DIC

AN IMPROVED STATISTICAL METHOD FOR MEASURING RELIABILITY WITH SPECIAL REFERENCE TO ROCKET ENGINES

by

Robert B. Abernethy

Submitted as a thesis for the Ph.D. Degree in the Faculty of Science at the Imperial College of Science and Technology in the University of London.

July 1965

ABSTRACT

The current procedure employed in industry to estimate rocket engine reliability is based entirely on discrete, success-failure, variables. The precision of this method is inadequate at high levels of reliability; large changes in reliability cannot be detected. A new method is developed based on treating the performance parameters as continuous, measured variables and the mechanical characteristics as discrete variables. The new method is more precise, more accurate, and has wider application to the complex problems of estimating vehicle and mission reliability. Procedures are provided for frequentist, likelihood and Bayesian reliability estimates. Estimates of the proportion of a normal distribution exceeding or satisfying a limit, or limits, are treated in detail and tables of these estimates are tabulated. Monte Carlo simulation is used to verify analytical results. Computer programs in Extended Mercury-Atlas Autocode and Fortran IV languages are included with typical computer output for the estimates developed.

ACKNOWLEDGEMENT

The author is deeply indebted to Professor G. A. Barnard and Professor E. H. Lloyd, his advisors, for their guidance and counsel; to Dr. D. J. Farlie for his many constructive comments; to Mr. T. E. Roughley for his help with computer services; and to his wife for her encouragement and patience.

INDEX

Part I Introduction

Chapter		
1	OBJECTIVE	11
	a. Requirements for Reliability Estimates	
	b. Rocket Engine Project Life Span	
	c. Development Program Data	
	d. Objective	
	e. Organization	
2	THE CURRENT METHOD	20
	a. Reliability Growth	
	b. High Failure Rate Components	
	c. Dangerous Test Objectives	
	d. Variable Firing Duration	
	e. Ground Test Versus Mission Populations	
	f. Trimming Propellant Mixture Ratio and Thrust Controls	
	g. Documentation	
	h. Criticism	
3	THE PROPOSED METHOD	34
	a. ~~Math Model~~ Mathematical Model	
	b. Overall Reliability Estimates	
	c. Independence and Normality	
	d. Data Description	

INDEX

Part II

Estimates of the Proportion of a Normal Distribution Defined by a Fixed Limit or Limits

Chapter

4 INTRODUCTION 43

a. Required Estimates

b. Current Estimates

c. Point Estimates

d. Interval Estimates

e. Likelihood Estimates

5 THE DISTRIBUTION OF $k = \frac{\bar{X}-L}{s}$ 47

a. Sampling Distribution

b. Tables

c. Simulation

d. Expectation of k

e. Variance of k

f. Comparison

g. Higher Moments

h. Median

TABLE III

6 THE DISTRIBUTION OF $\tilde{h} = \Phi(k)$ 87

a. The Sampling Distribution

b. Transformations

c. Expectation of $\tilde{h}$

d. Bias

Chapter

7 THE DISTRIBUTION OF h, AN UNBIASED ESTIMATE OF η 96

a. An Iterative Method

b. Exact Method

c. Interval Estimates of h

d. Sides and Limits

e. Examples

8 LIKELIHOOD ESTIMATES 109

a. Likelihood Function

b. Maximum Likelihood Estimate

c. Bayesian Estimates

d. Double Limits

9 THE DISTRIBUTION OF h', KNOWN VARIANCE 124

a. The Estimate h'

b. The Distribution of h'

c. Interval Estimates of h'

10 A RELATED PROBLEM-VARIABLE LIMITS 132

a. The Variable Limit

b. The Distribution

c. References

d. Discussion

INDEX

Part III

THE SIMPLE MODEL

	ABSTRACT	136
Chapter		
11	FUNCTIONAL RELIABILITY	137
	a. Definition	
	b. Functional Performance Limits	
	c. Weighting Factors	
	d. Estimates	
12	PERFORMANCE RELIABILITIES	144
	a. Specific Impulse	
	b. Mixture Ratio and Thrust	
	c. Known Variance	
13	OVERALL RELIABILITY	150
	a. Point Estimates	
	b. Interval Estimate	
	c. Overall Reliability Distribution	
	d. Simulation and Discretized Results	
	e. Confidence Interval	
	f. Approximations	
	g. Efficiency	
	h. Summary	

Chapter

14 LIKELIHOOD AND BAYESIAN ESTIMATES 174

a. Likelihood Function

b. Maximum Likelihood Estimate

c. Bayesian Confidence Interval

d. Simple Method

e. Complex Method

f. Choice of Estimates

g. Conclusion

Part IV

COMPLEX MODELS

ABSTRACT 190

Chapter

15 MULTIFIRING SAMPLE SPACE AND TRIMMING 191

a. Complex Versus Simple Models

b. The Complex Model

c. Engine-to-Engine Variation

d. Run-to-Run Variation

e. Noncentral t

f. σ_r Small Compared to σ_E

g. σ_r Not Small Compared to σ_E

h. Trimming

i. Stability of Variance Ratio

j. Approximate Degrees of Freedom

k. Current Method

16 PROBLEMS FOR FUTURE RESEARCH 218

a. Multifiring-Multi-engine Mission Space

b. Variable Number of Firings Per Engine Assembly

c. True Versus Indicated Reliabilities

d. Likelihood-Bayesian Estimates

APPENDIX

I	REFERENCES	228
II	NOTATION	232

VOLUME II

COMPUTER PROGRAMS AND RESULTS

5.1 Percentiles of the Noncentral t/sqrtN

7.1 Iterated Bias

7.2 Simulation Normal Double Limits

8.1 Normal Reliability Likelihood Contours

13.1 Binomial Trinormal Simulation

13.1A Binomial Trinormal Simulation with Variance, Efficiency

13.2 Overall Reliability Confidence Bound (Fortran)

14.1 Overall Reliability Likelihood Functions

14.2 Bayesian Confidence Bound Based on Chi-Squared Prior Distributions

15.1 Degrees of Freedom Estimates

CHAPTER I

OBJECTIVE

1.a. Requirements for Reliability Estimates

The first American space vehicles and military missiles were both costly and unreliable. The most unreliable component systems were propulsion and electronics and as a result the electronics and rocket engine industries were the first to be required by contract to monitor and report estimates of reliability. The statistical methods developed are reminiscent of early quality control methods; they are based on discrete random variables and generally suffer from poor precision. Although these methods were satisfactory at low levels of reliability, better methods are required at the current higher levels of reliability (Abernethy 1963 (b)).

In addition to production control, there are other requirements for rocket engine reliability estimates. The design of vehicle engine failure detection systems and abort systems and the use of redundant engines are based on engine reliability estimates. (Abernethy 1963 (a)). Many advanced missions involve multiple launches; the number of vehicles needed to complete the mission is a random variable that is dependent on engine reliability.

The optimum single engine thrust size for multi-engine vehicles is a function of total thrust required, reliability and cost: the smaller the engine, the more development tests can be made for a fixed cost and therefore, the higher reliability, but in production the cost per pound thrust favors larger engines. It follows that there is an optimum size single engine for each vehicle as a function of reliability, total cost and total thrust required. (Mulready et al)

1.b. Rocket Engine Project Life Span

The lifespan of a rocket engine project may be divided into three overlapping periods: design, development and production.

THE LIFESPAN OF A ROCKET ENGINE PROJECT

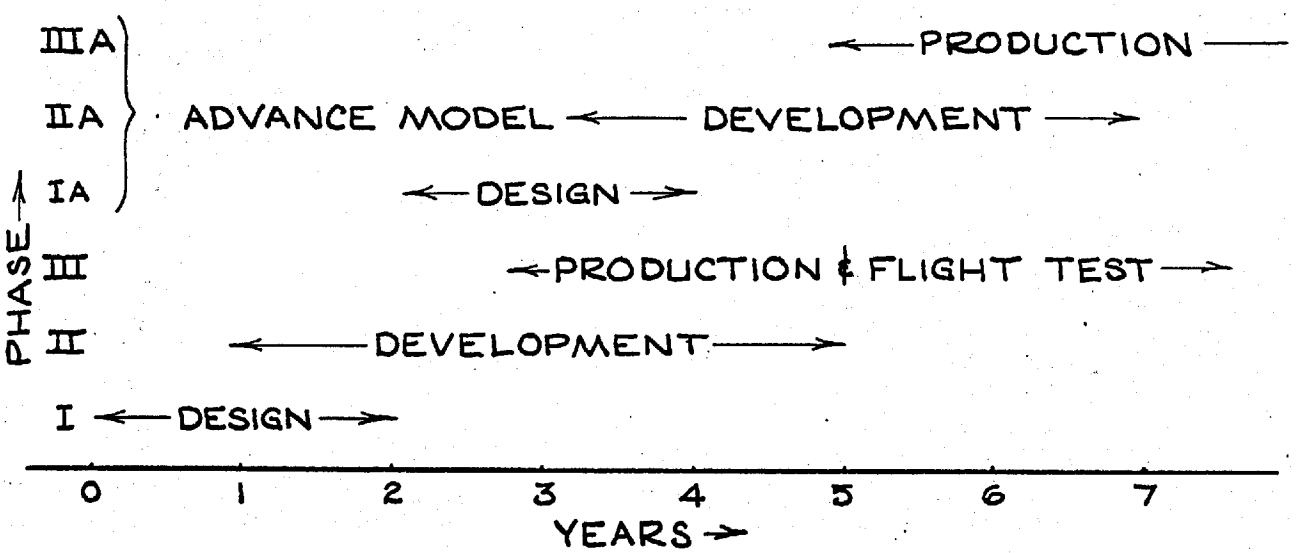


FIGURE 1.1

In the design period there are no full scale engine test data available for reliability estimation. In the same period with electronic systems it is common practice to make system reliability estimates based on component test data, but with rocket engines analogous methods have failed; the mechanical and thermodynamic interactions between components are significant and the engine environment of vibration, cryogenic and combustion temperatures, and transient shock is virtually impossible to simulate on a component test stand. Therefore, rocket engine reliability estimates in the design period will be largely guesses. Historically the best guess has been the reliability of the last engine produced by the same engineering team.

The production-flight test period is characterized by reasonably large quantities of valid data from flight tests and from production engine ground test. The statistical problems of making estimates in this period are trivial and the need for estimates is less than during the development program. The development period is the critical period for reliability estimates for two reasons: the need for reliability estimates is most acute at this time because the vehicle design and mission analysis is done concurrently and the quantity of reliability data

available is small and it is of poor quality.

1.c. Development Program Data

Development period data consists of the results of full scale tests of experimental engines constructed of the following types of components:

a. Experimental: untested new designs
b. Obsolete: inferior designs that have been replaced in the bill of materials
c. Overage: parts with test exposure exceeding ten or fifteen times the design life
d. Damaged or repaired parts
e. Green run parts - untested parts
f. Standard bill-of-materials parts.

Flight engines consist of bill-of-materials parts only. Reliability estimates for flight engines based on development data must utilize data from experimental, non-homogeneous engines as it is the only data available. Further, the test programs and test objectives of experimental engines test conflict with the objectives of reliability demonstration. A good development program will uncover as many weaknesses and failure modes as possible; failure should often be a test objective, either directly or indirectly. The tests that are particularly

dangerous must be excluded from the reliability evaluation as non-representative of a mission firing. The test firing duration may be less than or greater than a mission firing and therefore a system of weighting test data by test duration is required; a successful short duration test should not count as much as a successful long duration test. The weighting factors for short and long runs are added together to determine the number of equivalent full duration runs.

The statistical parameter to be estimated, reliability, is not constant during the development program but should increase in discrete steps as engineering changes are made to the engine. The growth of reliability is a measure of the success of the development program which in turn is a function of total test firing duration, the number of test firings, engineering proficiency, the state-of-the-art of the engine (very advanced versus similar to last engine produced), the initial design, and perhaps the total time allowed. (Figure 1.2) For a given engineering team, the total number of firings appears to be the most significant factor for rocket engines.

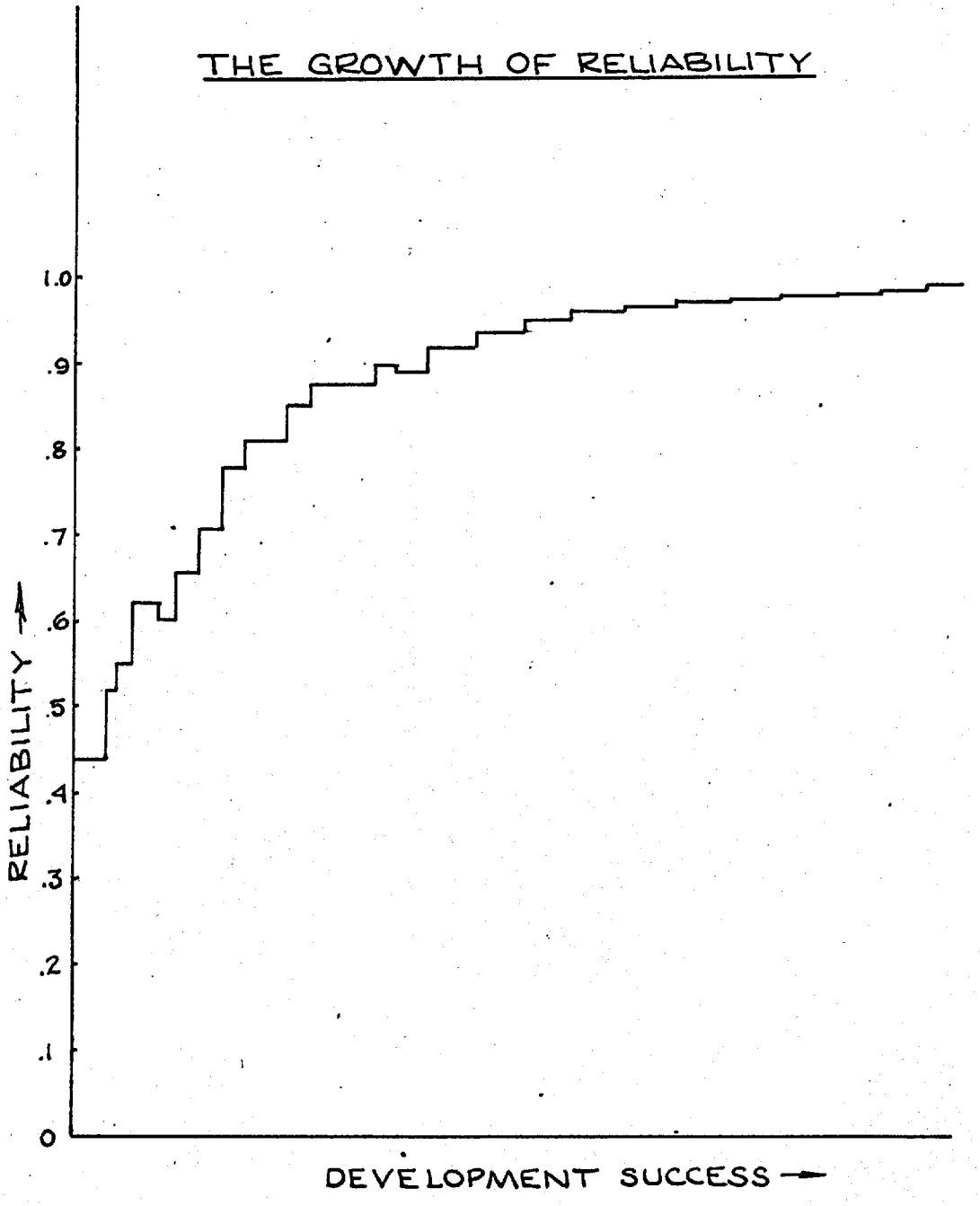

THE GROWTH OF RELIABILITY
RELIABILITY
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
DEVELOPMENT SUCCESS

FIGURE 1.2

There are also problems related to the performance parameters: specific impulse, propellant mixture ratio and thrust. The reliability estimate required is for a mission firing. The number of firings per engine will be determined by the mission requirements and will usually be one. The population sampled, ground test firings of experimental engines, will often consist of more than one firing per engine and may be as high as ten firings per engine. The variances of the performance parameters are composed of components of engine-to-engine variation and run-to-run variation. It follows that the ground test population may have different performance variances from the mission population.

Further, the propellant mixture ratio and thrust are usually controlled by hydro-mechanical controllers which are adjusted on flight engines during production acceptance testing. One or two firings are required to adjust or 'trim' the engine and the engine is not considered 'trimmed' until the trimming firings are completed. Most experimental engines are not trimmed for various reasons. The experimental engine which is not trimmed must be excluded from satisfying the propellant mixture ratio and thrust requirements. Even on trimmed engines the instrumentation will sometimes fail which also leads

to exclusion of one or all of the performance parameters. As a result the sample size associated with each performance parameter will usually be different and this problem should be reflected in the precision of the reliability estimates.

In view of these major problems it would seem appropriate to conduct reliability demonstration firings separate from development firings. Although it would be highly desirable and has often been suggested, it has rarely been done because of the enormous cost of rocket engine testing and because the correlation between development reliability estimates and later flight test results has been quite good.

1.d. Objective

The objective of the research project reported herein is to develop a statistical method to measure the reliability of rocket engines during the development program of the engine which will overcome the deficiencies associated with the current method. A single firing mission will be assumed as it has the widest application.

1.e. Organization

This thesis is divided into two volumes, the second containing computer programs and results. The first volume is subdivided into four major parts indicated by

Roman numerals. Each chapter is further divided into sections indicated by lower case English letters. For example the third section in Chapter 13 is indicated by (13c). Computer programs, equations, tables, and figures are numbered within chapters. For example, the second equation in Chapter 13 would be referred to as (13.2). References are listed in the appendix and are referenced by the surname of the author with the year indicated if there is more than one reference to the author. An effort has been made to use standard notation: a list of notation and definitions is provided in the appendix. The next chapter is devoted to a critical description of the current industry method.

CHAPTER 2

THE CURRENT METHOD

The method currently employed throughout the American rocket industry was developed in the late 1950's for application to intercontinental ballistic missile rocket engines. It appears to have been jointly developed at North American Aviation Rocketdyne Division and Aerojet-General Corporation. It is described in detail in Chapter 16 of (Lloyd and Lipow 1962). The treatment of the method here is intended to indicate the solutions to problems described earlier and the areas of similarity and difference between the current system and the new system.

The statistics of the current system would be simple if good valid data were available. A sample of full duration firings, one each on bill-of-materials, trimmed, engines would be used. Each firing would be classified a success if the engine started, ran for a full duration, provided specific impulse, propellant mixture ratio and thrust with specification limits and shutdown without incident; otherwise it would be classified a failure. Point estimates and interval estimates of reliability would be obtained using the binomial probability distribution.

The ratio of the number of successful firings to the total firings will be an unbiased, maximum likelihood, estimate of the true reliability, η, equivalent to the binomial parameter p.

A lower, one-sided, 90°/o confidence bound h_c for the true reliability, η, may be obtained from the binomial relation:

$$0.10 = \sum_{r=o}^{F} \frac{N!}{r!(N-r)!} h_c^{N-r} (1-h_c)^r \qquad (2.1)$$

where N ≡ number of firings, F ≡ number failures

The complexity of the current method is due to the modifications required to treat the following problems.

2.a. Reliability Growth

If all the development data available is utilized the lower reliability of earlier times will bias the estimate of current reliability. A moving average is employed to minimize the bias. One hundred equivalent full duration runs appears to be a good compromise between the large variance of smaller samples and the bias of larger samples. In a typical program, a sample of one hundred equivalent runs will extend over about three or four months. In the late portion of the development

program when the rate of change of reliability is small, larger sizes are advisable, perhaps 500 equivalent full duration runs.

2.b. High Failure Rate Components

Up to three minor and one major non-bill-of-material components may be pre-declared as dangerous components. If the firing is a success it is so classified; if a failure, it is excluded if the failure is due to the dangerous condition pre-declared and otherwise, it is classified a failure. If the engine contains more than the allowable number, the firing may or may not be a pre-test exclusion at the option of the engineer in-charge. If it is not excluded before the test, no more than allowable number of components may be pre-declared.

2.c. Dangerous Test Objectives

If the Pre-Test Declaration includes any of the following test objectives, the firing is automatically excluded before the test begins:

1. first firing of a newly assembled engine
2. test duration less than ten seconds
3. a test to probe the limits of operation
4. accelerated testing outside the design range of operation.

2.d. Variable Firing Duration

A short duration successful firing should not be credited as much as a long duration test. If the hazard rate were constant throughout the firing, weighting factors based on the exponential probability distribution could be employed; the weighting factors would be the ratio of the intended run duration to a full duration. Unfortunately rocket engine failure rates are not even approximately constant. More than half of all failures occur in the first ten seconds due to the combined effect of start transient shock and infant mortality. The failure rate continues to decrease for the first two minutes of operation. It increases again during the shutdown transient. A mathematical model to describe this complex mortality function might be difficult to treat. The method employed is related to actuarial insurance statistics. The shape of the mortality curve is estimated from a very large sample of past data. For this purpose the full duration is divided into twenty or thirty intervals.

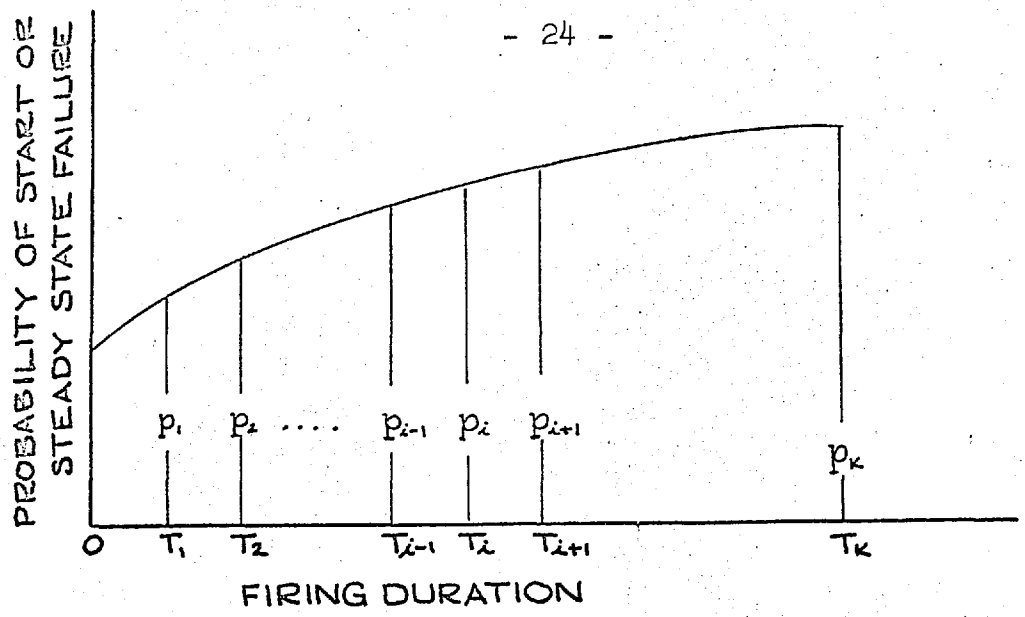


FIGURE 2.1

p_i is an estimate of the probability of start or steady state failure for runs with intended firing duration within interval $(T_i - T_{i-1})_i$.

If the full duration falls in interval k, the weighting factor for interval i is defined as:

$$w_i = \frac{p_i + (1-p_i)\epsilon}{p_k + (1-p_k)\epsilon} \qquad (2.2)$$

where ϵ is an estimate of the probability of a shutdown failure. The sampling characteristics of w_i are described in (Lloyd and Lipow1962). p_i and ϵ are maximum likelihood estimates.

2.e. Ground Test versus Mission Populations

The use of a binomial model for reliability implies that the binomial parameter, η, is constant for each trial. The ground test data may consist of five to ten firings per engine assembly with extreme cases up to about one hundred firings per assembly while the mission consists of a single firing per assembly. This difference produces two problems:

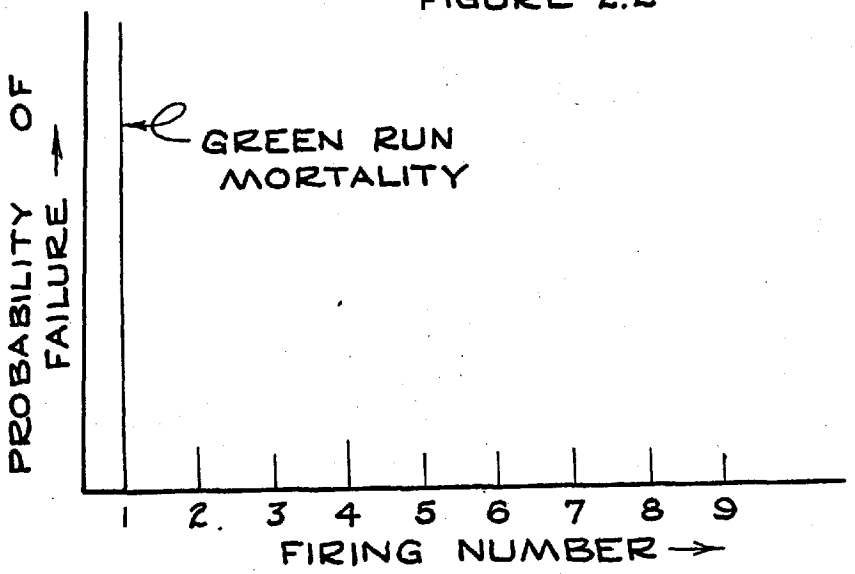
(1) If we omit the performance requirements from the definition of reliability, the remainder, the probability that the engine when operated in the manner and environment intended, will start, run for a full duration, and shutdown without incident, will be referred to as functional reliability*. If there is a run-to-run effect on functional reliability in ground test data, it must be eliminated before the ground test data is employed to estimate mission functional reliability. There are failure modes that occur during the shutdown transient which do not affect the firing just completed but would degrade the functional reliability of succeeding firings. Every attempt is made to eliminate the effect of these failures by rigid inspection between ground test firings, conducted exactly like the pre-launch, countdown inspection. The mortality curve as a function of firing number

---

*This definition will be treated in more detail in Chapter 11.

per engine assembly must be constant.

FIGURE 2.2



The first firing of a new engine called the green run, has a very high failure rate (infant mortality), perhaps twenty to thirty times that of succeeding firings and therefore, it is excluded from the ground test data. As all production engines are fired several times before leaving the production facility, green run mortality should not be experienced in the mission population. With some engines, the second firing may also be excluded due to infant mortality.

(2) The second problem is that of the performance parameter variances. In the mission population, the total variance is the sum of the engine-to-engine and run-to-run variances. If there is more than one firing per engine the ground test population variances will be

less than the mission variances because of a reduction in the engine-to-engine component; in the limit if all the firings in a sample were conducted on one engine assembly, the total variance would be the same as the run-to-run variance. In the current method, this problem is not treated statistically; instead a contractual clause is added to require that either the number of engine assemblies tested within the sample must exceed some value, say five, or the number of firings per engine assembly must not exceed some value, say twenty. With regard to the assumption that the mission environment consists of single firings on each production engine, the current method cannot be applied to multi-firing missions, nor was it intended to, without serious modification. Multi-firing missions will be discussed as an application of the proposed method in Chapter 16.

2.f. Trimming Propellant Mixture Ratio and Thrust Controls

An untrimmed engine is excluded from meeting the thrust and mixture ratio limits. In the current system there is usually a requirement that an arbitrary minimum number of firings must be trimmed, say 10% or 20%. Even such a low minimum cannot always be achieved during development programs; there have been times when the

entire reliability system was obstructed by this requirement.

2.g. Documentation

The accuracy and precision of statistical estimates is limited by the accuracy, validity and completeness of the data available. Large quantities of reliability data are required for both the current and the proposed methods. The data processing system used at Pratt and Whitney Aircraft for the current system (developed by the author) is based on the following documents:

1. Pre-Test Declaration. A description of the engine parts and components that are non-bill-of-materials, test objectives, trim condition, intended test duration, and any unusual aspects of the firing, completed by the test engineer and filed prior to the automatic countdown sequence.
2. Post-Test Report. This is a report of the results available immediately after the firing as filed by the test engineer.
3. Failure Report. This is a report that some type of failure has occurred. The failure may be minor, major, quality control, instrumentation, facilities, design, performance or even human.

4. Failure Analysis. An analysis of the failure (reported by (3)) completed by the Project Engineer responsible for the failure, describing the cause, the analysis, and if possible, the corrective action taken to prevent repetition of the failure.

5. Serialized Parts History. The critical parts in an engine are marked with serial numbers and the complete history of their test exposure on rigs and engines is maintained for failure analysis.

6. Performance Analysis Report. The final detailed performance analysis of the engine is reported by the Performance Analysis Group. The report is used for both firing classification and failure analysis.

The documentation system utilizes IBM 1401 and IBM 7090 computers. Monthly reliability reports are produced entirely on the IBM 7090 computer and consist of the following sections:

1. Statistical Estimates of Reliability: current point estimates, lower one-sided 90% confidence bounds, sample sizes of success, failure, pre-test and post test exclusions, corresponding statistics for the previous month.

2. Firing summary: a chronological summary of all firings included in the span of one hundred equivalent

full duration runs with pertinent results and cross-referenced to the failure reports submitted.

3. Failure Summary: a summary of all failures and analysis by component, cross referenced to the firing summary.

4. Weighting Factors: the weighting factors by time interval.

The reliability documentation system is a basic engineering record and is employed for many services other than reliability estimation. Duplicate information systems for other purposes are not maintained.

2.h. Criticism

The success of the current method as measured by the good correlation between reliability estimates and flight results is noteworthy and certainly encouraging to applied statisticians when they consider the complexity of this system. The modifications included in the proposed method are required, at least partially, because of the increase in rocket engine reliability level; at reliabilities above 95%, the precision of the current method is inadequate. A study of this specific problem is contained in (Abernethy 1963b), and inspired the project reported herein. With most of the current engines well

above 99°/o reliability, this problem is acute. For example, if the sample size is one hundred, and the true reliability, η, is 0.99, the standard deviation of the estimate h is:

$$\sigma = \sqrt{\frac{\eta(1-\eta)}{100}} = \sqrt{\frac{0.99(.01)}{100}} \cong 0.01 \qquad (2.3)$$

The unreliability $(1-\eta)$ and one standard deviation of the estimate of reliability are approximately equal. It follows that significant improvements in reliability above 99°/o will be totally obscured by the poor precision of the estimate. In the new system it is proposed to improve the precision by treating the three performance parameters as continuous measured variables rather than discrete (success-failure) variables. Precise and accurate measurements are available for these parameters but to date this information has not been utilized for reliability estimates except in a discrete sense.

When the performance parameters are treated as discrete variables there is no known way of adjusting the data from the ground test population to the mission population to properly account for the differences. For example, if the performance parameters have large engine-to-engine variances, reliability estimates based on several firings per engine will be biased higher than

the true reliability of the single-firing mission population. (This will be demonstrated in Chapter 15). By treating the performance parameters as variables, it is possible to make separate independent estimates of engine-to-engine and run-to-run variance and recombine these estimates in the correct proportion to estimate the mission variances. Although the poor precision of the current method at high reliability is well recognized, this improper treatment of variation is a more subtle criticism which is not widely understood.

The untrimmed engine performance exclusion also contributes bias to the current estimates. For example, if thrust reliability is 99°/o we would expect one thrust failure on the average in a sample of one hundred firings, but if ninety of the firings are excluded as untrimmed, the expectation would only be one tenth of a failure. Here again, by treating the thrust and propellant mixture ratio parameters as measured variables, this bias can be eliminated. The degrees of freedom of estimates of the performance variances will reflect the number of trimmed firings included in the sample.

The current method cannot be applied to multi-firing missions as there is no measure of the interaction between firings and as mentioned above, no method for adjusting

the data for the differences between the population sampled and the mission population. As the space projects become more complex, upper stage engines will have many requirements for multi-firing missions. The proposed method can be used to estimate multi-firing reliability. A new statistic, the multi-firing factor, will be described in Chapter 16; it is used to measure the conditional probability that given the engine has fired successfuly, that it is capable of firing again. However, the precision of multi-firing estimates will be less than single-firing estimates because certain errors will be compounded.

CHAPTER 3

THE PROPOSED METHOD

3.a. ~~Math Model~~ Mathematical Model

The math model for the new method is a product of four factors.

$$\eta_o = \eta_{fu}\ \eta_{ISP}\ \eta_{FN}\ \eta_{rm} \qquad (3.1)$$

where: $\eta_o \equiv$ overall reliability, the probability that the engine when operated in the manner and environment intended will start, run for a full duration, provide specification levels of specific impulse, propellant mixture ratio, and thrust, and shutdown without incident.

$\eta_{fu} \equiv$ functional reliability, the probability that the engine when operated in the manner and environment intended will start, run for a full duration with performance parameters within functional limits, and shutdown without incident. Functional limits of performance are arbitrary limits, considerably wider than specification limits. Functional limits correspond to an easy mission, a mission with relaxed engine requirements. The concept of functional reliability was originated by the author in response to numerous requests from mission analysts. It is now in general industry use. It has been monitored and reported since 1961 at

Pratt and Whitney Aircraft.

$\eta_{ISP} \equiv$ specific impulse reliability, the conditional probability, given functional success, that the steady state specific impulse corrected to nominal mixture ratio and thrust, equals or exceeds the specification minimum. Specific impulse is the ratio of thrust produced to the weight of propellant consumed per second. It has the units of seconds and for a given propellant combination, it is a measure of overall efficiency.

$\eta_{FN} \equiv$ thrust reliability, the conditional probability, given functional success, that the steady state thrust is within the specification limits. The units are pounds-force.

$\eta_{rm} \equiv$ propellant mixture ratio reliability, the conditional probability, given functional success, that the steady state propellant mixture ratio is within the specification limits. The ratio is pounds of oxidizer per unit time to pounds of fuel per unit time; it is dimensionless.

If the four factors, η_{fu}, η_{ISP}, η_{FN}, η_{rm}, are replaced by the sample estimates h_{fu}, h_{ISP}, h_{FN}, h_{rm}, then the product, h_o, will be an estimate of η_o. If the sample estimates are independent* and unbiased, then h_o will be an unbiased estimate of η_o. The proof of the following

*next page

statement is shown in (Cramer, page 173): 'the mean value of a product of independent random variables is equal to the product of the mean values of the factors, provided that the latter mean values exist.'

---

*Given that the more general condition that no relationship exists between the factors η_{fu}, η_{ISP}, η_{FN}, η_{rm} it may be shown that h_o will be an unbiased estimate of η_o, if and only if, the sample estimates are unbiased:

Suppose $Eh_o = E(h_{fu}h_{ISP}h_{FN}h_{rm}) = \eta_{fu}\eta_{ISP}\eta_{FN}\eta_{rm} = \eta_o$ (3.2)

As the estimates are independent, the expectation of the product equals the product of the expectations:

$$E(h_{fu}h_{ISP}h_{FN}h_{rm}) = (Eh_{fu})(Eh_{ISP})(Eh_{FN})(Eh_{rm})$$

$$= fn(\eta_{fu})fn(\eta_{ISP})fn(\eta_{FN})\ fn(\eta_{rm})$$

$$= \eta_{fu}\ \eta_{ISP}\ \eta_{FN}\ \eta_{rm} \qquad (3.3)$$

Now unless $fn(\eta_{fu}) = \eta_{fu}$, $fn(\eta_{ISP}) = \eta_{ISP}$, (3.4)

$fn(\eta_{FN}) = \eta_{FN}$, $fn(\eta_{rm}) = \eta_{rm}$ (3.5)

the above would constitute a relationship between the factors.

3.b. Overall Reliability Estimates

The current method was designed to produce unbiased point estimates and one-sided lower 90°/o confidence interval estimates of overall reliability. Current American rocket engine contracts contain requirements to compute and report these statistics monthly and the new method must have the same capability to satisfy these requirements. However, the choice of the type of estimate depends on the data available and on the application, and even then it is controversial; unbiased point estimates and confidence interval estimates are certainly not optimum for all applications. The chapters following will describe the application of the new math model to point and confidence interval estimates of η_o; other types of estimates will be discussed in Chapters 8 and 14.

3.c. Independence and Normality

If there is no physical relationship between the performance variables, they will be both functionally and statistically independent. The physical relationship between the performance parameters depends on the mechanics and thermodynamics of the particular engine model. In most engines mixture ratio and thrust will

be independently adjustable and controlled. These two parameters have precise double specification limits; interacting controls would make the production adjustment or 'trimming' very difficult. In addition; there are often vehicle requirements for independent controls. For example, it may be desired to vary mixture ratio during the mission firing so that the oxidizer and fuel tanks empty simultaneously. Thrust must be almost constant during this operation or the problem of predicting vehicle trajectory becomes impossible.

The relation between thrust and specific impulse is weak; a very large change in thrust is required to produce a small change in specific impulse. In general, mixture ratio and specific impulse are more strongly related but fortunately, rocket engines are designed to run at the maximum value of specific impulse and the peak is broad and flat. Therefore, over the operating range of the engine, specific impulse will be almost constant as a function of mixture ratio: it is not a coincidence that these relations are weak; if they were strong, a change in mixture ratio or thrust would have a severe effect on vehicle performance.

Even if a small residual correlation remains between the parameters on a series of firings on the same engine,

the correlation tends to disappear when each firing is made on a different engine because the engine-to-engine variances mask the relationship as illustrated in Figure 3.1.

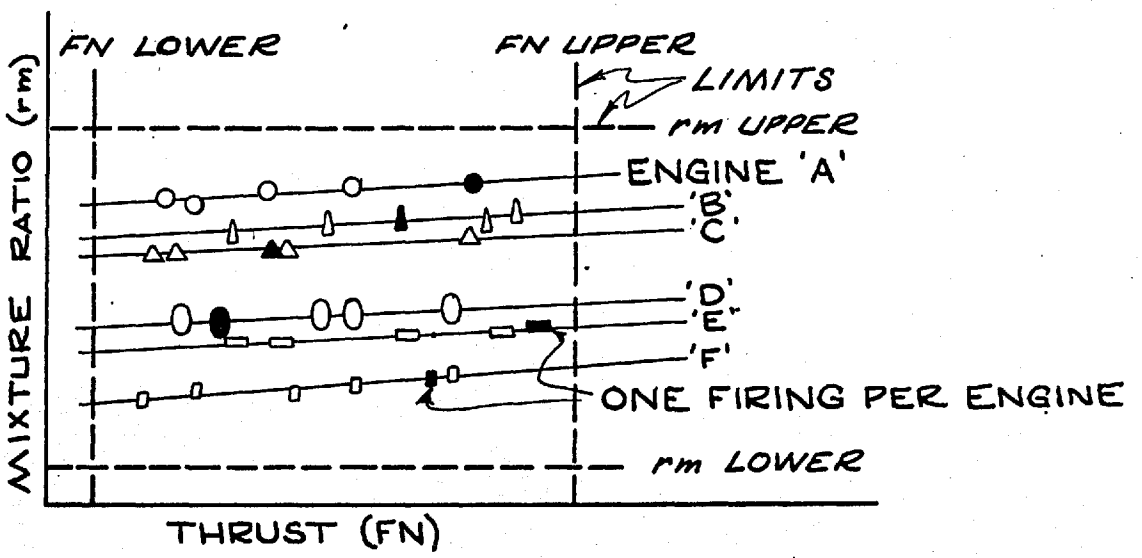


FIGURE 3.1

Performance data on the Pratt and Whitney Aircraft RL-10 rocket engine cannot be presented here as it has a security classification: however sample correlation coefficients for the three parameters taken in all pairs have been calculated for this study for several groups of single firings from single engines (the sample included 93 engines) and none of the coefficients was significantly different from zero at 0.05 level of significance. The

groups were determined by engine model. The estimates h_{ISP}, h_{FN}, h_{rm} are each functions of samples of their respective performance parameters. If the performance parameters are mutually independent, the estimates must also be mutually independent. It is concluded that for single firing missions, the three performance parameters may be considered statistically independent. Routine checks should be made on this assumption by correlation analysis prior to calculating reliability estimates.

All three performance parameters appear to be normally distributed; repeated tests of non-normality on large samples of data have failed to reject the null hypothesis that the data are normal.

3.d. Data Description

The data consist of a chronological record of all firings attempted documented on the records described in Chapter 1. In general the following numerical data are relevant to the problem of estimating reliability:

1. Engine assembly number
2. Firing number per assembly
3. Date, time
4. Intended firing duration
5. Actual firing duration
6. Functional Run Classification
    a. Success
    b. Failure

c. Exclusion

7. Trimmed or untrimmed status of mixture ratio and thrust controls
8. Specific Impulse (ISP) corrected to nominal mixture ratio and thrust
9. Mixture Ratio (rm)
10. Thrust (Fn)

Any of the performance parameters may be missing due to instrumentation malfunctions. The conversion of these data into estimates of reliability occupies the remainder of this thesis; Part II is devoted to the estimates based on the three normal variables, Part III treats the proposed method in terms of a simple model, and Part IV, more complex models.

PART II

ESTIMATES OF THE PROPORTION OF A NORMAL DISTRIBUTION

DEFINED BY A FIXED LIMIT OR LIMITS

CHAPTER 4

INTRODUCTION

4.a. Required Estimates

The reliability of a normally distributed parameter, η, is the proportion which satisfies the specification limits. For specific impulse reliability, η_{ISP}, this is the proportion exceeding the guaranteed minimum limit. (See Figure 12.1). For mixture ratio and thrust, η_{rm} and η_{FN}, it is the proportion of the distribution between the lower limit and the upper limit. (See Figure 12.2). The English letter h will be used to indicate an estimate of reliability, η, in conformance with current statistical practice. Part II of this thesis is devoted to the problem of estimating these proportions.

4.b. Current Estimates

One unbiased point estimate of these proportions is the ratio of the number of successes to the number of attempts, $\check{h}$; this is the estimate used in the current system described in (Lloyd and Lipow). Given the measured data and the distribution of the data, it is an inefficient estimate; otherwise, it is a most efficient, best, unbiased point estimate. It is often referred to in industry by this latter term, the best unbiased point estimate. The confidence interval estimate of overall reliability in the current system is determined using the binomial distribution.

4.c. Point Estimates

Given that the performance parameter distributions are normal, and given the specified limits, three cases are of interest:

(1) known mean and known variance

The interest here is academic as this case has little practical significance. The solution is to calculate the standard normal deviate: $Z_\eta = \frac{\mu - L}{\sigma}$ which is exceeded with probability η. The probability η may be determined from $\eta = \Phi(Z_\eta)$ (4.2)

where $\Phi(X)$ is the cumulative normal distribution

evaluated at X.

(2) unknown mean and unknown variance

It is logical in this case to replace μ and σ by the sample estimates, $\bar{X}$ and s:

$$\bar{X} \equiv \sum_{i=1}^{N} \frac{X_i}{N} \qquad (4.3) \qquad s \equiv \sqrt{\sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{N-1}} \qquad (4.4)$$

$$k \equiv \frac{(\bar{X}-L)}{s} \qquad (4.5) \qquad \tilde{h} \equiv \Phi(k) \qquad (4.6)$$

It will be shown that k is a biased estimate of Z_η and $\tilde{h}$ is a biased estimate of η. Chapters 5, 6 and 7 are devoted to this problem.

(3) unknown mean and known variance

This situation exists when the variance of a process is constant but changes in the mean are expected. It is an easier problem than (2) and will be left to Chapter 9.

The case of known mean and unknown variance has neither practical nor academic interest and will not be treated in this study.

4.d. Interval Estimates

In addition to unbiased estimates of the proportions and reliabilities, a measure of the precision of these

estimates should be produced to reflect their validity. As the distributions of both k and $\tilde{h}$ are generally skewed, it is more appropriate to employ confidence intervals as this gauge than to estimate the corresponding variances.

4.e. Likelihood Estimates

Chapter 8 will be devoted to estimates based on the likelihood function. The results will be used in Chapter 14 for both likelihood and Bayesian estimates of overall reliability.

CHAPTER 5

THE DISTRIBUTION OF $k = \frac{\bar{X}-L}{s}$

5.a. Sampling Distribution

Consider the random variable,

$$k = \frac{\bar{X}-L}{s} \tag{5.1}$$

where $\bar{X} \equiv \sum_{i=1}^{N} \frac{X_i}{N}$

$$s \equiv \sqrt{\sum_{i=1}^{N} \frac{(X_i-\bar{X})^2}{N-1}}$$

$L \equiv$ a point in the scale of X, a limit,

and the X_i are independent and normally distributed (μ, σ^2, both unknown). Multiplying by $\sqrt{N}$ on both sides, and dividing numerator and denominator of the right side by the true, but unknown, standard deviation σ:

$$k\sqrt{N} = \frac{\sqrt{N}\ (\frac{\bar{X}-L}{\sigma})}{\frac{s}{\sigma}} = \frac{\sqrt{N}\ (\frac{\bar{X}-\mu}{\sigma}) + \sqrt{N}\ Z_\eta}{\frac{s}{\sigma}} \tag{5.2}$$

where $Z_\eta \equiv \frac{\mu-L}{\sigma}$, the standard deviate exceeded with

probability η and $\sqrt{N}\ (\frac{\bar{X}-L}{\sigma})$ is a standard normal variate. s^2/σ^2 is distributed $\chi^2_{(N-1)}/(N-1)$. For the normal distribution it is well established that s is independent of $\bar{X}$. A statistic of the form:

$$t = \frac{(Y+\delta)}{\sqrt{W}} \tag{5.3}$$

where Y is a standard normal variate and W is an independent variate distributed as χ^2/f with f degrees of freedom, and δ is a constant, is distributed as the non-central t with δ as non-centrality parameter. Therefore, $k\sqrt{N}$ is distributed as the non-central t with N-1 degrees of freedom and non-centrality $\delta = \sqrt{N}\ Z_\eta$.

5.b. Tables

Seventeen percentiles of the sampling distribution of k were determined from the non-central t distribution using the above parameters and the method suggested by Resnikoff (1962) and due originally to (Johnson and Welch). Using their notation, the method is to calculate

$$r = \delta/\sqrt{2f} \quad \text{and} \quad \eta = r/\sqrt{1+R^2} \tag{5.4}$$

where δ is the noncentrality and f is the degrees of freedom (usually f = N-1). In their notation

$$\Pr\ (t > t_\epsilon \mid f,\delta) = \epsilon \tag{5.5}$$

The (Resnikoff) paper contains a triple entry table (f,δ,t) which provide values of λ. The required value of t_ϵ is determined from λ by the relation:

$$t = \frac{\delta + \sqrt{1+r^2 - \lambda^2/(2f)}}{1 - \lambda^2/(2f)} \qquad (5.6)$$

The calculations were originally done on an IBM 7090 computer at the Pratt and Whitney Aircraft, Florida Research and Development Center in July 1963 by the author. At that time the author was registered at Imperial College under program supervised by Professor G. A. Barnard. The results were published by Pratt and Whitney Aircraft in the author's name as Table III of report PWA FR 781, 'One Sided Statistical Tolerance Factors' and it is reproduced at the end of this Chapter. The body of Table III contains values of the statistic k and it is also a triple entry tabulation:

(1) The nine sections correspond to the percentiles (labelled Confidence Level): 50%, 60%, 70%, 80%, 90%, 95%, 97.5%, 99%, 99.5%.

(2) The rows are sample size (N).

(3) The columns indicate the noncentrality and are labelled with the corresponding value of η (returning to our notation). Interpolation on noncentrality should

be based on Z_η rather than η.

The fourth decimal place is usually not significant.

To obtain values at the complementary percentiles (50%, 40%, 30%, 20%, 10%, 5%, 2.5%, 1%, 0.5%), the following identity is useful and is taken from (Resnikoff 1962):

$$k_C(\eta) = - k_{1-C}\ (1-\eta) \qquad (5.7)$$

where C ≡ percentile/100. In (Resnikoff) notation the relation is $t_{1-\epsilon}\ (\delta) = - t_\epsilon\ (-\delta)$ (5.8)

Two examples of the application of the Table III are:

(1) Given a sample of the data from a normal distribution with unknown variance and unknown mean, what is the lower 90%/o confidence bound (one-sided) on the proportion of the distribution, exceeding a limit L?

Data from (Dixon and Massey, Table II), Random Normal Deviates ($\mu = 0.0$, $\sigma^2 = 1.0$) :

- 0.329
+ 0.085
+ 0.130
- 0.244
- 0.882
+ 0.472
+ 0.039
+ 1.420
- 1.033
+ 1.807

Given L = - 1.2816

$$\bar{X} = \sum_{i=1}^{N} \frac{X_i}{N} = 0.1465$$

$$s = \sqrt{\sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{N-1}} = 0.9022$$

$$k = \frac{\bar{X} - L}{s} = 1.582$$

From Table III (90% confidence):

$\eta = 0.80$, $k = 1.474$, $Z_{0.80} = 0.8416$

$\eta = 0.85$, $k = 1.733$, $Z_{0.85} = 1.0364$

By interpolation on Z_η :

$$h_C = 0.822$$

(2) What is the sampling distribution of k for sample size ten and noncentrality $\delta = \sqrt{10} \times 1.2816$?

In this example N = 10, η = .90. Values at the following percentiles can be read directly from the tables.

Percentile	k	
.50	1.3241	
60	1.4449	
70	1.5862	
80	1.7701	
90	2.0656	(The last
95	2.3547	digit is
97.5	2.6480	not signi-
99	3.0482	ficant.)
99.5	3.3685	

To obtain the lower percentiles, the identity from Resnikoff (1962) is used:

$$k_c(\eta) = -k_{1-c}(1-\eta)$$

For example, to obtain the tenth percentile for N = 10, η = .90

$$k_{.10}(0.9) = -k_{.90}(0.10) = 0.8284$$

The lower percentiles:

percentile	k
.5	.4358
1.0	.5075
2.5	.6152
5.0	.7115
10.0	.8284
20.	.9812
30.	1.1012
40.	1.2122

The distribution of k, described by the seventeen percentiles illustrated above, was required at several stages of this project and therefore, the Resnikoff procedure was programmed in both Extended Mercury-Atlas Autocode and in IBM 7090 Fortran IV languages. The Autocode program is Program 5.1 which is illustrated in Volume II of this thesis. The Fortran IV compilation was constructed as an independent subprogram of Program 13.2 so that it may be incorporated in other Fortran programs without modification.

5.c. Simulation

To verify that the above solution provides the sampling distribution of k, a Monte Carlo simulation

was done on the University of London Mercury computer. Random normal samples of size ten were generated ($\mu = 0.0$, $\sigma^2 = 1.0$) and the value of $k = \frac{\bar{X}-L}{s}$ was computed. The limit L was taken as $L = -1.2816$. ($\eta = 0.90$) The process was repeated 100 times and the ranked results are shown as Table 5.1.

TABLE 5.1

Simulation of $k = (\bar{X}-L)/s$

Limit $L = -1.2816$, sample size 10. simulation trials 100

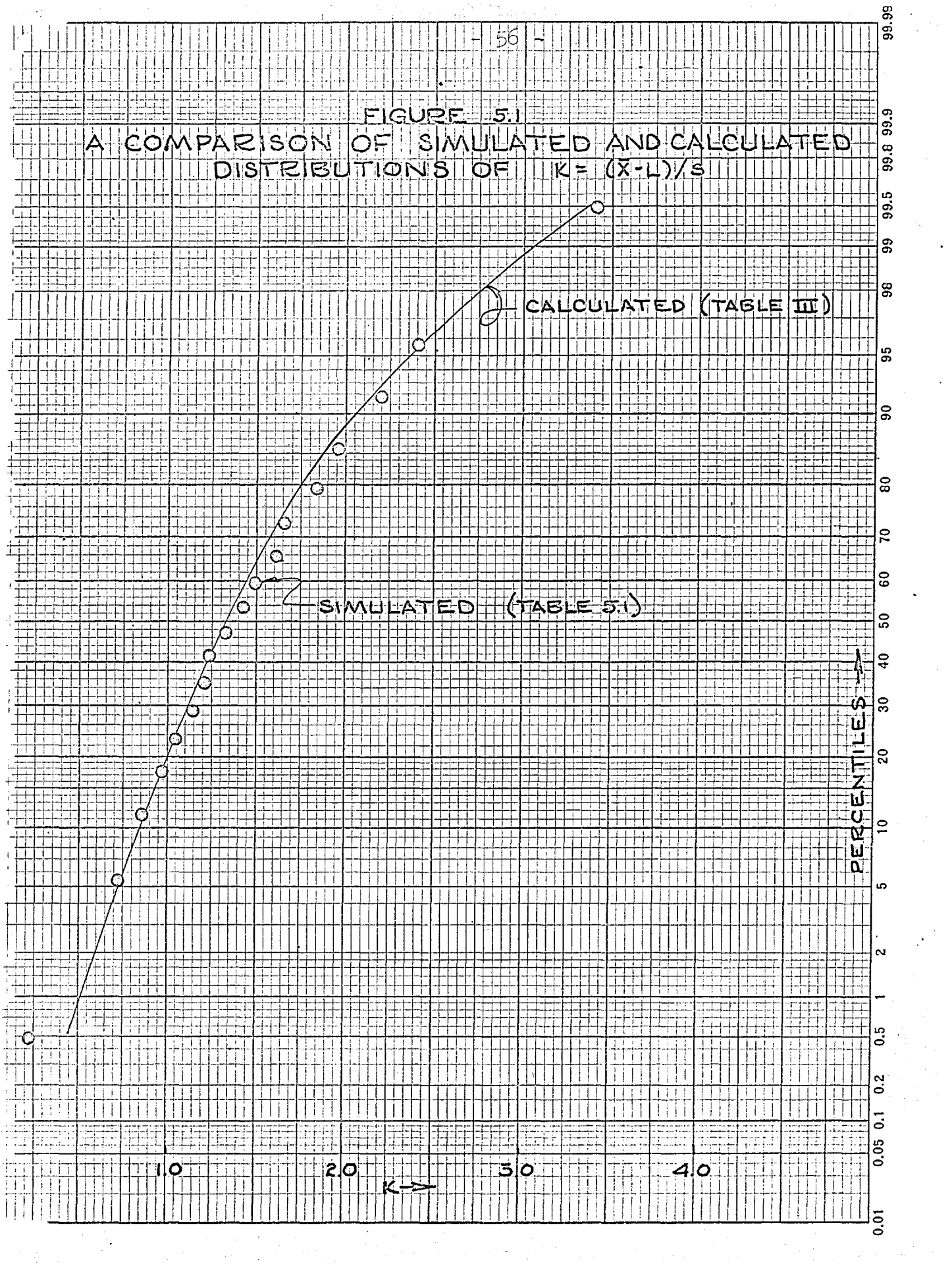
0.2130	0.5456	0.6428	0.6574	0.7190	0.7314
0.7700	0.7734	0.8124	0.8181	0.8572	0.8575
0.8667	0.8687	0.9256	0.9430	0.9446	0.9775
0.9813	0.9922	1.0093	1.0204	1.0596	1.0671
1.0985	1.1240	1.1268	1.1317	1.1453	1.1548
1.1580	1.1906	1.2006	1.2028	1.2124	1.2127
1.2136	1.2150	1.2159	1.2171	1.2310	1.2458
1.2523	1.2686	1.2752	1.2810	1.3071	1.3302
1.3886	1.3911	1.4217	1.4262	1.4271	1.4342
1.4369	1.4485	1.4500	1.4791	1.4835	1.4902
1.5069	1.5176	1.5431	1.5683	1.5817	1.6202
1.6232	1.6345	1.6462	1.6486	1.6619	1.6637
1.6914	1.7597	1.7807	1.7989	1.8289	1.8470
1.8478	1.8766	1.8857	1.8957	1.9304	1.9775
2.0113	2.0759	2.0807	2.1295	2.1932	2.2115
2.2130	2.3047	2.3269	2.3343	2.3496	2.4075
2.4215	2.4755	2.7625	3.4190		

A plot of the results, (every sixth point), on normal probability paper is shown as Figure 5.1. The plotting position (2i-1)2N was used. There is no significance in the use of normal probability paper except to provide a convenient scale (the distribution is non-central t). The comparison of the simulation results with the calculated results is quite good.

