

INTERACTIONS, CORRELATIONS AND COLLECTIVE BEHAVIOUR IN NON-EQUILIBRIUM SYSTEMS

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Interactions, correlations and collective behaviour in non-equilibrium systems

Rosalba García Millán

ABSTRACT

Non-equilibrium systems, which usually involve a large number of interacting particles, are ubiquitous in nature and society. Interactions, such as volume exclusion or branching, induce correlations in the system and often translate into emergent collective behaviour at macroscopic scales. This is the case, for instance, in *active matter*, where particles are subject to local non-thermal forces that are transformed into mechanical work. Establishing the relationship between macroscopic patterns and microscopic dynamics analytically is a challenge that has motivated different approaches in the community.

In this thesis I have focused on some statistical properties with an emphasis on correlations, both spatial and temporal, of six different non-equilibrium particle systems: the Oslo rice pile model, branching processes and applications, the voter model, and run-and-tumble motion. The first three systems display *critical phenomena* and the last one is an instance of *active motility*.

The approaches that I have followed are, on the one hand, one-to-one mappings between different, known, stochastic processes and, on the other hand, the Doi-Peliti field theory formalism. The point in common between both approaches is the ability to retain the particle entity, which proves essential when tackling observables that strongly depend on the microscopic dynamics of the system, such as correlation functions. However, the field-theoretic approach is much more systematic, as it becomes apparent in the applicability of this route to different kinds of reaction-diffusion particle systems.

To Raquel

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Preface

The content of this thesis is the collection of papers that I have published in peer-reviewed journals during my doctoral studies. This thesis also contains a paper currently under review and other work that is in preparation for publication, see Table 1. Each chapter corresponds to a paper and is therefore self-contained.

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Each chapter starts with its title and list of authors. Where applicable, the first lines of each chapter contain the full reference citation of the publication or preprint, the DOI with a hyperlink to the webpage where the work appears, and the copyright credit line with a reference to the reuse and permissions agreement in App. A. I also list the author contributions to the paper. Finally, at the beginning of each chapter, before the abstract of the paper, I have included an overview about how the chapter fits in the theme of this thesis.

Table 1: Summary of chapters in this thesis, specifying its publication status (published, submitted, accepted or in preparation), reference, whether I wrote the main body of the paper (first author), and the type of reproduction (verbatim, minor modifications, and new sections).

Chap.	Ref.	Status	I am first author	Reproduction
1	[78]	P: EPL 122 , 50003 (2018)	Yes	Verb.
2	[77]	P: PRE 98 , 062107 (2018)	Yes	Verb., minor mod.
3	[148]	S: arXiv:2003.03302 (2020)	No	Verb.
4	[21]	P: Sci. Rep. 9 , 15590 (2019)	No	Verb.
5	[40]	P: PRE 97 , 062156 (2018)	No	Verb., minor mod.
6	[75]	P: JSTAT 2020 , 053201 (2020)	Yes	Verb., new App. 6.F
7		In preparation	Yes	New

*The world, in case you haven't
noticed, is currently on fire.*

G. Thunberg [191]

Introduction

Equilibrium thermodynamics was largely developed during the 17th to the 19th century [1]. However, systems at equilibrium are rather exceptional in our quotidian experience. The generic term for systems that are not at equilibrium is, short of being redundant, non-equilibrium systems and are characterised for being permeable to their environment, often through the exchange of energy (or, equivalently, matter).

Consider, for instance, the Earth. Planet Earth is constantly receiving solar energy, most of which is reflected into space and a small part of which is absorbed by the atmosphere eventually reaching other, internal, layers of the Earth. Simultaneously, since the Earth has a positive temperature, it emits energy into space in the form of radiation, which, as a first approximation, is given by the blackbody radiation. This flow of energy between the Earth and its surroundings makes the Earth a non-equilibrium system. Equilibrium is not to be confused with energy balance: even if the influx equals the outflow, the flow of energy implies the permeability of the system with its environment [34, 114].

Non-equilibrium systems fall in two main categories: out of equilibrium, where a system initially at equilibrium is pushed away from it by a small perturbation; and far from equilibrium, where there is a perpetual flux of energy or matter between system and environment. The notion of detail balance in equilibrium systems does not apply to non-equilibrium systems, which implies the breaking time-reversal symmetry. In other words, while future and past are indistinguishable in systems at equilibrium, non-equilibrium systems do manifest an arrow of time. This is captured, for example, by a non-zero entropy production rate [67].

In this thesis, I focus on non-equilibrium particle systems far from equilibrium where particles may decay, diffuse or move ballistically and interact with each other via branching, volume-excluded interactions, transmutation or deposition. The reaction-diffusion particle systems in this thesis are: the Oslo rice pile model (Chapter 1) [33], branching processes with variations (Chapters 2 to 5) [99], the voter model (Chapter 6) [124] and

run-and-tumble motion (Chapter 7) [187].

For instance, ballistic motion and the creation of new particles through branching or other interactions are indicative of non-equilibrium processes because the particle density at stationarity does not obey a Boltzmann distribution [1, 67]. Pairwise interactions between particles play an important role in the microscopic dynamics of the system. An immediate consequence of interactions is the emergence of correlations between particles, which are visible in their state (particle numbers, position, velocity, etc.). For example, suppose that a mobile particle deposits immobile tracer particles at a certain rate. It is clear that finding one such tracer particle in a particular position is indicative of the presence of the mobile particle in that position at an earlier time, as discussed in Chapters 4 and 6. For this reason, interactions and correlations are closely related, where the former refers to the microscopic dynamics and the latter refers to an observable in the form of an expectation.

Another important consequence of interactions in reaction-diffusion particle systems is the emergence of collective behaviour. For instance, it is the case of an avalanche model in granular media, such as the Oslo model, where a local instability makes a particle topple to its neighbouring site, which, as a result, may become unstable as well, triggering an avalanche of a large number of particles, see Chapter 1. This is an example of a macroscopic phenomenon that has its origin in the microscopic dynamics.

Then, accounting for interactions in the microscopic dynamics of particle systems becomes crucial to discover and analyse the origin of fascinating emergent patterns and phenomena at large scales. This is often an arduous task and may even not be feasible. In some cases, the system can be mapped to another stochastic process which is already known and this allows to extend the properties of one system to the other. It is the case of the map between the Oslo model and the quenched Edwards-Wilkinson equation in Chapter 1 [140, 161] and the map between branching processes and Harris walks in Chapter 5 [98]. Given a system, however, there is no guarantee that a one-to-one map can be found with an already studied process. Instead, one can follow a path-integral approach, which is much more systematic. In this thesis I use the Doi-Peliti field theory [189], which is able to capture the particle entity and therefore it is adequate to account for interactions, see Chapters 2, 3, 4, 6 and 7.

OUTLINE OF THE THESIS

The underlying theme of most chapters in this thesis is avalanches and spreading. Chapter 1 is devoted to studying the temporal correlations in a model of granular media, the Oslo model. Next comes a module of four chapters that study different aspects of branching processes: a field theory of branching processes and universality class in Chapter 2, an extension of branching process with time-dependent extinction rate to model neuronal avalanches in Chapter 3, the spatial embedding of diffusive branching particles to model epidemic spreading in Chapter 4, and phase transition and finite-size scaling of the branching process avalanche size in Chapter 5. This is followed by Chapter 6, where I show that the Voter Model and the Concealed Voter Model belong to the same universality class and, finally, Chapter 7 contains the characterisation of run-and-tumble motion confined in a harmonic potential. In this thesis, the use of Doi-Peliti field theory as a tool is very prominent, since it is used in five out of seven chapters. In the other two chapters, 1 and 5, I use other methods based on stochastic theory and scaling theory. Most chapters present Monte Carlo simulations that support the analytical predictions. In Chapter 7, numerical results are based on the implementation of Hermite polynomials.

Each chapter starts with its title and list of authors. If the chapter corresponds to a paper that has been published or is publicly available, it follows its full reference citation and credit line. At the beginning of the chapter I have included a brief overview to contextualise it in the whole of this thesis. Throughout, I use the word *I* whenever I refer to work I have done. I use the word *we* almost always to refer to "the reader and I", but I also use *we* in collaborations to refer to "my collaborators and I", the distinction should be clear by the context.

1

Correlations and hyperuniformity in the avalanche size of the Oslo model

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OVERVIEW The Oslo rice pile model is a stochastic model of avalanches in granular media, where a new grain is added at each macroscopic time step triggering an avalanche that unfolds in the microscopic time scale [33]. The Oslo model is a non-equilibrium system in the class of Self-Organised Criticality. In this work we characterise the temporal correlations in the sequence of avalanche sizes and prove analytically that, although temporal correlations are short-ranged, the existence of a particular conservation law in the microscopic dynamics of this particle system gives rise to temporal long-range order. Surprisingly, this long-range order manifests in a change of exponent in the variance of the avalanche size, which corresponds to a phenomenon observed in spatial point patterns, known as *hyperuniformity* [193].

Author contributions: RGM conceived the idea to study hyperuniformity in avalanching; performed all numerical and analytical work presented in the paper, in particular Eqs. (4) and (5); did the literature review; and wrote the manuscript. GP explained the background and supervised the work. LP and KC explored the Oslo model along similar lines initially but made no discernible contribution to the published research.

ABSTRACT

Certain random processes display anticorrelations resulting in local Poisson-like disorder and global order, where correlations suppress fluctuations. Such processes are called hyperuniform. Using a map to an interface picture we show via analytic calculations that a sequence of avalanche sizes of the Oslo model is hyperuniform in the temporal domain with the minimal exponent $\lambda = 0$. We identify the conserved quantity in the interface picture that gives rise to the hyperuniformity in the avalanche size. We further discuss the fluctuations of the avalanche size in two variants of the Oslo model. We support our findings with numerical results.

1.1 INTRODUCTION

We study the fluctuations of the avalanche size s of the Oslo rice pile model [33], a paradigmatic example of a non-equilibrium system that evolves spontaneously into a scale-invariant state thus considered a representative case of Self-Organised Criticality [33, 140, 161, 92, 162, 36, 52]. Because of the complexity of the model and the number of random variables involved, little is understood about the correlations between avalanches. For example, when supposed asymptotics of moments are studied, often probability density functions are sampled by deliberately generating effectively independent samples [137]. As we show in the present work, correlations in the avalanche size of the Oslo model *suppress* fluctuations giving rise to a phenomenon that has been dubbed hyperuniformity, briefly outlined below.

The fluctuations in a time series of random variables s_t , $t \in \mathbb{N}$, may be characterised by the variance

$$\sigma^2(S(M)) \equiv \langle S^2(M) \rangle - \langle S(M) \rangle^2 \propto M^\lambda, \quad (1.1)$$

where $\langle \bullet \rangle$ denotes an expectation, S is the sum $S(M) = s_1 + \dots + s_M$ and λ is an

exponent to be determined. The variance of the unbiased [23] estimator $\bar{s} = S(M)/M$ of the mean $\langle s \rangle$ is $\sigma^2(\bar{s}(M)) = \sigma^2(S(M))/M^2 \propto M^{\lambda-2}$.

If s_t are independent and identically distributed, then $\lambda = 1$ [196], but in general, in the presence of correlations, λ is not known [144, 79, 84]. Nevertheless, in some particular stochastic processes [207] and one-dimensional point patterns [193, 101] it has been proved that anticorrelations suppress fluctuations in such a way that $\lambda \in [0, 1)$. Such processes are called hyperuniform [193, 101] or superhomogeneous [72], and their fingerprint is the suppression of fluctuations on large scales, manifesting a regularity that is not apparent on short scales [207, 219, 193, 101, 100, 72, 17, 71].

In this Letter, we demonstrate that the variance $\sigma^2(\bar{s}(M))$ of the estimator of the mean avalanche size $\bar{s}$ in the one-dimensional boundary driven Oslo model decays quadratically in M , hence showing that the avalanche size is hyperuniform with exponent $\lambda = 0$. *To our knowledge, this is the first time that hyperuniformity in the temporal domain has been identified and proven in a well-known interacting particle system.* Because of the intermittent nature of the dynamics of the Oslo model, the temporal anti-correlations are communicated from event to event via its configuration in space, rather than by direct interaction as is expected for purely spatial patterns [108]. We further show that the temporal correlations reduce the fluctuations in the estimates $\bar{s}^n$ of the moments $\langle s^n \rangle$. Consequently, our results imply that we obtain more precise estimates from a set of correlated avalanches than from a set of independent avalanches. We support our analytical findings with numerical results and we further explore two variants of the model, namely external drive uniformly distributed on the lattice and the two-dimensional pile.

In our numerical simulations, we compute the variance $\sigma^2(\bar{s}^n(M))$ from a sample of Q independent estimates $\{\bar{s}_1^n(M), \dots, \bar{s}_Q^n(M)\}$ of $\langle s^n \rangle$, where each $\bar{s}_q^n(M)$ is computed from M consecutive avalanches, $\bar{s}_q^n(M) = (s_{q,1}^n + \dots + s_{q,M}^n) / M$. All our measurements have been taken once the pile is in a recurrent configuration, that is when its statistical properties are stationary [52, 141, 36, 20, 140, 162].

Our proof of $\sigma^2(\bar{s}(M)) \propto M^{-2}$ has the following structure: starting from the description of the Oslo model [33] in an interface picture [140, 161] we are able to write the sum $S(M)$ of the sizes $s_1, \dots, s_M$ of M consecutive avalanches as the sum (1.4) of a deterministic term and a bounded noise asymptotically independent of M (1.5), so that the variance of the sum converges in large M . It follows that the avalanche size s is

hyperuniform for large M with $\lambda = 0$. Although our proof of hyperuniformity does not draw on scaling arguments, for the sake of completeness, we study the scaling form of $\sigma^2(\bar{s}(M))$ in the entire domain of M and show that it displays a crossover from Poisson-like behaviour (*i.e.* $\lambda = 1$) to hyperuniformity (with $\lambda = 0$). For simplicity, our argument is applied to $d = 1$ although it can be generalised to higher dimensions.

1.2 THE OSLO MODEL

In the Oslo model [33] n_x particles reside on site $x \in \{1, \dots, L\}$ of a one-dimensional lattice of size L and its configuration is described by the set of local slopes $z_x = n_x - n_{x+1}$ with $n_{L+1} \equiv 0$. Each site has associated with it a randomly chosen critical slope $z_x^c \in \{1, 2\}$. Given $z_x \leq z_x^c$ for all x , which is called a stable or quiescent configuration, the evolution of the pile follows the steps (i) *drive*, a grain is dropped at site $x = 1$ so that $z_1 \rightarrow z_1 + 1$; (ii) *relaxation*, every unstable site x , *i.e.* with $z_x > z_x^c$, is relaxed in parallel with all other unstable sites by toppling one grain to the neighbouring site on its right, resulting in the update rules $z_x \rightarrow z_x - 2$ and $z_{x+1} \rightarrow z_{x+1} + 1$ in the bulk. The update rules that apply to the boundaries are $z_1 \rightarrow z_1 - 2$, $z_2 \rightarrow z_2 + 1$ (open) and $z_L \rightarrow z_L - 1$, $z_{L-1} \rightarrow z_{L-1} + 1$ (closed). After every toppling, z_x^c is redrawn to be either 1 or 2 with equal probability. The relaxation process is repeated until the entire pile is in a stable configuration. The totality of the relaxations constitute an avalanche. We distinguish between the macroscopic timescale denoted by integer t , where the pile evolves from one stable configuration to another as t increases by 1, and a much smaller microscopic timescale in which the avalanches unfold.

The avalanche size s is defined as the total number of topplings that occur between two consecutive quiescent states. We find numerically that the temporal correlations between avalanches $\text{Cov}(s_i, s_j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$ are negative for $i \neq j$ and decay approximately exponentially (see App. 1.A), with a correlation time that is a power law in L . In that sense, s_i has short-ranged correlations. As we will show in the following, the anticorrelations exactly cancel the first term of order $1/M$ in the variance $\sigma^2(\bar{s}(M)) = \sigma^2(s)/M + \sum_{i,j=t+1; i \neq j}^{t+M} \text{Cov}(s_i, s_j)/M^2$ so that $\sigma^2(\bar{s}(M)) \sim M^{-2}$ in large M .

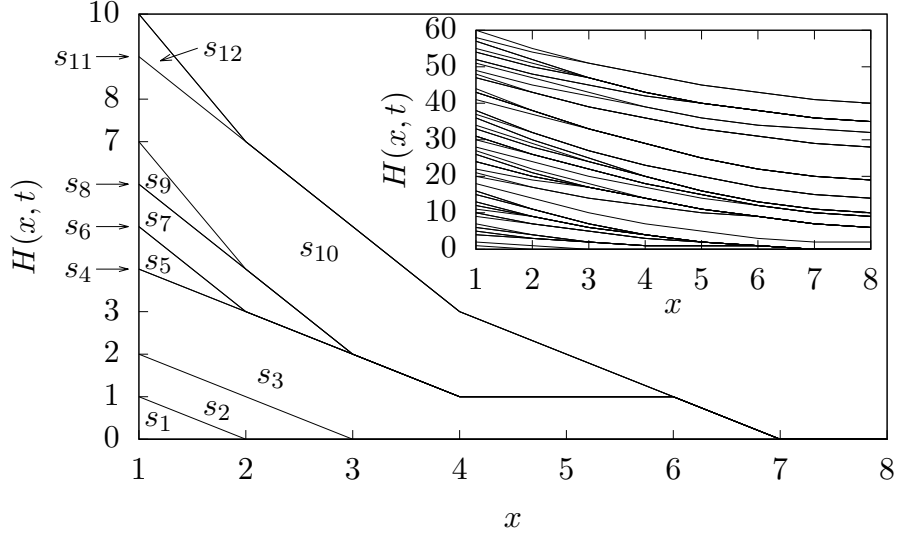


Figure 1.1: Oslo model interface picture. Example of the interface $H(x, t)$ of a one-dimensional boundary driven pile of size $L = 8$ with initial state $z_x = 1$ and z_x^c initially chosen at random for all x . In this realisation of the process, the series of avalanche sizes is $\{1, 2, 9, 0, 1, 0, 2, 0, 1, 11, 0, 1, \dots\}$, consistent with Eq. (1.2). The avalanche sizes are exactly the area between consecutive interface configurations if the interfaces are drawn using bars. Inset: $t \in [0, 65]$.

1.3 THE INTERFACE PICTURE

The dynamics of the pile can equivalently be described by the function $H(x, t)$, which gives the total number of topplings at site x up to time t [140, 161]. In this picture, the avalanche size s is equal to the area enclosed between two consecutive stable interface configurations,

$$s_t = \sum_{x=1}^L (H(x, t+1) - H(x, t)), \quad (1.2)$$

as illustrated in Fig. 1.1.

The temporal evolution of $H(x, t)$ was set out in [161] for one particular choice of the initial condition of the pile and two open boundaries. Here we use a generalised version for an arbitrary stable initial condition and one open and one closed boundary, see App. 1.B. In this setup, the spatial fluctuations of the interface $H(x, t)$ are confined because its curvature is bounded, Eq. (1.14).

We define the multiple drive avalanche size S as the sum $S(t, M) = \sum_{i=t+1}^{t+M} s_i$, based

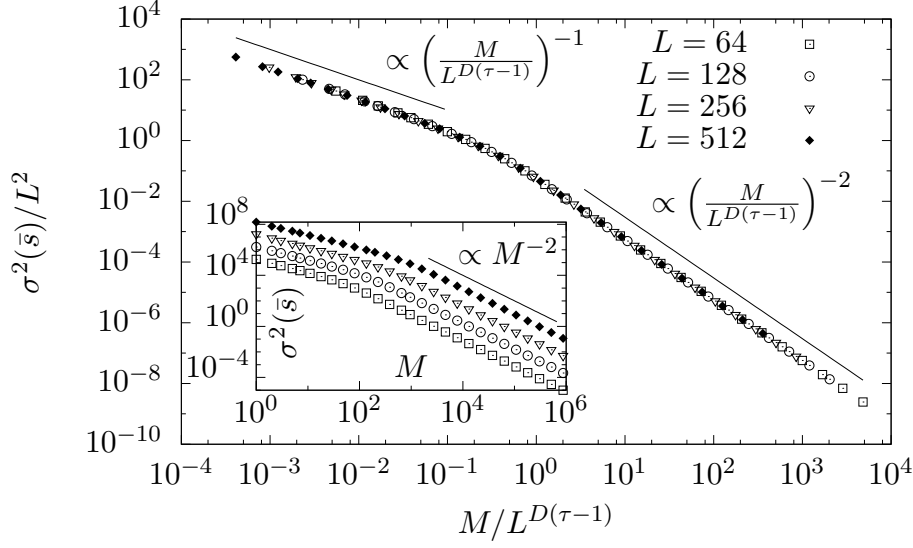


Figure 1.2: Variance of the estimated mean avalanche size. Data collapse of $\sigma^2(\bar{s})$ across many orders of magnitude as a function of $M/L^{D(\tau-1)}$ according to Eq. (1.7). The variance is compared to plain power laws, shown as straight lines, in the two regimes $M \ll L^{D(\tau-1)}$ and $M \gg L^{D(\tau-1)}$ with $M \in [1, 10^6]$. Inset: unscaled $\sigma^2(\bar{s})$ as a function of M .

on M consecutive avalanche sizes s_i , and the estimator $\bar{s}$ of $\langle s \rangle$ as $\bar{s} = S(t, M)/M$. By the Abelianess of the Oslo model [52], the total avalanche size triggered by M simultaneous charges is identical to the sum of M consecutive avalanche sizes. Since the macroscopic timescale is measured in units of the number of grains added to the pile, M is simultaneously a measure of time, drive size and sample size of $\bar{s}$.

Using Eq. (1.2), the multiple drive avalanche size S is equal to the area enclosed by the interfaces $H(x, t + M)$ and $H(x, t)$,

$$S(t, M) = \sum_{x=1}^L (H(x, t + M) - H(x, t)). \quad (1.3)$$

The function $H(x, t)$ can be determined in closed form in terms of the random variable $\eta_{x,t} \in \{-1, 0, 1\}$ [161], which encodes the configuration of the pile via the relation $\eta_{x,t} = 1 - z_x$ at each time t , see App. 1.B. We find that $S(t, M)$ can be expressed in closed form as the sum of a deterministic term and a noise

$$S(t, M) = ML + \xi(t, M), \quad (1.4)$$

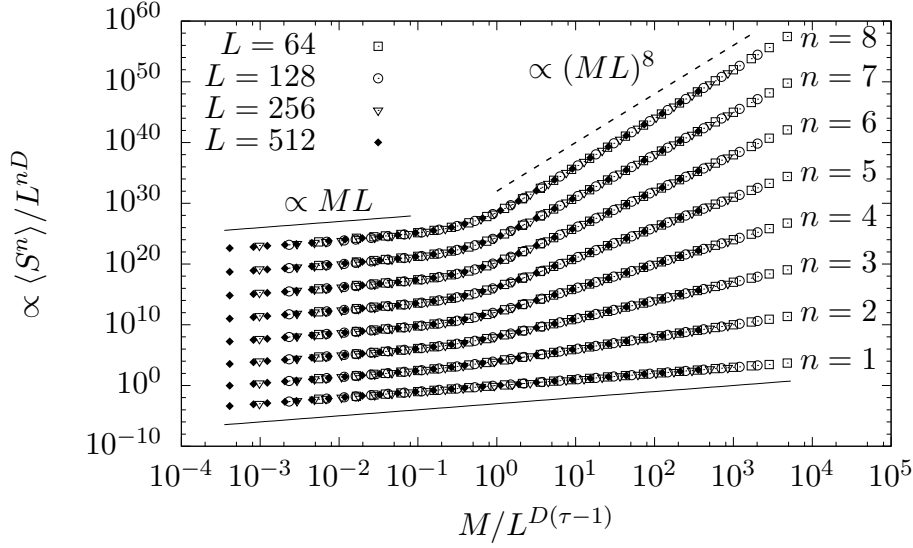


Figure 1.3: Moments of the multiple drive avalanche size. Data collapse of the moments $\langle S^n(M; L) \rangle$ across many orders of magnitude according to Eq. (1.6) with $n \in \{1, \dots, 8\}$ as a function of $M/L^{D(\tau-1)}$ for the system sizes $L \in \{64, 128, 256, 512\}$ and $M \in [1, 10^6]$. The collapsed curves for $n \geq 2$ have been shifted vertically for better visibility. The full line, with slope 1, is shown to emphasise the linearity of $\langle S^n \rangle$ in small M ; the dashed line, with slope 8 shows the proportionality $\langle S^8 \rangle \propto (ML)^8$ in large M .

where $\xi(t, M)$ can be written in the form Eq. (1.5),

$$\xi(t, M) = \frac{1}{2} \sum_{x=1}^L \left[\left(x(L+1-x) + \frac{1}{2}x(x-1) \right) (\eta_{x-1,t+M} + \eta_{x+1,t+M} - \eta_{x-1,t} - \eta_{x+1,t}) + \eta_{x,t+M} - \eta_{x,t} \right] \quad (1.5)$$

which is, in fact, a finite sum of bounded random variables. The strict bounds of ξ are $\pm \left[\frac{2}{3}L(L+1)(L+2) - L^2 \right] \in \mathbb{Z}$ and, asymptotically, it is easy to see that the distribution of ξ neither depends on M nor on t . In other words, the values that ξ can take do not scale with M for large M . From Eq. (1.5) we deduce that $\langle \xi \rangle = 0$, as it is the difference of two stationary expectations, and therefore $\langle S(t, M) \rangle = ML$. Moreover, Eq. (1.4) further implies the estimate of the mean is $\bar{s}(M) = L + \xi/M$. Crucially, $\langle \xi^2 \rangle$ does not scale in M , so that the variance of $\bar{s}$, which is $\sigma^2(\bar{s}(M)) = \langle \xi^2 \rangle / M^2$, decays quadratically in the limit of large M , proving that s is *hyperuniform in this limit* (see inset of Fig. 1.2) with exponent $\lambda = 0$, the fingerprint of enhanced hyperuniformity

[100].

1.3.1 SCALING OF $\sigma^2(\bar{s})$

In the following, we derive the scaling of the moments $\langle S^n \rangle$ of the total avalanche size S resulting from M drives and show that the hyperuniformity of s is consistent with its established scaling.

The single drive avalanche size s , which is S with $M = 1$, is power-law distributed [33, 92, 141, 162] and its moments are $\langle s^n \rangle \sim L^{D(1+n-\tau)}$, where $D = 2.250 \pm 0.002$ is the avalanche dimension and $\tau = 1.5556 \pm 0.0005$ [33, 92, 141] is the avalanche size exponent. The characteristic avalanche size is L^D . Since $\langle s \rangle = L$, the critical exponents satisfy $D(2 - \tau) = 1$.

Another characteristic scale entering the scaling of $\langle S^n \rangle$, and the n -point correlation function of H below, is the drive size M^* after which two interface configurations are effectively independent. The interfaces $H(x, t_1)$ and $H(x, t_2)$ can be considered independent if they are completely detached, *i.e.* $H(x, t_2) > H(x, t_1)$ for all x . This is typically the case when their separation exceeds the characteristic vertical fluctuations of the interface, which scale with L^χ [140, 162], so that $M^* \propto L^\chi$, where χ is the roughness exponent. Using $D = \chi + d$ [141, 140] and $D(2 - \tau) = d$ in the case of boundary driving, we have $\chi = D(\tau - 1)$.

In App. 1.C we discuss why the scale M^* is a necessary argument in the n -point correlation function of H . Assuming that the n -point correlation of H displays standard gap-scaling, *e.g.* $\langle H(x_1, t_1) \dots H(x_n, t_n) \rangle \propto L^{n\chi} \mathcal{C}_n(\dots, |t_2 - t_1|/aL^{D(\tau-1)}, \dots)$, Eq. (1.21), where the scale $L^{D(\tau-1)}$ enters explicitly and a is a metric factor, it follows that

$$\begin{aligned} \langle S^n \rangle &= \left\langle \int d^d x_1 \dots d^d x_n (H(x_1, t + M) - H(x_1, t)) \dots (H(x_n, t + M) - H(x_n, t)) \right\rangle \\ &= L^{nd+n\chi} \mathcal{G}_n \left(\frac{M}{L^{D(\tau-1)}} \right) = L^{nD} \mathcal{G}_n \left(\frac{M}{L^{D(\tau-1)}} \right), \end{aligned} \quad (1.6)$$

with $\mathcal{G}_n$ a scaling function. The scaling of $\langle S^n \rangle$ is compatible with the case $M = 1$, $\langle s^n \rangle \sim L^{D(1+n-\tau)}$, only if the scaling functions $\mathcal{G}_n$ are linear for small arguments (see Fig. 1.3), so that $\langle S^n \rangle \propto ML^{D(1+n-\tau)}$ in small M for all $n > \tau - 1$. For large arguments,

$\langle S^n \rangle \propto (ML)^n$ from Eq. (1.4), as ξ does not scale in M .

Writing $\mathcal{D}(x) = x^{-2} (\mathcal{G}_2(x) - \mathcal{G}_1^2(x))$, the scaling form of $\sigma^2(\bar{s})$ is thus

$$\sigma^2(\bar{s}) = \frac{\sigma^2(S(t, M))}{M^2} = L^2 \mathcal{D}\left(\frac{M}{L^{D(\tau-1)}}\right), \quad (1.7)$$

where $\mathcal{D}$ is a scaling function whose asymptotes we characterise in the following (see Fig. 1.2). Firstly, $\mathcal{D}(x) \sim x^{-1}$ for $x \ll 1$, as $\mathcal{G}_n(x) \propto x$ for small x . For large $M \gg M^*$, on the other hand, the boundedness of $\xi = S - ML$, Eq. (1.4), implies finite $\sigma^2(S)$ and thus hyperuniformity of s with exponent $\lambda = 0$, so $\mathcal{D}(x) \sim x^{-2}$ for $x \gg 1$. The crossover of the scaling takes place at $M^* \propto L^{D(\tau-1)}$. This concludes our derivation of the scaling of the variance $\sigma^2(\bar{s})$.

1.4 HIGHER MOMENTS

In the following, we investigate the fluctuations of higher moments. The estimator of $\langle s^n \rangle$ is $\bar{s}^n(t, M) = \left(\sum_{i=t+1}^{t+M} s_i^n\right) / M$. Fig. 1.4 shows the quantity

$$1 - M \left(\sigma^2(\bar{s}^n(t, M)) \right) / \left(\sigma^2(\bar{s}^n(t, 1)) \right), \quad (1.8)$$

which tends to 1 if s^n is hyperuniform, lies in $(0, 1)$ if there is suppression of fluctuations (in amplitude but not in rate of convergence) and tends to 0 if it shows Poisson-like behaviour. Although the random variables s^n with $n > 1$ are not hyperuniform, our results show that the correlations among consecutive avalanches still suppress fluctuations even in higher moments.

1.5 DISCUSSION

Our results show that the avalanche size of the Oslo model has the typical behaviour of hyperuniform systems [193, 101, 219, 17]: apparent Poisson-like disorder on short scales and suppression of fluctuations on large scales. The suppression of fluctuations is due to the negative correlations between consecutive avalanches that die off on the scale of $M^* \propto L^{D(\tau-1)}$, which is precisely proportional to the correlation time. We have identified the key ingredients of hyperuniformity in the Oslo model on the basis of the

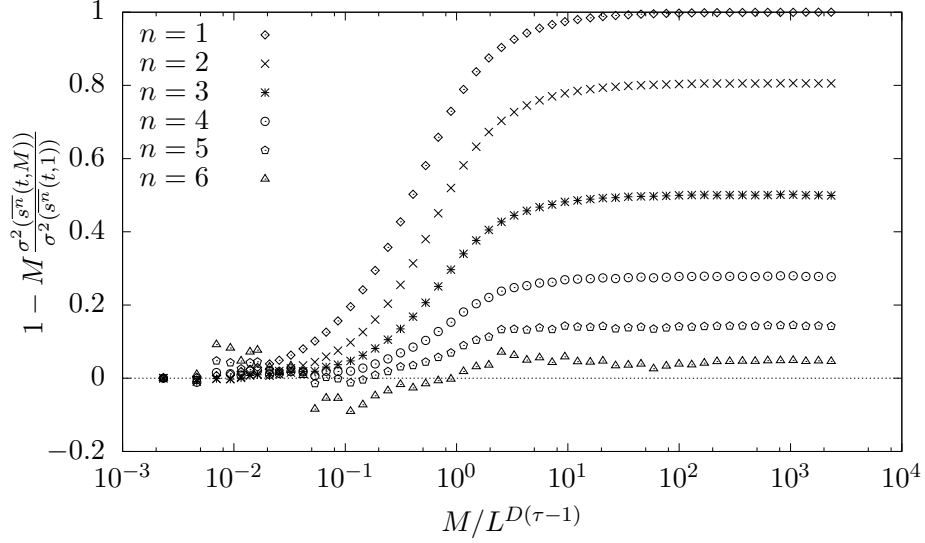


Figure 1.4: Comparison of the fluctuation scaling between independent and correlated samples. Quantity $1 - M (\sigma^2(\bar{s}^n(t, M))) / (\sigma^2(\bar{s}^n(t, 1)))$ that compares $\sigma^2(\bar{s}^n(t, M))$ with $\sigma^2(\bar{s}^n(t, 1)) / M$, which corresponds to the variance of the estimate $\bar{s}^n$ calculated from M independent samples, for the system size $L = 128$ and $M \in [1, 10^6]$.

interface picture (Fig. 1.1), namely the *conservation* of the area ML between interfaces, which is due to (a) the boundedness of the curvature of the interfaces and (b) the drive fixed on one site.

The boundedness of the curvature of the interfaces is explained by the stochastic dynamics that governs $H(x, t)$ (App. 1.B, in particular Eq. (1.14)) which encodes the distribution rule of slope units during the relaxation ($z_x \rightarrow z_x - 2$ and $z_{x\pm 1} \rightarrow z_{x\pm 1} + 1$). Further analysis shows that hyperuniformity is found provided only that the driving site is fixed and that at least one boundary is open, which is a necessary condition for stationarity. Regarding the numerical aspects of the Oslo model, the rapid convergence of $\bar{s}$ means that for a fixed sample size M , more precise statistics are obtained when the samples are taken sequentially rather than independently.

One may ask to what extent hyperuniformity is a particular feature of the Oslo model or a generic property of Self-Organised Criticality. To this end, we have studied the Oslo model in two other settings, namely the bulk-driven Oslo model and the two-dimensional Oslo model.

Firstly, we have derived the multiple drive avalanche size $S(M)$ for the bulk-driven

Oslo model [20, 162] in closed form, see App. 1.D. We find that the fluctuations due to the random drive (uniformly distributed in the bulk), whose variance can be calculated exactly, prevent hyperuniformity in the avalanche size. Because of this driving, the curvature of the interfaces is not bounded, so the area between them is no longer conserved. Asymptotically in large M , the variance of $\bar{s}$ is $\sigma(\bar{s}) \propto L^4 M^{-1}$; numerically we find its scaling form is $\sigma^2(\bar{s}) = L^{3.5} \tilde{\mathcal{D}}(M/L^{0.5})$, with $\tilde{\mathcal{D}}(x) \propto x^{-1}$ as $x \rightarrow \infty$ (see Tab. 1.1 and Fig. 1.7).

Secondly, we have studied numerically the two-dimensional Oslo model with fixed driving position (corner) and with uniformly distributed driving (bulk drive), see Tab. 1.1 and App. 1.E. In the former case, we recover hyperuniformity for the same reasons as in 1D and in the latter we find Poisson-like behaviour for the same reasons as in 1D.

Table 1.1: Summary of our results for each of the settings studied: the exponent σ_1 of $\langle s \rangle \sim L^{\sigma_1}$, the crossover point M^* of $\sigma^2(\bar{s})$ and its scaling behaviour for large M .

d	Drive	σ_1	M^*	Large M behaviour
1	corner	1	$L^{D(\tau-1)} = L^\chi$	$\sim M^{-2}$, hyperuniform
	uniform bulk	2	$L^{0.5}$	$\sim M^{-1}$, Poisson-like
2	corner	0	$L^{D(\tau-1)} = L^{\chi+2}$	$\sim M^{-2}$, hyperuniform
	uniform bulk	2	$L^{1.5}$	$\sim M^{-1}$, Poisson-like

These results, summarised in Tab. 1.1, answer the question of whether hyperuniformity is a particular or a generic feature. We have found two instances of the Oslo model, the 1D and the 2D Oslo model with bulk drive, that do not display hyperuniformity. In these cases, the Poisson-like fluctuations introduced by the randomness of the drive dominate over the suppression of fluctuations in the avalanches. The mechanism that causes the anomalous suppression of fluctuations is thus not present in all instances of Self-Organised Criticality.

In conclusion, we have proven hyperuniformity in the time series of avalanche sizes of the boundary-driven Oslo model. Its presence is a specific feature of the dynamics of this model. Identifying hyperuniformity in other models, such as the Manna model, will elucidate its role in interacting particle systems and advance our understanding of Self-Organised Criticality.

ACKNOWLEDGMENTS

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1.A TEMPORAL CORRELATIONS BETWEEN AVALANCHES

Our numerical results, Fig. 1.5, show that the temporal correlations between consecutive avalanches scale in

$$\text{Cov}(M) = \langle s_t s_{t+M} \rangle - \langle s_t \rangle \langle s_{t+M} \rangle = -L^2 \mathcal{F} \left(\frac{M}{L^{D(\tau-1)}} \right), \quad (1.9)$$

which we have fitted against $\mathcal{F}(x) = Ax^{-B}e^{-x/C}$. We find that the exponent B is very small. It decreases with L , taking values $B = 0.083$ at $L = 64$ and $B = 0.059$ at $L = 512$, which suggests that it may vanish in large L and the L -dependent deviation of $\mathcal{F}$ from a pure exponential may thus be regarded as a logarithmic correction. We therefore approximate that $\mathcal{F}(x) = Ae^{-x/C}$ with $A \simeq 0.36$ and $C \simeq 0.30$. Therefore, the correlation time is $CL^{D(\tau-1)}$, which is further discussed in this Letter and in App. 1.C.

To derive Eq. (1.9) we consider the correlation function

$$c(x, x', t, t') = \langle H(x, t) H(x', t') \rangle - \langle H(x, t) \rangle \langle H(x', t') \rangle, \quad (1.10)$$

whose scaling form follows Eq. (1.21) discussed further below. From Eq. (1.3) the correlation function between avalanches is calculated as in Eqs. (1.11a) to (1.11b) using that $c(x, x', t, t')$ is uniform in t and defining $L^{2D} \mathcal{D}(|t - t'|/L^{D(\tau-1)}) = \iint c(x, x', t, t') dx dx'$

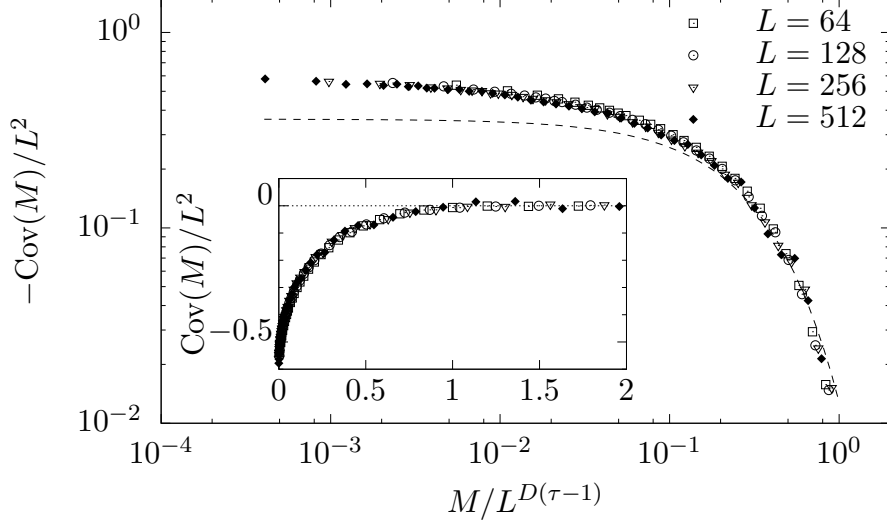


Figure 1.5: Rescaled correlation function $\text{Cov}(M) = \langle s_t s_{t+M} \rangle - \langle s_t \rangle \langle s_{t+M} \rangle$ Eq. (1.9) obtained numerically for $M > 0$, where s_t is taken when the pile is in the stationary regime. The correlations decay exponentially, the dashed line is the function $0.36e^{-x/0.3}$ with $x = M/L^{D(\tau-1)}$. Inset: same data but shown with linear axes, showing that the correlations are negative.

from Eq. (1.21).

$$\text{Cov}(M) = \langle s_t s_{t+M} \rangle - \langle s_t \rangle \langle s_{t+M} \rangle \quad (1.11a)$$

$$\begin{aligned} &= \left\langle \iint (H(x, t+1) - H(x, t)) (H(x', t+M+1) - H(x', t+M)) dx dx' \right\rangle \\ &\quad - \left\langle \int (H(x, t+1) - H(x, t)) dx \right\rangle \left\langle \int (H(x', t+M+1) - H(x', t+M)) dx' \right\rangle \\ &= \iint [2c(x, x', t, t+M) - c(x, x', t+1, t+M) - c(x, x', t, t+M+1)] dx dx' \\ &= aL^{2D} \left[2\mathcal{D}\left(\frac{M}{L^{D(\tau-1)}}\right) - \mathcal{D}\left(\frac{M-1}{L^{D(\tau-1)}}\right) - \mathcal{D}\left(\frac{M+1}{L^{D(\tau-1)}}\right) \right]. \end{aligned} \quad (1.11b)$$

Now, using $\varepsilon = 1/L^{D(\tau-1)}$ and expanding about $x = M/L^{D(\tau-1)}$ we have

$$2\mathcal{D}(x) - \mathcal{D}(x - \varepsilon) - \mathcal{D}(x + \varepsilon) = -\varepsilon^2 \mathcal{D}''(x) + \mathcal{O}(\varepsilon^4). \quad (1.12)$$

Using the relation $D(2 - \tau) = 1$, we obtain Eq. (1.9) with $a\mathcal{D}(x) = \mathcal{F}(x)$.

1.B INTERFACE PICTURE

In this section we demonstrate how the Oslo model is mapped to an interface depinning model in a generalisation of [161, 140], and solve the stochastic equation of motion to obtain the interfaces $H(x, t)$ (Fig. 1.1 and Fig. 1.6), which is needed to derive the exact form of the multiple drive avalanche size as given in Eqs. (1.4) and (1.5).

Let $h(x, t)$ count the number of slope units that fall on site x up to time t . The function $h(x, t)$ thus counts the number of times the slope z_x is incremented by 1 as the model evolves, as opposed to $H(x, t)$, which counts the number of times site x has toppled. Let $\eta(x, h(x, t))$ be a quenched noise that takes values 0 if $h(x, t)$ is even, and 1 or -1 with equal probability if $h(x, t)$ is odd [161]. Although η is therefore not explicitly a function of t , to ease notation we write $\eta_{x,t} \equiv \eta(x, h(x, t))$, as for each x and t there is a unique value of $h(x, t)$. The configuration of the pile in terms of local slopes z_x (we drop its time dependence solely to ease notation) can be read from η through the relation $\eta_{x,t} = 1 - z_x$. *Given that $\{z_x\}$ is Markovian by definition of the model, it is clear that $\{\eta_{x,t}\}$ is Markovian as well.*

For a given arbitrary initial configuration of the pile $\{z_x\}$, equivalently $\{\eta_{x,0}\}$, the microscopic temporal evolution of $h(x, t)$ in a microscopic time step of length $\delta t \ll 1$ is [161]

$$h(x, t + \delta t) - h(x, t) = \frac{1}{2} \left[h(x-1, t) - 2h(x, t) + h(x+1, t) + \eta_{x-1,t} + \eta_{x+1,t} - \eta_{x-1,0} - \eta_{x+1,0} \right]. \quad (1.13)$$

The boundary conditions that reproduce the original Oslo model [33] are: open on the left with an external source $h(0, t) = 2 \lfloor t \rfloor \in \mathbb{N}$ that accounts for the drive; and closed on the right $h(L, t) = h(L+1, t)$, giving the boundary conditions of $\eta_{x,t}$, namely $\eta_{0,t} = 0$ and $\eta_{L,t} = \eta_{L+1,t}$ for all t .

When the pile is in a quiescent state, the left-hand side of Eq. (1.13) vanishes for all x , which gives the expression that governs the function $h(x, t)|_{t \in \mathbb{N}}$ we are interested in,

$$\Delta h(x, t)|_{t \in \mathbb{N}} = a_{x,0} - a_{x,t}, \quad (1.14)$$

where $\Delta h(x, t)$ denotes the discrete Laplacian and $a_{x,t} = \eta_{x-1,t} + \eta_{x+1,t}$. As $a_{x,t}$ is bounded, Eq. (1.14) implies that $h(x, t)$ has finite curvature, which is, of course, con-

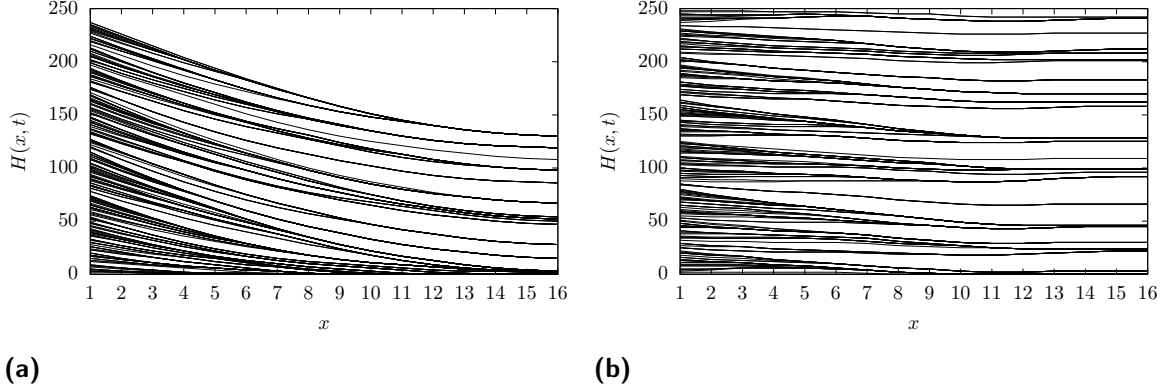


Figure 1.6: Interface picture with open and closed boundaries. Examples of the interfaces $H(x, t)$ on a one-dimensional boundary-driven pile of size $L = 16$ with $t \in [0, 250]$. In **(a)** the initial configuration is $z_x = 1$ and random z_x^c for all x and in **(b)** the initial state is a recurrent configuration taken at random (more specifically, the configuration of the pile after driving 1000 times the pile with $z_x = 1$ and random initial z_x^c).

sistent with our numerical simulations, where the spatial average of η in the recurrent state at quiescence is finite and negative. The solution of Eq. (1.14) with the boundary conditions as stated above is

$$h(x, t)|_{t \in \mathbb{N}} = 2t - \tilde{h}(x, 0) + \sum_{y=1}^x y a_{y,t} + x \sum_{y=x+1}^L a_{y,t}, \quad (1.15)$$

where $\tilde{h}(x, 0) = \sum_{y=1}^x y a_{y,0} + x \sum_{y=x+1}^L a_{y,0}$ is the shift in $h(x, t)$ that accounts for the initial state of the pile. If the initial condition is $\{z_x = 1\}$ for all x , then $\{\eta_{x,0} = 0\}$ and hence $\tilde{h}(x, 0) = 0$, which is the case studied in [161] (see Fig. 1.6a).

In [161] it is shown that $h(x, t)$ and $H(x, t)$ are related by

$$H(x, t) = \frac{1}{2} (h(x, t) + \eta_{x,t}). \quad (1.16)$$

The boundary conditions of h and η imply correspondingly $H(0, t) = \lfloor t \rfloor$ and $H(L, t) = H(L+1, t)$ for all t [140] (see Fig. 1.6).

Using Eqs. (1.15) and (1.16) we have

$$H(x, t)|_{t \in \mathbb{N}} = t + \frac{1}{2} \left[\eta_{x,t} - \tilde{h}(x, 0) + \sum_{y=1}^x y a_{y,t} + x \sum_{y=x+1}^L a_{y,t} \right], \quad (1.17)$$

and with Eq. (1.3) we obtain the expression of $S(t, M)$ in Eqs. (1.4) and (1.5).

1.C HIGHER CORRELATION FUNCTIONS OF THE INTERFACE

As discussed in [162], the scaling in system size L of the moments $\langle s^n \rangle$ of the avalanche size of the one-dimensional boundary-driven Oslo model is $\langle s^n \rangle \propto L^{\sigma_n}$ with $\sigma_n = D(1 + n - \tau)$ and L the linear extent of the system. This scaling is a straightforward consequence of the power-law distribution of the avalanche size,

$$P(s) = as^{-\tau} \mathcal{G}\left(\frac{s}{bL^D}\right), \quad (1.18)$$

which applies to all s beyond a lower cutoff s_0 . Here, a and b are metric factors and $\mathcal{G}$ is a scaling function [33, 92, 162]. However, as we show here, the scaling form stated above is not enough to characterise the moments of the multiple drive avalanche size $S(t, M)$. The scaling argument $M/L^{D(\tau-1)}$ is essential for the scaling functions considered in our Letter. The scale $L^{D(\tau-1)}$ is a correlation time as introduced in Eq. (1.9). We will demonstrate its necessity by firstly attempting to derive the moments of the multiple drive avalanches from the correlation function of the interface without this additional scale, then introduce it and show how it reconciles the interfacial correlation functions, the moments of S and the established scaling of s .

Firstly, we assume that the two-point, two-time connected correlation function of H has the usual (stationary) scaling form [115, 8]

$$\langle H(x, t) H(x', t') \rangle_c = a' L^{2\chi} \mathcal{C}_2\left(\frac{|t - t'|}{b' L^z}, \frac{x}{L}, \frac{x'}{L}\right), \quad (1.19)$$

where $\mathcal{C}_2$ is a scaling function, a' and b' are metric factors, χ is the roughness exponent and z is the dynamical exponent which gives rise to the characteristic avalanche duration $\sim L^z$ in units of microscopic time steps.

Using Eq. (1.3), we derive that the scaling of the second moment $\langle S^2(t, M) \rangle$ can be

calculated as

$$\langle S^2(t, M) \rangle = \iint \langle (H(x, t+M) - H(x, t))(H(x', t+M) - H(x', t)) \rangle dx dx' \quad (1.20a)$$

$$= 2 \iint (\langle H(x, t)H(x', t) \rangle - \langle H(x, t)H(x', t+M) \rangle) dx dx' \quad (1.20b)$$

using uniformity at stationarity in the form $\langle H(x, t) \rangle = \langle H(x, t+M) \rangle$ and $\langle H(x, t)H(x', t) \rangle = \langle H(x, t+M)H(x', t+M) \rangle$. Further using Eq. (1.19) and the relation $D = \chi + d$ we obtain the scaling $\langle S^2(t, M) \rangle = 2a'L^{2D} (\bar{\mathcal{C}}_2(0) - \bar{\mathcal{C}}_2(M/(b'L^z)))$ with $\bar{\mathcal{C}}_2(\tau) = \int dy dy' \mathcal{C}_2(\tau, y, y')$. The difference between the two scaling functions, $\bar{\mathcal{C}}_2(0) - \bar{\mathcal{C}}_2(M/(b'L^z))$, is, in fact, independent of M . This is, firstly, because $b'L^z$ is the characteristic duration of the avalanches. This is a *microscopic* time and therefore a multiple of $\delta t \ll 1$ and thus well separated from M which is a *macroscopic*, integer time. On that timescale, the pile is no longer evolving and the interface is no longer relaxing, so that $\bar{\mathcal{C}}_2$ has converged in that argument. In other words, $\bar{\mathcal{C}}_2(M/(b'L^z))$ should be considered as the limit $\tau \rightarrow \infty$ of $\bar{\mathcal{C}}_2(\tau)$ irrespective of M , provided M is a positive integer. Secondly, but closely related, L^z governs the relaxation process, *i.e.* the avalanching, not the time M that passes between drives. We conclude that $\langle S^2(t, M) \rangle = 2a'L^{2D} (\bar{\mathcal{C}}_2(0) - \lim_{\tau \rightarrow \infty} \bar{\mathcal{C}}_2(\tau)) \propto L^{2D}$.

This result is in contradiction with $\sigma_2 = D(3 - \tau)$ showing that the assumed scaling function in Eq. (1.19) is, at least, incomplete. To cure this contradiction, we need to consider the scaling of correlations on the macroscopic timescale. Its value $M^* \sim L^\chi$ is given by the characteristic number of avalanches needed to make interface configurations uncorrelated. In other words, M^* is proportional to the characteristic scale of the vertical fluctuations of the interfaces $H(x, t)$ [140]. Using $D = \chi + d$ and $\sigma_1 = D(2 - \tau) = d$ for $d = 1$, we have $\chi = D(\tau - 1)$. Hence, we amend the scaling function in (1.19) to obtain

$$\langle H(x, t)H(x', t') \rangle_c = a'L^{2\chi} \mathcal{C}_2 \left(\frac{|t - t'|}{b'L^z}, \frac{|t - t'|}{c'L^{D(\tau-1)}}, \frac{x}{L}, \frac{x'}{L} \right), \quad (1.21)$$

with c' a metric factor. The two timescales L^z and $L^{D(\tau-1)}$ are proportional to the microscopic and the macroscopic timescale respectively. As we consider $|t - t'| = M$ in the following, which is large compared to the microscopic timescale $b'L^z$ as discussed above, we will drop this argument in what follows.

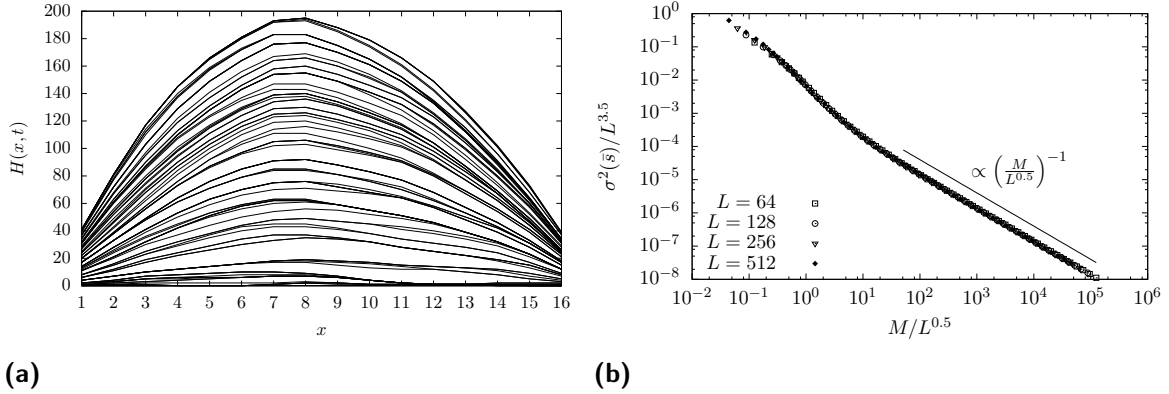


Figure 1.7: One-dimensional bulk driven Oslo model with open boundary conditions. (a) Typical interface configurations of topplings $H(x, t)$ in a system of size $L = 16$ with $t \in [0, 90]$ and initial condition $z_x = 1$ for all x ; **(b)** The scaling of the variance $\sigma^2(\bar{s})$ is asymptotically proportional to $M^{-1}L^4$ (measurements have been taken starting from recurrent configurations).

After performing the spatial integral (1.20b), the scaling of the second moment is

$$\langle S^2(t, M) \rangle = a' L^{2D} \left(\tilde{\mathcal{C}}_2(0) - \tilde{\mathcal{C}}_2 \left(M / (c' L^{D(\tau-1)}) \right) \right), \quad (1.22)$$

where $\tilde{\mathcal{C}}_2(\tau') = \int dy dy' \mathcal{C}_2(\tau, \tau', y, y')$. Eq. (1.22) can be reconciled with the scaling of $\langle S^2(t, 1) \rangle = \langle s^2 \rangle \propto L^{D(3-\tau)}$ only if $\tilde{\mathcal{C}}_2(0) - \tilde{\mathcal{C}}_2(\tau')$ is linear for small arguments $\tau' = 1/(c' L^{D(\tau-1)})$. Therefore, the scaling of the second moment is $\langle S^2(t, M) \rangle \propto M L^{D(3-\tau)} \hat{\mathcal{C}}_2(M/c' L^{D(\tau-1)})$ with $\hat{\mathcal{C}}_2(\tau') = (\tilde{\mathcal{C}}_2(0) - \tilde{\mathcal{C}}_2(\tau'))/\tau'$ convergent in small arguments τ' .

Indeed, applying this argument to the n th moment, we obtain the scaling $\langle S^n(t, M) \rangle \propto M L^{D(1+n-\tau)} \hat{\mathcal{C}}_n(M/c' L^{D(\tau-1)})$, which is consistent with the established exponents σ_n , see . Moreover, the numerics are consistent with that $\langle S^n(t, M) \rangle \propto M$ for small M and all n , as shown in .

1.D UNIFORM DRIVE IN THE BULK

In this section we derive the equations of motion of $h(x, t)$ and $H(x, t)$ and the multiple drive avalanche size $S(t, M)$ for the one-dimensional bulk driven Oslo model on a lattice with open boundaries [20, 162], along the lines of App. 1.B. The boundaries are now both open, *i.e.* $h(0, t) = h(L+1, t) = 0$ for all t and the equation of motion has a source

term given by $E(x, t) = \sum_{j=1}^t \theta(t-j) \delta_{x, x_j}$, where $\theta(t)$ is the Heaviside step function, $\delta_{x,y}$ is the Kronecker delta and $x_j \in \{1, \dots, L\}$ accounts for the randomly chosen site for driving the pile at each time step $j = 1, \dots, t$. Hence, the source function $E(x, t)$ counts the number of external grains that have been dropped by the external drive at x up to time t . The equation of motion now reads

$$h(x, t + \delta t) - h(x, t) = E(x, t) + \frac{1}{2} [h(x-1, t) - 2h(x, t) + h(x+1, t) + \eta_{x-1, t} + \eta_{x+1, t} - \eta_{x-1, 0} - \eta_{x+1, 0}]. \quad (1.23)$$

At quiescence, Eq. (1.23) with boundary conditions as stated above has the solution

$$h(x, t) = \left(\frac{L+1-x}{L+1} \right) \sum_{y=1}^{x-1} y b_{y, t} + \frac{x}{L+1} \sum_{y=x}^L (L+1-y) b_{y, t} - \hat{h}(x, 0), \quad (1.24)$$

where

$$b_{x, t} = -\Delta h(x, t)|_{t \in \mathbb{Z}} = 2E(x, t) + \eta_{x-1, t} + \eta_{x+1, t}, \quad (1.25)$$

and

$$\hat{h}(x, 0) = \left(\frac{L+1-x}{L+1} \right) \sum_{y=1}^{x-1} y b_{y, 0} + \frac{x}{L+1} \sum_{y=x}^L (L+1-y) b_{y, 0}, \quad (1.26)$$

accounting for the initialisation. From Eqs. (1.16) and (1.24) we can write $H(x, t)$ in closed form, see Fig. 1.7a, and further using Eq. (1.3) we derive the multiple drive avalanche size (*cf.* Eq. (1.4))

$$S(t, M) = \Lambda(t, M) + \xi'(t, M), \quad (1.27)$$

where the random variable

$$\Lambda(t, M) = \frac{1}{2} \sum_{x=1}^L x(L+1-x) (E(x, t+M) - E(x, t)) \quad (1.28)$$

is due to the drive and

$$\xi'(t, M) = \frac{1}{2} \sum_{x=1}^L \left[\eta_{x, t+M} - \eta_{x, t} + \frac{x}{2} (L+1-x) (\eta_{x-1, t+M} + \eta_{x+1, t+M} - \eta_{x-1, t} - \eta_{x+1, t}) \right] \quad (1.29)$$

is the new noise term. The difference between ξ' in (1.29) and the noise ξ in Eq. (1.5) is exclusively due to the different boundary conditions applied to the interfaces. It turns out that ξ' is also a finite sum of bounded random variables and thus ξ' is bounded itself. Therefore, ξ' is asymptotically independent of t and M implying that $\sigma^2(\xi')$ tends to a constant as $M \rightarrow \infty$.

From Eq. (1.27) we have that $\sigma^2(S(t, M)) = \sigma^2(\Lambda) + \sigma^2(\xi') + 2\text{Cov}(\Lambda, \xi')$. The variance of $\Lambda(t, M)$ can be calculated using the fact that the vector of random variables $\mathbf{E}(t + M) - \mathbf{E}(t)$ has a multinomial distribution independent of t with probabilities $p_x = 1/L$ for all x and M trials, which gives

$$\sigma^2(\Lambda(t, M)) = \frac{1}{720}M(L^4 - 5L^2 + 4). \quad (1.30)$$

Owing to the form of Λ and ξ' , it follows that $\text{Cov}(\Lambda, \xi')$ is subleading in M compared to $\sigma^2(\Lambda)$ as $M \rightarrow \infty$. As $\sigma^2(\xi')$ converges in M , the variance $\sigma^2(S(t, M)) \propto ML^4$ so that $\sigma^2(\bar{s}) \propto M^{-1}L^4$, implying that the avalanche size of the uniformly driven Oslo model does not display hyperuniformity, but rather Poisson-like behaviour.

Our numerical results, Fig. 1.7b, show that the variance follows the asymptotic scaling form $\sigma^2(\bar{s}) = L^{3.5}\tilde{\mathcal{D}}(M/L^{0.5})$, with $\tilde{\mathcal{D}}(x) \propto x^{-1}$ as $x \rightarrow \infty$. That is, asymptotically, $\sigma^2(\bar{s}) \propto M^{-1}L^4$. In fact, at large M the scaling is solely due to Eq. (1.30) as $\sigma^2(\bar{s})$ is dominated entirely by $\sigma^2(\Lambda)/M^2$. The interfaces detach dominantly due to the noise in the driving rather than the relaxation dynamics of the interface in response to the boundary driving.

In general, if the driving position x is fixed, then $\Lambda(t, M) = \frac{1}{2}x(L + 1 - x)M$ becomes deterministic so that $\sigma^2(\Lambda) = 0$ and the results for the boundary-driven Oslo model generalise to the Oslo model driven at a fixed (bulk) site. In this case, the avalanche size is hyperuniform on long timescales regardless of the driving position x and the boundary conditions. In particular, if the drive is located at the boundary $x = 1$ then $\Lambda(t, M) = \frac{1}{2}LM$, where the difference in the factor 1/2 in comparison to Eq. (1.4) is due to the different boundary conditions.

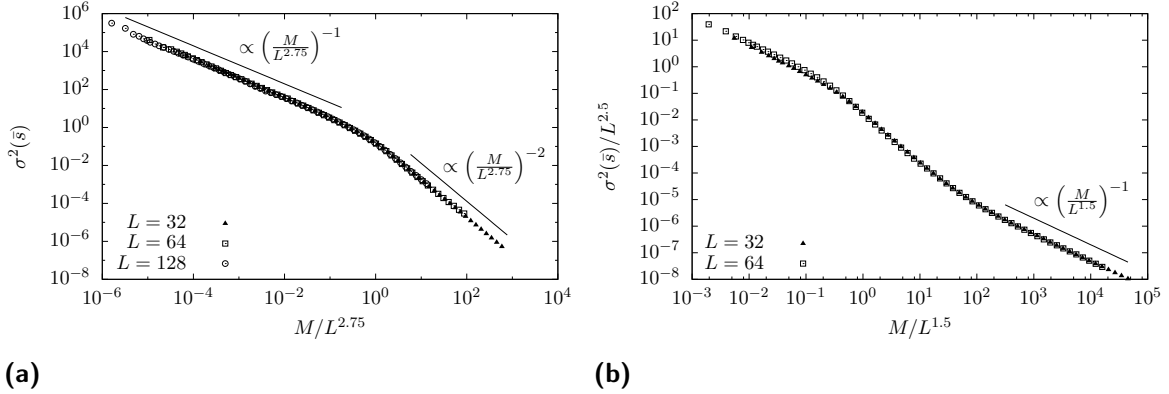


Figure 1.8: Scaling of $\sigma^2(\bar{s})$ in the two-dimensional Oslo model with (a) two open and two closed boundaries and the drive located at the intersection of the two open boundaries (corner drive, cf. Fig. 1.2) and (b) four open boundaries with drive uniformly distributed in the bulk (see Fig. 1.7b). In all cases, the pile is initialised in a recurrent configuration. Similarly to the one-dimensional cases, in (a) the fluctuations $\sigma^2(\bar{s})$ have two scaling regimes, displaying hyperuniformity for $M \gg L^{2.75}$, and in (b) $\sigma^2(\bar{s})$ is asymptotically proportional to $M^{-1}L^4$ indicating the absence of hyperuniformity.

1.E TWO-DIMENSIONAL OSLO MODEL

We have also studied the fluctuations in the avalanche size in the two following settings on a square lattice: (a) two open and two closed boundaries with the drive located at the intersection of two open boundaries (corner drive) and (b) four open boundaries and uniformly distributed drive on the lattice (bulk drive). These two settings resemble the one-dimensional cases studied in this Letter.

Our numerical results for setting (a) are shown in Fig. 1.8a. The variance has the scaling form

$$\sigma^2(\bar{s}) = \mathcal{D}\left(\frac{M}{L^{2.75}}\right), \quad (1.31)$$

with $\mathcal{D}(x) \sim x^{-1}$ for $x \ll 1$ and $\mathcal{D}(x) \sim x^{-2}$ for $x \gg 1$. The exponent 2.75 is in fact the avalanche dimension $D = 2.75 \pm 0.01$ [162, 20, 125, 141], which enters into the characteristic number of driving steps, $M^* \sim L^{D(\tau-1)}$. Since the average avalanche size diverges only logarithmically [162], $\sigma_1 = D(2 - \tau) = 0$, in the case of corner drive with boundary conditions as described above, it follows that $\tau = 2$, hence $D(\tau - 1) = D$ rather than $D(\tau - 1) = \chi$ as in the one-dimensional case. Further, the second moment

of S scales like $ML^{D(3-\tau)} = ML^D = M^2 (M/L^D)^{-1}$, so that

$$\sigma^2(\bar{s}) \propto \frac{1}{M^2} \langle S^2(t, M) \rangle = \mathcal{D} \left(\frac{M}{L^D} \right). \quad (1.32)$$

In setting (b), Fig. 1.8b, we find numerically that the variance has the scaling form $\sigma^2(\bar{s}) = L^{2.5} \tilde{\mathcal{D}}(M/L^{1.5})$, with $\tilde{\mathcal{D}}(x) \sim x^{-1}$ as $x \rightarrow \infty$, following the same scaling as the one-dimensional bulk driven Oslo model. We obtain $\sigma^2(\bar{s}) \propto M^{-1}L^4$ in large M , as, again, fluctuations of $\bar{s}$ are dominated by the driving rather than the interface roughness.

