

Floating body hydrodynamics: A load driven approach

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Philosophy in Civil Engineering of Imperial College London and the Diploma of
Imperial College London

To my loving family,

Declaration of originality

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Abstract

This thesis concerns a fundamental study of the loading and dynamic response truncated and domed cylinders. A potential application of the work is in the design of wave energy converters; therefore the body geometries chosen are of a similar size to such devices. Using a combination of a nonlinear potential flow model and an extensive experimental programme, nonlinear aspects of the body response in long irregular sea states have been established. Two primary sources of nonlinearity arise: the first due to nonlinear wave-structure interaction, and a second due to vortex shedding dissipation. The quantification of the relative importance of the two sources in the loading and dynamic response of the body is one of the principal achievements of this study.

The competition between these two effects is studied in the context of regular waves and irregular sea states. The regular wave investigation provides a detailed explanation of nonlinear wave-structure interaction using a newly-developed potential flow model, ICRD3D, in order to inform the results arising from the realistic irregular sea state experiments. Each of a number of irregular sea states captured in excess of 15,000 individual waves, resulting in a total data set of approximately 850,000 wave events.

The statistical distributions arising from the irregular sea state experiments demonstrate that nonlinear wave-structure interaction, the first source, dominates in the excitation problem. The nonlinear wave-structure interaction leads to amplifications in (absolute) force minima in excess of the linear prediction, whereas the force maxima are smaller than linearly predicted. This maxima-minima asymmetry arises from the set-down present under the largest incident wave events and increases with the underlying sea state steepness. The second source, due to vortex shedding dissipation, does not significantly affect the excitation problem, but is of crucial importance in the dynamic response. The effects are pronounced at the body resonance frequency, leading to reduced displacements with increasing sea state steepness despite the nonlinear force amplification. This second source is further examined by first considering a truncated cylinder with sharp corners, followed by a domed cylinder with a hemispherical base. The vortex shedding dissipation is much reduced in the latter case.

Perhaps of most importance to design practice is the specification of ‘design conditions’. The maximum body displacements were not found to arise from the largest wave crests. Instead, a wave group in which a large wave crest and trough precede the maximum surface elevation cause a gradual build of energy that permits the maximum dynamic response of the body.

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Nomenclature

g	Acceleration due to gravity
λ	Wavelength
KC	Keulegan-Carpenter number
ω	Wave circular frequency
ω_p	Spectral peak circular frequency
Φ	Total time-dependent velocity potential
ϕ	Total time-independent velocity potential
ϕ_D	Diffracted velocity potential, $\phi_I + \phi_S$
ϕ_I	Incident velocity potential
ϕ_R	Radiated velocity potential
ϕ_S	Scattered velocity potential
ρ	Density of water
θ	Azimuthal coordinate
ξ	Body displacement
a	Body radius
A_I	Incident wave amplitude
b	Body draught
C_D	Vortex-shedding damping coefficient
D	Body diameter
d_r	Hydrodynamic radiation damping coefficient
F	Total excitation force, $F^{(1)} + F^{(2)}$
f	Source strength
$F^{(1)}$	Total first-harmonic excitation force

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$F^{(2)}$	Total second-order excitation force, $F_q^{(2)} + F_s^{(2)}$
$F_q^{(2)}$	Second-harmonic excitation force due to quadratic products of linear potential
$F_s^{(2)}$	Second-harmonic excitation force due to the second-harmonic potential
G	Green's function
h	Water depth
H_s	Significant wave height
K	Deep water wavenumber, ω^2/g
k	Wavenumber
k_p	Spectral peak wavenumber
m	Fourier mode
m_r	Hydrodynamic added mass coefficient
n	Normal direction coordinate
Q	Probability of exceedance
r	Radial coordinate
T	Wave period
t	Time
T_p	Spectral peak period
T_r	Repeat time in irregular sea state
x, y, z	Cartesian coordinate directions

1

Introduction

1.1 Motivation

The deployment of structures in the offshore and coastal environments has long been of importance for human progress. The earliest example is, perhaps, the design of seaworthy ships to facilitate cross-continental trade and remains a key driver of the global economy. Later, the search for oil has resulted in a wide variety of structures located offshore, with such structures contributing to as much as 30% (U.S. Energy Information Administration, 2016) of crude oil production in 2015. Installations have been both fixed, such as in the North Sea, or floating in the Gulf of Mexico. More recently, the oceans are being seen as an important source of renewable energy by extracting wave energy or by installing fixed and floating wind turbines offshore.

Despite significant design and operational experience in the offshore engineering sector, a number of important areas are not fully explored or understood. One such area concerns the interplay between nonlinear wave mechanics and vortex shedding in cases where both the fluid particle excursion and the structure's motion excursion are large. This is, perhaps, one of the most challenging offshore engineering issues. Taken independently, nonlinear wave-structure interactions have the tendency to increase the dynamic response of a structure (compared to commonly adopted linear solutions), whereas vortex shedding damps the motion excursions. The relative contribution of these two phenomena is not currently well described for realistically sized highly-dynamic structures. Such structures could include wave energy converters (see Babarit, 2015 for examples), which generally

extract more energy with increasing displacement.

The strike price for wave energy currently stands at £305 MWh (DEC, 2013), triple that of onshore wind and double that of offshore wind. Therefore, for wave energy to be competitive, at least in the European continental electricity market, a significant cost reduction is required. This necessary cost reduction will likely come in large part from the convergence of a particular design and is being pursued through competitions such as the U.S. Department of Energy Wave Energy Prize and structured programmes for research and development by Wave Energy Scotland. Once the technology has been chosen, an improvement in engineering design and analysis could prove crucial to the overall success of such devices.

The primary motivation of this PhD is the detailed understanding of wave-structure interaction phenomena for highly dynamic structures. As an additional contribution, the structure sizing is consistent with wave energy devices with the intention that the results are informative to researchers and developers alike.

TO BE REMOVED

Given the great power observed in the oceans, it is perhaps unsurprising to find the first patent filed for a wave energy converter as far back as 1799 (Cruz, 2008). During the 1950s and 1960s, a substantial effort was then made by a Japanese team led by the “grandfather” of wave energy conversion (WEC) Yoshida Masuda (Falcão, 2010), resulting in an inexpensive buoy that generated its own energy for lighting safe routes through harbours (Charlier & Justus, 1993). Unfortunately, the buoy had an insufficient generation capacity for commercial installations (Charlier & Justus, 1993) and the ready availability of oil and other fossil fuels provided a low-cost means for the world’s energy production.

The 1973 oil crisis, triggered by political machinations in the Middle East, saw the quadrupling of the oil price in the space of six months (US Department of State, 2013) and spurred both the academic and commercial interest in wave energy devices for large-scale production. The most famous example of this time is the “Salter Duck” that both proved the potential of wave energy technology and raised the worldwide awareness of waves as an energy source (Salter, 1974). This interest was then bolstered by the “second oil shock” in 1979 in which the oil price further doubled in a twelve month period. During these boom years for the technology, a significant amount of progress was made in both the the-

oretical development (Evans, 1976, 1981; Budal & Falnes, 1975) and the laboratory testing of devices (Quarrell, 1978). Much of this work is still relevant in the design of WEC devices today.

In subsequent years, however, UK government funding for WEC research was significantly curtailed before any of the early prototypes could be commercially exploited. The appetite for wave energy conversion only remained in the academic community until its inclusion in the 1991 European Commission Research and Development Program under title “Preliminary Actions in Wave Energy R&D” (Falcão, 2010).

More recently, the growing spectre of climate change has prompted governments around the world to enact legislation aimed at reducing fossil-fuel energy dependence. Simultaneously, an increase in available funding has stimulated the development of a plethora of devices (Falcão, 2010; Babarit *et al.*, 2012), many of which are being developed commercially.

To date, none of the technological advances made to WEC devices has translated into commercial success. Unfortunately, some of the most promising developers are no longer active following liquidation, such as the Pelamis (BBC, 2014) and the Aquamarine Oyster (Aquamarine, 2015) devices. This uncertainty in the technology is reflected in the UK government renewable energy strike prices in which the price of wave power at £305/MWh is triple that of onshore wind and double that of offshore wind (DEC, 2013). There is no doubt that wave energy conversion, in principle, provides an efficient means of generating electricity, but the lack of commercial competitiveness is hampering its large scale adoption.

To reduce the cost of the technology, a number of factors come into play. These include both engineering and wider aspects (such as economic, financing, legislation, marine conservation), with the present thesis focusing on the core engineering discipline. To improve the performance of WEC devices, the wave-induced loading and associated dynamic response must be known in detail. Hitherto, much of the research in the offshore industry has predominantly focused on fixed structures, typically found in areas such the Gulf of Mexico or the North Sea. More recently, as oil and gas exploration platforms have moved into deeper water, structures such as tension-leg platforms and spars have been investigated; these structures, whilst floating, are generally much larger than a WEC device and, crucially, are designed in such a way to *minimise* their dynamic response. The motion and

dynamic response of WEC devices, on the other hand, must be *maximised* (or optimised) in order to extract as much power as possible.

In order to assert the commercial viability of WEC devices, the hydrodynamic analysis undertaken by developers has typically focused on power capture characteristics and on arrays using simplified solutions. These solutions most commonly rely on linear potential flow assuming small amplitude waves and device motions. This assumption is in direct confrontation with the large WEC motions required to generate the most power from the waves. Indeed, Rodríguez *et al.* (2016) have shown that, in the two-dimensional case, adopting a higher-order solution can lead to significant amplifications in body motions compared to the linear solution. Therefore adopting a higher order solution ought to improve the power capture estimates of a particular device.

Another key missing element in WEC device design is rigorous design guidance. The standards organisations are developing technical specifications for WEC development and design. Some of these specifications are now available, while others will not be available for some time, and only their day-to-day application will prove their suitability. The International Electrotechnical Commission TC114 is one such example. There is therefore a gap in knowledge of the “design conditions” of WEC devices—something in marked contrast to the oil and gas industry which has standards defining the precise analysis to be undertaken. Furthermore, load factors currently used in the design of floating offshore structures (taken primary from the oil and gas industry) must take into account the overwhelming priority of personnel safety, resulting in necessarily over-conservative force predictions. On the contrary, WEC devices are unlikely to be staffed thereby lessening the consequences of failure. Moreover, WEC installations usually comprise many devices so the loss of some of the devices may be acceptable. As a corollary, both the lack of a rigorous definition of the design conditions and the analysis method—design guidance—may be artificially increasing the cost of the technology and reducing its attractiveness to developers.

1.2 Research aims and objectives

As detailed above there are two principal topics currently lacking in the existing body of literature:

- i. A detailed understanding of the nonlinear wave-structure interaction of highly dy-

namic devices from a *physical* perspective, and

- ii. Design Guidance associated with the “design conditions” and solution methods appropriate to WEC devices.

To address these shortcomings, this thesis develops improved solutions and guidance by adopting the following intermediate steps:

- i. Develop a robust nonlinear radiation-diffraction solver for the floating body problem,
- ii. Undertake a detailed investigation into the dynamic floating body problem, specifically quantifying the consequences of adopting a nonlinear approach in both regular and irregular waves,
- iii. Commission an experimental investigation to determine the limits of both linear and nonlinear potential flow and to discover other physical phenomena that contribute to nonlinear loading,
- iv. Contribute and provide comparison data towards the development of fully nonlinear potential flow solvers and CFD, and
- v. Condense the findings from the above objectives into practical guidance for WEC design.

1.3 Thesis layout

This thesis is divided into seven main chapters:

Chapter 2 provides a review of the principles of wave-structure interaction and an exposition of the chosen body geometry. A critical assessment of the various analytic, semi-analytic, and numerical methods is undertaken, along with identifying relevant experimental studies. This assessment includes both regular wave and irregular sea state scenarios and highlights gaps in the existing knowledge. The chapter finishes with a description and validation of the semi-analytic model employed throughout the thesis.

Chapter 3 uses the semi-analytic model described in Chapter 2 to explore the excitation, forced radiation and floating body problems in regular waves. The focus of the chapter is the identification and physical explanation of the second-harmonic interactions that arise in the loading, and how these translate into the dynamic response of a truncated cylinder.

Chapter 4 concerns an experimental study of the heave excitation problem in irregular

sea states of varying steepness and peak period. The key features of the novel experimental setup are outlined. The nonlinear contributions are quantified using the force statistics, arising from 25,000 individual events, via comparisons to linear predictions. Again, the physical origin of these departures is described. Finally, the effect of the body draught is considered.

Chapter 5 experimentally examines the dynamic response of the truncated cylinder previously considered in Chapter 3 (nonlinear computations) and Chapter 4 (fixed excitation problem). Particular attention is paid to the competition of the nonlinearities arising from the wave-induced forcing and from vortex shedding damping. This is undertaken using the statistics gathered from long irregular sea state experiments. A method for predicting the dynamic response is proposed using results derived from a set of regular wave experiments.

Chapter 6 replaces the base of the truncated cylinder by a hemisphere. The impact of this change on the vortex shedding damping regime is considered in detail. Finally, the chapter presents the wave conditions that lead to the maximum body dynamic response. These observations may be used to specify design conditions appropriate to highly dynamic marine structures.

Chapter 7 outlines the principal achievements and engineering significance of this thesis. Elements for future investigation are also proposed.

2

Background

2.1 Geometry selection

In order to be concrete in the discussion that follows, it is helpful to first consider the body that will be investigated in this PhD. From the outset, general observations and conclusions about the physics driving wave-structure interaction were desired, rather than investigating a specific wave energy device. Therefore a “generic” device geometry which is widely representative has been explored.

This PhD has been undertaken as part of a wider EPSRC SUPERGEN Marine project in which Imperial College has investigated heaving buoy-type devices. The typical dimensions adopted in previous heaving buoy studies—both numerical and experimental—are presented in Figure 2.1, based upon the academic literature and WEC device manufacturer information. The list of references used can be found in Appendix D.

Clearly, the most common shape is a cylinder with a range of diameters between 2 and 20 m. Although a diameter of 4 m is the most prevalent, this is small for a full-scale WEC with a reasonable generation capacity—approximately 1 MW—meaning a larger diameter in the range of 10–20 m will be considered henceforth. Figure 2.1(c) shows a variety of draughts with the draught typically being half, or equal to, the body diameter. Finally, Figure 2.1(d) shows that these buoys typically operate in water depths up to 100 m. With these figures in mind, the “generic” device research presented here can be useful to the WEC community at large without restricting conclusions to a specific design. Such devices could include the PowerBuoy, CETO, Wavestar, Seabased, Oscilla Power, Seabased, and

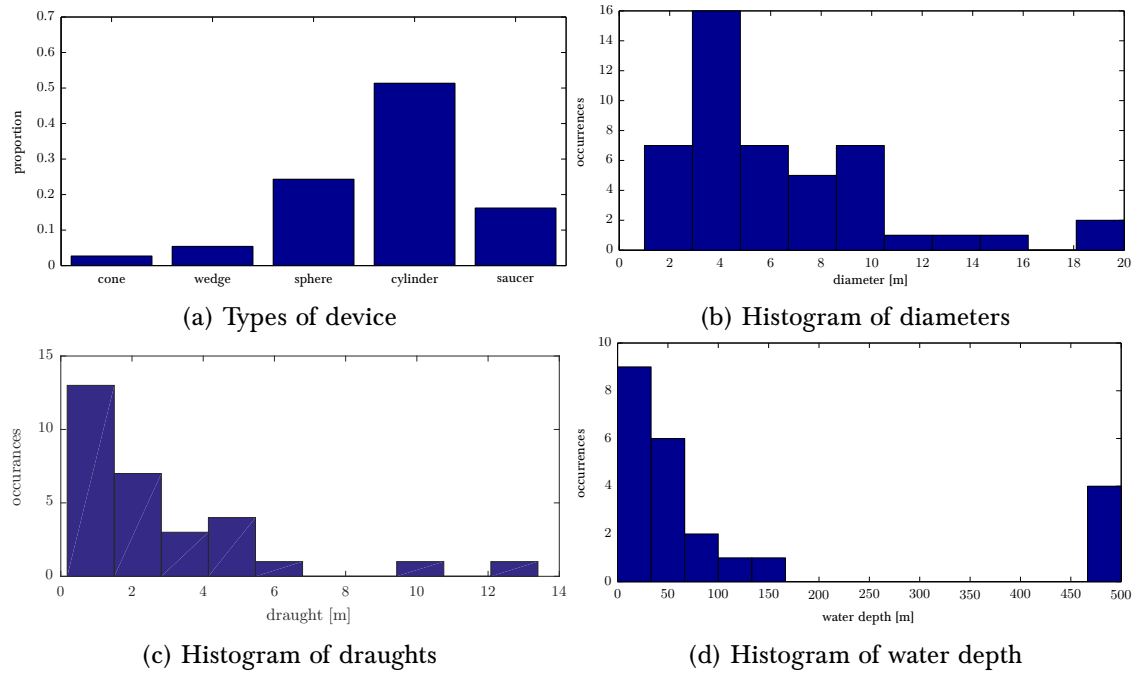


Figure 2.1: Consideration of the parameters of a generic heaving buoy WEC.

BOLT Lifesaver buoys.

2.2 Wave loading regimes

The wave loading regime fundamentally determines the physical processes that cause the loading and is expressed by the relative length scales of the waves, usually the wavelength, and the structure, characterised by the diameter. The classic authorities on offshore loading are Lighthill (1979, 1986), Newman (1977), Ogilvie (1983), and Faltinsen (1993).

When the wavelength is large compared to the diameter, the structure is considered slender, for example individual members on a jacket structure, for which Morison *et al.* (1950)—often referred to as Morison’s equation but hereafter as the MJOS equation—has proven of great utility in the design of offshore structures. The MJOS equation comprises two force components:

1. A drag component as a result of flow separation arising from the pressure gradient on the upstream and downstream faces of the body. The flow separation results in the development of a low pressure wake in the direction of instantaneous fluid motion. This force is proportional to the square of the fluid particle velocity.
2. An inertia component arising from the fluid particle acceleration in an unsteady flow. As the particles are accelerated, there must be a net force in action.

Keulegan & Carpenter (1958) considered the relative importance of these components by defining the Keulegan-Carpenter number as

$$KC = \frac{uT}{D}, \quad (2.1)$$

where u is a characteristic fluid particle velocity, T is the wave period and D is a characteristic dimension of the structure. The ratio expresses the distance the fluid particles travel relative to the structure which drives the formation of the wake. Sarpkaya (1986) notes that it is often assumed that for $KC < 4$ the drag force is considered unimportant for laboratory and ocean-based experiments, though this assumption “miss[es] a number of interesting flow phenomena”. A more thorough description of the MJOS equation can be found in Sumer & Fredsøe (2006).

One of the fundamental assumptions of the MJOS equation is that the kinematics of the fluid are calculated in the *absence* of the structure. A structure that is large compared to the wavelength will disturb the incident conditions to create a scattered, or diffracted,

wavefield, rendering the predictions of the MJOS equation inaccurate. Lighthill (1979) proposes a second-order correction to the inertia term to take account of the body, though this does not seem to have been incorporated into the design standards. Instead, the standards define the limit of applicability of the MJOS equation to be whenever

$$D \leq \frac{\lambda}{5}, \quad (2.2)$$

where λ is the incident wavelength.

The devices under consideration will operate in the near- to mid-shore environment. In this region, wave heights in the range 5-7 m and dominant wave periods of 4–12 s are typical. Under these wave conditions, the geometries considered in Figure 2.1 have

- $4 < KC < 60$ and
- $0.02 < D/\lambda < 0.4$,

indicating that both drag and diffraction forces are likely to be significant and should be taken into account. Furthermore, the wave theory validity chart of Dean (1974) suggests that these wave conditions require a nonlinear solution, at least from the perspective of the incident field, leading to potentially nonlinear forcing and body motions.

2.3 Review of fixed and floating body hydrodynamics

2.3.1 Introduction

The purpose of this section is to highlight the advances made in the research of floating body hydrodynamics over the last century. Despite their age, many of the historic discoveries still prove tremendously useful to investigators nowadays. Many of the themes in this section will be developed in detail in later chapters as well as providing justification for the proposed research methodology.

Arguably, the field started when G.G. Stokes derived closed form expressions of freely propagating free surface waves in his pioneering 1847 paper (Stokes, 1847). Stokes' perturbation expansion technique of the free-surface boundary conditions with the wave steepness—the ratio of wave amplitude to wavelength—under the assumptions of an inviscid, irrotational fluid, forms the bedrock of the vast majority of solutions thereafter. Solutions that retain only the first term in the perturbation expansion are referred to as “linear”

and can be solved either analytically or semi-analytically and are commonly adopted in design practice of WEC devices. Second-order solutions, which retain the first two terms in the expansion, are also common and are typically solved using semi-analytic techniques. Approaches without any approximations to the boundary conditions (fully nonlinear solutions) may only be solved using, often computationally expensive, numerical techniques.

Solutions to the diffraction problem, in which the body is fixed, or the forced oscillation radiation problem, require a solution that

- i. conserves mass and momentum,
- ii. incorporates the kinematic and dynamic free surface boundary condition,
- iii. defines a “radiation condition” in the far field, and
- iv. formulates the boundary condition on the body.

The body boundary condition expresses the notion that there can be no flow through the body. This results in a diffraction problem where the normal fluid velocity on the boundary is zero, and a radiation problem where fluid must have the same normal velocity as the body. The techniques employed to solve these two problems are often very similar, allowing the general discussion that follows.

2.3.2 Solution by eigenfunction expansions

The first solutions to the diffraction and radiation problems considered simple geometries in order to derive an analytic solution to the problem. Cases in which the body geometry aligns with the coordinate system can be readily solved using an eigenfunction expansion approach; where the water depth is constant, this amounts to solving the Helmholtz equation. If a cylindrical polar coordinate system is adopted, the geometry of a vertical, bottom-mounted cylinder is easily described, yielding the linear solution derived first by Omer & Hall (1949), though MacCamy & Fuchs (1954) is more widely known. A solution valid to second order for the diffraction around a vertical cylinder is given by Chau & Eatock Taylor (1992) which involves other challenges discussed in detail later.

The eigenfunction expansion technique is also useful for a geometry that can be divided into regions that align with the coordinate system. A classic example of this geometry is that of a truncated vertical cylinder in finite depth, where the domain can be divided into three regions:

1. upstream of the body;

2. under the body; and
3. downstream of the body.

The eigenfunction expansions appropriate to each region can be matched on the boundaries between each adjoining region. The method appears to have first been used by Takano (1960) in two dimensions and applied in three dimensions in response to the design of an artificial island by Miles & Gilbert (1968). A truncated cylinder immediately opens up the possibility of investigating moving bodies, with Yeung (1981) providing the linear solution for the heave, surge and roll degrees of freedom. Second-order solutions to the radiation-diffraction problem for the case of a truncated cylinder have been obtained by Moubayed & Williams (1994), Gören (1996) and Huang & Eatock Taylor (1996) among others. Finally, and remarkably, Eatock Taylor & Huang (1997) derived a second-order solution for any axisymmetric body using the eigenfunction expansion technique, though semi-analytic solutions were only obtained for the simple geometries considered above.

2.3.3 Solution by integral equations

Another classical solution to the radiation and diffraction problems is to use integral equations formed either by distribution of sources over the body or by using Green's second identity. In the source distribution technique, singular solutions which satisfy Laplace's equation and all boundary conditions except that on the body are placed on the boundary. The source strengths are then determined such that they satisfy the body boundary condition when taken together. It was this technique that was used by Havelock (1940) in the first three-dimensional solution of the diffraction problem. Once the strength of the sources has been found by solving the integral equations, the velocity potential throughout the domain is readily obtained.

The singular solutions are, in fact, a Green's function which can be derived according to the boundary conditions and the governing equation. John (1950) produced the seminal work on these functions in the context of floating body hydrodynamics as he proved the necessary conditions for the uniqueness of the solution. Both Black (1975) and Fenton (1978) built upon John's work and expanded the Green's function in a series of cylindrical harmonics. Crucially, this allows the Green's function to be written in such a way as to reduce the two-dimensional integral equation over the surface to line integrals by exploiting the axisymmetry of the body. This was particularly necessary as the pioneering work of

Milgram & Halkyard (1971) and Garrison & Chow (1972), who did not make such a reduction, was computationally very demanding at the time.

An alternative approach to solving the boundary value problem is to employ Green's second identity. In this case, the potential throughout the domain can be obtained directly without the intermediate step of solving for the sources first. Both approaches are outlined in Linton & McIver (2001, Chapter 4) along with further discussion in §2.4 of this thesis.

A particular problem when adopting integral equation solutions is calculating the Green's function numerically, particularly in proximity to the body. The problem was first highlighted by Fenton (1978) which then prompted Hulme (1983) to investigate a variety of Green's function representations in the context of the linear radiation and diffraction of a hemisphere. Newman (1985), Telste & Noblesse (1986), Liapis (1992) and Linton (1999) all sought to address the problem of the slowly converging Green's function as computational power grew, making numerical methods more attractive.

The body of research adopting the linear approach can be seen to culminate in the commercial radiation-diffraction solver WAMIT which has come to dominate in the design of offshore structures, though a similar code NEMOH (Babarit & Delhommeau, 2015) has recently been released as an open-source project. For small amplitude waves, the linear solution remains valid, though sea states in which WEC devices are expected to operate are rarely linear, particularly if the largest waves are to be modelled.

2.3.4 Consideration of second-order effects

The inadequacy of a linear solution has long been known and significant effort has been devoted to develop second-order accurate theories. The principal difficulty in finding a solution at second order is the inhomogeneous free-surface boundary condition. Independently, Molin (1979) and Lighthill (1979) calculated second-order forces without solving for the second-order potential explicitly, but rather by considering functions of the linear potential only. This so-called "indirect-method" can be used to calculate integrated quantities such as forces, and extended by Molin & Marion (1986) for motions, but cannot be used to calculate second-order pressures and surface elevations.

Finding an explicit solution to the inhomogeneous free-surface boundary condition requires evaluating an integral over the entire free surface of infinite extent. This free-surface integral is highly oscillatory, making its numerical evaluation extremely challeng-

ing. Eatock Taylor & Hung (1987) first tackled the free-surface integral through the use of assumptions about the behaviour of the far-field potential; this allows analytic expressions for the second-order force on a vertical bottom mounted cylinder to be derived. Later, Kim & Yue (1989) and Chau & Eatock Taylor (1992) developed methods relying on a numerical solution in the near field alongside an analytic far-field solution. The treatment of the free-surface integral is discussed further in §2.4.8.

Particularly due to the free-surface integral, the second-order solution is significantly more computationally demanding than the linear solution. It is perhaps unsurprising, therefore, that second-order calculations developed rapidly as computing power became more readily available. Solutions using an integral equation method were implemented by Hunt & Baddour (1981), Kim & Yue (1989, 1990), Chau & Eatock Taylor (1992), Eatock Taylor & Chau (1992), Eatock Taylor & Kernot (1999) and Zang *et al.* (2006) among many others. In the context of a bottom-mounted cylinder, Kim & Yue (1989) state forcefully that

“the second order potential cannot be neglected in favour of quadratic contributions of the linear potential. Double frequency results obtained without accounting for this [second-order] potential will probably be inadequate in all but very specialised cases”.

The pressing question now is whether effects beyond second order are important. Third-order solutions do exist, such as Malenica & Molin (1995) and Teng & Kato (2002), but incur a great increase in complexity (particularly in the free-surface integral) to the extent that the adoption of a fully-nonlinear numerical method may be preferable.

Of all the second-order references mentioned above, only a handful consider the floating body problem, including Kim & Yue (1990), Moubayed & Williams (1994, 1995), though Kim & Yue (1990) do not consider the motions arising at second order. There is therefore a limited understanding of the physics driving the response of a body in conditions where second-order quantities are deemed to be an important part of the overall problem.

2.3.5 Fully-nonlinear solutions

The solution methods mentioned above are solved in the frequency domain which allows an analytic solution for particular geometries, or semi-analytically in the more general case. The limitation, however, is the extent of nonlinearity that can be incorporated. In

contrast, solutions in the time domain can be fully nonlinear, which usually requires extensive computational resources, particularly in three dimensions. A popular example is the boundary element method which has been applied to the radiation-diffraction problem by Teng & Eatock Taylor (1995), Ferrant (1997), Liu *et al.* (2001), Bai & Eatock Taylor (2006), Zang *et al.* (2006) and Zhou *et al.* (2013, 2015). Unfortunately, many of these contributions are concerned with the numerical methods rather than exploring the physical explanations of nonlinear forcing. Such physical explanations could, for example, be achieved by validating nonlinear models against the linear solution, demonstrating where the latter becomes invalid. Explanations of this type are rarely reported, and no systematic comparison between linear and nonlinear floating body solutions is available.

The finite element method by Ma *et al.* (2001*a,b*), Hu *et al.* (2002) and Ma & Yan (2009), or even a hybrid model such as that by Wu & Eatock Taylor (2003), have also been applied to nonlinear floating body hydrodynamics. Again, these references do not provide detailed explanations of the underlying physics.

Rodríguez (2016) and Rodríguez *et al.* (2016), provide some of the missing physical explanations for the two-dimensional case in regular waves. In summary, they established three regions in which the nonlinear response of a body is governed by different sources:

- i. In the case where the body is small compared to the incident wavelength, the nonlinear forcing is dominated by the radiation problem, particularly if the second-harmonic wave frequency is in close proximity to the resonance frequency of the body;
- ii. In an intermediate range the second-harmonic motions are small due to a reduction in the forcing from the radiation problem; and
- iii. In the case where the body is of a comparable size to the incident wavelength, a large nonlinear forcing component is induced by a standing wavefield arising due to a combination of the radiation and excitation problems. The nonlinear motions, however, are smaller than in regime i., as the second-harmonic forcing component (occurring at a high frequency) cannot overcome the inertia of the body.

Rodríguez *et al.* (2016) rigorously quantified the nonlinear effects of the two-dimensional problem and, to the author's knowledge, no such detailed analysis has been performed for the floating body problem in three dimensions.

2.3.6 Floating body hydrodynamics in irregular waves

The discussion presented thus far has concerned the investigation of the forcing and response of a body subject to a regular wave train with a single frequency. While such investigations are undoubtedly useful, a realistic sea-state comprises many wave frequencies, characterised by a wave spectrum such as the JONSWAP distribution after Hasselmann *et al.* (1973), and best studied using a statistical representation. A statistical representation can, for example, take the form of a probability of exceedance distribution for a particular quantity by capturing a large number of individual wave events.

The simulation of the incident waves for irregular sea states is typically undertaken using a linear- or second-order random wave theory. The linear theory is fast to compute as the solution simply consists of a superposition, or summation, over all the components in the spectrum. The second-order solution for the interaction of two freely propagating waves was first determined by Longuet-Higgins & Stewart (1960) and later generalised for a spectrum by Sharma & Dean (1981). For any irregular sea state, the second-order solution requires a summation over all possible combinations of pairs of freely propagating waves. As a result, the second-order solution is much more computationally demanding than the linear solution. The interaction between component j and component l is the same as that of l with j , such that the number of computations scales with $(N + 1)N/2$, where N is the number of components in the spectrum. As an example, for a spectrum with a relatively modest frequency resolution comprising 180 components, 16290 interactions must be solved for at second-order, compared to the 180 required for the linear solution.

As the forces and motions are directly linked to the incident sea state, the same arguments apply. The problem is exacerbated due to the longer time required to compute interaction between two force/motion components compared to two incident components. Therefore, the vast majority of existing hydrodynamics literature has only considered bichromatic wave interactions. Examples include the calculation of forces on a truncated cylinder by Huang & Eatock Taylor (1997), the addition of motions by Moubayed & Williams (1995) and—additionally for a hemisphere— Kim & Yue (1990). These studies highlight the importance of considering the sum- and difference-frequencies, particularly in the case of motions where a frequency combination may be at the resonant frequency of the body leading to a large excitation.

The studies using bichromatic waves provide insight into the radiation-diffraction problem but are not directly applicable in the wider design context for realistic sea-states. In order to both optimise the operational working of WEC and to design for the extreme loading event, the probability of exceedance distributions of forces and motions are required. Kim & Yue (1991) appear to be the first to investigate forces arising in irregular sea states for three-dimensional bodies, building upon their earlier work (Kim & Yue, 1989, 1990). The forces for the irregular sea state were evaluated using a two-term Volterra series, for which Eatock Taylor & Kernot (1999) provide more detail. They demonstrated that the earlier approximation of the second-order forces by de Boom *et al.* (1983), Herford & Nielsen (1986) and Petrauskas & Liu (1987) were inadequate. The earlier approximation methods all involved neglecting the second-order scattered potential in favour of using products of the linear potential along with the second-order incident potential.

Eatock Taylor & Kernot (1999) examined the second order response of a body, also using a two-term Volterra series. The limitation of this method for motions, however, is the requirement of a linear transfer function linking the force and response. Rodríguez & Spinneken (2016) (in two-dimensions) demonstrated the importance of vortex shedding damping in the response statistics of the body. This damping was modelled successfully with a quadratic term, making the transfer function nonlinear. The inclusion of viscous effects is discussed in the next section.

2.3.7 Methods accounting for fluid viscosity

The potential flow methods described above cannot, by definition, take account of viscous effects such as vortex shedding. The significant reduction in body motions due to vortex shedding compared with a potential flow prediction is well known (Downie *et al.*, 1988; Vantorre *et al.*, 2004; Rodríguez & Spinneken, 2016), and is particularly acute at the body resonance frequency. In the context of the excitation forcing, Rodríguez & Spinneken (2016) demonstrate that viscous effects are relatively minor.

Computational fluid dynamics solvers may be used to take account of viscous effects. Examples include: (i) vortex methods employed by Downie *et al.* (1988), Yeung & Liao (1999), and Yeung & Jiang (2014), (ii) smoothed particle hydrodynamics by Bouscasse *et al.* (2013) and Baudry *et al.* (2015), (iii) Reynolds-averaged Navier-Stokes equations by Bhinder *et al.* (2009) and Bhinder *et al.* (2011), or (iv) volume of fluids methods by Chen

et al. (2014). While these methods have provided accurate predictions, this comes at great computational cost and cannot, therefore, be used for irregular sea-state simulation.

A second approach is to undertake an experimental investigation, though there are comparatively few studies for heaving buoys. The existing studies have been undertaken both in the laboratory at device scale (Vantorre *et al.*, 2004; Rodríguez & Spinneken, 2016) and array scale (Stratigaki *et al.*, 2015) and in the ocean at full scale (Eriksson *et al.*, 2007). There have also been a number of deployments of commercial heaving-buoy type devices such as the Wavebob in the Galway Bay, the AquaBuOY in the Pacific Ocean near Oregon, and the PowerBuoy in both Spain and Scotland (Falcão, 2010). Unfortunately, many of the results from these sea trials are not available in the public domain.

Section 5.3 provides a much more extensive discussion on the incorporation of viscous effects.

2.4 Development of a second-order radiation-diffraction solver

2.4.1 Introduction

As mentioned in the previous section, a second-order radiation-diffraction code is not currently available in the public domain. To advance the physical understanding in-line with the arguments described above, a new code has been developed from scratch as part of this PhD project. The new implementation builds upon theory already developed, particularly the work of Hulme (1983), Chau (1989), and Kim & Yue (1989, 1990), but makes use of modern programming techniques and hardware in order to conduct simulations that have previously not been possible with such models.

The code is dubbed ICRD3D and is seen as one of the major contributions of the present work. The description of the model that follows is kept brief as much of the material can be found in the papers noted in §2.3 along with the classic textbook by Linton & McIver (2001, Chapter 4). Within ICRD3D, the radiation and diffraction problems are solved analogously but with a different right hand side of the integral equation. The diffraction problem is considered first to illustrate the solution method before considering the combined problem in §2.4.10.

2.4.2 Formulation of the velocity potential

The basis of the method is most simply illustrated by considering the diffraction of a plane monochromatic incident wave characterised by circular frequency ω and a linear amplitude A , propagating in an laterally infinite water body $\mathbb{V}$ of constant depth h , interacting with a body with surface $\mathbb{S}$. A Cartesian coordinate system with the (x,y) -plane denoting the still water level and z pointing upwards is chosen, as shown in Figure 2.2. The azimuthal angle around the body is denoted θ .

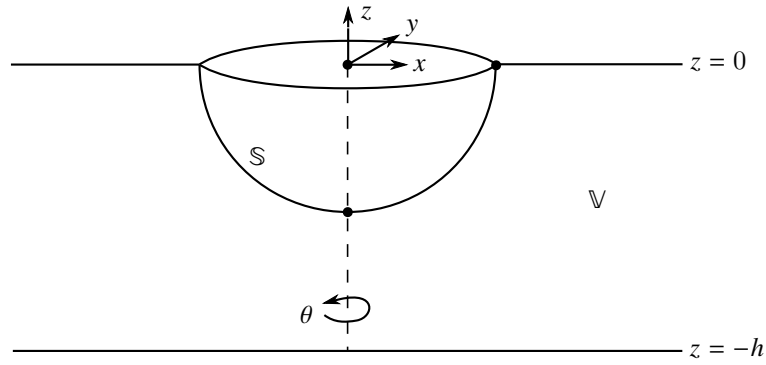


Figure 2.2: Coordinate system adopted

Assuming an inviscid, irrotational and incompressible fluid, a velocity potential, Φ , may be adopted and expanded in a perturbation series in the wave steepness $\varepsilon \equiv A_I k$, where k is the wavenumber given by the dispersion relationship

$$\omega^2 = gk \tanh(kh), \quad (2.3)$$

with g denoting the acceleration due to gravity. For weakly nonlinear waves, $\varepsilon \ll 1$ so that

$$\Phi = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \dots. \quad (2.4)$$

Given that the incident waves are monochromatic, the time dependence of the problem can be separated out explicitly with

$$\Phi^{(1)}(x, y, z, t) = \text{Re} \left\{ \phi^{(1)}(x, y, z) e^{-i\omega t} \right\} \quad (2.5)$$

$$\Phi^{(2)}(x, y, z, t) = \text{Re} \left\{ \phi^{(2)}(x, y, z) e^{-2i\omega t} \right\} + \bar{\Phi}^{(2)}(x, y, z), \quad (2.6)$$

in which the time independent term $\bar{\Phi}^{(2)}$ will contribute to surface-elevations or forces with

$O(\varepsilon^3)$ and is neglected henceforth (Eatock Taylor & Hung, 1987).

At each order the Boundary Value Problem (BVP) is linearly independent so that the diffracted potential ϕ_D may be decomposed into incident (ϕ_I) and scattered (ϕ_S) potentials so that $\phi_D = \phi_I + \phi_S$ at each order. The incident potentials for a monochromatic wave are given from Stokes's theory (Dean & Dalrymple, 1991) as

$$\phi_I^{(1)} = \frac{-igA_I}{\omega} \frac{\cosh k(z+h)}{\cosh kh} e^{ikx}, \quad (2.7)$$

and

$$\phi_I^{(2)} = \frac{-3i\omega A_I^2}{8} \frac{\cosh 2k(z+h)}{\sinh^4 kh} e^{i2kx}, \quad (2.8)$$

where the wave propagates from $x \sim -\infty$, into the positive x -direction. The purpose of the radiation-diffraction solver is to establish the solution to the scattered potential, ϕ_S , as a response to ϕ_I .

2.4.3 Definition of the boundary value problem

The BVP at each order, $\phi^{(j)}$, $j = 1, 2$ is defined by

$$\nabla^2 \phi_S^{(j)} = 0 \quad \text{in } \mathbb{V}; \quad (2.9)$$

$$\left(-(j\omega)^2 + g \frac{\partial}{\partial z} \right) \phi_S^{(j)} = q \quad \text{on } z = 0; \quad (2.10)$$

$$\frac{\partial \phi_S^{(j)}}{\partial z} = 0 \quad \text{on } z = -h; \quad (2.11)$$

$$\frac{\partial \phi_S^{(j)}}{\partial n} = -\frac{\partial \phi_I^{(j)}}{\partial n} \quad \text{on } \mathbb{S}; \quad (2.12)$$

where n is the normal pointing out of the fluid, and

$$q = \delta_{2j} \left\{ \left[-\frac{i\omega}{2g} \phi^{(1)} \left(-\omega^2 \frac{\partial \phi^{(1)}}{\partial z} + g \frac{\partial^2 \phi^{(1)}}{\partial z^2} \right) + i\omega \left(\nabla \phi^{(1)} \right)^2 \right]_{z=0} - q_{II} \right\}, \quad (2.13)$$

in which δ_{2j} defines the Kronecker delta. The q_{II} term takes account of the quadratic products of the incident potential which are already included in equation (2.8).

In addition to the boundary conditions above, the scattered potential must satisfy a radiation condition in the far field. For the linear problem ($j = 1$), the classic Sommerfield

radiation condition is used and given by

$$\lim_{kr \rightarrow \infty} \sqrt{r} \left(\frac{\partial}{\partial r} - ik \right) \phi_S^{(1)} = 0, \quad kr \gg 1, \quad (2.14)$$

where $r = \sqrt{x^2 + y^2}$. The radiation condition for the second-order problem is discussed in §2.4.8.

2.4.4 Formation of the linear solution

Following Kim & Yue (1989), the linear problem is solved using the source distribution technique. The linear and second-order solutions have many similarities so it is instructive to consider the simpler linear problem first. The outline that follows is taken primarily from Hulme (1983) with further details available in Linton & McIver (2001, §4.1).

A solution to the governing equation and boundary conditions given in equations (2.9–2.12) may be sought in the form

$$\phi_S(\mathbf{x}) = \iint_{\mathbb{S}} f(\mathbf{x}') G(\mathbf{x}; \mathbf{x}') dS', \quad (2.15)$$

which represents a distribution of sources over the body, each with a strength f . This solution satisfies the governing equation, the free-surface boundary condition and the radiation condition due to the properties of the Green's function G for a field point $\mathbf{x}$ and a source point $\mathbf{x}'$. The source strength can be found by differentiating $\phi_S(\mathbf{x})$ with respect to the normal direction on the boundary to give a Fredholm integral equation of the second kind for $f(\mathbf{x}')$:

$$2\pi f(\mathbf{x}) + \iint_{\mathbb{S}} f(\mathbf{x}') \frac{\partial G}{\partial n}(\mathbf{x}; \mathbf{x}') dS' = \frac{\partial \phi_I}{\partial n} \quad (\mathbf{x} \in \mathbb{S}). \quad (2.16)$$

For an axisymmetric problem, the solution may be reformulated in a cylindrical coordinate system. Subsequently, the integration in the circumferential coordinate, θ (Figure 2.2), can be analytically evaluated, thereby reducing the number of dimensions of the problem by one. This may be achieved by writing the surface element as $dS = r d\theta ds$, where ds is an arc length, and expressing the source strength and Green's function as Fourier cosine series. The cosine series arises by making use of the symmetry in the y -plane so that the

sine modes cancel. The resulting series are given by

$$f(r, \theta, z) = \sum_{m=0}^{\infty} f_m(r, z) \cos m\theta, \quad (2.17)$$

and

$$G(r, \theta, z; r', \theta', z') = \sum_{m=0}^{\infty} \epsilon_m g_m(r, z; r', z') \cos m(\theta - \theta'), \quad (2.18)$$

respectively. The circumferential coordinate is readily expressed as $\theta = \tan^{-1}(y/x)$ and Neumann's coefficient in the Green's function cosine series is given by

$$\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \geq 1. \end{cases} \quad (2.19)$$

Therefore, the integral equation for the strength of the source distribution can be written as

$$2\pi f_m(s) + \int_0^{s^*} 2\pi r' f_m(s') \frac{\partial g_m}{\partial n} ds = \frac{\partial \phi_{Im}}{\partial n}, \quad (2.20)$$

in which the formula for $\partial \phi_{Im}/\partial n$ can be found in Appendix A.2.1.

Applying a similar Fourier cosine series expansion to the velocity potential, the potential anywhere in the domain can be found using (again making use of the symmetry about $y = 0$)

$$\phi_S(r, \theta, z) = \sum_{m=0}^{\infty} \phi_{Sm}(r, z) \cos m\theta, \quad (2.21)$$

along with

$$\phi_{Sm}(r, z) = \int_0^{s^*} 2\pi r' f_m(s') g_m(r, z; r', z') ds'. \quad (2.22)$$

2.4.5 Formulation of the Green's function

As mentioned in §2.3.3, finding a suitable Green's function G is crucial to the solution of the radiation-diffraction problem using integral equation techniques. The Green's function must both be accurate and, as Newman (1985) notes, computationally efficient.

This section only considers the Green's function; the kernel of the integral equation $\partial g_m/\partial n$ can be found analytically using the operator derived in Appendix A.1 applied to the Green's function representations below. The final expressions for $\partial g_m/\partial n$ are contained

in Appendix A.3.

For ICRD3D, a combination of the cylindrical harmonics representation by John (1950) along with improvements made by Fenton (1978) has been adopted and termed the “subtracted singularities” technique. A full convergence investigation into various Green’s function representations along with derivations for the expressions that follow is included in Appendix B.

Following Hulme (1983), the Greens function in cylindrical harmonics is given by

$$g_m = \frac{1}{2}\pi i A_0 \cosh k(h+z) \cosh k(h+z') J_m(kr_<) H_m^{(1)}(kr_>) + \sum_{n=1}^{\infty} A_n \cos c_n(h+z) \cos c_n(h+z') I_m(c_n r_<) K_m(c_n r_>), \quad (2.23)$$

where the Bessel functions J_m , H_m (the first kind is implied henceforth), I_m and K_m have their usual meaning,

$$r_< = \min(r, r'), \quad r_> = \max(r, r'), \quad (2.24)$$

$$A_0 = \frac{4(k^2 - K^2)}{h(k^2 - K^2) + K}, \quad (2.25)$$

$$A_n = \frac{4(c_n^2 + K^2)}{h(c_n^2 + K^2) - K}, \quad (2.26)$$

in which $K = \omega^2/g$ and c_n is the solution to

$$\omega^2 = -g c_n \tan c_n h, \quad n = 1, 2, \dots \quad (2.27)$$

The computational difficulties that arise in the infinite summation term of equation (2.23) was first identified by Fenton (1978) due to the product

$$I_m(c_n r_<) K_m(c_n r_>) \rightarrow \frac{h}{2n\pi} (rr')^{-1/2} e^{-n\pi|r-r'|/h} \text{ as } n \rightarrow \infty \quad (2.28)$$

which will only converge rapidly if $|r - r'|/h$ is large, ie the source and field points are far apart. Fenton (1978) exploits the fact that the summation term in equation (2.23) has a closed form asymptotic expression. Therefore, the infinite summation may be rewritten as

$$\sum_{n=1}^{\infty} g_{mn} = \sum_{n=1}^{\infty} (g_{mn} - g_{mn}^*) + \sum_{n=1}^{\infty} g_{mn}^*, \quad (2.29)$$

in which g_{mn} is n -th term in g_m from equation (2.23), the $*$ terms are the asymptotic forms and the last term can be determined analytically. The full derivations of the asymptotic terms can be found in Appendix B.4 and B.5. The undoubted benefit of using the subtracted singularities technique is shown in Figure 2.3 in which the infinite summation converges after about 20 modes using subtracted singularities compared to the more than 100 modes required using the series of cylindrical harmonics.

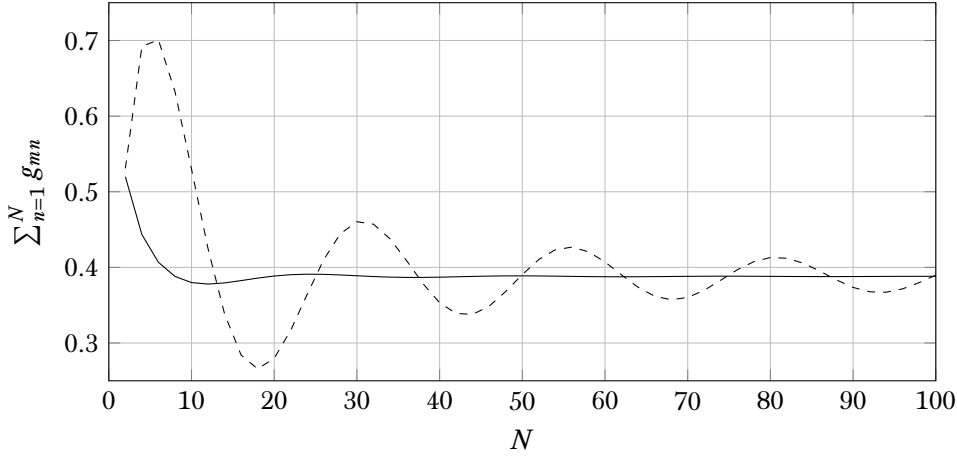


Figure 2.3: Comparison between Green’s function representations of cylindrical harmonics and subtracted singularities with $Kh = 3.0$, $m = 0$, $z = 0$, $z' = -0.1$ and $|r - r'|/h = 0.001$. Subtracted singularities (—) and cylindrical harmonics (---).

A further computational aspect of the Green’s function concerns numerical precision errors when the water becomes deep due to $(k^2 - K^2) \rightarrow 0$ in A_0 (equation 2.25). This is balanced, however, by the product of the two cosh terms in equation (2.23) which tend to infinity with increasing kh , leading to a finite result even at very large kh . A full discussion of the numerical errors associated with these two competing influences is provided in Appendix B.6.1. The large kh error is mitigated using the result of Hulme (1983) that

$$\lim_{h \rightarrow \infty} A_0 \cosh k(h + z) \cosh k(h + z') = 4Ke^{K(z+z')}, \quad (2.30)$$

which does not contain the problematic subtraction of $k^2 - K^2$.

2.4.6 Solution of the integral equations

At this stage, all terms in equation (2.20) can be calculated, leaving the solution of the integral equation as the barrier to finding the source distribution $f_m(s)$. The typical approach is to use the constant panel method of Hess & Smith (1964), which then allows the

application of well established computational methods for the solution of Fredholm integral equation of the second kind. Baker (1977) and Delves & Mohamed (1985) provide both the theory behind integral equations and numerous solution techniques.

The Nyström method is perhaps the simplest method of solving Fredholm integral equations of the second kind. The technique involves replacing the integral by quadrature of the form

$$\int_a^b F(t) dt = \sum_{j=1}^N W_j F(t_j) + E(N), \quad (2.31)$$

where the contour of integration $[a, b]$ is discretised into N -points, W_j are the ‘weights’ associated with the particular quadrature rule, and $E(N)$ is the error in the approximation. The substitution of the quadrature produces a linear set of equations

$$[A]\{\tilde{F}\} = \{b\}, \quad (2.32)$$

in which an approximation to the unknowns $\{\tilde{F}\}$ can be solved using standard numerical routines. The solution has the property that $|\tilde{F} - F| \rightarrow 0 \implies E(N) \rightarrow 0$ as $N \rightarrow \infty$.

Applying the quadrature rule to equation (2.20), and denoting a point on the body surface as s , gives

$$2\pi f_m(s_i) + \sum_{\substack{j=1 \\ (j \neq i)}}^N 2\pi r_j f_m(s_j) \frac{\partial g_m}{\partial n_i}(s_i, s_j) \Delta s = b_i \quad (i = 1, 2, \dots, N), \quad (2.33)$$

where Δs is the distance between point j and $j + 1$. Equation (2.33) may be expressed in terms of equation (2.32) as,

$$A_{ii} = 2\pi, \quad (2.34a)$$

$$A_{ij} = 2\pi r_j W_j \frac{\partial g_m}{\partial n_i}(s_i; s_j) \Delta s \quad (i \neq j), \quad (2.34b)$$

and

$$b_i = -\frac{\partial \phi_{Im}}{\partial n_i}. \quad (2.34c)$$

A number of different schemes may be used to represent the integral in equation (2.31), the principal choices being Newton-Cotes formulae or Gaussian quadrature. Newton-Cotes formulae, such as the trapezoid and Simpson’s $1/3$ rules, are preferred here as they may be

applied to any equally spaced discretisation of the body surface, as opposed to fixed function evaluation positions required by Gaussian quadrature. For ICRD3D a simple constant panel method is employed with the weighting scheme

$$W_j = \begin{cases} 0.5 & \text{for } j = 1 \\ 1 & \text{for } 2 \leq j \leq N - 1 \\ 0.5 & \text{for } j = N. \end{cases} \quad (2.35)$$

2.4.7 Calculation of hydrodynamic forces

The dynamic pressure, p , is given by the linearised Bernoulli equation as

$$p(\mathbf{r}, t) = -\rho \frac{\partial \phi}{\partial t} = \rho i \omega \phi(\mathbf{r}, t), \quad (2.36)$$

where ρ is the density of water, so that the total complex-valued force on the body is

$$\vec{F} = \rho i \omega \int_{\mathbb{S}} \phi \hat{n} dS, \quad (2.37)$$

in which, expressed via the unit vector tuple $(\hat{i}, \hat{j}, \hat{k})$, $\hat{n} = \hat{i} \cos \beta \cos \theta + \hat{j} \cos \beta \sin \theta + \hat{k} \sin \beta$.

The angle β is positioned between the normal direction and the r coordinate (Figure A.1).

Recalling the Fourier expansion

$$\phi = \sum_{m=0}^{\infty} \phi_m \cos m\theta, \quad (2.38)$$

and integrating equation (2.37) with respect to θ has the following implications for each coordinate direction:

$$\hat{i} : \int_0^{2\pi} \cos m\theta \cos \theta d\theta = \begin{cases} \pi & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}, \quad (2.39)$$

$$\hat{j} : \int_0^{2\pi} \cos m\theta \sin \theta d\theta = 0 \quad \forall m, \quad (2.40)$$

$$\hat{k} : \int_0^{2\pi} \cos m\theta d\theta = \begin{cases} 2\pi & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (2.41)$$

Therefore, the $m = 0$ term contributes to vertical forces, $m = 1$ contributes to horizontal forces, and there is no transverse force. Using these results, the total complex-valued force on the body may be written as

$$\vec{F} = \rho i \omega \pi \int_{\mathbb{S}} r \left(\hat{i} \cos \beta \phi_1(s) + \hat{k} 2 \sin \beta \phi_0(s) \right) ds. \quad (2.42)$$

2.4.8 Formation of the second-order solution

To solve the BVP introduced in §2.4.3 at second order ($j = 2$), the Green's function at double-frequency, G^+ , is used to apply Green's second identity. Following Linton & McIver (2001, §4.2), yields the integral equation for the second-order scattered potential $\phi_S^{(2)}$:

$$2\pi \phi_S^{(2)} + \iint_{\mathbb{S}} \phi_S^{(2)} \frac{\partial G^+}{\partial n} d\mathbf{x} = - \iint_{\mathbb{S}} G^+ \frac{\partial \phi_I^{(2)}}{\partial n} d\mathbf{x} + \frac{1}{g} \iint_{S_F} q G^+ d\mathbf{x}, \quad (2.43)$$

where S_F denotes the free-surface. The so-called “weak” radiation condition is satisfied as

$$\iint_{S_\infty} \left(\frac{\partial \phi_S^{(2)}}{\partial n} G^+ - \phi_S^{(2)} \frac{\partial G^+}{\partial n} \right) d\mathbf{x} = 0 \quad \text{as } r \rightarrow \infty \quad (2.44)$$

as discussed by Kim & Yue (1989, eq. (2.12)).

Similar to the linear solution, the two-dimensional integral equation (2.43) over the surface of the body may be reduced to a line integral by making use of the axisymmetry of the body. In addition to an expansion of the Green's function (equation 2.18), the scattered potential, the incident potential, and the free-surface forcing term are also expanded in the circumferential coordinate θ ,

$$\begin{Bmatrix} \phi_{I,S}^{(2)}(r, \theta, z) \\ q(r, \theta) \end{Bmatrix} = \sum_{m=0}^{\infty} \begin{Bmatrix} \phi_{I,Sm}^{(2)}(r, z) \\ q_m(r) \end{Bmatrix} \cos m\theta, \quad (2.45)$$

so that the one dimensional integral equation at second-order follows as

$$2\pi \phi_{Sm}^{(2)} + \int_0^{s^*} 2\pi r \phi_{Sm}^{(2)} \frac{\partial g_m^+}{\partial n} ds = - \int_0^{s^*} 2\pi r \frac{\partial \phi_{Im}^{(2)}}{\partial n} g_m^+ ds + \frac{1}{g} \int_{S_F} 2\pi r q_m g_m^+ dr, \quad (2.46)$$

with a full derivation included in Appendix A.4.1.

Equation (2.46) has a similar form to that of the linear problem (equation 2.20) meaning

the solution method outlined in §2.4.6 remains the same but with

$$b_i = - \int_0^{s^*} 2\pi r \frac{\partial \phi_{I_m}^{(2)}}{\partial n_i}(s) g_m^+(s, s_i) ds + \frac{1}{g} \int_{S_F} 2\pi r q_m g_m^+(r, 0; s_i) dr. \quad (2.47)$$

As noted by many authors (eg Eatock Taylor & Hung, 1987; Chau & Eatock Taylor, 1992; Kim & Yue, 1989; Teng & Kato, 1999) the evaluation of the free-surface integral in equation (2.47) is particularly challenging as it exhibits highly oscillatory and slow convergence characteristics. A method is therefore required to evaluate this term accurately and efficiently. Following the method of Chau & Eatock Taylor (1992), the integral is evaluated over two distinct intervals, $[a, b)$ and $[b, \infty]$, where the radius b is chosen so that the latter interval is evanescent-wave free. The final term in equation (2.47) may therefore be written as

$$I_m = \frac{2\pi}{g} \left[\int_a^b r q_m g_m^+ dr + \int_b^\infty r \hat{q}_m \hat{g}_m^+ dr + \int_b^\infty r (q_m g_m^+ - \hat{q}_m \hat{g}_m^+) dr \right], \quad (2.48)$$

where $\hat{\cdot}$ represents terms that contain contributions from progressive waves only. The third integral represents the difference between the progressive and evanescent modes and is included for mathematical consistency. If b is chosen such that the region $[b, \infty]$ is entirely evanescent free, the third integrand cancels and need not be evaluated.

The key advantage of such an approach is that the near-field behaviour, the first integral in equation (2.48) can be computed numerically, whereas the second integral of infinite extent can be computed analytically. The latter makes use of the far-field form of the double frequency Green's function

$$\hat{g}_m^+ = \frac{1}{2} C_0 \pi i \cosh k_2(z + h) \cosh k_2(z' + h) J_m(k_2 r') H_m(k_2 r), \quad (2.49)$$

where k_2 is the solution to $4\omega^2 = g k_2 \tanh k_2 h$, and the Kochin function

$$L_m = \int_0^{s^*} r' f_m(s') J_m(k r') \cosh k(z' + h) ds', \quad (2.50)$$

which results directly from the linear solution. The Kochin function allows an evanescent-

free form of the linear scattered potential to be written as

$$\hat{\phi}_{sm}^{(1)} = -4\pi^2 i C_0 \cosh k(z+h) H_n(kr) L_m \quad (2.51)$$

to then substitute into equation (2.13) to derive $\hat{q}_m$. The full derivations of q_m and $\hat{q}_m$ can be found in Appendix A.4.2 and A.4.3.

The possibility of an analytic integration in the far-field arises from the products of $\hat{\phi}_{sm}^{(1)}$ in q_m , with g_m to give an integral of the form

$$I_{lmn} = \int_b^\infty r^p \left[\frac{H_l(kr)}{H_l^*(kr)} \right] H_m(kr) H_n(k_2 r) dr, \quad p = \pm 1, \quad (2.52)$$

where $*$ denotes the complex conjugate. Detailed analyses of this so-called triple Hankel integral are given in Chau (1989), Chau & Eatock Taylor (1992) and Kim & Yue (1989). For the implementation of ICRD3D, the method of Chau (1989) is used as outlined in Appendix A.4.5–A.4.8.

2.4.9 Calculation of second-order forces

The second-order forces are derived in full in Appendix A.4.9 with the final expressions provided below. The total second-order force contains both a harmonic and a mean component

$$F^{(2)} = \text{Re} \left\{ F^{(2)} e^{-2i\omega t} \right\} + \bar{F}^{(2)} \quad (2.53)$$

The harmonic force component can be further decomposed into (i) contributions from quadratic products of the first-order potential and, (ii) contributions from the second-order potential. This gives $F^{(2)} = F_q^{(2)} + F_s^{(2)}$, in which

$$F_q^{(2)} = -\frac{\rho}{2} \left[\int_S (\nabla \phi^{(1)})^2 \hat{n} dS + \frac{\omega^2}{g} \int_{W_0} (\phi^{(1)})^2 \hat{n} dl \right], \quad (2.54)$$

and

$$F_s^{(2)} = 2i\omega\rho \int_S \phi^{(2)} \hat{n} dS, \quad (2.55)$$

where W_0 is the mean waterline. The mean second-order force component is given by

$$\bar{F}^{(2)} = -\frac{\rho}{2} \left[\int_S |\nabla \phi^{(1)}|^2 \hat{n} \, dS + \frac{\omega^2}{g} \int_{W_0} |\phi^{(1)}|^2 \hat{n} \, dl \right]. \quad (2.56)$$

2.4.10 The coupled radiation-diffraction problem

The discussion above has been limited to the diffraction problem to illustrate the solution method. Adopting a similar approach, the total potential for the combined problem may be expressed as

$$\phi = \varepsilon \left(\phi_I^{(1)} + \phi_S^{(1)} + \phi_R^{(1)} \right) + \varepsilon^2 \left(\phi_I^{(2)} + \phi_S^{(2)} + \phi_R^{(2)} \right), \quad (2.57)$$

where the new subscript R refers to the radiation potential.

The first-order radiation problem for $\phi_R^{(1)}$ is solved using exactly the same approach as the diffraction problem, using a body boundary condition of

$$\frac{\partial \phi_R^{(1)}}{\partial n} = -i\omega \hat{\xi}^{(1)} \cdot \hat{n} \quad \text{on } \mathbb{S}, \quad (2.58)$$

where $\hat{\xi}^{(1)}$ is the first-harmonic body displacement vector defined by

$$\hat{\xi}^{(1)} = (\xi_0 \hat{k} + \xi_1 \hat{i}). \quad (2.59)$$

The second-order solution requires the first-order potentials for the free surface and body boundary conditions. With the introduction of the radiation potentials in the combined problem, $\phi_R^{(1)}$ can either be incorporated either into $\phi_S^{(2)}$ or $\phi_R^{(2)}$. A commonly adopted approach (Kim & Yue, 1990; Moubayed & Williams, 1994) is the former. This makes the second-order solution analogous to the first-order solution, as $\phi_I^{(2)}$ and $\phi_S^{(2)}$ provide the excitation forcing for the second-harmonic motions. Physically, this means $\phi_S^{(2)}$ is the solution for scattering of second-harmonic incident waves while undergoing first harmonic motions (Kim & Yue, 1990), and $\phi_R^{(2)}$ represents the propagation of waves due to second-harmonic motions. Therefore, $\phi_R^{(2)}$ is solved identically to $\phi_R^{(1)}$ but at twice the frequency.

The second-order diffraction problem with first-harmonic radiation proceeds in the

same manner as §2.4.8 with

$$\frac{\partial \phi_S^{(2)}}{\partial n} = -\frac{\partial \phi_I^{(2)}}{\partial n} + B \quad \text{on } \mathbb{S} \quad (2.60)$$

as the body boundary condition. Considering only translation motions (surge or heave), the second-order body term is given by (Kim & Yue, 1990)

$$B = -\hat{n} \cdot (\hat{\xi}^{(1)} \cdot \nabla) \nabla \Phi^{(1)}, \quad (2.61)$$

where $\hat{\xi}^{(1)}$ are the first-harmonic translational motions. Furthermore, the total first-order potential is used when calculating the free surface forcing term q (equation 2.13).

