

SECONDARY MOTION AND DIFFUSION IN
UNSTEADY FLOW IN A CURVED PIPE

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ABSTRACT

The work in this thesis describes the fully developed velocity field generated in a curved pipe of circular cross-section when the pressure gradient along the pipe is any periodic function of time. The mean pressure gradient is chosen so that the interaction between the secondary motion in exactly steady flow and the oppositely directed steady streaming characteristic of high frequency oscillatory flow can be examined. When the Reynolds number for the steady streaming, R_s , is much larger than the Dean number, D , associated with the mean flow, the secondary motion is independent of the flow through the pipe, and an asymptotic theory is developed which shows that, away from the walls of the pipe, the mean axial velocity is proportional to the stream function describing the secondary flow. A numerical solution of the governing equations is obtained to verify the asymptotic results and also, to provide solutions for arbitrary values of D and R_s . The significant feature of the numerical results is the prediction of a range of values for the parameters where the solution to the equations is not unique. The velocity fields obtained are used to determine the influence of the secondary motion on the heat or mass transfer in unsteady flow in an attempt to explain the enhancement of oxygen transfer in respirators using pulsatile blood flow.

Finally, it is shown that for large enough frequencies, a purely oscillatory pressure gradient containing even harmonics of the fundamental frequency can produce a small mean axial flow.

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CHAPTER I

Introduction

The generation of a steady flow in a viscous fluid by a purely oscillatory external constraint is well known and has been extensively analysed. Typically the problem studied is the flow induced when a rigid body oscillates in an unbounded fluid; this flow is the subject of a review article by Riley (1967). Apart from this prototype flow, there are other instances where the interaction of harmonic components in an oscillatory flow leads to steady motions, and some of these are reviewed by Riley (1975). Of particular interest here is the oscillatory flow of a viscous liquid in a rigid curved pipe studied independently by Lyne (1970) and Zalosh and Nelson (1973) who predicted the existence of a steady flow in the plane of the cross-section of the pipe.

The assumption of oscillatory flow is equally restricting as the hypothesis of exactly steady flow, as in general, most of the situations of practical interest are unsteady. Clearly the analysis of such flows is complicated by the interaction of effects characteristic of oscillatory flows, and phenomena associated with the mean motion. It is these interactions, in the particular case of unsteady viscous flow in a curved pipe, which are studied in this thesis.

The properties of steady flow in a curved pipe are well known, the first analytical description of the motion being made by Dean (1927, 1928). Using numerical techniques the work of Dean has subsequently been extended by McConalogue and Srivastava (1968) and Collins and Dennis (1975). The notable feature of the flow is that, due to the curvature of the pipe, a motion in the plane of the cross-section of the pipe is established. This so called secondary motion takes the form of two vortices, one on each side of the axis of symmetry of the cross-section, with circulation such that particles near the axis of symmetry at the inside of the pipe bend drift across the pipe to the outside of the bend. The more recent work of Lyne and Zalosh and Nelson (1973) for oscillatory flow has shown that for sufficiently large

frequencies of oscillation, the secondary motion, which again consists of two symmetrically placed vortices, has circulation in the opposite sense to that found in exactly steady flow. For unsteady flow in a curved pipe we can expect some combination of these two effects.

Though the secondary velocities generated in both oscillatory and steady flow are generally very much smaller than the axial velocities, they vitally affect transport processes between the fluid and the pipe wall. The shear stress distribution around the wall of the pipe is largely controlled by the strength and direction of the secondary motion. In this context the work described here may be of relevance to the flow of blood in the arch of the aorta, in that it gives some indication of the magnitude and distribution of wall shear stresses. This will be of use when correlating arterial disease with physical factors, as now there is increasing evidence that mechanical forces are intimately connected with localization, and perhaps initiation, of various arterial disorders (Atherogenesis: Initiating factors. Ciba Foundation Symposium 12. 1973). Also the secondary motions in unsteady flow in curved pipes are important in the design of extracorporeal respirators. Experimental work by Melrose et al (1972) and Tanishita et al (1976) has shown the feasibility of utilizing such phenomena in improving the performance of blood oxygenators.

Experimental work has also been carried out to assess the theoretical results for oscillatory flow in a curved pipe. As part of his study of the problem, Lyne constructed an experiment which demonstrated qualitatively the existence of the steady inward secondary motion. More refined experimental work by Bertelsen (1974) enabled a quantitative comparison between theory and experiment to be made. His results demonstrated conclusively that the predictions of Lyne and Zalosh and Nelson (1973) were correct, and in particular, showed that the results from Lyne's analysis agreed better with experimental values than did the results of Zalosh and Nelson.

In order to obtain similar magnitudes for the outward

and inward secondary motions, it proved necessary to consider the unsteady flow as a perturbation of purely oscillatory flow. As the method of matched asymptotic expansions used by Lyne illustrates more clearly the physical structure of the flow, as well as giving the better agreement with experimental results, it is this technique we shall apply to the case of unsteady flow. The small parameter of the analysis is β , the thickness of the Stokes layers on the pipe wall compared to the radius of the pipe. The amplitude of the leading order oscillatory axial velocity is $\bar{W}$, and thus we seek a solution where the steady axial velocity is $O(\beta\bar{W})$, a small perturbation of the flow discussed by Lyne. The physical arguments in Chapter 2 show how this can be achieved, and also lead to the definition of two additional parameters.

The first of these parameters is the steady streaming Reynolds number, R_s , based on the steady velocities generated in the Stokes layer, and as shown by Lyne, R_s characterizes the steady streaming associated with purely oscillatory flow. The other parameter required is the Dean number, D , which is the quantity governing exactly steady flow through curved pipes. Finally the full Navier-Stokes and continuity equations are simplified by making the approximation $\frac{a}{R} \ll 1$, where R is the radius of curvature of the pipe, and a is the radius of the pipe cross-section.

In Chapter 3 the limit $\beta \rightarrow 0$ is taken and the method of matched asymptotic expansion used to solve for the flow in the Stokes layer, and to derive equations governing the motion in the outer region. As in the purely oscillatory flow, the leading order secondary flow in the outer region is steady, but due to the presence of a mean axial velocity, the motion is no longer governed by the 2-dimensional Navier-Stokes equations, but is coupled with the mean velocity to form the equations for exactly steady flow in a curved pipe. The boundary conditions are provided by matching with the flow in the Stokes layer, and the secondary flow is required to have a non-zero value at the edge of the outer region, just as in purely oscillatory flow.

The analysis of Chapter 3 is carried out for any periodic

pressure gradient which has a Fourier series representation. To $O(\beta^0)$ for the flow in the cross-section and $O(\beta)$ for the axial velocity, this is just a generalization of the results of Lyne for a sinusoidal pressure gradient. However at $O(\beta^2)$ in the axial velocity it is discovered that a steady velocity is generated from interaction of the harmonic components of the secondary flow and axial velocity, and therefore this steady velocity is independent of the mean pressure gradient. It is found that this steady axial velocity does not decay at the edge of the Stokes layer, but persists into the outer region. Chapter 9 takes up the analysis of this axial steady streaming in the outer region when the mean pressure gradient is zero, and we find that for both small and large R_s there is a non-zero flux of fluid through the pipe. Thus we have the unusual result that a purely oscillatory pressure gradient acting in a curved pipe can produce a mean flow!

Analytical solutions for the leading order secondary motion and mean axial flow are obtained in Chapters 4 and 5 for the cases of creeping flow and small mean flow respectively. Both of these conditions restrict the Dean number to be small in some sense. In Chapter 5 this restriction allows the flow in the cross-section to be independent of the axial velocity, and hence the solutions of Lyne for the secondary motion are valid. The axial flow is then given by a linear second order partial differential equation with the secondary flow velocities appearing as coefficients. For large values of R_s the secondary flow has an inviscid core/boundary layer structure, and by postulating a similar form for the axial velocity it is found that the axial velocity is constant on the streamlines of the secondary flow. An integral condition arising from the requirement that the solution be single-valued in the cross-section leads to the result that the axial velocity is linearly related to the stream function in the inviscid region.

The integral condition used in Chapter 5 to determine the axial velocity distribution is rederived in Chapter 6 by less formal arguments. A flow which consists of an axial velocity superimposed on a two-dimensional flow is considered,

and an exact integral condition on the flow obtained by integrating the equations of motion along a streamline. This integral condition is then used in conjunction with the inviscid equations of motion to determine the velocity distribution in the region of small viscous forces, and thereby extending the work of Batchelor (1956) to a flow without closed streamlines.

For arbitrary values of D and R_s , numerical methods were used to obtain solutions for the secondary motion and mean flow, particular attention being given to the parameter range where the secondary flow changes direction. Chapter 7 describes the development and testing of the program, based on the method of McConalogue and Srivastava (1968), used to verify the asymptotic results of Chapter 5, and its extension to obtain solutions for arbitrary D and R_s . The striking feature of these calculations is the prediction of regions in the $D - R_s$ plane where the governing equations and boundary condition have non-unique solutions.

In Chapter 8 the analytical results of Chapters 4 and 5, and the numerical results of Chapter 7 are brought together. The asymptotic results of Chapter 5 are shown to agree well with the numerical solutions at $R_s = 1800$; the numerical results also show the development, from small R_s , of the asymptotic state as R_s increases. The creeping flow solutions of Chapter 4 are used to illustrate the basic sequence of events in the transition from inward to outward secondary flow when non-linear interactions are weak. For $R_s = 100$ the character of the transition has changed markedly and it is postulated that the strong interaction of the two secondary motions is responsible for the multiple solutions obtained for certain values of D .

In Chapter 10 the heat transfer in unsteady flow in a curved pipe is studied when momentum and thermal Stokes layers are thin. The limit processes used in the momentum equations force the Peclet number for the heat transfer to be large and so heat transfer in steady flow in straight pipes is studied to illustrate the singular nature of the large Peclet number limit. The analysis parallels that of Van Dyke (1971) for entry flow in a channel. Based on the expansion scheme for

large Peclet number in steady flow, a similar scheme is employed to derive equations governing the temperature distribution in unsteady flow. It is shown that apart from a small region $O(\beta)$ around the step change in boundary conditions the temperature is steady and determined by the steady components of velocity. Using the velocity distribution of Chapter 5, we solve the resultant energy equations and calculate the increase in heat transfer over steady flow in a straight pipe. This is done in Chapter 11 as a model for the oxygenator experiments of Melrose et al (1972).

CHAPTER 2

The equations of motion

We consider the motion of a viscous incompressible fluid in a pipe, of radius a , whose centre line lies on a circle, of radius R and centre O , in the horizontal plane Oxy (see Fig. 2.1). The co-ordinates of an arbitrary point P inside the pipe are (r^*, α, θ) . $R\theta$ is the distance along the centre line from an arbitrary initial line through O , say Oy , to the plane of the cross-section OzP ; r^* and α are polar co-ordinates in the cross-section.

The velocity vector $\underline{u}$ has components (u^*, v^*, w^*) corresponding to the co-ordinate directions (r^*, α, θ) . It is assumed that the flow is fully developed so that $\underline{u}$ is independent of θ . The superscript $*$ is used to denote a dimensional variable.

In these co-ordinates the equations of motion are (Lyne 1970)

$$\begin{aligned} \frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial r^*} + \frac{v^*}{r^*} \frac{\partial u^*}{\partial \alpha} - \frac{v^{*2}}{r^*} - \frac{w^{*2} \cos \alpha}{R + r^* \cos \alpha} \\ = - \frac{\partial}{\partial r^*} \left(\frac{p^*}{\rho} \right) - \nu \left(\frac{1}{r^*} \frac{\partial}{\partial \alpha} - \frac{\sin \alpha}{R + r^* \cos \alpha} \right) \left(\frac{\partial v^*}{\partial r^*} + \frac{v^*}{r^*} - \frac{1}{r^*} \frac{\partial u^*}{\partial \alpha} \right), \end{aligned} \quad (2.1)$$

$$\begin{aligned} \frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial r^*} + \frac{v^*}{r^*} \frac{\partial v^*}{\partial \alpha} + \frac{u^* v^*}{r^*} + \frac{w^{*2} \sin \alpha}{R + r^* \cos \alpha} \\ = - \frac{1}{r^*} \frac{\partial}{\partial \alpha} \left(\frac{p^*}{\rho} \right) + \nu \left(\frac{\partial}{\partial r^*} + \frac{\cos \alpha}{R + r^* \cos \alpha} \right) \left(\frac{\partial v^*}{\partial r^*} + \frac{v^*}{r^*} - \frac{1}{r^*} \frac{\partial u^*}{\partial \alpha} \right) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial r^*} + \frac{v^*}{r^*} \frac{\partial w^*}{\partial \alpha} + \frac{u^* w^* \cos \alpha}{R + r^* \cos \alpha} - \frac{v^* w^* \sin \alpha}{R + r^* \cos \alpha} \\ = - \frac{1}{R + r^* \cos \alpha} \frac{\partial}{\partial \theta} \left(\frac{p^*}{\rho} \right) + \nu \left[\left(\frac{\partial}{\partial r^*} + \frac{1}{r^*} \right) \left(\frac{\partial w^*}{\partial r^*} + \frac{w^* \cos \alpha}{R + r^* \cos \alpha} \right) \right. \\ \left. + \frac{1}{r^*} \frac{\partial}{\partial \alpha} \left(\frac{1}{r^*} \frac{\partial w^*}{\partial \alpha} - \frac{w^* \sin \alpha}{R + r^* \cos \alpha} \right) \right]. \end{aligned} \quad (2.3)$$

The kinematic viscosity and density of the fluid are denoted by ν and ρ respectively, p^* denotes the pressure and t^* denotes time. The equation of continuity is

$$\frac{\partial u^*}{\partial r^*} + \frac{u^*}{r^*} + \frac{u^* \cos \alpha}{R + r^* \cos \alpha} + \frac{1}{r^*} \frac{\partial v^*}{\partial \alpha} - \frac{v^* \sin \alpha}{R + r^* \cos \alpha} = 0. \quad (2.4)$$

The assumption of fully developed flow requires the pressure to have the form

$$p^*(r^*, \alpha, \theta, t^*) = \theta A(t^*) + B(r^*, \alpha, t^*),$$

where $A(t^*)$ is proportional to the imposed pressure gradient along the pipe. We restrict $A(t^*)$ to be any period function of time and specifically assume that the pressure gradient can be Fourier analysed in time. Along the centre line of the pipe we assume that the pressure gradient can be written as

$$-\frac{1}{R} \frac{\partial}{\partial \theta} \left(\frac{p^*}{\rho} \right) = G + \bar{W} \omega \sum_{n=1}^{\infty} a_n \cos(n\omega t^* + \phi_n), \quad (2.5)$$

with $a_1 = 1$ and $\phi_1 = 0$,

where G is the (constant) mean pressure gradient and ω is the fundamental (angular) frequency of the oscillations. The Fourier coefficients, a_n are dimensionless as $\bar{W}$ has the units of velocity.

The size of the steady axial flow produced by the mean pressure gradient is determined by balancing the pressure drop along the pipe with the wall shear stress. This gives

$$\nu \alpha \frac{\partial w^*}{\partial r^*} \sim G \alpha^2$$

and if it is assumed that velocity gradients for the mean flow are $O(1/\alpha)$, then the mean axial velocity is $O(\frac{G\alpha^2}{\nu})$.

The order of magnitude of the oscillatory component of the axial velocity depends crucially on the ratio of the convective time ω^{-1} to the diffusive time α^2/ν . When this ratio is large, diffusion of momentum determines the flow and the oscillatory pressure gradient is balanced by the viscous term, resulting in a quasi steady velocity field. The case of interest though is when this ratio is small and the diffusion of momentum confined to a thin region $O(\sqrt{\nu/\omega})$ near to the wall of the pipe. If the velocity in the plane of the cross-section is small, then away from the walls the oscillatory

pressure gradient is balanced by the local acceleration resulting in an axial velocity of $O(\bar{W} \sum \frac{a_n}{n} \sin(n\omega t^2 + \phi_n))$ which is assumed to be $O(\bar{W})$.

The magnitude of the axial velocity can be seen more clearly by comparing the mean flow to the amplitude of the velocity fluctuations. This ratio is

$$\frac{Ga^2}{\nu} \cdot \frac{1}{\bar{W}} = \left[\frac{Ga^3}{\nu^2} \left(\frac{a}{R} \right)^{1/2} \right] \cdot \left[\frac{\bar{W}^2}{R\omega} \cdot \frac{a}{\nu} \right]^{-1/2} \cdot \left[\frac{\nu}{\omega a^2} \right]^{1/2} \quad (2.6)$$

The last term is recognized as the thickness of the viscous Stokes layer compared to the radius of the pipe, a parameter which is assumed to be small. Thus for order one values of the other two parameters the mean flow is a small perturbation of the basic oscillatory velocity. The structure of the oscillatory flow here is a generalization of that considered by Lyne (1970), and so the unsteady flow described above may be viewed as a perturbation of the solution for purely oscillatory flow.

The first parameter on the right hand side of (2.6) is proportional to the Dean number D for steady flow in curved pipes which is defined as

$$D = \frac{Ga^3}{4\nu^2} \left(\frac{a}{2R} \right)^{1/2} \quad (2.7)$$

This form of the Dean number was first introduced by McConalogue and Srivastava (1968) but we note that their Dean number is a factor of 8 larger than the Dean number defined by (2.7). For small values of D the axial velocity and secondary motion are dominated by viscous forces; for large D there is the possibility that the motion has a boundary layer/inviscid core structure, though this has not yet been properly clarified (Smith 1976).

In purely oscillatory flow the character of the secondary motion is determined by the steady streaming Reynolds number R_s defined by

$$R_s = \frac{\bar{W}^2}{R\omega} \cdot \frac{a}{\nu} \quad (2.8)$$

The relevance of this parameter to the problem was shown by

Lyne. The remaining term in (2.6) is then seen to be $R_s^{-1/2}$. The oscillatory axial velocity is

$$\frac{R\bar{W}}{R+r^2\cos\alpha} \sum_1^{\infty} \frac{\alpha_n}{n} \sin(n\omega t^* + \phi_n)$$

away from the walls of the pipe. This form for the axial velocity generates the centripetal pressure gradients necessary to keep fluid particles in circular paths centred on Oz. However the no slip boundary condition at the pipe wall requires the flow in the viscous region adjacent to the wall to be the sum of the Stokes shear wave corresponding to each component of the oscillatory axial velocity. As this viscous region is assumed to be thin, the pressures in the layer remain the same as in the inviscid flow. Thus the centripetal pressure gradients are no longer in balance with the smaller axial velocities, so a secondary flow in the plane of the cross-section is generated. The magnitude of these secondary velocities is given by balancing the centripetal pressure gradient term with the viscous terms, taking the velocity gradients (for the oscillatory components) to be $O(\frac{1}{\alpha}(\frac{\omega}{N})^{1/2})$ in the viscous Stokes layer. This leads to

$$v^* \sim O\left(\frac{\bar{W}^2}{R\omega}\right)$$

and

$$u^* \sim O\left(\frac{\bar{W}^2}{R\omega} \cdot \frac{1}{\alpha} \left(\frac{N}{\omega}\right)^{1/2}\right).$$

Provided the velocity gradients of the steady flow remain $O(\frac{1}{\alpha})$ in the Stokes layer, both of these components of the secondary motion are larger than the mean flow, and it is reasonable to consider the oscillatory velocity independently of the steady flow close to the wall.

To leading order then, the oscillatory flow is the same as considered by Lyne: the self interaction of the oscillatory velocity components produces a steady secondary motion in the Stokes layer which does not decay at the edge of the layer, but persists into the inviscid region. The secondary flow in the inviscid region is then $O(\frac{\bar{W}^2}{R\omega})$, which Lyne showed was small compared to the axial velocity away from the viscous

layer, and such that particles near the outside of the bend drift across the pipe to the inside of the bend.

The mean velocity distribution will not, in general, produce the centripetal pressure gradients necessary to maintain the fluid particles in uniform circular motion about the axis Oz. Consequently a motion in the plane of the cross-section of the pipe will be generated. For exactly steady flow in a curved pipe the secondary flow field is well known, and as a first estimate the secondary velocities due to the mean flow in an unsteady flow may be taken to be the same size as those in the corresponding steady flow. For small Dean number the secondary flow is $O(\frac{\omega}{a} D^2)$ while for large Dean number the secondary velocities are believed to be $O(\frac{\omega}{a} D^{2/3})$ away from the wall and $O(\frac{\omega}{a} D^{2/3})$ in the boundary layer on the pipe wall (Smith 1976). In both cases the secondary flow is directed oppositely to the secondary motion in purely oscillatory flow.

At this point it is useful to compare the magnitude of the secondary motions induced when the axial velocity is either purely oscillatory or exactly steady. In the former the steady streaming is $O(\frac{\bar{w}^2}{R\omega})$ while for steady flow in a curved pipe the secondary velocities can be written as $O(\frac{\omega}{a} D^n)$. The ratio of these quantities is $O(R, D^n)$. For small values of this ratio outward centrifuging associated with the mean flow would be expected, but when $R, \gg D^n$ then inward steady streaming would dominate. It is hoped that for $O(1)$ values of the ratio, the interaction of these two effects and the transition between the two extreme cases can be studied.

For the purpose of making the system of equations (2.1 to 2.4) non-dimensional we adopt the view that the unsteady flow is a perturbation of the purely oscillatory flow and introduce the non-dimensional quantities defined by Lyne (1970) for the oscillatory flow:

$$\left. \begin{aligned} r &= r^*/a, & w &= w^*/\bar{w}, & \tau &= \omega t^* \\ u &= u^*/(\frac{\bar{w}^2}{R\omega}), & v &= v^*/(\frac{\bar{w}^2}{R\omega}). \end{aligned} \right] \quad (2.9)$$

The equation of continuity becomes

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{u \delta \cos \alpha}{1 + \delta r \cos \alpha} + \frac{1}{r} \frac{\partial v}{\partial \alpha} - \frac{v \delta \sin \alpha}{1 + \delta r \cos \alpha} = 0. \quad (2.10)$$

The parameter δ is the curvature ratio of the pipe defined by

$$\delta = \frac{a}{R} , \quad (2.11)$$

and throughout this thesis it will be assumed that $\delta \ll 1$ so that (2.10) simplifies to

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \alpha} = 0 .$$

This simplified equation of continuity can be satisfied by introducing a stream function χ such that

$$u = \frac{1}{r} \frac{\partial \chi}{\partial \alpha} \quad \text{and} \quad v = - \frac{\partial \chi}{\partial r} \quad (2.12)$$

Eliminating the pressure between (2.1) and (2.2) and neglecting terms $O(\delta)$ we obtain the equation for the flow in the cross-section:

$$\frac{\partial}{\partial \tau} \nabla^2 \chi - \frac{R_s \beta^2}{2r} \frac{\partial (\chi, \nabla^2 \chi)}{\partial (r, \alpha)} - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) w^2 = \frac{1}{2} \beta^2 \nabla^4 \chi , \quad (2.13)$$

$$\text{where } \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} \quad \text{and} \quad \frac{\partial(a, b)}{\partial (r, \alpha)} = \frac{\partial a}{\partial r} \frac{\partial b}{\partial \alpha} - \frac{\partial a}{\partial \alpha} \frac{\partial b}{\partial r} . \quad (2.14)$$

Neglecting terms $O(\delta)$ in (2.3) and using (2.5), the axial momentum equation becomes

$$\frac{\partial w}{\partial \tau} - \frac{R_s \beta^2}{2r} \frac{\partial (\chi, w)}{\partial (r, \alpha)} = \sum_{n=1}^{\infty} a_n \cos(n\tau + \phi_n) + \frac{20\beta^3}{\sqrt{R_s}} + \frac{1}{2} \beta^2 \nabla^2 w . \quad (2.15)$$

The parameters D and R_s are defined in (2.7) and (2.8) respectively and β is given by

$$\beta = \left(\frac{2N}{\omega \alpha^2} \right)^{1/2} \quad (2.16)$$

Clearly β is the ratio of the Stokes layer thickness to the radius of the pipe, and as mentioned, we shall look for a solution when this ratio tends to zero.

The boundary conditions for (2.13) and (2.15) are those of no slip at the pipe wall, i.e.

$$\chi = \frac{\partial \chi}{\partial r} = w = 0 \quad \text{on } r = 1 , \quad (2.17)$$

together with the condition that the solution is regular within the pipe.

Finally it is emphasized that the equations (2.13) and (2.15) have been obtained from (2.1)-(2.4) via the limiting

process

limit $\delta \rightarrow 0$; D, R_s, β held fixed.

When dealing with the energy equation in Chapter 10 it will appear that this limit requires some special attention.

CHAPTER 3

The limit $\beta \rightarrow 0$

In this chapter, solutions to (2.13) and (2.15) are obtained in the limit $\beta \rightarrow 0$, R_s , D fixed. Physically this limit may be approximated by taking the frequency of the oscillations to be very large, and therefore this limit will often be referred to as the 'high frequency' case.

It is obvious that the limit $\beta \rightarrow 0$, R_s , D fixed is not the only possible limit and independently Smith (1975) has considered other limits of these equations. By taking $R_s \sim O(1)$ and letting one of the two parameters D, β take a small or large value, Smith constructs a series of asymptotic expansions which show the transition between the extreme cases of steady and oscillatory flow as the parameters D, β vary. He also arrives at the result that the change from inward to outward secondary motion occurs in the high frequency case. Thus the problem considered here is a close examination of the solutions in this high frequency limit and the determination of the details of the change in direction of the secondary motion.

Following the analysis of Lyne (1970) the axial velocity and stream function are expanded in series of ascending powers of the small parameter β . It has been shown in Chapter 2, that to leading order in β , the flow has an inviscid (outer) region surrounded by a viscous Stokes layer (inner region). Both w and χ are represented by different expansions in these two regions, but it is assumed that the different series have a common region of validity where the viscous region merges into the inviscid flow. The process of requiring the two solutions to coincide in this overlap region is called matching.

In the Stokes layer the relevant radial co-ordinate is the magnified distance from the wall

$$\eta = \frac{1-r}{\beta} \quad (3.1)$$

and the expansions for the axial velocity and stream function

$$\text{are: } w(r, \alpha, \tau; \beta, R_s, D) \sim W_0(\eta, \alpha, \tau; R_s, D) + \beta W_1 + \beta^2 W_2 + \dots \quad (3.2)$$

$$\text{and } \chi(r, \alpha, \tau; \beta, R_s, D) \sim \beta \Psi_0(\eta, \alpha, \tau; R_s, D) + \beta^2 \Psi_1 + \dots \quad (3.3)$$

The boundary conditions (2.17) on $r = 1$ become

$$W_i = \Psi_i = \frac{\partial \Psi_i}{\partial \eta} = 0, \quad i = 0, 1, 2, \dots \text{ on } \eta = 0. \quad (3.4)$$

In the outer region the axial velocity is expanded as

$$w(r, \alpha, \tau; \beta, R_s, D) \sim w_0(r, \alpha, \tau; R_s, D) + \beta w_1 + \beta^2 w_2 + \dots \quad (3.5)$$

and we require that this match with the solution for w in the inner region. Assuming that any secondary motion in the outer region is the same order as the tangential velocities generated in the Stokes layer we obtain the expansion

$$\chi(r, \alpha, \tau; \beta, R_s, D) \sim \chi_0(r, \alpha, \tau; R_s, D) + \beta \chi_1 + \beta^2 \chi_2 + \dots \quad (3.6)$$

for the stream function in the outer region.

Substituting (3.5) and (3.6) into the axial momentum equation (2.15) and equating like powers of β we derive the equations:

$$\frac{\partial w_0}{\partial \tau} = \sum_1 a_n \cos(n\tau + \phi_n), \quad (3.7)$$

$$\frac{\partial w_1}{\partial \tau} = 0, \quad (3.8)$$

$$\frac{\partial w_2}{\partial \tau} - \frac{R_s}{2r} \frac{\partial(\chi_0, w_0)}{\partial(r, \alpha)} = \frac{1}{2} \nabla^2 w_0, \quad (3.9)$$

$$\frac{\partial w_3}{\partial \tau} - \frac{R_s}{2r} \left[\frac{\partial(\chi_0, w_1)}{\partial(r, \alpha)} + \frac{\partial(\chi_1, w_0)}{\partial(r, \alpha)} \right] = \frac{2D}{\sqrt{R_s}} + \frac{1}{2} \nabla^2 w_1, \quad (3.10)$$

$$\text{and } \frac{\partial w_4}{\partial \tau} - \frac{R_s}{2r} \left[\frac{\partial(\chi_0, w_2)}{\partial(r, \alpha)} + \frac{\partial(\chi_1, w_1)}{\partial(r, \alpha)} + \frac{\partial(\chi_2, w_0)}{\partial(r, \alpha)} \right] = \frac{1}{2} \nabla^2 w_2. \quad (3.11)$$

The solution of (3.7) is

$$w_0 = \sum_1 \frac{a_n}{n} \sin(n\tau + \phi_n) + f_0(r, \alpha), \quad (3.12)$$

where f_0 is an arbitrary function. Equation (3.8) gives

$$w_1 = w_1(r, \alpha) \quad (3.13)$$

The equation for the leading order axial velocity in the Stokes layer,

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \right) W_0 = \sum_1 a_n \cos(n\tau + \phi_n), \quad (3.14)$$

is obtained by substituting (3.2) and (3.3) in (2.15) and collecting terms $O(\beta^0)$. The solution of (3.14) satisfying

the boundary conditions (3.4) and matching with the outer solution (3.12) is

$$w_0 = \sum_1^{\infty} \frac{a_n}{n} \left[\sin(n\tau + \phi_n) - e^{-\eta\sqrt{n}} \sin(n\tau - \eta\sqrt{n} + \phi_n) \right]. \quad (3.15)$$

Clearly the matching has required

$$f_0(1, \alpha) = 0. \quad (3.16)$$

At the next order of β , the equation for w_1 is

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \right) w_1 = -\frac{1}{2} \frac{\partial w_0}{\partial \eta}, \quad (3.17)$$

which has solution, satisfying (3.4)

$$w_1 = -\frac{1}{2} \sum_1^{\infty} \frac{a_n}{n} \eta e^{-\eta\sqrt{n}} \sin(n\tau - \eta\sqrt{n} + \phi_n) + A(\alpha)\eta. \quad (3.18)$$

Matching two terms of the axial velocity expansion in the outer region with two terms from the inner region we get

$$\left. \begin{aligned} A(\alpha) &\equiv 0, \\ \text{and } w_1(1, \alpha) &= 0. \end{aligned} \right\} \quad (3.19)$$

Returning to the flow in the outer region, the substitution of (3.5) and (3.6) into (2.13) yields the following equations:

$$\frac{\partial}{\partial \tau} \nabla^2 \chi_0 - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) w_0^2 = 0, \quad (3.20)$$

$$\frac{\partial}{\partial \tau} \nabla^2 \chi_1 - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (2w_0 w_1) = 0, \quad (3.21)$$

$$\frac{\partial}{\partial \tau} \nabla^2 \chi_2 - \frac{R_s}{2r} \frac{\partial(\chi_0, \nabla^2 \chi_0)}{\partial(r, \alpha)} - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (w_1^2 + 2w_0 w_2) = \frac{1}{2} \nabla^4 \chi_0 \quad (3.22)$$

$$\begin{aligned} \text{and } \frac{\partial}{\partial \tau} \nabla^2 \chi_3 - \frac{R_s}{2r} \left[\frac{\partial(\chi_0, \nabla^2 \chi_1)}{\partial(r, \alpha)} + \frac{\partial(\chi_1, \nabla^2 \chi_0)}{\partial(r, \alpha)} \right] \\ - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (2w_0 w_3 + 2w_1 w_2) = \frac{1}{2} \nabla^4 \chi_1. \end{aligned} \quad (3.23)$$

We require the solution to have only harmonic dependence on τ . Therefore to avoid secular terms in χ_n , the second term in (3.20) must be proportional to $e^{in\tau}$, $n = 1, 2, 3, \dots$. Using (3.12), this condition leads to

$$\frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) f_0^2 = 0,$$

which has solution $f_0(r, \alpha) = f_0(r \cos \alpha)$. The matching condition (3.16) gives $f_0(r \cos \alpha) \equiv 0$. Thus the axial velocity w_0 is now given by

$$w_0 = \sum_1^{\infty} \frac{a_n}{n} \sin(n\tau + \phi_n). \quad (3.24)$$

Equation (3.20) now simplifies to

$$\nabla^2 \chi_0 = g_0(r, \alpha) , \quad (3.25)$$

where g_0 is arbitrary, and using (3.13), a solution of (3.21) harmonically dependent on τ is guaranteed. Separating $\chi_0(r, \alpha, \tau)$ into steady, $\chi_0^{(s)}(r, \alpha)$ and unsteady, $\chi_0^{(u)}(r, \alpha, \tau)$ parts converts (3.25) into the two equations:

$$\nabla^2 \chi_0^{(u)} = 0 , \quad (3.26)$$

$$\text{and} \quad \nabla^2 \chi_0^{(s)} = g_0(r, \alpha) . \quad (3.27)$$

$\chi_0^{(u)}$ is harmonically dependent on τ .

At this stage the stream function in the Stokes layer may be determined. Substituting (3.2) and (3.3) into (2.13) and collecting terms $O(\beta^0)$ we obtain the equation for the leading term in the expansion for the stream function:

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \right) \frac{\partial^2 \Psi_0}{\partial \eta^2} = - \frac{\partial}{\partial \eta} W_0^2 \sin \alpha . \quad (3.28)$$

Using the solution (3.15) for W_0 , the general solution of (3.28) is

$$\begin{aligned} \Psi_0 = & \sin \alpha \sum_{n=1}^{\infty} \frac{a_n^2}{n^2} \left(\frac{2}{(1+i)^3 n \sqrt{n}} e^{-\eta(1+i)\sqrt{n}} - \frac{e^{-2\eta\sqrt{n}}}{8n\sqrt{n}} \right) \\ & - \sin \alpha \sum_{L=1}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} \frac{a_n a_{L-n}}{|n||L-n|} e^{iL\tau + i(\phi_n + \phi_{L-n})} \times \right. \\ & \left. \left[\frac{(b(n) + b(L-n)) e^{-\eta(b(n) + b(L-n))}}{16n(L-n) - 4iLb(n)b(L-n)} + \frac{b(L-n)}{2n(L-n)} e^{-\eta b(L-n)} \right] \right\} \\ & + \sum_{L=1}^{\infty} A_L^*(\alpha) e^{iL\tau - \eta(1+i)\sqrt{L}} \\ & + B(\alpha) \eta^3 + C(\alpha) \eta^2 + F(\alpha, \tau) \eta + G(\alpha, \tau) , \quad (3.29) \end{aligned}$$

$$\text{where} \quad b(m) = (2im)^{1/2} , \quad -\frac{3\pi}{2} < \arg m \leq \frac{\pi}{2} . \quad (3.30)$$

It is to be understood that we are only concerned with the real part of Ψ_0 as defined by (3.29). In evaluating the double summation in (3.29) the definitions

$$\left. \begin{aligned} a_{-m} &= -a_m , \\ \phi_{-m} &= -\phi_m , \\ a_0 &= 0 \end{aligned} \right\} \quad \text{and} \quad (3.31)$$

are used. If the Fourier series for the pressure gradient (2.5) is truncated after N terms then in the double summation

$$\left. \begin{array}{l} l = 1, 2, 3, \dots, 2N \\ \text{and } n = 1-N, 1-N+1, \dots, N-1, N. \end{array} \right] \quad (3.32)$$

In the case of a sinusoidal pressure gradient $N = 1$ and (3.29) reduces to the expression given in Lyne (1970).

The unknown constants in Ψ_0 can be determined by matching the stream function in the Stokes layer to that in the inviscid outer region. At the same time boundary conditions for the outer region will be obtained. The matching is affected by requiring that $\beta\Psi_0$ have the same behaviour for $\eta \rightarrow \infty$ as $\chi_0(r, \alpha, \tau)$ when $r \rightarrow 1$, i.e.

$$\chi_0(1, \alpha, \tau) + (-\eta\beta) \left. \frac{\partial \chi_0}{\partial r} \right|_{r=1} + \dots \sim B(\alpha)\eta^3\beta + C(\alpha)\eta^2\beta + F(\alpha, \tau)\eta\beta + G(\alpha, \tau)\beta.$$

This condition leads to

$$\left. \begin{array}{l} \chi_0(1, \alpha, \tau) = 0, \\ B(\alpha) \equiv C(\alpha) \equiv 0, \\ \text{and } \left. \frac{\partial \chi_0}{\partial r} \right|_{r=1} = -F(\alpha, \tau). \end{array} \right] \quad (3.33)$$

The first of (3.33) implies $\chi_0^{(u)}(1, \alpha, \tau) = 0$, which together with (3.26) gives

$$\chi_0^{(u)}(r, \alpha, \tau) \equiv 0. \quad (3.34)$$

Thus to leading order the secondary motion in the outer region is steady. The superscript (s) on χ_0 can now be omitted, it being understood that $\chi_0 = \chi_0(r, \alpha; R_1, 0)$. That B and C should be identically zero follows directly from the discussion in Chapter 2 where it was assumed that the velocity in the inner and outer regions should match.

As χ_0 is independent of time, the last of (3.33) gives that F is a function of α only, $F(\alpha)$. The boundary conditions (3.4) together with this result allow the unknown constants in Ψ_0 to be determined. It is found that

$$F(\alpha) = -\sin \alpha \sum_{n=1}^{\infty} \frac{a_n^2}{4n^3}, \quad (3.35a)$$

$$A_L^*(\alpha) = A_L \sin \alpha = \sin \alpha \sum_{n=-\infty}^{\infty} \frac{a_n a_{L-n}}{\ln |L-n|} \cdot \frac{e^{i(\phi_n + \phi_{L-n})}}{(1+i)\sqrt{L}} \times \left[\frac{2il + 2b(n)b(L-n)}{16n(L-n) - 4ilb(n)b(L-n)} + \frac{i}{n} \right], \quad (3.35b)$$

$$\begin{aligned}
 \text{and } G(\alpha, \tau) = & -\sin \alpha \sum_{n=1}^{\infty} \frac{a_n^2}{n^2} \left(\frac{2}{(1+i)^2 n \sqrt{n}} - \frac{1}{8 n \sqrt{n}} \right) \\
 & + \sin \alpha \sum_{L=1}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} \frac{a_n a_{L-n}}{|n| |L-n|} e^{iL\tau + i(\phi_n + \phi_{L-n})} \times \right. \\
 & \quad \left[\frac{b(n) + b(L-n)}{16n(L-n) - 4iLb(n)b(L-n)} + \frac{b(L-n)}{2n(L-n)} \right] \\
 & - \sin \alpha \sum_{L=1}^{\infty} A_L e^{iL\tau} . \quad (3.35c)
 \end{aligned}$$

The determination of Ψ_0 is now complete; we turn next to the derivation of the equations governing the steady streaming and the perturbation mean velocity in the outer region. The substitution of the solution for w_0 (3.24) into (3.9) gives the result that w_2 is independent of τ i.e.,

$$w_2 = w_2(r, \alpha). \quad (3.36)$$

Using the known properties of χ_0 , w_0 and w_1 ((3.34), (3.24) and (3.13) respectively) and insisting that the solutions have only harmonic dependence on τ , (3.10) can be separated into the two independent equations

$$\frac{\partial w_3}{\partial \tau} = 0, \quad (3.37)$$

$$\text{and } -\frac{R_s}{2r} \frac{\partial(\chi_0, w_1)}{\partial(r, \alpha)} = \frac{20}{\sqrt{R_1}} + \frac{1}{2} \nabla^2 w_1. \quad (3.38)$$

Similarly (3.22) can be decomposed into steady and unsteady parts giving the independent equations

$$\frac{\partial}{\partial \tau} \nabla^2 \chi_2 - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (2w_0 w_2) = 0, \quad (3.39)$$

$$\text{and } -\frac{R_s}{2r} \frac{\partial(\chi_0, \nabla^2 \chi_0)}{\partial(r, \alpha)} - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) w_1^2 = \frac{1}{2} \nabla^4 \chi_0. \quad (3.40)$$

Equations (3.38) and (3.40) are a simultaneous pair of equations for the leading order stream function and the mean velocity. Boundary conditions for these equations are given by (3.19), (3.33) and (3.35) which we collect here as

$$\left. \begin{aligned}
 w_1 = \chi_0 = 0, \\
 \frac{\partial \chi_0}{\partial r} = \sin \alpha \sum_{n=1}^{\infty} \frac{a_n^2}{4n^3} \equiv \mathcal{N} \sin \alpha
 \end{aligned} \right] \text{ on } r=1. \quad (3.41)$$

Apart from the presence of the parameter R_s , equations (3.38) and (3.40) are recognized as Dean's equations (1928)

for steady flow in a curved pipe. The boundary conditions (3.41) modify Dean's problem so as to include the effect of the Stokes layer on the wall. The equation governing χ_0 in the case of oscillatory flow is recovered by formally taking $D = 0$. Equation (3.40) then becomes the 2-dimensional Navier-Stokes equation with boundary conditions prescribing the velocity at the wall. In this case there is no mean flow and the secondary motion is directed towards the inside of the pipe bend. This is the case studied by Lyne. To obtain the equations for exactly steady flow in a curved pipe the stream function and axial velocity need to be rescaled. If χ_0 is replaced by $R_s^{-1} \chi_0$ and w_1 replaced by $R_s^{-1/2} w_1$, then R_s is removed from (3.38) and (3.40) and appears only in the boundary condition (3.41). By taking $R_s = 0$ the usual no slip conditions associated with the steady problem are obtained and the secondary motion is the well known outward centrifuging.

Proceeding to higher order terms, χ_1 is written as the sum of a steady term and a function harmonically dependent on τ , i.e.

$$\chi_1(r, \alpha, \tau) = \chi_1^{(u)}(r, \alpha) + \chi_1^{(w)}(r, \alpha, \tau). \quad (3.42)$$

One boundary condition for χ_1 is obtained by matching $\chi_0 + \beta \chi_1$ with the leading order solution for the stream function in the Stokes layer. The result is

$$\chi_1^{(u)}(1, \alpha, \tau) + \chi_1^{(w)}(1, \alpha) = G^{(u)}(\alpha, \tau) + G^{(w)}(\alpha), \quad (3.43)$$

where $G^{(u)}$ and $G^{(w)}$ are the unsteady and steady parts of $G(\alpha, \tau)$ defined by (3.35c). Clearly (3.43) leads to the condition

$$\chi_1^{(u)}(1, \alpha, \tau) = G^{(u)}(\alpha, \tau), \quad (3.44)$$

which together with (3.21) and the requirement that $\chi_1^{(u)}$ is regular within the pipe is sufficient to determine $\chi_1^{(u)}$ once w_1 is known.

One boundary condition for $\chi_1^{(w)}$ is given by (3.43), and a further condition comes from matching two terms of the outer solution with two terms of the inner solution for the stream function. The result is

$$\beta^2 \left[\frac{\eta^2}{2} \frac{\partial^2 \chi_0}{\partial r^2} \Big|_{r=1} - \eta \frac{\partial \chi_1^{(u)}}{\partial r} \Big|_{r=1} - \eta \frac{\partial \chi_1^{(w)}}{\partial r} \Big|_{r=1} \right] \sim \beta^2 \Psi, \quad (3.45)$$

and as $\chi_1^{(u)}$ may be considered known, only the steady part of Ψ_1 need be considered. The equation for Ψ_1 is obtained by collecting terms $O(\beta)$ when (3.2) and (3.3) are substituted in (2.13). This gives

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\partial^2}{\partial \eta^2}\right) \frac{\partial^2 \Psi_1}{\partial \eta^2} = \frac{\partial^2 \Psi_0}{\partial \tau \partial \eta} - \frac{\partial^3 \Psi_0}{\partial \eta^3} - 2 \frac{\partial}{\partial \eta} (w_0 w_1) \sin \alpha. \quad (3.46)$$

It is easily seen that terms in the particular solution which are harmonically dependent on τ are also exponentially small as $\eta \rightarrow \infty$, and thus the matching given by (3.45) is possible. The equation for the steady part of Ψ_1 , denoted $\Psi_1^{(u)}$ is then

$$-\frac{1}{2} \frac{\partial^4 \Psi_1^{(u)}}{\partial \eta^4} = -\frac{\partial^3 \Psi_0^{(u)}}{\partial \eta^3} - 2 \frac{\partial}{\partial \eta} (w_0 w_1)^{(u)} \sin \alpha, \quad (3.47)$$

where $\Psi_0^{(u)}$ is the steady part of Ψ_0 and $(w_0 w_1)^{(u)}$ is the steady part of the product $w_0 w_1$. The solution (3.47) satisfying the no slip conditions (3.4) is

$$\begin{aligned} \Psi_1^{(u)} = & \sin \alpha \sum_1 \frac{a_n^2}{n^2} \left(-\frac{\eta}{2n\sqrt{n}} e^{-2\eta\sqrt{n}} - \frac{1}{16n^2} e^{-2\eta\sqrt{n}} \right. \\ & \left. - \frac{(1+i)\eta}{4n\sqrt{n}} e^{-\eta(1+i)\sqrt{n}} + \frac{1}{4} e^{-\eta(1+i)\sqrt{n}} \right) \\ & + H(\alpha) \eta^3 + I(\alpha) \eta^2 \\ & + \eta \sin \alpha \sum_1 \frac{a_n^2}{4n^4} (1+i)\sqrt{n} (1+n^2) \\ & + \sin \alpha \sum_1 \frac{a_n^2}{4n^2} \left(\frac{1}{4n^2} - 1 \right). \end{aligned} \quad (3.48)$$

The matching condition (3.45) gives

$$\begin{aligned} H(\alpha) & \equiv 0, \\ I(\alpha) & = \frac{1}{2} \left. \frac{\partial^2 \chi_0}{\partial r^2} \right|_{r=1}, \end{aligned}$$

$$\text{and} \quad \left. \frac{\partial \chi_1^{(u)}}{\partial r} \right|_{r=1} = -\sin \alpha \sum_1 \frac{a_n^2}{4n^4} \sqrt{n} (1+n^2), \quad (3.49)$$

where only the real part of $\Psi_1^{(u)}$ has been used.

From (3.23) it is seen that w_2 will be involved in the equation for $\chi_1^{(u)}$. Therefore before deriving the equations for

$\chi_n^{(1)}$, boundary conditions for w_2 are derived. This entails the determination of W_2 in the Stokes layer. Substituting (3.2) and (3.3) in (2.15) and collecting terms $O(\beta^2)$ we get the equation for W_2 as

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2} \frac{\partial^2}{\partial \eta^2}\right) W_2 = \frac{R_s}{2} \frac{\partial \Phi_0}{\partial \alpha} \cdot \frac{\partial W_0}{\partial \eta} - \frac{1}{2} \frac{\partial W_1}{\partial \eta} - \frac{1}{2} \eta \frac{\partial W_0}{\partial \eta} \quad (3.50)$$

In the analysis so far the axial velocity in the Stokes layer has been purely oscillatory. Now there is the possibility of a steady axial velocity component arising from the interaction of the oscillatory parts of the leading order stream function and axial velocity. It is noted that this steady component would be independent of the mean pressure gradient. It is easily seen that terms in the particular solution of (3.50) which are harmonically dependent on τ are exponentially small as $\eta \rightarrow \infty$ and so cannot affect the outer flow. This agrees with the result (3.36) that w_2 is independent of time. It is necessary to determine the behaviour of the steady component $W_2^{(1)}$ of W_2 .

The equation governing $W_2^{(1)}$ is

$$\frac{\partial^2 W_2^{(1)}}{\partial \eta^2} = -R_s \left(\frac{\partial \Phi_0}{\partial \alpha} \cdot \frac{\partial W_0}{\partial \eta} \right)^{(1)} \quad (3.51)$$

with the obvious notation. The solution of (3.51) satisfying (3.4) is

$$\begin{aligned} W_2^{(1)} = & -\frac{R_s}{2} \cos \alpha \sum_{l=1}^{\infty} \left[\sum_{n=-\infty}^{\infty} \frac{a_n a_{l-n} a_l}{|n||l-n||l|} (1+i) e^{i(\phi_n + \phi_{l-n} - \phi_l)} \times \right. \\ & \left. \left(B_1 e^{-\eta(b(ln) + b(l-n) + (1-i)\sqrt{l})} + B_2 e^{-\eta(b(l-n) + (1-i)\sqrt{l})} \right. \right. \\ & \left. \left. - \frac{i B_2}{2l} e^{-\eta(1-i)\sqrt{l}} \right) \right] \\ & + \frac{R_s}{2} \cos \alpha \sum_{l=1}^{\infty} \frac{A_l a_l (1+i)}{l \sqrt{l}} e^{-i\phi_l} \left(\frac{e^{-2\eta\sqrt{l}}}{4} - \frac{i e^{-\eta(1-i)\sqrt{l}}}{2} \right) \\ & + L(\alpha) \eta + M^*(\alpha) , \end{aligned} \quad (3.52)$$

where $M^*(\alpha) = \mathcal{M} R_s \cos \alpha$

$$= \frac{R_s}{2} \cos \alpha \sum_{l=1}^{\infty} \left[\sum_{n=0}^{\infty} \frac{a_n a_{l-n} a_l}{|n| |l-n| |l|} (1+i) e^{i(\phi_n + \phi_{l-n} - \phi_l)} \left(B_1 + B_2 - \frac{i B_3}{2l} \right) \right] \\ - \frac{R_s}{2} \cos \alpha \sum_{l=1}^{\infty} \frac{A_l a_l (1+i)}{l \sqrt{l}} e^{-i \phi_l} \left(\frac{1}{4} - \frac{i}{2} \right) , \quad (3.53)$$

with $B_1 = \frac{b(n) + b(l-n)}{[16n(l-n) - 4iLb(n)b(l-n)][2b(n)b(l-n) + 2(1-i)\sqrt{l}(b(n) + b(l-n))]} ,$

$$B_2 = \frac{b(l-n)}{2n(l-n)[2(1-i)\sqrt{l}b(l-n) - 2in]} ,$$

and $B_3 = \frac{b(n) + b(l-n)}{16n(l-n) - 4iLb(n)b(l-n)} + \frac{b(l-n)}{2n(l-n)} . \quad (3.54)$

The unknown constant $L(\alpha)$ is found by matching the axial velocity in the outer and inner regions to $O(\beta^2)$. The matching condition obtained is

$$-\beta^2 \eta \frac{\partial w_1}{\partial r} \Big|_{r_1} + \beta^2 w_2(1, \alpha) \sim \beta^2 (L(\alpha) \eta + \mathcal{M} R_s \cos \alpha) ,$$

which gives
$$L(\alpha) = - \frac{\partial w_1}{\partial r} \Big|_{r_1} ,$$

and
$$w_2(1, \alpha) = \mathcal{M} R_s \cos \alpha . \quad (3.55)$$

Thus $w_2^{(1)}$ is completely determined and the last equation of (3.55) provides the necessary boundary condition for w_2 . It is seen that depending on the value of $\mathcal{M}$ (actually $\text{Re}\{\mathcal{M}\}$), the interaction of the oscillatory components of the secondary motion and axial velocity has produced a steady velocity which does not decay to zero at the edge of the Stokes layer, but persists into the outer region. Alternatively we may say that the Reynolds stresses in the Stokes layer act so as to generate an axial steady streaming over the whole flow.

The equations governing $\chi_1^{(1)}$ and w_2 are obtained from (3.23) and (3.11) by separating them into steady and unsteady components. For $\chi_1^{(1)}$ we obtain

$$-\frac{R_s}{2r} \left[\frac{\partial(\chi_0, \nabla^2 \chi_1^{(1)})}{\partial(r, \alpha)} + \frac{\partial(\chi_1^{(1)}, \nabla^2 \chi_0)}{\partial(r, \alpha)} \right] \\ - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (2w_1 w_2) = \frac{1}{2} \nabla^4 \chi_1^{(1)} \quad (3.56)$$

and for w_2 the equation is

$$-\frac{R_s}{r} \left[\frac{\partial(\chi_0, w_2)}{\partial(r, \alpha)} + \frac{\partial(\chi_1^{(1)}, w_1)}{\partial(r, \alpha)} \right] = \nabla^2 w_2 \quad (3.57)$$

Boundary conditions for these simultaneous equations are:

$$\left. \begin{aligned} \chi_1^{(1)} &= \frac{5}{8} \sin \alpha \sum_1^{\infty} \frac{a_n^2}{n^3 \sqrt{n}} , \\ \frac{\partial \chi_1^{(1)}}{\partial r} &= -\sin \alpha \sum_1^{\infty} \frac{a_n^2}{4n^4} \sqrt{n} (1+n^2) , \\ \text{and } w_2 &= \mathcal{M} R_s \cos \alpha \end{aligned} \right\} \text{ on } r=1 . \quad (3.58)$$

The first two equations of (3.58) are extensions to a general pressure gradient of the results of Lyne. The boundary condition on w_2 leads to the production of a steady axial velocity. As remarked earlier the boundary condition on w_2 is independent of the mean imposed pressure gradient. For the purpose of this thesis we shall restrict ourselves to the study of the structure of w_2 in the simpler case of zero mean pressure gradient. In this case $w_1 \equiv 0$ and the equation for w_2 simplifies to

$$-\frac{R_s}{r} \cdot \frac{\partial(\chi_0, w_2)}{\partial(r, \alpha)} = \nabla^2 w_2 \quad (3.59)$$

It is possible that $\mathcal{M}$ may be identically zero. In the case when the pressure gradient has only one harmonic term $\mathcal{M}$ is zero, which agrees with Lyne's calculations. Also if the amplitudes of the Fourier components are specified, then it may be possible to choose the phase terms ϕ_l so that $\mathcal{M}$ is zero. Finally, if there are no even harmonics, i.e. if $a_{2n} \equiv 0$ then the product $a_n a_{l-n} a_l$ is zero for all n and l and $\mathcal{M}$ is zero and there is no axial steady streaming.

In subsequent chapters, solutions for the mean pressure gradient driven flow, w_1 and the secondary motion χ_0 are obtained for various limits of R_s and D . Then for w_1 identically zero, the structure of the axial steady streaming is examined as a function of the parameter R_s .

CHAPTER 4

Creeping flow solutions

The leading order stream function, χ_0 , and the mean flow, w_1 , are governed by (3.38) and (3.40) together with the boundary conditions (3.41). The equations may be simplified by assuming the motion is so slow that the convective terms are negligible compared to the viscous terms. With this approximation the axial velocity will be Poiseuille flow, and due to the linearity of the slow motion equations, the secondary motion will be a linear superposition of inward and outward centrifuging.

The slow motion equations can be derived by taking the limit $R_s \rightarrow 0$ with

$$D = kR_s^{1/2}, \quad (4.1)$$

where k is a fixed constant, and using the expansions for χ_0 and w_1 :

$$\left. \begin{aligned} w_1(r, \alpha; R_s, 0) &\sim w_{10}(r, \alpha; k) + R_s w_{11} + R_s^2 w_{12} + \dots \\ \chi_0(r, \alpha; R_s, 0) &\sim \chi_{00}(r, \alpha; k) + R_s \chi_{01} + R_s^2 \chi_{02} + \dots \end{aligned} \right] \quad (4.2)$$

The boundary condition (3.41) can now be written

$$\left. \begin{aligned} w_{1i} &= \chi_{0i} = 0, \quad i = 0, 1, 2, 3, \dots \\ \frac{\partial \chi_{00}}{\partial r} &= \mathcal{N} \sin \alpha \\ \frac{\partial \chi_{0i}}{\partial r} &= 0, \quad i = 1, 2, 3, \dots \end{aligned} \right] \quad \text{on } r=1. \quad (4.3)$$

Substituting the expansions (4.2) into (3.38) and (3.40) and collecting like powers of R_s , we obtain the following equations for w_{10}, χ_{00} at $O(R_s^0)$:

$$\left. \begin{aligned} \nabla^2 w_{10} + 4k &= 0, \\ \frac{1}{2} \nabla^4 \chi_{00} &= -\frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) w_{10}^2. \end{aligned} \right] \quad (4.4)$$

The solution of (4.4) which is regular within the pipe and satisfies the conditions (4.3) is

$$w_{10} = k(1-r^2) \quad (4.5)$$

$$\begin{aligned} \chi_{00} &= \frac{k^2 \sin \alpha}{144} (4r - 9r^3 + 5r^5 - r^7) \\ &\quad + \frac{\mathcal{N} \sin \alpha}{2} (-r + r^3). \end{aligned} \quad (4.6)$$

The axial velocity is Poiseuille flow while the stream function is the sum of the secondary motion induced by the mean flow, and the steady streaming generated in the Stokes layer. The first term of χ_{00} in (4.6) is the secondary flow calculated by Dean (1928) for steady flow in a curved pipe, while the second term is similar to that given by Lyne for oscillatory flow. These results have also been derived by Smith (1975) for the case of $\mathcal{N} = 1/4$.

At $O(R_s)$ the equations governing w_{11} and χ_{01} are obtained:

$$\nabla^2 w_{11} = -\frac{1}{r} \frac{\partial (\chi_{00}, w_{10})}{\partial (r, \alpha)}, \quad (4.7)$$

$$\text{and } \frac{1}{2} \nabla^4 \chi_{01} = -\frac{1}{2r} \frac{\partial (\chi_{00}, \nabla^2 \chi_{00})}{\partial (r, \alpha)} - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (2w_{10} w_{11}). \quad (4.8)$$

The solutions of (4.7) and (4.8) satisfying (4.3) are

$$w_{11} = \frac{k^3 \cos \alpha}{5,760} (19r - 40r^3 + 30r^5 - 10r^7 + r^9) + \frac{k \mathcal{N} \cos \alpha}{24} (-2r + 3r^3 - r^5), \quad (4.9)$$

$$\begin{aligned} \text{and } \chi_{01} = & \frac{k^4 \sin 2\alpha}{144 \times 5,600} (4,979r^2 - 12,750r^4 + 11,340r^6 - 4,480r^8 + 1,050r^{10} - 144r^{12} + 5r^{14}) \\ & + \frac{\mathcal{N} k^2 \sin 2\alpha}{69,120} (-75r^2 + 184r^4 - 150r^6 + 48r^8 - 7r^{10}) \\ & + \frac{\mathcal{N}^2 \sin 2\alpha}{192} (-r^2 + 2r^4 - r^6). \end{aligned} \quad (4.10)$$

Again it is possible to identify terms given by previous workers for special cases of this unsteady flow. The first term in w_{11} was given by Dean (1928) and corresponds to a shift of the peak velocity towards the outside of the pipe; the second term represents the effect of the steady streaming, generated in the Stokes layer, on the axial velocity and so indicates a shift in the peak velocity towards the inside of the pipe bend. In the expression for χ_{01} , the first and last terms have been given by Dean (1928) and Lyne respectively.

The equations for w_{12} and χ_{02} , obtained at $O(R_s^2)$ are

$$\nabla^2 w_{12} = -\frac{1}{r} \left[\frac{\partial(\chi_{00}, w_{11})}{\partial(r, \alpha)} + \frac{\partial(\chi_{01}, w_{10})}{\partial(r, \alpha)} \right], \quad (4.11)$$

$$\text{and} \quad \frac{1}{2} \nabla^4 \chi_{02} = -\frac{1}{2r} \left[\frac{\partial(\chi_{00}, \nabla^2 \chi_{01})}{\partial(r, \alpha)} + \frac{\partial(\chi_{01}, \nabla^2 \chi_{00})}{\partial(r, \alpha)} \right] - \frac{1}{r} \left(r \sin \alpha \frac{\partial}{\partial r} + \cos \alpha \frac{\partial}{\partial \alpha} \right) (2w_{10}w_{12} + w_{11}^2). \quad (4.12)$$

From (4.11) we see that w_{12} will consist of a term proportional to $\cos 2\alpha$, plus a term independent of α . The former term will integrate to zero over the cross-section so that only the part of w_{12} independent of α is required to calculate the change in the flow rate. The result is

$$\begin{aligned} w_{12}^{(m)} = & \frac{k^5}{928,912,800} (-4,119 + 21,280r^2 - 46,340r^4 + 55,440r^6 - 39,830r^8 + 17,584r^{10} \\ & - 9,620r^{12} + 640r^{14} - 35r^{16}) \\ & + \frac{k^3 \mathcal{N}}{4,147,200} (953 - 4,110r^2 + 7,55r^4 - 6,400r^6 + 3,075r^8 \\ & - 738r^{10} + 65r^{12}) \\ & + \frac{k \mathcal{N}^2}{2,304} (-7 + 24r^2 - 30r^4 + 16r^6 - 3r^8). \end{aligned} \quad (4.13)$$

The flux ratio, Q_r is defined as

$$Q_r = \frac{\text{flow rate in curved pipe}}{\text{flow rate in straight pipe}} \quad (4.14)$$

when the mean pressure gradients in the curved and straight pipes are equal. Using (4.5) and (4.13) we find that

$$Q_r = 1 + R_s^2 \left(-k^4 1.474 \times 10^{-6} + k^2 \mathcal{N} 8.887 \times 10^{-5} - \mathcal{N}^2 1.389 \times 10^{-3} \right) \quad (4.15)$$

The quantity in brackets is always negative, but when

$$k^2 = 30.1 \mathcal{N}$$

the reduction in flow rate is least, being $5 R_s^2 \mathcal{N}^2 \times 10^{-5}$.

The result for χ_{02} will be omitted as it is excessively long and provides no new information.

In this limit of $R_s \rightarrow 0$ it is clear that the change in direction of the secondary motion depends on the parameter k . For small values of k the mean flow is small and so the outward centrifuging associated with it is small and the

secondary motion is as calculated by Lyne. As k increases the secondary flow generated by centripetal pressure gradient increases and eventually dominates. Further details of this process are given in Chapter 8.

CHAPTER 5

The limit $R_s \rightarrow \infty$

In Chapter 4 the mean velocity and secondary motion were calculated by assuming that viscous forces dominate the flow. In this chapter the possibility of obtaining solutions when the viscous forces are almost everywhere negligible is investigated.

The governing equations are the simultaneous equations (3.38) and (3.40) together with the boundary conditions (3.41). The problem would be considerably simplified if the stream function, χ_0 , were independent of the axial velocity. The equations would then be decoupled and the axial velocity calculated from the linear equation (3.38). This simplification can be achieved by noting that, for small D and R_s fixed, $w_1 \sim O(D)$ while χ_0 remains $O(1)$. Thus to leading order the stream function is independent of the axial velocity when the mean flow is small.

Formally it is assumed that χ_0 and w_1 have expansions

$$\left. \begin{aligned} \chi_0(r, \alpha; R_s, D) &\sim \chi_{00}(r, \alpha; R_s) + k^2 \chi_{01} + \dots \\ \text{and } w_1(r, \alpha; R_s, D) &\sim k w_{10}(r, \alpha; R_s) + k^3 w_{11} + \dots \end{aligned} \right\} \quad (5.1)$$

where k is defined by (4.1). The limit $k \rightarrow 0$ with R_s fixed is taken and the equations governing χ_{00} and w_{10} are found to be

$$-\frac{R_s}{r} \frac{\partial(\chi_{00}, \nabla^2 \chi_{00})}{\partial(r, \alpha)} = \nabla^4 \chi_{00}, \quad (5.2)$$

$$\text{and } -\frac{R_s}{r} \frac{\partial(\chi_{00}, w_{10})}{\partial(r, \alpha)} = 4 + \nabla^2 w_{10}. \quad (5.3)$$

The boundary conditions are

$$\chi_{00} = 0, \quad \frac{\partial \chi_{00}}{\partial r} = \mathcal{N} \sin \alpha \quad \text{on } r=1 \quad (5.4)$$

$$\text{and } w_{10} = 0 \quad \text{on } r=1. \quad (5.5)$$

Solutions of (5.2) and (5.3) for small values of R_s are contained in the analysis of Chapter 4; the limit $R_s \rightarrow \infty$ is examined in the rest of this chapter.