2

Field-theoretic approach to the universality of branching processes

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OVERVIEW A branching process is a simple model of population dynamics with plenty of applications in biological physics, social sciences, statistical mechanics, and other fields. Starting from one single individual, the population of a branching process grows stochastically, where each individual produces a random number of offspring. The interaction between particles is minimal, namely only that of creation (or lineage), but it is enough to induce correlations. This basic type of interaction appears in other, more complex, systems as in Chapters 3, 4 and 6, which are related to spreading and avalanching. In this work we study the universality class of branching processes and we characterise its statistical properties by calculating an extended list of observables, which illustrates the versatility and scope of Doi-Peliti field theory. In particular, the average temporal profile and the two-point temporal correlation function are observables

whose derivation becomes cumbersome through other methods.

Author contributions: All authors: derived the master equation, action and field theory of the continuous-time branching process. RGM & BW: estimated the two-point connected correlation function numerically. RGM & GP: calculated the expected avalanche shape. RGM: solved the recurrence relation that leads to the asymptotes of the moments; estimated the expected avalanche duration, and the distribution of the avalanche size numerically; did the literature review; made the figures; and wrote the body of the manuscript. JP: calculated the probability distribution of the particle number as a function of time, the expected avalanche duration, the two-point connected correlation function, and the distribution of the total avalanche size. GP: discussed the universality of branching processes based on a dimensional analysis argument; and supervised the work.

ABSTRACT

Branching processes are widely used to model phenomena from networks to neuronal avalanching. In a large class of continuous-time branching processes, we study the temporal scaling of the moments of the instant population size, the survival probability, expected avalanche duration, the so-called avalanche shape, the n -point correlation function and the probability density function of the total avalanche size. Previous studies have shown universality in certain observables of branching processes using probabilistic arguments, however, a comprehensive description is lacking. We derive the field theory that describes the process and demonstrate how to use it to calculate the relevant observables and their scaling to leading order in time, revealing the universality of the moments of the population size. Our results explain why the first and second moment of the offspring distribution are sufficient to fully characterise the process in the vicinity of criticality, regardless of the underlying offspring distribution. This finding implies that branching processes are universal. We illustrate our analytical results with computer simulations.

2.1 INTRODUCTION

Branching processes [99] are widely used for modelling phenomena in many different subject areas, such as avalanches [220, 123, 87], networks [85, 87, 123, 62], earthquakes

[132, 38], family names [165], populations of bacteria and cells [110, 63], nuclear reactions [209, 149], cultural evolution [167] and neuronal avalanches [174, 11]. Because of their mathematical simplicity they play an important role in statistical mechanics [189] and the theory of complex systems [38].

Branching processes are a paradigmatic example of a system displaying a second-order phase transition between extinction (absorbing state) with probability one and non-zero probability of survival (non-absorbing state) in the infinite time limit. The critical point in the parameter region at which this transition occurs is where branching and extinction rates exactly balance, namely when the expected number of offspring per particle is exactly unity [99, 38].

In the present work we study the continuous-time version of the Galton-Watson branching process [99], which is a generalisation of the birth-death process [80, 94]. In the continuous-time branching process, particles go extinct or replicate into a number of identical offspring at random and with constant Poissonian rates. Each of the new particles follows the same process. The difference between the original Galton-Watson branching process and the continuous-time branching process we consider here, lies in the waiting times between events. In the original Galton-Watson branching process, updates occur in discrete time steps, while in the continuous-time process we consider, waiting times follow a Poisson process [80, 94]. However, both processes share many asymptotics [99, 189], and therefore we regard the continuous-time branching process as the continuum limit of the Galton-Watson branching process.

By using field-theoretic methods, we provide a general framework to determine universal, finite-time scaling properties of a wide range of branching processes close to the critical point. The main advantages of this versatile approach are, on the one hand, the ease with which observables are calculated and, on the other hand, the use of diagrammatic language, which allows us to manipulate the sometimes cumbersome expressions in a neat and compact way. Other methods in the literature developed to study problems related to branching processes, in particular relating branching processes to different forms of motion, include the formalism based on the Pal-Bell equation [142, 13, 112].

Moreover, our framework allows us to determine systematically observables that are otherwise complicated to manipulate if possible at all. To illustrate this point, we have

calculated in closed form a number of observables that describe different aspects of the process in the vicinity of the critical point: we have calculated the moments of the population size as a function of time, the probability distribution of the population size as a function of time, the avalanche shape, the two-time and n -time correlation functions, and the total avalanche size and its moments.

Our results show that branching processes are universal in the vicinity of the critical point [119, 3] in the sense that exactly three quantities (the Poissonian rate and the first and second moments of the offspring distribution) are sufficient to describe the asymptotics of the process regardless of the underlying offspring distribution.

The contents of this paper are organised as follows. In Sec. 2.2 we derive the field theory of the continuous-time branching process. In Sec. 2.3 we use our formalism to calculate a number of observables in closed form, and in Sec. 2.4 we discuss our results and our conclusions. Further details of the calculations can be found in the appendices.

2.2 FIELD THEORY OF THE CONTINUOUS-TIME BRANCHING PROCESS

The continuous-time branching process is defined as follows. We consider a population of $N(t)$ identical particles at time $t \geq 0$ with initial condition $N(0) = 1$. Each particle is allowed to branch independently into κ offspring with Poissonian rate $s > 0$, where $\kappa \in \{0\} \cup \mathbb{N}$ is a random variable with probability distribution $P(\kappa = k) = p_k \in [0, 1]$ [80], Fig. 2.1. In the language of chemical reactions, this can be written as the reaction $A \rightarrow \kappa A$.

To derive the field theory of this process following the methods by Doi and Peliti [57, 150, 54, 189], we first write the master equation of the probability $P(N, t)$ to find N particles at time t ,

$$\frac{dP(N, t)}{dt} = s \sum_k p_k (N - k + 1) P(N - k + 1, t) - sNP(N, t), \quad (2.1)$$

with initial condition $P(N, 0) = \delta_{N,1}$. Following work by Doi [57], we cast the master equation in a second quantised form. A system with N particles is represented by a Fock-space vector $|N\rangle$. We use the ladder operators $a^\dagger$ (creation) and a (annihilation), which act on $|N\rangle$ such that $a|N\rangle = N|N-1\rangle$ and $a^\dagger|N\rangle = |N+1\rangle$, and satisfy the

commutation relation $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$. The probabilistic state of the system is given by

$$|\Psi(t)\rangle = \sum_N P(N, t) |N\rangle, \quad (2.2)$$

and its time evolution is determined by Eq. (2.1),

$$\frac{d|\Psi(t)\rangle}{dt} = s (f(a^\dagger) - a^\dagger) a |\Psi(t)\rangle, \quad (2.3)$$

using the probability generating function of κ ,

$$f(z) = \sum_{k=0}^{\infty} p_k z^k = \langle z^\kappa \rangle, \quad (2.4)$$

where $\langle \bullet \rangle$ denotes expectation. We define the *mass* r as the difference between the extinction and the net branching rates,

$$r = sp_0 - s \sum_{k \geq 2} (k-1)p_k = s(1 - \langle \kappa \rangle), \quad (2.5)$$

and the rates q_j as

$$q_j = s \sum_k \binom{k}{j} p_k = s \left\langle \binom{\kappa}{j} \right\rangle = \frac{s}{j!} f^{(j)}(1), \quad (2.6)$$

where $f^{(j)}(1)$ denotes the j th derivative of the probability generating function Eq. (2.4) evaluated at $z = 1$. We assume that the rates q_j are finite. In this notation, the time evolution in Eq. (2.3) can be written as

$$\tilde{\mathcal{A}} |\Psi(t)\rangle = \frac{d}{dt} |\Psi(t)\rangle \text{ and thus } |\Psi(t)\rangle = e^{\tilde{\mathcal{A}}t} |\Psi(0)\rangle, \quad (2.7)$$

where $\tilde{\mathcal{A}}$ is the operator

$$\tilde{\mathcal{A}} = \sum_{j \geq 2} q_j \tilde{a}^j a - r \tilde{a} a, \quad (2.8)$$

and $\tilde{a}$ denotes the Doi-shifted creation operator, $a^\dagger = 1 + \tilde{a}$.

The sign of the mass r , Eq. (2.5), determines in which regime a particular branching process is in; if $r = 0$ the process is at the critical point, if $r > 0$ the process is in the

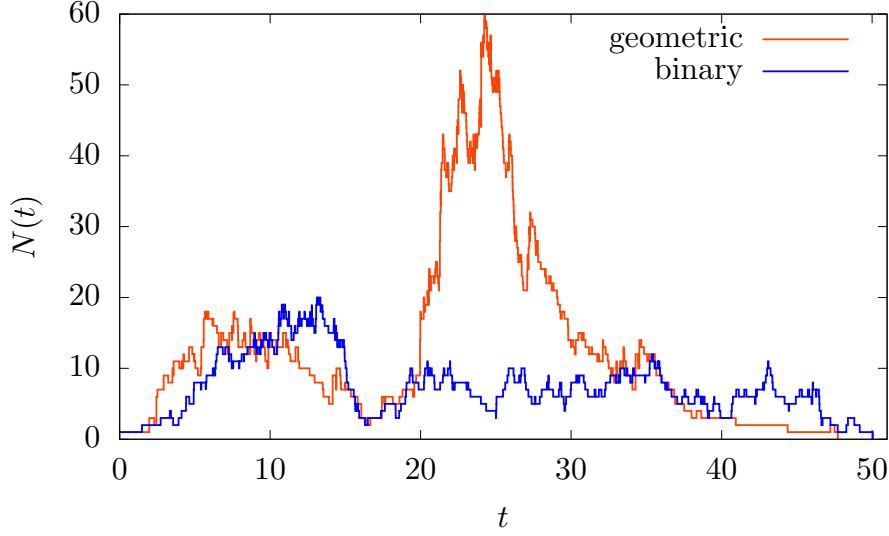


Figure 2.1: Typical avalanche profiles $N(t)$ of a binary branching process (blue) and a branching process with geometric distribution of the number of offspring (orange), both at criticality $r = 0$, and with Poissonian rate $s = 1$.

subcritical regime and if $r < 0$ the process is in the supercritical regime. Subcritical and critical processes are bound to go extinct in finite time, whereas supercritical process have a positive probability of survival [99].

Following the work by Peliti [150], Eq. (2.3) can be cast in path integral form. Here, the creation and annihilation operators $a^\dagger$ and a are transformed to time-dependent creation and annihilation fields $\phi^\dagger(t)$ and $\phi(t)$ respectively. Similarly, the Doi-shifted operator $\tilde{a}$ is transformed to the time-dependent Doi-shifted field $\tilde{\phi}(t) = \phi^\dagger(t) - 1$. The action functional of the resulting field theory is given by

$$\mathcal{A}[\tilde{\phi}, \phi] = \int_{-\infty}^{\infty} dt \left\{ \sum_{j \geq 2} q_j \tilde{\phi}^j(t) \phi(t) - \tilde{\phi}(t) \left(\frac{d}{dt} + r \right) \phi(t) \right\}. \quad (2.9)$$

Using the Fourier transform

$$\phi(t) = \int \mathfrak{d}\omega \phi(\omega) e^{-i\omega t} \quad \text{with} \quad \mathfrak{d}\omega = \frac{d\omega}{2\pi}, \quad (2.10a)$$

$$\phi(\omega) = \int dt \phi(t) e^{i\omega t}, \quad (2.10b)$$

Table 2.1: Summary of observables including their equation number and what cases and limits are exact. The expressions referred to are, in the limit $r \rightarrow 0$, either: *exact*, *exact for binary branching processes* (*i.e.* for other branching process, our result is the leading order in q_2/r for small r at fixed rt), or provide *exact asymptotes* (that is, our result is the leading order for any kind of branching process). The regime of validity near criticality is either critical and subcritical ($r \geq 0$) or all encompassing ($r \in \mathbb{R}$).

Observable	Symbol	Eq.	Expression	Regime	Fig.
Moments of the number of particles $N(t)$	$\langle N^n(t) \rangle$	(2.35), (2.36)	exact asymptote	$r \in \mathbb{R}$	2.2
Moments of the number of particles $N(t)$	$\langle N^n(t) \rangle, n \in \{1, 2, 3\}$	(2.64a)	exact	$r \in \mathbb{R}$	2.2
Moment generating function of $N(t)$	$\mathcal{M}_N(z)$	(2.38)	exact asymptote	$r \in \mathbb{R}$	
Probability distribution of $N(t)$	$P(N, t)$	(2.41)	exact for binary bp	$r \in \mathbb{R}$	
Probability of survival	$P_s(t)$	(2.42), (2.43)	exact for binary bp	$r \in \mathbb{R}$	2.3
Distribution of the avalanche duration T	$\mathcal{P}_T(t)$	(2.44), (2.45)	exact for binary bp	$r \geq 0$	2.4
Expected avalanche duration	$\langle T \rangle$	(2.46)	exact for binary bp	$r \geq 0$	
Avalanche shape	$V(t, T)$	(2.52), (2.53)	exact for binary bp	$r \in \mathbb{R}$	2.5a
Averaged avalanche shape	$\langle V(\tau) \rangle_T$	(2.71)	exact for binary bp	$r \in \mathbb{R}$	2.5b
Two-point connected correlation function	$\text{Cov}(N(t_1), N(t_2))$	(2.56)	exact	$r \in \mathbb{R}$	2.6
Three-point correlation function	$\langle N(t_1)N(t_2)N(t_3) \rangle$	(2.66)	exact	$r \in \mathbb{R}$	
Moments of the total avalanche size S	$\langle S^n \rangle$	(2.59b), (2.60c)	exact asymptote	$r \geq 0$	
Moment generating function of S	$\mathcal{M}_S(z)$	(2.62)	exact asymptote	$r \geq 0$	
Distribution of S	$\mathcal{P}_S(x)$	(2.63)	exact asymptote	$r \geq 0$	2.7

and identically for $\tilde{\phi}(t)$, the action Eq. (2.9) becomes local in ω and the bilinear, *i.e.* the Gaussian part

$$\mathcal{A}_0[\tilde{\phi}, \phi] = - \int d\omega \tilde{\phi}(-\omega)(-i\omega + r)\phi(\omega) \quad (2.11)$$

of the path integral can be determined in closed form. The Gaussian path integral is well-defined only when the mass is positive, $r > 0$. The non-linear terms, $j \geq 2$ in Eq. (2.9) are then treated as a perturbation about the Gaussian part, as commonly done in field theory [189, 190].

2.3 OBSERVABLES

We use the field theory described above to calculate a number of observables that have received attention in the literature in various settings. In Table 2.1 we list all the observables that we have calculated in closed form and the degree of approximation of our analytical result. Some results are exact for any kind of branching process and other results are only exact for binary branching processes. Those results that are an

approximation have the exact asymptotic behaviour.

All observables are constructed on the basis of the probability vector $|\Psi(t)\rangle$ which evolves according to Eq. (2.7). If the initial state, $t = 0$, consists of a single particle, then $|\Psi(0)\rangle = a^\dagger |0\rangle$ and $|\Psi(t)\rangle = \exp(\tilde{\mathcal{A}}t) a^\dagger |0\rangle$. Probing the particle number requires the action of the operator $a^\dagger a$, whose eigenvectors are the pure states $|N\rangle$, such that $a^\dagger a |N\rangle = N |N\rangle$. The components of the vector $a^\dagger a \exp(\tilde{\mathcal{A}}t) a^\dagger |0\rangle$ are thus the probability that the process has generated N particles weighted by N . To sum over all states, we further need the projection state

$$\langle \star | = \sum_{N=0}^{\infty} \langle N | = \sum_{N=0}^{\infty} \frac{1}{N!} \langle 0 | a^N = \langle 0 | e^a . \quad (2.12)$$

The expected particle number at time t may thus be written as

$$\langle N(t) \rangle = \langle \star | a^\dagger a e^{\tilde{\mathcal{A}}t} a^\dagger | 0 \rangle . \quad (2.13)$$

More complicated observables and intermediate temporal evolution can be compiled following the same pattern [190]. To perform any calculations, the operators need to be normal ordered and then mapped to fields as suggested above, $a^\dagger \rightarrow \phi^\dagger(t) = 1 + \tilde{\phi}(t)$ and $a \rightarrow \phi(t)$, where the time t corresponds to the total time the system has evolved for, Eq. (2.7). The expectation in Eq. (2.13) can thus be written as

$$\langle N(t) \rangle = \langle \phi^\dagger(t) \phi(t) \phi^\dagger(0) \rangle , \quad (2.14)$$

where $\langle \mathcal{O} \rangle$ denotes the path integral

$$\langle \mathcal{O} \rangle = \int \mathcal{D} [\tilde{\phi}, \phi] \mathcal{O} e^{\mathcal{A}[\tilde{\phi}, \phi]} . \quad (2.15)$$

The resulting expressions are most elegantly expressed in terms of Feynman diagrams [189]. The bare propagator of the field ϕ is read off from the bilinear part of the action which, in Fourier space, is

$$\langle \phi(\omega) \tilde{\phi}(\omega') \rangle = \frac{\delta(\omega + \omega')}{-i\omega + r} \hat{=} \text{---}, \quad (2.16)$$

where $\delta(\omega + \omega') = 2\pi\delta(\omega + \omega')$ denotes the scaled Dirac- δ function. Diagrammatically, the bare propagator is represented by a straight directed line. The directedness of the propagator reflects the causality (see Eq. (2.17)) of the process in the time domain as a particle has to be created before it can be annihilated but not vice versa. By convention, in our Feynman diagrams time proceeds from right to left.

Using the fact that the mass r is strictly positive for the Gaussian path integral to converge, we write the propagator in real time by Fourier transforming,

$$\langle \phi(t) \tilde{\phi}(t') \rangle = \int_{-\infty}^{\infty} d\omega d\omega' \frac{\delta(\omega + \omega')}{-i\omega + r} e^{-i\omega t} e^{-i\omega' t'} = \Theta(t - t') e^{-r(t-t')}, \quad (2.17)$$

where Θ is the Heaviside step function. If $r < 0$, the integral in Eq. (2.17) is only convergent for $t < t'$, which violates causality and therefore yields an unphysical result. For this reason, we will assume $r > 0$ and we will take the limit $r \rightarrow 0$ where possible. Therefore, in this paper, the analytical results obtained through this field theory hold for the critical and subcritical regimes only ($r \geq 0$). However, in some cases we may be able to use probabilistic arguments that allow us to extend our results to the supercritical case ($r < 0$), see Section 2.3.2. Furthermore, we will drop the cumbersome Heaviside Θ functions, assuming suitable choices for the times, such as $t > t'$ above.

Each of the interaction terms of the form $\tilde{\phi}^j \phi$ with $j \geq 2$ in the non-linear part of the action Eq. (2.8) come with individual couplings q_j , Eq. (2.6). These are to be expanded perturbatively in. Following the canonical field theoretic procedure [190, 189, 150], they are represented by (tree-like) amputated vertices such as

$$\begin{array}{c} q_2 \\ \diagup \quad \diagdown \\ \text{---} \end{array}, \quad \begin{array}{c} q_3 \\ \diagup \quad \diagdown \quad \diagup \\ \text{---} \end{array}, \quad \begin{array}{c} q_4 \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \text{---} \end{array}. \quad (2.18)$$

These vertices are *not* to be confused with the underlying branching process, because after the Doi-shift, lines are not representative of particles, but of their correlations. For example, the vertex with coupling q_2 in Eq. (2.18), accounts for density-density correlations due to *any* branching or extinction, just like the propagator Eq. (2.16) accounts for all particle density due to *any* branching or extinction. After Fourier transforming, these processes are accounted for regardless of when they take place.

The directionality of the diagrams allows us to define *incoming* legs and *outgoing* legs of a vertex [189]. In the present branching process, all vertices have one incoming leg and

j outgoing legs. We will refer to diagrams that are constructed solely from q_2 vertices as *binary tree diagrams*. The most basic such diagram is , which in real time reads

$$\begin{aligned}
\langle \phi^2(t) \phi^\dagger(0) \rangle &\doteq 2 \text{ 

where the pre-factor 2 is the combinatorial factor of this diagram.$$

The various observables that we calculate in the following are illustrated by numerics for two different kinds of continuous-time branching processes. Firstly, the binary branching process with probabilities $p_0, p_2 \geq 0$ such that $p_0 + p_2 = 1$, and secondly, the branching process with geometric offspring distribution $p_k = p(1-p)^k$ with $p \in [0, 1]$. The mass r (2.5) and the rates of the couplings q_j (2.6) are, in each case,

$$(\text{binary}) \quad r_B = s(1 - 2p_2), \quad q_{B2} = sp_2 = \frac{s - r_B}{2}, \quad q_{Bj} = 0 \text{ for } j \geq 3, \tag{2.20a}$$

$$(\text{geometric}) \quad r_G = s \frac{2p - 1}{p}, \quad q_{Gj} = s \left(\frac{1 - p}{p} \right)^j = s \left(1 - \frac{r_G}{s} \right)^j. \tag{2.20b}$$

Fig. 2.1 shows typical trajectories for each case.

2.3.1 MOMENTS $\langle N^n(t) \rangle$ AND THEIR UNIVERSALITY

In the following we will calculate the moments of the number of particles $N(t)$, which can be determined using the particle number operator $a^\dagger a$, as introduced above. The n th moment of $N(t)$ can be expressed as

$$\begin{aligned}
\langle N^n(t) \rangle &= \langle \star | (a^\dagger a)^n | \Psi(t) \rangle \\
&= \sum_{\ell=0}^n \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} \langle \star | a^\ell | \Psi(t) \rangle = \sum_{\ell=0}^n \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} \langle \phi^\ell(t) \tilde{\phi}(0) \rangle, \tag{2.21}
\end{aligned}$$

where $\left\{ \begin{smallmatrix} n \\ \ell \end{smallmatrix} \right\}$ denotes the Stirling number of the second kind for ℓ out of n *. We define the dimensionless function $\hat{g}_n(t)$ as the expectation

$$\hat{g}_n(t) = \left\langle \phi^n(t) \tilde{\phi}(0) \right\rangle \hat{=} \text{diagram with } n \text{ red lines entering a black circle from the left and one red line exiting to the right}, \quad (2.22)$$

with $\hat{g}_0(t) = \left\langle \tilde{\phi}(0) \right\rangle = 0$ and $\hat{g}_1(t) = \left\langle \phi(t) \tilde{\phi}(0) \right\rangle = e^{-rt}$. The black circle in the diagram of Eq. (2.22) represents the sum of all possible intermediate nodes, allowing for internal lines. For instance,

$$\hat{g}_1(t) \hat{=} \text{diagram with 1 red line entering a black circle from the left and one red line exiting to the right} = \text{single red line}, \quad (2.23a)$$

$$\hat{g}_2(t) \hat{=} \text{diagram with 2 red lines entering a black circle from the left and one red line exiting to the right} = 2 \times \text{diagram with two red lines meeting at a vertex and one red line exiting to the right}, \quad (2.23b)$$

$$\hat{g}_3(t) \hat{=} \text{diagram with 3 red lines entering a black circle from the left and one red line exiting to the right} = 6 \times \text{diagram with three red lines meeting at a vertex and one red line exiting to the right} + 12 \times \text{diagram with two red lines meeting at a vertex, which then connects to a third red line, and one red line exiting to the right}, \quad (2.23c)$$

$$\hat{g}_4(t) \hat{=} \text{diagram with 4 red lines entering a black circle from the left and one red line exiting to the right} = 24 \times \text{diagram with four red lines meeting at a vertex and one red line exiting to the right} + 48 \times \text{diagram with three red lines meeting at a vertex, which then connects to a fourth red line, and one red line exiting to the right} + 72 \times \text{diagram with two red lines meeting at a vertex, which then connects to two more red lines, and one red line exiting to the right} + 24 \times \text{diagram with two red lines meeting at a vertex, which then connects to a third red line, which then connects to a fourth red line, and one red line exiting to the right} + 96 \times \text{diagram with two red lines meeting at a vertex, which then connects to two more red lines, which then connect to a fourth red line, and one red line exiting to the right}, \quad (2.23d)$$

where the coefficient in front of each diagram is its symmetry factor, which is included in the representation involving the black circle, Eq. (2.22).

The tree diagrams follow a pattern, whereby $\hat{g}_n$ involves all $\hat{g}_m$ with $m < n$. For $n \geq 2$ this can be expressed as the recurrence relation

$$\hat{g}_n(t) \hat{=} \sum_{k=2}^n \sum_{m_1, \dots, m_k=1} \binom{n}{m_1, \dots, m_k} \text{diagram with } k \text{ black circles, each with } m_i \text{ red lines entering from the left, and } k \text{ red lines exiting to the right and meeting at a single vertex}, \quad (2.24)$$

where $\binom{n}{m_1, \dots, m_k}$ denotes the multinomial coefficient with the implicit constraint of $m_1 + \dots + m_k = n$.

*The Stirling numbers of the second kind can be calculated using the expression

$$\left\{ \begin{smallmatrix} n \\ \ell \end{smallmatrix} \right\} = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} j^n.$$

$\dots + m_k = n$. Including $\hat{g}_1(t)$ from Eqs. (2.23a) and (2.17), this may be written as

$$\hat{g}_n(t) = \delta_{n,1} e^{-rt} + \left(\sum_{k=2}^n q_k \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \times \int_0^t dt' e^{-r(t-t')} \hat{g}_{m_1}(t') \hat{g}_{m_2}(t') \cdots \hat{g}_{m_k}(t') \right), \quad (2.25)$$

where the integral accounts for the propagation up until time $t - t' \in [0, t]$ when a branching into (at least) k particles takes place, each of which will branch into (at least) m_k particles at some later time within $[t - t', t]$.

We proceed by determining the leading order behaviour of $\hat{g}_n$ in small r , starting with a dimensional argument. Since

$$\langle N^n(t) \rangle = \sum_{\ell=0}^n \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} \hat{g}_\ell(t) \quad (2.26)$$

from Eqs. (2.21) and (2.22), $\langle N^n(t) \rangle$ being dimensionless implies the same for $\hat{g}_n(t)$. Our notation for the latter obscures the fact that $\hat{g}_n(t)$ is also a function of r and all q_j , which, by virtue of s , are rates and thus have the inverse dimension of t . We may therefore write

$$\hat{g}_n(t) = \bar{g}_n(rt; \bar{q}_2, \bar{q}_3, \dots) \quad (2.27)$$

where $\bar{q}_j = q_j/r$ are dimensionless couplings. Dividing q_j by *any* rate renders the result dimensionless, but only the particular choice of dividing by r ensures that all couplings only ever enter multiplicatively (and never as an inverse), thereby enabling us to extract the asymptote of $\hat{g}_n(t)$ in the limit of small r , as we will see in the following.

Writing Eq. (2.25) as

$$\bar{g}_n(y; \bar{q}_2, \bar{q}_3, \dots) = \delta_{n,1} e^{-y} + \left(\sum_{k=2}^n \bar{q}_k \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \int_0^y dy' e^{-(y-y')} \times \bar{g}_{m_1}(y'; \bar{q}_2, \dots) \bar{g}_{m_2}(y'; \dots) \cdots \bar{g}_{m_k}(y'; \dots) \right), \quad (2.28)$$

the dominant terms in small r and fixed $y = rt$ are those that contain products involving

the largest number of factors of $\bar{q}_j \propto r^{-1}$. Since each $\bar{q}_j$ corresponds to a branching, diagrammatically these terms are those that contain the largest number of vertices, *i.e.* those that are entirely made up of binary branching vertices q_2 . This coupling, $q_2 = \langle \kappa(\kappa - 1) \rangle / 2$, cannot possibly vanish if there is any branching taking place at all. From Eqs. (2.27) and (2.28) it follows that $\hat{g}_n(t) \propto (q_2/r)^{(n-1)}$ to leading order in small r at fixed $y = rt$. Terms of that order are due to binary tree diagrams, whose contribution we denote by $g_n(t)$ in the following. For instance, $g_1(t) = \hat{g}_1(t)$, $g_2(t) = \hat{g}_2(t)$,

$$g_3(t) \hat{=} 12 \text{ (diagram: a vertex with three branches, one horizontal to the right and two diagonal down-left and down-right)}, \quad (2.29a)$$

$$g_4(t) \hat{=} 24 \text{ (diagram: a vertex with four branches, two horizontal to the right and two diagonal down-left and down-right)} + 96 \text{ (diagram: a vertex with three branches, one horizontal to the right and two diagonal down-left and down-right, each of which further branches into two)}. \quad (2.29b)$$

To summarise, $\hat{g}_n(t)$ is dominated by those terms that correspond to binary tree diagrams, which are the trees g_n that have the largest number of vertices for any fixed n , *i.e.*

$$\langle \phi^n(t) \tilde{\phi}(0) \rangle = \hat{g}_n(t) = g_n(t) + \mathcal{O} \left((1 - \langle \kappa \rangle)^{-(n-2)} \right), \quad (2.30)$$

where the correction in fact vanishes for $n < 3$.

As far as the asymptote in small r is concerned, we may thus replace $\hat{g}_\ell$ in Eq. (2.26) by g_ℓ . Among the $\hat{g}_\ell \sim r^{-(\ell-1)}$ with $\ell = 0, 1, \dots, n$, the dominating term is g_n so that the n th moment of the particle number N is, to leading order,

$$\langle N^n(t) \rangle \simeq g_n(t), \quad (2.31)$$

although exact results, as shown in Eq. (2.64a), are easily derived using Eqs. (2.21), (2.22), (2.23), and (2.25). On the basis of (2.25) the recurrence relation of g_n is give by

$$g_n(t) = \delta_{n,1} e^{-rt} + q_2 \sum_{m=1}^{n-1} \binom{n}{m} \int_0^t dt' e^{-r(t-t')} g_m(t') g_{n-m}(t'), \quad (2.32)$$

whose exact solution is

$$g_n(t) = n! e^{-rt} \left(\frac{q_2}{r} (1 - e^{-rt}) \right)^{n-1}. \quad (2.33)$$

We draw two main conclusions from our results. Firstly, that near the critical point

$r = 0$, the branching process is solely characterised by the two parameters r and q_2 . We therefore conclude that this process displays *universality*, in the sense that its asymptotia is exactly the same for any given r and q_2 regardless of the underlying offspring distribution. In particular, certain ratios of the moments of the particle number are universal constants (they do not depend on any parameters nor variables). For $k, \ell \in \mathbb{N}$ and $m \in 0, \dots, k-1$, we find the constant ratios

$$\frac{\langle N^k(t) \rangle \langle N^\ell(t) \rangle}{\langle N^{k-m}(t) \rangle \langle N^{\ell+m}(t) \rangle} = \frac{k! \ell!}{(k-m)! (\ell+m)!}. \quad (2.34)$$

Secondly, our results show that the scaling form of the moments is

$$\langle N^n(t) \rangle \simeq (q_2 t)^{n-1} \mathcal{G}_n(rt), \quad (2.35)$$

where $\mathcal{G}_n$ is the scaling function

$$\mathcal{G}_n(y) = n! e^{-y} \left(\frac{1 - e^{-y}}{y} \right)^{n-1}, \quad (2.36)$$

and the argument $y = rt$ is the rescaled time, Fig. 2.2. The asymptotes of $\mathcal{G}_n(y)$ characterise the behaviour of the branching process in each regime,

$$\mathcal{G}_n(y) \simeq \begin{cases} n! & \text{for } y = 0, \\ n! y^{-(n-1)} e^{-y} & \text{for } y \rightarrow \infty. \end{cases} \quad (2.37)$$

Moreover, from Eq. (2.35), we find that the moment generating function $\mathcal{M}_N(z) = \langle e^{Nz} \rangle$ is

$$\mathcal{M}_N(z) \simeq 1 + \frac{ze^{-rt}}{1 + z \frac{q_2}{r} (e^{-rt} - 1)}. \quad (2.38)$$

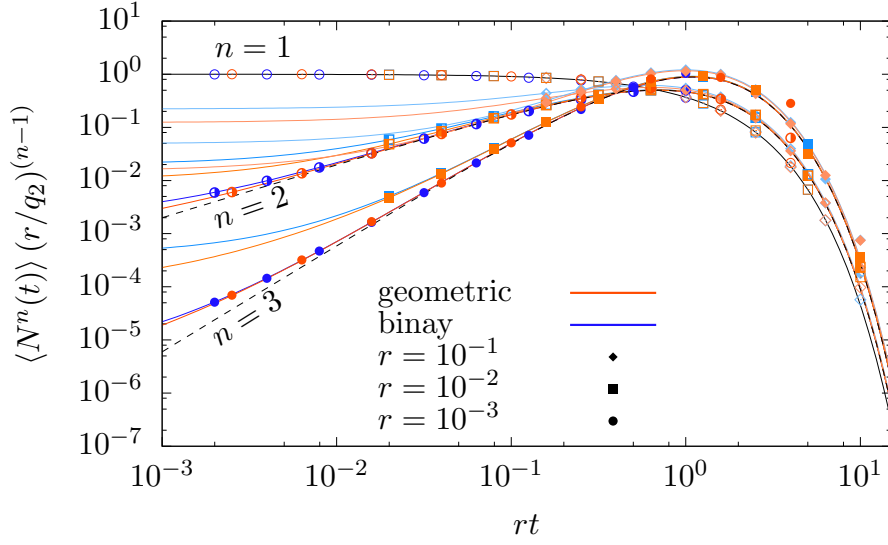


Figure 2.2: Data collapse of the moments $\langle N(t) \rangle$, $\langle N^2(t) \rangle$ and $\langle N^3(t) \rangle$, as a function of rescaled time rt as of Eq. (2.35). Symbols show results for the binary branching process (blue) and the branching process with geometric distribution of offspring (orange), both with $r \in \{10^{-3}, 10^{-2}, 10^{-1}\}$ and $s = 1$. Solid lines indicate the exact solution in Eq. (2.64a) and dashed lines indicate our approximation in Eq. (2.31).

2.3.2 PROBABILITY DISTRIBUTION OF $N(t)$, PROBABILITY OF SURVIVAL $P_s(t)$ AND EXPECTED AVALANCHE DURATION $\langle T \rangle$

Using Eq. (2.21) and the identity [†] of Stirling numbers of the second kind, we deduce that the falling factorial moments of $N(t)$ are

$$\langle \phi^\ell(t) \tilde{\phi}(0) \rangle = \langle N(t)(N(t) - 1) \dots (N(t) - \ell + 1) \rangle. \quad (2.39)$$

[†]The Stirling numbers of the second kind satisfy the identity

$$N^n = \sum_{\ell=0}^n \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} N(N-1) \dots (N-\ell+1).$$

Therefore, the probability generating function of $N(t)$ is

$$\begin{aligned}\mathcal{P}_{N(t)}(z) &= \sum_{\ell=0}^{\infty} \langle N(t)(N(t)-1)\dots(N(t)-\ell+1) \rangle \frac{(z-1)^\ell}{\ell!} \\ &= \sum_{\ell=0}^{\infty} \langle \phi^\ell(t) \tilde{\phi}(0) \rangle \frac{(z-1)^\ell}{\ell!},\end{aligned}\tag{2.40}$$

and the probability distribution of $N(t)$ is, using Eqs. (2.30) and (2.33),

$$P(N, t) = \frac{1}{N!} \frac{d^N}{dz^N} (\mathcal{P}_{N(t)}(z)) \Big|_{z=0} \tag{2.41a}$$

$$\simeq \sum_{\ell=N}^{\infty} \binom{\ell}{N} \frac{(-1)^{\ell-N}}{\ell!} g_\ell(t) \tag{2.41b}$$

$$= \begin{cases} 1 - \frac{e^{-rt}}{1 + \frac{q_2}{r}(1-e^{-rt})} & \text{if } N = 0, \\ \frac{e^{-rt} \left(\frac{q_2}{r}(1-e^{-rt}) \right)^{N-1}}{\left(1 + \frac{q_2}{r}(1-e^{-rt}) \right)^{N+1}} & \text{if } N > 0, \end{cases} \tag{2.41c}$$

which satisfies the initial condition $P(N, 0) = \delta_{N,1}$ and is an exact result for binary branching processes, consistent with [149].

It is straightforward to check that Eq. (2.41) satisfies the master equation (2.1) and the initial condition in the binary branching case. Due to the uniqueness of solutions of a system of coupled linear ordinary differential equations, the solution in Eq. (2.41) is the only solution. In particular, this solution holds in the supercritical case, $r < 0$. Reconstructing back the path that has lead us here, we find that $g_\ell(t)$ is the ℓ th falling factorial moment of $N(t)$, $\langle N(t)(N(t)-1)\dots(N(t)-\ell+1) \rangle$, for binary branching processes including the supercritical case and, therefore, most expressions derived from $g_\ell(t)$ can be extended to $r < 0$. In what follows, we will specify which expressions hold in the supercritical case.

The probability of survival $P_s(t)$ is the probability that there is at least one particle at time t , *i.e.* $P_s(t) = P(N(t) \geq 1)$. Therefore, from Eq. (2.41),

$$P_s(t) = 1 - P(0, t) = \frac{e^{-rt}}{1 + \frac{q_2}{r}(1-e^{-rt})}, \tag{2.42}$$

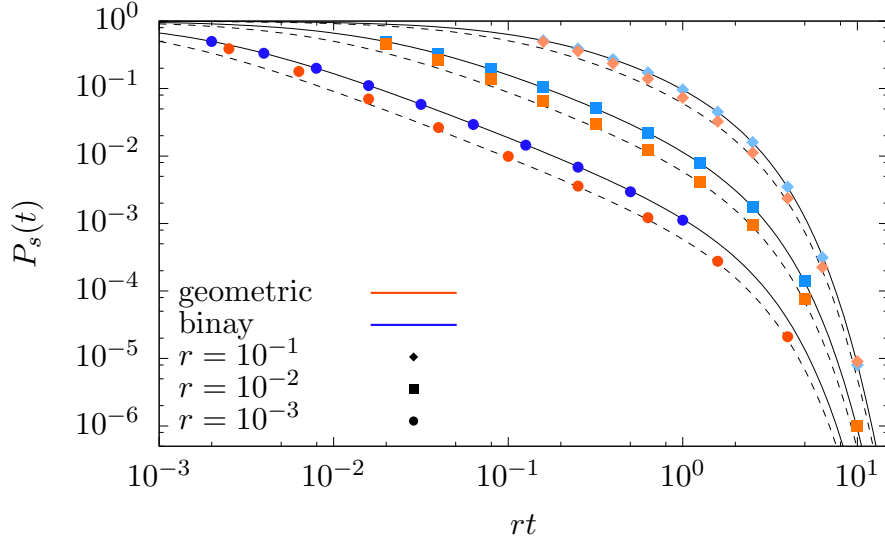


Figure 2.3: Probability of survival as a function of rescaled time rt as of Eq. (2.42). Symbols show numerical results for the binary branching process (blue) and the branching process with geometric distribution of offspring (orange), both with $r \in \{10^{-3}, 10^{-2}, 10^{-1}\}$ and $s = 1$. Lines indicate the result in Eq. (2.42), which is exact for binary branching (solid lines) and approximate otherwise (dashed lines). As r gets closer to the critical value, $r = 0$, the curves $P_s(t)$ flatten and resemble the power law in Eq. (2.43), which has exponent -1 .

and at the critical point,

$$\lim_{r \rightarrow 0} P_s(t) \simeq \frac{1}{1 + q_2 t}, \quad (2.43)$$

which is consistent with [76, 204, 39, 103], Fig. 2.3.

We define the avalanche duration T as the exact time where an avalanche dies, *i.e.* the time t when the process reaches the absorbing state, $T = \min\{t | N(t) = 0\}$. The probability of survival $P_s(t)$ gives the probability that $T > t$. Thus, $1 - P_s(t)$ is the cumulative distribution function of the time of death and its probability density function is

$$\mathcal{P}_T(t) = -\frac{dP_s(t)}{dt} \simeq \frac{r e^{rt} \left(1 + \frac{q_2}{r}\right)}{\left(\frac{q_2}{r} - e^{rt} \left(1 + \frac{q_2}{r}\right)\right)^2}, \quad (2.44)$$

and at the critical point,

$$\lim_{r \rightarrow 0} \mathcal{P}_T(t) \simeq \frac{q_2}{(1 + q_2 t)^2}, \quad (2.45)$$

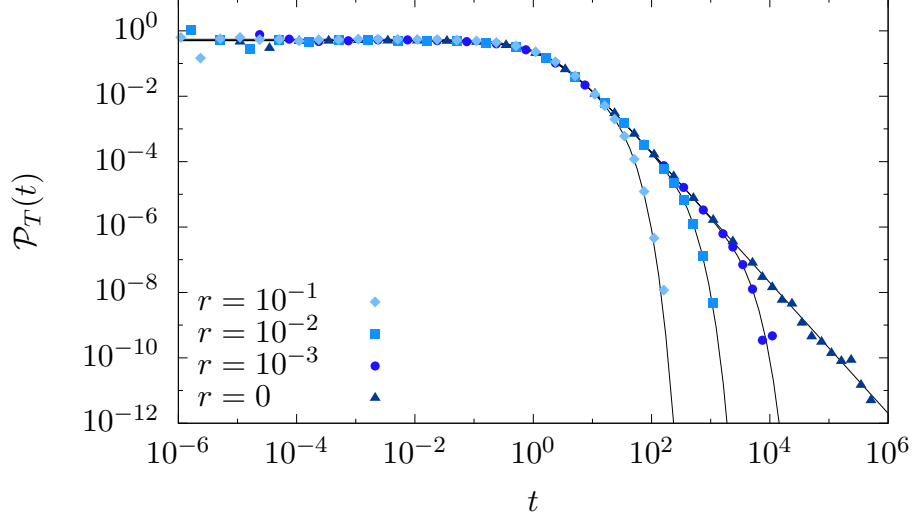


Figure 2.4: Probability density function of the avalanche duration $\mathcal{P}_T(t)$ for the binary branching process with $r \in \{0, 10^{-3}, 10^{-2}, 10^{-1}\}$ and $s = 1$. Solid lines represent our result in Eqs. (2.44) and (2.45), which is exact for binary branching. Symbols show numerical results.

see Fig. 2.4. It follows from (2.44) that the expected avalanche duration is

$$\langle T \rangle \simeq \frac{1}{q_2} \log \left(1 + \frac{q_2}{r} \right). \quad (2.46)$$

Because the derivation of Eq. (2.44) relies on a finite termination time, we cannot assume that it remains valid in the supercritical case, and similarly for (2.46).

2.3.3 AVALANCHE SHAPE $V(t, T)$

The avalanche shape $V(t, T)$ is defined as the average of the temporal profiles $N(t)$ conditioned to extinction at time T [87, 56, 177, 143, 5, 211, 117]. Closed form expressions of the avalanche shape have been calculated in other models such as avalanches in elastic interfaces [56], the Barkhausen noise [143], the discrete-time Ornstein-Uhlenbeck process [5]. An implicit expression of avalanche shape of branching processes is given in [87].

To produce an explicit expression we first calculate the joint expectation of number of particles at time t and the event that a branching process is extinct by time T ,

$\langle N(t), N(T) = 0 \rangle$. In terms of ladder operators,

$$\langle N(t), N(T) = 0 \rangle = \langle 0 | e^{\hat{A}(T-t)} a^\dagger a e^{\hat{A}t} a^\dagger | 0 \rangle, \quad (2.47)$$

which means that a particle is created from the vacuum, the system is allowed to evolve for time t , the number of particles is measured, and the system evolves further for time $T - t$. Finally, all possible trajectories are "sieved" so that only those avalanches whose number of particles is 0 at time T are taken into account. The path integral expression of Eq. (2.47) is

$$\begin{aligned} \langle N(t), N(T) = 0 \rangle &= \langle e^{-\phi(T)} \phi^\dagger(t) \phi(t) \phi^\dagger(0) \rangle \\ &= \langle \phi(t) \tilde{\phi}(0) \rangle + \sum_{n \geq 1} \frac{(-1)^n}{n!} \left(\langle \phi^n(T) \phi(t) \tilde{\phi}(0) \rangle + \langle \phi^n(T) \tilde{\phi}(t) \phi(t) \tilde{\phi}(0) \rangle \right). \end{aligned} \quad (2.48)$$

The two terms in the bracket have asymptotes

$$\langle \phi^n(T) \phi(t) \tilde{\phi}(0) \rangle \simeq \sum_{k=1}^n \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \frac{1}{k!} g_{m_1}(T-t) \cdots g_{m_k}(T-t) g_{k+1}(t), \quad (2.49)$$

and

$$\langle \phi^n(T) \tilde{\phi}(t) \phi(t) \tilde{\phi}(0) \rangle \simeq \sum_{k=1}^n \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \frac{1}{(k-1)!} g_{m_1}(T-t) \cdots g_{m_k}(T-t) g_k(t), \quad (2.50)$$

with the constraint $m_1 + \dots + m_k = n$ in both cases. Both expressions are exact in case of binary branching. Their diagrammatic representation and closed form expressions can be found in App. 2.B. Using the expression of $g_n(t)$ in Eq. (2.33) and the number of combinations of n legs into k groups, we have

$$\langle N(t), N(T) = 0 \rangle = e^{-rt} - P_s(T) \left[1 + \frac{q_2}{r} (1 - e^{-rt}) \left(2 - \frac{P_s(T)}{P_s(t)} \right) \right], \quad (2.51)$$

where $P_s(t)$ is given in Eq. (2.42).

In order to account solely for those instances that become extinct exactly at time T ,

the joint expectation $\langle N(t), N(T) = 0 \rangle$ is to be differentiated with respect to T , and in order to have a conditional expectation instead of a joint expectation, we need to divide the result by $-\frac{d}{dT}P_s(T)$, yielding,

$$\begin{aligned} V(t, T) &= \frac{\frac{d}{dT} \langle N(t), N(T) = 0 \rangle}{-\frac{d}{dT}P_s(T)} \\ &\simeq 1 + 2\frac{q_2}{r} (1 - e^{-rt}) \left(1 - \frac{P_s(T)}{P_s(t)}\right), \end{aligned} \quad (2.52)$$

Fig. 2.5a. Since the observable $V(t, T)$ suitably incorporates the condition $N(T) = 0$, the result in Eq. (2.52) holds for the supercritical case as well. At criticality, the avalanche shape is the parabola [87, 143, 5, 211]

$$\lim_{r \rightarrow 0} V(t, T) \simeq 1 + 2\frac{(q_2 T)^2}{1 + q_2 T} \left(1 - \frac{t}{T}\right) \frac{t}{T}. \quad (2.53)$$

The avalanche shape $V(t, T)$ in Eq. (2.52) is a symmetric function with its maximum at $t = T/2$, which is bounded [5] by

$$\lim_{T \rightarrow \infty} V\left(\frac{T}{2}, T\right) \simeq 1 + 2\frac{q_2}{r}. \quad (2.54)$$

2.3.4 CONNECTED CORRELATION FUNCTION $\text{COV}(N(t_1), N(t_2))$

To calculate the expectation $\langle N(t_1)N(t_2) \rangle$ we assume $0 < t_1 < t_2$ without loss of generality,

$$\langle N(t_1)N(t_2) \rangle = \left\langle \star \left| a^\dagger a e^{-\hat{A}(t_2-t_1)} a^\dagger a e^{-\hat{A}t_1} a^\dagger \right| 0 \right\rangle \quad (2.55a)$$

$$= \langle \phi(t_2) \phi^\dagger(t_1) \phi(t_1) \phi^\dagger(0) \rangle \quad (2.55b)$$

$$\begin{aligned} &= \langle \phi(t_2) \tilde{\phi}(t_1) \phi(t_1) \tilde{\phi}(0) \rangle + \langle \phi(t_2) \phi(t_1) \tilde{\phi}(0) \rangle \\ &\hat{=} \text{---} \text{---} + 2 \text{---} \text{---} \end{aligned} \quad (2.55c)$$

The diagram on the left consists of two separate components. We refer to diagrams of that kind as disconnected diagrams, in contrast to connected diagrams that only

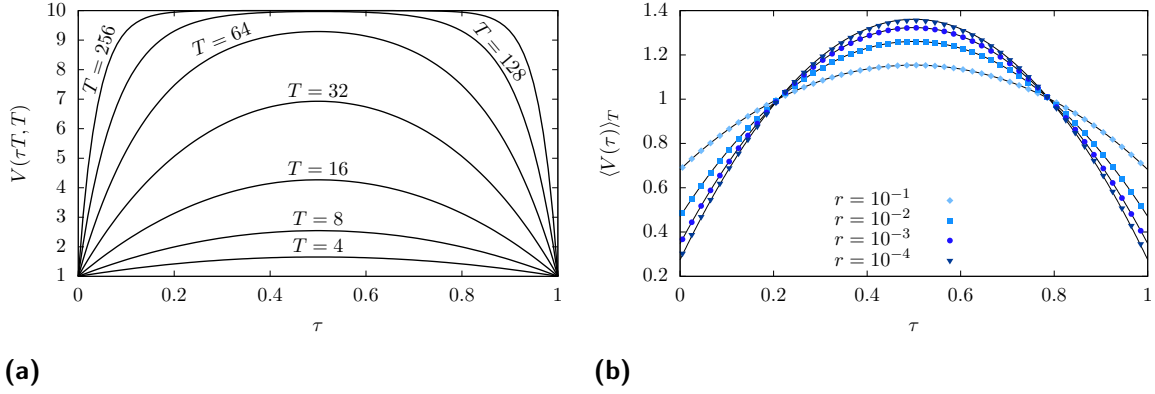


Figure 2.5: Average temporal profile. In **(a)**, avalanche shape V with rescaled time $\tau = t/T$ for different times of extinction T , $r = 10^{-1}$ and $q_2 = 0.45$ as of Eq. (2.52). The shapes are symmetric and flatten as T increases with the upper bound given in Eq. (2.54). However, this observable is numerically inaccessible because it is computationally unfeasible to obtain a large enough sample of avalanches in the subcritical regime conditioned to extinction at large times. Instead, in **(b)** we show an observable that is accessible both numerically and analytically, the averaged avalanche shape $\langle V(\tau) \rangle_T$, that is for each r , avalanches are rescaled in time to the interval $[0, 1]$, their shapes are averaged and normalised regardless of their extinction times T . Numerical results shown as symbols are for the binary branching process with $r \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$ and $s = 1$, and are in agreement with Eq. (2.71) (solid lines), which is an exact expression for binary branching processes. We find that the shape tends to a parabola as r approaches the critical point.

consist of one component as the one appearing on the right. The connected correlation function is

$$\begin{aligned} \text{Cov}(N(t_1), N(t_2)) &= \langle N(t_1)N(t_2) \rangle - \langle N(t_1) \rangle \langle N(t_2) \rangle \\ &= \left(2\frac{q_2}{r} + 1\right) e^{-r(t_1+t_2)} (e^{rt_1} - 1) \end{aligned} \quad (2.56)$$

which is an exact result independent of the type of branching process, (*i.e.* irrespective of the offspring distribution), Fig. 2.6. In particular, the variance is $\text{Var}(N(t)) = \text{Cov}(N(t), N(t))$ [189].

2.3.5 n -POINT CORRELATION FUNCTION

We call $\zeta_n(t_1, \dots, t_n)$, with $0 < t_1, \dots, t_n$ (not necessarily in order), the contribution of all binary, and therefore connected, diagrams to the n -point correlation function, where

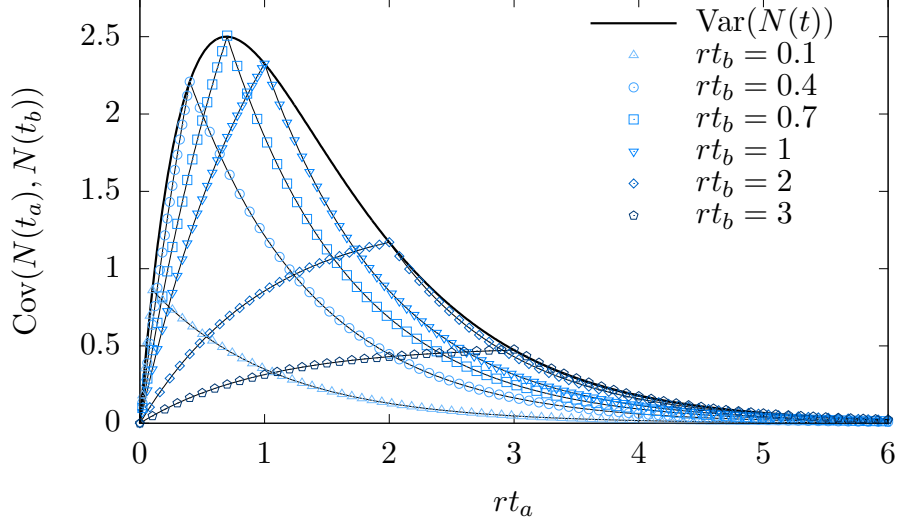


Figure 2.6: Two-point correlation function $\text{Cov}(N(t_a), N(t_b))$ of the binary continuous-time branching process with $r = 10^{-1}$ and $s = 1$. Our numerical results shown as symbols are in perfect agreement with the exact expression in Eq. (2.56) with $t_1 = \min(t_a, t_b)$ and $t_2 = \max(t_a, t_b)$ (solid lines). We also show $\text{Var}(N(t)) = \text{Cov}(N(t), N(t))$, which is the envelope.

the error term is controlled as

$$\langle N(t_1) \dots N(t_n) \rangle = \zeta_n(t_1, \dots, t_n) + \mathcal{O}\left((1 - \langle \kappa \rangle)^{-(n-2)}\right). \quad (2.57)$$

The leading order contribution ζ_n satisfies the following recurrence relation,

$$\zeta_n(t_1, \dots, t_n) = \sum_{m=1}^{n-1} \sum_{\substack{\sigma \subset \{t_1, \dots, t_n\} \\ |\sigma|=m}} \left\{ \begin{array}{c} \sigma \\ \sigma^C \end{array} \right\} \quad (2.58a)$$

$$= q_2 \sum_{m=1}^{n-1} \sum_{\substack{\sigma \subset \{t_1, \dots, t_n\} \\ |\sigma|=m}} \int_0^{t_{\min}} \zeta_m(t_{\sigma(1)} - t', \dots, t_{\sigma(m)} - t') \\ \times \zeta_{n-m}(t_{\sigma^C(m+1)} - t', \dots, t_{\sigma^C(n)} - t') e^{-rt'} dt', \quad (2.58b)$$

with $\zeta_0 = 0$ and $\zeta_1(t) = e^{-rt}$, and $t_{\min} = \min\{t_1, \dots, t_n\}$. Here, σ is a subset of the set of times $\{t_1, \dots, t_n\}$, whose size is $|\sigma|$, and $\sigma(1), \dots, \sigma(m)$ is a list of its distinct

elements. Its complementary set is $\sigma^c = \{t_1, \dots, t_n\} \setminus \sigma$, which contains the elements $\sigma^c(m+1), \dots, \sigma^c(n)$. Eq. (2.58b) is symmetric under exchange of any permutation of the times $t_1, \dots, t_n$, see the 3-point correlation function in App. 2.A.

This approximation is two-fold. First, it neglects higher order branching vertices proportional to $q_{j \geq 3}$, and secondly, it neglects contributions from disconnected diagrams, cf. Eq. (2.55). Latter contributions are dominant only when $t_{\max} = \max\{t_1, \dots, t_n\}$ is smaller than s^{-1} . For times in $(0, s^{-1})$, the branching process has typically not yet undergone a change in the particle number.

2.3.6 DISTRIBUTION OF THE TOTAL AVALANCHE SIZE S

We define the total avalanche size as the time-integrated activity $S = s \int dt N(t)$. Using $\langle N(t) \rangle = e^{-rt}$ and Eq. (2.56), the first and second moments of the total avalanche size [38, 40] (Chapter 5) read

$$\langle S \rangle = s \int dt \langle N(t) \rangle = \frac{s}{r} = \frac{1}{1 - \langle \kappa \rangle}, \quad (2.59a)$$

$$\langle S^2 \rangle = s^2 \int dt_1 dt_2 \langle N(t_1) N(t_2) \rangle = \frac{s^2}{r^2} \left(\frac{q_2}{r} + 1 \right). \quad (2.59b)$$

To calculate $\langle S^n \rangle$ close to criticality, we use the approximation to the n -point correlation function defined in Eq. (2.57) and find the following recurrence relation,

$$\langle S^n \rangle \simeq s^n \int dt_1 \dots dt_n \zeta_n(t_1, \dots, t_n) \quad (2.60a)$$

$$\simeq \frac{q_2}{r} \sum_{m=1}^{n-1} \binom{n}{m} \langle S^m \rangle \langle S^{n-m} \rangle \quad (2.60b)$$

$$\simeq \frac{s^n q_2^{n-1}}{r^{2n-1}} 2^{n-1} (2n-3)!!, \quad (2.60c)$$

see App. 2.D for a proof by induction of Eq. (2.60c). Similarly to Eq. (2.34), we find the universal constant ratios of the moments of S ,

$$\frac{\langle S^k \rangle \langle S^\ell \rangle}{\langle S^{k-m} \rangle \langle S^{\ell+m} \rangle} = \frac{(2k-3)!!(2\ell-3)!!}{(2(k-m)-3)!!(2(\ell+m)-3)!!} \quad (2.61)$$

with $k, \ell \in \mathbb{N}$ and $m \in \{0, \dots, k-1\}$. The moment generating function of S is

$$\mathcal{M}_S(z) \simeq 1 + \frac{r - \sqrt{r^2 - 4sq_2z}}{2q_2}, \quad (2.62)$$

and its probability density function $\mathcal{P}_S(x)$ is the inverse Laplace transform of $\mathcal{M}_S(-z)$,

$$\mathcal{P}_S(x) \simeq \frac{1}{2} \sqrt{\frac{s}{q_2\pi}} x^{-\frac{3}{2}} e^{-\frac{r^2x}{4q_2s}}, \quad (2.63)$$

which is a power law with exponent $-3/2$ with exponential decay, Fig. 2.7. At criticality, this distribution is a pure power law.

The approximation used to derive these results, Eq. (2.57), consists in neglecting contributions of disconnected diagrams to the n -point correlation function. This approximation is unjustified for total avalanche sizes corresponding those realisations of branching processes that underwent no branching but a single extinction event, and whose sizes are therefore typically smaller than 1, because their n -point correlation functions $\langle N(t_1) \dots N(t_n) \rangle$ vanish for $t_{\max} \gtrsim s^{-1}$. Consequently, the n -point correlation functions are dominated by purely disconnected diagrams (cf. Sec. 2.3.5). We therefore expect a breakdown of our approximation around $x = 1$. All three features of the distribution of the total avalanche size, the power-law behaviour, the exponential cutoff, and the breakdown of the approximation for $x < 1$, are in good agreement with numerical simulations as shown in Fig. 2.7.

2.4 DISCUSSION AND CONCLUSIONS

In this paper we study the continuous-time branching process following a field-theoretic approach. We build on the wealth of existing results in the literature obtained through other methods. Here, we demonstrate that the Doi-Peliti field theory provides an elegant, intuitive, and seemingly natural language for continuous-time branching processes.

We illustrate how to use the field theory to calculate a number of relevant observables, listed in Table 2.1. Our results are valid for any offspring distribution in the vicinity of the critical point and at large times. However, many of the results are exact for the binary branching process and others are exact for any branching process. In princi-

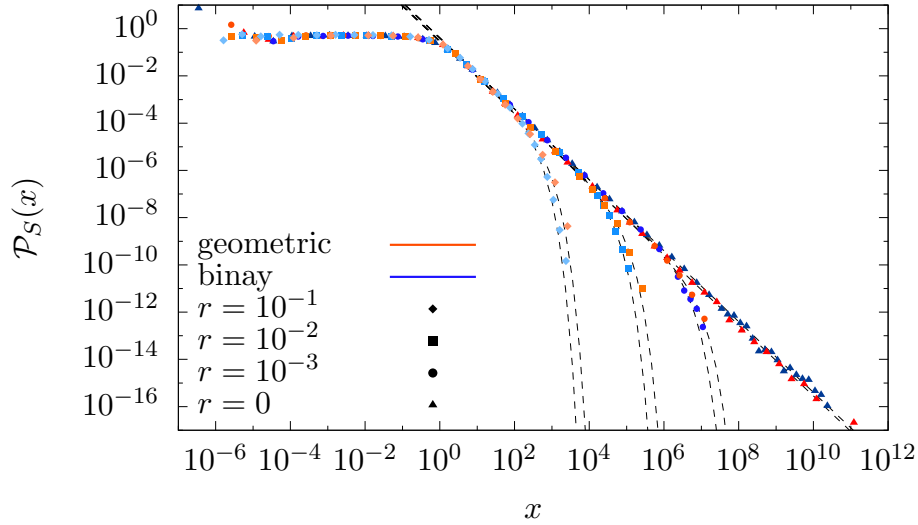


Figure 2.7: Probability density function of the total avalanche size $\mathcal{P}_S(x)$ for the binary branching process (blue) and the branching process with geometric distribution of offspring (orange), with $r \in \{0, 10^{-3}, 10^{-2}, 10^{-1}\}$ and $s = 1$. Dashed lines indicate our approximation in Eq. (2.63). This approximation is not valid for small times, which explains the disagreement between the numerical results and the dashed lines for small values of x .

ple, many observables can be calculated systematically using the field theory for any offspring distribution, for any time and any parameter set.

In this paper, we extend the existing results in the literature by finding explicit scaling functions and universal moment ratios for any offspring distribution. We find that all the scaling laws derived above depend on two parameters only, namely r and q_2 . Therefore, one may argue that the master equation of any branching process close to the critical point and asymptotically in large times is captured by the action Eq. (2.8) with couplings r and q_2 only.

Having established the field-theoretic ground work, in particular the basic formalism and range of relevant observables, we may now proceed by extending the basic branching process into more sophisticated models of natural phenomena. We hope that the methods established in this paper will help reaching new boughs, branches, and twigs of the many offspring of branching processes.

ACKNOWLEDGEMENTS

We thank Nanxin Wei, Stephanie Miller, Kay Wiese, Ignacio Bordeu Weldt, Eric Smith, David Krakauer, and Nicholas Moloney, for fruitful discussions. We extend our gratitude to Andy Thomas for invaluable computing support.

2.A EXACT EXPRESSIONS

The continuous-time branching process is exactly solvable, that is, in principle, all moments and correlation functions can be calculated in exact form if all the terms in the (possibly infinite) sums are taken into account. Here we show some exact expressions. The exact first three moments of $N(t)$ are

$$\langle N(t) \rangle = e^{-rt}, \quad (2.64a)$$

$$\langle N^2(t) \rangle = e^{-rt} \left(1 + \frac{2q_2}{r} (1 - e^{-rt}) \right), \quad (2.64b)$$

$$\langle N^3(t) \rangle = e^{-3rt} \left(\frac{6q_2^2}{r^2} - \frac{3q_3}{r} \right) - e^{-2rt} \left(\frac{12q_2^2}{r^2} + \frac{6q_2}{r} \right) + e^{-rt} \left(\frac{6q_2^2}{r^2} + \frac{3q_3}{r} + \frac{6q_2}{r} + 1 \right), \quad (2.64c)$$

and therefore the variance is

$$\text{Var}(N(t)) = \left(1 + 2\frac{q_2}{r} \right) e^{-rt} (1 - e^{-rt}), \quad (2.65)$$

which is consistent with Eq. (2.56) and [54, 189, 204, 99, 149]. The three-point correlation function is, assuming $0 \leq t_1 \leq t_2 \leq t_3$ and using Eq. (2.58b),

$$\begin{aligned} \langle N(t_1)N(t_2)N(t_3) \rangle &\simeq \zeta(t_1, t_2, t_3) \\ &= 2 \left(\frac{q_2}{r} \right)^2 e^{-r(t_1+t_2+t_3)} \left((e^{rt_1} - 1) (2e^{rt_1} + e^{rt_2}) - \frac{3}{2} (e^{2rt_1} - 1) \right). \end{aligned} \quad (2.66)$$

2.B DIAGRAMMATIC REPRESENTATION AND CLOSED FORM EXPRESSIONS OF EQS. (2.49) AND (2.50)

Defining

$$a = \frac{\exp(-r(T-t)) - \exp(-rT)}{1 - \exp(-r(T-t))}, \quad (2.67)$$

we have, firstly (2.49),

$$\begin{aligned} \langle \phi^n(T) \phi(t) \tilde{\phi}(0) \rangle &= \sum_{k=1}^n \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \frac{1}{k!} g_{m_1}(T-t) \cdots g_{m_k}(T-t) g_{k+1}(t) \\ &\doteq \sum_{k=1}^n \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \left\{ \begin{array}{c} \text{Diagram: A tree structure with } k \text{ nodes. The root node has } k \text{ children. The } i\text{-th child has } m_i \text{ children. The root node has } k \text{ incoming lines.} \end{array} \right\} \quad (2.68a) \end{aligned}$$

$$= n! e^{-rt} \left(\frac{q_2}{r} \right)^n (1 - e^{-rT})^n \left[\frac{a^2(n-1)}{(1+a)^2} + \frac{2a}{1+a} \right], \quad (2.68b)$$

and secondly (2.50),

$$\begin{aligned} \langle \phi^n(T) \tilde{\phi}(t) \phi(t) \tilde{\phi}(0) \rangle &= \sum_{k=1}^n \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \frac{1}{(k-1)!} g_{m_1}(T-t) \cdots g_{m_k}(T-t) g_k(t) \\ &\doteq \sum_{k=1}^n \sum_{m_1, \dots, m_k} \binom{n}{m_1, \dots, m_k} \left\{ \begin{array}{c} \text{Diagram: A tree structure with } k \text{ nodes. The root node has } k \text{ children. The } i\text{-th child has } m_i \text{ children. The root node has } k \text{ incoming lines.} \end{array} \right\} \quad (2.69a) \end{aligned}$$

$$= \frac{n! e^{-rt}}{1 - e^{-rt}} \left(\frac{q_2}{r} \right)^{n-1} (1 - e^{-rT})^n \left[\frac{a^2(n-1)}{(1+a)^2} + \frac{a}{1+a} \right]. \quad (2.69b)$$

2.C AVERAGED AVALANCHE SHAPE

In Section 2.3.3, we derive analytically the expected avalanche shape $V(t, T)$ for a specific time of death T . However, direct comparison with numerics is computationally very expensive as specific large times of death occur rarely for subcritical branching

processes. Here we describe an observable that is accessible both analytically and numerically: the averaged avalanche shape $\langle V(\tau) \rangle_T$. For a fixed parameter set, we first rescale time $\tau = t/T$ and then average all the avalanche profiles irrespectively of T . Finally, to achieve convergence to a shape comparable across parameter settings, we normalise the result by area [211],

$$\langle V(\tau) \rangle_T = \frac{1}{N_V} \int_0^\infty dT \mathcal{P}_T(T) V(\tau T, T), \quad (2.70a)$$

$$N_V = \int_0^1 \int_0^\infty d\tau dT \mathcal{P}_T(T) V(\tau T, T). \quad (2.70b)$$

The result [216] can be expressed with the Gaussian hypergeometric function ${}_2F_1(a, b, c, z)$,

$$\langle V(\tau) \rangle_T = \frac{1}{N_V} + \tau(\tau - 1) \frac{q_2 F(\tau, q_2, r)}{(q_2 + r)N_V}, \quad (2.71)$$

where

$$F(\tau, q_2, r) = \frac{{}_2F_1\left(1, 2 - \tau, 3 - \tau, \frac{q_2}{q_2 + r}\right)}{\tau - 2} - \frac{{}_2F_1\left(1, 1 + \tau, 2 + \tau, \frac{q_2}{q_2 + r}\right)}{\tau + 1}. \quad (2.72)$$

Both F and N_V diverge at the critical point with the limit

$$\lim_{r \rightarrow 0} \frac{F(\tau, q_2, r)}{N_V} = 6. \quad (2.73)$$

2.D PROOF OF EQ. (2.60C)

Eq. (2.60c) can be proved by induction. In Eq. (2.59b) we see that it applies to $\langle S \rangle$. The approximation of binary tree diagrams of $\langle N(t_1)N(t_2) \rangle$ gives $\langle S^2 \rangle = s^2 q_2 / r^3$, which

also satisfies Eq. (2.60c). The induction step is verified by

$$\begin{aligned}\langle S^n \rangle &= \frac{q_2}{r} \sum_{m=1}^{n-1} \binom{n}{m} \left(\frac{s^m q_2^{m-1} 2^{m-1} (2m-3)!!}{r^{2m-1}} \right) \left(\frac{s^{n-m} q_2^{n-m-1} 2^{n-m-1} (2(n-m)-3)!!}{r^{2(n-m)-1}} \right) \\ &= \frac{s^n q_2^{n-1}}{r^{2n-1}} 2^{n-2} \sum_{m=1}^{n-1} \binom{n}{m} (2m-3)!! (2(n-m)-3)!!.\end{aligned}\quad (2.74)$$

This sum is equivalent to

$$\begin{aligned}\sum_{m=1}^{n-1} \binom{n}{m} (2m-3)!! (2(n-m)-3)!! &= \frac{1}{n-1} \sum_{m=1}^{n-1} \binom{n}{m} (2m-3)!! (2(n-m)-1)!! \\ &= \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n}{k+1} (2k-1)!! (2(n-k)-3)!!\end{aligned}\quad (2.75a)$$

$$= 2(2n-3)!!,\quad (2.75b)$$

where we have used the identity [28],

$$\sum_{k=0}^{n-1} \binom{n}{k+1} (2k-1)!! (2(n-k)-3)!! = (2n-1)!!.\quad (2.76)$$

Using Eq. (2.75b) in Eq. (2.74) reproduces Eq. (2.60c), thereby completing the proof.

