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Turbulent plumes in a uniform crosswind

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Submitted in part fulfilment of the requirements for the degree of
Master of Philosophy in Civil and Environmental Engineering
of Imperial College London, and the Diploma of Imperial College London,
August 2020

Declaration

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.

Owen Jordan

February 11, 2021

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February 11, 2021

Acknowledgements

I would like to express my deep gratitude to Dr Maarten van Reeuwijk for his supervision over the past five years. This thesis has been made possible by his dedication and support. In addition I would like to thank Clodagh Li from the Centre for Doctoral Training in Fluid Dynamics across Scales, who helped and encouraged me through much of my time at Imperial. I also gratefully acknowledge the insightful conversations and advice received from Gabriel Rooney and Ben Devenish at the Met Office. This research was funded by the Centre for Doctoral Training (grant number EP/L016230/1). The computational resources required to perform the simulations were provided through the UK Turbulence Consortium (grant number EP/R029326/1). Finally, I would like to thank my parents for their invaluable support throughout this journey.

Abstract

Direct numerical simulation is used to investigate the integral behaviour of buoyant plumes subjected to a uniform crosswind. We find that the plume centreline is not aligned with the streamlines as is commonly assumed in integral plume models. Instead, the centreline is aligned with the total buoyancy flux, which implies that a model for the vertical turbulent buoyancy flux is required to faithfully predict the plume angle. A study of the volume conservation equation shows that the Leibniz term, i.e. entrainment due to incorporation of ambient fluid with non-zero velocity, is the dominant entrainment mechanism in strong crosswinds. The data indicate that pressure differences between the top and bottom of the plume play a leading role in the evolution of the horizontal and vertical momentum balances and are crucial to appropriately model plume rise. By parameterising the vertical buoyancy flux, the entrainment and the pressure, a novel integral plume model is developed which is in good agreement with the simulations for sufficiently strong crossflow.

Contents

1	An Introduction to Turbulent Plumes	8
2	Theory	13
2.1	Integral equations	13
2.2	Plume models	21
3	Numerical methods	26
3.1	Modelling and Simulation of Turbulent Flows	26
3.2	Description of the code	30
3.3	The nudging region	32
3.4	Plume thresholding	35
3.5	Case setup	39
3.6	Least-squares spline fitting for the plume centreline	42

4	Results	49
4.1	Plume evolution and geometry	49
4.2	Turbulent entrainment	61
4.3	The role of pressure in the momentum balances	66
4.4	Integral plume model	72
5	Conclusion	77

Chapter 1

An Introduction to Turbulent Plumes

Turbulent buoyant plumes have long been studied due to their ubiquity in nature and in industry, and their importance in fields such as disaster management and industrial pollution. Examples are volcanic eruptions, pollutant dispersion via chimneys, fires, ocean outfalls, and jet engines (Woods 2010; Mahesh 2013). A significant number of parameters play a role in the evolution of a buoyant plume. These include the source velocity, release angle, density and the state of the atmosphere the plume ascends through via the wind profile $\mathbf{U}(z)$ and the stratification, characterised by the (square) buoyancy frequency $N^2(z)$. Volcanic applications feature another level of complexity due to the initial ash volume fraction and its size distribution, ash reactivity and compressibility effects including shock waves due to supersonic conditions at the source.

Buoyant plumes are typically modelled via integral models. The now-classical plume equations were presented in Morton *et al.* (1956) (MTT), and comprise a set of three coupled ordinary differential equations for the volume flux Q , (streamwise) momentum flux M , and buoyancy flux F , respectively. This theory was formulated in the absence of wind such that the plume only evolves in the vertical coordinate z . One of the cornerstones of the theory is the entrainment assumption, which links the entrainment of fluid into the plume to the characteristic velocity inside the plume. MTT assumed that the entrainment coefficient α was constant, although different values of α are typically used for jets and plumes. Recent work by van Reeuwijk & Craske (2015) has clarified the relation between the MTT model and the Priestley & Ball (1955) model, who used a closure using the production of turbulent kinetic energy. Analysis of experimental and numerical data by van Reeuwijk & Craske (2015) indicated that for jets and plumes the more suitable model is that of Priestley & Ball. In fully-self-similar conditions, i.e. in the case when the plume velocity and buoyancy profiles may be represented by a function of a dimensionless similarity variable, and with these buoyancy and velocity profiles having an equal width, they found that $\alpha_{plume} \approx 5/3\alpha_{jet}$ (van Reeuwijk *et al.* 2016).

Briggs (1982) extended the plume model in a crossflow to account for behaviour of the plume after it surpasses the neutral buoyancy level, meaning that the plume’s final height and maximum height could be predicted. This involved adding an ODE for the horizontal momentum deficit M_x , and the work was developed further in Briggs (1984). The presence of wind can fundamentally alter the entrainment properties of the plume. Indeed, the

presence of the cross-wind sometimes causes the flow to bend over and organise itself into two turbulent counter-rotating vortices (Fischer *et al.* 1979; Weil 1988). For jets in a cross-wind, Mahesh (2013) summarises the research into counter-rotating vortices. In addition, he describes the horseshoe and wake vortices which form upstream of the jet’s leading edge as a result of adverse pressure gradients.

In parallel with the effort to develop a suitable integral plume model there is a large body of literature documenting experimental and numerical surveys of plumes in a crossflow (Cintolesi *et al.* 2019; Jiang *et al.* 2019). One of the earliest experimental studies (Fan 1967) investigates two categories of plumes: inclined plumes discharged into a stratified but quiescent environment, and plumes discharged into a homogeneous, uniform crossflow. Subsequent studies have been carried out for a wide range of initial and boundary conditions, resulting in a large catalogue of experimental work. For example, Gaskin (1995) compares a plume in a quiescent environment to a plume with a uniform crossflow, Huq & Stewart (1996) assess the effects of atmospheric turbulence on plume development and Jiang *et al.* (2019) carried out experiments to determine the effects of a confining boundary on a swept plume (in their case a tunnel ceiling). Since the widespread availability of high-powered computing, numerical studies and particularly Large-Eddy Simulations (LES) have been an additional powerful tool for investigating plume dynamics. Yuan *et al.* (1999) were early innovators in this regard, simulating jets in a crossflow (without buoyancy) by utilising a dynamic Smagorinsky model for the sub-grid scales. They used their LES to analyse the counter-rotating vortex structure, concluding that it was the result of

the hanging vortices which appear in the region close to the source of the jet. More recently, De Wit *et al.* (2014) examined a real-world application of LES to overflow dredging plumes resulting from a moving dredger. They discovered that the average horizontal velocity of the plume exceeds the velocity of the crossflow. We discuss this intriguing phenomenon in chapters 4.3 and 4.4. Another pertinent numerical study is the paper by Devenish *et al.* (2010) presenting LES of purely buoyant plumes released into stratified and uniform environments. They compare the results of their LES to plume integral models and proposed a modified entrainment assumption in which the contributions from the horizontal and vertical velocity components are not equally weighted.

Various entrainment assumptions exist (see §5), and all involve some mix of the plume velocity and the wind velocity. Most models neglect pressure differences between the plume and the environment, assuming that the pressure is hydrostatic within the plume at least when averaged over the plume cross-section. Most models also neglect the Leibniz term: the increase in (for example) integral volume flux as a result of the spreading of the plume into a non-quiescent atmosphere. Schatzmann (1978), on the other hand, fully integrated the governing equations and was the first to describe an integral plume model incorporating the Leibniz term. He argued that this term should not be counted towards entrainment, since it requires a fundamentally different interpretation of the entrainment coefficient. In addition, he noted the need for a pressure term, although at the time not enough was known to fully model it. A limitation of his analysis is the assumption of axisymmetry and the assumption that the turbulence terms are negligible. We

avoid these assumptions in our analysis by integrating the Reynolds-averaged equations directly, allowing us to access all terms in the integral budgets and to determine which are physically relevant and which can safely be ignored.

One pertinent example of an application in which integral models play a key role is in the determination of the mass flux of a volcanic eruption. It is not possible to measure this quantity directly, so the mass flux is often inferred from the rise height of the plume, which is where the plume has the same density as the surrounding atmosphere and starts spreading laterally like a gravity current. Turbulent entrainment thus plays a key role in the rise height, since this is one of the important mechanisms by which the plume dilutes. Appropriately incorporating the influence of wind in these estimates is key for appropriate estimation of the source mass flux. (Costa & *et al.* 2016; Woodhouse *et al.* 2013; Devenish 2013; Rossi *et al.* 2019).

The aim of this paper is to examine the assumptions underlying integral models of plumes in a crossflows, including entrainment, by means of direct numerical simulation. In order to manage the complexity of the problem, we will restrict ourselves to a neutral atmosphere ($N^2 = 0$) with a steady uniform wind velocity U and a momentumless source (i.e. an infinitely lazy plume). The only parameter that will be varied is the crossflow velocity. We will be focusing our attention on the behaviour exhibited by plumes with a strong crossflow, and developing a model for the entrainment and pressure drop over the plume.

Chapter 2

Theory

2.1 Integral equations

In this section, equations for the integral volume, momentum and buoyancy fluxes are derived. The derivation is consistent with that of Weil (1988) and Schatzmann (1978), but is performed here without any simplifications, such as the assumption of axisymmetry (Schatzmann 1978) or neglecting turbulence and pressure terms. A schematic detailing some of the nomenclature used here is given in Figure 2.1. We employ a plume-following coordinate system based on a plume centreline which makes an angle φ with the horizontal. Standard Cartesian coordinates (x, y, z) are used which represent the streamwise, transversal and vertical directions, respectively. It is convenient to introduce a plume-following coordinate system (s, y', η) , where s is the coordinate measuring distance along the plume centreline, η is the coordinate

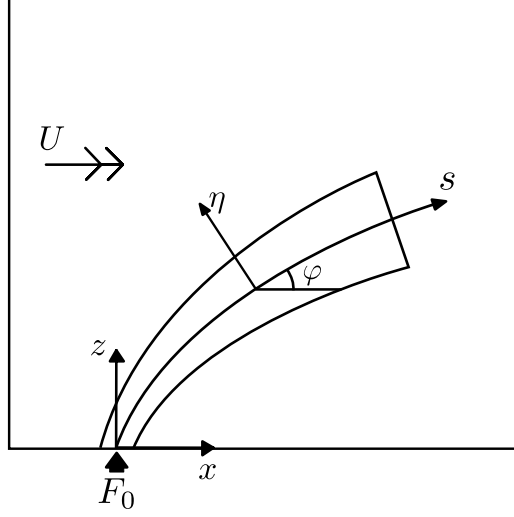


Figure 2.1: A schematic of the plume. The y direction is perpendicular to the plane of this cross-section.

perpendicular to s in the x - z plane and y' is the coordinate in the direction parallel with y . y' is equivalent to y except that y' is measured from the plume centreline at a distance s along the plume, whereas y is measured from the origin. However, since the plume centreline always remains in the $y = 0$ plane, we will regard these coordinates as identical and only use y for the remainder of the paper. The velocity in standard Cartesian coordinates will be denoted $\mathbf{u} = (u, v, w)$ and the velocity in the plume-following coordinates will be denoted $\mathbf{u} = (u_s, v, u_\eta)$. Again, we regard the velocity in the y -direction as being the same in both coordinate systems.

The incompressible Navier-Stokes equations with the Boussinesq approxima-

tion are

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + b \hat{\mathbf{k}}, \quad (2.1b)$$

$$\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b = \kappa \nabla^2 b, \quad (2.1c)$$

where $b = g(\rho_0 - \rho)/\rho_0$ is the buoyancy and $p = \tilde{p}/\rho_0 + gz$ is the kinematic pressure perturbation, where g is the gravitational acceleration, ρ_0 is the reference density and $\tilde{p}$ is the standard pressure. The momentum and scalar diffusivities are denoted ν and κ , respectively. Assuming high-Reynolds and -Péclet number flow, applying Reynolds-averaging to (2.1), and making use of the fact that the flow is statistically steady results in

$$\nabla \cdot \bar{\mathbf{u}} = 0, \quad (2.2a)$$

$$\nabla \cdot (\bar{\mathbf{u}} \bar{u}_i + \overline{\mathbf{u}' u'_i}) = -\frac{\partial \bar{p}}{\partial x_i} + \bar{b} \delta_{i3}, \quad (2.2b)$$

$$\nabla \cdot (\bar{\mathbf{u}} \bar{b} + \overline{\mathbf{u}' b'}) = 0. \quad (2.2c)$$

The equations above are all of the form (Weil 1988, e.g.)

$$\nabla \cdot \mathbf{F} = G \quad (2.3)$$

where $\mathbf{F} = \mathbf{u}\phi$, and $\phi(\mathbf{x})$, $G(\mathbf{x})$ are scalar functions. The position vector in the plume-coordinate system (s, y, η) is given by $\mathbf{x} = \mathbf{X}(s) + \mathbf{e}_y y + \mathbf{e}_\eta(s)\eta$

where $\mathbf{X}(s)$ is the plume-centreline and the unit vectors are given by

$$\mathbf{e}_s = \begin{pmatrix} \cos \varphi \\ 0 \\ \sin \varphi \end{pmatrix}, \quad \mathbf{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_\eta = \begin{pmatrix} -\sin \varphi \\ 0 \\ \cos \varphi \end{pmatrix}. \quad (2.4)$$

In the (s, y, η) system, the divergence becomes

$$\nabla \cdot \mathbf{F} = \frac{1}{J} \left(\frac{\partial F_s}{\partial s} + \frac{\partial F_y}{\partial y} + \frac{\partial F_\eta}{\partial \eta} \right) \quad (2.5)$$

where $J = 1 - \eta d\varphi/ds$ is the Jacobian, and the streamwise, spanwise and normal fluxes are defined as, respectively

$$F_s = F_x \cos \varphi + F_z \sin \varphi, \quad (2.6a)$$

$$F_{y'} = \left(1 - \eta \frac{d\varphi}{ds} \right) F_y \quad (2.6b)$$

$$F_\eta = (-F_x \sin \varphi + F_z \cos \varphi) \left(1 - \eta \frac{d\varphi}{ds} \right). \quad (2.6c)$$

To simplify further, we will ignore curvature effects so that $d\varphi/ds \ll 1$. This assumption can be justified by invoking theory for bent-over plumes. Indeed, assuming that the edge of the plume is at $\eta \sim r$, and using (2.21, 2.22), we have $\eta \propto x^{2/3}$. For bent-over plumes, the small-angle assumption $\tan \varphi \approx \varphi$ holds, which implies that $\varphi \approx dz_c/dx$. Since $z_c \propto x^{2/3}$, it follows that $\varphi \propto x^{-1/3}$. Once more invoking the small-angle assumption to note that $d\varphi/ds \approx d\varphi/dx$, we find that $d\varphi/ds \propto x^{-4/3}$ and thus that $\eta d\varphi/ds \sim x^{-2/3}$. This implies that this term can be safely ignored a few radii downstream of the source. With this simplification, the Jacobian becomes $J = 1 - \eta d\varphi/ds \approx$

1.

We will first integrate the equations over the area of the plume in the y - η plane and then differentiate the result in the s -direction. First, we need to note the Leibniz rule in 2 dimensions: if $A(s)$ is the region in the y - η plane within the plume boundary ∂A , and $\mathbf{\Delta}_A = (\Delta_y, \Delta_\eta)$ is the vector representing the change in the boundary as a function of s , then any scalar $\phi(\mathbf{x})$ obeys (Flanders 1973)

$$\frac{d}{ds} \iint_A u_s \phi dA = \iint_A \frac{\partial u_s \phi}{\partial s} dA + \mathcal{L}[\phi] \quad (2.7)$$

where the Leibniz term is defined as

$$\mathcal{L}[\phi] = \oint_{\partial A} u_s \phi (\Delta_y d\eta - \Delta_\eta dy). \quad (2.8)$$

We also note Green's theorem, which states that

$$\iint_A \frac{\partial v \phi}{\partial y} + \frac{\partial u_\eta \phi}{\partial \eta} dA = -E[\phi] \quad (2.9)$$

where the entrainment into the plume of the quantity ϕ is defined as

$$E[\phi] = - \oint_{\partial A} \phi (v d\eta - u_\eta dy) \quad (2.10)$$

We now observe that

$$\begin{aligned}
\iint_A \nabla \cdot (\mathbf{u}\phi) dA &= \iint_A \frac{\partial u_s \phi}{\partial s} + \frac{\partial v \phi}{\partial y} + \frac{\partial u_\eta \phi}{\partial \eta} dA \\
&= \frac{d}{ds} \iint_A u_s \phi dA - \mathcal{L}[\phi] + \iint_A \frac{\partial v \phi}{\partial y} + \frac{\partial u_\eta \phi}{\partial \eta} dA \\
&= \frac{d}{ds} \iint_A u_s \phi dA - \mathcal{L}[\phi] - E[\phi].
\end{aligned}$$

The equivalent result can be obtained using the recently developed unified description for turbulent entrainment (Van Reeuwijk *et al.* 2020). Using the expression above in combination with equation 2.3 results in

$$\frac{d}{ds} \iint_A u_s \phi dA = \iint_A G dA + E[\phi] + \mathcal{L}[\phi]. \quad (2.11)$$

To proceed further with an analytical approach, we need to apply Reynolds-averaging. The principal of Reynolds-averaging relies on the fact that all quantities f are composed of an average value and a fluctuating value, which we shall denote $\bar{f}$ and f' respectively. So, for example, $u = \bar{u} + u'$. The averaging in question may be time-averaging, ensemble-averaging (that being an average over many simulations or experiments with the same set-up), or some other averaging method. In this case, we choose to use time-averaged quantities. Upon averaging this becomes

$$\frac{d}{ds} \iint_A \bar{u}_s \bar{\phi} + \overline{u'_s \phi'} dA = \iint_A \bar{G} dA + \pi \overline{E[\phi]} + \pi \overline{\mathcal{L}[\phi]} \quad (2.12)$$

where $\bar{u}_s = \bar{u} \cos \varphi + \bar{w} \sin \varphi$ is the mean velocity in the s -direction, and $\bar{G}$ is the time-averaged quantity of G . In deriving the equation above it was

assumed that the curvature of the plume trajectory $d\varphi/ds \ll 1$. As we will no longer be concerned with the non-averaged quantities for E and $\mathcal{L}$, we may redefine these terms to include the time-averaging. Thus, the boundary terms $E[\phi]$ and $\mathcal{L}[\phi]$ can be redefined as:

$$E[\phi] = -\frac{1}{\pi} \oint_{\partial A} \overline{\phi} v d\eta - \overline{\phi} u_\eta dy, \quad \mathcal{L}[\phi] = \frac{1}{\pi} \oint_{\partial A} \overline{u_s \phi} (\Delta_y d\eta - \Delta_\eta dy). \quad (2.13)$$

These two terms represent distinct physical processes by which the quantity $\overline{\phi}$ enters the plume. The term $E[\phi]$ represents the advection of ϕ into the plume; the term $\mathcal{L}[\phi]$ represents incorporation of the quantity ϕ via expansion of the plume into a non-quiescent atmosphere – if the plume surface area A over which the Navier-Stokes equations are integrated changes as a function of s , and the value of ϕ at the boundary of the plume is nonzero, then the surface integral will be increased by a quantity equal to $[\overline{u_s \phi}]_b dA/ds$, where $[\overline{u_s \phi}]_b$ is the average value of $\overline{u_s \phi}$ at the boundary. Δ_y and Δ_η denote the change in the plume area. They are the speed at which the plume expands in the y and η directions respectively, as the integration domain moves along the plume centreline.