The procedure for calculating $\phi_S^{(2)}$ including linear-radiation is undertaken in the sequence:

- i. compute the linear diffraction problem,
- ii. compute the linear radiation problem for unit velocity,
- iii. compute the linear body velocity for the incident conditions using the equation of motion,
- iv. scale the linear radiation potential using the computed body velocity in iii., and
- v. compute the second-order diffraction problem including linear radiation.

2.5 Validation of ICRD3D

The semi-analytic solution of Eatock Taylor & Hung (1987), for the case of diffraction around a bottom-mounted cylinder, provides an effective means of validating ICRD3D. Their method was implemented and initially validated against tabulated data provided in Eatock Taylor & Hung (1987) as shown in Figure 2.4. The quantities plotted are the first-harmonic force F_1 , the mean second-harmonic force F_m , the force due to quadratic products of the linear potential F_θ , and the contributions due to the second-harmonic potential $F_{qI,II,III}$. Based on the excellent agreement in Figure 2.4, the semi-analytical solution was then taken forward to validate ICRD3D for a wide range of conditions. The calculations throughout this thesis have been undertaken on an HP Z420 workstation with a 12 thread Intel Xeon E5-1650 CPU @ 3.2 GHz and 16 GB of RAM. The computation of a single frequency takes in the order of one second for the linear solution, and approximately

two minutes at second-order.

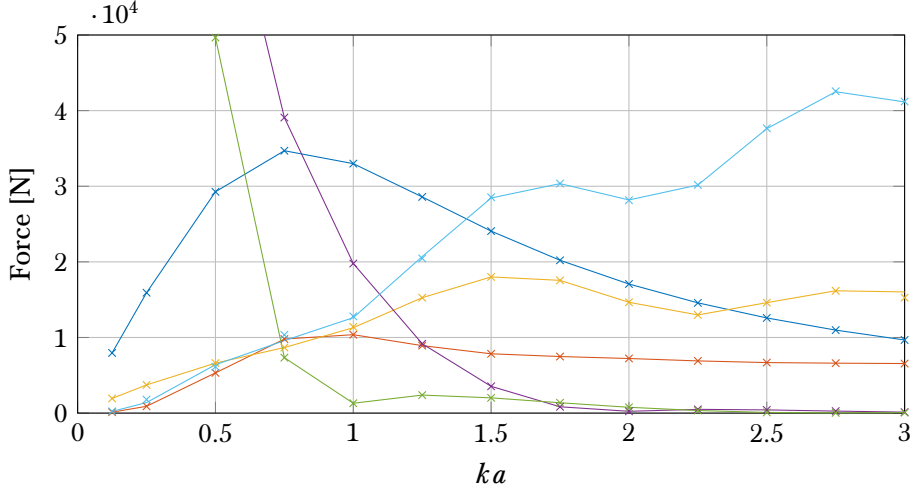


Figure 2.4: Comparison between author's implementation (symbols) and tabulated data of Eatock Taylor & Hung (1987) (lines) for $h/a = 1.0$ with $a = 1$ m: (—) F_1 , (—) F_m , (—) F_o , (—) F_{qI} , (—) F_{qII} , (—) F_{qIII} .

The validation exercise is undertaken by keeping the radius of the body a fixed, varying the wave frequency to achieve a range of Ka , and considering three water depths of $h = a$, $5a$, and $10a$. There are three primary parameters that control the accuracy of the solution:

- i. the body discretisation characterised by dS/S where S is the total arc length,
- ii. the number of evanescent modes used in the Green's function (equation 2.23), and
- iii. the number of Fourier modes used in the cosine expansion (equation 2.17).

Figure 2.5 shows the first-harmonic horizontal force on the cylinder as a function of Ka using the analytic solution of Eatock Taylor & Hung (1987) and ICRD3D. The agreement is excellent for all three geometries with a maximum departure of 0.6%. This was achieved following a convergence study for parameters i. and ii. above. Table 2.1 shows the percentage difference between the semi-analytic solution and ICRD3D with increasing body discretisation for $h/a = 1$ at three different wave frequencies. At $dS/S \leq 0.017$ the difference is approximately 0.5% or less for each of the frequencies. This discretisation was then taken forward to the $h/a = 5$ and 10 cases. A similar analysis was performed for the number of evanescent modes (parameter ii.), which determined that 20 modes are sufficient. Finally, for the linear horizontal forces, only the second Fourier mode is required (equation 2.39) so the Fourier modes do not form part of the convergence analysis.

Figure 2.6 demonstrates the ability of ICRD3D to calculate the second-harmonic force by

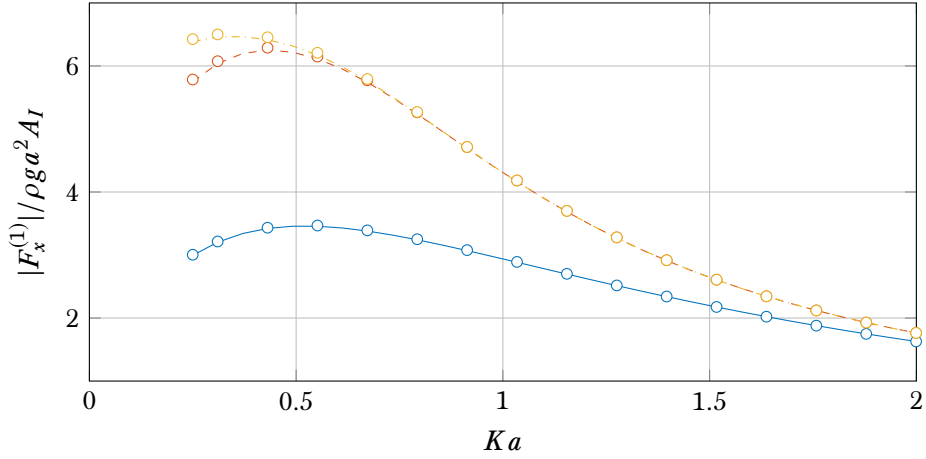


Figure 2.5: Normalised first-harmonic force on a bottom mounted cylinder for: (—) $h/a = 1$, (---) 5, (-.-) 10. (Lines) Eatock Taylor & Hung (1987), (symbols) ICRD3D.

Table 2.1: Percentage difference between $F_x^{(1)}$ calculated by ICRD3D and Eatock Taylor & Hung (1987) for $h/a = 1$.

dS/S	Ka		
	0.55	1.00	1.50
0.050	1.26	1.05	0.31
0.025	0.71	0.54	0.14
0.017	0.51	0.36	0.09
0.013	0.40	0.27	0.07

again comparing to the analytic solution of Eatock Taylor & Hung (1987). The test cases considered remain those adopted for the computation of the first-order forcing. Excluding the irregular frequency at $Ka \approx 1.85$, the solutions are generally within 5% for the three geometries, with a maximum departure of 12%. This agreement has been achieved using 8 Fourier modes to expand the linear solution. Furthermore, the second-order solution requires the partition radius for the free surface integral to be specified such that for $r > b$ the solution should be evanescent mode free (equation 2.48). Following Kim & Yue (1989) and independent analysis, $(b - a)/h = 4$ has been adopted.

Eatock Taylor & Hung (1987) employ the “indirect-method” (*cf* §2.3.4) in which only forces, and not pressures, can be calculated. A direct comparison of the pressures, however, provides a more stringent validation scenario as any errors in the pressure may be smeared out during integration to calculate the force. Figure 2.7 compares ICRD3D with Kim & Yue (1989, Figure 3a) for distribution of pressure with depth on the lee side ($\theta = 0$) of a bottom

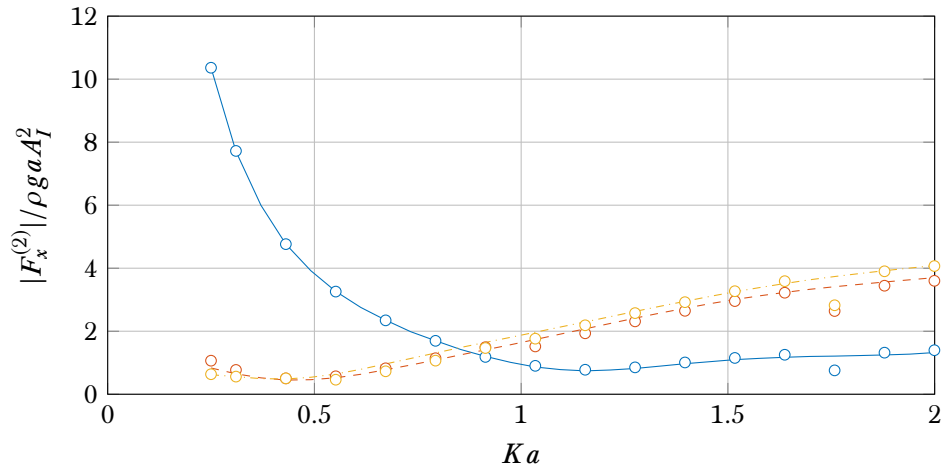


Figure 2.6: Normalised total second-harmonic force on a bottom mounted cylinder for: (—) $h/a = 1$, (---) 5, (-.-) 10. (Lines) Eatock Taylor & Hung (1987), (symbols) ICRD3D.

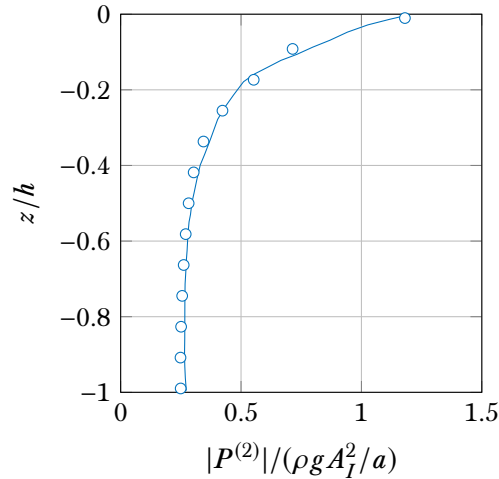


Figure 2.7: Second-harmonic pressure-depth distribution for $Ka = 2.00$ with $h/a = 1$. (—) Kim & Yue (1989), ($\circ$) ICRD3D

mounted cylinder. The agreement is within 5% for all points on the distribution. The exercise undertaken in this section demonstrate that ICRD3D can be relied upon to provide accurate predictions up to the second-order of wave steepness. Additional validation cases are presented in the course of the discussion in Chapter 3.

3

Nonlinear forcing and response in regular waves

3.1 Chapter overview

This chapter addresses the nonlinear forcing and associated motion response of a heaving truncated cylinder subject to regular waves. At this stage, the problem is assumed inviscid and irrotational such that the second-order code introduced in Chapter 2 may be adopted. The discussion (or inclusion) of viscous effects in the nonlinear forcing and response is delayed until Chapter 5.

As outlined in §2.3, the truncated cylinder problem has been a source of considerable research interest, particularly in the case of a fixed body. Building upon this solid foundation, this chapter enhances the understanding of the coupled radiation-diffraction problem for wave conditions and geometries which are representative of WEC devices. The aims of the chapter are to

- i. demonstrate the importance of using a second-order solution
- ii. quantify the extent of the nonlinear forcing and associated dynamic response, and
- iii. explain the physical origin of these nonlinearities.

Section 3.2 introduces the nonlinear radiation-diffraction problem using the physical insight gained in the two-dimensional scenario by Rodríguez *et al.* (2016). The section outlines the principal differences between two-dimensional and axisymmetric problems and the expected effect this has on the nonlinear forcing. The following section, 3.3,

then describes the geometries considered in this investigation, along with some additional validation cases for the new solver ICRD3D.

The motivation for an improved understanding of the combined radiation-diffraction problem is put forward in §3.4. The remainder of the chapter is then divided into three parts to isolate the sources of nonlinearity: the excitation problem in Part A (§3.5), the radiation problem in Part B (§3.6), and finally the coupled problem in Part C (§3.7). The chapter concludes with §3.8 which summarises the key findings and also indicates some limitations.

3.2 Introduction

A very recent and detailed explanation of the physical mechanisms driving the nonlinear forces and motion of a two-dimensional floating structure is provided in Rodríguez *et al.* (2016). This chapter extends the discussion from two-dimensions to three-dimensional axisymmetric structures, and provides additional physical insight. The principal differences between the two- and three-dimensional cases are the patterns of the scattered and radiated wave fields. In two-dimensions, the waves propagating away from the body into the far-field have constant amplitude with increasing distance. In three-dimensions, however, the progressive waves will spread radially in the coordinate r and decay, in amplitude, proportional to $r^{-1/2}$.

This radial decay renders the analysis of the resulting wave field much more difficult. For example, a harmonic decomposition of the wave field into its constituent components is readily achieved in two-dimensions (Lin & Huang, 2004), but this type of analysis proves very difficult in the three-dimensional case. This limitation provides one of the principal reasons for developing ICRD3D, where, in contrast to a time-domain numerical wave tank formulation, the individual terms in the solution can be examined in isolation. In applying ICRD3D, this chapter demonstrates that the consideration of second-order effects is an important part of the forcing and associated dynamic response of a heaving truncated cylinder.

3.3 Geometrical considerations

Chapters 3–5 consider a truncated cylinder, which is governed by two parameters: the radius a and the draught b . The radius is made dimensionless by multiplication with the deep water wavenumber $K = \omega^2/g$, where ω is the circular wave frequency, and g is the acceleration due to gravity. A dimensionless draught is commonly expressed as b/h , where the water depth h is constant for all cases considered. The ratio of radius to water depth is kept fixed at $a/h = 0.1$, which matches the experimental program detailed in Chapter 4; a full rationale for this ratio is provided in §4.2.2. The wave frequency is varied to achieve a broad range of Ka . The draught has also been varied to examine the depth dependence of the nonlinear effects, with $b/h = 0.1, 0.2$ and 0.4 . The resulting geometries are denoted TC1, TC2 and TC3 respectively and shown in Figure 3.1.

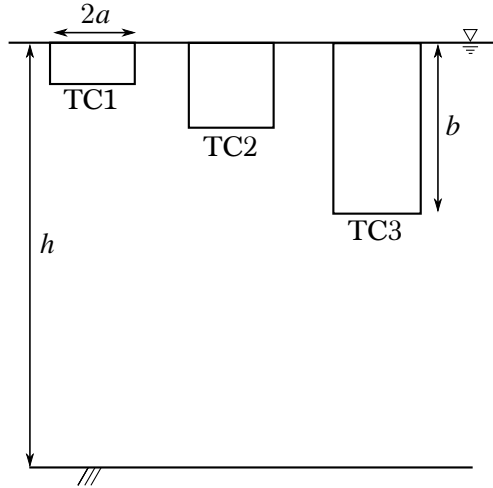


Figure 3.1: Truncated cylinder geometries under consideration.

3.4 An observation of the nonlinear response

To motivate the investigation into the physical origins of the nonlinearities arising in the radiation-diffraction problem, a series of illustrative examples follow. Firstly, Figure 3.2 compares ICRD3D to the open source linear radiation-diffraction solver NEMOH (Babarit & Delhommeau, 2015), including geometries TC1 and TC2. In terms of the response amplitude operator (RAO) considered in Figure 3.2, the two solutions are in very close agreement. This is expected, given that the second-order effects captured by ICRD3D do not affect the first-harmonic body motion (third-order, and higher, solutions will produce

contributions to the first-harmonic). Nevertheless, the RAO remains an important driver of nonlinearity. Around the resonant frequency, $Ka \approx 0.65$ for TC1 and $Ka \approx 0.4$ for TC2, the body undergoes large excursions, which Rodríguez *et al.* (2016) identified as being associated with significant nonlinearity. A primary aim of this chapter is to establish how RAOs of the types shown in Figure 3.2 govern these motion induced nonlinearities.

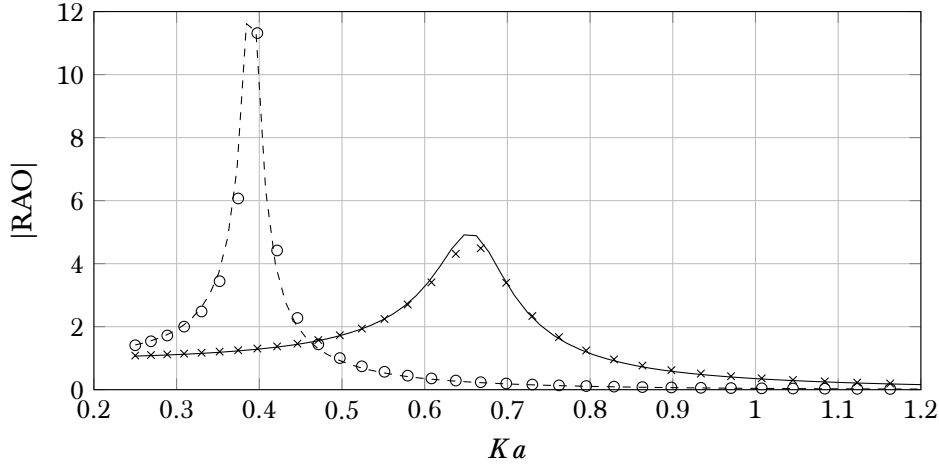


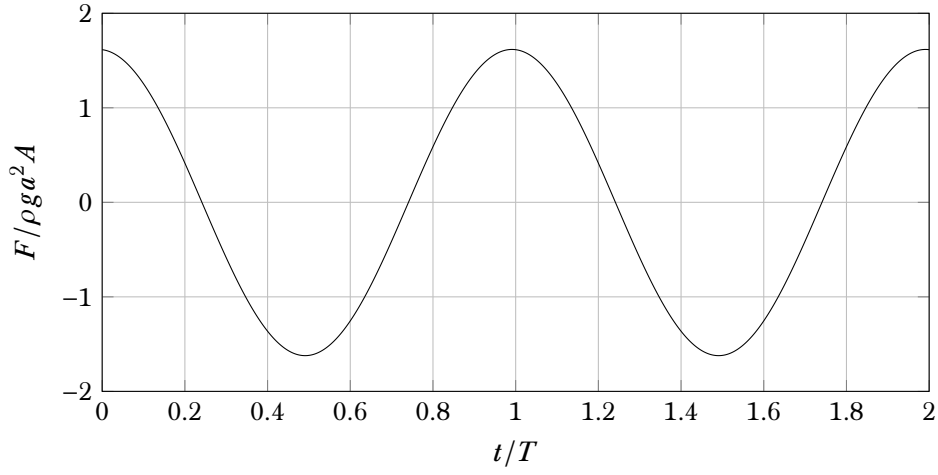
Figure 3.2: Heave RAO for truncated cylinders subject to regular waves. (—) TC1, (---) TC2; lines NEMOH, symbols ICRD3D.

In addition to nonlinearities driven by large motion excursions, the excitation forcing may also introduce significant second-order content. Adopting the solution given by ICRD3D, Figure 3.3 shows the time history of the excitation forcing on TC2 for three cases of constant wave steepness $A_I k = 0.12$ and:

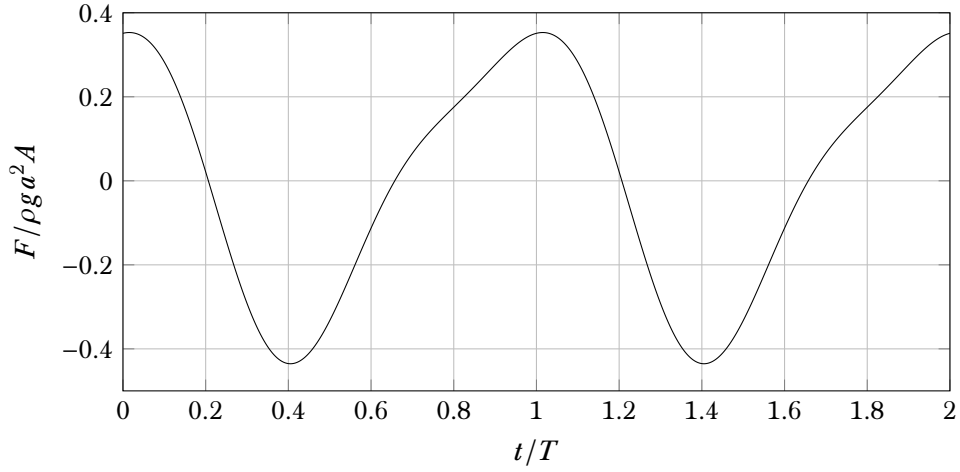
- i. Case 1: Figure 3.3(a): $Ka = 0.25$, $D/\lambda = 0.08$,
- ii. Case 2: Figure 3.3(b): $Ka = 0.80$, $D/\lambda = 0.25$, and
- iii. Case 3: Figure 3.3(c): $Ka = 1.20$, $D/\lambda = 0.38$,

where λ is the incident wavelength, and $D = 2a$ is the body diameter. Recalling the discussion in §2.2, linear diffraction effects are traditionally taken into account when $D/\lambda \geq 0.2$.

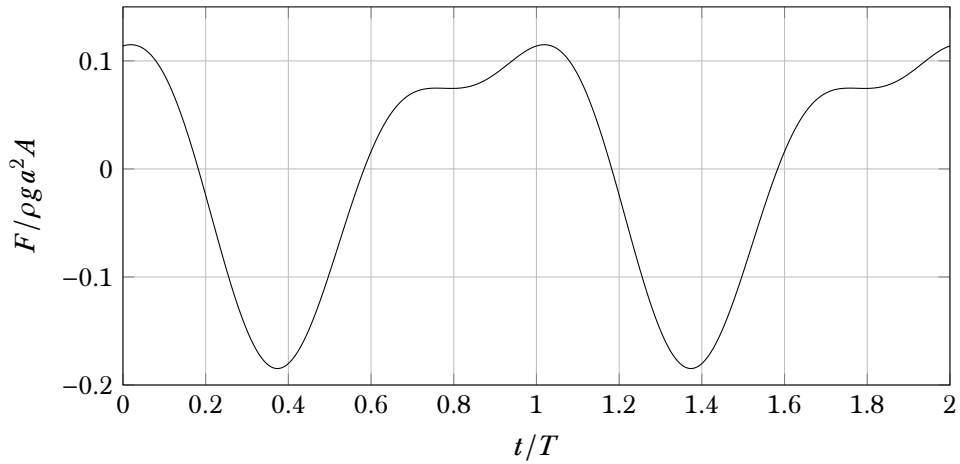
In considering the three cases in Figure 3.3, the second-harmonic component becomes more important with increasing Ka : in Case 1 (Figure 3.3(a)) the body is subject to near-linear (sinusoidal) forcing whereas in Case 3 (Figure 3.3(c)) a significant second harmonic is present, leading to greater force minima compared to maxima, and therefore introducing a strong asymmetry into the loading cycle. In this case, the second-harmonic component represents approximately 40% of the first-harmonic force.



(a) Case 1: $Ka = 0.25$ and $D/\lambda = 0.08$.



(b) Case 2: $Ka = 0.80$ and $D/\lambda = 0.25$.



(c) Case 3: $Ka = 1.20$ and $D/\lambda = 0.38$.

Figure 3.3: Time history of the normalised excitation force for TC2 subject to regular waves of steepness $A_I k = 0.12$.

In order to better understand the nonlinear behaviour observed in Figure 3.3(c), Figure 3.4 shows the first- and second-harmonic components of the total force. The phasing between the first, $F^{(1)}$, and the second-harmonic, $F^{(2)}$, force is particularly interesting. For the case observed in Figure 3.4, the second-harmonic component is aligned such that it consistently superimposes with the maximum negative first-harmonic force, while it destructively superimposes with the maximum positive force. Taken as a whole, significant nonlinearities in the coupled radiation-diffraction problem are expected to occur under large motion excursions, or in cases where large nonlinearities are induced in the excitation forcing. Perhaps the most severe cases are these where a combination of these two causes occurs.

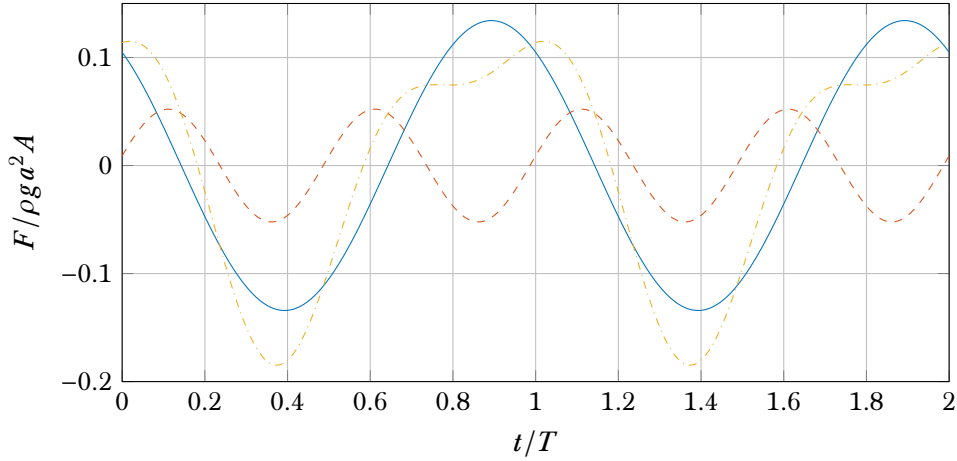


Figure 3.4: Time history of the force components for $Ka = 1.2$, $A_I k = 0.12$ with: (—) $F^{(1)}$, (---) $F^{(2)}$, (-.-) $F^{(1)} + F^{(2)}$.

3.5 Part A: The diffraction problem

3.5.1 First-harmonic excitation

To begin the discussion of the excitation force for the truncated cylinder, Figure 3.5 shows a comparison of the linear heave excitation force computed by ICRD3D and NEMOH (Babarit & Delhommeau, 2015). The codes are in excellent agreement, confirming to general trends:

- i. a decreasing force with increasing Ka , and
- ii. a decreasing force with increasing draught.

Both observations may be explained by considering the pressure distribution under the body. As shown in Figure 3.6, the linear pressure decays with depth and the decay is

more rapid with increasing Ka (or greater relative depth, kh). Therefore, as the pressure is smaller at deeper drafts and greater Ka , the corresponding force will consequently be smaller too.

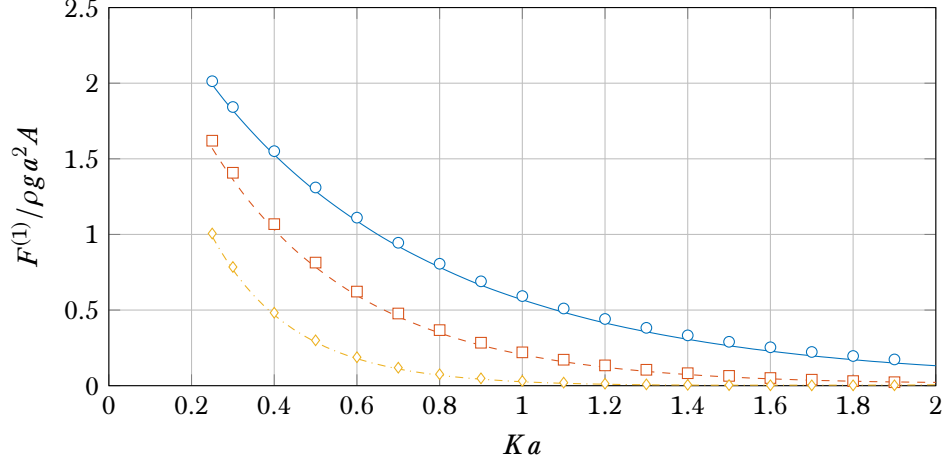


Figure 3.5: Normalised first-harmonic heave excitation force on truncated cylinders with: (—) TC1, (---) TC2, (---) TC3. (Lines) NEMOH, (symbols) ICRD3D.

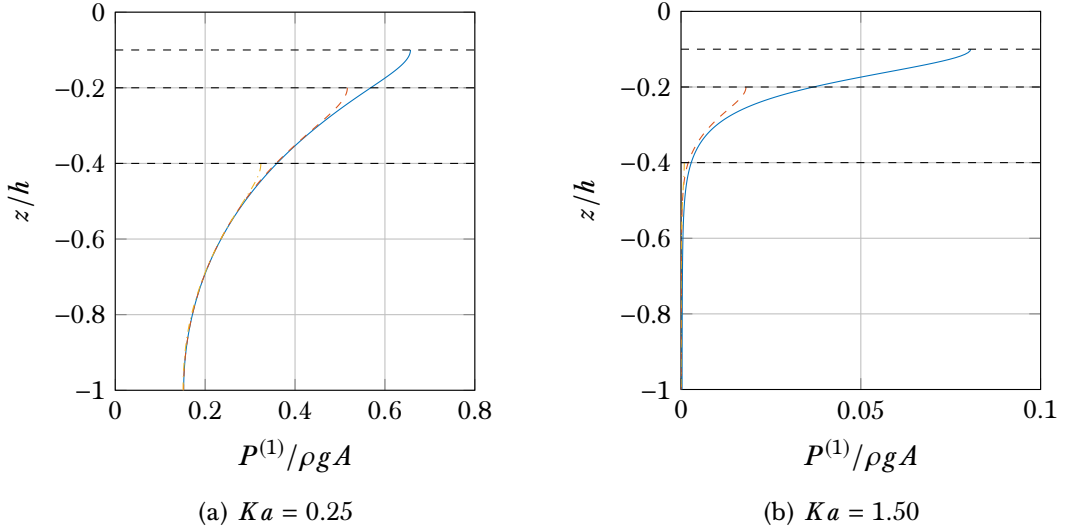


Figure 3.6: First-harmonic pressure distribution contributing to vertical forces under the body at $r/a = 0.5$: (—) TC1, (---) TC2, (---) TC3. The horizontal dashed lines indicate the draught of each geometry.

The phase relationship of the first-harmonic excitation force to the incident wave is best considered by separating the total excitation force into the Froude-Krylov and the scattered force components. The Froude-Krylov force arises from the incident-wave-pressure integration around the body and is therefore in phase with the incident wave at 0 rad, as shown by the solid line in Figure 3.7(a). Figure 3.7(a), for TC2, also shows that the scattered force

(dashed line) is approximately out of phase with the Froude-Krylov force at low Ka , and the phase of the scattered force remains relatively constant over the range of Ka under consideration.

The phase of the total excitation force (dot-dash line) can then be explained by considering Figure 3.7(b) which demonstrates that the magnitude of the Froude-Krylov force to scattered force decreases with increasing Ka . Therefore, the local maxima of the total excitation force are shifted towards the local maxima of the scattered force, as illustrated in Figures 3.7(c) and 3.7(d). In these latter figures, each component has been normalised by the scattered force to show it is the increases in the relative importance of the scattered force that causes the phase shift. A similar relationship is also seen for TC1 and TC3. This effect is important in the combined radiation-diffraction problem considered in §3.7.

3.5.2 Second-harmonic excitation

Turning now to the second-harmonic excitation force, Figure 3.8(a) shows the force for the three geometries over a range of Ka and includes the results of Teng & Kato (1999, Table 4). The results of ICRD3D are seen to be consistent with Teng & Kato (1999), which provides additional confidence as to the code's accuracy.

In examining the second-harmonic forcing, two underlying components of the force are considered separately. The notation of these force components comprises a superscript to denote the harmonic and a subscript to denote the type of potential causing the forcing. At second-harmonic, these two force components are:

- i. $F_q^{(2)}$ arising due to quadratic products of first-harmonic potentials $(\nabla\phi^{(1)})^2$, equation (2.54), and
 - ii. $F_s^{(2)}$ arising due to second-harmonic potentials, $\partial\phi^{(2)}/\partial t$, equation (2.55),
- to form the total $F^{(2)} = F_q^{(2)} + F_s^{(2)}$.

Figure 3.8(b) shows the forcing due to $F_q^{(2)}$. The magnitude of this force component generally decreases with increased draught, following a similar trend as the first harmonic force. However, unlike the first-harmonic force, $F_q^{(2)}$ does not monotonically decrease with increasing Ka . For TC1, the force component is small at small Ka , rises to a maximum at $Ka \approx 0.7$, and decreases relatively rapidly following this maximum. A similar trend is observed for TC2 and TC3, with the peak at a smaller Ka in each case.

An interpretation of the forcing term $F_s^{(2)}$, depicted in Figure 3.8(c) requires a more

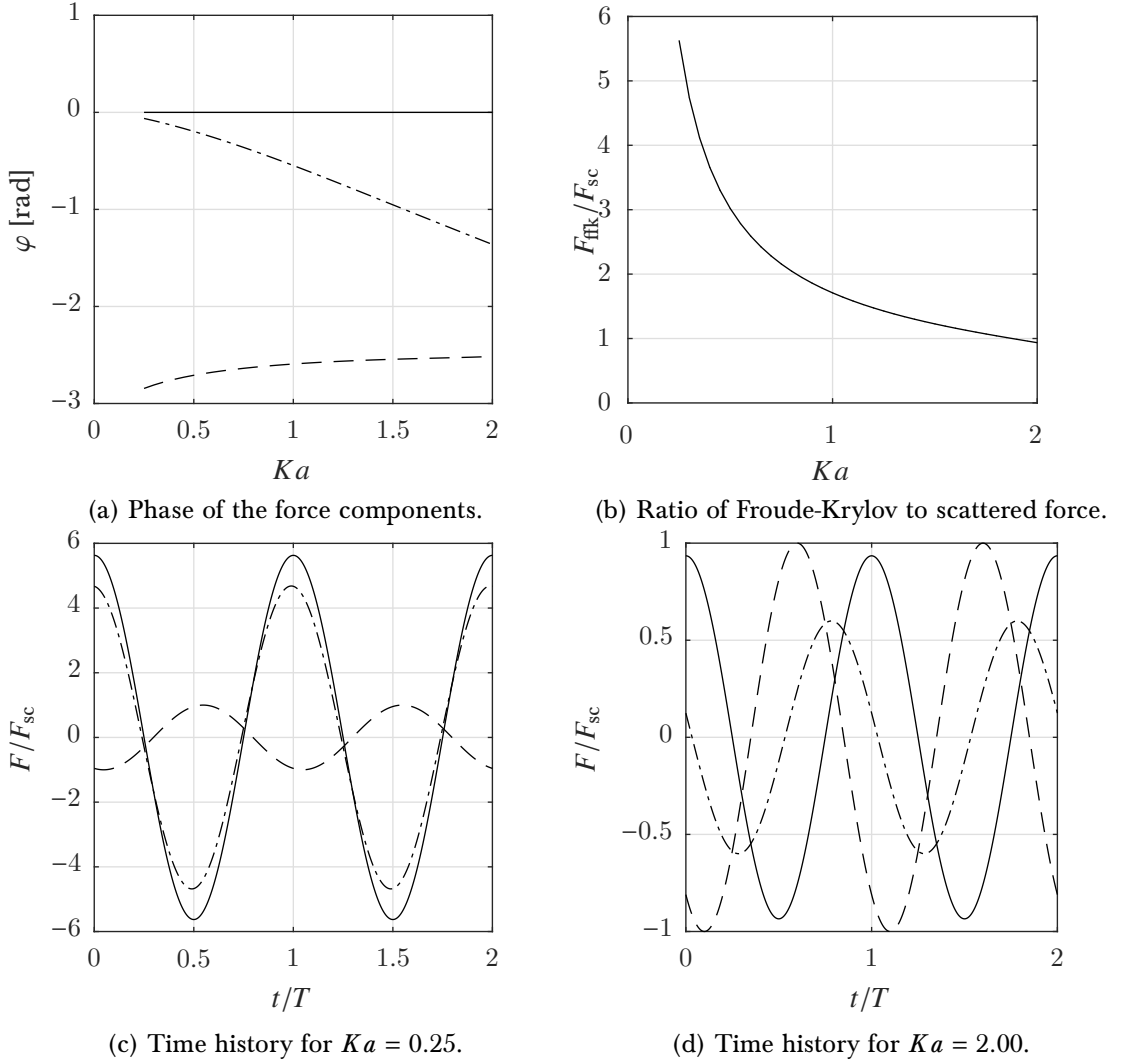
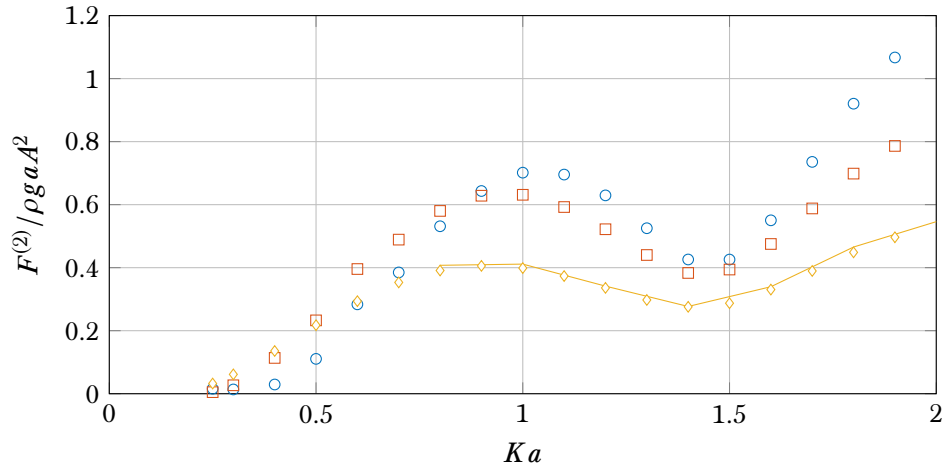


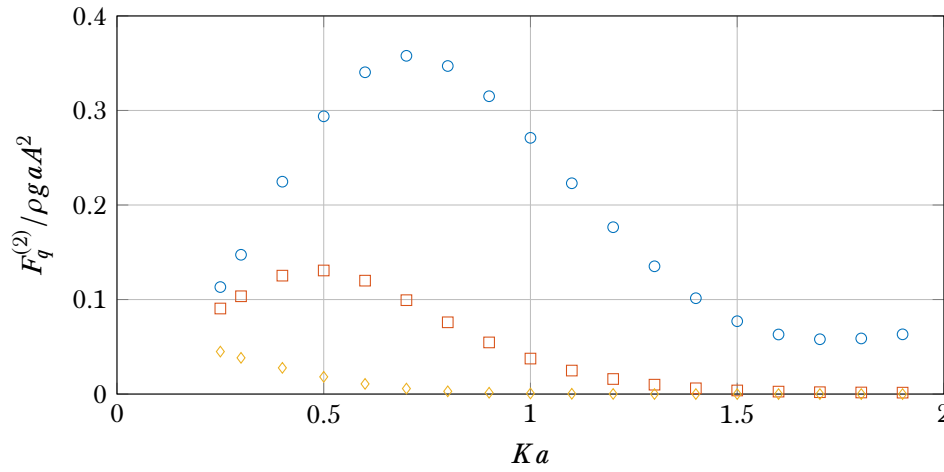
Figure 3.7: Properties of the first-harmonic force for TC2 separated into: (—) Froude-Krylov, (---) scattered, (- · -) total.

detailed discussion, which is provided in §3.5.3 below. At this stage it is important to highlight the relative contribution of $F_q^{(2)}$ and $F_s^{(2)}$.

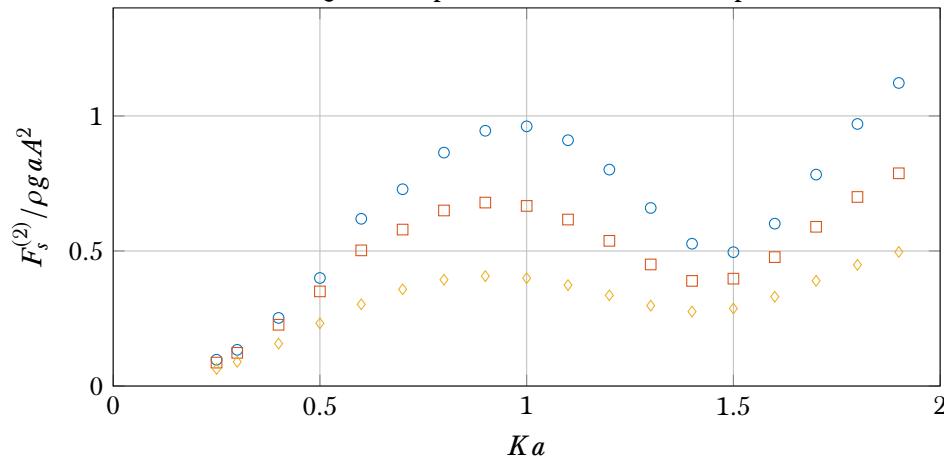
Figure 3.9(a) shows the ratio of $F_q^{(2)}$ to $F_s^{(2)}$ which is below 1.0 for all the draughts at $Ka > 0.35$. The ratio becomes smaller with increasing Ka . In the context that $F_s^{(2)}$ has historically often been neglected, Figure 3.9(a) re-emphasises the importance of calculating the full second-order solution. As Figure 3.9(b) demonstrates, the two components are approximately π rad out of phase over the entire range of Ka ; therefore the total second-harmonic force is generally smaller than the part due to the second-order scattered component alone.



(a) Total second-harmonic forcing



(b) Quadratic products of first-harmonic potentials



(c) Second-harmonic potentials

Figure 3.8: Components of the total second-harmonic excitation force for: (○) TC1, (□) TC2, (◇) TC3, (—) TC3 (Teng & Kato, 1999).

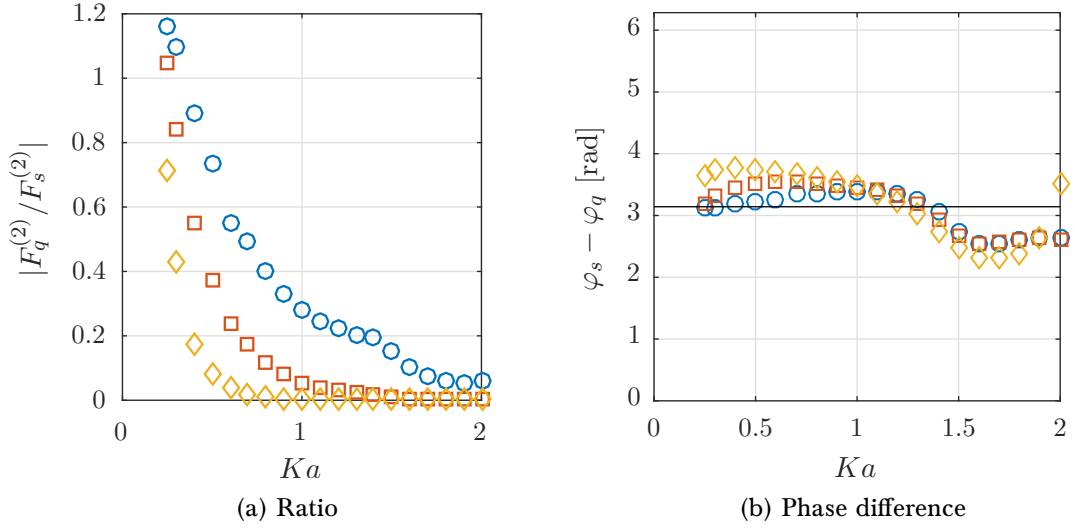


Figure 3.9: Relative quantities of the components of the second-harmonic forcing with: (○) TC1, (□) TC2, (◇) TC3, (—) $\varphi = \pi$.

3.5.3 An interpretation of the second-harmonic scattered force

As seen in §2.4.8, the second-order potential is computed from the solution of the integral equation (A.20) with a right hand side given by

$$-\int_{S_B} 2\pi r \frac{\partial \phi_{Im}^{(2)}}{\partial n} g_m^+ ds + \frac{1}{g} \int_{S_F} 2\pi r q_m g_m^+ dr, \quad (3.1)$$

in which the first term represents the no flow condition through the body and the second term, the so-called free-surface integral, represents the interaction of the first-harmonic incoming incident and the outgoing scattered waves (Lighthill, 1979, §12). Another interpretation, following Huang & Eatock Taylor (1996, §2.3), is that the former represents a free-wave due to the diffraction of a second-order incident potential and the latter a ‘locked’ or bound wave due to the first-order forcing of the free-surface. Rodríguez *et al.* (2016) show that the interaction between the incident and reflected wave in the two-dimensional case accounts for almost the entire second-harmonic forcing.

Longuet-Higgins (1950), in two-dimensions, showed that the deep water approximation of this forcing term is driven by the second-harmonic pressure

$$P_{\text{deep}} = -2\rho g A_1 A_2 k \cos(2\omega t), \quad (3.2)$$

where A_1 and A_2 are the amplitudes of two progressive waves travelling in opposite directions. Unfortunately, a similar closed form expression is not readily available in three dimensions (though Longuet-Higgins (1950) does note such a derivation is possible). For the purpose of the present investigation, the relative contribution to the second-order scattered potential due to the two terms in equation (3.1) is instead measured by taking their L^2 -norm. As described in §2.4.6, the integral equation for the second-harmonic potential may be solved by forming a matrix system

$$[A]\{\phi\} = \{BB\} + \{FS\}, \quad (3.3)$$

in which $\{BB\}$ is the first term in equation (3.1), and $\{FS\}$ the second, both discretised over the body. The ℓ^2 -norm for each term is then given by

$$\|\mathbf{x}\| = \sqrt{\sum_{k=1}^N |x_k|^2}, \quad (3.4)$$

where x_k is an element in $\{BB\}$ or $\{FS\}$.

Figure 3.10 shows the ℓ^2 -norm for each contribution for TC2, in which the free-surface integral term clearly dominates the total second-harmonic loading. The contribution from the second-harmonic incident potential is negligible, as it is governed by the ratio (*cf* equation 2.8)

$$\phi_I^{(2)} \propto \frac{\cosh 2k(z+h)}{\sinh^4 kh}, \quad (3.5)$$

which is small for the range $0.25 \leq Ka \leq 2$ ($2.5 \leq Kh \leq 20$) under consideration due to the $\sinh^4 kh$ term. Therefore, the scattering of second-harmonic free waves is not the primary driver of the second-order excitation problem, which is in keeping with the conclusions of Rodríguez *et al.* (2016).

A feature of the two-dimensional second-harmonic pressure term, equation (3.2), is that it does not attenuate with depth. In the three-dimensional problem of a bottom mounted cylinder, Molin (1979, eq 11) showed that, in the far field, the pressure associated with the free-surface integral is proportional to $\cosh[k\sqrt{2+2\cos\theta}(z+h)]$, where θ is the azimuthal angle measure from the lee-side of the cylinder. Therefore, when $\theta = \pi$ (the weather side), the pressure is depth-constant. Far away from the cylinder, the curvature of the radially

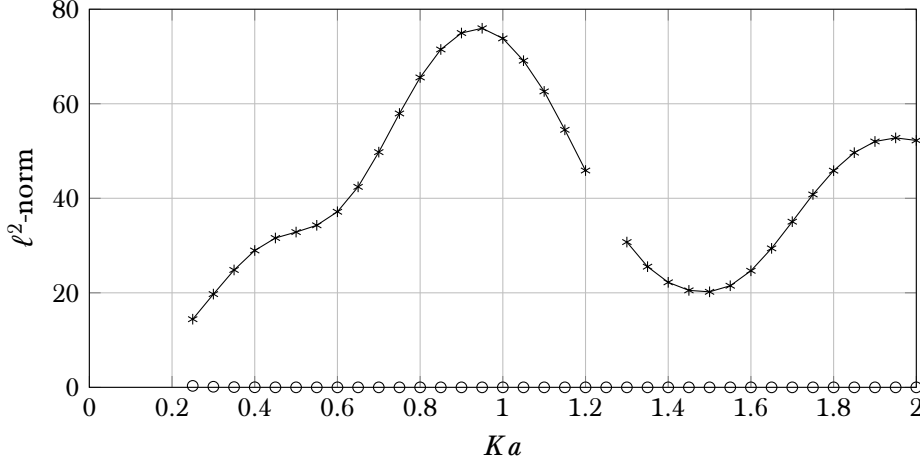


Figure 3.10: Contribution to the second-order scattered potential in terms of the ℓ^2 -norm for TC2: ($\circ$) body boundary condition, (*) free-surface integral, (—) total.

spreading scattered waves is small such that they appear long crested to the oncoming incident wave. The interaction mechanism, therefore, will be similar to the true two-dimensional case given by equation (3.2). Closer to the body, however, the scattered-wave curvature will be larger, thereby producing a different interaction mechanism in which the pressure is not depth-constant.

Figure 3.11 shows the second-harmonic pressure distribution under body TC2 at $r/a = 0.5$ for a range of Ka , separated by Fourier mode m . These Ka have been chosen as they coincide with the maxima and minima in the forcing shown in Figure 3.8(c). It is only mode 0 that contributes to the heave forcing, as shown by equation (2.41); all other modes have a symmetric distribution about $r = 0$ so that their individual pressure contributions cancel out when considering the vertical force.

The depth-dependence of the second-harmonic pressure is in marked contrast to the first-harmonic pressure. Although the pressure is not depth-constant (as in the two-dimensional case), $P_s^{(2)}$ (Figure 3.11) attenuates much more slowly compared to $P^{(1)}$ (Figure 3.6). The effect is particularly acute at larger Ka where $P_s^{(2)}|_{z=-h}$ (Figure 3.11(c)) is approximately half the value on the base of the cylinder. In contrast, the value for $P^{(1)}|_{z=-h}$ at $Ka = 1.5$ is negligible (Figure 3.6(b)).

In terms of the local pressure on the body, Figure 3.11 also demonstrates a partial cancellation between the Fourier modes. For the Ka under consideration, the zeroth and first modes are out of phase such that the total pressure is less than that due to the zeroth (heave force contributing) mode only. The cancellation effect creates a maximum in total

pressure some distance below the base of the cylinder. This again is in contrast with the linear pressure which decreases in a strict monotonic fashion. This local maximum could have important implications when considering auxiliary components that rest below the main structure.

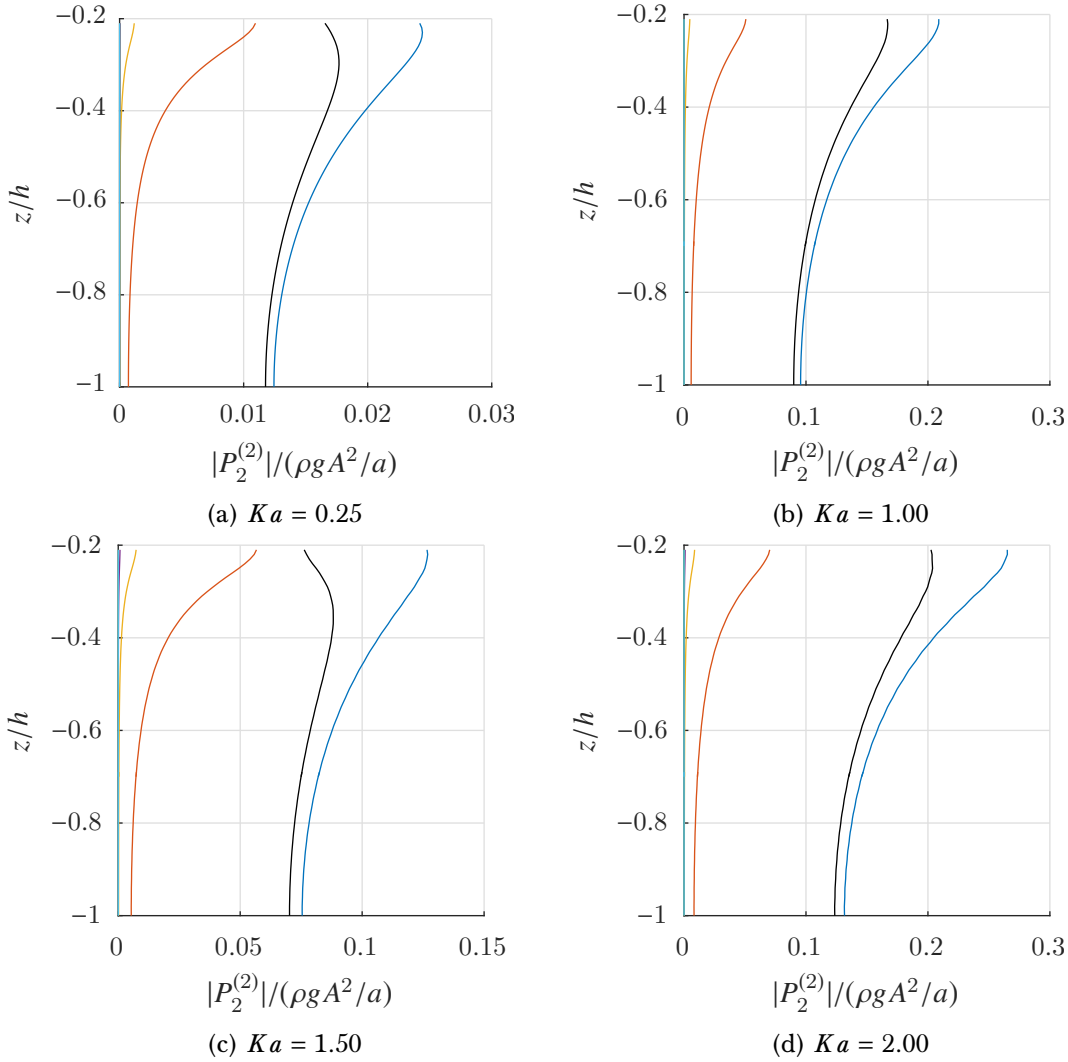


Figure 3.11: Section at $r/h = 0.5$ of the second-harmonic pressure distribution under TC2 at $\theta = 0$. Modes (—) 0, (—) 1, (—) 2, (—) 3, (—) 4, (—) 5, (—) total.

3.5.4 The effect of wave steepness

In the previous section, all figures were normalised with respect to the order of the wave amplitude. In this normalised form, it is difficult to express the absolute relevance of the second-harmonic force contribution. To overcome this, the present section provides computations of the force contributions for select values of the wave steepness, defined by

$A_I k$, varied between 0.02 and 0.18, corresponding to weakly to strongly nonlinear incident waves. Figure 3.12 shows the ratio of the total second-harmonic excitation force to the first-harmonic excitation force for $Ka = 1.00$. For both geometries the ratio scales linearly with $A_I k$ since $|F^{(2)}|/|F^{(1)}| \sim (A_I k)^2/A_I k$. The figure shows that the second-harmonic can account for as much as 35% of the first-harmonic at $A_I k = 0.12$, reaching in excess of 50% for $A_I k = 0.18$ and geometry TC2. In the case of TC3 (not shown in Figure 3.12), at $Ak = 0.125$, the second-harmonic forcing is 150% of the first-harmonic forcing, though given the deep draught, the total magnitude of the force is small. In any case, the nonlinear forcing is more pronounced for deeper drafts, which is due to the slower decay of the second-harmonic pressure (§3.5.2).

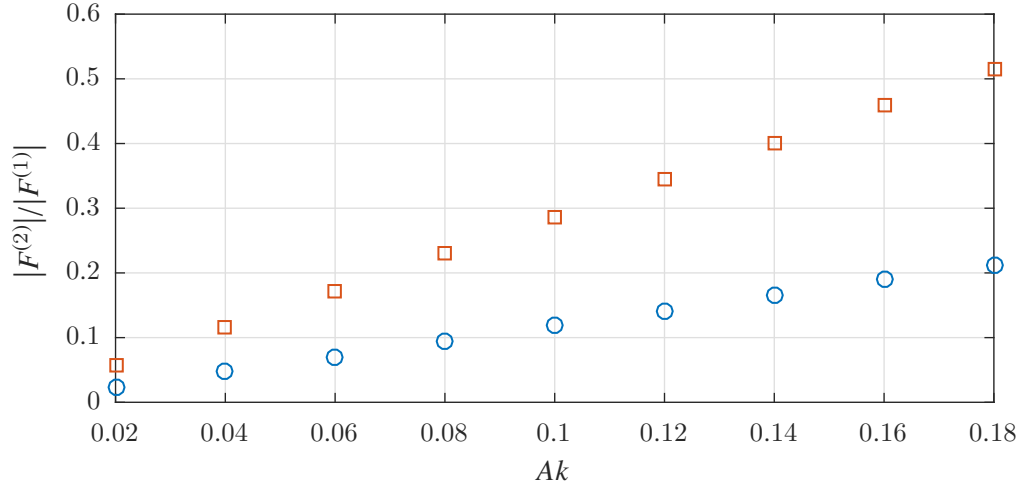


Figure 3.12: Ratio of total-second-harmonic to first-harmonic excitation force at $Ka = 1.00$: ($\circ$) TC1, ($\square$) TC2.

The solution implemented by ICRD3D is accurate up to the second order of wave steepness, and cannot account for effects at higher orders. In the two-dimensional case with similar draughts, Rodríguez *et al.* (2016) showed departures from the second-order solution of up to 10%. In adopting a fully nonlinear solver, they attribute these departures to effects beyond second order. For example, at third order, the first-harmonic force component is affected, thereby altering the ratio $F^{(2)}/F^{(1)}$.

3.6 Part B: The radiation problem

The coupled second-order radiation-diffraction problem may be specified in a number of ways as first explained in §2.4.10. To recapitulate, the combined radiation-diffraction

problem is decomposed such that the total potential can be expressed as

$$\Phi = \varepsilon \left(\phi_I^{(1)} + \phi_S^{(1)} + \phi_R^{(1)} \right) + \varepsilon^2 \left(\phi_I^{(2)} + \phi_S^{(2)} + \phi_R^{(2)} \right), \quad (3.6)$$

where the subscripts represent the incident, scattered, and radiated potentials respectively. At first order, the separation of these three potentials is clear as the scattered and radiated waves are independent. At second order, however, such a decomposition is not unique as the inhomogeneous free-surface boundary condition, equation (2.13), contains products of the linear potential. A commonly adopted (Kim & Yue, 1990; Moubayed & Williams, 1994) and consistent decomposition is to let

- i. $\phi_S^{(2)}$ represent the combined potential due to scattering of second-harmonic incident waves and due to the quadratic forcing of the first-harmonic diffraction and radiation problems, and
- ii. $\phi_R^{(2)}$ represent the radiated waves due to second-harmonic motions.

This decomposition is considered consistent as the combination of $\phi_I^{(2)}$ and $\phi_S^{(2)}$ form the excitation force for the second-harmonic motions, in an analogous fashion to the first-order problem. Therefore, $\phi_R^{(2)}$ solely concerns the radiation of second-harmonic free-waves into the far field due to second-harmonic motions.

3.6.1 First-harmonic wave generation

The perhaps simplest method to express the first-order radiation problem is to consider the so-called hydrodynamic coefficients, comprising the added mass, $m_r^{(1)}$, and the radiation damping, $d_r^{(1)}$. In terms of the wave field, the added mass arises from local evanescent modes required to satisfy the body boundary condition, whereas the radiation damping term quantifies the propagation of the free-wave into the far field.

Figure 3.13(a) shows the added mass coefficient for the three geometries under consideration. For the shallowest draught (solid line) $m_r^{(1)}$ decreases over the range $0.25 \leq Ka \leq 1.0$ before increasing slightly for $Ka \geq 1$. For the deeper two draughts however, the added mass coefficients remain relatively constant throughout the range of Ka considered, having already attained their added mass at infinite frequency.

The radiation damping coefficients are shown in Figure 3.13(b), in which $d_r^{(1)}$ decreases with draught. As the radiation damping coefficient expresses the energy propagated from

the body into the far field, the deeper draught therefore inhibits the generation of freely propagating waves. The radiation damping coefficient is approximately zero as $Ka \rightarrow 0$ and as $Ka \rightarrow \infty$ with the shape between these limits being dependent on the geometry. The lower limit physically represent the case in which the body is not moving, and no wave can be generated. The upper limit reflects the notion that the rapidly changing pressure field cannot generate waves. In the present case, the maximum value is attained for TC1, but not the for the other geometries which are poor wave generators throughout the Ka range in consideration.

Finally, Figure 3.13(c) shows the magnitude of total radiation force on the body calculated by

$$F^{(1)} = \xi^{(1)} \left(-m_r^{(1)} \omega^2 + d_r^{(1)} i \omega \right), \quad (3.7)$$

where $\xi^{(1)}$ is the first-harmonic displacement amplitude of the body. The trend of increasing $F^{(1)}$ with increasing Ka in Figure 3.13(c) is due to the dominance of the added mass term; at $Ka = 0.25$, the added mass force is already four times greater than the radiation damping force. The added mass force then increases with ω^2 , so that at $Ka = 2.0$, it is 65 times greater than the force due to radiation damping. In addition, the radiation damping coefficient also decreases with increasing Ka (Figure 3.13(b)).

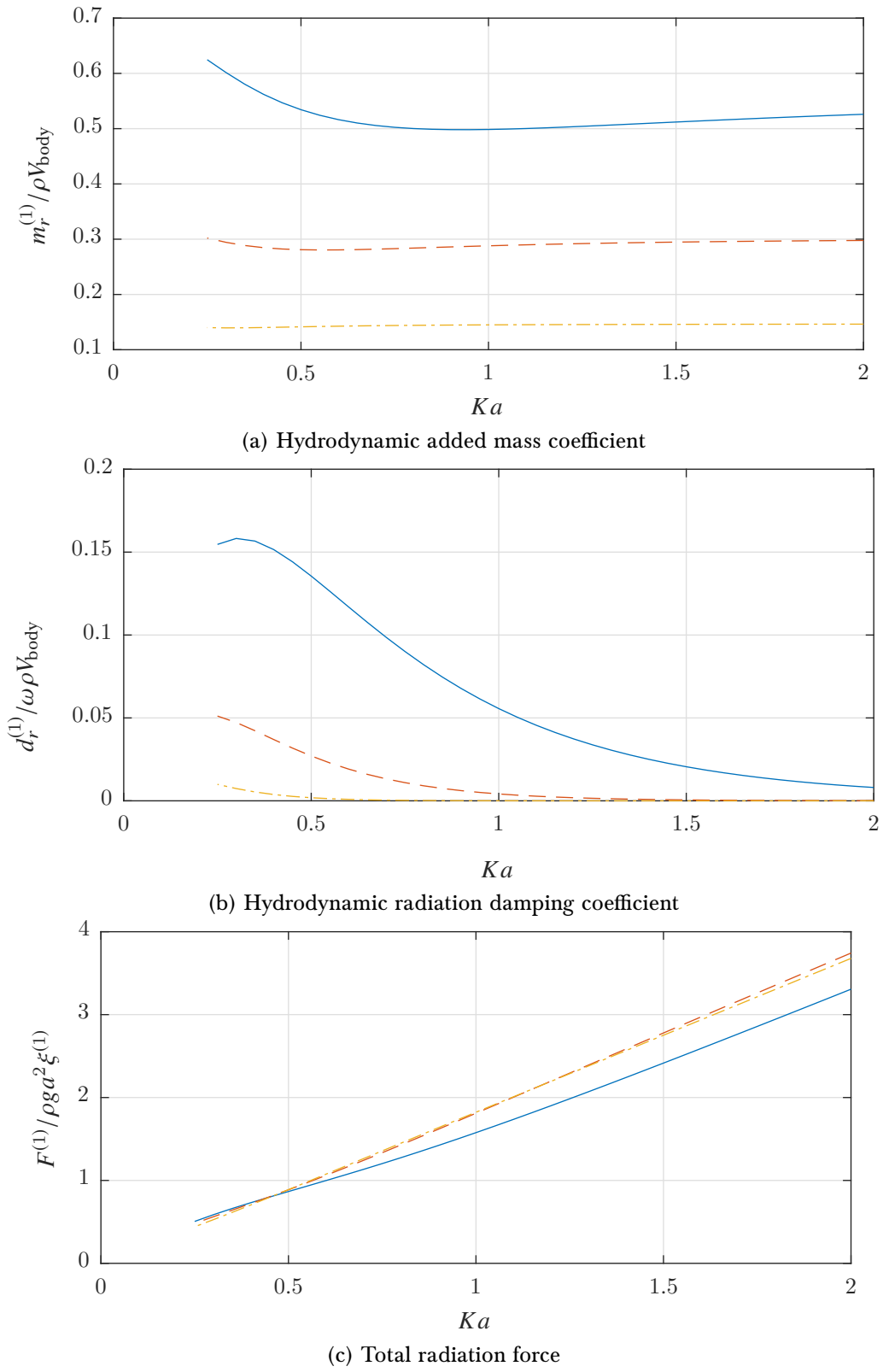


Figure 3.13: Quantities associated with the radiation problem calculated with ICRD3D, where V_{body} is the volume of the body: (—) TC1, (---) TC2, (-.-) TC3.