The distribution of k is skewed for all values of other than 0, 0.5 and 1.0. Several transforms of k were investigated and the results will be discussed in Chapter 6.

FIGURE 5.1

A COMPARISON OF SIMULATED AND CALCULATED DISTRIBUTIONS OF $K = (\bar{x}-L)/s$



5.d. Expectation of k

An expression for the expectation of k and also the variance of k was derived without using the non-central t probability distribution or density function:

$$E(k) = E\left(\frac{\bar{x}-L}{s}\right) = \left(E\bar{x} - (\mu - \sigma Z_\eta)\right) E\left(\frac{1}{s}\right)$$

$$E\bar{x} = \mu$$

$$\frac{(N-1)s^2}{\sigma^2} = \chi^2_{(N-1)}$$

$$\frac{1}{s} = \frac{\sqrt{N-1}}{\sigma\chi}$$

$$E\left(\frac{1}{s}\right) = \frac{\sqrt{N-1}}{\sigma} E\left(\frac{1}{\chi}\right)$$

$$E\left(\frac{1}{\chi}\right) = \frac{\Gamma\left(\frac{N-2}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{N-1}{2}\right)} \qquad \text{(HALD PAGE 299)}$$

$$E\left(\frac{1}{s}\right) = \frac{\sqrt{N-1}\,\Gamma\left(\frac{N-2}{2}\right)}{\sigma\sqrt{2}\,\Gamma\left(\frac{N-1}{2}\right)}$$

$$E(k) = (\mu + \sigma Z_\eta - \mu) \times E\left(\frac{1}{s}\right)$$

$$= Z_\eta \sqrt{\frac{N-1}{2}}\ \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}$$

5.e. VARIANCE OF K

$$\sigma_k^2 = E(K^2) - (E(K))^2$$

$$E(K^2) = E\left(\frac{\bar{x}-L}{s}\right)^2$$

$$= \left(E\bar{x}^2 - 2LE\bar{x} + L^2\right) E\left(\frac{1}{s^2}\right)$$

$$\sigma_{\bar{x}}^2 = E(\bar{x}^2) - (E\bar{x})^2 \quad , (E\bar{x})^2 = \mu^2$$

$$E(\bar{x}^2) = \sigma_{\bar{x}}^2 + \mu^2 = \frac{\sigma^2}{N} + \mu^2$$

$$E\left(\frac{1}{S^2}\right) = E\left(\frac{N-1}{\chi^2\sigma^2}\right) = \left(\frac{N-1}{\sigma^2}\right) E\left(\frac{1}{\chi^2}\right)$$

$$E\left(\frac{1}{\chi^2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{N-3}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} = \frac{1}{2}\frac{\left(\frac{N-3}{2}-1\right)!}{\left(\frac{N-1}{2}-1\right)!} \qquad \text{(HALD)}$$

$$= \frac{1}{2}\,\frac{2}{(N-3)} = \frac{1}{N-3}$$

$$E\left(\frac{1}{S^2}\right) = \frac{(N-1)}{(N-3)}\,\frac{1}{\sigma^2}$$

$$E(K^2) = \left(E\bar{X}^2 - 2LE\bar{X} + L^2\right) E\left(\frac{1}{S^2}\right)$$

$$= \left(\frac{\sigma^2}{N} + \mu - 2L\mu + L^2\right)\left(\frac{N-1}{N-3}\right)\frac{1}{\sigma^2}$$

$$= \left(\frac{1}{N} + \left(\frac{\mu - L}{\sigma}\right)^2\right)\left(\frac{N-1}{N-3}\right)$$

$$= \left(\frac{1}{N} + Z_\eta^2\right)\left(\frac{N-1}{N-3}\right)$$

$$\sigma_K^2 = E(K^2) - (EK)^2$$

$$= \left(\frac{1}{N} + Z_\eta^2\right)\left(\frac{N-1}{N-3}\right) - (EK)^2$$

EXAMPLE: $N = 10$, $\mu = 0$, $\eta = 0.90$, $Z_\eta = 1.2816$

$$EK = +1.2816\sqrt{\frac{9}{2}}\;\frac{\Gamma\left(\frac{8}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= +1.4025$$

$$\sigma_K^2 = \left(\frac{1}{10} + 1.2816^2\right)\frac{9}{7} - (+1.4025)^2$$

$$= .2723$$

$$\sigma_K = \sqrt{.2723} = 0.522$$

5.f. Comparison

The simulation results shown in Table 5.2 have the same noncentrality and sample size as the example above and therefore, the average value of the 100 times may be compared with the calculated expectation.

From Table 5.1:

$$k_{average} = \sum_{i=1}^{100} \frac{k_i}{100} = 1.454$$

From above: $\sigma_k = 0.522$

$$\sigma_{k\ average} \cong \sigma_k/\sqrt{100} = 0.052$$

$$\frac{1.454-1.4025}{.052} \cong 1.0$$

5.g. Higher Moments

Higher moments were not required in this study. However, if required, the third and fourth moments can be approximated using the methods in (Hogben et al, 1961 and 1964).

5.h. Median

The median value of k is by definition the 50°/o confidence tolerance factor Table III.

TABLE III

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.5000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-3.3626	-2.7989	-2.5254	-1.7784	-1.3817	-1.1144	-0.9028	-0.7221	-0.5603	-0.4108	-0.2695
6.	-3.3036	-2.7503	-2.4824	-1.7505	-1.3605	-1.0977	-0.8898	-0.7121	-0.5527	-0.4057	-0.2666
7.	-3.2663	-2.7197	-2.4547	-1.7314	-1.3466	-1.0868	-0.8814	-0.7058	-0.5483	-0.4024	-0.2643
8.	-3.2395	-2.6978	-2.4354	-1.7188	-1.3369	-1.0799	-0.8758	-0.7010	-0.5444	-0.3998	-0.2628
9.	-3.2200	-2.6820	-2.4212	-1.7091	-1.3299	-1.0739	-0.8712	-0.6977	-0.5420	-0.3979	-0.2615
10.	-3.2053	-2.6699	-2.4103	-1.7016	-1.3241	-1.0697	-0.8678	-0.6950	-0.5399	-0.3965	-0.2606
11.	-3.1934	-2.6601	-2.4016	-1.6957	-1.3197	-1.0662	-0.8652	-0.6929	-0.5383	-0.3954	-0.2598
12.	-3.1838	-2.6522	-2.3945	-1.6910	-1.3162	-1.0635	-0.8630	-0.6912	-0.5371	-0.3944	-0.2592
13.	-3.1758	-2.6457	-2.3887	-1.6870	-1.3132	-1.0611	-0.8612	-0.6898	-0.5360	-0.3937	-0.2588
14.	-3.1691	-2.6402	-2.3837	-1.6837	-1.3107	-1.0592	-0.8596	-0.6886	-0.5351	-0.3930	-0.2583
15.	-3.1633	-2.6355	-2.3795	-1.6808	-1.3085	-1.0575	-0.8583	-0.6875	-0.5343	-0.3925	-0.2580
16.	-3.1584	-2.6314	-2.3759	-1.6784	-1.3067	-1.0561	-0.8572	-0.6867	-0.5336	-0.3920	-0.2577
17.	-3.1541	-2.6278	-2.3727	-1.6762	-1.3051	-1.0548	-0.8562	-0.6859	-0.5331	-0.3916	-0.2574
18.	-3.1503	-2.6247	-2.3699	-1.6744	-1.3037	-1.0537	-0.8553	-0.6852	-0.5325	-0.3912	-0.2572
19.	-3.1469	-2.6220	-2.3675	-1.6727	-1.3024	-1.0528	-0.8545	-0.6846	-0.5321	-0.3909	-0.2569
20.	-3.1439	-2.6195	-2.3652	-1.6712	-1.3013	-1.0519	-0.8539	-0.6841	-0.5317	-0.3906	-0.2568
21.	-3.1412	-2.6173	-2.3633	-1.6699	-1.3003	-1.0511	-0.8532	-0.6836	-0.5313	-0.3903	-0.2566
22.	-3.1387	-2.6153	-2.3615	-1.6686	-1.2994	-1.0504	-0.8527	-0.6831	-0.5310	-0.3901	-0.2564
23.	-3.1365	-2.6135	-2.3599	-1.6675	-1.2986	-1.0498	-0.8522	-0.6827	-0.5307	-0.3898	-0.2563
24.	-3.1345	-2.6118	-2.3584	-1.6665	-1.2978	-1.0492	-0.8517	-0.6824	-0.5304	-0.3896	-0.2562
25.	-3.1327	-2.6103	-2.3570	-1.6656	-1.2971	-1.0486	-0.8513	-0.6821	-0.5302	-0.3895	-0.2560
26.	-3.1310	-2.6089	-2.3558	-1.6648	-1.2965	-1.0481	-0.8509	-0.6817	-0.5299	-0.3893	-0.2559
27.	-3.1294	-2.6076	-2.3546	-1.6640	-1.2959	-1.0477	-0.8505	-0.6815	-0.5297	-0.3891	-0.2558
28.	-3.1280	-2.6064	-2.3536	-1.6633	-1.2954	-1.0473	-0.8502	-0.6812	-0.5295	-0.3890	-0.2557
29.	-3.1266	-2.6053	-2.3526	-1.6626	-1.2949	-1.0469	-0.8499	-0.6810	-0.5293	-0.3889	-0.2557
30.	-3.1254	-2.6043	-2.3517	-1.6620	-1.2944	-1.0465	-0.8496	-0.6807	-0.5292	-0.3887	-0.2556
35.	-3.1202	-2.6001	-2.3479	-1.6594	-1.2925	-1.0450	-0.8484	-0.6798	-0.5284	-0.3882	-0.2552
40.	-3.1164	-2.5970	-2.3451	-1.6576	-1.2911	-1.0439	-0.8475	-0.6791	-0.5279	-0.3879	-0.2550
45.	-3.1135	-2.5946	-2.3430	-1.6561	-1.2900	-1.0431	-0.8469	-0.6786	-0.5275	-0.3876	-0.2548
50.	-3.1112	-2.5927	-2.3412	-1.6550	-1.2891	-1.0424	-0.8463	-0.6782	-0.5272	-0.3873	-0.2547
60.	-3.1077	-2.5898	-2.3387	-1.6532	-1.2878	-1.0414	-0.8455	-0.6775	-0.5267	-0.3870	-0.2544
70.	-3.1053	-2.5878	-2.3369	-1.6520	-1.2869	-1.0406	-0.8449	-0.6771	-0.5264	-0.3868	-0.2543
80.	-3.1034	-2.5863	-2.3356	-1.6511	-1.2862	-1.0401	-0.8445	-0.6768	-0.5261	-0.3866	-0.2542
90.	-3.1020	-2.5851	-2.3345	-1.6504	-1.2857	-1.0397	-0.8442	-0.6765	-0.5259	-0.3864	-0.2541
100.	-3.1009	-2.5842	-2.3337	-1.6498	-1.2853	-1.0394	-0.8439	-0.6763	-0.5258	-0.3863	-0.2540
150.	-3.0975	-2.5815	-2.3312	-1.6482	-1.2840	-1.0384	-0.8432	-0.6757	-0.5253	-0.3860	-0.2538
200.	-3.0959	-2.5801	-2.3300	-1.6473	-1.2834	-1.0379	-0.8428	-0.6754	-0.5251	-0.3858	-0.2537
250.	-3.0949	-2.5793	-2.3293	-1.6468	-1.2830	-1.0376	-0.8425	-0.6752	-0.5249	-0.3857	-0.2536
300.	-3.0942	-2.5787	-2.3288	-1.6465	-1.2828	-1.0374	-0.8424	-0.6751	-0.5249	-0.3857	-0.2536
350.	-3.0937	-2.5783	-2.3284	-1.6463	-1.2826	-1.0373	-0.8423	-0.6750	-0.5248	-0.3856	-0.2535
400.	-3.0934	-2.5780	-2.3282	-1.6461	-1.2825	-1.0372	-0.8422	-0.6749	-0.5247	-0.3856	-0.2535
450.	-3.0931	-2.5778	-2.3280	-1.6459	-1.2824	-1.0371	-0.8421	-0.6749	-0.5247	-0.3855	-0.2535
500.	-3.0929	-2.5776	-2.3278	-1.6458	-1.2823	-1.0370	-0.8421	-0.6748	-0.5247	-0.3855	-0.2535
1000.	-3.0919	-2.5768	-2.3271	-1.6453	-1.2819	-1.0367	-0.8418	-0.6747	-0.5245	-0.3854	-0.2534

TABLE III(CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.5000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{x}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	0.	0.2695	0.5603	0.7221	0.9028	1.1144	1.2386	1.3817	1.4463	1.5166	1.5938
6.	0.	0.2666	0.5527	0.7121	0.8898	1.0977	1.2198	1.3605	1.4241	1.4931	1.5691
7.	0.	0.2643	0.5483	0.7058	0.8814	1.0868	1.2075	1.3466	1.4093	1.4774	1.5523
8.	0.	0.2628	0.5444	0.7010	0.8758	1.0799	1.1993	1.3369	1.3992	1.4668	1.5411
9.	0.	0.2615	0.5420	0.6977	0.8712	1.0739	1.1929	1.3299	1.3917	1.4588	1.5327
10.	0.	0.2606	0.5399	0.6950	0.8678	1.0697	1.1879	1.3241	1.3856	1.4525	1.5260
11.	0.	0.2598	0.5383	0.6929	0.8652	1.0662	1.1841	1.3197	1.3810	1.4476	1.5208
12.	0.	0.2592	0.5371	0.6912	0.8630	1.0635	1.1809	1.3162	1.3772	1.4436	1.5166
13.	0.	0.2588	0.5360	0.6898	0.8612	1.0611	1.1783	1.3132	1.3741	1.4403	1.5131
14.	0.	0.2583	0.5351	0.6886	0.8596	1.0592	1.1761	1.3107	1.3714	1.4375	1.5101
15.	0.	0.2580	0.5343	0.6875	0.8583	1.0575	1.1742	1.3085	1.3692	1.4351	1.5076
16.	0.	0.2577	0.5336	0.6867	0.8572	1.0561	1.1726	1.3067	1.3672	1.4330	1.5054
17.	0.	0.2574	0.5331	0.6859	0.8562	1.0548	1.1711	1.3051	1.3655	1.4312	1.5035
18.	0.	0.2572	0.5325	0.6852	0.8553	1.0537	1.1699	1.3037	1.3640	1.4297	1.5018
19.	0.	0.2569	0.5321	0.6846	0.8545	1.0528	1.1688	1.3024	1.3627	1.4283	1.5004
20.	0.	0.2568	0.5317	0.6841	0.8539	1.0519	1.1678	1.3013	1.3615	1.4270	1.4991
21.	0.	0.2566	0.5313	0.6836	0.8532	1.0511	1.1669	1.3003	1.3605	1.4259	1.4979
22.	0.	0.2564	0.5310	0.6831	0.8527	1.0504	1.1661	1.2994	1.3595	1.4249	1.4968
23.	0.	0.2563	0.5307	0.6827	0.8522	1.0498	1.1654	1.2986	1.3587	1.4240	1.4958
24.	0.	0.2562	0.5304	0.6824	0.8517	1.0492	1.1647	1.2978	1.3579	1.4232	1.4949
25.	0.	0.2560	0.5302	0.6821	0.8513	1.0486	1.1641	1.2971	1.3572	1.4224	1.4941
26.	0.	0.2559	0.5299	0.6817	0.8509	1.0481	1.1636	1.2965	1.3565	1.4217	1.4934
27.	0.	0.2558	0.5297	0.6815	0.8505	1.0477	1.1631	1.2959	1.3559	1.4210	1.4927
28.	0.	0.2557	0.5295	0.6812	0.8502	1.0473	1.1626	1.2954	1.3553	1.4204	1.4921
29.	0.	0.2557	0.5293	0.6810	0.8499	1.0469	1.1621	1.2949	1.3548	1.4199	1.4915
30.	0.	0.2556	0.5292	0.6807	0.8496	1.0465	1.1617	1.2944	1.3543	1.4194	1.4909
35.	0.	0.2552	0.5284	0.6798	0.8484	1.0450	1.1600	1.2925	1.3523	1.4172	1.4887
40.	0.	0.2550	0.5279	0.6791	0.8475	1.0439	1.1588	1.2911	1.3508	1.4157	1.4870
45.	0.	0.2548	0.5275	0.6786	0.8469	1.0431	1.1578	1.2900	1.3496	1.4145	1.4857
50.	0.	0.2547	0.5272	0.6782	0.8463	1.0424	1.1571	1.2891	1.3487	1.4135	1.4847
60.	0.	0.2544	0.5267	0.6775	0.8455	1.0414	1.1559	1.2878	1.3474	1.4121	1.4832
70.	0.	0.2543	0.5264	0.6771	0.8449	1.0406	1.1551	1.2869	1.3464	1.4110	1.4821
80.	0.	0.2542	0.5261	0.6768	0.8445	1.0401	1.1545	1.2862	1.3457	1.4103	1.4813
90.	0.	0.2541	0.5259	0.6765	0.8442	1.0397	1.1540	1.2857	1.3451	1.4097	1.4807
100.	0.	0.2540	0.5258	0.6763	0.8439	1.0394	1.1537	1.2853	1.3447	1.4092	1.4802
150.	0.	0.2538	0.5253	0.6757	0.8432	1.0384	1.1525	1.2840	1.3434	1.4078	1.4787
200.	0.	0.2537	0.5251	0.6754	0.8428	1.0379	1.1520	1.2834	1.3427	1.4071	1.4780
250.	0.	0.2536	0.5249	0.6752	0.8425	1.0376	1.1517	1.2830	1.3423	1.4067	1.4775
300.	0.	0.2536	0.5249	0.6751	0.8424	1.0374	1.1514	1.2828	1.3421	1.4064	1.4772
350.	0.	0.2535	0.5248	0.6750	0.8423	1.0373	1.1513	1.2826	1.3419	1.4062	1.4770
400.	0.	0.2535	0.5247	0.6749	0.8422	1.0372	1.1512	1.2825	1.3417	1.4061	1.4769
450.	0.	0.2535	0.5247	0.6749	0.8421	1.0371	1.1511	1.2824	1.3416	1.4060	1.4768
500.	0.	0.2535	0.5247	0.6748	0.8421	1.0370	1.1510	1.2823	1.3415	1.4059	1.4767
1000.	0.	0.2534	0.5245	0.6747	0.8418	1.0367	1.1507	1.2819	1.3411	1.4055	1.4762

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.5000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	1.6800	1.7784	1.8943	2.0370	2.2266	2.5254	2.7989	2.9874	3.3626	3.5842	4.0535
6.	1.6539	1.7505	1.8642	2.0039	2.1897	2.4824	2.7503	2.9352	3.3036	3.5211	3.9818
7.	1.6360	1.7314	1.8435	1.9817	2.1654	2.4547	2.7197	2.9023	3.2663	3.4812	3.9364
8.	1.6242	1.7188	1.8301	1.9669	2.1488	2.4354	2.6978	2.8787	3.2395	3.4525	3.9037
9.	1.6151	1.7091	1.8196	1.9555	2.1363	2.4212	2.6820	2.8617	3.2200	3.4317	3.8799
10.	1.6080	1.7016	1.8116	1.9469	2.1268	2.4103	2.6699	2.8488	3.2053	3.4159	3.8620
11.	1.6025	1.6958	1.8053	1.9401	2.1193	2.4016	2.6602	2.8383	3.1935	3.4032	3.8475
12.	1.5981	1.6910	1.8002	1.9345	2.1131	2.3945	2.6522	2.8299	3.1838	3.3929	3.8357
13.	1.5943	1.6870	1.7959	1.9299	2.1080	2.3887	2.6457	2.8228	3.1758	3.3844	3.8260
14.	1.5912	1.6837	1.7924	1.9260	2.1037	2.3837	2.6402	2.8169	3.1691	3.3771	3.8178
15.	1.5885	1.6808	1.7893	1.9227	2.1000	2.3795	2.6355	2.8118	3.1633	3.3710	3.8108
16.	1.5862	1.6784	1.7867	1.9198	2.0969	2.3759	2.6314	2.8075	3.1584	3.3657	3.8047
17.	1.5842	1.6762	1.7844	1.9173	2.0941	2.3727	2.6278	2.8037	3.1541	3.3610	3.7995
18.	1.5824	1.6744	1.7823	1.9151	2.0917	2.3699	2.6247	2.8003	3.1503	3.3570	3.7948
19.	1.5809	1.6727	1.7805	1.9132	2.0895	2.3675	2.6220	2.7974	3.1469	3.3534	3.7907
20.	1.5795	1.6712	1.7790	1.9114	2.0876	2.3652	2.6195	2.7947	3.1439	3.3502	3.7870
21.	1.5782	1.6699	1.7775	1.9099	2.0859	2.3633	2.6173	2.7924	3.1412	3.3473	3.7838
22.	1.5771	1.6686	1.7762	1.9085	2.0843	2.3615	2.6153	2.7902	3.1387	3.3447	3.7808
23.	1.5761	1.6675	1.7750	1.9072	2.0829	2.3599	2.6135	2.7883	3.1365	3.3423	3.7781
24.	1.5751	1.6665	1.7740	1.9060	2.0816	2.3584	2.6118	2.7865	3.1345	3.3401	3.7756
25.	1.5743	1.6656	1.7730	1.9050	2.0805	2.3570	2.6103	2.7849	3.1327	3.3381	3.7734
26.	1.5735	1.6648	1.7721	1.9040	2.0794	2.3558	2.6089	2.7834	3.1310	3.3363	3.7713
27.	1.5727	1.6640	1.7712	1.9031	2.0784	2.3546	2.6076	2.7820	3.1294	3.3347	3.7694
28.	1.5721	1.6633	1.7705	1.9022	2.0774	2.3536	2.6064	2.7807	3.1280	3.3331	3.7676
29.	1.5714	1.6626	1.7698	1.9015	2.0766	2.3526	2.6053	2.7795	3.1266	3.3317	3.7660
30.	1.5709	1.6620	1.7691	1.9007	2.0758	2.3517	2.6043	2.7784	3.1254	3.3303	3.7645
35.	1.5685	1.6594	1.7663	1.8978	2.0725	2.3479	2.6001	2.7739	3.1202	3.3248	3.7582
40.	1.5667	1.6576	1.7643	1.8956	2.0701	2.3451	2.5970	2.7706	3.1164	3.3208	3.7536
45.	1.5653	1.6561	1.7628	1.8939	2.0682	2.3430	2.5946	2.7680	3.1135	3.3176	3.7500
50.	1.5642	1.6550	1.7615	1.8925	2.0667	2.3412	2.5927	2.7660	3.1112	3.3151	3.7471
60.	1.5626	1.6532	1.7597	1.8905	2.0645	2.3387	2.5898	2.7629	3.1077	3.3114	3.7429
70.	1.5615	1.6520	1.7584	1.8891	2.0629	2.3369	2.5878	2.7607	3.1053	3.3088	3.7399
80.	1.5606	1.6511	1.7574	1.8881	2.0618	2.3356	2.5863	2.7591	3.1034	3.3069	3.7377
90.	1.5600	1.6504	1.7566	1.8872	2.0609	2.3345	2.5851	2.7579	3.1020	3.3053	3.7360
100.	1.5594	1.6498	1.7560	1.8866	2.0602	2.3337	2.5842	2.7569	3.1009	3.3041	3.7346
150.	1.5579	1.6482	1.7542	1.8846	2.0580	2.3312	2.5815	2.7539	3.0975	3.3005	3.7305
200.	1.5571	1.6473	1.7533	1.8837	2.0569	2.3300	2.5801	2.7525	3.0959	3.2987	3.7285
250.	1.5566	1.6468	1.7528	1.8831	2.0563	2.3293	2.5793	2.7516	3.0949	3.2977	3.7272
300.	1.5563	1.6465	1.7525	1.8827	2.0559	2.3288	2.5787	2.7510	3.0942	3.2970	3.7264
350.	1.5561	1.6463	1.7522	1.8824	2.0556	2.3284	2.5783	2.7506	3.0937	3.2965	3.7259
400.	1.5559	1.6461	1.7520	1.8822	2.0553	2.3282	2.5780	2.7503	3.0934	3.2961	3.7254
450.	1.5558	1.6459	1.7519	1.8821	2.0552	2.3280	2.5778	2.7500	3.0931	3.2958	3.7251
500.	1.5557	1.6458	1.7517	1.8819	2.0550	2.3278	2.5776	2.7498	3.0929	3.2956	3.7248
1000.	1.5552	1.6453	1.7512	1.8814	2.0544	2.3271	2.5768	2.7490	3.0919	3.2945	3.7236

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.6000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-3.0419	-2.5243	-2.2727	-1.5824	-1.2120	-0.9609	-0.7605	-0.5878	-0.4320	-0.2869	-0.1481
6.	-3.0226	-2.5093	-2.2602	-1.5767	-1.2095	-0.9609	-0.7625	-0.5917	-0.4375	-0.2939	-0.1570
7.	-3.0122	-2.5016	-2.2536	-1.5737	-1.2090	-0.9621	-0.7651	-0.5954	-0.4423	-0.2997	-0.1636
8.	-3.0060	-2.4971	-2.2501	-1.5730	-1.2098	-0.9640	-0.7679	-0.5989	-0.4464	-0.3044	-0.1690
9.	-3.0027	-2.4949	-2.2483	-1.5728	-1.2107	-0.9658	-0.7703	-0.6018	-0.4498	-0.3084	-0.1738
10.	-3.0011	-2.4941	-2.2479	-1.5736	-1.2122	-0.9676	-0.7725	-0.6044	-0.4528	-0.3119	-0.1775
11.	-3.0003	-2.4938	-2.2479	-1.5744	-1.2135	-0.9694	-0.7745	-0.6068	-0.4555	-0.3149	-0.1808
12.	-3.0001	-2.4939	-2.2482	-1.5753	-1.2149	-0.9710	-0.7764	-0.6089	-0.4579	-0.3175	-0.1837
13.	-3.0002	-2.4944	-2.2488	-1.5764	-1.2163	-0.9726	-0.7782	-0.6109	-0.4601	-0.3199	-0.1863
14.	-3.0007	-2.4950	-2.2495	-1.5774	-1.2176	-0.9741	-0.7799	-0.6127	-0.4621	-0.3220	-0.1886
15.	-3.0012	-2.4957	-2.2502	-1.5784	-1.2188	-0.9754	-0.7814	-0.6144	-0.4639	-0.3240	-0.1907
16.	-3.0019	-2.4965	-2.2511	-1.5795	-1.2200	-0.9768	-0.7828	-0.6159	-0.4655	-0.3257	-0.1926
17.	-3.0027	-2.4973	-2.2519	-1.5805	-1.2211	-0.9780	-0.7841	-0.6173	-0.4671	-0.3274	-0.1943
18.	-3.0035	-2.4981	-2.2528	-1.5815	-1.2222	-0.9791	-0.7854	-0.6186	-0.4684	-0.3288	-0.1959
19.	-3.0044	-2.4990	-2.2537	-1.5824	-1.2232	-0.9802	-0.7865	-0.6198	-0.4697	-0.3302	-0.1974
20.	-3.0052	-2.4998	-2.2545	-1.5833	-1.2242	-0.9812	-0.7876	-0.6210	-0.4709	-0.3315	-0.1987
21.	-3.0061	-2.5007	-2.2554	-1.5842	-1.2251	-0.9822	-0.7886	-0.6221	-0.4721	-0.3327	-0.2000
22.	-3.0070	-2.5015	-2.2562	-1.5851	-1.2260	-0.9831	-0.7896	-0.6231	-0.4731	-0.3338	-0.2012
23.	-3.0078	-2.5024	-2.2570	-1.5859	-1.2269	-0.9840	-0.7905	-0.6240	-0.4741	-0.3348	-0.2023
24.	-3.0087	-2.5032	-2.2578	-1.5867	-1.2277	-0.9849	-0.7914	-0.6249	-0.4751	-0.3358	-0.2033
25.	-3.0095	-2.5040	-2.2586	-1.5874	-1.2285	-0.9857	-0.7922	-0.6258	-0.4760	-0.3367	-0.2043
26.	-3.0103	-2.5047	-2.2593	-1.5882	-1.2292	-0.9864	-0.7930	-0.6266	-0.4768	-0.3376	-0.2052
27.	-3.0111	-2.5055	-2.2601	-1.5889	-1.2299	-0.9871	-0.7937	-0.6274	-0.4776	-0.3384	-0.2060
28.	-3.0119	-2.5062	-2.2608	-1.5895	-1.2306	-0.9878	-0.7944	-0.6281	-0.4783	-0.3392	-0.2069
29.	-3.0127	-2.5069	-2.2615	-1.5902	-1.2313	-0.9885	-0.7951	-0.6288	-0.4791	-0.3400	-0.2076
30.	-3.0134	-2.5076	-2.2621	-1.5908	-1.2319	-0.9891	-0.7958	-0.6295	-0.4798	-0.3407	-0.2084
35.	-3.0169	-2.5108	-2.2652	-1.5937	-1.2347	-0.9920	-0.7986	-0.6324	-0.4828	-0.3438	-0.2116
40.	-3.0200	-2.5136	-2.2679	-1.5961	-1.2371	-0.9943	-0.8010	-0.6348	-0.4852	-0.3463	-0.2142
45.	-3.0227	-2.5161	-2.2703	-1.5982	-1.2391	-0.9963	-0.8030	-0.6368	-0.4873	-0.3484	-0.2164
50.	-3.0251	-2.5183	-2.2724	-1.6001	-1.2409	-0.9981	-0.8047	-0.6386	-0.4890	-0.3502	-0.2182
60.	-3.0293	-2.5220	-2.2759	-1.6032	-1.2438	-1.0010	-0.8076	-0.6414	-0.4919	-0.3531	-0.2212
70.	-3.0327	-2.5251	-2.2788	-1.6057	-1.2462	-1.0033	-0.8098	-0.6436	-0.4942	-0.3554	-0.2235
80.	-3.0356	-2.5277	-2.2812	-1.6078	-1.2482	-1.0051	-0.8117	-0.6455	-0.4960	-0.3572	-0.2254
90.	-3.0381	-2.5299	-2.2833	-1.6096	-1.2498	-1.0067	-0.8132	-0.6470	-0.4975	-0.3588	-0.2269
100.	-3.0403	-2.5318	-2.2851	-1.6111	-1.2513	-1.0081	-0.8146	-0.6483	-0.4988	-0.3601	-0.2283
150.	-3.0481	-2.5387	-2.2916	-1.6166	-1.2562	-1.0129	-0.8192	-0.6529	-0.5033	-0.3646	-0.2328
200.	-3.0530	-2.5430	-2.2957	-1.6200	-1.2593	-1.0158	-0.8220	-0.6556	-0.5060	-0.3673	-0.2355
250.	-3.0566	-2.5461	-2.2986	-1.6224	-1.2615	-1.0178	-0.8240	-0.6575	-0.5079	-0.3691	-0.2373
300.	-3.0592	-2.5485	-2.3008	-1.6242	-1.2631	-1.0193	-0.8254	-0.6589	-0.5093	-0.3705	-0.2387
350.	-3.0613	-2.5503	-2.3025	-1.6256	-1.2644	-1.0205	-0.8266	-0.6600	-0.5104	-0.3716	-0.2398
400.	-3.0631	-2.5518	-2.3039	-1.6267	-1.2654	-1.0215	-0.8275	-0.6609	-0.5113	-0.3725	-0.2406
450.	-3.0645	-2.5531	-2.3051	-1.6277	-1.2663	-1.0223	-0.8283	-0.6617	-0.5120	-0.3732	-0.2414
500.	-3.0658	-2.5542	-2.3061	-1.6285	-1.2671	-1.0230	-0.8289	-0.6623	-0.5126	-0.3738	-0.2420
1000.	-3.0727	-2.5602	-2.3117	-1.6331	-1.2711	-1.0268	-0.8325	-0.6658	-0.5160	-0.3771	-0.2453

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.6000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	0.1208	0.3960	0.6979	0.8686	1.0603	1.2863	1.4198	1.5740	1.6438	1.7198	1.8034
6.	0.1090	0.3797	0.6759	0.8420	1.0289	1.2492	1.3790	1.5289	1.5968	1.6706	1.7518
7.	0.1001	0.3678	0.6597	0.8236	1.0076	1.2237	1.3510	1.4980	1.5646	1.6370	1.7168
8.	0.0929	0.3590	0.6481	0.8101	0.9918	1.2052	1.3307	1.4756	1.5412	1.6126	1.6913
9.	0.0872	0.3517	0.6387	0.7993	0.9794	1.1907	1.3149	1.4583	1.5233	1.5940	1.6718
10.	0.0825	0.3457	0.6310	0.7907	0.9695	1.1792	1.3025	1.4449	1.5093	1.5794	1.6565
11.	0.0785	0.3407	0.6248	0.7836	0.9614	1.1698	1.2924	1.4339	1.4978	1.5674	1.6440
12.	0.0749	0.3364	0.6194	0.7775	0.9546	1.1620	1.2839	1.4246	1.4882	1.5574	1.6335
13.	0.0719	0.3327	0.6148	0.7724	0.9487	1.1552	1.2766	1.4167	1.4800	1.5489	1.6247
14.	0.0691	0.3294	0.6108	0.7679	0.9436	1.1494	1.2703	1.4099	1.4729	1.5415	1.6170
15.	0.0667	0.3265	0.6073	0.7639	0.9391	1.1443	1.2648	1.4039	1.4667	1.5351	1.6103
16.	0.0645	0.3239	0.6041	0.7604	0.9352	1.1398	1.2600	1.3987	1.4613	1.5294	1.6044
17.	0.0625	0.3215	0.6012	0.7572	0.9316	1.1358	1.2557	1.3940	1.4564	1.5244	1.5992
18.	0.0607	0.3194	0.5987	0.7544	0.9285	1.1322	1.2518	1.3898	1.4521	1.5199	1.5945
19.	0.0590	0.3174	0.5964	0.7518	0.9256	1.1289	1.2483	1.3860	1.4482	1.5158	1.5903
20.	0.0575	0.3157	0.5942	0.7494	0.9230	1.1260	1.2451	1.3825	1.4446	1.5121	1.5864
21.	0.0560	0.3140	0.5923	0.7473	0.9205	1.1232	1.2422	1.3794	1.4414	1.5088	1.5829
22.	0.0547	0.3125	0.5905	0.7453	0.9183	1.1207	1.2395	1.3765	1.4384	1.5057	1.5797
23.	0.0535	0.3111	0.5888	0.7434	0.9163	1.1184	1.2370	1.3738	1.4356	1.5028	1.5767
24.	0.0523	0.3098	0.5872	0.7417	0.9144	1.1163	1.2347	1.3714	1.4331	1.5002	1.5740
25.	0.0513	0.3085	0.5858	0.7401	0.9126	1.1143	1.2326	1.3691	1.4307	1.4977	1.5715
26.	0.0502	0.3074	0.5844	0.7386	0.9110	1.1124	1.2306	1.3669	1.4285	1.4954	1.5691
27.	0.0493	0.3063	0.5831	0.7372	0.9094	1.1107	1.2288	1.3649	1.4264	1.4933	1.5669
28.	0.0484	0.3052	0.5819	0.7359	0.9079	1.1091	1.2270	1.3630	1.4245	1.4913	1.5648
29.	0.0475	0.3043	0.5808	0.7347	0.9066	1.1075	1.2254	1.3613	1.4227	1.4894	1.5628
30.	0.0467	0.3033	0.5797	0.7335	0.9053	1.1061	1.2238	1.3596	1.4210	1.4876	1.5610
35.	0.0432	0.2994	0.5751	0.7285	0.8998	1.0999	1.2173	1.3526	1.4137	1.4801	1.5532
40.	0.0403	0.2962	0.5715	0.7246	0.8954	1.0951	1.2121	1.3471	1.4080	1.4742	1.5471
45.	0.0380	0.2936	0.5685	0.7213	0.8919	1.0912	1.2080	1.3426	1.4034	1.4695	1.5422
50.	0.0360	0.2915	0.5661	0.7187	0.8890	1.0880	1.2045	1.3389	1.3996	1.4656	1.5382
60.	0.0329	0.2880	0.5621	0.7144	0.8844	1.0828	1.1991	1.3331	1.3937	1.4594	1.5318
70.	0.0304	0.2853	0.5591	0.7112	0.8808	1.0790	1.1950	1.3287	1.3891	1.4548	1.5270
80.	0.0284	0.2832	0.5567	0.7086	0.8780	1.0759	1.1917	1.3253	1.3856	1.4511	1.5232
90.	0.0268	0.2814	0.5547	0.7065	0.8758	1.0734	1.1891	1.3225	1.3827	1.4481	1.5201
100.	0.0254	0.2799	0.5531	0.7047	0.8738	1.0713	1.1869	1.3201	1.3802	1.4456	1.5175
150.	0.0207	0.2749	0.5476	0.6988	0.8675	1.0643	1.1795	1.3123	1.3723	1.4374	1.5090
200.	0.0179	0.2719	0.5443	0.6954	0.8638	1.0603	1.1753	1.3079	1.3677	1.4327	1.5042
250.	0.0160	0.2699	0.5421	0.6931	0.8613	1.0576	1.1725	1.3049	1.3646	1.4295	1.5009
300.	0.0146	0.2685	0.5405	0.6914	0.8595	1.0557	1.1705	1.3027	1.3624	1.4272	1.4986
350.	0.0136	0.2673	0.5393	0.6901	0.8581	1.0542	1.1689	1.3010	1.3607	1.4255	1.4968
400.	0.0127	0.2664	0.5383	0.6890	0.8570	1.0530	1.1676	1.2997	1.3593	1.4241	1.4953
450.	0.0120	0.2657	0.5375	0.6882	0.8561	1.0520	1.1666	1.2986	1.3582	1.4229	1.4941
500.	0.0113	0.2650	0.5368	0.6875	0.8553	1.0511	1.1657	1.2977	1.3572	1.4220	1.4931
1000.	0.0080	0.2616	0.5331	0.6836	0.8512	1.0467	1.1610	1.2928	1.3522	1.4168	1.4879

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.6000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	1.8969	2.0037	2.1297	2.2850	2.4918	2.8183	3.1177	3.3243	3.7361	3.9795	4.4954
6.	1.8426	1.9463	2.0684	2.2190	2.4195	2.7359	3.0261	3.2264	3.6258	3.8619	4.3622
7.	1.8059	1.9077	2.0274	2.1749	2.3713	2.6812	2.9655	3.1617	3.5531	3.7844	4.2747
8.	1.7792	1.8795	1.9976	2.1430	2.3366	2.6421	2.9223	3.1156	3.5011	3.7290	4.2119
9.	1.7587	1.8580	1.9748	2.1186	2.3101	2.6122	2.8892	3.0804	3.4615	3.6868	4.1643
10.	1.7427	1.8411	1.9568	2.0993	2.2891	2.5886	2.8632	3.0526	3.4304	3.6537	4.1269
11.	1.7296	1.8273	1.9422	2.0837	2.2722	2.5695	2.8421	3.0302	3.4052	3.6268	4.0966
12.	1.7186	1.8158	1.9301	2.0708	2.2581	2.5536	2.8246	3.0116	3.3843	3.6046	4.0715
13.	1.7094	1.8061	1.9198	2.0598	2.2461	2.5402	2.8098	2.9958	3.3667	3.5859	4.0504
14.	1.7014	1.7977	1.9109	2.0503	2.2359	2.5287	2.7972	2.9823	3.3515	3.5698	4.0323
15.	1.6944	1.7903	1.9032	2.0421	2.2270	2.5187	2.7861	2.9706	3.3384	3.5558	4.0165
16.	1.6882	1.7839	1.8964	2.0348	2.2191	2.5099	2.7764	2.9603	3.3269	3.5436	4.0027
17.	1.6828	1.7782	1.8903	2.0284	2.2121	2.5021	2.7679	2.9512	3.3167	3.5327	3.9905
18.	1.6779	1.7730	1.8849	2.0226	2.2059	2.4951	2.7602	2.9430	3.3075	3.5230	3.9796
19.	1.6735	1.7684	1.8801	2.0174	2.2003	2.4888	2.7533	2.9357	3.2994	3.5143	3.9698
20.	1.6695	1.7642	1.8756	2.0127	2.1952	2.4831	2.7470	2.9291	3.2919	3.5064	3.9609
21.	1.6658	1.7604	1.8716	2.0084	2.1906	2.4779	2.7414	2.9230	3.2852	3.4992	3.9528
22.	1.6625	1.7569	1.8679	2.0045	2.1864	2.4732	2.7361	2.9175	3.2790	3.4927	3.9454
23.	1.6594	1.7537	1.8645	2.0009	2.1825	2.4688	2.7314	2.9124	3.2733	3.4866	3.9387
24.	1.6565	1.7507	1.8614	1.9976	2.1788	2.4648	2.7269	2.9077	3.2681	3.4811	3.9324
25.	1.6539	1.7479	1.8584	1.9945	2.1755	2.4611	2.7228	2.9034	3.2632	3.4759	3.9266
26.	1.6514	1.7453	1.8557	1.9916	2.1724	2.4576	2.7190	2.8993	3.2587	3.4711	3.9212
27.	1.6491	1.7429	1.8532	1.9889	2.1695	2.4543	2.7155	2.8955	3.2545	3.4666	3.9162
28.	1.6469	1.7406	1.8508	1.9864	2.1668	2.4513	2.7121	2.8920	3.2505	3.4625	3.9115
29.	1.6449	1.7385	1.8486	1.9840	2.1642	2.4485	2.7090	2.8887	3.2469	3.4585	3.9071
30.	1.6430	1.7365	1.8465	1.9818	2.1618	2.4458	2.7061	2.8856	3.2434	3.4549	3.9030
35.	1.6349	1.7280	1.8376	1.9723	2.1517	2.4345	2.6937	2.8725	3.2287	3.4393	3.8855
40.	1.6285	1.7214	1.8307	1.9650	2.1438	2.4257	2.6841	2.8623	3.2174	3.4273	3.8720
45.	1.6234	1.7161	1.8251	1.9591	2.1374	2.4187	2.6764	2.8541	3.2083	3.4176	3.8612
50.	1.6192	1.7118	1.8205	1.9542	2.1322	2.4128	2.6700	2.8474	3.2008	3.4097	3.8523
60.	1.6126	1.7049	1.8133	1.9466	2.1240	2.4038	2.6601	2.8369	3.1891	3.3973	3.8384
70.	1.6076	1.6997	1.8078	1.9409	2.1179	2.3969	2.6526	2.8290	3.1803	3.3880	3.8280
80.	1.6037	1.6956	1.8036	1.9364	2.1130	2.3915	2.6468	2.8228	3.1735	3.3807	3.8198
90.	1.6005	1.6922	1.8001	1.9327	2.1091	2.3872	2.6420	2.8177	3.1679	3.3748	3.8132
100.	1.5978	1.6895	1.7972	1.9296	2.1058	2.3836	2.6381	2.8136	3.1633	3.3699	3.8077
150.	1.5891	1.6804	1.7877	1.9196	2.0951	2.3717	2.6252	2.7999	3.1482	3.3539	3.7898
200.	1.5840	1.6752	1.7822	1.9139	2.0890	2.3650	2.6178	2.7922	3.1396	3.3448	3.7797
250.	1.5807	1.6717	1.7786	1.9101	2.0849	2.3605	2.6130	2.7871	3.1339	3.3388	3.7729
300.	1.5783	1.6692	1.7760	1.9073	2.0819	2.3572	2.6095	2.7833	3.1298	3.3345	3.7681
350.	1.5764	1.6672	1.7739	1.9052	2.0797	2.3548	2.6068	2.7805	3.1266	3.3311	3.7644
400.	1.5749	1.6657	1.7723	1.9035	2.0779	2.3528	2.6046	2.7782	3.1241	3.3285	3.7614
450.	1.5737	1.6644	1.7710	1.9021	2.0764	2.3511	2.6028	2.7764	3.1221	3.3263	3.7590
500.	1.5727	1.6633	1.7699	1.9009	2.0752	2.3498	2.6014	2.7748	3.1203	3.3245	3.7570
1000.	1.5672	1.6577	1.7640	1.8948	2.0686	2.3426	2.5936	2.7666	3.1113	3.3149	3.7463

TABLE III(CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.7000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-2.7461	-2.2702	-2.0382	-1.3980	-1.0507	-0.8134	-0.6217	-0.4544	-0.3021	-0.1587	-0.0200
6.	-2.7582	-2.2818	-2.0498	-1.4103	-1.0638	-0.8276	-0.6370	-0.4712	-0.3201	-0.1779	-0.0413
7.	-2.7707	-2.2936	-2.0613	-1.4214	-1.0751	-0.8395	-0.6492	-0.4841	-0.3340	-0.1929	-0.0576
8.	-2.7823	-2.3042	-2.0715	-1.4310	-1.0854	-0.8493	-0.6597	-0.4951	-0.3453	-0.2051	-0.0704
9.	-2.7933	-2.3144	-2.0812	-1.4399	-1.0939	-0.8579	-0.6683	-0.5039	-0.3548	-0.2151	-0.0808
10.	-2.8029	-2.3230	-2.0895	-1.4475	-1.1012	-0.8654	-0.6760	-0.5119	-0.3629	-0.2235	-0.0896
11.	-2.8118	-2.3311	-2.0972	-1.4544	-1.1079	-0.8720	-0.6826	-0.5186	-0.3699	-0.2307	-0.0972
12.	-2.8200	-2.3384	-2.1042	-1.4606	-1.1138	-0.8778	-0.6885	-0.5246	-0.3760	-0.2370	-0.1038
13.	-2.8275	-2.3451	-2.1105	-1.4662	-1.1192	-0.8831	-0.6937	-0.5299	-0.3814	-0.2426	-0.1097
14.	-2.8344	-2.3513	-2.1164	-1.4713	-1.1240	-0.8878	-0.6984	-0.5347	-0.3863	-0.2476	-0.1149
15.	-2.8407	-2.3570	-2.1217	-1.4760	-1.1285	-0.8921	-0.7027	-0.5390	-0.3907	-0.2522	-0.1195
16.	-2.8466	-2.3622	-2.1267	-1.4803	-1.1325	-0.8961	-0.7067	-0.5429	-0.3947	-0.2563	-0.1238
17.	-2.8521	-2.3671	-2.1313	-1.4843	-1.1363	-0.8997	-0.7103	-0.5465	-0.3984	-0.2600	-0.1276
18.	-2.8572	-2.3716	-2.1356	-1.4879	-1.1397	-0.9031	-0.7136	-0.5498	-0.4017	-0.2634	-0.1311
19.	-2.8619	-2.3759	-2.1396	-1.4913	-1.1429	-0.9061	-0.7166	-0.5529	-0.4048	-0.2665	-0.1343
20.	-2.8664	-2.3798	-2.1433	-1.4945	-1.1459	-0.9090	-0.7194	-0.5557	-0.4076	-0.2694	-0.1373
21.	-2.8706	-2.3836	-2.1468	-1.4975	-1.1487	-0.9117	-0.7221	-0.5583	-0.4103	-0.2721	-0.1400
22.	-2.8746	-2.3871	-2.1501	-1.5004	-1.1513	-0.9142	-0.7246	-0.5608	-0.4127	-0.2746	-0.1426
23.	-2.8784	-2.3904	-2.1533	-1.5030	-1.1538	-0.9166	-0.7269	-0.5631	-0.4151	-0.2770	-0.1450
24.	-2.8819	-2.3935	-2.1562	-1.5055	-1.1561	-0.9188	-0.7291	-0.5653	-0.4172	-0.2792	-0.1473
25.	-2.8853	-2.3965	-2.1590	-1.5079	-1.1583	-0.9210	-0.7311	-0.5673	-0.4193	-0.2812	-0.1494
26.	-2.8885	-2.3994	-2.1617	-1.5102	-1.1604	-0.9229	-0.7331	-0.5693	-0.4212	-0.2832	-0.1514
27.	-2.8915	-2.4021	-2.1642	-1.5123	-1.1623	-0.9248	-0.7349	-0.5711	-0.4231	-0.2850	-0.1533
28.	-2.8944	-2.4046	-2.1666	-1.5143	-1.1642	-0.9266	-0.7367	-0.5728	-0.4248	-0.2868	-0.1550
29.	-2.8972	-2.4071	-2.1689	-1.5163	-1.1660	-0.9284	-0.7384	-0.5745	-0.4264	-0.2885	-0.1567
30.	-2.8999	-2.4094	-2.1711	-1.5181	-1.1677	-0.9300	-0.7400	-0.5760	-0.4280	-0.2900	-0.1583
35.	-2.9116	-2.4198	-2.1809	-1.5263	-1.1752	-0.9371	-0.7469	-0.5829	-0.4348	-0.2969	-0.1653
40.	-2.9214	-2.4284	-2.1889	-1.5330	-1.1814	-0.9430	-0.7526	-0.5885	-0.4404	-0.3024	-0.1709
45.	-2.9296	-2.4356	-2.1957	-1.5386	-1.1865	-0.9479	-0.7573	-0.5931	-0.4450	-0.3070	-0.1755
50.	-2.9367	-2.4418	-2.2015	-1.5435	-1.1909	-0.9520	-0.7613	-0.5970	-0.4489	-0.3109	-0.1795
60.	-2.9484	-2.4521	-2.2111	-1.5514	-1.1981	-0.9588	-0.7679	-0.6034	-0.4552	-0.3173	-0.1858
70.	-2.9577	-2.4602	-2.2187	-1.5577	-1.2038	-0.9642	-0.7730	-0.6085	-0.4601	-0.3222	-0.1908
80.	-2.9654	-2.4669	-2.2249	-1.5628	-1.2085	-0.9686	-0.7772	-0.6125	-0.4642	-0.3262	-0.1948
90.	-2.9718	-2.4725	-2.2301	-1.5671	-1.2123	-0.9722	-0.7807	-0.6159	-0.4675	-0.3295	-0.1981
100.	-2.9773	-2.4773	-2.2346	-1.5708	-1.2157	-0.9753	-0.7837	-0.6188	-0.4703	-0.3323	-0.2009
150.	-2.9964	-2.4940	-2.2502	-1.5835	-1.2271	-0.9860	-0.7939	-0.6287	-0.4800	-0.3418	-0.2104
200.	-3.0081	-2.5042	-2.2597	-1.5912	-1.2340	-0.9924	-0.8001	-0.6347	-0.4858	-0.3476	-0.2161
250.	-3.0163	-2.5113	-2.2663	-1.5966	-1.2388	-0.9969	-0.8043	-0.6387	-0.4898	-0.3515	-0.2200
300.	-3.0224	-2.5166	-2.2712	-1.6006	-1.2424	-1.0002	-0.8075	-0.6418	-0.4927	-0.3544	-0.2229
350.	-3.0272	-2.5208	-2.2751	-1.6037	-1.2452	-1.0028	-0.8099	-0.6442	-0.4951	-0.3567	-0.2251
400.	-3.0311	-2.5242	-2.2783	-1.6063	-1.2474	-1.0049	-0.8119	-0.6461	-0.4969	-0.3585	-0.2269
450.	-3.0344	-2.5270	-2.2809	-1.6084	-1.2493	-1.0067	-0.8136	-0.6477	-0.4985	-0.3600	-0.2284
500.	-3.0371	-2.5294	-2.2831	-1.6102	-1.2509	-1.0082	-0.8150	-0.6490	-0.4998	-0.3613	-0.2297
1000.	-3.0524	-2.5426	-2.2954	-1.6201	-1.2597	-1.0163	-0.8226	-0.6564	-0.5069	-0.3683	-0.2366