3

Time-dependent branching processes: a model of oscillating neuronal avalanches

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OVERVIEW Branching processes are very widely used to model neuronal avalanches given that both have some statistical properties in common. It has long been debated whether the spreading of electrical signals in the brain is in fact captured by a branching process at criticality [11]. Recent experimental studies on the temporal profile of neuronal avalanches show the fingerprint of oscillations [134]. In this work we study a branching process with oscillatory extinction rate and calculate the average temporal profile based on a perturbative expansion of the Doi-Peliti field theory in Chapter 2. We show qualitatively that our model approximates the experimental findings. We also calculate the two-point correlation function.

Author contributions: JP and GP wrote the body of the manuscript. RGM had the idea of modelling oscillating neuronal avalanches. JP, RGM and GP derived the perturbative field theory and calculated the various observables. RGM performed the numerical simulations. JP compared qualitatively the model to available experimental results. GP supervised the project throughout.

ABSTRACT

Recently, neuronal avalanches have been observed to display oscillations, a phenomenon regarded as the co-existence of a scale-free behaviour (the avalanches close to criticality) and scale-dependent dynamics (the oscillations). Ordinary continuous-time branching processes with constant extinction and branching rates are commonly used as models of neuronal activity, yet they lack any such time-dependence. In the present work, we extend a basic branching process by allowing the extinction rate to oscillate in time as a new model to describe cortical dynamics. By means of a perturbative field theory, we derive relevant observables in closed form. We support our findings by quantitative comparison to numerics and qualitative comparison to available experimental results.

3.1 INTRODUCTION

In the brain, electrical signals propagate between neurons of the cortical network through action potentials, which are spikes of polarisation in the membrane of the neuron's axon. These spikes have an amplitude of about 100mV, typically last about 1ms [45] and can be recorded using micro-electrodes [154, 155]. In order to study the signaling in larger regions of neurons, multielectrode arrays, comprising about 60 electrodes spread across $\approx 4\text{mm}^2$, are used to capture the collective occurrence of spikes as local field potentials (LFPs). In this setting, the electrodes are extracellular and each is sensitive to electrical signals from several surrounding neurons [109, 11, 12, 158, 159].

The data of the LFP recordings are processed by putting them into time bins of a few milliseconds length and by introducing a voltage threshold. In addition, a refractory period is imposed to avoid counting large voltage excursions more than once. The details of processing can differ slightly between experiments [11, 12, 199, 158, 159]. However, after processing, the data is a time series of two values for each electrode: on (detected

signal above threshold) and off (no detected signal or signal below threshold). A *neuronal avalanche* is then defined as a set of uninterrupted signals detected across the micro-electrode array. Each avalanche is both preceded and succeeded by at least one time bin where none of the electrodes detected a signal, defining the avalanche duration as the number of time bins where avalanche unfolds. Which and how many electrodes detect a signal varies during the avalanche [11, 12]. The duration of avalanches typically ranges between a few milliseconds and 30ms [10]. A prominent observable is the avalanche size, which is the total number of recorded signals during the avalanche. If there is only one electrode detecting a signal in each time bin of an avalanche, its size and duration are equal. However, the size of an avalanche is usually larger than its duration due to the simultaneous detection of signals by different electrodes. Histograms of the avalanche size show an apparent *power-law distribution of sizes*, with common small avalanches and rare large avalanches [11, 12], the fingerprint of scale-free phenomena. The exponent of the power law was observed to be $-3/2$ in [11]. However, its exact value was found to be significantly sensitive to the choice of time bin size [159] and spatial sub-sampling of the neural network [158].

The observation of this power-law distribution led to the hypothesis that neuronal avalanches can be modelled adequately by models in the class of Self-Organised Criticality (SOC) [11, 159, 24] or models of critical branching processes [11, 96, 210, 215], because both are avalanche models showing power-law distribution of the avalanche size [162, 77] (Chapter 2). In this article, we focus on modelling neuronal avalanches as branching processes.

A *branching process* is a stochastic process in which a particle can either randomly create identical copies of itself or spontaneously go extinct, triggering an avalanche of particles [203, 99, 4, 149, 209, 132, 123, 179, 209, 63, 87, 174, 77] (Chapter 2). The particle number N in the branching process is interpreted as the number of electrodes that detect a signal. The creation of a new particle is interpreted as the change of an electrode from no detection to the detection of a signal [11], while the extinction of a particle corresponds to the change from detection to no detection [156, 159, 215]. Branching processes display a phase transition from asymptotic extinction with probability 1 to asymptotic survival with a positive probability, depending on the average number of created particles. At the critical point, the avalanche size of a branching process is power-law distributed [77] (Chapter 2).

Recently, there has been evidence that neuronal activity, when compared to branching processes, is not at criticality but in a reverberating regime close to criticality [212, 213, 215]. The reason for the strong interest in the system's distance to a critical point is the criticality hypothesis, which states that information processing in the brain might be optimized by the cortical network being close to a critical point [96, 10, 210, 192, 215]. However, fitting power laws is notoriously difficult [88, 213] and other means of verifying the criticality of the neuronal avalanches are essential. For this reason, the *avalanche shape*, defined as the average temporal profile of the avalanches, has received more attention in recent years. It is debated whether, at criticality, the neuronal avalanche shape takes the universal form of an inverse parabola [143, 70, 117, 169, 87, 77, 134] (Chapter 2), which is the case of a critical branching process. The universality of this shape has been particularly challenged in [134] by observations of oscillations that modulate the avalanche shape. In [134], the oscillations are identified as γ -oscillations, which are a particular frequency band of brain waves between 30 and 100Hz. Brain waves are electric oscillations spanning the entire brain that can be recorded using electroencephalography (EEG) [16, 26]. The oscillations are organized into bands covering frequencies between 0.05Hz (slow 4 band) and 500Hz (ultra fast band). Their power spectrum follows approximately a $1/f$ distribution [152].

The observation of oscillations in the EEG activity on the one hand, and the observation of scale-free avalanches of LFP activity on the other hand raises the question of how these two seemingly incompatible descriptions of the same phenomena can be reconciled. The experimental data in [134] indicates that γ -oscillations modulate the avalanche shape. Can branching processes incorporate oscillations as well? Will those oscillations modulate the avalanche shape, widely regarded as a universal feature?

To answer these questions, we extend in this paper the field theory in [77] (Chapter 2) by incorporating oscillatory extinction rates, Sec. 3.2. We then calculate observables such as moments of the particle number, Sec. 3.3, its covariance, Sec. 3.4.1, survival probability, Sec. 3.4.2, and the avalanche shape (or temporal profile), Sec. 3.4.3 and compare them qualitatively to experimental results from [134]. We conclude in Sec. 3.5.

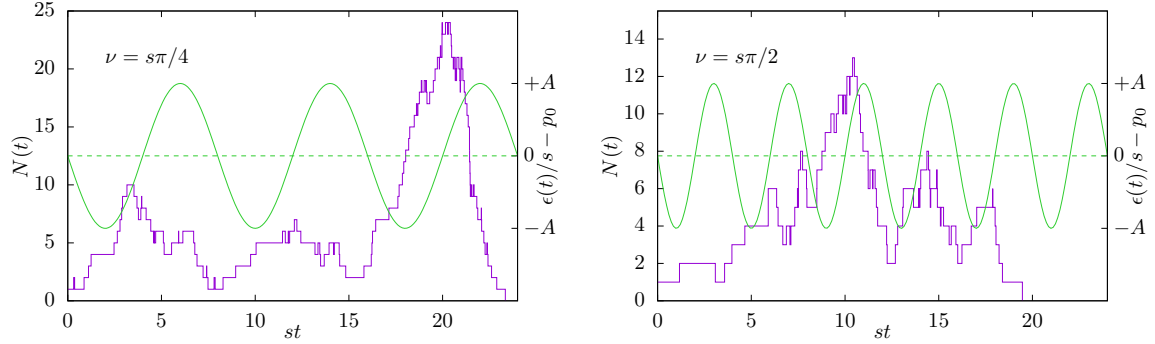


Figure 3.1: Two example trajectories of the number of particles $N(t)$ (purple, left ordinate) of branching processes with $r = 0$ and periodically varying extinction rate $\epsilon(t) = s(p_0 - A \sin(\nu t))$, Eq. (3.4). Both trajectories were initialized with one particle at $t_0 = 0$. The perturbation of the extinction rate, $-A \sin(\nu t)$ with $\nu/s \in \{\pi/4, \pi/2\}$, $A = 0.5$ is shown in green (right ordinate). When the extinction rate is lower, branching is effectively promoted.

3.2 MODEL

A branching process in continuous time t can be regarded as a reaction of a single particle type B that splits into $K \in \mathbb{N}_0$ copies of itself,



with Poissonian rate s [99, 4, 149]. The particle number at time t , which is also called population size, is denoted by $N(t)$. In the event of a reaction, the population size increases by $K - 1$ particles. The waiting time for an individual particle to undergo any such reaction is exponentially distributed with rate s . The number of offspring K may be a random variable itself, with probability distribution $P(K = k) = p_k$. We call the reaction in (3.1) a branching event if $K > 1$ and an extinction event if $K = 0$. Throughout the present work we initialize the process with one particle at time $t_0 = 0$, so that $N(0) = 1$. The totality of a realisation of a branching process, from the initialization until the termination after which the population size remains 0 indefinitely is referred to as an avalanche.

In [77] (Chapter 2), we introduced a Doi-Peliti field theory for the continuous-time branching process with time-independent but arbitrary offspring distribution p_k . We refer to such a branching process as *standard branching process*. The action functional

of this field theory is

$$\mathcal{A}_0[\phi, \tilde{\phi}] = \int \left\{ \tilde{\phi} \left(-\frac{d}{dt} - r \right) \phi + \sum_{j=2}^{\infty} q_j \tilde{\phi}^j \phi \right\} dt, \quad (3.2)$$

with

$$r = s(1 - \mathbb{E}[K]) \text{ and } q_j = s \sum_{k=j}^{\infty} \binom{k}{j} p_k, \quad (3.3)$$

and time-dependent Doi-shifted annihilation and creation fields $\phi(t)$ and $\tilde{\phi}(t)$ respectively. The parameter r is called the mass in the context of field theories and is, according to Eq. (3.3), closely linked to the first moment of the offspring distribution $\mathbb{E}[K]$. Traditionally [99], the branching process is called *sub-critical* if $\mathbb{E}[K] < 1$ (and thus $r > 0$), it is called *critical* if $\mathbb{E}[K] = 1$ (and thus $r = 0$), and *super-critical* if $\mathbb{E}[K] > 1$ (and thus $r < 0$).

The time-scale of this branching process is set by s , the rate with which *any* single-particle event takes place. The rate with which any particle spontaneously disappears, the extinction rate, is thus $\epsilon = sp_0$.

In the following we will focus on binary branching processes, *i.e.* $p_k = 0$ for all k except $k = 0$ and $k = 2$, so that p_2 and $p_0 = 1 - p_2$ in Eq. (3.3) are determined by $r/s = 1 - 2p_2$. The rest of parameters follow then immediately, such as $\mathbb{E}[K] = 2p_2$, $q_2/s = p_2 = 1/2(1 - r/s)$ and $q_j = 0$ for $j \geq 3$, as well as bounds such as $s \geq r + q_2 \geq 0$. In particular $r = \epsilon - q_2$, so that the branching process is critical if q_2 balances the extinction rate. Extensions to branching processes with other time-independent offspring distributions are straight forward [77] (Chapter 2).

Our extension to this model (3.2) consists of a *time-dependent*, oscillating extinction rate

$$\epsilon(t) = s(p_0 - A \sin(\nu t)), \quad (3.4)$$

where ν is the frequency of the oscillations. The dimensionless amplitude A is the small parameter of the field theoretic perturbation theory about $A = 0$. In the following, $r \geq 0$ is considered. The magnitude $|A|$ is bounded by p_0 because $\epsilon(t)$, as an extinction rate, has to be non-negative at all times. If the amplitude A is positive, then the extinction rate is initially suppressed, favouring more branching events. In the field

theory, this extension amounts to adding the term (details in App. 3.A),

$$As \int \left\{ \tilde{\phi}(t) \phi(t) \sin(\nu t) \right\} dt, \quad (3.5)$$

to the original action $\mathcal{A}_0$, Eq. (3.2), where $r = sp_0 - sp_2$ is still the "bare mass". Our extension can be summarised as a standard branching process where binary branching takes place with rate $q_2 = sp_2$ and extinction with rate $\epsilon(t)$, Eq. (3.4). We retain the parameterisation $p_0 + p_2 = 1$.

It will turn out that the perturbation remains noticeable at all times in the process in all observables considered. In particular, even at criticality, the shape of the avalanche, Sec. 3.4.3, carries a clear signature of the oscillations despite its expected universality [143]. The model and the analysis in the present work therefore provide an explanation for the shape of the temporal profile of neuronal avalanches recently reported by Miller *et al.* [134].

Figure 3.1 shows two example trajectories of the population size $N(t)$, together with the perturbation of the extinction rate, $\epsilon(t)/s - p_0 = -A \sin(\nu t)$. In all figures, the latter is shown in green with the ordinate on the right. All data in this work is presented in dimensionless form, in particular time as st .

In the following, we state the central results, whose field-theoretic derivation is relegated to the Appendix. In particular, we consider the avalanche shape at criticality, Sec. 3.4.3, which is a common observable in LFP recordings of the brain [11, 12, 158, 159] and the covariance, Sec. 3.4.1, which recently gained more attention in neuroscience as a tool to estimate the system's distance to the critical point [212, 213, 215].

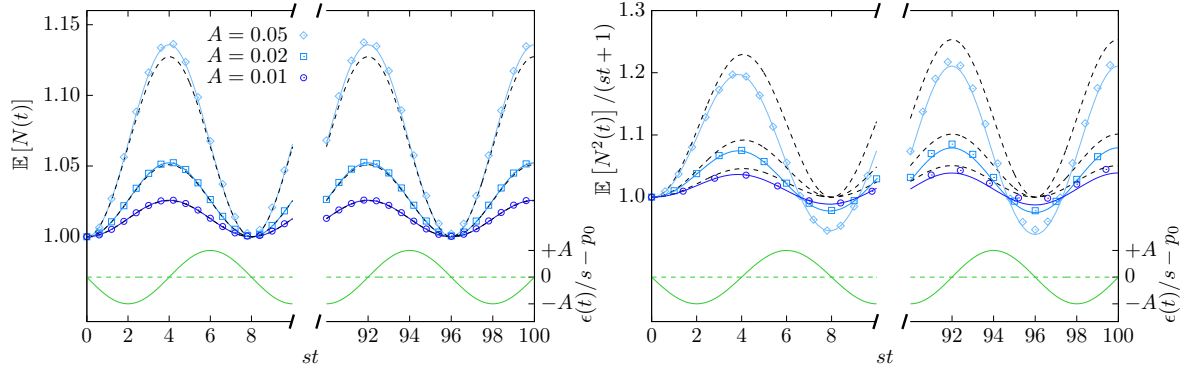


Figure 3.2: First and second moment of particle number for $A \in \{0.01, 0.02, 0.05\}$, $\nu/s = \pi/4$ and $r = 0$. The system was initialized with a single particle at $t_0 = 0$. Symbols: simulation results. Left: First moment (10^9 realizations). Right: Rescaled second moment (10^7 realizations). Full blue lines: exact analytic predictions, Eq. (3.6b) (left), Eq. (3.7) (right). Dashed black lines: analytic approximation, see Eqs. (3.10), (3.12), (3.16a), and Eq. (3.16b). The perturbation of the extinction rate, Eq. (3.4), is shown in green (right ordinate).

3.3 MOMENTS

3.3.1 FIRST MOMENT

Since the extinction rate varies periodically, we expect that the first moment varies accordingly. As shown in appendix 3.B.1, the expected particle number is,

$$\mathbb{E}[N(t)] = \Theta(t) \exp \left(-rt + \int_0^t A s \sin(\nu t') dt' \right) \quad (3.6a)$$

$$= \Theta(t) \exp \left(-rt - \frac{As}{\nu} (\cos(\nu t) - 1) \right), \quad (3.6b)$$

where $\Theta(t)$ is the Heaviside function reflecting the initialization at $t_0 = 0$, henceforth dropped from all expressions. This result is consistent with the known result of an inhomogenous Poisson process with a time-dependent event rate [111].

Fig. 3.2 shows an estimate of the first moment based on Monte-Carlo simulations together with the analytical result Eq. (3.6b) for $r = 0$. In addition to this exact result, the dashed line in Figure 3.2 shows the first order approximation $\mathbb{E}[N(t)] = g_1^{(0)}(0, t) + A g_1^{(1)}(0, t) + \mathcal{O}(A^2)$, Eq. (3.10), which is discussed below in Sec. 3.B.3.

Because the extinction rate is reduced during the first half-period, Eq. (3.4), branching dominates the process and the population size increases at first. During the second half-period, extinction dominates and the population size decreases, perfectly balancing, on average, the creation of particles in the first half of the period. According to Eq. (3.6b) and Figure 3.2, the expected population size $\mathbb{E}[N(t)]$ at $r = 0$ is always equal or larger than unity, which is the expected number of particles of a standard branching process at criticality. Thus, while the extinction rate is not shifted on average, the expected particle number is on average larger than in the process without oscillations.

Just like in a standard branching process, $\mathbb{E}[N(t)]$ converges to 0 for $r > 0$ and diverges for $r < 0$, indicating that the present process is critical at $r = 0$. However, unlike a standard branching process, the expected population size never converges for $r = 0$ as the oscillations never cease. It will turn out that the effective mass acquires no shift due to the perturbation.

3.3.2 SECOND MOMENT

The second moment can be calculated in closed form from a convolution integral involving the first moment only, App. 3.B.2. From Eqs. (3.31b), (3.32) and (3.34),

$$\begin{aligned} \mathbb{E}[N^2(t)] = & \exp\left(-rt - \frac{As}{\nu}(\cos(\nu t) - 1)\right) \\ & \times \left[1 + 2q_2 \int_0^t \exp\left(-r(t-t') - \frac{As}{\nu}(\cos(\nu t) - \cos(\nu t'))\right) dt'\right]. \end{aligned} \quad (3.7)$$

At $r = 0$ the second moment is to first order in A

$$\mathbb{E}[N^2(t)] = 1 + 2q_2 t + \frac{As}{\nu} \left[1 + 2q_2 t + \frac{2q_2}{\nu} \sin(\nu t) - (1 + 4q_2 t) \cos(\nu t)\right] + \mathcal{O}(A^2) \quad (3.8)$$

which diverges asymptotically linear in t . Because $\mathbb{E}[N(t)]$ is bounded, the variance also diverges linearly in t . Figure 3.2 shows the ratio $\mathbb{E}[N^2(t)]/(1 + 2q_2 t)$, which illustrates the deviation of the second moment at $A > 0$ from that at $A = 0$. For large t , the second moment shows a linear increase with an amplitude that oscillates mildly with period ν , so that $\mathbb{E}[N^2(t)]$ oscillates around the $A = 0$ behaviour. For large $q_2 t$ the ratio $\mathbb{E}[N^2(t)]/(1 + 2q_2 t)$ is to leading order $1 - (As/\nu)(2\cos(\nu t) - 1)$.

Although the extinction is not shifted on average, the second moment is shifted on average due to the oscillations, see Fig. 3.2.

3.3.3 N-TH MOMENT

In principle, all moments of $N(t)$ can be calculated exactly, following the same lines as in the previous Sec. 3.3.2 and in App. 3.B.2. However, even the second moment involves an integral that cannot be carried out in closed form and, for higher moments, the procedure quickly becomes unpleasantly complicated. To calculate higher moments of the population size $\mathbb{E}[N^n(t)]$, we resort to a perturbative expansion in powers of the amplitude A , which can be systematically expressed in terms of diagrams shown in the appendix.

Following [77] (Chapter 2), all moments can be written in terms of factorial moments, which are naturally produced by the diagrammatics of the field theory. We denote the n -th factorial moment by $g_n(t_0, t)$, so that

$$\mathbb{E}[N^n(t)|N(t_0) = 1] = \sum_{\ell=0}^n \left\{ \begin{matrix} n \\ \ell \end{matrix} \right\} g_\ell(t_0, t), \quad (3.9)$$

where $\left\{ \begin{matrix} n \\ \ell \end{matrix} \right\}$ are the Stirling numbers of the second kind. In the following, the factorial moments are expressed in orders of A ,

$$g_n(t_0, t) = g_n^{(0)}(t_0, t) + A g_n^{(1)}(t_0, t) + \mathcal{O}(A^2). \quad (3.10)$$

The n -th factorial moment at $A = 0$, given by

$$g_n^{(0)}(t_0, t) = n! e^{-r(t-t_0)} \left(\frac{q_2}{r} (1 - e^{-r(t-t_0)}) \right)^{n-1}, \quad (3.11)$$

was calculated in closed form from the diagrams as a matter of combinatorics [77] (Chapter 2). $g_n^{(0)}$ is dominated by $(q_2(t - t_0))^{n-1}$ for small $r(t - t_0) \ll 1$, while it is exponentially decaying for large $r(t - t_0) \gg 1$. In the present work, the factorial moments acquire a dependence on the initial time t_0 , when $N(t_0) = 1$. Only to zeroth order in A , at $A = 0$, do the factorial moments in the present work become time-

homogeneous and reduce to those calculated in [77] (Chapter 2), $g_n^{(0)}(t_0, t) = g_n^{(0)}(0, t - t_0)$. The next order term in the small- A expansion equals

$$g_n^{(1)}(t_0, t) = g_n^{(0)}(t_0, t)s \int_0^{t-t_0} \sin(\nu(t-t')) \left(1 + (n-1) \frac{e^{-rt'} - e^{-r(t-t_0)}}{1 - e^{-r(t-t_0)}} \right) dt', \quad (3.12)$$

whose derivation is explained in Appendix 3.B.3.

In the subcritical regime $r > 0$, in large $r(t - t_0) \gg 1$ all moments vanish exponentially, because $g_n^{(0)}$ vanishes exponentially, see Eq. (3.11). For small $r(t - t_0) \ll 1$ and large $q_2(t - t_0) \gg 1$, the moments are dominated by the largest factorial moment,

$$\text{for } rt \ll 1 : \quad \lim_{t \rightarrow \infty} \frac{\mathbb{E}[N^n(t)]}{g_n(0, t)} = 1. \quad (3.13)$$

This can be seen by expanding $\mathbb{E}[N^n(t)]$ in terms of the factorial moments which are asymptotically dominated by $g_n(t_0, t) \sim \mathcal{O}((q_2 t)^{n-1})$,

$$\mathbb{E}[N^n(t)] = g_n(0, t) + \left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} \underbrace{g_{n-1}(0, t)}_{\sim \mathcal{O}((q_2 t)^{n-2})} + \dots \quad (3.14)$$

Within the small- A expansion of the n -th factorial moment, the terms $g_n^{(0)}$ and $g_n^{(1)}$ dominate $\mathbb{E}[N^n(t)]$, such that

$$\mathbb{E}[N^n(t)] = g_n^{(0)}(0, t) + A g_n^{(1)}(0, t) + \mathcal{O}(A^2 (q_2 t)^{n-1}) + \mathcal{O}((q_2 t)^{n-2}) \quad (3.15)$$

However, from Eq. (3.12) it can be seen that the oscillations in the amplitude of $g_n^{(0)}(t_0, t)$ in $g_n^{(1)}(t_0, t)$ will never cease, so that the limit $\lim_{t \rightarrow \infty} \mathbb{E}[N^n(t)] / (q_2(t - t_0))^{n-1}$ strictly does not exist. In other words, $g_n(t_0, t)$ captures the leading order of $\mathbb{E}[N^n(t)]$ in t , but $(q_2(t - t_0))^{n-1}$ does not.

The first two moments at $r = 0$ can be approximated to first order in A by

$$\mathbb{E}[N(t)] = 1 - \frac{As}{\nu} (\cos(\nu t) - 1) + \mathcal{O}(A^2) \quad (3.16a)$$

$$\mathbb{E}[N^2(t)] = 2q_2 t \left(1 + \frac{As}{\nu} (1 - 2\cos(\nu t)) + \mathcal{O}(A^2) \right) + \mathcal{O}((q_2 t)^0) \quad (3.16b)$$

on the basis of Eq. (3.12) and Eq. (3.15). The expressions are consistent with the exact expressions Eq. (3.6b) and Eq. (3.7) (as given in Eq. (3.8)) respectively. Fig. 3.2 shows a comparison between the exact expressions (solid blue lines) and the approximation Eq. (3.16) (black dashed lines). Deviations are clearly noticeable only for large amplitudes A .

3.4 FURTHER OBSERVABLES

In the following, we analyse observables that are somewhat more involved to derive in the present framework. In particular, the avalanche shape and covariance are more of immediate interest to experimentalists because these are more accessible from LFP recordings of the brain [11, 12, 158, 159, 212, 213, 215].

3.4.1 COVARIANCE

The autocorrelation function

$$\text{Cov}(N(t_1), N(t_2)) = \mathbb{E}[N(t_1)N(t_2)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \quad (3.17)$$

is the covariance of $N(t_1)$ and $N(t_2)$ and a common way to quantify how strongly correlated data are at different points in time. If the population size at different times was independent, the autocorrelation function at times t_1 and t_2 , $t_1 \neq t_2$, would be zero (the converse does not hold in general [145]).

As shown in App. 3.C.1 the covariance can be calculated in closed form up to an integral,

$$\begin{aligned} \text{Cov}(N(t_1), N(t_2)) = & \exp\left(-rt_{\max} + \frac{As}{\nu}(\cos(\nu t_{\max}) - 1)\right) \\ & \times \left[1 + 2q_2 \int_0^{t_{\min}} \exp\left(r(t' - t_{\min}) + \frac{As}{\nu}(\cos(\nu t_{\min}) - \cos(\nu t'))\right) dt'\right] \\ & - \exp\left(-r(t_1 + t_2) + \frac{As}{\nu}(\cos(\nu t_1) + \cos(\nu t_2) - 2)\right), \end{aligned} \quad (3.18)$$

where $t_{\min} = \min\{t_1, t_2\}$ and $t_{\max} = \max\{t_1, t_2\}$.

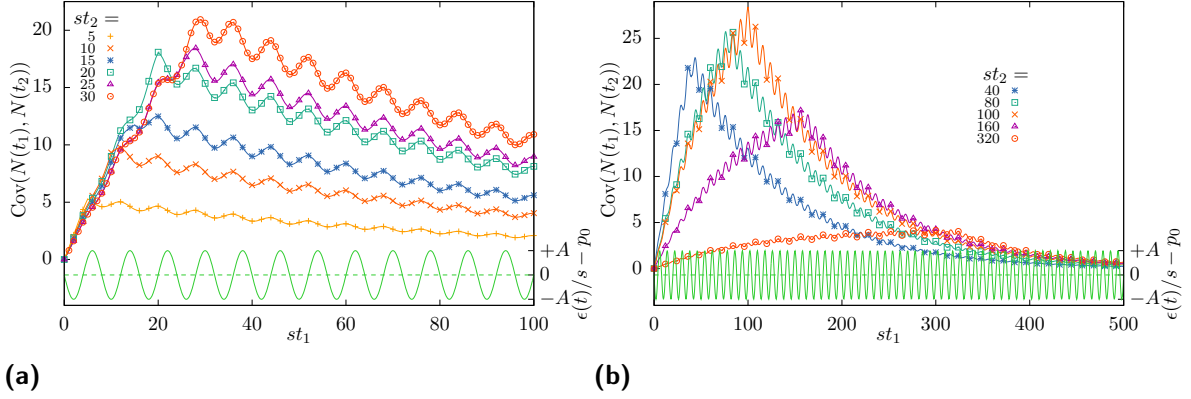


Figure 3.3: Covariance $\text{Cov}(N(t_1), N(t_2))$ Eq. (3.17) for $r/s = 0.01$, $\nu/s = \pi/4$, $A = 0.05$ and a $st_2 = 5, 10, 15, \dots, 30$, b $st_2 = 40, 80, 100, 160, 320$. The system was initialized with a single particle at time $t_0 = 0$. Symbols: Simulations results from 10^9 trajectories. Full lines: Analytic expression, Eq. (3.18). The perturbation of the extinction rate, Eq. (3.4), is shown in green at the bottom of the figure (right ordinate).

Simulation results and analytical expression Eq. (3.18) are shown together in Fig. 3.3a and Fig. 3.3b. For $A = 0$, the maximum of $\text{Cov}(N(t_1), N(t_2))$ occurs at $t_1 = t_2 = t$ and equals $(2q_2/r + 1)(\exp(-rt) - \exp(-2rt))$ [77] (Chapter 2). Furthermore, its maximum rises with increasing t , provided that $rt < \log(2)$. When $A \neq 0$, these results are only approximations and the exact position of the maximum depends on the phase and amplitude of the oscillations. In Figure 3.3a, all st_2 are chosen to fulfil that condition, while in Figure 3.3b, the larger st_2 don't meet this condition and show a decreasing maximum.

As an autocorrelation function, the covariance quantifies how the activity in the system at one instance influences the system at later instances. This has recently proven to be a valuable tool for determining the distance of neural networks from the critical point [212, 213, 215]. What the imposed oscillations imply for this tool is discussed in Sec. 3.5.

3.4.2 SURVIVAL PROBABILITY

The first moment, Sec. 3.3.1, shows that the expected number of particles is on average larger if $A > 0$, i.e. if the extinction rate drops before growing in every period, compared to the case without oscillations or reversed order of rise and fall of extinction, $A < 0$.

0. As noted in Sec. 3.3.1, the sign of the mass r still determines whether $\mathbb{E}[N(t)]$ ultimately vanishes or diverges, even when $\mathbb{E}[N(t)]$ oscillates indefinitely for $r = 0$. This observation raises the question whether the survival probability $P_s(t_0, t)$, that is the probability of $N(t) > 0$ at a given time t after initialization at t_0 , displays a corresponding behaviour.

Based on the derivation in App. 3.C.2, to leading order in A we find

$$\lim_{r \rightarrow 0} P_s(t_0 = 0, t) = \frac{1}{1 + q_2 t} + \frac{As}{\nu} \left(\frac{1}{1 + q_2 t} - \frac{\cos(\nu t) + \frac{q_2}{\nu} \sin(\nu t)}{(1 + q_2 t)^2} \right) + \mathcal{O}(A^2). \quad (3.19)$$

The first term, which is independent of A , is the probability of survival of the critical branching process with constant extinction rate [77] (Chapter 2), Figure 3.4. The second term is the first-order correction and indicates a shift of the probability of survival. For positive A (leading to an initial decrease of the extinction rate) the survival probability increases compared to the system without oscillations. For negative A (corresponding to an initial increase of the extinction rate) it decreases. For $A > 0$, the initial push into the supercritical phase seems to dominate the entire survival probability, even many oscillations later, see Fig. 3.4.

Despite the shifted survival probabilities, Fig. 3.4, and shifted average number of particles, Fig. 3.2, the avalanches do not survive indefinitely. They die eventually with probability 1 at $r = 0$. To appreciate better the effect of the extinction oscillations on the ultimate survival, we also consider the asymptotics of large t for $r > 0$ or equivalently $p_0 > p_2$,

$$\text{for } rt \gg 1 \quad P_s(0, t)|_{r>0} \simeq \frac{\exp(-rt)}{1 + q_2/r} \left(1 + \frac{As}{\nu} (1 - \cos(\nu t)) + \mathcal{O}(A^2) \right) \xrightarrow{t \rightarrow \infty} 0 \quad (3.20)$$

to leading order.

For $r < 0$, or equivalently $p_0 < p_2$, the limit is positive,

$$\lim_{t \rightarrow \infty} P_s(t_0, t)|_{r<0} = -\frac{r}{q_2} \left(1 + \frac{As\nu}{r^2 + \nu^2} \right) + \mathcal{O}(A^2), \quad (3.21)$$

where we have to rely on Eq. (3.55) being the analytic continuation for the result obtained at positive mass r . Eq. (3.20) implies that $P_s(t_0, t)$, with or without oscillations,

vanishes in the limit of large times as extinction prevails, since $r > 0$. Eq. (3.21) indicates that the linear increase in $-r$ of the ultimate survival probability is present with or without oscillations, however, that the amplitude of that increase depends on A . As far as the frequency ν is concerned, the effect of the oscillations on the ultimate survival is most pronounced for $\nu = \pm r$,

$$\lim_{t \rightarrow \infty} P_s(t_0, t)|_{r < 0} = -\frac{r}{q_2} \pm \frac{As}{2q_2} + \mathcal{O}(A^2), \quad (3.22)$$

with the minimum attained if $\nu = r$ and the maximum for $\nu = -r$. It is noteworthy that Eq. (3.22) no longer vanishes as $r \rightarrow 0$. The constraints mentioned above, such as $As \leq r + q_2$, do not affect this result, as $As = q_2/2$ still produces $\lim_{r \rightarrow 0^-} \lim_{t \rightarrow \infty} P_s(t_0, t)|_{r < 0} = 1/4$. Together with Eq. (3.19) this seems to suggest the possibility of a sudden onset survival, whereby $\lim_{t \rightarrow \infty} P_s(t_0, t)$ jumps from 0 at $r \rightarrow 0$ to a finite value.

However, it is crucial in which order limits are taken. Eq. (3.19) remains valid if $r \rightarrow 0$ (tying $\nu = \pm r$) *before* $t \rightarrow \infty$. When taken in this order, the ultimate survival probability is zero at the critical point.

We therefore conclude that the ultimate survival increases linearly and continuously from 0 for $r \geq 0$, Eqs. (3.19) and (3.20), to a finite value at $r < 0$, Eq. (3.21). If the critical point of the present process is defined as the onset of ultimate survival, then it is not shifted by the oscillations. Fig. 3.4 shows a comparison of the first-order corrected survival probability Eq. (3.19) as a function of time to simulation results.

3.4.3 AVALANCHE SHAPE

SHAPE DEPENDING ON TIME OF DEATH T

The (temporal) shape of the avalanche $V(t, T)$ is the expectation of $N(t)$ conditioned to the branching process going spontaneously extinct at some fixed termination time $t = T$, *i.e.* for all small $\delta > 0$, $N(T - \delta) > 0$ and $N(T + \delta) = 0$. Typically, time is rescaled to $\tau = t/T$ [116].

After suitable normalisation [77] (Chapter 2) the time-homogeneous version of the process processes displays a universal, parabolic shape. Not least because of its universality, the shape has gained some popularity to serve as a fingerprint of a process [56, 143, 5].

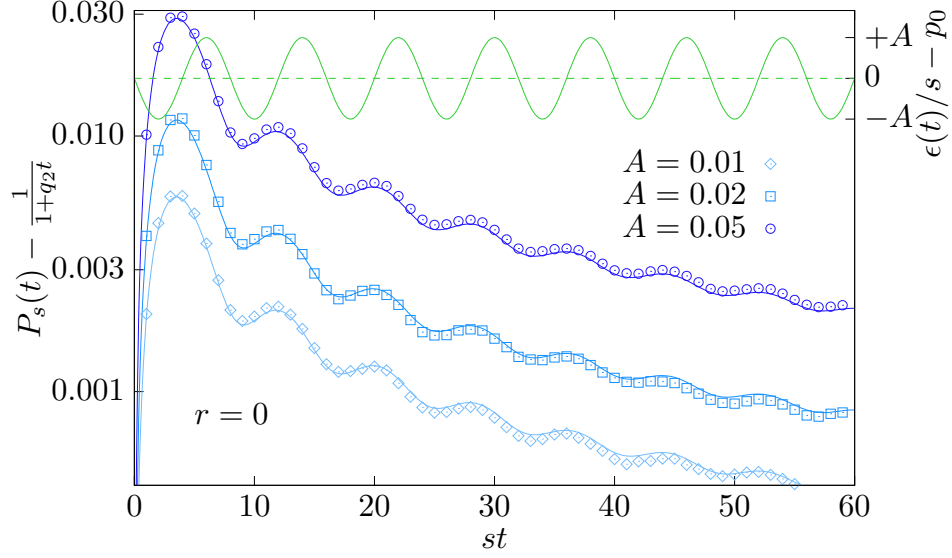


Figure 3.4: Difference between the probability of survival $P_s(0, t)$ at $r = 0$, Eq. (3.19) with and without oscillations, for $\nu/s = \pi/4$ and $A \in \{0.01, 0.02, 0.05\}$. The system was initialized with one particle at time $t_0 = 0$. Symbols: Simulations results using $3 \cdot 10^7$ trajectories for $A = 0.01$ and 10^7 otherwise. Full blue lines: Analytic prediction to first order in A . Full green lines: Perturbation of the extinction rate, Eq. (3.4) (right ordinate).

As shown in Figure 3.5, the periodic extinction imposes characteristic humps on the shape. They are of course rooted in the periodic extinction, as an analytical calculation shows, App. 3.C.3. These oscillations remain visible in the shape $V(t, T)$ at all T and all t , but become particularly vivid whenever the termination time T is commensurate with the period of the oscillations, $2\pi/\nu$. As we have used our field-theoretic scheme only to first order, the slight mismatch with simulation results at larger A , such as those shown in Figure 3.5b, is not surprising. However, at such large amplitudes, the resulting shape resembles that of recent experimental results, where γ -oscillation modulated the average shape of neuronal avalanches [134]. Furthermore, for larger amplitudes A , the avalanche shapes are asymmetric, which can be seen particularly well in Fig. 3.5b for $sT = 16$.

COMPARISON TO EXPERIMENTS

The mean avalanche shapes, such as in Figure 3.5 can be qualitatively compared to avalanche profiles recorded in the brain. We reproduce in Fig. 3.6 plots from Miller, Yu

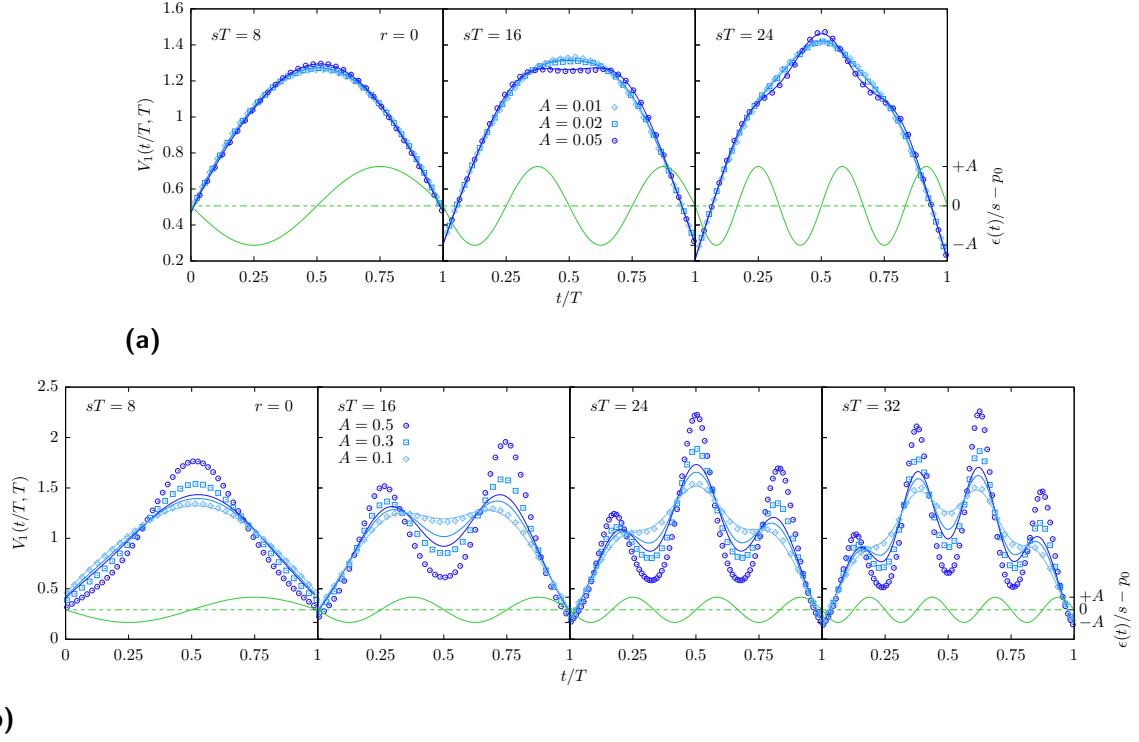


Figure 3.5: Area-normalised expected avalanche shapes $V_1(t, T)$, Eq. (3.66), for $r = 0$, $\nu/s = \pi/4$, (a) $sT \in \{8, 16, 24\}$ for $A \in \{0.01, 0.02, 0.05\}$ and (b) $sT \in \{8, 16, 24, 32\}$ for $A \in \{0.1, 0.3, 0.5\}$. The time is rescaled by the termination time T . The system was initialized with one particle at time $t_0 = 0$. Symbols: Simulation results obtained by averaging over 10^8 trajectories $N(t)$ with a termination time $sT \pm 0.2$. Full blue lines: Analytic prediction to first order in A . Full green lines: Perturbation of the extinction rate, Eq. (3.4) (right ordinate).

and Plenz (2019) (Figure 4(a) in reference [134]), and adapt it by overlaying our analytical plots from Fig. 3.5b, in accordance with the Creative Commons Attribution 4.0 International License. The data was collected through multielectrode arrays implanted in three adult nonhuman primates (see [134] for details).

The plots show qualitatively a good agreement between the data and our model. The general shape and periodicity is captured extremely well. Small qualitative disagreement occurs in the third and fourth plot in the relative heights of consecutive maxima and minima. This disagreement may be down to higher order correction terms in the amplitude A .

The figure illustrates that branching processes, which are commonly used to explain the statistical properties of avalanches recorded in the brain, can be extended to also

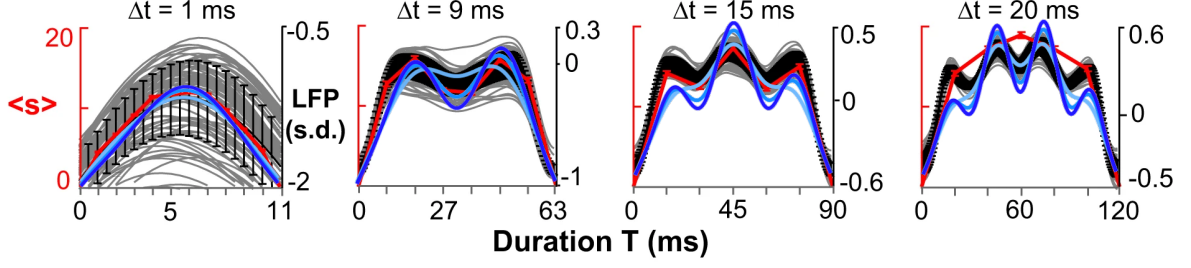


Figure 3.6: Comparison of model to experimental data. Underlying figure (grey, black and red colors) from Miller, Yu and Plenz (2019) [134], Fig. 4(a), reproduced and adapted under Creative Commons Attribution 4.0 International License. Adaptation: Overlay of blue colors: analytical results from Fig. 3.5b above. On y -axis, $\langle s \rangle$ is the mean profile in [134], corresponding to our $V(t, T)$; on x -axis, duration T from [134] is denoted by time t in our article. Data was collected through multielectrode arrays implanted in adult nonhuman primates (see [134] for details). Grey lines, single electrode data, black line mean of array, red mean size-per-timestep.

incorporate neuronal oscillations. Although the observation of non-universal avalanche shapes questioned the criticality hypothesis of the brain [10, 214, 134], our analytical results clearly show that criticality is compatible with avalanche profiles that are modulated by oscillations.

UNIVERSAL PARABOLIC SHAPE

The universal parabolic shape may be recovered by suitably averaging across different termination times T . Devising such a scheme in a given numerical or experimental setting may not always be feasible [211]. Figure 3.7 shows the T -averaged and T -rescaled expected avalanche shape,

$$\bar{V}(\tau) = \frac{\int_0^\infty P_T(T) V(\tau T, T) dT}{\int_0^1 \int_0^\infty P_T(T) V(\tau T, T) dT d\tau}. \quad (3.23)$$

Varying small amplitudes do not seem to alter this shape and appear to converge to the universal parabola shape when criticality is approached $r \rightarrow 0$.

3.5 DISCUSSION AND CONCLUSION

Our discussion is two-fold: First, we focus on aspects of interest to research in stochastic processes. We then follow with a discussion on the implications for research in the area

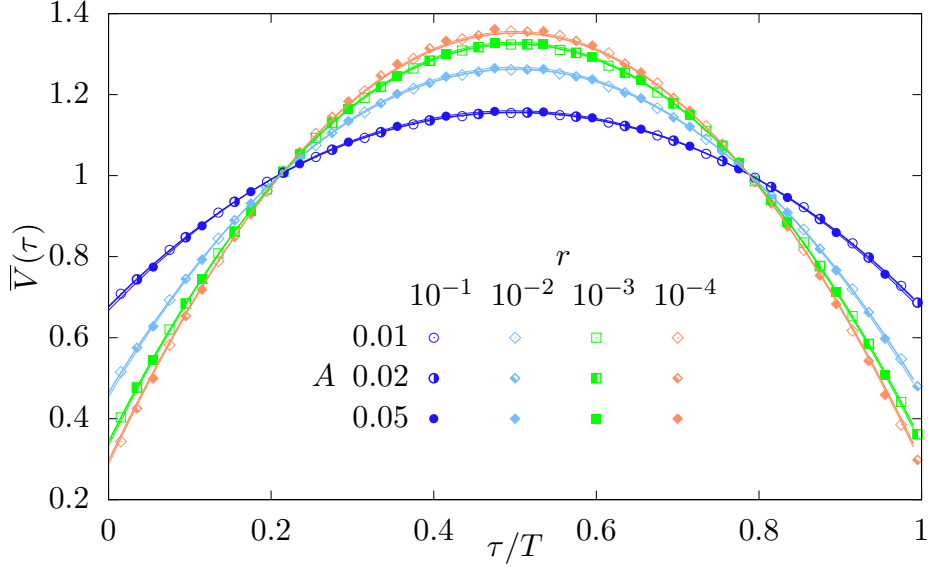


Figure 3.7: Area-normalized expected avalanche shapes averaged over time of death T , $\bar{V}(\tau)$, Eq. (3.23), for $r \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$, $\nu/s = \pi/4$ and $A \in \{0.01, 0.02, 0.05\}$. Before averaging, the time of each avalanche is rescaled by its termination time T . The system was initialized with one particle at time $t_0 = 0$. Symbols: Simulations results obtained by averaging over 10^7 trajectories $N(t)$. Full lines: Analytic prediction to first order in A .

of neuronal avalanches.

In this paper, we extend the standard branching process by including time-dependent, deterministic variations of a reaction rate. While we focus on varying the extinction rate only, our approach is applicable to any reaction rate in a Doi-Peliti field theory.

Neuronal avalanches can show oscillating behaviour, which we capture with a branching process model with oscillating extinction rate. These oscillations can be observed in any moment, of which we present the zeroth (the survival probability), first and second, and in any correlation function, of which we present the two-time covariance. All of these observables are calculated exactly. Furthermore, we introduce an approximation scheme for factorial moments to first order in the amplitude of the extinction rate oscillations. This allows approximating more complicated observables such as the survival probability and the avalanche shape in the limit of small oscillation amplitudes.

All analytically calculated observables are compared with simulation results from Monte Carlo simulations. While the exact analytic results match perfectly, we also evaluate the first-order approximation scheme and find good agreement in the limit of small

oscillation amplitudes.

Although the extinction rate is unchanged on average, its oscillation leads *on average* to a shift in the moments. For example, if the oscillations start with a decrease of the extinction rate, the expected particle number is always greater compared to the process without oscillations. Conversely, if the oscillations start by increasing the extinction rate, the expected number of particles is always lower than without oscillations. Despite the average shift of particle numbers, the onset of indefinite survival is not shifted by the oscillations. We therefore conclude that the critical point remains unchanged.

Both subcritical branching processes and neuronal activity show exponentially decaying auto-correlation functions whose decay rate is independent of the spatial sub-sampling of the neural network [215]. In particular, if the neural system is close to the critical point, in the reverberating regime, the exponential decay is slow and can be observed over more than 100ms even in a single neuron’s activity [215]. As the oscillations in the neuronal activity can have periods well below 100ms (*e.g.* in the spindle band), they should be visible in the data.

Similarly, both the oscillating branching process presented here and recordings of neuronal avalanches show oscillating avalanche shapes [134]. In our model and in the data, the oscillations modulate the shape and are most pronounced when the avalanche length is an integer multiple of the period of the oscillations. Surprisingly, these results defy the assumption that the avalanche shape is universal at criticality. However, when the shapes are also rescaled and averaged over the avalanche duration, our simulations indicate that the universal parabola shape is recovered.

In future research, the oscillating branching process should be compared quantitatively to local field potential recordings of oscillating neuronal avalanches. Thus, the presented model can contribute to the understanding of neuronal avalanches by rejecting or supporting the branching process picture of signal propagation in the brain. Furthermore, it would be interesting to find out whether other types of variation of the extinctions rate, such as random fluctuations, result in similar behaviour.

ACKNOWLEDGEMENTS

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3.A EXTENSION OF THE ACTION

The extension of the model consists of making the extinction rate time-dependent. Hence, in the following derivation, we focus on introducing the new extinction rate and ignore the branching process.

Let $P(n, t)$ be the probability that there are n particles in the system at time t , *i.e.* $N(t) = n$. Then, the master equation for the extinction process only is

$$\frac{d}{dt}P(n, t) = \epsilon(t)((n+1)P(n+1, t) - nP(n, t)). \quad (3.24)$$

The first step in the derivation of the field theory is the introduction of *bra-ket vectors* $\langle n|$ and $|n\rangle$, respectively, and *ladder operators* a and $a^\dagger$. The ket vector $|n\rangle$ represents that there are n particles in the system, while the bra vector $\langle n|$ project out the state of n particles, $\langle n|m\rangle = \delta_{n,m}$, the Kronecker function. The ladder operator $a^\dagger$ creates an additional particle, *i.e.* $a^\dagger|n\rangle = |n+1\rangle$, and the operator a annihilates a particle (with the additional prefactor n) $a|n\rangle = n|n-1\rangle$. Thus, the entire, probabilistic state of the system at time t can be represented by a ket vector $|\mathcal{M}(t)\rangle$,

$$|\mathcal{M}(t)\rangle = \sum_{n=0}^{\infty} P(n, t)|n\rangle, \quad (3.25)$$

which is nothing else than a representation of the probability generating function.

In the next step, following [57], the master equation (3.24) is used to derive a corre-

sponding equation for the state vector $|\mathcal{M}(t)\rangle$,

$$\frac{d}{dt}|\mathcal{M}(t)\rangle = \epsilon(t) (a - a^\dagger a) |\mathcal{M}(t)\rangle. \quad (3.26)$$

In the final step, based on work by Peliti [150], equations such as (3.26) can be solved by a path integral with an action $\mathcal{A}_\epsilon$, where in every normal-ordered compound operator, a is replaced by a field $\phi(t)$ and $a^\dagger$ is replaced by a field $\phi^\dagger(t)$, and where an additional time-derivative term is added:

$$\mathcal{A}_\epsilon = \int \left\{ (\phi^\dagger(t) - 1) \left(-\frac{d}{dt} \phi(t) \right) + \epsilon(t) (\phi(t) - \phi^\dagger(t) \phi(t)) \right\} dt. \quad (3.27)$$

In general, it is useful to shift the creation field $\phi^\dagger = \tilde{\phi} + 1$. An explanation of implications of the shift, as well as a more detailed derivation of the path integral can be found in [147]. Thus, a simpler action is obtained

$$\mathcal{A}_\epsilon = \int \tilde{\phi} \left(-\frac{d}{dt} - \epsilon(t) \right) \phi(t) dt. \quad (3.28)$$

The total action

$$\mathcal{A} = \int \left\{ \tilde{\phi} \left(-\frac{d}{dt} - \epsilon(t) \right) \phi(t) + \sum_{j=2}^{\infty} q_j \tilde{\phi}^j \phi \right\} dt \quad (3.29)$$

enters the path integral in the form $\exp(\mathcal{A})$ so that field theoretic expectations become

$$\langle \bullet \rangle = \int \exp(\mathcal{A}) \bullet \mathcal{D}\phi \mathcal{D}\tilde{\phi} \quad (3.30)$$

and are calculated by performing the path integral over the bilinear part of the action $\mathcal{A}_0$, Eq. (3.2), and expanding perturbatively for the term with coupling $sp_0 - \epsilon(t)$, which is unaccounted for in $\mathcal{A}_0$. In connection with the observation of oscillating neuronal avalanches, we choose $\epsilon(t) = sp_0 - A \sin(\nu t)$ and the perturbative part of the action becomes Eq. (3.5), giving rise to a perturbation theory in small A .

3.B DERIVATION OF MOMENTS

In order to keep a readable, consistent notation, we use the mathematical notation, *e.g.* $\mathbb{E}[N(t)]$, when we characterize the statistical properties of $N(t)$. When referring to field-theoretic calculations, we use the notation common in physics, *i.e.* $\langle \mathcal{O} \rangle$ for the expectation of an observable $\mathcal{O}$. Of course, the two are closely related, *e.g.* $\langle \phi(t)\phi^\dagger(t_0) \rangle = \mathbb{E}[N(t)|N(t_0) = 1]$ but $\langle \phi^2(t)\phi^\dagger(t_0) \rangle + \langle \phi(t)\phi^\dagger(t_0) \rangle = \mathbb{E}[N^2(t)|N(t_0) = 1]$, Eq. (3.32). Unless stated otherwise, observables are conditioned to the initialization of the branching process with one particle at time $t_0 = 0$, *i.e.* $N(0) = 1$, routinely omitted in the mathematical notation. For example, $\mathbb{E}[N(t)]$ is short for $\mathbb{E}[N(t)|N(0) = 1]$.

3.B.1 FIRST MOMENT

In the field theoretic calculation, the first moment is calculated as the propagator. Due to the varying extinction rate the propagator acquires corrections which schematically depicted as diagrams. The red, full lines $\overset{t_2}{\text{---}} \overset{t_1}{\text{---}}$ represent the bare propagator $\langle \phi(t_2)\tilde{\phi}(t_1) \rangle_0$, while the black dashed lines ending in circles $\overset{t'}{\text{---}} \bullet$ represent correction terms, where $t_2 \geq t' \geq t_1$. In a Doi-Peliti field theory, these diagrams are by convention read from right to left. The first moment is thus written as follows:

$$\mathbb{E}[N(t)|N(t_0) = 1] = \langle \phi(t)\phi^\dagger(t_0) \rangle = \langle \phi(t)\tilde{\phi}(t_0) \rangle$$

$$\hat{=} \text{---} + \text{---} \bullet + \text{---} \bullet \bullet + \dots \quad (3.31a)$$

$$= \Theta(t)e^{-rt} \left(1 + \int_{t_0}^t sA \sin(\nu t') dt' + \int_{t_0}^t sA \sin(\nu t') \int_{t_0}^{t'} sA \sin(\nu t'') dt'' dt' + \dots \right)$$

$$= \Theta(t)e^{-rt} \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int_{t_0}^t sA \sin(\nu t') dt' \right)$$

$$= \Theta(t) \exp \left(-r + \int_{t_0}^t sA \sin(\nu t') dt' \right), \quad (3.31b)$$

exactly as expected for an inhomogeneous Poisson process with time-varying intensity that governs the extinction [111]. For $t_0 = 0$, this reproduces the result for $\mathbb{E}[N(t)]$, Eq. (3.6b).

3.B.2 SECOND MOMENT

The second moment $\mathbb{E}[N^2(t)]$ is determined field-theoretically on the basis of the square of the particle number operator, $(a^\dagger a)^2 = (a^\dagger)^2 a^2 + a^\dagger a$, which is easily expressed in terms of fields $\phi^\dagger$ and ϕ once in normal-ordered form,

$$\mathbb{E}[N^2(t)] = \langle \phi^2(t) \tilde{\phi}(0) \rangle + \underbrace{\langle \phi(t) \tilde{\phi}(0) \rangle}_{=\mathbb{E}[N(t)]}. \quad (3.32)$$

Diagrammatically, the first term is

$$\langle \phi^2(t) \tilde{\phi}(0) \rangle \hat{=} \text{diagram} \quad (3.33)$$

which is a convolution in direct time,

$$\langle \phi^2(t) \tilde{\phi}(0) \rangle = 2q_2 \int_0^t \langle \phi(t) \tilde{\phi}(t') \rangle^2 \langle \phi(t') \tilde{\phi}(0) \rangle dt', \quad (3.34)$$

that can be calculated with the help of Eq. (3.31b) to produce Eq. (3.7).

3.B.3 n -TH FACTORIAL MOMENT TO FIRST ORDER

We define the n -th factorial moment of $N(t)$ [77] (Chapter 2) as the function

$$g_n(t_0, t) = \left\langle \phi(t)^n \tilde{\phi}(t_0) \right\rangle = g_n^{(0)}(t_0, t) + A g_n^{(1)}(t_0, t) + A^2 g_n^{(2)}(t_0, t) + \dots \quad (3.35)$$

where the additional extinction $-A \sin(\nu t)$, Eq. (3.4), is dealt with perturbatively about $A = 0$. The zeroth-order correction $g_n^{(0)}(t_0, t)$ corresponds to the sum of binary tree diagrams with one in-coming leg and n out-going legs. Since the zeroth-order correction does not include any perturbation in the extinction rate, the process is homogeneous in time and, therefore, it does not depend on the initial time t_0 , but rather on the amount

of time $t - t_0$.

In [77] (Chapter 2), the function $g_n^{(0)}(t_0, t)$ was derived from the following forest (meaning it contains a sum of tree-like diagrams)

$$g_n^{(0)}(t_0, t) \hat{=} \sum_{\ell=1}^n \sum_{m_1, \dots, m_\ell=1}^n \binom{n}{m_1, \dots, m_\ell} \text{diagram} \quad (3.36a)$$

$$\hat{=} n! e^{-r(t-t_0)} \left(\frac{q_2}{r} (1 - e^{-r(t-t_0)}) \right)^{n-1}. \quad (3.36b)$$

This zeroth-order correction satisfies the identity

$$g_n^{(0)}(t_0, t) = \sum_{\ell=1}^n g_\ell^{(0)}(t', t) \frac{1}{\ell!} \sum_{m_1=1, \dots, m_\ell=1}^n \binom{n}{m_1, \dots, m_\ell} g_{m_1}^{(0)}(t_0, t') \cdots g_{m_\ell}^{(0)}(t_0, t'), \quad (3.37)$$

which can be obtained by choosing an arbitrary time $t' \in [t_0, t]$ when a diagram has ℓ legs, *cf.* Eq. (3.39a), and reconstructing the entire forest. As the left-hand side of Eq. (3.37) does not depend on t' it might be tempting to differentiate with respect to t' , but for our purposes the sum

$$\frac{\partial}{\partial t} g_n^{(0)}(t_0, t) = \sum_{\ell=1}^n \frac{\partial}{\partial t} g_\ell^{(0)}(t', t) \frac{1}{\ell!} \sum_{m_1=1, \dots, m_\ell=1}^n \binom{n}{m_1, \dots, m_\ell} g_{m_1}^{(0)}(t_0, t') \cdots g_{m_\ell}^{(0)}(t_0, t') \quad (3.38)$$

will prove particularly useful.

We proceed with the construction of the first order correction $Ag_n^{(1)}(t)$ of $g_n(t)$. This correction contains the external perturbation $dt' A \sin(\nu t')$ once in each diagram. It needs to be attached to every tree diagram at time t' , as indicated by the vertical dotted

line in the diagram below,

$$g_n^{(1)}(t_0, t) \hat{=} \int_{t_0}^t \sum_{\ell=1}^n \sum_{m_1, \dots, m_\ell}^n \binom{n}{m_1, \dots, m_\ell} \begin{array}{c} \text{Diagram: A vertex at time } t' \text{ with } \ell \text{ legs (red lines) connecting to } \ell \text{ vertices at time } t_0. \text{ The legs are labeled } m_1, m_2, \dots, m_\ell. \text{ A dashed line connects the vertex at } t' \text{ to a black dot at time } t. \end{array} dt' \quad (3.39a)$$

$$\begin{aligned} &= s \int_{t_0}^t \sin(\nu t') \sum_{\ell=1}^n \ell g_\ell^{(0)}(t_0, t') \frac{1}{\ell!} \sum_{m_1=1, \dots, m_\ell=1}^n \binom{n}{m_1, \dots, m_\ell} g_{m_1}^{(0)}(t', t) \cdots g_{m_\ell}^{(0)}(t', t) dt' \\ &= s \int_0^{t-t_0} \sin(\nu(t-t')) \sum_{\ell=1}^n \ell g_\ell^{(0)}(0, t-t_0-t'') \frac{1}{\ell!} \sum_{m_1=1, \dots, m_\ell=1}^n \binom{n}{m_1, \dots, m_\ell} g_{m_1}^{(0)}(0, t'') \cdots g_{m_\ell}^{(0)}(0, t'') dt'', \end{aligned} \quad (3.39b)$$

where the last identity is obtained by the substitution $t'' = t - t'$ and using the time-homogeneity $g_\ell^{(0)}(t_0, t) = g_\ell^{(0)}(0, t - t_0)$. Defining the sum

$$S(t-t_0, t'') = \sum_{\ell=1}^n \ell g_\ell^{(0)}(0, t-t_0-t'') \frac{1}{\ell!} \sum_{m_1=1, \dots, m_\ell=1}^n \binom{n}{m_1, \dots, m_\ell} g_{m_1}^{(0)}(0, t'') \cdots g_{m_\ell}^{(0)}(0, t''), \quad (3.40)$$

the first order perturbation can be written as

$$g_n^{(1)}(t_0, t) = s \int_0^{t-t_0} \sin(\nu(t-t'')) S(t-t_0, t'') dt''. \quad (3.41)$$

The sum $S(t-t_0, t'')$, Eq. (3.40), would be equal to $g_n^{(0)}(t_0, t)$, Eq. (3.37), was it not for the factor ℓ in front of $g_\ell^{(0)}(0, t-t_0-t'')$, corresponding to the ℓ legs at time t' to choose from to insert the external perturbation. Nevertheless, the summation Eq. (3.40) can be carried out by using $g_n^{(0)}$ of Eq. (3.36b) as a generating function. The term $\ell g_\ell^{(0)}(t_0, t)$ can be obtained from the time-derivative of the function $g_n^{(0)}(t_0, t)$ in Eq. (3.36),

$$\frac{\partial}{\partial t} g_\ell^{(0)}(t_0, t) = r g_\ell^{(0)}(t_0, t) \frac{\ell e^{-r(t-t_0)} - 1}{1 - e^{-r(t-t_0)}}, \quad (3.42)$$

which produces the identity

$$\ell g_\ell^{(0)}(t_0, t) = e^{r(t-t_0)} \left(g_\ell^{(0)}(t_0, t) + \frac{1}{r} (1 - e^{-r(t-t_0)}) \frac{\partial}{\partial t} g_\ell^{(0)}(t_0, t) \right). \quad (3.43)$$

Rewriting $\ell g_\ell^{(0)}(0, t - t_0 - t'') = \ell g_\ell^{(0)}(t_0 + t'', t)$ in Eq. (3.40) in terms of $\frac{\partial}{\partial t} g_\ell^{(0)}(t_0, t)$ and multiples of $g_\ell^{(0)}(t_0, t)$ according to Eq. (3.43) allows the summation to be carried out, using Eq. (3.37) and Eq. (3.38), which eventually gives

$$S(t - t_0, t'') = g_n^{(0)}(0, t - t_0) \left(1 + (n - 1) \frac{e^{-rt''} - e^{-r(t-t_0)}}{1 - e^{-r(t-t_0)}} \right), \quad (3.44)$$

and therefore Eq. (3.12). For what follows, this is best re-written as

$$g_n^{(1)}(t_0, t) = g_n^{(0)}(t_0, t) \left(u(t_0, t) + (n - 1)v(t_0, t) \right) \quad (3.45)$$

with

$$u(t_0, t) = s \int_0^{t-t_0} \sin(\nu(t - t')) dt' = \frac{s}{\nu} (\cos(\nu t_0) - \cos(\nu t)) \quad (3.46a)$$

$$\begin{aligned} v(t_0, t) &= s \int_0^{t-t_0} \sin(\nu(t - t')) \frac{e^{-rt'} - e^{-r(t-t_0)}}{1 - e^{-r(t-t_0)}} dt' \\ &= \frac{s}{1 - e^{-r(t-t_0)}} \left[\frac{1}{\nu} e^{-r(t-t_0)} (\cos(\nu t) - \cos(\nu t_0)) \right. \\ &\quad \left. + \frac{1}{r^2 + \nu^2} \left(r \sin(\nu t) - \nu \cos(\nu t) + e^{-r(t-t_0)} (\nu \cos(\nu t_0) - r \sin(\nu t_0)) \right) \right]. \end{aligned} \quad (3.46b)$$

3.C FURTHER OBSERVABLES

3.C.1 COVARIANCE

The covariance is defined as

$$\text{Cov}(N(t_1), N(t_2)) = \mathbb{E}[N(t_1)N(t_2)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)], \quad (3.47)$$

where all expectations are conditioned to $N(0) = 1$. In the field theory, the term $\mathbb{E}[N(t_1)N(t_2)]$ is calculated as

$$\begin{aligned}\mathbb{E}[N(t_1)N(t_2)] &= \langle \phi(t_{\max})\phi^\dagger(t_{\min})\phi(t_{\min})\phi^\dagger(0) \rangle \\ &= \langle \phi(t_{\max})\phi(t_{\min})\tilde{\phi}(0) \rangle + \langle \phi(t_{\max})\tilde{\phi}(t_{\min})\phi(t_{\min})\tilde{\phi}(0) \rangle,\end{aligned}\quad (3.48)$$

where $t_{\max} = \max\{t_1, t_2\}$ and $t_{\min} = \min\{t_1, t_2\}$. The first term of Eq. (3.48) is a convolution similar to Eqs. (3.33) and (3.34),

$$\begin{aligned}\langle \phi(t_{\max})\phi(t_{\min})\tilde{\phi}(0) \rangle &\hat{=} \begin{array}{c} t_{\max} \\ \text{---} \diagdown \text{---} \\ \text{---} \diagup \text{---} \\ t_{\min} \end{array} \begin{array}{c} 0 \end{array} \\ &= 2q_2 \int \langle \phi(t_{\max})\tilde{\phi}(t') \rangle \langle \phi(t_{\min})\tilde{\phi}(t') \rangle \langle \phi(t')\tilde{\phi}(0) \rangle dt',\end{aligned}\quad (3.49)$$

while the second term is a product of two first moments,

$$\begin{aligned}\langle \phi(t_{\max})\tilde{\phi}(t_{\min})\phi(t_{\min})\tilde{\phi}(0) \rangle &\hat{=} \begin{array}{c} t_{\max} \quad t_{\min} \quad 0 \\ \text{---} \text{---} \text{---} \end{array} \\ &= \langle \phi(t_{\max})\tilde{\phi}(t_{\min}) \rangle \langle \phi(t_{\min})\tilde{\phi}(0) \rangle.\end{aligned}\quad (3.50)$$

At $t_1 = t_2$, when $\langle \phi(t_{\max})\tilde{\phi}(t_{\min}) \rangle = 1$, Eq. (3.48) recovers Eq. (3.32).