Equation (5.2), with the boundary conditions (5.4), has been solved by Lyne (1970). It is clear that χ_{∞} is independent of the mean flow and so the known solution for χ_{∞} may be used in (5.3) to calculate the axial velocity w_{∞} . For large values of R_s the viscous terms in (5.2) are important only in thin regions, $O(R_s^{-1/2})$, around the wall of pipe, $r = 1$ and along the axis of symmetry, $\alpha = 0, \pi$. Elsewhere the flow is inviscid and has closed stream lines. Postulating that no stream lines from the inviscid region enter the viscous layers we require the inviscid core to have uniform vorticity (Batchelor 1956). The boundary layer equations have been solved approximately by linearizing about the core velocity at the edge of the layer; matching the boundary layers to the core then determines the core vorticity. For the case of $\mathcal{N} = 1/4$ Lyne calculated the uniform vorticity in the core to be -0.56 in the upper semicircle. Numerical solution of this problem by Kuwahara and Imai (1969) confirmed Lyne's analysis and gave a limiting value of -0.54 for the core vorticity for computations up to R_s of 2064. For the case of general $\mathcal{N}$ it is easy to show the vorticity in the core is given by

$$\zeta_0 = -2.24 \mathcal{N} \quad (5.6)$$

This result is obtained by noting that $\mathcal{N}$ can be removed from the problem by basing the steady streaming Reynolds number on $\mathcal{N}\bar{W}$ rather than $\bar{W}$. (See Chapter 8 for further discussion.)

The possibility that the uniform vorticity theorem can be used to gain information about the structure of the axial velocity is investigated in the next chapter, but first a more direct asymptotic approach is used.

For large R_s the secondary motion has an inviscid core/boundary layer structure; away from the boundaries χ_{∞} has the form

$$\chi_{\infty}(r, \alpha; R_s) \sim \bar{\chi}_0(r, \alpha) + R_s^{-1/2} \bar{\chi}_1 + \dots \quad (5.7)$$

while near the wall $r = 1$

$$\chi_{\infty}(r, \alpha; R_s) \sim R_s^{-1/2} \chi_0(r, \alpha) + R_s^{-1} \chi_1 + \dots \quad (5.8)$$

and along the axis of symmetry $\alpha = 0, \pi$

$$\chi_{\infty}(r, s; R_s) \sim R_s^{-1/2} \chi_0(N, s) + R_s^{-1} \chi_1 + \dots \quad (5.9)$$

The boundary layer co-ordinates ρ and N are defined by

$$\rho = (1-r)R_s^{1/2} \quad \text{and} \quad N = nR_s^{1/2}. \quad (5.10)$$

The co-ordinate systems (n,s) and (N,s) and the corresponding velocity components U,V are shown in Fig. 5.1. The symbol X_i is used for the stream function in both boundary layers but no confusion should arise as it will be clear which region is under consideration. Similarly the symbols U_i, V_i will be used for the velocity components in either layer, thus at first order

$$\left. \begin{aligned} U_o &= \frac{\partial X_o}{\partial \alpha} \quad \text{and} \quad V_o = \frac{\partial X_o}{\partial \rho}, \\ \text{or} \quad U_o(N,s) &= \frac{\partial X_o}{\partial s} \quad \text{and} \quad V_o(N,s) = -\frac{\partial X_o}{\partial N}. \end{aligned} \right] \quad (5.11)$$

In the upper semicircle $\bar{X}_o$ is given by

$$\left. \begin{aligned} \nabla^2 \bar{X}_o &= -\zeta_o \\ \text{with } \bar{X}_o &= 0 \quad \text{on the boundary of the semicircle.} \end{aligned} \right] \quad (5.12)$$

The solution (Lyne 1970) is

$$\begin{aligned} \bar{X}_o &= \frac{\zeta_o}{2\pi} \left\{ \left[1 - \frac{1}{2} \left(r^2 + \frac{1}{r^2} \right) \cos 2\alpha \right] \arctan \left(\frac{2r \sin \alpha}{1-r^2} \right) \right. \\ &\quad - \frac{1}{4} \left(r^2 - \frac{1}{r^2} \right) \sin 2\alpha \log \left(\frac{1+2r \cos \alpha + r^2}{1-2r \cos \alpha + r^2} \right) \\ &\quad \left. + \left(r - \frac{1}{r} \right) \sin \alpha - \frac{\pi r^2}{2} (1 - \cos 2\alpha) \right\}, \quad (5.13) \end{aligned}$$

which gives the tangential velocity at the edge of the core as

$$\begin{aligned} \bar{V}_o(\alpha) &= -\frac{\partial \bar{X}_o}{\partial r} \Big|_{r=1} = \frac{\zeta_o}{\pi} \left[\pi \sin^2 \alpha - 2 \sin \alpha - \sin 2\alpha \log \tan \alpha/2 \right] \\ \text{and } \bar{V}_o(s) &= -\frac{\partial \bar{X}_o}{\partial n} \Big|_{n=0} = -\frac{\zeta_o}{\pi} \left[1 - \frac{1}{(1-s)^2} - \frac{1}{2(1-s)} \left((1-s)^2 - \frac{1}{(1-s)^2} \right) \log \left(\frac{2-s}{s} \right) \right]. \quad (5.14) \end{aligned}$$

The axial velocity must now be determined from (5.3). It is expected that the viscous terms will be important only near the walls of the pipe, and possibly in the viscous layer on the axis of symmetry. In these boundary layers the velocity

gradients are $O(R_s^{1/2})$. However the condition that the pressure drop along the pipe must balance the shear stress at the wall requires the velocity gradients to be $O(1)$ as $R_s \rightarrow \infty$. This is most easily seen by integrating (5.3) over the cross-section of the pipe; the exact relationship

$$0 = 4\pi + \int_0^{2\pi} \left. \frac{\partial w_{10}}{\partial r} \right|_{r=1} d\alpha \quad (5.15)$$

is obtained, and clearly $w_{10} \sim O(R_s^{1/2})$ near $r = 1$ in order to satisfy this condition. Thus we are led to the expansion

$$w_{10}(r, \alpha; R_s) \sim R_s^{-1/2} W_0(\rho, \alpha) + R_s^{-1} W_1 + \dots \quad (5.16)$$

for the axial velocity near $r = 1$. The boundary layer equation is then

$$\frac{\partial(X_0, W_0)}{\partial(\rho, \alpha)} = \frac{\partial^2 W_0}{\partial \rho^2}, \quad (5.17)$$

with boundary conditions $W_0 = 0$ on $\rho = 0$. (5.18)

Further conditions are obtained by matching W_0 to the core flow for large ρ . Consideration of the layer on the axis of symmetry is deferred until the axial core flow has been determined.

Away from the boundaries of the semicircle it is expected that the viscous term $R_s^{-1} \nabla^2 w_{10}$ is negligible compared to the inertia and pressure gradient terms. This leaves the flow in the core to be determined by a balance between the convective and pressure gradient terms and implies an axial velocity field $O(R_s^{-1})$. By converting to streamline co-ordinates it is found that this balance gives a result that is not a single-valued function of position and so cannot be the required solution. For an axial velocity $O(1)$ a single-valued solution can be found: here the velocity is constant on the streamlines of the secondary motion and the variation of the velocity across the streamlines is determined by a balance between the pressure gradient and viscous terms.

The streamline co-ordinates (ψ, η, z) to be used for the axial velocity equation are shown in Fig. 5.2. The plane $z = \text{constant}$ corresponds to a plane $\theta = \text{constant}$ in the (r, α, θ) co-ordinate system, and ψ, η are co-ordinates in this plane.

The direction of z increasing is the same as θ increasing.

The lines $\Psi = \text{constant}$ are given by the streamlines

$\chi_{\infty} = \text{constant}$, however in the analysis for large R_s the approximation $\bar{\chi}_0$ to χ_{∞} will be used and will replace χ_{∞} in the definitions below. The lines of constant η (the Ψ axis) are normal to the streamlines.

The velocity vector is

$$\underline{v} = [0, -q, w_{10}] \quad , \quad (5.19)$$

where

$$q = \left[\left(\frac{1}{r} \frac{\partial \chi_{\infty}}{\partial \alpha} \right)^2 + \left(- \frac{\partial \chi_{\infty}}{\partial r} \right)^2 \right]^{1/2} . \quad (5.20)$$

The non-dimensional equation of steady motion is

$$R_s (\underline{v} \times \underline{\omega}) = \nabla H + \text{curl} \underline{\omega} \quad , \quad (5.21)$$

where $\underline{\omega} = \text{curl} \underline{v} = [\omega_1, \omega_2, \omega_3]$

and the non-dimensional scale factors are

$$\left. \begin{aligned} h_1 &= 1/q \quad , \\ h_2 &= h_2(\psi, \eta) \quad , \\ \text{and} \quad h_3 &= 1 \quad . \end{aligned} \right\} \quad (5.22)$$

Also the total head, H is given by

$$H = \frac{p(\psi, \eta, z)}{\rho} + \frac{1}{2} R_s \underline{v} \cdot \underline{v} \quad . \quad (5.23)$$

The vorticity vector $\underline{\omega}$ is

$$\underline{\omega} = \left[\frac{1}{h_2} \frac{\partial w_{10}}{\partial \eta} , -q \frac{\partial w_{10}}{\partial \psi} , -\frac{q}{h_2} \frac{\partial}{\partial \psi} (q h_2) \right] \quad , \quad (5.24)$$

so that the third component of (5.21) is

$$-R_s \frac{q}{h_2} \frac{\partial w_{10}}{\partial \eta} = 4 + \frac{q}{h_2} \left[\frac{\partial}{\partial \psi} \left(q h_2 \frac{\partial w_{10}}{\partial \psi} \right) + \frac{\partial}{\partial \eta} \left(\frac{1}{q h_2} \frac{\partial w_{10}}{\partial \eta} \right) \right] \quad , \quad (5.25)$$

where the constant, 4, is the steady pressure gradient along the pipe. As no approximations have been made (5.25) corresponds to (5.3) exactly, and we note the transformation of the operator

$$-\frac{1}{r} \frac{\partial \chi_{\infty}}{\partial r} \frac{\partial}{\partial \alpha} + \frac{1}{r} \frac{\partial \chi_{\infty}}{\partial \alpha} \frac{\partial}{\partial r} \longrightarrow -\frac{q}{h_2} \frac{\partial}{\partial \eta} \quad . \quad (5.26)$$

The asymptotic solution for χ_0 is now used to define the streamlines and the co-ordinates ψ, η . In these co-ordinates (5.12) gives the important result

$$\omega_3 = \zeta_0 = -\frac{q}{h_2} \frac{\partial}{\partial \psi} (h_2 q) \quad (5.27)$$

where q, h_2 are now defined in terms of $\bar{\chi}_0$.

Neglecting the viscous terms in (5.3) and balancing the remaining terms we require an expansion

$$w_{10} \sim R_s^{-1} \phi(r, \alpha) + o(R_s^{-1/2}) .$$

The equation for ϕ is

$$-\frac{1}{r} \frac{\partial(\bar{\chi}_0, \phi)}{\partial(r, \alpha)} = 4 ,$$

or using the transformation (5.26)

$$-\frac{q}{h_2} \frac{\partial \phi}{\partial \eta} = 4 .$$

Only solutions which are single-valued functions of position can be accepted as physically reasonable. This condition can be expressed as an integral around a closed η -line in the form

$$\oint \frac{\partial w_{10}}{\partial \eta} d\eta = 0 . \quad (5.28)$$

Applying this condition to ϕ we get

$$-\oint \frac{\partial \phi}{\partial \eta} d\eta = \oint \frac{4h_2}{q} d\eta \neq 0$$

as $q \gg 0$ and $h_2 \gg 0$ by definition. (The case of $h_2 \equiv 0$ is of no interest.) Thus it is concluded that w_{10} is NOT $O(R_s^{-1})$; similar arguments apply when $w_{10} \sim O(R_s^{-1/2})$. If w_{10} is expanded

$$w_{10} \sim R_s^{-1/2} \bar{\Phi}_0 + R_s^{-1} \bar{\Phi}_1 + o(R_s^{-3/2})$$

then $\bar{\Phi}_0 = \bar{\Phi}_0(\bar{\chi}_0)$ and the equation for $\bar{\Phi}_1$, obtained from terms $O(R_s^0)$ in (5.3), is

$$-\frac{1}{r} \frac{\partial(\bar{\chi}_0, \bar{\Phi}_1)}{\partial(r, \alpha)} - \frac{1}{r} \frac{\partial(\bar{\chi}_1, \bar{\Phi}_0)}{\partial(r, \alpha)} = 4 .$$

In streamline co-ordinates this equation is

$$\frac{\partial \bar{\Phi}_1}{\partial \eta} = -\frac{4h_2}{q} + \frac{d\bar{\Phi}_0}{d\psi} \frac{\partial \bar{\chi}_1}{\partial \eta}$$

which shows that $\bar{\Phi}_1$ does not satisfy (5.28), and hence leads to the conclusion that w_{10} is NOT $O(R_s^2)$. However a single-valued solution can be obtained if $w_{10} \sim O(1)$ as $R_s \rightarrow \infty$.

Expanding w_{10} as

$$w_{10}(r, \alpha; R_s) \sim \bar{w}_0(r, \alpha) + R_s^{-1/2} \bar{w}_1 + \dots \quad (5.29)$$

and substituting in (5.3) we obtain the following equations. At leading order

$$-\frac{1}{r} \frac{\partial(\bar{\alpha}_0, \bar{w}_0)}{\partial(r, \alpha)} = 0,$$

which has solution

$$\bar{w}_0 = \bar{w}_0(\bar{\alpha}_0) \equiv \bar{w}_0(\psi) \quad (5.30)$$

Terms with coefficients $R_s^{1/2}$ give the equation

$$-\frac{1}{r} \frac{\partial(\bar{\alpha}_0, \bar{w}_1)}{\partial(r, \alpha)} - \frac{1}{r} \frac{\partial(\bar{\alpha}_1, \bar{w}_0)}{\partial(r, \alpha)} = 0.$$

The solution is

$$\bar{w}_1 = f_1(\bar{\alpha}_0) + \bar{\alpha}_1 \bar{w}_0', \quad (5.31)$$

where a prime denotes differentiation with respect to ψ (or $\bar{\alpha}_0$), and f_1 is an arbitrary function of $\bar{\alpha}_0$ (or ψ).

Similarly, terms independent of R_s give the equation

$$-\frac{1}{r} \frac{\partial(\bar{\alpha}_0, \bar{w}_2)}{\partial(r, \alpha)} - \frac{1}{r} \frac{\partial(\bar{\alpha}_1, \bar{w}_1)}{\partial(r, \alpha)} - \frac{1}{r} \frac{\partial(\bar{\alpha}_2, \bar{w}_0)}{\partial(r, \alpha)} = 4 + \nabla^2 \bar{w}_0. \quad (5.32)$$

Using the solutions (5.30) and (5.31), (5.32) becomes

$$\begin{aligned} -\frac{1}{r} \frac{\partial(\bar{\alpha}_0, \bar{w}_2)}{\partial(r, \alpha)} - \frac{f_1'}{r} \frac{\partial(\bar{\alpha}_1, \bar{\alpha}_0)}{\partial(r, \alpha)} - \frac{\bar{\alpha}_1 \bar{w}_0''}{r} \frac{\partial(\bar{\alpha}_1, \bar{\alpha}_0)}{\partial(r, \alpha)} - \frac{\bar{w}_0'}{r} \frac{\partial(\bar{\alpha}_2, \bar{\alpha}_0)}{\partial(r, \alpha)} \\ = 4 + \nabla^2 \bar{w}_0, \end{aligned}$$

which transforms to

$$\begin{aligned} -\frac{q}{h_2} \left[\frac{\partial \bar{w}_2}{\partial \eta} - \frac{\partial \bar{\alpha}_1}{\partial \eta} (f_1' + \bar{\alpha}_1 \bar{w}_0'') - \bar{w}_0' \frac{\partial \bar{\alpha}_2}{\partial \eta} \right] \\ = 4 + \frac{q}{h_2} \cdot \frac{\partial}{\partial \psi} (h_2 q \bar{w}_0') \end{aligned} \quad (5.33)$$

where the Laplacian has been simplified using (5.30).

Requiring $\bar{w}_2$ to satisfy the condition (5.28) we get an equation for $\bar{w}_0$ as follows:

$$\oint \left[\frac{4h_2}{q} + \frac{\partial}{\partial \psi} (h_2 q \bar{w}_0') \right] d\eta = 0. \quad (5.34)$$

This second order linear equation has the general solution

$$\bar{w}_0(\psi) = \frac{4\psi}{\beta_0} + A \int^{\psi} \frac{d\psi}{\oint h_2 q d\eta} + B, \quad (5.35)$$

where (5.27) has been used to simplify the particular solution. The quantity $\oint h_2 q d\eta$ is proportional to the circulation about the vortex centre, ψ_m and it can be shown that the circulation tends to zero like $(\psi - \psi_m)$ as $\psi \rightarrow \psi_m$. Thus for $\bar{w}_0$ to be regular we require $A \equiv 0$. Matching the core velocity to the boundary flow we get

$$\frac{4}{\beta_0} \left(\bar{\chi}_0(1, \alpha) - R_s^{-1/2} \rho \frac{\partial \bar{\chi}_0}{\partial r} \Big|_{r=R_s} + \dots \right) + B \sim R_s^{-1/2} W_0(\rho, \alpha).$$

The boundary condition on $\bar{\chi}_0$, (5.12), leads to

$$\left. \begin{aligned} B &\equiv 0, \\ W_0(\rho, \alpha) &\sim -\frac{4}{\beta_0} \rho \frac{\partial \bar{\chi}_0}{\partial r} \Big|_{r=R_s}, \text{ as } \rho \rightarrow \infty \end{aligned} \right] \quad (5.36)$$

and so the axial velocity in the core is

$$\bar{w}_0(r, \alpha) = \frac{4}{\beta_0} \bar{\chi}_0(r, \alpha). \quad (5.37)$$

The result (5.37) for the axial velocity is somewhat novel in that the velocity distribution has been determined mainly by viscous forces in an inviscid region. However this situation is closely analogous to the result of the uniform vorticity model for 2-dimensional flow with closed streamlines. Indeed omitting the pressure gradient term from (5.3) and interpreting w_{10} as the vorticity $-\nabla^2 \chi_0$ reduces the equation to (5.2); the integral condition (5.34) would then make $\bar{w}_0$ (the vorticity) constant across streamlines, which is the result of Batchelor (1956). Thus in both 2-dimensional flow with closed streamlines and the axial flow considered here, the

analysis shows that the dominance of the convective terms causes the vorticity and the axial velocity to be constant on the streamlines. Thereafter it is the smaller viscous terms in the equation that determine the variation of the vorticity or axial velocity across the streamlines. In the case of 2-dimensional flow, viscous diffusion acts to produce a region of uniform vorticity, whereas in the problem under consideration, diffusive effects result in a velocity profile such that viscous dissipation of energy in the inviscid region balances the energy supplied by the pressure gradient. This is readily seen by noting that $4 + \nabla^2 \bar{w}_0$ is identically zero when (5.37) is used.

A similar analysis for the flow in the lower semicircle gives (5.37) as the velocity distribution.

As $\bar{\chi}_0$ is zero on the boundary $r = 1$ (5.12) so too is the axial velocity $\bar{w}_0$. Thus the core flow satisfies the no slip condition (5.5) making boundary layers on the pipe wall appear irrelevant. However along the axis of symmetry the normal velocity gradient $\frac{\partial \bar{w}_0}{\partial n}$ is discontinuous. Symmetry conditions require zero normal derivative on $\alpha = 0, \pi$ and so a stress layer of thickness $O(R_s^{-1/2})$ is needed to remove the discontinuity and to transfer to the boundary layers on the wall the stress existing on the plane of symmetry. The core flow tends to zero on $\alpha = 0, \pi$ so that in the boundary layer the axial velocity is $O(R_s^{-1/2})$. The expansion

$$w_{00}(n, s; R_s) \sim R_s^{-1/2} W_0(N, s) + R_s^{-1} W_1 + \dots, \quad (5.38)$$

leads to the boundary layer equation

$$-\frac{\partial(\chi_0, W_0)}{\partial(N, s)} = \frac{\partial^2 W_0}{\partial N^2}. \quad (5.39)$$

Matching with the core flow as $N \rightarrow \infty$ we obtain

$$\left. \begin{aligned} W_0(N, s) &\sim \frac{4}{\pi} N \frac{\partial \bar{\chi}_0}{\partial n} \Big|_{n=0}, \\ \text{while the boundary condition at } N = 0 \text{ is} \\ \frac{\partial W_0}{\partial N} &= 0. \end{aligned} \right] \quad (5.40)$$

Before solving the boundary layer problems (5.17) and

(5.39), the solution in the core is continued, and $\bar{\chi}_1$ and $\bar{w}_1$ are determined in principle. From (5.2) the equation governing $\bar{\chi}_1$ is

$$-\frac{1}{r} \frac{\partial(\bar{\chi}_0, \nabla^2 \bar{\chi}_1)}{\partial(r, \alpha)} - \frac{1}{r} \frac{\partial(\bar{\chi}_1, \nabla^2 \bar{\chi}_0)}{\partial(r, \alpha)} = 0,$$

and using (5.12) to simplify the second Jacobian, the solution is seen to be

$$\nabla^2 \bar{\chi}_1 = g_1(\bar{\chi}_0). \quad (5.41)$$

As we proceed to an equation two orders of $R_s^{-1/2}$ higher, viscous terms will appear, and an integral condition analogous to (5.28) will show that $g_1(\bar{\chi}_0)$ must be a constant. Therefore we have

$$\nabla^2 \bar{\chi}_1 = -3. \quad (5.42)$$

Boundary conditions for $\bar{\chi}_1$ are obtained by matching $\bar{\chi}_0 + R_s^{-1/2} \bar{\chi}_1$ with the leading order boundary layer solutions, and in the usual way. $\bar{\chi}_1$ is the modification to the external flow due to the boundary layer displacement thickness (Van Dyke 1962). Specifically we have

$$\left. \begin{aligned} \bar{\chi}_1(1, \alpha) &= \lim_{p \rightarrow \infty} \left(X_0(p, \alpha) + p \frac{\partial \bar{\chi}_0}{\partial r} \Big|_{r=1} \right), \\ \bar{\chi}_1(0, s) &= \lim_{N \rightarrow \infty} \left(X_0(N, s) - N \frac{\partial \bar{\chi}_0}{\partial n} \Big|_{n=0} \right). \end{aligned} \right\} \quad (5.43)$$

Also the behaviour of X_0 at the edge of the boundary layer is

$$\left. \begin{aligned} X_0(p, \alpha) &\sim -p \frac{\partial \bar{\chi}_0}{\partial r} \Big|_{r=1} + o(1) \quad \text{as } p \rightarrow \infty, \\ X_0(N, s) &\sim N \frac{\partial \bar{\chi}_0}{\partial n} \Big|_{n=0} + o(1) \quad \text{as } N \rightarrow \infty. \end{aligned} \right\} \quad (5.44)$$

The axial velocity equation (5.33) gives the form of $\bar{w}_2$ as

$$\bar{w}_2 = f_2(\bar{\chi}_0) + f_1' \bar{\chi}_1 + \frac{4}{3} \bar{\chi}_2. \quad (5.45)$$

Terms with coefficient $R_s^{-1/2}$ in the axial momentum equation give an equation for $\bar{w}_3$:

$$-\frac{1}{r} \left[\frac{\partial(\bar{\chi}_0, \bar{w}_3)}{\partial(r, \alpha)} + \frac{\partial(\bar{\chi}_1, \bar{w}_2)}{\partial(r, \alpha)} + \frac{\partial(\bar{\chi}_2, \bar{w}_1)}{\partial(r, \alpha)} + \frac{\partial(\bar{\chi}_3, \bar{w}_0)}{\partial(r, \alpha)} \right] = \nabla^2 \bar{w}_1$$

which in streamline co-ordinates is

$$-\frac{q}{h_2} \left[\frac{\partial \bar{w}_3}{\partial \eta} - \frac{\partial \bar{x}_1}{\partial \eta} (f_2' + \bar{x}_1 f_1'') - f_1' \frac{\partial \bar{x}_2}{\partial \eta} - \frac{4}{S_0} \frac{\partial \bar{x}_3}{\partial \eta} \right] \\ = \frac{q}{h_2} \frac{\partial}{\partial \psi} (q h_2 f_1') - \frac{4}{S_0} \cdot S_1, \quad (5.46)$$

where (5.37) and (5.42) have been used to simplify the equation. This equation has the same form as (5.33) and it is clear that the integral condition (5.28) will lead to the result

$$f_1(\bar{x}_0) = -\frac{4S_1}{S_0^2} \bar{x}_0 + A \int \frac{d\psi}{\oint h_2 d\eta} + B.$$

To obtain a regular solution $A \equiv 0$ so that, from (5.31)

$$\bar{w}_1 = -\frac{4S_1}{S_0^2} \bar{x}_0 + \frac{4}{S_0} \bar{x}_1 + B. \quad (5.47)$$

The constant B is determined by matching two terms of the core flow with the leading order boundary layer term, and on the walls $r = 1$ this leads to

$$R_s^{1/2} \left(-\rho \frac{\partial \bar{w}_d}{\partial r} \Big|_{r=1} + \frac{4}{S_0} \bar{x}_1(1, \alpha) + B \right) \sim R_s^{1/2} W_0(\rho, \alpha),$$

i.e.
$$B = \lim_{\rho \rightarrow \infty} \left(W_0(\rho, \alpha) - \frac{4}{S_0} X_0(\rho, \alpha) \right) \quad (5.48)$$

after using (5.43). From (5.36) and (5.44) we see that the terms $O(\rho)$ in (5.48) will cancel, and thus for W_0 and $\frac{4}{S_0} X_0$ to differ by a constant, (5.48) is equivalent to the condition

$$\frac{\partial W_0}{\partial \alpha} \sim \frac{4}{S_0} \frac{\partial X_0}{\partial \alpha} \quad \text{as } \rho \rightarrow \infty. \quad (5.49)$$

In the course of the boundary layer solution this result will be derived thereby showing that to first order no inconsistencies have been introduced by the expansion procedure.

The boundary layer equations (5.17) and (5.39) constitute a linear problem for W_0 , though the complicated nature of the coefficients in the equations makes the solution difficult. An approximate analytical solution for W_0 can be found by assuming that convection of momentum in the boundary layers

takes place with the core velocity just outside the layer. This is the simplification used by Lyne to linearize the boundary layer equations for the flow in the cross-section, and the analysis below follows very closely that presented by Lyne.

For the boundary layer on $r = 1$, the velocities in the cross-section U_o , V_o are written as perturbations from the core velocity just outside the layer i.e.

$$\left. \begin{aligned} V_o &= \bar{V}_o(\alpha) + V_{op} \\ \text{and } U_o &= \rho \frac{d\bar{V}_o}{d\alpha} + U_{op} \end{aligned} \right] , \quad (5.50)$$

where V_{op} and U_{op} are assumed to be small compared with $\bar{V}_o$ and $\rho \frac{d\bar{V}_o}{d\alpha}$ respectively. For convenience W_o is written as a perturbation of the axial core velocity, but no assumption is made as to the size of the perturbation. Specifically we write

$$W_o = -\rho \left. \frac{\partial \bar{W}_o}{\partial r} \right|_{r=1} + W_{op} \quad (5.51)$$

which leads to the equation, from (5.16),

$$\begin{aligned} \bar{V}_o \frac{\partial W_{op}}{\partial \alpha} - \rho \frac{d\bar{V}_o}{d\alpha} \frac{\partial W_{op}}{\partial \rho} - \frac{4}{\bar{S}_o} \bar{V}_o U_{op} + \rho V_{op} \frac{4}{\bar{S}_o} \frac{d\bar{V}_o}{d\alpha} \\ + V_{op} \frac{\partial W_{op}}{\partial \alpha} - U_{op} \frac{\partial W_{op}}{\partial \rho} = \frac{\partial^2 W_{op}}{\partial \rho^2} , \end{aligned} \quad (5.52)$$

for W_{op} . The boundary conditions are

$$\left. \begin{aligned} W_{op} &= 0 \quad \text{on } \rho = 0 , \\ \text{and } \frac{\partial W_{op}}{\partial \rho} &\rightarrow 0 \quad \text{as } \rho \rightarrow \infty . \end{aligned} \right] \quad (5.53)$$

As $V_{op} \rightarrow 0$ as $\rho \rightarrow \infty$, for large values of ρ (5.52) reduces to

$$\bar{V}_o \left(\frac{\partial W_{op}}{\partial \alpha} - \frac{4}{\bar{S}_o} U_{op} \right) \rightarrow 0 ,$$

which proves (5.49), as $\bar{V}_o$ is non-zero.

The approximation that V_{op} and U_{op} are small allows the last two terms on the left hand side of (5.52) to be neglected compared to the first two. This, together with the transformation

$$\left. \begin{aligned} y &= (-\bar{S}_o)^{\frac{1}{2}} \bar{S}_o^{-1} \bar{V}_o \rho , \\ x &= -\bar{S}_o^{-1} \int_{\pi}^{\alpha} \bar{V}_o d\alpha , \\ \lambda(x,y) &= \bar{S}_o W_{op} . \end{aligned} \right] \quad (5.54)$$

changes (5.52) into the diffusion equation

$$\frac{\partial \lambda}{\partial x} = \frac{\partial^2 \lambda}{\partial y^2} + p(x, y) , \quad (5.55)$$

where

$$p(x, y) = \rho V_{op} \frac{4}{s_0} \left(\frac{s_0}{\bar{V}_0} \right)^2 \frac{d\bar{V}_0}{d\alpha} - 4 \left(\frac{s_0}{\bar{V}_0} \right) U_{op} . \quad (5.56)$$

The boundary conditions for (5.55) are

$$\left. \begin{aligned} \lambda(x, 0) &= 0 \\ \text{and} \quad \frac{\partial \lambda}{\partial y} &\rightarrow 0 \quad \text{as } y \rightarrow \infty , \end{aligned} \right] \quad (5.57)$$

so that the solution is

$$\begin{aligned} \lambda(x, y) = & \int_0^\infty \frac{\lambda(0, u)}{2\sqrt{\pi x}} \left[e^{-\frac{(y-u)^2}{4x}} - e^{-\frac{(y+u)^2}{4x}} \right] du \\ & + \int_0^\infty \int_0^x \frac{p(x-t, u)}{2\sqrt{\pi t}} \left[e^{-\frac{(y-u)^2}{4t}} - e^{-\frac{(y+u)^2}{4t}} \right] dt du , \end{aligned} \quad (5.58)$$

where $\lambda(0, u)$ is some initial profile.

A similar analysis may be performed for the boundary layer on the axis of symmetry. The velocity components in the cross-section, U_0 , V_0 are written as

$$\left. \begin{aligned} V_0 &= \bar{V}_0(s) + V_{op} \\ \text{and} \quad U_0 &= -N \frac{d\bar{V}_0}{ds} + U_{op} , \end{aligned} \right] \quad (5.59)$$

where V_{op} and U_{op} are small quantities. Writing the axial velocity as

$$W_0 = N \left. \frac{\partial \bar{W}_0}{\partial n} \right|_{n=0} + W_{op} , \quad (5.60)$$

(5.39) becomes

$$\begin{aligned} \bar{V}_0 \frac{\partial W_{op}}{\partial s} - N \frac{d\bar{V}_0}{ds} \frac{\partial W_{op}}{\partial N} - 4N \bar{s}_0^{-1} \frac{d\bar{V}_0}{ds} V_{op} - 4 \bar{s}_0^{-1} \bar{V}_0 U_{op} \\ + V_{op} \frac{\partial W_{op}}{\partial s} + U_{op} \frac{\partial W_{op}}{\partial N} = \frac{\partial^2 W_{op}}{\partial N^2} . \end{aligned} \quad (5.61)$$

The boundary conditions, from (5.40), are

$$\left. \begin{aligned} \frac{\partial W_{op}}{\partial N} &= 4 \bar{s}_0^{-1} \bar{V}_0 \quad \text{on } N=0 \\ \text{and} \quad \frac{\partial W_{op}}{\partial N} &\rightarrow 0 \quad \text{as } N \rightarrow \infty . \end{aligned} \right] \quad (5.62)$$

Letting $N \rightarrow \infty$ in (5.61) we obtain the result

$$\bar{V}_0 \left(\frac{\partial W_{0p}}{\partial s} - \frac{4}{\bar{V}_0} U_{0p} \right) \rightarrow 0 ,$$

which is needed to prove the result analogous to (5.49) for matching between the core and the layer on the axis of symmetry. Neglecting the last two terms on the left hand side of (5.61) and using the transformation

$$\left. \begin{aligned} Y &= -(-\zeta_0)^{1/2} \int_0^s \bar{V}_0 N \\ X &= -\zeta_0^{-1} \int_0^s \bar{V}_0 ds \\ \Lambda(X,Y) &= \zeta_0 W_{0p} \end{aligned} \right\} \quad (5.63)$$

we get the diffusion equation

$$\frac{\partial \Lambda}{\partial X} = \frac{\partial^2 \Lambda}{\partial Y^2} + P(X,Y) , \quad (5.64)$$

$$\text{where} \quad P(X,Y) = -\frac{4}{\zeta_0} \left(\frac{\zeta_0}{\bar{V}_0} \right)^2 N V_{0p} \frac{d\bar{V}_0}{ds} - \frac{4\zeta_0}{\bar{V}_0} U_{0p} . \quad (5.65)$$

The boundary conditions (5.62) become

$$\left. \begin{aligned} \frac{\partial \Lambda}{\partial Y} &= 4(-\zeta_0)^{1/2} \quad \text{on } Y=0 \\ \frac{\partial \Lambda}{\partial Y} &\rightarrow 0 \quad \text{as } Y \rightarrow \infty \end{aligned} \right\} \quad (5.66)$$

If $\Lambda(0,u)$ is some (unknown) initial profile then the solution of (5.64) satisfying (5.66) is

$$\begin{aligned} \Lambda(X,Y) &= \int_0^\infty \frac{\Lambda(0,u)}{2\sqrt{\pi X}} \left[e^{-\frac{(Y-u)^2}{4X}} + e^{-\frac{(Y+u)^2}{4X}} \right] du \\ &+ \int_0^\infty \int_0^X \frac{P(X-t,u)}{2\sqrt{\pi t}} \left[e^{-\frac{(Y-u)^2}{4t}} + e^{-\frac{(Y+u)^2}{4t}} \right] dt du \\ &+ 4(-\zeta_0)^{1/2} \int_0^Y \operatorname{erfc}\left(\frac{u}{2\sqrt{X}}\right) du . \end{aligned} \quad (5.67)$$

The solutions given by (5.58) and (5.67) have been obtained assuming the wall layer and the layer on the axis of symmetry are unrelated. However both solutions contain unknown initial profiles which must be determined by examining the flow where one layer ends and the other begins, thereby linking the two boundary regions. First, the results of

Lyne's analysis for the boundary layer flow in the cross-section are summarized.

As the layer on the wall approaches the corner $r = 1$, $\alpha = 0$ its thickness increases to $O(R_0^{1/2} \log^{1/2} R_0)$ and the viscous terms in the boundary layer equation become small compared to the inertia terms. It is assumed that the motion in the corner remains inviscid, so that due to the symmetry of the equations and boundary conditions, the velocity profile leaving the corner is the same as the velocity profile approaching the corner (Stewartson 1957). This result does not depend on the linearization of the equations. However the perturbation velocities V_{op} and U_{op} increase as the corner is approached, eventually causing the linearization of the boundary layer equations to be a poor approximation.

For the axial flow, it is clear that the failure of the linearization for the flow in the cross-section will also cause the simplification of (5.52) and (5.61) to be invalid before the corner is reached. We note that using (5.55) and (5.64) to describe the flow all the way to the corner is the same order of approximation as used by Lyne. Also, examination of the full axial momentum equation near the corner shows that, due to the thickening of the boundary layer, the viscous terms become small compared to the convective terms. Assuming the motion remains inviscid, then the axial velocity profile is convected around the corner on the streamlines of the flow in the cross-section. This leads to the result

$$\left. \begin{aligned} y &\equiv Y \\ \lambda(x_e, y) &= \Lambda(0, Y) \end{aligned} \right\} \quad (5.68a)$$

as $\bar{V}_0(\alpha)$ and $\bar{V}_0(s)$ have the same behaviour in the corner.

The end of the wall layer, $r = 1$, $\alpha = 0$ is denoted x_e .

Similarly, if x_e is the end of the layer $r = 1$, $\alpha = \pi$, then

$$\left. \begin{aligned} Y &\equiv y \\ \Lambda(x_e, y) &= \lambda(0, y) \end{aligned} \right\} \quad (5.68b)$$

Equations (5.58) and (5.67) can now be combined using (5.68) to get the integral equation for the initial profile $\lambda(0, y)$

$$\begin{aligned} \lambda(0, Y) = & \int_0^\infty \int_0^\infty \frac{\lambda(0, u)}{4\pi\sqrt{x_e x_e}} \left[e^{\frac{-(y-u)^2}{4x_e}} - e^{\frac{-(y+u)^2}{4x_e}} \right] \left[e^{\frac{-(Y-y)^2}{4x_e}} + e^{\frac{-(Y+y)^2}{4x_e}} \right] dy du \\ & + \int_0^\infty \int_0^\infty \int_0^{x_e} \frac{p(x_e - t, u)}{4\pi\sqrt{t x_e}} \left[e^{\frac{-(y-u)^2}{4t}} - e^{\frac{-(y+u)^2}{4t}} \right] \left[e^{\frac{-(Y-y)^2}{4x_e}} + e^{\frac{-(Y+y)^2}{4x_e}} \right] dt du dy \\ & + 4(-f_0)^{1/2} \int_0^Y \operatorname{erfc}\left(\frac{u}{2\sqrt{x_e}}\right) du \\ & + \int_0^\infty \int_0^{x_e} \frac{P(x_e - t, u)}{2\sqrt{\pi t}} \left[e^{\frac{-(Y-u)^2}{4t}} + e^{\frac{-(Y+u)^2}{4t}} \right] dt du . \end{aligned}$$

After performing the y integrations the equation becomes

$$\begin{aligned} \lambda(0, Y) = & \int_0^\infty \frac{\lambda(0, u)}{2\sqrt{\pi(x_e + x_e)}} \left[e^{\frac{-(Y-u)^2}{4(x_e + x_e)}} \operatorname{erf}(\alpha) + e^{\frac{-(Y+u)^2}{4(x_e + x_e)}} \operatorname{erf}(\beta) \right] du \\ & + \int_0^\infty \int_0^{x_e} \frac{p(x_e - t, u)}{2\sqrt{\pi(t + x_e)}} \left[e^{\frac{-(Y-u)^2}{4(t + x_e)}} \operatorname{erf}(\alpha_t) + e^{\frac{-(Y+u)^2}{4(t + x_e)}} \operatorname{erf}(\beta_t) \right] dt du \\ & + 4(-f_0)^{1/2} \int_0^Y \operatorname{erfc}\left(\frac{u}{2\sqrt{x_e}}\right) du \\ & + \int_0^\infty \int_0^{x_e} \frac{P(x_e - t, u)}{2\sqrt{\pi t}} \left[e^{\frac{-(Y-u)^2}{4t}} + e^{\frac{-(Y+u)^2}{4t}} \right] dt du , \quad (5.69) \end{aligned}$$

where

$$\alpha = \left(\frac{u}{4x_e} + \frac{Y}{4x_e} \right) \left(\frac{4x_e x_e}{x_e + x_e} \right)^{1/2} ,$$

and

$$\beta = \left(\frac{u}{4x_e} - \frac{Y}{4x_e} \right) \left(\frac{4x_e x_e}{x_e + x_e} \right)^{1/2} . \quad (5.70)$$

α_t and β_t are obtained from (5.70) by replacing x_e with t in the definitions (5.70).

The numerical solution of (5.69) was attempted but a reasonable result could not be obtained. The difficulty of evaluating the second and fourth terms in (5.69) sufficiently accurately in a reasonable amount of time proved too great; both p and P involved integrals over an infinite range, and integrals with singular integrands. By neglecting these two terms an approximate solution for $(-f_0)^{1/2} \lambda(0, Y)$ was obtained and is given in Table 1. The integrals were evaluated using Simpson's rule and a step length of 0.1. Infinity was taken to be 12.8, and this was more than adequate. However as the major contribution to the first integral in (5.69) comes from the region $u = Y$, it was necessary to take the upper limit of integration in this integral as 25.6. As the profile tends to a constant as $Y \rightarrow \infty$, the value of λ at $Y = 12.8$ was

used as the value of λ for $12.8 \leq Y \leq 25.6$. The integral equation was solved by an iterative technique, the initial approximation to the profile being $\lambda \equiv 0$. The iteration process was terminated when two successive iterations differed by less than 0.005 % .

A useful check on the numerical solution is provided by requiring the shear stress at the wall of the pipe to balance the applied pressure gradient; for the mean flow, this relationship is expressed by (5.15). From (5.29), (5.51) and (5.54) the axial velocity gradient at the wall is given by

$$\left. \frac{\partial w_{10}}{\partial r} \right|_{r=1} = - \left(4 \zeta_0^{-1} \bar{V}_0 + \zeta_0^{-1} \frac{\partial \lambda}{\partial y} \left((-\zeta_0)^{1/2} \zeta_0^{-1} \bar{V}_0 \right) \right) + O(R_3^{-1/2})$$

If the contribution of the term p in (5.58) is neglected when calculating $\frac{\partial \lambda}{\partial y}$, we obtain

$$\left. \frac{\partial w_{10}}{\partial r} \right|_{r=1} = - \left(4 \zeta_0^{-1} \bar{V}_0 - \zeta_0^{-1} \bar{V}_0 \int_0^\infty \frac{(-\zeta_0)^{1/2} \lambda(0,u)}{2 \sqrt{\pi x}} \cdot \frac{u}{x} \cdot e^{-u^2/4x} du \right), \quad (5.71)$$

which, when substituted in (5.15) gives the condition

$$- \int_0^\infty (-\zeta_0)^{1/2} \lambda(0,u) \operatorname{erfc} \left(\frac{u}{2 \sqrt{x_c}} \right) du = 2\pi - 4x_c. \quad (5.72)$$

This condition was satisfied to 4 significant figures when the converged profile $\lambda(0,u)$ was used. (The integral was evaluated over $0 \leq u \leq 12.8$ using Simpson's rule and a step length of 0.1.)

The flow rate of fluid through the pipe can now be calculated. The velocities in the boundary layers are $O(R_3^{-1/2})$ and so, to leading order, do not contribute to the flow through the pipe. The flux ratio $Q_{r\infty}$, defined by (4.14), is therefore

$$Q_{r\infty} = \frac{\int_A \bar{w}_0 dA}{\pi/2} + O(R_3^{-1/2})$$

$$\text{i.e.} \quad Q_{r\infty} = 0.379 + O(R_3^{-1/2}) \quad (5.73)$$

CHAPTER 6

The uniform vorticity model

For steady laminar flow with closed streamlines Batchelor (1956) has shown how an integral condition arising from the effect of viscosity can be used with the inviscid flow equations to determine the vorticity distribution when the Reynolds number is large. The integral condition is an exact result and is derived by integrating the full equations of motion around a closed streamline (or vortex line). The method was applied successfully to two-dimensional flows, axisymmetric flows without swirl and a limited class of axisymmetric flows with swirl. In this chapter a condition analogous to that used by Batchelor will be applied to a flow where the streamlines are not closed, but lie on the surface of a cylinder. The results of this chapter have already been derived in Chapter 5 by formal methods; the alternative derivation used here provides some information not readily obtainable from the previous method. In particular we show that the single-valuedness condition introduced in an ad hoc manner in Chapter 5 can be derived from an exact line integral of the equations of motion. With this interpretation it is then possible to use the integral condition to get a priori estimates of the size of the velocity in the inviscid region.

The flow studied may be constructed as follows. Consider a steady two-dimensional flow in a confined region, the motion being generated by the boundaries. A stream function ψ can be introduced, and in the plane of the motion a useful co-ordinate system is the orthogonal curvilinear co-ordinates (η, ξ) , the lines $\xi = \text{constant}$ being everywhere normal to the closed lines $\psi = \text{constant}$. The axis normal to the (η, ξ) plane is the z axis, and by definition the velocity components are independent of z . Now impose a velocity in the z -direction such that the stream function describing the previously existing two-dimensional flow is unaltered. This is achieved if the axial velocity is independent of z and the pressure has the form

$$\frac{p(\eta, \xi, z)}{\rho} = \frac{p'(\eta, \xi)}{\rho} - Gz, \quad (6.1)$$

where G is a constant. The condition on the axial velocity comes directly from continuity requirements, and the equations of motion dictate the form of the pressure. The streamlines of this motion are not closed, but lie on right cylinders; the projection of a streamline onto the plane $z = \text{constant}$ gives the closed curve $\Psi = \text{constant}$.

From the definition of the velocity field the equations of motion will separate into an equation, independent of the axial flow, for the motion in the z -plane, and an equation for the axial velocity. Similarly an integral of the full equations of motion along a streamline can be expected to give two independent integrals, one representing the properties of the axial velocity, the other a property of the motion in the z -plane. The latter integral should be the same as used by Batchelor in two-dimensional flow when the streamlines are actually closed. We shall show that this is the case, and also derive a condition on the axial velocity.

The equation of steady motion for a viscous, incompressible fluid is

$$\underline{v} \times \underline{\omega} = \nabla H + \nu \text{curl} \underline{\omega} \quad (6.2)$$

where $\underline{\omega}$ is the vorticity vector and is the curl of the velocity field $\underline{v}$. The quantity H is the total head and is given by

$$H = \frac{p}{\rho} + \frac{1}{2} \underline{v} \cdot \underline{v}$$

In the (ψ, ζ, z) co-ordinate system the velocity vector $\underline{v}$ is $[0, q, w]$ and the infinitesimal line element is $[d\psi/q, h_2 d\zeta, dz]$ where h_2 is an unknown function of ψ and ζ . The vorticity $\underline{\omega}$ is then

$$\underline{\omega} = [\omega_1, \omega_2, \omega_3] = \left[\frac{1}{h_2} \frac{\partial w}{\partial \eta}, -q \frac{\partial w}{\partial \psi}, \frac{q}{h_2} \frac{\partial}{\partial \psi} (h_2 q) \right] \quad (6.3)$$

When the viscous forces are small (6.2) may be replaced by the statement that H is constant on streamlines. Using (6.1), the total head can be written

$$H = \frac{p'(\psi, \zeta)}{\rho} + \frac{1}{2} q^2 - Gz + \frac{1}{2} w^2 \quad (6.4)$$

When G and w are zero the flow is two-dimensional and H becomes $H' = \frac{p'}{\rho} + \frac{1}{2} q^2 \equiv H'(\psi)$ in the region of small viscous forces. By definition, G and w do not alter the flow described by ψ , so that in the region of small viscous forces (6.4) becomes

$$H = H'(\psi) - Gz + \frac{1}{2} w^2, \quad (6.5)$$

which must be constant on streamlines.

The inviscid equations give no information as to the variation of H across the streamlines. To obtain such information we follow Batchelor and derive an exact result by integrating (6.2) along a streamline. The endpoints A and B of the path of integration are chosen so that one circuit of the projection of the streamline in the z -plane is made. We obtain

$$\int_{AB} \nabla H \cdot d\mathbf{s} + \nu \int_{AB} \text{curl} \boldsymbol{\omega} \cdot d\mathbf{s} = 0, \quad (6.6)$$

where $d\mathbf{s}$ is the line element in the streamline.

The first integral in (6.6) is independent of the path of integration, and so may be taken along the straight line AB , which, from the definition of the endpoints, is in the direction of the z -axis. Thus only $\frac{\partial H}{\partial z}$ will contribute to this term. The second integral may be expanded by expressing $d\mathbf{s}$ in terms of the (ψ, ξ, z) co-ordinate system, where the streamline element is $[0, h_2 d\xi, dz]$ and the equation of the streamline is

$$\frac{\gamma q d\psi}{0} = \frac{h_2 d\xi}{q} = \frac{dz}{w}. \quad (6.7)$$

Thus (6.6) becomes

$$\begin{aligned} -G \int_A^B dz + \nu \int_A^B \frac{q}{h_2} \left(\frac{\partial}{\partial \psi} (\omega_2 h_2) - \frac{\partial}{\partial \xi} \left(\frac{\omega_1}{q} \right) \right) dz \\ + \nu \oint \left(-q \frac{\partial \omega_3}{\partial \psi} \right) h_2 d\xi = 0. \end{aligned} \quad (6.8)$$

The path of integration for the last integral in (6.8) is the closed curve $\psi = \text{constant}$ in the z -plane, and clearly

this integral is independent of both G and w . Thus (6.8) gives the two independent relationships

$$-\nu \oint \eta \frac{\partial \omega_3}{\partial \psi} h_2 d\zeta = 0, \quad (6.9)$$

and
$$-G \int_A^B dz + \nu \int_A^B \frac{\eta}{h_2} \left(\frac{\partial}{\partial \psi} (\omega_2 h_2) - \frac{\partial}{\partial \zeta} \left(\frac{\omega_1}{\eta} \right) \right) dz = 0. \quad (6.10)$$

Both (6.9) and (6.10) are exact results. Equation (6.9) was derived by Batchelor (1956) for two-dimensional flow with closed streamlines, and when the path of integration lies wholly in the region of small viscous forces, the inviscid flow approximation $\omega_3 = \omega_3(\psi)$ may be used with (6.9) to derive the result that $\omega_3(\psi)$ is a constant, say J . We are interested in the conditions imposed by (6.10) on G and w as $\nu \rightarrow 0$, when the streamline path of integration lies wholly in the region of small viscous forces. Before turning away from the flow in the cross-section we note the result, given by (6.3)

$$J = \frac{\eta}{h_2} \frac{\partial}{\partial \psi} (h_2 \eta). \quad (6.11)$$

A more useful form of (6.10) can be obtained by using (6.7) to replace the z integral by an integral around the closed contour $\psi = \text{constant}$. The balance expressed by (6.10) is then

$$-G \oint \frac{w h_2}{\eta} d\zeta + \nu \oint w \left(\frac{\partial}{\partial \psi} (\omega_2 h_2) - \frac{\partial}{\partial \zeta} \left(\frac{\omega_1}{\eta} \right) \right) d\zeta = 0, \quad (6.12)$$

where the path of integration is in the region of inviscid flow. By definition, velocity gradients remain $O(1)$ in an inviscid flow as $\nu \rightarrow 0$, and thus we can discard the possibility that large velocity gradients are responsible for the balance of terms in (6.12). Excluding the possibility that the first term in (6.12) is identically zero, this relationship forces the ordering

$$G \sim O(\nu w), \quad \nu \rightarrow 0. \quad (6.13)$$

Using this in (6.5) and letting $\nu \rightarrow 0$ gives

$$H = H'(\psi) + \frac{1}{2} w^2(\psi, \zeta), \quad (6.14)$$

and for H to be constant on streamlines in the inviscid region, w must be a function of ψ only, $w(\psi)$. This conclusion is consistent with the requirement that the first term of (6.12) is non-zero. The neglect of the pressure gradient, $-Gz$, in going from (6.5) to (6.14) is consistent with the definition of H . Essentially (6.4) is an energy equation derived by integrating the exact equations of motion and neglecting the $O(v)$ viscous terms as $v \rightarrow 0$. Here G is $O(v)$ compared to other terms in (6.5) and so may be neglected. The form (6.14) of the energy equation is not immediately apparent from the derivation of Chapter 5, and also we see that integrating the equations of motion has given the correct order of the pressure gradient and velocity directly.

Substituting $w = w(\psi)$ in (6.3) and using this to simplify (6.12) we obtain

$$w(\psi) \times \left[G \oint \frac{h_2}{q} d\zeta + \nu \oint \frac{\partial}{\partial \psi} (h_2 q \frac{dw}{d\psi}) d\zeta \right] = 0. \quad (6.15)$$

The non-trivial solution of (6.15) is

$$w(\psi) = - \frac{G\psi}{\nu \zeta} + C \int \frac{d\psi}{\oint h_2 q d\zeta} + D, \quad (6.16)$$

where C and D are arbitrary constants and (6.11) has been used to simplify the first term. The unknown constants will be determined by requiring the flow in the inviscid region to match onto the flow at the edge of the boundary layers.

The axial velocity may be generated by a pressure gradient (as in Chapter 5) or by some axial motion at the boundaries (as in Chapter 9) or by some combination of both. In the case of $G \equiv 0$ the equation of motion for w , in cartesian co-ordinates, is the same as the equation for ω_3 . When a two-dimensional flow has closed streamlines we have that ω_3 is constant in any connected inviscid region. For an axial flow through such a region it may be tempting to say that by analogy with ω_3 , w is also constant in the inviscid region. Equation (6.16) shows this analogy to be incorrect, there being the possibility that w varies across the streamlines when C is non-zero.

Apart from matching conditions, there is also the condition that w must be regular which can be used to determine the unknown constants. In particular, in any region where ψ has a local extrema, then C must be zero for w to be regular. This is readily shown by noting that in a flow of uniform vorticity the circulation $\oint \mathbf{h} \cdot d\mathbf{s}$ along a line $\psi = \text{constant}$ goes to zero like $(\psi - \psi_m)$ as $\psi \rightarrow \psi_m$, the local extrema, so that the integral in (6.16) is logarithmically singular at $\psi = \psi_m$.

We conclude this chapter by noting that the equations of steady heat and mass transfer have the same structure as the equation governing the axial flow considered here. It is expected then that certain line integrals of the exact equations will be useful in determining the temperature and concentration distributions in regions where diffusive effects are small and the fluid flow similar to the fully developed motion studied in this chapter.

CHAPTER 7

A numerical approach

Equations (3.38) and (3.40) for the leading order stream function and the mean flow contain all the nonlinearity present in the full Navier-Stokes equations, so that an analytic solution is not generally possible. Thus in order to gain some knowledge of the structure of the stream function and axial velocity for arbitrary values of the parameters D and R_s , a numerical solution was obtained. For certain ranges of D and R_s , the behaviour of χ_0 and w_1 is already known: for example Chapter 5 has dealt with the case $D \rightarrow 0$, R_s fixed, and as will be shown later the results can be extended to the parameter range $D \sim o(R_s)$; also when $D \gg R_s$, Smith (1975) has shown that the steady streaming generated by the Stokes layer is swamped by the secondary motion due to centripetal pressure forces, and that χ_0 and w_1 have the same form as for exactly steady flow in a curved pipe. Detailed numerical calculation in these ranges is not necessary, though known solutions provide a useful check for any numerical scheme. Instead attention is focused on the intermediate parameter range where the secondary flow changes its direction. Finally, as the results of Chapter 5 seemed somewhat novel, these numerical calculations were carried out so as to check the predictions of the asymptotic solution.

The numerical method used was that devised by McConalogue and Srivastava (1968) for steady flow in a curved pipe. Here the stream function χ_0 , third component of vorticity $-\nabla^2 \chi_0$ and the axial velocity w_1 are expanded in a Fourier series

$$\left. \begin{aligned} \chi_0 &= \frac{A_0(r)}{2} + \sum_{n=1}^{\infty} \left(A_n(r) \cos n\alpha + B_n(r) \sin n\alpha \right), \\ -\nabla^2 \chi_0 &= \frac{C_0(r)}{2} + \sum_{n=1}^{\infty} \left(C_n(r) \cos n\alpha + D_n(r) \sin n\alpha \right), \\ \text{and } w_1 &= \frac{E_0(r)}{2} + \sum_{n=1}^{\infty} \left(E_n(r) \cos n\alpha + G_n(r) \sin n\alpha \right), \end{aligned} \right] \quad (7.1)$$

where the Fourier coefficients are functions of r only. These series are then substituted into the governing

equations (3.38) and (3.40) and equations for the Fourier coefficients obtained by equating the coefficients of $\cos n\alpha$ and $\sin n\alpha$. The nonlinear terms in (3.38) and (3.40) produce terms which are the summation of the interaction of all harmonic components. These interaction terms are assumed known and the resulting linear ordinary differential equation for the Fourier coefficient is solved by variation of parameters. Thus each of the A_n etc. are known in terms of integrals of themselves and all other Fourier coefficients, so that the problem is reduced to the solution of a set of coupled integral equations.

To evaluate the integrals numerically the $A_n(r)$ etc. were represented by their values at discrete points in the interval $[0,1]$. In the original work of McConalogue and Srivastava a least squares polynomial was fitted to these values and the integrals evaluated assuming that this polynomial adequately represented the variation of the function between the mesh points. This approach had the advantage of tending to damp any oscillations introduced by the iterative nature of the solution algorithm, but the accuracy of the approximation was limited by the order of the polynomial chosen. Also, the polynomial approximation to a given function would not be able to satisfy the relevant boundary conditions, thereby introducing further errors. To overcome these difficulties cubic splines (Ahlberg, Nilson and Walsh 1967) were used to represent the integrands between the mesh points. In this way, not only were the boundary conditions satisfied, but use was made of first and second derivatives of the coefficients at the endpoints of the interval. This modification proved to be most successful and solutions were obtained at Dean numbers well above the value at which the original technique failed.

The coupled integral equations for the Fourier coefficients were solved by an iterative technique. An initial approximation to the A_n etc. was provided by analytic solutions (for low R_s and D), or more usually by a previously found numerical solution at another point in the D - R_s plane. From this initial guess for all the A_n etc. a new approximation

to a given coefficient was found by evaluating the integrals defining that coefficient. Thus in the calculation of subsequent coefficients there is now a choice between two approximations to the same coefficient. Brief initial experimentation showed that the faster convergence was obtained by using the most recently determined approximation for each coefficient. The iteration was terminated when new approximations for all the coefficients agreed to 4 significant figures with the immediately preceding estimates. These converged values of the A_n etc. were then taken as the solution and the various properties of χ_0 and w_1 were obtained by summing the series (7.1).

For all values of D and R_s convergence to the solution was not monotonic, but the approximations to the A_n etc. oscillated about the final result. For large values of D and R_s these oscillations were rather wild and the iteration technique became divergent. By taking a weighted average of the initial and just calculated approximation to each coefficient the oscillations were damped, and a convergent iteration scheme obtained. The weight factors in the average were such that as the parameters were increased, less and less importance was placed on the latest approximation to each coefficient. This had the effect of producing a slow rate of convergence, but at least the process remained stable and a converged solution obtained in a reasonable amount of computer time. The number of subdivisions of the interval $[0,1]$, NR , and the number of terms taken in the series (7.1), NCF , also had a small effect on the stability of the iteration scheme. Increasing both NR and NCF tended to stabilize the algorithm, but this was obtained at the expense of computer storage and time.

The number of Fourier terms and the step length necessary to get an accurate representation of χ_0 and w_1 was determined by experimentation. Initially a step length of 0.1 (corresponding to $NR = 11$) and terms to $\sin 11\alpha$ and $\cos 11\alpha$ (corresponding to $NCF = 12$) were used. This was more than adequate for low values of D and R_s , but at larger values of these parameters the rate of convergence of (7.1) became

slower and more terms were needed to maintain sufficient accuracy. Similarly as boundary layer structures began to develop the step length had to be decreased so as to obtain reasonable accuracy in regions of large gradients.

Throughout the work the values of $\chi_0, -\nabla^2 \chi_0$ and w_1 for $\alpha = 0, \pi/2, \pi, 3\pi/2$ were used to monitor the accuracy of the results. At selected values of D and R_s two calculations of these quantities were performed: the second calculation having NOF or NR, or both, increased by at least 50%. If the results of the two calculations agreed to 4 significant figures, the smaller values of NOF and NR were deemed adequate. When the two results differed, the larger NOF and NR were adopted for further calculation. The values of NOF and NR used for each calculation are shown with the results. Also, some typical results of the iteration process are given along with details of the solution by the method of McConalogue and Srivastava in Appendix A.

The program was originally formulated to solve the decoupled equations (5.2) and (5.3) and thereby check the asymptotic solution for the axial velocity given in Chapter 5. A simple problem governed by the decoupled equations for which an analytical solution can be obtained was used to test the numerical scheme. The problem is governed by (5.2) and (5.3) and has boundary conditions

$$\left. \begin{aligned} \chi_0 &= 0 \\ \frac{\partial \chi_0}{\partial r} &= 1 + \epsilon \sin \alpha \\ w_1 &= 0 \end{aligned} \right\} \quad \text{on } r=1. \quad (7.2)$$

The motion corresponds to flow in a straight pipe when the pipe wall 'rotates' about the axis of the pipe with velocity $-(1+\epsilon \sin \alpha)$. For small ϵ this problem is a perturbation of solid body rotation and Poiseuille flow, and analytic solutions can be obtained by linearizing (5.2) and (5.3) about this basic flow. Burggraf (1966) calculated the perturbation to solid body rotation using this linearization, but applied the results outside the limits of validity of the approximation. The basic flow and $O(\epsilon)$ perturbation form an accurate approximation to the solution for $\epsilon \leq 0.05$.

Neglecting $O(\epsilon^2)$ the solution for the stream function is

$$\chi_0 = \frac{1}{2}(r^2-1) + \epsilon \left[a_1 r + b_1 I_1(r\sqrt{-iR_s}) \right] e^{i\alpha} \quad (7.3a)$$

Using this result in (5.3) and neglecting $O(\epsilon^2)$ again we get

$$w_1 = (r^2-1) + \epsilon \left[-2a_1 r + b_1 r \sqrt{-iR_s} I_0(r\sqrt{-iR_s}) + c_1 I_1(r\sqrt{-iR_s}) \right] e^{i\alpha} \quad (7.3b)$$

where only the real parts of (7.3a,b) have physical significance. Here I_0, I_1, I_2 are the modified Bessel functions (Abramowitz and Stegun 1965) and the constants a_1, b_1 and c_1 are:

$$a_1 = \frac{i I_1(\sqrt{-iR_s})}{\sqrt{-iR_s} I_2(\sqrt{-iR_s})} \quad (7.3c)$$

$$b_1 = -\frac{i}{\sqrt{-iR_s} I_2(\sqrt{-iR_s})} \quad (7.3d)$$

and
$$c_1 = \frac{i I_0(\sqrt{-iR_s})}{I_1(\sqrt{-iR_s}) I_2(\sqrt{-iR_s})} + \frac{2i}{\sqrt{-iR_s} I_2(\sqrt{-iR_s})} \quad (7.3e)$$

Solutions from the numerical procedure for w_1, χ_0 and $-\nabla^2 \chi_0$ on $\alpha = 0, \pi/2, \pi, 3\pi/2$ and for the flow rate compared to Poiseuille flow were compared with the results of the analytical solution in the range $0 < R_s \leq 400$ and $\epsilon \leq 0.01$. For the numerical scheme NOF was 8 and NR was 11. Agreement between the two solutions was good, generally 4 significant figures, and as the flow had developed a boundary layer/ core structure at $R_s = 400$ further testing of the program was not carried out.

With the computational procedure verified, the problem posed in Chapter 5 could now be attacked. A minor change in the program was all that was necessary for the numerical scheme to now solve (5.2) and (5.3) with boundary conditions (5.4) and (5.5). For definiteness the $a_n, n = 2, 3, \dots$ in the pressure gradient (2.5) were taken as zero, resulting in a value of 0.25 for $\mathcal{W}$. Thus (5.4) and (5.5) are now

$$\left. \begin{aligned} \chi_0 &= 0, \\ \frac{\partial \chi_0}{\partial r} &= \frac{1}{4} \sin \alpha, \\ w_1 &= 0 \end{aligned} \right\} \quad \text{on } r=1. \quad (7.4)$$

A reduction in numerical labour is achieved by noting that the flow is symmetrical about the line $\alpha = 0, \pi$, so that the A_n , C_n and G_n are identically zero.

The stream function now corresponds to that calculated by Lyne (1970) and Kuwahara and Imai (1969) and these published results were used as another check on the numerical method. Fig. 7.1 shows $-\nabla^2 \chi$, predicted by the present work for $R_s = 1,800$ compared with Lyne's asymptotic result, and the numerical result of Kuwahara and Imai at $R_s = 2,048$. The agreement is satisfactory.

The results of this program are presented pictorially in Figs. 7.2 to 7.11 as plots of lines of constant stream function (upper semi-circle) and lines of constant axial velocity (lower semi-circle). The stream function and axial velocity change by one tenth of their respective maximum value from contour to contour. Also indicated on each figure is the position of the extrema of the stream function and the axial velocity and the values of NOF and NR used in the calculation. In Fig. 7.12 the axial velocity contours are shown superimposed on the streamlines for $R_s = 1,800$, and it is clearly seen that the two sets of curves are tending to coincide, thus confirming the analysis of Chapter 5. Also, a graph of the flow rate compared to Poiseuille flow for the same pressure gradient, Fig. 7.13, shows the tendency to the asymptotic solution of Chapter 5.