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.7000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	0.2545	0.5405	0.8609	1.0438	1.2512	1.4973	1.6431	1.8122	1.8898	1.9724	2.0644
6.	0.2284	0.5082	0.8185	0.9948	1.1947	1.4315	1.5712	1.7331	1.8066	1.8867	1.9749
7.	0.2090	0.4845	0.7887	0.9610	1.1558	1.3861	1.5222	1.6799	1.7512	1.8289	1.9145
8.	0.1940	0.4664	0.7661	0.9359	1.1274	1.3531	1.4860	1.6400	1.7098	1.7859	1.8698
9.	0.1820	0.4516	0.7484	0.9158	1.1045	1.3270	1.4583	1.6103	1.6791	1.7540	1.8366
10.	0.1719	0.4400	0.7341	0.8998	1.0865	1.3065	1.4361	1.5862	1.6542	1.7282	1.8097
11.	0.1633	0.4300	0.7221	0.8866	1.0716	1.2895	1.4180	1.5666	1.6338	1.7071	1.7878
12.	0.1559	0.4215	0.7119	0.8753	1.0591	1.2753	1.4027	1.5501	1.6168	1.6894	1.7694
13.	0.1494	0.4140	0.7031	0.8656	1.0483	1.2631	1.3896	1.5360	1.6022	1.6743	1.7538
14.	0.1437	0.4075	0.6954	0.8571	1.0389	1.2525	1.3783	1.5238	1.5896	1.6613	1.7402
15.	0.1386	0.4017	0.6886	0.8497	1.0306	1.2432	1.3684	1.5131	1.5786	1.6499	1.7284
16.	0.1339	0.3964	0.6825	0.8430	1.0232	1.2349	1.3596	1.5037	1.5688	1.6398	1.7179
17.	0.1298	0.3917	0.6770	0.8370	1.0167	1.2276	1.3517	1.4952	1.5601	1.6308	1.7086
18.	0.1260	0.3875	0.6721	0.8317	1.0107	1.2209	1.3447	1.4876	1.5523	1.6227	1.7002
19.	0.1225	0.3836	0.6676	0.8268	1.0053	1.2149	1.3383	1.4808	1.5452	1.6154	1.6926
20.	0.1193	0.3800	0.6635	0.8223	1.0004	1.2095	1.3324	1.4745	1.5388	1.6087	1.6857
21.	0.1163	0.3767	0.6597	0.8182	0.9959	1.2044	1.3271	1.4688	1.5329	1.6026	1.6794
22.	0.1136	0.3737	0.6562	0.8144	0.9917	1.1998	1.3222	1.4635	1.5274	1.5970	1.6736
23.	0.1110	0.3708	0.6530	0.8109	0.9879	1.1955	1.3176	1.4586	1.5224	1.5918	1.6682
24.	0.1086	0.3682	0.6499	0.8076	0.9843	1.1915	1.3134	1.4541	1.5178	1.5870	1.6632
25.	0.1063	0.3657	0.6471	0.8045	0.9810	1.1878	1.3095	1.4499	1.5134	1.5825	1.6586
26.	0.1042	0.3633	0.6445	0.8017	0.9778	1.1844	1.3058	1.4460	1.5094	1.5784	1.6543
27.	0.1022	0.3611	0.6420	0.7990	0.9749	1.1811	1.3023	1.4423	1.5056	1.5745	1.6503
28.	0.1003	0.3591	0.6396	0.7964	0.9721	1.1781	1.2991	1.4389	1.5021	1.5708	1.6465
29.	0.0985	0.3571	0.6374	0.7941	0.9695	1.1752	1.2961	1.4356	1.4987	1.5674	1.6429
30.	0.0968	0.3553	0.6353	0.7918	0.9671	1.1725	1.2932	1.4326	1.4956	1.5641	1.6396
35.	0.0895	0.3473	0.6263	0.7821	0.9566	1.1609	1.2809	1.4195	1.4822	1.5503	1.6253
40.	0.0836	0.3409	0.6192	0.7745	0.9483	1.1518	1.2713	1.4093	1.4716	1.5394	1.6141
45.	0.0788	0.3357	0.6134	0.7683	0.9415	1.1444	1.2635	1.4010	1.4631	1.5306	1.6050
50.	0.0747	0.3313	0.6085	0.7630	0.9359	1.1383	1.2570	1.3940	1.4560	1.5233	1.5974
60.	0.0681	0.3242	0.6007	0.7547	0.9270	1.1285	1.2467	1.3831	1.4447	1.5118	1.5855
70.	0.0630	0.3188	0.5947	0.7484	0.9201	1.1210	1.2388	1.3748	1.4362	1.5030	1.5764
80.	0.0589	0.3144	0.5899	0.7433	0.9147	1.1151	1.2326	1.3682	1.4294	1.4960	1.5692
90.	0.0555	0.3109	0.5860	0.7391	0.9102	1.1102	1.2275	1.3628	1.4239	1.4903	1.5634
100.	0.0526	0.3078	0.5827	0.7356	0.9064	1.1062	1.2232	1.3582	1.4192	1.4855	1.5585
150.	0.0429	0.2976	0.5716	0.7239	0.8939	1.0926	1.2090	1.3432	1.4038	1.4697	1.5422
200.	0.0371	0.2916	0.5651	0.7170	0.8866	1.0847	1.2007	1.3345	1.3949	1.4605	1.5327
250.	0.0332	0.2875	0.5607	0.7124	0.8817	1.0794	1.1951	1.3286	1.3888	1.4543	1.5264
300.	0.0303	0.2845	0.5574	0.7090	0.8781	1.0755	1.1911	1.3243	1.3845	1.4498	1.5217
350.	0.0281	0.2821	0.5549	0.7064	0.8753	1.0725	1.1879	1.3210	1.3811	1.4464	1.5182
400.	0.0263	0.2803	0.5529	0.7043	0.8730	1.0701	1.1854	1.3183	1.3784	1.4436	1.5153
450.	0.0247	0.2787	0.5513	0.7025	0.8712	1.0681	1.1833	1.3162	1.3761	1.4413	1.5130
500.	0.0235	0.2774	0.5499	0.7010	0.8696	1.0664	1.1816	1.3143	1.3742	1.4394	1.5110
1000.	0.0166	0.2703	0.5423	0.6931	0.8613	1.0574	1.1722	1.3045	1.3642	1.4291	1.5004

TABLE III(CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.7000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	2.1676	2.2855	2.4245	2.5958	2.8243	3.1857	3.5178	3.7475	4.2060	4.4773	5.0526
6.	2.0737	2.1867	2.3197	2.4838	2.7025	3.0483	3.3660	3.5856	4.0238	4.2831	4.8330
7.	2.0103	2.1198	2.2488	2.4082	2.6207	2.9566	3.2651	3.4781	3.9033	4.1548	4.6883
8.	1.9637	2.0710	2.1974	2.3533	2.5611	2.8895	3.1912	3.3995	3.8154	4.0614	4.5832
9.	1.9289	2.0345	2.1588	2.3121	2.5164	2.8393	3.1359	3.3407	3.7494	3.9912	4.5041
10.	1.9009	2.0052	2.1279	2.2792	2.4808	2.7995	3.0921	3.2941	3.6974	3.9359	4.4417
11.	1.8781	1.9812	2.1027	2.2523	2.4518	2.7670	3.0564	3.2562	3.6549	3.8908	4.3909
12.	1.8589	1.9612	2.0815	2.2299	2.4275	2.7399	3.0266	3.2246	3.6196	3.8533	4.3487
13.	1.8426	1.9441	2.0636	2.2108	2.4069	2.7169	3.0014	3.1978	3.5897	3.8215	4.3130
14.	1.8285	1.9294	2.0481	2.1943	2.3892	2.6971	2.9797	3.1748	3.5640	3.7942	4.2824
15.	1.8162	1.9165	2.0345	2.1799	2.3737	2.6798	2.9608	3.1547	3.5416	3.7705	4.2557
16.	1.8053	1.9051	2.0226	2.1673	2.3601	2.6646	2.9441	3.1370	3.5219	3.7495	4.2322
17.	1.7956	1.8950	2.0119	2.1560	2.3479	2.6510	2.9293	3.1213	3.5044	3.7310	4.2114
18.	1.7869	1.8859	2.0024	2.1458	2.3370	2.6389	2.9160	3.1072	3.4887	3.7143	4.1927
19.	1.7790	1.8776	1.9937	2.1367	2.3272	2.6280	2.9040	3.0945	3.4745	3.6993	4.1758
20.	1.7718	1.8702	1.9859	2.1284	2.3182	2.6180	2.8931	3.0830	3.4617	3.6857	4.1605
21.	1.7652	1.8633	1.9787	2.1208	2.3100	2.6089	2.8832	3.0725	3.4500	3.6732	4.1466
22.	1.7592	1.8570	1.9721	2.1138	2.3025	2.6006	2.8741	3.0628	3.4392	3.6619	4.1338
23.	1.7537	1.8512	1.9660	2.1074	2.2956	2.5929	2.8657	3.0539	3.4293	3.6514	4.1220
24.	1.7485	1.8458	1.9604	2.1014	2.2892	2.5858	2.8579	3.0457	3.4202	3.6417	4.1112
25.	1.7437	1.8408	1.9551	2.0958	2.2833	2.5792	2.8507	3.0380	3.4117	3.6327	4.1011
26.	1.7392	1.8362	1.9502	2.0907	2.2777	2.5730	2.8440	3.0309	3.4038	3.6243	4.0917
27.	1.7350	1.8318	1.9457	2.0859	2.2725	2.5673	2.8377	3.0243	3.3964	3.6164	4.0829
28.	1.7311	1.8277	1.9414	2.0813	2.2677	2.5619	2.8318	3.0180	3.3895	3.6091	4.0747
29.	1.7274	1.8239	1.9374	2.0771	2.2631	2.5568	2.8263	3.0122	3.3830	3.6022	4.0670
30.	1.7239	1.8203	1.9336	2.0731	2.2588	2.5521	2.8211	3.0067	3.3769	3.5957	4.0597
35.	1.7091	1.8048	1.9174	2.0560	2.2405	2.5318	2.7990	2.9833	3.3509	3.5683	4.0290
40.	1.6975	1.7927	1.9048	2.0427	2.2262	2.5160	2.7817	2.9651	3.3307	3.5468	4.0050
45.	1.6881	1.7830	1.8945	2.0319	2.2147	2.5032	2.7678	2.9504	3.3144	3.5296	3.9857
50.	1.6803	1.7748	1.8860	2.0229	2.2051	2.4926	2.7563	2.9382	3.3008	3.5152	3.9697
60.	1.6679	1.7620	1.8727	2.0088	2.1900	2.4760	2.7382	2.9190	3.2796	3.4928	3.9446
70.	1.6586	1.7523	1.8625	1.9981	2.1786	2.4633	2.7244	2.9045	3.2636	3.4758	3.9256
80.	1.6511	1.7446	1.8544	1.9896	2.1695	2.4534	2.7136	2.8931	3.2509	3.4624	3.9106
90.	1.6451	1.7383	1.8479	1.9827	2.1622	2.4452	2.7048	2.8838	3.2406	3.4515	3.8984
100.	1.6400	1.7330	1.8424	1.9770	2.1560	2.4385	2.6974	2.8760	3.2320	3.4424	3.8883
150.	1.6232	1.7156	1.8242	1.9578	2.1356	2.4161	2.6731	2.8503	3.2036	3.4124	3.8549
200.	1.6134	1.7055	1.8137	1.9468	2.1239	2.4031	2.6591	2.8356	3.1873	3.3952	3.8356
250.	1.6069	1.6987	1.8066	1.9394	2.1160	2.3945	2.6497	2.8257	3.1764	3.3836	3.8228
300.	1.6021	1.6938	1.8015	1.9340	2.1103	2.3882	2.6429	2.8185	3.1685	3.3753	3.8134
350.	1.5984	1.6899	1.7975	1.9299	2.1058	2.3833	2.6376	2.8130	3.1624	3.3688	3.8063
400.	1.5955	1.6869	1.7944	1.9265	2.1023	2.3795	2.6334	2.8085	3.1575	3.3637	3.8005
450.	1.5930	1.6844	1.7918	1.9238	2.0994	2.3763	2.6300	2.8049	3.1535	3.3594	3.7958
500.	1.5910	1.6823	1.7896	1.9215	2.0970	2.3736	2.6271	2.8019	3.1501	3.3559	3.7919
1000.	1.5801	1.6710	1.7778	1.9092	2.0839	2.3593	2.6116	2.7856	3.1322	3.3370	3.7708

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.8000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-2.4489	-2.0137	-1.8009	-1.2091	-0.8828	-0.6561	-0.4705	-0.3065	-0.1550	-0.0094	0.1328
6.	-2.4888	-2.0492	-1.8343	-1.2378	-0.9105	-0.6838	-0.4992	-0.3367	-0.1867	-0.0436	0.0959
7.	-2.5214	-2.0781	-1.8616	-1.2616	-0.9334	-0.7065	-0.5220	-0.3600	-0.2109	-0.0697	0.0681
8.	-2.5489	-2.1024	-1.8843	-1.2809	-0.9515	-0.7246	-0.5403	-0.3787	-0.2306	-0.0904	0.0462
9.	-2.5729	-2.1236	-1.9043	-1.2980	-0.9676	-0.7401	-0.5559	-0.3947	-0.2471	-0.1073	0.0283
10.	-2.5936	-2.1419	-1.9215	-1.3126	-0.9812	-0.7534	-0.5690	-0.4080	-0.2606	-0.1215	0.0131
11.	-2.6119	-2.1580	-1.9366	-1.3254	-0.9931	-0.7649	-0.5804	-0.4195	-0.2724	-0.1336	0.0005
12.	-2.6281	-2.1723	-1.9500	-1.3367	-1.0036	-0.7751	-0.5905	-0.4296	-0.2826	-0.1442	-0.0105
13.	-2.6427	-2.1851	-1.9620	-1.3468	-1.0130	-0.7841	-0.5994	-0.4385	-0.2917	-0.1535	-0.0202
14.	-2.6558	-2.1967	-1.9729	-1.3560	-1.0215	-0.7923	-0.6075	-0.4466	-0.2999	-0.1619	-0.0288
15.	-2.6679	-2.2072	-1.9828	-1.3643	-1.0291	-0.7997	-0.6147	-0.4538	-0.3072	-0.1693	-0.0366
16.	-2.6789	-2.2169	-1.9919	-1.3719	-1.0361	-0.8064	-0.6213	-0.4604	-0.3138	-0.1761	-0.0435
17.	-2.6890	-2.2258	-2.0002	-1.3789	-1.0426	-0.8126	-0.6274	-0.4664	-0.3199	-0.1823	-0.0499
18.	-2.6984	-2.2341	-2.0079	-1.3853	-1.0485	-0.8182	-0.6329	-0.4719	-0.3254	-0.1879	-0.0556
19.	-2.7071	-2.2417	-2.0151	-1.3913	-1.0540	-0.8235	-0.6380	-0.4769	-0.3305	-0.1930	-0.0609
20.	-2.7152	-2.2488	-2.0217	-1.3968	-1.0591	-0.8283	-0.6427	-0.4816	-0.3352	-0.1978	-0.0658
21.	-2.7228	-2.2555	-2.0280	-1.4020	-1.0638	-0.8329	-0.6472	-0.4860	-0.3395	-0.2022	-0.0703
22.	-2.7299	-2.2617	-2.0338	-1.4069	-1.0682	-0.8371	-0.6513	-0.4900	-0.3436	-0.2063	-0.0745
23.	-2.7366	-2.2676	-2.0393	-1.4115	-1.0724	-0.8411	-0.6551	-0.4939	-0.3474	-0.2102	-0.0784
24.	-2.7429	-2.2731	-2.0445	-1.4158	-1.0764	-0.8448	-0.6588	-0.4974	-0.3510	-0.2138	-0.0821
25.	-2.7489	-2.2784	-2.0494	-1.4198	-1.0801	-0.8483	-0.6622	-0.5008	-0.3544	-0.2172	-0.0855
26.	-2.7546	-2.2833	-2.0540	-1.4237	-1.0836	-0.8517	-0.6654	-0.5040	-0.3575	-0.2204	-0.0888
27.	-2.7599	-2.2880	-2.0584	-1.4273	-1.0869	-0.8548	-0.6685	-0.5070	-0.3605	-0.2234	-0.0919
28.	-2.7650	-2.2925	-2.0626	-1.4308	-1.0901	-0.8578	-0.6714	-0.5099	-0.3634	-0.2262	-0.0948
29.	-2.7699	-2.2968	-2.0666	-1.4341	-1.0931	-0.8607	-0.6741	-0.5126	-0.3661	-0.2289	-0.0975
30.	-2.7745	-2.3008	-2.0704	-1.4372	-1.0959	-0.8634	-0.6768	-0.5152	-0.3686	-0.2315	-0.1001
35.	-2.7949	-2.3187	-2.0871	-1.4510	-1.1084	-0.8752	-0.6882	-0.5264	-0.3798	-0.2427	-0.1114
40.	-2.8117	-2.3333	-2.1007	-1.4623	-1.1186	-0.8848	-0.6975	-0.5355	-0.3888	-0.2517	-0.1205
45.	-2.8258	-2.3457	-2.1122	-1.4717	-1.1272	-0.8929	-0.7052	-0.5430	-0.3963	-0.2592	-0.1281
50.	-2.8379	-2.3562	-2.1220	-1.4798	-1.1345	-0.8997	-0.7118	-0.5495	-0.4026	-0.2655	-0.1344
60.	-2.8576	-2.3734	-2.1381	-1.4929	-1.1463	-0.9109	-0.7225	-0.5599	-0.4129	-0.2757	-0.1447
70.	-2.8733	-2.3871	-2.1508	-1.5033	-1.1557	-0.9196	-0.7309	-0.5681	-0.4209	-0.2837	-0.1527
80.	-2.8860	-2.3982	-2.1612	-1.5118	-1.1633	-0.9268	-0.7377	-0.5747	-0.4274	-0.2901	-0.1591
90.	-2.8967	-2.4075	-2.1699	-1.5188	-1.1696	-0.9327	-0.7434	-0.5802	-0.4328	-0.2955	-0.1645
100.	-2.9059	-2.4155	-2.1773	-1.5249	-1.1750	-0.9378	-0.7482	-0.5848	-0.4374	-0.3000	-0.1690
150.	-2.9375	-2.4430	-2.2029	-1.5456	-1.1936	-0.9551	-0.7648	-0.6008	-0.4530	-0.3154	-0.1844
200.	-2.9568	-2.4597	-2.2185	-1.5583	-1.2048	-0.9653	-0.7747	-0.6104	-0.4624	-0.3247	-0.1935
250.	-2.9701	-2.4714	-2.2293	-1.5670	-1.2125	-0.9725	-0.7815	-0.6170	-0.4688	-0.3310	-0.1998
300.	-2.9801	-2.4800	-2.2373	-1.5735	-1.2183	-0.9778	-0.7866	-0.6219	-0.4735	-0.3356	-0.2044
350.	-2.9880	-2.4868	-2.2436	-1.5786	-1.2228	-0.9819	-0.7906	-0.6257	-0.4772	-0.3393	-0.2080
400.	-2.9943	-2.4924	-2.2488	-1.5827	-1.2265	-0.9853	-0.7938	-0.6288	-0.4802	-0.3422	-0.2109
450.	-2.9996	-2.4969	-2.2530	-1.5861	-1.2295	-0.9881	-0.7965	-0.6314	-0.4827	-0.3447	-0.2133
500.	-3.0041	-2.5009	-2.2566	-1.5890	-1.2321	-0.9905	-0.7987	-0.6336	-0.4848	-0.3467	-0.2154
1000.	-3.0288	-2.5223	-2.2765	-1.6050	-1.2462	-1.0036	-0.8111	-0.6454	-0.4963	-0.3579	-0.2264

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.8000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	0.4204	0.7284	1.0795	1.2823	1.5140	1.7903	1.9545	2.1456	2.2322	2.3267	2.4310
6.	0.3754	0.6717	1.0059	1.1984	1.4173	1.6776	1.8324	2.0122	2.0937	2.1826	2.2807
7.	0.3422	0.6313	0.9549	1.1403	1.3514	1.6021	1.7505	1.9228	2.0011	2.0864	2.1805
8.	0.3166	0.6003	0.9171	1.0978	1.3030	1.5464	1.6906	1.8579	1.9339	2.0167	2.1080
9.	0.2964	0.5760	0.8879	1.0651	1.2661	1.5045	1.6456	1.8093	1.8835	1.9643	2.0535
10.	0.2793	0.5563	0.8639	1.0387	1.2365	1.4708	1.6094	1.7701	1.8430	1.9224	2.0099
11.	0.2651	0.5397	0.8441	1.0168	1.2120	1.4432	1.5797	1.7381	1.8099	1.8881	1.9744
12.	0.2528	0.5256	0.8273	0.9983	1.1915	1.4200	1.5549	1.7114	1.7823	1.8595	1.9447
13.	0.2421	0.5133	0.8129	0.9824	1.1739	1.4002	1.5338	1.6886	1.7588	1.8352	1.9195
14.	0.2326	0.5025	0.8003	0.9686	1.1586	1.3830	1.5154	1.6689	1.7385	1.8142	1.8977
15.	0.2242	0.4930	0.7891	0.9564	1.1452	1.3680	1.4994	1.6517	1.7207	1.7959	1.8787
16.	0.2166	0.4844	0.7792	0.9456	1.1332	1.3546	1.4852	1.6365	1.7051	1.7797	1.8619
17.	0.2097	0.4767	0.7703	0.9359	1.1226	1.3427	1.4726	1.6230	1.6911	1.7652	1.8470
18.	0.2035	0.4698	0.7622	0.9272	1.1130	1.3321	1.4612	1.6108	1.6785	1.7523	1.8336
19.	0.1978	0.4634	0.7549	0.9192	1.1043	1.3224	1.4509	1.5998	1.6672	1.7406	1.8214
20.	0.1926	0.4576	0.7483	0.9120	1.0963	1.3135	1.4416	1.5897	1.6568	1.7299	1.8104
21.	0.1877	0.4522	0.7421	0.9053	1.0891	1.3055	1.4330	1.5806	1.6474	1.7202	1.8003
22.	0.1832	0.4472	0.7364	0.8992	1.0823	1.2980	1.4251	1.5721	1.6387	1.7112	1.7911
23.	0.1790	0.4426	0.7312	0.8935	1.0761	1.2911	1.4178	1.5643	1.6307	1.7029	1.7825
24.	0.1751	0.4383	0.7262	0.8882	1.0704	1.2847	1.4110	1.5571	1.6233	1.6953	1.7746
25.	0.1714	0.4343	0.7217	0.8833	1.0650	1.2788	1.4047	1.5504	1.6163	1.6881	1.7672
26.	0.1680	0.4305	0.7174	0.8786	1.0599	1.2732	1.3988	1.5441	1.6099	1.6815	1.7603
27.	0.1647	0.4269	0.7134	0.8743	1.0552	1.2680	1.3933	1.5382	1.6038	1.6752	1.7539
28.	0.1617	0.4236	0.7096	0.8702	1.0508	1.2631	1.3881	1.5327	1.5982	1.6694	1.7479
29.	0.1588	0.4204	0.7060	0.8664	1.0466	1.2585	1.3833	1.5276	1.5929	1.6639	1.7422
30.	0.1560	0.4174	0.7026	0.8628	1.0427	1.2542	1.3787	1.5227	1.5878	1.6587	1.7369
35.	0.1441	0.4044	0.6881	0.8472	1.0258	1.2357	1.3591	1.5018	1.5664	1.6367	1.7141
40.	0.1346	0.3941	0.6766	0.8349	1.0125	1.2211	1.3437	1.4855	1.5496	1.6194	1.6962
45.	0.1267	0.3857	0.6672	0.8249	1.0017	1.2092	1.3312	1.4722	1.5360	1.6054	1.6818
50.	0.1201	0.3786	0.6594	0.8165	0.9927	1.1993	1.3208	1.4612	1.5247	1.5937	1.6698
60.	0.1095	0.3673	0.6468	0.8032	0.9783	1.1837	1.3044	1.4437	1.5068	1.5753	1.6508
70.	0.1013	0.3585	0.6372	0.7929	0.9673	1.1718	1.2918	1.4305	1.4931	1.5613	1.6364
80.	0.0946	0.3515	0.6295	0.7848	0.9586	1.1623	1.2818	1.4199	1.4823	1.5502	1.6249
90.	0.0892	0.3457	0.6232	0.7781	0.9515	1.1545	1.2737	1.4113	1.4735	1.5411	1.6156
100.	0.0846	0.3408	0.6179	0.7725	0.9455	1.1480	1.2669	1.4041	1.4661	1.5335	1.6078
150.	0.0689	0.3244	0.6001	0.7537	0.9254	1.1263	1.2441	1.3801	1.4415	1.5083	1.5818
200.	0.0597	0.3147	0.5897	0.7427	0.9137	1.1137	1.2309	1.3662	1.4273	1.4937	1.5668
250.	0.0533	0.3082	0.5826	0.7353	0.9058	1.1052	1.2220	1.3568	1.4177	1.4839	1.5567
300.	0.0487	0.3033	0.5774	0.7298	0.9000	1.0990	1.2155	1.3500	1.4107	1.4767	1.5493
350.	0.0451	0.2996	0.5734	0.7256	0.8956	1.0942	1.2105	1.3447	1.4053	1.4711	1.5436
400.	0.0421	0.2965	0.5702	0.7222	0.8920	1.0903	1.2065	1.3404	1.4009	1.4667	1.5390
450.	0.0397	0.2941	0.5675	0.7194	0.8890	1.0871	1.2032	1.3370	1.3974	1.4630	1.5353
500.	0.0377	0.2919	0.5652	0.7171	0.8865	1.0844	1.2004	1.3340	1.3944	1.4600	1.5321
1000.	0.0266	0.2806	0.5531	0.7044	0.8731	1.0701	1.1854	1.3183	1.3783	1.4435	1.5152

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.8000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	2.5480	2.6820	2.8402	3.0357	3.2968	3.7104	4.0911	4.3543	4.8801	5.1914	5.8523
6.	2.3907	2.5166	2.6651	2.8484	3.0932	3.4809	3.8378	4.0845	4.5777	4.8697	5.4895
7.	2.2860	2.4067	2.5490	2.7247	2.9592	3.3307	3.6724	3.9087	4.3809	4.6604	5.2538
8.	2.2103	2.3274	2.4654	2.6357	2.8630	3.2229	3.5540	3.7829	4.2402	4.5110	5.0856
9.	2.1533	2.2676	2.4023	2.5685	2.7905	3.1419	3.4650	3.6884	4.1344	4.3984	4.9588
10.	2.1080	2.2202	2.3524	2.5155	2.7332	3.0776	3.3944	3.6133	4.0506	4.3095	4.8588
11.	2.0710	2.1814	2.3116	2.4722	2.6864	3.0255	3.3372	3.5526	3.9828	4.2375	4.7779
12.	2.0401	2.1491	2.2777	2.4362	2.6476	2.9822	3.2897	3.5023	3.9267	4.1779	4.7110
13.	2.0138	2.1217	2.2489	2.4056	2.6147	2.9455	3.2496	3.4597	3.8792	4.1276	4.6545
14.	1.9912	2.0981	2.2240	2.3793	2.5864	2.9140	3.2151	3.4232	3.8385	4.0844	4.6060
15.	1.9714	2.0775	2.2024	2.3564	2.5617	2.8866	3.1851	3.3914	3.8031	4.0468	4.5639
16.	1.9540	2.0593	2.1833	2.3362	2.5400	2.8624	3.1587	3.3634	3.7720	4.0138	4.5269
17.	1.9385	2.0431	2.1663	2.3182	2.5207	2.8410	3.1353	3.3386	3.7444	3.9845	4.4941
18.	1.9246	2.0286	2.1511	2.3021	2.5034	2.8217	3.1143	3.3163	3.7196	3.9583	4.4646
19.	1.9120	2.0154	2.1373	2.2875	2.4877	2.8044	3.0953	3.2962	3.6973	3.9346	4.4380
20.	1.9005	2.0035	2.1248	2.2743	2.4735	2.7886	3.0781	3.2780	3.6770	3.9131	4.4140
21.	1.8900	1.9926	2.1134	2.2622	2.4606	2.7742	3.0624	3.2614	3.6585	3.8935	4.3921
22.	1.8804	1.9826	2.1029	2.2511	2.4487	2.7610	3.0480	3.2461	3.6416	3.8756	4.3720
23.	1.8716	1.9733	2.0932	2.2409	2.4377	2.7489	3.0347	3.2321	3.6260	3.8591	4.3535
24.	1.8633	1.9648	2.0842	2.2314	2.4276	2.7376	3.0224	3.2191	3.6116	3.8438	4.3364
25.	1.8557	1.9568	2.0759	2.2226	2.4181	2.7272	3.0111	3.2071	3.5982	3.8297	4.3205
26.	1.8486	1.9494	2.0681	2.2144	2.4094	2.7175	3.0005	3.1959	3.5858	3.8165	4.3058
27.	1.8419	1.9425	2.0609	2.2067	2.4011	2.7084	2.9906	3.1854	3.5742	3.8042	4.2920
28.	1.8357	1.9360	2.0541	2.1996	2.3935	2.6999	2.9813	3.1756	3.5633	3.7926	4.2791
29.	1.8298	1.9299	2.0477	2.1928	2.3862	2.6919	2.9726	3.1664	3.5531	3.7818	4.2670
30.	1.8243	1.9241	2.0417	2.1865	2.3794	2.6844	2.9644	3.1577	3.5434	3.7716	4.2556
35.	1.8007	1.8996	2.0160	2.1594	2.3505	2.6524	2.9296	3.1209	3.5026	3.7284	4.2073
40.	1.7822	1.8804	1.9960	2.1383	2.3279	2.6274	2.9024	3.0922	3.4708	3.6947	4.1696
45.	1.7672	1.8649	1.9797	2.1212	2.3096	2.6072	2.8804	3.0689	3.4451	3.6675	4.1392
50.	1.7548	1.8519	1.9662	2.1070	2.2944	2.5905	2.8622	3.0497	3.4238	3.6450	4.1140
60.	1.7352	1.8316	1.9450	2.0846	2.2705	2.5641	2.8336	3.0195	3.3903	3.6096	4.0745
70.	1.7203	1.8161	1.9289	2.0676	2.2524	2.5442	2.8119	2.9966	3.3650	3.5828	4.0447
80.	1.7085	1.8039	1.9161	2.0542	2.2381	2.5284	2.7948	2.9785	3.3450	3.5617	4.0211
90.	1.6988	1.7939	1.9057	2.0432	2.2264	2.5156	2.7808	2.9638	3.3288	3.5445	4.0019
100.	1.6908	1.7855	1.8969	2.0341	2.2167	2.5048	2.7692	2.9515	3.3152	3.5302	3.9859
150.	1.6640	1.7578	1.8680	2.0037	2.1843	2.4693	2.7307	2.9110	3.2704	3.4828	3.9331
200.	1.6485	1.7417	1.8513	1.9862	2.1657	2.4488	2.7084	2.8875	3.2445	3.4555	3.9027
250.	1.6380	1.7309	1.8401	1.9744	2.1531	2.4351	2.6936	2.8718	3.2272	3.4372	3.8824
300.	1.6304	1.7230	1.8319	1.9658	2.1440	2.4251	2.6827	2.8604	3.2146	3.4240	3.8676
350.	1.6246	1.7170	1.8256	1.9592	2.1370	2.4174	2.6744	2.8516	3.2049	3.4137	3.8562
400.	1.6199	1.7121	1.8205	1.9539	2.1313	2.4112	2.6677	2.8446	3.1972	3.4056	3.8471
450.	1.6160	1.7081	1.8164	1.9496	2.1267	2.4061	2.6622	2.8388	3.1908	3.3988	3.8396
500.	1.6127	1.7047	1.8129	1.9459	2.1228	2.4019	2.6576	2.8340	3.1855	3.3932	3.8333
1000.	1.5953	1.6867	1.7942	1.9263	2.1020	2.3791	2.6330	2.8080	3.1569	3.3630	3.7998

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9000

PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks

SAMPLE SIZE	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-2.1065	-1.7168	-1.5242	-0.9818	-0.6751	-0.4562	-0.2726	-0.1060	0.0524	0.2072	0.3623
6.	-2.1716	-1.7737	-1.5778	-1.0279	-0.7194	-0.5008	-0.3189	-0.1554	-0.0011	0.1485	0.2967
7.	-2.2246	-1.8202	-1.6213	-1.0648	-0.7545	-0.5357	-0.3547	-0.1930	-0.0415	0.1048	0.2493
8.	-2.2682	-1.8585	-1.6573	-1.0954	-0.7832	-0.5645	-0.3838	-0.2230	-0.0732	0.0709	0.2124
9.	-2.3055	-1.8912	-1.6879	-1.1214	-0.8076	-0.5881	-0.4079	-0.2479	-0.0991	0.0435	0.1833
10.	-2.3375	-1.9193	-1.7143	-1.1436	-0.8284	-0.6085	-0.4283	-0.2687	-0.1209	0.0205	0.1590
11.	-2.3656	-1.9439	-1.7374	-1.1630	-0.8464	-0.6261	-0.4458	-0.2868	-0.1393	0.0012	0.1385
12.	-2.3906	-1.9658	-1.7579	-1.1802	-0.8624	-0.6416	-0.4613	-0.3022	-0.1554	-0.0156	0.1209
13.	-2.4129	-1.9854	-1.7762	-1.1956	-0.8766	-0.6554	-0.4749	-0.3161	-0.1696	-0.0303	0.1054
14.	-2.4330	-2.0031	-1.7928	-1.2095	-0.8894	-0.6678	-0.4872	-0.3284	-0.1822	-0.0433	0.0918
15.	-2.4513	-2.0191	-1.8078	-1.2220	-0.9010	-0.6790	-0.4983	-0.3396	-0.1936	-0.0551	0.0796
16.	-2.4681	-2.0338	-1.8216	-1.2335	-0.9116	-0.6892	-0.5083	-0.3497	-0.2039	-0.0656	0.0686
17.	-2.4835	-2.0473	-1.8342	-1.2441	-0.9213	-0.6985	-0.5175	-0.3589	-0.2132	-0.0752	0.0587
18.	-2.4977	-2.0597	-1.8458	-1.2537	-0.9302	-0.7071	-0.5259	-0.3672	-0.2217	-0.0839	0.0497
19.	-2.5109	-2.0713	-1.8566	-1.2627	-0.9385	-0.7150	-0.5337	-0.3749	-0.2295	-0.0919	0.0414
20.	-2.5232	-2.0821	-1.8667	-1.2711	-0.9461	-0.7223	-0.5408	-0.3821	-0.2367	-0.0992	0.0338
21.	-2.5347	-2.0921	-1.8761	-1.2789	-0.9533	-0.7291	-0.5475	-0.3887	-0.2434	-0.1061	0.0268
22.	-2.5455	-2.1016	-1.8849	-1.2862	-0.9600	-0.7355	-0.5538	-0.3949	-0.2496	-0.1124	0.0203
23.	-2.5556	-2.1104	-1.8932	-1.2930	-0.9662	-0.7415	-0.5596	-0.4007	-0.2555	-0.1184	0.0142
24.	-2.5652	-2.1188	-1.9010	-1.2995	-0.9721	-0.7471	-0.5651	-0.4062	-0.2609	-0.1239	0.0085
25.	-2.5742	-2.1267	-1.9084	-1.3056	-0.9777	-0.7524	-0.5703	-0.4113	-0.2661	-0.1291	0.0031
26.	-2.5828	-2.1341	-1.9153	-1.3114	-0.9830	-0.7575	-0.5752	-0.4162	-0.2709	-0.1341	-0.0019
27.	-2.5909	-2.1412	-1.9220	-1.3168	-0.9880	-0.7622	-0.5798	-0.4207	-0.2755	-0.1387	-0.0067
28.	-2.5986	-2.1480	-1.9283	-1.3220	-0.9927	-0.7667	-0.5842	-0.4251	-0.2799	-0.1431	-0.0112
29.	-2.6059	-2.1544	-1.9342	-1.3270	-0.9972	-0.7710	-0.5883	-0.4292	-0.2840	-0.1473	-0.0154
30.	-2.6129	-2.1605	-1.9400	-1.3317	-1.0015	-0.7751	-0.5923	-0.4331	-0.2879	-0.1512	-0.0194
35.	-2.6437	-2.1874	-1.9650	-1.3524	-1.0203	-0.7929	-0.6096	-0.4501	-0.3049	-0.1684	-0.0369
40.	-2.6689	-2.2094	-1.9856	-1.3693	-1.0356	-0.8074	-0.6236	-0.4639	-0.3186	-0.1821	-0.0509
45.	-2.6902	-2.2280	-2.0029	-1.3834	-1.0484	-0.8195	-0.6353	-0.4754	-0.3299	-0.1935	-0.0624
50.	-2.7084	-2.2438	-2.0177	-1.3955	-1.0594	-0.8298	-0.6453	-0.4851	-0.3396	-0.2032	-0.0722
60.	-2.7381	-2.2698	-2.0418	-1.4153	-1.0772	-0.8466	-0.6614	-0.5008	-0.3552	-0.2187	-0.0879
70.	-2.7617	-2.2903	-2.0609	-1.4309	-1.0913	-0.8598	-0.6741	-0.5132	-0.3673	-0.2309	-0.1001
80.	-2.7809	-2.3070	-2.0765	-1.4436	-1.1027	-0.8705	-0.6843	-0.5232	-0.3772	-0.2407	-0.1099
90.	-2.7970	-2.3211	-2.0896	-1.4543	-1.1122	-0.8795	-0.6929	-0.5315	-0.3854	-0.2488	-0.1181
100.	-2.8108	-2.3330	-2.1007	-1.4633	-1.1204	-0.8871	-0.7002	-0.5386	-0.3923	-0.2557	-0.1250
150.	-2.8584	-2.3745	-2.1393	-1.4946	-1.1484	-0.9132	-0.7251	-0.5627	-0.4160	-0.2791	-0.1484
200.	-2.8876	-2.3998	-2.1628	-1.5136	-1.1654	-0.9290	-0.7402	-0.5773	-0.4302	-0.2931	-0.1623
250.	-2.9078	-2.4174	-2.1792	-1.5268	-1.1771	-0.9399	-0.7505	-0.5873	-0.4399	-0.3027	-0.1718
300.	-2.9229	-2.4305	-2.1913	-1.5366	-1.1858	-0.9480	-0.7582	-0.5947	-0.4471	-0.3098	-0.1789
350.	-2.9347	-2.4407	-2.2009	-1.5443	-1.1927	-0.9544	-0.7642	-0.6005	-0.4528	-0.3153	-0.1844
400.	-2.9443	-2.4491	-2.2086	-1.5506	-1.1982	-0.9595	-0.7691	-0.6051	-0.4573	-0.3198	-0.1888
450.	-2.9523	-2.4560	-2.2151	-1.5558	-1.2028	-0.9638	-0.7731	-0.6090	-0.4611	-0.3235	-0.1925
500.	-2.9592	-2.4619	-2.2205	-1.5602	-1.2067	-0.9674	-0.7766	-0.6123	-0.4643	-0.3266	-0.1956
1000.	-2.9966	-2.4944	-2.2506	-1.5843	-1.2281	-0.9872	-0.7953	-0.6303	-0.4817	-0.3437	-0.2124

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	0.6857	1.0409	1.4554	1.6974	1.9760	2.3109	2.5109	2.7440	2.8499	2.9656	3.0934
6.	0.6023	0.9345	1.3172	1.5393	1.7943	2.0999	2.2817	2.4937	2.5902	2.6955	2.8119
7.	0.5439	0.8621	1.2247	1.4346	1.6753	1.9629	2.1336	2.3324	2.4229	2.5216	2.6307
8.	0.5001	0.8077	1.1580	1.3598	1.5899	1.8647	2.0284	2.2191	2.3056	2.3998	2.5039
9.	0.4657	0.7662	1.1064	1.3021	1.5254	1.7914	1.9492	2.1329	2.2164	2.3075	2.4081
10.	0.4373	0.7325	1.0654	1.2565	1.4742	1.7332	1.8868	2.0656	2.1468	2.2353	2.3331
11.	0.4138	0.7046	1.0317	1.2192	1.4324	1.6859	1.8362	2.0110	2.0904	2.1770	2.2725
12.	0.3937	0.6811	1.0036	1.1880	1.3977	1.6467	1.7943	1.9659	2.0438	2.1287	2.2225
13.	0.3763	0.6608	0.9795	1.1615	1.3681	1.6135	1.7589	1.9278	2.0045	2.0880	2.1803
14.	0.3609	0.6431	0.9586	1.1385	1.3427	1.5850	1.7284	1.8951	1.9707	2.0532	2.1441
15.	0.3473	0.6276	0.9403	1.1185	1.3205	1.5601	1.7019	1.8666	1.9414	2.0229	2.1127
16.	0.3352	0.6137	0.9240	1.1007	1.3009	1.5382	1.6786	1.8416	1.9156	1.9962	2.0851
17.	0.3242	0.6012	0.9095	1.0848	1.2834	1.5187	1.6579	1.8194	1.8927	1.9726	2.0607
18.	0.3143	0.5900	0.8964	1.0706	1.2677	1.5012	1.6392	1.7994	1.8721	1.9513	2.0387
19.	0.3053	0.5798	0.8846	1.0576	1.2535	1.4853	1.6224	1.7814	1.8536	1.9321	2.0188
20.	0.2970	0.5704	0.8737	1.0458	1.2405	1.4709	1.6071	1.7651	1.8367	1.9148	2.0008
21.	0.2893	0.5618	0.8638	1.0351	1.2287	1.4578	1.5931	1.7501	1.8214	1.8989	1.9845
22.	0.2822	0.5539	0.8546	1.0251	1.2178	1.4457	1.5803	1.7365	1.8073	1.8844	1.9694
23.	0.2756	0.5465	0.8462	1.0160	1.2078	1.4346	1.5685	1.7239	1.7943	1.8710	1.9556
24.	0.2694	0.5396	0.8383	1.0074	1.1985	1.4243	1.5576	1.7122	1.7823	1.8586	1.9428
25.	0.2637	0.5332	0.8310	0.9995	1.1898	1.4147	1.5475	1.7014	1.7712	1.8472	1.9310
26.	0.2582	0.5272	0.8241	0.9921	1.1818	1.4058	1.5380	1.6913	1.7608	1.8365	1.9199
27.	0.2531	0.5216	0.8177	0.9852	1.1742	1.3974	1.5292	1.6819	1.7511	1.8265	1.9096
28.	0.2483	0.5162	0.8117	0.9787	1.1671	1.3896	1.5209	1.6731	1.7420	1.8171	1.8999
29.	0.2438	0.5112	0.8060	0.9725	1.1604	1.3822	1.5131	1.6648	1.7335	1.8084	1.8909
30.	0.2395	0.5065	0.8006	0.9668	1.1541	1.3753	1.5058	1.6570	1.7255	1.8001	1.8823
35.	0.2209	0.4861	0.7776	0.9421	1.1273	1.3458	1.4746	1.6238	1.6914	1.7650	1.8461
40.	0.2061	0.4700	0.7595	0.9226	1.1063	1.3227	1.4502	1.5978	1.6647	1.7375	1.8177
45.	0.1940	0.4568	0.7447	0.9068	1.0892	1.3039	1.4304	1.5768	1.6431	1.7153	1.7948
50.	0.1838	0.4458	0.7324	0.8936	1.0749	1.2883	1.4140	1.5594	1.6253	1.6969	1.7759
60.	0.1674	0.4281	0.7128	0.8727	1.0524	1.2637	1.3881	1.5320	1.5971	1.6680	1.7460
70.	0.1547	0.4146	0.6978	0.8568	1.0353	1.2451	1.3685	1.5112	1.5758	1.6460	1.7234
80.	0.1445	0.4037	0.6859	0.8441	1.0217	1.2302	1.3529	1.4947	1.5589	1.6287	1.7056
90.	0.1361	0.3948	0.6761	0.8337	1.0105	1.2181	1.3402	1.4813	1.5451	1.6145	1.6910
100.	0.1291	0.3872	0.6679	0.8250	1.0012	1.2080	1.3296	1.4700	1.5336	1.6027	1.6788
150.	0.1051	0.3620	0.6403	0.7959	0.9701	1.1744	1.2943	1.4328	1.4955	1.5636	1.6386
200.	0.0909	0.3471	0.6243	0.7789	0.9520	1.1548	1.2739	1.4113	1.4734	1.5409	1.6153
250.	0.0813	0.3370	0.6134	0.7675	0.9399	1.1417	1.2601	1.3968	1.4586	1.5258	1.5997
300.	0.0742	0.3296	0.6054	0.7591	0.9310	1.1321	1.2501	1.3863	1.4478	1.5147	1.5883
350.	0.0687	0.3239	0.5992	0.7526	0.9241	1.1247	1.2424	1.3781	1.4395	1.5062	1.5795
400.	0.0642	0.3193	0.5943	0.7474	0.9186	1.1188	1.2362	1.3716	1.4328	1.4993	1.5725
450.	0.0605	0.3154	0.5902	0.7432	0.9141	1.1139	1.2311	1.3663	1.4273	1.4937	1.5668
500.	0.0574	0.3122	0.5868	0.7395	0.9102	1.1098	1.2268	1.3618	1.4227	1.4890	1.5619
1000.	0.0406	0.2948	0.5682	0.7202	0.8897	1.0878	1.2039	1.3376	1.3981	1.4637	1.5360

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9000

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	3.2368	3.4013	3.5958	3.8364	4.1583	4.6690	5.1398	5.4653	6.1160	6.5017	7.3210
6.	2.9425	3.0922	3.2690	3.4875	3.7796	4.2433	4.6707	4.9665	5.5579	5.9084	6.6530
7.	2.7531	2.8933	3.0589	3.2636	3.5372	3.9713	4.3714	4.6482	5.2018	5.5299	6.2268
8.	2.6207	2.7546	2.9125	3.1077	3.3686	3.7825	4.1639	4.4278	4.9554	5.2681	5.9323
9.	2.5210	2.6502	2.8028	2.9911	3.2427	3.6417	4.0093	4.2636	4.7721	5.0734	5.7133
10.	2.4427	2.5683	2.7165	2.8995	3.1440	3.5316	3.8887	4.1357	4.6294	4.9220	5.5434
11.	2.3796	2.5023	2.6471	2.8258	3.0645	3.4430	3.7916	4.0327	4.5146	4.8001	5.4065
12.	2.3276	2.4479	2.5899	2.7651	2.9992	3.3702	3.7118	3.9481	4.4203	4.7001	5.2942
13.	2.2837	2.4021	2.5417	2.7141	2.9442	3.3090	3.6448	3.8771	4.3413	4.6162	5.2001
14.	2.2461	2.3628	2.5005	2.6704	2.8973	3.2568	3.5877	3.8165	4.2738	4.5447	5.1199
15.	2.2135	2.3288	2.4647	2.6325	2.8566	3.2115	3.5382	3.7641	4.2155	4.4829	5.0506
16.	2.1848	2.2989	2.4334	2.5993	2.8209	3.1719	3.4949	3.7182	4.1645	4.4288	4.9899
17.	2.1594	2.2724	2.4056	2.5699	2.7893	3.1368	3.4566	3.6777	4.1194	4.3810	4.9363
18.	2.1365	2.2485	2.3806	2.5435	2.7609	3.1053	3.4221	3.6412	4.0788	4.3380	4.8882
19.	2.1159	2.2271	2.3581	2.5197	2.7354	3.0770	3.3912	3.6085	4.0425	4.2995	4.8450
20.	2.0973	2.2076	2.3377	2.4982	2.7123	3.0513	3.3632	3.5789	4.0096	4.2646	4.8060
21.	2.0803	2.1899	2.3191	2.4785	2.6912	3.0280	3.3378	3.5519	3.9796	4.2329	4.7705
22.	2.0647	2.1737	2.3021	2.4606	2.6720	3.0067	3.3145	3.5273	3.9523	4.2039	4.7381
23.	2.0503	2.1587	2.2865	2.4440	2.6542	2.9870	3.2931	3.5047	3.9272	4.1773	4.7083
24.	2.0371	2.1449	2.2720	2.4288	2.6379	2.9689	3.2734	3.4838	3.9040	4.1528	4.6808
25.	2.0248	2.1321	2.2586	2.4146	2.6227	2.9521	3.2551	3.4644	3.8825	4.1300	4.6553
26.	2.0133	2.1202	2.2462	2.4015	2.6086	2.9365	3.2381	3.4464	3.8625	4.1089	4.6317
27.	2.0027	2.1091	2.2345	2.3892	2.5955	2.9220	3.2222	3.4297	3.8439	4.0892	4.6097
28.	1.9926	2.0987	2.2236	2.3777	2.5832	2.9084	3.2074	3.4140	3.8265	4.0708	4.5891
29.	1.9832	2.0889	2.2134	2.3669	2.5716	2.8956	3.1934	3.3993	3.8102	4.0535	4.5697
30.	1.9744	2.0797	2.2038	2.3567	2.5608	2.8836	3.1804	3.3854	3.7949	4.0373	4.5516
35.	1.9369	2.0407	2.1630	2.3137	2.5147	2.8327	3.1250	3.3269	3.7300	3.9686	4.4749
40.	1.9075	2.0102	2.1311	2.2802	2.4789	2.7931	3.0819	3.2814	3.6796	3.9153	4.4152
45.	1.8839	1.9856	2.1054	2.2531	2.4499	2.7612	3.0472	3.2447	3.6390	3.8723	4.3672
50.	1.8642	1.9652	2.0841	2.2307	2.4260	2.7348	3.0185	3.2144	3.6054	3.8368	4.3276
60.	1.8334	1.9332	2.0507	2.1955	2.3885	2.6934	2.9735	3.1670	3.5529	3.7813	4.2657
70.	1.8100	1.9090	2.0254	2.1689	2.3601	2.6622	2.9396	3.1312	3.5134	3.7395	4.2190
80.	1.7916	1.8898	2.0055	2.1479	2.3377	2.6376	2.9129	3.1030	3.4822	3.7065	4.1822
90.	1.7765	1.8742	1.9892	2.1309	2.3195	2.6176	2.8912	3.0801	3.4569	3.6797	4.1524
100.	1.7640	1.8612	1.9757	2.1166	2.3043	2.6009	2.8731	3.0610	3.4358	3.6574	4.1275
150.	1.7224	1.8182	1.9308	2.0695	2.2542	2.5458	2.8134	2.9980	3.3663	3.5840	4.0457
200.	1.6984	1.7933	1.9049	2.0424	2.2253	2.5141	2.7790	2.9618	3.3263	3.5418	3.9987
250.	1.6823	1.7767	1.8876	2.0242	2.2060	2.4929	2.7561	2.9376	3.2996	3.5137	3.9673
300.	1.6706	1.7645	1.8750	2.0110	2.1919	2.4775	2.7394	2.9200	3.2802	3.4932	3.9445
350.	1.6616	1.7552	1.8653	2.0008	2.1811	2.4656	2.7265	2.9065	3.2653	3.4775	3.9270
400.	1.6543	1.7477	1.8575	1.9926	2.1724	2.4561	2.7163	2.8957	3.2534	3.4649	3.9130
450.	1.6484	1.7416	1.8511	1.9859	2.1653	2.4483	2.7078	2.8868	3.2436	3.4546	3.9016
500.	1.6434	1.7364	1.8458	1.9803	2.1593	2.4418	2.7008	2.8794	3.2354	3.4459	3.8919
1000.	1.6167	1.7088	1.8171	1.9503	2.1274	2.4069	2.6630	2.8396	3.1916	3.3997	3.8405

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9500

PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks

	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-1.8697	-1.5096	-1.3300	-0.8177	-0.5187	-0.2994	-0.1102	0.0658	0.2368	0.4083	0.5828
6.	-1.9495	-1.5794	-1.3957	-0.8746	-0.5750	-0.3578	-0.1729	-0.0033	0.1594	0.3204	0.4826
7.	-2.0139	-1.6360	-1.4489	-0.9202	-0.6189	-0.4025	-0.2205	-0.0549	0.1030	0.2577	0.4128
8.	-2.0669	-1.6825	-1.4926	-0.9577	-0.6551	-0.4393	-0.2583	-0.0949	0.0597	0.2106	0.3605
9.	-2.1122	-1.7222	-1.5300	-0.9897	-0.6855	-0.4694	-0.2896	-0.1277	0.0247	0.1726	0.3195
10.	-2.1512	-1.7565	-1.5621	-1.0170	-0.7115	-0.4955	-0.3160	-0.1552	-0.0044	0.1414	0.2857
11.	-2.1855	-1.7866	-1.5904	-1.0410	-0.7340	-0.5177	-0.3386	-0.1786	-0.0290	0.1153	0.2576
12.	-2.2160	-1.8133	-1.6155	-1.0622	-0.7539	-0.5373	-0.3583	-0.1989	-0.0502	0.0928	0.2336
13.	-2.2432	-1.8373	-1.6379	-1.0811	-0.7717	-0.5547	-0.3759	-0.2168	-0.0689	0.0732	0.2127
14.	-2.2679	-1.8589	-1.6582	-1.0982	-0.7877	-0.5703	-0.3915	-0.2328	-0.0854	0.0558	0.1943
15.	-2.2903	-1.8785	-1.6766	-1.1137	-0.8021	-0.5844	-0.4056	-0.2472	-0.1002	0.0404	0.1780
16.	-2.3108	-1.8965	-1.6935	-1.1279	-0.8153	-0.5973	-0.4184	-0.2602	-0.1136	0.0264	0.1634
17.	-2.3296	-1.9131	-1.7090	-1.1410	-0.8274	-0.6090	-0.4301	-0.2720	-0.1258	0.0138	0.1502
18.	-2.3471	-1.9284	-1.7233	-1.1529	-0.8385	-0.6198	-0.4408	-0.2828	-0.1368	0.0024	0.1383
19.	-2.3633	-1.9426	-1.7366	-1.1641	-0.8488	-0.6297	-0.4506	-0.2927	-0.1469	-0.0080	0.1274
20.	-2.3785	-1.9559	-1.7490	-1.1744	-0.8583	-0.6389	-0.4597	-0.3018	-0.1562	-0.0176	0.1174
21.	-2.3926	-1.9683	-1.7606	-1.1841	-0.8673	-0.6475	-0.4682	-0.3103	-0.1648	-0.0265	0.1081
22.	-2.4059	-1.9799	-1.7715	-1.1932	-0.8756	-0.6555	-0.4761	-0.3182	-0.1729	-0.0348	0.0995
23.	-2.4184	-1.9908	-1.7817	-1.2017	-0.8834	-0.6630	-0.4835	-0.3257	-0.1804	-0.0425	0.0915
24.	-2.4302	-2.0011	-1.7914	-1.2097	-0.8908	-0.6701	-0.4904	-0.3326	-0.1875	-0.0497	0.0840
25.	-2.4413	-2.0109	-1.8005	-1.2173	-0.8978	-0.6768	-0.4970	-0.3392	-0.1941	-0.0565	0.0770
26.	-2.4518	-2.0201	-1.8091	-1.2244	-0.9043	-0.6831	-0.5032	-0.3453	-0.2003	-0.0629	0.0705
27.	-2.4618	-2.0289	-1.8173	-1.2312	-0.9106	-0.6891	-0.5090	-0.3512	-0.2062	-0.0689	0.0643
28.	-2.4714	-2.0372	-1.8251	-1.2377	-0.9165	-0.6947	-0.5146	-0.3567	-0.2118	-0.0746	0.0584
29.	-2.4804	-2.0451	-1.8325	-1.2438	-0.9221	-0.7001	-0.5199	-0.3619	-0.2171	-0.0800	0.0529
30.	-2.4891	-2.0527	-1.8396	-1.2497	-0.9275	-0.7052	-0.5249	-0.3669	-0.2221	-0.0851	0.0476
35.	-2.5272	-2.0860	-1.8707	-1.2754	-0.9510	-0.7277	-0.5468	-0.3886	-0.2439	-0.1072	0.0249
40.	-2.5585	-2.1134	-1.8962	-1.2965	-0.9702	-0.7459	-0.5645	-0.4062	-0.2614	-0.1250	0.0068
45.	-2.5849	-2.1364	-1.9177	-1.3142	-0.9863	-0.7611	-0.5793	-0.4207	-0.2760	-0.1396	-0.0081
50.	-2.6075	-2.1562	-1.9361	-1.3293	-1.0000	-0.7741	-0.5918	-0.4331	-0.2883	-0.1520	-0.0207
60.	-2.6446	-2.1885	-1.9663	-1.3541	-1.0224	-0.7953	-0.6123	-0.4531	-0.3082	-0.1720	-0.0410
70.	-2.6740	-2.2141	-1.9902	-1.3736	-1.0400	-0.8119	-0.6283	-0.4688	-0.3237	-0.1876	-0.0567
80.	-2.6981	-2.2351	-2.0097	-1.3896	-1.0544	-0.8254	-0.6413	-0.4815	-0.3363	-0.2001	-0.0694
90.	-2.7182	-2.2527	-2.0261	-1.4029	-1.0664	-0.8367	-0.6521	-0.4921	-0.3467	-0.2105	-0.0799
100.	-2.7355	-2.2677	-2.0401	-1.4143	-1.0766	-0.8463	-0.6614	-0.5011	-0.3556	-0.2194	-0.0887
150.	-2.7954	-2.3198	-2.0885	-1.4537	-1.1120	-0.8794	-0.6930	-0.5318	-0.3858	-0.2493	-0.1188
200.	-2.8321	-2.3517	-2.1182	-1.4778	-1.1335	-0.8994	-0.7121	-0.5503	-0.4039	-0.2673	-0.1366
250.	-2.8577	-2.3739	-2.1388	-1.4944	-1.1483	-0.9133	-0.7253	-0.5630	-0.4163	-0.2795	-0.1489
300.	-2.8768	-2.3905	-2.1543	-1.5069	-1.1594	-0.9236	-0.7351	-0.5724	-0.4255	-0.2886	-0.1579
350.	-2.8918	-2.4035	-2.1663	-1.5166	-1.1681	-0.9317	-0.7427	-0.5798	-0.4327	-0.2957	-0.1649
400.	-2.9040	-2.4141	-2.1762	-1.5245	-1.1751	-0.9382	-0.7489	-0.5858	-0.4385	-0.3014	-0.1706
450.	-2.9141	-2.4229	-2.1843	-1.5311	-1.1810	-0.9436	-0.7541	-0.5907	-0.4433	-0.3061	-0.1753
500.	-2.9227	-2.4304	-2.1913	-1.5367	-1.1860	-0.9482	-0.7584	-0.5949	-0.4474	-0.3101	-0.1793
1000.	-2.9703	-2.4716	-2.2296	-1.5675	-1.2132	-0.9735	-0.7823	-0.6178	-0.4697	-0.3320	-0.2008

