

Harmonic-measure distribution functions for a class of multiply connected symmetrical slit domains

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The harmonic-measure distribution function, or h -function, of a planar domain $\Omega \subset \mathbb{C}$ with respect to a basepoint $z_0 \in \Omega$ is a signature that profiles the behaviour in Ω of a Brownian particle starting from z_0 . Explicit calculation of h -functions for a wide array of simply connected domains using conformal mapping techniques has allowed many rich connections to be made between the geometry of the domain and the behaviour of its h -function. Until now, almost all h -function computations have been confined to simply connected domains. In this work, we apply the theory of the Schottky-Klein prime function to explicitly compute the h -function of the doubly connected slit domain $\mathbb{C} \setminus ([-1/2, -1/6] \cup [1/6, 1/2])$. In view of the connection between the middle-thirds Cantor set and highly multiply connected symmetric slit domains, we then extend our methodology to explicitly construct the h -functions associated with symmetric slit domains of arbitrary even connectivity. To highlight both the versatility and generality of our results, we graph the h -functions associated with quadruply and octuply connected slit domains.

1 Introduction

This paper employs the Schottky-Klein prime function [8, 9, 10, 11] to construct analytical formulae for h -functions associated with a symmetric class of multiply connected slit domains.

First we explain the notion of an h -function. Given a domain Ω in the complex plane and a fixed basepoint z_0 in Ω , the harmonic-measure distribution function, or h -function for short, is the non-decreasing, piecewise smooth function $h : [0, \infty) \rightarrow [0, 1]$ defined by

$$h(r) = h_{\Omega, z_0}(r) := \omega_{\text{hm}}(z_0, \partial\Omega \cap \overline{B(z_0, r)}, \Omega), \quad (1)$$

where ω_{hm} denotes harmonic measure. Specifically, for each $r > 0$, the quantity $h(r)$ is the harmonic measure in Ω at z_0 of the part of the boundary $\partial\Omega$ that lies within distance r of z_0 . Thus $h(r)$ is equal to $u(z_0)$, where $u(z)$ is the solution to the following Dirichlet problem:

$$\begin{cases} \nabla^2 u = 0 & \text{on } \Omega; \\ u \equiv 1 & \text{on } \partial\Omega \cap \overline{B(z_0, r)}; \\ u \equiv 0 & \text{on } \partial\Omega \setminus \overline{B(z_0, r)}. \end{cases}$$

Kakutani established a deep connection between the Dirichlet problem and Brownian motion [20]. In our situation, consider a Brownian particle released from the basepoint z_0 and travelling in the domain Ω . Kakutani's result shows that the quantity $h(r) = u(z_0)$ is equal to the probability that the first place where the Brownian particle exits the domain Ω is within the 'capture circle' of radius r centered at the basepoint z_0 . One can think of the graph of a particular h -function as a signature that encodes information about the geometry of the boundary $\partial\Omega$ of the domain Ω it is associated with.

Harmonic-measure distribution functions were first defined in [27], motivated by a problem posed by Stephenson in [7], where the study of a closely related function $g_{\Omega, z_0}(r)$ was proposed. In [26], Tsuji proved an estimate for a function closely related to $g_{\Omega, z_0}(r)$. The theory and properties of the h -function are developed in the papers [27, 28, 6, 23, 29, 24, 4], and summarised in the recent survey paper [25]. It should also be noted that, in the case of a half-plane, the harmonic measure of a subset $E \in \mathbb{R}$ is known and given by the normalised 'angle of sight' i.e. the angle subtended by E at z_0 divided by π .

Using the conformal invariance of harmonic measure, given a conformal map between a domain Ω and a simpler domain whose h -function is already known, one can find an explicit formula for the h -function of Ω . A wide class of simply connected domains for which this has been done can be found in [25]. Toward calculating h -functions for doubly connected domains, Aarão *et al.* [1] used a variety of techniques to calculate the value of the h -function of the symmetric doubly connected domain $\mathbb{C} \setminus ([0, 1/3] \cup [2/3, 1])$ at a single point. However, for domains of higher connectivity, until now the only technique for finding the h -function explicitly was approximation through simulation of Brownian motion in the domain.

The present paper will demonstrate the relevance, and usefulness, of the Schottky-Klein prime function, referred to henceforth simply as the 'prime function', defined on the Schottky double of a multiply connected circular domain that is conformally equivalent to the slit domains whose h -functions will be constructed. All the necessary background on the prime function is given in a recent monograph [8] on the theory and applications of it to solving problems involving multiply

connected domains. Since our focus in this paper is on deploying these very general methods for the specific purpose of computing h -functions, herein we will state without proof the necessary function theoretic properties and refer readers to [8], and to the earlier review articles [9, 11] for additional background. On a practical level, it is important to emphasise that freely downloadable software resources now exist [19], based on numerical algorithms described in [15], for fast and efficient computation of the prime function. Therefore, for the purposes of the present article, we view the prime function as a readily computable special function. It is also worth mentioning that the prime function has recently been used to make progress constructing another function associated with multiply connected domains, the so-called ‘squeezing function’ [21], underlining the significance of the prime function as a valuable function theoretical tool.

To motivate the ensuing presentation, we briefly consider the h -function for a simply connected domain. Take $\mathbb{C} \setminus [0, 1]$, and fix the basepoint to be $z_0 = -1$. Using a standard explicit conformal mapping from $\mathbb{C} \setminus [0, 1]$ to a half-plane, it is straightforward to calculate the h -function in this simply connected situation to be

$$h(r) = \frac{2}{\pi} \arctan \left[\sqrt{2} \sqrt{\frac{1-r}{r-2}} \right]. \quad (2)$$

It is significant that exactly the same formula for this h -function associated with $\mathbb{C} \setminus [0, 1]$ can be obtained using the prime function and the methodology described in this paper (refer to Section 5 for more detail). The prime function relevant for this is a simple monomial; however, the general theory of the prime function extends naturally to higher connected domains [8] making it the natural choice for devising a generalised construction of h -functions associated with multiply connected slit domains.

In the current paper, we present a method to compute the h -function $h(r)$, at all radii r , of a multiply connected domain whose boundary consists of an even number of slits lying on the real line that are symmetric about the imaginary axis, such as the domain Ω_4 depicted in Figure 3. It must be emphasised that the prime function formulation works more generally, to *any* number of slits, but we restrict interest here to an *even* number of slits because of our interest in Cantor sets, as we now explain. For each of the slit domains we consider, we take a basepoint that is situated strictly to the left (or right) of all the slits *and* lies on the real axis. This critical positioning of the basepoint simplifies all calculations in this paper (work is already underway examining the effect on h -functions of different basepoint locations). We consider symmetric slit domains in particular owing to their simple geometry and their relationship with Cantor sets (we could, of course, consider other domains, but this will be left for future endeavours). Indeed, the current work is motivated in part by the open problem of finding the h -function of the infinitely connected domain $\Omega_c = \mathbb{C} \setminus \mathcal{C}$, where $\mathcal{C}$ is the middle-thirds Cantor set. Moreover, the current work opens the door to making progress in the context of multiply connected domains on many of the facets of the h -function theory, such as uniqueness, characterisation of the class of h -functions, asymptotic behaviour, and convergence of sequences of h -functions, that have already been investigated for simply connected domains [25].

To illustrate our general procedure for calculating h -functions for slit domains, let us provide an overview of the doubly connected case where the domain is defined to be $\Omega_2 = \mathbb{C} \setminus ([-1/2, -1/6] \cup [1/6, 1/2])$, and the basepoint is set at $z_0 = -3/2$. Figure 1 shows a schematic of this domain Ω_2 and a typical capture circle of radius r used to define its h -function. Figure 2 shows the graph of the h -function for Ω_2 with basepoint $z_0 = -3/2$. It should be noted that this h -function has a single horizontal ‘step’, in the sense that it is constant for r in the interval $[4/3, 5/3]$. In earlier work [1], several different methods are presented for finding this constant value of the h -function on $[4/3, 5/3]$. However, the methods in that paper do not extend to the radii $r \in (1, 4/3)$ and $r \in (5/3, 2)$, where $h(r)$ is not constant. For slit domains of connectivity $2N$, the associated graph of the h -function will feature $2N - 1$ steps, provided the basepoint is located strictly to the left (or right) of all the slits and lies on the real axis. This statement would not be true for an arbitrary positioning of basepoint, and emphasises the intrinsic nature of the h -function’s dependence on *both* the domain and basepoint for the slit domains considered in this paper.

The fundamental importance of the prime function is reflected here in its use for two distinct purposes. First, we use it to derive an explicit formula for the conformal map which transplants a bounded circular domain (a disc with $2N - 1$ discs deleted from its interior) to Ω_{2N} , the plane with $2N$ line segments symmetrically deleted from the real axis (see Section 3). Second, for carefully chosen sets E_r on the boundary of the circular domain linked to the definition of the h -function (see Figure 5), we use the prime function to construct a new analytic function from the circular domain to $\mathbb{C}$ whose imaginary part has boundary values 1 on E_r and 0 elsewhere. This procedure yields the solution to the Dirichlet problem in the circular domain, which pushes forward to the domain Ω_{2N} to give the value of the h -function at a single radius r . Section 5 outlines the method for producing the values of the entire h -function at all radii $r > 0$. Moreover, all functions constructed using the prime function admit the same functional form regardless of the connectivity of the domains being considered.

The paper is organised as follows. In Section 3, we derive the canonical conformal map from a bounded circular domain to a symmetric slit domain Ω_{2N} of arbitrary even connectivity $2N$. In Section 4, we derive analytical formulae for the step heights of the h -function associated with Ω_{2N} and a suitable real basepoint. We then proceed to derive analytical formulae for the actual h -function associated with these symmetric slit domains in Section 5. Throughout, we readily appeal to the function theory of the prime function, an overview of which is given in Section 2.

