

VARIATIONS AND GENERALIZATIONS OF PREPROJECTIVE ALGEBRAS

A THESIS PRESENTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY OF IMPERIAL COLLEGE LONDON
AND THE
DIPLOMA OF IMPERIAL COLLEGE
BY
DANIEL KAPLAN

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE
180 QUEEN'S GATE, LONDON SW7 2BZ

JULY 2019

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed: _____

COPYRIGHT

The copyright of this thesis rests with the author and is made available under a Creative Commons Attribution Non-Commercial No Derivatives licence. Researchers are free to copy, distribute or transmit the thesis on the condition that they attribute it, that they do not use it for commercial purposes and that they do not alter, transform or build upon it. For any reuse or redistribution, researchers must make clear to others the licence terms of this work.

Variations and Generalizations of Preprojective Algebras

ABSTRACT

The preprojective algebra, defined by Gelfand and Ponomarev, is the algebraic precursor to the study of Nakajima quiver varieties, which are pervasive in geometric representation theory. Examples of such varieties include flag varieties (for the type A quivers) and the Hilbert schemes of points in the plane (for the Jordan quiver.) In this thesis, we take the *algebra* as the fundamental object of study. We produce interesting flat degenerations of the preprojective algebra, in the direction of Geiß, Leclerc, and Schröer following de Thanhoffer de Völcsey and Presotto. Additionally, we study a multiplicative analog of the preprojective algebra, and prove that a large class of such algebras are 2-Calabi-Yau.

To my parents, for their continued support.

ACKNOWLEDGMENTS

My senior year at Northwestern, I asked then graduate student Joel Spector what the differences are between being an undergraduate and graduate student. He said, “as a graduate student, you’re lucky if anyone ever teaches you anything.” I found this comment puzzling at the time. And while I now understand completely what he meant, I really am one of the lucky ones.

Travis Schedler is extremely patient and generous with his time. He is an excellent role model, possessing a rare combination of vision and technical prowess. Yanki Lekili has been incredibly supportive and fosters an excellent learning environment. Richard Thomas continually goes above and beyond the call of duty and seems to be carrying Imperial’s Geometry group on his back.

I’ve enjoyed numerous mathematical discussions with Lee Cohn, Eric Korman, Sam Gunningham, Pavel Safronov, and José Simental. I’ve taken insightful and carefully-planned courses from Travis, Sam, Owen Gwillam and Claudia Scheimbuer, and Tye Lidman. I’ve worked closely and fruitfully with Michael Wong and Haiping Yang.

My PhD was transatlantic, beginning in Austin and ending in London, with the occasional visit to Bonn. Amid this nomadic lifestyle, crêpe nights with Travis and Marie-Amélie were a pleasant staple.

In Bonn, both the Max Planck Institute and Hausdorff Institute provided fantastic work environments. In London, for all practical purposes my office was Chief Coffee.

The PhD can often feel like an exercise in staying sane and fending off insecurities. I was fortunate to receive emotional support from my family and friends.

Finally, if you’re reading this entire thesis, and not just the acknowledgements, then you’re undoubtably Sue Sierra. You deserve a medal, but will have to settle for a thank you.

Note to the reader:

This thesis is the culmination of two papers, both of which can be found on ArXiv:

Frobenius Degenerations of Preprojective Algebras:

<https://arxiv.org/pdf/1803.09267.pdf>

The 2-Calabi-Yau property for multiplicative preprojective algebras:

<https://arxiv.org/pdf/1905.12025.pdf>

The latter paper, whose content makes up the second chapter of this thesis, is joint work with my supervisor, Travis Schedler. Since much of that paper appears here verbatim, in this sense I'm not the sole author of my thesis.

CONTENTS

0	INTRODUCTION	1
0.1	Motivation and Summary of Results	1
0.2	Background	4
0.2.1	Representations of Quivers	4
0.2.2	Preprojective Algebras	6
0.2.3	Calabi-Yau algebras	10
0.3	Techniques	14
0.3.1	Diamond Lemma	14
0.3.2	Deformation Theory and Frobenius Algebras	19
1	FROBENIUS DEGENERATIONS OF PREPROJECTIVE ALGEBRAS	25
1.1	Introduction	25
1.2	Representations of Decorated Quivers	33
1.3	Representations of Decorated Preprojective Algebras	37
1.4	Decorated Preprojective Algebras as Degenerations	46
1.4.1	Frobenius Degenerations	46
1.4.2	Degenerations of Preprojective Algebras	56
1.4.3	Flatness Conjecture	58
1.5	The Dynkin Case	63
1.5.1	Case: $\Pi(G_2)$	65
1.5.2	Case: $\Pi(F_4)$	70
1.5.3	Case: $\Pi(B_n)$	75
1.5.4	Case: $\Pi(C_n)$	83
2	THE 2-CALABI-YAU PROPERTY FOR MULTIPLICATIVE PREPROJECTIVE ALGEBRAS	88

2.1	The multiplicative preprojective algebra	95
2.1.1	Definitions	95
2.1.2	The map θ	97
2.2	The 2-Calabi–Yau property	98
2.2.1	Free product properties	99
2.2.2	A bimodule resolution of Λ	101
2.2.3	The dual complex	103
2.3	Formality of the multiplicative Ginzburg dg-algebras	107
2.4	The center and primality of multiplicative preprojective algebras	109
2.4.1	The center of free products	109
2.4.2	The center is trivial for quivers containing a cycle	114
2.5	The strong free product property	115
2.5.1	The case of cycles	115
2.5.2	Partial multiplicative preprojective algebras	119
2.5.3	A convenient substitution	123
2.5.4	Quivers containing cycles	125



INTRODUCTION

0.1 MOTIVATION AND SUMMARY OF RESULTS

The preprojective algebra is ubiquitous in geometry. A few different guises of these algebras include:

- (1) (Symplectic Geometry / Quiver Varieties) an algebra associated to a quiver whose modules naturally correspond to symplectic reduction on the tangent space to the moduli space of representations of the quiver,
- (2) (Algebraic Geometry) a non-commutative (crepant) resolution of Kleinian singularities¹⁵,
- (3) (Homological Algebra) an explicit description of any graded 2-Calabi-Yau algebra⁶,
- (4) (Quantum Groups) an algebra whose irreducible representations correspond to Lusztig's semicanonical basis for certain quantum enveloping algebras⁴⁴,
- (5) (Linear Algebra) an algebra with deformations whose irreducible representations give solutions to the additive Deligne-Simpson problem¹³,
- (6) (Historical) and an algebra containing each irreducible preprojective module for the path algebra exactly once, as a summand³⁷.

Preprojective algebras are often studied through their variety of representations and as an intermediary for more geometric explorations, such as Nakajima quiver varieties, ALE

spaces^{47,42}, or semicanonical bases⁴⁴. However, even in the pursuit of geometry, one often needs to answer more fundamental algebraic questions. For instance, Crawley-Boevey and Holland realized the preprojective algebra of a cycle as a resolution of the type A Kleinian singularity via an identification of algebras $\Pi(\tilde{A}_n) \cong \mathbb{C}[x, y] \# \mathbb{Z}/(n+1)\mathbb{Z}$ ¹⁵. Additionally, one can study the category (or variety) of modules of an algebra via additional structures on the algebra. For instance, a Poisson structure on the moduli space of representations of the preprojective algebra, is determined by a double Poisson bracket the preprojective algebra itself,⁵⁴. Therefore, in this thesis we embark on a largely algebraic study of the preprojective algebra and its variants, following^{50,28,7,20,16,52,36,21}.

One explanation for the diversity of the preprojective algebra in geometry is the large extent to which its behavior depends on the quiver. There is a natural trichotomy of quivers into ADE Dynkin, extended ADE Dynkin, and other quivers. Their sets of indecomposable representations are respectively: finite, in finitely many one-parameter families, and containing a 2-parameter family. Notice that the representations of a subquiver $Q' \subset Q$ can be identified with the representations of Q that are supported on Q' . In this way, one can see that ADE Dynkin quivers are the smallest, with every subquiver ADE Dynkin. Extended ADE Dynkin are larger, in fact with all strict subquivers ADE Dynkin. And every other quiver contains an extended ADE Dynkin subquiver. See Figure 1 for illustrations of all Dynkin and extended Dynkin quivers.

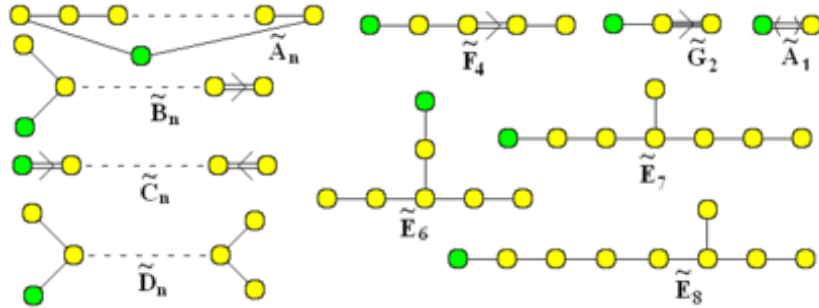


Figure 1: Extended Dynkin diagrams, with the Dynkin diagrams obtained by removing the green vertex.

Revisiting (1)-(6) with this trichotomy in mind, one notices that only (1) and (6) are general descriptions. Indeed (2) is specific to the extended Dynkin case, (3) does not hold in the Dynkin setting, (4) requires the connection to Kac-Moody Lie algebras, and (5) concerns star-shaped quivers. Therefore, we view (1) as the fundamental geometric property of the preprojective algebra, proven in the next section.

We proceed by considering two variants of the preprojective algebra, both satisfying some form of (1), with the hope of establishing some of the other connections. That is, we have the guiding meta-result:

Property $\mathcal{P}$, which holds for the preprojective algebra of a quiver, also holds for $\mathcal{M}$, a modified version of the preprojective algebra.

Indeed, in Chapter 1, $\mathcal{P}$ is the Hilbert series (and the Frobenius property) and $\mathcal{M}$ is certain degenerations. In Chapter 2, $\mathcal{P}$ is the Calabi-Yau property and $\mathcal{M}$ is the multiplicative preprojective algebra.

In Chapter 1, we study a generalization of the usual notion of a representation of a quiver to an algebra-valued representation of a quiver. Using this notion, and a symmetric Frobenius form on the algebras, one can define a notion of preprojective algebra, which we denote $\Pi(Q, D_Q)$,^{36,43}. Examples of these algebras appear in a series of papers by Geiß, Leclerc, and Schröer who establish a version of (1) in Proposition 5.2 of³⁶ and a version of (4) as the main result in³⁵. We introduce the perspective that these algebras can be viewed as *degenerations* of ordinary preprojective algebras. Using this approach, one can compute the Hilbert series of such algebras using the Hilbert series of ordinary preprojective algebras as a lower bound and directly establishing an upper bound.

In more detail, we conjecture:

Conjecture 0.1.1 (1.4.21). *The Hilbert–Poincaré series of $\Pi(Q, D_Q)$, with $D_Q = (A_i, M^\alpha)$ is given by,*

$$h_{\Pi(Q, D_Q)}(t) = \begin{cases} (I + t^{\max_t + 2}P)D(1 - At + t^2)^{-1} & \text{if } (Q, D_Q) \text{ is Dynkin} \\ D(1 - At + t^2)^{-1} & \text{otherwise} \end{cases}$$

where, in the Dynkin case, $\max_t$ is the maximum path length of a non-zero homogeneous element in $\Pi(Q, D_Q)$ and P is the permutation matrix for the permutation of the set $\{1_{A_i}\}_{i \in Q_0}$ given by the Nakayama automorphism. Here $A = [a_{ij}]$ with $a_{ij} := \sum_{\alpha: i \rightarrow j} \dim_{A_i}(M^\alpha)$ and $D = [d_{ij}]$ with $d_{ij} := \delta_{ij} \dim_k(A_i)$

And we prove this conjecture in the Dynkin case:

Theorem 0.1.2 (1.4.26). *The Hilbert–Poincaré series of $\Pi(Q, D_Q)$, with (Q, D_Q) Dynkin (i.e. $2I - A$ positive-definite) is given by,*

$$h_{\Pi(Q, D_Q)}(t) = (I + t^{\max_t + 2}P)D(1 - At + t^2)^{-1}.$$

In Chapter 2, we study a multiplicative version of the preprojective algebra, denoted $\Lambda^q(Q)$. It was first defined by Crawley-Boevey and Shaw in light of (5): its indecomposable

representations give solutions to the *multiplicative* Deligne-Simpson problem¹⁶. Van den Bergh showed that these algebras satisfy a version of (1), where the symplectic reduction is replaced by a quasi-Hamiltonian reduction for the *group*-valued moment map,⁵³. We shift attention to (3) conjecturing these algebras are 2-Calabi-Yau, and (2) conjecturing they are non-commutative resolutions in the extended Dynkin cases. We prove these conjectures in the case of a cycle and use an induction to show the 2-Calabi-Yau property for quivers containing a cycle. Moreover, we view these properties as consequences of a decomposition, we call the strong free product property.

In more detail, or at least more notation, we conjecture:

Conjecture 0.1.3 (2.0.1). $\Lambda^q(Q)$ is 2-Calabi-Yau and prime for all $q \in (k^\times)^{Q_0}$ and all Q connected and not ADE. If Q is furthermore not extended ADE, then $Z(\Lambda^q(Q)) = k$, and the Calabi-Yau structure is unique.

And we prove this conjecture in the case of quivers containing a cycle:

Theorem 0.1.4 (2.0.2). $\Lambda^q(Q)$ is 2-Calabi-Yau and prime for any $q \in (k^\times)^{Q_0}$, and Q connected and containing an unoriented cycle. If the containment is proper, then $Z(\Lambda^q(Q)) = k$, and the Calabi-Yau structure is unique.

0.2 BACKGROUND

0.2.1 REPRESENTATIONS OF QUIVERS

For completeness of exposition and to fix notation, we recall the definition of a representation of a quiver, then show how the theory is subsumed by the representation theory of associative algebras.

Fix a ground field k .

Definition 0.2.1. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ where $s, t : Q_1 \rightarrow Q_0$ are the source and target maps respectively from the set of arrows, Q_1 , to the set of vertices, Q_0 .

A quiver is merely a directed graph, but the terminology comes from Gabriel, who considered representations of such graphs, defined as follows:

Definition 0.2.2. A *representation* of a quiver Q is $\rho = (V_i, \rho_\alpha)_{i \in Q_0, \alpha \in Q_1}$ where $\rho_\alpha : V_{s(\alpha)} \rightarrow V_{t(\alpha)}$ is a k -linear map of vector spaces, for each $\alpha \in Q_1$.

We consider exclusively quivers with finitely many vertices and arrows and representations with each vector space finite-dimensional. The function $\dim(V_{(-)}) : Q_0 \rightarrow \mathbb{N}$ defines an $\mathbb{N}^{Q_0}$ -grading on the set of representations. Fixing $d = (d_i) \in \mathbb{N}^{Q_0}$ we define

$$\text{Rep}_d(Q) := \{\rho = (k^{d_i}, \rho_\alpha)\} = \bigoplus_{\alpha \in Q_1} \text{Hom}(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}).$$

$\text{Rep}_d(Q)$ is a finite-dimensional vector space with an action of $G_d := \prod_{i \in Q_0} GL_{d_i}(k)$ given by

$$(g_i)_{i \in Q_0} \cdot (V_i, \rho_\alpha)_{i \in Q_0, \alpha \in Q_1} = (V_i, g_{t(\alpha)} \circ \rho_\alpha \circ g_{s(\alpha)}^{-1}).$$

One says two representations are isomorphic if they lie in the same G_d -orbit.

Alternatively, one can collectively study representations of Q by considering the category $\mathcal{R}(Q)$ whose objects are representations of Q and morphisms are commuting diagrams between representations. Invertible morphisms are given by $|Q_0|$ -tuples of (compatible) invertible linear maps. Hence, for the full subcategory $\mathcal{R}_d(Q) \subset \mathcal{R}(Q)$ whose objects are d -dimensional representations, one has a natural identification between the moduli space of isomorphism classes of semisimple representations in $\mathcal{R}_d(Q)$ and the GIT quotient $\text{Rep}_d(Q)//G$.

The category $\mathcal{R}(Q)$ is equivalent to the category of finitely-generated modules over an algebra, as we now explain.

Definition 0.2.3. The *path algebra* $P_k(Q)$ for a fixed quiver $Q = (Q_0, Q_1, s, t)$ is defined to be

$$P_k(Q) := T_{kQ_0}(kQ_1),$$

where kS denotes the k -vector space with basis S and kQ_1 is made into a kQ_0 -bimodule by k -linear extension of the action

$$e_i \cdot \alpha \cdot e_j := \delta_{is(\alpha)} \alpha \delta_{t(\alpha)j}.$$

Here $e_{(\cdot)} : Q_0 \rightarrow P_k(Q)$ regards a vertex as a length zero path. We will usually write $P(Q) := P_k(Q)$ suppressing the dependence on the field.

Remark 0.2.4. As defined above, multiplication in $P(Q)$ is identified with concatenation of tensors in $T_{kQ_0}(kQ_1)$ and hence is read left to right. This choice is convenient when performing computations with long paths, but requires considering right modules, as explained below.

The path algebra is graded with n th graded piece given by k -linear combinations of

paths of length n , i.e. n -tuples of composable arrows in the quiver. Since vertices are idempotents in $P(Q)$ summing to the identity element, there is a further decomposition of the path algebra by source and target vertices,

$$P(Q) = 1P(Q)1 = \left(\sum_{i \in Q_0} e_i \right) P(Q) \left(\sum_{j \in Q_0} e_j \right) = \bigoplus_{i,j \in Q_0} e_i P(Q) e_j.$$

Representations of Q can be identified with right modules for the path algebra, $P(Q)$. Explicitly, if V is a right $P(Q)$ -module then one can form the representation $\rho = (V_i, \rho_\alpha)$ where $V_i := V \cdot e_i$ and ρ_α is the right action by α , a linear map $V_{s(\alpha)} \rightarrow V_{t(\alpha)}$. Conversely, if (V_i, ρ_α) is a representation of Q one can form the right module $V := \bigoplus_{i \in Q_0} V_i$ with length one paths α acting by ρ_α and paths $\gamma = \alpha_1 \cdots \alpha_n$ acting by $\rho_{\alpha_n} \circ \cdots \circ \rho_{\alpha_1}$.

To describe equivalence classes of representations one needs to compute the orbit space of the action of G on $\text{Rep}_d(Q)$.

0.2.2 PREPROJECTIVE ALGEBRAS

Given a quiver $Q = (Q_0, Q_1, s, t)$ one can define the opposite quiver to be $Q^{op} := (Q_0, Q_1, t, s)$, which has the same vertices but with all arrows reversed. From this the double quiver is $\overline{Q} := (Q_0, Q_1 \sqcup Q_1, s \sqcup t, t \sqcup s)$, which has the same vertices as Q but with both the arrows and the inverted arrows and consequently is insensitive to the orientation of the quiver Q . If $\alpha \in Q_1$, we write α^* for the corresponding arrow in Q_1^{op} , and vice versa.

The representation theory of the double quiver can be described in terms of the representation theory of the original quiver. Fix a dimension vector d and denote by $\text{Rep}_d(Q)$ the vector space of representations of Q with dimension vector d . Then

$$\begin{aligned} \text{Rep}_d(\overline{Q}) &\xrightarrow{\cong} \text{Rep}_d(Q) \oplus \text{Rep}_d(Q^{op}) \\ \rho &\longmapsto \rho|_Q \oplus \rho|_{Q^{op}}. \end{aligned}$$

One has the non-degenerate pairing

$$\text{Rep}_d(Q) \times \text{Rep}_d(Q^{op}) \rightarrow k \quad (\rho, \rho') \mapsto \sum_{\alpha \in Q_1} \text{tr}(\rho(\alpha) \circ \rho'(\alpha^*))$$

which induces a k -linear isomorphism

$$\mathrm{Rep}_d(Q^{op}) \cong \mathrm{Rep}_d(Q)^* \quad (V_i, \rho_\alpha) \mapsto \left(\psi \mapsto \sum_{\alpha \in Q_1} \mathrm{tr}(\psi(\alpha^*) \circ \rho_\alpha(\alpha)) \right).$$

We conclude that,

$$\mathrm{Rep}_d(\overline{Q}) \cong \mathrm{Rep}_d(Q) \oplus \mathrm{Rep}_d(Q^{op}) \cong \mathrm{Rep}_d(Q) \oplus \mathrm{Rep}_d(Q)^* =: T^*(\mathrm{Rep}_d(Q)).$$

For any finite-dimensional vector space V , the vector space $V \oplus V^* \cong T^*(V)$ carries a canonical symplectic form. Taking $V := \mathrm{Rep}_d(Q)$, the symplectic form on $\mathrm{Rep}_d(\overline{Q}) \cong T^*(\mathrm{Rep}_d(Q))$ is explicitly given by:

$$\omega_d : \mathrm{Rep}_d(\overline{Q}) \times \mathrm{Rep}_d(\overline{Q}) \rightarrow k$$

$$\omega_d(\rho, \rho') = \sum_{\alpha \in Q_1} \mathrm{tr}(\rho(\alpha^*)\rho'(\alpha)) - \mathrm{tr}(\rho(\alpha)\rho'(\alpha^*))$$

where $\mathrm{tr} : \mathrm{End}_k(\oplus_{i \in Q_0} k^{d_i}) \rightarrow k$ is the usual trace of matrices.

The action of $G_d = \prod_{i \in Q_0} GL_{d_i}(k)$ on $\mathrm{Rep}_d(Q)$ is extended to $\mathrm{Rep}_d(\overline{Q})$ by,

$$g \cdot (x, \phi) = (g \cdot x, \phi \circ (d_x \varphi_g)^{-1})$$

where $d_x \varphi_g$ is the differential of the action map. This action is Hamiltonian with G_d -equivariant moment map $\mu_d : \mathrm{Rep}_d(\overline{Q}) \rightarrow \mathfrak{g}_d^*$ given by

$$\begin{aligned} \mu_d(\rho)(x) &= \frac{1}{2} \omega(\rho, x \cdot \rho) = \frac{1}{2} \sum_{\alpha \in Q_1} \mathrm{tr}(\rho(\alpha^*)x \cdot \rho(\alpha) - x \cdot \rho(\alpha)\rho(\alpha^*)) \\ &= \frac{1}{2} \sum_{\alpha \in Q_1} \mathrm{tr}(x \cdot \rho([\alpha, \alpha^*])), \end{aligned}$$

where $\mathfrak{g}_d := \mathrm{Lie}(G_d)$. We view the moment map as valued in $\mathfrak{g}_d$ by post-composing with twice the trace form,

$$\rho \in \mathrm{Rep}_d(\overline{Q}) \longrightarrow \frac{1}{2} \mathrm{tr} \left(- \cdot \sum_{\alpha \in Q_1} \rho([\alpha, \alpha^*]) \right) \in \mathfrak{g}^* \longrightarrow \sum_{\alpha \in Q_1} \rho([\alpha, \alpha^*]) \in \mathfrak{g}$$

Since the second map is an isomorphism, we can identify the moment map with evaluation at $r := \sum_{\alpha \in Q_1} [\alpha, \alpha^*]$.

Returning to the algebraic setting, representations of $\overline{Q}$ are right modules over the path

algebra

$$P(\overline{Q}) = T_{kQ_0}(kQ_1 \oplus kQ_1^{op}).$$

The condition that a representation, ρ , is in the zero fiber of the moment map is precisely the condition that $\rho(r) = 0$, i.e. the module descends to a module for the quotient algebra $P(\overline{Q})/\langle r \rangle$, where $\langle r \rangle$ denotes the two-sided ideal generated by r . This algebra was originally defined and studied by Gelfand and Ponomarev, prior to the moment map perspective, since in the Dynkin case it contains the path algebra as a subalgebra as well as all indecomposable preprojective modules for the path algebra³⁷.

Summarizing this discussion:

Definition 0.2.5. The *preprojective algebra* associated to a quiver Q and a field k is

$$\Pi_k(Q) := P(\overline{Q}) / \left\langle \sum_{\alpha \in Q_1} [\alpha, \alpha^*] \right\rangle.$$

We will usually write $\Pi(Q) := \Pi_k(Q)$ suppressing the dependence on the field.

Proposition 0.2.6. *With the identifications induced from the non-degenerate pairings,*

$$\text{Rep}_d(Q) \otimes \text{Rep}_d(Q^{op}) \rightarrow k \quad (\rho, \rho') \mapsto \sum_{\alpha \in Q_1} \text{tr}(\rho(\alpha) \circ \rho'(\alpha^*))$$

and

$$\mathfrak{g}_d \otimes \mathfrak{g}_d \rightarrow k \quad (X, Y) \mapsto \text{tr}(XY)$$

the following diagram commutes,

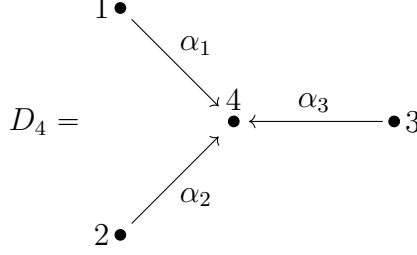
$$\begin{array}{ccc} & T^* \text{Rep}_d(Q) & \xrightarrow{\mu_d} \mathfrak{g}_d^* \\ \nearrow \cong & & \searrow \cong \\ \text{Rep}_d(\overline{Q}) & \xrightarrow{\rho \mapsto \rho(\sum_{\alpha \in Q_1} [\alpha, \alpha^*])} & \mathfrak{g}_d \end{array}$$

and hence one can view the moment map as evaluation on the element $\sum_{\alpha \in Q_1} [\alpha, \alpha^*]$.

Corollary 0.2.7. *The set of representations of dimension vector d in $\Pi(Q)$ is the zero fiber of the moment map μ_d .*

Remark 0.2.8. Crucially, one needs to identify $\text{Rep}_d(\overline{Q}) \cong T^* \text{Rep}_d(Q)$ as symplectic representations of G_d . The symplectic structure on $\text{Rep}_d(\overline{Q})$ is defined through the equivalence. The preservation of the G_d -action follows since G_d acts on $\text{Rep}_d(\overline{Q})$ by conjugation and the pairing uses the trace, which is conjugation-invariant. Additionally, $\mathfrak{g}_d \cong \mathfrak{g}_d^*$ as G_d -representations with the adjoint and coadjoint actions.

Example 0.2.9. We conclude this section with a computation of a basis for $\Pi_{\mathbb{C}}(D_4)$, where



$\Pi_{\mathbb{C}}(D_4)$ is a quotient of the path algebra $P_{\mathbb{C}}(D_4) = T_{\mathbb{C}\{e_1, e_2, e_3, e_4\}}(\mathbb{C}\{\alpha_1, \alpha_1^*, \alpha_2, \alpha_2^*, \alpha_3, \alpha_3^*\})$ by the two-sided ideal generated by the relation

$$R = \sum_{i=1,2,3} \alpha_i \alpha_i^* - \alpha_i^* \alpha_i.$$

This ideal contains $e_i R = \alpha_i^* \alpha_i$ for $i = 1, 2, 3$ and $e_4 R = \sum_{i=1,2,3} \alpha_i \alpha_i^*$.

Notice any path of length ≥ 1 must pass through vertex 4, and hence we first compute all paths starting and ending at vertex 4. Such a path is equal to one avoiding vertex 3 as

$$\alpha_3 \alpha_3^* = -\alpha_1 \alpha_1^* - \alpha_2 \alpha_2^*$$

and cannot revisit the same vertex twice as

$$(\alpha_i \alpha_i^*)(\alpha_i \alpha_i^*) = \alpha_i (\alpha_i^* \alpha_i) \alpha_i^* = 0,$$

for $i = 1, 2, 3$. Rewriting the condition that visiting vertex 3 twice is zero gives the relation,

$$0 = (\alpha_3 \alpha_3^*)(\alpha_3 \alpha_3^*) = (\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^*)(\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^*) = \alpha_1 \alpha_1^* \alpha_2 \alpha_2^* + \alpha_2 \alpha_2^* \alpha_1 \alpha_1^*.$$

So there is a single path of length 4 and none of greater length. In particular,

$$e_4 \Pi_{\mathbb{C}}(D_4) e_4 = \text{Span}\{e_4, \alpha_1 \alpha_1^*, \alpha_2 \alpha_2^*, \alpha_1 \alpha_1^* \alpha_2 \alpha_2^*\}$$

Now, to obtain all paths from 1 to 2 one applies $\alpha_2^*(\cdot)\alpha_1$ to $e_4 \Pi_{\mathbb{C}}(D_4) e_4$, and all terms vanish except $\alpha_2^* e_4 \alpha_1$. For paths from 1 to 4, $e_4 \alpha_1$ and $\alpha_2 \alpha_2^* \alpha_1$ are both non-zero from which one can determine the only path from 1 to 1, besides e_1 , is $\alpha_1^* \alpha_2 \alpha_2^* \alpha_1$. By symmetry,

both in the quiver and by reversing paths, one can compute the full matrix-valued Hilbert-Poincaré series to be,

$$\begin{pmatrix} 1+t^4 & t^2 & t^2 & t(1+t^2) \\ t^2 & 1+t^4 & t^2 & t(1+t^2) \\ t^2 & t^2 & 1+t^4 & t(1+t^2) \\ t(1+t^2) & t(1+t^2) & t(1+t^2) & (1+t^2)^2 \end{pmatrix}$$

where the (i, j) th entry is the polynomial $h_{e_i \Pi_{\mathbb{C}}(D_4) e_j}(t)$ whose t^m coefficient is $\dim_{\mathbb{C}}(e_i \Pi_{\mathbb{C}}^m(D_4) e_j)$.

The four maximum length paths,

$$\begin{aligned} \alpha_1^* \alpha_2 \alpha_2^* \alpha_1 &= -\alpha_1^* \alpha_3 \alpha_3^* \alpha_1 \\ \alpha_2^* \alpha_1 \alpha_1^* \alpha_2 &= -\alpha_2^* \alpha_3 \alpha_3^* \alpha_2 \\ \alpha_3^* \alpha_1 \alpha_1^* \alpha_3 &= -\alpha_3^* \alpha_2 \alpha_2^* \alpha_3 \\ \alpha_1 \alpha_2^* \alpha_2 \alpha_1^* &= -\alpha_1 \alpha_3^* \alpha_3 \alpha_1^* = \alpha_2 \alpha_1^* \alpha_1 \alpha_2^* = -\alpha_2 \alpha_3^* \alpha_3 \alpha_2^* = \alpha_3 \alpha_1^* \alpha_1 \alpha_3^* = -\alpha_3 \alpha_2^* \alpha_2 \alpha_3^*, \end{aligned}$$

gives rise to a Frobenius form

$$\lambda : \Pi_{\mathbb{C}}(D_4) \rightarrow \mathbb{C}$$

$$\gamma \mapsto [(\alpha_1^* \alpha_2 \alpha_2^* \alpha_1)^{\vee} + (\alpha_2^* \alpha_1 \alpha_1^* \alpha_2)^{\vee} + (\alpha_3^* \alpha_1 \alpha_1^* \alpha_3)^{\vee} + (\alpha_1 \alpha_2^* \alpha_2 \alpha_1^*)^{\vee}](\gamma)$$

where $\alpha^{\vee}$ denotes the dual of α .

This is well-defined since each cycle lies in a one-dimensional graded component of the algebra. One needs to choose this sum of cycles, as opposed to any path, to get non-degeneracy of the form.

0.2.3 CALABI-YAU ALGEBRAS

Calabi-Yau algebras were defined by Kontsevich and Soibelman⁴¹ and studied by Ginzburg³⁸. Here we provide a terse treatment containing mainly ideas needed in the rest of the thesis.

Definition 0.2.10. An algebra A is *d-Calabi-Yau* if A is homologically smooth (i.e. A has a finite length resolution by finitely-generated projective A -bimodules) and there exists

$$\eta : \mathrm{RHom}_{A^e}(A, A^e) =: A^{\vee} \xrightarrow{\text{quasi-iso}} A[-d]$$

where $A^e := A \otimes A^{\mathrm{op}}$ is an algebra whose left modules are naturally A -bimodules.

We motivate this definition by relation to other fields in the next two remarks.

Remark 0.2.11. If A is a Calabi-Yau algebra, then the bounded derived category of finite-dimensional left A -modules, $D^b(A - \text{mod}^{fd})$, is a *Calabi-Yau* category, meaning it has a Serre functor* given by a shift. Calabi-Yau categories arise in algebraic geometry as the bounded derived category of coherent sheaves on a Calabi-Yau manifold. In this case, the Calabi-Yau structure is thought of as a trivialization of the canonical bundle, and the Serre functor is tensoring by the canonical bundle. Calabi-Yau categories arise in symplectic topology as the wrapped Fukaya category of a Liouville manifold³³.

Remark 0.2.12. A d -Calabi-Yau algebra induces a $(2 - d)$ -shifted symplectic structure on the moduli space of finite-dimensional representations of the algebra⁸. In particular, 2-Calabi-Yau algebras induce ordinary symplectic structures.

A large class of Calabi-Yau algebras arise from a superpotential, including all complete Calabi-Yau algebras⁵⁵. We recall the construction here. Consider the path algebra of a quiver, kQ , and the quotient $kQ_{\text{cyclic}} := kQ/[kQ, kQ]$. Notice if $p = a_1 a_2 \cdots a_n$ is a path in kQ with different source and target, written $s(p) \neq t(p)$, then $p = s(p)p = [s(p), p]$ is zero in kQ_{cyclic} . Next notice that for any path $[a_1, a_2 a_3 \cdots a_n] = p - a_2 \cdots a_n a_1$ is zero in kQ_{cyclic} , so kQ_{cyclic} is the vector space of cyclic paths in Q , up to cyclic permutation. Given $\omega \in kQ_{\text{cyclic}}$, we can regard it as an element $\tilde{\omega} \in kQ$ by summing over all cyclic permutations of ω . That is, $\tilde{p} := a_1 a_2 \cdots a_n + a_2 a_3 \cdots a_1 + \cdots + a_n a_1 \cdots a_{n-1}$. Therefore, given an element $\omega \in kQ_{\text{cyclic}} := kQ/[kQ, kQ]$, called a superpotential, one can define the Jacobi algebra to be,

$$J(Q, \omega) := kQ / \left(\left\{ \frac{\partial \tilde{\omega}}{\partial a} \right\}_{a \in Q_1} \right)$$

where $\frac{\partial \tilde{p}}{\partial a} = b$ if $p = ab$ and zero otherwise. The Jacobi algebra, denoted A for brevity below, has a natural self-dual complex:

$$\begin{aligned} \bigoplus_{i \in Q_0} e_i A \otimes A e_i &\xrightarrow{(a \otimes 1 - 1 \otimes a)_{a \in Q_1}} \bigoplus_{a \in Q_1} e_{s(a)} A \otimes A e_{t(a)} \\ &\xrightarrow{\left(\frac{\partial \tilde{\omega}}{\partial a \partial b} \right)_{a, b \in Q_1}} \bigoplus_{a \in Q_1} e_{s(a)} A \otimes A e_{t(a)} \xrightarrow{(a \otimes 1 - 1 \otimes a)_{a \in Q_1}} \bigoplus_{i \in Q_0} e_i A \otimes A e_i. \end{aligned}$$

If this complex is a resolution, then one can use it to show A is 3-Calabi-Yau. That is, one can show $\text{Ext}_{A \otimes A^{op}}^3(A, A \otimes A^{op}) \cong A[-3]$ by using this projective resolution to resolve A on both sides, together with the observation that this resolution is self-dual.

*A Serre functor on a category $\mathcal{C}$ is an autoequivalence, S , on objects of $\mathcal{C}$ defined such that $\text{Hom}_{\mathcal{C}}(X, Y) \cong \text{Hom}_{\mathcal{C}}(Y, S(X))^*$.

Example 0.2.13. If Q is the quiver with a single vertex and three arrows, x, y, z , then $kQ = k\langle x, y, z \rangle$. Let $\omega = xyz - yxz$ so $\tilde{\omega} = xyz + zxy + yzx - yxz - zyx - xzy$. The partial derivatives are:

$$\frac{\partial \tilde{\omega}}{\partial x} = yz - zy \quad \frac{\partial \tilde{\omega}}{\partial y} = zx - xz \quad \frac{\partial \tilde{\omega}}{\partial z} = xy - yx$$

so the Jacobi algebra $J(Q, \omega) = k[x, y, z]$, the polynomial algebra in three variables. The self-dual complex in this case agrees with a bimodule version of the Koszul complex:

$$k[x, y, z] \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} k[x, y, z]^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}} k[x, y, z]^{\oplus 3} \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} k[x, y, z]$$

obtained by tensoring the Koszul complex by $- \otimes_k k[x, y, z]$. From this perspective, the self-duality of the bimodule resolution comes from a Frobenius structure on the Koszul dual algebra $\bigwedge^\bullet(k_{x,y,z}^3)$.

Davison showed that the group algebra of the fundamental group of a hyperbolic manifold of dimension at least two is Calabi-Yau but does not arise from a superpotential, see¹⁹. His key observation is that Calabi-Yau algebras coming from a superpotential satisfy a stronger condition, called *exact* Calabi-Yau, which implies the map η which can be viewed as an element of $\mathrm{HH}_d(A, A)$ lies in the image of the Connes map from cyclic homology.

Theorem 0.2.14. $\Pi(Q)$ is 2-Calabi-Yau for Q non-Dynkin.

Proof. (Sketch, see Etingof and Eu's paper for a complete proof²⁸)

Define $V := kQ_1$ and $R := \mathrm{Span}\{\sum_{a \in Q_1} [a, a^*]\}$, so $\Pi(Q) = T_{kQ_0}(V)/(R)$. Then one has an $\Pi(Q)$ -bimodule complex[†] given by

$$\Pi(Q) \otimes R \otimes \Pi(Q) \xrightarrow{\alpha} \Pi(Q) \otimes V \otimes \Pi(Q) \xrightarrow{\beta} \Pi(Q) \otimes \Pi(Q) \xrightarrow{\mu} \Pi(Q)$$

where

$$\alpha \left(x \otimes \sum_{a \in Q_1} [a, a^*] \otimes y \right) = \sum_{a \in Q_1} xa \otimes a^* \otimes y - x \otimes a \otimes a^*y - xa^* \otimes a \otimes y + x \otimes a^* \otimes ay$$

[†]This complex is the Koszul complex, a subcomplex of the bar resolution, which is itself a resolution precisely for Koszul algebras.

and

$$\beta(x \otimes v \otimes y) = xv \otimes y - x \otimes vy \quad \mu(x \otimes y) = xy,$$

and all tensor products are taken over kQ_0 . This complex is a resolution, i.e. exact, if and only if α is injective. And this complex is clearly self-dual, so the 2-Calabi-Yau property follows from the injectivity of α .

This boils down to an elementary but difficult question about non-trivial elements of the preprojective algebra: can they be extended to longer non-trivial elements. A path that cannot be extended necessarily lies in the kernel of α . So in the Dynkin case α is not injective as $\Pi(Q)$ is finite-dimensional, see Theorem 0.3.6. Conversely, if Q is non-Dynkin then we claim α is injective and will outline an inductive proof given in Etingof and Eu²⁸.

Crucially, any non-Dynkin quiver, Q , contains an extended Dynkin quiver, Q_E , from which one gets a decomposition $kQ \cong kQ_E *_{kQ_0} k(Q \setminus Q_E)$. But already in the example of the Jordan quiver sitting inside the two-loop quiver, one can see the relationship on the level of preprojective algebras is more subtle: $\Pi(Q_E)$ is not a subspace of $\Pi(Q)$ but rather a subquotient. Define a filtration on $\Pi(Q)$ by putting arrows in $\overline{Q}_E$ in degree one and all other arrows in degree zero. Then the associated graded with respect to this filtration:

$$\text{gr}(\Pi(Q)) \cong \Pi(Q_E) *_{kQ_0} \Pi(Q \setminus Q_E, \partial)$$

where $\partial \subset (Q_E)_0$ is the vertices in Q_E that have an outgoing arrow to $Q \setminus Q_E$ or an incoming arrow from $Q \setminus Q_E$. Here $\Pi(Q, J)$ for a set $J \subset Q_0$ denotes the *partial* preprojective algebra

$$\Pi(Q, J) := k\overline{Q} / \left(\left(\sum_{i \in Q_0 \setminus J} e_i \right) \cdot \sum_{a \in Q_1} [a, a^*] \right)$$

with the usual preprojective relations at the vertices in $Q_0 \setminus J$ but no relations at vertices in J . Now one can establish the injectivity of α (and the 2-Calabi-Yau property) for any quiver, using exclusively information about extended Dynkin quivers, which are well studied, and partial preprojective algebras, which are simpler. In more detail, the preprojective algebra for extended Dynkin quivers can be described as skew group algebras using the McKay correspondence¹⁵, from which one can produce a basis (or write down a Hilbert series) and check that α is injective. Heuristically, α is injective for partial preprojective algebras since any path can be extended to one starting at a vertex with no relation, by taking the direct path. We conclude that α is injective on $\text{gr}(\Pi(Q))$ and hence on $\Pi(Q)$. $\square$

Remark 0.2.15. The contrast between the Dynkin and non-Dynkin cases dissipates if

one considers a suitable dg-replacement of the preprojective algebra, which is always 2-Calabi-Yau. The general idea here is to consider the semi-free algebra $k\overline{Q} *_k k[t]$ with t in degree one and the differential defined on generators by $d(t) = \sum_{a \in Q_1} [a, a^*]$ and $d(a) = 0 = d(a^*)$. It is easy to see that the homology of this algebra is the ordinary preprojective algebra, as $\ker(d) = k\overline{Q}$ and $\operatorname{im}(d) = (\sum_{a \in Q_1} [a, a^*])$, the two-sided ideal generated by the preprojective relation.

0.3 TECHNIQUES

Our main theorems involve computing bases for both multiplicative and degenerate preprojective algebras. In this pursuit, we employ two main techniques liberally: the Diamond Lemma and the upper-semicontinuity of deformations.

0.3.1 DIAMOND LEMMA

Algebras often arise in practice through *presentations*, i.e. quotients of a free algebra $T_k(V)/(R)$ with generators in V and relations in R , or *representations*, i.e. as subsets of $\operatorname{End}_k(V)$ for some V . For instance, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is defined to be a quotient

$$\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g}) / (x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}).$$

The Weyl algebra, denoted $\operatorname{Weyl}(\mathbb{A}_k^1)$, is often defined through an action of $k\langle x, y \rangle$ on the vector space of polynomials $k[x]$, with x acting by left multiplication by x and y acting by differentiation $\partial/\partial x$. The power rule says $\partial/\partial x \circ x - x \circ \partial/\partial x$ acts as the identity, so one obtains a presentation $\operatorname{Weyl}(\mathbb{A}_k^1) \cong k\langle x, y \rangle / (xy - yx - 1)$.

While presentations and representations provide an explicit description of an algebra, often faithful to its geometric origins, certain properties are harder to ascertain. For example, how does one determine when such an algebra is the zero algebra? A more refined question, at the heart of this thesis, is the following:

Given an presentation of an algebra $A \cong T_k(V)/(R)$, how does one compute $\dim_k(A)$?

In practice, our algebras will come equipped with an additional grading, $A \cong \bigoplus_{n \in \mathbb{N}} A_n$, in

which case one would like to compute the Hilbert series, defined by

$$h_t(A) := \sum_{n \geq 0} \dim(A_n) t^n.$$

In the presence of additional decompositions, such as $A = \sum_{i,j} e_i A e_j$, as is the case for algebras arising from quivers, one can define a matrix-valued Hilbert series, where the (i, j) th-entry is the Hilbert series $h_t(e_i A e_j)$.

The key idea is the Diamond Lemma for ring theory,³ which produces a k -linear decomposition:

$$T_k(V) \cong A \oplus (R)$$

by identifying $A \cong T(V)_{\text{irr}} \subset T(V)$ as a k -linear *subset* rather than a quotient of $T(V)$. Before explaining the general theory, here are a few examples of this procedure in practice:

- Example 0.3.1.** (1) $k[x]/(x^2)$ can be viewed as the subset of linear polynomials in $k[x]$. The multiplication of two such polynomials is again a linear polynomial after truncating the quadratic term. This realizes $k[x] \cong k[x]/(x^2) \oplus (x^2)$, $p(x) \mapsto (q(x), p(x) - q(x))$ where $q(x) \equiv p(x)$ modulo (x^2) . In general, in the one-variable case $k[x]/(p(x))$, one can perform the Euclidean algorithm to rewrite any $a(x) \in k[x]$ as $a(x) = p(x)q(x) + r(x) \in (p(x)) \oplus k[x]/(p(x))$.
- (2) The polynomial algebra $k[x_1, x_2, \dots, x_n]$ can be viewed as a subset of non-commutative polynomials $k\langle x_1, x_2, \dots, x_n \rangle$ where each monomial occurs “in order” $x_1 < x_2 < x_3 < \dots < x_n$. So,

$$k\langle x_1, x_2, \dots, x_n \rangle \cong [x_1, x_2, \dots, x_n] \oplus (x_i x_j - x_j x_i \mid 1 \leq i, j \leq n)$$

where $x_2 x_1$ is sent to $x_1 x_2 + (x_2 x_1 - x_1 x_2)$, for instance.

Roughly speaking, in these examples we define an ordering on the set of non-commutative monomials in $T_k(V)$. Then we read the relation as a prescription to rewrite the leading monomial of the relation in terms of strictly smaller monomials. Then we extend this to be the identity on all other monomials to get a reduction: $\langle x_1, x_2, \dots, x_n \rangle \rightarrow k\langle x_1, x_2, \dots, x_n \rangle$. Now an element is *irreducible* if every reduction acts as the identity and we write $T_k(V)_{\text{irr}}$ for the k -linear subspace of k -linear combinations of irreducible monomials in $T_k(V)$. The map $\text{red} : T_k(V) \rightarrow T_k(V)_{\text{irr}}$ is the one realizing $T_k(V)_{\text{irr}}$ as a retract, and hence a direct summand, of $T_k(V)$. It is defined by taking *any* (finite) sequence of reductions resulting in an irreducible polynomial. We now see, by example, that we need not have existence nor uniqueness of such an irreducible polynomial.

