\Omega = \Omega(s,\cdot)$$

and thus a dual symplectic pairing $\Omega^\vee$ on $\mathsf{F}[z^{-1}]^\vee$:

$$\Omega^\vee \colon \mathsf{F}[z^{-1}]^\vee \otimes_{\mathcal{O}_{\mathcal{M}}} \mathsf{F}[z^{-1}]^\vee \to \mathcal{O}_{\mathcal{M}}$$

The dual flat connection $\nabla^\vee$ is defined by:

$$\nabla^\vee \colon (z^{-1}\mathsf{F})^\vee \to \Omega^1_{\mathcal{M}}(\log D) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathsf{F}^\vee, \quad \langle \nabla^\vee \varphi, s\rangle := d\langle \varphi, s\rangle - \langle \varphi, \nabla s\rangle$$

6.2. **The Total Space of a log-cTEP Structure.** We now consider the total space L of a log-cTEP structure. This is an algebraic analogue of Givental's Lagrangian submanifold [43].

Definition 6.2.1. Let $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ be a log-cTEP structure. The *total space* L of $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ is the total space of the infinite dimensional vector bundle associated to $z\mathsf{F}$.

As a set, $\mathsf{L} = \{(t, \boldsymbol{x}) : t \in \mathcal{M}, \boldsymbol{x} \in z\mathsf{F}_t\}$. Let $\mathrm{pr}\colon \mathsf{L} \to \mathcal{M}$ denote the natural projection. We regard L as a "fiberwise algebraic variety" over $\mathcal{M}$, endowing it with the structure of a ringed space exactly as in [25, Definition 4.7]. Let $\boldsymbol{\mathcal{O}}$ denote the structure sheaf of L. For a connected open set $U \subset \mathcal{M}$ such that $\mathsf{F}|_U$ is a free $\mathcal{O}_U[\![z]\!]$-module, the ring of regular functions on $\mathrm{pr}^{-1}(U)$ is the polynomial ring over $\mathcal{O}(U)$:

$$\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U)) := \mathrm{Sym}^\bullet_{\mathcal{O}(U)} \Gamma(U, (z\mathsf{F})^\vee). \tag{54}$$

To make this concrete, take a trivialization $\mathsf{F}|_U \cong \mathbb{C}^{N+1} \otimes \mathcal{O}_U[\![z]\!]$. Consider the induced trivialization $\mathsf{F}[z^{-1}]|_U \cong \mathbb{C}^{N+1} \otimes \mathcal{O}_U(\!(z)\!)$, and the dual frame $x^i_n \in \mathsf{F}[z^{-1}]^\vee$, $n \in \mathbb{Z}$, $0 \le i \le N$, defined by:

$$\begin{aligned} x^i_n \colon \mathsf{F}[z^{-1}]\Big|_U \cong \mathbb{C}^{N+1} \otimes \mathcal{O}_U(\!(z)\!) &\longrightarrow \mathcal{O}_U \\ \sum_{m\in\mathbb{Z}} \sum_{j=0}^{N} a^j_m e_j z^m &\longmapsto a^i_n \end{aligned} \tag{55}$$

where e_i, $0 \le i \le N$, denotes the standard basis of $\mathbb{C}^{N+1}$. Restricting x^i_n to $z\mathsf{F}$, we obtain co-ordinates x^i_n, $n \ge 1$, $0 \le i \le N$, on the fibers of $\mathsf{L}|_U$.

Definition 6.2.2. Let $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ be a log-cTEP structure of rank $N+1$ with base $(\mathcal{M}, D)$. Let $t^0, q_1, \dots, q_r, t^{r+1}, \dots, t^R$ be local co-ordinates on an open set $U \subset \mathcal{M}$ such that $D \cap U$ is given by $(q_1 q_2 \cdots q_r = 0)$. We call the co-ordinate system

$$\{(t^j, q_k, x^i_n) : j \in \{0, r+1, \dots, R\},\ 1 \le k \le r,\ 0 \le i \le N,\ n \ge 1\}$$

an *algebraic local co-ordinate system* on L. We also set $q_k = e^{t^k}$ for $k = 1, \dots, r$ so that $(t^0, t^1, \dots, t^r, t^{r+1}, \dots, t^R)$ gives a multi-valued co-ordinate system on $U \setminus D$. We write t for a point on $\mathcal{M}$; this is a slight abuse of notation.

We have:

$$\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U)) = \mathcal{O}(U)\left[x_n^i \,|\, n \geq 1, 0 \leq i \leq N\right].$$

We equip $\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U))$ with a grading and filtration as follows. The grading on $\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U))$ is given by the degree as polynomials in the variables x_n^i. The lth part of the filtration, $l \geq 0$, is given by:

$$\boldsymbol{\mathcal{O}}_l(\mathrm{pr}^{-1}(U)) = \left\{ \sum_{n\geq 0} \sum_{\substack{l_1,\dots,l_n\geq 0\\ l_1+\cdots+l_n\leq l}} \sum_{i_1,\dots,i_n\geq 0} f^{l_1,\dots,l_n}_{i_1,\dots,i_n}(t) x^{i_1}_{l_1+1}\cdots x^{i_n}_{l_n+1} \;\Big|\; f^{l_1,\dots,l_n}_{i_1,\dots,i_n}(t) \in \mathcal{O}(U) \right\}. \tag{56}$$

This is an increasing filtration $\boldsymbol{\mathcal{O}}_l(\mathrm{pr}^{-1}(U)) \subset \boldsymbol{\mathcal{O}}_{l+1}(\mathrm{pr}^{-1}(U))$.

6.3. **Miniversality of** log**-cTEP structures.** Suppose now that $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ is a log-cTEP structure with base $(\mathcal{M}, D)$. Writing the connection ∇ in terms of our trivialization $\mathsf{F}|_U \cong \mathbb{C}^{N+1}\otimes \mathcal{O}_U[\![z]\!]$ gives:

$$\nabla s = ds - \frac{1}{z}\mathcal{C}(t,z)s \tag{57}$$

where $s \in \mathbb{C}^{N+1}\otimes\mathcal{O}_U[\![z]\!] \cong \mathsf{F}|_U$ and $\mathcal{C}(t,z) \in \mathrm{End}(\mathbb{C}^{N+1})\otimes\Omega^1_U(\log D)[\![z]\!]$. The residual part $\mathcal{C}(t,0) = (-z\nabla)|_{z=0}$ determines a section of $\mathrm{End}(\mathsf{F}_0)\otimes\Omega^1_U(\log D)$ which is independent of choice of trivialization.

Example 6.3.1. In the case of the A-model log-cTEP structure (Example 6.1.5), we have $\mathcal{C}(t,0) = (\phi_0 *)dt^0 + \sum_{i=1}^r (\phi_i *)\frac{dq_i}{q_i} + \sum_{i=r+1}^N (\phi_i *)dt^i$.

Definition 6.3.2. Let $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ be a log-cTEP structure. Let:

$$\begin{aligned}
\mathsf{F}^\circ_{0,t} &:= \{x_1 \in \mathsf{F}_{0,t} \,|\, \Theta_{\mathcal{M}}(\log D)_t \to \mathsf{F}_{0,t},\ v \mapsto \iota_v \mathcal{C}(t,0)x_1 \text{ is an isomorphism}\}\\
\mathbf{L}^\circ &:= \{(t,\boldsymbol{x}) \in \mathbf{L} \,|\, t\in\mathcal{M},\ \boldsymbol{x} \in z\mathsf{F}_t,\ (\boldsymbol{x}/z)|_{z=0} \in \mathsf{F}^\circ_{0,t}\}\\
\mathsf{F}^\circ_0 &:= \bigcup_{t\in\mathcal{M}} \mathsf{F}^\circ_{0,t}
\end{aligned}$$

These are open subsets of, respectively, $\mathsf{F}_{0,t}$, $\mathbf{L}$, and F_0. If for every point $t \in \mathcal{M}$, $\mathsf{F}^\circ_{0,t}$ is a non-empty Zariski open subset of $\mathsf{F}_{0,t}$, then we say that $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ is *miniversal*.

Remark 6.3.3. A miniversal log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ with base $(\mathcal{M}, D)$ satisfies $\dim\mathcal{M} = \mathrm{rank}\,\mathsf{F}$.

We henceforth assume that our log-cTEP structure is miniversal. Let $\{t^j, q_k, x^i_n\}$ be an algebraic local co-ordinate system on $\mathbf{L}$, and write $\mathcal{C}(t,z) = \sum_{i=0}^N \mathcal{C}_i(t,z)dt^i$. Here recall that $dt^i = \frac{dq_i}{q_i}$ for $1 \leq i \leq r$. Consider:

$$P(t,x_1) := (-1)^{N+1}\det(\mathcal{C}_0(t,0)x_1, \mathcal{C}_1(t,0)x_1, \dots, \mathcal{C}_N(t,0)x_1) \tag{58}$$

This is a polynomial of degree $N+1$, $P(t,x_1) \in \mathcal{O}(U)[x_1^0,\dots,x_1^N]$, called the *discriminant*. The set $\mathbf{L}^\circ$ is the complement[10] of the zero-locus of $P(t,x_1)$. The ring of regular functions over $\mathrm{pr}^{-1}(U)^\circ := \mathrm{pr}^{-1}(U)\cap\mathbf{L}^\circ$ is:

$$\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U)^\circ) = \mathcal{O}(U)[\{x^i_n\}_{n\geq 1, 0\leq i\leq N}, P(t,x_1)^{-1}]$$

Since $P(t,x_1)$ is homogeneous in x_1 and lies in the zeroth filter, the grading and filtration on $\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U))$ extend canonically to $\boldsymbol{\mathcal{O}}(\mathrm{pr}^{-1}(U)^\circ)$.

[10]More invariantly, we can think of $P(t,x_1)dt^0\wedge\cdots\wedge dt^N$ as a section of the line bundle $\mathrm{pr}^\star(\det(\mathsf{F}_0)\otimes\Omega^{N+1}_{\mathcal{M}}(\log D))$ over $\mathbf{L}$, and of $\mathbf{L}^\circ$ as the complement to the zero locus of that section.

6.4. **One-Forms and Vector Fields on L.** The sheaf $\boldsymbol{\Omega}^1(\log D)$ of logarithmic one-forms on L is defined in algebraic local co-ordinates $\{t^j, q_k, x^i_n\}$ as:

$$\boldsymbol{\Omega}^1(\log D) = \bigoplus_{j=0}^{N} \boldsymbol{\mathcal{O}} dt^j \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{i=0}^{N} \boldsymbol{\mathcal{O}} dx^i_n$$

where recall again that $dt^i = \frac{dq_i}{q_i}$ for $1 \le i \le r$. The grading on $\boldsymbol{\Omega}^1(\log D)$ is determined by[11]:

$$\deg(dt^j) = 0, \qquad \deg(dx^i_n) = 1$$

The filtration on $\boldsymbol{\Omega}^1(\log D)$ is determined by putting dt^i in the (-1)st filter but not the zeroth filter, and putting dx^i_n in the $(n-1)$st filter but not the nth filter. We write $\boldsymbol{\mathcal{O}}^d_e$ and $\boldsymbol{\Omega}^1(\log D)^d_e$ for the eth filter of the dth graded piece of $\boldsymbol{\mathcal{O}}$ and $\boldsymbol{\Omega}^1(\log D)$ respectively, so that

$$\boldsymbol{\Omega}^1(\log D)^d_e = \bigoplus_{j=0}^{N} \boldsymbol{\mathcal{O}}^d_{e+1} dt^j \oplus \bigoplus_{e_1+e_2 \le e} \bigoplus_{i=0}^{N} \boldsymbol{\mathcal{O}}^{d-1}_{e_1} dx^i_{e_2+1}$$

We also set:

$$\big((\boldsymbol{\Omega}^1(\log D))^{\otimes k}\big)^d_e = \sum_{e_1+\cdots+e_k \le e} \sum_{d_1+\cdots+d_k=d} \boldsymbol{\Omega}^1(\log D)^{d_1}_{e_1} \otimes \cdots \otimes \boldsymbol{\Omega}^1(\log D)^{d_k}_{e_k}$$

The sheaf $\boldsymbol{\Theta}(\log D)$ of logarithmic vector fields on L is defined by

$$\boldsymbol{\Theta}(\log D) = \mathrm{Hom}_{\boldsymbol{\mathcal{O}}}(\boldsymbol{\Omega}^1(\log D), \boldsymbol{\mathcal{O}}).$$

In algebraic local co-ordinates $\{t^j, q_k = e^{t^k}, x^i_n\}$, with $\partial_j := \frac{\partial}{\partial t^j}$ and $\partial_{n,i} := \frac{\partial}{\partial x^i_n}$, we have:

$$\boldsymbol{\Theta}(\log D) = \prod_{j=0}^{N} \boldsymbol{\mathcal{O}} \partial_j \times \prod_{n=1}^{\infty} \prod_{i=0}^{N} \boldsymbol{\mathcal{O}} \partial_{n,i}$$

Note that $\boldsymbol{\Omega}^1(\log D)$ is the direct sum whereas $\boldsymbol{\Theta}(\log D)$ is the direct product.

6.5. **The Yukawa Coupling and the Kodaira–Spencer Map.** As above, let $\{t^j, q_k, x^i_n\}$ be an algebraic local co-ordinate system on L, and write $\mathcal{C}(t,z) = \sum_{i=0}^{N} \mathcal{C}_i(t,z) dt^i$. From flatness of ∇ and flatness of the pairing we have that:

$$\big[\mathcal{C}_i(t,0), \mathcal{C}_j(t,0)\big] = 0$$
$$\big(\mathcal{C}_i(t,0)s_1, s_2\big)_{\mathsf{F}_0} = \big(s_1, \mathcal{C}_i(t,0)s_2\big)_{\mathsf{F}_0}$$

for all i, j and all s_1, $s_2 \in \mathsf{F}_0$. Thus the endomorphisms $\mathcal{C}_i(t,0)$ equip the fibers of F_0 with a structure similar to that of a Frobenius algebra (we need to choose an identity element here; cf. [25, §4.4]).

Definition 6.5.1. The *Yukawa coupling* is a cubic tensor:

$$\boldsymbol{Y} = \sum_{i,j,k} C^{(0)}_{ijk} dt^i \otimes dt^j \otimes dt^k \in \big((\boldsymbol{\Omega}^1(\log D))^{\otimes 3}\big)^2_{-3}$$

defined in algebraic local co-ordinates $\{t^j, q_k, x^i_n\}$ by:

$$C^{(0)}_{ijk}(t, \boldsymbol{x}) = \big(\mathcal{C}_i(t,0)x_1, \mathcal{C}_j(t,0)\mathcal{C}_k(t,0)x_1\big)_{\mathsf{F}_0} \qquad \text{where } x_1 = (\boldsymbol{x}/z)|_{z=0}$$

[11] More precisely, the grading and the filtration are defined on the module $\boldsymbol{\Omega}^1(\log D)(\mathrm{pr}^{-1}(U))$ or on $\boldsymbol{\Omega}^1(\log D)(\mathrm{pr}^{-1}(U)^\circ)$ for an open set $U \subset \mathcal{M}$. We will omit the domains $\mathrm{pr}^{-1}(U)$, $\mathrm{pr}^{-1}(U)^\circ$, to ease the notation.

Recall again that $q_k = e^{t^k}$ for $1 \le k \le r$.

Remark 6.5.2. The Yukawa coupling is a symmetric cubic tensor on $\mathbf{L}$ that is pulled back from F_0.

Let $\mathrm{pr}\colon \mathbf{L} \to \mathcal{M}$ denote the natural projection. We define:

$$\mathrm{pr}^\star(z^n\mathsf{F}) := \varprojlim_l \mathrm{pr}^\star(z^n\mathsf{F}/z^l\mathsf{F}) \cong (\mathrm{pr}^{-1}\, z^n\mathsf{F}) \otimes_{\mathrm{pr}^{-1}\,\mathcal{O}_{\mathcal{M}}[\![z]\!]} \boldsymbol{\mathcal{O}}[\![z]\!]$$

$$\mathrm{pr}^\star\, \mathsf{F}[z^{-1}] := \varprojlim_l \mathrm{pr}^\star(\mathsf{F}[z^{-1}]/z^l\mathsf{F}) \cong (\mathrm{pr}^{-1}\, \mathsf{F}[z^{-1}]) \otimes_{\mathrm{pr}^{-1}\,\mathcal{O}_{\mathcal{M}}(\!(z)\!)} \boldsymbol{\mathcal{O}}(\!(z)\!)$$

$$\mathrm{pr}^\star(z^n\mathsf{F})^\vee := (\mathrm{pr}^{-1}(z^n\mathsf{F})^\vee) \otimes_{\mathrm{pr}^{-1}\,\mathcal{O}_{\mathcal{M}}} \boldsymbol{\mathcal{O}}$$

$$\mathrm{pr}^\star\, \mathsf{F}[z^{-1}]^\vee := \varprojlim_l \mathrm{pr}^\star(z^{-l}\mathsf{F})^\vee \cong (\mathrm{pr}^{-1}\, \mathsf{F}[z^{-1}]^\vee) \otimes_{\mathrm{pr}^{-1}\,\mathcal{O}_{\mathcal{M}}(\!(z)\!)} \boldsymbol{\mathcal{O}}(\!(z)\!).$$

These are locally free modules over $\boldsymbol{\mathcal{O}}[\![z]\!]$, $\boldsymbol{\mathcal{O}}(\!(z)\!)$, $\boldsymbol{\mathcal{O}}$, $\boldsymbol{\mathcal{O}}(\!(z)\!)$ respectively. Note that, with the exception of $\mathrm{pr}^\star(z^n\mathsf{F})^\vee$, these differ from the standard notion of pullback. For example, $\mathrm{pr}^\star(z^n\mathsf{F})$ is the completion of the standard pull-back $\mathrm{pr}^{-1}(z^n\mathsf{F}) \otimes_{\mathrm{pr}^{-1}\,\mathcal{O}_{\mathcal{M}}} \boldsymbol{\mathcal{O}}$ of $z^n\mathsf{F}$ with respect to the z-adic topology.

The pull-back $\mathrm{pr}^\star\, \mathsf{F}$ admits a flat connection $\widetilde{\nabla} := \mathrm{pr}^\star\, \nabla$:

$$\widetilde{\nabla}\colon\ \mathrm{pr}^\star\, \mathsf{F} \to \boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star(z^{-1}\mathsf{F}). \tag{59}$$

where $\widehat{\otimes}$ is the completed tensor product:

$$\boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star(z^{-1}\mathsf{F}) := \varprojlim_n (\boldsymbol{\Omega}^1(\log D) \otimes \mathrm{pr}^\star(z^{-1}\mathsf{F}/z^n\mathsf{F}))$$

Let $\{t^j, q_k, x^i_n\}$ be an algebraic local co-ordinate system on $\mathbf{L}$, where $(t^0, q_1, \dots, q_r, t^{r+1}, \dots, t^N)$ are local co-ordinates on an open subset U of $\mathcal{M}$, and consider a local trivialization $\mathsf{F}|_U \cong \mathbb{C}^{N+1} \otimes \mathcal{O}_U[\![z]\!]$. The trivialization of $\mathsf{F}|_U$ allows us to write:

$$\nabla s = ds - \frac{1}{z}\mathcal{C}(t,z)s$$

where $\mathcal{C}(t,z) = \sum_{i=0}^N \mathcal{C}_i(t,z)dt^i$ (recall that $dt^i = \frac{dq_i}{q_i}$ for $1 \le i \le r$). The trivialization of $\mathsf{F}|_U$ also induces a trivialization $\mathrm{pr}^\star\, \mathsf{F}|_{\mathrm{pr}^{-1}(U)} \cong \mathbb{C}^{N+1} \otimes \boldsymbol{\mathcal{O}}[\![z]\!]$, and with respect to this trivialization we have:

$$\widetilde{\nabla}_{\partial_j} = \partial_i - \frac{1}{z}\mathcal{C}_i(t,z) \qquad 0 \le i \le N$$

$$\widetilde{\nabla}_{\partial_{n,i}} = \partial_{n,i} \qquad 0 \le i \le N,\ 1 \le n < \infty$$

Definition 6.5.3. The *tautological section* $\boldsymbol{x}$ of $\mathrm{pr}^\star(z\mathsf{F})$ is defined by

$$\boldsymbol{x}(t, \boldsymbol{x}) = \boldsymbol{x}$$

where $(t, \boldsymbol{x})$ denotes the point $\boldsymbol{x} \in z\mathsf{F}_t$ on $\mathbf{L}$.

Definition 6.5.4. The *Kodaira–Spencer map* $\mathrm{KS}\colon \boldsymbol{\Theta}(\log D) \to \mathrm{pr}^\star\, \mathsf{F}$ is defined by:

$$\mathrm{KS}(v) = \widetilde{\nabla}_v \boldsymbol{x}$$

The *dual Kodaira–Spencer map* $\mathrm{KS}^\star\colon\ \mathrm{pr}^\star\, \mathsf{F}^\vee \to \boldsymbol{\Omega}^1(\log D)$ is defined by:

$$\mathrm{KS}^\star(\varphi) = \varphi(\widetilde{\nabla}\boldsymbol{x}), \quad \varphi \in \mathrm{pr}^*\, \mathsf{F}^\vee.$$

Remark 6.5.5. The maps KS and $\mathrm{KS}^\star$ are isomorphisms over $\mathbf{L}^\circ \subset \mathbf{L}$.

Definition 6.5.6. Let $\boldsymbol{\Theta}_\circ(\log D)$ denote the restriction of $\boldsymbol{\Theta}(\log D)$ to $\mathsf{L}^\circ \subset \mathsf{L}$, and let $\boldsymbol{\Omega}^1_\circ(\log D)$ denote the restriction of $\boldsymbol{\Omega}^1(\log D)$ to $\mathsf{L}^\circ \subset \mathsf{L}$.

Remark 6.5.7. Using the connection $\widetilde{\nabla}$ on $\mathrm{pr}^\star \mathsf{F}$ and the tautological section $\boldsymbol{x} \in \mathrm{pr}^\star \mathsf{F}$, we can write the Yukawa coupling as follows:

$$\boldsymbol{Y}(X,Y,Z) = \Omega(\widetilde{\nabla}_X \widetilde{\nabla}_Y \boldsymbol{x}, \widetilde{\nabla}_Z \boldsymbol{x}).$$

6.6. **The Euler Vector Field and Grading Operators.**

Definition 6.6.1. An *Euler vector field* for a log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ with base $(\mathcal{M}, D)$ is a logarithmic vector field E on $\mathcal{M}$ such that $\nabla_{z\partial_z + E}$ is regular at $z = 0$.

Remark 6.6.2. A miniversal log-cTEP structure always admits an Euler vector field, and this Euler vector field is unique.

Definition 6.6.3. Suppose that $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ is a log-cTEP structure with Euler vector field E. Define the grading operator $\mathrm{gr} \in \mathrm{End}_{\mathbb{C}}(\mathsf{F}[z^{-1}])$ by

$$\mathrm{gr} := \nabla_{z\partial_z + E}$$

The grading operator gr preserves $\mathsf{F} \subset \mathsf{F}[z^{-1}]$. For $\varphi \in \mathsf{F}[z^{-1}]^\vee$, define $\mathrm{gr}^\vee(\varphi)$ by $\mathrm{gr}^\vee(\varphi)(x) = E(\varphi(x)) - \varphi(\mathrm{gr}(x))$.

Lemma 6.6.4. $\mathrm{gr}^\vee$ *is a well-defined element of* $\mathrm{End}_{\mathbb{C}}(\mathsf{F}[z^{-1}]^\vee)$.

Proof. Let $(\mathcal{M}, D)$ be the base of the log-cTEP structure and suppose that $\varphi \in \mathsf{F}[z^{-1}]^\vee$. We need to show that $\mathrm{gr}^\vee(\varphi) \in \mathsf{F}[z^{-1}]^\vee$, i.e. that $\mathrm{gr}^\vee(\varphi)$ is $\mathcal{O}_{\mathcal{M}}$-linear. Let $f \in \mathcal{O}_{\mathcal{M}}$ and $x \in \mathsf{F}[z^{-1}]$. Then:

$$\begin{aligned}
\mathrm{gr}^\vee(\varphi)(fx) &= E(f\varphi(x)) - \varphi(\mathrm{gr}(fx)) \\
&= E(f)\varphi(x) + fE(\varphi(x)) - E(f)\varphi(\mathrm{gr}(x)) - f\varphi(\mathrm{gr}(x)) \\
&= f\,\mathrm{gr}^\vee(\varphi)(x)
\end{aligned}$$

as required. □

Example 6.6.5. Consider the big B-model log-cTEP structure $(\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ (Example 6.1.6). Then the grading operator $\mathrm{gr} \in \mathrm{End}(\mathsf{F}_{\mathrm{B}}^{\mathrm{big}})$, when restricted to the small parameter space $\mathcal{M}_{\mathrm{B}}^\times$, coincides with $\mathrm{Gr} - \frac{3}{2}$ where Gr is the grading operator on the GKZ system (Definition 3.4.4). The shift by $\frac{3}{2}$ here reflects the shift by $\frac{3}{2}$ in Definition 3.2.2, which was made to ensure that the B-model TEP structure had weight zero.

Definition 6.6.6. Let $\sharp\colon \mathsf{F}[z^{-1}] \to \mathsf{F}[z^{-1}]^\vee$ be the map $\alpha \mapsto \Omega(\alpha, -)$, and let $\flat\colon \mathsf{F}[z^{-1}]^\vee \to \mathsf{F}[z^{-1}]$ be the inverse map. Write $\alpha^\sharp$ for $\sharp(\alpha)$, and $\varphi^\flat$ for $\flat(\varphi)$.

Lemma 6.6.7. *We have:*

(a) $\mathrm{gr}^\vee\big(\alpha^\sharp\big) = \big((\mathrm{gr}+1)\alpha\big)^\sharp$;
(b) $\big(\mathrm{gr}^\vee(\varphi)\big)^\flat = (\mathrm{gr}+1)\big(\varphi^\flat\big)$;
(c) $\big(\mathrm{gr}^\vee \otimes 1 + 1 \otimes \mathrm{gr}^\vee\big)\Omega = \Omega$;
(d) $\big(\mathrm{gr} \otimes 1 + 1 \otimes \mathrm{gr}\big)\Omega^\vee = -\Omega^\vee$.

Proof. For α, $\beta \in \mathsf{F}[z^{-1}]$, we have:

$$\big(z\partial_z + E\big)\big((-)^\star\alpha, \beta\big) = \big((-)^\star\,\mathrm{gr}(\alpha), \beta\big) + \big((-)^\star\alpha, \mathrm{gr}(\beta)\big)$$

and hence:

$$(E-1)\Omega(\alpha,\beta) = \Omega\big(\operatorname{gr}(\alpha),\beta\big) + \Omega\big(\alpha,\operatorname{gr}(\beta)\big) \tag{60}$$

Rearranging gives:

$$E\Omega(\alpha,\beta) - \Omega\big(\alpha,\operatorname{gr}(\beta)\big) = \Omega\big(\operatorname{gr}(\alpha),\beta\big) + \Omega(\alpha,\beta)$$

which is (a). Part (b) follows immediately. Rearranging (60) again gives:

$$E\Omega(\alpha,\beta) - \Omega\big(\alpha,\operatorname{gr}(\beta)\big) - \Omega\big(\operatorname{gr}(\alpha),\beta\big) = \Omega(\alpha,\beta)$$

which is (c). For (d), we have:

$$\begin{aligned}
\big(\operatorname{gr}\otimes 1 + 1\otimes\operatorname{gr}\big)\Omega^\vee &= \big(\operatorname{gr}\otimes 1 + 1\otimes\operatorname{gr}\big)(\flat\otimes\flat)\Omega \\
&= (\flat\otimes\flat)\big((\operatorname{gr}^\vee - 1)\otimes 1 + 1\otimes(\operatorname{gr}^\vee - 1)\big)\Omega \\
&= (\flat\otimes\flat)(-\Omega) && \text{by (c)} \\
&= -\Omega^\vee
\end{aligned}$$

□

6.7. **Opposite Modules and Propagators.**

Definition 6.7.1 (cf. Definition 2.8.5)**.** Let $(\mathsf{F},\nabla,(\cdot,\cdot)_{\mathsf{F}})$ be a log-cTEP structure with base $(\mathcal{M},D)$. Let P be a locally free $\mathcal{O}_{\mathcal{M}}[z^{-1}]$-submodule P of $\mathsf{F}[z^{-1}]$. We say that:

(1) P is opposite to F if $\mathsf{F}[z^{-1}] = \mathsf{F}\oplus\mathsf{P}$;
(2) P is isotropic if $\Omega(s_1,s_2) = 0$ for all $s_1,\, s_2 \in \mathsf{P}$;
(3) P is parallel if $\nabla_X\mathsf{P}\subset\mathsf{P}$ for all $X\in\Theta_{\mathcal{M}}(\log D)$;
(4) P is homogeneous if $\nabla_{z\partial_z}\mathsf{P}\subset\mathsf{P}$.

An *opposite module* for $(\mathsf{F},\nabla,(\cdot,\cdot)_{\mathsf{F}})$ is a locally free $\mathcal{O}_{\mathcal{M}}[z^{-1}]$-submodule P of $\mathsf{F}[z^{-1}]$ such that P is opposite to F, isotropic, parallel, and homogeneous. Let U be an open subset of $\mathcal{M}$. We say that P is an *opposite module over* U if P is an opposite module for the restriction $(\mathsf{F},\nabla,(\cdot,\cdot)_{\mathsf{F}})\big|_U$.

Remark 6.7.2. Conditions (3) and (4) here imply that an opposite module P is preserved by the grading operator gr.

Example 6.7.3 (opposites compatible with Deligne give opposites for log-cTEP structures)**.** Let $(\mathcal{F},\nabla,(\cdot,\cdot)_{\overline{\mathcal{F}}})$ be a log-TEP structure with base $(\mathcal{M},D)$ which is the Deligne extension of a TEP structure $(\mathcal{F}^\times,\nabla,(\cdot,\cdot)_{\mathcal{F}})$ with base $\mathcal{M}\setminus D$. Let $(\mathsf{F},\nabla,(\cdot,\cdot)_{\mathsf{F}})$ be the log-cTEP structure with base $(\mathcal{M},D)$ obtained from $(\mathcal{F},\nabla,(\cdot,\cdot)_{\overline{\mathcal{F}}})$ by taking the formal completion along the divisor $z=0$ in $\mathcal{M}\times\mathbb{C}$. Suppose that $\mathbf{P}$ is an opposite module for $(\mathcal{F}^\times,\nabla,(\cdot,\cdot)_{\mathcal{F}})$ which is compatible with the Deligne extension (Definition 5.1.1). Then $\mathbf{P}$ determines a trivialization of $\mathcal{F}$ and hence a trivialization of F. Thus $\mathbf{P}$ determines an opposite module P for $(\mathsf{F},\nabla,(\cdot,\cdot)_{\mathsf{F}})$.

Example 6.7.4. In particular, Proposition 5.1.7 determines opposite modules for the big B-model log-cTEP structure:

P_{LR}, defined near the large-radius limit point
$\mathsf{P}_{\mathrm{con}}$, defined near the conifold point
$\mathsf{P}_{\mathrm{orb}}$, defined near the orbifold point.

Example 6.7.5. The canonical opposite module $\mathbf{P}_{\mathrm{A}}$ for the A-model TEP structure defined in Example 2.8.6 is compatible with the Deligne extension. It thus determines a canonical opposite submodule P_{A} for the A-model log-cTEP structure.

An opposite module P determines flat connections on the logarithmic tangent sheaf and logarithmic cotangent sheaf of $\mathbf{L}^\circ$, as follows. The connection $\widetilde{\nabla}$ on $\mathrm{pr}^\star\,\mathsf{F}$ (equation (59)) extends z^{-1}-linearly to a flat connection $\widetilde{\nabla}\colon\ \mathrm{pr}^\star\,\mathsf{F}[z^{-1}] \to \boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star(\mathsf{F}[z^{-1}])$, where:

$$\boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star(\mathsf{F}[z^{-1}]) := \varprojlim_n (\boldsymbol{\Omega}^1(\log D) \otimes \mathrm{pr}^\star(\mathsf{F}[z^{-1}]/z^n\mathsf{F}))$$

The dual flat connection $\widetilde{\nabla}^\vee\colon\ \mathrm{pr}^\star\,\mathsf{F}[z^{-1}]^\vee \to \boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star\,\mathsf{F}[z^{-1}]^\vee$ is defined by:

$$\big\langle \widetilde{\nabla}^\vee \varphi, s \big\rangle := d\langle \varphi, s\rangle - \big\langle \varphi, \widetilde{\nabla} s \big\rangle \qquad\qquad s \in \mathrm{pr}^\star\,\mathsf{F}[z^{-1}],\ \varphi \in \mathrm{pr}^\star\,\mathsf{F}[z^{-1}]^\vee$$

where $\boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star(\mathsf{F}[z^{-1}]^\vee) := \varprojlim_n (\boldsymbol{\Omega}^1(\log D) \otimes \mathrm{pr}^\star(z^{-n}\mathsf{F})^\vee)$. This induces flat connections $\widetilde{\nabla}^\vee\colon\ \mathrm{pr}^\star(z^n\mathsf{F})^\vee \to \boldsymbol{\Omega}^1 \otimes \mathrm{pr}^\star(z^{n+1}\mathsf{F})^\vee$ for each $n \in \mathbb{Z}$.

Definition 6.7.6. Let P be an opposite module for the log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_\mathsf{F})$, and let $\Pi\colon \mathsf{F}[z^{-1}] \to \mathsf{F}$ be the projection along P. The composition of the maps:

$$\mathrm{pr}^\star\,\mathsf{F} \xrightarrow{\ \widetilde{\nabla}\ } \boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star(z^{-1}\mathsf{F}) \xrightarrow{\ \mathrm{id}\otimes\Pi\ } \boldsymbol{\Omega}^1(\log D) \mathbin{\widehat{\otimes}} \mathrm{pr}^\star\,\mathsf{F}$$

$$\mathrm{pr}^\star\,\mathsf{F}^\vee \xrightarrow{\ \Pi^\vee\ } \mathrm{pr}^\star(z^{-1}\mathsf{F})^\vee \xrightarrow{\ \widetilde{\nabla}^\vee\ } \boldsymbol{\Omega}^1(\log D) \otimes \mathrm{pr}^\star\,\mathsf{F}^\vee$$

(restricted to $\mathbf{L}^\circ$) with the Kodaira–Spencer isomorphisms $\mathrm{KS}\colon \boldsymbol{\Theta}_\circ(\log D) \to \mathrm{pr}^\star\,\mathsf{F}$, $\mathrm{KS}^\star\colon\ \mathrm{pr}^\star\,\mathsf{F}^\vee \to \boldsymbol{\Omega}^1_\circ(\log D)$ induces connections:

$$\begin{aligned} &\boldsymbol{\nabla}\colon \boldsymbol{\Theta}_\circ(\log D) \to \boldsymbol{\Omega}^1_\circ(\log D) \mathbin{\widehat{\otimes}} \boldsymbol{\Theta}_\circ(\log D) \\ &\boldsymbol{\nabla}\colon \boldsymbol{\Omega}_\circ(\log D) \to \boldsymbol{\Omega}_\circ(\log D) \otimes \boldsymbol{\Omega}_\circ(\log D) \end{aligned} \tag{61}$$

where $\boldsymbol{\Omega}^1_\circ(\log D) \mathbin{\widehat{\otimes}} \boldsymbol{\Theta}_\circ(\log D) := \varprojlim_n \big(\boldsymbol{\Omega}^1_\circ(\log D) \otimes \big(\boldsymbol{\Theta}_\circ(\log D) / \mathrm{KS}^{-1}(\mathrm{pr}^\star(z^n\mathsf{F}))\big)\big)$.

The connections in (61) are dual to each other. Proposition 4.108 in [25] shows that they are flat.

Definition 6.7.7. Let P_1, P_2 be opposite modules for the log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_\mathsf{F})$. Let $\Pi_i\colon \mathsf{F}[z^{-1}] \to \mathsf{F}$, $i \in \{1,2\}$, be the projection along P_i defined by the decomposition $\mathsf{F}[z^{-1}] = \mathsf{P}_i \oplus \mathsf{F}$. The *propagator* $\Delta = \Delta(\mathsf{P}_1, \mathsf{P}_2) \in \mathscr{H}om_{\boldsymbol{\mathcal{O}}}(\boldsymbol{\Omega}^1_\circ(\log D) \otimes \boldsymbol{\Omega}^1_\circ(\log D), \boldsymbol{\mathcal{O}})$ is defined by:

$$\Delta(\omega_1, \omega_2) = \Omega^\vee\big(\Pi_1^\star(\mathrm{KS}^\star)^{-1}\omega_1, \Pi_2^\star(\mathrm{KS}^\star)^{-1}\omega_2\big), \qquad\qquad \omega_1, \omega_2 \in \boldsymbol{\Omega}^1_\circ(\log D).$$

The logarithmic bivector field Δ coincides, via the Kodaira–Spencer isomorphism $\mathrm{KS}^\star$, with the push-forward along $\Pi_1 \times \Pi_2$ of the Poisson bivector field on $\mathsf{F}[z^{-1}]$ defined by $\Omega^\vee$.

The propagator $\Delta := \Delta(\mathsf{P}_1, \mathsf{P}_2)$ is symmetric, i.e. $\Delta(\omega_1, \omega_2) = \Delta(\omega_2, \omega_1)$ for all $\omega_1, \omega_2 \in \boldsymbol{\Omega}^1_\circ$ [25, Proposition 4.110]. Furthermore, if P_1, P_2, P_3 are opposite modules for the log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_\mathsf{F})$ and $\Delta_{ij} := \Delta(\mathsf{P}_i, \mathsf{P}_j)$ then [25, Proposition 4.111]:

$$\Delta_{13} = \Delta_{12} + \Delta_{23}$$

In particular, $\Delta(\mathsf{P}_1, \mathsf{P}_2) = -\Delta(\mathsf{P}_2, \mathsf{P}_1)$.

Lemma 6.7.8. *Let* P *be an opposite module for the* log-*cTEP structure* $(\mathsf{F}, \nabla, (\cdot,\cdot)_\mathsf{F})$*, and let* $\Pi\colon \mathsf{F}[z^{-1}] \to \mathsf{F}$ *be the projection along* P*. Then* $\mathrm{gr} \circ \Pi = \Pi \circ \mathrm{gr}$.

Proof. Let $\alpha \in \mathsf{F}[z^{-1}]$, and write $\alpha = \alpha_\mathsf{F} + \alpha_\mathsf{P}$ with $\alpha_\mathsf{F} \in \mathsf{F}$ and $\alpha_\mathsf{P} \in \mathsf{P}$. Then $\mathrm{gr}(\alpha) = \mathrm{gr}(\alpha_\mathsf{F}) + \mathrm{gr}(\alpha_\mathsf{P})$. The operator gr preserves both F and P, so $\mathrm{gr}(\alpha_\mathsf{F}) \in \mathsf{F}$ and $\mathrm{gr}(\alpha_\mathsf{P}) \in \mathsf{P}$. Thus $\Pi \circ \mathrm{gr}(\alpha) = \mathrm{gr}(\alpha_F) = \mathrm{gr} \circ \Pi(\alpha)$, as required. □

Lemma 6.7.9. *Let* P_1, P_2 *be opposite modules for the* log-*cTEP structure* $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$, *and let* $\Pi_i \colon \mathsf{F}[z^{-1}] \to \mathsf{F}$ *be the projection along* P_i, $i \in \{1,2\}$. *Let* $V = (\Pi_1 \otimes \Pi_2)\Omega^\vee$. *Then* $(\mathrm{gr} \otimes 1 + 1 \otimes \mathrm{gr})V = -V$.

Proof. Combine Lemma 6.6.7 and Lemma 6.7.8:

$$\begin{aligned}(\mathrm{gr}\otimes 1 + 1 \otimes \mathrm{gr})(\Pi_1 \otimes \Pi_2)\Omega^\vee &= (\Pi_1 \otimes \Pi_2)(\mathrm{gr}\otimes 1 + 1 \otimes \mathrm{gr})\Omega^\vee \\ &= -(\Pi_1 \otimes \Pi_2)\Omega^\vee = -V\end{aligned}$$

□

6.8. **The Fock Sheaf.** Consider a miniversal log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ with base $(\mathcal{M}, D)$. As before, let $\{t^j, q_k, x^i_n\}$ be an algebraic local co-ordinate system on L (see Definition 6.2.2) where $\{t^j, q_k\}$ are co-ordinates on an open set $U \subset \mathcal{M}$. Write the co-ordinates $\{t^0, \log q_1, \dots, \log q_r, t^{r+1}, \dots, t^N, x^i_n\}$ as $\{\mathsf{x}^\mu\}$, so that:

$$\begin{cases} d\mathsf{x}^\mu = \frac{dq_j}{q_j} \text{ and } \frac{\partial}{\partial \mathsf{x}^\mu} = q_j \frac{\partial}{\partial q_j} & \text{if } \mathsf{x}^\mu = \log q_j \text{ and } 1 \le j \le r \\ d\mathsf{x}^\mu = dt^j \text{ and } \frac{\partial}{\partial \mathsf{x}^\mu} = \frac{\partial}{\partial t^j} & \text{if } \mathsf{x}^\mu = t^j \text{ and } j = 0 \text{ or } r < j \le N \\ d\mathsf{x}^\mu = dx^i_n \text{ and } \frac{\partial}{\partial \mathsf{x}^\mu} = \frac{\partial}{\partial x^i_n} & \text{if } \mathsf{x}^\mu = x^i_n \end{cases}$$

We use Einstein's summation convention for repeated indices, expressing the Yukawa coupling and propagator $\Delta = \Delta(\mathsf{P}_1, \mathsf{P}_2)$ as:

$$\boldsymbol{Y} = C^{(0)}_{\mu\nu\rho} d\mathsf{x}^\mu \otimes d\mathsf{x}^\nu \otimes d\mathsf{x}^\rho \qquad\qquad \Delta = \Delta^{\mu\nu}\partial_\mu \otimes \partial_\nu$$

where $\partial_\mu := \frac{\partial}{\partial \mathsf{x}^\mu}$. Let P be an opposite module for $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ and consider the flat connection $\boldsymbol{\nabla}$ on $\boldsymbol{\Omega}^1_\circ(\log D)$ determined by P (Definition 6.7.6). The Christoffel symbols of $\boldsymbol{\nabla}$ are defined by:

$$\boldsymbol{\nabla}_{\partial_\nu} d\mathsf{x}^\mu = -\Gamma^\mu_{\nu\rho} d\mathsf{x}^\rho$$

The flat connection $\boldsymbol{\nabla}$ acts on n-tensors $C_{\mu_1\cdots\mu_n} d\mathsf{x}^{\mu_1} \otimes \cdots \otimes d\mathsf{x}^{\mu_n} \in \big(\boldsymbol{\Omega}^1_\circ(\log D)\big)^{\otimes n}$ by:

$$\boldsymbol{\nabla}(C_{\mu_1\cdots\mu_n} d\mathsf{x}^{\mu_1} \otimes \cdots \otimes d\mathsf{x}^{\mu_n}) = (\boldsymbol{\nabla}_\nu C_{\mu_1\cdots\mu_n}) d\mathsf{x}^\nu \otimes d\mathsf{x}^{\mu_1} \otimes \cdots \otimes d\mathsf{x}^{\mu_n}$$

where:

$$(\boldsymbol{\nabla}_\nu C_{\mu_1\cdots\mu_n}) := \partial_\nu C_{\mu_1\cdots\mu_n} - \sum_{i=1}^n C_{\mu_1\cdots\underset{i}{\rho}\cdots\mu_n} \Gamma^\rho_{\mu_i\nu} \tag{62}$$

Definition 6.8.1 (local Fock space)**.** The *local Fock space* $\mathfrak{Fock}(U;\mathsf{P})$ consists of collections:

$$\big\{ \boldsymbol{\nabla}^n C^{(g)} \in \big(\boldsymbol{\Omega}^1(\log D)\big)^{\otimes n}\big(\mathrm{pr}^{-1}(U)^\circ\big) : g \ge 0, n \ge 0, 2g-2+n > 0 \big\}$$

of completely symmetric logarithmic n-tensors on $\mathrm{pr}^{-1}(U)^\circ$ such that the following conditions hold:

- (Yukawa) $\boldsymbol{\nabla}^3 C^{(0)}$ is the Yukawa coupling $\boldsymbol{Y}$;
- (Jetness) $\boldsymbol{\nabla}(\boldsymbol{\nabla}^n C^{(g)}) = \boldsymbol{\nabla}^{n+1} C^{(g)}$;
- (Grading and Filtration) $\boldsymbol{\nabla}^n C^{(g)} \in \big(\big(\boldsymbol{\Omega}^1(\log D)\big)^{\otimes n}\big(\mathrm{pr}^{-1}(U)^\circ\big)\big)^{2-2g}_{3g-3}$;
- (Pole) $P\boldsymbol{\nabla} C^{(1)}$ extends to a regular 1-form on $\mathrm{pr}^{-1}(U)$, where P is the discriminant (58). Furthermore for $g \ge 2$ we have:

$$C^{(g)} \in P^{5-5g}\mathcal{O}(U)[x_1, x_2, Px_3, \dots, P^{3g-4}x_{3g-2}]$$

Writing:

$$\boldsymbol{\nabla}^n C^{(g)} = C^{(g)}_{\mu_1 \cdots \mu_n} d\mathsf{x}^{\mu_1} \otimes \cdots \otimes d\mathsf{x}^{\mu_n}$$

we refer to $\boldsymbol{\nabla}^n C^{(g)}$ or $C^{(g)}_{\mu_1 \cdots \mu_n}$ as *n-point correlation functions.*

We encode elements of the local Fock space $\mathfrak{Fock}(U;\mathsf{P})$ as formal functions on the total space of the logarithmic tangent bundle $\boldsymbol{\Theta}(\log D)|_{\mathrm{pr}^{-1}(U)^\circ}$, called jet potentials. Let $\{\mathsf{y}^\mu\}$ denote the fiber co-ordinates of the logarithmic tangent bundle $\boldsymbol{\Theta}(\log D)$ dual to $\{\frac{\partial}{\partial \mathsf{x}^\mu}\}$, so that (x,y) denotes a point in the total space of $\boldsymbol{\Theta}(\log D)|_{\mathrm{pr}^{-1}(U)^\circ}$.