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9500

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	0.9539	1.3693	1.8615	2.1519	2.4865	2.8894	3.1308	3.4127	3.5411	3.6813	3.8364
6.	0.8224	1.1981	1.6377	1.8950	2.1907	2.5463	2.7592	3.0076	3.1205	3.2438	3.3801
7.	0.7340	1.0863	1.4943	1.7319	2.0050	2.3329	2.5281	2.7561	2.8599	2.9732	3.0985
8.	0.6697	1.0062	1.3924	1.6173	1.8752	2.1835	2.3674	2.5820	2.6797	2.7864	2.9042
9.	0.6197	0.9448	1.3168	1.5321	1.7783	2.0732	2.2490	2.4541	2.5473	2.6489	2.7613
10.	0.5797	0.8957	1.2566	1.4650	1.7033	1.9880	2.1573	2.3547	2.4444	2.5423	2.6505
11.	0.5465	0.8558	1.2077	1.4106	1.6423	1.9189	2.0833	2.2748	2.3619	2.4568	2.5617
12.	0.5185	0.8224	1.1672	1.3657	1.5922	1.8622	2.0226	2.2094	2.2944	2.3870	2.4892
13.	0.4943	0.7939	1.1329	1.3279	1.5500	1.8147	1.9718	2.1547	2.2379	2.3286	2.4287
14.	0.4733	0.7693	1.1035	1.2954	1.5139	1.7741	1.9285	2.1082	2.1899	2.2789	2.3772
15.	0.4548	0.7477	1.0779	1.2672	1.4826	1.7390	1.8910	2.0679	2.1483	2.2360	2.3327
16.	0.4383	0.7286	1.0553	1.2424	1.4552	1.7082	1.8582	2.0327	2.1121	2.1985	2.2938
17.	0.4234	0.7115	1.0352	1.2203	1.4308	1.6810	1.8292	2.0016	2.0800	2.1653	2.2595
18.	0.4101	0.6961	1.0170	1.2004	1.4089	1.6564	1.8031	1.9736	2.0511	2.1355	2.2286
19.	0.3979	0.6822	1.0006	1.1825	1.3891	1.6343	1.7796	1.9484	2.0251	2.1087	2.2009
20.	0.3867	0.6694	0.9857	1.1662	1.3711	1.6143	1.7583	1.9257	2.0017	2.0845	2.1759
21.	0.3764	0.6577	0.9721	1.1514	1.3548	1.5961	1.7390	1.9050	1.9804	2.0625	2.1531
22.	0.3669	0.6470	0.9595	1.1377	1.3398	1.5794	1.7213	1.8861	1.9609	2.0424	2.1323
23.	0.3581	0.6370	0.9480	1.1251	1.3260	1.5641	1.7050	1.8687	1.9430	2.0239	2.1132
24.	0.3499	0.6277	0.9373	1.1135	1.3132	1.5500	1.6900	1.8526	1.9265	2.0069	2.0956
25.	0.3423	0.6191	0.9273	1.1027	1.3014	1.5368	1.6761	1.8378	1.9112	1.9911	2.0793
26.	0.3351	0.6110	0.9180	1.0926	1.2904	1.5246	1.6631	1.8240	1.8970	1.9765	2.0642
27.	0.3283	0.6035	0.9094	1.0832	1.2801	1.5132	1.6511	1.8111	1.8837	1.9628	2.0500
28.	0.3220	0.5964	0.9012	1.0744	1.2704	1.5026	1.6398	1.7991	1.8713	1.9500	2.0368
29.	0.3160	0.5897	0.8935	1.0661	1.2614	1.4926	1.6292	1.7878	1.8597	1.9381	2.0245
30.	0.3103	0.5834	0.8863	1.0583	1.2529	1.4832	1.6192	1.7771	1.8488	1.9268	2.0128
35.	0.2858	0.5563	0.8555	1.0251	1.2167	1.4432	1.5770	1.7322	1.8025	1.8792	1.9637
40.	0.2664	0.5349	0.8313	0.9990	1.1883	1.4120	1.5440	1.6971	1.7665	1.8421	1.9254
45.	0.2505	0.5175	0.8116	0.9778	1.1654	1.3868	1.5174	1.6688	1.7374	1.8122	1.8945
50.	0.2372	0.5029	0.7952	0.9603	1.1464	1.3660	1.4954	1.6454	1.7134	1.7875	1.8690
60.	0.2158	0.4797	0.7693	0.9325	1.1164	1.3332	1.4609	1.6088	1.6758	1.7487	1.8291
70.	0.1993	0.4619	0.7495	0.9115	1.0937	1.3083	1.4348	1.5811	1.6474	1.7195	1.7990
80.	0.1861	0.4477	0.7338	0.8947	1.0757	1.2887	1.4141	1.5593	1.6250	1.6965	1.7753
90.	0.1753	0.4360	0.7210	0.8811	1.0610	1.2727	1.3973	1.5415	1.6068	1.6778	1.7560
100.	0.1661	0.4262	0.7102	0.8696	1.0487	1.2594	1.3833	1.5267	1.5916	1.6622	1.7400
150.	0.1351	0.3934	0.6743	0.8316	1.0080	1.2152	1.3370	1.4778	1.5414	1.6107	1.6870
200.	0.1169	0.3741	0.6533	0.8094	0.9844	1.1897	1.3102	1.4495	1.5125	1.5810	1.6565
250.	0.1045	0.3611	0.6392	0.7945	0.9686	1.1725	1.2923	1.4306	1.4932	1.5612	1.6361
300.	0.0953	0.3515	0.6288	0.7837	0.9570	1.1601	1.2793	1.4169	1.4791	1.5468	1.6213
350.	0.0882	0.3441	0.6209	0.7753	0.9481	1.1504	1.2692	1.4063	1.4683	1.5357	1.6098
400.	0.0825	0.3381	0.6144	0.7685	0.9409	1.1428	1.2612	1.3979	1.4596	1.5268	1.6007
450.	0.0777	0.3332	0.6092	0.7630	0.9350	1.1364	1.2545	1.3909	1.4525	1.5195	1.5932
500.	0.0737	0.3291	0.6047	0.7583	0.9301	1.1311	1.2490	1.3850	1.4465	1.5134	1.5869
1000.	0.0521	0.3067	0.5808	0.7333	0.9035	1.1026	1.2193	1.3538	1.4146	1.4806	1.5533

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9500

PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks

	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	4.0105	4.2101	4.4461	4.7379	5.1284	5.7487	6.3210	6.7172	7.5101	7.9802	8.9794
6.	3.5331	3.7087	3.9162	4.1733	4.5174	5.0639	5.5680	5.9170	6.6147	7.0285	7.9079
7.	3.2392	3.4005	3.5912	3.8269	4.1424	4.6434	5.1056	5.4256	6.0654	6.4448	7.2512
8.	3.0366	3.1883	3.3676	3.5889	3.8850	4.3553	4.7891	5.0894	5.6903	6.0467	6.8039
9.	2.8874	3.0320	3.2027	3.4139	3.6963	4.1448	4.5584	4.8448	5.4167	5.7559	6.4767
10.	2.7719	2.9110	3.0754	3.2785	3.5501	3.9813	4.3790	4.6542	5.2047	5.5310	6.2246
11.	2.6795	2.8144	2.9737	3.1706	3.4338	3.8517	4.2369	4.5035	5.0367	5.3528	6.0245
12.	2.6040	2.7355	2.8908	3.0826	3.3391	3.7461	4.1213	4.3810	4.9002	5.2080	5.8619
13.	2.5410	2.6697	2.8216	3.0093	3.2602	3.6582	4.0251	4.2790	4.7867	5.0875	5.7268
14.	2.4874	2.6137	2.7629	2.9470	3.1932	3.5837	3.9436	4.1926	4.6905	4.9855	5.6124
15.	2.4412	2.5655	2.7122	2.8934	3.1355	3.5196	3.8734	4.1183	4.6078	4.8979	5.5141
16.	2.4008	2.5234	2.6680	2.8466	3.0852	3.4636	3.8123	4.0535	4.5357	4.8215	5.4285
17.	2.3652	2.4862	2.6290	2.8053	3.0408	3.4144	3.7584	3.9965	4.4723	4.7543	5.3532
18.	2.3331	2.4527	2.5939	2.7682	3.0010	3.3701	3.7101	3.9453	4.4154	4.6940	5.2856
19.	2.3043	2.4227	2.5624	2.7349	2.9652	3.3305	3.6668	3.8995	4.3645	4.6400	5.2252
20.	2.2783	2.3956	2.5340	2.7049	2.9330	3.2947	3.6278	3.8582	4.3186	4.5914	5.1707
21.	2.2547	2.3710	2.5082	2.6776	2.9038	3.2623	3.5924	3.8207	4.2770	4.5473	5.1214
22.	2.2331	2.3485	2.4847	2.6527	2.8771	3.2327	3.5601	3.7866	4.2391	4.5071	5.0764
23.	2.2133	2.3279	2.4631	2.6299	2.8526	3.2056	3.5305	3.7552	4.2043	4.4703	5.0352
24.	2.1951	2.3089	2.4431	2.6088	2.8300	3.1806	3.5033	3.7264	4.1723	4.4364	4.9973
25.	2.1782	2.2913	2.4247	2.5894	2.8092	3.1575	3.4781	3.6998	4.1427	4.4051	4.9622
26.	2.1625	2.2749	2.4076	2.5713	2.7898	3.1360	3.4547	3.6751	4.1153	4.3761	4.9297
27.	2.1478	2.2597	2.3916	2.5545	2.7718	3.1161	3.4329	3.6521	4.0898	4.3490	4.8995
28.	2.1341	2.2455	2.3767	2.5387	2.7549	3.0974	3.4126	3.6306	4.0660	4.3238	4.8713
29.	2.1213	2.2321	2.3628	2.5240	2.7392	3.0800	3.3936	3.6105	4.0436	4.3002	4.8449
30.	2.1093	2.2196	2.3497	2.5102	2.7243	3.0636	3.3758	3.5916	4.0227	4.2780	4.8201
35.	2.0583	2.1666	2.2943	2.4517	2.6618	2.9945	3.3005	3.5120	3.9345	4.1847	4.7157
40.	2.0187	2.1254	2.2512	2.4063	2.6132	2.9408	3.2421	3.4503	3.8662	4.1124	4.6349
45.	1.9867	2.0922	2.2165	2.3698	2.5742	2.8977	3.1952	3.4008	3.8113	4.0543	4.5701
50.	1.9604	2.0648	2.1879	2.3397	2.5420	2.8622	3.1566	3.3601	3.7662	4.0066	4.5168
60.	1.9191	2.0220	2.1432	2.2926	2.4918	2.8068	3.0964	3.2965	3.6959	3.9323	4.4338
70.	1.8880	1.9897	2.1095	2.2572	2.4540	2.7652	3.0512	3.2488	3.6431	3.8764	4.3715
80.	1.8635	1.9643	2.0830	2.2293	2.4242	2.7325	3.0157	3.2113	3.6017	3.8326	4.3227
90.	1.8436	1.9437	2.0615	2.2067	2.4001	2.7059	2.9869	3.1809	3.5681	3.7971	4.2831
100.	1.8270	1.9265	2.0435	2.1878	2.3800	2.6839	2.9629	3.1556	3.5401	3.7676	4.2502
150.	1.7723	1.8698	1.9845	2.1258	2.3140	2.6113	2.8843	3.0727	3.4486	3.6709	4.1424
200.	1.7408	1.8372	1.9506	2.0902	2.2761	2.5697	2.8392	3.0251	3.3961	3.6155	4.0806
250.	1.7198	1.8154	1.9279	2.0664	2.2508	2.5420	2.8091	2.9935	3.3612	3.5786	4.0396
300.	1.7045	1.7996	1.9115	2.0491	2.2324	2.5218	2.7873	2.9705	3.3359	3.5519	4.0098
350.	1.6928	1.7874	1.8988	2.0359	2.2183	2.5064	2.7706	2.9529	3.3164	3.5314	3.9870
400.	1.6834	1.7777	1.8887	2.0253	2.2071	2.4941	2.7573	2.9389	3.3009	3.5150	3.9688
450.	1.6757	1.7697	1.8804	2.0166	2.1978	2.4839	2.7463	2.9273	3.2882	3.5016	3.9538
500.	1.6692	1.7630	1.8734	2.0093	2.1901	2.4754	2.7371	2.9176	3.2775	3.4903	3.9413
1000.	1.6346	1.7272	1.8362	1.9703	2.1487	2.4301	2.6881	2.8661	3.2207	3.4304	3.8746

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9750

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-1.6911	-1.3523	-1.1811	-0.6862	-0.3883	-0.1631	0.0368	0.2278	0.4176	0.6108	0.8103
6.	-1.7795	-1.4299	-1.2546	-0.7516	-0.4552	-0.2350	-0.0434	0.1360	0.3112	0.4873	0.6676
7.	-1.8509	-1.4926	-1.3140	-0.8042	-0.5070	-0.2896	-0.1030	0.0698	0.2369	0.4029	0.5710
8.	-1.9104	-1.5449	-1.3633	-0.8472	-0.5496	-0.3336	-0.1496	0.0191	0.1809	0.3403	0.5007
9.	-1.9608	-1.5892	-1.4051	-0.8836	-0.5851	-0.3697	-0.1877	-0.0218	0.1363	0.2914	0.4468
10.	-2.0048	-1.6279	-1.4415	-0.9149	-0.6152	-0.4001	-0.2197	-0.0558	0.0997	0.2515	0.4030
11.	-2.0435	-1.6619	-1.4735	-0.9422	-0.6414	-0.4265	-0.2468	-0.0844	0.0691	0.2184	0.3669
12.	-2.0778	-1.6921	-1.5018	-0.9665	-0.6645	-0.4496	-0.2705	-0.1092	0.0428	0.1902	0.3363
13.	-2.1085	-1.7192	-1.5273	-0.9882	-0.6850	-0.4701	-0.2914	-0.1309	0.0198	0.1657	0.3099
14.	-2.1364	-1.7436	-1.5503	-1.0078	-0.7036	-0.4885	-0.3100	-0.1502	-0.0004	0.1442	0.2869
15.	-2.1617	-1.7659	-1.5712	-1.0256	-0.7203	-0.5050	-0.3268	-0.1675	-0.0185	0.1251	0.2666
16.	-2.1849	-1.7863	-1.5904	-1.0418	-0.7357	-0.5201	-0.3420	-0.1831	-0.0348	0.1080	0.2484
17.	-2.2063	-1.8051	-1.6080	-1.0568	-0.7497	-0.5339	-0.3559	-0.1973	-0.0495	0.0925	0.2320
18.	-2.2262	-1.8226	-1.6243	-1.0705	-0.7625	-0.5465	-0.3685	-0.2102	-0.0628	0.0786	0.2173
19.	-2.2446	-1.8388	-1.6395	-1.0833	-0.7745	-0.5581	-0.3801	-0.2220	-0.0750	0.0658	0.2039
20.	-2.2618	-1.8539	-1.6537	-1.0952	-0.7855	-0.5689	-0.3909	-0.2329	-0.0863	0.0541	0.1916
21.	-2.2779	-1.8680	-1.6669	-1.1064	-0.7959	-0.5790	-0.4009	-0.2431	-0.0967	0.0433	0.1802
22.	-2.2930	-1.8813	-1.6793	-1.1168	-0.8055	-0.5884	-0.4102	-0.2525	-0.1064	0.0332	0.1697
23.	-2.3073	-1.8937	-1.6910	-1.1266	-0.8146	-0.5972	-0.4190	-0.2614	-0.1155	0.0239	0.1599
24.	-2.3207	-1.9055	-1.7021	-1.1358	-0.8232	-0.6054	-0.4272	-0.2696	-0.1239	0.0151	0.1508
25.	-2.3334	-1.9167	-1.7125	-1.1445	-0.8313	-0.6132	-0.4349	-0.2774	-0.1319	0.0069	0.1423
26.	-2.3455	-1.9272	-1.7224	-1.1528	-0.8389	-0.6206	-0.4422	-0.2848	-0.1393	-0.0008	0.1343
27.	-2.3569	-1.9373	-1.7318	-1.1606	-0.8461	-0.6276	-0.4491	-0.2917	-0.1464	-0.0080	0.1267
28.	-2.3678	-1.9468	-1.7407	-1.1681	-0.8530	-0.6342	-0.4556	-0.2983	-0.1531	-0.0149	0.1196
29.	-2.3782	-1.9559	-1.7492	-1.1752	-0.8595	-0.6405	-0.4619	-0.3045	-0.1594	-0.0214	0.1129
30.	-2.3881	-1.9646	-1.7573	-1.1819	-0.8658	-0.6465	-0.4678	-0.3104	-0.1654	-0.0275	0.1066
35.	-2.4317	-2.0028	-1.7930	-1.2117	-0.8931	-0.6727	-0.4935	-0.3361	-0.1914	-0.0541	0.0791
40.	-2.4677	-2.0343	-1.8225	-1.2360	-0.9154	-0.6941	-0.5144	-0.3569	-0.2123	-0.0754	0.0573
45.	-2.4981	-2.0608	-1.8473	-1.2566	-0.9341	-0.7119	-0.5317	-0.3741	-0.2296	-0.0930	0.0393
50.	-2.5242	-2.0837	-1.8686	-1.2741	-0.9501	-0.7271	-0.5465	-0.3887	-0.2442	-0.1078	0.0241
60.	-2.5671	-2.1211	-1.9035	-1.3029	-0.9762	-0.7519	-0.5705	-0.4123	-0.2678	-0.1316	-0.0002
70.	-2.6011	-2.1508	-1.9312	-1.3256	-0.9968	-0.7714	-0.5894	-0.4308	-0.2863	-0.1502	-0.0190
80.	-2.6290	-2.1751	-1.9539	-1.3442	-1.0136	-0.7872	-0.6047	-0.4458	-0.3011	-0.1651	-0.0342
90.	-2.6525	-2.1955	-1.9729	-1.3598	-1.0277	-0.8005	-0.6174	-0.4583	-0.3135	-0.1775	-0.0467
100.	-2.6725	-2.2130	-1.9892	-1.3731	-1.0397	-0.8118	-0.6283	-0.4689	-0.3240	-0.1880	-0.0573
150.	-2.7423	-2.2738	-2.0457	-1.4191	-1.0811	-0.8506	-0.6656	-0.5052	-0.3598	-0.2237	-0.0932
200.	-2.7853	-2.3111	-2.0805	-1.4473	-1.1063	-0.8742	-0.6881	-0.5271	-0.3813	-0.2450	-0.1144
250.	-2.8152	-2.3371	-2.1047	-1.4669	-1.1238	-0.8905	-0.7037	-0.5421	-0.3960	-0.2595	-0.1290
300.	-2.8376	-2.3566	-2.1228	-1.4815	-1.1369	-0.9027	-0.7152	-0.5533	-0.4069	-0.2703	-0.1397
350.	-2.8553	-2.3719	-2.1370	-1.4930	-1.1471	-0.9122	-0.7243	-0.5620	-0.4154	-0.2787	-0.1481
400.	-2.8696	-2.3843	-2.1485	-1.5023	-1.1554	-0.9199	-0.7316	-0.5691	-0.4223	-0.2855	-0.1548
450.	-2.8815	-2.3947	-2.1582	-1.5101	-1.1623	-0.9263	-0.7377	-0.5750	-0.4281	-0.2911	-0.1604
500.	-2.8917	-2.4035	-2.1663	-1.5167	-1.1682	-0.9318	-0.7429	-0.5800	-0.4329	-0.2959	-0.1652
1000.	-2.9478	-2.4521	-2.2115	-1.5530	-1.2004	-0.9616	-0.7712	-0.6071	-0.4593	-0.3218	-0.1908

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9750

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	1.2412	1.7329	2.3167	2.6621	3.0631	3.5469	3.8356	4.1738	4.3287	4.4979	4.6850
6.	1.0488	1.4774	1.9826	2.2801	2.6233	3.0365	3.2841	3.5733	3.7050	3.8489	4.0080
7.	0.9244	1.3168	1.7750	2.0435	2.3527	2.7245	2.9466	3.2061	3.3242	3.4534	3.5961
8.	0.8361	1.2042	1.6324	1.8826	2.1698	2.5141	2.7201	2.9607	3.0703	3.1900	3.3224
9.	0.7688	1.1200	1.5273	1.7644	2.0363	2.3617	2.5564	2.7837	2.8871	2.9999	3.1246
10.	0.7148	1.0548	1.4458	1.6727	1.9333	2.2453	2.4311	2.6480	2.7466	2.8542	2.9732
11.	0.6712	1.0014	1.3798	1.5991	1.8505	2.1513	2.3304	2.5392	2.6342	2.7378	2.8524
12.	0.6348	0.9573	1.3258	1.5391	1.7833	2.0752	2.2488	2.4513	2.5434	2.6438	2.7548
13.	0.6039	0.9201	1.2807	1.4890	1.7273	2.0119	2.1811	2.3784	2.4682	2.5659	2.6740
14.	0.5771	0.8882	1.2422	1.4465	1.6799	1.9583	2.1239	2.3168	2.4046	2.5002	2.6058
15.	0.5536	0.8604	1.2089	1.4097	1.6389	1.9123	2.0747	2.2639	2.3500	2.4437	2.5473
16.	0.5328	0.8360	1.1797	1.3775	1.6032	1.8721	2.0319	2.2179	2.3026	2.3947	2.4965
17.	0.5141	0.8143	1.1539	1.3491	1.5717	1.8368	1.9942	2.1775	2.2608	2.3516	2.4518
18.	0.4973	0.7946	1.1304	1.3233	1.5432	1.8048	1.9601	2.1409	2.2231	2.3126	2.4114
19.	0.4820	0.7768	1.1094	1.3002	1.5176	1.7762	1.9296	2.1082	2.1894	2.2778	2.3754
20.	0.4680	0.7606	1.0903	1.2793	1.4945	1.7504	1.9021	2.0787	2.1590	2.2464	2.3430
21.	0.4552	0.7458	1.0729	1.2603	1.4735	1.7270	1.8772	2.0520	2.1315	2.2181	2.3136
22.	0.4433	0.7323	1.0570	1.2428	1.4543	1.7056	1.8545	2.0277	2.1064	2.1922	2.2869
23.	0.4324	0.7198	1.0424	1.2269	1.4367	1.6860	1.8337	2.0054	2.0835	2.1685	2.2624
24.	0.4222	0.7081	1.0288	1.2121	1.4205	1.6680	1.8145	1.9849	2.0623	2.1467	2.2398
25.	0.4127	0.6974	1.0163	1.1984	1.4054	1.6513	1.7968	1.9660	2.0428	2.1266	2.2190
26.	0.4039	0.6873	1.0046	1.1857	1.3915	1.6358	1.7803	1.9484	2.0247	2.1079	2.1997
27.	0.3955	0.6779	0.9937	1.1738	1.3784	1.6213	1.7650	1.9320	2.0078	2.0905	2.1817
28.	0.3877	0.6690	0.9834	1.1627	1.3663	1.6079	1.7507	1.9167	1.9921	2.0743	2.1650
29.	0.3803	0.6607	0.9738	1.1523	1.3549	1.5952	1.7373	1.9024	1.9774	2.0591	2.1493
30.	0.3734	0.6529	0.9648	1.1425	1.3441	1.5833	1.7247	1.8890	1.9636	2.0449	2.1346
35.	0.3434	0.6194	0.9264	1.1010	1.2988	1.5332	1.6716	1.8324	1.9053	1.9849	2.0726
40.	0.3198	0.5930	0.8964	1.0685	1.2634	1.4941	1.6303	1.7884	1.8601	1.9383	2.0245
45.	0.3004	0.5716	0.8720	1.0422	1.2348	1.4626	1.5971	1.7531	1.8238	1.9009	1.9859
50.	0.2842	0.5537	0.8518	1.0205	1.2112	1.4367	1.5697	1.7240	1.7939	1.8701	1.9542
60.	0.2584	0.5254	0.8199	0.9863	1.1742	1.3960	1.5269	1.6785	1.7473	1.8221	1.9047
70.	0.2385	0.5038	0.7957	0.9605	1.1463	1.3654	1.4946	1.6443	1.7122	1.7861	1.8675
80.	0.2226	0.4866	0.7766	0.9400	1.1242	1.3413	1.4693	1.6175	1.6846	1.7577	1.8383
90.	0.2095	0.4725	0.7609	0.9233	1.1062	1.3217	1.4487	1.5957	1.6623	1.7348	1.8146
100.	0.1984	0.4606	0.7478	0.9094	1.0912	1.3054	1.4315	1.5775	1.6436	1.7156	1.7949
150.	0.1613	0.4210	0.7042	0.8632	1.0417	1.2515	1.3750	1.5178	1.5825	1.6528	1.7303
200.	0.1394	0.3978	0.6789	0.8364	1.0131	1.2205	1.3425	1.4835	1.5473	1.6167	1.6931
250.	0.1246	0.3821	0.6619	0.8184	0.9939	1.1998	1.3208	1.4606	1.5238	1.5926	1.6684
300.	0.1136	0.3707	0.6494	0.8053	0.9799	1.1847	1.3050	1.4439	1.5068	1.5751	1.6504
350.	0.1052	0.3618	0.6398	0.7952	0.9691	1.1731	1.2928	1.4312	1.4937	1.5617	1.6366
400.	0.0983	0.3546	0.6321	0.7871	0.9605	1.1638	1.2831	1.4210	1.4833	1.5510	1.6256
450.	0.0927	0.3487	0.6258	0.7804	0.9534	1.1562	1.2752	1.4126	1.4747	1.5422	1.6166
500.	0.0879	0.3437	0.6204	0.7748	0.9475	1.1498	1.2685	1.4055	1.4675	1.5348	1.6089
1000.	0.0621	0.3170	0.5917	0.7447	0.9156	1.1155	1.2327	1.3680	1.4290	1.4955	1.5685

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9750

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	4.8950	5.1359	5.4212	5.7745	6.2474	6.9986	7.6918	8.1704	9.1278	9.6957	10.9032
6.	4.1867	4.3917	4.6340	4.9338	5.3352	5.9730	6.5619	6.9698	7.7863	8.2707	9.3006
7.	3.7565	3.9404	4.1578	4.4274	4.7883	5.3618	5.8912	6.2578	6.9910	7.4260	8.3508
8.	3.4710	3.6415	3.8431	4.0924	4.4262	4.9566	5.4462	5.7853	6.4634	6.8657	7.7210
9.	3.2647	3.4253	3.6151	3.8500	4.1643	4.6639	5.1249	5.4442	6.0828	6.4616	7.2670
10.	3.1067	3.2599	3.4409	3.6649	3.9645	4.4407	4.8801	5.1844	5.7935	6.1547	6.9227
11.	2.9809	3.1283	3.3025	3.5180	3.8063	4.2642	4.6868	4.9795	5.5650	5.9123	6.6506
12.	2.8793	3.0221	3.1909	3.3995	3.6787	4.1221	4.5312	4.8145	5.3812	5.7173	6.4317
13.	2.7953	2.9343	3.0985	3.3016	3.5733	4.0048	4.4028	4.6784	5.2296	5.5565	6.2513
14.	2.7243	2.8602	3.0207	3.2191	3.4845	3.9060	4.2947	4.5638	5.1021	5.4212	6.0996
15.	2.6635	2.7967	2.9540	3.1484	3.4085	3.8215	4.2022	4.4659	4.9931	5.3056	5.9699
16.	2.6107	2.7415	2.8961	3.0871	3.3426	3.7482	4.1221	4.3810	4.8986	5.2055	5.8576
17.	2.5642	2.6931	2.8452	3.0333	3.2847	3.6839	4.0518	4.3065	4.8158	5.1177	5.7592
18.	2.5223	2.6494	2.7994	2.9848	3.2327	3.6261	3.9887	4.2397	4.7416	5.0390	5.6711
19.	2.4849	2.6104	2.7586	2.9416	3.1863	3.5746	3.9325	4.1802	4.6754	4.9689	5.5925
20.	2.4513	2.5753	2.7218	2.9027	3.1446	3.5283	3.8819	4.1267	4.6159	4.9059	5.5220
21.	2.4208	2.5436	2.6885	2.8675	3.1068	3.4864	3.8362	4.0783	4.5622	4.8490	5.4582
22.	2.3930	2.5147	2.6582	2.8355	3.0725	3.4483	3.7947	4.0343	4.5133	4.7972	5.4003
23.	2.3676	2.4882	2.6305	2.8062	3.0410	3.4135	3.7567	3.9941	4.4687	4.7499	5.3473
24.	2.3442	2.4638	2.6050	2.7793	3.0122	3.3815	3.7217	3.9572	4.4277	4.7065	5.2987
25.	2.3227	2.4414	2.5814	2.7544	2.9855	3.3520	3.6895	3.9231	4.3899	4.6664	5.2539
26.	2.3026	2.4205	2.5596	2.7314	2.9608	3.3246	3.6597	3.8916	4.3548	4.6293	5.2124
27.	2.2840	2.4011	2.5393	2.7100	2.9379	3.2992	3.6320	3.8622	4.3223	4.5949	5.1738
28.	2.2666	2.3831	2.5204	2.6900	2.9165	3.2755	3.6062	3.8349	4.2920	4.5628	5.1379
29.	2.2504	2.3661	2.5027	2.6713	2.8964	3.2533	3.5820	3.8094	4.2637	4.5328	5.1044
30.	2.2351	2.3503	2.4861	2.6538	2.8777	3.2326	3.5594	3.7854	4.2371	4.5047	5.0729
35.	2.1709	2.2835	2.4162	2.5801	2.7988	3.1453	3.4643	3.6849	4.1257	4.3867	4.9410
40.	2.1211	2.2317	2.3621	2.5230	2.7377	3.0778	3.3909	3.6073	4.0396	4.2957	4.8393
45.	2.0812	2.1902	2.3187	2.4772	2.6888	3.0238	3.3321	3.5452	3.9709	4.2229	4.7580
50.	2.0483	2.1561	2.2830	2.4397	2.6486	2.9795	3.2839	3.4943	3.9145	4.1633	4.6914
60.	1.9971	2.1029	2.2275	2.3812	2.5861	2.9106	3.2090	3.4152	3.8269	4.0707	4.5880
70.	1.9587	2.0630	2.1858	2.3373	2.5394	2.8590	3.1529	3.3560	3.7615	4.0015	4.5108
80.	1.9285	2.0317	2.1532	2.3030	2.5027	2.8186	3.1091	3.3097	3.7103	3.9474	4.4504
90.	1.9041	2.0063	2.1267	2.2751	2.4730	2.7860	3.0736	3.2723	3.6689	3.9036	4.4016
100.	1.8837	1.9852	2.1047	2.2520	2.4483	2.7588	3.0441	3.2412	3.6346	3.8673	4.3612
150.	1.8169	1.9160	2.0326	2.1762	2.3676	2.6701	2.9479	3.1397	3.5224	3.7489	4.2291
200.	1.7786	1.8763	1.9912	2.1328	2.3214	2.6193	2.8929	3.0817	3.4584	3.6813	4.1538
250.	1.7531	1.8498	1.9637	2.1039	2.2906	2.5856	2.8564	3.0432	3.4160	3.6364	4.1039
300.	1.7346	1.8307	1.9437	2.0830	2.2684	2.5612	2.8299	3.0154	3.3853	3.6040	4.0678
350.	1.7203	1.8160	1.9284	2.0669	2.2513	2.5425	2.8097	2.9940	3.3617	3.5792	4.0401
400.	1.7090	1.8042	1.9162	2.0541	2.2377	2.5276	2.7935	2.9770	3.3430	3.5594	4.0181
450.	1.6997	1.7946	1.9062	2.0436	2.2265	2.5153	2.7803	2.9631	3.3276	3.5431	4.0000
500.	1.6918	1.7865	1.8978	2.0348	2.2172	2.5050	2.7691	2.9514	3.3147	3.5295	3.9849
1000.	1.6502	1.7434	1.8530	1.9879	2.1674	2.4505	2.7102	2.8893	3.2463	3.4573	3.9046

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9900

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-1.5077	-1.1889	-1.0253	-0.5434	-0.2372	0.0040	0.2255	0.4441	0.6685	0.9060	1.1461
6.	-1.6040	-1.2742	-1.1070	-0.6196	-0.3191	-0.0873	0.1180	0.3145	0.5121	0.7156	0.9270
7.	-1.6817	-1.3423	-1.1715	-0.6774	-0.3802	-0.1585	0.0386	0.2263	0.4104	0.5942	0.7862
8.	-1.7482	-1.4016	-1.2269	-0.7248	-0.4299	-0.2106	-0.0203	0.1584	0.3336	0.5086	0.6861
9.	-1.7992	-1.4462	-1.2704	-0.7671	-0.4716	-0.2562	-0.0673	0.1080	0.2755	0.4404	0.6117
10.	-1.8504	-1.4916	-1.3129	-0.8032	-0.5075	-0.2914	-0.1066	0.0635	0.2270	0.3888	0.5519
11.	-1.8929	-1.5291	-1.3484	-0.8343	-0.5379	-0.3227	-0.1396	0.0278	0.1880	0.3456	0.5039
12.	-1.9308	-1.5626	-1.3799	-0.8618	-0.5646	-0.3500	-0.1683	-0.0028	0.1548	0.3093	0.4639
13.	-1.9648	-1.5926	-1.4083	-0.8865	-0.5884	-0.3742	-0.1934	-0.0295	0.1261	0.2782	0.4300
14.	-1.9956	-1.6198	-1.4339	-0.9087	-0.6098	-0.3957	-0.2158	-0.0531	0.1009	0.2511	0.4006
15.	-2.0237	-1.6446	-1.4573	-0.9289	-0.6292	-0.4152	-0.2358	-0.0741	0.0786	0.2272	0.3748
16.	-2.0495	-1.6673	-1.4787	-0.9473	-0.6468	-0.4329	-0.2540	-0.0930	0.0587	0.2059	0.3520
17.	-2.0733	-1.6883	-1.4984	-0.9643	-0.6630	-0.4490	-0.2705	-0.1101	0.0406	0.1868	0.3315
18.	-2.0955	-1.7078	-1.5168	-0.9800	-0.6778	-0.4637	-0.2854	-0.1256	0.0244	0.1695	0.3130
19.	-2.1161	-1.7260	-1.5339	-0.9945	-0.6916	-0.4773	-0.2992	-0.1398	0.0095	0.1538	0.2962
20.	-2.1354	-1.7429	-1.5498	-1.0080	-0.7043	-0.4899	-0.3119	-0.1529	-0.0041	0.1394	0.2808
21.	-2.1534	-1.7588	-1.5647	-1.0207	-0.7162	-0.5016	-0.3237	-0.1650	-0.0168	0.1261	0.2667
22.	-2.1704	-1.7737	-1.5787	-1.0325	-0.7274	-0.5125	-0.3348	-0.1763	-0.0285	0.1138	0.2537
23.	-2.1863	-1.7878	-1.5919	-1.0437	-0.7378	-0.5228	-0.3451	-0.1869	-0.0394	0.1023	0.2416
24.	-2.2014	-1.8010	-1.6043	-1.0542	-0.7477	-0.5324	-0.3547	-0.1967	-0.0496	0.0917	0.2304
25.	-2.2157	-1.8136	-1.6161	-1.0642	-0.7570	-0.5415	-0.3638	-0.2060	-0.0591	0.0817	0.2199
26.	-2.2293	-1.8255	-1.6272	-1.0736	-0.7658	-0.5501	-0.3724	-0.2147	-0.0681	0.0724	0.2101
27.	-2.2422	-1.8368	-1.6379	-1.0825	-0.7741	-0.5582	-0.3805	-0.2229	-0.0766	0.0636	0.2009
28.	-2.2544	-1.8476	-1.6480	-1.0910	-0.7820	-0.5659	-0.3882	-0.2307	-0.0846	0.0553	0.1922
29.	-2.2661	-1.8578	-1.6576	-1.0991	-0.7896	-0.5733	-0.3955	-0.2381	-0.0921	0.0474	0.1840
30.	-2.2773	-1.8676	-1.6668	-1.1068	-0.7967	-0.5802	-0.4025	-0.2451	-0.0993	0.0400	0.1762
35.	-2.3266	-1.9109	-1.7073	-1.1408	-0.8282	-0.6107	-0.4327	-0.2756	-0.1304	0.0080	0.1429
40.	-2.3675	-1.9467	-1.7408	-1.1687	-0.8540	-0.6355	-0.4572	-0.3001	-0.1553	-0.0176	0.1165
45.	-2.4021	-1.9770	-1.7691	-1.1922	-0.8755	-0.6562	-0.4775	-0.3204	-0.1758	-0.0385	0.0948
50.	-2.4318	-2.0030	-1.7934	-1.2124	-0.8940	-0.6739	-0.4948	-0.3376	-0.1931	-0.0562	0.0766
60.	-2.4808	-2.0458	-1.8334	-1.2454	-0.9242	-0.7027	-0.5229	-0.3654	-0.2211	-0.0847	0.0474
70.	-2.5197	-2.0798	-1.8651	-1.2716	-0.9480	-0.7253	-0.5449	-0.3872	-0.2429	-0.1067	0.0249
80.	-2.5517	-2.1078	-1.8912	-1.2930	-0.9675	-0.7438	-0.5628	-0.4049	-0.2605	-0.1245	0.0068
90.	-2.5786	-2.1312	-1.9131	-1.3110	-0.9838	-0.7592	-0.5778	-0.4196	-0.2752	-0.1392	-0.0081
100.	-2.6017	-2.1514	-1.9318	-1.3264	-0.9977	-0.7724	-0.5905	-0.4320	-0.2876	-0.1516	-0.0207
150.	-2.6823	-2.2216	-1.9972	-1.3798	-1.0459	-0.8177	-0.6342	-0.4747	-0.3298	-0.1939	-0.0633
200.	-2.7322	-2.2650	-2.0376	-1.4126	-1.0754	-0.8453	-0.6606	-0.5005	-0.3551	-0.2191	-0.0886
250.	-2.7670	-2.2952	-2.0657	-1.4355	-1.0958	-0.8644	-0.6789	-0.5182	-0.3725	-0.2363	-0.1059
300.	-2.7931	-2.3179	-2.0868	-1.4525	-1.1111	-0.8787	-0.6924	-0.5313	-0.3854	-0.2491	-0.1186
350.	-2.8136	-2.3357	-2.1034	-1.4660	-1.1231	-0.8898	-0.7031	-0.5416	-0.3955	-0.2590	-0.1285
400.	-2.8303	-2.3503	-2.1169	-1.4769	-1.1328	-0.8989	-0.7117	-0.5499	-0.4036	-0.2671	-0.1365
450.	-2.8443	-2.3624	-2.1282	-1.4860	-1.1409	-0.9064	-0.7189	-0.5568	-0.4104	-0.2737	-0.1432
500.	-2.8562	-2.3727	-2.1378	-1.4937	-1.1478	-0.9128	-0.7249	-0.5627	-0.4161	-0.2794	-0.1488
1000.	-2.9220	-2.4298	-2.1907	-1.5363	-1.1857	-0.9480	-0.7583	-0.5948	-0.4473	-0.3101	-0.1792

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9900

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	1.6809	2.2875	3.0106	3.4478	3.9473	4.5635	4.9448	5.3895	5.5918	5.8127	6.0568
6.	1.3682	1.8742	2.4867	2.8462	3.2590	3.7541	4.0493	4.3957	4.5596	4.7385	4.9363
7.	1.1901	1.6438	2.1806	2.4955	2.8578	3.2930	3.5529	3.8563	3.9992	4.1555	4.3284
8.	1.0602	1.4791	1.9660	2.2509	2.5788	2.9760	3.2197	3.5044	3.6314	3.7691	3.9214
9.	0.9617	1.3577	1.8198	2.0850	2.3902	2.7591	2.9849	3.2487	3.3672	3.4952	3.6368
10.	0.8915	1.2664	1.7006	1.9539	2.2455	2.5957	2.8044	3.0482	3.1594	3.2810	3.4155
11.	0.8320	1.1922	1.6084	1.8508	2.1293	2.4637	2.6630	2.8956	3.0016	3.1175	3.2457
12.	0.7833	1.1322	1.5343	1.7681	2.0365	2.3584	2.5501	2.7740	2.8759	2.9873	3.1105
13.	0.7424	1.0823	1.4732	1.7001	1.9602	2.2720	2.4576	2.6743	2.7730	2.8808	2.9999
14.	0.7075	1.0400	1.4216	1.6429	1.8962	2.1996	2.3802	2.5910	2.6869	2.7917	2.9075
15.	0.6771	1.0036	1.3774	1.5939	1.8415	2.1378	2.3142	2.5200	2.6136	2.7158	2.8288
16.	0.6503	0.9717	1.3390	1.5514	1.7941	2.0844	2.2572	2.4586	2.5503	2.6504	2.7609
17.	0.6266	0.9436	1.3052	1.5141	1.7526	2.0377	2.2072	2.4050	2.4950	2.5931	2.7016
18.	0.6050	0.9179	1.2744	1.4801	1.7148	1.9953	2.1620	2.3564	2.4449	2.5413	2.6479
19.	0.5854	0.8949	1.2468	1.4497	1.6812	1.9575	2.1218	2.3132	2.4003	2.4953	2.6002
20.	0.5677	0.8741	1.2220	1.4224	1.6509	1.9237	2.0857	2.2745	2.3604	2.4541	2.5575
21.	0.5515	0.8552	1.1995	1.3977	1.6236	1.8931	2.0531	2.2396	2.3244	2.4169	2.5190
22.	0.5366	0.8378	1.1790	1.3752	1.5987	1.8653	2.0235	2.2078	2.2917	2.3831	2.4840
23.	0.5229	0.8219	1.1602	1.3546	1.5760	1.8399	1.9965	2.1789	2.2618	2.3523	2.4521
24.	0.5101	0.8072	1.1428	1.3356	1.5551	1.8165	1.9717	2.1523	2.2344	2.3240	2.4229
25.	0.4983	0.7935	1.1268	1.3181	1.5358	1.7950	1.9488	2.1278	2.2092	2.2980	2.3960
26.	0.4872	0.7808	1.1119	1.3018	1.5179	1.7751	1.9276	2.1052	2.1859	2.2739	2.3711
27.	0.4769	0.7690	1.0981	1.2867	1.5013	1.7566	1.9080	2.0841	2.1642	2.2515	2.3479
28.	0.4672	0.7579	1.0851	1.2726	1.4858	1.7393	1.8896	2.0646	2.1441	2.2307	2.3264
29.	0.4580	0.7475	1.0730	1.2594	1.4713	1.7232	1.8725	2.0463	2.1252	2.2113	2.3063
30.	0.4494	0.7377	1.0616	1.2470	1.4576	1.7081	1.8565	2.0291	2.1076	2.1931	2.2875
35.	0.4126	0.6960	1.0134	1.1946	1.4003	1.6445	1.7891	1.9572	2.0335	2.1167	2.2086
40.	0.3836	0.6633	0.9758	1.1538	1.3557	1.5952	1.7369	1.9015	1.9763	2.0577	2.1476
45.	0.3599	0.6368	0.9454	1.1209	1.3198	1.5555	1.6950	1.8569	1.9304	2.0105	2.0989
50.	0.3401	0.6148	0.9202	1.0938	1.2903	1.5230	1.6606	1.8204	1.8929	1.9718	2.0590
60.	0.3087	0.5801	0.8809	1.0514	1.2443	1.4724	1.6072	1.7636	1.8346	1.9118	1.9971
70.	0.2847	0.5537	0.8511	1.0195	1.2097	1.4345	1.5672	1.7212	1.7910	1.8670	1.9508
80.	0.2655	0.5328	0.8276	0.9944	1.1826	1.4047	1.5359	1.6880	1.7569	1.8320	1.9147
90.	0.2497	0.5156	0.8085	0.9739	1.1605	1.3806	1.5105	1.6611	1.7293	1.8036	1.8855
100.	0.2365	0.5013	0.7925	0.9569	1.1422	1.3606	1.4895	1.6388	1.7064	1.7801	1.8612
150.	0.1920	0.4535	0.7397	0.9007	1.0818	1.2949	1.4204	1.5658	1.6316	1.7032	1.7820
200.	0.1658	0.4256	0.7091	0.8682	1.0470	1.2572	1.3808	1.5239	1.5887	1.6591	1.7367
250.	0.1481	0.4068	0.6886	0.8465	1.0238	1.2320	1.3545	1.4961	1.5602	1.6299	1.7067
300.	0.1351	0.3931	0.6736	0.8307	1.0069	1.2138	1.3354	1.4760	1.5396	1.6088	1.6849
350.	0.1250	0.3824	0.6621	0.8185	0.9940	1.1998	1.3208	1.4606	1.5238	1.5926	1.6683
400.	0.1168	0.3739	0.6528	0.8088	0.9836	1.1886	1.3091	1.4482	1.5112	1.5796	1.6550
450.	0.1101	0.3668	0.6452	0.8008	0.9751	1.1795	1.2995	1.4381	1.5008	1.5690	1.6441
500.	0.1044	0.3609	0.6388	0.7941	0.9679	1.1717	1.2914	1.4296	1.4921	1.5601	1.6349
1000.	0.0737	0.3290	0.6045	0.7581	0.9298	1.1307	1.2486	1.3846	1.4460	1.5128	1.5864

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9900

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	6.3307	6.6448	7.0157	7.4743	8.0878	9.0621	9.9607	10.5712	11.7883	12.5111	14.0490
6.	5.1585	5.4132	5.7142	6.0866	6.5849	7.3765	8.1069	8.6086	9.6042	10.1953	11.4528
7.	4.5225	4.7451	5.0079	5.3253	5.7505	6.4269	7.0517	7.4864	8.3656	8.8871	9.9958
8.	4.0924	4.2887	4.5208	4.8136	5.2064	5.8307	6.4069	6.8056	7.5956	8.0645	9.0619
9.	3.7959	3.9785	4.1943	4.4614	4.8192	5.3883	5.9139	6.2783	7.0150	7.4521	8.3813
10.	3.5665	3.7398	3.9446	4.1978	4.5367	5.0755	5.5731	5.9179	6.6083	7.0179	7.8891
11.	3.3896	3.5548	3.7500	3.9912	4.3142	4.8275	5.3015	5.6300	6.2873	6.6774	7.5068
12.	3.2489	3.4075	3.5951	3.8270	4.1373	4.6305	5.0858	5.4012	6.0325	6.4070	7.2033
13.	3.1337	3.2872	3.4685	3.6928	3.9927	4.4695	4.9096	5.2145	5.8245	6.1864	6.9558
14.	3.0375	3.1866	3.3628	3.5807	3.8721	4.3352	4.7627	5.0588	5.6511	6.0024	6.7494
15.	2.9556	3.1011	3.2729	3.4854	3.7696	4.2212	4.6379	4.9266	5.5040	5.8464	6.5744
16.	2.8850	3.0273	3.1954	3.4033	3.6813	4.1230	4.5305	4.8128	5.3773	5.7121	6.4238
17.	2.8233	2.9629	3.1278	3.3316	3.6042	4.0373	4.4368	4.7136	5.2669	5.5950	6.2926
18.	2.7675	2.9047	3.0667	3.2670	3.5348	3.9601	4.3525	4.6243	5.1677	5.4899	6.1748
19.	2.7180	2.8530	3.0125	3.2096	3.4731	3.8917	4.2777	4.5451	5.0796	5.3966	6.0704
20.	2.6736	2.8067	2.9640	3.1583	3.4180	3.8304	4.2108	4.4742	5.0010	5.3133	5.9771
21.	2.6336	2.7650	2.9202	3.1120	3.3683	3.7753	4.1506	4.4105	4.9301	5.2382	5.8931
22.	2.5973	2.7272	2.8805	3.0700	3.3232	3.7253	4.0960	4.3527	4.8660	5.1703	5.8170
23.	2.5642	2.6927	2.8443	3.0317	3.2822	3.6797	4.0463	4.3001	4.8075	5.1084	5.7477
24.	2.5339	2.6610	2.8112	2.9967	3.2446	3.6380	4.0008	4.2519	4.7540	5.0517	5.6842
25.	2.5059	2.6319	2.7806	2.9644	3.2099	3.5996	3.9589	4.2076	4.7048	4.9996	5.6259
26.	2.4801	2.6050	2.7524	2.9346	3.1780	3.5642	3.9202	4.1666	4.6594	4.9514	5.5721
27.	2.4561	2.5800	2.7262	2.9069	3.1483	3.5313	3.8843	4.1287	4.6173	4.9069	5.5222
28.	2.4338	2.5567	2.7019	2.8812	3.1207	3.5007	3.8510	4.0934	4.5781	4.8654	5.4758
29.	2.4129	2.5350	2.6791	2.8572	3.0949	3.4722	3.8199	4.0605	4.5416	4.8267	5.4326
30.	2.3934	2.5147	2.6578	2.8347	3.0708	3.4455	3.7908	4.0297	4.5074	4.7906	5.3921
35.	2.3116	2.4295	2.5687	2.7406	2.9700	3.3339	3.6692	3.9011	4.3648	4.6396	5.2232
40.	2.2484	2.3638	2.5000	2.6680	2.8924	3.2481	3.5757	3.8023	4.2553	4.5237	5.0937
45.	2.1979	2.3113	2.4451	2.6102	2.8305	3.1797	3.5013	3.7237	4.1681	4.4315	4.9907
50.	2.1566	2.2684	2.4002	2.5628	2.7799	3.1239	3.4405	3.6595	4.0971	4.3563	4.9067
60.	2.0925	2.2018	2.3306	2.4896	2.7016	3.0375	3.3466	3.5603	3.9873	4.2401	4.7770
70.	2.0447	2.1522	2.2788	2.4350	2.6433	2.9732	3.2768	3.4866	3.9057	4.1539	4.6807
80.	2.0074	2.1134	2.2383	2.3924	2.5979	2.9231	3.2223	3.4291	3.8421	4.0867	4.6057
90.	1.9772	2.0820	2.2056	2.3580	2.5612	2.8827	3.1784	3.3828	3.7909	4.0325	4.5453
100.	1.9521	2.0560	2.1785	2.3295	2.5307	2.8493	3.1421	3.3445	3.7485	3.9877	4.4953
150.	1.8703	1.9712	2.0901	2.2365	2.4317	2.7404	3.0240	3.2198	3.6108	3.8422	4.3331
200.	1.8236	1.9228	2.0396	2.1836	2.3753	2.6784	2.9568	3.1491	3.5327	3.7597	4.2411
250.	1.7926	1.8907	2.0062	2.1485	2.3380	2.6374	2.9124	3.1023	3.4811	3.7052	4.1804
300.	1.7701	1.8675	1.9820	2.1231	2.3110	2.6078	2.8804	3.0685	3.4438	3.6658	4.1366
350.	1.7530	1.8497	1.9635	2.1037	2.2903	2.5852	2.8559	3.0427	3.4153	3.6358	4.1031
400.	1.7393	1.8355	1.9488	2.0882	2.2739	2.5672	2.8364	3.0221	3.3927	3.6118	4.0765
450.	1.7280	1.8239	1.9367	2.0755	2.2604	2.5524	2.8204	3.0053	3.3741	3.5922	4.0547
500.	1.7186	1.8141	1.9265	2.0649	2.2491	2.5400	2.8069	2.9912	3.3585	3.5758	4.0364
1000.	1.6686	1.7624	1.8727	2.0085	2.1893	2.4745	2.7361	2.9165	3.2763	3.4890	3.9398