Finally the program was modified so as to solve the simultaneous equations (3.38) and (3.40) with boundary conditions (7.4). This involved the inclusion of a subroutine to evaluate the centripetal terms in (3.40) and a slight reorganization of the iteration scheme. Two extreme cases of the D , R_s parameter range were used to check the program: very small D and arbitrary R_s ; and very small R_s and arbitrary D . In the first test case the program produced the results already obtained for the decoupled equations, as for small D , the stream function is independent of the flow down the pipe. The second test case corresponds to the classical Dean problem of steady flow in a curved pipe, as the motion generated by the Stokes layer is swamped by the secondary velocities

induced by centripetal pressure forces. The predictions of this calculation were in excellent agreement with the results of McConalogue and Srivastava (1968) and the more recent numerical work of Collins and Dennis (1975). Table 3 contains a comparison of the results for the flow through the pipe compared to Poiseuille flow.

Solutions of (3.38) and (3.40) were obtained at values of D and R_s that illustrated the structure of the flow during the transition from inward to outward secondary motion. This change was followed by fixing R_s and allowing D to vary over a sufficiently large range. The characteristic of the flow which proved most useful in monitoring the motion was the flux of fluid through the pipe compared to Poiseuille flow in a straight pipe with the same pressure gradient, which for conciseness we will call the flux ratio and denote by Q_r .

From the creeping flow solutions of Chapter 4 we know that for (small) fixed R_s the flux ratio increases smoothly to a maximum and then decreases as D increases from zero to values greater than $\sim 2\sqrt{2R_s}$. It was expected that at larger values of D and R_s inertial effects would modify this result in a trivial manner, and that the change in direction of the secondary motion would occur via a continuous process.

Calculations at $R_s = 100$ did not confirm the above expectations. Instead the graph of flux ratio against D consisted of two distinct curves positioned such that for $28 \leq D \leq 31$ there were two values of the flux ratio for the same value of D . Support for this result was provided by calculations at $R_s = 30, 50$ and 75 . For these values of R_s the flux ratio versus D curve was continuous, but it was observed that the slope $\frac{\partial Q_r}{\partial D}$ at the inflection point increased with increasing R_s . (See Fig. 7.14.)

Further evidence of the validity of this result was provided when it was found that altering the iteration scheme, in a manner suggested by Dr. L.M. Hocking, produced results which allowed the two curves to be smoothly connected. Instead of fixing D and calculating the flow field, the axial velocity at the centre of the pipe was specified and the associated velocity field and Dean number calculated

(see Appendix A). It had been observed that the axial velocity on the axis of the pipe was a monotonic increasing function of D on each branch of the flux ratio graph, and that the extreme points of each branch defined a range of axial velocities for which no solutions had been obtained. Setting the axial velocity to values in this interval produced solutions with D in the range $28 \leq D \leq 31$ and flux ratios between the two supposedly distinct branches. It was then seen that the two branches of the flux ratio curve were the upper and lower sections of a continuous 'S' shaped curve, Fig. 7.15, thereby indicating the existence of 3 solutions to the equations in this range.

Finally, each of the 3 possible solutions for a given D in the range $[28, 31]$ were checked by increasing the value of NR from 11 to 17. No significant changes in the results were obtained. It was deemed unnecessary to increase NOF , as for $NOF = 12$ the ratio of the last to the first Fourier coefficient was $O(10^{-10})$.

Calculations at $Re = 150$ again produced surprising results, though less confidence can be placed in them than could for those at $Re = 100$. The flux ratio versus D graph again indicates the non-uniqueness of the solutions, but it appears that rather than being confined to a bounded interval on the D axis, there is the possibility of multiple solutions in the unbounded interval $[D_c, \infty)$ (Fig. 7.16). The modified iteration technique was not able to join the two distinct curves and on one branch of the curve the Fourier series representation (7.1) became quite poor. Further discussion of these results is postponed to the next chapter.

By now it was obvious that the change of direction of the secondary motion was not as straight forward as anticipated, and that due to the latent numerical difficulties, a considerable amount of computational labour would be involved in obtaining further solutions. As the results already obtained were of some interest it was decided to present them and leave numerical treatment of (3.38) and (3.40) unfinished. The results are presented in Figs. 7.17 to 7.48 as plots of lines of constant stream function and lines of constant axial

velocity. From contour to contour the stream function and axial velocity change by one tenth of their respective range, and the positions of the vortex centre(s) and axial velocity maximum are accurate to approximately $\pm 1\frac{1}{2}^\circ$ azimuthally and ± 0.05 radially.

CHAPTER 8

Discussion of results

The determination of the leading order steady secondary motion and the mean axial flow produced by the unsteady pressure gradient (2.5) acting along a curved pipe has been our primary objective. The quantities χ_0 and w_1 are governed by equations (3.38) and (3.40) together with boundary conditions (3.41). Both numerical and analytical solutions have been obtained for χ_0 and w_1 .

Where it has been possible to obtain analytical solutions, the constant $\mathcal{N}$ in (3.41) appears as a parameter in the final result. However, for the numerical solutions $\mathcal{N}$ was taken to be 0.25, the value corresponding to the pressure gradient consisting of a mean and sinusoidal term. This is not as severe a limitation as it seems. In Chapter 3 it was mentioned in passing that χ_0 and w_1 could be rescaled in such a way that R_s appeared only in the boundary conditions as (say)

$$\frac{\partial \chi_0'}{\partial r} = R_s \mathcal{N} \sin \alpha \quad \text{on } r=1.$$

By defining a new steady streaming Reynolds number R_s' to be $R_s \mathcal{N}$ the constant $\mathcal{N}$ no longer appears explicitly in the problem, and so the non-dimensional solution (for χ_0' and w_1' say) must be independent of $\mathcal{N}$. Using this device it is possible to apply the numerical results obtained when $\mathcal{N}$ is 0.25 to arbitrary $\mathcal{N}$. Despite the generality conferred on the numerical work by using R_s' as the Reynolds number, the results will be presented and discussed for $\mathcal{N} = 0.25$, the Reynolds number being R_s .

Analytical solutions for χ_0 and w_1 are available only in the limit $D \rightarrow 0$ when, either the viscous terms dominate the motion, or when the stream function becomes independent of the axial velocity. The creeping flow solutions of Chapter 4 will be discussed later when the emphasis will be on the change of direction of the secondary motion. First we wish to examine the asymptotic analysis of Chapter 5 and compare the results with the numerical solutions of the

equations.

Chapter 5 deals with the limiting form of (3.38) and (3.40) when the stream function is independent of the axial velocity, and hence as given by Lyne. The principal result of the analysis for large R_s is that, away from the boundaries, the axial velocity is proportional to the stream function. This result is confirmed quite well by the numerical calculations at $R_s = 1,800$. The streamlines and axial velocity contours are shown superimposed in Fig. 7.12 and the two sets of curves are very nearly coincident at this value for R_s . The constant of proportionality predicted by the analysis (for the upper semi-circle) is $4/s_\infty = -7.14$ while from the numerical results the axial velocity divided by the stream function, along the line $\alpha = \pi/2$, has the approximate value of -8.1 . This discrepancy is attributed to the rather slow approach of the flow field to its asymptotic state. For example, at $R_s = 1,800$ we see that the vortex centre has not yet reached its asymptotic location of $(0.48, \pi/2)$ (Lyne 1970). Also the flux ratio versus R_s graph (Fig. 7.13) shows a slow approach to the limiting solution. However, if we consider values of R_s between 600 and 1,800 and plot the logarithm of the difference between the asymptotic and actual flux ratios against $\log R_s$, the points fall on a straight line of slope very close to -0.5 , and thus indicating that the terms $O(R_s^{-1/2})$ are still significant at $R_s = 1,800$. Therefore, the difference between numerical and asymptotic results can be attributed to the higher order terms in the asymptotic analysis.

Solution of the integral equation (5.69) for the axial boundary layer proved quite difficult, and only by neglecting the terms involving p and P was it possible to obtain an approximate axial velocity profile in the boundary layer. Though the neglect of these terms has no rational basis, the results obtained agree well with the computational results. Figure 5.3 shows the axial velocity gradient at the wall of the pipe as a function of α for both the asymptotic and numerical ($R_s = 1,800$) solutions. Ignoring the regions near $\alpha = 0, \pi$, where the boundary layer analysis is invalid, despite having slightly different shapes, the two curves agree satis-

factorily. Thus overall we have satisfactory agreement between the asymptotic analysis and the computational results.

In Chapter 5 the full equations (3.38) and (3.40) were simplified by taking the limit $k \rightarrow 0$, R_s fixed so that the stream function became independent of the axial velocity. An alternative way of decoupling the equations, and one which eases the restrictions on the size of D is to consider the limit $R_s \rightarrow \infty$ first. The equations will decouple if the centripetal terms in (3.40) are much less than the convective terms, i.e. if $O(w_1^2) \ll R_s$. When this condition is satisfied, the secondary flow has uniform vorticity, and the analysis of Chapter 5 or 6 then indicates that the size of w_1 is given by a balance between the viscous and pressure gradient terms in the axial momentum equation, and therefore $w_1 \sim O(D R_s^{1/2})$. Hence to satisfy the decoupling condition for large R_s we must have $D \sim o(R_s)$, which considerably extends the range of parameters to which the results of Chapter 5 can be applied.

At this point it is appropriate to attempt to apply our results to the flow of blood in the arch of the (human) aorta. The simplest model for flow in the aortic arch is fully developed flow in a curved pipe, however due to the differences between this model and the physiological conditions it is expected that our results will only provide estimates for shear stresses and their distributions. It becomes clearer that our work is of limited value when it is seen that not all the parameters fall in the correct range for the theory to be valid.

The main approximation to the physiological conditions made in modelling the aorta is the assumption of fully developed flow. In reality a fully developed configuration would not exist due to the repeated branching of the aorta and its changing cross-section. Also the close proximity of the left ventricle means that in the aortic arch entry length effects are likely to be important. It is expected then that shear stresses calculated on the hypothesis of fully developed flow will under estimate the physiological values, and that the branching of the aorta will perturb the shear stress

distribution predicted by the model. The neglect of the elasticity of the artery will be of minor importance, as in the case of the straight elastic pipe (Womersley 1956), and the approximation of small a/R is fair. Finally, for simplicity we assume the pressure gradient which has $\mathcal{N} = 1/4$

Values for the blood velocities and artery size for man are given in MacDonald (1973). Typical values for resting conditions are:

$$\begin{aligned} a &= 1.25 \text{ cm} \\ \text{mean blood flow of } 6\text{L/min} \\ v &= 0.04 \text{ cm}^2 \text{ s}^{-1} \end{aligned} \quad (8.1)$$

while a reasonable estimate of the maximum velocity is 100 cm/sec. Data for the mean pressure gradient in the aorta is scarce and subject to large errors of measurement (MacDonald 1973) so that we can only estimate the Dean number of the flow. If we assume that the mean flow has a parabolic velocity distribution, then for $a/R = 0.25$, D is 440. This is an under estimate of the true value of D . As a first estimate of the amplitude of the velocity fluctuations we subtract the maximum mean velocity from the peak velocity, and therefore we take $\bar{W}$ to be 60 cm/sec. Assuming a heart rate of one beat per second then $\omega = 2\pi$ rad/sec, which gives R_s the value 2,690 and β the value 0.08.

For the asymptotic analysis of Chapter 3 to be valid it is necessary that the secondary velocities be smaller than the leading order oscillatory axial velocity; this condition is expressed as

$$R_s \beta^2 \ll 1$$

For the values of R_s and β given above this condition is not met as $R_s \beta^2$ is 18. Clearly to quote numerical values for quantities associated with the secondary motion is misleading as, for example, shear stresses due to the secondary velocities will be larger than those caused by the axial velocity when in fact they should be asymptotically smaller. Thus in keeping with the limitations already mentioned, only quantities associated with the axial velocity will be considered.

As R_s is large, and also much larger than D , the flow should have the asymptotic form given by Chapter 5. Assuming this to be so, new estimates of D and R_s can be obtained. The estimate of D can be improved by dividing the previous value by the asymptotic flux ratio; this has the effect of basing D on the pressure gradient in a curved pipe required to produce the asymptotic flow. The new value for D is 1,100. Similarly a more accurate value for $\bar{W}$ can be obtained as we have a better knowledge of the axial velocity distribution. In fact the maximum steady velocity is 1.1 times the peak velocity in Poiseuille flow (at the same flow rate) so that $\bar{W}$ is now $(100 - 1.1 \times 40)$ cm/sec, i.e. 56 cm/sec, resulting in a new value of 2500 for R_s . These new values of R_s and D still satisfy the conditions required by Chapter 5.

The oscillatory velocity gradient at the wall is calculated using the first two terms of the Stokes layer expansion (3.15) and (3.18), and this gives a peak shear stress of 32 dynes/cm which is uniform around the pipe wall. For the steady shear stresses, the velocity gradients calculated at $R = 1,800$ will be taken as the asymptotic values and we see that the leading order non-dimensional velocity gradient varies from -0.34 at the outside of the bend to -2.58 at the inside. Thus the magnitude of the leading order steady shear stresses ranges from 1 dyn/cm to 8 dyn/cm around the pipe.

As expected the mean shear stresses are quite small compared to the peak shear stress. It seems likely therefore that the effect of curvature on the shear stress distribution is small compared to the highly localized effects of branching of the aorta.

The flow conditions discussed up to now have only allowed the secondary motion to affect the axial velocity. We now turn to situations where it is possible for the axial velocity to influence the flow in the cross-section, and possibly change the direction of the secondary velocities.

The creeping flow solutions of Chapter 4 clearly illustrate the development of the secondary flow when inertial effects are small compared to viscous effects. The axial

velocity is a small perturbation away from Poiseuille flow, the location of the axial velocity maximum being dictated by the direction of the secondary motion. The leading order stream function χ_0 , (4.6) is the sum of two terms, one of which represents the motion generated by the boundary conditions, and the other the motion due to the centripetal pressure gradients. In this slow flow limit the balance between these two oppositely directed flows is determined by the parameter $k = 0/\sqrt{R_s}$ (4.1).

For $k^2 \ll 18\mathcal{N} + O(R_s^2)$ the flow in the cross-section is dominated by the motion due to the boundary conditions, and particles near the axis drift across the pipe from the outside to the inside of the bend. As k increases (i.e. the axial velocity increases) the centripetal pressure gradients become stronger, and as these forces have their maximum value near the axial velocity maximum it is in this region we expect the outward secondary motion to first appear. (See, for example, the numerical results at $R_s = 30$ shown in Figs. 7.17 to 7.22.) The inward secondary velocities near the axial velocity peak are decreased, and at $k^2 = 18\mathcal{N} + O(R_s^2)$ a stagnation point appears at $r = 0.0174\mathcal{N}R_s + O(R_s^3)$ on the line $\alpha = \pi$. Under these conditions the axial velocity maximum occurs at $r = 0.0120R_s\mathcal{N} + O(R_s^3)$ and $\alpha = \pi$.

As k^2 is increased above $18\mathcal{N}$, the stagnation point becomes a streamline $\chi_0 = 0$ separating two regions of oppositely directed circulation. Inside the separating streamline we have secondary motion analogous to that found in exactly steady flow in curved pipes, while outside there is the steady streaming associated with oscillatory flow. The region of Dean type flow continues to increase with increasing k , and as it does, the axial velocity maximum is forced by the secondary flow to move towards the outside of the pipe bend. For $k^2 = 25.2\mathcal{N}$ the peak axial velocity occurs at the origin, and for larger k^2 , moves onto the line $\alpha = 0$.

Dean type secondary flow fills the whole cross-section for $k^2 \gg 18\mathcal{N} + O(R_s^2)$. The motion due to the boundary conditions is confined to a thin layer on the wall and the velocity field is essentially that obtained in exactly steady flow.

Changes of the flux of fluid through the pipe, though associated with the above phenomena, are not manifested until $O(R_s^2)$ as shown by (4.15). It is possible to identify the terms appearing in (4.15): the quantities $O(R_s^2 k^4)$ and $O(R_s^2 \mathcal{N}^2)$ are the reduction in flux ratio due to the shifting of the axial velocity peak to the outside, and to the inside of the pipe bend respectively, the former being first calculated by Dean (1928); the term $O(R_s^2 k^2 \mathcal{N})$ represents the increase in flux ratio as the axial velocity peak moves across the pipe. At $k = 0$ the flux ratio has a local minimum, and as k is increased Q_r increases to a maximum of $1-5R_s^2 \mathcal{N}^2 \times 10^{-5}$ at $k^2 = 30.1 \mathcal{N}$ and then decreases for further increases in k . The maximum rate of increase of the flux ratio, i.e. the inflection point in the Q_r versus k curve (at constant R_s), occurs at $k^2 = 10.05 \mathcal{N}$ and we note that the slope at the inflection point increases as R_s increases.

For values of R_s and D not small the creeping flow solutions are no longer appropriate and numerical methods have been used to obtain solutions. These solutions do not give the same detailed information as the analytical results, but from Fig. 7.17 to 7.27 for $R_s = 30$ and 75 it is seen that the transition from inward to outward secondary flow occurs in much the same manner as for the viscous flow solutions: a stagnation point appears on the line $\alpha = \pi$ and grows into a region of Dean type secondary motion which eventually fills the cross-section as D is increased; similarly the axial velocity peak moves from the inside to the outside of the pipe bend while the flux ratio gradually rises to a maximum and then decreases. Also we note that the slope at the inflection point of the flux ratio versus D graph (Fig. 7.14) follows the result of the viscous flow solution and continues to increase as R_s increases, while the locations of the inflection point and flux ratio maximum move closer together. It is not unreasonable to predict then that for some R_s between 75 and 100 the slope at the inflection point will become infinite, and at some slightly larger value of R_s , this slope will become negative, thereby leading to an 'S' shaped curve. This is the result obtained for $R_s = 100$.

When any numerical scheme produces initially unexpected results the first thing that is questioned must be the reliability of the method of solution. However, as the program was checked several times during its development against analytical solutions and known numerical results, and the agreement was found to be satisfactory, some confidence can be placed in the solution technique. The actual behaviour of the iterations during the calculations at $R_s = 100$ is also important. It was found that weight factors of 1 to 2 were quite sufficient to keep the process stable, and depending on the initial approximation, weight factors of 0.5 or less were used without detrimental effects. Also the iterations behaved in the same manner as when testing the program to compare with known results. Finally the rapid convergence of the Fourier series (7.1) and the agreement of results obtained on different mesh sizes also contribute to our belief that the results obtained for $R_s = 100$ are good numerical approximations to the exact solutions of the governing equations and boundary conditions.

The plots of streamlines and axial velocity contours for $R_s = 100$ are shown in Figs. 7.28 to 7.37. The graphs are arranged in order of increasing axial velocity at the origin, which also corresponds to increasing distance, measured from $D = 0$, along the 'S' flux ratio curve Figs. 7.14 and 7.15.

In an experiment where the pressure gradient determines the flow, only the sections AB and CE of the flux ratio graph would be obtained. For small values of D such that the flux ratio lies on the lower branch AB, the flow in the cross-section is inward with some weakening of the secondary velocities near the peak axial velocity; there is no sign of any Dean type secondary flow (Figs. 7.28 to 7.30). Once at B, any increase in D causes the velocity field to change dramatically with the flux ratio jumping to B' on the upper curve. The streamline pattern now shows a large Dean type secondary flow surrounded by the steady streaming associated with the boundary conditions (Fig. 7.35). Further increases in D from these conditions tend to make the velocity distributions more like those found in exactly steady flow in curved pipes.

Starting from a flow with flux ratio on the upper branch CE and decreasing D , we find that the flow changes dramatically at $D_c < D_a$. The Dean type vortex present at C (Fig. 7.33) disappears completely at C' and the flux ratio is markedly lower due to the sudden jump of the axial velocity maximum from the origin to the inside edge of the pipe (Fig. 7.28). For further decreases of D the flux ratio follows the curve C'A.

The flow configurations in Fig. 7.31 to 7.32, corresponding to the section BC of the flux ratio curve, have been obtained by imposing additional constraints on the velocity field. In the absence of these constraints the flow cannot be physically realized. Some indication as to why it is necessary to introduce an extra force into the flow in order to obtain solutions on the curve BC can be obtained by examining the streamline and axial velocity contours of the flows studied.

Those flows where the change over of the secondary motion takes place smoothly result essentially from the linear combination of the two opposing circulations, just as in the creeping flow solutions. This can be seen from the streamline patterns by noting that the small Dean vortex, once formed, has little effect on axial velocity gradients at the wall, which (at this stage) are controlled largely by the inward secondary velocities. However at $R_s = 100$, where the axial velocity maximum is very close to the wall, there is the possibility for the latent Dean vortex to interact strongly with the inward motion and alter the shear stresses on the wall, especially near $\alpha = \pi$. Thus a small change in the Dean circulation could produce large changes in the flow as a whole.

Increasing R_s to 125 does not significantly alter the velocity field. The 'S' shaped flux ratio graph is again obtained, though the region of multiple solutions has greatly increased (Fig. 7.16). This similarity is not surprising as the flow fields at $R_s = 100$ and $R_s = 125$ in the limit $D \rightarrow 0$ are very similar. However for $R_s = 150$, $D \rightarrow 0$ a new feature emerges: the axial velocity maximum does not occur on the

axis of symmetry, as in the lower R_s cases, but now appears at some angle, say α_w , less than π . We can expect the centripetal pressure forces for this velocity distribution to be different from those when the axial velocity decreases away from the axis of symmetry. In fact the initial increase in the axial velocity away from the axis will cause pressure forces that strengthen the inward secondary motion along the axis, thereby preventing the axial velocity maximum returning to the axis and leading to a new pattern of flow development with increasing D .

The streamline and axial velocity contours for $R_s = 150$ are shown in Fig. 7.38 to 7.48, and the flux ratio versus D curve in Fig. 7.16. The results obtained for the flux ratio suggest that the two branches of the curve are completely distinct. On the upper curve, the flow is essentially the same as exactly steady flow in a curved pipe (Figs. 7.46 to 7.48), while on the lower branch, before the local maxima in the flux ratio, the secondary motion is the inward steady streaming typical of flows dominated by the boundary conditions (Figs. 7.38 to 7.39). Flow configurations on the lower branch past the flux ratio maximum reflect the strong interaction of the centripetal pressure gradient forces and the inward steady streaming, and Figs. 7.39 to 7.45 show the complicated nature of the motion. It should be admitted immediately that the results in this range of D on the lower branch are to be viewed with some suspicion: these results could be due solely to the poor convergence of the Fourier series (7.1) in this interval. However experience gained with the numerical scheme when insufficient terms in (7.1) have been used tempts us to believe that, at least qualitatively, the results are correct.

The solutions represented by the upper curve could only be obtained when the initial approximation to the solution was a flow with outward secondary motion. If the initial approximation was a flow with inward steady streaming, then no matter how high D was taken (with $R_s = 150$), the solution obtained always fell on the lower curve. This is in contrast to the result at $R_s = 100$ where it was possible to jump from one branch to the other in either direction. For $R_s = 150$ it

appears that it is only possible to jump down from the upper to lower curve.

The modified iteration technique where the axial velocity at the centre of the pipe is specified failed to locate any link between the two branches. Indeed it was found that this technique was more useful in determining solutions on the lower branch than if D had been specified.

Examination of the streamlines for flows on the lower branch show clearly the reason for the poor representation afforded by (7.1) when NOF is 18. As D is increased the inward steady streaming forms itself into a small vortex near $\alpha = 0$, causing the stream function to have large variations over a small range of α . Also, the strength of this vortex increases as D increases.

The axial velocity maximum never returns to the axis of symmetry, though at one stage α_w did increase slightly with increasing D . As anticipated, the angle α_w generally decreased with increases in D until the axial velocity maximum was quite near $\alpha = 0$. The slight change in direction of the movement of the peak axial velocity appeared to be typical of those flows, which in the limit $D \rightarrow 0$, had the velocity maximum just off the axis. Brief exploration also showed that it was possible for the axial velocity maximum to return to the axis of symmetry, and then leave it again as D was increased.

Finally this chapter is ended with a conjecture as to how the flux ratio curve for $R_s = 150$ has developed from that at $R_s = 100$, and a guess at the regions in the D - R_s plane where solutions of the equations are non-unique. As R_s is increased from 100 to 125, the region of non-unique solutions increased markedly. This was mainly due to the lower jump point B occurring at a larger D . Coupled with the results for $R_s = 150$, this suggests that this lower jump point has moved to very large, if not infinite, values of D , while the upper jump point remains virtually unaltered. In this case the portion BC of the flux ratio curve at $R_s = 100$ would become the curve shown by the broken line in Fig. 8.1 when R_s is 150.

In the $D-R_s$ plane the region of multiple solutions is bounded by the loci of the upper and lower jump points, and for R_s less than 125, this region is reasonably well defined. From the results at $R_s = 150$ the locus of the lower jump point is almost parallel to the D axis; if it is assumed that the lower jump point has moved out to infinity, then its locus asymptotes a line parallel to the D axis. Only a rough approximation to the locus of the upper jump point can be obtained. For large R_s there is no interaction between the two secondary flows provided $D \ll R_s$. As it appears that a strong interaction between these effects is necessary for multiple solutions to exist, then the locus of the upper jump point must lie somewhere to the right of a line $R_s = aD$, where $a \sim o(1)$, as $R_s \rightarrow \infty$. The region of non-uniqueness is then the area shown shaded in Fig. 8.2.

Verification of the position of the locus of the upper jump point is a lengthy computational task due to the large parameters involved; the conjecture about the locus of the lower jump point can be checked by calculations in which D is held fixed and R_s varied. This work remains to be carried out.

CHAPTER 9

The axial steady streaming

When the pressure gradient along the pipe is a general periodic function of time which contains even harmonics of the fundamental frequency, the analysis of Chapter 3 shows that a steady axial velocity exists at the edge of the Stokes layer. The generation of this axial velocity is similar to the mechanism producing the steady velocity at the edge of the Stokes layer for the flow in the cross-section. In both instances the oscillatory components of the flow in the Stokes layer interact to produce a steady velocity component. When the no slip condition at the wall is enforced it is found that these steady velocity components do not decay to zero at the edge of the Stokes layer, but persist into the outer region, driving steady streaming motions.

Here we shall study this axial flow for the case when the mean pressure gradient is zero, so that w_1 is identically zero. The equations governing w_2 , obtained from (3.40) and (3.57), are then

$$-\frac{R_s}{r} \frac{\partial(\chi_0, \nabla^2 \chi_0)}{\partial(r, \alpha)} = \nabla^4 \chi_0, \quad (9.1)$$

and
$$-\frac{R_s}{r} \frac{\partial(\chi_0, w_2)}{\partial(r, \alpha)} = \nabla^2 w_2. \quad (9.2)$$

The boundary conditions are given by (3.41) and (3.58) and are repeated here:

$$\left. \begin{aligned} \chi_0 &= 0 \\ \frac{\partial \chi_0}{\partial r} &= \mathcal{W} \sin \alpha \end{aligned} \right\} \text{ on } r=1 \quad (9.3)$$

and
$$w_2 = R_s \mathcal{M} \cos \alpha \quad \text{on } r=1. \quad (9.4)$$

The problem described by the equations and boundary conditions (9.1) to (9.4) may be artificially interpreted as the fully developed flow of a viscous, incompressible fluid in a straight pipe when the walls of the pipe have the velocities given by the boundary conditions (9.3) and (9.4). At first sight it appears as though there will be no net flux

of fluid through the pipe as the mean axial velocity around the wall is zero. However closer examination shows that the boundary conditions cause the mean value of the azimuthal component of vorticity at $r = 1$ to be non-zero, thereby implying a net flux through the pipe in the direction dictated by the sign of the azimuthal vorticity.

For the artificial problem the vorticity vector, $\underline{\omega}$ is

$$\underline{\omega} = \left[\frac{1}{r} \frac{\partial w_2}{\partial \alpha}, - \frac{\partial w_2}{\partial r}, - \nabla^2 \chi_0 \right] . \quad (9.5)$$

The third component of the vorticity vector at the wall is

$$\omega_3 = - \mathcal{W} \sin \alpha + \left. \frac{\partial v_0}{\partial r} \right|_{r=1} , \quad (9.6)$$

and for w_2 zero, the vortex lines are straight lines parallel to the axis of the pipe. The imposition of an axial velocity which varies around the circumference of the pipe, e.g. (9.4), tends to bend any given originally straight vortex line about a radial line, thereby inducing an azimuthal component of vorticity. The bending of the original axial vortex lines is associated with the radial vorticity $\omega_1 = -\mathcal{M}R_s \sin \alpha$, and the non-zero mean value for the azimuthal vorticity on $r = 1$ comes from the interaction of the $\sin \alpha$ terms in ω_3 and ω_1 . The mean azimuthal vorticity source strength is obtained by evaluating the axial momentum equation (9.2) at the boundary and integrating the result around the circumference. We obtain the result

$$\frac{1}{2\pi} \int_0^{2\pi} \left. \frac{\partial \omega_2}{\partial r} \right|_{r=1} d\alpha = - \frac{R_s^2 \mathcal{W} \mathcal{M}}{2} , \quad (9.7)$$

where $\left. \frac{\partial \omega_2}{\partial r} \right|_{r=1}$ is the azimuthal vorticity source strength. In deriving (9.7) we have used the fact that

$$\int_0^{2\pi} \left. \frac{\partial w_2}{\partial r} \right|_{r=1} d\alpha = 0 , \quad (9.8)$$

which is easily obtained by integrating (9.2) over the cross-section.

Associated with this azimuthal vorticity is a mean flux of fluid along the pipe. As $\mathcal{W}$ is always positive, the sign of the vorticity source strength, and hence the direction of

the mean flux, is determined by the sign of $\mathcal{M}$.

By using an artificial interpretation of (9.1) to (9.4) we have shown that there will be a net flux of fluid through the pipe. The presence of a mean axial flow implies that the directions $+\theta$ and $-\theta$ are in some way different, so that now we must return to the original problem of an oscillatory pressure gradient acting along a curved pipe and determine if it is possible for a preferential axial direction to be established. Clearly the geometry of the pipe gives no preferential axial direction, thus leaving the imposed pressure gradient as the only possible source. This is the result obtained in the previous paragraph, as the quantity $\mathcal{M}$, which was shown to determine the direction of the mean flow, depends only on the pressure gradient.

The pressure gradient distinguishes one axial direction from the opposite direction, so that any asymmetry in the pressure gradient will indicate that a preferential axial direction exists. The analysis of Chapter 3 showed that $\mathcal{M}$ is zero if the pressure gradient consisted of only odd harmonics of the fundamental frequency. Denoting such a pressure gradient by $PG_o(\tau)$, it is easy to see that

$$PG_o(\tau+\pi) = -PG_o(\tau).$$

Clearly, the positive and negative excursions of $PG_o(\tau)$ are identical, and also the time spent with the pressure gradient positive is equal to the time for which the pressure gradient is negative. Such symmetry of the pressure gradient can produce no preferential direction, so that $\mathcal{M}$ being zero for $PG_o(\tau)$ is a necessary consequence of the non-existence of a preferential direction.

A pressure gradient containing even harmonics, $PG_e(\tau)$, does not have the symmetry properties of $PG_o(\tau)$. The positive and negative excursions of $PG_e(\tau)$ are not the same, and more importantly, the time spent with a positive pressure gradient is not the same as the time for which the pressure gradient is negative (see Fig. 9.1a,b). Thus from this asymmetry of $PG_e(\tau)$ it is possible to infer a preferential axial direction. The presence of a mean flow of fluid through the curved pipe

is then consistent with the existence of this preferred axial direction.

The preferential axial direction comes from considering the pressure gradient per se; this property of the pressure gradient is not a property of any of the harmonic components of the pressure gradient. In a linear system (e.g. oscillatory flow in a straight pipe) where each frequency component can be treated independently of all the others, the concept of a preferred axial direction has no consequence. It is only where nonlinearities preclude consideration of each frequency component separately that the directional properties of the pressure gradient as a whole become relevant.

We return to the problem of obtaining solutions of (9.1) to (9.4) for the axial steady streaming. The parameter governing the distribution of the azimuthal vorticity produced at the boundary is R_s . For small values of R_s we expect the vorticity to be diffused over the cross-section, while for large R_s we expect the vorticity to be confined to a region adjacent to the wall. In both cases the stream function χ_0 is given by generalizations of the solution of Lyne (1970).

In the limit $R_s \rightarrow 0$ the expansions

$$\left. \begin{aligned} \chi_0 &\sim \chi_{00} + R_s \chi_{01} + \dots \\ \text{and} \quad w_z &\sim R_s w_{z0} + R_s^2 w_{z1} + \dots \end{aligned} \right] \quad (9.9)$$

are used. For the stream function we have

$$\begin{aligned} \chi_0 = & \frac{\mathcal{M} \sin \alpha}{2} (-r + r^3) + R_s \frac{\mathcal{M}^2 \sin 2\alpha}{192} (-r^2 + 2r^4 - r^6) \\ & + R_s^2 \left[\frac{\mathcal{M}^3 \sin \alpha}{45080} (-2r + 11r^3 - 20r^5 + 15r^7 - 4r^9) \right. \\ & \left. + \frac{\mathcal{M}^3 \sin 3\alpha}{23040} (-r^3 + 3r^5 - 3r^7 + r^9) \right]. \quad (9.10) \end{aligned}$$

The boundary conditions (9.4) become

$$\left. \begin{aligned} w_{z0} &= \mathcal{M} \cos \alpha \\ \text{and} \quad w_{zi} &= 0, \quad i = 1, 2, 3, \dots \end{aligned} \right] \quad \text{on } r=1. \quad (9.11)$$

Substituting the expansions (9.9) into (9.2) and equating like powers of R_s we get

$$\nabla^2 w_{20} = 0$$

as the equation for w_{20} . The regular solution, satisfying (9.11) is

$$w_{20} = \mathcal{M} \cos \alpha \cdot r \quad . \quad (9.12)$$

At $O(R_s^2)$ we obtain the equation for w_{21} as

$$\nabla^2 w_{21} = -\frac{1}{r} \frac{\partial(\chi_{00}, w_{20})}{\partial(r, \alpha)} \quad ,$$

which has solution

$$w_{21} = \frac{\mathcal{M}\mathcal{N}}{16} (1-2r^2+r^4) + \frac{\mathcal{M}\mathcal{N} \cos 2\alpha}{24} (r^2-r^4) \quad . \quad (9.13)$$

Evaluation of $\left. \frac{\partial w_{21}}{\partial r} \right|_{n_0}$ shows that (9.8) is satisfied as the mean component of the velocity gradient is identically zero on $r = 1$. Also integration of w_{21} over the cross-section shows that the flow rate along the pipe is non-zero.

Proceeding to higher orders we obtain

$$w_{22} = \frac{\mathcal{M}\mathcal{N}^2 \cos \alpha}{23,040} (-14r + 21r^3 - 7r^5) + \frac{\mathcal{M}\mathcal{N}^2 \cos 3\alpha}{3560} (3r^3 - 5r^5 - 2r^7) \quad (9.14)$$

$$\begin{aligned} \text{and } w_{23} = & \frac{\mathcal{M}\mathcal{N}^3}{12^3 \times 6,400} (-2,371 + 8,280r^2 - 10,770r^4 + 6,400r^6 - 1,725r^8 + 192r^{10}) \\ & + \frac{\mathcal{M}\mathcal{N}^3 \cos 2\alpha}{12^3 \times 1,280} (39r^2 - 192r^4 + 279r^6 - 152r^8 + 26r^{10}) \\ & + \frac{\mathcal{M}\mathcal{N}^3 \cos 4\alpha}{12^3 \times 22,400} (1,189r^4 - 2,604r^6 + 1,855r^8 - 440r^{10}) \quad . \quad (9.15) \end{aligned}$$

Integrating the expression for w_2 over the cross-section we obtain the flux of fluid down the tube, Q_2 , as

$$Q_2 = \frac{R_s^2 \mathcal{M}\mathcal{N}}{96} \left(1 - \frac{3\mathcal{N}^2 R_s^2}{128} \right) + O(R_s^4) \quad . \quad (9.16)$$

We now turn to the determination of the axial steady streaming in the limit $R_s \rightarrow \infty$. In the notation of Chapter 5,

the stream function is represented by the expansions:

$$\chi_o(r, \alpha; R_s) \sim \bar{\chi}_o(r, \alpha) + R_s^{-\frac{1}{2}} \bar{\chi}_1 + \dots \quad (9.17)$$

away from the boundaries of the semicircle; and

$$\left. \begin{aligned} \chi_o(r, \alpha; R_s) &\sim R_s^{\frac{1}{2}} \chi_o(\rho, \alpha) + R_s^{-1} \chi_1 + \dots \text{ on } r=1 \\ \text{and } \chi_o(n, s; R_s) &\sim R_s^{\frac{1}{2}} \chi_o(N, s) + R_s^{-1} \chi_1 + \dots \text{ on } n=0 \end{aligned} \right] , \quad (9.18)$$

where ρ, N are the $O(1)$ boundary layer co-ordinates defined by (5.10). It is assumed that w_2 will have an inviscid core/boundary layer structure similar to that of the stream function. Thus in the regions away from the boundaries of the semicircles the axial velocity has the form

$$w_2(r, \alpha; R_s) \sim R_s^m \left(\bar{w}_o(r, \alpha) + R_s^{-\frac{1}{2}} \bar{w}_1 + \dots \right), \quad (9.19)$$

where the constant m is to be determined by matching with the solution for w_2 in the boundary layer. (The notation of Chapter 5 is being used for the expansions, but as this chapter and Chapter 5 are sufficiently self contained, no confusion should arise. For example, the symbol $\bar{w}_o$ is used for the leading term in the expansion of w_{1o} in Chapter 5 and for the leading term of w_2 , and clearly represents two different axial velocity distributions.)

For large R_s the convective terms dominate the viscous terms in (9.2) so that, away from the boundaries, the axial velocity is constant on the streamlines of the flow in the cross-section. As these streamlines are closed a single-valuedness condition analogous to (5.28) may be applied to determine the variation of the axial velocity across the streamlines. Alternatively the results of Chapter 6 may be applied, and with either technique the leading order solution for w_2 is

$$\bar{w}_o = \text{constant}. \quad (9.20)$$

The radial and azimuthal vorticity components are given by (9.5), and using the leading order solution (9.20) for w_2 it is seen that these components of vorticity are zero in the core. The radial vorticity produced by the boundary condition (9.4) and the induced azimuthal vorticity are then confined to the boundary layer surrounding the inviscid core.

The boundary layer on the axis of symmetry is due to convection across the pipe of vorticity produced at the walls.

The boundary condition (9.4) on the axial velocity at the pipe wall $r = 1$ requires that $w_2 \sim O(R_s)$ in the boundary layer. The thickness of the layers is $O(R_s^{1/2})$, as usual, and to ensure that the vorticity in the layer on the axis of symmetry is the same size as in the layer on the wall, we must have $w_2 \sim O(R_s)$ on $\alpha = 0, \pi$. The axial velocity in the boundary layers then has the form

$$w_2(r, \alpha; R_s) \sim R_s W_0(\rho, \alpha) + R_s^{1/2} W_1 + \dots \text{ on } r=1 \quad (9.21)$$

and
$$w_2(n, s; R_s) \sim R_s W_0(N, s) + R_s^{1/2} W_1 + \dots \text{ on } n=0. \quad (9.22)$$

For the boundary layer flow to match the core flow W_0 must tend to a constant as ρ or N tend to infinity. Thus at the edge of the boundary layer the axial velocity is $O(R_s)$ at most, and this must also be the magnitude of w_2 away from the boundaries. Thus the matching has given the results

$$w_2 \sim R_s \left(\bar{W}_0 + R_s^{1/2} \bar{W}_1 + \dots \right) \quad (9.23)$$

and $W_0 \rightarrow \text{a constant, } \bar{W}_0 \text{ at the edge of the boundary layers} \quad (9.24)$

The boundary layer equations for W_0 are the same as (5.16) and (5.39), but rather than determine W_0 we define W_{0p} as a function of (ρ, α) or (N, s) by

$$W_{0p} = W_0 - \bar{W}_0, \quad (9.25)$$

and derive the equations and boundary conditions below:

$$\left. \begin{aligned} \frac{\partial(X_0, W_{0p})}{\partial(\rho, \alpha)} &= \frac{\partial^2 W_{0p}}{\partial \rho^2}, \\ W_{0p} &= \mathcal{M} \cos \alpha - \bar{W}_0 \text{ on } \rho=0, \\ W_{0p} &\rightarrow 0 \text{ as } \rho \rightarrow \infty; \end{aligned} \right\} \quad (9.26)$$

and

$$\left. \begin{aligned} -\frac{\partial(X_0, W_{0p})}{\partial(N, s)} &= \frac{\partial^2 W_{0p}}{\partial N^2}, \\ \frac{\partial W_{0p}}{\partial N} &= 0 \text{ on } N=0, \\ W_{0p} &\rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned} \right\} \quad (9.27)$$

The problems (9.26) and (9.27) may be simplified by employing the approximation that the axial velocity perturbation W_{0p} is convected around the boundary layer with the velocity of the inviscid motion in the cross-section just outside the boundary layer. This approximation is expressed in (5.50) and (5.59) and when used here leads to the equations

$$\left. \begin{aligned} \bar{V}_0(\alpha) \frac{\partial W_{0p}}{\partial \alpha} - \rho \frac{d\bar{V}_0}{d\alpha} \frac{\partial W_{0p}}{\partial \rho} &= \frac{\partial^2 W_{0p}}{\partial \rho^2} \\ \text{and} \quad \bar{V}_0(s) \frac{\partial W_{0p}}{\partial s} - N \frac{d\bar{V}_0}{ds} \frac{\partial W_{0p}}{\partial N} &= \frac{\partial^2 W_{0p}}{\partial N^2} \end{aligned} \right\} \quad (9.28)$$

The transformations (5.54) and (5.63), used with the redefinition of λ and Λ

$$\left. \begin{aligned} \lambda(x, y) &= W_{0p}(\rho, \alpha) \\ \Lambda(x, \gamma) &= W_{0p}(N, s) \end{aligned} \right\} \quad (9.29)$$

lead to the diffusion equations

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial x} &= \frac{\partial^2 \lambda}{\partial y^2} \\ \text{and} \quad \frac{\partial \Lambda}{\partial x} &= \frac{\partial^2 \Lambda}{\partial \gamma^2} \end{aligned} \right\} \quad (9.30)$$

The transformed boundary conditions are:

$$\begin{aligned} \lambda &= \lambda(x, 0) \\ \text{and} \quad \lambda &\rightarrow 0 \quad \text{as } y \rightarrow \infty ; \\ \text{and} \quad \frac{\partial \Lambda}{\partial N} &= 0 \quad \text{on } N = 0 \\ \text{and} \quad \Lambda &\rightarrow 0 \quad \text{as } N \rightarrow \infty \end{aligned} \quad (9.31)$$

The solutions for $\lambda(x, y)$ and $\Lambda(x, \gamma)$ are:

$$\left. \begin{aligned} \lambda(x, y) &= \int_0^\infty \frac{\lambda(0, u)}{2\sqrt{\pi x}} \left[e^{-(y-u)^2/4x} - e^{-(y+u)^2/4x} \right] du \\ &\quad + \frac{2}{\sqrt{\pi}} \int_{y/2\sqrt{x}}^\infty \lambda\left(x - \frac{y^2}{4u^2}, 0\right) e^{-u^2} du, \\ \text{and} \quad \Lambda(x, \gamma) &= \int_0^\infty \frac{\Lambda(0, u)}{2\sqrt{\pi x}} \left[e^{-(\gamma-u)^2/4x} + e^{-(\gamma+u)^2/4x} \right] du, \end{aligned} \right\} \quad (9.32)$$

where $\lambda(0, u)$ and $\Lambda(0, u)$ are initial profiles for each layer.

These two separate solutions may be combined into one integral equation by assuming that the velocity profile at the end of one layer is convected around the corner, and becomes the initial profile for the other layer. Denoting the end of the layer on the wall by x_e and the end of the layer on the axis of symmetry by λ_e , the above assumption gives

$$\lambda(x_e, y) = \Lambda(0, y)$$

and

$$\Lambda(x_e, y) = \lambda(0, y) \quad (9.33)$$

as in each corner $y = Y$. By focusing attention on the corner $x = x_e$ we obtain an integral equation for the profile $\lambda(x_e, y)$ which is analogous to the equation derived by Lyne for the boundary layer profile in this corner. We have

$$\lambda(x_e, y) = \frac{1}{2\sqrt{\pi(x_e + x_e)}} \int_0^\infty \lambda(x_e, u) e^{-\frac{u^2}{4x_e} - \frac{y^2}{4x_e}} \left(e^{\alpha^2} \operatorname{erf}(\alpha) - e^{\beta^2} \operatorname{erf}(\beta) \right) du + \frac{2}{\sqrt{\pi}} \int_{\frac{y}{2\sqrt{x_e}}}^\infty \lambda(x_e - \frac{y^2}{4u^2}, 0) e^{-u^2} du, \quad (9.34)$$

where $\alpha = \left(\frac{u}{4x_e} + \frac{y}{4x_e} \right) \left(\frac{4x_e \lambda_e}{x_e + x_e} \right)^{1/2}$, $\beta = \left(\frac{u}{4x_e} - \frac{y}{4x_e} \right) \left(\frac{4x_e \lambda_e}{x_e + x_e} \right)^{1/2}$.

The last integral in (9.34) can be transformed to

$$\frac{y}{2\sqrt{\pi}} \int_\pi^0 \frac{(-\zeta_0)^{-1} \bar{V}_0(\alpha) (\mathcal{M} \cos \alpha - \bar{w}_0)}{(x_e - x(\alpha))^{3/2}} e^{-\frac{y^2}{4(x_e - x(\alpha))}} d\alpha, \quad (9.35)$$

which is more suitable for numerical evaluation. This integral is written in the form

$$\mathcal{M} \left(\lambda_0^*(y) + \frac{\bar{w}_0}{\mathcal{M}} \lambda_1^*(y) \right), \quad (9.36)$$

where λ_0^* and λ_1^* are known functions. Similarly, $\lambda(x_e, y)$ is defined as

$$\mathcal{M} \left(\lambda_0(y) + \frac{\bar{w}_0}{\mathcal{M}} \lambda_1(y) \right) \quad (9.37)$$

so that (9.34) gives the two integral equations

$$\lambda_i(y) = \int_0^\infty \lambda_i(u) K(u; y) du + \lambda_i^*(y), \quad i=0,1. \quad (9.38)$$

These two equations can be solved independently for the $\lambda_i(y)$ and the constant $\bar{w}_0/\mathcal{M}$ obtained by applying the condition that $\lambda(x_e, y) \rightarrow 0$ as $y \rightarrow \infty$. Thus we have

$$\frac{\bar{w}_0}{\mu} = \lim_{y \rightarrow \infty} \left[\frac{-\lambda_0(y)}{\lambda_1(y)} \right] . \quad (9.39)$$

The two integral equations (9.38) were solved numerically using the iterative technique described in Chapter 5 for the integral equation (5.69). When successive approximations to the functions λ_0 and λ_1 differed by less than $10^{-3} \%$, the iteration process was terminated and the value of $\bar{w}_0/\mu$ determined from (9.39). To 3 significant figures the result is

$$\frac{\bar{w}_0}{\mu} = 4.65 . \quad (9.40)$$

The functions λ_0 , λ_1 , λ_0^* and λ_1^* are given in Table 4.

Some feel for the constant μ can be obtained by replacing (2.5) by

$$-\frac{1}{R} \frac{\partial}{\partial \theta} \left(\frac{p^*}{\rho} \right) = \omega \bar{W} \left(\cos \tau + a_2 \cos(2\tau + \phi_2) \right) ,$$

the simplest pressure gradient for which μ is non-zero. From (3.53) we obtain the result

$$\mu = a_2 e^{i\phi_2} \left[\frac{162 - 142\sqrt{2} + i(101 - 36\sqrt{2})}{1.152} \right] \quad (9.41)$$

where it is understood that only the real part of μ is of interest. Thus from (9.41), the largest value of μ is approximately $0.055|a_2|$. Also, the importance of the phase ϕ_2 is clearly shown, and as suggested in Chapter 3, by proper choice of the phase the steady axial velocities can be made to disappear.

Finally we note that the flow in an oscillating torus can be analyzed by the methods used in this thesis and analogous results derived. In particular there will be a mean axial motion if the angular velocity contains even harmonics of the fundamental frequency of oscillation, though the value of μ obtained will be different from (3.53). As the leading and first order axial velocity away from the walls of the torus are zero, the dominant axial velocity is the steady streaming predicted here. Thus from an experimental

point of view, an oscillating torus is more likely to be of use in examining the predictions of the theory than pressure gradient driven flow in a stationary curved pipe.

CHAPTER 10

Heat transfer in unsteady flow

In this chapter we consider the transport of a scalar quantity, denoted T^* , in a curved pipe when the velocity field is fully developed and the pressure gradient driving the fluid motion is given by (2.5). The quantity T^* may be thought of as the temperature of the fluid or the concentration of a species diffusing in the fluid, but throughout this section the terminology of the heat transfer literature will be used.

The dimensional equation governing the distribution of T^* in the fluid is

$$\begin{aligned} \frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial r^*} + \frac{v^*}{r^*} \frac{\partial T^*}{\partial \alpha} + \frac{w^*}{R+r^* \cos \alpha} \frac{\partial T^*}{\partial \theta} \\ = K \left[\frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T^*}{\partial r^*} + \frac{1}{r^{*2}} \frac{\partial^2 T^*}{\partial \alpha^2} + \frac{1}{(R+r^* \cos \alpha)^2} \frac{\partial^2 T^*}{\partial \theta^2} \right. \\ \left. + \frac{\cos \alpha}{R+r^* \cos \alpha} \frac{\partial T^*}{\partial r^*} - \frac{\sin \alpha}{R+r^* \cos \alpha} \cdot \frac{1}{r^*} \frac{\partial T^*}{\partial \alpha} \right], \quad (10.1) \end{aligned}$$

where the co-ordinate system used is shown in Fig. 2.1, and K is the diffusivity of T^* in the fluid. The properties of the fluid are assumed to be constant. For simplicity we shall only consider boundary conditions which require a step change in T^* at the pipe wall, i.e.

$$\left. \begin{aligned} T^* &= T_0 \quad \text{for } \theta < 0, \\ T^* &= T_1 \quad \text{for } \theta > 0 \end{aligned} \right] \quad \text{on } r^* = a. \quad (10.2)$$

Expressions for u^*, v^*, w^* are available for the case when the curvature of the pipe is small. When the non-dimensionalization (2.9) is used and the limit $\epsilon \rightarrow 0$ attempted in (10.1), it is found that the ratio of the axial convection of heat (by the mean velocity) to the axial conduction of heat tends to infinity. The ratio of convection to conduction terms defines a thermal Reynolds number, usually called a Peclet number, Pe which has the form of a Reynolds number for the fluid flow multiplied by the ratio of the kinematic viscosity to the thermal diffusivity. Thus we see that small ϵ implies large Pe ; this is the difficulty referred

to at the end of Chapter 2.

The reason for the large Pe can be traced to the limiting process used to obtain the momentum equations (2.13) and (2.15). There the limit $\epsilon \rightarrow 0$, D fixed (among others) was taken. As D may be regarded as the product of the Reynolds number for the axial flow and $\epsilon^{1/2}$, it follows that to keep D fixed, the Reynolds number must increase as ϵ decreases. The Reynolds number in D is that used to define the Peclet number, Pe , so that Pe must also increase as ϵ decreases. Alternatively, we can interpret $\epsilon^{1/2}$ as a Reynolds number for the axial flow.

A large Peclet number, for whatever reason, is usually taken as sufficient justification for the neglect of the axial conduction of heat. In steady, forced convective heat transfer in a straight pipe this approximation leads to the classical Graetz-Nusselt problem for which analytical solutions are well known (Jakob 1949). Even though solutions, both analytic and numerical, have been obtained when axial conduction is included (Tan and Hsu 1972 and Michelsen and Villadsen 1974), little work has gone into understanding the effect of the axial conduction term on the solution, or why its neglect is very often a good approximation. Therefore, before attempting to calculate the heat transfer in unsteady flow in a curved pipe, the simpler problem of steady, forced convective heat transfer in a straight pipe is studied in order to elucidate the nature of the large Peclet number limit. From these results an expansion scheme is developed for heat transfer in curved pipes for both steady and unsteady flow fields.

We consider an infinitely long straight tube of circular cross-section with a step change in the wall temperature at $s^* = 0$ (see Fig. 10.1). Fluid is flowing steadily through the pipe with a parabolic velocity distribution, and as the problem is axisymmetric, the energy equation is

$$\frac{G a^2}{4\nu} \left(1 - \frac{r^{*2}}{a^2}\right) \frac{\partial T}{\partial s^*} = \kappa \left(\frac{\partial^2 T}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T}{\partial r^*} + \frac{\partial^2 T}{\partial s^{*2}} \right), \quad (10.3)$$

and the boundary conditions are

$$\left. \begin{array}{ll} T = 0 & \text{for } s^* < 0 \\ T = 1 & \text{for } s^* > 0 \end{array} \right] \text{ on } r^* = a. \quad (10.4)$$

The pressure gradient along the pipe is $G = -\frac{\partial}{\partial s^*} \left(\frac{p^*}{\rho} \right)$ and T is considered to be non-dimensional.

The Peclet number is defined by

$$Pe = \frac{Ga^2}{4\nu} \cdot \frac{a}{K} \quad (10.5)$$

and when this is large it is usually assumed that the axial conduction term is negligible compared to the convection term. This approximation changes (10.3) from an elliptic equation to a parabolic equation, thus allowing the solution for $s^* < 0$ to be written down at once as

$$T(r^*, s^*) = 0, \quad s^* < 0. \quad (10.6)$$

An attempt to calculate the effect of axial conduction by a regular expansion of T in powers of Pe^{-2} has been made by Jones (1971). Clearly the equations for higher order terms will be parabolic with forcing terms confined to the region $s^* > 0$, so the result (10.6) remains unchanged. As we expect the inclusion of axial conduction to allow some diffusion of heat upstream of $s^* = 0$, we are led to examine the validity of the expansion technique. We find that for $r^* \simeq a$ and $s^* \simeq 0$, the neglected axial conduction term is the same size as the terms retained at leading order, thus indicating that the expansion is not uniformly valid. This result would be expected on physical grounds, as here the axial temperature gradient is large and the fluid velocity small.

Thus the solutions of Graetz and Nusselt are valid downstream of the origin, while near $s^* = 0$ new approximations are required[†]. The problem is analogous to the entry flow in a channel where Van Dyke (1970) showed that the solution obtained by neglecting axial diffusion of vorticity (Schlichting 1934) is only valid downstream of the origin, and that a different approximation is needed for regions close to the origin.

[†] While this work was in preparation the author learnt that A.S. Jones treated this problem in an, as yet, unpublished paper.

Consequently the approach used here closely parallels Van Dyke's treatment of the entry flow problem.

The co-ordinates are made non-dimensional by scaling with respect to the pipe radius, a :

$$\left. \begin{aligned} r &= r^*/a \\ s &= s^*/a \end{aligned} \right] \quad (10.7)$$

Equation (10.3) becomes

$$(1-r^2) \frac{\partial T}{\partial s} = Pe^{-1} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial s^2} \right) \quad (10.8)$$

with boundary conditions

$$\left. \begin{aligned} T &= 0 \quad \text{for } s < 0, \\ T &= 1 \quad \text{for } s > 0 \end{aligned} \right] \quad \text{on } r=1. \quad (10.9)$$

In a region around $s = 0$ we expect that the thermal boundary layer will be thin and that the radial and axial temperature gradients large. It is found, by requiring the convective term to balance the largest diffusion terms, that the thickness and length of the region is $O(Pe^{-1/2})$. New co-ordinates ρ, S are introduced for this region and defined by

$$\left. \begin{aligned} \rho &= (1-r) Pe^{1/2} \\ S &= s Pe^{1/2} \end{aligned} \right] \quad (10.10)$$

The temperature is assumed to have the form

$$T(r, s; Pe) \sim \bar{T}_0(r, s) + Pe^{-1/2} \bar{T}_1 + \dots \quad (10.11)$$

away from the wall, and

$$T(r, s; Pe) \sim T_0(\rho, S) + Pe^{-1/2} T_1 + \dots \quad (10.12)$$

near $r = 1$. For future reference the region away from the wall is labelled as region II and the region near the wall as region I.

Substituting (10.11) into (10.8) and equating equal powers of Pe we obtain at leading order

$$(1-r^2) \frac{\partial \bar{T}_0}{\partial s} = 0.$$

The solution satisfying the condition at $s = -\infty$ is

$$\bar{T}_0(r,s) = 0.$$

Continuing this process we obtain the result that

$$\bar{T}_i(r,s) = 0, \quad i = 0, 1, 2, \dots$$

so that

$$T(r,s;\rho_c) \equiv 0 \quad \text{in region II} \quad (10.13)$$

The equation governing $T_0(\rho, S)$ in the boundary layer is found to be

$$2\rho \frac{\partial T_0}{\partial S} = \frac{\partial^2 T_0}{\partial \rho^2} + \frac{\partial^2 T_0}{\partial S^2}, \quad (10.14)$$

and the boundary conditions are

$$\left. \begin{array}{l} T_0 = 0 \quad \text{for } s < 0, \\ T_0 = 1 \quad \text{for } s > 0 \end{array} \right\} \quad \text{on } \rho = 0. \quad (10.15)$$

Conditions on T_0 as ρ becomes large are found by matching the boundary layer to region II. Using (10.13) in the matching we find that

$$T_0 \rightarrow 0 \quad \text{exponentially as } \rho \rightarrow \infty. \quad (10.16)$$

The condition that T_0 and $\frac{\partial T_0}{\partial S}$ are bounded as $|S| \rightarrow \infty$ are sufficient to complete the formulation of the problem for T_0 .

The more difficult mixed boundary value problem, where the condition $\frac{\partial T_0}{\partial \rho} = 0$ replaces $T_0 = 0$ on $\rho = 0, s < 0$, has been studied by Springer and Pedley (1973). However they were interested only in the heat flux through the wall $s > 0$ and did not determine the temperature field. Rather than extend their analysis for the physically more realistic boundary conditions, we will illustrate the essential features of the temperature field near $s = 0$ by the simpler problem (10.14) to (10.16).

Introducing the Fourier transform, $\hat{T}_0$ of T_0 defined by

$$\hat{T}_0(\rho, k) = \int_{-\infty}^{\infty} T_0(\rho, S) e^{ikS} dS,$$

(10.14) becomes

$$\frac{d^2 \hat{T}_0}{dz^2} - z \hat{T}_0 = 0, \quad (10.17)$$

where

$$z = e^{-i\pi/6} k^{1/3} (2\rho + ik)^{2/3}. \quad (10.18)$$

To obtain convergence of the Fourier transform, the boundary conditions (10.15) are replaced by

$$\left. \begin{aligned} T_0 &= 0 & \text{for } S < 0, \\ T_0 &= e^{-bS} & \text{for } S > 0, b > 0 \end{aligned} \right\} \quad \text{on } \rho = 0, \quad (10.19)$$

which gives boundary conditions for $\hat{T}_0$ as

$$\hat{T}_0 = \frac{1}{b - ik} \quad \text{on } z = e^{-i\pi/6} 4^{-1/3} ik^{4/3}. \quad (10.20)$$

This device has been used by Springer and Pedley (1973); the limit $b \rightarrow 0$ is taken in the solution for $T_0(\rho, S)$.

The solution of (10.17) satisfying (10.20) is

$$\hat{T}_0(z, k) = \frac{Ai(z)}{(b - ik) Ai(e^{-i\pi/3} 4^{-1/3} k^{4/3})}, \quad (10.21)$$

where $Ai(z)$ is the Airy function of the first kind (Abramowitz and Stegun, 1965, p. 446). This solution for $\hat{T}_0$ is exponentially small as $\rho \rightarrow \infty$ provided the k -plane is cut along the negative imaginary axis, i.e.

$$-\pi/2 < \arg k < 3\pi/2. \quad (10.22)$$

The inversion integral is

$$T_0(\rho, S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikS} \frac{Ai(4^{-1/3} e^{-i\pi/6} k^{4/3} (2\rho + ik))}{(b - ik) Ai(4^{-1/3} e^{-i\pi/3} k^{4/3})} dk. \quad (10.23)$$

The integrand has simple poles at $k = -ib$ and at the zeros of $Ai(4^{-1/3} e^{-i\pi/3} k^{4/3})$. The zeros of $Ai(z)$ occur for $\arg z = \pm\pi$, but due to the branch cut in the k -plane, only those zeros with $\arg k = \pi/2$ are relevant.

For $S > 0$ the inversion contour, C_+ is shown in Fig. 10.2; the integrand is exponentially small on the semicircular arcs and so the only contributions to T_0 come from integrating around the branch cut and from the pole at $k = -ib$. It can be shown that the branch point contributes nothing to T_0 so we get

$$\begin{aligned} T_0(\rho, S) &= \frac{1}{2} e^{-bS} \left[\frac{Ai(4^{-1/3} e^{-i\pi/3} b^{4/3} (2\rho + b))}{Ai(4^{-1/3} e^{-i\pi/3} b^{4/3})} + \text{c.c.} \right] \\ &\quad - \frac{i}{2\pi} \int_0^{\infty} \frac{e^{-rS}}{b - r} \left[\frac{Ai(4^{-1/3} e^{-i\pi/3} r^{4/3} (2\rho + r))}{Ai(4^{-1/3} e^{-i\pi/3} r^{4/3})} - \text{c.c.} \right] dr, \end{aligned} \quad (10.25)$$

where the second term is a Cauchy Principle Value and c.c. denotes complex conjugate.

To obtain the solution for T_0 satisfying the original boundary conditions (10.15) we let $b \rightarrow 0$ in (10.25). The first term gives no trouble, but the Cauchy Principle Value integral requires closer examination as the location of the singularity tends to an end point of the interval as $b \rightarrow 0$. Near $r = 0$ the integrand behaves like $r^{1/3}/(b-r)$, so that as $b \rightarrow 0$ the integrand is $O(r^{2/3})$ near the origin, and is integrable. Thus we obtain

$$T_0(\rho, s) = 1 - \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{e^{-rs}}{r} \left(\frac{2\rho+r}{r} \right)^{1/2} \times \left[\frac{J_{1/3}\left(\frac{r^{3/2}}{3}(2\rho+r)^{3/2}\right) J_{1/3}\left(\frac{r^2}{3}\right) - J_{1/3}\left(\frac{r^{3/2}}{3}(2\rho+r)^{3/2}\right) J_{1/3}\left(\frac{r^2}{3}\right)}{J_{1/3}^2\left(\frac{r^2}{3}\right) - J_{1/3}\left(\frac{r^2}{3}\right) J_{1/3}\left(\frac{r^2}{3}\right) + J_{1/3}^2\left(\frac{r^2}{3}\right)} \right] dr \quad (10.26)$$

where the Airy functions have been transformed to Bessel functions using the relations (Abramowitz and Stegun, 1965, p. 447 and p. 375)

$$Ai(z) = \frac{1}{3}\sqrt{z} \left[I_{1/3}\left(\frac{2}{3}z^{3/2}\right) - I_{5/3}\left(\frac{2}{3}z^{3/2}\right) \right]$$

and
$$I_n(z) = e^{-\frac{1}{2}\pi ni} J_n(z e^{i\pi/2}) \quad , \quad -\pi < \arg z \leq \frac{\pi}{2} \quad .$$

For $S < 0$ the inversion contour is C_- in Fig. 10.2. The integrand is exponentially small on the semicircular arc, so that after letting $b \rightarrow 0$, the only contribution to T_0 comes from the residues of the enclosed poles. If l_n , $n = 1, 2, 3, \dots$ are such that

$$Ai\left(4^{1/3} e^{i\pi/3} (il_n)^{2/3}\right) = 0 \quad , \quad (10.27)$$

then
$$T_0(\rho, s) = \frac{3}{4^{1/3}} \sum_{n=1}^{\infty} \frac{e^{l_n s}}{l_n^{2/3}} \frac{Ai\left(4^{1/3} l_n^{1/3} (2\rho - l_n)\right)}{Ai'(-4^{1/3} l_n^{2/3})} \quad , \quad \text{for } s < 0 \quad , \quad (10.28)$$

where ' denotes differentiation with respect to the argument.