Definition 6.8.2 (jet potential)**.** Given an element $\mathscr{C} = \{\boldsymbol{\nabla}^n C^{(g)}\}_{g,n}$ of $\mathfrak{Fock}(U;\mathsf{P})$, set:

$$\begin{aligned} \mathcal{W}^g(\mathsf{x},\mathsf{y}) &= \sum_{n=\max(0,3-2g)}^{\infty} \frac{1}{n!} C^{(g)}_{\mu_1,\dots,\mu_n}(\mathsf{x}) \mathsf{y}^{\mu_1} \cdots \mathsf{y}^{\mu_n} \\ \mathcal{W}(\mathsf{x},\mathsf{y}) &= \sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{W}^g(\mathsf{x},\mathsf{y}) \end{aligned} \tag{63}$$

We call $\mathcal{W}^g$ the *genus-g jet potential* and $\exp(\mathcal{W})$ the *total jet potential* associated to $\mathscr{C}$.

Remark 6.8.3. $\exp(\mathcal{W})$ is well-defined as a power series in $\hbar$ and $\hbar^{-1}$: cf. [25, Remark 4.63(2)].

The Fock sheaf is constructed by gluing local Fock spaces $\mathfrak{Fock}(U;\mathsf{P}_1)$, $\mathfrak{Fock}(U;\mathsf{P}_2)$ according to the following *transformation rule*. Let Δ denote the propagator $\Delta(\mathsf{P}_1,\mathsf{P}_2)$. The transformation rule $T(\mathsf{P}_1,\mathsf{P}_2)\colon \mathfrak{Fock}(U;\mathsf{P}_1) \to \mathfrak{Fock}(U;\mathsf{P}_2)$ is a map which assigns to a jet potential $\exp(\mathcal{W})$ for an element of $\mathfrak{Fock}(U;\mathsf{P}_1)$, the jet potential $\exp(\widehat{\mathcal{W}})$ for an element of $\mathfrak{Fock}(U;\mathsf{P}_2)$ given by:

$$\exp\big(\widehat{\mathcal{W}}(\mathsf{x},\mathsf{y})\big) = \exp\left(\frac{\hbar}{2}\Delta^{\mu\nu}\partial_{\mathsf{y}^\mu}\partial_{\mathsf{y}^\nu}\right)\exp\big(\mathcal{W}(\mathsf{x},\mathsf{y})\big). \tag{64}$$

This is equivalent to expressing the correlation functions $\{\widehat{C}^{(g)}_{\mu_1,\dots,\mu_n}\}_{g,n}$ for $\widehat{\mathcal{W}}$ in terms of sums over Feynman graphs, the vertex terms of which are the correlation functions $\{C^{(g)}_{\mu_1,\dots,\mu_n}\}_{g,n}$ for $\mathcal{W}$. We use the notation for graphs established in Appendix B. The transformation rule (64) is equivalent to the *Feynman rule*:

$$\widehat{C}^{(g)}_{\mu_1,\dots,\mu_n} = \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}\}_{h\le g})_{\mu_1,\dots,\mu_n}$$

Here the summation is over all connected decorated graphs Γ such that

- To each vertex $v \in V(\Gamma)$ is assigned a non-negative integer $g_v \ge 0$, called genus;
- Γ has labelled n-legs: an isomorphism $L(\Gamma) \cong \{1,2,\dots,n\}$ is given;
- Γ is stable, i.e. $2g_v - 2 + n_v > 0$ for every vertex v. Here $n_v = |\pi_V^{-1}(v)|$ denotes the number of edges or legs incident to v;
- $g = \sum_v g_v + 1 - \chi(\Gamma)$.

We put the index μ_i on the ith leg, the correlation function $\boldsymbol{\nabla}^{n_v} C^{(g_v)}$ on the vertex v, and the propagator Δ on every edge. Then $\operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}\}_{h\le g})_{\mu_1,\dots,\mu_n}$ is defined to be the contraction of all these tensors with the indices $\mu_1,\dots,\mu_n$ on the legs fixed. Here $\operatorname{Aut}(\Gamma)$ denotes the automorphism group of the decorated graph Γ.

Remark 6.8.4. We showed in [25, Proposition 4.115] that the transformation rule (64) is well-defined, i.e. that it preserves the conditions (Yukawa), (Jetness), (Grading and Filtration), and (Pole) in the definition of the local Fock space $\mathfrak{Fock}(U;\mathsf{P}_i)$.

Remark 6.8.5. The transformation rule (64) satisfies the cocycle condition [25, Proposition 4.111] : if P_1, P_2, P_3 are opposite modules for F over U and $T_{ij} = T(\mathsf{P}_i, \mathsf{P}_j)$ is the transformation rule from $\mathfrak{Fock}(U;\mathsf{P}_i)$ to $\mathfrak{Fock}(U;\mathsf{P}_j)$ then $T_{13} = T_{23} \circ T_{12}$.

Assumption 6.8.6 (Covering Assumption)**.** *There is an open covering $\{U_a : a \in A\}$ of $\mathcal{M}$ such that for each $a \in A$ there exists an opposite module P_a for F over U_a.*

Definition 6.8.7 (Fock sheaf)**.** If Assumption 6.8.6 holds, then we define the *Fock sheaf* to be the sheaf of sets on $\mathcal{M}$ obtained by gluing the local Fock spaces $\mathfrak{Fock}(U_a;\mathsf{P}_a)$, $a \in A$, using the transformation rule

$$T(\mathsf{P}_a, \mathsf{P}_b) : \mathfrak{Fock}(U_a \cap U_b; \mathsf{P}_a) \to \mathfrak{Fock}(U_a \cap U_b; \mathsf{P}_b) \qquad a, b \in A$$

over $U_a \cap U_b$.

Remark 6.8.8. Note that the Fock sheaf is a sheaf over all of $\mathcal{M}$, not just over $\mathcal{M} \setminus D$.

Remark 6.8.9. We can define the Fock sheaf without the covering assumption: see [25, §4.13]. The definition there requires an analysis of anomaly equations for curved (i.e. non-parallel) opposite modules.

Definition 6.8.10 (Gromov–Witten wave function)**.** Let X denote either $\overline{\mathcal{X}}$ or $\overline{Y}$, and consider the A-model log-cTEP structure for X defined in Example 6.1.5. The base of this log-cTEP structure is $(\mathcal{M}_{\mathrm{A},X}, D_{\mathrm{A},X})$, and we denote the corresponding Fock sheaf on $\mathcal{M}_{\mathrm{A},X}$ by $\mathfrak{Fock}_{\mathrm{A},X}$. The Gromov–Witten ancestor potentials of X define a global section $\mathscr{C}_X$ of $\mathfrak{Fock}_{\mathrm{A},X}$, the *Gromov–Witten wave function*, as we now explain.

Let $\{\phi_i\}_{i=0}^N$ be a homogeneous basis of H_X as in §2.1, and write a general point $t \in \mathcal{M}_{\mathrm{A},X}$ as $t = \sum_{i=0}^N t^i \phi_i$. Recalling that $\phi_1, \dots, \phi_r$ form a basis for $H^2(X)$, set $q_i = e^{t^i}$, $1 \le i \le r$, and write $\{t^j, q_k, x_n^i\}$ for the corresponding algebraic local co-ordinate system on the total space L of the A-model log-cTEP structure. Let P_{A} denote the canonical opposite module defined in Example 6.7.5. The Gromov–Witten wave-function $\mathscr{C}_X$ is defined by the element $\{\boldsymbol{\nabla}^n C_X^{(g)}\}_{g,n} \in \mathfrak{Fock}_{\mathrm{A}}(\mathcal{M}_{\mathrm{A},X}; \mathsf{P}_{\mathrm{A}})$ where:

$$\begin{aligned}
\boldsymbol{\nabla}^3 C_X^{(0)} &= \boldsymbol{Y} = \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N dt^i \otimes dt^j \otimes dt^k \Big(\phi_i * \phi_j * x_1, \phi_k * x_1 \Big) \\
\boldsymbol{\nabla} C_X^{(1)} &= d(F_X^1(t) + \bar{\mathcal{F}}_X^1)\Big|_{a_0=0, Q_1=\dots=Q_r=1} \\
C_X^{(g)} &= \bar{\mathcal{F}}_X^g\Big|_{a_0=0, Q_1=\dots=Q_r=1} \qquad \text{for } g \ge 2
\end{aligned} \tag{65}$$

and $\boldsymbol{\nabla}$ denotes the covariant derivative $\boldsymbol{\nabla}^{\mathsf{P}_{\mathrm{A}}}$ from Definition 6.7.6. Here $*$ is the quantum product (9), $\bar{\mathcal{F}}_X^g$ is the genus-g ancestor potential (17), $F_X^1(t)$ is the non-descendant genus-one Gromov–Witten potential:

$$F_X^1(t) = \sum_{n=0}^\infty \sum_{\substack{d \in \mathrm{NE}(X) \\ (n,d) \neq (0,0)}} \frac{Q^d}{n!} \langle t, \dots, t \rangle_{1,n,d}$$

and we used the Dilaton shift:

$$a_n^i = x_n^i + \delta_n^1 \delta_0^i \qquad n \ge 1 \tag{66}$$

to identify the variables $\{t^i, q_k, a_n^i\}$ on the right-hand side with the co-ordinates $\{t^i, q_k, x_n^i\}$ on L. Our convergence results in [23] imply that the Gromov–Witten wave-function is well-defined – that is, that the specialization $Q_1 = \dots = Q_r = 1$ in (65) makes sense and yields

an analytic function, and the resulting correlation functions satisfy the conditions (Yukawa), (Jetness), (Grading & Filtration) and (Pole). See [25, Section 6] for details.

Remark 6.8.11 ([25, Theorem 6.8])**.** The Ancestor–Descendant relation [64, Theorem 2.1], [42, §5] implies that for $t \in \mathcal{M}_{\mathrm{A},X}$ sufficiently close to the large-radius limit point for X and $\boldsymbol{x}$ sufficiently close to $-1z$, there are flat co-ordinates $\mathbf{q} = (q_n^i)$ on a neighbourhood of $(t, \boldsymbol{x})$ in L such that:

$$\begin{aligned} \boldsymbol{\nabla}^3 C_X^{(0)} = \boldsymbol{Y} &= \sum_{l,m,n=0}^{\infty} \sum_{i,j,k=0}^{N} \frac{\partial^3 \mathcal{F}_X^0}{\partial q_l^i \partial q_m^j \partial q_n^k}(\mathbf{q})\, dq_l^i \otimes dq_m^j \otimes dq_n^k \\ \boldsymbol{\nabla} C_X^{(1)} &= d\mathcal{F}_X^1(\mathbf{q}) \\ C_X^{(g)} &= \mathcal{F}_X^g(\mathbf{q}) \qquad \text{for } g \geq 2. \end{aligned}$$

Here we regard the genus-g descendant potential $\mathcal{F}_X^g$, which was defined in §2.6 as a function of variables t_n^i, as a function of q_n^i via the Dilaton Shift $t_n^i = q_n^i + \delta_n^1 \delta_0^i$. The flat co-ordinates $\mathbf{q}$ and the algebraic co-ordinates $(t, \boldsymbol{x})$ are related by

$$\mathbf{q}(z) = \big[L(t,-z)^{-1} \boldsymbol{x}(z)\big]_+$$

where $L(t,-z)$ is the fundamental solution (12), $[\cdots]_+$ denotes the non-negative part as a z-series, $\mathbf{q}(z) = \sum_{n=0}^{\infty} q_n z^n$, $\boldsymbol{x}(z) = \sum_{n=1}^{\infty} x_n z^n$, $q_n = \sum_{i=0}^{N} q_n^i \phi_i$, and $x_n = \sum_{i=0}^{N} x_n^i \phi_i$. Thus one can think of the Gromov–Witten wave function $\mathscr{C}_X$ as encoding the total descendant potential $\mathcal{Z}_X$ of X.

6.9. **A Global Section of the Fock Sheaf for the Big B-Model** log**-cTEP Structure.** We now construct a global section of the Fock sheaf for the big B-model log-cTEP structure. This global section coincides under mirror symmetry with the Gromov–Witten wave functions $\mathscr{C}_{\overline{Y}}$ and $\mathscr{C}_{\overline{\mathcal{X}}}$.

Proposition 6.9.1. *The Covering Assumption (Assumption 6.8.6) holds for the big B-model* log*-cTEP structure.*

Proof. Let $y \in \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ be a point of $\mathcal{M}_{\mathrm{B}}^{\circ} \subset \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. In §5 we constructed, for a sufficiently small neighbourhood U_y^{sm} of y in $\mathcal{M}_{\mathrm{B}}$, an opposite module $\mathbf{P}_y^{\mathrm{sm}}$ for the B-model TEP structure on $U_y^{\mathrm{sm}} \setminus D$ which is compatible with the Deligne extension. After shrinking U_y^{sm} if necessary, we may assume that $U_y^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{B}}^{\circ}$. By the construction of $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ in §5, the opposite module $\mathbf{P}_y^{\mathrm{sm}}$ extends to an opposite module $\mathbf{P}_y$ for the big B-model TEP structure on $U_y \setminus D^{\mathrm{big}}$ which is compatible with the Deligne extension, for some neighbourhood U_y of y in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. Recall that $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ was constructed as the germ of a thickening of $\mathcal{M}_{\mathrm{B}}^{\circ}$; after shrinking $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ if necessary we may assume that the open sets $\{U_y : y \in \mathcal{M}_{\mathrm{B}}^{\circ}\}$ just constructed form an open covering of $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. By Example 6.7.3, the opposite module $\mathbf{P}_y$ over U_y determines an opposite module P_y for the big B-model log-cTEP structure over U_y. Thus Assumption 6.8.6 holds for the big B-model log-cTEP structure. □

Definition 6.9.2. In view of Proposition 6.9.1, there is a Fock sheaf on $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ determined by the big B-model log-cTEP structure. We denote this by $\mathfrak{Fock}_{\mathrm{B}}$.

Definition 6.9.3. Recall the definition of $\mathcal{C}(t,z)$ from equation (57). We say that a log-cTEP structure $(\mathsf{F}, \nabla, (\cdot,\cdot)_{\mathsf{F}})$ with Euler field E and base $(\mathcal{M}, D)$ is *tame semisimple* at $t \in \mathcal{M}$ if the endomorphism $\iota_E \mathcal{C}(t,0) \in \mathrm{End}_{\mathbb{C}}(\mathsf{F}_{0,t})$ is semisimple with pairwise distinct eigenvalues. This endomorphism is "multiplication by the Euler field" and coincides with the action of $\nabla_{z^2\partial_z}$ on

$\mathsf{F}_{0,t}$. Tame semisimplicity implies that all operators $\iota_v\mathcal{C}(t,0) \in \mathrm{End}(\mathsf{F}_{0,t})$ with $v \in T\mathcal{M}(\log D)$ are semisimple.

Consider now the big B-model log-cTEP structure $(\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ with base $(\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}, D^{\mathrm{big}})$. Let U_{ss} denote the set of points at which this log-cTEP structure is tame semisimple. The complement of U_{ss} in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ is a union of divisors $\{B_i : i \in I\}$ and, after shrinking the thickening $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ of $\mathcal{M}_{\mathrm{B}}^{\circ}$ if necessary, we may insist that each irreducible component B_i meets $\mathcal{M}_{\mathrm{B}}^{\circ} \subset \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. The critical values of the superpotential W_y are distinct for $y \in \mathcal{M}_{\mathrm{B}}^{\times} = \mathcal{M}_{\mathrm{B}}^{\circ} \setminus D$ (see equation (85)), and so the tame semisimple locus U_{ss} contains $\mathcal{M}_{\mathrm{B}}^{\times}$. This implies that, for each $i \in I$, the intersection of the divisor B_i with $\mathcal{M}_{\mathrm{B}}^{\circ}$ either contains the component $(y_1 = 0)$ or the component $(y_2 = 0)$ of $D \cap \mathcal{M}_{\mathrm{B}}^{\circ}$. In particular, each divisor B_i contains the large-radius limit point $y_1 = y_2 = 0$. Moreover, we can also see that U_{ss} does not intersect with D^{big}. In fact, by our local construction of $\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}$ in §5.2, the residues of $z\nabla$ along D^{big} define nilpotent operators (see Proposition A.0.4) in $\mathrm{End}(\mathsf{F}_{\mathrm{B},0}^{\mathrm{big}})$, which are non-zero by miniversality. Therefore a point on D^{big} cannot be tame semisimple[12].

In previous work we have shown – see [25, Definition 7.9] – that Givental's formula [41] for higher-genus potentials defines a section $\mathscr{C}_{\mathrm{ss}}$ of the B-model Fock sheaf $\mathfrak{Fock}_{\mathrm{B}}$ over the tame semisimple locus $U_{\mathrm{ss}} \subset \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. The mirror isomorphism of log-TEP structures from Theorem 5.4.1:

$$\big(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}}\big)\Big|_{U^{\mathrm{big}}\times\mathbb{C}} \cong \mathrm{Mir}^{\star}\left(\mathcal{F}_{\mathrm{A},\overline{Y}}, \nabla^{\mathrm{A},\overline{Y}}, (\cdot,\cdot)_{\mathrm{A},\overline{Y}}\right)$$

induces an isomorphism of Fock sheaves

$$\mathfrak{Fock}_{\mathrm{B}}\big|_{U^{\mathrm{big}}} \cong \mathrm{Mir}^{\star}\,\mathfrak{Fock}_{\mathrm{A},\overline{Y}}$$

and Teleman's theorem [80] implies that, under this isomorphism, $\mathscr{C}_{\mathrm{ss}}$ corresponds to the Gromov–Witten wave function $\mathrm{Mir}^{\star}\mathscr{C}_{\overline{Y}}$ (see Definition 6.8.10) over $U_{\mathrm{ss}} \cap U^{\mathrm{big}}$. (This is explained in detail in [25, Theorem 7.15].) The same is true when we replace $\overline{Y}$ with $\overline{\mathcal{X}}$ and work near the orbifold point, which is the large-radius limit point for $\overline{\mathcal{X}}$. We obtain the following:

Theorem 6.9.4. *After shrinking the thickening $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ of $\mathcal{M}_{\mathrm{B}}^{\circ}$ if necessary, there exists a global section $\mathscr{C}_{\mathrm{B}}$ of $\mathfrak{Fock}_{\mathrm{B}}$ over $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ extending $\mathscr{C}_{\mathrm{ss}}$ such that the following holds:*

(a) *near the large radius limit point for $\overline{Y}$, $\mathscr{C}_{\mathrm{B}}$ corresponds to the Gromov–Witten wave function of $\overline{Y}$ under the mirror isomorphism in Theorem 5.4.1;*
(b) *near the large radius limit point for $\overline{\mathcal{X}}$, $\mathscr{C}_{\mathrm{B}}$ corresponds to the Gromov–Witten wave function of $\overline{\mathcal{X}}$ under the mirror isomorphism in Remark 5.4.3.*

Proof. In view of our discussion, it suffices to show that the section $\mathscr{C}_{\mathrm{ss}}$ extends holomorphically across the divisors B_i, $i \in I$. The divisors B_i, $i \in I$, all meet the open set U^{big}. By Hartog's Principle, it suffices to check that the correlation functions for $\mathscr{C}_{\mathrm{ss}}$ with respect to one opposite module extend to holomorphic functions on all of U^{big}. We check this using the opposite module P_{LR} from Example 6.7.4; under mirror symmetry, this corresponds to the canonical opposite module P_{A} from Example 6.7.5 (see Theorem 5.4.1) and $\mathscr{C}_{\mathrm{ss}}$ corresponds to $\mathscr{C}_{\overline{Y}}$. But the correlation functions (65) for $\mathscr{C}_{\overline{Y}}$ are evidently holomorphic on all of $\mathrm{Mir}(U^{\mathrm{big}})$. □

[12]We can also check that this holds for the big quantum cohomology of $\overline{Y}$: the quantum product $h_i\star$ coincides with the nilpotent operator $h_i\cup$ along $q_i = 0$ because of the Divisor Equation.

Remark 6.9.5. The existence of a global section $\mathscr{C}_{\mathrm{B}}$ with these properties establishes a higher-genus version of the Crepant Resolution Conjecture for $\overline{\mathcal{X}} = \mathbb{P}(1,1,1,3)$. See Theorem 8.1 and Corollary 8.2 in [25] for a more general result for weak-Fano toric orbifolds.

7. The Finite-Dimensional Fock Sheaf

In this section we construct a finite-dimensional version of the Fock sheaf, which one can think of as arising from the big B-model Fock sheaf by taking the conformal limit. Recall from §4 that we have a three-dimensional vector bundle $\overline{H} \to \mathcal{M}_{\mathrm{CY}}$ equipped with a logarithmic flat connection ∇, a two-dimensional flat subbundle H_{vec} of $\overline{H}$, a two-dimensional flat affine subbundle H_{aff} of $\overline{H}$, and a distinguished section ζ of H_{aff}. There is a canonical identification, for each $y \in \mathcal{M}_{\mathrm{CY}}$, between any tangent space to $H_{\mathrm{aff}}|_y$ and the fiber $H_{\mathrm{vec}}|_y$, so H_{aff} is parallel to H_{vec}; H_{aff} is a symplectic affine bundle, H_{vec} is a symplectic vector bundle, and this identification between H_{aff} and H_{vec} intertwines the symplectic structures. The base $\mathcal{M}_{\mathrm{CY}}$ of $\overline{H}$, H_{aff}, and H_{vec} is isomorphic to $\mathbb{P}(3,1)$, and the flat connection ∇ has logarithmic poles at the divisor $D_{\mathrm{CY}} = \{0, -\frac{1}{27}\}$. The finite-dimensional Fock sheaf that we will construct has base $\mathcal{M}_{\mathrm{CY}}$.

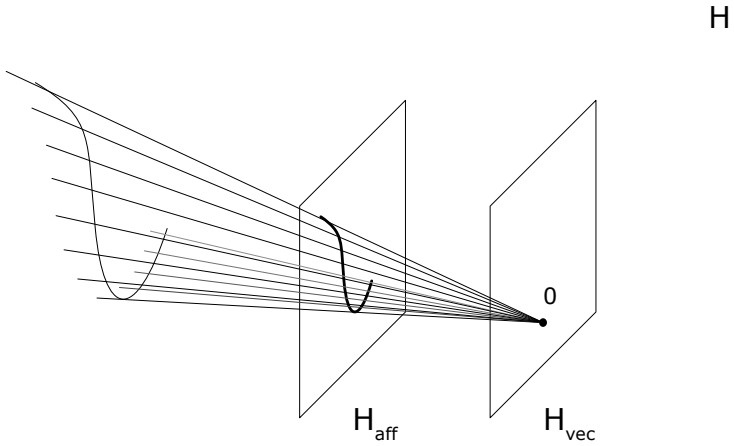


FIGURE 4. The finite-dimensional cone $\widehat{\mathcal{L}}$: the primitive section ζ sweeps out a Lagrangian curve $\mathcal{L} \subset H_{\mathrm{aff}}|_{t_0}$ via parallel translation to the fibre at t_0.

7.1. The Yukawa Coupling and the Kodaira–Spencer Map.

Notation 7.1.1. We denote by $\Theta(\log D_{\mathrm{CY}})$ the sheaf of tangent vector fields on $\mathcal{M}_{\mathrm{CY}}$ logarithmic along D_{CY} and by $\Omega^1(\log D_{\mathrm{CY}})$ the sheaf of 1-forms on $\mathcal{M}_{\mathrm{CY}}$ logarithmic along D_{CY}. Similarly, we denote by $\Theta(\log\{0\})$ (respectively $\Omega^1(\log\{0\})$) the sheaf of tangent vector fields (respectively 1-forms) logarithmic only at $0 \in D_{\mathrm{CY}}$.

Definition 7.1.2 (cf. Definition 6.5.1)**.** The *Yukawa coupling* $Y_{\mathrm{CY}} \in \big(\Omega^1(\log D_{\mathrm{CY}})\big)^{\otimes 3}$ is defined by:

$$Y_{\mathrm{CY}}(X_1, X_2, X_3) = \Omega\big(\nabla_{X_1}\nabla_{X_2}\zeta, \nabla_{X_3}\zeta\big) \qquad X_1,\, X_2,\, X_3 \in \Theta(\log D_{\mathrm{CY}})$$

Here we regard $\nabla_{X_3}\zeta$ and $\nabla_{X_1}\nabla_{X_2}\zeta$ as sections of H_{vec}, via the identification of tangent spaces to H_{aff} with fibers of H_{vec} discussed above.

Definition 7.1.3 (Definition 6.5.4)**.** The *Kodaira–Spencer map* is:

$$\begin{aligned} \mathrm{KS}\colon \Theta(\log\{0\}) &\to \mathcal{O}(H_{\mathrm{vec}}) \\ X &\mapsto \nabla_X \zeta \end{aligned}$$

Remark 7.1.4. The Kodaira–Spencer map gives an isomorphism between the logarithmic tangent bundle $\Theta(\log\{0\})$ of $(\mathcal{M}_{\mathrm{CY}}, \{0\})$ and the subbundle $F^2_{\mathrm{vec}} \subset H_{\mathrm{vec}}$ defined in §4.3.

Remark 7.1.5. From the previous remark, it follows that the Yukawa coupling has a pole of order 3 at the large-radius limit point $y_1 = 0$ and a pole of order 1 at the conifold point $y_1 = -\frac{1}{27}$. We give an explicit formula for Y_{CY} in Example 8.1.2.

Let $t_0 \in \mathcal{M}_{\mathrm{CY}}$ be a point away from D_{CY}. Locally near t_0, we can encode the information of the filtered flat bundle $(\overline{H}, \overline{F}^1 \subset \overline{F}^2 \subset \overline{H}, \nabla)$ discussed in §4.3 as a finite-dimensional cone $\widehat{\mathcal{L}}$ in $\overline{H}|_{t_0}$. Parallel translation defines an isomorphism $\overline{H}|_t \cong \overline{H}|_{t_0}$ for t in a small neighbourhood of t_0. Via this isomorphism, the flag $(0 \subset \overline{F}^1_t \subset \overline{F}^2_t \subset \overline{H}|_t)$ can be identified with a flag[13] in $\overline{H}|_{t_0}$. With this identification in mind, we define the finite dimensional cone $\widehat{\mathcal{L}} \subset \overline{H}|_{t_0}$ to be:

$$\widehat{\mathcal{L}} = \bigcup_t \overline{F}^1_t$$

where t varies in a neighbourhood of t_0. See Figure 4. Recall from §4.3 that $\overline{F}^1_t$ is a line generated by the primitive section $\zeta = -z$, that $\overline{F}^2_t$ is generated by ζ and $\theta\zeta$, and that $\theta\zeta = \nabla_{y_1 \frac{\partial}{\partial y_1}} \zeta$. The tangent space of $\widehat{\mathcal{L}}$ along the line $\overline{F}^1_t$ is therefore $\overline{F}^2_t$. Recall also that ζ lies in the affine subbundle H_{aff}. Under the above identification, $t \mapsto \zeta(t)$ sweeps out a one-dimensional Lagrangian submanifold $\mathcal{L}$ in $H_{\mathrm{aff}}|_{t_0}$:

$$\mathcal{L} = \widehat{\mathcal{L}} \cap H_{\mathrm{aff}}|_{t_0} = \{\zeta(t) : t \text{ in a neighbourhood of } t_0\}.$$

In other words, $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ can be locally embedded into a fiber $H_{\mathrm{aff}}|_{t_0}$ as a Lagrangian submanifold. The tangent space of $\mathcal{L}$ at $\zeta(t)$ is identified with $F^2_{\mathrm{vec}}|_t \subset H_{\mathrm{vec}}|_t \cong T_{\zeta(t)} H_{\mathrm{aff}}|_t$. By the same construction, we can realize the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ as an immersed Lagrangian submanifold in $H_{\mathrm{aff}}|_{t_0}$.

7.2. **Opposite Line Bundles and Propagators.**

Definition 7.2.1 (cf. Definitions 2.8.5, 6.7.1)**.** Let U be an open subset of $\mathcal{M}_{\mathrm{CY}}$. An *opposite line bundle* over U is a one-dimensional subbundle P of $H_{\mathrm{vec}}|_U$ such that

(1) P is flat, i.e. $\nabla\mathcal{O}(P) \subset \Omega^1(\log D_{\mathrm{CY}}) \otimes \mathcal{O}(P)$;
(2) for each $t \in U$, we have $P_t \oplus F^2_{\mathrm{vec}}|_t = H_{\mathrm{vec}}|_t$.

Definition 7.2.2 (Flat connection $\boldsymbol{\nabla}^P$, cf. Definition 6.7.6)**.** Let P be an opposite line bundle over $U \subset \mathcal{M}_{\mathrm{CY}}$. Let $\Pi\colon H_{\mathrm{vec}} \to H_{\mathrm{vec}}/P \cong F^2_{\mathrm{vec}}$ denote the projection along P. The flat connection ∇ on H_{vec} induces a flat connection on F^2_{vec} over U:

$$\mathcal{O}(F^2_{\mathrm{vec}}) \xrightarrow{\nabla} \Omega^1(\log D_{\mathrm{CY}}) \otimes \mathcal{O}(H_{\mathrm{vec}}) \xrightarrow{\mathrm{id}\otimes\Pi} \Omega^1(\log D_{\mathrm{CY}}) \otimes \mathcal{O}(F^2_{\mathrm{vec}}) \tag{67}$$

Composing this with the Kodaira-Spencer isomorphism $\Theta(\log\{0\}) \cong \mathcal{O}(F^2_{\mathrm{vec}})$, we obtain a logarithmic flat connection on $U \subset \mathcal{M}_{\mathrm{CY}}$:

$$\boldsymbol{\nabla}^P\colon \Theta(\log\{0\}) \to \Omega^1(\log D_{\mathrm{CY}}) \otimes \Theta(\log\{0\})$$

[13]The map $t \mapsto (0 \subset \overline{F}^1_t \subset \overline{F}^2_t \subset \overline{H}|_t) \in \mathrm{Fl}_{1,2,3}(\overline{H}|_{t_0})$ can be viewed as a period map.

We denote the dual connection

$$\boldsymbol{\nabla}^P \colon \Omega^1(\log\{0\}) \to \Omega^1(\log D_{\mathrm{CY}}) \otimes \Omega^1(\log\{0\})$$

by the same symbol.

Remark 7.2.3. The description of H_{vec} in §4.2 shows that the residue endomorphism N of ∇ at a point $t_0 \in D_{\mathrm{CY}}$ is nilpotent. Monodromy invariance then forces that an opposite line bundle P around t_0 is unique and has $\operatorname{Im} N$ as the fiber at t_0; such a line bundle will be denoted by P_{LR} for $t_0 = 0$ and by P_{con} for $t_0 = -\frac{1}{27}$ – see Notation 7.2.7 below. Therefore the connection (67) has no logarithmic singularities along D_{CY} for such P, i.e. gives a map

$$\mathcal{O}(F^2_{\mathrm{vec}}) \to \Omega^1 \otimes \mathcal{O}(F^2_{\mathrm{vec}})$$

Consequently, the connection $\boldsymbol{\nabla}^P$ gives a map:

$$\boldsymbol{\nabla}^P \colon \Omega^1(\log\{0\}) \to \Omega^1 \otimes \Omega^1(\log\{0\}) \subset \Omega^1(\log\{0\})^{\otimes 2}.$$

In particular, a flat co-ordinate associated with $\boldsymbol{\nabla}^{P_{\mathrm{con}}}$ is *holomorphic* at the conifold point, whereas a flat co-ordinate for $\boldsymbol{\nabla}^{P_{\mathrm{LR}}}$ is *logarithmic* at the large-radius limit point. Note however that the connection (67) can have poles along D_{CY} if we do not require the opposite line bundle P to be flat (see §7.4 for curved opposite line bundles).

Recall from the previous section that a neighbourhood of $t_0 \in \mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ can be embedded into $H_{\mathrm{aff}}|_{t_0}$ as a Lagrangian submanifold $\mathcal{L}$. Choose affine Darboux co-ordinates (p, x) on $H_{\mathrm{aff}}|_{t_0}$ such that $\partial/\partial p$ is parallel to P_{t_0} and that $\Omega = \frac{1}{3} dp \wedge dx$. The fact that P is an opposite line bundle implies that $P_{t_0} = \langle \partial/\partial p \rangle$ is transversal to the tangent space $T_{\zeta(t)}\mathcal{L} = F^2_{\mathrm{vec}}|_t$ (note that P_t is independent of t when transported to the fiber $H_{\mathrm{vec}}|_{t_0}$). Therefore $\mathcal{L}$ can be written as the graph of a function (see Figure 5), $p = p(x)$. We may regard a function $\mathcal{F}^0_{\mathrm{B}}(x)$ satisfying

$$p(x) = 3\frac{\partial \mathcal{F}^0_{\mathrm{B}}}{\partial x}$$

as a "genus-zero potential" for the B-model; this depends on the choice of P. The co-ordinate x restricted to $\mathcal{L} \subset H_{\mathrm{aff}}|_{t_0}$ defines an affine flat co-ordinate with respect to $\boldsymbol{\nabla}^P$. More invariantly, the affine flat structure is given by the projection along the linear foliation P_{t_0}:

$$\mathcal{L} \subset H_{\mathrm{aff}}|_{t_0} \longrightarrow H_{\mathrm{aff}}|_{t_0}/P_{t_0}$$

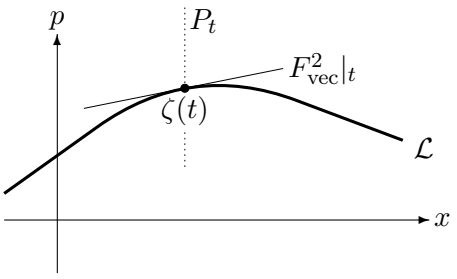


FIGURE 5. Writing $\mathcal{L} \subset H_{\mathrm{aff}}|_{t_0}$ as a graph of $p = p(x)$: Darboux co-ordinates (p, x) are chosen so that $P_{t_0} = \langle \partial/\partial p \rangle$; the co-ordinate x on $\mathcal{L}$ then defines the affine flat structure associated to P.

Example 7.2.4 (The Yukawa coupling in flat co-ordinates)**.** Let P be an opposite line bundle in a neighbourhood of $t_0 \in \mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$. As above, choose affine Darboux co-ordinates (p, x) of $H_{\mathrm{aff}}|_{t_0}$ such that $P_{t_0} = \langle \partial/\partial p \rangle$ and that $\Omega = \frac{1}{3} dp \wedge dx$, and write $\mathcal{L}$ as the graph of a function $p = p(x)$. We set:

$$\tau = \frac{\partial p}{\partial x} \tag{68}$$

Then the Yukawa coupling is given by:

$$\begin{aligned} Y_{\mathrm{CY}} &= Y_{\mathrm{CY}}(\partial_x, \partial_x, \partial_x) dx^{\otimes 3} = \Omega(\nabla_{\partial_x} \nabla_{\partial_x} \zeta, \nabla_{\partial_x} \zeta) dx^{\otimes 3} \\ &= \Omega\left(\begin{pmatrix} \frac{\partial \tau}{\partial x} \\ 0 \end{pmatrix}, \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right) dx^{\otimes 3} = \frac{1}{3} \frac{\partial \tau}{\partial x} dx^{\otimes 3} \end{aligned}$$

Remark 7.2.5. This calculation shows that the Yukawa coupling is given by the 3rd derivative of a generating function $\mathcal{F}_{\mathrm{B}}^0$ for $\mathcal{L}$. Note that the Yukawa coupling is independent of the choice of P whereas $\mathcal{F}_{\mathrm{B}}^0$ depends on P. For Givental's infinite-dimensional Lagrangian cone, this is explained in [28, §6.1; 37, §6].

Proposition 7.2.6. *For every $t \in \mathcal{M}_{\mathrm{CY}}$, there exists an opposite line bundle P in a neighbourhood of t.*

Proof. When $t \in \mathcal{M}_{\mathrm{CY}}$ is not equal to the large-radius, conifold, or orbifold points, one can construct an opposite line bundle P on a neighbourhood U of t by choosing a one-dimensional subspace P_t of $H_{\mathrm{vec}}|_t$ that is opposite to F^2_{vec}, and then extending P_t to a line bundle over U by parallel translation.

When t is equal to the large radius, conifold, or orbifold points, we have a canonical choice for an opposite line bundle near t. As shown in Proposition 5.1.7, there are unique opposite modules $\mathbf{P}_{\mathrm{LR}}$, $\mathbf{P}_{\mathrm{con}}$, $\mathbf{P}_{\mathrm{orb}}$ for $\mathcal{F}_{\mathrm{B}}^{\times}$ near the large radius, conifold and orbifold points, which are compatible with the Deligne extension $\mathcal{F}_{\mathrm{B}}$. These opposite modules induce, via the correspondence in Proposition 4.3.1, opposite line bundles over neighbourhoods of the large radius, conifold and orbifold points respectively. □

Notation 7.2.7. We write P_{LR}, P_{con}, P_{orb} for the opposite line bundles in a neighbourhood of the large-radius, conifold and orbifold points (respectively) discussed in the proof above. The fibres of P_{LR}, P_{con}, P_{orb} at the respective limit points are (see Proposition 4.3.2):

$$\begin{aligned} P_{\mathrm{LR}}\big|_{y_1=0} &= \langle \theta^2 \zeta \rangle, \\ P_{\mathrm{con}}\big|_{y_1=-\frac{1}{27}} &= \langle (1 + 27 y_1) \theta^2 \zeta \rangle, \\ P_{\mathrm{orb}}\big|_{y_1=\infty} &= \langle z^{-1} \mathfrak{d}_1^2 \rangle = \langle \mathfrak{y}_1^{-2} \theta(\theta + \tfrac{1}{3}) \zeta \rangle \end{aligned}$$

These are unique opposite line bundles, respectively, around the large-radius, conifold and orbifold points: see Remark 7.2.3 and Proposition 10.3.2.

Definition 7.2.8 (cf. Definition 6.7.7)**.** Let P_1, P_2 be opposite line bundles over U, and let $\Pi_i \colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ be the projection along P_i, $i \in \{1, 2\}$. The *propagator* $\Delta = \Delta(P_1, P_2)$ is the logarithmic bivector field $\Delta \in \big(\Theta(\log\{0\})\big)^{\otimes 2}$ defined by:

$$\Delta := (\mathrm{KS} \otimes \mathrm{KS})^{-1} (\Pi_1 \otimes \Pi_2) \Omega^{\vee}$$

where $\Omega^{\vee} \in H_{\mathrm{vec}} \otimes H_{\mathrm{vec}}$ is the dual symplectic form on $H^{\vee}_{\mathrm{vec}}$.

Example 7.2.9 (The propagator in flat co-ordinates, cf. [25, Lemma 5.22])**.** Let P_1, P_2 be opposite line bundles over a neighbourhood U of $t_0 \in \mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$. We embed U into $H_{\mathrm{aff}}|_{t_0}$ as a Lagrangian curve $\mathcal{L}$ as above. Let (p, x) and (p', x') denote affine Darboux co-ordinates on $H_{\mathrm{aff}}|_{t_0}$ associated to P_1 and P_2 respectively as in Example 7.2.4, so that

$$P_1|_{t_0} = \left\langle \frac{\partial}{\partial p} \right\rangle, \quad P_2|_{t_0} = \left\langle \frac{\partial}{\partial p'} \right\rangle, \quad \Omega = \frac{1}{3} dp \wedge dx = \frac{1}{3} dp' \wedge dx'.$$

Then x and x' restricted to $\mathcal{L}$ give flat co-ordinates for P_1 and P_2 respectively. If

$$\begin{aligned} p' &= ap + bx + e \\ x' &= cp + dx + f \end{aligned} \qquad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

is the affine symplectic co-ordinate change between (p, x) and (p', x'), the slope parameters (68) of $\mathcal{L}$ are related by

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

and the flat vector fields ∂_x, $\partial_{x'}$ on $\mathcal{L} \cong U$ are related by

$$\partial_x = \frac{\partial x'}{\partial x} \partial_{x'} = (c\tau + d) \partial_{x'}. \tag{69}$$

Let $\Pi_i \colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ denote the projection along P_i for $i = 1, 2$. Then we have

$$\mathrm{KS}^{-1} \Pi_1(\partial_p) = 0 \qquad \mathrm{KS}^{-1} \Pi_2(\partial_p) = \mathrm{KS}^{-1} \Pi_2(a\partial_{p'} + c\partial_{x'}) = c\partial_{x'} = \frac{c}{c\tau + d} \partial_x$$

$$\mathrm{KS}^{-1} \Pi_1(\partial_x) = \partial_x \qquad \mathrm{KS}^{-1} \Pi_2(\partial_x) = \mathrm{KS}^{-1} \Pi_2(b\partial_{p'} + d\partial x') = d\partial_{x'} = \frac{d}{c\tau + d} \partial_x$$

and the propagator is:

$$\begin{aligned} \Delta(P_1, P_2) &= (\mathrm{KS} \otimes \mathrm{KS})^{-1} (\Pi_1 \otimes \Pi_2)(3\partial_p \otimes \partial_x - 3\partial_x \otimes \partial_p) \\ &= -\frac{3c}{c\tau + d} \partial_x \otimes \partial_x. \end{aligned}$$

Lemma 7.2.10 (Propagator calculus, cf. [25, Proposition 4.45])**.** *Let P_1, P_2 be opposite line bundles over U, let $t_0 \in U$, and let x be a flat co-ordinate on U corresponding to P_1. Write the propagator $\Delta(P_1, P_2)$ as $\Delta(x) \partial_x \otimes \partial_x$, and write the Yukawa coupling as $Y_{\mathrm{CY}}(x)\, dx \otimes dx \otimes dx$. Then:*

$$\begin{aligned} \boldsymbol{\nabla}^{P_2} - \boldsymbol{\nabla}^{P_1} &= \Delta(P_1, P_2) \cdot Y_{\mathrm{CY}} = \Delta(x) Y_{\mathrm{CY}}(x) dx \\ \frac{\partial \Delta}{\partial x} &= \Delta(x)^2 Y_{\mathrm{CY}}(x) \end{aligned}$$

where we regard $\boldsymbol{\nabla}^{P_1}$, $\boldsymbol{\nabla}^{P_2}$ as connections on $\Omega^1(\log\{0\})$.

Proof. Choose co-ordinates as in Examples 7.2.4, 7.2.9 so that, with notation as there,

$$Y_{\mathrm{CY}}(x) = \frac{1}{3} \frac{\partial \tau}{\partial x}, \qquad \Delta(x) = -\frac{3c}{c\tau + d}.$$

Let us denote derivatives with respect to x by subscripts. Recalling that x and x' are flat co-ordinates for P_1 and P_2 respectively, and using (69), we have:

$$\begin{aligned} &\boldsymbol{\nabla}^{P_1}(dx) = 0 \\ &\boldsymbol{\nabla}^{P_2}(dx) = \boldsymbol{\nabla}^{P_2} \left(\frac{dx'}{c\tau + d} \right) = -\frac{c\tau_x}{(c\tau + d)^2} dx \otimes dx' = -\frac{c\tau_x}{c\tau + d} dx \otimes dx. \end{aligned}$$

Hence

$$\boldsymbol{\nabla}^{P_2}(dx) - \boldsymbol{\nabla}^{P_1}(dx) = \Delta(x) Y_{\mathrm{CY}}(x) dx \otimes dx$$

and:

$$\frac{\partial \Delta}{\partial x} = \frac{3c^2 \tau_x}{(c\tau + d)^2} = \Delta(x)^2 Y_{\mathrm{CY}}(x)$$

as claimed. □

7.3. **The Fock Sheaf.** We describe the Fock sheaf in the finite-dimensional setting. The construction is almost parallel to §6.8.

Definition 7.3.1. Let U be an open subset of $\mathcal{M}_{\mathrm{CY}}$ and let P be an opposite line bundle over U. First suppose that U does not contain the conifold point $y_1 = -\frac{1}{27}$. The *local Fock space* $\mathfrak{Fock}_{\mathrm{CY}}(U; P)$ consists of collections:

$$\left\{ \boldsymbol{\nabla}^n C^{(g)} \in \left(\Omega^1_U(\log\{0\}) \right)^{\otimes n} : g \geq 0,\, n \geq 0,\, 2g - 2 + n > 0 \right\}$$

of completely symmetric logarithmic n-tensors on U such that:

- (Yukawa) $\boldsymbol{\nabla}^3 C^{(0)}$ is the Yukawa coupling Y_{CY};
- (Jetness) $\boldsymbol{\nabla}(\boldsymbol{\nabla}^n C^{(g)}) = \boldsymbol{\nabla}^{n+1} C^{(g)}$.

Here $\boldsymbol{\nabla} = \boldsymbol{\nabla}^P$ is the flat connection on $\Omega^1_U(\log\{0\})$ defined by P, extended to logarithmic n-tensors in the obvious way; cf. the discussion in §6.8. When U contains the conifold point, we define $\mathfrak{Fock}_{\mathrm{CY}}(U; P)$ similarly except that we allow $\boldsymbol{\nabla}^n C^{(g)}$ to have poles of order at most $2g - 2 + n$ at the conifold point, and impose the same conditions (Yukawa) and (Jetness).

Remark 7.3.2. The Yukawa coupling $Y_{\mathrm{CY}} = \boldsymbol{\nabla}^3 C^{(0)}$ has a pole of order 1 at the conifold point – see Remark 7.1.5 – and thus satisfies the last condition in the definition.

Let t be a co-ordinate on U. If the point $t = 0$ is the large-radius limit point then write $u = \log t$, $du = \frac{dt}{t}$, and $\partial_u = t\frac{\partial}{\partial t}$; otherwise write $u = t$, $du = dt$, $\partial_u = \frac{\partial}{\partial t}$. Then, as in the infinite-dimensional case, if:

$$\boldsymbol{\nabla}^n C^{(g)} = C^{(g)}_n du^{\otimes n}$$

then we refer to the tensors $\boldsymbol{\nabla}^n C^{(g)}$ or the functions $C^{(g)}_n$ as *n-point correlation functions*. We again encode elements of the local Fock space $\mathfrak{Fock}_{\mathrm{CY}}(U; P)$ as formal functions on the logarithmic tangent bundle. Let v be the fiber co-ordinate on the logarithmic tangent bundle $\Theta_U(\log\{0\})$ that is dual to ∂_u, so that (u, v) denotes a point in the total space of $\Theta_U(\log\{0\})$.