Example 0.3.2. (Existence of an irreducible polynomial)

Consider $k[x]$ with the ordering on monomials $1 > x > x^2 > \dots$ and the relation $x^2 - x$ viewed as a reduction by $x \mapsto x^2$. The expression x can be reduced to x^2 then to x^3 (in two ways) then to x^4 , etc. But it will never be irreducible. One can instead consider the order $1 < x < x^2 < \dots$, which satisfies the descending chain condition, and hence every polynomial $f(x)$ is irreducible after applying $\deg(f(x)) - 1$ reductions.

Example 0.3.3. (Uniqueness of an irreducible polynomial)

Consider the algebra $k\langle x, y, z \rangle / (x^2, zy, xy - yx)$. Order the monomials in $k\langle x, y, z \rangle$ by

$$x < y < z < x^2 < xy < xz < yx < y^2 < yz < zx < \dots$$

The leading terms of the relations are x^2, zy , and yx , giving reductions $r_1 : x^2 \mapsto 0$, $r_2 : zy \mapsto 0$ and $r_3 : yx \mapsto xy$. Now one has two choices for how to reduce the term zyx . Unlike for the term zyx^2 , which also can be reduced in two ways, the reductions *overlap* so one cannot apply both reductions. Indeed the two reductions give rise to different irreducible expressions: 0 and zxy . The map $\text{red} : \langle x, y, z \rangle \rightarrow k\langle x, y, z \rangle_{\text{irr}}$ is not well-defined. Fortunately, one can correct this problem by adding an additional reduction. One should think of this reduction as prioritizing r_2 over r_3 in this instance, by further reducing zxy to zero. In the presence of this new reduction, $r_4 : zxy \mapsto 0$, one has an additional ambiguity, since the term $zxyx$ can be reduced in two overlapping ways. But the reduction r_1 ensures both ways further reduce to zero, so the ambiguity did not persist to the reduced form. In this case, we say the ambiguity *resolves*.

The Diamond Lemma says that $\text{red} : T(V) \rightarrow T(V)_{\text{irr}}$ is well-defined if (1) every monomial reduces to an irreducible polynomial after applying *finitely many* reductions and (2) every overlapping ambiguity resolves.

In practice, we will define an ordering on monomials in $T(V)$ through a disorder function taking values in $\mathbb{N}^n$ for some n . In this case, the descending chain condition on $\mathbb{N}^n$ with the lexicographical ordering (i.e. $(a, b) < (c, d)$ if $a < c$ or $a = c$ and $b < d$), ensures that (1) is satisfied. Even restricting to lexicographical orderings on n letters, one has $n!$ choices, and some choices are better than others, as we now demonstrate.

Example 0.3.4. Consider a modification of the above example where $A = k\langle x, y, z \rangle / (zy, xy - yx)$, with again the ordering $x < y < z$ and the reductions:

$$r_2 : zy \mapsto 0 \quad r_3 : yx \mapsto xy \quad r_4 : zxy \mapsto 0$$

Now the overlap ambiguity $zxyx \mapsto 0$ under r_4 and the irreducible monomial zx^2y under

r_3 . So one defines an additional reduction $r_5 : zx^2y \mapsto 0$ so that $zxyx$ now uniquely reduces to zero. But r_5 creates the overlap zx^2yx which maps to zero under r_5 but zx^3y under r_3 . Now you probably see the pattern. One needs to introduce the reduction $r_{n+3} : zx^ny \mapsto 0$. And indeed after appending infinitely many new reductions, now every expression uniquely reduces. However, as we move to more complicated examples, we need these calculations to be as efficient as possible in order to stay manageable. So in this case, one can simply change the order to $z < y < x$, so that the relation $xy - yx$ now has leading term xy and hence the reduction $r_3 : xy \mapsto yx$. In this case zx^ny can be unambiguously reduced to zyx^n and then to zero. So now the reductions r_2 and r_3 are sufficient for each monomial to reduce to a unique irreducible polynomial.

Throughout this thesis we will continue to refine this technique: using it in the setting of modules, using it in tandem with deformation theoretic techniques, and using it in the presence of additional decompositions. But for the reader uninterested in our chosen applications, we wish to emphasize that this is a powerful tool. As a proof of concept, we prove the PBW theorem.

Theorem 0.3.5. (*Poincaré-Birkhoff-Witt*)

Let $\mathfrak{g}$ be a k -vector space with bilinear form $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow k$. Define the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ to be the quotient $\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g}) / (x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g})$. Then the Hilbert series $h_{\text{Sym}(\mathfrak{g})}(t)$ and $h_{\text{gr}(\mathfrak{U}(\mathfrak{g}))}(t)$ agree in the linear term if and only if $[-, -]$ is a Lie bracket, in which case

$$h_{\text{Sym}(\mathfrak{g})}(t) = \frac{1}{(1-t)^{\dim(\mathfrak{g})}} = h_{\text{gr}(\mathfrak{U}(\mathfrak{g}))}(t).$$

Proof. We can rephrase the above theorem to say that the PBW map, defined by the factoring of the quotient map through $\text{Sym}(\mathfrak{g})$:

$$\begin{array}{ccccc} T(\mathfrak{g}) & \xrightarrow{\pi} & \mathfrak{U}(\mathfrak{g}) & \longrightarrow & \text{gr}(\mathfrak{U}(\mathfrak{g})) \\ & \searrow & & \nearrow \text{PBW} & \\ & & \text{Sym}(\mathfrak{g}) & & \end{array}$$

is surjective and an isomorphism if $[-, -]$ is a Lie bracket on $\mathfrak{g}$.

First notice that for any pair $x, y \in \mathfrak{g}$,

$$[x, y] + [y, x] := (x \otimes y - y \otimes x) + (y \otimes x - x \otimes y) = 0$$

so either $[-, -]$ is skew-symmetric or every element of $\mathfrak{g}$ that can be written as a sum

$[x, y] + [y, x]$ is zero in $\mathfrak{U}(\mathfrak{g})$.

Next pick a basis $\{x_1, x_2, \dots, x_n\}$ of $\mathfrak{g}$ and consider the lexicographical ordering with $x_i \leq x_j$ if $i \leq j$. Then each relation $x_j \otimes x_i - x_i \otimes x_j = [x_j, x_i]$ is read as a reduction,

$$r_{ji} : x_j \otimes x_i \mapsto x_i \otimes x_j - [x_j, x_i]$$

for $1 \leq i < j \leq n$. Now observe that given $i < j < k$, one can consider the cubic term $x_k \otimes x_j \otimes x_i$, which can be reduced using either r_{ji} or r_{kj} . Performing each reduction and then further reducing gives the polynomials,

$$\begin{aligned} x_k \otimes x_j \otimes x_i &= x_k \otimes (x_i \otimes x_j + [x_j, x_i]) = (x_i \otimes x_k + [x_k, x_i]) \otimes x_j + x_k \otimes [x_j, x_i] \\ &= x_i \otimes (x_j \otimes x_k + [x_k, x_j]) + [x_k, x_i] \otimes x_j + x_k \otimes [x_j, x_i] \\ &= x_i \otimes x_j \otimes x_k + x_i \otimes [x_k, x_j] + [x_k, x_i] \otimes x_j + x_k \otimes [x_j, x_i] \end{aligned}$$

and

$$\begin{aligned} x_k \otimes x_j \otimes x_i &= (x_j \otimes x_k + [x_k, x_j]) \otimes x_i = x_j(x_i \otimes x_k + [x_k, x_i]) + [x_k, x_j] \otimes x_i \\ &= (x_i \otimes x_j + [x_j, x_i]) \otimes x_k + x_j \otimes [x_k, x_i] + [x_k, x_j] \otimes x_i \\ &= x_i \otimes x_j \otimes x_k + [x_j, x_i] \otimes x_k + x_j \otimes [x_k, x_i] + [x_k, x_j] \otimes x_i, \end{aligned}$$

which differ by $[x_i, [x_j, x_k]] + [x_k, [x_i, x_j]] + [x_j, [x_k, x_i]]$. In the presence of the Jacobi identity, this term is zero, and the two ways of reducing $x_k \otimes x_j \otimes x_i$ agree. This implies any reordering of higher order polynomials produces the same result, and the PBW map is an isomorphism. And $\mathfrak{U}(\mathfrak{g})$ has a basis $\{x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \mid m_i \in \mathbb{N}\}$. Without the Jacobi identity, this quotient kills any element in the Lie algebra of the form $[x_i, [x_j, x_k]] + [x_k, [x_i, x_j]] + [x_j, [x_k, x_i]]$ and so $\mathfrak{g}$ no longer sits inside $\mathfrak{U}(\mathfrak{g})$. $\square$

Theorem 0.3.6. $\Pi(Q)$ is finite-dimensional for Q ADE Dynkin.

Proof. Write $Q_{p,q,r}$ for the quiver with three branches of length $p-1, q-1$, and $r-1$ with $p \leq q \leq r$. Notice that the Dynkin quivers satisfy $1/p + 1/q + 1/r > 1$ and hence include two infinite families $(1, q, r)$, type A_{q+r-1} , $(2, 2, r)$, type D_{r+2} , and $(2, 3, r)$ with $r = 3, 4, 5$, types E_6, E_7 , and E_8 .

Since there are finitely many paths of each length, it suffices to show that arbitrary large paths are zero. Such paths are equal to paths passing through the central vertex twice and hence can be written as a composition $\alpha\beta\gamma$ with $\beta \in e_v \Pi(Q_{p,q,r}) e_v$ with v central. So it suffices to show $e_v \Pi^{\geq N}(Q_{p,q,r}) e_v = 0$ for some large N . Next observe that the relations

in the preprojective algebras in for $Q_{p,q,r}$ imply

$$A_{p,q,r} := e_v \Pi(Q_{p,q,r}) e_v \cong k\langle x, y, z \rangle / (x^p, y^q, z^r, x + y + z) \cong k\langle x, y \rangle / (x^p, y^q, (x + y)^r).$$

If $p = 1$, then $x = 0$ and so $A_{1,q,r} \cong k[x]/(x^q)$ is q -dimensional. If $p = 2$ and $q = 2$ then $A_{2,2,r}$ is a quotient of the algebra of alternating words $k\langle x, y \rangle / (x^2, y^2)$, where the two words of length r are equal. Hence this algebra is $2r$ -dimensional. Since $\dim A_{2,3,3} < \dim A_{2,3,4} < \dim A_{2,3,5}$, it suffices to show $A_{2,3,5}$ is finite-dimensional.

We have the system of reductions for $A_{2,3,5}$ under the lexicographical ordering with $x < y$:

$$\begin{aligned} r_1(x^2) &= 0, \quad r_2(y^3) = 0, \quad r_3(y^2xy^2) = -xyxyx - xyxy^2 - xy^2xy - yxyxy - yxy^2x - y^2xyx \\ r_4(y^2xyxy) &= -yxy^2xy - yxyxy^2 + xyxy^2x + xy^2xyx, \quad r_5((yx)^5y) = (xy)^5x. \end{aligned}$$

Reductions 1 and 2 say that reduced words are alternating in x and either y or y^2 . Reductions 3 and 4 say that y^2 can only appear in the last 6 letters of a reduced word, and consequently $A_{2,3,5}$ is finite-dimensional over the subspace of alternating words in x and y . Finally, Reduction 5 says that this subspace of alternating words is at most 22-dimensional since $(yx)^6 = 0 = (xy)^6$. This completes the proof. $\square$

Remark 0.3.7. $\Pi(Q)$ is infinite-dimensional for all connected non-Dynkin quivers. The key idea is that any larger quiver contains an extended Dynkin subquiver, so it suffices to show $\Pi(Q)$ is infinite-dimensional in the extended Dynkin setting. In the three (or more) branch case, the length four path xy is not nilpotent while in the cycle (i.e. $\tilde{A}_n$) case, no clockwise cycle is nilpotent.

0.3.2 DEFORMATION THEORY AND FROBENIUS ALGEBRAS

The second technique exploited throughout the thesis is the deformation theory of associative algebras, and the corresponding theory for algebras with additional structures. The fundamental idea here is introduce a topology to the (usually discrete) set of n -dimensional k -algebras. After which, one can speak of the notion of limits of algebras and exploit relationships among algebras and their limits.

As a first pass (and computationally manageable entry point) we will consider *filtered* deformations. Recall an $\mathbb{N}$ -graded algebra A is one with a vector space decomposition $A = \bigoplus_{n \in \mathbb{N}} A_n$ such that $A_i \cdot A_j \subset A_{i+j}$.[‡] A filtered algebra A is an algebra equipped with a

[‡]One often speaks of an algebra as graded, without specifying the grading. Indeed given two choices of $\mathbb{N}$ -gradings $\bigoplus A_n = A = \bigoplus B_m$, then $A = \bigoplus_{n,m} A_n \cap B_m$ is an $\mathbb{N}^2$ -grading, so either one grading is a

sequence of inclusions $A_{\leq 0} \subset A_{\leq 1} \subset A_{\leq 2} \subset \cdots \subset A_{\leq n} \subset \cdots$ such that $A_{\leq n} A_{\leq m} \subset A_{\leq n+m}$.

A graded algebra $A = \bigoplus_n A_n$ has the natural filtration $A_{\leq m} := \bigoplus_{n \leq m} A_n$. Conversely, given a filtered algebra A with filtration $\mathcal{F} = A_{\leq 0} \subset A_{\leq 1} \subset \cdots \subset A_{\leq n} \subset \cdots$ one can form the associated graded algebra

$$\text{gr}(A) := \bigoplus_{n \in \mathbb{N}} \frac{A_{\leq n}}{A_{\leq n-1}}$$

where $A_{\leq -1} = 0$. We will refer to A as a filtered deformation of $\text{gr}(A)$. There is a natural surjection, $A \rightarrow \text{gr}(A)$, which implies, if A is finite-dimensional, that $\dim_k(A) \geq \dim(\text{gr}(A))$. This idea is essential to this thesis: we can bound the dimension of an algebra from below using the dimension of its associated graded algebra. If the dimensions are equal, then we say A is a *flat* filtered deformation of $\text{gr}(A)$.

An algebra can have many (non-isomorphic) filtrations, which give rise to (non-isomorphic) deformations, as we now show.

Example 0.3.8. Consider the following filtrations of the algebra $\mathbb{C}^4$ (with pointwise multiplication) where $\langle x \rangle$ denotes the $\mathbb{C}$ -span of $x \in \mathbb{C}^4$, 1 denotes the unit $(1, 1, 1, 1) \in \mathbb{C}^4$, and ζ_n denotes a primitive n th root of unity:

- (1) $\mathcal{F}_1 = \{0\} \subset \mathbb{C}^4$
- (2) $\mathcal{F}_2 = \{0\} \subset \langle 1 \rangle \subset \langle 1, (1, -1, 1, 1) \rangle \subset \mathbb{C}^4$
- (3) $\mathcal{F}_3 = \{0\} \subset \langle 1 \rangle \subset \langle 1, (\zeta_3, \zeta_3^2, 1, 1) \rangle \subset \langle 1, (\zeta_3, \zeta_3^2, 1, 1), (\zeta_3^2, \zeta_3, 1, 1) \rangle \subset \mathbb{C}^4$
- (4) $\mathcal{F}_4 = \{0\} \subset \langle 1 \rangle \subset \langle 1, (1, -1, 1, 1), (1, 1, 1, -1) \rangle \subset \mathbb{C}^4$
- (5) $\mathcal{F}_5 = \{0\} \subset \langle 1 \rangle \subset \langle 1, (\zeta_4, \zeta_4^2, \zeta_4^3, 1) \rangle \subset \mathbb{C}^4$
- (6) $\mathcal{F}_6 = \{0\} \subset \langle 1 \rangle \subset \{(x_i) \in \mathbb{C}^4 : \sum x_i = 0\} \subset \mathbb{C}^4$
- (7) $\mathcal{F}_7 = \{0\} \subset \langle 1 \rangle \subset \mathbb{C}^4$

Now,

- (1) $\text{gr}(\mathbb{C}^4, \mathcal{F}_1) \cong \mathbb{C}^4$
- (2) $\text{gr}(\mathbb{C}^4, \mathcal{F}_2) \cong \mathbb{C}^2 \oplus \mathbb{C}[x]/(x^2)$
- (3) $\text{gr}(\mathbb{C}^4, \mathcal{F}_3) \cong \mathbb{C} \oplus \mathbb{C}[x]/(x^3)$

refinement of the other, or both are refinable.

$$(4) \text{ gr}(\mathbb{C}^4, \mathcal{F}_4) \cong \mathbb{C}[x]/(x^2) \oplus \mathbb{C}[y]/(y^2)$$

$$(5) \text{ gr}(\mathbb{C}^4, \mathcal{F}_5) \cong \mathbb{C}[x]/(x^4)$$

$$(6) \text{ gr}(\mathbb{C}^4, \mathcal{F}_6) \cong \mathbb{C}[x, y]/(x^2, y^2)$$

$$(7) \text{ gr}(\mathbb{C}^4, \mathcal{F}_7) \cong \mathbb{C}[x, y, z]/(x, y, z)^2$$

The main difficulty in constructing these examples lies in the identification, $\mathbb{C}[x]/(x^n - 1) \cong \mathbb{C}^n$ sending x to $(\zeta_n^i)_{i=1, \dots, n}$, for $n = 2, 3, 4$.

In Chapter 1, we will largely work in the setting of Frobenius algebras.

Definition 0.3.9. A Frobenius algebra is a triple (A, μ, λ) where A is a vector space, $\mu : A \otimes A \rightarrow A$ is a linear map making (A, μ) an associative algebras, and $\lambda : A \rightarrow k$ is a non-degenerate form, in the sense that if $\ker(\lambda) \subset I$ a left ideal, then $I = \{0\}$ is the zero ideal.

Proposition 0.3.10. *The following data are equivalent:*

(1) (A, μ, λ) a Frobenius algebra

(2) $(A, \mu, (-, -))$ with $(-, -)$ a non-degenerate bilinear form satisfying associativity, i.e. $(ab, c) = (a, bc)$ for all $a, b, c \in A$

(3) (A, μ, Φ) with $\Phi : A \rightarrow \text{Hom}_k(A, k)$ an isomorphism of left A -modules

Proof. (1) $\iff$ (2) Given a Frobenius algebra build the form $(a, b) := (\lambda \circ \mu)(a, b)$ for all $a, b \in A$. We need to show this form is associative and non-degenerate. Using associativity of the multiplication,

$$(ab, c) := \lambda \circ \mu(\mu(a, b), c) = \lambda \circ \mu(a, \mu(b, c)) =: (a, bc).$$

If $(a, b) = 0$ is zero for all $a \in A$, then

$$0 = (a, b) = (1, ab) = \lambda(ab)$$

and hence λ vanishes on the left ideal generated by b , which implies $b = 0$, and hence $(-, -)$ is right non-degenerate. Left non-degeneracy follows similarly from the fact that λ

doesn't vanish on non-trivial right ideals. Since if λ vanished on the right ideal generated by $b \in A$ then,

$$0 = \lambda(ba) = (1, ba) = (b, a) = (a, \eta(b)) = (a\eta(b), 1) = \lambda(a\eta(b))$$

so λ vanishes on the left ideal generated by $\eta(b) \in A$. Here $\eta : A \rightarrow A$ is the algebra automorphism defined through the equality of linear functionals $(b, -) \equiv (-, \eta(b))$, known as the Nakayama automorphism. Conversely, one can show that $\lambda := (-, 1)$ is non-degenerate following from the non-degeneracy of $(-, -)$.

$$(1) \iff (3)$$

Given Φ define $\lambda := \Phi(1)$, and assume it vanishes on a right ideal I . If $a \in I$ then

$$0 = \lambda(a \cdot -) = a \cdot \lambda(-) = a \cdot \Phi(1) = \Phi(a)$$

hence the injectivity of Φ implies that $I = \{0\}$ and λ is (right) non-degenerate. Conversely, one can build Φ uniquely from the conditions that $\Phi(1) = \lambda$ and Φ is a left A -module map. $\square$

Definition 0.3.11. A *symmetric* Frobenius algebra is a Frobenius algebra (A, λ, μ) such that

$$\lambda \circ \mu(a, b) = \lambda \circ \mu(b, a)$$

for all $a, b \in A$.

Remark 0.3.12. In terms of the equivalent structures $(-, -)$ and Φ from Proposition 0.3.10, symmetric Frobenius algebras have $(-, -)$ symmetric and Φ a *bimodule*-isomorphism.

Remark 0.3.13. If there is ever a time to listen to the homotopy theorists, it is with the careful distinction between a structure and a condition. The unit in an algebra is a structure, while associativity is a condition. For the homotopy theorists everything is thought of as a structure and if morphisms don't preserve a given structure then the choice is irrelevant, and the "structure" plays more like the existence of such a structure, which is a condition. For our purposes, the key point here is that the functor $Forget : Frob \rightarrow Ass$ taking a Frobenius algebra (with structure λ) $(A, \mu, \lambda) \rightarrow (A, \mu)$ to an associative algebra possessing a Frobenius form is an equivalence of categories. So it is purely stylistic to think about λ as a fixed structure when the morphisms are not altered to preserve it.

Example 0.3.14. Here are my five favorite classes of examples of Frobenius algebras:

- (1) Given a group G with identity $e \in G$, the group algebra $k[G]$ is a Frobenius k -algebra with Frobenius form $\lambda(\sum \alpha_g g) = \alpha_e$.
- (2) Given a compact oriented manifold M , the cohomology $H^*(M, \mathbb{Q})$ with the cup product is a Frobenius $\mathbb{Q}$ -algebra with Frobenius form given by projecting onto $H^{\dim(M)}(M, \mathbb{Q}) \cong \mathbb{Q}_{[M]}$. Using $\mathbb{Z}/2\mathbb{Z}$ instead of $\mathbb{Q}$ (or taking values in the orientation sheaf instead of the constant sheaf) allows one to drop the oriented assumption.
- (3) Any matrix algebra can be made into a Frobenius algebra using the usual trace. Since a sum of Frobenius algebras is a Frobenius algebra by adding the forms and finite products agree with finite sums, every finite product of matrix algebras is Frobenius. In particular, by the Artin-Wedderburn theorem, every finite-dimensional semisimple algebra over an algebraically closed field is Frobenius.
- (4) Given an ADE quiver, Q , one can form the preprojective algebra as a quotient

$$k\overline{Q}/\sum_{\alpha \in Q_1} [\alpha, \alpha^*].$$

These algebras are finite-dimensional and in fact Frobenius, with form given by projecting onto a sum of maximum length paths.

- (5) Given a vector space V and a non-degenerate bilinear form $(-, -) : V \otimes V \rightarrow k$, one can encode this form as an algebra on the underlying vector space $V \oplus k$. That is, define $A(V, (-, -)) := k \oplus V \oplus k$ with multiplication

$$(a, b, c) \cdot (a', b', c') = (aa', ab' + ba', ac' + c'a + (b, b'))$$

and form given by projecting onto the second factor of k .

How many Frobenius forms can one put on an associative algebra? The set of Frobenius forms is a subset of the dual space $\text{Hom}_k(A, k)$ and when A has only the left ideals $\{0\}$ and A , e.g. if A is a field extension of k , then the Frobenius condition on the form is vacuous and every element of $\text{Hom}_k(A, k)$ is a Frobenius form.

At the other extreme, many associative algebras have no Frobenius forms. Forgetting the algebra structure, the existence of an isomorphism $A \cong \text{Hom}_k(A, k)$ in definition (3) implies that A is finite-dimensional. Remembering the algebra structure, since every algebra A is self-projective (i.e. projective as an A -module) one has that Frobenius algebras are self-injective (i.e. injective as an A -module.) This can be expressed as the vanishing of $\text{Ext}_A^i(-, A) = 0$ for $i \geq 0$, and hence includes semisimple algebras.

Provided (A, μ) has a single Frobenius form $\lambda : A \rightarrow k$, then precomposition by R_u , i.e. right multiplication by a unit $u \in A^\times$, produces another Frobenius form $\lambda \circ R_u$. In fact, every two Frobenius forms can be related in this way. Using definition (3) one can relate Φ_1 and Φ_2 by observing $\Phi_1 \circ \Phi_2^{-1} : A \rightarrow A$ is determined by the image of 1, which lies in $A^\times$. The (non-empty) set of Frobenius forms is an $A^\times$ -torsor. Note that non-invertible elements $a \in A$ satisfy $\det(L_a) = 0$ and hence form a closed subset of A . Over a field of characteristic zero, one has a large supply of invertible elements by taking a neighborhood around the identity element, and consequently a large supply of Frobenius forms. However, as the next example illustrates, there exists an algebras with only one or two Frobenius forms.

Example 0.3.15. Consider the two algebras: $\mathbb{F}_2[x]/(x^2)$ and $\mathbb{F}_2 \oplus \mathbb{F}_2$.

- $\mathbb{F}_2[x]/(x^2)$ has non-trivial left ideal $(x) = \{0, x\}$ and hence any Frobenius form, λ , determined by its value on $\{1, x\}$, must have $\lambda(x) \neq 0$ and hence $\lambda(x) = 1$. Therefore, such a form is determined by its value on 1, and hence the space of Frobenius forms can be identified with $\mathbb{F}_2$ by the map $\lambda \mapsto \lambda(1)$. Notice that the two forms differ by right multiplication by the unit $1 + x$.
- $\mathbb{F}_2 \oplus \mathbb{F}_2$ has non-trivial left ideals generated by the elements $(1, 0)$ and $(0, 1)$. Hence there is a unique Frobenius form, taking $(0, 1)$ and $(1, 0)$ to one and $(0, 0)$ and $(1, 1)$ to zero. The only unit here is the identity.

A large source of Frobenius algebras arise as filtered deformations of Frobenius algebras. Indeed if B is a filtered deformation of a finite-dimensional algebra $A = \bigoplus_{i=0}^n A_i$, then a Frobenius form on A induces a Frobenius form on B . To see this, observe $\lambda : A \rightarrow k$ non-degenerate implies $\ker(\lambda) \not\subset A_n \cong B/B_{\leq(n-1)}$. Hence the lift $\tilde{\lambda} : B \rightarrow k$ to B by defining $\tilde{\lambda}(B_{\leq(n-1)}) = 0$ is non-degenerate.

Example 0.3.16. To illustrate the utility of this point, consider the Frobenius form on $\mathbb{C}[x]/(x^4)$ given by projecting onto the cubic term of any polynomial. This algebra deforms to $\mathbb{C}^4, \mathbb{C}^2 \oplus \mathbb{C}[x]/(x^2), \mathbb{C} \oplus \mathbb{C}[x]/(x^3)$, and $\mathbb{C}[x]/(x^2) \oplus \mathbb{C}[y]/(y^2)$ as associative algebras and hence each of these algebras inherits a Frobenius form, and is therefore Frobenius. This seems to be a somewhat new perspective on Frobenius algebras, all arise as filtered deformations of the algebras $A(V, (-, -))$ in (5). This is explained in detail in Section 4 of Chapter 1.

1

FROBENIUS DEGENERATIONS OF PREPROJECTIVE ALGEBRAS

1.1 INTRODUCTION

A quiver, coined by Peter Gabriel, is a directed graph, whose representations are defined by placing a vector space at each vertex and a linear transformation at each arrow. Gabriel classified connected quivers with finite representation type: these are the quivers whose underlying graph is given by the simply-laced Dynkin diagrams, two infinite families $A_n, n \geq 1$ and $D_n, n \geq 4$, and three exceptions E_6, E_7, E_8 ³². Subsequently, Dlab and Ringel proved a more general result for “species”, quivers labelled with a division ring at each vertex and a bimodule at each edge. They extended Gabriel’s classification by showing that species with finite representation type are precisely the Dynkin ones²¹. This chapter explores a further generalization to decorated quivers – quivers with vertices labelled by k -algebras and arrows labelled by bimodules. An original motivation for this work is²⁰, which in our language considers the affine D_4 case, while we focus on the Dynkin case.

One approach to representations of quivers involves studying the variety of all such representations and considering its cotangent bundle. More precisely, fixing a field k , a quiver Q with vertices Q_0 and arrows Q_1 , and a dimension vector $d = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$, define $\text{Rep}_d(Q)$ to be the vector space of k -representations of Q with vertex i labelled by the vector space k^{d_i} . There is a natural action of the group $GL_d(k) := \prod_{i \in Q_0} GL_{d_i}(k)$ on

$\text{Rep}_d(Q)$. Letting $\mathfrak{gl}_d(k) := \oplus_{i \in Q_0} \mathfrak{gl}_{d_i}(k)$ denote the Lie algebra of $GL_d(k)$, one studies the moduli space

$$\mathcal{M}_d := T^*\text{Rep}_d(Q) // GL_d(k) := \mu^{-1}(0) // GL_d(k)$$

where $\mu : T^*\text{Rep}_d(Q) \rightarrow \mathfrak{gl}_d(k)^*$ is the moment map for the associated action.

The preprojective algebra gives an algebraic construction of $\mathcal{M}_d$. First, one defines the double quiver $\overline{Q}$ to have the same vertices as Q but with a reverse arrow $\alpha^* \in \overline{Q}_1$ for each $\alpha \in Q_1$. Then the preprojective algebra, $\Pi(Q)$ is defined as the quotient

$$\Pi(Q) := P(\overline{Q}) / \left\langle \sum_{\alpha \in Q_1} \alpha \alpha^* - \alpha^* \alpha \right\rangle$$

where $P(\overline{Q})$ is the path algebra of $\overline{Q}$ ³⁷. Then we recover $\mathcal{M}_d$ as the moduli space of representations of $\Pi(Q)$ with dimension vector d :

$$\mathcal{M}_d \cong \text{Rep}_d \Pi(Q) // GL_d(k). \quad (1.1.1)$$

Anachronistically, one can view this as a litmus test for a correct definition of a preprojective algebra. For a quiver Q together with a decoration D_Q consisting of a k -algebra at each vertex and a bimodule at each arrow, we would like a notion of preprojective algebra $\Pi(Q, D_Q)$ such that the analogue of (1.1.1) holds:

$$\text{Rep}_d \Pi(Q, D_Q) // G_d \cong \mu^{-1}(0) // G_d. \quad (1.1.2)$$

We will define an algebra $\Pi(Q, D_Q)$ and a group G_d such that (1.1.2) holds in the presence of certain restrictions on the representations and the decorations.

Remark 1.1.1. A definition of $\Pi(Q, D_Q)$ was recently given in⁴³. Thanks to Külshammer for pointing this out. Under condition (F) below this recovers our definition. Note that in⁴³, the viewpoint of representation varieties was not considered and it's an open question to what extent condition (F) can be relaxed while retaining a moment map interpretation. See Remark 1.3.14 for a more precise discussion.

In more detail, a decoration D_Q consists of

- $\{A_i\}_{i \in Q_0}$ with each A_i a k -algebra and
- $\{M^\alpha\}_{\alpha \in Q_1}$ with M^α an (A_i, A_j) -bimodule, if α is an arrow from i to j .

A decoration satisfies condition (F) if each A_i is symmetric Frobenius and, for every α , M^α is either $A_i \otimes_k A_j$, or $M^\alpha = A_i = A_j$.

A representation of a decorated quiver (Q, D_Q) with $D_Q = (A_i, M^\alpha)$ consists of

- $\{V_i\}_{i \in Q_0}$ with each V_i a *right* A_i -module and
- $\{\rho_\alpha : M^\alpha \rightarrow \text{Hom}_k(V_i, V_j)\}_{\alpha: i \rightarrow j \in Q_1, i, j \in Q_0}$ with each ρ_α an (A_i, A_j) -bimodule map.

We fix a dimension vector $d = (d_i) \in \mathbb{N}^{Q_0}$ and restrict to representations where $V_i = A_i^{d_i}$ for each $i \in Q_0$. Denote this vector space by $\text{Rep}_d(Q, D_Q)$.

We define the decorated path algebra to be $P(Q, D_Q) := T_{\oplus_i A_i}(\oplus_\alpha M^\alpha)$, so modules for the decorated path algebra are representations of the decorated quiver. We then double the quiver by adding an arrow $\alpha^* : j \rightarrow i$ for each $\alpha : i \rightarrow j \in Q_1$ and decorate the new arrow $\alpha^* : j \rightarrow i$ with the bimodule dual,

$$M^{\alpha^*} := \text{Hom}_{A_i \otimes A_j}(M^\alpha, A_i \otimes A_j).$$

A decorated preprojective algebra satisfying (1.1.2) cannot be defined in this level of generality. Hence we restrict to decorations satisfying condition (F) and representations with $V_i := A_i^{d_i}$ for each $i \in Q_0$. The decorated preprojective algebra $\Pi(Q, D_Q)$ is the quotient of the path algebra for the double quiver by the two-sided ideal generated by the element $r = \sum_{\alpha \in Q_1} r_\alpha$. Here r_α for $\alpha : i \rightarrow j$ is a commutator $[1_{M^\alpha}, 1_{M^{\alpha^*}}]$ in the case $M^\alpha = A_i = A_j$, and $\sum_l e_l^i \otimes 1 \otimes f_l^i - \sum_m e_m^j \otimes 1 \otimes f_m^j$ in the case $M^\alpha = A_i \otimes A_j$, where $\{e_l^i\}$ and $\{f_l^i\}$ are dual bases of A_i under the Frobenius form, for all $i \in Q_0$. See Definition 1.3.2 for more detail and Remark 1.3.3 for independence on the choice of basis.

There is a natural action of $G_d := \prod_i GL_{A_i}(V_i) \cong \prod_i GL_{A_i}(A_i^{d_i})$ on $\Pi(Q, D_Q)$. Denoting the corresponding Lie algebra by $\mathfrak{g}_d = \oplus_i \mathfrak{gl}_{d_i}(A_i)$ one can again consider the moment map $\mu_d : T^*\text{Rep}_d(Q, D_Q) \rightarrow \mathfrak{g}_d^*$. Finally, we can state (1.1.2) precisely as a theorem.

Theorem 1.1.2. *$\text{Rep}_d(\Pi(Q, D_Q))$ is the zero fiber of the moment map*

$$\mu_d : T^*\text{Rep}_d(Q, D_Q) \rightarrow \mathfrak{g}_d^*.$$

Moreover, the coarse moduli space of representations of $\Pi(Q, D_Q)$ of dimension vector d is given by the Hamiltonian reduction $\mu_d^{-1}(0)//G_d$.

This theorem follows from a more precise statement, Theorem 1.3.6, given in Section 3.

A second goal of this chapter is to study 1-parameter families of decorated preprojective algebras whose generic fibers are ordinary preprojective algebras. In this sense, decorated preprojective algebras are often (formal) degenerations of preprojective algebras, another motivation for their study.

This perspective is best understood in a simple example. Consider the decorated quiver

$$(Q, D_Q) = k \xleftarrow{k} k \xrightarrow{k} k.$$

Since right k -modules are vector spaces and k -bimodule maps are linear transformations, $\text{Rep}_d(Q, D_Q) \cong \text{Rep}_d(A_3)$ and $\Pi(Q, D_Q) = \Pi(A_3)$ is an ordinary preprojective algebra. View these two arrows as a single arrow into the direct sum, folding the decorated quiver as follows:

$$k \xleftarrow{k} k \xrightarrow{k} k \quad \xRightarrow{\text{fold}} \quad k \xrightarrow{k \oplus k} k \oplus k.$$

Folding the decorated quiver does not change the decorated preprojective algebra, $\Pi(A_3)$. Furthermore, $k \oplus k$ can be viewed as a 2-dimensional k -algebra with pointwise multiplication where it has a non-trivial flat degeneration to the k -algebra $S = k[x]/(x^2)$. Hence one can degenerate the decoration

$$k \xrightarrow{k \oplus k} k \oplus k \quad \xrightarrow{\text{deg}} \quad k \xrightarrow{S} S$$

which induces a degeneration of $\Pi(A_3) = \Pi(k \xrightarrow{k \oplus k} k \oplus k)$ to $\Pi(k \xrightarrow{S} S)$. We denote this degeneration by $\Pi(B_2)$ since the B_2 quiver is obtained from A_3 by folding the arrows as above.

Both the formal and filtered perspectives of this degeneration prove fruitful. In the formal setting, $k[[t]][x]/(x^2 - t)$ is an explicit formal degeneration from $k \oplus k$ to S . The corresponding formal degeneration of decorated quivers is

$$\begin{array}{ccc} & (k[[t]] \xrightarrow{k[x][[t]]/(x^2-t)} k[x][[t]]/(x^2-t)) & \\ \swarrow \text{"t=1"} & & \searrow \text{t=0} \\ (k \xrightarrow{k \oplus k} k \oplus k) & & (k \xrightarrow{S} S), \end{array}$$

which in turn gives a formal degeneration $\Pi_{k[[t]]}(k[[t]] \rightarrow k[x][[t]]/(x^2 - t))$ from $\Pi(A_3)$ to $\Pi(k \rightarrow S)$. Alternatively, viewing $k \oplus k \cong k[x]/(x^2 - 1)$ as a filtered algebra with associated graded algebra S , realizes $\Pi(A_3)$ as a filtered algebra with associated graded algebra $\Pi(k \rightarrow S)$. The filtration on $\Pi(A_3)$ is given by $\{0\} \subset \Pi(A_3)^\Phi \subset \Pi(A_3)$ where $\Phi : A_3 \rightarrow A_3$ is the graph automorphism exchanging the outer vertices and $\Pi(A_3)^\Phi$ is the

subspace of Φ -invariant elements.

This motivates the study of decorated quivers as a means of interpreting a large class of degenerations of preprojective algebras. But one should not conflate these notions. The decorated preprojective algebra $\Pi(k \rightarrow \text{Mat}_2(k))$ is *not* a degeneration of $\Pi(\tilde{D}_4)$, where $\tilde{D}_4$ is the affine D_4 quiver. Conversely, $\Pi(A_2)$ has a flat Frobenius degeneration to $\bigwedge(k^2)$, see Example 1.4.14, which is *not* a decorated preprojective algebra for any decorated quiver except $Q = A_1$ and $D_Q = (\bigwedge(k^2))$.

Most of the decorated quivers appearing in this chapter arise as degenerations of ordinary quivers by folding vertices and then degenerating, as we now explain. In general, a group H acts on a quiver Q if H acts on the sets Q_1 and Q_0 such that the source and target maps are H -equivariant. In this case, the source and target maps descend to the quotient quiver $Q/H := (Q_0/H, Q_1/H, s, t)$. In the case $H = \text{Aut}(Q)$ is the group of graph automorphisms of the underlying undirected graph of Q , we call the quotient $Q/\text{Aut}(Q)$ the folding of Q by automorphisms.

If $D_Q = (A_i, M^\alpha)$ is a decoration of Q then one can decorate $Q/\text{Aut}(Q)$ by $D_Q/\text{Aut}(Q) = (B_j, N^\beta)$ defined by:

$$B_j := \bigoplus_{i \in \text{Aut}(Q) \cdot j} A_i \quad \text{and} \quad N^\beta := \bigoplus_{\alpha \in \text{Aut}(Q) \cdot \beta} M^\alpha.$$

In this case, we say (Q, D_Q) *folds* to $(Q/\text{Aut}(Q), D_Q/\text{Aut}(Q))$. Notice that a folding of decorated quivers induces an equivalence

$$\text{Rep}(Q, D_Q) \cong \text{Rep}(Q/\text{Aut}(Q), D_Q/\text{Aut}(Q))$$

and an isomorphism

$$\Pi(Q, D_Q) \cong \Pi(Q/\text{Aut}(Q), D_Q/\text{Aut}(Q)).$$

The second aspect of this procedure is degenerating the decoration, which amounts to degenerating the algebras at each vertex and the bimodules at each arrow. Since the definition of the preprojective algebra requires each algebra to be Frobenius, one needs to degenerate the algebras in a way that preserves a Frobenius form, see Section 4 for more detail.

We suspect that these degenerations of preprojective algebras are always flat. In other words, we present the following conjecture in Section 4.

Conjecture 1.4.20. Suppose D_Q flatly Frobenius deforms to D'_Q . Then with respect to the path length grading,

$$h_{\Pi(Q, D_Q)}(t) = h_{\Pi(Q, D'_Q)}(t).$$

Remark 1.1.3. The conjecture was proven in the case $Q = \tilde{D}_4$ for *commutative* Frobenius algebras A_i in²⁰, by computing an explicit basis for the maximally folded and maximally degenerated decoration

$$\begin{array}{ccccccc} & & k & & & & \\ & & \downarrow & & & & \\ k & \longrightarrow & k & \longleftarrow & k & \xrightarrow{\text{fold}} & k^{\oplus 4} \longrightarrow k \xrightarrow{\text{deg}} k[x, y]/(xy, x^2 - y^2) \longrightarrow k. \\ & & \uparrow & & & & \\ & & k & & & & \end{array}$$

This work may be considered a sequel, in that we generalize their statement to arbitrary Frobenius decorated quivers. Moreover, we prove the conjecture in the Dynkin case.

In fact, we suspect that the existence of a deformation, while crucial for our techniques, may be an artifact of the proof style and unnecessary for the result. That is, assuming condition (F), the Hilbert–Poincaré series of the decorated preprojective algebra may depend purely combinatorially on the dimensions of the algebras and bimodules in the decoration.

One can define a (not-necessarily symmetric) Cartan matrix $2I - A$ where $A = [a_{ij}]$ is a $Q_0 \times Q_0$ matrix with entries

$$a_{ij} := \sum_{\alpha: i \rightarrow j} \frac{\dim_k(M^\alpha)}{\dim_k(A_i)} = \sum_{\alpha: i \rightarrow j} \dim_{A_i}(M^\alpha) \in \mathbb{Z}.$$

We say (Q, D_Q) is Dynkin if the Cartan matrix $2I - A$ is positive-definite, which implies that $2I - A$ is a Cartan matrix for a Dynkin diagram.

Observe, that if $D = [d_{ij}]$ with $d_{ij} := \delta_{ij} \dim_k(A_i)$ then DA is symmetric as

$$(DA)_{ij} = \sum_{\alpha: i \rightarrow j} \dim_k(M^\alpha) = \sum_{\alpha^*: j \rightarrow i} \dim_k(M^{\alpha^*}) = (DA)_{ji},$$

and therefore, from the perspective of³⁶, A is a Cartan matrix with symmetrizer D .

Conjecture 1.4.21. The Hilbert–Poincaré series of $\Pi(Q, D_Q)$, with $D_Q = (A_i, M^\alpha)$ is

given by,

$$h_{\Pi(Q, D_Q)}(t) = \begin{cases} (I + t^{\max_t + 2} P) D(1 - At + t^2)^{-1} & \text{if } (Q, D_Q) \text{ is Dynkin} \\ D(1 - At + t^2)^{-1} & \text{otherwise} \end{cases}$$

where, in the Dynkin case, $\max_t$ is the maximum path length of a non-zero homogeneous element in $\Pi(Q, D_Q)$ and P is the permutation matrix for the permutation of the set $\{1_{A_i}\}_{i \in Q_0}$ given by the Nakayama automorphism.

Moreover, we will often have an additional grading on the decorated path algebra coming from placing a graded algebra at one or more vertices. Throughout the chapter we call this the x -grading defined as the number of occurrences of x in the path, since the graded algebras we consider at the vertices are of the form $k[x]/(x^n)$ for some $n \in \mathbb{N}$. The x -grading descends from the decorated path algebra to the decorated preprojective algebra in the C_n , G_2 , and B_2 cases. The x -grading can be adjusted to descend in the B_n with $n \geq 3$, and F_4 cases. With respect to path length and an additional x -grading the Hilbert–Poincaré series is conjecturally given by

$$h_{\Pi(Q, D_Q)}(t, s) = \begin{cases} (1 + s^{\max_s} t^{\max_t + 2} P) D(1 - At + Bt^2)^{-1} & \text{if } (Q, D_Q) \text{ is Dynkin} \\ D(1 - At + Bt^2)^{-1} & \text{otherwise} \end{cases}$$

where $\max_s$ is the maximum x -graded of a non-zero homogeneous element of $\Pi(Q, D_Q)$,

$$A = [a_{ij}], \quad a_{ij} := \sum_{\alpha: i \rightarrow j} \frac{h_{M^\alpha}(s)}{h_{A_i}(s)},$$

and

$$B = [b_{ij}], \quad b_{ij} := \delta_{ij} h_{k \cdot 1_{A_i} r}(s).$$

Here we say (Q, D_Q) is *Dynkin* if, as before, the Cartan matrix $2I - A|_{s=1}$ is positive-definite.

In Sections 4 and 5 we prove the conjecture in the Dynkin case and show that the Frobenius form on the preprojective algebra degenerates to one on the decorated preprojective algebras. More precisely, we prove

Theorem 1.4.26. Let (Q, D_Q) be Dynkin. Suppose D_Q Frobenius degenerates flatly to D'_Q . Then, $\Pi(Q, D_Q)$ and $\Pi(Q, D'_Q)$ are both Frobenius algebras and $\Pi(Q, D_Q)$ Frobenius degenerates flatly to $\Pi(Q, D'_Q)$.

Moreover, letting $S := k[x]/(x^2)$ and $S' := k[x]/(x^3)$, the most degenerate decorated non-simply laced Dynkin quivers are:

$$\begin{aligned} B_n &:= (A_{n+1}, (\{k, S, \dots, S\}, \{{}_k S_{S,S} S_S, \dots, {}_S S_S\})), \quad n \geq 2 \\ C_n &:= (A_{n+1}, (\{k, \dots, k, S\}, \{{}_k k_k, \dots, {}_k k_k, {}_k S_S\})), \quad n \geq 4 \\ F_4 &:= (A_4, (\{k, k, S, S\}, \{{}_k k_k, {}_k k_S, {}_S S_S\})) \\ G_2 &:= (A_2, (\{k, S'\}, \{{}_k S'_{S'}\})). \end{aligned}$$

The (path, x)-bigraded Hilbert–Poincaré series of the corresponding decorated preprojective algebras are:

$$\begin{aligned} h_{e_i \Pi(B_n) e_j}(s, t) &= t^{|i-j|} s^{|i-j|/2} (1 + t^{2 \min\{i,j\}-2} s^{\min\{i,j\}-1}) (1 + s) \\ &\quad \cdot (1 + st^2 + s^2 t^4 + \dots + s^{n-\max\{i,j\}} t^{2n-2 \max\{i,j\}}), \quad i, j \geq 2 \\ h_{e_1 \Pi(B_n) e_j}(s, t) &= h_{e_j \Pi(B_n) e_1}(s, t) = t^{j-1} s^{(j-1)/2} (1 + s) \\ &\quad \cdot (1 + st^2 + s^2 t^4 + \dots + s^{n-j} t^{2n-2j}), \quad j \geq 2 \\ h_{e_1 \Pi(B_n) e_1}(s, t) &= 1 + st^2 + s^2 t^4 + \dots + s^{n-1} t^{2n-2} \\ h_{e_i \Pi(C_n) e_j}(s, t) &= (1 + st^{2|n-\max\{i,j\}|}) t^{|i-j|} (1 + t^2 + t^4 + \dots + t^{2 \min\{i,j\}-2}) \end{aligned}$$

$$h_{\Pi(F_4)}(s, t) = (1 + t^4 s).$$

$$\begin{pmatrix} (1+t^6 s) & t(1+t^2 s+t^4 s^2) & t^2(1+s)(1+t^2 s) & t^3 s^{1/2}(1+s) \\ t(1+t^2 s+t^4 s^2) & (1+t^2)(1+t^2 s+t^4 s) & t(1+s)(1+t^2 s) & t^2 s^{1/2}(1+s)(1+t^2) \\ t^2(1+s)(1+t^2 s) & t(1+s)(1+t^2 s) & (1+s)(1+t^2+t^4)(1+t^2 s) & t s^{1/2}(1+s)(1+t^2+t^4) \\ t^3 s^{1/2}(1+s) & t^2 s^{1/2}(1+s)(1+t^2) & t s^{1/2}(1+s)(1+t^2+t^4) & (1+s)(1+t^6 s) \end{pmatrix}$$

$$h_{\Pi(G_2)}(s, t) = (1 + t^2 s) \begin{pmatrix} 1 + t^2 s^2 & (1 + s + s^2)t \\ (1 + s + s^2)t & (1 + s + s^2)(1 + t^2) \end{pmatrix}$$

For more detail, see Section 5.