The heat transfer through the walls is

$$H(s) = - \frac{\partial T_0}{\partial \rho} \Big|_{\rho=0} \quad \text{and for } s > 0$$

$$H(s) = \frac{q}{2\pi^2} \int_0^\infty \frac{e^{-rs}}{r^2} \frac{dr}{J_{1/3}^2(u) - J_{1/3}(u) J_{5/3}(u) + J_{5/3}^2(u)} \quad , \quad u = \frac{r^2}{3} \quad , \quad (10.29)$$

while for $S < 0$ the heat transfer is given by

$$H(s) = -\frac{3}{2} \sum_{n=1}^{\infty} \frac{e^{L_n s}}{L_n} \quad (10.30)$$

Of the solutions (10.26) and (10.28) for $T_0(\rho, S)$, only the solution for $S < 0$ satisfies the matching condition that the temperature be exponentially small as $\rho \rightarrow \infty$. The expression (10.26) for T_0 satisfies the differential equation (10.14) and boundary conditions (10.15), but not the matching condition (10.16). This has arisen due to the limiting process $|k| \rightarrow \infty$, ρ, S fixed implicit in the inversion of the Fourier transform by contour integration. For $S > 0$ the contour is such that T_0 is determined by integrating along a path where the Airy function decays only algebraically for large argument, while for $S < 0$ the residues contributing to T_0 decay exponentially for large arguments. Examination of the inversion formula (10.23) shows that the order of the limits $|k| \rightarrow \infty$, $\rho \rightarrow \infty$ is not interchangeable in the double limit process, so we conclude that (10.26) is valid for $\rho \sim O(1)$.

An asymptotic representation for $T_0(\rho, S)$, $S > 0$ can be found by considering ρ to be large, and then using the method of steepest descent to evaluate (10.23). Using the asymptotic formula for the Airy function (Abramowitz and Stegun, 1965, p. 448)

$$Ai(z) \sim \frac{1}{2} \pi^{1/2} z^{-1/4} e^{-\frac{2}{3} z^{3/2}}, \quad |\arg z| < \pi,$$

the inversion integral (10.23) becomes

$$T_0(\rho, S) \sim \frac{e^{i\pi/24}}{2^{1/12} \pi^{3/2} \rho^{1/4}} \int_{-\infty}^{\infty} \frac{\exp \left[i \rho^{3/2} \left(-k/\mu - \frac{\sqrt{3}}{3} e^{-3\pi i/4} k^{1/2} \right) \right]}{k^{1/2} (b - ik) Ai(4^{1/3} e^{\pi i/3} k^{4/3})} dk, \quad (10.31)$$

where
$$\mu = \frac{S}{\rho^{3/2}}, \quad (10.32)$$

and is regarded as fixed as $\rho \rightarrow \infty$. The integrand has a saddle point at $k = 2i/9\mu^2$, so that (10.31) can be evaluated along the contour C_s (Fig. 10.2) where the steepest descent path through the saddle point is parallel to the real axis. As the inversion contour has been deformed through several poles, the residues there will also contribute to T_0 . The pole at

$k = -ib$ does not contribute to T_0 , and so we may set $b \equiv 0$.

We define il_m to be the location of the first pole above C_s , i.e.

$$L_{m-1} < \frac{2}{9\mu^2} < L_m, \quad m = 1, 2, 3, \dots, \quad (10.33)$$

then for $\rho \rightarrow \infty$, T_0 is given by

$$\begin{aligned} T_0(\rho, S) \sim & \frac{3}{2^{2/3} \rho^{1/2} \sqrt{\pi}} \sum_{n=1}^M \frac{\exp \left[-\rho^{3/2} \left(\frac{\sqrt{8}}{3} L_n^{1/2} - \mu L_n \right) \right]}{L_n^{17/12} \text{Ai}' \left(-4^{1/3} L_n^{4/3} \right)} \\ & + \frac{3^{7/6} \mu^{2/3}}{2^{2/3} \pi \rho} \cdot \frac{e^{-2\rho^{3/2}/9\mu}}{\text{Ai} \left(-4^{1/3} \left(\frac{2}{9\mu^2} \right)^{4/3} \right)}, \end{aligned} \quad (10.34)$$

provided that the saddle point does not coincide with a pole.

This form of the solution clearly satisfies the condition of exponential decay at the edge of the layer; it also gives some new information about the structure of the outer regions of the boundary layer. First we note that the contributions from the poles decay less rapidly as $\rho \rightarrow \infty$ than does the term arising from the saddle point. This follows directly from the fact that the saddle point is located where the coefficient of $\rho^{3/2}$, in the exponential term, has its maximum value.

For $S \rightarrow 0^+$, i.e. $\mu \rightarrow 0$, the position of the saddle point tends to $+i\infty$ and more and more poles are included under the contour C_s . The saddle point contribution may be neglected, and then (10.34) reduces to (10.28) when the Airy functions are expanded for large ρ . The solution (10.28), in terms of the sum of residues, represents the diffusion of heat upstream from $s = 0$. Thus in the asymptotic representation (10.34) for $S > 0$, the sum of the finite number of residues may be identified with the convection downstream of heat that had diffused upstream. As we move further downstream, i.e. as μ increases, the number of poles included under C_s decreases, thus indicating the tendency for the downstream solution to become independent of the details of the upstream conditions. The temperature distribution in the outer edges of the boundary layer becomes completely independent of the upstream diffusion of heat when the saddle point lies below the pole at $k = il_1$. This occurs when

$\mu > \frac{1}{3} \sqrt{\frac{2}{L_1}}$ and then (10.34) becomes

$$T_o(\rho, s) \sim \frac{3^{1/6} \mu^{2/3}}{2^{2/3} \pi \rho} \cdot \frac{e^{-2\rho^{3/2}/9\mu^2}}{Ai\left(-\frac{1}{4}^{1/3}\left(\frac{2}{9\mu^2}\right)^{1/3}\right)} \quad (10.35)$$

Thus far the asymptotic analysis has been carried out assuming $\mu \sim O(1)$. The above form for T_o shows that this condition can be considerably relaxed: T_o will decay exponentially as $\rho \rightarrow \infty$ provided μ increases slower than $\rho^{3/2}$. This then limits the value of S , and hence s , for which the boundary layer solution is valid. The values of ρ are limited by requiring that the parabolic velocity distribution be adequately represented by a linear profile, which gives that $\rho \sim O(Pe^{1/2})$. Thus using the definition of μ we derive the result that the inner solution is valid downstream for $s \sim O(Pe)$, despite initially assuming a region of validity of $O(Pe^{1/2})$. The inner solution in entry flow in a channel also has an analogous property (Van Dyke 1971).

We can also estimate how far upstream the Graetz solution (see below) is valid. The Graetz solution completely neglects the effects of axial conduction, which is valid when $\mu > \frac{1}{3} \sqrt{\frac{2}{L_1}}$. Substituting for S and ρ in the definition of μ , (10.32), and using the known value for l , (Abramowitz and Stegun 1965, p. 448) we find that the downstream solution is valid for

$$s > 0.3 Pe^{1/4} \quad (10.36)$$

Thus increasing Pe increases, albeit weakly, the extent of the region influenced by axial conduction of heat. It should be pointed out that (10.36) is derived from considerations at the very edge of the boundary layer; closer to the wall the downstream solution is applicable even closer to the origin, being valid for $s \sim O(Pe^{1/2})$, thereby accounting for its usefulness in heat transfer calculations.

Before formulating the downstream problem, equation (10.35) is applied to the situation where the velocity profile is linear. If the axial conduction term in (10.14) is neglected, implying the initial condition $T_o(\rho, 0) = 0$, we obtain the classical Leveque problem (Bird, Stewart and Lightfoot 1960, p. 363) which has solution

$$\left. \begin{aligned} T_{0L} &= 1 - c_0 \int_0^\phi e^{-t^3} dt, \\ \text{where } c_0^{-1} &= \int_0^\infty e^{-t^3} dt = \Gamma(4/3), \\ \text{and } \phi &= r \left(\frac{2}{9s} \right)^{1/3}. \end{aligned} \right] \quad (10.37)$$

$\Gamma(z)$ is the Gamma function (Abramowitz and Stegun 1965, p. 255).

Integrating (10.37) by parts we obtain the asymptotic form for T_{0L} for large ϕ

$$\begin{aligned} T_{0L} &\sim \frac{1}{3} c_0 \phi^{-2} e^{-\phi^3} \\ &= \frac{\Gamma(2/3)\sqrt{3}}{2\pi} \phi^{-2} e^{-\phi^3}, \end{aligned} \quad (10.38)$$

where we have used the identity $\Gamma(1/3)\Gamma(2/3) = 2\pi/\sqrt{3}$ (Abramowitz and Stegun 1965, p. 256). This is the asymptotic form of the temperature distribution at the edge of the boundary layer when axial conduction is neglected. It is easily seen that the asymptotic form (10.35) obtained by retaining axial conduction terms reduces to (10.38).

We introduce a new variable Φ such that

$$\Phi^3 = \frac{2r^{3/2}}{9\mu}$$

and replace $\text{Ai}\left(-4^{1/3}\left(\frac{2}{9\mu}\right)^{1/3}\right)$ by $\text{Ai}(0) = \left(3^{2/3}\Gamma(2/3)\right)^{-1}$ thereby transforming (10.35) to

$$T_0 \sim \frac{\Gamma(2/3)\sqrt{3}}{2\pi} \Phi^{-2} e^{-\Phi^3},$$

where

$$\Phi = r \left(\frac{2}{9s} \right)^{1/3} \equiv \phi.$$

Similarly the expression for the heat transfer at $\rho=0$, (10.29) can be evaluated asymptotically for large distances downstream, it being found that the heat transfer given by the Leveque solution is the leading term in the expansion. This result was also obtained by Springer and Pedley (1973).

We continue now with the determination of a solution valid when $s \sim O(\text{Pe})$. This downstream region will later be referred to as region III. Here the axial conduction of heat is much smaller than the axial convection of heat, and it is natural to represent this fact by introducing a new axial co-ordinate s_1 defined by

$$s_1 = s / Pe \quad (10.39)$$

Thus s_1 is a contracted downstream co-ordinate and in this system the energy equation becomes

$$(1-r^2) \frac{\partial T}{\partial s_1} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{Pe^2} \frac{\partial^2 T}{\partial s_1^2} \quad (10.40)$$

The temperature is expanded in powers of Pe^{-2} as follows

$$T(r, s; Pe) \sim \mathcal{T}_0(r, s_1) + Pe^{-2} \mathcal{T}_1 + \dots \quad (10.41)$$

and a set of equations for the $\mathcal{T}_i$ obtained by equating powers of Pe^{-2} . We have

$$\left. \begin{aligned} (1-r^2) \frac{\partial \mathcal{T}_0}{\partial s_1} &= \frac{\partial^2 \mathcal{T}_0}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{T}_0}{\partial r} , \\ \text{and } (1-r^2) \frac{\partial \mathcal{T}_i}{\partial s_1} &= \frac{\partial^2 \mathcal{T}_i}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{T}_i}{\partial r} + \frac{\partial^2 \mathcal{T}_{i-1}}{\partial s_1^2} , \quad i=1,2,3,\dots \end{aligned} \right] \quad (10.42)$$

Initial conditions for these parabolic equations are obtained by matching the solution for $s_1 \rightarrow 0$ with the solutions in regions I and II as $S \rightarrow \infty$. At leading order this matching will be automatic as the solution in region I for large S becomes the solution obtained by neglecting axial conduction, the approximation made in deriving (10.42). For $\mathcal{T}_0$ we have the conditions

$$\left. \begin{aligned} \mathcal{T}_0(r, 0) &= 0 \\ \mathcal{T}_0(1, s_1) &= 1 \end{aligned} \right] , \quad (10.43)$$

so that with (10.42), the problem for $\mathcal{T}_0$ is the classical Graetz-Nusselt problem. The solution is

$$\mathcal{T}_0(r, s_1) = 1 - \sum_{n=0}^{\infty} B_n f_n(r) e^{-\beta_n^2 s_1} ,$$

where the β_n^2 and f_n are the eigenvalues and eigenfunctions, respectively, of a Sturm-Louville boundary value problem (Jakob 1949, §22-2). Further terms may be calculated following Jones (1971).

The eigenvalue β_0^2 determines the thermal entry length, i.e. the distance down the pipe necessary for the fluid temperature to become uniform. We shall be concerned mainly with

this inlet region in subsequent calculations as it is here that the heat transfer is greatest.

Having developed an expansion procedure for large Pe heat transfer with a steady velocity field, we now turn to the construction of a method for calculating heat transfer in a curved pipe with a time dependent velocity field. This problem is represented by (10.1) and (10.2). Introducing the non-dimensional temperature T defined by

$$T = \frac{T^* - T_0}{T_1 - T_0} \quad (10.44)$$

and employing the scaling used for the momentum equations, the energy equation becomes

$$\begin{aligned} \frac{\partial T}{\partial \tau} + \frac{R_s \beta^2}{2} \left(u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \alpha} \right) + \beta \sqrt{\frac{R_s}{2\delta}} \frac{w}{1 + \delta r \cos \alpha} \frac{\partial T}{\partial s} \\ = \frac{\beta^2}{2\sigma} \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \alpha^2} + \frac{1}{(1 + \delta r \cos \alpha)^2} \frac{\partial^2 T}{\partial s^2} \right. \\ \left. + \frac{\delta \cos \alpha}{1 + \delta r \cos \alpha} \frac{\partial T}{\partial r} - \frac{\delta \sin \alpha}{1 + \delta r \cos \alpha} \cdot \frac{1}{r} \frac{\partial T}{\partial \alpha} \right] \quad (10.45) \end{aligned}$$

with boundary conditions

$$\left. \begin{aligned} T &= 0 \quad \text{for } s < 0, \\ T &= 1 \quad \text{for } s > 0 \end{aligned} \right] \quad \text{on } r=1. \quad (10.46)$$

The axial co-ordinate is now

$$s = R\theta/a, \quad (10.47a)$$

and σ is the Prandtl number defined by

$$\sigma = \left(\frac{\kappa}{\lambda} \right)^{-1}. \quad (10.47b)$$

Some information about the temperature distribution can be obtained by comparing the axial conduction of heat to the axial convection of heat. The axial velocity has an $O(1)$ oscillatory component plus an $O(\beta)$ mean component. It is not useful to compare convection of heat by an oscillatory flow to conduction, as over most of the flow, the oscillatory velocity will produce no net effect, whereas the conduction terms acting over long periods of time can produce large effects.

It is more useful to construct a Peclet number by comparing convection by the mean velocity to axial conduction. Anticipating that these expansions will be used in the case when $w_1 \sim O(R_1^* D)$ we obtain a Peclet number

$$Pe = 2\sigma D / \sqrt{2\delta} \quad (10.48)$$

As $\delta \rightarrow 0$ the Peclet number becomes large, which was the original motivation for studying the large Pe limit.

As we are primarily interested in the effect of the secondary motion on the thermal inlet length, a full solution of (10.45) will not be attempted. Rather we shall content ourselves with obtaining the leading order solution almost everywhere in the flow field. The first step in this direction is to quantify the effect of the oscillatory axial velocity. We note that the amplitude of oscillation of a fluid particle is $O(\beta)$, so that for distances $s > O(\beta)$ away from $s = 0$, the fluid particles are not convected past the origin. In particular, for regions close to the wall, the axial temperature gradient is small (due to the boundary conditions) so that the oscillatory flow has little effect on the temperature distribution. Similarly, far away from $s = 0$ where axial temperature gradients are small over the whole cross-section the oscillatory velocity will not affect the temperature field. However in a region $O(\beta)$ around $s=0$ the temperature gradients are large and here transfer of heat by oscillatory convection would be important.

Some support for this hypothesis can be obtained from the work of Davidson and Parker (1971) on heat transfer to unsteady flow between parallel plates. Their analysis can only be regarded as approximate as the true velocity profile is replaced by a uniform flow of the same volume flux at each instant in time, and only the mean and fundamental terms are retained for the calculation of the temperature variation with time. Nevertheless their results indicate that as the frequency of the pulsations is increased the size of the region around the origin where the heat transfer is different from its steady state value is decreased.

The unsteady secondary velocities present in this curved pipe flow are $O(\beta\delta^{1/2})$ compared to the oscillatory axial velocity, so that it may be assumed that the large temperature gradients near $r = 1$, $s = 0$ are not convected into the centre of the pipe. Thus we conclude by postulating that the oscillatory velocity field is important in determining the heat transfer only in a region $O(\beta)$ around $s = 0$, $r = 1$. Therefore we write

$$T(r, \alpha, s, \tau) \sim T(r, \alpha, s) + T_p(\eta, \alpha, \xi) + o(1), \quad (10.49)$$

where to leading order in β , T satisfies

$$\begin{aligned} -\frac{R_s}{2r} \frac{\partial (W_0 T)}{\partial (r, \alpha)} + \sqrt{\frac{R_s}{2\delta}} \frac{W_1}{1 + \delta r \cos \alpha} \frac{\partial T}{\partial s} \\ = \frac{1}{2\sigma} \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \alpha^2} + \frac{1}{(1 + \delta r \cos \alpha)^2} \frac{\partial^2 T}{\partial s^2} \right. \\ \left. + \frac{\delta}{1 + \delta r \cos \alpha} \left(\cos \alpha \frac{\partial T}{\partial r} - \frac{\sin \alpha}{r} \frac{\partial T}{\partial \alpha} \right) \right], \quad (10.50a) \end{aligned}$$

with boundary conditions

$$\left. \begin{aligned} T &= 0 & \text{for } s < 0, \\ T &= 1 & \text{for } s > 0 \end{aligned} \right] \quad \text{on } r=1, \quad (10.50b)$$

and T_p is governed by

$$\begin{aligned} \frac{\partial T_p}{\partial \tau} + \sqrt{\frac{R_s}{2\delta}} \frac{W_0}{1 + \delta r \cos \alpha} \frac{\partial T_p}{\partial \xi} - \frac{1}{2\sigma} \left[\frac{\partial^2 T_p}{\partial \eta^2} + \frac{1}{(1 + \delta r \cos \alpha)^2} \frac{\partial^2 T_p}{\partial \xi^2} \right] \\ = \beta \sqrt{\frac{R_s}{2\delta}} \frac{W_0}{1 + \delta r \cos \alpha} \frac{\partial T}{\partial s}, \quad (10.51a) \end{aligned}$$

and boundary conditions

$$T_p = 0 \quad \text{on } \eta = 0. \quad (10.51b)$$

The Stokes layer co-ordinates are

$$\eta = (1-r)/\beta$$

and

$$\xi = s/\beta.$$

The formulation of the problem has lead to the result that over most of the field the temperature is steady, and its distribution is determined by the steady components of

velocity. The effect of unsteadiness, represented by T_p , is confined to a small region and does not influence the steady temperature T . Clearly $T_p \sim O(\beta)$ for all regions where $\frac{\partial T}{\partial s} \sim O(1)$, i.e. for $\eta, \xi \rightarrow \infty$, so that the boundary conditions (10.50b) need not be modified by any contributions from T_p due to matching. Finally we assume that $T_p \sim O(1)$ at most as $\xi \rightarrow 0$.

Implicit in the above formulation is that the order of the limits $\beta \rightarrow 0$ and $\xi \rightarrow 0$ can be interchanged in the analysis of the equations of motion. Examination of the momentum equations shows that this is the case. Also we have assumed, implicitly, that the Prandtl number σ of the fluid is not small, so that the thermal Stokes layers formed are no thicker than the momentum Stokes layers.

The temperature distribution is described by (10.50a,b); the stream function $\chi_0(r, \alpha; R_s, 0)$ and axial velocity $w_1(r, \alpha; R_s, 0)$ are known, and so a solution in the limit $\xi \rightarrow 0$, R_s, D fixed can be obtained. As $\xi \rightarrow 0$, a parameter interpreted as a Peclet number (10.48) becomes large, and the temperature T is expanded in terms of this Peclet number following the scheme devised for the straight pipe.

In region I we have

$$T(r, \alpha, s) \sim T_0(\rho, \alpha, S) + \left(\frac{2\sigma D}{\sqrt{2}\xi}\right)^{1/2} T_1 + \dots,$$

where the equation for T_0 is

$$\frac{\sqrt{R_s}}{D} \rho \frac{\partial w_1}{\partial r} \bigg|_{r=R_s} \frac{\partial T_0}{\partial S} = \frac{\partial^2 T_0}{\partial \rho^2} + \frac{\partial^2 T_0}{\partial S^2}, \quad (10.52)$$

with boundary conditions

$$\begin{aligned} T_0 &= 0 & \text{for } S < 0 \\ T_0 &= 1 & \text{for } S > 0 \quad \text{on } \rho = 0. \end{aligned}$$

The boundary layer co-ordinates ρ, S are as defined in (10.10). Apart from a parametric dependence on α introduced via $\frac{\partial w_1}{\partial r} \bigg|_{r=R_s}$ (10.52) is the same as (10.14), so the solution for T_0 in this inner region may be considered known. In passing we note that temperature gradients here are $O\left(\frac{2\sigma D}{\sqrt{2}\xi}\right)^{1/2}$; thus we have the restriction that $\left(\frac{2\sigma D}{\sqrt{2}\xi}\right)^{1/2} \ll \beta^{-1}$ for the

formulation (10.49) to be valid. Physically this corresponds to requiring the momentum Stokes layers to be much thinner than the steady thermal boundary layer near the origin.

Near $s = 0$, but not near the wall we have region II. Here the temperature is determined by the upstream conditions, i.e.

$$T \equiv 0 \quad (10.53)$$

Downstream of the origin the contracted variable s_1 is introduced and the temperature expanded in powers of $\left(\frac{\sqrt{2s}}{2\sigma D}\right)^2$. We have

$$s_1 = s / \left(\frac{2\sigma D}{\sqrt{2s}}\right) \quad (10.54)$$

and $T(r, \alpha, s) \sim \mathcal{T}_0(r, \alpha, s_1) + \left(\frac{\sqrt{2s}}{2\sigma D}\right)^2 \mathcal{T}_1 + \dots$

At leading order $\mathcal{T}_0$ is governed by

$$-\frac{R_s}{2r} \frac{\partial(\chi_0, \mathcal{T}_0)}{\partial(r, \alpha)} + \frac{w_1 \sqrt{R_s}}{2\sigma D} \frac{\partial \mathcal{T}_0}{\partial s_1} = \frac{1}{2\sigma} \left[\frac{\partial^2 \mathcal{T}_0}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{T}_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \mathcal{T}_0}{\partial \alpha^2} \right] \quad (10.55)$$

with boundary and initial conditions

$$\left. \begin{aligned} \mathcal{T}_0(1, \alpha, s_1) &= 1 \\ \text{and } \mathcal{T}_0(r, \alpha, 0) &= 0 \end{aligned} \right] \quad (10.56)$$

The problem of heat transfer in steady flow in a curved pipe has received considerable attention in the literature, though almost invariably solutions have been obtained by numerical techniques. The difficulty of deriving the energy equation as $\delta \rightarrow 0$ if D and δ are used as parameters for the velocity field has been circumvented in several ways, but it does not seem to have been noticed that the limit $\delta \rightarrow 0$ with D fixed forces the Peclet number to be large in the heat transfer problem. Typical of the approach used in this field is the work of Kalb and Seader (1972) who solved the momentum and energy equations without taking the limit $\delta \rightarrow 0$, and Akiyama and Cheng (1974) who simply neglected the axial conduction terms and applied an initial condition at $s = 0$. The problem posed by Akiyama and Cheng gives the temperature distribution some distance downstream of the origin, as can be shown following the technique used here. Thus this chapter is concluded with a brief derivation of the equation for

heat transfer, valid downstream of the origin, when the flow in a curved pipe is exactly steady.

In steady flow in a curved pipe the velocity components are scaled as follows

$$(u^*, v^*, w^*) = \frac{\mathcal{N}}{\sigma} (u, v, \delta^{-1/2} w)$$

and then the steady energy equation becomes

$$\begin{aligned} -\frac{1}{r} \frac{\partial(\chi, T)}{\partial(r, \alpha)} + \delta^{-1/2} \frac{w}{1 + \delta r \cos \alpha} \frac{\partial T}{\partial s} \\ = \frac{1}{\sigma} \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \alpha^2} + \frac{1}{(1 + \delta r \cos \alpha)^2} \frac{\partial^2 T}{\partial s^2} \right. \\ \left. + \frac{\delta}{1 + \delta r \cos \alpha} \left(\cos \alpha \frac{\partial T}{\partial r} - \frac{\sin \alpha}{r} \frac{\partial T}{\partial \alpha} \right) \right] \end{aligned}$$

with boundary conditions (10.50b). As $\delta \rightarrow 0$ we expand the temperature in terms of $(\sigma/\sqrt{\delta})$ and divide the flow into regions I, II and III as in the previous two examples.

Here we are only interested in the downstream region where axial conduction of heat is negligible, so we introduce the contracted axial co-ordinate $s_1 = s(\sigma/\sqrt{\delta})^{-1}$ and obtain at leading order

$$-\frac{1}{r} \frac{\partial(\chi, T_0)}{\partial(r, \alpha)} + \frac{w}{\sigma} \frac{\partial T_0}{\partial s_1} = \frac{1}{\sigma} \left[\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r} \frac{\partial T_0}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T_0}{\partial \alpha^2} \right]. \quad (10.57a)$$

The boundary and initial conditions are

$$\left. \begin{aligned} T_0(1, \alpha, s_1) &= 1 \\ T_0(r, \alpha, 0) &= 0 \end{aligned} \right] \quad (10.57b)$$

CHAPTER 11

Application of results to artificial respirators

The enhancement of heat and mass transfer by secondary motion is well known in engineering practice. Typically, the random mixing processes occurring in turbulent flow are used to increase the heat transfer above that obtained in a steady laminar flow. In bio-engineering applications where the working fluid is blood, the high shear stresses associated with the turbulent flow could not be tolerated and so laminar flow situations must be used. In the search for means of increasing heat transfer, interest has been centred on utilizing the secondary motions that occur when a fluid flows in a curved pipe. The case of heat transfer to a fluid flowing steadily in a curved pipe has been extensively studied (Akiyama and Cheng 1974 and Kalb and Seader 1972), and more recently there has been some experimental work on mass transfer to fluids flowing unsteadily in curved pipes (Tanishita et al 1976 and Melrose et al 1972). It is this latter problem we wish to consider here.

The experiments of Melrose et al were concerned with the transfer of oxygen to venous blood flowing through silicone membrane tubes coiled around a drum as shown in Fig. 11.1. When the drum was stationary the oxygen transferred was only one tenth of the oxygen capable of crossing the membrane. However when the drum was oscillated sinusoidally about its axis an increase in the oxygen transfer of five to ten times the steady flow value was obtained for the highest frequency of oscillation.

The results reported by Tanishita et al are less dramatic. They considered oxygen transfer in pulsatile and steady flow through curved micro-porous tubing for both blood and water. Their results showed that the gas exchange in unsteady flow was up to twice the steady flow gas transfer.

The experimental conditions used by Melrose et al correspond quite closely with the assumptions made in this thesis (see Table 5); also, as R_s is large and D is small, the steady

velocity field will have the asymptotic form given in Chapter 5. However the application of the heat transfer analysis in Chapter 10 requires some qualification. Implicit in Chapter 10 was the assumption that, for mass transfer, the diffusing species does not react chemically with the fluid. For oxygen transfer to blood this condition is not met as oxygen combines chemically with the haemoglobin in red blood cells. However some indication of the effect of secondary motion on gas exchange in the unsteady flow of blood can be obtained by considering oxygen transfer to water (say) where the oxygen just goes into physical solution. In the heat transfer context, the calculation described below is valid without further restrictions.

The values of the parameters describing the experimental conditions used by Tanishita et al are slightly outside the range of validity of the theory developed here. In particular the Stokes layers are not thin compared to the pipe radius, and as the assumption of thin Stokes layers is fundamental to the method of matched asymptotic expansions used to derive the governing equations, our results cannot be expected to apply to these experiments. Thus we shall be more concerned with conditions relevant to the work of Melrose et al.

Before returning to the terminology of heat transfer, we note that for mass transfer the quantity σ is the Schmidt number, defined as the ratio of the kinematic viscosity of the fluid to the diffusion coefficient of the substance being transported. For oxygen diffusing in water the Schmidt number is $O(10^3)$, so that even for moderately small curvature, the mass transfer Peclet number $(O(\sqrt{R}))$ will be quite large. This allows the whole length of the gas exchanger to be treated as the far downstream region III where axial diffusion is negligible. Also the steady flow experiments, used by Melrose et al as a comparison for the unsteady flow experiments, were conducted at low Dean number; the largest value of D was 4.3. As D is small, the secondary motions will be very weak and have little effect on the oxygen transfer. Therefore, in the

experiments of Melrose et al the oxygen uptake in unsteady flow in a curved pipe is compared to the gas exchange obtained in steady flow in a straight pipe, and so for the analytical work it is reasonable to use the known heat transfer results for steady flow in a straight pipe (Jakob 1949) as the basis for comparison with heat transfer in the unsteady flow situation.

From §22-2 of Jakob, the non-dimensional temperature distribution in Poiseuille flow through a circular pipe is

$$\phi(r, \alpha, s_1) = 1 + \sum_{n=0}^{\infty} N_n R_n(r; \beta_n) e^{-\beta_n^2 s_1}, \quad (11.1)$$

when the temperature at the pipe wall is

$$\begin{aligned} \phi &= 0 & \text{for } s_1 < 0 \\ \phi &= 1 & \text{for } s_1 > 0 \end{aligned}$$

Here s_1 is measured from the start of the heated section and is the contracted downstream variable defined by

$$s_1 = \frac{s^*}{a} \left(\frac{2 V_m a}{K} \right)^{-1}, \quad (11.2)$$

and where V_m is the mean velocity. The functions $R_n(r; \beta_n)$ are the eigenfunctions corresponding to the eigenvalues β_n^2 of the governing Sturm-Liouville problem

$$\frac{d}{dr} \left(r \frac{dR}{dr} \right) + \beta^2 r (1-r^2) R = 0, \quad (11.3a)$$

$$\left. \begin{aligned} R(1) &= 0, \\ R \text{ is regular at } r &= 0. \end{aligned} \right] \quad (11.3b)$$

The equation has a regular solution

$$R(r) = 1 + \sum_{n=1}^{\infty} \mathcal{A}_{2n} r^{2n}, \quad (11.4a)$$

where

$$\mathcal{A}_2 = -\beta^2/4$$

and

$$\mathcal{A}_{2n+4} = \frac{\beta^2}{(2n+4)^2} \left[\mathcal{A}_{2n} - \mathcal{A}_{2n+2} \right], \quad n=0,1,2,\dots \quad (11.4b)$$

Thus the $\mathcal{A}_{2n}$ are polynomials of degree $2n$ in β , and the eigenvalues β_m^2 are the roots of the equation

$$1 + \sum_{n=1}^{\infty} \mathcal{A}_{2n}(\beta_m^2) = 0. \quad (11.5)$$

The eigenfunctions are orthogonal with respect to the weight function $r(1-r^2)$, so the Fourier coefficients N_n are given by

$$N_n = - \frac{\int_0^1 R_n r (1-r^2) dr}{\int_0^1 R_n^2 r (1-r^2) dr} \quad (11.6)$$

The heat transferred to the fluid from a length s_1 of the heated section of the pipe is equal to the energy convected through the cross-section at s_1 , i.e.

$$- \int_0^{s_1} \int_0^{2\pi} \frac{\partial \phi}{\partial r} \bigg|_{r=1} d\alpha ds_1 = \int_0^{s_1} \int_0^{2\pi} w \phi r d\alpha dr \quad (11.7)$$

where the axial velocity w is $(1-r^2)$. A quantity related to the energy flux at s_1 is the bulk, or cup mixing temperature ϕ_m defined by

$$\phi_m(s_1) = \frac{\int_0^{s_1} \int_0^{2\pi} w \phi r d\alpha dr}{\int_0^{s_1} \int_0^{2\pi} w r d\alpha dr} = 1 + \sum_{n=0}^{\infty} M_n e^{-\beta_n^2 s_1} \quad (11.8)$$

The cup mixing temperature $\phi_m(s_1)$ is the temperature the fluid would obtain if the tube were chopped off at s_1 and if the fluid leaving the pipe were collected in a container and thoroughly mixed. (See Bird, Stewart and Lightfoot, 1960, p. 297.)

The local rate of heat transfer is quantified by the local Nusselt number $Nu(s_1)$ defined as

$$Nu(s_1) = \frac{- \int_0^{2\pi} \frac{\partial \phi}{\partial r} \bigg|_{r=1} d\alpha}{\pi (1 - \phi_m)} \quad ,$$

which, using (11.7) and (11.8) becomes

$$Nu(s_1) = \frac{1}{2} \frac{d\phi_m}{ds_1} / (1 - \phi_m) \quad (11.9)$$

The Nusselt number is infinite at $s_1 = 0$, and when the temperature field is fully developed the Nusselt number is

independent of s_1 ; here, as $s_1 \rightarrow \infty$ the Nusselt number tends to the constant value of 3.66. The variation of $Nu(s_1)$ with s_1 is shown in Fig. 11.2.

The equation governing the heat transfer in the far downstream region in unsteady flow in a curved pipe is (10.55); the boundary and initial conditions are (10.56). When D is very much less than R_s , as in the experiments of Melrose et al, the velocity has the expansion (5.1), and following the straight pipe solution, the temperature $\mathcal{T}$ is written as

$$\mathcal{T}_0(r, \alpha, s_1) = 1 + e^{-s_1 \gamma} T(r, \alpha) \quad (11.10)$$

Thus, at leading order the equation for T is

$$-\frac{R_s}{r} \frac{\partial(\chi_{00}, T)}{\partial(r, \alpha)} - \frac{\gamma w_{10}}{\sigma} T = \frac{1}{\sigma} \nabla^2 T \quad (11.11a)$$

and the boundary conditions are

$$T(1, \alpha) = 0. \quad (11.11b)$$

We are primarily interested in the form of T when R_s is large and both χ_{00} and w_{10} have inviscid core/boundary layer structure. It is expected that T will also consist of a convection dominated core surrounded by a boundary layer where conduction of heat is important. The thickness of the thermal boundary layer will be $O(R_s^{-1/2})$ (assuming $\sigma \sim O(1)$ as $R_s \rightarrow \infty$), as for the momentum boundary layers, and for the heat transfer to remain finite as $R_s \rightarrow \infty$, the temperature in the layer must be $O(R_s^{-1/2})$. If the leading order temperature in the core is $R_s^m T(r, \alpha)$, then in order to match to the boundary layer we require

$$m = 0, \quad (11.12a)$$

$$\text{and} \quad T(1, \alpha) = 0. \quad (11.12b)$$

The size of the constant γ can be determined by examining the integral condition arising when (11.11a) is integrated around a closed streamline, which for large R_s is the curve $\psi = \bar{\psi}_0 = \text{constant}$. Using the asymptotic result (5.37) for w_{10} and the streamline co-ordinate system of Fig. 5.1, the integral condition is

$$-\gamma \bar{w}_0 \oint \tau \frac{h_2}{q} d\eta = \oint \frac{h_2}{q} \nabla^2 T d\eta \quad (11.13)$$

and provided none of the integrals are identically zero, γ must be $O(1)$ as $R_s \rightarrow \infty$. For R_s large, the expansions for χ_0 and w_0 are (5.7) and (5.29), while for T and γ we have

$$\left. \begin{aligned} T(r, \alpha; R_s) &\sim \bar{T}_0(r, \alpha) + R_s^{-1/2} \bar{T}_1 + \dots \\ \gamma &\sim \gamma_0 + R_s^{-1/2} \gamma_1 + \dots \end{aligned} \right\} \quad (11.14)$$

The energy balance (11.7) indicates that the heat transfer to the fluid can be evaluated in one of two ways: by solving the boundary layer equations and evaluating the temperature gradient at the wall; or by calculating the energy flux through the pipe. As the temperature in the core is much larger than in the boundary layer, the energy flux through the pipe is equal to the energy flux through the core to leading order. Thus the leading order temperature field in the core gives sufficient information to evaluate the heat transfer.

The leading order temperature in the core, $\bar{T}_0$ is found by substituting (5.7), (5.29) and (11.14) into (11.11a) and equating like powers of R_s . The equation occurring at $O(R_s)$ shows that $\bar{T}_0$ is constant on the streamlines of the secondary motion, i.e.

$$\bar{T}_0 = \bar{T}_0(\bar{\chi}_0) \quad (11.15)$$

The variation of $\bar{T}_0$ across the streamlines is obtained by combining (11.15) with the integral condition (11.13), and this gives

$$-\bar{w}_0 \bar{T}_0 \gamma_0 \oint \frac{h_2}{q} d\eta = \oint \frac{\partial}{\partial \eta} (q h_2 \bar{T}_0') d\eta, \quad (11.16)$$

where ' denotes differentiation with respect to the argument. Introducing the circulation $\Gamma(\psi)$ defined by

$$\Gamma(\psi) = - \oint q h_2 d\eta, \quad (11.17)$$

and using (5.27) to simplify the left hand side, (11.16) becomes

$$\gamma_0' \bar{w}_0 \bar{T}_0 \gamma_0 \Gamma = (\bar{T}_0' \Gamma)' \quad (11.18)$$

Boundary conditions for $\bar{T}_0$ are

$$\left. \begin{aligned} \bar{T}_0 &= 0 \text{ on } \psi = 0 \\ \bar{T}_0 &\text{ is regular in the cross-section} \end{aligned} \right] \quad (11.19)$$

so that γ_0 , $\bar{T}_0$ are the eigenvalues and eigenfunctions of a Sturm-Louville boundary value problem.

Initially the circulation $\Gamma(\psi)$ was evaluated numerically by calculating the area inside a closed streamline $\psi = \text{constant}$ and using the result

$$\Gamma(\psi) = \gamma_0 \iint dA$$

which is obtained from the general definition of Γ by noting that the vorticity is constant. However it was found that to a good approximation, the area inside the closed streamline was proportional to the stream function (see Fig. 11.3), so that Γ became

$$\Gamma(\psi) = \frac{\pi}{2} \left(1 - \frac{\psi}{\psi_m} \right) \quad , \quad (11.20)$$

where ψ_m is the value of the stream function at the vortex centre. We note that for a uniform vorticity flow in an elliptical region, the linear dependence of the circulation on ψ , expressed by (11.20), is an exact result (see Batchelor, 1967, p. 539). Substituting (11.20) into (11.17) and using the transformation

$$x^2 = 1 - \frac{\psi}{\psi_m} \quad , \quad (11.21)$$

we obtain

$$\frac{d}{dx} \left(x \frac{d\bar{T}_0}{dx} \right) + \gamma_0 \frac{16\psi_m^2}{\gamma_0^2} x(1-x^2) \bar{T}_0 = 0 \quad , \quad (11.22a)$$

with boundary conditions

$$\left. \begin{aligned} \bar{T}_0(1) &= 0 \\ \bar{T}_0 &\text{ regular in the cross-section} \end{aligned} \right] \quad . \quad (11.22b)$$

The Sturm-Louville problem (11.22) is the same as (11.3) so the solution for γ_0 satisfying the initial condition (10.56) can be written down at once as

$$\gamma_0 = 1 + \sum_0^\infty N_n R_n e^{-\gamma_{0n} \zeta_1} \quad , \quad (11.23)$$

where the eigenvalues γ_{0n} and β_n^2 are linked by

$$\beta_n^2 = \frac{16\psi_m^2}{\gamma_0^2} \gamma_{0n} \quad . \quad (11.24)$$

Also, it is clear that the cup mixing temperature $\mathcal{J}_{om}(s_1)$ is

$$\mathcal{J}_{om}(s_1) = 1 + \sum_{n=0}^{\infty} M_n e^{-\gamma_{on} s_1} \quad (11.25)$$

Jakob (1949) gives only the first three eigenvalues and eigenfunctions; due to the slow convergence of the Fourier series (11.22) or (11.24) more eigenvalues are required to get sufficient accuracy near the origin $s_1 = 0$, so four more were evaluated. The eigenfunctions and the quantity on the left hand side of (11.5) were evaluated by summing a finite number of terms. From (11.4b) it is seen that the $\mathcal{A}_{1n}$ alternate in sign so that the error made in truncating the infinite series in (11.4) and (11.5) is less than the first term neglected. In the numerical evaluation of R_n terms greater than 10^{-14} were included in the sum, so that R_n should be in error by at most $O(10^{-14})$. However, as the evaluation of the sums involved the difference of nearly equal numbers, several significant figures were lost, and R_n is then accurate to about 9 significant figures. The zeros of (11.5) were obtained by a standard secant method; the left hand side of (11.5) summed to less than 10^{-7} at the eigenvalues β_n^2 . The first 7 values for β_n^2 , γ_{on} , N_n and M_n are given in Table 6. Also included in this table are the eigenvalues calculated by Jones (1971) and it is seen that there is good agreement between the two sets of eigenvalues. The integrals involved in the definitions of M_n and N_n were evaluated using Simpson's rule and a step length of $1/80$.

From (11.25) it is possible to calculate the local Nusselt number and the total amount of heat transferred to the fluid. The comparison between the latter quantity and the heat transferred in a straight pipe is best made at the same value of dimensional distance from the start of the heated region, all other quantities being held fixed. When the mean pressure gradient in the two pipes is the same, s_1 for the straight pipe (11.2) is the same as s_1 for the curved pipe for equal physical distances along the pipe. Thus using (11.25), (11.7) and (11.8) the ratio of heat transferred at equal pressure

gradients is

$$RG(s_1) = Q_{r\infty} \frac{1 + \sum M_n e^{-\gamma_{0n} s_1}}{1 + \sum M_n e^{-\beta_n^2 s_1}}, \quad (11.26)$$

where $Q_{r\infty}$ is the volume flux ratio as $R_s \rightarrow \infty$ from Chapter 5. However, if the heat transfer comparison is made for the same volume flow rate through the two pipes, equal dimensional distances along the straight and curved pipes give s_1 and $Q_{r\infty} s_1$ as the respective non-dimensional distance. Thus using (11.7), (11.8) and (11.25) the ratio of heat transfer at equal volume flow rates is

$$RF(s_1) = \frac{1 + \sum M_n e^{-\gamma_{0n} s_1 Q_{r\infty}}}{1 + \sum M_n e^{-\beta_n^2 s_1}} \quad (11.27)$$

The local Nusselt number for the curved pipe, defined using (11.9), is shown plotted against s_1 in Fig. 11.2; also plotted in Fig. 11.2 is $Nu(s_1)$ for the straight pipe. When Nu is constant, the temperature field is fully developed, and in this region Nu is simply related to the first eigenvalue:

$$\left. \begin{array}{l} Nu \sim \frac{1}{2} \beta_0^2 \\ \text{or } Nu \sim \frac{1}{2} \gamma_{00} \end{array} \right] \quad \text{as } s_1 \rightarrow \infty \quad (11.28)$$

Also, as $\gamma_{00} > \beta_0^2$, the distance along the pipe necessary to reach the fully developed state is smaller in the curved pipe than in the straight pipe. It is interesting to compare the Nusselt numbers obtained for small D and large R_s in an unsteady flow to those obtained for arbitrary D in steady flow in a curved pipe (see Fig. 11.2, results from Akiyama and Cheng 1974). It is readily seen that even for moderate D , the secondary motion in steady flow in a curved pipe increases Nu more than an infinite R_s for unsteady flow⁺. This is due to the nature of the secondary flow: in steady flow in a curved pipe the secondary vortex is such that all fluid particles spend some time near the wall of the pipe, while in unsteady flow at large R_s fluid particles move on almost concentric orbits and so remain a fixed distance away from the pipe wall.

⁺ The results in Fig. 11.2, quoted from Akiyama and Cheng for $D=135$, do not show Nusselt numbers larger than those obtained in unsteady flow at infinite R_s . Though they did not consider higher values of D their paper indicates that for larger D , Nu will continue to increase.

In Fig. 11.4 the quantities RF and RG are plotted against s_1 . We see that as $s_1 \rightarrow \infty$, $RG \rightarrow Q_{r,\infty}$ and $RF \rightarrow 1$, and as expected RG tends to its asymptotic value more quickly than RF. The maximum values of RF and RG occur at $s_1 = 0$, but they cannot be determined as the series (11.1) and (11.23) are not sufficiently accurate at the origin. However, it seems likely that the heat transfer to fluid in a curved pipe will be no greater than twice the heat transfer to fluid in a straight pipe. Thus our results are not able to explain the experimental observations of Melrose et al (1972). At the exit of the gas exchanger in Melrose's experiment $s_1 \approx 0.005$, so when the volume flux is the same, the heat transfer should be ~ 1.7 times the heat transfer in a straight pipe. The large increases in mass transfer found by Melrose et al must be due to effects other than the secondary motion. A possibility is that oscillating the tubes caused the cross-section of the pipe to change periodically (as the pipe wall was very thin) and so keep the blood well mixed.

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APPENDIX A

The method of McConalogue and Srivastava

In this appendix we seek solutions of (3.38) and (3.40) with boundary conditions (3.41) by the method of McConalogue and Srivastava (1968). As the problem is symmetrical about $\alpha = 0, \pi$ the stream function, third component of vorticity and the axial velocity can be represented by the Fourier series

$$\left. \begin{aligned} \chi_0 &= \sum_1^\infty B_n(r) \sin n\alpha, \\ -\nabla^2 \chi_0 &= \sum_1^\infty D_n(r) \sin n\alpha, \\ w_1 &= \frac{E_0(r)}{2} + \sum_1^\infty E_n(r) \cos n\alpha. \end{aligned} \right\} \quad (A.1)$$

If the problem is not symmetrical about $\alpha = 0, \pi$, as in the problem governed by (5.3) and (5.2) and (7.2), then the Fourier series (7.1) must be used; for this case the analysis follows closely that described below.

Substituting (A.1) into the equations of motion and equating coefficients of $\cos n\alpha$ in (3.38) and $\sin n\alpha$ in (3.40) we obtain the following equations

$$\left. \begin{aligned} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) E_0 &= 2 f_0(r) \\ \text{and for } n \geq 1, \quad \mathcal{L}_n(E_n) &= f_n(r), \\ \mathcal{L}_n(B_n) &= -D_n(r), \\ \mathcal{L}_n(D_n) &= F_n(r) + S_n(r), \\ \text{where } \mathcal{L}_n &\equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2}. \end{aligned} \right\} \quad (A.2)$$

The functions f_n , F_n and S_n come from the nonlinear terms in the momentum equations and are given by

$$\left. \begin{aligned} f_0 &= \frac{R_s}{2r} \sum_{m=1}^\infty \left(m E_m \frac{dB_m}{dr} + m B_m \frac{dE_m}{dr} \right) - \frac{4D}{\sqrt{R_s}}, \\ \text{and for } n \geq 1, \quad f_n &= \frac{R_s}{2r} \sum_{m=-\infty}^\infty \left(m B_m \frac{dE_{n-m}}{dr} - m E_m \frac{dB_{n-m}}{dr} \right), \end{aligned} \right\}$$

$$F_n = \frac{R_s}{2r} \sum_{m=-\infty}^{\infty} \left(m B_m \frac{dD_{n-m}}{dr} - m D_m \frac{dB_{n-m}}{dr} \right) ,$$

and

$$S_n = \sum_{m=-\infty}^{\infty} \frac{dE_m}{dr} (E_{n-m+1} - E_{n-m+1}) - \frac{1}{r} \sum_{m=-\infty}^{\infty} m E_m (E_{n-m-1} + E_{n-m+1}) \quad (A.3)$$

In the summations the following relations have been used

$$\left. \begin{aligned} B_{-n} &= -B_n \\ D_{-n} &= -D_n \\ B_0 &\equiv D_0 \equiv 0 \\ E_{-n} &= E_n \end{aligned} \right\} \quad (A.4)$$

The solutions of (A.2) satisfying the boundary conditions (3.41) are

$$E_0(r) = 2 \log r \int_0^r u f_0 du - 2 \int_1^r u \log u f_0 du ,$$

and for $n \geq 1$, $E_n(r) = \frac{r^n}{2n} \int_0^r f_n u^{n+1} du - \frac{r^{-n}}{2n} \int_0^r f_n u^{n+1} du$ (A.5)

and

$$B_1(r) = -\frac{r}{2} \int_1^r D_1 du + \frac{1}{2r} \int_0^r u^2 D_1 du + \frac{r \mathcal{M}}{2} , \quad \left[+ \frac{r^n}{2n} \int_1^r f_n u^{n+1} du \right]$$

$$D_1(r) = \frac{r}{2} \int_1^r (F_1 + S_1) du - \frac{1}{2r} \int_0^r u^2 (F_1 + S_1) du - \frac{r}{2} \int_0^1 u^4 (F_1 + S_1) du$$

$$+ r \int_0^1 u^2 (F_1 + S_1) du - \mathcal{M} r ,$$

and for $n \geq 2$, $B_n(r) = -\frac{r^n}{2n} \int_1^r D_n u^{n+1} du + \frac{r^{-n}}{2n} \int_0^r D_n u^{n+1} du ,$

$$D_n(r) = \frac{r^n}{2n} \int_1^r u^{n+1} (F_n + S_n) du - \frac{r^{-n}}{2} \int_0^1 u^{n+1} (F_n + S_n) du$$

$$+ \frac{(n+1)r^n}{2n} \int_0^1 u^{n+1} (F_n + S_n) du - \frac{r^{-n}}{2n} \int_0^r u^{n+1} (F_n + S_n) du . \quad (A.6)$$

Derivatives of the Fourier coefficients, needed to evaluate f_n etc., can be calculated from (A.5) and (A.6). At $r = 0$ (A.5) and (A.6) take the special forms:

$$\left. \begin{aligned} E_0(0) &= 2 \int_0^1 u \log u f_0 du , \\ E_n(0) &= 0 , \quad n \geq 1 , \\ B_n(0) &= 0 , \quad n \geq 1 , \end{aligned} \right\} \downarrow$$

$$\begin{aligned}
 D_n(0) &= 0, \quad n \geq 1, \\
 f_0(0) &= R_s \left. \frac{dB_1}{dr} \cdot \frac{dE_1}{dr} \right|_{r=0} - \frac{4D}{\sqrt{R_s}}, \\
 f_n(0) &= 0, \quad n \geq 1, \\
 F_n(0) &= 0, \quad n \geq 1, \\
 \text{and } S_n(0) &= 0, \quad n \geq 1.
 \end{aligned} \tag{A.7}$$

In the numerical determination of the Fourier coefficients, the f_n , F_n and S_n are given by their values at discrete points across the interval. The integrals in (A.5) and (A.6) are evaluated by fitting a cubic spline function to the f_n etc. and assuming that the spline adequately represents the variation of the functions between the mesh points. To determine the spline function uniquely on $[0,1]$ it is necessary to supply the derivatives of f_n etc. at $r = 0$ and $r = 1$, as well as the values of the f_n etc. at the mesh points. After some analysis, the slopes of the f_n etc. at $r = 0$ can be shown to be given by

$$\begin{aligned}
 \frac{df_0}{dr} &= 0, \\
 \frac{df_1}{dr} &= R_s \left(\frac{dB_1}{dr} \frac{d^2E_2}{dr^2} + \frac{d^2B_2}{dr^2} \frac{dE_1}{dr} + \frac{1}{2} \frac{dB_1}{dr} \frac{d^2E_0}{dr^2} \right) \Big|_{r=0}, \\
 \frac{df_n}{dr} &= 0, \quad n \geq 2, \\
 \frac{dF_1}{dr} &= R_s \left(\frac{dB_1}{dr} \frac{d^2D_2}{dr^2} - \frac{d^2B_2}{dr^2} \frac{dD_1}{dr} \right) \Big|_{r=0}, \\
 \frac{dF_n}{dr} &= 0, \quad n \geq 2, \\
 \frac{dS_1}{dr} &= E_0 \left(\frac{d^2E_0}{dr^2} - 2 \frac{d^2E_2}{dr^2} \right) \Big|_{r=0}, \\
 \text{and } \frac{dS_n}{dr} &= 0, \quad n \geq 2.
 \end{aligned} \tag{A.8}$$

The slopes at $r = 1$ are easily obtained by differentiation of (A.3).

In the straight forward iteration technique the values of R_s and D are given together with an initial approximation

for the Fourier coefficients. From this initial approximation the f_n , F_n and S_n are evaluated and new estimates of the Fourier coefficients obtained from (A.5) and (A.6). In the modified iteration technique an initial approximation to the Fourier coefficients, a value for R_s and the value of the axial velocity at $r = 0$, $\frac{1}{2}E_0(0)$ are given. From the first relationship in (A.7) an estimate of D is given by

$$D = \left(\frac{4}{\sqrt{R_s}} \int_0^1 u \log u \, du \right)^{-1} \left[\int_0^1 u \log u \frac{R_s}{2u} \sum_{n=0}^{\infty} \left(m E_m \frac{dB_m}{d\alpha} + m B_m \frac{dE_m}{du} \right) du - \frac{E_0(0)}{2} \right] \quad (A.9)$$

which is then used in the subsequent calculation of the new estimate of the Fourier coefficients.

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OREYNOLDS NO. OF 100.00 DEAN NO. OF 29.50

WT FACTS 0 .50

OIAL DS (M, 2) DS (M, N2) DS (M, NF) DDS (M, 2)
(M, NF)

OIAL ES (1, 1) ES (M, N2) ES (M, NF) DES (M, 1)
(M, NF)

Iteration number	$D_1(\frac{1}{2})$ $E_0(10)$	$D_9(\frac{1}{2})$ $E_9(\frac{1}{2})$	$D_n(\frac{1}{2})$ $E_n(\frac{1}{2})$
1	.59355601E+00	.31462871E-06	.77807800E-03
1	.54974669E+01	-.20378935E-07	.26706158E-09
2	.56817863E+00	.31154059E-06	.79298168E-08
2	.54308002E+01	-.20791591E-07	.25119396E-09
3	.56339229E+00	.32970764E-06	.84639140E-08
3	.54601651E+01	-.22066702E-07	.23035904E-09
4	.56248429E+00	.33878197E-06	.88360053E-08
4	.54752493E+01	-.24856425E-07	.18916973E-09
5	.56230283E+00	.34281152E-06	.90892928E-08
5	.54775439E+01	-.28148943E-07	.12636110E-09
6	.56234893E+00	.34364794E-06	.92359925E-08
6	.54766897E+01	-.30804265E-07	.54445128E-10
7	.56244895E+00	.34307403E-06	.93015154E-08
7	.54758655E+01	-.32363502E-07	-.35768815E-11
8	.56252140E+00	.34217531E-06	.93167231E-08
8	.54754038E+01	-.32993060E-07	-.55082538E-10
9	.56255504E+00	.34146246E-06	.93095937E-08
9	.54751767E+01	-.33076256E-07	-.80897970E-10
10	.56256493E+00	.34104315E-06	.92973736E-08
10	.54750824E+01	-.32949864E-07	-.91938448E-10
11	.56256523E+00	.34084956E-06	.92875028E-08
11	.54750557E+01	-.32801851E-07	-.94541146E-10
12	.56256328E+00	.34078328E-06	.92815509E-08
12	.54750557E+01	-.32699410E-07	-.93612997E-10
13	.56256163E+00	.34077255E-06	.92786788E-08
13	.54750619E+01	-.32646101E-07	-.91922085E-10
14	.56256072E+00	.34077892E-06	.92776032E-08
14	.54750669E+01	-.32625053E-07	-.90580690E-10
15	.56256035E+00	.34078672E-06	.92773563E-08
15	.54750695E+01	-.32619882E-07	-.89795904E-10
16	.56256025E+00	.34079176E-06	.92773961E-08
16	.54750705E+01	-.32620475E-07	-.89432233E-10
17	.56256025E+00	.34079417E-06	.92774857E-08
17	.54750708E+01	-.32622170E-07	-.89304581E-10
18	.56256027E+00	.34079502E-06	.92775510E-08
18	.54750707E+01	-.32623440E-07	-.89290549E-10
19	.56256029E+00	.34079519E-06	.92775848E-08
19	.54750707E+01	-.32624102E-07	-.89289509E-10
20	.56256030E+00	.34079514E-06	.92775980E-08
20	.54750706E+01	-.32624359E-07	-.89302348E-10
21	.56256031E+00	.34079506E-06	.92776012E-08
21	.54750706E+01	-.32624421E-07	-.89310913E-10
22	.56256031E+00	.34079501E-06	.92776010E-08
22	.54750706E+01	-.32624415E-07	-.89315072E-10

Typical behaviour
of iteration
scheme.

TABLES

TABLE 1: The function $(-s_0)^{-1/2} \lambda(0, y)$

y	$(-s_0)^{-1/2} \lambda(0, y)$	y	$(-s_0)^{-1/2} \lambda(0, y)$
0.0	4.212	2.6	1.107
0.1	3.827	2.8	1.116
0.2	3.474	3.0	1.125
0.3	3.152	3.2	1.134
0.4	2.860	3.4	1.141
0.5	2.596	3.6	1.146
0.6	2.360	3.8	1.150
0.7	2.152	4.0	1.152
0.8	1.969	4.5	1.154
0.9	1.809	5.0	1.153
1.0	1.671	5.5	1.152
1.2	1.454	6.0	1.152
1.4	1.303	6.5	1.152
1.6	1.205	7.0	1.151
1.8	1.146	7.5	--
2.0	1.114	8.0	--
2.2	1.102	9.0	--
2.4	1.102	10.0	1.151

TABLE 2: Comparison of analytical and numerical solutions of (5.2), (5.3) and (7.2) for $R_s = 200$ and $\epsilon = 0.01$.

$-\chi_{00}(r, 0)$		$\nabla^2 \chi_{00}(r, \alpha) \Big _{\alpha = \pi/2}$	
Analytical	Numerical	Analytical	Numerical
0.5000	0.5000	2.000	2.000
0.4951	0.4951	2.000	2.000
0.4801	0.4801	2.000	2.001
0.4552	0.4552	2.000	2.000
0.4202	0.4202	2.000	2.000
0.3753	0.3753	1.999	1.999
0.3204	0.3204	1.996	1.997
0.2554	0.2554	1.994	1.994
0.1804	0.1804	2.006	2.006
0.0952	0.0952	2.057	2.056
0.0000	0.0000	2.116	2.116

$w_{10}(r, \pi)$	
Analytical	Numerical
1.0000	1.0000
0.9901	0.9899
0.9602	0.9598
0.9103	0.9097
0.8405	0.8395
0.7506	0.7494
0.6408	0.6392
0.5109	0.5092
0.3606	0.3595
0.1901	0.1899
0.0000	0.0000

TABLE 3: Comparison of flux ratios with McConalogue and Srivastava (1968) and Collins and Dennis (1975).

Dean number	FLUX		RATIO
	Present calculations	McConalogue and Srivastava	Collins and Dennis
10	0.9876	0.987	
20	0.9217	0.921	
30	0.8598	0.861	
40	0.8153	0.815	
50	0.7821	0.781	0.787
60	0.7555	0.751	
70	0.7335	0.729	
100	0.6807		0.681
200	0.5760		0.570
400	0.4794		0.481

TABLE 4: The functions $\lambda_i^*(y)$ and $\lambda_i(y)$

y	$-\lambda_0^*(y)$	$-\lambda_1^*(y)$	$\lambda_0(y)$	$-\lambda_1(y)$
0.0	-1.0000	1.0000	1.000	1.000
0.1	-0.8610	0.9432	0.878	--
0.2	-0.7273	0.8836	0.761	--
0.3	-0.6063	0.8263	0.656	--
0.4	-0.4970	0.7698	0.563	--
0.5	-0.3999	0.7145	0.482	--
0.6	-0.3148	0.6607	0.412	--
0.7	-0.2412	0.6086	0.353	--
0.8	-0.1785	0.5584	0.304	--
0.9	-0.1258	0.5102	0.264	--
1.0	-0.0821	0.4644	0.233	--
1.1	-0.0466	0.4210	0.208	--
1.2	-0.0183	0.3800	0.190	--
1.3	0.0037	0.3416	0.178	--
1.4	0.0204	0.3057	0.170	--
1.5	0.0325	0.2725	0.165	--
1.6	0.0408	0.2418	0.163	--
1.7	0.0460	0.2136	0.164	--
1.8	0.0487	0.1879	0.166	--
1.9	0.0494	0.1645	0.170	--
2.0	0.0486	0.1434	0.174	--
2.1	0.0469	0.1244	0.178	--
2.2	0.0440	0.1075	0.183	--
2.3	0.0408	0.0924	0.188	--
2.4	0.0373	0.0791	0.192	--
2.5	0.0336	0.0674	0.196	1.000
2.6	0.0300	0.0572	0.200	0.999
2.7	0.0265	0.0482	0.204	--
2.8	0.0232	0.0405	0.207	--
2.9	0.0202	0.0339	0.209	--
3.0	0.0174	0.0282	0.211	--
3.1	0.0148	0.0233	0.213	--
3.2	0.0126	0.0192	0.214	--
3.3	0.0106	0.0158	0.215	--

y	$-\lambda_0^*(y)$	$-\lambda_1^*(y)$	$\lambda_0(y)$	$-\lambda_1(y)$
3.4	0.0088	0.0129	0.216	0.999
3.5	0.0073	0.0104	0.217	--
3.6	0.0060	0.0084	0.217	--
3.7	0.0050	0.0068	--	--
3.8	0.0040	0.0054	--	--
3.9	0.0033	0.0043	--	--
4.0	0.0026	0.0034	--	--
4.1	0.0021	0.0027	0.217	--
4.2	0.0017	0.0021	0.216	--
4.3	0.0013	0.0017	--	--
4.4	0.0010	0.0013	--	--
4.5	0.0008	0.0010	--	--
4.6	0.0006	0.0008	--	--
4.7	0.0005	0.0006	0.216	--
4.8	0.0004	0.0004	0.215	--
4.9	0.0003	0.0003	--	--
5.0	0.0002	0.0002	--	--
5.1	0.0002	0.0002	--	--
5.2	0.0001	0.0001	--	--
5.3	0.0001	0.0001	--	--
5.4	0.0001	0.0001	--	--
5.5	0.0000	0.0000	--	--
5.6	0.0000	0.0000	0.215	--
5.7	0.0000	0.0000	0.214	--
5.8	--	--	--	--
5.9	--	--	--	--
6.0	0.0000	0.0000	0.214	0.999

TABLE 5: Parameters describing experiments of Melrose et al (1972) and Tanishita et al (1976).