3.C.2 PROBABILITY OF SURVIVAL TO FIRST ORDER

The survival probability $P_s(t_0, t)$ is determined via the probability $P(N(t) = 0 | N(t_0) = 1) = 1 - P_s(t_0, t)$ that the system contains no particles at time t . Field-theoretically that is $\langle \exp(-\phi(t)) \phi^\dagger(t_0) \rangle$ [77, 147] (Chapter 2) so that

$$\begin{aligned}P_s(t_0, t) &= 1 - \langle e^{-\phi(t)} \phi^\dagger(t_0) \rangle \\ &= - \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \langle \phi^n(t) \tilde{\phi}(t_0) \rangle \\ &= \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n!} g_n(t_0, t).\end{aligned}\quad (3.51)$$

To zeroth order the summation is easily carried out, since $g_n^{(0)}(t_0, t)$, Eq. (3.36b), may be written as $g_n(t_0, t) = n! \alpha \beta^{n-1}$ with $\alpha = \exp(-r(t - t_0))$ and $\beta = q_2(1 - \exp(-r(t - t_0)))/r$,

$$P_s(t_0, t) = \alpha \sum_{m=0}^{\infty} (-\beta)^m + \mathcal{O}(A) = \frac{\alpha}{1 + \beta} + \mathcal{O}(A) , \quad (3.52)$$

as shown in Eq. (42) in [77] (Chapter 2). Using the first-order corrected $g_n(t_0, t)$, Eq. (3.45), the summation does not change significantly as far as the term $Au(t_0, t)$ is concerned, but $A(n - 1)v(t_0, t)$ requires some additional work. Recognising that Eq. (3.52) is effectively a generating function, the required sum is easily obtained from Eq. (3.52),

$$P_s(t_0, t) = \frac{\alpha}{1 + \beta} \left(1 + Au(t_0, t) \right) + \beta \frac{\partial}{\partial \beta} \frac{\alpha}{1 + \beta} Av(t_0, t) \quad (3.53)$$

To reduce clutter, we expand $u(t_0, t)$ and $v(t_0, t)$ only for the specific case of $t_0 = 0$. Using

$$P_s^{(0)}(0, t) = \frac{\exp(-rt)}{1 + \frac{q_2}{r}(1 - \exp(-rt))} \quad (3.54)$$

for the zeroth order approximation, the survival probability to first order reads

$$\begin{aligned} P_s(0, t) = & P_s^{(0)}(0, t) \left\{ 1 + As \left[\frac{1}{\nu} + P_s^{(0)}(0, t) \frac{q_2 r}{(r^2 + \nu^2) \nu} \right. \right. \\ & - \cos(\nu t) \left(\frac{1}{\nu} + P_s^{(0)}(0, t) \frac{q_2}{r} \left(\frac{1}{\nu} - \frac{\nu}{\exp(-rt)(r^2 + \nu^2)} \right) \right) \\ & \left. \left. - \sin(\nu t) \left(P_s^{(0)}(0, t) \frac{q_2}{\exp(-rt)(r^2 + \nu^2)} \right) \right] \right\} + \mathcal{O}(A^2) . \end{aligned} \quad (3.55)$$

3.C.3 AVALANCHE SHAPE TO FIRST ORDER

In [77, 147] (Chapter 2), it is shown that the avalanche shape (temporal profile) of a branching process can be related to the incompletely normalised average

$$\mathbb{E}[N(t); N(T) = 0 | N(t_0) = 1] = \sum_N NP(N(t) = N, N(T) = 0 | N(t_0) = 1) , \quad (3.56)$$

which is the average population size N at time t taken over the joint probability $P(N(t) = N, N(T) = 0 | N(t_0) = 1)$ of having size N at time t and size 0 at time T , conditioned to having size 1 at time t_0 . This quantity is easily captured in the field theory,

$$\begin{aligned} \mathbb{E}[N(t); N(T) = 0 | N(t_0) = 1] & \\ &= \langle e^{-\phi(T)} \phi^\dagger(t) \phi(t) \phi^\dagger(t_0) \rangle \\ &= \langle \phi(t) \tilde{\phi}(t_0) \rangle + \sum_{n=1} \frac{(-1)^n}{n!} \left(\langle \phi^n(T) \phi(t) \tilde{\phi}(t_0) \rangle + \langle \phi^n(T) \tilde{\phi}(t) \phi(t) \tilde{\phi}(t_0) \rangle \right). \end{aligned} \quad (3.57)$$

Using identities (B2) and (B3) of [77] (Chapter 2) in the form

$$\begin{aligned} \langle \phi^n(T) \phi(t) \tilde{\phi}(t_0) \rangle &= \sum_{k=1}^n \sum_{m_1=1, \dots, m_k=1}^n \binom{n}{m_1, \dots, m_k} \\ &\quad \times g_{m_1}(t, T) \dots g_{m_k}(t, T) g_{k+1}(t_0, t) \frac{1}{k!} \end{aligned} \quad (3.58)$$

and

$$\begin{aligned} \langle \phi^n(T) \tilde{\phi}(t) \phi(t) \tilde{\phi}(t_0) \rangle &= \sum_{k=1}^n \sum_{m_1=1, \dots, m_k=1}^n \binom{n}{m_1, \dots, m_k} \\ &\quad \times g_{m_1}(t, T) \dots g_{m_k}(t, T) g_k(t_0, t) \frac{1}{(k-1)!}, \end{aligned} \quad (3.59)$$

and writing $g_n(t_0, t) = n! \alpha \beta^{n-1}$ and $g_n(t, T) = n! a b^{n-1}$, Eq. (3.57) can be written at $A = 0$ after some tedious algebra

$$\begin{aligned} \mathcal{S} &= \mathbb{E}[N(t); N(T) = 0 | N(t_0) = 1] \\ &= \alpha - \frac{a\alpha}{1+b(1+\frac{a\beta}{b})} \left[1 + \beta \left(2 - \frac{a(1+\beta)}{1+b(1+\frac{a\beta}{b})} \right) \right]. \end{aligned} \quad (3.60)$$

To zeroth order in A , we recover the result in [77] (Chapter 2) with $\alpha = \exp(-r(t - t_0))$, $\beta = \frac{q_2}{r} (1 - \exp(-r(t - t_0)))$, as used in Sec. 3.C.2, as well as $a = \exp(-r(T - t))$ and $b = \frac{q_2}{r} (1 - \exp(-r(T - t)))$. To incorporate the first-order correction, we adjust α , β , a and b so that $g_n(t_0, t)$ and $\langle N(t), N(t_0) = 1, N(T) = 0 \rangle$ can be calculated order by

order via generating functions. Defining

$$\alpha = e^{-r(t-t_0)}(1 + A_1), \quad (3.61a)$$

$$\beta = \frac{q_2}{r} (1 - e^{-r(t-t_0)}) z_1, \quad (3.61b)$$

$$a = e^{-r(T-t)}(1 + A_2), \quad (3.61c)$$

$$b = \frac{q_2}{r} (1 - e^{-r(T-t)}) z_2, \quad (3.61d)$$

and the generating function $\mathcal{G} = n!\alpha\beta^{n-1}$, we recover $g_n(t_0, t)$ to first order according to Eq. (3.45) in the form

$$g_n(t_0, t) = \left\{ \mathcal{G} + A \left(u(t_0, t) \frac{\partial \mathcal{G}}{\partial A_1} + v(t_0, t) \frac{\partial \mathcal{G}}{\partial z_1} \right) \right\} \Big|_{A_1=0, z_1=1} + \mathcal{O}(A^2), \quad (3.62)$$

and using the generating function $\mathcal{S}$ as defined in Eq. (3.60), similarly for the avalanche shape,

$$\begin{aligned} \mathbb{E}[N(t); N(T) = 0 | N(t_0) = 1] = & \left\{ \mathcal{S} + A \left(u(t_0, t) \frac{\partial \mathcal{S}}{\partial A_1} + u(t, T) \frac{\partial \mathcal{S}}{\partial A_2} + v(t_0, t) \frac{\partial \mathcal{S}}{\partial z_1} \right. \right. \\ & \left. \left. + v(t, T) \frac{\partial \mathcal{S}}{\partial z_2} \right) \right\} \Big|_{A_1=0, z_1=1, A_2=0, z_2=1} + \mathcal{O}(A^2). \end{aligned} \quad (3.63)$$

At this point, it is worth noting that Eq. (3.63) is a general expression whose dependence on the time-dependent extinction rate lies in the functions u and v . Hence, this expression of the avalanche shape holds for different time-dependent extinction rates as long as u and v are calculated along the lines of Eq. (3.46). As indicated earlier, the expression in Eq. (3.63) is the expectation of N over the *joint* probability of $N(t) = N$ and $N(T) = 0$. This includes all trajectories that become extinct before time T . To calculate the avalanche shape of those instances that become extinct exactly at time T , we need the expectation of $N(t)$ *conditioned* to $N(T) = 0$ and $N(t) > 0$ for $t < T$,

$$V(t, T) = \mathbb{E}[N(t) | N(T) > t > t_0 > 0, N(T) = 0, N(t_0) = 1]. \quad (3.64)$$

Assuming $dT > 0$, those trajectories that have $N(T + dT) = 0$ but $N(T) \neq 0$, have the desired feature of going extinct precisely at time T . Since $N(T) = 0$ implies $N(T + dT) = 0$, the shape $V(t, T)$ is equal to the expectation $\mathbb{E}[N(t); N(T) = 0 | N(t_0) = 1]$

differentiated with respect to T , and normalised by the probability density of extinction at exactly time T ,

$$V(t, T) = \frac{\frac{d}{dT} \mathbb{E}[N(t); N(T) = 0 | N(t_0) = 1]}{-\frac{d}{dT} P_s(0, T)}, \quad (3.65)$$

which can be calculated to first order in A by using Eqs. (3.55) and (3.63). However, numerically it is much easier to normalise the avalanche shape $V(t, T)$ by the area under its curve

$$V_1(t, T) = \frac{V(t, T)}{\int_0^1 V(\tau T, T) d\tau}, \quad (3.66)$$

which is what is shown in the plots in Figure 3.5.

4

Volume explored by a branching random walk on general graphs

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OVERVIEW The branching random walk is a branching process embedded in space, where each particle diffuses following a random walk. This process models for instance, the spreading of an infection in space [61]. We are interested in questions such as how does the volume explored by the infection evolve in time, or whether it is affected by finite-size effects, and how it depends on the topology of the embedding space. Since a diffusive particle may visit the same site infinitely many times, calculating the explored volume can be very challenging. To avoid double-counting, we introduce a new species: immobile tracer particles. These are deposited by diffusive particles whenever they visit a site for the first time. Because each site can be occupied by at most one tracer particle, these interact with each other through volume-exclusion interactions.

The explored volume and higher moments are then equivalent to correlation functions between mobile particles and immobile tracer particles.

Author contributions: IB and GP wrote the paper. SA wrote the algorithms for lattices of dimensions 2, 3, and 5, and networks. RGM wrote the algorithm for the 1 dimensional lattice. RGM and BW wrote the fitting routines. SA, RGM, BW and IB analysed the numerical results with support of GP. NW contributed to the extension of the theory to general graphs. All authors worked on the theory, read and approved the manuscript.

ABSTRACT

Branching processes are used to model diverse social and physical scenarios, from extinction of family names to nuclear fission. However, for a better description of natural phenomena, such as viral epidemics in cellular tissues, animal populations and social networks, a spatial embedding—the branching random walk (BRW)—is required. Despite its wide range of applications, the properties of the volume explored by the BRW so far remained elusive, with exact results limited to one dimension. Here we present analytical results, supported by numerical simulations, on the scaling of the volume explored by a BRW in the critical regime, the onset of epidemics, in general environments. Our results characterise the spreading dynamics on regular lattices and general graphs, such as fractals, random trees and scale-free networks, revealing the direct relation between the graphs’ dimensionality and the rate of propagation of the viral process. Furthermore, we use the BRW to determine the spectral properties of real social and metabolic networks, where we observe that a lack of information of the network structure can lead to differences in the observed behaviour of the spreading process. Our results provide observables of broad interest for the characterisation of real world lattices, tissues, and networks.

4.1 INTRODUCTION

Modern models of disease propagation incorporate spatial interaction by allowing a pathogen to be passed on only to the neighbours of an infected host [64, 146]. A virus may multiply at a host cell and then infect any of the neighbouring ones at random

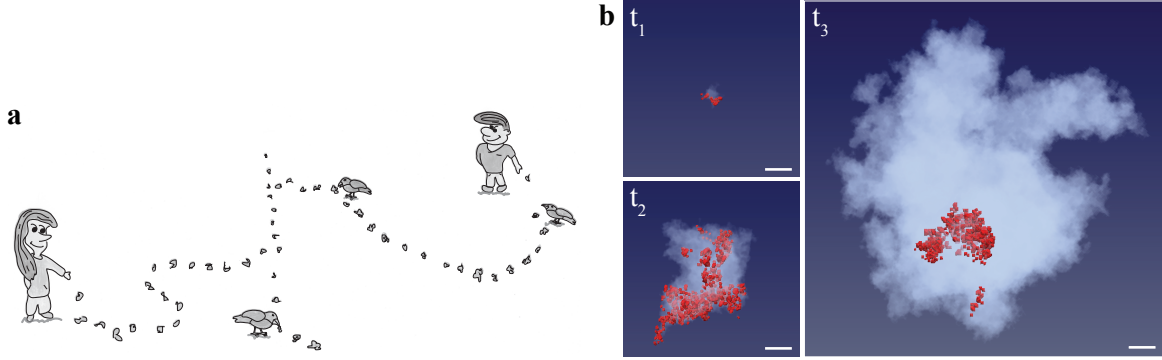


Figure 4.1: Tracing the path. **a**, The active walkers, Hänsel and Gretel, leave a trace of bread-crumbs along their way to mark the path they have taken. Birds slowly remove the breadcrumbs, as if they were subject to decay (regularisation, see main text). **b**, Time evolution of branching random walkers (red) and the cloud of visited sites on a 3d regular lattice at times $t_1 \sim 10^2$, $t_2 \sim 10^3$, and $t_3 \sim 10^4$. Scale bars are equal for all time points.

[171]. The total number of infected cells therefore corresponds to the number of distinct sites visited by a branching random walk (BRW) [61], also referred to as the Branching Wiener Sausage [138, 15]. In this process active random walkers spontaneously produce descendants that carry on hopping from site to site. At the same time, the walkers are subject to spontaneous extinction, for example, by immune-response, healing or decay. The average number of descendants produced during any of these events, branching and extinction, is known to control a transition from a subcritical phase, where the disease ultimately infects only a finite number of sites, to a supercritical phase, where the exponential growth of the virus eventually engulfs almost all available tissue [99]. The expected fraction of distinct sites visited or the size of the epidemic outbreak can be seen as the order parameter of the process.

The characterisation of the distribution of distinct sites visited by a BRW is a long-standing problem of branching processes and random walk theory [172, 61, 163]. Exact results have been obtained for one-dimensional systems [163]. However, extending such results to higher dimensional lattices and networks is met with major technical obstacles, some of which have been addressed over the past decade [61, 120, 121].

In the present work, we characterise analytically and, to confirm our findings, numerically the epidemic spreading in general graphs, including regular lattices, fractal, and artificial and real complex networks, at the onset of epidemics. At this point fluctuations are of crucial importance, dominating the dynamics.

4.2 THE MODEL

We model the epidemic as a Poisson process by considering a reaction-diffusion system of a population of active (mobile, branching, spawning) **walkers** that hop from their current location $\mathbf{x}$ on a graph to any adjacent site $\mathbf{y}$ with rate H , and have occupation numbers $n_{\mathbf{x}}$. Walkers are further subject to two concurrent Poisson processes, namely extinction with rate e and binary branching with rate s , thereby producing **descendants**, which are indistinguishable from their ancestors.

To extract the number of distinct sites visited, we introduce an immobile **tracer** particle species with occupation numbers $m_{\mathbf{x}}$. They are spawned as **offspring** by the active walkers with rate γ at the sites they are visiting, thereby leaving a trail of tracers behind, similar to the breadcrumbs left by Hänsel and Gretel [93], Fig. 4.1a. We impose the constraint that at most a single tracer can reside at any given site, which means that the spawning of a tracer is suppressed in the presence of another tracer. It is that suppression that generates significant complications from the point of view of the stochastic process. Yet, only with this restriction in place is the number of tracers a measure of the number of distinct sites visited by the walkers, as pictured by the *cloud* of visited sites in Fig. 4.1b.

There is no interaction between active and tracer particles, other than at the spawning of immobile tracers by active walkers. In principle, the spawning (attempt) rate γ has to diverge in order to mark every single site visited by the walkers. However, it turns out that this limit is irrelevant as far as the asymptotic features of this process at large system sizes and long times are concerned [138].

By definition, the sets $\{n\}$ and $\{m\}$ of occupation numbers $n_{\mathbf{x}}$ and $m_{\mathbf{x}}$, respectively, for each site $\mathbf{x}$ of a given graph, are Markovian and a master equation can be written for the joint probability $\mathcal{P}(\{n\}, \{m\}; t)$ to find the graph in a certain configuration of occupation numbers at time t

$$\dot{\mathcal{P}} = \dot{\mathcal{P}}_s + \dot{\mathcal{P}}_e + \dot{\mathcal{P}}_{e'} + \dot{\mathcal{P}}_H + \dot{\mathcal{P}}_{\gamma}, \quad (4.1)$$

where $\dot{\mathcal{P}}$ corresponds to the time derivative of the (joint) probability $\mathcal{P}(\{n\}, \{m\}; t)$, and the terms on the right-hand side, $\dot{\mathcal{P}}_{\bullet} = \dot{\mathcal{P}}_{\bullet}(\{n\}, \{m\}; t)$, indicate the contributions from branching s , extinction of active walkers e and tracer particles e' , hopping H and

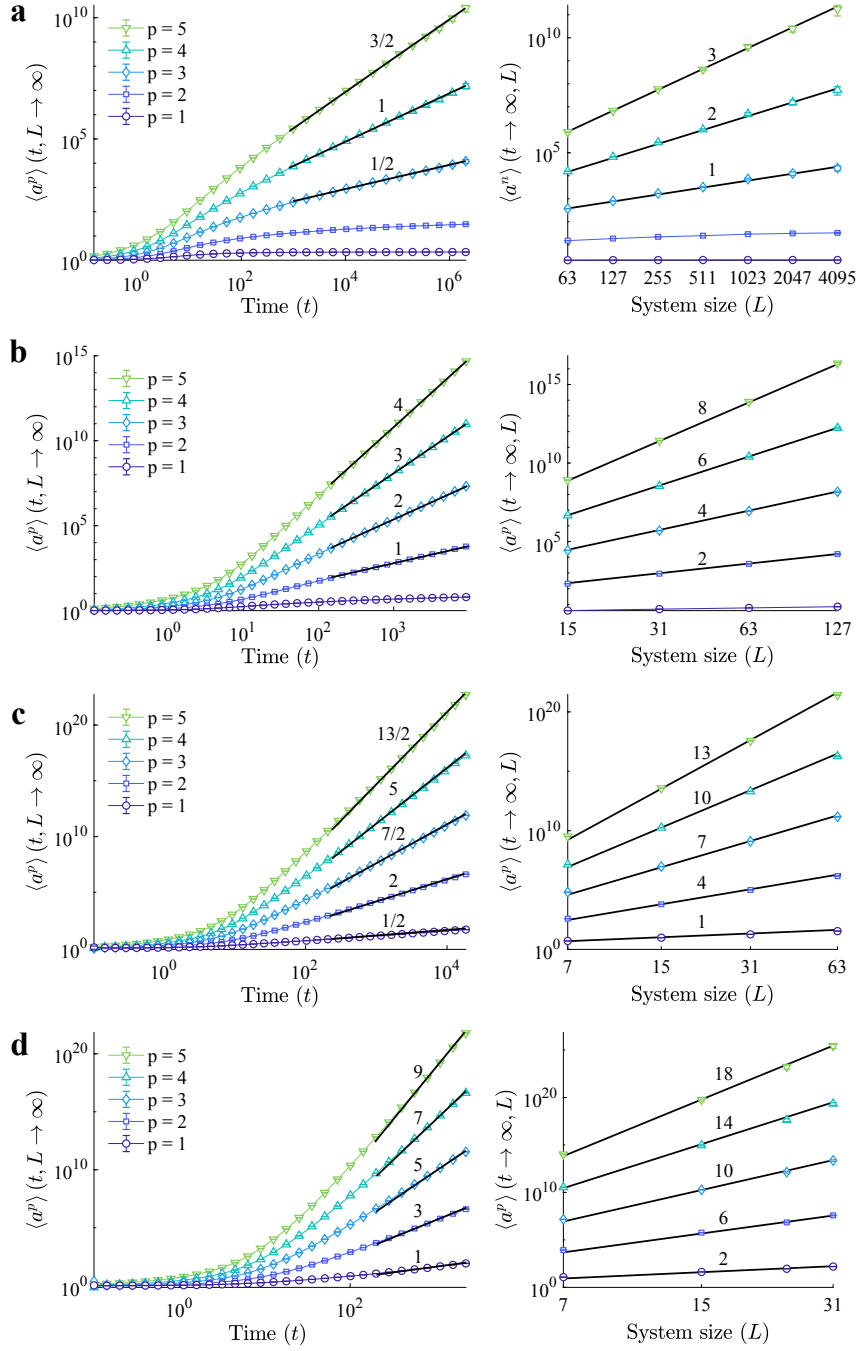


Figure 4.2: Distinct sites visited on regular lattices. Scaling of the moments of the numbers of distinct sites visited in time (left) and system size (right) for **a**, 1d, **b**, 2d, **c**, 3d, and **d** 5d regular lattices. Solid black lines represent the theoretical exponents given by Eq. (4.2) for $d < 4$, and Eq. (4.3) for $d > 4$. Simulations parameters: $H = 0.1$, $s = e = 0.45$, $\epsilon' = 0$, and $\gamma \rightarrow \infty$.

deposition γ , respectively (see Sec. 4.A for details). We constructed a statistical field theory from the master Eq. (4.1) using the ladder operators introduced by Doi [57] and Peliti [150] (methods Sec. 4.6.1). To regularise the propagators of the immobile particles in the field theory, we allow for the extinction of immobile particles with rate ϵ' in Eq. (4.1), not dissimilar to the birds that foiled Hänsel and Gretel's plans (Fig. 4.1a). The propagators for active and tracer particles do not renormalise, and the limit $\epsilon' \rightarrow 0$ is taken before any observable is evaluated. Through field-theoretic renormalisation in dimensions $d = 4 - \varepsilon$ we can then determine the exact scaling behaviour of the number of distinct sites visited by the walkers.

The branching process described by Eq. (4.1) has three regimes, as becomes evident in the field-theoretic formulation, where a *net* extinction rate $r = e - s$ appears. This net extinction rate is not renormalised in the field-theory and therefore no mass shift appears. The BRW is subcritical for $r > 0$, critical for $r = 0$ (onset of epidemics) and supercritical for $r < 0$. Hereafter, we focus on the critical case, where fluctuations dominate the dynamics, and the behaviour becomes unpredictable and highly volatile. Furthermore, for both analytical and numerical computations we consider the initial condition of a single walker at $t = 0$. Extensions to different initial conditions are straight-forward.

4.3 RESULTS FOR REGULAR LATTICES

Following the field theoretic approach (details in Secs. 4.6.1 and 4.B) of the bulk critical behaviour in the continuum limit, where hopping is replaced by diffusion by introducing a diffusion constant D , we find that in the thermodynamic limit at long times t , the expected number of distinct sites visited or the volume explored, $\langle a \rangle(t, L)$, scales like $t^{(d-2)/2}$ in dimensions $d < 4$. In dimensions $d < 2$ this volume remains finite in large t . The scaling of the p -th moment of the number of distinct sites visited follows,

$$\langle a^p \rangle(t, L) \propto t^{(pd-2)/2} \quad \text{for} \quad Dt \ll L^2 \quad (4.2a)$$

$$\langle a^p \rangle(t, L) \propto L^{(pd-2)} \quad \text{for} \quad Dt \gg L^2 \quad (4.2b)$$

in $d < 4$ provided that $pd - 2 > 0$. The gap-exponent [153] of $\langle a^{p+1} \rangle / \langle a^p \rangle$ for the scaling in L , which can be thought of as the effective dimension of the cluster of visited

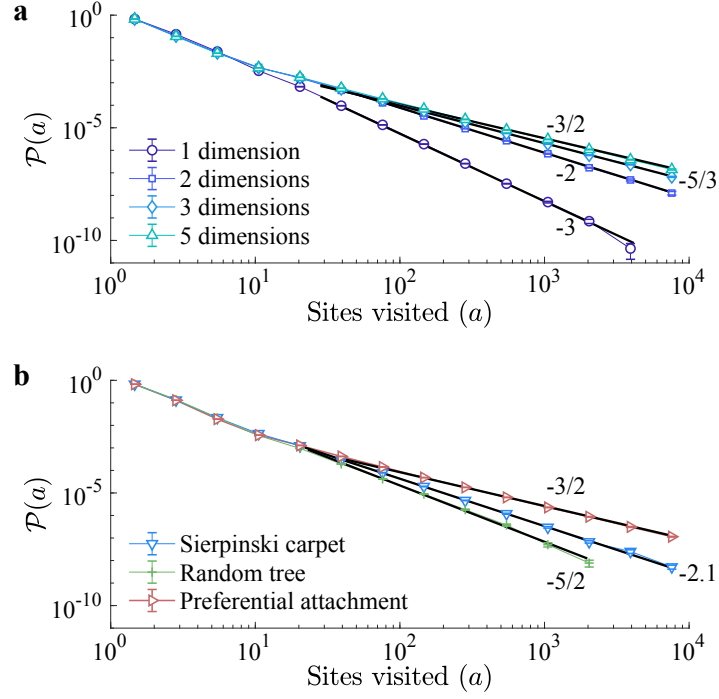


Figure 4.3: Probability distribution of the number of visited sites, a for regular lattices of dimensions $d = 1, 2, 3$ and 5 , and for **b** Sierpinski carpet, random tree, and a preferential attachment (scale-free) networks. The solid black lines represent the predicted scaling given by Eq. (4.4). Simulations parameters: $H = 0.1$, $s = e = 0.45$, $\epsilon' = 0$, and $\gamma \rightarrow \infty$.

sites, is therefore d in dimensions less than $d_c = 4$. These results describe the numerical observations on regular lattices in dimensions $d = 1, 2$, and 3 (see Sec. 4.6.2, as shown in Figs. 4.2a, 4.2b, and 4.2c, respectively, where, after an initial transient, the moments scale according to Eq. (4.2) in time and system size (see Tabs. 4.1 and 4.2). The process is *free* beyond $d_c = 4$ dimensions, where the probability of any walker or any of its ancestors or descendants ever to return to a previously visited site drops below unity, and the scaling becomes independent of the dimension,

$$\langle a^p \rangle(t, L) \propto t^{2p-1} \quad \text{for} \quad Dt \ll L^2 \quad (4.3a)$$

$$\langle a^p \rangle(t, L) \propto L^{4p-2} \quad \text{for} \quad Dt \gg L^2 \quad (4.3b)$$

with logarithmic corrections in $d = d_c = 4$. The gap-exponent in dimensions greater than $d_c = 4$ is thus 4 , as confirmed by numerical observations in dimension $d = 5$ (see Fig. 4.2d and Tab. 4.2). As correlations become irrelevant, this is usually referred

to as mean-field behaviour. The set of sites visited may thus be regarded as a four-dimensional object, projected into the d -dimensional lattice considered. Focusing on dimensions below $d_c = 4$, the distribution of the number of distinct sites visited, a , follows a power law,

$$\mathcal{P}(a) = Aa^{-(1+2/d)}\mathcal{G}(a/a_c) \quad (4.4)$$

with metric factor A and cutoff $a_c \sim (Dt)^{d/2}$ for $Dt \ll L^2$ and $a_c \sim L^d$ otherwise. These results show how increasing the dimensionality of the lattice promotes the appearance of larger events, evidencing the relevance of dimension on the spreading.

In dimensions $d \geq d_c = 4$ the resulting scaling of the distribution is that of Eq. (4.4) at $d = 4$, where the probability distribution decays like $a^{-3/2}$. Numerically, we recorded, for each realisation, the total number of distinct sites visited by the process in order to construct the distribution, $\mathcal{P}(a)$, of sites visited. The numerical results coincide with our theoretical predictions, as shown in Fig. 4.3a.

The exponents found above for $d = 1$ are in agreement with the exact solution by Ramola et al. [163], where $\mathcal{P}(a)$ decays as a^{-3} . In two dimensions the power-law tail decays as a^{-2} , which coincides with the decay of the 2d convex hull area distribution [61].

4.4 EXTENSION TO GENERAL GRAPHS

In the field theoretic approach followed to find the scaling in Sec. 4.3 the spatial dimension of the lattice enters only in as far as its spectral dimension is concerned, which characterises the density of eigenvalues of the Laplace operator on the graph given. Our results extend naturally to all translational invariant lattices and graphs, by replacing the dimension d of the lattice in Eqs. (4.2), (4.3) and (4.4) by the spectral dimension d_s of the graph, as detailed in Sec. 4.E. This holds true more generally as long as the lattice Laplacian itself does not undergo renormalisation, *i.e.* in the absence of an anomalous dimension [25]. In the study of networks the number of nodes N , is a more natural measure of the size of the network than the linear size L . Using $L \sim N^{1/d_s}$ we can write

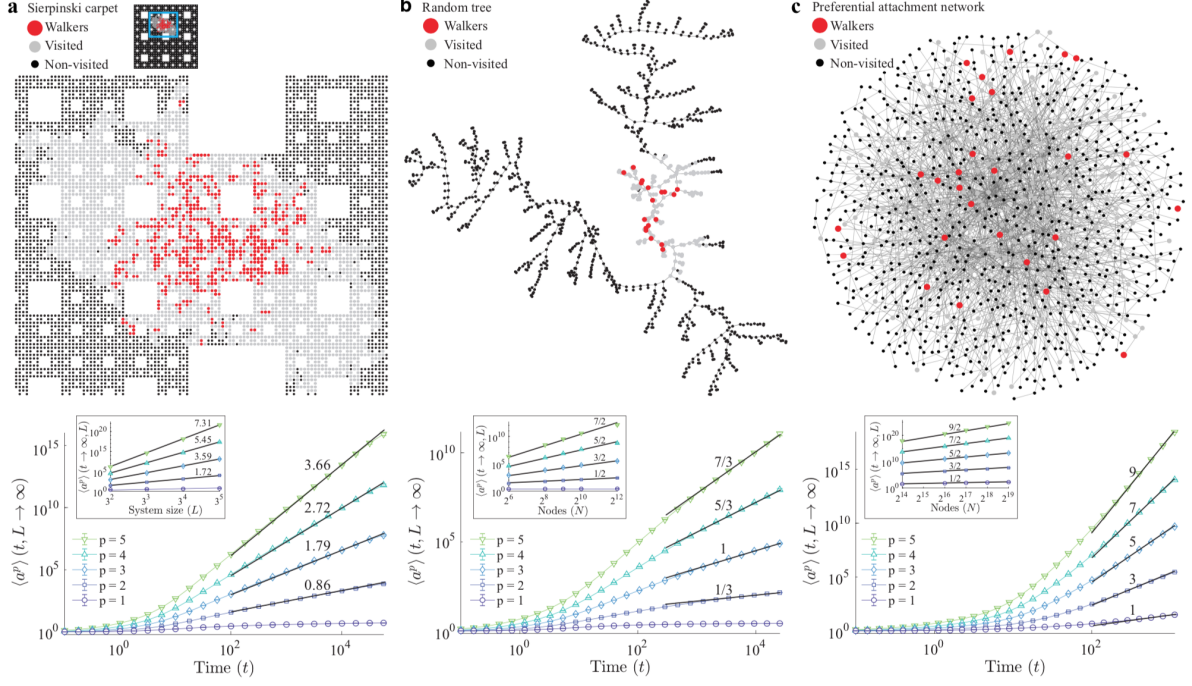


Figure 4.4: Scaling on general graphs on **a**, the Sierpinski carpet, **b**, random tree and, **c**, preferential attachment networks. The top row shows representative states (full Sierpinski carpet shown on inset), indicating walkers (red), visited sites (grey) and non-visited sites (black). The bottom row shows the scaling of moments of the number of distinct sites visited as a function of time, and linear system size (inset), or number of nodes, in the case of networks. The solid black lines represent the predicted scaling from Eqs. (4.5). Simulation parameters: $H = 0.1$, $s = e = 0.45$, $\epsilon' = 0$, and $\gamma \rightarrow \infty$.

the scaling of the BRW in time and number of nodes as

$$\langle a^p \rangle(t, N) \propto t^{(pd_s-2)/2} \quad \text{for} \quad Dt \ll N^{2/d_s} \quad (4.5a)$$

$$\langle a^p \rangle(t, N) \propto N^{(p-2/d_s)} \quad \text{for} \quad Dt \gg N^{2/d_s}. \quad (4.5b)$$

Here, the gap-exponent for the scaling in number of nodes is always unity. This extension to graphs allowed us to predict the behaviour of the BRW spreading in both artificial networks relevant for social and biological sciences, and complex systems in general [7, 2, 146, 200], as well as real networks. To illustrate this, we considered first the Sierpinski carpet (SCs) (Fig. 4.4a, methods Sec. 4.6.2), and random trees (RTs) (Fig. 4.4b and methods Sec. 4.6.2). Both of these graphs are widely applied in the context of porous media [218] and percolation [126], and have known spectral dimension: $d_s \approx 1.86$ for the SC [201], and $d_s = 4/3$ for RTs [49]. Considering Eq. (4.2)

with $d = d_s$, for the SC, and (4.5) for the RT we obtain accurate predictions for the spreading dynamics as confirmed by numerical simulations, Figs. 4.4d and 4.4e. These theoretical predictions extend also to the distribution of visited sites (see Fig. 4.3b), by setting $d = d_s$ in (4.4).

Furthermore, we studied the BRW behaviour on a class of scale free networks [6]. Since their introduction, scale free graphs have been observed to describe a plethora of natural phenomena, including the World-Wide-Web [7], transportation [95], and metabolic networks [107], to name but a few. We considered a preferential attachment scheme [6] (Fig. 4.4c, see methods Sec. 4.6.2), to construct networks with power-law degree distribution (Fig. 4.6). The existence of a finite spectral gap in these networks, which indicates slow decay of return times [170, 133], suggested that the BRW process is bound to exhibit mean-field behaviour, i.e. $d_s \geq 4$. This was confirmed by numerical simulations, where the probability distribution of visited sites (Fig. 4.3b) has a power-law decay with exponent $-1.52(2) \approx -3/2$, and the scaling in time and system size (Fig. 4.4f and Tab. 4.1) follow mean-field behaviour as predicted by (4.5) for $d_s = 4$.

The spectral dimension gives information on the behaviour of dynamical processes on graphs. Here we use the BRW to characterise real-world networks through the power-law decay of the distribution of visited sites $\mathcal{P}(a)$, which according to Eq. (4.4) is $a^{-(1+2/d_s)}$ provided $d_s \leq 4$. For example, the BRW exhibits near mean field-behaviour on a subset of the Facebook network, which has been characterised as scale-free [198]. Hence, we derived a large effective (spectral) dimension, $d_s = 3.9(1)$, indicating a fast spreading of the viral process in this network (see Fig. 4.5).

We should emphasize that the spectral dimension is sensitive to changes in network topology and connectivity. To exemplify this we have considered two publicly available datasets for the yeast protein interaction network (see Fig. 4.5). We found that even though both network describe subsets of the same biochemical network, namely the complete yeast protein interactome, the spectral dimensions in both cases are significantly different, $d_s = 3.0(1)$ for the network with $N = 1870$ nodes [106], and $d_s = 3.8(1)$ for the larger network of $N = 2559$ nodes [74], leading to differences in properties of the spreading process among the two. The discrepancy points to differences in the connectivities of both networks and shows the importance of having access to the complete network in order provide a reliable analysis of its properties, which may have biological implications [97, 186].

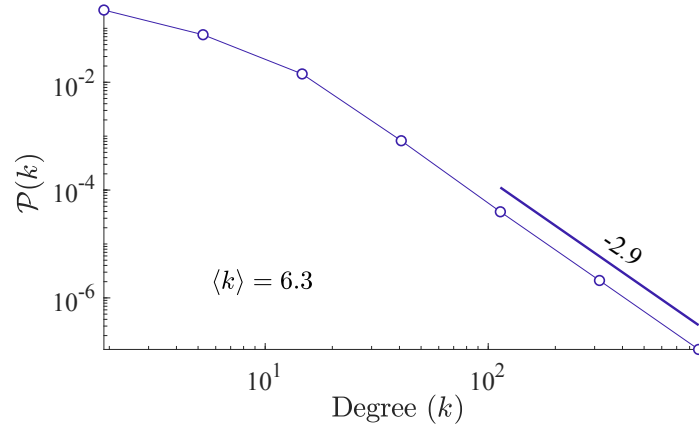


Figure 4.5: Probability distribution of number of distinct sites visited $\mathcal{P}(a)$, for the Facebook network ($L = 63730$ nodes) [198], and yeast protein interaction networks with $L = 1870$ [106], and $L = 2559$ nodes [74]. The data was obtained from simulations of the BRW on each graph, with parameters $H = 0.1$, $s = e = 0.45$, $\epsilon' = 0$, and $\gamma \rightarrow \infty$.

4.5 OUTLOOK

The results presented above for the binary branching process, where walkers branch into exactly two new walkers, apply equally to more general branching processes, where the number of offspring in each birth event is given by a distribution (for details see Sec. 4.D). This can be seen, for example, in real-world scenarios where a single infected individual or device infects a whole neighbourhood around them, or in the case of signal propagation in protein networks, where the activation of one node (or chemical reaction) can in turn activate a whole fraction of its neighboring nodes.

While the scaling behaviour does not depend on the initial position $\mathbf{x}_0$ of a walker, provided it is located in the bulk and remains there as the thermodynamic limit is taken, the field theory has to be adjusted to account for more complicated boundary conditions [138] or the walker starting close to any such boundary. It may also be interesting to consider the case of initialising each site with an independent Poisson distributions of walkers [29].

The approach followed in the present work provides a quantitative measure to explore and determine the spectral dimension of artificial and real networks. This is of particular interest when the spectral dimension is greater or equal to 2, where the traditional approach of exploring graphs, based on simple random walks [180, 133], fails. When

simulating the BRW we made the observation that robust scaling is more easily obtained on small lattices if the hopping rate H is clearly smaller than the rates of branching s and extinction e . For large values of the hopping rate particles leave the system during the initial transient, as seen in Fig. 4.2, thus boundary effects appear before any robust scaling can be observed. In graphs such as the PA network (Fig. 4.4), that does not have any boundaries, these artefacts are much less pronounced. In summary our results shed new light on the properties of spatial branching processes on general graphs, and their applicability in the study of real complex networks, and provide observables of broad interest for the characterisation of real world lattices, tissues, and networks.

4.6 METHODS

4.6.1 FIELD THEORY OF THE BRW

In order to derive the main results for the scaling of distinct sites visited by the BRW (Sec. 4.3) we work along established lines [190], casting the master equation in a field theory of the annihilation fields $\phi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ for the active and the immobile particles, respectively, and of the corresponding (Doi-shifted) creation fields $\tilde{\phi}(\mathbf{x}, t)$ and $\tilde{\psi}(\mathbf{x}, t)$. The governing Liouvillian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ consists of a harmonic part,

$$\mathcal{L}_0(\phi, \psi, \tilde{\phi}, \tilde{\psi}) = -\tilde{\phi}\partial_t\phi + D\tilde{\phi}\nabla^2\phi - r\tilde{\phi}\phi - \tilde{\psi}\partial_t\psi - \epsilon'\tilde{\psi}\psi + \tau\tilde{\psi}\phi, \quad (4.6)$$

and a non-linear part,

$$\mathcal{L}_1(\phi, \psi, \tilde{\phi}, \tilde{\psi}) = s\tilde{\phi}^2\phi + \sigma\tilde{\psi}\tilde{\phi}\phi - \lambda\tilde{\psi}\psi\phi - \xi\tilde{\psi}^2\psi\tilde{\phi}\phi - \kappa\tilde{\psi}\psi\tilde{\phi}\phi - \chi\tilde{\psi}^2\psi\phi, \quad (4.7)$$

where we have taken the continuum limit. The space and time integrated Liouvillian produces the field-theoretic action $\mathcal{A} = \int d^d x dt \mathcal{L}$, whose exponential $e^{\mathcal{A}}$ enters into the path integral formulation. The couplings in the Liouvillian are related to the rates in the master equation as follows: D is a diffusion constant $D = H\Delta x^2$, where Δx is the lattice spacing, and $H \propto \Delta x^{-2}$ when the limit $\Delta x \rightarrow 0$ is taken, in order to maintain finite diffusivity. At bare level the non-linear couplings, with the exception of the branching rate s , are equal to spawning rate γ , i.e. $\tau = \sigma = \lambda = \xi = \kappa = \chi = \gamma$. This follows from translating the master Eq. (4.1) into field-theoretic language (see

Sec. 4.A for details).

At the same time the *net* extinction rate $r = e - s$, the field-theoretic mass of the walkers, has to be kept finite. In this parameterisation, there are three regimes, as described in the main text: a subcritical one for $r > 0$, a critical for $r = 0$ and a supercritical for $r < 0$. In the field theory, all large scale (infrared) phenomena will be controlled by $r \rightarrow 0^+$, which corresponds to the onset of epidemics, the limit studied in this work. The mass of the tracers, ϵ' , serves merely as a regularisation, and is removed by taking the limit $\epsilon' \rightarrow 0$. The bare transmutation rate τ , corresponding to γ on the lattice, and the bare branching rate s of the active particles (s on the lattice) are the two processes that we expect will govern all infrared behaviour in all dimensions and are therefore assumed to be dimensionless. These two choices determine the engineering dimension [189] of all other bare couplings, resulting in ξ , κ and χ being infrared irrelevant. Together with λ , these four couplings are due to the suppression of the spawning of tracers when a site is occupied already. At the upper critical dimension, $d_c = 4$, the coupling λ is marginally relevant, being infrared irrelevant above and relevant below. The minimal subtraction scheme [189] we have used will produce results in terms of $\varepsilon = 4 - d$.

The Liouvillian constructed above is the object that allows the exact calculation of the scaling exponents of the p -th moment of the volume explored by a branching random walk $\langle a^p \rangle(t, L)$, in time t , and linear system size L . Initialising the system at time $t_0 = 0$ with a single active walker at position $\mathbf{x}_0$, field-theoretically implemented by the creation field $\tilde{\phi}(\mathbf{x}_0, 0)$, the ensemble average $\langle a \rangle(t, L)$ of the volume explored by the BRW is determined by

$$\langle a \rangle(t, L) = \int d^d x \left\langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, 0) \right\rangle, \quad (4.8)$$

where the density of tracers particles at position $\mathbf{x}$ and time $t > 0$ is measured by $\psi(\mathbf{x}, t)$ and integrated over all space. Similarly [138], higher moments are dominated by integrals of the form

$$\langle a^p \rangle(t, L) \sim \int d^d x_p \dots d^d x_1 \left\langle \psi(\mathbf{x}_p, t) \dots \psi(\mathbf{x}_1, t) \tilde{\phi}(\mathbf{x}_0, 0) \right\rangle, \quad (4.9)$$

or equivalently, by evaluating the Fourier transform at spatial momentum $\mathbf{k} = 0$. These are functions of the couplings introduced above, but to leading order not of the walker's

initial position $\mathbf{x}_0$, provided it is located in the bulk. We implement this numerically by always placing the walker initially at the centre site of odd-sized regular lattices, see Sec. 4.6.2. The average $\langle \bullet \rangle$ introduced on the right hand side of Eq. (4.8) correspond to the path integral

$$\left\langle \psi(\mathbf{x}_p, t) \dots \psi(\mathbf{x}_1, t) \tilde{\phi}(\mathbf{x}_0, 0) \right\rangle = \int \mathcal{D}\Pi \left(\psi(\mathbf{x}_p, t) \dots \psi(\mathbf{x}_1, t) \tilde{\phi}(\mathbf{x}_0, 0) \right) e^{\int d^d x dt \mathcal{L}}, \quad (4.10)$$

which measures the p -point correlation function of tracers at $(\mathbf{x}_i, t)$, $i = 1, 2, \dots, p$ in response to the creation of a walker at $(\mathbf{x}_0, t = 0)$. Here, the integration measure is $\mathcal{D}\Pi = \mathcal{D}\phi \mathcal{D}\tilde{\phi} \mathcal{D}\psi \mathcal{D}\tilde{\psi}$. Field theoretic renormalisation in dimensions $d = 4 - \varepsilon$ then allows us to derive the scaling of the number of distinct sites visited (see Sec. 4.B for more details).

4.6.2 NUMERICAL IMPLEMENTATION

In the numerical implementation, an active particle is allowed to diffuse by hopping from the site it resides on to a nearest neighbouring site with rate H , branch with rate s by placing an identical offspring at the present site or become extinct with rate e . Each distinct site visited is recorded, equivalent to taking the limit $\gamma \rightarrow \infty$ in the theory. The instantaneous number $a(t, L)$ of distinct sites visited up to time t is therefore the number of sites recorded. Parameters were chosen such that $H + s + e = 1$, H was set to 0.1, and $e = s = 0.45$. If M walkers are present in the system at a given time the waiting time for the next event (hopping, branching or extinction) is determined by $-\ln(1 - u)/M$ where $[0, 1) \ni u \sim U(0, 1)$ is a uniformly distributed random variable. For every lattice size we performed $10^6 - 10^9$ realisations of the process.

REGULAR, INTEGER-DIMENSIONAL LATTICES

The regular lattices studied here are hypercubic d -dimensional lattices, characterised by their linear size $L = 2^m - 1$, $m \geq 4$, which is chosen to be odd so that it contains a well-defined centre site, on which the single active walker is initially placed. To study finite-size scaling, absorbing boundary conditions were applied. However, we observed that the boundary conditions have no effect on the scaling (data not shown). The numerical results were fitted to a power-law as described in Sec. 4.F, to obtain the

values in Tabs. 4.1 and 4.2.

SIERPINSKI CARPET

The Sierpinski carpets were constructed from two dimensional lattices of linear dimension 3^m , $m \geq 2$. The lattice was divided into 3^2 equal sub-squares each of size 3^{m-1} , the central square was removed, leaving $3^2 - 1$ sub-squares. The procedure is iterated over the remaining sub-squares. The spectral dimension of the Sierpinski carpet has been estimated to be $d_s = 1.86$ [201, 43]. A random point around the central hole of the fractal was used as the initial location of the walker in every realisation.

RANDOM TREES

The critical random tree networks [48] were constructed as a critical Galton-Watson process, where every node has either 0, 1, or 2 descendants, such that the mean degree of the network is 2. We generated networks with $2^6 - 2^{12}$ nodes. These graphs have no closed loops. The spectral dimension of the random tree ensemble is $d_s = 4/3$ [49]. For every realisation of the process, a new random tree was generated, and a node was selected at random as the starting location of the initial walker.

PREFERENTIAL-ATTACHMENT NETWORK

A preferential attachment (PA) network is a class of scale-free networks, characterised by a power-law degree distribution. We followed the Barabási–Albert model of preferential attachment [6] initialised with a single node to generate networks with $2^{12} - 2^{19}$ nodes. The networks have power-law degree distribution with exponent -2.9 and mean degree $\langle k \rangle = 6.3$ (see Fig. 4.6). For every realisation of the process, a new network was constructed, and a node was selected at random as the starting location of the initial walker.

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4.A MASTER EQUATION FOR THE BRANCHING RANDOM WALK

In the following we describe the contributions to the master Eq. (4.1) from each of the processes the branching random walk comprises. The contribution to the master equation for the joint probability $\mathcal{P}(\{n\}, \{m\}; t)$ from the spawning of immobile tracer particles by active walkers must take into account the finite carrying capacity $\bar{m}_0$ of each lattice site. To account for a finite carrying capacity an effective deposition rate is introduced that decays linearly with the number of tracer particles already present at the site of interest [138],

$$\gamma_{\text{eff}} = \gamma \frac{\bar{m}_0 - m_{\mathbf{x}}}{\bar{m}_0}. \quad (4.11)$$

To study the number of distinct sites visited $\bar{m}_0$ is set to 1. With this constraint in place, each site visited is *marked* with a tracer particle at most once, so that their total number is that of *distinct* sites visited by the BRW. With these considerations, the contributions to the master equation from deposition of tracer particles read

$$\begin{aligned} \dot{\mathcal{P}}_{\gamma}(\{n\}, \{m\}; t) = & \gamma \sum_{\mathbf{x}} \left((1 - (m_{\mathbf{x}} - 1)) n_{\mathbf{x}} \mathcal{P}(\{n\}, \{\dots, m_{\mathbf{x}} - 1, \dots\}; t) \right. \\ & \left. - (1 - m_{\mathbf{x}}) n_{\mathbf{x}} \mathcal{P}(\{n\}, \{m\}; t) \right), \end{aligned} \quad (4.12)$$

where $n_{\mathbf{x}}$ and $m_{\mathbf{x}}$ correspond to the number of active and immobile particles at site $\mathbf{x}$. The sum $\sum_{\mathbf{x}}$ runs over all lattice sites. The contribution from branching of active

walkers reads

$$\dot{\mathcal{P}}_s(\{n\}, \{m\}; t) = s \sum_{\mathbf{x}} \left((n_{\mathbf{x}} - 1) \mathcal{P}(\{\dots, n_{\mathbf{x}} - 1, \dots\}, \{m\}; t) - n_{\mathbf{x}} \mathcal{P}(\{n\}, \{m\}; t) \right). \quad (4.13)$$

The contributions from extinction are

$$\dot{\mathcal{P}}_e(\{n\}, \{m\}; t) = e \sum_{\mathbf{x}} \left((n_{\mathbf{x}} + 1) \mathcal{P}(\{\dots, n_{\mathbf{x}} + 1, \dots\}, \{m\}; t) - n_{\mathbf{x}} \mathcal{P}(\{n\}, \{m\}; t) \right) \quad (4.14)$$

for active particles, and

$$\dot{\mathcal{P}}_{e'}(\{n\}, \{m\}; t) = e' \sum_{\mathbf{x}} \left((m_{\mathbf{x}} + 1) \mathcal{P}(\{n\}, \{\dots, m_{\mathbf{x}} + 1, \dots\}; t) - m_{\mathbf{x}} \mathcal{P}(\{n\}, \{m\}; t) \right) \quad (4.15)$$

for immobile particles. Finally, the contribution to the joint probability from the hopping of active walkers reads

$$\begin{aligned} \dot{\mathcal{P}}_H(\{n\}, \{m\}; t) = & \frac{H}{q} \sum_{\mathbf{x}} \sum_{\mathbf{y}, nn.\mathbf{x}} \left((n_{\mathbf{y}} + 1) \mathcal{P}(\{\dots, n_{\mathbf{x}} - 1, \dots, n_{\mathbf{y}} + 1, \dots\}, \{m\}; t) \right. \\ & \left. - n_{\mathbf{x}} \mathcal{P}(\{n\}, \{m\}; t) \right), \end{aligned} \quad (4.16)$$

where the sum $\sum_{\mathbf{y}, nn.\mathbf{x}}$ runs over all q nearest neighbouring (nn) sites $\mathbf{y}$ of $\mathbf{x}$.

Combining the contributions from all the subprocesses, the master equation for the joint probability $\mathcal{P} = \mathcal{P}(\{n\}, \{m\}; t)$ reads

$$\dot{\mathcal{P}} = \dot{\mathcal{P}}_s + \dot{\mathcal{P}}_e + \dot{\mathcal{P}}_{e'} + \dot{\mathcal{P}}_H + \dot{\mathcal{P}}_{\gamma} \quad (4.17)$$

as shown in Eq. (4.1).

4.B FIELD-THEORY OF THE BRW

In the following, we show the details of the field-theoretical calculations performed to obtain the main results of the article, Eqs. (4.2), (4.4), and (4.3). In Sec. 4.B.1 we describe the dimensional analysis of the bare couplings. In Sec. 4.B.3, we introduce a diagrammatic representation of the propagators and couplings, and in Sec. 4.B.4 we

determine the relevant interactions. In Sec. 4.B.5, we perform the renormalisation of the couplings, and finally calculate the higher order correlations that give rise to the scaling of the moments of the number of distinct sites visited in Sec. 4.B.7.

4.B.1 DIMENSIONAL ANALYSIS OF THE BARE COUPLINGS

To compute the critical dimension of the process described by the Liouvillian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, Eqs. (4.6) and (4.7), and to extract the relevant interactions, i.e the couplings that remain relevant in every spatial dimension, we study the engineering dimensions (here, represented by $[\cdot]$) of every coupling in the action. We expect that the long range physics in time and space is governed by three processes: diffusion with constant D , branching with rate s , and transmutation with rate τ . Introducing three independent dimensions, namely $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, we impose

$$[\tau] = \mathbf{A}, \quad [s] = \mathbf{B}, \quad \text{and} \quad [D] = \mathbf{C}. \quad (4.18)$$

With $[\mathbf{x}] = \mathbf{L}$, $[t] = \mathbf{T}$, and $[\partial_t] = [D\nabla^2]$ it follows that $\mathbf{T} = \mathbf{C}\mathbf{L}^2$ is not an independent dimension. As the action, $\mathcal{A} = \int d^d x dt \mathcal{L}$, itself must be dimensionless, i.e. $[\mathcal{A}] = 1$, we obtain $[r] = \mathbf{T}^{-1} = \mathbf{C}\mathbf{L}^{-2}$ and

$$[\tilde{\phi}] = \mathbf{B}^{-1}\mathbf{C}\mathbf{L}^{-2}, \quad [\phi] = \mathbf{B}\mathbf{C}^{-1}\mathbf{L}^{2-d}, \quad [\tilde{\psi}] = \mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{C}^2\mathbf{T}^{-2}, \quad [\psi] = \mathbf{A}\mathbf{B}\mathbf{C}^{-2}\mathbf{L}^{4-d}, \quad (4.19)$$

for the fields in real time and space, such that $[\tilde{\phi}\phi] = [\tilde{\psi}\psi] = \mathbf{L}^{-d}$. The engineering dimensions of the couplings follow:

$$[\lambda] = \mathbf{B}^{-1}\mathbf{C}^2\mathbf{L}^{d-4}, \quad [\sigma] = \mathbf{A}\mathbf{B}\mathbf{C}^{-1}\mathbf{L}^2, \quad [\chi] = \mathbf{A}\mathbf{L}^d, \quad [\kappa] = \mathbf{C}\mathbf{L}^{d-2}, \quad [\xi] = \mathbf{A}\mathbf{B}\mathbf{C}^{-1}\mathbf{L}^{d+2}. \quad (4.20)$$

Setting $\mathbf{A} = \mathbf{B} = \mathbf{C} = 1$, we find a critical dimension $d_c = 4$, above which all of these interactions become irrelevant. At the critical dimension $d = d_c = 4$ the couplings σ , χ , κ , and ξ remain irrelevant, while λ becomes marginal. To regularise the ultraviolet we work in dimensions $d = 4 - \epsilon < 4$.

As a point of discussion, we note that other choices of independent dimensions are possible, limited only by the symmetries to be preserved. Initially we considered σ , rather than τ to have an independent dimension. The resulting (very messy) field

theory depends on the non-universal, bare value of s and produces no renormalisation of τ , which, however, *must* renormalise as $\langle a \rangle(t, L) \sim \tau_{\text{eff}} L^2$ (see Section 4.B.5) and cannot scale faster than the volume of the system, L^d .

A coupling with independent dimension is saved from changing relevance and thus from possible irrelevance in the infrared limit of large space and long time. The choice of dimensions is therefore a choice of interactions that ultimately govern the infrared. If the stochastic process under consideration takes place on the lattice, this may be determined by taking the continuum limit, provided the process does not possess any competing scales, in which case the continuum limit coincides with the thermodynamic limit of infinite system size. However, as soon as different processes and scales compete, such as hopping, branching, spawning and extinction rates in the present case, the continuum limit is a mere approximation of the original process on the lattice and the choice of (independent) dimensions becomes a claim about which interactions govern the infrared. For example, considering a biased random walk, letting space scale linear in time preserves a drift but removes diffusion, while letting space scale quadratically in time preserves the latter, while the drift velocity diverges.

4.B.2 FOURIER TRANSFORM

Throughout the manuscript, we denote the Fourier transform $\mathcal{F}[f(\mathbf{x}, t)]$ of a function $f(\mathbf{x}, t)$ in space $\mathbf{x}$ and time t simply as $f(\mathbf{k}, \omega)$, where the spatial momentum $\mathbf{k}$ is the conjugate of the position $\mathbf{x}$, and the frequency ω is the conjugate of time t . The direct Fourier transform is defined as

$$f(\mathbf{k}, \omega) = \int e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}, t) d^d \mathbf{x} dt, \quad (4.21)$$

so that the inverse Fourier transform is

$$f(\mathbf{x}, t) = \int e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{k}, \omega) d^d \mathbf{k} d\omega, \quad (4.22)$$

where $d^d \mathbf{k} = (1/2\pi)^d d\mathbf{k}$, $d\omega = (1/2\pi) d\omega$, and d is the spatial dimension.

4.B.3 PROPAGATORS AND COUPLINGS

We begin by considering the field-theoretic action $\mathcal{A} = - \int d^d x dt \mathcal{L}$, where the terms in the Liouvillian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ are given by Eqs. (4.6) and (4.7), respectively. In order to render the Laplacian term local the action is rewritten in Fourier space, where the momentum $\mathbf{k}$ is the conjugate of position $\mathbf{x}$ and the frequency ω is the conjugate of time t (as defined in Sec. 4.B.2). The perturbative renormalisation scheme starts by reading off the propagators from the bilinear part, introducing a diagrammatic language as we proceed. For the walkers the bare propagator reads

$$\left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \right\rangle_0 = \frac{\delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega')}{-i\omega + Dk^2 + r} \hat{=} \text{---}, \quad (4.23)$$

where $\delta(\mathbf{k} + \mathbf{k}') = (2\pi)^d \delta(\mathbf{k} + \mathbf{k}')$ denotes a scaled d -dimensional Dirac- δ function, and correspondingly for $\delta(\omega + \omega')$. Diagrammatically, the bare propagator is shown as a straight line. For the tracers the bare propagator becomes

$$\left\langle \psi(\mathbf{k}, \omega) \tilde{\psi}(\mathbf{k}', \omega') \right\rangle_0 = \frac{\delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega')}{-i\omega + \epsilon'} \hat{=} \text{~~~~~}, \quad (4.24)$$

diagrammatically shown as a wavy line. Both bare propagators carry a positive mass, $r = e - s$ in Eq. (4.23) and ϵ' in Eq. (4.24), which guarantees causality as the inverse Fourier transform will generate a Heaviside- θ function in time. Both propagators Eqs. (4.23) and (4.24) do not undergo renormalisation. Finally, the transmutation vertex features in

$$\left\langle \psi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \right\rangle_0 = \tau \frac{\delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega')}{(-i\omega + \epsilon')(-i\omega + Dk^2 + r)} \hat{=} \text{~~~~~} \tau \text{---} \quad (4.25)$$

and signals the appearance of a tracer in response to the presence of a walker, as time is to be read from right to left. The non-linear part of the Liouvillian, $\mathcal{L}_1$, contributes with six interaction vertices, which diagrammatically read

$$\begin{array}{cccccc} \text{---}^s & , & \text{---}^\sigma & , & \text{~~~~~}^\xi & , & \text{~~~~~}^\kappa & , & \text{~~~~~}^\chi & , & \text{~~~~~}^{-\lambda} & . \end{array} \quad (4.26)$$

Finally, the observables of the form of Eq. (4.10) have the diagrammatic structure

$$p \left\{ \begin{array}{c} \text{wavy line} \\ \vdots \\ \text{wavy line} \end{array} \right\} \text{---} \bigcirc \text{---} . \quad (4.27)$$

Their scaling in time and finite-size can be extracted from the scaling of the vertex generating function, which is the standard object of field-theoretic renormalisation. In the next section we describe all possible infrared-relevant interactions.

4.B.4 RELEVANT INTERACTIONS

Whether a particular interaction is allowed by the basic process introduces above is a matter of some topological constraints, which we will discuss in the first part of this section. Whether it is infrared-relevant is determined by its engineering dimension, which we discuss in the second part of this section. Combining topological and engineering constraints will then produce a finite number of interaction vertices to consider. Constraints that avoid certain, otherwise relevant vertices from being generated are preserved under renormalisation.

The general proper vertex

$$\Gamma \left[\begin{array}{cc} m & n \\ p & q \end{array} \right] = \text{diagram of a circle with } m \text{ red lines and } n \text{ red lines entering, and } p \text{ wavy lines and } q \text{ wavy lines exiting} \quad (4.28)$$

are the one-particle irreducible graphs of the amputated correlation function

$$\begin{aligned} G \left[\begin{array}{cc} m & n \\ p & q \end{array} \right] (r, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \{\mathbf{k}_1, \dots, \mathbf{k}_{m+n+p+q}; \omega_1, \dots, \omega_{m+n+p+q}\}) \\ = \left\langle \underbrace{\phi(\mathbf{k}_1, \omega_1) \dots \phi(\mathbf{k}_m, \omega_m)}_{m \text{ terms}} \underbrace{\psi \dots \psi}_{p \text{ terms}} \underbrace{\tilde{\phi} \dots \tilde{\phi}}_{n \text{ terms}} \underbrace{\tilde{\psi} \dots \tilde{\psi}}_{q \text{ terms}} \right\rangle \end{aligned} \quad (4.29)$$

Denoting, where applicable, terms of higher order in non-linear couplings by h.o.t., the bare couplings are the tree-level contributions to the proper vertices:

$$\tau = \Gamma \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] + \text{h.o.t.} \quad s = \Gamma \left[\begin{array}{cc} 2 & 1 \\ 0 & 0 \end{array} \right] \quad \lambda = \Gamma \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right] + \text{h.o.t.} \quad (4.30a)$$

$$\sigma = \Gamma\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] + \text{h.o.t.} \quad \chi = \Gamma\left[\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right] + \text{h.o.t.} \quad \kappa = \Gamma\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] + \text{h.o.t.} \quad \xi = \Gamma\left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right] + \text{h.o.t.} \quad (4.30b)$$

Every proper vertex has a number of topological constraints, since any such term needs to arise from the perturbative expansion of the action as a one-particle irreducible (connected, amputated) diagram made from the bare vertices available in the theory. By inspection, we found the following constraints, which we will use to determine all relevant, possible couplings below: Firstly, all non-linear vertices in the field theory (all diagrams except the bare propagator of the tracer particles) have at least one straight leg coming in, $n \geq 1$. Secondly, all vertices have at least as many wavy legs coming out, as come in, $p \geq q$. Thirdly, there are at least as many outgoing legs (wavy or straight), as there are incoming straight legs, $m + p \geq n$.

The engineering dimension of the general proper vertex can be determined from the considerations at the beginning of Section 4.B.1, using the fact that each proper vertex may be seen as an effective coupling, which, after integration over real time and space, gives rise to a dimensionless contribution to the action, $L^d T[\Gamma\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right] \tilde{\phi}^m \tilde{\psi}^p \phi^n \psi^q] = 1$, so that

$$[\Gamma\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right]] = L^{d(n+q-1)+2(m-n+2p-2q-1)} A^{p-q} B^{m-n+p-q} C^{n-m-2p+2q+1}. \quad (4.31)$$

Demanding that (effective) transmutation τ , branching s and diffusion D may remain relevant at any scale (which amounts to a suitable continuum limit), we set the independent dimensions A , B and C , respectively, to unity $A = B = C = 1$. The (marginally) infrared-relevant couplings are those whose engineering dimension (in L) is non-positive. At the upper critical dimension $d = d_c = 4$, the inequality $d(n + q - 1) + 2(m - n + 2p - 2q - 1) \leq 0$ gives

$$m + n + 2p \leq 3. \quad (4.32)$$

The field theory needs to include all vertices $\Gamma\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right]$ with (non-negative) integers m , n , p and q that fulfill Eq. (4.32) together with the topological constraints $n \geq 1$, $p \geq q$ and $m + p \geq n$. To find them, we distinguish two cases for Eq. (4.32):

- $p = 0 \xrightarrow{p \geq q} q = 0$, then $m + n \leq 3$. Under the topological constraint $m + p \geq n$ there are only two viable solutions: $m = n = 1$, or $m = 2$ and $n = 1$, that

correspond to

$$\text{---} \quad \text{and} \quad \text{---} \begin{array}{c} \diagup \\ \diagdown \end{array} s, \quad (4.33)$$

the bare propagator for active walkers, and branching of active walkers, respectively.