3.6.2 Second-harmonic wave generation

As noted at the start of this section, the second-order radiation problem solely concerns the generation of second-harmonic free-waves due to second-harmonic body motions. These body motions, in the coupled problem, will be driven by a second-order forcing term.

The second-order radiation problem is, therefore, characterised by the hydrodynamic coefficients in a near-identical fashion to the first-harmonic problem. The difference is that the added mass, $m_r^{(2)}$, and radiation damping, $d_r^{(2)}$, are calculated at twice the first-harmonic frequency. Figure 3.14 shows the total second-harmonic radiation force for TC1 and the contributions of the radiation damping and added mass. The total force is dominated by the added mass. As a result, this force is a factor 4 greater than $F^{(1)}$ due to the $(2\omega)^2$ term associated with the added mass force. The negligible radiation damping term demonstrates that the body will not generate significant second-harmonic free waves.

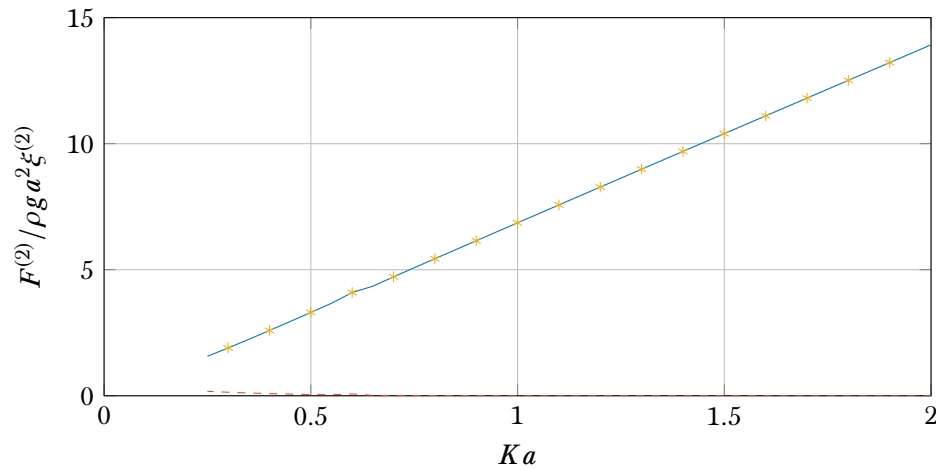


Figure 3.14: Second harmonic radiation force for TC1 showing contributions from (—) added mass, (---) radiation damping, and (*) total.

3.7 Part C: The floating body problem

The discussion above has highlighted sources of nonlinear forcing, both from the excitation and radiation problems. In the excitation problem, the nonlinear forcing is primarily due to $F_s^{(2)}$, which arises from the interaction of first-harmonic incoming incident and outgoing scattered waves. The nonlinear forcing from the radiation problem is driven by the local wavefield (or added mass term). This Part C explores how these nonlinearities combine

in the floating body problem.

To facilitate the discussion that follow, Table 3.1 defines the various forces for the combined problem and their constituent velocity potentials. This breakdown was explained fully at the start of §3.6. A force related to the combined problem has the subscript C .

Table 3.1: Forces for the combined floating body problem discussion

Symbol	Constituent terms	Harmonic	Force
$F_S^{(1)}$	$\phi_S^{(1)}$	1	excitation
$F_R^{(1)}$	$\phi_R^{(1)}$	1	radiation
$F_C^{(1)}$	$\phi_S^{(1)} + \phi_R^{(1)}$	1	combined
$F_S^{(2)}$	$\phi_S^{(2)}, \phi_S^{(1)}$	2	excitation from $\phi_S^{(2)}$
$F_{SC}^{(2)}$	$\phi_S^{(2)}, \phi_S^{(1)} + \phi_R^{(1)}$	2	combined from $\phi_S^{(2)}$
$F_q^{(2)}$	$\phi_S^{(1)}$	2	excitation from products of $\phi_S^{(1)}$
$F_{qC}^{(2)}$	$\phi_S^{(1)} + \phi_R^{(1)}$	2	combined from products of $\phi_S^{(1)} + \phi_R^{(1)}$
$F^{(2)}$	$F_S^{(2)} + F_q^{(2)}$	2	total 2nd-order excitation
$F_C^{(2)}$	$F_{SC}^{(2)} + F_{qC}^{(2)}$	2	total 2nd-order combined

3.7.1 Combining the first-harmonic excitation and radiation forces

Figure 3.15 compares $F_S^{(1)}$ and $F_C^{(1)}$. For Ka less than the resonance frequency, the relative phasing between the $F_S^{(1)}$ and $F_R^{(1)}$ causes a force reduction in $F_C^{(1)}$ compared to $F_S^{(1)}$. As shown in Figure 3.7(a), $F_S^{(1)}$ is approximately in phase with the surface elevation in this Ka region. In long waves, the body follows the surface such that the body displacement is in phase with the surface elevation. As discussed in §3.6.1, the added mass term dominates $F_R^{(1)}$. Consequently, the component $F_R^{(1)}$ is approximately in phase with the body acceleration, and π rad out of phase with the surface elevation.

In proximity of the body resonance, the displacements are maximum and $F_R^{(1)}$ dominates the overall forcing, $F_C^{(1)}$. For Ka greater than the resonance frequency, the body displacements start to reduce, thereby reducing $F_R^{(1)}$, until at approximately $Ka = 2.0$ for TC1 and at $Ka = 1.5$ for TC2 $F_C^{(1)}$ is dominated by $F_S^{(1)}$.

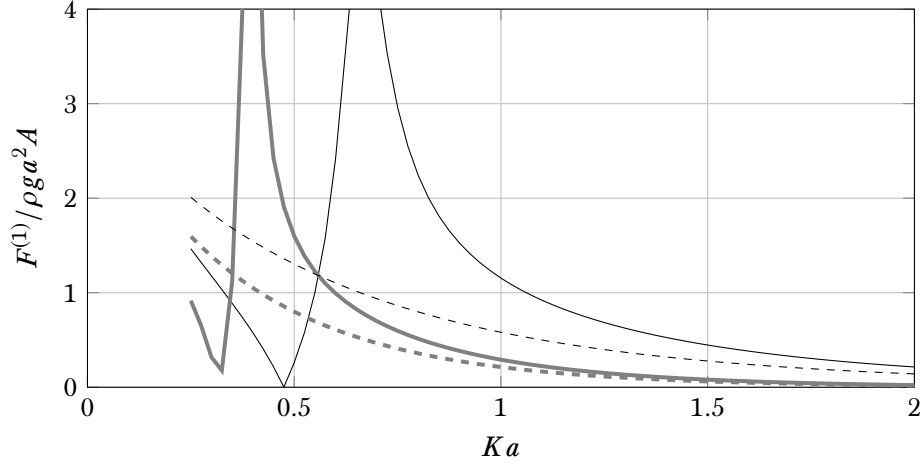


Figure 3.15: First-harmonic force for (—) TC1 $F_C^{(1)}$ and (—) TC2 $F_C^{(1)}$. (---) TC1 $F_S^{(1)}$ and (- - -) TC2 $F_S^{(1)}$.

3.7.2 Second-harmonic excitation force

Figure 3.16 shows the impact on the second-harmonic forcing components when the body undergoes first-harmonic motions. For $F_{qC}^{(2)}$, the behaviour contrasts between the two geometries: TC1 generally shows a reduction in the force magnitude around the resonance frequency of the body whereas TC2 shows an increase in force magnitude. Furthermore, at high Ka , $F_{qC}^{(2)}$ is significantly greater than $F_q^{(2)}$, and reflects the ratio of the magnitude of $F_R^{(1)}$ (Figure 3.13(c)) to $F_S^{(1)}$ (Figure 3.5).

Figure 3.16(b) shows the dramatic impact the body undergoing first-harmonic displacements has on $F_S^{(2)}$. Around the first-harmonic resonance period for both geometries, the body is undergoing large displacements meaning $\phi_R^{(1)}$ makes significant contributions to $F_{SC}^{(2)}$. At higher Ka , where the first-harmonic displacements are small, $F_{SC}^{(2)}$ converges to $F_S^{(2)}$. In close similarity to the diffraction only case (Part A §3.5), the magnitude of $F_{SC}^{(2)}$ is greater than $F_{qC}^{(2)}$ and therefore dominates the total second-harmonic excitation force $F_C^{(2)}$ as shown in Figure 3.16(c).

3.7.3 Second-harmonic motions

Once $F_{SC}^{(2)}$ has been calculated, the second-harmonic response amplitude operator is readily obtained after computing the second-harmonic hydrodynamic coefficients as outlined in §3.6.2. Figure 3.17 shows this second-harmonic response along with the linear RAO. Both geometries exhibit similar behaviour: the maximum second-harmonic response occurs

at the linear resonant frequency and is generally smaller due to the inertia of the body resisting the higher frequency motions.

Moubayed & Williams (1994) undertook second-order computations at low Ka and found that there was a second-harmonic resonance at a frequency approximately half that of the first-harmonic resonance. Indeed, at this frequency, they found the second-harmonic motions to be significantly greater than those at first-harmonic. This effect can be seen for TC1 in Figure 3.17(a), where an additional second-harmonic resonance frequency occurs at $Ka = 0.16$. For $Ka < 0.4$, the second-harmonic RAO begins to increase to approach this resonance frequency. Moubayed & Williams (1994) show multiple second-harmonic resonances at low Ka , which is in marked contrast to the linear response which attains a value of 1 when Ka is small.

The ratio of the first- and second-harmonic response combines the essence of the discussion above. The second-harmonic response is dominated by the radiation problem in low and intermediate Ka , and leads to a significant second-harmonic contribution to the overall motion particularly at low Ka . At resonance, the body motions will be at their largest, but due to the added mass (Figure 3.14) and inertia components, the second-harmonic response is smaller compared to the first-harmonic. At larger values of Ka , the radiation problem plays almost no part, meaning the problem is solely driven by $F_S^{(2)}$. At these higher Ka , however, the overall inertia of the system prevents the motion at both first- and second-harmonic.

In order to investigate the phasing of the first- and second-harmonic motion components, Figure 3.18 shows the time history for a range of Ka at steepness $A_I = 0.14$ for TC1. In the $Ka = 0.25$ case, Figure 3.18(a), the magnitude of the second-harmonic motions are significant and are in phase with the first-harmonic motions, leading to a strong crest-trough asymmetry. At $Ka = 0.75$, the phasing of the components is such that the second-harmonic motions add to the troughs and subtract from the crests. The phase relationship at $Ka = 1.00$ is similar to that of $Ka = 0.75$, but due to the smaller relative magnitude of the second-harmonic component, the effect is less pronounced. At $Ka = 1.00$ the second-harmonic motion plays a negligible role at this steepness.

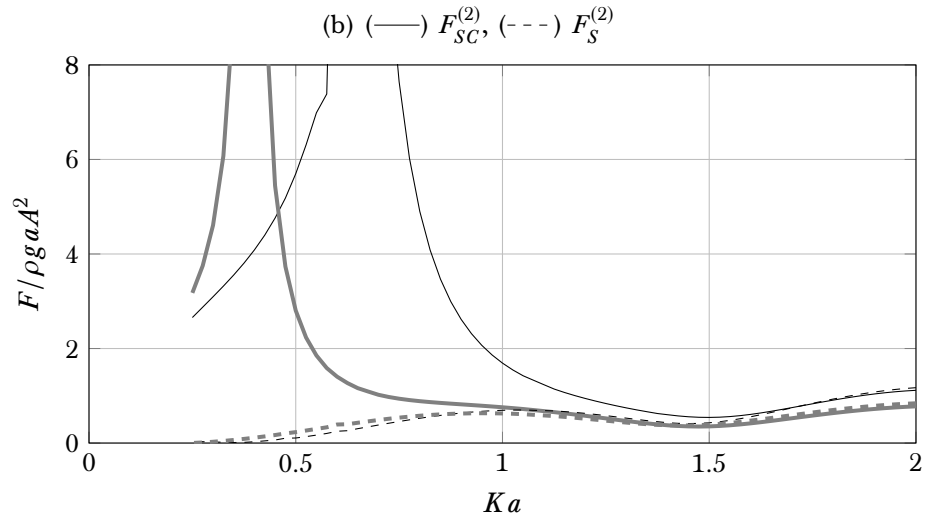
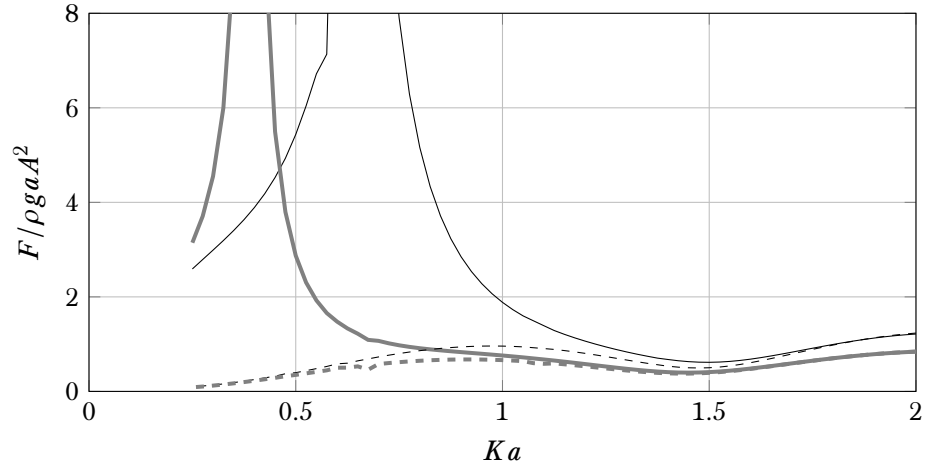
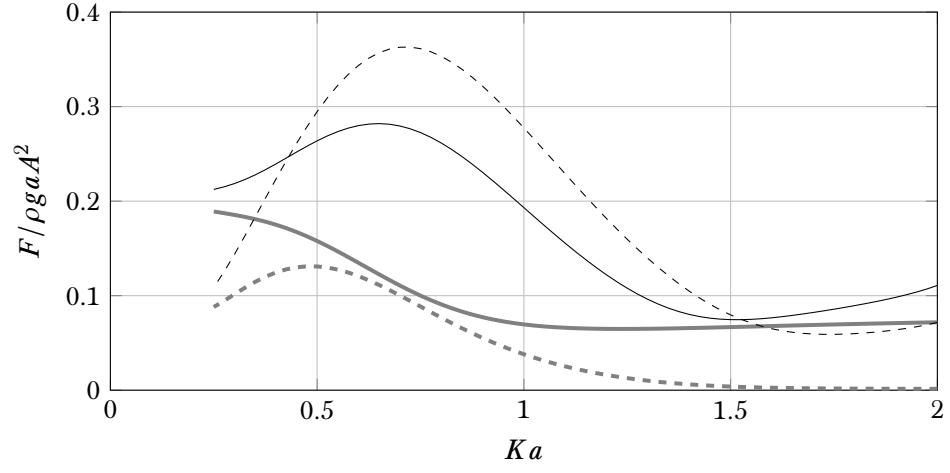


Figure 3.16: Second-order forces for (—) TC1 and (—) TC2 for the combined problem

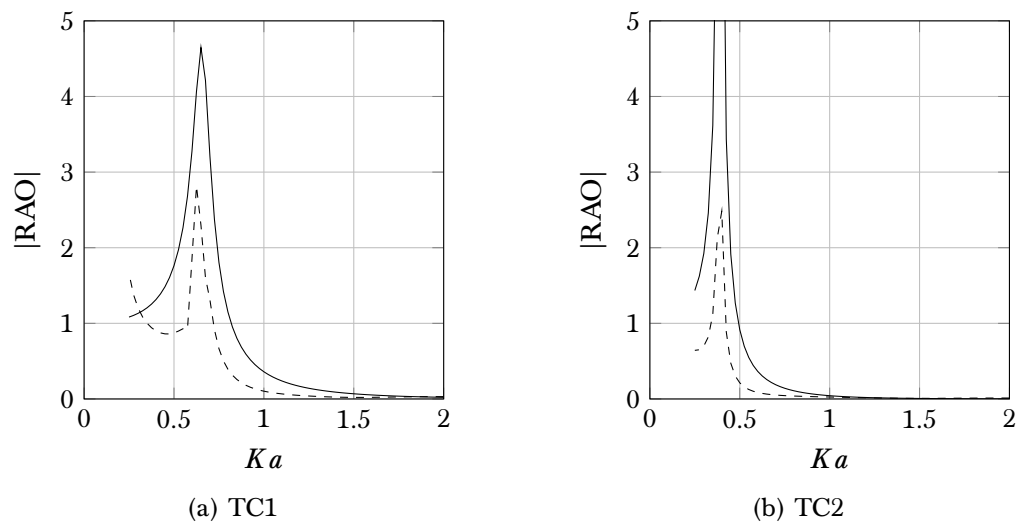


Figure 3.17: First-harmonic (—) and second-harmonic (---) RAOs for TC1 and TC2.

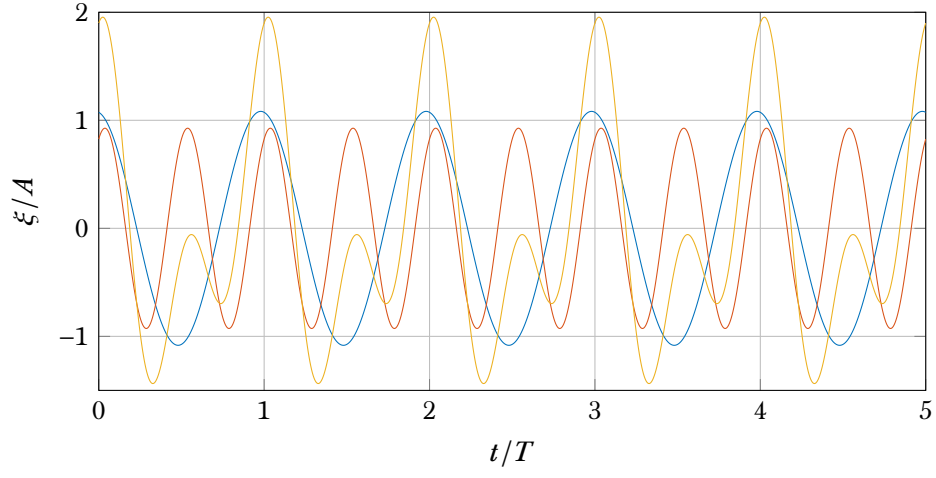
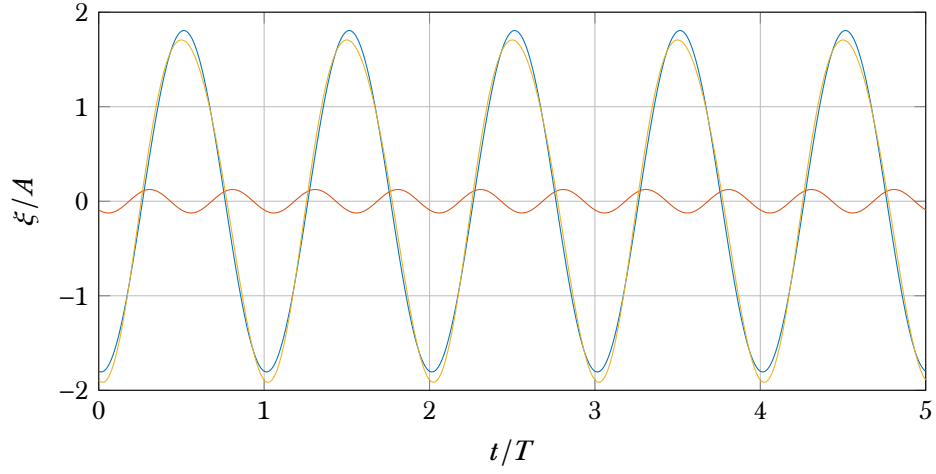
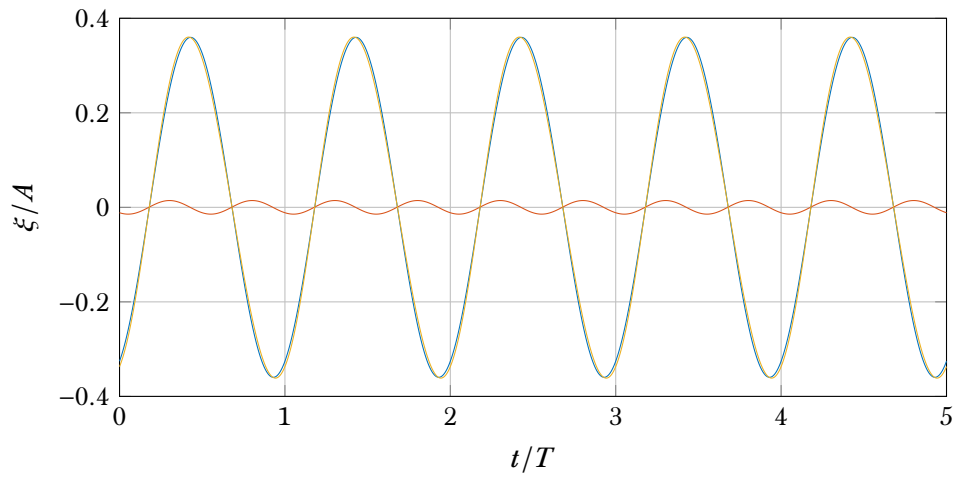
(a) $Ka = 0.25$ (b) $Ka = 0.75$ (c) $Ka = 1.00$

Figure 3.18: First-harmonic, second-harmonic, and total TC1 response in time for $A_I k = 0.14$:
 (—) $\xi^{(1)}$, (—) $\xi^{(2)}$, (—) ξ .

3.8 Concluding remarks

This chapter has highlighted the importance of nonlinearity in the description of a heaving truncated cylinder. The sources of nonlinear excitation have been shown to result in significant second-harmonic motion.

The nonlinear contribution in the fixed body excitation problem is significant both in terms of magnitude and phase. Of the two components in the second-harmonic force, the interaction between the first-harmonic scattered and incident wave dominates. The force due to the quadratic products of the linear potential is smaller in magnitude, but is approximately π rad out of phase with $F_s^{(2)}$, thereby reducing the overall force. Nevertheless, the second-harmonic force makes a significant contribution to the total excitation force.

Once the body starts to undergo first-harmonic motion, the second-harmonic excitation force is dominated by the addition of the first-harmonic radiation potential. This is due to the interaction of the outgoing first-harmonic radiated wave with the incident and scattered waves. At the first-harmonic resonance, the contribution is greatest, leading to significant second harmonic motions. The results thus far have been calculated using potential flow theory in the absence of viscous effects: the following chapters explore the forcing and dynamic response in a real fluid.

4

Nonlinear forcing in irregular sea states

4.1 Chapter overview

Chapter 3 established and explained the occurrence of nonlinear forcing and motion in regular waves. This chapter extends that discussion into long duration irregular sea states using an extensive experimental programme. A novel experimental setup was designed and manufactured specifically for this task. The key benefit of this experimental apparatus is that it features an innovative internal layout to accommodate (i) the measurement of forces in this chapter, and, (ii) the measurements of displacements in the next.

The quantification of the nonlinear forcing in irregular sea states is achieved by examining the force statistics on a truncated cylinder. To generate these statistics, in excess of 300,000 individual wave events have been captured, comprising sea states of varying steepness and peak frequency. This large, new, data set is one of the major achievements of this PhD project.

The chapter begins with an exposition of the experimental model and wave flume setup in §4.2 followed by an outline of the wave conditions in sections 4.3 and 4.4. The analysis of the nonlinear heave force commences with §4.5 by examining the excitation force statistics. Comparisons are made with ICRD3D along side the observations from Chapter 3 throughout. A spectral analysis in §4.6 definitively establishes the source of the nonlinear forcing and is subsequently investigated by examining individual excitation force

events in §4.7. The chapter ends (§4.8) with an exploration of the effect of the truncated cylinder draught on the nonlinear forcing.

4.2 Experimental setup

4.2.1 Wave flume setup

The experimental programme has been carried out in the Long Flume located in the Hydrodynamics Laboratory within the Department of Civil and Environmental Engineering at Imperial College London. The flume is 63 m long, 2.79 m wide and 1.25 m deep, as shown in Figure 4.1. The uni-directional incident waves are generated at one end of the flume using four absorbing flap-type wavemakers running Edinburgh Designs software. At the other end, a highly optimised parabolic beach (Heller *et al.*, 2016) ensures that the wave reflections are minimised, which is crucial for the long irregular sea state experiments.

The truncated cylinder (full description in §4.2.2), was located approximately half-way along the tank, 29 m downstream of the wavemakers, and along the flume centreline. A series of resistance-type wave gauges were also placed along the flume centreline at discrete points to measure the water surface elevation. These gauges comprise two vertical, 1.5 mm diameter stainless steel rods spaced 10 mm apart, and can measure the surface elevation to an accuracy of ± 0.5 mm.

4.2.2 Physical model and motion constraint apparatus

A novel physical model has been developed for the experimental programme, for which full engineering drawings are included in Appendix C. As mentioned in §2.1, a truncated cylinder has been selected for the geometry. The setup, as a whole, must balance the following competing interests:

- i. scale to a realistically sized highly-dynamic structure,
- ii. be readily constructable,
- iii. be of suitable size for the experimental facility, and
- iv. allow accurate force measurements to be taken.

As detailed in §2.1, a typical WEC device has a diameter in the order of 10–20 m, operating in water depths between 50 and 100 m. Given the tank has a fixed depth of 1.25 m, the corresponding model scale factors are between 40 and 80; applying these scale

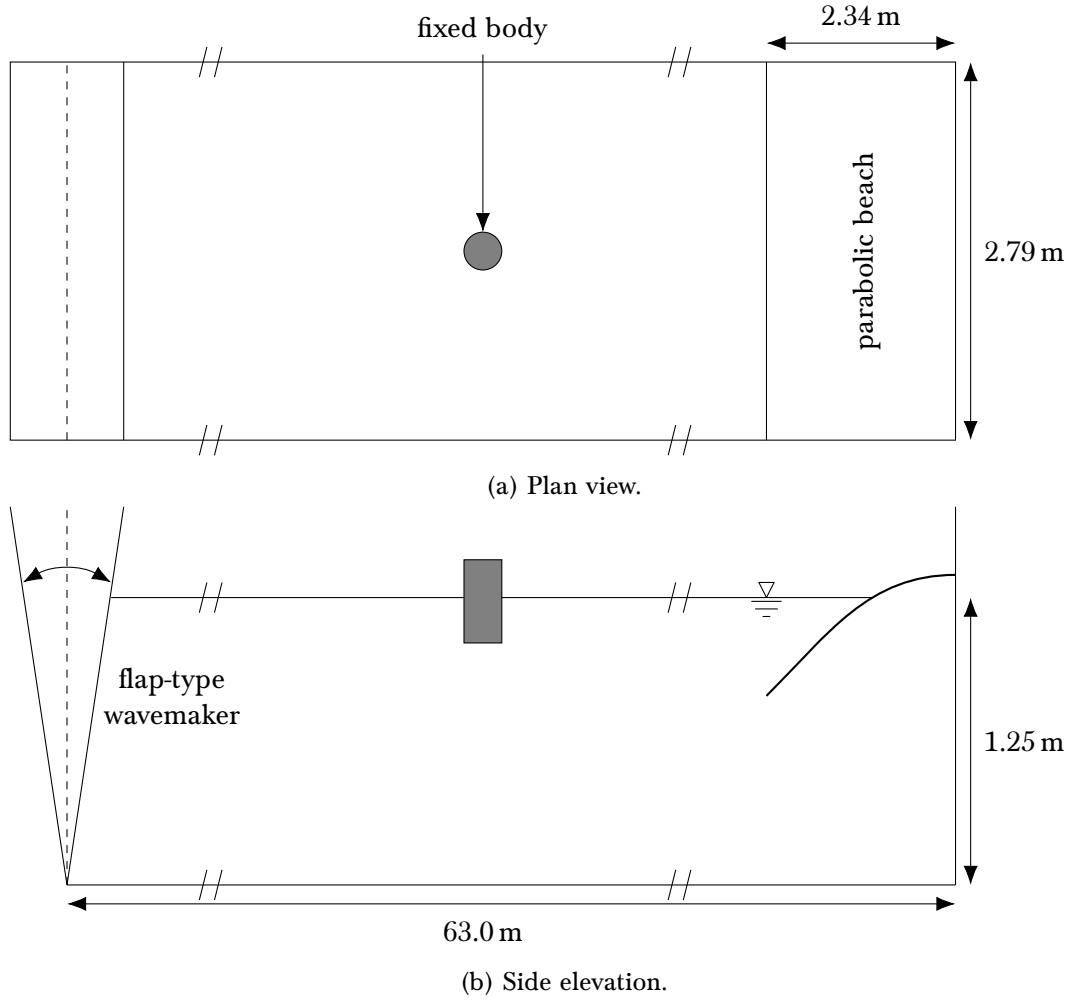


Figure 4.1: Schematic of the wave flume setup.

factors gives a model diameter, D , of 0.25 m. The side walls of the tank are therefore more than $5D$ away on both sides of the body. The effect of the side walls in the scattering and radiation of waves has been studied by several authors such as Yeung & Sphaier (1989), Thomas (1991), and Linton & Evans (1992). Yeung & Sphaier (1989) identified ‘spikes’ occurring in the excitation force and hydrodynamic coefficients using a linear potential solution and the method of images. These ‘spikes’ occur due to “symmetric transverse resonance modes” at $kw/2\pi = 1, 2, 3, \dots$, where w is the channel width. In practice, only the first three modes are important and correspond to $Ka = 0.28, 0.56$, and 0.84 . The effect is further discussed in the context of the present experimental results in §4.6 for forces and §5.3.3 for displacements.

A schematic of the truncated cylinder is illustrated in Figure 4.2, showing the key fea-

tures of the design. At the highest level, the superstructure (point 1) is mounted on beams which rest on the side walls of the tank. The buoy is confined to the heave degree of freedom by means of a fixed, 20 mm diameter, stainless steel shaft (part 3). The model can be used in one of two ways:

- i. For the purposes of the present chapter, the loads are recorded by a load cell connected between the fixed shaft and the model (part 10). The load cell reacts against any external forces acting on the buoy. To ensure that the entire heave load is transferred via the load cell, the fixed shaft is connected to the model via two linear NewWay air bushings. These air bushings operate at 413 kPa (60 psi) to provide near-zero friction resistance, and are placed away from each other to maximise the moment resisting couple.
- ii. For the purposes of the following Chapter 5, the load cell (part 10) can simply be removed to allow the truncated cylinder to move freely in heave. In this second scenario, the use of the air bushings is again crucial to ensure minimal friction resistance.

All parts of the model and support structure were manufactured by the technicians in the Hydrodynamics Laboratory.

Fixing the shaft and allowing the truncated cylinder to move or react against the load cell has two principal advantages: first the moment couple about the air bearings is reduced, and, second, the buoyancy force of the buoy is not required to act against the mass of the shaft. As a result of the latter, and the chosen material of Ebaboard L-1, the buoy's equilibrium position is at 0.125 m submergence, corresponding to geometry TC1 in the previous chapter. The TC2 geometry can be achieved by simply moving the shaft down in the case of forces (this chapter) or by adding lead-weights in the case of motions (Chapter 5).

The excitation forces are measured using an HBM S2M/200 N load cell. Load cell measurements are taken using a National Instruments 9237 input module and collected by bespoke LabView interface at 100 Hz. Before conducting the experiments, the load cell calibration was tested using a set of high precision calibration weights. This calibration was undertaken in the dry for tension and compression, and in-situ using the hydrostatic force at different submergence depths. The calibration factor certified by the manufacturer was confirmed.

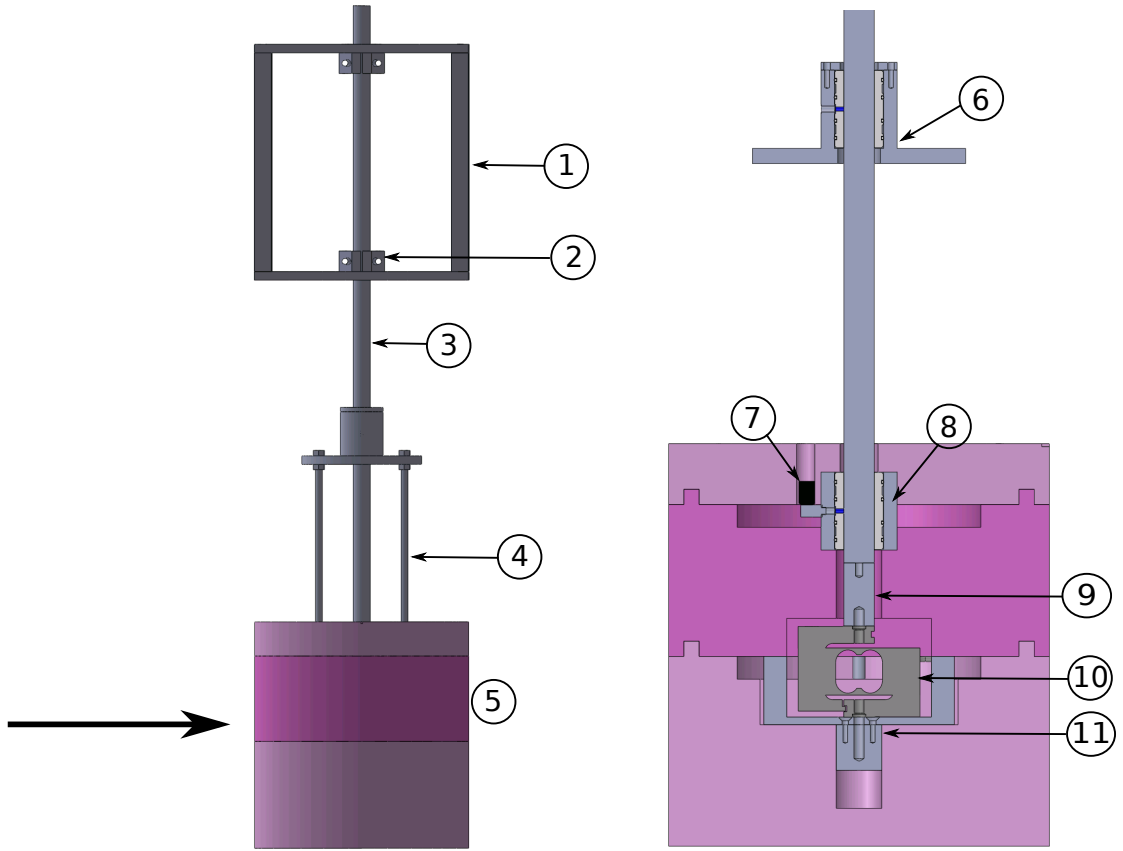


Figure 4.2: Physical model for the fixed truncated cylinder where the large arrow indicates the direction of the incident waves. Key: (1) aluminium superstructure, (2) rigid clamp blocks, (3) 20mm stainless steel shaft, (4) 2xM8 studding, (5) Ebaboard L1 buoy in three sections, (6) air bushing and housing, (7) push-fit compressed air adaptor, (8) air bushing and housing, (9) M8 to M6 adaptor, (10) bi-directional load cell, (11) load cell-buoy connector.

4.3 Wave generation procedure

In contrast to the single frequency cases considered in the previous chapter, a realistic sea state comprises multiple wave components over a range of frequencies. A plethora of distributions are available; the JONSWAP distribution (Hasselmann *et al.*, 1973) has been chosen herein as it is perhaps the most widely used of these distributions. The JONSWAP spectrum is typically defined by two parameters: the peak period, T_p , which characterises the frequency bin which contains the most energy, and the significant wave height, defined by $H_s = \sqrt{4m_0}$, where m_0 is the zeroth spectral moment. Furthermore, each frequency component is given a random phase angle in the interval $[0, 2\pi]$.

The JONSWAP energy spectrum, illustrated in Figure 4.3, is defined by

$$S_{\eta\eta}(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left(-\frac{5}{4} \frac{\omega_p^4}{\omega^4}\right) \gamma^\beta, \quad (4.1)$$

where

$$\beta = \exp\left(\frac{(\omega - \omega_p)^2}{2\omega_p^2\sigma^2}\right), \quad (4.2)$$

with

$$\sigma = \begin{cases} 0.07 & \text{for } \omega \leq \omega_p \\ 0.09 & \text{for } \omega > \omega_p. \end{cases} \quad (4.3)$$

Moreover, $\omega_p = 2\pi/T_p$ is the peak circular frequency, γ is the peak enhancement factor, and α is the Phillips parameter. The choice of γ defines the relationship between ω_p and H_s : a large value of γ concentrates the energy around the peak frequency (with a single frequency sea state being the limiting case) whereas a smaller value of γ spreads the energy more broadly across the range of frequencies (Figure 4.3). These are referred to as ‘narrow’ and ‘broad’ banded seas respectively. Realistic sea states tend to be broad-banded, with a value of $\gamma = 3.3$ regularly used in design, and adopted for the remainder of this thesis. Finally, the spectrum can be scaled using α to achieve the desired H_s .

Theoretically, the JONSWAP spectrum is defined over an infinite frequency range. Due to the physical limitation of the wavemakers, the range of generated frequencies is limited to between 0.2 and 2 Hz. These limits are indicated by the vertical dashed lines in Figure 4.3, which clearly show that for $T_p = 1.43$ s the vast majority of the sea state energy lies within the limiting band. This is also true of the remaining sea states characterised by $1.61 \leq T_p \leq 1.20$, with the maximum amount of energy truncation being less than 2.4%.

For the laboratory generation of waves, a repeat time T_r is often introduced. The repeat time ensures that the wave generation is periodic over a certain time frame which, in turn, allows a harmonic discrete Fourier transform analysis to be readily undertaken. In order to obtain statistically significant data, the experiments must be undertaken for a sufficiently long time to capture a large number of wave events. Despite the absorbing wavemakers and optimised parabolic beach, there is a progressive, small, build-up of wave reflections in the tank which prevents very long runs from being achieved. Instead, following Latheef & Swan (2013), the entire experiment comprises a number of shorter runs. Each of these

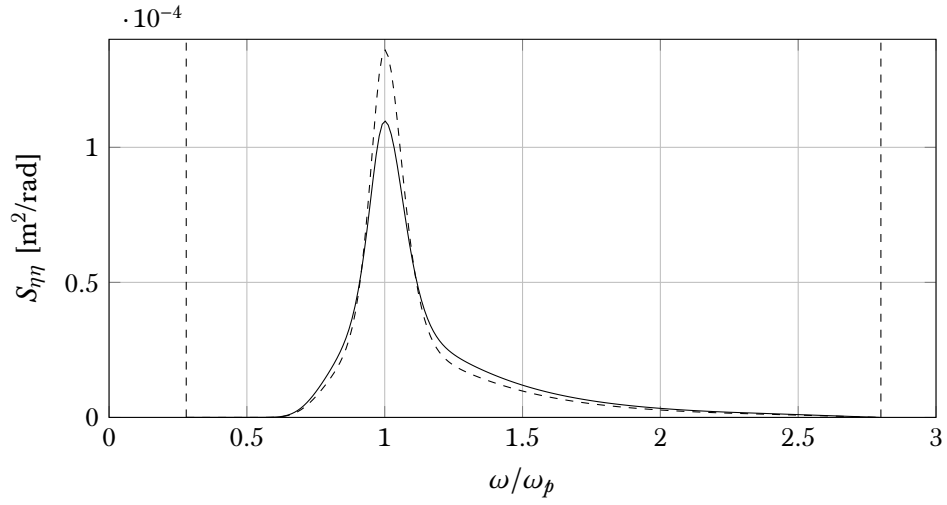


Figure 4.3: Example JONSWAP energy spectrum for $T_p = 1.43$ s, $H_s = 0.050$ m and γ (—) 3.3 and (---) 5.0.

runs carries the same energy spectrum, but is associated with a different set of random phase angles for the individual wave components (seeds). These runs are then combined to acquire the desired long duration sea state statistics. In the context of the present work, each run is undertaken for a repeat time of 1500 s, plus an additional 100 s to allow all the wave components in the spectrum to reach the body in advance of the data sampling.

To obtain statistically significant data, approximately 25000 waves have been captured per sea state, which requires approximately 23 individual runs. This allows the probability of exceedance distribution to be defined down to 10^{-4} without extrapolation. Therefore, including a settling time of 20 minutes between runs, a single sea state requires approximately 18 hours of testing time.

4.4 Incident wave conditions

Two principal parameters govern an irregular sea state:

- i. the steepness given by $\frac{1}{2}H_s k_p$, and
- ii. the peak frequency, given in non-dimensional form by $k_p h$,

where k_p is the solution of the dispersion equation at the peak circular frequency. The importance of these parameters on the sea state has been previously demonstrated by Latheef & Swan (2013). Steepness values of $\frac{1}{2}H_s k_p = 0.050, 0.075, 0.100$ have been chosen along with $k_p h = 2.0, 2.5, 3.0, 3.5$, so that a total of 12 sea states have been investigated.

At full scale (using a factor of 1:56), these nondimensional parameters result in $9.0 \leq T_p \leq 12.1$ s and $2.0 \leq H_s \leq 7$ m. In all, these 12 sea states require 216 hours of testing per geometry.

To confirm the wave generation process, the incident conditions in the absence of the structure were first investigated. Figure 4.4 shows an example measured incident spectrum compared to the corresponding linear input spectrum provided to the wavemakers. Similar comparisons were also undertaken for the remaining 11 incident sea states. In each case, the measured spectrum is found to adequately achieve the target spectrum, and is further used as the input to the numerical modelling that follows.

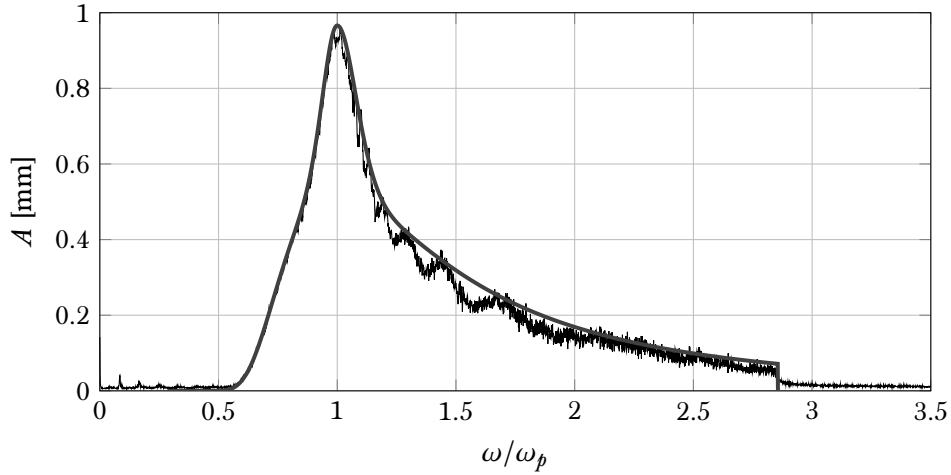


Figure 4.4: Incident wave amplitude JONSWAP spectrum for $k_p h = 2.5$, $\frac{1}{2}H_s k_p = 0.05$, and $\gamma = 3.3$. (—) theoretical input and (—) experimental measurements.

In order to quantify the nonlinear content in the incident waves, the probability of exceedance, Q , of the wave crest-heights, η_c , has been examined. Figure 4.5(a) compares experimental data for $k_p h = 2.5$ and $\frac{1}{2}H_s k_p = 0.05$ to the theoretical linear (Rayleigh) and second-order (Forristall, 2000) distributions. The experimental data can be seen to follow the Forristall (2000) distribution suggesting there is some second-order content in the incident sea state. At $Q \leq 10^{-3}$, the experimental data depart from the Forristall distribution: this only represents 0.1% of the distribution so cannot be absolutely relied on. This departure, however, is commensurate with the results of Latheef & Swan (2013) in unidirectional waves. At $k_p h = 3.5$ and $\frac{1}{2}H_s k_p = 0.100$, Figure 4.5(b) shows that there is a significant departure from the linear and second-order predictions from $Q < 10^{-1}$, which is again consistent with Latheef & Swan (2013).

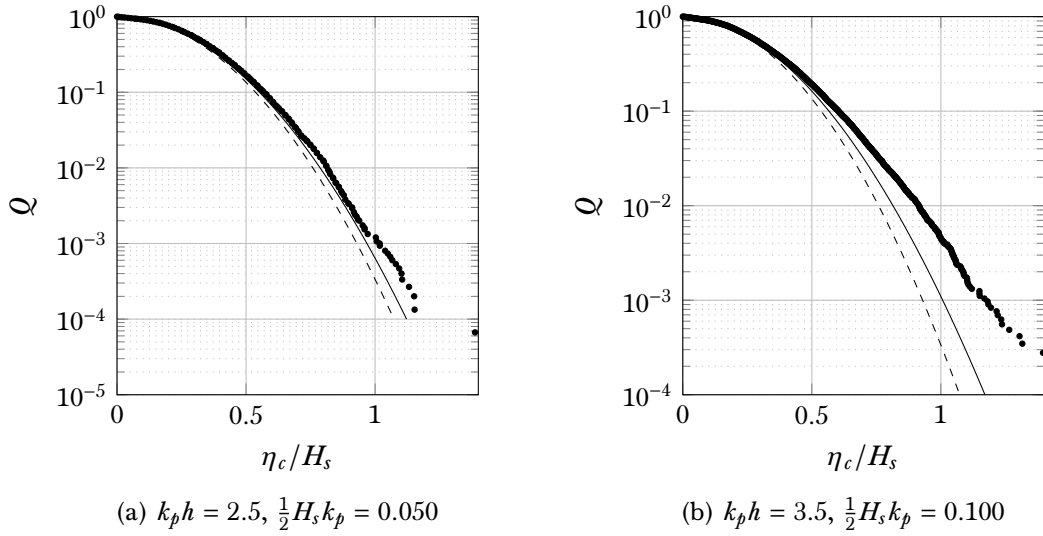


Figure 4.5: Probability of exceedance of the incident wave crests, comparing (---) Rayleigh distribution, (—) Forristall (2000) distribution, and (●) experimental data.

4.5 Nonlinear excitation forcing of TC1

Having established the incident wave conditions, the truncated cylinder was placed in the wave flume as described in §4.2.1 and §4.2.2. The discussion in this section is focused on geometry TC1; §4.8 goes on to discuss the effect of the draught using geometry TC2. Figure 4.6(a) shows a sample force time-history for $k_p h = 2.5$ and $\frac{1}{2} H_s k_p = 0.05$ over a portion of the seed 1 run. Inevitably, in a busy laboratory environment, there are a number of sources of electrical noise, located at high frequencies above those which can be attributed to wave effects. To achieve the records in Figure 4.6, a low-pass FFT filter has been applied with a cut-off frequency of 5 Hz. This type of filter is characterised by zero phase shift and no signal attenuation in the passband. Applying this filter did not affect the observed force maxima or minima but did prevent a large number of artificial ‘events’ from being included in the subsequent up-crossing analysis.

The calculations undertaken in the previous chapter assumed an inviscid and irrotational flow field. This assumption no longer holds in an experimental context meaning vortex-shedding effects may be present. Vortices are likely to be shed both due to separation of the flow around the circumference of the body as well as due to flow around the sharp corners of the truncated cylinder. Adopting the definition of the Keulegan-Carpenter number in irregular seas from Longoria *et al.* (1991) gives $KC_r \approx 1$ for the sea states under consideration, suggesting that vortex-shedding around the circumference of the body will

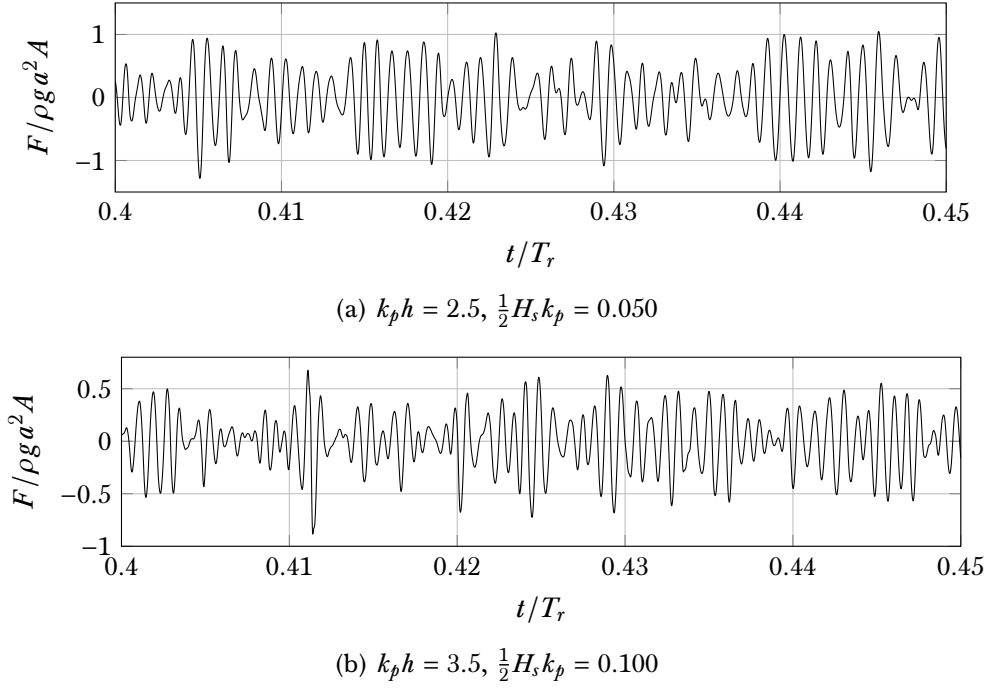


Figure 4.6: Sample time-history of the heave excitation force in irregular seas.

be small. Flow around sharp corners induces vortex shedding, even at low KC as described by Graham (1980), however following the work of Rodríguez & Spenneken (2016) the effect is anticipated to be small. The effect of the sharp corners on the excitation force is discussed further in Chapter 6 and confirmed to be negligible.

The probability of exceedance is obtained for both the force maxima and minima and shown in Figure 4.7 for all the sea states under consideration. In Figure 4.7 the force maxima are shown by the dashed, coloured lines, the force minima by the solid, coloured lines and the linear, analytical solution by the solid black line. The linear analytical solution predicts equal distribution of force minima and maxima, and is therefore represented with a single line. The difference in the experimental distributions of maxima and minima is the first, qualitative, evidence of the nonlinear excitation forcing on TC1.

The evidence of the nonlinearity arising in the excitation forcing is provided by the increase in (absolute) force minima and the decrease of the force maxima with increasing sea state steepness. For $k_p h = 2.0$ the force minima increases by 16% from $\frac{1}{2} H_s k_p = 0.050$ to 0.100 at $Q = 10^{-3}$. As the largest absolute forces are usually the most interesting in design, Figure 4.8 quantifies the increase of the force minima by normalising each of the measured force minima distributions in Figure 4.7 by the corresponding linear prediction. Even in

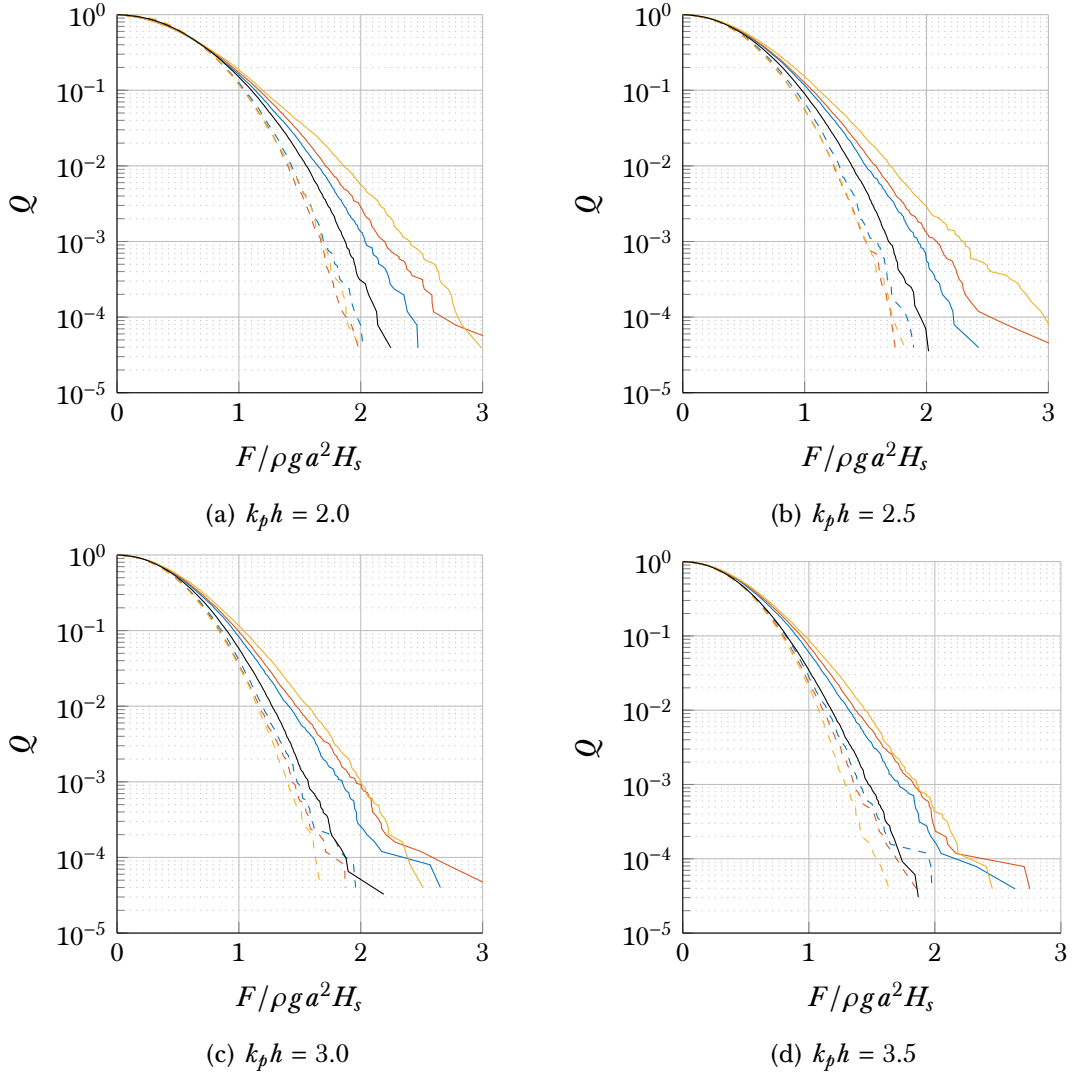


Figure 4.7: Probability of exceedance of the heave force TC1 subject to $\frac{1}{2}H_s k_p =$ (—) 0.05, (—) 0.075, and (—) 0.100 over a range of $k_p h$. The solid lines refer to the force minima and the dashed lines to the maxima. (—) Linear analytical solution from ICRD3D.

the smallest sea state, the measured force is some 15% larger than the linear prediction across all $k_p h$. In the largest sea states, the forces at $Q = 10^{-3}$ are up to 30% larger than linearly predicted. The effect on the force maxima, however, is more limited: across all $k_p h$ the difference between $\frac{1}{2}H_s k_p = 0.050$ and 0.100 is only approximately 1%.

The asymmetry between force maxima and minima has been observed in the context of the regular wave calculations. Figure 3.4 showed that the second-harmonic force minima coincides both with the first-harmonic minimum *and* maximum. Therefore, in that case, the total force minimum was approximately 70% larger (in absolute terms) than the maximum. Furthermore, Rodríguez & Spinneken (2016), in the two-dimensional case, at-

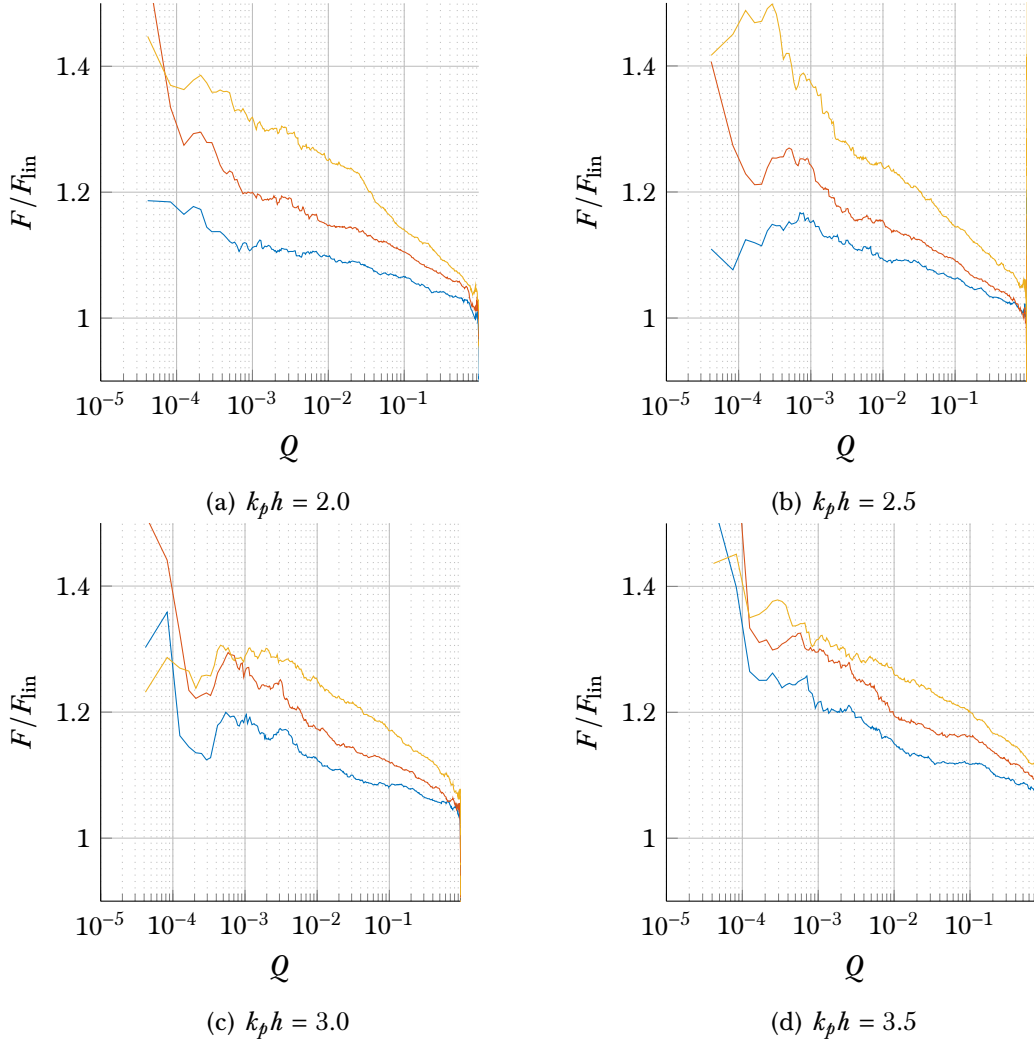


Figure 4.8: Nonlinear amplification of the heave force minima for TC1 subject to $\frac{1}{2}H_s k_p =$ (—) 0.05, (—) 0.075, and (—) 0.100 over a range $k_p h$.

tributed a similar observation of the maxima-minima asymmetry in irregular sea states to sum frequency-forcing.

In a first step to confirm this attribution in the present case, ICRD3D was extended to simulate irregular sum-frequency interactions. The method of Kim & Yue (1990) was adopted and follows the solution methodology outlined in §2.4. A repeat time of 128 s was chosen to create a wave spectrum of 180 components (corresponding to 16290 interactions) to compute the “quadratic transfer function” (QTF). Each QTF took approximately 72 hours to compute and expresses the amplitude-normalised force for each interaction pair. The QTF may then be used with the incident amplitude spectrum to generate irregular sea state statistics, using a number of seeds in a similar manner to the experiments.

Following this process, the second-harmonic sum-frequency content was found to be very small, suggesting an alternative source of the asymmetry. Sections 4.6 and 4.7 provide a detailed investigation and identify the source of the nonlinear forcing.

Turning to the dependence of the excitation force on the sea state peak frequency, Figure 4.7 shows a general trend of decreasing force maxima and minima as $k_p h$ increases. This can be attributed to the behaviour of the first-harmonic force which comprises the majority of the total force. Section 3.5.1 demonstrated that with increasing relative water depth, kh , the first harmonic pressure decays faster with depth, thereby reducing the force on the bottom of the truncated cylinder. In an irregular sea state, the argument still holds as the wavelengths in the sea state as a whole will be dominated by that corresponding to k_p .

To assess the effect of sea state peak frequency on the nonlinear force amplifications, Figure 4.9 re-presents Figure 4.7 by showing the force minima normalised by the linear prediction as a function of $k_p h$. This involves taking one steepness distribution from each of the sub-figures in Figure 4.7 and plotting them on the same figure. For $\frac{1}{2}H_s k_p = 0.075$ the nonlinear amplification of the force minima increases with $k_p h$ whereas for $\frac{1}{2}H_s k_p = 0.100$, there is little variation. This suggests that effects beyond second-order could be present as the force is no longer scaling with the sea state steepness. This can be observed directly by comparing the difference between the force maxima for $\frac{1}{2}H_s k_p = 0.050$ and 0.100 in Figure 4.7(a) and (d).

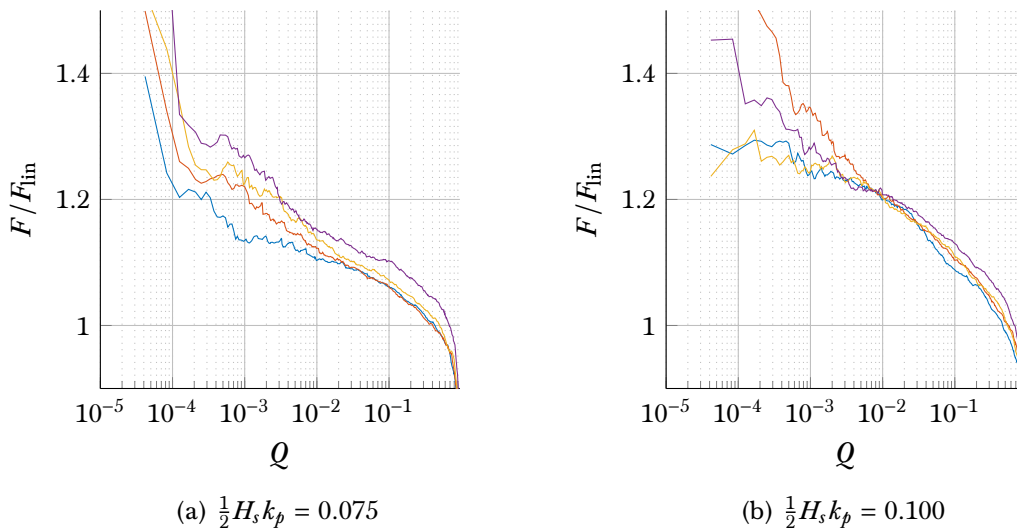


Figure 4.9: Nonlinear amplification of the heave force minima for TC1 with $k_p h =$ (—) 2.0, (—) 2.5, (—) 3.0, and (—) 3.5.