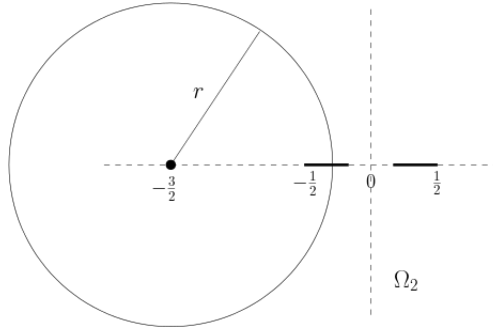


Figure 1: The domain $\Omega_2 = \mathbb{C} \setminus ([-1/2, -1/6] \cup [1/6, 1/2])$ with basepoint $z_0 = -3/2$, and a typical capture circle of radius r , with $1 < r < 4/3$, used in the definition of the h -function $h(r) = h_{\Omega_2, z_0}(r)$.

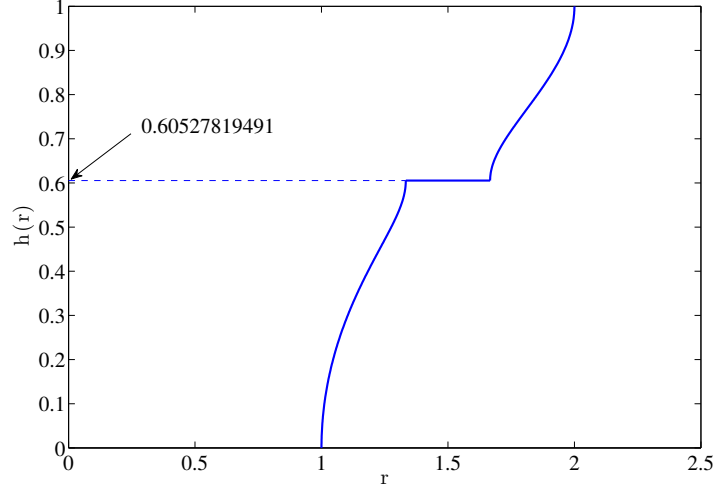


Figure 2: The h -function $h(r)$ plotted against the capture circle radius r , with basepoint $z_0 = -3/2$, for the double-slit domain Ω_2 of Figure 1.

2 The prime function

We first present some necessary prerequisites on the prime function required to establish all subsequent results in this paper. Products and quotients of the prime function evaluated at various points in $\mathbb{C}^2$ will be used in two ways: to find the conformal map from the bounded circular domain D_ζ to the slit domain Ω_{2N} (Section 3), and to construct certain harmonic functions on D_ζ (Section 5), which in turn lead to the explicit values of the h -function.

Let D_ζ be the bounded $2N$ -connected circular domain that is the exterior of $2N - 1$ discs each lying within the unit ζ -disc. Given such a domain D_ζ , the associated prime function is a function $\omega : \mathbb{C}^2 \rightarrow \mathbb{C}$ of two complex variables, usually denoted $\omega(\zeta, \alpha)$. The prime function $\omega(\zeta, \alpha)$ associated with D_ζ is an analytic and single-valued function of ζ , for all values $\zeta, \alpha \in D_\zeta$, having a simple zero when $\zeta = \alpha$.

The following important functional relation involves the Schwarz conjugate of the prime function and was first established by Crowdy & Marshall [12] but is justified more generally in [8]:

$$\overline{\omega}(1/\zeta, 1/\alpha) = -\frac{1}{\zeta\alpha}\omega(\zeta, \alpha). \quad (3)$$

Recall that the Schwarz conjugate of a function $f(\zeta, \alpha)$ is given by $\overline{f}(\zeta, \alpha) = \overline{f(\zeta, \overline{\alpha})}$.

An important property of the prime function is

$$\omega(\theta_k(\zeta), \alpha) = \Gamma_k(\zeta, \alpha)\omega(\zeta, \alpha), \quad k = 1, \dots, 2N - 1, \quad (4)$$

where

$$\Gamma_k(\zeta, \alpha) = \exp \left[-2\pi i \left(v_k(\zeta) - v_k(\alpha) + \frac{1}{2}\tau_{kk} \right) \right] \sqrt{\theta'_k(\zeta)}. \quad (5)$$

Here, $\{v_k(\zeta)\}_{j=k}^{2N-1}$ are the integrals of the first kind associated with D_ζ [8]. The integrals of the first kind are analytic multi-valued functions everywhere in D_ζ with the following properties:

$$\oint_{C_k} d[v_p(\zeta)] = \delta_{kp}, \quad v_k(\theta_p(\zeta)) - v_k(\zeta) = \tau_{kp}, \quad \text{Im}[v_k(\zeta)] = \begin{cases} 0 & \zeta \in C_0; \\ \gamma_{kp} & \zeta \in C_p. \end{cases} \quad (6)$$

Here, δ_{kp} denotes the Kronecker delta function, $\{\tau_{kp}\}_{k,p=1}^{2N-1}$ are purely imaginary constants and $\{\gamma_{kp}\}_{k,p=1}^{2N-1}$ are purely real constants, both sets of which can be calculated once the data $\{\delta_k, q_k\}_{k=1}^{2N-1}$ of D_ζ is known. The Möbius transformations $\{\theta_k(\zeta)\}_{k=0}^{2N-1}$ are given by

$$\theta_k(\zeta) = \delta_k + \frac{q_k^2 \zeta}{1 - \bar{\delta}_k \zeta}, \quad k = 1, \dots, 2N-1, \quad (7)$$

where δ_k is the centre and q_k is the radius of the k th boundary circle C_k of D_ζ (see Section 3). It then follows from (4) that

$$\frac{\omega(\theta_k(\zeta), \alpha_1)}{\omega(\theta_k(\zeta), \alpha_2)} = \beta_k(\alpha_1, \alpha_2) \frac{\omega(\zeta, \alpha_1)}{\omega(\zeta, \alpha_2)} \quad (8)$$

where

$$\beta_k(\alpha_1, \alpha_2) = \exp[2\pi i (v_k(\alpha_1) - v_k(\alpha_2))]. \quad (9)$$

We will also need formulae for a particular set of harmonic functions associated with bounded multiply connected circular domains. Crowdy & Marshall [14] showed how to construct the following $2N-1$ harmonic functions $\{\sigma_k(\zeta, \alpha)\}_{k=1}^{2N-1}$ associated with D_ζ :

$$\sigma_k(\zeta, \alpha) = \sum_{p=1}^{2N-1} a_{kp} \log \left| |\beta_p(\theta_p(1/\bar{\alpha}), \alpha)|^{1/2} \left(\frac{R_0(\zeta, \alpha)}{R_p(\zeta, \alpha)} \right)^{1/2} \right|, \quad k = 1, \dots, 2N-1, \quad (10)$$

where

$$a_{kp} = (\mathbf{A}^{-1})_{kp}, \quad A_{kp} = \frac{1}{2} \log |\beta_p(\theta_k(1/\bar{\alpha}), 1/\bar{\alpha})|, \quad (11)$$

and

$$R_0(\zeta, \alpha) = \frac{\omega(\zeta, \alpha) \bar{\omega}(1/\zeta, 1/\alpha)}{\omega(\zeta, 1/\bar{\alpha}) \bar{\omega}(1/\zeta, \bar{\alpha})}, \quad R_p(\zeta, \alpha) = \frac{\omega(\zeta, \alpha) \bar{\omega}(\bar{\theta}_p(1/\zeta), \bar{\theta}_p(1/\alpha))}{\omega(\zeta, \theta_p(1/\bar{\alpha})) \bar{\omega}(\bar{\theta}_p(1/\zeta), \bar{\alpha})}. \quad (12)$$

The functions $\{\sigma_k(\zeta, \bar{\zeta})\}_{k=1}^{2N-1}$ are harmonic everywhere in D_ζ and satisfy the boundary conditions

$$\sigma_k(\zeta, \bar{\zeta}) = \begin{cases} 1 & \zeta \in C_k; \\ 0 & \zeta \in C_j, \quad j \neq k; \\ 0 & \zeta \in C_0. \end{cases} \quad (13)$$

For more background on all the general properties just described, the reader is referred to [8].

3 The conformal mapping

We will now describe the multiply connected domains Ω_{2N} and D_ζ , setting some notation, and then proceed to construct the conformal mapping $f(\zeta)$ from the circular domain D_ζ onto Ω_{2N} , relying on the radial-slit conformal map presented in Crowdy & Marshall [13]. See Figure 3 for an illustration of the quadruply connected case, where $2N = 4$. Let the domain D_ζ lie in the ζ -plane and the domain Ω_{2N} lie in the z -plane.

3.1 Description of the domains Ω_{2N} and D_ζ

We start with the domain Ω_{2N} shown on the right in Figure 3 (where $2N = 4$). The domain Ω_{2N} is defined to be the complement in the complex plane of $2N$ intervals located on the real line. In particular, the point 0 lies in Ω_{2N} and not in any boundary component, and we set the basepoint in Ω_{2N} at $z_0 = -3/2$ (say). We number the $2N$ boundary components *from left to right* as $I_0, I_1, \dots, I_{2N-1}$. In this paper, we assume that $2N$ is even and that the domain Ω_{2N} is symmetric under reflection across the imaginary axis, indicated by a dashed line in Figure 3. We denote the endpoints of the intervals by $\ell_1, \ell_2, \dots, \ell_{2N}$, reading from left to right, so that $I_0 = [\ell_1, \ell_2]$, $I_1 = [\ell_3, \ell_4]$, $\dots$, $I_{2N-1} = [\ell_{4N-1}, \ell_{4N}]$. We also normalise so that the left endpoint of the left-most interval I_0 is at the point $\ell_1 = -1/2$ on the real axis, and the right endpoint of the right-most interval I_{2N-1} is at $\ell_{4N} = 1/2$. The h -functions of Ω_4 and Ω_8 with basepoint $z_0 = -3/2$ are calculated below and plotted in Figure 7 and Figure 8, respectively.