- $p = 1 \implies m + n \leq 1$. Only the propagator of the immobile particles allows for $n = 0$. Otherwise, $n \geq 1$ requires $m = 0$. The constraint $p \geq q$ leaves only $q = 0$ and $q = 1$. As a result, there are three viable combinations: Firstly, $m = n = 0$ and $q = 1$, secondly, $m = q = 0$ and $n = 1$, thirdly, $m = 0$ and $n = q = 1$, which correspond to

$$\text{---} \quad \text{---} \begin{array}{c} \tau \\ \text{---} \end{array} \quad \text{and} \quad \text{---} \begin{array}{c} -\lambda \\ \text{---} \end{array}, \quad (4.34)$$

the bare propagator of immobile tracer particles, the transmutation vertex and hindrance of spawning, respectively.

Together with the propagators, the vertices in (4.33) and (4.34) represent all (marginally) relevant couplings at $d = d_c = 4$, consisting of the (bilinear) transmutation, τ , and the interaction vertices s of branching and $-\lambda$ of suppression of spawning.

In the following we perform the renormalisation of the couplings τ and $-\lambda$.

4.B.5 RENORMALISATION OF THE COUPLINGS

As far as the observables in the present work are concerned, the only couplings to consider are τ and λ . Both are renormalised by the same set of loops

$$\tau_R \hat{=} \tau = \tau + \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \dots \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \dots \quad (4.35)$$

and

$$-\lambda_R \hat{=} -\lambda = -\lambda + \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \dots \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \dots \quad (4.36)$$

where all diagrams are amputated. The subscript R indicates a renormalised quantity, which may still be dimensionfull as in the expression above. Only the non-crossing loop diagrams, such as the first three in Eqs. (4.35) and (4.36), are easily calculated (see Sec. 4.C for details). Of the diagrams in Eqs. (4.35) and (4.36), the non-crossing

ones are summed over by virtue of field-theoretic renormalisation. The last diagram in both Eq. (4.35) and Eq. (4.36), on the other hand, require further explicit calculation and subsequent summation. The same applies to an infinite number of further crossing diagrams. And yet, because of the Ward-identity (Sec. 4.B.6)

$$\frac{\partial \tau_R}{\partial \tau} = \frac{\lambda_R}{\lambda} \quad (4.37)$$

all exponents can be determined without calculating any of the diagrams explicitly.

As usual in perturbative field theory [189, 118], the governing non-linearity, here λ , becomes spatially dimensionless by multiplying it by $\mu^{-\epsilon}$, where μ is an arbitrary inverse length scale. In fact, *any* dimensionless coupling involving λ , τ , s , D and μ is proportional to a power of $\lambda s D^{-2} \mu^{-\epsilon}$. Introducing $g = \lambda s \mathcal{U} \mu^{-\epsilon} D^{-2} \Gamma(\epsilon/2)$ with suitable numerical factor $\mathcal{U}$, both couplings λ and τ renormalise identically

$$\tau_R = \tau Z(g) \quad \text{and} \quad \lambda_R = \lambda Z(g) \quad (4.38)$$

with $Z(g)$ governing the renormalisation of both λ and τ . To one loop and with suitable $\mathcal{U}$, the Z -factor becomes $Z(g) = 1 - g$, see Eqs. (4.35) and (4.36), and Sec. 4.C. However, there is no need to determine the precise dependence of Z on g as far as scaling is concerned. It suffices to know that the renormalised, dimensionless

$$\begin{aligned} g_R &= \lambda_R s \mathcal{U} \mu^{-\epsilon} D^{-2} \Gamma(\epsilon/2) \\ &= Z \lambda s \mathcal{U} \mu^{-\epsilon} D^{-2} \Gamma(\epsilon/2) \end{aligned} \quad (4.39)$$

has β -function

$$\beta_g = \frac{dg_R}{d \ln \mu} = -\epsilon g_R + g_R \frac{d \ln Z}{d \ln \mu} \quad (4.40)$$

which implies $d \ln Z / d \ln \mu = \epsilon$ at the root $\beta_g(g = g^*) = 0$, irrespective of $\mathcal{U}$ and therefore irrespective of the presence or absence of the crossing diagrams. It follows that $Z \sim \mu^\epsilon$ in $d \leq 4$ and therefore the effective transmutation rate is $\tau_{\text{eff}} \sim \tau Z \sim \mu^\epsilon$. In the limit of $t \rightarrow \infty$, for systems of linear size L , the characteristic scale is $\mu \sim L^{-1}$ and thus $\tau_{\text{eff}} \sim L^{-\epsilon}$. With open boundary conditions, the branching walkers visit $\sim L^2$ sites during the course of their lifetimes, leaving behind $\sim \tau_{\text{eff}} L^2 \sim L^{2-\epsilon}$ immobile tracer particles in dimensions greater than 2, so that $\langle a \rangle(t, L) \sim L^{d-2}$. This average

is bounded from below by a constant, as at least one site is always visited, so that $\langle a \rangle(t, L)$ approaches a constant below 2 dimensions. As for the time-dependence, the characteristic inverse scale μ is proportional to $t^{-1/2}$ because the dynamical exponent $z = 2$ in $\mu \sim t^{-1/z}$ remains unchanged. It follows that $\langle a \rangle(t, L) \sim t^{(d-2)/2}$.

In the following section, the mean $\langle a \rangle(t, L)$ and higher moments are calculated in greater detail.

4.B.6 WARD IDENTITY

To identify the Ward-identity, we first state the action

$$\mathcal{A}([\phi, \psi, \tilde{\phi}, \tilde{\psi}]; D, r, \epsilon', \tau, s, \lambda) = \int d^d x dt \left(-\tilde{\phi} \partial_t \phi + D \tilde{\phi} \nabla^2 \phi - r \tilde{\phi} \phi - \tilde{\psi} \partial_t \psi - \epsilon' \tilde{\psi} \psi + \tau \tilde{\psi} \phi + s \tilde{\phi}^2 \phi - \lambda \tilde{\psi} \psi \phi \right) \quad (4.41)$$

after having removed the irrelevant couplings from the Liouvillian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, Eqs. (4.6) and (4.7). The Ward-identity Eq. (4.37) is rooted in a symmetry of the action under shifting $\psi(\mathbf{x}, t)$ by a constant Σ ,

$$\mathcal{A}([\phi, \psi + \Sigma, \tilde{\phi}, \tilde{\psi}]; D, r, \epsilon', \tau, s, \lambda) = \mathcal{A}([\phi, \psi, \tilde{\phi}, \tilde{\psi}]; D, r, \epsilon', \tau - \lambda \Sigma, s, \lambda) + \Sigma \int d^d x dt (-\epsilon') \tilde{\psi}(\mathbf{x}, t), \quad (4.42)$$

where the last term amounts to a source term, which maintains a density of Σ of immobile particles throughout time and space, as they are subject to continuous decay with (matching) rate ϵ' . To ease notation we write

$$\mathcal{A} = \mathcal{A}([\phi, \psi, \tilde{\phi}, \tilde{\psi}]; D, r, \epsilon', \tau, s, \lambda) \quad (4.43a)$$

$$\mathcal{A}' = \mathcal{A}([\phi, \psi + \Sigma, \tilde{\phi}, \tilde{\psi}]; D, r, \epsilon', \tau, s, \lambda) \quad (4.43b)$$

$$\mathcal{A}'' = \mathcal{A}([\phi, \psi, \tilde{\phi}, \tilde{\psi}]; D, r, \epsilon', \tau - \lambda \Sigma, s, \lambda) \quad (4.43c)$$

so that

$$\mathcal{A}' = \mathcal{A}'' - \epsilon' \Sigma \int d^d x dt \tilde{\psi}(\mathbf{x}, t), \quad (4.44)$$

as well as

$$\langle \bullet \rangle_{\mathcal{A}} = \int \mathcal{D}\Pi \bullet e^{\mathcal{A}}, \quad (4.45)$$

and similarly for the actions $\mathcal{A}'$ and $\mathcal{A}''$. Since ψ is only a dummy variable in this path integral, any expectation over the action $\mathcal{A}$ of an observable involving the field ψ , is identical to the expectation of an observable involving the shifted field $\psi + \Sigma$ over the action $\mathcal{A}'$, for example

$$\left\langle \psi(\mathbf{x}_3, t_3) \psi(\mathbf{x}_2, t_2) \tilde{\psi}(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}} = \left\langle (\psi(\mathbf{x}_3, t_3) + \Sigma) (\psi(\mathbf{x}_2, t_2) + \Sigma) \tilde{\psi}(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}'} . \quad (4.46)$$

To derive the Ward-identity (4.37), we consider

$$\begin{aligned} \left\langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}} &= \left\langle (\psi(\mathbf{x}, t) + \Sigma) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}'} \\ &= \left\langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}'} + \Sigma \left\langle \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}'} \\ &= \left\langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) e^{-\epsilon' \Sigma \int d^d x dt \tilde{\psi}(\mathbf{x}, t)} \right\rangle_{\mathcal{A}''} \end{aligned} \quad (4.47)$$

using $\left\langle \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}'} = 0$ and Eq. (4.44) in the last line. Differentiation with respect to Σ and evaluating at $\Sigma = 0$ then gives

$$0 = -\lambda \partial_{\tau} \left\langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}} - \epsilon' \int d^d x' dt' \left\langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) \tilde{\psi}(\mathbf{x}', t') \right\rangle \quad (4.48)$$

as $\mathcal{A}'' = \mathcal{A}$ at $\Sigma = 0$ and the left-hand side of Eq. (4.47) is independent of Σ . The integral is most efficiently evaluated after Fourier-transforming, as $\int d^d x' dt' \tilde{\psi}(\mathbf{x}', t') = \tilde{\psi}(\mathbf{k} = 0, \omega = 0)$ and noting that

$$\left\langle \psi(\mathbf{k}', \omega') \tilde{\psi}(\mathbf{k} = 0, \omega = 0) \right\rangle = \frac{1}{\epsilon'} \delta(\omega') \delta(\mathbf{k}') \quad (4.49)$$

whenever $\tilde{\psi}(\mathbf{k} = 0, \omega = 0)$ is paired up with any internal field $\psi(\mathbf{k}', \omega')$. Dividing out two bare propagators, the right-hand side of Eq. (4.48) consists of the amputated diagrams shown in Eq. (4.35) and Eq. (4.36), so that

$$0 = -\lambda \partial_{\tau} \tau_R + \frac{\epsilon'}{\epsilon} \lambda_R, \quad (4.50)$$

the desired identity Eq. (4.37).

4.B.7 CALCULATING SCALING OF HIGHER-ORDER CORRELATION FUNCTIONS

The scaling of higher-order correlation functions is derived, within the field theory, from the solution of the Callan-Symanzik equation [189] for the general proper vertex Eq. (4.28), from which the scaling of the moments of the total number of distinct sites visited follow, Eq. (4.2). From dimensional analysis (Sec. 4.B.1), and by introducing a bare scale μ_0 , related to μ by $\mu = \mu_0 \ell$, the general proper vertex, Eq. (4.28), then satisfies

$$\begin{aligned} & \Gamma\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right](r, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \{\mathbf{k}; \omega\}) \\ &= \ell^{-d(n+q-1)-2(m-n+2p-2q-1)+(p-q)\gamma_\tau} \Gamma\left[\begin{smallmatrix} m & n \\ p & q \end{smallmatrix}\right] \left(\frac{r}{\ell^2}, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \left\{ \frac{\mathbf{k}}{\ell}; \frac{\omega}{\ell^2} \right\} \right), \quad (4.51) \end{aligned}$$

asymptotically in small ℓ and provided that r is close enough to the critical point, $r_c = 0$. For the transmutation vertex, where $p = n = 1$ and $q = m = 0$, we find

$$\Gamma^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}(r, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \{\mathbf{k}; \omega\}) = \ell^{\gamma_\tau} \Gamma^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\left(\frac{r}{\ell^2}, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \left\{\frac{\mathbf{k}}{\ell}; \frac{\omega}{\ell^2}\right\}\right), \quad (4.52)$$

with $\gamma_\tau = \varepsilon = 4 - d$. Generally, for observables of the form Eq. (4.27), where $n = 1$ and $q = m = 0$ we have

$$\Gamma_{\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}}(r, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \{\mathbf{k}; \omega\}) = \ell^{4(1-p)+p\gamma\tau} \Gamma_{\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}}\left(\frac{r}{\ell^2}, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \left\{\frac{\mathbf{k}}{\ell}; \frac{\omega}{\ell^2}\right\}\right). \quad (4.53)$$

The scaling of the first moment of the number of distinct sites visited, $\langle a(t) \rangle$, as function of time, t , can be obtained by analysing the scaling of

$$\begin{aligned}\langle a(t) \rangle &= \int d^d \mathbf{x} \langle \psi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, 0) \rangle \\ &\hat{=} \int d\omega d\omega_0 e^{-i\omega t} \left. \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \right|_{\mathbf{k}=0} \\ &= \int d\omega e^{-i\omega t} \frac{1}{-i\omega + \epsilon'} \Gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{-i\omega + r}. \end{aligned} \quad (4.54)$$

According to Eq. (4.52), $\Gamma[\overset{0}{1}\overset{1}{0}]$ scales like

$$\Gamma[\overset{0}{1}\overset{1}{0}](L^{-2}, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \{\mathbf{k}; \omega\}) = L^{-\gamma_\tau} \Gamma[\overset{0}{1}\overset{1}{0}]\left(1, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \left\{\mathbf{k}L; \omega L^2\right\}\right), \quad (4.55)$$

if we identify $r \sim L^{-2}$ and $\ell \sim L^{-1}$, which means that the effective transmutation rate scales like $L^{-\varepsilon}$ in large linear system size L , as $\gamma_\tau = \varepsilon = 4 - d$. In long time t , the integral over ω in Eq. (4.54) has the effect of evaluating $\Gamma[\overset{0}{1}\overset{1}{0}]_{\frac{1}{i\omega+r}}$ at $\omega = 0$, because

$$\lim_{t \rightarrow \infty} \lim_{\epsilon' \rightarrow 0} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{1}{-i\omega + \epsilon'} f(\omega) = f(0) \quad (4.56)$$

provided $f(\omega)$ has no pole at 0.

It follows that

$$\lim_{t \rightarrow \infty} \langle a(t) \rangle \propto L^{2-\epsilon}. \quad (4.57)$$

For higher moments, on the basis of Eq. (4.53) we find

$$\lim_{t \rightarrow \infty} \langle a^p(t) \rangle \propto L^2 L^{pd-4} \Gamma[\overset{0}{p}\overset{1}{0}](1, D, \tau, s, \sigma, \lambda, \kappa, \chi, \xi; \{0, 0\}). \quad (4.58)$$

We thus recover the finite-size scaling results Eqs. (4.2b) and (4.3b) of Section 4.3 for the p -th moment of the volume explored by a branching random walk

$$\lim_{t \rightarrow \infty} \langle a^p(t) \rangle \propto \begin{cases} L^{dp-2} & \text{if } \varepsilon > 0 \\ L^{4p-2} & \text{if } \varepsilon < 0 \end{cases} \quad (4.59)$$

where $\varepsilon > 0$ and $\varepsilon < 0$ separate regions below and above the upper critical dimension, $d_c = 4$, respectively. The dimensionality of the embedding space enters only below the upper critical dimension. Above the upper critical dimension, fluctuations and interactions become asymptotically irrelevant and the process can be considered as free.

The above analysis is easily extended to scaling in time, using $t \propto \mu^{-z}$ with $z = 2$ as the relevant scale, thereby reproducing Eqs. (4.2a) and (4.3a).

4.C LOOP INTEGRALS

The non-crossing diagrams, such as the first three in Eqs. (4.35) and (4.36), are calculated through the integral

$$I_\tau = \text{diagram} = \int \mathrm{d}^d k \, \mathrm{d}\omega \frac{\tau}{-i\omega + \epsilon'} \frac{1}{\omega^2 + (Dk^2 + r)^2} = \tau \frac{1}{2} \frac{r^{-\varepsilon/2}}{(4\pi D)^{d/2}} \Gamma(\varepsilon/2) , \quad (4.60)$$

and (essentially identical)

$$I_{-\lambda} = \text{diagram} = \int \mathrm{d}^d k \, \mathrm{d}\omega \frac{-\lambda}{-i\omega + \epsilon'} \frac{1}{\omega^2 + (Dk^2 + r)^2} = -\lambda \frac{1}{2} \frac{r^{-\varepsilon/2}}{(4\pi D)^{d/2}} \Gamma(\varepsilon/2) , \quad (4.61)$$

where the lower part of the loop carries the coupling τ in case of contributing to τ or the coupling $-\lambda$ and an incoming wavy leg in case of contributing to λ . The integration measure is $\mathrm{d}^d k \, \mathrm{d}\omega = \mathrm{d}^d k \, \mathrm{d}\omega / (2\pi)^{d+1}$.

4.D GENERALISATION TO k OFFSPRING

In this section we extend the field-theoretic results presented above to the case where the offspring number is a random number and show that it lies in the same universality class as binary branching [3, 119]. Instead of two distinct processes for branching into two active walkers (with rate s above) and getting extinguished (with rate e above) we consider the latter as branching into $k = 0$ walkers and generalise the former to branching into any number k of walkers. Each of these processes may occur with rate σ_k , which can always be written as $\sigma_k = \sigma p_k$ with p_k the normalised probability for branching into k walkers and σ the rate with which any such processes take place.

The two contributions $\mathcal{P}_s$, Eq. (4.13), and $\mathcal{P}_e$, Eq. (4.14), are thus subsumed and generalised by

$$\dot{\mathcal{P}}_c(\{n\}, \{m\}; t) = \sigma \sum_{k=0}^{\infty} \sum_{\mathbf{x}} p_k \left((n_{\mathbf{x}} - k + 1) \mathcal{P}(\{\dots, n_{\mathbf{x}} - k + 1, \dots\}, \{m\}; t) - n_{\mathbf{x}} \mathcal{P}(\{n\}, \{m\}; t) \right), \quad (4.62)$$

which allows for p_1 , but the process of branching into a single particle has no bearing on the master equation.

In the field theory, the mass of the bare propagator for active walkers becomes [77] (Chapter 2)

$$r = -\sigma \sum_{k=0}^{\infty} p_k(k-1) = \sigma(1 - \bar{k}), \quad (4.63)$$

where $\bar{k} = \sum_{k=0}^{\infty} p_k k$ is the average offspring number, which again, defines a subcritical ($r > 0$), a critical ($r = 0$), and a supercritical ($r < 0$) regime.

In the case of generalised branching, the non-linear part of the action contains contributions of the form $\tilde{\phi}^k \phi$ for all $k \geq 2$ as soon as there is any $k \geq 2$ with $p_k > 0$ [77]. Terms with $k > 2$, however, turn out to be infrared irrelevant, as their couplings have dimension $\mathbf{B}^{k-1} \mathbf{C}^{2-k} \mathbf{L}^{2(k-2)}$. The field theoretic results above for binary branching therefore govern also branching processes with generalised offspring distribution.

4.E EXTENSION TO GENERAL GRAPHS

In this section we provide further details about the extension of our results to general graphs. The loops integrated over in Eqs. (4.60) and (4.61) are in fact integrals over the spectrum of the Laplacian accounting for the diffusion on the graph considered. Generalising to arbitrary graphs, the Laplacian is to be replaced by a lattice-Laplacian and the integral in Eqs. (4.60) and (4.61) by a suitable sum or, equivalently, an integral with suitable spectral density. In fact, the d -dimensional integral in Eqs. (4.60) and (4.61) can be seen as an integral over all distinct eigenvalues $\mathbf{k}^2$ of the Laplacian entering with weight $w(k)dk = S_d k^{d-1} dk$ with $S_d = 2\pi^{d/2}/\Gamma(d/2)$. On regular lattices, their Hausdorff dimension d coincides with the spectral dimension d_s characterising, in particular, the small k asymptote of $w(k) \sim k^{d_s-1}$. Replacing $\int d^d k$ by $\int dk w(k)$ suggests that the results derived above remain valid by replacing d by d_s , in order to recover the scaling of the various observables in arbitrary graphs with spectral dimension d_s . The replacement $d \rightarrow d_s$ hinges crucially on the fact that d_s characterises the scaling of the spectral density of the Laplacian. If this operator itself renormalises, then a different spectral density may be needed. In other words, d_s may not be the correct dimension if the Laplacian renormalises, *i.e.* if the anomalous dimension does not vanish, $\eta \neq 0$

[25]. This argument relies on the assumption that vertices such as Eq. (4.28) preserve momentum, that is integrals of the form

$$I_n(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n) = \int d^d x u_{\mathbf{k}_1}(x) u_{\mathbf{k}_2}(x) \dots u_{\mathbf{k}_n}(x) \quad (4.64)$$

over eigenfunctions $u_{\mathbf{k}}(x)$ of the Laplacian with eigenvalue $\mathbf{k} \cdot \mathbf{k}$ vanish for off-diagonal terms, i.e. whenever $\mathbf{k}_1 + \mathbf{k}_2 + \dots + \mathbf{k}_n \neq 0$. This condition can be further relaxed by demanding merely that off-diagonal terms are sub-leading as observed in the presence of boundaries [55, 138].

Considering only graphs which are translationally invariant such that the indices $\mathbf{j}_m$ of the q neighbours $m = 1, \dots, q$ of any node $\mathbf{i}$ can be determined by adding the same set of translational lattice vectors, $\mathbf{d}_1, \dots, \mathbf{d}_q$, such that $\mathbf{j}_m = \mathbf{i} + \mathbf{d}_m$, it is easy to show that the Laplacian has exponential eigenfunctions and any of their products are an eigenfunction as well, so that $I_n(\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n) = I_2(\mathbf{k}_1, \mathbf{k}_2 + \dots + \mathbf{k}_n)$, which vanishes by orthogonality for any $\mathbf{k}_1 + \mathbf{k}_2 + \dots + \mathbf{k}_n \neq \mathbf{0}$, i.e. the assumption of momentum conservation mentioned above is fulfilled.

4.F NUMERICS FOR THE SCALING OF MOMENTS

The scaling of the moments $\langle a^p \rangle(t, L)$ for $p = 1, 2, 3, \dots, 5$, as function of time t in the limit $L \rightarrow \infty$, and as function of the system size L in the limit $t \rightarrow \infty$ were obtained from numerical Monte Carlo simulations and fitted against a power-law

$$f(x) = Ax^B \quad (4.65)$$

and a power-law with corrections of the form

$$g(x) = Ax^B + Cx^{B-1/2}. \quad (4.66)$$

The fitting parameter B in Eqs. (4.65) and (4.66) provides the estimates of the exponents that characterise the scaling of the moments in time t and system size L (or N , see main text), by fitting the numerical estimates against $f(x)$ and $g(x)$, with x replaced by t and L , respectively. At large times the moments display plateauing due to finite-size effects.

For the scaling in system size L , we fitted the data for the latest time point available against Eq. (4.65) and used the estimates of A and B as the initial values for a fit against Eq. (4.66), which gave the final estimates of the finite-size scaling exponents.

For the scaling in time t , we fitted data for the largest system, of size $L = L_{\max}$. The fitting range in t for each moment was determined systematically as follows. To remove the time-point affected by the finite-size effects, we defined the upper bound of the fitting range as the time t^{up} for which the lowest moment displaying algebraic divergence ($p = p_{\text{low}}$) reached a value of half the maximum value in the plateau, *i.e.* $\langle a^{p_{\text{low}}} \rangle(t^{\text{up}}, L_{\max}) = \max_t (\langle a^{p_{\text{low}}} \rangle(t, L_{\max})) / 2$. For the preferential attachment network the plateau was observed to occur at an earlier time point than t^{up} , probably due to the high connectivity of the networks, so we set the upper bound to $t_{pa}^{\text{up}} = (1/5) \max(\langle a^k \rangle)$, in this case.

To find the lower bound t_{low} of the fitting range in t we fitted both equations, (4.65) and (4.66), to the data for $L_{\max}$. We define $\hat{f}_{[t^*, t^{\text{up}}]}(t)$ and $\sigma_{[t^*, t^{\text{up}}]}^{\hat{f}}(t)$ as the values and errors, respectively, of fitting Eq. (4.65) to the data in the range $t \in [t^*, t^{\text{up}}]$, and $\hat{g}_{[t^*, t^{\text{up}}]}(t)$ and $\sigma_{[t^*, t^{\text{up}}]}^{\hat{g}}(t)$ as the values and errors, respectively, of fitting Eq. (4.66) to the same data set and range. Further, we define $N_{[t^*, t^{\text{up}}]}$ as the number of data points within the fitting interval $[t^*, t^{\text{up}}]$. The lower bound for the time range t_{low} is the earliest time at which both fitting models (4.65) and (4.66) agree within errors, that is

$$t_{\text{low}} = \min \left\{ t^* : |\hat{f}_{[t^*, t^{\text{up}}]}(t^*) - \hat{g}_{[t^*, t^{\text{up}}]}(t^*)| \leq \sqrt{N_{[t^*, t^{\text{up}}]}} \max \left(\sigma_{[t^*, t^{\text{up}}]}^{\hat{f}}(t^*), \sigma_{[t^*, t^{\text{up}}]}^{\hat{g}}(t^*) \right) \right\}. \quad (4.67)$$

Where we account for correlations between estimates of moments by rescaling the error by the square root of the number of data points in the fitting range, $N_{[t^*, t^{\text{up}}]}$. The exponents characterising the time dependence of the moments are determined by fitting the data in the range $[t^*, t^{\text{up}}]$ against Eq. (4.66).

The fitting of the power laws, Eqs. (4.65) and (4.66), was done by means of the Levenberg-Marquardt algorithm [157]. In table 4.1 and 4.2 we report the numerical results for the asymptotic scaling in time, $\langle a^p \rangle(t) \sim t^{\alpha_p}$, and in system size, $\langle a^p \rangle(t) \sim L^{\beta_p}$, provided these observables display an algebraic divergence.

Table 4.1: Scaling of visited sites in time. Scaling in time, $\langle a^p \rangle(t) \sim t^{\alpha_p}$, of the p -th moment of the number of distinct sites visited for regular lattices of integer dimension, d , as indicated, and for the Sierpinski carpet (S.C., $d_s \approx 1.86$), the random tree (R.T., $d_s = 4/3$), and preferential attachment (P.A., $d_s > 4$) networks. The columns marked *num* shows the numerical results, the columns marked *theo* show theoretical results according to Eqs. (4.2a) with d replaced by the spectral dimension d_s where applicable. The row marked *mean gap* show the average gap-exponent, $(1/p) \sum_{i=1}^p (\alpha_{i+1} - \alpha_i)$, for the corresponding lattice.

Exponent	$d = 1$		$d = 2$		$d = 3$		$d = 5$		S.C.		R.T.		P.A.	
	num	theo	num	theo	num	theo	num	theo	num	theo	num	theo	num	theo
α_1					0.47(2)	1/2	1.0(2)	1					1.0(1)	1
α_2			0.98(3)	1	2.0(1)	2	2.9(3)	3	0.81(5)	0.86	0.35(7)	1/3	2.8(2)	3
α_3	0.48(4)	1/2	2.0(1)	2	3.5(1)	7/2	4.8(4)	5	1.71(7)	1.79	0.9(1)	1	4.8(4)	5
α_4	1.0(1)	1	2.9(1)	3	5.0(2)	5	6.7(7)	7	2.62(10)	2.72	1.6(2)	5/3	6.6(5)	7
α_5	1.5(1)	3/2	3.9(1)	4	6.4(2)	13/2	9(1)	9	3.54(14)	3.66	2.2(4)	7/3	8.5(9)	9
Mean gap	0.5(1)	1/2	1.0(1)	1	1.5(2)	3/2	2.0(5)	2	0.91(9)	0.93	0.6(2)	2/3	1.9(4)	2

Table 4.2: Scaling of visited sites by a BRW as function of the system size. Scaling, $\langle a^p \rangle(t) \sim L^{\beta_p}$, of the p -th moment of the number of distinct sites visited as function of the system size L , for regular lattices of integer dimension d as indicated and for the Sierpinski carpet (S.C., $d_s \approx 1.86$). The columns marked *num* show the numerical results, the columns marked *theo* show theoretical results according to Eq. (4.2b), for regular lattices, and S.C. (with d replaced by the spectral dimension d_s), and according to Eq. (4.5) for random tree (R.T.) and preferential attachment (P.A.) The row marked *mean gap* shows the average gap-exponent, $(1/p) \sum_{i=1}^p (\beta_{i+1} - \beta_i)$, for the corresponding lattice. *Goodness of fit < 0.05 .

Exponent	$d = 1$		$d = 2$		$d = 3$		$d = 5$		S.C.		R.T.		P.A.	
	num	theo	num	theo	num	theo	num	theo	num	theo	num	theo	num	theo
β_1					0.97(4)	1	1.9(2)*	2					0.49(1)	1/2
β_2			2.1(2)	2	3.9(1)*	4	5.7(5)*	6	1.9(1)	1.72	0.58(6)	1/2	1.49(1)	3/2
β_3	0.96(4)	1	4.2(3)	4	6.8(2)*	7	10(1)	10	3.8(2)	3.59	1.6(1)	3/2	2.49(2)	5/2
β_4	1.93(7)	2	6.1(3)	6	9.8(3)*	10	14(2)	14	5.7(3)	5.45	2.6(2)	5/2	3.49(2)	7/2
β_5	2.93(8)	3	8.1(4)	8	12.7(4)*	13	17(3)	18	7.5(4)	7.31	3.7(2)	7/2	4.49(2)	9/2
Mean gap	0.9(1)	1	2.0(3)	2	2.9(2)	3	4(1)	4	1.9(3)	1.86	1.0(1)	1	1.00(2)	1
Fit range	[255, 4095]		[15, 127]		[7, 127]		[7, 31]		[9, 243]		[2 ⁶ - 1, 2 ¹² - 1]		[2 ¹⁴ - 1, 2 ¹⁹ - 1]	

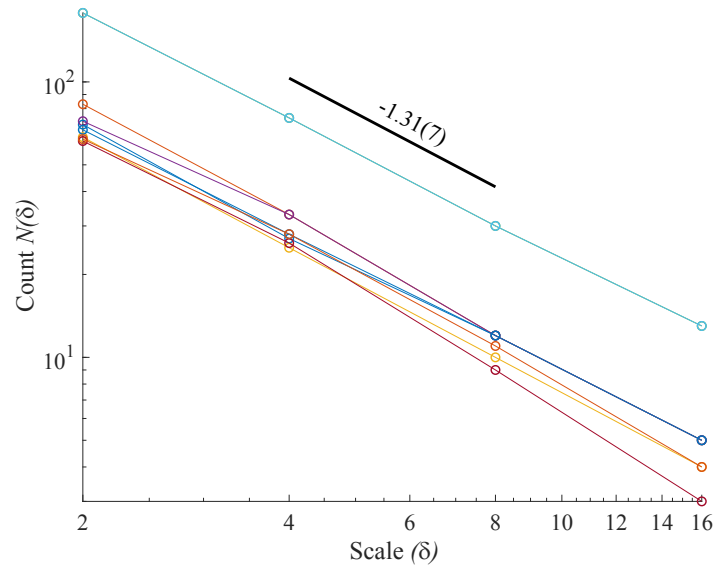


Figure 4.6: Degree distribution of the preferential attachment networks used for the simulations presented in Sec. 4.4

5

Phase transition, scaling of moments, and order-parameter distributions in Brownian particles and branching processes with finite-size effects

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OVERVIEW A Galton-Watson branching process (which is a branching process in discrete time) displays finite-size effects if it is stopped at generation L . This process is equivalent to a (biased) random walk through the Harris map [98]. At the same time, in the continuum limit, the random walk can be approximated by diffusive motion (with drift). Then, making use of these relations, the link between the avalanche size of the Galton-Watson branching process and the first-passage time of a diffusive particle is established [68]. We use this map to study the distribution, first and second

moments of the avalanche size in each regime (critical, subcritical and supercritical) of the branching process phase transition. We further study the finite-size scaling of the size moments in the three regimes.

Author contributions: AC wrote the paper. AC conceived the idea of using the Harris map to characterise branching processes in the continuum limit and derived the analytical results; RGM did the numerical simulations and data analysis; NM gave background on first-passage times; FFC advised about the numerical inversion of the Laplace transform and first-passage times.

ABSTRACT

We revisit the problem of Brownian diffusion with drift in order to study finite-size effects in the geometric Galton-Watson branching process. This is possible because of an exact mapping between one-dimensional random walks and geometric branching processes, known as the Harris walk. In this way, first-passage times of Brownian particles are equivalent to sizes of trees in the branching process (up to a factor of proportionality). Brownian particles that reach a distant boundary correspond to percolating trees, and those that do not reach the boundary correspond to non-percolating trees. In fact, both systems display a second-order phase transition between “insulating” and “conducting” phases, controlled by the drift velocity in the Brownian system. In the limit of large system size, we obtain exact expressions for the Laplace transforms of the probability distributions and their first and second moments. We also show that these quantities obey finite-size scaling laws.

5.1 INTRODUCTION

Random walks and rooted trees are important models in probability theory and statistical physics [98]. Random walks [113, 205, 206] provide a microscopic model for diffusion processes [42], and rooted trees are the geometric representation of branching processes [99, 110]. The Galton-Watson process, which is the simplest branching process and is at the heart of self-organized-critical behaviour [38, 220], is essentially the same as mean-field percolation [34, 185] when the offspring distribution of the former is binomial. Percolation, meanwhile, is one of the simplest examples of a second-order phase

transition [135, 184, 217]. Moreover, important characteristics of phase transitions also show up in bifurcations in low-dimensional dynamical systems [41]. An exact, purely geometric mapping between walks and rooted trees was presented in [98], thereby connecting results from random walks to branching processes and beyond [68]. Fig. 5.1 offers a scheme for these relations.

This mapping, known as the Harris walk, implies that a realization of a finite-size geometric Galton-Watson branching process with no more than L generations corresponds (exactly) to a random walker confined between absorbing and reflecting boundaries (at $X = 0$ and $X = L$, respectively). Recently, Font-Clos and Moloney [68] applied this mapping to derive the distribution of the size of the percolating clusters in a finite Bethe lattice, by using the first-passage time to the origin of a Brownian particle conditioned to first reach L . In the critical case (unbiased diffusion), a Kolmogorov-Smirnov distribution is obtained (as in [22]). In the subcritical case (diffusion with negative drift) these authors find that the size of the percolating cluster follows Gaussian statistics, whereas in the supercritical case (diffusion with positive drift), an exponential-like distribution is reported.

In this work we follow the approach of [68] and use the first-passage time in a diffusion process to calculate the size distribution of both percolating and non-percolating clusters in a geometric Galton-Watson process with a finite number of generations. First, we solve the corresponding diffusion problem and derive analytical expressions and scaling laws (Sec. 5.2); then we translate the results to the random-walk picture (Sec. 5.3); and finally, by means of the mapping from trees to walks [98], we obtain the properties of the associated branching process. A discussion about the most appropriate definition of an order parameter in these systems is also provided. Our main focus is the size distribution of all clusters (whether they percolate or not). Given that the percolating case was thoroughly studied in [68], we will only provide details for the non-percolating case.

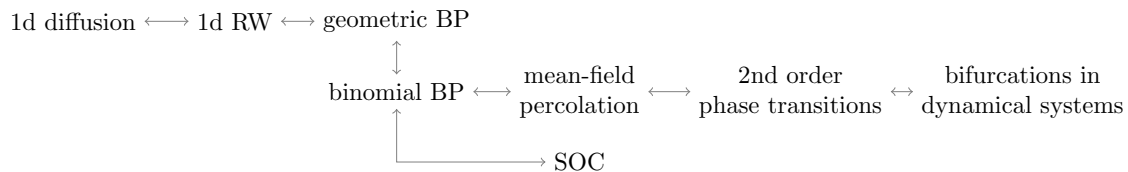


Figure 5.1: Relations between different physical processes and models, with abbreviations RW (random walk), BP (Galton-Watson branching process), SOC (self-organized critical phenomena) [34, 105, 202]. BP with geometric and binomial offspring distribution share the same critical properties due to universality.

5.2 FIRST-PASSAGE TIMES VIA THE DIFFUSION EQUATION

Consider the one-dimensional diffusion equation with drift,

$$\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2} \quad (5.1)$$

which describes the evolution of the concentration $c(x, t)$ of particles at position x and time t , with drift velocity v and diffusion constant D . Both position and time are continuous. We will work in a finite interval, $0 \leq x \leq \ell$, with ℓ playing the role of system size (X and L mentioned in the introduction are dimensionless versions of x and ℓ). Following Redner [164], first-passage times are most readily obtained by applying the Laplace transform $c(x, s) = \int_0^\infty e^{-st} c(x, t) dt$ to the diffusion equation, yielding

$$sc(x, s) - c(x, t = 0) + vc'(x, s) = Dc''(x, s), \quad (5.2)$$

where the prime denotes a derivative with respect to x .

5.2.1 ABSORBING BOUNDARIES

First-passage time probability densities $f(t)$ are obtained from spatial concentration gradients at absorbing boundaries [164]. To see this, we track the rate of particle loss

from the interval $0 \leq x \leq \ell$,

$$f(t) = -\frac{d}{dt} \int_0^\ell c(x, t) dx = -\int_0^\ell D \frac{\partial^2 c}{\partial x^2} dx + \int_0^\ell v \frac{\partial c}{\partial x} dx = -D \frac{\partial c}{\partial x} \Big|_{x=\ell} + D \frac{\partial c}{\partial x} \Big|_{x=0}, \quad (5.3)$$

where we have made use of the normalization of $c(x, t)$ for $t = 0$ and of the absorbing boundary conditions at $x = 0$ and $x = \ell$,

$$c(x = 0, t) = c(x = \ell, t) = 0. \quad (5.4)$$

Note that the terms in the sum above are not probability densities themselves (because they are not normalized). Rather, they are the net outflux of particles at each boundary, so that $f(t) = j_\ell(t) + j_0(t)$. The Laplace transform of the probability density can therefore be written as

$$f(s) = j_\ell(s) + j_0(s) = -Dc'(x = \ell, s) + Dc'(x = 0, s). \quad (5.5)$$

With Dirac- δ initial condition centered at x_0 ,

$$c(x, t = 0) = \delta(x - x_0), \quad (5.6)$$

the solution of the Laplace-transformed diffusion equation with two absorbing boundaries is [164]

$$c(x, s) = \frac{e^{(x/\ell - u_0)P_e} \sinh(Rx_</\ell) \sinh[R(\ell - x_>)/\ell]}{DR \sinh(R)/\ell}, \quad (5.7)$$

where P_e is a dimensionless parameter known as the Péclet number (up to a factor of $1/2$ according to convention),

$$P_e = \frac{\ell v}{2D}, \quad (5.8)$$

u_0 is the dimensionless initial position and τ a diffusion time,

$$u_0 = \frac{x_0}{\ell}, \quad \tau = \frac{\ell^2}{D}, \quad (5.9)$$

and, for convenience of notation,

$$R = \sqrt{P_e^2 + \tau s}, \quad \text{and} \quad x_< = \min(x, x_0), \quad x_> = \max(x, x_0). \quad (5.10)$$

5.2.2 ABSORPTION AT $x = 0$

Using the above formalism, Redner [164] examines the driftless case, $v = 0$, in full detail. For completeness, we provide the calculation for $v \neq 0$ in App. 5.A. In summary, the Laplace transform of the first-passage time density, $f_0(s)$, for absorption at $x = 0$ can be expanded in powers of u_0 as

$$f_0(s) = 1 - \left(\frac{R}{\tanh R} - \frac{P_e}{\tanh P_e} \right) u_0 + \mathcal{O}(u_0^2). \quad (5.11)$$

A numerical inversion of $f_0(s)$ is shown in Fig. 5.2.

The first two moments can be similarly expanded as

$$\langle t_0 \rangle = \frac{1}{2P_e \tanh P_e} \left(1 + P_e \tanh P_e - \frac{P_e}{\tanh P_e} \right) \tau u_0 + \mathcal{O}(u_0^2), \quad (5.12a)$$

$$\langle t_0^2 \rangle = \frac{1}{2P_e^3 \tanh P_e} \left(\frac{1}{2} - \frac{P_e \tanh P_e}{2} + \frac{P_e}{2 \tanh P_e} + P_e^2 - \frac{P_e^2}{\tanh^2 P_e} \right) \tau^2 u_0 + \mathcal{O}(u_0^2). \quad (5.12b)$$

A plot of $\langle t_0 \rangle$ as a function of P_e is shown in Fig. 5.3.

Note that these expressions are even in P_e . The expansions are valid in the limit of small u_0 , that is $x_0 \ll \ell$ (more precisely, $u_0 P_e \ll 1$ and $u_0 R \ll 1$). See Eqs. (5.66), (5.67), and (5.68) in App. 5.A for their full derivation.

5.2.3 CRITICAL POINT

The critical point corresponds to diffusion with no drift, $P_e = 0$ (i.e. $v = 0$). Using the results of App. 5.A, the exact distribution reads

$$f_0^*(s) = \frac{\sinh[(1 - u_0)\sqrt{\tau s}]}{(1 - u_0) \sinh \sqrt{\tau s}}, \quad (5.13)$$

where the asterisk denotes the critical point. To first order in u_0

$$f_0^*(s) = 1 - \left(\frac{\sqrt{\tau s}}{\tanh \sqrt{\tau s}} - 1 \right) u_0 + \mathcal{O}(u_0^2). \quad (5.14)$$

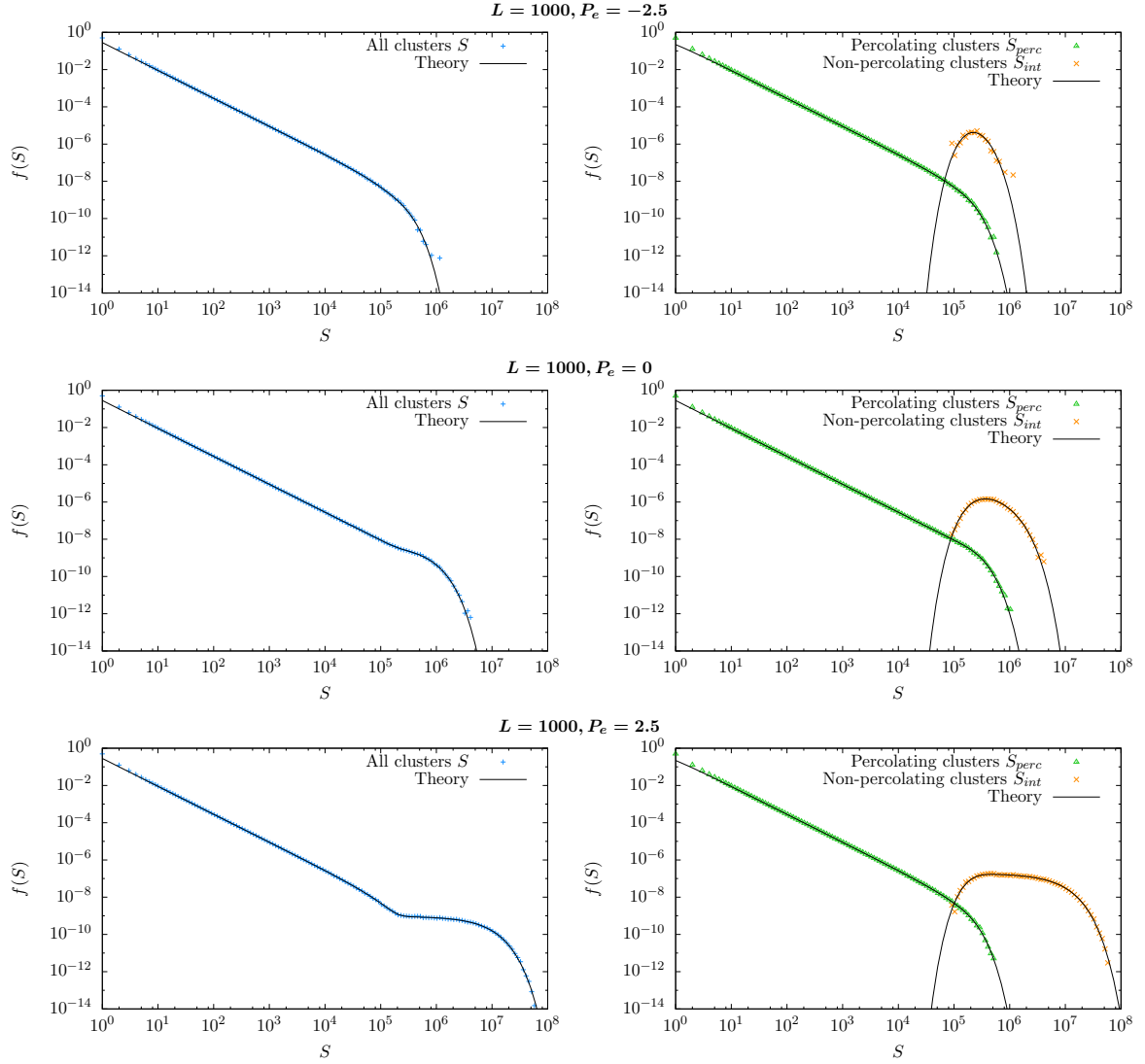


Figure 5.2: Distribution of sizes in the geometric Galton-Watson process in the three regimes: subcritical ($P_e = -2.5$, top), critical ($P_e = 0$, middle), and supercritical ($P_e = 2.5$, bottom), together with theoretical predictions from the first-passage times of a diffusion process (numerical inversion of the Laplace transform is done by the Talbot method using the routine in [139]). Plots on the left show the full distribution. Plots on the right show the distribution decomposed into its percolating and non-percolating contributions. In the critical regime, the complete distribution $f(S)$ (middle left) shows a small bump for large sizes, which is much more pronounced in the supercritical regime. The subcritical distribution, having no bump (top left), can be visually confused with a critical distribution. Logarithmic binning has been applied [34, 46].

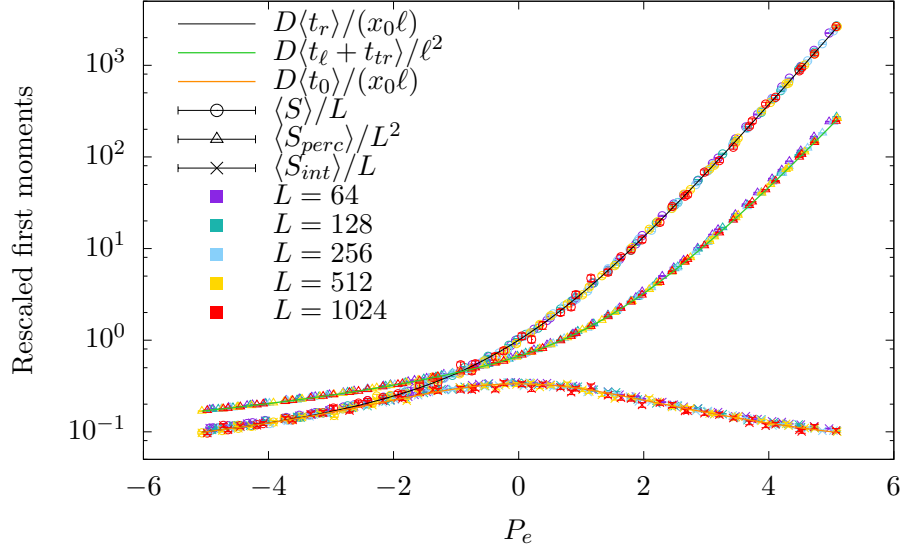


Figure 5.3: Rescaled average avalanche size (simulations, 10^5 realisations, markers) **and rescaled first-passage times** (theory, solid lines) as a function of the Péclet number for different systems sizes L .

Fig. 5.2 shows a plot of the numerical inverse Laplace transform of $f_0^*(s)$. The case $P_e = 0$ and ℓ infinite corresponds to $\tau s \gg 1$, for which

$$f_0^*(s) = j_0^*(s) \simeq 1 - u_0 \sqrt{\tau s} + \mathcal{O}(u_0^2). \quad (5.15)$$

After inverting the Laplace transform (see for instance Eq. (4.6.23) of [19]),

$$f_0^*(t) = j_0^*(t) \simeq \frac{u_0}{2} \sqrt{\frac{\tau}{\pi t^3}} + \mathcal{O}(u_0^2), \quad (5.16)$$

for $P_e = 0$ and $t \ll \tau = \ell^2/D$, thereby recovering the well-known $t^{-3/2}$ behaviour of first-passage to the origin in an infinite system.

To lowest order in u_0 , the moments are given by

$$\langle t_0^* \rangle = \frac{\tau u_0}{3} = \frac{x_0 \ell}{3D}, \quad (5.17a)$$

$$\sigma_0^* \simeq \sqrt{\langle t_0^{2*} \rangle} = \sqrt{\frac{2\tau^2 u_0}{45}} = \sqrt{\frac{2x_0 \ell^3}{45D^2}}, \quad (5.17b)$$

expanding Eq. (5.14) for small $\sqrt{\tau s} = \ell \sqrt{s/D}$.

5.2.4 ABSORBING BOUNDARY AT $x = \ell$ AND SPLITTING PROBABILITY

Although already studied in [68], for completeness we summarize the results for absorption at $x = \ell$. In the notation of this article, we extract from Eq. (5.7) the expansion

$$j_\ell(s) = -Dc'(x = \ell, s) = \frac{e^{(1-u_0)P_e} \sinh(u_0 R)}{\sinh R} = \frac{e^{P_e} R u_0}{\sinh R} + \mathcal{O}(u_0^2), \quad (5.18)$$

valid for $u_0 \ll 1$. For $z = \tau s / P_e^2 \ll 1$,

$$j_\ell(s) = \frac{e^{P_e} P_e u_0}{\sinh P_e} \left[1 - \left(-1 + \frac{P_e}{\tanh P_e} \right) \frac{z}{2} + \frac{1}{2} \left(\frac{2P_e^2}{\tanh^2 P_e} - \frac{P_e}{\tanh P_e} - P_e^2 - 1 \right) \frac{z^2}{4} + \mathcal{O}(z^3) \right] + \mathcal{O}(u_0^2), \quad (5.19)$$

So, to zeroth order in u_0 ,

$$\langle t_\ell \rangle = \frac{\tau}{2P_e^2} \left(\frac{P_e}{\tanh P_e} - 1 \right) + \mathcal{O}(u_0), \quad (5.20a)$$

$$\langle t_\ell^2 \rangle = \frac{\tau^2}{4P_e^4} \left(\frac{2P_e^2}{\tanh^2 P_e} - \frac{P_e}{\tanh P_e} - P_e^2 - 1 \right) + \mathcal{O}(u_0). \quad (5.20b)$$

Note that the common multiplying factor in Eq. (5.19),

$$C_\ell = j_\ell(0) = \frac{e^{(1-u_0)P_e} \sinh(u_0 P_e)}{\sinh P_e} = \frac{e^{P_e} P_e u_0}{\sinh P_e} + \mathcal{O}(u_0^2) \quad (5.21)$$

gives the ratio between the outflux of particles at ℓ , denoted by $j_\ell(t)$, and the probability density $f_\ell(t)$ of the first-passage time to the boundary at $x = \ell$; i.e. $j_\ell(t) = C_\ell f_\ell(t)$, so that,

$$f_\ell(s) = \frac{j_\ell(s)}{C_\ell} = \frac{\sinh P_e}{\sinh R} \frac{\sinh(u_0 R)}{\sinh(u_0 P_e)} = \frac{R \sinh P_e}{P_e \sinh R} + \mathcal{O}(u_0). \quad (5.22)$$

C_ℓ is known as the splitting probability [65, 164]. Note that $\langle t_\ell \rangle$, $\langle t_\ell^2 \rangle$, and $f_\ell(s)$ (and therefore $f_\ell(t)$) are even for P_e (but not $j_\ell(s)$ and C_ℓ). For $P_e = 0$ (at the critical point) one has that $f_\ell^*(s)$, to first order in u_0 , is the Laplace transform of the celebrated

Kolmogorov-Smirnov distribution [22, 68],

$$f_\ell^*(s) = \frac{\sinh(u_0\sqrt{\tau s})}{u_0 \sinh \sqrt{\tau s}} = \frac{\sqrt{\tau s}}{\sinh \sqrt{\tau s}} + \mathcal{O}(u_0). \quad (5.23)$$

The factor C_ℓ is also the probability that a particle is absorbed at $x = \ell$. Eq. (5.21) is the same scaling law found in [39, 76], as explained in the next section. A formula for C_ℓ also appears in [65, 68, 164], but without expanding in u_0 , therefore hiding its finite-size scaling. The probability that a particle is instead absorbed at $x = 0$ is $C_0 = 1 - C_\ell \simeq 1$ (to zeroth order in u_0).

Combining the results for t_ℓ with those obtained previously for t_0 , we have the solution of one-dimensional diffusion between two absorbing boundaries. In App. 5.B, we show that it displays a phase transition with finite-size scaling when $x_0 \ll \ell$ [65].

5.2.5 REFLECTING BOUNDARY AT $x = \ell$

We now consider first-passage times to $x = 0$, starting from a reflecting boundary at $x = \ell$. The initial condition is most conveniently handled by injecting into an empty interval a single particle at $x = \ell$. Together with a zero flux condition at this boundary for $t > 0$, this stipulates that

$$j_{tr\ell}(t) = \left[vc(x, t) - D \frac{\partial c(x, t)}{\partial x} \right]_{x=\ell} = -\delta(t), \quad (5.24)$$

where the $-\delta(t)$ term ensures that the injection (hence minus sign) occurs at $t = 0$. In Laplace space, this boundary condition reads

$$vc(x = \ell, s) - Dc'(x = \ell, s) = -1. \quad (5.25)$$

Note that in this case there is no dependence on u_0 , since the particle is injected precisely at $x = \ell$. The absorbing boundary condition at $x = 0$ remains unchanged.

The solution of the diffusion equation Eq. (5.2) with these boundary conditions is

provided in detail in [68, 164]. The outflux at the absorbing boundary is

$$f_{tr}(s) = j_{tr0}(s) = Dc'(x=0, s) = \frac{e^{-P_e R}}{R \cosh R - P_e \sinh R}, \quad (5.26)$$

where the subscript tr denotes *transmission* from the reflecting to the absorbing boundary [164]. In contrast to the case of two absorbing boundaries, this Laplace transform is not even in P_e . The exact moments are given by

$$\langle t_{tr} \rangle = \frac{\tau}{4P_e^2} (e^{2P_e} - 2P_e - 1), \quad (5.27a)$$

$$\langle t_{tr}^2 \rangle = \frac{\tau^2}{8P_e^4} [e^{4P_e} + (1 - 6P_e)e^{2P_e} + 2(P_e^2 - 1)], \quad (5.27b)$$

see [164]. For the critical case ($P_e = 0$), we recover

$$f_{tr}^*(s) = j_{tr}^*(s) = \frac{1}{\cosh \sqrt{\tau s}}, \quad (5.28)$$

as in [68].

Since t_ℓ and t_{tr} are independent random variables, the total time $t_{\ell tr} = t_\ell + t_{tr}$ until absorption, having first reached $x = \ell$ from the origin, Laplace transforms as [68]

$$f_{\ell tr}(s) = f_\ell(s) f_{tr}(s) = \frac{R^2 e^{-P_e} \sinh P_e}{P_e R \sinh R \cosh R - P_e^2 \sinh^2 R} + \mathcal{O}(u_0). \quad (5.29)$$

The average first-passage time $\langle t_{\ell tr} \rangle$ and the inverse Laplace transforms of $f_{\ell tr}(s)$ and $f_{\ell tr}^*(s)$ are shown in Figs. 5.2 and 5.3.

5.2.6 ENTIRE DIFFUSION PROBLEM: MOMENTS

We are now in a position to study the entire diffusion problem. The quantity of primary interest is the first-passage time to $x = 0$, denoted t_r , where the subscript r refers to the so-called reflection mode [164]. This time is a mixture of two times: t_0 (for realizations that do not reach $x = \ell$ before absorption at $x = 0$), and $t_{\ell tr} = t_\ell + t_{tr}$ (for realizations that do reach $x = \ell$ before absorption at $x = 0$). The weight of each time is given by 1 (to zeroth order in u_0) and C_ℓ (from Eq. (5.21)), respectively. Thus, to lowest order in

u_0 , the expected value of t_r is

$$\begin{aligned}\langle t_r \rangle &= \langle t_0 \rangle + C_\ell(\langle t_\ell \rangle + \langle t_{tr} \rangle) \\ &= \left\{ \frac{1}{2P_e \tanh P_e} \left(1 + P_e \tanh P_e - \frac{P_e}{\tanh P_e} \right) \right. \\ &\quad \left. + \frac{e^{P_e} P_e}{\sinh P_e} \left[\frac{1}{2P_e^2} \left(\frac{P_e}{\tanh P_e} - 1 \right) + \frac{e^{2P_e} - 2P_e - 1}{4P_e^2} \right] \right\} \tau u_0 + \mathcal{O}(u_0^2),\end{aligned}\quad (5.30)$$

see Fig. 5.3. For the second moment we find

$$\langle t_r^2 \rangle = \langle t_0^2 \rangle + C_\ell(\langle t_\ell^2 \rangle + \langle t_{tr}^2 \rangle + 2\langle t_\ell \rangle \langle t_{tr} \rangle), \quad (5.31)$$

to lowest order in u_0 . This result is due to the fact that the moments of any order (with respect to the origin $t = 0$) are additive in a mixture of random variables (including the corresponding weights), but not for the sum of independent random variables (t_ℓ and t_{tr}), for which only the variances are additive. The particular form of $\langle t_r^2 \rangle$ can be obtained directly from Eqs. (5.12b), (5.20b), and (5.27b). To first order in u_0 , $\sigma_r^2 = \langle t_r^2 \rangle$.

5.2.7 POSSIBLE ORDER PARAMETERS

Asymptotically, the first moment of t_r , Eq. (5.30), behaves as

$$\langle t_r \rangle \sim \langle t_0 \rangle \sim \frac{\tau u_0}{2|P_e|} = \frac{x_0}{|v|} \quad \text{for } P_e \rightarrow -\infty, \quad (5.32a)$$

$$\langle t_r^* \rangle = \left[\frac{1}{3} + 1 \cdot \left(\frac{1}{6} + \frac{1}{2} \right) \right] \tau u_0 = \frac{\ell x_0}{D} \quad \text{for } P_e = 0, \quad (5.32b)$$

$$\langle t_r \rangle \sim C_\ell \langle t_{tr} \rangle \sim \frac{\tau u_0 e^{2P_e}}{2P_e} = \frac{x_0 e^{\ell v/D}}{v} \quad \text{for } P_e \rightarrow \infty, \quad (5.32c)$$

as obtained immediately from Eqs. (5.12a), (5.20a), (5.21), and (5.27a)e. On the other hand, the order parameter considered in [22, 68], $t_{\ell tr} = t_\ell + t_{tr}$, behaves as

$$\langle t_{\ell tr} \rangle \sim \frac{\tau}{|P_e|} = \frac{2\ell}{|v|} \quad \text{for } P_e \rightarrow -\infty, \quad (5.33a)$$

$$\langle t_{\ell tr}^* \rangle = \left(\frac{1}{6} + \frac{1}{2} \right) \tau = \frac{2\ell^2}{3D} \quad \text{for } P_e = 0, \quad (5.33b)$$

$$\langle t_{\ell tr} \rangle \sim \langle t_{tr} \rangle \sim \frac{\tau e^{2P_e}}{4P_e^2} = \frac{De^{\ell v/D}}{v^2} \quad \text{for } P_e \rightarrow \infty. \quad (5.33c)$$

Finally, the order parameter in [39, 76] behaves as

$$C_\ell \sim 2u_0|P_e|e^{-2|P_e|} = \frac{x_0|v|}{D}e^{-\ell|v|/D} \quad \text{for } P_e \rightarrow -\infty, \quad (5.34a)$$

$$C_\ell^* = u_0 = \frac{x_0}{\ell} \quad \text{for } P_e = 0, \quad (5.34b)$$

$$C_\ell \sim 2u_0P_e = \frac{x_0v}{D} \quad \text{for } P_e \rightarrow \infty. \quad (5.34c)$$

All three quantities, $\langle t_r \rangle$, $\langle t_{\ell tr} \rangle$, and C_ℓ , are reasonable candidates for order parameters — finding the order parameter of a phase transition is often not obvious [66]: citing J. P. Sethna, “there is often more than one sensible choice” [175].

Typically, one expects that in the thermodynamic limit ($\ell \rightarrow \infty$) the order parameter goes to zero for $P_e < 0$ and scales with a power of ℓ for $P_e > 0$ (if the order parameter is extensive). This is not the case for $\langle t_r \rangle$ or $\langle t_{\ell tr} \rangle$, and there is no simple rescaling to bring about the desired behaviour. Instead, one could redefine the order parameters somewhat artificially as $\ell^{-1} \ln(|v|\langle t_r \rangle/x_0)$ and $\ell^{-1} \ln(v^2\langle t_{\ell tr} \rangle/D)$, which below and above the critical point behave as intensive order parameters — but they are not well defined at the critical point. C_ℓ , meanwhile, undergoes a transcritical bifurcation at the critical point [41], but below the transition it goes to zero exponentially rather than as a power law of $1/\ell$. A drawback of all three order parameters is the lack of an associated variance that diverges at the critical point.

5.2.8 FINITE-SIZE SCALING FOR THE MOMENTS OF THE DISTRIBUTIONS

The equations in the previous subsections show that, when u_0 is small, all the first-passage times ($t_0, t_\ell, t_{tr}, t_{\ell tr}$, and t_r) obey finite-size scaling laws [160]. Indeed, starting with t_ℓ ,

$$\langle t_\ell \rangle = \tau G_{\ell 1}(P_e) = \frac{\ell^2}{D} G_{\ell 1}\left(\frac{\ell v}{2D}\right), \quad (5.35a)$$

$$\langle t_\ell^2 \rangle = \tau^2 G_{\ell 2}(P_e) = \frac{\ell^4}{D^2} G_{\ell 2}\left(\frac{\ell v}{2D}\right), \quad (5.35b)$$

to zeroth order in u_0 , where $G_{\ell 1}(P_e)$ and $G_{\ell 2}(P_e)$ are scaling functions completely determined by Eqs. (5.20a) and (5.20b). The same scaling holds for t_{tr} and $t_{\ell tr}$ (but with different scaling functions). In contrast, from Eqs. (5.30) and (5.31),

$$\langle t_r \rangle = u_0 \tau G_{r 1}(P_e) = \frac{x_0 \ell}{D} G_{r 1}\left(\frac{\ell v}{2D}\right), \quad (5.36a)$$

$$\langle t_r^2 \rangle = u_0 \tau^2 G_{r 2}(P_e) = \frac{x_0 \ell^3}{D^2} G_{r 2}\left(\frac{\ell v}{2D}\right), \quad (5.36b)$$

to first order in u_0 . The moments of t_0 share the same scaling (but with different scaling functions). For further comparison,

$$C_\ell = u_0 G_c(P_e) = \frac{x_0}{\ell} G_c\left(\frac{\ell v}{2D}\right), \quad (5.37)$$

to lowest order in u_0 . Note that the position of the critical point does not shift with ℓ : it remains at $P_e = 0$ (or $v = 0$) for finite ℓ .

5.2.9 ENTIRE DIFFUSION PROBLEM: DISTRIBUTION

The Laplace transform of the probability density of the first-passage time t_r , can be written as the weighted sum

$$f_r(s) = (1 - C_\ell) f_0(s) + C_\ell f_\ell(s) f_{tr}(s) = j_0(s) + j_\ell(s) f_{tr}(s). \quad (5.38)$$

From Eqs. (5.62), (5.18), and (5.26), we thus obtain

$$f_r(s) = j_0(s) + j_\ell(s)f_{tr}(s) = \frac{e^{-P_e u_0} \sinh[(1 - u_0)R]}{\sinh R} + \frac{2e^{-u_0 P_e} R \sinh(u_0 R)}{R \sinh(2R) - 2P_e \sinh^2 R}, \quad (5.39)$$

which, at the critical point $P_e = 0$, reduces to

$$f_r^*(s) = \frac{\sinh[(1 - u_0)\sqrt{\tau s}]}{\sinh \sqrt{\tau s}} + \frac{2 \sinh(u_0 \sqrt{\tau s})}{\sinh(2\sqrt{\tau s})}. \quad (5.40)$$

Expanding in u_0 , we find (see Eqs. (5.64), (5.18), and (5.26))

$$\begin{aligned} f_r(s) &= 1 - \left(P_e + \frac{R}{\tanh R} \right) u_0 + \left(\frac{e^{P_e} R}{\sinh R} \right) \left(\frac{e^{-P_e} R}{R \cosh R - P_e \sinh R} \right) u_0 + \mathcal{O}(u_0^2) \\ &= 1 - \left(P_e + \frac{R}{\tanh R} - \frac{2R^2}{R \sinh(2R) - 2P_e \sinh^2 R} \right) u_0 + \mathcal{O}(u_0^2), \end{aligned} \quad (5.41)$$

valid for $u_0 R \ll 1$. At the critical point ($P_e = 0$),

$$\begin{aligned} f_r^*(s) &= 1 - \frac{\sqrt{\tau s}}{\tanh \sqrt{\tau s}} u_0 + \left(\frac{\sqrt{\tau s}}{\sinh \sqrt{\tau s}} \right) \left(\frac{1}{\cosh \sqrt{\tau s}} \right) u_0 + \mathcal{O}(u_0^2) \\ &= 1 - \left(\frac{\sqrt{\tau s}}{\tanh \sqrt{\tau s}} - \frac{\sqrt{4\tau s}}{\sinh \sqrt{4\tau s}} \right) u_0 + \mathcal{O}(u_0^2). \end{aligned} \quad (5.42)$$

Plots of the inverse Laplace transforms of $f_r(s)$ and $f_r^*(s)$ are shown in Fig. 5.2. Note that all the results presented here for diffusion processes are valid for $x_0 \ll \ell$.

5.2.10 FINITE-SIZE SCALING FOR THE DISTRIBUTIONS

The Laplace transforms of the probability densities of t_ℓ , t_{tr} , and $t_{\ell tr}$ obey simple finite-size scaling laws (to zeroth order in u_0),

$$f_\ell(s) = \hat{F}_\ell(\tau s, P_e) = \hat{F}_\ell\left(\frac{\ell^2 s}{D}, \frac{\ell v}{2D}\right), \quad (5.43)$$

from Eq. (5.23), where the scaling function $\hat{F}_\ell$ is exactly known. Again, $f_{tr}(s)$ and $f_{\ell tr}(s)$ scale in the same way as $f_\ell(s)$ (with different scaling functions). Inverting the Laplace transforms, we see that the probability densities also obey simple finite-size

scaling laws for fixed P_e ,

$$f_\ell(t) = \frac{1}{\tau} F_\ell \left(\frac{t}{\tau}, P_e \right) = \frac{D}{\ell^2} F_\ell \left(\frac{Dt}{\ell^2}, \frac{\ell v}{2D} \right), \quad (5.44)$$

with $f_{tr}(t)$ and $f_{\ell tr}(t)$ scaling in the same way.