Definition 7.3.3 (jet potential)**.** Given an element $\mathscr{C} = \{\boldsymbol{\nabla}^n C^{(g)}\}_{g,n}$ of $\mathfrak{Fock}_{\mathrm{CY}}(U; P)$, set:

$$\mathcal{W}^g(u, v) = \sum_{n = \max(0, 3-2g)}^{\infty} \frac{C^{(g)}_n(u)}{n!} v^n \qquad \text{and} \qquad \mathcal{W}(u, v) = \sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{W}^g(u, v)$$

We call $\mathcal{W}^g$ the *genus-g jet potential* and $\exp(\mathcal{W})$ the *total jet potential* associated to $\mathscr{C}$.

Remark 7.3.4. We regard $\mathcal{W}^{(g)}(u, v)$ as a formal function on the total space of the logarithmic tangent bundle. As in the infinite-dimensional case, $\exp(\mathcal{W})$ is well-defined as a power series in $\hbar$ and $\hbar^{-1}$.

Definition 7.3.5 (transformation rule)**.** Let P_1 and P_2 be opposite line bundles over U, and consider the propagator $\Delta(P_1, P_2) = \Delta(u)\partial_u \otimes \partial_u$. The *transformation rule*

$$T(P_1, P_2) \colon \mathfrak{Fock}_{\mathrm{CY}}(U; P_1) \to \mathfrak{Fock}_{\mathrm{CY}}(U; P_2)$$

assigns to a jet potential $\mathcal{W}$ for an element $\mathscr{C} \in \mathfrak{Fock}_{\mathrm{CY}}(U, P_1)$, the jet potential $\widehat{\mathcal{W}}$ for an element $\widehat{\mathscr{C}} \in \mathfrak{Fock}_{\mathrm{CY}}(U, P_2)$ given by:

$$\exp\big(\widehat{\mathcal{W}}(u,v)\big) = \exp\left(\frac{\hbar}{2}\Delta(u)\frac{\partial^2}{\partial v^2}\right)\exp\big(\mathcal{W}(u,v)\big)$$

Suppose that $\mathscr{C} = \{C_n^{(g)} du^{\otimes n} : g \geq 0,\ n \geq 0,\ 2g-2+n>0\}$ are the correlation functions for $\mathcal{W}$ and that $\widehat{\mathscr{C}} = \{\widehat{C}_n^{(g)} du^{\otimes n} : g \geq 0,\ n \geq 0,\ 2g-2+n>0\}$ are the correlation functions for $\widehat{\mathcal{W}}$. The transformation rule in Definition 7.3.5 is equivalent to the Feynman rule:

$$\widehat{C}_n^{(g)} du^{\otimes n} = \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}\}_{h\leq g}) \tag{70}$$

where the summation is over all connected decorated graphs Γ such that:

- To each vertex $v \in V(\Gamma)$ is assigned a non-negative integer $g_v \geq 0$, called genus;
- Γ has n labelled legs: an isomorphism $L(\Gamma) \cong \{1,2,\dots,n\}$ is given;
- Γ is stable, i.e. $2g_v - 2 + n_v > 0$ for every vertex v. Here $n_v = |\pi_V^{-1}(v)|$ denotes the number of edges or legs incident to v;
- $g = \sum_v g_v + 1 - \chi(\Gamma)$.

(See Appendix B for our notation for graphs.) We put the correlation function $C_{n_v}^{(g_v)} du^{\otimes n_v}$ on the vertex v and put the propagator $\Delta(P_1,P_2)$ on every edge. Then $\operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}\}_{h\leq g})$ is defined to be the contraction of all these tensors; the result is an n-tensor with the n tensor indices corresponding to the n labelled legs[14]. As before, $\operatorname{Aut}(\Gamma)$ denotes the automorphism group of the decorated graph Γ.

Proposition 7.3.6. *The transformation rule is well-defined. In other words, if:*

$$\mathscr{C} = \{C_n^{(g)} du^{\otimes n} : g \geq 0,\ n \geq 0,\ 2g-2+n>0\}$$

is an element of $\mathfrak{Fock}_{\mathrm{CY}}(U, P_1)$ *and:*

$$\widehat{\mathscr{C}} = \{\widehat{C}_n^{(g)} du^{\otimes n} : g \geq 0,\ n \geq 0,\ 2g-2+n>0\}$$

is defined by the Feynman rule (70) *then* $\widehat{\mathscr{C}} \in \mathfrak{Fock}_{\mathrm{CY}}(U, P_2)$.

Proof. First note that if U contains either large-radius or conifold points, there is a unique opposite line bundle over U – see Remark 7.2.3. Therefore the transformation rule is trivial and there is nothing to prove. In particular, we do not need to discuss the 'pole order $2g-2+n$' condition at the conifold point. (When we consider curved opposite line bundles, however, this condition matters: see §7.4.)

We need to show that $\widehat{\mathscr{C}}$ satisfies the properties (Yukawa) and (Jetness) in Definition 7.3.1. (Yukawa) is obvious, as there is exactly one stable 3-valent graph with $g=0$ and so $\widehat{C}_3^{(0)} = C_3^{(0)}$. To establish (Jetness), we shall differentiate the right-hand side of (70) with respect to $\boldsymbol{\nabla}^{P_2}$ and check if it coincides with the Feynman rule for $\widehat{C}_{n+1}^{(g)} du^{\otimes(n+1)}$.

As discussed, it suffices to check (Jetness) away from D_{CY}. Therefore we may choose the co-ordinate u to be a flat co-ordinate x associated with P_1 and use notation as in Lemma 7.2.10. Since $\boldsymbol{\nabla}^{P_2} = \boldsymbol{\nabla}^{P_1} + \Delta(x) Y_{\mathrm{CY}}(x) dx$ by Lemma 7.2.10, $\boldsymbol{\nabla}^{P_2}$ applied to (70) yields a sum over stable Feynman graphs as above, but with either:

[14]Since the base $\mathcal{M}_{\mathrm{CY}}$ is one-dimensional, there is only one kind of tensor indices; the labelling of legs still plays a role in reducing automorphisms of Γ.

(a) a distinguished vertex that carries $\boldsymbol{\nabla}^{P_1}\big(C_n^{(g_v)}dx^{\otimes n}\big) = C_{n_v+1}^{(g_v)}dx^{\otimes(n_v+1)}$; or
(b) a distinguished edge which carries $d\Delta(x) = Y_{\mathrm{CY}}(x)\Delta(x)^2dx$; or
(c) a distinguished leg that carries $Y_{\mathrm{CY}}(x)\Delta(x)dx$ in place of dx.

The first possibility here arises from differentiating a vertex term in (70); the second possibility arises from differentiating an edge term; and the third possibility arises from the difference of $\boldsymbol{\nabla}^{P_1}$ and $\boldsymbol{\nabla}^{P_2}$ – recall how $\boldsymbol{\nabla}$ acts on n-tensors from Equation (62). Note that we have used (Jetness) for $\mathscr{C}$ in (a) and Lemma 7.2.10 in (b). Observe that these are precisely the contributions appearing in the Feynman sum for $\widehat{C}_{n+1}^{(g)}dx^{\otimes(n+1)}$; in fact ($a$)–($c$) correspond respectively to Feynman graphs such that

(a') the leg labelled by $n+1$ is on a vertex v such that $2g_v - 2 + n_v > 1$;
(b') the leg labelled by $n+1$ is on a genus-zero vertex with 1 leg and 2 adjacent edges;
(c') the leg labelled by $n+1$ is on a genus-zero vertex with 2 legs and 1 adjacent edge.

The proposition follows. □

We now show that the transformation rule satisfies the cocycle condition.

Proposition 7.3.7. *Let P_1, P_2, and P_3 be opposite line bundles over U, and let $\Delta_{ij} = \Delta(P_i, P_j)$ be the corresponding propagators. We have:*

$$\Delta_{13} = \Delta_{12} + \Delta_{23}$$

In particular, $\Delta_{12} = -\Delta_{21}$.

Proof. Let $\Pi_i \colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ be the projection along P_i. Then, for any sections ω, ω' of $(F^2_{\mathrm{vec}})^\vee$, we have:

$$\begin{aligned}
\Delta_{13}(\mathrm{KS}^\star\,\omega, \mathrm{KS}^\star\,\omega') &= \Omega^\vee\big((\Pi_1^\star - \Pi_3^\star)\omega, \Pi_3^\star\omega'\big) \\
&= \Omega^\vee\big((\Pi_1^\star - \Pi_2^\star)\omega, \Pi_3^\star\omega'\big) + \Omega^\vee\big((\Pi_2^\star - \Pi_3^\star)\omega, \Pi_3^\star\omega'\big) \\
&= \Omega^\vee\big((\Pi_1^\star - \Pi_2^\star)\omega, \Pi_2^\star\omega'\big) + \Delta_{23}(\mathrm{KS}^\star\,\omega, \mathrm{KS}^\star\,\omega') \\
&= \Delta_{13}(\mathrm{KS}^\star\,\omega, \mathrm{KS}^\star\,\omega') + \Delta_{23}(\mathrm{KS}^\star\,\omega, \mathrm{KS}^\star\,\omega')
\end{aligned}$$

For the first equality here we used the fact that $\operatorname{Im}\Pi_3^\star = P_3^\perp$ is isotropic; for the third equality we used the fact that $\operatorname{Im}(\Pi_2^\star - \Pi_3^\star)$ and $\operatorname{Im}(\Pi_1^\star - \Pi_2^\star)$ are contained in the isotropic subspace $(F^2_{\mathrm{vec}})^\perp$. □

Corollary 7.3.8. *The transformation rule (Definition 7.3.5) satisfies the cocycle condition: if P_1, P_2, and P_3 are opposite line bundles over U then:*

$$T(P_1, P_3) = T(P_2, P_3) \circ T(P_1, P_2)$$

Thus the following Definition makes sense.

Definition 7.3.9 (Fock sheaf)**.** From Proposition 7.2.6 we know that there is an open covering $\{U_a : a \in A\}$ of $\mathcal{M}_{\mathrm{CY}}$ such that for each $a \in A$ there exists an opposite line bundle P_a over U_a. The *Fock sheaf* $\mathfrak{Fock}_{\mathrm{CY}}$ is defined to be the sheaf of sets over $\mathcal{M}_{\mathrm{CY}}$ obtained by gluing the local Fock spaces $\mathfrak{Fock}_{\mathrm{CY}}(U_a; P_a)$, $a \in A$, using the transformation rule

$$T(P_a, P_b) : \mathfrak{Fock}_{\mathrm{CY}}(U_a \cap U_b; P_a) \to \mathfrak{Fock}_{\mathrm{CY}}(U_a \cap U_b; P_b) \qquad a, b \in A$$

over $U_a \cap U_b$.

Remark 7.3.10. The Feynman rule (70) coincides with that used by Aganagic–Bouchard–Klemm [2, §2]. It arises there through stationary phase approximation of certain integral operators acting on wave functions, which suggests a possible non-perturbative extension of the quantization formalism. Our approach here emphasizes rigorous mathematical constructions, but in doing so hides this possible link to a non-perturbative theory.

7.4. **Curved Opposite Line Bundles.** We discuss a generalization of the previous framework to possibly curved (i.e. not necessarily parallel) opposite line bundles. Of particular interest to us are the complex conjugate line bundle and the algebraic opposite line bundles which will be introduced later in §10.4. A general theory for curved opposite modules in the infinite-dimensional setting was developed in [25, §4.13 and §9], and the discussion here is parallel to that.

Definition 7.4.1. Let $U \subset \mathcal{M}_{\mathrm{CY}}$ be an open set. A *possibly curved opposite line bundle* over U is a topological (or C^∞) line subbundle P of $H_{\mathrm{vec}}|_U$ such that $H_{\mathrm{vec}}|_U = F^2_{\mathrm{vec}}|_U \oplus P$.

For possibly curved opposite line bundles P_1, P_2, the propagator

$$\Delta(P_1, P_2) := (\mathrm{KS}^{-1} \otimes \mathrm{KS}^{-1})(\Pi_1 \times \Pi_2)_* \Omega^\vee$$

is still well-defined as a *continuous* (or C^∞) section of $(\Theta(\log\{0\}))^{\otimes 2}$ (see Definition 7.2.8). Let P_0 be an opposite line bundle and suppose that an element of the local Fock space for P_0

$$\mathscr{C} = \{C_n^{(g)} du^{\otimes n} : 2g - 2 + n > 0\} \in \mathfrak{Fock}_{\mathrm{CY}}(U; P_0)$$

is given. For a possibly curved opposite line bundle P over U, we define *genus-g, n-point correlation functions $\widehat{C}_n^{(g)} du^{\otimes n}$ with respect to P* by the same Feynman rule as before

$$\widehat{C}_n^{(g)} du^{\otimes n} = \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta(P_0, P), \{C^{(h)}\}_{h \le g}) \tag{71}$$

where Γ ranges over all connected, decorated, genus-g stable graphs (see the list of conditions below equation (70)). Note that $\widehat{C}_3^{(0)} du^{\otimes 3} = C_3^{(0)} du^{\otimes 3}$ is the Yukawa coupling.

Lemma 7.4.2. *If U does not contain the conifold point, then the correlation functions* (71) *are continuous sections of $\Omega^1(\log\{0\})^{\otimes n}$. If U contains the conifold point, $(1+27y)^{2g-2+n} \widehat{C}_n^{(g)} du^{\otimes n}$ extends continuously across the conifold point.*

Proof. The former statement is obvious from the definition. The latter statement follows from the condition that $C_n^{(g)} du^{\otimes n}$ has a pole of order $2g - 2 + n$ at the conifold point (see Definition 7.3.1), and the fact that the "Euler number" $2g - 2 + n$ is additive under graph contractions. □

Remark 7.4.3. The Feynman rule involving curved opposite modules still satisfies the cocycle condition. This is because Proposition 7.3.7 and its proof are valid also for curved opposite line bundles. In particular, we can invert the Feynman rule (71) to get

$$C_n^{(g)} du^{\otimes n} = \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta(P, P_0), \{\widehat{C}^{(h)}\}_{h \le g}).$$

The main difference from the parallel case is that correlation functions with respect to a possibly curved opposite line bundle *do not satisfy (Jetness)* in general. In place of (Jetness), they satisfy certain anomaly equations. We assume henceforth that a possibly curved opposite line bundle P is a C^∞ subbundle of H_{vec}. Also, for simplicity, we work over the locus $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ where the connection ∇ has no singularities. We can define a (not necessarily

flat) connection $\boldsymbol{\nabla}^P$ associated with a curved P by the same formula as in Definition 7.2.2. Namely, the projection $\Pi\colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ along P induces a (not necessarily flat) connection

$$C^\infty(F^2_{\mathrm{vec}}) \xrightarrow{\nabla} C^\infty(T^\vee_{\mathbb{C}} \otimes H_{\mathrm{vec}}) \xrightarrow{\mathrm{id}\otimes\Pi} C^\infty(T^\vee_{\mathbb{C}} \otimes F^2_{\mathrm{vec}})$$

which in turn defines the connection:

$$\boldsymbol{\nabla}^P\colon C^\infty(T^{1,0}) \to C^\infty(T^\vee_{\mathbb{C}} \otimes T^{1,0})$$

and its dual:

$$\boldsymbol{\nabla}^P\colon C^\infty((T^{1,0})^\vee) \to C^\infty(T^\vee_{\mathbb{C}} \otimes (T^{1,0})^\vee)$$

via the Kodaira–Spencer isomorphism $F^2_{\mathrm{vec}} \cong T^{1,0}$. Here $T_{\mathbb{C}}$ denotes the complexified tangent bundle of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$, $T^\vee_{\mathbb{C}}$ its dual, and $T_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$ the type decomposition. The connection $\boldsymbol{\nabla}^P$ can be naturally extended to n-tensors:

$$\boldsymbol{\nabla}^P\colon C^\infty((T^{1,0})^\vee \otimes \cdots \otimes (T^{1,0})^\vee) \to C^\infty(T^\vee_{\mathbb{C}} \otimes (T^{1,0})^\vee \otimes \cdots \otimes (T^{1,0})^\vee)$$

Note that the $(0,1)$-part of $\boldsymbol{\nabla}^P$ is the standard Dolbeault operator.

Definition 7.4.4 (torsion)**.** Let P be a possibly curved opposite line bundle. The *torsion* of P is the C^∞ tensor $\Lambda\colon (T^{1,0})^\vee \otimes (T^{1,0})^\vee \to T^\vee_{\mathbb{C}}$ defined by

$$\Lambda(\omega_1,\omega_2) = \Omega^\vee(\nabla^\vee \Pi^*(\mathrm{KS}^*)^{-1}\omega_1, \Pi^*(\mathrm{KS}^*)^{-1}\omega_2) \qquad \omega_1,\omega_2 \in C^\infty((T^{1,0})^\vee)$$

where $\nabla^\vee$ is the connection on $H^\vee_{\mathrm{vec}}$ dual to ∇, $\Pi^*\colon (F^2_{\mathrm{vec}})^\vee \to H^\vee_{\mathrm{vec}}$ is the dual of the projection $\Pi\colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$, and $\mathrm{KS}^*\colon (F^2_{\mathrm{vec}})^\vee \cong (T^{1,0})^\vee$ is the dual of the Kodaira–Spencer isomorphism.

Note that Λ vanishes if and only if P is parallel for ∇. We can generalize the propagator calculus in Lemma 7.2.10 as follows:

Lemma 7.4.5. *Let P_0 be a (parallel) opposite line bundle over U and let P be a possibly curved opposite line bundle over U. Let x be a flat co-ordinate associated with P_0. Write the propagator $\Delta(P_0,P)$ as $\Delta(x)\partial_x \otimes \partial_x$, the Yukawa coupling as $Y_{\mathrm{CY}}(x)dx\otimes dx\otimes dx$, and the torsion of P as $\Lambda = \Lambda(dx,dx) = \Lambda_x dx + \Lambda_{\overline{x}} d\overline{x} \in C^\infty(T^\vee_{\mathbb{C}})$. Then:*

$$\boldsymbol{\nabla}^P - \boldsymbol{\nabla}^{P_0} = \Delta(P_0,P)\cdot Y_{\mathrm{CY}} = \Delta(x) Y_{\mathrm{CY}}(x) dx$$
$$d\Delta(x) = \Delta(x)^2 Y_{\mathrm{CY}}(x)dx - \Lambda_x dx - \Lambda_{\overline{x}} d\overline{x}.$$

Moreover, the curvature of $\boldsymbol{\nabla}^P$ on the cotangent bundle $(T^{1,0})^\vee$ is $Y_{\mathrm{CY}}(x)\Lambda_{\overline{x}}\, dx\wedge d\overline{x}$.

Proof. We use notation as in Examples 7.2.4 and 7.2.9. Take a point $t_0 \in U$ and embed a neighbourhood of t_0 as a Lagrangian curve $\mathcal{L} \subset H_{\mathrm{aff}}|_{t_0}$. Fix Darboux co-ordinates (p,x) on $H_{\mathrm{aff}}|_{t_0}$ such that $P_0 = \langle \partial/\partial p\rangle$, $\Omega = \frac{1}{3} dp\wedge dx$, and that x coincides with the given flat co-ordinate when restricted to $\mathcal{L}$. In terms of the co-ordinates (p,x), we have

$$\mathrm{KS}(\partial_x) = \tau\frac{\partial}{\partial p} + \frac{\partial}{\partial x} = \begin{pmatrix}\tau\\1\end{pmatrix}$$

where τ is the slope parameter of $\mathcal{L}$ in (68). Suppose that the fiber P_x at x is written in the form (via parallel translation to $H_{\mathrm{vec}}|_{t_0}$):

$$P_x = \mathbb{C}\begin{pmatrix}c\\1\end{pmatrix}$$

for a smooth function $c = c(x)$. Then a computation similar to Example 7.2.9 gives:

$$\Delta(P_0,P) = \Delta(x)\partial_x\otimes\partial_x = -\frac{3}{\tau - c}\partial_x\otimes\partial_x.$$

In terms of the dual frame $\{dp, dx\}$ of $H^\vee_{\mathrm{vec}}|_{t_0}$, we have:

$$\Pi^*(\mathrm{KS}^*)^{-1}(dx) = \frac{1}{\tau - c}(dp - c dx) = \frac{1}{\tau - c}(1, -c)$$

Hence:

$$\Lambda = \Lambda(dx, dx) = \Omega^\vee\left(d\left[\frac{1}{\tau - c}(1,-c)\right], \frac{1}{\tau - c}(1,-c)\right) = \frac{3}{(\tau - c)^2} dc \tag{72}$$

and:

$$\boldsymbol{\nabla}^P dx = \mathrm{KS}^*\left(\nabla^\vee \Pi^* (\mathrm{KS}^*)^{-1}(dx)|_{F^2_{\mathrm{vec}}}\right) = \mathrm{KS}^*\left[d\left(\frac{1}{\tau - c}(1,-c)\right)\Big|_{F^2_{\mathrm{vec}}}\right] = -\frac{d\tau \otimes dx}{\tau - c}.$$

Therefore, using $Y_{\mathrm{CY}}(x) = \frac{1}{3}\tau_x$, we find:

$$\boldsymbol{\nabla}^P dx - \boldsymbol{\nabla}^{P_0} dx = -\frac{3}{\tau - c} Y_{\mathrm{CY}}(x) dx \otimes dx = \Delta(x) Y_{\mathrm{CY}}(x) dx \otimes dx$$

and:

$$d\Delta = \frac{3 d\tau}{(\tau - c)^2} - \frac{3dc}{(\tau - c)^2} = \Delta(x)^2 Y_{\mathrm{CY}}(x) dx - \Lambda.$$

Finally, the curvature of $\boldsymbol{\nabla}^P$ is given by:

$$\left(\boldsymbol{\nabla}^P\right)^2 (dx) = \boldsymbol{\nabla}^P\left(-\frac{\tau_x}{\tau - c} dx \otimes dx\right) = -\frac{\tau_x c_{\overline{x}}}{(\tau - c)^2}(d\overline{x} \wedge dx) \otimes dx = Y_{\mathrm{CY}}(x) \Lambda_{\overline{x}} (dx \wedge d\overline{x}) \otimes dx$$

as claimed. □

Using Lemma 7.4.5, we deduce the following anomaly equation. We omit the proof since the argument is very similar to that proving (Jetness) in Proposition 7.3.6.

Proposition 7.4.6 (anomaly equation, cf. [25, Theorem 4.86])**.** *Let P_0 be a (parallel) opposite line bundle and P be a possibly curved opposite line bundle. Let x denote a flat co-ordinate associated with P_0 and let $\mathscr{C} = \{C_n^{(g)} dx^{\otimes n}\}_{2g-2+n>0}$ be an element of the local Fock space $\mathfrak{Fock}_{\mathrm{CY}}(U; P_0)$. Let $\widehat{C}_n^{(g)} dx^{\otimes n}$ denote the genus-g, n-point correlation functions with respect to P produced from $\mathscr{C}$ by the Feynman rule* (71)*. Let $\Lambda = \Lambda(dx, dx) = \Lambda_x dx + \Lambda_{\overline{x}} d\overline{x}$ denote the torsion of P. Then we have:*

$$\widehat{C}^{(g)}_{n+1} dx^{\otimes(n+1)} = \boldsymbol{\nabla}^P(\widehat{C}^{(g)}_n dx^{\otimes n}) + \frac{1}{2} \sum_{\substack{h+k=g \\ i+j=n}} \binom{n}{i} \Lambda \otimes \widehat{C}^{(h)}_{i+1} \widehat{C}^{(k)}_{j+1} dx^{\otimes n} + \frac{1}{2} \Lambda \otimes \widehat{C}^{(g-1)}_{n+2} dx^{\otimes n}$$

Equivalently:

$$\widehat{C}^{(g)}_{n+1} = \frac{\partial \widehat{C}^{(g)}_n}{\partial x} + n \Delta(x) Y_{\mathrm{CY}}(x) \widehat{C}^{(g)}_n + \frac{1}{2} \sum_{\substack{h+k=g \\ i+j=n}} \binom{n}{i} \Lambda_x \widehat{C}^{(h)}_{i+1} \widehat{C}^{(k)}_{j+1} + \frac{1}{2} \Lambda_x \widehat{C}^{(g-1)}_{n+2},$$

$$0 = \frac{\partial \widehat{C}^{(g)}_n}{\partial \overline{x}} + \frac{1}{2} \sum_{\substack{h+k=g \\ i+j=n}} \binom{n}{i} \Lambda_{\overline{x}} \widehat{C}^{(h)}_{i+1} \widehat{C}^{(k)}_{j+1} + \frac{1}{2} \Lambda_{\overline{x}} \widehat{C}^{(g-1)}_{n+2}$$

where we use notation from Lemma 7.4.5*.*

Remark 7.4.7. When we apply this in §10, we shall restrict to the following special cases:

(a) P is an anti-holomorphic line subbundle, i.e. preserved by the $(1,0)$-part of ∇ (e.g. the complex conjugate opposite line bundle in Definition 10.4.1);

(b) P is a holomorphic line subbundle which is not flat (e.g. the algebraic opposite line bundle in Definition 10.4.3).

In the case (a), we have $\Lambda_x = 0$. Therefore the correlation functions satisfy 'partial' jetness $(\boldsymbol{\nabla}^P)^{1,0}(\widehat{C}_n^{(g)} dx^{\otimes n}) = \widehat{C}_{n+1}^{(g)} dx^{\otimes(n+1)}$ and the holomorphic anomaly equations:

$$0 = \frac{\partial \widehat{C}_1^{(1)}}{\partial \overline{x}} + \frac{1}{2}\Lambda_{\overline{x}} Y_{\mathrm{CY}}(x) \qquad \text{for } g = 1$$

$$0 = \frac{\partial \widehat{C}_0^{(g)}}{\partial \overline{x}} + \frac{1}{2}\sum_{h=1}^{g-1} \Lambda_{\overline{x}} \widehat{C}_1^{(h)} \widehat{C}_1^{(g-h)} + \frac{1}{2}\Lambda_{\overline{x}} \widehat{C}_2^{(g-1)} \qquad \text{for } g \geq 2$$

where $\widehat{C}_1^{(g)} = \partial_x \widehat{C}_0^{(g)}$ for $g \geq 2$ and $\widehat{C}_2^{(g)} = \partial_x \widehat{C}_1^{(g)} + \Delta(x) Y_{\mathrm{CY}}(x) \widehat{C}_1^{(g)}$ for $g \geq 1$. Note that the holomorphic anomaly equation in genus 1 says that $\widehat{C}_1^{(1)} dx$ behaves as a 'connection 1-form' on the square root of the canonical bundle:

$$d(\widehat{C}_1^{(1)} dx) = \frac{1}{2}\Lambda_{\overline{x}} Y_{\mathrm{CY}}(x) dx \wedge d\overline{x} = \frac{1}{2}(\boldsymbol{\nabla}^P)^2.$$

In the case (b), we have $\Lambda_{\overline{x}} = 0$. Thus the correlation functions $\widehat{C}_n^{(g)}$ are holomorphic, but do not satisfy (Jetness).

8. The Conformal Limit of the Fock Sheaf

Let $\mathcal{M}_{\mathrm{CY}}^\circ$ denote the complement of the conifold locus:

$$\mathcal{M}_{\mathrm{CY}}^\circ := \mathcal{M}_{\mathrm{CY}} \setminus \{-\tfrac{1}{27}\}$$

and let $\mathfrak{Fock}_{\mathrm{CY}}^\circ$ denote the restriction to $\mathcal{M}_{\mathrm{CY}}^\circ$ of the finite-dimensional Fock sheaf $\mathfrak{Fock}_{\mathrm{CY}}$ from §7. Recall from §6 that the B-model Fock sheaf is defined on $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ and let $i\colon \mathcal{M}_{\mathrm{CY}}^\circ \to \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ denote the inclusion. In this section, we prove:

Theorem 8.0.8. *There exists a restriction map of Fock sheaves, $i^{-1}\mathfrak{Fock}_{\mathrm{B}} \to \mathfrak{Fock}_{\mathrm{CY}}^\circ$.*

There are several things to understand:

- how correlation functions for $\mathfrak{Fock}_{\mathrm{B}}$ give rise to correlation functions for $\mathfrak{Fock}_{\mathrm{CY}}^\circ$ (§8.1);
- how opposite modules for the big B-model log-cTEP structure that are compatible with the Deligne extension give rise to opposite line bundles for H_{vec} (§8.2);
- how the transformation rule used to assemble $\mathfrak{Fock}_{\mathrm{B}}$ out of local Fock spaces gives rise to the transformation rule used to assemble $\mathfrak{Fock}_{\mathrm{CY}}^\circ$ out of local Fock spaces (§§8.3–8.4).

With this material in place, we define the restriction map $i^{-1}\mathfrak{Fock}_{\mathrm{B}} \to \mathfrak{Fock}_{\mathrm{CY}}^\circ$ in §8.5. In §8.6 we show that there is a global section $\mathscr{C}_{\mathrm{CY}}$ of the finite-dimensional Fock sheaf $\mathfrak{Fock}_{\mathrm{CY}}^\circ$, which arises via restriction from the global section $\mathscr{C}_{\mathrm{B}}$ of $\mathfrak{Fock}_{\mathrm{B}}$. Near the large-radius limit point, correlation functions of $\mathscr{C}_{\mathrm{CY}}$ encode Gromov–Witten invariants of the non-compact Calabi–Yau 3-fold $Y = K_{\mathbb{P}^2}$ and near the orbifold point, the correlation functions of $\mathscr{C}_{\mathrm{CY}}$ encode Gromov–Witten invariants of the non-compact orbifold $\mathcal{X} = \left[\mathbb{C}^3/\mu_3\right]$. We will see in §9 that the genus-g, n-point correlation functions of $\mathscr{C}_{\mathrm{CY}}$, which *a priori* are holomorphic functions on $\mathcal{M}_{\mathrm{CY}}^\circ$, are in fact meromorphic functions on $\mathcal{M}_{\mathrm{CY}}$ with poles at the conifold point $-\frac{1}{27} \in \mathcal{M}_{\mathrm{CY}}$ of order at most $2g-2+n$. Thus we can think of $\mathscr{C}_{\mathrm{CY}}$ as a global section of the finite-dimensional Fock sheaf $\mathfrak{Fock}_{\mathrm{CY}}$.

8.1. **Correlation Functions in the Conformal Limit.** Recall from Theorem 5.0.3 that the big B-model log-TEP structure $\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}$ (and hence the big B-model log-cTEP structure $\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}$) has logarithmic singularities along $D^{\mathrm{big}} \subset \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. Let $\mathbf{L}$ denote the total space of the big B-model log-cTEP structure and $\mathbf{L}^\circ$ denote its open subset as defined in Definition 6.3.2. Correlation functions for the B-model Fock sheaf $\mathfrak{Fock}_{\mathrm{B}}$ are local sections $\boldsymbol{\nabla}^n C^{(g)}$ of $\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})^{\otimes n}$, where $\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})$ is the sheaf of one-forms on $\mathbf{L}^\circ$ logarithmic along $\mathrm{pr}^{-1} D^{\mathrm{big}}$, satisfying the conditions (Yukawa), (Jetness), (Grading and Filtration), and (Pole). Correlation functions for the finite-dimensional Fock sheaf $\mathfrak{Fock}_{\mathrm{CY}}$ are local sections $\boldsymbol{\nabla}^n C^{(g)}$ of $\Omega^1(\log\{0\})^{\otimes n}$, where $\Omega^1(\log\{0\})$ is the sheaf of one-forms on $\mathcal{M}_{\mathrm{CY}}$ logarithmic at $y_1 = 0$, satisfying the conditions (Yukawa) and (Jetness). Roughly speaking, to relate $\mathfrak{Fock}_{\mathrm{B}}$ to $\mathfrak{Fock}_{\mathrm{CY}}$, we want to pull back correlation functions for $\mathfrak{Fock}_{\mathrm{B}}$ along[15] the primitive section $\zeta\colon \mathcal{M}^\circ_{\mathrm{CY}} \to \mathbf{L}^\circ$. This requires care, because in general there is no canonical way to restrict logarithmic forms to the logarithmic locus.

Example 8.1.1. Let $i\colon D \to \mathcal{M}$ be the inclusion of a normal crossing divisor into a complex manifold. Then there is no canonical map $i^\star\Omega^1_{\mathcal{M}}(\log D) \to \Omega^1_D$; indeed the canonical map goes in the other direction, and fits into an exact sequence

$$0 \longrightarrow \Omega^1_D \longrightarrow i^\star\Omega^1_{\mathcal{M}}(\log D) \xrightarrow{\mathrm{res}} \mathcal{O}_D \longrightarrow 0$$

where the map res takes the residue along D.

To pull back correlation functions for $\mathfrak{Fock}_{\mathrm{B}}$, we first restrict to the image of $\zeta\colon \mathcal{M}^\circ_{\mathrm{B}} \to \mathbf{L}^\circ$. Here there is a well-defined pullback, as $\zeta\big|_{\mathcal{M}^\circ_{\mathrm{B}}}$ is transverse to the logarithmic locus in $\mathbf{L}^\circ$; over $\mathcal{M}^\circ_{\mathrm{CY}}$, it defines a map:

$$\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \to \Omega^1_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D)\big|_{\mathcal{M}^\circ_{\mathrm{CY}}},$$

where here and hereafter $\zeta^\star$ means the pull-back by $\zeta\colon \mathcal{M}^\circ_{\mathrm{CY}} \to \mathbf{L}^\circ$ and $D \subset \mathcal{M}_{\mathrm{B}}$ is the divisor (40). Then we choose a splitting of

$$0 \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D)\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} \dashrightarrow \mathcal{O}_{\mathcal{M}^\circ_{\mathrm{CY}}} \longrightarrow 0. \tag{73}$$

Here the dashed arrow is multiplication by

$$du_2 = \frac{1}{3}\frac{dy_1}{y_1} + \frac{dy_2}{y_2} = \frac{d\mathfrak{y}_2}{\mathfrak{y}_2}.$$

As we will see in §8.3, in our situation this choice of splitting *is* canonical. Combining, we get a restriction map

$$\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \tag{74}$$

as the composition

$$\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D)\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \oplus \mathcal{O}_{\mathcal{M}^\circ_{\mathrm{CY}}} \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\})$$

where the middle arrow is the splitting and the right-hand arrow is projection to the first factor.

[15]The primitive section ζ lands in $\mathbf{L}^\circ \subset \mathbf{L}$ because Reichelt's conditions (IC), (GC) hold along $\mathcal{M}^\circ_{\mathrm{CY}}$: see §5.2.

Example 8.1.2. We can compute the B-model Yukawa coupling $\boldsymbol{Y}$ (Definition 6.5.1) using Proposition 3.5.1. Restricting the result to $\zeta(\mathcal{M}_{\mathrm{B}}^\circ)$, which is possible because $\zeta\big|_{\mathcal{M}_{\mathrm{B}}^\circ}$ is transverse to the logarithmic locus, yields

$$\boldsymbol{Y}\big|_{\zeta(\mathcal{M}_{\mathrm{B}}^\circ)} = -\frac{1}{3(1+27y_1)}\Big(\frac{dy_1}{y_1}\Big)^{\otimes 3} + 9(du_2)^{\otimes 3}.$$

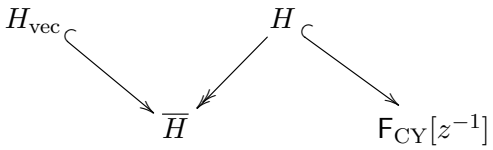
But the Yukawa coupling in the finite-dimensional setting (Definition 7.1.2) is

$$Y_{\mathrm{CY}} = \Omega(\theta^2\zeta, \theta\zeta)\Big(\frac{dy_1}{y_1}\Big)^{\otimes 3} = -\frac{1}{3(1+27y_1)}\Big(\frac{dy_1}{y_1}\Big)^{\otimes 3}$$

– see (47). Thus the restriction map (74) takes the Yukawa coupling $\boldsymbol{Y}$ to Y_{CY}.

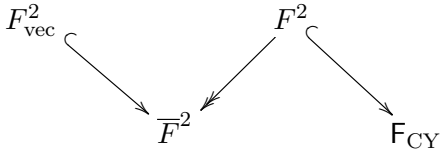
8.2. **Opposite Modules in the Conformal Limit.** We now discuss how opposite modules for the big B-model log-cTEP structure that are compatible with the Deligne extension give rise to opposite line bundles in the conformal limit. This is largely a summary of material from §§4–5. In §4, we considered the restriction $\mathcal{F}_{\mathrm{CY}}$ of the B-model log-TEP structure to $\mathcal{M}_{\mathrm{CY}}\times\mathbb{C}$. This is a log-TEP structure with base $(\mathcal{M}_{\mathrm{CY}}, D_{\mathrm{CY}})$, which carries an endomorphism $N\colon \mathcal{F}_{\mathrm{CY}} \to z^{-1}\mathcal{F}_{\mathrm{CY}}$ given by the residue of the B-model connection along the divisor $\mathcal{M}_{\mathrm{CY}} \times \mathbb{C} \subset \mathcal{M}_{\mathrm{B}} \times \mathbb{C}$. Taking the formalization[16] of $\mathcal{F}_{\mathrm{CY}}$ at $z=0$ defines a log-cTEP structure F_{CY} with base $(\mathcal{M}_{\mathrm{CY}}, D_{\mathrm{CY}})$, equipped with a residue endomorphism $N\colon \mathsf{F}_{\mathrm{CY}} \to z^{-1}\mathsf{F}_{\mathrm{CY}}$ induced by that on $\mathcal{F}_{\mathrm{CY}}$. In §4 we considered a six-dimensional vector bundle H, a three-dimensional vector bundle $\overline{H}$, and a two-dimensional vector bundle H_{vec}; these are related to the log-cTEP structure F_{CY} as follows:

(75)



The vector bundle H is included in $\mathsf{F}_{\mathrm{CY}}[z^{-1}]$ as the degree-1 part; it is preserved by the action of the residue endomorphism N, so we can regard N as an endomorphism of H. There is a canonical surjection from H onto $\overline{H} = H/\operatorname{Im} N$, and $H_{\mathrm{vec}} = (\operatorname{Ker} N)/(\operatorname{Im} N \cap \operatorname{Ker} N)$ sits canonically as a subbundle of $\overline{H}$. The diagram (75) induces the following diagram of Hodge subbundles:

(76)



Here $F^2 \subset H$ is three-dimensional, $\overline{F}^2 \subset \overline{H}$ is two-dimensional, and $F^2_{\mathrm{vec}} \subset H_{\mathrm{vec}}$ is one-dimensional. These were introduced in §4.1 and §4.3; we give explicit bases for them below. Let U be an open neighbourhood of $y \in \mathcal{M}_{\mathrm{CY}}^\circ$ in $\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. Let $\mathbf{P}$ be an opposite module for the big B-model log-TEP structure $(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ over $U \setminus D^{\mathrm{big}}$ such that $\mathbf{P}$ is compatible with the Deligne extension $(\mathcal{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})|_U$ in the sense of Definition 5.1.1. The opposite module $\mathbf{P}$ naturally yields an opposite module P for the big B-model log-cTEP structure $(\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$ over U. By a slight abuse of language, we call such a P a *Deligne-extension-compatible opposite module* for the big B-model log-cTEP structure $\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}$. Let P_{CY} denote the

[16]Since $\mathcal{F}_{\mathrm{CY}}$ is graded – see (42) – no information is lost by the formalization $(\mathcal{F}_{\mathrm{CY}}, \nabla) \rightsquigarrow (\mathsf{F}_{\mathrm{CY}}, \nabla)$.

restriction P to $\mathcal{M}^\circ_{\mathrm{CY}} \cap U$. Combining Propositions 5.1.5, 4.3.1 and 4.3.2, we find that $\mathbf{P}$ (or P_{CY}) induces an opposite line bundle P over $U \cap \mathcal{M}^\circ_{\mathrm{CY}}$ and that (75) induces the following diagram of opposite modules, filters, and line bundles:

(77)
$$\begin{array}{ccccc} P & & & & U_1 \\ & \searrow & & \swarrow & \\ & & \overline{U}_1 & & \\ & & & \searrow & \\ & & & & \mathsf{P}_{\mathrm{CY}} \end{array}$$

Here U_1 is the degree-one part of P_{CY}, which is three-dimensional; $\overline{U}_1$ is the image of U_1 under the projection to $\overline{H}$, which is one-dimensional; and the opposite line bundle P is equal to $\overline{U}_1$. We have that $U_1 = \langle z^{-1}D_2^2 \rangle + \{s \in \operatorname{Ker} N : [s] \in P\}$.

Let us now give explicit bases for the bundles in (76) and (77), summarizing the discussion in §4. We have, in the manifold chart $\mathcal{M}_{\mathrm{CY}} \setminus \{y_1 = \infty\}$:

$$\begin{aligned} \operatorname{Ker} N &= \langle D_1 - \tfrac{1}{3}D_2, z^{-2}D_2^3, z^{-1}(1+27y_1)(D_1 - \tfrac{1}{3}D_2)^2 \rangle \\ \operatorname{Im} N &= \langle D_2, z^{-1}D_2^2, z^{-2}D_2^3 \rangle \end{aligned}$$

Furthermore:

$$\begin{aligned} F^2 &= \langle z, D_1 - \tfrac{1}{3}D_2, D_2 \rangle \\ \overline{F}^2 &= \langle [z], [D_1 - \tfrac{1}{3}D_2] \rangle = \langle [z], [D_1] \rangle = \langle \zeta, \theta\zeta \rangle \\ F^2_{\mathrm{vec}} &= \langle [D_1 - \tfrac{1}{3}D_2] \rangle = \langle [D_1] \rangle = \langle \theta\zeta \rangle \end{aligned}$$

and:

$$\begin{aligned} U_1 &= \langle z^{-1}D_2^2, z^{-2}D_2^3, z^{-1}(1+27y_1)(D_1 - \tfrac{1}{3}D_2)^2 + a(D_1 - \tfrac{1}{3}D_2) \rangle \\ P &= \langle (1+27y_1)\theta^2\zeta + a\theta\zeta \rangle \end{aligned}$$

where a is a scalar-valued function of y_1 that parameterizes the opposite filter U_1 or line bundle P.

A key observation is that the surjection $\operatorname{Ker} N \twoheadrightarrow H_{\mathrm{vec}}$ induces an isomorphism $F^2 \cap \operatorname{Ker} N \xrightarrow{\sim} F^2_{\mathrm{vec}}$. That is, the residue endomorphism N singles out a canonical lift of F^2_{vec} to H. This is also true near the orbifold point. As we now explain, it is this that makes our choice of splitting in (73) canonical. Note that the Kodaira–Spencer map (see Definition 6.5.4) gives an isomorphism $\zeta^\star \boldsymbol{\Theta}_\circ(\log D^{\mathrm{big}}) \xrightarrow{\sim} \zeta^\star \operatorname{pr}^\star \mathsf{F}^{\mathrm{big}}_{\mathrm{B}} = \mathsf{F}_{\mathrm{CY}}|_{\mathcal{M}^\circ_{\mathrm{CY}}}$, and consider the diagram

(78)
$$\begin{array}{ccccc} \zeta^\star\boldsymbol{\Theta}_\circ(\log D^{\mathrm{big}}) & \xrightarrow[\sim]{\mathrm{KS}} & \mathsf{F}_{\mathrm{CY}}\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} & & \\ \uparrow & & \uparrow & & \\ \Theta_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D)\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} & \xrightarrow{\mathrm{KS}} & F^2\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} & = & \langle z, D_1 - \tfrac{1}{3}D_2, D_2 \rangle \\ \vdots & & \uparrow & & \\ & & \big(F^2 \cap \operatorname{Ker} N\big)\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} & = & \langle D_1 - \tfrac{1}{3}D_2 \rangle \\ & & \wr\uparrow & & \\ \Theta_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) & \xrightarrow[\sim]{\mathrm{KS}} & F^2_{\mathrm{vec}}\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} & = & \langle [D_1 - \tfrac{1}{3}D_2] \rangle \end{array}$$

where the lower-right vertical isomorphism is the canonical lift of F^2_{vec} to H and KS denotes the Kodaira–Spencer map. There is a unique choice for the dashed arrow that makes the diagram commute: the bottom horizontal map takes $y_1\frac{\partial}{\partial y_1}$ to $[D_1] = [D_1 - \frac{1}{3}D_2]$, and so the dashed map must take $y_1\frac{\partial}{\partial y_1}$ to $y_1\frac{\partial}{\partial y_1} - \frac{1}{3}y_2\frac{\partial}{\partial y_2}$. Thus our choice of splitting in (73) is the unique choice such that this diagram commutes. Dualizing gives:

Lemma 8.2.1. *The restriction map* (74) *is the unique map that makes the following diagram commute:*

$$\begin{array}{ccc}
\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) & \xleftarrow[\sim]{\mathrm{KS}^\star} & \mathsf{F}^\vee_{\mathrm{CY}}\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} \\
\vdots & & \downarrow \\
 & & \big(F^2 \cap \operatorname{Ker} N\big)^\vee\big|_{\mathcal{M}^\circ_{\mathrm{CY}}} \\
 & & \downarrow \wr \\
\Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) & \xleftarrow[\sim]{\mathrm{KS}^\star} & \big(F^2_{\mathrm{vec}}\big)^\vee\big|_{\mathcal{M}^\circ_{\mathrm{CY}}}
\end{array}$$

8.3. **Connections in the Conformal Limit.** In this section we will show that the restriction map (74) sends the connection (Definition 6.7.6)

$$\boldsymbol{\nabla}^{\mathsf{P}} \colon \boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \longrightarrow \boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \otimes \boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \tag{79}$$

to the connection (Definition 7.2.2)

$$\boldsymbol{\nabla}^{P} \colon \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \otimes \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \tag{80}$$

where the opposite line bundle P is induced by the Deligne-extension-compatible opposite module P as in §8.2. More precisely, these connections are defined on open sets where P or P are defined, but we shall omit the restriction signs to ease the notation. Note that it suffices to check the correspondence between the connections (79), (80) on the manifold chart $\{y_1 \neq \infty\}$; we will work only with this chart.