The circular domain D_ζ shown on the left in Figure 3 is defined to be a subset of the open unit disc, formed by deleting $2N-1$ closed disjoint sub-discs. For brevity, we omit mention of $2N$ from the symbol D_ζ . For convenience, and due to the

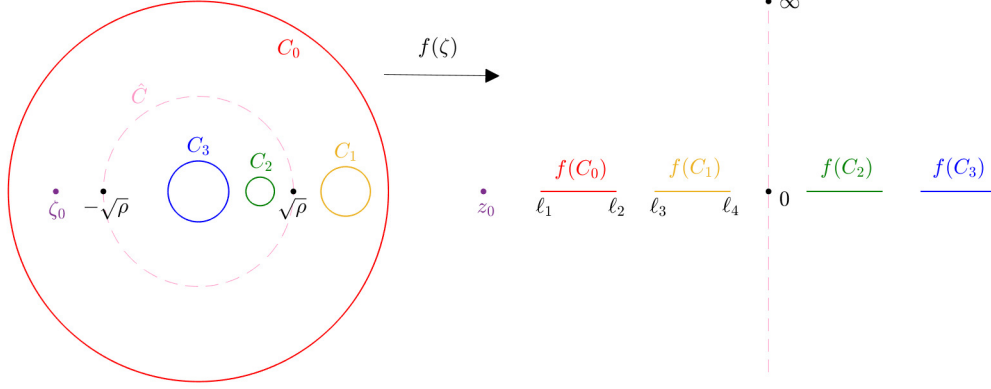


Figure 3: Schematic (not to scale) of the preimage bounded circular domain D_ζ in the quadruply connected setting (where $2N = 4$) (left) and the target symmetric slit domain Ω_4 (right) which is the image of D_ζ under the conformal map $f(\zeta)$. The basepoint $z_0 = -3/2$ in Ω_4 has preimage $\zeta_0 = f^{-1}(z_0)$ in D_ζ .

symmetry across the imaginary axis of Ω_{2N} , we take these sub-discs to be centred on the non-negative real axis, with one of them centred at 0. Two real degrees of freedom permitted by the Riemann mapping theorem justify centering one sub-disc at 0, while the other remaining rotational freedom allows the other sub-discs to be centred on the positive real axis.

The outer boundary circle of D_ζ is the unit circle; we denote it by C_0 . We number the remaining $2N - 1$ boundary circles *from right to left* as $C_1, C_2, \dots, C_{2N-1}$. The circles C_0 and C_{2N-1} are concentric, being centred at 0. The outer boundary circle C_0 meets the real line at -1 and 1 . We denote the points where the circle C_{2N-1} meets the real axis by $-\rho$ and ρ , with $0 < \rho < 1$. We denote the centre of the k th circle C_k by δ_k , and the radius of C_k by q_k , with $\delta_0 = \delta_{2N-1} = 0$ and $q_0 = 1$. We refer to the collection of $4N$ numbers $\delta_0, \delta_1, \dots, \delta_{2N-1}, q_0, q_1, \dots, q_{2N-1}$ as the ‘data’ of the circular domain D_ζ .

The preimage under the conformal map f of the line of symmetry is a circle $\hat{C}$ in D_ζ , concentric with the boundary circles C_0 and C_{2N-1} , and shown as a dashed circle in Figure 3. The symmetry of Ω_{2N} means that reflection in the circle $\hat{C}$ exchanges the boundary circles C_0 and C_{2N-1} . Hence, the radius of $\hat{C}$ is $\sqrt{\rho}$. Note that the reflection in the circle $\hat{C}$ is given by $\zeta \mapsto \rho/\bar{\zeta}$.

Geometrically (see Figure 3), the conformal map f sends the bounded domain D_ζ *interior* to the circle C_0 to the unbounded domain Ω_{2N} *exterior* to the interval $f(C_0)$. Moving from right to left in D_ζ through the circles $C_1, C_2, \dots, C_{2N-1}$ in order corresponds to moving from left to right in Ω_{2N} through the intervals $f(C_1), f(C_2), \dots, f(C_{2N-1})$ in that order. This is why the circles and intervals are numbered the way they are.

3.2 Conformal map from D_ζ to Ω_{2N}

Let $\omega(\zeta, \alpha)$ be the prime function associated with the bounded $2N$ -connected circular domain D_ζ . We will show that the conformal map $f(\zeta)$ from the circular domain D_ζ in the ζ -plane onto the domain Ω_{2N} in the z -plane, where Ω_{2N} is symmetric under reflection across the imaginary axis, is given by

$$f(\zeta) = -\frac{\omega(-1, -\sqrt{\rho})\omega(-1, -1/\sqrt{\rho})}{2\omega(-1, \sqrt{\rho})\omega(-1, 1/\sqrt{\rho})} \left[\frac{\omega(\zeta, \sqrt{\rho})\omega(\zeta, 1/\sqrt{\rho})}{\omega(\zeta, -\sqrt{\rho})\omega(\zeta, -1/\sqrt{\rho})} \right]. \quad (14)$$

This mapping $f(\zeta)$ is an instance of the radial slit map from D_ζ onto a domain Ω whose boundary components are line segments lying on rays emanating from 0, as given in Crowdy & Marshall [13]. In their formulation, the number of boundary components need not be even, and the boundary components need not be symmetrically arranged. The general form given in [13] is

$$f(\zeta) = A \left[\frac{\omega(\zeta, a)\omega(\zeta, 1/\bar{a})}{\omega(\zeta, b)\omega(\zeta, 1/\bar{b})} \right], \quad (15)$$

where $a \in D_\zeta$ is the preimage of $0 \in \Omega$, $b \in D_\zeta$ is the preimage of $\infty \in \Omega$, and A is a normalisation constant. In our case, Ω_{2N} has $2N$ boundary intervals which lie on two rays emanating from 0: the positive real axis and the negative real axis. The preimage of 0 is $a = \sqrt{\rho}$ and the preimage of ∞ is $b = -\sqrt{\rho}$.

It remains to determine the normalisation constant A in terms of ρ . Normalising such that $f(-1) = \ell_1 = -1/2$, it follows that

$$A = -\frac{\omega(-1, -\sqrt{\rho})\omega(-1, -1/\sqrt{\rho})}{2\omega(-1, \sqrt{\rho})\omega(-1, 1/\sqrt{\rho})} \quad (16)$$

This establishes the formula in (14) for the conformal map $f(\zeta)$.

Given the geometry of the domain Ω_{2N} , we want to deduce the geometry of the domain D_ζ . Specifically, given the endpoints $\ell_1, \dots, \ell_{4N}$ of the boundary intervals $I_0, \dots, I_{2N-1}$ of Ω_{2N} , we want to find the centres $\delta_0, \dots, \delta_{2N-1}$ and radii $q_0, \dots, q_{2N-1}$ of the boundary circles $C_0, \dots, C_{2N-1}$ of the domain D_ζ .

We may choose the data $\delta_0, \dots, \delta_{2N-1}$ and $q_0, \dots, q_{2N-1}$ so that the boundary circles $C_0, \dots, C_{2N-1}$ of D_ζ are mapped by $f(\zeta)$ onto the boundary intervals $I_0, \dots, I_{2N-1}$, respectively, with the points $\delta_k \pm q_k$ on the circle C_k being mapped to the endpoints of the interval I_k :

$$f(-1) = \ell_1, \quad f(1) = \ell_2; \quad f(\delta_k - q_k) = \ell_{2k+2}, \quad f(\delta_k + q_k) = \ell_{2k+1}, \quad k = 1, \dots, 2N-1. \quad (17)$$

It is then straightforward to determine the preimage ζ_0 of the basepoint z_0 *viz.* $\zeta_0 = f^{-1}(z_0)$. Note that we may exploit the symmetry of D_ζ across the circle $\widehat{C}$ to determine half of the data of D_ζ , with the remaining half of the data provided via the following formulae:

$$\delta'_k = \frac{\rho\delta_k}{\delta_k^2 - q_k^2}, \quad q'_k = \frac{\rho q_k}{\delta_k^2 - q_k^2}. \quad (18)$$

Here, δ'_k and q'_k denote the centre and radius of the reflection of circle C_k in $\widehat{C}$. Given our choice of normalisation and the symmetry of Ω_{2N} , and for general even $2N$, there will be $2N-1$ parameters to calculate in order to determine all the data of the preimage domain D_ζ .

4 Step heights of the h -function

Once the data found by enforcing the equations in (17) and (18) is determined for a particular target slit domain Ω_{2N} , the domain D_ζ is completely specified, and the conformal map (14) is entirely known, as shown in Section 3. We can then proceed to determine the step heights of the h -function. Let $z_r \in \mathbb{R}$ be the point of intersection of the capture circle of radius r with $\partial\Omega_{2N}$. By ‘step heights’, we mean the constant values of $h(r)$ when r is such that z_r lies in between a pair of slits. In addition, $h(r) = 0$ when r is such that z_r lies strictly to the left (or right) of all $2N$ slits, and $h(r) = 1$ when r is such that z_r lies strictly to the right (or left) of all $2N$ slits. In the case of a single slit, there will of course be no step height to determine.

In the h -function of the doubly connected domain Ω_2 with basepoint $z_0 = -3/2$, the single step occurs when $r \in [7/3, 8/3]$ and has step height 0.60528 to 5 decimal places (see Figure 2). Each value of $r \in [7/3, 8/3]$ corresponds to a capture circle that encloses all of the left-hand boundary interval I_0 of Ω_2 , but does not enclose any of the other right-hand boundary interval I_1 (see Figure 1). Thus the h -function is constant for $r \in [7/3, 8/3]$. Analogously, in the h -function of the quadruply connected domain Ω_4 with basepoint $z_0 = -3/2$, the three steps occur when $r \in [10/9, 11/9]$, $r \in [10/9, 11/9]$, and $r \in [10/9, 11/9]$, corresponding to capture circles that cross the gap between I_0 and I_1 , the gap between I_1 and I_2 , and the gap between I_2 and I_3 , respectively (see Figures 3 and 7).