Here, the factor π in the definition of $\mathcal{L}[\phi]$ and $E[\phi]$ is introduced to stay consistent with the convention in plume theory of dividing the integral quantities through by π (see below).

Integrating the Navier-Stokes equations over the plume cross-section and

applying (2.12) with $\phi = 1, u - U, w$ and $\bar{b}$ respectively, results in

$$\frac{dQ}{ds} = E[1] + \mathcal{L}[1] = E + \mathcal{L} \quad (2.14a)$$

$$\frac{dM_x}{ds} + \frac{dM'_x}{ds} = -P_x + E[u - U] + \mathcal{L}[u - U] \quad (2.14b)$$

$$\frac{dM_z}{ds} + \frac{dM'_z}{ds} = -P_z + B + E[w] + \mathcal{L}[w] \quad (2.14c)$$

$$\frac{dF}{ds} + \frac{dF'}{ds} = E[b] + \mathcal{L}[b] \quad (2.14d)$$

where the integral volume flux Q , the horizontal momentum excess flux M_x , the vertical momentum flux M_z , the buoyancy flux F and the integral horizontal and vertical pressures P_x and P_z are defined as, respectively

$$\begin{aligned} Q &= \frac{1}{\pi} \iint_A \bar{u}_s dA, & F &= \frac{1}{\pi} \iint_A \bar{u}_s \bar{b} dA \\ M_x &= \frac{1}{\pi} \iint_A \bar{u}_s (\bar{u} - U) dA, & P_x &= \frac{1}{\pi} \iint_A \frac{\partial \bar{p}}{\partial x} dA, \\ M_z &= \frac{1}{\pi} \iint_A \bar{u}_s \bar{w} dA, & P_z &= \frac{1}{\pi} \iint_A \frac{\partial \bar{p}}{\partial z} dA, \\ M'_x &= \frac{1}{\pi} \iint_A \overline{u'_s u'} dA, & F' &= \frac{1}{\pi} \iint_A \overline{u'_s b'} dA, \\ M'_z &= \frac{1}{\pi} \iint_A \overline{u'_s w'} dA, & B &= \frac{1}{\pi} \iint_A \bar{b} dA \end{aligned}$$

The scalar $\bar{u} - U$ is chosen for the x -momentum equation so as to be consistent with the work of Weil in the definition of M_x . The integral therefore represents the momentum in the x -direction in excess of the momentum present in the ambient atmosphere. A consequence of this choice of variable is that the Leibniz and entrainment terms for the x -momentum equation are significantly reduced, due to the fact that to first order $u \approx U$ at the boundary

of the plume. Integrals equivalent to the fluxes but with fluctuating parts in the integrand instead of mean quantities are denoted with an apostrophe '.

2.2 Plume models

In conventional plume theory $\bar{u} - U$, $\bar{w}$, and $\bar{b}$ are assumed to vanish at the plume boundary. In addition $\mathcal{L}[1]$ is assumed to be small. It thus follows that all entrainment and Leibniz terms are negligible except for $E \equiv E[1]$. Furthermore, it is commonly assumed that the pressure at the plume boundary is equal to the ambient pressure, which implies that P_x and P_z are zero. We will show in §4.3 that this assumption is violated. The buoyancy integral $B = \pi^{-1} \int \bar{b} dA$ is generally modelled as $B = F/U_s$, where U_s is the characteristic velocity of the plume (defined rigorously further in this section). The entrainment term $E + \mathcal{L}$ is parameterised as (Weil 1988; Devenish *et al.* 2010)

$$E + \mathcal{L} = 2\gamma r_m w_m \quad (2.15)$$

where γ is the entrainment coefficient, and r_m and w_m are a characteristic radius and vertical velocity respectively. There is flexibility in which choice to make for these variables as long as the (η, y) -dependence of r_m and w_m (and the other characteristic plume variables to be introduced below) is the same at a given value of s . A conventional choice is to use top-hat quantities, which are linked to the integral quantities as

$$Q = r_m^2 U_s, \quad M_x = r_m^2 U_s u_m, \quad M_z = r_m^2 U_s w_m, \quad F = r_m^2 U_s b_m. \quad (2.16)$$

Here, U_s is the characteristic velocity along the plume, u_m is the characteristic horizontal plume velocity relative to the wind (horizontal velocity excess), and b_m is the characteristic buoyancy or reduced gravity. The velocity U_s is the characteristic total velocity component in the s -direction and is given by

$$U_s = (U + u_m) \cos \varphi + w_m \sin \varphi \quad (2.17)$$

The characteristic radius r_m is defined as $r_m = (Q/U_s)^{1/2}$ (i.e. using the definition of Q ; 2.16), which encapsulates both the pure plume definition $r_m = Q/M_z^{1/2}$ for $U \ll w_m$, $u_m \ll w_m$, and the bent-over solution $r_m = (Q/U)^{1/2}$ for $U \gg u_m$, $U \gg w_m$ (see below).

Substitution of the stated assumptions at the start of this section and (2.15) into the integral equations (2.14a-2.14d) results in:

$$\frac{dQ}{ds} = 2\gamma \frac{M_z}{r_m U_s}, \quad \frac{dM_x}{ds} = 0, \quad (1+k) \frac{dM_z}{ds} = \frac{F}{U_s}, \quad \frac{dF}{ds} = 0. \quad (2.18)$$

where k is the added-mass constant which takes into account the effects of an adverse pressure gradient on the vertical motion (Weil 1988, e.g.). It is introduced by analogy with the motion of a solid body and is equivalent to using a drag term (see Briggs 1975; Schatzmann 1978; Davidson 1989). These equations need to be augmented by kinematic relations for the plume location

$$\frac{dx_c}{ds} = \cos \varphi, \quad \frac{dz_c}{ds} = \sin \varphi. \quad (2.19)$$

Here x_c and z_c determine the plume centreline. We can take x_c to be equal to x and therefore $z_c(x)$ is a function giving the plume height as a function

of x . To close the set of equations, the angle φ is conventionally assumed to be the angle of the local streamlines φ_u , which is given by

$$\tan \varphi_u = \frac{w_m}{U + u_m}. \quad (2.20)$$

In this case, i.e. with the assumption that $\varphi = \varphi_u$ and therefore that $\frac{dz_c}{dx} = \frac{w_m}{U + u_m}$, U_s represents the velocity magnitude $\sqrt{(U + u_m)^2 + w_m^2}$, which can then be equivalently expressed in terms of the integral quantities as $Q^{-1} \sqrt{(QU + M_x)^2 + M_z^2}$. Here, we make an explicit difference between φ and φ_u since it will turn out that these quantities are not identical (see §4.1). When $\varphi \neq \varphi_u$, there will be a component of velocity in the η direction

$$U_\eta = -(U + u_m) \sin \varphi + w_m \cos \varphi.$$

We briefly provide an overview of analytical results for bent-over plumes in a neutral stratification and a uniform crossflow. Bent-over plumes are defined to have their centreline inclined at a very small angle to the horizontal (in which case the equations are formally equal to those for a line thermal; Briggs 1984; Lee & Chu 2003). This implies that $w_m \ll U$, $u_m \ll U$ and $ds \approx dx$. Assuming that in this case, the entrainment rate is a constant $\gamma = \beta'$, the following relations can be derived (Weil 1988, e.g.): on integrating the volume flux equation, (2.18a), we get

$$r_m = \beta' z_c + r_0 \quad (2.21)$$

where r_0 is the source radius. On integrating the vertical momentum flux

equation, (2.18c), twice and making use of (2.21), (2.19) and (2.20), assuming $\varphi = \varphi_u$, we get

$$\frac{z_c}{\ell_b} = \left(\frac{3}{2\beta^2} \right)^{1/3} \left(\frac{x - x_v}{\ell_b} \right)^{2/3} \quad (2.22)$$

for $r_0 \ll \beta' z_c$ where β' is a constant coefficient, $\ell_b = F/U^3$ is the buoyancy length scale, and x_v is a virtual origin correction. The value of β is not equal to β' because of added-mass effects (Lee & Chu 2003), and the two can be related as $\beta = \beta' \sqrt{1+k}$ (Weil 1988, p.164). Experiments indicate that $0.3 \lesssim k \lesssim 1.3$ (e.g. Briggs 1975; Contini *et al.* 2011; Weil 1988) and $0.45 \lesssim \beta' \lesssim 0.75$ (e.g. Briggs 1984; Weil 1988; Huq & Stewart 1996; Hoult & Weil 1972; Hewett *et al.* 1971). It should be noted that Lee & Chu (2003) do not divide their integral fluxes by π , which results in a factor π appearing in their expression for z_c ; this however will not influence the value of β' . Although β' is a constant coefficient, it may vary between simulations or experiments dependent on factors such as the strength of the crossflow and the initial buoyancy flux, F_0 (Briggs 1984).

Several entrainment models are in use for plumes in a crossflow (Costa & *et al.* 2016). The classical model for a bent-over plume (Briggs 1984; Weil 1988, e.g.) assumes $\gamma = \beta'$. An entrainment model which is also valid for weak-crossflow plumes is given by (DRWT; Devenish *et al.* 2010)

$$\gamma = \frac{\sqrt[n]{(\alpha w_m)^n + (\beta' U)^n}}{U_s} \quad (2.23)$$

The DRWT model uses two parameters, α and β' , to account for entrainment due to vertical and horizontal motion respectively and assumes that

the added-mass coefficient $k = 0$. A further parameter n determines the function used to combine the entrainment resulting from the vertical and horizontal momentum into a single entrainment quantity. When comparing the DRWT model with our own results (see §4.4) we implemented the model with $n = 3/2$, a value suggested by the authors. This value for n indicates that vertical and horizontal motion contribute to entrainment in a weakly non-linear manner. If the plume is bent-over, $w_m \ll U$ and $U_s \approx U$, so the DRWT model is equivalent to the Weil model in this limit, independent of the value of n . Similarly, in the limit of no ambient wind, $U \ll w_m$ and $U_s \approx w_m$, so the DRWT model gives $\gamma = \alpha$, independent of the value for n . Thus we should expect the value of α in the DRWT model to be equal to the entrainment coefficient for vertical axisymmetric plumes.

Chapter 3

Numerical methods

3.1 Modelling and Simulation of Turbulent Flows

In general, partial differential equations (PDEs) such as 2.1 do not have analytical solutions. In these instances, solution of the PDEs by computer is often the only method which can be utilised. However, a notable feature of turbulent flows is the vast disparity between the largest and smallest scales of the flow. Due to the need to resolve both of these scales simultaneously, numerical computations of turbulent flows are frequently prohibitively expensive. The field of computational fluid dynamics (CFD) has been developed to tackle the problem of efficiently simulating flows, and several different approaches can be taken.

The first method that we will discuss is the simulation of the Reynolds-averaged Navier-Stokes (RANS) equations. Instead of solving the original

PDEs, RANS simulations solve modified PDEs governing the behaviour of Reynolds-averaged quantities. This implies that none of the smaller turbulent length-scales need to be resolved, and as a result RANS simulations can be drastically cheaper than other numerical simulations methods. For a general statistically 3-dimensional flow, four independent equations govern the mean flow. However, these 4 equations contain more than 4 unknowns. A key component to the success of a RANS simulation is the determination of the Reynolds stresses, $\overline{u'_i u'_j}$. These stresses are not directly given by the RANS equations, and are typically modelled by some other means (Pope 2001). The gradient-diffusion hypothesis and eddy-viscosity hypothesis are two notable early attempts at closing the RANS equations. The gradient-diffusion hypothesis makes the claim that the turbulent transport of a scalar ϕ , denoted $\overline{\mathbf{u}'\phi'}$ is aligned with the mean scalar gradient, $-\nabla\overline{\phi}$, so

$$\overline{\mathbf{u}'\phi'} = -\Gamma_T \nabla \overline{\phi} \quad (3.1)$$

where $\Gamma_T(\mathbf{x}, t)$ is a positive scalar known as the turbulent diffusivity. The eddy-viscosity instead proposes that the deviatoric Reynolds stress, $\overline{\rho u'_i u'_j} - \frac{2}{3}\rho k\delta_{ij}$, is proportional to the rate-of-strain tensor, $S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$. Here, $k = \frac{1}{2}\overline{u'_i u'_i}$ is the turbulent kinetic energy. The hypothesis can be stated as

$$\overline{\rho u'_i u'_j} - \frac{2}{3}\rho k\delta_{ij} = 2\rho\nu_T S_{ij} \quad (3.2)$$

where $\nu_T(\mathbf{x}, t)$ is a positive scalar known as the turbulent viscosity. These are just two possible methods by which the RANS equations could be closed.

In contrast to RANS, Large-Eddy simulations (LES) directly represent the larger, three-dimensional unsteady turbulent features of a flow. LES are simulations in which the turbulent eddies are resolved down to a certain size, the grid lengthscale. In order to accurately represent the physics of eddies smaller than the grid lengthscale, energy dispersion effects below this grid lengthscale need to be explicitly modelled. LES represent the turbulent flow as the sum of a filtered component and a residual component. The filtered component represents the motion of the large-scale eddies. The equations for the evolution of the filtered velocity are derived from the Navier-Stokes equations, and include a residual-stress tensor. To close the equations, the residual-stress tensor must be modelled. For example, Smagorinsky (1963) proposed a closure via an eddy-viscosity model. LES are always computationally cheaper than DNS, which must have a grid-scale smaller than the turbulence length-scale, but are more costly than RANS simulations.

DNS is the most computationally costly approach to CFD, but also provides the most reliable results due to the fact that none of the turbulence needs to be modelled. In order for simulations to be considered true DNS, the grid resolution Δ_x must be sufficiently small that the smallest turbulent eddies are represented. The conventional measure for the size of the smallest turbulent eddies is the Kolmogorov length-scale. The Kolmogorov length-scale is defined as:

$$\eta_K = (\nu^3/\varepsilon)^{1/4}. \quad (3.3)$$

where ν is the kinematic viscosity and ε is the dissipation (Pope 2001). Thus for a simulation to be considered DNS we require $\Delta_x < \eta_K$. DNS in the

past has been impossible for the vast majority of applications due to how computationally expensive it is, but advances in super-computing facilities mean that DNS is becoming more and more feasible for a greater number of problems.

A further method for numerical modelling of fluid dynamics is the use of semi-analytical procedures. For example, Lai & Lee (2012) conducted a study in which they split a jet into a series of thin jet elements. Each jet element was modelled as a 3D point sink. The irrotational velocity field induced by a point sink is known analytically, and the induced velocity from all the jet elements were superimposed numerically to produce an overall solution.

The final method for CFD that we will discuss is a hybridisation of LES and RANS. One method to hybridise these techniques is by resolving in great detail using LES the areas of the flow which are predicted to be highly turbulent (adjacent to a wall, for example), but using RANS for the remaining regions of the flow, then matching boundary conditions at the interface between the two regions (Temmerman *et al.* 2005; Fröhlich & von Terzi 2008). This method can be incredibly effective and computationally efficient, but requires a deep understanding of the underlying physics governing the characteristics of the flow, and adds complexity to the coding implementation.

For this investigation we chose to use DNS, due to the fact that the simple box geometry of our simulations allows us to utilise spectral methods (as expanded upon in section 3.2), which when implemented correctly can be highly accurate and efficient, thus reducing the computational cost of the simulations and making DNS a possibility.

3.2 Description of the code

For our simulations, we utilised the SPARKLE code, as described in Craske & van Reeuwijk (2015). This is based on a finite volume discretisation, using the symmetry-preserving methods of Verstappen & Veldman (2003). It conserves mass, momentum and energy, and is fourth-order accurate. In order to accomplish conservation with a fourth-order numerical scheme, Verstappen & Veldman implemented a finite volume discretisation over two nested control volumes. Central differencing is applied to the boundary fluxes of each control volume independently and then combined to remove their second order error.