Example 1.1.4. We conclude the introduction by returning to our simple yet illustrative B_2 example, i.e., the A_2 quiver with vertices labelled by the k -algebras k and $S = k[x]/(x^2)$ and the arrow labelled by S viewed as a (k, S) -bimodule,

$$k \xrightarrow{S} S.$$

S has Frobenius form $\lambda : S \rightarrow k$ given by $s + tx \mapsto t$, which is non-degenerate since the

kernel is the subspace of constant polynomials, which does not contain any non-trivial left ideals. With respect to this form, the basis $\{1, x\}$ has dual basis $\{x, 1\}$ giving rise to the non-degenerate, S -central element $x \otimes_k 1 + 1 \otimes_k x$.

Using this structure, the decorated preprojective algebra is defined as

$$\Pi(Q, D_Q) := T_{k \oplus S}(S \oplus S) / \langle 1 \otimes_S 1, 1 \otimes_k x + x \otimes_k 1 \rangle$$

where $S \oplus S$ is made into a $(k \oplus S)$ -bimodule by viewing one S as an (S, k) -bimodule and the other a (k, S) -bimodule, and defining all other actions to be zero. More explicitly, one can see that $\Pi^{\geq 3}(Q, D_Q) = 0$ and compute,

$$\begin{aligned} \Pi(Q, D_Q) &= \Pi^0(Q, D_Q) \oplus \Pi^1(Q, D_Q) \oplus \Pi^2(Q, D_Q) \\ &= (S \oplus k) \oplus (S \oplus S) \oplus (S \oplus k) \end{aligned}$$

where the second graded piece is computed using the identifications

$$S \otimes_S S / \langle 1 \otimes_S 1 \rangle \cong S / \langle 1 \rangle \cong k$$

as vector spaces and

$$S \otimes_k S / \langle 1 \otimes_k x + x \otimes_k 1, x \otimes_k x \rangle \cong S$$

as S -bimodules. Consequently, the matrix-valued Hilbert–Poincaré series in variables t and s corresponding to path length and x -grading respectively is given by

$$h_{\Pi(B_2)}(t, s) = \begin{pmatrix} 1 + t^2 s & t(1 + s) \\ t(1 + s) & (1 + t^2)(1 + s) \end{pmatrix}.$$

This agrees with the formula in Conjecture 1.4.21.

1.2 REPRESENTATIONS OF DECORATED QUIVERS

Definition 1.2.1. A *decorated quiver* is a pair (Q, D_Q) where $Q = (Q_0, Q_1, s, t)$ is a quiver and

$$D_Q = (A_i, M^\alpha)_{i \in Q_0, \alpha \in Q_1}$$

where A_i is a k -algebra and M^α is an $(A_{s(\alpha)}, A_{t(\alpha)})$ -bimodule. Fixing Q , a *decoration* is a choice of D_Q . The decoration D_Q is *finite* if each A_i is a finitely-generated k -algebra and each M^α is finitely generated as a left $A_{s(\alpha)}$ -module and as a right $A_{t(\alpha)}$ -module.

Remark 1.2.2. This notion appears in the literature under the names k -species or modulated graph when each A_i is a division ring²¹. One can alternatively choose a presentation for each A_i and instead consider the quiver with relations where a loop is added for each generator of each algebra and relations are imposed to capture identical information: see³⁶ and its sequels.

Remark 1.2.3. One can take a categorical perspective by viewing Q as a category where:

- objects are vertices Q_0 ,
- morphisms are paths, and
- composition is given by concatenation of paths, when possible, and zero otherwise.

We denote this category by $F(Q)$ to emphasize the structural differences from the original quiver Q . Using this language, a decoration is a functor $D_{(-)} : F(Q) \rightarrow \text{Alg}$ into the Morita category of algebras, objects are k -algebras, morphisms are bimodules, and composition of morphisms is given by the tensor product. This is the approach taken in⁴³ under the name *prospecies of algebras*, in the case where the bimodule is projective as a left module and as a right module.

We now proceed to develop all of the notions in the previous section for decorated quivers. The original source for much of this material in the setting of species is Dlab and Ringel²¹.

Definition 1.2.4. A *representation* of a decorated quiver (Q, D_Q) is

$$\rho = (V_i, \rho_\alpha)_{i \in Q_0, \alpha \in Q_1}$$

where V_i is a right A_i -module and $\rho_\alpha : M^\alpha \rightarrow \text{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)})$ is an $(A_{s(\alpha)}, A_{t(\alpha)})$ -bimodule map.

Note that $\text{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)})$ has a right $A_{t(\alpha)}$ -action by postcomposition and a left $A_{s(\alpha)}$ -action by precomposition.

Remark 1.2.5. This definition reduces to the ordinary definition of a representation of an (undecorated) quiver when each $A_i = k$ and $M^\alpha = {}_k k_k$, in which case each ρ_α is determined by a single linear map $\rho_\alpha({}_k 1_k) : V_{s(\alpha)} \rightarrow V_{t(\alpha)}$.

Fixing a quiver and a finite decoration (Q, D_Q) , we denote by $\mathcal{R}(Q, D_Q)$ the category of representations of finite type: each V_i is finitely generated as a right A_i -module. Morphisms from ρ to ρ' in this category are given by a collection of right A_i -module maps

$(f_i : V_i \rightarrow V'_i)$ such that for all $\alpha \in Q_1$ and for all $m \in M^\alpha$, the diagram of right $(A_{s(\alpha)} \oplus A_{t(\alpha)}^{op})$ -modules

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{\rho_\alpha(m)} & V_{t(\alpha)} \\ \downarrow f_{s(\alpha)} & & \downarrow f_{t(\alpha)} \\ V'_{s(\alpha)} & \xrightarrow{\rho'_\alpha(m)} & V'_{t(\alpha)} \end{array}$$

commutes. As in the undecorated case, for a fixed $d \in \mathbb{N}^{Q_0}$, we consider the full subcategory of “locally free” representations $\mathcal{R}_d(Q, D_Q)$ whose objects are representations $\rho = (V_i, M^\alpha)$ with $V_i = A_i^{d_i}$ for all $i \in Q_0$.

As before, one can compute equivalence classes of objects in $\mathcal{R}_d(Q, D_Q)$ as orbits under a group action on the vector space of all representations with $V_i = A_i^{d_i}$ for all $i \in Q_0$, denoted $\text{Rep}_d(Q, D_Q)$. In more detail, there is an action of $G_d = \prod_{i \in Q_0} GL_{A_i}(A_i^{d_i})$ on $\text{Rep}_d(Q, D_Q)$ by

$$(g_i)_{i \in Q_0} \cdot (V_i, \rho_\alpha) = (V_i, g_{t(\alpha)} \rho_\alpha g_{s(\alpha)}^{-1}).$$

One says two representations are isomorphic if they lie in the same G_d -orbit. Isomorphism classes of semisimple objects in $\mathcal{R}_d(Q, D_Q)$ can be identified with $\text{Rep}_d(Q, D_Q) // G_d$.

The category $\mathcal{R}_d(Q, D_Q)$ is equivalent to a category of modules for an algebra, generalizing the path algebra in the previous subsection.

Definition 1.2.6. The *decorated path algebra* $P_k(Q, D_Q)$ for $D_Q = (A_i, M^\alpha)$ is the tensor algebra $T_{\oplus_i A_i}(\oplus_\alpha M^\alpha)$ where each M^α is viewed as an $(\oplus_i A_i, \oplus_i A_i)$ -bimodule by defining

$$\left(\sum_i a_i \right) \cdot m_\alpha \cdot \left(\sum_i a'_i \right) := a_{s(\alpha)} \cdot m_\alpha \cdot a'_{t(\alpha)}$$

for $a_i, a'_i \in A_i$ and $m_\alpha \in M^\alpha$. We will usually write $P(Q, D_Q) := P_k(Q, D_Q)$ suppressing the dependence on the field.

Proposition 1.2.7. *The category of representations of the decorated quiver (Q, D_Q) is equivalent to the category of right modules over the path algebra $P(Q, D_Q)$.*

Proof. Given a representation $\rho = (V_i, \rho_\alpha)$ of the decorated quiver $D_Q = (A_i, M^\alpha)$ one can form a right module $V := \oplus_{i \in Q_0} V_i$ of $P(Q, D_Q)$ with the action defined on generators

by

$$\begin{aligned} \left(\sum_{i \in Q_0} v_i \right) \cdot a_j &:= v_j \cdot a_j & a_j \in A_j, v_i \in V_i \\ \left(\sum_{i \in Q_0} v_i \right) \cdot m^\alpha &:= \rho_\alpha(m^\alpha)(v_{s(\alpha)}) & m^\alpha \in M^\alpha. \end{aligned}$$

By the universal property of the tensor algebra, this extends to a right module over all of $P(Q, D_Q)$ since ρ_α is an $(A_{s(\alpha)}, A_{t(\alpha)})$ -bimodule map, which can be viewed as an $\oplus_i A_i$ -bimodule map.

One can upgrade this map of objects to a functor defined by taking ρ to V as above and by taking a morphism $(f_i : V_i \rightarrow V'_i)$ from ρ to ρ' to the map $\oplus_i f_i : \oplus_i V_i \rightarrow \oplus_i V'_i$. The condition that the diagrams commutes (i.e. $\rho'_\alpha \circ f_{s(\alpha)} = f_{t(\alpha)} \circ \rho_\alpha$) implies that $\oplus_i f_i$, apriori a $\oplus_i A_i$ -module map, is a right $P(Q, D_Q)$ -module map. We now construct the inverse functor on objects, which in fact gives an *isomorphism* of categories in the sense that the compositions equal the identity functor, as opposed to merely being naturally isomorphic.

A right module for the decorated path algebra is a map $\varphi : T_{\oplus_i A_i}(\oplus_\alpha M^\alpha) \rightarrow \text{End}_k(V)$, which is determined by its restriction to generators

$$T_{\oplus_i A_i}^{\leq 1}(\oplus_\alpha M^\alpha) = \bigoplus_i A_i \oplus \bigoplus_\alpha M^\alpha.$$

Then one defines $\rho := (\varphi(1_{A_i})V, \varphi|_{M^\alpha})$ where $\varphi(1_{A_i})V$ is given a right A_i -module structure by

$$\varphi(1_{A_i})(v) \cdot a_i := \varphi(a_i)(v)$$

for $v \in V, a_i \in A_i$. Clearly $\varphi|_{M^\alpha}$ is an $(A_{s(\alpha)}, A_{t(\alpha)})$ -bimodule map since φ is an $(\oplus_i A_i)$ -bimodule map.

□

As in the undecorated case, one would like to study the quotient $\text{Rep}_d(Q, D_Q)/G_d$. In the next section we consider the symplectic action of G on the cotangent bundle to $\text{Rep}_d(Q, D_Q)$, but in order to describe the moment map we need to impose restrictions on the decoration. This definition will be justified a posteriori by our study of the zero fiber of the moment map.

Definition 1.2.8. A decoration $D_Q = (A_i.M^\alpha)$ satisfies *condition (F)* if

- A_i is symmetric Frobenius (i.e. $A_i \cong \text{Hom}_k(A_i, k)$ as A_i -bimodules) or all $i \in Q_0$.
- $M^\alpha = A_i \otimes_k A_j$ or $M^\alpha = A_i = A_j$, for all $\alpha \in Q_1$ with $\alpha : i \rightarrow j$.

We will have occasional need to distinguish between the two allowable bimodule decorations in condition (F) and hence write $Q_1 = Q'_1 \sqcup Q''_1$ where the decorations for Q'_1 are of the form $A_j \otimes_k A_i$ and the decorations for Q''_1 are of the form $A_i = A_j$.

Remark 1.2.9. In subsection 1.3, we explain why the Frobenius condition is necessary to define a preprojective algebra satisfying Theorem 1.1.2. One can alternatively consider additional structure on the bimodules instead of the algebras, as is done in²¹. Symmetry of the Frobenius form is needed to identify the moment map with an evaluation map equivariantly, see Remark 1.3.15.

1.3 REPRESENTATIONS OF DECORATED PREPROJECTIVE ALGEBRAS

In this section we extend the definition of the preprojective algebra of a quiver to the decorated setting. We justify the term preprojective algebra using the moment map for the action of $G = \bigoplus_{i \in Q_0} GL_{A_i}(V_i)$ on the cotangent bundle of the space of all representations. Namely, we show that the set of representations of the decorated preprojective algebra is the zero fiber of the moment map.

Recall the decorated path algebra for a decorated quiver (Q, D_Q) with decoration $D_Q = (A_i, M^\alpha)$ is $P(Q, D_Q) = T_{\bigoplus A_i}(\bigoplus_\alpha M^\alpha)$. The decorated preprojective algebra is a quotient of the decorated path algebra for the *double* decorated quiver, which we define now.

Definition 1.3.1. Given a decorated quiver (Q, D_Q) , the *opposite* decorated quiver (Q^{op}, D_Q^{op}) is defined by

$$D_Q^{op} := (A_i, M^{\alpha*}), \quad M^{\alpha*} := \text{Hom}_{A_{s(\alpha)} \otimes A_{t(\alpha)}}(M^\alpha, A_{s(\alpha)} \otimes A_{t(\alpha)})$$

and the *double* decorated quiver $(\overline{Q}, \overline{D_Q})$ is defined by $\overline{D_Q} := D_Q \cup_{D_{Q_0}} D_Q^{op}$.

Suppose D_Q satisfies condition (F) and partition the arrows into $Q'_1 \sqcup Q''_1$ where the former has label $M^\alpha = A_{s(\alpha)} \otimes A_{t(\alpha)}$ and the latter has label $M^\alpha = A_{s(\alpha)} = A_{t(\alpha)}$. Additionally, pick a basis $\{e_j^i\}_j$ for A_i as a vector space and use the Frobenius form $\lambda_i : A_i \rightarrow k$ to build a dual basis $\{f_l^i\}_l$ such that $\lambda_i(e_j^i f_l^i) = \delta_{j,l}$, for all $i \in Q_0$.

Definition 1.3.2. Using notation from the previous paragraph, the *decorated preprojective algebra* $\Pi_k(Q, D_Q)$ is defined to be the quotient of the decorated path algebra of the

double $P(\overline{Q}, \overline{D_Q})$ by the two-sided ideal generated by

$$r := \sum_{\alpha \in Q'_1} \left(\sum_i e_i^{s(\alpha)} \otimes 1 \otimes f_i^{s(\alpha)} - \sum_j e_j^{t(\alpha)} \otimes 1 \otimes f_j^{t(\alpha)} \right) + \sum_{\alpha \in Q''_1} [1_{M^\alpha}, 1_{M^{\alpha*}}].$$

We will usually write $\Pi(Q, D_Q) := \Pi_k(Q, D_Q)$, suppressing the dependence on the field.

The usual preprojective relation in $k\overline{Q}_1 \otimes_{kQ_0} k\overline{Q}_0$ is kQ_0 -central, since it is concentrated on the diagonal. Similarly, we want r to be an $(\oplus A_i)$ -central element of $\oplus M^\alpha \otimes_{(\oplus A_i)} \oplus M^{\alpha*}$, so we choose for each i , the dual element to the Frobenius pairing, $k \rightarrow A_i \otimes A_i$, which when written in bases $\{e_j^i\}$ and dual bases $\{f_j^i\}$ is the A_i -central element $\sum_j e_j^i \otimes_k f_j^i$. For arrows in Q''_1 , $M^\alpha \otimes_A M^{\alpha*} \cong A \otimes_A A \cong A$ has the A -central element $1_{M^\alpha \otimes_A M^{\alpha*}} = 1_{M^\alpha} \otimes 1_{M^{\alpha*}}$, so one need not use the Frobenius form in the definition of the decorated preprojective algebras, yet we still require the existence of such a form for desirable properties, see Remark 1.3.15.

Remark 1.3.3. The above definition depends only on the decorated quiver (Q, D_Q) and the field k . To see this, observe that for each $\alpha : i \rightarrow j \in Q'_1$ the summand of r corresponding to α lies in

$$A_i \otimes A_i \cong A_i \otimes k \otimes A_i \subset A_i \otimes A_j \otimes A_i \cong (A_i \otimes A_j) \otimes_{A_j} (A_j \otimes A_i) \cong M^\alpha \otimes_{A_j} M^{\alpha*}$$

and hence it suffices to show that, in any Frobenius algebra A with basis $\{e_i\}$ and dual basis $\{f_i\}$, the ideal generated by $c := \sum_i e_i \otimes f_i$ doesn't depend on the choice of basis nor the form. For independence of basis, notice that c can be represented canonically as the comultiplication of the unit. Next, observe that a Frobenius form can be viewed as the image of 1_A under a left A -module isomorphism $\varphi : A \rightarrow A^*$. Hence, two Frobenius forms $\lambda_1, \lambda_2 : A \rightarrow k$ each give rise to $\varphi_1, \varphi_2 : A \rightarrow A^*$ such that $\varphi_2^{-1} \circ \varphi_1 : A \rightarrow A$ is a left A -module isomorphism. Such a map is determined by $u := \varphi_2^{-1} \circ \varphi_1(1_A)$, which is invertible. Hence

$$\lambda_1 = \varphi_1(1) = \varphi_2(u) = \lambda_2 \circ R_u$$

where R_u is right multiplication by $u \in A^\times$. We conclude that the canonical sums c_1 and c_2 satisfy $c_1 = c_2 u$ and hence generate the same two-sided ideal in $A \otimes_k A$.

Example 1.3.4. This phenomenon is already visible in the setting of the four element algebra $A = \mathbb{F}_2[x]/(x^2)$. Here $\lambda : A \rightarrow \mathbb{F}_2$ is Frobenius if $(x) \not\subset \ker(\lambda)$ or equivalently if $\lambda(x) \neq 0$ and hence is 1. So there are two Frobenius forms: $\lambda_1(1) = 0$ and $\lambda_2(1) = 1$.

They are related by the invertible element $1 + x \in A$ as $\lambda_1(1) = 0 = \lambda_2(1(1 + x))$ and $\lambda_1(x) = 1 = \lambda_2(x(1 + x))$. The basis $\{1, x\}$ has dual basis $\{x, 1\}$ under λ_1 and dual basis $\{1, 1 + x\}$ under λ_2 . Hence $c_1 = 1 \otimes x + x \otimes 1$ and

$$c_2 = 1 \otimes x + x \otimes (1 + x) = 1 \otimes x(1 + x) + x \otimes 1(1 + x) = c_1(1 + x)$$

both generate the ideal $(c_1) = \{c_1, c_2, c_1 + c_2\} \subset A \otimes A$.

Remark 1.3.5. This explains the reason for the dichotomy between types of arrows and the resulting asymmetry in the definition. Namely, it's an artifact of the identifications,

$$\text{End}_A(A) \cong A \cong M^\alpha \otimes_A M^{\alpha*} \quad id \mapsto 1 \mapsto 1 \otimes_A 1$$

as k -bimodules for $M^\alpha = {}_A A_A \cong M^{\alpha*}$ and

$$\text{End}_k(A) \cong A \otimes_k A^* \cong M^\alpha \otimes_k M^{\alpha*} \quad id \mapsto c = \sum_i e_i \otimes_k f_i \mapsto \sum_i e_i \otimes 1 \otimes f_i$$

as A -bimodules for $M^\alpha = {}_A A_k$ an (A, k) -bimodule.

The key result motivating the definition is the following. Recall fixing (Q, D_Q) and a dimension vector d , we have the action of $G_d = \prod_{i \in Q_0} GL_{A_i}(A_i^{d_i})$ on $\text{Rep}_d(Q, D_Q)$ with Lie algebra $\mathfrak{g}_d = \bigoplus_{i \in Q_0} \mathfrak{gl}_{A_i}(A_i^{d_i})$. Let $\mu_d : T^*(\text{Rep}_d(Q, D_Q)) \rightarrow \mathfrak{g}_d^*$ denote the moment map corresponding to the Hamiltonian action on the cotangent space coming from the action on the base. In this setting, we can generalize Proposition 0.2.6 as follows:

Theorem 1.3.6. *With the identifications induced from the non-degenerate pairings,*

$$\begin{aligned} \text{Rep}_d(Q, D_Q) \otimes \text{Rep}_d(Q^{op}, D_Q^{op}) &\rightarrow k & \mathfrak{g}_d \otimes \mathfrak{g}_d &\rightarrow k \\ (\rho, \rho') &\mapsto \sum_{\alpha \in Q_1} \lambda_{A_{s(\alpha)}} \circ \text{tr}_{A_{s(\alpha)}}(\rho(1_{M^\alpha}) \circ \rho'(1_{M^{\alpha*}})) & (X, Y) &\mapsto \sum_{i \in Q_0} \lambda_i \circ \text{tr}_{A_i}(XY) \end{aligned}$$

the following diagram commutes,

$$\begin{array}{ccc} T^* \text{Rep}_d(Q, D_Q) & \xrightarrow{\mu_d} & \mathfrak{g}_d^* \\ \cong \nearrow & & \searrow \cong \\ \text{Rep}_d(\overline{Q}, \overline{D_Q}) & \xrightarrow{(\rho(1_{M^\alpha}), \rho(1_{M^{\alpha*}})) \mapsto \rho(r)} & \mathfrak{g}_d \end{array}$$

and hence one can view the moment map as evaluation on the element r .

Note that $\rho(r)$ is, a priori, an element of $\bigoplus_{i \in Q_0} \mathfrak{gl}_k(A_i^{d_i})$, but in fact is $\bigoplus A_i$ -linear since $e_i r$

is A_i -central, hence an element of $\mathfrak{g}_d$. Additionally, since each pairing is G_d -invariant, we have identified the moment map with evaluation on r , G_d -equivariantly

Remark 1.3.7. For each A_i , to apply $tr_{A_i} : \text{Mat}_{d_i}(A_i) \rightarrow A_i$ one needs to precompose with an identification $\mathfrak{gl}_{A_i}(A_i^{d_i}) \cong \text{Mat}_{d_i}(A_i)$. Although tr_{A_i} depends on the choice of identification, $\lambda_{A_i} \circ tr_{A_i}$ does not, as λ_{A_i} is symmetric.

Corollary 1.3.8. *The set of representations of dimension vector d in $\Pi(Q, D_Q)$ is the zero fiber of the moment map μ_d .*

The remainder of this section is dedicated to a proof of the above theorem, justifying the terminology preprojective algebra in a geometric context.

Since Q is a finite quiver, $r = \sum_{i \in Q_0} e_i r$ can be computed locally at each vertex, where it is given by a sum over all incoming arrows. Hence it suffices to prove the above result in the context of a quiver with a single decorated arrow. Such decorated quivers satisfying condition (F) take the form,

$$\begin{aligned} \text{(I)} \quad A_1 &\xrightarrow{A_1 \otimes_k A_2} A_2 & \text{(II)} \quad A &\xrightarrow{A} A \\ \text{(III)} \quad A &\xrightarrow{A \otimes_k A} A & \text{(IV)} \quad A &\cup A & \text{(V)} \quad A &\cup A \otimes_k A \end{aligned}$$

where A, A_1 , and A_2 are symmetric Frobenius algebras. The formula for the moment map in cases (IV) and (V) follow from the (II) and (III) cases, respectively, by post-composing with the map $d\Delta^* : \mathfrak{g}_d^* \oplus \mathfrak{g}_d^* \rightarrow \mathfrak{g}_d^*$, induced from the diagonal map $\Delta : G_d \rightarrow G_d \times G_d$. Next notice (III) is a special case of (I), when $A_1 = A_2 = A$. Moreover the definition of $e_1 r$ uses only the unit $k \subset A_2$ and hence one can further reduce to verifying the theorem in the two cases:

$$Q' := A \xrightarrow{A A_k} k \quad \text{or} \quad Q'' := A \xrightarrow{A} A.$$

In these cases, representations of the double decorated quiver have a concrete description, owing to the cyclicity of the bimodules. That is, if $d = (d_1, d_2)$, then

$$\begin{aligned} \text{Rep}_d(\overline{Q'}) &= \text{Hom}_{A\text{-mod}}({}_A A_k, \text{Hom}_k(A^{d_1}, k^{d_2})) \oplus \text{Hom}_{\text{mod-}A}({}_k A_A, \text{Hom}_k(A^{d_1}, k^{d_2})) \\ &\cong \text{Hom}_k(A^{d_1}, k^{d_2}) \oplus \text{Hom}_k(k^{d_2}, A^{d_1}) \end{aligned}$$

and

$$\begin{aligned}
& \text{Rep}_d(\overline{Q''}) \\
&= \text{Hom}_{A\text{-bimod}}({}_A A_A, \text{Hom}_k(A^{d_1}, A^{d_2})) \oplus \text{Hom}_{A\text{-bimod}}({}_A A_A, \text{Hom}_k(A^{d_2}, A^{d_1})) \\
&\cong \text{Hom}_A(A^{d_1}, A^{d_2}) \oplus \text{Hom}_A(A^{d_2}, A^{d_1}).
\end{aligned}$$

In each case, the theorem reduces to much more concrete statements, given respectively by Proposition 1.3.9 and Proposition 1.3.13.

Proposition 1.3.9. *With the identifications induced from the non-degenerate pairings,*

$$\begin{aligned}
\text{Hom}_k(A^{d_1}, k^{d_2}) \otimes \text{Hom}_k(k^{d_2}, A^{d_1}) &\rightarrow k & (f, g) &\mapsto \text{tr}(f \circ g) \\
\text{End}_A(A^{d_1}) \otimes \text{End}_A(A^{d_1}) &\rightarrow k & (X, Y) &\mapsto \lambda_A \circ \text{tr}_A(X \circ Y) \\
\text{End}_k(k^{d_2}) \otimes \text{End}_k(k^{d_2}) &\rightarrow k & (X, Y) &\mapsto \text{tr}(X \circ Y)
\end{aligned}$$

the following diagram commutes:

$$\begin{array}{ccc}
\text{Hom}_k(A^{d_1}, k^{d_2}) \oplus \text{Hom}_k(A^{d_1}, k^{d_2})^* & \xrightarrow{\mu_d} & \text{End}_A(A^{d_1})^* \oplus \text{End}_k(k^{d_2})^* \\
\cong \nearrow & & \nwarrow \cong \\
\text{Hom}_k(A^{d_1}, k^{d_2}) \oplus \text{Hom}_k(k^{d_2}, A^{d_1}) & \xrightarrow{\rho(A^{d_1} 1_k, \rho(k 1_A)) \mapsto \rho(r)} & \text{End}_A(A^{d_1}) \oplus \text{End}_k(k^{d_2})
\end{array}$$

where $r = 1 \otimes_A 1 - \sum e_i \otimes_k f_i$.

Roughly speaking, in the ordinary case two applications of the trace pairing cancel, yielding a simplified expression for the moment map. From this perspective, the main technical difficulty arises from the need to relate the trace of an A -linear map (regarded as a k -linear map) with the Frobenius form.

More precisely, if $(f, g) \in \text{Hom}_k(A^{d_1}, k^{d_2}) \oplus \text{Hom}_k(k^{d_2}, A^{d_1})$ and $(\varphi, \psi) \in \text{End}_A(A^{d_1}) \oplus \text{End}_k(k^{d_2})$ then in the above diagram,

$$\begin{array}{ccc}
(f, \text{tr}(g \circ -)) & \xrightarrow{\quad} & \text{tr}(f \circ g \circ -) + \text{tr}(-g \circ f \circ -) \\
\nearrow & & \parallel \\
& & \lambda_A \circ \text{tr}_A(\varphi \circ -) + \text{tr}(\psi \circ -) \\
(f, g) & \xrightarrow{\quad ? \quad} & (\varphi, \psi)
\end{array}$$

In our setting, $f = \rho(A 1_k)$ and $g = \rho(k 1_A)$ and so we need to find $\varphi \in \text{End}_A(A^{d_1})$ and

$\psi \in \text{End}_k(k^{d_2})$ such that,

$$\text{tr}(\rho(-_A 1 \otimes_k 1_A) \circ -) = \lambda_A \circ \text{tr}_A(\varphi \circ -)$$

and

$$\text{tr}(\rho(_k 1 \otimes_A 1_k) \circ -) = \text{tr}(\psi \circ -).$$

Clearly, $\psi = \rho(_k 1 \otimes_A 1_k)$ and hence the moment map at the vertex decorated with k is evaluation at $_k 1 \otimes_A 1_k$, playing the role of $\alpha \circ \alpha^*$ in the ordinary case.

However, $\varphi \neq \rho(-1 \otimes_k 1)$, which need not be A -linear in general. To remedy the situation, we will define a map $\Phi : \text{End}_k(A) \rightarrow \text{End}_A(A)$ such that

$$\varphi = \text{Mat}_d(\Phi)(\rho(-1 \otimes_k 1)) = \rho\left(-\sum e_i \otimes f_i\right),$$

completing the proof of the proposition.

We now present a series of lemmas about Frobenius algebras. Since the subject matter is classical and the proofs are elementary, the following is probably well known. Nonetheless we provide short proofs for completeness of exposition.

Lemma 1.3.10. *Let A be a Frobenius algebra with multiplication μ and form $\lambda : A \rightarrow k$. Define $\Phi : \text{End}_k(A) \rightarrow \text{End}_A(A)$ by $\Phi(\phi)(a) = \sum_i \phi(e_i) f_i a$, where $\{e_i\}$ is a basis for A over k with dual basis $\{f_i\}$. Then*

$$\begin{array}{ccccc} \text{End}_k(A) & \xrightarrow{\text{tr}} & k \\ \cong \downarrow & \searrow \Phi & \nearrow \lambda & & \\ A \otimes_k A^* & & \text{End}_A(A) & & \\ \cong \downarrow & & \cong \downarrow & & \\ A \otimes_k A & \xrightarrow{\mu} & A & & \end{array}$$

commutes. In particular, $\text{tr}(\phi) = \lambda(\Phi(\phi)(1))$ for $\phi \in \text{End}_k(A)$.

Proof. Φ is defined so the diagram commutes. In more detail,

$$\begin{array}{ccccccc} \text{End}_k(A) & \longrightarrow & A \otimes_k A^* & \longrightarrow & A \otimes_k A & \longrightarrow & A \\ \phi & \longmapsto & \sum_i \phi(e_i) \otimes_k \lambda \circ \mu(-, f_i) & \longmapsto & \sum_i \phi(e_i) \otimes_k f_i & \longmapsto & \sum_i \phi(e_i) f_i \end{array}$$

Since $\Phi(\phi)(1) := \sum_i \phi(e_i) f_i$ the lower left half of the diagram commutes.

For the rest of the diagram, compute tr using the basis $\{e_i\}$. If $\phi(e_i)$ is expanded as $\sum_j \alpha_{ij} e_j$ then,

$$\alpha_{ii} = \lambda \circ \mu(\sum_j \alpha_{ij} e_j, f_i) = \lambda \circ \mu(\phi(e_i), f_i).$$

and hence $tr(\phi) = \lambda \circ \mu(\sum_i \phi(e_i), f_i) = \lambda(\Phi(\phi)(1))$ as desired. $\square$

Lemma 1.3.11. *The following diagram commutes:*

$$\begin{array}{ccccc}
\text{End}_k(A^d) & & & & \\
\downarrow \cong & & \text{tr} & & \\
\text{Mat}_d(\text{End}_k(A)) & \xrightarrow{\text{Mat}_d(tr)} & \text{Mat}_d(k) & \xrightarrow{tr} & k \\
\downarrow \text{Mat}_d(\Phi) & & \nearrow \text{Mat}_d(\lambda) & & \\
\text{Mat}_d(\text{End}_A(A)) & \xrightarrow{\cong} & \text{Mat}_d(A) & \xrightarrow{tr_A} & A \\
& & \nearrow \lambda & &
\end{array}$$

Proof. Applying $\text{Mat}_d(-)$ to the commutative diagram in Lemma 1.3.10 gives commutativity of the lower left trapezoid. The commutativity of the upper triangle is the fact that the trace of a block matrix is the sum of the traces of the diagonal blocks. The final quadrilateral commutes since if $M = (a_{i,j}) \in \text{Mat}_d(A)$ then

$$\lambda \circ tr_A(M) = \sum_i \lambda(a_{i,i}) = tr((\lambda(a_{i,j}))) = tr \circ \text{Mat}_d(\lambda)(M). \quad \square$$

Lemma 1.3.12. *The following diagram commutes:*

$$\begin{array}{ccc}
\text{End}_k(A^d) & \xrightarrow{\cong} & \text{End}_k(A^d)^* \\
\downarrow \text{Mat}_d(\Phi) & & \downarrow \text{rest.} \\
\text{End}_A(A^d) & \xrightarrow{\cong} & \text{End}_A(A^d)^*
\end{array}$$

where the horizontal identifications are given by the non-degenerate pairings,

$$\text{End}_k(A^d) \otimes \text{End}_k(A^d) \rightarrow k \quad (\phi_1 \otimes \phi_2) \mapsto tr(\phi_1 \circ \phi_2)$$

$$\text{End}_A(A^d) \otimes \text{End}_A(A^d) \rightarrow k \quad (\psi_1 \otimes \psi_2) \mapsto \lambda \circ tr_A(\psi_1 \circ \psi_2).$$

Proof. Commutativity amounts to the equivalence

$$tr(\phi \circ \psi) = \lambda \circ tr_A(\text{Mat}_d(\Phi)(\varphi) \circ \psi)$$

for all $\phi \in \text{End}_k(A^d)$ and $\psi \in \text{End}_A(A^d)$. By Lemma 1.3.11, one can factor trace as,

$$tr = \lambda \circ tr_A \circ \text{Mat}_d(\Phi)$$

and hence it suffices to show $\text{Mat}_d(\Phi)(\phi \circ \psi) = \text{Mat}_d(\Phi)(\phi) \circ \psi$. By comparing components of the mappings, one reduces to the case $d = 1$, where the result follows from the fact that Φ is an A -bimodule map and ψ can be viewed as an element of A . $\square$

Note that this completes the proof of Proposition 1.3.9, since

$$\text{Mat}_d(\Phi)\rho(1 \otimes_k 1) = \rho(\Phi(1 \otimes_k 1)) = \rho\left(\sum e_i \otimes_k f_i\right).$$

Proposition 1.3.13. *With the identifications induced from the non-degenerate pairings,*

$$\begin{aligned} \text{Hom}_A(A^{d_1}, A^{d_2}) \otimes \text{Hom}_A(A^{d_2}, A^{d_1}) &\rightarrow k & \text{End}_A(A^{d_j}) \otimes \text{End}_A(A^{d_j}) &\rightarrow k \\ (\rho, \rho') &\mapsto \sum_{\alpha \in Q_1} \lambda_A \circ tr_A(\rho(1_{M^\alpha}) \circ \rho'(1_{M^\alpha}{}^*)) & (X, Y) &\mapsto \sum_{i \in Q_0} \lambda_A \circ tr_A(XY) \end{aligned}$$

for $j \in \{1, 2\}$, the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_A(A^{d_1}, A^{d_2}) \oplus \text{Hom}_A(A^{d_1}, A^{d_2})^{*\mu_d} & \xrightarrow{\quad} & \text{End}_A(A^{d_1})^* \oplus \text{End}_A(A^{d_2})^* \\ \cong \nearrow & & \nwarrow \cong \\ \text{Hom}_A(A^{d_1}, A^{d_2}) \oplus \text{Hom}_A(A^{d_2}, A^{d_1}) & \xrightarrow{\rho(A^{1_A}), \rho(A^{1_A}) \mapsto \rho(r)} & \text{End}_A(A^{d_1}) \oplus \text{End}_A(A^{d_2}) \end{array}$$

where $r = e_1 r + e_2 r = 1 \otimes_A^2 1 - 1 \otimes_A^1 1 = [{}_A 1_{A,A} 1_A]$ where the labels 1 and 2 are used to distinguish between the two copies of A , one at each vertex.

The difficulty in the previous proposition was relating, for a given $\alpha : i \rightarrow j$, the expressions $\lambda_{A_i} \circ tr_{A_i}$ and $\lambda_{A_j} \circ tr_{A_j}$. In this case, $A_i = A = A_j$ and hence that issue is not present.

Proof. The result follows from showing that the moment map still has the same form as before. Whenever G acts on a vector space V , the moment map for the action of G on $T^*(V) = V \oplus V^*$ is given by

$$\mu : T^*(V) \rightarrow \mathfrak{g}^* \quad \mu((v, \varphi))(X) = \varphi(X \cdot v).$$

In this case $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a sum and the action $X = (X_1, X_2)$ on v defined by differentiating

the conjugation action is

$$X \cdot v = X_2 \cdot v - v \cdot X_1.$$

Therefore,

$$\begin{array}{ccc} (f, \lambda_A \circ \text{tr}_A(g \circ -)) & \longmapsto & \lambda_A \circ \text{tr}_A(-g \circ f \circ -) + \lambda_A \circ \text{tr}_A(f \circ g \circ -) \\ \nearrow (f, g) & & \nwarrow (-g \circ f + f \circ g) \end{array}$$

and hence the composition is giving by commutator and plugging in $f = \rho({}_A 1_A)$ and $g = \rho({}_A 1_A)$ gives the desired result. $\square$

This completes the proof of Theorem 1.3.6.

Remark 1.3.14. A more general definition of preprojective algebra appears in⁴³, which puts a weaker restriction on the decorations than condition (F), although the Frobenius condition still arises naturally as we explain below. When can the representations of such algebras be identified with the zero fiber of a moment map, as in Theorem 1.3.6?

In more detail, in⁴³, decorations are required instead to have the property that each M^α is projective as a left $A_{t(\alpha)}$ -module and a projective as a right $A_{s(\alpha)}$ -module. In this setting, M^α has both left and right duals, and it is further required that these duals agree,

$$\text{Hom}_{A_{s(\alpha)}^{\text{op}}}(M^\alpha, A_{s(\alpha)}) \cong \text{Hom}_{A_{t(\alpha)}}(M^\alpha, A_{t(\alpha)})$$

as $(A_{t(\alpha)}, A_{s(\alpha)})$ -bimodules. One says M^α is *dualizable* in this case, and one says D_Q is *dualizable* if each M^α is dualizable.

Notice that if a quiver has an arrow labelled with $M := {}_A A_k$ then dualizability at that arrow says,

$$A \cong \text{Hom}_A(M, A) \cong \text{Hom}_k(M, k) \cong \text{Hom}_k(A, k)$$

as an (k, A) -bimodule, and hence A is Frobenius. More generally, if $M := {}_A A_R$ where R is a k -algebra and A is an algebra over R then dualizability says

$$A \cong \text{Hom}_R(A, R)$$

and hence A is Frobenius relative to R , the assumption used in²⁰.

Remark 1.3.15. In the case $M = {}_A A_A$ dualizability is vacuous, yet we still need A to

be Frobenius in order to have a non-degenerate pairing

$$\mathrm{Hom}_A(A^{d_1}, A^{d_2}) \otimes \mathrm{Hom}_A(A^{d_2}, A^{d_1}) \xrightarrow{\circ} \mathrm{End}_A(A^{d_1}) \xrightarrow{\mathrm{tr}_A} A \xrightarrow{\lambda} k.$$

Further, for the pairing to be $\mathrm{GL}_A(A^{d_1})$ -invariant then λ needs to be symmetric. This justifies the assumption that the algebra at each vertex be symmetric Frobenius.

Remark 1.3.16. If $\alpha : i \rightarrow j$ is decorated with M^α an (A_i, A_j) -bimodule, then

$$\mathrm{RHom}_{A_i \otimes_k A_j^{\mathrm{op}}}(M^\alpha, \mathrm{Hom}_k(V_i, V_j)) \cong \mathrm{Hom}_{A_i \otimes_k A_j^{\mathrm{op}}}(M^\alpha, \mathrm{Hom}_k(V_i, V_j))$$

if M^α is a projective (A_i, A_j) -bimodule or $\mathrm{Hom}_k(V_i, V_j)$ is an injective (A_i, A_j) -bimodule. Hence the restriction to free A_i -modules, V_i , implies that the underived representation spaces are actually derived, hence well-behaved and natural to study. One could alternatively consider M^α perfect and remove restrictions on the representations. Since our main objective is to study preprojective algebras, we prefer to allow for general M^α .

1.4 DECORATED PREPROJECTIVE ALGEBRAS AS DEGENERATIONS

Many of the examples of decorated quivers in this thesis have an interpretation as degenerations of ordinary quivers. In this section we explain this interpretation by first defining a notion of Frobenius deformation and proving that every finite-dimensional Frobenius algebra degenerates to a Frobenius algebra determined by a vector space with a non-degenerate bilinear form. In the commutative case, over an algebraically closed field of characteristic not two, there is a unique such degenerate algebra. Then we define a notion of Frobenius deformation for decorated quivers and use this to realize decorated preprojective algebras as degenerations. Finally, we conjecture that these degenerations are flat, and use upper semi-continuity of dimension under deformations to reduce the conjecture to a single verification for each quiver.

1.4.1 FROBENIUS DEGENERATIONS

We want to degenerate (or deform) a decoration by degenerating both the k -algebras labelling each vertex and the bimodules labelling each edge. In the presence of condition (F), the degeneration of the decoration is determined by the degeneration of the algebras at the vertices. Moreover, to ensure condition (F) is satisfied when degenerating a decoration satisfying condition (F) we need to ensure each symmetric Frobenius algebra degenerates to a symmetric Frobenius algebra.

We first recall the notions of formal and filtered deformations of associative algebras and bimodules.

Definition 1.4.1. Let A and B be unital associative algebras over a field k . One says B is a *formal deformation* of A and A is a *formal degeneration* of B if there is a $k[[t]]$ -algebra D such that

- $D/Dt \cong A$ as k -algebras and
- $D[t^{-1}] \cong B((t))$ as $k((t))$ -algebras.

Moreover, one says a formal deformation is *flat* if $D \cong A[[t]]$ as left $k[[t]]$ -modules.

Definition 1.4.2. Let A and B be unital associative k -algebras. One says B is a *filtered deformation* of A and A is a *filtered degeneration* of B if there exists:

- a filtration $B_0 \subset B_{\leq 1} \subset B_{\leq 2} \subset \cdots \subset B_{\leq n} = B$,
- a grading $A \cong \bigoplus_{i=0}^n A_i$, and
- an isomorphism of graded algebras,

$$gr(B) := \bigoplus_{i=0}^n B_{\leq i}/B_{\leq (i-1)} \cong \bigoplus_{i=0}^n A_i \cong A.$$

Any filtered deformation B with $gr(B) \cong A$ gives rise to a formal deformation,

$$\hat{R}B := \left\{ \sum_{i \geq 0} b_i t^i : b_i \in B_{\leq i} \right\},$$

called the Rees algebra. We present theory in the more general setting of formal deformations, but strengthen results by presenting proofs in the more specific setting of filtered deformations.

Since our objective is to degenerate decorated quivers, we want to consider degenerations preserving condition (F). Notice that a (symmetric) Frobenius algebra can degenerate to a non-Frobenius algebra, e.g $k[[t]][x, y]/(x^2 - t, y^2 - t, xy)$ degenerates $k^{\oplus 3}$ to $k[x, y]/(x^2, y^2, xy)$, which is not Frobenius as it is not self-injective since $xA \rightarrow A$ sending $x \mapsto y$ does not extend to all of A .

Hence, at each vertex we need to degenerate the symmetric Frobenius algebra to another symmetric Frobenius algebra.

Definition 1.4.3. Let D be a formal deformation of associative k -algebras from A to B . We say the deformation is *Frobenius* if D is Frobenius as a $k[[t]]$ -algebra and A and B are Frobenius as k -algebras. Further we say the deformation is *symmetric Frobenius* if D , A , and B are all symmetric Frobenius.

Remark 1.4.4. This definition appears in²⁰ and the notion was studied in¹², where it is shown that the cohomology of the dg-Lie algebra controlling Frobenius deformations of a fixed Frobenius algebra (A, λ) is the cyclic cohomology of A , see¹².

Notice that we do *not* require that the isomorphisms $A \cong D/tD$ or $B((t)) \cong D[t^{-1}]$ preserve the Frobenius forms. This added flexibility is convenient and innocuous for our purposes since different Frobenius forms on the same decoration yield isomorphic preprojective algebras, as explained in Remark 1.3.3. However, one should be mindful about some non-intuitive consequences of not fixing Frobenius forms, as explained in Example 1.4.7 (4).