Remark 8.3.1. Since $\boldsymbol{\nabla}^{\mathsf{P}}$ is not $\mathcal{O}_{\mathsf{L}^\circ}$-linear, it *does not* induce a map from $\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})$ to $\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})^{\otimes 2}$

Since $\zeta|_{\mathcal{M}^\circ_{\mathrm{B}}}$ is transverse to the logarithmic locus, we can pull back the connection (79) to get a connection

$$\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})\big|_{\zeta(\mathcal{M}^\circ_{\mathrm{B}})} \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D) \otimes \Big(\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})\big|_{\zeta(\mathcal{M}^\circ_{\mathrm{B}})}\Big)$$

Restricting to $\mathcal{M}^\circ_{\mathrm{CY}}$ gives

$$\zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \longrightarrow \Big(\Omega^1_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D)\Big|_{\mathcal{M}^\circ_{\mathrm{CY}}}\Big) \otimes \zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})$$

and using the splitting (73) gives a connection

$$\nabla' \colon \zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}) \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \otimes \zeta^\star\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}}).$$

Explicitly:

$$\nabla'\alpha = \frac{dy_1}{y_1} \otimes \Big(\boldsymbol{\nabla}^{\mathsf{P}}_{\left(y_1\frac{\partial}{\partial y_1} - \frac{1}{3}y_2\frac{\partial}{\partial y_2}\right)}\alpha\Big).$$

Let us identify $\zeta^\star \mathbf{\Omega}^1_\circ(\log D^{\mathrm{big}})$ with $\mathsf{F}^\vee_{\mathrm{CY}}$ using the Kodaira–Spencer map, so that

$$\nabla' \colon \mathsf{F}^\vee_{\mathrm{CY}} \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \otimes \mathsf{F}^\vee_{\mathrm{CY}}$$

We need to show that ∇' induces a connection on $(F^2_{\mathrm{vec}})^\vee$ via the map $\mathsf{F}^\vee_{\mathrm{CY}} \to (F^2 \cap \operatorname{Ker} N)^\vee \cong (F^2_{\mathrm{vec}})^\vee$ – see Lemma 8.2.1 – and that this induced connection coincides, via the Kodaira–Spencer map, with $\boldsymbol{\nabla}^P$. To see this, consider the dual connection

$$\nabla' \colon \mathsf{F}_{\mathrm{CY}} \longrightarrow \Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}) \otimes \mathsf{F}_{\mathrm{CY}}$$

and compute:

$$\begin{aligned}
\nabla'(D_1 - \tfrac{1}{3}D_2) &= \frac{dy_1}{y_1} \otimes \boldsymbol{\nabla}^{\mathsf{P}}_{\left(y_1 \frac{\partial}{\partial y_1} - \frac{1}{3} y_2 \frac{\partial}{\partial y_2}\right)} (D_1 - \tfrac{1}{3}D_2) \\
&= \frac{dy_1}{y_1} \otimes \Pi_{\mathsf{P}}\Big(z^{-1} (D_1 - \tfrac{1}{3}D_2)^2 \Big) \\
&= -\frac{dy_1}{y_1} \otimes \frac{a}{1+27y_1} (D_1 - \tfrac{1}{3}D_2)
\end{aligned}$$

where $\Pi_{\mathsf{P}} \colon \mathsf{F}_{\mathrm{CY}}[z^{-1}] \to \mathsf{F}_{\mathrm{CY}}$ is the projection along P. Here we used the fact that Π_{P} on H is the same as projection $H \to F^2$ along U_1, together with the explicit bases from §8.2. Thus ∇' preserves $F^2 \cap \operatorname{Ker} N$, and so induces a connection on F^2_{vec}. It remains to show that this induced connection is $\boldsymbol{\nabla}^P$. But this is obvious:

$$\begin{aligned}
\boldsymbol{\nabla}^P\big(\theta\zeta\big) &= \frac{dy_1}{y_1} \otimes \Pi_P(\theta^2 \zeta) \\
&= -\frac{dy_1}{y_1} \otimes \frac{a}{1+27y_1} \big(\theta\zeta\big)
\end{aligned}$$

where we again used the explicit bases in §8.2. So under the identification $F^2_{\mathrm{vec}} \xrightarrow{\sim} F^2 \cap \operatorname{Ker} N$, which sends $\theta\zeta$ to $D_1 - \frac{1}{3}D_2$, ∇' coincides with $\boldsymbol{\nabla}^P$. Thus we have shown that the restriction map (74) sends the connection (79) to the connection (80).

8.4. **The Propagators Agree in the Conformal Limit.** In this section, we prove:

Proposition 8.4.1. *Let P_1, P_2 be Deligne-extension-compatible opposite modules for the big B-model* log-*cTEP structure $(\mathsf{F}^{\mathrm{big}}_{\mathrm{B}}, \nabla^{\mathrm{B}}, (\cdot,\cdot)_{\mathrm{B}})$. Let P_1, P_2 be the corresponding opposite line bundles. The pull-back by $\zeta \colon \mathcal{M}^\circ_{\mathrm{CY}} \to \mathsf{L}^\circ$ of the propagator in the infinite-dimensional setting* (*Definition* 6.7.7)

$$\zeta^\star \Delta(\mathsf{P}_1, \mathsf{P}_2) \in \mathscr{H}om(\zeta^\star \mathbf{\Omega}_\circ(\log D^{\mathrm{big}})^{\otimes 2}, \mathcal{O}_{\mathcal{M}^\circ_{\mathrm{CY}}})$$

is induced from the propagator in the finite-dimensional setting (*Definition* 7.2.8)

$$\Delta(P_1, P_2) \in \Theta_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\})^{\otimes 2} = \mathscr{H}om((\Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\}))^{\otimes 2}, \mathcal{O}_{\mathcal{M}^\circ_{\mathrm{CY}}})$$

via the restriction map (74).

The log-cTEP structure $(\mathsf{F}_{\mathrm{CY}}, \nabla, (\cdot,\cdot))$ with base $(\mathcal{M}_{\mathrm{CY}}, D_{\mathrm{CY}})$ carries an $\mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}$-linear grading operator:

$$\mathrm{Gr}(P) = \big[z\tfrac{\partial}{\partial z}, P \big]$$

This is the grading inherited from the GKZ system (Definition 3.4.4). It is a shift of the grading operator gr inherited from the big B-model log-cTEP structure so that $\mathrm{Gr} = \mathrm{gr} + \frac{3}{2}$:

see Example 6.6.5. The $\mathcal{O}_{\mathcal{M}_{\mathrm{CY}}}[\![z]\!]$-module F_{CY} decomposes as:

$$\mathsf{F}_{\mathrm{CY}} = \prod_{i=0}^{\infty} \mathsf{F}_{\mathrm{CY}}^{(i)}$$

where $\mathsf{F}_{\mathrm{CY}}^{(i)}$ is the sub-bundle of degree i with respect to Gr. We have:

$$\mathsf{F}_{\mathrm{CY}}^{(i)} = \begin{cases} \big\langle 1 \big\rangle & i = 0 \\ \big\langle z, D_2, D_2 - 3D_1 \big\rangle & i = 1 \\ \big\langle z^2, zD_2, z(D_2 - 3D_1), D_2^2, D_1(D_2 - 3D_1) \big\rangle & i = 2 \\ \big\langle z^3, z^2 D_2, z^2(D_2 - 3D_1), zD_2^2, zD_1(D_2 - 3D_1), D_2^3 \big\rangle & i = 3 \\ z^{i-3}\mathsf{F}_{\mathrm{CY}}^{(3)} & i \geq 4 \end{cases}$$

Note that $\mathsf{F}_{\mathrm{CY}}^{(1)} = F^2 \subset H$. Recall from Definition 6.7.7 that the propagator $\Delta(\mathsf{P}_1, \mathsf{P}_2)$ is induced from the tensor $V \in \mathsf{F}_{\mathrm{B}}^{\mathrm{big}} \mathbin{\widehat{\otimes}} \mathsf{F}_{\mathrm{B}}^{\mathrm{big}} = \mathscr{H}om(\mathsf{F}_{\mathrm{B}}^{\mathrm{big}\vee} \otimes \mathsf{F}_{\mathrm{B}}^{\mathrm{big}\vee}, \mathcal{O}_{\mathcal{M}_{\mathrm{B}}^{\mathrm{big}}})$:

$$V(\varphi_1, \varphi_2) := \Omega^\vee(\Pi_1^\star \varphi_1, \Pi_2^\star \varphi_2)$$

as $\Delta(\mathsf{P}_1, \mathsf{P}_2) = (\mathrm{KS} \otimes \mathrm{KS})\,\mathrm{pr}^\star(V)$, where $\Pi_i \colon \mathsf{F}_{\mathrm{B}}^{\mathrm{big}}[z^{-1}] \to \mathsf{F}_{\mathrm{B}}^{\mathrm{big}}$ is the projection along P_i.

Proposition 8.4.2. *Let V_{CY} denote the restriction of V to $\mathcal{M}_{\mathrm{CY}}^\circ$. Then we have:*

$$V_{\mathrm{CY}} \in \big\langle (D_2 - 3D_1)^{\otimes 2} \big\rangle = (F^2 \cap \operatorname{Ker} N)^{\otimes 2} \subset \mathsf{F}_{\mathrm{CY}}^{(1)} \otimes \mathsf{F}_{\mathrm{CY}}^{(1)}.$$

Proof. Lemma 6.7.9 implies that $(\mathrm{Gr} \otimes 1 + 1 \otimes \mathrm{Gr})V_{\mathrm{CY}} = 2V_{\mathrm{CY}}$, and therefore that:

$$V_{\mathrm{CY}} \in \Big(\mathsf{F}_{\mathrm{CY}}^{(0)} \otimes \mathsf{F}_{\mathrm{CY}}^{(2)}\Big) \oplus \Big(\mathsf{F}_{\mathrm{CY}}^{(1)} \otimes \mathsf{F}_{\mathrm{CY}}^{(1)}\Big) \oplus \Big(\mathsf{F}_{\mathrm{CY}}^{(2)} \otimes \mathsf{F}_{\mathrm{CY}}^{(0)}\Big)$$

Let us write:

$$V_{\mathrm{CY}} = 1 \otimes a_2 + \sum \gamma_{ij} \phi_i \otimes \phi_j + a_2 \otimes 1$$

where $a_2 \in \mathsf{F}_{\mathrm{CY}}^{(2)}$, γ_{ij} is symmetric in i and j, and $(\phi_1, \phi_2, \phi_3) = (z, D_2, D_2 - 3D_1)$ is a basis for $\mathsf{F}_{\mathrm{CY}}^{(1)}$. We claim that the following equation holds:

$$\big(D_2 \otimes 1 - 1 \otimes D_2\big) V_{\mathrm{CY}} = 0 \tag{81}$$

where $D_2 = -zN$ is the endomorphism of F_{CY} (see §4.1). To see this, note that, since $D_2 = -zN$ preserves both F_{CY} and $\mathsf{P}_i|_{\mathcal{M}_{\mathrm{CY}}}$, we have that $D_2 \Pi_i = \Pi_i D_2$. Thus:

$$\begin{aligned} \big(D_2 \otimes 1 - 1 \otimes D_2\big) V_{\mathrm{CY}} &= \big(D_2 \otimes 1 - 1 \otimes D_2\big)\big(\Pi_1 \otimes \Pi_2\big)\Omega^\vee \\ &= \big(\Pi_1 \otimes \Pi_2\big)\big(D_2 \otimes 1 - 1 \otimes D_2\big)\Omega^\vee \end{aligned}$$

which is zero as D_2 is self-adjoint with respect to Ω. Writing out the graded pieces of (81) yields:

$$0 = 1 \otimes D_2 a_2 \qquad (0,3) \text{ component}$$

$$0 = D_2 \otimes a_2 - \sum \gamma_{ij} \phi_i \otimes D_2 \phi_j \qquad (1,2) \text{ component}$$

The first equation shows that $a_2 \in \operatorname{Ker} D_2$. The second equation gives $\phi_i \neq D_2 \implies D_2 \phi_j = 0$, i.e. $\phi_j = D_2 - 3D_1$, and thus $\gamma_{11} = \gamma_{12} = \gamma_{31} = \gamma_{32} = 0$. Symmetry of γ gives

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix}$$

The second equation now becomes $D_2 \otimes a_2 = \gamma_{22} D_2 \otimes D_2^2$, and thus $a_2 = \gamma_{22} D_2^2$. Since $D_2 a_2 = 0$, we conclude that $\gamma_{22} = 0$ and $a_2 = 0$. Thus $V_{\mathrm{CY}} = \gamma_{33}(D_2 - 3D_1) \otimes (D_2 - 3D_1)$ and the Proposition follows. □

Proof of Proposition 8.4.1. In view of the diagram (78) and Proposition 8.4.2, it suffices to show that the element $V_{\mathrm{fd}} \in F^2_{\mathrm{vec}} \otimes F^2_{\mathrm{vec}}$ defined by

$$V_{\mathrm{fd}}(\varphi_1, \varphi_2) := \Omega^\vee(\pi_1^\star \varphi_1, \pi_2^\star \varphi_2)$$

coincides with V_{CY} under the identification $F^2_{\mathrm{vec}} \cong F^2 \cap \operatorname{Ker} N$, where $\pi_i \colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ is the projection along P_i and $\varphi_i \in (F^2_{\mathrm{vec}})^\vee$. Take $\varphi \in \mathsf{F}^\vee_{\mathrm{CY}}$ and choose $v_i \in \mathsf{P}_i|_{\mathcal{M}^\circ_{\mathrm{CY}}}$ such that $\varphi = \Omega(v_i, \cdot)$ for $i = 1, 2$; then we have

$$\iota_\varphi V_{\mathrm{CY}} = v_1 - v_2 \in \mathsf{F}_{\mathrm{CY}}.$$

We know that $v_1 - v_2$ lies in $F^2 \cap \operatorname{Ker} N$ by Proposition 8.4.2. On the other hand, let $\overline{\varphi} \in (F_2 \cap \operatorname{Ker} N)^\vee \cong (F^2_{\mathrm{vec}})^\vee$ be the image of φ and let $w_i \in H$ be the degree-1 part of v_i. Then we have $\overline{\varphi} = \Omega(w_i, \cdot)$ on $F^2_{\mathrm{vec}} \cap \operatorname{Ker} N$. By the correspondence between P_i and P_i in §8.2, the image $[w_i]$ of w_i in $\overline{H} = \operatorname{Cok} N$ lies in P_i and thus:

$$\iota_{\overline{\varphi}} V_{\mathrm{fd}} = [w_1] - [w_2] \in F^2_{\mathrm{vec}} \subset \overline{H}.$$

Since $v_1 - v_2$ is of degree 1, we have $v_1 - v_2 = w_1 - w_2$. Thus $v_1 - v_2$ corresponds to $[w_1] - [w_2]$ under the isomorphism $F^2 \cap \operatorname{Ker} N \cong F^2_{\mathrm{vec}}$. The conclusion follows. □

8.5. **The Restriction Map on Fock Sheaves.** As discussed, correlation functions for the B-model Fock sheaf $\mathfrak{Fock}_{\mathrm{B}}$ are local sections $\boldsymbol{\nabla}^n C^{(g)}$ of $\boldsymbol{\Omega}^1_\circ(\log D^{\mathrm{big}})^{\otimes n}$ satisfying the conditions (Yukawa), (Jetness), (Grading and Filtration), and (Pole). Applying the restriction map (74) to such correlation functions $\{\boldsymbol{\nabla}^n C^{(g)}\}_{g,n}$ yields local sections of $\Omega^1_{\mathcal{M}^\circ_{\mathrm{CY}}}(\log\{0\})^{\otimes n}$ which satisfy (Yukawa), by Example 8.1.2, and (Jetness), by §8.3. To show that we get a restriction map on Fock sheaves

$$i^{-1}\mathfrak{Fock}_{\mathrm{B}} \to \mathfrak{Fock}^\circ_{\mathrm{CY}} \tag{82}$$

it remains only to check that the restriction map takes the propagator for the big B-model Fock sheaf $\mathfrak{Fock}_{\mathrm{B}}$ to the propagator for the finite-dimensional Fock sheaf $\mathfrak{Fock}^\circ_{\mathrm{CY}}$. This is the content of §8.4. Theorem 8.0.8 is proved.

8.6. **A Global Section of the Finite-Dimensional Fock Sheaf.** Applying the restriction map (82) to the global section $\mathscr{C}_{\mathrm{B}}$ of $\mathfrak{Fock}_{\mathrm{B}}$ (see Theorem 6.9.4) gives a global section $\mathscr{C}_{\mathrm{CY}}$ of $\mathfrak{Fock}^\circ_{\mathrm{CY}}$. We can compute the correlation functions of $\mathscr{C}_{\mathrm{CY}}$ with respect to the opposite line bundle P_{LR} by applying the restriction map to the Gromov–Witten wave function $\mathscr{C}_{\overline{Y}}$, which is a global section of $\mathfrak{Fock}_{\mathrm{A},\overline{Y}}$. With notation as in Definition 6.8.10, the mirror map (see Theorem 3.3.1) gives:

$$\begin{aligned} t^1 &= \log q_1 = \log y_1 - 3g(y_1) \\ t^2 &= \log q_2 = \log y_2 + g(y_1) \end{aligned} \qquad \text{where} \qquad g(y_1) = \sum_{d=1}^{\infty} \frac{(3d-1)!}{(d!)^3} (-1)^{d+1} y_1^d.$$

Thus $\frac{1}{3} dt^1 + dt^2 = \frac{1}{3} d\log y_1 + d\log y_2$, and the splitting (73) in these co-ordinates is given by $du_2 = \frac{1}{3} dt^1 + dt^2$. Restricting to the image of $\zeta\big|_{\mathcal{M}^\circ_{\mathrm{CY}}}$ sets

$$t^0 = t^3 = t^4 = t^5 = 0 \qquad q_2 = 0 \qquad x^i_n = \begin{cases} -1 & \text{if } n = 1 \text{ and } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathscr{C}_{\overline{Y}} = \{\boldsymbol{\nabla}^n C^{(g)}_{\overline{Y}}\}_{g,n}$ denote the Gromov–Witten wave function of $\overline{Y}$. Then the restriction of $\boldsymbol{\nabla}^n C^{(g)}_{\overline{Y}}$ to $\zeta(\mathcal{M}^\circ_{\mathrm{CY}})$ under the map (74) is given by:

$$\left(\partial_{t^1} - \tfrac{1}{3}\partial_{t^2}\right)^n F^{(g)}_{\overline{Y}}(t)\Big|_{Q_1=Q_2=1, t^0=t^3=t^4=t^5=0, q_2=0} (dt^1)^{\otimes n}$$

where $F^{(g)}_{\overline{Y}}$ is the Gromov–Witten potential (7) of $\overline{Y}$. Writing

$$n_{g,d} = \big\langle\big\rangle^{\overline{Y}}_{g,n,(d,0)} = \big\langle\big\rangle^{Y}_{g,n,d} \qquad\qquad d>0,$$

we have

$$\begin{aligned}
\text{the restriction of } \boldsymbol{\nabla}^3 C^{(0)}_{\overline{Y}} &= \left(-\frac{1}{3} + \sum_{d=1}^{\infty} d^3 n_{0,d}\, q_1^d\right)\left(\frac{dq_1}{q_1}\right)^{\otimes 3} \\
\text{the restriction of } \boldsymbol{\nabla} C^{(1)}_{\overline{Y}} &= \left(-\frac{1}{12} + \sum_{d=1}^{\infty} d n_{1,d}\, q_1^d\right)\frac{dq_1}{q_1} \qquad (83)\\
\text{the restriction of } C^{(g)}_{\overline{Y}} &= \sum_{d=0}^{\infty} n_{g,d}\, q_1^d \qquad\qquad \text{for } g\geq 2
\end{aligned}$$

where $q_1 = e^{t^1}$ and we used the fact that $\big\langle (h_1 - \frac{1}{3}h_2)^3\big\rangle^{\overline{Y}}_{1,3,0} = \int_{\overline{Y}} (h_1 - \frac{1}{3}h_2)^3 = -\frac{1}{3}$ and $\big\langle h_1 - \frac{1}{3}h_2\big\rangle^{\overline{Y}}_{1,1,0} = -\frac{1}{24}\int_{\overline{Y}} (h_1 - \frac{1}{3}h_2)\cup c_2(\overline{Y}) = -\frac{1}{12}$. These are the correlation functions of $\mathscr{C}_{\mathrm{CY}}$ with respect to the opposite line bundle P_{LR} and coincide with (the derivatives of) the Gromov–Witten potentials of Y.

In a similar way, we can compute the correlation functions of $\mathscr{C}_{\mathrm{CY}}$ with respect to the opposite line bundle P_{orb} by applying the restriction map (82) to the Gromov–Witten wave function $\mathscr{C}_{\overline{\mathcal{X}}}$. Let $\{\mathfrak{t} = t^4, \log q = t^1\}$ denote the co-ordinates on $H^2_{\mathrm{orb}}(\overline{\mathcal{X}})$ dual to $\{\mathbf{1}_{\frac{1}{3}}, h\}$ defined in §2.1. Recall that the mirror map in Theorem 3.3.2 gives

$$\mathfrak{t} = \sum_{n=0}^{\infty} (-1)^n \frac{\prod_{j=0}^{n-1}(\frac{1}{3}+j)^3}{(3n+1)!}\mathfrak{y}_1^{3n+1}, \qquad\qquad \log q = 3\log \mathfrak{y}_2.$$

Thus the splitting (73) is given in these co-ordinates by $du_2 = \frac{1}{3} d\log q$. Writing

$$n^{\mathrm{orb}}_{g,k} = \left\langle \mathbf{1}_{\frac{1}{3}}, \dots, \mathbf{1}_{\frac{1}{3}}\right\rangle^{\overline{\mathcal{X}}}_{g,k,0} = \left\langle \mathbf{1}_{\frac{1}{3}}, \dots, \mathbf{1}_{\frac{1}{3}}\right\rangle^{\mathcal{X}}_{g,k,0} \qquad \text{when } 2g-2+k>0$$

we have:

$$\begin{aligned}
\text{the restriction of } \boldsymbol{\nabla}^3 C^{(0)}_{\overline{\mathcal{X}}} &= \left(\sum_{k=0}^{\infty} n^{\mathrm{orb}}_{0,k+3}\frac{\mathfrak{t}^k}{k!}\right)(d\mathfrak{t})^{\otimes 3} \\
\text{the restriction of } \boldsymbol{\nabla} C^{(1)}_{\overline{\mathcal{X}}} &= \left(\sum_{k=0}^{\infty} n^{\mathrm{orb}}_{1,k+1}\frac{\mathfrak{t}^k}{k!}\right)d\mathfrak{t} \qquad (84)\\
\text{the restriction of } C^{(g)}_{\overline{\mathcal{X}}} &= \sum_{k=0}^{\infty} n^{\mathrm{orb}}_{g,k}\frac{\mathfrak{t}^k}{k!} \qquad\qquad \text{for } g\geq 2.
\end{aligned}$$

These are the correlation functions for $\mathscr{C}_{\mathrm{CY}}$ with respect to the opposite line bundle P_{orb} and coincide with (the derivatives of) the Gromov–Witten potentials of $\mathcal{X}$. This proves:

Theorem 8.6.1 (cf. Theorem 6.9.4)**.** *Let $\mathscr{C}_{\mathrm{CY}}$ be the section of the Fock sheaf $\mathfrak{Fock}^\circ_{\mathrm{CY}}$ over $\mathcal{M}^\circ_{\mathrm{CY}}$ given as the restriction of the global section $\mathscr{C}_{\mathrm{B}} \in \mathfrak{Fock}_{\mathrm{B}}$ under the map* (82)*. Then:*

(a) *around $y_1 = 0$ and with respect to the opposite line bundle P_{LR}, the correlation functions of $\mathscr{C}_{\mathrm{CY}}$ are given by the Gromov–Witten potential of Y as in* (83)*;*
(b) *around $y_1 = \infty$ and with respect to the opposite line bundle P_{orb}, the correlation functions of $\mathscr{C}_{\mathrm{CY}}$ are given by the Gromov–Witten potential of $\mathcal{X}$ as in* (84)*.*

Remark 8.6.2 (cf. Remark 6.9.5)**.** The existence of a global section $\mathscr{C}_{\mathrm{CY}}$ with these properties establishes a higher-genus version of the Crepant Resolution Conjecture [15, 27, 28, 55, 73] for the crepant resolution $Y \to \mathcal{X}$.

9. Estimates at the Conifold Point

Given an open set $U \subset \mathcal{M}^\circ_{\mathrm{CY}}$ and an opposite line bundle P over U, correlation functions with respect to P for the global section $\mathscr{C}_{\mathrm{CY}}$ of $\mathfrak{Fock}^\circ_{\mathrm{CY}}$ are holomorphic functions on U. Recall that there is a unique opposite line bundle P_{con} near the conifold point $-\frac{1}{27} \in \mathcal{M}_{\mathrm{CY}}$: see Notation 7.2.7. In this section we show that genus-g, m-point correlation functions for $\mathscr{C}_{\mathrm{CY}}$ with respect to P_{con} extend meromorphically across the conifold point and have a pole of order at most $2g-2+m$ there. This shows that $\mathscr{C}_{\mathrm{CY}}$ satisfies the conifold pole condition in Definition 7.3.1, and thus that $\mathscr{C}_{\mathrm{CY}}$ extends to a global section of $\mathfrak{Fock}_{\mathrm{CY}}$ over $\mathcal{M}_{\mathrm{CY}}$. This follows immediately from the corresponding statement about $\mathscr{C}_{\mathrm{B}}$:

Theorem 9.0.3. *Let $\mathbf{P}_{\mathrm{con}}$ denote the unique opposite module for $\mathcal{F}^{\mathrm{big}}_{\mathrm{B}}$ at the conifold point that is compatible with the Deligne extension (see Proposition* 5.1.7*) and let $\mathsf{P}_{\mathrm{con}}$ denote the corresponding opposite module for $\mathsf{F}^{\mathrm{big}}_{\mathrm{B}}$ (see Example* 6.7.4*). Consider the pull-back of the genus-g, m-point correlation function for $\mathscr{C}_{\mathrm{B}}$ with respect to $\mathsf{P}_{\mathrm{con}}$ along the primitive section $\zeta\colon \mathcal{M}^\circ_{\mathrm{B}} \to \mathbf{L}^\circ$; this gives a local section of $\Omega^1_{\mathcal{M}^\circ_{\mathrm{B}}}(\log D)^{\otimes m}$ which is defined in a neighbourhood of the conifold divisor ($y_1 = -\frac{1}{27}$) but is not defined on the divisor itself. This extends meromorphically across the conifold divisor, and has a pole of order at most $2g-2+m$ there.*

In outline: this will follow from the Givental's higher-genus formula – which we used to define the B-model global section $\mathscr{C}_{\mathrm{B}}$, and which expresses each genus-g correlation function as a finite sum over Feynman graphs – together with an analysis of the stationary phase asymptotics of various oscillating integrals. The stationary phase analysis will allow us to estimate the pole order of each ingredient of Givental's formula.

9.1. **Critical Points.** Consider the Landau–Ginzburg mirror (π, W, ω) from §3.1, and identify the fiber of π with $(\mathbb{C}^\times)^3$ by setting

$$w_1 = x_1x_3, \qquad w_2 = x_2x_3, \qquad w_3 = \frac{y_1x_3}{x_1x_2}, \qquad w_4 = x_3, \qquad w_5 = \frac{y_2}{x_3}$$

where $(x_1, x_2, x_3) \in (\mathbb{C}^\times)^3$. Then the superpotential becomes

$$W(x_1, x_2, x_3) = x_1x_3 + x_2x_3 + \frac{y_1x_3}{x_1x_2} + x_3 + \frac{y_2}{x_3}$$

and there are six critical points:

$$(x_1^c, x_2^c, x_3^c) = \left(\sqrt[3]{y_1}, \sqrt[3]{y_1}, \sqrt{\frac{y_2}{1 + 3\sqrt[3]{y_1}}} \right)$$

Writing $T = y_1 + \frac{1}{27}$ for the co-ordinate near the conifold point, we see that four of the critical points extend holomorphically across $T = 0$ and the other two escape to infinity there. The divergent critical points are those for which the critical value

$$W(x_1^c, x_2^c, x_3^c) = 2\sqrt{y_2(1 + 3\sqrt[3]{y_1})} \tag{85}$$

approaches zero as $T \to 0$. We also note that $x_3^c = O(T^{-1/2})$ for a divergent c.

Introduce logarithmic co-ordinates near a critical point c, setting

$$x_1 = x_1^c \exp((x_3^c)^{-1/2}\theta_1), \qquad x_2 = x_2^c \exp((x_3^c)^{-1/2}\theta_2), \qquad x_3 = x_3^c \exp((x_3^c)^{1/2}\theta_3),$$

and writing

$$W_{ij\cdots k}(c) = \left(\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}\cdots\frac{\partial}{\partial\theta_k}W\right)(x_1^c, x_2^c, x_3^c)$$

for the multiple logarithmic derivative of W at c. Then the logarithmic Hessian at c satisfies:

$$H_c = \begin{pmatrix} 2\sqrt[3]{y_1} & \sqrt[3]{y_1} & 0 \\ \sqrt[3]{y_1} & 2\sqrt[3]{y_1} & 0 \\ 0 & 0 & 2y_2 \end{pmatrix} \qquad H_c^{-1} = \begin{pmatrix} \frac{2}{3\sqrt[3]{y_1}} & -\frac{1}{3\sqrt[3]{y_1}} & 0 \\ -\frac{1}{3\sqrt[3]{y_1}} & \frac{2}{3\sqrt[3]{y_1}} & 0 \\ 0 & 0 & \frac{1}{2y_2} \end{pmatrix}$$

and $\det(H_c) = 6y_1^{2/3}y_2$. These quantities are holomorphic at $T = 0$. For a divergent critical point c, and for m, $n \geq 1$ and $l \geq 0$, we have:

$$W_{\underbrace{1\cdots1}_{m}\underbrace{3\cdots3}_{l}}(c) = W_{\underbrace{2\cdots2}_{m}\underbrace{3\cdots3}_{l}}(c) = \begin{cases} 0 & m \text{ odd} \\ O(T^{\frac{m-l-2}{4}}) & m \text{ even} \end{cases}$$

$$W_{\underbrace{3\cdots3}_{m}}(c) = \begin{cases} 0 & m \text{ odd} \\ O(T^{\frac{-m+2}{4}}) & m \text{ even} \end{cases} \qquad W_{\underbrace{1\cdots1}_{n}\underbrace{2\cdots2}_{m}\underbrace{3\cdots3}_{l}}(c) = O(T^{\frac{n+m-l-2}{4}})$$

as $T \to 0$. At non-divergent critical points, the multiple logarithmic derivatives of W are holomorphic at $T = 0$. Altogether, we get

$$W_{i_1\ldots i_k}(c) = \begin{cases} O(T^{-\frac{k}{4}+\frac{1}{2}}) & \text{if } c \text{ is divergent;} \\ O(1) & \text{if } c \text{ is non-divergent.} \end{cases} \tag{86}$$

9.2. **Givental's Higher-Genus Formula.** Choose a point $t \in \mathcal{M}_{\mathrm{B}} \setminus D \subset \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. The B-model log-cTEP structure is tame semisimple at t because W has pairwise distinct eigenvalues. Correlation functions for the B-model wave function $\mathscr{C}_{\mathrm{B}}$ with respect to $\mathsf{P}_{\mathrm{con}}$ are obtained by applying a certain quantized operator $\widehat{R}_t$ to the product of Kontsevich–Witten tau-functions

$$\mathcal{T} = \prod_c \tau(\mathbf{q}^c) \qquad \text{where } \mathbf{q}^c = q_0^c + q_1^c z + q_2^c z^2 + \cdots \in \mathbb{C}[\![z]\!].$$

Here c ranges over critical points of W and R_t is an invertible $\mathbb{C}[\![z]\!]$-linear operator:

$$R_t \colon \prod_c \mathbb{C}[\![z]\!] \longrightarrow \mathsf{F}_{\mathrm{B}}^{\mathrm{big}}|_t$$

This is Givental's formula for the ancestor potentials of a semisimple Frobenius manifold. It is discussed, in a notation and framework convenient for our setting, in [23, §§3–4]; the original reference is [41]. The operator R_t here is a certain "asymptotic fundamental solution" for the connection ∇^{B}, whose existence near t is guaranteed in general by [40, Proposition 1.1] and which in our setting we can obtain from a genuine fundamental solution matrix by taking stationary phase asymptotics.

Recall from Proposition 5.1.7 that the flat trivialization of $\mathcal{F}_{\mathrm{B}}$ corresponding to $\mathbf{P}_{\mathrm{con}}$ is given by the frame 1, D_2, D_2^2, D_2^3, D_1, $(1+27y_1)D_1^2$ at the conifold point $(y_1,y_2)=(-1/27,0)$. Let $\mathcal{D}_1,\dots,\mathcal{D}_6$ denote differential operators whose classes in the GKZ system give the flat trivialization associated with $\mathbf{P}_{\mathrm{con}}$ and coincide with the above frame at the conifold point. Let c be a critical point, and let $\Gamma_+(c)$ denote the Lefschetz thimble given by upward gradient flow from c of the function $x \mapsto \Re\bigl(\frac{W(x)}{z}\bigr)$. Let $\{s_c\}$ denote the flat sections of $\mathcal{F}_{\mathrm{B}}$ dual to the cycles $\{\Gamma_+(c)\}$ so that

$$\frac{1}{(2\pi z)^{3/2}} \int_{\Gamma_+(c)} s_{c'} = \delta_{c,c'} \qquad \text{(cf. §3.2).}$$

Define a matrix $S_t = (s_{jc})$ with rows indexed by $j \in \{1,2,\dots,6\}$ and columns indexed by critical points c of W, by expressing the sections s_c with respect to the frame $\mathcal{D}_1,\dots,\mathcal{D}_6$:

$$s_c = \sum_{j=1}^{6} s_{jc} \mathcal{D}_j$$

The matrix S_t is a fundamental solution matrix for ∇^{B}; its entries are multi-valued holomorphic functions on $\mathcal{M}_{\mathrm{B}}^{\circ} \times \mathbb{C}^{\times}$. The duality between the sections $\{s_c\}$ and the cycles $\{\Gamma_c\}$ implies that the (c,j) entry of the inverse matrix S_t^{-1} is the oscillating integral

$$[S_t^{-1}]_{(c,j)} = \frac{1}{(2\pi z)^{3/2}} \mathcal{D}_j \int_{\Gamma_+(c)} e^{-W/z} \omega. \tag{87}$$

In the basis $\{\mathcal{D}_i\}$ of $\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}|_t$, the linear operator R_t^{-1} is represented by a formal power series in z with coefficients in 6 by 6 matrices. The (c,j)-entry of R_t^{-1} is obtained from the (c,j)-entry (87) of S_t^{-1} by stationary phase expansion:

$$e^{W(c)/z} [S_t^{-1}]_{(c,j)} \;\sim\; [R_t^{-1}]_{(c,j)} \qquad \text{as } z \to +0.$$

The basis $\mathcal{D}_i$ can be calculated explicitly up to order $O(T)$.

Lemma 9.2.1. *Define* $(\mathcal{D}_1',\dots,\mathcal{D}_6') := (1, D_2, D_2^2 - y_2, D_2^3 + zy_2 - 2y_2D_2, D_1, (1+27y_1)D_1^2)$. *Then* $\mathcal{D}_i = \mathcal{D}_i' + O(T)$ *in the GKZ system.*

Proof. We need to show that $\mathcal{D}_i'$ gives a flat trivialization associated with $\mathbf{P}_{\mathrm{con}}$ along the divisor $(y_1 = -1/27)$. Since $\{\mathcal{D}_i'\}$ coincides with the frame $1, D_2, D_2^2, D_2^3, D_1, (1+27y_1)D_1^2$ at $(y_1,y_2)=(-1/27,0)$, it suffices to check that the action of D_2 in the basis $\{\mathcal{D}_i'\}$ is represented by a matrix independent of z. Indeed, the action of D_2 in the basis $\{\mathcal{D}_i'\}$ is

$$\begin{pmatrix} 0 & y_2 & 0 & -2y_2^2 & 0 & 0 \\ 1 & 0 & y_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_2 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9y_2 & 0 & 0 \end{pmatrix}$$

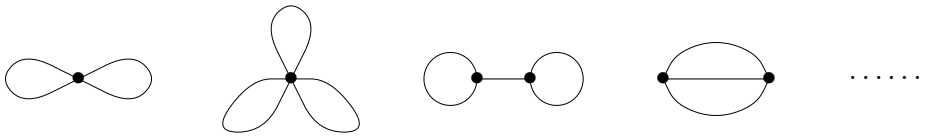
along the divisor $y_1 = -1/27$. The lemma follows. □

9.3. **Stationary Phase Asymptotics.** We say that a function f of T *has T-order α* if $f(T) = O(T^{\alpha})$ as $T \to 0$. We evaluate the T-order of R_t^{-1} by examining the stationary phase asymptotics of (87), where $T = y_1 + \frac{1}{27}$ is the co-ordinate of t and we shall keep $y_2 \neq 0$ fixed.

Let c be a divergent critical point, and start with the oscillatory integral associated with c. We have

$$\frac{1}{(2\pi z)^{3/2}}\int_{\Gamma_+(c)} e^{-W/z}\omega \sim \frac{e^{-W(c)/z}}{\sqrt{x_3^c}\sqrt{\det H_c}}\left[e^{\frac{z}{2}\Delta}\exp\left(-\frac{1}{z}\sum_{k=3}^{\infty}\frac{1}{k!}\sum_{i_1,\dots,i_k} W_{i_1\cdots i_k}(c)\theta_{i_1}\cdots\theta_{i_k}\right)\right]_{\theta_1=\theta_2=\theta_3=0}$$

where $\Delta = \sum_{i,j=1}^{3} H_c^{ij}\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j}$ and H_c^{ij} are the matrix entries of the inverse Hessian. We can obtain this asymptotic expansion by expanding the integrand in Taylor series in $\theta_1,\theta_2,\theta_3$ and performing termwise Gaussian integrals: see [20, §6.2; 66, Appendix A]. The factor $(x_3^c)^{-1/2}$ comes from $\omega = (x_3^c)^{-1/2}d\theta_1 d\theta_2 d\theta_3$. By Wick's theorem, the term in square brackets is the sum over graphs[17]



where each vertex has valency at least 3, we place the tensor $-\frac{1}{z}\sum_{i_1,\dots,i_k} W_{i_1\cdots i_k}d\theta_{i_1}\cdots d\theta_{i_k}$ at a vertex of valency k, and contract using the bivector field $z\Delta$ on each edge. A graph with E edges and V vertices contributes to the coefficient of z^{E-V} in the asymptotic expansion, and if the graph has vertices of valencies $k_1,\dots,k_V$, then its contribution has T-order $\sum_{i=1}^{V}(-\frac{k_i}{4}+\frac{1}{2}) = -\frac{E-V}{2}$; here we used (86). Thus

$$\frac{1}{(2\pi z)^{3/2}}\int_{\Gamma_+(c)} e^{-W/z}\omega \sim e^{-W(c)/z}\sum_{n=0}^{\infty} a_n z^n \qquad \text{with } a_n = O(T^{\frac{1}{4}-\frac{n}{2}}) \text{ as } T\to 0.$$

The stationary phase asymptotics of $\mathcal{D}\int_{\Gamma_+(c)} e^{-W/z}\omega$, with $\mathcal{D}$ a differential operator in y_1 and y_2, can be computed similarly. For example, if $\mathcal{D} = D_2 = -zy_2\frac{\partial}{\partial y_2}$, we have

$$\frac{1}{(2\pi z)^{3/2}}D_2\int_{\Gamma_+(c)} e^{-W/z}\omega = \frac{1}{(2\pi z)^{3/2}}\frac{y_2}{x_3^c}\int_{\Gamma_+(c)} e^{-W/z-\sqrt{x_3^c}\theta_3}\omega$$
$$\sim \frac{e^{-W(c)/z}}{\sqrt{x_3^c}\sqrt{\det(H_c)}}\frac{y_2}{x_3^c}\left[e^{\frac{z}{2}\Delta}\exp\left(-\sqrt{x_3^c}\theta_3 - \frac{1}{z}\sum_{k=3}^{\infty}\frac{1}{k!}\sum_{i_1,\dots,i_k} W_{i_1\dots i_k}(c)\theta_{i_1}\cdots\theta_{i_k}\right)\right]_{\theta_1=\theta_2=\theta_3=0}$$

We apply Wick's theorem again to express this as a sum over graphs. In this case, we allow graphs to have additional vertices of valency 1 where we place the tensor $(-\sqrt{x_3^c}d\theta_3)$. If a graph has V vertices of valencies $k_1,\dots,k_V \geq 3$, L vertices of valency 1, and E edges, its contribution has T-order $\sum_{i=1}^{V}(-\frac{k_i}{4}+\frac{1}{2}) - \frac{L}{4} = -\frac{E-V}{2}$. Hence

$$\frac{1}{(2\pi z)^{3/2}}D_2\int_{\Gamma_+(c)} e^{-W/z}\omega \sim e^{-W(c)/z}\sum_{n=0}^{\infty} b_n z^n \qquad \text{with } b_n = O(T^{\frac{3}{4}-\frac{n}{2}}) \text{ as } T\to 0.$$

Combining this method with Lemma 9.2.1, we can compute the T-orders of the asymptotic expansion of $\mathcal{D}_i\int_{\Gamma_+(c)} e^{-W/z}\omega$ for all i. The analysis for a non-divergent critical point is

[17] As an illustration, we listed all graphs that contribute whose number of edges is less than or equal to 3. Such a graphical technique is standard in quantum field theory: see e.g. [10, §2; 82, Chapter I.7].

identical except for the fact that everything is holomorphic at $T = 0$. Ordering the critical points so that the first two are divergent and the last four are non-divergent, we see that

$$R_t^{-1} = \sum_{n=0}^{\infty} \begin{pmatrix} [\frac{1}{4} - \frac{n}{2}] & [\frac{3}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] & [\frac{3}{4} - \frac{n}{2}] & [-\frac{1}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] \\ [\frac{1}{4} - \frac{n}{2}] & [\frac{3}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] & [\frac{3}{4} - \frac{n}{2}] & [-\frac{1}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] \\ [0] & [0] & [0] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [0] & [1] \end{pmatrix} z^n \tag{88}$$

where $[\alpha]$ denotes a term that has T-order α.

We will need similar estimates for the matrix entries of R_t. For this we use the unitarity condition

$$R_t(-z)^{-\mathrm{T}} R_t(z)^{-1} = G \quad \text{or equivalently,} \quad \sum_c [R_t(-z)^{-1}]_{(c,i)} [R_t(z)^{-1}]_{(c,j)} = g_{ij}$$

from [41, §1.3]; here $G = (g_{ij}) = (((-)^\star \mathcal{D}_i, \mathcal{D}_j)_{\mathrm{B}})$ is the Gram matrix in Proposition 3.5.1(b) evaluated at the conifold point $(y_1, y_2) = (-1/27, 0)$. The unitarity follows directly from the description (26) of the B-model pairing. Thus

$$R_t(z) = G^{-1} R_t(-z)^{-\mathrm{T}}$$

and since

$$G^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 & 0 & 1 \\ 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -3 & 0 \end{pmatrix}$$

we conclude that

$$R_t = \sum_{k=0}^{\infty} \begin{pmatrix} [\frac{3}{4} - \frac{n}{2}] & [\frac{3}{4} - \frac{n}{2}] & [0] & [0] & [0] & [0] \\ [\frac{1}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] & [0] & [0] & [0] & [0] \\ [\frac{3}{4} - \frac{n}{2}] & [\frac{3}{4} - \frac{n}{2}] & [0] & [0] & [0] & [0] \\ [\frac{1}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] & [0] & [0] & [0] & [0] \\ [\frac{1}{4} - \frac{n}{2}] & [\frac{1}{4} - \frac{n}{2}] & [1] & [1] & [1] & [1] \\ [-\frac{1}{4} - \frac{n}{2}] & [-\frac{1}{4} - \frac{n}{2}] & [0] & [0] & [0] & [0] \end{pmatrix} z^n. \tag{89}$$

9.4. **The Proof of Theorem 9.0.3.** It remains to translate these estimates for the pole order in T of stationary phase asymptotics to estimates for the pole order in T of correlation functions. Choose a point $t \in \mathcal{M}_{\mathrm{B}} \setminus D \subset \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$. As before, we fix the co-ordinate $y_2 \neq 0$ of the point t and study the asymptotics of correlation functions as $T = y_1 + \frac{1}{27}$ goes to zero. Introduce algebraic co-ordinates $(\tilde{t}, \boldsymbol{x})$ on the total space L of the big B-model log-cTEP structure, where

- $\tilde{t} \in \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ represents a point in a neighbourhood of t; and
- $\boldsymbol{x} = \sum_{n=1}^{\infty} \sum_{i=1}^{6} x_n^i e_i z^n \in z\mathbb{C}^6[\![z]\!]$ are co-ordinates along the fiber of $\mathsf{L} \to \mathcal{M}_{\mathrm{B}}^{\mathrm{big}}$ associated with the frame $\mathcal{D}_1, \dots, \mathcal{D}_6$ of $\mathsf{F}_{\mathrm{B}}^{\mathrm{big}}$.

See Definition 6.2.2. There is a distinguished flat co-ordinate system[18] $\hat{\mathbf{q}} \in \mathbb{C}^6[\![z]\!]$ in the formal neighbourhood of the fiber L_t° in L°, associated with the frame $\{\mathcal{D}_i\}$: see [25, Definition 4.28].

[18]Note that the flat co-ordinate system $\hat{\mathbf{q}}$ depends on the choice of t.

This is given by

$$\hat{\mathbf{q}} = [M(\tilde{t}, z)\boldsymbol{x}]_+ \tag{90}$$

where $[\cdots]_+$ means the non-negative part as a z-series. The inverse fundamental solution matrix $M(\tilde{t}, z)$ here is characterized by the conditions $M(t, z) = \mathrm{Id}$ and $dM(\tilde{t}, z) = \frac{1}{z} M(\tilde{t}, z) A(\tilde{t})$, where d is the differential in the $\tilde{t}$-direction and $\frac{1}{z} A(\tilde{t})$ is the matrix-valued connection 1-form for ∇^{B} written in the frame $\{\mathcal{D}_i\}$.