The control volumes are centred on the transported quantities, and fluxes are calculated at the face of each volume, meaning that the transporting velocities, transported variables and fluxes are arranged in a staggered grid pattern. Consequently, some care must be taken to ensure that velocities and transported quantities are both interpolated to the centre of the control volume face before taking their product to obtain the flux.

The equations 2.1 when discretised can be expressed as

$$\frac{\delta \mathbf{u}}{\delta t} + N(\mathbf{u})\mathbf{u} = -\mathbf{G}p + L\mathbf{u} \quad (3.4)$$

$$D\mathbf{u} = 0 \quad (3.5)$$

where capital letters denote spatial discretisation operators and $\frac{\delta}{\delta t}$ denotes a discrete time derivative. The discretisation designed by Verstappen &

Veldman is such that, so long as the boundary conditions are periodic in the x and y directions, the discrete operators satisfy the skew-symmetric condition

$$\langle D\mathbf{u}, \phi \rangle = -\langle \mathbf{u}, \mathbf{G}\phi \rangle, \quad \langle N(\mathbf{u})\mathbf{v}, \mathbf{w} \rangle = -\langle \mathbf{v}, N(\mathbf{u})\mathbf{w} \rangle. \quad (3.6)$$

The result of this is that in periodic domains total energy is conserved except for viscous dissipation and explicit external forcing. The diffusion operator is defined by it's relationship to D and $\mathbf{G}$ as $L \equiv D\mathbf{G}$.

To advance the solution in time, a third-order variable-time-step Adams-Bashforth method is used. At each time-step, an intermediate velocity $\mathbf{u}^*$ is calculated by applying the forcing terms and pressure gradient to the previous velocity, so

$$\mathbf{u}^* = \mathbf{u}^n + \Delta t_n \sum_i \alpha_i f^{n-i} - \Delta t_n \mathbf{G} p^{n-1}. \quad (3.7)$$

Here, $\mathbf{u}^n$ is the velocity field at time n , Δt_n is the length of the time-step and p^{n-1} is the pressure field at the previous time-step. f^{n-i} are explicit acceleration terms and α_i are coefficients dependent on Δt_n , Δt_{n-1} and Δt_{n-2} . The α_i terms are defined in such a way that the leading order error terms cancel, thus ensuring third-order accuracy of the Adams-Bashforth method. Defining the current pressure field to be $p^n = p^{n-1} + \pi^n / \Delta t_n$ results in the velocity correction

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \mathbf{G}\pi^n \quad (3.8)$$

In order to maintain a divergence-free velocity field, the new velocity must

satisfy $D\mathbf{u}^{n+1} = 0$. This results in a Poisson equation for the pressure,

$$L\pi^n = D\mathbf{u}^*, \quad (3.9)$$

which can be solved through the use of Fourier modes in the x and y directions. The original SPARKLE code required several modifications for the present simulations, the most notable of which was the addition of a nudging region, to account for the crosswind whilst maintaining periodic boundary conditions. This method is detailed in the next section. In addition to this, the code was extended to output 3-dimensional time-averaged statistical budgets for each variable. Prior to this, the code had been used for vertical plumes and, in addition to time-averaging, statistical budgets had been averaged in the azimuthal direction. However, since our plumes are not axisymmetric, azimuthal averaging is inappropriate so the full 3-dimensional statistics must be obtained.

3.3 The nudging region

Due to the need for a periodic boundary condition, the addition of crosswind to the SPARKLE code is non-trivial. If the velocity at the $x = 0$ boundary were simply to be assigned as the crossflow velocity with no further modification, the periodic condition would imply a discontinuous velocity at the $x = L_x$ boundary. This interferes significantly with the Fourier transform method needed to solve the Poisson equation for the pressure. Consequently some method is required to enforce a uniform crosswind at the $x = 0$ bound-

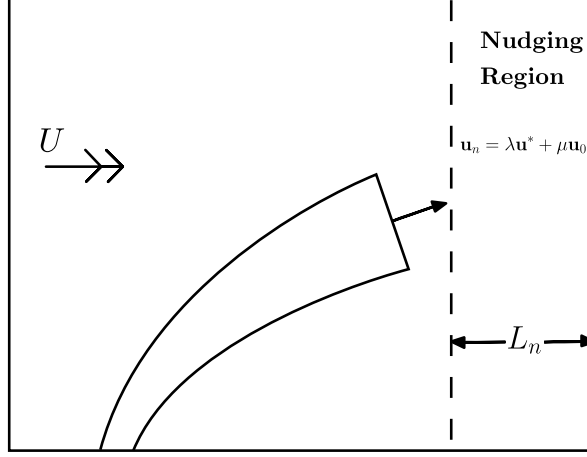


Figure 3.1: A schematic illustrating the nudging region.

ary, without introducing a discontinuity into the flow.

One solution to this problem, proposed by Stevens *et al.* (2014), is to smoothly transition from the simulated velocity $\mathbf{u}$ to the boundary velocity $\mathbf{u}_0$. In our simulations $\mathbf{u}_0 = U\hat{\mathbf{i}}$ where U is the chosen crossflow velocity for that simulation and $\hat{\mathbf{i}}$ is the unit vector in the x -direction. However, the same method could be applied to generate any wind profile at the $x = 0$ boundary. This smooth transition occurs in a region of the simulation called the nudging region, which is the volume of the simulation space located between the plane $x = L_x - L_n$ and the downwind boundary itself at $x = L_x$. This region is illustrated in figure 3.1. L_n is the parameter governing the width of the nudging region, and must be determined suitably for each crossflow speed. In particular, the nudging region width must be sufficiently large that the tran-

sition between the two velocities is gradual. For the presented simulations we chose L_n to be approximately 5-10% of the total domain length.

Within the nudging region itself, the process of transitioning from the simulated velocity to the desired boundary wind-profile is applied after the intermediate velocity $\mathbf{u}^*$ is found but before the pressure correction is applied, so as to maintain a divergence-free velocity field. This ‘nudging’ of the velocity towards the wind profile is implemented as a linear interpolation between the simulated velocity and desired boundary condition. The velocity in the nudging region, $\mathbf{u}_n$, can be written:

$$\mathbf{u}_n = \lambda \mathbf{u}^* + \mu \mathbf{u}_0, \quad (3.10)$$

where $\lambda = 1 - x^*/L_n$ and $\mu = x^*/L_n$ are interpolating parameters ranging between 0 and 1, and $x^* = x - (L_n - L_x)$ is the distance from the beginning of the nudging region. Provided that the nudging region is specified to be large enough, this linear interpolation allows the simulation velocity to assume the wind-profile without introducing a sharp gradient in the velocity field, and therefore without having a negative impact on the Fourier transform method.

We have taken care to ensure that the nudging region does not affect the upstream statistics, which would occur by pressure gradients generated by the nudging influencing upwind regions. In figure 3.2, we present the pressure gradient $d\langle p \rangle/dx$ as a function of the downstream distance for all five simulations. Here, $\langle p \rangle$ is the average pressure inside the plume. The black dashed vertical line denotes the start of the nudging region, while the red

dash-dotted vertical lines show the x -interval that is analysed in the current manuscript. It is clear that the nudging region does not impose significant adverse pressure gradients, as can be observed by the absence of deviations from zero near the black dashed line.

3.4 Plume thresholding

Plume thresholding is the process of defining the area $A(s)$ of the plume. This is an important aspect of our statistical analysis, as an overestimation or underestimation of the plume area at this stage of the data processing will negatively impact all of the subsequent results. The plume boundary is conventionally established on the basis of a plume scalar, which we call C . In our simulations, this scalar is introduced at the source and travels with the flow. To define the boundary of the plume area, we choose a threshold of the scalar, above which fluid is considered to be within the plume. The threshold in any given y - η plane is a certain percentage of the maximum value of C in that plane, denoted $C_m \equiv \max_{s=s'}\{C\}$. As the threshold percentage governs the position of the plume boundary, it is reasonable to expect that the choice of the threshold percentage will have a particularly large impact on the boundary integrals, namely the entrainment and Leibniz terms. In figure 3.4 we illustrate plume boundaries resulting from threshold percentages of $C_t = 1\%$, 0.1% and 0.01% of C_m , for the instantaneous and time-averaged plume scalars. Intuitively, it seems reasonable that a low threshold percentage, which gives plume boundaries further out into the domain, should reduce

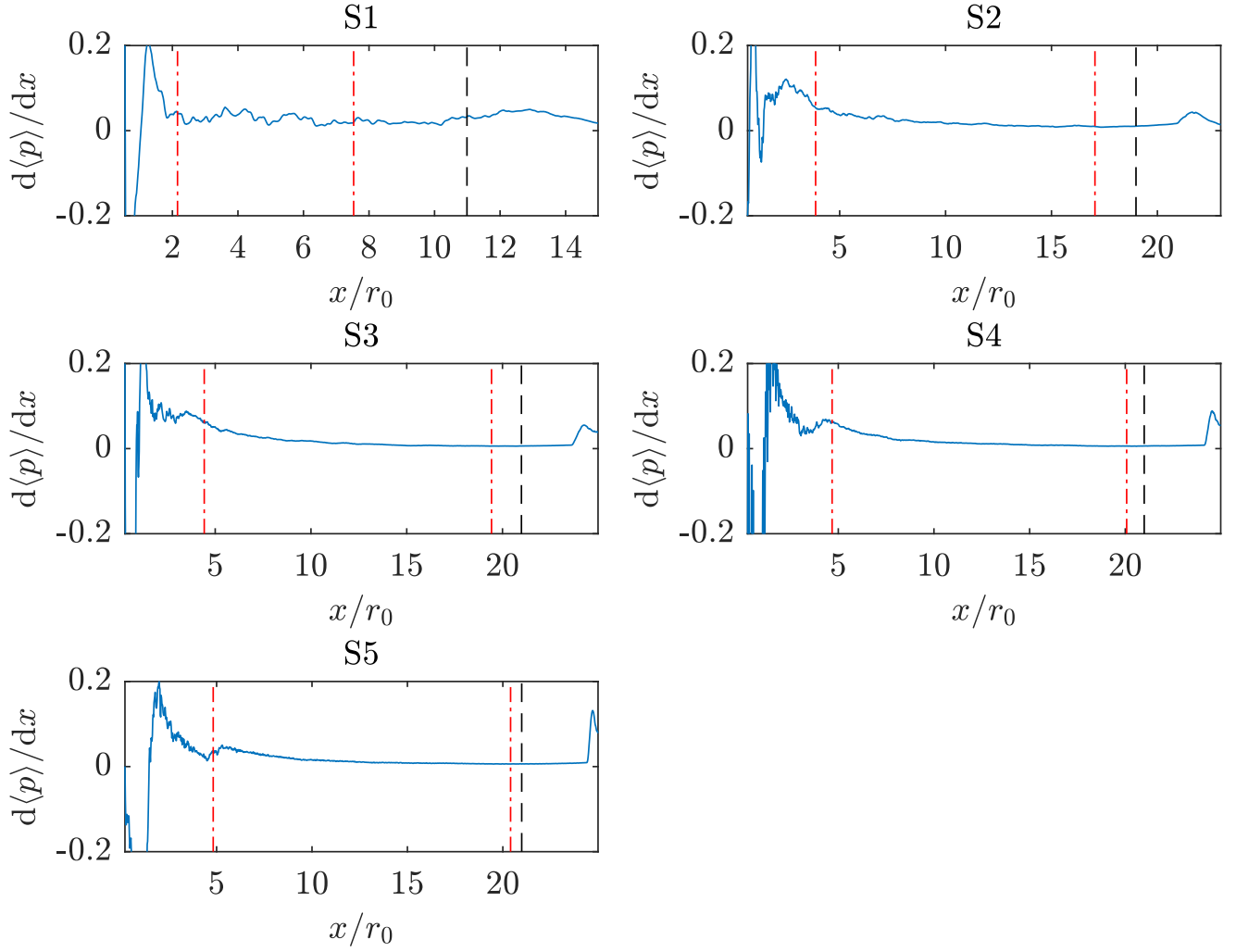


Figure 3.2: Pressure gradient $d\langle p \rangle / dx$ as a function of the downstream distance. Black dashed line: start of the nudging region. Red dash-dotted lines: the data analysed in the manuscript.

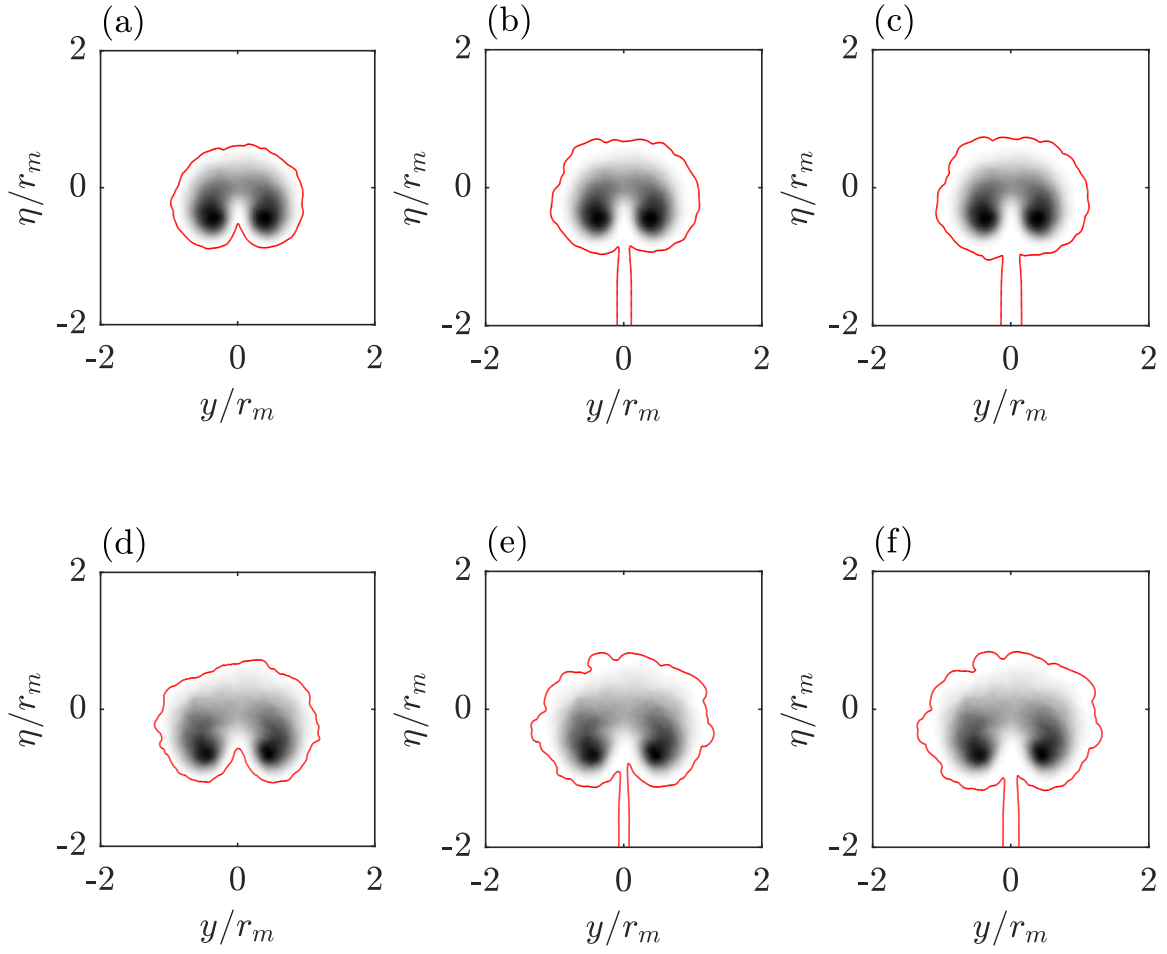


Figure 3.3: A visualised slice of the time-averaged concentration fields of the plume through the $y - \eta$ plane at $s = 3$ (a-c) and $s = 10$ (d-f). Different threshold percentages are shown in red: (a,d) 1% threshold, (b,e) 0.1% threshold, (c,f) 0.01% threshold.

the size of the Leibniz terms, due to the fact that we would be integrating in regions where the integrand is relatively small. While this may seem desirable as we would then be able to neglect the Leibniz terms from our plume equations safely, a low threshold percentage also implies that relatively quiescent fluid is defined as being within the plume. Given that this fluid is not turbulent it is unreasonable to define it as being contained within the plume. Such a low threshold value should therefore be avoided. Similarly, we should reject threshold values which are too high, as this will restrict the definition of the plume to a very narrow plume width, which would exclude significant regions of the turbulent motion. Based on this criteria, we have chosen 1% as our threshold percentage, so that at any particular s -value, fluid with a plume scalar concentration of greater than $0.01C_m$ is considered to be within the plume. It can be seen from figure 3.4 that the qualitative behaviour at thresholds of 0.1% and 0.01% is significantly different to the behaviour at 1%. In these cases there appears a notable region under the main plume. We attribute this region to plume concentration which has been swept out from the main plume by the crossflow, leading to fluid with increased concentrations of the plume scalar appearing beneath the main body of the plume. Fluid in this region is relatively non-turbulent and travels with the ambient crossflow, indicating that we should not consider this region to be part of the plume. The desire to exclude this region from our integral calculations is the primary reason for choosing the 1% threshold.

Another scalar that can be utilised to establish the position of the plume boundary is the enstrophy. This is sometimes used as a means of estimating turbulent intensity (Holzner *et al.* 2008) and would therefore be a suitable

potential candidate for use as a thresholding scalar. We implemented boundaries resulting from thresholds of enstrophy and found that they did not differ significantly from boundaries resulting from thresholds of plume concentration scalar. This suggests that the plume concentration scalar which we have chosen to employ offers a physically useful definition of the plume area.