Remark 1.4.5. A Frobenius form $\lambda : D \rightarrow k[[t]]$ on D gives rise to Frobenius forms λ_0 on the k -algebra $A \cong D/tD$ and λ_1 on the $k((t))$ -algebra $B((t)) \cong D[t^{-1}]$ defined so that the following diagram commutes:

$$\begin{array}{ccccc}
 & & D/tD & \xrightarrow{\lambda_0} & k \\
 & \nearrow \text{mod } t & & & \nearrow \text{mod } t \\
 D & \xrightarrow{\lambda} & k[[t]] & & \\
 & \searrow & & & \searrow \\
 & & D[t^{-1}] & \xrightarrow{\lambda_1} & k((t)).
 \end{array}$$

Remark 1.4.6. If B is a filtered deformation of a finite-dimensional algebra $A = \bigoplus_{i=0}^n A_i$, then a Frobenius form on A induces a Frobenius form on B . To see this, observe $\lambda : A \rightarrow k$ non-degenerate implies $\ker(\lambda) \not\subset A_n \cong B/B_{\leq(n-1)}$. Hence the lift $\tilde{\lambda} : B \rightarrow k$ to B by defining $\tilde{\lambda}(B_{\leq(n-1)}) = 0$ is non-degenerate.

Example 1.4.7.

- (1) Assume k contains n th roots of unity, write ζ for a primitive n th root of unity, and define the surjective algebra homomorphism

$$k[x] \rightarrow k^{\oplus n} \quad p(x) \mapsto (p(\zeta), p(\zeta^2), \dots, p(\zeta^n))$$

with kernel $(x^n - 1)$. Using this map one can view $k^{\oplus n} \cong k[x]/(x^n - 1)$ as a filtered deformation of $k[x]/(x^n)$. The Frobenius form $\lambda : k[x]/(x^n) \rightarrow k$ given by

$\lambda(\sum_j a_j x^j) = a_{n-1}$ lifts to $\tilde{\lambda}(\sum_j a_j x^j) = a_{n-1}$. Precomposing with $k^{\oplus n} \cong k[x]/(x^n - 1)$ gives the Frobenius form

$$k^{\oplus n} \rightarrow k \quad (a_1, a_2, \dots, a_n) \mapsto \frac{1}{n} \sum_{j=1}^n \zeta^j a_j$$

- (2) Fix a field k of characteristic not two, a k -vector space V , and a quadratic form $Q : V \rightarrow k$. Then define the Clifford algebra by

$$Cl_k(V, Q) := T_k(V) / \langle v \otimes_k v - Q(v)1 \rangle_{v \in V}.$$

This is a filtered algebra with associated graded algebra given by the exterior algebra, $\bigwedge(V) = Cl_k(V, 0)$. The Frobenius form on the exterior algebra given by projecting onto the top wedge power remains non-degenerate when viewed as a form on $Cl(V, Q)$.

- (3) $A \cong Mat_{2 \times 2}(k)$ is separable as an algebra over k so

$$HH^2(A, A) := Ext_{A \otimes A^{op} - mod}^2(A, A) = 0.$$

We conclude that A has no deformations as a (symmetric Frobenius) algebra. In particular, A does not deform to k^4 and more generally in all dimensions $n \geq 4$, $A \oplus k^{n-4}$ does not deform to k^n .

- (4) $Mat_{2 \times 2}(k)$ can be viewed as a Clifford algebra, and hence has a degeneration to the exterior algebra. However, it does *not* degenerate as a Frobenius algebra using the trace form, and in fact does not degenerate as a symmetric Frobenius algebra.

In more detail, if k is a field of characteristic not two and containing $i := \sqrt{-1}$, then one realizes $Mat_{2 \times 2}(k)$ as a Clifford algebra using the surjective algebra map

$$\varphi : k\langle x, y \rangle \rightarrow Mat_2(k) \quad \text{given by} \quad x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad y \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with kernel $(x^2 + 1, y^2 + 1, xy + yx)$. Therefore,

$$Mat_{2 \times 2}(k) \cong k\langle x, y \rangle / (x^2 + 1, y^2 + 1, xy + yx) \cong Cl(V, Q)$$

where $V \cong k^2$ with basis $\{x, y\}$ and $Q : V \rightarrow k$ is determined by $Q(x) = Q(y) = Q(x + y)/2 = -1$. Hence $Mat_{2 \times 2}(k)$ degenerates as an associative algebra to $\bigwedge(V)$.

Consider two Frobenius forms $\lambda_1, \lambda_2 : \text{Mat}_{2 \times 2}(k) \rightarrow k$ given by,

$$\lambda_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d \quad \lambda_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = b + c.$$

Observe λ_1 is the usual trace and hence symmetric, but $\lambda_1(x) = \lambda_1(xy) = 0$ implies it degenerates to a form on $\bigwedge(V)$ whose kernel contains the left ideal generated by x . Therefore, the degeneration as associative algebras from $\text{Mat}_2(k)$ to $\bigwedge(V)$ cannot be made into a Frobenius degeneration when $\text{Mat}_2(k)$ is given the Frobenius form λ_1 . In fact, one can show that there is no associative algebra degeneration taking λ_1 to a non-degenerate form on $\bigwedge(V)$.

However, λ_2 degenerates to $\pi^2 : \bigwedge(V) \rightarrow \bigwedge^2(V) \cong k$, the usual Frobenius form on $\bigwedge(V)$. We conclude that $(\text{Mat}_2(k), \lambda_2)$ Frobenius degenerates to $(\bigwedge(V), \pi^2)$.

In the remainder of this subsection we address the following questions:

Consider the directed graph $G_{n,k}$ for each $n \in \mathbb{N}$ and field k with a vertex for each isomorphism class of n -dimensional Frobenius k -algebra and an arrow from A to B if A deforms to B as a Frobenius k -algebra, with some choice of forms. What is the shape of $G_{n,k}$? More specifically, viewed as a partially ordered set, does it have a greatest and least element? If not, what are the minimal and maximal elements?

Example 1.4.8. The subgraph of commutative Frobenius algebras, denoted $G_{4,k}^{\text{com}} \subset G_{4,k}$, for k algebraically closed appears in²⁰. $G_{4,k}^{\text{com}}$ has least element $k[x, y]/(x^2 - y^2, xy)$ and greatest element $k^{\oplus 4}$, and hence, by composing deformations, every four-dimensional commutative Frobenius k -algebra deforms to $k^{\oplus 4}$ and degenerates to $k[x, y]/(x^2 - y^2, xy)$. See Example 1.4.14 for the entire $G_{4,k}$, with characteristic of k not two.

By Remark 1.4.4, if an algebra has vanishing second cyclic cohomology, then it has no deformations as a Frobenius algebra, and hence is maximal. Any Frobenius deformation gives rise to a deformation of associative algebras by forgetting the Frobenius form and hence rigid algebras are maximal. In particular, this includes all semisimple algebras.

A commutative n -dimensional algebra is called *smoothable* if it deforms to $k^{\oplus n}$. The smallest non-smoothable commutative algebra is 8-dimensional, e.g.

$$k[x, y, z, w]/(x^2, xy, y^2, z^2, zw, w^2, xw - yz),$$

see⁴⁸. The smallest non-smoothable commutative Frobenius algebra is 14-dimensional, see¹⁰. Said differently, $k^{\oplus n}$ is the unique n -dimensional, rigid commutative algebra if $n < 8$ and the unique n -dimensional rigid commutative Frobenius algebra if $n < 14$. Classifying rigid algebras becomes intractable in higher dimensions. For a nice survey of the techniques used and difficulties encountered, see¹⁸, where 8-dimensional rigid algebras are classified.

One can generalize this notion to the non-commutative case by asking when an algebra deforms to a semisimple algebra. The smallest non-smoothable algebra is the 3-dimensional path algebra $P(A_2)$. The smallest non-smoothable Frobenius algebras are 4-dimensional, e.g. $k\langle x, y \rangle / (x^2, y^2, xy - 2yx)$, see Example 1.4.14.

We now classify minimal elements in $G_{n,k}$, i.e. most degenerate Frobenius algebras. Additionally, we characterize Frobenius algebras among associative algebras, as filtered deformations of this class of algebras.

One can view a finite-dimensional vector space V with a bilinear form $(-, -) : V \times V \rightarrow k$ as a multiplication μ on the graded vector space $V \oplus k$ with V in degree 1 and k in degree 2. The multiplication is trivially associative since $\mu \circ (\mu \otimes 1) = 0 = \mu \circ (1 \otimes \mu)$ and commutative if $(-, -)$ is symmetric. One can add a unit, a copy of the ground field in degree 0, to obtain a $\dim(V) + 2$ dimensional algebra,

$$A(V, (-, -)) := k \oplus V \oplus k$$

as a graded vector space with multiplication,

$$(a, b, c) \cdot (a', b', c') := (aa', ab' + ba', (b, b') + ac' + ca').$$

Moreover, if $(-, -)$ is non-degenerate, then projecting onto the copy of k in the second graded piece is a Frobenius form. These algebras are the only candidates for the most degenerate Frobenius algebras, as made precise in the following proposition.

Proposition 1.4.9. *Let A be a Frobenius algebra of dimension $n > 1$ over any field k . Then there exists a Frobenius form λ on A such that A degenerates as a Frobenius algebra to $A(V, (-, -))$ for some non-degenerate, bilinear form $(-, -) : V \otimes V \rightarrow k$.*

How general is this class of algebras? If V is finite-dimensional then choose a basis $\{e_i\}$ for V and write the bilinear form $(-, -)$ as a matrix M with (i, j) th entry $m_{ij} := (e_i, e_j)$. Over a field of characteristic different from 2, if $(-, -)$ is symmetric then one can change bases so that M is diagonal. Moreover, if all square roots of the diagonal entries are

contained in the field, then one can rescale any element a to $a/\sqrt{(a,a)}$ so that the matrix is given by the identity. Hence $A(V, (-, -)) \cong A(k^n, (e_i, e_j) = \delta_{i,j})$ as Frobenius algebras over an algebraically closed field of characteristic not two. To condense notation we write $Z_n := A(k^n, (e_i, e_j) = \delta_{i,j})$ and note that,

$$Z_n := k[1] \oplus k[x_1, \dots, x_{n-2}] \oplus k[w] \quad x_i \cdot x_j = \delta_{ij}w, \quad x_i \cdot w = 0 \quad \lambda : Z_n \rightarrow k[w].$$

for $n \geq 2$ and we define $Z_1 := k$. Therefore, in the commutative case, over an algebraically closed field of characteristic not two, there is a single most degenerate Frobenius algebra.

Corollary 1.4.10. *Every n -dimensional commutative Frobenius algebra over an algebraically closed field k with characteristic not two Frobenius degenerates, with some choice of form, to Z_n .*

The idea of the proof of the proposition is to write down an explicit 3-term filtration on the Frobenius algebra A with form λ by,

$$k1 \subset \ker(\lambda) \subset A$$

whose associated graded algebra is of the form $A(V, \lambda \circ \mu)$. Therefore, we first establish a technical lemma that says one can always find a Frobenius form such that $k1 \subset \ker(\lambda)$.

Lemma 1.4.11. *Let A be a Frobenius algebra with form λ over $k \neq \mathbb{F}_2$. Then either $A = k$ or there exists a unit $u \in A^\times$ such that $\lambda(u) = 0$. Consequently, there exists a Frobenius form λ' on A such that $\lambda'(1) = 0$.*

Proof. If λ vanishes on a unit $u \in A^\times$ then define $\lambda' := \lambda \circ L_u$ where L_u is left multiplication by $u \in A$, so $\lambda'(1) = \lambda(u) = 0$ as desired.

Suppose λ is non-zero on every unit. Notice that each element $x \in A$ with $x^n = 0$ gives rise to a unit $1 + x \in A^\times$ as

$$(1 + x) \left(\sum_{j=0}^{n-1} (-x)^j \right) = (1 \pm x^n) = 1.$$

Then $\lambda(1 \cdot \lambda(x) - \lambda(1) \cdot x) = 0$ implies $1 \cdot \lambda(x) - \lambda(1) \cdot x$ is not a unit and hence $\lambda(x) = 0$. We conclude that λ vanishes on all nilpotent elements and hence λ vanishes on the Jacobson radical, $J(A)$. Non-degeneracy of λ implies $J(A) = 0$. The vanishing of the Jacobson radical and the fact that A is Artinian together imply that A is semisimple. By the

Artin-Wedderburn Theorem, $A \cong \prod_{i=1}^m \text{Mat}_{n_i}(D_i)$ is a product of matrix algebras over division algebras over k .

Therefore, A has a subalgebra $k^{\sum_{i=1}^m n_i}$ of diagonal matrices with coefficients in k . Splitting the inclusion of this subalgebra and summing the components gives a Frobenius form

$$\lambda' : A \cong \prod_{i=1}^m \text{Mat}_{n_i}(D_i) \longrightarrow k^{\sum n_i} \longrightarrow k,$$

which by construction agrees with the usual trace for matrices in $\prod_i \text{Mat}_{n_i}(k)$. In particular, λ' vanishes on the units given by products of diagonal matrices with non-zero entries summing to zero. Such entries can be found since $k \neq \mathbb{F}_2$. As any two Frobenius forms differ by multiplication by a unit, see Remark 1.3.3, we have $\lambda' = \lambda \circ L_u$ for some $u \in A^\times$. In particular, λ vanishes on a unit if and only if λ' does. So we conclude that $m = 1$ and $n_1 = 1$.

Now $A \cong D_1$ and $\lambda : A \rightarrow k$ doesn't vanish on a unit only if λ is injective. Hence $A = k$, completing the proof. $\square$

Remark 1.4.12. If $k = \mathbb{F}_2$, the above proof holds for m even, and for m odd can be modified to show $A = \mathbb{F}_2^m$, a product of copies of $\mathbb{F}_2$ with pointwise multiplication. Non-degeneracy of a form $\lambda : A \rightarrow \mathbb{F}_2$ implies $\lambda(1, 0, \dots, 0) = \lambda(0, 1, 0, \dots, 0) = \dots = \lambda(0, \dots, 0, 1) = 1$, in which case $\lambda(1, 1, \dots, 1) = m \equiv 1 \pmod{2}$. Hence A has a unique Frobenius form, which takes the unit to one. Therefore, the assumption that $k \neq \mathbb{F}_2$ in Lemma 1.4.11 is necessary. However, if $m \geq 3$ is odd, then $\mathbb{F}_2^m \cong \mathbb{F}_2^2 \oplus \mathbb{F}_2^{m-2}$ Frobenius degenerates to $\mathbb{F}_2[x]/(x^2) \oplus \mathbb{F}_2^{m-2}$, which has Frobenius form $\lambda(b_1 + b_2x, a_1, \dots, a_{m-2}) = b_1 + b_2 + a_1 + \dots + a_{m-2}$ sending the unit to zero. Therefore, every Frobenius algebra Frobenius degenerates to a one whose form vanishes on a unit.

Proof. (of Prop 1.4.9) If $k \neq \mathbb{F}_2$, then by Lemma 1.4.11 A has a Frobenius form which vanishes on the unit. If $k = \mathbb{F}_2$ then A either has such a form, or can be Frobenius degenerated to an algebra B with form λ satisfying $\lambda(1) = 0$, by Remark 1.4.12. Then the filtration,

$$k1 \subset \ker(\lambda) \subset B$$

has associated graded algebra

$$\text{gr}(B) = k1 \oplus \ker(\lambda)/k1 \oplus B/\ker(\lambda).$$

Define $V := \ker(\lambda)/k1$ and observe $B/\ker(\lambda) \cong \text{im}(\lambda) \cong k$, so $\text{gr}(B) = k \oplus V \oplus k$ as a

graded vector space. λ is still a well-defined Frobenius form on $gr(B)$. And multiplication is defined on $gr(B)$ so that $(1, 0, 0)$ is the unit and $(0, v, 0) \cdot (0, w, 0) = \lambda(\mu(v, w))$. Hence A degenerates as a Frobenius algebra to $A(V, \lambda \circ \mu)$. $\square$

The upshot is that every n -dimensional Frobenius algebra has a filtered Frobenius degeneration to some $A(V, (-, -))$. In the commutative case, over an algebraically closed field of characteristic not two, $A(V, (-, -)) \cong Z_n$. Conversely, Frobenius algebras are characterized among associative algebras as being filtered deformations of some $A(V, (-, -))$, which necessarily preserves the Frobenius form by Remark 1.4.5.

Remark 1.4.13. Frobenius degenerations from $A(V, (-, -))$ to $A(V, (-, -)')$ by definition preserve the unit and top-graded piece and hence necessarily arise as degenerations of the underlying bilinear form. In terms of the corresponding matrices M and M' of the forms, one must have M' in the closure of the congruence class of M , i.e.

$$M' = \lim_{t \rightarrow 0} N_t M N_t^\top,$$

for some $N_t \in Mat_{|V|}(k((t)))$. For instance,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \lim_{t \rightarrow 0} \begin{pmatrix} t^2 & 1 \\ -1 & 0 \end{pmatrix} = \lim_{t \rightarrow 0} \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$$

realizes the skew-symmetric bilinear form as a degeneration of the form on $k_{x,y}^2$ given by $(x, x) = (x, y) = -(y, x) = 1 + (y, y) = 1$. This is the only non-trivial deformation among non-degenerate bilinear forms on k^2 , see²². The question of determining the closures of congruence classes of matrices is studied in²³, by first establishing a canonical form for every congruence class of matrices.

Example 1.4.14. Here is $G_{4,k}$ for characteristic of k not 2, where we have omitted arrows that can be obtained as a composition of the given arrows. The commutative component

appeared in²⁰, and is classical.

$$\begin{array}{c}
\frac{k[x, y]}{(x^2 - y^2, xy)} \longrightarrow \frac{k[x]}{(x^4)} \begin{array}{l} \nearrow \frac{k[x]}{(x^3)} \oplus k \\ \searrow \frac{k[x]}{(x^2)} \oplus \frac{k[y]}{(y^2)} \end{array} \begin{array}{l} \nearrow \frac{k[x]}{(x^2)} \oplus k \oplus k \\ \searrow \end{array} \longrightarrow k^{\oplus 4} \\
\\
\bigwedge(k^2) \longrightarrow \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} : a, b, c, d \in k \right\} \cong \Pi(A_2) \longrightarrow \text{Mat}_2(k) \\
\searrow \\
\frac{k\langle x, y \rangle}{(y^2, x^2 + yx, xy + yx)} \\
\\
\frac{k\langle x, y \rangle}{(x^2, y^2, xy - \lambda yx)}, \lambda \in k \setminus \{-1, 0, 1\}
\end{array}$$

Notice that the algebras on the far left are of the form $A(k^2, (-, -))$:

$$\begin{aligned}
A(k^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) &\cong k[x, y]/(x^2 - y^2, xy) \\
A(k^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) &\cong \bigwedge(k^2) \\
A(k^2, \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}) &\cong k\langle x, y \rangle/(x^2, y^2, xy - \lambda yx) \\
A(k^2, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}) &\cong k\langle x, y \rangle/(y^2, x^2 + yx, xy + yx).
\end{aligned}$$

The deformation of $\Pi(A_2) = k[e_1, e_2] \oplus k[\alpha, \alpha^*]$ to the matrix algebra is given by deforming the preprojective relation $[\alpha, \alpha^*] = t(e_1 - e_2)$. The degeneration to $\bigwedge(k^2)$ is the filtered Frobenius degeneration produced in the proof of Proposition 1.4.9, with λ given by summing the non-diagonal entries, which vanishes on the identity. The deformation from $\bigwedge(k^2)$ to $k\langle x, y \rangle/(y^2, x^2 + yx, xy + yx)$ comes from a deformation of the underlying bilinear forms, as explained in Remark 1.4.13.

1.4.2 DEGENERATIONS OF PREPROJECTIVE ALGEBRAS

We define a notion of degeneration of decorations, which gives rise to degenerations at the level of decorated path and preprojective algebras. We use this notion to view decorated preprojective algebras as degenerations of ordinary preprojective algebras.

We want to deform pairs (A, M) where A is a k -algebra and M is an A -bimodule.

Definition 1.4.15. Let A, B be associative algebras and M an A -bimodule and N a B -bimodule. One says (A, M) *deforms* to (B, N) if there exists a pair (D, P) where D is a $k[[t]]$ -module deforming A to B and P is a D -bimodule such that,

- $P/tP \cong M$ as A -bimodules
- $P[t^{-1}] \cong N((t))$ as $B((t))$ -bimodules

Moreover, one says the deformation is *flat* if $(D, P) \cong (A[[t]], M[[t]])$ as $k[[t]]$ -modules.

One can modify this definition to work for left or right modules as well. Additionally, if D is a symmetric Frobenius deformation then we say the pair (D, P) is a symmetric Frobenius deformation.

Remark 1.4.16. In the presence of condition (F), one only needs to degenerate A as an (A, k) -bimodule, (k, A) -bimodule, or (A, A) -bimodule. All such degenerations come from degenerations of A as an algebra over k . Moreover, flatness of the module deformation is precisely flatness of the algebra deformation.

Hence, we arrive at the notion of degeneration of a decoration (Q, D_Q) by degenerating each k -algebra and bimodule.

Definition 1.4.17. Let $D_Q = (A_i, M^\alpha)$ and $D'_Q = (B_i, N^\alpha)$ be decorations of Q . One says D_Q *deforms* to D'_Q if the pair $(\oplus_{i \in Q_0} A_i, \oplus_{\alpha \in Q_1} M^\alpha)$ deforms to $(\oplus_{i \in Q_0} B_i, \oplus_{\alpha \in Q_1} N^\alpha)$.

Notice that degenerations of decorated quivers need not preserve condition (F), even for the quiver with a single vertex and no arrows, see Example 1.4.7. Consequently, we exclusively consider *symmetric* Frobenius degenerations of decorations, which do preserve condition (F).

Conceptually, one can view the pair $(\oplus_{i \in Q_0} D_i, \oplus_{\alpha \in Q_1} P^\alpha)$ as a decoration for Q by symmetric Frobenius $k[[t]]$ -algebras. Now the path algebra, $P_{k[[t]]}(Q, (D_i, P^\alpha))$, is a deformation

from $P_k(Q, (A_i, M^\alpha))$ to $P_k(Q, (B_i, N^\alpha))$. And, using the Frobenius form on the deformation, the preprojective algebra $\Pi_k[[t]](Q, (D_i, P^\alpha))$ is a deformation from $\Pi_k(Q, (A_i, M^\alpha))$ to $\Pi_k(Q, (B_i, N^\alpha))$.

Having defined degenerations of decorations we now define foldings of decorations. If Q is a quiver with graph automorphisms $\text{Aut}(Q)$, then one can form the quotient quiver $Q/\text{Aut}(Q)$ and decorate it with decoration $D_Q/\text{Aut}(Q) = (B_j, N^\beta)$ defined by:

$$B_j := \bigoplus_{i \in \text{Aut}(Q) \cdot j} A_i \quad \text{and} \quad N^\beta := \bigoplus_{\alpha \in \text{Aut}(Q) \cdot \beta} M^\alpha.$$

In this case, we say (Q, D_Q) *folds* to $(Q/\text{Aut}(Q), D_Q/\text{Aut}(Q))$.

Consider Q a quiver with the decoration

$$C_Q := \{A_i = k, M^\alpha = k\},$$

known as the *constant k -decoration*. With this decoration, one recovers the classical notions,

$$\text{Rep}(Q, C_Q) \cong \text{Rep}(Q) \quad P(Q, C_Q) = P(Q) \quad \Pi(Q, C_Q) = \Pi(Q).$$

Next suppose Q has non-trivial graph automorphisms and fold (Q, C_Q) to $(Q/\text{Aut}(Q), C_Q/\text{Aut}(Q))$, so the decoration consists of sums of copies of k at each vertex. Notice,

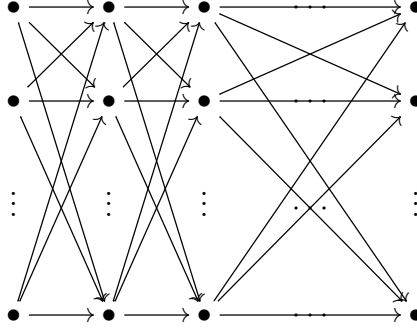
$$\begin{aligned} \text{Rep}(Q) &\cong \text{Rep}(Q, C_Q) \cong \text{Rep}(Q/\text{Aut}(Q), C_Q/\text{Aut}(Q)) \\ P(Q) &= P(Q, C_Q) \cong P(Q/\text{Aut}(Q), C_Q/\text{Aut}(Q)) \\ \Pi(Q) &= \Pi(Q, C_Q) \cong \Pi(Q/\text{Aut}(Q), C_Q/\text{Aut}(Q)) \end{aligned}$$

so we've built a decorated quiver with (1) the same preprojective algebra as an ordinary quiver and (2) a non-trivial degeneration to a decorated quiver for each symmetric Frobenius degeneration at each folded vertex.

It is natural to ask which decorated quivers, and hence decorated preprojective algebras, arise from ordinary quivers, and hence ordinary preprojective algebras, in this way. In the case of a quiver with no arrows, this is the class of Frobenius algebras with a Frobenius deformation to $k^{\oplus n}$, denoted $\mathcal{F}_n$. More generally, any decorated quiver (Q, D_Q) satisfying condition (F) with $\bigoplus_i A_i \in \mathcal{F}_{\sum_i \dim_k(A_i)}$ deforms to an ordinary quiver. Consequently, a large class of decorated preprojective algebras deform to ordinary preprojective algebras,

including many in³⁶.

Example 1.4.18. For any dimension vector $d = (d_i) \in \mathbb{N}^n$, and choice of Frobenius algebra $F_i \in \mathcal{F}_{d_i}$ for $i \in \{1, \dots, n\}$ can obtain a decorated A_n quiver with the k -algebras given by F_i , and the arrows decorated by $F_i \otimes F_{i+1}$ as a degeneration of the k -constant decoration of the quiver,



with d_i vertices in the i th column.

1.4.3 FLATNESS CONJECTURE

In this subsection, we conjecture that all flat Frobenius degenerations of decorations induce flat degenerations of the preprojective algebras.

Recall that a deformation of associative algebras D deforming A to B is flat if $D \cong A[[t]]$ as $k[[t]]$ -modules. If D is finitely-generated over $k[[t]]$ then flatness is equivalent to $\dim_k(A) = \dim_k(B)$. In the absence of flatness, the vector space underlying B is no larger than that of A .

Proposition 1.4.19. *Let A and B be finite-dimensional k -algebras. If A deforms to B as k -algebras, then $\dim_k(A) \geq \dim_k(B)$.*

This Proposition is a special case of the upper semicontinuity of the fiber, see Theorem 12.8 in³⁹.

In all of the cases where we use this inequality one has a finite filtration on B , $B_{\leq 0} \subset B_{\leq 1} \subset \dots \subset B_{\leq m} = B$ such that A is the associated graded algebra and hence there is a surjection, $\varphi_n : A_n \rightarrow B_{\leq n}/B_{\leq n-1}$ for each n and hence,

$$\begin{aligned} \dim(A) &= \sum_{n=0}^m \dim(A_n) \geq \sum_{n=0}^m \dim(B_{\leq n}/B_{\leq n-1}) = \sum_{n=0}^m \dim(B_{\leq n}) - \dim(B_{\leq n-1}) \\ &= \dim(B_{\leq m}) - \dim(B_{\leq -1}) = \dim(B). \end{aligned}$$

In general, the inequality can be strict, e.g. $k[[t]][x]/(xt, x^2)$ deforms $k[x]/(x^2)$ to k .

Additionally, in our set-up A and B both have $\mathbb{N}$ -gradings, distinct from the grading already on A as a filtered degeneration, by path length. The deformation preserves the length of each element and so we will have the more refined inequality on the level of Hilbert–Poincaré series

$$h_A(t) := \sum_{j \in \mathbb{N}} \dim(A_j) t^j$$

given by $h_A(t) \geq h_B(t)$ in the sense that the difference $h_A(t) - h_B(t)$ has non-negative coefficients.

In the interest of computing the Hilbert–Poincaré series of decoration preprojective algebras, we would like to know when the degeneration from the ordinary preprojective algebra, defined in the previous section, is flat. Flatness of the length zero and one paths says the algebras and the bimodules in the decoration have the same dimension. We conjecture that this is all that is needed to ensure flatness in any length.

Conjecture 1.4.20. *Suppose D_Q flatly Frobenius deforms to D'_Q . Then with respect to the path length grading,*

$$h_{\Pi(Q, D_Q)}(t) = h_{\Pi(Q, D'_Q)}(t).$$

For ordinary preprojective algebras of Dynkin quivers, one can use the Euler–Poincaré principle applied to the Schofield resolution to get a formula for the Hilbert–Poincaré series:

$$h_{\Pi(Q)}(t) = \frac{1 - Pt^{\max}}{1 - At + t^2}$$

where A is the adjacency matrix for Q , and P is the permutation matrix of the vertices induced from the Nakayama automorphism, see⁴⁵. In the non-Dynkin case, one can again use the Euler–Poincaré principle, now applied to the Koszul resolution, to get a formula for the Hilbert–Poincaré series:

$$h_{\Pi(Q)}(t) = \frac{1}{1 - At + t^2}.$$

If we assume the Schofield and Koszul resolutions degenerate to *resolutions* in the decorated setting, then the same analysis produces the following formula. Let $A = (a_{i,j})$ be a $|Q_0| \times |Q_0|$ matrix with coefficients in $\mathbb{Z}$ defined by

$$a_{i,j} := \sum_{\alpha: i \rightarrow j \in Q_1} \frac{\dim_k(M^\alpha)}{\dim_k(A_i)}.$$

Let $D = (d_{i,j})$ be a diagonal matrix defined by $d_{i,j} = \delta_{ij} \dim_k(A_i)$. We conjecture the following formula holds:

Conjecture 1.4.21. *The Hilbert–Poincaré series of $\Pi(Q, D_Q)$, with $D_Q = (A_i, M^\alpha)$ is given by,*

$$h_{\Pi(Q, D_Q)}(t) = \begin{cases} (I + t^{\max_i} P) D (1 - At + t^2)^{-1} & \text{if } (Q, D_Q) \text{ is Dynkin} \\ D (1 - At + t^2)^{-1} & \text{otherwise} \end{cases}$$

where, in the Dynkin case, $\max_i$ is the maximum path length of a non-zero homogeneous element in $\Pi(Q, D_Q)$ and P is the permutation matrix for the permutation of the set $\{1_{A_i}\}_{i \in Q_0}$ given by the Nakayama automorphism.

By Proposition 1.4.19, it suffices to prove Conjecture 1.4.20, at the extremes. That is, for a fixed quiver Q , it suffices to prove the conjecture for $D_Q = (A_i, M^\alpha)$ with each A_i a most degenerate Frobenius algebra and with each A_i a most deformed Frobenius algebra. By Proposition 1.4.9 and Remark 1.4.13, the most degenerate Frobenius algebras are given by a vector space with a non-degenerate bilinear form without degenerations to an inequivalent non-degenerate bilinear form.

In the commutative case, if k is algebraically closed and characteristic zero, with $n < 14$, $k^{\oplus n}$ is the unique most deformed Frobenius algebra and Z_n is the unique most degenerate Frobenius algebra, by¹⁰ and Corollary 1.4.10. Therefore, to prove the conjecture for commutative decorations of total dimension less than 14, one only needs to establish a single equality of Hilbert–Poincaré series. Additionally, one Hilbert–Poincaré series is equal to an ordinary preprojective algebra and hence is known and the other cannot have any smaller coefficients. Therefore, the conjecture reduces to finding a spanning set (of a specified size) for a single algebra.

In particular, in the Dynkin setting, the automorphism groups of the underlying diagrams are: (1) trivial in types B, C, F, G , (2) $\mathbb{Z}/2\mathbb{Z}$ in types A, D, E except D_4 and (3) the symmetric group on the three valence one vertices in type D_4 . Therefore, by folding the constant decoration at these vertices, one gets the Frobenius algebras $k, k^{\oplus 2}, k^{\oplus 3}$. Consequently, in the Dynkin setting, we only need to consider the maximal Frobenius degeneration of decorations $k, k^{\oplus 2}, k^{\oplus 3}$ to $k, k[x]/(x^2)$, and $k[x]/(x^3)$ respectively. In Section 5, we exploit these observations to prove the conjecture in the Dynkin case.

Remark 1.4.22. By generalizing a result of Etingof and Eu, in²⁹, we believe that the conjecture can be reduced to the Dynkin and extended Dynkin cases. In more detail, for any connected non-Dynkin quiver Q there exists an extended Dynkin subquiver Q_E .

Filter $\Pi(Q)$ by giving arrows in $\overline{Q_E}$ degree one, while all others are degree zero. For any subquiver of $Q' \subset Q$, denote by $\widehat{Q'}$ the quiver with arrow set Q'_1 but vertex set Q_0 . Then,

$$gr(\Pi(Q)) = \Pi(\widehat{Q_E}) * \Pi(\widehat{Q \setminus Q_E}, (Q_E)_0)$$

where $\Pi(\widehat{Q \setminus Q_E}, (Q_E)_0)$ is the algebra obtained from $\Pi(\widehat{Q \setminus Q_E})$ by deleting the relation at each vertex in Q_E . This determines the Hilbert–Poincaré series of $\Pi(Q)$ in terms of the known polynomial for Q_E and a partial preprojective algebra. Future work will attempt to generalize this argument to the decorated setting.

Evidence for Conjecture 1.4.20 consists of the following:

- (1) a proof of the conjecture in the case Q is Dynkin, see Theorem 1.4.26,
- (2) a proof of the conjecture in the case $Q = \tilde{D}_4$ with commutative algebras at the vertices see²⁰, and computer evidence suggesting that commutativity can be removed,
- (3) and computer evidence in the case

$$Q = (\{1, \dots, n\}, \{\alpha_1, \dots, \alpha_{n-1}\}, s(\alpha_i) = i, t(\alpha_i) = n)$$

for $n \leq 50$, in low path-graded pieces.

Here we present computations in Magma providing evidence for the conjecture.

Let Q_{n+1} be a quiver with $n + 1$ vertices, n sources and one sink. Q_{n+1} can be folded to A_2 with decoration $k^{\oplus n}$ and k at the vertices. Then one can maximally degenerate the decoration to Z_n and k . Denote the decorated preprojective algebra by $\Pi(Z_n)$ and its j -path graded piece by $\Pi^j(Z_n)$.

Using Magma we've shown,

Proposition 1.4.23.

$$\dim \Pi^j(Q_{n+1}) = \dim \Pi^j(Z_n)$$

for $(j, n) \in \{1, \dots, 10\} \times \{1, \dots, 10\} \cup \{1, \dots, 5\} \times \{11, \dots, 50\}$.

Corollary 1.4.24. *Every degeneration of $\Pi(Q_n)$ by degenerating the decoration is flat in the j th graded piece, for $j \leq 5$ and $n \leq 50$.*

Additionally, we computed the dimension of the decorated preprojective algebra of the A_2 quiver, with labels k and $\text{Mat}_n(k)$, denoted $\Pi(\text{Mat}_n(k))$.

Proposition 1.4.25.

$$\dim \Pi^j(\text{Mat}_n(k)) = \dim \Pi^j(Z_{n^2})$$

for $(j, n) \in \{1, \dots, 10\} \times \{1, 2, 3\} \cup \{1, \dots, 5\} \times \{4, 5, 6, 7\}$.

Combining the propositions, one has

$$\dim \Pi^j(\text{Mat}_n(k)) = \dim \Pi^j(Q_{n^2+1})$$

despite there being no deformation of $k \rightarrow \text{Mat}_n(k)$ to $k \rightarrow k^{\oplus n^2}$.

This suggests that Theorem 3.8 in²⁰, giving the dimensions $\dim \Pi^j(F)$ for all $j \in \mathbb{N}$ and for any F , a four-dimensional *commutative* Frobenius algebra, may hold for non-commutative Frobenius algebras as well.

In the next section we prove the conjecture in the case of degenerations of Dynkin quivers.

Theorem 1.4.26. *Let (Q, D_Q) be Dynkin. Suppose D_Q Frobenius degenerates flatly to D'_Q . Then, $\Pi(Q, D_Q)$ and $\Pi(Q, D'_Q)$ are both Frobenius algebras and $\Pi(Q, D_Q)$ Frobenius degenerates flatly to $\Pi(Q, D'_Q)$.*

Remark 1.4.27. Theorem 1.4.26 only addresses Frobenius degenerations of preprojective algebras coming from degenerations of decorated quivers. This does not include, for instance, the degeneration of $\Pi(A_2)$ to $\bigwedge(k^2)$ from Example 1.4.14, which in degree zero degenerates $ke_1 \oplus ke_2$ to k .

Proof. For each non-simply laced Dynkin diagram X obtained by folding Q denote by $\Pi(X)$ the quiver obtained in the statement above with the most degenerate decorations. To show flatness it suffices to compute the dimension of $\Pi(X)$, apriori no smaller than the dimension of $\Pi(Q)$ by Proposition 1.4.19. Hence we need to show:

- $\dim(\Pi(G_2)) \leq \dim(\Pi(D_4)) = 28$
- $\dim(\Pi(F_4)) \leq \dim(\Pi(E_6)) = 156$
- $\dim(\Pi(B_n)) \leq \dim(\Pi(A_{2n-1})) = \frac{1}{3}n(2n-1)(2n+1)$
- $\dim(\Pi(C_n)) \leq \dim(\Pi(D_{n+1})) = \frac{1}{3}n(n+1)(2n+1)$

This is done in each subsection of Section 5.

In each case, the Frobenius structure on $\Pi(X)$ can be viewed as a composition:

$$\lambda : \Pi(X) \xrightarrow{\pi} \Pi^{\max_t}(X) \cong \Pi^0(X) \xrightarrow{\oplus \lambda_i} k,$$

where π is the projection to the maximum length paths and $\oplus_i \lambda_i$ is the Frobenius form on the sum of the decorations at the vertices. Since $\oplus \lambda_i$ is non-degenerate by assumption, it suffices to show π is non-degenerate. In other words, we need to show any path can be extended to a maximum length path, i.e. for all $q \in \Pi(X)$ there exists $p \in \Pi(X)$ such that $pq \in \Pi^{\max_t}(X)$. This is apparent in the explicit bases we produce, where each basis element is visibly a subpath of a maximum length basis element. From this perspective the deformation to an ordinary preprojective algebra is a Frobenius deformation, coming from the Frobenius deformations of the decorations at the vertices. $\square$

Remark 1.4.28. These arguments hold over any field, as the classification of 3-dimensional Frobenius algebras over k is independent of k , and the calculations do not involve dividing by integers.

1.5 THE DYNKIN CASE

This section is dedicated to a proof of Theorem 1.4.26, which proves Conjecture 1.4.20 in the case of Q Dynkin. In the previous section, we reduced Theorem 1.4.26 to computing spanning sets for the decorated quivers,

$$\Pi(B_n), \Pi(C_m), \Pi(F_4), \Pi(G_2)$$

for $n \geq 2$ and $m \geq 4$, defined by folding and maximally degenerating (Q, C_Q) with Q respectively A_{2n-1} , D_{m+1} , E_6 , and D_4 . In diagrams,

$$\begin{array}{ccc} \begin{array}{c} k \rightarrow \cdots \rightarrow k \\ \nearrow \\ A_{2n-1} = k \\ \searrow \\ k \rightarrow \cdots \rightarrow k \end{array} & \xRightarrow{\text{fold}} & k \rightarrow k \oplus k \rightarrow \cdots \rightarrow k \oplus k \xrightarrow{\text{deg}} B_n = k \rightarrow S \rightarrow \cdots \rightarrow S \\ \\ \begin{array}{c} k \\ \nearrow \\ D_{m+1} = k \rightarrow \cdots \rightarrow k \\ \searrow \\ k \end{array} & \xRightarrow{\text{fold}} & k \rightarrow \cdots \rightarrow k \rightarrow k \oplus k \xrightarrow{\text{deg}} C_m = k \rightarrow \cdots \rightarrow k \rightarrow S \end{array}$$

$$\begin{array}{ccc}
E_6 = k \rightarrow k & \xrightarrow{\text{fold}} & k \rightarrow k \rightarrow k \oplus k \rightarrow k \oplus k \xrightarrow{\text{deg}} F_4 = k \rightarrow k \rightarrow S \rightarrow S \\
\begin{array}{c} \nearrow k \\ \searrow k \end{array} & & \\
D_4 = k \rightarrow k & \xrightarrow{\text{fold}} & k \rightarrow k \oplus k \oplus k \xrightarrow{\text{deg}} G_2 = k \rightarrow S' \\
\begin{array}{c} \nearrow k \\ \searrow k \end{array} & &
\end{array}$$

where $S := k[x]/(x^2)$ and $S' := k[x]/(x^3)$ are the most degenerate Frobenius algebras of dimension 2 and 3, respectively. Now,

$$\begin{array}{ll}
\Pi(B_n) := \Pi(k \rightarrow S \rightarrow \cdots \rightarrow S) & \Pi(F_4) := \Pi(k \rightarrow k \rightarrow S \rightarrow S) \\
\Pi(C_n) := \Pi(k \rightarrow \cdots \rightarrow k \rightarrow S) & \Pi(G_2) := \Pi(k \rightarrow S')
\end{array}$$

We present these examples in a pedagogical order. The first example, $\Pi(G_2)$ can be carefully worked out by hand without machinery and mirrors the computations of $\Pi(D_4)$. The second example, $\Pi(F_4)$ requires more knowledge of preprojective algebras and is more difficult. The third and fourth examples, $\Pi(B_n)$ and $\Pi(C_n)$ use Gröbner bases.

Remark 1.5.1. The B_n and C_n examples are long since we provide the reader with the additional information of how to arrive at a basis, instead of merely checking a set is a basis, in order to demystify the computation. We also give a proof of linear independence by applying the Buchberger's algorithm and showing that all ambiguities are resolvable. In light of the inequality in Proposition 1.4.19 this is not strictly necessary, but highlights a technique available for computing the Hilbert–Poincaré series of decorated quivers not arising as degenerations. The reader can distill each subsection to half a page by first listing the Gröbner basis of the ideal $(r) \subset P(Q, D_Q)$ and the proposed spanning set. Then one can verify that the elements of the Gröbner basis lie in the ideal generated by the relations and show that the proposed set spans modulo (multiples of) leading terms of the Gröbner basis.

Remark 1.5.2. Recall that flipping the orientation of an arrow $\alpha \in Q_1$ in a quiver Q gives a new quiver, denote it Q^α , with the same double $\overline{Q} = \overline{Q}^\alpha$. Moreover, one can

identify the path algebras

$$P(\overline{Q}) \rightarrow P(\overline{Q^\alpha})$$

by sending

$$\alpha \mapsto \alpha^* \quad \alpha^* \mapsto -\alpha.$$

Now let Q be a bipartite graph and fix a decomposition $Q_0 = Q'_0 \sqcup Q''_0$. By applying orientation flips, one can arrange for each source to lie in Q'_0 and each target in Q''_0 . Such flips give equivalences on the path algebras of the doubled quivers, described above. Then observe for any arrow $\alpha \in Q_1$, one has

$$\left(\sum_{e' \in Q'_0} e' - \sum_{e'' \in Q''_0} e'' \right) (\alpha \alpha^* - \alpha^* \alpha) = \alpha \alpha^* + \alpha^* \alpha$$

and

$$\left(\sum_{e' \in Q'_0} e' - \sum_{e'' \in Q''_0} e'' \right) (\alpha \alpha^* + \alpha^* \alpha) = \alpha \alpha^* - \alpha^* \alpha.$$

Therefore, the two-sided ideal generated by $(\alpha \alpha^* - \alpha^* \alpha)$ equals that generated by $(\alpha \alpha^* + \alpha^* \alpha)$, and consequently one can drop the signs in the definition of the preprojective algebra. This argument applies equally well in the setting of decorated bipartite quivers, which includes all examples in the next section. Therefore, for computations in Section 5 we will drop all minus signs in the relations for convenience.

1.5.1 CASE: $\Pi(G_2)$

Let $Q = A_2$ with $D_Q = (\{k, S := k[x]/(x^3)\}, \{{}_k S_S\})$, visualized as

$$k \xrightarrow{{}_k \otimes_k S} S.$$

To compute the preprojective algebra one doubles the quiver and doubles the decoration by adding the bimodule $\text{Hom}_k({}_k S_S, k) \cong {}_S S_k$.

We will begin by describing an explicit basis for the algebra $\Pi(G_2) := \Pi(Q, D_Q)$. Since a degeneration cannot have smaller dimension, we have

$$\dim \Pi(G_2) \geq \dim \Pi(D_4) = 28$$

So we will produce a spanning set with 28 elements, which is automatically linearly independent by the above discussion on degenerations.

Remark 1.5.3. To distinguish between the length zero paths $1 \in k$ and $1 \in S$ as well as the length one paths $1 \in {}_S S_k$ and $1 \in {}_k S_S$ we use subscripts: ${}_k 1_k, {}_S 1_k, {}_k 1_S, {}_S 1_S$. Often, the path γ will be written ${}_{s(\gamma)} \gamma_{t(\gamma)}$, to delineate its location.

Proposition 1.5.4. *The following set is a basis of the algebra*

$$\Pi(G_2) = T_{S \oplus k}({}_k S_S \oplus {}_S S_k) / \langle {}_k 1 \otimes_S {}_S 1_k + 1 \otimes_k x^2 + x \otimes_k x + x^2 \otimes_k 1 \rangle$$

as a vector space over k , presented by length with $\alpha := {}_k 1 \otimes_S x_k$, $\beta := {}_k 1 \otimes_S x_k^2$ and $\mathcal{B} := \{1, x, x^2\} \subset S$ a basis for S over k :

Length 0: $\mathcal{B}, {}_k 1_k$

Length 3: $\alpha \otimes_k 1 \cdot \mathcal{B}, \mathcal{B} \cdot 1 \otimes_k \alpha$

Length 1: $\mathcal{B} \cdot {}_S 1_k, {}_k 1_S \cdot \mathcal{B}$

Length 4: $\alpha \otimes_k \beta, 1 \otimes_k \alpha \otimes_k 1 \cdot \mathcal{B}$

Length 2: $\alpha, \beta, 1 \otimes_k 1 \cdot \mathcal{B}, x \otimes_k 1 \cdot \mathcal{B}$.

Corollary 1.5.5. ${}_S 1_S \cdot \Pi(G_2)$ is a free left S -module and $\Pi(G_2) \cdot {}_S 1_S$ is a free right S -module

Corollary 1.5.6. *The Hilbert–Poincaré series for $\Pi(G_2)$ is*

$$h_{\Pi(G_2)}(t) = 4 + 6t + 8t^2 + 6t^3 + 4t^4$$

where the coefficient of t^i is the dimension $\Pi^i(G_2)$, the subspace spanned by length i paths. Using the bigrading (t, s) by path length and x -degree, one obtains the matrix-valued Hilbert–Poincaré series

$$(1 + t^2 s) \begin{pmatrix} 1 + t^2 s^2 & (1 + s + s^2)t \\ (1 + s + s^2)t & (1 + s + s^2)(1 + t^2) \end{pmatrix}$$

where enumerating the vertices the (i, j) th entry is the Hilbert–Poincaré series $h_{e_i \Pi(G_2) e_j}(t, s)$.