In contrast, the Laplace transforms associated with t_r and t_0 obey

$$f_r(s) = \hat{F}_r(\tau s, u_0, P_e) = \hat{F}_r \left(\frac{\ell^2 s}{D}, \frac{x_0}{\ell}, \frac{\ell v}{2D} \right), \quad (5.45)$$

to first order in u_0 , from Eq. (5.41). This is not finite-size scaling for fixed x_0 , due to the dependence on u_0 ; f_0 obeys an analogous equation. The corresponding densities obey

$$f_r(t) = \frac{1}{\tau} F_r \left(\frac{t}{\tau}, u_0, P_e \right) = \frac{D}{\ell^2} F_r \left(\frac{Dt}{\ell^2}, \frac{x_0}{\ell}, \frac{\ell v}{2D} \right), \quad (5.46)$$

with $f_0(t)$ scaling in the same way, but with its own scaling function. However, as the moments of t_r and t_0 have an extra factor u_0 in comparison with t_ℓ and t_{tr} , this suggests that the densities and their transforms can be written with an extra factor u_0 as well, i.e.,

$$f_r(t) = \frac{u_0}{\tau} F_r \left(\frac{t}{\tau}, P_e \right) = \frac{D x_0}{\ell^3} F_r \left(\frac{Dt}{\ell^2}, \frac{\ell v}{2D} \right), \quad (5.47)$$

where in a slight abuse of notation we have recycled the symbol F_r , which now refers to a different scaling function. The first-passage time t_0 obeys an analogous equation (now truly a finite-size scaling law). The scaling law for $f_r(t)$ is compatible with the form

$$f_r(t) = \frac{1}{m} \left(\frac{m}{t} \right)^\alpha F \left(\frac{t}{\tau}, P_e \right) \quad (5.48)$$

(and similarly for $f_0(t)$) with the new scaling function F going to a constant for small arguments and decaying very fast for large arguments, and with $\alpha = 3/2$ and $m = u_0^2 \tau$ the minimum value of t (that is, for $t < m$ the probability density $f_0(t)$ can be considered as zero). This is in agreement with the power-law behaviour shown in Eq. (5.16). An alternative way to write the scaling law (5.48) is

$$f_r(t) = \frac{1}{m} \left(\frac{m}{\tau} \right)^\alpha \tilde{F} \left(\frac{t}{\tau}, P_e \right), \quad (5.49)$$

with the scaling function $\tilde{F}$ absorbing the power-law part with exponent $\alpha = 3/2$. For $P_e = 0$ this leads to the same scaling law as in [37]. In fact, that reference gives a more direct derivation of the scaling laws (5.48) and (5.49), but only for $f_0(t)$ with $P_e = 0$.

5.3 BRANCHING PROCESS AND SIZE DISTRIBUTIONS

5.3.1 DIFFUSION AND RANDOM WALKS

First, we recall the connection between diffusion and random walks. Consider a random walk described by a position X at time T , which are both discrete and dimensionless. At each time step T , the position X increases by one unit with probability q , or decreases by one unit with probability $1 - q$. The continuum limit of the random walk is a diffusion process, with $x = X\delta_x$ and $t = T\delta_t$, where δ_x and δ_t are elementary space and time units, which tend to zero [164]. The limiting process is described by the diffusion equation, Eq. (5.1), with $v = (2q - 1)\delta_x/\delta_t$ and $D = 2q(1 - q)\delta_x^2/\delta_t$. In this limit, the results obtained for moments and probability densities of diffusion processes are also valid for random walks [164]. Since all the relevant equations of the previous section can be written in terms of dimensionless quantities, we can make the substitutions

$$v \rightarrow 2\left(q - \frac{1}{2}\right), \quad D \rightarrow 2q(1 - q), \quad \text{and} \quad \tau \rightarrow \frac{L^2}{2q(1 - q)}, \quad (5.50)$$

together with $u_0 = X_0/L$, with $X_0 = x_0/\delta_x$ and $L = \ell/\delta_x$; in particular

$$P_e = \frac{L(q - 1/2)}{2q(1 - q)}. \quad (5.51)$$

As an illustration, Eq. (5.30) becomes

$$\langle T_r \rangle = \frac{1}{4P_e} \left(\frac{e^{3P_e}}{\sinh P_e} - \frac{1}{\tanh P_e} - 3 \right) \frac{X_0 L}{2q(1 - q)} + \mathcal{O}(u_0^2). \quad (5.52)$$

Care is required approximating the (dimensionless) probability mass functions of random walk times T , with the probability densities for diffusion times t , which have the dimensions of inverse time. Note that, in the former case, first-passage times are discretised in steps of 2 (since boundaries can only be reached in either an odd or even

number of steps). Thus, the two functions are related via

$$f_T(T) = 2\delta_t f_t(T\delta_t), \quad (5.53)$$

where f_T denotes the probability mass function for the random walk and f_t the probability density for the diffusion process. The extra factor 2 with respect a standard change of variables comes from the discretization of T . In this way, rewriting Eqs. (5.44) and (5.47) for random walk scaling laws, we obtain

$$f_L(T) = \frac{4q(1-q)}{L^2} F_\ell \left(\frac{2q(1-q)T}{L^2}, P_e \right) \simeq \frac{1}{L^2} F_\ell \left(\frac{T}{2L^2}, P_e \right), \quad (5.54)$$

with an analogous expression for $f_{TR}(T)$, while

$$f_R(T) = \frac{4q(1-q)X_0}{L^3} F_r \left(\frac{2q(1-q)T}{L^2}, P_e \right) \simeq \frac{X_0}{L^3} F_r \left(\frac{T}{2L^2}, P_e \right), \quad (5.55)$$

where we have used $q \simeq 1/2$ close to the critical point. The scaling functions F_ℓ and F_r are the same as for the diffusion process of the previous section.

5.3.2 BRANCHING PROCESSES

We now consider the Galton-Watson branching process associated with the random walk. The branching process starts with one single member (also known as the root), defining the first generation. It produces a random number of offspring drawn from a geometric distribution, i.e. the probability of k offspring is $(1-q)q^k$, $k = 0, 1, 2, \dots$, where $1-q$ is the success probability. Each of these second generation offspring produce their own (third generation) offspring, and so on, independently and identically. The process can be visualised as a rooted tree. In principle, q is the only parameter of the model, but one can introduce finite-size effects by stopping the branching process at generation L .

The size S of the branching process, or the size of the tree or cluster, is given by its total population (total number of offspring plus root, or, equivalently, the total number of vertices in the tree) [38, 99]. In a finite system, clusters can be classified into two types: percolating clusters, which reach generation L , and non-percolating clusters, which do

not reach L . We denote the size of each of these by the random variables S_{perc} (for percolating) and S_{int} (internal, for non-percolating). Overall, S is a mixture of S_{perc} and S_{int} .

5.3.3 MAPPING FROM BRANCHING PROCESSES TO RANDOM WALKS

Harris' mapping from trees to walks proceeds as follows (for complete details see [39, 50, 68, 98], for a visual explanation, see Fig. 5.4). A (deterministic) walker is placed at the root of the tree and carries out a so-called depth-first search by traversing each branch in turn to its very end, starting with the leftmost branch. Whenever a choice of unvisited branches presents itself, the walker traverses them in the order left to right. Eventually, the walker will return to the root, having traversed all branches. In doing so, the walker will have visited each member of the tree twice. To define a positive walk, it is convenient to append a final step to a "generation zero", see Fig. 5.4. In this way, one obtains a one-dimensional, positive walk (or excursion), starting at $X = 1$ and ending at $X = 0$, where X corresponds to the generation number as the walker traverses the tree. The size of the tree S is then seen to be the duration of the walk (plus one) divided by two — the division by two takes care of the fact that each member of the tree is visited twice.

In addition, a probability measure over the set of all positive walks is inherited from the probability measure over the set of rooted trees. When the tree is generated with a geometric offspring distribution with success probability p , one obtains the standard random walk, where $X \rightarrow X + 1$ with probability $q = 1 - p$, and $X \rightarrow X - 1$ with probability p .

In this way, a realization of a Galton-Watson process with geometric offspring distribution is equivalent to a realization of a random walk that starts at $X = 1$ and ends at $X = 0$, staying positive in between. The stipulation that a branching process cannot exceed L generations is effected, in the random walk, by a reflecting boundary at $X = L$.

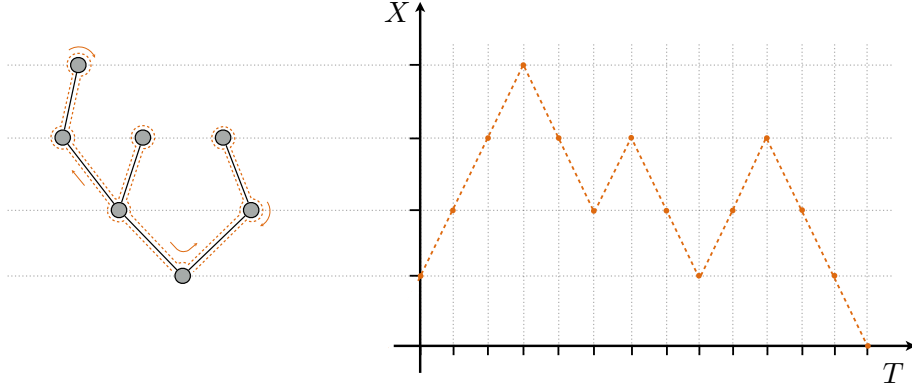


Figure 5.4: Harris map. Correspondence between trees and positive walks (excursions), known as a Harris walk [98]. A walker starts at the root of the tree (the vertex at the bottom) and follows each branch in turn to its end, starting with the left-most branch. If the tree is generated from a geometric offspring distribution, the resulting path is a simple random walk.

5.3.4 MOMENTS AND SPLITTING PROBABILITY

One can calculate the avalanche size of percolating clusters S_{perc} from the first-passage time of a diffusing particle that first reaches $x = \ell$, before being absorbed at $x = 0$ (as in [68]). The total time is the sum of two independent times: the first, t_ℓ , is the time to reach $x = \ell$ starting from $x = x_0$ (and not touch $x = 0$), and the second, t_{tr} , is the time to reach $x = 0$ starting from $x = \ell$, see [68]. Thus, in terms of a random walk, $S_{perc} = (T_L + T_{tr} + 1)/2$, where the discrete and dimensionless times T_L and T_{tr} are analogs of t_ℓ and t_{tr} for the diffusion process. Similarly, S_{int} is obtained from the first-passage time to $x = 0$ of a diffusing particle that does not reach $x = \ell$, denoted t_0 . Thus, $S_{int} = (T_0 + 1)/2$, with $T_0 = t_0/\delta_t$. The total size S (percolating or not) can be obtained directly from the total first-passage time t_r , via $S = (T_r + 1)/2$, with $T_r = t_r/\delta_t$. The moments of S are

$$\langle S \rangle = \frac{\langle T_r \rangle + 1}{2} \simeq \frac{\langle t_r \rangle}{2\delta_t}, \quad (5.56a)$$

$$\langle S^2 \rangle = \frac{\langle T_r^2 \rangle + 2\langle T_r \rangle + 1}{4} \simeq \frac{\langle t_r^2 \rangle}{4\delta_t^2}, \quad (5.56b)$$

for $L \gg 1$. Thus, from Eq. (5.52)

$$\langle S \rangle = \frac{1}{8P_e} \left(\frac{e^{3P_e}}{\sinh P_e} - \frac{1}{\tanh P_e} - 3 \right) 2L + \frac{1}{2}, \quad (5.57)$$

where we have used the fact that, close to the critical point, $q \simeq 0.5$. In the limit of large system size, the moments of S obey the same scaling laws as those of t_r , Eqs. (5.36a) and (5.36b), i.e.

$$\langle S \rangle = \frac{L}{4q(1-q)} G_{r1} \left(\frac{L(q-1/2)}{2q(1-q)} \right) \simeq L G_{r1} (2L(q-1/2)), \quad (5.58)$$

Fig. 5.3 shows good agreement between theory and realizations of branching processes from computer simulations.

Note that the scaling law for C_ℓ , Eq. (5.21), derived in the diffusion framework, is the same as that obtained for branching processes in [39, 76]. Indeed, transcribing ℓ , D and v into their discrete versions, Eq. (5.50), we get

$$2P_e = \frac{\ell v}{D} \simeq 4L \left(q - \frac{1}{2} \right) \simeq L(\langle k \rangle - 1) \quad (5.59)$$

to first order in $q - 1/2$, with $\langle k \rangle = q/p \simeq 1 + 4(q - 1/2)$ for the geometric distribution. Substituting the previous expression for $2P_e$ into Eq. (5.21), with $x_0/\ell = 1/L$, leads to the result in [39, 76].

5.3.5 DISTRIBUTION OF SIZES

The distributions of sizes $f_S(S)$ (for S_{int} , S_{perc} or total S) are related to the distributions of first-passage times in the random walk $f_T(T)$ and in the diffusion process $f_t(t)$ via

$$f_S(S) = f_T(2S - 1) = 2\delta_t f_t((2S - 1)\delta_t), \quad (5.60)$$

using Eq. (5.53). While we do not have explicit formulas for $f_T(T)$ and $f_t(t)$, we do have their Laplace transforms $f_t(s)$. Thus, given a finite-size Galton-Watson branching process, with parameters q and L , we can calculate τ , P_e , and u_0 for the equivalent diffusion process, using Eqs. (5.50) and (5.51), and then perform the numerical inversion

of the Laplace-transformed expressions (5.11), (5.29), and (5.41). This yields a nearly perfect agreement between computer simulations of the Galton-Watson process and theory, based on diffusion processes, as Fig. 5.2 illustrates.

Note from the figure that the critical case, $P_e = 0$, displays a bump before the exponential decay at large sizes, which comes from the Kolmogorov-Smirnov distribution associated with percolating clusters. Similar bumps have been observed in paradigmatic models of critical phenomena, such as the Oslo sandpile model [35]. Therefore, although deviations from criticality ($P_e \neq 0$) in infinite systems lead to a $f(S)$ given by a power-law multiplied by an exponential tail ($f(S) \propto e^{-S/\xi}/S^{3/2}$, see [38]), this parameterization is not valid for finite-size effects at the critical point, since it does not reproduce the bump for large S . A much larger bump is present in the supercritical regime.

This behaviour can be taken as an instance of the so-called dragon-king effect [183], in which events at the tail of a distribution have a much larger probability than that given by the extrapolation of a power-law central part. This would correspond to the so-called characteristic-earthquake scenario in statistical seismology [14], although from our results and those in [68] it seems clear that the bump cannot be described by Gaussian-like statistics (which is only applicable in the subcritical regime, where the bump is negligible or very small, depending on P_e), contrary to the statement in [14].

Finally, from Eqs. (5.44) and (5.47), the scaling laws for the size distributions can be written as

$$f_{perc}(S) = \frac{1}{L^2} F_{\ell tr} \left(\frac{2S-1}{2L^2}, P_e \right), \quad (5.61a)$$

$$f_{int}(S) = \frac{1}{L^3} F_0 \left(\frac{2S-1}{2L^2}, P_e \right), \quad (5.61b)$$

$$f(S) = \frac{1}{L^3} F_r \left(\frac{2S-1}{2L^2}, P_e \right), \quad (5.61c)$$

where, again, $q \simeq 1/2$ close to the critical point. The scaling functions F_ℓ , F_0 , and F_r are the same as for the diffusion process.

5.4 SUMMARY AND CONCLUSIONS

The Harris walk mapping establishes a direct correspondence between finite-size branching processes (in which the number of generations cannot exceed L) and one-dimensional random walks between $X = 0$ and $X = L$. By approximating random walks with diffusions, techniques from the latter can be applied to branching processes, such that first-passage times of Brownian particles with drift correspond to sizes of trees generated by the branching process (up to a proportionality factor).

We solved the ensuing diffusion equations, arriving, in the limit of large system size, at exact expressions for the Laplace transforms of the probability densities of first-passage times. The drift term is a control parameter, bringing about a second-order phase transition as it changes sign. This transition separates a regime in which diffusing particles (starting at the origin) barely reach the distant boundary, from a regime in which particles reach the boundary with non-zero probability. In the context of branching processes, the transition separates subcritical and supercritical phases. In the latter case, trees percolate (i.e. reach generation L) with non-zero probability. In the limit of infinite system size, these transitions are sharp. We obtained finite-size scaling laws for probability densities, and discussed possible choices of order parameter.

Our approach allows us to treat separately the contribution from particles that do reach the further boundary (corresponding to percolating trees) and particles that do not. In the latter case, the distribution is governed by a power law with exponent $3/2$ (except for very large and very short times), whereas in the former case we recover the results in [68], which give a Kolmogorov-Smirnov distribution in the critical case.

An important lesson from this study is that truncated gamma distributions [173] (power laws multiplied by an exponential decay term), although valid for modeling off-critical effects in infinite systems, are not appropriate for modeling finite-size effects in critical systems, due to a large-size bump in the distribution coming from system-spanning clusters. Another point to bear in mind is that the existence of finite-size scaling in the distributions of some observable is not a guarantee that the system under consideration is at a critical point. It could be that the system is simply close to but not at the critical point, in such a way that the rescaled control parameter (P_e in our case) takes a constant value.

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5.A FIRST-PASSAGE TIME t_0

We provide details of the calculation of the outflux at $x = 0$ in a system with two absorbing boundaries, see Eqs. (5.1), (5.4), and (5.6). From Eq. (5.7), we arrive at the exact expression

$$j_0(s) = Dc'(x = 0, s) = \frac{e^{-P_e u_0} \sinh[(1 - u_0)R]}{\sinh R}, \quad (5.62)$$

from which

$$f_0(s) = \frac{j_0(s)}{j_0(s = 0)} = \frac{\sinh[(1 - u_0)R] \sinh P_e}{\sinh[(1 - u_0)P_e] \sinh R}. \quad (5.63)$$

We are interested in particles starting very close to the $x = 0$ boundary, i.e. $x_0 \ll \ell$ and $u_0 \ll 1$. Expanding Eq. (5.62) to first order in u_0 ,

$$j_0(s) = 1 - \left(P_e + \frac{R}{\tanh R} \right) u_0 + \mathcal{O}(u_0^2) \quad (5.64)$$

(in fact, we require $u_0 P_e \ll 1$ and $u_0 R \ll 1$). The properties of the first-passage time to $x = 0$ will arise from the Taylor expansion of $j_0(s)$ around $s = 0$. If we write $R = P_e \sqrt{1 + z}$ with $z = \tau s / P_e^2$, then, to second order in z ,

$$\begin{aligned} j_0(s) = & 1 - \left(P_e + \frac{P_e}{\tanh P_e} \right) u_0 - \frac{P_e}{2 \tanh P_e} \left(1 + P_e \tanh P_e - \frac{P_e}{\tanh P_e} \right) u_0 z \\ & - \frac{P_e}{2 \tanh P_e} \left(-\frac{1}{2} + \frac{P_e \tanh P_e}{2} - \frac{P_e}{2 \tanh P_e} - P_e^2 + \frac{P_e^2}{\tanh^2 P_e} \right) \frac{u_0 z^2}{2} + \mathcal{O}(z^3) u_0 + \mathcal{O}(u_0^2). \end{aligned} \quad (5.65)$$

Since $j(s)$ is the moment generating function for first passage times, i.e. $j(s) = C(1 -$

$\langle t \rangle s + \langle t^2 \rangle s^2/2 + \mathcal{O}(s^3)$), for some normalization constant C , we read off from the above expansion

$$\langle t_0 \rangle = \frac{1}{2P_e \tanh P_e} \left(1 + P_e \tanh P_e - \frac{P_e}{\tanh P_e} \right) \tau u_0 + \mathcal{O}(u_0^2), \quad (5.66)$$

$$\langle t_0^2 \rangle = \frac{1}{2P_e^3 \tanh P_e} \left(\frac{1}{2} - \frac{P_e \tanh P_e}{2} + \frac{P_e}{2 \tanh P_e} + P_e^2 - \frac{P_e^2}{\tanh^2 P_e} \right) \tau^2 u_0 + \mathcal{O}(u_0^2), \quad (5.67)$$

where the subscript 0 in t denotes first-passage to the boundary at $x = 0$. To first order in u_0 , the variance coincides with the second moment, i.e. $\sigma_0^2 = \langle t_0^2 \rangle - \langle t_0 \rangle^2 \simeq \langle t_0^2 \rangle$.

The zeroth-order term in s (namely the constant $C = C_0 = j_0(s = 0)$) is only one in the limit $u_0 \rightarrow 0$. To first order in u_0 , the coefficients in the expansion of $j_0(s)$ yield the moments of t_0 . Using Eq. (5.64), the Laplace-transformed probability density is

$$f_0(s) = \frac{j_0(s)}{j_0(s = 0)} = 1 - \left(\frac{R}{\tanh R} - \frac{P_e}{\tanh P_e} \right) u_0 + \mathcal{O}(u_0^2), \quad (5.68)$$

to first order in u_0 .

5.B FIRST-PASSAGE TIME $t_{0\ell}$

The problem of one-dimensional diffusion between two absorbing boundaries (analysed in [65]) displays a phase transition in the same way as diffusion between absorbing and reflecting boundaries. The calculation of first-passage times is analogous to that of the absorbing-reflecting system, but with the contribution from t_{tr} excluded.

The exact Laplace-transformed probability density for $t_{0\ell}$, the first-passage time to either boundary, reads

$$f_{0\ell}(s) = \frac{e^{-P_e u_0} \sinh[(1 - u_0)R]}{\sinh R} + \frac{e^{(1-u_0)P_e} \sinh(u_0 R)}{\sinh R}, \quad (5.69)$$

which, at the critical point ($P_e = 0$), reduces to

$$f_{0\ell}^*(s) = \frac{\sinh[(1 - u_0)\sqrt{\tau s}] + \sinh(u_0 \sqrt{\tau s})}{\sinh \sqrt{\tau s}}. \quad (5.70)$$

These expressions may be expanded in u_0 as

$$f_{0\ell}(s) = j_0(s) + j_\ell(s) = 1 - u_0 \left(P_e + \frac{R}{\tanh R} - \frac{e^{P_e R}}{\sinh R} \right) + \mathcal{O}(u_0^2), \quad (5.71)$$

and, at the critical point,

$$f_{0\ell}^*(s) = 1 - u_0 \left(\frac{\sqrt{\tau s}}{\tanh \sqrt{\tau s}} - \frac{\sqrt{\tau s}}{\sinh \sqrt{\tau s}} \right) + \mathcal{O}(u_0^2), \quad (5.72)$$

in which the Kolmogorov-Smirnov distribution again appears (corresponding to particles that reach $x = \ell$). Note that $t_{0\ell}$ scales in the same way as t_r and t_0 (but with different scaling functions). Thus, the scaling laws in the core of the paper also apply here (but with different scaling functions).

To first order in u_0 , we have $\langle t_{0\ell} \rangle = \langle t_0 \rangle + C_\ell \langle t_\ell \rangle$, where C_ℓ is the same as in the absorbing-reflecting system. Thus, the first moment is given by

$$\langle t_{0\ell} \rangle = \frac{\tau u_0}{2} \left(1 + \frac{1}{\tanh P_e} - \frac{1}{P_e} \right), \quad (5.73)$$

which is the expansion to first order in u_0 of Eq. (2.3.10) in [164], equivalently Eq. (6) in [65].

6

The Concealed Voter Model is in the Voter Model universality class

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OVERVIEW The Voter Model is a simple stochastic process motivated by the dynamics of public opinion formation, where two complementary opinions compete for space until the system reaches consensus [124]. The competition happens locally through the interaction between neighbouring sites. The Concealed Voter Model is an extension of the Voter Model, where there is an additional layer of private opinions that interact locally with the external layer of public opinions [82]. In this work I show that the fundamental interactions in both processes are effective branching and volume-exclusion interactions, and that both models belong to the same universality class.

ABSTRACT

The Concealed Voter Model (CVM) is an extension of the original Voter Model, where two different opinions compete until consensus is reached. In the CVM, agents express an opinion not only publicly, they also hold a private opinion, which they may disclose or change. In this paper I derive the critical exponents of both models via renormalised field theory and show that both belong to the compact directed percolation universality class.

6.1 INTRODUCTION

The Concealed Voter Model (CVM) is a simple agent-based model of opinion formation in social networks that takes into account the consistency or the lack thereof between a voter's private and publicly expressed opinions [82, 83]. In the original Voter Model (VM), agents adopt the expressed opinion of one of their neighbours at random until the system reaches consensus [124]. Modelling the private opinion of agents in the CVM is introduced as an extension to the VM in [82].

The VM is a special case of the Domany-Kinzel cellular automaton, which is in the universality class of compact directed percolation, also known as the VM universality class [58, 53, 60]. To determine whether the CVM belongs to the same universality class, I calculate the critical exponents characterising the spreading of an opinion from a localised source via a path integral approach. In [102], a path-integral method was used to compute the persistence probability on a slight variation of the VM.

In the VM, each agent expresses one of two opinions, which may be thought of as particles and empty sites on a d -dimensional lattice, Fig. 6.1. At rate $\alpha > 0$, every agent **copies** the opinion expressed by one of their randomly chosen neighbours. The system is initialised with a single particle, and the process of copying is repeated until the system reaches one of the two absorbing states: all sites either empty or occupied. In the VM, activity only happens at the boundaries between clusters of empty and occupied sites. The critical point of the VM separates the two phases where, in the thermodynamic limit, particles are subject to extinction with probability 1 and where there is a positive probability of indefinite survival. Microscopically, the VM is at

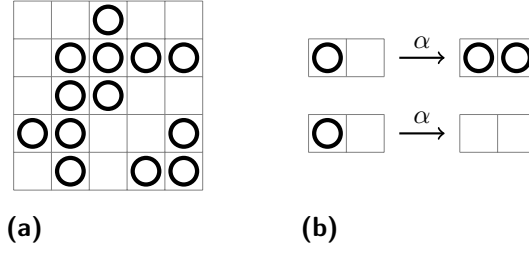


Figure 6.1: The voter model (opinions are represented by circles and empty spaces). (a) Sample of a particle configuration in the VM in $d = 2$. (b) The process of copying at the cluster boundary.

criticality if given a pair of neighbours at the interface between clusters $A + \emptyset$, the two possible outcomes after copying, $\emptyset + \emptyset$ and $A + A$, are equally likely. If, for instance, the former is more likely, then, on average, there is a net loss of particles that gives rise to the mass in the field theory. The VM at criticality is also known as unbiased voter [53] and it is representative of so-called neutral theories [130].

Observables commonly considered to characterise such critical processes are [91, 136]: the expected number of particles $n(t)$ at time t , the mean square spread of particles $R^2(t)$ (or radius of gyration), and the survival probability $P_s(t)$, namely, the probability that there is at least one particle in the system. At criticality, these three observables have an asymptotic algebraic scaling in time,

$$\langle n(t) \rangle \propto t^\eta, \quad R^2(t) \propto t^z, \quad P_s(t) \propto t^{-\delta}, \quad (6.1)$$

whose exponents η , z and δ characterise the VM universality class. The critical exponent z is closely related to the dynamical critical exponent $\hat{z}$, defined as $\tau_t \propto \xi^{\hat{z}}$, where τ_t and ξ are temporal and spatial correlation lengths, via the relation $z = 2/\hat{z}$ [90].

Due to the duality between the VM and coalescent random walks [53, 60, 136], it is known that $\eta = 0$ and $z = 1$ in all dimensions, and $\delta = 1/2$ at dimension $d = 1$ and $\delta = 1$ for dimensions $d \geq 2$, satisfying the hyperscaling relation [53]

$$\delta + \eta = \frac{1}{2}dz \quad (6.2)$$

in dimensions $d \leq 2$.

In Sec. 6.2, I derive the field theory of the VM and calculate the critical exponents; in Sec. 6.3, I incorporate the dynamics of the CVM in the field theory and calculate the

critical exponents. Finally, I discuss the results in Sec. 6.4.

6.2 CRITICAL EXPONENTS OF THE VOTER MODEL

The critical exponents of the VM are well understood [53, 136]. Nevertheless, in this section, the exponents of the VM are derived via a path integral approach to lay the foundations of the derivation of the critical exponents of the CVM in Sec. 6.3.

The configuration of the system is given by the set $\{m\}$ of occupation numbers $m_{\mathbf{x}}$, which is the number of particles at lattice site $\mathbf{x}$. The dynamics of the VM are condensed in the following master equation,

$$\begin{aligned} \dot{P}(\{m\}; t) = \sum_{\mathbf{x}, \mathbf{y}} \bigg\{ & \alpha (m_{\mathbf{x}} + 1) \left(1 - \frac{m_{\mathbf{y}}}{c}\right) P(m_{\mathbf{x}} + 1, m_{\mathbf{y}}; t) \\ & + \alpha m_{\mathbf{y}} \left(1 - \frac{m_{\mathbf{x}} - 1}{c}\right) P(m_{\mathbf{x}} - 1, m_{\mathbf{y}}; t) \\ & - \alpha \left[m_{\mathbf{x}} \left(1 - \frac{m_{\mathbf{y}}}{c}\right) + m_{\mathbf{y}} \left(1 - \frac{m_{\mathbf{x}}}{c}\right) \right] P(m_{\mathbf{x}}, m_{\mathbf{y}}; t) \bigg\}, \end{aligned} \quad (6.3)$$

where $P(\{m\}; t)$ is the probability of finding the system in microstate $\{m\}$ at time t , $\mathbf{x}$ and $\mathbf{y}$ are nearest neighbours, and c is the local carrying capacity, the maximum number of particles allowed in each site. This carrying capacity implements the fermionic nature of opinions and may be set to $c = 1$. However, it is useful to parametrise this excluded volume constraint as it is the only interaction that keeps the spreading of particles at bay [138, 21, 208] (Chapter 4). The first term in Eq. (6.3) describes the elimination of a particle at $\mathbf{x}$ by an empty space at $\mathbf{y}$; and the second, the production of a particle at $\mathbf{x}$ by a particle at $\mathbf{y}$. Initially, at time $t_0 = 0$, a single particle is placed at $\mathbf{x}_0$. Following the Doi-Peliti formalism, the master equation in (6.3) is cast in path integral form [189]. After taking the continuum limit and introducing the annihilation field $\phi(\mathbf{x}, t)$ and the Doi-shifted creation field $\tilde{\phi}(\mathbf{x}, t)$, the action functional of the resulting field theory is

$\mathcal{A}_0 + \mathcal{A}_{\text{int}}$, where

$$\mathcal{A}_0 = \int d^d \mathbf{x} dt \left\{ \tilde{\phi} \partial_t \phi - D \tilde{\phi} \nabla^2 \phi + r \tilde{\phi} \phi \right\}, \quad (6.4a)$$

$$\mathcal{A}_{\text{int}} = \int d^d \mathbf{x} dt \left\{ -s \tilde{\phi}^2 \phi + \chi \tilde{\phi}^2 \phi^2 + w_1 \tilde{\phi} (\nabla^2 \phi) \tilde{\phi} - w_2 \tilde{\phi} (\nabla^2 \phi) \tilde{\phi}^2 \phi + w_3 \tilde{\phi}^3 \phi^2 \right\}, \quad (6.4b)$$

$D = \alpha$ is the effective diffusion constant, $s = \alpha d$ is the effective branching rate, and the other non-linearities are $\chi = \alpha d/c$, $w_1 = \alpha$, $w_2 = \alpha/c$ and $w_3 = \alpha d$. The mass $r > 0$ of the particles is added to regularise the infrared, ensuring causality in the time domain $t \geq t_0$, and is later removed by taking the limit $r \rightarrow 0$ at criticality. In fact, if the two processes $A \rightarrow A + A$ and $A \rightarrow \emptyset$ in Fig. 6.1b are set to happen at rates α_1 and α_2 respectively, with $\alpha_1 \neq \alpha_2$, a mass term $(\alpha_2 - \alpha_1) d \tilde{\phi} \phi$ emerges in the action naturally. The action functional allows the calculation of an observable $\mathcal{O}$, via the path integral

$$\langle \mathcal{O} \rangle = \int \mathcal{D}[\phi, \tilde{\phi}] \mathcal{O} e^{-(\mathcal{A}_0 + \mathcal{A}_{\text{int}})}, \quad (6.5)$$

which satisfies the normalisation condition $\langle 1 \rangle = 1$.

The long-range behaviour in space and time of the spreading of particles in the VM is governed by the processes of effective branching and effective diffusion. Consequently, I demand that s and D have independent dimensions, $[s] = \mathbf{A}$ and $[D] = \mathbf{B}$. This choice of engineering dimensions is ultimately a decision about how the continuum limit is taken, which, in the present case, is done by demanding that neither branching nor diffusion are irrelevant in any dimension. Along with $[\mathbf{x}] = \mathbf{L}$ and the dimensionlessness of the action functional, $[\mathcal{A}_0 + \mathcal{A}_{\text{int}}] = 1$, it follows that

$$[\phi] = \mathbf{L}^{-d+2} \mathbf{A} \mathbf{B}^{-1}, \quad [\tilde{\phi}] = \mathbf{L}^{-2} \mathbf{A}^{-1} \mathbf{B}, \quad [t] = \mathbf{L}^2 \mathbf{B}^{-1},$$

$$[r] = \mathbf{L}^{-2} \mathbf{B}, \quad [\chi] = \mathbf{L}^{d-2} \mathbf{B}, \quad [w_1] = \mathbf{L}^2 \mathbf{A}, \quad [w_2] = \mathbf{L}^{d+2} \mathbf{A}, \quad [w_3] = \mathbf{L}^d \mathbf{A}, \quad (6.6)$$

implying that the couplings w_1 , w_2 and w_3 are irrelevant, which yields the action functional in Eq. (6.4) consistent with [53, 136]. Moreover, the action of the VM is equivalent to the action of the diffusion-limited pair annihilation process [151] under rapidity

reversal [189], that is under time inversion and the exchange of fields

$$\phi(x, t) \leftrightarrow -\tilde{\phi}(x, -t). \quad (6.7)$$

As a result, the renormalisations of both field theories are intimately related. The choice of engineering dimensions above is not unique. An alternative choice [151] is, for example, $[s] = [\chi]$.

The coupling χ has critical dimension $d_c = 2$ such that χ is relevant for $d < d_c$, marginal at $d = d_c$ and irrelevant for $d > d_c$. The fact that, in terms of the original parameters, the coupling χ is inversely proportional to the carrying capacity c indicates that χ arises from the excluded-volume interactions between particles, which, according to the critical dimension d_c become irrelevant for dimensions $d > d_c$.

The Fourier transform is defined, by convention, as

$$\phi(\mathbf{k}, \omega) = \int d^d \mathbf{x} dt e^{i\omega t - i\mathbf{k}\mathbf{x}} \phi(\mathbf{x}, t), \quad (6.8a)$$

$$\phi(\mathbf{x}, t) = \int d^d \mathbf{k} d\omega e^{-i\omega t + i\mathbf{k}\mathbf{x}} \phi(\mathbf{k}, \omega), \quad (6.8b)$$

where the spatial momentum $\mathbf{k}$ is conjugate of the position $\mathbf{x}$, the frequency ω is conjugate of time t , $d^d \mathbf{k} = d^d \mathbf{k}/(2\pi)^d$ and $d\omega = d\omega/2\pi$. The same convention applies to $\tilde{\phi}(\mathbf{x}, t)$. It is often convenient to use Feynman diagrams to express lengthy expressions [189]. In the diagrams, time, or the direction of causality as implemented by the positive mass r , is read from right to left. The bare propagator of the ϕ field is then, from Eq. (6.4a),

$$\begin{aligned} \phi \text{ --- } \tilde{\phi} &\triangleq \langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle \\ &= \frac{\delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}')}{-i\omega + D\mathbf{k}^2 + r}, \end{aligned} \quad (6.9)$$

where $\delta(\omega + \omega') = \delta(\omega + \omega')/(2\pi)$, $\delta(\mathbf{k} + \mathbf{k}') = \delta(\mathbf{k} + \mathbf{k}')/(2\pi)^d$ and $\delta(x)$ is the Dirac δ function.

6.2.1 EXPONENTS η AND z

The expected number of particles in the system is

$$n(t) = \int d^d \mathbf{x} \langle \phi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle = \Theta(t - t_0) e^{-r(t-t_0)}, \quad (6.10)$$

where Θ is the Heaviside step function and $r > 0$. At criticality, $r = 0$, the expected number of particles is $n(t) = 1$, so that $\eta = 0$ according to Eq. (6.1). The mean square spread of particles is

$$\begin{aligned} R^2(t) &= \int d^d \mathbf{x} (\mathbf{x} - \mathbf{x}_0)^2 \langle \phi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle \\ &= - \frac{d^2}{d\mathbf{k}^2} \left(\int d\mathbf{k}' d\omega d\omega' e^{-i\omega t + i\mathbf{k}\mathbf{x}} e^{-i\omega' t_0 + i\mathbf{k}'\mathbf{x}_0} \langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle \right) \Big|_{\mathbf{k}=\mathbf{0}} \\ &= 2dDt e^{-rt}, \end{aligned} \quad (6.11)$$

where $t_0 = 0$ and $r > 0$. At the critical point, $R^2(t) \propto t$, so that $z = 1$. The reason why both exponents, η and z , attain their trivial values is the non-renormalisation of the propagator and the absence of a temporal singularity at time $t = t_0$ that would require an independent renormalisation constant [189, 104].

6.2.2 EXPONENT δ

According to [103, 104, 77] (Chapter 2), the survival probability is given by

$$P_s(r, D, s, \chi; t) = - \left\langle e^{-\int d^d \mathbf{x} \phi(\mathbf{x}, t)} \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle \quad (6.12a)$$

$$= - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \left(\int d^d \mathbf{x} \phi(\mathbf{x}, t) \right)^n \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle \quad (6.12b)$$

$$\hat{=} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} n \left\{ \begin{array}{c} \text{diagram with one incoming leg and } n \text{ outgoing legs} \end{array} \right\} \quad (6.12c)$$

where the hatched circle in Eq. (6.12c) indicates the sum over all diagrams that have one incoming leg and n outgoing legs. For instance, the two tree-level contributions to

$n = 4$ are

$$\begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array} \text{---} , \quad \begin{array}{c} \diagup \\ \diagup \\ \diagup \end{array} \text{---} . \quad (6.13)$$

At tree level, the sum (6.12c) is easily performed to produce

$$P_s(r, D, s, \chi; t) = \frac{\exp(-rt)}{1 + \frac{s}{r} (1 - \exp(-rt))} , \quad (6.14)$$

which becomes

$$P_s(0, D, s, \chi; t) = \frac{1}{1 + st} \quad (6.15)$$

in the limit of $r \rightarrow 0$. However, in $d < 2$, where χ is relevant, loop diagrams contribute to the survival probability, whose effect is captured by replacing the bare s in (6.14) by an effective (renormalised) coupling. This non-linearity renormalises under dimensional regularisation at $d = 2 - \epsilon$, by means of the governing non-linearity χ , which renormalises itself.

The effective, renormalised, couplings χ_{eff} and s_{eff} are, diagrammatically,

$$\chi_{\text{eff}} = \chi + \chi \text{---} \text{---} \chi + \chi \text{---} \text{---} \chi \text{---} \text{---} \chi + \dots \quad (6.16a)$$

$$s_{\text{eff}} = s + \chi \text{---} s + \chi \text{---} \chi \text{---} s + \dots \quad (6.16b)$$

which can be summed exactly,

$$\chi_{\text{eff}} = \frac{\chi}{1 + \mathcal{J}_\chi} , \quad (6.17a)$$

$$s_{\text{eff}} = \frac{s}{1 + \mathcal{J}_\chi} , \quad (6.17b)$$

taking into account that s enters in the expansion of the exponential of the action in Eq. (6.5) with a positive sign and χ with a negative sign, and where $\mathcal{J}$ is the loop

integral

$$\mathcal{J} = \text{circle} \quad (6.18a)$$

$$\begin{aligned} &\doteq 2 \int \mathrm{d}^d \mathbf{k}_1 \mathrm{d}^d \mathbf{k}_2 \mathrm{d}^d \mathbf{k}_3 \mathrm{d}^d \mathbf{k}_4 \mathrm{d} \omega_1 \mathrm{d} \omega_2 \mathrm{d} \omega_3 \mathrm{d} \omega_4 \delta(\mathbf{k}_2 + \mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_3) \\ &\quad \times \delta(\omega_2 + \omega_4) \delta(\omega_1 + \omega_3) \left\langle \phi(\mathbf{k}_2, \omega_2) \tilde{\phi}(\mathbf{k}_1, \omega_1) \right\rangle \left\langle \phi(\mathbf{k}_4, \omega_4) \tilde{\phi}(\mathbf{k}_3, \omega_3) \right\rangle \\ &= \frac{r^{\frac{d}{2}-1}}{(4\pi D)^{\frac{d}{2}}} \Gamma\left(1 - \frac{d}{2}\right) \simeq \frac{2r^{-\frac{\epsilon}{2}}}{\epsilon(4\pi D)^{\frac{d}{2}}}, \end{aligned} \quad (6.18b)$$

with $d = 2 - \epsilon$.

Defining the Z -factors such that $\chi_{\text{eff}} = \chi Z_\chi$ and $s_{\text{eff}} = s Z_s$, from Eq. (6.17a) it follows that the two Z -factors are equal, $Z_\chi = Z_s$, yielding the renormalisation of the coupling s entirely driven by the renormalisation of χ . It is noteworthy that the renormalisation of this pair of couplings satisfies the Ward identity [21] (Chapter 4)

$$\frac{ds_{\text{eff}}}{ds} = \frac{\chi_{\text{eff}}}{\chi}, \quad (6.19)$$

derived in 6.A.

The renormalisation of the dimensionless coupling $\chi_{\mathcal{R}} = \mu^{-\epsilon} \mathcal{U} \chi_{\text{eff}}$, where $\mu = \sqrt{r/D}$ is an arbitrary inverse length scale and $\mathcal{U} = 2/(D(4\pi)^{d/2})$, is, using Eqs. (6.17a), (6.18a) and the identity $x = y/(1+ay) = y(1-ax)$, exactly

$$\chi_{\mathcal{R}} = \mu^{-\epsilon} \mathcal{U} \chi \left(1 - \frac{1}{\epsilon} \chi_{\mathcal{R}} \right). \quad (6.20)$$

The β -function of the renormalised governing coupling $\chi_{\mathcal{R}}$ is

$$\beta_\chi = \mu \frac{d}{d\mu} \chi_{\mathcal{R}} = -\epsilon \chi_{\mathcal{R}} + \chi_{\mathcal{R}}^2, \quad (6.21)$$

whose infrared-stable fixed point is $\chi_{\mathcal{R}}^* = \epsilon$ since $d\beta_{\chi}(\chi_{\mathcal{R}}^* = \epsilon)/d\chi_{\mathcal{R}} = \epsilon > 0$. Since both Z-factors are identical, $Z_{\chi} = Z_s = 1 - \chi_{\mathcal{R}}/\epsilon$, the Wilson γ -functions of the couplings are then identical as well,

$$\gamma_s = \gamma_\chi = \mu \frac{d}{d\mu} \log Z_\chi = \frac{-1}{\epsilon Z_\chi} \beta_\chi = \chi_{\mathcal{R}} \rightarrow \chi_{\mathcal{R}}^*. \quad (6.22)$$

Then, the anomalous dimensions of s and χ are $\gamma_s = \gamma_\chi = \epsilon$ for $\epsilon > 0$. For $\epsilon < 0$, the infrared-stable fixed point of β_χ is $\chi^* = 0$ as $d\beta_\chi(\chi_\mathcal{R}^* = 0)/d\chi_\mathcal{R} = -\epsilon > 0$, and therefore $\gamma_s = \gamma_\chi = 0$. As a result, in dimensions $d > 2$ the anomalous dimensions of s and χ vanishes and the process exhibits mean-field behaviour as correlations due to volume exclusion become irrelevant.

Dimensional consistency in Eq. (6.14) requires $[s] = [r]$ and therefore $\mathbf{A} = \mathbf{L}^{-2}\mathbf{B}$. As $[P_s] = 1$ it follows that

$$P_s(r, D, s, \chi; t) = P_s \left(\frac{r}{L^{-2}B}, \frac{D}{B}, \frac{s}{L^{-2}B}, \frac{\chi}{L^{-\epsilon}B}, \frac{t}{L^2B^{-1}} \right), \quad (6.23)$$

for any positive, real L, B . A similar expression can be found for all vertex functions. Following the usual process [189] of using (6.23) with $B = D$ and $L = \sqrt{Dt}$ in conjunction with the solution of the Callan-Symanzik equation on the basis of the Wilson γ -functions (6.22) produces

$$P_s(r, D, s, \chi; t) = P_s \left(rt, 1, st\ell^{\gamma_s}, \frac{\chi\ell^{\gamma_\chi}}{D(Dt)^{-\epsilon/2}}; 1 \right), \quad (6.24)$$

with a dimensionless, small scale parameter ℓ , which one may choose to be

$$\ell = (D/\chi \cdot (Dt)^{-\epsilon/2})^{1/\gamma_\chi}, \quad (6.25)$$

such that at $\epsilon > 0$, when $\gamma_s = \gamma_\chi = \epsilon$,

$$P_s(r, D, s, \chi; t) = P_s \left(rt, 1, \frac{s}{\chi}(Dt)^{1-\frac{\epsilon}{2}}, 1; 1 \right). \quad (6.26)$$

As the accounts for the effect of the interaction that takes hold at large times by adjusting parameters in an expression involving (arbitrarily) short times, when the tree-level theory applies, the survival probability at criticality (6.15) is expected to be

$$P_s(0, D, s, \chi; t) = \frac{1}{1 + \frac{s}{\chi}(Dt)^{1-\frac{\epsilon}{2}}}. \quad (6.27)$$

Asymptotically in t ,

$$P_s(0, D, s, \chi; t) \propto t^{-(1-\frac{\epsilon}{2})}, \quad (6.28)$$

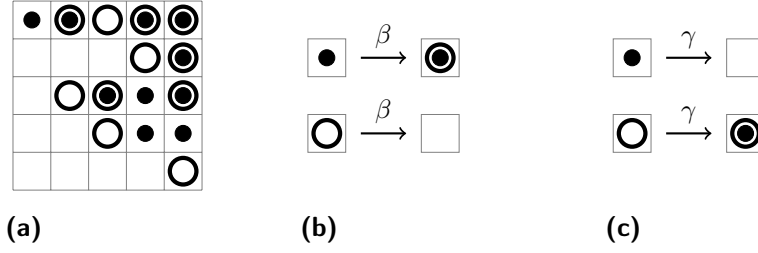


Figure 6.2: The concealed voter model (opinions in the outer layer are represented by circles and empty spaces, and, in the inner layer, by solid disks and empty spaces). (a) Sample of a particle configuration in the CVM in $d = 2$. (b) Externalisation process between the two layers. (c) Internalisation process between the two layers. The process of copying in the outer layer is represented in Fig. 6.1b.

which implies by Eq. (6.1) that $\delta = 1 - \epsilon/2$ for $d < 2$. For $d \geq 2$ the γ -functions vanish so that $\delta = 1$, with logarithmic corrections at $d = d_c = 2$, recovering the results in [53].

6.3 CRITICAL EXPONENTS OF THE CONCEALED VOTER MODEL

In addition to the lattice (outer layer) of the voter model above, where opinions are copied to nearest neighbours with rate α , the CVM has an inner layer, that represents the private opinion of each agent, which may equal or differ from the expressed opinion in the outer layer, Fig. 6.2. The private and publicly expressed opinions interact via two new processes: an agent may **externalise** their private opinion with rate β (the opinion in the inner layer is copied into the outer layer) and **internalise** their expressed opinion with rate γ (the opinion in the outer layer is copied into the inner layer). In total, there are three concurrent Poisson processes: copying at rate α , externalisation at rate β and internalisation at rate γ . The original VM is recovered if $\beta = 0$ [82].

As in the VM, the interaction between two competing opinions can be represented by the interactions between particles and empty spaces. The particles in the two layers may be considered two distinct species, namely, mobile particles with occupation numbers $\{m\}$ and immobile particles with occupation numbers $\{n\}$ [138, 21] (Chapter 4).

The master equation of the CVM comprises in addition to Eq. (6.3) for the process of copying (the left-hand side of the equation becomes $\dot{P}_\alpha(\{m\}; t)$) the following contribu-

tion

$$\begin{aligned} \dot{P}_\beta(\{m\}, \{n\}; t) = \sum_{\mathbf{x}} \Big\{ & \beta(m_{\mathbf{x}} + 1) \left(1 - \frac{n_{\mathbf{x}}}{c}\right) P(n_{\mathbf{x}}, m_{\mathbf{x}} + 1; t) \\ & + \beta n_{\mathbf{x}} \left(1 - \frac{m_{\mathbf{x}} - 1}{c}\right) P(n_{\mathbf{x}}, m_{\mathbf{x}} - 1; t) \\ & - \beta \left[m_{\mathbf{x}} \left(1 - \frac{n_{\mathbf{x}}}{c}\right) + n_{\mathbf{x}} \left(1 - \frac{m_{\mathbf{x}}}{c}\right) \right] P(n_{\mathbf{x}}, m_{\mathbf{x}}; t) \Big\}, \end{aligned} \quad (6.29)$$

for the process of externalisation with rate β , and a symmetric transformation of the above for the process of internalisation obtained by swapping $m_{\mathbf{x}}$ for $n_{\mathbf{x}}$, the rate β for γ . Hence, the time evolution of the microstate probability is $\dot{P}(\{m\}, \{n\}; t) = \dot{P}_\alpha(\{m\}; t) + \dot{P}_\beta(\{m\}, \{n\}; t) + \dot{P}_\gamma(\{m\}, \{n\}; t)$. At time $t_0 = 0$, the system is initialised with one particle of either species with equal probability at $\mathbf{x}_0$.

It follows that the action functional $\mathcal{A}'_0 + \mathcal{A}'_{\text{int}}$ of the CVM is

$$\begin{aligned} \mathcal{A}'_0 &= \int d^d \mathbf{x} dt \left\{ \tilde{\phi} \partial_t \phi - D \tilde{\phi} \nabla^2 \phi + r_1 \tilde{\phi} \phi + \tilde{\psi} \partial_t \psi + r_2 \tilde{\psi} \psi - \tau_1 \psi \tilde{\phi} - \tau_2 \phi \tilde{\psi} \right\} \quad (6.30a) \\ \mathcal{A}'_{\text{int}} &= \int d^d \mathbf{x} dt \left\{ -s \tilde{\phi}^2 \phi - \sigma_1 \tilde{\psi} \psi \tilde{\phi} - \sigma_2 \tilde{\phi} \phi \tilde{\psi} + \chi \tilde{\phi}^2 \phi^2 + \kappa_1 \psi \tilde{\phi}^2 \phi + \kappa_2 \phi \tilde{\psi}^2 \psi \right. \\ &\quad \left. + w_1 \tilde{\phi} (\nabla^2 \phi) \tilde{\phi} - w_2 \tilde{\phi} (\nabla^2 \phi) \tilde{\phi}^2 \phi + w_3 \tilde{\phi}^3 \phi^2 + w_4 \tilde{\psi} \psi \tilde{\phi}^2 \phi + w_5 \tilde{\phi} \phi \tilde{\psi}^2 \psi \right\} \end{aligned}$$

where ψ is the annihilation and $\tilde{\psi}$ is the Doi-shifted creation field of the immobile species, and where, in terms of the model parameters, $r_1 = \tau_1 = \sigma_1 = \beta$ (associated with externalisation), $r_2 = \tau_2 = \sigma_2 = \gamma$ (associated with internalisation), $\kappa_1 = w_4 = \beta/c$ and $\kappa_2 = w_5 = \gamma/c$. The mass terms r_1 and r_2 emerge from the microscopic dynamics of the process: r_1 derives from the depletion of mobile particles due to externalisation (Fig. 6.2b bottom) and r_2 from the depletion of immobile particles due to internalisation (Fig. 6.2c up). The couplings τ_1 and τ_2 are commonly referred to as transmutations [138, 21] (Chapter 4), as they provide causal correlations between fields of different species (Fig. 6.2b up and Fig. 6.2c bottom, respectively). This terminology may lead to confusion, though, if one thinks of particles in one species changing into the other species. In the CVM it is more accurate to say that immobile particles deposit mobile particles with rate β and mobile particles deposit immobile particles with rate γ . Assuming that $\mathbf{L}$, $\mathbf{A}$ and $\mathbf{B}$ are independent dimensions, and that the two fields ϕ and ψ have the same dimensions, $[\phi] = [\psi] = \mathbf{L}^{-d+2} \mathbf{A} \mathbf{B}^{-1}$, the dimensions of the new couplings

are

$$\begin{aligned} [r_1] = [r_2] = [\tau_1] = [\tau_2] &= \mathbf{B}\mathbf{L}^{-2}, \quad [\sigma_1] = [\sigma_2] = \mathbf{A}, \\ [\kappa_1] = [\kappa_2] &= \mathbf{L}^{d-2}\mathbf{B}, \quad [w_4] = [w_5] = \mathbf{L}^d\mathbf{A}. \end{aligned} \quad (6.31)$$

This implies that w_4 and w_5 are irrelevant non-linearities, while all the others are either relevant or marginal.

The couplings s and χ stay as in the VM, but the new couplings σ_1 , σ_2 , κ_1 and κ_2 couple the new ψ , $\tilde{\psi}$ fields to the old ϕ , $\tilde{\phi}$. Diagrammatically,

$$\begin{array}{cccc} \sigma_1 & \sigma_2 & \kappa_1 & \kappa_2 \\ \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} \end{array} \quad (6.32)$$

where $\text{---} \text{---} \text{---}$ represents the propagator $\langle \psi \tilde{\psi} \rangle$. This set of mixed couplings encode correlations between the two species due to transmutation and volume exclusion. From Eq. (6.30a), the four response propagators $\langle \phi \tilde{\phi} \rangle$, $\langle \psi \tilde{\psi} \rangle$, $\langle \phi \tilde{\psi} \rangle$ and $\langle \psi \tilde{\phi} \rangle$ are determined, 6.B.

In fact, all six s -like couplings (one incoming leg and two outgoing legs) are effectively available by combining s , σ_1 or σ_2 with suitable propagators, and are denoted by $\sigma_3, \dots, \sigma_5$. Similarly, the nine χ -like couplings (two incoming and two outgoing legs) are available, by combining χ , κ_1 or κ_2 with the propagators, and are denoted by $\kappa_3, \dots, \kappa_8$. Therefore, the one-loop renormalisation of each of the available couplings may need to consider sixteen loop integrals, which are reduced to ten integrals due to their symmetries, 6.C.

However, to define the observables $n(t)$, $R^2(t)$ and $P_s(t)$ in the CVM, it is convenient to consider the total number of particles, irrespectively of their species. Indeed, the system will reach an absorbing state whenever the total population is zero. This motivates the definition of the annihilation *density* field $\rho = \phi + \psi$ (and the Doi-shifted creation density field $\tilde{\rho} = \tilde{\phi} + \tilde{\psi}$), whose propagator is, using the propagators in 6.B,

$$\begin{aligned} \rho \text{---} \text{---} \text{---} \tilde{\rho} &\triangleq \langle \rho(\mathbf{k}, \omega) \tilde{\rho}(\mathbf{k}', \omega') \rangle \\ &= \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \frac{-2i\omega + D\mathbf{k}^2 + r_1 + r_2 + \tau_1 + \tau_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1\tau_2}. \end{aligned} \quad (6.33)$$

The critical point of the CVM happens when particles, *i.e.* one of the two opinions, are no longer subject to *spontaneous* extinction, $r_1 = \tau_1$ and $r_2 = \tau_2$. That is, when the gain and loss of particles due to deposition and decay are balanced: immobile particles through the internalisation process, and mobile particles through the externalisation process. To deal with the infrared divergence at criticality, it can be assumed that $r_1 = \tau_1 + r$ and $r_2 = \tau_2 + r$, taking the limit $r \rightarrow 0$ to reach the critical point. This parametrisation of r_1 and r_2 is allowed because the critical point $r \rightarrow 0$ of the CVM is unique, as shown in 6.D.

The loop integrals in 6.C contain two infrared divergencies, $r \rightarrow 0$ and $\tau_1 + \tau_2 \rightarrow 0$, where the second one corresponds to the original VM at criticality. Henceforth, I assume $\tau_1 + \tau_2 > 0$ and refer to $r \rightarrow 0$ as the critical point of the CVM. Moreover, the loop integral I_{10} in Eq. (6.58j) has a quadratic divergence that results from taking the continuum limit. Since the process is originally defined on the lattice, this divergence can be regularised by introducing an ultraviolet cutoff. Usually, such a quadratic divergence can be absorbed into a fluctuation-induced shift of the critical point by means of additive renormalisation. However, in this case the quadratic divergence is associated with an infrared divergence in τ_2 , *i.e.* the critical point of the original VM within the CVM.

6.3.1 EXPONENTS η AND z

The expected number of particles is

$$n(t) = \frac{1}{2} \int d^d \mathbf{x} \langle \rho(\mathbf{x}, t) \tilde{\rho}(\mathbf{x}_0, t_0) \rangle = \Theta(t - t_0) e^{-r(t-t_0)}, \quad (6.34)$$

where the factor of $1/2$ accounts for a particle of either species being created with equal probability. At $r = 0$, *i.e.* at the critical point $n(t) = 1$, and the critical exponent in Eq. (6.1) is $\eta = 0$. The mean square spread of mobile particles is

$$\begin{aligned} R^2(t) &= \frac{1}{2} \int d^d \mathbf{x} (\mathbf{x} - \mathbf{x}_0)^2 \langle \rho(\mathbf{x}, t) \tilde{\rho}(\mathbf{x}_0, t_0) \rangle \\ &= \frac{dD}{\tau_1 + \tau_2} e^{-rt} \left[2\tau_2 t + \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 - e^{-(\tau_1 + \tau_2)t}) \right], \end{aligned} \quad (6.35)$$

where $t_0 = 0$. At the critical point $R^2(t) \propto t$, so the critical exponent is $z = 1$.

Eq. (6.35) recovers the behaviour of the original voter model, Eq. (6.11), for vanishing transmutation, depending on how this limit is taken. For $\tau_1 = 0$, $\tau_2 \rightarrow 0$ and $\tau_2 = 0$, $\tau_1 \rightarrow 0$ the mean square spread is half of that of Eq. (6.11), as the spread is zero if the initial particle happens to be immobile.

6.3.2 EXPONENT δ

The survival probability in the CVM is

$$P_s(r, \tau_1, \tau_2, D, s, \sigma_1, \dots, \sigma_5, \chi, \kappa_1, \dots, \kappa_8; t) = -\frac{1}{2} \left\langle e^{-\int d^d \mathbf{x} \rho(\mathbf{x}, t)} \tilde{\rho}(\mathbf{x}_0, t_0) \right\rangle \quad (6.36a)$$

$$\begin{aligned} &= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \left(\int d^d \mathbf{x} \rho(\mathbf{x}, t) \right)^n \tilde{\rho}(\mathbf{x}_0, t_0) \right\rangle \\ &\hat{=} -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \quad n \left\{ \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right\} \quad (6.36b) \end{aligned}$$

where the sum contains all binary tree diagrams with one incoming $\tilde{\rho}$ field and n outgoing ρ fields.

To account for the transmutations between the ϕ and ψ fields in the sum in Eq. (6.36a), the density field ρ alone is not enough. To make use of the symmetries that will later allow for simplifications, I consider the *polarity* fields $\nu = \phi - \psi$ and $\tilde{\nu} = (\tau_1 \tilde{\phi} - \tau_2 \tilde{\psi}) / (\tau_1 + \tau_2)$. The fields ϕ , $\tilde{\phi}$, ψ , $\tilde{\psi}$ are then mapped to the fields ρ , $\tilde{\rho}$, ν , $\tilde{\nu}$, by means of linear combinations. As will become clearer, this is a particularly simple mapping that allows the loop correction analysis. The propagators $\langle \nu \tilde{\nu} \rangle$, $\langle \rho \tilde{\nu} \rangle$ and $\langle \nu \tilde{\rho} \rangle$ are stated explicitly in 6.E. Similarly, the couplings involving ρ , $\tilde{\rho}$, ν , $\tilde{\nu}$, are weighted linear combinations of s , σ_1 , $\dots$, σ_5 , χ , κ_1 , $\dots$, κ_8 and all observables may be parametrised in terms of these linear combinations weighted by powers of $\tau_1 / (\tau_1 + \tau_2)$ and $\tau_2 / (\tau_1 + \tau_2)$ as they enter $\tilde{\nu}$. Eq. (6.36a) is thus simply a reparametrisation of the survival probability in terms of linear combinations of couplings. Consider, for example, the branching non-linearity involving only ρ and $\tilde{\rho}$ fields

$$\begin{array}{c} \zeta \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \quad (6.37)$$

with $\zeta = \tau_2 (s\tau_2 + \tau_1(\sigma_1 + \sigma_2)) / (2(\tau_1 + \tau_2)^2)$, and the four-legged non-linearity involving

only ρ and $\tilde{\rho}$ fields

$$\begin{array}{c} \lambda \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \quad (6.38)$$

with $\lambda = (\tau_2^2(\chi + \kappa_1) + \tau_1^2\kappa_2) / (4(\tau_1 + \tau_2)^2)$.

For example, for $n = 3$, the sum in Eq. (6.36a) has contributions from diagrams such as

$$\begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}, \quad \begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}, \quad \begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}, \quad (6.39)$$

where $\dots\dots\dots$ represents the propagator $\langle \nu \tilde{\nu} \rangle$.

In what follows I show that any addend in Eq. (6.36a) containing a ν or a $\tilde{\nu}$ field is asymptotically sub-leading and, therefore, its contribution can be neglected. From the propagators in 6.E, it follows that the propagators in real time, integrated over all space, are

$$\int d^d \mathbf{x} \langle \nu(\mathbf{x}, t) \tilde{\nu}(\mathbf{x}_0, t_0) \rangle = -e^{-(\tau_1 + \tau_2 + r)t}, \quad (6.40a)$$

$$\int d^d \mathbf{x} \langle \rho(\mathbf{x}, t) \tilde{\nu}(\mathbf{x}_0, t_0) \rangle = \frac{\tau_2 - \tau_1}{\tau_1 + \tau_2} e^{-(\tau_1 + \tau_2 + r)t}, \quad (6.40b)$$

$$\int d^d \mathbf{x} \langle \nu(\mathbf{x}, t) \tilde{\rho}(\mathbf{x}_0, t_0) \rangle = 0, \quad (6.40c)$$

which decay exponentially faster in time t than $\int d^d \mathbf{x} \langle \rho(\mathbf{x}, t) \tilde{\rho}(\mathbf{x}_0, t_0) \rangle = 2e^{-rt}$ in Eq. (6.34), since $r \rightarrow 0$ while $\tau_1, \tau_2 > 0$. Hence, only binary tree diagrams containing ρ and $\tilde{\rho}$ asymptotically contribute to the sum in Eq. (6.36a), which gives

$$P_s(r, \tau_1, \tau_2, D, s, \sigma_1, \dots, \sigma_5, \chi, \kappa_1, \dots, \kappa_8; t) \simeq \frac{\exp(-rt)}{2 \left(1 + \frac{\zeta}{r} (1 - \exp(-rt))\right)}. \quad (6.41)$$

Further, the fields ν and $\tilde{\nu}$ are also present in the renormalisation of the non-linearities. For example, ζ is renormalised diagrammatically by

$$\begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \mathcal{R} = \begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \lambda \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \zeta \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} + \dots \quad (6.42)$$

and similarly for the four-legged diagrams, which are, in fact, the governing non-

linearities. However, the choice of mapping between fields suitably fits the underlying symmetries of the original loops, Eqs. (6.58a)–(6.58j), and one can show that, crucially, any loops containing ν or $\tilde{\nu}$ are ultraviolet convergent as far as the phase transition at $r \rightarrow 0$ is concerned. Expressing the loops formed with $\rho, \tilde{\rho}, \nu, \tilde{\nu}$ as linear combinations of the loops formed with $\phi, \tilde{\phi}, \psi, \tilde{\psi}$ in Appendix 6.C (derived in Appendix 6.F), and assuming $\tau_1 + \tau_2 > 0$, it follows that the only ultraviolet divergence to be taken into account is

$$\mathcal{K} = \text{[Diagram: A circle with two external lines on the left and right, and a dashed line on the top half]} \quad (6.43a)$$

$$\begin{aligned} &\hat{=} 2 \int \mathrm{d}^d \mathbf{k}_1 \mathrm{d}^d \mathbf{k}_2 \mathrm{d}^d \mathbf{k}_3 \mathrm{d}^d \mathbf{k}_4 \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}\omega_3 \mathrm{d}\omega_4 \delta(\mathbf{k}_2 + \mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_3) \\ &\quad \times \delta(\omega_2 + \omega_4) \delta(\omega_1 + \omega_3) \langle \rho(\mathbf{k}_2, \omega_2) \tilde{\rho}(\mathbf{k}_1, \omega_1) \rangle \langle \rho(\mathbf{k}_4, \omega_4) \tilde{\rho}(\mathbf{k}_3, \omega_3) \rangle \\ &= 4 \left(\frac{\tau_1 + \tau_2}{\tau_2 4\pi D} \right)^{\frac{d}{2}} r^{\frac{d}{2}-1} \Gamma \left(1 - \frac{d}{2} \right), \end{aligned} \quad (6.43b)$$

whereas all the other loop integrals are ultraviolet convergent,

$$0 \hat{=} \text{[Diagram: Circle with red dashed top and black dashed bottom]} = \text{[Diagram: Circle with red dashed top and red dashed bottom]} = \text{[Diagram: Circle with red dashed top and black dashed bottom]} = \text{[Diagram: Circle with red dashed top and black dashed bottom]} = \text{[Diagram: Circle with red dashed top and red dashed bottom]} = \text{[Diagram: Circle with red dashed top and red dashed bottom]} = \text{[Diagram: Circle with red dashed top and black dashed bottom]} = \text{[Diagram: Circle with red dashed top and black dashed bottom]} = \text{[Diagram: Circle with red dashed top and red dashed bottom]}. \quad (6.44)$$

In conclusion, the renormalisation of the reparametrised non-linearities is solely run by λ , in the same way that, in the VM, χ is the governing non-linearity. It follows that the arguments in Sec. 6.2.2 for the VM apply equally to the CVM, producing the same result for the critical exponent δ in the CVM as in the VM.

6.4 DISCUSSION AND CONCLUSIONS

In summary, the CVM has the same asymptotic behaviour as the VM at criticality and, therefore, the CVM belongs to the VM universality class. The addition of the inner layer and the new interactions with the external layer introduce changes in the short-term dynamics of the process, where the inner layer acts as an internal memory of previous states. However, the new interactions are not strong enough to significantly change the long-range and long-term behaviour of the process.