In this section, we compute the step heights of the h -functions for Ω_{2N} with basepoint location $z_0 = -3/2$. That is, we compute the values of these h -functions at those values of r corresponding to capture circles that pass through gaps between the boundary intervals of Ω_{2N} . In Section 5, we compute the values of the h -functions for Ω_{2N} with basepoint $z_0 = -3/2$ for the remaining values of r , namely those r corresponding to capture circles that cross a boundary interval of Ω_{2N} . In that case, a more delicate analysis is required.

4.1 The doubly connected case

Let $\Omega_2 = \mathbb{C} \setminus ([-1/6, -1/2] \cup [1/6, 1/2])$. The step height in this case is equal to the harmonic measure at the basepoint $z_0 = -3/2$ of the left-hand boundary component $[-1/2, -1/6]$ of Ω_2 . This quantity is denoted $\omega_{\text{hm}}(-3/2, [-1/2, -1/6], \Omega_2)$. In terms of Brownian motion, this number is the probability that the first exit from the domain Ω_2 of a Brownian particle released at $z_0 = -3/2$ will be on $[-1/2, -1/6]$, rather than on $[1/6, 1/2]$. In terms of the h -function, this number is the constant height, for r in the range $4/3 \leq r \leq 5/3$, of the h -function $h(r) = h_{\Omega_2, -3/2}(r)$.

In this doubly connected case, the function theory simplifies pleasantly. Using the notation and results of Section 3, the preimage circular domain D_ζ is the concentric annulus $\rho \leq |\zeta| \leq 1$, with data $\delta_0 = \delta_1 = 0$, $q_0 = 1$, $q_1 = \rho$. The associated

prime function is given by

$$P(\zeta/\tau, \rho) = (1 - \zeta/\tau) \prod_{j=1}^{\infty} (1 - \rho^{2j} \zeta/\tau)(1 - \rho^{2j} \tau/\zeta) \quad (19)$$

up to a pre-multiplicative constant, and the conformal mapping transplanting D_ζ onto Ω_2 is

$$f(\zeta) = -\frac{P(1/\sqrt{\rho}, \rho)P(\sqrt{\rho}, \rho)}{2P(-1/\sqrt{\rho}, \rho)P(-\sqrt{\rho}, \rho)} \left[\frac{P(\zeta/\sqrt{\rho}, \rho)P(\zeta\sqrt{\rho}, \rho)}{P(-\zeta/\sqrt{\rho}, \rho)P(-\zeta\sqrt{\rho}, \rho)} \right], \quad (20)$$

where ρ is found so that $f(-1) = -1/2$, and $\zeta_0 = f^{-1}(-3/2)$.

The solution to the Dirichlet problem in the annulus D_ζ with boundary values 1 on C_0 and 0 on C_1 is the harmonic function $u(\zeta) = 1 - (\log |\zeta| / \log \rho)$. Using this fact, Kakutani's result linking the hitting probabilities of Brownian motion with the Dirichlet problem, and the conformal invariance of harmonic measure, we see that the quantity we seek is given by

$$\omega_{\text{hm}}(-3/2, [-1/2, -1/6], \Omega_2) = \omega_{\text{hm}}(\zeta_0, C_0, D_\zeta) = u(\zeta_0). \quad (21)$$

We obtain that the single step height in this doubly connected case is given by $u(\zeta_0) = 0.60528$ to 5 decimal places, which is consistent with the graph of the h -function in Figure 2.

4.2 The higher connectivity case

For a $2N$ -connected symmetric slit domain Ω_{2N} and a basepoint z_0 positioned on the real line strictly to the left (or right) of all $2N$ slits, calculating the $2N - 1$ step heights of the associated h -function can be performed as follows. Refer to Figure 3 for an illustration of the domains D_ζ and Ω_4 in the quadruply connected case.

The key idea is to define the set of functions $\{\mathcal{V}_k(\zeta)\}_{k=1}^{2N-1}$ to be particular linear combinations of the integrals of the first kind $\{v_k(\zeta)\}_{k=1}^{2N-1}$ associated with D_ζ :

$$\mathcal{V}_k(\zeta) = \sum_{p=1}^{2N-1} c_{kp} v_p(\zeta), \quad k = 1, \dots, 2N - 1. \quad (22)$$

Each $\mathcal{V}_k(\zeta)$ is analytic and multi-valued everywhere in D_ζ . By analogy with the doubly connected case, the real numbers $\{c_{kp}\}$ are found by solving a linear system of equations resulting from the enforcement of the following boundary conditions for each k :

$$\text{Im}[\mathcal{V}_k(\zeta)] = \begin{cases} 1 & \zeta \in C_k; \\ 0 & \zeta \in C_j, j \neq k; \\ 0 & \zeta \in C_0. \end{cases} \quad (23)$$

The height $h_{\Omega_{2N}}^{(m)}$ of the m th step of the h -function associated with Ω_{2N} is thus

$$h_{\Omega_{2N}}^{(m)} = 1 - \text{Im} \left[\sum_{p=m}^{2N-1} \mathcal{V}_p(\zeta_0) \right], \quad m = 1, \dots, 2N - 1. \quad (24)$$

Figure 4 shows plots of step height as a function of position along the slits for the domains Ω_2 , Ω_4 and Ω_8 , and Table 1 lists the numerical values for these step heights computed from (24). Motivated by the middle-thirds Cantor set, we take

$$\Omega_4 = \mathbb{C} \setminus ([-1/2, -7/18] \cup [-5/18, -1/6] \cup [1/6, 5/18] \cup [7/18, 1/2]). \quad (25)$$

Our definition of Ω_8 can be attained by removing the middle-third portions of each of the four intervals of Ω_4 . It should be noted that the values given in Table 1 are consistent with the result of a short calculation that shows that the height of the middle step in the h -function of the domain Ω_{2N} must decrease monotonically as $2N$ increases.

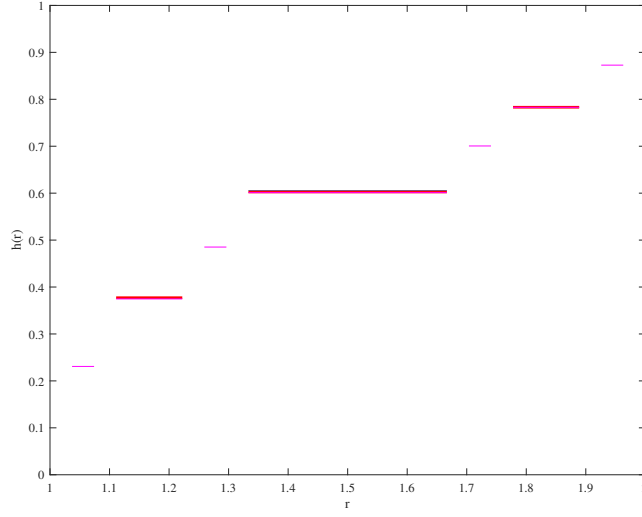


Figure 4: Step heights for the domains Ω_2 (blue), Ω_4 (red), and Ω_8 (magenta).

Table 1: Step heights (to 5 decimal places) for the three slit domains Ω_2 , Ω_4 and Ω_8 . Entries in the same row refer to step heights of the same step for different h -functions. For example, in the middle row the entries 0.60528, 0.60255 and 0.60117 are the values for $4/3 \leq r \leq 5/3$ of the h -functions $h_{\Omega_2}(r)$, $h_{\Omega_4}(r)$, and $h_{\Omega_8}(r)$, respectively.

Ω_2	Ω_4	Ω_8
		0.23082
	0.37725	0.37469
		0.48516
0.60528	0.60255	0.60117
		0.70057
	0.78307	0.78289
		0.87277

5 Construction of the full h -function for all values of r

In this section, we construct explicit analytical formulae for the non-constant components of the h -function for the domain Ω_{2N} and real basepoint z_0 lying to the left of all $2N$ slits, where $2N$ is an *arbitrary* positive even number. We will appeal to conformal maps of ‘Cayley-type’, defined in [8] and explained below, to build $2N$ ‘potential functions’. These potential functions describe the non-constant components of the h -function associated with the domain Ω_{2N} and basepoint z_0 . It is important to observe in this section that the Cayley-type maps $\mathcal{S}_k(\zeta, \tau, \bar{\tau})$ and the potential functions $\mathcal{W}_k(\zeta, \tau, \bar{\tau})$ admit the same functional forms regardless of the connectivity, another advantage afforded to us by the prime function.

5.1 Cayley-type maps

Let us first introduce what are referred to in [8] (see §5.10) as ‘Cayley-type mappings’. These mappings are such that bounded $2N$ -connected circular domains are transplanted onto half-planes with $2N - 1$ finite-length slits located on rays that emanate from the origin. Let τ and $\bar{\tau}$ be the two preimages on ∂D_ζ of the point z_τ , the point of intersection on one of the slits of the leading edge of the capture circle of radius r centred at z_0 ; this is illustrated in Figure 5. It should be noted that the dependence of the h -function $h_{\Omega_{2N}, z_0}(r)$ on the capture circle radius r is precisely through τ which is located on one of the $2N$ boundary circles of ∂D_ζ determined by the magnitude of r .