TABLE III(CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9950

SAMPLE SIZE	PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks										
	0.00100	0.00500	0.01000	0.05000	0.10000	0.15000	0.20000	0.25000	0.30000	0.35000	0.40000
5.	-1.3957	-1.0889	-0.9284	-0.4477	-0.1289	0.1277	0.3727	0.6242	0.8841	1.1445	1.4386
6.	-1.4937	-1.1747	-1.0116	-0.5308	-0.2253	0.0168	0.2380	0.4554	0.6745	0.8936	1.1277
7.	-1.5770	-1.2494	-1.0822	-0.5935	-0.2937	-0.0625	0.1439	0.3430	0.5422	0.7431	0.9509
8.	-1.6441	-1.3088	-1.1394	-0.6480	-0.3518	-0.1255	0.0729	0.2604	0.4456	0.6341	0.8313
9.	-1.7009	-1.3592	-1.1878	-0.6934	-0.3976	-0.1766	0.0184	0.2002	0.3761	0.5529	0.7361
10.	-1.7530	-1.4051	-1.2309	-0.7306	-0.4358	-0.2170	-0.0278	0.1485	0.3197	0.4903	0.6635
11.	-1.7978	-1.4448	-1.2685	-0.7640	-0.4691	-0.2520	-0.0655	0.1070	0.2735	0.4385	0.6052
12.	-1.8378	-1.4802	-1.3020	-0.7936	-0.4984	-0.2825	-0.0980	0.0717	0.2347	0.3954	0.5574
13.	-1.8738	-1.5120	-1.3321	-0.8201	-0.5244	-0.3094	-0.1264	0.0412	0.2014	0.3589	0.5172
14.	-1.9064	-1.5409	-1.3594	-0.8440	-0.5478	-0.3333	-0.1516	0.0144	0.1724	0.3274	0.4827
15.	-1.9361	-1.5672	-1.3843	-0.8657	-0.5689	-0.3549	-0.1740	-0.0095	0.1469	0.2997	0.4526
16.	-1.9635	-1.5913	-1.4071	-0.8856	-0.5881	-0.3744	-0.1943	-0.0308	0.1241	0.2752	0.4261
17.	-1.9887	-1.6136	-1.4281	-0.9038	-0.6057	-0.3922	-0.2127	-0.0501	0.1036	0.2532	0.4024
18.	-2.0122	-1.6344	-1.4477	-0.9207	-0.6219	-0.4084	-0.2293	-0.0675	0.0851	0.2334	0.3810
19.	-2.0341	-1.6537	-1.4659	-0.9364	-0.6368	-0.4233	-0.2446	-0.0834	0.0682	0.2154	0.3616
20.	-2.0546	-1.6718	-1.4829	-0.9509	-0.6506	-0.4371	-0.2587	-0.0981	0.0528	0.1989	0.3439
21.	-2.0738	-1.6887	-1.4988	-0.9646	-0.6636	-0.4499	-0.2718	-0.1117	0.0385	0.1838	0.3277
22.	-2.0918	-1.7046	-1.5137	-0.9774	-0.6757	-0.4619	-0.2839	-0.1243	0.0253	0.1698	0.3128
23.	-2.1088	-1.7196	-1.5278	-0.9894	-0.6871	-0.4732	-0.2953	-0.1361	0.0130	0.1569	0.2990
24.	-2.1249	-1.7337	-1.5411	-1.0007	-0.6977	-0.4837	-0.3060	-0.1471	0.0016	0.1448	0.2862
25.	-2.1401	-1.7471	-1.5537	-1.0115	-0.7078	-0.4937	-0.3161	-0.1574	-0.0092	0.1336	0.2742
26.	-2.1546	-1.7598	-1.5657	-1.0216	-0.7174	-0.5031	-0.3255	-0.1671	-0.0192	0.1230	0.2631
27.	-2.1683	-1.7719	-1.5770	-1.0313	-0.7265	-0.5120	-0.3345	-0.1762	-0.0287	0.1131	0.2526
28.	-2.1814	-1.7835	-1.5878	-1.0404	-0.7351	-0.5204	-0.3429	-0.1849	-0.0377	0.1038	0.2428
29.	-2.1939	-1.7944	-1.5982	-1.0492	-0.7432	-0.5284	-0.3510	-0.1931	-0.0461	0.0949	0.2335
30.	-2.2058	-1.8049	-1.6080	-1.0575	-0.7510	-0.5361	-0.3586	-0.2009	-0.0541	0.0866	0.2247
35.	-2.2586	-1.8513	-1.6515	-1.0942	-0.7852	-0.5694	-0.3919	-0.2346	-0.0888	0.0507	0.1871
40.	-2.3024	-1.8897	-1.6875	-1.1243	-0.8132	-0.5965	-0.4187	-0.2617	-0.1164	0.0221	0.1574
45.	-2.3395	-1.9223	-1.7179	-1.1498	-0.8367	-0.6191	-0.4411	-0.2841	-0.1392	-0.0013	0.1331
50.	-2.3715	-1.9503	-1.7442	-1.1716	-0.8568	-0.6384	-0.4601	-0.3031	-0.1585	-0.0210	0.1127
60.	-2.4242	-1.9964	-1.7873	-1.2075	-0.8896	-0.6698	-0.4909	-0.3338	-0.1895	-0.0527	0.0801
70.	-2.4662	-2.0332	-1.8216	-1.2359	-0.9156	-0.6946	-0.5151	-0.3578	-0.2136	-0.0772	0.0550
80.	-2.5008	-2.0634	-1.8498	-1.2591	-0.9368	-0.7148	-0.5348	-0.3773	-0.2331	-0.0969	0.0349
90.	-2.5300	-2.0888	-1.8735	-1.2787	-0.9546	-0.7316	-0.5512	-0.3934	-0.2492	-0.1132	0.0183
100.	-2.5550	-2.1107	-1.8939	-1.2954	-0.9698	-0.7460	-0.5651	-0.4072	-0.2629	-0.1270	0.0043
150.	-2.6424	-2.1869	-1.9650	-1.3536	-1.0224	-0.7957	-0.6130	-0.4542	-0.3095	-0.1737	-0.0430
200.	-2.6967	-2.2342	-2.0090	-1.3894	-1.0546	-0.8259	-0.6421	-0.4825	-0.3375	-0.2016	-0.0710
250.	-2.7347	-2.2672	-2.0397	-1.4144	-1.0770	-0.8469	-0.6621	-0.5020	-0.3567	-0.2206	-0.0901
300.	-2.7632	-2.2920	-2.0628	-1.4331	-1.0938	-0.8625	-0.6771	-0.5165	-0.3709	-0.2347	-0.1043
350.	-2.7857	-2.3115	-2.0809	-1.4478	-1.1069	-0.8748	-0.6888	-0.5278	-0.3820	-0.2457	-0.1152
400.	-2.8040	-2.3274	-2.0957	-1.4598	-1.1176	-0.8847	-0.6982	-0.5369	-0.3909	-0.2546	-0.1241
450.	-2.8193	-2.3407	-2.1080	-1.4697	-1.1265	-0.8930	-0.7061	-0.5446	-0.3984	-0.2619	-0.1314
500.	-2.8323	-2.3520	-2.1185	-1.4782	-1.1340	-0.9001	-0.7128	-0.5510	-0.4047	-0.2682	-0.1376
1000.	-2.9045	-2.4146	-2.1767	-1.5251	-1.1757	-0.9388	-0.7495	-0.5864	-0.4392	-0.3021	-0.1713

TABLE III (CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9950

PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks

SAMPLE SIZE	0.50000	0.60000	0.70000	0.75000	0.80000	0.85000	0.87500	0.90000	0.91000	0.92000	0.93000
5.	2.0590	2.7854	3.6302	4.1505	4.7523	5.4964	5.9595	6.4987	6.7438	7.0113	7.3068
6.	1.6491	2.2213	2.9262	3.3528	3.8413	4.4255	4.7734	5.1790	5.3636	5.5651	5.7880
7.	1.3999	1.9015	2.5074	2.8637	3.2804	3.7802	4.0784	4.4265	4.5850	4.7582	4.9496
8.	1.2320	1.6887	2.2380	2.5517	2.9133	3.3577	3.6231	3.9330	4.0777	4.2370	4.4131
9.	1.1179	1.5460	2.0390	2.3343	2.6744	3.0832	3.3274	3.6127	3.7427	3.8847	4.0417
10.	1.0277	1.4307	1.9011	2.1769	2.4939	2.8750	3.1026	3.3685	3.4898	3.6225	3.7692
11.	0.9541	1.3384	1.7859	2.0476	2.3484	2.7095	2.9251	3.1769	3.2917	3.4173	3.5561
12.	0.8947	1.2648	1.6946	1.9454	2.2334	2.5789	2.7851	3.0259	3.1357	3.2557	3.3884
13.	0.8454	1.2044	1.6200	1.8621	2.1399	2.4729	2.6715	2.9034	3.0091	3.1246	3.2523
14.	0.8036	1.1535	1.5576	1.7926	2.0620	2.3847	2.5770	2.8016	2.9039	3.0157	3.1393
15.	0.7676	1.1100	1.5045	1.7335	1.9959	2.3099	2.4970	2.7155	2.8150	2.9237	3.0438
16.	0.7361	1.0723	1.4585	1.6825	1.9389	2.2456	2.4283	2.6415	2.7385	2.8446	2.9618
17.	0.7083	1.0390	1.4183	1.6380	1.8892	2.1896	2.3684	2.5770	2.6720	2.7758	2.8905
18.	0.6829	1.0085	1.3814	1.5971	1.8438	2.1385	2.3140	2.5186	2.6118	2.7136	2.8260
19.	0.6600	0.9813	1.3485	1.5608	1.8035	2.0933	2.2658	2.4670	2.5585	2.6585	2.7689
20.	0.6393	0.9568	1.3190	1.5283	1.7674	2.0529	2.2227	2.4208	2.5109	2.6093	2.7180
21.	0.6205	0.9345	1.2924	1.4989	1.7349	2.0165	2.1840	2.3793	2.4681	2.5651	2.6722
22.	0.6032	0.9142	1.2681	1.4723	1.7054	1.9835	2.1489	2.3417	2.4294	2.5250	2.6307
23.	0.5874	0.8956	1.2460	1.4480	1.6786	1.9535	2.1169	2.3074	2.3941	2.4886	2.5930
24.	0.5727	0.8785	1.2257	1.4257	1.6539	1.9259	2.0876	2.2761	2.3618	2.4552	2.5585
25.	0.5591	0.8627	1.2069	1.4051	1.6312	1.9006	2.0607	2.2472	2.3321	2.4246	2.5267
26.	0.5464	0.8480	1.1896	1.3861	1.6103	1.8772	2.0358	2.2206	2.3047	2.3963	2.4974
27.	0.5345	0.8343	1.1734	1.3684	1.5908	1.8555	2.0128	2.1960	2.2793	2.3701	2.4703
28.	0.5234	0.8214	1.1583	1.3520	1.5727	1.8353	1.9913	2.1730	2.2557	2.3457	2.4451
29.	0.5129	0.8094	1.1442	1.3366	1.5557	1.8165	1.9713	2.1516	2.2336	2.3230	2.4216
30.	0.5031	0.7981	1.1310	1.3222	1.5399	1.7988	1.9526	2.1316	2.2130	2.3017	2.3996
35.	0.4612	0.7504	1.0754	1.2616	1.4734	1.7250	1.8742	2.0479	2.1269	2.2129	2.3077
40.	0.4282	0.7130	1.0321	1.2145	1.4218	1.6678	1.8137	1.9833	2.0603	2.1443	2.2369
45.	0.4014	0.6827	0.9972	1.1767	1.3804	1.6220	1.7652	1.9316	2.0072	2.0895	2.1803
50.	0.3791	0.6576	0.9685	1.1456	1.3465	1.5846	1.7256	1.8894	1.9638	2.0449	2.1342
60.	0.3437	0.6183	0.9236	1.0973	1.2939	1.5266	1.6643	1.8242	1.8968	1.9758	2.0630
70.	0.3167	0.5884	0.8899	1.0610	1.2545	1.4833	1.6186	1.7757	1.8469	1.9245	2.0100
80.	0.2951	0.5648	0.8633	1.0325	1.2236	1.4495	1.5829	1.7378	1.8080	1.8845	1.9688
90.	0.2775	0.5456	0.8417	1.0093	1.1986	1.4221	1.5541	1.7072	1.7766	1.8522	1.9355
100.	0.2627	0.5294	0.8237	0.9901	1.1779	1.3994	1.5302	1.6818	1.7506	1.8254	1.9079
150.	0.2130	0.4759	0.7644	0.9268	1.1098	1.3252	1.4521	1.5992	1.6658	1.7383	1.8182
200.	0.1839	0.4448	0.7300	0.8903	1.0706	1.2827	1.4075	1.5520	1.6175	1.6887	1.7671
250.	0.1642	0.4238	0.7070	0.8660	1.0446	1.2544	1.3779	1.5208	1.5855	1.6558	1.7333
300.	0.1497	0.4085	0.6903	0.8483	1.0257	1.2340	1.3565	1.4982	1.5623	1.6321	1.7089
350.	0.1385	0.3966	0.6774	0.8347	1.0112	1.2183	1.3401	1.4809	1.5446	1.6139	1.6902
400.	0.1295	0.3871	0.6671	0.8238	0.9996	1.2058	1.3270	1.4671	1.5305	1.5994	1.6753
450.	0.1220	0.3793	0.6586	0.8149	0.9901	1.1955	1.3163	1.4558	1.5189	1.5875	1.6631
500.	0.1157	0.3727	0.6515	0.8074	0.9821	1.1869	1.3073	1.4463	1.5092	1.5776	1.6529
1000.	0.0816	0.3372	0.6133	0.7673	0.9395	1.1411	1.2594	1.3960	1.4577	1.5248	1.5987

TABLE III(CONTINUED)

STATISTICAL TOLERANCE FACTORS

CONFIDENCE LEVEL = 0.9950

PROPORTION OF THE DISTRIBUTION GREATER THAN (LESS THAN) $\bar{X}$ MINUS (PLUS) Ks

SAMPLE SIZE	0.94000	0.95000	0.96000	0.97000	0.98000	0.99000	0.99500	0.99700	0.99900	0.99950	0.99990
5.	7.6382	8.0181	8.4523	8.9781	9.6824	10.8031	11.8390	12.5740	14.0532	14.9304	16.7948
6.	6.0382	6.3252	6.6681	7.1037	7.6865	8.6121	9.4660	10.0515	11.2118	11.9006	13.3663
7.	5.1647	5.4114	5.7030	6.0640	6.5475	7.3165	8.0269	8.5193	9.5058	10.0913	11.3370
8.	4.6110	4.8379	5.1062	5.4240	5.8480	6.5230	7.1473	7.5815	8.4718	8.9999	10.1225
9.	4.2181	4.4206	4.6599	4.9562	5.3530	5.9842	6.5672	6.9714	7.7808	8.2612	9.2831
10.	3.9341	4.1233	4.3469	4.6234	4.9936	5.5824	6.1263	6.5033	7.2597	7.7087	8.6637
11.	3.7122	3.8912	4.1028	4.3645	4.7148	5.2718	5.7864	6.1430	6.8580	7.2824	8.1850
12.	3.5374	3.7085	3.9107	4.1607	4.4953	5.0275	5.5190	5.8596	6.5421	6.9472	7.8087
13.	3.3958	3.5604	3.7549	3.9956	4.3176	4.8296	5.3025	5.6301	6.2864	6.6759	7.5041
14.	3.2782	3.4375	3.6258	3.8586	4.1702	4.6656	5.1230	5.4399	6.0746	6.4511	7.2519
15.	3.1788	3.3336	3.5166	3.7429	4.0457	4.5271	4.9715	5.2794	5.8958	6.2615	7.0392
16.	3.0935	3.2445	3.4230	3.6437	3.9390	4.4083	4.8416	5.1418	5.7426	6.0990	6.8570
17.	3.0193	3.1670	3.3416	3.5574	3.8462	4.3052	4.7288	5.0223	5.6096	5.9580	6.6988
18.	2.9522	3.0970	3.2681	3.4797	3.7626	4.2124	4.6274	4.9149	5.4902	5.8315	6.5570
19.	2.8929	3.0351	3.2031	3.4109	3.6888	4.1303	4.5378	4.8200	5.3847	5.7196	6.4317
20.	2.8400	2.9799	3.1452	3.3496	3.6229	4.0572	4.4579	4.7355	5.2907	5.6200	6.3200
21.	2.7924	2.9302	3.0931	3.2944	3.5637	3.9915	4.3862	4.6595	5.2063	5.5305	6.2198
22.	2.7493	2.8853	3.0460	3.2446	3.5102	3.9321	4.3213	4.5909	5.1300	5.4497	6.1293
23.	2.7101	2.8445	3.0031	3.1993	3.4616	3.8781	4.2624	4.5285	5.0607	5.3763	6.0471
24.	2.6743	2.8071	2.9639	3.1578	3.4171	3.8288	4.2085	4.4715	4.9974	5.3092	5.9720
25.	2.6414	2.7728	2.9279	3.1198	3.3762	3.7835	4.1591	4.4192	4.9393	5.2476	5.9030
26.	2.6110	2.7411	2.8947	3.0847	3.3386	3.7417	4.1135	4.3709	4.8857	5.1909	5.8395
27.	2.5828	2.7117	2.8640	3.0521	3.3037	3.7030	4.0713	4.3263	4.8361	5.1384	5.7807
28.	2.5566	2.6845	2.8354	3.0219	3.2713	3.6671	4.0322	4.2849	4.7901	5.0896	5.7262
29.	2.5322	2.6590	2.8088	2.9938	3.2411	3.6337	3.9957	4.2463	4.7473	5.0443	5.6754
30.	2.5094	2.6353	2.7839	2.9675	3.2129	3.6024	3.9616	4.2102	4.7073	5.0019	5.6280
35.	2.4141	2.5360	2.6799	2.8577	3.0953	3.4722	3.8196	4.0600	4.5406	4.8254	5.4306
40.	2.3407	2.4596	2.6000	2.7734	3.0049	3.3723	3.7108	3.9450	4.4131	4.6904	5.2796
45.	2.2821	2.3987	2.5363	2.7062	2.9331	3.2929	3.6243	3.8536	4.3118	4.5833	5.1599
50.	2.2344	2.3491	2.4844	2.6515	2.8746	3.2282	3.5540	3.7793	4.2295	4.4962	5.0627
60.	2.1607	2.2725	2.4043	2.5671	2.7844	3.1287	3.4457	3.6649	4.1029	4.3623	4.9131
70.	2.1059	2.2156	2.3449	2.5045	2.7175	3.0550	3.3655	3.5803	4.0092	4.2632	4.8025
80.	2.0632	2.1713	2.2986	2.4558	2.6655	2.9976	3.3032	3.5145	3.9364	4.1863	4.7166
90.	2.0288	2.1355	2.2614	2.4166	2.6236	2.9515	3.2531	3.4616	3.8779	4.1244	4.6476
100.	2.0003	2.1060	2.2305	2.3841	2.5890	2.9134	3.2117	3.4179	3.8296	4.0734	4.5907
150.	1.9077	2.0099	2.1303	2.2788	2.4767	2.7899	3.0776	3.2765	3.6733	3.9082	4.4065
200.	1.8549	1.9552	2.0734	2.2190	2.4130	2.7198	3.0017	3.1964	3.5849	3.8148	4.3024
250.	1.8200	1.9191	2.0357	2.1795	2.3710	2.6737	2.9517	3.1436	3.5267	3.7533	4.2340
300.	1.7948	1.8930	2.0086	2.1510	2.3406	2.6404	2.9156	3.1056	3.4847	3.7090	4.1846
350.	1.7755	1.8731	1.9878	2.1292	2.3174	2.6149	2.8881	3.0766	3.4527	3.6752	4.1470
400.	1.7602	1.8572	1.9713	2.1118	2.2990	2.5947	2.8662	3.0535	3.4273	3.6484	4.1171
450.	1.7476	1.8441	1.9577	2.0976	2.2839	2.5782	2.8482	3.0346	3.4065	3.6264	4.0926
500.	1.7371	1.8332	1.9464	2.0857	2.2712	2.5643	2.8332	3.0188	3.3890	3.6079	4.0721
1000.	1.6812	1.7755	1.8863	2.0228	2.2044	2.4910	2.7539	2.9353	3.2970	3.5108	3.9641

CHAPTER 6

THE DISTRIBUTION OF $\tilde{h} = \Phi(k)$

6.a. Sampling Distribution

The popular method of estimating η is to employ $h = \Phi(k)$; the cumulative normal distribution with argument k. Given the sampling distribution of k (from the previous chapter), the sampling distribution of $\tilde{h}$ can be obtained directly. See Table 6.1. Sketches of the probability density functions for k and $\tilde{h}$ are shown in Figure 6.1 for the same noncentrality and sample size used in Table 6.1. The function is the noncentral t density which is tabulated in (Resnikoff and Lieberman, 1957).

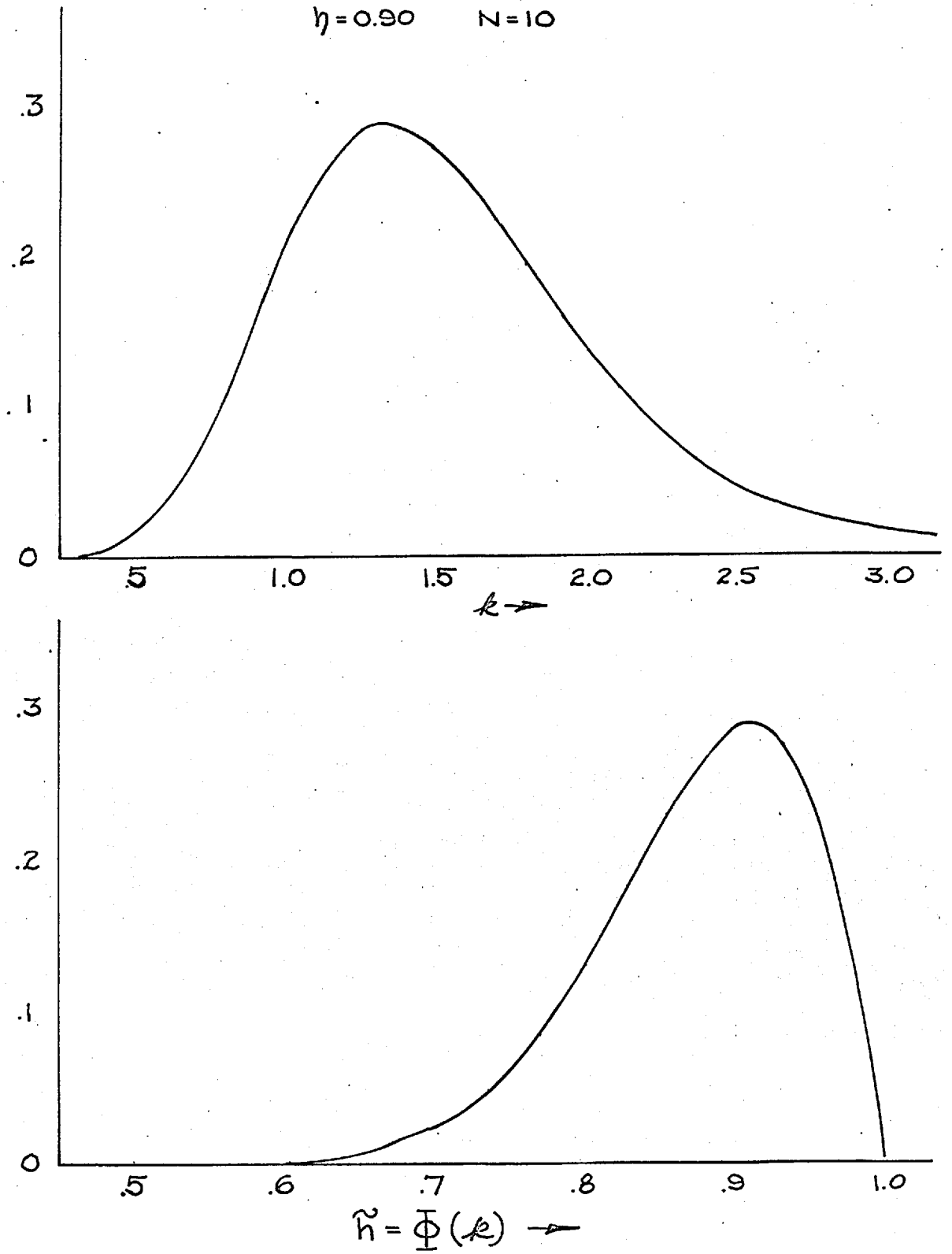
TABLE 6.1

Sampling Distribution of $\tilde{h} = \Phi(k)$, $N = 10$, $\eta = 0.90$

Percentiles	k	$\tilde{h} = \Phi(k)$
0.0050	0.4358	0.6688
0.0100	0.5075	0.6942
0.0250	0.6152	0.7309
0.0500	0.7115	0.7617
0.1000	0.8284	0.7964
0.2000	0.9812	0.8368
0.3000	1.1012	0.8646
0.4000	1.2122	0.8873
0.5000	1.3241	0.9073
0.6000	1.4449	0.9258
0.7000	1.5862	0.9437
0.8000	1.7701	0.9617
0.9000	2.0656	0.9806
0.9500	2.3547	0.9907
0.9700	2.6480	0.9958
0.9900	3.0482	0.9988
0.9950	3.3685	0.9996

FIGURE 6.1

PROBABILITY DENSITY FUNCTIONS OF k AND $\tilde{\eta}$



6.b. Transformations

The effect of the transformation of k to $\tilde{h}$ is to convert a right skewed distribution to a left skewed distribution. See Figures 6.1 and 6.2. Several additional transformations of both k and $\tilde{h}$ were studied with the objective of finding a symmetrical sampling distribution. For k, the most effective transformation was log k which is roughly linear when plotted on normal probability paper. See Figure 6.3. For $\tilde{h}$, the transformation to $\left(\log \log \frac{\tilde{h}}{1-\tilde{h}}\right)$ produces a distribution which is approximately normal and also appears to stabilize the variance of $\tilde{h}$, at least in the range from $\eta = .9$ to $\eta = .999$. See Figure 6.4. Note that this transformation is related to the Fisher-Yates 'Logit' which is $1/2 \log \frac{\tilde{h}}{1-\tilde{h}}$; it is the equivalent of the log 'Logit'.

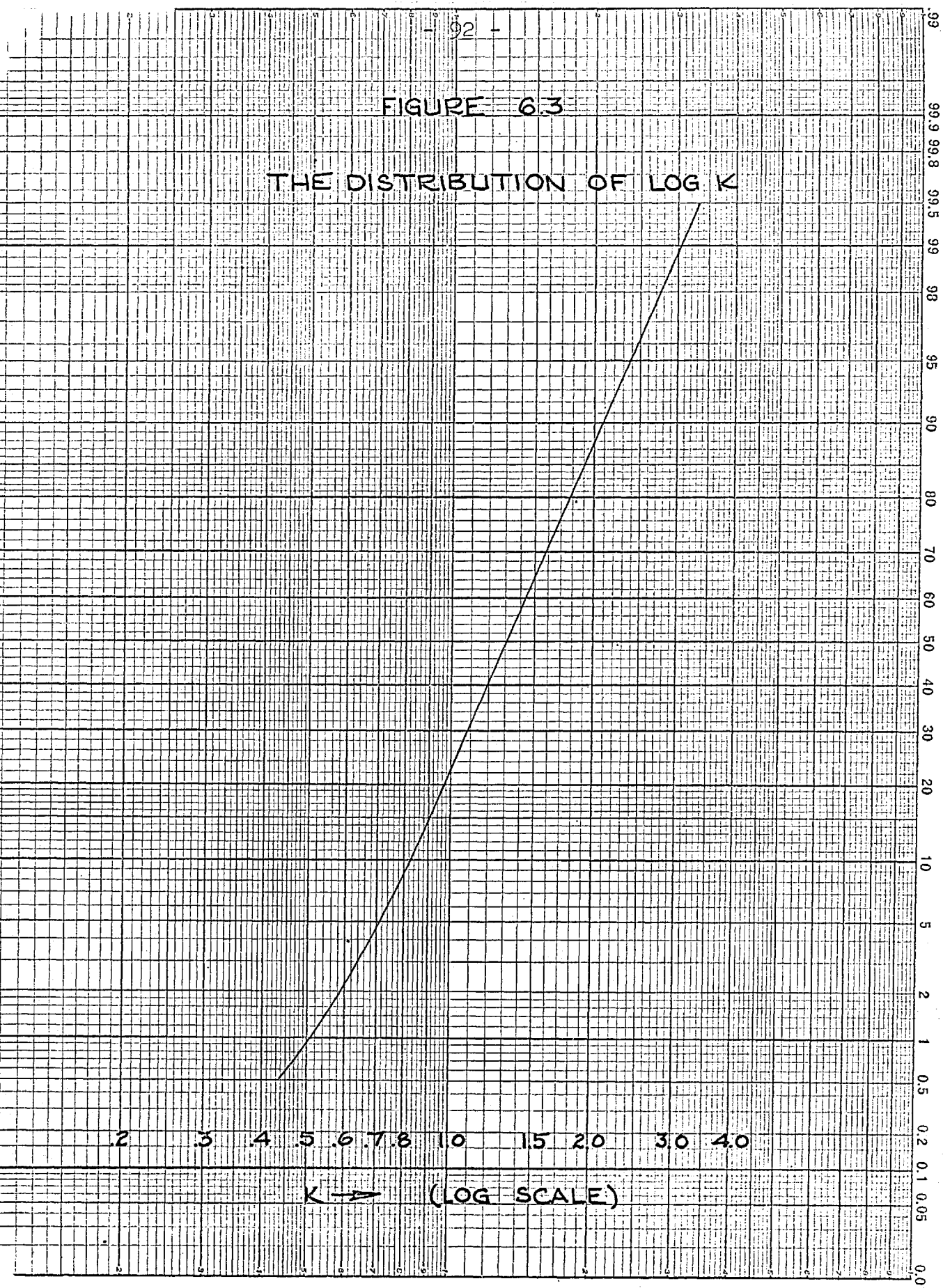
6.c. Expectation of $\tilde{h}$

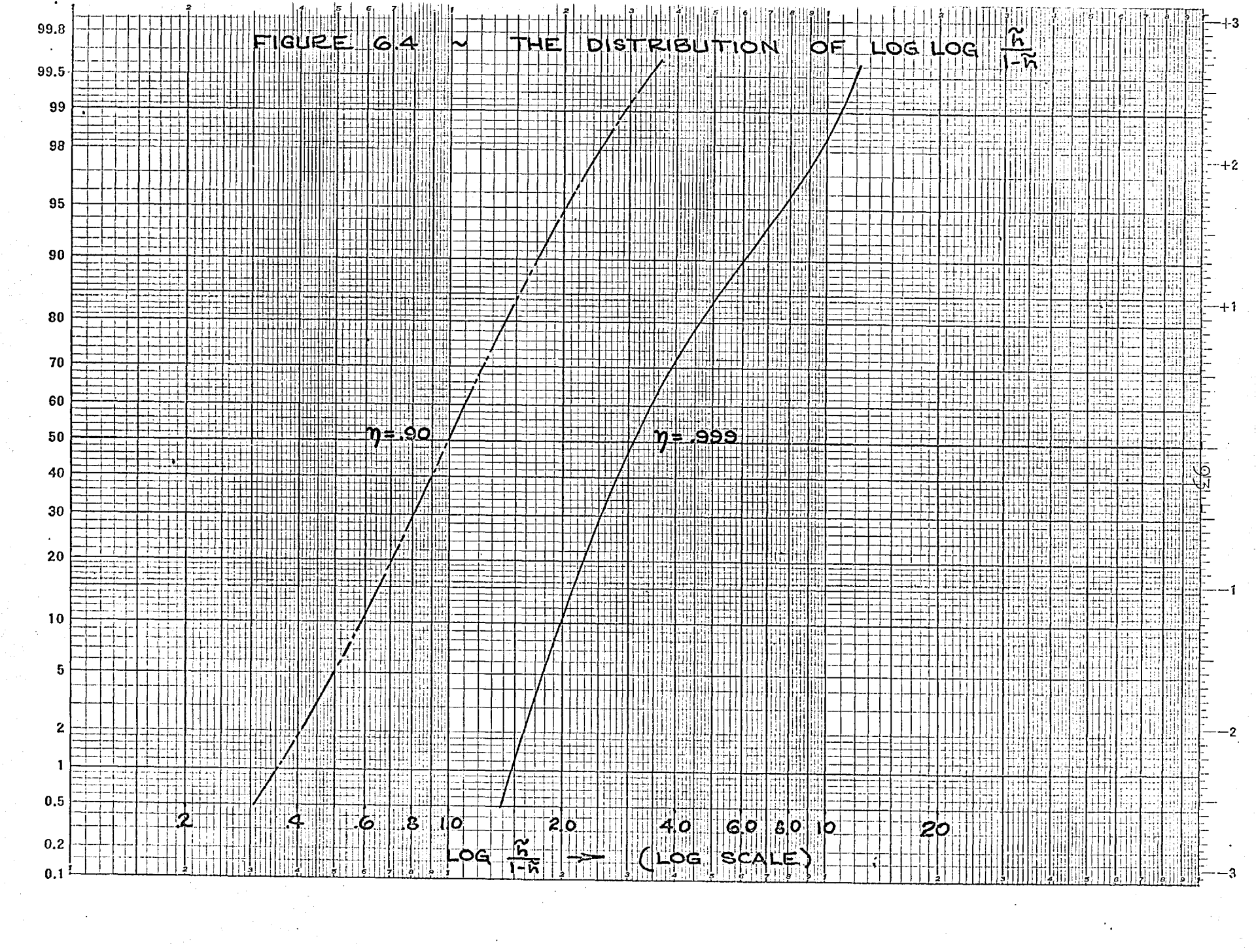
The expected value of $\tilde{h}$ was obtained by numerical integration from Table 6.1 over the range available, 0.5°/o to 99.5°/o. In the interval from 10°/o to 90°/o, Simpson's Rule was employed; from .5°/o to 10°/o and from 90°/o to 99.5°/o the trapezoidal rule was required due to the unequal spacing of the percentiles. The

FIGURE 6.2
THE DISTRIBUTIONS OF K AND $\hat{\eta}^2$

FIGURE 6.3

THE DISTRIBUTION OF LOG K




FIGURE 6.4 ~ THE DISTRIBUTION OF LOG LOG $\frac{\pi^2}{1-\pi^2}$
$\eta = .90$
$\eta = .999$
99.8
99.5
99
98
95
90
80
70
60
50
40
30
20
10
5
2
1
0.5
0.2
0.1
+3
+2
+1
0
−1
−2
−3
.2
.4
.6
.8
1.0
2.0
4.0
6.0
8.0
10
20
LOG $\frac{\pi^2}{1-\pi^2}$ → (LOG SCALE)

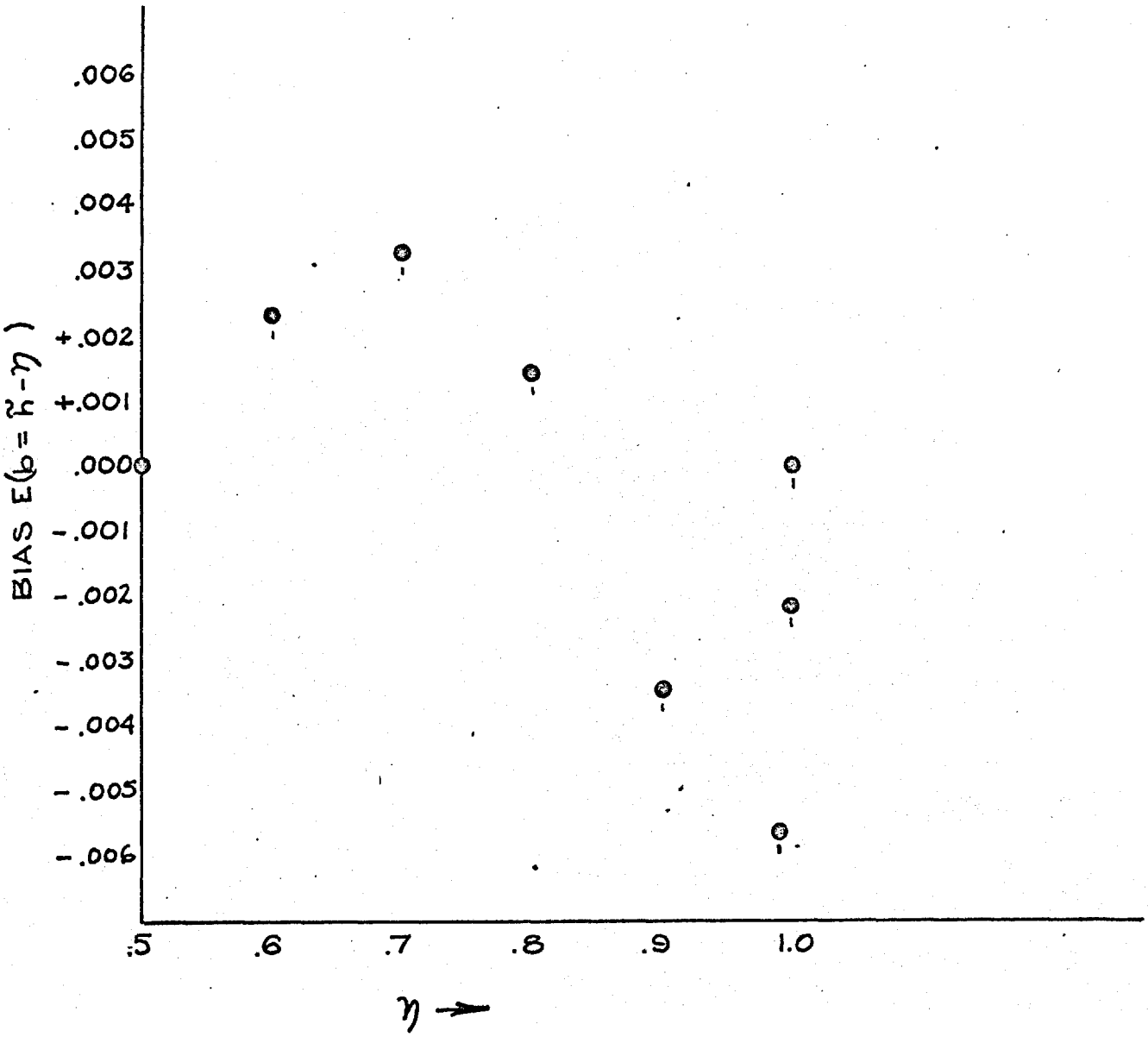
average obtained was divided by 0.99 to account for the missing 1°/o.

6.d. Bias

The bias b of $\tilde{\eta}$ was determined by the above procedure for sample size ten for eight values of η and is plotted as Figure 6.5.

FIGURE 6.5

THE BIAS IN THE ESTIMATE $\tilde{h}$



CHAPTER 7

THE DISTRIBUTION OF h, AN UNBIASED ESTIMATE OF η

7.a. An Iterative Method

If the bias in the estimate $\tilde{h}$ were constant over the sampling distribution, an unbiased estimate could be produced by correcting $\tilde{h}$ for its bias, b. Unfortunately, the bias is not constant (Figure 6.5). The author experimented with iterative computer methods with the hope that there exists a bias function which could be used to correct $\tilde{h}$ to an unbiased estimate, and that if the function does exist, a computer program could be made to converge to it. The following method was successful and the results are shown in Volume II as Program 7.1.

(1) Select a range of values of η between 0.5 and 1.0. See for example those indicated in Figure 6.5.

(2) For each value of η, generate the distribution of k.

(3) First trial. Transform the k values to $\tilde{h}$ as illustrated in Chapter 6. $h = \Phi(k)$. Determine the bias b in estimating η and plot 1/2b versus $\eta + \frac{1}{2}b$. Let $\bar{b} = \frac{1}{2}b$. Connect these points

with straight lines. See Figure 7.1.

(4) Second trial. Transform each k distribution to a distribution of $\bar{h}$, where the bar indicates interim values less biased than h.

$$\bar{h} = \Phi(k) - \bar{b} \qquad (7.1)$$

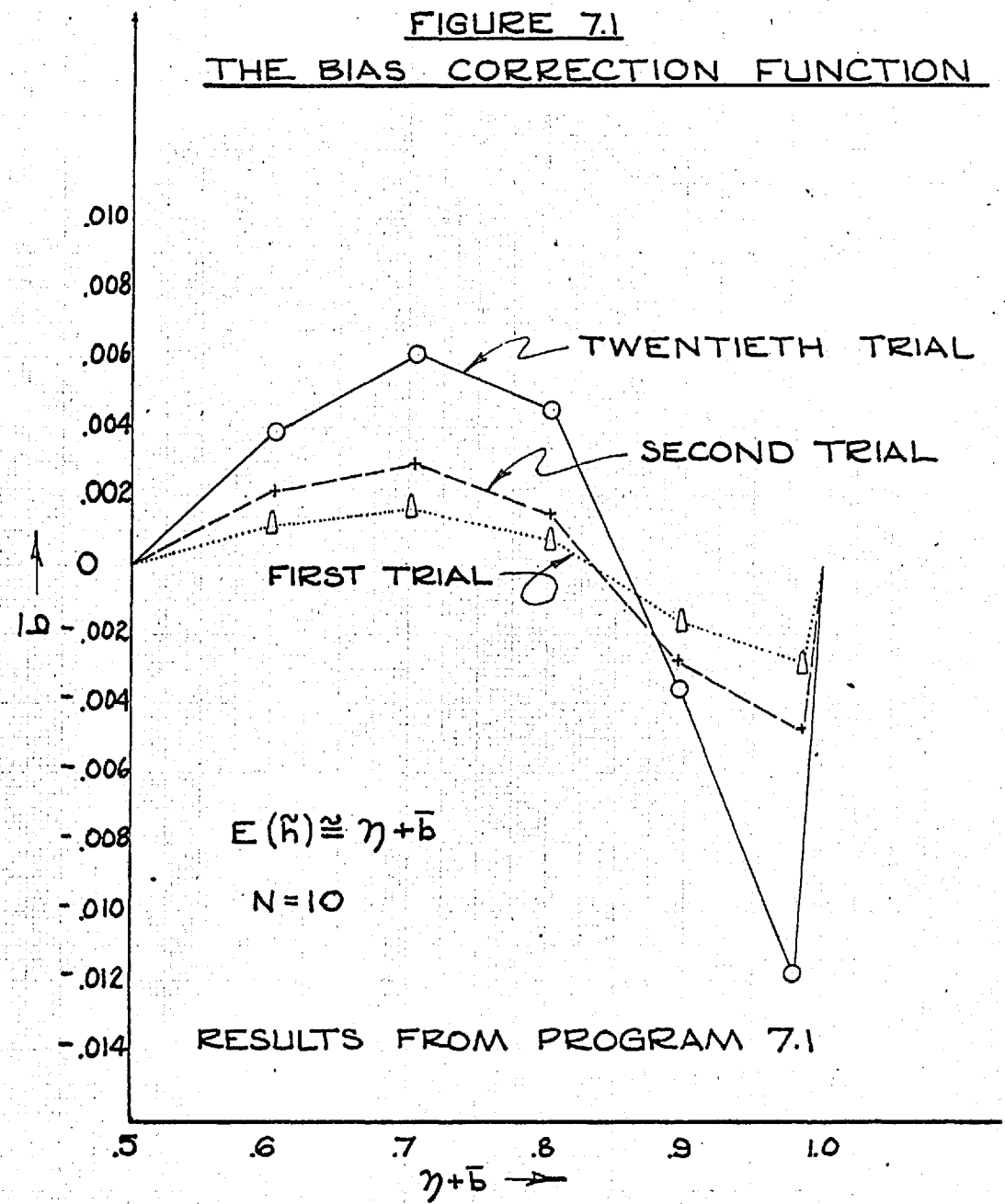
and the interim bias function, $\bar{b}$, is read from the dotted line in Figure 7.1. Determine the new bias in the distribution of $\bar{h}$ by numerical integration and plot the sum of $\bar{b}$ and one half of the bias as the second trial in Figure 7.1.

(5) Repeat (4) until the maximum (absolute) bias is less than any required small tolerance; The value 0.0001 was used in Program 7.1 and the function converged in twenty trials as shown in Figure 7.1. The final result is a correction bias function (Figure 7.1) which results in an unbiased estimate, h, (within 0.0001) when employed in the relation:

$$h = \Phi(k) - \bar{b} \qquad (7.2)$$

FIGURE 7.1

THE BIAS CORRECTION FUNCTION



The use of one-half the indicated bias instead of the entire bias avoided interim values of $\bar{h}$ greater than one and may have improved the stability of convergence. Additional accuracy could be obtained by refining the normal table approximation and using more values of η in step (1). However at this point the paper by (Lieberman and Resnikoff, 1955) was discovered.

7.b. Exact Method

This paper contains a formulation for exact values of h.

It is shown that if:

$$1 - h = \begin{cases} I_Z\left(\frac{N-2}{2}, \frac{N-2}{2}\right) & \text{for } 0 < Z < 1 \\ 0 & \text{for } Z \leq 0 \\ 1 & \text{for } Z \geq 1 \end{cases} \qquad (7.3)$$

and I_Z (a,b) is the Incomplete Beta Function ratio with parameters a, and b, then (1-h) is a uniformly minimum variance unbiased estimator. The proof is based on the Blackwell-Rao theorem extended by (Lehman and Scheffe) and uses the fact that another unbiased estimator exists, $\check{h}$, which is not a function of the normal sufficient statistics, while h is a function of the sufficient statistics. In this case the attribute ratio, $\check{h}$, of the number of number values exceeding the

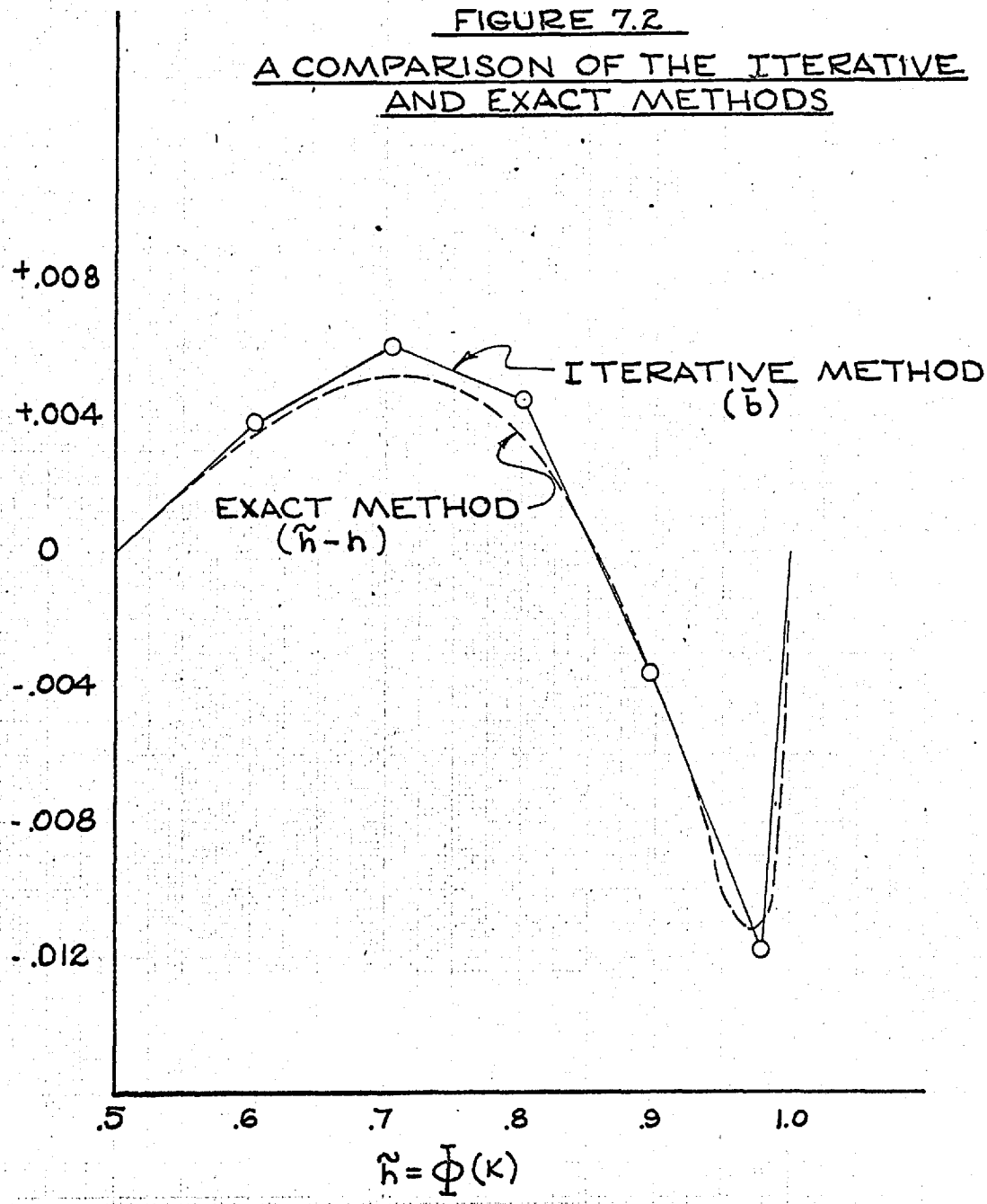
limit L, to the sample size, N, is an unbiased estimate (binomial) which is not a function of the sufficient statistics of the normal distribution. (Lehman and Scheffe) have further shown that if the sufficient statistic is complete, the unbiased estimate which is a function of this statistic will have uniformly smallest variance. The remainder of the derivation consists of evaluating the expression

$$1 - h = E(1-\check{h} \mid \bar{X}, s) \qquad (7.4)$$

using the joint density of $\bar{X}$ and s, and transforming the result to the density of the symmetrical Beta distribution. Tables of values of (1-h) are given in (Lieberman and Resnikoff, 1955) and more completely in (Bowker and Lieberman, 1959). The excellent agreement between the results from the iterative procedure and the referenced tables is illustrated in Figure 7.2. It is stated in (Lieberman and Resnikoff, 1957) that $\tilde{h}$ and h are asymptotically equivalent and asymptotically efficient. It is also mentioned that for small sample sizes, the minimum mean square error, $E(h-\eta)^2$ versus $E(\tilde{h}-\eta)^2$, depends on η and that neither h nor $\tilde{h}$ is clearly 'best' in this sense, although h is 'best' among unbiased estimators. Throughout the referenced texts, (1-h) is

FIGURE 7.2

A COMPARISON OF THE ITERATIVE AND EXACT METHODS



the estimator rather than h as the parameter is per cent defective rather than reliability which is the complement of per cent defective. However, the methods are directly applicable as an unbiased estimate of (1-h) immediately gives an unbiased estimate of h.

7.c. Interval Estimates of η

The sampling distribution of h is skewed for all values of η except 0, 0.5, and 1.0. Although it is possible to estimate the variance of h, this measure of variability has less significance with a skewed distribution than with a symmetric distribution and a confidence interval estimate is a more meaningful measure of precision. An analogy can be made with the problem of estimating the binomial parameter p, where the usual measure of precision is a confidence interval. For very small or very large p, the limiting value of zero and one are used for the one bound of the interval and the other bound is determined from sampling distribution. The interval determined by these confidence bounds has the characteristic that if an infinite number of samples were drawn, and an interval calculated from each sample, then a proportion α of the intervals would contain the true value being estimated.

7.d. Sides and Limits

If one of the confidence interval bounds is set equal to the maximum or minimum value possible, the interval is described as one-sided. This is equivalent to stating that the parameter being estimated is greater than (less than) the lower (upper) one-sided bound with confidence α.

If the interval is not bounded by a maximum or minimum value, it is said to be two-sided. The risk of error may be partitioned in any manner between the upper and lower sides, but the most popular split is even, $(\frac{1-\alpha}{2})$ above and below.

The study reported herein involves fixed limits on the scale of measured variables. There may be single or double limits and there may be one-sided or two-sided confidence intervals; all four combinations are possible. To avoid confusion full nomenclature such as, one-sided, double limit, confidence interval, will be maintained hereafter, even though it is awkward.

7.e. Examples

Example (1) of Chapter 5 illustrates the method for determining a lower, one-sided, single limit, confidence bound, the upper bound taken as one. Another

example will illustrate the application of the method to reliability analysis:

(1) Assume the specific impulse reliability of a rocket engine is defined as the probability that the specific impulse, on any randomly selected firing, will exceed or equal the guaranteed value. Specific impulse is a measure of efficiency and is defined as pounds of thrust produced per pound of propellant per second.

Given: (1) a guaranteed limit of 400 seconds,

(2) that specific impulse data are normally distributed.

(3) a sample of ten firings with an average of 410.0 seconds and a sample standard deviation of 4.29 seconds. What is an unbiased estimate and a one-sided, single limit 90°/o confidence bound for impulse reliability?

$$k = \frac{\bar{X}-L}{s} = \frac{410 - 400}{4.29} = 2.33$$

From Table III

$$h_{C=90°/o} = 93.0°/o$$

From (Resnikoff and Lieberman, 1957) $1-h = 0.00189$

$$h = 99.811°/o$$

An exact solution for the one-sided, double limit, confidence interval is not available but (Resnikoff and Lieberman, 1955) state: 'For the case where there is a double specification limit one of the authors made numerical investigations of the OC curves for various divisions of the per cent defective. It was evident that the band was so narrow that for all practical purposes it can be assumed to be a single OC curve, i.e. the OC of the one-sided plan.' Their discussion is in terms of a double limit test of significance but it is applicable to the problem of a one-sided, double limit confidence interval. The investigation referred to in the quotation is (Resnikoff, 1952). The method is also described under variables sampling plans in (Bowker and Lieberman, 1959); it is stated that the operating characteristic curve for the equivalent single limit is contained within the narrow bound of the various possible double limits. The procedure is as follows:

(1) Given a sample of size N from a normal distribution, an upper limit U, and a lower limit L, calculate $k_U = \frac{U-\bar{X}}{s}$, $k_L = \frac{\bar{X}-L}{s}$

(2) Determine h_U and h_L from (Bowker and Lieberman, 1959) and sum $h = h_U + h_L - 1$.

(3) Determine the equivalent one-sided k from the same table entering with h

(4) From Table III determine the lower confidence bound. (The upper bound is 1.0)

To illustrate assume in example (1)

that $h = h_L + h_U - 1 = 99.811\%$

then $k = 2.33$

and the rest of the solution is identical to (1).

To verify that this approximation is accurate, a Monte Carlo simulation was conducted using a replicated, two factor, three level factorial. (Program 7.2). The factors were the partition and the magnitude of η. The required levels of both factors were obtained by selecting the limits, L and U, from a table of the cumulative probability distribution of the normal distribution. A comparison between the ranked, ninetieth observations from 100 trials is shown in Table 7.1. All of the 1800 trials are shown in Volume II, Program 7.2. It may be concluded from these results, that the Resnikoff and Lieberman conclusion is also correct for confidence intervals; the approximation is good, particularly at high reliability, the area of interest, and the error appears to be conservative,

that is the one-sided, single limit confidence interval is longer than the one-sided, double interval. For the single limit case, column three in Table 7.1, the upper ninetieth percentile of distribution from Table III is indicated for comparison. (Unfortunately, it is an approximate comparison as the expected value of the ninetieth ranked observation in 100 trials is only approximately the ninetieth percentile of the distribution).

