

Comment on “On the Lagrangian and Hamiltonian description of the damped linear harmonic oscillator”

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In a remarkable paper Chandrasekar *et al.* showed that the (second-order constant-coefficient) classical equation of motion for a damped harmonic oscillator can be derived from a Hamiltonian having one degree of freedom. This paper gives a simple derivation of their result and generalizes it to the case of an n th-order constant-coefficient differential equation.

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I. INTRODUCTION

It seems unlikely that the equation for the damped classical harmonic oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega^2x = 0 \quad (\gamma > 0), \quad (1)$$

a dissipative system, could be derived from a Hamiltonian. This is because $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2x^2$, the standard expression for the total energy, is not conserved for $\gamma \neq 0$. Indeed, it satisfies the equation

$$\frac{d}{dt} \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2x^2 \right) = -2\gamma\dot{x}^2, \quad (2)$$

showing that it decreases with time.

Nonetheless, Bateman [1] made the remarkable observation that if one appends the time-reversed oscillator equation with undamping (gain) instead of damping,

$$\ddot{y} - 2\gamma\dot{y} + \omega^2y = 0 \quad (\gamma > 0), \quad (3)$$

then even though the two oscillators are independent and noninteracting [the two evolution equations (1) and (3) are not coupled], the equations of motion (1) and (3) can be derived from the time-independent quadratic Hamiltonian

$$H = pq + \gamma(yq - xp) + (\omega^2 - \gamma^2)xy \quad (4)$$

involving the two pairs of canonical variables (x, p) and (y, q) . To derive the original equation (1) we use the pair of Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} = q - \gamma x, \quad (5)$$

$$\dot{q} = -\frac{\partial H}{\partial y} = -\gamma q - (\omega^2 - \gamma^2)x. \quad (6)$$

Substituting for q in (6) gives $\dot{q} = -\gamma\dot{x} - \omega^2x$, and then differentiating (5) with respect to t gives (1). The partner equation (2) can be similarly obtained from the remaining Hamilton's equations

$$\dot{y} = \frac{\partial H}{\partial q} = p + \gamma y, \quad (7)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = \gamma p - (\omega^2 - \gamma^2)y, \quad (8)$$

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It is worth noting that the Hamiltonian (4) is $\mathcal{PT}$ symmetric [2] if we define the action of $\mathcal{P}$ as interchanging the two oscillators

$$\mathcal{P} : x \rightarrow y, \quad y \rightarrow x, \quad p \rightarrow q, \quad q \rightarrow p, \quad (9)$$

while $\mathcal{T}$ reverses the signs of the momenta,

$$\mathcal{T} : x \rightarrow x, \quad y \rightarrow y, \quad p \rightarrow -p, \quad q \rightarrow -q. \quad (10)$$

The $\mathcal{PT}$ symmetry of H in (4) and the success of Bateman's strategy depend crucially on the gain and loss terms in (1) and (3) being exactly balanced. As a consequence of the gain/loss balance, the system possesses a conserved quantity, namely the value of H . However, the energy has the complicated form (4) and is not a simple sum of kinetic and potential energies.

Recently it was shown [3] that the equation of motion (1) of the damped oscillator can be derived from a (non-quadratic) time-independent Hamiltonian depending on only a *single canonical pair* (x, p) . This remarkable result was proved by using a modification [4] of the Prolle-Singer approach [5] to identify integrals of motion of dynamical systems. This paper is a comment on the interesting work in Refs. [3, 4].

In [3] different forms of the Hamiltonian were given depending on whether the system was overdamped, underdamped, or critically damped. In particular, for the overdamped case ($\gamma > \omega$) the Hamiltonian takes the unconventional form

$$H = Axp + Bp^\delta, \quad (11)$$

where A , B , and δ are constants that we will specify later. Different functional forms were given for the other cases in order to have a real Hamiltonian. However, since this is not a concern for us, the functional form of (11) serves for all cases (apart from an obvious modification in the case of critical damping).

In this paper we show in Sec. II how the Hamiltonian (11) can be derived by elementary means and we identify the coefficients therein in terms of the (generally complex) eigenfrequencies of the problem. The conserved quantity, written in terms of $\dot{x}$ and x , is similarly identified. In Sec. III we go on to apply the procedure to the third-order linear equation with constant coefficients, a particular example of which describes the nonrelativistic self-acceleration of a charged oscillating particle [6]. The additional conceptual problems that arise in this case are discussed in detail. In Sec. IV we generalize the procedure to an arbitrary n th-order constant-coefficient equation. Section V discusses the difficult problem of quantizing Hamiltonians of this type. Finally, Sec. VI gives a brief summary.