Parameter	Melrose <u>et al</u>	Tanishita <u>et al</u>
δ	0.00133	0.067 to 0.25
β	0.11 (min)	0. 28
D	4.3 (max)	15 to 60 (estimated)
R_s	860 (max)	Not available

TABLE 6: Eigenvalues and Fourier coefficients from the heat transfer calculation.

n	β_n^2	γ_n	N_n	$-M_n$	β_n^2 Jones (1971).
0	7.31358	47.9680	-1.476	0.819	7.31358
1	44.6095	292.582	0.806	0.0975	44.6096
2	113.921	747.180	-0.589	0.0325	113.921
3	215.240	1411.71	0.476	0.0154	215.240
4	348.564	2286.15	-0.405	0.00879	348.564
5	513.890	3370.48	0.356	0.00558	513.888
6	711.333	4665.46	-0.320	0.00383	711.216

FIGURES

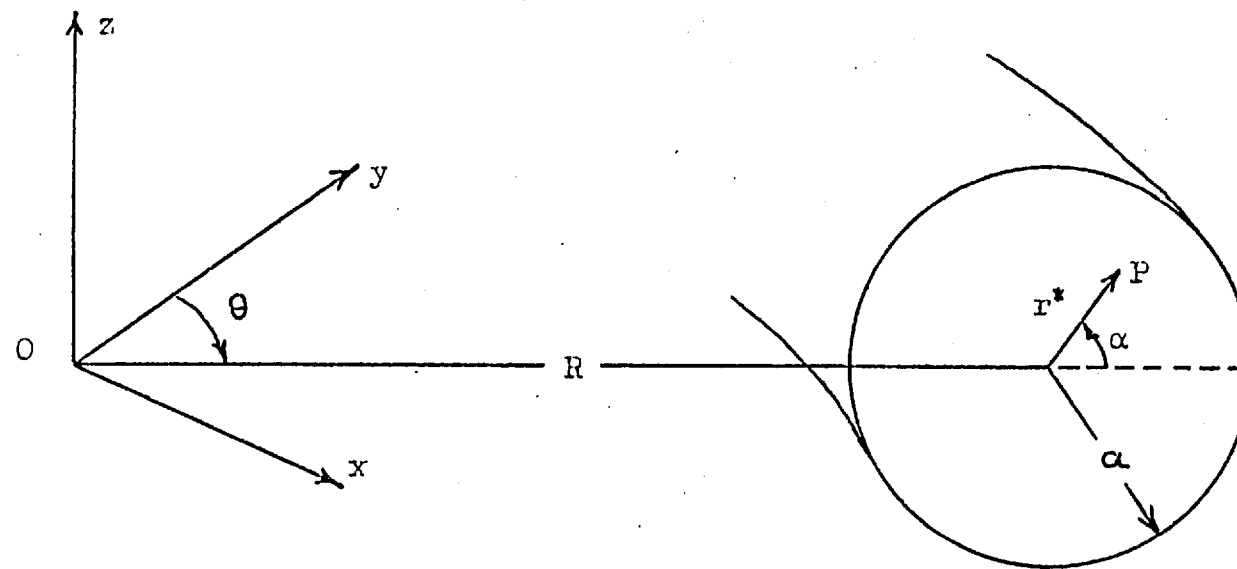


Fig. 2.1: The (r^*, α, θ) co-ordinate system for the curved pipe.

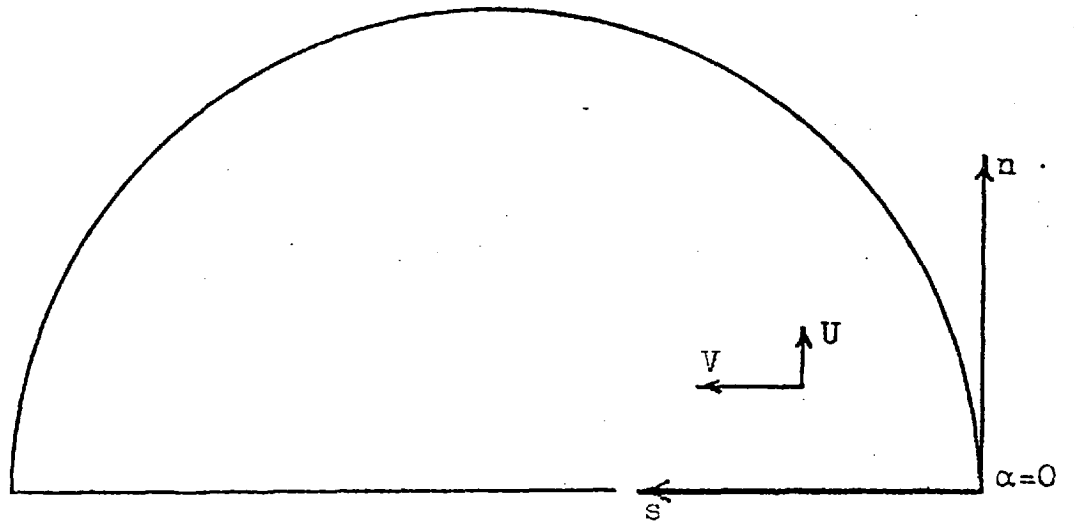


Fig. 5.1: The co-ordinate system for the boundary layers on the axis of symmetry.

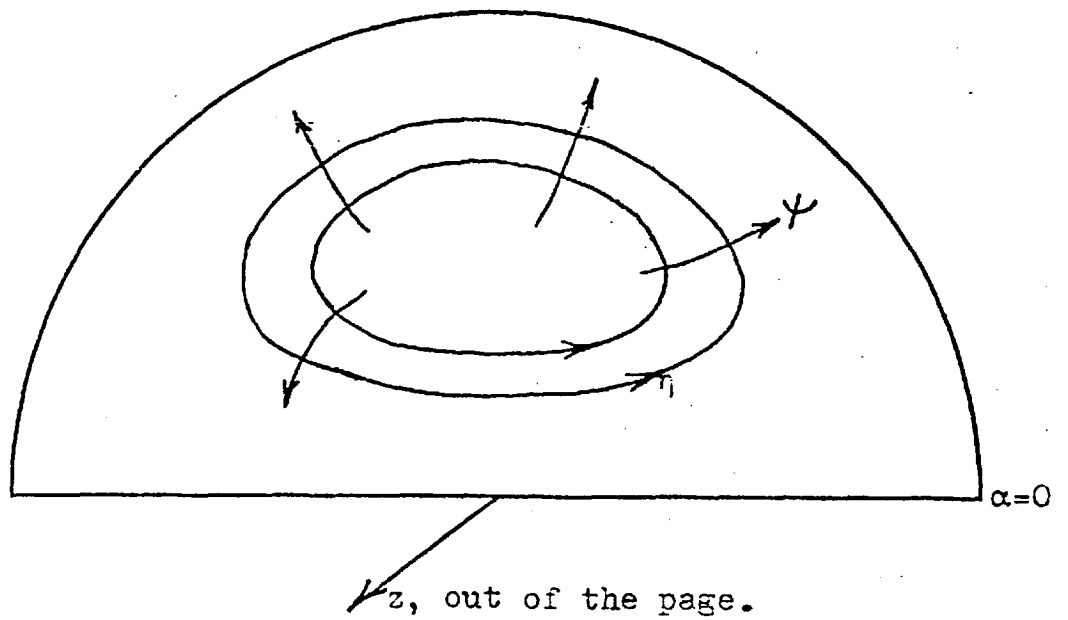


Fig. 5.2: The streamline co-ordinate system.

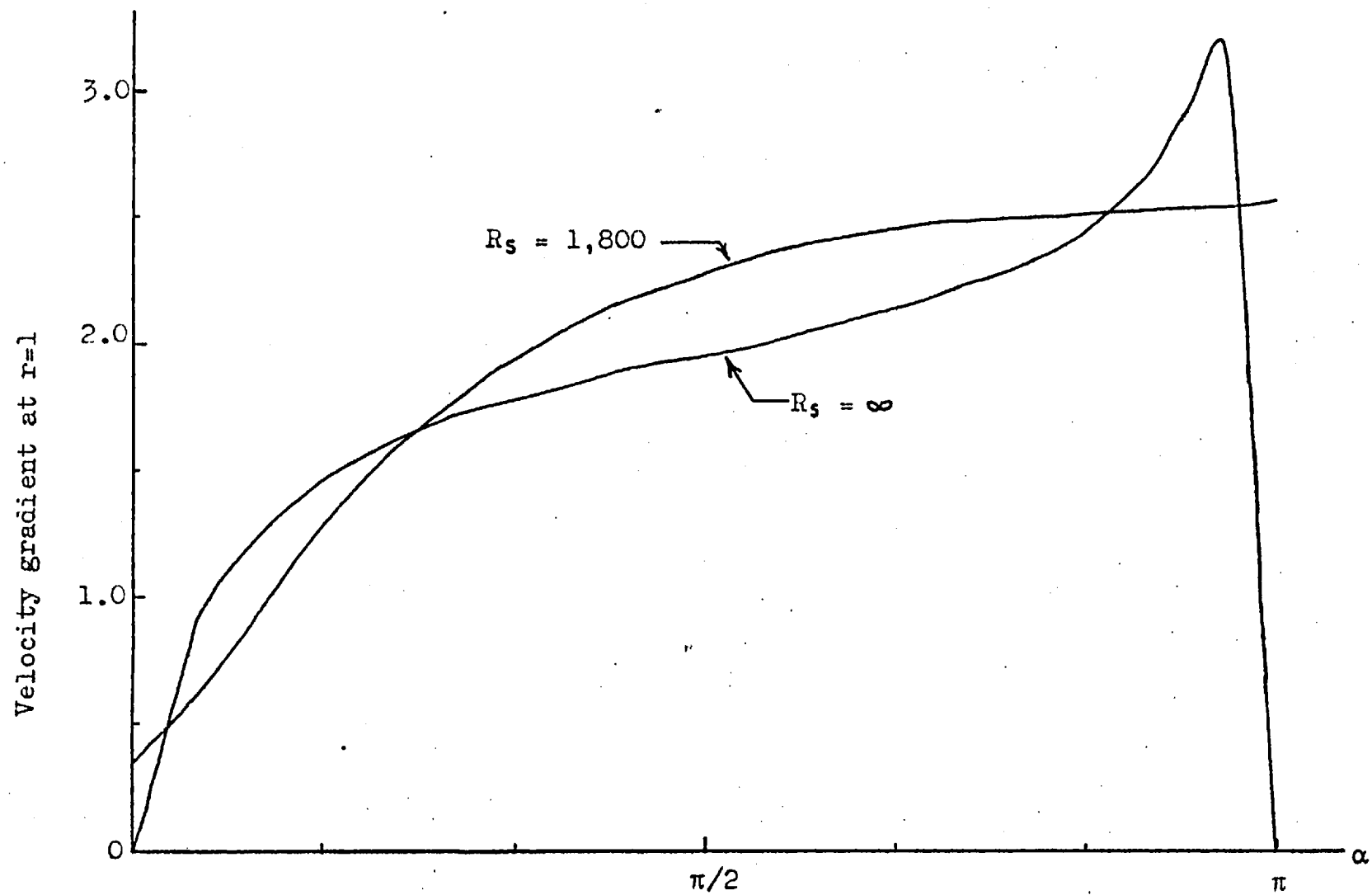


Fig. 5.3: Axial velocity gradient at $r=1$, from solution of (5.2), (5.3) and (7.4).

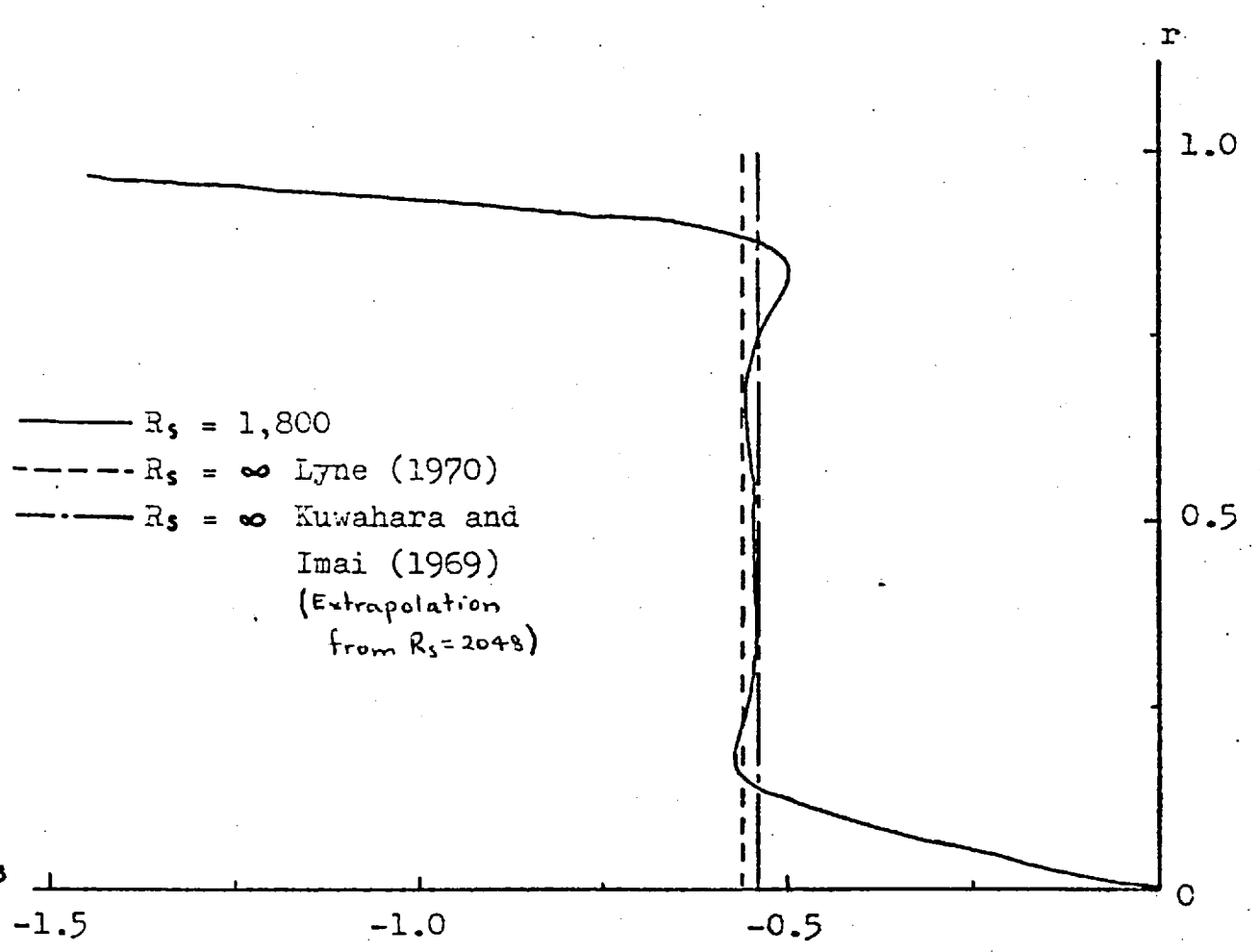
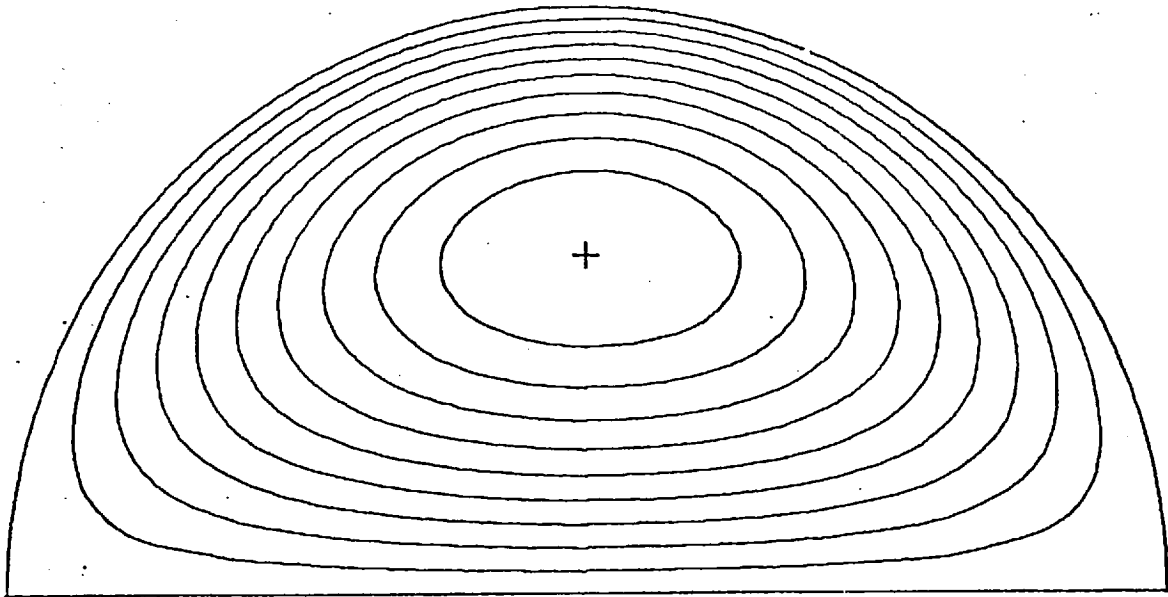


Fig. 7.1: The third component of vorticity on $\alpha=\pi/2$.



NOF = 12

NR = 11

$\chi_{\text{nom}} = -0.0481$

$w_{\text{nom}} = 0.992$

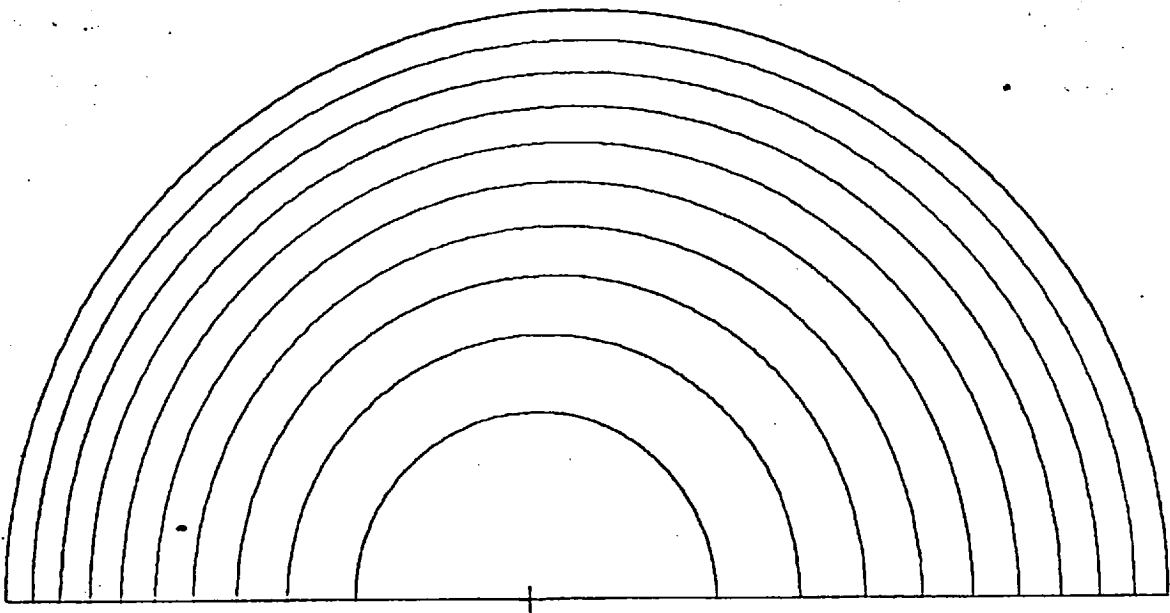
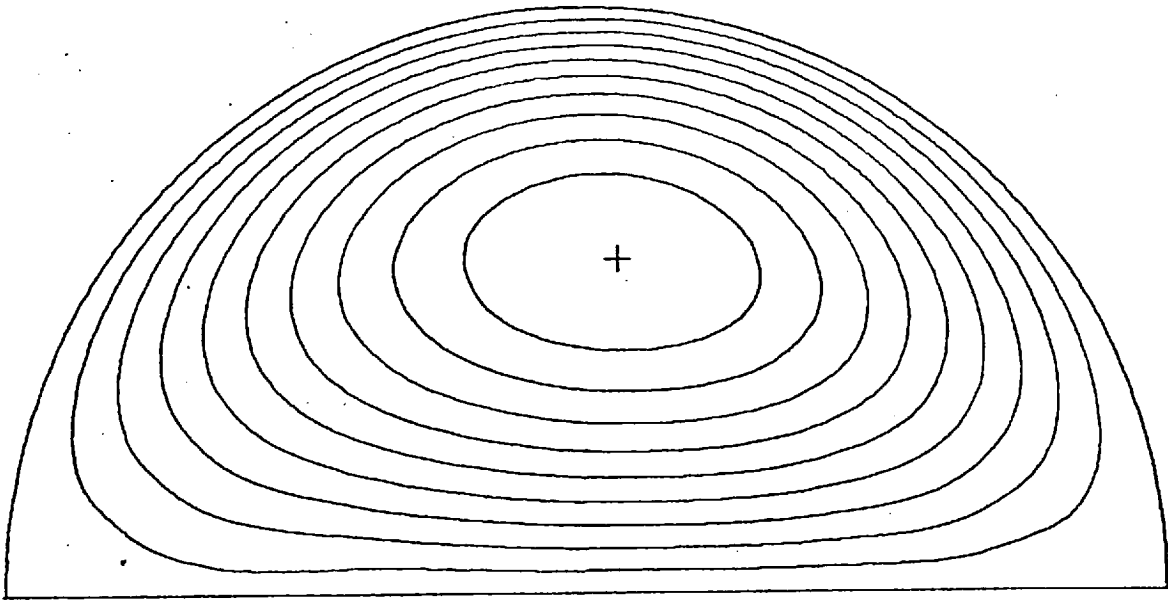


Fig. 7.2

$R_s = 10$



NOF = 12

NR = 11

$\lambda_{oom} = -0.0483$

$w_{iom} = 0.853$

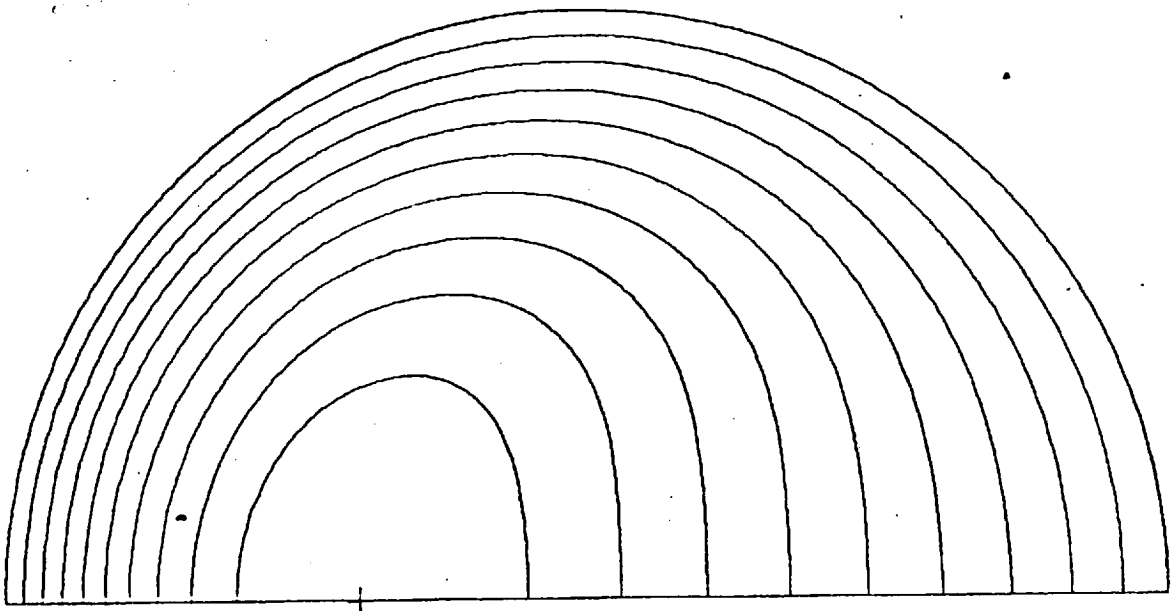
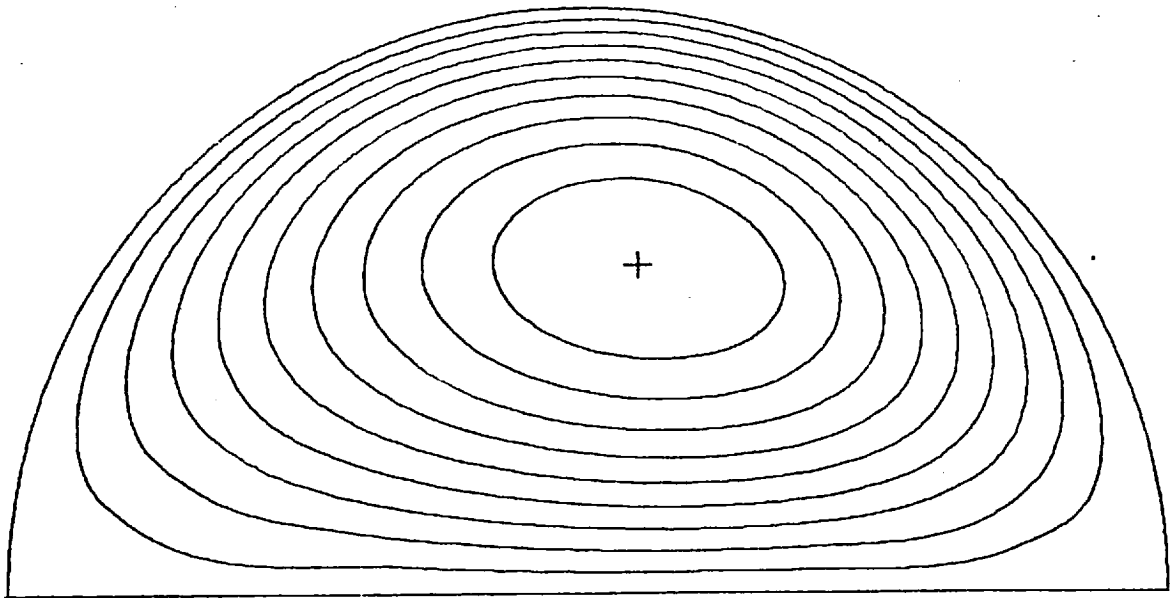


Fig. 7.3

$R_s = 50$



NOF = 12

NR = 11

$\lambda_{\text{boom}} = -0.0487$

$w_{\text{oom}} = 0.664$

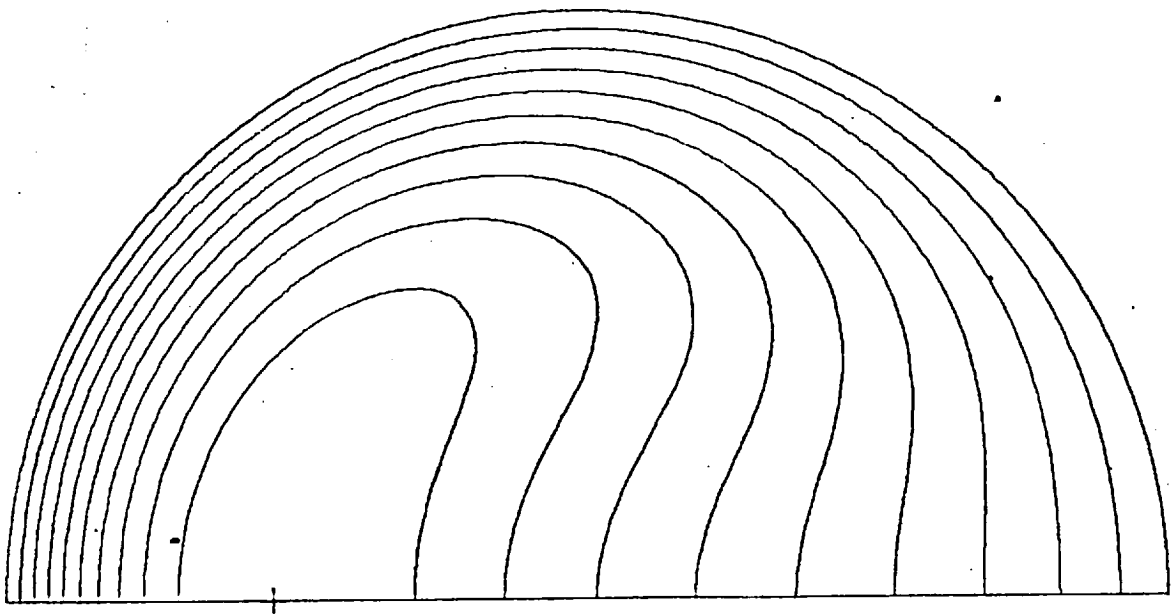
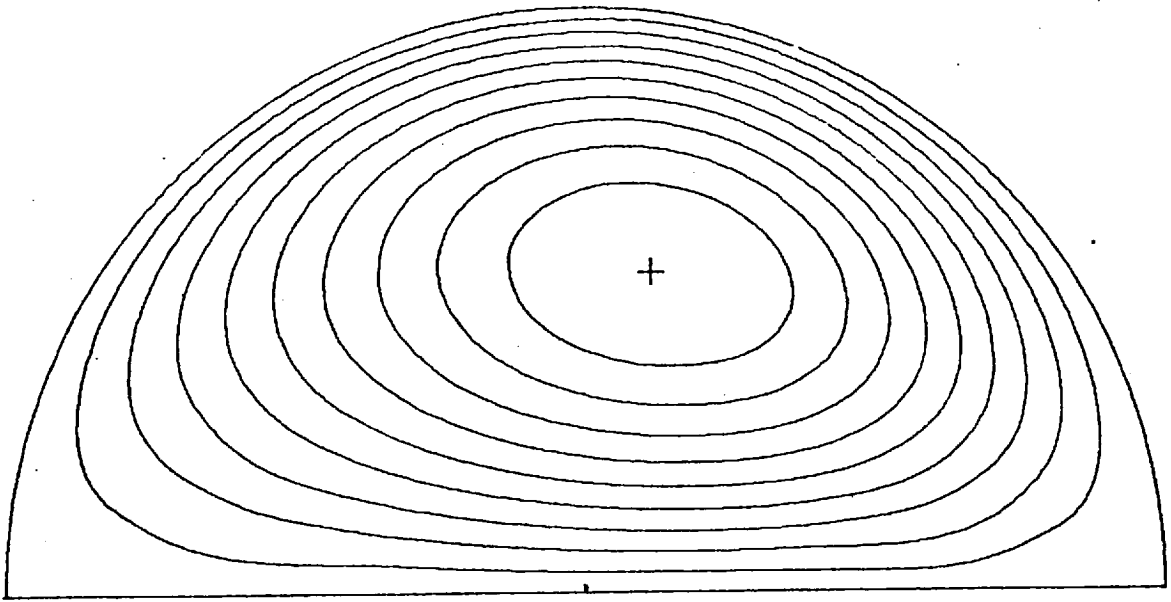


Fig. 7.4

$R_3 = 100$



NOF = 12

NR = 11

$\lambda_{\text{oom}} = -0.0492$

$w_{\text{iom}} = 0.540$

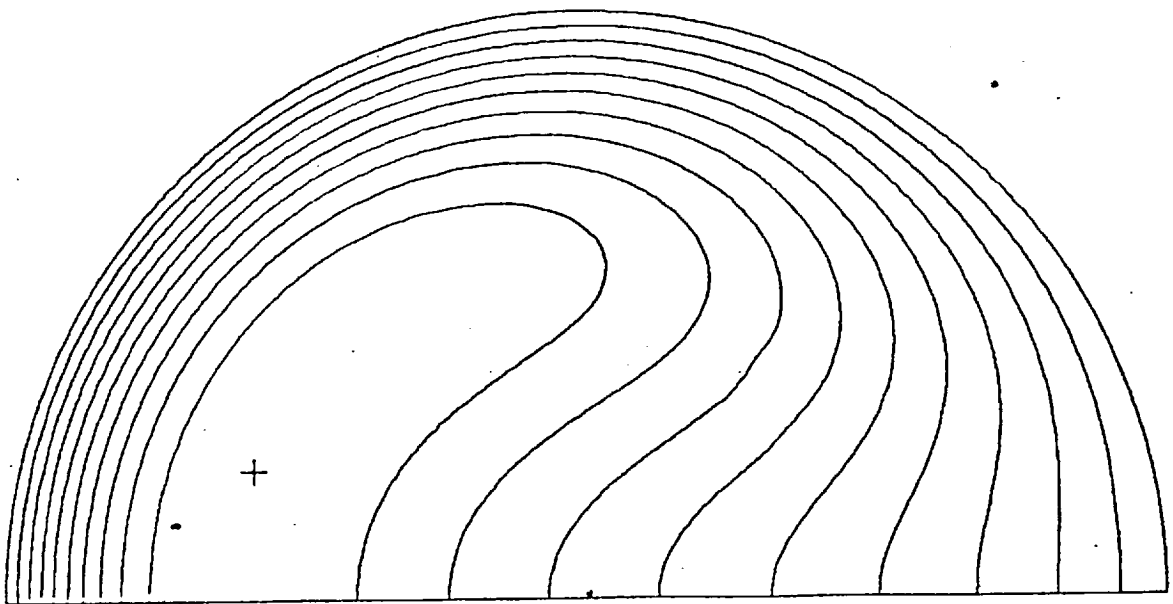
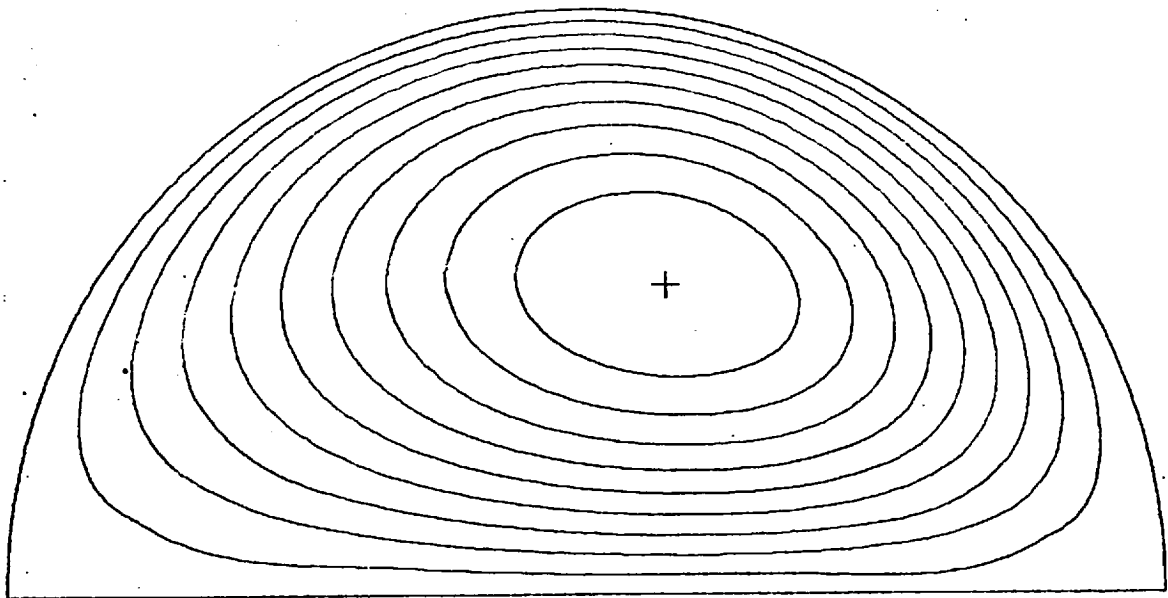


Fig. 7.5

$R_s = 150$



NOF = 12

NR = 11

$\gamma_{00m} = -0.0500$

$w_{10m} = 0.482$

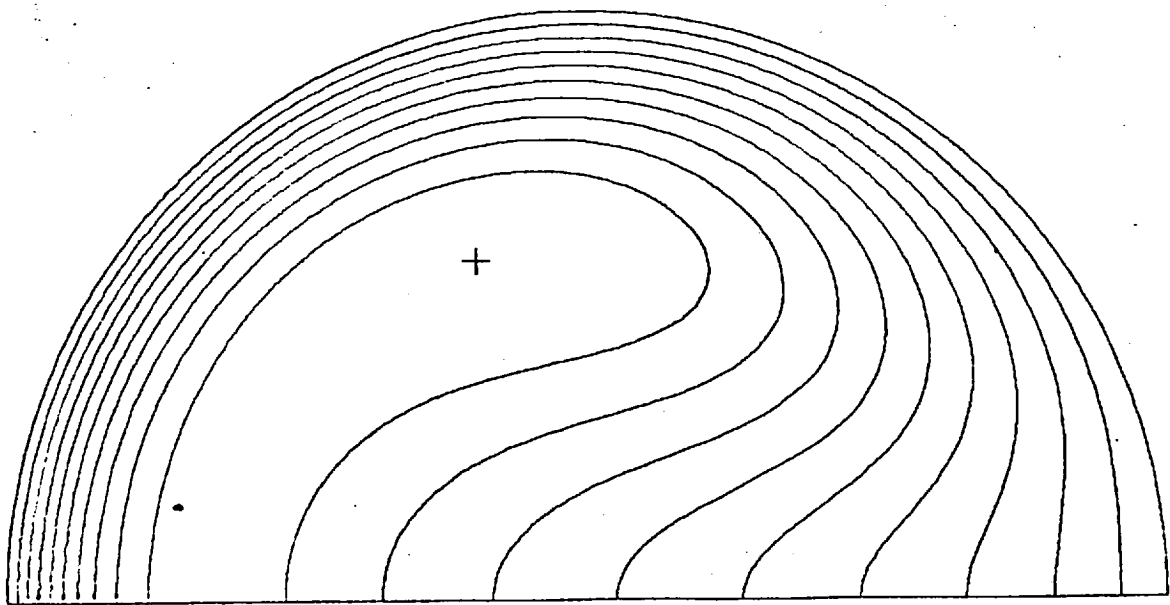
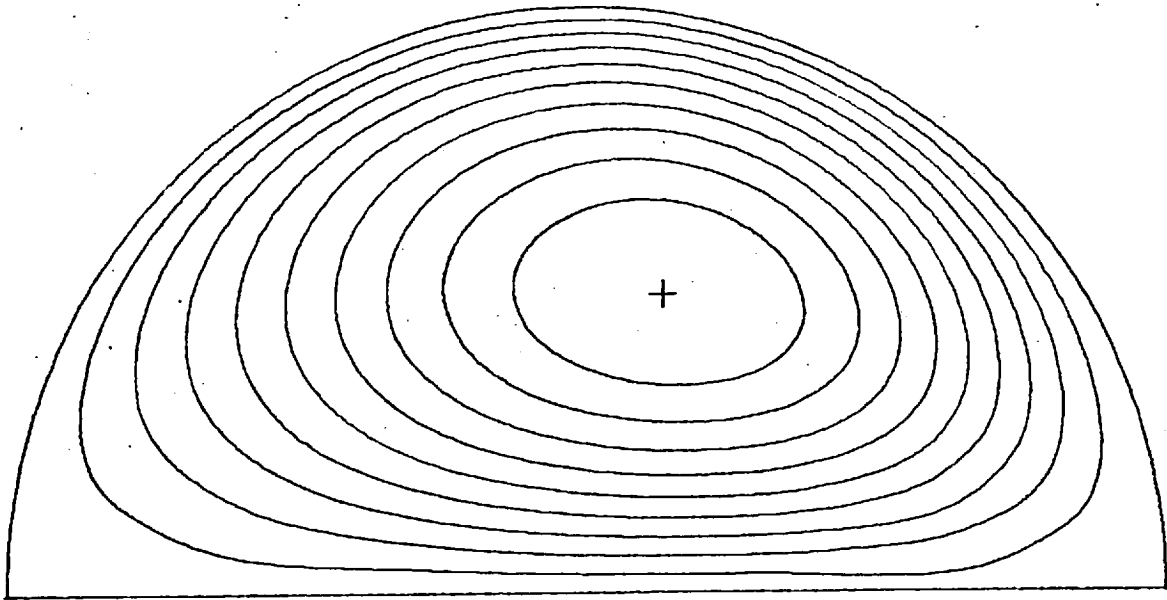


Fig. 7.6

$R_s = 200$



NOF = 12

NR = 11

$\lambda_{100m} = -0.0506$

$w_{10m} = 0.454$

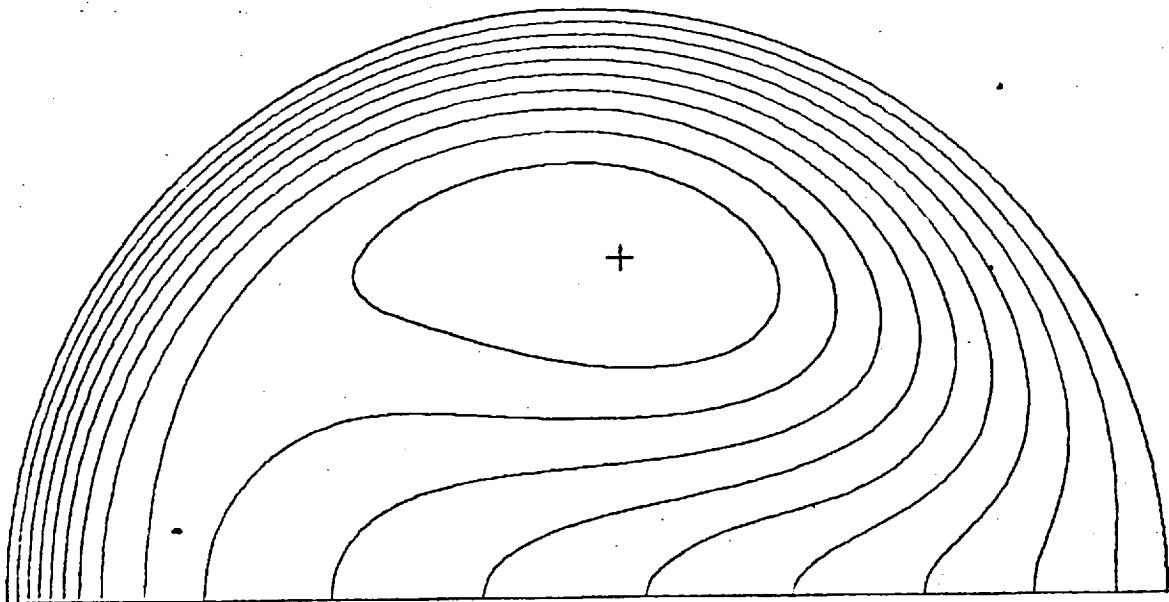
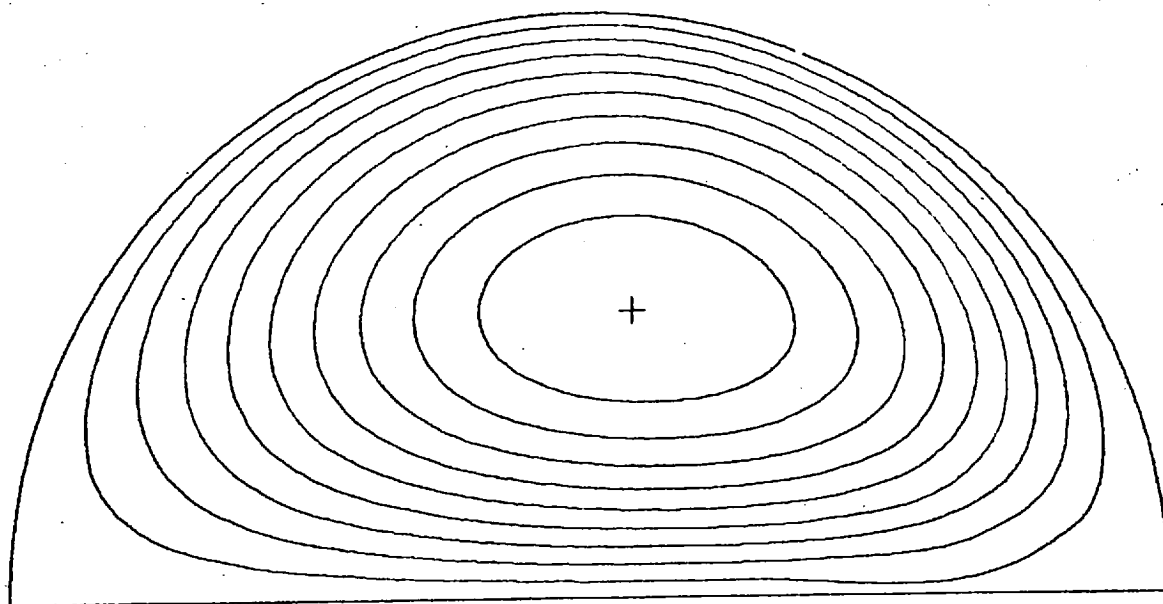


Fig. 7.7

$R_s = 300$



NOF = 18

NR = 11

$\chi_{\text{oom}} = -0.0519$

$w_{\text{oom}} = 0.443$

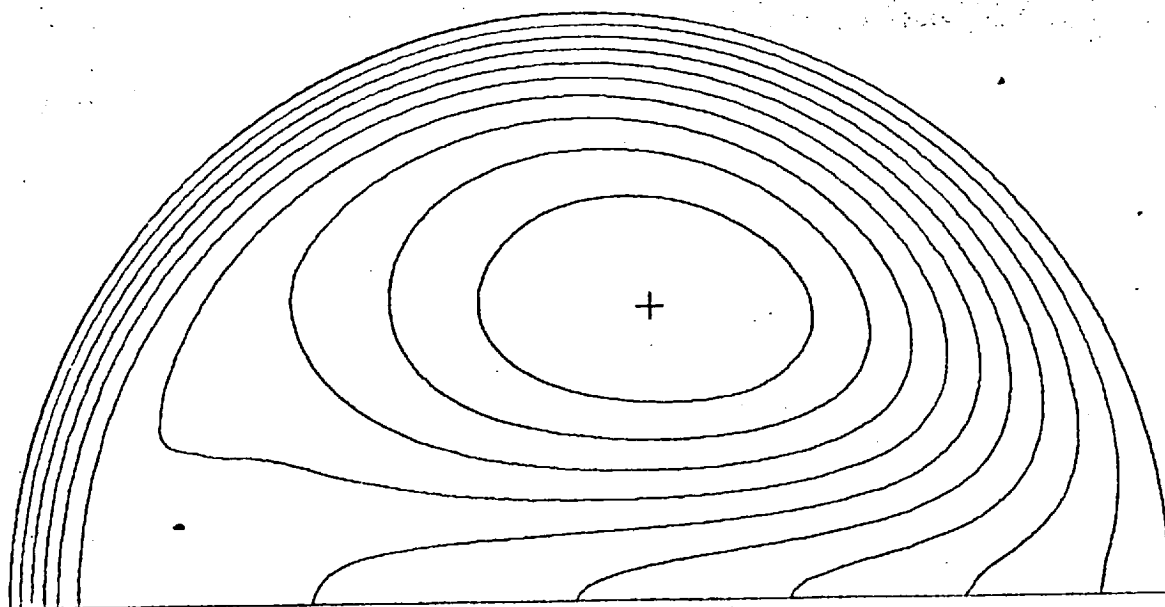
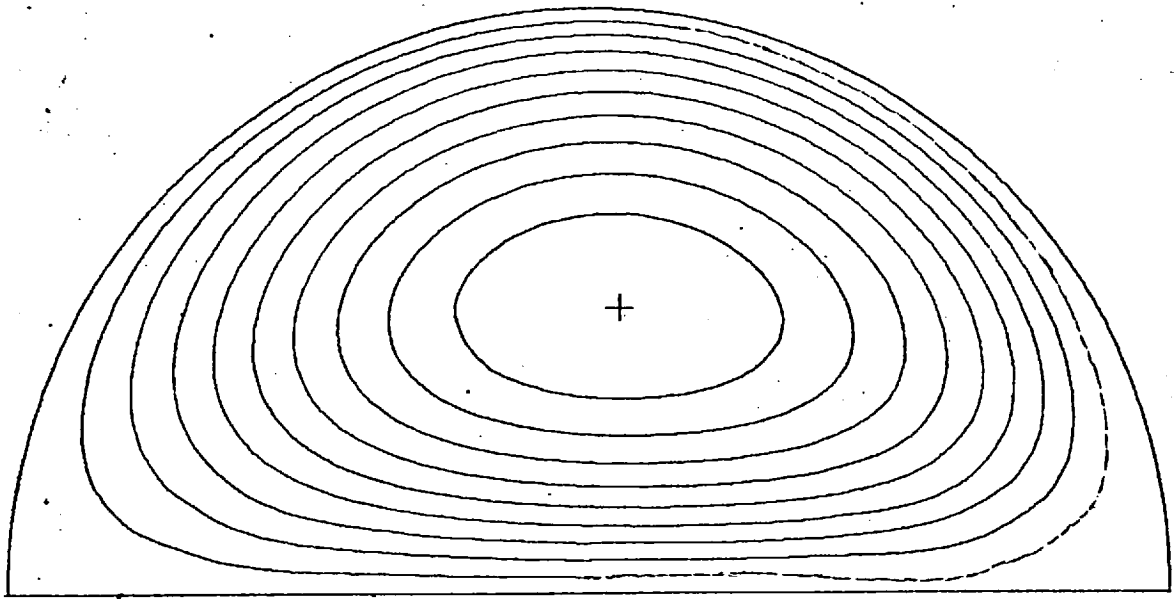


Fig. 7.8

$R_s = 600$



NOF = 24

NR = 33

$\gamma_{oom} = -0.0524$

$w_{oom} = 0.435$

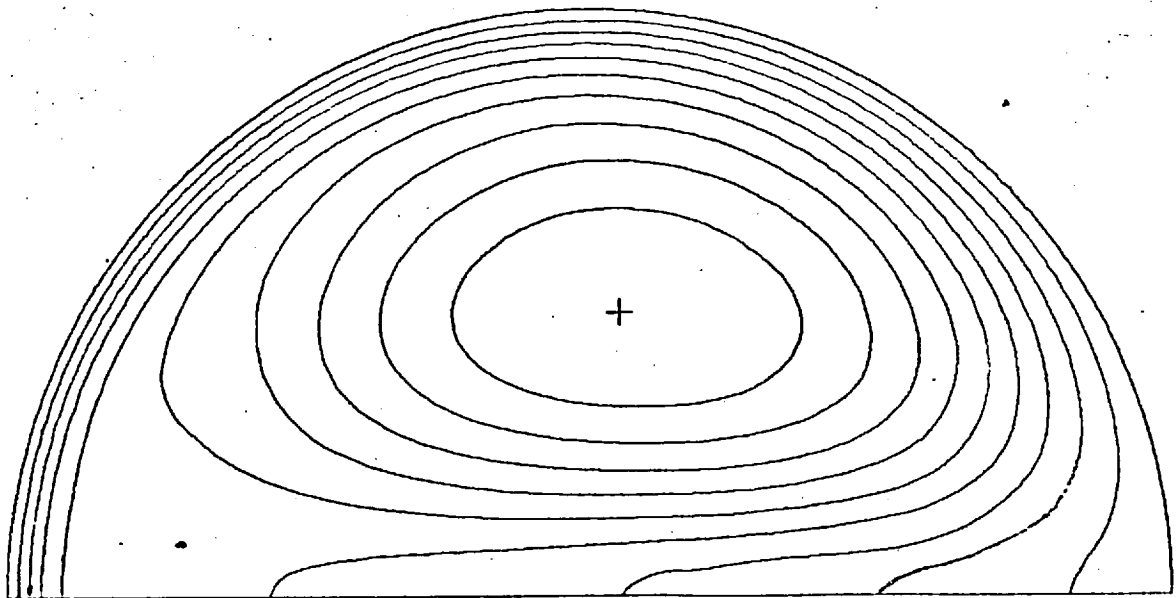
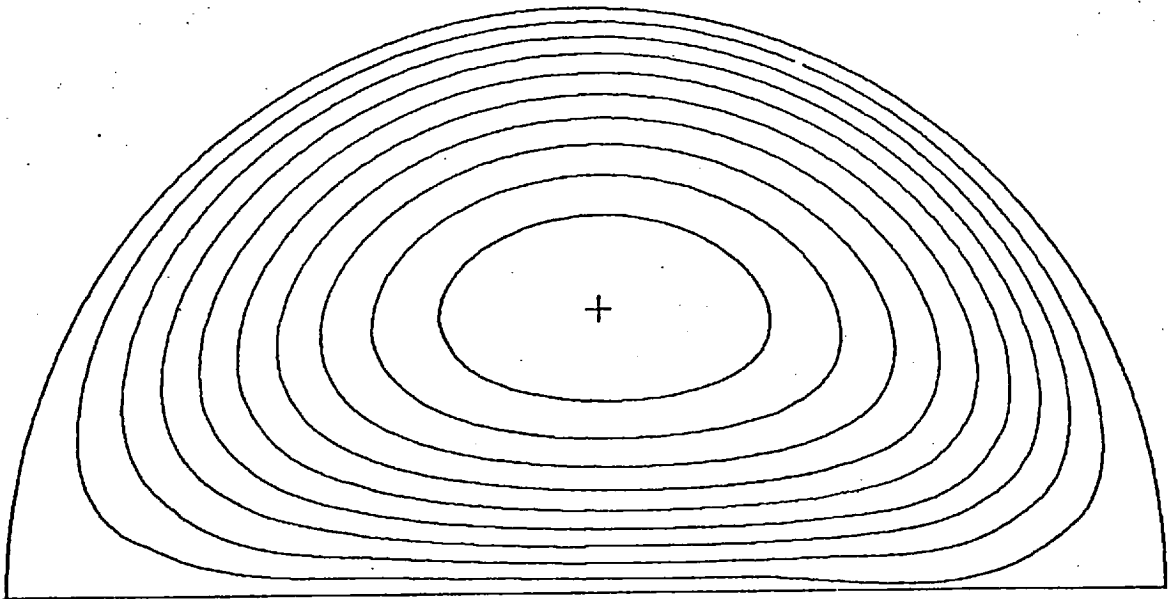


Fig. 7.9

$R_s = 1,000$



NOF = 24

NR = 33

$\gamma_{oom} = -0.0528$

$w_{iom} = 0.430$

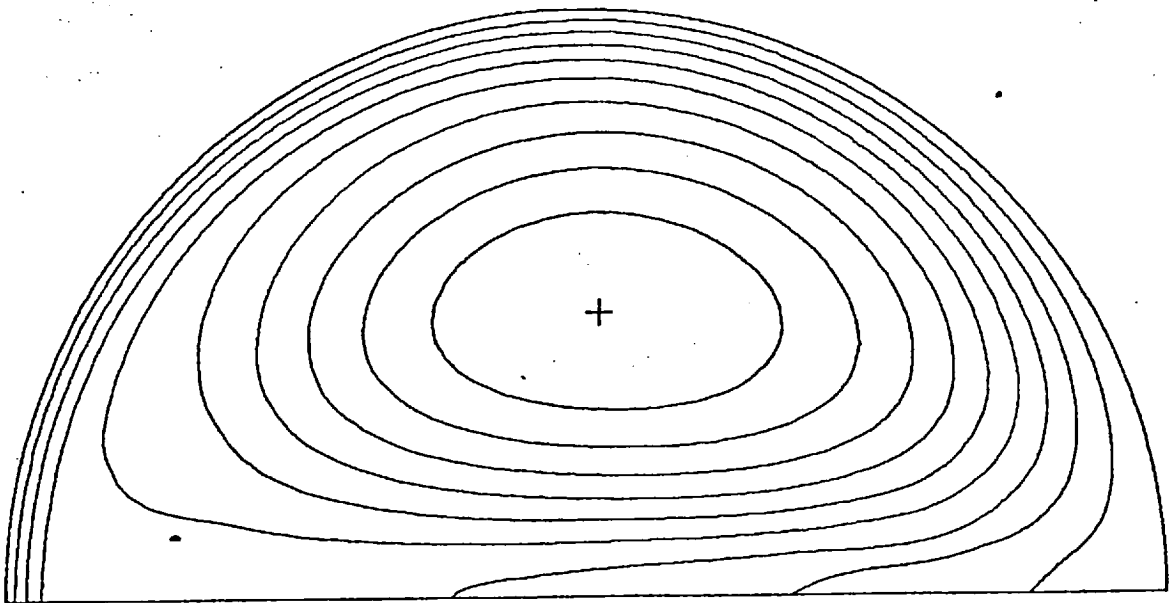
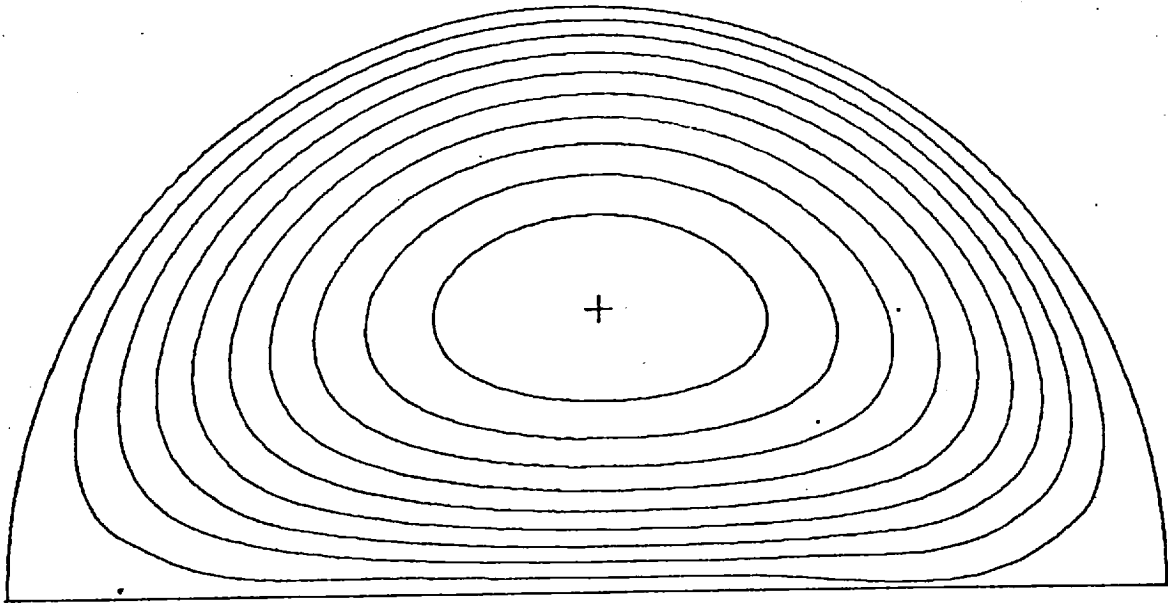


Fig. 7.10

$R_s = 1,400$



NOF = 24

NR = 33

$\chi_{\text{oom}} = -0.0530$

$w_{\text{oom}} = 0.426$

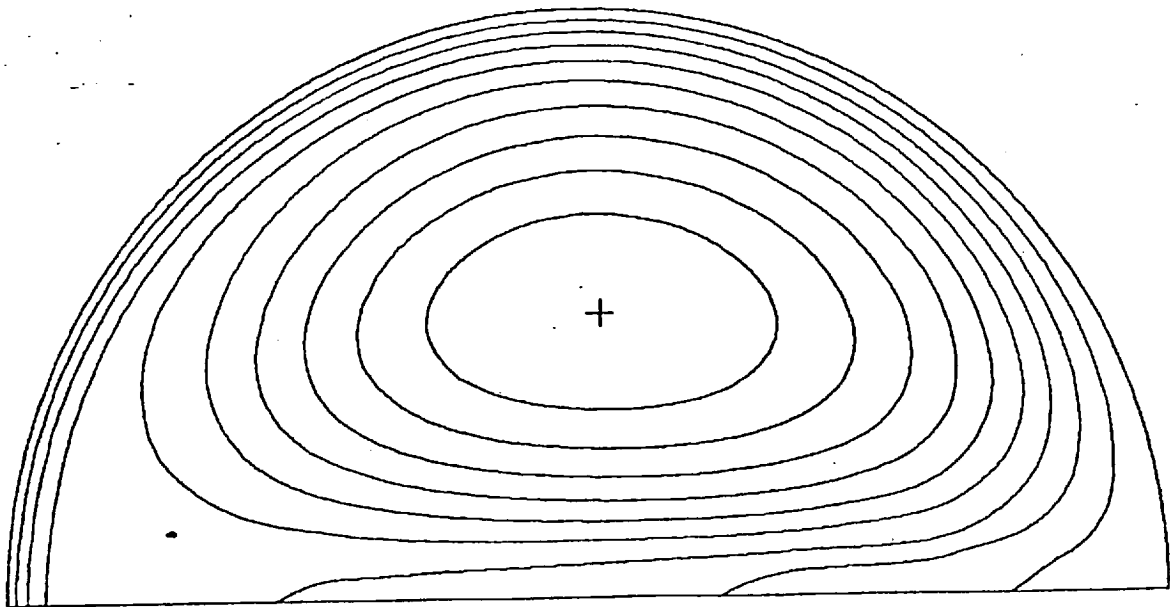


Fig. 7.11 $R_s = 1,800$

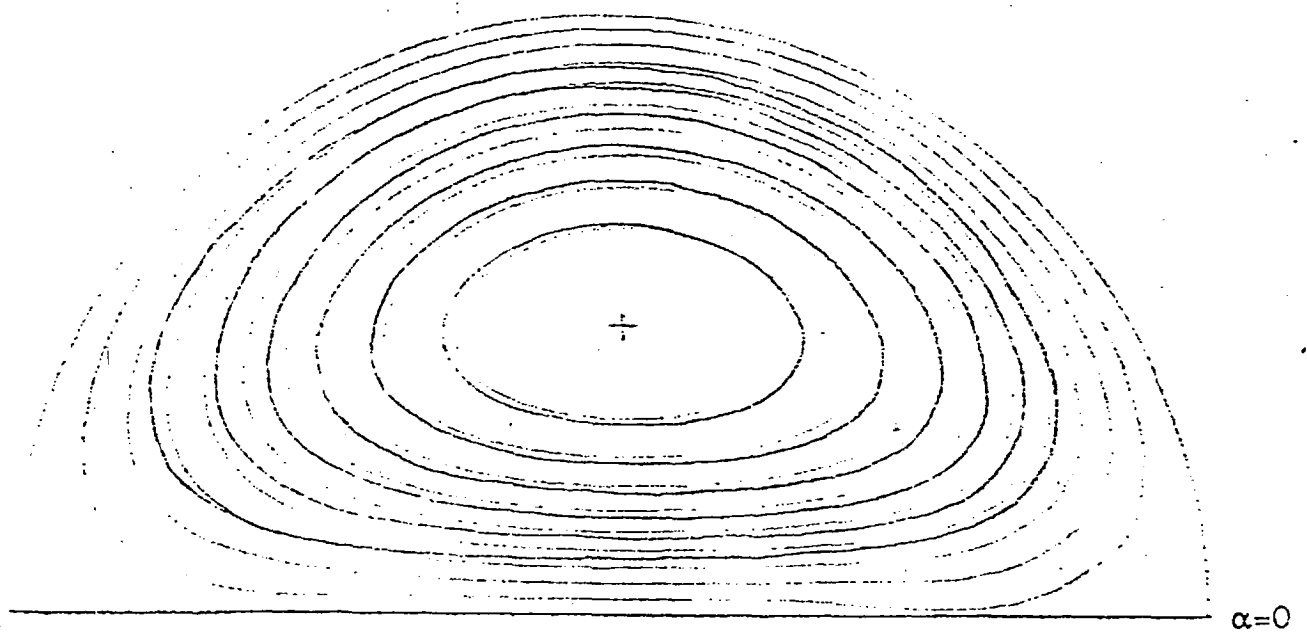


Fig. 7.12: Superposition of streamlines and axial velocity contours at $R_s = 1,800$. Lighter lines are streamlines.