Let $\{\boldsymbol{\nabla}^m C^{(g)}\}_{g,m}$ denote the correlation functions for $\mathscr{C}_{\mathrm{B}}$ with respect to $\mathsf{P}_{\mathrm{con}}$. Givental's higher genus formula discussed in §9.2 gives correlation functions along $\mathbb{L}_t^\circ$, expressed in terms of the flat co-ordinate system $\hat{\mathbf{q}} = \sum_{n=0}^\infty \sum_{i=1}^6 \hat{q}_n^i e_i z^n$. Writing

$$\boldsymbol{\nabla}^m C^{(g)}\big|_{\mathbb{L}_t^\circ} = \sum C^{(g)}_{(n_1,i_1),\dots,(n_m,i_m)}(t, \boldsymbol{x})\, dq_{n_1}^{i_1} \otimes \cdots \otimes d\hat{q}_{n_m}^{i_m}$$

and setting[19]

$$\mathcal{A}^{\mathrm{con}}_{(t,\boldsymbol{x})} = \exp\left(\sum_{g=0}^\infty \sum_{m : 2g-2+m>0} \frac{\hbar^{g-1}}{m!} \sum_{n_1,\dots,n_m} \sum_{1 \le i_1,\dots,i_m \le 6} C^{(g)}_{(n_1,i_1),\dots,(n_m,i_m)}(t, \boldsymbol{x}) \hat{a}_{n_1}^{i_1} \cdots \hat{a}_{n_m}^{i_m} \right),$$

we have

$$\mathcal{A}^{\mathrm{con}}_{(t,\boldsymbol{x})} = \left[\exp\left(\frac{\hbar}{2} \sum V_t^{(n,c),(n',c')} \frac{\partial}{\partial q_n^c} \frac{\partial}{\partial q_{n'}^{c'}} \right) \mathcal{T}' \right]_{\mathbf{q}^c = [R_t^{-1}(\boldsymbol{x} + \hat{\mathbf{a}})]^c} \tag{91}$$

where $V_t^{(n,c),(n',c')}$ are coefficients of Givental's propagator defined below, and $\mathcal{T}'$ is the product of the Kontsevich–Witten tau function modified at genus 1:

$$\mathcal{T}'(\mathbf{q}) = \prod_c \exp\left(\sum_{g=0}^\infty \hbar^{g-1} \left(\mathcal{F}^g_{\mathrm{pt}}(\mathbf{q}^c) - \delta_{g,1} \mathcal{F}^1_{\mathrm{pt}}([R_t^{-1}\boldsymbol{x}]^c) \right) \right).$$

Recall that $\mathcal{F}^g_{\mathrm{pt}}$ is the genus-g descendant potential (16) of a point; we regard it as a function of the dilaton-shifted co-ordinate $\mathbf{q}^c = -z + \mathbf{t}$. Formula (91) follows from the definition of $\mathscr{C}_{\mathrm{ss}}$ [25, Definition 7.9], the fact that $\{\boldsymbol{\nabla}^m C^{(g)}\}$ can be obtained from $\mathscr{C}_{\mathrm{ss}}$ by the transformation rule $T(\mathsf{P}_{\mathrm{ss}}, \mathsf{P}_{\mathrm{con}})$, and the following facts:

- the 'conifold ancestor potential' $\mathcal{A}^{\mathrm{con}}_{(t,\boldsymbol{x})}$ is the image under the formalization map of $\{\boldsymbol{\nabla}^m C^{(g)}\}$ at $(t, \boldsymbol{x})$, see [25, Definition 5.11], where the formalization map is the one associated with the frame $\{\mathcal{D}_i\}$.
- $\mathcal{T}'(\mathbf{q})$ is the image under the formalization map of $\mathscr{C}_{\mathrm{ss}}$ at $(t, R_t^{-1}\boldsymbol{x})$, where the formalization map is the one associated with the semisimple trivialization. To see this combine Lemma 5.13 and Lemma 7.13 of [25].
- the transformation rule $T(\mathsf{P}_{\mathrm{ss}}, \mathsf{P}_{\mathrm{con}})$ is expressed in terms of the action of Givental's quantized operator $\widehat{R}_t$ through the formalization map; see [25, Theorem 5.14].

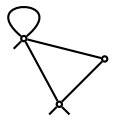
Definition 9.4.1. *Givental's propagator* $\{V_t^{(n,c),(n',c')}\}$ associated with R_t is defined by

$$\sum_{n=0}^\infty \sum_{n'=0}^\infty (-1)^{n+n'} V_t^{(n,c),(n',c')} w^n z^{n'} = \left[\frac{R_t(-w)^{-1} R_t(z) - \mathrm{Id}}{z + w} \right]_{(c,c')}$$

where c, c' range over critical points of W.

[19] $\mathcal{A}^{\mathrm{con}}_{(t,\boldsymbol{x})}$ is a formal power series in the shifted flat co-ordinate $\hat{\mathbf{a}} = \sum_{n=0}^\infty \sum_{i=1}^6 \hat{a}_n^i e_i z^n := \hat{\mathbf{q}} - \boldsymbol{x}$ on a neighbourhood of $(t, \boldsymbol{x})$.

The formula (91) together with the discussion in [23, §3] implies that $C^{(g)}_{(n_1,i_1),\dots,(n_m,i_m)}(t,\boldsymbol{x})$ is given by the sum over decorated connected Feynman graphs



where

- each vertex v is labelled by an integer $g_v \geq 0$;
- the graph has m external half-edges, called *legs*, labelled by $\{1,\dots,m\}$;
- a label $(l,c) \in \mathbb{Z}_{\geq 0} \times \{\text{critical points of } W\}$ is assigned to each pair of a vertex and a half-edge incident to it; note that we assign a label (l,c) to legs (external half-edges) too so that legs have two different kinds of labels;
- the Euler number χ of the graph satisfies $g = 1 - \chi + \sum_{v:\text{vertex}} g_v$;

and we require that, for each vertex v, if $(l_1,c_1),\dots,(l_k,c_k)$ are all the labels attached to half-edges incident to v, then

$$l_1 + \cdots + l_k \leq 3g_v - 3 + k \qquad \text{and} \qquad 2g_v - 2 + k > 0$$

There are finitely many such decorated Feynman graphs [41]. The contribution of such a graph Γ to $C^{(g)}_{(n_1,i_1),\dots,(n_m,i_m)}(t,\boldsymbol{x})$ is:

$$\frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} (\text{edge term for } e) \prod_{v \in V(\Gamma)} (\text{vertex term for } v) \prod_{\ell \in L(\Gamma)} (\text{leg term for } \ell) \tag{92}$$

where the edge term for an edge with labels (l,c), (l',c') is the coefficient $V_t^{(l,c),(l',c')}$ of Givental's propagator; the vertex term for a vertex v incident to half-edges with labels $(l_1,c_1),\dots,(l_k,c_k)$ is[20]

$$\begin{aligned} \frac{\partial^k \mathcal{F}^{g_v}_{\mathrm{pt}}}{\partial q^c_{l_1} \partial q^c_{l_2} \cdots \partial q^c_{l_k}}\bigg|_{\mathbf{q}^c = [R_t^{-1}\boldsymbol{x}]^c} &= \sum_{p=0}^{\infty} \frac{1}{p!} \left\langle \psi_1^{l_1}, \dots, \psi_k^{l_k}, \mathbf{t}^c(\psi_{k+1}), \dots \mathbf{t}^c(\psi_{k+p}) \right\rangle^{\mathrm{pt}}_{g_v,k+p,0} \\ &= \sum_{p=0}^{\infty} (-q_1^c)^{-(2g_v-2+k+p)} \frac{1}{p!} \left\langle \psi_1^{l_1}, \dots, \psi_k^{l_k}, \mathbf{q}^c_{\geq 2}(\psi_{k+1}), \dots, \mathbf{q}^c_{\geq 2}(\psi_{k+p}) \right\rangle^{\mathrm{pt}}_{g_v,k+p,0} \end{aligned} \tag{93}$$

if $c_1 = \cdots = c_k = c$ and zero otherwise, where we set

$$\mathbf{q}^c(z) = -z + \mathbf{t}^c(z) = [R_t(z)^{-1}\boldsymbol{x}(z)]^c, \tag{94}$$

and $\mathbf{q}^c_{\geq 2}(z)$ denotes the truncation of the z-series $\mathbf{q}^c(z)$ at degree two; and finally the leg term of a leg ℓ with labels $s \in \{1,\dots,m\}$ and (l,c) is $[R_t^{-1} e_{i_s} z^{n_s}]^c_l$, where $[\cdots]^c_l$ denotes the coefficient of z^l in the cth component.

Now we restrict $(t,\boldsymbol{x})$ to lie on the image of the primitive section $\zeta\colon \mathcal{M}^\circ_{\mathrm{B}} \to \mathbf{L}^\circ$, and evaluate the T-order of $C^{(g)}_{(n_1,i_1),\dots,(n_m,i_m)}(t,\boldsymbol{x})$. The primitive section ζ is given by $-z = -z\mathcal{D}'_1$ in the GKZ system. Since the frame $\{\mathcal{D}_i\}$ is homogeneous and $\mathcal{D}'_1 = 1$ has the lowest possible degree, it follows that $\mathcal{D}_1 = f(T)\mathcal{D}'_1$ for some holomorphic function $f(T)$ such that $f(0) = 1$ and f is independent of y_2 or z. Therefore the section ζ is given in terms of $\boldsymbol{x}$ by

$$x^1_1 = -1 + O(T), \quad x^2_1 = \cdots = x^6_1 = 0, \quad x^1_n = \cdots = x^6_n = 0 \quad \text{for all } n \geq 2. \tag{95}$$

[20] We used the Dilaton equation in the second line.

On this locus, using (94) and (88), we have for $n \geq 1$,

$$q_n^c = \begin{cases} O(T^{\frac{3}{4}-\frac{n}{2}}) & \text{if } c \text{ is divergent;} \\ O(1) & \text{if } c \text{ is non-divergent.} \end{cases}$$

A more careful analysis of the first column of R_t^{-1} shows that q_1^c is exactly of T-order $\frac{1}{4}$ if c is divergent, and is exactly of T-order 0 if c is non-divergent. Thus we find for every critical point c,

$$(q_1^c)^{-1} = O(T^{-1/4}) \qquad \text{and} \qquad q_n^c = O(T^{\frac{3}{4}-\frac{n}{2}}) \quad \text{for } n \geq 2.$$

We can estimate the T-order of the vertex term (93) from this. Using the fact that the coefficient of $q_{j_1}^c \cdots q_{j_p}^c$ (with $j_1, \ldots, j_p \geq 2$) in (93) is non-zero only when $l_1 + \cdots + l_k + j_1 + \cdots + j_p = 3g_v - 3 + k + p$, we find that the T-order of the vertex term (93) is at least

$$\frac{-(2g_v - 2 + k + p)}{4} + \sum_{r=1}^{p} \left(\frac{3}{4} - \frac{j_r}{2} \right) = -(2g_v - 2) - \sum_{r=1}^{k} \left(\frac{3}{4} - \frac{l_r}{2} \right). \tag{96}$$

The T-order of the leg term $[R_t^{-1} e_{i_s} z^{n_s}]_l^c$ is $-\frac{1}{4} - \frac{l-n_s}{2}$ by (88). To compute the T-order of the edge term, writing

$$\begin{aligned} \sum_{l,l' \geq 0} (-1)^{l+l'} V_t^{(l,c),(l',c')} w^l z^{l'} &= \left[R_t(-w)^{-1} \frac{R_t(z) - R_t(-w)}{z+w} \right]_{(c,c')} \\ &= \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} (-1)^l w^l z^{l'} \sum_{e+e'=l+l'+1, e \leq n} \left[(R_t^{-1})_e (R_t)_{e'} \right]_{(c,c')} \end{aligned}$$

with $R_t(z)^{-1} = \sum_{e \geq 0} (R_t^{-1})_e z^e$ and $R_t(z) = \sum_{e \geq 0} (R_t)_e z^e$, and using (88) and (89), we find that $V^{(l,c),(l',c')} = O\big(T^{-\frac{l+l'+1}{2}}\big)$ as $T \to 0$, for all pairs (c, c') of critical points. Since

$$-\frac{l+l'+1}{2} = \left(\frac{3}{4} - \frac{l}{2} \right) + \left(\frac{3}{4} - \frac{l'}{2} \right) - 2$$

let us split the contribution $O\big(T^{-\frac{l+l'+1}{2}}\big)$ from an edge e with labels (l, c) and (l', c') up into contributions $O(T^{\frac{3}{4}-\frac{l}{2}})$ and $O(T^{\frac{3}{4}-\frac{l'}{2}})$ carried by the two half-edges given by e and a contribution $O(T^{-2})$ carried by e itself. We include this new contribution $O(T^{\frac{3}{4}-\frac{l}{2}})$ from a half-edge with label (l, c) into the T-order of vertices or legs incident to it. Then, the new T-order of the vertex term of a vertex v becomes $-(2g_v - 2)$ – see (96) – and the new T-order of the leg term of a leg labelled by $s \in \{1, \ldots, m\}$ is $-1 + \frac{n_s}{2}$. Therefore the total T-order of the contribution from a graph Γ is at least

$$\overbrace{-2|E(\Gamma)|}^{\text{edge terms}} \overbrace{- \sum_{v \in V(\Gamma)} (2g_v - 2)}^{\text{vertex terms}} \overbrace{- \sum_{1 \leq s \leq m} \left(1 - \frac{n_s}{2} \right)}^{\text{leg terms}} \geq -(2g - 2 + m)$$

where we used $g = \sum_{v \in \Gamma(V)} g_v + 1 - \chi$. Summing over all graphs, we find that

$$C^{(g)}_{(n_1,i_1),\ldots,(n_m,i_m)}(t, \boldsymbol{x}) = O(T^{-(2g-2+m)})$$

as $T \to 0$ on the image of ζ.

We need to check that the change of co-ordinates (90) does not affect the pole order in T. Recall that $C^{(g)}_{(n_1,i_1),\ldots,(n_m,i_m)}(t, \boldsymbol{x})$ is an m-tensor written in the basis $\{d\hat{q}_n^i\}$ of 1-forms. Write

$\tilde{t} = (\tilde{t}^1, \dots, \tilde{t}^6)$ for a co-ordinate system centered at t, and write $\frac{1}{z}A = \frac{1}{z}\sum_{\alpha=1}^{6} A_\alpha(\tilde{t})d\tilde{t}^\alpha$ for the connection 1-form of ∇^{B}. Equation (90) gives

$$\begin{aligned} \hat{q}_0^i &= \sum_\alpha \tilde{t}^\alpha [A_\alpha(t)x_1]^i + O(|\tilde{t}|^2) \\ \hat{q}_n^i &= x_n^i + \sum_\alpha \tilde{t}^\alpha [A_\alpha(t)x_{n+1}]^i + O(|\tilde{t}|^2) \qquad \text{for } n \geq 1. \end{aligned}$$

Since the section $\zeta = -z$ has co-ordinates $x_n^i = 0$ for $n \geq 2$ – see (95) – we have:

$$\begin{aligned} \zeta^\star(d\hat{q}_0^i)\big|_t &= [\nabla^{\mathrm{B}}\zeta]_0^i = \delta_{1,i}\frac{dy_1}{y_1} + \delta_{2,i}\frac{dy_2}{y_2} + O(T) \\ \zeta^\star(d\hat{q}_n^i)\big|_t &= dx_n^i \qquad \text{for } n \geq 1 \end{aligned}$$

where $[\cdots]_0^i$ means the coefficient in front of $z^0\mathcal{D}_i$ when expanded in the basis $\{z^n\mathcal{D}_i\}$. These 1-forms are regular along $T = 0$. This means that $\zeta^\star(\boldsymbol{\nabla}^m C^{(g)})$ has poles of order $2g-2+m$ along $y_1 = -1/27$, for any fixed $y_2 \neq 0$. We already know from Theorem 6.9.4 that $\boldsymbol{\nabla}^m C^{(g)}$ extends regularly across $y_2 = 0$ as a logarithmic tensor; Hartog's Principle applied to a section of $\Omega^1_{\mathcal{M}_{\mathrm{B}}}(\log D)^{\otimes m}$ thus proves Theorem 9.0.3.

Remark 9.4.2. In this section, we studied correlation functions on the image of ζ, but the pole order along $T = y_1 + \frac{1}{27} = 0$ depends on the choice of slice. A similar analysis shows that $C^{(g)}(t, \boldsymbol{x})$ (the 0-point correlation function, with $g \geq 2$) has pole of order $g-1$ along $T = 0$ for a fixed *generic* $\boldsymbol{x}$. The restriction to the image of ζ is special because ζ touches the discriminant divisor $P(t, x_1) = 0$ (see equation (58)) at the conifold point; this follows from $q_1^c = [R_t^{-1}x_1]^c = O(T^{1/4})$ on the image of ζ for a divergent c. We have the $5g-5$ pole order condition along the discriminant (Definition 6.8.1), and correlation functions on the image of ζ acquire part of their poles from this.

10. Modularity

We now apply the theory developed in the preceding sections to show that the Gromov–Witten potential of local $\mathbb{P}^2$ is a quasi-modular function with respect to the congruence subgroup $\Gamma_1(3)$ of $SL(2,\mathbb{Z})$:

$$\Gamma_1(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : a \equiv d \equiv 1,\ c \equiv 0 \bmod 3 \right\}.$$

10.1. **The Mirror Family for Local $\mathbb{P}^2$.** As discussed in the Introduction, the mirror to the non-compact Calabi–Yau manifold Y is a certain family of elliptic curves $\{E_y : y \in \mathcal{M}_{\mathrm{CY}}\}$. This family has been studied by many authors: see for example [2, 16, 31, 51, 62, 77, 78]. We summarize the aspects of this work that we need.

10.1.1. *A Family of Elliptic Curves with $\Gamma_1(3)$-Level Structure.* Recall that $\mathcal{M}_{\mathrm{CY}} = \mathbb{P}(3,1)$ and $D_{\mathrm{CY}} = \{-\frac{1}{27}, 0\}$. We will see the mirror family of Y emerging in the conformal limit $y_2 \to 0$ of the Landau–Ginzburg potential mirror to $\overline{Y} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2))$ from §3.1:

$$W_y = w_1 + w_2 + w_3 + w_4 + w_5 \text{ with } w_1w_2w_3 = y_1w_4^3,\ w_4w_5 = y_2.$$

Setting the last co-ordinate $w_5 = y_2/w_4$ to zero and considering the zero locus of W_y in the projective space with co-ordinates $[w_1, w_2, w_3, w_4]$, we obtain a family of elliptic curves

$$\begin{aligned} E_y &= \big\{[w_1,w_2,w_3,w_4] \in \mathbb{P}^3 : w_1w_2w_3 = yw_4^3,\ w_1+w_2+w_3+w_4 = 0\big\} \\ &= \text{compactification of } \left\{(w_1,w_2) \in (\mathbb{C}^\times)^2 : w_1+w_2+\frac{y}{w_1w_2}+1=0\right\} \end{aligned}$$

parametrized by $y = y_1 \in \mathbb{C} \subset \mathcal{M}_{\mathrm{CY}}$. The second line is a presentation in the affine chart $w_4 = 1$. The curve E_y has singularities when $y \in D_{\mathrm{CY}}$. By introducing a co-ordinate $v = (w_1w_2w_3)^{1/3}$, we can extend the family across the orbifold point $y = \infty$ as

$$E_y = \big\{[w_1,w_2,w_3,v] \in \mathbb{P}^3 : w_1w_2w_3 = v^3,\ w_1+w_2+w_3+\mathfrak{y}v = 0\big\}$$

with $\mathfrak{y} = \mathfrak{y}_1 = y^{-1/3}$. The isotropy group μ_3 at $y = \infty \in \mathbb{P}(3,1)$ acts on the family as $v \mapsto \xi^{-1}v$, $\mathfrak{y} \mapsto \xi\mathfrak{y}$. A holomorphic volume form on E_y is given by the one-form

$$\lambda_y = \frac{1}{3}\frac{d\log w_1 \wedge d\log w_2}{d(w_1+w_2+\frac{y}{w_1w_2}+1)} = \frac{dw_1}{3w_1(w_2-\frac{y}{w_1w_2}))}$$

where $(w_1, w_2) \in (\mathbb{C}^\times)^2$ are co-ordinates on the affine chart.

Remark 10.1.1. Aganagic–Bouchard–Klemm [2] worked with a 3-fold covering $\pi\colon \widetilde{E}_y \to E_y$ given by

$$\widetilde{E}_y = \big\{[X,Y,Z] \in \mathbb{P}^2 : X^3+Y^3+Z^3+\mathfrak{y}XYZ = 0\big\}$$

where π maps $[X, Y, Z]$ to $[w_1, w_2, w_3, v] = [X^3, Y^3, Z^3, XYZ]$.

A $\Gamma_1(3)$-*level structure* on an elliptic curve E (equipped with a group structure) is by definition choice of a 3-torsion point $\mathtt{t}$ on E. This is equivalent to the choice of an order-3 automorphism σ of E without fixed points, or to a non-zero element ℓ in $H_1(E, \mathbb{Z}/3\mathbb{Z})$. We introduce a group structure on E_y such that $[w_1, w_2, w_3, w_4] = [1, -1, 0, 0] \in E_y$ is the identity element, and define a $\Gamma_1(3)$-structure on E_y by the order 3 automorphism σ:

$$\sigma\colon [w_1,w_2,w_3,w_4] \mapsto [w_3,w_1,w_2,w_4]$$

The corresponding 3-torsion point is $\mathtt{t} = \sigma(0) = [0, 1, -1, 0] \in E_y$. For a path γ connecting 0 and $\mathtt{t}$, 3γ defines a non-zero element $\ell \in H_1(E_y, \mathbb{Z}/3\mathbb{Z})$, which is independent of the choice of the path γ. The set of ordered bases $\{\alpha, \beta\}$ for $H_1(E_y, \mathbb{Z})$ satisfying $\alpha \cdot \beta = 1$ and $[\alpha] = \ell$ is a torsor over $\Gamma_1(3)$, via change of basis.

A *marked elliptic curve* is a pair $(E, \{\alpha, \beta\})$ of an elliptic curve E (with group structure) and a symplectic basis, also called a marking, $\{\alpha, \beta\} \subset H_1(E, \mathbb{Z})$ with $\alpha \cdot \beta = 1$. The moduli space of marked elliptic curves can be identified with the upper-half plane $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ via the period map $(E, \{\alpha, \beta\}) \mapsto \tau = \int_\beta \lambda / \int_\alpha \lambda \in \mathbb{H}$, where λ is a non-zero holomorphic one-form on E. We call τ a *modular parameter*. We let $\mathrm{SL}(2, \mathbb{Z})$, and hence $\mathrm{PSL}(2, \mathbb{Z})$, act on the upper-half plane by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau+b}{c\tau+d}$$

which corresponds to the change of markings

$$(\alpha,\beta) \mapsto (\alpha',\beta') = (\alpha,\beta)\begin{pmatrix} d & b \\ c & a \end{pmatrix}. \tag{97}$$

The moduli stack of elliptic curves with $\Gamma_1(3)$-level structure is identified with the quotient:

$$[\mathbb{H}/\Gamma_1(3)] \tag{98}$$

The $\Gamma_1(3)$-orbit of a marked elliptic curve $(E, \{\alpha, \beta\})$ corresponds to the elliptic curve E with the $\Gamma_1(3)$-level structure $\ell = [\alpha] \in H_1(E, \mathbb{Z}/3\mathbb{Z})$.

Remark 10.1.2. The $\Gamma_1(3)$-structure on E_y lifts to a level-3 structure on $\widetilde{E}_y$, i.e. to a basis of 3-torsion points. The corresponding order-3 automorphisms are given by $[X, Y, Z] \mapsto [Z, X, Y]$ and $[X, Y, Z] \mapsto [X, \xi Y, \xi^2 Z]$ with $\xi \in \mu_3$.

Proposition 10.1.3. *The base space $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ of the mirror family can be identified with the moduli stack* (98) *of elliptic curves with $\Gamma_1(3)$-level structure.*

Proof. As we saw, $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ is equipped with a family of elliptic curves with $\Gamma_1(3)$-level structure. Hence we have a canonical map $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}} \to [\mathbb{H}/\Gamma_1(3)]$. The j-invariant of E_y is given by

$$j(E_y) = -\frac{(1+24y)^3}{y^3(1+27y)}$$

and this gives the composition $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}} \to [\mathbb{H}/\Gamma_1(3)] \to \mathbb{H}/\operatorname{PSL}(2,\mathbb{Z}) \cong \mathbb{C}$. We can easily see that this has the same degree ($= 4$) and ramification data (at $j = 0, 1728$) as the covering $[\mathbb{H}/\Gamma_1(3)] \to \mathbb{H}/\operatorname{PSL}(2,\mathbb{Z})$. Thus the coarse moduli spaces of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ and $[\mathbb{H}/\Gamma_1(3)]$ are the same. The μ_3-orbifold structures at $y = \infty$, $\tau = e^{2\pi \mathtt{i}/3}$ also match. □

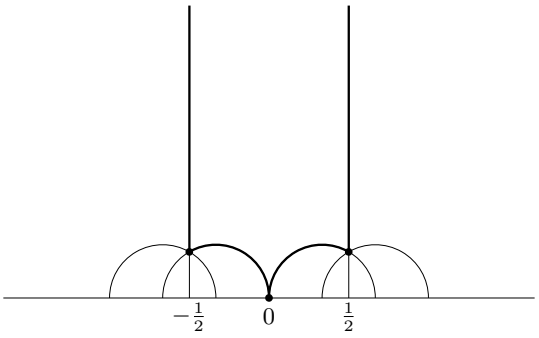


FIGURE 6. A fundamental domain for $\mathbb{H}/\Gamma_1(3)$. Note that $\Gamma_1(3)$ is generated by $\tau \mapsto \tau + 1$ and $\tau \mapsto \tau/(3\tau+1)$. The large-radius limit point is $\tau = +\infty\mathtt{i}$, the conifold point is $\tau = 0$, and the orbifold point is $\tau = -\frac{1}{2} + \frac{\mathtt{i}}{2\sqrt{3}} = \frac{\xi-1}{3}$, where the parameter τ is as in Corollary 10.2.10.

10.1.2. *A Relative Cohomology Mirror and the Picard–Fuchs Equation.* Let $F_y = F_y(w_1, w_2)$ denote the defining equation of E_y on the affine chart $(w_1, w_2) \in (\mathbb{C}^\times)^2$:

$$F_y = w_1 + w_2 + \frac{y}{w_1 w_2} + 1.$$

The corresponding affine elliptic curve

$$E_y^\circ = \big\{(w_1, w_2) \in (\mathbb{C}^\times)^2 : F_y(w_1, w_2) = 0\big\}$$

is $E_y \setminus \{0, \mathtt{t}, 2\mathtt{t}\}$, where $\mathtt{t}$ is the 3-torsion point as before. Near $y = \infty$, by introducing variables $v_1 = \mathfrak{y} w_1$, $v_2 = \mathfrak{y} w_2$, we define

$$E_y^\circ = \big\{(v_1, v_2) \in (\mathbb{C}^\times)^2 : v_1 + v_2 + 1/(v_1 v_2) + \mathfrak{y} = 0\big\} = E_y \setminus \{0, \mathtt{t}, 2\mathtt{t}\}, \tag{99}$$

where $\mathfrak{y} = y^{-1/3}$. A mirror for Y is given by the relative cohomology of the pair $\big((\mathbb{C}^\times)^2, E_y^\circ\big)$; such a mirror has been analysed by Stienstra [77], N. Takahashi [78] and Konishi–Minabe [62].

We shall see that the variation of Hodge structure on $H^1(E_y)$ corresponds to the rank 2 vector bundle H_{vec} from §4.2, and that the variation of mixed Hodge structure on $H^2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$ corresponds to the rank 3 vector bundle $\overline{H}$ there. Let $\zeta_y \in H^2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$ denote the relative cohomology class given by

$$\zeta_y = \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} = \frac{dv_1}{v_1} \wedge \frac{dv_2}{v_2}.$$

Proposition 10.1.4 ([6, 62, 77, 78])**.** *The classes $\zeta_y \in H^2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$, $\lambda_y \in H^1(E_y)$ satisfy*

$$\theta\zeta_y = \delta\left(\lambda_y\big|_{E_y^\circ}\right)$$

where $\theta = \nabla_{y\frac{\partial}{\partial y}}$ is the Gauss–Manin connection and $\delta\colon H^1(E_y^\circ) \to H^2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$ is the connecting homomorphism. They satisfy the Picard–Fuchs equations:

$$\begin{aligned} \big(\theta^3 + 3y\theta(3\theta+1)(3\theta+2)\big)\,\zeta_y &= 0 \\ \big(\theta^2 + 3y(3\theta+1)(3\theta+2)\big)\,\lambda_y &= 0 \end{aligned} \tag{100}$$

Proof. Let $C \in H_2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$ be a relative cycle. Working in the chart near $y = \infty$, we find

$$3y\frac{\partial}{\partial y}\int_C \zeta_y = -\mathfrak{y}\frac{\partial}{\partial \mathfrak{y}}\int_C \zeta_y = \int_{\partial C} \mathfrak{y}\frac{d\log v_1 \wedge d\log v_2}{d(v_1 + v_2 + \frac{1}{v_1 v_2})} = 3\int_{\partial C} \lambda_y$$

(see [78, Lemma 1.8], [62, Lemma 4.3]). This gives the first equation. The Picard–Fuchs equations are well-known: see [6, Theorem 14.2] and [77, §6]. □

Corollary 10.1.5. *We have the following isomorphisms.*

1. *The rank 3 vector bundle $\bigcup_y H^2((\mathbb{C}^\times)^2, E_y^\circ)$ over $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ equipped with the Gauss–Manin connection is isomorphic to the vector bundle $(\overline{H}, \nabla)$ from §4.2 .*
2. *The rank 2 vector bundle $\bigcup_y H^1(E_y)$ over $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ equipped with the Gauss–Manin connection is isomorphic to the vector bundle $(H_{\mathrm{vec}}, \nabla)$ from §4.2.*

These isomorphisms map $\zeta_y \in H^2((\mathbb{C}^\times)^2, E_y^\circ)$ to $\zeta \in \overline{H}$ and $\lambda_y \in H^1(E_y)$ to $\theta\zeta \in H_{\mathrm{vec}}$.

Proof. The vector bundles $(\overline{H}, \nabla)$, $(H_{\mathrm{vec}}, \nabla)$ are described by the same Picard–Fuchs equations (100); see (45). □

Consider now the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2\big((\mathbb{C}^\times)^2\big) & \longrightarrow & H_2\big((\mathbb{C}^\times)^2, E_y^\circ\big) & \xrightarrow{\partial} & H_1(E_y^\circ) & \longrightarrow & H_1\big((\mathbb{C}^\times)^2\big) & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle i_*} & & & & \\ & & & & & & H_1(E_y) & & & & \end{array}$$

where we use $\mathbb{Z}$ coefficients and the top row is exact. Since $\Re(F_y)\colon (\mathbb{C}^\times)^2 \to \mathbb{R}$ is a Morse function with 3 critical points of Morse index 2, it follows from Morse theory that

$$H_1((\mathbb{C}^\times)^2, E_y^\circ) = 0, \qquad H_2((\mathbb{C}^\times)^2, E_y^\circ) \cong \mathbb{Z}^3;$$

see e.g. [54, §3.3.1]. Generators of $H_2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$ are given by 3 Lefschetz thimbles emanating from critical points of F_y. We define the *lattice of vanishing cycles* to be

$$\mathrm{VC}_y := \mathrm{Im}\left(i_* \circ \partial\colon H_2\big((\mathbb{C}^\times)^2, E_y^\circ; \mathbb{Z}\big) \to H_1(E_y; \mathbb{Z})\right).$$

Proposition 10.1.6. *The sublattice* $\mathrm{VC}_y \subset H_1(E_y;\mathbb{Z})$ *is of index* 3 *and is given by*

$$\mathrm{VC}_y = 3H_1(E_y;\mathbb{Z}) + \big\{\alpha \in H_1(E_y;\mathbb{Z}) : [\alpha] = \ell\big\} = \pi_* H_1(\widetilde{E}_y;\mathbb{Z})$$

where $\ell \in H_1(E_y, \mathbb{Z}/3\mathbb{Z})$ *is the* $\Gamma_1(3)$*-level structure of* E_y *and* $\pi\colon \widetilde{E}_y \to E_y$ *is the* 3*-fold covering described in Remark 10.1.1.*

Proof. We work in the chart near $y = \infty$ and use the presentation (99) of E_y°. Consider the projection $E_y^\circ \to \mathbb{C}$, $(v_1, v_2) \mapsto v_1$ to the v_1-plane, which extends to a ramified covering $E_y \to \mathbb{P}^1$. This has 4 branch points given by $v_1 = 0$ and $v_1(v_1 + \mathfrak{y})^2 = 4$; note that $v_1 = \infty$ is not a branch point. The branch points move as $\mathfrak{y}$ varies, and two of them coalesce when $\mathfrak{y} = -3\xi^j$, $j \in \{0,1,2\}$, with $\xi = e^{2\pi\mathtt{i}/3}$, where E_y is singular. The three vanishing cycles on $E_{y=\infty}$ associated with three paths $[0, -3\xi^i]$, $i \in \{0,1,2\}$, on the $\mathfrak{y}$-plane are given by the trajectories of coalescing branch points: see Figure 7. It is then easy to see that these vanishing cycles generate a sublattice of index 3. Thus VC_y is of index 3.

On the other hand, the sublattice VC_y is clearly invariant under monodromy. Since we have $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}} \cong [\mathbb{H}/\Gamma_1(3)]$, the monodromy group is $\Gamma_1(3)$ and acts on symplectic bases of $H_1(E_y,\mathbb{Z})$ by (97). It is easy to see that there is a unique sublattice of index 3 which is invariant under $\Gamma_1(3)$. The conclusion follows. □

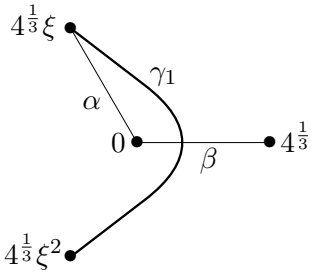


FIGURE 7. A vanishing cycle γ_1 on E_∞, pictured on the v_1-plane. The black dots are branch points. Two other vanishing cycles γ_2, γ_3 are obtained from γ_1 by $2\pi/3$, $4\pi/3$ rotations respectively. The cycles $\{\alpha, \beta\}$ give a symplectic basis. With some choice of orientations, we find $\gamma_1 = 2\alpha + \beta$, $\gamma_2 = \alpha + 2\beta$, $\gamma_3 = -\alpha + \beta$.

Remark 10.1.7. As a mirror to Y, Chiang–Klemm–Yau–Zaslow [16] considered periods of a multi-valued one-form

$$\int_{\gamma \subset E_y} \log(w_1) \frac{dw_2}{w_2}$$

and periods of the 3-fold $\check{Y} = \big\{(w_1, w_2, u, v) \in (\mathbb{C}^\times)^2 \times \mathbb{C}^2 : F_y(w_1, w_2) + uv = 0\big\}$:

$$\int_{S \subset \check{Y}} \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \frac{du}{u}.$$

These are equivalent, up to a Tate twist, to the relative cohomology mirror [62].

10.2. **Periods and Compactly Supported K-theory.** We next compute periods of the mirror family as explicit hypergeometric series. To do this, we identify periods over integral cycles with elements of the compactly supported K-group of Y (or $\mathcal{X}$) via the $\widehat{\Gamma}$-integral structure [54–56]. We then identify the modular parameter τ with the second derivative of the genus-zero Gromov–Witten potential of Y. Most of the computations in this section are already in the literature, in particular in work of Hosono [51].

10.2.1. *I-function, $\widehat{\Gamma}$-Integral Structure and Monodromy.* The I-functions [18,39] of Y and $\mathcal{X}$ are the power series

$$I_Y(y,z) = \sum_{d=0}^{\infty} y^{h/z+d} \frac{\prod_{m=0}^{3d-1}(-3h-mz)}{\prod_{m=1}^{d}(h+mz)^3}$$
$$I_{\mathcal{X}}(\mathfrak{y},z) = \sum_{d=0}^{\infty} \mathfrak{y}^d \frac{\prod_{0\le m<d/3, \langle d/3\rangle=\langle m\rangle}(-mz)^3}{d!z^d} \mathbf{1}_{\langle d/3\rangle}$$

which take values, respectively, in $H_Y = H^\bullet(Y)$ and $H_{\mathcal{X}} = H^\bullet_{\mathrm{orb}}(\mathcal{X})$. In the second line, $\langle r\rangle$ denotes the fractional part of a real number r. The components of I_Y written in the basis $\{1,h,h^2\}$, or the components of $I_{\mathcal{X}}$ written in the basis $\{\mathbf{1}, \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}\}$, form a basis of solutions to the Picard–Fuchs equation (100) satisfied by ζ_y. Therefore periods of ζ_y can be written as certain linear combinations of these hypergeometric series. In what follows, we set $z=1$ and write $I_Y(y) = I_Y(y,1)$ and $I_{\mathcal{X}}(\mathfrak{y}) = I_{\mathcal{X}}(\mathfrak{y},1)$. The Mirror Theorem [40, Theorem 4.2] implies that the I-function of Y can be expanded as

$$I_Y(y) = 1 + th + \frac{\partial F_Y^0}{\partial t}(-3h^2)$$

where $t = t(y)$ is the mirror map for Y, given by $t(y) = \log y + g(y)$ with $g(y)$ as in Theorem 3.3.1, and

$$F_Y^0(t) = -\frac{1}{18}t^3 + \sum_{d=1}^{\infty} \langle\rangle^Y_{0,0,d}\, e^{td} \tag{101}$$

is the genus-zero Gromov–Witten potential[21] restricted to $H^2(Y)$. The Mirror Theorem [18, Theorem 4.6] implies that the I-function of $\mathcal{X}$ can be expanded as

$$I_{\mathcal{X}}(\mathfrak{y}) = 1 + \mathfrak{t}\mathbf{1}_{\frac{1}{3}} + \frac{\partial F^0_{\mathcal{X}}}{\partial \mathfrak{t}}(3\mathbf{1}_{\frac{2}{3}})$$

where $\mathfrak{t} = \mathfrak{t}(\mathfrak{y})$ is the mirror map for $\mathcal{X}$, which is the same map as appeared in Theorem 3.3.2, and

$$F^0_{\mathcal{X}}(\mathfrak{t}) = \sum_{n=3}^{\infty} \left\langle \mathbf{1}_{\frac{1}{3}}, \dots, \mathbf{1}_{\frac{1}{3}} \right\rangle^{\mathcal{X}}_{0,n,0} \frac{\mathfrak{t}^n}{n!}. \tag{102}$$

is the genus-zero Gromov–Witten potential restricted to $H^2_{\mathrm{orb}}(\mathcal{X})$.

Consider now the $\widehat{\Gamma}$-integral structure [54, §2.4; 55, §2]. The classes $\widehat{\Gamma}_Y \in H_Y$, $\widehat{\Gamma}_{\mathcal{X}} \in H_{\mathcal{X}}$ are defined by:

$$\widehat{\Gamma}_Y := \Gamma(1+h)^3\,\Gamma(1-3h) = 1 + \pi^2 h^2, \qquad \widehat{\Gamma}_{\mathcal{X}} := \bigoplus_{i=0}^{2} \Gamma(1-\tfrac{i}{3})^3 \mathbf{1}_{\frac{i}{3}}.$$

Let X denote either Y or $\mathcal{X}$ and consider the K-group $K_c(X)$ of coherent sheaves on X with compact support. The groups $K_c(Y)$, $K_c(\mathcal{X})$ are freely generated by 3 coherent sheaves:

$$K_c(Y) = \langle \mathcal{O}_{\mathrm{pt}}, \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^2}(-1)\rangle, \qquad K_c(\mathcal{X}) = \langle \mathcal{O}_0, \mathcal{O}_0\otimes\varrho, \mathcal{O}_0\otimes\varrho^2\rangle$$

[21] We added a cubic term to F_Y^0 which is responsible for the cup product.

where $\mathbb{P}^1 \subset \mathbb{P}^2$ denotes a line and ϱ is the standard one-dimensional representation of μ_3. For $V \in K_c(X)$, we define a vector $\Psi(V)$ lying in the compactly supported (orbifold) cohomology $H_{X,c}$ of X by

$$\Psi(V) = \widehat{\Gamma}_X \cup (2\pi\mathtt{i})^{\frac{\deg}{2}} \operatorname{inv}^* \widetilde{\mathrm{ch}}(V).$$

This is an analogue of the Mukai vector. For a precise definition of the right-hand side, we refer the reader to [54, §2.4] and [55, §2.5]. In the case at hand, we have:

$$\begin{aligned}
\Psi(\mathcal{O}_{\mathrm{pt}}) &= (2\pi\mathtt{i})^3[\mathrm{pt}] \\
\Psi(\mathcal{O}_{\mathbb{P}^1}(-1)) &= (2\pi\mathtt{i})^2[\mathbb{P}^1] \\
\Psi(\mathcal{O}_{\mathbb{P}^2}(-1)) &= (2\pi\mathtt{i})\left(1 + \pi\mathtt{i}h - \pi^2h^2\right) \cap [\mathbb{P}^2]
\end{aligned}$$

for Y and

$$\Psi(\mathcal{O}_0 \otimes \varrho^i) = (2\pi\mathtt{i})^3 \left(\frac{1}{3}[\mathrm{pt}] + \frac{\xi^{-i}}{\Gamma(\frac{1}{3})^3}\mathbf{1}_{\frac{1}{3}} - \frac{\xi^{-2i}}{\Gamma(\frac{2}{3})^3}\mathbf{1}_{\frac{2}{3}} \right) \qquad i \in \{0,1,2\}$$

for $\mathcal{X}$, where $[\mathrm{pt}] \in H^6_c(\mathcal{X}) \subset H^6_{\mathrm{orb},c}(\mathcal{X})$ is the class of a non-stacky point, so that $(1,[\mathrm{pt}]) = 1$, and $\xi = e^{2\pi\mathtt{i}/3}$. Cf. [55, Example 2.16].

Definition 10.2.1 ([54, 55])**.** Let X be Y or $\mathcal{X}$. We define the *quantum cohomology central charge* of $V \in K_c(X)$ to be

$$\Pi_X(V) = \Big((-1)^{\deg/2} I_X, \Psi(V) \Big)$$

where I_X is the I-function of X and $(\cdot,\cdot)$ is the natural pairing between (orbifold) cohomology and compactly supported (orbifold) cohomology.

Remark 10.2.2. The quantum cohomology central charge in [54, 55] is a function of the A-model co-ordinates (Kähler parameters) and is related to the present one by a change of co-ordinate given by the mirror map, together with a multiplicative factor of $(2\pi\mathtt{i})^{-3}$. Under the mirror map $t = t(y)$ for Y, we have

$$\begin{aligned}
\Pi_Y(\mathcal{O}_{\mathrm{pt}}) &= (2\pi\mathtt{i})^3 \\
\Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1)) &= -(2\pi\mathtt{i})^2 t \\
\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1)) &= -(2\pi\mathtt{i})\left(\pi^2 + \pi\mathtt{i}t + 3\frac{\partial F_Y^0}{\partial t} \right).
\end{aligned} \tag{103}$$

Similarly, under the mirror map $\mathfrak{t} = \mathfrak{t}(\mathfrak{y})$ for $\mathcal{X}$, we have

$$\Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho^i) = (2\pi\mathtt{i})^3 \left(\frac{1}{3} + \frac{\xi^{-2i}}{3\Gamma(\frac{2}{3})^3}\mathfrak{t} + \frac{\xi^{-i}}{\Gamma(\frac{1}{3})^3}\frac{\partial F_{\mathcal{X}}^0}{\partial \mathfrak{t}} \right), \qquad i \in \{0,1,2\}. \tag{104}$$

We introduce period vectors $\overrightarrow{\Pi}_Y$ and $\overrightarrow{\Pi}_{\mathcal{X}}$ as follows:

$$\begin{aligned}
\overrightarrow{\Pi}_Y &:= \big(\Pi_Y(\mathcal{O}_{\mathrm{pt}}), \Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1)), \Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1)) \big), \\
\overrightarrow{\Pi}_{\mathcal{X}} &:= \big(\Pi_{\mathcal{X}}(\mathcal{O}_0), \Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho), \Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho^2) \big).
\end{aligned}$$

They are power series solutions defined near $y = 0$, $\mathfrak{y} = y^{-1/3} = 0$ respectively; since they satisfy the Picard–Fuchs equation, they analytically continue to the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$. Take a base point $y_0 \in \mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ such that $0 < y_0 \ll 1$. We choose a branch of $\overrightarrow{\Pi}_Y$ around y_0 by requiring that $\log y_0 \in \mathbb{R}$.

Proposition 10.2.3 ([32,50,51])**.** *Under analytic continuation along the positive real line in the y-plane, we have*

$$\overrightarrow{\Pi}_Y = \overrightarrow{\Pi}_{\mathcal{X}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Moreover, the analytic continuation of $\overrightarrow{\Pi}_Y$ along the loops γ_{LR}, γ_{con}, γ_{orb} in Figure 8 are given by $\overrightarrow{\Pi}_Y M_{\mathrm{LR}}$, $\overrightarrow{\Pi}_Y M_{\mathrm{con}}$, $\overrightarrow{\Pi}_Y M_{\mathrm{orb}}$ respectively, where

$$M_{\mathrm{LR}} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_{\mathrm{con}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \qquad M_{\mathrm{orb}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix}.$$

Proof. The analytic continuation has been computed in [32,50,51] in a slightly different basis. The Barnes integral representation for the I-function yields the connection formula between $\overrightarrow{\Pi}_Y$ and $\overrightarrow{\Pi}_{\mathcal{X}}$: see e.g. [27, Appendix; 50, Appendix A]. It is easy to see that the monodromy around the orbifold point $y = \infty$ corresponds to $(-) \otimes \varrho$ on $K_c(\mathcal{X})$ and that the monodromy around the large radius limit point $y = 0$ corresponds to $(-) \otimes \mathcal{O}(-1)$ on $K_c(Y)$. This together with the connection formula yields M_{LR} and M_{orb}. The conifold monodromy M_{con} is then given by $M_{\mathrm{LR}}^{-1} M_{\mathrm{orb}}^{-1}$. □

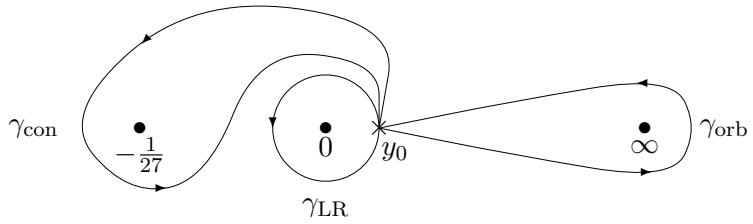


FIGURE 8. Paths in $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$. The base point y_0 of the analytic continuation is chosen so that $0 < y_0 \ll 1$.