3.5 Case setup

Consistent with the theory presented in the previous chapter and the methodology set out above, we simulated an unstratified environment with a uniform crossflow velocity U . We imposed a buoyancy flux F_0 on a circular area source of diameter $2r_0$, which was placed at $x = 0$. Due to the fact that the strength of the crossflow varies the behaviour of the plume, we optimised the sizes of the domains such that each plume fit comfortably within the simulation domain without wasting space in the domain calculating regions where only ambient flow is present. The domain dimensions are given in Table 3.1, with the note that to accommodate the source of the momentum flux being located at the origin of the domain, the x -coordinate ranges from $x = -5$ to $x = L_x - 5$.

The source is momentumless with $M_0 = 0$, which implies the plume is infinitely lazy (Hunt & Kaye 2005). The relevant parameters for the direct numerical simulations are given in Table 3.1 and an illustration of the regime is shown in figure 2.1.

The relevant non-dimensional parameters for buoyant jets in a crossflow can

be formed from the jet exit velocity $w_0 = M_0^{1/2}/r_0$, the buoyancy velocity scale $F_0^{1/3}/r_0^{1/3}$, the crossflow velocity U , and for completeness, the viscous and thermal velocity scales ν/r_0 and κ/r_0 respectively. From these five velocity scales, four dimensionless quantities can be constructed,

$$R_0 = \frac{w_0}{U}, \quad Ri_0 = \frac{F}{r_0 w_0^3} = \frac{b_0 r_0}{w_0^2}, \quad Re_0 = \frac{2F_0^{1/3} r_0^{2/3}}{\nu}, \quad Pr = \frac{\nu}{\kappa}, \quad (3.11)$$

where R_0 is the jet-to-crossflow velocity ratio, Ri_0 is the source Richardson number, Re_0 is the source Reynolds number and Pr is the Prandtl number. Here, we have chosen to stick with conventional dimensionless quantities (note that the Froude number, which is also commonly used, is the square root of the inverse of Ri_0), even though these are not ideal for the current simulation set up since $w_0 = 0$ which implies that $w_0/U = 0$ and $Ri_0 = \infty$. In Table 3.1 we present $Ri_U = Ri_0 R_0^3 = F/(U^3 r_0)$, which is a measure of the strength of the buoyancy relative to the crossflow velocity. Equivalently, by introducing the buoyancy length scale $\ell_b = F/U^3$, Ri_U can be interpreted as a ratio of lengthscales: $Ri_U = \ell_b/r_0$. This is one possible dimensionless parameter that we can use instead of R_0 to determine a relation similar to 4.2. We tested what the relation between β' and Ri_U for our simulations and we found that the empirical model corresponding to Briggs' relation in our case was $\beta' \approx 0.88 - 0.049 Ri_U$. The source Reynolds number and Prandtl number for all of our simulations are, respectively, $Re_0 = 1000$ and $Pr = 1$.

As discussed in section 3.1, we need to ensure that the simulations are DNS by requiring that $\Delta_x < \eta_K$. Since the dissipation is a local quantity, the definition for η_K varies, spatially and temporally. We could require that the

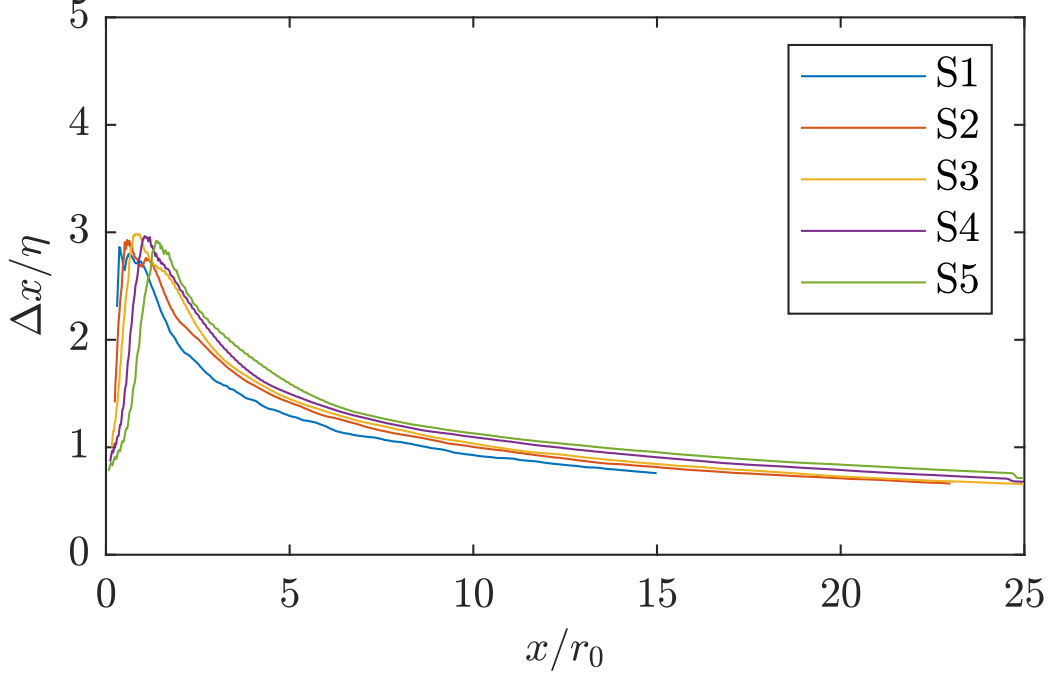


Figure 3.4: Resolution relative to Kolmogorov scale η as a function of x for the five simulations.

grid resolution is less than the Kolmogorov lengthscale at all points in space, and for the duration of the simulation, but this would require significant computational resources. A more pragmatic alternative is to minimise the regions where $\eta_K > \Delta_x$. In all of our simulations the ratio $\Delta x/\eta_K$ reaches a maximum value of approximately 3 just downstream of the source and quickly decreases thereafter. This is depicted in Figure 3.4, and is clear evidence that the flow is fully resolved and thus that Direct Numerical Simulation (DNS) can be claimed. Because the regions downstream are well-resolved, we can reasonably expect the plume evolution in those regions to be accurately represented, even if the region close to the source is somewhat underresolved.

Table 3.1: Simulation details. $Re_0 = 1000$ and $Pr = 1$ for all simulations.

sim. id	Ri_U	$N_x \times N_y \times N_z$	$L_x/r_0 \times L_y/r_0 \times L_z/r_0$	β	β'	k	x_v/r_0
S1	8.0	$1152 \times 768 \times 1152$	$24 \times 24 \times 36$	0.68	0.49	0.91	0.45
S2	2.4	$1536 \times 768 \times 960$	$32 \times 24 \times 30$	0.97	0.76	0.62	1.24
S3	1.0	$1728 \times 768 \times 768$	$36 \times 24 \times 24$	1.05	0.83	0.62	1.85
S4	0.5	$1728 \times 576 \times 576$	$36 \times 18 \times 18$	1.15	0.88	0.71	1.91
S5	0.3	$1728 \times 576 \times 576$	$36 \times 18 \times 18$	1.15	0.84	0.86	2.95

Attempting to calculate the entrainment and Leibniz terms directly from the definition in equation 2.13 is difficult, since it requires a boundary integral in the plane perpendicular to the plume. Instead, we choose to use Green's theorem in reverse to calculate E via

$$E[\phi] = - \iint_A \frac{\partial v \phi}{\partial y} + \frac{\partial u_\eta \phi}{\partial \eta} dA \quad (3.12)$$

and we use the original definition of $\mathcal{L}$

$$\mathcal{L}[\phi] = \frac{d}{ds} \iint_A u_s \phi dA - \iint_A \frac{\partial u_s \phi}{\partial s} dA. \quad (3.13)$$

3.6 Least-squares spline fitting for the plume centreline

The analysis requires derivatives of the plume angle, $d\varphi/ds$. This is a particularly sensitive quantity, as the angle φ is itself based on the derivative

dz_c/dx . The thresholding causes small discontinuities in the statistics which will amplify and contaminate the signal upon repeated differentiation. Therefore, we approximate the plume trajectory $(x(s), z_c(s))$ with a least-squares spline-fitting method, allowing us to use several cubic polynomials to represent the centreline. The primary aim of this is to reduce noise in $\frac{d\varphi}{ds}$, while maintaining a centreline as close to the original as possible.

The unprocessed centreline, $z(x)$ is determined by taking the centre of mass of the plume concentration scalar in the z -direction at each x -location, which numerically results in a vector $\mathbf{z}_c$, with N_x elements where N_x is the length of the position vector $\mathbf{x}$. We wish to represent this vector as N cubic splines of the form:

$$p_j(x) = \lambda_{j,0} + \lambda_{j,1}x + \lambda_{j,2}x^2 + \lambda_{j,3}x^3, \quad (3.14)$$

where $\lambda_{j,k}$ are the coefficients required to determine p_j . We require these polynomials and their first and second derivatives to match in value at the locations where we transition between polynomials. Assuming that the optimal x -locations for the transition between p_j and p_{j+1} are evenly spaced, and denoting the number of x -locations in each polynomial as $n \equiv N_x/N$, the transition location between p_j and p_{j+1} is at x_{jn} . The continuity and smoothness conditions can now be written as

$$\begin{aligned} (\lambda_{j,0} - \lambda_{j+1,0}) + (\lambda_{j,1} - \lambda_{j+1,1})x_{jn} + (\lambda_{j,2} - \lambda_{j+1,2})x_{jn}^2 + (\lambda_{j,3} - \lambda_{j+1,3})x_{jn}^3 &= 0 \\ (\lambda_{j,1} - \lambda_{j+1,1}) + 2(\lambda_{j,2} - \lambda_{j+1,2})x_{jn} + 3(\lambda_{j,3} - \lambda_{j+1,3})x_{jn}^2 &= 0 \\ 2(\lambda_{j,2} - \lambda_{j+1,2}) + 6(\lambda_{j,3} - \lambda_{j+1,3})x_{jn} &= 0 \end{aligned}$$

for $1 \leq j \leq N - 1$. We call the first and last points of the spline x_0 and x_{N_x} , consistent with the previous notation.

The spline $\hat{z}(x)$ will be the stitched-together function consisting of these polynomials, i.e.

$$\hat{z}(x) = \begin{cases} p_1(x) & x_0 \leq x \leq x_n, \\ p_2(x) & x_n \leq x \leq x_{2n}, \\ \vdots & \\ p_N(x) & x_{N_x-n} \leq x \leq x_{N_x}. \end{cases}$$

Now, $p_j(x_i)$ can be written as

$$p_j(x_i) = [1, x_i, x_i^2, x_i^3] \begin{bmatrix} \lambda_{j,0} \\ \lambda_{j,1} \\ \lambda_{j,2} \\ \lambda_{j,3} \end{bmatrix} \quad (3.15)$$

and by extension, the spline-approximation for each element x_i in the range of polynomial j is given by

$$\hat{\mathbf{z}}_j = \begin{pmatrix} 1 & x_{jn} & x_{jn}^2 & x_{jn}^3 \\ 1 & x_{jn+1} & x_{jn+1}^2 & x_{jn+1}^3 \\ \vdots & & & \\ 1 & x_{(j+1)n} & x_{(j+1)n}^2 & x_{(j+1)n}^3 \end{pmatrix} \begin{bmatrix} \lambda_{j,0} \\ \lambda_{j,1} \\ \lambda_{j,2} \\ \lambda_{j,3} \end{bmatrix} \equiv X_j \boldsymbol{\lambda}_j \quad (3.16)$$

Putting all polynomials evaluations into a single block-matrix leads to the

estimator function

$$\hat{\mathbf{z}} = \begin{pmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X_N \end{pmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \vdots \\ \boldsymbol{\lambda}_N \end{bmatrix} \equiv X\boldsymbol{\lambda} \quad (3.17)$$

The coefficients $\boldsymbol{\lambda}$ can be found by minimising $\|\hat{\mathbf{z}} - \mathbf{z}_c\|^2$ subject to the continuity and smoothness constraints on $\boldsymbol{\lambda}$. These constraints can be written

$$C\boldsymbol{\lambda} = \mathbf{0} \quad (3.18)$$

where

$$C = \begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{N-1} \end{pmatrix} \quad (3.19)$$

and

$$C_j = \begin{pmatrix} 1 & x_{jn} & x_{jn}^2 & x_{jn}^3 & -1 & -x_{jn} & -x_{jn}^2 & -x_{jn}^3 \\ 0 & 1 & 2x_{jn} & 3x_{jn}^2 & 0 & -1 & -2x_{jn} & -3x_{jn}^2 \\ 0 & 0 & 2 & 6x_{jn} & 0 & 0 & -2 & -6x_{jn} \end{pmatrix}. \quad (3.20)$$

Thus, the problem of finding the optimal spline has been reduced to a constrained least squares (CLS) problem, to which the solution $\hat{\boldsymbol{\lambda}}$ is (Boyd &

Vandenberghe 2018)

$$\begin{bmatrix} \hat{\boldsymbol{\lambda}} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{pmatrix} 2X^T X & C^T \\ C & 0 \end{pmatrix}^{-1} \begin{bmatrix} 2X^T \mathbf{z}_c \\ \mathbf{0} \end{bmatrix} \quad (3.21)$$

where $\boldsymbol{\theta}$ is a vector of Lagrange multipliers. We implemented this method with $N = 4$ polynomials. The resulting spline is shown in figure 3.5 and demonstrates good agreement with the original centreline whilst reducing the noise in $\frac{d\varphi}{ds}$ significantly. One further complication which appears during this spline-creation process is the fact that the matrix requiring inversion in 3.21 is poorly scaled. This is apparent when considering that the matrix contains a wide variation in values. Specifically it must contain zeros, ones and the value x_{jn}^6 , which could be on the order of 10^7 . This presents a problem with floating point precision when calculating the inverse matrix. A possible solution to this scaling problem would be to assume for the purposes of calculating $\hat{\boldsymbol{\lambda}}$ that the x axis ranges between 0 and 1. An elegant way to approach this is to scale the matrices C and X by a scaling matrix S such that

$$S = \begin{pmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_N \end{pmatrix} \quad (3.22)$$

and

$$S_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{x_{j(n+1)}} & 0 & 0 \\ 0 & 0 & \frac{1}{x_{j(n+1)}}^2 & 0 \\ 0 & 0 & 0 & \frac{1}{x_{j(n+1)}}^3 \end{pmatrix} \quad (3.23)$$

With a small amount of matrix manipulation, equation 3.21 can then be written

$$\begin{bmatrix} \hat{\boldsymbol{\lambda}} \\ \boldsymbol{\theta} \end{bmatrix} = S \begin{pmatrix} 2SX^TXS & SC^T \\ CS & 0 \end{pmatrix}^{-1} \begin{bmatrix} 2SX^T\mathbf{z}_c \\ \mathbf{0} \end{bmatrix} \quad (3.24)$$

where the matrix to be inverted is now suitably scaled.

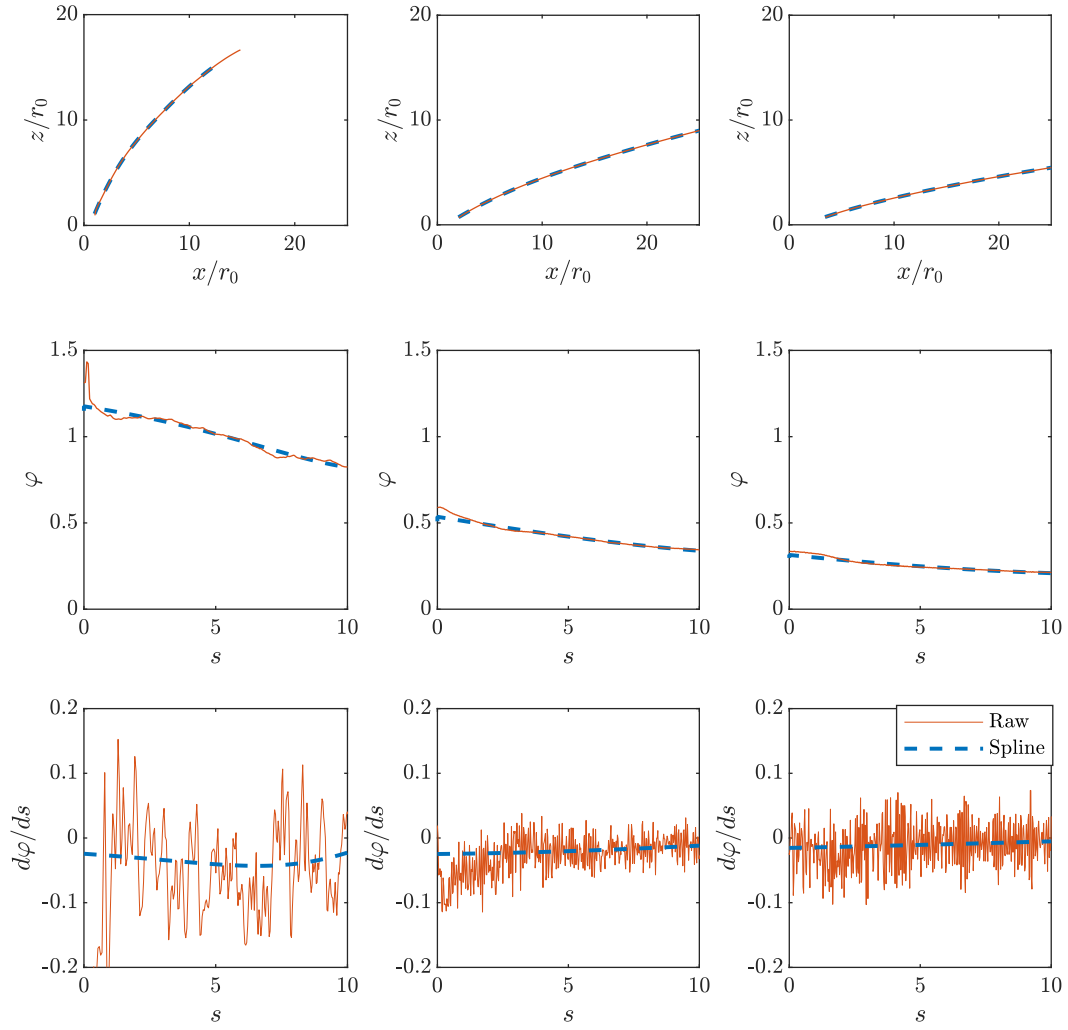


Figure 3.5: A figure demonstrating the noisy signal resulting from taking the derivative of φ , and the smoother signal generated by computing the derivative of the spline.