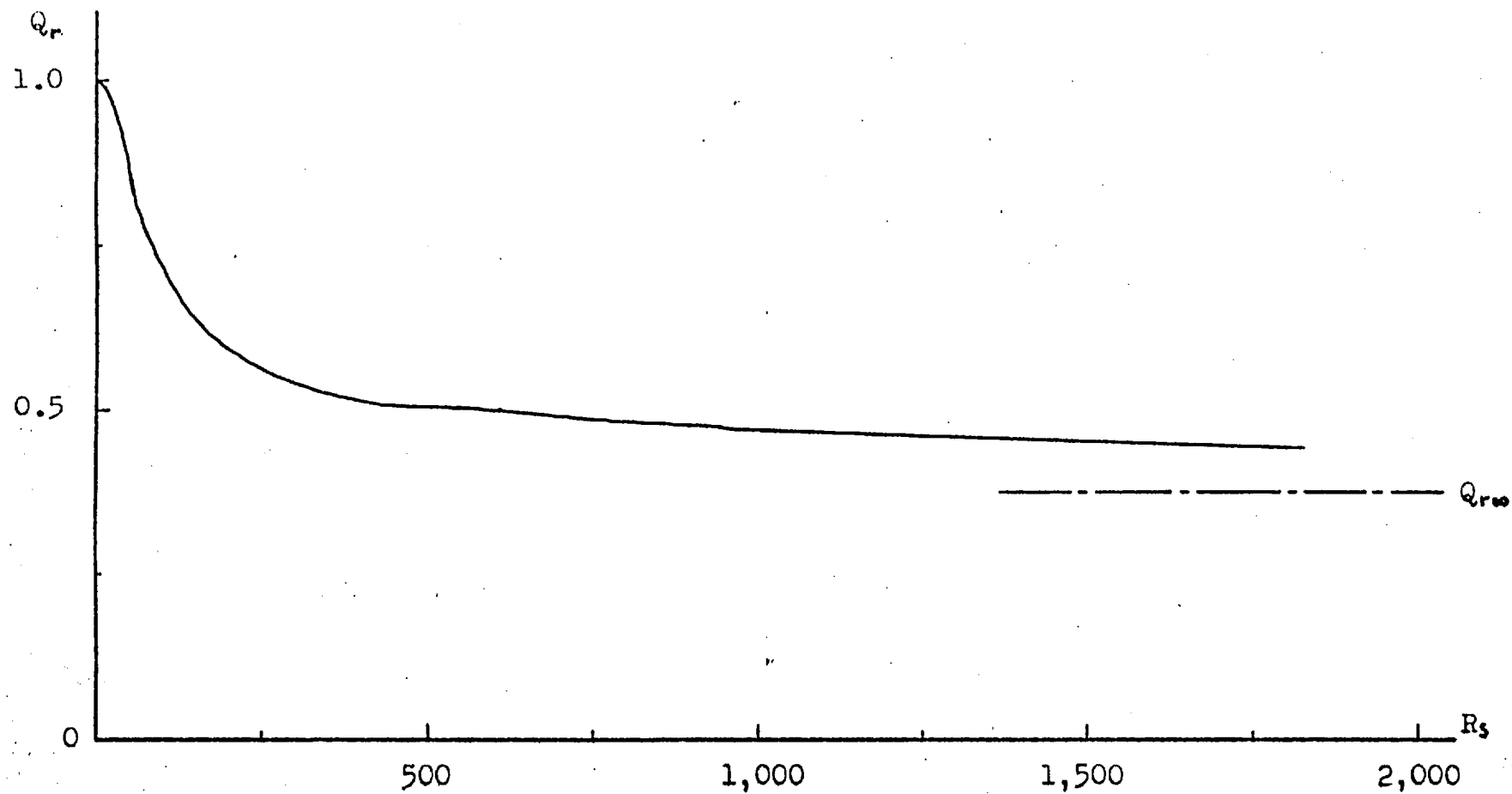


Fig. 7.13: The flux ratio, Q_r from numerical solution of (5.2), (5.3) and (7.4).

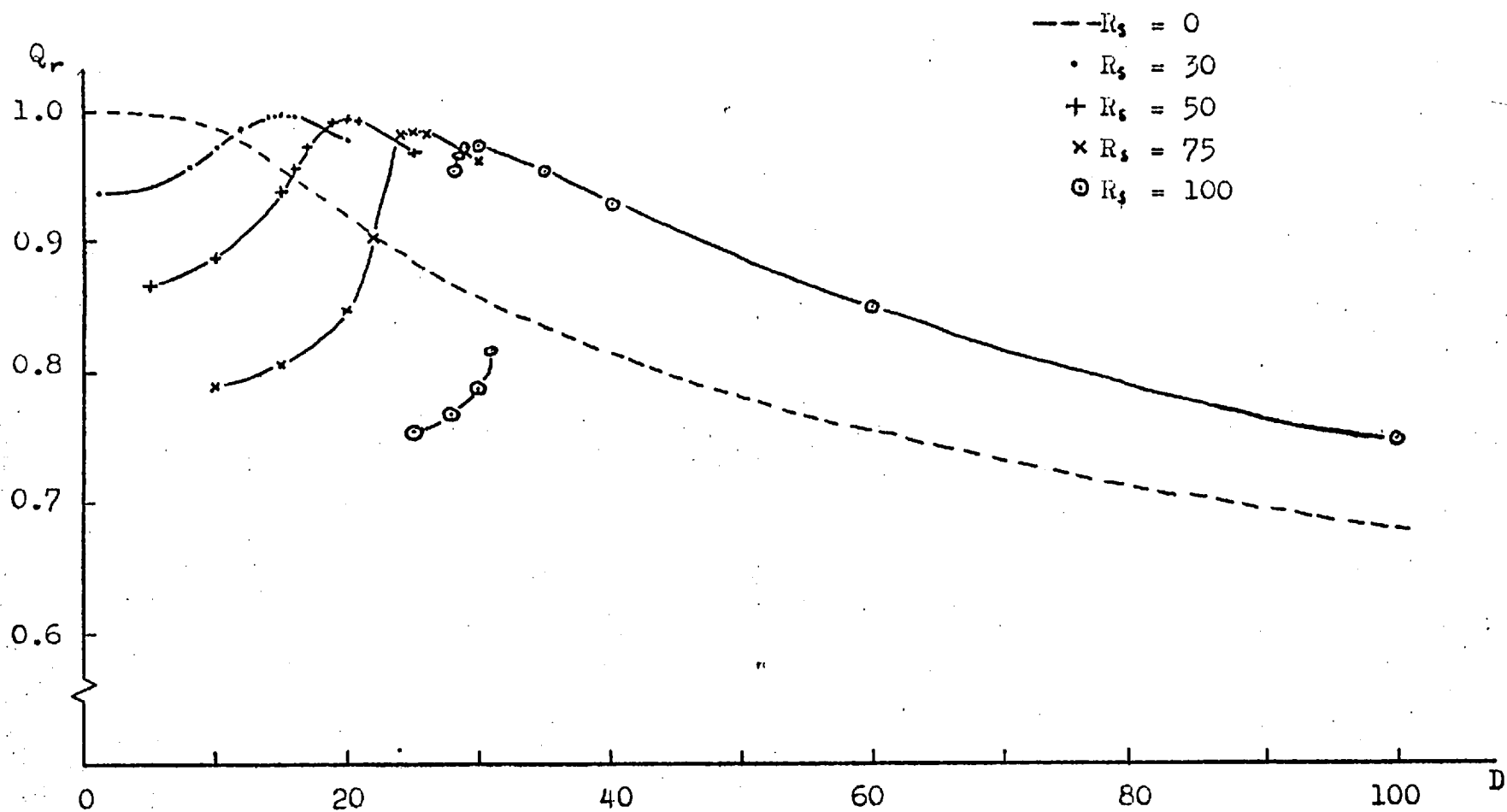


Fig. 7.14: Flux ratio from numerical solution of (3.38), (3.40) and (7.4).

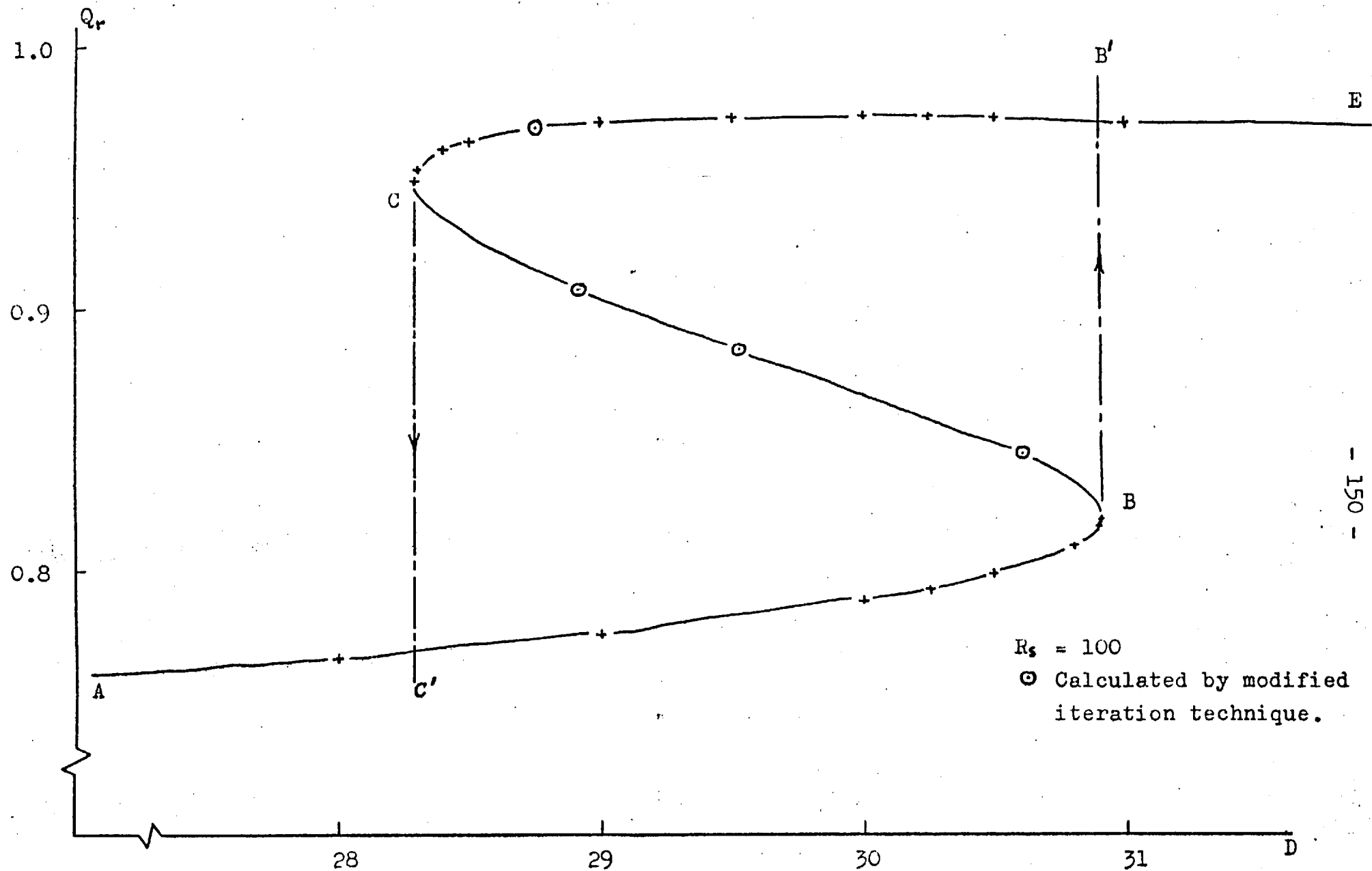


Fig. 7.15: Flux ratio Q_r from numerical solution of (3.38), (3.40) and (7.4).

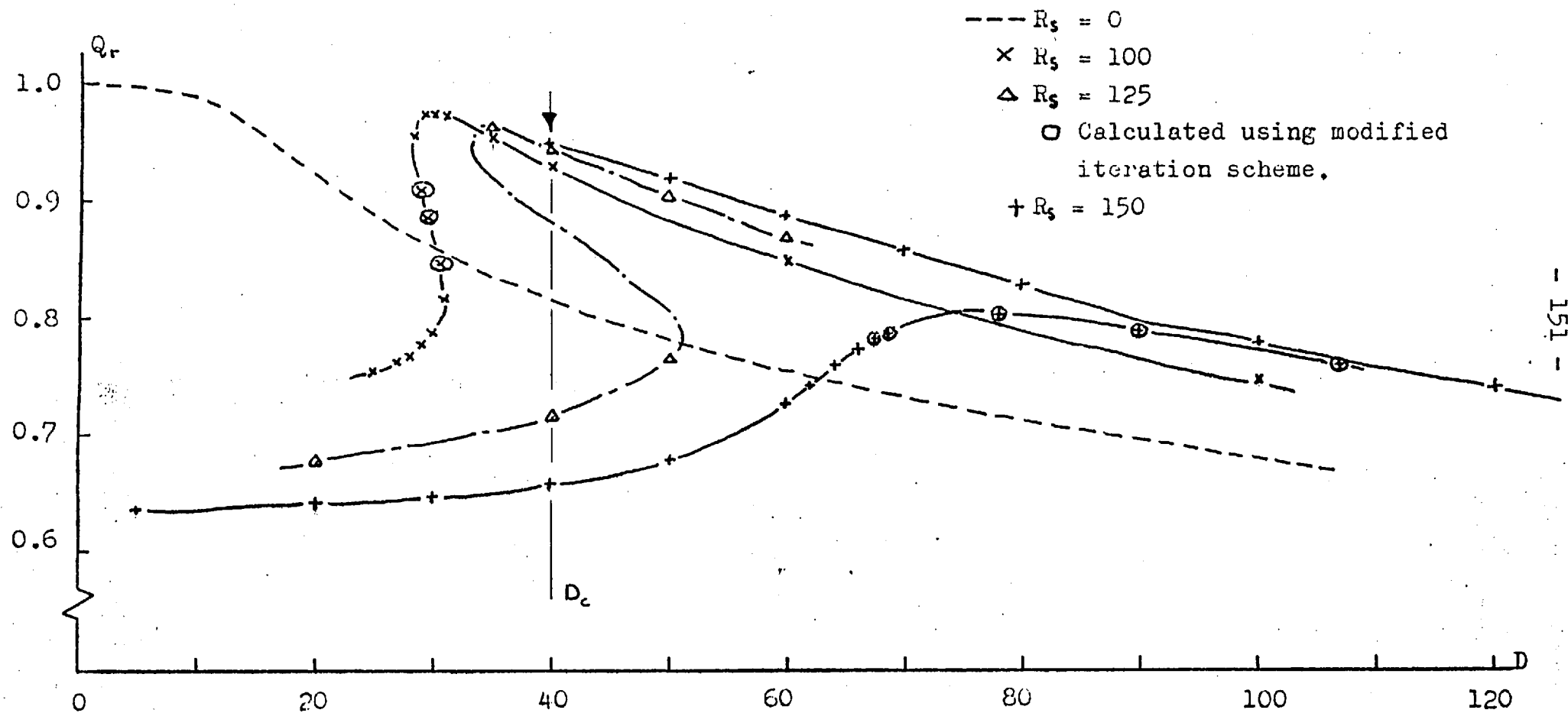
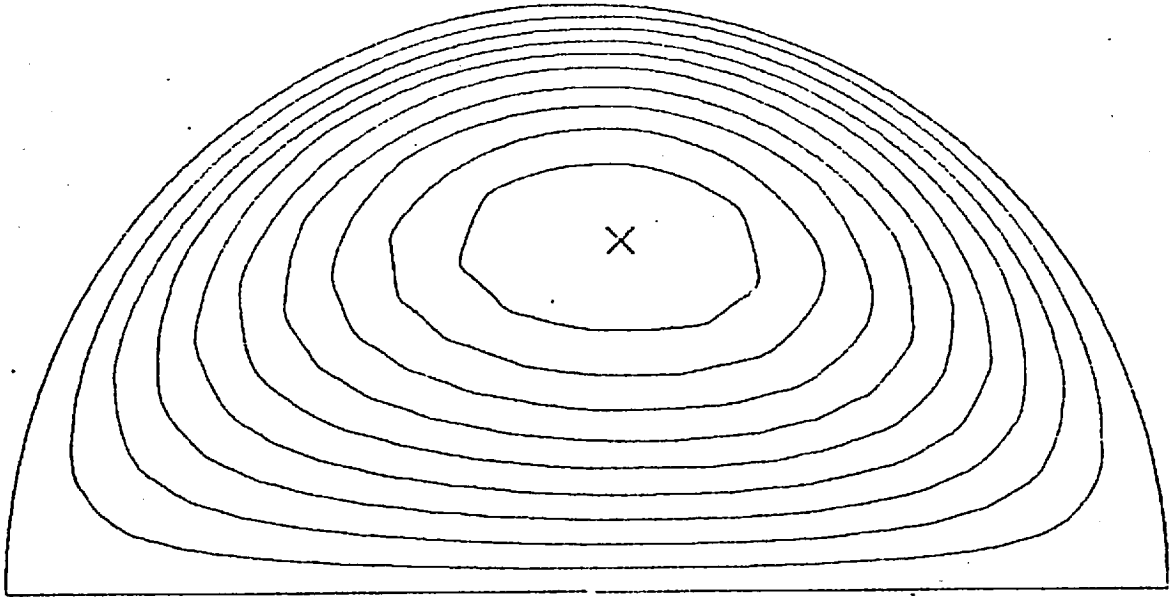


Fig. 7.16: Flux ratio from numerical solution of (3.38), (3.40) and (7.4).



NOF = 12

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0440$

$w_{\min} = 0.861$

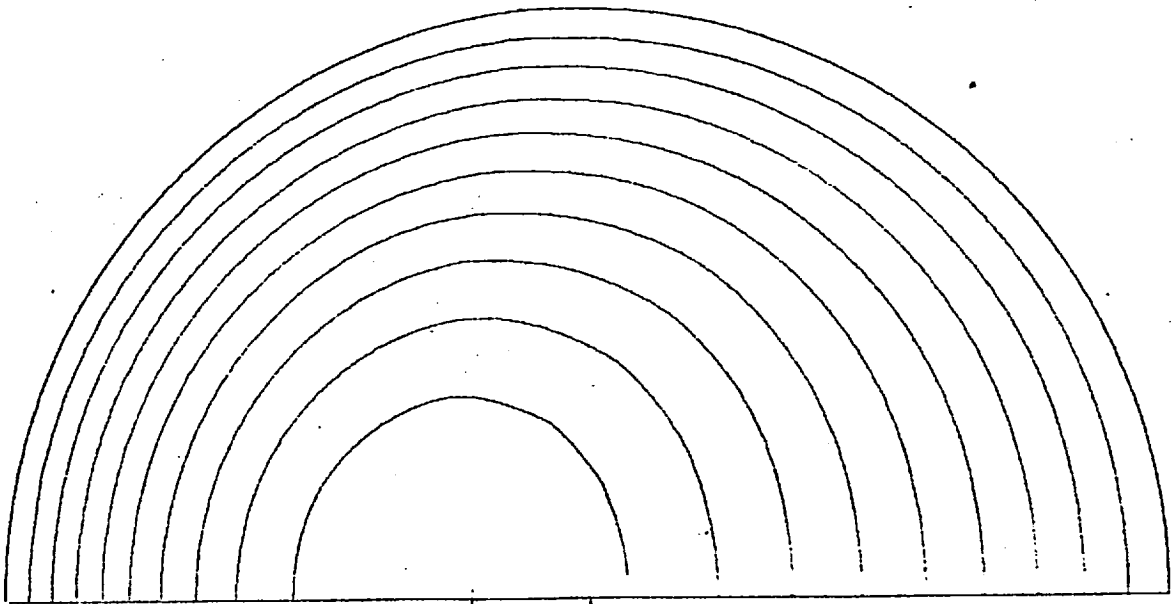
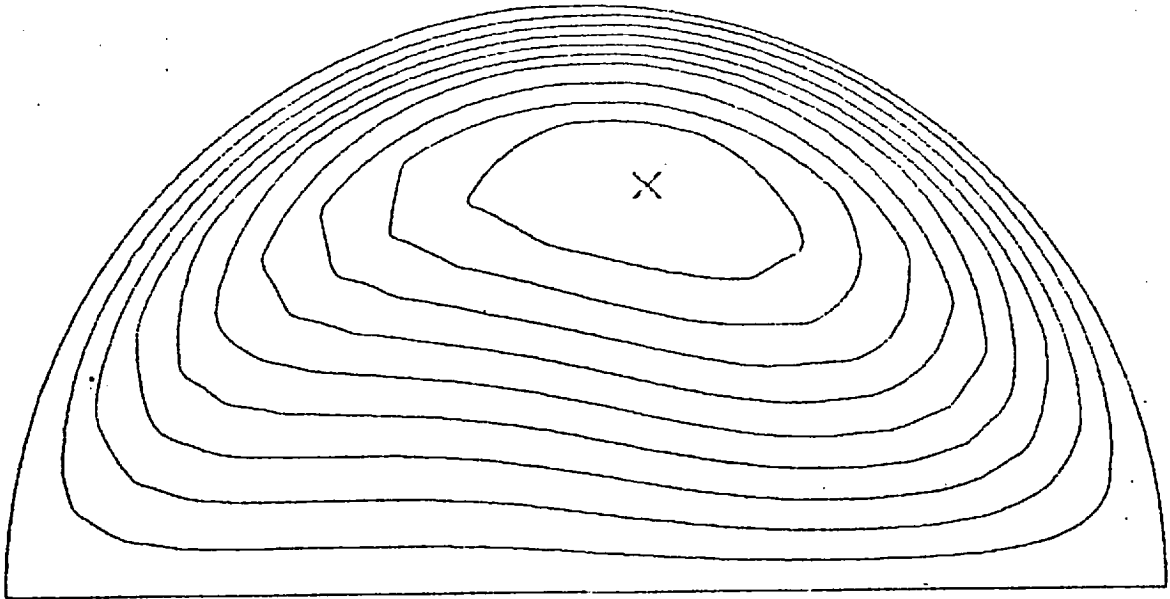


Fig. 7.17

$R_s = 30, D = 5$



NOF = 12

NR = 11

$\chi_{\text{max}} = 0.$

$\chi_{\text{min}} = -0.0315$

$w_{\text{im}} = 1.78$

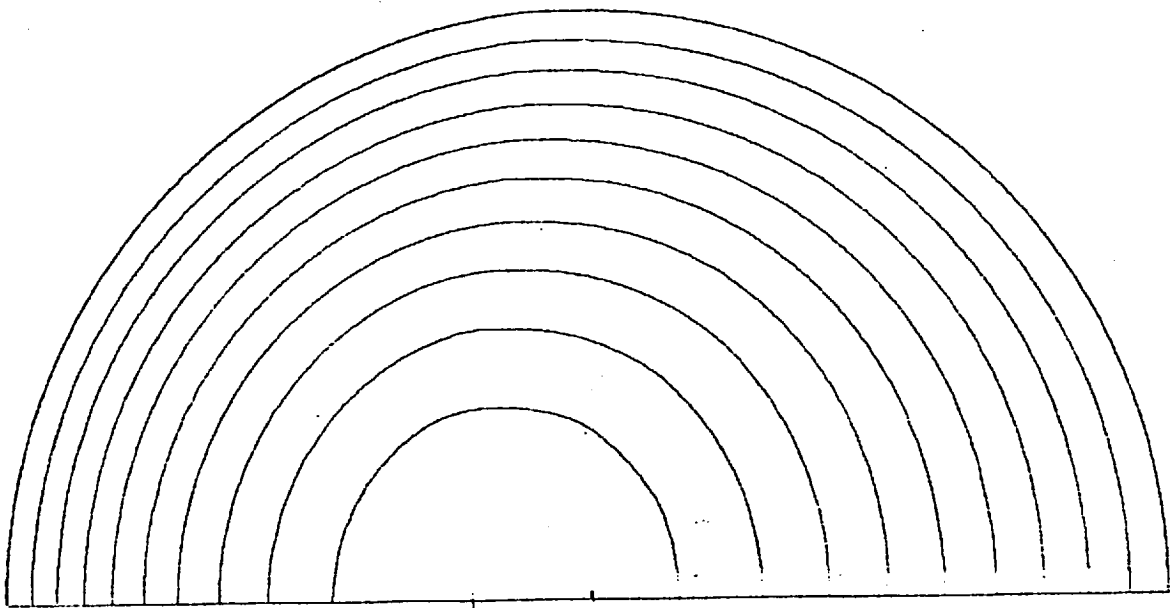
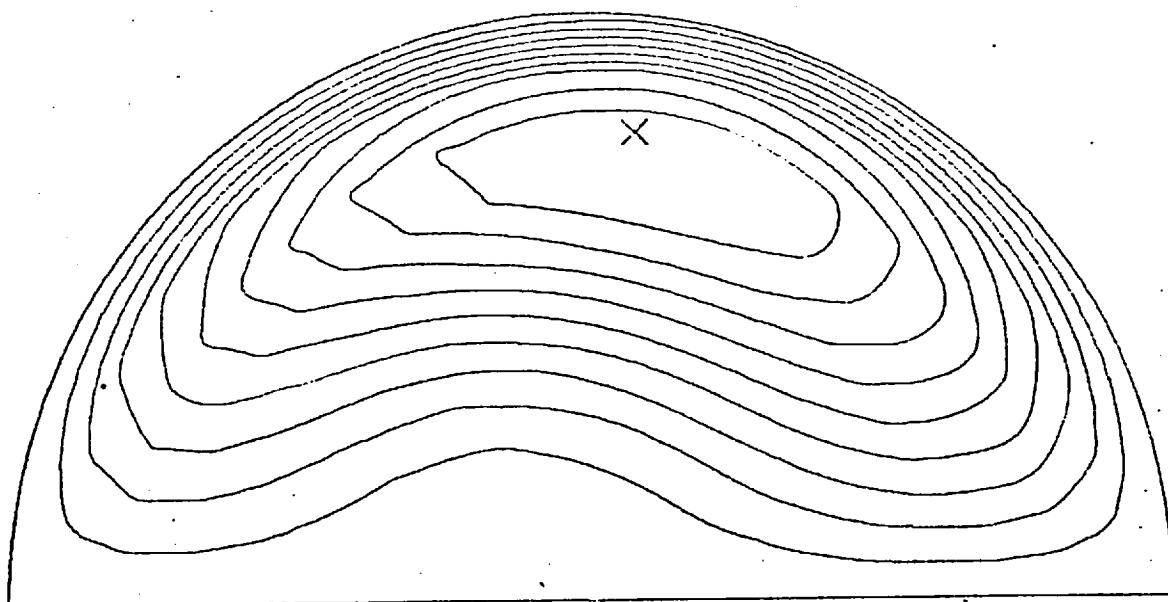


Fig. 7.18

$R_s = 30, D = 10$



NOF = 12

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0252$

$w_{\min} = 2.14$

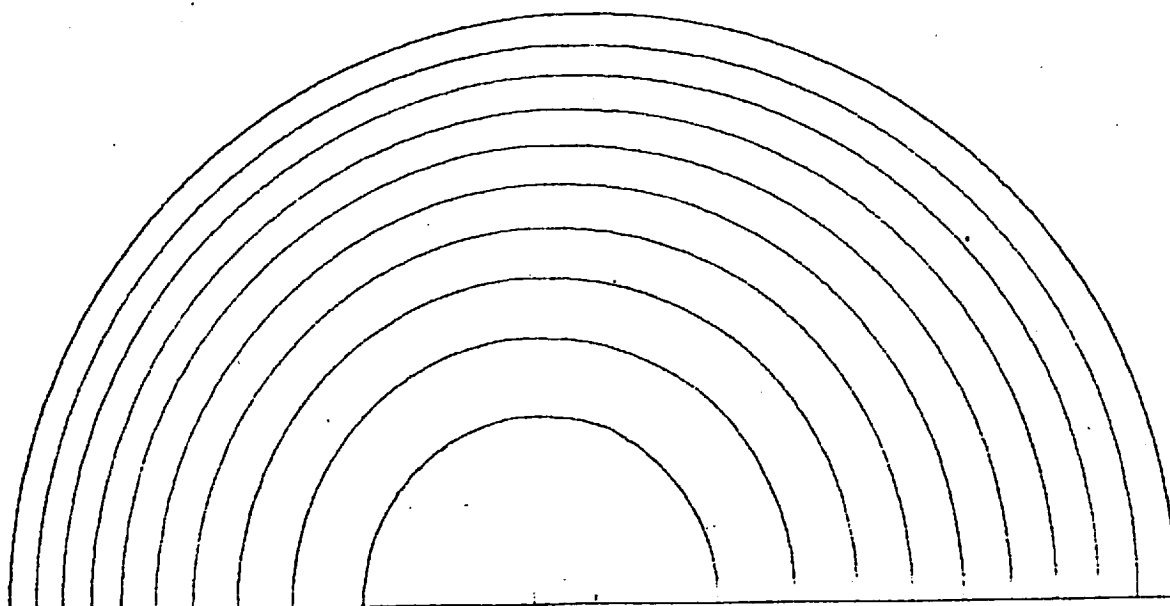
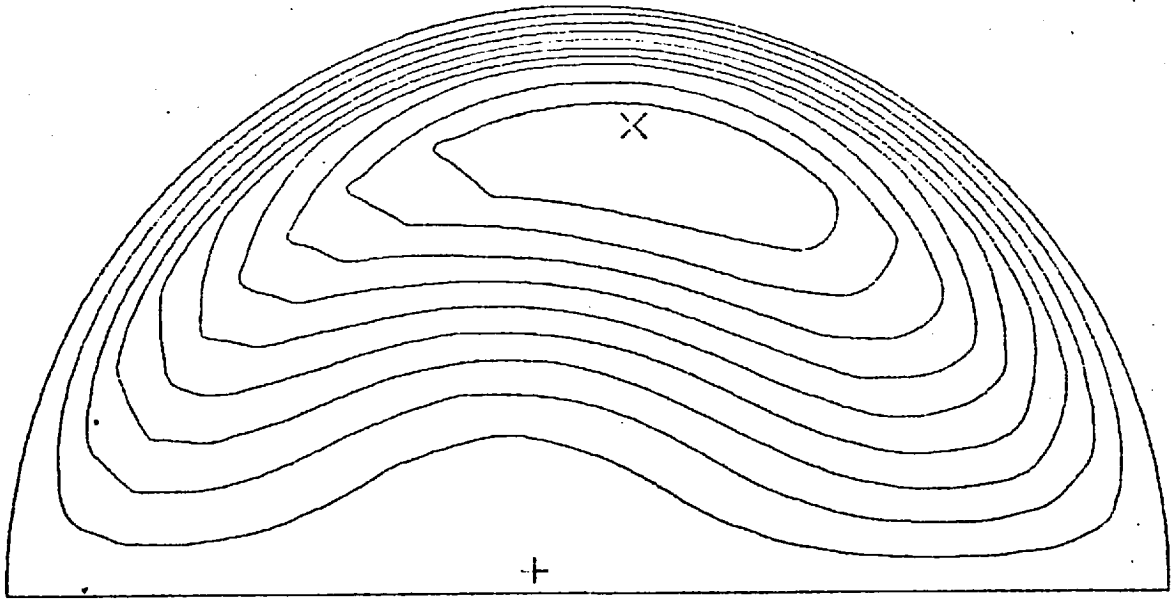


Fig. 7.19

$R_s = 30, D = 11.87$



NOF = 12

NR = 11

$\chi_{\max} = 0.311 \times 10^{-4}$

$\chi_{\min} = -0.0251$

$w_{\min} = 2.51$

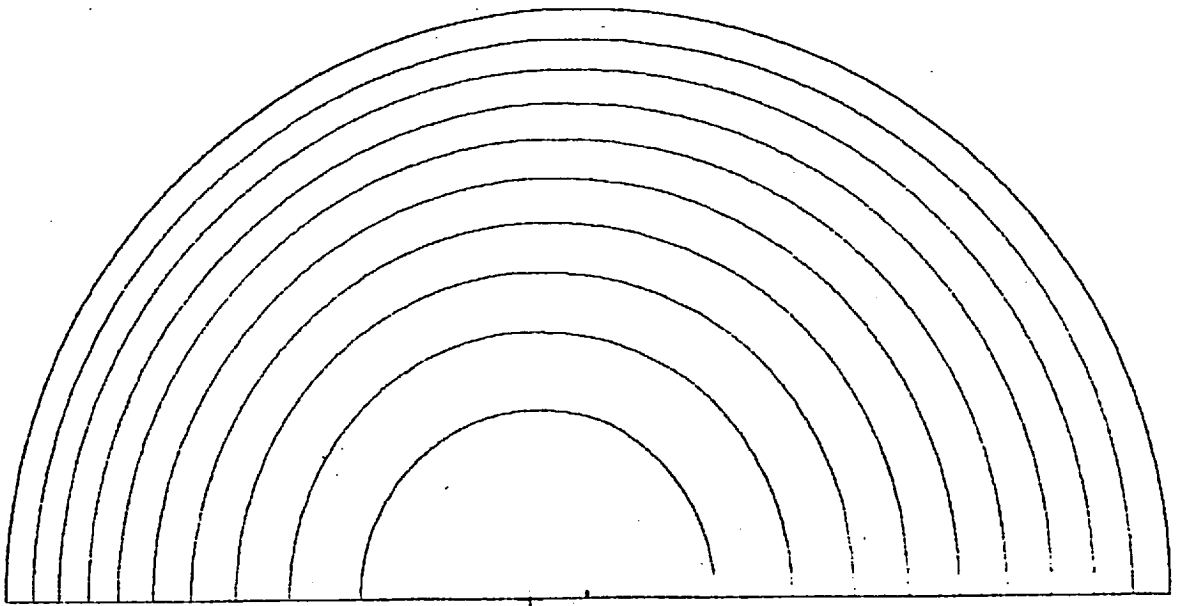
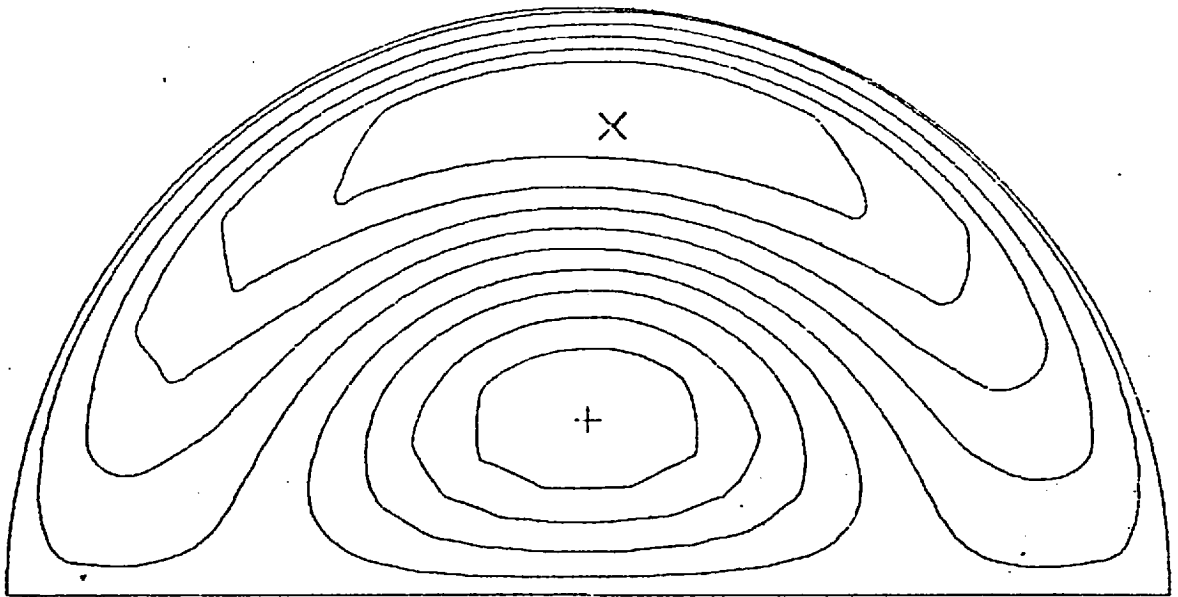


Fig. 7.20

$R_s = 30, D = 11.93$



NOF = 12

NR = 11

$\lambda_{\max} = 0.0159$

$\lambda_{\min} = -0.0180$

$w_{\min} = 2.72$

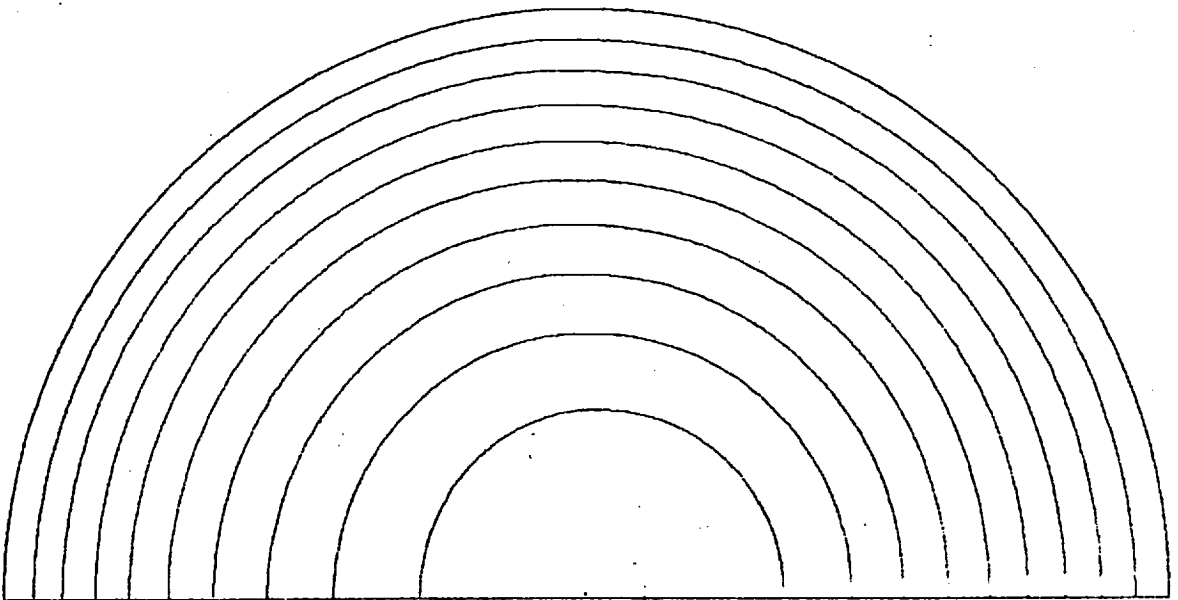
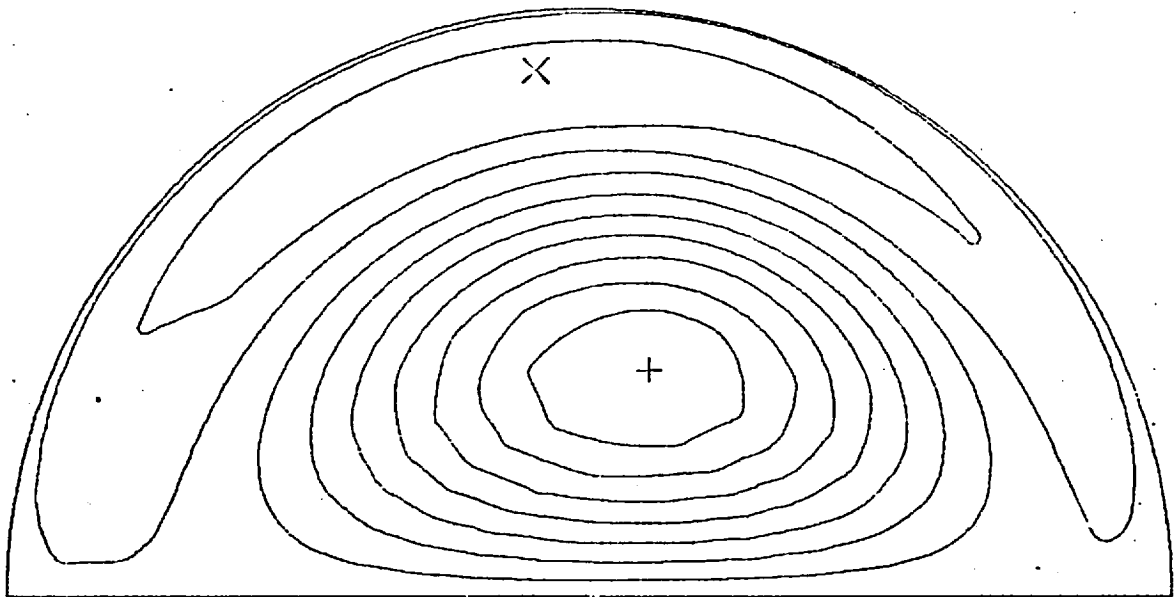


Fig. 7.21

$R_s = 30, D = 15$



NOF = 12

NR = 11

$\lambda_{\max} = 0.0474$

$\lambda_{\min} = -0.0129$

$w_{\min} = 3.51$

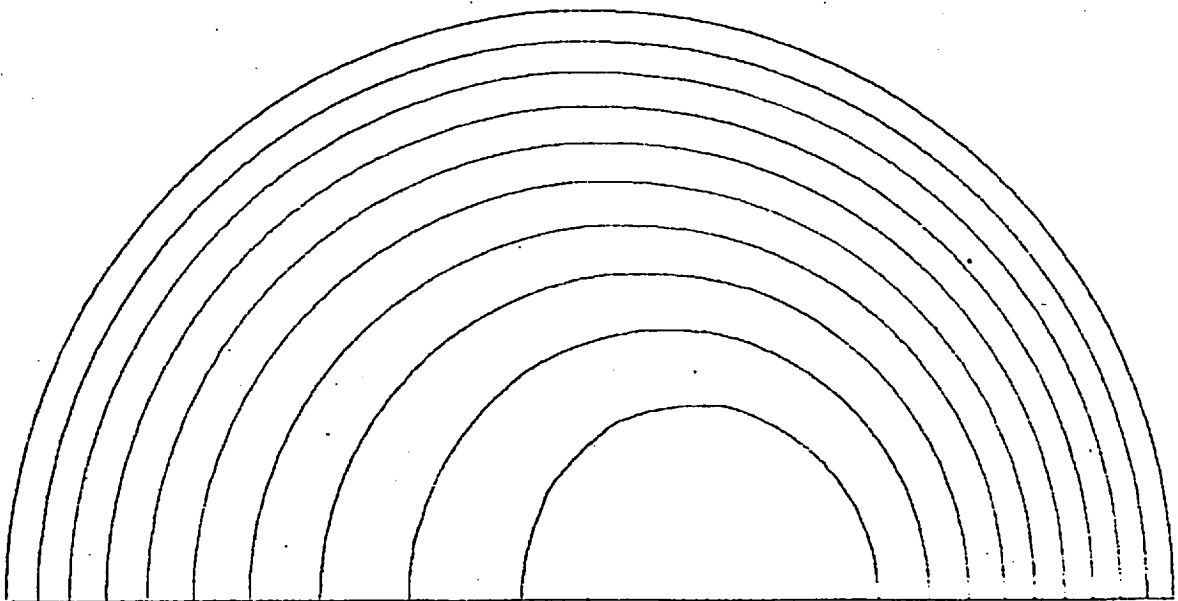
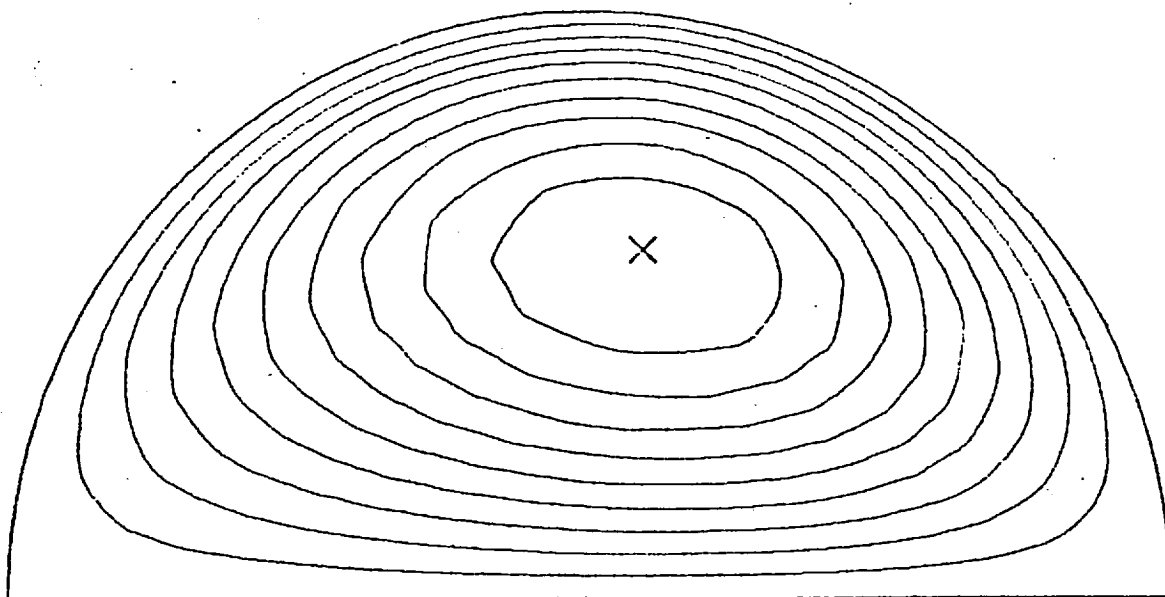


Fig. 7.22

$R_s = 30, D = 20$



NOF = 12

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0456$

$w_{\min} = 0.882$

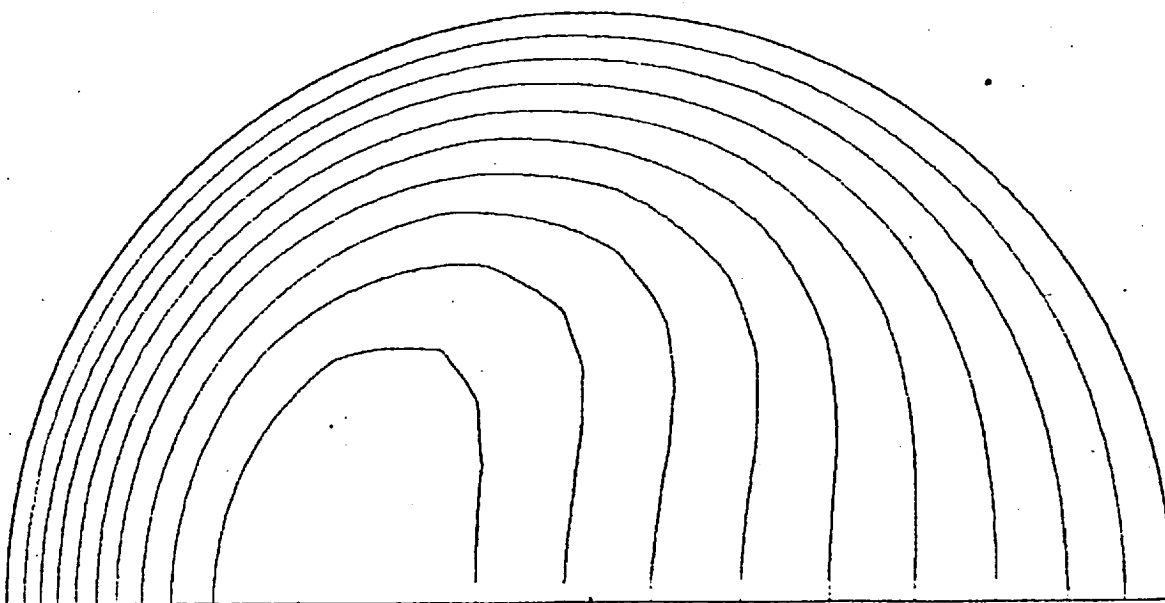
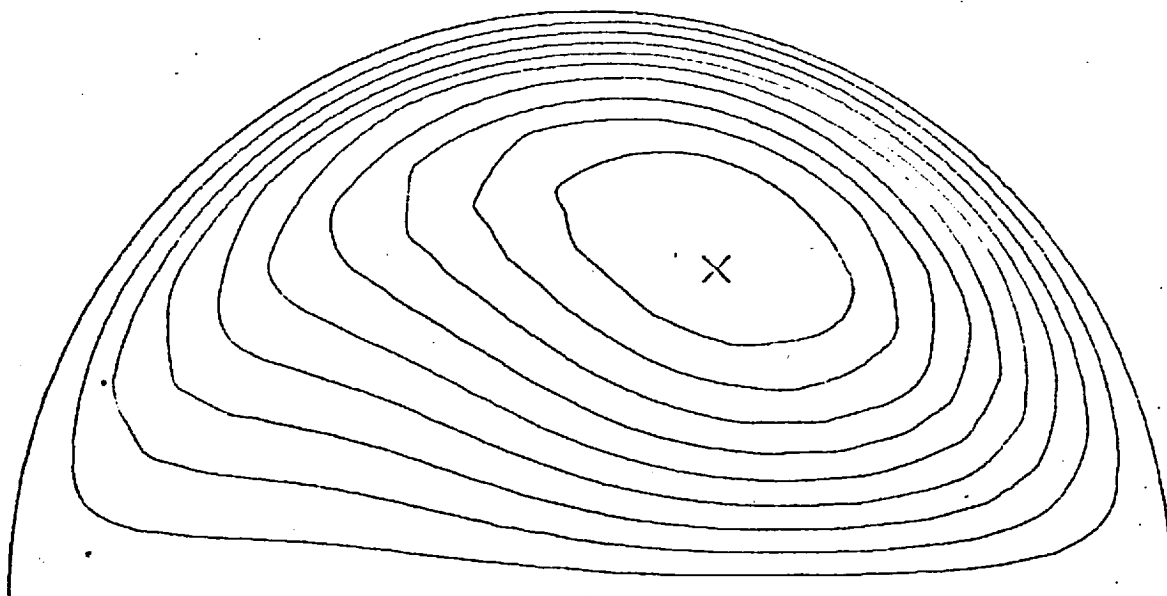


Fig. 7.23

$R_s = 75, D = 10$



NOF = 12

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0338$

$w_{\text{im}} = 1.95$

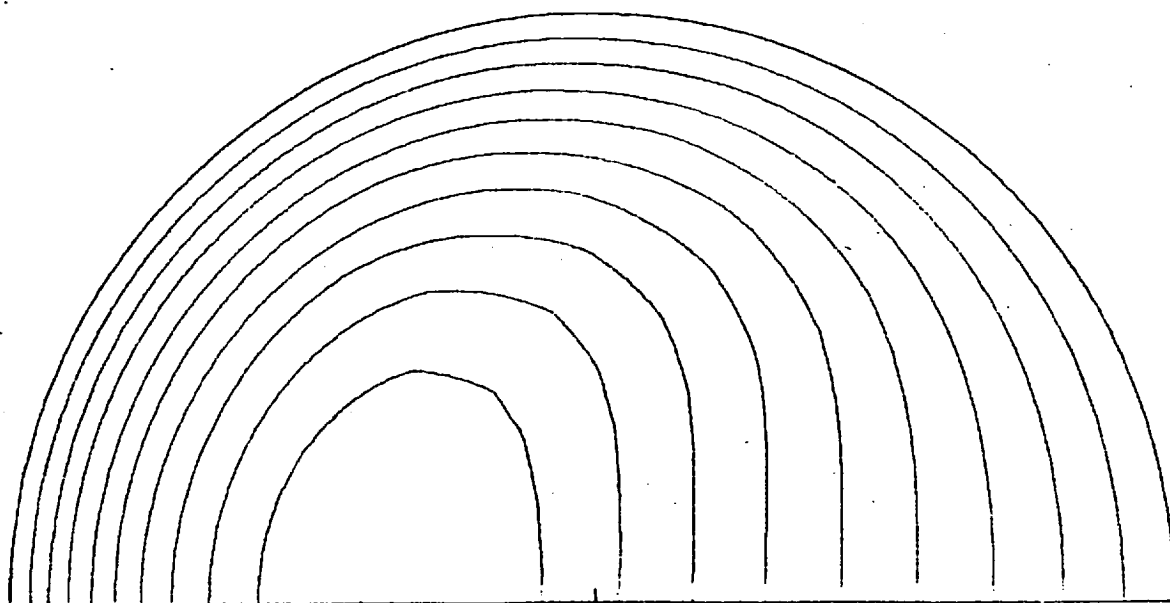
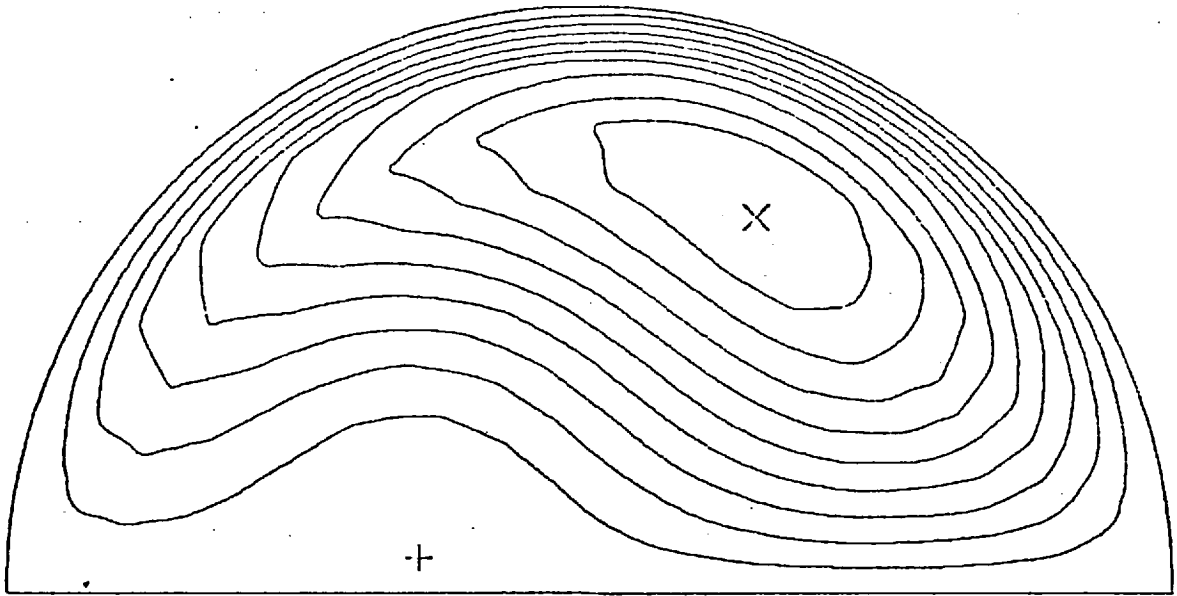


Fig. 7.24

$R_s = 75, D = 20$



NOF = 12

NR = 11

$\chi_{\max} = 0.673 \times 10^{-4}$

$\chi_{\min} = -0.0270$

$w_{\min} = 2.32$

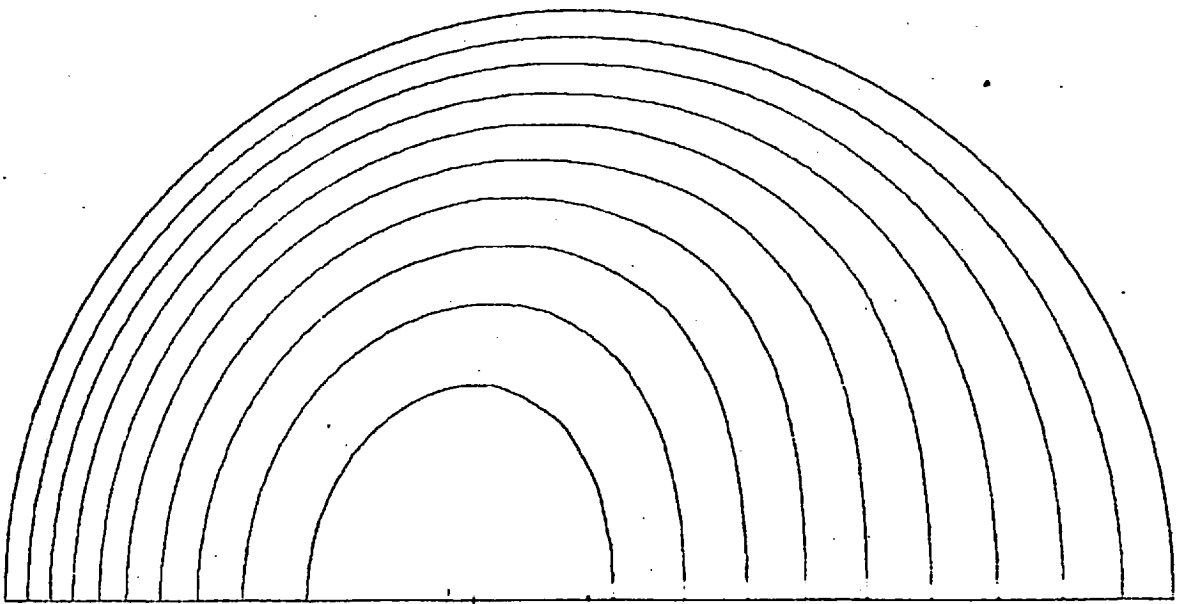
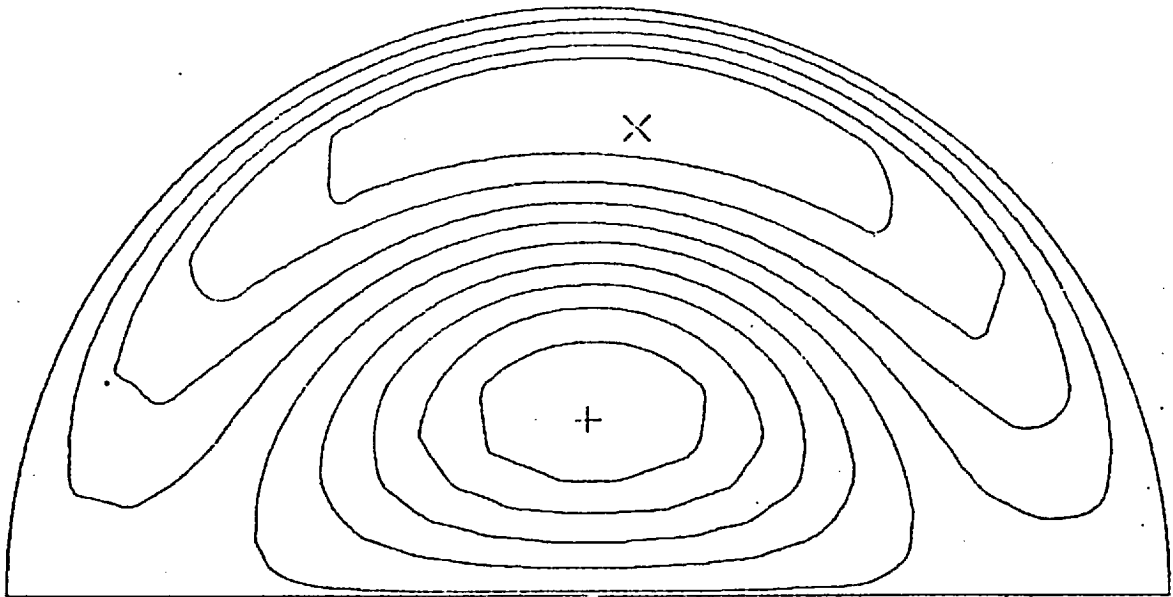


Fig. 7.25

$R_s = 75, D = 22$



NOF = 12

NR = 11

$\chi_{\max} = 0.0179$

$\chi_{\min} = -0.0168$

$w_{im} = 2.80$

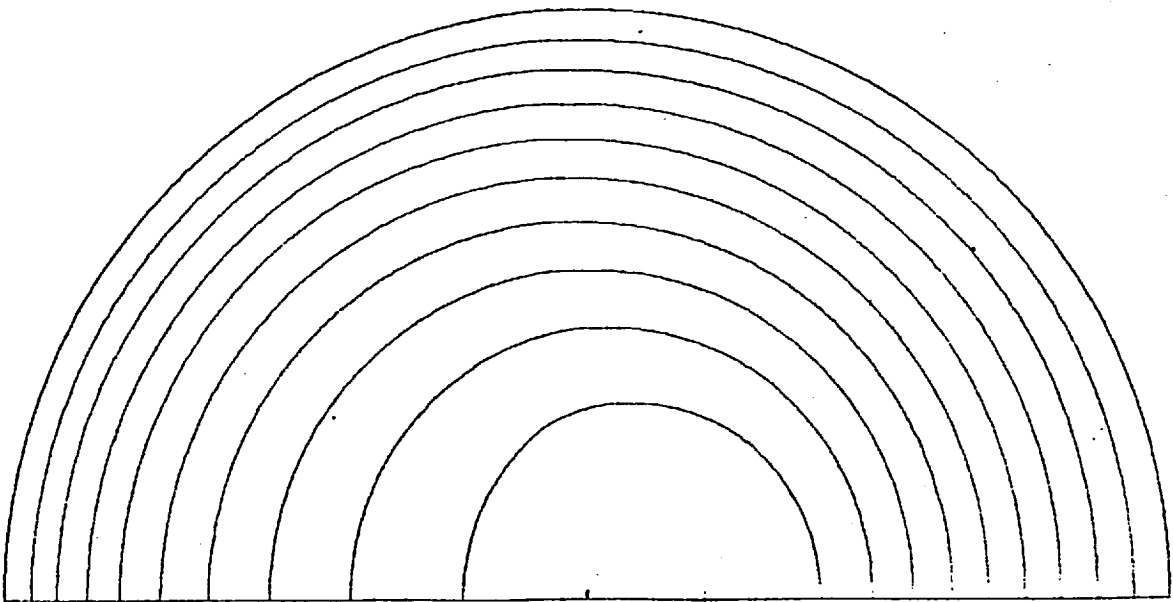
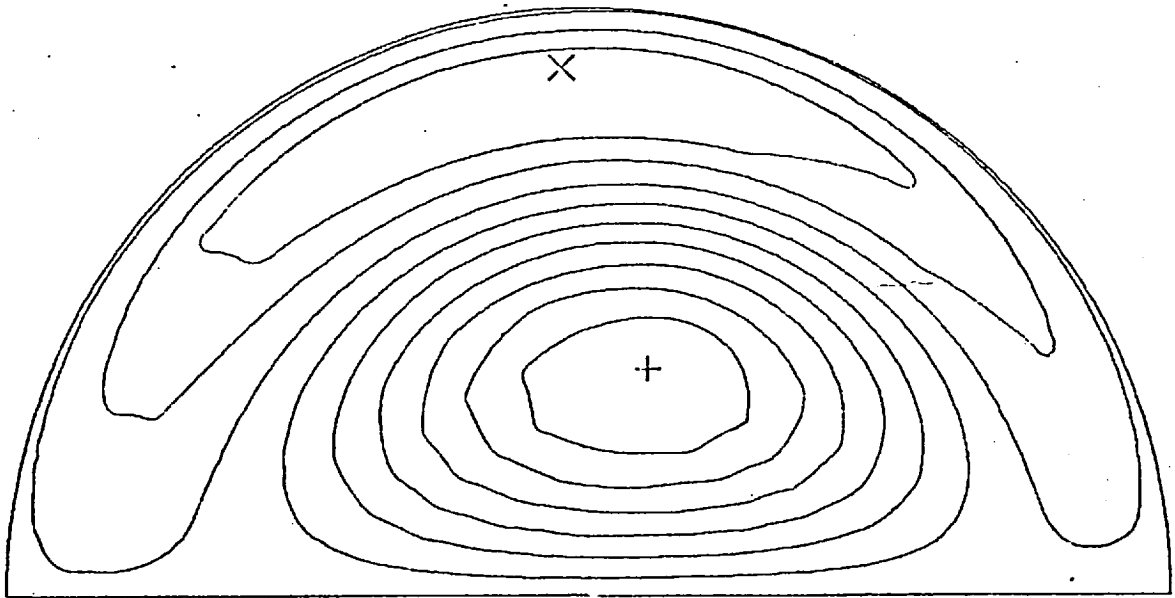