Remark 10.2.4 ([12, 26, 49–51])**.** The connection matrix relating $\overrightarrow{\Pi}_Y$ and $\overrightarrow{\Pi}_{\mathcal{X}}$ coincides with the Fourier–Mukai transformation between compactly supported K-groups. Consider the diagram

$$\begin{array}{ccc} & [\mathcal{O}_{\mathbb{P}^2}(-1)/\mu_3] & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ Y & & \mathcal{X} \end{array}$$

and the Fourier–Mukai transformation $\Phi(-) = \boldsymbol{R}g_*(f^*(-) \otimes \mathcal{O}(-1) \otimes \varrho)$. Then we have:

$$(\Phi(\mathcal{O}_{\mathrm{pt}}), \Phi(\mathcal{O}_{\mathbb{P}^1}(-1)), \Phi(\mathcal{O}_{\mathbb{P}^2}(-1))) = \left(\mathcal{O}_0, \mathcal{O}_0 \otimes \varrho, \mathcal{O}_0 \otimes \varrho^2\right) \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

As we remarked in the proof, M_{LR} and M_{orb} correspond to the autoequivalences $(-) \otimes \mathcal{O}(-1)$, $(-) \otimes \varrho$ respectively. The inverse conifold monodromy M_{con}^{-1} corresponds to the Seidel–Thomas spherical twist by the object $\mathcal{O}_{\mathbb{P}^2}(-1)$. Observe also that the 2×2 right-lower submatrices of $M_{\mathrm{LR}}, M_{\mathrm{con}}, M_{\mathrm{orb}}$ generate $\Gamma_1(3)$.

Remark 10.2.5. The above matrices $M_{\mathrm{LR}}, M_{\mathrm{con}}, M_{\mathrm{orb}}$ represent the monodromy acting on *homology* $H_2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$; the monodromy acting on cohomology $H^2\big((\mathbb{C}^\times)^2, E_y^\circ\big)$, or equivalently the monodromy of $\big(\overline{H}, \nabla\big)$, is given by the adjoint-inverse of these matrices.

10.2.2. *Identification of periods with hypergeometric series.* We show that quantum cohomology central charges are periods of ζ_y over integral cycles, and vice versa.

Lemma 10.2.6 ([51, Appendix A])**.** *For $0 < y < \frac{1}{27}$, let $\Gamma_{\mathbb{R}} \in H_2\big((\mathbb{C}^\times)^2, E_{-y}^\circ\big)$ denote the class of a Lefschetz thimble associated to the critical value $1 - 3y^{1/3}$ of F_{-y} and the straight path $[0, 1 - 3y^{1/3}]$, i.e.*

$$\Gamma_{\mathbb{R}} = \left\{ (w_1, w_2) \in (\mathbb{C}^\times)^2 : w_1 < 0, w_2 < 0, w_1 + w_2 + \frac{-y}{w_1 w_2} + 1 \geq 0 \right\}.$$

Then we have

$$\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))(e^{\pi \mathtt{i}} y) = 2\pi\mathtt{i} \int_{\Gamma_{\mathbb{R}}} \zeta_{-y}.$$

Remark 10.2.7. The $(2\pi\mathtt{i})$ factor on the right-hand side here reflects the fact that we are working with a 2-dimensional relative cohomology mirror model, instead of a 3-dimensional mirror.

Proof of Lemma 10.2.6. Hosono [51, equation (A.4)] evaluated the period integral over a vanishing sphere in the 3-dimensional mirror model (see Remark 10.1.7), and his computation implies the lemma. We give another proof using the Mellin transform, which was used by Katzarkov–Kontsevich–Pantev [58] to compute oscillatory integrals mirror to $\mathbb{P}^n$. Via the co-ordinate change $u_1 = -w_1$, $u_2 = -w_2$, $u_3 = -y/(w_1 w_2)$, we write, for $0 < y < \frac{1}{27}$,

$$\varphi(y) := \int_{\Gamma_{\mathbb{R}}} \zeta_{-y} = \int_{\substack{u_1>0, u_2>0, u_3>0, u_1+u_2+u_3 \leq 1 \\ u_1 u_2 u_3 = y}} \frac{d\log u_1 \wedge d\log u_2 \wedge d\log u_3}{d\log y}.$$

We set $\varphi(y) = 0$ for $y \geq \frac{1}{27}$. The Mellin transform of $\varphi(y)$ can be computed as the Euler integral:

$$\int_0^\infty y^s \frac{dy}{y} \varphi(y) = \int_{\substack{u_1+u_2+u_3 \leq 1 \\ u_1>0, u_2>0, u_3>0}} (u_1 u_2 u_3)^s \frac{du_1}{u_1} \frac{du_2}{u_2} \frac{du_3}{u_3} = \frac{\Gamma(s)^3}{\Gamma(1+3s)}$$

for $\Re(s) > 0$. The Mellin inversion formula gives

$$\varphi(y) = \frac{1}{2\pi\mathtt{i}} \int_{c-\mathtt{i}\infty}^{c+\mathtt{i}\infty} \frac{\Gamma(s)^3}{\Gamma(1+3s)} y^{-s} ds$$

for $c > 0$. Closing the contour to the left, we can write $\varphi(y)$ as the sum of residues at $s = -n$, $n = 0, 1, 2, \ldots$. Thus

$$\begin{aligned}
\varphi(y) &= \sum_{n=0}^\infty \mathrm{Res}_{h=0} \frac{\Gamma(h-n)^3}{\Gamma(1+3h-3n)} y^{n-h} dh \\
&= \int_{\mathbb{P}^2} \sum_{n=0}^\infty h^3 \frac{\Gamma(h-n)^3}{\Gamma(1+3h-3n)} y^{n-h} \\
&= \frac{1}{2\pi\mathtt{i}} \Big((-1)^{\deg/2} I_Y(e^{\pi\mathtt{i}} y), \Psi(\mathcal{O}_{\mathbb{P}^2}(-1)) \Big) = \frac{1}{2\pi\mathtt{i}} \Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))(e^{\pi\mathtt{i}} y).
\end{aligned}$$

In the second line here, h is regarded as a cohomology class on $\mathbb{P}^2$. The lemma follows. □

Proposition 10.2.8. *Let X denote either Y or $\mathcal{X}$. We have an isomorphism* $\mathrm{Mir}\colon K_c(X) \cong H_2\big((\mathbb{C}^\times)^2, E_y^\circ; \mathbb{Z}\big)$ *of integral lattices such that for $V \in K_c(X)$,*

$$\Pi_X(V) = 2\pi\mathtt{i} \int_{\mathrm{Mir}(V)} \zeta_y. \tag{105}$$

Proof. It suffices to prove this for $X = Y$. We saw in Lemma 10.2.6 that the identity (105) holds for $V = \mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathrm{Mir}(V) = \Gamma_{\mathbb{R}}$. Recall from Proposition 10.2.3 that monodromy M_{LR} around the large radius limit $y = 0$ corresponds to $(-)\otimes\mathcal{O}(-1)$ on $K_c(Y)$. Since $K_c(Y)$ is generated by $\mathcal{O}_{\mathbb{P}^2}(-1)$ under $(-)\otimes\mathcal{O}(-1)$, and Lefschetz thimbles are generated by $\Gamma_{\mathbb{R}}$ under monodromy around $y = 0$, the conclusion follows. □

Next we describe cycles $\partial\,\mathrm{Mir}(V)$ on the elliptic curve E_y in terms of the level structure.

Proposition 10.2.9. *Let* $\mathrm{Mir}\colon K_c(Y) \cong H_2\big((\mathbb{C}^\times)^2, E_y^\circ; \mathbb{Z}\big)$ *as in Proposition 10.2.8 and set $\Gamma_1 = \mathrm{Mir}(\mathcal{O}_{\mathbb{P}^1}(-1))$, $\Gamma_2 = \mathrm{Mir}(\mathcal{O}_{\mathbb{P}^2}(-1))$. There exist a symplectic basis $\{\alpha, \beta\}$ of $H_1(E_y;\mathbb{Z})$ and a sign $\varepsilon \in \{\pm 1\}$ such that $[\alpha]$ is the level structure $\ell \in H_1(E_y; \mathbb{Z}/3\mathbb{Z})$ and that*

$$\partial\Gamma_1 = \varepsilon 3\beta, \qquad \partial\Gamma_2 = \varepsilon\alpha.$$

Proof. By differentiating (105) and using Proposition 10.1.4, we obtain

$$y\frac{\partial}{\partial y}\Pi_X(V) = 2\pi\mathtt{i}\int_{\partial\,\mathrm{Mir}(V)} \lambda_y, \tag{106}$$

that is, the derivatives of the quantum cohomology central charges are precisely periods over cycles from VC_y. Since $\{\mathcal{O}_{\mathrm{pt}}, \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^2}(-1)\}$ is a basis of $K_c(Y)$ and $y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathrm{pt}}) = 0$, $y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1))$ and $y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))$ form a basis of periods over vanishing cycles, i.e.

$$\mathrm{VC}_y = \langle \partial\Gamma_1, \partial\Gamma_2 \rangle .$$

The monodromy of $y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1))$, $y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))$ is given by the 2×2 right-lower submatrices of $M_{\mathrm{LR}}, M_{\mathrm{con}}, M_{\mathrm{orb}}$ in Proposition 10.2.3. By reducing the monodromy modulo 3, we find that the class of $\partial\Gamma_1$ in $\mathrm{VC}_y/3\,\mathrm{VC}_y$ generates a monodromy-invariant line over $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

Let us choose a symplectic basis $\{\alpha, \beta\}$ of $H_1(E_y;\mathbb{Z})$ such that $[\alpha]$ is the given $\Gamma_1(3)$-level structure. Then $\{-3\beta, \alpha\}$ forms a basis of VC_y by Proposition 10.1.6. The monodromy in this basis is given by (see (97)):

$$(-3\beta, \alpha) \mapsto (-3\beta', \alpha') = (-3\beta, \alpha)\begin{pmatrix} a & -c/3 \\ -3b & d \end{pmatrix}.$$

Thus the basis $\{-3\beta, \alpha\}$ also transforms under $\Gamma_1(3)$, and we see that -3β generates a monodromy-invariant line of $\mathrm{VC}_y/3\,\mathrm{VC}_y$. The discussion in the previous paragraph implies

$$\partial\Gamma_1 \equiv \pm 3\beta \mod 3\,\mathrm{VC}_y\,. \tag{107}$$

Since $\partial\Gamma_1, \partial\Gamma_2$ are a basis of VC_y, this implies that $[\partial\Gamma_2] = n[\alpha] + m[3\beta]$ in $\mathrm{VC}_y/3\,\mathrm{VC}_y$ for some $n \in \mathbb{F}_3^\times$ and $m \in \mathbb{F}_3$; in particular

$$\partial\Gamma_2 \equiv \pm\alpha \mod 3H_1(E_y;\mathbb{Z}).$$

Thus the class of $\partial\Gamma_2$ in $H_1(E_y;\mathbb{Z}/3\mathbb{Z})$ equals $\varepsilon\ell$ for some $\varepsilon \in \{\pm 1\}$. Equation (107) implies that $\partial\Gamma_1$ is divisible by 3 in $H_1(E_y;\mathbb{Z})$ and thus $\{\partial\Gamma_2, \partial\Gamma_1/3\}$ gives a basis of $H_1(E_y;\mathbb{Z})$. It now suffices to show that this is a *symplectic* basis: $\partial\Gamma_2 \cdot (\partial\Gamma_1/3) = 1$. We will discuss this in the proof of the following Corollary 10.2.10. □

Corollary 10.2.10 (cf. Proposition 10.1.3)**.** *The multi-valued function*

$$\tau = -\frac{y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))}{y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1))} = -\frac{1}{2} - \frac{3}{2\pi\mathtt{i}}\frac{\partial^2 F_Y^0}{\partial t^2}$$

takes values in the upper-half plane $\mathbb{H}$ *and induces an isomorphism* $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}} \cong [\mathbb{H}/\Gamma_1(3)]$*, where* $t = t(y)$ *is the mirror map for* Y*.*

Proof. We have shown that there exist a symplectic basis $\{\alpha, \beta\}$ of $H_1(E_y;\mathbb{Z})$ and $\varepsilon \in \{\pm 1\}$ such that $[\alpha] = \ell$ and $\partial\Gamma_1 = \pm\varepsilon 3\beta$ and $\partial\Gamma_2 = \varepsilon\alpha$. (The sign $\pm$ was not determined in the above discussion.) Recall from §10.1.1 that the modular parameter for E_y, with respect to this marking, is given by $\tau' = \int_\beta \lambda_y / \int_\alpha \lambda_y$. On the other hand, by (106), we have

$$\tau = -\frac{y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))}{y\frac{\partial}{\partial y}\Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1))} = \frac{-\int_{\partial\Gamma_2}\lambda_y}{\int_{\partial\Gamma_1}\lambda_y} = \pm\frac{1}{-3\tau'}.$$

This quantity satisfies

$$\tau \sim -\frac{1}{2} + \frac{\log y}{2\pi\mathtt{i}} + O(y) \qquad\qquad \text{as } y \to 0$$

which lies in the upper-half plane $\mathbb{H}$ when $|y|$ is sufficiently small. The Riemann bilinear inequality then implies that $(\partial\Gamma_1, -\partial\Gamma_2)$ is positively oriented, i.e. that $\partial\Gamma_1 = \varepsilon 3\beta$ and $\tau = 1/(-3\tau')$. The isomorphism $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}} \cong [\mathbb{H}/\Gamma_1(3)]$ in Proposition 10.1.3 was given by the parameter τ'; it now suffices to observe that the map $\tau' \mapsto \tau = 1/(-3\tau')$ induces an isomorphism $[\mathbb{H}/\Gamma_1(3)] \cong [\mathbb{H}/\Gamma_1(3)]$ via the involution on $\Gamma_1(3)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c/3 \\ -3b & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}^{-1}.$$

The second expression for τ follows from (103). □

Remark 10.2.11. The parameter τ in Corollary 10.2.10 is a modular parameter for $\widetilde{E}_y$ rather than for E_y. The map $\tau' \mapsto 1/(-3\tau')$ exchanging the modular parameters of E_y and $\widetilde{E}_y$ is known as the Fricke involution. The Fricke involution exchanges the large-radius ($\tau = +\infty\mathtt{i}$) and conifold ($\tau = 0$) points, and preserves the orbifold point ($\tau = \frac{1-\xi}{3}, \frac{\xi^2-1}{3}$). The role of Fricke involution in this context has been studied extensively by Alim–Scheidegger–Yau–Zhou [4].

Let $\chi(V_1, V_2) = \sum_{i=0}^3 (-1)^i \dim \mathrm{Ext}^i(V_1, V_2)$ denote the Euler pairing of coherent sheaves V_1, V_2 with compact support. Since we have $\partial\Gamma_1 \cdot \partial\Gamma_2 = -3$ and $\chi(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^2}(-1)) = 3$, we conclude:

Corollary 10.2.12. *Let* X *denote either* Y *or* $\mathcal{X}$*. For* $V_1, V_2 \in K_c(X)$*, we have*

$$\chi(V_1, V_2) = -(\partial\,\mathrm{Mir}(V_1)) \cdot (\partial\,\mathrm{Mir}(V_2)).$$

10.3. **Opposite Line Bundles at Cusps and the Crepant Resolution Conjecture.** Recall the opposite line bundles P_{LR}, P_{con}, P_{orb} associated with large-radius, conifold and orbifold points that were defined in Notation 7.2.7. We next describe these opposite line bundles in terms of flat co-ordinates given by central charge functions, and obtain an explicit Feynman rule relating the Gromov–Witten potentials of $\mathcal{X}$ and Y.

As discussed in §4.2.2, any (local) function ψ satisfying the Picard–Fuchs equation:

$$\left[\theta^3 + 3y\theta(3\theta+1)(3\theta+2)\right]\psi = 0 \qquad \text{with } \theta = y\frac{\partial}{\partial y} \tag{108}$$

defines (locally) a D-module homomorphism $\psi^\sharp\colon \mathcal{O}(\overline{H}) \to \mathcal{O}$ sending ζ to ψ. In particular, the central charge functions $\Pi_Y(V)$, $\Pi_{\mathcal{X}}(V)$ define "flat co-ordinates" on $\overline{H}$ – that is, flat sections of the dual bundle $\overline{H}^\vee$. Recall from §§4.2.2–4.2.3 that the subbundles $H_{\mathrm{aff}}, H_{\mathrm{vec}} \subset \overline{H}$ are cut out, respectively, by the equations

$$\Pi_Y(\mathcal{O}_{\mathrm{pt}})^\sharp = (2\pi\mathtt{i})^3, \qquad \Pi_Y(\mathcal{O}_{\mathrm{pt}})^\sharp = 0;$$

see also (103). Introduce the following flat co-ordinates on $\overline{H}$:

$$\begin{aligned} x &= \mathtt{i}(2\pi\mathtt{i})^{-3/2}\Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1))^\sharp \\ p &= -\mathtt{i}(2\pi\mathtt{i})^{-3/2}\Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))^\sharp \end{aligned} \tag{109}$$

where we set $\mathtt{i}^{1/2} = e^{\pi\mathtt{i}/4} = (1+\mathtt{i})/\sqrt{2}$. These co-ordinates are multi-valued: they are originally defined near a point y_0 with $0 < y_0 \ll 1$, and then analytically continued over the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$. For example, if we analytically continue them to the orbifold point along the positive real line in the y-plane, we have, from the connection formula in Proposition 10.2.3:

$$\begin{aligned} x &= \mathtt{i}(2\pi\mathtt{i})^{-3/2}\left(-\Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho)^\sharp + \Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho^2)^\sharp\right) \\ p &= -\mathtt{i}(2\pi\mathtt{i})^{-3/2}\Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho^2)^\sharp \end{aligned} \tag{110}$$

x and p give Darboux co-ordinates corresponding to an *integral* basis of $K_c(Y)$ or $K_c(\mathcal{X})$.

Lemma 10.3.1. *When we restrict (p,x) to H_{aff}, we have $\Omega = \frac{1}{3}dp \wedge dx$.*

Proof. Set $\Pi_1 = \Pi_Y(\mathcal{O}_{\mathbb{P}^1}(-1))$ and $\Pi_2 = \Pi_Y(\mathcal{O}_{\mathbb{P}^2}(-1))$. As $y \to 0$, we have (see equation (103))

$$\begin{aligned} \Pi_1 &= -(2\pi\mathtt{i})^2 \log y + O(y) \\ \Pi_2 &= -(2\pi\mathtt{i})\left(\pi^2 + \pi\mathtt{i}\log y - \frac{1}{2}(\log y)^2\right) + O(y\log y) \end{aligned}$$

The sections $\theta\zeta$, $\theta^2\zeta \in \mathcal{O}(H_{\mathrm{vec}})$ form a fiberwise tangent frame of H_{aff} near $y = 0$. Since x and p are flat, it suffices to check that the asymptotics of $\Omega(\theta\zeta, \theta^2\zeta)$ and $\frac{1}{3}(dp \wedge dx)(\theta\zeta, \theta^2\zeta)$ agree. We have (see equation (47)):

$$\begin{aligned} \Omega(\theta\zeta, \theta^2\zeta) &= \frac{1}{3(1+27y)} \sim \frac{1}{3} \\ (dp \wedge dx)(\theta\zeta, \theta^2\zeta) &= (2\pi\mathtt{i})^{-3}\left(\theta\Pi_2 \cdot \theta^2\Pi_1 - \theta^2\Pi_2 \cdot \theta\Pi_1\right) \sim 1 \end{aligned}$$

as $y \to 0$. The conclusion follows. □

Proposition 10.3.2. *Let (p,x) be the coordinates of $\overline{H}$ given by* (109)*. If we analytically continue the co-ordinates (p,x) along the paths shown in Figure* 8*, we have that:*

(a) *the opposite line bundle $P_{\mathrm{LR}} \subset H_{\mathrm{vec}}$ is cut out by $x = 0$;*
(b) *the opposite line bundle $P_{\mathrm{con}} \subset H_{\mathrm{vec}}$ is cut out by $p = 0$;*
(c) *the opposite line bundle $P_{\mathrm{orb}} \subset H_{\mathrm{vec}}$ is cut out by $x + (1-\xi)p = 0$, where $\xi = e^{2\pi\mathtt{i}/3}$.*

Proof. The opposite line bundles P_{LR}, P_{con}, P_{orb} are flat subbundles of H_{vec} around the large-radius, conifold, and orbifold points respectively, and as such, they are necessarily invariant under the corresponding local monodromy. From the computation in Proposition 10.2.3, we find that $\{x = 0\}$ is a unique invariant line in $H_{\mathrm{vec}} = \{\Pi_Y(\mathcal{O}_{\mathrm{pt}})^\sharp = 0\}$ around the large-radius limit point; similarly $\{p = 0\}$ is a unique invariant line around the conifold point. Parts (a)

and (b) follow. The monodromy around the orbifold point is semisimple with eigenvalues $\{\xi, \xi^2\}$ and we have precisely two invariant lines given by $x + (1-\xi^i)p = 0$, $i \in \{1,2\}$. On the other hand, the generator $v := \nabla_{\partial/\partial\mathfrak{y}}\zeta$ of F^2_{vec} near the orbifold point has co-ordinates

$$x(v) = \mathtt{i}(2\pi\mathtt{i})^{-3/2}\partial_{\mathfrak{y}}\left(-\Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho) + \Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho^2)\right) = \mathtt{i}\frac{(2\pi\mathtt{i})^{3/2}}{3\Gamma(\frac{2}{3})^3}(-\xi + \xi^2) + O(\mathfrak{y})$$

$$p(v) = -\mathtt{i}(2\pi\mathtt{i})^{-3/2}\partial_{\mathfrak{y}}\Pi_{\mathcal{X}}(\mathcal{O}_0 \otimes \varrho^2) = -\mathtt{i}\frac{(2\pi\mathtt{i})^{3/2}}{3\Gamma(\frac{2}{3})^3}\xi^2 + O(\mathfrak{y})$$

where we used (110) and the formula (104) for $\overrightarrow{\Pi}_{\mathcal{X}}$. Therefore $F^2_{\mathrm{vec}}|_{\mathfrak{y}=0}$ lies in the subspace $x + (1-\xi^2)p = 0$. Part (c) follows since P_{orb} is transversal to F^2_{vec} near $\mathfrak{y} = 0$. □

Recapitulation 10.3.3. Recall from §7.1 that we can immerse the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ into the fiber $H_{\mathrm{aff}}|_{y_0}$ as an (immersed) Lagrangian submanifold $\mathcal{L}$, by parallel translation of the primitive section ζ. In terms of the "integral" co-ordinates (p,x) on $H_{\mathrm{aff}}|_{y_0}$ – see equation (109) – $\mathcal{L}$ is given by:

$$p = p(\zeta) = \mathtt{i}(2\pi\mathtt{i})^{-1/2}\left(\pi^2 + \pi\mathtt{i}t + 3\frac{\partial F^0_Y}{\partial t}\right), \qquad x = x(\zeta) = -\mathtt{i}(2\pi\mathtt{i})^{1/2}t$$

where $t = t(y)$ is the mirror map for Y (see equation (103)), and the tangent space is:

$$T_{(x(\zeta),p(\zeta))}\mathcal{L} = \mathbb{C}\begin{pmatrix}\tau\\1\end{pmatrix} \quad \text{with } \tau = \frac{\partial p(\zeta)}{\partial x(\zeta)} = -\frac{1}{2} - \frac{3}{2\pi\mathtt{i}}\frac{\partial^2 F^0_Y}{\partial t^2}\ . \tag{111}$$

We saw in Corollary 10.2.10 that the slope τ lies in $\mathbb{H}$ and identifies the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ with $\mathbb{H}$.

Notation 10.3.4. In this section, we denote by F^g_X the genus-g Gromov–Witten potential (7) of $X = \mathcal{X}$ or Y, restricted to the second cohomology and with Novikov parameters specialized to $Q = 1$. As before, we write $t \mapsto th \in H^2(Y)$, $\mathfrak{t} \mapsto \mathfrak{t}\mathbf{1}_{\frac{1}{3}} \in H^2_{\mathrm{orb}}(\mathcal{X})$ for parameters on the second cohomology. Explicitly, we have (see equations (83)–(84) and (101)–(102)):

$$F^0_Y(t) = -\frac{1}{18}t^3 + \sum_{d=1}^{\infty}\langle\rangle^Y_{0,0,d}\,e^{dt}$$

$$F^1_Y(t) = -\frac{1}{12}t + \sum_{d=1}^{\infty}\langle\rangle^Y_{1,0,d}\,e^{dt}$$

$$F^g_Y(t) = \sum_{d=0}^{\infty}\langle\rangle^Y_{g,0,d}\,e^{dt} \qquad \text{for } g \geq 2$$

and

$$F^g_{\mathcal{X}}(\mathfrak{t}) = \sum_{n:2g-2+n>0}\left\langle \mathbf{1}_{\frac{1}{3}}, \dots, \mathbf{1}_{\frac{1}{3}}\right\rangle^{\mathcal{X}}_{g,n,0}\frac{\mathfrak{t}^n}{n!} \qquad \text{for } g \geq 0.$$

Corollary 10.3.5. *The following objects can be analytically continued to the universal cover $(\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}})^\sim \cong \mathbb{H}$ of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$:*

(a) *the opposite line bundles P_{LR}, P_{con}, P_{orb};*
(b) *the Gromov–Witten potential $F^g_Y(t)$ of Y, when regarded as a function near the large-radius limit point $y = 0$ via the mirror map $t = t(y)$;*

(c) *the Gromov–Witten potential $F^g_{\mathcal{X}}(\mathfrak{t})$ of $\mathcal{X}$, when regarded as a function near the orbifold point $\mathfrak{y}=0$ via the mirror map $\mathfrak{t}=\mathfrak{t}(y)$.*

Proof. We use notation as in Recapitulation 10.3.3. Since τ is a non-zero holomorphic function on $(\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}})^\sim$, it follows from Proposition 10.3.2 that $\mathcal{L}$ is transversal to both P_{LR} and P_{con} *everywhere* on $(\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}})^\sim$, i.e. P_{LR} and P_{con} extend to opposite line bundles over $(\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}})^\sim$. Similarly, since $1+(1-\xi)\tau$ never vanishes for $\tau\in\mathbb{H}$, P_{orb} also extends to the universal cover. Part (a) follows. Parts (b) and (c) follow from Part (a) and the fact that the Gromov–Witten potentials of Y and $\mathcal{X}$ extend to a global section $\mathscr{C}_{\mathrm{CY}}$ of $\mathfrak{Fock}^\circ_{\mathrm{CY}}$: see Theorem 8.6.1. □

Notation 10.3.6 (Darboux co-ordinates at cusps)**.** Let (p,x) be the "integral" Darboux co-ordinates on H_{aff} given in (109) and let τ be the slope (111) of $\mathcal{L}$ in these co-ordinates. In view of Proposition 10.3.2, we introduce the following Darboux co-ordinates on $H_{\mathrm{aff}}|_{y_0}$ associated to the large-radius, conifold and orbifold points (cf. Examples 7.2.4, 7.2.9). Here all Darboux co-ordinates (p',x') are normalized so that $\Omega=\frac{1}{3}dp'\wedge dx'$.

(1) To the large-radius limit point, we associate the Darboux co-ordinates

$$\begin{pmatrix}p_{\mathrm{LR}}\\ x_{\mathrm{LR}}\end{pmatrix}=\mathtt{i}(2\pi\mathtt{i})^{-1/2}\begin{pmatrix}-2\pi\mathtt{i} & -\pi\mathtt{i}\\ 0 & 1\end{pmatrix}\begin{pmatrix}p\\ x\end{pmatrix}+\begin{pmatrix}-\pi^2\\ 0\end{pmatrix}$$

such that $P_{\mathrm{LR}}=\langle\partial/\partial p_{\mathrm{LR}}\rangle$. In these co-ordinates, $\mathcal{L}$ and its slope are given by:

$$\begin{cases}p_{\mathrm{LR}}=3\frac{\partial F^0_Y}{\partial t}\\ x_{\mathrm{LR}}=t\end{cases}\quad\text{and}\quad \tau_{\mathrm{LR}}=\frac{\partial p_{\mathrm{LR}}}{\partial x_{\mathrm{LR}}}=-2\pi\mathtt{i}\left(\tau+\frac{1}{2}\right)=3\frac{\partial^2F^0_Y}{\partial t^2}$$

where $t=t(y)$ is the mirror map for Y (see §10.2.1).

(2) To the conifold point, we associate the Darboux co-ordinates:

$$\begin{pmatrix}p_{\mathrm{con}}\\ x_{\mathrm{con}}\end{pmatrix}=\frac{1}{\sqrt{3}}(2\pi\mathtt{i})^{-1/2}\begin{pmatrix}0 & 2\pi\mathtt{i}\\ -3 & 0\end{pmatrix}\begin{pmatrix}p\\ x\end{pmatrix}$$

such that $P_{\mathrm{con}}=\langle\partial/\partial p_{\mathrm{con}}\rangle$. In these co-ordinates, the slope of $\mathcal{L}$ is given by:

$$\tau_{\mathrm{con}}=\frac{\partial p_{\mathrm{con}}}{\partial x_{\mathrm{con}}}=-\frac{2\pi\mathtt{i}}{3\tau}$$

(3) To the orbifold point, we associate the Darboux co-ordinates

$$\begin{pmatrix}p_{\mathrm{orb}}\\ x_{\mathrm{orb}}\end{pmatrix}=\frac{1}{\mathtt{i}(2\pi\mathtt{i})^{3/2}}\begin{pmatrix}3\Gamma(\frac{1}{3})^3 & (1-\xi)\Gamma(\frac{1}{3})^3\\ 3\Gamma(\frac{2}{3})^3 & (1-\xi^2)\Gamma(\frac{2}{3})^3\end{pmatrix}\begin{pmatrix}p\\ x\end{pmatrix}+\begin{pmatrix}\Gamma(\frac{1}{3})^3\\ \Gamma(\frac{2}{3})^3\end{pmatrix}$$

such that $P_{\mathrm{orb}}=\langle\partial/\partial p_{\mathrm{orb}}\rangle$. In these co-ordinates, $\mathcal{L}$ and its slope are given by:

$$\begin{cases}p_{\mathrm{orb}}=3\frac{\partial F^0_{\mathcal{X}}}{\partial \mathfrak{t}}\\ x_{\mathrm{orb}}=\mathfrak{t}\end{cases}\quad\text{and}\quad \tau_{\mathrm{orb}}=\frac{\partial p_{\mathrm{orb}}}{\partial x_{\mathrm{orb}}}=\frac{\Gamma(\frac{1}{3})^3}{\Gamma(\frac{2}{3})^3}\cdot\frac{3\tau+1-\xi}{3\tau+1-\xi^2}=3\frac{\partial^2F^0_{\mathcal{X}}}{\partial\mathfrak{t}^2}$$

where $\mathfrak{t}=\mathfrak{t}(\mathfrak{y})$ is the mirror map for $\mathcal{X}$ (see §10.2.1).

Remark 10.3.7. The slope parameters $\tau_{\mathrm{LR}},\tau_{\mathrm{con}},\tau_{\mathrm{orb}}$ take values, respectively, in the right-half plane $\{\tau_{\mathrm{LR}}\in\mathbb{C}:\Re(\tau_{\mathrm{LR}})>0\}$, the left-half plane $\{\tau_{\mathrm{con}}\in\mathbb{C}:\Re(\tau_{\mathrm{con}})<0\}$ and the disc $\{\tau_{\mathrm{orb}}\in\mathbb{C}:|\tau_{\mathrm{orb}}|<\Gamma(\frac{1}{3})^3/\Gamma(\frac{2}{3})^3\}$. In particular, the inequalities

$$\Re\left(\frac{\partial^2F^0_Y}{\partial t^2}\right)>0,\qquad\left|\frac{\partial^2F^0_{\mathcal{X}}}{\partial\mathfrak{t}^2}\right|<\frac{\Gamma(\frac{1}{3})^3}{3\Gamma(\frac{2}{3})^3}$$

hold.

Remark 10.3.8. The conifold co-ordinate $x_{\mathrm{con}} = \sqrt{3}(2\pi\mathtt{i})^{-1/2}p$ restricted to $\mathcal{L}$ can be written as the integral:

$$x_{\mathrm{con}} = \frac{\sqrt{3}}{2\pi}\int_{\Gamma_{\mathbb{R}}} \zeta$$

for $-1/27 < y < 0$ by Lemma 10.2.6. In particular, θx_{con} is a period over a vanishing cycle at the conifold point (see (106)). We also obtain the asymptotics $x_{\mathrm{con}} \sim 1 + 27y$ near the conifold point by approximating the above integral by the area of an ellipse. Since x_{con} is invariant under the conifold monodromy, it is holomorphic near $y = -1/27$.

The slope parameters in Notation 10.3.6 are related to each other by:

$$\tau_{\mathrm{LR}} = \frac{-3\pi\mathtt{i}\tau_{\mathrm{con}} - 4\pi^2}{3\tau_{\mathrm{con}}}, \qquad \tau_{\mathrm{con}} = \frac{-4\pi^2}{3(\tau_{\mathrm{LR}} + \pi\mathtt{i})}$$

$$\tau_{\mathrm{LR}} = -\frac{\pi}{\sqrt{3}}\frac{\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} + \Gamma(\frac{1}{3})^3}{\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} - \Gamma(\frac{1}{3})^3} \qquad \tau_{\mathrm{orb}} = \frac{\Gamma(\frac{1}{3})^3}{\Gamma(\frac{2}{3})^3}\frac{\sqrt{3}\tau_{\mathrm{LR}} - \pi}{\sqrt{3}\tau_{\mathrm{LR}} + \pi}$$

$$\tau_{\mathrm{con}} = 2\pi\mathtt{i}\frac{\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} - \Gamma(\frac{1}{3})^3}{(1-\xi^2)\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} - (1-\xi)\Gamma(\frac{1}{3})^3} \qquad \tau_{\mathrm{orb}} = \frac{\Gamma(\frac{1}{3})^3}{\Gamma(\frac{2}{3})^3}\frac{(1-\xi)\tau_{\mathrm{con}} - 2\pi\mathtt{i}}{(1-\xi^2)\tau_{\mathrm{con}} - 2\pi\mathtt{i}}$$

Therefore, by Example 7.2.9, the propagators among the opposite line bundles P_{LR}, P_{con}, P_{orb} are given as follows:

$$\begin{aligned}
&\Delta(P_{\mathrm{LR}}, P_{\mathrm{con}}) = \frac{-3}{\tau_{\mathrm{LR}} + \pi\mathtt{i}}(\partial_{x_{\mathrm{LR}}})^{\otimes 2} && \Delta(P_{\mathrm{orb}}, P_{\mathrm{con}}) = \frac{-3\Gamma(\frac{2}{3})^3}{\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} + \xi\Gamma(\frac{1}{3})^3}(\partial_{x_{\mathrm{orb}}})^{\otimes 2} \\
&\Delta(P_{\mathrm{con}}, P_{\mathrm{LR}}) = -\frac{3}{\tau_{\mathrm{con}}}(\partial_{x_{\mathrm{con}}})^{\otimes 2} && \Delta(P_{\mathrm{con}}, P_{\mathrm{orb}}) = \frac{-3(1-\xi^2)}{(1-\xi^2)\tau_{\mathrm{con}} - 2\pi\mathtt{i}}(\partial_{x_{\mathrm{con}}})^{\otimes 2} \\
&\Delta(P_{\mathrm{LR}}, P_{\mathrm{orb}}) = \frac{-3\sqrt{3}}{\sqrt{3}\tau_{\mathrm{LR}} + \pi}(\partial_{x_{\mathrm{LR}}})^{\otimes 2} && \Delta(P_{\mathrm{orb}}, P_{\mathrm{LR}}) = \frac{-3\Gamma(\frac{2}{3})^3}{\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} - \Gamma(\frac{1}{3})^3}(\partial_{x_{\mathrm{orb}}})^{\otimes 2}
\end{aligned} \tag{112}$$

where $x_{\mathrm{LR}} = t(y)$, $x_{\mathrm{orb}} = \mathfrak{t}(\mathfrak{y})$ are the mirror maps for Y and $\mathcal{X}$ respectively. The correlation functions of $\mathscr{C}_{\mathrm{CY}}$ with respect to P_{LR}, P_{con}, P_{orb} are related by Feynman rules given by these propagators. In particular, we get:

Theorem 10.3.9 (Crepant Resolution Conjecture for $\mathcal{X}$: explicit form)**.** *As in Corollary* 10.3.5*, we regard the Gromov–Witten potentials of Y and $\mathcal{X}$ as holomorphic functions on the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$, and we use Notation* 10.3.4*. After analytic continuation along the positive real line in the y-plane, the Gromov–Witten potentials of Y and $\mathcal{X}$ are related by a Feynman rule as in* §7.3*:*

$$(\partial_{\mathfrak{t}}^3 F_{\mathcal{X}}^0)\cdot(d\mathfrak{t})^{\otimes 3} = (\partial_t^3 F_Y^0)\cdot(dt)^{\otimes 3} = -\frac{1}{3(1+27y)}\left(\frac{dy}{y}\right)^{\otimes 3}$$

$$(\partial_{\mathfrak{t}} F_{\mathcal{X}}^1)d\mathfrak{t} = \left(\partial_t F_Y^1 + \frac{1}{2}(\partial_t^3 F_Y^0)\Delta\right)dt$$

$$F_{\mathcal{X}}^g = \sum_{\Gamma}\frac{1}{|\operatorname{Aut}(\Gamma)|}\operatorname{Cont}_\Gamma\left(\Delta, \{\partial_t^\bullet \mathcal{F}_Y^h : h \le g\}\right) \qquad \text{for } g \ge 2$$

where Γ in the third line ranges over all connected stable decorated genus-g graphs without legs, Δ is the propagator from P_{LR} to P_{orb}:

$$\Delta = -\frac{3\sqrt{3}}{\sqrt{3}\tau_{\mathrm{LR}} + \pi} \qquad \text{with } \tau_{\mathrm{LR}} = 3\frac{\partial^2 F_Y^0}{\partial t^2},$$

$\mathrm{Cont}_\Gamma(\Delta, \{\partial_t^\bullet \mathcal{F}_Y^h : h \le g\})$ *denotes the contraction along the graph* Γ *with edge terms* Δ *and vertex terms* $\partial_t^\bullet \mathcal{F}_Y^h$ *– see the explanation after* (70) *– and*

$$d\mathfrak{t} = -\frac{\sqrt{3}}{4\pi^2}\Gamma\left(\frac{2}{3}\right)^3\left(\sqrt{3}\tau_{\mathrm{LR}} + \pi\right)dt.$$

Proof. This follows from Theorem 8.6.1 and the definition of the finite-dimensional Fock sheaf $\mathfrak{Fock}^\circ_{\mathrm{CY}}$. See Example 8.1.2 for the Yukawa coupling (genus-zero term) and (112) for the propagator. □

Remark 10.3.10. By the general theory developed in §7.3, we can invert the Feynman rule in the above theorem, by exchanging $F_{\mathcal{X}}^g$ and F_Y^g, $x_{\mathrm{orb}} = \mathfrak{t}$ and $x_{\mathrm{LR}} = t$, and replacing Δ with $-3\Gamma(\frac{2}{3})^3/(\Gamma(\frac{2}{3})^3\tau_{\mathrm{orb}} - \Gamma(\frac{1}{3})^3)$.

Example 10.3.11. In Theorem 10.3.9, the Feynman rule at genus two takes the form:

$$F_{\mathcal{X}}^2 = F_Y^2 + \frac{1}{2}\Delta(\partial_t^2 F_Y^1) + \frac{1}{2}\Delta(\partial_t F_Y^1)^2 + \frac{1}{2}\Delta^2(\partial_t F_Y^1)(\partial_t^3 F_Y^0) + \frac{1}{8}\Delta^2(\partial_t^4 F_Y^0) + \frac{5}{24}\Delta^3(\partial_t^3 F_Y^0)^2.$$

10.4. **Algebraic and Complex Conjugate Opposite Line Bundles.** Recall the notion of curved opposite line bundle from §7.4. In this section we introduce two curved opposite line bundles P_{alg} and P_{cc}. The algebraic opposite line bundle P_{alg} is a holomorphic subbundle of H_{vec} which is opposite to F^2_{vec} but is not flat; the complex conjugate opposite line bundle P_{cc} is a C^∞-subbundle of H_{vec} which is opposite to F^2_{vec} but is not flat in the antiholomorphic direction. The key property of these line bundles is that they are *single-valued* over $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$, and therefore they yield single-valued correlation functions of the global section $\mathscr{C}_{\mathrm{CY}}$ of $\mathfrak{Fock}^\circ_{\mathrm{CY}}$. This property plays a crucial role in the next section.

As explained in §10.3, the central charges $\Pi_Y(V)$ of $V \in K_c(Y)$ give flat co-ordinates $\Pi_Y(V)^\sharp$ on $\overline{H}$, and thus on H_{vec}. Since the $\mathbb{Z}$-lattice formed by these co-ordinates $\Pi_Y(V)^\sharp$ is preserved under monodromy, they determine a real flat subbundle of $H_{\mathrm{vec}}|_{\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}}}$

$$H_{\mathrm{vec},\mathbb{R}} := \left\{v \in H_{\mathrm{vec}}\big|_{\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}}} : \mathtt{i}(2\pi\mathtt{i})^{-3/2}\Pi_Y(V)^\sharp(v) \in \mathbb{R} \text{ for all } V \in K_c(Y)\right\}$$

with the property that $H_{\mathrm{vec}} = H_{\mathrm{vec},\mathbb{R}} \oplus \mathtt{i}H_{\mathrm{vec},\mathbb{R}}$. Recall from Corollary 10.1.5 that $H_{\mathrm{vec}}|_y$ is isomorphic to $H^1(E_y, \mathbb{C})$ for $y \in \mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$, and that via this isomorphism, $\Pi_Y(V)^\sharp$ corresponds[22] to the integration over the integral cycle $\partial\,\mathrm{Mir}(V)$. By scaling the isomorphism $H_{\mathrm{vec}} \cong \bigcup_y H^1(E_y, \mathbb{C})$ by a constant, therefore, we have that $H_{\mathrm{vec},\mathbb{R}} \cong H^1(E_y, \mathbb{R})$.

Definition 10.4.1 (complex conjugate opposite)**.** The *complex conjugate opposite line bundle* P_{cc} is defined to be the C^∞ complex subbundle of $H_{\mathrm{vec}}|_{\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}}}$ given as the complex conjugate of F^2_{vec} with respect to the real form $H_{\mathrm{vec},\mathbb{R}}$.

Since P_{cc} is the complex conjugate of a holomorphic subbundle, P_{cc} is flat in the holomorphic direction – that is, $\nabla_v C^\infty(P_{\mathrm{cc}}) \subset C^\infty(P_{\mathrm{cc}})$ for any $(1,0)$-vector field v. It is not flat in the antiholomorphic direction.

Lemma 10.4.2. *The line bundle* $P_{\mathrm{cc}} \subset H_{\mathrm{vec}}|_{\mathcal{M}_{\mathrm{CY}}\setminus D_{\mathrm{CY}}}$ *extends to a topological line subbundle of* H_{vec} *over* $\mathcal{M}_{\mathrm{CY}}$ *such that* $F^2_{\mathrm{vec}} \oplus P_{\mathrm{cc}} = H_{\mathrm{vec}}$ *holds globally. Moreover, we have*

$$P_{\mathrm{cc}}|_{y=0} = P_{\mathrm{LR}}|_{y=0}, \qquad P_{\mathrm{cc}}|_{y=-\frac{1}{27}} = P_{\mathrm{con}}|_{y=-\frac{1}{27}}, \qquad P_{\mathrm{cc}}|_{y=\infty} = P_{\mathrm{orb}}|_{y=\infty}.$$

[22]Up to a constant – see equations (105) and (106).

Proof. Under the isomorphism $H_{\mathrm{vec}}|_y \cong H^1(E_y,\mathbb{C})$, $F^2_{\mathrm{vec}}|_y$ corresponds to $H^{1,0}(E_y,\mathbb{C}) \cong \mathbb{C}\lambda_y$. As discussed above, we have $H_{\mathrm{vec},\mathbb{R}}|_y \cong H^1(E_y,\mathbb{R})$. The Hodge decomposition implies that $P_{\mathrm{cc}}|_y$ corresponds to $H^{0,1}(E_y,\mathbb{C})$ and is opposite to $F^2_{\mathrm{vec}}|_y$ over $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$.