Chapter 4

Results

4.1 Plume evolution and geometry

Instantaneous and time-averaged snapshots of the buoyancy field b in the x - z plane are shown in Figure 4.1. In order to enhance the clarity of the figures, the concentrations are normalised by $b_{\max}(x) = \max_{y,z} \bar{b}$. It is clear from the snapshots that the plume takes a while to diffuse through the thermal boundary layer at the wall, during which time it is advected downstream by the crossflow. This implies that a virtual source correction will be necessary (see below). Once the plume has separated from the boundary, it gradually changes angle as it rises through the ambient, increasing its radius along the way due to turbulent entrainment. This is clearly dependent on the flow velocity, as the plumes are observed to be narrower for the higher cross flow velocities. Another notable feature of Figure 4.1 is that there is a tendency

for a small amount of tracer to be swept into the wake of the plume. This requires some careful attention. When choosing the methods of numerical analysis for the simulations, as it is in general inappropriate to regard this region, which is otherwise quiescent, as being within the plume. For clarity we have removed the concentration that would be in these quiescent regions from the figure. Further quantitative snapshots are shown in figure 4.2, this time in the η - y plane. These clearly show a buoyant core structure with twin vortices, with a development time dependent on the crossflow velocity.

Shown in figure 4.3 are the centrelines of the plume. It is well known that there are some subtle differences depending on whether these are inferred from the maximum of velocity or scalar (Mahesh 2013). Here, we infer the trajectory as a function of x from by calculating the centre of mass of the buoyancy:

$$z_c(x) = \frac{\iint_A z \bar{b} dy dz}{\iint_A \bar{b} dy dz} \quad (4.1)$$

This approach will work particularly well for simulations with large crossflow velocity U (i.e. bent-over plumes). The threshold value to identify the plume was taken to be 1% of the maximum buoyancy at fixed x . The plume centrelines are shown as a function of x/r_0 in Figure 4.3(a). In order to validate the simulations, we compare these centrelines with the analytical prediction (2.22) of a bent-over plume. Shown in Figure 4.3(b) is $(z_c/\ell_b)^{3/2}$ against x . As can be observed, all plume trajectories evolve in an approximately linear fashion, confirming scaling consistent with expectation. By fitting a line through the last part of the trajectory, β' , as well as the virtual origin x_v can be identified. The value of the virtual origin is identified by fitting a

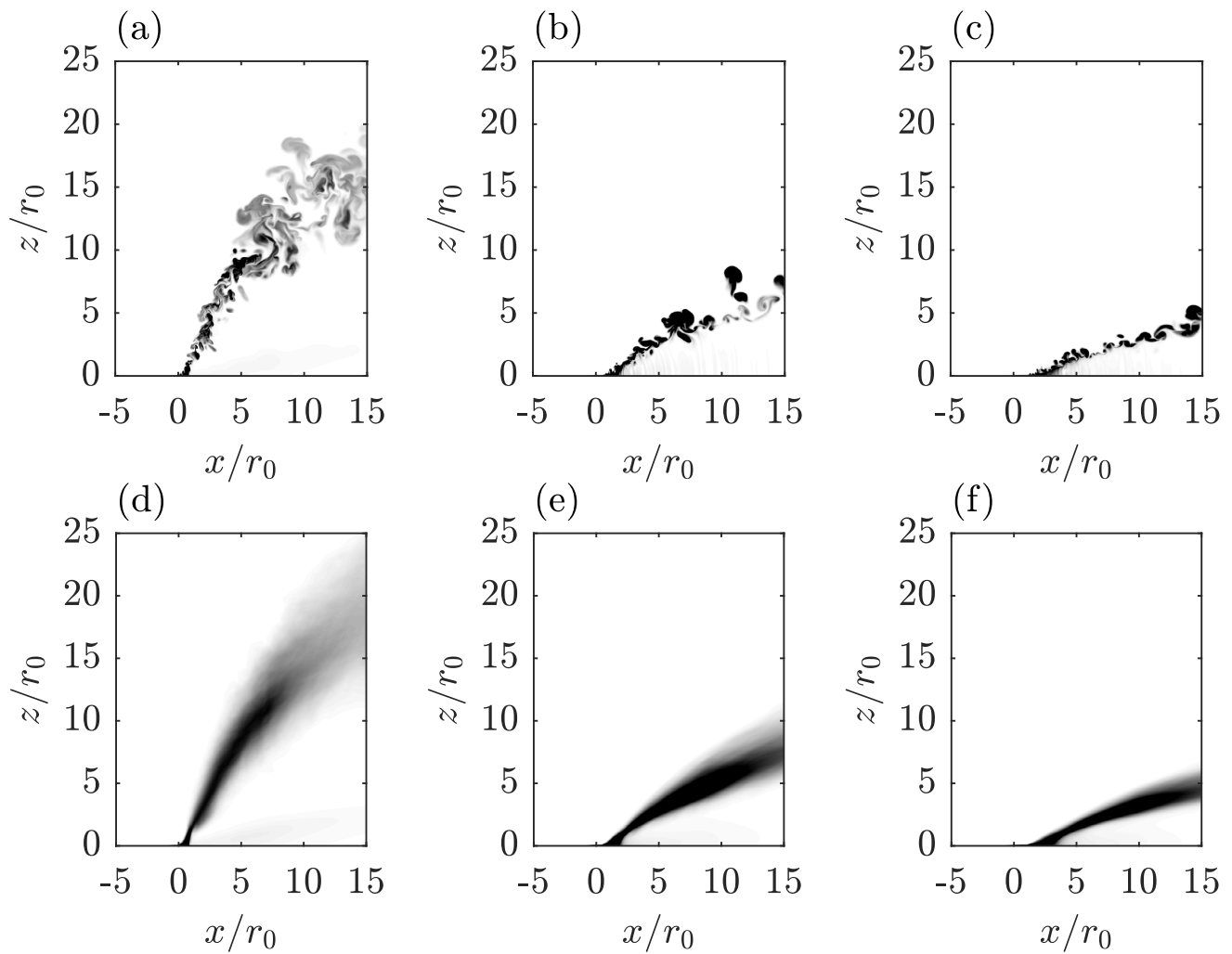


Figure 4.1: A slice through the $y = 0$ plane of the plume from the a) instantaneous and b) time-averaged concentration fields.

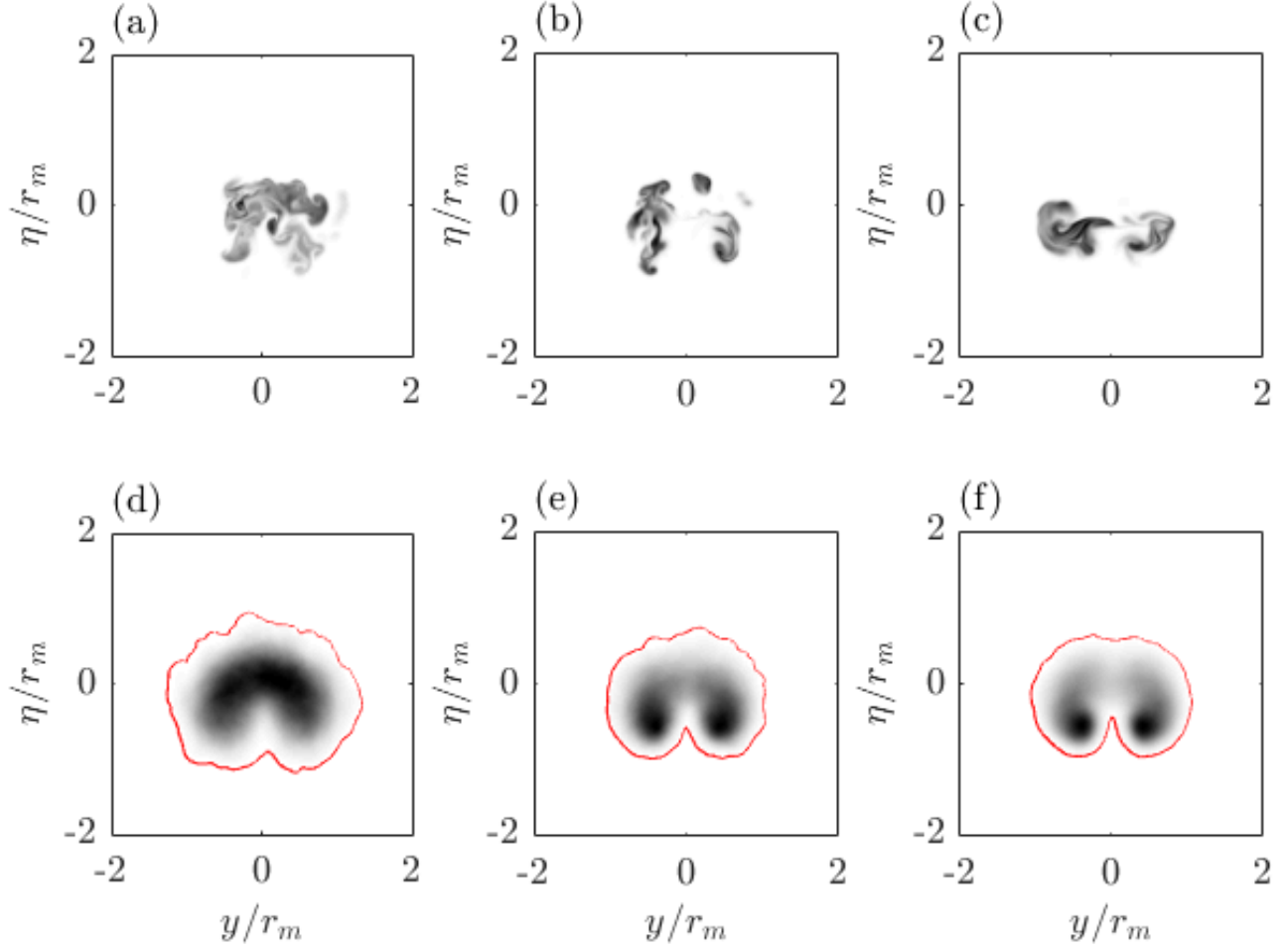


Figure 4.2: A visualised slice of the plume through the $y - \eta$ plane at $s = 5$, for the (a-c) instantaneous and (d-f) time-averaged concentration fields. (a,d) simulation S1. (b,e) simulation S3. (c,f) simulation S5. The red line in figures (d-f) is the applied threshold on buoyancy at 1% of the maximum of the time-averaged value.

straight line through the trajectories in Figure 4.3(b) and calculating where the fitted line intercepts with the x -axis. x_v is only used to compare the plume trajectories with equation 2.22, and does not need to be subtracted from x for the other figures (such as Figure 4.4). Figure 4.3(c) shows the linear dependence between r_m and z_c , conform with the bent-over plume predictions (2.22). The values of β , β' , k and x_v/r_0 are presented in Table 3.1 for each simulations. We find values of β' in the range $\beta' \in [0.49, 0.88]$ which is higher than in the experiments of De Wit *et al.* (2014), and the LES simulations of Cintolesi *et al.* (2019), suggesting the plumes considered here ascend less rapidly. We suspect these higher values of β' are due to the fact that the plume under consideration is infinitely lazy at the source. Indeed, Briggs (1984) developed an empirical prediction

$$\beta' = 0.4 + 1.2/R_0 \quad (4.2)$$

for the dependence of β' on $R_0 = w_0/U$, demonstrating that this parameter depends on conditions at the source, particularly at low values of R_0 .

Shown in figure 4.4 are the evolution of the integral volume flux Q , horizontal momentum excess flux M_x , vertical momentum flux M_z and buoyancy flux F as a function of s . The volume flux Q corresponds almost entirely with $U \frac{dA}{ds}$, indicating that the Leibniz term is the primary driver of volume flux growth. Strikingly, the momentum excess flux M_x is not conserved for the plumes. Instead, it grows strongly, particularly for the lower windspeeds (simulations S1 and S2). The vertical momentum flux also grows more strongly as a function of s for lower windspeeds. We believe the reason for this is that

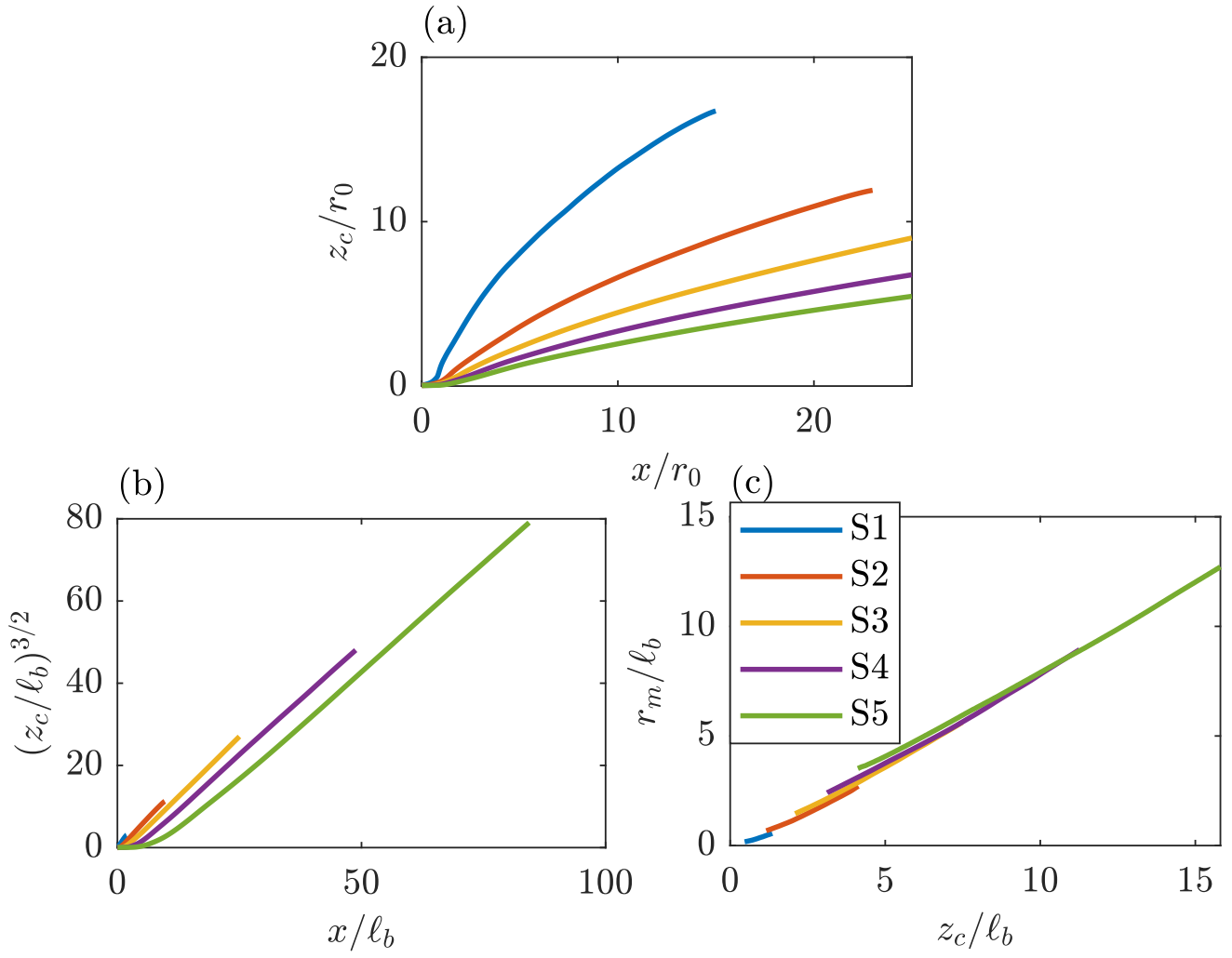


Figure 4.3: Plume trajectories. (a) unnormalised. (b) centreline location z_c as a function of x . (c) plume radius r_m as a function of z .