Define $M := S$ as a (k, S) -bimodule and $N := S$ as a (S, k) -bimodule. Before presenting a proof, we make a few useful observations. Notice that in length 2,

$$T_{S \oplus k}^2(M \oplus N) = (M \oplus N) \otimes_{S \oplus k} (M \oplus N) = M \otimes_S N \oplus N \otimes_k M$$

and similarly for higher length terms: the two non-vanishing tensor products alternate in M and N .

Moreover, in odd length the two non-zero tensor products are isomorphic via a map reading each tensor product from right to left. In even length $2m$, the term starting with M is a vector space generated by applying $1 \otimes_S -$ to paths of length $2m - 1$. The term starting with N is an S -bimodule generated as left S -module by applying $1 \otimes_k -$ to paths of length $2m - 1$ and hence generated as a vector space by left multiplication by ${}_S 1_k$, ${}_S x_k$, and ${}_S x_k^2$.

Finally, notice that the tensor algebra $T_{S \oplus k}(M \oplus N)$ has a decomposition

$$\begin{aligned} T_{S \oplus k}(M \oplus N) &= (1_k + 1_S) T_{S \oplus k}(M \oplus N) (1_k + 1_S) \\ &= {}_k T_{S \oplus k}(M \oplus N)_k \oplus {}_k T_{S \oplus k}(M \oplus N)_S \\ &\quad \oplus {}_S T_{S \oplus k}(M \oplus N)_k \oplus {}_S T_{S \oplus k}(M \oplus N)_S \end{aligned}$$

and an $\mathbb{N}$ -grading given by the number of times x appears in a monomial expression. Since the relations $1 \otimes_S 1$ and $Rel := 1 \otimes_k x^2 + x \otimes_k x + x^2 \otimes_k 1$ are homogeneous with respect to this grading and respect the above decomposition, the quotient $\Pi(G_2)$ inherits both. We conclude that all relations are generated by relations involving only paths with the same starting and ending vertex of the same length and x -degree. We write ${}_i \Pi^d(G_2)_{A_j}$ for the subspace of linear combinations of length d paths starting at vertex i and ending at vertex j .

Proof of Proposition 1.5.4. Once we show that the given set spans, then linear independence will follow from Proposition 1.4.19 together with the fact that $\Pi(D_4)$ is 28-dimensional. So we will only need to show the spanning property.

For lengths 0 and 1, it is clear that these elements span. For higher lengths we now explain how any tensor product of lower length basis elements can be realized in the span of our given basis.

In length 2, $M \otimes_S N \cong S$ is three dimensional generated by $1 \otimes_S 1$, α , and β . Since $1 \otimes_S 1$ is a relation in $\Pi(G_2)$, it follows that α and β generate ${}_k \Pi^2(G_2)_k$. $N \otimes_k M$ is generated by $1 \otimes_k 1$ as an S -bimodule and hence is 9-dimensional as a vector space, generated by the six elements $1 \otimes_k 1 \cdot \mathcal{B}$ and $x \otimes_k 1 \cdot \mathcal{B}$ together with the elements

$$\begin{aligned} Rel &:= 1 \otimes_k x^2 + x \otimes_k x + x^2 \otimes_k 1 \\ x \cdot Rel &= x \otimes_k x^2 + x^2 \otimes_k x \\ x^2 \cdot Rel &= x^2 \otimes_k x^2 \end{aligned}$$

all relations in $\Pi(G_2)$.

In length 3, both direct summands have the same dimension, and applying $-\otimes_k M$ to the 2-dimensional vector space $\text{Span}\{\alpha, \beta\} = M \otimes_S N / \langle 1 \otimes_S 1 \rangle$ yields the spanning set,

$$\alpha \otimes_k \mathcal{B} \cup \beta \otimes_k \mathcal{B}.$$

For each $n \in \{2, 3, 4\}$, if we sum all of the above elements of x -degree equal to n , we obtain the relation $x^{n-2} \otimes_S \text{Rel}$. Hence the smaller set $\alpha \otimes_k \mathcal{B}$ spans ${}_k\Pi^3(G_2)_S$, as desired.

In length 4, first notice that the quotient ${}_k\Pi^4(G_2)_k$ of $M \otimes_S N \otimes_k M \otimes_S N$ has a spanning set

$$\alpha \otimes_k \mathcal{B} \otimes_S 1,$$

the image of the spanning set $\alpha \otimes_k \mathcal{B}$ under the surjective map $-\otimes_S 1$. The term $\alpha \otimes_k 1 \otimes_S 1$ is zero in the quotient, as is the term

$$\alpha \otimes x \otimes_S 1 \equiv 1 \otimes_S \text{Rel} \otimes_S 1 \bmod 1 \otimes_S 1.$$

Next the quotient ${}_S\Pi^4(G_2)_k$ of $N \otimes_k M \otimes_S N \otimes_k M$ is generated as a left S -module by

$$1 \otimes_k 1 \otimes_S x \otimes_k \mathcal{B}.$$

But, this set also generates as a vector space since,

$$x \cdot (1 \otimes_k 1 \otimes_S x \otimes_k 1) - (1 \otimes_k 1 \otimes_S x \otimes_k 1) \cdot x \equiv 1 \otimes_k 1 \otimes_S \text{Rel} - \text{Rel} \otimes_S 1 \otimes_k 1$$

are congruent modulo $1 \otimes_S 1$, so the left action is equivalent to a right action, which produces no new terms as $S \otimes_S S \cong S$.

To show $\Pi^{\geq 5}(G_2) = 0$, it suffices to show that $\Pi^5(G_2) = 0$, and by the isomorphism between the direct summands, it suffices to show that ${}_k\Pi^5(G_2)_S$ is zero. As a right S -module ${}_k\Pi^5(G_2)_S$ is generated by the element,

$$1 \otimes_S x \otimes_k 1 \otimes_S x^2 \otimes_k 1,$$

which is the relation $1 \otimes_S x \otimes_k 1 \otimes_S \text{Rel} + 1 \otimes_S \text{Rel} \otimes_S 1 \otimes_k x$ modulo $1 \otimes_S 1$, and hence zero in $\Pi(G_2)$. We conclude that our given set spans and hence forms a basis for $\Pi(G_2)$. $\square$

Remark 1.5.7. Here is an alternative proof of linear independence that does not appeal to degeneration arguments, but rather examines the relations carefully. The technical computations are left to the reader.

Notice any two basis elements are in different bigradings or start at different vertices and

hence are linearly independent with the following exceptions

$$x \otimes_k 1, 1 \otimes_k x \quad \text{and} \quad x \otimes_k x, 1 \otimes_k x^2$$

which can be seen to be linearly independent directly from the definition. To see that all listed paths are non-zero observe each is a subpath of $1 \otimes_S x \otimes_k 1 \otimes_S x^2$ or $1 \otimes_k 1 \otimes_S x \otimes_k x^2$, which can be shown non-zero by writing out the length four elements in the two-sided idea generated by the relations.

Next notice that $\Pi(G_2)$ is Frobenius with the form given by projecting to the one-dimensional space, spanned by the element,

$$1 \otimes_S x \otimes_k 1 \otimes_S x^2 + 1 \otimes_k 1 \otimes_S x \otimes_k x^2$$

summing over both cycles of maximal bigrading (4, 3). Non-degeneracy of this form amounts to the fact that every path is a subpath of a maximal length cycle, which can be checked by hand. This Frobenius form deforms to the usual one on $\Pi(D_4)$.

Corollary 1.5.8. $\Pi(G_2)$ is a flat Frobenius degeneration of $\Pi(D_4)$.

Restricting to the above basis, the Frobenius pairing, given by first multiplying and then applying the Frobenius form, takes values in $\{-1, 0, 1\}$ with the -1 values given by the pairs:

$$\begin{array}{lll} (1 \otimes_S x, 1 \otimes_S x^2) & (1 \otimes_k x, 1 \otimes_k x^2) & (x \otimes_k 1, x \otimes_k x) \\ ({}_S x_k, 1 \otimes_S x \otimes_k x) & ({}_k x_S^2, {}_S 1 \otimes_k 1 \otimes_S x) & ({}_k x_S, {}_S x \otimes_k 1 \otimes_S x) \\ ({}_k 1_S, {}_S x^2 \otimes_k 1 \otimes_S x) & & \end{array}$$

The Nakayama automorphism is given by $a \mapsto -a$ on the elements appearing in the above pairings and is the identity on all other basis elements, and hence squares to the identity.

Using computations from the proof of Proposition 1.5.4, it is not hard to compute the center, $Z(\Pi(G_2))$.

Proposition 1.5.9. $Z(\Pi(G_2)) = k \cdot 1 \oplus \Pi^4(G_2) \cong k[0] \oplus (k \oplus S)[4]$.

Proof. The Frobenius structure allows one to identify $\Pi^i(G_2)$ with $\Pi^{4-i}(G_2)^*$ for $i \in \{0, 1, 2, 3, 4\}$. In particular, the Frobenius pairing identifies $\Pi^4(G_2)$ with $S \oplus k$, explaining the second isomorphism.

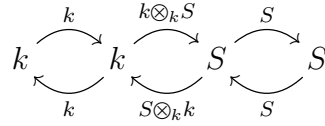
$\Pi^4(G_2) \subset Z(\Pi(G_2))$ as one only needs to check that the commutator $[{}_S x_S, 1 \otimes_k 1 \otimes_S x \otimes_k 1]$ vanishes, which is done in the proof of Proposition 1.5.4 above. Conversely, $Z(\Pi(G_2)) \subset$

$k[0] \oplus k[4] \oplus S[4]$ as the centralizer of ${}_k 1_k$ excludes all paths of odd length, the centralizer of ${}_k 1_S$ excludes all paths of even length less than 4 except the the span of the identity and the span of $\{1 \otimes_k \gamma\}_{\gamma \in \{1, x, x^2\}}$. The latter span is not in the centralizer of ${}_k x_S$. $\square$

1.5.2 CASE: $\Pi(F_4)$

Now we consider the example $Q = A_4$ with decoration

$D_Q = (\{k, k, S, S\}, \{k, k \otimes_k S, S\})$. The doubled decorated quiver is,



We want to compute a basis for the decorated preprojective algebra

$\Pi(F_4) := \Pi(Q, D_Q)$. The length zero paths

$$e_1 := {}_1 1_1, \quad e_2 := {}_2 1_2, \quad e_3 := {}_3 1_3, \quad e_4 := {}_4 1_4, \quad {}_3 x_3, \quad {}_4 x_4$$

and length one paths

$${}_1 1_2, \quad {}_2 1_1, \quad {}_2 1_3, \quad {}_3 1_2, \quad {}_3 1_4, \quad {}_4 1_3$$

form a generating set. The quadratic relation R can be multiplied by the length zero path e_i to get a relation Ri at each vertex i , given by,

$$\begin{cases} R1 := {}_1 1_2 \otimes_k {}_2 1_1 & \text{at vertex 1} \\ R2 := {}_2 1_1 \otimes_k {}_2 1_1 + {}_2 1_3 \otimes_S {}_3 1_2 & \text{at vertex 2} \\ R3 := {}_3 1_2 \otimes_k {}_2 x_3 + {}_3 x_2 \otimes_k {}_2 1_3 + {}_3 1_4 \otimes_S {}_4 1_3 & \text{at vertex 3} \\ R4 := {}_4 1_3 \otimes_S {}_3 1_4 & \text{at vertex 4.} \end{cases}$$

Appealing directly to the definition, the relations at vertices 1, 2, and 4 are the usual commutators, ignoring signs in light of Remark 1.5.2. At vertex 3, one needs to use the Frobenius form $\lambda : S = k[x]/(x^2) \rightarrow k$ by $\lambda(a + bx) := b$ to get the dual basis $\{x, 1\}$ to the basis $\{1, x\}$, and hence the canonical element ${}_3 1_2 \otimes_k {}_2 x_3 + {}_3 x_2 \otimes_k {}_2 1_3$.

Remark 1.5.10. Consider ${}_3 1_4$ and ${}_4 1_3$ to have x -grading $1/2$, so that the relation at vertex 3 is homogeneous in the x -grading. We make this choice so the x -grading is symmetric, but if one would like to realize this algebra as a filtered degeneration of $\Pi(E_6)$ then one needs to choose integral x -gradings, such as $|{}_3 1_4|_x = 1$ and $|{}_4 1_3|_x = 0$.

Here we explain the methodology to compute a basis for this algebra, leaving the details to the reader. Recall it suffices to provide a spanning set with exactly $\dim(\Pi(E_6)) = 156$ elements.

Remark 1.5.11. To simplify notation for longer paths we define

$$\gamma_{a,b} := \begin{cases} {}_a 1_{a+1} \cdot {}_{a+1} 1_{a+2} \cdots {}_{b-1} 1_b & \text{if } a < b \\ {}_a 1_a & \text{if } a = b \\ {}_a 1_{a-1} \cdot {}_{a-1} 1_{a-2} \cdots {}_{b+1} 1_b & \text{if } a > b \end{cases}$$

using an ordering on the vertices Q_0 . Then define

$$\gamma_{i_1, i_2, \dots, i_n} := \gamma_{i_1, i_2} \gamma_{i_2, i_3} \cdots \gamma_{i_{n-1}, i_n}$$

where if $i_j < i_{j+1} < i_{j+2}$ or $i_j > i_{j+1} > i_{j+2}$ then we remove i_{j+1} from the notation in order to solely delineate changes in direction.

Observe that every path is equivalent to one passing through vertex 3 except those belonging to $X := \{{}_1 1_1, {}_2 1_2, {}_4 1_4, {}_4 x_4, {}_1 1_2, {}_2 1_1\}$. Exploiting this observation, for any pair of vertices i and j the map,

$$\Phi_{i,j} : e_3 \Pi(F_4) e_3 \mapsto e_i \Pi(F_4) e_j / k \cdot e_i X e_j \quad \alpha \mapsto \gamma_{i,3} \alpha \gamma_{3,j}$$

is surjective, where $k \cdot e_i X e_j$ denotes the k -span of paths in X from i to j . Therefore,

$$e_i \Pi(F_4) e_j \cong \Phi_{i,j}(e_3 \Pi(F_4) e_3) \oplus k \cdot e_i X e_j$$

as k -vector spaces and we have the equality

$$\begin{aligned} \dim(\Pi(F_4)) &= \sum_{i,j} \dim(e_i \Pi(F_4) e_j) \\ &= \dim k \cdot X + \sum_{i,j} \dim(\Phi_{i,j}(e_3 \Pi(F_4) e_3)) \\ &= \dim k \cdot X + \sum_{i,j} \dim(e_3 \Pi(F_4) e_3) - \dim(\ker(\Phi_{i,j})) \\ &= 6 + 16 \dim(e_3 \Pi(F_4) e_3) - \sum_{i,j} \dim(\ker(\Phi_{i,j})). \end{aligned}$$

Since $\dim(\ker(\Phi_{j,i})) = \dim(\ker(\Phi_{i,j}))$, we've reduced showing the inequality $\dim(\Pi(F_4)) \leq 156$ to computing a spanning set for $e_3 \Pi(F_4) e_3$ and linearly independent subsets for ten

subspaces, $\ker(\Phi_{i,j})$ for $i \leq j$.

We compute a spanning set for $e_3\Pi(F_4)e_3$ by building a surjection $q : k\langle\alpha, \beta\rangle \twoheadrightarrow e_3\Pi(F_4)e_3$, and then reducing the spanning set by identifying elements in the kernel of q . Every path in $e_3\Pi(F_4)e_3$ is equivalent to one avoiding vertex 1 and vertex 4, using the relations at vertex 2 and 3, respectively. Therefore, $e_3\Pi(F_4)e_3$ is generated by $q(\alpha) := {}_31_2 \otimes_k {}_21_3$ and $q(\beta) := {}_3x_3$. They satisfy the relations:

$$\begin{aligned} q(\alpha)^3 &= {}_31_2 \otimes_k {}_21_3 \otimes_S {}_31_2 \otimes_k {}_21_3 \otimes_S {}_31_2 \otimes_k {}_21_3 \\ &\stackrel{R2}{=} {}_31_2 \otimes_k {}_21_1 \otimes_k {}_11_2 \otimes_k {}_21_1 \otimes_k {}_11_2 \otimes_k {}_21_3 \\ &\stackrel{R1}{=} 0, \end{aligned}$$

$q(\alpha\beta + \beta\alpha)^2 \stackrel{R3}{=} ({}_31_4 \otimes {}_41_3)^2 \stackrel{R4}{=} 0$, and $q(\beta^2) = x^2 = 0$. Hence q factors through the quotient,

$$q' : \frac{k\langle\alpha, \beta\rangle}{(\alpha^3, \beta^2, (\alpha\beta + \beta\alpha)^2)} \twoheadrightarrow e_3\Pi(F_4)e_3.$$

Let C denote $k\langle\alpha, \beta\rangle/(\alpha^3, \beta^2, (\alpha\beta + \beta\alpha)^2)$. We find a k -basis for C in Example 1.5.16, after introducing the Diamond Lemma, given by:

- (0) $1, \beta$
- (2) $\alpha, \alpha\beta, \beta\alpha, \beta\alpha\beta$
- (4) $\alpha^2, \alpha^2\beta, \alpha\beta\alpha, \alpha\beta\alpha\beta, \beta\alpha^2, \beta\alpha^2\beta$
- (6) $\alpha^2\beta\alpha, \alpha^2\beta\alpha\beta, \alpha\beta\alpha^2, \alpha\beta\alpha^2\beta, \beta\alpha^2\beta\alpha, \beta\alpha^2\beta\alpha\beta$
- (8) $\alpha^2\beta\alpha^2, \alpha^2\beta\alpha^2\beta, \alpha^2\beta\alpha^2\beta, \alpha\beta\alpha^2\beta\alpha, \alpha\beta\alpha^2\beta\alpha\beta$
- (10) $\alpha^2\beta\alpha^2\beta\alpha, \alpha^2\beta\alpha^2\beta\alpha\beta.$

Here we have ordered the elements first by number of occurrences of α and then by alphabetical order. We conclude that C is a free rank 12 right S -module and has dimension 24 as a k -module. This shows q' is an isomorphism since,

$$24 = \dim((e_3 + e_5)\Pi(E_6)(e_3 + e_5)) \leq \dim(e_3\Pi(F_4)e_3) \leq \dim(C) = 24.$$

The first inequality is a refinement of $\dim(\Pi(E_6)) \leq \dim(\Pi(F_4))$ formed by keeping track of the decomposition of paths by starting and ending vertices. Here E_6 is folded to F_4 by

identifying vertices 3 and 5 and vertices 4 and 6. In particular,

$$\dim(\Pi(F_4)) \leq 6 + 16(24) - \sum_{i,j} \dim(\ker(\Phi_{i,j})) = 390 - \sum_{i,j} \dim(\ker(\Phi_{i,j}))$$

and it remains to bound the dimension of the kernel of each map $\Phi_{i,j}$ from below.

Observe that $\gamma_{3,j}\alpha^j = 0 = \alpha^j\gamma_{j,3}$ for all $j \in \{1, 2, 3\}$ and $\gamma_{3,4}(\alpha\beta + \beta\alpha) = 0 = (\alpha\beta + \beta\alpha)\gamma_{3,4}$. Hence we have the following containment,

$$\begin{aligned} \ker(\Phi_{i,j}) &= \ker(\Phi_{i,3} \circ \Phi_{3,j}) = \ker(L_{\gamma_{i,3}} \circ R_{\gamma_{3,j}}) = \ker(R_{\gamma_{3,j}} \circ L_{\gamma_{i,3}}) \\ &\supset \ker(R_{\gamma_{3,j}}) \cup \ker(L_{\gamma_{i,3}}) \\ &\supset \langle \alpha^i, (\alpha\beta + \beta\alpha)^{5-i} \rangle_R \cup \langle \alpha^j, (\alpha\beta + \beta\alpha)^{5-j} \rangle_L, \end{aligned}$$

where $\langle s \rangle_R$ and $\langle s \rangle_L$ denote the right and left ideals in C generated by s and L_s and R_s denote left and right multiplication by $s \in C$.

For each i, j , one can find linearly independent elements of the above ideals such that the total number of elements is 390-156. For instance, if $i = 4$ and $j = 3$, then $\ker(\Phi_{4,3}) \supset \langle \alpha^i, (\alpha\beta + \beta\alpha)^{5-i} \rangle_R \cup \langle \alpha^j, (\alpha\beta + \beta\alpha)^{5-j} \rangle_L = \langle \alpha\beta + \beta\alpha \rangle_R$.

Using a basis for C , it is not hard to find a basis for this ideal, given by

$$\begin{aligned} \langle \alpha\beta + \beta\alpha \rangle_R &= \text{Span}\{\alpha\beta + \beta\alpha, \beta\alpha\beta, \alpha^2\beta + \alpha\beta\alpha, \alpha\beta\alpha\beta, \alpha^2\beta\alpha, \alpha^2\beta\alpha\beta, \beta\alpha^2\beta\alpha, \\ &\quad \alpha\beta\alpha^2\beta\alpha, \alpha\beta\alpha^2\beta\alpha\beta, \alpha\beta\alpha^2\beta\alpha\beta, \alpha^2\beta\alpha^2\beta\alpha, \alpha^2\beta\alpha^2\beta\alpha\beta\} \end{aligned}$$

which is 12-dimensional. Hence $\Phi_{4,3}(C)$ is at most 12-dimensional.

Note that the number of linearly independent elements needed for each i, j is

$$\dim C - \sum_{i' \mapsto i} \sum_{j' \mapsto j} \dim(e_{i'}\Pi(E_6)e_{j'}) - \dim(e_{i'}\Pi(E_6 \setminus \{3, 5\}))e_{j'})$$

where the sum is taken over all vertices i' and j' that fold to i and j , respectively. So $E_6 \setminus \{3, 5\} = A_1 \sqcup A_1 \sqcup A_2$ is the disconnected quiver obtained by discarding vertices 3 and 5 and all arrows with either 3 or 5 as source or target. If $i = 4$ and $j = 3$ this is precisely

$$24 - 2 \dim(e_3\Pi(E_6)e_4) - 2 \dim(e_3\Pi(E_6)e_6) = 12,$$

as desired.

The remaining computations are done similarly and are left to the reader. The upshot is

that the 4×4 matrix with (i, j) th entry given by $e_i \Pi(F_4) e_j$ is

$$\begin{pmatrix} 4 & 6 & 8 & 4 \\ 6 & 12 & 16 & 8 \\ 8 & 16 & 24 & 12 \\ 4 & 8 & 12 & 8 \end{pmatrix}.$$

For each entry of this matrix, we compute the Hilbert–Poincaré series, a polynomial in variables s and t defined so the coefficient of $t^n s^m$ is dimension of $\Pi^{(n,m)}(F_4)$, i.e. the path length n and x -grading m component of $\Pi(F_4)$. That is,

Proposition 1.5.12. $h_{\Pi(F_4)}(s, t) = (1 + t^4 s)$.

$$\begin{pmatrix} (1+t^6 s) & t(1+t^2 s+t^4 s^2) & t^2(1+s)(1+t^2 s) & t^3 s^{1/2}(1+s) \\ t(1+t^2 s+t^4 s^2) & (1+t^2)(1+t^2 s+t^4 s) & t(1+s)(1+t^2 s) & t^2 s^{1/2}(1+s)(1+t^2) \\ t^2(1+s)(1+t^2 s) & t(1+s)(1+t^2 s) & (1+s)(1+t^2+t^4)(1+t^2 s) & t s^{1/2}(1+s)(1+t^2+t^4) \\ t^3 s^{1/2}(1+s) & t^2 s^{1/2}(1+s)(1+t^2) & t s^{1/2}(1+s)(1+t^2+t^4) & (1+s)(1+t^6 s) \end{pmatrix}$$

and

$$h_{\Pi(F_4)}(t, 1) = 6 + 10t + 14t^2 + 18t^3 + 20t^4 + 20t^5 + 20t^6 + 18t^7 + 14t^8 + 10t^9 + 6t^{10}$$

where the coefficient in front of t^n is the dimension of the vector space of paths of length n in $\Pi(F_4)$.

As before, the Frobenius pairing is defined by projecting onto the $(10, 3)$ -bigraded piece and adding the coefficients. In more detail, if $\mathcal{B}$ is a basis of $\Pi(F_4)$ with dual basis $\mathcal{B}^\vee = \{\gamma^\vee : \gamma \in \mathcal{B}\}$, then

$$(\alpha, \beta) := \sum_{\gamma^\vee \in Y^\vee} \gamma^\vee(\alpha\beta)$$

where

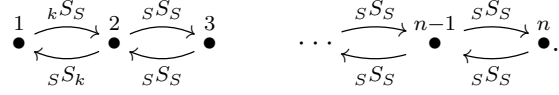
$$Y := \{1_1 1_2 \beta \alpha \beta \beta_2 1_1, \alpha \beta \beta \alpha \beta, {}_3 1_2 \beta \alpha \beta \beta_2 1_3, {}_4 1_3 \cdot {}_3 1_2 \alpha \beta \beta_2 1_3 \cdot {}_3 1_4\}$$

is the set of paths of $(10, 3)$ -bidegree in the basis for $\Pi(F_4)$. This Frobenius pairing deforms to the usual one in $\Pi(E_6)$, defined analogously.

Corollary 1.5.13. $\Pi(F_4)$ is a flat Frobenius degeneration of $\Pi(E_6)$.

1.5.3 CASE: $\Pi(B_n)$

Consider the decorated double quiver



The relations in the preprojective algebra are

$$\left\{ \begin{array}{ll} R1 := {}_1 1_2 \otimes_S {}_2 1_1 & \text{at vertex 1} \\ R2 := {}_2 1_1 \otimes_k {}_1 x_2 + {}_2 x_1 \otimes_k {}_1 1_2 + {}_2 1_3 \otimes_S {}_3 1_2 & \text{at vertex 2} \\ Rj := {}_j 1_{j+1} \otimes_S {}_{j+1} 1_j + {}_j 1_{j-1} \otimes_S {}_{j-1} 1_j & \text{at vertex } j \in \{3, 4, \dots, n-1\} \\ Rn := {}_n 1_{n-1} \otimes_S {}_{n-1} 1_n & \text{at vertex } n. \end{array} \right.$$

These relations are homogeneous in path length and can be made homogeneous in x -degree by declaring the x -grading of ${}_i 1_{i+1}$ and ${}_{i+1} 1_i$ to be $1/2$ for $i > 1$. Therefore, the preprojective algebra inherits both gradings from the the path algebra.

The idea is to produce a basis for $\Pi(B_n)$ by realizing it as a subalgebra of $P(Q, D_Q)$, denoted $P(Q, D_Q)_{\text{irr}}$. The key ingredient is an algebra map

$$r : P(Q, D_Q) \rightarrow P(Q, D_Q)_{\text{irr}}$$

satisfying:

$$(1) \quad \ker(r) = \ker(P(Q, D_Q) \rightarrow \Pi(B_n)) \quad \text{and} \quad (2) \quad r \circ i = \text{id}_{P(Q, D_Q)_{\text{irr}}},$$

where $i : P(Q, D_Q)_{\text{irr}} \rightarrow P(Q, D_Q)$ is the natural inclusion. The first condition says r descends to $\Pi(B_n)$ and the second condition implies r is surjective. Therefore r induces an identification $\Pi(B_n) \cong P(Q, D_Q)_{\text{irr}}$ as algebras.

To define r , we choose a partial ordering $\leq$ on paths in $P(B_n)$. This gives rise to a notion of leading term, for any linear combination of paths, denoted lt . Then, for each relation Rj in $\Pi(B_n)$, define a k -linear map r_j which is the identity on all monomials except taking the leading term $\text{lt}(Rj)$ to the strictly smaller element $\text{lt}(Rj) - Rj$. This map is called a reduction and informally it imposes the relation Rj only in the direction that makes the path smaller with respect to the partial order.

Remark 1.5.14. For any pair of monomials $A, B \in P(Q, D_Q)$, $ARjB$ is also a relation and hence one defines the reduction r_{AjB} taking $A(\text{lt}(Rj))B$ to $A(\text{lt}(Rj) - Rj)B$. Since A

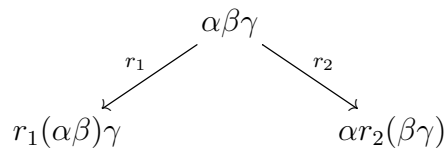
and B will be clear in context, we simplify but abuse notation and write r_j for any such reduction.

One would like to define r to be any finite composition of these reductions such that the resulting expression is irreducible, i.e. every reduction acts as the identity. The existence of r will be clear in our context since each path will have a finite disorder index. This index is a natural number measuring the failure of a path to be irreducible and it strictly decreases under reductions, and hence is eventually zero. However, r is, in general, *not* well-defined, as it depends on the order that the reductions are applied.

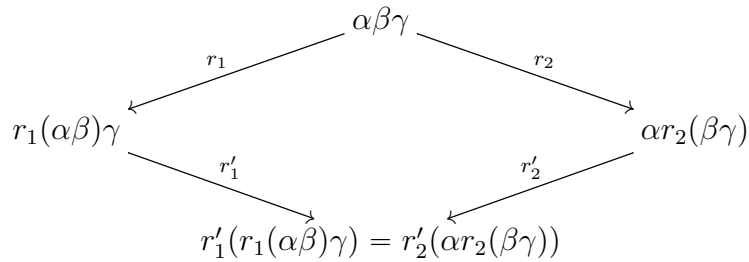
The Diamond Lemma in³ says that r is well-defined, if whenever a product of monomials $\alpha\beta\gamma$ is reduced by r_1 and r_2 to elements $r_1(\alpha\beta)\gamma$ and $\alpha r_2(\beta\gamma)$, each further reduces to the same irreducible element. Such a 5-tuple $(\alpha, \beta, \gamma, r_1, r_2)$ where $\alpha\beta$ is the leading term for r_1 and $\beta\gamma$ is the leading term for r_2 , is called an *ambiguity*. And we say the ambiguity *resolves* if $r_1(\alpha\beta)\gamma$ and $\alpha r_2(\beta\gamma)$ have the same reduced form.

Theorem 1.5.15 (Theorem 1.2 in³). *Let $P(Q, D_Q)$ be the decorated path algebra and consider an ordering $\leq$ on path in $P(Q, D_Q)$ that satisfies the descending chain condition (i.e every sequence of paths $p_1 \geq p_2 \geq \dots \geq p_n \geq \dots$ eventually stabilizes to some path p_N .) Suppose $\{r_i\}$ is a set of kQ_0 -linear reductions compatible with the ordering (i.e. $r_i(p) \leq p$ for all p and all i .) Then every path reduces uniquely to normal form (i.e. a sum of irreducible paths) after applying finitely many reductions if and only if every ambiguity resolves.*

The name Diamond Lemma comes from the visual that every diagram



called an ambiguity, can be completed to a diagram,



for some composition of reductions, r'_1 and r'_2 , called a resolution.

If every ambiguity is resolvable then r is well-defined and clearly satisfies (1) and (2) above. Hence i is a splitting for r and realizes,

$$P(Q, D_Q) \cong P(Q, D_Q)_{\text{irr}} \oplus I$$

the decorated preprojective algebras as a *subalgebra*, as opposed to a quotient, of the decorated path algebra. From this perspective a basis for the decorated preprojective algebras is given by a basis for the irreducible paths.

Example 1.5.16. Consider $C = k\langle\alpha, \beta\rangle/(\alpha^3, \beta^2, (\alpha\beta + \beta\alpha)^2)$ from the previous section. One defines a partial order on monomials in $k\langle\alpha, \beta\rangle$ by $\gamma \leq \gamma'$ if the length of γ is less than the length of γ' or they have the same length but γ is first alphabetically. Then the three relations are viewed as reductions:

$$r_1(\alpha^3) = 0, \quad r_2(\beta^2) = 0, \quad r_3(\beta\alpha\beta\alpha) = -\alpha\beta\alpha\beta - \beta\alpha\alpha\beta.$$

Reducing $\beta\alpha\beta\alpha^2$ using r_1 gives zero and using $r_3 \circ r_3 \circ r_3$ gives $\beta\alpha^2\beta\alpha^2 - \alpha^2\beta\alpha^2\beta + \alpha\beta\alpha^2\beta\alpha$. And hence we add the reduction, $r_4(\beta\alpha^2\beta\alpha^2) = \alpha^2\beta\alpha^2\beta - \alpha\beta\alpha^2\beta\alpha$, and the remaining ambiguities, $\beta\beta\alpha^2\beta\alpha^2$ and $\beta\alpha^2\beta\alpha^2\alpha$, are resolvable. We conclude that a word is reduced if it has no occurrences of $\alpha^3, \beta^2, \beta\alpha\beta\alpha$, or $\beta\alpha^2\beta\alpha^2$.

Reductions r_1 and r_2 reduce any word to one alternating between β and either α or α^2 . The reduction r_3 says a reduced word has only a single $\beta\alpha$, which necessarily occurs at the end. The reduction r_4 says any reduced word starting with β cannot have repeating α^2 . Hence the longest reduced word is $\alpha^2\beta\alpha^2\beta\alpha\beta$ and it is not hard to realize all 24 basis elements of C as subwords.

Define the following ordering of the generators at each vertex i

$${}_ix_{i-1} > {}_ix_{i+1} > {}_i1_{i-1} > {}_i1_{i+1}$$

with $1 < i < n$ and ${}_nx_{n-1} > {}_n1_{n-1}$ and extend this to a partial ordering on all paths lexicographically. That is, view each path γ of length l as a product of length one paths $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_l$ and define $\alpha > \beta$ for paths α and β if

- (I) α is longer than β or
- (II) they are the same length with α having the larger x -grading or
- (III) they have the same length and x -grading with $\alpha_i > \beta_i$ for the smallest i where $\alpha_i \neq \beta_i$.

To make this definition well-defined one should write the any subpath $x \otimes_S 1$ as $1 \otimes_S x$, always moving the x 's towards the end of the path.

The relations $R1, R2, \dots, Rn$ give rise to a system of reductions, with respect to the lexicographical ordering.

Upward Reductions:

- $r_j^U(j1_{j-1} \otimes_S j-11_j) = j1_{j+1} \otimes_S j+11_j$
for $3 \leq j \leq n-1$

- $r_n^U(n1 \otimes_S 1_n) = 0$

X-Reductions:

- $r_2^X(2x \otimes_k 1_2) = 21 \otimes_k x_2 + 21 \otimes_S 1_2$

- $r_{2,1}^X(21_1 \otimes_k 1_2 \otimes_S 2x_1) = \gamma_{2,3,1}$

Integral Reductions:

- $r_j^I(\gamma_{2,1,j+1,j}) = \gamma_{2,3,1,j}$
for $2 \leq j \leq n-1$

Death Reductions:

- $r_{i,j}^D(\gamma_{i,n-j+i+1,1,j}) = 0$
for $2 \leq i < j \leq n$

Mound Reductions:

- $r_j^M(\gamma_{1,j,1}) = 0$, for $2 \leq j \leq n$

where we use the notation $\gamma_{i_1, i_2, \dots, i_n}$ with $i_1, \dots, i_n \in Q_0$ introduced in Remark 1.5.11.

Upward Reductions pull paths towards vertex n and realize paths visiting vertex n consecutively to be zero, see Figure 1.4. The X -reductions remove occurrences of x when possible and moves them to the end of the path otherwise, see Figure 1.3. The Integral Reductions straighten paths after visiting vertex 1, thus removing any instances of the shape $\int$, see Figure 1.1. The Death Reductions identify when paths visiting vertex 1 are too long and hence zero and the Mound Reductions identify paths of the shape $\cap$ starting and ending at vertex 1 to be zero, see Figure 1.2.

One reads these figures as reductions according to the following rules:

- The paths are depicted in two dimensions with integer vertical component indicating the vertices of the quiver and the horizontal component increases when the path changes directions, for visualization.
- Unbroken paths are products of the generators ${}_i1_{i+1}, {}_{i+1}1_i$ for $i \in \{1, \dots, n-1\}$. Paths involving generators ${}_ix_i$ for $i \in \{2, 3, \dots, n\}$ are broken by x at vertex i , i.e. $-x-$ indicates an insertion of the length zero path ${}_ix_i$ at this vertex.
- Caution: Signs are not depicted.

Remark 1.5.17. Starting from the relations of the preprojective algebra, viewed as reductions $r_2^M, r_2^X, r_j^U, r_n^U$ one realizes the need for the remaining reductions in light of the following ambiguities.

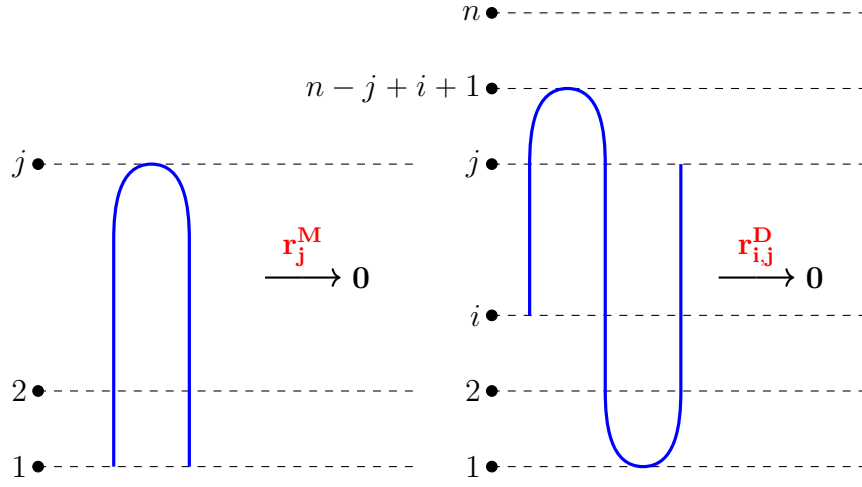


Figure 1.2: Mound Reduction r_j^M for $1 \leq j \leq n$ on the left and Death Reduction $r_{i,j}^D$ on the right

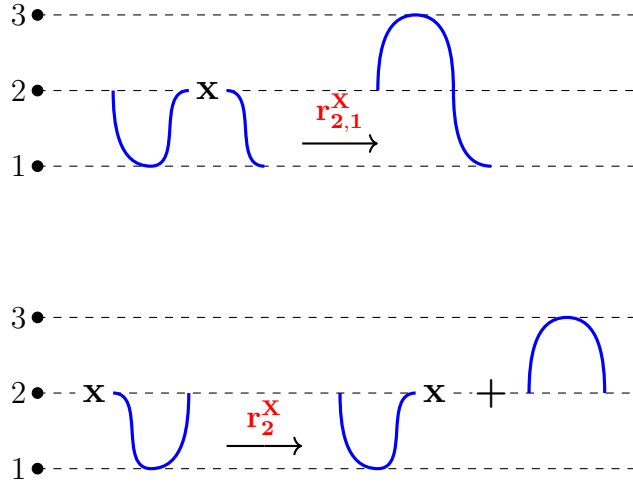


Figure 1.3: X-reductions $r_{2,1}^X$ and r_2^X

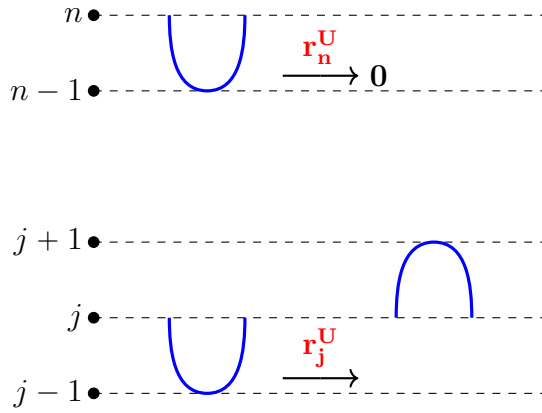


Figure 1.4: Upward Reduction r_j^U for $3 \leq j \leq n-1$ and r_n^U

Corollary 1.5.19. *The bigraded Hilbert–Poincaré series are:*

$$\begin{aligned}
h_{e_i \Pi(B_n) e_j}(s, t) &= t^{|i-j|} s^{|i-j|/2} (1 + s) \\
&\quad \cdot (1 + st^2 + s^2 t^4 + \dots + s^{n-\max\{i,j\}} t^{2n-2\max\{i,j\}}) \quad \text{if } i, j \geq 2 \\
&\quad + t^{i+j-2} s^{(i+j-4)/2} (1 + s) \\
&\quad \cdot (1 + st^2 + s^2 t^4 + \dots + s^{(n-j-1)/2} t^{n-j-1}) \\
h_{e_1 \Pi(B_n) e_j}(s, t) &= h_{e_j \Pi(B_n) e_1}(s, t) = t^{j-1} s^{(j-1)/2} (1 + s) \\
&\quad \cdot (1 + st^2 + s^2 t^4 + \dots + s^{n-j} t^{2n-2j}) \quad \text{if } j \geq 2 \\
h_{e_1 \Pi(B_n) e_1}(s, t) &= 1 + st^2 + s^2 t^4 + \dots + s^{n-1} t^{2n-2}
\end{aligned}$$

and the x -degree of ${}_i 1_{i+1}$ and ${}_{i+1} 1_i$ are $1/2$ to ensure the relations $R2, R3, \dots, R(n-1)$ are homogeneous in the x -degree.

Proof. We first show that any path can be reduced to a linear combination of paths in the proposed basis. Let α be a path from i to j .

Case: α doesn't visit vertex 1

If α does not contain any occurrences of x then applying the Upward Reductions $r_n^U \circ r_{n-1}^U \circ \dots \circ r_3^U$ is either zero or $\gamma_{i,k,j}$ for $k \geq \max\{i, j\}$. Any occurrence of x can be moved to the end as all tensor products in the path are over S and hence two such occurrences gives zero.

Case: α visits vertex 1 once

If α doesn't contain an occurrence of x then by the above the subpaths before and after vertex 1 can be reduced to the form $\gamma_{i,k,2}$ and $\gamma_{2,l,j}$ and hence α can be reduced to $\gamma_{i,k,1,l,j}$. If $i = 1$ then such a path has the form $\gamma_{1,l,j}$ for $l \geq j$. Similarly, if $j = 1$ reading the path backward must have this form.

If $i, j > 1$ then applying the Integral Reductions $r_j^I \circ r_{j+1}^I \circ \dots \circ r_{l-1}^I$ gives $\gamma_{i,m,1,j}$, where the path is now direct after visiting vertex 1. Finally, applying the Death Reductions $r_{i,j}$ gives zero unless $m \leq n - j + i$.

If x occurs in α before visiting vertex 1, then one can arrange for the path to contain ${}_2 x_1$, since all tensor products are over S before 1. The X-Reduction r_2^X gives a linear combination of paths where x occurs after vertex 1. So the x can be taken to occur at the end, giving the desired form.

Case: α visits vertex 1 at least twice

If $j > 1$ then by applying the X-reductions $r_{2,1}^X \circ r_2^X$ successively one can ensure the x 's appear at the end and so the Mound Reductions $r_n^M \circ \dots \circ r_2^M$ send the subpath starting and ending at vertex 1 to zero. So the path must end at vertex 1 and cannot visit vertex 1 more than twice. Consequently, the path is concatenation of a path in the previous case ending at vertex 2 (i.e. $\gamma_{i,m,1,2}$, $\gamma_{i,m,1,2}x$, $\gamma_{1,k,2}$, $\gamma_{1,k,2}x$) with 21_1 and therefore, after applying the Mound Reductions $r_n^M \circ \dots \circ r_3^M \circ r_2^M$, is zero or of the form $\gamma_{i,m,1,22}x_1$ or $\gamma_{1,k}x\gamma_{k,1}$. The X-Reduction $r_{2,1}^X$ sends $\gamma_{i,m,1,22}x_1$ to a path visiting vertex 1 a single time and hence any path visiting vertex 1 twice is of the form $\gamma_{1,k}x\gamma_{k,1}$, for $k > 1$.

This shows that the above set is a spanning set. It follows from the Diamond Lemma that for linear independence one only needs to show that all ambiguities are resolvable.

The family of Integral Reductions, Mound Reductions, and Death Reductions all arise as ambiguities with Upward Reductions. Moreover, Upward Reductions cannot overlap with X-Reductions nor with other Upward Reductions and hence there are no overlap ambiguities involving them. Clearly there are no ambiguities among two reductions sending the path to zero and so it remains to check Integral Reductions and X-reductions.

Moreover, Integral Reductions overlap with Mound and Death Reductions in the same way that Upward Reductions do, with the resolution given by other Mound and Death Reductions. Integral Reductions cannot self overlap and the only ambiguity with X-reductions is r_2^X and r_2^I applied to $x\gamma_{2,1,3,2}$ which both further reduce to $\gamma_{2,3,1,2}x + \gamma_{2,4,2}$ by applying $r_3^U \circ r_2^X \circ r_2^I$. The final ambiguity to resolve is r_2^X and $r_{2,1}^X$ applied to $\gamma_{2,1,2}x\gamma_{2,1,2}$, with each term further resolving to zero by $r_2^M \circ r_3^M$. $\square$

Remark 1.5.20. Carefully reading the above argument, and defining:

$$r^U := r_n^U \circ \dots \circ r_3^U, \quad r^M := r_n^M \circ \dots \circ r_2^M, \quad r^I := r_{n-1}^I \circ \dots \circ r_2^I,$$

$$r^D := r_{n-1}^D \circ \dots \circ r_2^D \quad \text{where} \quad r_i^D := r_{i,i+1}^D \circ \dots \circ r_{i,n}^D$$

one can define $r : P(Q, D_Q) \rightarrow P(Q, D_Q)_{\text{irr}}$ to be the composition

$$r := r^M \circ r^D \circ r^U \circ r^I \circ r^U \circ (r_2^X \circ r_{2,1}^X)^n.$$

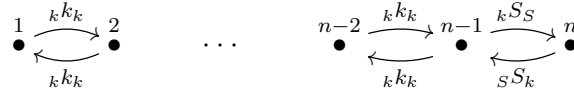
That is, one can first reduce a path by moving the x 's to the end then pushing upward until Integral Reductions arise and then further pushing upward until Mound or Death Reductions arise.

As before, the functional giving the sum of the coefficients of all the maximum length cycles containing an x gives a Frobenius form.