The extension of P_{cc} across D_{CY} and the oppositeness there follow from a property of the nilpotent orbit associated to a degeneration of Hodge structure (see [76]). We will give an elementary account below. Choose one of the limit points from $D_{\mathrm{CY}} = \{0, -\frac{1}{27}\}$ and let t denote a local co-ordinate centred at that point. From the description of $\overline{H}$ in §4.2, we can find a local basis $\{s_0, s_1\}$ of H_{vec} near $t = 0$ such that $F^2_{\mathrm{vec}} = \langle s_1 \rangle$ and that the connection ∇ is of the form:

$$(\nabla s_0, \nabla s_1) = (s_0, s_1) A(t) \frac{dt}{t} \qquad \text{with } A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then we can find a basis $\{f_0, f_1\}$ of flat sections of the form (see e.g. Proposition A.0.2):

$$(f_0, f_1) = (s_0, s_1) G(t) \begin{pmatrix} 1 & -\log t \\ 0 & 1 \end{pmatrix}$$

where $G(t)$ is a holomorphic matrix-valued function near $t = 0$ with $G(0) = I_2$. The flat section f_0 spans a monodromy-invariant line; hence after scaling $\{s_0, s_1\}$ by a constant, we may assume that $f_0 \in H_{\mathrm{vec},\mathbb{R}}$. Let $f_2 = a f_0 + b f_1$, $b \neq 0$ be another flat section taking values in $H_{\mathrm{vec},\mathbb{R}}$ and linearly independent of f_0. The monodromy acts on f_2 as $f_2 \mapsto f_2 - 2\pi \mathtt{i} b f_0$; reality of the monodromy then implies $b \in \mathtt{i}\mathbb{R}$. The complex conjugate $\overline{f_1}$ of f_1 with respect to $H_{\mathrm{vec},\mathbb{R}}$ is then computed as:

$$\overline{f_1} = \frac{\overline{a} - a}{b} f_0 - f_1.$$

Thus the complex conjugate of $\{s_0, s_1\}$ with respect to $H_{\mathrm{vec},\mathbb{R}}$ is:

$$\begin{aligned}
(\overline{s_0}, \overline{s_1}) &= (\overline{f_0}, \overline{f_1}) \begin{pmatrix} 1 & \overline{\log t} \\ 0 & 1 \end{pmatrix} \overline{G(t)}^{-1} = (f_0, f_1) \begin{pmatrix} 1 & \frac{\overline{a}-a}{b} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & \overline{\log t} \\ 0 & 1 \end{pmatrix} \overline{G(t)}^{-1} \\
&= (s_0, s_1) G(t) \begin{pmatrix} 1 & -\log t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\overline{a}-a}{b} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & \overline{\log t} \\ 0 & 1 \end{pmatrix} \overline{G(t)}^{-1} \\
&= (s_0, s_1) \left[\begin{pmatrix} 1 & \frac{\overline{a}-a}{b} + 2\log|t| \\ 0 & -1 \end{pmatrix} + O(|t| \log|t|) \right]
\end{aligned}$$

Since $P_{\mathrm{cc}} = \langle \overline{s_1} \rangle$, this implies that P_{cc} extends across $t = 0$ as a topological line bundle, and that the fiber at $t = 0$ is spanned by the invariant section $f_0|_{t=0} = s_0|_{t=0}$. This also shows the oppositeness of P_{cc} along D_{CY} and that $P_{\mathrm{cc}}|_{y=0} = P_{\mathrm{LR}}|_{y=0}$ and $P_{\mathrm{cc}}|_{y=-\frac{1}{27}} = P_{\mathrm{con}}|_{y=-\frac{1}{27}}$ (we showed in the proof of Proposition 10.3.2 that P_{LR} and P_{con} are spanned by invariant sections near cusps). To show that $P_{\mathrm{orb}}|_{y=\infty} = P_{\mathrm{cc}}|_{y=\infty}$, it suffices to note that these subspaces are uniquely characterized by invariance under μ_3-monodromy and oppositeness to $F^2_{\mathrm{vec}}|_{y=\infty}$. □

Recall from §4.2 that $\mathcal{O}(H_{\mathrm{vec}})$ is isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ as a vector bundle over $\mathcal{M}_{\mathrm{CY}} \cong \mathbb{P}(3,1)$, and that the subsheaf $\mathcal{O}(F^2_{\mathrm{vec}}) \cong \Theta(\log\{0\})$ is isomorphic to $\mathcal{O}(1)$. There is a (precisely) one-dimensional family of holomorphic line subbundles of H_{vec} which correspond to the factor $\mathcal{O}(-1)$.

Definition 10.4.3 (algebraic opposite)**.** The *algebraic opposite line bundle* $P_{\mathrm{alg}} = P_{\mathrm{alg}}(a)$, where $a \in \mathbb{C}$, is the holomorphic line subbundle of H_{vec} with basis given by:

$$\begin{aligned}
s_0 &= (9y + a)\theta\zeta + (1 + 27y)\theta^2\zeta && \text{over } \mathcal{M}_{\mathrm{CY}} \setminus \{y = \infty\} \\
s_\infty &= (1 - 3a)\mathfrak{y}^2 \partial_{\mathfrak{y}}\zeta + (27 + \mathfrak{y}^3)\partial^2_{\mathfrak{y}}\zeta = 9\mathfrak{y} s_0 && \text{over } \mathcal{M}_{\mathrm{CY}} \setminus \{y = 0\}
\end{aligned}$$

where $\theta = y\frac{\partial}{\partial y}$ and $\partial_{\mathfrak{y}} = \frac{\partial}{\partial \mathfrak{y}}$ act via the connection ∇. Any holomorphic line subbundle of H_{vec} which is globally complementary to F^2_{vec} is of the form $P_{\mathrm{alg}}(a)$ for some $a \in \mathbb{C}$.

Lemma 10.4.4. *Let $P_{\mathrm{alg}}(a)$ be the algebraic opposite line in Definition* 10.4.3. *We have*

$$P_{\mathrm{alg}}(0)|_{y=0} = P_{\mathrm{LR}}|_{y=0}, \quad P_{\mathrm{alg}}(\tfrac{1}{3})|_{y=-\frac{1}{27}} = P_{\mathrm{con}}|_{y=-\frac{1}{27}}, \quad P_{\mathrm{alg}}(a)|_{y=\infty} = P_{\mathrm{orb}}|_{y=\infty},$$

for all $a \in \mathbb{C}$.

Proof. This follows from Notation 7.2.7 and Proposition 4.3.2. □

Proposition 10.4.5. *We use Notation* 10.3.6. *The propagators between P_{LR}, P_{con}, P_{orb} and P_{cc} are given as follows:*

$$\begin{aligned}
\Delta(P_{\mathrm{LR}}, P_{\mathrm{cc}}) &= \frac{-3}{\tau-\overline{\tau}}(\partial_x)^{\otimes 2} = \frac{3}{2\pi\mathtt{i}(\tau-\overline{\tau})}(\partial_{x_{\mathrm{LR}}})^{\otimes 2} = -\frac{3}{\tau_{\mathrm{LR}}+\overline{\tau_{\mathrm{LR}}}}(\partial_{x_{\mathrm{LR}}})^{\otimes 2}\\
\Delta(P_{\mathrm{con}}, P_{\mathrm{cc}}) &= -3\frac{\overline{\tau}}{\tau}\frac{1}{\tau-\overline{\tau}}(\partial_x)^{\otimes 2} = -\frac{9}{2\pi\mathtt{i}}\frac{|\tau|^2}{\tau-\overline{\tau}}(\partial_{x_{\mathrm{con}}})^{\otimes 2} = -\frac{3}{\tau_{\mathrm{con}}+\overline{\tau_{\mathrm{con}}}}(\partial_{x_{\mathrm{con}}})^{\otimes 2}\\
\Delta(P_{\mathrm{orb}}, P_{\mathrm{cc}}) &= -3\frac{1+\overline{\tau}(1-\xi)}{1+\tau(1-\xi)}\frac{1}{\tau-\overline{\tau}}(\partial_x)^{\otimes 2} = \frac{3\Gamma(\frac{2}{3})^6}{(2\pi\mathtt{i})^3}\frac{(3\tau+1-\xi^2)(3\overline{\tau}+1-\xi^2)}{\tau-\overline{\tau}}(\partial_{x_{\mathrm{orb}}})^{\otimes 2}\\
&= \frac{3\overline{\tau_{\mathrm{orb}}}}{\Gamma(\frac{1}{3})^6\Gamma(\frac{2}{3})^{-6}-|\tau_{\mathrm{orb}}|^2}(\partial_{x_{\mathrm{orb}}})^{\otimes 2}
\end{aligned}$$

where we regard x, x_{LR}, x_{con}, x_{orb} as co-ordinates on the immersed submanifold $\mathcal{L} \looparrowright H_{\mathrm{aff}}$, or on the universal cover of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ (see Recapitulation 10.3.3*).*

Proof. Since (p, x) are real co-ordinates with respect to $H_{\mathrm{vec},\mathbb{R}}$, the complex conjugation in these co-ordinates is the ordinary one. Written in this frame, we have:

$$F^2_{\mathrm{vec}} = \mathbb{C}\begin{pmatrix}\tau\\1\end{pmatrix}, \qquad P_{\mathrm{cc}} = \mathbb{C}\begin{pmatrix}\overline{\tau}\\1\end{pmatrix}, \quad \text{and} \quad \mathrm{KS}(\partial_x) = \begin{pmatrix}\tau\\1\end{pmatrix}.$$

Hence by writing $\Pi_{\mathrm{cc}}\colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ for the projection along P_{cc}, we have

$$\mathrm{KS}^{-1}\,\Pi_{\mathrm{cc}}(\partial_p) = \frac{1}{\tau-\overline{\tau}}\partial_x, \qquad \mathrm{KS}^{-1}\,\Pi_{\mathrm{cc}}(\partial_x) = -\frac{\overline{\tau}}{\tau-\overline{\tau}}\partial_x.$$

Let Π_{LR}, Π_{orb}, $\Pi_{\mathrm{con}}\colon H_{\mathrm{vec}} \to F^2_{\mathrm{vec}}$ denote the projections along P_{LR}, P_{orb}, P_{con} respectively. Since $\mathrm{KS}^{-1}\,\Pi_{\mathrm{LR}}(\partial_p) = 0$, $\mathrm{KS}^{-1}\,\Pi_{\mathrm{LR}}(\partial_x) = \partial_x$, we have (see Definition 7.2.8)

$$\Delta(P_{\mathrm{LR}}, P_{\mathrm{cc}}) = (\mathrm{KS}^{-1}\otimes\mathrm{KS}^{-1})(\Pi_{\mathrm{LR}}\otimes\Pi_{\mathrm{cc}})(3\partial_p\otimes\partial_x - 3\partial_x\otimes\partial_p) = -\frac{3}{\tau-\overline{\tau}}(\partial_x)^{\otimes 2}.$$

The other formulae can be obtained similarly using Notation 10.3.6 and

$$\begin{aligned}
\mathrm{KS}^{-1}\,\Pi_{\mathrm{con}}(\partial_p) &= \frac{1}{\tau}\partial_x, & \mathrm{KS}^{-1}\,\Pi_{\mathrm{con}}(\partial_x) &= 0,\\
\mathrm{KS}^{-1}\,\Pi_{\mathrm{orb}}(\partial_p) &= \frac{1-\xi}{1+\tau(1-\xi)}\partial_x, & \mathrm{KS}^{-1}\,\Pi_{\mathrm{orb}}(\partial_x) &= \frac{1}{1+\tau(1-\xi)}\partial_x
\end{aligned}$$

which we deduce easily from Proposition 10.3.2. □

Remark 10.4.6. The propagators $\Delta(P, P_{\mathrm{cc}})$ with $P = P_{\mathrm{LR}}$, P_{con}, P_{orb} approach zero at the corresponding limit points, confirming again the conclusion of Lemma 10.4.2.

Lemma 10.4.7. *For any flat affine Darboux co-ordinates $(\tilde{p}, \tilde{x})$ on H_{aff} with $\Omega = \frac{1}{3}d\tilde{p}\wedge d\tilde{x}$, we have $\theta\tilde{p}(\zeta)\cdot\theta^2\tilde{x}(\zeta) - \theta^2\tilde{p}(\zeta)\cdot\theta\tilde{x}(\zeta) = (1+27y)^{-1}$, where $\theta = y\frac{\partial}{\partial y}$.*

Proof. This follows from $3\Omega(\theta\zeta, \theta^2\zeta) = (1+27y)^{-1}$: see (47). □

Let $E_2(\tau)$ and $\widehat{E}_2(\tau)$ denote the second Eisenstein series and its modular counterpart:

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nQ^n}{1-Q^n} \qquad\qquad \widehat{E}_2(\tau) = E_2(\tau) + \frac{6}{\pi\mathtt{i}}\frac{1}{\tau-\overline{\tau}}$$

with $Q = e^{2\pi\mathtt{i}\tau}$. Then we have [57]:

$$\begin{aligned} E_2\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^2 E_2(\tau) + \frac{6c(c\tau+d)}{\pi\mathtt{i}} \\ \widehat{E}_2\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^2 \widehat{E}_2(\tau) \end{aligned} \tag{113}$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

Proposition 10.4.8. *Let $P_{\mathrm{alg}} = P_{\mathrm{alg}}(a)$ be the algebraic opposite line bundle in Definition* 10.4.3*. Use Notation* 10.3.6*. The propagator between P_{cc} and P_{alg} is given by*

$$\Delta(P_{\mathrm{cc}}, P_{\mathrm{alg}}) = 3\left(\frac{1}{\tau-\overline{\tau}} - \theta x \cdot \big((9y+a)\theta x + (1+27y)\theta^2 x\big)\right)\partial_x \otimes \partial_x \tag{114}$$

$$= \frac{\pi\mathtt{i}}{2}\widehat{E}_2(\tau)\partial_x \otimes \partial_x + 3\left(\frac{1}{12} - a\right)\theta \otimes \theta. \tag{115}$$

where we regard $x = x(\zeta)$ as a co-ordinate on the immersed submanifold $\mathcal{L} \looparrowright H_{\mathrm{aff}}$, or on the universal cover $\mathbb{H}$ of $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$ (see Recapitulation 10.3.3*).*

Proof. In terms of the integral Darboux co-ordinates (p, x) in (109), P_{alg} is given by

$$P_{\mathrm{alg}} = \mathbb{C}\begin{pmatrix} (9y+a)\theta p(\zeta) + (1+27y)\theta^2 p(\zeta) \\ (9y+a)\theta x(\zeta) + (1+27y)\theta^2 x(\zeta) \end{pmatrix}$$

Set $A := \theta x(\zeta) \cdot ((9y+a)\theta x(\zeta) + (1+27y)\theta^2 x(\zeta))$. Using Lemma 10.4.7, we find

$$P_{\mathrm{alg}} = \mathbb{C}\begin{pmatrix} \frac{\theta p(\zeta)}{\theta x(\zeta)} A - 1 \\ A \end{pmatrix} = \mathbb{C}\begin{pmatrix} \tau A - 1 \\ A \end{pmatrix}.$$

Arguing as in Proposition 10.4.5, we find:

$$\Delta(P_{\mathrm{cc}}, P_{\mathrm{alg}}) = 3\left(\frac{1}{\tau-\overline{\tau}} - A\right)\partial_x \otimes \partial_x.$$

This shows (114). Next we show that the expressions (114) and (115) coincide. Recall that (p, x) and τ transform under monodromy as (see Proposition 10.2.3)

$$\begin{pmatrix} p \\ x \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p \\ x \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix}, \qquad \tau \mapsto \frac{a\tau+b}{c\tau+d}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(3)$, whence $\theta x = \theta x(\zeta)$ transforms as:

$$\theta x \mapsto c(\theta p) + d(\theta x) = (c\tau+d)\theta x. \tag{116}$$

Therefore the modular transformation property (113) implies that

$$\widehat{E}_2(\tau)\partial_x \otimes \partial_x = \widehat{E}_2(\tau)(\theta x)^{-2}\theta \otimes \theta$$

is invariant under monodromy, and that the expression (115) descends to a single-valued bivector field on $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$. Moreover, (115) extends to a continuous global section of $\Theta(\log\{0\})^{\otimes 2}$ since $\widehat{E}_2(\tau) \to 1$ and $\theta x(\zeta) \to -\mathtt{i}(2\pi\mathtt{i})^{1/2}$ in the large-radius limit $\tau \to +\infty\mathtt{i}$, and

$$\widehat{E}_2(\tau)\partial_x \otimes \partial_x = \widehat{E}_2(\tau)\frac{-3\tau^2}{2\pi\mathtt{i}}\partial_{x_{\mathrm{con}}} \otimes \partial_{x_{\mathrm{con}}} = -\frac{3}{2\pi\mathtt{i}}\widehat{E}_2(-1/\tau)\partial_{x_{\mathrm{con}}} \otimes \partial_{x_{\mathrm{con}}}$$

tends to $-3/(2\pi\mathtt{i})(\partial_{x_{\mathrm{con}}})^{\otimes 2}$ in the conifold limit $\tau \to 0$. Note that $\Delta(P_{\mathrm{cc}}, P_{\mathrm{alg}})$ is a global continuous section of $\Theta(\log\{0\})^{\otimes 2}$ as P_{cc}, P_{alg} are globally defined; moreover the difference between (114) and (115) is holomorphic. Thus the difference between (114) and (115) is a global holomorphic section of $\Theta(\log\{0\})^{\otimes 2} \cong \mathcal{O}(2)$. Such a section is unique up to a constant, so it suffices now to check that (114) and (115) have the same value $-3a\theta \otimes \theta$ at $y = 0$. □

Comparing (114) and (115), we obtain:

Corollary 10.4.9. $-2\pi\mathtt{i}E_2(\tau) = (\theta x) \cdot \big((1 + 108y)\,\theta x + 12(1 + 27y)\theta^2 x\big)$, *where* $x = x(\zeta)$.

Corollary 10.4.10. *Let* $\eta(\tau) = e^{\pi\mathtt{i}\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi\mathtt{i}\tau n})$ *denote the Dedekind eta function. We have* $\eta(\tau) = e^{-\frac{\pi\mathtt{i}}{24}} y^{\frac{1}{24}} (1 + 27y)^{\frac{1}{8}} \sqrt{\mathtt{i}(2\pi\mathtt{i})^{-1/2}\theta x}$.

Proof. Using the identity $\frac{\partial}{\partial\tau}\log\eta(\tau) = \frac{\pi\mathtt{i}}{12}E_2(\tau)$ and the above corollary, we have

$$\theta \log\eta(\tau) = -\frac{1}{(1 + 27y)(\theta x)^2}\frac{\pi\mathtt{i}}{12}E_2(\tau) = \frac{1}{24} + \frac{1}{8}\frac{27y}{1 + 27y} + \frac{1}{2}\frac{\theta^2 x}{\theta x}.$$

We arrive at the formula by integrating this. □

Combining Propositions 10.4.5, 10.4.8, we obtain:

Corollary 10.4.11. *With notation as in Propositions* 10.4.5 *and* 10.4.8, *we have*

$$\begin{aligned}
\Delta(P_{\mathrm{LR}}, P_{\mathrm{alg}}) - \Delta_a &= \frac{\pi\mathtt{i}}{2}E_2(\tau)(\partial_x)^{\otimes 2} = -\frac{1}{4}E_2(\tau)(\partial_{x_{\mathrm{LR}}})^{\otimes 2} \\
\Delta(P_{\mathrm{con}}, P_{\mathrm{alg}}) - \Delta_a &= \left(\frac{3}{\tau} + \frac{\pi\mathtt{i}}{2}E_2(\tau)\right)(\partial_x)^{\otimes 2} = \frac{3}{4}E_2\left(\frac{3\tau_{\mathrm{con}}}{2\pi\mathtt{i}}\right)(\partial_{x_{\mathrm{con}}})^{\otimes 2} \\
\Delta(P_{\mathrm{orb}}, P_{\mathrm{alg}}) - \Delta_a &= \left(\frac{3(1-\xi)}{1 + \tau(1-\xi)} + \frac{\pi\mathtt{i}}{2}E_2(\tau)\right)(\partial_x)^{\otimes 2} \\
&= \partial_{\mathfrak{y}} x_{\mathrm{orb}} \cdot \left(\frac{1}{12}\mathfrak{y}^2\partial_{\mathfrak{y}} x_{\mathrm{orb}} + 3\left(1 + \frac{\mathfrak{y}^3}{27}\right)\partial_{\mathfrak{y}}^2 x_{\mathrm{orb}}\right)(\partial_{x_{\mathrm{orb}}})^{\otimes 2}
\end{aligned}$$

where $\Delta_a = 3\left(\frac{1}{12} - a\right)\theta \otimes \theta$.

Proof. We use $\Delta(P_{\mathrm{LR}}, P_{\mathrm{alg}}) = \Delta(P_{\mathrm{LR}}, P_{\mathrm{cc}}) + \Delta(P_{\mathrm{cc}}, P_{\mathrm{alg}})$ etc. from Proposition 7.3.7. We also use Notation 10.3.6, equation (113), Corollary 10.4.9, and Lemma 10.4.7 for $(p_{\mathrm{orb}}, x_{\mathrm{orb}})$. Another way to compute these quantities will be explained in Lemma 10.7.1. □

10.5. **Quasi-modularity of Gromov–Witten Potentials.** In this section we prove that the Gromov–Witten potential F_Y^g is a quasi-modular function. Let us begin by reviewing the theory of quasi-modular forms introduced[23] by Kaneko–Zagier [57]. We say that a holomorphic

[23]To be more precise, Kaneko–Zagier considered quasi-modular forms which satisfy a standard growth condition at cusps. We do not impose the growth condition, since we deal with (quasi-)modular forms with non-positive weight.

function $f\colon \mathbb{H} \to \mathbb{C}$ is a *quasi-modular form of weight* k for $\Gamma_1(3)$ if there exist finitely many holomorphic functions $f_i\colon \mathbb{H} \to \mathbb{C}$, $i = 1, \dots, n$ such that

$$\hat{f}(\tau) = f(\tau) + \frac{f_1(\tau)}{\tau - \overline{\tau}} + \dots + \frac{f_n(\tau)}{(\tau - \overline{\tau})^n}$$

is modular of weight k, i.e.

$$\hat{f}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \hat{f}(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(3) \text{ and all } \tau \in \mathbb{H}.$$

When $n = 0$, f is a (holomorphic) modular form of weight k. It is known that $f_1, \dots, f_n$ (and hence $\hat{f}$) are uniquely determined by f [57, Proposition 1]; see [11, Proposition 3.4] for a proof. The function $\hat{f}$ is called the *almost holomorphic modular form* associated with f, and f is called the *holomorphic limit* of $\hat{f}$. Equation (113) shows that E_2 is a quasi-modular form of weight 2 and $\widehat{E}_2$ is the associated almost holomorphic modular form. Every almost holomorphic modular form of weight k can be uniquely expanded in the form:

$$\hat{f}(\tau) = \sum_{j=0}^{n} g_j(\tau) \widehat{E}_2(\tau)^j$$

where g_j is holomorphic modular of weight $k - 2j$. Taking the holomorphic limit, we find that the corresponding quasi-modular form f admits a unique expansion:

$$f(\tau) = \sum_{j=0}^{n} g_j(\tau) E_2(\tau)^j \tag{117}$$

with g_j holomorphic modular of weight $k - 2j$. The ring of quasi-modular forms is therefore generated by modular forms and E_2 (see [11, 57]).

Remark 10.5.1 (modular quantities)**.** Let (p, x) be the Darboux co-ordinates from (109), regarded as functions on $\mathcal{L} \cong (\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}})^\sim \cong \mathbb{H}$ as in Recapitulation 10.3.3. The following quantities are holomorphic modular for $\Gamma_1(3)$:

- the rational co-ordinates y, $\mathfrak{y}$, which are of weight 0;
- θx, which is of weight 1 (see equation (116));
- $\theta\tau = \theta(\theta p/\theta x) = -(1 + 27y)^{-1}(\theta x)^{-2}$, which is of weight -2;
- the Yukawa coupling $Y_{\mathrm{CY}}(x) = \frac{1}{3}\partial_x \tau = \frac{1}{3}\theta\tau/\theta x$, which is of weight -3.

We also note the following:

- $f(\tau)(d\tau)^{\otimes k}$ is $\Gamma_1(3)$-invariant $\iff$ $f(\tau)$ is of weight $2k$;
- $f(\tau)(\partial_x)^{\otimes k}$ is $\Gamma_1(3)$-invariant $\iff$ $f(\tau)$ is of weight k;
- $f(\tau)(dx)^{\otimes k}$ is $\Gamma_1(3)$-invariant $\iff$ $f(\tau)$ is of weight $-k$.

These follow from $d\tau = (\theta\tau)\frac{dy}{y}$, $\partial_x = (\theta x)^{-1}\theta$ and the above computation.

Notation 10.5.2 (correlation functions for $\mathscr{C}_{\mathrm{CY}}$)**.** Let $x = x(\zeta)$ denote the co-ordinate on $(\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}})^\sim$ induced by the integral Darboux co-ordinates (109). We represent the global section $\mathscr{C}_{\mathrm{CY}}$ of the finite-dimensional Fock sheaf by correlation functions as follows (see §§8–9):

(1) the correlation functions $C^{(g)}_{Y,n} dx^{\otimes n}$ with respect to P_{LR};
(2) the correlation functions $C^{(g)}_{\mathcal{X},n} dx^{\otimes n}_{\mathrm{orb}}$ with respect to P_{orb};
(3) the correlation functions $C^{(g)}_{\mathrm{con},n} dx^{\otimes n}_{\mathrm{con}}$ with respect to P_{con};
(4) the correlation functions $C^{(g)}_{\mathrm{cc},n} dx^{\otimes n}$ with respect to P_{cc};

(5) the correlation functions $C^{(g)}_{\mathrm{alg},n}dx^{\otimes n}$ with respect to $P_{\mathrm{alg}}(a)$.

The co-ordinates $\mathfrak{t} = x_{\mathrm{orb}}$ and x_{con} here were defined in Notation 10.3.6. We will set $a = \frac{1}{12}$ unless otherwise specified, and will write $F^g_{\mathrm{con}} = C^{(g)}_{\mathrm{con},0}$. Theorem 8.6.1 gives

$$C^{(g)}_{Y,n} = \frac{\partial^n F^g_Y}{\partial x^n} = \mathtt{i}^n(2\pi\mathtt{i})^{-n/2}\frac{\partial^n F^g_Y}{\partial t^n} \qquad \text{and} \qquad C^{(g)}_{\mathcal{X},n} = \frac{\partial^n F^g_{\mathcal{X}}}{\partial \mathfrak{t}^n} \tag{118}$$

where F^g_Y and $F^g_{\mathcal{X}}$ were defined in Notation 10.3.4.

Theorem 10.5.3. *Let g and n be non-negative integers satisfying $2g-2+n>0$. We have the following (quasi-)modularity with respect to the group $\Gamma_1(3)$ and the modular parameter τ from Corollary* 10.2.10.

(a) $C^{(g)}_{Y,n}$ *is a quasi-modular form of weight $-n$;*

(b) $C^{(g)}_{\mathrm{cc},n}$ *is the almost-holomorphic modular form of weight $-n$ associated with $C^{(g)}_{Y,n}$;*

(c) $C^{(g)}_{\mathrm{alg},n}$ *is the holomorphic modular form of weight $-n$ which appears as the constant term of the E_2-expansion* (117) *of $C^{(g)}_{Y,n}$.*

Proof. The correlation functions $\{C^{(g)}_{Y,n}dx^{\otimes n}\}$, $\{C^{(g)}_{\mathrm{cc},n}dx^{\otimes n}\}$, $\{C^{(g)}_{\mathrm{alg},n}dx^{\otimes n}\}$ are different realizations of the same section $\mathscr{C}_{\mathrm{CY}}$ of the Fock sheaf, and therefore they are related by the Feynman rule. The relationship between these correlation functions is shown in Figure 9: the propagators recorded there were computed in Proposition 10.4.5, Proposition 10.4.8, and Corollary 10.4.11. Since P_{cc} and P_{alg} are single-valued subbundles on $\mathcal{M}_{\mathrm{CY}}$, we know that $C^{(g)}_{\mathrm{cc},n}dx^{\otimes n}$ and $C^{(g)}_{\mathrm{alg},n}dx^{\otimes n}$ are $\Gamma_1(3)$-invariant. Remark 10.5.1 then implies that $C^{(g)}_{\mathrm{cc},n}$ and $C^{(g)}_{\mathrm{alg},n}$ are modular of weight $-n$. The Feynman rules between P_{LR} and $P_{\mathrm{cc}}/P_{\mathrm{alg}}$ imply that $C^{(g)}_{\mathrm{cc},n}$ and $C^{(g)}_{\mathrm{alg},n}$ can be written in the form:

$$C^{(g)}_{\mathrm{cc},n} = C^{(g)}_{Y,n} + \sum_{i=1}^{3g-3+n} f_i(\tau)\left(\frac{-3}{\tau-\overline{\tau}}\right)^i$$

$$C^{(g)}_{Y,n} = C^{(g)}_{\mathrm{alg},n} + \sum_{i=1}^{3g-3+n} \tilde{f}_i(\tau)\left(-\frac{\pi\mathtt{i}}{2}E_2(\tau)\right)^i$$

for some holomorphic functions f_i, $\tilde{f}_i$ on $\mathbb{H}$. Moreover $\tilde{f}_i$ is modular because it consists of products of several $C^{(h)}_{\mathrm{alg},m}$'s (with total weight $-n-2i$). This implies that $C^{(g)}_{Y,n}$ is a quasi-modular form, that $C^{(g)}_{\mathrm{cc},n}$ is the corresponding almost holomorphic modular form, and that $C^{(g)}_{\mathrm{alg},n}$ is the constant term of the E_2-expansion of $C^{(g)}_{Y,n}$, as claimed. □

Essentially the same argument shows the parallel results for conifold correlation functions:

Proposition 10.5.4. *Let g and n be non-negative integers satisfying $2g-2+n>0$. Let $\heartsuit$ denote one of* alg, con *and* cc, *and write $\widetilde{C}^{(g)}_{\heartsuit,n} := (\frac{\partial x}{\partial x_{\mathrm{con}}})^n C^{(g)}_{\heartsuit,n}$ for the correlation functions in the frame $(dx_{\mathrm{con}})^{\otimes n}$. We have the following (quasi-)modularity with respect to the group $\Gamma_1(3)$ and the modular parameter $\tau' := -1/(3\tau) = \tau_{\mathrm{con}}/(2\pi\mathtt{i})$ discussed in Remark 10.2.11.*

(a) $C^{(g)}_{\mathrm{con},n}$ *is a quasi-modular form of weight $-n$;*

(b) $\widetilde{C}^{(g)}_{\mathrm{cc},n} = (6\pi\mathtt{i})^{n/2}(\tau')^n C^{(g)}_{\mathrm{cc},n}$ *is the associated almost-holomorphic modular form;*

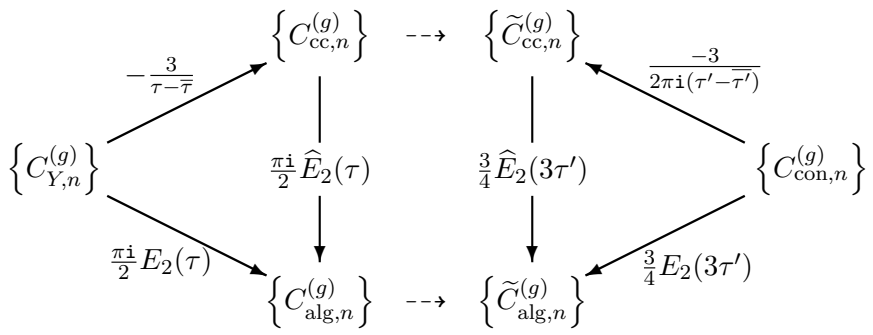


FIGURE 9. Triples of correlation functions related by Feynman rules, with the arrows labelled by propagators. The dashed arrows ⇢ mean a change of frames (multiplication by $(6\pi\mathtt{i})^{n/2}(\tau')^n$) and change of variables $\tau' = -1/(3\tau)$.

(c) $\widetilde{C}^{(g)}_{\mathrm{alg},n} = (6\pi\mathtt{i})^{n/2}(\tau')^n C^{(g)}_{\mathrm{alg},n}$ *is the holomorphic modular form which appears as the constant term of the* $E_2(3\tau')$*-expansion of* $C^{(g)}_{\mathrm{con},n}$.

10.6. **The Holomorphic Anomaly Equation and the Anomaly Equation.**

Proposition 10.6.1. *Let* (p, x) *denote the integral Darboux co-ordinates* (109)*, and regard* $x = x(\zeta)$ *as a co-ordinate on the universal cover of* $\mathcal{M}_{\mathrm{CY}} \setminus D_{\mathrm{CY}}$*. The torsion forms (see Definition 7.4.4) of* P_{cc} *and* $P_{\mathrm{alg}}(a)$ *are given respectively as follows:*

$$\Lambda_{\mathrm{cc}}(dx, dx) = \frac{3}{(\tau-\overline{\tau})^2} d\overline{\tau} = -\frac{3}{(\tau-\overline{\tau})^2} \frac{d\overline{x}}{(1+27\overline{y})(\overline{\theta x})^3}$$

$$\Lambda_{\mathrm{alg}}\left(\frac{dy}{y}, \frac{dy}{y}\right) = \frac{9(1-6a)y - 3a^2}{1+27y} \frac{dy}{y}$$

Proof. Use (72). In the notation there, we have $c = \overline{\tau}$ for P_{cc} and $c = \tau - 1/A$ for P_{alg}, where A is as in the proof of Proposition 10.4.8. We also use:

$$d\tau = \tau_x dx = 3Y_{\mathrm{CY}}(x)dx = -\frac{1}{(1+27y)(\theta x)^3} dx \qquad \text{(see Example 8.1.2)},$$

$$(1+27y)\theta^3 x = -6y\theta x - 27y\theta^2 x \qquad \text{(since } x \text{ is a solution to (108))}$$

where $Y_{\mathrm{CY}} = Y_{\mathrm{CY}}(x)(dx)^{\otimes 3}$. □

Remark 10.6.2. The connection $\boldsymbol{\nabla}^{P_{\mathrm{cc}}}$ associated with the complex conjugate opposite line bundle (see §7.4) respects the positive definite Hermitian metric h on $\Theta \cong F^2_{\mathrm{vec}} \cong \bigcup_y H^{1,0}(E_y)$ given by $h(\lambda_1, \lambda_2) = \mathtt{i}\int_{E_y} \lambda_1 \cup \overline{\lambda_2}$ for $\lambda_i \in H^{1,0}(E_y)$. Therefore $\boldsymbol{\nabla}^{P_{\mathrm{cc}}}$ is the Chern connection associated with this Hermitian metric. Recall also from Lemma 7.4.5 that

$$Y_{\mathrm{CY}}(x)\Lambda_{\overline{x}}\, dx \wedge d\overline{x} = \frac{dx \wedge d\overline{x}}{(\tau-\overline{\tau})^2|1+27y|^2|\theta x|^6} = \frac{d\tau \wedge d\overline{\tau}}{(\tau-\overline{\tau})^2}$$

is the curvature of $\boldsymbol{\nabla}^{P_{\mathrm{cc}}}$ and gives the Poincaré metric on $\mathbb{H}$.

From Proposition 7.4.6 and Remark 7.4.7, we obtain the following:

Proposition 10.6.3. *The correlation functions $C^{(g)}_{\mathrm{cc},n}dx^{\otimes n}$ associated with P_{cc} satisfy the following holomorphic anomaly equation:*

$$C^{(g)}_{\mathrm{cc},n+1} = \boldsymbol{\nabla}^{P_{\mathrm{cc}}}_x C^{(g)}_{\mathrm{cc},n} = \frac{\partial C^{(g)}_{\mathrm{cc},n}}{\partial x} - \frac{n\tau_x}{(\tau-\overline{\tau})} C^{(g)}_{\mathrm{cc},n}$$

$$\frac{\partial C^{(1)}_{\mathrm{cc},1}}{\partial \overline{x}} = -\frac{1}{2}\frac{|\tau_x|^2}{(\tau-\overline{\tau})^2}$$

$$\frac{\partial C^{(g)}_{\mathrm{cc},0}}{\partial \overline{x}} = -\frac{1}{2}\sum_{h=1}^{g-1}\frac{3\overline{\tau_x}}{(\tau-\overline{\tau})^2} C^{(h)}_{\mathrm{cc},1} C^{(g-h)}_{\mathrm{cc},1} - \frac{1}{2}\frac{3\overline{\tau_x}}{(\tau-\overline{\tau})^2} C^{(g-1)}_{\mathrm{cc},2}.$$

Proposition 10.6.4. *Let $\widehat{C}^{(g)}_{\mathrm{alg},n} := C^{(g)}_{\mathrm{alg},n}(\theta x)^n$ denote the correlation function associated with $P_{\mathrm{alg}}(a)$ written in the frame $(\frac{dy}{y})^{\otimes n}$. Then $\widehat{C}^{(g)}_{\mathrm{alg},n}$ is a rational function of y of the form:*

$$\widehat{C}^{(g)}_{\mathrm{alg},n} = \sum_{i=\lceil n/3\rceil}^{2g-2+n} \frac{c_i}{(1+27y)^i}, \qquad c_i \in \mathbb{C}, \tag{119}$$

and satisfies the following anomaly equation:

$$\widehat{C}^{(g)}_{\mathrm{alg},n+1} = \theta\widehat{C}^{(g)}_{\mathrm{alg},n} + \frac{n(a+9y)}{1+27y}\widehat{C}^{(g)}_{\mathrm{alg},n} + \frac{9(1-6a)y-3a^2}{2(1+27y)}\left(\sum_{\substack{h+k=g\\ i+j=n}}\binom{n}{i}\widehat{C}^{(h)}_{\mathrm{alg},i+1}\widehat{C}^{(k)}_{\mathrm{alg},j+1} + \widehat{C}^{(g-1)}_{\mathrm{alg},n+2}\right)$$

with $\theta = y\frac{\partial}{\partial y}$. (In this proposition, we do not specialize a to $\frac{1}{12}$.)

Proof. By Theorem 9.0.3 and Lemma 7.4.2, the n-tensor $\widehat{C}^{(g)}_{\mathrm{alg},n}(\frac{dy}{y})^{\otimes n} = \widehat{C}^{(g)}_{\mathrm{alg},n}(-3\frac{d\mathfrak{y}}{\mathfrak{y}})^{\otimes n}$ has poles of order at most $2g-2+n$ at the conifold point and is regular elsewhere. Thus $\widehat{C}^{(g)}_{\mathrm{alg},n}$ is of the form (119). The anomaly equation follows from Propositions 7.4.6 and 10.6.1: it suffices to note that

$$\boldsymbol{\nabla}^{P_{\mathrm{alg}}(a)}\left(\frac{dy}{y}\right) = \frac{a+9y}{1+27y}\frac{dy}{y}\otimes\frac{dy}{y}, \tag{120}$$

which follows by combining Lemma 7.4.5, Corollaries 10.4.11 and 10.4.9, and $\frac{dy}{y} = \frac{dx}{\theta x}$. □

Remark 10.6.5. The above anomaly equation reconstructs n-point correlation functions $\widehat{C}^{(g)}_{\mathrm{alg},n}$ from the base cases $\widehat{C}^{(0)}_{\mathrm{alg},3}$, $\widehat{C}^{(1)}_{\mathrm{alg},1}$ and $\widehat{C}^{(h)}_{\mathrm{alg},0}$, $h \geq 2$. The anomaly equation preserves the pole order condition at $y = -1/27$ and the vanishing condition at $y = \infty$.

Example 10.6.6 (genus-one potential)**.** In this example we set $a = -1/12$. The genus-one, one-point function $\widehat{C}^{(1)}_{\mathrm{alg},1}$ is of the form:

$$\widehat{C}^{(1)}_{\mathrm{alg},1} = \frac{c}{1+27y}$$

for some $c \in \mathbb{C}$. The transformation rule between P_{alg} and P_{LR} implies:

$$\frac{\mathtt{i}}{\sqrt{2\pi\mathtt{i}}}\partial_t F^1_Y = C^{(1)}_{Y,1} = C^{(1)}_{\mathrm{alg},1} - \frac{1}{2}\frac{\pi\mathtt{i}}{2}E_2(\tau)Y_{\mathrm{CY}}(x). \tag{121}$$

Since $\partial_t F^1_Y|_{y=0} = -1/12$, it follows that $c = -1/24$. Using Corollary 10.4.9, we get:

$$(\theta x) \cdot C^{(1)}_{Y,1} = -\frac{1}{2}\frac{\theta^2 x}{\theta x} - \frac{1}{12}\left(1 + \frac{27y}{1+27y}\right).$$

Integrating, we have

$$\begin{aligned} F^1_Y = \int C^{(1)}_{Y,1} dx &= \log\left((\theta x)^{-\frac{1}{2}} y^{-\frac{1}{12}} (1+27y)^{-\frac{1}{12}}\right) + \text{const} \\ &= -\log\left(e^{\frac{\pi \mathtt{i}}{24}} y^{\frac{1}{24}} (1+27y)^{-\frac{1}{24}} \eta(\tau)\right) \qquad \text{(by Corollary 10.4.10)} \end{aligned}$$

or equivalently,

$$e^{F^1_Y}\sqrt{dx} = \text{const.}\frac{\sqrt{\frac{dy}{y}}}{y^{\frac{1}{12}}(1+27y)^{\frac{1}{12}}}$$

which shows that $e^{F^1_Y}$ is modular of weight $-1/2$ (with automorphic factors). Equation (121) also implies that:

$$dF^1_Y = \left(-\frac{\pi \mathtt{i}}{12} E_2(\tau) + \frac{1}{24}(\theta x)^2\right) d\tau.$$

Remark 10.6.7 (solving the holomorphic anomaly equation)**.** Bershadsky–Cecotti–Ooguri–Vafa introduced a Feynman diagram technique to solve the holomorphic anomaly equation [8, 9]. For example, in our case, a solution at genus one (see Proposition 10.6.3) is

$$C^{(1)}_{\mathrm{cc},1} = -\frac{\tau_x}{2(\tau - \overline{\tau})} + f(x) \tag{122}$$

for some holomorphic function $f(x)$. On the other hand, the transformation rule from P_{cc} to $P_{\mathrm{alg}}(\frac{1}{12})$ gives:

$$C^{(1)}_{\mathrm{alg},1} = C^{(1)}_{\mathrm{cc},1} + \frac{\pi \mathtt{i}}{4}\widehat{E}_2(\tau) Y_{\mathrm{CY}}(x) = C^{(1)}_{\mathrm{cc},1} + \frac{\tau_x}{2(\tau - \overline{\tau})} + \frac{d}{dx}\log(\eta(\tau));$$

see Proposition 10.4.8. Therefore the holomorphic ambiguity $f(x) = C^{(1)}_{\mathrm{alg},1} - \partial_x \log \eta(\tau)$ essentially corresponds to the algebraic potential $C^{(1)}_{\mathrm{alg},1}$. More generally, for $g \geq 2$, the transformation rule gives:

$$C^{(g)}_{\mathrm{alg},0} = C^{(g)}_{\mathrm{cc},0} + \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta(P_{\mathrm{cc}}, P_{\mathrm{alg}}), \{C^{(h)}_{\mathrm{cc},\bullet}\}_{h<g})$$

where we separate the Feynman rule (70) into the leading term $C^{(g)}_{\mathrm{cc},0}$ and the lower-genus contribution. When viewed as an expression for $C^{(g)}_{\mathrm{cc},0}$, this formula solves the holomorphic anomaly equation recursively in genus, with holomorphic ambiguity $C^{(g)}_{\mathrm{alg},0}$.

Remark 10.6.8. Integrating (122), we find that $F^1_{\mathrm{cc}} = \log(|\tau - \overline{\tau}|^{-\frac{1}{2}} |y|^{-\frac{1}{12}} |1+27y|^{\frac{1}{12}} |\eta(\tau)|^{-2})$ satisfies $\partial_x F^1_{\mathrm{cc}} = C^{(1)}_{\mathrm{cc},1}$. Then F^1_{cc} is $\Gamma_1(3)$-invariant and $e^{F^1_{\mathrm{cc}}} = |e^{F^1_Y}|^2 |\tau - \overline{\tau}|^{-\frac{1}{2}}$. We may view $e^{F^1_{\mathrm{cc}}}$ as a 'norm' of the exponentiated genus-one potential, see [8], [25, §9.4].

10.7. **Algebraic Opposite and Finite Generation.** As we saw in Proposition 10.6.4, correlation functions with respect to the algebraic opposite line bundle $P_{\mathrm{alg}}(a)$ belong to the polynomial ring $\mathbb{C}[(1+27y)^{-1}]$. Using the transformation rule from the algebraic opposite to other opposites, we conclude that correlation functions with respect to $P_{\mathrm{LR}}, P_{\mathrm{orb}}, P_{\mathrm{con}}$ belong to the polynomial ring generated by $(1+27y)^{-1}$ and the respective propagators, and that they are related to each other by 'interchanging the propagators'. This recovers the results of Lho and Pandharipande [65], and their version of the Crepant Resolution Conjecture [66].

Let us write the propagators between $P_{\mathrm{alg}}(a)$ and P_{LR}, P_{con}, P_{orb} in the frame $\theta\otimes\theta$, setting

$$\Delta(P_{\mathrm{alg}}(a), P_\heartsuit) = \Delta_{\mathrm{alg},\heartsuit}\,\theta\otimes\theta$$

where $\heartsuit$ is one of 'LR', 'con', 'orb'.

Lemma 10.7.1 (cf. Corollary 10.4.11)**.** *Using Notation* 10.3.6*, we have*

$$\Delta_{\mathrm{alg},\heartsuit} = 3(1+27y)\frac{\theta^2 x_\heartsuit}{\theta x_\heartsuit} + 27y + 3a, \qquad \heartsuit \in \{\mathrm{LR}, \mathrm{con}, \mathrm{orb}\}.$$

Proof. We can deduce this from the previous computation (Corollary 10.4.11), but here we outline a simpler derivation. By definition – see Definition 10.4.3 – P_{alg} is spanned by:

$$v_{\mathrm{alg}} = (-27y-3a)\theta\zeta - 3(1+27y)\theta^2\zeta.$$

On the other hand $P_\heartsuit$ is cut out by $x_\heartsuit = 0$ (see Notation 10.3.6) and is therefore spanned by

$$v_\heartsuit = 3(1+27y)\frac{\theta^2 x_\heartsuit}{\theta x_\heartsuit}\theta\zeta - 3(1+27y)\theta^2\zeta.$$

Since $\Omega(v_{\mathrm{alg}}, \theta\zeta) = \Omega(v_\heartsuit, \theta\zeta) = 1$, it follows that $v_\heartsuit - v_{\mathrm{alg}} = \Delta_{\mathrm{alg},\heartsuit}(\theta\zeta)$. The conclusion follows. □

Lemma 10.7.2 (cf. Lemma 7.4.5)**.**

$$\theta((1+27y)^{-1}) = (1+27y)^{-2} - (1+27y)^{-1},$$
$$\theta(\Delta_{\mathrm{alg},\heartsuit}) = -\frac{1}{3(1+27y)}(\Delta_{\mathrm{alg},\heartsuit})^2 + \frac{2(a+9y)}{1+27y}\Delta_{\mathrm{alg},\heartsuit} + \frac{9(1-6a)y - 3a^2}{1+27y}.$$

Proof. The first equation is obvious. The second equation is an analogue of Lemma 7.4.5 (see also Proposition 10.6.1 and equation (120)) and follows immediately from the Picard–Fuchs equation (108) for $x_\heartsuit$. □

The lemma shows that the ring $\mathbb{C}[\Delta_{\mathrm{alg},\heartsuit}, (1+27y)^{-1}]$ is closed under the differential $\theta = y\frac{\partial}{\partial y}$, so that it is a differential ring. The following theorem shows that the Gromov–Witten potentials of $\mathcal{X}$ and Y belong respectively to the differential rings $\mathbb{C}[\Delta_{\mathrm{alg,orb}}, (1+27y)^{-1}]$ and $\mathbb{C}[\Delta_{\mathrm{alg,LR}}, (1+27y)^{-1}]$.