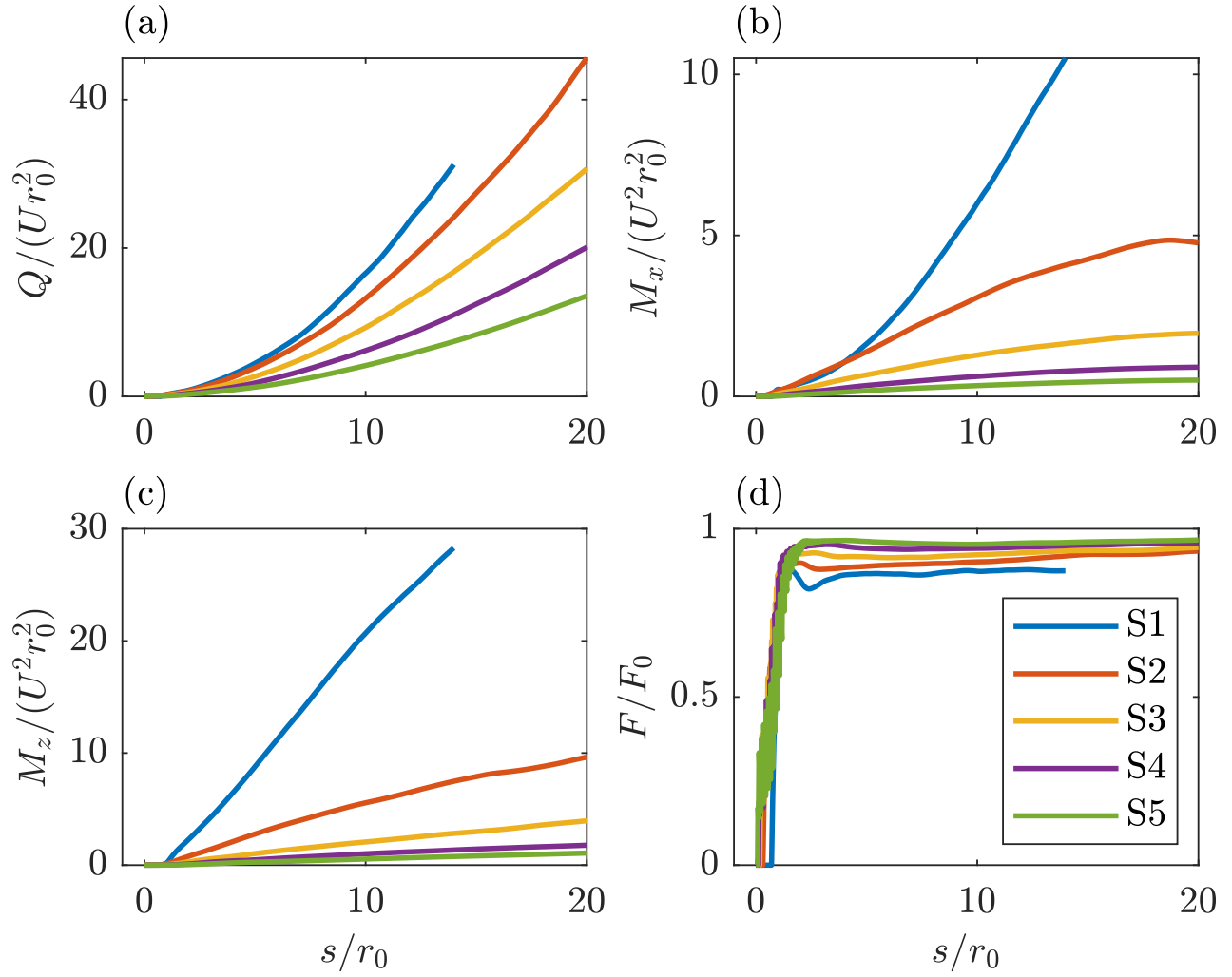


Figure 4.4: Integral quantities Q (a), M_x (b), M_z (c) and F (d) as a function of s for each simulation.

there is a high-pressure region above the plume, which is greater in the high crossflow cases, causing the vertical rise to be damped more. We explain this further in §4.3. The buoyancy flux F is approximately constant across all simulations, and remains close to its initial value, F_0 . This is consistent with the buoyancy flux equation from 2.18.

Having calculated the integral plume quantities, we verify whether the plume slope dz_c/dx is consistent with the conventional definitions. From (2.19) and taking $\varphi = \varphi_u$ it follows that

$$\frac{dz_c}{dx} \stackrel{?}{=} \frac{w_m}{U + u_m} = \frac{M_z}{QU + M_x} \quad (4.3)$$

This relation is plotted in Figure 4.5 for simulation S3, together with the gradient of z_c . In order to avoid the amplification of noise due to repeated differencing (we will also use $d\theta/ds$ later on), z_c was approximated with a cubic spline before differentiating. The spline approximation was obtained using a least-squares method which is described in Appendix 3.6. The spline representation using two splines displays only very small deviations from the original. In the appendix we show a comparison of the raw derivatives with the approximated derivatives, showing the benefits of utilising the spline representation.

Figure 4.5 shows that $w_m/(U + u_m)$ is nearly half of dz_c/dx , and therefore implies that the ratio of the integral momentum fluxes is a very poor estimate of the plume slope. Given that this is one of the basic assumptions of the integral theory of plumes in a cross-wind, the implications of this observation are wide-ranging. The basic image of a plume in a cross-wind is that in which

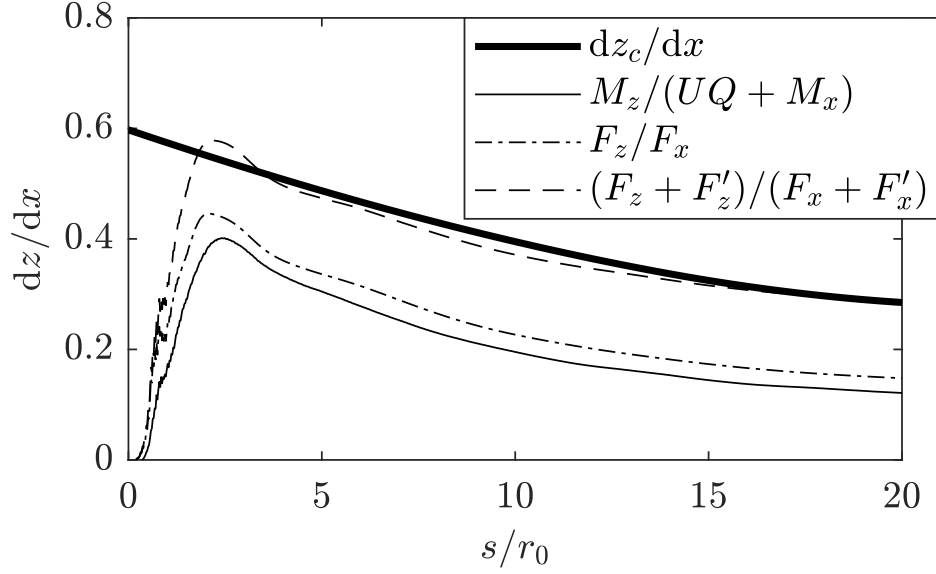


Figure 4.5: A comparison of different estimates for the slope of the plume, along with the actual value.

all quantities of relevance are advected by the streamwise velocity U_s inside the plume. Indeed, the plume centreline is often envisaged as the central streamline of the plume Weil (1988). The result in Figure 4.5 is thus mildly disturbing, as it indicates that this image is incorrect.

In Figure 4.6, streamlines of various quantities are shown through the plume. The extent of the plots in the x -direction is kept small in order to display a detailed comparison of the three types of streamline presented, but the characteristic behaviour of the streamlines remains similar downstream of the displayed extent of the figures. The streamlines for Figure 4.6(a) were constructed by averaging the velocity over the y -direction within the plume. The resulting velocity vectors were input into the MATLAB streamline func-

tion, which calculates paths by starting at a fixed point and moving along the path a small increment in the direction of the velocity vector. Figure 4.6(a) is consistent with Figure 4.5 in that it is clear that the angle of the streamlines is substantially different from the plume centreline (thick black line). For the case under consideration, the streamlines enter the top of the plume (indicated by a dashed line) at nearly 45° for the case under consideration (S3). Interestingly though, the streamlines are reasonably parallel at the bottom boundary. Importantly though, the centre of mass of the plume, as indicated by the thick black line, is not parallel with any streamline. Thus, the plume angle φ is different from the velocity angle φ_u , which implies that the conventional assumption of the plume "going with the flow" ($\varphi = \varphi_u$) is violated.

Since the centreline was determined from the centre of mass of the buoyancy of the plume, it stands to reason to investigate whether it is more suitable to consider streamlines based on the local mean buoyancy flux $\bar{\mathbf{f}} = (\bar{u}\bar{b}, \bar{w}\bar{b})$, which are shown in Figure 4.6(b). These "buoyancy streamlines" are constructed by the same method as that used for the velocity streamlines, but with the y -averaged $\bar{u}\bar{b}$ and $\bar{w}\bar{b}$ used as inputs for the MATLAB streamline function instead of $\bar{u}$ and $\bar{w}$, respectively. Although the mean buoyancy streamlines follow the bottom boundary almost perfectly, it is clear that the behaviour is nearly identical to Figure 4.6(a). This is also evident in Figure 4.5, where the ratio of F_z/F_x , which is the approximation of dz_c/dx based on mean buoyancy fluxes in the horizontal and vertical direction, produces results that are very similar to those given by the $w_m/(U + u_m)$ estimate for

dz_c/dx . In this plot, we define

$$F_x = \frac{1}{\pi} \iint_A \bar{u}\bar{b}dA, \quad F_z = \frac{1}{\pi} \iint_A \bar{w}\bar{b}dA, \quad (4.4)$$

$$F'_x = \frac{1}{\pi} \iint_A \overline{u'b'}dA, \quad F'_z = \frac{1}{\pi} \iint_A \overline{w'b'}dA. \quad (4.5)$$

It is only when we turn to the streamlines of the *total* buoyancy flux, using the same method but with $\mathbf{f} = (\bar{u}\bar{b} + \overline{u'b'}, \bar{w}\bar{b} + \overline{w'b'})$, that the streamlines are found to be fully representative of the plume evolution. A secondary conclusion that can be drawn from comparing Figs 4.6(b,c) is that the region below the plume centreline is dominated by mean buoyancy transport and the top by a combination of mean and turbulent buoyancy fluxes. Furthermore this suggests that the plume is entraining strongly from the top. The fact that the total buoyancy transport determines the angle of the plume is verified in Figure 4.5, which is shown to follow dz_c/dx very closely.

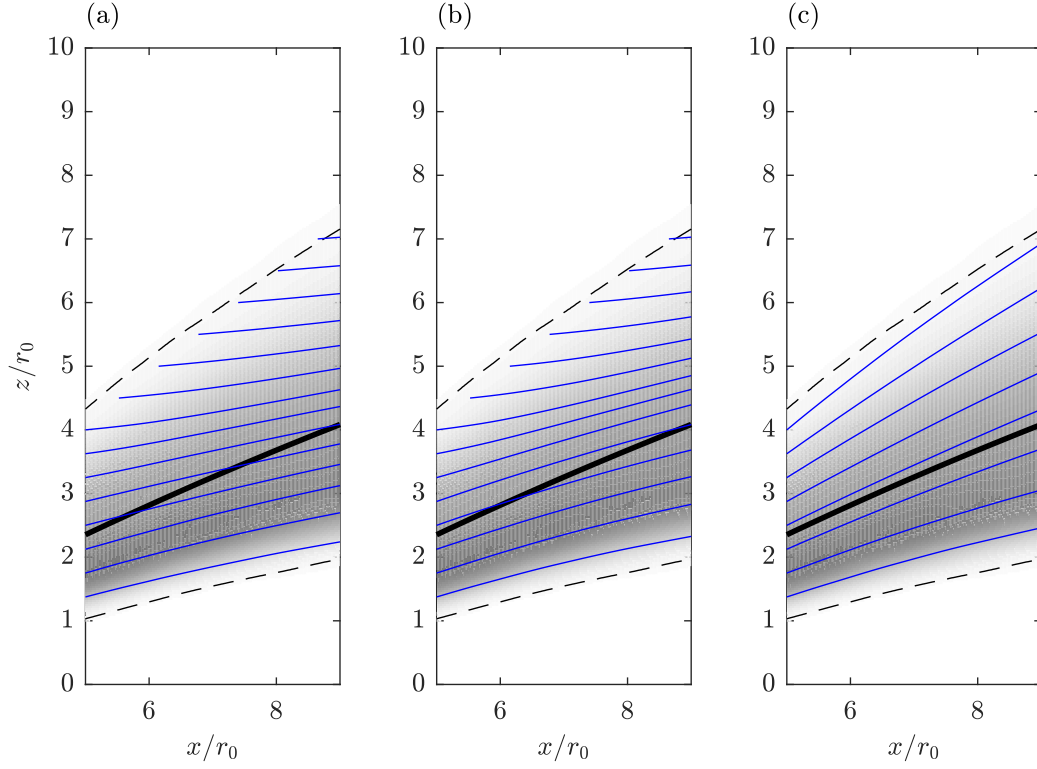


Figure 4.6: Streamlines (blue) through a section of simulation S3, together with a contour of the normalised buoyancy (in grey), the plume centreline (thick red line) and the plume boundaries (dashed line). (a) streamlines of $\mathbf{u} = (\bar{u}, \bar{w})$. (b) (a) streamlines of $\bar{\mathbf{f}} = (\bar{u}\bar{b}, \bar{w}\bar{b})$. (c) streamlines of $\mathbf{f} = (\bar{u}\bar{b} + \overline{u'b'}, \bar{w}\bar{b} + \overline{w'b'})$.

4.2 Turbulent entrainment

The DNS data allows us to directly probe the relationship between the entrainment flux and the plume properties, i.e. equation (2.15). The expectation is that if this equation holds, then the entrainment coefficient γ will be constant and the same for all simulations. Figure 4.7(a) convincingly shows that this is not the case. Indeed, γ is an increasing function of s for all simulations and there is no sign of reaching a far-field constant. This raises the question of whether w_m is the appropriate velocity scale for entrainment as is conventionally suggested. Indeed, an equally logical choice of the characteristic plume velocity is to use U_s , which implies an entrainment closure given by

$$E = 2\gamma_U r_m U_s \quad (4.6)$$

where γ_U is the entrainment coefficient based on U_s . From figure 4.7, it can be seen that γ_U is more or less constant for all simulations, although the value of γ_u decreases with increasing wind speed. The ansatz (4.6), therefore, can serve as the basis for a parameterisation.

Shown in Figure 4.8 is the integral budget for the continuity equation (2.14a) for simulation S1, S3 and S5, which provide insight into the two identified modes of entrainment: flow of fluid into the plume (E) and incorporation of ambient fluid due to expansion of the plume (L). As the wind velocity increases, L becomes increasingly dominant. Physically, this implies that for sufficiently large wind speeds, the main contribution to the volume flux is *not* flow directly into or perpendicular to the plume, but rather incorporation as

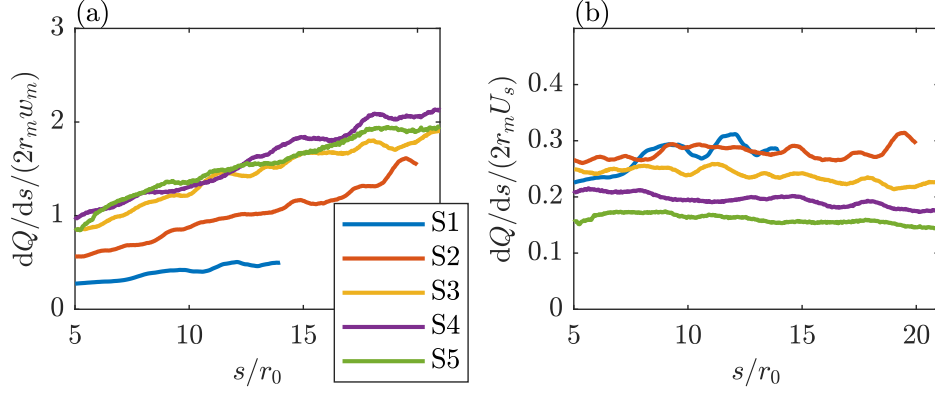


Figure 4.7: Entrainment parameterisations. (a) $\gamma = dQ/ds/(2r_m w_m)$. (b) $\gamma_U = dQ/ds/(2r_m U_s)$.

the plume expands of new ambient fluid with non-zero velocity. Schatzmann (1978) argued that this term should not be considered entrainment as it is not caused by turbulence but rather by the convection of the ambient fluid. We may nevertheless assume that the Leibniz term is the dominant contribution to the volume flux, implying that dQ equals Lds . As the local plume angle is φ , the height increment of the plume is $dz = \sin \varphi ds$. Assuming that the plume width is $2ar_m$ where a is an empirical constant, the volume flux increment $Lds = 2aUr_m dz = 2Uar_m \sin \varphi ds$, and thus

$$L = 2aUr_m \sin \varphi. \quad (4.7)$$

Upon equating the expression for L with the entrainment ansatz (4.6), we obtain

$$\gamma_U = a \frac{U}{U_s} \sin \varphi = a \frac{U}{U_s} \frac{dz_c}{ds}. \quad (4.8)$$

where a is a parameter. As can be seen from figure 4.9(b), a is bounded

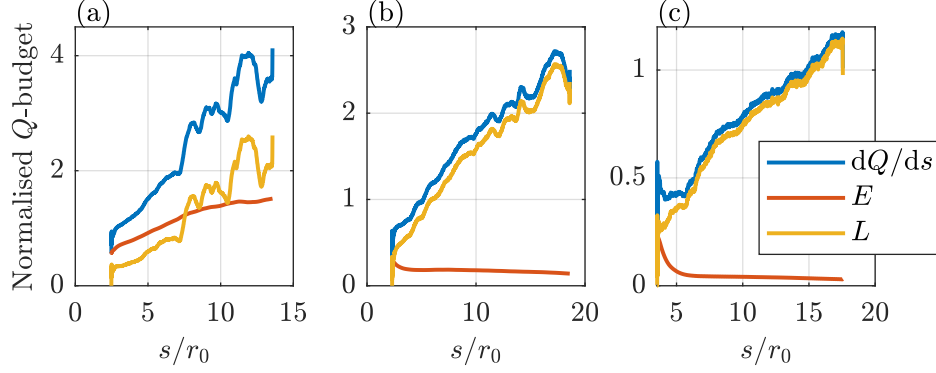


Figure 4.8: Budget of the integral continuity equation for simulations S1, S3 and S5, normalised by Ur_0 .

between 0.5 and 1. We will use the value $a = 0.9$ here.