Fig. 7.26

$R_s = 75, D = 25$



NCF = 12

NR = 11

$\chi_{\text{omax}} = 0.0308$

$\chi_{\text{omin}} = -0.0143$

$w_{im} = 3.21$

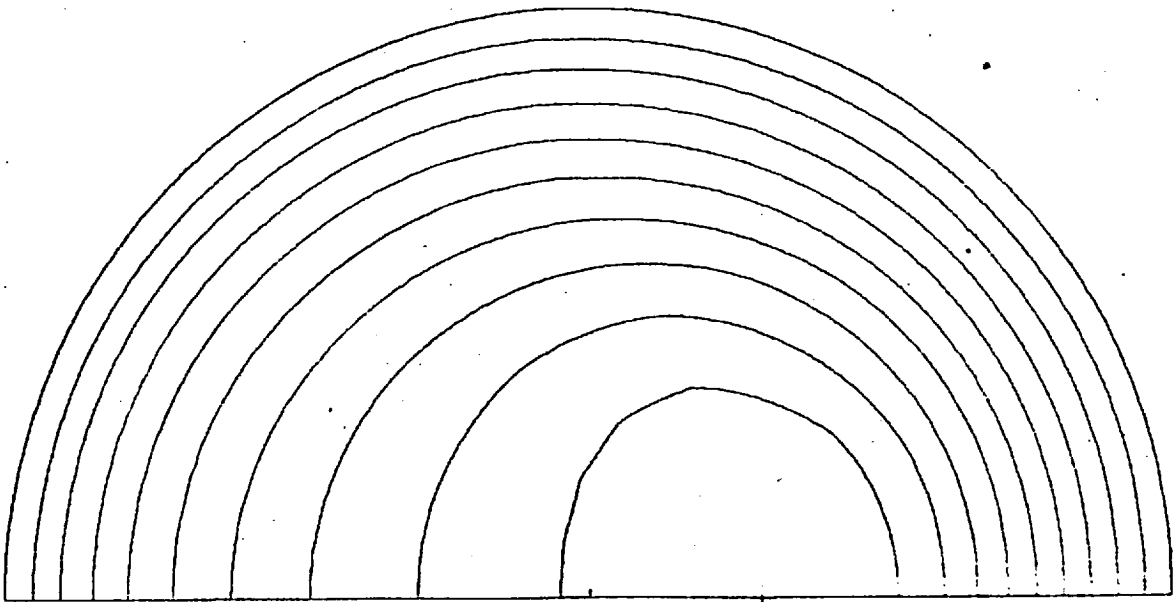
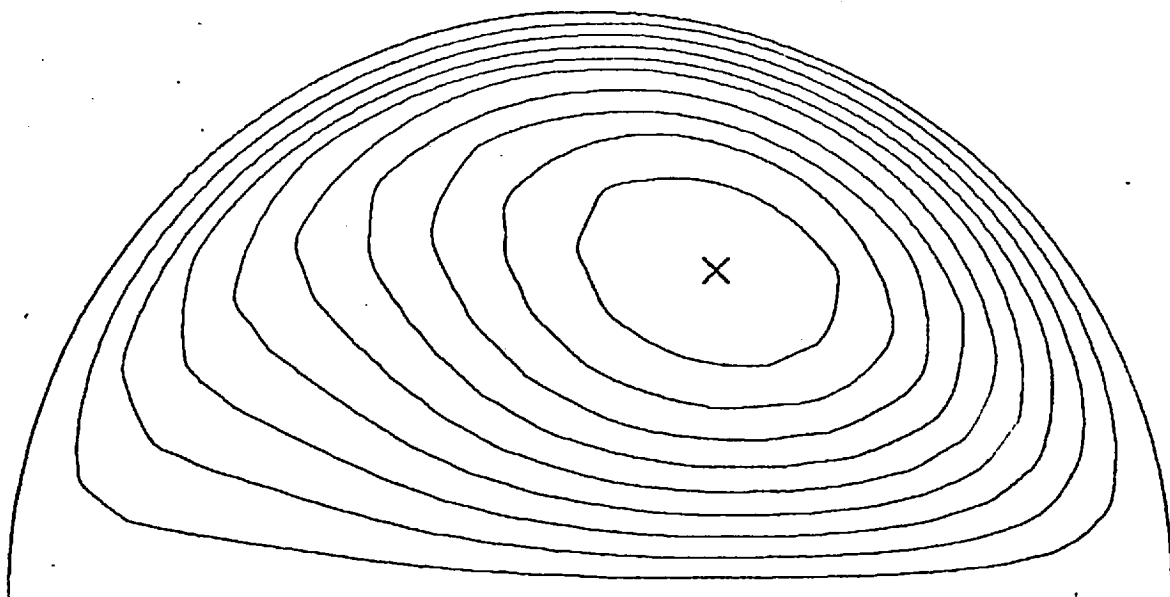


Fig. 7.27

$R_s = 75, D = 30$



NOF = 12

NR = 11

$\chi_{\text{max}} = 0.$

$\chi_{\text{min}} = -0.0369$

$w_{\text{im}} = 2.06$

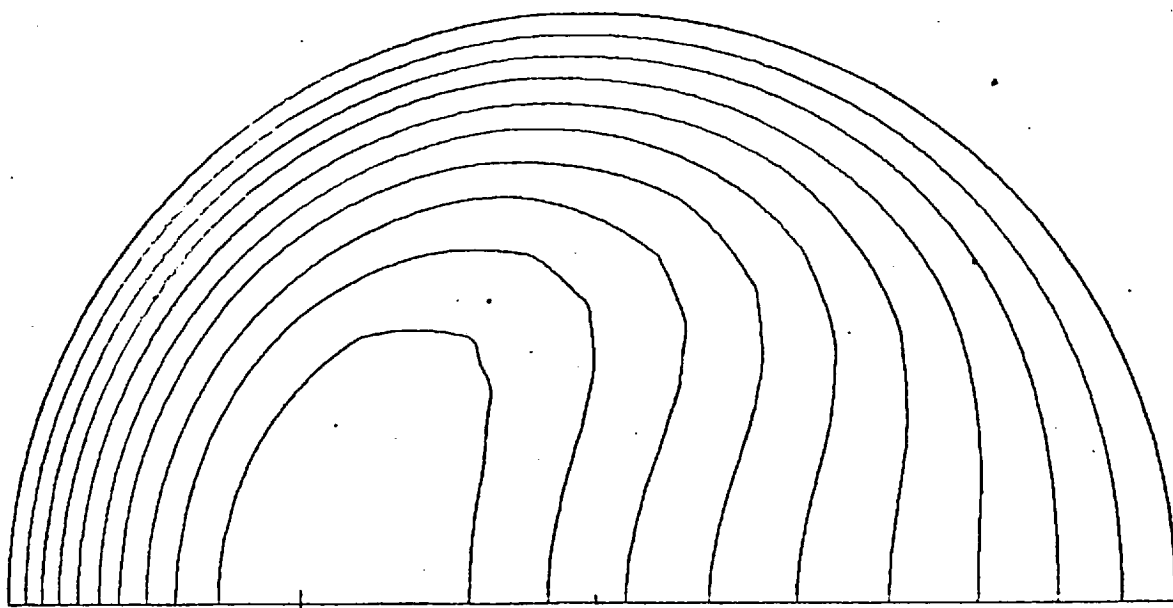
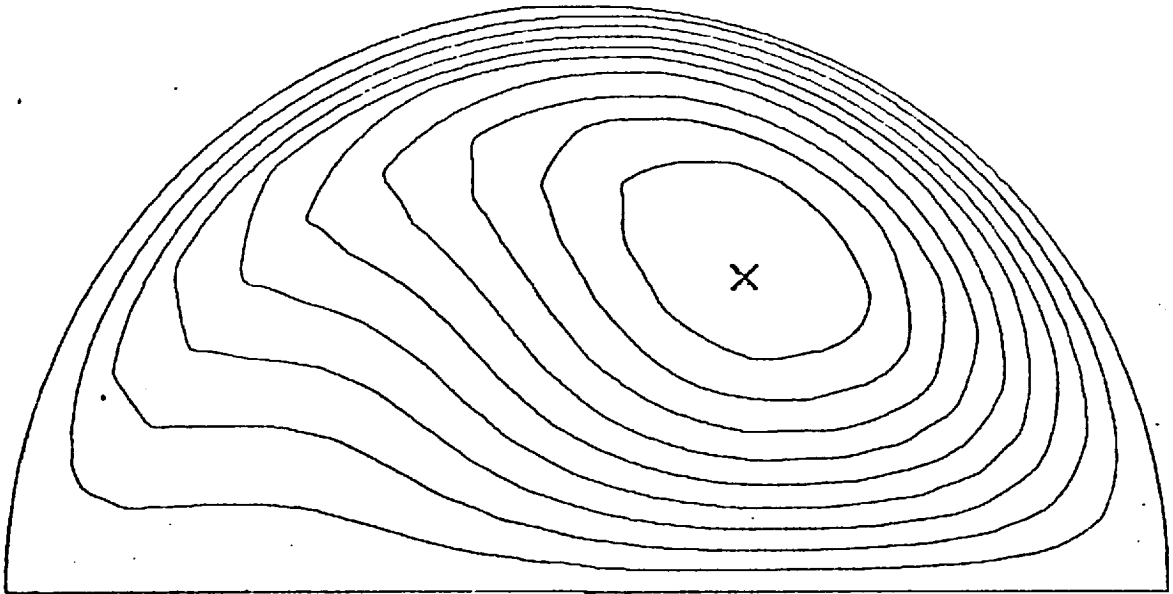


Fig. 7.28

$R_s = 100, D = 28$



NOF = 12

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0322$

$w_m = 2.39$

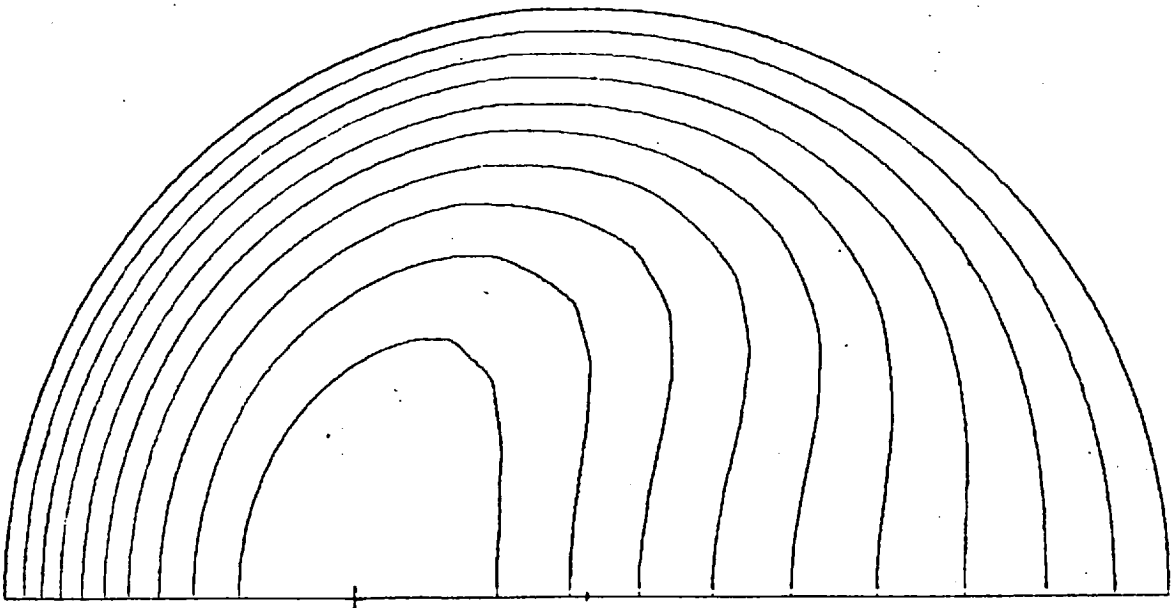
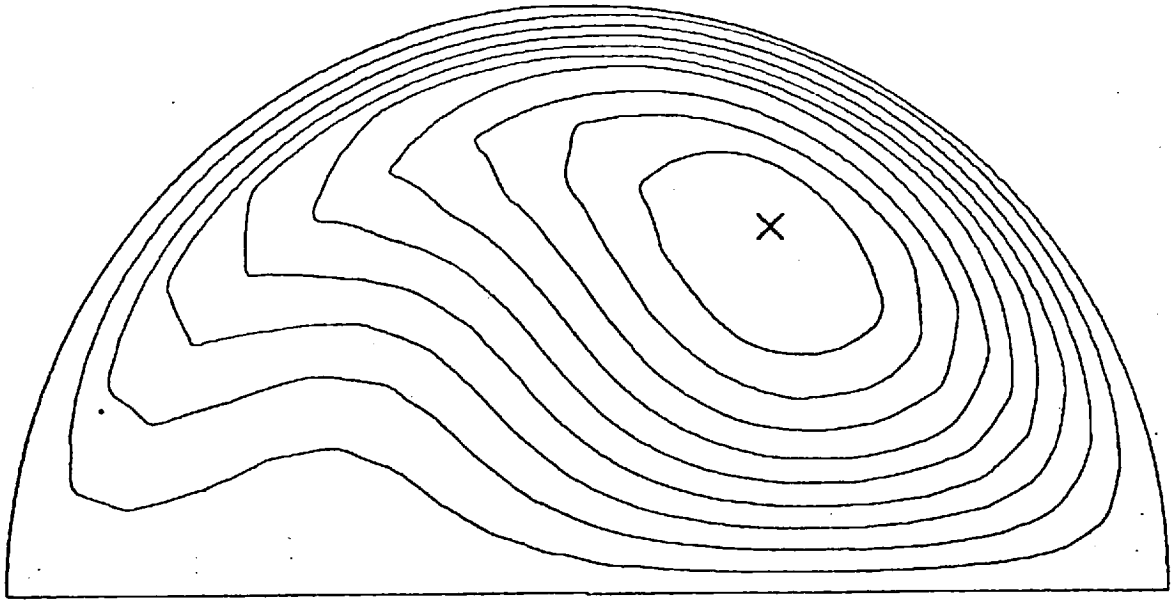


Fig. 7.29

$R_s = 100, D = 30.5$



NOF = 12

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0297$

$w_{im} = 2.52$

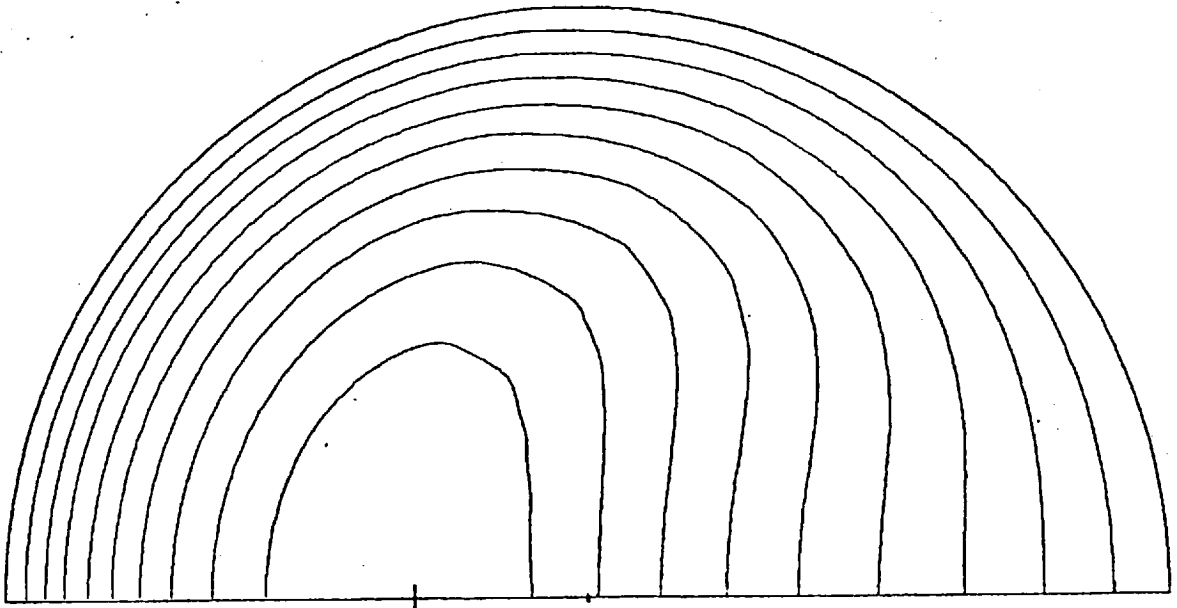
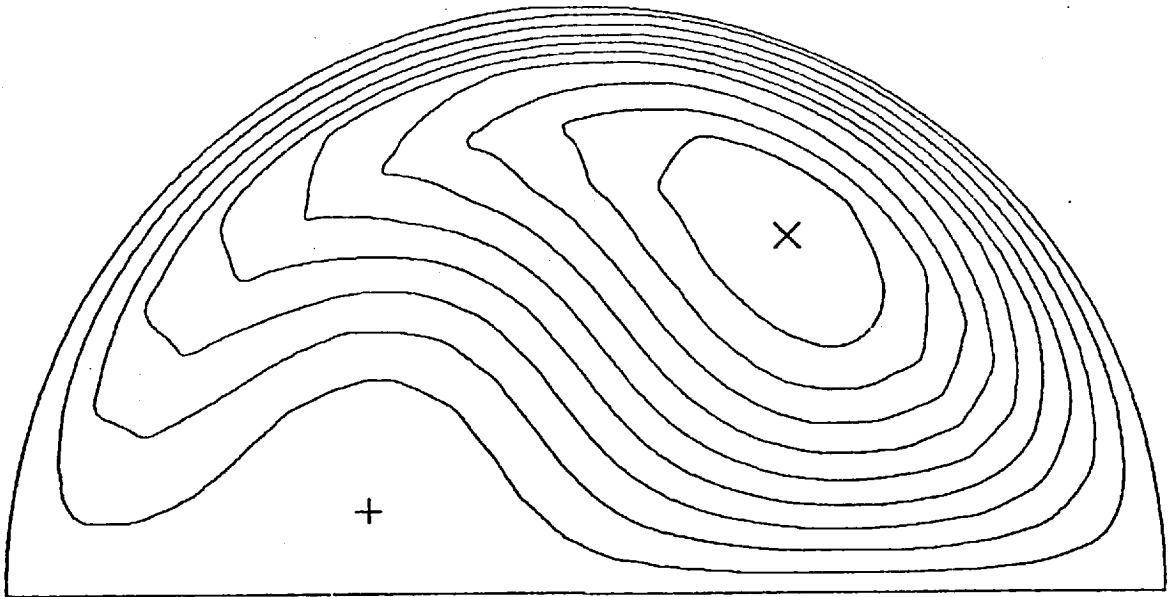


Fig. 7.30

$R_s = 100, D = 30.91$



NOF = 12

NR = 11

$\eta_{\max} = 0.565 \times 10^{-3}$

$\eta_{\min} = -0.0275$

$w_{im} = 2.60$

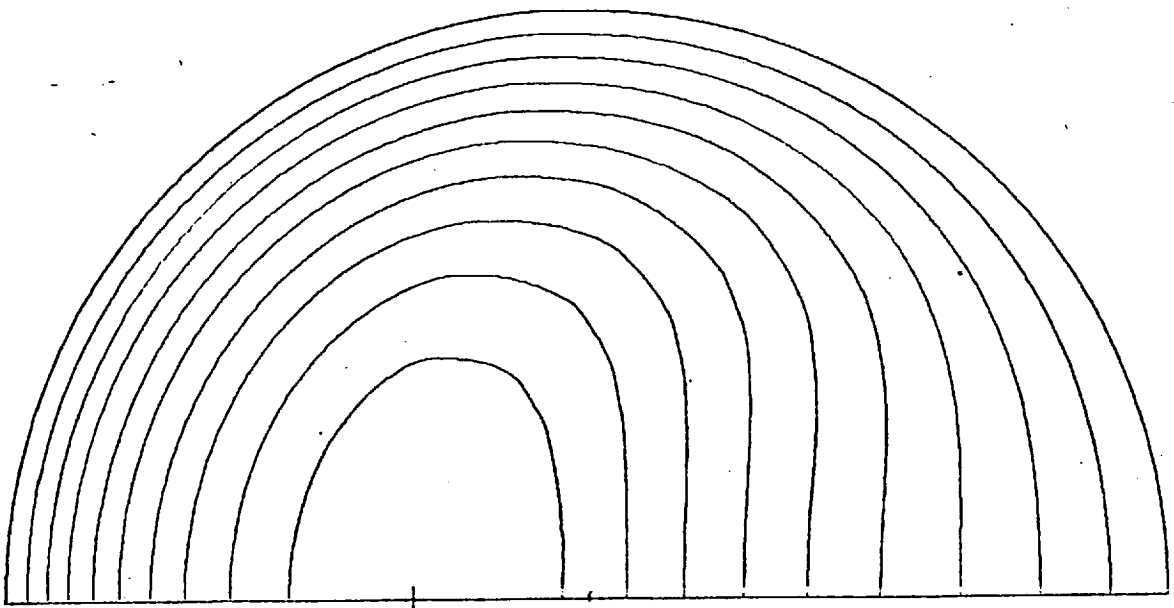
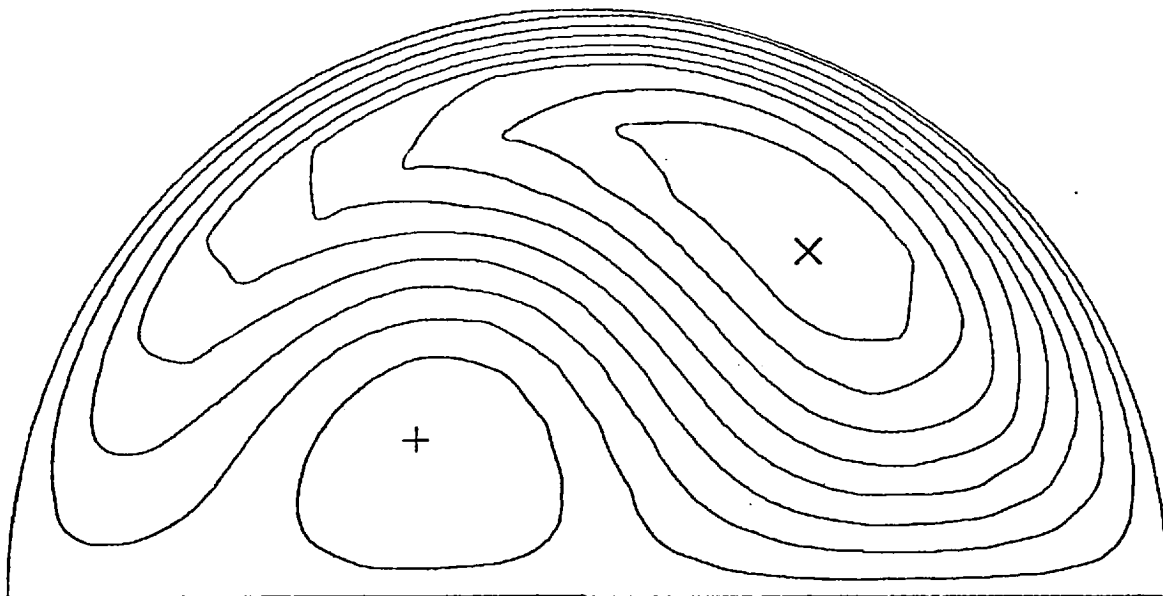


Fig. 7.31

$R_s = 100$, $D = 30.6$



NOF = 12

NR = 11

$\chi_{\max} = 0.00397$

$\chi_{\min} = -0.0241$

$w_{\min} = 2.65$

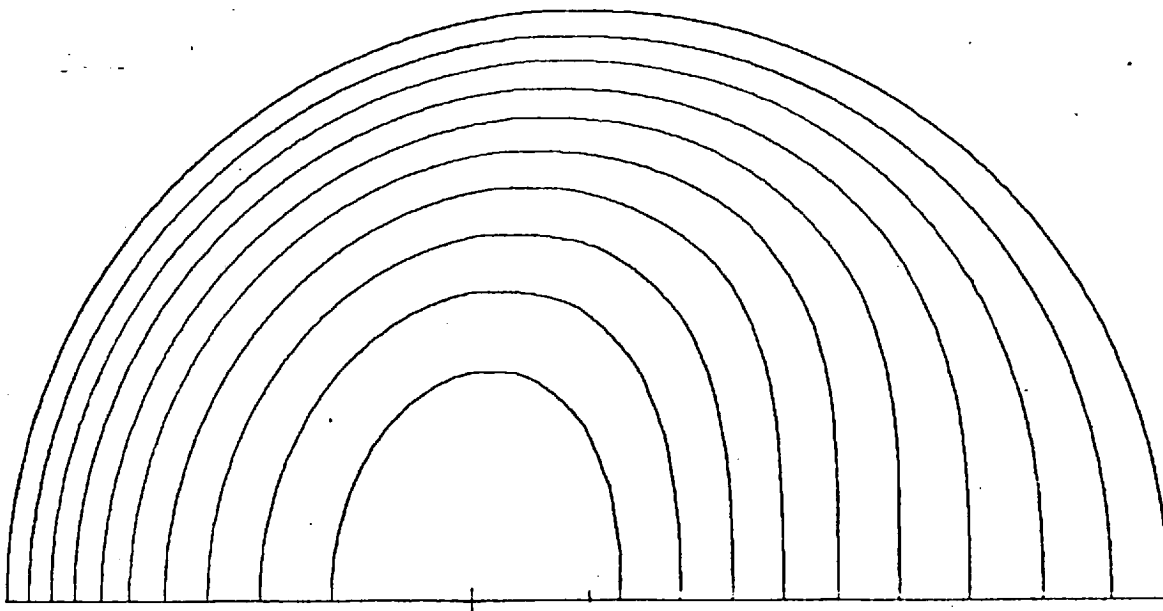
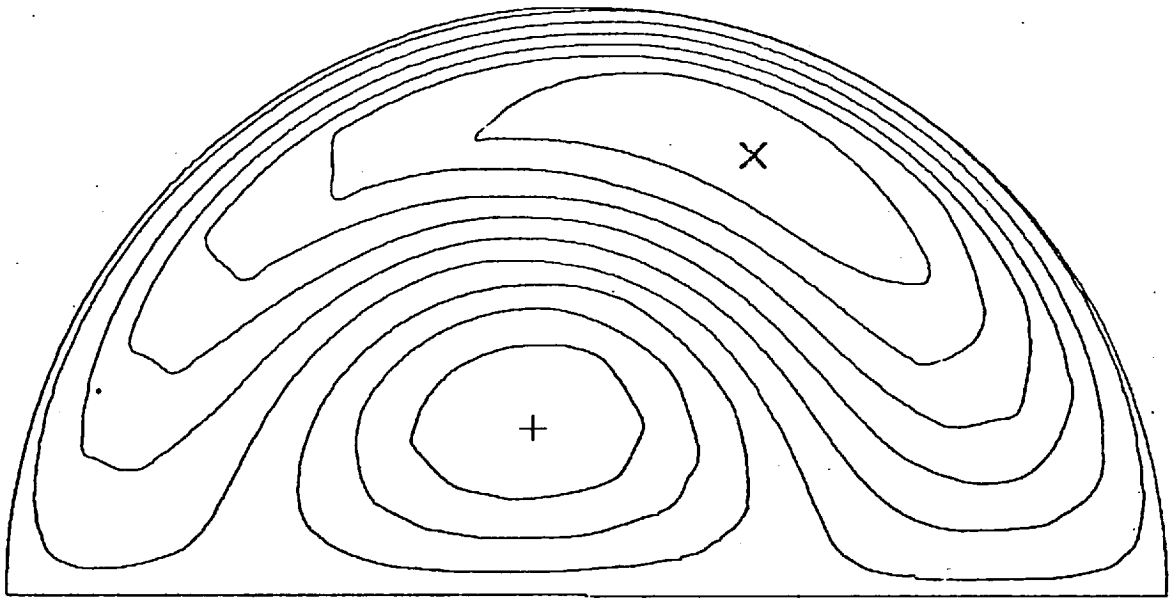


Fig. 7.32

$R_s = 100, D = 29.5$



NOF = 12

NR = 11

$\chi_{\text{omax}} = 0.0115$

$\chi_{\text{omin}} = -0.0194$

$w_{\text{im}} = 2.70$

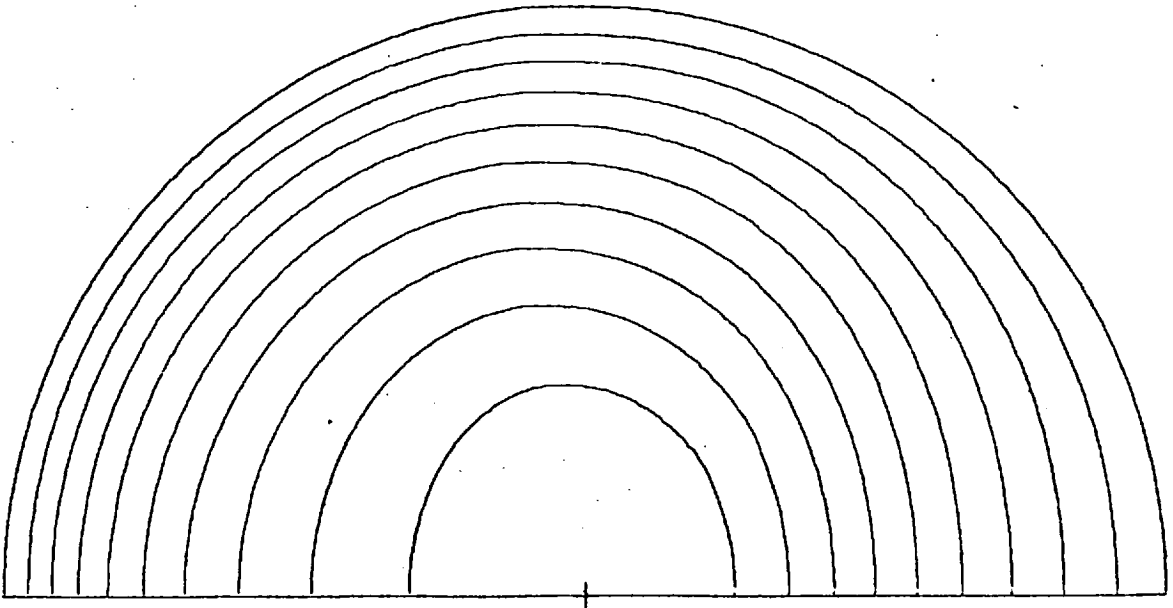
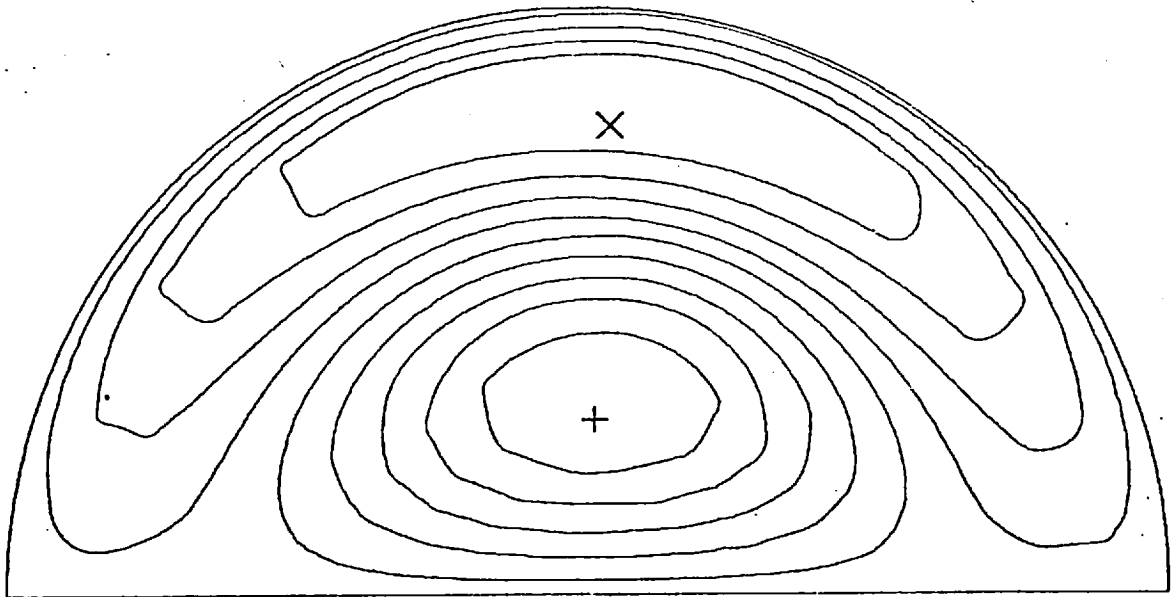


Fig. 7.33

$R_s = 100, D = 28.3$



NOF = 12

NR = 11

$\chi_{\max} = 0.0197$

$\chi_{\min} = -0.0156$

$w_{im} = 2.89$

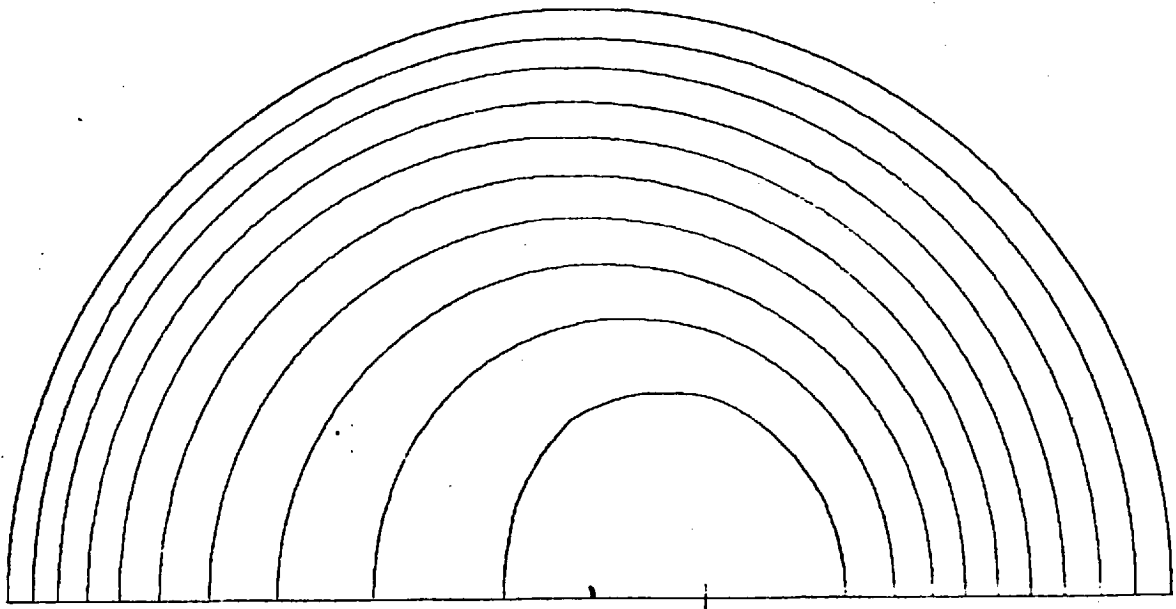
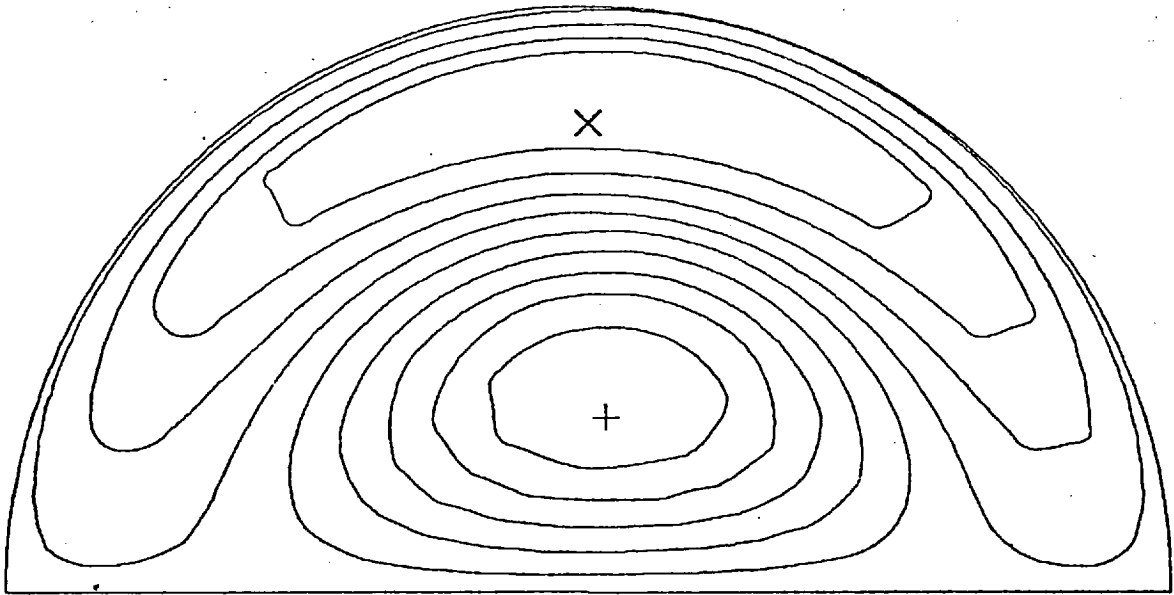


Fig. 7.34

$R_s = 100, D = 30.5$



NOF = 12

NR = 11

$\chi_{\text{max}} = 0.0209$

$\chi_{\text{min}} = -0.0152$

$w_{\text{im}} = 2.92$

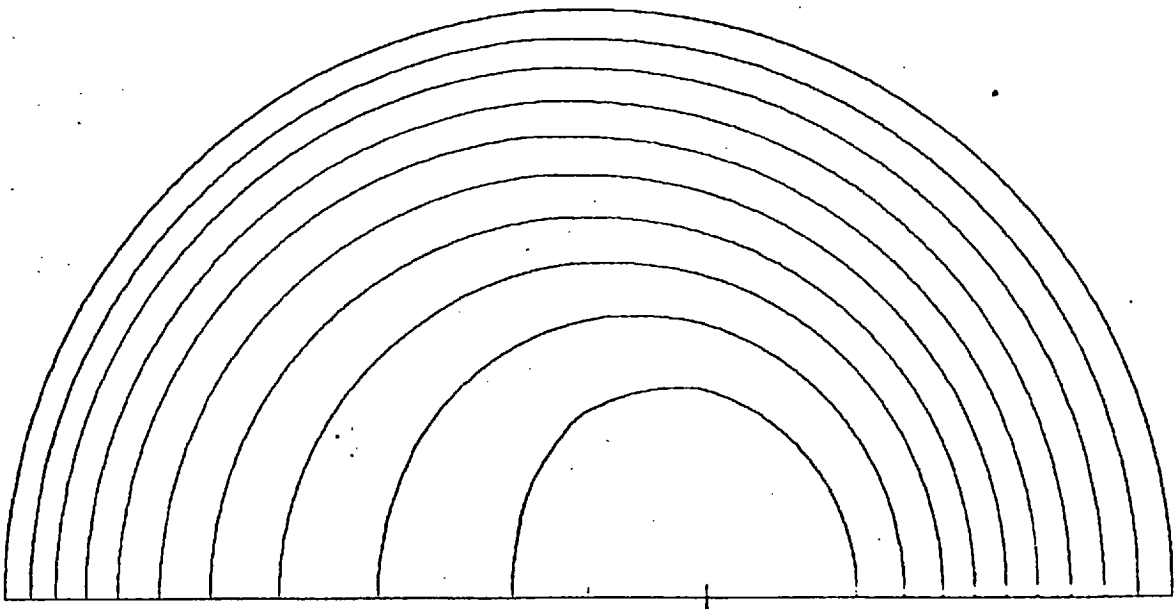
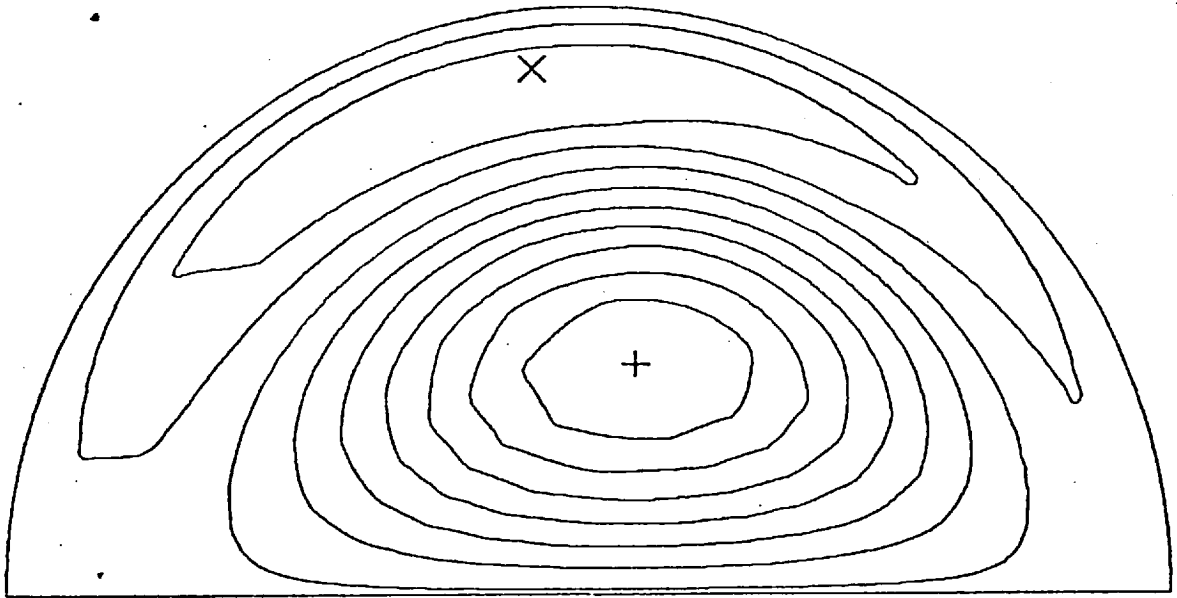


Fig. 7.35

$R_s = 100, D = 31$



NOF = 12

NR = 11

$\chi_{\text{omax}} = 0.0352$

$\chi_{\text{omin}} = -0.0134$

$w_{\text{im}} = 3.47$

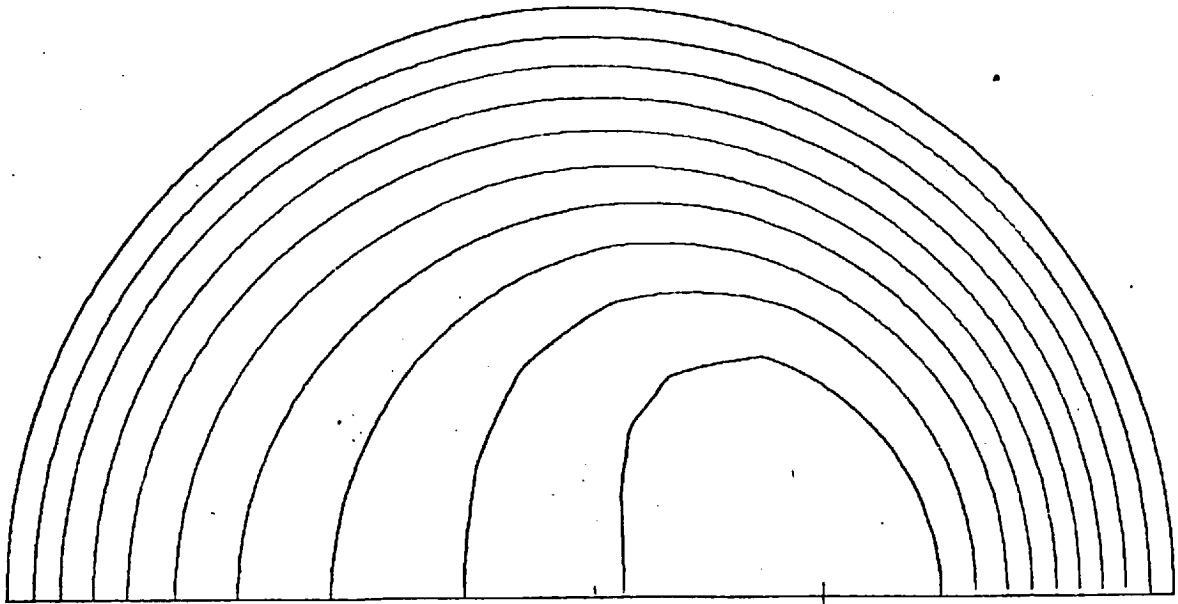
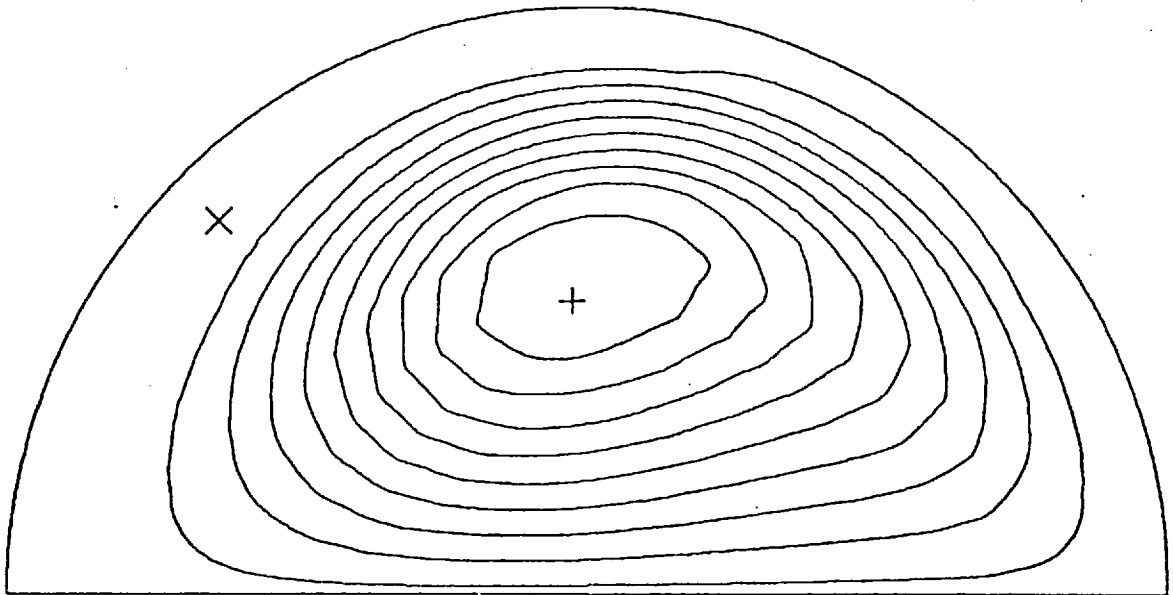


Fig. 7.36

$R_s = 100, D = 40$



NOF = 12

NR = 11

$\chi_{\max} = 0.0779$

$\chi_{\min} = -0.0057$

$w_{\min} = 6.43$

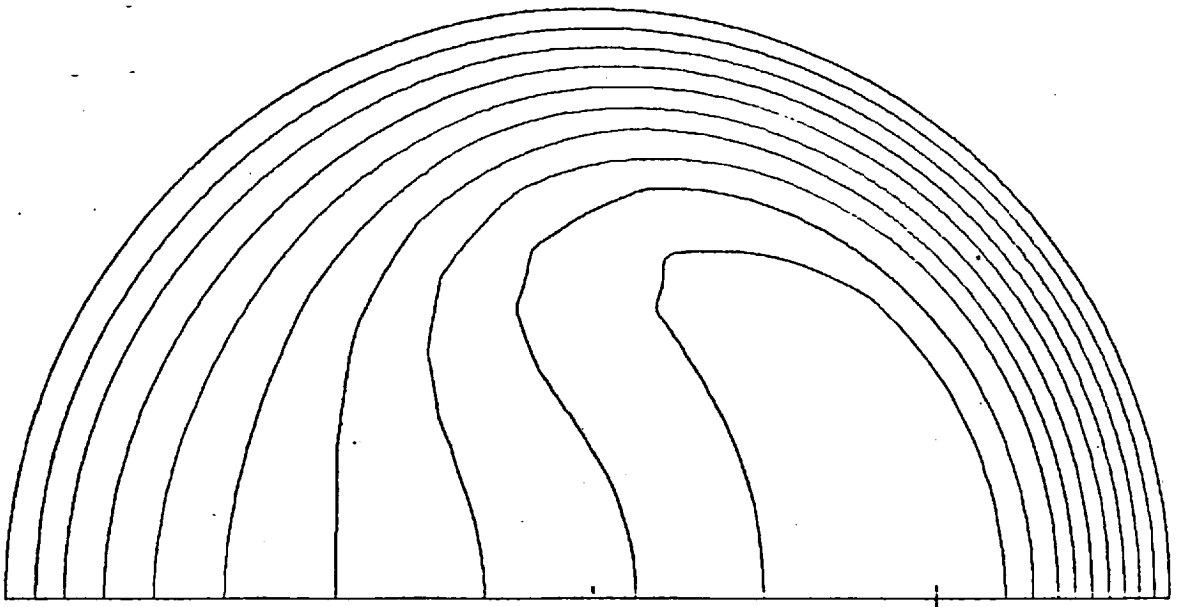
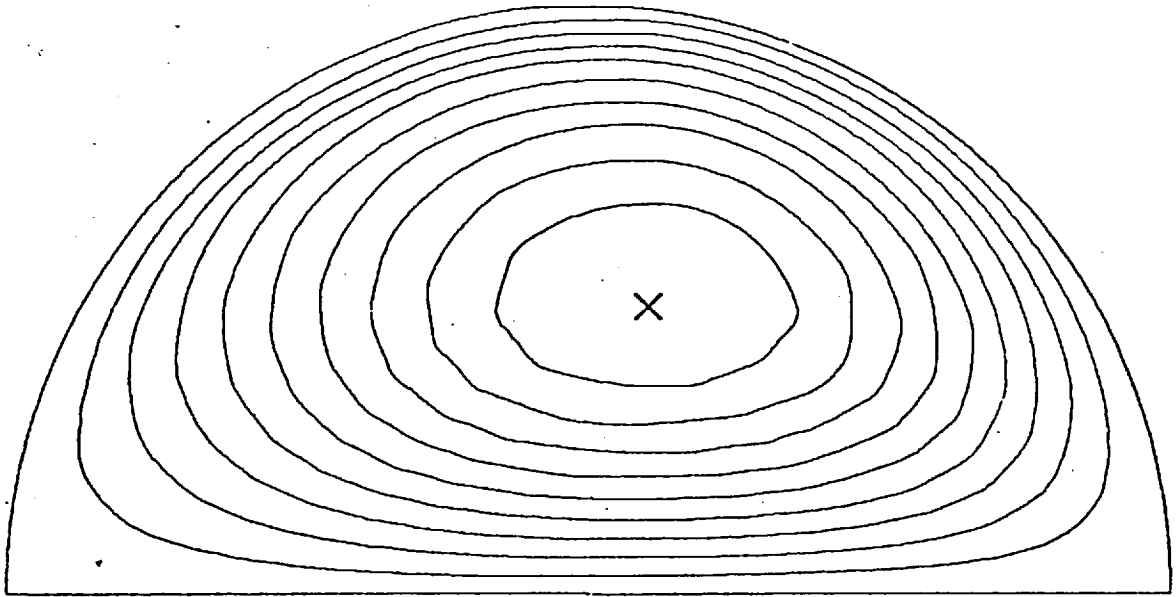


Fig. 7.37

$R_s = 100, D = 100$



NOF = 18

NR = 11

$\chi_{\text{max}} = 0.$

$\chi_{\text{min}} = -0.0451$

$w_{\text{im}} = 1.82$

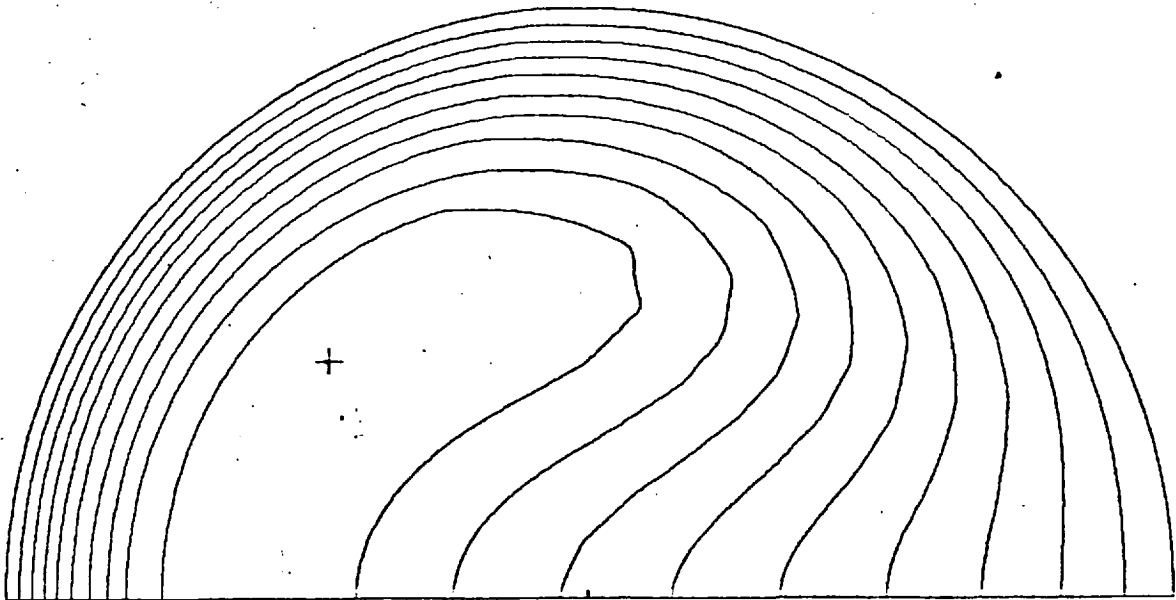
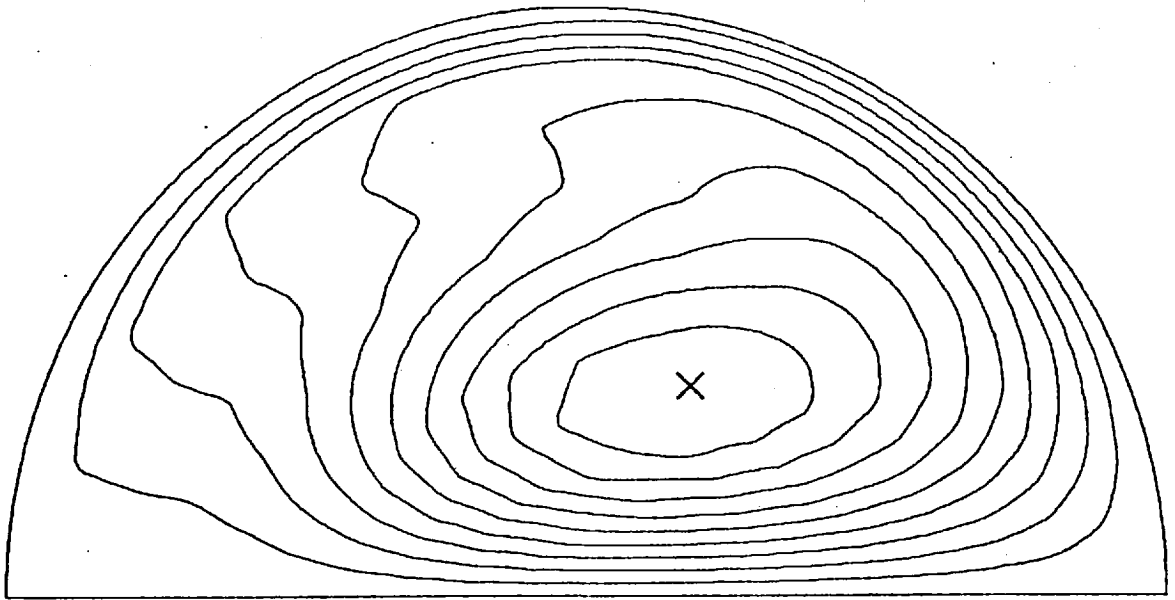


Fig. 7.38

$R_s = 150, D = 40$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0328$

$w_{\min} = 3.31$

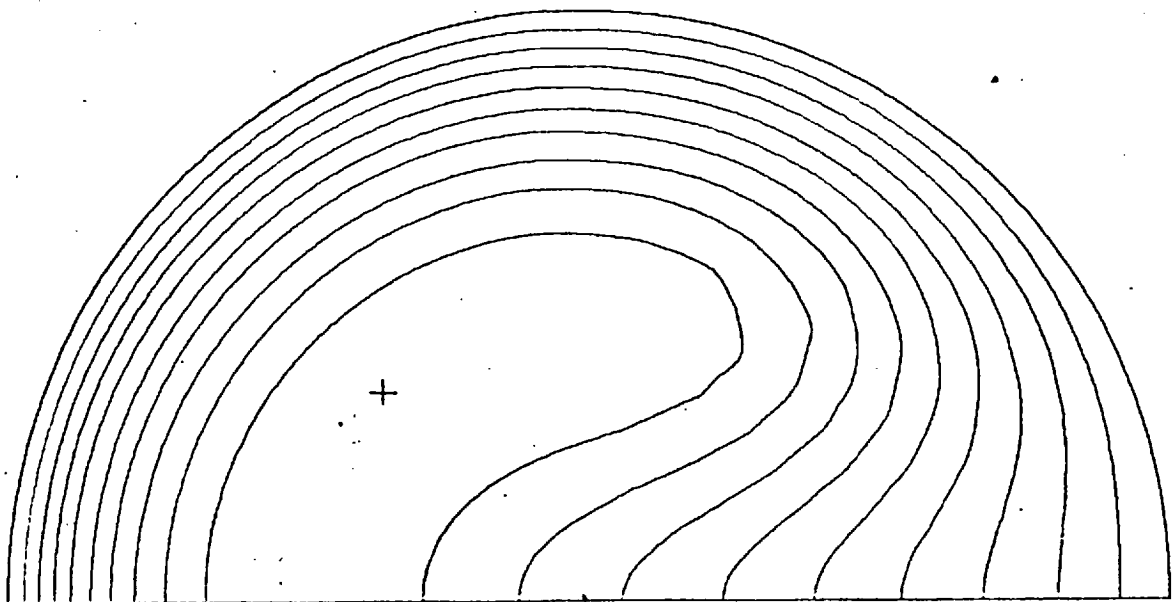
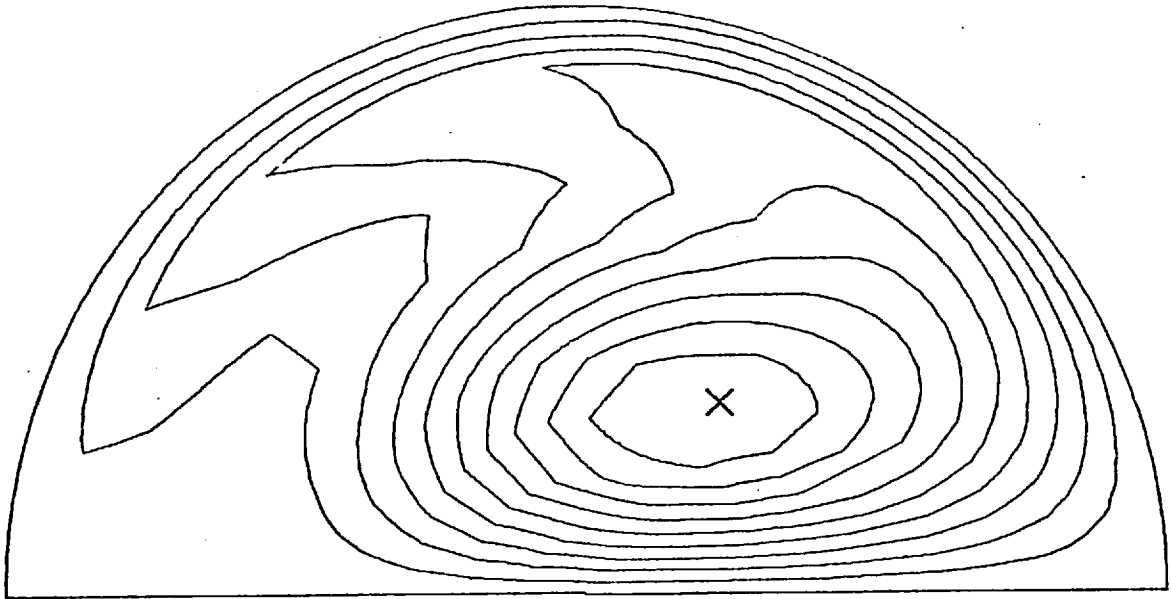


Fig. 7.39

$R_s = 150, D = 64$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.$

$\chi_{\min} = -0.0326$

$w_{\min} = 3.62$

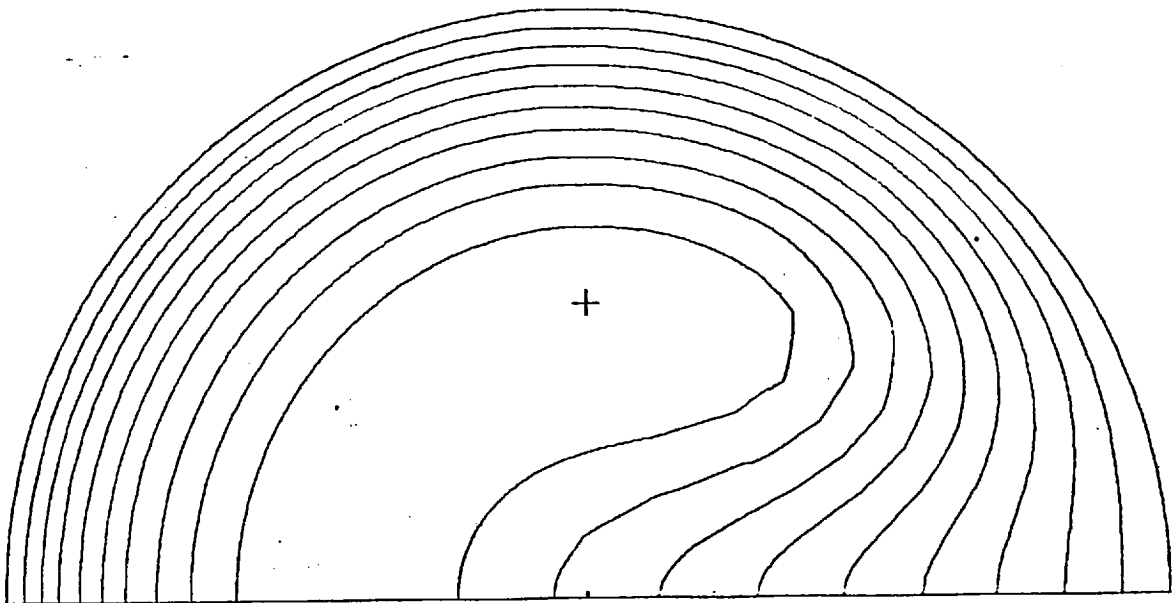
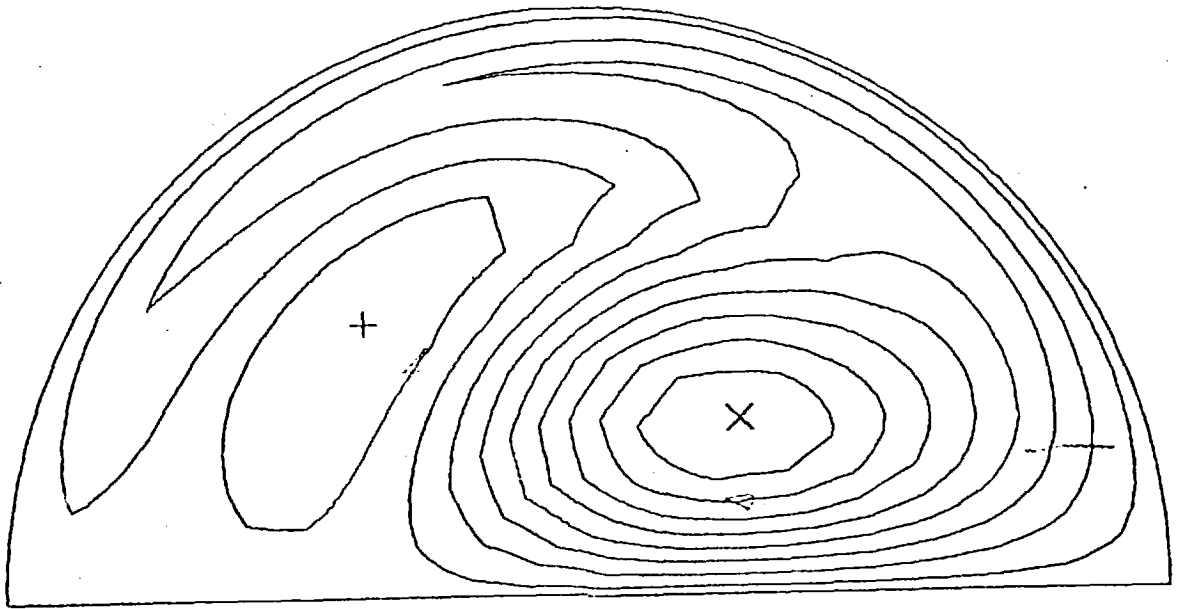