Yet, the interactions in the CVM manage to cause havoc to the renormalisation cal-

ulation and require linear combinations of fields and couplings to be considered. The field theory encounters a range of interesting technical challenges. For example, since the VM at criticality is a particular case of the CVM, the loop integrals involved in the CVM present multiple infrared divergencies. Moreover, the renormalisation of the field theory in ϕ and ψ involves ten loop integrals for a set of fifteen coupled non-linearities. Nevertheless, the renormalisation scheme greatly simplifies by reparametrising the non-linearities by suitable linear transformations of the fields that make use of the intrinsic symmetries of the field theory of the CVM.

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6.A WARD IDENTITY

In this section I adapt the derivation of the Ward identity in [21] (Chapter 4) to the Voter Model. The action of the VM without irrelevant terms is, from Eqs. (6.4) and (6.4b),

$$\mathcal{A}([\phi, \tilde{\phi}]; D, r, s, \chi) = \int d^d \mathbf{x} dt \left\{ \tilde{\phi}(\partial_t - D\nabla^2 + r)\phi - s\tilde{\phi}^2\phi + \chi\tilde{\phi}^2\phi^2 \right\}. \quad (6.45)$$

The Ward identity, Eq. (6.19), has its origin in a symmetry of the action when shifting the field $\phi(x, t)$ by a constant Σ ,

$$\mathcal{A}([\phi + \Sigma, \tilde{\phi}]; D, r, s, \chi) = \mathcal{A}([\phi, \tilde{\phi}]; D, r, s - 2\chi\Sigma, \chi) + \int d^d \mathbf{x} dt \left\{ r\Sigma\tilde{\phi} - s\Sigma\tilde{\phi}^2 + \chi\Sigma^2\tilde{\phi}^2 \right\}. \quad (6.46)$$

To simplify the notation, let

$$\mathcal{A} = \mathcal{A}([\phi, \tilde{\phi}]; D, r, s, \chi) \quad (6.47a)$$

$$\mathcal{A}' = \mathcal{A}([\phi + \Sigma, \tilde{\phi}]; D, r, s, \chi) \quad (6.47b)$$

$$\mathcal{A}'' = \mathcal{A}([\phi, \tilde{\phi}]; D, r, s - 2\chi\Sigma, \chi) \quad (6.47c)$$

so that Eq. (6.46) reads

$$\mathcal{A}' = \mathcal{A}'' + \int d^d \mathbf{x} dt \left\{ r \Sigma \tilde{\phi} - s \Sigma \tilde{\phi}^2 + \chi \Sigma^2 \tilde{\phi}^2 \right\}, \quad (6.48)$$

and define the path integral

$$\langle \mathcal{O} \rangle_{\mathcal{A}} = \int \mathcal{D}[\phi, \tilde{\phi}] \mathcal{O} e^{-\mathcal{A}}, \quad (6.49)$$

and equivalently for the actions $\mathcal{A}'$ and $\mathcal{A}''$, for any observable $\mathcal{O}$. To derive the Ward identity, consider the observable $\langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}}$. Since ϕ is a dummy variable in the path integral, the integral is invariant under a shift of the field by a constant Σ ,

$$\begin{aligned} \langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}} &= \left\langle \left(\phi(\mathbf{x}_2, t_2) + \Sigma \right) \left(\phi(\mathbf{x}_1, t_1) + \Sigma \right) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}'} \\ &= \langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}'} + \Sigma \langle \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}'} \\ &\quad + \Sigma \langle \phi(\mathbf{x}_2, t_2) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}'} + \Sigma^2 \langle \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}'} . \end{aligned} \quad (6.50)$$

Differentiating Eq. (6.50) with respect to Σ and evaluating at $\Sigma = 0$ gives

$$\begin{aligned} 0 &= -2\chi \partial_s \langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}} \\ &\quad - r \int d^d \mathbf{x}' dt' \langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \tilde{\phi}(\mathbf{x}', t') \rangle_{\mathcal{A}} \\ &\quad + s \int d^d \mathbf{x}' dt' \langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \tilde{\phi}^2(\mathbf{x}', t') \rangle_{\mathcal{A}} \\ &\quad + \langle \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}} + \langle \phi(\mathbf{x}_2, t_2) \tilde{\phi}(\mathbf{x}_0, t_0) \rangle_{\mathcal{A}}, \end{aligned} \quad (6.51)$$

since $\mathcal{A}' = \mathcal{A}$ at $\Sigma = 0$. The path integral in the first term of the right-hand side of Eq. (6.51) is

$$\begin{aligned}
\left\langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle_{\mathcal{A}} &\hat{=} 2 \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ 2 \end{array} \begin{array}{c} 0 \end{array} \\
&= 2s_{\text{eff}} \int \tilde{d}^d \mathbf{k}_0 \tilde{d}^d \mathbf{k}'_0 \tilde{d}^d \mathbf{k}_1 \tilde{d}^d \mathbf{k}'_1 \tilde{d}^d \mathbf{k}_2 \tilde{d}^d \mathbf{k}'_2 \tilde{d} \omega_0 \tilde{d} \omega'_0 \tilde{d} \omega_1 \tilde{d} \omega'_1 \tilde{d} \omega_2 \tilde{d} \omega'_2 \\
&\quad \times \delta(\omega'_0 + \omega'_1 + \omega'_2) \delta(\mathbf{k}'_0 + \mathbf{k}'_1 + \mathbf{k}'_2) e^{-i(\omega_0 t_0 + \omega_1 t_1 + \omega_2 t_2) + i(\mathbf{k}_0 \mathbf{x}_0 + \mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2)} \\
&\quad \times \left\langle \phi(\omega'_0, \mathbf{k}'_0) \tilde{\phi}(\omega_0, \mathbf{k}_0) \right\rangle \left\langle \phi(\omega_1, \mathbf{k}_1) \tilde{\phi}(\omega'_1, \mathbf{k}'_1) \right\rangle \left\langle \phi(\omega_2, \mathbf{k}_2) \tilde{\phi}(\omega'_2, \mathbf{k}'_2) \right\rangle \\
&= 2s_{\text{eff}} \int \tilde{d}^d \mathbf{k}_1 \tilde{d}^d \mathbf{k}_2 \tilde{d} \omega_1 \tilde{d} \omega_2 e^{-i(\omega_1(t_1 - t_0) + \omega_2(t_2 - t_0)) + i(\mathbf{k}_1(\mathbf{x}_1 - \mathbf{x}_0) + \mathbf{k}_2(\mathbf{x}_2 - \mathbf{x}_0))} \\
&\quad \times \left\langle \phi(\omega_1 + \omega_2, \mathbf{k}_1 + \mathbf{k}_2) \tilde{\phi}(-\omega_1 - \omega_2, -\mathbf{k}_1 - \mathbf{k}_2) \right\rangle \\
&\quad \times \left\langle \phi(\omega_1, \mathbf{k}_1) \tilde{\phi}(-\omega_1, -\mathbf{k}_1) \right\rangle \left\langle \phi(\omega_2, \mathbf{k}_2) \tilde{\phi}(-\omega_2, -\mathbf{k}_2) \right\rangle. \tag{6.52}
\end{aligned}$$

The second path integral in Eq. (6.51) has three contributions,

$$\begin{aligned}
&\int d^d \mathbf{x}' dt' \left\langle \phi(\mathbf{x}_2, t_2) \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \tilde{\phi}(\mathbf{x}', t') \right\rangle_{\mathcal{A}} \\
&\hat{=} \int d^d \mathbf{x}' dt' \left\{ \begin{array}{c} 1 \quad 0 \\ \text{---} \end{array} \times \begin{array}{c} 2 \quad ' \\ \text{---} \end{array} + \begin{array}{c} 2 \quad 0 \\ \text{---} \end{array} \times \begin{array}{c} 1 \quad ' \\ \text{---} \end{array} + 4 \begin{array}{c} 1 \quad 0 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ 2 \quad ' \end{array} \right\} \\
&= \frac{1}{r} \left\langle \phi(\mathbf{x}_1, t_1) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle + \frac{1}{r} \left\langle \phi(\mathbf{x}_2, t_2) \tilde{\phi}(\mathbf{x}_0, t_0) \right\rangle \\
&\quad - \frac{4\chi_{\text{eff}}}{r} \int \tilde{d}^d \mathbf{k}_1 \tilde{d}^d \mathbf{k}_2 \tilde{d} \omega_1 \tilde{d} \omega_2 e^{-i(\omega_1(t_1 - t_0) + \omega_2(t_2 - t_0)) + i(\mathbf{k}_1(\mathbf{x}_1 - \mathbf{x}_0) + \mathbf{k}_2(\mathbf{x}_2 - \mathbf{x}_0))} \\
&\quad \times \left\langle \phi(\omega_1 + \omega_2, \mathbf{k}_1 + \mathbf{k}_2) \tilde{\phi}(-\omega_1 - \omega_2, -\mathbf{k}_1 - \mathbf{k}_2) \right\rangle \\
&\quad \times \left\langle \phi(\omega_1, \mathbf{k}_1) \tilde{\phi}(-\omega_1, -\mathbf{k}_1) \right\rangle \left\langle \phi(\omega_2, \mathbf{k}_2) \tilde{\phi}(-\omega_2, -\mathbf{k}_2) \right\rangle, \tag{6.53}
\end{aligned}$$

where integrating over space and time amounts to evaluating the propagator in Fourier space at $\omega' = 0$ and $\mathbf{k}' = \mathbf{0}$,

$$\int d^d \mathbf{x}' dt' \left\langle \phi(\mathbf{x}, t) \tilde{\phi}(\mathbf{x}', t') \right\rangle = \int \tilde{d}^d \mathbf{k} \tilde{d} \omega \left\langle \phi(\omega, \mathbf{k}) \tilde{\phi}(\omega' = 0, \mathbf{k}' = \mathbf{0}) \right\rangle = \frac{1}{r}. \tag{6.54}$$

The third term in Eq. (6.51), with coupling s , vanishes because the action does not provide any vertex that allows to pair each creator field with an annihilator field. Substituting Eqs. (6.52) and (6.53) in Eq. (6.51), and dividing out the common integral, which amounts to amputating diagrams, gives the Ward identity,

$$0 = -\chi \partial_s s_{\text{eff}} + \chi_{\text{eff}}, \quad (6.55)$$

which is a generalisation of the VM Lagrangian invariance in [136].

The Ward identity in Eq. (6.55) also holds in the diffusion-limited pair annihilation process field theory [151] by virtue of the rapidity reversal of the VM field theory, Eq. (6.7). In this case, the origin of the Ward identity lies in the symmetry of the action under a shift of the creation field by a constant (instead of the annihilation field).

6.B PROPAGATORS INVOLVING ϕ AND ψ

From Eq. (6.30a), the four response propagators $\langle \phi \tilde{\phi} \rangle$, $\langle \psi \tilde{\psi} \rangle$, $\langle \phi \tilde{\psi} \rangle$, $\langle \psi \tilde{\phi} \rangle$, are read off from the inverse of the bilinear interaction matrix as follows:

$$\begin{aligned} \phi \quad \tilde{\phi} &= \text{---} + \text{---} + \text{---} + \dots \\ &\hat{=} \langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle \\ &= \frac{\delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') (-i\omega + r_2)}{(-i\omega + D\mathbf{k}^2 + r_1) (-i\omega + r_2) - \tau_1 \tau_2}, \end{aligned} \quad (6.56a)$$

$$\begin{aligned} \psi \quad \tilde{\psi} &= \text{---} + \text{---} + \text{---} + \dots \\ &\hat{=} \langle \psi(\mathbf{k}, \omega) \tilde{\psi}(\mathbf{k}', \omega') \rangle \\ &= \frac{\delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') (-i\omega + D\mathbf{k}^2 + r_1)}{(-i\omega + D\mathbf{k}^2 + r_1) (-i\omega + r_2) - \tau_1 \tau_2}, \end{aligned} \quad (6.56b)$$

$$\begin{aligned}
\phi \text{---} \tilde{\psi} &= \text{---} + \text{---} + \text{---} + \dots \\
&\hat{=} \langle \phi(\mathbf{k}, \omega) \tilde{\psi}(\mathbf{k}', \omega') \rangle \\
&= \frac{\delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \tau_1}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2},
\end{aligned} \tag{6.56c}$$

$$\begin{aligned}
\psi \text{---} \tilde{\phi} &= \text{---} + \text{---} + \text{---} + \dots \\
&\hat{=} \langle \psi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle \\
&= \frac{\delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \tau_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2}.
\end{aligned} \tag{6.56d}$$

6.C LOOP INTEGRALS WITH ϕ AND ψ

The derivation of the loop integrals with ϕ and ψ is in Appendix 6.F. The loops in Eq. (6.58) are written by substituting $r_i = \tau_i + r$, $i \in \{1, 2\}$, in Eq. (6.79), to leading order (assuming $0 < r \ll \tau_i$). Let

$$\tilde{\mathcal{U}} = \frac{(\tau_1 + \tau_2)^{\frac{d}{2}-2}}{2\tau_2(4\pi D)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right), \tag{6.57}$$

then the loop integrals involving the fields $\phi, \tilde{\phi}, \psi, \tilde{\psi}$, are

$$I_1 = \text{---} \hat{=} \tilde{\mathcal{U}} \left(\tau_1 \tau_2 + \tau_2^2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \tag{6.58a}$$

$$I_2 = \text{---} \hat{=} \tilde{\mathcal{U}} \left(-\tau_1 \tau_2 + \tau_1 \tau_2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \tag{6.58b}$$

$$I_3 = \text{---} \hat{=} \tilde{\mathcal{U}} \left(-\tau_2^2 + \tau_2^2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \tag{6.58c}$$

$$I_4 = \text{---} \hat{=} \tilde{\mathcal{U}} \left(\tau_1 \tau_2 + 2\tau_2^2 + \tau_1 \tau_2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \tag{6.58d}$$

$$I_5 = \text{---} \hat{=} \tilde{\mathcal{U}} \left(-\tau_1^2 + \tau_1^2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \tag{6.58e}$$

$$I_6 = \text{loop diagram} \hat{=} I_2, \quad (6.58f)$$

$$I_7 = \text{loop diagram} \hat{=} I_3, \quad (6.58g)$$

$$I_8 = \text{loop diagram} \hat{=} \tilde{\mathcal{U}} \left(\tau_2^2 + \tau_1 \tau_2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \quad (6.58h)$$

$$I_9 = \text{loop diagram} \hat{=} \tilde{\mathcal{U}} \left(\tau_1 \tau_2 + \tau_1^2 \left(\frac{r}{\tau_2} \right)^{\frac{d}{2}-1} \right), \quad (6.58i)$$

$$I_{10} = \text{loop diagram} \hat{=} \frac{\Lambda^d}{2\tau_2} + I_9, \quad (6.58j)$$

where $\Lambda < \infty$ is an ultraviolet cutoff to regularise the quadratic divergence of I_{10} at $d = d_c$ as $|\mathbf{k}| \rightarrow \infty$. The loop I_{10} is UV finite with a quadratic divergence entering only as the internal layer becomes inert in the absence of internalisation, $\tau_2 \rightarrow 0$. All loops above have the same divergence as $r \rightarrow 0$, the critical point of the CVM, up to an amplitude in powers of τ_1 and τ_2 that is effectively accounted for by the linear combination of fields introduced with $\tilde{\nu}$. In any loops that contain ν and $\tilde{\nu}$ fields, which are constructed by subtracting fields ϕ, ψ and $\tilde{\phi}, \tilde{\psi}$ respectively, UV divergencies will therefore by symmetry vanish.

6.D UNIQUENESS OF THE CRITICAL POINT OF THE CVM

The critical point is found in the divergence of the loop integrals. For example, the integral I_1 in Eq. (6.58a) is, assuming $r_1, r_2, \tau_1, \tau_2 > 0$,

$$I_1 = \frac{\Gamma(1 - \frac{d}{2})}{2(r_2^2 + \tau_1 \tau_2)(4\pi D)^{d/2}} \left[\tau_1 \tau_2 (r_1 + r_2)^{d/2-1} + r_2^2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} \right]. \quad (6.59)$$

This indicates the presence of two divergencies: $r_1 + r_2 \rightarrow 0$, which recovers the original VM, and $\bar{\varepsilon} = r_1 - \tau_1 \tau_2 / r_2 \rightarrow 0$, which defines the parameter regime of the CVM at criticality. Let the masses r_1 and r_2 be parametrised by $r_i = \tau_i + \theta_i$ with $i \in \{1, 2\}$. The distance to the critical point $\bar{\varepsilon}$ reads

$$\bar{\varepsilon} = \tau_1 + \theta_1 - \frac{\tau_1 \tau_2}{\tau_2 + \theta_2} = \frac{\theta_1 \tau_2 + \theta_2 \tau_1 + \theta_1 \theta_2}{\tau_2 + \theta_2}. \quad (6.60)$$

Taking the limit $\theta_i \rightarrow 0$ while keeping $\theta_j > 0$, with $j \in \{1, 2\}$ and $j \neq i$, gives

$$\lim_{\theta_i \rightarrow 0} \bar{\varepsilon} = \frac{\theta_j \tau_i}{\tau_2 + \theta_2} > 0. \quad (6.61)$$

Since it is necessary to take both limits, $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow 0$, simultaneously in order to find the critical point,

$$\lim_{\theta_1 \rightarrow 0} \lim_{\theta_2 \rightarrow 0} \bar{\varepsilon} = 0, \quad (6.62)$$

the critical point of the CVM is unique and therefore r_i can be parametrised by $r_i = \tau_i + r$.

6.E PROPAGATORS INVOLVING ρ AND ν

The propagators $\langle \nu \tilde{\nu} \rangle$, $\langle \rho \tilde{\nu} \rangle$, $\langle \nu \tilde{\rho} \rangle$, are

$$\begin{aligned} \nu \cdots \tilde{\nu} &\hat{=} \langle \nu(\mathbf{k}, \omega) \tilde{\nu}(\mathbf{k}', \omega') \rangle \\ &= \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \frac{(\tau_1 + \tau_2)(-i\omega) + \tau_2 D\mathbf{k}^2 + \tau_1 r_2 + \tau_2 r_1 - 2\tau_1 \tau_2}{(\tau_1 + \tau_2) [(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2]}, \end{aligned} \quad (6.63a)$$

$$\begin{aligned} \rho \cdots \tilde{\nu} &\hat{=} \langle \rho(\mathbf{k}, \omega) \tilde{\nu}(\mathbf{k}', \omega') \rangle \\ &= \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \frac{(\tau_1 - \tau_2)(-i\omega) - \tau_2 D\mathbf{k}^2 + \tau_1 r_2 - \tau_2 r_1}{(\tau_1 + \tau_2) [(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2]}, \end{aligned} \quad (6.63b)$$

$$\begin{aligned} \nu \cdots \tilde{\rho} &\hat{=} \langle \nu(\mathbf{k}, \omega) \tilde{\rho}(\mathbf{k}', \omega') \rangle \\ &= \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \frac{-D\mathbf{k}^2 + r_2 - r_1 + \tau_1 - \tau_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2}. \end{aligned} \quad (6.63c)$$

6.F CALCULATION OF THE LOOPS WITH ϕ AND ψ

This Appendix contains the calculation of the loop integrals that are listed in Appendix 6.C. First, a clarification is needed about the symmetry factors of the loops and their

diagrammatic representation. In Eq. (6.18), the loop diagram represent the integral $\mathcal{J}$

$$\text{---}\bigcirc\text{---} \hat{=} \int \mathrm{d}^d \mathbf{k}_1 \mathrm{d}^d \mathbf{k}_2 \mathrm{d} \omega_1 \mathrm{d} \omega_2 \left\langle \phi(\mathbf{k}_2, \omega_2) \phi(-\mathbf{k}_2, -\omega_2) \tilde{\phi}(\mathbf{k}_1, \omega_1) \tilde{\phi}(-\mathbf{k}_1, -\omega_1) \right\rangle \quad (6.64a)$$

$$= 2 \int \mathrm{d}^d \mathbf{k} \mathrm{d} \omega \left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(-\mathbf{k}, -\omega) \right\rangle \left\langle \phi(-\mathbf{k}, -\omega) \tilde{\phi}(\mathbf{k}, \omega) \right\rangle \quad (6.64b)$$

and similarly in Eq. (6.43) for the field ρ . Eq. (6.64b) derives from (6.64a) by applying Wick's Theorem [189] and realising that one of the dummy variables can be transformed so that the two terms in the sum are equal, and hence the origin of symmetry factor 2 in (6.64b). This convention eases the notation in equations involving the renormalisation group such as (6.16) and (6.17), however it may lead to confusion when new fields are introduced. Consider, for instance, the integral

$$\int \mathrm{d}^d \mathbf{k}_1 \mathrm{d}^d \mathbf{k}_2 \mathrm{d} \omega_1 \mathrm{d} \omega_2 \left\langle \phi(\mathbf{k}_2, \omega_2) \psi(-\mathbf{k}_2, -\omega_2) \tilde{\phi}(\mathbf{k}_1, \omega_1) \tilde{\psi}(-\mathbf{k}_1, -\omega_1) \right\rangle \quad (6.65a)$$

$$= \int \mathrm{d}^d \mathbf{k} \mathrm{d} \omega \left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(-\mathbf{k}, -\omega) \right\rangle \left\langle \psi(-\mathbf{k}, -\omega) \tilde{\psi}(\mathbf{k}, \omega) \right\rangle \quad (6.65b)$$

$$+ \int \mathrm{d}^d \mathbf{k} \mathrm{d} \omega \left\langle \phi(\mathbf{k}, \omega) \tilde{\psi}(-\mathbf{k}, -\omega) \right\rangle \left\langle \psi(-\mathbf{k}, -\omega) \tilde{\phi}(\mathbf{k}, \omega) \right\rangle . \quad (6.65c)$$

Clearly, the proper diagrammatic representation of Eq. (6.65a) is

$$\text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---}, \quad (6.66)$$

namely, after applying Wick's theorem. Henceforth, when combining more than one field, I use the diagrammatic representation of the loops *after* applying Wick's theorem, although to be clear of any ambiguity, each diagram is defined when introduced for the first time.

Using the four propagators in Eq. (6.56), the loop integrals are then,

$$I_1 = \text{---}\bigcirc\text{---} \hat{=} \int \mathrm{d}^d \mathbf{k} \mathrm{d} \omega \frac{-i\omega + r_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{i\omega + r_2}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67a)$$

$$I_2 = \text{---}\bigcirc\text{---} \hat{=} \int \mathrm{d}^d \mathbf{k} \mathrm{d} \omega \frac{-i\omega + r_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_1}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67b)$$

$$I_3 = \text{Diagram 1} \hat{=} \int d^d \mathbf{k} d\omega \frac{-i\omega + r_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_2}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67c)$$

$$I_4 = \text{Diagram 2} \hat{=} \int d^d \mathbf{k} d\omega \frac{-i\omega + D\mathbf{k}^2 + r_1}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{i\omega + r_2}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67d)$$

$$I_5 = \text{Diagram 3} \hat{=} \int d^d \mathbf{k} d\omega \frac{\tau_1}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_1}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67e)$$

$$I_6 = \text{Diagram 4} \hat{=} \int d^d \mathbf{k} d\omega \frac{\tau_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_1}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2} \quad (6.67f)$$

$$I_7 = \text{Diagram 5} \hat{=} \int d^d \mathbf{k} d\omega \frac{\tau_2}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_2}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67g)$$

$$I_8 = \text{Diagram 6} \hat{=} \int d^d \mathbf{k} d\omega \frac{-i\omega + D\mathbf{k}^2 + r_1}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_2}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67h)$$

$$I_9 = \text{Diagram 7} \hat{=} \int d^d \mathbf{k} d\omega \frac{-i\omega + D\mathbf{k}^2 + r_1}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{\tau_1}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}, \quad (6.67i)$$

$$I_{10} = \text{Diagram 8} \hat{=} \int d^d \mathbf{k} d\omega \frac{-i\omega + D\mathbf{k}^2 + r_1}{(-i\omega + D\mathbf{k}^2 + r_1)(-i\omega + r_2) - \tau_1 \tau_2} \frac{-i\omega + D\mathbf{k}^2 + r_1}{(i\omega + D\mathbf{k}^2 + r_1)(i\omega + r_2) - \tau_1 \tau_2}. \quad (6.67j)$$

Some of the integrals are proportional to others, which brings down to six the number of integrals to be calculated,

$$\frac{I_2}{\tau_1} = \frac{I_3}{\tau_2}, \quad \frac{I_5}{\tau_1^2} = \frac{I_6}{\tau_1 \tau_2} = \frac{I_7}{\tau_2^2}, \quad \frac{I_8}{\tau_2} = \frac{I_9}{\tau_1}. \quad (6.68)$$

To calculate the integrals, it seems more convenient to first integrate over ω using the Residue Theorem and then integrate over $\mathbf{k}$. All integrands have the same denominator so that, when integrating over ω , all have the same four poles,

$$\omega = \pm \frac{i}{2} (D\mathbf{k}^2 + r_1 + r_2) \pm \frac{i}{2} \sqrt{(D\mathbf{k}^2 + r_1 + r_2)^2 - 4((D\mathbf{k}^2 + r_1)r_2 - \tau_1 \tau_2)}. \quad (6.69)$$

To simplify the notation, let A and B such that the poles are $\omega = \frac{i}{2}(\pm A \pm B)$. Closing the contour of the contour integral in the complex upper half-plane encloses the poles $\omega = \frac{i}{2}(A \pm B)$.

Starting from the simplest integral over ω and progressing towards the most complex, the integrals read

$$\begin{aligned} I_5 &= \tau_1^2 \int \mathrm{d}^d \mathbf{k} \mathrm{d}\omega \frac{1}{(\omega - \frac{i}{2}(A+B)) (\omega - \frac{i}{2}(A-B)) (\omega - \frac{i}{2}(-A+B)) (\omega - \frac{i}{2}(-A-B))} \\ &= \tau_1^2 \int \mathrm{d}^d \mathbf{k} \frac{i^2}{AB} \left(\frac{1}{A+B} + \frac{1}{B-A} \right) \\ &= \tau_1^2 2 \int \mathrm{d}^d \mathbf{k} \frac{1}{A(A^2 - B^2)}, \end{aligned} \quad (6.70a)$$

$$\begin{aligned} I_2 &= \tau_1 \int \mathrm{d}^d \mathbf{k} \frac{i^2}{AB} \left(\frac{\frac{1}{2}(A+B) + r_2}{A+B} + \frac{\frac{1}{2}(A-B) + r_2}{B-A} \right) \\ &= \tau_1 r_2 2 \int \mathrm{d}^d \mathbf{k} \frac{1}{A(A^2 - B^2)}, \end{aligned} \quad (6.70b)$$

$$\begin{aligned} I_8 &= \tau_2 2 \int \mathrm{d}^d \mathbf{k} \frac{D\mathbf{k}^2 + r_1}{A(A^2 - B^2)} \\ &= \tau_2 2 \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{A^2 - B^2} - r_2 \frac{1}{A(A^2 - B^2)} \right), \end{aligned} \quad (6.70c)$$

$$\begin{aligned} I_4 &= \int \mathrm{d}^d \mathbf{k} \frac{i^2}{AB} \left(\frac{\frac{i^2}{4}(A+B)^2 - \frac{1}{2}(A+B)(D\mathbf{k}^2 + r_1 - r_2) + r_2(D\mathbf{k}^2 + r_1)}{A+B} \right. \\ &\quad \left. + \frac{\frac{i^2}{4}(A-B)^2 - \frac{1}{2}(A-B)(D\mathbf{k}^2 + r_1 - r_2) + r_2(D\mathbf{k}^2 + r_1)}{B-A} \right) \\ &= \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{2A} + 2r_2 \frac{D\mathbf{k}^2 + r_1}{A(A^2 - B^2)} \right) \\ &= \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{2A} + 2r_2 \frac{1}{A^2 - B^2} - 2r_2^2 \frac{1}{A(A^2 - B^2)} \right), \end{aligned} \quad (6.70d)$$

$$I_1 = \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{2A} + 2r_2^2 \frac{1}{A(A^2 - B^2)} \right), \quad (6.70e)$$

$$\begin{aligned}
I_{10} &= \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{2A} + 2 \frac{(D\mathbf{k}^2 + r_1)^2}{A(A^2 - B^2)} \right) \\
&= \int \mathrm{d}^d \mathbf{k} \frac{(D\mathbf{k}^2 + r_1)^2 + (D\mathbf{k}^2 + r_1)r_2 - \tau_1\tau_2}{2A(A^2 - B^2)} \\
&= \int \mathrm{d}^d \mathbf{k} \frac{(D\mathbf{k}^2 + r_1 + r_2)^2 - (D\mathbf{k}^2 + r_1 + r_2)r_2 - \tau_1\tau_2}{2A(A^2 - B^2)} \\
&= \frac{1}{2} \int \mathrm{d}^d \mathbf{k} \left(\frac{D\mathbf{k}^2 + r_1 + r_2}{A^2 - B^2} - \frac{r_2}{A^2 - B^2} - \frac{\tau_1\tau_2}{A(A^2 - B^2)} \right) \\
&= \frac{1}{2} \int \mathrm{d}^d \mathbf{k} \left(\frac{D\mathbf{k}^2 + r_1}{A^2 - B^2} - \frac{\tau_1\tau_2}{A(A^2 - B^2)} \right) \\
&= \frac{1}{2} \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{r_2} \frac{(D\mathbf{k}^2 + r_1)r_2 - \tau_1\tau_2 + \tau_1\tau_2}{A^2 - B^2} - \frac{\tau_1\tau_2}{A(A^2 - B^2)} \right) \\
&= \frac{1}{2} \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{r_2} + \frac{\tau_1\tau_2}{r_2} \frac{1}{A^2 - B^2} - \frac{\tau_1\tau_2}{A(A^2 - B^2)} \right). \tag{6.70f}
\end{aligned}$$

After integrating over ω , the integrals have reduced to linear combinations of a handful of integrals in $\mathbf{k}$, namely

$$\int \mathrm{d}^d \mathbf{k} \frac{1}{A}, \quad \int \mathrm{d}^d \mathbf{k} \frac{1}{A^2 - B^2}, \quad \int \mathrm{d}^d \mathbf{k} \frac{1}{A(A^2 - B^2)}, \quad \text{and} \quad \int \mathrm{d}^d \mathbf{k} 1. \tag{6.71}$$

To calculate the first three integrals in (6.71), the following identities are useful [127],

$$\int \mathrm{d}^d \mathbf{k} = \mathcal{S}_d \int_0^\infty dk k^{d-1}, \quad \mathcal{S}_d = \frac{\mathcal{S}_d}{(2\pi)^d}, \quad \mathcal{S}_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{6.72a}$$

$$\int_0^\infty dx \frac{x^\alpha}{(x+r)^\beta} = r^{1+\alpha-\beta} \frac{\Gamma(\beta - \alpha - 1)\Gamma(\alpha + 1)}{\Gamma(\beta)}, \tag{6.72b}$$

$$\Gamma(1) = 1, \quad \Gamma(n+1) = n\Gamma(n), \tag{6.72c}$$

$$\frac{1}{ab} = \int_0^1 \frac{dx}{(ax + b(1-x))^2} \quad (\text{Feynman parametrisation}). \tag{6.72d}$$

Firstly,

$$\begin{aligned}
\int \mathrm{d}^d \mathbf{k} \frac{1}{A} &= \int \mathrm{d}^d \mathbf{k} \frac{1}{D\mathbf{k}^2 + r_1 + r_2} = \frac{1}{D} \int \mathrm{d}^d \mathbf{k} \frac{1}{\mathbf{k}^2 + \frac{r_1+r_2}{D}} = \frac{\mathcal{S}_d}{D} \int_0^\infty dk \frac{k^{d-1}}{\mathbf{k}^2 + \frac{r_1+r_2}{D}} \\
&= \frac{\mathcal{S}_d}{2D} \int_0^\infty dx \frac{x^{d/2-1}}{x + \frac{r_1+r_2}{D}} = \frac{\mathcal{S}_d}{2D} \left(\frac{r_1 + r_2}{D} \right)^{d/2-1} \frac{\Gamma(1-d/2)\Gamma(d/2)}{\Gamma(1)} \\
&= \frac{\Gamma(1-d/2)}{(4\pi D)^{d/2}} (r_1 + r_2)^{d/2-1}.
\end{aligned} \tag{6.73}$$

And secondly,

$$\begin{aligned}
\int \mathrm{d}^d \mathbf{k} \frac{1}{A^2 - B^2} &= \int \mathrm{d}^d \mathbf{k} \frac{1}{4((D\mathbf{k}^2 + r_1)r_2 - \tau_1\tau_2)} = \frac{1}{4r_2} \int \mathrm{d}^d \mathbf{k} \frac{1}{D\mathbf{k}^2 + r_1 - \frac{\tau_1\tau_2}{r_2}} \\
&= \frac{\Gamma(1-d/2)}{4r_2(4\pi D)^{d/2}} \left(r_1 - \frac{\tau_1\tau_2}{r_2} \right)^{d/2-1}.
\end{aligned} \tag{6.74}$$

To calculate the third integral in (6.71),

$$\int \mathrm{d}^d \mathbf{k} \frac{1}{A(A^2 - B^2)} = \frac{1}{4r_2} \int \mathrm{d}^d \mathbf{k} \frac{1}{(D\mathbf{k}^2 + r_1 + r_2) \left(D\mathbf{k}^2 + r_1 - \frac{\tau_1\tau_2}{r_2} \right)}, \tag{6.75}$$

consider

$$\begin{aligned}
\int \mathrm{d}^d \mathbf{k} \frac{1}{(D\mathbf{k}^2 + \varepsilon)(D\mathbf{k}^2 + \sigma)} &= \int \mathrm{d}^d \mathbf{k} \int_0^1 dx \frac{1}{((D\mathbf{k}^2 + \varepsilon)x + (D\mathbf{k}^2 + \sigma)(1-x))^2} \\
&= \int \mathrm{d}^d \mathbf{k} \int_0^1 dx \frac{1}{((\varepsilon - \sigma)x + D\mathbf{k}^2 + \sigma)^2} = \frac{1}{(\varepsilon - \sigma)^2} \int \mathrm{d}^d \mathbf{k} \int_0^1 dx \frac{1}{\left(x + \frac{D\mathbf{k}^2 + \sigma}{\varepsilon - \sigma}\right)^2} \\
&= \frac{1}{\varepsilon - \sigma} \int \mathrm{d}^d \mathbf{k} \left(\frac{1}{D\mathbf{k}^2 + \sigma} - \frac{1}{D\mathbf{k}^2 + \varepsilon} \right) = \frac{\Gamma(1-d/2)}{(4\pi D)^{d/2}} \left(\frac{\sigma^{d/2-1} - \varepsilon^{d/2-1}}{\varepsilon - \sigma} \right).
\end{aligned} \tag{6.76}$$

Then, Eq. (6.75) is,

$$\int \mathrm{d}^d \mathbf{k} \frac{1}{A(A^2 - B^2)} = \frac{\Gamma(1-d/2)}{4(4\pi D)^{d/2} (r_2^2 + \tau_1\tau_2)} \left(\left(r_1 - \frac{\tau_1\tau_2}{r_2} \right)^{d/2-1} - (r_1 + r_2)^{d/2-1} \right). \tag{6.77}$$

Finally, the integral $\int \mathrm{d}^d \mathbf{k} 1$ is ultraviolet divergent, however, this divergence can be

regularised by setting an ultraviolet cutoff $\Lambda < \infty$, so that

$$\int \mathrm{d}^d \mathbf{k} \, 1 = \Lambda^d, \quad (6.78)$$

which is justified by the finiteness of the lattice spacing. Combining Eqs. (6.68), (6.70), (6.73), (6.74), (6.77) and (6.78), the loop integrals $I_1, \dots, I_{10}$ are

$$I_1 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(r_2^2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} + \tau_1 \tau_2 (r_1 + r_2)^{d/2-1} \right), \quad (6.79a)$$

$$I_2 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(r_2 \tau_1 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} - r_2 \tau_1 (r_1 + r_2)^{d/2-1} \right), \quad (6.79b)$$

$$I_3 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(r_2 \tau_2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} - r_2 \tau_2 (r_1 + r_2)^{d/2-1} \right), \quad (6.79c)$$

$$I_4 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\tau_1 \tau_2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} + (2r_2^2 + \tau_1 \tau_2)(r_1 + r_2)^{d/2-1} \right), \quad (6.79d)$$

$$I_5 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\tau_1^2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} - \tau_1^2 (r_1 + r_2)^{d/2-1} \right), \quad (6.79e)$$

$$I_6 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\tau_1 \tau_2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} - \tau_1 \tau_2 (r_1 + r_2)^{d/2-1} \right), \quad (6.79f)$$

$$I_7 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\tau_2^2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} - \tau_2^2 (r_1 + r_2)^{d/2-1} \right), \quad (6.79g)$$

$$I_8 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\frac{\tau_1 \tau_2^2}{r_2} \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} + r_2 \tau_2 (r_1 + r_2)^{d/2-1} \right), \quad (6.79h)$$

$$I_9 = \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\frac{\tau_1^2 \tau_2}{r_2} \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} + r_2 \tau_1 (r_1 + r_2)^{d/2-1} \right), \quad (6.79i)$$

$$I_{10} = \frac{\Lambda^2}{2r_2} + \frac{\Gamma(1 - d/2)}{2(4\pi D)^{d/2}(r_2^2 + \tau_1 \tau_2)} \left(\left(\frac{\tau_1 \tau_2}{r_2} \right)^2 \left(r_1 - \frac{\tau_1 \tau_2}{r_2} \right)^{d/2-1} + \tau_1 \tau_2 (r_1 + r_2)^{d/2-1} \right). \quad (6.79j)$$

7

Run-and-tumble motion: microscopic dynamics and entropy production

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OVERVIEW Run-and-tumble motion is an example of active motility, where particles move at constant speed and change direction at random times [187]. In one dimension, this process can be represented by the interaction between right- and left-moving particles, which transmutate into one another following a Poisson process. In this work we characterise run-and-tumble motion in one dimension both in free space and in a harmonic potential. In the second case, we also add diffusion. By means of a Doi-Peliti field theory, we calculate a list of observables in closed form, of which the mean square displacement, the stationary distribution, the two-point correlation function and the entropy production are particularly relevant. All our results are exact.

7.1 INTRODUCTION

Active matter has a research focus in statistical mechanics over recent years. Active matter refers to reaction-diffusion systems out of equilibrium where particles are subject

to local non-thermal forces [67]. Examples are active nematics [131], active emulsions [122], and active motility [44, 18], amongst many others. Among an extraordinary large number of examples of active matter, self-propelled particles have received a lot of attention.

One type of self-propelled particles are motile bacteria such as *Escherichia coli*, which move in a sequence of *runs* at constant speed w interrupted by sudden changes in the orientation of motion that happen at rate τ , the *tumbles* [181, 166, 176, 182, 129, 178]. This motion pattern is known as run-and-tumble motion, and one of its main characteristic features is that it resembles ballistic motion at the microscopic scale, but diffusive motion at the large scale.

In the present work we study a run-and-tumble particle in one dimension via a path integral approach [166] in two different settings, where (i) the particle is unconstrained on the real line and (ii) the particle is confined by an external harmonic potential and is subject to diffusion. The first setting is presented for illustration purposes and as a basis for the second setting, which requires a technically more involved derivation due to the external potential. In each case, we calculate a number of observables in closed form, such as the mean square displacement, the expected velocity, the particle distribution at stationarity, the two-point correlation function, and the entropy production. The results we provide are exact. In particular, we calculate the entropy production drawing on the microscopic dynamics of the stochastic process without resorting to coarse-graining.

The contents are organised as follows: in Sec. 7.2, we derive the field theory of a free run-and-tumble particle in one dimension; in Sec. 7.3, we introduce the harmonic potential and add the diffusion, we also derive the field theory and calculate the observables in closed form. Finally, in Sec. 7.4, we discuss the main results of this work.

7.2 FIELD THEORY OF RUN-AND-TUMBLE MOTION

A particle undergoing run-and-tumble motion moves at constant speed w (the *run*) during a random lapse of time, of Poissonian distributed length. At the end of the run, the particle changes the direction of motion to a new random direction, uniformly distributed amongst all the available directions of motion (the *tumble*). The particle then initiates the following *run*, which is followed by a *tumble*, and so on. The tumbling

events constitute a Poisson process in time with tumbling rate τ . In one dimension, the particle can only move in two directions: rightwards and leftwards.

To study this process analytically, it is convenient to consider two species of particles, right-moving particles and left-moving particles, which transmute into one another whenever a tumbling event happens. The Langevin equations of these two types of motion are $\dot{x}_R = w$ and $\dot{x}_L = -w$, where $\dot{x}_R$ and $\dot{x}_L$ are the velocities of a right-moving and a left-moving particle respectively. The transmutation of either species into the other at rate τ turns into the coupling of their respective Fokker-Planck equations,

$$\partial_t P_R = -w \partial_x P_R - \tau P_R + \tau P_L, \quad (7.1a)$$

$$\partial_t P_L = w \partial_x P_L - \tau P_L + \tau P_R, \quad (7.1b)$$

where $P_R(x, t)$ and $P_L(x, t)$ are the probability function of finding a right-moving and a left-moving particle respectively at position x at time t . It follows that the action functional [189] of this process is

$$\mathcal{A}([\phi, \tilde{\phi}, \psi, \tilde{\psi}]) = \int dx dt \left\{ \tilde{\phi} (\partial_t \phi + w \partial_x \phi) + \tilde{\psi} (\partial_t \psi - w \partial_x \psi) + \tau (\tilde{\phi} - \tilde{\psi}) (\phi - \psi) \right\}, \quad (7.2)$$

where $\tilde{\phi}$ is the Doi-shifted creation field $\phi^\dagger$ of right-moving particles, $\phi^\dagger = \tilde{\phi} + 1$; ϕ is the annihilation field of right-moving particles; $\tilde{\psi}$ is the Doi-shifted creation field $\psi^\dagger$ of left-moving particles, $\psi^\dagger = \tilde{\psi} + 1$; and ψ is the annihilation field of left-moving particles. The action functional allows to calculate an observable $\bullet$ via the path integral

$$\langle \bullet \rangle = \int \mathcal{D}[\phi, \tilde{\phi}, \psi, \tilde{\psi}] \bullet e^{-\mathcal{A}([\phi, \tilde{\phi}, \psi, \tilde{\psi}])}. \quad (7.3)$$

The action in Eq. (7.2) contains only bilinear terms, where the only interaction between particles is due to the tumbling events. This action does not have any non-linear couplings.

The Fourier transform is defined, by convention, as

$$\phi(\mathbf{k}, \omega) = \int d^d \mathbf{x} dt e^{i\omega t - i\mathbf{k}\mathbf{x}} \phi(\mathbf{x}, t), \quad (7.4a)$$

$$\phi(\mathbf{x}, t) = \int d^d \mathbf{k} d\omega e^{-i\omega t + i\mathbf{k}\mathbf{x}} \phi(\mathbf{k}, \omega), \quad (7.4b)$$

where the spatial momentum $\mathbf{k}$ is conjugate of the position $\mathbf{x}$, the frequency ω is conjugate of time t , $\mathrm{d}^d\mathbf{k} = \mathrm{d}^d\mathbf{k}/(2\pi)^d$ and $\mathrm{d}\omega = \mathrm{d}\omega/2\pi$. The same convention applies to $\tilde{\phi}(\mathbf{x}, t)$, $\psi(\mathbf{x}, t)$ and $\tilde{\psi}(\mathbf{x}, t)$. In Fourier space, the action functional is

$$\mathcal{A}([\phi, \tilde{\phi}, \psi, \tilde{\psi}]) = \int \mathrm{d}\mathbf{k} \mathrm{d}\omega \left\{ (-i\omega + i\mathbf{k}w + \tau) \tilde{\phi}\phi + (-i\omega - i\mathbf{k}w + \tau) \tilde{\psi}\psi - \tau (\tilde{\phi}\psi + \tilde{\psi}\phi) \right\}, \quad (7.5)$$

which is local in both $\mathbf{k}$ and ω . Then, the propagator matrix reads from the inverse of the action (7.5) in matrix form [189],

$$\begin{pmatrix} \langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle & \langle \phi(\mathbf{k}, \omega) \tilde{\psi}(\mathbf{k}', \omega') \rangle \\ \langle \psi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle & \langle \psi(\mathbf{k}, \omega) \tilde{\psi}(\mathbf{k}', \omega') \rangle \end{pmatrix} = \frac{\delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}')}{-\omega^2 - 2i\tau\omega + (\mathbf{k}w)^2} \begin{pmatrix} -i\omega - i\mathbf{k}w + \tau & \tau \\ \tau & -i\omega + i\mathbf{k}w + \tau \end{pmatrix}. \quad (7.6)$$

7.2.1 MEAN SQUARE DISPLACEMENT

Suppose that the system is initialised with one right-moving particle at position x_0 and time $t_0 = 0$. To calculate the mean square displacement,

$$R^2(t) = \langle (x(t) - x_0)^2 \rangle, \quad (7.7)$$

we need the first and second moments of the position of particles,

$$\langle x(t) \rangle = \int \mathrm{d}x x \langle (\phi(x, t) + \psi(x, t)) \tilde{\phi}(x_0, t_0) \rangle, \quad (7.8a)$$

$$\langle x^2(t) \rangle = \int \mathrm{d}x x^2 \langle (\phi(x, t) + \psi(x, t)) \tilde{\phi}(x_0, t_0) \rangle, \quad (7.8b)$$

which derive from the first and second moments of right- and left-moving particles separately,

$$\langle x_\phi(t) \rangle = \int \mathrm{d}x x \langle \phi(x, t) \tilde{\phi}(x_0, t_0) \rangle, \quad \langle x_\psi^2(t) \rangle = \int \mathrm{d}x x^2 \langle \psi(x, t) \tilde{\phi}(x_0, t_0) \rangle, \quad (7.9a)$$

$$\langle x_\psi(t) \rangle = \int \mathrm{d}x x \langle \psi(x, t) \tilde{\phi}(x_0, t_0) \rangle, \quad \langle x_\phi^2(t) \rangle = \int \mathrm{d}x x^2 \langle \phi(x, t) \tilde{\phi}(x_0, t_0) \rangle. \quad (7.9b)$$

The expressions in Eq. (7.9) derive from the *joint* probability that the run-and-tumble particle is of either type *and* it is found at position x at time t , the expressions do not derive from the *conditional* probability. The difference between the two is a normalisation factor (either $\int dx \langle \phi(x, t) \tilde{\phi}(x_0, t_0) \rangle$ or $\int dx \langle \psi(x, t) \tilde{\phi}(x_0, t_0) \rangle$).

Following the method in App. 7.A, the expected position of right-moving particles is

$$\begin{aligned} \langle x_\phi(t) \rangle &= i \frac{d}{d\mathbf{k}} \int d\omega d\omega' d\mathbf{k}' e^{-i(\omega t + \omega' t_0)} e^{i\mathbf{k}' x_0} \left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \right\rangle \Big|_{\mathbf{k}=0} \\ &= \frac{x_0}{2} (1 + e^{-2\tau t}) + \frac{w}{2\tau} (1 - e^{-2\tau t}) , \end{aligned} \quad (7.10)$$

as expected and, similarly,

$$\begin{aligned} \langle x_\psi(t) \rangle &= i \frac{d}{d\mathbf{k}} \int d\omega d\omega' d\mathbf{k}' e^{-i(\omega t + \omega' t_0)} e^{i\mathbf{k}' x_0} \left\langle \psi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \right\rangle \Big|_{\mathbf{k}=0} \\ &= \frac{x_0}{2} (1 - e^{-2\tau t}) , \end{aligned} \quad (7.11)$$

so the overall expected position is

$$\langle x(t) \rangle = \langle x_\phi(t) \rangle + \langle x_\psi(t) \rangle = x_0 + \frac{w}{2\tau} (1 - e^{-2\tau t}) . \quad (7.12)$$

The second moment of the position of right-moving particles is

$$\begin{aligned} \langle x_\phi^2(t) \rangle &= - \frac{d^2}{d\mathbf{k}^2} \int d\omega d\omega' d\mathbf{k}' e^{-i(\omega t + \omega' t_0)} e^{i\mathbf{k}' x_0} \left\langle \phi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \right\rangle \Big|_{\mathbf{k}=0} \\ &= \frac{x_0^2}{2} (1 + e^{-2\tau t}) + \left(\frac{w^2 t}{2\tau} - \frac{x_0 w}{\tau} \right) (1 - e^{-2\tau t}) , \end{aligned} \quad (7.13)$$

and similarly,

$$\begin{aligned} \langle x_\psi^2(t) \rangle &= - \frac{d^2}{d\mathbf{k}^2} \int d\omega d\omega' d\mathbf{k}' e^{-i(\omega t + \omega' t_0)} e^{i\mathbf{k}' x_0} \left\langle \psi(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \right\rangle \Big|_{\mathbf{k}=0} \\ &= \frac{w^2 t}{2\tau} (1 + e^{-2\tau t}) + \frac{1}{2} \left(x_0^2 - \frac{w^2}{\tau^2} \right) (1 - e^{-2\tau t}) , \end{aligned} \quad (7.14)$$

then, the second moment of the overall position is,

$$\langle x^2(t) \rangle = \langle x_\phi^2(t) \rangle + \langle x_\psi^2(t) \rangle = x_0^2 + \frac{w^2 t}{\tau} - \left(\frac{x_0 w}{\tau} + \frac{w^2}{2\tau^2} \right) (1 - e^{-2\tau t}) . \quad (7.15)$$

The mean square displacement is therefore

$$R^2(t) = \frac{w^2 t}{\tau} - \frac{w^2}{2\tau^2} (1 - e^{-2\tau t}) , \quad (7.16)$$

which is consistent with [128] and only differs by a factor 2 from the mean square displacement in two dimensions [197].

EFFECTIVE DIFFUSION AND BALLISTIC MOTION

On the short time scale, typically before the particle tumbles for the first time $\tau t \ll 1$, expanding the mean square displacement gives

$$R^2(t) \simeq (wt)^2 , \quad (7.17)$$

which corresponds to ballistic motion. On the other hand, on the long time scale $\tau t \gg 1$, $R^2(t)$ scales linearly in time, implying that the particle moves effectively diffusively. Defining the effective diffusion constant D_{eff} such that $R^2(t) \simeq 2tD_{\text{eff}}$, the effective diffusion constant of a run-and-tumble particle is

$$D_{\text{eff}} = \frac{w^2}{2\tau} , \quad (7.18)$$

consistent with [128, 31, 197, 9, 187].

7.3 FIELD THEORY OF RUN-AND-TUMBLE MOTION WITH DIFFUSION IN A HARMONIC POTENTIAL

In this section we consider a run-and-tumble particle confined in a harmonic potential $V(x) = \frac{1}{2}kx^2$ and subject to thermal noise. The Langevin equations are then $\dot{x}_R = w - \partial_x V(x) + \eta(t)$ and $\dot{x}_L = -w - \partial_x V(x) + \eta(t)$, where η is a Gaussian white noise with mean $\langle \eta(t) \rangle = 0$ and correlation function $\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$, and where D is the diffusion constant. Therefore, the Fokker-Planck equations read

$$\partial_t P_R = -\partial_x [(w - kx)P_R] + D\partial_x^2 P_R - \tau P_R + \tau P_L , \quad (7.19a)$$

$$\partial_t P_L = -\partial_x [(-w - kx)P_L] + D\partial_x^2 P_L - \tau P_L + \tau P_R , \quad (7.19b)$$

(cf. Eq. (7.1)) and so, the action functional is

$$\begin{aligned}\mathcal{A}([\phi, \tilde{\phi}, \psi, \tilde{\psi}]) = \int dx dt \{ & \tilde{\phi} (\partial_t \phi + \partial_x [(w - kx)\phi] - D\partial_x^2 \phi) \\ & + \tilde{\psi} (\partial_t \psi + \partial_x [(-w - kx)\psi] - D\partial_x^2 \psi) \\ & + \tau(\tilde{\phi} - \tilde{\psi})(\phi - \psi) \},\end{aligned}\quad (7.20)$$

where ϕ , $\tilde{\phi}$, ψ and $\tilde{\psi}$ are defined as in Sec. 7.2. In Fourier space, the action functional (cf. Eq. (7.5)) reads

$$\begin{aligned}\mathcal{A}([\phi, \tilde{\phi}, \psi, \tilde{\psi}]) = \int d\mathbf{k} d\omega \{ & \tilde{\phi} (-i\omega + i\mathbf{k}w + k\mathbf{k}\partial_{\mathbf{k}} + D\mathbf{k}^2 + \tau) \phi \\ & + \tilde{\psi} (-i\omega - i\mathbf{k}w + k\mathbf{k}\partial_{\mathbf{k}} + D\mathbf{k}^2 + \tau) \psi \\ & - \tau(\tilde{\phi}\psi + \tilde{\psi}\phi) \},\end{aligned}\quad (7.21)$$

which has two terms that are not local in the spatial momentum component, namely $\partial_{\mathbf{k}}\phi$ and $\partial_{\mathbf{k}}\psi$. All terms are local in the frequency ω . As a consequence, this action written in Fourier space cannot be used to calculate the path integral that gives the propagators. To do this, we need to find a representation of the action where all terms included in the Gaussian integral are in local form. Any other term that is not in local form, may be dealt with perturbatively. Finding such a representation of the fields is equivalent to solving an eigenvalue problem $\mathcal{L}[u] = \lambda u$, where the operator $\mathcal{L}[u]$ is derived from the diagonal terms of the action. However, as the action stands, Eqs. (7.20) and (7.21), this would mean to solve different eigenvalue problems for ϕ and ψ and therefore having to handle different representations for each of the four fields. Instead, we can introduce a rotation of the fields, where we reparametrise the fields and consider the density field $\alpha = \frac{1}{\sqrt{2}}(\phi + \psi)$ and the polarity field $\beta = \frac{1}{\sqrt{2}}(\phi - \psi)$. The action is then,

$$\begin{aligned}\mathcal{A}([\alpha, \tilde{\alpha}, \beta, \tilde{\beta}]) = \int dx d\omega \{ & -i\omega \tilde{\alpha}\alpha - k\tilde{\alpha}\partial_x(x\alpha) - D\tilde{\alpha}\partial_x^2\alpha \\ & + (2\tau - i\omega)\tilde{\beta}\beta - k\tilde{\beta}\partial_x(x\beta) - D\tilde{\beta}\partial_x^2\beta \\ & + w\tilde{\alpha}\partial_x\beta + w\tilde{\beta}\partial_x\alpha \},\end{aligned}\quad (7.22)$$

which has the same operator structure for α as it does for β . The set of orthogonal functions that make the diagonal terms local in space x solve the eigenvalue problem $\mathcal{L}[u] = u'' + (xu')/L^2 = \lambda u$, with $L = \sqrt{D/k}$. The solution to this eigenvalue problem is the set of Hermite functions

$$u_n(x) = e^{-\frac{x^2}{2L^2}} H_n\left(\frac{x}{L}\right), \quad (7.23)$$

with $\lambda_n = -n/L^2$, where $H_n(x)$ is the n -th Hermite polynomial (Sec. 7.C, which also contains a list of useful properties of Hermite polynomials). Defining the set of functions

$$v_n(x) = \frac{1}{\sqrt{2\pi} n!} H_n\left(\frac{x}{L}\right), \quad (7.24)$$

it follows from the orthogonality of Hermite polynomials in Eq. (7.93), the orthogonality relation between $u_n(x)$ and $v_m(x)$,

$$\int_{-\infty}^{\infty} u_n(x) v_m(x) dx = L \delta_{n,m}, \quad (7.25)$$

where $\delta_{n,m}$ is the Kronecker δ . We can now use the eigenfunctions $u_n(x)$ and $v_m(x)$ as a basis for the fields α , $\tilde{\alpha}$, β and $\tilde{\beta}$,

$$\alpha(x, t) = \int d\omega e^{-i\omega t} \frac{1}{L} \sum_n \alpha_n(\omega) u_n(x), \quad \tilde{\alpha}(x, t) = \int d\omega e^{-i\omega t} \frac{1}{L} \sum_n \tilde{\alpha}_n(\omega) v_n(x), \quad (7.26a)$$

$$\beta(x, t) = \int d\omega e^{-i\omega t} \frac{1}{L} \sum_n \beta_n(\omega) u_n(x), \quad \tilde{\beta}(x, t) = \int d\omega e^{-i\omega t} \frac{1}{L} \sum_n \tilde{\beta}_n(\omega) v_n(x). \quad (7.26b)$$

Using the orthogonality relation in Eq. (7.25), the action is then,

$$\begin{aligned} \mathcal{A}([a_m, \tilde{\alpha}_n, b_m, \tilde{\beta}_n]) &= \int \mathfrak{d}\omega \mathfrak{d}\omega' \delta(\omega + \omega') \\ &\times \frac{1}{L^2} \sum_{n,m} \left\{ L\delta_{n,m} \left[(-i\omega + kn) \tilde{\alpha}_n(\omega') \alpha_m(\omega) + (-i\omega + kn + 2\tau) \tilde{\beta}_n(\omega') \beta_m(\omega) \right] \right. \\ &\quad \left. - \frac{w}{L} L\delta_{n-1,m} \left(\tilde{\alpha}_n(\omega') \beta_m(\omega) + \tilde{\beta}_n(\omega') \alpha_m(\omega) \right) \right\}, \end{aligned} \quad (7.27)$$

whose diagonal terms are in local form both in the frequency ω as in the index n , whereas the off-diagonal terms are not. We can deal with the off-diagonal terms perturbatively in w by splitting the action in $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, where $\mathcal{A}_0$ is the harmonic part and $\mathcal{A}_1$ is the perturbation,

$$\begin{aligned} \mathcal{A}_0 &= \int \mathfrak{d}\omega \mathfrak{d}\omega' \delta(\omega + \omega') \\ &\quad \frac{1}{L^2} \sum_{n,m} L\delta_{n,m} \left[(-i\omega + kn) \tilde{\alpha}_n(\omega') \alpha_m(\omega) + (-i\omega + kn + 2\tau) \tilde{\beta}_n(\omega') \beta_m(\omega) \right], \end{aligned} \quad (7.28a)$$

$$\mathcal{A}_1 = \int \mathfrak{d}\omega \mathfrak{d}\omega' \delta(\omega + \omega') \frac{1}{L^2} \sum_{n,m} \left(-\frac{w}{L} \right) L\delta_{n-1,m} \left(\tilde{\alpha}_n(\omega') \beta_m(\omega) + \tilde{\beta}_n(\omega') \alpha_m(\omega) \right). \quad (7.28b)$$

The bare propagators then read

$$\underline{\alpha_n(\omega)} \quad \tilde{\alpha}_m(\omega') \hat{=} \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') \rangle_0 = \frac{L\delta_{n,m} \delta(\omega + \omega')}{-i\omega + kn + r}, \quad (7.29a)$$

$$\beta_n(\omega) \quad \tilde{\beta}_m(\omega') \hat{=} \langle \beta_n(\omega) \tilde{\beta}_m(\omega') \rangle_0 = \frac{L\delta_{n,m} \delta(\omega + \omega')}{-i\omega + kn + r + 2\tau}, \quad (7.29b)$$

where r is a mass term added to regularise the infrared divergence and that is to be taken to 0 after calculating any observable, whereas

$$\langle \alpha_n(\omega) \tilde{\beta}_m(\omega') \rangle_0 = 0, \quad (7.30a)$$

$$\langle \beta_n(\omega) \tilde{\alpha}_m(\omega') \rangle_0 = 0, \quad (7.30b)$$

whenever $n = m$.

7.3.1 FULL PROPAGATOR

Before diving into calculating any observables, it is worth analysing the four full propagators. Consider, for instance,

$$\langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') \rangle = \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') e^{-\mathcal{A}_1} \rangle_0 = \sum_N \frac{1}{N!} \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') (-\mathcal{A}_1)^N \rangle_0. \quad (7.31)$$

The zeroth order $N = 0$ of this expansion is clearly Eq. (7.29a). The first order term, $N = 1$, is

$$\begin{aligned} \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') (-\mathcal{A}_1) \rangle_0 &= \frac{w}{L^2} \sum_{i,j} \delta_{i-1,j} \int \mathfrak{d}\omega_1 \mathfrak{d}\omega'_1 \delta(\omega_1 + \omega'_1) \\ &\quad \left\langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') \left(\tilde{\alpha}_i(\omega'_1) \beta_j(\omega_1) + \tilde{\beta}_i(\omega'_1) \alpha_j(\omega_1) \right) \right\rangle_0 \\ &= 0, \end{aligned} \quad (7.32)$$

which does not contribute to the full propagator because there is no way to arrange the fields such that α is paired with $\tilde{\alpha}$ and β is paired with $\tilde{\beta}$, which is imposed by Eqs. (7.30a) and (7.30b). The second order term is,

$$\begin{aligned} \frac{1}{2} \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') (-\mathcal{A}_1)^2 \rangle_0 &= \frac{1}{2} \frac{w^2}{L^4} \sum_{i,j,k,\ell} \delta_{i-1,j} \delta_{k-1,\ell} \int \mathfrak{d}\omega_1 \mathfrak{d}\omega'_1 \mathfrak{d}\omega_2 \mathfrak{d}\omega'_2 \delta(\omega_1 + \omega'_1) \delta(\omega_2 + \omega'_2) \\ &\quad \times \left\langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') \left(\tilde{\alpha}_i(\omega'_1) \beta_j(\omega_1) + \tilde{\beta}_i(\omega'_1) \alpha_j(\omega_1) \right) \left(\tilde{\alpha}_k(\omega'_2) \beta_\ell(\omega_2) + \tilde{\beta}_k(\omega'_2) \alpha_\ell(\omega_2) \right) \right\rangle_0 \\ &= \frac{w^2}{L^4} \sum_{i,j,k,\ell} \delta_{i-1,j} \delta_{k-1,\ell} \left\langle \alpha_n(\omega) \tilde{\alpha}_i(-\omega_1) \beta_j(\omega_1) \tilde{\beta}_k(-\omega_2) \alpha_\ell(\omega_2) \tilde{\alpha}_m(\omega') \right\rangle_0 \\ &= \frac{w^2}{L^4} \delta_{n,m+2} \langle \alpha_n(\omega) \tilde{\alpha}_{m+2}(-\omega_1) \rangle_0 \langle \beta_{m+1}(\omega_1) \tilde{\beta}_{m+1}(-\omega_2) \rangle_0 \langle \alpha_m(\omega_2) \tilde{\alpha}_m(\omega') \rangle_0 \\ &= \frac{w^2}{L} \frac{\delta_{n,m+2} \delta(\omega + \omega')}{(-i\omega + k(m+2) + r) (-i\omega + k(m+1) + r + 2\tau) (-i\omega + km + r)}. \end{aligned} \quad (7.33)$$

It is then easy to see that any term of odd order N vanishes because the α , $\tilde{\alpha}$, β and $\tilde{\beta}$ cannot be paired appropriately. The terms of even order N are

$$\begin{aligned} & \frac{1}{N!} \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') (-\mathcal{A}_1)^N \rangle_0 \\ &= \left(\frac{w}{L}\right)^N \frac{L \delta_{n,m+N} \delta(\omega + \omega')}{-i\omega + km + r} \prod_{j=1}^{N/2} \frac{1}{(-i\omega + k(m+2j-1) + r + 2\tau) (-i\omega + k(m+2j) + r)} . \end{aligned} \quad (7.34)$$

Following a similar argument, the contributions to the full propagator $\langle \beta_n(\omega) \tilde{\beta}_m(\omega') \rangle = \sum_N \frac{1}{N!} \langle \beta_n(\omega) \tilde{\beta}_m(\omega') (-\mathcal{A}_1)^N \rangle_0$ are 0 if N is odd, and

$$\begin{aligned} & \frac{1}{N!} \langle \beta_n(\omega) \tilde{\beta}_m(\omega') (-\mathcal{A}_1)^N \rangle_0 \\ &= \left(\frac{w}{L}\right)^N \frac{L \delta_{n,m+N} \delta(\omega + \omega')}{-i\omega + km + r + 2\tau} \prod_{j=1}^{N/2} \frac{1}{(-i\omega + k(m+2j-1) + r) (-i\omega + k(m+2j) + r + 2\tau)} , \end{aligned} \quad (7.35)$$

if N is even. The contributions to the full propagator $\langle \alpha_n(\omega) \tilde{\beta}_m(\omega') \rangle$ are 0 if N is even and

$$\begin{aligned} & \frac{1}{N!} \langle \alpha_n(\omega) \tilde{\beta}_m(\omega') (-\mathcal{A}_1)^N \rangle_0 \\ &= \left(\frac{w}{L}\right)^N L \delta_{n,m+N} \delta(\omega + \omega') \prod_{j=0}^{(N+1)/2} \frac{1}{(-i\omega + k(m+2j) + r + 2\tau) (-i\omega + k(m+2j+1) + r)} , \end{aligned} \quad (7.36)$$

if N is odd. Similarly, the contributions to the full propagator $\langle \beta_n(\omega) \tilde{\alpha}_m(\omega') \rangle$ are 0 if N is even and

$$\begin{aligned} & \frac{1}{N!} \langle \beta_n(\omega) \tilde{\alpha}_m(\omega') (-\mathcal{A}_1)^N \rangle_0 \\ &= \left(\frac{w}{L}\right)^N L \delta_{n,m+N} \delta(\omega + \omega') \prod_{j=0}^{(N+1)/2} \frac{1}{(-i\omega + k(m+2j) + r) (-i\omega + k(m+2j+1) + r + 2\tau)} , \end{aligned} \quad (7.37)$$

if N is odd. Diagrammatically, the four full propagators are

$$\begin{array}{c} n, \omega \\ \text{---} \bullet \text{---} \\ m, \omega' \end{array} = \begin{array}{c} n, \omega \\ \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \dots, \quad (7.38a)$$

$$\begin{array}{c} n, \omega \\ \text{---} \bullet \text{---} \\ m, \omega' \end{array} = \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \dots, \quad (7.38b)$$

$$\begin{array}{c} n, \omega \\ \text{---} \bullet \text{---} \\ m, \omega' \end{array} = \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \dots, \quad (7.38c)$$

$$\begin{array}{c} n, \omega \\ \text{---} \bullet \text{---} \\ m, \omega' \end{array} = \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \begin{array}{c} n, \omega \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ m, \omega' \end{array} + \dots. \quad (7.38d)$$

Some of these propagators are calculated in closed form in App. 7.E.