From the analysis in the previous section, we know that

$$\frac{dz_c}{dx} \approx \frac{F_z + F'_z}{F_x + F'_x} = \frac{F_z/F + F'_z/F}{F_x/F + F'_x/F} \approx \frac{\sin \varphi_u + \theta_f}{\cos \varphi_u} \quad (4.9)$$

where $\tan \varphi_u = w_m/(U + u_m)$, $\theta_f = F'_z/F$. In obtaining the expression above, it was assumed that $F_x/F \approx \sin \varphi_u$ and $F_z/F \approx \cos \varphi_u$ (the velocity and mean buoyancy flux have approximately the same angle) and that $F'_x \ll F_x$. The ratio of the turbulent buoyancy flux to the mean buoyancy flux F'_z/F_z was denoted θ_f in van Reeuwijk & Craske (2015), which is what will be used here.

In the expression above the only term that needs to be parameterised is θ_f , since the velocity components will be known as part of the integral plume model. In general, we expect that F'_z will depend on the initial conditions Ri_U and the bent-overness of the plume $R = w_m/U$. A detailed analysis of

the DNS data shows that θ_f can be modelled as:

$$\theta_f = \theta_{fb} + (\theta_{fp} - \theta_{fb}) \left(1 - \exp \left[- \left(\frac{Ri_U}{Ri_{U0}} \right) \right] \right); \quad (4.10)$$

where $\theta_{fp} = 0.20$ is the value for the pure plume case ($Ri_U \rightarrow \infty$) and $\theta_{fb} = 0.06$ is the value for the bent-over case ($Ri_U = 0$). The cross-over parameter Ri_{U0} is a constant which determines how the two regimes compare. Empirically, we find this value to be 1.2. For the cases under consideration, θ_f is predominantly determined by the initial condition Ri_U and not R . It would stand to reason that at some point far downstream of the source, the plume will forget about its origins and θ_f scales with R , but this is not observed in the limited downstream range simulated here. Further work is needed to provide a more complete model of θ_f .

Figure 4.9(a) shows the actual (solid line) and predicted (dashed line) plume-centreline slope dz_c/dx for all five simulations. As can be seen, the parameterisation predicts the slopes reasonably well, perhaps with exception of the slope for S1 which is underpredicted. The prediction for γ_U is shown in 4.9(b). The entrainment coefficient γ_U is predicted well for simulations S3-5 but is overpredicted for the weaker crossflow simulations S1-2. This is due to the fact that the entrainment model was formulated using the assumption that the Leibniz term $\mathcal{L}$ dominates over the radial entrainment term E . This assumption is only valid for plumes in a strong crossflow $Ri_U \ll 1$, so the overprediction of γ_U at low flowspeeds is to be expected. Given that the prediction for S3 is accurate, we postulate that our model is valid for $Ri_U \lesssim 1$

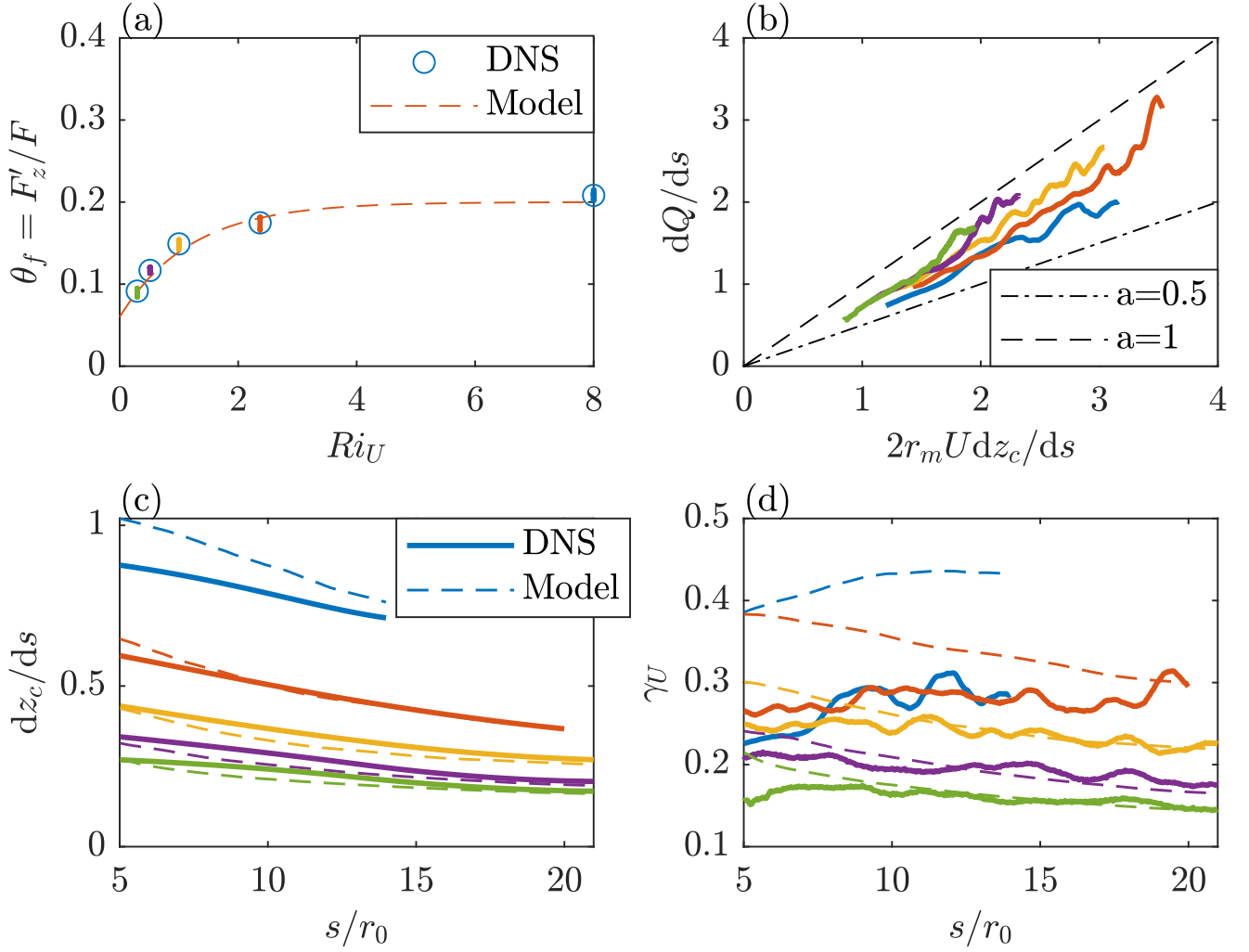


Figure 4.9: Entrainment parameterisation. (a) θ_f as a function of Ri_U . (b) Entrainment flux. (c). Slope dz_c/ds . (d) γ_U . Color scheme consistent with previous figures. DNS data and the parameterisation are shown in solid and dashed lines, respectively.

4.3 The role of pressure in the momentum balances

In this section, the behaviour of the momentum balances (2.14b)-(2.14c) are explored. Figure 4.10 shows the momentum budget in the x -direction (Figs a-c) and the z -direction (Figs d-f) for simulation S1 (column 1), S3 (column 2) and S5 (column 3). The entrainment and Leibniz terms have been combined, since it has been established that $\mathcal{L}$ is the dominant term for bent-over plumes. Figures 4.10(a-c) show clearly that M_x is heavily influenced by the integral pressure gradient P_x , and somewhat influenced by the entrainment and Leibniz terms representing additional momentum being incorporated from outside the plume.

The budget of the integral vertical momentum flux is clearly dominated by the buoyancy term which is the only term that most plume models incorporate (see chapter 2). However, the integral pressure gradient term also plays a key role in this balance and acts to counteract a large proportion the buoyancy term. Integral models that do not take into account this term can therefore be expected to overestimate the vertical velocity inside the plume. The Leibniz terms are negligible.

In order to gain further understanding into what underlies the observed behaviour of P_x and P_z , we display the pressure in the $\eta - y$ plane in figure 4.11 together with the velocity vectors. Here, the cross-flow contribution $U \sin \varphi$ has been subtracted from the velocity vectors to emphasise the net circulation inside the plume (it would otherwise be dominated by $U \sin \varphi$).

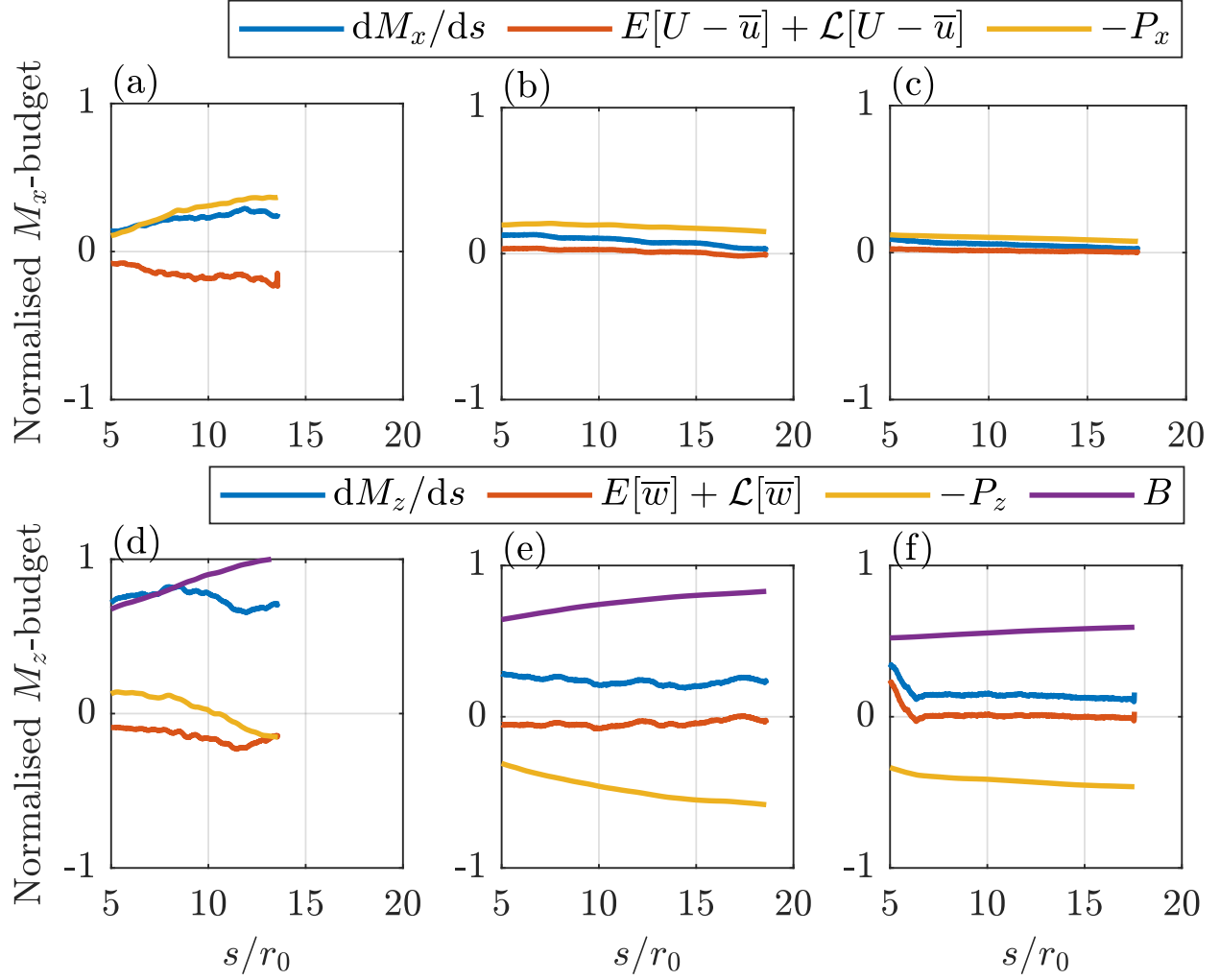


Figure 4.10: Budgets for the integral momentum equations in the x -direction (a-c) and z -directions (d-f), normalised by U^2/r_0 . (a,d) simulation S1. (b,e) simulation S3. (c,f) simulation S5.

The corrected velocity vectors display the canonical two-vortex geometry of a turbulent plume in a cross-flow. At the windward side, a positive pressure is observed whilst there is a low pressure region on the leeward side. This pressure arrangement causes the horizontal integral pressure gradient forcing $-P_x$ to be positive and thus acts to accelerate the plume, whilst it causes $-P_z$ to be negative and thus acts to oppose the buoyancy forcing. Large negative pressures can be observed in the core of the two vortices. The pressure field also demonstrates that the origin of the two-roll structure of plumes in cross-flows is the high pressure area on the upstream side of the plume.

The observations above make clear that a model is required for the pressure terms P_x and P_z . Here, the momentum equation in the η -direction is a convenient starting point, since this balance can be expected to be dominated by buoyancy and pressure. The integral momentum flux in the η -direction is given by $M_\eta = M_z \cos \varphi - (QU + M_x) \sin \varphi$. Differentiating this expression with respect to s results in

$$\frac{dM_\eta}{ds} = \frac{dM_z}{ds} \cos \varphi - \left(\frac{dQ}{ds} U + \frac{dM_x}{ds} \right) \sin \varphi - M_s \frac{d\varphi}{ds}, \quad (4.11)$$

where $M_s = M_z \sin \varphi + (QU + M_x) \cos \varphi$ is the streamwise momentum flux. Now, substituting (2.14a)-(2.14d) results in

$$\frac{dM_\eta}{ds} = -P_\eta + B \cos \varphi - \mathcal{L}[1]U \sin \varphi - M_s \frac{d\varphi}{ds}, \quad (4.12)$$

where $P_\eta = P_z \cos \varphi - P_x \sin \varphi$ is the integral pressure gradient in the η -

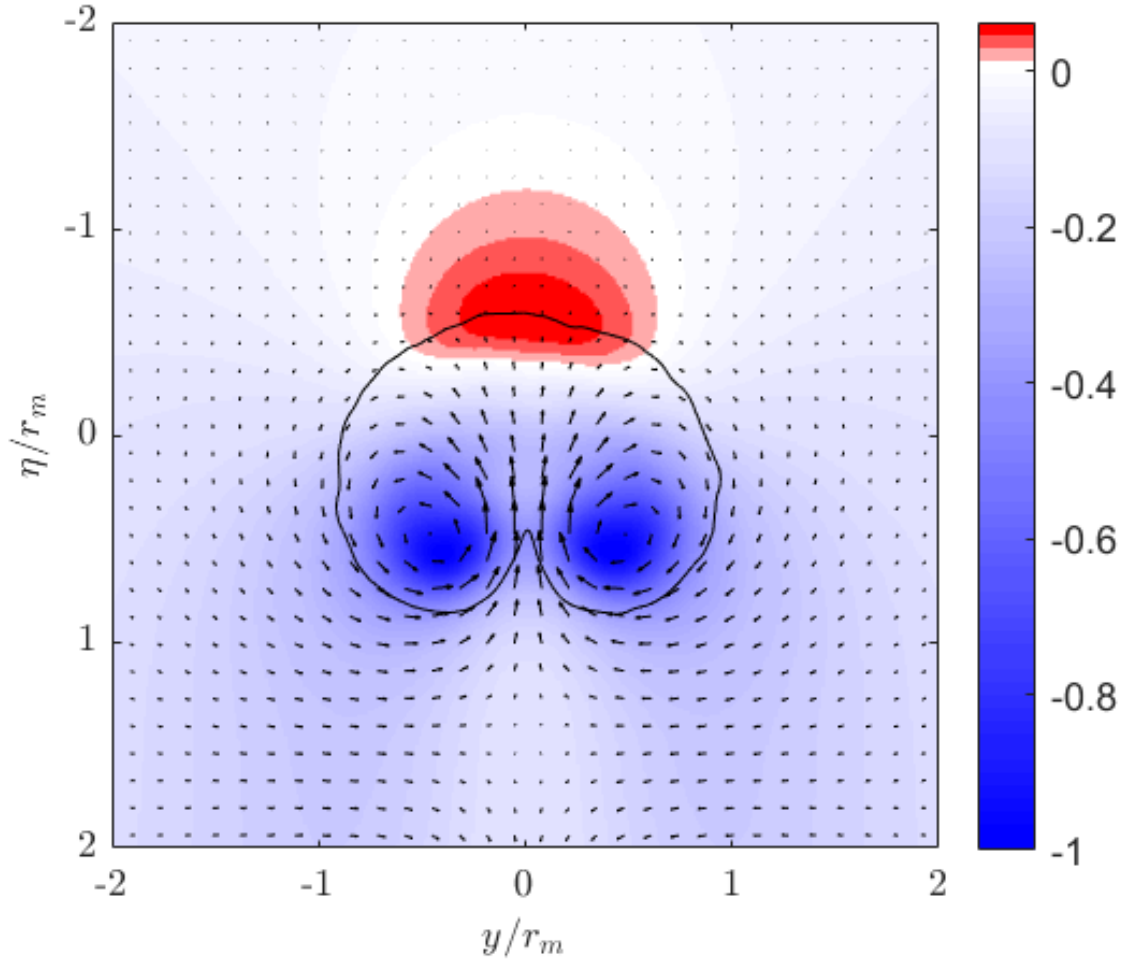


Figure 4.11: Pressure in a plane perpendicular to the plume at $s/r_0 = 3$ for simulation S3. The crossflow has been subtracted from the wind velocity vectors, revealing the vortical structure in the velocity surpluses. The pressure has been normalised by it's maximum absolute value.