Corollary 1.5.21. $\Pi(B_n)$ is a flat Frobenius degeneration of $\Pi(A_{2n-1})$.

1.5.4 CASE: $\Pi(C_n)$

Consider the decorated quiver



The goal of this section is to compute a basis for the decorated preprojective algebra using the Diamond Lemma for rings, developed in³ and explained in the previous subsection.

Proposition 1.5.22. A basis for $e_i \Pi(C_n) e_j$ is given by

$$\{\gamma_{i,m,j}; \quad \gamma_{i,l,nn} x_n \gamma_{n,j} \mid m \leq \min\{i, j\}, i \geq l > \max\{i - j, 0\}\}$$

and hence $\dim(e_i \Pi(C_n) e_j) = 2 \min\{i, j\}$. In particular, $\Pi(C_n)$ is a flat degeneration of $\Pi(D_{n+1})$.

The relations in the preprojective algebra are

$$\begin{cases} 1 1_2 \otimes_{k_2} 2 1_1 & \text{at vertex 1} \\ j 1_{j+1} \otimes_{k_{j+1}} 1_j + j 1_{j-1} \otimes_{k_{j-1}} 1_j & \text{at vertex } j \in \{2, 3, 4, \dots, n-1\} \\ n-1 1_n \otimes_{S_n} 1_{n-1} + n-1 1_{n-2} \otimes_{k_{n-2}} 1_{n-1} & \text{at vertex } n-1 \\ n 1_{n-1} \otimes_{k_{n-1}} x_n + n x_{n-1} \otimes_{k_{n-1}} 1_n & \text{at vertex } n. \end{cases}$$

One views each relation as a reduction reducing the length or the number of times a path changes direction by pulling all paths towards vertex 1. More formally, one defines a partial order, determined on length one paths by

$$_i x_{i-1} > _i x_{i+1} > _i 1_{i-1} > _i 1_{i+1}$$

and extended to all paths lexicographically, as described in the previous subsection. Hence

the reductions are determined by:

$$\begin{aligned}
r_1(1_2 \otimes_{k_2} 2_1) &= 0 \\
r_j(1_{j+1} \otimes_{k_{j+1}} 1_j) &= -1_{j-1} \otimes_{k_{j-1}} 1_j \text{ for } j \in \{2, \dots, n-2\} \\
r_{n-1}(1_n \otimes_S 1_{n-1}) &= -1_{n-2} \otimes_{k_{n-2}} 1_{n-1} \\
r_n(x_{n-1} \otimes_{k_{n-1}} 1_n) &= -1_{n-1} \otimes_{k_{n-1}} x_n.
\end{aligned}$$

With this set of reductions there are unresolvable ambiguities and hence we introduce additional reductions:

Downward Reductions

- $r_1(1_2 \otimes_k 2_1) = 0$
- $r_j(1_{j+1} \otimes_{k_{j+1}} 1_j) = -1_{j-1} \otimes_{k_{j-1}} 1_j$ for $l < j < n-1$
- $r_{n-1}(1_n \otimes_S 1_{n-1}) = -1_{n-2} \otimes_{k_{n-2}} 1_{n-1}$

X-Reductions

- $r_k^X(n x_n \gamma_{n,k-1,k}) = \gamma_{n,n-1,n} x_n \gamma_{n,k}$ for $1 < k \leq n$

Death Reductions

- $r_{i,j}^D(\gamma_{i,i-j,n} x_n \gamma_{n,j}) = 0$ for $i > j$.

Visually, Figures 1.5, 1.6, and 1.7 provide schematics for the reductions, where paths are depicted with vertical position giving the vertex of the quiver and the horizontal component is strictly for visualization, as explained in the previous subsection.

Remark 1.5.23. The X-reductions are introduced to resolve the ambiguity created by reducing $n x_n \gamma_{n,n-1,n,k}$ using $r_k \circ \dots \circ r_{n-2} \circ r_{n-1}$ as opposed to r_n . The Death Reductions are then introduced to resolve the ambiguity created by reducing $\gamma_{i,n} x_n (\gamma_{n,n-1,n})^j \gamma_{n,j}$ using $r_{i-j+1} \circ r_{i-j+2} \circ \dots \circ r_n \circ (r_n^X)^j$ as opposed to $r_1 \circ r_2 \circ \dots \circ r_n$.

All ambiguities are resolvable:

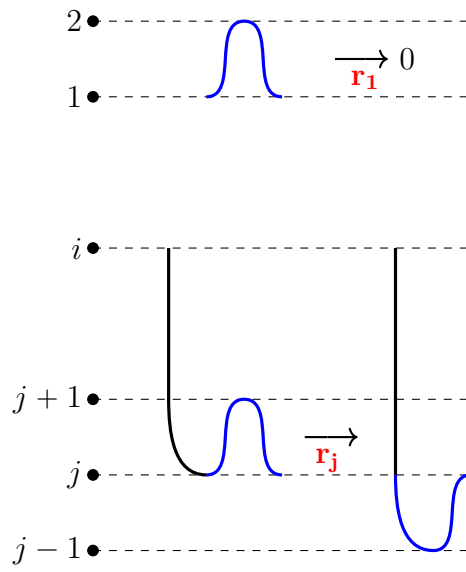


Figure 1.5: Downward Reductions

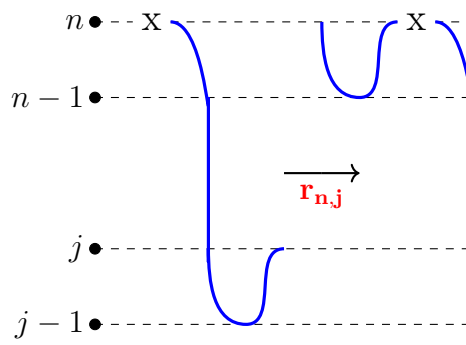


Figure 1.6: X-Reductions

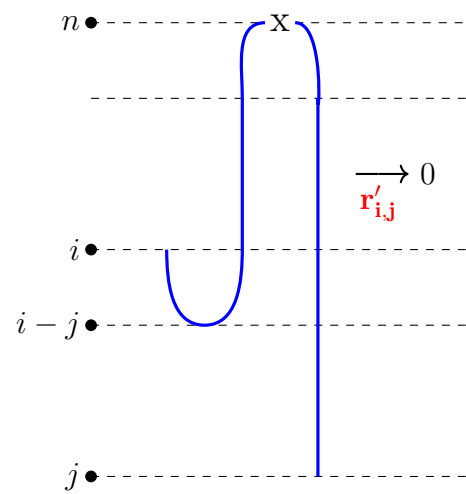


Figure 1.7: Death Reductions

(I) The ambiguity giving rise to additional X -reductions:

$$r_j^X - r_{j-1}^X \circ r_{j-1}(nx_n \gamma_{n,j-1,j,j-1}) = 0$$

(II) The ambiguity giving rise to the Death reductions:

$$r_1 - r_2^X \circ r_1^X(nx_n \gamma_{n,1,2,1}) = 0$$

(III) The Downward-Death ambiguity:

$$r_{i,j}^D - r_{i-1,j}^D \circ r_{i-j} \circ \cdots \circ r_{i-2} \circ r_{i-1}(\gamma_{i-1,i,i-j,nn} x_n \gamma_{n,i-j}) = 0$$

(IV) The X -Death ambiguity:

$$r_{i,j}^D - r_{i,j+1}^D \circ r_{i-j} \circ \cdots \circ r_{n-2} \circ r_{n-1} \circ r_{j+1}^X(\gamma_{i,i-j,nn} x_n \gamma_{n,j,j+1}) = 0$$

(V) The Double X -Death ambiguity:

$$r_{i,j}^D - r_{n-1}^X \circ r_{n-2}^X \circ \cdots \circ r_j^X(nx_n \gamma_{n,i-j-1,nn} x_n \gamma_{n,j}) = 0$$

With these reductions one can put any path in normal form. According to the Downward Reductions such a path cannot go from increasing to decreasing, except at vertex n . Consequently, paths not containing x must be of the form $\gamma_{i,m,j}$ for $m \leq \min\{i, j\}$, where equality gives the direct path. Moreover, no path can visit vertex n more than once since between the final two such visits the path can be reduced to $\gamma_{n,m,n}$ at which point iterated application of the X -reductions collides the two x 's giving zero.

Applying the X -reductions to a path with one instance of x allows for the path after x to be direct. Therefore, every such path can be reduced to one of the form $\gamma_{i,l,nn} x_n \gamma_{n,j}$ and the Death Reductions say that these are non-zero if $l > i - j$. This proves Proposition 1.5.22.

Corollary 1.5.24. *The matrix-valued Hilbert–Poincaré series of $\Pi(C_n)$ is of the form,*

$$\begin{pmatrix} 2 & 2 & 2 & \cdots & 2 \\ 2 & 4 & 4 & \cdots & 4 \\ 2 & 4 & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 4 & 6 & \cdots & 2n \end{pmatrix}$$

The bigraded Hilbert–Poincaré series

$$h_{e_i \Pi(C_n) e_j}(s, t) = t^{d(i,j)}(1 + t^2 + t^4 + \cdots + t^{2\min\{i,j\}-2})(1 + st^{2d(\max\{i,j\},n)})$$

where $d(a, b) = |a - b|$ is the distance between the vertices a and b .

As above, one can define a Frobenius form on the degenerate preprojective algebra using the cycles of maximal length, $\gamma_{i,nn}x_n\alpha^{i-1}\gamma_{n,i}$ for each $i \in \{1, \dots, n\}$.

Corollary 1.5.25. $\Pi(C_n)$ is a flat Frobenius degeneration of $\Pi(D_{n+1})$.

This is the final case of maximal degenerations of folding ADE quivers and hence we've completed the proof of Theorem 1.4.26.

2

THE 2-CALABI-YAU PROPERTY FOR MULTIPLICATIVE PREPROJECTIVE ALGEBRAS

Multiplicative preprojective algebras have recently gained attention in geometry and topology. These algebras appear in the study of certain wrapped Fukaya categories, see^{26, 25}, in the study of microlocal sheaves on rational curves, see⁴, and in the study of generalized affine Hecke algebras, see Appendix 1 in³¹. Their moduli spaces of representations are called multiplicative quiver varieties, an analogue of Nakajima's quiver varieties, which includes character varieties of rank n local systems on closed Riemann surfaces, or on open Riemann surfaces with punctures and monodromy conditions^{16, 57, 50}. Multiplicative quiver varieties have also been studied from various viewpoints in^{53, 5, 14}, and¹¹. A quantization was defined in⁴⁰ and further studied in³⁴.

Historically, the multiplicative preprojective algebra was defined by Crawley-Boevey and Shaw, in¹⁶, to view solutions of the Deligne–Simpson problem as irreducible representations of multiplicative preprojective algebras of certain star-shaped quivers. Their paper establishes the foundations for much of this work. For a fixed field k and a quiver Q with vertex set Q_0 and arrow set Q_1 and $q \in (k^\times)^{Q_0}$, Crawley-Boevey and Shaw define

$$\Lambda^q(Q) := \frac{L_Q}{J_Q} := \frac{k\overline{Q}[(1 + aa^*)^{-1}]_{a \in \overline{Q}}}{\left(r := \prod_{a \in Q_1} (1 + aa^*)(1 + a^*a)^{-1} - q\right)},$$

a quotient of the localized path algebra of the double quiver, L_Q , by the two-sided ideal J_Q generated by a single relation, r .

In section 7.5 of⁵⁰, Tirelli and the second author conjecture that multiplicative preprojective algebras of connected, non-ADE quivers are 2-Calabi–Yau. They observe, following⁶, that the 2-Calabi–Yau property would determine the (formal) local structure of the moduli space of representations. Namely, any formal neighborhood can be identified with the formal neighborhood of the zero representation of the moduli space of representations of some (additive) preprojective algebra. Among other applications mentioned in⁵⁰, it would follow that multiplicative quiver varieties are normal. In light of these applications, and others mentioned below, the goal of this paper is to work towards a proof of the following conjecture:

Conjecture 2.0.1. *$\Lambda^q(Q)$ is 2-Calabi–Yau for all $q \in (k^\times)^{Q_0}$ and all Q connected and not ADE; moreover, it is a prime ring.* If Q is furthermore not extended ADE, then $Z(\Lambda^q(Q)) = k$, and the Calabi–Yau structure is unique.*

We explain how one can reduce these conjectures to the extended ADE cases below, see Section 2.5.4 for more detail. We carry this procedure out for $Q = \widetilde{A}_n$, thereby proving the conjecture for all connected quivers containing it:

Theorem 2.0.2. *$\Lambda^q(Q)$ is 2-Calabi–Yau and prime for any $q \in (k^\times)^{Q_0}$ and any k a field, and Q connected and containing an unoriented cycle. If the containment is proper, then $Z(\Lambda^q(Q)) = k$, and the Calabi–Yau structure is unique.*

As observed, it follows the multiplicative quiver varieties for quivers containing cycles are formally locally isomorphic to ordinary quiver varieties, and in particular are normal. This includes (as an open subset) character varieties of Riemann surfaces of positive genus with punctures and prescribed monodromy conditions, as explained in⁵⁰ §3 (following^{16,57}). (In the case of closed Riemann surfaces, as observed in² Remark 8.8, this statement does not require our result, since the group algebra $\mathbb{C}[\pi_1(\Sigma)]$ is well-known to be 2-Calabi–Yau.)

In addition to the above perspective of giving a (formal) local description of multiplicative quiver varieties, there are various other perspectives on this work. Namely, multiplicative preprojective algebras arise from studying certain wrapped Fukaya categories. If X_Γ is a plumbing of cotangent bundles according to the graph Γ , then Ekholm and Lekili²⁴ and

*Recall that a prime ring is a noncommutative analogue of an integral domain, being a ring R in which $aRb = 0$ implies $a = 0$ or $b = 0$.

Etgü and Lekili^{26, 25} produced quasi-isomorphisms,

$$\mathcal{W}(X_\Gamma) \xrightarrow[\cong]{24} \mathfrak{B}_\Gamma \xrightarrow[\cong]{26, 25} \mathfrak{L}_\Gamma$$

where $\mathcal{W}(X_\Gamma)$ denotes the partially wrapped Fukaya category of X_Γ , $\mathfrak{B}_\Gamma$ denotes the Chekanov and Eliashberg dg-algebra and $\mathfrak{L}_\Gamma$ is a derived multiplicative preprojective algebra. It follows from Ganatra's thesis³³ that $\mathfrak{L}_\Gamma$ is 2-Calabi–Yau, as a dg-algebra. We establish this result purely algebraically, in the case Γ contains a cycle. We show that

$$H^*(\mathfrak{L}_\Gamma) \xrightarrow[\cong]{\text{Section 2.3}} H^0(\mathfrak{L}_\Gamma) = \Lambda^1(\Gamma)$$

and hence $\mathfrak{L}_\Gamma$ is formal. Note that for the ordinary preprojective algebra, $\Pi(Q)$, the Ginzburg dg-algebra has homology concentrated in degree zero: $\Pi(Q)$ has a length two bimodule resolution (see^{9,46} for the characteristic zero case, and²⁸ in general) which Anick shows is equivalent for *graded* connected algebras in¹ Theorems 2.6 & 2.9, and³⁰ observes this extends to the quiver case.

Consequently, deformations of the wrapped Fukaya category $\mathcal{W}(X_\Gamma)$ as an A_∞ (respectively A_∞ Calabi–Yau) category over a degree zero base are given by deformations of $\Lambda^1(\Gamma)$ as an associative (respectively Calabi–Yau) algebra. The infinitesimal deformations can be identified with $\mathrm{HH}^2(\Lambda^1(\Gamma))$, which by Van den Bergh duality assuming Conjecture 2.0.1 is isomorphic to $\mathrm{HH}_0(\Lambda^1(\Gamma))$. The techniques in⁴⁹ can likely be adapted to compute the zeroth Hochschild homology using an explicit basis for $\Lambda^q(Q)$ computed here. Furthermore, by the 2-Calabi–Yau property, $\mathrm{HH}^3(\Lambda^q(Q)) = 0$, so there are no obstructions to extending to infinite order deformations. We conjecture that the same holds for every connected, non-ADE quiver. More precisely:

Conjecture 2.0.3. *If Q is connected and not ADE, then the dg multiplicative preprojective algebra $\Lambda^{dg,q}(Q)$ is quasi-isomorphic to $\Lambda^q(Q)$, in degree zero.*

We give the precise definitions and details, as well as proof in the case Q contains a cycle in Section 2.3.

Another perspective on this work involves specializing to Q_E an extended ADE quiver and $q = 1$. In Shaw's thesis,⁵² he shows that for the extended vertex v , in analogy with the additive case, the subalgebra $e_v \Lambda^1(Q_E) e_v$ is commutative of dimension 2 and is smooth away from the origin. In further analogy, it is reasonable to pose the following conjecture:

Conjecture 2.0.4. *The algebra $\Lambda^1(Q_E)$ is a 2-dimensional NCCR (non-commutative*

crepant resolution) over its center, $Z(\Lambda^1(Q_E))$. The Satake map[†] $Z(\Lambda^1(Q_E)) \rightarrow e_v \Lambda^1(Q_E) e_v$, $z \mapsto e_v z$, is an isomorphism.

For $Q_E = \widetilde{A}_n$, our results imply the conjecture as follows: One can identify

$$\Lambda^1(\widetilde{A}_n) \xrightarrow{\cong} \text{End}_{e_0 \Lambda^1(\widetilde{A}_n) e_0}(e_0 \Lambda^1(\widetilde{A}_n)),$$

by identifying $e_0 \Lambda^1(\widetilde{A}_n) e_0 \cong k[X, Y, Z]/(Z^{n+1} + XY + XYZ)$ as an algebra (which is commutative) and $e_0 \Lambda^1(\widetilde{A}_n) e_i \cong M_i := (Z^i, Y)$, the ideal. The isomorphism is then explicitly given:

- on vertices by $i \mapsto \text{id} : M_i \rightarrow M_i$;
- on arrows by $a_i \mapsto Z : M_i \rightarrow M_{i+1}$ for $i < n$ and $a_n \mapsto Z/(1+Y)Z : M_n \rightarrow M_0$;
- on reverse arrows by $a_i^* \mapsto \text{inc} : M_i \hookrightarrow M_{i-1}$ for $i > 0$ and $a_0^* \mapsto Y : M_0 \rightarrow M_n$.

The surjectivity follows from the observation that every $e_0 \Lambda^1(\widetilde{A}_n) e_0$ -module map of ideals is given by left multiplication by some element of the field of fractions of $e_0 \Lambda^1(\widetilde{A}_n) e_0$. The injectivity follows from the fact that $\Lambda^1(\widetilde{A}_n)$ is prime (see Proposition 2.5.3) and injectivity on $e_0 \Lambda^1(\widetilde{A}_n) e_0$.

Now, to prove the second statement, generally, if A is an algebra and e an idempotent, then $\text{End}_{A^{\text{op}}}(eA) \cong eAe$. In the case $\text{End}_{eAe}(eA) \cong A^{\text{op}}$ also, we obtain an identification $Z(eAe) \cong \text{End}_{eAe \otimes A^{\text{op}}}(eA) \cong Z(A)$. Explicitly, the map $Z(A) \rightarrow Z(eAe)$ is given by $z \mapsto ez$. If in addition, as here, eAe is commutative, we obtain $Z(A) \cong eAe$, via the Satake map.

Finally, to complete the proof of the first statement, the computation above shows that $\text{Hom}_{e_0 \Lambda^1(\widetilde{A}_n) e_0}(M_i, M_j) \cong e_i \Lambda^1(\widetilde{A}_n) e_j \cong M_{j-i}$ as a module over $e_0 \Lambda^1(\widetilde{A}_n) e_0 \cong Z(\Lambda^1(\widetilde{A}_n)) \cong e_i \Lambda^1(\widetilde{A}_n) e_i$, so in particular $e_0 \Lambda^1(\widetilde{A}_n)$ is self-dual and hence reflexive as a $e_0 \Lambda^1(\widetilde{A}_n) e_0$ -module. Since furthermore $\Lambda^1(\widetilde{A}_n)$ is 2-Calabi–Yau (by Theorem 2.0.2) and $e_0 \Lambda^1(\widetilde{A}_n) e_0$ is Gorenstein (as it is a hypersurface in $\mathbb{A}^3$), it follows that $\Lambda^1(\widetilde{A}_n)$ is an NCCR over $e_0 \Lambda^1(\widetilde{A}_n) e_0$.

Note that $\Lambda^1(Q_E)$ can only be an NCCR in the extended ADE case, as in the ADE case it has infinite global dimension, and in the non-ADE, non-extended ADE case it is expected to have trivial center (by Conjecture 2.0.1); see Section 2.4.

[†]We use the terminology “Satake” following the analogous one for symplectic reflection algebras at $t = 0$ of Etingof–Ginzburg, itself coming from the map for affine Hecke algebras proposed by Lusztig.

A final perspective on this work involves the Kontsevich–Rosenberg Principle which says: a non-commutative geometric structure on an associative algebra A should induce a geometric structure on the representation spaces $\text{Rep}_n(A)$, for all $n \geq 1$. This principle needs adjusting for structures living in the *derived* category of A -modules, as the representation functor is not exact. For a d-Calabi–Yau structure on A , it is shown in⁸ and⁵⁸ that the derived moduli stack of perfect complexes of A -modules, $\mathbb{R}\text{Perf}(A)$, has a canonical $(2 - d)$ -shifted symplectic structure. Since the dg Ginzburg algebra is 2-Calabi–Yau, this implies that its moduli stack of representations has a 0-shifted symplectic structure. By Conjecture 2.0.1, it is the same as the moduli stack of representations of $\Lambda^q(Q)$ itself. Note that the multiplicative quiver variety can be viewed as a coarse moduli space of stemistable representations; so the aforementioned result that this variety locally has the structure of an ordinary quiver variety is a singular analogue of the statement on the moduli stack.

We prove Theorem 2.0.2 using a complex:

$$P_\bullet := \Lambda^q(Q) \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda^q(Q) \xrightarrow{\alpha} \Lambda^q(Q) \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} \Lambda^q(Q) \xrightarrow{\beta} \Lambda^q(Q) \otimes \Lambda^q(Q)$$

defined originally in¹⁶ and shown to resolve $\Lambda^q(Q)$, *except* for the injectivity of the map α . We show α is injective and then show the dual complex $P_\bullet^\vee$ is a resolution of $\Lambda^q(Q)[-2]$, which implies $\Lambda^q(Q)$ is 2-Calabi–Yau.

First, we establish a chain of implications to reduce the proof to a presentation of the localization L_Q that we call the *strong free product property*. (The strong free product property is a version of Anick’s weak summand property in the ungraded case, see¹.) We explain a procedure to show that the strong free product property holds for any connected quiver containing a subquiver where the strong free product property holds. Then we establish the strong free product property explicitly using the Diamond Lemma in the case $Q = \widetilde{A}_n$ and carry out the procedure in this case.

To prove the 2-Calabi–Yau property from the strong free product property we establish

the chain of implications:

$$\begin{array}{c}
\text{Strong Free Product Property for } Q : \\
\sigma' : \Lambda^q(Q) *_{kQ_0} kQ_0[t, (q+t)^{-1}] \rightarrow L_Q \text{ is a linear isomorphism} \\
\Downarrow \text{Section 2.2.1} \\
\text{Weak Free Product Property for } Q : \\
\text{gr}(\sigma') : \text{gr}(\Lambda^q(Q) *_{kQ_0} kQ_0[t]) \rightarrow \text{gr}(L_Q) \text{ is an algebra isomorphism} \\
\Downarrow \text{Prop 2.2.11} \\
\text{gr}(\sigma')_1 : \Lambda^q(Q) \otimes_{kQ_0} kQ_0[t] \otimes_{kQ_0} \Lambda^q(Q) \rightarrow J_Q/J_Q^2 \text{ is an isomorphism of } \Lambda^q(Q)\text{-bimodules} \\
\Downarrow \text{Prop 2.2.10 \& Prop 2.2.11} \\
P_\bullet \text{ is a length two projective } \Lambda^q(Q)\text{-bimodule resolution of } \Lambda^q(Q) \\
\Downarrow \text{Thm 2.2.13} \\
\Lambda^q(Q) \text{ is 2-Calabi-Yau}
\end{array}$$

Here the isomorphism σ' is determined by a choice of kQ_0 -bimodule section $\Lambda^q(Q) \rightarrow L_Q$ of the quotient map $L_Q \twoheadrightarrow \Lambda^q(Q)$, but $\text{gr}(\sigma')$ is independent of this choice. The element t maps to the relation, and the filtrations used are the t -adic one on the source and the J_Q -adic filtration on L_Q .

Remark 2.0.5. Note that for ordinary preprojective algebras, the free product property was observed in⁴⁹. In fact, as these algebras are nonnegatively graded augmented with positively graded augmentation ideal, the strong and weak free product properties are equivalent (and independent of the choice of graded section σ), and equivalent to the condition that the algebra has a projective bimodule resolution of length two, as was already observed by Anick in¹. Such algebras were called *noncommutative complete intersections* in³⁰ due to their close relationship to the condition that representation varieties be complete intersections.

However, in the ungraded case, we only have the implication above, that the free product property implies the existence of a length-two projective bimodule resolution. Indeed, as we see above, the latter property only depends on a piece of the associated graded algebra with respect to the (r) -adic filtration, and in the ungraded case this filtration need not even be Hausdorff. In contrast, the strong free product property implies the Hausdorff condition and gives information about the algebra itself.

Motivated by this, we believe that the strong free product property can be viewed as an ungraded analogue of the noncommutative complete intersection property. Following³⁰, it makes sense to ask if the strong free product property can be used to show that representation varieties are (asymptotic) complete intersections under certain conditions.

In Section 2.3, we show that the strong free product property implies that the dg multiplicative preprojective algebra $\Lambda^{dg,q}$ is formal. Therefore, by the results of this chapter, both Conjectures 2.0.1 and 2.0.3 would follow from the following more general statement (see Section 2.5 for precise details):

Conjecture 2.0.6. *If Q is a connected and not ADE quiver, then σ' as above is a linear isomorphism $((L_Q, r, \sigma, kQ_0[t, (t+q)^{-1}])$ satisfies the strong free product property).*

Let us outline an inductive strategy to prove the strong free product property. Let $Q = Q' \cup Q''$ be a connected quiver containing Q' . Q'' is chosen so that $Q'_1 \cap Q''_1 = \emptyset$ and define $\mathcal{W} := Q'_0 \cap Q''_0$. Assume we have the strong free product property for Q' . Filter $\Lambda^q(Q)$ by the descending filtration given by powers of the ideal generated by those arrows in Q'' incident to $\mathcal{W}$. We expect that this filtration is Hausdorff and that the associated graded algebra with respect to this filtration is isomorphic to a free product $\Lambda^q(Q') *_{kQ'_0} \Lambda^q(Q'', \mathcal{W})$. Here $\Lambda^q(Q'', \mathcal{W})$ is the *partial* preprojective algebra with no relations at the vertices in $\mathcal{W} \subset Q_0$. We prove the strong free product property for partial multiplicative preprojective algebras with $Q'' \neq \emptyset$. Put together we obtain linear isomorphisms

$$\begin{aligned} L_Q &\cong L_{Q'} *_{kQ'_0} L_{Q''} \\ &\cong \Lambda^q(Q') *_{kQ'_0} kQ'_0[t, (q+t)^{-1}] *_{kQ_0} \Lambda^q(Q'', \mathcal{W}) *_{kQ''_0} kQ''_0[t, (q+t)^{-1}] \\ &\cong \text{gr } \Lambda^q(Q) *_{kQ_0} kQ_0[t, (q+t)^{-1}], \end{aligned}$$

and since the filtration is Hausdorff we can also pick a section, eliminating the gr .

In Section 2.5, we prove the strong free product property when $Q = \tilde{A}_n$ is a cycle. That is, we build a section $\Lambda^q(\tilde{A}_n) \hookrightarrow L_{\tilde{A}_n}$ and hence obtain a map

$$\varphi : \Lambda^q(\tilde{A}_n) *_{k(\tilde{A}_n)_0} k(\tilde{A}_n)_0[t, (q+t)^{-1}] \rightarrow L_{\tilde{A}_n}.$$

We show that φ is an isomorphism with a carefully chosen basis for $L_{\tilde{A}_n}$, whose linear independence and spanning are proven with the Diamond Lemma. Putting this together with the strong free product property for *partial* multiplicative preprojective algebras, we obtain the strong free product property, and hence the 2-Calabi-Yau property, for all

connected quivers Q containing $Q' = \widetilde{A}_n$. As a schematic,

$$\begin{aligned} & \text{Strong Free Product Property for } \widetilde{A}_n \text{ (Prop 2.5.1) :} \\ & \sigma'_n : \Lambda^q(\widetilde{A}_n) *_{k(\widetilde{A}_n)_0} k(\widetilde{A}_n)_0[t, (q+t)^{-1}] \rightarrow L_{\widetilde{A}_n} \text{ is a linear isomorphism} \\ & \qquad \qquad \qquad \& \end{aligned}$$

Strong Free Product Property for partial multiplicative preprojective algebras (Prop 2.5.4) :

$$\begin{aligned} & \sigma'_{\text{partial}} : \Lambda^q(Q, \mathcal{W}) *_{kQ_0} kQ_0[t, (q+t)^{-1}] \rightarrow L_Q \text{ is a linear isomorphism} \\ & \qquad \qquad \qquad \Downarrow \text{Prop 2.5.5} \end{aligned}$$

Strong Free Product Property for $Q \supset \widetilde{A}_n$:

$$\sigma' : \Lambda^q(Q) *_{kQ_0} kQ_0[t, (q+t)^{-1}] \rightarrow L_Q \text{ is a linear isomorphism}$$

This is the technical heart of the chapter.

2.1 THE MULTIPLICATIVE PREPROJECTIVE ALGEBRA

2.1.1 DEFINITIONS

Throughout the chapter we fix an arbitrary field k . For each quiver (i.e. directed graph) Q , let Q_0 be the vertex set, Q_1 be the arrow set, and $h, t : Q_1 \rightarrow Q_0$ the head and tail maps, respectively. We will assume that Q_0 and Q_1 are finite for convenience, but really only need finitely many arrows incident to each vertex. Q^{op} denotes the quiver with the same underlying graph of vertices and arrows, but with every arrow in the opposite direction. $\overline{Q}$ denotes the quiver with the same vertex set as Q and Q^{op} and with arrow set $Q_1 \sqcup Q_1^{\text{op}}$. For each arrow $a \in Q_1$, we write a^* for the corresponding arrow in Q_1^{op} , and vice versa. In $\overline{Q}$ we distinguish between arrows in Q and Q^{op} using a function $\epsilon : \overline{Q}_1 \rightarrow \{\pm 1\}$ defined by $\epsilon(a) = 1$ if $a \in Q_1$ and $\epsilon(a) = -1$ if $a \in Q_1^{\text{op}}$.

Next define $g_a := 1 + aa^*$, viewed as elements in the algebra of paths, $k\overline{Q}$, and consider the localization $L_Q := k\overline{Q}[g_a^{-1}]_{a \in \overline{Q}_1}$. We write $L := L_Q$, when the quiver is clear from context. Notice, for all $a \in \overline{Q}_1$:

$$g_a a = a + aa^* a = ag_{a^*}. \tag{2.1.1}$$

This implies:

$$g_{a^*} a^* = a^* g_a \qquad g_a^{-1} a = a g_{a^*}^{-1} \qquad g_{a^*}^{-1} a^* = a^* g_a^{-1}.$$

Fixing a total ordering $\leq$ on the set of arrows $\overline{Q}_1$, one can make sense of a product over

(subsets of) the arrow set. Using $\leq$ and ϵ we define:

$$\rho := \prod_{a \in \overline{Q}} g_a^{\epsilon(a)} \quad l_a := \prod_{b \in \overline{Q}, b < a} g_b^{\epsilon(b)} \quad r_a := \prod_{b \in \overline{Q}, b > a} g_b^{\epsilon(b)}.$$

When we need to make the role of the total ordering $\leq$ more explicit, we write $\rho_{\leq}, l_{a, \leq},$ and $r_{a, \leq}$ for $\rho, l_a,$ and r_a respectively. By definition, l_a and r_a are truncations of ρ to the right and left of a , respectively. Therefore,

$$\rho = l_a g_a^{\epsilon(a)} r_a \quad (2.1.2)$$

for all $a \in \overline{Q}_1$.

Definition 2.1.1. Let k be a field, Q a quiver, and $q \in (k^\times)^{Q_0}$. Fixing an ordering $\leq$ on the arrows of $\overline{Q}$ and ϵ as defined above, the *multiplicative preprojective algebra*, $\Lambda^q(Q)$, is defined to be

$$\Lambda^q(Q) := L/J$$

where $L = k\overline{Q}[g_a^{-1}]_{a \in \overline{Q}_1}$ is the localization and J is the two-sided ideal generated by the element $\rho - q$.

Note that q is viewed as an element of $k\overline{Q}$ via $\sum_{i \in Q_0} q_i e_i \in kQ_0 \subset k\overline{Q}$, and as ρ is invertible we need $q_i \neq 0$ so $e_i \Lambda^q(Q) \neq 0$, for all i .

Remark 2.1.2. The isomorphism class of $\Lambda^q(Q)$ is independent of both the orientation of the quiver and the choice of an ordering on the arrows, by Section 2 in ¹⁶.

Example 2.1.3. (ADE Case)

If $k = \mathbb{C}$, Crawley-Boevey non-constructively showed that $\Lambda^1(Q) \cong \Pi^0(Q) := k\overline{Q}/(\sum_{a \in Q_1} [a, a^*])$ for Q ADE, ¹⁴ Corollary 1. This implies, viewing $\Lambda^q(Q)$ as a filtered deformation of $\Lambda^1(Q)$ under the path-length filtration, that

$$\Lambda^q(Q) \cong \Pi^\lambda(Q) := k\overline{Q} / \left(\sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \lambda_i e_i \right)$$

for some $\lambda = (\lambda_i) \in k^{Q_0}$, by ²⁸ Proposition 5.0.2. In type A, this isomorphism holds over any field k , and is achieved by a change of variables. Namely, identifying $(A_n)_0 = \{1, 2, \dots, n\}$ and $(A_n)_1 = \{a_1, a_2, \dots, a_{n-1}\}$ the isomorphism is given by,

$$\Lambda^q(A_n) \cong \Pi^\lambda(A_n) \quad e_i \mapsto e_i, \quad a_i \mapsto a_i, \quad a_i^* \mapsto \left(\prod_{j>i} q_j \right) a_i^*$$

where $\lambda_i := (q_i - 1) \prod_{j>i} q_j$. In particular, since $\Pi^\lambda(A_n)$ is non-zero if and only if $\sum_{i=1}^n \lambda_i = 0$, it follows that $\Lambda^q(A_n)$ is non-zero if and only if $\prod_{i=1}^n q_i = 1$. In the D_4 case – with arrows a, b, c all pointing towards the central vertex – Shaw constructions an isomorphism,

$$\Lambda^1(D_4) \cong \Pi^0(D_4) \quad \begin{aligned} a &\mapsto \left(1 - \frac{1}{2}bb^*\right)a, \quad a^* \mapsto a^* \left(1 + \frac{1}{2}bb^*\right) \\ a' &\mapsto a' \text{ for } a' \in \{e_i, b, b^*, c, c^*\}_{i=1,2,3,4} \end{aligned}$$

which holds over any field of characteristic not 2. Additionally, he proves that no such isomorphism exists over any field of characteristic 2, ⁵² Lemma 5.2.1.

In the multiplicative preprojective algebra, (2.1.2) becomes the identity:

$$l_a g_a^{\epsilon(a)} r_a = q.$$

Hence:

$$r_a l_a = q g_a^{-\epsilon(a)}. \quad (2.1.3)$$

2.1.2 THE MAP θ

In ¹⁶, the important map $\theta : \overline{Q}_1 \rightarrow \Lambda^q(Q)$ is defined by $\theta(a) = q^{-1} l_a a r_a^*$ and extended to $k\overline{Q}$ by the identity on Q_0 and by requiring θ to be an algebra map. Then Lemma 3.3 in ¹⁶ shows that

$$\theta(g_a) = l_a g_a l_a^{-1} \quad (2.1.4)$$

$$= r_a^{-1} g_a r_a \quad (2.1.5)$$

is invertible, and hence θ factors through the localization $L := k\overline{Q}[g_a^{-1}]_{a \in \overline{Q}_1}$. We will show θ descends to the quotient $\Lambda^q(Q)$, with the ordering of the arrows reversed, using the following result.

Lemma 2.1.4. *Let $\leq$ denote a total order on $\overline{Q}_1$ and let $\geq$ denote its opposite ordering, i.e. $a \geq b$ if $b \leq a$. Such an order fixes a bijection $\overline{Q}_1 \cong \{a_1, a_2, \dots, a_{|\overline{Q}_1|}\}$. Then*

$$\theta(r_{a_j, \geq}) = l_{a_j, \leq} =: l_{a_j} \quad \text{and} \quad \theta(l_{a_j, \geq}) = r_{a_j, \leq} =: r_{a_j}$$

for any $a_j \in \overline{Q}_1$.

Proof. We prove $\theta(r_{a_j, \geq}) = l_{a_j}$ by induction on j , where $j = 1$ is the identity $\theta(1) = 1$.

Then,

$$\begin{aligned}
\theta(r_{a_{j+1}, \geq}) &= \theta(g_{a_j}^{\epsilon(a_j)})\theta(r_{a_j, \geq}) \\
&\stackrel{(\text{IH})}{=} \theta(g_{a_j}^{\epsilon(a_j)})l_{a_j} \\
&\stackrel{(2.1.4)}{=} l_{a_j}g_{a_j}^{\epsilon(a_j)}l_{a_j}^{-1}l_{a_j} \\
&= l_{a_j}g_{a_j}^{\epsilon(a_j)} \\
&= l_{a_{j+1}}.
\end{aligned}$$

The second identity is similar and one can formally obtain a proof from the above by exchanging the symbols r and l , the identity 2.1.4 for 2.1.5, and the order of the multiplication. $\square$

Corollary 2.1.5. $\theta(\rho_{\geq}) = \rho$.

This corollary implies θ descends to a map $\Lambda^q(Q, \geq) \rightarrow \Lambda^q(Q, \leq)$. Notice that we can similarly define $\theta_{\geq} : \Lambda^q(Q, \leq) \rightarrow \Lambda^q(Q, \geq)$.

Proposition 2.1.6. $\theta_{\geq} \circ \theta = \text{Id}_{\Lambda^q(Q, \leq)}$.

Proof. It suffices to check $\theta_{\geq} \circ \theta$ is the identity on arrows in Q_1 and indeed:

$$\begin{aligned}
\theta_{\geq}(\theta(a)) &= \theta(q^{-1}l_a a r_{a^*}) \\
&= q^{-1}\theta_{\geq}(l_a)\theta_{\geq}(a)\theta_{\geq}(r_{a^*}) \\
(2.1.4) \quad &= q^{-1}r_a\theta_{\geq}(a)l_{a^*} \\
&= q^{-1}r_a(q^{-1}l_a a r_{a^*})l_{a^*} \\
(2.1.3) \quad &= g_a^{-\epsilon(a)} a g_{a^*}^{-\epsilon(a^*)} \\
(2.1.1) \quad &= a g_{a^*}^{-\epsilon(a)} g_{a^*}^{-\epsilon(a^*)} \\
&= a
\end{aligned}$$

$\square$

2.2 THE 2-CALABI–YAU PROPERTY

The goal of this section is to prove that $\Lambda^q(Q)$ is 2-Calabi–Yau for Q containing an unoriented cycle. We do so by exhibiting a length two, projective, $\Lambda^q(Q)$ -bimodule resolution

$P_\bullet$ of $\Lambda^q(Q)$, whose bimodule dual complex $P_\bullet^\vee$ is quasi-isomorphic to $\Lambda^q(Q)$. This resolution is due to Crawley-Boevey and Shaw, but they don't state nor prove that it is exact. The main new ingredient we provide is the injectivity of α which relies on the weak free product property. Hence we begin this section with a short digression explaining the strong and weak free product properties.

2.2.1 FREE PRODUCT PROPERTIES

Generally, if A is an algebra over a semisimple ring S and $J = (r)$ an ideal generated by a single relation r , we can form a canonical algebra map,

$$\Phi : A/J *_S S[t] \rightarrow \text{gr}_J A, \quad \Phi|_{A/J} = \text{Id}, \Phi(t) = r, \quad (2.2.1)$$

where gr_J means the associated graded algebra with respect to the J -adic filtration.

Definition 2.2.1. The pair (A, r) satisfies the *weak free product property* if Φ is an isomorphism.

We abuse notion and say the quotient A/J satisfies the weak free product property when r is clear from context and (A, r) satisfies the weak free product property.

Next, given an S -bimodule section $\sigma : A/J \rightarrow A$ of the quotient map that is the identity on S , we can form an associated linear map,

$$\tilde{\sigma} : A/J *_S S[t] \rightarrow A, \quad (a_0 t^{m_1} a_1 t^{m_2} \dots t^{m_n} a_n) \mapsto \sigma(a_0) r^{m_1} \sigma(a_1) r^{m_2} \dots r^{m_n} \sigma(a_n), \quad (2.2.2)$$

for $m_i > 0$, for all i .

By construction, this is $S[t]$ -bilinear, where t acts on A by multiplication by r . It also reduces to the identity modulo (t) on the source and J on the target. If (A, r) satisfies the weak free product property, then moreover the completion

$$\widehat{\tilde{\sigma}} : \widehat{A/J *_S S[t]} \rightarrow \widehat{A}, \quad (2.2.3)$$

with respect to the t -adic and J -adic filtrations, is an isomorphism.

In nice cases we can identify a subalgebra on the left-hand side mapping isomorphically to A (which is a subalgebra of $\widehat{A}$ under the Hausdorff condition $\bigcap_{m \geq 0} J^m = \{0\}$):

Definition 2.2.2. (A, r, σ) satisfies the *strong free product property* if $\tilde{\sigma} : A/J *_S S[t] \rightarrow A$ is an S -bimodule isomorphism.

Since the t -adic filtration on $S[t]$ is Hausdorff and $\sigma(t) = r$ it follows that the J -adic filtration on A is Hausdorff if (A, r, σ) satisfies the strong free product property.

Remark 2.2.3. The choice of σ is important. Let $A = k\langle x, y \rangle$ and $J = (y)$ so $A/J \cong k[x]$. Consider two different choices

$$\sigma_1, \sigma_2 : k[x] \rightarrow k\langle x, y \rangle \quad \sigma_1(x + (y)) = x, \quad \sigma_2(x + (y)) = x - yx.$$

Then σ'_1 is a linear isomorphism while σ'_2 is not surjective as

$$x = \sigma_2(x + (y))(1 - y)^{-1} = \sigma_2(x + (y)) \sum_{i \geq 0} y^i \notin \tilde{\sigma}_2(k[x] *_k k[t]).$$

Remark 2.2.4. The strong free product property has already appeared, albeit in the graded context working over a field, in¹: Anick defines a notion of *weak summand* $B \subset A$, meaning there exists a linear isomorphism $A \cong B * A/(\overline{B})$, where $\overline{B} := \bigoplus_{i > 0} B_i$ is the augmentation ideal. In the graded setting he observes that the strong free product property does not depend on the choice of σ and it is easy to see it is implied by the weak free product property.

Next, let B be a localization of $S[t]$ obtained by inverting finitely many elements of $S^\times + (t)$, such that $S[t] \rightarrow A$ extends (uniquely) to an algebra map $\tau : B \rightarrow A$. Let $\overline{B} := tB$, so that we have an S -bimodule decomposition $B = S \oplus \overline{B}$. Then $\tilde{\sigma}$ extends to a map $\sigma' : A/J *_S B \rightarrow A$, which has the form

$$a_0 b_1 a_1 \cdots b_n a_n \mapsto \sigma(a_0) \tau(b_1) \cdots \tau(b_n) \sigma(a_n), a_i \in A/J, b_i \in \overline{B}. \quad (2.2.4)$$

Definition 2.2.5. The data (A, r, σ, B) satisfies the *strong free product property* if σ' is a linear isomorphism.

This definition generalizes the above definition of the strong free product property for (A, r, σ) , which is the special case $B = S[t]$, $\tau(t) = r$.

In this case, it follows by taking associated graded algebras that (A, r) satisfies the weak free product property. Moreover, A is Hausdorff in the J -adic filtration (because the source of σ' is Hausdorff in the t -adic filtration), and σ' is indeed a restriction of $\hat{\tilde{\sigma}}$.

Remark 2.2.6. The definition of σ' in (2.2.4) depends on the choice $\overline{B} = tB$ of complement. In fact the property of being an isomorphism, i.e. the strong free product property, depends on this choice.

To see this return to the example of $(A = k\langle x, y \rangle, r = y, \sigma_2(x + (y)) = x - yx)$ in Remark 2.2.3. Let $B = k[t, (1 - t)^{-1}]$ with $\overline{B}_1 := tB$ and $\overline{B}_2 := (1 - t)^{-1}B$. Then σ'_2 as defined using $\overline{B}_1$ is not surjective, as $x \notin \sigma'_2(k[x] * k[t, (1 - t)^{-1}])$, as before. But σ'_2 as defined using $\overline{B}_2$ is an isomorphism. See Remark 2.5.2 for another example.

Now back to the setup of Q a connected, non-ADE quiver and $q = (k^\times)^{Q_0}$. Let $B := kQ_0[t, (q + t)^{-1}]$ and $\overline{B} := tB = \text{Span}(t^m, (t')^m \mid m \geq 1)$, for $t' := (q + t)^{-1} - q^{-1}$. We conjecture that, for every connected, non-ADE quiver Q , there exists σ such that the quadruple (L_Q, r, σ, B) satisfies the strong free product property. Moreover, in Section 2.5 we prove this conjecture in the case of quivers containing a cycle:

Theorem 2.2.7 (Proposition 2.5.5 in Section 2.5). *Let Q be a connected quiver containing an unoriented cycle. Let $B = kQ_0[t, (q + t)^{-1}]$ and let r denote the multiplicative preprojective relation. There exists a section σ such that (L_Q, r, σ, B) satisfies the strong free product property.*

Remark 2.2.8. The connectedness assumption can be weakened as follows. If $Q = Q' \sqcup Q''$ then $L_Q = L_{Q'} \oplus L_{Q''}$ and $\Lambda^q(Q) = \Lambda^q(Q') \oplus \Lambda^q(Q'')$ and so by adding sections, the strong free product property for

$$(L_{Q'} \oplus L_{Q''}, r' + r'', \sigma' + \sigma'', B' \oplus B'') = (L_Q, r, \sigma, B)$$

follows from the strong free properties for $(L_{Q'}, r', \sigma', B')$ and $(L_{Q''}, r'', \sigma'', B'')$. So one only needs the weaker assumption that Q is a quiver with *each* component containing an unoriented cycle. But we state results in the connected setting to simplify the hypotheses.