The classical Cayley map transplanting the interior of the unit ζ -disc onto the lower half-plane is given by the Möbius transformation

$$\mathcal{S}_0(\zeta, \tau, \bar{\tau}) = -\frac{1}{\tau} \frac{\zeta - \tau}{\zeta - \bar{\tau}} \quad (26)$$

for some $\tau \in C_0$ such that $\text{Im } \tau > 0$. The unit ζ -circle is mapped onto the real line. As ζ traverses the unit circle C_0 in an anti-clockwise direction, its image $z = \mathcal{S}_0(\zeta, \tau, \bar{\tau})$ traverses the real line from right to left. As ζ moves anti-clockwise through τ along the circle C_0 , its image $\mathcal{S}_0(\zeta, \tau, \bar{\tau})$ moves from right to left through 0 along the real line, and hence $\arg \mathcal{S}_0(\zeta, \tau, \bar{\tau})$ decreases by π .

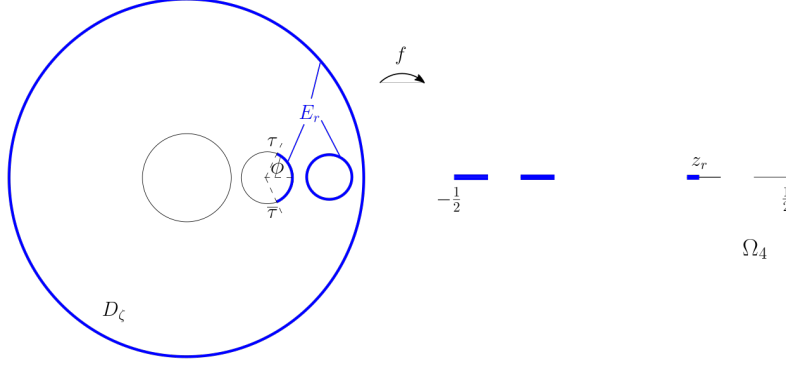


Figure 5: Illustration of the conformal mapping $z = f(\zeta)$ from D_ζ , the unit disc with 3 deleted sub-discs in the ζ -plane, to the region Ω_4 in the z -plane. The subset E_r of the boundary ∂D_ζ is the preimage of the portion of the boundary $\partial\Omega_4$ lying inside a capture circle of radius r centred at the basepoint $z_0 = -3/2$. Here, $z_r = f(\tau) = f(\bar{\tau})$.

We note that the classical Cayley map (26) has the following equivalent expression in terms of the prime function:

$$\mathcal{S}_0(\zeta, \tau, \bar{\tau}) = -\frac{1}{\tau} \frac{\omega(\zeta, \tau)}{\omega(\zeta, \bar{\tau})}. \quad (27)$$

Here, $\omega(\zeta, \tau) = \zeta - \tau$ for the unit ζ -disc. This equivalent expression (27) motivates the Cayley-type maps for multiply connected circular domains. Indeed, for a general prime function $\omega(\zeta, \tau)$, we will show that the map (27) transplants the interior of a bounded $2N$ -connected circular domain D_ζ onto the lower half-plane with $2N - 1$ finite-length slits located on rays that emanate from the origin, with the unit circle C_0 mapping onto the real line.

On the other hand, for $\tau \in C_k$ such that $\text{Im } \tau > 0$ and $k \neq 0$, we will show that the following Cayley-type map (28) transplants the interior of a bounded $2N$ -connected circular domain D_ζ onto the lower half-plane with $2N - 1$ finite-length slits located on rays that emanate from the origin, with circle C_k mapping onto the real line:

$$\mathcal{S}_k(\zeta, \tau, \bar{\tau}) = \frac{1}{\tau - \delta_k} \frac{\omega(\zeta, \tau)}{\omega(\zeta, \bar{\tau})}. \quad (28)$$

In this case, as ζ traverses the circle C_k in a clockwise direction, its image $z = \mathcal{S}_k(\zeta, \tau, \bar{\tau})$ traverses the real line from left to right. As ζ moves clockwise through τ along the circle C_k , its image $\mathcal{S}_k(\zeta, \tau, \bar{\tau})$ moves from left to right through 0 along the real line, and hence $\arg \mathcal{S}_0(\zeta, \tau, \bar{\tau})$ *increases* by π . A derivation of the pre-factors of (27) and (28) is given below.

Let us now discuss the two Cayley-type maps (27) and (28) in turn.

5.1.1 $\tau, \bar{\tau} \in C_0$

Here, $\tau, \bar{\tau} \in C_0$ correspond to an intersection point z_r of the capture circle on the left-most slit $I_0 \subset \partial\Omega_{2N}$. We require a Cayley-type map $\mathcal{S}_0(\zeta, a, b)$ which maps C_0 onto the real line and the other $2N - 1$ boundary circles of D_ζ onto $2N - 1$ finite-length slits in the lower half-plane lying on rays emanating from the origin. The reasoning for this becomes apparent when $\arg[\mathcal{S}_0(\zeta, a, b)]$ is calculated. Such a Cayley-type map will have the following functional form, where $a, b \in C_0$:

$$\mathcal{S}_0(\zeta, a, b) = A_0 \frac{\omega(\zeta, a)}{\omega(\zeta, b)}. \quad (29)$$

It is required to determine the constant pre-factor A_0 . Using the properties (3), (4) and (8) of the prime function, it can be shown that, for $a, b \in C_0$:

$$\overline{\mathcal{S}_0(\zeta, a, b)} = \begin{cases} \overline{A_0 b} \mathcal{S}_0(\zeta, a, b) & \zeta \in C_0; \\ \frac{\overline{A_0 a}}{A_0 b} e^{-2\pi i(v_j(a) - v_j(b))} \mathcal{S}_0(\zeta, a, b) & \zeta \in C_j, j \neq 0. \end{cases} \quad (30)$$

Setting $a = \bar{b} = \tau$, we require that

$$\frac{\overline{A_0\tau}}{A_0\tau} = 1. \quad (31)$$

That is, $A_0\tau = \lambda$ for some real constant λ . We may select $\lambda = -1$ so that

$$A_0 = -\frac{1}{\tau} \quad (32)$$

and the interior of D_ζ is mapped into the lower half-plane. Thus, we have established formula (27) for $2N$ -connected domains. Figure 6 illustrates a typical image domain after D_ζ has been conformally mapped by $\mathcal{S}_0(\zeta, \tau, \bar{\tau})$ in the quadruply connected setting.

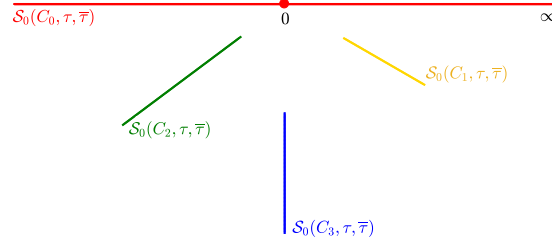


Figure 6: A typical quadruply connected image domain illustrating the action of the Cayley-type conformal mapping $z = \mathcal{S}_0(\zeta, \tau, \bar{\tau})$ on D_ζ . Here, $\tau \in C_0$, $\text{Im } \tau > 0$. The unit circle C_0 is mapped onto the real line, $\zeta = \tau$ is mapped to zero, $\zeta = \bar{\tau}$ is mapped to infinity, and the circles C_1 , C_2 and C_3 are mapped onto finite-length slits on rays emanating from zero in the lower half-plane.

5.1.2 $\tau, \bar{\tau} \in C_k$

In this case, $\tau, \bar{\tau} \in C_k$ correspond to an intersection point z_r of the capture circle on the k th slit $I_k \subset \partial\Omega_{2N}$, $k \neq 0$. We require a Cayley-type map $\mathcal{S}_k(\zeta, c, d)$ which maps C_k onto the real line and the other $2N - 1$ boundary circles of D_ζ onto $2N - 1$ finite-length slits in the lower half-plane lying on rays emanating from the origin. As before, the reasoning for this becomes apparent when $\arg[\mathcal{S}_k(\zeta, c, d)]$ is calculated. Such a Cayley-type map will have the following functional form, where $c, d \in C_k$:

$$\mathcal{S}_k(\zeta, c, d) = A_k \frac{\omega(\zeta, c)}{\omega(\zeta, d)}. \quad (33)$$

We must again determine the constant pre-factor A_k . Using the properties (3), (4) and (8) of the prime function, it can be shown that, for $c, d \in C_k$:

$$\overline{\mathcal{S}_k(\zeta, c, d)} = \begin{cases} \frac{\overline{A_k}(d - \delta_k)}{A_k(c - \delta_k)} e^{2\pi i(v_k(c) - v_k(d))} \mathcal{S}_k(\zeta, c, d) & \zeta \in C_0; \\ \frac{\overline{A_k}(d - \delta_k)}{A_k(c - \delta_k)} \mathcal{S}_k(\zeta, c, d) & \zeta \in C_k; \\ \frac{\overline{A_k}(d - \delta_k)}{A_k(c - \delta_k)} e^{-2\pi i(v_j(c) - v_j(d))} e^{2\pi i(v_k(c) - v_k(d))} \mathcal{S}_k(\zeta, c, d) & \zeta \in C_j, j \neq k, j \neq 0. \end{cases} \quad (34)$$

Setting $c = \bar{d} = \tau$, we require that

$$\frac{\overline{A_k}(\bar{\tau} - \delta_k)}{A_k(\tau - \delta_k)} = 1. \quad (35)$$

That is, $A_k(\tau - \delta_k) = \lambda$ for some real constant λ . We may select $\lambda = 1$ so that

$$A_k = \frac{1}{\tau - \delta_j} \quad (36)$$

and the interior of D_ζ is mapped into the lower half-plane. We have thus established formula (28) for N -connected domains.