4.6 Spectral analysis

In order to understand the asymmetry between the force maxima and the force minima observed in the probability of exceedance plots, Figure 4.10 presents the spectral content of the ensemble averaged experimental data for $k_p h = 2.5$ and $\frac{1}{2}H_s k_p = 0.050, 0.075$ and 0.100 . After ensemble averaging, a 20 point moving average smoothing is further applied, which corresponds to a frequency band of 0.0133 Hz. To allow a direct comparison, the linear potential flow prediction using ICRD3D is also included in Figure 4.10.

The spectral content arising from the experiments in Figure 4.10 agrees well with the linear prediction for $\omega/\omega_p > 0.6$, and, most notably, there is near-absence of sum-frequency forcing at $\omega/\omega_p > 2.0$. This confirms the calculations of ICRD3D and suggests that the low-frequency difference terms are mostly responsible for the maxima-minima asymmetry. In Figure 4.10, these difference components are most pronounced in the range $\omega/\omega_p \leq 0.6$. Indeed, with increasing $\frac{1}{2}H_s k_p$ the force asymmetry increases, as does the spectral content of the low-frequency components. To further demonstrate the role of the low-frequencies terms, Figure 4.11 shows the probability of exceedance in which the excitation force has been reconstructed from the measured experimental spectra *with the low-frequency components subtracted*. To achieve this subtraction, the magnitude of the components for which $\omega/\omega_p < 0.6$ was set to zero and the force record was then reconstructed using an IFFT. Figure 4.11 finally arises after performing the same up-crossing analysis procedure previously described on this reconstructed force.

Figure 4.11 clearly demonstrates that the low-frequency components are responsible for the majority of the asymmetry between the force maxima and minima. For $\frac{1}{2}H_s k_p = 0.075$, at $Q = 10^{-3}$ the discrepancy is approximately 7% without the low-frequency difference terms compared to the 32% difference when these terms are included. The remaining 7% most likely arises due to the other nonlinear interactions which occur throughout the spectrum.

The effect of the low-frequency difference terms may be attributed to the underlying wave field. These nonlinear terms were first identified by Longuet-Higgins & Stewart (1960), and are associated with a set-down under the largest waves in the incident sea state (Mansard *et al.*, 1988; Baldock & Swan, 1996). As the largest force events will be correlated to the largest wave events, the difference terms from the wave field are carried through into the force record. A detailed analysis of the difference-frequency forcing using

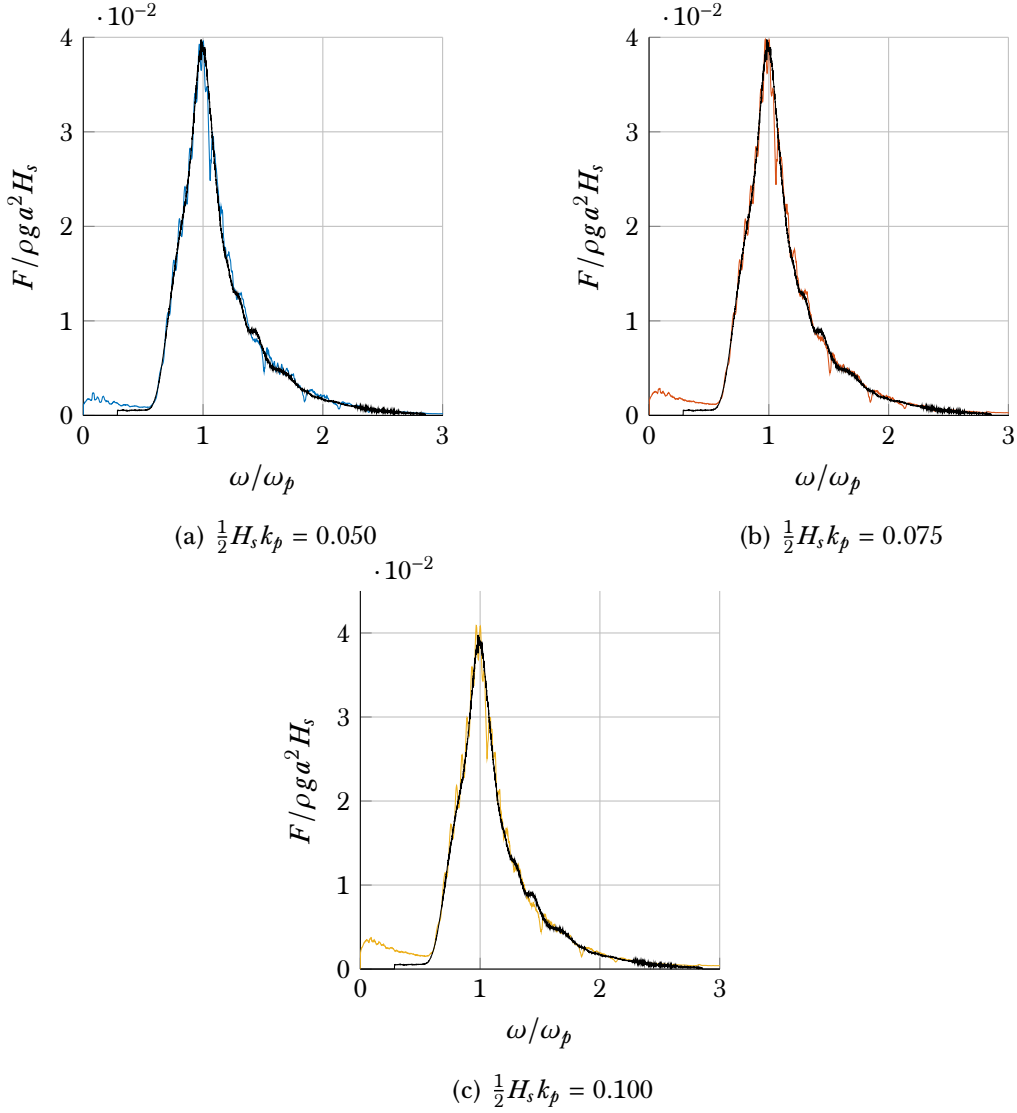


Figure 4.10: Analysis of the spectral content of the force measurements for $k_p h = 2.5$. (—) Linear solution, (—, —, —) Ensemble average of experimental data.

ICRD3D is beyond the scope of this study.

The lack of sum-frequency forcing at $\omega/\omega_p > 2.0$ contrasts with the experimental observations of Rodríguez & Spinneken (2016). In the two-dimensional case, a similar phenomena of the asymmetry between the force minima and maxima was observed in the probability of exceedance distributions. This asymmetry was up to 70% for $k_p h = 2.0$ and $\frac{1}{2} H_s k_p = 0.075$ compared to 26% for TC1 (Figure 4.7(a)). In the spectral analysis by Rodríguez & Spinneken (2016), with increasing $\frac{1}{2} H_s k_p$, the spectral content at the peak frequency reduced whereas the content at the higher harmonics increased, indicating the presence of sum-frequency forcing. The low-frequency difference terms also increased, as

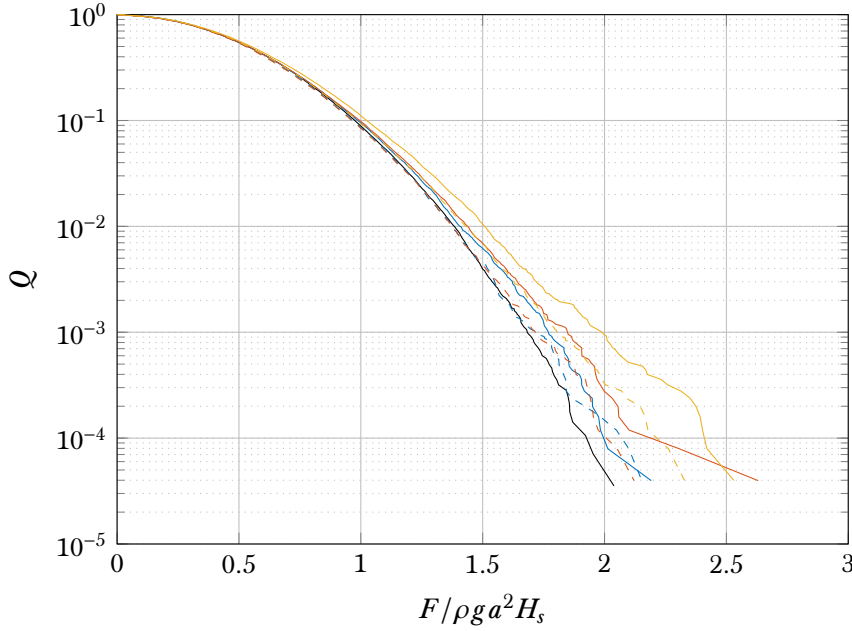


Figure 4.11: Probability of exceedance for $k_p h = 2.5$ reconstructed from the experimental spectrum with the difference frequency components subtracted. Sea state steepness $\frac{1}{2}H_s k_p =$ (—) 0.05, (—) 0.075, and (—) 0.100. Dashed lines refer to force maxima and solid lines to force minima.

in the present case, but the maxima-minima asymmetry was attributed solely to the sum-frequency terms, and the low-frequency terms were not discussed further. The arguments put forward for this attribution to the sum-frequency terms are compelling suggesting that, given the evidence herein of the role of the low-frequency terms, both sum-frequency and difference-frequency terms were important in the Rodríguez & Spinneken (2016) scenarios.

The lack of sum-frequency second-harmonic forcing in the present case compared to the two-dimensional study of Rodríguez & Spinneken (2016) can be attributed to interactions of the scattered and incident waves. Both Rodríguez & Spinneken (2016) and Chapter 3 identified this interaction being the primary driver of the second-harmonic forcing. The two-dimensional case will cause a greater disturbance to the waves as the whole channel width is taken up by the body. Therefore, this interaction mechanism will be stronger in 2D. A secondary factor is that the pressure associated with this mechanism is depth-independent in 2D and is not in 3D. The pressure on the underside of the body is likely to be smaller relative to the surface in 3D compared to having the same value in 2D.

Finally, evidence of the ‘spikes’ in the excitation force discussed in §4.2.2 and identified by Yeung & Sphaier (1989) is also present. The modes are predicted to occur at $Ka = 0.28, 0.56$ and 0.84 which corresponds to $\omega/\omega_p = 1.06, 1.51$ and 1.83 . At each of these

frequencies in Figure 4.10, a clear reduction in the spectral content can be seen. These reductions are localised around the mode frequency; a general reduction in the excitation force posited by Yeung & Sphaier (1989) is not observed. Indeed the overall agreement with the linear potential solution calculated in the absence of side walls remains good throughout. This confirms that the presence of side walls has small localised effects, but does not carry through into the bulk of the data. As a result, the side walls are believed to have a negligible effect on the up-crossing statistics.

4.7 Individual force events

In order to further understand the nonlinearities associated with the heave excitation forcing, the time histories associated with individual wave events have been examined. Figure 4.12 shows the force maxima and minima events for a sea state with $k_p h = 2.5$ and $\frac{1}{2}H_s k_p = 0.100$. The individual events shown are representative for probabilities of exceedance of 10^{-2} and 10^{-3} , based on the distributions in Figure 4.7(b). For all the events, the time axis has been shifted such that the event occurs at $t = 0$. In each case, forces due to the low-frequency terms have been separated out using a procedure similar to that described in §4.6. The force is reconstructed using an IFFT, having set the magnitude of all the components in the spectrum to zero for $\omega/\omega_p > 0.6$. In all cases, for both the maxima and minima, the difference frequency has a minimum value at the instant of the event. Therefore, the maximum event is smaller than linearly predicted and the minimum event is larger, leading to the discrepancy observed in the probability of exceedance plots (Figure 4.7). As described in §4.6 this corresponds to the set-down in the wave field under the largest wave events.

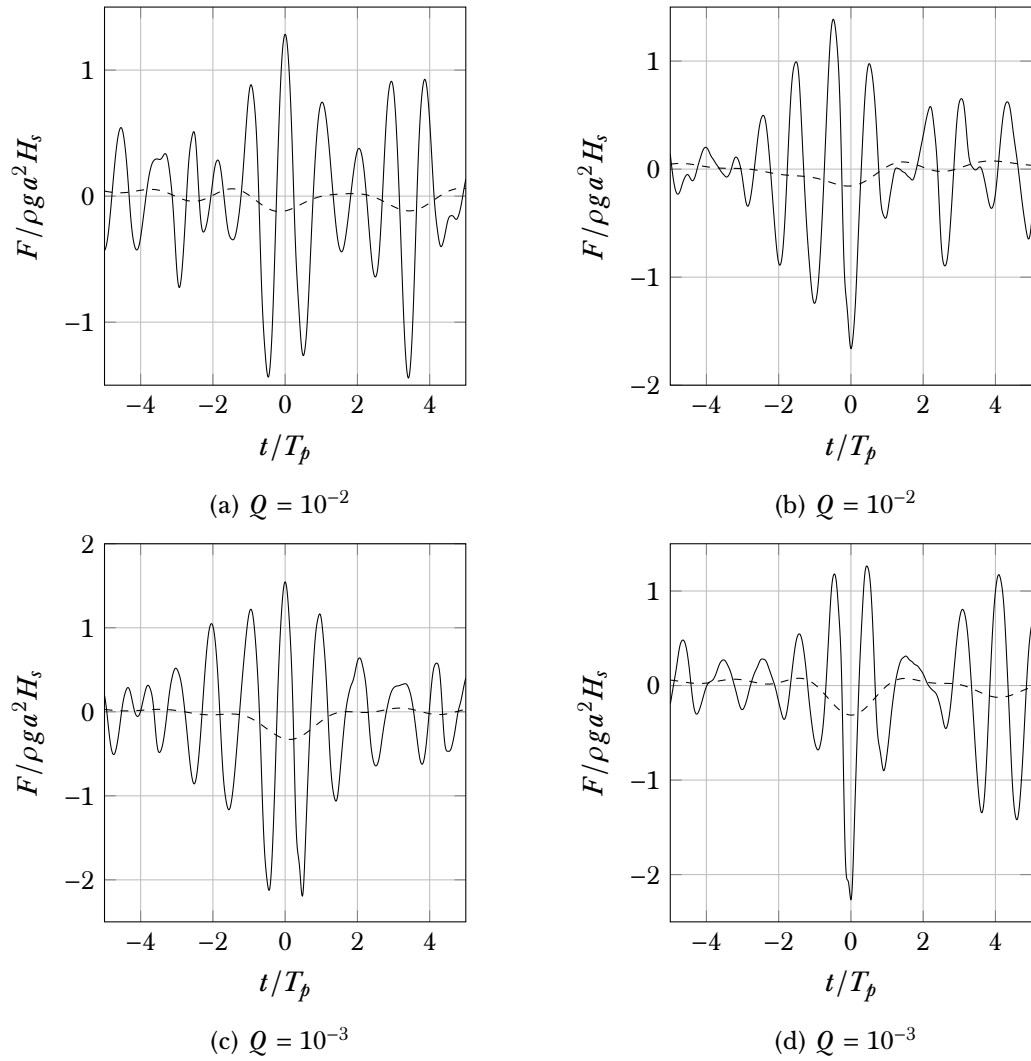


Figure 4.12: Time history of the heave excitation force, including the low frequency difference terms, taken from a sea state with $k_p h = 2.5$ and $\frac{1}{2}H_s k_p = 0.100$. The left column corresponds to force maxima and right column to force minima. (—) Total force and (---) low-frequency difference terms.

4.8 Effect of body draught

In order to examine the effect of the draught, Figure 4.13 shows a comparison of the force minima between TC1 and TC2 for $k_p h = 2.0$ and 3.5 . Considering first the relative magnitudes of the forces, for both values of $k_p h$ the force minima are smaller for the deeper draft TC2 geometry. Section 4.5 demonstrated that the force minima and maxima lie above and below the linear prediction, respectively. Furthermore, §3.5.1 demonstrated the rapid decay of the first-harmonic force with increasing draught. Therefore, as the linear component of the force has decreased, so must the force minima. This reduction is exacerbated for $k_p h = 3.5$ as the first-harmonic pressure decays more rapidly in greater relative water depth (§3.5.1).

Figure 4.14 concerns the nonlinear amplifications for the TC1 and TC2 geometries by normalising the experimental force minima by the linear prediction. For the wave conditions under consideration, the deepening of the draught does not cause an increase in the nonlinearity of the excitation force. In §3.5.4, the relative importance of the second-harmonic excitation force was seen to increase with increasing draught due to the slower decay of the second-harmonic pressure term. In the irregular wave case, however, Figure 4.10 demonstrated the lack of sum-frequency forcing. This is believed to eventually lead to the small difference between TC1 and TC2 observed in Figure 4.14. The importance of the difference terms is re-emphasised in Figure 4.15 which shows the low-frequency force on TC2 for a maxima event at $Q = 10^{-3}$.

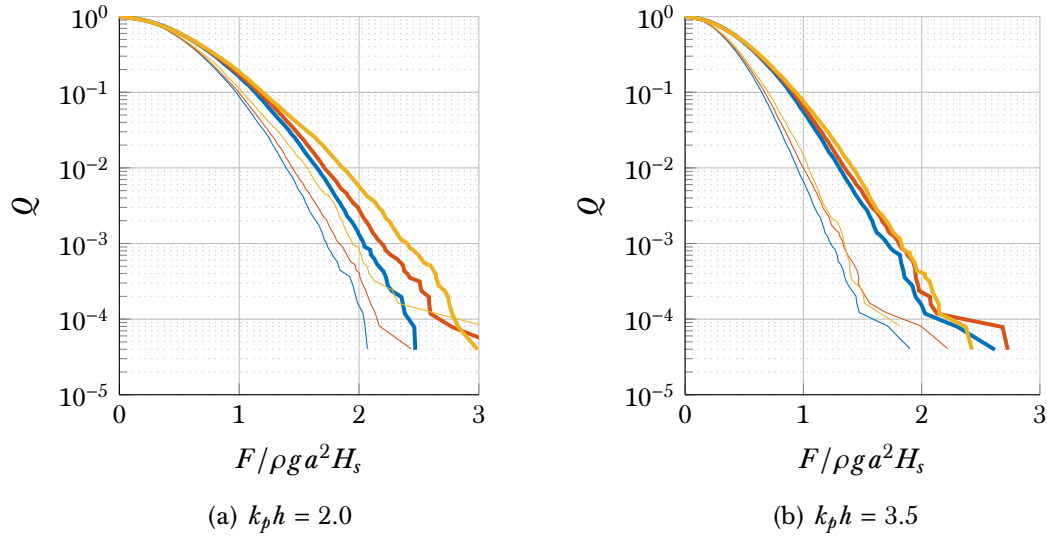


Figure 4.13: Comparison of force minima between TC1 (thick) and TC2 (thin) for $\frac{1}{2} H_s k_p =$ (—) 0.050, (—) 0.075, (—) 0.100.

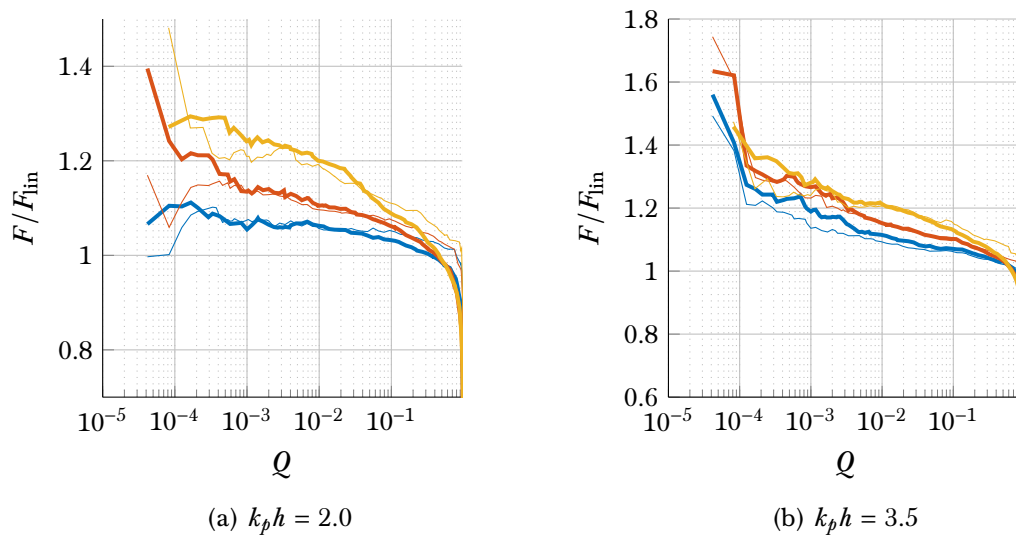


Figure 4.14: Comparison of force minima normalised by the linear solution between TC1 (thick) and TC2 (thin) for $\frac{1}{2} H_s k_p =$ (—) 0.050, (—) 0.075, (—) 0.100.

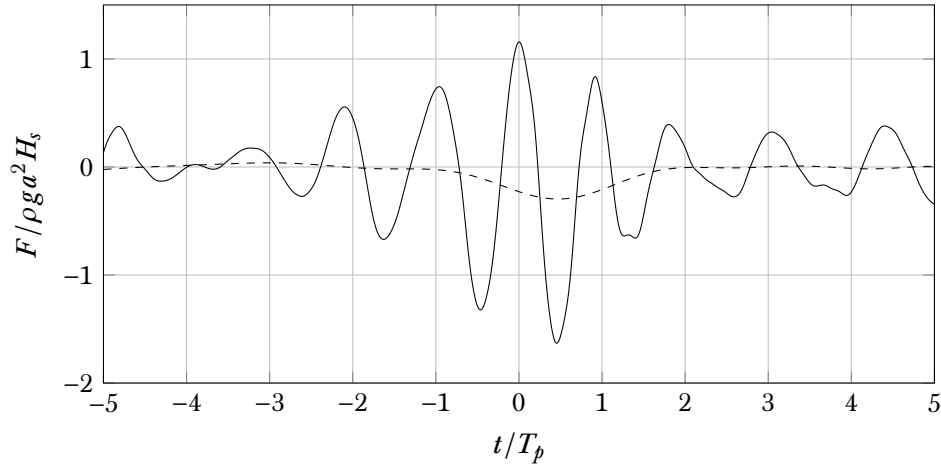


Figure 4.15: Time history of the TC2 excitation force for $k_p h = 2.5$, $\frac{1}{2} H_s k_p = 0.100$ maxima event at $Q = 10^{-3}$, including the forcing due to the low frequency difference terms. (—) Total force and (---) low frequency difference terms.

4.9 Conclusions

The principal contribution of this chapter is the demonstration of the importance of low-frequency difference terms on the excitation force in irregular sea states. For all of the sea states under consideration, the excitation force maxima were seen to be smaller than linearly predicted and the force minima were larger than the linear prediction. Furthermore, the under-prediction of the force minima was larger than the over-prediction of the force maxima. The greatest discrepancy occurred for a sea state characterised by $k_p h = 2.5$ and $\frac{1}{2} H_s k_p = 0.100$ in which the difference between the maxima and the minima was 34% at a probability of exceedance of 10^{-3} . This discrepancy decreases with decreasing $\frac{1}{2} H_s k_p$ but remains relatively constant with $k_p h$. The change with $k_p h$, however, is for the force maxima to lie closer to the linear prediction.

Following an analysis of the spectral content of the heave excitation force, the discrepancy between the force minima and maxima has been attributed to the low-frequency difference components. After subtracting these difference frequencies from the spectrum and reconstructing the force trace, the discrepancy is reduced from up to 34% to approximately 7%. An examination of the time-history of these low-frequency terms confirmed that in the largest waves, the force maxima are reduced and the minima increased, thereby leading to the discrepancy. In contrast to the two-dimensional case studied by Rodríguez & Spinneken (2016), in which significant sum-frequency terms were observed, these terms

are almost entirely non-existent for the geometry herein.

Increasing the draught of the body has a limited impact on the nonlinear amplification due to the low-frequency difference terms. Both the force maxima and force minima were seen to reduce across all sea-states, however, the largest forces were seen to scale with the linear prediction. This is due to lack of sum-frequency forcing, which was found to become more important with increasing draft in Chapter 3.

5

Nonlinear heave motions in irregular sea states

5.1 Chapter overview

Chapters 3 and 4 have shown the importance of nonlinear heave forcing and displacements for a truncated cylinder. This chapter aims to demonstrate how these findings translate into the heave displacements in irregular sea states.

The experimental setup described in Chapter 4 is adapted to remove the motion constraint, allowing the truncated cylinder to move (or heave) freely. The use of air bushings is particularly important in this case as they do not provide any impediment to the vertical body motion. As a result, all observed effects may be directly attributed to the wave action.

Allowing the body to freely respond significantly increases the importance of the nonlinear vortex-shedding damping. This has the general effect of reducing the motions predicted using a potential flow solution. Therefore, there is a competition between (i) the nonlinear amplifications in the excitation forcing to increase the displacements, and (ii) the vortex shedding damping to reduce the displacements. It is this competition that is investigated throughout the present chapter.

This chapter also proposes and demonstrates a methodology for taking this vortex shedding damping into account. It involves the use of a time domain simulation with a quadratic damping term, for which the coefficient is empirically determined from a set of regular wave experiments.

The chapter continues as follows. Section 5.2 describes the modifications to the experimental setup from Chapter 4 and provides an overview of heave displacements. This overview takes the form of probability of exceedance distributions and direct wave-by-wave comparisons for the excitation force and resulting displacements. The chapter then proceeds in two main parts. The first part, §5.3, quantifies the effect of the vortex shedding damping and examines the displacement spectral content. The second part, §5.4 quantifies the nonlinear motions arising from the nonlinear excitation force components identified in Chapter 4.

5.2 Heave displacements in irregular sea states

5.2.1 Physical model setup

The truncated cylinder used for the floating body tests was identical to that used for the fixed body tests (§4.2.2), with the internal arrangement altered to remove the motion constraint. This is shown in Figure 5.1. The truncated cylinder is now free to move in the heave degree of freedom in a ± 75 mm range, enforced by a plastic end-stop at the base of the shaft. Once again, it should be noted that the presence of the air-bearings in the setup allow for near-zero friction movement of the truncated cylinder. The TC1 geometry, $b/a = 1$, has been considered throughout this chapter.

The displacement of the buoy is measured using a MICRO-EPSILON optoNCDT 1402 laser distance sensor. The sensor is mounted on the aluminium superstructure (item 1 in Figure 4.2) and recorded at 100 Hz using a National Instruments 9205 input module. Before testing commenced, the manufacturer's calibration coefficient was confirmed.

5.2.2 Test conditions

In order to investigate the heave motion response of the buoy, the same model location and wave conditions were used as the fixed-body experiments (§4.4). These conditions comprise all the combinations of the (non-dimensional) peak frequency $k_p h = 2.0, 2.5, 3.0$, and 3.5 and the steepness defined by $\frac{1}{2}H_s k_p = 0.05, 0.075$, and 0.100 . Therefore, the means by which the excitation forces are carried through in the motions can be examined. Once again, the sea states were run for 1600 s, comprising 100 s for all the wave components to reach the truncated cylinder and a further repeat time of 1500 s.

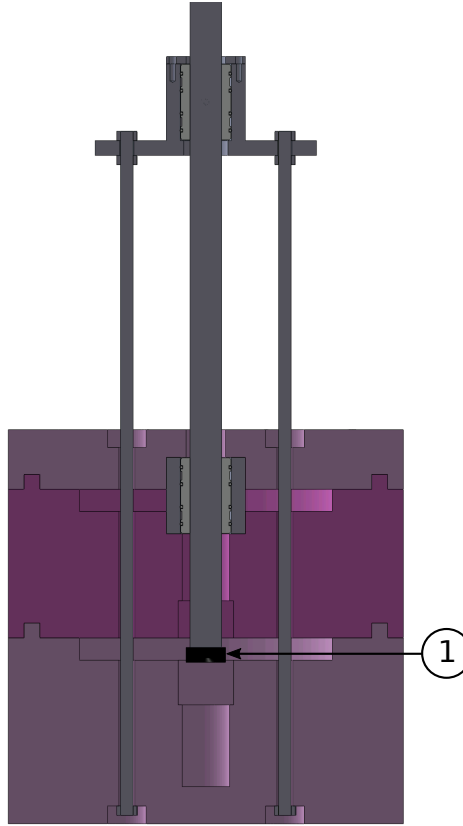


Figure 5.1: Physical model of the floating truncated cylinder. Items identified in Figure 4.2 remain the same. Key: (1) End-stop constraint.

5.2.3 Heave motion probability of exceedance

Figure 5.2 shows a sample time-history for the first seed of $k_p = 2.5$, $\frac{1}{2}H_s k_p = 0.500$ and $k_p h = 3.5$, $\frac{1}{2}H_s k_p = 0.100$, corresponding to the same conditions and time-windows as the forces in Figure 4.6. In contrast to the force records, the signal from the laser distance sensor did not require any filtering during post-processing. Qualitatively, the largest absolute forces in the time window of Figure 4.6 correspond with the largest absolute displacements in Figure 5.2.

In order to generate the probability of exceedance distributions for the heave motion of the truncated cylinder, each sea state was run for a number of seeds (a set of random phase angles) to capture in excess of 15000 wave events. The same input files as used for the fixed-body experiments (Chapter 4) were provided to the wavemakers. The probability of exceedance distributions arise after performing an up-crossing analysis on the each of the motion records for a particular sea state and sorting the displacement maxima and minima into ascending order.

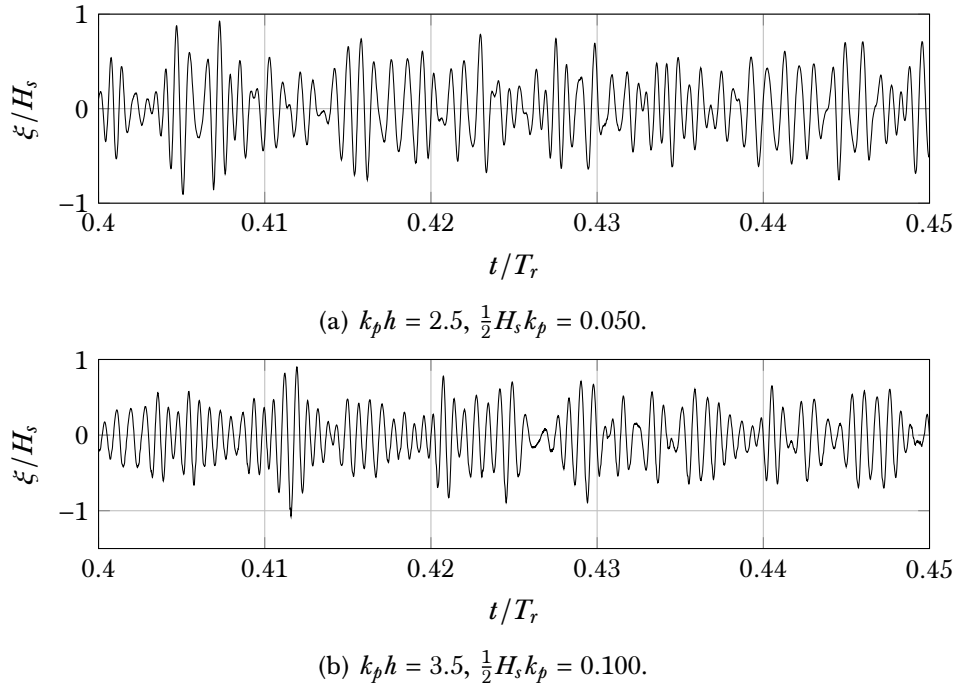


Figure 5.2: Sample time-history of the displacements in irregular sea states

Figure 5.3 presents the motion probability of exceedance distributions, and demonstrates that the trends are in marked contrast to those for the heave excitation forces in Figure 4.7. The force maxima were generally over-predicted by the potential-flow analytic solution, whereas the force minima were under-predicted, with the difference between the maxima and minima increasing with increasing $\frac{1}{2} H_s k_p$. In contrast, the motion maxima and minima are approximately equal and substantially over predicted by the potential flow solution. This is an important finding as nonlinear amplifications in the forcing *do not* translate into amplifications in the displacements.

An illustration of the behaviour leading to the force and displacement probability of exceedance distributions is given in Figure 5.4, which superimposes the time-history of the excitation force and corresponding displacement. Considering the figure as a whole, the body displacements generally follow the excitation force, although evidence of the role of the body dynamics is evident. For example at $t/T_p \approx 1$ in Figure 5.4(b) the displacements do not follow the excitation force and the body is still dynamically responding to the previous event.

Figures 5.4(a) and (b) show the force maxima and minima at $Q = 10^{-2}$ occurring at $t/T_p = 0$. These correspond to a single point on the dashed and solid lines in Figure 4.7(b)

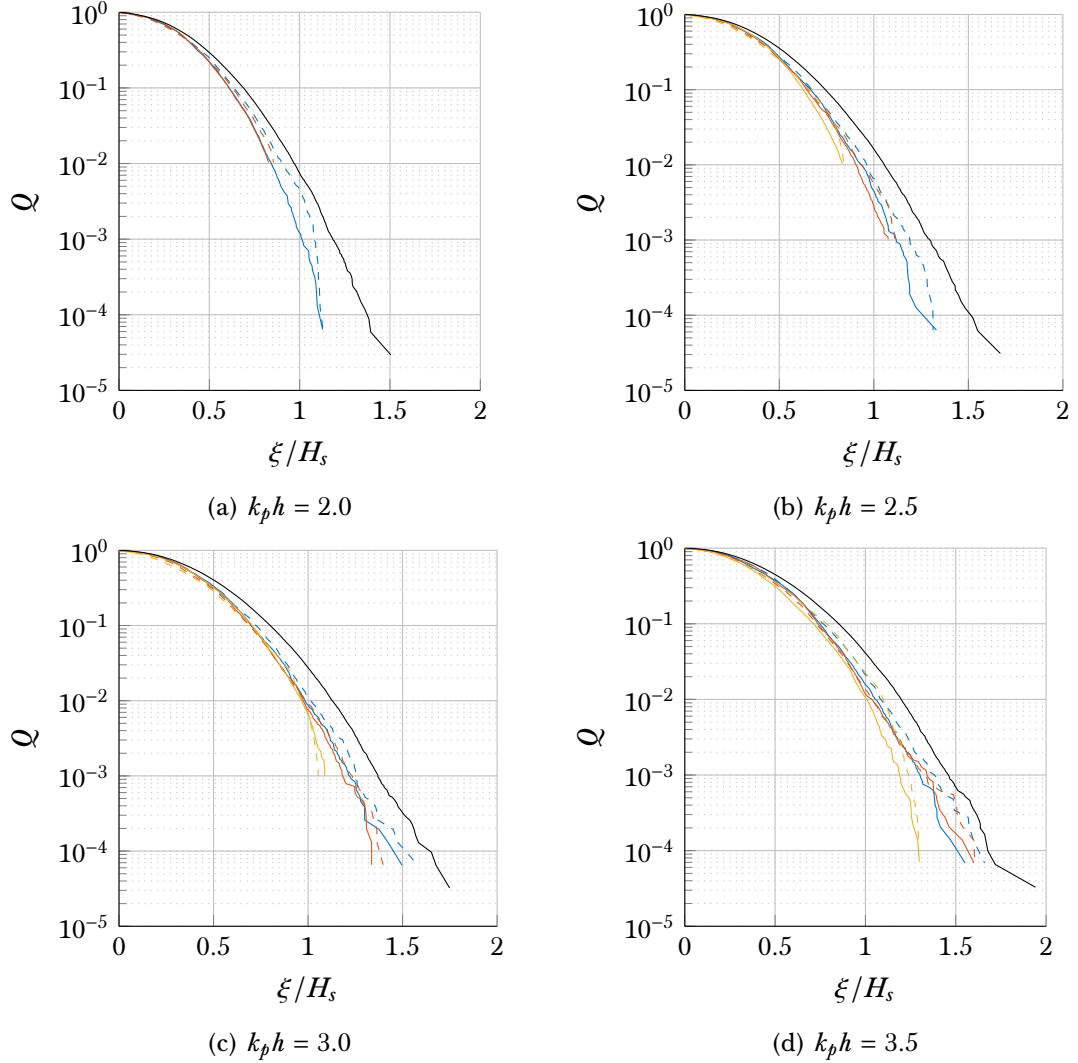


Figure 5.3: Heave displacement probability of exceedance subject to $\frac{1}{2}H_s k_p =$ (—) 0.05, (—) 0.075 and (—) 0.100. The solid lines refer to motion minima and dashed lines to maxima. (—) Linear analytical solution. Note: The truncated lines indicate the motions were in excess of the travel distance.

respectively. The asymmetry is evident in the excitation force as the absolute value of the force minimum in (b) is greater than the maximum in (a). The asymmetry in the displacement traces for this event is much reduced: in both cases $|\xi|/H_s \approx 1.0$. This suggests that a larger force does not necessarily translate directly into larger displacements.

This lack of direct correlation between the largest forces and displacements is further examined in Figures 5.4(c) and (d) which show the traces for the *displacements* corresponding to $Q = 10^{-2}$. The absolute values of the displacements at $t/T_p = 0$ are very similar, despite the excitation force for the maximum being 18% smaller than for the minimum.

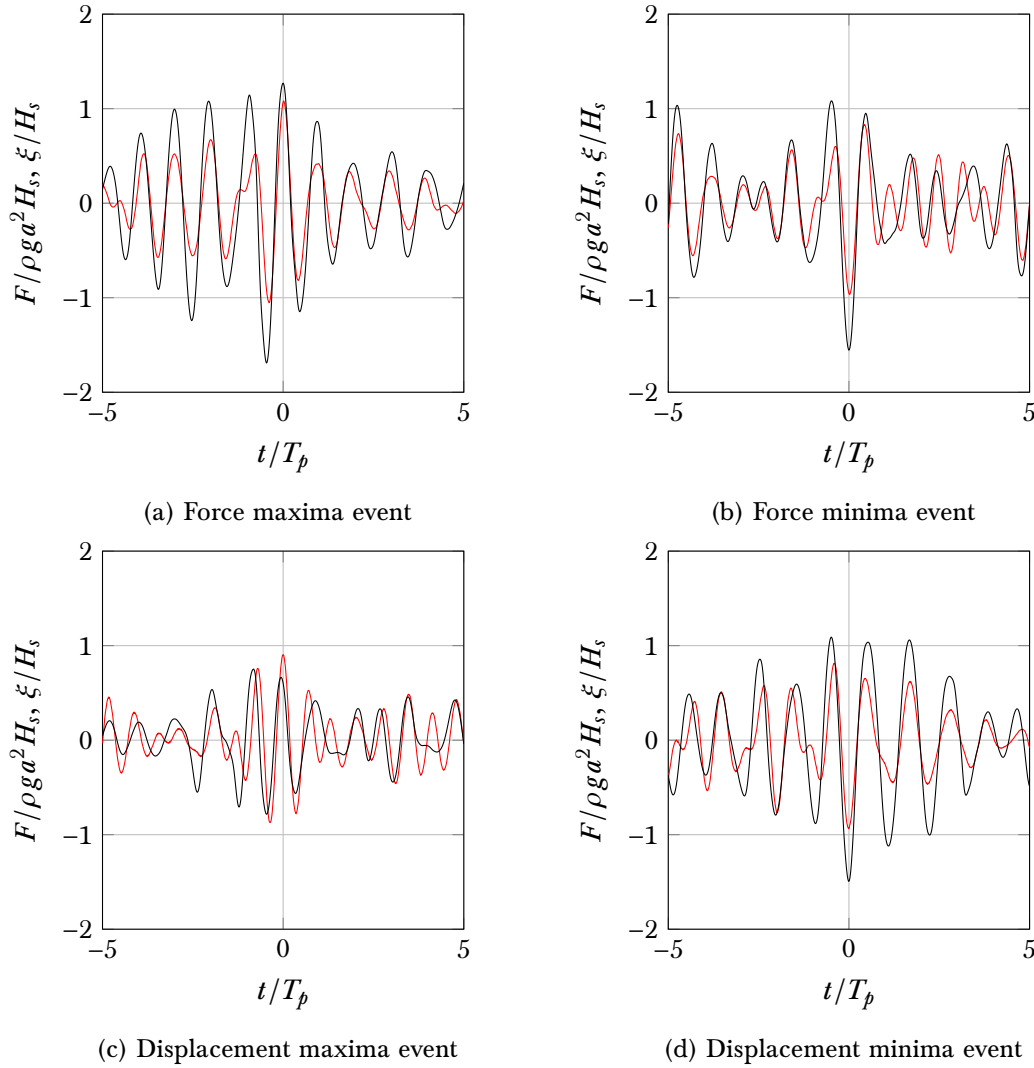


Figure 5.4: Comparison of the force and displacement maxima and minima events at $Q = 10^{-2}$ for $k_p h = 2.5$ and $\frac{1}{2}H_s k_p = 0.075$. (—) Excitation force, (—) displacement.

The overall reduction of the displacements compared to the linear solution is quantified in Figure 5.5. For example at $Q = 10^{-3}$, $k_p h = 3.5$ and $\frac{1}{2}H_s k_p = 0.100$, the displacement is 25% smaller than the potential-flow prediction (Figure 5.5(d)). For comparison, the corresponding force is 30% larger (Figure 4.8(d)). Figure 5.5 demonstrates a further contrast with the force results: with increasing $\frac{1}{2}H_s k_p$, the displacements reduce despite an increasing excitation force. This effect is quantified in Figure 5.6, which plots the ratio of the normalised displacement over the normalised force for $k_p h = 3.5$. This ratio is well below unity for all the $\frac{1}{2}H_s k_p$ under consideration and becomes smaller with increasing $\frac{1}{2}H_s k_p$.

For the largest sea states the truncated cylinder exceeded the travel distance and therefore encountered the end-stop constraint. All end-stop data were subsequently removed in

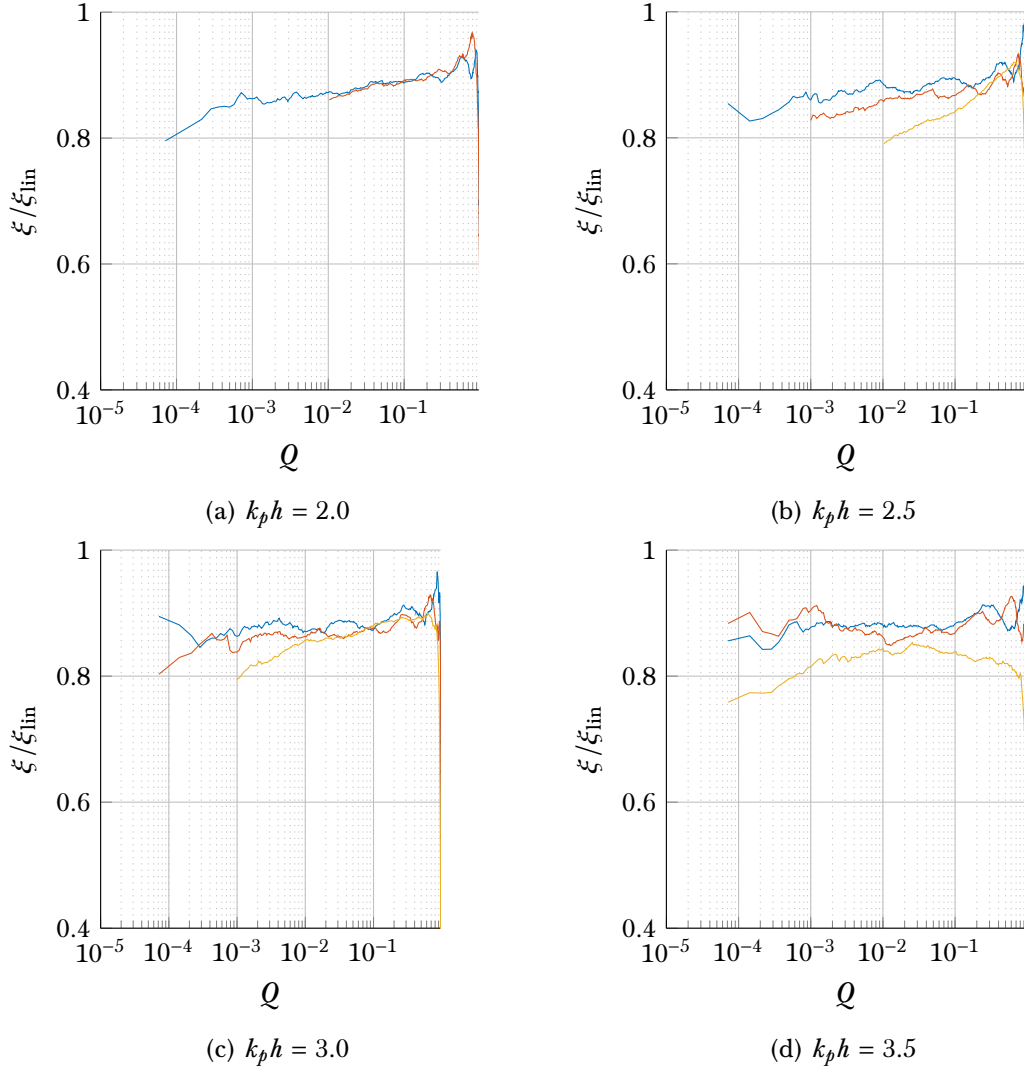


Figure 5.5: Ratio of the heave displacement experimental data to the linear potential-flow prediction for displacement minima $\frac{1}{2}H_s k_p =$ (—) 0.05, (—) 0.075 and (—) 0.100.

the post processing. This gives rise to the shortened lines seen in Figures 5.3(a)–(c). The effect on the distribution of removing these events is exaggerated due to the logarithmic scale. For example, for $k_p h = 2.5$, $\frac{1}{2}H_s k_p = 0.100$, 154 events were removed from a total of 15430 individual wave events, representing just 1% of the total distribution. Therefore, the data above the cut-off point are still statistically significant.

From the discussion above, two effects must be explained: the overall reduction of the motions compared to the linear solution, particularly with increasing $\frac{1}{2}H_s k_p$, and the lack of motion maxima-minima asymmetry. A systematic investigation of the causes is undertaken in the following sections by examining the dynamic properties of the truncated cylinder

and the onset of vortex-shedding damping.

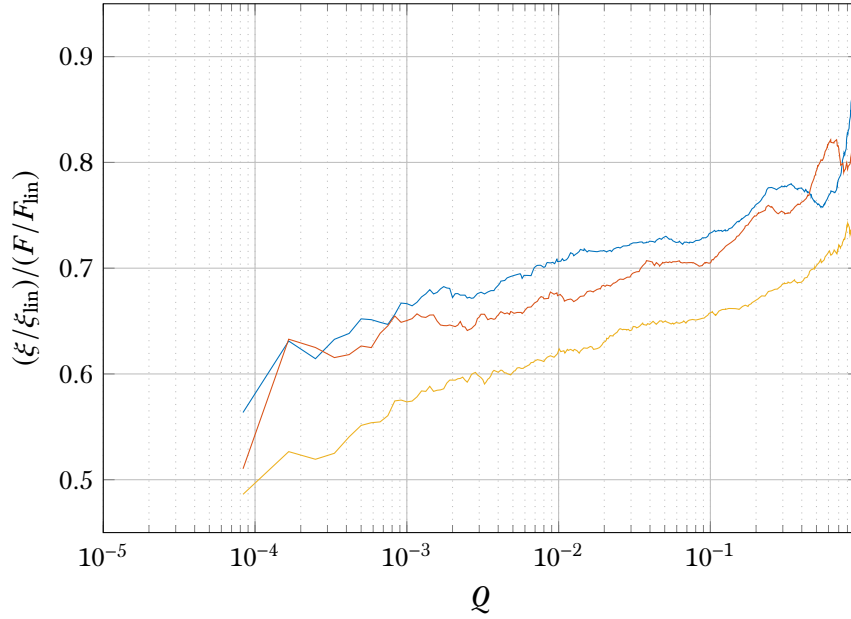


Figure 5.6: Ratio of the heave displacement to the heave force, each normalised by their linear prediction, for $k_p h = 3.5$ and $\frac{1}{2} H_s k_p =$ (—) 0.05, (—) 0.075 and (—) 0.100.

5.3 Quantification of vortex shedding damping

5.3.1 Determination of a damping coefficient

In the context of body motions, the effect of vortex-shedding damping is of well known importance (Newman, 1977; Downie *et al.*, 1988; Yeung & Jiang, 2014). The term “vortex shedding damping” is used to mean the damping of the body’s mechanical energy due to the shedding of vortices. The study of loading arising from vortex-shedding damping (drag forces) on offshore structures is also well established and was outlined in §2.2. In regular waves, the vortex-shedding damping takes the form of vortex shedding which establishes a repeating pattern as the flow oscillates about the body. Therefore, the coefficient depends on both the Reynolds and Keulegan-Carpenter numbers. A detailed description of the phenomena can be found in Sarpkaya & Isaacson (1981).

In irregular wave conditions, however, the vortex-flow regimes are disrupted, leading to a different distribution of C_D as a function of KC compared to the regular wave case (Sumer & Fredsøe, 2006). In irregular wave conditions the Keulegan-Carpenter number

has been defined by Longoria *et al.* (1991), among others, as

$$KC_r = \frac{\sqrt{2}\sigma_U T_z}{D}, \quad (5.1)$$

where σ_U is the root-mean square of the fluid velocity for the total length of the measured record, and T_z is the mean zero up-crossing period. Sumer & Kozakiewicz (1995), using the results of Longoria *et al.* (1991), explain the different in the vortex shedding regimes between the regular and irregular cases, and conclude that for $KC_r > 7$, the vortex shedding occurs in “repeatable interactions between vortices of two successive half periods in sinusoidal flows”. This interaction is destroyed in irregular waves due to non-constant half-period of oscillation. For $KC_r < 3$, the vortex shedding pattern in irregular waves is substantially similar to the regular wave case as the half-period is too short to be subjected to a such disruption (Sumer & Kozakiewicz, 1995).

The vast majority of the studies for the drag coefficient consider the case of a cylinder either forced to oscillate in an irregular fashion in otherwise still water, or a fixed cylinder subject to irregular wave groups. Examples of such work includes Longoria *et al.* (1991), Sumer & Kozakiewicz (1994, 1995), Mackwood *et al.* (1997), Burrows *et al.* (1997) for a vertical cylinder and Chaplin & Subbiah (1997) for a horizontal cylinder. Furthermore, Vengatesan *et al.* (2000) considered a truncated vertical cylinder, and Stallard *et al.* (2009) examined the evolution of C_D in a focused, transient wave group.

Isaacson *et al.* (1991) outlines some methods by which C_D may be estimated from the irregular wave record:

- i. least-squares fit of the force time series,
- ii. least-squares fit of the force spectrum,
- iii. method of moments applied to the force probability distribution,
- iv. cross-spectrum fitting, and
- v. least squares fit on a wave-by-wave basis.

For all the methods above, an accurate prediction of the wave particle velocity and acceleration is required, which is a significant challenge in itself, particularly in the largest waves which give rise to the largest forces (Sobey, 1992). The offshore standards specify that a second-order random wave theory performs well in such conditions (DNV, 2014, 3.3.3.4).

In the context of floating bodies, Rodríguez & Spinneken (2016) outlined a pragmatic

approach to model the vortex shedding damping. This approach involves determining a wave-parameter independent damping coefficient, C_D , from regular wave experiments which is then taken forward into irregular sea states. In the present case, this approach may be initially justified by considering KC_r , calculated using the body velocity. For all the sea states under consideration $0.41 \leq KC_r \leq 1.04$, which is within the range Sumer & Kozakiewicz (1995) suggest the vortex shedding structures between regular and irregular waves are substantially similar. Furthermore, studies such as those Bhinder *et al.* (2015), Combourieu *et al.* (2015) and Zurkinden *et al.* (2014) have demonstrated good agreement in irregular waves with a single value of C_D compared to RANS modelling for the former and experiments for the latter two.

Therefore, regular wave tests have been undertaken over $0.2 \leq ka \leq 1.0$, which captures the behaviour of the truncated cylinder for frequencies both smaller and larger than the resonance frequency (*cf.* Figure 3.2). In addition, four values of the incident wave steepness, $A_I k = 0.050, 0.075, 0.100$, and 0.125 have been considered. The motion response of the body is quantified by normalising the first-harmonic motions by the incident wave amplitude to produce the response amplitude operator (RAO).

Figure 5.7 compares the RAO calculated using ICRD3D and the results of the model tests. The agreement between the experimental data and the potential flow results is good away from the resonance frequency; however, between $ka = 0.6$ and 0.7 , significant motion reductions are observed in the experiments. By definition at the resonance frequency, the inertia and spring components cancel so that the system is primarily driven by the damping component. A shift of the resonance towards a lower frequency is also observed, which is consistent with increased damping (Wylie & Barrett, 1995) caused by the vortex shedding. Finally, the nonlinear nature of the vortex shedding is demonstrated by the decreasing motions with increasing steepness near the resonance frequency.

In order to take account of the vortex shedding, a simple MJOS approach is adopted, which assumes that the forcing associated with vortex-shedding damping is proportional to the body velocity squared. Introducing this additional term into the equation of motion for the buoy gives

$$(M + m_r)\ddot{\xi} + d_r\dot{\xi} + C\xi + \frac{1}{2}\rho AC_D|\dot{\xi}|\dot{\xi} = F(t), \quad (5.2)$$

where $F(t)$ is the time-dependent linear excitation force, M is the mass of the body, C is

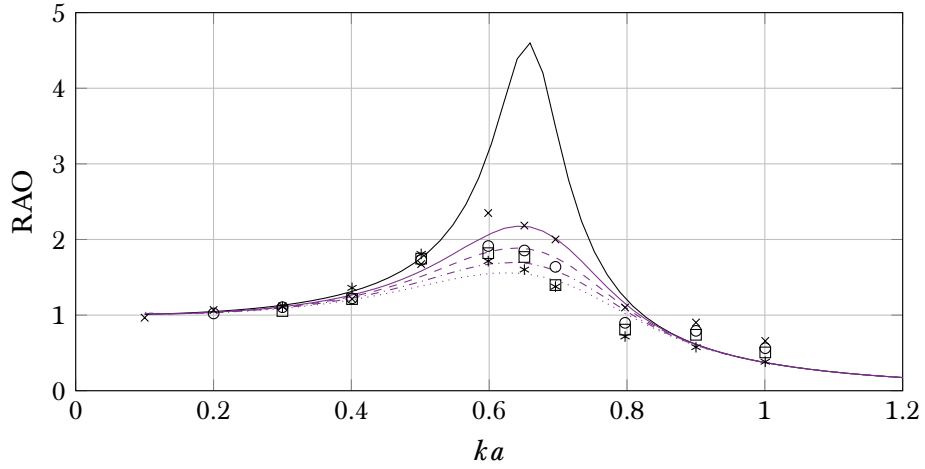


Figure 5.7: Heave response amplitude operator for experimental incident waves with steepness, $A_I k$ ($\times$) 0.05, ($\circ$) 0.075, ($\square$) 0.100, ($*$) 0.125, and (—) ICRD3D. Time domain simulations with $C_D = 1.7$ for $A_I k =$ (—) 0.05, (---) 0.075, (-.-) 0.100, (.....) 0.125.

the hydrostatic restoring coefficient, A is the body area, and C_D is the damping coefficient. The excitation force and the hydrodynamic coefficients are computed using ICRD3D. The system is then readily solved using the ode45 routine in MATLAB with C_D varied to best match the experimental data.

The purple lines in Figure 5.7 have been obtained by solving equation (5.2) with $C_D = 1.7$ for *all* the values of incident wave steepness. The agreement for each case is good and is further illustrated in Figure 5.8 which shows a wave by wave comparison of the experimental time-history and the time domain simulation at the resonance frequency. Given the good agreement in each case, this procedure for taking account of the vortex shedding damping is considered adequate.

5.3.2 Time domain simulation with quadratic damping

In order to model the vortex shedding damping in the irregular sea state, the following procedure was employed:

- i. Calculate the linear excitation force and hydrodynamic coefficients using ICRD3D over the range of frequencies under consideration.
- ii. Calculate the force amplitude spectrum by multiplying the normalised force from ICRD3D with the input amplitude spectrum.
- iii. Run a series of time domain simulations, with a different set of random phases for the force spectrum, until 15000 waves have been captured.

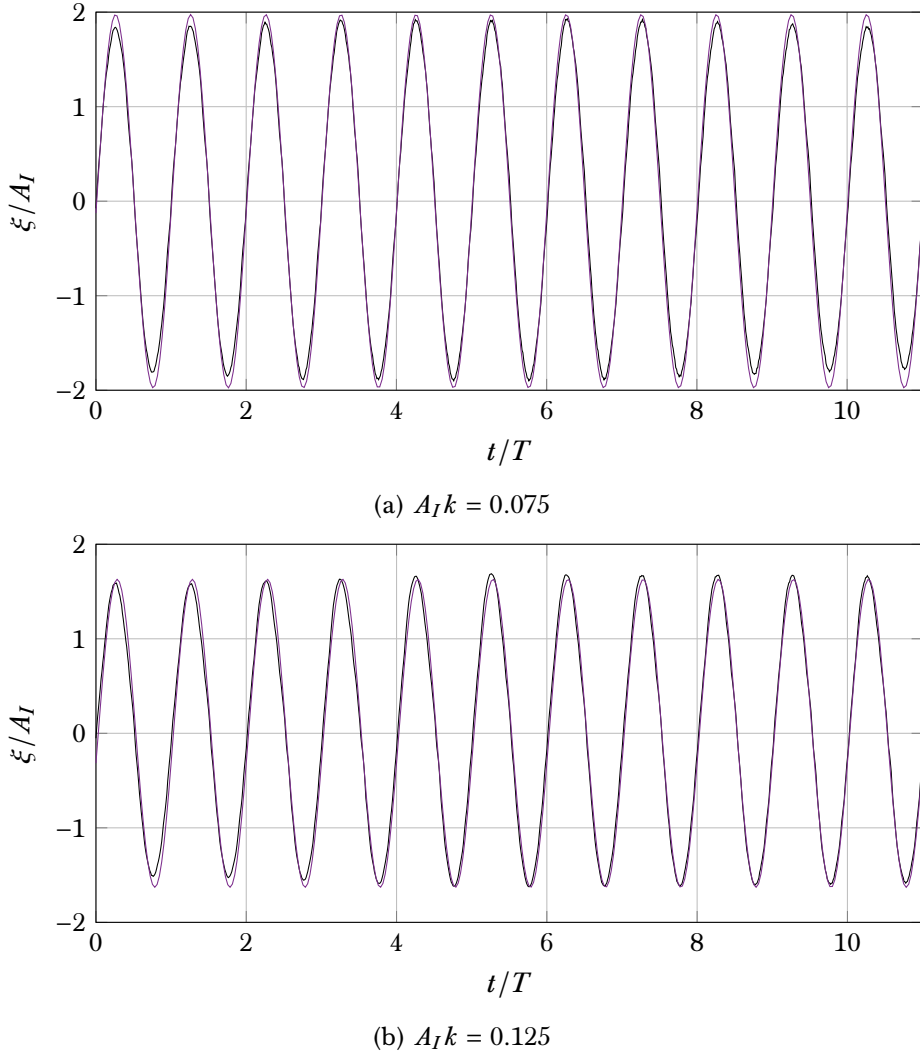


Figure 5.8: Time-history of heave displacement in regular waves with $ka = 0.65$: (—) experimental data, (---) time domain simulation with $C_D = 1.7$.

- iv. Each run can then be processed to generate the probability of exceedance distributions and the underlying spectra (§5.3.3), similarly to the experiments.

The time domain simulation in irregular waves must take into account the full frequency range, in contrast to the single frequency regular wave cases. In the irregular wave case, the hydrodynamic coefficients are expressed as an impulse response function, which upon the convolution with the heave velocity, yields the radiation force (Jefferys, 1984). The convolution captures the fluid memory effects. The buoyancy stiffness, inertia and hydrodynamic damping terms are also included in the equation of motion in the same manner as in the regular wave case (§5.3.1). Crucially, the same damping coefficient of $C_D = 1.7$ from the regular wave simulations has been used for *all* the irregular sea states. The equa-

tion of motion is solved using a fixed-step solver with 5th order accurate Dormand-Price integration.

Figure 5.9 compares the linear potential solution, the experimental data and the time domain simulation. To capture both the effect of sea state steepness and of the peak frequency, the left column, (a)–(c), considers $k_p h = 3.5$ and varies $\frac{1}{2}H_s k_p$ whereas the right column, (d)–(f), considers $\frac{1}{2}H_s k_p = 0.075$ and varies $k_p h$. In each case the time domain model reduces the motions of the linear potential flow solution suggesting that the vortex shedding damping plays an important role in the reduction in motions observed in the experiments. Overall, the time domain solution provides reasonable predictions, particularly for $Q \geq 10^{-1}$, and even at smaller probabilities of exceedance the agreement is within 9% and 15% at $Q = 10^{-2}$ and 10^{-3} respectively.

To further understand the relationship between the experimental results and the time domain simulation seen in Figure 5.9, a wave-by-wave comparison is provided in Figure 5.10. The comparison is made for a time-window around the displacement minima event at $Q = 10^{-2}$ (shifted to occur at $t/T_p = 0$) for $k_p h = 2.5$ and 3.5 with $\frac{1}{2}H_s k_p = 0.075$. Each event at $t = T_p = 0$ corresponds to a single point in Figures 5.9(d) and (f) respectively. For this wave-by-wave comparison, the measured excitation force is used as the input to the time domain model and the initial conditions, at $t/T_p = -8$, are the measured body displacement and velocity. The use of the measured excitation force introduces the low-frequency difference terms studied in Chapter 4 which are not present in the calculated excitation force used to generate Figure 5.9 (the effects of the force nonlinearity on the motions are discussed in detail in §5.4).

Figure 5.10 demonstrates the ability of the time domain simulation to capture the wave-by-wave displacement behaviour of the truncated cylinder as well as the statistical representations in Figure 5.9. For smaller displacements, such as those for $-8 \leq T/T_p \leq -2$ in Figure 5.10(a), the time domain simulation provides an accurate prediction of the measured displacements. The departures arise, however, in the larger wave events for example at $t/T_p = -1.8$, Figure 5.10(a), as observed more generally in the distributions. Similar behaviour between the time domain solution and the experimental data is seen in Figure 5.10(b) with slightly larger departures and is again in keeping with the overall distributions. Despite these relatively small departures, these wave-by-wave comparisons demonstrate the approach adopted herein for modelling the vortex shedding damping

provides a faithful representation of the measured behaviour.