Corollary 2.2.9. *Let Q be a connected quiver containing an unoriented cycle. Then $\Lambda^q(Q)$ satisfies the weak free product property. In particular, there exists an isomorphism of graded algebras:*

$$\sum_i \varphi_i : \text{gr}(\Lambda^q(Q) *_{kQ_0} kQ_0[t]) \rightarrow \text{gr}(L_Q)$$

where the associated graded algebras are taken with respect to the t -adic and J_Q -adic filtrations on $\Lambda^q(Q) *_{kQ_0} kQ_0[t]$ and L_Q respectively.

2.2.2 A BIMODULE RESOLUTION OF Λ

In this subsection, we show that for *any* quiver, the weak free product property for $\Lambda^q(Q)$ implies $\Lambda^q(Q)$ has a length two projective bimodule resolution. Consequently, since we establish the weak free product property for connected quivers containing a cycle, we

prove $\Lambda^q(Q)$ has Hochschild dimension at most two for connected quivers containing a cycle.

Crawley-Boevey and Shaw build a chain complex of Λ -bimodules $P_\bullet = P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0$ where,

$$P_2 = P_0 := \Lambda \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda = \langle \eta_v \rangle_{v \in Q_0} \quad P_1 := \Lambda \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda = \langle \eta_a \rangle_{a \in \overline{Q}_1}$$

and

$$\alpha(\eta_v) := \sum_{a \in \overline{Q}_1: t(a)=v} l_a \Delta_a r_a \quad \text{where} \quad \Delta_a = \begin{cases} \eta_a a^* + a \eta_a^* & \text{if } a \in Q_1 \\ -g_a^{-1}(\eta_a a^* + a \eta_a^*) g_a^{-1} & \text{if } a \in Q_1^{op}. \end{cases}$$

$$\beta(\eta_a) := a \eta_{t(a)} - \eta_{h(a)} a.$$

We claim that it is a resolution of Λ . To see this, following¹⁶, we first write down an explicit chain map of Λ -bimodule complexes $\psi : P_\bullet \rightarrow Q_\bullet$, where $Q_\bullet$ is quasi-isomorphic to Λ ; we then prove it is an isomorphism. $Q_\bullet$ is the cotangent exact sequence in Corollary 2.11 in¹⁷, but in this context it was defined earlier (and shown quasi-isomorphic to Λ) by Schofield in⁵¹. So we have the maps:

$$P_\bullet \xrightarrow[\text{CBS}]{\psi} Q_\bullet \xrightarrow[\text{Schofield}]{q.i.} \Lambda.$$

Proposition 2.2.10 (Lemma 3.1 in¹⁶). *For any quiver Q , the following diagram commutes:*

$$\begin{array}{ccccccc} P_0 & \xrightarrow{\alpha} & P_1 & \xrightarrow{\beta} & P_0 & \xrightarrow{\gamma} & \Lambda \\ \downarrow \psi_2 & & \cong \downarrow \psi_1 & & \cong \downarrow \psi_0 & & \downarrow id \\ J/J^2 & \xrightarrow{\kappa} & \Lambda \otimes_L \Omega_{kQ_0}(\Lambda) \otimes_L \Lambda & \xrightarrow{\lambda} & \Lambda \otimes_{kQ_0} \Lambda & \xrightarrow{\mu} & \Lambda \end{array}$$

where the vertical maps are Λ -bimodule maps defined on generators by,

$$\psi_2(\eta_v) := \rho e_v - q e_v \quad \psi_1(\eta_a) := 1 \otimes_L [a \otimes_{kQ_0} 1 - 1 \otimes_{kQ_0} a] \otimes_L 1 \quad \psi_0(\eta_v) = e_v \otimes e_v$$

and the horizontal maps are defined by,

$$\begin{aligned} \kappa(x + J^2) &= 1 \otimes_L \delta(x) \otimes 1 \quad \text{for } x \in J \quad \text{where } \delta(x) = x \otimes 1 - 1 \otimes x \\ \lambda(1 \otimes_L [ab \otimes_{kQ_0} c - a \otimes_{kQ_0} bc] \otimes_L 1) &= ab \otimes_{kQ_0} c - a \otimes_{kQ_0} bc \quad \text{for } a, b, c \in L \end{aligned}$$

and μ is the multiplication map in Λ .

Since ψ_0 and ψ_1 are Λ -bimodule isomorphisms, it remains to show ψ_2 is a Λ -bimodule isomorphism. We show this using the weak free product property.

Proposition 2.2.11. *Suppose Λ satisfies the weak free product property. Then $P_\bullet$ is a bimodule resolution of Λ .*

Proof. Taking the $i = 1$ piece of the graded isomorphism

$$\mathrm{gr}(\varphi) = \sum_i \varphi_i : \mathrm{gr}(\Lambda *_{kQ_0} kQ_0[t, (t+q)^{-1}]) \longrightarrow \mathrm{gr}(L_Q)$$

gives an isomorphism of Λ -bimodules,

$$\varphi_1 : \Lambda \otimes_{kQ_0} kQ_0 \cdot t \otimes_{kQ_0} \Lambda \rightarrow J_Q/J_Q^2.$$

Since φ_1 sends $t \mapsto r$, it sends $te_v \mapsto re_v = (\rho - q)e_v$ and hence $\varphi_1 = \psi_2$. We conclude that ψ_2 is an isomorphism of Λ -bimodules and hence $\psi_\bullet : P_\bullet \rightarrow Q_\bullet$ is an isomorphism of Λ -bimodule complexes. In particular, $P_\bullet$ is a resolution since $Q_\bullet$ is a resolution. $\square$

Therefore, Λ has Hochschild dimension two. Every left Λ -module, M , has a length two projective left Λ -module resolution $P_\bullet \otimes_\Lambda M$, and hence Λ has global dimension *at most* two.

Example 2.2.12. Note that the inequality may be strict, since if Q is the Jordan quiver (i.e the quiver with one vertex with a loop) then

$$\Lambda^q(Q) \cong k\langle a, a^* \rangle[(1 + a^*a)^{-1}]/(aa^* - qa^*a - (q-1)).$$

The change of variables $x := a$ and $y := a^*/(q-1)$ when $q \neq 1$, identifies $\Lambda^q(Q)$ with a localization of the first quantum Weyl algebra, $k\langle x, y \rangle/(xy - qyx - 1)$, which has global dimension one.

2.2.3 THE DUAL COMPLEX

In this subsection, we show for *any* quiver that if $P_\bullet$ is a resolution of $\Lambda^q(Q)$, then $\Lambda^q(Q)$ is 2-Calabi–Yau. Combining this with the previous subsection, we get that if $\Lambda^q(Q)$ satisfies the weak free product property then $\Lambda^q Q$ is 2-Calabi–Yau. In particular, this shows that $\Lambda^q(Q)$ is 2-Calabi–Yau for connected quivers containing a cycle.

Recall that a d -Calabi–Yau structure is an A -bimodule isomorphism

$$\eta : A \rightarrow \mathrm{Ext}_{A \otimes A^{\mathrm{op}}}^d(A, A \otimes A^{\mathrm{op}}).$$

One can view such an η as an element of $HH_d(A, A)$ by taking $M = N = A$ in the quasi-isomorphism,

$$\mathrm{RHom}_{A \otimes A^{\mathrm{op}}}(M, N) \xrightarrow{\cong} \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(M, A \otimes A) \otimes_{A \otimes A^{\mathrm{op}}}^L N.$$

For dg-algebras, one further equips this structure with a class in negative cyclic homology that lifts the Hochschild homology class of the isomorphism. But, as shown in Proposition 5.7 and explained in Definition 5.9 of⁵⁶, for ordinary algebras this requirement is automatic.

We have established $P_\bullet$ as a Λ -bimodule resolution of Λ , if Q is connected and contains a cycle. To show Λ is 2-Calabi–Yau, it suffices to show that its dual complex

$$\mathrm{RHom}_{\Lambda\text{-bimod}}(\Lambda, \Lambda \otimes \Lambda) := \mathrm{Hom}_{\Lambda\text{-bimod}}(P_\bullet, \Lambda \otimes \Lambda) =: P_\bullet^\vee$$

is quasi-isomorphic to $\Lambda[-2]$.

Define $\eta_v^\vee \in \mathrm{Hom}_{\Lambda\text{-bimod}}(P_0, \Lambda \otimes \Lambda)$ and $\eta_a^\vee \in \mathrm{Hom}_{\Lambda\text{-bimod}}(P_1, \Lambda \otimes \Lambda)$ by,

$$\eta_v^\vee(\eta_w) := \begin{cases} e_v \otimes e_v & \text{if } v = w \\ 0 & \text{otherwise} \end{cases} \quad \eta_a^\vee(\eta_b) := \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } b = a^* \\ 0 & \text{otherwise} \end{cases}.$$

These are generators of $P_0^\vee$ and $P_1^\vee$ respectively and give isomorphisms,

$$P_0^\vee \cong \Lambda \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda = \langle \eta_v^\vee \rangle, \quad P_1^\vee \cong \Lambda \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda = \langle \eta_a^\vee \rangle.$$

Rather than directly study the dual complex $P_\bullet^\vee$, we modify the formulas for $\alpha^\vee$ and $\beta^\vee$ using the map θ , in a way that doesn't affect the homology of the complex. Namely, after choosing generators $\{\xi_v\}$ for $P_0^\vee$ and $\{\xi_a\}$ for $P_1^\vee$, defined below, one can expand:

$$\alpha^\vee(\xi_a) = \sum_{v \in Q_0} a'_v \xi_v a''_v \quad \beta^\vee(\xi_v) = \sum_{a \in \overline{Q}_1} b'_a \xi_a b''_a,$$

for some $a'_v, a''_v, b'_a, b''_a \in \Lambda$ and then define

$$\alpha_\theta^\vee(\xi_a) := \sum_{v \in Q_0} \theta(a'_v) \xi_v \theta(a''_v) \quad \beta_\theta^\vee(\xi_v) := \sum_{a \in \overline{Q}_1} \theta(b'_a) \xi_a \theta(b''_a).$$

With this choice of generators

$$\alpha_\theta^\vee = (\theta \otimes 1 \otimes \theta) \circ \alpha^\vee \quad \alpha^\vee = (\theta \otimes 1 \otimes \theta) \circ \alpha_\theta^\vee \quad \beta_\theta^\vee = (\theta \otimes 1 \otimes \theta) \circ \beta^\vee \quad \beta^\vee = (\theta \otimes 1 \otimes \theta) \circ \beta_\theta^\vee$$

and therefore,

$$\ker(\alpha_\theta^\vee) = \ker(\alpha^\vee) \quad \text{im}(\alpha_\theta^\vee) = \text{im}(\alpha^\vee) \quad \ker(\beta_\theta^\vee) = \ker(\beta^\vee) \quad \text{im}(\beta_\theta^\vee) = \text{im}(\beta^\vee).$$

It suffices to show that

$$(P_\bullet^\vee)_\theta \quad := \quad P_0^\vee \xrightarrow{-\beta_\theta^\vee} P_1^\vee \xrightarrow{\alpha_\theta^\vee} P_0^\vee$$

is quasi-isomorphic to $\Lambda[-2]$.

We prove this by establishing an isomorphism of Λ -bimodule complexes $\varphi_\bullet : P_\bullet[2] \rightarrow (P_\bullet^\vee)_\theta$ following Crawley-Boevey and Shaw, so

$$(P_\bullet^\vee)_\theta \xrightarrow{\varphi_\bullet^{-1}} P_\bullet[2] \xrightarrow{\psi_\bullet[2]} Q_\bullet[2] \xrightarrow{\text{quasi-iso}} \Lambda[2].$$

Theorem 2.2.13. *The following diagram commutes:*

$$\begin{array}{ccccccc} P_0^\vee & \xrightarrow{-\beta_\theta^\vee} & P_1^\vee & \xrightarrow{\alpha_\theta^\vee} & P_0^\vee & \xrightarrow{\gamma \circ \varphi_0^{-1}} & \Lambda \\ \varphi_0 \uparrow \cong & & \varphi_1 \uparrow \cong & & \varphi_0 \uparrow \cong & & \text{id} \uparrow = \\ P_0 & \xrightarrow{\alpha} & P_1 & \xrightarrow{\beta} & P_0 & \xrightarrow{\gamma} & \Lambda \end{array}$$

where the vertical maps are Λ -bimodule isomorphisms defined on generators by,

$$\varphi_0(\eta_v) := \xi_v := q\eta_v^\vee \quad \varphi_1(\eta_a) := \xi_{a^*} := \begin{cases} l_a \eta_{a^*}^\vee l_{a^*}^{-1} & \text{if } a \in Q_1^{op} \\ -r_{a^*}^{-1} \eta_{a^*}^\vee r_a & \text{if } a \in Q_1. \end{cases}$$

Note that φ_1 is an invertible map since r_a and l_a are invertible elements of Λ for all $a \in \overline{Q}_1$. The commuting of the above diagram becomes clear once we compute the maps $\alpha^\vee$ and $\beta^\vee$ explicitly, the content of the next two lemmas.

Lemma 2.2.14 (Lemma 3.2 in ¹⁶).

$$\begin{aligned}\alpha^\vee(\eta_a^\vee) &= \begin{cases} a^* r_a \eta_{h(a)}^\vee l_a - g_a^{-1} r_a^* \eta_{t(a)}^\vee l_a^* g_a^{-1} a^* & \text{if } a \in Q_1 \\ r_a^* \eta_{t(a)}^\vee l_a^* a^* - a^* g_a^{-1} r_a \eta_{h(a)}^\vee l_a g_a^{-1} & \text{if } a \in Q_1^{op} \end{cases} \\ \alpha^\vee(\xi_a) &= \theta(a^*) \xi_{t(a^*)} - \xi_{h(a^*)} \theta(a^*) \\ \alpha_\theta^\vee(\xi_{a^*}) &= a \xi_{t(a)} - \xi_{h(a)} a \\ \beta &= \varphi_0^{-1} \circ \alpha_\theta^\vee \circ \varphi_1\end{aligned}$$

So square **(I)** in Theorem 2.2.13 commutes.

Proof. The first two equalities are shown directly in ¹⁶ and the last two are clear from the definitions together with Proposition 2.1.6. $\square$

Theorem 2.2.13. By Lemma 2.2.14, it suffices to show that **(II)** commutes. While one can similarly compute $\beta^\vee$ directly, such a calculation is unnecessary as the commuting of **(II)** follows from that of **(I)**.

Indeed, dualizing and applying $(-)_\theta$ to the maps in **(I)**, produces a still commuting diagram:

$$\begin{array}{ccc} P_1 & \xleftarrow{(\alpha_\theta^\vee)_\theta^\vee} & P_0 \\ (\varphi_1)_\theta^\vee \downarrow & \textbf{(I)}_\theta^\vee & \downarrow (\varphi_0)_\theta^\vee \\ P_1^\vee & \xleftarrow{\beta_\theta^\vee} & P_0^\vee \end{array} = \begin{array}{ccc} P_1 & \xleftarrow{\alpha} & P_0 \\ -\varphi_1 \downarrow & \textbf{(I)}_\theta^\vee & \downarrow \varphi_0 \\ P_1^\vee & \xleftarrow{\beta_\theta^\vee} & P_0^\vee \end{array}$$

which indeed shows $\varphi_1 \circ \alpha = -\beta_\theta^\vee \circ \varphi_0$, i.e. **(II)** commutes.

The equality of maps $(\alpha_\theta^\vee)_\theta^\vee = \alpha$ follows from Proposition 2.1.6, and $(\varphi_0)_\theta^\vee = \varphi_0^\vee = \varphi_0$ follows from the definitions. For $(\varphi_1)_\theta^\vee = -\varphi_1$, observe that it suffices to show $(\varphi_1)_\theta = (\varphi_1)^\vee$ and indeed,

$$\begin{aligned}(\varphi_1)_\theta(\eta_{a^*}) &= (\xi_a)_\theta = \begin{cases} \theta(l_a^*) \eta_a^\vee \theta(l_a^{-1}) & \text{if } \epsilon(a) = 1 \\ -\theta(r_a^{-1}) \eta_a^\vee \theta(r_a) & \text{if } \epsilon(a) = -1 \end{cases} \\ &= \begin{cases} r_a^* \eta_a^\vee r_a^{-1} & \text{if } \epsilon(a) = 1 \\ -l_a^{-1} \eta_a^\vee l_a & \text{if } \epsilon(a) = -1 \end{cases} \\ &= -(\varphi_1)^\vee(\eta_{a^*}). \end{aligned} \quad \square$$

So without conditions on the quiver, we have established:

Corollary 2.2.15. *If $P_\bullet \rightarrow \Lambda$ is exact then $(P_\bullet^\vee)_\theta \rightarrow \Lambda[-2]$ is exact and $P_\bullet^\vee \rightarrow \Lambda[-2]$ is*

exact.

Therefore, the 2-CY property for Λ follows from the – apriori weaker – homological dimension two property. In the previous subsection, we showed that Λ has homological dimension two for Q connected and containing a cycle.

Corollary 2.2.16. *If Q is connected and contains a cycle then $\Lambda^q(Q)$ is 2-Calabi–Yau.*

2.3 FORMALITY OF THE MULTIPLICATIVE GINZBURG DG-ALGEBRAS

In this section we show that if Q satisfies the strong free product property, then the multiplicative Ginzburg dg-algebra is formal. In particular this proves Conjecture 2.0.3 in the case Q is connected and contains a cycle. Moreover, it reduces the conjecture to the remaining extended ADE cases and Conjecture 2.0.6.

If one views the dg multiplicative preprojective algebra as the central object of study, as in²⁶ and²⁵, then we are showing one can formally replace it by the non-dg version.

We begin with an elementary lemma.

Lemma 2.3.1. *Let k be a commutative ring. Let A be the dg-algebra defined as a graded algebra to be $k[r] * k[s]$ with $|r| = 0$ and $|s| = -1$, product given by concatenation of words, and differential extended as a derivation from the generators $d(s) = r$ and $d(r) = 0$. Then A is quasi-isomorphic (in fact chain homotopy equivalent) to its cohomology $H^*(A) = k$ concentrated in degree zero.*

Proof. Let $h : A \rightarrow A[-1]$ be the homotopy with the property $h(rf) = sf$ and $h(sf) = 0$ for all $f \in A$, and $h(k) = 0$. Then $h \circ d + d \circ h - 1_A$ is the projection with kernel k to the augmentation ideal of A . Therefore, it defines a contracting homotopy from A to k . $\square$

In other words, the lemma is observing that A , as the tensor algebra on an acyclic complex $kr \oplus ks$, is itself quasi-isomorphic to k .

Corollary 2.3.2. *The dg-algebra A given by*

$$\Lambda^q(Q) *_{kQ_0} kQ_0[r, (r + q)^{-1}] *_{kQ_0} kQ_0[s] \quad \text{with } |r| = 0 \text{ and } |s| = 1$$

and with differential determined by $d(s) = r$ is quasi-isomorphic to $\Lambda^q(Q)$ concentrated in degree zero.

Proof. Define a homotopy by, for $f, g \in \Lambda^q(Q)$:

$$h(frg) = fsg, \quad h(fsg) = 0, \quad h(f(r+q)^{-1}g) = q^{-1}h(fg) - q^{-1}fs(r+q)^{-1}g$$

where the definition of $h(f(r+q)^{-1}g)$ is chosen to match the formula for $h(frg)$ in the r -adic completion. There is an augmentation $A \twoheadrightarrow \Lambda^q(Q)$ with kernel $(r, s, r' := (r+q)^{-1} - q^{-1})$. Notice that $h \circ d + d \circ h$ is a homotopy from the identity on A to the augmentation $\Lambda^q(Q)$, as it annihilates $\Lambda^q(Q)$ and is the identity on s, r , and r' . $\square$

Definition 2.3.3. The *Ginzburg dg-algebra* $\Lambda^{dg,q}(Q)$ is defined to be

$$\Lambda^{dg,q}(Q) := (L_Q *_{kQ_0} kQ_0[s], \quad d(s) = \rho - q).$$

Proposition 2.3.4. *If $\Lambda^q(Q)$ satisfies the strong free product property (meaning $L_Q, r, \sigma, kQ_0[t, (t+q)^{-1}]$) satisfies the strong free product property for some choice of σ) then*

$$H_*(\Lambda^{dg,q}(Q)) = H_0(\Lambda^{dg,q}(Q)) \cong \Lambda^q(Q)$$

so in particular $\Lambda^{dg,q}(Q)$ is formal.

Proof. The strong free product property yields an isomorphism of graded vector spaces,

$$L_Q \cong \Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}].$$

Hence, as complexes

$$\Lambda^{dg,q}(Q) \cong L_Q *_{kQ_0} kQ_0[s] \cong \Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}] *_{kQ_0} kQ_0[s],$$

which by Corollary 2.3.2 is quasi-isomorphic to $\Lambda^q(Q)$, concentrated in degree zero. It follows that,

$$\Lambda^{dg,q}(Q) \cong H_*(\Lambda^{dg,q}(Q)) \cong H_0(\Lambda^{dg,q}(Q)) \cong \Lambda^q(Q)$$

as dg-algebras. $\square$

Remark 2.3.5. In the presence of Conjecture 2.0.1, formality of $\Lambda^{dg,q}(Q)$ implies $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau. Hence by Theorem 2.0.2, we have shown that $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau, when Q is connected and contains a cycle. One may be able to adapt the techniques in Section 2.2 to prove that $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau, in general. In more detail, the role of the $\Lambda^q(Q)$ -bimodule resolution, $P_\bullet$, should now be played by the Λ^{dg} -dg-bimodule given

by the total complex of:

$$\Lambda^{dg} \otimes_{kQ_0} k \cdot s \otimes_{kQ_0} \Lambda^{dg} \xrightarrow{\alpha^{dg}} \Lambda^{dg} \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} \Lambda^{dg} \xrightarrow{\beta_0^{dg}} \Lambda^{dg} \otimes_{kQ_0} \Lambda^{dg},$$

$$\underbrace{\hspace{10em}}_{\beta_1^{dg}}$$

where $\beta_1(a \otimes s \otimes b) = as \otimes b - a \otimes sb$ and $\beta_0(a \otimes x \otimes b) = ax \otimes b - a \otimes xb$.

2.4 THE CENTER AND PRIMALITY OF MULTIPLICATIVE PREPROJECTIVE ALGEBRAS

In the ADE case, for $q = 1$ and k characteristic zero, one can compute the center using the isomorphism $\Lambda^1(Q) \cong \Pi^0(Q)$ first established in¹⁴. Conjecture 2.0.4 predicts $Z(\Lambda^1(Q)) \cong e_v \Lambda^1(Q) e_v$ for Q extended ADE and $q = 1$ (proved in the case $Q = \widetilde{A}_n$). In the remaining cases, Conjecture 2.0.1 predicts $Z(\Lambda^q(Q)) = k$ for Q non-ADE and non-extended ADE. The goal of this section is to establish the conjecture in the case Q contains a cycle. We begin with a general discussion of the center of free products.

2.4.1 THE CENTER OF FREE PRODUCTS

Let R and S be rings containing $B = k^I$ as not-necessarily unital subrings. Write e_i for $\delta_{i,j} : I \rightarrow k \in B$. Let $A := R *_B S := T_B(R \oplus S)$ be the free product over B . Notice that A can have non-trivial central elements:

Example 2.4.1.

- (1) (trivial product: $S = B$) Let $R = \mathbb{C}[x]$ be a one-dimensional polynomial algebra and take $S = B = \mathbb{C} \cdot 1_R$, so $A = R = \mathbb{C}[x]$ is commutative.
- (2) (dimension 2) Let $R = \mathbb{R}[x]/(x^2 + 1)$, $S = \mathbb{R}[y]/(y^2 + 1)$ and $B = \mathbb{R}1_R = \mathbb{R}1_S$ then $xy + yx \in Z(A)$
- (3) (matrix exception) Let $R = S = \text{Mat}_2(k)$. Let $B = k^2$ be diagonal matrices in each. Writing r for a strictly upper triangular matrix in R and s for a strictly lower triangular matrix in S , we have $rs + sr \in Z(A)$.
- (4) (disconnected piece) Write kQ for the path algebra of the quiver Q over a field k . Let $R = k(\bullet_1 \rightarrow \bullet_2)$, $S = k(\bullet_2 \rightarrow \bullet_3)$, and $B = k(\bullet_2)$ be path algebras so $A = R *_B S = k(\bullet_1 \rightarrow \bullet_2 \rightarrow \bullet_3)$. A has the non-trivial central element $e_1 = 1_R - 1_B$, since this piece is disconnected from the rest of the algebra.

- (5) (preprojective algebra example) Let $S = \Pi(\bullet_1 \xrightarrow{\alpha} \bullet_2 \xleftarrow{\beta} \bullet_3)$ be the preprojective algebra of the A_3 quiver. Let $R = \Pi(\bullet_4 \xrightarrow{\gamma} \bullet_2 \xleftarrow{\delta} \bullet_5, \{2\})$ be the partial preprojective algebra of the A_3 quiver, where recall we don't enforce the relation at vertex 2. Let $B = k(\bullet_2)$. Notice that $A = R *_B S$ is the associated graded algebra to $\Pi(\widetilde{D}_4)$ with descending filtration $\mathcal{F}$, defined by $\mathcal{F}^{\leq n}$ is the subspace of sum of paths with at most n occurrences of γ or δ . So,

$$Z(\text{gr}(A)) = Z(\Pi(\widetilde{D}_4)) \cong e_1 \Pi(\widetilde{D}_4) e_1 \ni \alpha \beta^* \beta \alpha^*$$

from which we conclude that $Z(A) \supset Z(\text{gr}(A)) \neq k$.

In this section we ask and answer the question:

When is the center of a free product trivial? What conditions on R, S , and B ensure $R *_B S$ has center given by scalar multiples of the identity?

In order to rule out the first example we require $R \neq B$ and $S \neq B$. Examples (2) and (3) are regarded as low-dimensional exceptions, so we rule them out individually. To exclude examples (4) and (5), we require certain connectivity assumptions.

Definition 2.4.2. Let R be a ring. We say $r \in R$ is a *non-commutative non-zero divisor* if for all $r' \in R$, we have $rRr' \neq 0$ and $r'Rr \neq 0$. We say R is *prime* if $rRr' = 0$ implies $r = 0$ or $r' = 0$.

Note that a ring is prime if and only if every nonzero element is a non-commutative non-zero divisor.

We rule out Examples (4) and (5) by requiring that R be prime and 1_B be a non-commutative non-zero divisor in S . Notice in Example (4) that $1_B R e_1 = 0$ so R is not prime and in Example (5) that $1_B S(\alpha \beta^*) = 0$ so 1_B is a non-commutative zero divisor in S .

Finally, note that R prime is a stronger condition than needed for triviality of the center (e.g., $\mathbb{C}[x]/(x^3) *_\mathbb{C} \mathbb{C}[y]/(y^2)$.) So see Lemma 2.4.5 for weaker conditions, that only require the prime condition on elements in B together with some size constraints to avoid low-dimensional counterexamples, which we introduce now.

Definition 2.4.3. We say $i \in I$ is a *source* (respectively *sink*) for a vector subspace $V \subset A$ if $e_i V \neq 0$ (respectively $V e_i \neq 0$.) We say $i \in I$ is a *double source* (respectively *double sink*) if $\dim_k(e_i V) > 1$ (respectively $\dim_k(V e_i) > 1$).

Lemma 2.4.4. *Let R be a prime ring with $B = k^I$ a not-necessarily unital subring and a decomposition $R = \overline{R} \oplus B$. Then, every e_i is both a source and a sink for $\overline{R}$. Moreover, if $R \not\cong \text{Mat}_2(k)$ and R is not a degree two field extension of k , then I has a double source and a double sink.*

Proof. If $e_i = 1_R$ then $e_i \overline{R} = \overline{R} = \overline{R} e_i$ is non-zero by assumption. Otherwise, $1_R - e_i \neq 0$ and

$$e_i \overline{R} (1_R - e_i) = e_i R (1_R - e_i) \neq 0 \quad (1_R - e_i) \overline{R} e_i = (1_R - e_i) R e_i \neq 0$$

since R is prime. In either case, i is both a source and a sink for $\overline{R}$.

Case: $|I| \geq 3$

Pick $1, 2, 3 \in I$ and R prime implies

$$e_1 \overline{R} e_2 = e_1 R e_2 \neq 0 \quad e_1 \overline{R} e_3 = e_1 R e_3 \neq 0.$$

So $\dim_k(e_1 R) \geq 2$ and 1 is a double source for $\overline{R}$. Similarly 1 is a double sink for $\overline{R}$.

Case: $|I| = 2$

R prime implies each $d_{i,j} := \dim(e_i R e_j) \neq 0$ for $i, j \in I$. If any pair (i, j) has $d_{i,j} > 1$ then $d_{i,j'} > 1$ for some $j' \neq j$. So $2 \leq \dim(e_i R e_{j'}) = \dim(e_i \overline{R} e_{j'})$ and i is a double source and j' is a double sink for $\overline{R}$. Otherwise each $d_{i,j} = 1$ and pick $\alpha \in e_1 R e_2$ and $\beta \in e_2 R e_1$ both non-zero. Notice $\alpha \beta \in e_1 R e_1 = e_1 B e_1 = k e_1$ implies $\alpha \beta = c_1 e_1$ and similarly $\beta \alpha = c_2 e_2$. R prime implies $c_1 \alpha = \alpha \beta \alpha = \alpha c_2$ is non-zero and so $c_1 = c_2 \neq 0$. We conclude that the map

$$\varphi : \text{Mat}_2(k) \rightarrow R \quad e_{11} \mapsto e_1, \quad e_{22} \mapsto e_2, \quad e_{12} \mapsto \alpha, \quad e_{21} \mapsto \beta/c_1$$

is an isomorphism.

Case: $|I| = 1$

If $e_1 \neq 1_R$ then $e_2 := 1_R - e_1$ is an idempotent and R prime forces the isomorphism $\text{Mat}_2(k) \cong R$ as above.

If $e_1 = 1_R$ then 1 is both a double source and double sink for $\overline{R}$ unless $\dim_k(R) = 2$, in which case R prime implies R is the splitting field for some quadratic polynomial over k . $\square$

Lemma 2.4.5. *Define $J := \{i \in I : e_i \text{ is a non-commutative non-zero divisor in } R\}$. Assume:*

- (a) J has a source or sink for $\overline{S}$
- (b) J has a double source and double sink for $\overline{R}$
- (c) Every $i \in I$ is a source or sink for $\overline{R}$

Then $Z(A) \subset k1_A \oplus K$ where $K := \ker(1_B \cdot - : Z(A) \rightarrow k1_B)$. In particular, if 1_B is a non-commutative non-zero divisor in both S and R , then $K = 0$ and $Z(A) = k$ is trivial.

Proof. Fix $z \in Z(A)$ and write $z = z_B + z_R + z_S + z_{RS} + z_{SR}$ according to the decomposition of the free product into pieces:

$$A = B \oplus A_R \oplus A_S \oplus A_{RS} \oplus A_{SR}$$

where

$$A_{RS} := \sum_{m \geq 1} (\overline{RS})^m, \quad A_{SR} := \sum_{m \geq 1} (\overline{SR})^m, \quad A_R := \overline{R} \oplus \overline{R} \cdot A_{SR}, \quad A_S := \overline{S} \oplus \overline{S} \cdot A_{RS}.$$

We first show $1_B z \in B$ by showing it individually for each term in the decomposition.

$1_B z_R \in B$

Suppose $j \in J$ is a source for $\overline{S}$. So there exists $s \in S$ such that $e_j s \neq 0$. Then

$$A_{RS} \oplus A_S \oplus B \ni z(e_j s) = (e_j s)z \in A_{SR} \oplus A_S \oplus B$$

so this element is in the intersection $A_S \oplus B$. Hence $e_j z_R \in B$, so $e_j z_R = \alpha e_j$ for $\alpha \in k$. Since e_j is a non-commutative non-zero divisor in R , for all $i \in I$, there exists $r_i \in R$ such that

$$0 \neq e_j r_i (z_R e_i) = (e_j z_R) r_i e_i = \alpha r_i e_i \implies e_i z_R \in B$$

for all $i \in I$. Hence $1_B z_R = \sum_i e_i z_R \in B$.

$1_B z_S \in B$

Fix $i \in I$, necessarily a source or sink for $\overline{R}$. Without loss of generality, suppose i is a source. So there exists $r \in \overline{R}$ such that $e_i r \neq 0$. Then

$$A_{SR} \oplus A_R \oplus B \ni (e_i r)z = z(e_i r) \in A_{SR} \oplus A_R \oplus B$$

so $e_i r z \in A_R \oplus B$ and $e_i z_S = z_S e_i \in B$. This holds for all $i \in I$ and hence the sum $1_B z_S \in B$.

$$\underline{1_B z_{RS} \in B}$$

Write $A_{RS} = \overline{R}A_S$ and $A_{SR} = A_S\overline{R}$. Pick a basis for A_S , $\{f_m\}_{m \in M}$ such that each $f_m \in e_{i'}A_S e_{i''}$. Here $i', i'' \in I \sqcup \{0\}$ where $e_0 := 1_R - 1_B$. Write

$$z_{RS} = \sum r_m f_m \quad z_{SR} = \sum f_m r'_m \quad r_m, r'_m \in \overline{R}.$$

Suppose $i \in I$ is a double source for $\overline{R}$ and $z_{RS}e_i \notin B$. Then there exists $n \in M$ such that $r_n f_n e_i \notin B$. Pick $r''_n \in e_i \overline{R} \setminus \{\alpha r'_n\}_{\alpha \in k}$ not a scalar multiple of r'_n . This ensures that the sums,

$$B \oplus A_R \ni c + \sum_m r_m f_m r''_m = z r''_m = r''_m z = c' + \sum_m r''_m f_m r'_m \in B \oplus A_R$$

disagree on $\overline{R}f_n \overline{R}$ (i.e. $r_n f_n r''_n \neq r''_n f_n r'_n$). Hence we've contradicted the assumption $z_{RS}e_i \notin B$.

$$\underline{1_B z_{SR} \in B}$$

This argument is identical by examining z_{SR} instead of z_{RS} .

Next notice that $1_B z \in B$ implies $1_B z = \alpha 1_B$ for $\alpha \in k$. To see this, pick $i \in I$ and define $\alpha \in k$ such that $\alpha e_i = z e_i$. For $j \in I$, write β for the scalar such that $\beta_j e_j = z e_j$. Then, since e_i is a non-commutative non-zero divisor in R , there exists $r \in R$ such that $e_i r e_j \neq 0$. Hence

$$\alpha e_i r e_j = z e_i r e_j = e_i r z e_j = \beta e_i r e_j$$

so $\alpha = \beta$ and $1_B z = \alpha 1_B$, as desired.

Finally, for $z \in Z(A)$ pick $\alpha \in k$ such that $1_B(z - \alpha) = 0$. Since 1_B is non-commutative non-zero divisor in both R and S , and for all $r \in R$ and all $s \in S$,

$$1_B r(z - \alpha) = 1_B(z - \alpha)r = 0 \quad \text{and} \quad 1_B s(z - \alpha) = 1_B(z - \alpha)s = 0 \quad \implies \quad z - \alpha = 0,$$

completing the proof. □

Combining Lemma 2.4.5 and Lemma 2.4.4

Corollary 2.4.6. *Let R be a prime ring, $R \not\cong \text{Mat}_2(k)$ nor a degree two field extension of k . Let 1_B be a non-commutative non-zero divisor in S . Then $Z(A) = k$.*

Proof. Notice that $J = I$ since R is prime and I has a source for $\overline{S}$ since

$$1_B \overline{S} \neq 0 \implies e_i \overline{S} \neq 0 \text{ for some } i \in I.$$

So condition (a) is satisfied. Condition (b) follows from Lemma 2.4.4, which applies as we have assumed R is prime and not $\text{Mat}_2(k)$ nor a degree two field extension of k . Condition (c) follows from R prime. Finally, 1_B is a non-commutative non-zero divisor in S by assumption and in R since R is prime. We conclude that $Z(A) = k$. $\square$

Notice that under the hypotheses R prime and 1_B a non-commutative non-zero divisor in S , one has that A is prime. That is, given non-zero words $w_1, w_2 \in A$ one can find $w' \in A$ such that $w_1 w w_2 \neq 0$. In the case that w_1 ends in $\overline{S}$ and w_2 starts in $\overline{S}$ one has $s, s' \in S$ such that $w_1 s 1_B \neq 0$ and $1_B s' w_2 \neq 0$. Then using R prime, one can choose $r \in \overline{R}$ and there exists $r', r'' \in R$ such that $w_1 s 1_B r' r r'' 1_B s' w_2 \neq 0$. The other cases are easier and we leave them to the reader.

Lemma 2.4.7. *If R is prime and 1_B a two-sided non-commutative non-zero divisor in S , then $A = R *_B S$ is prime.*

2.4.2 THE CENTER IS TRIVIAL FOR QUIVERS CONTAINING A CYCLE

Using a decomposition for $\Lambda^q(Q)$ as a free product from Corollary 2.5.6 together with Corollary 2.4.6 we get:

Corollary 2.4.8. *If Q is connected and contains an unoriented cycle, then $\Lambda^q(Q)$ is prime. If the containment is proper, then $Z(\Lambda^q(Q)) = k$.*

Proof. Suppose $\widetilde{A}_n \subsetneq Q$. By Corollary 2.5.6, one has a decomposition,

$$\text{gr } \Lambda^q(Q) \cong \Lambda^{q_n}(\widetilde{A}_n) *_k Q_0 \Lambda^{q'}(Q \setminus \widetilde{A}_n, (\widetilde{A}_n)_0),$$

where $q = (q_n, q')$ and $\Lambda^q(Q)$ is given the descending filtration by powers of the ideal generated by those arrows in $\overline{Q} \setminus \overline{\widetilde{A}_n}$ which are incident to $(\widetilde{A}_n)_0$. Therefore

$$\text{gr}(Z(\Lambda^q(Q))) \subseteq Z(\text{gr}(\Lambda^q(Q))) = Z(\Lambda^{q_n}(\widetilde{A}_n) *_k Q_0 \Lambda^{q'}(Q \setminus \widetilde{A}_n, (\widetilde{A}_n)_0)),$$

and so to show $\text{gr}(Z(\Lambda^q(Q))) = k$, it remains to check the hypotheses of Corollary 2.4.6, with $R = \Lambda^{q_n}(\widetilde{A}_n)$, $S = \Lambda^{q'}(Q \setminus \widetilde{A}_n, (\widetilde{A}_n)_0)$ and $B = kQ_0$. Indeed,

- $\Lambda^{q_n}(\widetilde{A}_n)$ is prime by Proposition 2.5.3, and clearly $\Lambda^{q_n}(\widetilde{A}_n) \neq kQ_0$.
- $\Lambda^{q'}(\widetilde{A}_n)$, being infinite-dimensional, is not isomorphic to $\text{Mat}_2(k)$, nor $k[x]/(x^2 + bx + c)$.

- $\Lambda^q(Q \setminus \widetilde{A}_n, (\widetilde{A}_n)_0) \neq kQ_0$ is the statement that the containment $\widetilde{A}_n \subsetneq Q$ is proper.
- 1_{kQ_0} is a non-commutative non-zero divisor in $\Lambda^q(Q \setminus \widetilde{A}_n, (\widetilde{A}_n)_0)$ since Q is connected, so every vertex in $Q \setminus \widetilde{A}_n$ is connected to some vertex in $(\widetilde{A}_n)_0$.

So $\text{gr}(Z(\Lambda^q(Q))) = k$. And by Lemma 2.4.7, $\text{gr}(Z(\Lambda^q(Q)))$ is prime. Since the filtration is Hausdorff (again by Corollary 2.5.6), we conclude also that $Z(\Lambda^q(Q)) = k$ and $\Lambda^q(Q)$ is prime. $\square$

Corollary 2.4.9. *If Q is connected and properly contains a non-oriented cycle, then $\Lambda^q(Q)$ has a unique, up to scaling, Calabi–Yau structure.*

Proof. Any two Calabi–Yau structures differ by an invertible map $\text{Hom}_{\Lambda\text{-bimod}}(\Lambda, \Lambda)$, which is determined by the image of the unit, a central invertible element. So the set of Calabi–Yau structures, when non-empty, forms a $Z(\Lambda)^\times$ -torsor, which in this case is $k^\times$. So any two Calabi–Yau structures differ by an invertible scalar. $\square$

2.5 THE STRONG FREE PRODUCT PROPERTY

The goal of this section is to prove the strong free product property for connected quivers containing a cycle. We first establish the strong free product property for the quivers $\widetilde{A}_n$ for $n \geq 0$ using the Diamond Lemma to build a section of the quotient map $\pi : L \rightarrow \Lambda^q(\widetilde{A}_n)$. Then we establish the more general result using the corresponding result for *partial* multiplicative preprojective algebras. See Subsection 2.2.1 for the prerequisite definitions.

As results in Sections 3, 4, and 5 rely on results established in this section, the reader should note that we do not use any results from previous sections.

2.5.1 THE CASE OF CYCLES

Consider the quiver $\widetilde{A}_{n-1}$: with vertex set $(\widetilde{A}_{n-1})_0 := \{0, 1, \dots, n-1\}$ and arrow set $(\widetilde{A}_{n-1})_1 = \{a_0, a_0^*, a_1, a_1^*, \dots, a_{n-1}, a_{n-1}^*\}$ with $s(a_i) = i$ and $t(a_i) = i+1 \pmod{n}$. The multiplicative preprojective algebra for this quiver is defined, with respect to the ordering given by $a_i < a_{i+1} < a_j^* < a_{j+1}^*$ for all $i, j \in \{0, 1, \dots, n-2\}$, to be

$$\Lambda^q(\widetilde{A}_{n-1}) := \frac{k\widetilde{A}_{n-1}[(1 + a_i a_i^*)^{-1}, (1 + a_i^* a_i)^{-1}]_{i=0, \dots, n-1}}{\langle \prod_{i=0}^n (1 + a_i a_i^*) \prod_{i=0}^{n-1} (1 + a_i^* a_i)^{-1} - \sum_{i=1}^n q_i e_i \rangle} =: \frac{L}{J}$$

Writing $a := \sum_i a_i$, $a^* := \sum_i a_i^*$, and $q = \sum_i q_i e_i$ since

$$1 + aa^* = 1 + \sum_i a_i a_i^* = \prod_{i=0}^{n-1} (1 + a_i a_i^*) \quad 1 + a^* a = 1 + \sum_i a_i^* a_i = \prod_{i=0}^{n-1} (1 + a_i^* a_i)$$

we have,

$$\Lambda^q(\tilde{A}_{n-1}) := \frac{k\tilde{A}_{n-1}[(1 + aa^*)^{-1}, (1 + a^*a)^{-1}]}{\langle (1 + aa^*)(1 + a^*a)^{-1} - q \rangle}.$$

We write $r := (1 + aa^*)(1 + a^*a)^{-1} - q$ for this relation, S for the degree zero piece $k(\tilde{A}_{n-1})_0$ of $\Lambda^q(\tilde{A}_{n-1})$. As before let $B := kQ_0[t, (q + t)^{-1}]$ and $\bar{B} = tB$, spanned over kQ_0 by $t^m, (t')^m, m \geq 1$, for $t' := (q + t)^{-1} - q^{-1}$. Let $r' := (q + r)^{-1} - q^{-1}$.

We construct $\sigma : L/(r) *_{kQ_0} B \rightarrow L$ so that (L, r, σ', B) satisfies the strong free product property using an explicit basis:

Proposition 2.5.1. *L is a free left kQ_0 -module with basis consisting of 1 together with all alternating products of elements of the following two sets, for $x := (1 + aa^*)$:*

$$\mathfrak{B} := \{x^m a^\ell, x^m (a^*)^\ell \mid m \in \mathbb{Z}, \ell \in \mathbb{N}\}, \quad \mathfrak{R} := \{r^m, (r')^m \mid m \in \mathbb{N}\}.$$

In particular, $\mathfrak{B}$ forms a basis for $\Lambda^q(\tilde{A}_{n-1}) = L/(r)$, and (L, r, σ, B) satisfies the strong free product property, with σ induced from the inclusion of $\mathfrak{B}$ into L .

Proof. Note that, for every vertex i , we have $e_i a = a e_j$ for a unique j , and similarly for the elements $a^*, x, y := (1 + a^* a)^{-1}, x^{-1}, y^{-1}$, and by definition, $e_i r = r e_i$. Therefore L is spanned as a left S -module by words in $a, a^*, x, y, x^{-1}, y^{-1}$, and r, r' (of course, this set is redundant, but we will need it this way to obtain the desired basis). The relations are:

$$xx^{-1} = 1 = x^{-1}x, \quad yy^{-1} = 1 = y^{-1}y, \quad x = 1 + aa^*, y = 1 + a^*a, \quad (2.5.1)$$

$$r = xy^{-1} - q, r' = yx^{-1} - q^{-1}. \quad (2.5.2)$$

The first step is to show that we can get into a normal form in finitely many steps. Afterwards, we will show that this form is unique. We can form the following system of reductions which implement these relations:

- Inverse Reductions: $xx^{-1}, x^{-1}x, yy^{-1}, y^{-1}y \mapsto 1$.
- Short Cycle Reductions: $aa^* \mapsto x - 1, a^*a \mapsto y - 1$.
- Reordering Reductions: $a^*x^{\pm 1} \mapsto y^{\pm 1}a^*, ay^{\pm 1} \mapsto x^{\pm 1}a$.

- Substitution Reductions: $y^{-1} \mapsto x^{-1}(r + q)$, $y \mapsto (r' + q^{-1})x$ (if not preceded by a);
 $ax \mapsto a(r + q)y$, $ax^{-1} \mapsto ay^{-1}(r' + q^{-1})$.
- Reductions in B : $rr', r'r \mapsto -qr' - q^{-1}r$.