Theorem 10.7.3. *Let $C^{(g)}_{Y,n}, C^{(g)}_{\mathcal{X},n}, C^{(g)}_{\mathrm{con},n}$ be as in Notation* 10.5.2*. For a pair (g,n) of non-negative integers with $2g-2+n>0$, there exists a polynomial $f_{g,n}(\Delta, R) \in \mathbb{C}[\Delta, R]$ such that:*

$$C^{(g)}_{Y,n} = f_{g,n}(\Delta_{\mathrm{alg,LR}}, (1+27y)^{-1})(\theta x)^{-n},$$
$$C^{(g)}_{\mathcal{X},n} = f_{g,n}(\Delta_{\mathrm{alg,orb}}, (1+27y)^{-1})(\theta x_{\mathrm{orb}})^{-n},$$
$$C^{(g)}_{\mathrm{con},n} = f_{g,n}(\Delta_{\mathrm{alg,con}}, (1+27y)^{-1})(\theta x_{\mathrm{con}})^{-n}.$$

Moreover, we have $\deg_\Delta f_{g,n} \leq 3g-3+n$, $\deg_R f_{g,n} \leq 2g-2+n$ *and*

$$\frac{\partial f_{g,n}}{\partial \Delta} = \frac{1}{2} \sum_{\substack{n_1+n_2=n \\ g_1+g_2=g}} \binom{n}{n_1} f_{g_1,n_1+1} f_{g_2,n_2+1} + \frac{1}{2} f_{g-1,n+2}. \tag{123}$$

Proof. This follows from the Feynman rule relating $\widehat{C}^{(g)}_{\mathrm{alg},n}$ to each of $C^{(g)}_{Y,n}$, $C^{(g)}_{\mathcal{X},n}$, $C^{(g)}_{\mathrm{con},n}$ together with Proposition 10.6.4. Note that the Feynman rule for $C^{(g)}_{Y,n}$, written in the frame $(\frac{dy}{y})^{\otimes n}$, is of the form:

$$C^{(g)}_{Y,n}(\theta x)^n = \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma \left(\Delta_{\mathrm{alg,LR}}, \{\widehat{C}^{(h)}_{\mathrm{alg},\bullet}\}_{h\leq g} \right)$$

and the Feynman rules for $C^{(g)}_{\mathcal{X},n}$ and $C^{(g)}_{\mathrm{con},n}$ have the same shape. By Proposition 10.6.4, $\widehat{C}^{(h)}_{\mathrm{alg},m}$ is a polynomial in $(1+27y)^{-1}$ of degree $\leq 2h-2+m$. Thus the right-hand side can be written as a polynomial $f_{g,n}(\Delta_{\mathrm{alg,LR}}, (1+27y)^{-1})$ such that $f_{g,n}(\Delta, R)$ has degree $\leq 2g-2+n$ in R. The degree of $f_{g,n}$ as a polynomial in Δ is bounded by $3g-3+n$, which is the maximum possible number of edges appearing in Feynman graphs. The differential equation for $f_{g,n}$ follows from the Feynman rule: the first term corresponds to separating edges and the second term corresponds to non-separating edges. □

Remark 10.7.4. Lho–Pandharipande [65, Theorems 1,2], [66, Theorems 1,2] showed essentially the same result for $F^g_Y, F^g_{\mathcal{X}}$ using stable quotient invariants. The differential equation (123) together with $\partial_x C^{(g)}_{Y,n} = C^{(g)}_{Y,n+1}$ implies a "holomorphic version" of holomorphic anomaly equation proved by Lho–Pandharipande [65, Theorem 2], [66, Theorem 2]; such equations are sometimes referred to as "modular anomaly equations". The above theorem implies that the Gromov–Witten potentials $F^{(g)}_Y = C^{(g)}_{Y,0}$, $F^{(g)}_{\mathcal{X}} = C^{(g)}_{\mathcal{X},0}$ are related by a change of generators:

$$\mathcal{F}^{(g)}_Y = \mathcal{F}^{(g)}_{\mathcal{X}}\Big|_{\Delta_{\mathrm{alg,orb}}\to\Delta_{\mathrm{alg,LR}}}$$

for $g \geq 2$. This is a formulation of Crepant Resolution Conjecture due to Lho and Pandharipande [66].

10.8. **Calculation of Gromov–Witten Invariants and the Conifold Gap.** One can combine knowledge of the first $2g-1$ genus-g Gromov–Witten invariants of Y – for instance from the topological vertex [3, 67] – with the modularity results from §10.5 to determine all genus-g Gromov–Witten invariants of Y (and $\mathcal{X}$), as we now explain. This is essentially the calculation of Aganagic–Bouchard–Klemm [2, Section 6.4], placed in our rigorous mathematical setting.

From Example 8.1.2 and Example 10.6.6 we know the Yukawa coupling Y_{CY} and the genus-one data dF^1_Y exactly. Let $g \geq 2$, suppose by induction that we know F^h_Y exactly for $h < g$, and suppose that we know the first $2g-1$ genus-g Gromov–Witten invariants: $n_{g,d}$, $0 \leq d \leq 2g-2$. Consider the transformation rule

$$C^{(g)}_{\mathrm{alg},0} = C^{(g)}_{Y,0} + \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}_{Y,\bullet}\}_{h<g}) \tag{124}$$

between the correlation functions $C^{(g)}_{\mathrm{alg},0}$ with respect to the algebraic opposite line bundle $P_{\mathrm{alg}}(a)$ and the correlation functions $C^{(g)}_{Y,\bullet}$ with respect to P_{LR} (see Notation 10.5.2). Here the precise value of a is not important, but it is convenient to take $a = \frac{1}{12}$. In (124) we have divided the terms in the transformation rule (70) into the main term $C^{(g)}_{Y,0}$ and the

sum over all Feyman graphs with a non-zero number of edges; also $\Delta = \Delta(P_{\mathrm{LR}}, P_{\mathrm{alg}})$ from Corollary 10.4.11. Equation (124) determines the first $2g-1$ terms of the Taylor series expansion of $C^{(g)}_{\mathrm{alg},0}$ in y. On the other hand, we know that $C^{(g)}_{\mathrm{alg},0}$ has a pole of order at most $2g-2$ at the conifold point, and is regular elsewhere, so (see Proposition 10.6.4)

$$C^{(g)}_{\mathrm{alg},0} = \sum_{i=0}^{2g-2} \frac{a_i}{(27y+1)^i}$$

and the first $2g-1$ Taylor coefficients of $C^{(g)}_{\mathrm{alg},0}$ determine $C^{(g)}_{\mathrm{alg},0}$ exactly. Reading equation (124) as an expression for $C^{(g)}_{Y,0}$ now determines $C^{(g)}_{Y,0}$, and hence all genus-g Gromov–Witten invariants of Y, exactly.

Remark 10.8.1. The correlation function $C^{(g)}_{Y,0}$ here is a 'holomorphic ambiguity' of Bershadsky–Cecotti–Ooguri–Vafa [9]; Aganagic–Bouchard-Klemm [2] denote it by $h^{(0)}_g$, see Remark 10.6.7.

With the holomorphic ambiguities in hand, we can compute higher-genus Gromov–Witten invariants of $\mathcal{X}$ too. From Example 8.1.2 we have that the Yukawa coupling is

$$Y_{\mathrm{CY}} = \frac{9}{\mathfrak{y}^3+27}\, d\mathfrak{y}^{\otimes 3},$$

where $\mathfrak{y} = y^{-1/3}$, and by arguing as in Example 10.6.6 we have that

$$dF^1_{\mathcal{X}} = \frac{1}{8}\frac{\mathfrak{y}^2}{\mathfrak{y}^3+27}\, d\mathfrak{y} + \frac{1}{2}\Delta_{\mathrm{alg,orb}} Y_{\mathrm{CY}}$$

where $\Delta_{\mathrm{alg,orb}}$ is the propagator from Lemma 10.7.1. By inverting the mirror map $\mathfrak{t} = \mathfrak{t}(\mathfrak{y})$ from Theorem 3.3.2, we can write Y_{CY} and $dF^1_{\mathcal{X}}$ in terms of the orbifold flat co-ordinate $\mathfrak{t}$; thus we know both the Yukawa coupling and $dF^1_{\mathcal{X}}$ near the orbifold point exactly. Consider now the transformation rule

$$C^{(g)}_{\mathrm{alg},0} = C^{(g)}_{\mathcal{X},0} + \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}_{\mathcal{X},\bullet}\}_{h<g}) \tag{125}$$

between the correlation functions $C^{(g)}_{\mathrm{alg},0}$ and the correlation functions $C^{(g)}_{\mathcal{X},\bullet}$ with respect to P_{orb}. Here $\Delta = \Delta(P_{\mathrm{orb}}, P_{\mathrm{alg}})$ from Corollary 10.4.11. Assuming by induction that we know the genus-h orbifold Gromov–Witten invariants of $\mathcal{X}$ for all $h < g$, or equivalently that we know $C^{(h)}_{\mathcal{X},n}$ for all $h < g$ and all n, equation (125) determines $C^{(g)}_{\mathcal{X},0}$, and hence all genus-g orbifold Gromov–Witten invariants of $\mathcal{X}$, exactly.

We can perform the same analysis at the conifold point, obtaining 'conifold Gromov–Witten invariants'. The conifold flat co-ordinate x_{con} from Notation 10.3.6 is the holomorphic solution to the Picard–Fuchs equations that satisfies $x_{\mathrm{con}} \sim 27y+1$ near the conifold point (see Remark 10.3.8). Thus we can find x_{con} by solving the Picard–Fuchs equations in power series:

$$x_{\mathrm{con}} = (27y+1) + \tfrac{11}{18}(27y+1)^2 + \tfrac{109}{243}(27y+1)^3 + \tfrac{9389}{26244}(27y+1)^4 + \tfrac{88351}{295245}(27y+1)^5 + \cdots$$

and

$$27y+1 = x_{\mathrm{con}} - \tfrac{11}{18}x^2_{\mathrm{con}} + \tfrac{145}{486}x^3_{\mathrm{con}} - \tfrac{6733}{52488}x^4_{\mathrm{con}} + \tfrac{120127}{2361960}x^5_{\mathrm{con}} - \cdots$$

Example 8.1.2 determines the Yukawa coupling near the conifold point:

$$Y_{\mathrm{CY}} = -\frac{1}{3(1+27y)}\Big(\frac{dy}{y}\Big)^{\otimes 3} =$$
$$\Big(\tfrac{1}{3}\tfrac{1}{x_{\mathrm{con}}} - \tfrac{1}{54} + \tfrac{1}{2916}x_{\mathrm{con}} + \tfrac{7}{19683}\tfrac{x_{\mathrm{con}}^2}{2!} - \tfrac{529}{2361960}\tfrac{x_{\mathrm{con}}^3}{3!} + \tfrac{53}{531441}\tfrac{x_{\mathrm{con}}^4}{4!} - \tfrac{26093}{1205308188}\tfrac{x_{\mathrm{con}}^5}{5!} - \cdots\Big)dx_{\mathrm{con}}^{\otimes 3}$$

Arguing as in Example 10.6.6 we have that

$$dF^1_{\mathrm{con}} = -\frac{1}{24}\frac{1}{1+27y}\frac{dy}{y} + \frac{1}{2}\Delta_{\mathrm{alg,con}}Y_{\mathrm{CY}}$$

where $\Delta_{\mathrm{alg,con}}$ is as in Lemma 10.7.1, so that dF^1_{con} is

$$\Big(-\tfrac{1}{12}\tfrac{1}{x_{\mathrm{con}}} + \tfrac{5}{216} - \tfrac{1}{11664}x_{con} - \tfrac{5}{26244}\tfrac{x_{con}^2}{2!} + \tfrac{283}{3149280}\tfrac{x_{con}^3}{3!} - \tfrac{215}{6377292}\tfrac{x_{con}^4}{4!} + \tfrac{4517}{535692528}\tfrac{x_{con}^5}{5!} + \cdots\Big)dx_{\mathrm{con}}$$

Note that both the Yukawa coupling and dF^1_{con} have a simple pole at the conifold point. The transformation rule

$$C^{(g)}_{\mathrm{alg},0} = C^{(g)}_{\mathrm{con},0} + \sum_{\Gamma}\frac{1}{|\operatorname{Aut}(\Gamma)|}\operatorname{Cont}_\Gamma(\Delta, \{C^{(h)}_{\mathrm{con},\bullet}\}_{h<g}) \tag{126}$$

between the correlation functions $C^{(g)}_{\mathrm{alg},0}$ and the correlation functions $C^{(g)}_{\mathrm{con},\bullet}$ with respect to P_{con} inductively determines all the correlation functions $C^{(g)}_{\mathrm{con},\bullet}$ from the holomorphic ambiguities $C^{(g)}_{\mathrm{alg},0}$, exactly as above. Here $\Delta = \Delta(P_{\mathrm{con}}, P_{\mathrm{alg}})$ from Corollary 10.4.11.

The results of these calculations can be found in Appendix C. We determine Gromov–Witten and Gopakumar–Vafa invariants of Y up to genus 7 and degree 15, as well as orbifold Gromov–Witten invariants of $\mathcal{X}$ up to genus 7 with up to 27 insertions of the orbifold class $\mathbf{1}_{\frac{1}{3}}$, and conifold Gromov–Witten invariants up to genus 7 and degree 4. In particular we find that the genus-g correlation function with respect to the conifold opposite line bundle P_{con}, for $2 \le g \le 7$, has a pole of order of order $2g-2$ at the conifold point:

$$C^{(g)}_{\mathrm{con},0} = \frac{B_{2g}}{2g(2g-2)}3^{g-1}x_{\mathrm{con}}^{2-2g} + \cdots \tag{127}$$

and that no other negative powers of x_{con} occur in the Laurent expansion of $C^{(g)}_{\mathrm{con},0}$. Thus we verify the "conifold gap" conjecture of Huang–Klemm [52, 53] up to genus 7.

Source Code. This paper is accompanied by fully-commented source code[24], written in the computer algebra system Sage [79]. This should allow the reader to verify the calculations presented here, and to perform similar calculations. The source code, but not the text of this paper, is released under a Creative Commons CC0 license [29]: see the included file `LICENSE` for details. If you make use of the source code in an academic or commercial context, please acknowledge this by including a reference or citation to this paper. Part of the code, a Sage package for performing sums over Feynman graphs, makes use of data files produced by the program 'boundary' by Stefano Maggiolo and Nicola Pagani [69].

[24] `https://arxiv.org/src/1804.03292/anc`

Appendix A. Basic Facts About Connections With Logarithmic Singularities

Proposition A.0.2. *Consider $\mathbb{C}^r \times \mathbb{C}^s$ with standard co-ordinates $(x_1, \dots, x_r, y_1, \dots, y_s)$, a contractible open neighbourhood U of $(0,0)$ in $\mathbb{C}^r \times \mathbb{C}^s$, and a trivial holomorphic vector bundle $E = \mathbb{C}^{N+1} \times (U \times \mathbb{C}) \to U \times \mathbb{C}$. Let z denote the standard co-ordinate on the second factor $\mathbb{C}$ of $U \times \mathbb{C}$. Suppose that E has a "partial" meromorphic flat connection ∇ in the directions of x and y:*

$$\nabla = d + \frac{1}{z}\left(\sum_{i=1}^{r} A_i(x,y,z)\frac{dx_i}{x_i} + \sum_{j=1}^{s} B_j(x,y,z)dy_j \right)$$

where $A_1, \dots, A_r$, $B_1, \dots, B_s$ are matrix-valued holomorphic functions on $U \times \mathbb{C}$ such that, for $1 \le i \le r$, $A_i(0,0,z)$ is nilpotent. Then there exists a unique matrix-valued function $L(x,y,z)$ of the form:

$$L(x,y,z) = \widetilde{L}(x,y,z)e^{-\sum_{i=1}^{r} A_i(0,0,z)\log x_i/z}$$

with $\widetilde{L}$ regular along $U \times \mathbb{C}^\times$ and that $\widetilde{L}(0,0,z) = \mathrm{id}$, the identity matrix, such that:

$$\begin{aligned} &\nabla_{x_k\partial_{x_k}} L(x,y,z)v = 0 \qquad 1 \le k \le r \\ &\nabla_{\partial_{y_k}} L(x,y,z)v = 0 \qquad 1 \le k \le s \end{aligned} \tag{128}$$

for every $v \in \mathbb{C}^{N+1}$.

Proof. By the assumption, the residue endomorphism $N_i = A_i(0,0,z)/z$ along $x_i = 0$ is non-resonant, i.e. the eigenvalues of N_i do not differ by positive integers. In this case, for each $z \in \mathbb{C}^\times$, the connection $\nabla|_{U\times\{z\}}$ is gauge equivalent to the connection $d + \sum_{i=1}^{r} N_i \frac{dx_i}{x_i}$ [30, 5.4, 5.5]. The required gauge transformation is $\widetilde{L}$ in the proposition. □

Remark A.0.3. The flat connection in the above proposition is only a "partial" connection defined in the directions of x and y. In what follows, we consider a "full" flat connection extended in the direction of z: even in such a situation we still consider a matrix-valued function L which solves the equations (128) only in the directions of x and y. Informally, we call such an L a fundamental solution in the directions of x and y.

Let us recall Birkhoff factorization in the theory of loop groups (see [70]). A smooth loop $z \mapsto L(z)$ in GL_{N+1} which is sufficiently close to the identity (in the "big cell" of LGL_{N+1}) admits a unique factorization

$$L = L_+ L_-$$

where L_+ is a holomorphic map from $\{z \in \mathbb{C} : |z| < 1\}$ to GL_{N+1} with smooth boundary values and L_- is a holomorphic map from $\{z \in \mathbb{P}^1 : |z| > 1\}$ to GL_{N+1} with smooth boundary values which equals the identity at $z = \infty$. In the following proposition, we regard the fundamental solution L as an element of the loop group LGL_{N+1} by restricting z to lie in S^1 and consider its Birkhoff factorization. This method has been used in quantum cohomology in [22, 44].

Proposition A.0.4. *Suppose that the partial meromorphic flat connection ∇ on E in Proposition A.0.2 is extended in the z-direction to a meromorphic flat connection of the form:*

$$\nabla = d + \frac{1}{z}\left(\sum_{i=1}^{r} A_i(x,y,z)\frac{dx_i}{x_i} + \sum_{j=1}^{s} B_j(x,y,z)dy_j + C(x,y,z)\frac{dz}{z} \right) \tag{129}$$

where $C(x,y,z)$ is a matrix-valued holomorphic function on $U \times \mathbb{C}$ such that $C(0,0,z)$ depends linearly on z, i.e. $C(0,0,z) = C_0 + C_1 z$ for some constant matrices C_0 and C_1. Assume moreover that $A_i^\circ = A_i(0,0,z)$ is both nilpotent and independent of z. Let L be the fundamental

solution in the directions of x and y in Proposition A.0.2. After shrinking U if necessary, L admits a Birkhoff factorization $L = L_+L_-$ such that L_+ is holomorphic on $U \times \mathbb{C}$, and after gauge transformation by L_+, the connection ∇ takes the form:

$$L_+^{-1} \circ \nabla \circ L_+ = d + \frac{1}{z}\left(\sum_{i=1}^{r} \widetilde{A}_i(x,y)\frac{dx_i}{x_i} + \sum_{j=1}^{s} \widetilde{B}_j(x,y)dy_j + \widetilde{C}(x,y,z)\frac{dz}{z} \right) \tag{130}$$

where $\widetilde{A}_1, \dots, \widetilde{A}_r$, $\widetilde{B}_1, \dots, \widetilde{B}_s, \widetilde{C}$ are matrix-valued holomorphic functions on U such that, for $1 \le i \le r$, $\widetilde{A}_i|_{x_i=0} = A_i^\circ$ is independent of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s$ and nilpotent and that $\widetilde{C}(x,y,z)$ depends linearly on z: $\widetilde{C}(x,y,z) = \widetilde{C}_0(x,y) + C_1 z$ for some matrix-valued regular function $\widetilde{C}_0$ on U and some constant matrix C_1.

Proof. Recall that the fundamental solution is of the form $L = \widetilde{L}e^{-\sum_{i=1}^r A_i^\circ \log x_i/z}$ with $\widetilde{L}$ holomorphic on $U \times \mathbb{C}^\times$. Because $\widetilde{L}(0,0,z) = \mathrm{id}$, $\widetilde{L}$ admits the Birkhofff factorization $\widetilde{L} = \widetilde{L}_+\widetilde{L}_-$, shrinking U if necessary. Because $A_i^\circ = A_i(0,0,z)$ is independent of z, this gives the Birkhoff factorization of L: $L_+ := \widetilde{L}_+$ and $L_- := \widetilde{L}_- e^{-\sum_{i=1}^r A_i^\circ \log x_i/z}$. Note that L_+ is holomorphic on $U \times \{z \in \mathbb{C} : |z| < 1\}$ with smooth boundary values. The fundamental solution L transforms the connection ∇ to $L^{-1} \circ \nabla \circ L = d + Ddz/z$ with D given by

$$\begin{aligned} D &= L^{-1}(z\partial_z L + z^{-1}CL) \\ &= e^{\sum_{i=1}^r A_i^\circ \log x_i/z}\widetilde{L}^{-1}\left(z\partial_z\widetilde{L} + \widetilde{L}z^{-1}\sum_{i=1}^r A_i^\circ \log x_i + z^{-1}C\widetilde{L} \right) e^{-\sum_{i=1}^r A_i^\circ \log x_i/z}. \end{aligned}$$

The flatness of $d + D\frac{dz}{z}$ implies that D is independent of x and y. The above expression for D is polynomial in $\log x_1, \dots, \log x_r$ as A_i° is nilpotent. Therefore taking the constant term in $\log x$, x and y, we obtain

$$D = z^{-1}C(0,0,z).$$

Substituting L_+L_- for L in the equation $L^{-1} \circ \nabla \circ L = d + Ddz/z$, we obtain

$$\begin{aligned} zL_+^{-1}(x_i\partial_{x_i}L_+) + L_+^{-1}A_iL_+ &= zL_-(x_i\partial_{x_i}L_-^{-1}) \\ zL_+^{-1}(\partial_{y_j}L_+) + L_+^{-1}B_jL_+ &= zL_-(\partial_{y_j}L_-^{-1}) \\ L_+^{-1}(z\partial_z L_+) + L_+^{-1}z^{-1}CL_+ &= L_-(z\partial_z L_-^{-1}) + L_-z^{-1}C(0,0,z)L_-^{-1} \end{aligned}$$

The first two equations show that the left-hand sides are analytically continued to $U \times \mathbb{P}^1$ and thus are independent of z. Therefore, the gauge transformation L_+ transforms the connection matrices A_i, B_j into z-independent connection matrices. The third equation implies that

$$L_+^{-1}(z\partial_z L_+) + L_+^{-1}z^{-1}CL_+ = z^{-1}\left[L_+^{-1}CL_+\right]_{z=0} + C_1$$

where $C(0,0,z) = C_0 + C_1 z$. Therefore L_+ transforms the connection matrix $z^{-1}C$ into a connection matrix of the form $z^{-1}\widetilde{C} = z^{-1}\widetilde{C}_0(x,y) + C_1$. On the other hand, this equation can be viewed as a differential equation for L_+ in the z-direction. Since the differential equation has no singularities on $\mathbb{C}^\times$, L_+ is analytically continued to a holomorphic function on $U \times \mathbb{C}$.

Now we know that L_+ is holomorphic on $U \times \mathbb{C}$, and after gauge transformation by L_+ the connection ∇ remains flat and takes the form:

$$\widetilde{\nabla} = L_+^{-1} \circ \nabla \circ L_+ = d + \frac{1}{z}\left(\sum_{i=1}^{r} \widetilde{A}_i(x,y)\frac{dx_i}{x_i} + \sum_{j=1}^{s} \widetilde{B}_j(x,y)dy_j + \widetilde{C}(x,y,z)\frac{dz}{z} \right)$$

where $\widetilde{A}_1, \dots, \widetilde{A}_r$, $\widetilde{B}_1, \dots, \widetilde{B}_s$ are independent of z, $\widetilde{A}_i(0,0) = A_i^\circ$ is nilpotent for $1 \le i \le r$, and $\widetilde{C}$ is linear in z: $\widetilde{C} = \widetilde{C}_0(x,y) + zC_1$. It remains to show that $\widetilde{A}_i|_{x_i=0}$ is independent of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s$; then it coincides with the nilpotent matrix A_i°. Flatness of $\widetilde{\nabla}$ yields:

$$\partial_{y_j} \widetilde{A}_i = x_i \partial_{x_i} \widetilde{B}_j + z^{-1}[\widetilde{A}_i, \widetilde{B}_j]$$
$$x_j \partial_{x_j} \widetilde{A}_i = x_i \partial_{x_i} \widetilde{A}_j + z^{-1}[\widetilde{A}_i, \widetilde{A}_j]$$

This implies that $\widetilde{A}_i|_{x_i=0}$ is independent of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s$. □

Proposition A.0.5. *Let (E, ∇) be the meromorphic flat connection in Proposition A.0.4. Suppose that E is equipped with a holomorphic non-degenerate pairing*

$$(\cdot,\cdot)_E \colon (-1)^*\mathcal{O}(E) \otimes_{U\times\mathbb{C}} \mathcal{O}(E) \to \mathcal{O}_{U\times\mathbb{C}}$$

such that $\big(\mathcal{O}(E), \nabla, (\cdot,\cdot)_E\big)$ is a log*-TEP structure with base (U, D), where $(-)\colon U\times\mathbb{C} \to U\times\mathbb{C}$ is the map sending (x,y,z) to $(x,y,-z)$ (see Definition 2.7.2) and $D = \{x_1 \cdots x_r = 0\}$ is the normal crossing divisor. Suppose also that the Gram matrix of the pairing $(\cdot,\cdot)_E$ is independent of z along $\{(0,0)\}\times\mathbb{C}$. After the gauge transformation by L_+ in Proposition A.0.4, the Gram matrix of the pairing $(\cdot,\cdot)_E$ with respect to the new trivialization is constant on $U \times \mathbb{C}$.*

Proof. In the new trivialization after the gauge transformation by L_+, the connection takes the form (130) and the pairing is flat with respect to it. Let G be the Gram matrix of $(\cdot,\cdot)_E$ in the new trivialization. We expand $G = \sum_{n\ge 0} G^{(n)}(x,y)z^n$. The flatness of the pairing with respect to the connection (130) implies that

$$x_i \frac{\partial}{\partial x_i} G^{(n)} = -A_i^{\mathrm{T}} G^{(n+1)} + G^{(n+1)} A_i$$
$$\frac{\partial}{\partial y_i} G^{(n)} = -B_i^{\mathrm{T}} G^{(n+1)} + G^{(n+1)} B_i$$

By assumption, we have $G^{(n)}(0,0) = 0$ for $n > 0$. The second equation then implies that $G^{(n)}(0,y)$ is independent of y and is zero for $n > 0$. Expand:

$$A_i(x,y) = \sum_I A_i^I(y)x^I, \quad G^{(n)}(x,y) = \sum_I G^{n,I}(y)x^I$$

where $I \in \mathbb{N}^r$ is a multi-index. We have from the first equation that

$$k_i G^{n,K} = \sum_{K=I+J} (-(A_i^J)^{\mathrm{T}} G^{n+1,I} + G^{n+1,I} A_i^J)$$

Suppose by induction that $G^{n,K} = 0$ for all K with $0 \le |K| \le m$ and all $n \ge 0$ except for the case $(n,K) = (0,0)$. For a multi-index K with $|K| = m+1$, we have

$$k_i G^{n,K} = -(A_i^0)^{\mathrm{T}} G^{n+1,K} + G^{n+1,K} A_i^0.$$

We can choose $1 \le i \le r$ such that $k_i \ne 0$, because $|K| = m+1 > 0$. Note that $A_i^0 = A_i(0,y) = A_i(0,0)$ is nilpotent. Using the above equation recursively, we find that $G^{n,K} = 0$ using the nilpotence of A_i^0. This completes the induction step and we have that G is constant. This completes the proof. □

Appendix B. Notation for Graphs

We fix terminology for graphs as follows. A graph Γ is given by four finite sets $V(\Gamma)$, $E(\Gamma)$, $L(\Gamma)$, $F(\Gamma)$ called (the set of) *vertices*, *edges*, *legs* and *flags* respectively, together with incidence maps

$$\pi_V \colon F(\Gamma) \to V(\Gamma), \qquad \pi_E \colon F(\Gamma) \to E(\Gamma) \sqcup L(\Gamma)$$

such that $|\pi_E^{-1}(e)| = 2$ for each $e \in E(\Gamma)$ and $|\pi_E^{-1}(l)| = 1$ for each $l \in L(\Gamma)$. We assign to an edge e a closed interval $I_e \cong [0,1]$, to a leg l a half-open interval $H_l \cong [0,1)$ and to a vertex v a point p_v, and fix identifications $\pi_E^{-1}(e) \cong \partial I_e$, $\pi_L^{-1}(l) \cong \partial H_l$. By identifying I_e, H_l, p_v via the map $\pi_V \colon F(\Gamma) \cong \bigsqcup \partial I_e \sqcup \bigsqcup \partial H_l \to V(\Gamma) \cong \{p_v\}$, we get a topological realization $|\Gamma|$ of the graph Γ. We say that Γ is connected if $|\Gamma|$ is connected, and write $\chi(\Gamma) = \chi(|\Gamma|)$ for the topological Euler characteristic of $|\Gamma|$.

Appendix C. Tables of Gromov–Witten and Gopakumar–Vafa Invariants

This section records the results of the calculations described in §10.8. Entries in **bold face** are input to the calculation: everything else is derived from these. Our results are in agreement with calculations and conjectures in the literature, except for a handful of cases where we correct typographical errors. These are indicated in `typewriter font`.

	Degree														
Genus	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	3	$-\frac{45}{8}$	$\frac{244}{9}$	$-\frac{12333}{64}$	$\frac{211878}{125}$	$-\frac{102365}{8}$	$\frac{64639725}{343}$	$-\frac{1140830253}{512}$	$\frac{6742982701}{243}$	$-\frac{3690119381}{200}$	$\frac{642598273215}{1331}$	$-\frac{95814310549}{144}$	$\frac{20613867993236}{2197}$	$-\frac{3702105516692659}{2744}$	$\frac{739828382713942}{125}$
1	$\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{23}{3}$	$\frac{3437}{16}$	$-\frac{43107}{10}$	79522	$-\frac{39826681}{28}$	$\frac{803703117}{32}$	$-\frac{15878598303}{66}$	$\frac{1546102432}{20}$	$-\frac{297994973}{22}$	$\frac{7124283102275}{3}$	$-\frac{5418144496746}{13}$	$\frac{4101371483470148}{56}$	$-\frac{644362702900656}{5}$
2	$\frac{1}{80}$	0	$\frac{3}{20}$	$-\frac{514}{5}$	$\frac{43497}{8}$	$-\frac{1552743}{5}$	$\frac{92569957}{16}$	$-\frac{776658618}{5}$	$\frac{311565686229}{80}$	$-\frac{18610371037}{2}$	$\frac{1716132926015}{8}$	$-\frac{962191179023583}{20}$	$\frac{52781214823353}{5}$	$-\frac{9102066555959750921}{40}$	$\frac{9667253840148948851}{2}$
3	$\frac{1}{2016}$	$\frac{1}{336}$	$\frac{1}{56}$	$\frac{1480}{63}$	$-\frac{138571}{336}$	$\frac{3438610}{112}$	$-\frac{45630561485}{288}$	$\frac{4781660931}{42}$	$-\frac{771922095237}{32}$	$\frac{400254094073885}{504}$	$-\frac{27226141576196371}{112}$	$\frac{983475982888097}{14}$	$-\frac{162843995042411187}{8}$	$\frac{41214863871270091}{84}$	$-\frac{13266967197002009748}{1}$
4	$\frac{1}{57600}$	$\frac{1}{1920}$	$\frac{7}{1600}$	$-\frac{900}{61}$	$\frac{2491}{1920}$	$-\frac{2172225227}{640}$	$\frac{364416184799}{11520}$	$-\frac{31680669367}{150}$	$\frac{7262000603582}{6400}$	$-\frac{150516582117817}{2880}$	$\frac{13729997706813910}{640}$	$-\frac{3220666414546452353}{400}$	$\frac{843823756379706895}{300}$	$-\frac{14824239312581305514377}{1600}$	$\frac{46493722208852997775773}{160}$
5	$\frac{1}{1774080}$	$\frac{1}{14080}$	$\frac{61}{49280}$	$\frac{4471}{22176}$	$-\frac{653083197}{98560}$	$\frac{5383395285}{19712}$	$-\frac{1701298787451}{354816}$	$\frac{64688948714407}{12320}$	$-\frac{11945278310269797}{28160}$	$\frac{3595959101526103397}{12672}$	$-\frac{13785482612596271967}{8960}$	$\frac{46573191135827599411}{6160}$	$-\frac{2487850368779387078761}{73920}$	$\frac{68093696367936600227475877}{49280}$	$-\frac{6533200987152510743952890}{12320}$
6	$\frac{691}{39626496000}$	$\frac{11747}{1320883200}$	$\frac{377977}{1100736000}$	$-\frac{1238328800}{1238328800}$	$\frac{202790371913}{1320883200}$	$-\frac{2416371485701}{140764800}$	$\frac{6413977747469033}{1132185600}$	$-\frac{431003499904037995}{412776000}$	$\frac{9919004156917847477}{768000}$	$-\frac{23950107031305313197100}{1981324800}$	$\frac{13502525377246412604194}{140764800}$	$-\frac{16421278887619903601053357}{27518,4000}$	$\frac{21640191374402736001856549777}{63504000}$	$-\frac{3029955595813431541386906295}{1728000}$	$\frac{43364714546345870048459770393}{5241600}$
7	$\frac{1}{19160064000}$	$\frac{31}{29030400}$	$\frac{703}{7603200}$	$\frac{11705}{4790016}$	$\frac{8293308997}{319334400}$	$\frac{5408101039343}{7096320}$	$-\frac{203904956647321131}{383201280}$	$\frac{67514633329270311}{39916800}$	$-\frac{771053050449734496 11}{29654400}$	$\frac{42491482875357032433349}{95800320}$	$-\frac{16559929312899212458246279}{35481600}$	$\frac{13400468324230111992071993}{3326400}$	$-\frac{23713389495101796065291526451}{79833600}$	$\frac{10251435487725124852427652944187}{53222400}$	$-\frac{24893430253088273695533668867 57}{2217600}$

Table 2. Some Gromov–Witten invariants of $Y = K_{\mathbb{P}^2}$

	Degree														
Genus	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	3	−6	27	−192	1695	−17064	188454	−2228160	27748899	−360012150	4827935937	−66537713520	938273463465	−13491638200194	197287568723655
1	0	0	−10	231	−4452	80948	−1438086	25301295	−443384578	7760515332	−135854179422	2380305803719	−41756224045650	733512068799924	−12903696488738656
2	**0**	**0**	0	−102	5430	−194022	5784837	−155322234	3894455457	−93050366010	2145146041119	−48109281322212	1055620386953940	−22755110195405850	483361869975894765
3	**0**	**0**	**0**	**15**	−3672	290853	−15363990	649358826	−23769907110	786400843911	−24130293606924	698473748830878	−19298221675559646	513289541565539286	−13226687073790872894
4	**0**	**0**	**0**	**0**	**1386**	**−290400**	29056614	−2003386626	109496290149	−5094944994204	210503102300868	−7935125096754762	278055282896359878	−917953248073048 4952	288379973286696180135
5	**0**	**0**	**0**	**0**	**−270**	**196857**	**−40492272**	**4741754985**	−396521732268	26383404443193	−1485630816648252	736133151485863 17	−3295843339183602162	135875843241729533613	−52306625282958887 02200
6	**0**	**0**	**0**	**0**	**21**	**−90390**	**42297741**	**−8802201084**	**1156156082181**	**−111935744536416**	8698748079113310	−572001241783007370	329701597168366634586	−17078865527050775 81628	809798545044560652 93006
7	**0**	**0**	**0**	**0**	**0**	**27538**	**−33388020**	**12991744968**	**−2756768768616**	**395499033672279**	**−42968546119317066**	**3786284014554551293**	−28312309926620079 9858	185426954126006603 15361	−10885209636994534 40916068

Table 3. Some Gopakumar–Vafa invariants of $Y = K_{\mathbb{P}^2}$

The input values for $2 \le g \le 4$ are taken from work of Klemm–Zaslow [61]. The input values for $5 \le g \le 7$ are taken from work of Haghighat–Klemm–Rauch [45]. The $g = 3$, $d = 7$ Gopakumar–Vafa invariant corrects a typographical error in [61, Figure 2].

	i												
Genus	0	1	2	3	4	5	6	7	8	9	10	11	12
2	$-\frac{1}{2160}$	$\frac{1}{4320}$	$-\frac{1}{7680}$	0	0	0	0	0	0	0	0	0	0
3	$\frac{1}{544320}$	$-\frac{41}{1088640}$	$\frac{673}{4354560}$	$-\frac{809}{3870720}$	$\frac{373}{4128768}$	0	0	0	0	0	0	0	0
4	$-\frac{7}{41990400}$	$\frac{863}{38888300}$	$-\frac{37403}{209018880}$	$\frac{727859}{1175731200}$	$-\frac{5397557}{5573836800}$	$\frac{10531379}{14863564800}$	$-\frac{10450031}{52842304000}$	0	0	0	0	0	0
5	$\frac{3161}{7759259200}$	$-\frac{223829}{19399564800}$	$\frac{517321728000}{11888603}$	$-\frac{249278237}{1655429529600}$	$\frac{46781311369}{9932577177600}$	$-\frac{140380033}{17517773280}$	$\frac{20544043433}{269072960800}$	$-\frac{2921058683}{761014517760}$	$\frac{117292943}{147639500800}$	0	0	0	0
6	$-\frac{6261257}{3177648714240000}$	$\frac{13190237077}{1271059485696000}$	$-\frac{7195198914889}{2033695177713600}$	$\frac{1950434482756}{5084237942784000}$	$-\frac{1299672653035618}{650782456676352000}$	$\frac{182570868110632}{30989640794112000}$	$-\frac{24530495575515863}{2313893179283696000}$	$\frac{2937733938899971387}{2468152724579942400}$	$-\frac{81366182834551611}{997233424072704000}$	$\frac{22892959680834949}{7313045109866496000}$	$-\frac{2683097043975380}{5200378633682841600}$	0	0
7	$\frac{263323}{17958454272000}$	$-\frac{3447502789}{2773220696064000}$	$\frac{242936799515323}{3660651318804800}$	$-\frac{53485220467509}{48808682507264000}$	$\frac{2886125863756135}{3346812057640960}$	$-\frac{506419154595610651}{1301564913352704000}$	$\frac{78865187127719825617}{7140013238963404800}$	$-\frac{6890676332776271251}{3332006178182922400}$	$\frac{1282973510134786701}{4936305449159848000}$	$-\frac{2386115547816695505}{1096956766479974000}$	$\frac{490668126443679677363}{4212313983283101696000}$	$-\frac{67696526638789604113}{1872139548125822976000}$	$\frac{11779039621933585193}{2377320061112151616000}$

TABLE 4. The coefficient of $(27y+1)^{-i}$ in the expansion of the genus-g holomorphic ambiguity

This table records the Laurent expansion at the conifold point of the genus-g holomorphic ambiguity $\widehat{C}^{(g)}_{\mathrm{alg},0}$. Here $a = \frac{1}{12}$.

	Number of insertions of $\mathbf{1}_{\frac{1}{3}}$								
Genus	3	6	9	12	15	18	21	24	27
0	$\frac{1}{3}$	$-\frac{1}{27}$	$\frac{1}{9}$	$-\frac{1093}{729}$	$\frac{119401}{2187}$	$-\frac{27428707}{6561}$	$\frac{102777653467}{177147}$	$-\frac{210755831694887}{1594323}$	$\frac{2448769004921523 5}{531441}$
1	0	$\frac{1}{243}$	$-\frac{14}{243}$	$\frac{13007}{6561}$	$-\frac{8354164}{59049}$	$\frac{9730293415}{531441}$	$-\frac{6226371315626}{1594323}$	$\frac{1836708604854737}{14348907}$	$-\frac{877447958012634736 0}{14348907}$
2	$\frac{1}{19440}$	$\frac{13}{11664}$	$\frac{20693}{524880}$	$-\frac{12803923}{4723920}$	$\frac{314291111}{944784}$	$-\frac{8557024202467}{127545840}$	$\frac{7964469005139389}{3836375 20}$	$-\frac{216565282609522258 9}{2395825 12}$	$\frac{123817684497686855335 97}{3096242608}$
3	$-\frac{31}{2449440}$	$\frac{11569}{22044960}$	$-\frac{2429003}{66134880}$	$\frac{871749323}{198404640}$	$-\frac{152004598488 7}{1785641760}$	$\frac{450933448038569}{1785641760}$	$-\frac{39050288662607269}{357128352}$	$\frac{960710990792716223 7163}{144636982560}$	$-\frac{142627453213813541344 70275}{260346568608}$
4	$\frac{313}{62985600}$	$-\frac{1889}{5038848}$	$\frac{115647179}{2550916800}$	$-\frac{29321809247}{34012224000}$	$\frac{22766570703031}{918330048 0}$	$-\frac{855627159576453613}{8264970432 00}$	$\frac{562917323177869058989}{9298091736 00}$	$-\frac{214511332407818424698522 3}{4463084032 80}$	$\frac{101317832251195109678555855547}{2008387814976 0}$
5	$-\frac{174596083200}{519961}$	$\frac{196898123}{523788249600}$	$-\frac{33915798378 1}{4714094246400}$	$\frac{7865894778214 7}{3856986201600}$	$-\frac{1057430723091383537}{1272805446528 00}$	$\frac{5402182862315780935057}{1145524901875 200}$	$-\frac{373827360739829909299220839}{1030972411678 7800}$	$\frac{151886918476841080302517121 9}{4123889646757 072}$	$-\frac{179481198406406304666721460292 49}{3711500682075648}$
6	$\frac{146097306 07}{571976768563200 0}$	$-\frac{258703053013}{514779001706880}$	$\frac{24536786546443 13}{1715030305689600 0}$	$-\frac{400157741939699601803}{69495177380428800 0}$	$\frac{534247019795165421373 9}{166788425713029120}$	$-\frac{375638250799690605317846887}{1563641491059648000}$	$\frac{801978001440525496948611918311 9}{3577465626988839680000}$	$-\frac{307050231671971275365086731000961 7}{1013239686206651904000}$	$\frac{298104636878022635887325818821845255 23}{607943811723991142400}$
7	$-\frac{1122101011}{377127539712000}$	$\frac{219679341420 1}{2545610893056000}$	$-\frac{21275260973695 39}{6109466143334400}$	$\frac{2637337512443986991 3}{13746298225024000}$	$-\frac{3500876264323814391009 11}{2474333788050432000}$	$\frac{1155026829373310723028673}{8435228822899200}$	$-\frac{1140326860613031125619939416251 37}{6680701227736166 4000}$	$\frac{161996369442994217877658453070121}{6012631104962549 7600}$	$-\frac{425117914436352886295747496799184581 1}{8016841473283996 80}$

TABLE 5. Some Gromov–Witten invariants of $\mathcal{X} = \left[\mathbb{C}^3/\mu_3\right]$

The five entries in `typewriter font` here correct typographical errors in the corresponding table on page 808 of [2].

	Degree																
Genus	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
2	0	0	0	0	0	0	0	0	0	0	$-\frac{1}{80}$	0	$-\frac{13}{25920}$	$\frac{1}{19440}$	$-\frac{3187}{377913600}$	$\frac{239}{255091680}$	$-\frac{19151}{257132413440}$
3	0	0	0	0	0	0	0	0	$\frac{1}{112}$	0	0	0	$\frac{121}{58786560}$	$-\frac{1}{1469664}$	$\frac{23855}{179992689408}$	$-\frac{557}{24794911296}$	$\frac{15575867}{4498788705546240}$
4	0	0	0	0	0	0	$-\frac{3}{160}$	0	0	0	0	0	$-\frac{1093}{31744742400}$	$\frac{7}{377913600}$	$-\frac{6830569}{1190155742208000}$	$\frac{1561279}{120503268898560 0}$	$-\frac{1444309519}{57263153380595712 00}$
5	0	0	0	0	$\frac{27}{352}$	0	0	0	0	0	0	0	$\frac{9841}{8380611993600}$	$-\frac{809}{942818849280}$	$\frac{118418785}{3266120600226570 24}$	$-\frac{113975899}{1002105184160424960}$	$\frac{11188464609233}{3939475899971462602752 00}$
6	0	0	$-\frac{18657}{36400}$	0	0	0	0	0	0	0	0	0	$-\frac{61203943}{9266023650723840 00}$	$\frac{1276277}{2105914466073600 0}$	$-\frac{279842720162009}{9052836032762704465920000}$	$\frac{9844865 11}{8199042415858022 4000}$	$-\frac{338480893523407}{8702660397209685567897 6000}$
7	$\frac{81}{16}$	0	0	0	0	0	0	0	0	0	0	0	$\frac{550838251}{1000730554278174 72000}$	$-\frac{7943}{1309171316428800}$	$\frac{27776712091}{7792369912031464488960}$	$-\frac{1177971963811}{7889774535931855795072 00}$	$\frac{11286380576743987}{2107000152432129587163955 2000}$

TABLE 6. Some conifold Gromov–Witten invariants

This table records the expansion coefficients of the genus-g conifold correlation function $C^{(g)}_{\mathrm{con},0}$ as a Laurent series in the conifold flat co-ordinate:

$$C^{(g)}_{\mathrm{con},0} = \sum_{d\in\mathbb{Z}} n^{\mathrm{con}}_{g,d}\, x^d_{\mathrm{con}}$$

See Notation 10.3.6 and Notation 10.5.2 for precise definitions. The genus-g conifold correlation function has a pole of order $2g-2$ at the conifold point, so $n^{\mathrm{con}}_{g,d}$ vanishes for $d < 2-2g$ and $n^{\mathrm{con}}_{g,2-2g}$ is non-zero. The leading term

$$C^{(g)}_{\mathrm{con},0} = \frac{B_{2g}}{2g(2g-2)} 3^{g-1} x^{2-2g}_{\mathrm{con}} + \cdots$$

agrees with predictions in the literature, up to rescaling x_{con} by $\sqrt{3}$ to match with [9, 36] or $\sqrt{-3}$ to match with [53]. Note that no other negative powers of x_{con} occur, as predicted by [52, 53]: this is the "conifold gap".

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