5.2 Potential functions

Using the $2N$ Cayley-type maps $\{\mathcal{S}_k(\zeta, \tau, \bar{\tau})\}_{k=0}^{2N-1}$ introduced above, we are now equipped to construct the $2N$ potential functions $\{\mathcal{W}_k(\zeta, \tau, \bar{\tau})\}_{k=0}^{2N-1}$. We will show that the imaginary part $\text{Im}[\mathcal{W}_k(\zeta, \tau, \bar{\tau})]$ of each potential function is harmonic on D_ζ , constant on each boundary component C_j of D_ζ other than C_k , and piecewise constant on the k th boundary component C_k of D_ζ . With the $2N$ potential functions in hand, graphing of the h -function associated with Ω_{2N} and real basepoint z_0 lying strictly to the left of all $2N$ slits can be performed. The resulting h -function $h(r)$ is defined piecewise: it is constant on each of $2N + 1$ intervals in the r -axis, interspersed with $2N$ intervals where the h -function is non-constant and equal to the imaginary part of the relevant potential function.

5.2.1 $\tau, \bar{\tau} \in C_0$

With $A_0 = -1/\tau$ set, consider the logarithm of the Cayley-type map $\mathcal{S}_0(\zeta, \tau, \bar{\tau})$:

$$\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau}) = -\frac{1}{\pi} \log [\mathcal{S}_0(\zeta, \tau, \bar{\tau})] = -\frac{1}{\pi} \log \left[-\frac{1}{\tau} \frac{\omega(\zeta, \tau)}{\omega(\zeta, \bar{\tau})} \right]. \quad (37)$$

We select the branch of the logarithm such that $-\pi \leq \arg[\mathcal{S}_0(\zeta, \tau, \bar{\tau})] < \pi$. By (30) and our choice of A_0 , we have that

$$e^{2i \arg[\mathcal{S}_0(\zeta, \tau, \bar{\tau})]} = \frac{\mathcal{S}_0(\zeta, \tau, \bar{\tau})}{\overline{\mathcal{S}_0(\zeta, \tau, \bar{\tau})}} = \begin{cases} 1 & \zeta \in C_0; \\ e^{2\pi i(v_j(\tau) - v_j(\bar{\tau}))} & \zeta \in C_j, j \neq 0. \end{cases} \quad (38)$$

It then follows that

$$\text{Im}[\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau})] = \begin{cases} 0, 1 & \zeta \in C_0; \\ -(v_j(\tau) - v_j(\bar{\tau})) & \zeta \in C_j, j \neq 0. \end{cases} \quad (39)$$

Importantly, it should be observed here that the value of the harmonic function $\text{Im}[\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau})]$ *increases* by 1 as $\zeta = \tau$ is traversed in an anti-clockwise sense, which is as desired. It is also important to observe that the quantity $(v_j(\tau) - v_j(\bar{\tau}))$ is purely real.

In the above, we constructed the function $\text{Im}[\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau})]$ which has most of the properties we seek. It remains to adjust the boundary values of $\text{Im}[\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau})]$ on each circle C_j other than C_0 . We do so by finding another harmonic function to add to $\text{Im}[\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau})]$ to ensure that this combined function is zero if $\zeta \in C_j$, $j \neq 0$, while retaining the same values as $\text{Im}[\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau})]$ for $\zeta \in C_0$. To this end, consider the following potential function defined on D_ζ :

$$\mathcal{W}_0(\zeta, \tau, \bar{\tau}) = \widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau}) + i \sum_{j=1}^{2N-1} (v_j(\tau) - v_j(\bar{\tau})) \tilde{\sigma}_j(\zeta). \quad (40)$$

Here, $\{v_j(\zeta)\}_{j=1}^{2N-1}$ are the integrals of the first kind associated with D_ζ (6), and $\{\tilde{\sigma}_j(\zeta, \bar{\zeta})\}_{j=1}^{2N-1}$ are the analytic extensions of the harmonic functions $\{\sigma_j(\zeta, \bar{\zeta})\}_{j=1}^{2N-1}$ associated with D_ζ (10):

$$\sigma_j(\zeta, \bar{\zeta}) = \text{Re}[\tilde{\sigma}_j(\zeta)], \quad j = 1, \dots, 2N-1. \quad (41)$$

It is straightforward to verify that $\text{Im}[\mathcal{W}_0(\zeta, \tau, \bar{\tau})]$ has the desired boundary values.

The h -function in the case when $\tau, \bar{\tau} \in C_0$ is thus

$$h_0(\zeta_0, \tau, \bar{\tau}) = \text{Im}[\mathcal{W}_0(\zeta_0, \tau, \bar{\tau})] = \text{Im}[\widetilde{\mathcal{W}}_0(\zeta_0, \tau, \bar{\tau})] + \sum_{j=1}^{2N-1} (v_j(\tau) - v_j(\bar{\tau})) \sigma_j(\zeta_0, \bar{\zeta}_0). \quad (42)$$

5.2.2 $\tau, \bar{\tau} \in C_k$

With $A_k = 1/(\tau - \delta_k)$ set, consider the logarithm of the Cayley-type map $\mathcal{S}_k(\zeta, \tau, \bar{\tau})$:

$$\widetilde{\mathcal{W}}_k(\zeta, \tau, \bar{\tau}) = -\frac{1}{\pi} \log[\mathcal{S}_k(\zeta, \tau, \bar{\tau})] = -\frac{1}{\pi} \log \left[\frac{1}{\tau - \delta_k} \frac{\omega(\zeta, \tau)}{\omega(\zeta, \bar{\tau})} \right]. \quad (43)$$

The choice of logarithmic branch $-\pi \leq \arg[\widetilde{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})] < \pi$ is as before. By (34) and our choice of A_k , we have that

$$e^{2i \arg[\mathcal{S}_k(\zeta, \tau, \bar{\tau})]} = \frac{\mathcal{S}_k(\zeta, \tau, \bar{\tau})}{\overline{\mathcal{S}_k(\zeta, \tau, \bar{\tau})}} = \begin{cases} e^{-2\pi i(v_k(\tau) - v_k(\bar{\tau}))} & \zeta \in C_0; \\ 1 & \zeta \in C_k; \\ e^{2\pi i(v_j(\tau) - v_j(\bar{\tau}))} e^{-2\pi i(v_k(\tau) - v_k(\bar{\tau}))} & \zeta \in C_j, j \neq k, j \neq 0. \end{cases} \quad (44)$$

It then follows that

$$\operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})] = \begin{cases} v_k(\tau) - v_k(\bar{\tau}) & \zeta \in C_0; \\ 0, 1 & \zeta \in C_k; \\ -(v_j(\tau) - v_j(\bar{\tau})) + (v_k(\tau) - v_k(\bar{\tau})) & \zeta \in C_j, j \neq k, j \neq 0. \end{cases} \quad (45)$$

Importantly, it should be observed here that the value of the harmonic function $\operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})]$ *decreases* by 1 as $\zeta = \tau$ is traversed in an anti-clockwise sense, which is as desired.

We previously constructed the function $\operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})]$ which has most of the required properties. We now need to alter the boundary values of $\operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})]$ on each circle C_j , $j \neq k$, $j \neq 0$. We do so by finding another harmonic function to add to $\operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})]$ to ensure that this combined function has the correct distribution of zeroes and ones on the circles C_j , $j \neq k$, $j \neq 0$, while retaining the same values as $\operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau})]$ on C_k . Let us first zero-out all non-zero values on the circles C_j , $j \neq k$, and then add in the boundary values of one on the circles where they are required. To this end, consider the following potential function defined on D_ζ :

$$\widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau}) = \widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau}) - i(v_k(\tau) - v_k(\bar{\tau}))(1 - \widetilde{\sigma}_k(\zeta)) + i \sum_{j \neq k}^{2N-1} (v_j(\tau) - v_j(\bar{\tau})) \widetilde{\sigma}_j(\zeta). \quad (46)$$

Next, consider

$$\mathcal{W}_k(\zeta, \tau, \bar{\tau}) = \widehat{\mathcal{W}}_k(\zeta, \tau, \bar{\tau}) + i \sum_{n=0}^{k-1} \widetilde{\sigma}_n(\zeta) \quad (47)$$

where

$$\widetilde{\sigma}_0(\zeta) = 1 - \sum_{p=1}^{2N-1} \widetilde{\sigma}_p(\zeta). \quad (48)$$

It is straightforward to verify that $\operatorname{Im}[\mathcal{W}_k(\zeta, \tau, \bar{\tau})]$ has the desired boundary values.

The h -function in the case when $\tau, \bar{\tau} \in C_k$, for $k = 1, \dots, 2N-1$, is thus

$$h_k(\zeta_0, \tau, \bar{\tau}) = \operatorname{Im}[\mathcal{W}_k(\zeta_0, \tau, \bar{\tau})] = \operatorname{Im}[\widehat{\mathcal{W}}_k(\zeta_0, \tau, \bar{\tau})] + \sum_{n=0}^{k-1} \sigma_n(\zeta_0, \bar{\zeta}_0) \quad (49)$$

where

$$\sigma_0(\zeta, \bar{\zeta}) = 1 - \sum_{p=1}^{2N-1} \sigma_p(\zeta, \bar{\zeta}). \quad (50)$$

5.3 Examples

We now finally show how to calculate the h -function of the domain Ω_{2N} , with basepoint z_0 located on the real axis to the left of all the boundary intervals of Ω_{2N} , from the collection of functions $\{\operatorname{Im}[\mathcal{W}_k(\zeta, \tau, \bar{\tau})]\}_{k=0}^{2N-1}$. By way of illustration of the full versatility of our results, let us consider the three domains Ω_2 , Ω_4 and Ω_8 .