Observe that we can apply a reduction to a monomial unless the monomial has the form $w_1 u_1 w_2 \cdots u_n w_{\ell+1}$ for some $\ell \in \mathbb{Z}_{\geq 0}$ where $u_i \in \mathfrak{A}$ and $w_i \in \mathfrak{B}$.

We can form an ordering on monomials such that these reductions always replace monomials with a linear combination of smaller monomials. The ordering is as follows. For every generator or word in generators z , let $n_z(w)$ denote the number of occurrences of z in w . For every pair of generators (or words in generators) z, z' , let $n_{z,z'}(w)$ denote the number of pairs of occurrences of z and z' in w with z appearing to the left of z' . For subsets of generators (or words in generators) Z, Z' let $n_Z := \sum_{z \in Z} n_z$ and $n_{Z,Z'} := \sum_{z \in Z, z' \in Z'} n_{z,z'}$.

For a word w in the generators, define the function

$$N(w) := (n_a(w), n_{\{a, a^*\}, \{x, x^{-1}, y, y^{-1}\}}(w), n_{\{ax, ax^{-1}\}}(w), n_{\{y, y^{-1}\}}(w), n_{\{r, r'\}}(w)) \in \mathbb{N}^5. \quad (2.5.3)$$

Now we say that $w' < w$ if $N(w') < N(w)$ in the lexicographical ordering on $\mathbb{N}^5$. Observe that, if w' is obtained from w by applying a reduction, then $N(w') < N(w)$. This implies that any sequence of reductions terminates in finitely many steps.

It remains to show that the procedure above produces a unique result. More precisely, we wish to show that any nontrivial linear combination of monomials in normal form is nonzero. First note that, because the generators $y^{\pm 1}$ are nonreduced, they can be substituted out everywhere (except in the substitution reductions for $y^{\pm 1}$ themselves). As long as we check that the defining relations are still satisfied, this will produce another valid reduction system. The result is:

Inverse Reductions:	Reordering Reductions:	(9) $y^{-1} \mapsto x^{-1}(r + q)$
(1) $xx^{-1} \xrightarrow{r_1} 1$	(5) $a^*x \xrightarrow{r_5} (r' + q^{-1})xa^*$	(10) $y \mapsto (r' + q^{-1})x$
(2) $x^{-1}x \xrightarrow{r_2} 1$	(6) $ax \xrightarrow{r_6} qxa - qar'x$	
	(7) $ax^{-1} \xrightarrow{r_7} x^{-1}a(r' + q^{-1})$	Reductions in B :
Short Cycle Reductions:	(8) $a^*x^{-1} \xrightarrow{r_8} x^{-1}(r + q)a^*$	(11) $rr' \mapsto -qr' - q^{-1}r$
(3) $aa^* \xrightarrow{r_3} x - 1$		
(4) $a^*a \xrightarrow{r_4} (r' + q^{-1})x - 1$	Substitution Reductions:	(12) $r'r \mapsto -qr' - q^{-1}r$

It is clear that this still implies all of the defining relations: for instance, $(1 + aa^*)$ still reduces to x and the relations for r and r' are implied by substitution.

The advantage of doing it this way is that the final reductions (substitution and B) don't overlap with any others, so the only overlaps between the nonreduced words above are amongst the (1)–(8), involving the generators a, a^*, x, x^{-1} only. (As we stated, such a procedure works in general, whenever some generators are nonreduced.)

To prove uniqueness, we apply the Diamond Lemma (see³). The Diamond Lemma says it suffices to check overlap ambiguities, i.e., where two non-reduced subwords overlap in a larger word. In this case, it suffices to show that the 12 cubic terms formed by overlapping the leading terms of the quadratic reductions uniquely resolve. These terms are:

$$\begin{array}{llll}
\text{(I)} & xx^{-1}x & \text{(IV)} & a^*aa^* & \text{(VII)} & a^*x^{-1}x & \text{(X)} & a^*ax \\
\text{(II)} & x^{-1}xx^{-1} & \text{(V)} & a^*xx^{-1} & \text{(VIII)} & ax^{-1}x & \text{(XI)} & aa^*x^{-1} \\
\text{(III)} & aa^*a & \text{(VI)} & axx^{-1} & \text{(IX)} & aa^*x & \text{(XII)} & a^*ax^{-1}.
\end{array}$$

The resolution of (I) and (II) are immediate (and are completely general, having to do with a basis for $k[x, x^{-1}]$). Here is a summary of the remaining resolutions of ambiguities:

$$\begin{array}{ll}
\text{(III)} & (r_3 - r_6 \circ r_4)(aa^*a) = 0 \\
\text{(IV)} & (r_4 - r_5 \circ r_3)(a^*aa^*) = 0 \\
\text{(V)} & (r_8 \circ r_5 - r_1)(a^*xx^{-1}) = 0 \\
\text{(VI)} & (r_7 \circ r_6 - r_1)(axx^{-1}) = 0 \\
\text{(VII)} & (r_5 \circ r_8 - r_2)(a^*x^{-1}x) = 0 \\
\text{(VIII)} & (r_8 \circ r_7 - r_2)(ax^{-1}x) = 0 \\
\text{(IX)} & (r_3 - r_3 \circ r_6 \circ r_5)(aa^*x) = 0 \\
\text{(X)} & (r_4 - r_4 \circ r_4 \circ r_5 \circ r_6)(a^*ax) = 0 \\
\text{(XI)} & (r_3 - r_3 \circ r_7 \circ r_8)(aa^*x^{-1}) = 0 \\
\text{(XII)} & (r_4 - r_4 \circ r_8 \circ r_7)(a^*ax^{-1}) = 0.
\end{array}$$

We explicitly demonstrate (X), one of the more involved resolutions.

$$a^*ax = (a^*a)x \xrightarrow{r_4} [(r' + q^{-1})x - 1]x = (r' + q^{-1})x^2 - x$$

and

$$\begin{aligned}
a^*ax &= a^*(ax) \xrightarrow{r_6} a^*(qxa - qar'x) \\
&\xrightarrow{r_4 \circ r_5} q(r' + q^{-1})xa^*a - q((r' + q^{-1})x - 1)r'x \\
&\xrightarrow{r_4} q(r' + q^{-1})x((r' + q^{-1})x - 1) - q((r' + q^{-1})x - 1)r'x \\
&= q(r' + q^{-1})x(q^{-1}x - 1) + qr'x = (r' + q^{-1})x^2 - x.
\end{aligned}$$

□

Remark 2.5.2. The choice of $\overline{B}$ was important here. If we instead had defined it so that $(q+t)^{-1} \in \overline{B}$, i.e., if we replace $r' = (q+r)^{-1} - q^{-1} \in \mathfrak{R}$ by $(q+r)^{-1}$, then our desired basis would no longer be linearly independent. Indeed, reducing aa^*a one way, we get $(x-1)a = xa - a$, which is reduced, whereas the other way we get $a(y-1) = a(q+r)^{-1}x - a$, also reduced. That is, $xa = a(q+r)^{-1}x$, an equality of two distinct reduced elements.

Proposition 2.5.3. $\Lambda^q(\widetilde{A}_n)$ is prime for all $n \geq 0$ and all $q \in (k^\times)^{n+1}$.

Proof. We need to show, for every pair $f, g \in \Lambda^q(\widetilde{A}_n)$, both nonzero, there exists some $h \in \Lambda^q(\widetilde{A}_n)$ such that $fhg \neq 0$. It suffices to take f and g to be linear combinations of basis elements that all begin at some vertex i and end at some vertex j . By right multiplication by a^{n-j} or $(a^*)^j$, one can take f and g to be linear combinations of *basis* elements ending at vertex 0. By left multiplication by a^i or $(a^*)^{n-i}$ and then applying Reordering reductions, one can take f or g to be linear combinations of basis elements starting and ending at vertex 0. In fact by doing this we can assume that f is of the form $e_0 f_1(x, x^{-1}) f_2(a^{n+1})$, where $f_1 \neq 0$ and f_2 has nonzero constant term, and similarly for g . Then their product is of the same form, and hence is nonzero. □

2.5.2 PARTIAL MULTIPLICATIVE PREPROJECTIVE ALGEBRAS

Generally, for any quiver Q , let $\mathcal{W} \subseteq Q_0$ be a nonempty subset of *white* vertices, and $\mathcal{B} := Q_0 \setminus \mathcal{W}$ the *black* vertices. Then the *partial multiplicative preprojective algebras* of $(Q, \mathcal{W})$ is $\Lambda^q(Q, \mathcal{W}) := L/(r_{\mathcal{B}})$, where $r_{\mathcal{B}} = 1_{\mathcal{B}} r 1_{\mathcal{B}}$, for $1_{\mathcal{B}} := \sum_{j \in \mathcal{B}} e_j$. This algebra interpolates between $\Lambda^q(Q, Q_0) = L$ and $\Lambda^q(Q, \emptyset) = \Lambda^q(Q)$. Let $B := \mathcal{B}[t, (t+q)^{-1}] = 1_{\mathcal{B}} k Q_0[t, (q+t)^{-1}]$.

Proposition 2.5.4. Let Q be a connected quiver and $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ a decomposition into black and white vertices with $\mathcal{W} \neq \emptyset$. Then $(L, r_{\mathcal{B}}, \sigma, B)$ satisfies the strong free product property for some choice of σ .

In more detail, let $F \subseteq \overline{Q}$ be a union of trees rooted at the vertices of $\mathcal{W}$ with:

- $F_0 = Q_0$,
- no arrows between vertices in $\mathcal{W}$, and
- arrows directed towards the roots.

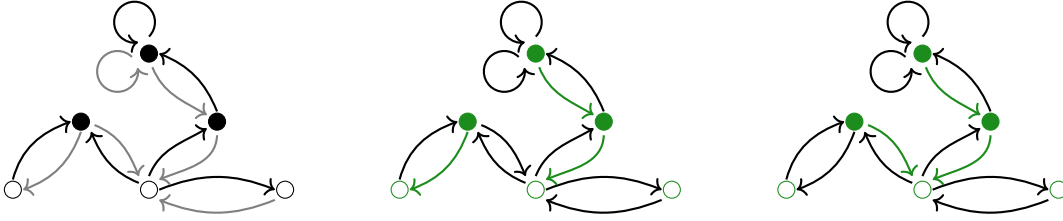


Figure 2.1: The quiver on the left is a doubled quiver, obtained by adding the grey arrows. It has three white vertices and three black vertices. The middle and right diagrams show two inequivalent spanning forests, in green, with roots at the white vertices.

We call such an F a spanning forest, see Figure 2.1. For convenience we assume (up to reorienting the arrows of Q) that $F_1 \subseteq Q_1$.

A basis for L is given by concatenable words in the arrows $a \in \overline{Q}_1$, the elements $x_a^{\pm 1} := (1 + aa^*)^{\pm 1}$ for $a \in \overline{Q}_1 \setminus \overline{F}_1$, and the elements $r_B, r'_B := (q + r_B)^{-1} - q^{-1}$, such that the following subwords do not occur:

$$x_a x_a^{-1}, x_a^{-1} x_a, aa^*, ax_{a^*}^{\pm 1}, \text{ for } a \in \overline{Q}_1; \quad x_{a^*}^{-1}, x_a^* a, x_{a^*}^2, \text{ for } a \in F_1. \quad (2.5.4)$$

The words in which r_B do not occur form a basis for $\Lambda^q(Q, \mathcal{W}) = L/(r_B)$, and the section σ is given by the inclusion of these elements.

Proof of Proposition 2.5.4. The proof parallels that of Proposition 2.5.1.

Note that L is spanned by concatenable words in $a, x_a := (1 + aa^*), x_a^{-1}, r, r'$ subject to the following relations, depending on the choice of ordering $\leq$ on the arrows $a \in \overline{Q}_1$:

$$x_a x_a^{-1} = 1 = x_a^{-1} x_a, \quad x_a = 1 + aa^*, \quad (2.5.5)$$

$$r = \prod_{\leq} x_a^{\epsilon(a)} - q, \quad r' = \prod_{a, \geq} x_a^{-\epsilon(a)} - q^{-1} \quad (2.5.6)$$

$$rr' = r'r = -qr' - q^{-1}r. \quad (2.5.7)$$

Recall $l_a := \prod_{b \in \overline{Q}, b < a} g_b^{\epsilon(b)}$ and $r_a := \prod_{b \in \overline{Q}, b > a} g_b^{\epsilon(b)}$. So

$$l_a(1 + aa^*)^{\epsilon(a)} r_a = r + q \implies x_a^{\epsilon(a)} := (1 + aa^*)^{\epsilon(a)} = l_a^{-1}(r + q)r_a^{-1}.$$

So define $\text{red}_a^{\epsilon(a)} = l_a^{-1}(r + q)r_a^{-1}$. We implement the above relations with the following reductions:

- Inverse Reductions: $x_a x_a^{-1}, x_a^{-1} x_a \mapsto 1$ for $a \in \overline{Q}_1$.

- Short Cycle Reductions: $aa^* \mapsto x_a - 1$ for $a \in \overline{Q}_1$.
- Reordering Reductions: $a^*x_a^\pm \mapsto x_{a^*}^\pm a^*$ for $a \in \overline{Q}_1$.
- Substitution Reductions: $x_a^\pm \mapsto \text{red}_a^\pm$, $x_{a^*}^{-1} \mapsto 1 - a^* \text{red}_a^{-1} a$,
 $x_{a^*}^2 \mapsto x_{a^*} + a^* \text{red}_a a$, $x_{a^*} a^* \mapsto a^* \text{red}_a$, for $a \in F_1$
- Reductions in B : $rr', r'r \mapsto -qr' - q^{-1}r$.

For each monomial in L , define a weighted size,

$$\varphi_a := n_{\{a, a^*\}} + \frac{3}{2}n_{\{x_a, x_{a^*}\}} + 3n_{\{x_a^{-1}, x_{a^*}^{-1}\}}$$

for each $a \in \overline{Q}_1$. From this, one can define a notion of size among monomials in $\langle x_a^\pm, a \mid a \in \overline{Q}_1 \rangle$, by first defining a total ordering on the arrows $(\overline{Q}_1, <)$ such that,

$$\begin{aligned} a < a' & \text{ if } a \in F_1, a' \in \overline{Q}_1 \setminus F_1, \\ & \text{ or if } a, a' \in F_1 \text{ with } a' \text{ disconnected from } \mathcal{W} \text{ in } F_1 \setminus \{a\}. \end{aligned}$$

Intuitively, we are saying that arrows in the spanning forest come before the rest in the ordering, with arrows closer to the white vertices coming first. Using $<$, define

$$\begin{aligned} N' : \langle x_a^\pm, a, r, r' \mid a \in \overline{Q} \rangle & \rightarrow \mathbb{N}^{(\overline{Q}_1, <)} \times \mathbb{N}^2 \\ N'(w) & := (2\varphi_a(w), n_{\{a \mid a \in \overline{Q}_1\}, \{x_a \mid a \in \overline{Q}_1\}}(w), n_{\{r, r'\}}(w)), \end{aligned}$$

from which we say $w < w'$ if $N(w) < N(w')$ in the lexicographical ordering on $\mathbb{N}^{|\overline{Q}_1|+2}$. Notice, as in Proposition 2.5.1, that $N'(r_i(w)) < N'(w)$ for any word w and reduction r_i with $r_i(w) \neq w$. First notice that, by design, φ_a decreases under the following reductions:

- Inverse Reductions: $\varphi_a(x_a x_a^{-1}) = \varphi_a(x_a^{-1} x_a) = 3 + 3/2 > 0 = \varphi_a(1)$
- Short Cycle Reductions: $\varphi_a(aa^*) = 2 > 3/2 = \varphi_a(x_a)$
- Substitution Reductions: $\varphi_a(x_a) = 3/2 > 0 = \varphi_a(\text{red}_a)$,
 $\varphi_a(x_a^{-1}) = 3 > 0 = \varphi_a(\text{red}_a^{-1})$, $\varphi(x_{a^*}^{-1}) = 3 > 2 = \varphi_a(a^* \text{red}_a^{-1} a)$,
 $\varphi(x_{a^*}^2) = 3 > 2 = \varphi_a(a^* \text{red}_a a)$ and $\varphi(x_{a^*}^2) = 3 > 3/2 = \varphi_a(x_{a^*})$,
 $\varphi_a(x_{a^*} a^*) = 5/2 > 1 = \varphi_a(a^* \text{red}_a)$.

For the Substitution Reductions observe that red_a for $a \in F_1$ has subwords x_b, x_{b^*} for only a single $b \in F_1$ which is necessarily *farther* from the root than a and the remaining

arrows are not in the spanning forest. Consequently, φ_a decreasing, despite φ_b increasing for larger b , implies that N decreases. The Reordering Reductions preserve all φ_a but decrease $n_{\{a|a \in \overline{Q}_1\}, \{x_a|a \in \overline{Q}_1\}}(w)$ by definition, and hence decrease N' . The reductions in B preserve all φ_a and $n_{\{a|a \in \overline{Q}_1\}, \{x_a|a \in \overline{Q}_1\}}(w)$ but decrease $n_{\{r, r'\}}(w)$.

We conclude that every monomial in L reduces to a linear combination of monomials without subwords in

$$\{x_a x_a^{-1}, x_a^{-1} x_a, aa^*, ax_a^*, ax_a^{-1} \mid a \in \overline{Q}_1\} \cup \{x_{a^*}^{-1}, x_{a^*} a, x_{a^*}^2 \mid a \in F_1\}$$

after applying *finitely* many reductions.

Note that some generators are nonreduced: x_a , x_a^{-1} , and $x_{a^*}^{-1}$ for $a \in F_1$. Therefore, we can put in reductions for each of these and throw out all other reductions involving these generators, provided we check that all the defining relations still reduce to zero. We have the reductions:

$$\begin{array}{ll} (1) \ x_a x_a^{-1} \xrightarrow{r_1} 1 \text{ for } a \in \overline{Q}_1 & (5) \ ax_{a^*}^{\pm} \xrightarrow{r_5} x_a^{\pm} a \text{ for } a \notin \overline{F}_1 \\ (2) \ x_a^{-1} x_a \xrightarrow{r_2} 1 \text{ for } a \in \overline{Q}_1 & (6) \ ax_{a^*} \xrightarrow{r_6} \text{red}_a a \text{ for } a \in F_1 \\ (3) \ aa^* \xrightarrow{r_3} \text{red}_a - 1 \text{ for } a \in F_1 & (7) \ x_{a^*}^2 \xrightarrow{r_7} x_{a^*} + a^* \text{red}_a a \text{ for } a \in F_1 \\ (4) \ aa^* \xrightarrow{r_4} x_a - 1 \text{ for } a \notin F_1 & (8) \ x_{a^*} a^* \xrightarrow{r_8} a^* \text{red}_a \text{ for } a \in F_1 \end{array}$$

which don't overlap with the remaining reductions:

$$\text{Substitution reductions: } x_a^{\pm 1} \mapsto \text{red}_a^{\pm 1}, x_{a^*}^{-1} \mapsto 1 - a^* x_a^{-1} a, \quad a \in F_1;$$

$$\text{Reductions in B: } rr', r'r \mapsto -qr' - q^{-1}r.$$

As before, reductions (3) and (4) imply the relations $x_a = 1 + aa^*$, whereas the substitution reductions imply the defining relations for r, r' . So this is a valid reduction system.

This reduction system has thirteen ambiguities:

$$\begin{array}{lll} \text{(I)} \ x_a x_a^{-1} x_a \text{ for } a \notin \overline{F}_1 & \text{(V)} \ ax_{a^*}^2 \text{ for } a \in F_1 & \text{(IX)} \ a^* ax_{a^*} \text{ for } a \in F_1 \\ \text{(II)} \ x_a^{-1} x_a x_a^{-1} \text{ for } a \notin \overline{F}_1 & \text{(VI)} \ x_{a^*}^2 a^* \text{ for } a \in F_1 & \text{(X)} \ a^* ax_{a^*} \text{ for } a \in \overline{Q} \setminus F_1 \\ \text{(III)} \ ax_{a^*} x_{a^*}^{-1} \text{ for } a \notin \overline{F}_1 & \text{(VII)} \ x_{a^*} a^* a \text{ for } a \in F_1 & \text{(XI)} \ aa^* a \text{ for } a \in \overline{Q}_1 \setminus \overline{F}_1 \\ \text{(IV)} \ ax_{a^*}^{-1} x_{a^*} \text{ for } a \notin \overline{F}_1 & \text{(VIII)} \ ax_{a^*} a^* \text{ for } a \in F_1 & \text{(XII)} \ aa^* a \text{ for } a \in F_1 \end{array}$$

(XIII) aa^*a for $a^* \in F_1$

which all resolve by the resolutions

$$\begin{aligned}
\text{(I)} \quad & (r_1 - r_2)(x_a x_a^{-1} x_a) = 0 & \text{(VIII)} \quad & (r_3 \circ r_6 - r_3 \circ r_8)(ax_a^* a^*) = 0 \\
\text{(II)} \quad & (r_2 - r_1)(x_a^{-1} x_a x_a^{-1}) = 0 & \text{(IX)} \quad & (r_7 \circ r_4 - r_6)(a^* a x_a^*) = 0 \\
\text{(III)} \quad & (r_1 - r_1 \circ r_5 \circ r_5)(ax_a^* x_a^{-1}) = 0 & \text{(X)} \quad & (r_4 \circ r_5 \circ r_5 - r_4)(a^* a x_a^*) = 0 \\
\text{(IV)} \quad & (r_2 - r_2 \circ r_5 \circ r_5)(ax_a^{-1} x_a^*) = 0 & \text{(XI)} \quad & (r_4 - r_5 \circ r_4)(aa^* a) = 0 \\
\text{(V)} \quad & (r_3 \circ r_6 \circ r_7 - r_6 \circ r_6)(ax_a^2) = 0 & \text{(XII)} \quad & (r_6 \circ r_4 - r_3)(aa^* a) = 0 \\
\text{(VI)} \quad & (r_3 \circ r_8 \circ r_7 - r_8 \circ r_8)(x_a^2 a^*) = 0 & \text{(XIII)} \quad & (r_8 \circ r_4 - r_3)(aa^* a) = 0. \\
\text{(VII)} \quad & (r_7 \circ r_4 - r_8)(x_a^* a^* a) = 0
\end{aligned}$$

The resolutions of the ambiguities (I)–(IV) and (X)–(XIII) are quick, leaving the computational heart of the calculations with the five resolutions (V)–(IX). Note that the resolutions for (V) and (VI) are identical after swapping the roles of reductions r_6 and r_8 , and similarly for (IX) and (VII), leaving three calculations: (V), (VIII), and (IX). These ambiguities express the overlap of r_6 with r_7 , r_8 , and r_4 respectively and further reduce uniquely to $\text{red}_a^2 a$, $\text{red}_a(\text{red}_a - 1)$, and $a^* \text{red}_a a$. $\square$

2.5.3 A CONVENIENT SUBSTITUTION

It will be convenient for us to make the substitutions:

$$\overline{x_a^\pm} := x_a^{\pm 1} - 1. \quad (2.5.8)$$

The motivation for this is as follows: let A be either a multiplicative preprojective algebra or a partial version for a connected quiver Q . Let I be the ideal generated by all paths beginning and ending at vertices having either $q = 1$ or the color white (the latter applies only to the setting of partial multiplicative preprojective algebras). Then A/I is nonzero, and we can make use of the I -adic filtration. The modified generators $\overline{x_a^\pm}$, for a an arrow in I , have the advantage of lying in the ideal I . As we will see, at least in the case where Q either contains a cycle or there is a white vertex, the I -adic filtration is Hausdorff. Thus, we get an embedding of A into the completion $\widehat{A}_I$, realising $\overline{x_a^\pm}$ as power series with zero constant term. In the special case where $q = 1$ (at all black vertices), this

completion means we are working with noncommutative power series in arrows, and all of the modified generators have zero constant term. This completion is closely related to the completion of (partial) additive preprojective algebras at $\lambda = 0$ (at all black vertices).

Practically speaking, we only require the above substitution in the next subsection in the partial case, at white vertices, for the purpose of obtaining a basis for quivers containing cycles. But the substitution probably is worth making as well at any vertex where we want to think of q as a deformation parameter based at $q = 1$.

Let us explain how this substitution works in the case of the cycle $\widetilde{A}_n$ (although we do not strictly need it in that case). We formally set $\overline{x^\pm} := x^{\pm 1} - 1$ and $\overline{y^\pm} := y^{\pm 1} - 1$; then the modified reductions from Section 2.5.1 are the following ones:

- Inverse Reductions: $\overline{x^+x^-}, \overline{x^-x^+} \mapsto -\overline{x^+} - \overline{x^-}$ and $\overline{y^+y^-}, \overline{y^-y^+} \mapsto -\overline{y^+} - \overline{y^-}$
- Short Cycle Reductions: $aa^* \mapsto \overline{x^+}, a^*a \mapsto \overline{y^+}$.
- Reordering Reductions: $a^*\overline{x^\pm} \mapsto \overline{x^\pm}a^*, a\overline{y^\pm} \mapsto \overline{y^\pm}a$.
- Substitution Reductions: $\overline{y^-} \mapsto \overline{x^-}(r+q)+r+(q-1), \overline{y^+} \mapsto (r'+q^{-1})\overline{x^+}+r'+(q^{-1}-1)$ (if not preceded by a);
 $a\overline{x^+} \mapsto a(r+q)\overline{y^+} + ar + (q-1)a, a\overline{x^-} \mapsto a\overline{y^-}(r'+q^{-1}) + ar' + (q^{-1}-1)a$.

This produces the same ambiguities as before, which resolve in the same way after eliminating the nonreduced generators $\overline{y^\pm}$ (another way to say this is that the reductions are the same up to the change of variables, so ambiguities resolve if and only if they did before). The modified ordering function,

$$N^z(w) := (n_a(w), n_{\{a, a^*\}, \{\overline{x^+}, \overline{x^-}, \overline{y^+}, \overline{y^-}\}}(w), n_{\{a\overline{x^+}, a\overline{x^-}\}}(w), n_{\{\overline{y^+}, \overline{y^-}\}}(w)),$$

is strictly decreasing under applications of reductions and hence every term reduces after applying finitely many reductions. The new left free kQ_0 -module basis, as in Proposition 2.5.1, is given by alternating words in $\mathfrak{R}$ and

$$\mathfrak{B}' := \{(\overline{x^\pm})^m a^\ell, (\overline{x^\pm})^m (a^*)^\ell \mid m \in \mathbb{N}, \ell \in \mathbb{N}\}.$$

In the case of the partial multiplicative preprojective algebra, the modified reductions are as follows:

- Inverse Reductions: $\overline{x_a^+x_a^-}, x_a^{-1}x_a \mapsto -\overline{x_a^+} - \overline{x_a^-}$ for $a \in \overline{Q}_1$.

- Short Cycle Reductions: $aa^* \mapsto \overline{x_a}$ for $a \in \overline{Q_1}$.
- Reordering Reductions: $a^* \overline{x_a^\pm} \mapsto \overline{x_{a^*}^\pm} a^*$ for $a \in \overline{Q_1}$.
- Substitution Reductions: $\overline{x_a^\pm} \mapsto \text{red}_a^\pm - 1$, $\overline{x_{a^*}^-} \mapsto -a^* \text{red}_a^{-1} a$,
 $\overline{x_{a^*}^+}^2 \mapsto -\overline{x_{a^*}^-} + a^* \text{red}_a a$, $\overline{x_{a^*}^-} a^* \mapsto a^* (\text{red}_a - 1)$, for $a \in F_1$.

Again, the same ordering function applies here and strictly decreases under these reductions. The ambiguities must resolve since they did before. Note that, for the following subsection, we only require the substitutions $\overline{x_a}$ in the case where the arrow a begins at a white vertex (which in particular implies that $a \notin F_1$, although it could be that $a^* \in F_1$). If we only make these substitutions, it is similarly easy to write the above reductions in the case where for certain arrows $\overline{x_a}$ appears and for others x_a appears; we leave this to the reader.

The only thing that we require from the above in the next subsection is the following observation: *reductions on $1_{\mathcal{W}} L_Q 1_{\mathcal{W}}$ preserve the augmentation ideal*, that is to say, the ideal generated by the generators. In other words, any monomial of positive length beginning and ending at white vertices reduces to a linear combination of other such monomials. This was not true with the original generators (e.g., looking at the inverse reductions).

2.5.4 QUIVERS CONTAINING CYCLES

Finally, we prove the strong free product property for a connected quiver containing a cycle, along with providing a natural decomposition and basis for its multiplicative preprojective algebra. The technique involves showing that, in these cases, the multiplicative preprojective algebra decomposes (as a vector space) into a free product of the multiplicative preprojective algebra for the cycle and a *partial* multiplicative preprojective algebra for the complement of the cycle, where “partial” means that we turn off the relations at the vertices of the cycle. This technique should extend to the case of general extended ADE quivers, hence reducing Conjecture 2.0.1 to the extended ADE case.

We will show furthermore that, if Q is a connected quiver containing a cycle Q_E , with $Q' := Q \setminus Q_E$, then there is a linear isomorphism, for $\mathcal{W} := (Q_E)_0$:

$$\Lambda^{q_E}(Q_E) *_{k_{Q_0}} \Lambda^{q'}(Q', \mathcal{W}) \rightarrow \Lambda^q(Q). \quad (2.5.9)$$

Let us explain how such an isomorphism arises. Let $Q''_1 \subseteq Q'_1$ be the subset of arrows incident to $\mathcal{W}$. Consider the $(\overline{Q''_1})$ -adic filtration on both sides. Then the associated

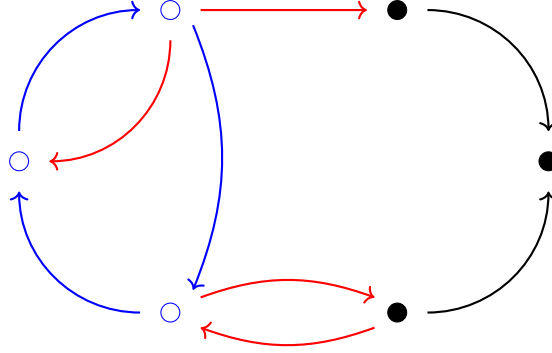


Figure 2.2: This quiver, Q , contains the cycle $Q_E = \widetilde{A}_2$, in blue, with complement, Q' – a quiver with underlying graph B_6 – in black and red. The red arrows are the arrows in Q' with either head or tail on the cycle. After doubling the quiver, one considers the descending filtration by the red arrows and their opposites on the multiplicative preprojective algebra, $\Lambda^q(Q)$, whose associated graded is $\Lambda^{q_2}(\widetilde{A}_2) *_{kQ} \Lambda^{q'}(Q', \mathcal{W})$.

graded relation to r_Q is the sum $r_{Q_E} + r_{Q', \mathcal{B}}$. As a result, there is a canonical surjection

$$\Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^{q'}(Q', \mathcal{W}) \twoheadrightarrow \text{gr}_{(\overline{Q'}_1)} \Lambda^q(Q), \quad (2.5.10)$$

see Figure 2.2.

The isomorphism (2.5.9) is filtered and induces the surjection 2.5.10 on associated graded algebras, which is hence also an isomorphism. We note that, in the additive case, the isomorphism (2.5.9) follows from the proof of²⁸ Theorem 3.4.2 (see also⁴⁹ Section 5, particularly Corollary 5.2.9.(ii)). Also, note that the fact that the associated graded of (2.5.9) is an isomorphism does not imply that the original map is an isomorphism (there would be no way to deduce surjectivity, since the descending filtration is not finite; if we knew a priori that the filtration were Hausdorff then we could deduce injectivity, although we only know this as a consequence of the proof of the isomorphism (2.5.9)).

As before, let $B := kQ_0[t, (q+t)^{-1}]$ and $\overline{B} = tB$, which is spanned by elements $t^m, (t')^m, m \geq 1$ where $t' := (q+t)^{-1} - q^{-1}$.

Proposition 2.5.5. *Let Q be a connected quiver containing a cycle $Q_E \subseteq Q$ ($Q_E \cong \widetilde{A}_{n-1}$). Then a k -linear basis for L is obtained by concatenable products of arrows a and elements x_a , for $a \in (\overline{Q_E})_1$, and $\overline{x}_a$, for $a \in \overline{Q'}_1$, and the elements $r, r' := (q+r)^{-1} - q^{-1}$, such that maximal subwords in $a, x_a^{\pm 1}, a \in (Q_E)_1$ are basis elements for $\Lambda^{q_E}(Q_E)$, and maximal subwords in $a, \overline{x}_a^{\pm 1}, a \in Q'_1$ are basis elements of $\Lambda^q(Q', \mathcal{W})$.*

In particular, the inclusion on basis elements not containing r, r' defines a filtered isomorphism (2.5.9), as well as a section $\sigma : \Lambda^q(Q) \rightarrow L$, such that (L, r, σ, B) satisfies the strong free product property.

Note in the proposition that we only need the elements $\overline{x_a}$ instead of x_a for $a \in \overline{Q'}_1$ beginning at a vertex of Q_E (and making this change to the statement does not affect the proof). On the other hand, we could freely replace all generators by the modified ones $\overline{x_a}$, even for $a \in (Q_E)_1$, again without changing the proof.

Corollary 2.5.6. *The filtration on $\Lambda^q(Q)$ generated by arrows in $\overline{Q \setminus Q_E}$ incident to $(Q_E)_0$ is Hausdorff, and the associated graded map of (2.5.9) (using the same filtration on the LHS) is an algebra isomorphism,*

$$\Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^{q'}(Q', (Q_E)_0) \rightarrow \text{gr } \Lambda^q(Q). \quad (2.5.11)$$

Proof of Corollary 2.5.6. Let $X \subseteq \overline{Q \setminus Q_{E1}}$ be the set of arrows incident to $(Q_E)_0$ not in Q_E itself. Note that $\text{gr } \Lambda^{q'}(Q', (Q_E)_0) \cong \Lambda^{q'}(Q', (Q_E)_0)$, since there are no relations at $(Q_E)_0$; also the filtration in $\Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^{q'}(Q', (Q_E)_0)$ is entirely induced by the second factor, so the associated graded algebra of the free product is again the free product itself.

Next, observe that the filtration on $\Lambda^{q'}(Q', (Q_E)_0)$ is Hausdorff, as again there are no relations at $(Q_E)_0$. So the filtration is Hausdorff on the source of (2.5.9), hence also on the target, as the isomorphism is filtered. This yields the desired statement. $\square$

Proof of Proposition 2.5.5. Using the strong free product properties for $L(Q_E)$ and $L(Q')$, we obtain the strong free product property for $(L, r(Q_E) + r(Q')_{\mathcal{B}}, \sigma, B)$, with σ defined by extending the sections $\sigma(Q_E), \sigma(Q')$. In more detail, combining the reduction systems for L_{Q_E} in Proposition 2.5.1 and for $L_{Q'}$ in Proposition 2.5.4, one gets a reduction system for $L_Q \cong L_{Q_E} *_{kQ_0} L_{Q'}$. However, $\Lambda^q(Q) = L/(\rho(Q_E)\rho(Q') - q)$ is *not* the quotient $L/(\rho(Q_E) + \rho(Q')_{\mathcal{B}} - q)$. So we need to perturb the relation to $r = \rho(Q')\rho(Q_E) - q$.

First observe that this change does nothing to the reductions for $L_{Q'}$, since there the relation r is unchanged. For L_{Q_E} , this change perturbs only the substitution reductions, which become the following (applying a reordering relation as well for clarity). For convenience we use the original generators x, y, a, a^* for Q_E but the modified ones $\overline{x_a}$ for L'_Q (we only need this for a beginning at a vertex of Q_E):

- $y^{-1} \mapsto x^{-1}(\rho(Q')^{-1} - 1)(r + q) + x^{-1}(r + q),$
 $y \mapsto (r' + q^{-1})(\rho(Q') - 1)x + (r' + q^{-1})x$ (if not preceded by a);
- $ax \mapsto a(\rho(Q')^{-1} - 1)(r + q)y + ary + qya,$
 $ax^{-1} \mapsto ay^{-1}(r' + q^{-1})(\rho(Q') - 1) + ay^{-1}r' + q^{-1}y^{-1}a.$

The reason why we wrote these relations as above is because, at every vertex $v \in (Q_E)_0$, $e_v(\rho(Q')^{\pm 1} - 1)$ is a linear combination of positive-length monomials in the generators $\overline{x_a^\pm}, a, a \in Q'$ (in fact we only need the $\overline{x_a^\pm}$).

Combining these systems of reductions one gets a system of reductions for $L = L_{Q_E} *_{kQ_0} L_{Q'}$. Order monomials in L lexicographically in the orderings N and N' of Propositions 2.5.1 and 2.5.4. Then the above reductions strictly decrease the ordering (we use here the fact that the ideal of positive-length monomials beginning and ending at vertices of $(Q_E)_0$ is preserved under reductions). All ambiguities lie either entirely in L_{Q_E} or entirely in $L_{Q'}$ and hence they all resolve.

We conclude that the strong free product property holds for $(L, \sigma, \rho(Q_E)\rho(Q') - q, B)$ and hence 2.5.9 is an isomorphism. $\square$

REFERENCES

- [1] David J. Anick. Noncommutative graded algebras and their Hilbert series. *J. Algebra*, 78(1):120–140, 1982.
- [2] G. Bellamy and Travis Schedler. Symplectic resolutions of quiver varieties and character varieties. *arXiv:1602.00164*, 2016.
- [3] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.
- [4] Roman Bezrukavnikov and Mikhail Kapranov. Microlocal sheaves and quiver varieties. *Ann. Fac. Sci. Toulouse Math. (6)*, 25(2-3):473–516, 2016.
- [5] Philip Boalch. Global Weyl groups and a new theory of multiplicative quiver varieties. *Geom. Topol.*, 19, 07 2013.
- [6] Raf Bocklandt. Graded Calabi Yau algebras of dimension 3. *J. Pure Appl. Algebra*, 212(1):14–32, 2008.
- [7] Raf Bocklandt, Federica Galluzzi, and Francesco Vaccarino. The Nori-Hilbert scheme is not smooth for 2-Calabi-Yau algebras. *J. Noncommut. Geom.*, 10(2):745–774, 2016.
- [8] Christopher Brav and Tobias Dyckerhoff. Relative Calabi-Yau structures II: Shifted Lagrangians in the moduli of objects. *arXiv*, page arXiv:1812.11913, Dec 2018.
- [9] Sheila Brenner, Michael C. R. Butler, and Alastair D. King. Periodic algebras which are almost Koszul. *Algebr. Represent. Theory*, 5(4):331–367, 2002.
- [10] Gianfranco Casnati, Joachim Jelisiejew, and Roberto Notari. Irreducibility of the gorenstein loci of hilbert schemes via ray families. *Algebra Number Theory*, 9(7):1525–1570, 2015.
- [11] Oleg Chalykh and Maxime Fairon. Multiplicative quiver varieties and generalised Ruijsenaars-Schneider models. *J. Geom. Phys.*, 121, 04 2017.
- [12] Alain Connes, Moshé Flato, and Daniel Sternheimer. Closed star products and cyclic cohomology. *Lett. Math. Phys.*, 24(1):1–12, 1992.

- [13] William Crawley-Boevey. On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero. *Duke Math. J.*, 118(2):339–352, 2003.
- [14] William Crawley-Boevey. Monodromy for systems of vector bundles and multiplicative preprojective algebras. *Bull. Lond. Math. Soc.*, 45(2):309–317, 2013.
- [15] William Crawley-Boevey and Martin P. Holland. Noncommutative deformations of Kleinian singularities. *Duke Math. J.*, 92(3):605–635, 04 1998.
- [16] William Crawley-Boevey and Peter Shaw. Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem. *Adv. Math.*, 201(1):180–208, 2006.
- [17] Joachim Cuntz and Daniel Quillen. Algebra extensions and nonsingularity. *J. Amer. Math. Soc.*, 8(2):251–289, 1995.
- [18] Thierry Dana-Picard. Generic 8-dimensional algebras with mixed basis-graph. *Pacific J. Math.*, 164(2):229–261, 1994.
- [19] Ben Davison. Superpotential algebras and manifolds. *Adv. Math.*, 231(2):879–912, 2012.
- [20] Louis de Thanhoffer de Völcsey and Dennis Presotto. Some generalizations of preprojective algebras and their properties. *J. Algebra*, 470:450–483, 2017.
- [21] Vlastimil Dlab and Claus Michael Ringel. On algebras of finite representation type. *Journal of Algebra*, 33(2):306 – 394, 1975.
- [22] Andrii Dmytryshyn, Vyacheslav Futorny, Bo Kågström, Lena Klimenko, and Vladimir V. Sergeichuk. Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence. *Linear Algebra Appl.*, 469:305–334, 2015.
- [23] Andrii R. Dmytryshyn, Vyacheslav Futorny, and Vladimir V. Sergeichuk. Miniversal deformations of matrices of bilinear forms. *Linear Algebra and its Applications*, 436(7):2670 – 2700, 2012.
- [24] Tobias Ekholm and Yanki Lekili. Duality between Lagrangian and Legendrian invariants. *arXiv*, page arXiv:1701.01284, Jan 2017.
- [25] T. Evgü and Y. Lekili. Fukaya categories of plumbings and multiplicative preprojective algebras. *arXiv*, March 2017.

- [26] T. Etingö and Y. Lekili. Koszul duality patterns in Floer theory. *Geom. Topol.*, 21(6):3313–3389, 2017.
- [27] P. Etingof and C.-H. Eu. Hochschild and cyclic homology of preprojective algebras of ADE quivers. *arXiv*, August 2006.
- [28] Pavel Etingof and Ching-Hwa Eu. Koszulity and the Hilbert series of preprojective algebras. *Math. Res. Lett.*, 14(4):589–596, 2007.
- [29] Pavel Etingof and Ching-Hwa Eu. Koszulity and the Hilbert series of preprojective algebras. *Math. Res. Lett.*, 14(4):589–596, 2007.
- [30] Pavel Etingof and Victor Ginzburg. Noncommutative complete intersections and matrix integrals. *Pure Appl. Math. Q.*, 3(1, Special Issue: In honor of Robert D. MacPherson. Part 3):107–151, 2007.
- [31] Pavel Etingof, Alexei Oblomkov, and Eric Rains. Generalized double affine Hecke algebras of rank 1 and quantized del Pezzo surfaces. *Adv. Math.*, 212(2):749–796, 2007.
- [32] Peter Gabriel. Unzerlegbare darstellungen i. *manuscripta mathematica*, 6(1):71–103, 1972.
- [33] Sheel Ganatra. *Symplectic Cohomology and Duality for the Wrapped Fukaya Category*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Massachusetts Institute of Technology.
- [34] Iordan Ganev, David Jordan, and Pavel Safronov. The quantum Frobenius for character varieties and multiplicative quiver varieties. *arXiv*, page arXiv:1901.11450, Jan 2019.
- [35] Christof Geiß, Bernard Leclerc, and Jan Schröer. Quivers with relations for symmetrizable Cartan matrices III: Convolution algebras. *Represent. Theory*, 20:375–413, 2016.
- [36] Christof Geiss, Bernard Leclerc, and Jan Schröer. Quivers with relations for symmetrizable Cartan matrices I: Foundations. *Invent. Math.*, 209(1):61–158, 2017.
- [37] I. M. Gelfand and V. A. Ponomarev. Model algebras and representations of graphs. *Funktsional. Anal. i Prilozhen*, 13:1–12, 1979.
- [38] Victor Ginzburg. Calabi-Yau algebras. 01 2007.

- [39] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [40] David Jordan. Quantized multiplicative quiver varieties. *Adv. Math.*, 250:420 – 466, 2014.
- [41] M. Kontsevich and Y. Soibelman. Notes on A_∞ -algebras, A_∞ -categories and non-commutative geometry. In *Homological mirror symmetry*, volume 757 of *Lecture Notes in Phys.*, pages 153–219. Springer, Berlin, 2009.
- [42] P. B. Kronheimer. The construction of ALE spaces as hyper-Kähler quotients. *J. Differential Geom.*, 29(3):665–683, 1989.
- [43] Julian Külshammer. Pro-species of algebras I: Basic properties. *Algebr. Represent. Theory*, 20(5):1215–1238, 2017.
- [44] G. Lusztig. Semicanonical bases arising from enveloping algebras. *Adv. Math.*, 151(2):129–139, 2000.
- [45] Anton Malkin, Viktor Ostrik, and Maxim Vybornov. Quiver varieties and Lusztig’s algebra. *Adv. Math.*, 203(2):514–536, 2006.
- [46] Roberto Martínez-Villa. Applications of Koszul algebras: the preprojective algebra. In *Representation theory of algebras (Cocoyoc, 1994)*, volume 18 of *CMS Conf. Proc.*, pages 487–504. Amer. Math. Soc., Providence, RI, 1996.
- [47] Hiraku Nakajima. Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Math. J.*, 76(2):365–416, 1994.
- [48] Bjorn Poonen. The moduli space of commutative algebras of finite rank. *J. Eur. Math. Soc. (JEMS)*, 10(3):817–836, 2008.
- [49] Travis Schedler. Zeroth Hochschild homology of preprojective algebras over the integers. *Adv. Math.*, 299:451–542, 2016.
- [50] Travis Schedler and Andrea Tirelli. Symplectic resolutions for multiplicative quiver varieties and character varieties for punctured surfaces. *arXiv*, page arXiv:1812.07687, Dec 2018.
- [51] A. H. Schofield. *Representation of rings over skew fields*, volume 92 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1985.

- [52] Peter Shaw. Generalisations of preprojective algebras. Ph.D. thesis, Univeristy of Leeds, 2005.
- [53] Michel Van den Bergh. Non-commutative quasi-Hamiltonian spaces. *Contemp. Math.*, 450, 04 2007.
- [54] Michel Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360(11):5711–5769, 2008.
- [55] Michel Van den Bergh. Calabi-Yau algebras and superpotentials. *Selecta Math. (N.S.)*, 21(2):555–603, 2015.
- [56] Michel Van den Bergh and Louis de Thanhoffer de Volcsey. Calabi-yau deformations and negative cyclic homology. *J. Noncommut. Geom.*, 12, 01 2012.
- [57] Daisuke Yamakawa. Geometry of multiplicative preprojective algebra. *IMRP*, 2008:rpn008, 2008.
- [58] Wai-kit Yeung. Pre-Calabi-Yau structures and moduli of representations. *arXiv*, page arXiv:1802.05398, Feb 2018.