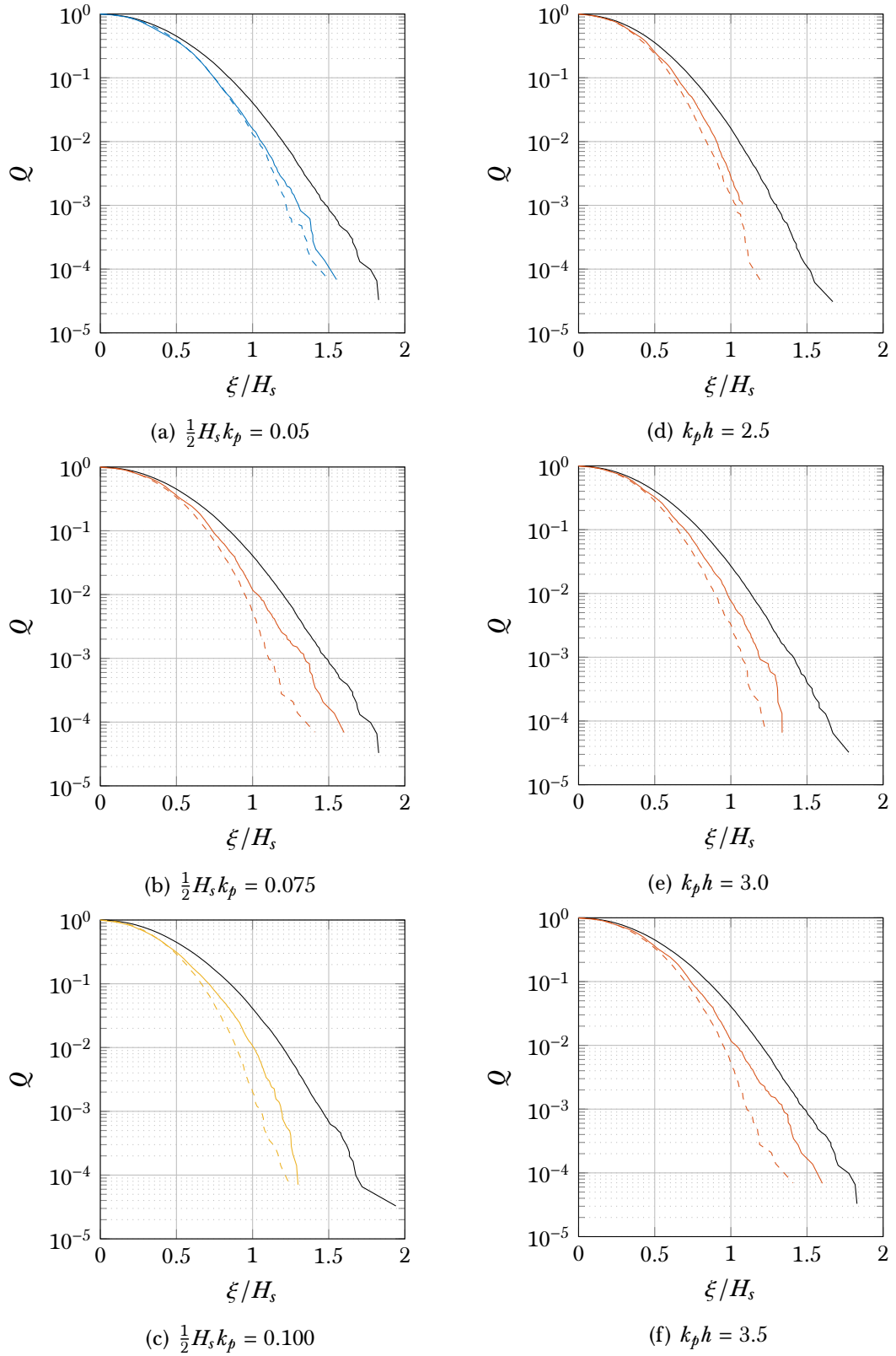


Figure 5.9: Probability of exceedance of the heave displacement showing (—) linear analytical solution, (solid coloured line) experimental data, and (dashed line) time domain simulation. Left column for $k_p h = 3.5$ and varying $\frac{1}{2}H_s k_p$, right column for $\frac{1}{2}H_s k_p = 0.075$ and varying $k_p h$.

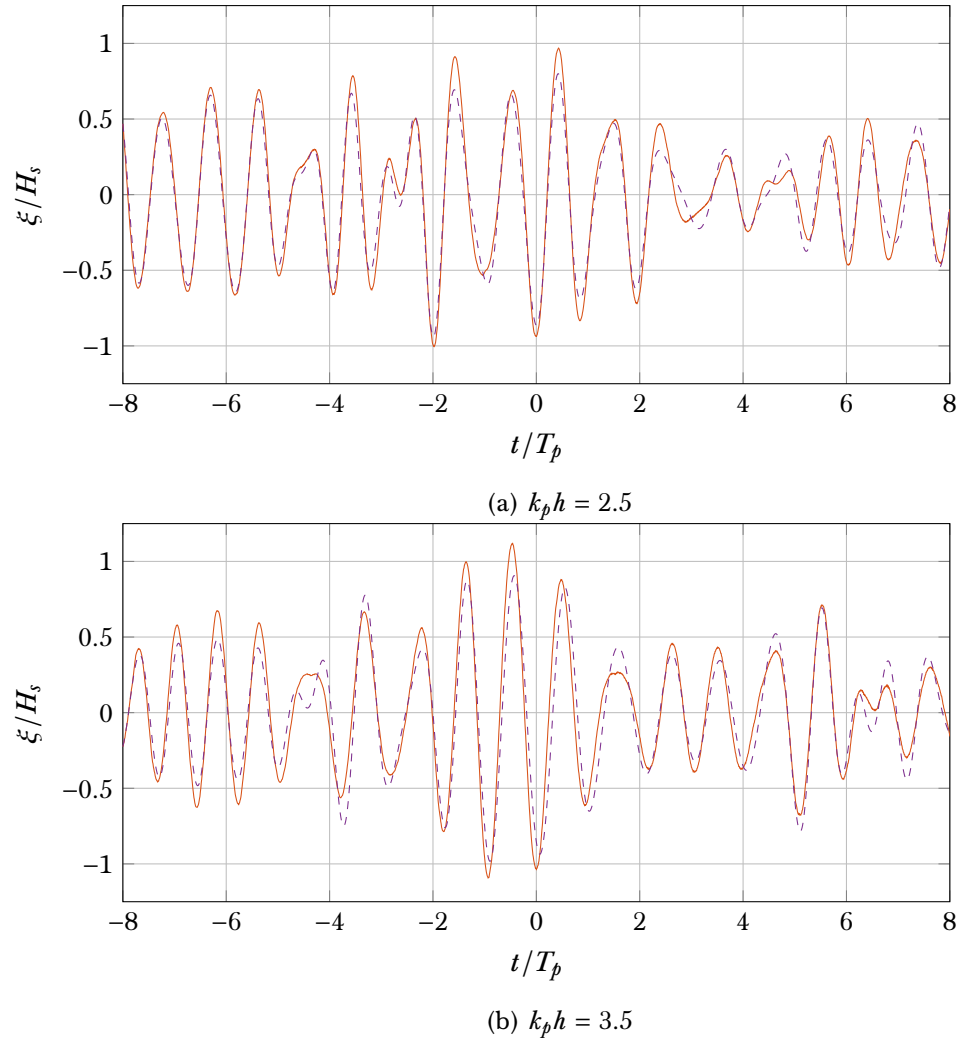


Figure 5.10: Comparison of measured displacements and the time domain simulation for displacement minima at $Q = 10^{-2}$ for $\frac{1}{2}H_s k_p = 0.075$. (—) Experimental measurement and time domain simulations with (---) $C_D = 1.7$.

5.3.3 Spectral analysis

Overview of the response

In a similar fashion to the excitation problem, the spectral content of the heave displacement has been calculated. This is achieved by applying a FFT over the 1500 s repeat time of the measured displacements. For all the cases under consideration, the resonance frequency of the body remains the same at $\omega_r = 7.14$ rad/s. The sea state peak frequency, however, varies between $\omega_p = 3.89$ and 5.23 rad/s for $k_p h = 2.0$ and 3.5 , respectively. The response, therefore, will have two peaks: one at ω_p , where the RAO ≈ 1 , and a second at ω_r . The relationship of the incident amplitude spectrum, the RAO, and the linear potential flow prediction (simply the product of these two) is shown in Figure 5.11 for $k_p h = 3.5$.

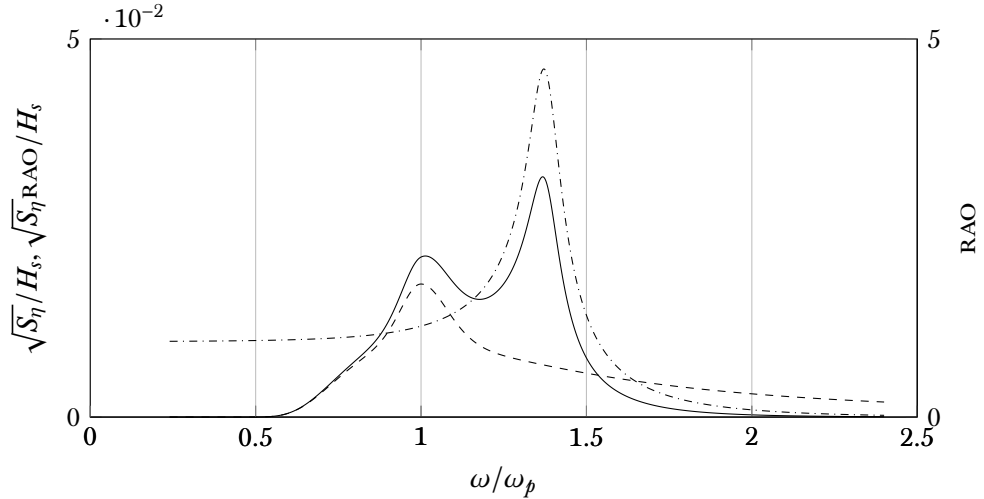


Figure 5.11: Components of the linear displacement spectrum for $k_p h = 3.5$ with (---) JONSWAP incident amplitude spectrum, (- · - · -) RAO, and (—) product of incident spectrum and body RAO.

Effect of sea state peak frequency

Figure 5.12 considers $\frac{1}{2}H_s k_p = 0.075$ and varies $k_p h$ to demonstrate the relationship between the linear potential flow prediction, the experimental data and the time domain simulation. This corresponds to the right column of Figure 5.9 with the addition of $k_p h = 2.0$. As $k_p h$ increases, the ratio between the peak and resonance frequencies reduces, corresponding to $\omega/\omega_p = 1.84$ and 1.4 for $k_p h = 2.0$ and 3.5 respectively. This has the effect of increasing the ratio between the peaks at these two frequencies. As the plots in Figure 5.12

are normalised by H_s the peak at ω/ω_p does not significantly change with increasing $k_p h$. Conversely, with increasing $k_p h$, the maximum of the RAO amplifies a greater value of the underlying incident amplitude-spectrum, thereby increasing the peak at higher ω/ω_p .

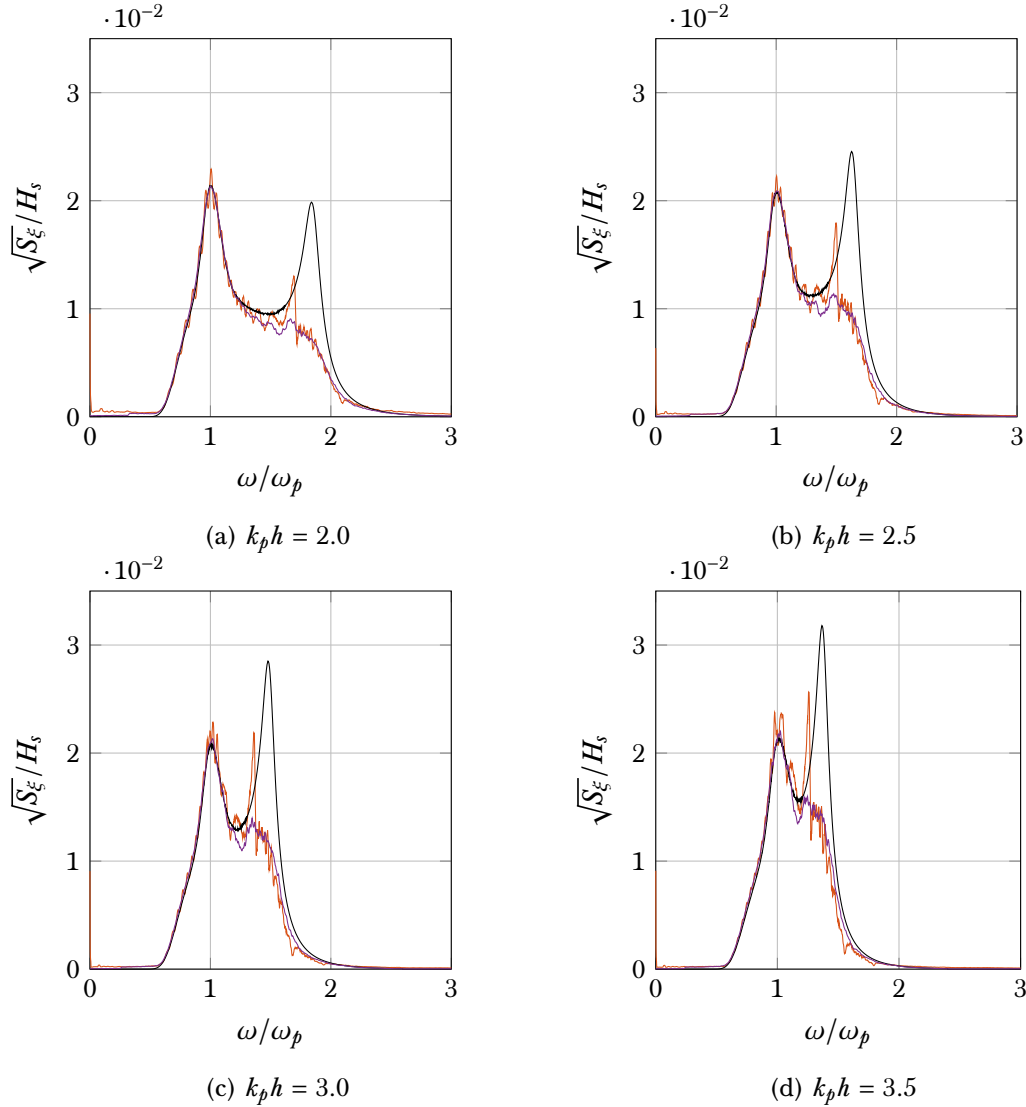


Figure 5.12: Heave displacement spectra for $\frac{1}{2}H_s k_p = 0.075$ for (—) linear potential flow solution, (—) ensemble averaged experimental data, and (—) time domain simulation. Note: The spike in the spectra near the resonance frequency is explained in the text.

The relationship between the experimental data and the linear potential flow prediction in Figure 5.12 is consistent with increasing $k_p h$: around the spectral peak, $\omega/\omega_p < 1.2$, the agreement between the two is good whereas around the resonance frequency, the experimental data are well below the linear prediction. In each case, this reduction is approximately 60%, which is consistent with the reduction observed in the regular wave

experiments (Figure 5.7) at $ka = 0.65$ and $A_I k = 0.075$. Therefore, the difference between linear potential prediction and the experimental measurements in the probability of exceedance distributions (Figure 5.3) may be, almost exclusively, attributed to vortex shedding damping at the body resonance frequency.

The close agreement between the experimental data and the potential flow prediction at the peak frequency is initially surprising as vortex shedding damping could be expected to reduce the displacements across the spectrum. The agreement between the time domain simulation in Figure 5.12 and the experimental data, however, confirms that such behaviour may be captured using a quadratic damping model for all sea state frequencies under consideration. This may be explained by considering the velocity spectra.

Figure 5.13 presents the potential-prediction velocity spectra for the same cases as Figure 5.12. Due to the factor ω between the displacement and velocity spectra, the peak at the resonance frequency is amplified in the velocity spectra. Given that the force associated with the vortex-shedding damping is proportional to the velocity squared, the difference between the peaks increases further and, consequently, motions for the peak at ω_r are more heavily dissipated.

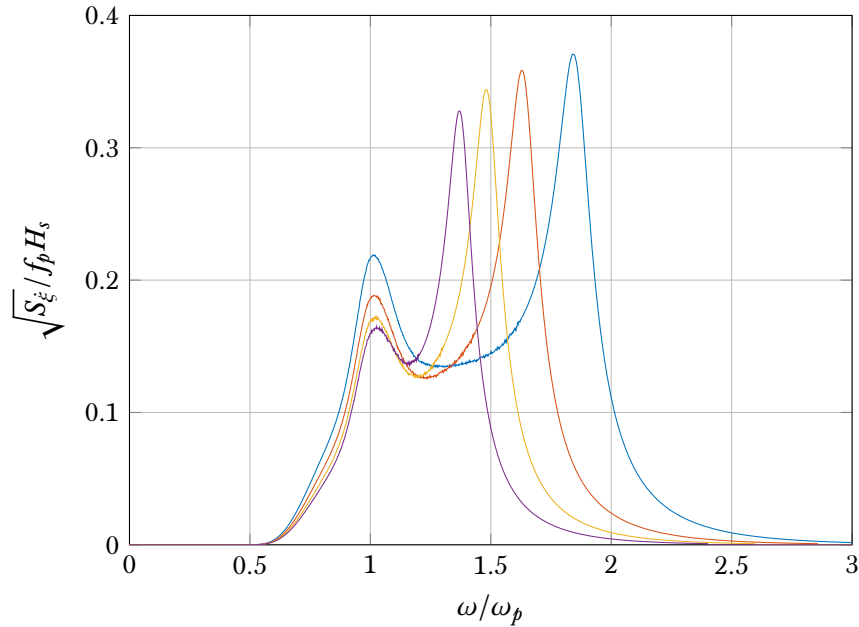


Figure 5.13: Heave velocity spectra for $\frac{1}{2}H_s k_p = 0.075$ and (—) $k_p h = 2.0$, (—) 2.5 , (—) 3.0 , and (—) 3.5

Effect of sea state steepness

In order to further investigate the behaviour of the damping mechanism, Figure 5.14 presents the spectral content for $k_p h = 3.5$ with increasing sea state steepness. The spectra, therefore, correspond to the left column of distributions in Figure 5.9.

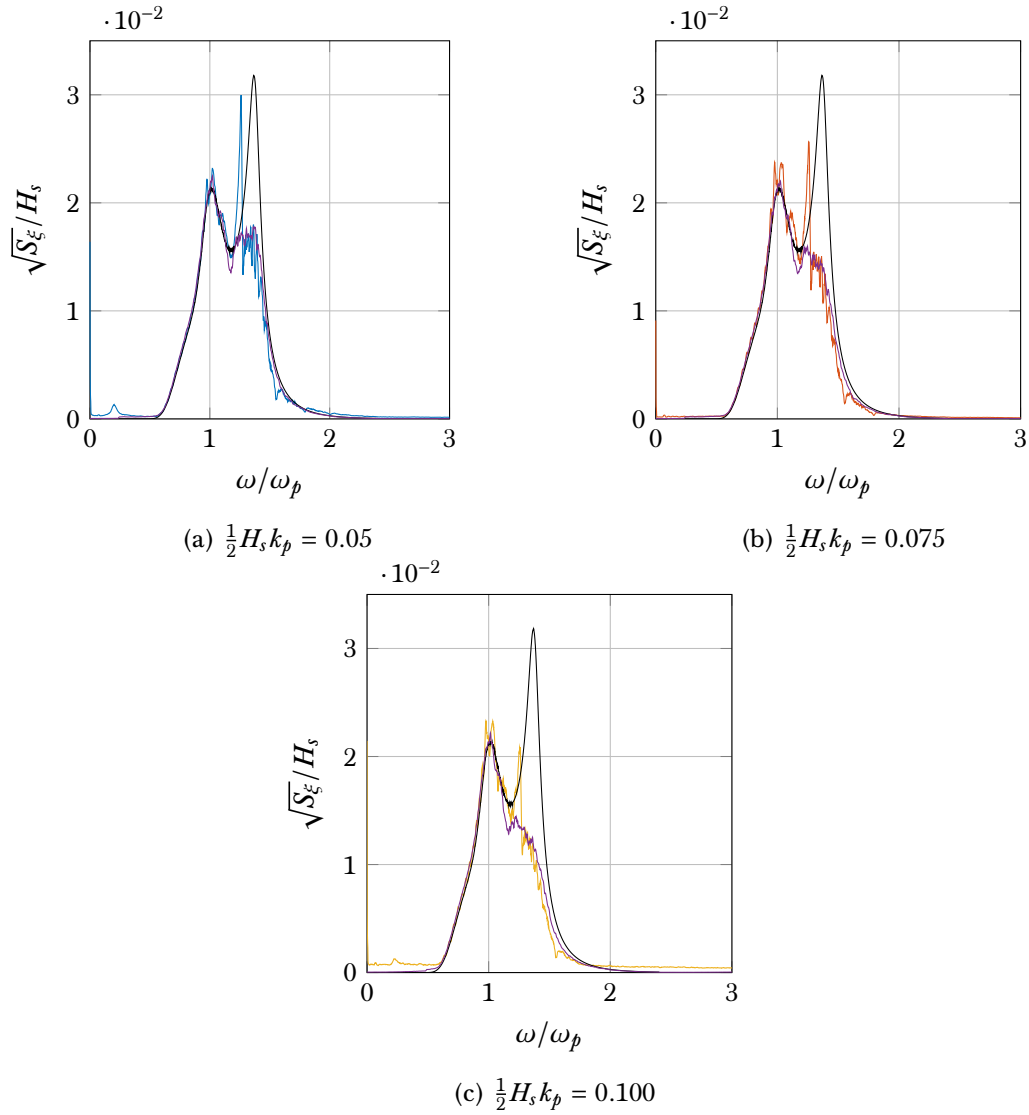


Figure 5.14: Heave displacement spectra for $k_p h = 3.5$ for (—) linear potential flow solution, (—, —, —) ensemble averaged experimental data, and (—) time domain simulation. The spike in the experimental data at $\omega/\omega_p \approx 1.25$ is due to the side-walls and discussed in the text.

With increasing $\frac{1}{2} H_s k_p$, Figure 5.14 demonstrates that the trends between the experimental data and the linear potential flow prediction are consistent. As the spectrum is normalised by H_s , the linear potential prediction is the same for each value of $\frac{1}{2} H_s k_p$. Once again, around the sea state peak frequency the experimental data and linear poten-

tial prediction are in close agreement. At the resonance frequency, however, the nonlinear effect of the vortex-shedding damping is evident as the reduction in the experimental spectral content increases with increasing sea state steepness. This is expressed by the ratio $(\text{Lin}/\text{Exp})|_{\omega_r}$, which is 1.93 for $\frac{1}{2}H_s k_p = 0.050$ and rises to 2.58 for $\frac{1}{2}H_s k_p = 0.100$. Once again, the time domain model provides accurate predictions across the range of sea state steepness. This is particularly remarkable given a single vortex-shedding damping coefficient is being used from the regular wave experiments.

Side-wall effects

In all the spectra, a spike is present just before the resonance frequency, which corresponds to the second side-wall induced mode identified by Yeung & Sphaier (1989) and discussed for the excitation force in §4.6. Although the change to the excitation force is small, the proximity of the mode frequency to the resonance frequency greatly amplifies its effect in the motion response of the body. Furthermore, the resonance identified by the potential flow solver is, by definition, computed in the absence of vortex-shedding damping. For a general dynamic system, increasing the damping lowers the resonance frequency which would have the effect of exacerbating the spike in the force spectrum. The spike does not appear in the time domain simulation spectrum as this spectrum has been computed using the potential flow excitation force.

Yeung & Sphaier (1989) also concluded that the hydrodynamic added mass is significantly altered at low frequencies; however, given the good agreement between the linear potential flow and the experimental results at these frequencies, this effect is deemed to be small. The spike in the experimental spectra suggests the observed displacements are slightly greater than if the side walls were not present.

To understand the effect of this spike, the probability of exceedance distribution for $k_p h = 3.0$ and $\frac{1}{2}H_s k_p = 0.075$ has been recreated from the experimental spectra with the spike artificially removed. This has been achieved by taking the displacement amplitude spectrum (Figure 5.13) and replacing the spectral content across the spike with a straight line as shown in Figure 5.15. The displacement record is then recreated by assigning a random phase to each component in the modified spectrum and performing an IFFT. Figure 5.16 shows the result of the process: using the modified spectrum reduces the displacements by approximately 6% at $Q = 10^{-3}$, but does not take account of the full

overdamping of the displacements by the time domain solution which are a further 10% smaller. This demonstrates that although present, the side-wall effect is relatively modest.

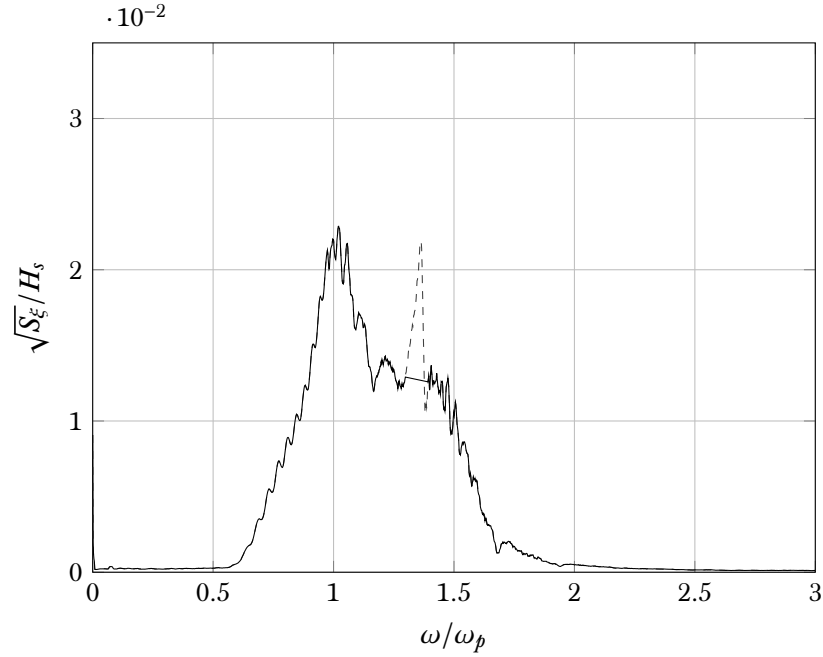


Figure 5.15: Experimental spectra for $k_p h = 3.0$, $\frac{1}{2} H_s k_p = 0.075$ with the spike due to sidewall effects removed. (---) Original spectrum, and (—) with spike removed.

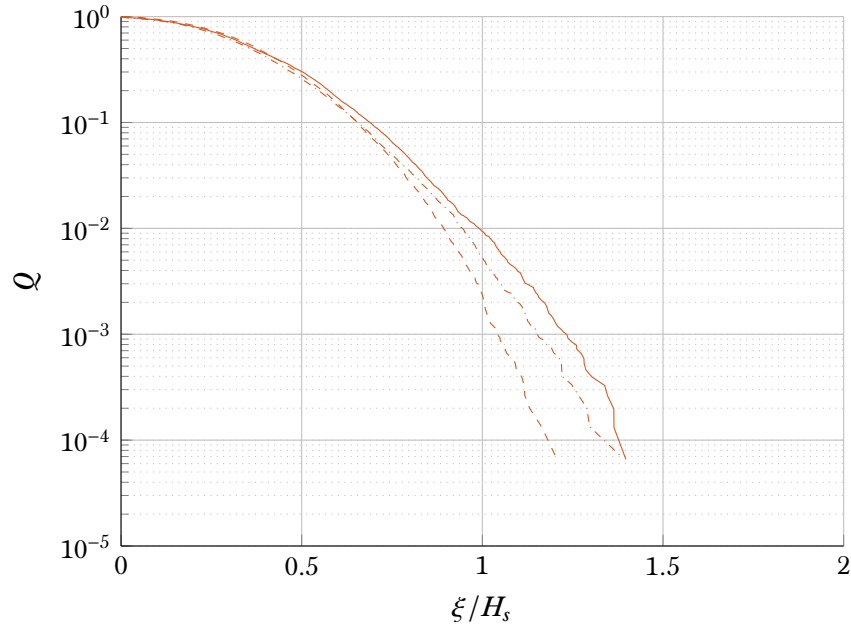


Figure 5.16: Heave displacement probability of exceedance distribution with spike due to side-wall effects removed. (—) Original experimental spectrum, (---) Modified spectrum, and (-.-) time domain simulation with linear analytic spectrum.

5.4 Quantification of nonlinearity

As was established in §4.5, an asymmetry in the force maxima and minima was observed in the experimental probability of exceedance distributions. Such an asymmetry was not translated into the displacements, however, as shown in Figure 5.3. Possible reasons for the lack of asymmetry could be:

- i. the presence of vortex shedding damping (§5.3), or
- ii. the dynamic properties of the body.

The asymmetry in the heave excitation force was attributed to the low-frequency difference terms (§4.6). In the context of body motions, the spectral analysis demonstrated that the content at these frequencies was small, as seen in Figures 5.12 and 5.14. Given the spectral analysis also demonstrated that the vortex shedding damping overwhelming occurs at the resonance frequency, the low-frequency terms are not suspected to be dissipated by vortex shedding.

To confirm that vortex shedding damping plays a small role in inhibiting the transfer of the low-frequency difference terms from the excitation force, the time domain solution developed in the previous section is used. For this analysis, however, the measured exci-

tation force is used as the input, rather than a linearly calculated force. This measured force includes the full nonlinear content discussed in the previous chapter. Furthermore, $C_D = 0$ to remove vortex-shedding damping effects.

Figure 5.17 shows the heave probability of exceedance distributions with $k_p h = 2.5$, $\frac{1}{2} H_s k_p = 0.075$ including the linear potential prediction, the experimental data and the time domain solution. For the time domain simulation, both the motion minima and maxima lie above the experimental data, and close to the linear potential prediction, indicating the lack of vortex-shedding damping in the simulation. Similar to the experimental data, however, is the relatively small displacement minima-maxima asymmetry, corresponding to approximately 8% at $Q = 10^{-3}$. For the same conditions, the force minima are 33% larger than the force maxima.

In the long wave regime (small ω) the value of the RAO is 1 (Figure 5.7), suggesting that the forcing at these low frequencies ought to translate into the displacements. The presence of the second peak in the response spectrum, due to the value of the RAO at the resonance frequency (Figure 5.11), however, decreases the relative importance of the low-frequency components in the overall response. This reduced importance, therefore, goes some way to explain the reduced minima-maxima asymmetry in the displacement probability of exceedance plots. Given that the RAO at resonance is between 1.5 and 2.0 (Figure 5.7), however, suggests that four-fold decrease in the asymmetry may not be entirely attributed to this reason. Chapter 3 demonstrated the importance of the radiation potential on the second-harmonic motions for the sum-frequency (self-interaction) displacements and may also be important for the difference-frequency displacements. This could be investigated further using ICRD3D in the future to draw a firmer conclusion.

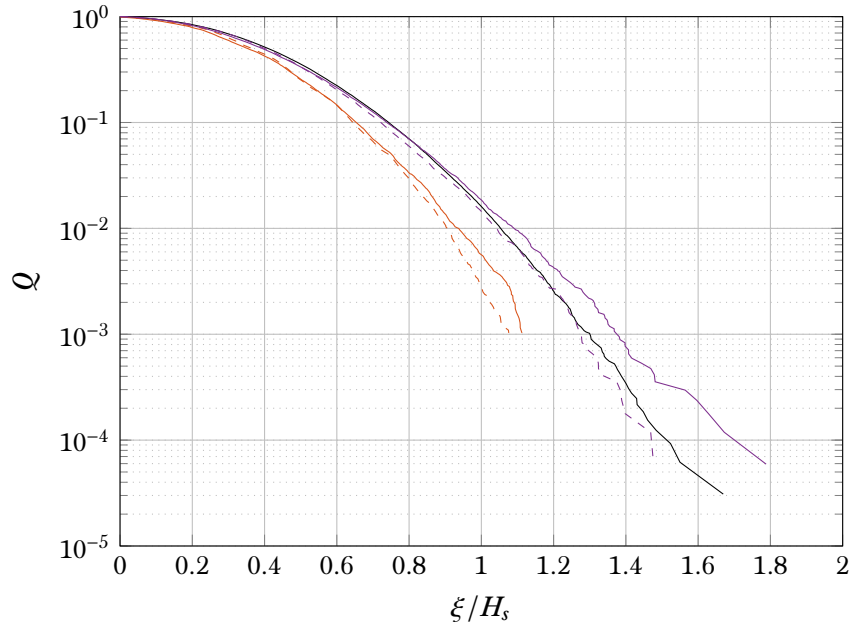


Figure 5.17: Heave motion probability of exceedance distribution for $k_p h = 2.5$, $\frac{1}{2} H_s k_p = 0.075$ using time domain simulation with measured excitation force and $C_D = 0$. (—) Linear potential prediction, (—) experimental data and (—) time domain simulation. Solid lines refer to displacement minima and dashed lines to displacement maxima.

5.5 Concluding remarks

This chapter has examined how the excitation forces described in Chapter 4 translate into truncated cylinder displacements. The displacement behaviour displays a number of distinct characteristics compared to the excitation force.

Firstly, the measured displacements were smaller than predicted using linear potential flow theory. This is despite the nonlinear amplifications in the excitation force which could be expected to increase the motions. The contrast is particularly acute when considering an increase in sea state steepness: the excitation force increases yet the displacements decrease. Identifying and quantifying this effect is one of the principal achievements of this PhD.

The reduction in displacement is attributed to vortex shedding damping. A simple methodology using a time domain simulation with a quadratic damping term was found to accurately predict the experimental measurements. Notably, a linear potential flow excitation force was used as the input. Bhinder *et al.* (2011) also took the power-take-off (PTO) mechanism into account and found the damping associated with the PTO “overshadowed”

that due to vortex shedding. The relative importance of these two sources of damping will be dependent on the specifics of a particular device (such as the control strategy) and can be investigated in the future.

The second distinct characteristic in the displacement data is the lack of maxima-minima asymmetry. Such an asymmetry was observed in the excitation force. Demonstrating that the low-frequency difference terms present in the forcing do not translate into the motions is the second key contribution of this chapter. This is due to the introduction of the second peak (at the resonance frequency) in the displacement response reducing the overall importance of the low-frequency components.

Finally, it should be noted that the methodology presented herein of predicting the truncated cylinder displacements is well within the capability of design practice. The time domain simulation is computationally efficient and, even with a linear excitation force input, provides accurate results.

6

Hydrodynamics of a domed cylinder

6.1 Chapter overview

The previous chapter has highlighted the importance of vortex shedding damping in predicting the displacement of a heaving truncated cylinder. In the context of WEC devices, it is advantageous to minimise these losses in order to maximise the body displacements. This chapter, therefore, considers a cylinder with a hemispherical base. The domed cylinder is chosen to have the same dynamic properties—predicted by potential flow—as the truncated cylinder so that any differences in the observations are due to non-potential effects.

Once again, this chapter comprises analysis using new experimental data and time domain simulations and has two principal aims:

- i. quantify the effect of rounding the sharp corners, and
- ii. investigate the conditions that cause maximum displacement to develop an understanding of an appropriate design wave.

Despite experiencing near-identical excitation forcing, the experimentally observed displacements are significantly larger for the domed cylinder. This is a testament to a reduction in the vortex shedding from the sharp edges of the truncated cylinder. In common with the previous chapter, a detailed spectral analysis is undertaken, which reveals a different damping mechanism to that observed for the truncated cylinder.

Finally, the wave conditions which lead to the largest domed cylinder displacements are explored. It is established that these large displacements do not necessarily occur in the individual largest waves, but rather for wave groups that have a large crest immediately preceding the main event. Once the underlying incident conditions leading to such large displacement events are known, the time domain solution can accurately reproduce the measured displacement traces.

6.2 Experimental setup

The physical model, described in §4.2.2 for fixed tests and §5.2.1 for floating tests, was designed from the outset to accommodate a hemispherical base. Figure 6.1 shows the domed cylinder configured for the floating experiments, in which the bottom section of the cylinder has been replaced with a hemisphere. The internal layout of the hemispherical base is identical to the truncated base, so that the arrangement for the fixed-body tests with the load cell is the same as that in Figure 4.2.



Figure 6.1: Physical model for the domed cylinder. The internal layout is identical to the truncated base so that the arrangement with the load cell for the fixed-body experiments is identical to that depicted in Figure 4.2. Key: 1. Hemispherical base

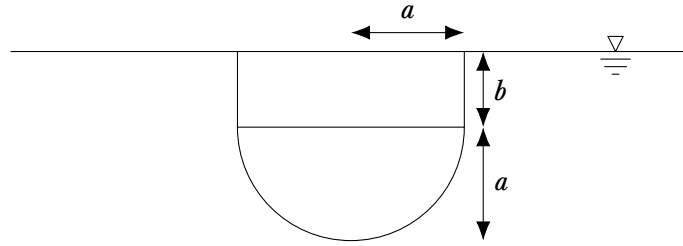


Figure 6.2: Definition of the draught for the domed cylinder.

In order to make comparisons with the previous chapters, an ‘equivalent draught’ must be defined. As the vertical dimension of the hemisphere is fixed as the radius of the cylinder, the draught, b , refers to the distance of submergence of the cylindrical part of the buoy. This definition is consistent with Yeung & Jiang (2014), and is shown in Figure 6.2. A draught of $b/a = 0$ implies a ‘pure’ hemisphere. For the floating experiments, the mass of the body is increased using an even distribution of lead weights on the top surface to achieve this additional submergence.

The ‘equivalent draught’ is defined as the draught which causes the domed cylinder, denoted DC1, to have similar dynamic properties to TC1 in potential flow conditions. Therefore, the difference in the experimentally measured responses of the two geometries will be solely due to vortex-shedding effects. Figure 6.3 shows the RAO for a range of b/a for the domed cylinder (solid lines) along with the RAO for TC1 (dashed line). For the ‘pure’ hemisphere case, the resonance frequency is much greater, $ka > 1$, than TC1 at $ka = 0.65$; however with increasing b/a , the resonance frequency reduces and the response becomes larger. At $b/a = 0.56$, corresponding to $b = 70$ mm, a balance is struck between the resonance frequency and response amplitude, and is used henceforth. The total draft for DC1 is therefore 195 mm compared to 125 mm for TC1.

For $b/a = 0.56$, Figure 6.4 also shows that the first-harmonic excitation force for TC1 and DC1 are near-identical. This may be explained by observing that the average force application point is higher for a domed cylinder than a truncated cylinder. This is illustrated schematically in Figure 6.5. An increase in draught brings the application points closer together, which ultimately leads to the near-identical force characteristic for $b/a = 0.56$ (Figure 6.4).

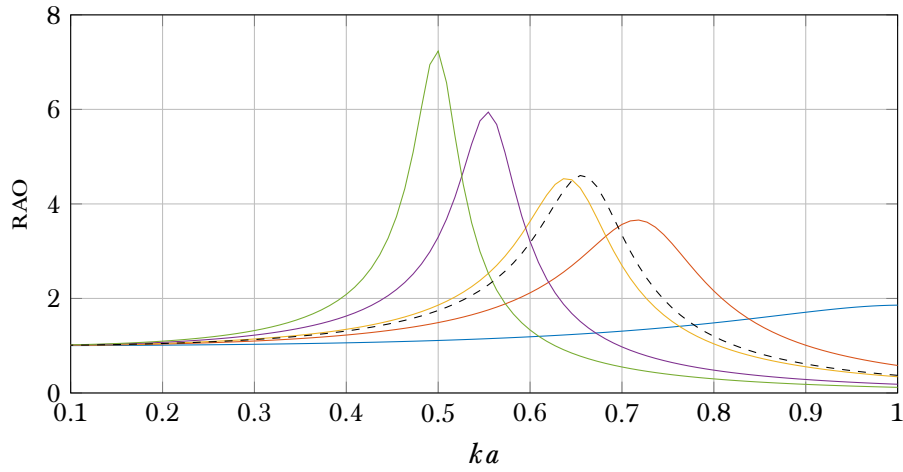


Figure 6.3: Heave response amplitude operator for a domed cylinder with $b/a =$ (—) 0.00, (—) 0.40, (—) 0.56, (—) 0.80, (—) 1.00, and (---) TC1, calculated using ICRD3D.

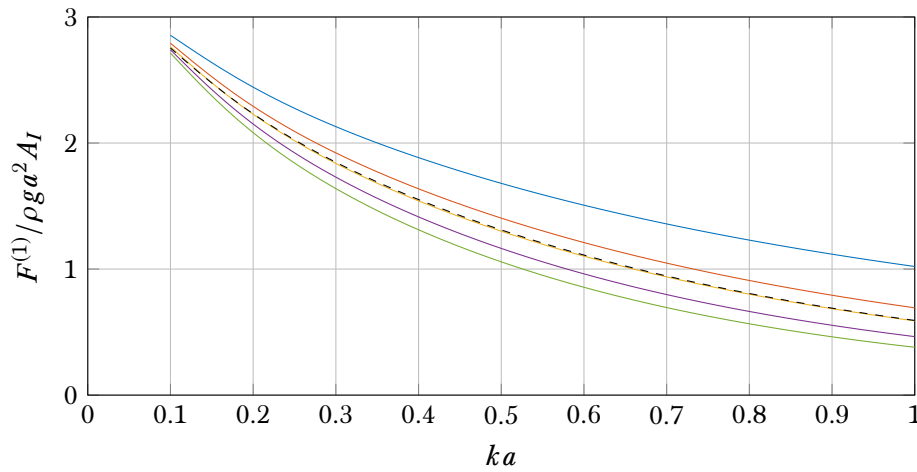


Figure 6.4: First-harmonic heave excitation force for a domed cylinder $b/a =$ (—) 0.00, (—) 0.40, (—) 0.56, (—) 0.80, (—) 1.00, and (---) TC1, calculated using ICRD3D.

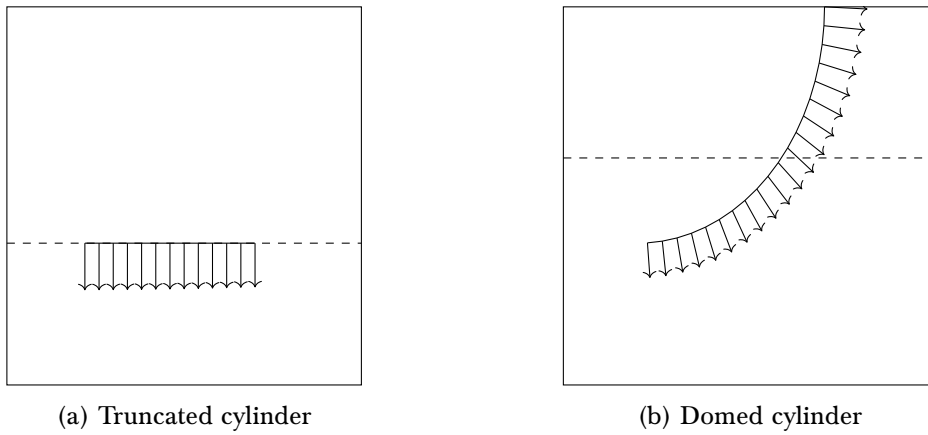


Figure 6.5: Comparison of the point of average force application. Arrows indicate direction of normal force and dashed line the point of average application.

6.3 Determination of the damping coefficient

The procedure for determining the vortex shedding damping coefficient is identical to that outlined in §5.3.1. At each wave frequency, the equation of motion is solved in the time domain, including an additional damping term proportional to the square of the body velocity. A single damping coefficient for all ka and $A_I k$ is adjusted to provide a good prediction of the experimental results.

Figure 6.6 shows the heave response amplitude operator for DC1 obtained using a potential flow solution (ICRD3D), experiments of wave steepness $A_I k = 0.05, 0.075$ and 0.100 , and the corresponding time domain simulations with $C_D = 0.33$. As seen previously with the truncated cylinder, the experimental data and potential flow solution are in good agreement away from the resonance frequency. At the resonance frequency, vortex shedding damping reduces the motions compared to the potential flow solution. The time domain model captures this reduction, but the quadratic damping term leads to larger variation in displacement at the frequencies around the resonance compared to the experimental data. Indeed, the experimental data scales near-linearly with the incident amplitude which is explored further in §6.4.5. Therefore, using $C_D = 0.33$ for all the incident wave steepness leads to an over estimation of the displacements for $A_I k = 0.05$ and an under estimation for $A_I k = 0.100$.

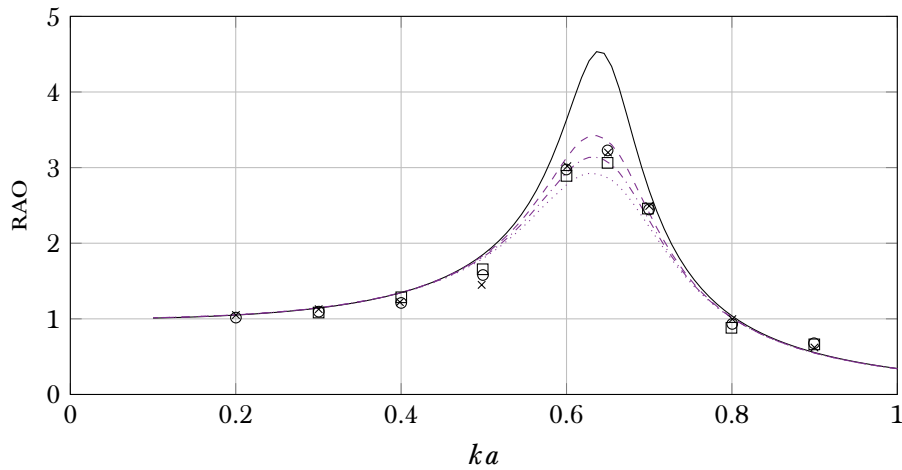


Figure 6.6: Heave response amplitude operator for DC1 with incident waves of steepness $A_I k = (\times)$ 0.05, $(\circ)$ 0.075, and $(\square)$ 0.100. Potential flow simulation using (—) ICRD3D and quadratic-damping time domain simulations with $C_D = 0.33$ for $A_I k = (- - -)$ 0.05, $(- \cdot -)$ 0.075, and $(\cdots)$ 0.100.

The damping coefficient obtained for the domed cylinder is a factor 5.2 smaller than for

the truncated cylinder. To illustrate this reduced damping, Figure 6.7 compares the RAO for TC1 and DC1 obtained from the experiments with $A_I k = 0.075$, and from the potential flow solution. Near the resonant frequency, at $ka = 0.65$, the measured displacement is 60% lower than the potential flow solution for TC1 compared to a reduction of 29% for DC1.

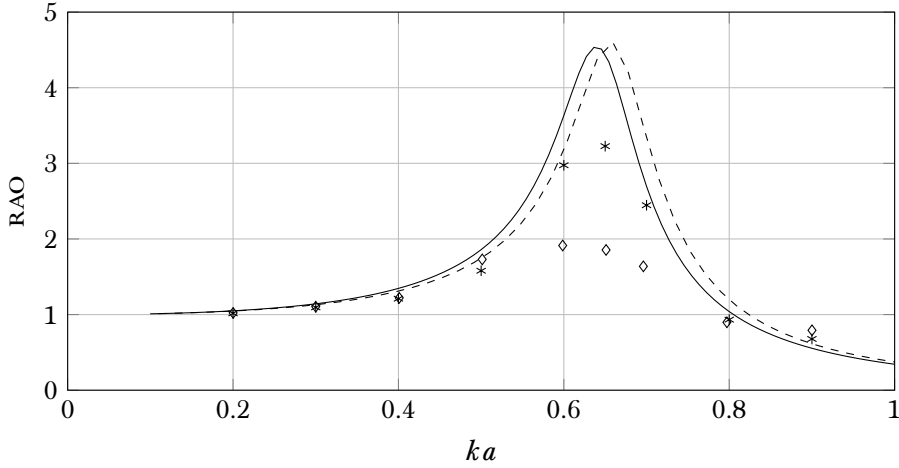


Figure 6.7: Heave response amplitude operator comparison for TC1 and DC1. ICRD3D: (---) TC1 and (—) DC1. Experiment with $A_I k = 0.075$: (◇) TC1 and (*) DC1.

6.4 Hydrodynamics in irregular sea states

6.4.1 Test conditions

As seen in the previous chapter, both the sea-state peak frequency, $k_p h$, and the steepness, $\frac{1}{2} H_s k_p$, play a role in the displacement response of the truncated cylinder. Therefore, two series of tests have been undertaken:

- i. $k_p h = 3.5$ for $\frac{1}{2} H_s k_p = 0.05, 0.075$ and 0.100 , and
- ii. $\frac{1}{2} H_s k_p = 0.05$ for $k_p h = 2.0, 2.5, 3.0$ and 3.5 ,

Once again, the input files to the wavemakers were identical to those used for the TC1 excitation force and displacement tests.

6.4.2 Preliminary observations

To start to understand the effect of changing from a domed to a truncated cylinder in irregular sea states, direct time-history comparisons of the excitation force and displace-

ment have been made in Figures 6.8 and 6.9 respectively. The comparison is made for sea states characterised by $k_p h = 3.5$ of wave steepness $\frac{1}{2}H_s k_p = 0.050$ and 0.100 . A time window of $0.48 \leq t/T_r \leq 0.50$ is used for the first sea state and $0.59 \leq t/T_r \leq 0.61$ for the second.

The excitation force traces in Figure 6.8 are very similar in both wave steepness cases, thereby indicating that the close comparison of the excitation force in the regular wave case (Figure 6.4) carries through into irregular sea states. Differences of approximately 14% are measured for the largest maximum for $\frac{1}{2}H_s k_p = 0.05$ at $t/T_r = 0.491$, and of 12% between the largest minimum for $\frac{1}{2}H_s k_p = 0.100$ at $t/T_r = 0.600$. The generally smaller excitation force in Figure 6.8(b) despite the steeper sea state is simply due to the portion of the trace being examined. The consistency between the two geometries provides some additional evidence that vortex shedding damping has a minor impact on the excitation forcing.

Figure 6.9 compares the displacement time-history of DC1 and TC1 corresponding to the excitation force traces in Figure 6.8. For $\frac{1}{2}H_s k_p = 0.05$ (sub-figure (a)) the pattern of the displacements for DC1 is broadly similar with TC1, but experiencing slightly greater motion amplitudes. These departures occur, for example for the wave group in the range $0.480 \leq t/T_r \leq 0.485$ where the excitation force is near-identical. The difference between the two geometries is significantly greater for $\frac{1}{2}H_s k_p = 0.100$ in Figure 6.9(b), where differences of up to 70% are measured. Again, these large differences occur at times for which the excitation forces are near-identical. Overall, the displacements are substantially smaller for TC1 indicating the effect of reducing the vortex shedding damping is also important in irregular waves.

6.4.3 Heave force and displacement statistics

Figure 6.10 presents the force and displacement statistics for DC1. The excitation force probability of exceedance, Figure 6.10(a) demonstrates that the distributions for DC1 are similar in magnitude to TC1 (Figure 4.7(d)), and also exhibit the maxima-minima asymmetry attributed to the low-frequency difference terms. This supports the observations in Figures 6.4 and 6.8 that a domed cylinder draft of $b/a = 0.56$ does not result in a significant change to the excitation force. Furthermore, the results in Figure 6.10(a) clearly demonstrate that vortex damping effects do not play a large role in the excitation force.

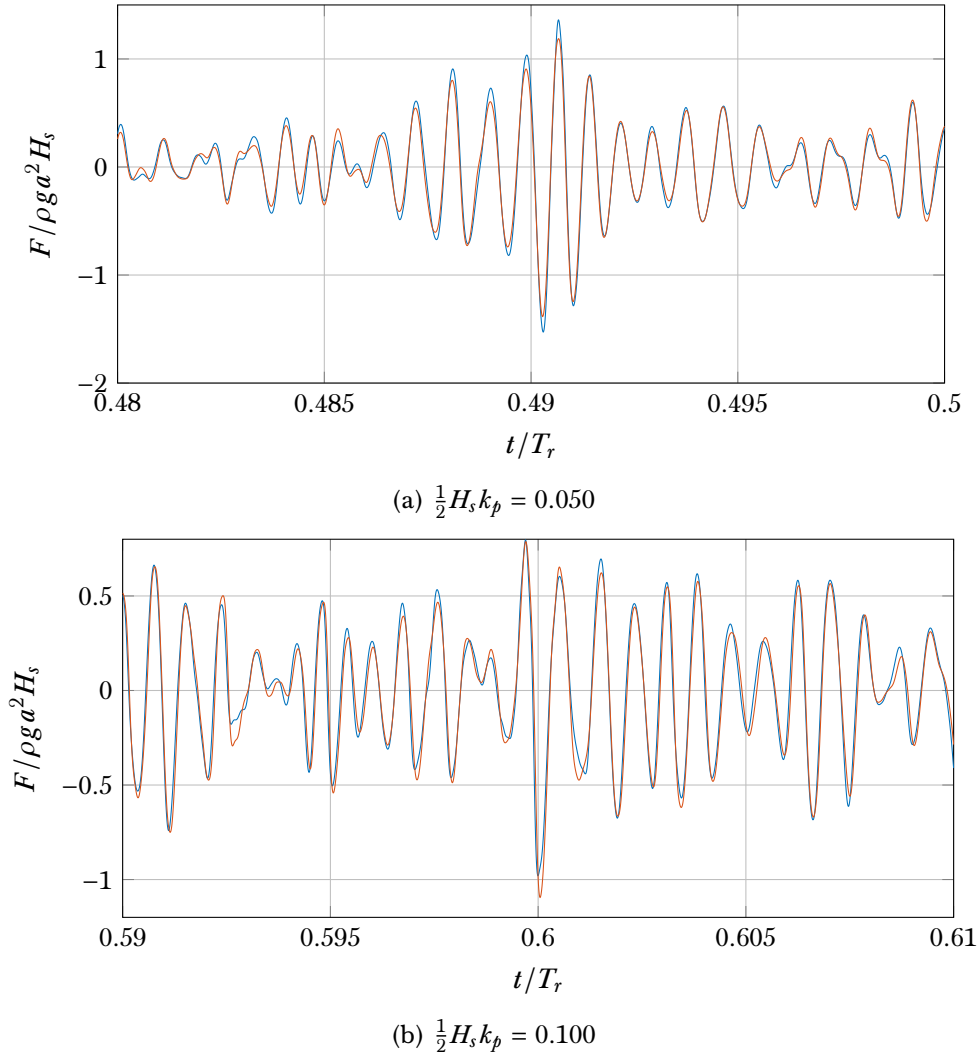
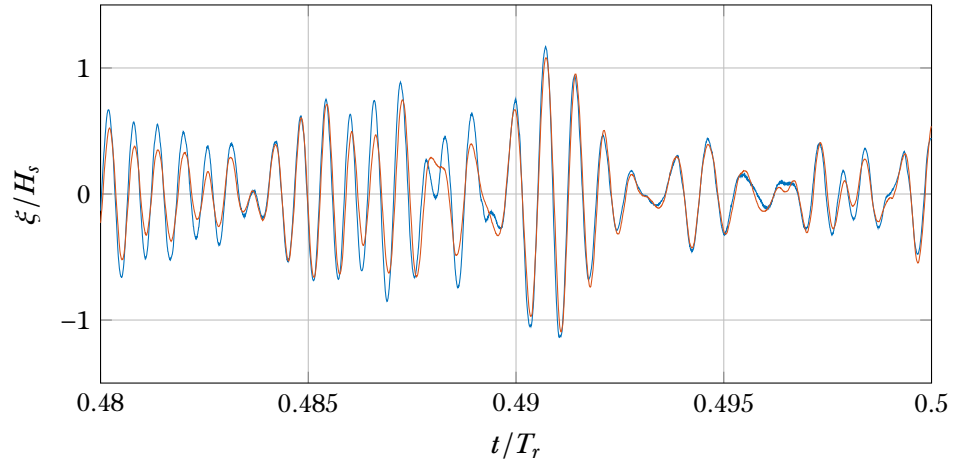
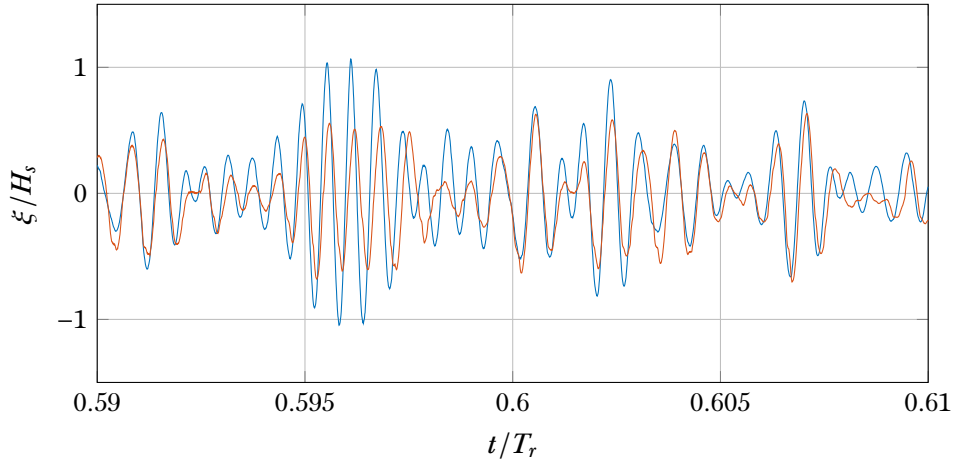


Figure 6.8: Excitation force time history comparison of (—) DC1 and (—) TC1 for $k_p h = 3.5$

The effect of using the hemispherical base on the displacements is evident from the probability of exceedance distribution shown in Figure 6.10(b). For all of the $\frac{1}{2} H_s k_p$ values under consideration, the experimental results either lie close to the linear prediction, or exceed the prediction. Figure 6.10(c) demonstrates that for $\frac{1}{2} H_s k_p = 0.05$ the observed displacements remain close to the linear prediction but increase to become about 8% larger than the prediction at $Q = 10^{-3}$. Furthermore, the displacements for $\frac{1}{2} H_s k_p = 0.075$ and 0.100 both generally lie between approximately 4–8% above the linear prediction. This is in stark contrast to TC1, where the observed displacements were generally 10% less than the linear potential flow prediction (Figure 5.5(d)). As with TC1, the asymmetry of the force maxima and minima is not translated into the displacements.

A second feature of the displacement statistics for the cylinder with hemispherical base

(a) $\frac{1}{2}H_s k_p = 0.050$ (b) $\frac{1}{2}H_s k_p = 0.100$ Figure 6.9: Displacement time history comparison of (—) DC1 and (—) TC1 for $k_p h = 3.5$

is the broadly increasing displacements with increasing $\frac{1}{2}H_s k_p$. Again, this contrasts with TC1, where the relative (or normalised) displacements reduced with increasing $\frac{1}{2}H_s k_p$ (Figure 5.5(d)). To explore both these features in further detail, a spectral analysis has been undertaken as described in the following section.

6.4.4 Spectral analysis

Adopting the same form of analysis as in the context of Chapter 5, Figure 6.11 shows the spectral content of (i) the experimentally observed heave motions for DC1, (ii) the linear analytical solution, and (iii) the time domain solution with $C_D = 0.33$. The left column, sub-figures (a)–(c), shows the effect of increasing $\frac{1}{2}H_s k_p$ for $k_p h = 3.5$ and the right column,

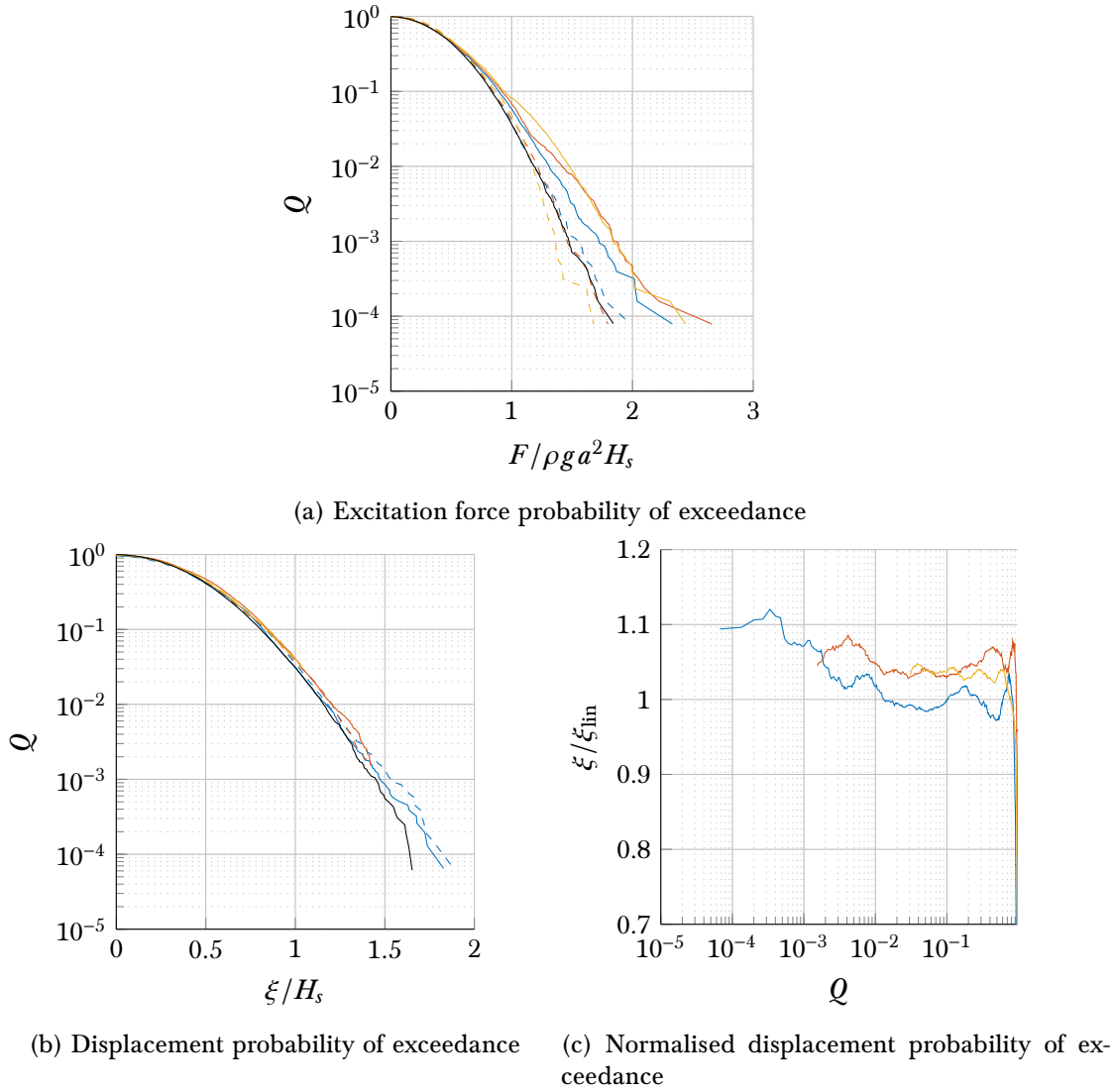


Figure 6.10: Heave excitation force and displacement statistics for $k_p h = 3.5$ and $\frac{1}{2}H_s k_p =$ (—) 0.050, (—) 0.075, and (—) 0.100. Solid lines refer to force/motion minima and dashed lines to the force/motion maxima. (—) Linear analytical solution.

sub-figures (d)–(f), the effect of increasing $k_p h$ for $\frac{1}{2}H_s k_p = 0.050$. As with TC1, a spike appears in the motion spectra due to the side walls (see §5.3.3).