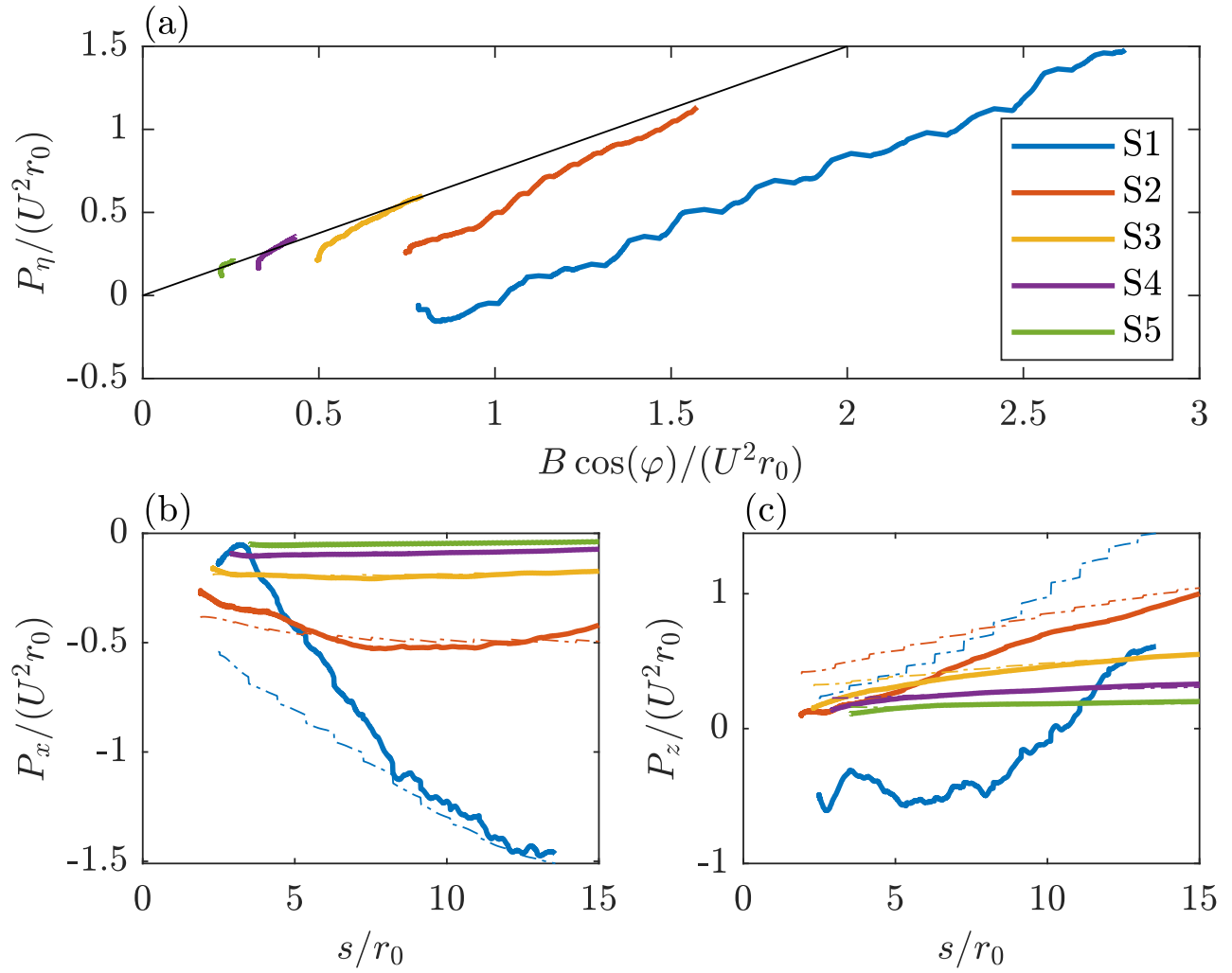


Figure 4.12: Pressure parameterisation. a) the relation between P_η and B . The solid line is the relation $P_\eta = cB \cos \varphi$, with $c = 0.8$. b) Comparison of simulations and model predictions for P_x . c) Comparison of simulations and model predictions for P_z .

direction. In the equation above, entrainment contributions in the momentum equations were neglected, consistent with the observations in §4.4. For the x-momentum equation this is justified by assuming that $M_x \ll QU$. Furthermore, the fluctuating terms were neglected.

We expect that the main balance of forces in the η -direction is between pressure and buoyancy and will assume that all the other terms act in proportion to the buoyancy, suggesting a model as follows:

$$P_\eta \sim B \cos \varphi. \quad (4.13)$$

The relation between P_η and $B \cos \varphi$ is plotted in Figure 4.12(a), and shows that these variables are strongly correlated as expected. The parameterisation $P_\eta = cB \cos \varphi$ with $c = 0.8$ fits all simulations reasonably well, particularly the most bent-over cases S3-S5. For the lower windspeeds, more sophisticated models will be needed. Upon assuming that the integral streamwise pressure gradient P_s is negligible compared to P_η , P_x and P_z are given by

$$P_x = -P_\eta \sin \varphi = -cB \cos \varphi \sin \varphi, \quad (4.14)$$

$$P_z = P_\eta \cos \varphi = cB \cos^2 \varphi. \quad (4.15)$$

These relations are plotted in Figure 4.12(b,c) for all simulations and it can be seen that it captures the pressure contributions well, particularly those for S3-S5, where it is hard to distinguish between the simulation data and the model prediction. For simulation S1, the horizontal pressure gradient is predicted remarkably well, but the prediction for the vertical pressure

gradient P_z is off to the extent that even the sign is incorrect. This implies that the pressure model presented here is incapable of representing plumes in weak cross-flows.

4.4 Integral plume model

In this section, existing integral models are compared to the model with the parameterisations developed here. To be precise, the governing equations for the new model are

$$\frac{dQ}{ds} = \mathcal{L}, \quad \frac{dM_x}{ds} = -P_x, \quad \frac{dM_z}{ds} = -P_z + B, \quad \frac{dF}{ds} = 0,$$

with entrainment closure (4.6, 4.8) and pressure closure (4.15). The coefficients used are $a = 0.9$ and $c = 0.8$. The DRWT model uses parameters $\alpha = 0.1$, $\beta' = 0.5$ and $n = 3/2$ and as stated earlier, the added-mass coefficient $k = 0$. Figure 4.13 shows the comparison for simulations S1, S3 and S5 between the DNS data (thick black line), the DRWT model (green line), the model with new pressure and entrainment parameterisations (NM; blue line), the new model without horizontal pressure (NM-EPz; blue dashed line), and the new model without the pressure terms (NM-E; blue dash-dotted line).

The first thing to note is that z_c is much better predicted with NM than with the DRWT model for the high cross-wind cases. This is directly associated with the parameterisation of the vertical pressure term P_z . Indeed, neglecting the horizontal pressure term (NM-EPz) can be seen to have very

similar behaviour to the NM model. Upon neglecting both pressure terms (NM-E), the model behaves very similarly to the DRWT model. The fact that P_x is dynamically insignificant can be understood from the fact that it appears in the M_x -equation, which represents the momentum surplus. In the integral equations M_x appears only in terms of its contribution to M_s , the integral streamwise momentum flux. Thus, it always appears in a term of the form $(QU + M_x)^{1/2}$, but $QU \gg M_x$, so the contribution from this term is dominated by QU . It is therefore unsurprising that M_x may be dynamically neglected. It should be noted that the DRWT model makes this assumption by default.

The DRWT model performs very well in terms of the evolution of r_m . NM can be seen to overpredict r_m for the weak-crossflow case S1, but to accurately predict r_m for the higher-crossflow cases S3 and S5. Here, it should be noted that the DRWT entrainment model uses β' as its bent-over-limit entrainment coefficient, so appropriate scaling of r_m should be expected. For the volume flux Q , the DRWT performs an excellent job. Perhaps surprisingly, the NM-E model – despite doing an excellent job at predicting the entrainment – severely overpredicts Q for all simulations. Once more, this highlights the importance of the pressure terms, as the inclusion of pressure (NM-Pz and NM) results in excellent predictions for S3 and S5. In the weak-crossflow case S1, the volume flux is overpredicted, demonstrating that the model needs to be augmented with a weak-crossflow parameterisation in future work.

The equation for M_z provides the key to understanding the disparities in the plume trajectories. It can be seen that M_z is dramatically overestimated by

the DRWT model, as well as the NM-E model for cases S3 and S5. Since $w_m = M_z/Q$, this leads to an overestimation of the plume angle ϕ_u . A model for the integral vertical pressure difference over the plume, as given in equation 4.15, is essential in order to capture the damping effect that pressure has on the increase of vertical momentum. With the inclusion of P_z , we can see a dramatic improvement over the DRWT model for S3 and S5. Consistent with earlier observations, NM fails to predict the behaviour of the integral quantities at low windspeeds. At these windspeeds, the pressure term becomes much less significant in reducing the upwards momentum of the plume. The plume is accurately modelled by the DRWT model in the lower crossflow speed case, so a further improvement of the model could be the amalgamation of the two models, but that is beyond the scope of this thesis.

Schatzmann (1978) argued that the Leibniz term should not be considered entrainment as it is not caused by turbulence but rather by the “convection” of the ambient fluid. This argument can be defended from the point of view of conserving the classical notion of entrainment at that time, i.e. of ambient fluid flowing into a turbulent free shear flow as is the case for turbulent jets and plumes without crossflow. However, it has become clear that this definition of entrainment is incomplete. Indeed, the Leibniz term is another process through which entrainment can occur. It is driven by the increase of the plume-surface area which is due to turbulence. Furthermore, there is an entire class of problems in which the turbulence interface moves without fluid flowing into it. A canonical example is the atmospheric boundary layer (Kato & Phillips 1969; Deardorff *et al.* 1980; Sullivan *et al.* 1998; Jonker *et al.* 2013),

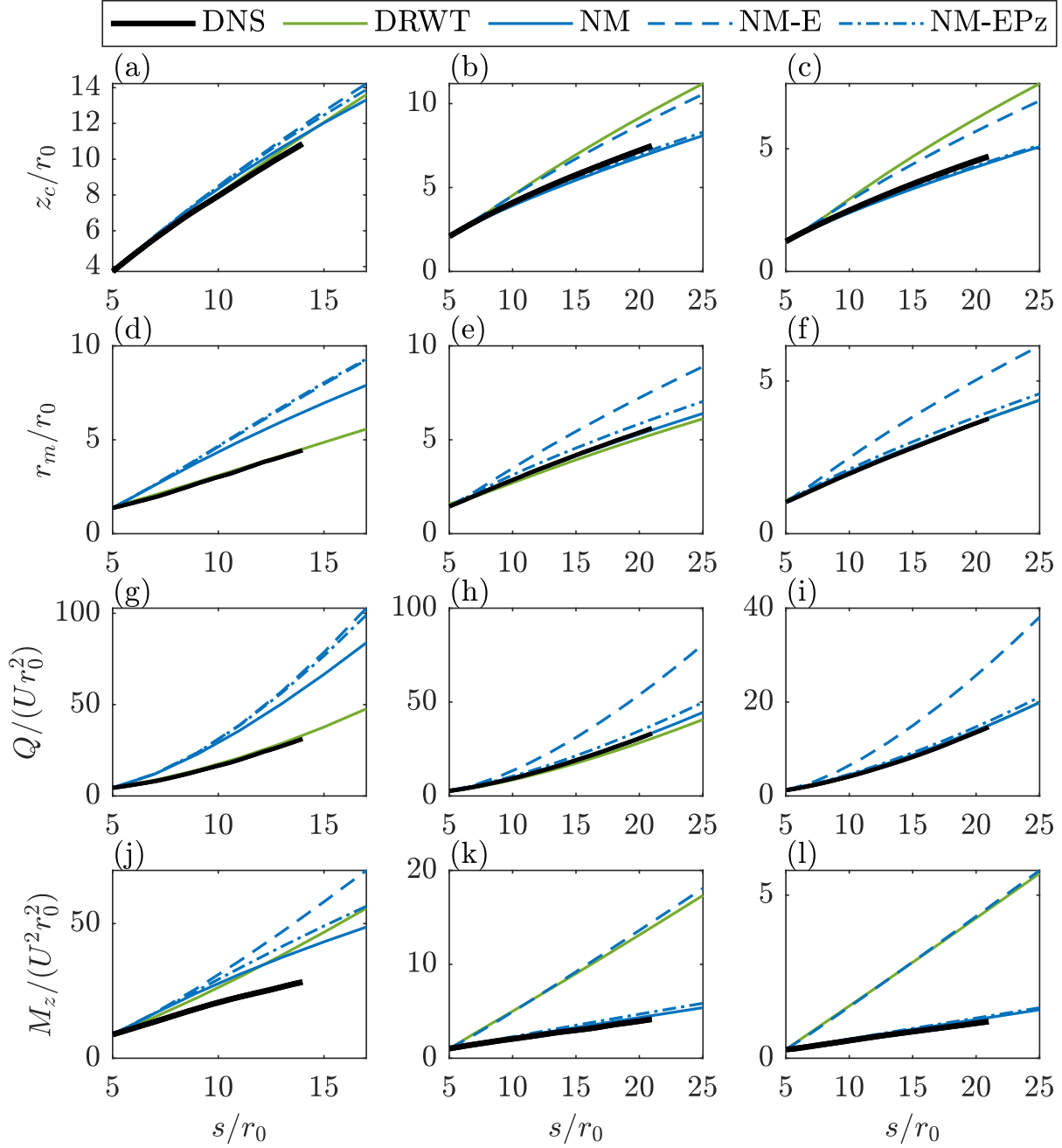


Figure 4.13: Comparison of DNS data for simulations S1 (a,d,g,j), S3 (b,e,h,k) and S5 (c,f,i,l) with the DRWT plume model (DRWT), the new model (NM), the new model without pressure terms (NM-E), and the new model without the horizontal pressure term (NM-EPz). (a-c) plume centreline z_c . (d-f) plume radius r_m . (g-i) volume flux Q . (j-l) vertical momentum flux M_z .

where the turbulence created by heating of the surface causes deepening of the mixed layer in time. Entrainment is defined as the net exchange across an interface, which is the relative velocity between the interface and the fluid. In classical problems, the interface is (statistically static) and the fluid is moving in, in temporal problems, the interface is moving without actual fluid movement. Both describe the same process. Other examples of temporal problems include temporal jets, plumes, gravity currents, mixing layers and wakes (e.g. Da Silva & Pereira 2008; van Reeuwijk & Holzner 2014; Krug *et al.* 2017; van Reeuwijk *et al.* 2018, 2019; Redford *et al.* 2012; Watanabe *et al.* 2016, 2018; Jahanbakhshi & Madnia 2018). A unified description of entrainment which brings together all these different entrainment processes into a single framework has recently been proposed (Van Reeuwijk *et al.* 2020).

Chapter 5

Conclusion

In this thesis we performed and analysed a set of Direct Numerical Simulations of turbulent plumes in a crossflow for a range of Ri_U between 0.3 and 8.0. A striking finding was that the plume angle is not only determined by the ratio of vertical to horizontal velocity as is traditionally assumed, but crucially depends on the vertical turbulent buoyancy flux. This overturns one of the cornerstones of many plume models. Indeed, it was shown that the streamlines enter the plume at its windward side, and have a different angle to the plume centreline. Instead, it is the streamlines of the *total* (mean + turbulent) buoyancy flux that align with the plume centreline. This has important implications for the plume trajectory predictions and particularly impacts the predicted rise height of the plume.

Detailed analysis of the entrainment into the plume showed that the Leibniz terms dominate for strong crossflow plumes. Furthermore, it was shown that

standard entrainment closures do not accurately represent the entrainment flux and a new closure was presented. However, in comparing integral plume models to the DNS data, it was observed that the entrainment flux was in excellent agreement for the classical models, suggesting that the coefficients are tuned to correct for other unrepresented terms in the integral equations (e.g. pressure.).

Through a scrutiny of the integral momentum budgets it was shown that some existing integral models significantly and systematically underestimate the horizontal momentum flux excess and overestimate the vertical momentum flux. The reason for this is that without an added-mass term, pressure differences across the plume cross-section are neglected in the plume equations. However, there is a significant, high pressure region at the leading edge of the plume which cannot be ignored when computing momentum budgets. This high pressure region has the effect of damping the vertical momentum flux by exerting a downward force on the plume, whilst simultaneously exerting a force in the horizontal direction. The former leads to reduced vertical plume velocity and the latter leads to a horizontal plume velocity greater than the free stream velocity. This is a clear mechanism by which the work done by the buoyancy flux can generate a horizontal momentum flux.

Parameterisations were developed for the turbulent vertical buoyancy flux, the entrainment and the integral pressure. It was shown that the new model was able to reproduce the plume trajectories much better than existing models for the sufficiently strong cross-flows ($Ri_U \lesssim 1$). It was furthermore shown that the horizontal pressure contribution could be safely neglected

and that the secret to the success of the model was the combination of the new entrainment model and the vertical pressure parameterisation.

Further work is needed to provide insight into the accuracy of the new parameterisations in other parameter regimes. There are many possible extensions, including plumes with nonzero exit velocities, plumes in stably stratified environments, plumes with linear wind shear and plumes in turbulent atmospheric conditions. DNS of these different parameter scenarios would all be valuable additions to our current knowledge of plumes in a crossflow.

A further line of enquiry would be the consideration of a Lagrangian/particle-tracking method, which would give additional insight into the turbulent/non-turbulent interface in these flows. This would allow a more detailed interrogation of the methods by which fluid is entrained into the plume, and may provide further understanding of the entrainment and Leibniz terms.

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