TABLE 7.1

PARTITION OF TAIL AREAS

η	$1-\eta_L = 1-\eta_U$		$9(1-\eta_L) = 1-\eta_U$		$\eta_U = \eta$	
	k	h	k	h	k	h
0.90	1.98	.9873	2.17	.9948	2.28	.99733
	2.07	.9914	1.92	.9838	2.00	.9983
					(2.07)	(.9914)
0.95	2.66	.99996	2.45	.99931	2.61	.99991
	2.48	.99949	2.44	.99924	2.63	.99993
					(2.57)	(.99983)
0.99	2.75	1.0	2.75	1.0	2.75	1.0
	2.75	1.0	2.75	1.0	2.75	1.0
					(3.53)	(1.0)

*Because the last step in the procedure is an inverse table 'lookup', the equivalent $k \leq 2.75$ for N = 10.

With regard to the effect of the double-single limit procedure on point estimates, the analysis is easier. Both h_L and h_U are unbiased point estimates (single limits) and therefore, their sum is also an unbiased estimate even though h_L and h_U are negatively correlated.

CHAPTER 8

LIKELIHOOD ESTIMATES

8.a. Likelihood Function

Given observations X_1, X_2, X_3,....,X_N from a normal distribution with unknown mean, μ, and variance, σ^2, the joint likelihood function of the mean and standard deviation is well known:

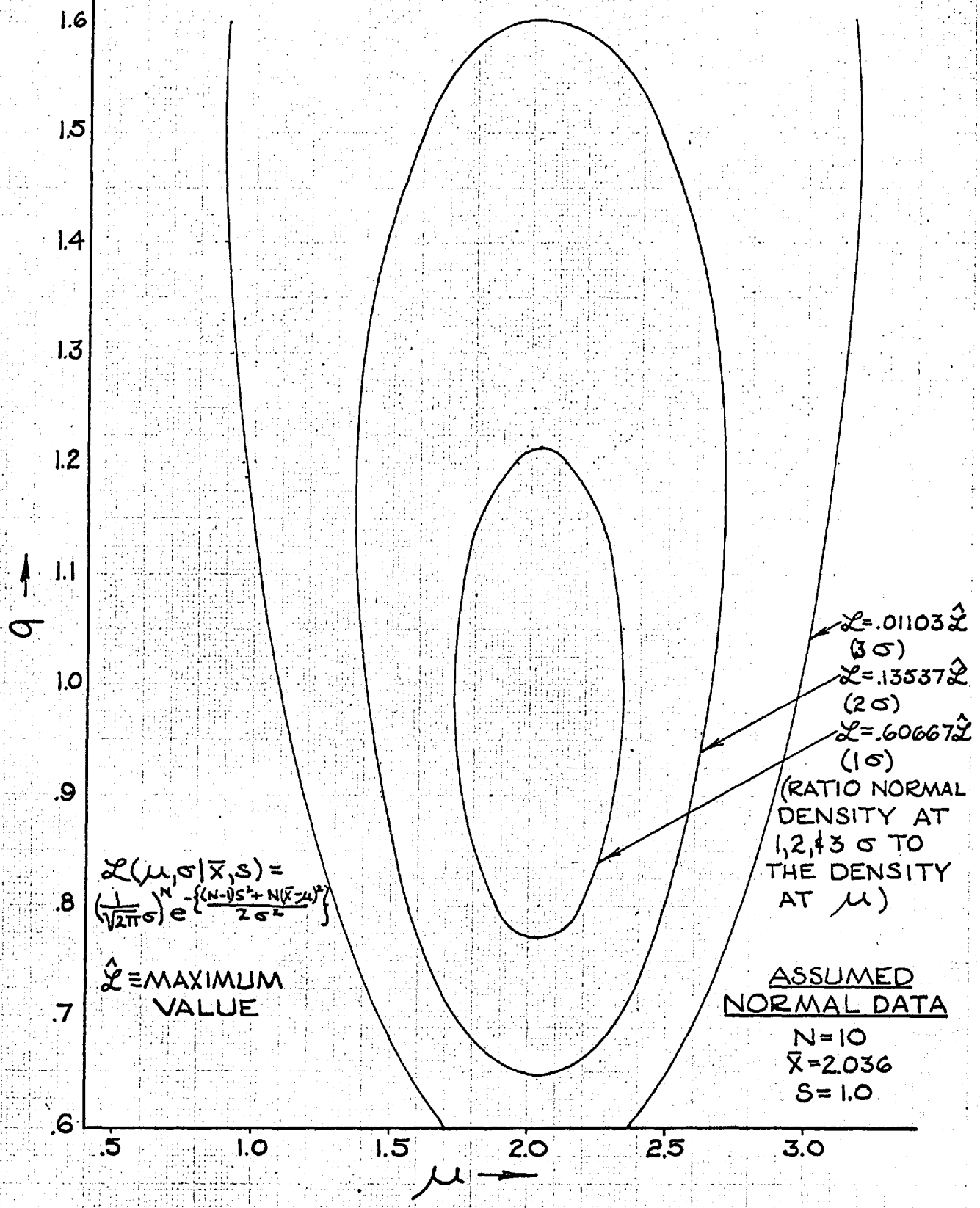
$$\mathcal{L}\left(\mu,\sigma \,\middle|\, \bar{X},s\right) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N e^{-\left(\frac{(N-1)s^2+N(\bar{X}-\mu)^2}{2\sigma^2}\right)} \qquad (8.1)$$

A plot of the likelihood contours for a typical sample of data is shown as Figure 8.1. These results were calculated in Program 8.1.

Given a limit L, taken as the origin without loss of generality, the parameter of interest is the ratio μ/σ, because the proportion of the distribution exceeding the limit is the value of the cumulative probability distribution of the normal distribution with argument, μ/σ. A joint likelihood function involving the ratio may be obtained from the equation (8.1) by substituting:

$$\begin{aligned} \mu &= (\mu/\sigma)\ \sigma \\ &= (Z_\eta)\ \sigma \\ &= (\Phi^{-1}(\eta))\ \sigma \end{aligned} \qquad (8.2)$$

FIGURE 8.1
JOINT LIKELIHOOD FUNCTION OF μ & σ



$$\mathcal{L}(\eta,\sigma \mid \bar{X},s) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N e^{-\left(\frac{(N-1)s^2+N(\bar{X}-\sigma\Phi^{-1}(\eta))}{2\sigma^2}\right)} \quad (8.3)$$

A contour plot of this function for the same data is shown as Figure 8.2 and was also calculated in Program 8.1. Unfortunately, this likelihood function is a joint expression for the likelihood of the reliability, η, which is of interest and the standard deviation σ, which is irrelevant to the problem at hand. The latter parameter has been called a nuisance parameter by R. A. Fisher for obvious reasons; it is not clear how the standard deviation, σ, should be dealt with. At least three treatments seem to have some merit:

(1) If in the future, knowledge of the standard deviation may be available, perhaps from further experimentation, then the best course of action is probably to retain the entire joint likelihood function.

When the standard deviation is known, the likelihood function of η is obtained from Equation (8.3) by substitution. See Figure 8.3 (bottom) which represents cross-sections of Figure 8.2 at constant σ. A considerable improvement in the precision of estimating the ratio might be obtained by this method.

FIGURE 8.2
JOINT LIKELIHOOD FUNCTION OF $\Phi\left(\frac{\mu-L}{\sigma}\right)$ & σ

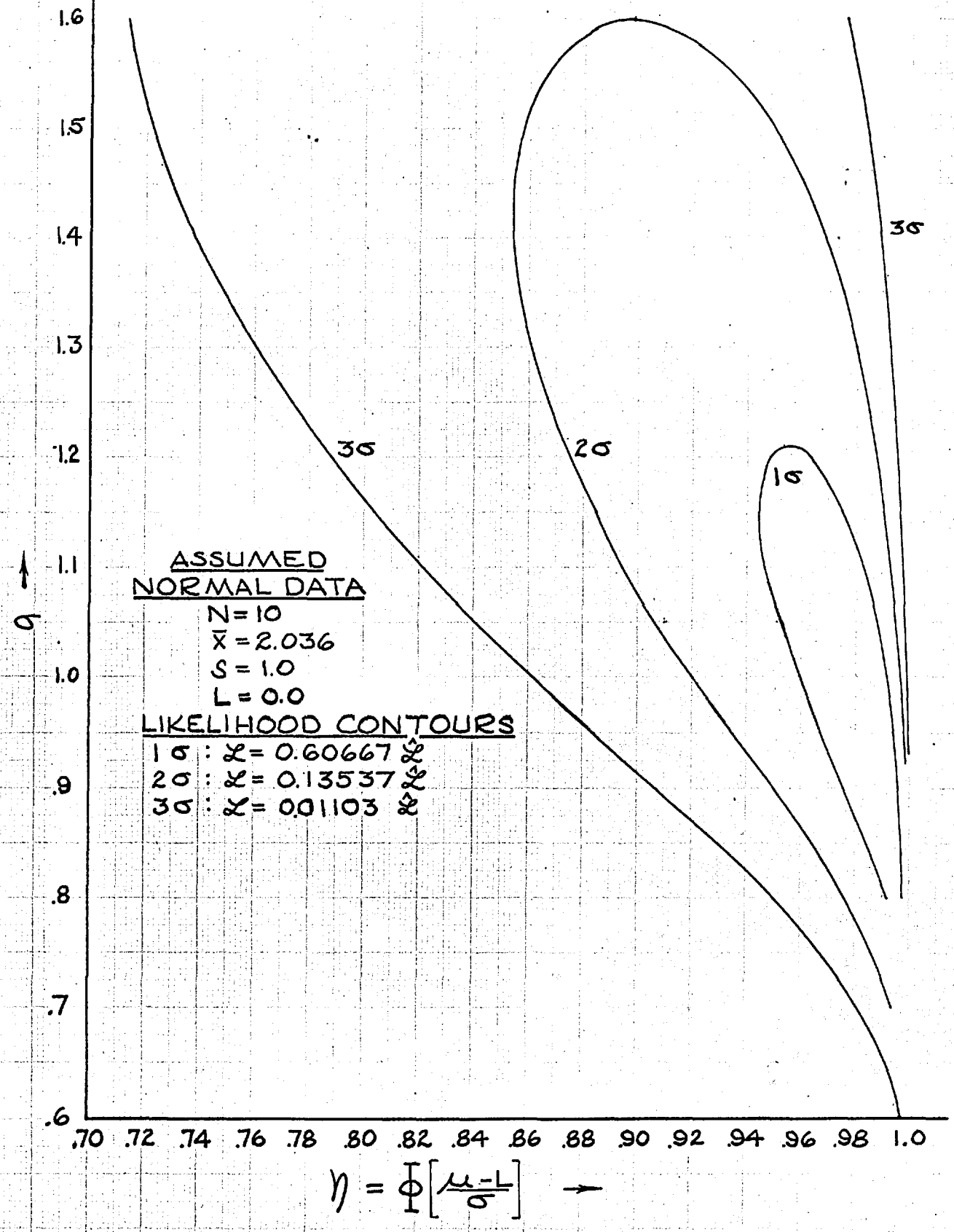
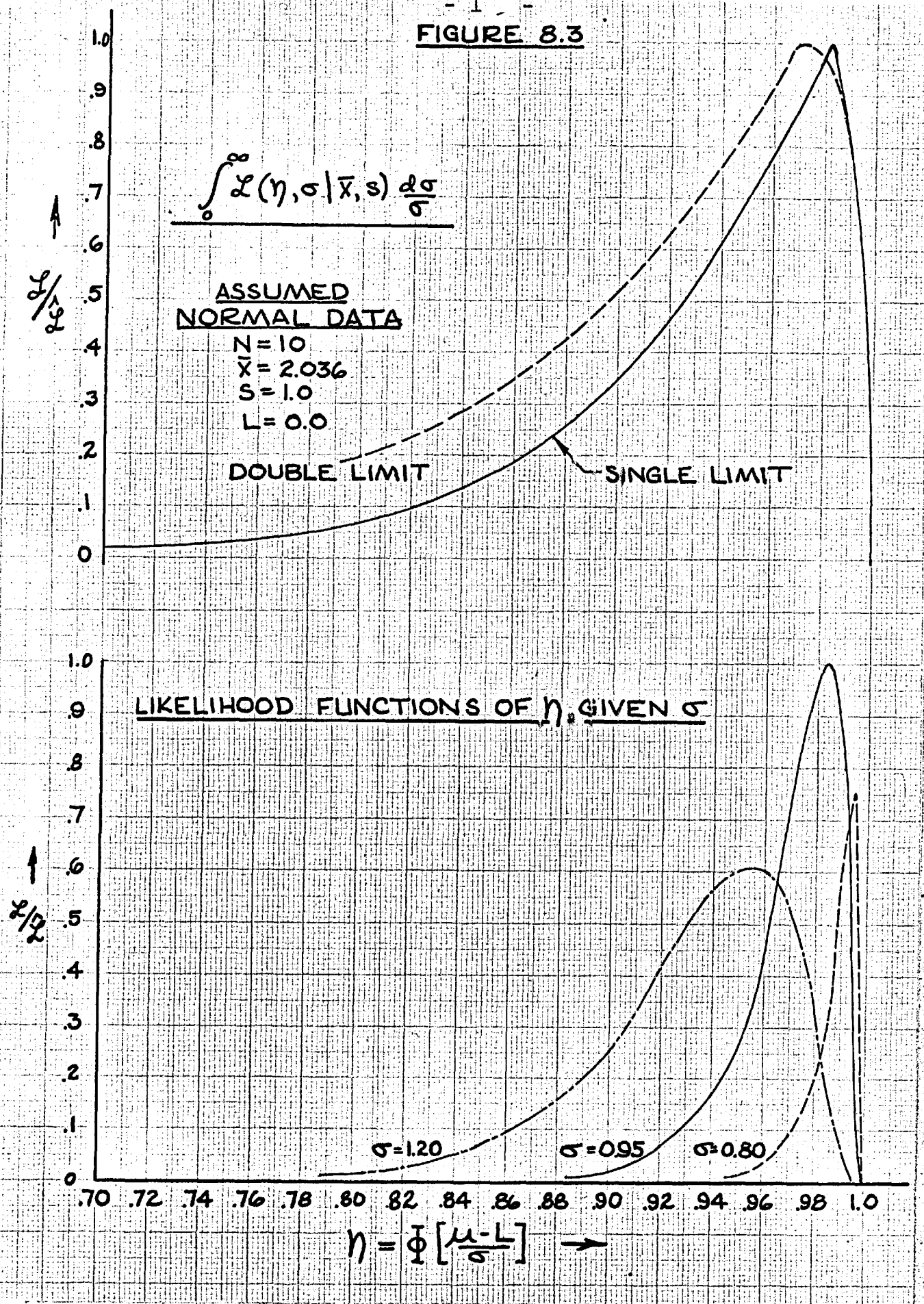


FIGURE 8.3


$\int_0^\infty \mathcal{L}(\eta, \sigma | \bar{x}, s)\, \frac{d\sigma}{\sigma}$
$\mathcal{L}/\hat{\mathcal{L}}$
ASSUMED NORMAL DATA
N = 10
$\bar{x}$ = 2.036
S = 1.0
L = 0.0
DOUBLE LIMIT
SINGLE LIMIT
LIKELIHOOD FUNCTIONS OF η, GIVEN σ
σ = 1.20
σ = 0.95
σ = 0.80
$\eta = \Phi\left[\frac{\mu - L}{\sigma}\right]$

(2) The standard deviation may be taken as the maximum likelihood value. Although this procedure has some justification, if the true value is significantly different from the maximum likelihood estimate, the likelihood function may be grossly in error as shown by Figure 8.3.

(3) The standard deviation may be integrated out using a weighting function. The problem here is to select the correct weighting. (Jeffries) suggests the reciprocal of the standard deviation. This function is equivalent to uniform weighting for the logarithm of the standard deviation. Note that although the weighting function has been taken from Bayesian literature, the result is a likelihood function rather than a posterior distribution because the likelihood of η has not been weighted.

$\int \mathcal{L}(\eta,\sigma|\bar{X},s)\frac{d\sigma}{\sigma}$: USING TRANSFORMATIONS

$$U=\frac{s}{\sigma},\ \sigma=\frac{s}{U},\ dU=-\frac{s}{\sigma^2}d\sigma,\ \frac{dU}{s}=-\frac{d\sigma}{\sigma^2}$$

$$t=\frac{\bar{X}\sqrt{N}}{s}=k\sqrt{N},\ (L=0.0),\ t\sim \text{NONCENTRAL } t\ (\delta=Z_\eta\sqrt{N}, f=N-1)$$

$$\int_0^\infty \mathcal{L}(\eta,\sigma|\bar{X},s)\frac{d\sigma}{\sigma}=\int_0^\infty \mathcal{L}(Z_\eta,\sigma|\bar{X},s)\frac{d\sigma}{\sigma}=$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^N\int_0^\infty\left(\frac{U}{s}\right)^{N-1}\exp^{-\frac{1}{2}\frac{U^2}{s^2}\left\{N\left(\frac{st}{\sqrt{N}}-\frac{Z_\eta s}{U}\right)^2+(N-1)s^2\right\}}\frac{dU}{s}=$$

$$\left(\frac{1}{\sqrt{2\pi}\,s}\right)^N\int_0^\infty U^{N-1}\exp^{-\frac{1}{2}\left\{N\left(\frac{Ut}{\sqrt{N}}-Z_\eta\right)^2+(N-1)U^2\right\}}dU=$$

$$\left(\frac{1}{\sqrt{2\pi}\,s}\right)^N\int_0^\infty U^{N-1}\exp^{-\frac{1}{2}\left\{U^2(t^2+N-1)-2Ut Z_\eta\sqrt{N}+NZ_\eta^2\right\}}dU$$

(8.4)

It will be shown that this likelihood function, Equation (8.4), is related to the noncentral t density in the same sense that the joint likelihood function, Equation (8.3), is related to the normal density. This link between the likelihood, Bayesian and frequentist methods was found through analysis of the likelihood function, Figure 8.2.

Joint Density of $\bar{X}$ & s:

$$g(\bar{X},s) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp - \left(\frac{(N-1)s^2+N(\bar{X}-\mu)^2}{2\sigma^2}\right) \frac{s^{N-2}}{K_N}$$

where K_N is a function of N.

Joint Density of t & s:

$$g(t,s) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp - \left(\frac{(N-1)s^2+N\left(\frac{st}{\sqrt{N}}-Z_\eta\sigma\right)^2}{2\sigma^2}\right) \frac{s^{N-1}}{K'_N}$$

Density of t:

$$g(t) = \int_0^\infty \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \frac{1}{K'_N} \exp - \left(\frac{(N-1)s^2+(st-Z_\eta\sigma\sqrt{N})^2}{2\sigma^2}\right) s^{N-1}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^N \frac{1}{K'_N} \int_0^\infty u^{N-1} \exp - \left(\frac{(N-1)u^2+(ut-Z_\eta\sqrt{N})^2}{2\sigma^2}\right) du$$

using: $U = \frac{S}{\sigma}$, $dU = \frac{dS}{\sigma}$, $\frac{S^{N-1} dS}{\sigma^N} = U^{N-1} dU$

$$g(t)\,dt = \left|\frac{1}{\sqrt{2\pi}}\right|^N \frac{1}{K'_N} \int_0^\infty U^{N-1} \exp -\frac{1}{2}\left|U^2(t^2+N-1) - 2Ut\,Z_\eta\sqrt{N} + N Z_\eta^2\right| dU \qquad (8.5)$$

Equations (8.4) and (8.5) are of the same form.

To illustrate the effect of integrating out the standard deviation, the resulting likelihood function, Equation (8.4), has been evaluated for the sample data and is plotted on Figure 8.3 (labelled 'single limit') for comparison with the joint likelihood function, Figure 8.2.

The three treatments of the joint likelihood function are not equivalent. For example, in general the maximum likelihood estimates, $\hat{h}$, of η, will be different for each of the three methods. The choice is controversial; the author's recommendation is to first plot the contours of the joint likelihood function as this was most enlightening with the sample data. and to employ the other two methods only if necessary, that is, if the true value of the standard deviation will remain unknown. The second method is not recommended unless the variance can be

estimated with considerable precision; that is a sharp, peaked likelihood curve must result from sectioning the joint likelihood at constant η.

8.b. Maximum Likelihood Estimates

As previously mentioned, in general, the three procedures for treating the likelihood function lead to different maximum likelihood estimates although they will be approximately equal. The maximum likelihood estimate for methods (1) and (2) are determined by substituting the value of σ in the joint normal likelihood expression, Equation (8.2), and determining the value of η that maximizes the function. For method (3), the estimate is determined from the noncentral t likelihood, Equation (8,4) or the following equivalent expression.

$$g(f,\delta,t) = \frac{f!}{2^{\frac{f-1}{2}}\Gamma(\frac{f}{2})\sqrt{\pi f}} \left(\frac{f}{f+t^2}\right)^{\frac{f+1}{2}} \exp -\frac{1}{2}\frac{f\delta^2}{f+t^2} \, Hh_f\left(\frac{-\delta t}{\sqrt{f+t^2}}\right) \tag{8.6}$$

where: $f = N-1$

$$\delta = Z_\eta \sqrt{N}$$

$$Hh_f(y) = \int_0^\infty \frac{v^f}{f!} \exp -\frac{1}{2}(v+y)^2 \, dv \quad \text{(Airey's function)}$$

The noncentrality parameter that maximizes this expression for a given sample may be found by trial and error using the (Mathematical Tables) of Airey's function. However, it is not an easy task. In the section written by R. A. Fisher, a large sample approximation for Airey's function is given and it is shown that for a given noncentrality using the approximation, the modal value of the noncentral t density occurs at

$$\hat{k} = \frac{\bar{X}-L}{\hat{s}} = \frac{\bar{X}-L}{s} \times \sqrt{\frac{N}{N-1}} = k\sqrt{\frac{N}{N-1}} \tag{8.7}$$

The modal value was obtained by setting the first derivative equal to zero. The author repeated the derivation holding k constant and taking the derivative with respect to noncentrality. The result is a large sample approximation for the maximum likelihood estimate of η.

$$\hat{h} \cong \Phi\left(k\sqrt{\frac{N}{N-1}}\right) \tag{8.8}$$

Comparing the exact trial and error results with this approximation, it appears to be reasonably accurate for sample sizes as small as ten.

8.c. Bayesian Estimates

Within this thesis there is no requirement for Bayesian estimates of the performance reliabilities but Chapter 14 is devoted to Bayesian estimates of overall reliability and, as an interim step in the procedure, the performance reliability functions are weighted with prior probability functions. Discussion will therefore be postponed until Chapter 14. Here it is sufficient to indicate that the posterior probability of a performance reliability is proportional to the product of its likelihood and prior probability. The difficulties with the procedure are associated with selecting the correct prior distribution; the integration of the likelihood function and the prior probability is fairly easy.

8.d. Double Limits

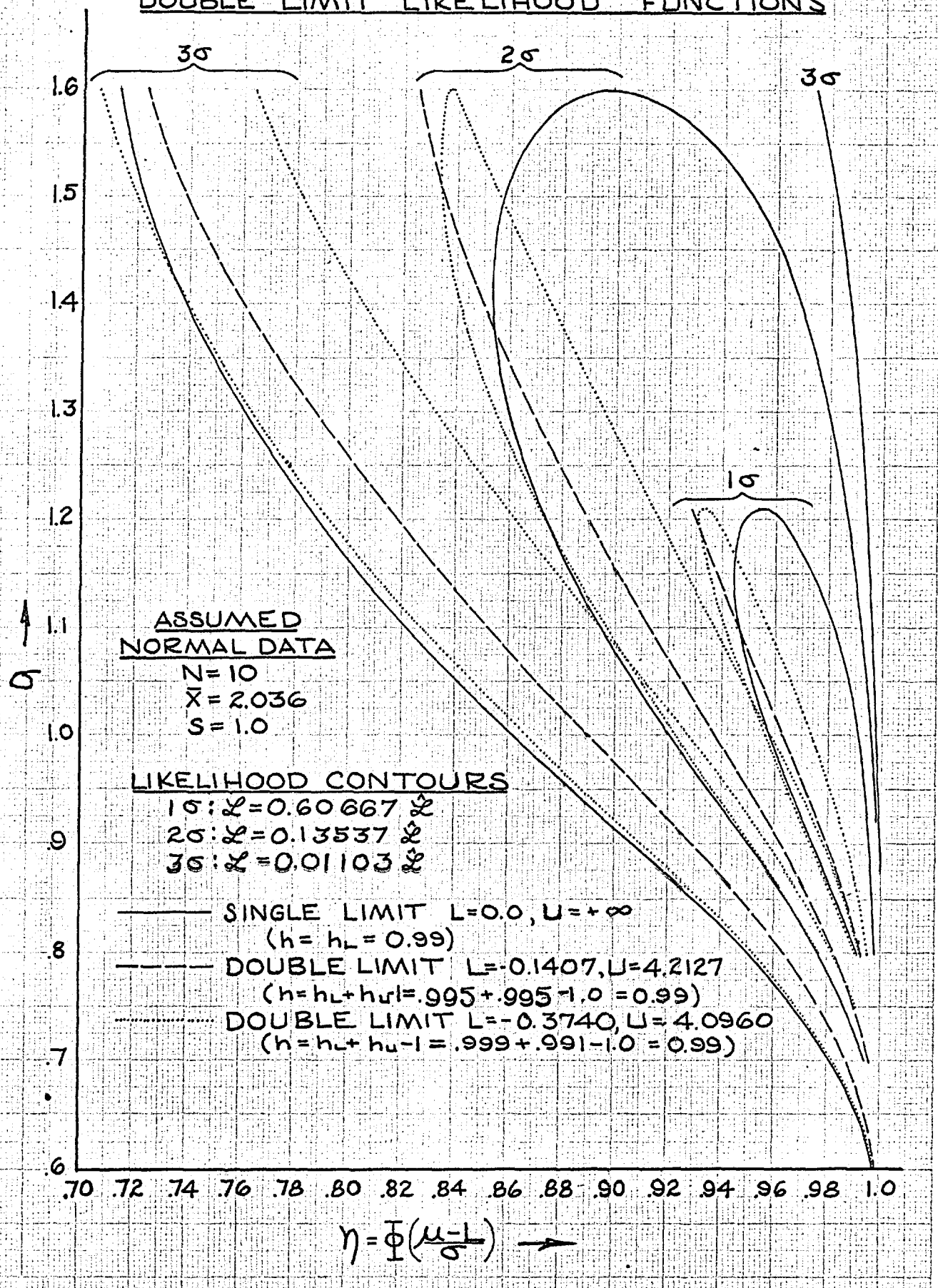
In Chapter 7, the problem of double limits was discussed and in summary, the procedure adopted was to estimate the areas in the tails of the distribution below the lower limit and above the upper limit and add these estimates together. The sum and the corresponding equivalent, single limit, k, are then treated as single limit results. It was shown by simulation Program 7.2 that single limit and double limit performance parameters

with the same reliability have similar sampling distributions under this treatment and therefore it was confirmed that the procedure is approximately valid as reported by (Resnikoff and Lieberman 1955 and 1957).

It is interesting to re-examine this approximation using the likelihood function. Under the general likelihood principle (Barnard et al), two experiments which produce equivalent likelihood functions are equivalent with regard to inference. The double limit procedure leads to the same inference if the sum of the estimates of the tail areas in the double limit case is the same as the estimates of the tail area in the single limit case. The joint likelihood of a given μ and σ imply the likelihood of both tail areas as well as the single tail area. The double limits were determined, given $\bar{X}$ and s, so that the estimate h would be the same as the single limit estimate. The joint likelihood function, Equation (8.2), was used to produce the plot shown as Figure 8.4. Three different partitions of the estimates of tail areas were calculated: $100^{o}/o:0^{o}/o$ (single limit), $50^{o}/o:50^{o}/o$, and $90^{o}/o:10^{o}/o$. The symmetric split ($\bar{X} = \frac{L+U}{2}$) represents one extreme while the single limit is the other extreme. The three dimensional figure that represents the likelihood function changes from a teardrop shaped hill (single limit)

FIGURE 8.4

DOUBLE LIMIT LIKELIHOOD FUNCTIONS



to a curved plane (symmetric limits). In the transition, illustrated by the 90°/o 10°/o split, the function has the curious property of two likelihood values for each value of η. The explanation is that there are two values of μ for each σ that produce the same value of η with different likelihoods. The symmetric split also has this property but fortunately the dual likelihood values are equal. Therefore the analysis, which follows, was conducted on the symmetric split as the limiting case of the double limit procedure. Methods for dealing with a nonsymmetric split, dual valued, likelihood function are not known to the author.

The standard deviation was integrated out with the reciprocal weighting previously described. Trapezoidal numerical integration was used based on the additional contours produced by Program 8.1 but not plotted in Figure 8.4. The results are shown in Figure 8.3 (labelled 'Double Limit'). The two likelihood functions, single and double limit, are equivalent in an approximate sense. If the single limit likelihood function is used for a double limit inference the results would be optimistic. For η values less than the maximum likelihood values, the double limit likelihood is greater than the single limit likelihood

(when both likelihood functions are normalized by ratio to their maximum value). Given double limits and a normal sample, the choice between using a single limit or a symmetric double limit likelihood function will depend on the degree of offset of the sample mean.

It should be noted that the double limit problem presented in Chapter 7 is not exactly equivalent to the problem analyzed here. In Chapter 7 the sampling distributions of double and single limit variables with the same η were compared while in this chapter the sample estimates were taken as equal and the inferences about η compared.

CHAPTER 9

THE DISTRIBUTION OF h', KNOWN VARIANCE

If variations in production processes are likely to cause changes in the level of a normally distributed parameter, but not affect the variance of the parameter, and if historical data verify this hypothesis, the variance may be assumed known. The accumulated data should be quite large, at least fifty to one hundred observations spaced over a fairly long period of time and a large number of batches.

The problem of making an unbiased estimate of the proportion η less than, greater than, or between limits is considerably reduced as the statistic $k' = \frac{\overline{X}-L}{\sigma}$ can be estimated with greater precision than $k = \frac{\overline{X}-L}{s}$. The primed superscript indicates known variance.

9.a. The Estimate h'

Again the ratio of the number of observations satisfying the limit (s) to the total observations is an unbiased (but inefficient) estimate. (Resnikoff and Lieberman, 1955) show that given the above unbiased estimator, not dependent on the sufficient statistic, $\overline{X}$, that a uniformly minimum variance unbiased estimator,

h', exists by using the Blackwell-Rao theorem. The probability that any single observation y, in the random sample $X_1, X_2, X_3, \ldots, X_N$ satisfies the specification limits is h'. If $g(y,\bar{X})$ is the joint density of y and $\bar{X}$, and $h(\bar{X})$ is the density of $\bar{X}$,

$$h' = \int_L^U \frac{g(Y,\bar{X})}{h(\bar{X})}\, dY \qquad (9.1)$$

where U and L are the upper and the lower limits as before. By substituting the appropriate density functions and using a transformation, (Resnikoff and Lieberman, 1955) derive the following expression:

$$h' = \int_L^U \sqrt{\frac{N}{N-1}}\, \frac{1}{\sqrt{2\pi}\sigma}\, e^{-N(Y-\bar{X})^2/2\sigma^2(N-1)}\, dY$$

$$= 1 - \int_{-\infty}^{\sqrt{\frac{N}{N-1}}\,\frac{(L-\bar{X})}{\sigma}} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt - \int_{\sqrt{\frac{N}{N-1}}\,\frac{(U-\bar{X})}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt \qquad (9.2)$$

Therefore if $C_L = \frac{\bar{X}-L}{\sigma}\sqrt{\frac{N}{N-1}} = k_L'\sqrt{\frac{N}{N-1}}$ (9.3)

$$C_U = \frac{U-\bar{X}}{\sigma}\sqrt{\frac{N}{N-1}} = k_U'\sqrt{\frac{N}{N-1}} \qquad (9.4)$$

Single lower limit: $h'_L = \Phi(C_L)$ (9.5)

Single upper limit: $h'_U = \Phi(C_U)$ (9.6)

Double limit: $h' = h'_L + h'_U - 1$ (9.7)

In summary, to make an unbiased estimate of h' given a sample of normal data with known variance, σ^2, and a fixed limit (s), calculate C_L and/or C_U as appropriate, and using a table of the cumulative normal distribution determine h'_L and/or h'_U. For the two-sided limit,

$$h' = h'_L + h'_U - 1 . \quad (9.8)$$

9.b. The Distribution of h'

To illustrate the improvement in precision due to the known variance, Table 9.1 has been constructed. It shows the sampling percentiles of k_L, h_L, $C_L = \sqrt{\frac{N}{N-1}}\, k'$, and h'_L, for $\eta = 0.90$ and $N = 10$.

TABLE 9.1

THE SAMPLING DISTRIBUTIONS OF k, h, $\sqrt{\frac{N}{N-1}}\,k'$, h'

$\eta = 0.90, \quad N = 10$

Percentiles	Unknown Variance		Known Variance	
	k	h	$\sqrt{\frac{N}{N-1}}\,k'$	h'
.5	.4358	.6636	.4924	.6888
1.0	.5075	.6889	.5755	.7175
2.5	.6152	.7256	.6976	.7573
5.0	.7115	.7569	.8026	.7889
10.0	.8284	.7927	.9237	.8222
20.0	.9814	.8355	1.0704	.8578
30.0	1.1012	.8653	1.1762	.8802
40.0	1.2122	.8893	1.267	.8975
50.0	1.3241	.9123	1.351	.9118
60.0	1.4449	.9330	1.435	.9244
70.0	1.5862	.9529	1.526	.9361
80.0	1.7701	.9726	1.631	.9485
90.0	2.0656	.9912	1.778	.9623
95.0	2.3547	.99843	1.899	.9720
97.5	2.6480	.99995	2.004	.9793
99.0	3.0482	1.00000	2.126	.9833
99.5	3.3685	1.00000	2.209	.9864

9.c. Interval Estimates

A procedure for determining confidence intervals was developed as follows:

(A) One sided-single limit: The subscript c implies the lower one sided confidence bound while the subscript o is used to indicate the observed value. The subscript L (or U) is omitted throughout.

$$\Pr(h'_c \leq \eta < 1.0 \mid h'_o) = \alpha$$

$$\Pr(h' \geq h'_o \mid \eta = h'_c) = 1-\alpha =$$

$$\Pr(\bar{X} \geq \bar{X}_o \mid \mu = \mu_c) = 1 - \Phi(Z_\alpha)$$

$$C = \Phi^{-1}(h') = \sqrt{\frac{N}{N-1}}\, k' = \sqrt{\frac{N}{N-1}}\,\left(\frac{\bar{X}-L}{\sigma}\right)$$

$$\bar{X}_o = \mu_c + Z_\alpha \frac{\sigma}{\sqrt{N}}$$

$$C = \sqrt{\frac{N}{N-1}}\left(\frac{\mu_c + Z_\alpha \frac{\sigma}{\sqrt{N}} - L}{\sigma}\right)$$

$$\left(\frac{\mu_c - L}{\sigma}\right) = C\sqrt{\frac{N-1}{N}} - Z_\alpha/\sqrt{N}$$

$$= k' - Z_\alpha/\sqrt{N}$$

$$h'_c = \Phi\left(\frac{\mu_c - L}{\sigma}\right) = \Phi\left(C\sqrt{\frac{N-1}{N}} - Z_\alpha/\sqrt{N}\right)$$

$$h'_c = \Phi\left(k' - Z_\alpha/\sqrt{N}\right) \qquad (9.9)$$

Example: Given $h'_L = 0.99$, $N = 10$

$\sigma = 1.0$, $L = 100$

what is $h_{c=90\%} = ?$

$$Z_{.90} = 1.2816, \quad c_L = \Phi^{-1}(.99) = 2.326$$

$$h_c' = \Phi\left(2.326\sqrt{\frac{9}{10}} - 1.2816/\sqrt{10}\right) = 0.9641$$

(B) One-sided-double limit. The discussion in (Resnikoff and Lieberman, 1955 and 1957) and also in (Bowker and Lieberman, 1959), although not intended for confidence intervals, implies that the errors in treating $h' = h_L' + h_U - 1$ as a one-sided, single limit problem are small. Further, the 1955 paper comments that treating the double limit result as a single limit is a conservative procedure for quality control, accept-reject, problems; with quality better than the AQL the double limit material has a greater chance of acceptance than the single limit and at inferior quality levels there is almost no difference between the two results. To study this curious comment in its relation to confidence intervals, Figure 9.1 was constructed. This is analogous to the

simulation Program 7.2 except that with the variance known, the sampling distributions may be calculated.

Analysis of Figure 9.1 supports the (Resnikoff and Lieberman, 1955) conclusions; if the single limit distribution is used to calculate a lower confidence bound for a double limit variable, the answer will be conservative, and over most of the range, even at this small sample size, the distributions are similar. It should be noted that the upper bound of the one-sided interval in this case is not 1.0, but rather $\eta_{max} = 2 \times \Phi\left(\frac{\mu-L}{\sigma}\right) - 1.0$ with $\mu = \frac{L+U}{2}$. The maximum proportion of the normal distribution is limited by the known variance and the limits, L and U, and occurs when the distribution is located symmetrically between the limits. This limit does not exist when the variance is unknown. However, point estimates may exceed η_{max} if $\bar{X}$ falls close to (L+U)/2. $h_L' + h_U' - 1$ will exceed the maximum possible η because $\frac{\bar{X}-L}{\sigma}\sqrt{\frac{N}{N-1}} > \frac{\mu-L}{\sigma}$, where $\mu = \frac{L+U}{2}$. There is no way of avoiding this paradox, known to the author, without destroying the unbiased characteristic of h'.

FIGURE 9.1

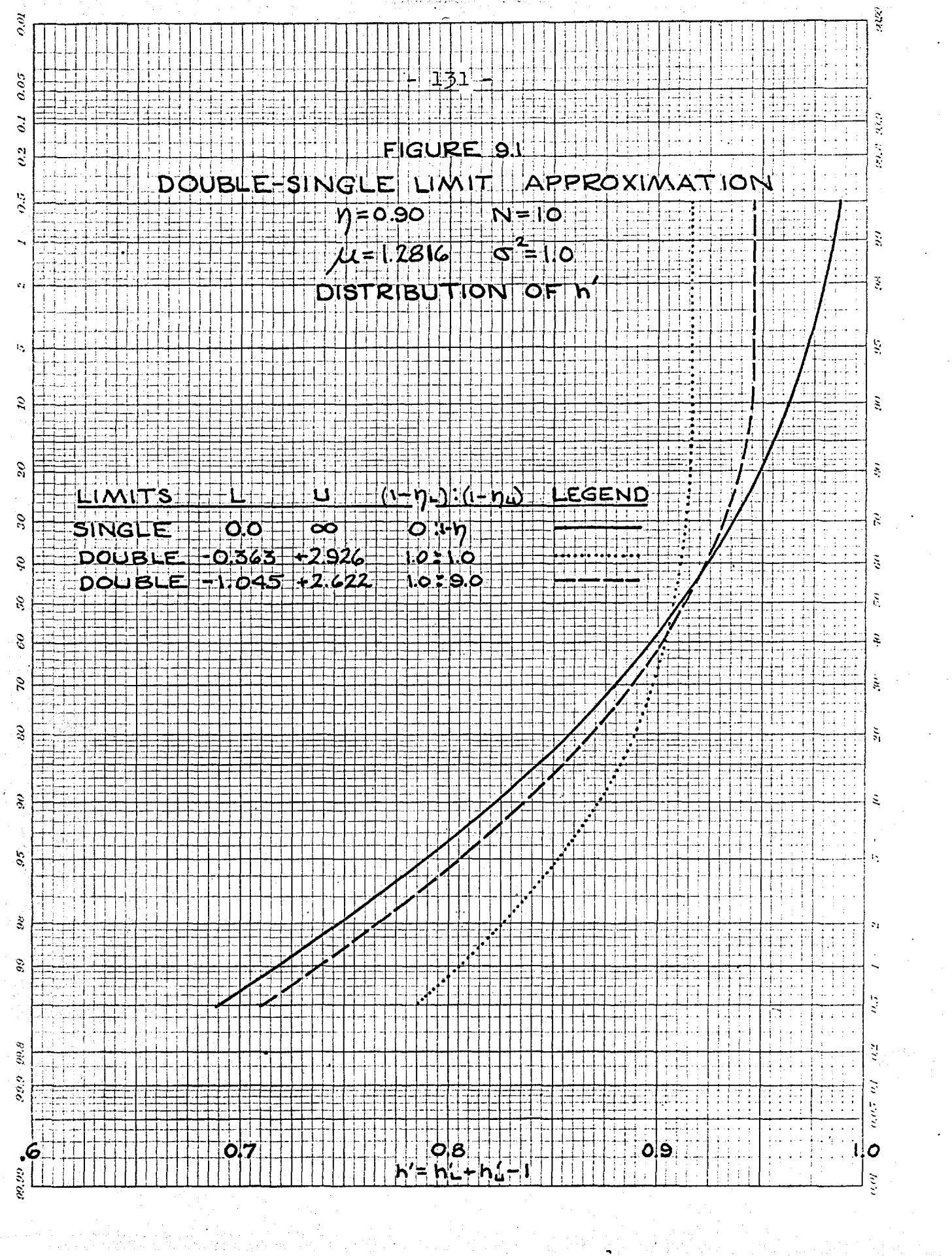
DOUBLE-SINGLE LIMIT APPROXIMATION

$\eta = 0.90$ $N = 10$

$\mu = 1.2816$ $\sigma^2 = 1.0$

DISTRIBUTION OF h'

LIMITS	L	U	$(1-\eta_L):(1-\eta_U)$	LEGEND
SINGLE	0.0	∞	$0:1-\eta$	———
DOUBLE	-0.363	+2.926	1.0 : 1.0	.....
DOUBLE	-1.045	+2.622	1.0 : 9.0	– – – –



CHAPTER 10

A RELATED PROBLEM-VARIANCE LIMITS

10.a. The Variable Limit

There is a related problem which is easily confused with the problem discussed previously in this report. Given sample data from a normal distribution, what is the value $\mathcal{K}$, such that $\bar{X} - \mathcal{K}\, s$ defines the lower bound of an interval, bounded by plus infinity above, which will include on the average a proportion, η, of the normal distribution? Note that the lower limit is a variable here and is not fixed as in the other problem.

$$\Pr(\bar{X} - X_i \leq \mathcal{K}\, s) = \eta \qquad (10.1)$$

where X_i is any randomly selected future observation distributed $N(\mu, \sigma^2)$.

10.b. The Distribution

It will be shown that $\mathcal{K}$ is distributed as the central t distribution with N-1 degrees of freedom times $\sqrt{\frac{N+1}{N}}$. If U is normally distributed with zero mean and unit variance and v has a χ^2 distribution with f degrees of freedom and U and v are independently distributed, then the variable $t = \frac{U\sqrt{f}}{v}$ has a central t (Student's) distribution.

$$\text{Variance of}\left(\bar{X} - X_i\right) = \frac{\sigma^2}{N} + \sigma^2 = \left(\frac{N+1}{N}\right)\sigma^2$$

$$\text{let } U = \frac{\bar{X}-X_i}{\sigma}\sqrt{\frac{N}{N+1}} \sim N(0,1)$$

$$v = \frac{(N-1)\,s^2}{\sigma^2} \sim \chi^2_f$$

$$f = N-1$$

$$\frac{U\sqrt{v}}{v} = \frac{\bar{X}-X_i}{s}\sqrt{\frac{N}{N+1}} \sim t_{(f)}$$

$$\frac{\bar{X}-X_i}{s} \sim \sqrt{\frac{N+1}{N}}\; t_{(f)}$$

$$\mathcal{K} \sim t_{(f)}\sqrt{\frac{N+1}{N}} \qquad (10.2)$$

10.c. References

The original derivation is shown in (Fisher, 1935), again in (Wilks, 1941) and more completely in (Proschan, 1953). This latter paper contains an excellent summary of this type with various combinations of known and unknown, variance and mean. It also includes tables of values of $\mathcal{K}$. (Fraser and Guttman, 1956) and (Guttman, 1957) obtain similar results using measure theory notation and discuss the optimum characteristics of these intervals.

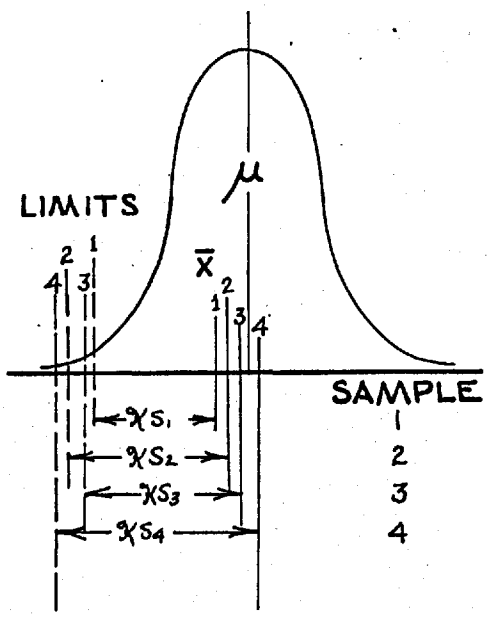
10.d. Discussion

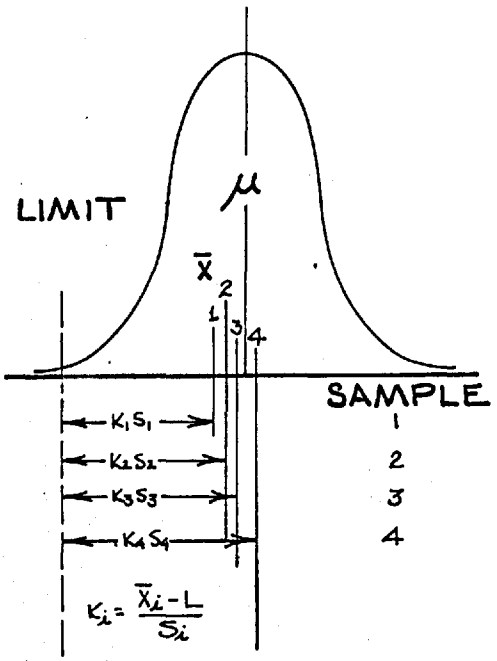
The distinction between these two problems is subtle and is illustrated in Figure 10.1.

The double variable limit estimate is easily obtained because the central t distribution is symmetric. The interval $\bar{X} \pm \mathcal{K}$ s where $\mathcal{K} = \sqrt{\frac{N+1}{N}}\ t_{(N-1,\frac{1-\eta}{2})}$ (two-sided t) which contain a proportion η of the normal distribution on the average.

FIGURE 10.1

VARIABLE LIMITS VERSUS FIXED LIMITS





a) $h_1 = h_2 = h_3 = h_4$

b) The expected area to the right of L_i is η

c) χ is a constant for a given η & n

d) As a function of η & n, χ is distributed as the central $t \times \sqrt{\frac{n+1}{n}}$

a) $h_1 \neq h_2 \neq h_3 \neq h_4$

b) The expected value of h is η

c) k is a random variable for a given L & n

d) As a function of L & n, k is distribution as the non-central $t/\sqrt{n}$

Part III

THE SIMPLE MODEL

Statistical inference usually involves making estimates in a parameter space from observations in a sample space where both spaces are based on the same distribution. Within the problem treated here there are two sample spaces and two parameter spaces, one of each for the population of ground test firings and for the population of mission firings. These two populations differ for all the reasons stated in Part I including the fact that several firings may be made on each ground test engines and the thrust and mixture ratio controls may not be trimmed properly. The mission model is defined as a single trimmed firing made on each of the flight engines. If the ground test data are similar to the extent that only one firing is made per engine and some or all engines are trimmed, they will be referred to as the simple model and will be treated in this part. Fortunately, this simple model is often realistic: if the data consist of the final acceptance firing of each production engine these conditions will be satisfied. Functional, performance and overall reliabilities will be treated in succession.

CHAPTER 11

FUNCTIONAL RELIABILITY

11.a. Definition

Functional reliability is a measure of the mechanical perfection of the engine. From inspection of the definition in Chapter 3, functional reliability may be divided into three factors for discussion:

$$\eta_{fu} = \eta_{start} \times \eta_{steady\ state} \times \eta_{shutdown} \qquad (11.1)$$

Start reliability (η_{start}) may be defined as the probability that the engine, when operated in the manner and environment intended, will start and run for ten seconds. The start transient exposes the engine to severe thermal and mechanical shock; a typical transient from zero to maximum thrust takes about one second. As a result of this shock, more than half of all rocket engine failures occur during the start transient. Shutdown reliability ($\eta_{shutdown}$) may be defined as the conditional probability that given the engine has started and satisfied steady state functional requirements, that it will shutdown without damage to the vehicle or abnormal loss of propellants.

Steady state reliability ($\eta_{steady\ state}$) is the conditional probability that the engine will run for the

intended run duration or longer with performance parameters within functional limits, given that the engine has started.

11.b. Functional Performance Limits

The requirements for satisfying performance limits within the definition of functional reliability may appear strange and warrants additional explanation. There are two reasons for functional performance limits, one physical, the other statistical:

1. There are many missions which do not require the tight, difficult specification performance limits. The functional limits correspond to these easy missions; the functional reliability estimate will be used as the overall reliability estimate for these missions.

2. The performance reliability estimates are based on the assumption of normally distributed data. If a functional failure occurs, the result will often be an outlier among the performance variables: that is, a performance value at an abnormal deviation from the mean. If rocket engine failure analysis were an exact science, the analysis would establish the difference between random and causal dispersions. Unfortunately, failure analysis is often a vain effort: many times the only

indication of functional failure will be an unusual performance datum and there is a mass of evidence to indicate that even when failure analysis produces a plausible hypothesis, the hypothesis will later be proved false. The functional performance limits lie outside the specification performance limits and therefore, should be at least four or five standard deviations from the population mean as the alternative would imply that overall reliability would be very low indeed. It is therefore proposed that the functional performance limits be employed as the criteria for outliers. If one or more of the performance values lies outside the functional limits, the firing will be classified as a functional failure. Note that this procedure does not discard outliers (the usual practice) but rather treats them as functional failures rather than including them in the performance sample. This is an arbitrary decision but so is the use of a t statistic as suggested by (W. R. Thompson) which requires an arbitrary selection of the level of significance. Another procedure which eliminates outliers is that proposed by (Tukey 1965) who suggests routine elimination or 'trimming' of the largest and smallest values in <u>every</u> sample. In the author's opinion the loss of information would be too

severe to use this procedure here and it is at least as arbitrary as the procedure recommended.

11.c. Weighting Factors

The system of weighting factors briefly described in Chapter 2 and in detail in Chapter 16 of (Lloyd and Lipow) appears to be the best method for determining the equivalence of short and long duration firings. Serious consideration was given to the alternative: the assumption of a constant hazard rate. The simplicity that this assumption allows is very attractive as the weighting factor calculation is both complex and laborious: the many advocates of the assumption provide support by arguing that as the reliability approaches unity, the hazard rate must approach a constant value. The author cannot subscribe to this argument because:

(1) it is not intuitively clear why a very small hazard rate implies a constant hazard rate

(2) the early portion of the firing follows the shock of the start transient and will often be affected by control and combustion instability for as long as two minutes after the start transient. To assume the hazard rate is constant throughout the firing is contrary to this knowledge.

Therefore, although it would be mathematically convenient to assume a constant hazard rate, the assumption does not appear to be justified.

(Lloyd and Lipow) suggest using the previous nine months data including a minimum of thirty failures to establish the weighting factors. At high levels of reliability, failures are rare events, and thirty failures may require several years of firing data; such data are still applicable and should be used regardless of their calendar age because as the reliability approaches one, its growth rate must decrease and therefore, two year old data may be as applicable to the current engine as nine month old data are at lower levels. The criterion is not one of calendar age but rather the change in the shape of the engine mortality curve as a function of engine development. If the level of reliability has changed little it is probably safe to assume the shape of the mortality curve has also changed little and, further, there seems to be no alternative.

11.d. Estimates

The application of the weighting factors, W_i (where i indicates the time interval) to estimate functional reliability, uses the model:

$$h_{fu} = 1.0 - (\text{Total number of failures}/(\sum_{i=1}^{k} N_i W_i)) \qquad (11.2)$$

where $N_i \equiv$ the number of firings with intended duration within interval i

$k \equiv$ the interval which includes the full duration

It is shown in (Lloyd and Lipow) that h_{fu} is an unbiased estimate of η_{fu} if the weighting factors are known; actually the weighting factors are estimated from very large samples of data, perhaps two or three thousand firings, and so the estimate h_{fu} is approximately, if not absolutely, unbiased. The weighting factors are functions of maximum likelihood estimates of the probability of a shutdown failure and the joint probability of a start and steady state failure. h_{fu} will be distributed as a binomial variable with parameters $\left\{p = \eta_{fu}, \quad N = \sum_{i=1}^{k} N_i W_i\right\}$. The summation, $\sum_{i=1}^{k} N_i W_i$ is the number of equivalent fall duration firings: it is usually fixed by contract as a moving average of one hundred during the development period and perhaps five hundred equivalent full duration firings during the

production period, when more data are available and reliability growth has slowed down. An interval estimate of η_{fu} may be obtained in the same manner as an interval estimate of the binomial parameter. One-sided intervals, bounded above by unity, are standard in the rocket engine industry with the confidence coefficient set at 0.90. The following relationship may be used to determine the lower, one-sided, confidence bound, h_c:

$$1.0 - 0.90 = \sum_{r=o}^{F} \frac{N!}{r!(N-r)!} \; h_c^{N-r} \; (1-h_c)^r \qquad (11.3)$$

where: $F \equiv$ observed number of failures

$N \equiv$ equivalent full duration firings

The estimate h_{fu} is a maximum likelihood estimate of η_{fu} for the same reasons that the ratio of observed events to total events is a maximum likelihood estimate of the binomial p. The likelihood function and Bayesian estimates will be discussed in Chapter 14.

CHAPTER 12

PERFORMANCE RELIABILITIES

The three performance parameters are assumed to be independently and normally distributed for the reasons presented in Chapter 3. Reliability estimates derived from these data will also be independent. Part II is devoted to this problem; the following is a summary of the results.