Figures 6.11(a)–(c) display some similar characteristics to the truncated cylinder: the experimental data and the linear potential prediction are in close agreement at the sea state peak frequency, whereas the experimental data lie well below the potential solution at the resonant frequency. The damping effect at ω_r is much reduced in the case of DC1, however. For example, in a sea state characterised by $k_p h = 3.5$, $\frac{1}{2}H_s k_p = 0.075$, the displacement reduction at ω_r for DC1 (Figure 6.11(b)) is approximately 24% compared

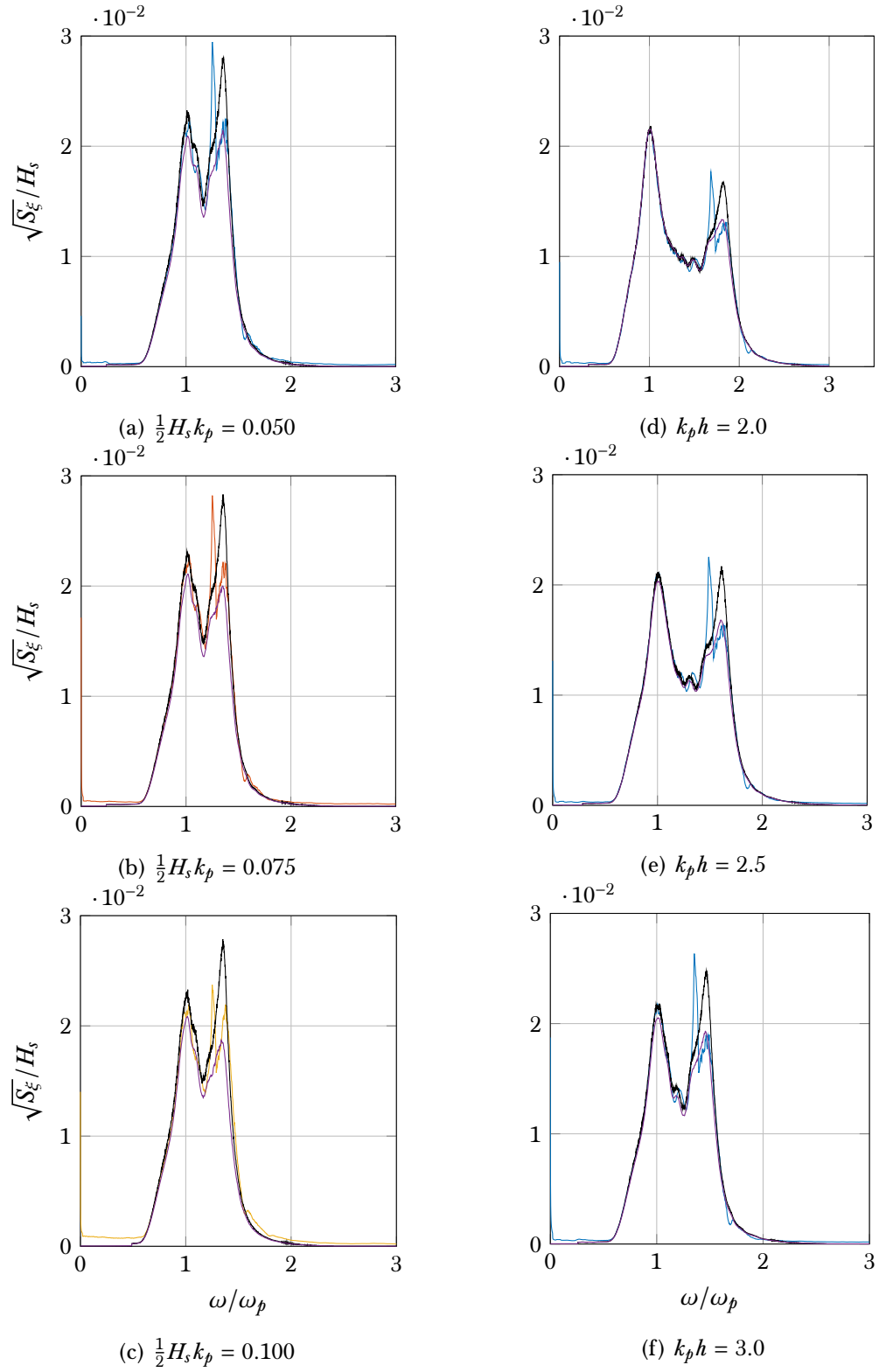


Figure 6.11: Heave displacement spectra for DC1 showing (—) linear analytical solution, (—, —, —) experimental data, and (—) time domain solution with $C_D = 0.33$. Left column: $k_p h = 3.5$. Right column: $\frac{1}{2}H_s k_p = 0.050$.

Table 6.1: Ratios of spectral content at ω_r for DC1 and TC1 with $k_p h = 3.5$.

$\frac{1}{2}H_s k_p$	TC1: Lin/Exp	DC1: Lin/Exp	DC1: Lin/TD
0.050	2.00	1.28	1.18
0.075	2.46	1.26	1.27
0.100	2.91	1.29	1.35

to 56% for TC1 (Figure 5.14(b)). This reduction in damping at ω_r increases the overall domed cylinder velocity which results in some damping at ω_p . At this spectral peak, for $k_p h = 3.5$, $\frac{1}{2}H_s k_p = 0.100$ (Figure 6.11(c)) the experimental data lie approximately 6.6% below the linear potential prediction.

The distinguishing feature of the experimental data in Figures 6.11(a)–(c) is the lack of increased damping with increasing sea state steepness. This is demonstrated in Table 6.1 which compares the ratio of the linear potential prediction to the experimental data at the resonance frequency. For TC1, the ratio clearly increases, indicating increased vortex damping, whereas the ratio remains at approximately 1.28 for DC1. As the figures are normalised by the significant wave height, this suggests that the vortex damping scales linearly with the domed cylinder velocity. This linear scaling cannot be captured by the time domain simulation which assumes a quadratic-damping model and predicts a (small) increase in the damping as shown in the third column of Table 6.1.

The inability of the time domain simulation to fully capture the vortex damping is further observed in Figures 6.11(d)–(f) which shows the variation of the spectra with sea state peak frequency with $\frac{1}{2}H_s k_p = 0.050$. The simulation consistently over estimates the displacements at ω_r by approximately 10%. This result arises directly from the observations in determining the damping coefficient in §6.3, as $C_D = 0.33$ was found to over predict the motions for regular waves with $A_I k = 0.050$. Overall, however, the time simulation is once again proven to be a very useful method of reproducing the experimental data. The damping mechanism encountered with DC1 is explored further in the next section.

Despite the reduction in spectral content at the resonance frequency, the heave motion probability of exceedance plots, Figures 6.10(b)–(c), showed the linear potential flow solution under predicting the experimental motions by up to 8%. This difference can, at least in part, be attributed to the spike induced by the side walls of the tank. In each case, the energy contained within this spike slightly exceeds that between the experimental result

and the linear potential solution at the resonance frequency. Therefore, if the spike were to be removed, the experimental probability of exceedance distributions would lie just below the linear prediction, with the difference due to the vortex shedding damping at the resonance frequency.

6.4.5 Damping mechanism

Both the motion probability of exceedance results and the spectral analysis indicate that the damping mechanism for a domed cylinder scales linearly with the significant wave height of the sea state. In a linear setting, this is equivalent to scaling linearly with body velocity. This contrasts with a drag-force based approach which assumes the force due to vortex shedding scales with the square of the body velocity. A linearly scaling drag force term only occurs in laminar flows so is not applicable in the conditions herein.

Yeung & Jiang (2014) examined a number of cylinder base shapes for heaving cylinders in two dimensions using a discrete vortex method (DVM). These shapes included both a truncated and domed cylinder. Yeung & Jiang (2014) sought to characterise the heave behaviour of the body, including viscosity, using hydrodynamic coefficients calculated using a viscous solver rather than the approach proposed herein of including classic drag-force term in the equation of motion. Including viscosity was found to change the coefficients by $\pm 15\%$

Figure 6.12 reproduces Fig 10(a) by Yeung & Jiang (2014) to show the vorticity distribution around two cylinders undergoing heave motions in waves using their DVM solver. The red patches indicate clockwise vorticity and the blue patches denote anti-clockwise vorticity. The truncated cylinder in the left image shows vortex blobs that are convected away from the body, primarily due to flow separation at the sharp corners. This mechanism of vortex shedding is somewhat similar to the pattern associated with a typical drag force that has been modelled with the quadratic term. This may suggest why the time domain model was able to accurately predict the measured displacements at the resonant frequency in the spectral analysis for TC1 (§5.3.3).

The right image in Figure 6.12, however, shows that vorticity is generated in the vicinity of the domed cylinder and is not convected away. The energy damping mechanism in this scenario will be different and suggests a reason as to why the quadratic damping model assumed in this chapter does not fully capture the damping behaviour for a body with a

hemispherical base. A detailed investigation of these vortex shedding patterns is beyond the scope of this thesis and requires an extensive study in its own right. Further detail of such an investigation is included in the further work outlook in §7.3.3.

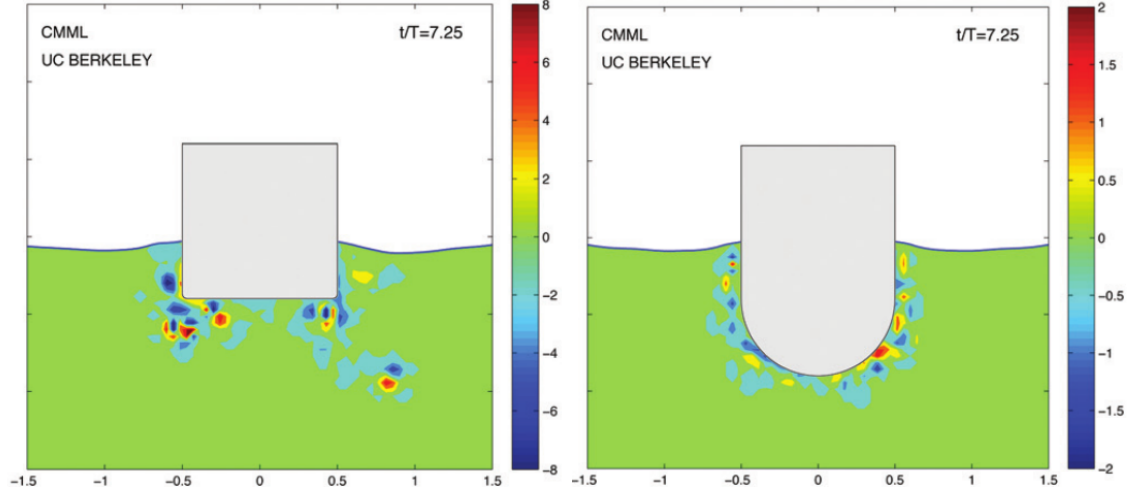


Figure 6.12: Visualisation of vortex blobs where the contour scale is based on a nondimensional vorticity defined by $\zeta/\sqrt{2g/B}$ where ζ is the surface elevation and B is the body diameter. Extract of Fig. 10(a) by Yeung & Jiang (2014)

6.5 Towards a design condition

6.5.1 Introduction

A key missing component in design practice for floating offshore structures is the specification of the conditions that lead to the maximum body displacement. In contrast, the conditions for the maximum force on fixed structures are well established and codified (DNV, 2014).

For the fixed-structure problem, the maximum force is calculated using the MJOS equation and is governed by the particle velocity and acceleration. The particle kinematics are proportional to the wave amplitude meaning that the largest (or steepest) waves necessarily lead to the largest forces. The specification of the design condition, therefore, is determining the shape of the wave at a particular probability of exceedance. This is typically a two stage process:

- i. the wave height or crest elevation is determined from a statistical distribution such as that of Forristall (2000), and

- ii. using the parameters established under (i), a focused wave group is generated using the NewWave spectrum defined by Tromans *et al.* (1991).

This provides a deterministic means of generating realistic large-amplitude wave events without needing to undertake long-duration irregular sea state simulations or testing.

The key feature of a focused wave group is that all the components come into phase at a particular point in time and space. An example group for the surface elevation, η , is shown in Figure 6.13 for $k_p h = 3.5$ and a non-dimensional crest elevation of $A_{\text{sum}} k_p = 0.125$, where A_{sum} is the amplitude of the focus event. This surface elevation is based on a linear calculation, though nonlinear calculations are also common. The frequency focussing effect is characterised by the maximum elevation occurring at $t/T_p = 0$, and a symmetric pattern before and after the maximum. As all of the components are in phase at the focus location, this peak represents the largest possible wave in the spectrum. A detailed discussion of focus wave groups in the context of surface elevations can be found in Gibson & Swan (2007).

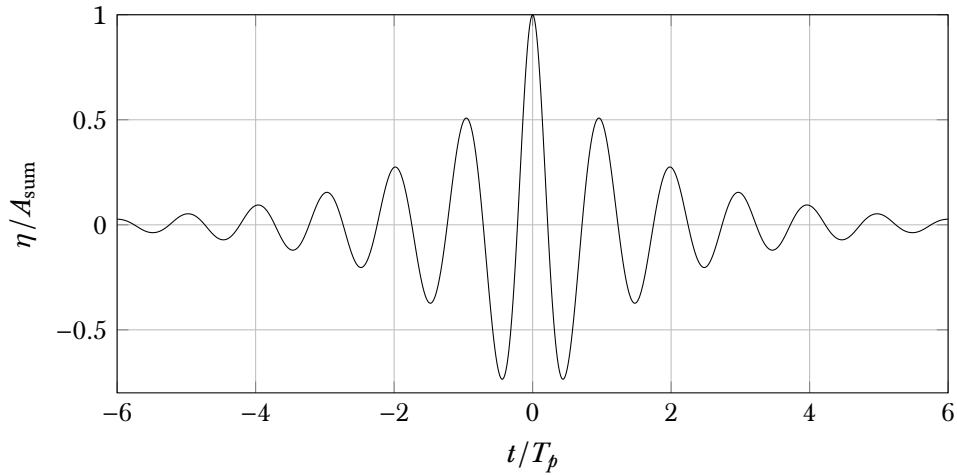


Figure 6.13: Example surface elevation for a focused wave group for $k_p h = 3.5$, $A_{\text{sum}} k_p = 0.125$ with Tromans *et al.* (1991) spectrum.

The use of focused wave groups for the determination of the maximum forces on a fixed structure has proven of great utility. In the case of body displacements, however, a similar approach may not be suitable. This is primarily due to the short duration of the focused event, which may not allow the body to reach its maximum acceleration. The remainder of this section explores the design condition for a sea state with $k_p h = 3.5$ and $\frac{1}{2} H_s k_p = 0.075$ for an event with $Q = 10^{-3}$.

6.5.2 Heave displacement and underlying incident waves

In order to assess the conditions that lead to the largest body motions in a particular sea state, nine displacement maxima events closest to $Q = 10^{-3}$ have been isolated from the time history. The ensemble average of these events is then taken, representing the *expected* displacement time history at $Q = 10^{-3}$. Figure 6.14 shows the nine displacement events and the expected trace.

The expected trace in Figure 6.14 has, broadly, the shape of a focused wave discussed in the previous subsection. The maxima and minima immediately preceding and succeeding the event are near-symmetric, with those before the event slightly larger (in absolute terms) compared to those after. This shows that the components of the displacement spectrum are indeed in phase at the time of the event.

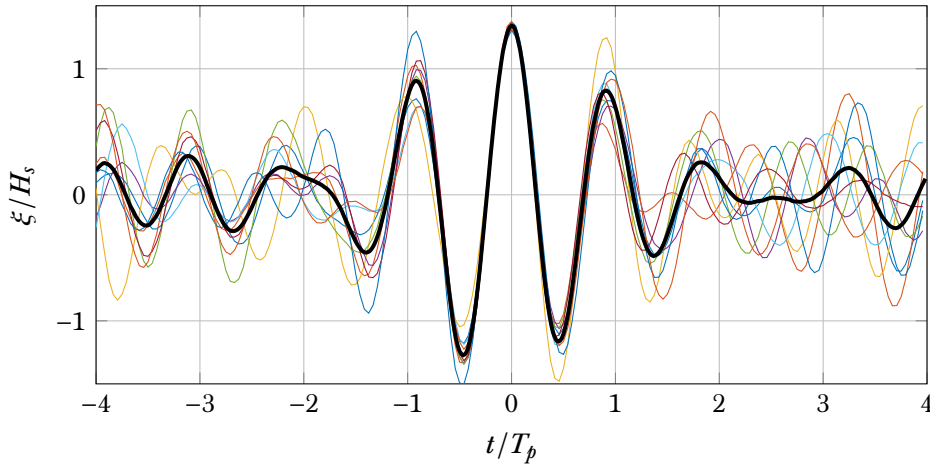


Figure 6.14: Heave displacement traces for $k_p h = 3.5$, $\frac{1}{2} H_s k_p = 0.075$ for events with $Q = 10^{-3}$. (coloured lines) Individual event traces, (—) ensemble average.

Turning to the underlying surface elevation that produces these motions, Figure 6.15 isolates the surface elevation (in the absence of the body) corresponding to the *displacement* events close to $Q = 10^{-3}$ seen in Figure 6.14. The expected surface elevation trace does not, however, resemble that of a focused wave, but rather has a large maximum and minimum immediately preceding the event. These large preceding waves excite the body in advance of the main event thereby inducing the greatest dynamic response for the largest wave in the group. Figure 6.16 compares these expected traces of the surface elevations which cause *displacement* and *crest* amplitudes at the 10^{-3} probability of exceedance. These two surface elevation histories are referred to as η_ξ and η_c respectively.

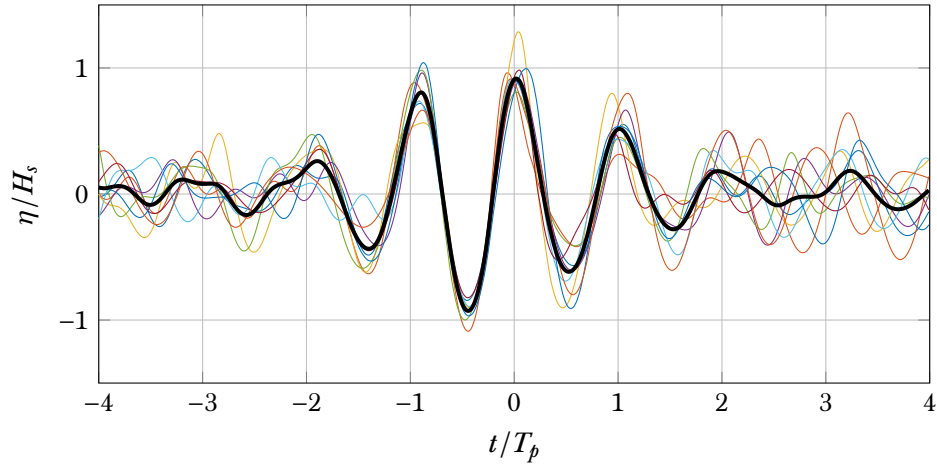


Figure 6.15: Incident wave surface elevation traces (in the absence of the body) for $k_p h = 3.5$, $\frac{1}{2} H_s k_p = 0.075$ for displacement events with $Q = 10^{-3}$. (coloured lines) Individual event traces, (—) ensemble average.

The profile of η_c in Figure 6.16 is one of a typical focused wave. This focussing effect leads to a maximum amplitude that is 17% larger than the maximum amplitude of η_ξ but also to a proceeding crest that is 56% smaller than that of η_ξ . This result provides definitive evidence that the largest body displacements do not necessarily occur when subjected to the largest incident waves.

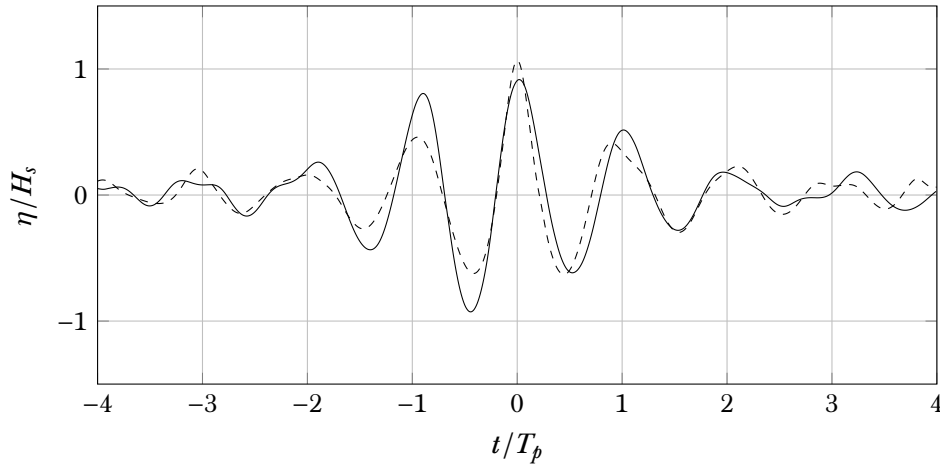


Figure 6.16: Comparison between expected surface elevation for $k_p h = 3.5$, $\frac{1}{2} H_s k_p = 0.075$, causing (—) displacements, η_ξ , and (---) wave-crests, η_c , arising at $Q = 10^{-3}$.

6.5.3 Effect of spectral bandwidth

In the context of predicting the design wave for the fixed-body problem, the spectral shape determined by Tromans *et al.* (1991) is typically employed to generate a focused wave

group. To predict the most probable shape of the largest waves within an irregular sea state, Tromans *et al.* (1991) found that the spectral energy scales with the amplitude spectrum rather than with the square root, as for the underlying JONSWAP spectrum. To generate a focus group with the same maximum amplitude, this scaling relationship has the effect of making the Tromans *et al.* (1991) spectrum more narrow banded compared to the JONSWAP spectrum. The different spectra and resulting focused groups are shown in Figure 6.17, in which the narrower bandwidth causes a more gradual build up of the surface elevation to the maximum. Rodríguez (2016) studied the nonlinear body displacements arising from these spectra in detail for two-dimensional potential flow, and found that the Tromans *et al.* (1991) spectrum led to greater displacements.

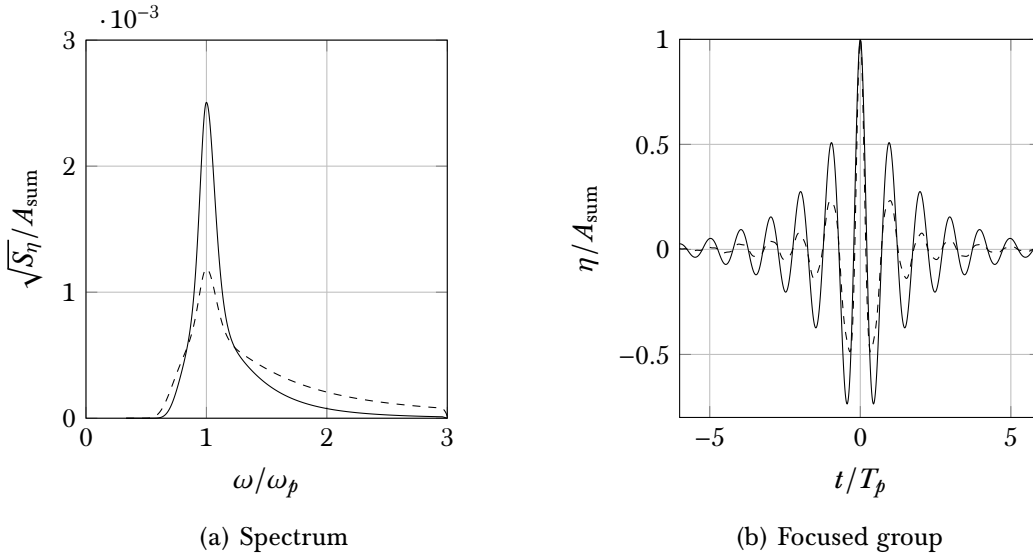


Figure 6.17: Comparison of spectrum and resulting focused group for (—) Tromans *et al.* (1991), and (---) JONSWAP spectra.

Figure 6.18 compares the displacement and surface elevation spectra arising from the expected $Q = 10^{-3}$ displacement event (Figures 6.14–6.15) and the underlying spectra from the sea-state record. For both the displacements and surface elevations, the spectra associated with the 10^{-3} displacement event is significantly more narrow banded than that for the underlying sea state. Therefore, a more gradual build up of energy is critical to achieving the largest body displacements.

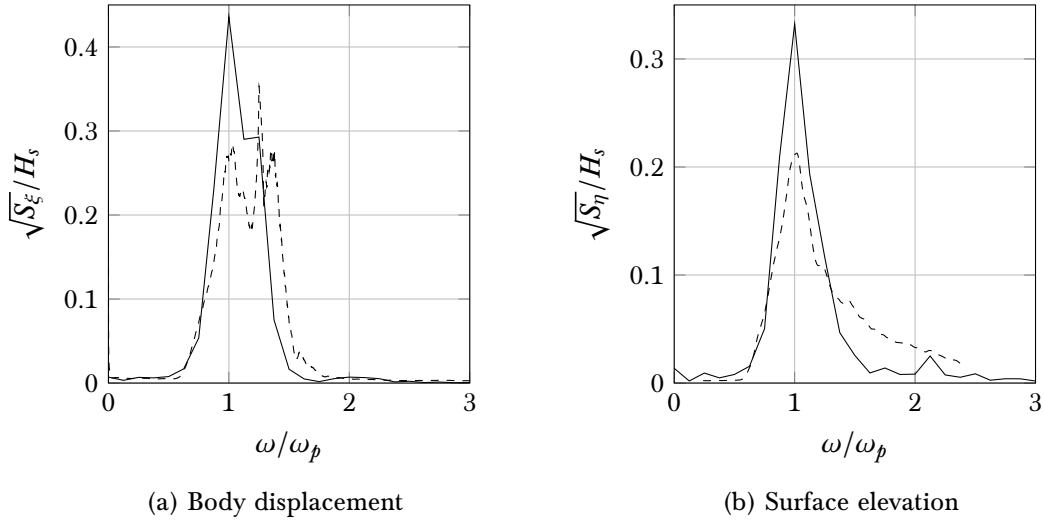


Figure 6.18: Comparison of the spectra of expected event at $Q = 10^{-3}$ and the underlying sea state. (—) Event, (---) sea state.

6.5.4 Design condition methodology

A number of approaches exist to specify the design condition. One approach is to undertake long irregular experiments for a geometry to measure the displacements arising at a particular probability of exceedance, as demonstrated in the present thesis. Such an approach requires the construction of a model and the use of relatively expensive model testing facilities. It is, therefore, not conducive to design iteration or exploring a parameter space. In lieu of the experiments, this thesis has demonstrated the applicability of a time domain simulation to faithfully model the body displacements. For the present work, simulation of a 1500 s return period sea state required approximately 580 s of computing time per seed, and therefore approximately 1.5 h per sea state using standard MATLAB routines on a 12 core workstation. For a more computationally expensive system and to reach smaller probabilities of exceedance the ‘naive’ time domain approach may become untenable.

Another approach to specifying the design condition is to empirically fit a distribution to the displacement probability of exceedance (Rodríguez & Spinneken, 2016). This allows an extensive experimental investigation to be of broader use and is successfully adopted in the context of incident waves following Forristall (2000). The shortcoming of this approach, in the context of floating offshore structures, is that it cannot properly take account of the innumerable body geometries and other particulars in the dynamic system, for example

mooring or power-take-off forces.

The more general approach is to specify the design incident wave which is then used as an input to a system-specific dynamics model. The basic feasibility of this latter approach, in the current context, is demonstrated in Figure 6.19, which compares the displacement trace from the time domain simulation using η_ξ as the input, and calculating the resulting excitation force using ICRD3D. The agreement in the magnitude of the displacement event is excellent.

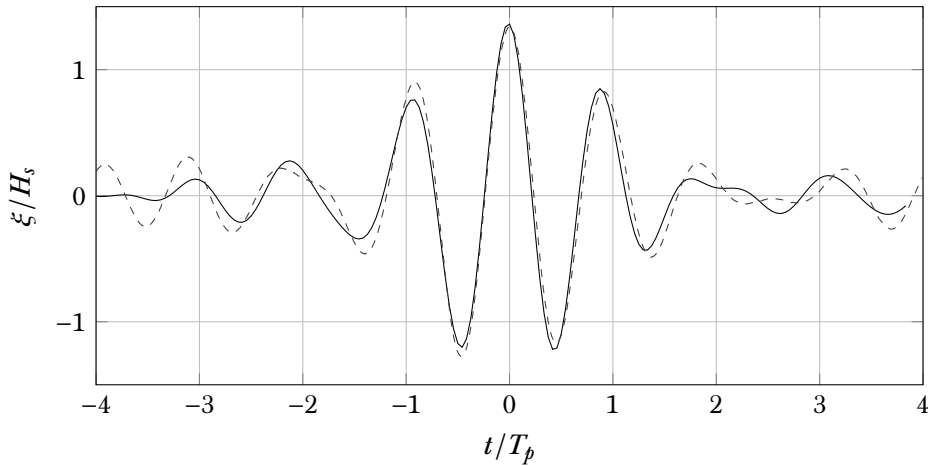


Figure 6.19: Re-creation of the displacement using the measured experimental design wave for $k_p h = 3.5$, $\frac{1}{2}H_s k_p = 0.075$ at $Q = 10^{-3}$. (---), Experimental data, and (—) time domain simulation.

The challenge, therefore, is to specify the spectrum of the underlying incident wave; this is made more difficult by the wave not being in focus at the time of maximum displacement. This requires further detailed analytical, empirical and computational study and is discussed in the context of further work in §7.3.4.

6.6 Conclusions

This chapter has experimentally investigated the heave forces and displacements of a cylinder with a hemispherical base. This domed cylinder was chosen to have near-identical dynamic properties to the truncated cylinder used in the previous chapter. Removing the sharp corners of the truncated cylinder was found not to significantly alter the excitation forcing but does have a profound effect on the displacements. The displacements of the domed cylinder are much greater than those for the truncated cylinder, which may be directly attributed to a reduction in vortex shedding damping. Further investigation is

required to establish the degree of corner-rounding to achieve the displacement increases observed for the fully rounded cylinder base.

One of the principal achievements of this PhD is the demonstration of the incident wave conditions that give rise to the maximum body displacements. These maximum displacements do not necessarily occur in the incident wave group that has the largest crest, but rather in a group which has a large crest immediately preceding the maximum. This is indicative of a group that is more narrow banded than the underlying sea state and was confirmed by examining the spectra of both the displacement and incident wave events. The displacement trace itself resembled a focused event.

Defining wave conditions for the motion trace directly is problematic as the number of parameters makes any distribution specific to that particular set up. The approach proposed herein, therefore, is to define the conditions of incident wave group, then apply the time domain simulation to predict the displacement trace. The feasibility of this approach was demonstrated using the experimentally observed incident group. A systematic determination of the incident group is made particularly challenging due to its unfocused nature, and requires further detailed investigation.

7

Engineering significance and conclusions

Recalling the research aims in §1.2, the purpose of this thesis has been to gain a detailed understanding of the “nonlinear wave-structure interaction for highly dynamic devices from a physical perspective”. To gain this understanding, two principal sources of nonlinearity have been studied: firstly that from the underlying wave mechanics as manifested in the excitation force, and, secondly, that arising from vortex-shedding damping when the body is free to respond. To address this challenge in a logical fashion, the following approach was adopted:

- i. identify a realistically sized geometry,
- ii. develop a nonlinear potential flow solver to study the wave-structure interaction mechanisms,
- iii. create an experimental rig capable of measuring both forces and displacements,
- iv. use the experimental apparatus to obtain data in long duration irregular sea states to examine force and displacement statistics, including both nonlinear force amplification and vortex shedding damping, and
- v. use the force and displacement statistics to propose design conditions.

7.1 Principal achievements

In pursuit of achieving these goals, this thesis has delivered a number of advances. These are:

1. A thorough review of previous research and of deployed devices to select a generic geometry that will provide insight for WEC design.
2. The development of a second-order radiation-diffraction solver for axisymmetric bodies, named ICRD3D:
 - (a) Implemented from scratch using modern C++14.
 - (b) Based principally upon the formulation of Kim & Yue (1989, 1990), but drawing in best numerical practices from a multitude of sources.
 - (c) Applicable to regular waves and random sea states.
3. Applying ICRD3D to regular waves established the following:
 - (a) The second-harmonic excitation force is driven primarily by the free surface forcing term.
 - (b) The second-harmonic excitation force increases with body draught and may account for up to 30% of the total force even in moderately steep waves.
 - (c) The greatest second-harmonic displacements occur at the first-harmonic resonance frequency.
 - (d) The second-harmonic displacements have a number of resonance frequencies and in short waves may exceed the first-harmonic motions.
4. A novel experimental apparatus was designed and manufactured in-house. The physical model had a diameter of 0.25 m, and was restricted to the heave degree of freedom using air bushings incorporated into the buoy. These bushings provided near-zero friction resistance. Irregular sea states were generated to capture in excess of 850,000 individual wave events. This data set represents one of the most extensive in the discipline and is a major accomplishment of this PhD project.
5. The experimental programme has made the following contributions:
 - (a) The excitation force in irregular sea states was found to be nonlinear, with the force minima being greater than linearly predicted and the force maxima smaller. These nonlinear amplifications can be attributed to the low-frequency difference terms and are as much as 30%.
 - (b) For a truncated cylinder, the displacements are smaller than linearly predicted by up to 20%. This is primarily due to vortex shedding at the resonance frequency, with the displacements at the spectral peak frequency predicted well by a linear potential flow solution. A time domain solution with a quadratic

damping term provided accurate predictions of the experimental data.

- (c) Changing the base shape of the cylinder to a hemisphere made a substantial difference to heave displacements. The damping due to vortex shedding is greatly reduced so that the observed motions lie close to the linear potential flow prediction.
- (d) The vortex shedding damping regime for the hemispherical bottom appears to scale linearly with the body velocity. A time domain solution with a quadratic damping term was not able to capture the behaviour at the resonant frequency. This feature is discussed further in §7.3.3.
- (e) By examining the conditions that led to the maximum body displacements, it was determined that the maximum displacements do not necessarily occur in waves with the largest crest amplitudes. Instead, a large crest and trough in advance of the maximum elevation excite the body and cause a greater dynamic response than due to a large crest alone.

7.2 Engineering significance

7.2.1 Design methodology

The analysis of the truncated and domed cylinders in this thesis has proven that using relatively simple models, great insight may be gained. The current tendency to employ highly sophisticated—and inevitably computationally expensive—numerical models is sometimes at odds with the needs of design. For example, long duration irregular sea state simulations are well beyond the capability of computational fluid dynamics solvers. The thesis has demonstrated that using a potential flow solver and a time domain model on typical computing hardware can accurately capture the complex mechanisms inherent in floating body motion. From a design perspective, this should not be underestimated, as WEC developers can now have confidence in adopting such an analysis. The approach outlined herein, along with the observations of the design conditions, offers a practical design solution.

7.2.2 Scale effects

In any laboratory investigation the question of scale effects arises. For wave mechanics problems, Froude scaling is typically applied (Hughes, 1993) in conditions where effects arising due to the fluid viscosity, such as vortex shedding, are considered small. This implies that the data arising from the heave force probability of exceedance distributions may be up-scaled, whereas the application of the displacement data at full scale should be done with care. The methodology adopted herein could also be adopted at larger scale with a change of the damping coefficient.

7.3 Suggestions for further work

7.3.1 Experiments in six degree of freedom

The present study has been limited to heave forces and motions. A floating offshore structure, however, is likely to be free to move in all six degrees of freedom (6DOF) which creates additional complexities. For example, the presence of a mooring line may significantly alter the body dynamics and introduce additional nonlinear sources into the problem. Furthermore, allowing the body to respond in all DOFs may induce cross coupling between the various modes, particularly when subjected to a directional sea state. The measurements from this thesis provides a solid platform on which to base future comparisons with 6DOF results.

7.3.2 Testing in directionally spread sea states

The experimental programme undertaken herein was conducted in a wave flume using unidirectional waves. Following the work of Latheef & Swan (2013) the maximum wave crest elevations in directionally spread seas are smaller than for unidirectional seas. Furthermore, Latheef & Swan (2013) established that increasing the directional spread hastened the onset of wave breaking and therefore also reduced the crest-height statistics. This change of the underlying sea state may have significant consequences on the distributions of both forces and displacements, and can build upon the understanding of the wave-structure interaction mechanisms identified and studied in this PhD.

7.3.3 Vortex shedding regime for floating bodies

Chapter 6 identified the importance of sharp corners in the role of vortex shedding damping of the body motions. For the domed cylinder, the damping behaviour could not be characterised by a model proportional to the square of the body velocity, which was found to accurately model the regime for the truncated cylinder. A detailed study using techniques that can capture vortex shedding behaviour is required to illuminate this aspect of the hydrodynamic response of the body.

This PhD has considered the two extreme cases of a truncated and a domed cylinder. Intermediate geometries exist, however, in which the sharp corners become a fillet with a particular radius. Even a slight rounding of the corners may create near-equivalent conditions to the hemisphere case. Rigorously establishing the transition point from truncated-like vortex shedding behaviour to domed-like behaviour would be of particular interest to developers of fully-submerged, heaving, wave energy converters. This interest lies in wanting to optimise the shape of the top of the device to maximise the excitation force and minimise vortex shedding.

7.3.4 Specification of a design wave

This thesis has offered definitive evidence that the largest body displacements arise from a wave that has a large crest and trough immediately preceding the maximum wave crest in the group. It has also demonstrated that once this incident wave is known, the resulting body displacements may be accurately predicted. The deterministic specification of this incident wave, however, remains outstanding and requires a thorough investigation. Research is currently being undertaken within the Hydrodynamics group at Imperial College to achieve this aim and will, using insights from this PhD, signify a major breakthrough in the design of WEC devices.

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A

Additional derivations

A.1 Taking derivatives with respect to the normal

A general operator may be derived by chain rule differentiation as

$$\partial_n = \frac{\partial r}{\partial n} \partial_r + \frac{\partial z}{\partial n} \partial_z. \quad (\text{A.1})$$

The derivatives of r and z with respect to n can be determined by examining the co-ordinate transform from (r, z) to (n, s) , as shown in Figure A.1. The transformation may be expressed in terms of the rotation matrix

$$\begin{pmatrix} \partial_n \\ \partial_s \end{pmatrix} = \begin{bmatrix} \partial_n r & \partial_n z \\ \partial_s r & \partial_s z \end{bmatrix} \begin{pmatrix} \partial_r \\ \partial_z \end{pmatrix} = \begin{bmatrix} \cos \beta & \sin \beta \\ \sin \beta & -\cos \beta \end{bmatrix} \begin{pmatrix} \partial_r \\ \partial_z \end{pmatrix}, \quad (\text{A.2})$$

where $\beta = \pi - \tan^{-1}(dx/dz)$, in which dx, dz are the local gradients in the x and z directions at the point under consideration. For the example of ∂_{n_p} these gradients would be taken at the r, z location of the field point p .

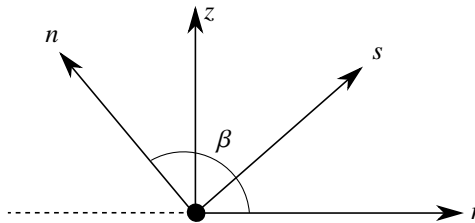


Figure A.1: Co-ordinate transformation adopted

A.2 Source distribution technique

A.2.1 Normal derivative of an axisymmetric incident potential

The derivative of the incident potential may be calculated analytically. In cylindrical coordinates, ϕ^i may be written as

$$\phi^i = -\frac{igA \cosh k(h+z)}{\omega \cosh kh} e^{ikr \cos \theta}, \quad (\text{A.3})$$

where A is the amplitude of the incident wave. In order to make use of the axisymmetry of the problem, the potential is most easily expanded as a Fourier series by making use of the Jacobi-Anger expansion (Cuyt *et al.*, 2008, eq 17.1.7)

$$\begin{aligned} e^{ikr \cos \theta} &= \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\theta} = J_0(kr) + 2 \sum_{m=1}^{\infty} i^m J_m(kr) \cos m\theta \\ &= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta, \end{aligned} \quad (\text{A.4})$$

where $\epsilon = 1$ for $m = 0$ and 2 otherwise, so that

$$\phi^i = -\frac{igA \cosh k(h+z)}{\omega \cosh kh} \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta. \quad (\text{A.5})$$

A comparison of (A.5) and (A.3) immediately yields

$$\phi_m^i = -\frac{igA \cosh k(h+z)}{\omega \cosh kh} \epsilon_m i^m J_m(kr). \quad (\text{A.6})$$

Finally, by applying the operator derived in (A.1) gives the derivative of the incident potential with respect to the normal as

$$\frac{\partial \phi_m^i}{\partial n} = -\frac{gAk\epsilon_m i^m}{\omega \cosh kd} \left[\frac{\partial r}{\partial n} \cosh k(h+z) J'_m(kr) + \frac{\partial z}{\partial n} \sinh k(h+z) J_m(kr) \right], \quad (\text{A.7})$$

where J_m is the Bessel function of the first kind, and the prime denotes the derivative with respect to the argument.

A.3 Normal derivatives of the Green's function

Applying the operator derived in (A.1) to the expressions for the Green functions yields the following results.

A.3.1 Cylindrical harmonics

The derivatives with respect to r and z are

$$\begin{aligned} \frac{\partial g_m}{\partial r} = & \frac{1}{2}\pi i A_0 \cosh k(h+z)k \left\{ \frac{J_m(kr')H'_m(kr)}{J'_m(kr)H_m(kr')} \right\} \\ & + \sum_{n=1}^{\infty} A_n \cos c_n(h+z)c_n \left\{ \frac{I_m(c_nr')K'_m(c_nr)}{I'_m(c_nr)K_m(c_nr')} \right\}, \end{aligned} \quad (\text{A.8})$$

and

$$\begin{aligned} \frac{\partial g_m}{\partial z} = & \frac{1}{2}\pi i A_0 k \sinh k(h+z) \left\{ \frac{J_m(kr')H_m(kr)}{J_m(kr)H_m(kr')} \right\} \\ & - \sum_{n=1}^{\infty} A_n c_n \sin c_n(h+z) \left\{ \frac{I_m(c_nr')K_m(c_nr)}{I_m(c_nr)K_m(c_nr')} \right\}, \end{aligned} \quad (\text{A.9})$$

respectively. In the expressions above, the top line is used when $r' < r$, the bottom line when $r' > r$ and the mean of the two lines when $r = r'$.

A.3.2 Subtracted singularities terms

Derivative with respect to r

Asymptotic form for a particular mode n :

$$\begin{aligned} \frac{\partial g_{mn}^*}{\partial r} = & -2h^{-1}(rr')^{-1/2}e^{-n\pi|r-r'|/h} \left\{ \begin{aligned} & \left\{ \begin{array}{l} + \\ - \end{array} \right\} \frac{Kz}{n\pi} \cos\left(\frac{n\pi}{h}(h+z')\right) \sin\left(\frac{n\pi}{h}(h+z)\right) \\ & \left\{ \begin{array}{l} + \\ - \end{array} \right\} \frac{Kz'}{n\pi} \sin\left(\frac{n\pi}{h}(h+z')\right) \cos\left(\frac{n\pi}{h}(h+z)\right) \end{aligned} \right. \\ & \left. \left(\left\{ \begin{array}{l} + \\ - \end{array} \right\} 1 + \frac{K(r-r')}{n\pi} - \frac{h}{R} \frac{4m^2-1}{8n\pi} + \frac{h}{r} \frac{4m^2+3}{8n\pi} \right) \cos\left(\frac{n\pi}{h}(h+z')\right) \cos\left(\frac{n\pi}{h}(h+z)\right) \right\} \end{aligned} \quad (\text{A.10})$$

The analytic summation of the asymptotic form:

$$\begin{aligned}
 \sum \frac{\partial g_{mn}^*}{\partial r} = & h^{-1}(rr')^{-1/2} \left\{ \begin{aligned} & \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{\gamma \cos \Sigma}{1 - 2\gamma \cos \Sigma + \gamma^2} \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{\gamma(\cos \Delta - \gamma)}{1 - 2\gamma \cos \Delta + \gamma^2} \\ & + \left(\frac{K}{2\pi}(r - r') - \frac{h}{r'} \frac{4m^2 - 1}{16\pi} + \frac{h}{r} \frac{4m^2 + 3}{16\pi} \right) \ln(1 - 2\gamma \cos \Sigma + \gamma^2) \\ & + \left(\frac{K}{2\pi}(r - r') - \frac{h}{r'} \frac{4m^2 - 1}{16\pi} + \frac{h}{r} \frac{4m^2 + 3}{16\pi} \right) \ln(1 - 2\gamma \cos \Delta + \gamma^2) \\ & \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{K}{\pi}(z' + z) \tan^{-1} \left(\frac{\gamma \sin \Sigma}{1 - \gamma \cos \Sigma} \right) \\ & \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{K}{\pi}(z' - z) \tan^{-1} \left(\frac{\gamma \sin \Delta}{1 - \gamma \cos \Delta} \right) \end{aligned} \right\}
 \end{aligned}
 \tag{A.11}$$

Derivative with respect to z

Fenton (1978) writes the derivative wrt the normal in a different way so there is a factor -1 difference between his expressions and the ones below.

Asymptotic form for a particular mode n :

$$\begin{aligned}
 \frac{\partial g_{mn}^*}{\partial z} = & 2h^{-1}(r'r)^{-1/2} e^{-n\pi|r'-r|/h} \left\{ \begin{aligned} & - \frac{Kz'}{n\pi} \sin \left(\frac{n\pi}{h}(h + z') \right) \sin \left(\frac{n\pi}{h}(h + z) \right) \\ & - \left(1 + \frac{K|r' - r|}{n\pi} - \frac{4m^2 - 1}{8n\pi} \frac{h|r' - r|}{r'r} \right) \cos \left(\frac{n\pi}{h}(h + z') \right) \sin \left(\frac{n\pi}{h}(h + z) \right) \\ & \frac{Kz}{n\pi} \cos \left(\frac{n\pi}{h}(h + z') \right) \cos \left(\frac{n\pi}{h}(h + z) \right) \end{aligned} \right\}
 \end{aligned}
 \tag{A.12}$$

The analytic summation of the asymptotic form

$$\begin{aligned}
 \sum \frac{\partial g_{mn}^*}{\partial z} = & -h^{-1}(r'r)^{-1/2} \left\{ \right. \\
 & \frac{\gamma \sin \Sigma}{1 - 2\gamma \cos \Sigma + \gamma^2} - \frac{\gamma \sin \Delta}{1 - 2\gamma \cos \Delta + \gamma^2} \\
 & + \frac{K}{2\pi}(z' + z) \ln(1 - 2\gamma \cos \Sigma + \gamma^2) \\
 & + \frac{K}{2\pi}(z' - z) \ln(1 - 2\gamma \cos \Delta + \gamma^2) \\
 & + \left(\frac{K}{\pi} |r' - r| \begin{Bmatrix} + \\ - \end{Bmatrix} \right) h \left(\frac{1}{r} - \frac{1}{r'} \right) \frac{4m^2 - 1}{8\pi} \tan^{-1} \frac{\gamma \sin \Sigma}{1 - \gamma \cos \Sigma} \\
 & + \left(-\frac{K}{\pi} |r' - r| \begin{Bmatrix} - \\ + \end{Bmatrix} \right) h \left(\frac{1}{r} - \frac{1}{r'} \right) \frac{4m^2 - 1}{8\pi} \tan^{-1} \frac{\gamma \sin \Delta}{1 - \gamma \cos \Delta} \left. \right\}
 \end{aligned} \tag{A.13}$$

A.4 Second-order terms

A.4.1 Reduction of the integral equation to one dimension

As in the linear problem, axisymmetry is exploited to reduce the integral equation over a surface to a sequence of boundary-integral problems over a line in the r - z plane. To achieve this, $\phi_i^{(2)}$, $\phi_s^{(2)}$, q and G^+ are expanded into Fourier-cosine series in the circumferential coordinate θ (Figure 2.2) so that

$$\begin{Bmatrix} \phi_{i,s}^{(2)}(r, \theta, z) \\ q(r, \theta) \end{Bmatrix} = \sum_{m=0}^{\infty} \begin{Bmatrix} \phi_{i,s,m}^{(2)}(r, z) \\ q_n(r) \end{Bmatrix} \cos m\theta, \tag{A.14}$$

and

$$G^+(r, \theta, z; r', \theta', z') = \sum_{m=0}^{\infty} \epsilon_m g_m^+(r, z; r', z') \cos m(\theta - \theta'), \tag{A.15}$$

Substituting (A.14) and (A.15) into (2.43) yields

$$\begin{aligned}
 & 2\pi \sum_{n=0}^{\infty} \phi_{Dn}^{(2)} \cos n\theta' + \int \int_{S_B} \sum_{n=0}^{\infty} \phi_{Dn}^{(2)} \cos n\theta \frac{\partial G_n^+}{\partial n} \frac{\epsilon_n}{2\pi} \cos n(\theta - \theta') d\mathbf{x} \\
 & = - \int \int_{S_B} \sum_{n=0}^{\infty} \frac{\epsilon_n}{2\pi} G_n^+ \cos n(\theta - \theta') \frac{\partial \phi_{In}^{(2)}}{\partial n} \cos n\theta d\mathbf{x} \\
 & + \frac{1}{g} \int \int_{S_F} \sum_{n=0}^{\infty} q_n \cos n\theta \frac{\epsilon_n}{2\pi} G_n^+ \cos n(\theta - \theta') d\mathbf{x}.
 \end{aligned} \tag{A.16}$$

In order to reduce the problem to one of line integrals rather than surface integrals, an integration in θ must be performed. In all the θ containing terms, the same product appears. Defining,

$$I_\theta = \int_0^{2\pi} \cos n\theta \cos n(\theta - \theta') d\theta, \quad (\text{A.17})$$

the integral is most easily performed using the product-to-sum formula (Abramowitz & Stegun, 1972, 4.3.32)

$$2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta), \quad (\text{A.18})$$

so that

$$\begin{aligned} I_\theta &= \frac{1}{2} \int_0^{2\pi} [\cos n\theta' + \cos n(2\theta - \theta')] d\theta \\ &= \pi \cos n\theta'. \end{aligned} \quad (\text{A.19})$$

Substituting (A.19) into (A.16) and equating the Fourier coefficients in θ' yields

$$2\pi\phi_{sm}^{(2)} + \oint_{S_B} 2\pi r \phi_{sm}^{(2)} \frac{\partial g_m^+}{\partial n} ds = - \int_{S_B} 2\pi r \frac{\partial \phi_{im}^{(2)}}{\partial n} g_m^+ ds + \frac{1}{g} \int_{S_F} 2\pi r q_m g_m^+ dr. \quad (\text{A.20})$$

The form is very similar to the integral equation in the linear problem but with the addition of the free surface integral.

A.4.2 Near-field free-surface forcing terms

The q_m terms are the Fourier coefficients required to make the axisymmetric reduction of the problem, introduced in (A.14). Using the standard definition of the Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n \cos nx \quad (\text{A.21})$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad (\text{A.22})$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \quad (\text{A.23})$$

Therefore, to obtain the term q_0 , (A.22) must be applied to the general definition of q in (2.13). Expanding the final term in (2.13) gives (momentarily omitting the q_{II} term)

$$q = \left[-\frac{i\omega}{2g} \phi^{(1)} \left(-\omega^2 \frac{\partial \phi^{(1)}}{\partial z} + g \frac{\partial^2 \phi^{(1)}}{\partial z^2} \right) + i\omega \left[\left(\frac{\partial \phi^{(1)}}{\partial x} \right)^2 + \left(\frac{\partial \phi^{(1)}}{\partial y} \right)^2 + \left(\frac{\partial \phi^{(1)}}{\partial z} \right)^2 \right] \right]_{z=0}, \quad (\text{A.24})$$

which may be taken into cylindrical-polar coordinates using the Jacobian

$$J = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta \\ \frac{\cos \theta}{r} & -\frac{\sin \theta}{r} \end{bmatrix}, \quad (\text{A.25})$$

so that

$$\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 = \left(\frac{\partial \phi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2. \quad (\text{A.26})$$

Therefore, (A.24), can be written in cylindrical-polar coordinates as

$$q = \frac{i\omega}{2} \left\{ \left(v \frac{\partial \phi}{\partial z} - \frac{\partial^2 \phi}{\partial z^2} \right) \phi + 2 \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right] + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right\}, \quad (\text{A.27})$$

where the ⁽¹⁾ superscript is assumed. Each term in (A.27) is a quadratic product meaning they can be dealt with in broadly the same way. Taking the final term as an example, mode zero requires the evaluation of

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 d\theta. \quad (\text{A.28})$$

Using the Fourier cosine series expansion in (A.14),

$$\begin{aligned} \left(\frac{\partial \phi}{\partial \theta} \right)^2 &= \left(\sum_{m=0}^{\infty} -m \phi_m \sin m\theta \right)^2 \\ &= \sum_{m=0}^{\infty} m^2 \phi_m^2 \sin^2 m\theta + \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} 2ij \phi_i \phi_j \sin i\theta \sin j\theta. \end{aligned} \quad (\text{A.29})$$

Substituting into (A.30) gives

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 d\theta &= \frac{1}{\pi r^2} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{\infty} m^2 \phi_m^2 \sin^2 m\theta + \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} 2ij \phi_i \phi_j \sin i\theta \sin j\theta \right] d\theta \\ &= \sum_{m=0}^{\infty} \frac{m^2}{r^2} \phi_m^2, \end{aligned} \quad (\text{A.30})$$

as the second term, the double summation, is zero when integrated over the $-\pi$ to π range due to orthogonality of sine terms

$$\int_{-\pi}^{\pi} \sin i\theta \sin j\theta d\theta = 0, \quad i \neq j. \quad (\text{A.31})$$

Undertaking the same procedure with the other terms in (A.27) yields

$$q_0 = \frac{i\omega}{2} \sum_{l=0}^{\infty} \frac{1}{\epsilon_l} \left\{ \left(\nu \frac{\partial \phi_l^{(1)}}{\partial z} - \frac{\partial^2 \phi_l^{(1)}}{\partial z^2} \right) \phi_l^{(1)} + 2 \left[\left(\frac{\partial \phi_l^{(1)}}{\partial r} \right)^2 + \left(\frac{\partial \phi_l^{(1)}}{\partial z} \right)^2 \right] + \frac{l^2 \epsilon_l}{r^2} \phi_l^{(1)2} \right\} - q_{110}. \quad (\text{A.32})$$

The procedure is similar, though more lengthy, with the other Fourier coefficients. Using the standard definition (A.23) and the first product in (A.27) as an example:

$$\frac{\partial \phi}{\partial z} \phi = \left(\sum_{m=0}^{\infty} \frac{\partial \phi_m}{\partial z} \cos m\theta \right) \left(\sum_{m=0}^{\infty} \phi_m \cos m\theta \right), \quad (\text{A.33})$$

so that the Fourier coefficient is given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{\infty} \frac{\partial \phi_m}{\partial z} \phi_m \cos^2 m\theta + 2 \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} \frac{\partial \phi_i}{\partial z} \phi_j \cos i\theta \cos j\theta \right] \cos n\theta d\theta. \quad (\text{A.34})$$

Taking the θ dependant terms in the first summation

$$\begin{aligned} \int_{-\pi}^{\pi} \cos^2 m\theta \cos n\theta \, d\theta &= \int_{-\pi}^{\pi} \left(\frac{1 + \cos 2m\theta}{2} \right) \cos n\theta \, d\theta \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \cos 2m\theta \cos n\theta \, d\theta \\ &= \begin{cases} \pi/2 & \text{if } 2m = n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (\text{A.35})$$

again due the orthogonality relationship of cosines. Looking now at the second summation, and only the θ dependant terms

$$\begin{aligned} \int_{-\pi}^{\pi} \cos i\theta \cos j\theta \cos n\theta \, d\theta &= \int_{-\pi}^{\pi} [\cos \theta(i-j) + \cos \theta(i+j)] \cos n\theta \\ &= \begin{cases} \pi/2 & \text{if } i+j = n \\ \pi/2 & \text{if } i-j = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{A.36})$$

As the values of i and j that give non-zero results for the integral (A.34) are known, the double sum can be collapsed into a summation in one variable. Denoting the value of integration for an i - j combination I_{ij} , (A.34) becomes

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\partial \phi_i}{\partial z} \phi_j I_{ij} &= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{n-j} \frac{\partial \phi_i}{\partial z} \phi_j I_{ij} + \sum_{i=n-j+1}^{\infty} \frac{\partial \phi_i}{\partial z} \phi_j I_{ij} \right) \\ &= \sum_{j=0}^n \frac{\pi}{2} \frac{\partial \phi_{n-j}}{\partial z} \phi_j + \sum_{j=0}^{\infty} \frac{\pi}{2} \frac{\partial \phi_{n+j}}{\partial z} \phi_j \end{aligned} \quad (\text{A.37})$$

Finally, applying this result to the other terms in (A.27) gives

$$\begin{aligned} q_m &= \frac{i\omega}{2} \sum_{l=0}^m \left\{ \frac{1}{2} \left(v \frac{\partial \phi_{m-l}^{(1)}}{\partial z} - \frac{\partial^2 \phi_{m-l}^{(1)}}{\partial z^2} \right) \phi_l^{(1)} + \frac{\partial \phi_{m-l}^{(1)}}{\partial r} \frac{\partial \phi_l^{(1)}}{\partial r} + \frac{\partial \phi_{m-l}^{(1)}}{\partial z} \frac{\partial \phi_l^{(1)}}{\partial z} - \frac{(m-l)l}{r^2} \phi_{m-l}^{(1)} \phi_l^{(1)} \right\} \\ &+ \frac{i\omega}{2} \sum_{l=0}^{\infty} \left\{ \frac{1}{2} \left(v \frac{\partial \phi_l^{(1)}}{\partial z} - \frac{\partial^2 \phi_l^{(1)}}{\partial z^2} \right) \phi_{m+l}^{(1)} + \frac{1}{2} \left(v \frac{\partial \phi_{m+l}^{(1)}}{\partial z} - \frac{\partial^2 \phi_{m+l}^{(1)}}{\partial z^2} \right) \phi_l^{(1)} \right. \\ &\left. + 2 \left(\frac{\partial \phi_l^{(1)}}{\partial r} \frac{\partial \phi_{m+l}^{(1)}}{\partial r} + \frac{\partial \phi_l^{(1)}}{\partial z} \frac{\partial \phi_{m+l}^{(1)}}{\partial z} \right) + \frac{2l(m+l)}{r^2} \phi_l^{(1)} \phi_{m+l}^{(1)} \right\} - q_{11m}, \end{aligned} \quad (\text{A.38})$$

where q_{11m} represents the quadratic products of the linear incident potential and are given below. These latter terms must be subtracted out as the second-order incident potential (2.8) satisfies the free surface boundary condition itself. Each term in the above equation involving $\phi^{(1)}$ refers to the total potential, ie $\phi_m^{(1)} = \phi_{im}^{(1)} + \phi_{sm}^{(1)}$, calculated over a spatial r - z grid with the derivatives taken numerically and evaluated at the free surface ($z = 0$).

To obtain the q_{IIIm} terms in (A.38), the quadratic products of the incident potential are calculated such that

$$q_{IIIm} = \left[-\frac{i\omega}{2g} \phi_{im}^{(1)} \left(-\omega^2 \frac{\partial \phi_{im}^{(1)}}{\partial z} + g \frac{\partial^2 \phi_{im}^{(1)}}{\partial z^2} \right) + i\omega (\nabla \phi_{im}^{(1)})^2 \right]_{z=0}, \quad (\text{A.39})$$

where, upon the application of the Jacobi-Anger expansion,

$$\phi_{im}^{(1)}|_{z=0} = -\frac{igA}{\omega} \epsilon_m i^m J_m(kx), \quad (\text{A.40})$$

$$\frac{\partial \phi_{im}^{(1)}}{\partial z} \Big|_{z=0} = -\frac{igAk}{\omega} \tanh(kh) \epsilon_m I^m J_m(kx), \quad (\text{A.41})$$

$$\frac{\partial^2 \phi_{im}^{(1)}}{\partial z^2} \Big|_{z=0} = -\frac{igAk^2}{\omega} \epsilon_m i^m J_m(kx), \quad (\text{A.42})$$

$$\frac{\partial \phi_{im}^{(1)}}{\partial x} \Big|_{z=0} = -\frac{igAk}{\omega} \epsilon_m i^m J'_m(kx). \quad (\text{A.43})$$

A.4.3 Far-field forcing terms

The far-field forcing terms are found using by substituting the evanescent-free potentials into the free surface forcing terms (A.32) and (A.38). The resulting expressions allow the far-field integral to infinity to be performed analytically.

The linear evanescent-free diffracted potential is given by

$$\hat{\phi}_{Dm}^{(1)} = -4\pi^2 i C_0 \cosh k(z+h) H_n(k\rho) L_n, \quad (\text{A.44})$$

where L_n is the n -th mode Kochin function, given by

$$L_m = \int_{S_B} r' \sigma_m(\mathbf{x}') J_m(kr') \cosh k(z'+h) ds'. \quad (\text{A.45})$$

Introducing the coefficients independent of z and ρ ,

$$\alpha_m \equiv -4\pi^2 i \frac{i\omega}{Ag} C_0 L_m, \quad \beta_m \equiv \epsilon_m i^m / \cosh(kh), \quad (\text{A.46})$$

which corrects for a typographic error in Kim & Yue (1989), allows the diffracted and incident potentials can be re-written as

$$\hat{\phi}_{Dm}^{(1)} = \frac{-iAg}{\omega} \alpha_m \cosh k(z+h) H_m(k\rho), \quad (\text{A.47})$$

and

$$\phi_{Im}^{(1)} = \frac{-IAg}{\omega} \beta_m \cosh k(z+h) H_m(k\rho), \quad (\text{A.48})$$

respectively.

Looking at the free surface forcing term for the 0-th mode (A.32), the terms required, having dropped the ⁽¹⁾ superscript and evaluating at $z = 0$

$$\hat{\phi}_m = \frac{-iAg}{\omega} \cosh kh (\alpha_m H_m + \beta_m J_m) \quad (\text{A.49})$$

$$\frac{\partial \hat{\phi}_m}{\partial z} = \frac{-iAg}{\omega} k \sinh kh (\alpha_m H_m + \beta_m J_m) \quad (\text{A.50})$$

$$\frac{\partial^2 \hat{\phi}_m}{\partial z^2} = \frac{-iAg}{\omega} k^2 \sinh kh (\alpha_m H_m + \beta_m J_m) \quad (\text{A.51})$$

$$\frac{\partial \hat{\phi}_m}{\partial \rho} = \frac{-iAg}{\omega} k^2 \cosh kh (\alpha_m H'_m + \beta_m J'_m), \quad (\text{A.52})$$

where the primes denote the derivative with respect to the whole argument. Taking each of the products in turn:

$$\begin{aligned} \hat{\phi}_m \frac{\partial \hat{\phi}_m}{\partial z} &= \left(\frac{-iAg}{\omega} \right)^2 k \cosh kh \sinh kh (\alpha_m \alpha_m H_m H_m + \alpha_m \beta_m H_m J_m \\ &\quad + \alpha_m \beta_m J_m H_m + \beta_m \beta_m J_m J_m), \end{aligned} \quad (\text{A.53})$$

which may be simplified by introducing

$$S_{l,m}(kr) \equiv \alpha_l \alpha_m H_l(kr) H_m(kr) + \beta_l \alpha_m J_l(kr) H_m(kr) + \alpha_l \beta_m H_l(kr) J_m(kr), \quad (\text{A.54})$$

so that

$$\hat{\phi}_m \frac{\partial \hat{\phi}_m}{\partial z} = \left(\frac{-iAg}{\omega} \right)^2 k \cosh kh \sinh kh S_{m,m}, \quad (\text{A.55})$$

and similarly,

$$\hat{\phi}_m \frac{\partial^2 \hat{\phi}_m}{\partial z^2} = \left(\frac{-iAg}{\omega} \right)^2 k^2 \cosh^2 kh S_{m,m}. \quad (\text{A.56})$$

The term involving the product of Bessel functions of the first kind cancels when subtracting the q_{II0} term.