7.3.2 STATIONARY DISTRIBUTION

Assuming that the system is initialised with a right-moving particle at position x_0 at time $t_0 = 0$, the particle distribution $\rho(x, t)$ can be calculated via the observable $\rho(x, t) = \langle (\phi(x, t) + \psi(x, t)) \tilde{\phi}(x_0, t_0) \rangle = \langle \alpha(x, t) \tilde{\alpha}(x_0, t_0) \rangle + \langle \alpha(x, t) \tilde{\beta}(x_0, t_0) \rangle$. From Eq. (7.26a),

$$\langle \alpha(x, t) \tilde{\alpha}(x_0, t_0) \rangle = \int d\omega d\omega' e^{-i\omega t} e^{-i\omega' t_0} \frac{1}{L^2} \sum_{n,m} u_n(x) v_m(x_0) \langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') \rangle, \quad (7.39)$$

and using Eqs. (7.31) and (7.34), we see that the poles of the propagator $\langle \alpha_n(\omega) \tilde{\alpha}_m(\omega') \rangle$ when integrating over ω are of the form

$$\omega_j = i(kj + r) \quad \text{if } j \text{ even}, \quad (7.40a)$$

$$\omega_j = i(kj + r + 2\tau) \quad \text{if } j \text{ odd}, \quad (7.40b)$$

where $m \leq j \leq n$ in Eq. (7.39). Using the Residue Theorem to integrate the right-hand side of Eq. (7.39) over ω , we find that there are different contributions with the prefactor $\exp(i\omega_j t)$. The stationary distribution $\lim_{t \rightarrow \infty} \rho(x, t)$ will be given by those contributions such that $\Im[\omega_j] \rightarrow 0$ as $r \rightarrow 0$. From Eq. (7.40), this is only the case for

$\omega_0 = ir$. This fixes $j = m = 0$ in Eq. (7.39). Then, in the limit $t \rightarrow \infty$,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \langle \alpha(x, t) \tilde{\alpha}(x_0, t_0) \rangle &= \lim_{t \rightarrow \infty} \int d\omega d\omega' e^{-i\omega t} e^{-i\omega' t_0} \frac{1}{L^2} \sum_n u_n(x) v_0(x_0) \langle \alpha_n(\omega) \tilde{\alpha}_0(\omega') \rangle \\
&= \lim_{t \rightarrow \infty} \frac{v_0(x_0)}{L^2} \left(\overset{0, t}{\text{---}} \overset{0, t_0}{\text{---}} u_0(x) + \overset{2, t}{\text{---}} \overset{0, t_0}{\text{---}} u_2(x) + \overset{4, t}{\text{---}} \overset{0, t_0}{\text{---}} u_4(x) + \dots \right) \\
&= \lim_{t \rightarrow \infty} \frac{v_0(x_0)}{L^2} e^{-r(t-t_0)} \left(u_0(x) + \frac{w}{L} \frac{u_2(x)}{2k(k+2\tau)} + \left(\frac{w}{L} \right)^2 \frac{u_4(x)}{4k(3k+2\tau)2k(k+2\tau)} + \dots \right). \tag{7.41}
\end{aligned}$$

On the other hand, we can see from Eq. (7.36) that the contribution from $\langle \alpha(x, t) \tilde{\beta}(x_0, t_0) \rangle$ will vanish as $t \rightarrow \infty$ because the pole with smallest imaginary part is $\omega = i(r + 2\tau)$ for finite transmutation rate $\tau > 0$. In summary, the stationary particle distribution is, from Eq. (7.41),

$$\lim_{t \rightarrow \infty} \rho(x, t) = \sqrt{\frac{k}{2\pi D}} e^{-\frac{kx^2}{2D}} \left(1 + \sum_{n=1}^{\infty} \left(\frac{w^2}{kD} \right)^n H_{2n} \left(\sqrt{\frac{k}{D}} x \right) \prod_{j=1}^n \frac{1}{2j((2j-1) + 2\tau/k)} \right), \tag{7.42}$$

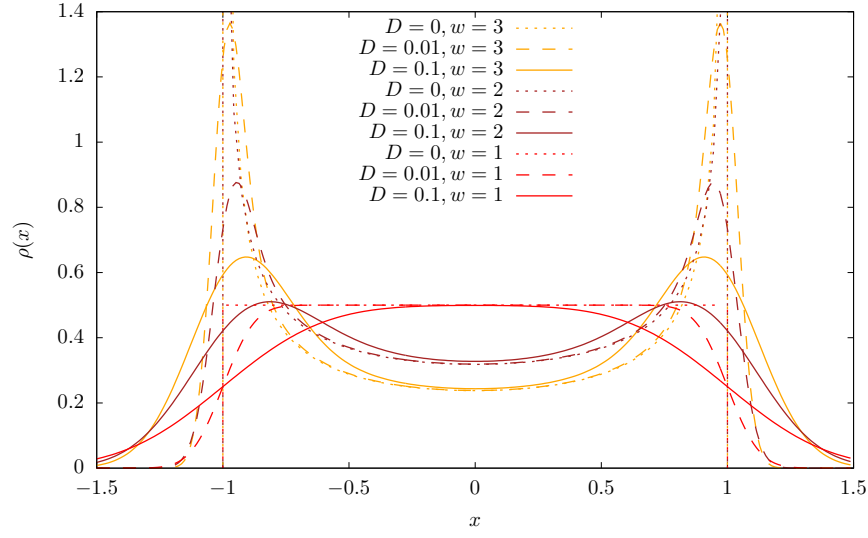
see Fig. 7.1. In the case where the particle is not subject to diffusion, $D = 0$, the stationary distribution can be derived from the coupled Fokker-Planck equations in Eq. (7.19) [47, 51] and is

$$\lim_{t \rightarrow \infty} \rho(x, t) = \frac{k \Gamma(1/2 + \tau/k)}{\sqrt{\pi} w \Gamma(\tau/k)} \left(1 - \left(\frac{kx}{w} \right)^2 \right)^{\tau/k-1}, \tag{7.43}$$

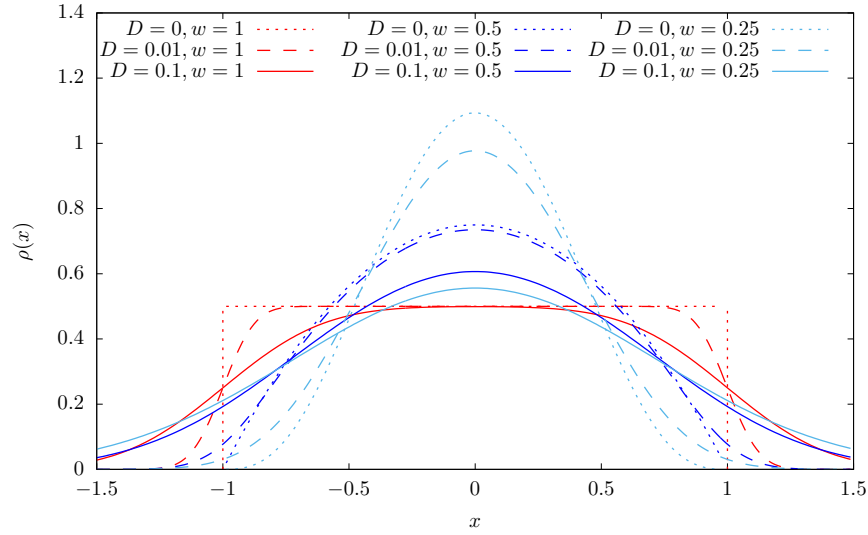
where $x \in [-w/k, w/k]$, see Fig. 7.1 [187, 188, 30, 9].

NUMERICAL IMPLEMENTATION

The numerical implementation of Eq. (7.42) is not straightforward because the divergence in n of some of the terms in its summands is balanced by the faster convergence to zero by others. The divergent terms are, for instance, any fraction with D in the denominator, whose limit $D \rightarrow 0$ we are interested in. Hermite polynomials are also divergent at $x \rightarrow \pm\infty$, and Hermite polynomials of even order are increasingly large



(a)



(b)

Figure 7.1: Stationary distribution $\rho(x)$ of a run-and-tumble particle in a harmonic potential, according to Eqs. (7.42) and (7.43), with $\tau = 1$ and a range of values for w , k and D . To ease comparison, we chose $w = k$ so that the limits of the space explored by the confined particle in the diffusionless case are $x_M = \pm 1$. In (a), where $k \geq \tau$, the *active* behaviour of the particle is manifested for increasing velocity w in the pronounced presence of the particle around the limits x_M , which appear diffuse whenever $D > 0$. In (b), where $k \leq \tau$, the activity of the particle is less prominent and the behaviour of the particle resembles that of a *passive* particle as the velocity w decreases. In fact, for $w = 0$, the particle is simply a diffusive particle confined in a harmonic potential, which is known as Ornstein-Uhlenbeck process [194]. Our implementation is based on [73, 69].

at $x = 0$ because they involve the double factorial, $H_{2n}(0) = (-1)^n(2n-1)!!$. These divergences are compensated by the weight $\exp(-kx^2/(2D))$ as well as by the product $F_n = \prod_{j=1}^n 1/[2j k((2j-1)k + 2\tau)]$, which is bounded by $F_n \leq 1/[k^{2n}(2n)!]$ and therefore has the limit $F_n \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, the sum over n needs to be truncated at some n such that the desired accuracy is achieved. Although Eq. (7.42) poses no formal challenge, when it comes to its numerical implementation the contributions in higher n entail products of very large numbers and very small numbers, which, at some point, exceed the numerical precision at either end and introduce errors due to rounding.

At first, we used the open source implementation of Hermite polynomials in the GNU Scientific Library (GSL) on Hermite Polynomials and Functions [73]. However, it showed that for relatively small values of D , even `long double` precision was not enough, see Fig. 7.2. Moreover, the use of the double factorial in the library is limited to $298!!$. See App. 7.D for a discussion on the technical challenges.

To solve the problem on the numerical convergence of Eq. (7.42), we implemented the Hermite polynomials using multiple-precision floating-point arithmetic [69] based on the GSL implementation [73]. Following this route, we can reach an arbitrary truncation point. In Fig. 7.1, the truncation point is $n = 5000$ for all curves (this means, the highest-order polynomial has order $2n = 10^4$).

7.3.3 EXPECTED POSITION

The expected position in Eq. (7.8a) is $\langle x(t) \rangle = \int dx x \left\langle \alpha(x, t) (\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \right\rangle$. Since x is itself a Hermite polynomial, $H_1(x) = x$, and we are integrating over space, the contributions to this expression are determined by the orthogonality relation in (7.25). In particular, using $\alpha(x, t) = \int d\omega \exp(-i\omega t) \sum_n \alpha_n(\omega) u_n(x)/L$, the orthogonality relation

$$\int dx \frac{x}{L} e^{-\frac{x^2}{2L^2}} H_n\left(\frac{x}{L}\right) = \sqrt{2\pi} L \delta_{1,n} n!, \quad (7.44)$$

fixes $n = 1$. Therefore, there are only two contributions to $\langle x(t) \rangle$,

$$\langle x(t) \rangle \hat{=} \overset{1, t}{\text{---}} \overset{1, t_0}{\text{---}} H_1\left(\frac{x_0}{L}\right) \frac{1!}{1!} + \overset{1, t}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{1!}{0!}, \quad (7.45)$$

which give,

$$\langle x(t) \rangle = x_0 e^{-kt} + \frac{w}{k - 2\tau} (e^{-2\tau t} - e^{-kt}) , \quad (7.46)$$

using Eqs. (7.95b) and (7.95g), and letting $r \rightarrow 0$ and $t_0 = 0$.

Due to the completeness of Hermite polynomials [127], any observable that is a polynomial in x can be written as a linear combination of Hermite polynomials and, due to the orthogonality relation, this turns out to be very advantageous because we can determine from the observable which of the infinitely many terms in the full propagator contribute. When integrating over space, we are making a projection of the full propagator into a finite number of modes. The expressions are further simplified by the use of Feynman diagrams.

7.3.4 EXPECTED VELOCITY

To calculate the expected velocity, one could naïvely differentiate the expected position $\langle x(t) \rangle$ in Eq. (7.46) with respect to time. However, analysing it with more detail we see that

$$\partial_t \langle x(t) \rangle = \int dx x \partial_t P(x, t) , \quad (7.47)$$

which will be 0 in the limit $t \rightarrow \infty$ because the distribution $P(x, t)$ at stationarity does not depend on t . Moreover, the expected position $\langle x(t) \rangle$ also tends to 0 due to the symmetry between right-moving and left-moving particles.

Instead, the expected velocity $\langle \mathcal{V}(t) \rangle$ derives from the Langevin equation, which gives $\mathcal{V}(x) = \pm w - \partial_x V(x)$. To obtain a meaningful expression, we calculate the expected velocity of a right-moving particle, which is the conditional expectation

$$\langle \mathcal{V}_\phi(t) \rangle = \frac{\int dx (w - kx) \langle \phi(x, t) \tilde{\phi}(x_0, t_0) \rangle}{\int dx' \langle \phi(x', t) \tilde{\phi}(x_0, t_0) \rangle} , \quad (7.48)$$

where the probability that the particle is a right-moving particle is

$$\begin{aligned}
\int dx \langle \phi(x, t) \tilde{\phi}(x_0, t_0) \rangle &= \frac{1}{2} \int dx \langle (\alpha(x, t) + \beta(x, t))(\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \rangle \\
&= \frac{1}{2L} \left(\overset{0, t}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{0!}{0!} + \overset{0, t}{\text{~~~~~}} \overset{0, t_0}{\text{~~~~~}} H_0\left(\frac{x_0}{L}\right) \frac{0!}{0!} \right) \\
&= \frac{1}{2} (1 + e^{-2\tau t})
\end{aligned} \tag{7.49}$$

and the expected position of right-moving particles is

$$\begin{aligned}
\langle x_\phi(t) \rangle &= \frac{1}{2} \int dx x \langle (\alpha(x, t) + \beta(x, t))(\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \rangle \\
&= \frac{1}{2} \left(\overset{1, t}{\text{---}} \overset{1, t_0}{\text{---}} H_1\left(\frac{x_0}{L}\right) \frac{1!}{1!} + \overset{1, t}{\text{---}} \overset{0, t_0}{\text{~~~~~}} H_0\left(\frac{x_0}{L}\right) \frac{1!}{0!} \right. \\
&\quad \left. + \overset{1, t}{\text{~~~~~}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{1!}{0!} + \overset{1, t}{\text{~~~~~}} \overset{1, t_0}{\text{~~~~~}} H_1\left(\frac{x_0}{L}\right) \frac{1!}{1!} \right) \\
&= \frac{1}{2} \left(\frac{w}{k+2\tau} + \frac{w}{k-2\tau} e^{-2\tau t} + \left(x_0 - \frac{w}{k-2\tau} \right) e^{-kt} + \left(x_0 - \frac{w}{k+2\tau} \right) e^{-(k+2\tau)t} \right).
\end{aligned} \tag{7.50}$$

Then, the expected velocity in Eq. (7.48) at stationarity is, using Eqs. (7.49) and (7.50),

$$\lim_{t \rightarrow \infty} \langle \mathcal{V}_\phi(t) \rangle = \frac{2\tau w}{k+2\tau}. \tag{7.51}$$

This means that in the presence of the potential, the velocity of a particle is on average lower than the in-built velocity w . If the tumbling rate is low, then the particle spends a lot of time stuck running against the potential walls. But if the tumbling rate is high, then the particle does not explore the steepest regions of the potential and runs almost as fast as its in-built velocity.

7.3.5 SECOND MOMENT OF THE POSITION $x(t)$

To calculate the second moment, $\langle x^2(t) \rangle = \int dx x^2 \langle \alpha(x, t) (\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \rangle$, we represent x^2 in the basis of the Hermite polynomials, $x^2 = H_0(x) + H_2(x)$, so that

$$\int dx \left(\frac{x}{L} \right)^2 e^{-\frac{x^2}{2L^2}} H_n \left(\frac{x}{L} \right) = \sqrt{2\pi} L (\delta_{0,n} + \delta_{2,n}) n! \quad (7.52)$$

and thus, the allowed modes are $n = 0$ and $n = 2$. Then,

$$\begin{aligned} \langle x^2(t) \rangle \hat{=} L \left(\overset{0,t}{\text{---}} \overset{0,t_0}{\text{---}} H_0 \left(\frac{x_0}{L} \right) \frac{0!}{0!} + \overset{2,t}{\text{---}} \overset{2,t_0}{\text{---}} H_2 \left(\frac{x_0}{L} \right) \frac{2!}{2!} \right. \\ \left. + \overset{2,t}{\text{---}} \overset{1,t_0}{\text{---}} H_1 \left(\frac{x_0}{L} \right) \frac{2!}{1!} + \overset{2,t}{\text{---}} \overset{0,t_0}{\text{---}} H_0 \left(\frac{x_0}{L} \right) \frac{2!}{0!} \right) \end{aligned} \quad (7.53a)$$

$$\begin{aligned} = x_0^2 e^{-2kt} + \frac{D}{k} (1 - e^{-2kt}) + 2 \frac{x_0 w}{k - 2\tau} (e^{-(k+2\tau)t} - e^{-2kt}) \\ + 2w^2 \left(e^{-2kt} \frac{1}{2k(k-2\tau)} + e^{-(k+2\tau)t} \frac{1}{(k+2\tau)(2\tau-k)} + \frac{1}{2k(k+2\tau)} \right), \end{aligned} \quad (7.53b)$$

letting $t_0 = 0$ and $r \rightarrow 0$. See in App. 7.E the contributions in Eq. (7.53a) in closed form.

7.3.6 MEAN SQUARE DISPLACEMENT

The mean square displacement $R^2(t)$ defined in Eq. (7.7), can be calculated using Eqs. (7.46) and (7.53b). In the limit $t \rightarrow \infty$, the mean square displacement is

$$\lim_{t \rightarrow \infty} R^2(t) = x_0^2 + \frac{w^2}{k(k+2\tau)} + \frac{D}{k}, \quad (7.54)$$

a constant that shows the confinement of the particle in the potential. In the short time scale, the diffusive behaviour of the particle is captured by

$$\lim_{t \rightarrow t_0} \frac{1}{2} \partial_t R^2(t) = D, \quad (7.55)$$

which shows that there is no additional factor that contributes to the diffusive behaviour of the particle other than the diffusion itself.

7.3.7 THE TWO-POINT CORRELATION FUNCTION

The correlation function $\langle x(t)x(t') \rangle$, with $t_0 < t' < t$, is given by the observable

$$\langle x(t)x(t') \rangle = \int dx dx' x x' \left\langle [\phi(x, t) + \psi(x, t)] \left[\tilde{\phi}(x', t')\phi(x', t') + \tilde{\psi}(x', t')\psi(x', t') \right] \tilde{\phi}(x_0, t_0) \right\rangle. \quad (7.56)$$

In words, this expression means that the system is initialised with a right-moving particle at time t_0 and position x_0 , and it is let to evolve by an interval of time $t' - t_0$. At time t' , the expression measures the presence of a particle, which involves its annihilation, and recreates it. The system is then let to evolve an interval of time $t - t'$, at which point the presence of either species is measured again. The propagator is integrated over x and x' to obtain the two-point correlation function. In terms of the density and polarity fields,

$$\langle x(t)x(t') \rangle = \int dx dx' x x' \langle \alpha(x, t) \tilde{\alpha}(x', t') \rangle \left\langle \alpha(x', t') (\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \right\rangle \quad (7.57a)$$

$$+ \int dx dx' x x' \left\langle \alpha(x, t) \tilde{\beta}(x', t') \right\rangle \left\langle \beta(x', t') (\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \right\rangle \quad (7.57b)$$

Starting with the term in Eq. (7.57a), the integral over x fixes the indices of the propagator $\langle \alpha_n(\omega) a_m(\omega') \rangle$ of $\langle \alpha(x, t) \tilde{\alpha}(x', t') \rangle$ to $\overset{1, t}{\text{---}} \overset{1, t'}{\text{---}} H_1(x'/L)$ via Eq. (7.44). Using the result in Eq. (7.52), the propagator $\langle \alpha(x', t') \tilde{\alpha}(x_0, t_0) \rangle$ has the contributions

$$\overset{0, t'}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{0!}{0!} + \overset{2, t'}{\text{---}} \overset{2, t_0}{\text{---}} H_2\left(\frac{x_0}{L}\right) \frac{2!}{2!} + \overset{2, t'}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{2!}{0!} \quad (7.58)$$

and the propagator $\langle \alpha(x', t') \tilde{\beta}(x_0, t_0) \rangle$ has the sole contribution $\overset{2, t'}{\text{---}} \overset{1, t_0}{\text{---}} H_1(x_0/L) 2!/1!$.

On the other hand, in the term in Eq. (7.57b), the propagator $\langle \alpha(x, t) \tilde{\beta}(x', t') \rangle$ has a contribution from $\overset{1, t}{\text{---}} \overset{0, t'}{\text{---}} H_0(x'/L)$, which turns the integral over x' into an expression like the one in Eq. (7.44). The propagator $\langle \beta(x', t') \tilde{\alpha}(x_0, t_0) \rangle$ has a contri-

bution from $\overset{1, t'}{\text{---}} \overset{0, t_0}{\text{---}} H_0(x_0/L) 1!/0!$ and the propagator $\langle \beta(x', t') \tilde{\beta}(x_0, t_0) \rangle$ from $\overset{1, t'}{\text{---}} \overset{1, t_0}{\text{---}} H_1(x_0/L) 1!/1!$. Therefore, the two-point correlation function reads

$$\begin{aligned} \langle x(t)x(t') \rangle \hat{=} & \overset{1, t}{\text{---}} \overset{1, t'}{\text{---}} \times \left[\overset{0, t'}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{0!}{0!} + \overset{2, t'}{\text{---}} \overset{2, t_0}{\text{---}} H_2\left(\frac{x_0}{L}\right) \frac{2!}{2!} \right. \\ & + \overset{2, t'}{\text{---}} \overset{1, t_0}{\text{---}} H_1\left(\frac{x_0}{L}\right) \frac{2!}{1!} + \overset{2, t'}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{2!}{0!} \left. \right] \\ & + \overset{1, t}{\text{---}} \overset{0, t'}{\text{---}} \times \left[\overset{1, t'}{\text{---}} \overset{0, t_0}{\text{---}} H_0\left(\frac{x_0}{L}\right) \frac{1!}{0!} + \overset{1, t'}{\text{---}} \overset{1, t_0}{\text{---}} H_1\left(\frac{x_0}{L}\right) \frac{1!}{1!} \right] \end{aligned} \quad (7.59a)$$

$$\begin{aligned} = & e^{-k(t-t')} \left[x_0^2 e^{-2kt'} + \frac{D}{k} \left(1 - e^{-2kt'} \right) + 2 \frac{x_0 w}{k - 2\tau} \left(e^{-(k+2\tau)t'} - e^{-2kt'} \right) \right. \\ & + 2w^2 \left(e^{-2kt'} \frac{1}{2k(k-2\tau)} + e^{-(k+2\tau)t'} \frac{1}{(k+2\tau)(2\tau-k)} + \frac{1}{2k(k+2\tau)} \right) \left. \right] \\ & + \frac{w}{k+2\tau} \left(1 - e^{-(k+2\tau)(t-t')} \right) \left[\frac{w}{k+2\tau} \left(1 - e^{-(k+2\tau)t'} \right) + x_0 e^{-(k+2\tau)t'} \right], \end{aligned} \quad (7.59b)$$

see App. 7.E. It is easy to see that $\lim_{t' \rightarrow t} \langle x(t)x(t') \rangle = \langle x^2(t) \rangle$ in Eq. (7.53b) since $\lim_{t' \rightarrow t} \langle \alpha_1(t) \tilde{\alpha}_1(t') \rangle = L$ and $\lim_{t' \rightarrow t} \langle \alpha_1(t) \tilde{\beta}_0(t') \rangle = 0$.

7.3.8 ENTROPY PRODUCTION

The entropy production [81] is defined as

$$\dot{S}_i = \frac{1}{2} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int dx dy (W(x \rightarrow y; \Delta t) P(x) - W(y \rightarrow x; \Delta t) P(y)) \log \left(\frac{W(x \rightarrow y; \Delta t) P(x)}{W(y \rightarrow x; \Delta t) P(y)} \right), \quad (7.60)$$

where $W(x \rightarrow y; \Delta t)$ is the transition rate of a particle to go from x to y in an interval of time Δt . The following expressions are useful to simplify Eq. (7.60). First, for any Markov process at stationarity we have

$$\int dx W(x \rightarrow y; \Delta t) P(x) = P(y), \quad (7.61)$$

and second, integrating the transition rate $W(y \rightarrow x; \Delta t)$ over x gives

$$\int dx W(y \rightarrow x; \Delta t) = 1. \quad (7.62)$$

Using Eqs. (7.61) and (7.62), we find that Eq. (7.60) simplifies to

$$\dot{S}_i = \frac{1}{2} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int dx dy (W(x \rightarrow y; \Delta t)P(x) - W(y \rightarrow x; \Delta t)P(y)) \log \left(\frac{W(x \rightarrow y; \Delta t)}{W(y \rightarrow x; \Delta t)} \right). \quad (7.63)$$

On the other hand, when taking the limit $\Delta t \rightarrow 0$ in Eq. (7.60) we have the term

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int dx dy (W(x \rightarrow y; \Delta t)P(x) - W(y \rightarrow x; \Delta t)P(y)) \quad (7.64a)$$

$$\begin{aligned} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int dx dy (W(x \rightarrow y; \Delta t)P(x) - W(x \rightarrow y; 0)P(x) + W(x \rightarrow y; 0)P(x) \\ &\quad - W(y \rightarrow x; \Delta t)P(y) + W(y \rightarrow x; 0)P(y) - W(y \rightarrow x; 0)P(y)) \\ &= \int dx dy \left(\dot{W}(x \rightarrow y; 0)P(x) - \dot{W}(y \rightarrow x; 0)P(y) + \delta(x - y)(P(x) - P(y)) \right) \\ &= \int dx dy \left(\dot{W}(x \rightarrow y; 0)P(x) - \dot{W}(y \rightarrow x; 0)P(y) \right), \end{aligned} \quad (7.64b)$$

using $W(y \rightarrow x; 0) = \delta(x - y)$. Since the entropy production draws on the microscopic dynamics, namely for $\Delta t \rightarrow 0$, we can approximate the transition rate $W(x \rightarrow y; \Delta t)$ of a run-and-tumble particle by that of a drift-diffusion particle, where the drift is modulated by the potential $V(x)$,

$$W_R(x \rightarrow y; \Delta t) = \frac{1}{\sqrt{4\pi D \Delta t}} e^{-\frac{(y-x-(w-kx)\Delta t)}{4\pi D \Delta t}}, \quad (7.65)$$

$$W_L(x \rightarrow y; \Delta t) = \frac{1}{\sqrt{4\pi D \Delta t}} e^{-\frac{(y-x-(-w-kx)\Delta t)}{4\pi D \Delta t}}. \quad (7.66)$$

The transition rates that involve a transmutation of a right-moving particle into a left-moving particle and the converse, are of higher order in Δt because both the rate of transmutation and the rate of translation from one position to another are proportional to Δt . Namely, the overall transition rate is quadratic in Δt , $W(x \rightarrow y, R \rightarrow L; \Delta t) \propto \Delta t^2$, and similarly, $W(y \rightarrow x, L \rightarrow R; \Delta t)$, $W(x \rightarrow y, L \rightarrow R; \Delta t)$, $W(y \rightarrow x, R \rightarrow L; \Delta t)$. Therefore none of these state transitions contribute when taking the limit

$\Delta t \rightarrow 0$ in Eq. (7.63). Given the symmetry between right-moving and left-moving particles, we can first calculate the entropy production of a right-moving particle. The logarithm in Eq. (7.63) is then

$$\lim_{\Delta t \rightarrow 0} \log \left(\frac{W_R(x \rightarrow y; \Delta t)}{W_R(y \rightarrow x; \Delta t)} \right) = \frac{1}{D}(y - x) \left(w - k \frac{x + y}{2} \right). \quad (7.67)$$

The time derivative of

$$P(x, t + \Delta t) = \int dy W(y \rightarrow x; \Delta t) P(y), \quad (7.68)$$

reads

$$\frac{\partial}{\partial \Delta t} P_R(x, t + \Delta t) \Big|_{\Delta t \rightarrow 0} = \int dy \dot{W}_R(y \rightarrow x; 0) P_R(y), \quad (7.69)$$

where $\partial_t P_R$ on the left-hands side satisfies the Fokker-Planck Eq. (7.19a). Also, Eq. (7.19a) can be written as

$$\partial_t P_R(x, r) = \int dy \delta(x - y) (D \partial_y^2 - \partial_y(w - ky)) P_R(y, t) \quad (7.70)$$

and by integration by parts as

$$\begin{aligned} \partial_t P_R(x, r) &= \int dy (\delta''(x - y) D - (w - ky) \delta'(x - y)) P_R(y, t) \\ &= \int dy \dot{W}_R(y \rightarrow x; 0) P_R(y). \end{aligned} \quad (7.71)$$

It follows that the kernel $\dot{W}_R$ is $\dot{W}_R(y \rightarrow x; 0) = D \delta''(x - y) - (w - ky) \delta'(x - y)$. Then the entropy production $\dot{S}_i$ in Eq. (7.63) for right-moving particles only is

$$\begin{aligned} \dot{S}_i &= \frac{1}{2D} \int dx dy \left\{ (\delta''(x - y) D - (w - ky) \delta'(x - y)) P_R(x) \right. \\ &\quad \left. - (\delta''(x - y) D - (w - ky) \delta'(x - y)) P_R(y, t) \right\} (y - x) \left(w - k \frac{x + y}{2} \right) \\ &= \int dx P_R(x) \left(\frac{1}{D} (w - kx)^2 - k \right). \end{aligned} \quad (7.72)$$

By symmetry, the total entropy production of right-moving and left-moving particles is

$$\dot{S}_i = 2 \int dx P_R(x) \left(\frac{1}{D} (w - kx)^2 - k \right) \quad (7.73)$$

From Eq. (7.49),

$$\lim_{t \rightarrow \infty} \int dx P_R(x, t) = \frac{1}{2}, \quad (7.74)$$

and from Eq. (7.50),

$$\lim_{t \rightarrow \infty} \langle x_\phi(t) \rangle = \frac{w}{2(k + 2\tau)}. \quad (7.75)$$

The second moment of the position of a right-moving particle at stationarity is

$$\lim_{t \rightarrow \infty} \langle x_\phi^2(t) \rangle = \lim_{t \rightarrow \infty} \int dx x^2 \langle \phi(x, t) \tilde{\phi}(x_0, t_0) \rangle \quad (7.76a)$$

$$\begin{aligned} &= \lim_{t \rightarrow \infty} \frac{1}{2} \int dx x^2 \langle (\alpha(x, t) + \beta(x, t)) (\tilde{\alpha}(x_0, t_0) + \tilde{\beta}(x_0, t_0)) \rangle \\ &= \lim_{t \rightarrow \infty} \frac{L}{2} \left(\begin{array}{c} \textcolor{blue}{0, t} \quad \textcolor{blue}{0, t_0} \\ \text{---} \end{array} H_0 \left(\frac{x_0}{L} \right) \frac{0!}{0!} + \begin{array}{c} \textcolor{blue}{2, t} \quad \textcolor{blue}{2, t_0} \\ \text{---} \end{array} H_2 \left(\frac{x_0}{L} \right) \frac{2!}{2!} \right. \\ &\quad + \begin{array}{c} \textcolor{blue}{2, t} \quad \textcolor{orange}{1, t_0} \\ \text{---} \textcolor{orange}{\sim} \end{array} H_1 \left(\frac{x_0}{L} \right) \frac{2!}{1!} + \begin{array}{c} \textcolor{blue}{2, t} \quad \textcolor{orange}{\sim} \quad \textcolor{blue}{0, t_0} \\ \text{---} \end{array} H_0 \left(\frac{x_0}{L} \right) \frac{2!}{0!} \\ &\quad + \begin{array}{c} \textcolor{orange}{0, t} \quad \textcolor{orange}{0, t_0} \\ \textcolor{orange}{\sim} \end{array} H_0 \left(\frac{x_0}{L} \right) \frac{0!}{0!} + \begin{array}{c} \textcolor{orange}{2, t} \quad \textcolor{orange}{2, t_0} \\ \textcolor{orange}{\sim} \end{array} H_2 \left(\frac{x_0}{L} \right) \frac{2!}{2!} \\ &\quad \left. + \begin{array}{c} \textcolor{orange}{2, t} \quad \textcolor{orange}{1, t_0} \\ \textcolor{orange}{\sim} \text{---} \end{array} H_1 \left(\frac{x_0}{L} \right) \frac{2!}{1!} + \begin{array}{c} \textcolor{orange}{2, t} \quad \textcolor{orange}{\sim} \quad \textcolor{orange}{0, t_0} \\ \textcolor{orange}{\sim} \text{---} \end{array} H_0 \left(\frac{x_0}{L} \right) \frac{2!}{0!} \right) \\ &= \lim_{t \rightarrow \infty} \frac{L}{2} \left(\begin{array}{c} \textcolor{blue}{0, t} \quad \textcolor{blue}{0, t_0} \\ \text{---} \end{array} + 2 \times \begin{array}{c} \textcolor{blue}{2, t} \quad \textcolor{orange}{\sim} \quad \textcolor{blue}{0, t_0} \\ \text{---} \end{array} \right) \\ &= \frac{L^2}{2} + \frac{w^2}{2k(k + 2\tau)} \quad (7.76b) \end{aligned}$$

Then, the entropy production in Eq. (7.73) is

$$\dot{S}_i = \frac{2\tau w^2}{D(k + 2\tau)}. \quad (7.77)$$

7.4 DISCUSSION AND CONCLUSIONS

The main contribution of this work is the derivation of the field theory of a diffusive run-and-tumble in a harmonic potential. This is an example of how to introduce an external potential in the field theory and how to deal with it given that, in this case, the potential rendered the action functional semi-local. By means of a Doi-Peliti field theory, we calculated the first and second moments of the position of the particle as a function of time, its mean square displacement, the expected velocity, the two-point correlation function and, most importantly, the entropy production. In this work, entropy production is derived from the microscopic dynamics of the system without resorting to coarse-graining, nor to an effective theory. On the contrary, the expression we provide is an exact result.

ACKNOWLEDGEMENTS

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7.A DERIVING THE n TH MOMENT OF $\mathbf{x}$

The Fourier transform can be used as a generating function to calculate the moments of $\mathbf{x}$. Let $G(\mathbf{k}, \omega; \mathbf{k}', \omega')$ be the propagator of a process in Fourier space. Then, the n th moment of the position $\mathbf{x}$ is

$$\langle \mathbf{x}^n(t) \rangle = \int d^d \mathbf{x} d^d \mathbf{k} d^d \mathbf{k}' d\omega d\omega' \mathbf{x}^n e^{-i\omega t - i\omega' t_0 + i\mathbf{k}\mathbf{x} + i\mathbf{k}'\mathbf{x}_0} G(\mathbf{k}, \omega; \mathbf{k}', \omega'), \quad (7.78)$$

which, defining $F(\mathbf{k}) = \int d^d \mathbf{k}' d\omega d\omega' \exp(-i\omega t - i\omega' t_0 + i\mathbf{k}' \mathbf{x}_0) G(\mathbf{k}, \omega; \mathbf{k}', \omega')$, reduces to

$$\langle \mathbf{x}^n(t) \rangle = \int d^d \mathbf{x} d^d \mathbf{k} \mathbf{x}^n e^{i\mathbf{k}\mathbf{x}} F(\mathbf{k}) . \quad (7.79)$$

The n th power of $\mathbf{x}$ can be obtained by taking n times the gradient of $\exp(i\mathbf{k}\mathbf{x})$ with respect to $\mathbf{k}$,

$$\langle \mathbf{x}^n(t) \rangle = i^{-n} \int d^d \mathbf{x} d^d \mathbf{k} (\nabla_{\mathbf{k}}^n e^{i\mathbf{k}\mathbf{x}}) F(\mathbf{k}) , \quad (7.80)$$

and, using integration by parts,

$$\langle \mathbf{x}^n(t) \rangle = i^n \int d^d \mathbf{x} d^d \mathbf{k} e^{i\mathbf{k}\mathbf{x}} (\nabla_{\mathbf{k}}^n F(\mathbf{k})) . \quad (7.81)$$

The integral over $\mathbf{x}$ gives a Dirac δ function,

$$\langle \mathbf{x}^n(t) \rangle = i^n \int d^d \mathbf{k} \delta(\mathbf{k}) (\nabla_{\mathbf{k}}^n F(\mathbf{k})) , \quad (7.82)$$

which, integrating over $\mathbf{k}$, gives

$$\langle \mathbf{x}^n(t) \rangle = i^n \nabla_{\mathbf{k}}^n F(\mathbf{k}) \Big|_{\mathbf{k}=\mathbf{0}} . \quad (7.83)$$

7.B MEAN SQUARE DISPLACEMENT OF A DIFFUSIVE PARTICLE

The effective diffusion constant of a process in the large time limit is calculated from the linear term that governs the mean square displacement. A purely diffusive process has a mean square displacement linear in time. Since the diffusion process is simple and very well understood it is often used as a benchmark for other less-known processes.

The Fokker-Planck equation of the diffusion process in $\mathbb{R}^d$ is

$$\partial_t P(\mathbf{x}, t) = D \partial_{\mathbf{x}}^2 P(\mathbf{x}, t) , \quad (7.84a)$$

$$P(\mathbf{x}, t_0) = \delta(\mathbf{x} - \mathbf{x}_0) , \quad (7.84b)$$

where $P(\mathbf{x}, t)$ is the probability of finding a particle at position $\mathbf{x}$ and time t , D is the diffusion constant, $t_0 = 0$ is the initial time and $\mathbf{x}_0$ is the initial position of the particle.

The solution to this partial differential equation is the Gaussian distribution,

$$P(\mathbf{x}, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(\mathbf{x}-\mathbf{x}_0)^2}{4Dt}}, \quad (7.85)$$

which can be used to calculate the mean square displacement $R^2(t) = \langle (\mathbf{x}(t) - \mathbf{x}_0)^2 \rangle$. Alternatively, using the method outlined in App. 7.A, and knowing that the propagator in Fourier space is, from Eq. (7.84a),

$$G(\mathbf{k}, \omega; \mathbf{k}', \omega') = \frac{\delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega')}{-i\omega + D\mathbf{k}^2 + r}, \quad (7.86)$$

where r is a mass term to regularise the infrared divergence, and which is to be taken to 0. Then,

$$F(\mathbf{k}) = e^{-i\mathbf{k}\mathbf{x}_0 - D\mathbf{k}^2 t}, \quad (7.87)$$

so the first and second moments of the position are

$$\langle \mathbf{x}(t) \rangle = i \nabla_{\mathbf{k}} F(\mathbf{k}) \Big|_{\mathbf{k}=0} = \mathbf{x}_0, \quad (7.88a)$$

$$\langle \mathbf{x}^2(t) \rangle = - \nabla_{\mathbf{k}}^2 F(\mathbf{k}) \Big|_{\mathbf{k}=0} = \mathbf{x}_0^2 + 2dDt, \quad (7.88b)$$

and the mean square displacement, Eq. (7.7), is

$$R^2(t) = \langle (x(t) - x_0)^2 \rangle = 2dDt. \quad (7.89)$$

Taking a diffusive particle as reference, the effective diffusion constant for any process is thus sensibly defined as

$$D_{\text{eff}} = \frac{1}{2d} \partial_t R^2(t) \Big|_{t=0}. \quad (7.90)$$

7.C HERMITE POLYNOMIALS

The definition of Hermite polynomials [127] used in this chapter is

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \partial_x^n \left(e^{-\frac{x^2}{2}} \right), \quad (7.91)$$

so that the first few Hermite polynomials are

$$H_0(x) = 1, \quad (7.92a)$$

$$H_1(x) = x, \quad (7.92b)$$

$$H_2(x) = x^2 - 1, \quad (7.92c)$$

$$H_3(x) = x^3 - 3x, \quad (7.92d)$$

$$H_4(x) = x^4 - 6x^2 + 3, \quad (7.92e)$$

$$H_5(x) = x^5 - 10x^3 + 15x, \quad (7.92f)$$

$$H_6(x) = x^6 - 15x^4 + 45x^2 - 15, \quad (7.92g)$$

$$H_7(x) = x^7 - 21x^5 + 105x^3 - 105x, \quad (7.92h)$$

$$H_8(x) = x^8 - 28x^6 + 210x^4 - 420x^2 + 105. \quad (7.92i)$$

The Hermite polynomials are orthogonal with respect to the weight function $w(x) = \exp(-x^2/2)$, that is,

$$\int e^{-\frac{x^2}{2}} H_n(x) H_m(x) dx = \sqrt{2\pi} n! \delta_{n,m}, \quad (7.93)$$

where $\delta_{n,m}$ is the Kronecker delta.

Hermite polynomials $H_n(x)$ are eigenfunctions of the differential operator

$$\mathcal{L}[u] = u'' - xu' = -\mu u, \quad (7.94)$$

also known as Hermite's differential equation, for non-negative integer eigenvalue $\mu_n = n$. Using this equation, it is possible to show that the solution to the eigenvalue problem $\mathcal{L}[u] = u'' + (xu)'/L^2 = \lambda u$ in Sec. 7.3 are the rescaled Hermite polynomials multiplied by the weight function $w(x)$, namely $u_n(x) = \exp(-x^2/(2L^2)) H_n(x/L)$, with eigenvalues $\lambda_n = -\mu_n/L^2 = -n/L^2$ [168].

7.D NUMERICAL ERROR IN THE IMPLEMENTATION OF $\rho(x)$

As mentioned in Sec. 7.3.2 the numerical implementation of the higher-order Hermite polynomials presented some technical challenges, which we mention here briefly. First,

the GSL library dedicated to Hermite polynomials uses `double` precision throughout, so we changed it to `long double` [73]. Second, the GSL deals with $H_{2n}(0) = (-1)^n(2n-1)!!$ as a special case, where the double factorials are drawn from a table where $298!!$ is the largest number available. We then extended the table to $500!!$ but still failed to obtain an accurate result, see Fig. 7.2a. The origin of the bug was the compiler, which read the numbers in the table in `double` precision (instead of `long double`) and the rounding applied by the compiler, which is platform dependent, was producing an error. We found that even if $H_{2n}(0)$ was not dealt with as a special case, the numerical precision `long double` was not enough, see Fig. 7.2b. We then decided to implement Hermite polynomials using multiple-precision floating-point arithmetic based on the GSL implementation [69].

7.E SOME PROPAGATORS IN CLOSED FORM USED ABOVE

The propagators used above to calculate the different observables of a run-and-tumble particle in a harmonic potential with diffusion are listed here.

$${}_{0,t} \text{---} {}_{0,t'} \hat{=} \langle \alpha_0(t) \tilde{\alpha}_0(t') \rangle = L e^{-r(t-t')} \quad (7.95a)$$

$${}_{1,t} \text{---} {}_{1,t'} \hat{=} \langle \alpha_1(t) \tilde{\alpha}_1(t') \rangle = L e^{-(k+r)(t-t')} \quad (7.95b)$$

$${}_{2,t} \text{---} {}_{2,t'} \hat{=} \langle \alpha_2(t) \tilde{\alpha}_2(t') \rangle = L e^{-(2k+r)(t-t')} \quad (7.95c)$$

$${}_{0,t} \text{---} {}_{0,t'} \hat{=} \langle \beta_0(t) \tilde{\beta}_0(t') \rangle = L e^{-(2\tau+r)(t-t')} \quad (7.95d)$$

$${}_{1,t} \text{---} {}_{1,t'} \hat{=} \langle \beta_1(t) \tilde{\beta}_1(t') \rangle = L e^{-(k+2\tau+r)(t-t')} \quad (7.95e)$$

$${}_{2,t} \text{---} {}_{2,t'} \hat{=} \langle \beta_2(t) \tilde{\beta}_2(t') \rangle = L e^{-(2k+2\tau+r)(t-t')} \quad (7.95f)$$

$${}_{1,t} \text{---} {}_{0,t'} \hat{=} \langle \alpha_1(t) \tilde{\beta}_0(t') \rangle = \frac{w}{k-2\tau} e^{-r(t'-t)} \left(e^{-2\tau(t-t')} - e^{-k(t-t')} \right) \quad (7.95g)$$

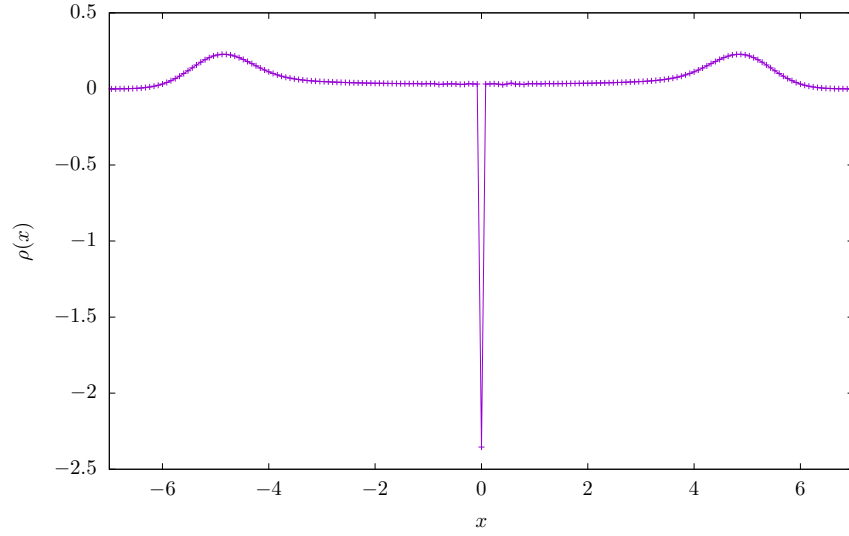
$${}_{2,t} \text{---} {}_{1,t'} \hat{=} \langle \alpha_2(t) \tilde{\beta}_1(t') \rangle = \frac{w}{k-2\tau} e^{-r(t'-t)} \left(e^{-(k+2\tau)(t-t')} - e^{-2k(t-t')} \right) \quad (7.95h)$$

$$1, \overset{t'}{\text{orange wavy}} \text{---} \overset{0, t'}{\text{blue}} \hat{=} \langle \beta_1(t) \tilde{\alpha}_0(t') \rangle = \frac{w}{k+2\tau} e^{-r(t'-t)} \left(1 - e^{-(k+2\tau)(t'-t)} \right) \quad (7.95i)$$

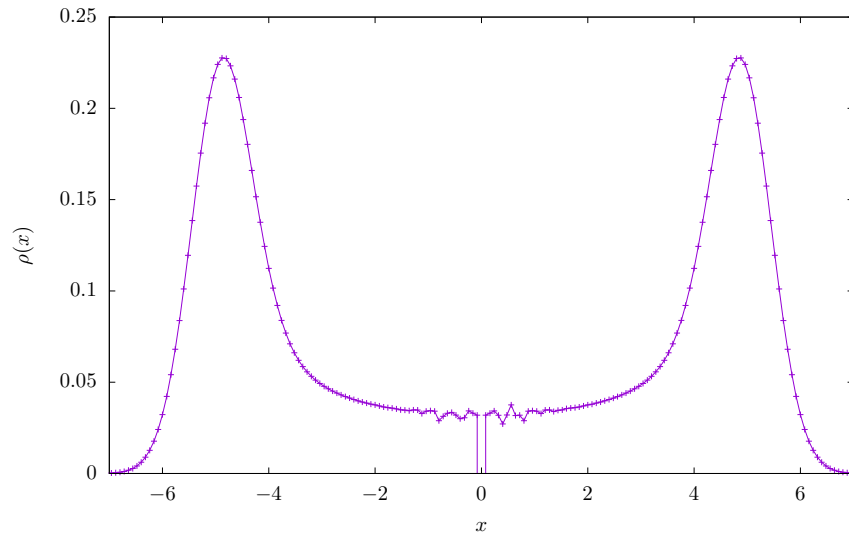
$$2, \overset{t'}{\text{orange wavy}} \text{---} \overset{1, t'}{\text{blue}} \hat{=} \langle \beta_2(t) \tilde{\alpha}_1(t') \rangle = \frac{w}{k+2\tau} e^{-r(t'-t)} \left(e^{-k(t-t')} - e^{-(2k+2\tau)(t-t')} \right) \quad (7.95j)$$

$$\begin{aligned} 2, \overset{t}{\text{blue}} \text{---} \overset{0, t'}{\text{orange wavy}} &\hat{=} \langle \alpha_2(t) \tilde{\alpha}_0(t') \rangle \\ &= \frac{w^2}{L} e^{-r(t-t')} \left(\frac{e^{-2k(t-t')}}{2k(k-2\tau)} + \frac{e^{-(k+2\tau)(t-t')}}{(k+2\tau)(2\tau-k)} + \frac{1}{2k(k+2\tau)} \right) \end{aligned} \quad (7.95k)$$

$$\begin{aligned} 2, \overset{t}{\text{orange wavy}} \text{---} \overset{0, t'}{\text{blue}} &\hat{=} \langle \beta_2(t) \tilde{\beta}_0(t') \rangle \\ &= \frac{w^2}{L} e^{-r(t-t')} \left(\frac{e^{-(2k+2\tau)(t-t')}}{2k(k+2\tau)} + \frac{e^{-k(t-t')}}{(k+2\tau)(2\tau-k)} + \frac{e^{-2\tau(t-t')}}{2k(k-2\tau)} \right) \end{aligned} \quad (7.95l)$$



(a)



(b)

Figure 7.2: Two types of numerical error encountered when implementing $\rho(x)$ in Eq. (7.42) with parameters $\tau = 0.2$, $k = 1$, $w = 5$, $D = 0.3$ and truncation point $2n = 412$. In (a), there is a visible error at $x = 0$, where the evaluation of $H_{2n}(0)$ as a double factorial fails. In (b), the roughness of the curve around $x = 0$ indicates its lack of precision.

Discussion and outlook

The abundance of non-equilibrium systems in nature and society makes our study and understanding of such systems vitally important. Non-equilibrium systems are so prevalent that systems naturally at equilibrium are actually the exception. Often we find ourselves perplexed by the drastic, far-reaching effects of small actions, apparently innocent, of millions and billions of people in a short period of time. Choose your favourite economic example — the most immediate example that comes to my mind is the repeatedly reported *panic buying* during the COVID-19 pandemic [195].

Reality is far too complex to be studied as a whole and this is why understanding much smaller, simple and accessible systems is so necessary. This is the main motivation of statistical mechanics, where systems involve a large collection of particles that move and interact following the same, very simple, rules. The collective behaviour of these particles results in emergent phenomena and patterns in the system as a whole. Establishing the link between the microscopic dynamics, at the particle level, and the macroscopic features, say at the level of pattern formation, remains an open problem that has motivated a huge variety of approaches and methods in statistical mechanics.

In this thesis, the non-equilibrium systems that I have studied are: the Oslo model (Chapter 1), branching processes and applications (Chapters 2 to 5), the voter model (Chapter 6) and run-and-tumble motion (Chapter 7). The analytical tools that I have used are a one-to-one map between two different systems (Chapters 1 and 5) and Doi-Peliti field theory (DPFT) (all other chapters). I have also performed computer simulations and numerical fittings to support the analytical findings.

The path integral approach based on DPFT has proven to be a hugely versatile and systematic tool. Despite its achievements (see for instance the average temporal profile of a time-dependent branching process in Chapter 3, or the scaling of visited sites by a branching random walk in Chapter 4), DPFT can present some major challenges that require a sense of perspective — and much thinking and trying very many different routes — to be solved (see the parametrisation of fields in the concealed voter model

in Chapter 6, or the diagonalisation of the action functional in Chapter 7 to write it in local form).

Since DPFT retains the particle entity through the second quantised form, where the ladder operators are then transformed into creation and annihilation fields, it is the ideal tool to tackle new, complex, problems in statistical mechanics. There are fascinating open question in active matter that are attracting a lot of attention. As an example, many efforts are being devoted to understanding motility-induced phase separation (MIPS) in many-particle systems with active motility and volume-excluded interactions [89]. MIPS is one of the most fascinating emergent phenomena [32] as, counter-intuitively, particles collect and separate due to their ability to move away without the apparent need of attractive forces. Active matter depends heavily on the microscopic dynamics of the system and therefore effective theories, which do not retain the particle entity, have a limited scope.

An illustrative example of active matter is active nematics [86], where self-driven rods are in constant motion on 2D interfaces. The interplay between interaction-induced alignment of motile particles and their self-propulsion gives rise to so-called comet-like and trefoil-like topological defects with topological charges $+1/2$ and $-1/2$ respectively [59]. Experimentally, active nematics have been observed in the interactions between microtubules and kinesin molecular motors in liquid crystals, generating intricate patterns such as active turbulence [131]. This instance of collective motion displays MIPS.

An interesting problem for future research is to use DPFT as a framework to describe the dynamics of topological defects in active nematics. There are primarily two antagonistic types of topological defects involved, which may be thought of as particles with equal topological charges but opposed sign. Defects create and annihilate in antagonistic pairs, so that there is no net gain nor loss of topological charge. Due to their shape, the two types of defects obey different motility patterns: active Brownian motion ($+1/2$) and plain diffusion ($-1/2$). A field-theoretic description of this process constructed from first principles would, in principle, allow us to calculate characteristic correlation lengths, the active length scale, velocity fields and entropy production. The two types of defects can be dealt with in the field theory as two interacting species of particles, as Chapters 4 and 6.

Another open problem is to understand and calculate the entropy production, which

is indicative of how far from equilibrium the system is, at different levels of coarse graining. In the absence of an exact solution, many approaches make use of an effective theory to calculate the entropy production [27]. However, as seen in Chapter 7, the entropy production is an expression of the microscopic dynamics and therefore any coarse-graining results in a loss of information. Trivially, if we represent the system by one state, the maximum coarse-graining possible, then the entropy production will be zero because the system cannot flow between different states.

In conclusion, retaining the particle entity is crucial in the study of non-equilibrium systems, in particular when analysing correlations and collective behaviour. This has been central in the different statistical properties and aspects of the six different systems that I have presented in this thesis. The methods shown here lay the foundations of future research.

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Most of the work presented in this thesis has been previously published in peer-reviewed journals. This appendix contains the proof of permission to reproduce all third-party copyrighted work and the approval of all co-authors.

A.1 COPYRIGHT PERMISSION

A.1.1 *Europhysics Letters* PUBLISHED BY THE INSTITUTE OF PHYSICS

Chapter 1 was originally published in *Europhysics Letters*. Following the instructions on their website, I requested permission by email. This was the answer from the editorial office.

From: Editorial Office <editorial.office@epletters.net>

Sent: 28 Feb 2020, at 13:24

To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>

Subject: Re: Permission to use EPL copyrighted material

Dear Dr. Garcia-Millan,

Many thanks for your message. In answer to your request, we are pleased to inform you that you are allowed to use the following paper (text and figures) in your PhD thesis:

R. Garcia-Millan, G. Pruessner, L. Pickering and K. Christensen
"Correlations and hyperuniformity in the avalanche size of the Oslo model"
EPL 122 (2018) 50003, doi:10.1209/0295-5075/122/50003
The references of the sources must be given (title, year, issue).

Thank you.

Sincerely yours,

Kevin Desse
Editorial Assistant

A.1.2 *Physical Review E* PUBLISHED BY THE AMERICAN PHYSICAL SOCIETY

In Chapters 2 and 5, text and figures were originally published in *Physical Review E*. The APS has the following statement on their [website](#) (updated December 2017, accessed February 2020).

As the author of an APS-published article, may I include my article or a portion of my article in my thesis or dissertation?

Yes, the author has the right to use the article or a portion of the article in a thesis or dissertation without requesting permission from APS, provided the bibliographic citation and the APS copyright credit line are given on the appropriate pages.

A.1.3 *Scientific Reports* PUBLISHED BY SPRINGER NATURE

Chapter 4 was originally published in *Scientific Reports*. Following the [link](#) to **Rights and permissions** in the article website leads to the following.

Volume explored by a branching random walk on general graphs

Author: Ignacio Bordeu et al

Publication: Scientific Reports

Publisher: Springer Nature

Date: Oct 30, 2019

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A.1.4 *Journal of Statistical Mechanics: Theory and Experiment* PUBLISHED BY THE INSTITUTE OF PHYSICS

Chapter 6 was published in *Journal of Statistical Mechanics: Theory and Experiment (JSTAT)* on 13 May 2020. The decision letter that I received by email contains the permission for reproduction of the manuscript.

From: JSTAT Editorial Office <jstat-eo@jstat.sissa.it>

Sent: 12 February 2020 at 22:02:02 GMT

To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>

Subject: [JSTAT] Editor decision on document JSTAT_009P_1019

Authors: Rosalba Garcia-Millan

Title: The Concealed Voter Model is in the Voter Model universality class

Dear Dr. Rosalba Garcia Millan,

We are pleased to inform you that your submission JSTA_009P_1019 has been accepted for publication in JSTAT. [...]

You are encouraged to post to arXiv and/or to another institutional archive the version of this article accepted for publication including all changes made as a result of the peer review process, but excluding any other editing, typesetting or other changes made by IOP Publishing and/or its licensors. Please select the non-exclusive licence to distribute offered by arXiv when uploading the article. We recommend the inclusion of the DOI when citing the published manuscript.

The publishers current authors rights policy can be found at
<https://publishingsupport.iopscience.iop.org/current-policy-author-rights...>

Sincerely yours,
JSTAT Editor

A.2 APPROVAL OF CO-AUTHORS

This section contains the proof of approval of all co-authors to reproduce published or submitted work. At the time of request, the title of this thesis used to be "Correlations and fluctuations in non-equilibrium systems". Authors appear in alphabetic order.

A.2.1 SAOIRSE AMARTEIFIO

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020, at 15:59
To: Amarteifio, Saoirse <s.amarteifio14@imperial.ac.uk>
Subject: Permission to reproduce

Dear Saoirse,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Sci. Rep. **9**, 15590 (2019). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Amarteifio, Saoirse <s.amarteifio14@imperial.ac.uk>
Sent: 29 Feb 2020, at 13:50
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Dear Rosalba,

Permission granted, please freely reproduce this publication (Sci. Rep. **9**, 15590 (2019) in your thesis, possibly verbatim and without restriction.

Best wishes

Saoirse

A.2.2 IGNACIO BORDEU

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020, at 15:57
To: Bordeu, Ignacio <ib443@maths.cam.ac.uk>
Subject: Permission to reproduce

Dear Ignacio,

I am completing my PhD thesis at Imperial College London entitled "Cor-

relations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Sci. Rep. **9**, 15590 (2019). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Bordeu, Ignacio <ibordeu@gurdon.cam.ac.uk>
Sent: 29 Feb 2020, at 13:50
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

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Please find attached the copyright declaration for this article.

All the best,
Ignacio

A.2.3 KIM CHRISTENSEN

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020, at 14:39
To: Christensen, Kim <k.christensen@imperial.ac.uk>
Subject: Permission to reproduce

Dear Kim,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Europhys. Lett. **122**, 50003, (2018). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Christensen, Kim <k.christensen@imperial.ac.uk>
Sent: 27 Feb 2020, at 17:30
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Dear Rosalba,

Yes of course I am happy for you to reproduce this publication in your thesis. Please do not hesitate to contact me should you need further proof.

Best wishes,

Prof. Kim Christensen

Head of Condensed Matter Theory

Imperial College London

E-mail: k.christensen@imperial.ac.uk

Phone: +44 (0)20 7594 7574

A.2.4 ALVARO CORRAL

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>

Sent: 27 Feb 2020, at 14:43

To: Alvaro Corral <acorral@crm.cat>

Subject: Permission to reproduce

Dear Alvaro,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Phys. Rev. E 97, 062156, (2018). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,

Rosalba

From: Alvaro Corral <acorral@crm.cat>

Sent: 27 Feb 2020, at 23:23

To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>

Subject: Re: Permission to reproduce

Hola Ros,

espero que todo bien por UK. Lo de reproducir la publicacion en la tesis, "possible verbatim", lo has pensado bien? Me da la impresion que no haria justicia a lo que ha sido tu trabajo de estos anyos, que ha sido mucho, y bueno, mientras que ese paper yo lo veo como una cosa puramente colateral.

Un saludo

Alv

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>

Sent: 28 Feb 2020, at 17:52

To: Alvaro Corral <acorral@crm.cat>

Subject: Re: Permission to reproduce

Hola Alv,

Entiendo lo que dices. Mi idea es hacer una tesis lo más parecida a una tesis por compendio de publicaciones, modelo que aquí no existe (por eso el contactar con mis coautores). Al principio de cada capítulo, que corresponde a una publicación, enumero los autores, cito la publicación, pongo una referencia a los permisos cedidos por la organización/editor/revista que tiene los derechos de autor (en este caso APS) y, por último, enumero las contribuciones de cada autor (en realidad, la única contribución relevante para mi tesis es la que yo he hecho).

Creo que el PRE del que hablamos contiene parte de mi trabajo (simulaciones) y por tanto que es buen candidato a formar parte de mi tesis. No pretendo atribuirme ningún mérito que no sea mío, por eso creo que es importante explicar cuál ha sido mi contribución (en esto sigo el ejemplo de otras tesis). Adjunto la primera página del capítulo tal como lo tengo ahora

mismo (te puedo adelantar que no sería el capítulo 1). ¿Qué te parece?
Puedo modificar las contribuciones si es conveniente.

Un saludo
Ros

From: Alvaro Corral <acorral@crm.cat>
Sent: 1 Mar 2020, at 14:49
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Hey Ros,
las simus que hiciste estaban muy bien, pero encajan bien en tu supertesis?
No son poca cosa unas simus comparado con todo el rollo fieldteoretico?
Pero si las quieres poner y a Gunnar le parece bien, entonces a mi tambien.
A veces un error es querer poner demasiadas cosas en la tesis, cuantas mas cosas mas facil es que algo no le guste a los evaluadores. Lo dicho, si crees que encaja bien, adelante, pero lo mas importante de la tesis es que, si no es un bloque, al menos tenga un hilo comun.

Saludos y buen fin de fin de
alv

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 14 Mar 2020, at 12:22
To: Alvaro Corral <acorral@crm.cat>
Subject: Re: Permission to reproduce

Hola Alv,

Gracias, entiendo lo que dices. Lo hablé con Gunnar y me dijo que le parece bien que incluya el artículo en mi tesis. Él me recomendó que en caso de incluirlo, escriba una introducción sobre cómo encaja en la tesis. Mi idea era incluirlo en el bloque de branching processes, donde que tengo tres

artículos más. ¿Te parecería bien así?

Saludos,
Ros

From: Alvaro Corral <acorral@crm.cat>
Sent: 14 Mar 2020, at 12:42
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Bien, estaria bien que pudieras encontrar un hilo conductor y unificador, una historia, vaya, que no parezca recorta y pega de unos articulos de aqui y de alli. Si a los British les gustan las tesis-tesis el formato de articulos no se corresponde mucho. Pero Gunnar sabe del tema...
Alv

A.2.5 FRANCESC FONT-CLOS

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020, at 15:51
To:
Subject: Permission to reproduce

Dear Francesc,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Phys. Rev. E **97**, 062156, (2018). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Font, Francesc <francesc.font@gmail.com>
Sent: 27 Feb 2020, at 15:50
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Dear Rosalba,

Congratulations on completing your PhD. I hereby give my consent as a coauthor of Phys. Rev. E **97**, 062156 (2018) for its reproduction in your PhD thesis.

Best wishes,
Francesc

A.2.6 NICHOLAS MOLONEY

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020 at 14:52
To: Moloney, Nicholas <n.moloney.lml@gmail.com>
Subject: Permission to reproduce

Dear Nicholas,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Phys. Rev. E **97**, 062156, (2018). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Moloney, Nicholas <n.moloney.lml@gmail.com>

Sent: 27 Feb 2020, at 15:02

To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>

Subject: Re: Permission to reproduce

Dear Rosalba,

I am happy for you to reproduce this publication in your thesis.

Best,

Nicholas

A.2.7 JOHANNES PAUSCH

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>

Sent: 14 Mar 2020, at 13:26

To: Johannes Pausch <jp634@cam.ac.uk>

Subject: Permission to reproduce

Dear Johannes,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Phys. Rev. E **98**, 062107 (2018) and the preprint arXiv:2003.03302 (2020). Please, can you confirm that you are happy for me to reproduce these publications in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,

Rosalba

From: Johannes Pausch <jp634@cam.ac.uk>

Sent: 15 Mar 2020, at 22:01

To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>

Subject: Re: Permission to reproduce

Dear Rosalba,

I'm happy with the reproduction (including verbatim) of Phys. Rev. E **98**, 062107 (2018) and the preprint arXiv:2003.03302 (2020).

Best wishes,
Johannes

A.2.8 LUKE PICKERING

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020, at 08:41
To: Luke Pickering <luketpickering@googlemail.com>
Subject: Permission to reproduce

Dear Luke,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Europhys. Lett. **122**, 50003, (2018). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Pickering, Luke <luketpickering@googlemail.com>
Sent: 3 Mar 2020, at 20:59
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Dear Rosalba,

I am happy for you to reproduce the publication, "Correlations and hyperuniformity in the avalanche size of the OsloModel, Europhys. Lett. **122**, 50003, (2018)", which we are both listed as authors of, in your thesis.

Thanks

Dr. Luke Pickering,
Postdoctoral Research Assistant
Michigan State University

A.2.9 GUNNAR PRUESSNER

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 14 Mar 2020, at 13:32
To: Gunnar Pruessner <g.pruessner@imperial.ac.uk>
Subject: Permission to reproduce

Dear Gunnar,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Phys. Rev. E **98**, 062107 (2018); Europhys. Lett. **122**, 50003, (2018); Sci. Rep. **9**, 15590 (2019) and the preprint arXiv:2003.03302 (2020). Please, can you confirm that you are happy for me to reproduce these publications in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Gunnar Pruessner <g.pruessner@imperial.ac.uk>

Sent: 14 Mar 2020, at 17:07
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Of course, I am happy for you to reproduce the material.

All the best,
Gunnar

A.2.10 BENJAMIN WALTER

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>
Sent: 27 Feb 2020, at 16:01
To: Walter, Benjamin <b.walter16@imperial.ac.uk>
Subject: Permission to reproduce

Dear Benjamin,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Phys. Rev. E **98**, 062107, (2018) and Sci. Rep. **9**, 15590 (2019). Please, can you confirm that you are happy for me to reproduce these publications in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,
Rosalba

From: Walter, Benjamin <b.walter16@imperial.ac.uk>
Sent: 27 Feb 2020, at 20:06
To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>
Subject: Re: Permission to reproduce

Dear Rosalba,

Thank you for contacting me. Of course I am happy for you to reproduce both Phys. Rev. E **98**, 062107, (2018) and Sci. Rep. **9**, 15590 (2019) in full length, verbatim, or as you like in your thesis.

Best,

Benjamin

A.2.11 NANXIN WEI

From: Garcia Millan, Rosalba <garciamillan16@imperial.ac.uk>

Sent: 27 Feb 2020, at 15:59

To: Wei, Nanxin <n.wei14@imperial.ac.uk>

Subject: Permission to reproduce

Dear Nanxin,

I am completing my PhD thesis at Imperial College London entitled "Correlations and fluctuations in non-equilibrium systems". I am contacting you as a co-author of Sci. Rep. **9**, 15590 (2019). Please, can you confirm that you are happy for me to reproduce this publication in my thesis, possibly verbatim, and indicating so where appropriate. I will include this email and your response in the appendix "Right to reproduce" of my thesis.

Best wishes,

Rosalba

From: Wei, Nanxin <n.wei14@imperial.ac.uk>

Sent: 27 Feb 2020, at 16:16

To: Rosalba Garcia Millan <garciamillan16@imperial.ac.uk>

Subject: Re: Permission to reproduce

Of course, best of luck for the thesis :)

Cheers,

Nanxin