12.a. Specific Impulse

The problem of estimating specific impulse reliability, η_{ISP}, is equivalent to estimating the shaded area in the following figure:

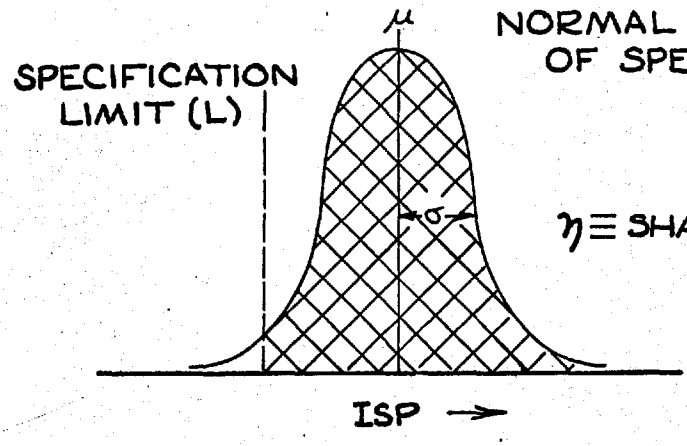


FIGURE 12.1
NORMAL DISTRIBUTION OF SPECIFIC IMPULSE

$\eta \equiv$ SHADED AREA

The statistical methods for determining h_{ISP}, a minimum

variance, unbiased, point estimate of η_{ISP}, are developed in Part II of this thesis in detail; one-sided, lower confidence bounds for η_{ISP} are also described.

A summary of the procedure follows:

(1) From the sample values, $(X_1, X_2, X_3, \ldots, X_N)$ of specific impulse calculate:

$$\bar{X} = \sum_{i=1}^{N} \frac{X_i}{N}, \quad S = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N - 1}}$$

(2) calculate $k = \frac{\bar{X} - L}{S}$, where L is the specification minimum limit. The statistic $k\sqrt{N}$ is distributed as the non-central t with (N-1) degrees of freedom and non-centrality $\delta = Z_\eta \sqrt{N}$

(3) Transform k to a point estimate h_{ISP} of η_{ISP} by entering Table 13.16 of (Bowker and Lieberman). h_{ISP} will be a minimum unbiased estimate.

(4) Transform k to a one-sided, 90°/o confidence interval estimate of η_{ISP} using Table III of this thesis.

Example: Given limit L = 400 seconds

$\bar{X}$ = 402.036 seconds, S = 1.00 seconds

N = 10

h_{ISP} Enter Table 13.16 with $k = \frac{\bar{X} - L}{S} = 2.036$, N = 10

Read p = 1.00°/o, $h_{ISP} = 0.99$

$\underline{h_{ISPc}}$ Enter Table III with k = 2.036, N = 10

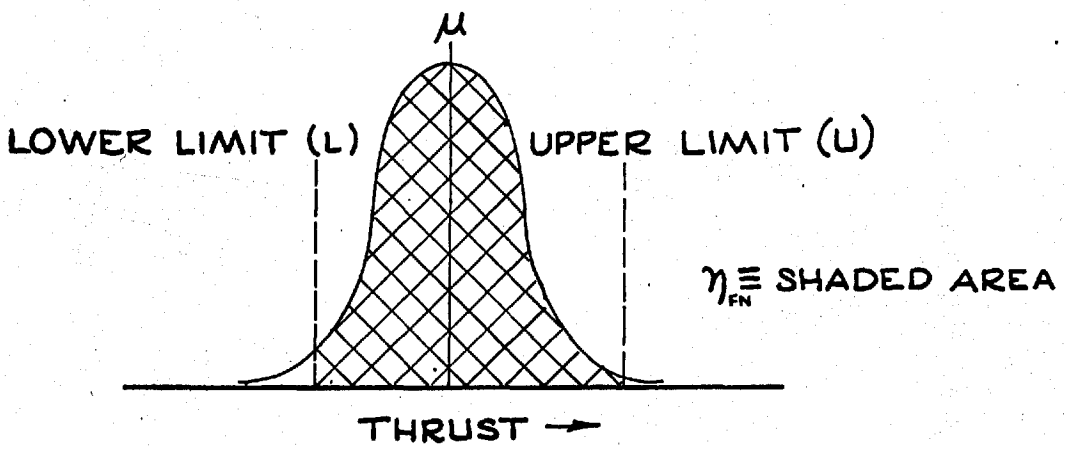
Interpolate* for $h_{ISPc} = 0.896$

where $p\left[h_{ISPc} \leq \eta_{ISP} < 1.0\right] = 0.90$

12.b. Mixture Ratio and Thrust

Compared with specific impulse these parameters are more complex because of their double limits. Both reliability estimates are obtained with the same procedure; thrust will be used to illustrate:

FIGURE 12.2.
NORMAL DISTRIBUTION OF THRUST (FN)



Again the statistical methods are described in detail in Part II and will only be summarized here:

*Interpolation in Table III should be based on the standard normal deviate Z_p rather than $p = \Phi(Z_p)$.

(1) From the sample values, $(X_1, X_2, X_3, \ldots, X_N)$ of thrust calculate:

$$\bar{X} = \sum_{i=1}^{N} \frac{X_i}{N}, \qquad S = \sqrt{\sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{N-1}}$$

(2) Calculate:

$$K_L = \frac{\bar{X} - L}{S}, \qquad K_u = \frac{U - \bar{X}}{S}$$

(3) Transform K_L and K_u to p_L and p_u by entering Table 13.16 of (Bowker and Lieberman).

$$h_{FN} = 1.0 - p_L - p_u$$

(4) Determine an equivalent single limit K, say K_{EQ}, by entering the body of Table 13.16 with $(1.0 - h_{FN})$, and N. Enter Table III with K_{EQ} and N, and interpolate (on Z_p) for a lower, one-sided, 90°/o confidence bound.

Example: Limits: L = 14700 pounds, U = 15300 pounds

Data: $\bar{X}$ = 15020 pounds, S = 120 pounds

N = 10

$$K_L = \frac{15020 - 14700}{120} = 2.66 \qquad K_u = \frac{15300 - 15020}{120} = 2.33$$

Table 13.16: $p_L = 0.00004$ $\qquad p_u = 0.00189$

Unbiased point estimate: $h_{FN} = 1.0 - 0.00193$

$= 0.99807$

$K_{EQ} = 2.327$

Table III Lower 90°/o confidence bound

$h_{FNC} = 0.929$

12.c. Known Variance

If the data, past history and engineering knowledge support the assumption of a known, constant variance, the problem is considerably reduced and the precision improved as shown in Chapter 9.

Example: Repeating the previous example but assuming the standard deviation is known rather than estimated:

$$K_L' = \frac{\bar{X}-L}{\sigma} \qquad K_u' = \frac{U-\bar{X}}{\sigma}$$

$$= 2.66 \qquad = 2.33$$

$$C_L = \sqrt{\frac{N}{N-1}}\ K_L' \qquad C_u = \sqrt{\frac{N}{N-1}}\ K_u'$$

$$= 2.805 \qquad = 2.458$$

$$h_L' = \Phi(C_L) \qquad h_u' = \Phi(C_u)$$

$$= 0.9975 \qquad = 0.9930$$

Unbiased point estimate: $h_{FN}' = h_L' + h_u' - 1.0$

$= 0.9905$

$$C_{EQ} = \Phi^{-1}(h_{FN}')$$

$$= \Phi^{-1}(0.9905)$$

$$= 2.3455$$

$$\frac{\mu_c - L}{\sigma} = C_{EQ}\sqrt{\frac{N-1}{N}} - Z_\alpha/\sqrt{N}$$

$$= 2.3455\sqrt{\frac{9}{10}} - 1.2816/3.1623$$

$$= 1.820$$

Lower 90% confidence bound: $h_{FN}'_C = \Phi(1.820)$

$$= 0.9656$$

CHAPTER 13 .

OVERALL RELIABILITY

13.a. Point Estimate

The product of the four reliability estimates will be an unbiased point estimate of overall reliability, given that the estimates are independent and unbiased.

$$h_o = h_{fu}\, h_{ISP}\, h_{FN}\, h_{rm} \qquad (13.1)$$

The precision of the estimate h_o may be measured by an interval estimate of η_o. An estimate of the variance of h_o has less significance as a measure of precision because the distribution is highly skewed; it is useful, however, for computing the efficiency of the proposed method compared to the current method. Both interval and variance estimates will be treated.

13.b. Interval Estimate

The distribution of h_o, given η_o, may be used to establish a lower, one sided, 90°/o confidence bound. The distribution is complex, a hybrid product of one binomial and three beta transforms of non-central t variables. Further, the relationship between the distribution of h_o and the distributions of the factors,

h_{fu}, h_{ISP}, h_{FN}, h_{rm}, was not obvious to the author. It was helpful to consider the problem in two parts:

1. Given values of η_{fu}^{*}, η_{ISP}^{*}, η_{FN}^{*}, η_{rm}^{*} (the asterisks are used to indicate this particular set) and their respective sample sizes, what is the distribution of h_o^{*}?
2. What is the appropriate relationship between the observed point estimates, h_{fu}, h_{ISP}, h_{FN}, h_{rm} and η_{fu}^{*}, η_{ISP}^{*}, η_{FN}^{*}, η_{rm}^{*}, such that the ninetieth percentile of the distribution of h_o^{*} equals the observed h_o?

With the solutions to these problems, the desired lower, one-sided, 90°/o confidence bound, h_{oc} will be the product (η_{fu}^{*} η_{ISP}^{*} η_{FN}^{*} η_{rm}^{*}) and will satisfy the relationship $P \quad h_{oc} \leq \eta_o < 1.0 \quad = 0.90$ (13.2) where the interval, h_{oc} to 1.0, is considered a random variable.

13.c. Overall Reliability Distribution

The variable h_o^{*} is the product of the four factors:

$$h_o^{*} = h_{fu}^{*} h_{ISP}^{*} h_{FN}^{*} h_{rm}^{*} \qquad (13.3)$$

The distribution of h_o^{*} may be obtained from the distributions of: h_{fu}^{*}, h_{ISP}^{*}, h_{FN}^{*}, h_{rm}^{*}.

$$P\left[h_o^{*} \leq \lambda\right] = P\left[(h_{fu}^{*}\, h_{ISP}^{*}\, h_{FN}^{*}\, h_{rm}^{*}) \leq \lambda\right] \quad (13.4)$$

h_{fu}^{*} is a discrete variable with the binomial distribution:

$$h_{fu}^{*} \sim \text{binomial } (p = \eta_{fu}^{*},\ N = N_{fu})$$

h_{ISP}^{*} is a continuous variable which is a beta transformation of a variable distributed as the non-central t times the square root of N_{ISP}. Methods for determining seventeen percentiles of the distribution of h_{ISP}^{*} are illustrated in Chapter 7.

The distributions of h_{FN}^{*} and h_{rm}^{*} are obtained in the same manner as h_{ISP}^{*} but represent approximate rather than exact solutions because the double limit is treated as the equivalent single limit. The accuracy of this approximation is discussed in Chapters 7 and 8.

A good approximation to the distribution of h_o^{*} was obtained by considering the three performance reliabilities to be discrete rather than continuous variables. Further, the error in the discrete approximation is bounded and may be determined. Consider the seventeen percentiles of h_{ISP}^{*}, given $\eta_{ISP}^{*} = 0.90$, $n = 10$, (from Table 9.1) in the following Table 13.1.

TABLE 13.1

DISCRETIZED PERFORMANCE RELIABILITY DISTRIBUTION

				BOUNDS	
	%	h_{ISP}^{*}	$P(h_{ISP}^{*})$	$\underline{h}$	$\overline{h}$
	0.0	0.0	.005	0.0	0.6636
1.	.5	0.6636	.005	0.6636	0.6889
2.	1.0	0.6889	.015	0.6889	0.7256
3.	2.5	0.7256	.025	0.7256	0.7569
4.	5.0	0.7569	.05	0.7569	0.7927
5.	10.0	0.7927	.10	0.7927	0.8355
6.	20.0	0.8355	.10	0.8355	0.8653
7.	30.0	0.8653	.10	0.8653	0.8893
8.	40.0	0.8893	.10	0.8893	0.9123
9.	50.0	0.9123	.10	0.9123	0.9330
10.	60.0	0.9330	.10	0.9330	0.9529
11.	70.0	0.9529	.10	0.9529	0.9726
12.	80.0	0.9726	.10	0.9726	0.9912
13.	90.0	0.9912	.05	0.9912	0.99843
14.	95.0	0.99843	.025	0.99843	0.99995
15.	97.5	0.99995	.015	0.99995	1.0000
16.	99.0	1.00000	.005	1.0000	1.0000
17.	99.5	1.00000	.005	1.0000	1.0000
	100.0	1.00000			

As indicated in Table 13.1, the variable h_{ISP}^{*} is bounded below by zero and above by one. The distribution may be 'discretized' by considering the eighteen intervals between zero and one as eighteen probabilities associated with eighteen values of h_{ISP}^{*}, each value typical within its corresponding interval. The bounds of the typical values are indicated as $\underline{h}$ and $\overline{h}$. Mixture ratio and thrust are treated similarly. The result is four discrete distributions. The bounds of the distribution of h_o^{*} are determined by taking all combinations of the factors and their corresponding probabilities:

$$\text{Upper bound} \quad \overline{h}_o^{*}(ijk\ell) = h_{fui}^{*}\, \overline{h}_{ISPj}\, \overline{h}_{FNk}\, \overline{h}_{rm\ell} \tag{13.5}$$

$$\text{Lower bound} \quad \underline{h}_o^{*}(ijk\ell) = h_{fui}^{*}\, \underline{h}_{ISPj}\, \underline{h}_{FNk}\, \underline{h}_{rm\ell} \tag{13.6}$$

$$\text{Probability of} \quad h_o^{*}(ijk\ell) = P_i\, P_j\, P_k\, P_\ell \tag{13.7}$$

where: $P_i \equiv$ probability of h_{fui}^{*}

$$P_j \equiv P\left[\underline{h}_{ISPj} \le h_{ISP}^{*} < \overline{h}_{ISPj}\right]$$

$$P_k \equiv P\left[\underline{h}_{FNk} \le h_{FN}^{*} < \overline{h}_{FNk}\right]$$

$$P_\ell \equiv P\left[\underline{h}_{rm\ell} \le h_{rm}^{*} < \overline{h}_{rm\ell}\right]$$

$$i = 1,2,3,\ldots,N_{fu},(N+1)_{fu}$$

$$k = 1,2,3,\ldots,18$$

$j = 1,2,3,\ldots,18$

$\ell = 1,2,3,\ldots,18$

If the typical values of the three performance reliabilities are taken as the lower bounds, the values, h_o^*, will be lower bounds and similarly for upper bounds.

The average value $h^*_{oijk\ell} = (\overline{h}_o + \underline{h}_o)_{ijk\ell}/2$ (13.8)

will have a maximum error $\epsilon = (\overline{h}_o - \underline{h}_o)_{ijk\ell}/2$ (13.9)

The cumulative distribution of h_o^* may be determined by ranking the values $h^*_{oijk\ell}$ and summing their respective probabilities.

Although eighteen discrete values of each performance reliability may appear to be a crude division, the distribution of h_o^* will have thousands of discrete values, say m values.

where $m = 18^3 \times (N_{fu} + 1)$ (13.10)

$= 5832 \times (N_{fu} + 1)$

the maximum $P\left[h^*_{oijk\ell}\right] = (0.1)^3 \times P_{i\ max}$ (13.11)

$= 0.001\ P_{i\ max}$

From a practical standpoint a discrete distribution with this fine division of probability approaches a continuous distribution.

13.d. Simulation and Discretized Results

The results of a sample calculation of the distribution of overall reliability are plotted as Figure 13.1. The calculation was done on an IBM 7090 computer as part of Program 13.2 which will be described at the end of this chapter.

For comparison and as a check on the method, a Monte Carlo simulation of the same test case was programmed on the ICT Atlas Computer, Program 13.1. The program is shown (EMA Autocode) with the results in Volume II. The functional data were simulated by generating twenty random numbers in the range from zero to one; if the random number exceeded the functional reliability, η_{fu}^{*}, a failure was indicated, otherwise a success was counted. The performance data were simulated by taking samples of size ten from a normal random number generator for each of the three performance variables. The performance reliabilities were calculated from the non-central t variable, K, transformed to an unbiased estimate of the true reliability using (Table 13.16 Bowker and Lieberman). Standard normal distributions, mean zero, variance one, were used and the specification limits were located such that the proportion of the normal satisfying the limit equalled the desired true reliability, η^{*}. The overall

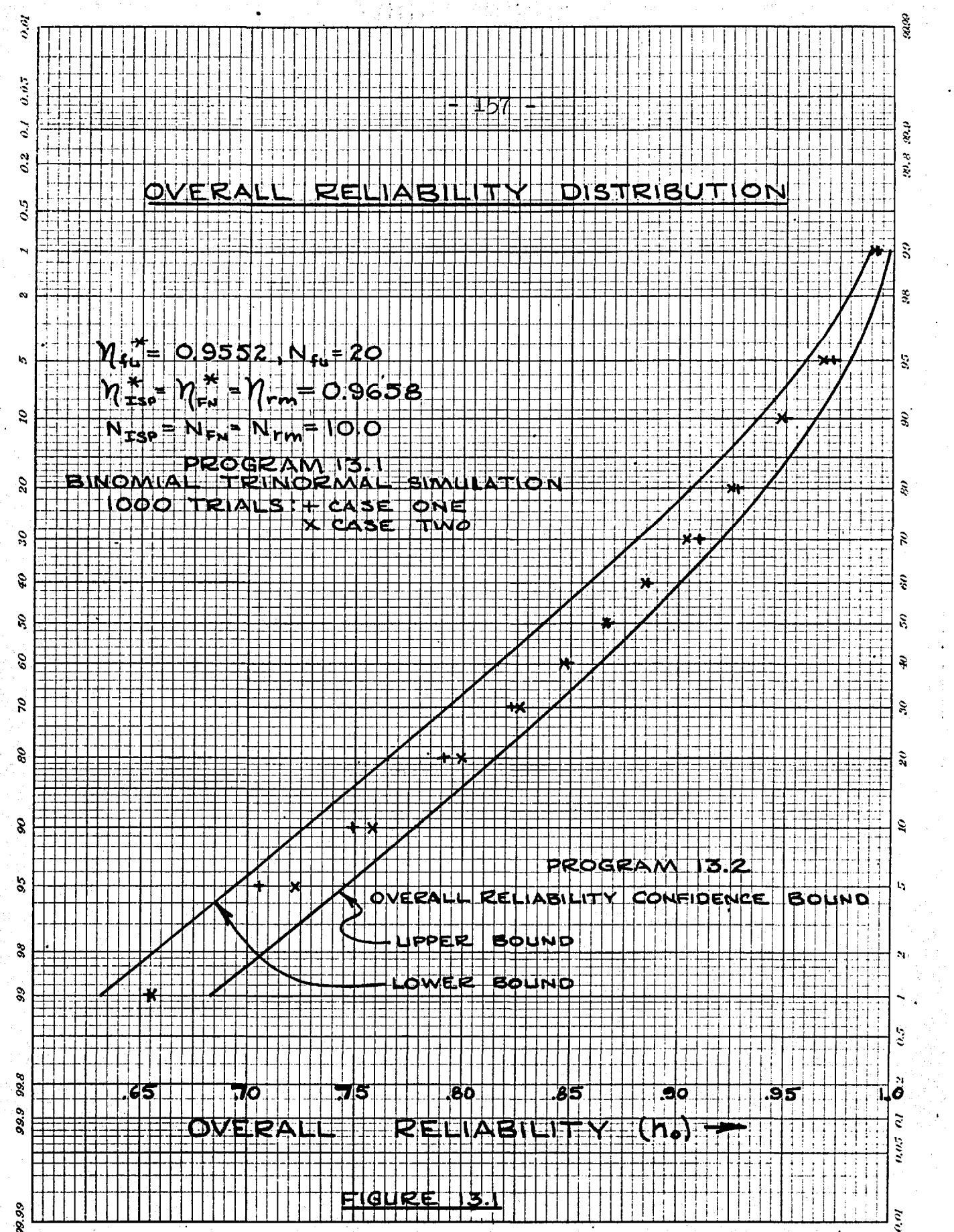

OVERALL RELIABILITY DISTRIBUTION
$\eta_{fu}^* = 0.9552$, $N_{fu} = 20$
$\eta_{ISP}^* = \eta_{FN}^* = \eta_{rm} = 0.9658$
$N_{ISP} = N_{FN} = N_{rm} = 10.0$
PROGRAM 13.1
BINOMIAL TRINORMAL SIMULATION
1000 TRIALS: + CASE ONE
X CASE TWO
PROGRAM 13.2
OVERALL RELIABILITY CONFIDENCE BOUND
UPPER BOUND
LOWER BOUND
.65
.70
.75
.80
.85
.90
.95
1.0
OVERALL RELIABILITY (h_o) →

FIGURE 13.1

reliability estimate was taken as the product of the estimates. The entire process was repeated 1000 times to simulate 1000 estimates. The ranked results are shown as Program 13.1 in Volume II. To provide a rough estimate of experimental simulation precision, each run is repeated as shown in Figure 13.1.

13.e. Confidence Interval

A one-dimensional confidence interval for overall reliability, η_o, represents a complex region in eight dimensions; η_{fu}, μ_{ISP}, σ^2_{ISP}, μ_{FN}, σ^2_{FN}, μ_{rm}, σ^2_{rm} and η_o. The price paid for this simplification is a loss of information and some arbitrariness. The transition from eight dimensions to five is accomplished by integrating out the three variances, a process described in Chapter 8. The results of this reduction are the five dimensions; η_{fu}, δ_{ISP}, δ_{FN}, δ_{rm} and η_o, where δ is the noncentrality parameter of the non-central t distribution. To reduce these five dimensions to one, an arbitrary relationship between the observed point estimates and the values η_{fu}^*, η_{ISP}^*, η_{FN}^*, η_{rm}^* may be employed. Arguments may be stated against this reduction but the confidence interval must be understood by non-statisticians: engineers, management and customer representatives. A multi-dimensional representation of even four dimensions seems to be well beyond the present state-of-the-art of graphic display, so there appears to be no alternative to a one-dimensional solution at this time.

The only arbitrary relationship that seems defensible is the following:

$$P\left[h_{fu}^{*} \geq h_{fu} \,\middle|\, \eta_{fu}^{*}\right] = P\left[h_{ISP}^{*} \geq h_{ISP} \,\middle|\, \eta_{ISP}^{*}\right] =$$
$$P\left[h_{FN}^{*} \geq h_{FN} \,\middle|\, \eta_{FN}^{*}\right] = P\left[h_{rm}^{*} \geq h_{rm} \,\middle|\, \eta_{rm}^{*}\right] = A \qquad (13.12)$$

The set of four η values are selected such that the observations are equally likely. Any other relationship is weakened by the fact that the variances of the four distributions are unequal; the producer would favour a large value of A for the performance variables which have small variances and a small value of A for functional reliability as the result would be to increase the confidence bound. The customer might have a contradictory preference. Further, the rule does seem to bear an intuitive relation to the one parameter, two-sided procedure of splitting the probability area equally between the tails of the distribution. This procedure is also arbitrary in the general case but it is widely accepted.

The additional requirement

$$P\left[h_{o}^{*} < h_{o} \,\middle|\, \eta_{fu}^{*}\, \eta_{ISP}^{*}\, \eta_{FN}^{*}\, \eta_{rm}^{*}\right] = 0.90 \qquad (13.13)$$

is sufficient to determine the constant A and the desired lower, one sided 90°/o confidence bound:

$$h_{oc} = \eta_o^{*} = \eta_{fu}^{*}\ \eta_{ISP}^{*}\ \eta_{FN}^{*}\ \eta_{rm}^{*} \qquad (13.14)$$

The random interval bounded by h_{oc} and unity is not the shortest (or the longest) possible interval. It may be considered to be a representative section from the higher dimensional region, a section such that the observations are equally likely.

The calculations required to solve the above relationship are tedious and could not be considered without a high speed digital computer. The procedure is a trial and error calculation based on assuming the constant A, calculating the distribution of overall reliability and comparing the upper ninetieth percentile with observed overall reliability, selecting a new value of A based on the error and repeating. Computer Program 13.2 accomplishes this task and will be described in detail at the end of this chapter.

The lower confidence bound marks the bottom of a wide interval and as such the precision provided by this method is more than adequate. Further, the error is bounded and known.

13.g. Efficiency

If the performance and functional sample sizes are equal, the ratio of the variance of h_o to the variance

of the binomial estimate of the current method will be a measure of the relative improvement in precision, a measure sometimes called efficiency.

Unfortunately the discretized distribution cannot be used to estimate the variance of h_o because the variance estimates from the upper and lower bounds will not necessarily bracket the true variance. However, the one thousand values of h_o from each simulation (Program 13.1) provide a good alternative. The following relation was employed:

$$S^2 = \frac{\sum_{i=1}^{1000} h_i^2 - \left[\sum_{i=1}^{1000} h_i\right]^2 \Big/ 1000}{999} \qquad (13.15)$$

If h_b is the binomial estimate of η_o using the current method, the variance of h_b will be

$$\sigma_b^2 = \eta_o (1-\eta_o)/N, \text{ where } N = \text{sample size} \qquad (13.16)$$

An estimate of efficiency will be the ratio, (S^2/σ_b^2).

A plot of efficiency is shown as Table 13.2. These results were calculated using Program 13.1.

TABLE 13.2

EFFICIENCY RESULTS

N = 10

η_{fu}	$\eta_{ESP} = \eta_{FN} = \eta_{rm}$ → .90	.99	.999
.90	(η_o = .656) .631 .617	(.873) .844 .782	(.897) .983 .918
.99	(.722) .584 .568	(.961) .503 .465	(.987) .843 .804
.999	(.728) .559 .565	(.969) .321 .357	(.996) .399 .452

13.h. Summary

Having attacked the problems of the simple model one at a time some clarity may be achieved by summarizing the procedure for obtaining point and interval estimates of overall reliability and illustrating the application of the procedure to an example.

Step 1. Functional data. Calculate weighting factors from several years of previous data including at least thirty failures. Assign these weighting factors by actual firing duration to each firing in this period. Sum the weighting factors in reverse chronological order until the desired number of equivalent full duration firings is reached. This defines the sample of both functional and performance data. The number of functional failures observed in the sample divided by the number of equivalent full duration firings (the sum of the weighting factors) is best unbiased, maximum likelihood estimate of functional reliability. Refer Sections 11.c. and 11.d.

Step 2. For each performance variable : Calculate the average, $\bar{X}$, and the square root of the unbiased estimate of the variance, S. Refer Section 12.a. For specific impulse, calculate $k_{ISP} = (\bar{X} - L)/S$.

Step 3. For mixture ratio and thrust: Using the

lower, L, and upper, U, specification limits, determine the lower and upper values of the statistic k.

$$k_L = \frac{\bar{X} - L}{S} \qquad k_U = \frac{U - \bar{X}}{S} \tag{13.17}$$

Using (Table 13.16 of Bowker and Lieberman) transform these k values to unbiased point estimates of h_i and h_u

$$h = h_L + h_u - 1 \tag{13.18}$$

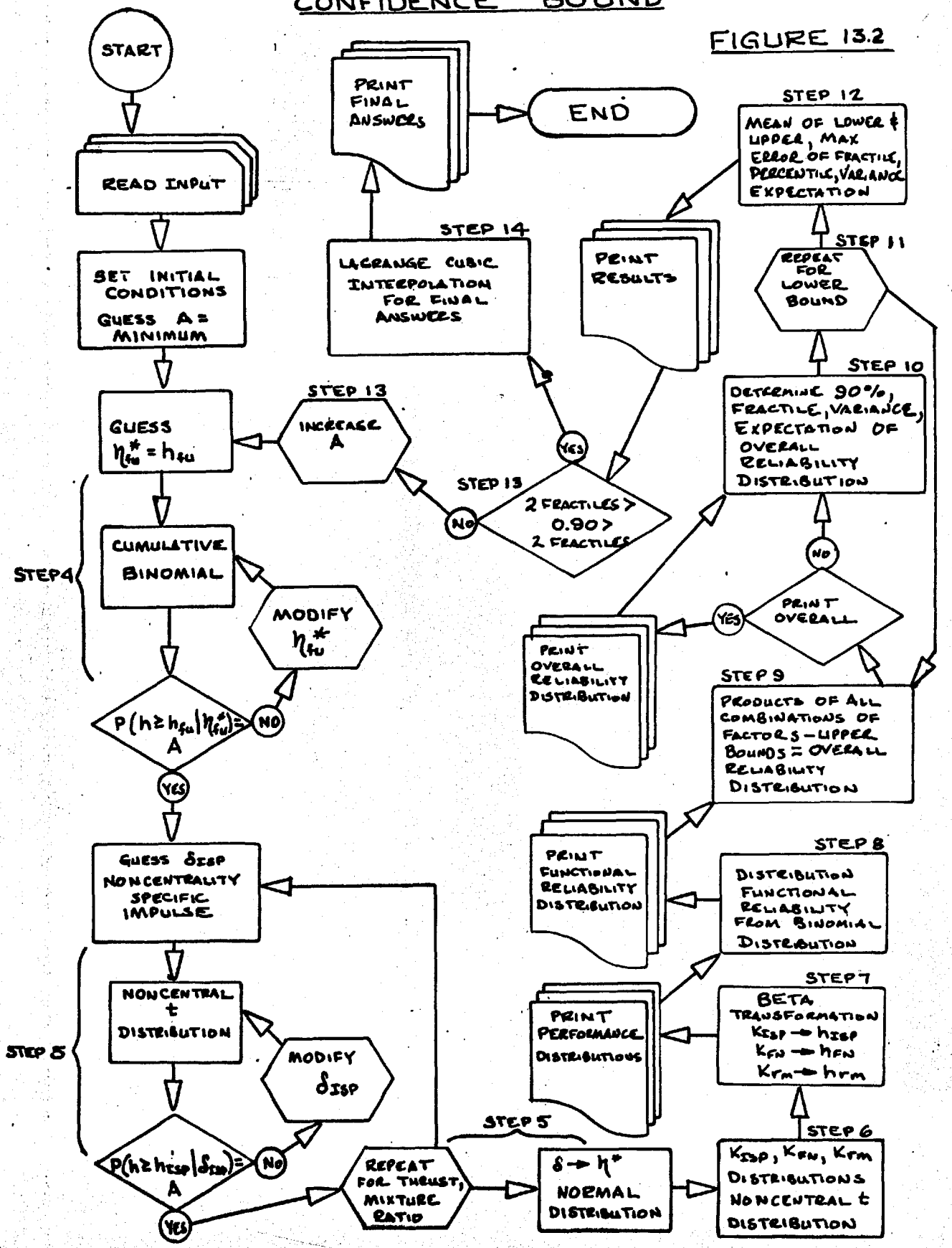
Interpolate in Table 13.16 again, to determine the equivalent, one-sided, k. Refer Section 12.b.

Step 4. The remaining steps are also illustrated by computer Program 13.2 in Fortran IV language, prepared for an IBM 7090 computer. The step numbers are indicated on the flow diagram, Figure 13.2. Program 13.2 in Fortran IV language with a sample printout of results is contained in Volume II. The following sample data will be used to illustrate the procedure and the program:

Functional reliability: 0 failures
20 equivalent full duration firings
$h_{fu} = 1.0$

FLOW DIAGRAM - OVERALL RELIABILITY CONFIDENCE BOUND

FIGURE 13.2



Performance reliabilities: $N_{ISP} = N_{FN} = N_{rm} = 10$

$k_{ISP} = k_{FN} = k_{rm} = 2.036$

$h_{ISP} = h_{FN} = h_{rm} = 0.99$

Overall reliability: $h_o = 0.9703$

Begin an iterative calculation by assuming a value of the constant A. It is convenient to select the complement of one of the seventeen percentiles used to calculate the k distributions in Section 5.a. The value η_{fu}^{*} is obtained from the cumulative form of the binomial distribution:

$$P\left[C \leq F \,\middle|\, \eta_{fu}^{*}\right] = A \tag{13.19}$$

$$A = \sum_{r=o}^{F} \frac{N!}{r!(N-r)!} \left(\eta_{fu}^{*}\right)^{N-r} \left(1-\eta_{fu}^{*}\right)^{r} \tag{13.20}$$

Step 5. The value η_{ISP}^{*} is obtained from the noncentrality of the noncentral t distribution. Using Table III, entering with confidence level set equal to the complement of A and the observed k, determine η_{ISP}^{*} from the noncentrality of the distribution that satisfies the relationship:

$$P\left[h_{ISP}^{*} \geq h_{ISP} \,\middle|\, \eta_{ISP}^{*}\right] = A = \tag{13.21}$$

$$P\left(k^{*} \geq k_{ISP} \,\middle|\, \delta = \sqrt{N}\, Z_p\right)$$

$$\eta_{ISP}^{*} = \Phi(Z_p) \qquad (13.22)$$

η_{FN}^{*} and η_{rm}^{*} are obtained similarly.

Step 6. Performance reliability distributions. Given the parameters η_{ISP}^{*}, η_{FN}^{*}, $\eta_{rm.}^{*}$ determine seventeen percentiles of the distribution of the corresponding k^{*} values from the noncentral t distribution as tabulated in Table III. Refer Chapter 5.

Step 7. Transform the k distribution from Step 6 to reliability values using Table 13.16 of (Bowker and Lieberman). Refer Chapter 7. The results of steps 5, 6 and 7 applied to the sample data with A = 0.3000 are shown in Program 13.2 Volume II

Step 8. Functional Reliability Distribution. Determine the distribution of functional reliability using the non-cumulative form of the binomial distribution. Refer Chapter 11.

$$h_{fu}^{*} = r/N \ , \ P(h_{fu}^{*} = r/N) = \frac{N!}{r!(N-r)!} \ (\eta_{fu}^{*})^{N-r} (1-\eta_{fu}^{*})^{r} \qquad (13.23)$$

Step 9. Consider the performance reliability distributions as discrete with eighteen values. Assign the upper bound value for interval as shown in Table

13.1. Take the product of every combination of one value from each of the four distributions and rank the results to obtain the overall reliability distribution. Refer Sections 13.c. Computer program results for the sample data are shown for A = 0.40 in Volume II. There is an option in the program at this point that prints the cumulative distribution using 1001 increments from zero to one if desired.

Step 10. From the ranked distribution determine the ninetieth percentile of overall reliability and the fractile of the observed overall reliability. Refer Sections 13.d and 13.g.

Step 11. Repeat steps 9 and 10 for the lower bounds of the eighteen intervals of the performance reliabilities.

Step 12. Determine the mean of the lower and upper values of statistics in Step 10 and the maximum error as half the difference between the lower and upper. Arrange the answers on the ninetieth percentile, fractile, as n x 4 matrices where the columns are lower, upper, mean, maximum error and the rows correspond to values of the constant A. Store the values η_{fu}^{*}, η_{ISP}^{*}, η_{FN}^{*}, η_{rm}^{*} and η_{o}^{*} as a m x 5 matrix; where m is the jth value of the constant A.

Step 13. If steps 4 - 12 have been repeated sufficient

times so that there are at least two values of the fractile (Step 10) above and below 0.90, continue on to Step 14. Otherwise increase the constant A and return to Step 4. A summary of the results of Steps 10, 11 and 12 for the sample data is shown in Volume II.

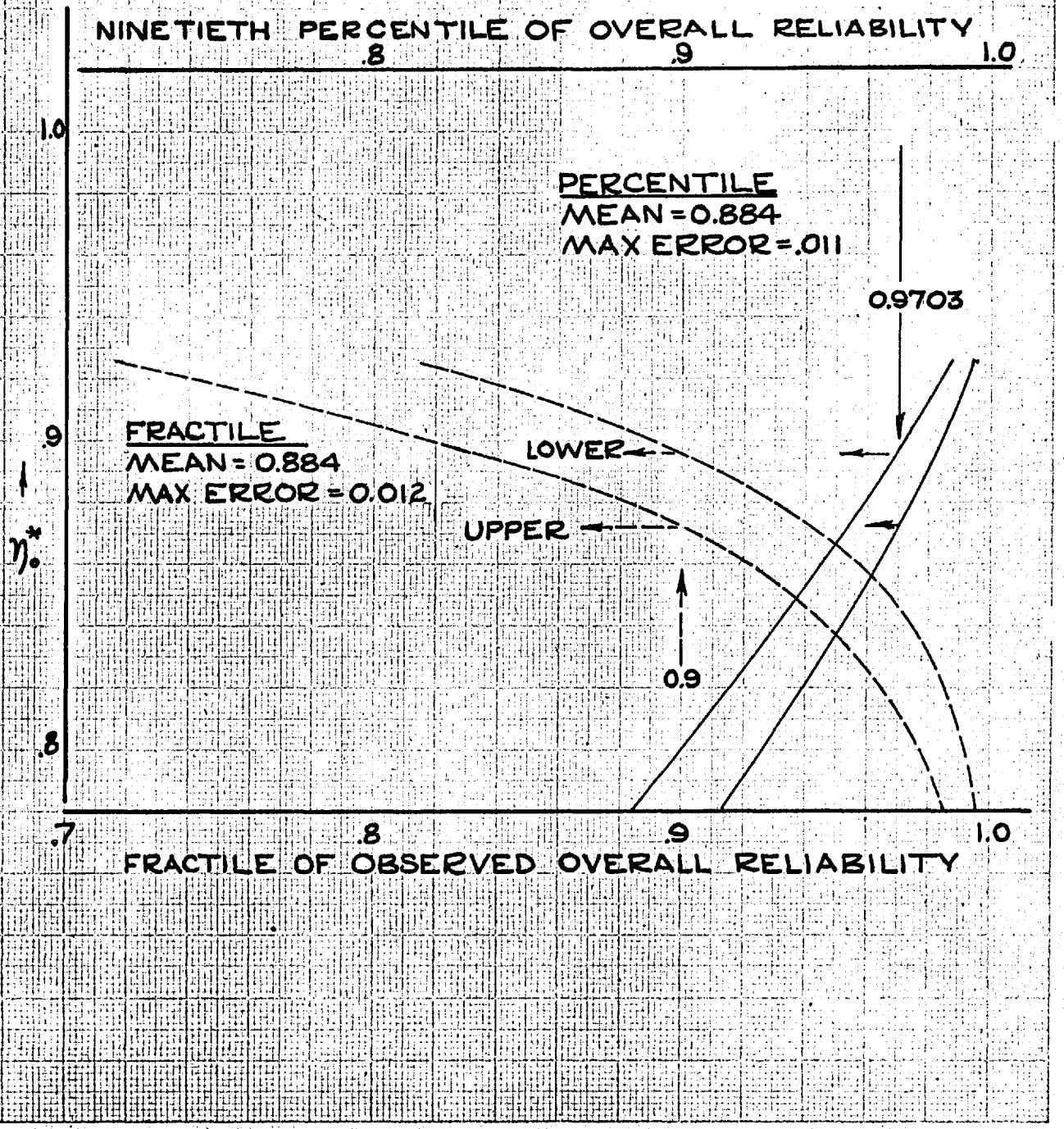
Step 14. The final answer, the lower 90°/o confidence bound for overall reliability, may be obtained by two different methods: both are illustrated in Figure 13.3. One method is to interpolate on the upper and lower bounds of the fractile of observed overall reliability, entering with the value 0.90; the other method is to interpolate on the bounds of the ninetieth percentile of the distribution of overall reliability. In both methods the answer is the midvalue of the interpolations. Both methods were programmed. The percentile method appears to be superior at low levels reliability, as $\eta_o^{\mathcal{H}}$ is almost a linear function of percentile while the fractile function is better at high reliability ($h_o > 0.99$). Lagrange third order interpolation is employed in the program to determine the final results shown in Volume II. The value 0.884 is the required lower 90°/o confidence bound for overall reliability for the sample data.

It should be mentioned that the mean value of the

FIGURE 13.3

OVERALL RELIABILITY CONFIDENCE BOUND

RESULTS FROM PROGRAM 13.2



expectations of the lower and upper distributions does not equal $\eta_o^{\star}$. The difference is a measure of the bias in the method of using the midvalue of the upper and lower bounds. Some consideration was given to applying the bias as a correction to the distribution of overall reliability but the idea was rejected because there is no reason to believe the bias is constant over the range of the distribution. The expectation may be calculated using a modification of Program 13.2.

A tabulation of typical computer results is shown in Table 13.3.

The solution to the known variance problem was not programmed but may be obtained by a simple modification of Program 13.2; instead of using the non-central t subroutine to obtain the performance reliability distributions a cumulative normal subroutine is required. Both subroutines are available in Program 13.2.

TABLE 13.3
LOWER 90% FREQUENTIST CONFIDENCE BOUND

OBSERVED DATA					BOUND
← PERFORMANCE →		← FUNCTIONAL →			
$N_{ISP} = N_{FN} = N_{rm}$	$h_{ISP} = h_{FN} = h_{rm}$	FAILURES	TESTS	h_{fu}	h_c
10	0.99	0	20	1.0	.884 ± .0108
	0.90	0	10	1.0	.567 ± .0288
		1		.90	.492 ± .0252
	0.99	0		1.0	.851 ± .0134
		1		.90	.714 ± .0141
	0.999	0		1.0	.913 ± .0114
		1		.90	.751 ± .0099
20	0.90	0	20	1.0	.621 ± .0216
	0.99				.906 ± .0089
	0.999				.952 ± .0043
100	0.90		100		.676 ± .0006
	0.99				.952 ± .0025
	0.999				.987 ± .0014

CHAPTER 14

LIKELIHOOD AND BAYESIAN ESTIMATES

Two estimates of overall reliability have been developed; an unbiased point estimate and a lower, one-sided, confidence interval. These estimates are required by most rocket engine contracts and they provide a means of comparison between the proposed method and the current method. However, whether these are the optimum type of estimates for this problem is questionable and it is appropriate that other estimates should be treated here.

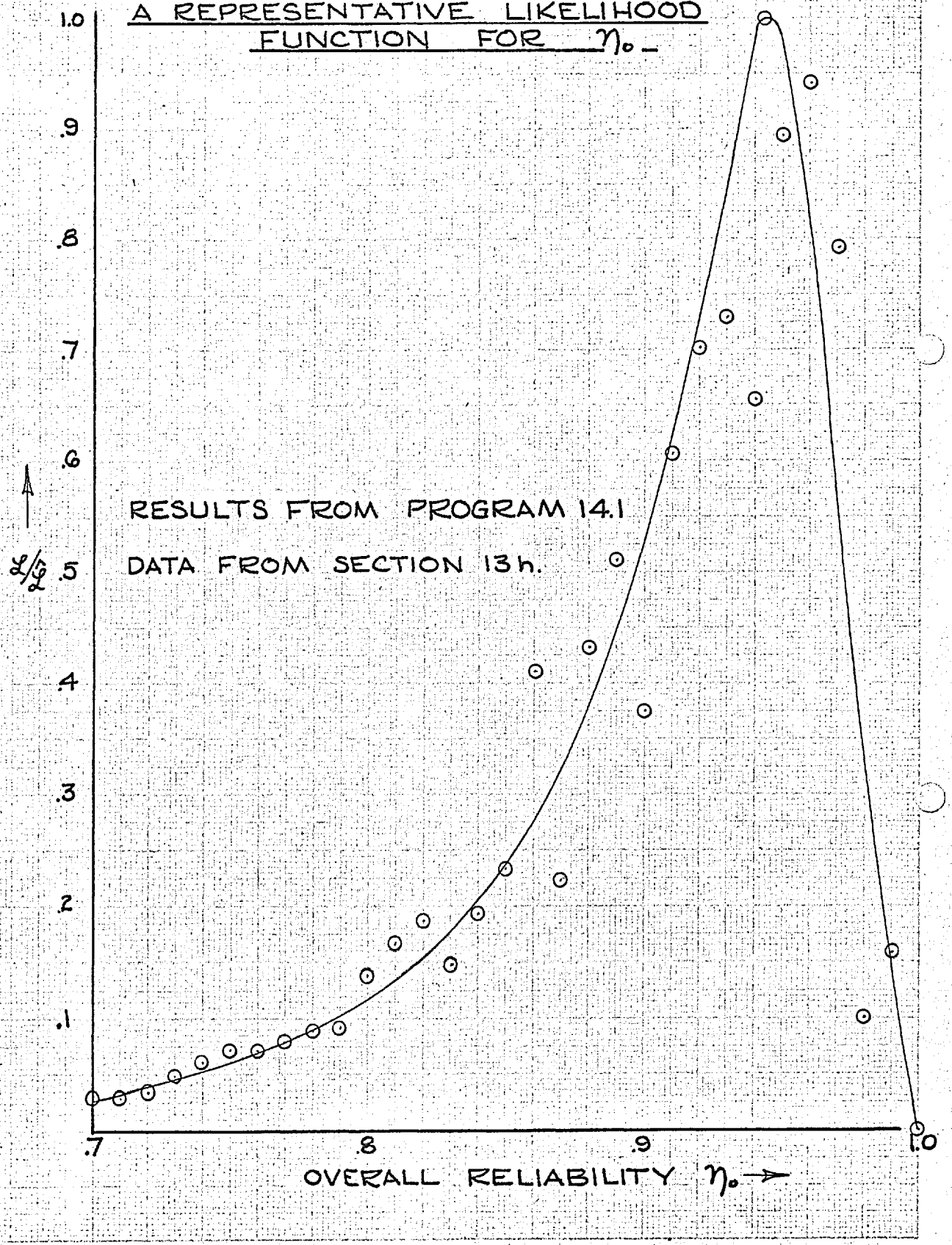
14.a. Likelihood Function

It is usually informative to plot the likelihood function but unfortunately the function for overall reliability involves seven parameters and therefore, the plot requires eight dimensional space. Using Method (3) of Section 8.a. (integrating out the standard deviation with reciprocal weighting) the seven parameters may be reduced to four but a five dimensional plot is still too complex to be easily understood. The problem is analogous to that of the confidence interval discussed in Chapter 13; a further simplification is required.

Program 14.1 was developed to investigate several alternative solutions. The objective was to produce a two dimensional plot of a representative likelihood function of overall reliability.

The following method was used. The interval from zero to one was divided into 1001 intervals or cells. The likelihood function of each of the four reliabilities was represented by twenty more values between reliabilities of zero to one. All combinations (about a quarter of a million) of the likelihood functions were calculated and stored in the appropriate cell determined by the corresponding value of overall reliability. The result was a sample of likelihood values, an average of several hundred, in each cell. The problem was then to select a representative value; although both the maximum and the midrange have some merit, the author believes the average value is the best choice. The latter is plotted in Figure 14.1 for the assumed sample data. If the average value curve is taken as typical or representative, the ratio of likelihood values from the curve may be used as an estimate of the odds in favour of one value over another value of overall reliability. The average value will be affected by the choice and number

FIGURE 14.1


A REPRESENTATIVE LIKELIHOOD FUNCTION FOR η_o -
RESULTS FROM PROGRAM 14.1
DATA FROM SECTION 13h.
$\mathcal{L}/\hat{\mathcal{L}}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
.7
.8
.9
1.0
OVERALL RELIABILITY η_o →

of values for the four likelihood functions; there should be a sufficient number to define each curve, particularly in areas where the slope changes rapidly. Values of the non-central t probability density function were taken from two different tables for the example illustrated, (Resnikoff and Lieberman 1957) and (Owens). Fortunately these tables have different increments of the argument of the function.

This approach is equally applicable to Method (1) and (2) likelihood functions of Section 8.a.

14.b. Maximum Likelihood Estimate

The maximum likelihood estimate of η_o is $\hat{h}_o$. As the factors in the product h_o are independent, the maximum likelihood estimate is the product of the maximum likelihood estimates of the factors

$$\hat{h}_o = \hat{h}_{fU}\ \hat{h}_{ISP}\ \hat{h}_{FN}\ \hat{h}_{rm} \qquad (14.1)$$

The estimate h_{fU} defined in Chapter 11 is a maximum likelihood estimate. The maximum likelihood estimates of the three performance reliabilities are based on the non-central t density function as described in Chapter 8.

Example: For the same assumed data used in the previous chapter;

Unbiased point estimates:

$$h_{fU} = 1.0, \quad h_{ISP} = h_{fN} = h_{rm} = 0.99$$

$$N_{fU} = 20, \quad N_{ISP} = N_{fN} = N_{rm} = 10$$

$$\hat{h}_o = \hat{h}_{fU}\ \hat{h}_{ISP}\ \hat{h}_{fU}\ \hat{h}_{rm}$$

$$= 1.0\ (0.9825)^3$$

$$= 0.9484$$

14.c. Bayesian Confidence Interval

The Bayesian posterior probability is proportional to the product of the likelihood function, $\mathcal{L}(\eta_o)$, and the Bayesian prior probability density, $b(\eta_o)$. There appeared to be two different approaches to developing a lower 90% Bayesian confidence interval for overall reliability, each with advantages over the other, and therefore both were developed.

14.d. Simple Method

A simple but approximate solution may be obtained from the representative likelihood function for overall reliability developed previously (Section 14.a). The following equation is used.

$$0.90 = \frac{\int_{h_c}^{1.0} \mathcal{L}(\eta_o)\ b\ (\eta_o)\ d\ \eta_o}{\int_{o}^{1.0} \mathcal{L}(\eta_o)\ b\ (\eta_o\ d\ \eta_o} \qquad (14.2)$$

The best choice for the prior distribution, b (η_o), is one established from past data, but lacking this, the most popular choice is the uniform distribution in the interval zero to one. It is used herein as a means of developing a Bayesian procedure but it is not supported by the author as a logical distribution assumption for this application; if it may be assumed that the overall reliability is equally likely to lie anywhere in the interval, zero to one, consideration should probably be given to cancelling the engine contract.

Based on the uniform prior distribution, zero to one, solution of Equation (14.2) is simple; the 1001 cells of Program 14.1 are summed, cumulatively, and the one-sided 90°/o Bayesian confidence interval is bounded above by 1.0 and below by the cell containing the first decile of the cumulative distribution. For the same assumed sample data used in the previous example, and the likelihood function from the non-central t, (Method 3), the value 0.804 was determined by Program 14.1 using the average likelihood. It is interesting to note that the

three methods, average, maximum and midrange likelihood functions, give approximately the same answer.

The advantage of this solution is its simplicity; any prior distribution may be easily combined with the two-dimensional likelihood function. For example, if a uniform prior distribution in the interval 0.80 to 1.0 is assumed the one-sided lower 90°/o confidence bound is 0.844. The disadvantage is that the answer is approximate because a two-dimensional likelihood function is used to represent an eight dimensional function. However, the results are encouraging when compared with the more exact solution that follows.

14.e. Complex Method

A much more precise solution is obtained by considering the product of the likelihood functions and the prior distributions of the four independent reliability factors.

$$0.90 = \frac{\iiiint_{(\eta_{fu}\eta_{ISP}\eta_{FN}\eta_{rm} > h_c)} \mathcal{L}(\eta_{fu})\mathcal{L}(\eta_{ISP})\mathcal{L}(\eta_{FN})\mathcal{L}(\eta_{rm})\, b(\eta_{fu})\, b(\eta_{ISP})\, b(\eta_{FN})\, b(\eta_{rm})\, d\eta_{fu}\, d\eta_{ISP}\, d\eta_{FN}\, d\eta_{rm}}{\iiiint_{(\text{ALL } \eta_{fu}\eta_{ISP}\eta_{FN}\eta_{rm})} \mathcal{L}(\eta_{fu})\mathcal{L}(\eta_{ISP})\mathcal{L}(\eta_{FN})\mathcal{L}(\eta_{rm})\, b(\eta_{fu})\, b(\eta_{ISP})\, b(\eta_{FN})\, b(\eta_{rm})\, d\eta_{fu}\, d\eta_{ISP}\, d\eta_{FN}\, d\eta_{rm}}$$

(14.3)

Each of the four likelihood functions is described by twenty to thirty intervals covering the range of reliability from zero to one. In the manner of trapezoidal integration, the likelihood value for each interval is taken as the average of the likelihood at the two extremes of the interval. The prior probability of the interval is obtained as the difference between cumulative prior probability evaluated at the extremes of the interval. Each interval is indexed by reliability; to obtain a solution with bounded error the calculation is done twice, first using the reliability at the upper bound of the interval and then the lower bound. For each combination of the four reliabilities; a weight is determined as the product of the four likelihoods and the four prior probabilities. (Again there are about a quarter of a million combinations). The weights are accumulated in 1001 cells of reliability between zero and one. The cumulative sum of these cells is used to find the cell containing the first decile; then the values within that cell are ranked to find the exact combination that determines the tenth decile. The average of the upper and lower bound solutions will have an error bounded by half the difference between the solutions. The method is illustrated by Program 14.2.

The first case investigated was based on assuming

a uniform prior distribution in the interval zero to one, for the four reliability factors. The result was a lower 90°/o bound of 0.613 ± .046 for the same data used with the simple method. The explanation for this surprisingly low result is that the assumption of uniform prior distributions for the four reliability factors is not equivalent to a uniform prior distribution for overall reliability. If t_1, t_2, t_3, t_4 are independent random variables with identical uniform distributions in the interval zero-one, the product $t_o = t_1 \times t_2 \times t_3 \times t_4$ will have expectation:

$$E(t_o) = E(t_1) \times E(t_2) \times E(t_3) \times E(t_4)$$
$$= (0.5)^4 = 0.0625 \qquad (14.4)$$

This pessimistic weight function is reflected in the very low 90°/o Bayesian confidence bound obtained above.

Therefore, to use this method, it is necessary to relate the factor prior distributions to the equivalent prior distribution for the product. Starting with a given prior for the product, overall reliability, it is an interesting problem in inverse convolution to determine the prior distribution of the independent and identically distributed factors. (The identically distributed characteristic is assumed here but may not be required).