Substituting these into the first term of (A.32), gives

$$\frac{i\omega}{2} \sum_{m=0}^{\infty} \frac{1}{\epsilon_m} \left(\frac{\omega^2}{g} \frac{\partial \hat{\phi}_m}{\partial z} - \frac{\partial^2 \hat{\phi}_m}{\partial z^2} \right) = \frac{-i\omega}{2} \left(\frac{iAg}{\omega} \right)^2 \sum_{m=0}^{\infty} \frac{1}{\epsilon_m} \quad (\text{A.57})$$

$$\left[\omega^2 S_{m,m} k \cosh kh \sinh kh - S_{m,m} k^2 \cosh^2 kh \right]. \quad (\text{A.58})$$

After introducing the dispersion equation, and multiplying the first term by $\cosh kh / \cosh kh$ in (A.58) yields

$$\frac{-ik^2 A^2 g^2 \cosh^2 kh}{2\omega} \sum_{m=0}^{\infty} \frac{1}{\epsilon_m} (\tanh^2 kh - 1) S_{m,m} \quad (\text{A.59})$$

Looking the next terms in (A.32):

$$\left(\frac{\partial \hat{\phi}_m}{\partial \rho} \right)^2 = \left(\frac{-iAg}{\omega} \right)^2 k^2 \cosh^2 kh T_{m,m} \quad (\text{A.60})$$

in which

$$T_{l,m}(kr) \equiv \alpha_l \alpha_m H'_l(kr) H'_m(kr) + \beta_l \alpha_m J'_l(kr) H'_l(kr) + \alpha_l \beta_m H'_m(kr) J'_m(kr), \quad (\text{A.61})$$

and

$$\begin{aligned} \left(\frac{\partial \hat{\phi}_m}{\partial z} \right)^2 &= \left(\frac{-iAg}{\omega} \right)^2 k^2 \sinh^2 kh S_{m,m} \\ &= \left(\frac{-iAg}{\omega} \right)^2 k^2 \cosh^2 kh \tanh^2 kh S_{m,m}. \end{aligned} \quad (\text{A.62})$$

Therefore, the second term in (A.32) is given by

$$\frac{i\omega}{2} \sum_{m=0}^{\infty} \frac{2}{\epsilon_m} \left[\left(\frac{\partial \hat{\phi}_m}{\partial \rho} \right)^2 + \left(\frac{\partial \hat{\phi}_m}{\partial z} \right)^2 \right] = \frac{-ik^2 A^2 g^2 \cosh^2 kh}{2\omega} \sum_{m=0}^{\infty} \frac{2}{\epsilon_m} \left(\tanh^2 kh S_{m,m} + T_{m,m} \right). \quad (\text{A.63})$$

The final term in (A.32) proceeds in the same manner:

$$\frac{i\omega}{2} \sum_{m=0}^{\infty} \frac{1}{\epsilon_m} \frac{\epsilon_m m^2}{\rho^2} (\hat{\phi}_m)^2 = \frac{ik^2 g^2 A^2 \cosh^2 kh}{2\omega} \sum_{m=0}^{\infty} \frac{1}{\epsilon_m} \frac{\epsilon_m m^2}{k^2 \rho^2} S_{m,m} \quad (\text{A.64})$$

Pulling all the above together the free surface forcing modes are given by

$$\begin{aligned} \hat{q}_0 &= \frac{-ik^2 g^2 A^2 \cosh^2 kh}{2\omega} \sum_{l=0}^{\infty} \frac{2}{\epsilon_l} \left[\left(\frac{3}{2} \tanh^2 kh - \frac{1}{2} + \frac{\epsilon_l l^2}{2(kr)^2} \right) S_{l,l}(kr) + T_{l,l}(kr) \right], \\ \hat{q}_m &= \frac{-ik^2 g^2 A^2 \cosh^2 kh}{2\omega} \left\{ \sum_{l=0}^m \left[\left(\frac{3}{2} \tanh^2 kh - \frac{1}{2} - \frac{(m-l)l}{(kr)^2} \right) S_{m-l,l}(kr) + T_{m-l,l}(kr) \right] \right. \\ &\quad \left. + 2 \sum_{l=0}^{\infty} \left[\left(\frac{3}{2} \tanh^2 kh - \frac{1}{2} + \frac{(m+l)l}{(kr)^2} \right) S_{m+l,l}(kr) + T_{m+l,l}(kr) \right] \right\}, \end{aligned} \quad (\text{A.65})$$

The far-field Green's function, $\hat{g}_m$ can be expressed as

$$g_m^+ = \frac{1}{2} C_0 \pi i \cosh k_2(z+h) \cosh k_2(z'+h) J_m(k_2 x') H_m(k_2 x), \quad (\text{A.66})$$

where k_2 is the double frequency wave number.

A.4.4 Evaluating the integrals

Looking back to (2.48) we now have all the terms required to perform the integration. The first integral over $[a, b]$ is performed numerically using the derivatives of the terms calculated at first order. The second integral is given an analytic treatment which is described below.

Looking at the product of $\hat{q}_m g_m^+$, there will terms of involving $J_m(kr)H_m(kr)$ from $S_{l,m}(kr)$ and $T_{l,m}(kr)$ in $\hat{q}_m$ multiplied by $H_m(k_2 r)$ from g_m^+ . Therefore, we need an ana-

lytic method to deal with integrals involving these terms. The first identity required is

$$J_m(x) = \frac{1}{2}(H_m(x) + H_m^*(x)), \quad (\text{A.67})$$

which allows all the products to be written in terms of triple Hankel functions. The second identity is the derivative with respect to the argument,

$$\mathcal{C}'_\nu(z) = \frac{1}{2} [\mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z)], \quad (\text{A.68})$$

where $\mathcal{C}$ denotes either J or H .

Examining the S_{lm} and T_{lm} terms, there are six combinations of Bessel and Hankel functions:

$$H_l H_m H_n, \quad (\text{A.69a})$$

$$J_l H_m H_n = \frac{1}{2} (H_l + H_l^*) H_m H_n, \quad (\text{A.69b})$$

$$H_l J_m H_n = \frac{1}{2} H_l (H_m + H_m^*) H_n, \quad (\text{A.69c})$$

$$H'_l H'_m H_n = \frac{1}{4} (H_{l-1} H_{m-1} - H_{l-1} H_{m+1} - H_{l+1} H_{m-1} + H_{l+1} H_{m+1}) H_n, \quad (\text{A.69d})$$

$$\begin{aligned} J'_l H'_m H_n &= \frac{1}{8} (H_{l-1} H_{m-1} - H_{l-1} H_{m+1} - H_{l+1} H_{m-1} + H_{l+1} H_{m+1}) H_n \\ &\quad + \frac{1}{8} (H_{l-1}^* H_{m-1} - H_{l-1}^* H_{m+1} - H_{l+1}^* H_{m-1} + H_{l+1}^* H_{m+1}) H_n, \end{aligned} \quad (\text{A.69e})$$

$$\begin{aligned} H'_l J'_m H_n &= \frac{1}{8} (H_{l-1} H_{m-1} - H_{l-1} H_{m+1} - H_{l+1} H_{m-1} + H_{l+1} H_{m+1}) H_n \\ &\quad + \frac{1}{8} (H_{l-1} H_{m-1}^* - H_{l-1} H_{m+1}^* - H_{l+1} H_{m-1}^* + H_{l+1} H_{m+1}^*) H_n. \end{aligned} \quad (\text{A.69f})$$

A.4.5 Evaluation of triple Hankel integral

The triple Hankel integrals have the general form

$$I_{lmn}^\mu = \int_b^\infty r^s \left[\frac{H_l(kr)}{H_l^*(kr)} \right] H_m(kr) H_n(kr) dr, \quad s = \pm 1, \quad (\text{A.70})$$

resulting from the presence of Bessel functions and a $1/r^2$ in the $\hat{q}$ terms. The case where $s = -1$ can be dealt with using numerical quadrature as the integral will converge as

$r \rightarrow \infty$, whereas the $s = 1$ case exhibits very slow convergence and therefore requires the special treatment outlined below.

The two integrals that need to be evaluated are

$$I_{lmn}^{(1)} = \int_{R_J}^{\infty} r \, dr H_l(kr) H_m(kr) H_n(\kappa r), \quad (\text{A.71})$$

and

$$I_{lmn}^{(2)} = \int_{R_J}^{\infty} r \, dr H_l^*(kr) H_m(kr) H_n(\kappa r), \quad (\text{A.72})$$

which are solved using the same technique. Each Hankel function is approximated using the Hankel asymptotic expansion (Abramowitz & Stegun, 1972, eq. 9.2.7)

$$H_m(x) = \sqrt{\frac{2}{\pi x}} [P_m(x) + iQ_m(x)] e^{i(x-\gamma_m)}, \quad (\text{A.73})$$

where

$$P_m(x) = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(1/2 + m + 2i)}{(2i)! \Gamma(1/2 + m - 2i) (2x)^{2i}} \quad (\text{A.74})$$

$$Q_m(x) = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(3/2 + m + 2i)}{(2i + 1)! \Gamma(-1/2 + m - 2i) (2x)^{2i+1}} \quad (\text{A.75})$$

A simple estimate for the truncation of these infinite summations is given as

$$M_m = m/2 + 2. \quad (\text{A.76})$$

Chau (1989) now gives detailed consideration to the properties of using this approximation. What is very important is that “there is an upper limit to accuracy that cannot be improved taking more terms in the sums”. Therefore, to obtain the required accuracy, the value of x must be increased, and increases with the value of m . As a corollary a different value of R_J must be chosen for each m .

An adaptive integration technique is therefore employed whereby

$$\int_{R_J}^{\infty} = \int_{R_J}^{R_S} + \int_{R_S}^{\infty}, \quad (\text{A.77})$$

where the first integral is computed numerically with exact Hankel functions and the second using the process outlined below. The integration limit of the finite integral is then

increased so the new value can be added to integration from R_J to R_S , such that

$$\int_{R_J} = \int_{R_J}^{R_{S1}} + \int_{R_{S1}}^{R_{S2}} + \int_{R_{S2}}^{\infty}. \quad (\text{A.78})$$

This process is then repeated until convergence is obtained by comparing the values of several successive approximations. Chau (1989) eq 4.149 suggests a step size of

$$\Delta s = R_{s,j+1} - R_{s,j} = \frac{2\pi}{2k + \kappa} \quad (\text{A.79})$$

A.4.6 Approximating the Hankel function

The Hankel function can be expanded as a polynomial of the form

$$H_n(kr) = \sqrt{\frac{2}{\pi kr}} e^{i(kr - \gamma_n)} \sum_{l=0}^{M_n} C_{nl}(kr)^{-l}, \quad (\text{A.80})$$

where

$$C_{n0} = 1 \quad (\text{A.81})$$

$$C_{nl} = \frac{i^l (4n^2 - 1)(4n^2 - 3) \dots (4n^2 - (2l - 1)^2)}{l! 8^l}. \quad (\text{A.82})$$

This latter term may be evaluated more efficiently using the recurrence relationship

$$C_{n,l+1} = \frac{i(4n^2 - (2l + 1)^2)}{8(l + 1)} C_{nl}. \quad (\text{A.83})$$

A.4.7 Solving the integral

The original integral (A.72) may now be written as

$$I_{lmn}^{\mu} = D_{lmn}^{\mu} \sum_{p=0}^{M_l} \sum_{q=0}^{M_m} \sum_{s=0}^{M_n} B^p C_{lp} C_{mq} C_{ns} k^{-(p+q)} \kappa^{-s} \int_{R_S}^{\infty} dr \frac{e^{i\alpha r}}{r^{p+q+s+1/2}} \quad (\text{A.84})$$

where

$$D_{lmn}^{\mu} = \begin{cases} \frac{2}{\pi k} \sqrt{\frac{2}{\pi \kappa}} e^{-i(\gamma_l + \gamma_m + \gamma_n)} & \mu = 1 \\ \frac{2}{\pi k} \sqrt{\frac{2}{\pi \kappa}} e^{-i(-\gamma_l + \gamma_m + \gamma_n)} & \mu = 2 \end{cases}, \quad (\text{A.85})$$

$$B = \begin{cases} 1 & \mu = 1 \\ -1 & \mu = 2 \end{cases}, \quad (\text{A.86})$$

and

$$\alpha = \begin{cases} 2k + \kappa & \mu = 1 \\ \kappa & \mu = 2 \end{cases}. \quad (\text{A.87})$$

The final integral now takes the form

$$\Pi_n = \int_{R_S}^{\infty} dr \frac{e^{i\alpha r}}{r^{n+1/2}}; \quad \alpha = 2k + \kappa, \quad (\text{A.88})$$

which now may be evaluated by the method of asymptotic expansion of integrals. First, integrating by parts derives the recurrence formula

$$\Pi_n = \frac{i}{\alpha} \frac{e^{i\alpha R_S}}{R_S^{n+1/2}} + \frac{n + 1/2}{i\alpha} \Pi_{n-1}, \quad (\text{A.89})$$

which then leads to the power series

$$\Pi_n = - \sum_{j=1}^N \frac{(n + j - 3/2)!}{(n - 1/2)!} \frac{1}{(i\alpha)^j} \frac{e^{i\alpha R_S}}{R_S^{n+j-1/2}}. \quad (\text{A.90})$$

The factorial of a non-integer can be found by using the Gamma function (Abramowitz & Stegun, 1972, eq. 6.1.5)

$$n! = \Gamma(n + 1). \quad (\text{A.91})$$

A.4.8 Comparison with direct quadrature

In order to determine the accuracy of this method, the integration over a finite range can be compared against direct quadrature. A finite range for I is obtained by two evaluations of (A.84) to yield $I(R_1) - I(R_2)$. Table A.1 gives the comparison for $k = 1$, $\kappa = 4$, $R_1 = 10$, $R_2 = 11$.

l	m	n	Eq (A.84)		Quadrature	
1	2	3	+0.001886	+0.003742	+0.001886	+0.003742
5	3	6	-0.006758	+0.002120	-0.006758	+0.002120
9	9	12	-0.011812	-0.034072	-0.011812	-0.034072
11	11	14	+0.109891	+0.000471	+0.109891	+0.000471
14	9	17	-0.361133	+0.022987	-0.361132	+0.022987
12	12	15	-0.040434	-0.246565	-0.040433	-0.246565
13	13	17	-0.240851	+0.712165	-0.240853	+0.712166

Table A.1: Comparison between outlined by Chau (1989) and implemented herein

A.4.9 Derivation of the second-order force due to quadratic products of linear potentials

This section outlines the derivation for Kim & Yue (1989) eq 3.7. They only provide the expression for the surge force meaning the heave force must be independently derived. Start with the surge force to be prove the correctness of the method.

Start with the waterline force

$$F_{W_0} = -\frac{\rho\omega^2}{4g} \int_{W_0} \left(\phi^{(1)}\right)^2 n_x dl, \quad (\text{A.92})$$

where $n_x = n_r \cos \theta$ is the normal in the x direction and dl is a portion of the arc around the body $r d\theta$. The task is to integrate (A.92) with respect to theta. Recall that the linear potential is given by

$$\phi(r, \theta, z) = \sum_{m=0}^{\infty} \phi_m \cos m\theta, \quad (\text{A.93})$$

so that the integral part becomes

$$I = \int_0^{2\pi} \left(\sum_{m=0}^{\infty} \phi_m \cos m\theta \right) \left(\sum_{m=0}^{\infty} \phi_m \cos m\theta \right) \cos m\theta d\theta. \quad (\text{A.94})$$

Expanding out the summations gives

$$I = \int_0^{2\pi} \left(\sum_{m=0}^{\infty} \phi_m^2 \cos^2 m\theta + \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} \phi_i \phi_j \cos i\theta \cos j\theta \right) \cos \theta d\theta, \quad (\text{A.95})$$

in which

$$\int_0^{2\pi} \cos^2 m\theta \cos \theta d\theta = 0 \quad \forall m, \quad (\text{A.96})$$

and

$$\int_0^{2\pi} \cos i\theta \cos j\theta \cos \theta = \begin{cases} \pi/2 & \text{for } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}. \quad (\text{A.97})$$

This then allows the double summation to be written as a single summation upon integration, viz

$$\begin{aligned} I &= 2 \sum_{j=0}^1 \frac{\pi}{2} \phi_{1-j} \phi_j + 2 \sum_{j=0}^{\infty} \frac{\pi}{2} \phi_{1+j} \phi_j \\ &= 2 \cdot 2 \frac{\pi}{2} \phi_0 \phi_1 + 2 \sum_{j=0}^{\infty} \frac{\pi}{2} \phi_{1+j} \phi_j. \end{aligned} \quad (\text{A.98})$$

Introducing $\epsilon_j = 1$ for $m = 0, 2$ otherwise, we have

$$I = \sum_{j=0}^{\infty} \frac{2\pi}{\epsilon_j} \phi_{j+1} \phi_j \quad (\text{A.99})$$

Now, bringing back all the constants and evaluating the quantities on the waterline,

$$F_{W_0} = -\frac{\rho a n_r \omega^2}{4g} \sum_{n=0}^{\infty} \frac{2\pi}{\epsilon_n} \phi_{n+1} \phi_n \Big|_{z=0} \quad (\text{A.100})$$

as required.

Now turning to the part arising from the linear velocities:

$$F_{\text{vel}} = \iint_{S_B} \left(\nabla \phi^{(1)} \right)^2 n_x \, dS \quad (\text{A.101})$$

Expanding the potential into cylindrical coordinates and noting that $dS = r \, d\theta \, ds$, we have

$$F_{\text{vel}} = \int_{\delta B} \int_0^{2\pi} \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right] n_r \cos \theta \, r \, d\theta \, ds \quad (\text{A.102})$$

Dealing with the integral in θ , starting with the $\partial \phi / \partial r$ term

$$I_r = \int_0^{2\pi} \left(\sum_{m=0}^{\infty} \frac{\partial \phi}{\partial r} \cos m\theta \right)^2 \cos \theta \, d\theta \quad (\text{A.103})$$

proceeds in the exactly the same manner as the waterline force, as does the $\partial \phi / \partial z$ term,

so that

$$\int_0^{2\pi} \left(\frac{\partial \phi}{\partial r} \right)^2 \cos \theta \, d\theta = \sum_{m=0}^{\infty} \frac{2\pi}{\epsilon_m} \frac{\partial \phi_m}{\partial r} \frac{\partial \phi_{m+1}}{\partial r}, \quad (\text{A.104})$$

$$\int_0^{2\pi} \left(\frac{\partial \phi}{\partial z} \right)^2 \cos \theta \, d\theta = \sum_{m=0}^{\infty} \frac{2\pi}{\epsilon_m} \frac{\partial \phi_m}{\partial z} \frac{\partial \phi_{m+1}}{\partial z}. \quad (\text{A.105})$$

Turning now the $\partial \phi / \partial \theta$ term

$$\frac{\partial \phi}{\partial \theta} = \sum_{m=0}^{\infty} -m \phi_m \sin m\theta, \quad (\text{A.106})$$

so that

$$\int_0^{2\pi} \left(\frac{\partial \phi}{\partial \theta} \right)^2 \cos \theta \, d\theta = \int_0^{2\pi} \left(\sum_{m=0}^{\infty} m \phi_m \sin m\theta \right) \left(\sum_{m=0}^{\infty} m \phi_m \sin m\theta \right) \cos \theta \, d\theta \quad (\text{A.107})$$

$$= \int_0^{2\pi} \left(\sum_{m=0}^{\infty} m^2 \phi_m^2 \sin^2 m\theta + \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} i j \phi_i \phi_j \sin i\theta \sin j\theta \right) \cos \theta \, d\theta \quad (\text{A.108})$$

in which the first term is always zero and the second is given by

$$\int_0^{2\pi} \sin i\theta \sin j\theta \cos \theta \, d\theta = \begin{cases} \pi/2 & \text{for } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}. \quad (\text{A.109})$$

Therefore

$$\int_0^{2\pi} \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} i j \phi_i \phi_j \cos \theta \, d\theta \sin i\theta \sin j\theta = \sum_{m=0}^{\infty} 2 \frac{\pi}{2} m(m+1) \phi_m \phi_{m+1} \quad (\text{A.110})$$

Bringing all the terms together

$$F_{\text{vel}} = -\frac{\rho}{4} \sum_{n=0}^{\infty} \int_{\delta_B} \, dl r n_r \left[\frac{2\pi}{\epsilon_m} \left(\frac{\partial \phi_m}{\partial r} \frac{\partial \phi_{m+1}}{\partial r} + \frac{\partial \phi_m}{\partial z} \frac{\partial \phi_{m+1}}{\partial z} \right) + \frac{\pi n(n+1)}{r^2} \phi_n \phi_{n+1} \right] \quad (\text{A.111})$$

as required.

Kim & Yue (1989) do not give the form of the heave force. The derivation proceeds in much the same way. Ignoring the waterline force, as we always have a wall sided body, the

force is given by

$$F_{z1}^{(2)} = \int_{\delta B} \int_0^{2\pi} \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right] n_z r \, d\theta \, dl. \quad (\text{A.112})$$

(note there is no $\cos \theta$ as the end as we do not need to resolve the force into a particular direction as we go around the cylinder like we do with the horizontal force)

Dealing with the $\partial \phi / \partial r$ term,

$$\int_0^{2\pi} \left(\sum_{m=0}^{\infty} \frac{\partial \phi}{\partial r} \cos m\theta \right) \left(\sum_{m=0}^{\infty} \frac{\partial \phi}{\partial r} \cos m\theta \right) d\theta \quad (\text{A.113})$$

$$= \int_0^{2\pi} \left[\sum_{m=0}^{\infty} \left(\frac{\partial \phi_m}{\partial r} \right)^2 \cos^2 m\theta + \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} \frac{\partial \phi_i}{\partial r} \frac{\partial \phi_j}{\partial r} \cos i\theta \cos j\theta \right] d\theta \quad (\text{A.114})$$

$$= \sum_{m=0}^{\infty} \frac{2\pi}{\epsilon_m} \left(\frac{\partial \phi_m}{\partial r} \right)^2 \quad (\text{A.115})$$

as the double summation is always zero.

Looking now at the $\partial \phi / \partial \theta$ term which proceeds in much the same manner in the surge case, we end up with

$$\int_0^{2\pi} \left(\frac{\partial \phi}{\partial \theta} \right)^2 = \int_0^{2\pi} \left[\sum_{m=0}^{\infty} m^2 \phi_m^2 \sin^2 m\theta + \sum_{i=0}^{\infty} \sum_{\substack{j=0 \\ j \neq i}}^{\infty} i j \phi_i \phi_j \sin i\theta \sin j\theta \right] d\theta \quad (\text{A.116})$$

$$= \sum_{m=0}^{\infty} m^2 \pi \phi_m^2 \quad (\text{A.117})$$

as the double summation is always zero.

Bringing all the terms together:

$$F_{z1}^{(2)} = -\frac{\rho}{4} \sum_{n=0}^{\infty} \int_{\delta B} dl r n_z \left\{ \frac{2\pi}{\epsilon_n} \left[\left(\frac{\partial \phi_n}{\partial r} \right)^2 + \left(\frac{\partial \phi_n}{\partial \theta} \right)^2 \right] + \frac{n^2 \pi}{r^2} \phi_n^2 \right\} \quad (\text{A.118})$$

B

An investigation into a Green's function for radiation-diffraction problems

B.1 Introduction

The solution of the radiation-diffraction problem has long been of interest, both to the scientific and industrial communities. A typical approach to the problem is to adopt a frequency domain solution which involves solving a boundary integral equation. The boundary integral equation arises from applying Green's theorem to yield an expression for the velocity potential in terms of a Green's function G and the known derivative of the potential, in this case on the body boundary. There are many different possible representations of the Greens function. This Appendix considers several and evaluates their use for solving the boundary integral equation for radiation-diffraction problems.

A rise in the use of boundary integral methods for the radiation diffraction problem occurred in the 1970s as computer power was starting to make such methods tractable. The early examples of Milgram & Halkyard (1971) and Garrison & Chow (1972) demonstrated the feasibility of the method but both comment on the difficulties associated with the numerical evaluation of the Green's function and its normal derivative, known as the kernel. The problem principally lies in the slow convergence of the expressions for the Green's function as the source point approaches the field point.

APPENDIX B. AN INVESTIGATION INTO A GREEN'S FUNCTION FOR RADIATION-DIFFRACTION PROBLEMS

Following the work of Black (1975), Fenton (1978) paid attention to the properties of the Green's function and proposed a method of improving convergence, known for the remainder of this chapter as the method of 'subtracted singularities'. Later Hulme (1983) was perhaps the first to specifically investigate the computational properties of different representations of the Green's function by considering three forms: first the classical integral representation by John (1950), second a series of cylindrical harmonics as used by Black (1975) and Fenton (1978), and thirdly a series of toroidal harmonics. The last form improves the convergence of the functions but introduces other difficulties such as numerically evaluating Cauchy principal value integrals numerically. Ultimately, Hulme (1983) concludes that there is no one formulation that is computationally useful everywhere.

The idea of having different representations for different regions is employed directly by authors such as Newman (1985) and Telste & Noblesse (1986) who decomposed the domain into four and five regions respectively, each with a different expression for the Green's function. Although such a technique provides improves computational characteristics, having many expressions for the Green's functions increases the complexity of the numerical model particularly when considering that the derivative of each with respect to the normal must be evaluated. Furthermore, in the case of Newman's methodology, 8000 Chebyshev coefficients must be evaluated to avoid the numerical integration inherent in the Green's function representations for each region.

Given the computing power available to the author's mentioned so far, detailed consideration of the computational merits of the Green's function were absolutely necessary. Newman (1985) notes that the Green's function is evaluated approximately 10^6 times for a single wave frequency. Nowadays, however, computing power is comparably inexpensive leading to the question of whether the considerable effort involved in both understanding and implementing the improvements outlined above is actually necessary for a new radiation-diffraction solver. For the purposes of this new radiation-diffraction solver, accuracy is the primary concern and therefore the task of this Appendix is to find the representation that is the most appropriate for today's computing environment. Ease of implementation is considered above computational efficiency in order to produce a working, accurate and, importantly, robust solver.

B.2 Problem definition

A coordinate system has been adopted with (x, y) -plane in the quiescent free surface with z pointing vertically upwards in a constant water depth h , as shown in Figure B.1. The Greens function is denoted $G(x, y, z; x', y', z')$ for a field-point (x, y, z) and source-point (x', y', z') , and represents the source potential pulsating with a frequency ω . In the context of radiation-diffraction problems, the boundary conditions for the Green's function are identical to those for the velocity potential except for the condition on the body.

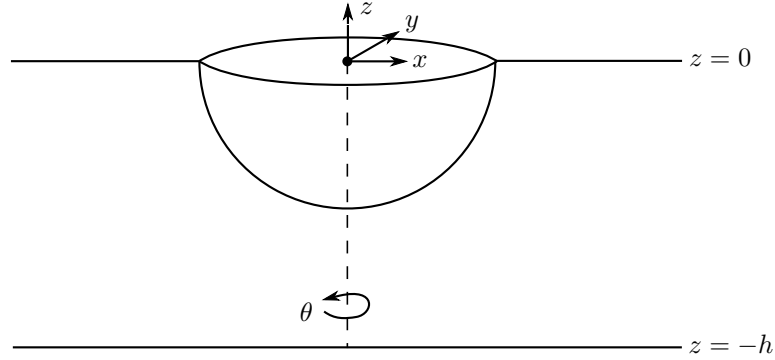


Figure B.1: Coordinate system adopted

The starting point for an analysis of the Green's function is the classical representation of the free-surface formulation by John (1950). All the forms of the Green's function that follow are derived from this formulation, given by

$$G = \frac{1}{R_1} + \frac{1}{R_2} + \oint_0^\infty \frac{2(k+K)e^{-kh}}{k \sinh kh - K \cosh kh} \cosh k(z+h) \cosh k(\zeta+h) J_0(kR) dk, \quad (\text{B.1a})$$

$$= -2 \oint_0^\infty \frac{K \sinh kz_> + k \cosh kz_>}{k \sinh kh - K \cosh kh} \cosh k(z_<+h) J_0(kR) dk, \quad (\text{B.1b})$$

where $K = \omega^2/g$, and $\oint$ denotes the principal value integral. Additionally the various coordinates used are given by

$$R_1 = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}, \quad (\text{B.2a})$$

$$R_2 = [(x-x')^2 + (y-y')^2 + (z+z'+2h)^2]^{1/2}, \quad (\text{B.2b})$$

$$R = [(x-x')^2 + (y-y')^2]^{1/2}, \quad (\text{B.2c})$$

APPENDIX B. AN INVESTIGATION INTO A GREEN'S FUNCTION FOR
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$$z_{>} = \max(z, z'), \quad z_{<} = \min(z, z'). \quad (\text{B.2d})$$

There are two problems inherent to the integral representation in (B.1b): firstly numerical integration is computationally expensive to perform and, secondly, the convergence of the integral is slow when R is small, ie when the field and source points are close together.

In order to overcome the first problem, John (1950, eq A.16) proposed an expansion into an progressive component and an infinite number of local components, given by

$$G = \frac{1}{2}\pi i A_0 \cosh k(h+z) \cosh k(h+\zeta) [J_0(kR) + iY_0(kR)] \\ + 4 \sum_{n=1}^{\infty} A_n \cos c_n(h+z) \cos c_n(h+\zeta) K_0(c_n R), \quad (\text{B.3})$$

where

$$A_0 = \frac{4(k^2 - K^2)}{h(k^2 - K^2) + K}, \quad (\text{B.4})$$

$$A_n = \frac{4(c_n^2 + K^2)}{h(c_n^2 + K^2) - K}, \quad (\text{B.5})$$

in which k is the solution to

$$\omega^2 = gk \tanh kh, \quad (\text{B.6})$$

and c_n to

$$\omega^2 = -g c_n \tan c_n h, \quad n = 1, 2, \dots \quad (\text{B.7})$$

Finally, J_0 and Y_0 are the Bessel functions of the first and second kind respectively, and K_0 is the modified Bessel function of the second kind.

Clearly when implementing a radiation-diffraction solution, the infinite sum in (B.3) must be truncated at a finite value N . Therefore, the rate of convergence of the summation is critical to achieving an accurate solution and is the focus of the rest of this appendix. The imaginary part (the first term in B.3) is uniformly convergent throughout the domain, and does not require particular attention.

B.3 Green's Function for the axisymmetric problem

In keeping with the aims of the present PhD project a Green's function will be developed for the axisymmetric problem. Adopting polar co-ordinates, (r, θ, z) , where $r^2 = x^2 + y^2$, the Green's function (B.3) can be expanded as in the azimuthal, $\theta = \tan^{-1}(y/x)$, coordinate direction. Watson (1944), and applied by Fenton (1978), gives the expansions for the Bessel functions in (B.3) using Graf's addition theorem,

$$J_0(kq) = \sum_{m=-\infty}^{\infty} J_m(kr')J_m(kr) \cos m(\theta - \theta') \quad (\text{B.8a})$$

$$Y_0(kq) = \sum_{m=-\infty}^{\infty} Y_m\left(k \begin{smallmatrix} r \\ r' \end{smallmatrix}\right) J_m\left(k \begin{smallmatrix} r' \\ r \end{smallmatrix}\right) \cos m(\theta - \theta') \quad (\text{B.8b})$$

$$K_0(c_n q) = \sum_{m=-\infty}^{\infty} K_m\left(c_n \begin{smallmatrix} r \\ r' \end{smallmatrix}\right) I_m\left(c_n \begin{smallmatrix} r' \\ r \end{smallmatrix}\right) \cos m(\theta - \theta'), \quad (\text{B.8c})$$

in which the top row is taken if $r \geq r'$ and the lower otherwise, and I_m is the modified Bessel function of the first kind. Finally noting that (Abramowitz & Stegun, 1972, 9.1.5)

$$J_{-m}(q) = (-)^m J_m(q), \quad Y_{-m}(q) = Y_m(q) \quad (\text{B.9})$$

and (Abramowitz & Stegun, 1972, 9.6.6)

$$K_{-m}(q) = K_m(q), \quad I_{-m}(q) = I_m(q), \quad (\text{B.10})$$

Neumann's coefficient, ϵ , can be introduced to the expansions above to give

$$J_0(kq) = \sum_{m=0}^{\infty} \epsilon J_m(kr')J_m(kr) \cos m(\theta - \theta'), \quad (\text{B.11})$$

where

$$\epsilon_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{otherwise} \end{cases}, \quad (\text{B.12})$$

and a similar relation for the other Bessel functions.

Therefore, the Green's function may be written as

$$G(r, \theta, z; r', \theta', z') = \sum_{m=0}^{\infty} \epsilon_m g_m(r, z; r', z') \cos m(\theta - \theta'), \quad (\text{B.13})$$

in which g_m is determined by substituting the results of Graf's addition (B.8), to give

$$\begin{aligned} g_m &= \frac{1}{2} \pi i A_0 \cosh k(h+z) \cosh k(h+z') J_m(kr_<) H_m^{(1)}(kr_>) \\ &+ \sum_{n=1}^{\infty} A_n \cos c_n(h+z) \cos c_n(h+z') I_m(c_n r_<) K_m(c_n r_>) \end{aligned} \quad (\text{B.14})$$

where $H_m^{(1)}$ is the Hankel function of the first kind, and

$$r_< = \min(r, r'), \quad r_> = \max(r, r') \quad (\text{B.15})$$

For consistency with Hulme (1983) the representation in (2.23) will be termed the 'series of cylindrical harmonics'.

B.4 Convergence of the series of cylindrical harmonics

Having established a Green's function for the axisymmetric problem in (2.23), the convergence properties of this representation can be investigated. In particular the nature of the singularity as $(x, z) \rightarrow (x', z')$ in the Greens function will be examined. The large m behaviour of the summation of (2.23) can be easily established by considering (B.7), where

$$c_n \rightarrow \frac{n\pi}{h} + O\left(\frac{1}{n}\right) \text{ as } n \rightarrow \infty, \quad (\text{B.16})$$

leading to

$$A_n \rightarrow \frac{4}{h} + O\left(\frac{1}{n^2}\right) \text{ as } n \rightarrow \infty. \quad (\text{B.17})$$

Combining the results above and the asymptotic forms of I_m and K_m from Watson (1944, §7.23), yields

$$I_m(c_n r_<) K_m(c_n r_>) \rightarrow \frac{h}{2n\pi} (rr')^{-1/2} e^{-n\pi|r-r'|/h} \text{ as } n \rightarrow \infty \quad (\text{B.18})$$

As Hulme (1983) notes (B.18) will be rapidly convergent whenever $|r-r'|/h$ is not small. Fenton (1978), however, goes farther to categorise the singularity explicitly. Substituting

these asymptotic expressions into the summation part of g_m gives

$$g_{mn}^* = \frac{2}{n\pi} (rr')^{-1/2} \cos\left(\frac{n\pi}{h}(h+z)\right) \cos\left(\frac{n\pi}{h}(h+z')\right) e^{-n\pi|r-r'|/h} \quad (\text{B.19})$$

where the $*$ denotes the result as $n \rightarrow \infty$. Then, by applying the product-to-sum formula for the cosine terms from Abramowitz & Stegun (1972, 4.3.32)

$$g_{mn}^* = \frac{1}{n\pi} (rr')^{-1/2} (\cos n\Sigma + \cos n\Delta) e^{-n\pi|r-r'|/h}, \quad (\text{B.20})$$

where $\Sigma = \pi(z - z')/h$ and $\Delta = \pi(2h + z + z')/h$. Fenton (1978) derives the result of (B.20) as $z \rightarrow z'$, giving

$$g_{mn} \rightarrow \frac{1}{n\pi} \cos(n\Sigma) e^{-n\pi|R-r'|/h} \quad \text{as } z \rightarrow z', \quad (\text{B.21})$$

so that, using Jolley (1961, eq 536),

$$\sum_{m=1}^{\infty} g_{mn} \approx -\frac{1}{2} \ln [1 - 2\gamma \cos \Sigma + \gamma^2], \quad (\text{B.22})$$

where $\gamma = \exp(-\pi|r - r'|/h)$, leading to

$$\sum_{m=1}^{\infty} g_{mn} \approx \frac{1}{2} \ln [(r' - r)^2 + (z' - z)^2], \quad \text{as } z \rightarrow z' \text{ and } r \rightarrow r', \quad (\text{B.23})$$

thereby showing a logarithmic singularity. This logarithmic singularity clearly demonstrates that there will be slow convergence of the infinite sum when the source and field point are close together. This has important implications for the accuracy of the integral equations as source and field points will be necessarily close together in order to achieve a reasonable discretisation of the body. Furthermore, in the second order problem, the free surface integral must be calculated in the vicinity of the body making R small; accurate evaluation in this region is vital for quantities such as the runup on the body.

In order to understand the implications of the logarithmic singularity, the following figures show the convergence firstly of the infinite series with number of mode N and then how this corresponds to radiation-diffraction problem.

Figure B.2 shows the convergence of $\sum_{n=1}^{\infty} g_{mn}$, with $kh = 3.0$, $m = 0$ and $1 \leq N \leq 200$. A range of cases is considered with $0.001 \leq R/h \leq 0.4$. As R/h reduces the infinite sum in the

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Greens function clearly takes more modes to converge. Adopting the slowest converging case ($R/h = 0.001$), Figure B.3 shows that the convergence does not heavily depend on Kh . Ratios in the order of $R/h = 1E-3$ are realistic for floating body problems and will certainly be encountered during this PhD project. Figure B.2 suggests that in excess of 200 modes are required for $R/h = 1E-3$ which drastically increases computational run time and become intractable for irregular waves.

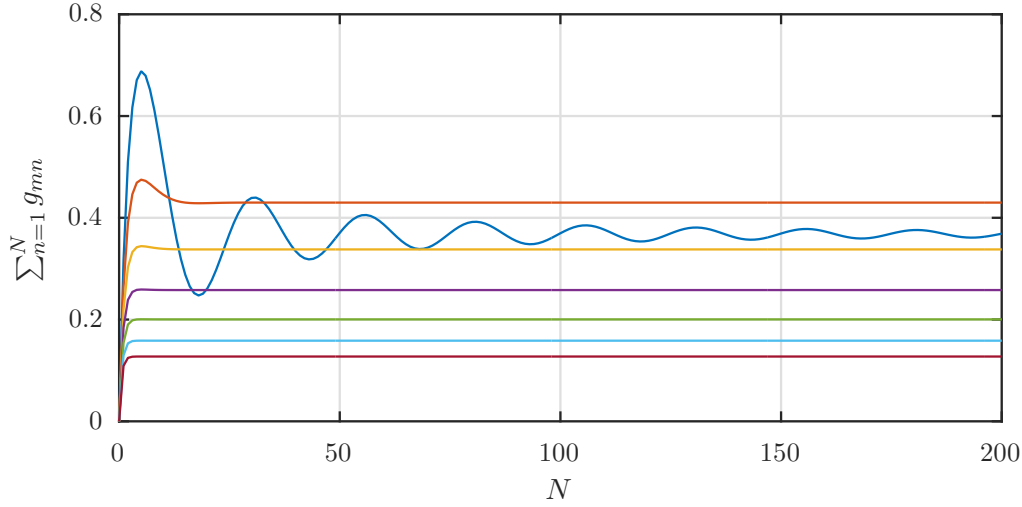


Figure B.2: Convergence of the series of cylindrical harmonics with $Kh = 3.0$, $m = 0$, $z = 0$, $z' = -0.1$, for $R/h = 0.001$ (—), 0.068 (—), 0.134 (—), 0.201 (—), 0.267 (—), 0.334 (—), 0.400 (—).

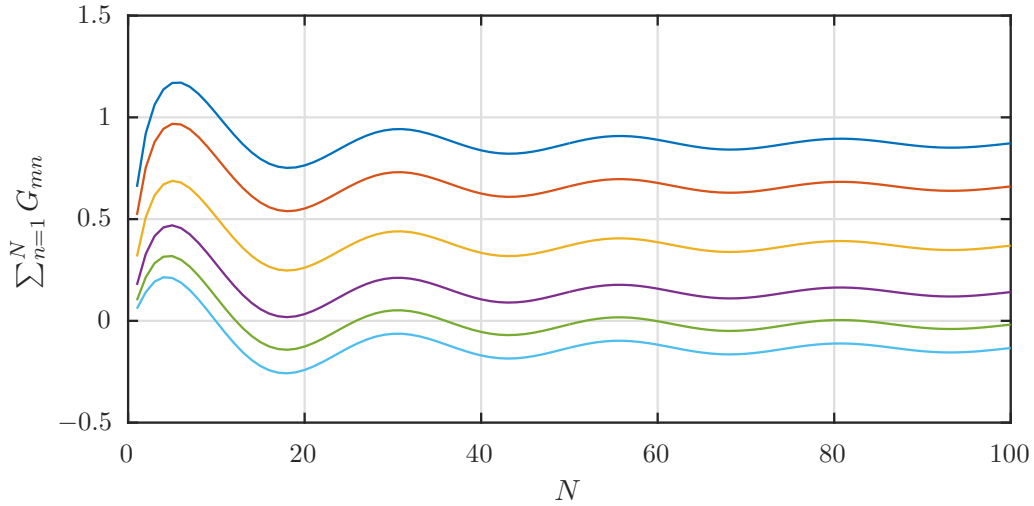


Figure B.3: Convergence of the series of cylindrical harmonics with, $m = 0$, $R/h = 0.001$, $z = 0$, $z' = -0.1$, for $Kh = 1.0$ (—), 2.0 (—), 3.0 (—), 4.0 (—), 5.0 (—), 6.0 (—).

Figures B.4 and B.5 shows the convergence characteristics by varying m and z' respectively and support the trend shown above in which the R/h ratio is the primary mechanism

for the slow convergence of the infinite sum. Therefore a special treatment is necessary for R/h only.

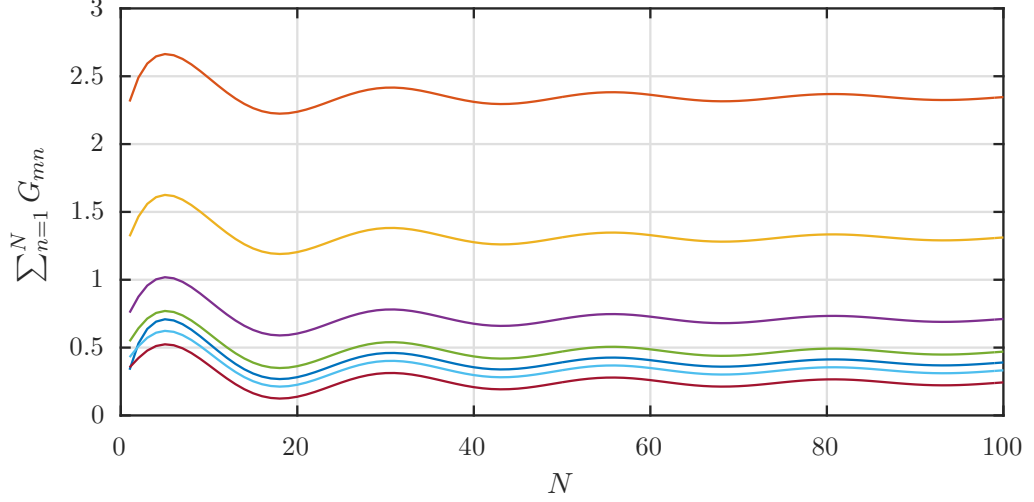


Figure B.4: Convergence of the series of cylindrical harmonics with, $Kh = 3.0$, $R/h = 0.001$, $z = 0$, $z' = -0.1$, for $m = 0$ (—), 2 (—), 4 (—), 6 (—), 8 (—), 10 (—), 12 (—).

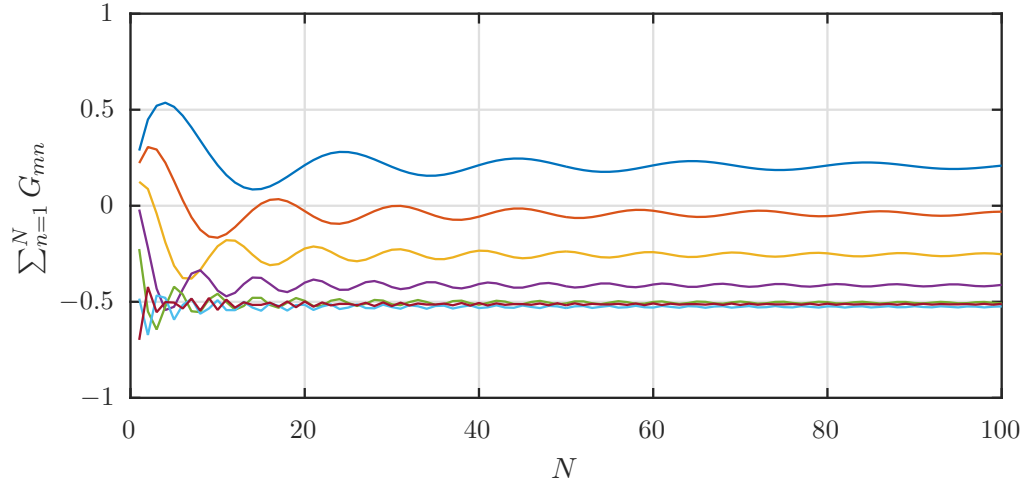


Figure B.5: Convergence of the series of cylindrical harmonics with, $Kh = 3.0$, $R/h = 0.001$, $z = 0$, $m = 0$ for $z'/h = -0.100$ (—), -0.144 (—), -0.208 (—), -0.300 (—), -0.433 (—), -0.624 (—), -0.900 (—).

Turning to the effect this has on the radiation-diffraction solution, Figure B.6 concerns the linear heave excitation force ($F_z^{(1)}$) (due to scattered potential only) with increasing numbers of modes in the infinite sum of the Greens function. Figure B.6 shows the overall convergence profiles for a hemispherical body and supports the observation made above that the convergence of the Green's function is not heavily dependent on the value of ka . Therefore, Figure B.7 shows a more detailed view of increasing the number of modes for

three different geometries.

The first observation from Figure B.7 is that the truncated cylinder with the shallowest draft takes in excess of 80 modes for convergence, with the other cylinders taking fewer modes with increasing draft. This behaviour is expected as the shallower draft reduces the $(z - z')/h$ ratio for which Figure B.5 showed a slow rate of convergence. Secondly, the hemispherical geometry convergences much faster than the truncated cylinder of 'equivalent' draft (ie $d = a$); again this behaviour is expected as the vertical side of the cylinder will have $R/h = 0$ along its length and is therefore the extreme case in Figure B.2 where the convergence of the Green's function is at its worst.

The convergence at second order is presented in Figure B.8 and shows a broadly similar trend to the linear force. The principal difference, however is the higher dependence on ka where the higher values of ka (shorter waves) take longer to converge.

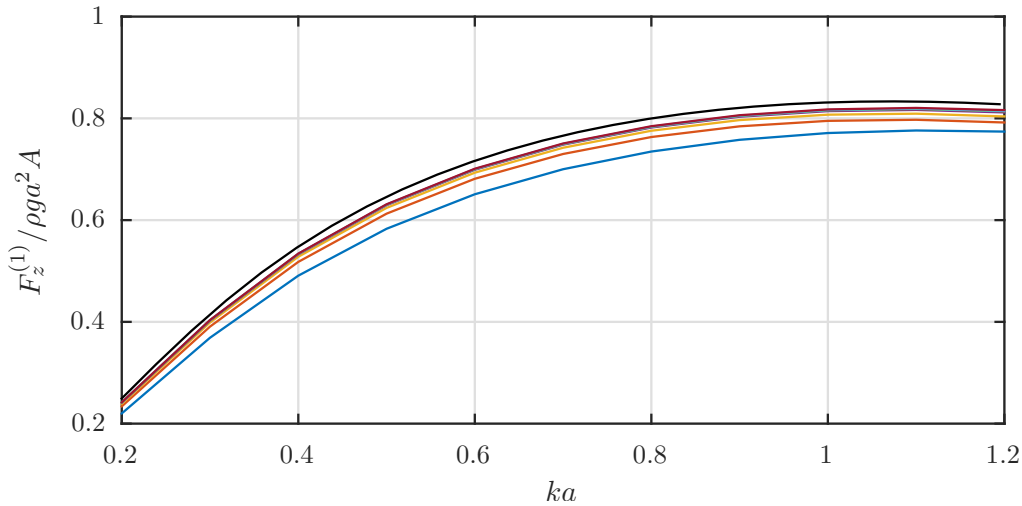


Figure B.6: Convergence of the linear heave excitation force on a hemisphere. WAMIT (—) and ICRD3D with $N = 5$ (—), 10 (—), 20 (—), 50 (—), 75 (—), 100 (—), 150 (—).

Within the context of this PhD and the investigation of heaving buoy-type devices, truncated cylinders form a critical part of the study meaning that the accurate of the excitation forces is paramount. For both confidence in the results, particularly when extending the code to irregular waves, the convergence of forces ought faster than the 80 modes shown in Figure B.7. Therefore, the simple method of the cylindrical harmonics is deemed unsuitable; the following sections outline a method of achieving faster convergence.

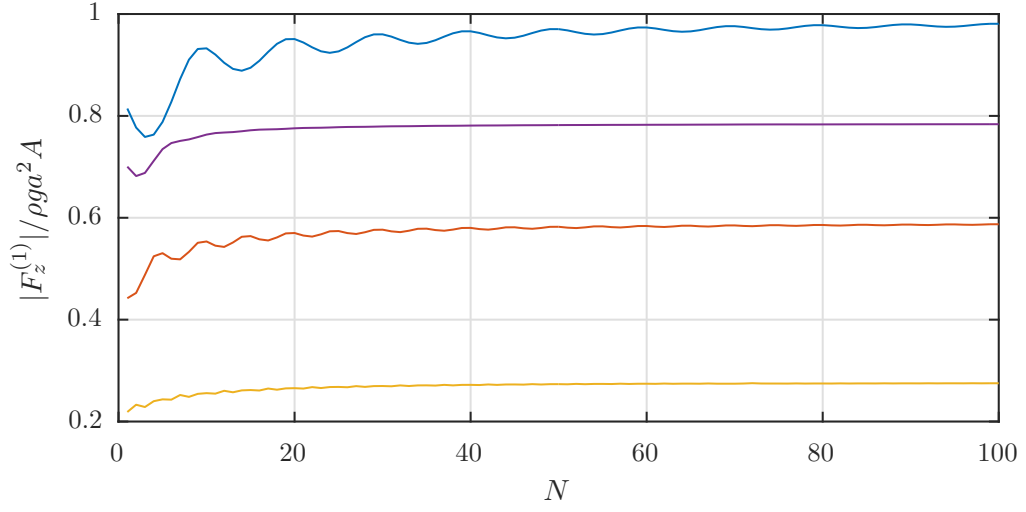


Figure B.7: Convergence profiles for $ka = 0.8$ for truncated cylinders with $a = 0.25$ m and draft $d = a/2$ (—), a (—), $2a$ (—), and a hemisphere also with $a = 0.25$ m (—) using the series of cylindrical harmonics.

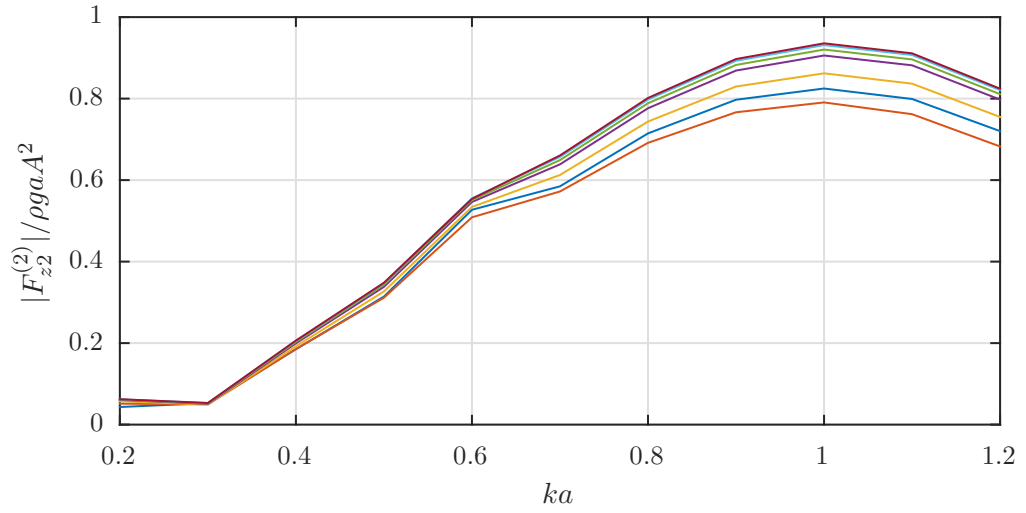


Figure B.8: Convergence of the second order heave excitation force due to second order potentials on a truncated cylinder with $a = 0.25$ m and $d = a$ for $N = 5$ (—), 10 (—), 20 (—), 50 (—), 75 (—), 100 (—), 150 (—).

B.5 Improving convergence: Subtraction of singularities

As noted in the previous section, Fenton (1978) proved the existence of a logarithmic singularity in the series of cylindrical harmonics expression (B.3). The method Fenton (1978) proposes to remove these singularities involves the realisation that the summation of the asymptotic form of (B.3) given in (B.20) has a closed form expression. Therefore,

the infinite summation may be rewritten as

$$\sum_{n=1}^{\infty} g_{mn} = \sum_{n=1}^{\infty} (g_{mn} - g_{mn}^*) + \sum_{n=1}^{\infty} g_{mn}^*, \quad (\text{B.24})$$

where g_{mn}^* is given in (B.20). Equation (B.24) removes the oscillatory behaviour of the Green's function due to cancellation of the first term as n increases. The difference between the near and asymptotic forms when n is small and is given by

$$\sum_{n=1}^{\infty} g_{mn}^* = \frac{(xx')^{1/2}}{2\pi} [\ln(1 - 2\gamma \cos \Sigma + \gamma^2) + \ln(1 - 2\gamma \cos \delta + \gamma^2)], \quad (\text{B.25})$$

in which Jolley (1961, eq 536) has been used.

Figure 2.3 shows the dramatic improvement to the Green's function convergence using the subtracted singularities method for the same case presented in Figure B.2. Figure B.10 then presents the effect the improvement in the subtracted singularities has on the absolute value of the heave excitation force using the same cases as Figure B.7. The improvement is very clear as convergence in the case of the subtracted singularities is achieved after 10 modes compared to the nearly 70 required for the cylindrical harmonics formulation. Figure B.10 also lends confidence in the implementation of both formulations as the solutions converge in the case of large N as they should. The subtracted singularities also leads to an improvement in the surge excitation force though the effect is not as pronounced and so not included herein.

Figure B.11 presents the same wave case and geometries (except hemisphere) as Figure B.10 but examines the second-order scattered excitation force due to second-order potentials rather than the linear force. A similar trend is observed at second order in which the subtracted singularities converges much faster than series of cylindrical harmonics, though both forms take longer to converge at second order. In the case of subtracted singularities, convergence is achieved after 25 modes, which is considered acceptable for the purposes of this project.

It should be noted that Fenton (1978) includes an additional analytic sum, S_6 , of the logarithmic singularities in (B.25); this adds additional complexity, particularly as additional terms are introduced to the integral equation. Given, the results obtained without the S_6 term, the S_6 term is not included at the present time.

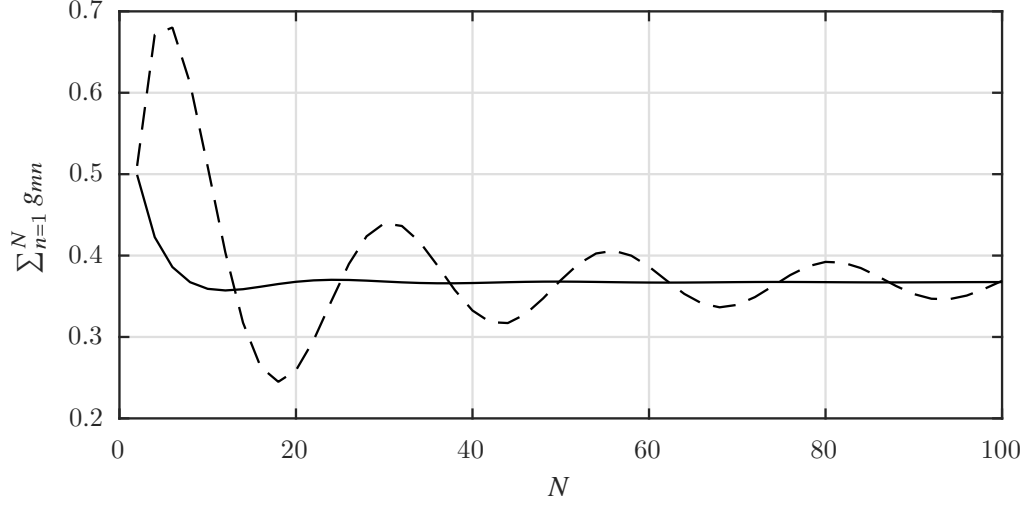


Figure B.9: Comparison between cylindrical harmonics and subtracted singularities with $Kh = 3.0$, $m = 0$, $z = 0$, $z' = -0.1$ and $R/h = 0.001$. Subtracted singularities (—) and Cylindrical harmonics (---).

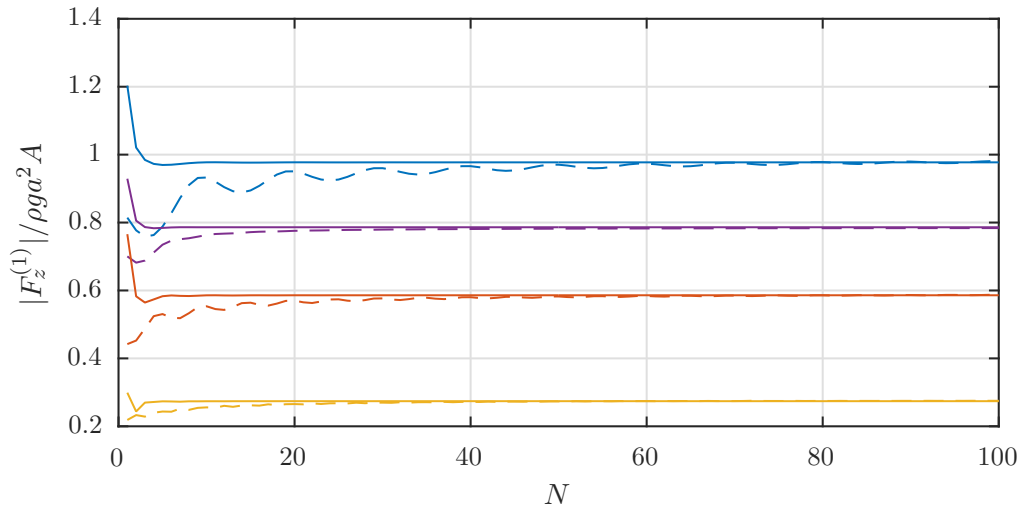


Figure B.10: Convergence profiles for $ka = 0.8$ for truncated cylinders with $a = 0.25$ m and draft $d = a/2$ (—), a (—), $2a$ (—), and a hemisphere also with $a = 0.25$ m (—) using subtracted singularities (—) and cylindrical harmonics (---)

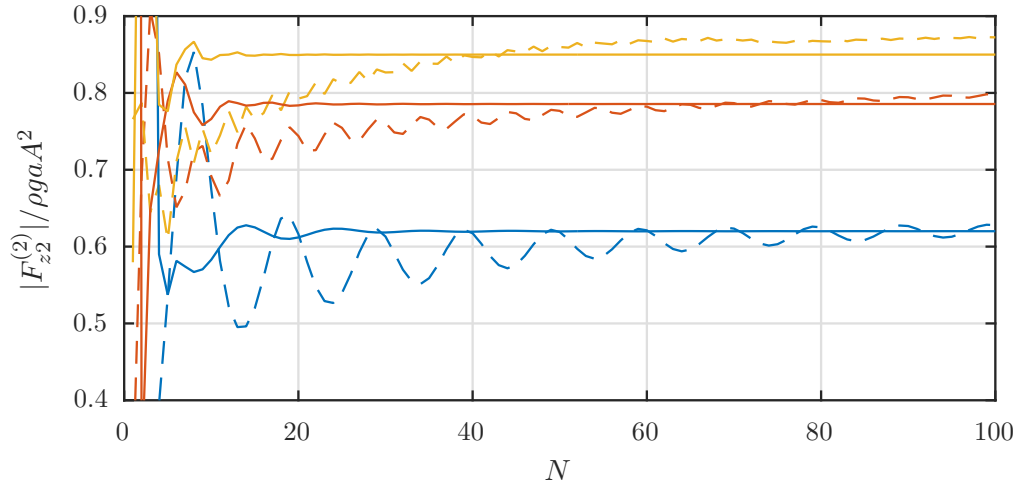


Figure B.11: Convergence profiles for $ka = 0.8$ for truncated cylinders with $a = 0.25$ m and draft $d = a/2$ (—), a (—), $2a$ (—), using subtracted singularities (—) and cylindrical harmonics (---)

B.6 The special case: $R/h = 0$

As mentioned previously, the truncated cylinder poses particular problems for the convergence of the Green's function as $R/h = 0$ along the entire vertical side. In looking more generally at the convergence of the Green's function, Linton (1999) developed a specific representation for this case which can be used to evaluate the accuracy of the subtracted singularities formulation. Linton (1999), later reformulated by Linton & McIver (2001), starts by taking the infinite sum in the expansion (B.3) and substitutes the modified Bessel function of the first kind K_0 for integral representation given by Gradshteyn & Ryzhik (2007, eq 8.432(7)) to give the real part of the Green's function as

$$\begin{aligned} \operatorname{Re} G = & \frac{1}{2}\pi Z_0 Y_0(kR) \\ & + 2 \sum_{n=1}^{\infty} Z_n \int_0^{\infty} e^{-R^2/4\nu} e^{-c_n^2 \nu} \frac{d\nu}{\nu}, \end{aligned} \quad (\text{B.26})$$

where ν is the dummy variable of integration and

$$Z_n = \begin{cases} A_0 \cosh k(h+z) \cosh k(h+z') & n = 0, \\ A_n \cos k_n(h+z) \cos k_n(h+z') & n > 0. \end{cases} \quad (\text{B.27})$$

Linton (1999) then continues by splitting the infinite integral at an arbitrary point α^2 to then define

$$\Lambda_0 = \frac{1}{2}\pi Y_0(kR) - \int_0^{\alpha^2} e^{-R^2/4\nu} e^{k^2 \nu} \frac{d\nu}{\nu}, \quad (\text{B.28a})$$

$$\Lambda_n = \int_{\alpha^2}^{\infty} e^{-R^2/4\nu} e^{-c_n^2 \nu} \frac{d\nu}{\nu}, \quad m \geq 1 \quad (\text{B.28b})$$

so that

$$G = \sum_{m=0}^{\infty} \Lambda_m Z_m + \int_0^{\alpha^2} e^{-R^2/4\nu} w(\nu) \frac{d\nu}{\nu}, \quad (\text{B.29})$$

where

$$w(\nu) = \sum_{m=0}^{\infty} Z_m e^{-c_m^2 \nu}. \quad (\text{B.30})$$

The function $w(\nu)$ is rapidly convergent for large ν , however in practice, α will be small meaning ν will not be large. Therefore, Linton (1999) derives an expansion for small ν

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leading to a Green's function representation of

$$G = \sum_{m=0}^{\infty} \Lambda_m Z_m + \frac{1}{r} \operatorname{erfc}\left(\frac{r}{2a}\right) + \frac{1}{r_2} \operatorname{erfc}\left(\frac{r_2}{2a}\right) \quad (\text{B.31})$$

$$+ \sum_{i=1}^4 \frac{1}{(R^2 + \chi_i^2)^{1/2}} \operatorname{erfc}\left(\frac{(R^2 + \chi_i^2)^{1/2}}{2a}\right) \quad (\text{B.32})$$

$$+ 2K \int_0^a \left[e^{K^2 u^2 - R^2/4u^2} \sum_{i=0}^4 e^{-K\chi_i} \operatorname{erfc}\left(\frac{\chi_i}{2u} - Ku\right) + Q \right] \frac{du}{u}, \quad (\text{B.33})$$

where

$$\begin{aligned} \chi_1 &= -\zeta - z, & \chi_2 &= 2h - \zeta + z, \\ \chi_3 &= 2h + \zeta - z, & \chi_4 &= 4h + \zeta + z, \end{aligned} \quad (\text{B.34})$$

erfc is the complementary error function, and Q may be neglected as described in Linton & McIver (2001). Critically, this representation of G is not singular as $R \rightarrow 0$ but instead, when $R = 0$,

$$\Lambda_0 = -\operatorname{Ei}(k^2 a^2), \quad \Lambda_n = \operatorname{E}_1(k_n^2 a^2), \quad (\text{B.35})$$

where Ei and E_1 are the exponential integrals as defined in Abramowitz & Stegun (1972).