Fig. 7.40

$R_s = 150, D = 68.5$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.00634$

$\chi_{\min} = -0.0349$

$w_{1m} = 4.19$

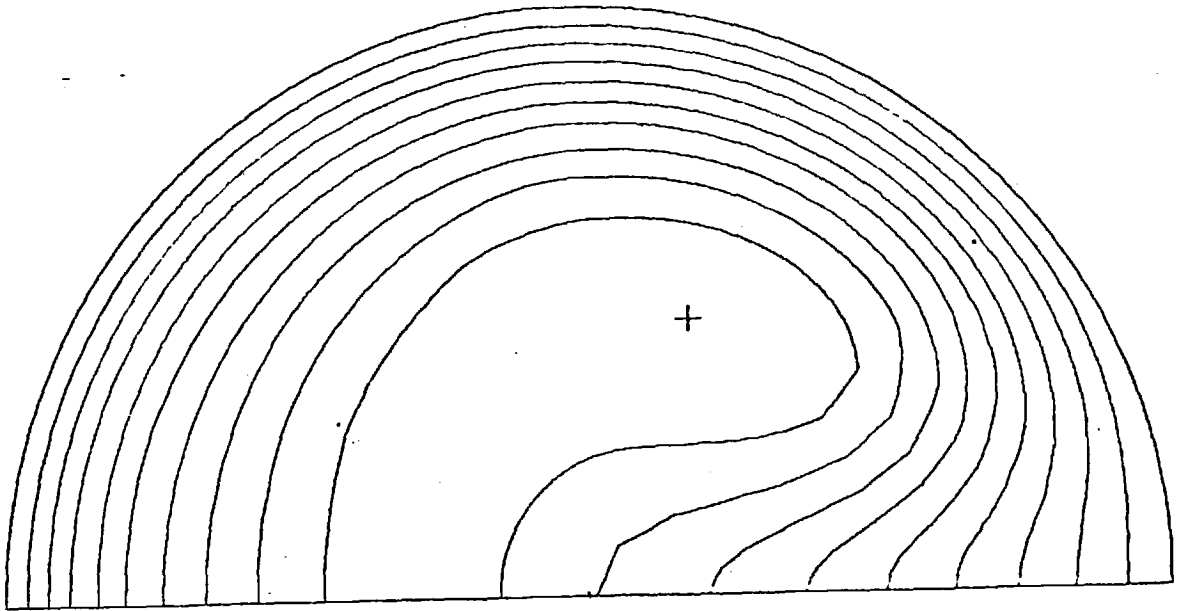
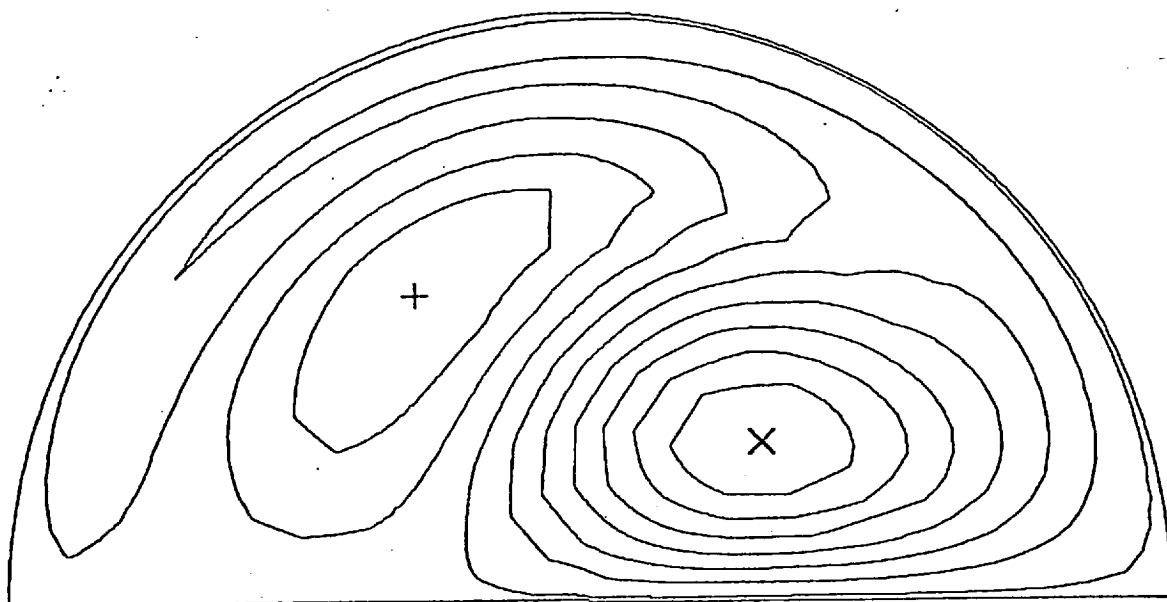


Fig. 7.41 $R_s = 150, D = 78$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.0149$

$\chi_{\min} = -0.0379$

$w_{\min} = 4.74$

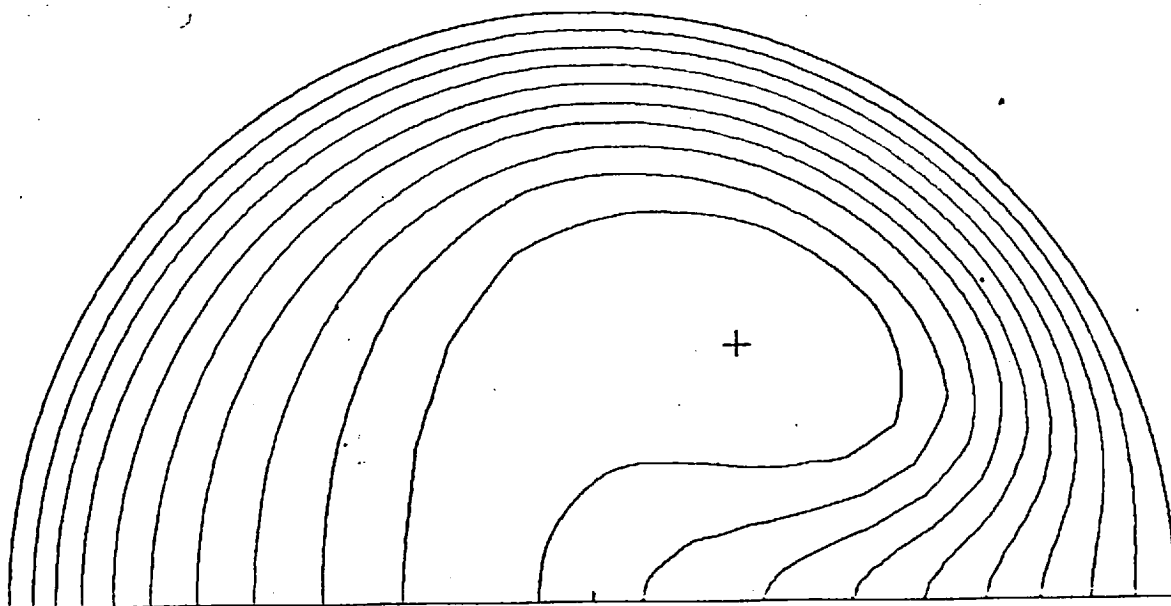
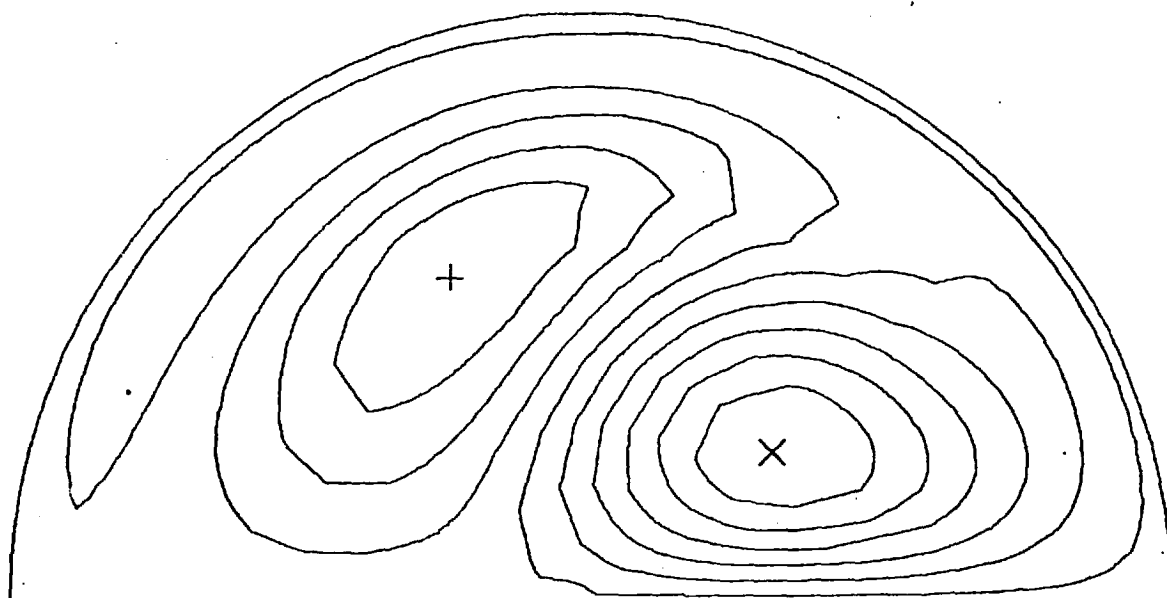


Fig. 7.42 $R_3 = 150, D = 90$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.0241$

$\chi_{\min} = -0.0407$

$w_{\min} = 5.43$

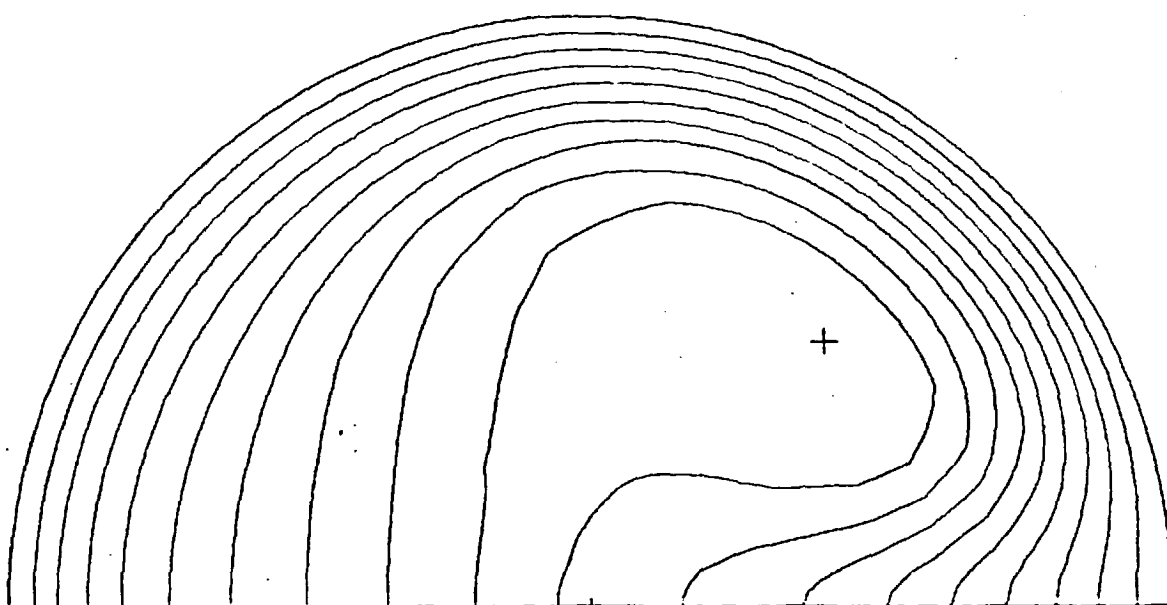
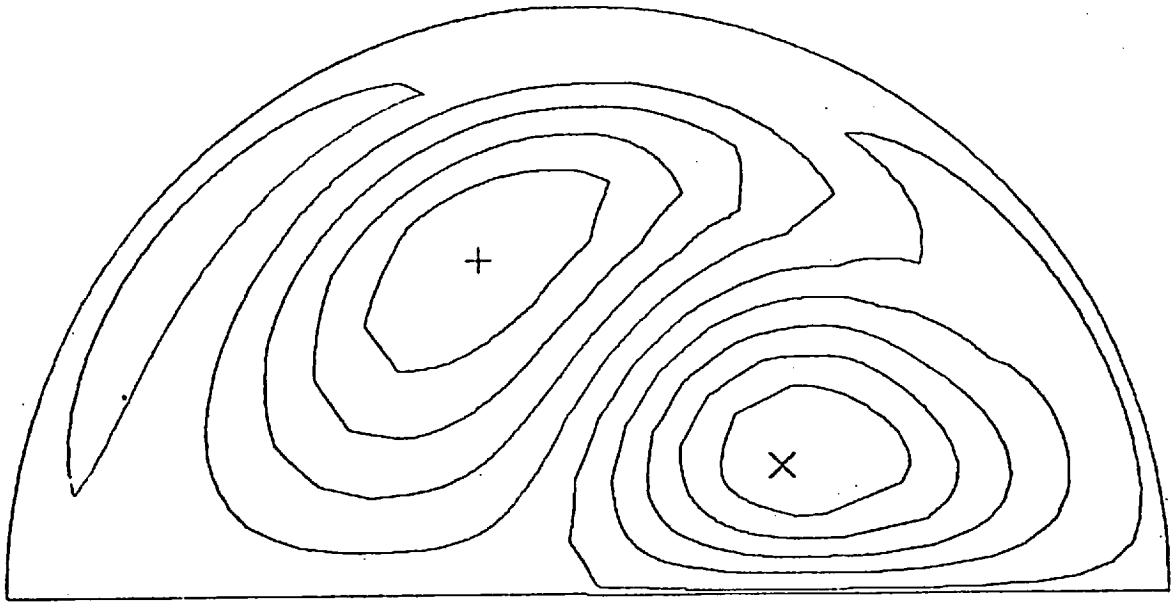


Fig. 7.43

$R_s = 150$, $D = 107$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.0393$

$\chi_{\min} = -0.0430$

$w_{\min} = 6.66$

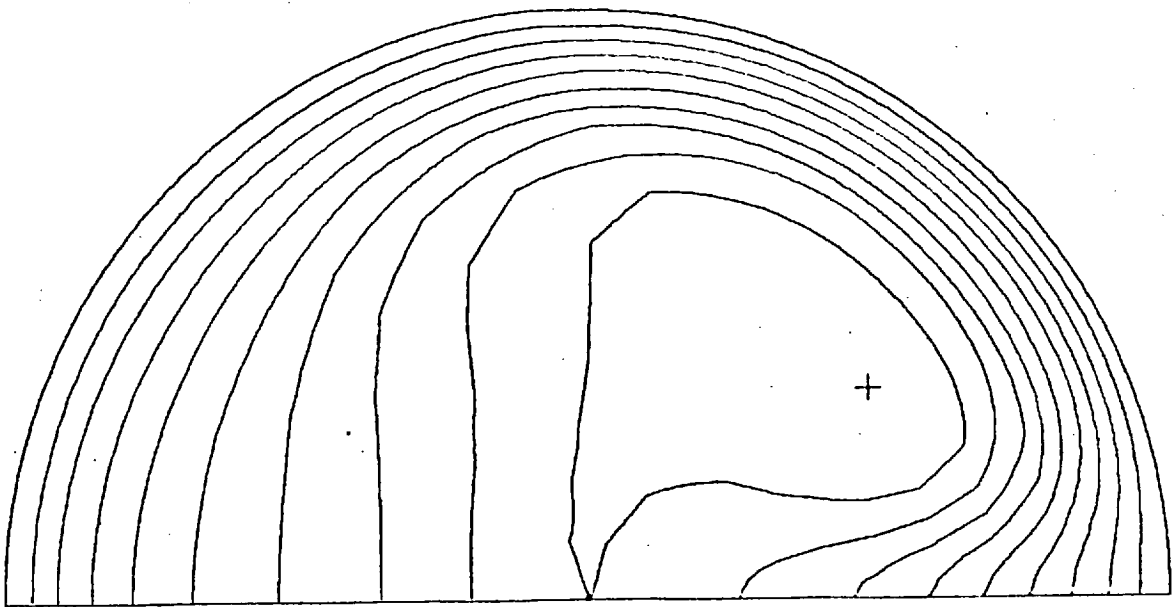
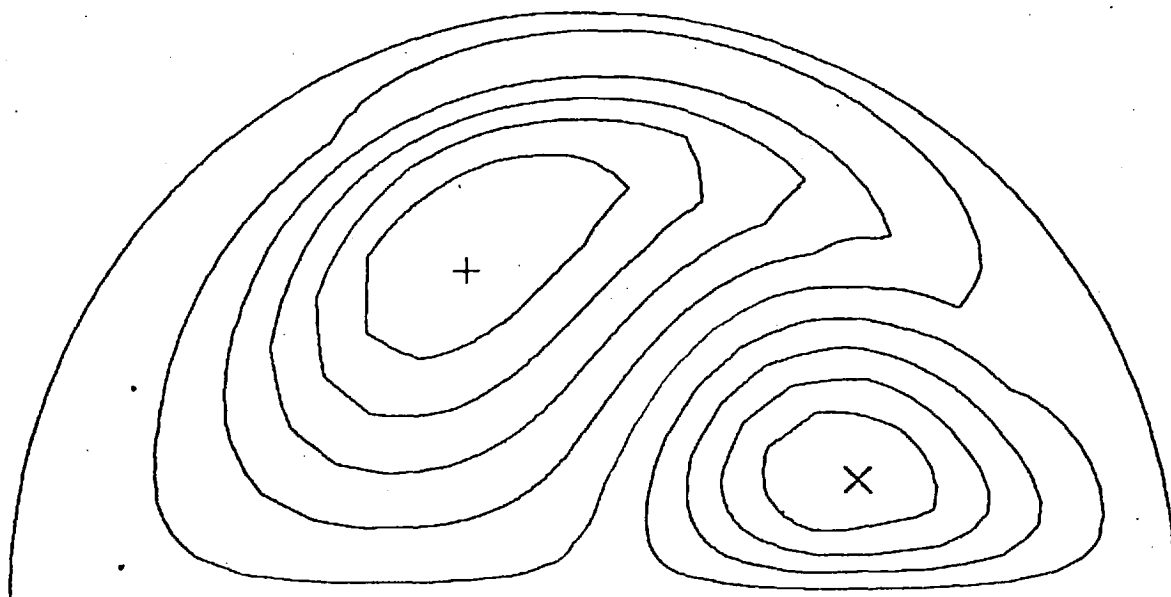


Fig. 7.44 $R_s = 150, D = 140$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.0642$

$\chi_{\min} = -0.0564$

$w_{\min} = 9.58$

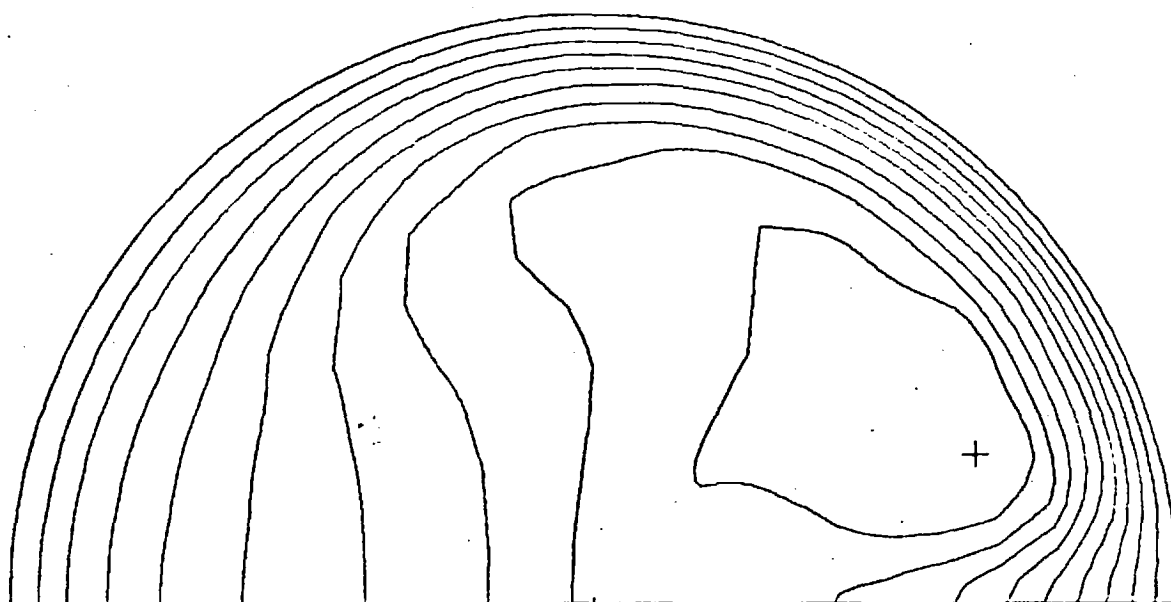
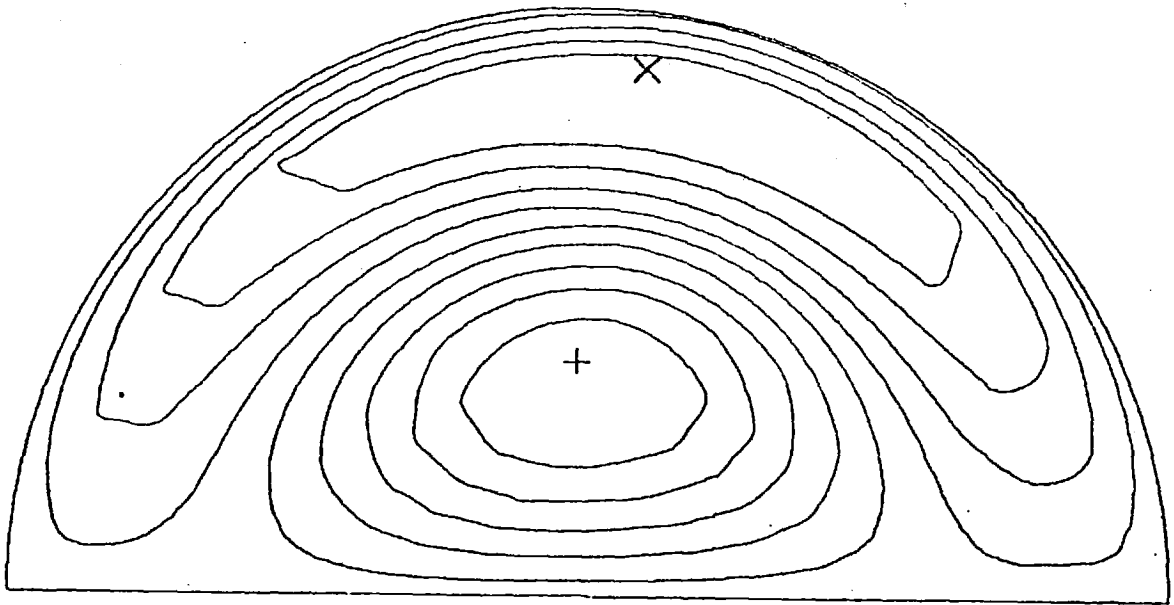


Fig. 7.45 $R_s = 150, D = 235$ (Lower branch)



NOF = 18

NR = 11

$\chi_{\text{omax}} = 0.0187$

$\chi_{\text{omin}} = -0.0148$

$w_{\text{im}} = 2.96$

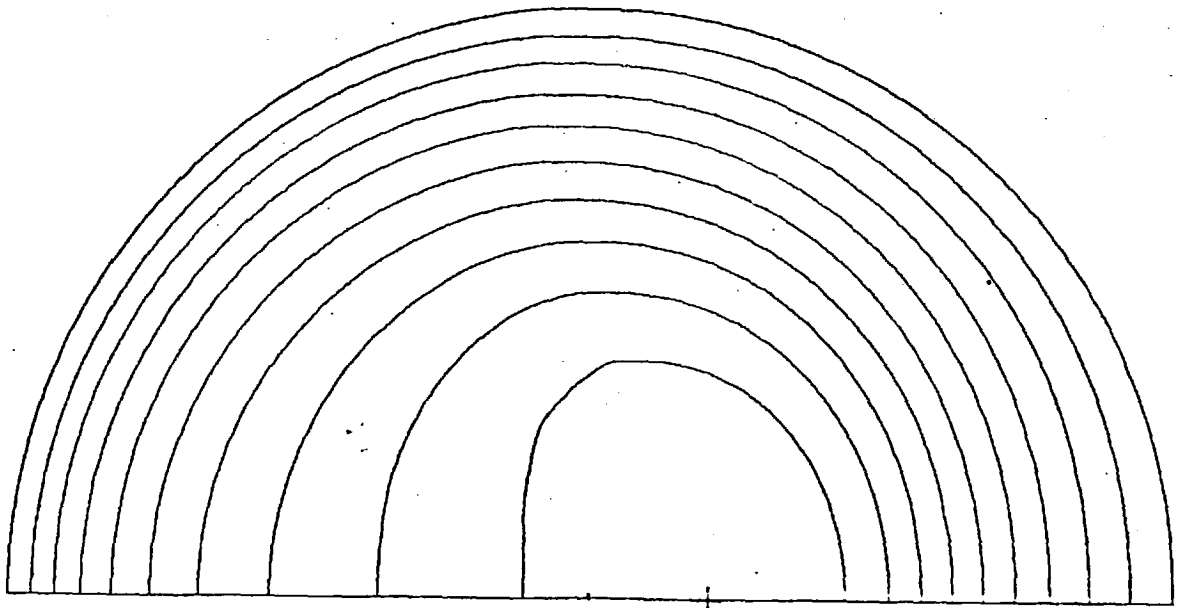
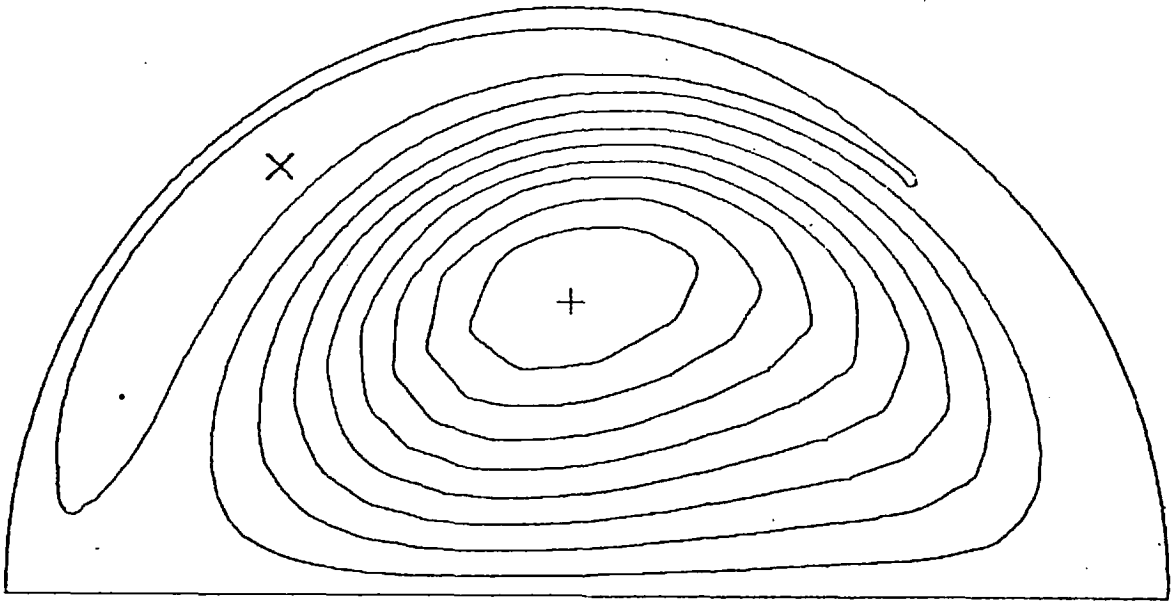


Fig. 7.46 $R_s = 150$, $D = 40$ (Upper branch)



NOF = 18

NR = 11

$\chi_{\max} = 0.0519$

$\chi_{\min} = -0.00761$

$w_{\min} = 5.41$

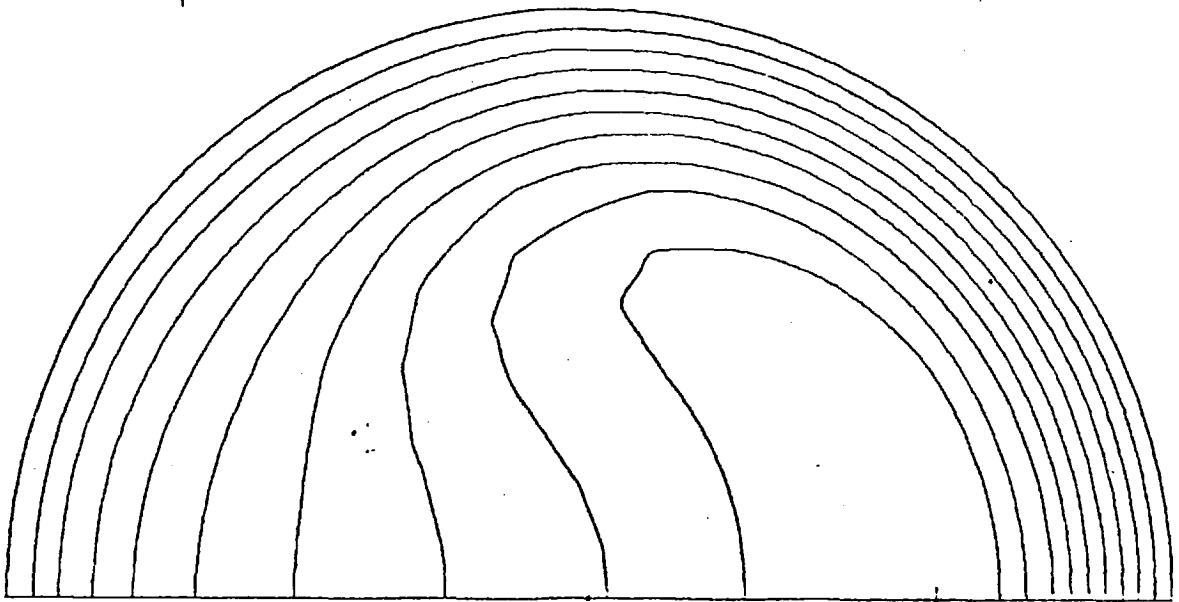
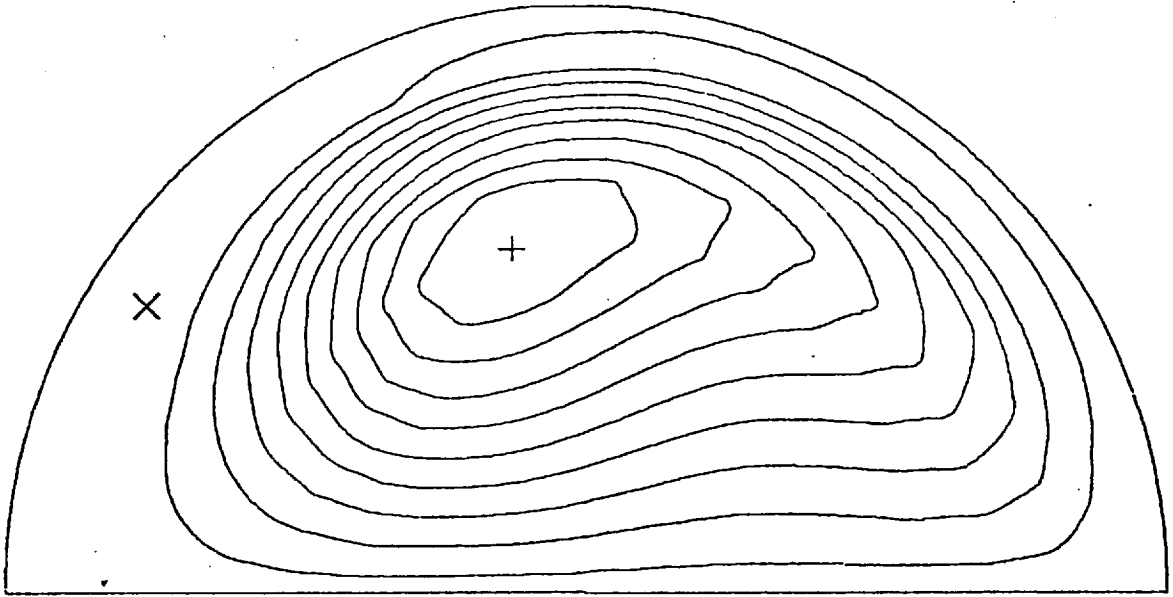


Fig. 7.47 $R_s = 150, D = 100$ (Upper branch)



NOF = 18

NR = 11

$\gamma_{\max} = 0.0778$

$\gamma_{\min} = -0.00371$

$w_{im} = 8.79$

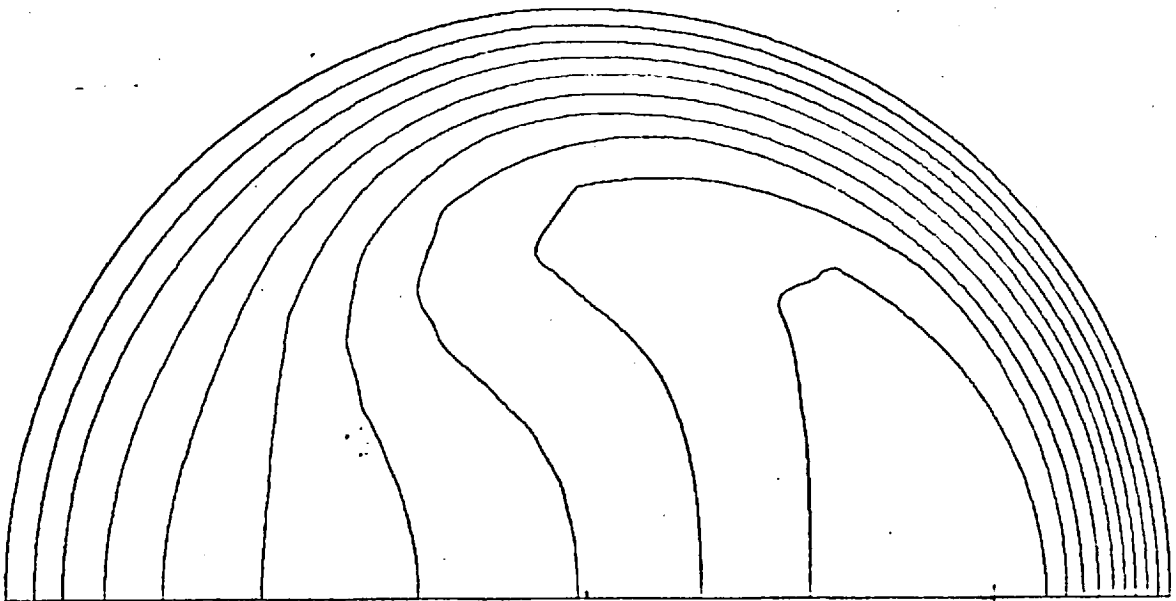


Fig. 7.48 $R_s = 150$, $D = 200$ (Upper branch)

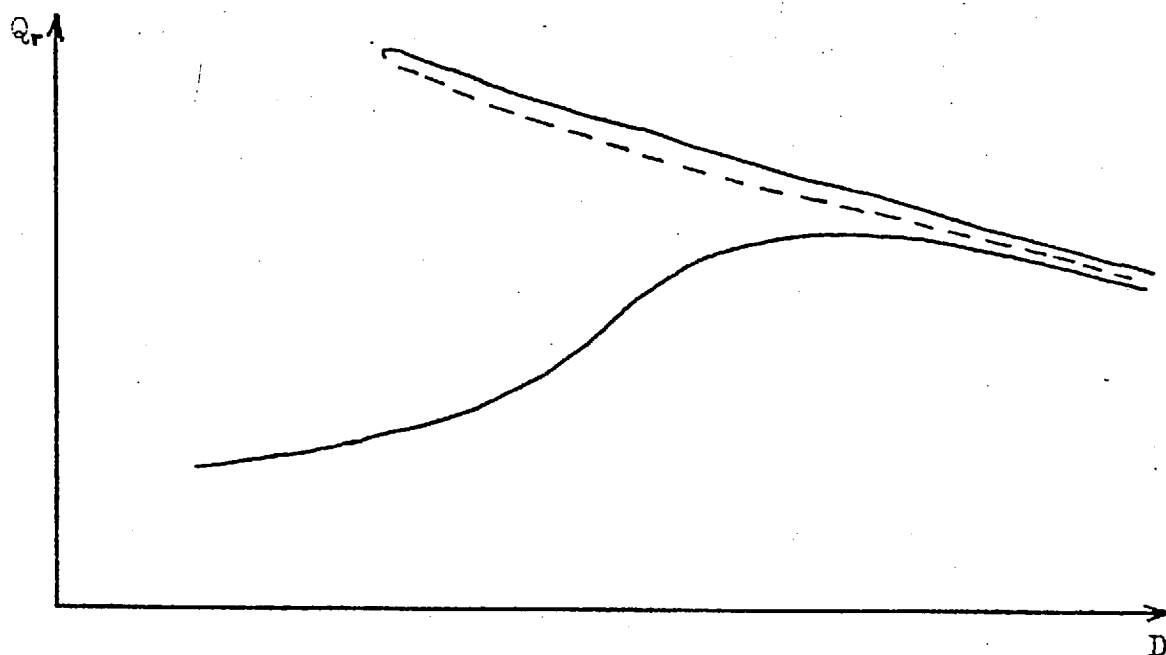


Fig. 8.1: Postulated shape of flux ratio curve for $R_s = 150$.

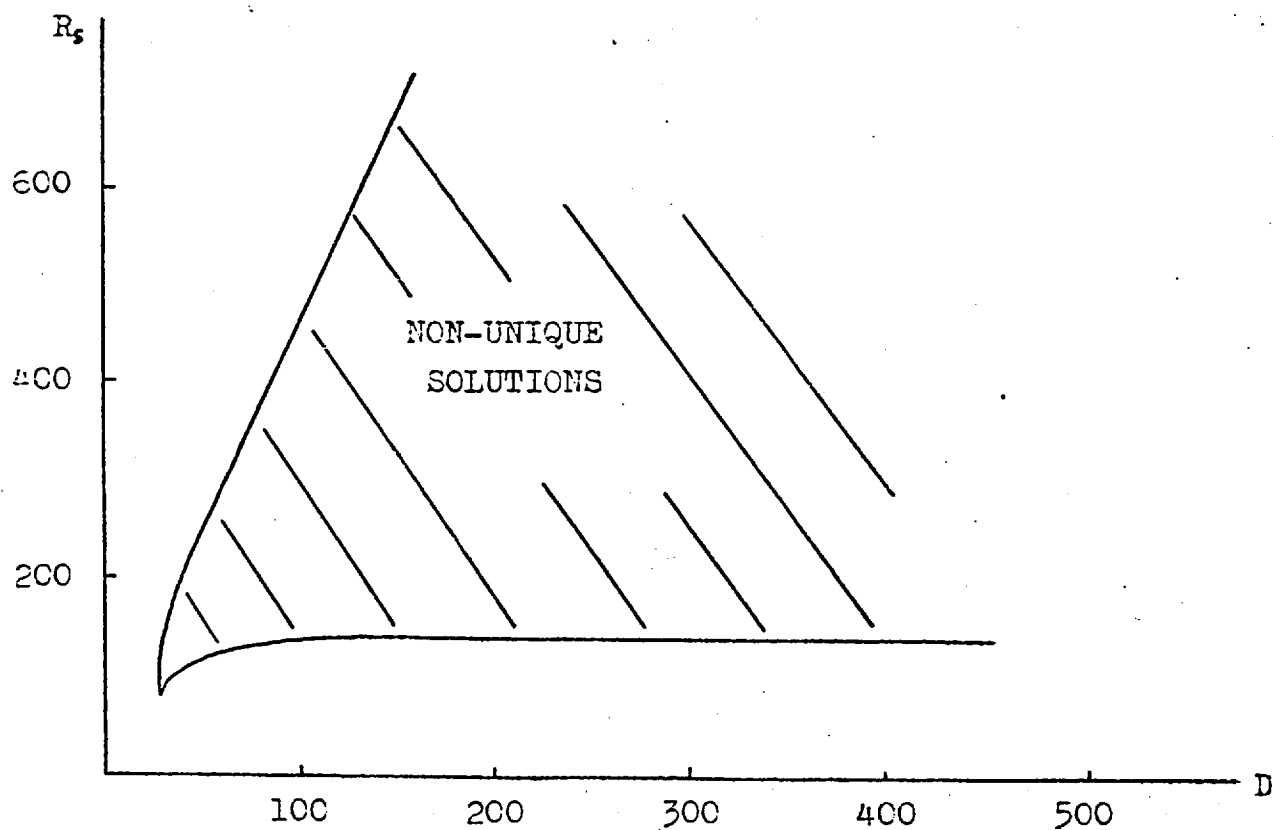


Fig. 8.2: Postulated region of non-unique solutions for (3.38), (3.40) and (7.4).

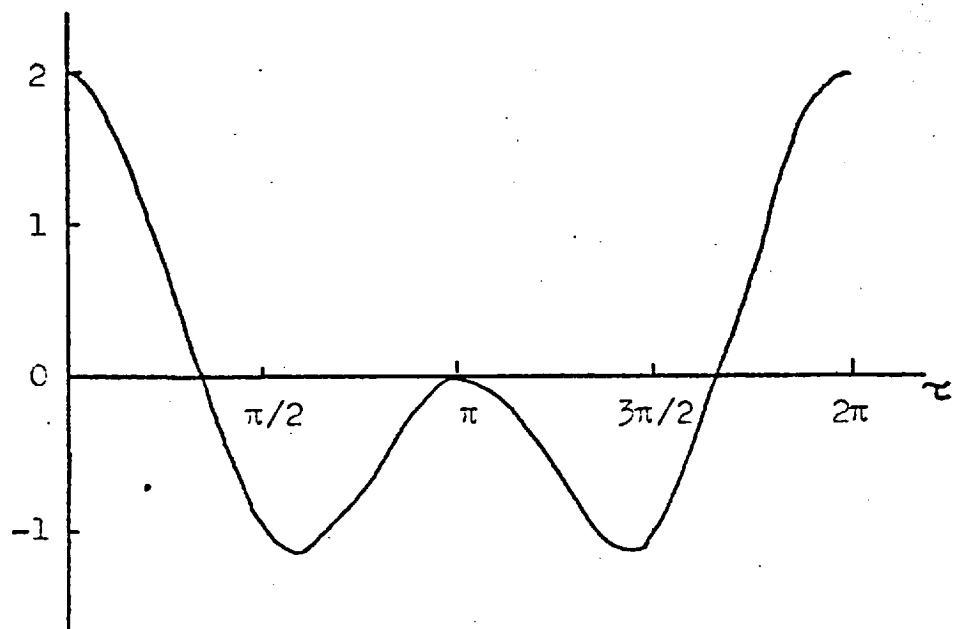


Fig. 9.1a: $\cos \tau + \cos 2\tau$

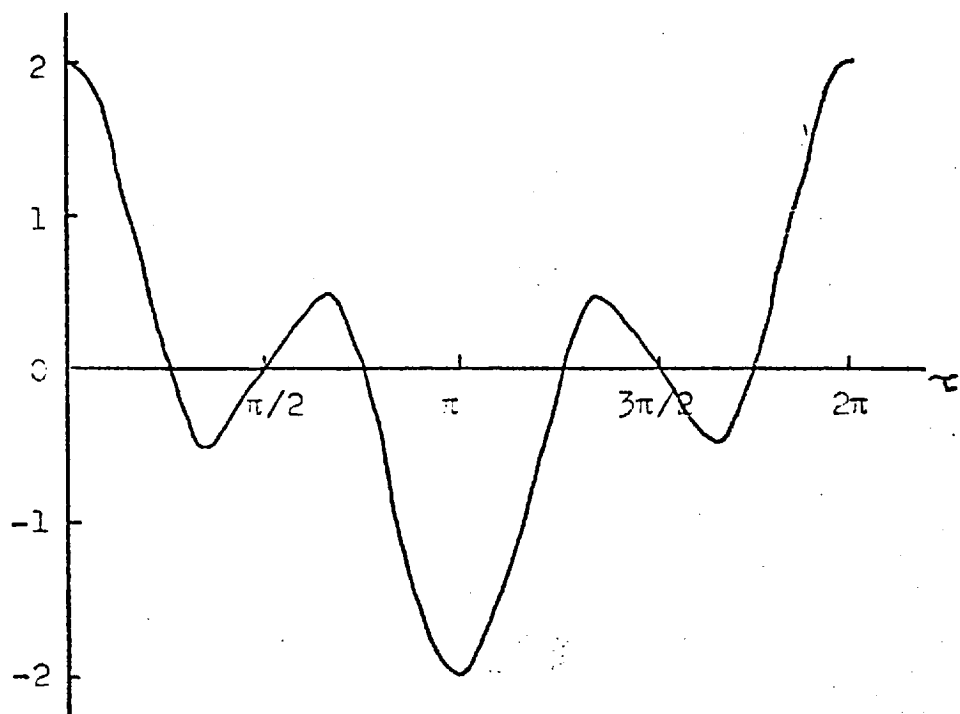


Fig. 9.1b: $\cos \tau + \cos 3\tau$

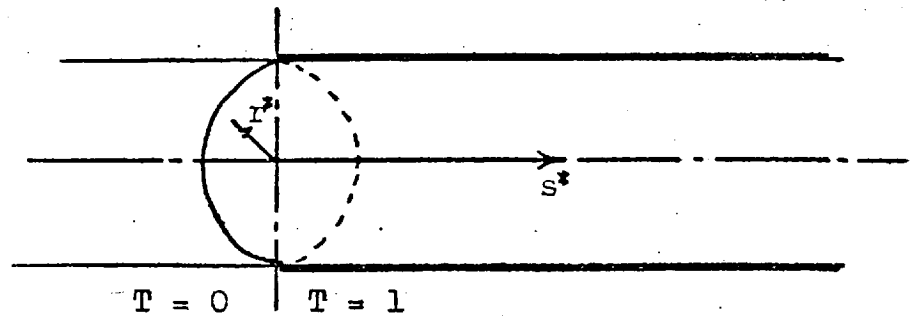


Fig. 10.1: The co-ordinate system for heat transfer in a straight pipe.

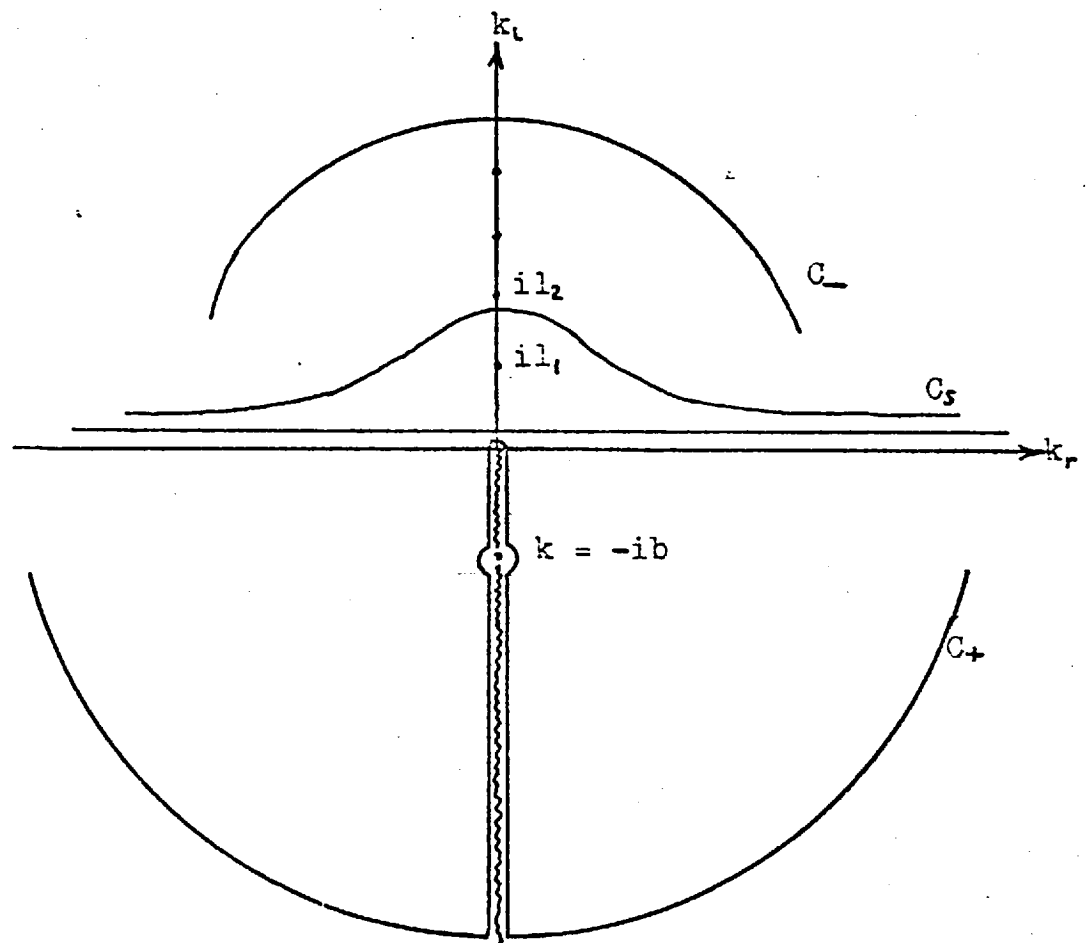
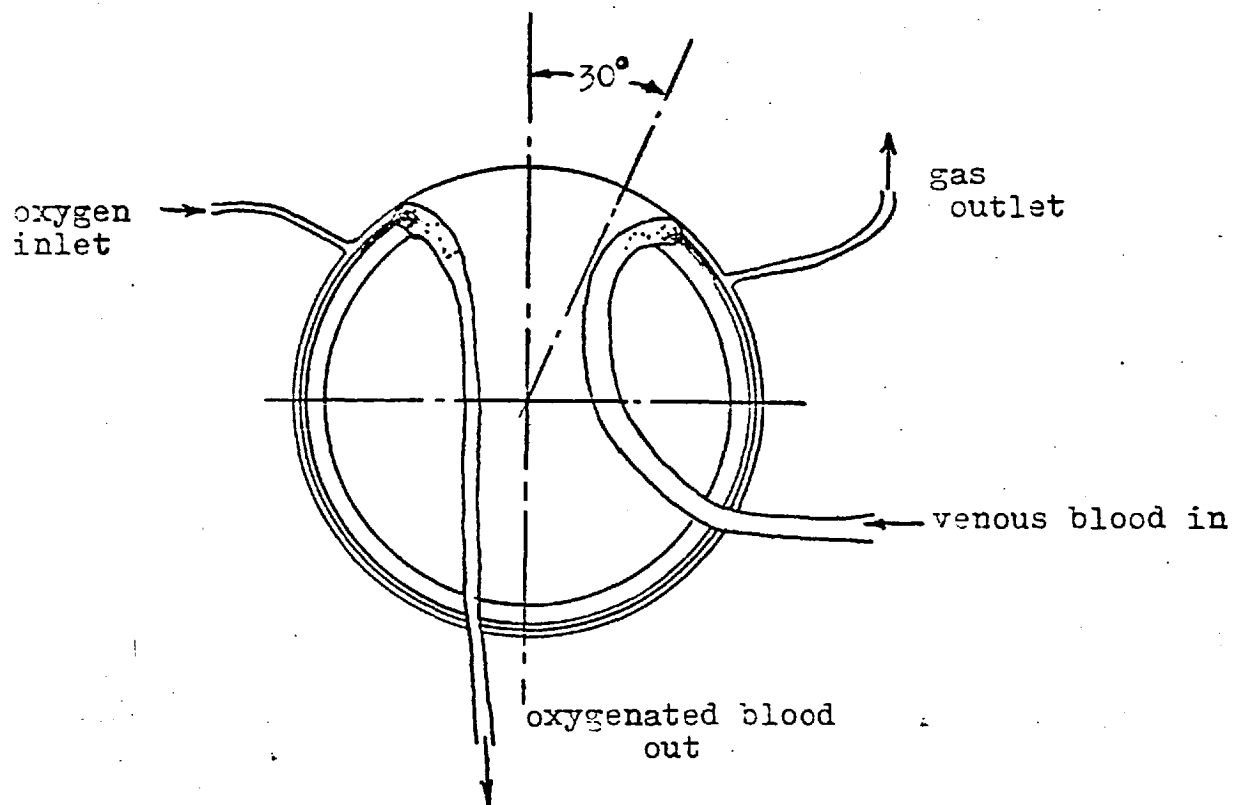


Fig. 10.2: The complex k plane.



Diameter of drum 30cm
Diameter of tubes 4mm
Max. blood flow rate 40ml/min

Fig. 11.1: Experimental apparatus of Melrose et al (1972).

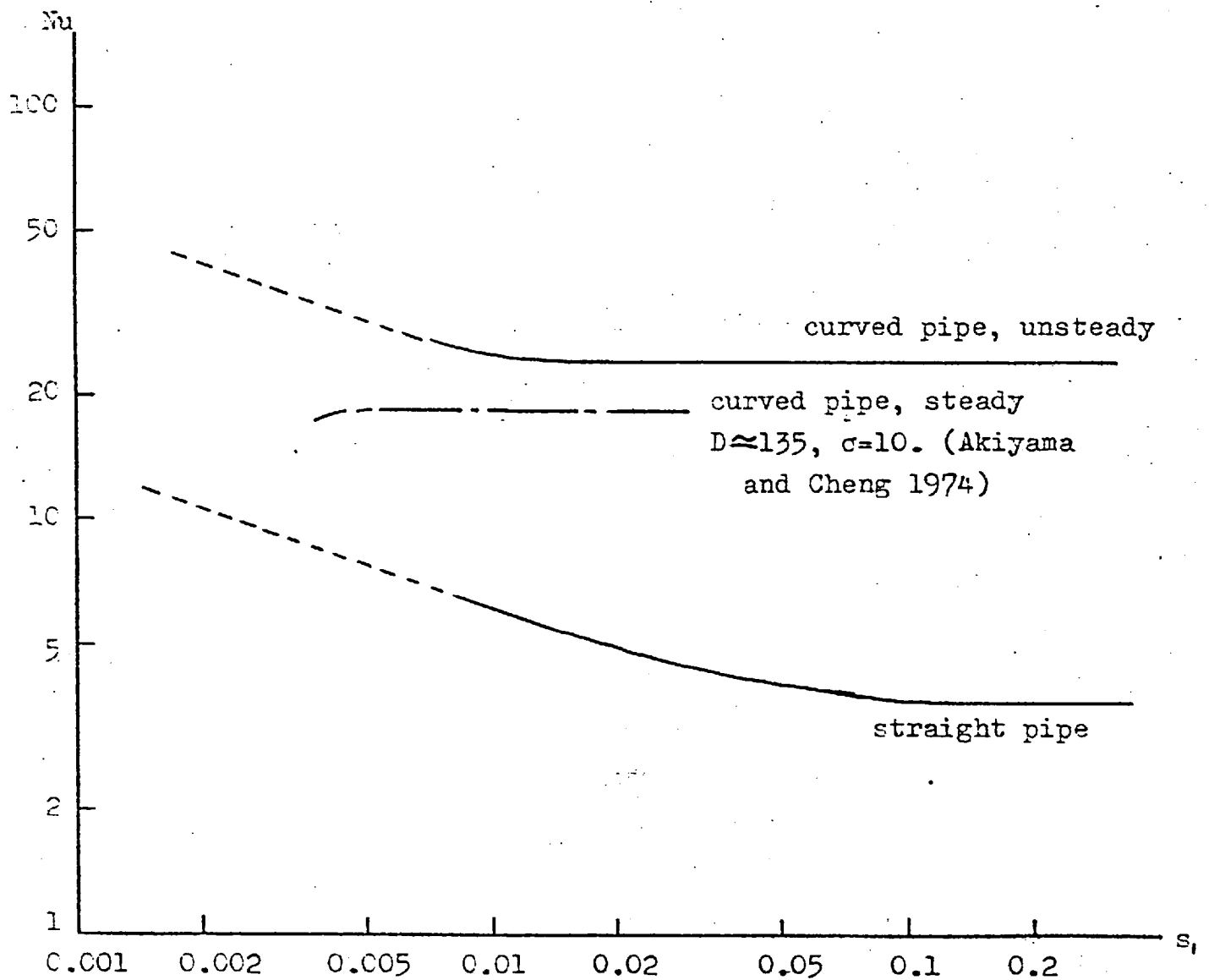


Fig. 11.2: Nusselt number for straight and curved pipes at same pressure gradient.

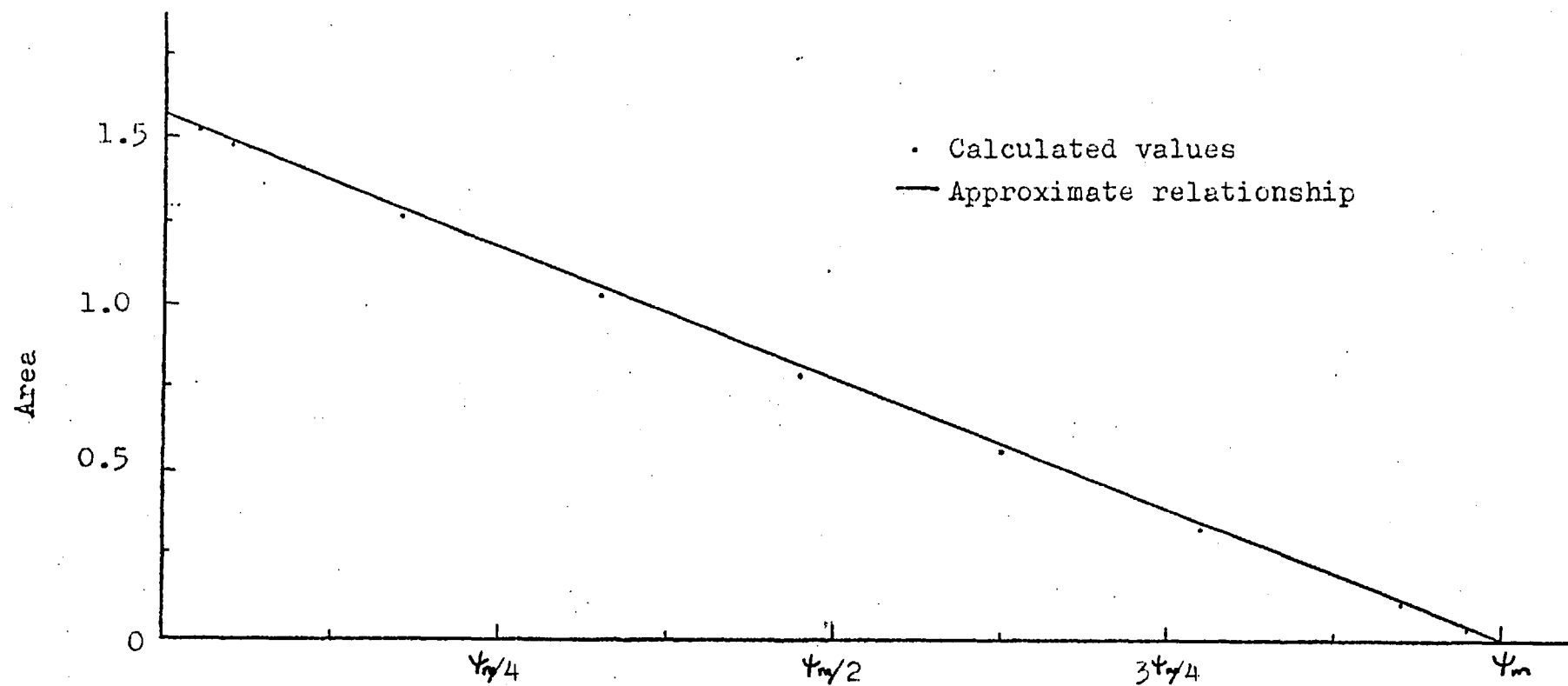


Fig. 11.3: Area enclosed by a streamline $\psi = \text{constant}$ as a function of the stream function.

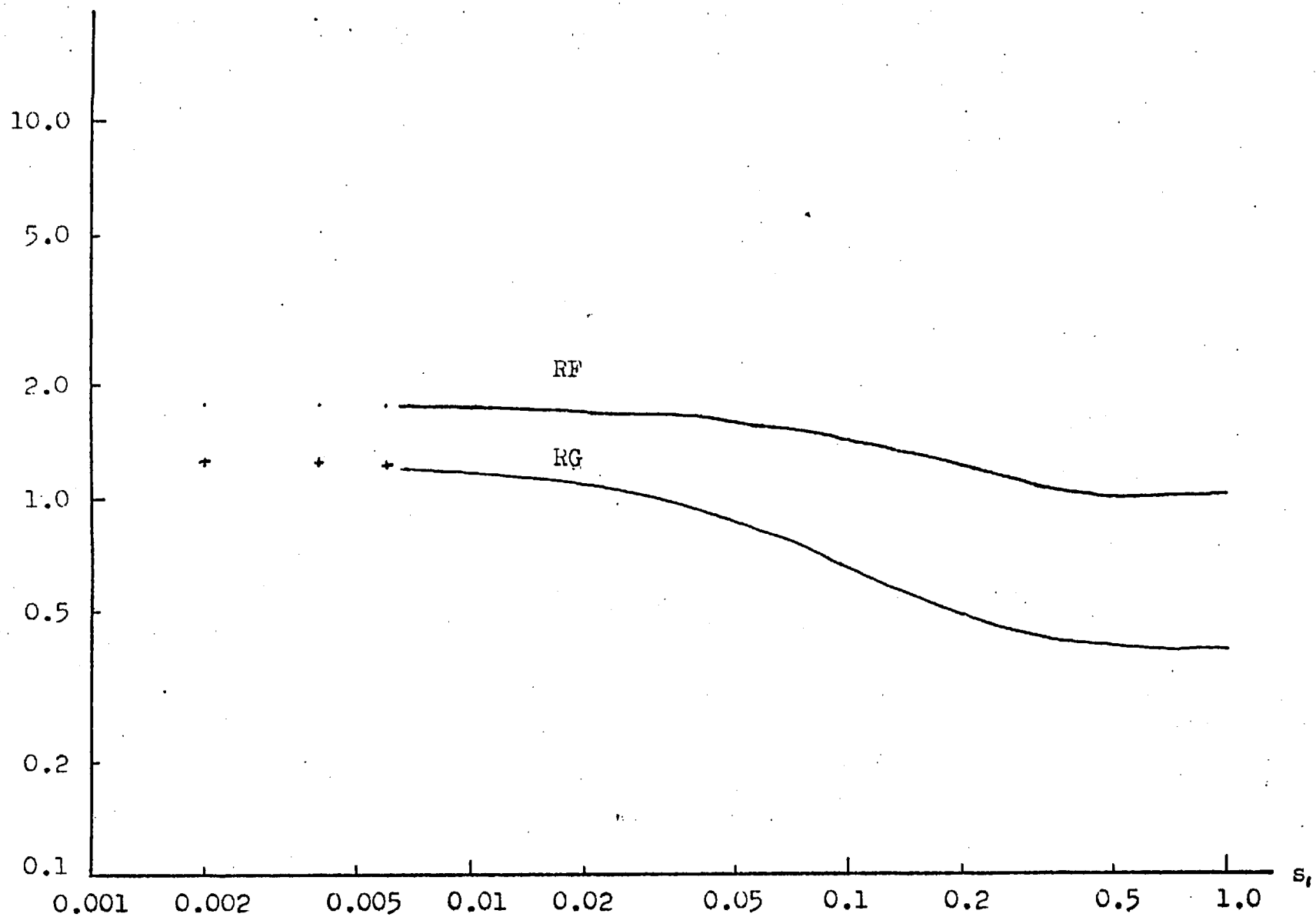


Fig. 11.4: Ratio of heat transfer in a curved pipe to that obtained in the same length of straight pipe.