The following solution was suggested by Professor G. A. Barnard for the case of the uniform distribution of the product and it will be useful in a more general sense:

If the prior distribution of $\eta_o = \eta_{fu}\ \eta_{ISP}\ \eta_{FN}\ \eta_{rm}$ is uniform in the interval zero to one, with

prior density: $b(\eta_o) = 1.0, \quad 0 < \eta_o < 1.0$

$$= 0, \quad \text{otherwise} \tag{14.5}$$

and if $\theta_o \equiv -2 \log \eta_o$, the transformed variable θ will have prior density:

$$b(\theta_o) = \frac{1}{2} \exp^{-\theta_o/2}, \quad 0 < \theta_o < \infty \tag{14.6}$$

This is the exponential density function with parameter $\lambda = \frac{1}{2}$ and expectation $\frac{1}{\lambda} = 2$. It is also the χ^2 density function with two degrees of freedom.

$$\theta \equiv -2 \log \eta_o = -2 \log(\eta_{fu}\ \eta_{ISP}\ \eta_{FN}\ \eta_{rm})$$

$$= -2 \log \eta_{fu} - 2 \log \eta_{ISP} - 2 \log \eta_{FN} - 2 \log \eta_{rm}$$

$$= \theta_{fu} + \theta_{ISP} + \theta_{FN} + \theta_{rm} \tag{14.7}$$

If the four variables on the right side of Equation (14.7) are independent and identically distributed as χ^2 with degrees of freedom $f = 1/2$, θ_o will be distributed as χ^2

with two degrees of freedom because of the additive characteristic of independent χ^2 distributions.

The results of applying this solution to the sample data are illustrated in Volume II as Program 14.2, and in Table 14.1, a summary of results. The agreement with the simple method is reasonable.

This method may be employed with other degrees of freedom to simulate prior distributions for overall reliability over the wide range of χ^2 distributions. The Russian tables by (Sluckii, E.E.) provide a simple, accurate method of generating χ^2 distributions with fractional degrees of freedom. Further the value 2 in the transformation from η to θ may also be varied.

TABLE 14.1

SUMMARY OF RELIABILITY ESTIMATES

ASSUMED DATA

Functional: 20 tests 0 failures

Performance: $N_{ISP} = N_{FN} = N_{rm} = 10$

$\bar{X}_{ISP} = \bar{X}_{FN} = \bar{X}_{rm} = 2.036$

$s_{ISP} = s_{FN} = s_{rm} = 1.0$

$L_{ISP} = (L_{EQ})_{FN} = (L_{EQ})_{rm} = 0.0$

η_{fu}, η_{ISP}, η_{FN}, η_{rm}

(A) Unbiased Point Estimates (Chapters 11, 12)

$h_{fu} = 1.0$, $h_{ISP} = h_{FN} = h_{rm} = 0.99$

(B) Maximum Likelihood Estimates (Chapters 8, 11, 14)

$h_{fu} = 1.0$, $h_{ISP} = h_{FN} = h_{rm} = 0.9825$

(C) Frequency Interval Estimates (Chapters 7, 11, 12)

90°/o confidence: $h_{fuc} = 0.891$

$h_{ISPc} = h_{FNc} = h_{rmc} = 0.896$

(D) Likelihood Functions (Chapters 8, 14)

See Figures, Chapter 8 & Programs 14.1, 14.2

η_{ov}

(A) Unbiased Point Estimate (Chapter 13)

$$h_{ov} = 0.9703$$

(B) Maximum Likelihood Estimate (Chapter 14)

$$\hat{h}_{ov} = 0.9484$$

(C) Frequency Interval Estimate (Chapter 13)

$$h_{ovc} = 0.884 \pm 0.011$$

(D) Likelihood Function (Chapter 14)

Figure 14.1

(E) Bayesian Interval Estimates (Chapter 14)

90% Confidence

METHOD	PERFORMANCE LIKELIHOOD	PRIOR DISTRIBUTION	LOWER BOUND
Simple (14d)	Method (3)(8a)	η_o uniform 0-1.0	0.804
Simple (14d)	Method (3)(8a)	η_o uniform 0.8-1.0	0.844
Complex(14e)	Method (3)(8a)	$\eta_{ISP}, \eta_{FN}, \eta_{rm}, \eta f_u$ uniform 0-1.0	$0.613 \pm .046$
Complex(14e)	Method (3)(8a)	η_o uniform 0-1.0	$0.776 \pm .05$
Complex(14e)	Method (2)(8a)	η_o uniform 0-1.0	$0.873 \pm .025$

14.f. Choice of Estimates

Methods for estimating overall reliability and its components have been developed under the three schools of inference: frequentist, likelihood and Bayesian. This might have been avoided if the author believed that one of these approaches is clearly superior to the other two; all three have advantages and disadvantages.

The final choice will be made by the customer, probably in a written contract, after considering the producer's recommendations. The choice should be based on the requirements of the application and the quality of the data. The requirements might be defined as:

(1) a need for a point estimate to indicate the level of reliability achieved

(2) some measure of uncertainty as a gauge of precision

(3) a method of comparing the estimate with reliability requirements or goals.

(4) a means of combining engine estimates with other system estimates to estimate vehicle reliability.

(5) perhaps most important, the estimates must be readily understood by a large audience of non-statisticians.

The point estimate employed in the current method is both unbiased and also a maximum likelihood estimate. The point estimate of overall reliability developed in Chapter 13 is unbiased, but it is not a maximum likelihood estimate, while the maximum likelihood estimate developed in this chapter is not unbiased. The choice between the two is difficult. The good statistical characteristics of maximum likelihood estimates are well known to statisticians and in general, they are better than unbiased estimates; unbiased estimates are more popular with non-statisticians and some education would be required to change to maximum likelihood estimates.

As a measure of precision the competitors are the (1) frequentist and (2) Bayesian confidence intervals and (3) the likelihood function. A great deal has been said and written about this choice based on theoretical considerations. The author's experience is entirely with the frequentist approach; it is a subtle concept which is both popular and generally misunderstood by project engineers and managers. This last is a serious drawback and it is sufficient to inspire consideration of alternatives.

14.g. Conclusion

For a trial period, the author proposes to conduct a competition between the methods. The criteria will be the reaction and comprehension of the producer and customer project engineers and management. These are the people who take action on the estimates and they are not statisticians. It is intended to publish both:

(1) unbiased point estimates and a frequentist confidence interval, and

(2) the maximum likelihood estimate with the likelihood function, or if a good prior distribution exists, a Bayesian confidence interval. The likelihood function will be presented as a means of evaluating progress toward reliability objectives, in its relation to the maximum likelihood estimate, and as a measure of precision.

PART IV

COMPLEX MODELS

The simple model described in Part III will often be appropriate. There are several other situations however, which occur often enough to warrant consideration even though they are much more complex: a sample space in which there is more than one firing per engine assembly, the partition of the sample space due to trimming, a mission space with multiple firings and multiple engines, and the effect of instrumentation precision. All of these cases share the common problem of dissimilar mission and sample populations.

CHAPTER 15

MULTIFIRING SAMPLE SPACE AND TRIMMING

15.a. Complex Versus Simple Models

Experimental engine assemblies are usually fired more than once; the cost of installing and removing engines from a test stand expressed in money and time is large and multiple firings may expose wearout failure modes to correction. It is in the best interest of reliability improvement to encourage this form of accelerated testing even though it makes reliability estimation more difficult. Similar reasoning follows for untrimmed firings; intentionally probing the limits of engine operation by running at extreme values of thrust and mixture ratio produces both performance and failure information that is invaluable for improving the engine.

Unfortunately, this type of data does not provide as much information for reliability estimates as the simple model data described in Part III and the solutions which follow are more complex and more approximate than the simple model solution described in Part III. Fortunately multifiring data can always be reduced to simple model data by randomly selecting a single firing from each

engine and discarding the remaining firings. The decision to either (1) discard data and use the simple, more exact, methods or (2) use the complex models with more approximate methods should be based on the improvement in precision versus the improvement in accuracy. Two guidelines may be suggested. If the number of engines in the sample is large, say fifty or more, the precision of the proposed method will be improved very little by larger performance samples and therefore, the simple model should be used. If the run-to-run (within engine) variance, σ_R^2, is extremely small, compared to the engine-to-engine variance, σ_E^2, the gain in precision obtained by considering more than one firing per engine will be very small and again, the simple model should be used. Otherwise, the decrease in the length of the confidence interval is recommended as a guide, perhaps 5°/o. Whatever criteria are employed it should be clearly understood between producer and consumer prior to its application. In general, each performance parameter will have different degrees of freedom and variances and therefore, three decisions will be required, one for each parameter.

15.b. The Complex Model

Assume the data sample consists of r firings (runs) made on each of e engines and the mission consists of a single firing. The standard Model II analysis of variance may be used to describe the sample data. The following procedures are structurally based on the literature available on the Model II ANOVA particularly (Davies), (Keeping), (Owens), and (Brownlee); however the application is different from the usual one described in the references. Consider the model:

$$X_{ij} = \mu + E_i + R_{ij} \qquad (15.1)$$

where μ is a constant

$E_i \sim N(0, \sigma_E^2)$

$R_{ij} \sim N(0, \sigma_R^2)$

$i = 1,2,3,\ldots,e$

$j = 1,2,3,\ldots,r$

15.c. Engine-to-Engine Variation

The grand mean of all engines is μ. The mean value of engine assembly i is $\mu + E_i$ with variance σ_E^2. The magnitude of the engine-to-engine variance, σ_E^2, is a measure of quality control; with perfect quality control and ideal raw materials this variance would tend to small

values. This is particularly true of specific impulse, which is independent of trimming but the mean values of mixture ratio and thrust are determined by adjusting or trimming their respective controls. The trimming procedure is part of the production quality control but even under ideal conditions the variances of these two parameters will not vanish as they are also a function of the run-to-run variance. If the trimming is done on the first firing, the procedure would be to observe the value,

$X'_{il} = \mu + E'_i + R_{il}$, (where primes indicate before trimming) and adjust the control during the firing until the observed value

$$X_{il} = \mu = \mu + E_i + R_{il} \qquad (15.2)$$

which implies that $E_i = -R_{il}$. It follows that for mixture ratio and thrust the minimum value of the variance σ_E^2 will be σ_R^2 unless a more complex trimming procedure is employed.

15.d. Run-to-Run Variation

The run-to-run or within engine variance, σ_R^2, will be a function of engine repeatability and instrumentation precision. The problem of true versus indicated estimates

will be discussed in the next chapter. Here it will be assumed that either the instrumentation precision is small or unknown or that the required mission estimate is to be based on indicated values (including instrumentation precision) in contrast to true values. The variances, σ_R^2 and σ_E^2, are usually referred to as components of variance. Estimates of the components of variance may be obtained from the following table:

TABLE 15.1

ANALYSIS OF VARIANCE

Source	Sum-of-Squares	D.F.	E(Mean Square)
Engine-to-Engine	$r\sum_{i=1}^{e}(\bar{X}_{i.}-\bar{X}_{..})^2$	(e-1)	$\sigma_R^2 + r\,\sigma_E^2$
Within Engine (Run-to-Run)	$\sum_{i=1}^{e}\sum_{j=1}^{r}(X_{ij}-\bar{X}_{i.})^2$	e(r-1)	σ_R^2
TOTAL	$\sum_{i=1}^{e}\sum_{j=1}^{r}(X_{ij}-\bar{X}_{..})^2$	(er-1)	

(A dot subscript indicates an average over that subscript)

15.e. Non Central t

The variance of the firing in the mission population is the sum of the engine-to-engine variance and the run-to-run variance.

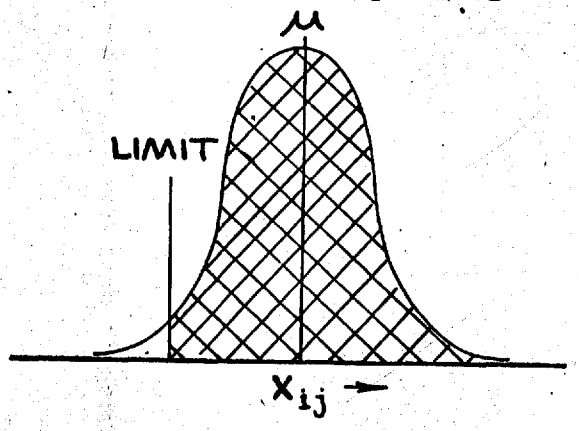
$$X_{ij} = \mu + E_i + R_{ij}$$

$$j = 1.0$$

$$X_{il} = \mu + E_i + R_{il} \qquad (15.3)$$

$$\sigma^2 = \sigma_E^2 + \sigma_R^2$$

The problem of estimating the shaded area in the figure below is equivalent to estimating the noncentrality of the corresponding noncentral t distribution.



$$\sigma = \sqrt{\sigma_E^2 + \sigma_R^2}$$

$$\delta \equiv \text{noncentrality} = \left(\frac{\mu - L}{\sigma}\right)\sqrt{N}$$

Returning to the statistic k treated previously in Part II:

$$\sqrt{N}\, k = \sqrt{N}\, \left(\frac{\bar{X} - L}{s}\right)$$

$$= \frac{\sqrt{N}\, \left(\frac{\bar{X} - \mu}{\sigma}\right) - \sqrt{N}\, \left(\frac{\mu - L}{\sigma}\right)}{\frac{s}{\sigma}}$$

Define σ_1^2 such that $\bar{X} \sim N(\mu, \frac{\sigma_1^2}{N})$

σ_2^2 such that $\frac{fs^2}{\sigma_2^2} \sim \chi_f^2$

Then the following variable is distributed as a non-central t:

$$t = \frac{\sqrt{N}\,\frac{\bar{X}-\mu}{\sigma_1} - \sqrt{N}\,(\frac{\mu-L}{\sigma})\,\frac{\sigma}{\sigma_1}}{\frac{s}{\sigma_2}} = \sqrt{N}\,\frac{\sigma_2}{\sigma_1}\,k \qquad (15.4)$$

If $Z_\eta \equiv \frac{\mu-L}{\sigma} \equiv \Phi^{-1}(\eta)$

$$t = \frac{\sqrt{N}\,\frac{\bar{X}-\mu}{\sigma_1} - \sqrt{N}\,\frac{\sigma}{\sigma_1}\,Z_\eta}{\frac{s}{\sigma_2}} = \sqrt{N}\,\frac{\sigma_2}{\sigma_1}\,k \qquad (15.5)$$

where: $\sigma^2 = \sigma_R^2 + \sigma_E^2$ (15.6)

$\sigma_1^2 = \sigma_R^2 + r\,\sigma_E^2$ (15.7)

$s/\sigma_2 \sim \sqrt{\chi_f^2 / f}$ (15.8)

$t \sim$ Noncentral t (noncentrality $\delta = \sqrt{er}\,\frac{\sigma}{\sigma_1}\,Z_\eta$,

Degrees of freedom = f)

With three unknown parameters σ_R^2, σ_E^2 and μ there does not appear to be an exact solution. The problem

is related to the celebrated Fisher-Behrens problem. If the past history of the engines shows that the ratio σ_R^2/σ_E^2 is reasonably stable, the same large batch of data that is used to estimate weighting factors may be used to establish the ratio. Methods of judging stability will be discussed later. Two different approximate solutions will be presented, both based on a known ratio, σ_R^2/σ_E^2; if the ratio is not known or unstable, the problem appears to be impossible and the data should be reduced to single firing per engine data by random trimming.

15.f. σ_R Small Compared to σ_E

If σ_R is extremely small compared to σ_E there will be little to be gained by considering the replication firings on each engine and as suggested in Section 15.a, the simple model should be used. However, if the number of engines is small, the number of firings per engine is large and σ_R is small compared to σ_E, the improvement in precision may be significant and the following method will be appropriate.

The best estimate of μ is the grand average of all the data, $\bar{X}_{..}$; the improved precision of $\bar{X}_{..}$ over $\bar{X}_{.1}$

is the reason for following this alternative to the simple model.

$$\bar{X}_{.1} \sim N(\mu, \frac{\sigma_R^2 + \sigma_E^2}{e})$$

$$\bar{X}_{..} \sim N(\mu, \frac{\sigma_r^2 + r\,\sigma_E^2}{re})$$

therefore: $\sigma_1^2 = \sigma_R^2 + r\,\sigma_E^2$ (15.9)

The mean square on the first line of the ANOVA, divided by r, is attractive as a choice for s^2 because it leads to simple noncentral t distribution and its expectation approximates $\sigma^2 = \sigma_R^2 + \sigma_E^2$

$$E(s^2) = \frac{\sigma_R^2}{r} + \sigma_E^2 = \sigma_R^2 + \sigma_E^2, \text{ if } \sigma_R << \sigma_E \quad (15.10)$$

$$= \sigma_2^2 = \frac{\sigma_1^2}{r} \quad (15.11)$$

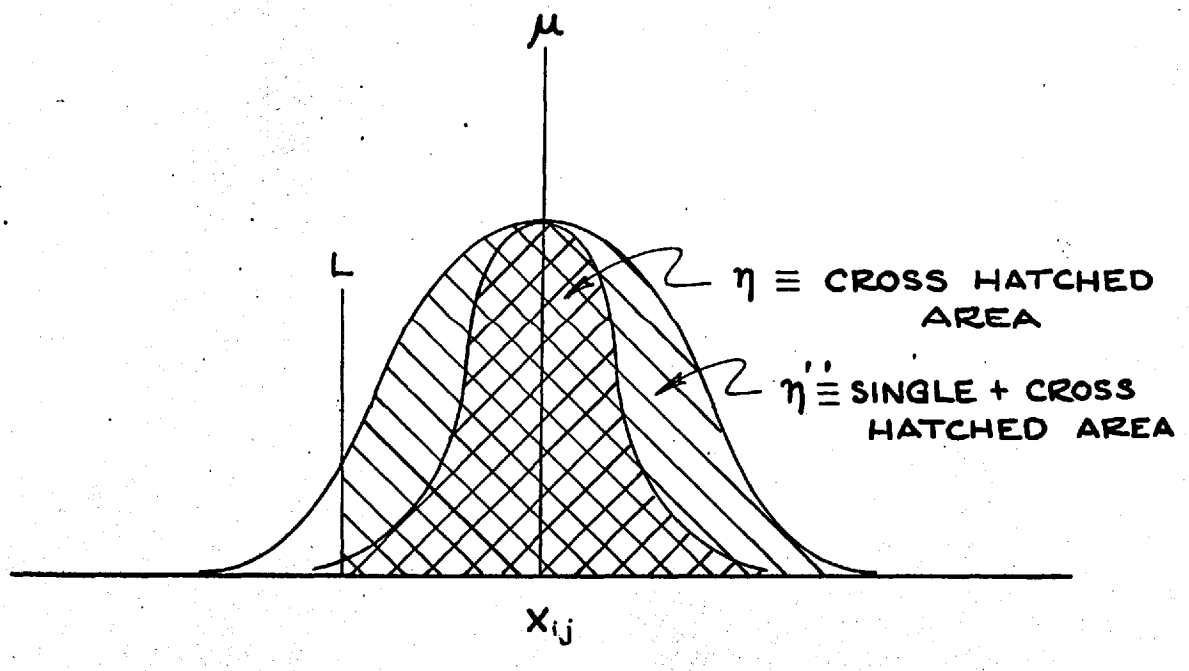
$$\sigma_2/\sigma_1 = 1/\sqrt{r} \quad (15.12)$$

$$\frac{(e-1)\,s^2}{\sigma_2^2} = \chi^2(e-1) \quad (15.13)$$

$$t = \sqrt{er}\ \frac{\sigma_2}{\sigma_1}\,k = \sqrt{e}\,k \quad (15.14)$$

with noncentrality $\delta = \sqrt{er}\sqrt{\frac{\sigma_R^2 + \sigma_E^2}{\sigma_R^2 + r\,\sigma_E^2}}\ Z_\eta$ (15.15)

Procedure: Calculate $\bar{X}_{..} = \sum_{i=1}^{e}\sum_{j=1}^{r}\frac{X_{ij}}{er}$ (15.16)



h'' = Estimate of η''

$$\eta'' = \Phi\left(\frac{\mu - L}{\sqrt{\sigma_R^2 + r\,\sigma_E^2}}\right) \tag{15.21}$$

h = Estimate of η

$$\eta = \Phi\left(\frac{\mu - L}{\sqrt{\sigma_R^2 + \sigma_E^2}}\right) \tag{15.22}$$

$$Z_\eta / Z_{\eta''} = \sqrt{\frac{\sigma_R^2 + r\,\sigma_E^2}{\sigma_R^2 + \sigma_E^2}} \tag{15.23}$$

$$= \sqrt{\frac{\theta + r}{\theta + 1}}$$

$$h = \Phi\left((Z_\eta / Z_{\eta''}) \, \Phi^{-1} (h'') \right) \tag{15.24}$$

$$s = \sqrt{\frac{\sum_{i=1}^{e} (\bar{X}_{i.} - \bar{X}_{..})^2}{e - 1}} \qquad (15.17)$$

$$k = \frac{\bar{X}_{..} - L}{s} \quad , \quad f = e-1 \qquad (15.18)$$

From (Bowker and Lieberman) and Table III, Part II, determine an unbiased point estimate, h'' and a lower one-sided 90°/o confidence bound, h_c'', using k and the degrees of freedom (e-1). Transform these estimates to the required estimates using the following relationship:

$$h = \Phi \left\{ \sqrt{\frac{\sigma_R^2 + r\,\sigma_E^2}{\sigma_R^2 + \sigma_E^2}} \; \Phi^{-1} (h'') \right\}$$

$$= \Phi \left[\sqrt{\frac{\theta + r}{\theta + 1}} \; \Phi^{-1} (h'') \right] \qquad (15.19)$$

where $\theta \equiv \sigma_R^2/\sigma_E^2$

$$h_c = \Phi \left[\sqrt{\frac{\theta + r}{\theta + 1}} \; \Phi^{-1} (h_c'') \right] \qquad (15.20)$$

An illustration may be helpful to demonstrate the effect of the above transformations.

The accuracy of this approximation will increase as the ratio θ approaches zero and as the proportion, η, approaches one. Given values of r, θ and η, the bias in the estimate h may be determined by numerical integration as illustrated below. Given r, θ, and h'' the bias in h may be estimated by assuming that $\eta'' = h''$. Program 13.2 was used to obtain eleven percentiles of the distribution of k'', given η''. These eleven were selected as they are typical (median) values within the ranges indicated. If we treat the distribution as discrete with these eleven values, the results will be sufficiently accurate for estimating error and the computations are considerably reduced. (See Table 15.2).

TABLE 15.2 CALCULATION OF BIAS

ASSUME $\eta'' = h'' = 0.9826$, $N = 10$, $\sqrt{\frac{\theta+r}{\theta+1}} = 1.4$

%	k''	h''	$Pr(h'')$	$h'' \times Pr(h'')$	$\Phi^{-1}(h'')$	$\sqrt{\frac{\theta+r}{\theta+1}}\,\Phi^{-1}(h'')$	h	$h \times Pr(h)$
2.5	1.279	0.9037	.05	.04518	1.3029	1.824	.9659	.04829
10.0	1.537	0.9464	.10	.09464	1.6108	2.255	.9879	.09879
20.0	1.731	0.9689	.10	.09689	1.8650	2.611	.9955	.09955
30.0	1.888	0.9817	.10	.09817	2.0902	2.926	.9983	.09983
40.0	2.036	0.9900	.10	.09900	2.3263	3.257	.99942	.099942
50.0	2.187	0.9953	.10	.09953	2.5972	3.636	.99985	.099985
60.0	2.353	0.9984	.10	.09984	2.9478	4.127	.99998	.099998
70.0	2.549	0.99977	.10	.099977	3.51	4.914	1.0	.10
80.0	2.806	1.0	.10	.10	∞	∞	1.0	.10
90.0	3.226	1.0	.10	.10	∞	∞	1.0	.10
97.5	4.066	1.0	.05	.05	∞	∞	1.0	.05

$\eta'' = 0.9826$ $\qquad \sum_{1}^{11} (h'' \times Pr(h'')) = 0.9832$

$$\eta = \Phi\left\{\sqrt{\frac{\theta+r}{\theta+1}}\,\Phi^{-1}(\eta'')\right\} = \Phi\left\{1.4 \times 2.1107\right\} = 0.9984$$

$$\sum_{1}^{11} (h \times Pr(h)) = 0.9964$$

ESTIMATE OF BIAS $= 0.9964 - 0.9984$
$= -0.0020$

The advantage of this method over the next method is that the distribution is exact, even though the variance is only approximately the correct variance and that a good estimate of the error or bias can be determined. If this method is used frequently, it may be possible to develop an unbiased estimate by iterative correction of the bias as illustrated in Chapter 7.

15.g. σ_R Not Small Compared to σ_E

The required variance, σ^2, may be estimated as a linear combination of the two independent mean squares in the ANOVA table:

$$\sigma^2 \equiv \sigma_R^2 + \sigma_E^2 = \frac{(\sigma_R^2 + r\,\sigma_E^2) + (r-1)\,\sigma_R^2}{r} \qquad (15.25)$$

$$s^2 = \sum_{i=1}^{e} \frac{(\bar{X}_{i.} - \bar{X}_{..})^2}{e-1} + \sum_{i=1}^{e}\sum_{j=1}^{r} \frac{(X_{ij} - \bar{X}_{i.})^2}{er} \qquad (15.26)$$

Unfortunately, the variable $(\frac{fs^2}{\sigma^2})$ is only approximately a χ_f^2 variable and therefore, the degrees of freedom cannot be determined exactly. The approximation in this method is to assume the above variable is distributed as a χ_f^2. Based on this assumption, two solutions for the degrees of freedom appear in current literature,

the more popular is due to (Welch and Satterthwaite), the other is due to (Dixon and Massey). A detailed discussion of these estimates will be postponed until later; it is sufficient here to state that the Welch-Satterthwaite solution appears to be superior.

Welch-Satterthwaite estimate:

if $s^2 = a_A S_A^2 + a_B s_B^2$

$$f \cong \frac{(a_A s_A^2 + a_B s_B^2)^2}{\frac{a_A^2 (s_A^2)^2}{f_A} + \frac{a_B^2 (s_B^2)^2}{f_B}} \qquad (15.27)$$

In our case, $s_A^2 = \sum_{i=1}^{e} \left(\frac{\bar{X}_{i.} - \bar{X}_{..}}{e - 1}\right)^2$, $f_A = e-1$, (15.28)

and $s_B^2 = \sum_{i=1}^{e} \sum_{j=1}^{r} \frac{(\bar{X}_{ij} - \bar{X}_{i.})^2}{er}$,

$f_B = e(r-1)$ (15.29)

$a_A = a_B = 1$

$$f \cong \frac{(s_A^2 + s_B^2)^2}{\frac{s_A^2}{e-1} + \frac{s_B^2}{e(r-1)}} \qquad (15.30)$$

Returning to the general formula for a non central t used earlier:

$$t = \frac{\sqrt{N}\ \frac{\bar{X}-\mu}{\sigma_1} - \sqrt{N}\ \frac{\sigma}{\sigma_1} Z_\eta}{\frac{s}{\sigma_2}} = \sqrt{N}\ \frac{\sigma_2}{\sigma_1}\ k \qquad (15.31)$$

$$\cdot\ \sigma_2^2 = \sigma^2 = \sigma_R^2 + \sigma_E^2 \qquad (15.32)$$

$$t = \frac{\sqrt{N}\ \frac{\bar{X}-\mu}{\sigma_1} - \sqrt{N}\ \frac{\sigma}{\sigma_1} Z_\eta}{\frac{s}{\sigma}} = \sqrt{N}\ \frac{\sigma}{\sigma_1}\ k \qquad (15.33)$$

$$N = er$$

Procedure: Calculate $\bar{X}_{..} = \sum_{i=1}^{e} \sum_{j=1}^{r} \frac{X_{ij}}{er}$ (15.35)

$$s^2 = \sum_{i=1}^{e} \frac{(\bar{X}_{i.} - \bar{X}_{..})^2}{e-1} + \sum_{i=1}^{e} \sum_{j=1}^{r} \frac{(X_{ij} - \bar{X}_{i.})^2}{er} \qquad (15.36)$$

$$f = \frac{(s_A^2 + s_B^2)^2}{\frac{s_A^2}{e-1} + \frac{s_B^2}{e(r-1)}} \qquad (15.37)$$

$$\text{if } \theta \equiv \sigma_R^2/\sigma_E^2\ ,\ \sigma_1/\sigma = \sqrt{\frac{\theta + r}{\theta + 1}} \qquad (15.38)$$

$$k''' = \sqrt{\frac{\theta + 1}{\theta + r}}\ \left(\frac{\bar{X}_{..} - L}{s}\right),\ \text{degrees of freedom} = f \qquad (15.39)$$

Read h''' and h_c''' from (Bowker and Lieberman) and Table III interpolating on both k''' and f. (f will not necessarily be an integer).

$$h = \Phi \left(\sqrt{\frac{\theta + r}{\theta + 1}}\, \Phi^{-1} (h''') \right) \tag{15.40}$$

$$h_c = \Phi \left(\sqrt{\frac{\theta + r}{\theta + 1}}\, \Phi^{-1} (h_c''') \right) \tag{15.41}$$

This solution is based on two approximations: the sum of mean squares is only approximately a χ^2 variable and the approximation that h''', an unbiased estimate of η''', may be transformed to an unbiased estimate of η. The error introduced in the latter assumption may be estimated by the same procedure used in the previous section, but the error from the former is unknown. This solution is therefore less attractive than the first solution (given σ_r small compared to σ_E) which is based on a single assumption with a good estimate of error available.

It may be noted that if σ_E is small compared to σ_R, a method very similar to first approximation may be developed. Under these conditions the mean square on the first line of the ANOVA table has expectation:

$$\sigma_R^2 + r\,\sigma_E^2 = \sigma_R^2 + \sigma_E^2 \text{ , if } \sigma_E << \sigma_R$$

This method is not presented because it would be unusual with rocket engine performance parameters to find $\sigma_E << \sigma_R$.

15.h. Trimming

As mentioned previously, mixture ratio and thrust share a special problem that does not affect specific impulse. If an engine has not been trimmed, mixture ratio and thrust are not required to fall within specification limits. Under the current method, the data on thrust and mixture ratio are discarded but if these untrimmed engines are fired several times the data can contribute to estimating the run-to-run variance, σ_R^2, even though it contributes nothing to the estimate of σ_E^2 because σ_R^2 is independent of the control settings. This information may be utilized if the components of variance are estimated separately as in the second method above, but not otherwise. A decision will have to be made to use the untrimmed data with the more approximate methods of this chapter or to discard it and use the simple model. The same criteria discussed in Section 15.a are applicable here.

If it is decided to incorporate the untrimmed data the second method above must be modified to accommodate the additional information by pooling untrimmed and trimmed run-to-run data. Two modifications are required: Assume the untrimmed engines are numbered e+1, e+2,....,g with r runs on each engine.

For the trimmed data: $$E\left[\sum_{i=1}^{e} \frac{(\bar{X}_{i.}-\bar{X}_{..})^2}{e-1}\right] = \frac{\sigma_R^2}{r} + \sigma_E^2 \qquad (15.42)$$

$$E\left[\left(\frac{r-1}{r}\right) \sum_{i=1}^{e} \sum_{j=1}^{r} \frac{(X_{ij}-\bar{X}_{i.})^2}{e(r-1)}\right] = \left(\frac{r-1}{r}\right) \sigma_R^2 \qquad (15.43)$$

For the untrimmed data:

$$E\left[\left(\frac{r-1}{r}\right) \sum_{i=e+1}^{g} \sum_{j=1}^{r} \frac{(X_{ij}-\bar{X}_{i.})^2}{(g-e)(r-1)}\right] = \left(\frac{r-1}{r}\right) \sigma_R^2 \qquad (15.44)$$

Pooling sums of squares, trimmed and untrimmed:

$$E\left[\left(\frac{r-1}{r}\right) \sum_{i=1}^{g} \sum_{j=1}^{r} \frac{(X_{ij}-\bar{X}_{i.})^2}{g(r-1)}\right] = \left(\frac{r-1}{r}\right) \sigma_R^2 \qquad (15.45)$$

$$E(s^2) = E\left[\sum_{i=1}^{e} \frac{(\bar{X}_{i.}-\bar{X}_{..})^2}{e-1} + \sum_{i=1}^{g} \sum_{j=1}^{r} \frac{(X_{ij}-\bar{X}_{i.})^2}{g(r)}\right] \qquad (15.46)$$

$$= \sigma_R^2 + \sigma_E^2$$

$$\text{Degrees of freedom, } f \simeq \frac{(s_A^2 + s_B^2)^2}{\frac{s_A^2}{e-1} + \frac{s_B^2}{q(r-1)}} \qquad (15.47)$$

$$\text{where } s_A^2 = \sum_{i=1}^{e} \frac{(\bar{X}_{i.} - \bar{X}_{..})^2}{e-1} , \qquad (15.48)$$

$$s_B^2 = \sum_{i=1}^{q} \sum_{j=1}^{r} \frac{(X_{ij} - \bar{X}_{i.})^2}{qr} \qquad (15.49)$$

15.i. Stability of Variance Ratio

The ratio of variances σ_R^2/σ_E^2 must be known and stable to provide the key to the solutions suggested in this chapter. If the ANOVA table mean square which has expectation $\sigma_R^2 + r\,\sigma_E^2$ is designated s_A^2, and the mean square with expectation σ_R^2 is designated s_B^2, an exact confidence interval for the ratio σ_R^2/σ_E^2 can be determined from the F statistic. The derivation is shown in (Brownlee). If the data used to calculate the weighting factors are divided up into equal sized samples and intervals calculated, at confidence (α), approximately (α) fraction of the intervals should include the ratio σ_R^2/σ_E^2. If the fraction of intervals is much less than (α), the assumption that the ratio is constant is questionable.

It would also be advisable to plot the sample ratios in chronological order to detect time trends.

15.j. Approximate Degrees of Freedom

The Welch-Satterthwaite estimate employed in Section 15.g is derived in (Kendall and Stuart pp.146) and (Brownlee pp.235). It is also discussed in (Davies a. pp.54). The derivation is based on equating expressions for the variance of the sum of sample variances and the variance of the equivalent variable and substituting sample estimates.

$$\text{If } s^2 = a_1 s_1^2 + a_2 s_2^2 \tag{15.50}$$

$$f \cong \frac{(a_1 s_1^2 + a_2 s_2^2)^2}{\frac{a_1^2 (s_1^2)^2}{f_1} + \frac{a_2^2 (s_2^2)^2}{f_2}} \tag{15.51}$$

For the application herein, $a_1 = a_2 = 1.0$, but the usual application is to estimate the degrees of freedom of the variance of the difference between two sample means.

$$\text{In that case } a_1 = {}^1/N_1, \quad a_2 = {}^1/N_2 \tag{15.52}$$

$$\text{and} \quad f = \frac{\left(\frac{s_1^2}{N_1} + \frac{s_2^2}{N_2}\right)^2}{\frac{(s_1^2/N_1)^2}{N_1 - 1} + \frac{(s_2^2/N_2)}{N_2 - 1}} \tag{15.53}$$

A different approximation for this latter problem was originally published in (Dixon and Massey) and is also given in (Hoel) and (Resnikoff and Lieberman 1959).

$$f = \frac{\left(\frac{s_1^2}{N_1} + \frac{s_2^2}{N_2}\right)^2}{\frac{(s_1^2/N_1)^2}{N_1 + 1} + \frac{(s_2^2/N_2)}{N_2+1}} - 2 \tag{15.54}$$

(Anderson and Bancroft) give both approximations. Unfortunately the derivation of the (Dixon and Massey) relation is not available and therefore, a comparison of the two estimates was made (Program 15.1) as a basis for selecting one in preference to the other. Monte Carlo simulation was employed; for each trial samples of random normal numbers, size N_1 and N_2, were generated using variances σ_1^2 and σ_2^2. The following statistics were calculated from the simulated data:

1. Unbiased estimates of the variances s_1^2 and s_2^2.
2. An estimate of total variance $s^2 = s_1^2 + s_2^2$.
3. The ratio s^2/σ^2 where $\sigma^2 = \sigma_1^2 + \sigma_2^2$.
4. Both approximations, Satterthwaite-Welch and Dixon-Massey.

From the results of fifty trials, the average and variance of the statistics calculated in steps 2, 3 and 4 were calculated and printed. The entire program was repeated (Case 2) to obtain a measure of simulation precision. A summary of typical results is shown in Table 15.3. The value f' was obtained by substituting the true values (the input to the simulator) in the expression:

$$f' = \left[\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}\right] \Big/ \left[\frac{\sigma_1^2}{N_1^2(N_1-1)} + \frac{\sigma_2^2}{N_2^2(N_2-1)}\right] \qquad (15.55)$$

The Welch-Satterthwaite estimate is obtained from this expression by substituting sample values for the true values. If the value f' is taken as the true value (which is equivalent to assuming the sum of variances is distributed as the χ^2) the Welch-Satterthwaite approximation is less biased, at least for the cases investigated and the variance of the Welch-Satterthwaite estimate is less than the Dixon-Massey estimate. For these two reasons, less bias and smaller variance, the Welch-Satterthwaite estimate is recommended in preference to the Dixon-Massey estimate. There may be other inference criteria that would favour the Dixon-Massey estimate.

TABLE 15.3 DEGREES OF FREEDOM

Results from Program 15.1

N_1		5	10	10	5	5	10
N_2		10	10	10	15	15	10
σ_1/σ_2		2	2	1.0	1.0	3.0	3.0
f'		5.05	13.2	18.0	6.9	4.3	11.0
	case	Welch-Satterthwaite Estimate					
Mean	1	5.7	13.6	16.5	9.2	4.7	11.6
	2	5.6	13.3	16.9	8.4	4.5	11.5
Variance	1	3.7	5.1	1.9	13.	0.60	2.9
	2	2.9	3.6	2.2	11.2	0.24	2.0
		Dixon-Massey Estimate					
Mean	1	6.4	14.7	18.3	11.0	5.0	12.1
	2	6.2	14.2	18.6	10.0	4.8	12.0
Variance	1	6.6	7.6	2.9	19.0	1.3	4.4
	2	5.2	5.4	3.2	18.0	0.51	3.0

The simulation results are based on the formulae applicable to the difference between two means rather than the problem contained in this thesis because without the derivation of the Dixon-Massey estimate, it is not known how to modify the expression to solve the problem herein.

15.k. Current Method

The problem of multiple firings per engine introduces much complexity in the proposed method. In contrast given the same data the current method is attractively simple. The only recognition of this problem is the restriction on the maximum number of firings per engine assembly sometimes present in the 'ground rules'. It is appropriate to reflect on whether or not the complexity is required. There are two reasons to support it.

There is a loss of information from discarding untrimmed mixture ratio and thrust data in the current method. If the sample sizes are small, this loss could be serious.

More significant is the error introduced by assuming the firings are equivalent to single-firing-per-engine data. If the same assumption were made in the proposed method it would be equivalent to assuming

that the expectation of the mean square on the last line of the ANOVA table equals $\sigma^2 = \sigma_E^2 + \sigma_R^2$.

The correct expectation:

$$E(s^2) = E\left(\sum_{i=1}^{e} \sum_{j=1}^{r} \frac{(X_{ij} - \bar{X}_{..})^2}{er - 1} \right)$$

$$E\left(\sum_{i=1}^{e} \sum_{j=1}^{r} (X_{ij} - \bar{X}_{..})^2 \right) =$$

$$E\left(r \sum_{i=1}^{e} (\bar{X}_{i.} - \bar{X}_{..})^2 \right) + E\left(\sum_{i=1}^{e} \sum_{j=1}^{r} (X_{ij} - \bar{X}_{i.})^2 \right) =$$

$$(e-1)(\sigma_R^2 + r\,\sigma_E^2) + e(r-1)\,\sigma_R^2$$

$$E(s^2) = \frac{(e-1)(\sigma_R^2 + r\,\sigma_E^2) + e(r-1)\,\sigma_R^2}{e\,r - 1} \qquad (15.56)$$

$$= \frac{r(e-1)}{er - 1}\,\sigma_E^2 + \sigma_R^2$$

$$\text{Ratio correct } \left(E(s^2)/\text{incorrect } E(s^2) \right) =$$

$$\frac{\frac{r(e-1)}{er - 1} + \theta}{1 + \theta} \qquad (15.57)$$

The result is to underestimate the variance and therefore, overestimate the reliability. This last error applies to all three performance parameters and

therefore, it is compounded in the estimate of overall reliability. The error depends on r, e and θ; the largest error occurs as $\theta \rightarrow 0$ and r is large, when the ratio above tends to $1/r$.

CHAPTER 16

PROBLEMS FOR FUTURE RESEARCH

There are several rocket engine reliability problems which have not been solved with the current method but which may possibly be solved with the new proposed method treated in this thesis. It is the author's intention to proceed with his research on these problems as soon as possible. The purpose of this chapter is to present these problems and the work completed towards their solution.

16.a. Multifiring-Multiengine Vehicles

Perhaps the most important is the multifiring, multiengine mission space. From studies of optimizing cost, performance and reliability of advanced vehicles, it appears that the current trend toward clusters of small engines rather than a single large engine will increase during the next decade. Vehicles with thirty or forty engines per stage are predicted. (Mulready et al). Further, as the design chamber pressure increases there will be less reason to design vehicles with single-firing, multiple stages and more reason to

support multiple-firing, single stage vehicles. Several current liquid rocket engines already have multiple firing capability (the Pratt and Whitney Aircraft RL-10 and the Bell Aircraft Agena rocket engines are two examples). In addition, as space missions increase in complexity there is no alternative to multifiring missions; to inject a communication satellite in a 'stationary' orbit or to impact an instrumented vehicle within a small radius on the surface of Mars requires very accurate, reliable rocket engines with multiple firing capability.

There are several problems which must be treated to obtain a reliability estimate for the multifiring rocket cluster. The first is the interaction between firings on a single engine. Functional reliability will be degraded by failures which do not affect single firing reliability but would affect multifiring reliability. For example, if an igniter burns out after an engine has started, it will not affect that firing but would prevent any additional firings. Another example might be a stress failure in a turbopump casting during the shutdown transient. Neither of these events would be treated as failures when computing a single firing reliability estimate but they must be accounted for when estimating multifiring

reliability. In anticipation of this problem, the author redesigned the reliability data system at Pratt and Whitney Aircraft several years ago to gather data to make estimates of a new parameter, the multifiring factor. This parameter is defined as the conditional probability, given the engine has fired successfully during the mission, that no failure has occurred prior to the next firing which degrades the reliability of the engine on that firing. Rocket engines are rigorously inspected before and after every ground test firing and a considerable volume of data should be available to estimate this factor. The multifiring factor may also be a function of the duration and path of the coast periods; meteorite damage, vacuum welding, and vacuum aging are examples of failure modes which cannot be estimated from ground test data: Some coast periods may be measured in years. If the mission is such that these effects are significant, the multifiring factor will have to be modified in some way to account for them. The multifiring factor will be included in the mathematical model as a product term once for each period between firings: for example, it would be used three times on a four firing mission.

The interaction between firings for performance

reliabilities will depend on the definition of performance requirements for the mission: difficult missions will require the performance parameter to satisfy the limits on every firing while easier missions will only require the average performance to satisfy the limits. In either case it should be possible, although perhaps difficult, to build an appropriate model based on ANOVA methods. Particular attention must be paid to correlation between performance parameters on firings made on the same engine. The best ground test data (the simple model in this situation) will have the same number of multiple firings per engine as the mission: given these data and well defined performance-functional requirements, the problem should be no more difficult than the single firing mission treated in this thesis.

There will also be an interaction between engines; given an engine fails it may or may not affect the other engines. If there are no redundant engines and no separate abort systems, single engine failure implies vehicle failure and the interaction is irrelevant: otherwise, it will be significant. One approach to this problem that appears promising is to classify single engine failures into two groups: those that will affect the

other engines (catastrophic failures) and those that will not. From this classification a multiengine factor might be estimated as the conditional probability, given a failure has occurred, that the other engines will not be affected. The application of this factor will be largely determined by vehicle design and the reliability and response of the engine failure detection system. (Abernethy 1963 A)

With all of these estimates, careful study must be made of the applicability of the ground test data to the mission environment. There are failure modes in both environments that do not exist in the other; only the best combined engineering and statistical analysis can hope to produce valid estimates. As the model becomes more and more complex, the precision will be degraded and this should be reflected in the size of the confidence interval and the shape of the likelihood function. Every effort must be made to keep the system simple. All procedures should be well defined and understood between customer and producer before their application to data.

16.b. Variable Firings per Engine

If the number of firings per engine is a variable, several methods appear to have some merit and unfortunately some deficiencies. In this case the model is still more complex than that treated in Chapter 15.

(a) Combined Estimates. Perhaps the best solution is to group the data by the number of firings per engine and obtain separate, independent, unbiased point estimates from each group. These estimates may be weighted and summed to provide the required point estimate. The choice of weights is discussed in (Brownlee Section 2.4) as well as other texts. The drawback is the interval estimate. The confidence bound from the largest group should be a conservative estimate on the average; however, in a specific case, if the point estimate from the largest group is greater than the combined point estimate this procedure may be optimistic. Perhaps with more complex model building it will be possible to estimate a lower bound based on all the data.

(b) Equivalent Firing per Engine. Equivalent sample sizes for balanced data may be developed using ANOVA methods described by (Brownlee, Davies et al and other). The algebra here is tedious, complex and approximate; if the components of variance have to be estimated

separately the results will have doubtful value. (Brownlee) says of the analogous Model II ANOVA, 'This is clearly an awkward business of unknown accuracy. All this trouble can be avoided by the use of equal sample sizes.' The author is in accord with Brownlee; it is most desirable to remain on the firm ground of valid statistics.

(c) Trimming. If most of the data is of one constant (j) number of firings per engine, random trimming of firings from engines with more than (j) firings, and elimination of engines with less than (j), may be the best course of action. This is particularly true if there is a fair size sample of data left after the trimming.

16.c. True Versus Indicated Reliability

The sample estimates of the performance parameter variances include the effect of instrumentation precision errors. The mission variances do not include instrumentation error; during the mission performance parameters are usually not measured and even if they are measured, flight instrumentation is quite different from ground test instrumentation. Customers have sometimes requested reliability estimates based on true rather than indicated values. In the past the request could not be satisfied

as there was no known method for modifying the current method but the proposed method may be developed to have this capability. If all pertinent ground test instrumentation is duplicated using independently calibrated and installed measuring devices, a statistical estimate of instrumentation precision may be produced. If this instrumentation variance is stable, it may be accurately estimated from the same large batch of data used to estimate the weighting factors and treated as a known constant. The true variance is equal to the indicated variance minus the known instrumentation variance for the simple model (single firing per engine); the reduction is more complicated for the models in Part IV.

A word of caution about bias error is in order here. It is quite common to assume that instrumentation bias errors are zero because all known bias errors are calibrated out and because there is no method of estimating instrumentation bias. This assumption is not conservative, although it is widely employed. If indicated variances are reduced by precision variances, the author recommends adding some variance to the result in recognition of the unknown bias errors. It is very difficult, if not impossible, to even make an intelligent guess

at the size of bias errors; perhaps a collection of engineering guesses could be obtained by survey and the most pessimistic employed. Alternatively, if the bias errors are expected (guessed) to be of the same magnitude as precision errors, it may be wise to use the indicated variances (reduced by the precision and increased by the same amount for bias errors) as the true variances.

16.d. Likelihood-Bayesian Estimates

Likelihood and Bayesian estimates were developed for the simple model in this thesis. The results were informative and competitive with the frequentist estimates for the author's recommendation. Analogous methods should be developed for the complex model. This has not been attempted and it is possible that some simplification and other advantages would result. The customer, particularly, should be interested in these methods as they provide a means for including a statistical expression for the customer's evaluation of the producer's capability. This is extremely important: one of the primary considerations in predicting the reliability of a new engine is the quality and maturity of the design development engineering team. Only very large expenditures of

money and time in testing can overcome deficiencies in experience. In the early development period a Bayesian analysis is probably the only quantitative way that the customer can evaluate competitive engines incorporating his opinion of the quality of engineering.

APPENDIX I

REFERENCES

Abernethy, R. B., Effects of parallel versus standby redundancy of the reliability of rocket propulsion systems, Pratt and Whitney Aircraft Report, PWA FR-479A, June, 1963(a).

Abernethy, R. B., The precision of reliability estimates based on small sample sizes, Pratt and Whitney Aircraft Report PWA FR 799, October, 1963(b).

Anderson, R. L., and Bancroft, T. A., Statistical Theory in Research, McGraw-Hill Book Co., 1952.

Barnard, G. A., Jenkins, G. M. and Winsten, C. B., Likelihood inference and time series, J. Roy. Statist. Soc., Series A, 125, part 3, 1962.

Bowker, H., and Lieberman, J., Engineering Statistics, Prentice Hall Inc., 1959.

British Association for the Advancement of Science, Mathematical Tables, 1, Cambridge University Press, 1946.

Brownlee, K. A., Statistical Theory and Methodology in Science and Engineering, John Wiley and Sons Ltd., 1960.

Davies, O. L., Statistical Methods in Research and Production, Hafner Publishing Co., 3rd Ed., 1961.

Davies, O. L., Design and Analysis of Industrial Experiments, Hafner Publishing Co., 1960.

Dixon, W. J. and Massey, F.J., Jr., Introduction to Statistical Analysis, McGraw Hill Book Co., 1957.

Fisher, R. A., The fiducial argument in statistical inference, Annals of Eugenics, VI, part IV, pp.391-398, 1935.

Fraser and Guttman, Tolerance Regions, Annals of Math. Stat., 27, p.162., 1956.

Guttman, Optimum tolerance region, Annals of Math. Stat., 1957, 28, pp.773.

Hald, A., Statistical Theory with Engineering Applications, John Wiley and Sons, Inc., London.

Hoel, P. G., Introduction to Mathematical Statistics, 3rd Ed., John Wiley and Sons, Inc., 1963.

Hogben, D., Pinkham, R. S. and Wilk, M. B., The moments of the non-central t distribution, Biometrika 48, 1961, pp.465-468.

Hogben, D., Pinkham, R.S. and Wilk, M. B., The moments of a variate related to the non-central t, Annals of Math. Stat., 35, 1964, pp.298-315.

Jeffreys, H., Theory of Probability, 2nd Ed., 1948, Oxford University Press.

Johnson, N. L., and Welch, B. L., Applications of the non-central t-distribution, Biometrika 31, 1940.

Keeping, E. S., Introduction to Statistical Inference, Van Nostrand

Kendall, M. G. and Stuart, A., The Advanced Theory of Statistics, Vol. 2, Inference and Relationship, Charles Griffin and Co. Ltd., (Three Volume Edition), 1961.

Lehmann, E. L. and Scheffe, H., Completeness, similar regions and unbiased estimation, Part 1, Sankhya, 10, 1950, pp.305-340.

Lieberman, G. J. and Resnikoff, G. J., Sampling Plans for inspection by variables, J. Amer. Statist. Assoc., 1955, vol.50, pp.457-516.

Lloyd, D. K. and Lipow, M., Reliability: Management, Methods and Mathematics, Prentice-Hall Inc., 1962.

Mulready, R. C., Atherton, R. R., Bike, M., Chamay, A., Engine installations for advanced vehicles, Pratt and Whitney Aircraft Report PWA FR-643, March, 1963.

Owen, D. B., Factors for one-sided tolerance limits and for variables sampling plans, Sandia Corporation Monograph, SCR-607, 1963.

Proschan, F., Confidence and tolerance intervals for the normal distribution, J. Amer. Statist. Assoc., 48, 1953, pp.550-564.

Resnikoff, G. J., A new two-sided acceptance region for sampling by variables, Technical Report no.8, Applied Mathematics and Statistics Laboratory, Stanford University, 1952.

Resnikoff, G. J., Tables to facilitate the computation of percentage points of the non-central t-distribution, Annals of Math. Stat. 32, no.2, 1962, pp. 580-586.

Resnikoff, G. J. and Lieberman, G. J., Tables of the non-central t distribution, Stanford University Press, Stanford, California, 1957.

Thompson, W. R., On a criterion for the rejection of observations and the distribution of the ratio of deviation to sample standard deviation, Annals of Math. Stat., 1935, pp.214-219.

Slutski, E. E., (СЛУЦКИЙ, Е.Е.) ТАБЛИЦЫ ДЛЯ ВЫЧИСЛЕНИЯ НЕПОЛНОЙ Г-ФУНКЦИИ И ФУНКЦИИ ВЕРОЯТНОСТИ χ^2, АКАДЕМИЯ НАУК СССР, 1950

Wilks, S. S., Determination of sample sizes for setting tolerance limits, Annals of Math. Stat., 1941, pp.91-96.

APPENDIX II

NOTATION

—— GENERAL ——

η RELIABILITY, THE PROBABILITY OF A SUCCESSFUL EVENT

h AN UNBIASED ESTIMATE OF RELIABILITY η

$\sim$ DISTRIBUTED AS

$\cong$ APPROXIMATELY EQUALS

$\equiv$ DEFINED AS OR EQUIVALENT TO

Φ CUMULATIVE NORMAL PROBABILITY WITH STANDARDIZED ARGUMENT

$\mathcal{L}$ LIKELIHOOD FUNCTION

Pr or p PROBABILITY

N SAMPLE SIZE EXCEPT FOLLOWING $\sim$ WHEN N INDICATES NORMAL DISTRIBUTION

δ THE NONCENTRALITY PARAMETER OF THE NONCENTRAL t

f DEGREES OF FREEDOM

Γ THE GAMMA FUNCTION

L, U LOWER, UPPER LIMITS, FIXED KNOWN POINTS ON A CONTINOUS SCALE

fn FUNCTION

E EXPECTATION OF A RANDOM VARIABLE

μ MEAN

σ^2 VARIANCE

z_η STANDARD NORMAL DEVIATE EXCEEDED WITH PROBABILITY $(1-\eta)$

k A BIASED ESTIMATE OF z_η, $k_L \equiv \frac{\bar{X}-L}{s}$

s SQUARE ROOT OF AN UNBIASED ESTIMATE OF σ^2

$\bar{X}$ or $\bar{X}_.$ THE AVERAGE VALUE OF X_i WITHIN A GIVEN SAMPLE

—— SUPERSCRIPTS ——

$\tilde{h}$ THE BIASED ESTIMATE OF η, $\tilde{h} \equiv \Phi(k)$

$\bar{h}$ THE UPPER BOUND OF AN INTERVAL IN THE SCALE OF h; IN CHAPTER 7 AN INTERIM VALUE OF h IN THE ITERATIVE METHOD

$'$ PRIME INDICATES KNOWN VARIANCE

$\hat{}$ A MAXIMUM LIKELIHOOD VALUE

$^{-1}$ INVERSE FUNCTION

$\overset{\times}{h}$ AN ESTIMATE OF η TAKEN AS THE RATIO OF OBSERVED SUCCESSES TO ATTEMPTS

—— SUBSCRIPTS ——

$\underline{h}$ THE LOWER BOUND OF AN INTERVAL

fu FUNCTIONAL

ISP SPECIFIC IMPULSE

FN THRUST

rm PROPELLANT MIXTURE RATIO

o OVERALL

c THE LOWER BOUND OF A ONE SIDED UPPER CONFIDENCE INTERVAL ESTIMATE