The new representation provided by (B.33) and (B.35) can now be used to compare the convergence of the subtracted singularities formulation as $R \rightarrow 0$. Linton & McIver (2001) suggest by taking a value of $a = h/4$, the infinite series in (B.33) can be truncated with $N = 3$. Clearly this is a significant improvement on the initial representations outlined above.

There are, however, drawbacks about using the Linton (1999) formulation. Principally, the radiation-diffraction problem has been derived herein for axisymmetric bodies which involves expanding the problem in the azimuthal coordinate θ to which the subtracted singularities formulation lends itself well. Therefore, the summation in (B.13) will be used for the subtracted singularities to return to the 3D case. The comparison aims to establish the adequacy of the subtracted singularities approach in the case of very small R/h ratios.

Figures B.12(a) and B.12(b) shows the comparison between the cylindrical harmonics, the subtracted singularities and the Linton (1999) expression as $R/h \rightarrow 0$. Firstly comparing the cylindrical harmonics and subtracted singularities, the improvement in using the latter formulation is clear in terms of convergence. The subtracted singularities converges towards the Linton (1999) expression, albeit at large N as discussed next, whereas the

cylindrical harmonics converges very slowly.

Turning to the subtracted singularities in Figure B.12(b), two trends are apparent: firstly, the Linton (1999) expression has already converged after just 5 modes, and secondly, the subtracted singularities tends towards the Linton (1999) expression as R/h decreases and takes longer to converge, after about 70 modes. This rate of convergence is slower than seen in the earlier figures, in particular Figure 2.3, for a single mode $m = 0$ suggesting there are problems higher modes and the summation in (B.13). An investigation into this problem is ongoing.

Looking back to the heave force convergence profiles of Figure B.10 in which convergence for the truncated cylinders was achieved after 5 modes, the importance of the $R/h = 0$ case does not appear to be too significant in the overall solution.

B.6.1 Deep water formulation

As can be seen in the first term of A_0 (2.25) as the water becomes deep (ie large kh), $k \rightarrow \nu$ so that $k^2 - \nu^2 \rightarrow 0$. This is balanced however by the product of the two cosh terms in the Greens function, which tend to infinity with increasing kh , leading finite result even at very large kh .

Figure B.13 shows the problems of numerically calculating A_0 as for a 64bit floating point number as the blue line takes finite values at particular values of kh where the numerical precision means $k^2 - \nu^2$ is finite. The effect of this can then be seen in looking at Figure B.14 where the blue line tends to infinity at the same kh values.

In order to overcome these problems calculations have been undertaken using a `boost::multiprecision` library 448bit floating point number. Figure B.13 shows the behaviour of A_0 beyond the point the 64bit number can capture resulting in Figure B.14 for the whole expression.

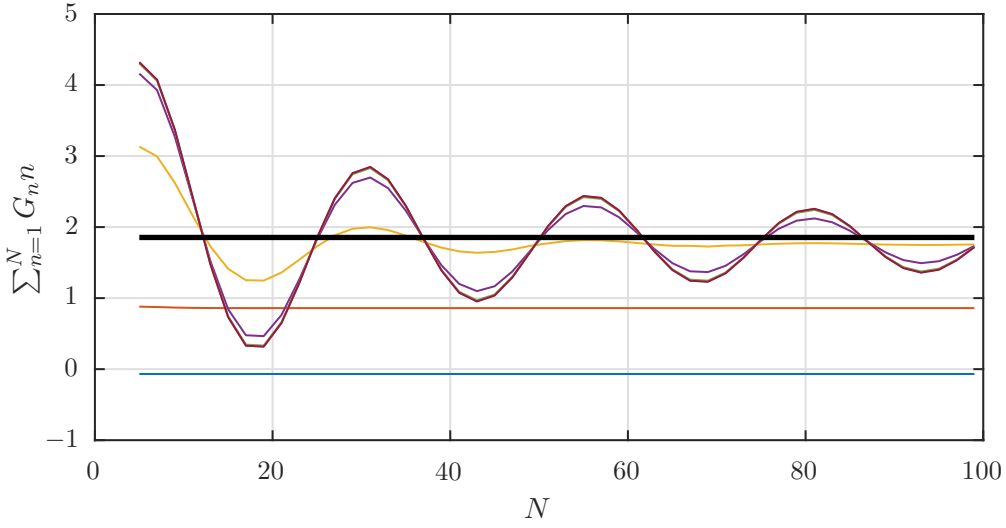
For deep water, Hulme (1983) gives the analytic simplification in equation 3.10:

$$\lim_{h \rightarrow \infty} A_0 \cosh k(h + zp) \cosh k(h + zq) = 4\nu e^{K(zp+zq)}, \quad (\text{B.36})$$

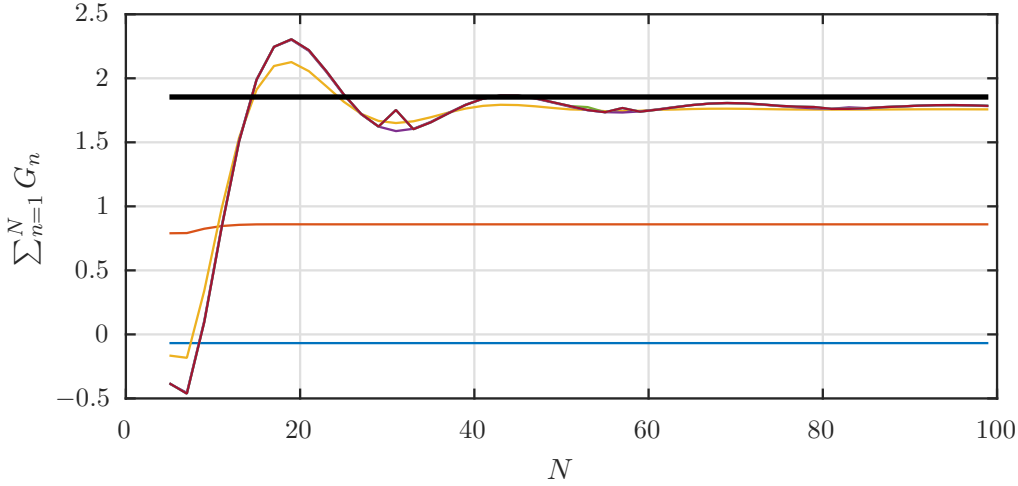
which is seen to compare well with the extended precision calculations shown in Figure B.15. Therefore the analytic expression is preferable and will be used whenever $kh > 10$.

A similar situation does not occur in the real part of the Greens function, the term

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(a) Cylindrical Harmonics



(b) Subtracted Singularities

Figure B.12: Comparison between the Green's functions as $R/h \rightarrow 0$: the expression of Linton (1999) (—), cylindrical harmonics and subtracted singularities for $R/h = 1.0$ (—), $1.0E-1$ (—), $1.0E-2$ (—), $1.0E-3$ (—), $1.0E-4$ (—), $1.0E-5$ (—), $1.0E-6$ (—).

involving A_n , as there is no subtraction operation. This is shown in Figure B.16: the 64bit and 448bit calculations agree and tend to the value of the analytic deep formulation in deep water.

Similar expressions for the imaginary part of the kernel should also be used. The only difference to the deep water expression above is to include a further factor ν which arises from the differentiation. The exponential part remains the same as:

$$\cosh x = \sinh x \text{ as } x \rightarrow \infty \quad (\text{B.37})$$

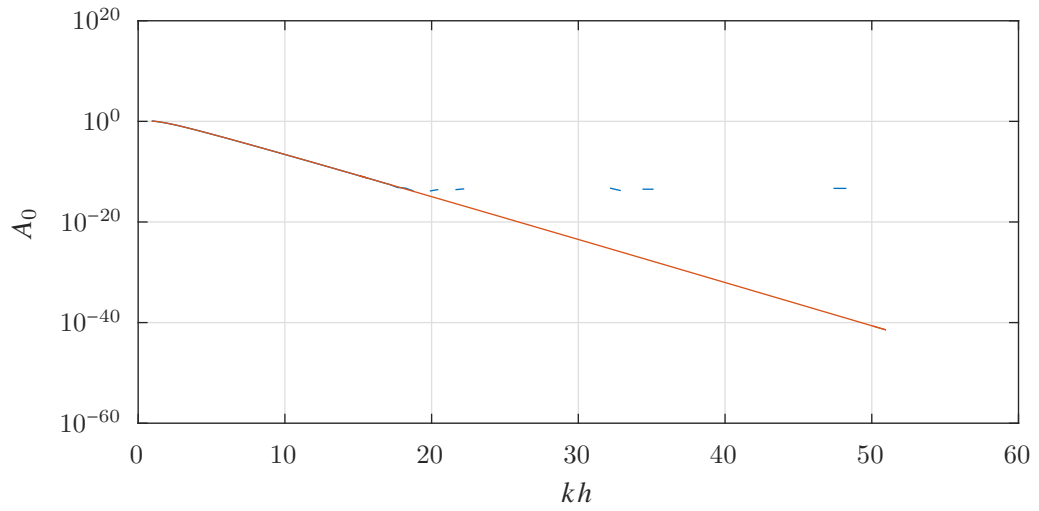


Figure B.13: Numerical precision problems in A_0 . Calculations using floating point numbers with: (—) 64bit, (—) 448bit.

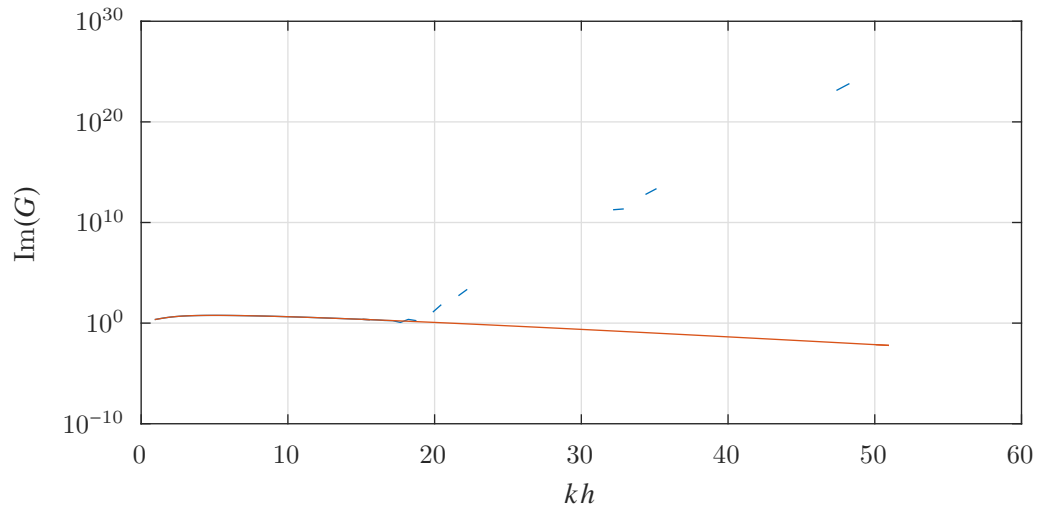


Figure B.14: Numerical precision problems the imaginary part of the Greens function. Calculations using floating point numbers with: (—) 64bit, (—) 448bit.

which can be easily shown by looking at the exponential forms of $\cosh$ and $\sinh$. The agreement of the analytic deep water expression with the numerical calculations is shown in Figure B.17 and again shows the 64bit calculations blowing up at large kh for the same reasons as outlined above.

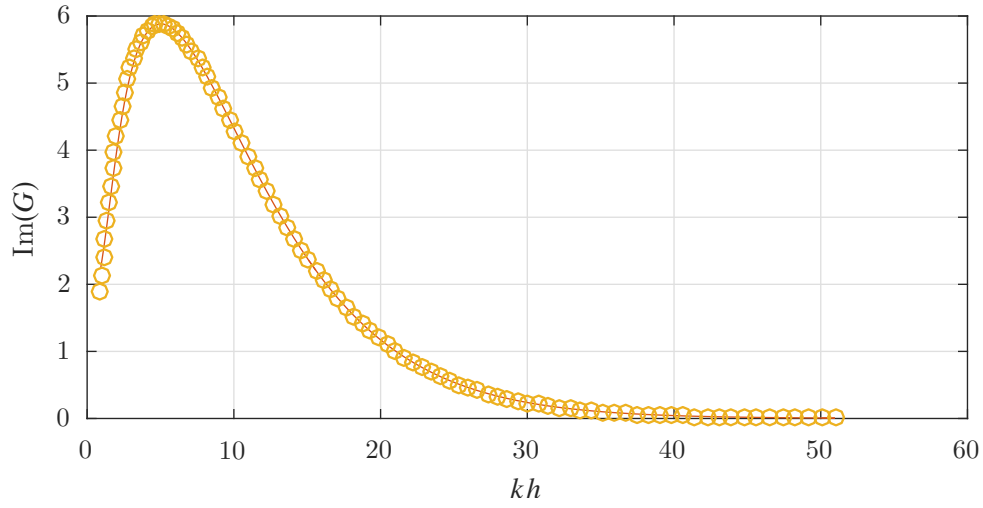


Figure B.15: Comparison between the extended precision calculation and the analytic deep water expression using double precision: (—) 448bit, (o) analytic.

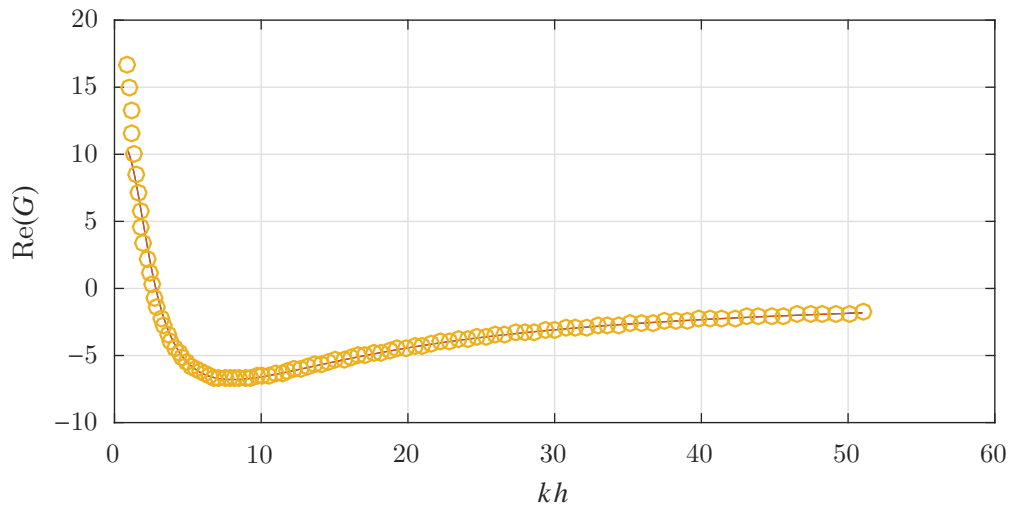


Figure B.16: Comparison of real part calculations of the Greens function. Calculations using floating point numbers with: (—) 64bit, (—) 448bit, (o) analytic.

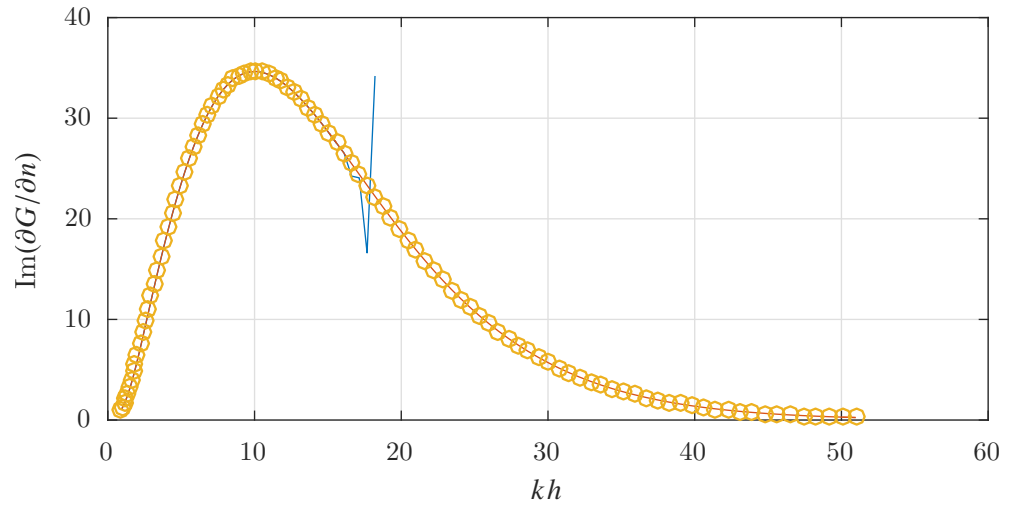


Figure B.17: Comparison of real part calculations of the ring-source kernel. Calculations using floating point numbers with: (—) 64bit, (—) 448bit, (○) analytic.

B.7 Conclusions

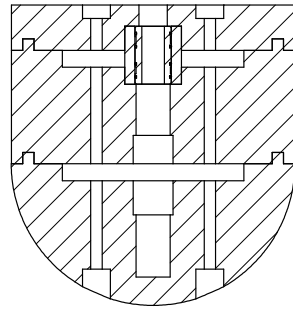
This appendix has presented two Greens function representations in detail: the series of cylindrical harmonics and the subtracted singularities. The former arises from the classical work of John (1950) and provides an expression which is straightforward to implement. The problem, however, for the series of cylindrical harmonics is the slow convergence rate of the infinite sum term when the distance between the source and field term becomes small. The evidence for this singular phenomenon was clear when considering both the linear and second order scattered excitation forces where up to 70 modes were required. The effect was particularly pronounced in the cases of truncated cylinders where along the vertical wall of the cylinder the radial distance between source and field point is zero.

In order to improve the convergence of the series of cylindrical harmonics Fenton (1978) proposed a subtraction of singularities technique. Adopting the technique improved the converge dramatically as the linear excitation force only requires approximately 5 modes to converge for all of the geometries considered. The second order force takes longer to converge, even with the subtracted singularities, requiring about 25 modes, though this is still considered acceptable for the purposes of this PhD project. Further improvement is expected to second-order convergence with some additional numerical considerations to the free-surface-integral.

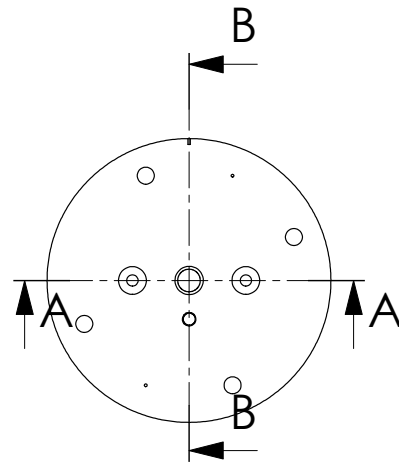
In order to investigate the limiting case of the radial distance between the source and field point being zero, both the cylindrical harmonics and subtracted singularities were compared against a specific form developed by Linton (1999) for this limiting case. Unfortunately, the expression by Linton (1999) cannot be applied directly in the current code as it does not take into account the axisymmetric reduction of the problem. Nonetheless the comparison was still instructive as even after 100 modes, the series of cylindrical harmonics was highly oscillatory and showed no sign of being close to convergence. The subtracted singularities, on the other hand, required around 70 modes to converge; despite this slow rate in the limiting case, as the overall convergence of the solution is adequate, the subtracted singularities is considered the most appropriate Green's function representation for this PhD project.

C

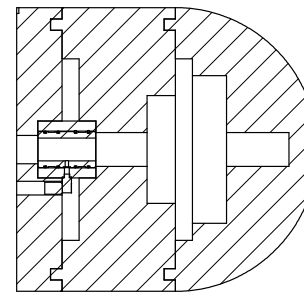
Physical model technical drawings



SECTION A-A



Refer to individual component drawings
for dimensions etc



SECTION B-B

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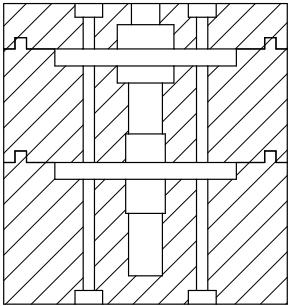
DRAWING BY: MAB

DATE:29/09/2014

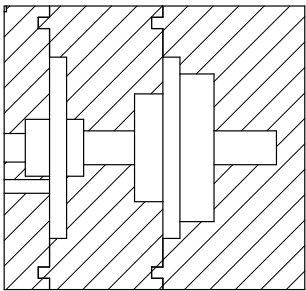
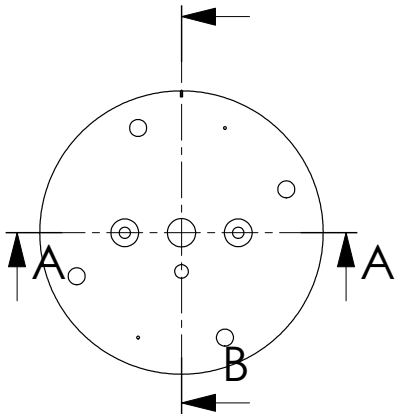
TITLE: Buoy assembly - hemisphere

SCALE:1:5

Revision: 1



SECTION A-A
B



SECTION B-B

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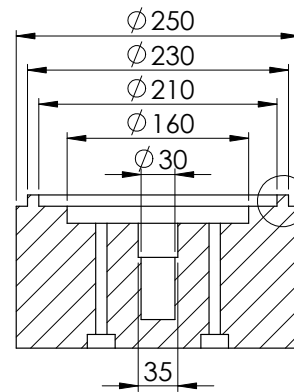
DATE:29/09/2014

TITLE: bottom_cylinder_assembly

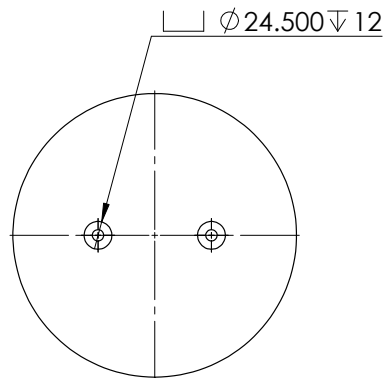
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Revision: 0

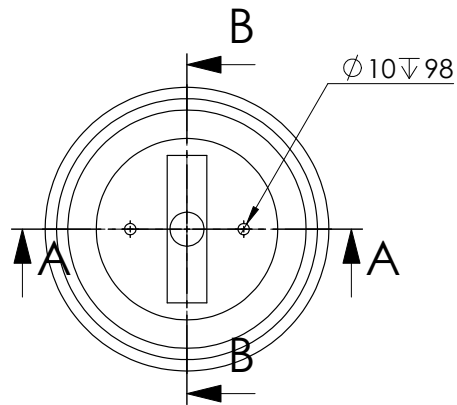
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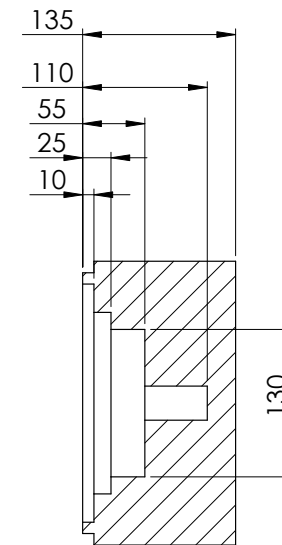
SECTION A-A



Bottom View



Top View



SECTION B-B

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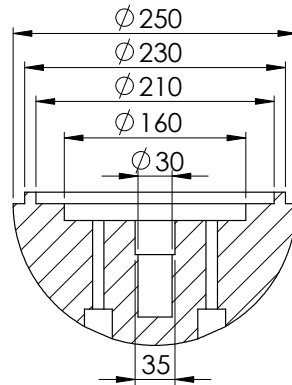
DRAWING BY: MAB

DATE: 04/12/2014

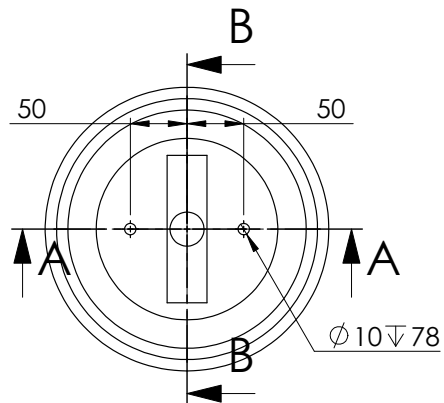
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SCALE:1:5

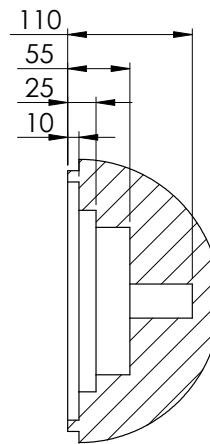
Revision: 1



SECTION A-A

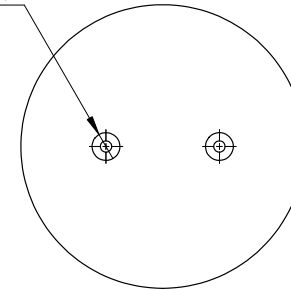


Top View

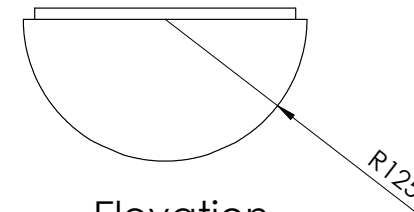


SECTION B-B

$\square$ $\phi 10$ THRU
 $\square$ $\phi 24.500 \nabla 16.569$



Bottom View



Elevation

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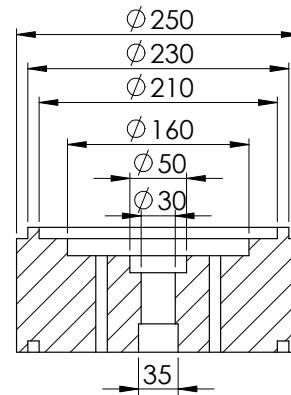
DATE: 29/09/2014

TITLE: bottom hemisphere

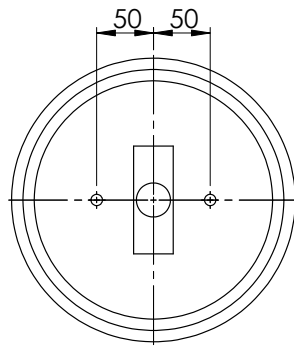
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Revision: 1

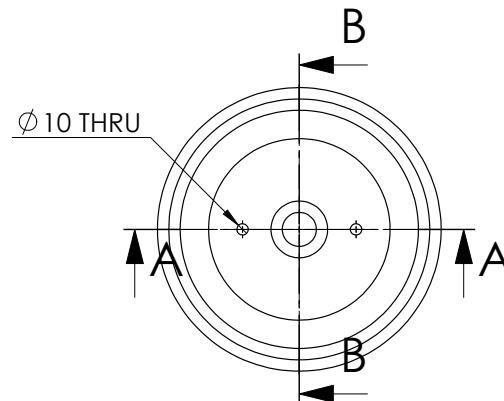
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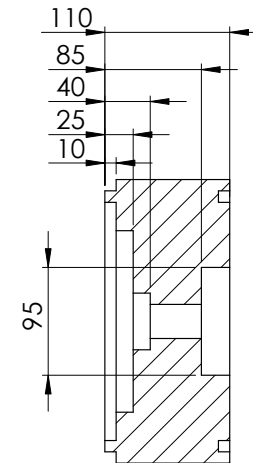
SECTION A-A



Bottom View



Top View



SECTION B-B

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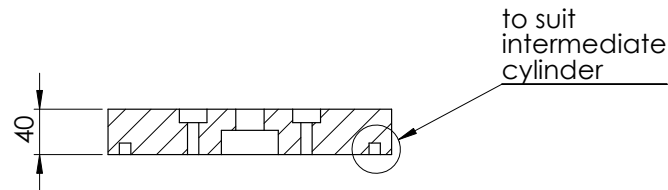
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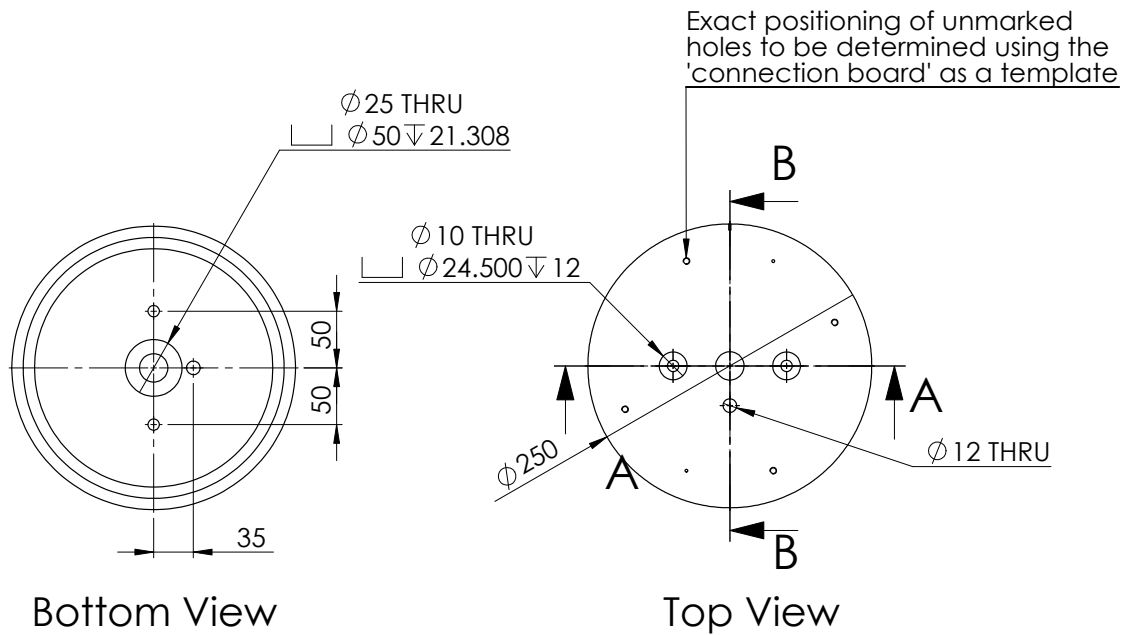
TITLE: intermediate cylinder

SCALE:1:5

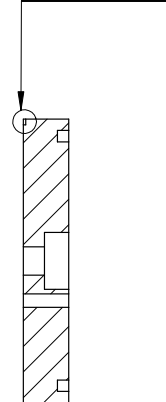
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SECTION A-A



10Lx5Wx5D
notch



SECTION B-B

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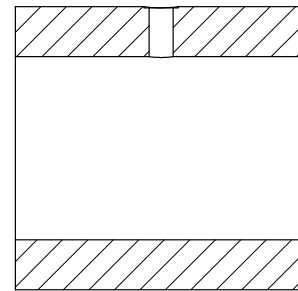
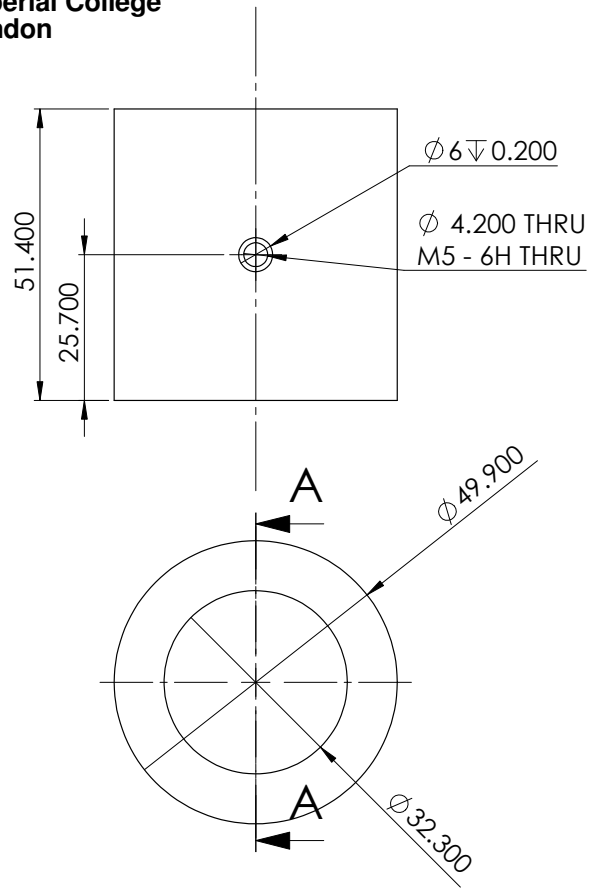
TITLE: lid

SCALE:1:5

Revision: 4

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SECTION A-A

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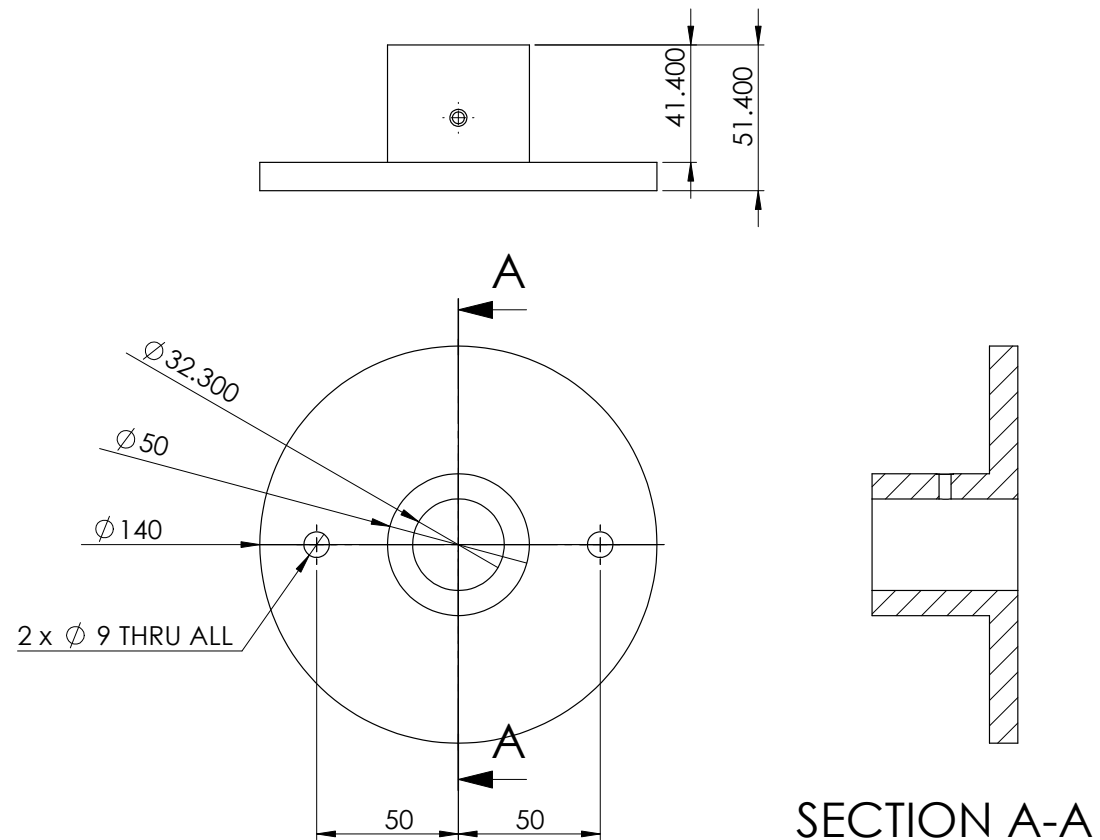
DRAWING BY: MAB

DATE: 04/11/2014

TITLE: air bearing housing

SCALE: 1:1

Revision: 3



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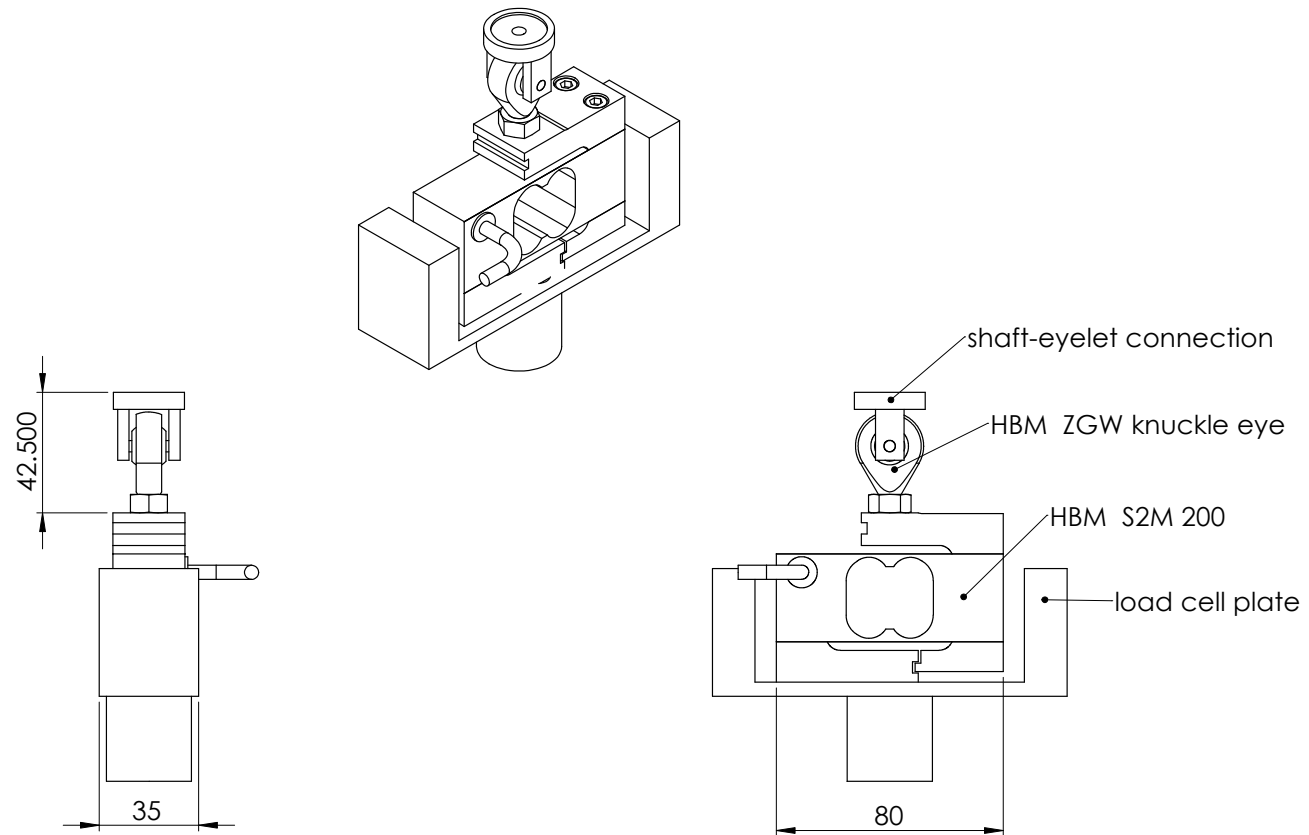
DATE:14/05/2015

TITLE: top_bearing_assembly

SCALE:1:2

Revision: 1

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Load cell and knuckle eye taken directly from HBM CAD database

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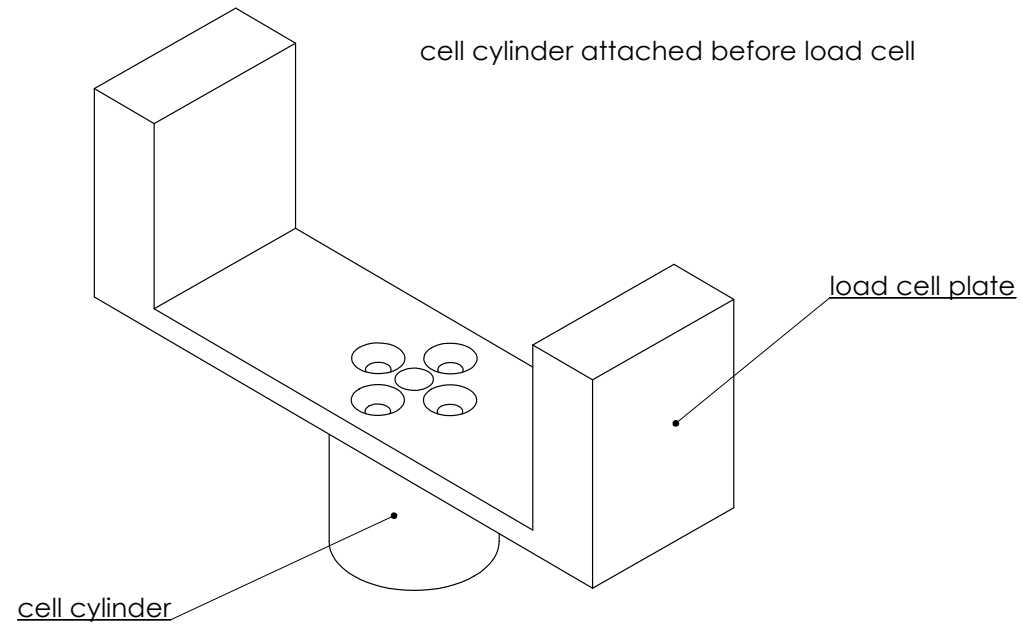
DRAWING BY: MAB

DATE:29/09/2014

TITLE: load cell assembly

SCALE:1:2

Revision: 1



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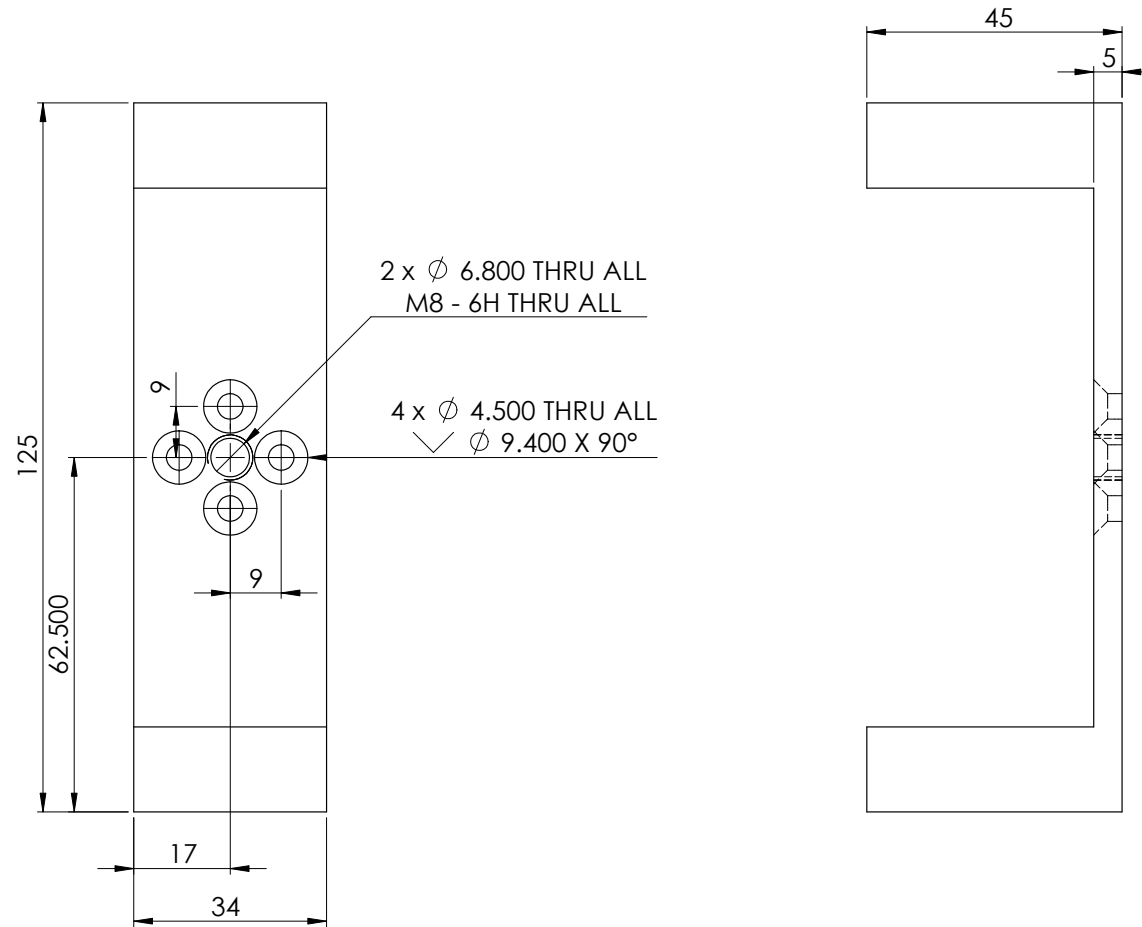
DATE: 29/09/2014

TITLE: load_cell_connection

SCALE:1:1

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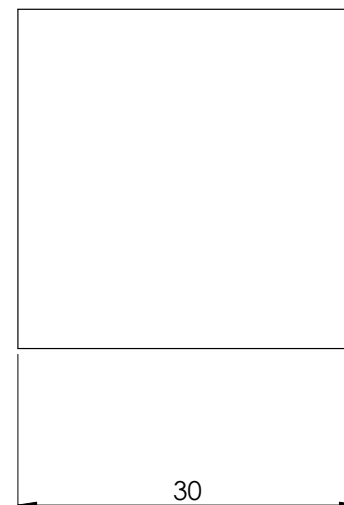
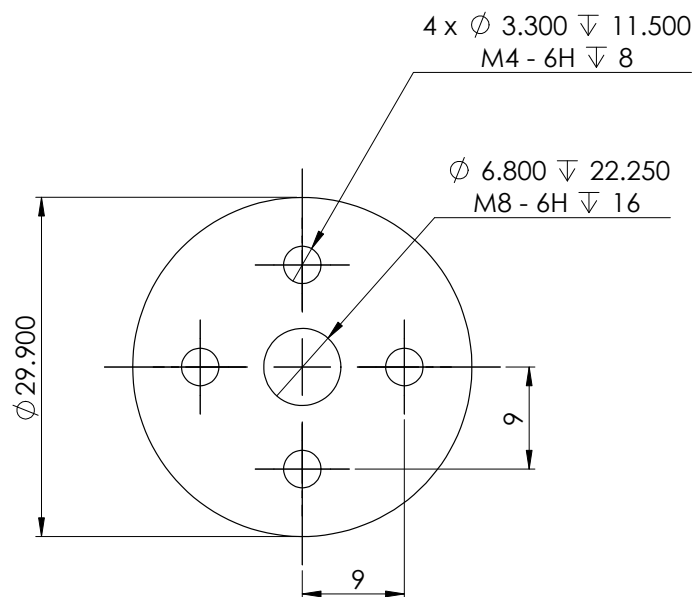
DRAWING BY: MAB

DATE: 11/04/2014

TITLE: load_cell_plate

SCALE:1:1

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DATE:04/11/2014

TITLE: cell_cylinder

SCALE:2:1

Revision: 2

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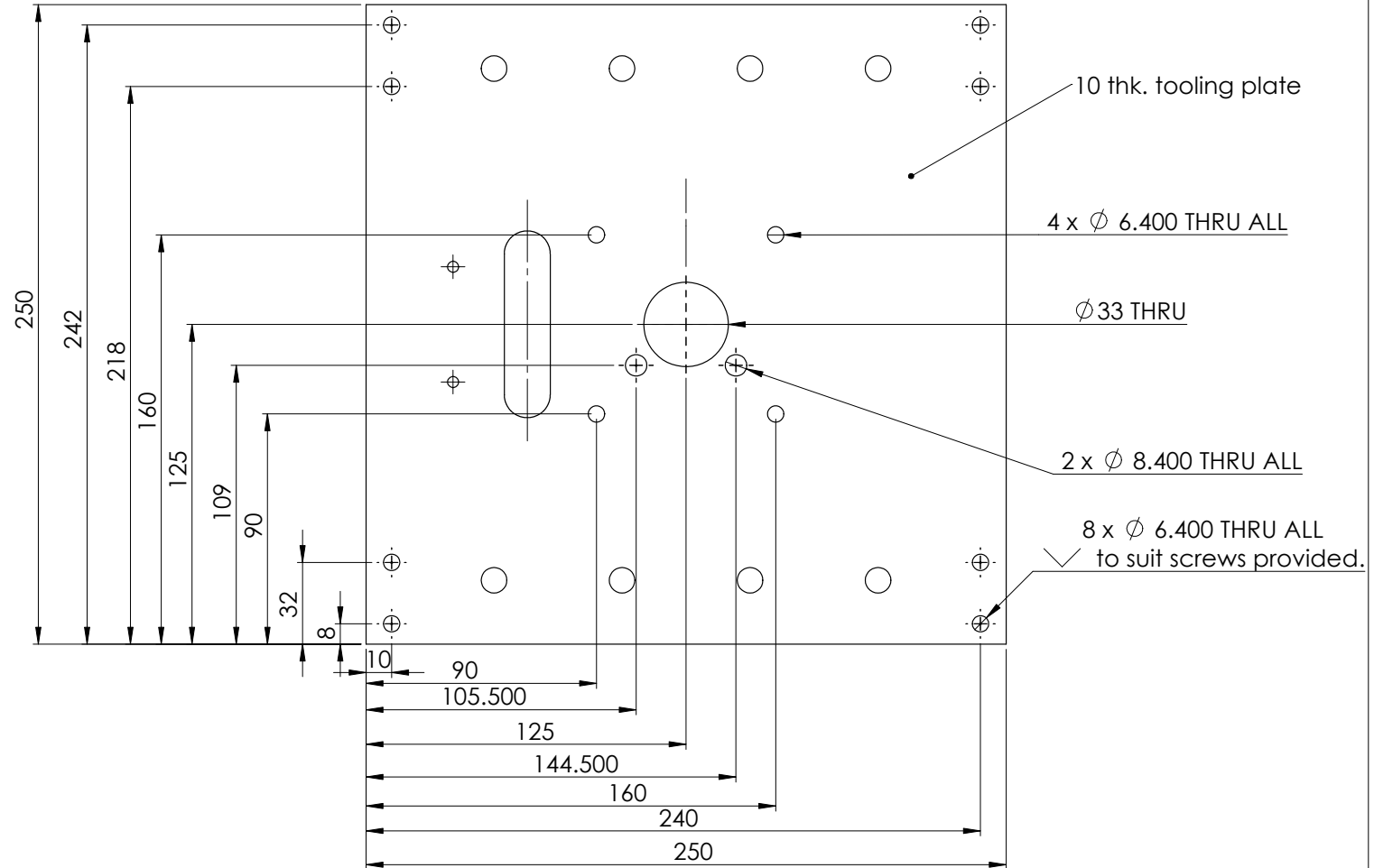
TITLE: top_support

SCALE:1:5

Revision: 1

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DATE: 04/12/2014

TITLE: bottom_plate sheet 1 of 2

SCALE: 1:2

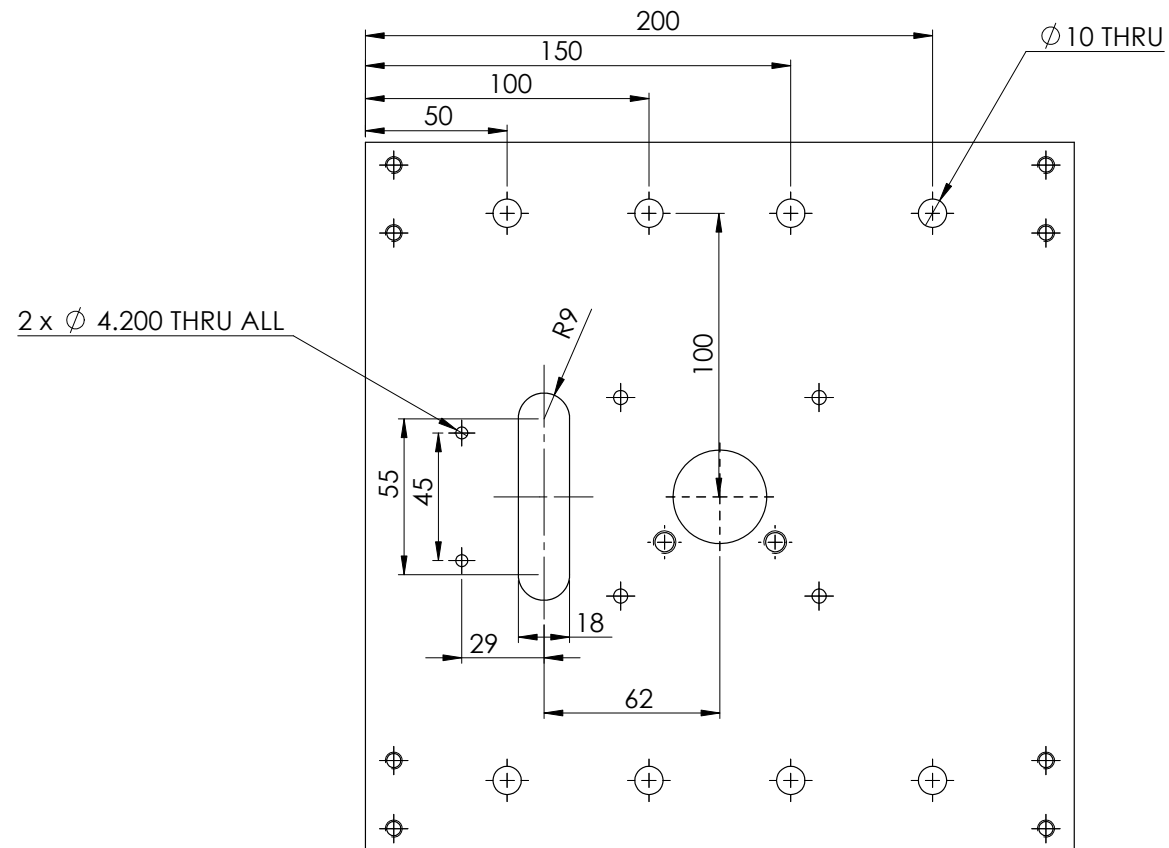
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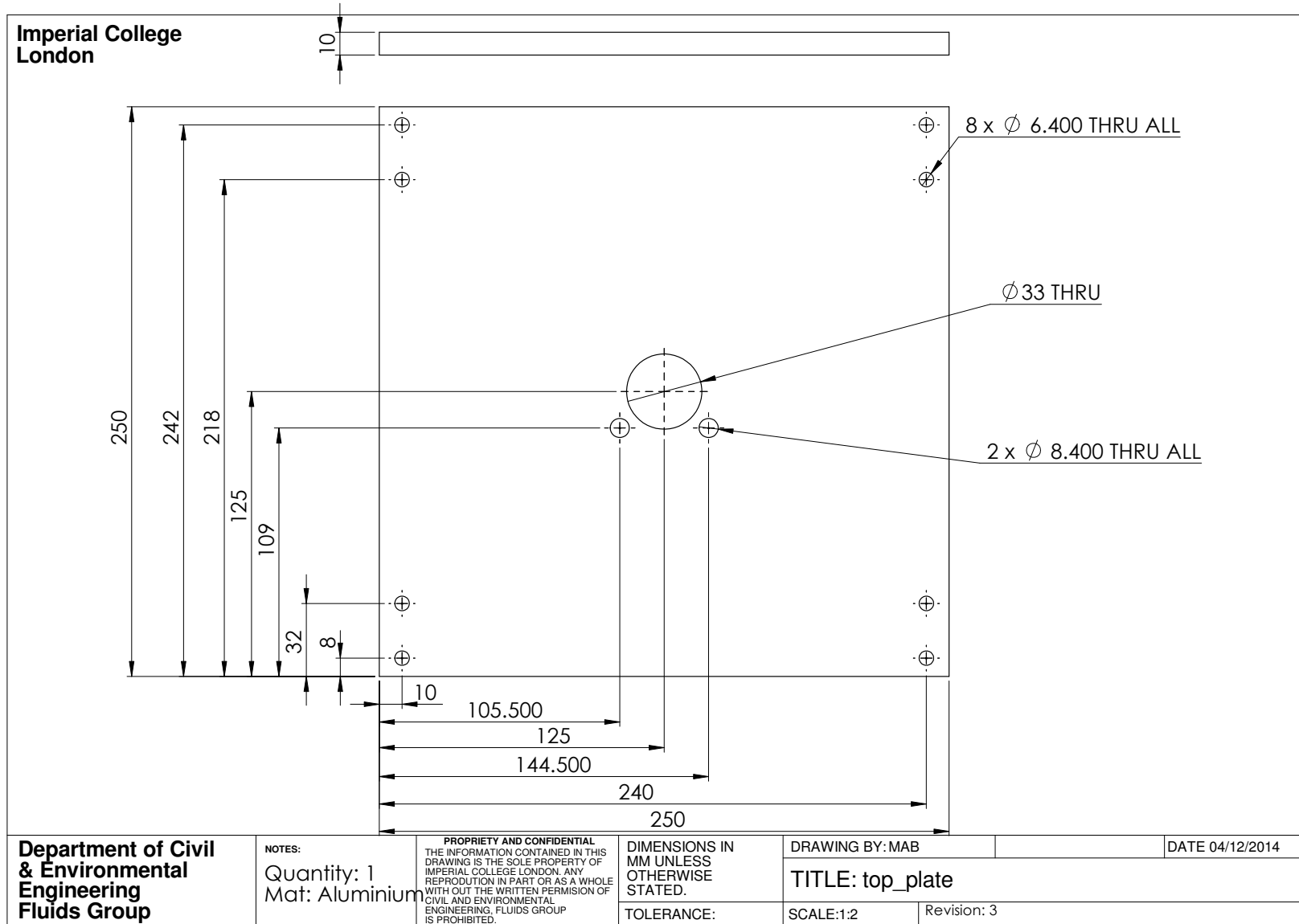
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TITLE: bottom_plate sheet 2 of 2

Revision: 1

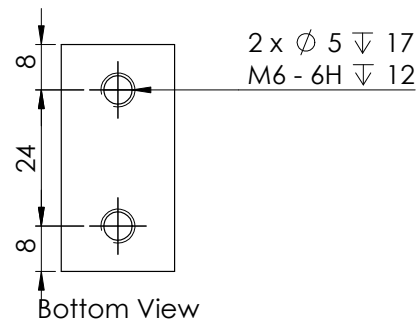




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Top View



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DATE: 09/02/2015

TITLE: inter_plate_columns

SCALE:1:1

Revision: 2

D

References used for geometry selection exercise

References used to generate geometrical study

Author	Year	Source	Shape	Dia	Draught	Depth
Vantorre et al	2004	Applied Ocean Research	coned cylinder	5	1.0, 2.0	-
Budal et al	1982	2 nd International Symposium on Wave Energy Utilization	sphere	10	10	40
Waters et al	2007	Applied Physics Letters	cylinder	3	0.4	25
Eriksson et al	2005	International Journal of Engineering Science	cylinder	2-10	1.5	23
Elwood et al	2009	Proc. 28th International Conference on Ocean Offshore and Arctic engineering	spar with annular shaped buoy	7	3.5?	-
Weinstein et al	2004	Proc. MTTS/IEEE Techno-Ocean 04 Conference	cylinder	4.5-6	2.5	45-76
Weber et al	2009	Proc 8 th European wave and tidal energy conference	hemisphere	5	5	-
PowerBuoy Mk3	-	website	saucer	11	~2	55
PowerBuoy Autonomous	-	website	cylinder	2.7	9.3	43
Nolan et al	2005	Proc. 15 th International Offshore and Polar engineering conference	cylinder	3.3	3.1	-
Bjarte-Larsson, Falnes	2006	Ocean Engineering	domed cylinder	3.5	4	18
Greenhow, White	1997	Applied Ocean Research	hemisphere	4	4	30
			cylinder	4	1.33	30
			cylinder	3	1.5	30

Berggren, Johansson	1992	Applied Ocean Research	cylinder	20	10	40
Babarit et al	2012	Renewable energy	saucer	3	0.6	40-100
			cylinder	7	5	70
			saucer	20	2	100
			hemisphere	5	5	13
			coned cylinder	8	13.4	70
Eriksson et al	2006	IEEE Journal of Oceanic Engineering	cylinder	3	0.25	25
Skjervhieux et al	2008	2 nd International Conference on Ocean Energy	cylinder	4.5	0.18	-
Sarlak et al	2010	Journal of Marine Engineering	cylinder	0.4	0.1	1.3
			hemisphere	0.8	0.1	1.3
Ricci et al	2009	Proc. 8 th European Wave and Tidal Energy Conference	cylinder	10	5	-
Korde	1999	Applied Ocean Research	hemisphere	2	2	-
Leonard et al	1993	Proc 3 rd International Offshore and Polar Engineering Conference	saucer	3	0.9	160
			prolate spheroid	0.9	7.6	91
Halls et al	2011	Journal of Offshore Mechanics and Arctic Engineering	sphere	10	10	-
Eidsmoen	1995	Applied Ocean Research	cylinder	7	7	70
Mei	2012	Philosophical Transactions of the Royal Society A	cylinder	5	2.5	15
			cylinder	8	4	15
			cylinder	13	6.5	15
Gore	2006	MSc Dissertation	cylinder	4.5	10	120
Engstrom	2011	PhD thesis	cylinder	3	0.4	25
			cylinder	6	0.72	25