5.3.1 The simply connected case

Take $\mathbb{C} \setminus [0, 1]$ with basepoint $z_0 = -1$. In this case, the preimage circular domain D_ζ is just the unit disc. The unit disc can be mapped to $\mathbb{C} \setminus [0, 1]$ via the Joukowski map $f(\zeta) = \frac{1}{4}(\zeta + \frac{1}{\zeta}) + \frac{1}{2}$. The preimage of the basepoint is $\zeta_0 = -3 + 2\sqrt{2}$. The non-constant part of the h -function is for those r with $1 \leq r \leq 2$. For such an r , let z_r be the point where the capture circle of radius r centred at $z_0 = -1$ intersects the boundary $[0, 1]$ of $\mathbb{C} \setminus [0, 1]$. Let τ and $\bar{\tau}$ be the preimages of z_r , on the boundary of the unit disc, with $\tau = e^{i\phi}$ lying in the upper half-plane. Then the h -function of $\mathbb{C} \setminus [0, 1]$ with respect to $z_0 = -1$ is given by

$$h(r) = h_{\Omega_1, z_0}(r) = -\frac{1}{\pi} \log[\mathcal{S}_0(\zeta_0, \tau, \bar{\tau})] = -\frac{1}{\pi} \arg \left[\frac{\zeta_0 - e^{i\phi}}{1 - e^{i\phi}\zeta_0} \right]. \quad (51)$$

This formula (51) is equivalent to formula (2) presented previously.

5.3.2 The doubly connected case

Take $\Omega_2 = \mathbb{C} \setminus ([-1/2, -1/6] \cup [1/6, 1/2])$ and basepoint $z_0 = -3/2$. As mentioned previously, the function theory simplifies pleasantly in the doubly connected situation and here serves to illustrate our general theory for higher connected domains. The preimage circular domain D_ζ is the concentric annulus $\rho \leq |\zeta| \leq 1$, and the conformal mapping $f(\zeta)$ of D_ζ onto Ω_2 was given in (20) in terms of the function (19), the prime function of D_ζ up to a pre-multiplicative constant. Further, the single harmonic function (10) associated with the concentric annulus is

$$\sigma_1(\zeta, \bar{\zeta}) = \frac{\log |\zeta|}{\log \rho} \quad (52)$$

and the single integral of the first kind is

$$v_1(\zeta) = \frac{1}{2\pi i} \log \zeta. \quad (53)$$

Setting $\tau = e^{i\phi}$, $0 < \phi < \pi$, and selecting the logarithmic branch such that $-\pi \leq \arg[\mathcal{S}_0(\zeta, \tau, \bar{\tau})] < \pi$, we have

$$\widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau}) = -\frac{1}{\pi} \log[\mathcal{S}_0(\zeta, \tau, \bar{\tau})] = -\frac{1}{\pi} \log \left[-e^{i\phi} \frac{P(\zeta e^{-i\phi}, \rho)}{P(\zeta e^{i\phi}, \rho)} \right] \quad (54)$$

and the potential function

$$\mathcal{W}_0(\zeta, \tau, \bar{\tau}) = \widetilde{\mathcal{W}}_0(\zeta, \tau, \bar{\tau}) + i \frac{\phi \log \zeta}{\pi \log \rho}. \quad (55)$$

Hence

$$h_0(\zeta_0, \tau, \bar{\tau}) = \text{Im}[\mathcal{W}_0(\zeta_0, \tau, \bar{\tau})] = -\frac{1}{\pi} \left(\arg \left[-e^{i\phi} \frac{P(\zeta_0 e^{-i\phi}, \rho)}{P(\zeta_0 e^{i\phi}, \rho)} \right] - \phi \frac{\log |\zeta_0|}{\log \rho} \right). \quad (56)$$

Setting $\tau = \rho e^{i\phi}$, $0 < \phi < \pi$, with the logarithmic branch as above, we have

$$\widetilde{\mathcal{W}}_1(\zeta, \tau, \bar{\tau}) = -\frac{1}{\pi} \log[\mathcal{S}_1(\zeta, \tau, \bar{\tau})] = -\frac{1}{\pi} \log \left[e^{i\phi} \frac{P(\zeta e^{-i\phi}/\rho, \rho)}{P(\zeta e^{i\phi}/\rho, \rho)} \right] \quad (57)$$

and the potential function

$$\mathcal{W}_1(\zeta, \tau, \bar{\tau}) = \widetilde{\mathcal{W}}_1(\zeta, \tau, \bar{\tau}) + i \left(1 - \frac{\phi}{\pi} \right) \left(1 - \frac{\log \zeta}{\log \rho} \right). \quad (58)$$

Hence

$$h_1(\zeta_0, \tau, \bar{\tau}) = \text{Im}[\mathcal{W}_1(\zeta_0, \tau, \bar{\tau})] = -\frac{1}{\pi} \left(\arg \left[e^{i\phi} \frac{P(\zeta_0 e^{-i\phi}/\rho, \rho)}{P(\zeta_0 e^{i\phi}/\rho, \rho)} \right] - (\phi - \pi) \left(1 - \frac{\log |\zeta_0|}{\log \rho} \right) \right). \quad (59)$$

The h -function, written out in full, is thus

$$h(r) = h_{\Omega_2, z_0}(r) = \begin{cases} 0 & 0 \leq r \leq 1; \\ h_0(\zeta_0, \tau, \bar{\tau}) & 1 < r < 4/3, |\tau| = 1, \text{Im } \tau > 0; \\ 1 - \text{Im}[\mathcal{V}_1(\zeta_0)] & 4/3 \leq r \leq 5/3; \\ h_1(\zeta_0, \tau, \bar{\tau}) & 5/3 < r < 2, |\tau| = \rho, \text{Im } \tau > 0; \\ 1 & r \geq 2. \end{cases} \quad (60)$$

Here, ζ_0 denotes the preimage of the basepoint $z_0 = -3/2$. The graph of the h -function (60) was shown previously in Figure 2.

5.3.3 The higher connectivity case

We include here plots of the h -function for the domains Ω_4 and Ω_8 with basepoint $z_0 = -3/2$ in Figures 7 and 8, respectively. The plots display only the values of r between 1 and 2; these h -functions are identically zero for $r \leq 1$, and identically 1 for $r \geq 2$. For completeness, in the case of Ω_4 (where the boundary consists of four intervals of equal length), we write out the

h -function in full as follows:

$$h(r) = h_{\Omega_4, z_0}(r) = \begin{cases} 0 & 0 \leq r \leq 1; \\ h_0(\zeta_0, \tau, \bar{\tau}) & 1 < r < 10/9, |\tau| = 1, \text{Im } \tau > 0; \\ 1 - \text{Im}[\mathcal{V}_1(\zeta_0) + \mathcal{V}_2(\zeta_0) + \mathcal{V}_3(\zeta_0)] & 10/9 \leq r \leq 11/9; \\ h_1(\zeta_0, \tau, \bar{\tau}) & 11/9 < r < 4/3, |\tau - \delta_1| = q_1, \text{Im } \tau > 0; \\ 1 - \text{Im}[\mathcal{V}_2(\zeta_0) + \mathcal{V}_3(\zeta_0)] & 4/3 \leq r \leq 5/3; \\ h_2(\zeta_0, \tau, \bar{\tau}) & 5/3 < r < 16/9, |\tau - \delta_2| = q_2, \text{Im } \tau > 0; \\ 1 - \text{Im}[\mathcal{V}_3(\zeta_0)] & 16/9 \leq r \leq 17/9; \\ h_3(\zeta_0, \tau, \bar{\tau}) & 17/9 < r < 2, |\tau| = q_3, \text{Im } \tau > 0; \\ 1 & r \geq 2; \end{cases} \quad (61)$$

Again, ζ_0 denotes the pre-image of the basepoint $z_0 = -3/2$. Note the constant values of this h -function corresponding to its three steps; moreover, these constant values coincide with those previously obtained from calculations in Section 4. The formulae for the h -functions associated with Ω_8 , and more generally Ω_{2N} , are analogous to (61) when the basepoint z_0 is located strictly to the left of all the slits and lying on the real line. Indeed, we have shown above how to compute each of its $2N + 1$ constituent parts.

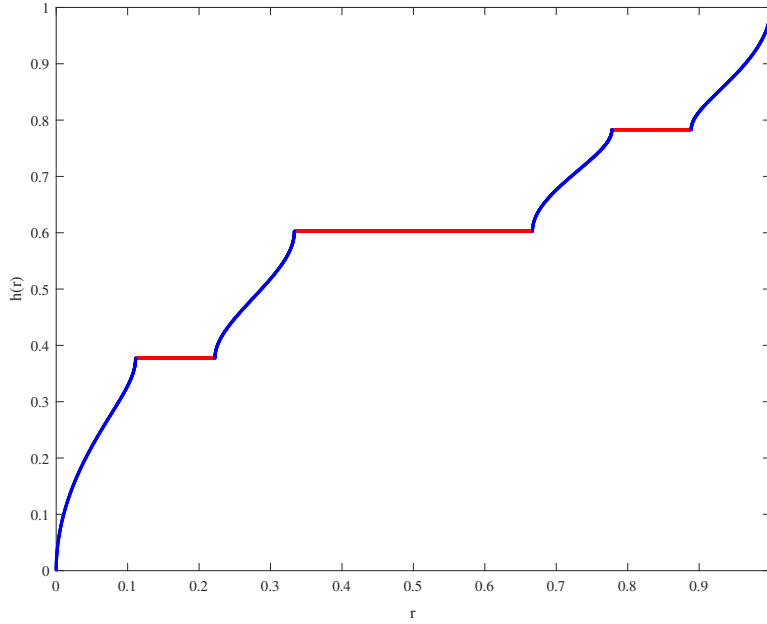


Figure 7: The h -function $h(r) = h_{\Omega_4, z_0}(r)$ plotted against the capture circle radius r , for the quadruply connected domain $\Omega_4 = \mathbb{C} \setminus ([-1/2, -7/18] \cup [-5/18, -1/6] \cup [1/6, 5/18] \cup [7/18, 1/2])$, with basepoint $z_0 = -3/2$. Here we plot only for $1 \leq r \leq 2$.

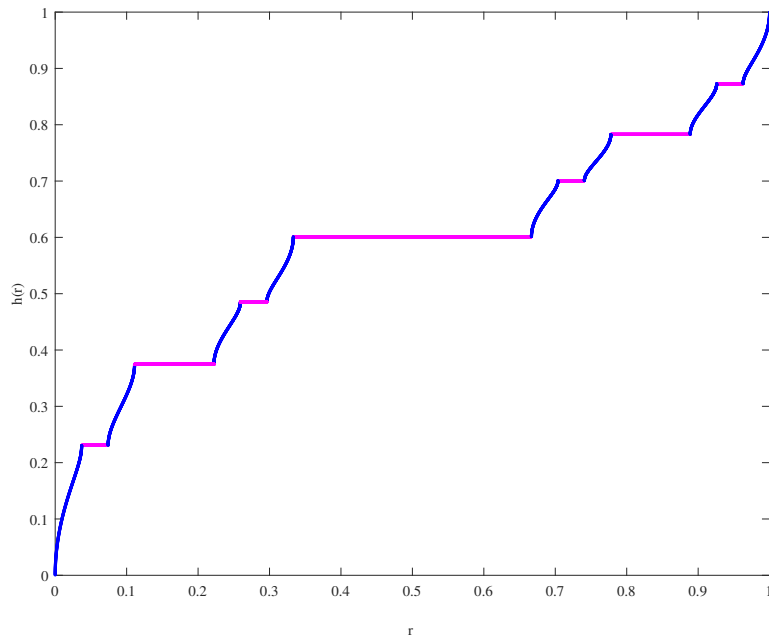


Figure 8: The h -function $h(r) = h_{\Omega_8, z_0}(r)$ plotted against the capture circle radius r , for the octuply connected domain $\Omega_8 = \mathbb{C} \setminus ([-1/2, -25/54] \cup [-23/54, -7/18] \cup [-5/18, -13/54] \cup [-11/54, -1/6] \cup [1/6, 11/54] \cup [13/54, 5/18] \cup [7/18, 23/54] \cup [25/54, 1/2])$, with basepoint $z_0 = -3/2$. Here we plot only for $1 \leq r \leq 2$.

6 Conclusions

Each planar domain Ω and specified basepoint z_0 in Ω uniquely determines a harmonic-measure distribution function $h(r) = h_{\Omega, z_0}(r)$. This function quantifies for each $r > 0$ the probability that a Brownian particle released from z_0 first hits the boundary of the domain Ω within distance r of the basepoint z_0 . Explicit calculation of h -functions is challenging. In this paper, we have shown how to construct explicit formulae for h -functions associated with symmetrical multiply connected slit domains, couched within the framework of the prime function, and making use of the conformal invariance of the underlying Dirichlet problems for each h -function. This work facilitates extending to general multiply connected domains the existing theory of h -functions, which until now has been almost entirely restricted to simply connected domains.

The main contribution of this work is the calculation of h -functions associated with a specific class of multiply connected domain, the slit domains being perhaps the easiest to treat during a first attempt of extending the theory to multiply connected domains. Previous studies calculating h -functions of doubly connected slit domains used different methods to those we have presented, for example, Monte Carlo simulation. The family of symmetrical slit domains we have considered also has the interesting limiting property of approaching the middle-thirds Cantor set, and it is of interest to calculate h -functions associated with such fractal domains.

The h -functions we have calculated have been for symmetrical slit domains with the basepoint located strictly to the left of all the slits and lying on the real line. An obvious and important question is how to adapt the techniques of this paper to cater for real basepoints positioned *between* any two slits. Non-real locations of the basepoint will likely only be amenable to numerical treatment. There is also the issue of relaxation of the left-right symmetry and the even connectivity of the target domains that we have so far imposed; for example, consider the calculation of h -functions for domains with an odd number of collinear slits of different lengths. Work in these directions is underway.

At the heart of this paper lies the function theory of the prime function which has been shown increasingly in recent years to be a useful object when deriving solutions to various problems involving multiply connected planar domains [8]. All the major results in this paper are explicitly given in terms of this special computable function, and thus pertain to *any* symmetric slit domain of arbitrary even finite connectivity. Further, the restriction to even connectivity can be removed by a suitable modification of our method, and our method can be adapted to geometries of boundary components other than collinear line segments. The prime function has facilitated treating both the multiply connected nature of the slit domains and the construction of general formulae for the h -functions associated with these domains, regardless of the connectivity. We have also highlighted the important constructive role played by conformal maps of Cayley-type, in particular when selecting an appropriate choice of logarithmic branch to render the h -functions single-valued. It is also evident from our explicit

construction that the analytic extensions of the harmonic functions from which the h -function is derived generically have logarithmic singularities at the point where the capture circle intersects the domain boundary, a property which is made clear by the prime function having a simple zero in its second argument.

This work has established a robust methodology for constructing h -functions for other types of planar multiply connected domains. It is clear from our explicit construction that the analytic extensions of the harmonic functions from which the h -function is derived generically have logarithmic singularities at the pair of preimage conjugate points where the capture circle intersects a particular boundary component. For example, the placement of the basepoint between two boundary components will result in two pairs of preimage conjugate points at which there will be logarithmic singularities. Indeed, all the formulae we have derived pertain, in principle, to *any* multiply connected domain of even connectivity that is reflectionally symmetric about the imaginary axis (subject to, perhaps, particular locations of the basepoint); the necessary change will be the conformal mapping function from a bounded circular domain. Work in these directions is already in progress.

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References

- [1] J. Aarão, M. A. Snipes, B. L. Walden & L. A. Ward, Four ways to calculate harmonic measure in a doubly connected domain. In preparation.
- [2] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill, New York, 1973.
- [3] L. V. Ahlfors, *Complex Analysis: An introduction to the theory of analytic functions of one complex variable*, 3/e, McGraw-Hill, New York, (1979).
- [4] A. Barton & L. A. Ward, A new class of harmonic measure distribution functions, *J. Geom. Anal.* **24** (2014), no. 4, 2035–2071.
- [5] D. Betsakos, Harmonic measure on simply connected domains of fixed inradius, *Ark. Mat.* **36** (1998), 275–306.
- [6] D. Betsakos & A. Solynin, On the distribution of harmonic measure on simply connected planar domains, *J. Aust. Math. Soc.* **75** (2003), 145–151.
- [7] D. A. Brannan & W. K. Hayman, Research problems in complex analysis, *Bull. London Math. Soc.* **21** (1989), 1–35.
- [8] D. G. Crowdy, *Solving problems in multiply connected domains*, SIAM, Philadelphia, 2020.
- [9] D. G. Crowdy, Conformal slit maps in applied mathematics, *ANZIAM J.* **53**(3) (2012), 171–189.
- [10] D. G. Crowdy, The Schottky-Klein prime function on the Schottky double of planar domains, *Comput. Methods Funct. Theory* **10**(2) (2010), 501–517.
- [11] D. G. Crowdy, Geometric function theory: a modern view of a classical subject, *Nonlinearity* **21**(10) (2008), T205–T219.
- [12] D. G. Crowdy & J. S. Marshall, Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461** (2005), 2477–2501.
- [13] D. G. Crowdy & J. S. Marshall, Conformal mappings between canonical multiply connected domains, *Comput. Methods Funct. Theory* **6** (2006), no. 1, 59–76.
- [14] D. G. Crowdy & J. S. Marshall, Green’s functions for Laplace’s equation in multiply connected domains, *IMA J. Appl. Math.* **72** (2007), no. 3, 278–301.
- [15] D. G. Crowdy, E. H. Kropf, C. C. Green & M. M. S. Nasser, The Schottky-Klein prime function: a theoretical and computational tool for applications. *IMA J. App. Math* **81** (2016), 589–628.

- [16] M. Fréchet, Sur l'écart de deux courbes et sur les courbes limites, *Trans. Amer. Math. Soc.* **VI** (1905), 435–449.
- [17] J. B. Garnett & D. E. Marshall, *Harmonic measure*, Cambridge Univ. Press, Cambridge, 2005.
- [18] G. M. Goluzin, *Geometric theory of functions of a complex variable*, American Mathematical Society, Providence, RI, 1969.
- [19] Imperial College London Applied & Computational Complex Analysis group GitHub website: <https://github.com/ACCA-Imperial>.
- [20] S. Kakutani, Two-dimensional Brownian motion and harmonic functions, *Proc. Imp. Acad. Tokyo* **20** (1944), 706–714.
- [21] T. W. Ng, C. C. Tang & J. Tsai, The squeezing function on doubly-connected domains via the Loewner differential equation, *Math. Annal.* **380** (2021), 1741–1766.
- [22] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992.
- [23] M. A. Snipes & L. A. Ward, Realizing step functions as harmonic measure distributions of planar domains, *Ann. Acad. Sci. Fenn. Math.* **30** (2005), no. 2, 353–360.
- [24] M.A. Snipes & L.A. Ward, Convergence properties of harmonic measure distributions for planar domains, *Complex Var. Elliptic Equ.* **53** (2008), no. 10, 897–913.
- [25] M. A. Snipes & L. A. Ward, Harmonic measure distribution functions of planar domains: A survey, *J. Anal.* **24** (2016), no. 2, 293–330.
- [26] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.
- [27] B. L. Walden & L. A. Ward, Distributions of harmonic measure for planar domains, in *Proceedings of 16th Nevanlinna Colloquium*, Joensuu, ed. I. Laine and O. Martio, Walter de Gruyter, Berlin (1996), 289–299.
- [28] B. L. Walden & L. A. Ward, Asymptotic behaviour of distributions of harmonic measure for planar domains, *Complex Variables: Theory and Applications* **46** (2001), 157–177.
- [29] B. L. Walden & L. A. Ward, A harmonic measure interpretation of the arithmetic-geometric mean, *Amer. Math. Monthly* **114** (2007), no. 7, 621–622.