II. HAMILTONIAN FOR THE DAMPED OSCILLATOR

We begin with (1) and substitute $x(t) = e^{-i\nu t}$. This gives a quadratic equation for the frequency ν :

$$\nu^2 + 2i\gamma\nu - \omega^2 = 0, \quad (12)$$

which factors as

$$(\nu - \omega_1)(\nu - \omega_2) = 0, \quad (13)$$

where

$$\omega_1 + \omega_2 = -2i\gamma, \quad \omega_1\omega_2 = -\omega^2, \quad (14)$$

with

$$\omega_{1,2} = -i\gamma \pm \Omega = -i\gamma \pm \sqrt{\omega^2 - \gamma^2}. \quad (15)$$

The generic form of a Hamiltonian $H(x, p)$ that can generate the evolution equation (1) is given in (11). There are two such Hamiltonians, corresponding to the two eigenfrequencies in (15). The first is

$$H_1 = -i\omega_1 xp + \frac{\omega_1}{\omega_1 - \omega_2} p^{1-\omega_2/\omega_1}. \quad (16)$$

A second and equally effective Hamiltonian is obtained by interchanging ω_1 and ω_2 :

$$H_2 = -i\omega_2 xp + \frac{\omega_2}{\omega_2 - \omega_1} p^{1-\omega_1/\omega_2}. \quad (17)$$

As mentioned in Sec. I, Hamiltonians of this form appear in Ref. [3] for the case of over-damping ($\gamma^2 > \omega^2$), where they are real since in that case ω_1 and ω_2 are pure imaginary, but they apply equally well when $\gamma^2 < \omega^2$ if we are not concerned with the reality of the Hamiltonian. Indeed, the Hamiltonian is no longer the standard real energy, which is not conserved. Rather, it is a complex quantity which is conserved and from which the equations of motion can be derived in the standard way.

For the Hamiltonian H_1 , Hamilton's equations read

$$\dot{x} = \frac{\partial H_1}{\partial p} = -i\omega_1 x + p^{-\omega_2/\omega_1}, \quad (18)$$

$$\dot{p} = -\frac{\partial H_1}{\partial x} = i\omega_1 p. \quad (19)$$

We then take a time derivative of (18) and simplify the resulting equation first by using (19) and then by using (18):

$$\ddot{x} + i\omega_1 \dot{x} = -\frac{\omega_2}{\omega_1} p^{-1-\omega_2/\omega_1} \dot{p} = -i\omega_2 p^{-\omega_2/\omega_1} = -i\omega_2 (\dot{x} + i\omega_1 x). \quad (20)$$

Thus,

$$\ddot{x} + i(\omega_1 + \omega_2) \dot{x} - \omega_1 \omega_2 x = 0, \quad (21)$$

which reduces to (1) upon using (14).

The evolution equation (1) has one conserved (time-independent) quantity, which can be expressed in terms of $x(t)$ only. To find this quantity, we begin with (18) and solve for p :

$$p = (\dot{x} + i\omega_1 x)^{-\omega_1/\omega_2}. \quad (22)$$

We then use this result to eliminate p from H_1 . Since H_1 is time independent, we conclude that

$$C_1 = (\dot{x} + i\omega_2 x)^{\omega_2} / (\dot{x} + i\omega_1 x)^{\omega_1} \quad (23)$$

is conserved. Had we started with the Hamiltonian H_2 we would have obtained the conserved quantity

$$C_2 = (\dot{x} + i\omega_1 x)^{\omega_1} / (\dot{x} + i\omega_2 x)^{\omega_2}, \quad (24)$$

but this is not an independent conserved quantity because $C_2 = 1/C_1$. These conserved quantities were also found in Ref. [3] for the case of over-damping.

When $\gamma = 0$, these results reduce to the familiar expressions for the simple harmonic oscillator. In that case we let $\omega = \omega_1 = -\omega_2$, so that H_1 becomes

$$H_1 = -i\omega x p + \frac{1}{2} p^2, \quad (25)$$

which is related to the standard Hamiltonian for the simple harmonic oscillator by the change of variable $p \rightarrow p - i\omega x$. The conserved quantities C_2 and C_1 become simply $(\dot{x}^2 + \omega^2 x^2)^{\pm\omega}$, in which we recognize the usual conserved total energy.

III. HAMILTONIAN FOR A LINEAR CONSTANT-COEFFICIENT THIRD-ORDER EQUATION

In this section we show how to construct a Hamiltonian that gives rise to the general third-order constant-coefficient evolution equation

$$(D + i\omega_1)(D + i\omega_2)(D + i\omega_3)x = 0, \quad (26)$$

where $D \equiv d/dt$. The Hamiltonian that we will construct has just one degree of freedom.

A physical example of such an equation is the third-order differential equation

$$m\ddot{x} + kx - m\tau\ddot{x} = 0 \quad (27)$$

that describes an oscillating charged particle subject to a radiative back-reaction force [6]. Following Bateman's approach for the damped harmonic oscillator, Englert [7] showed that the pair of noninteracting equations (27) and

$$m\ddot{y} + ky + m\tau\ddot{y} = 0 \quad (28)$$

can be derived from the quadratic Hamiltonian

$$H = \frac{ps - rq}{m\tau} + \frac{2rs}{m\tau^2} + \frac{pz + qw}{2} - \frac{mzw}{2} + kxy. \quad (29)$$

This Hamiltonian contains the *four* degrees of freedom (x, p) , (y, q) , (z, r) , and (w, s) . An interacting version of this model was studied in Ref. [8]. In fact, we find that the two equations of motion (27) and (28) can be derived from the simpler quadratic Hamiltonian

$$H = \frac{pr + qz}{\sqrt{m\tau}} - \frac{rz}{\tau} + kxy,$$

which has only the three degrees of freedom (x, p) , (y, q) , and (z, r) . A similar three-degree of freedom Hamiltonian was also found in Ref. [7].

Our objective here is to find a *one*-degree-of-freedom Hamiltonian that can be used to derive the third-order differential equation (26). Note that the general solution to (26) is

$$x = a_1 e^{-i\omega_1 t} + a_2 e^{-i\omega_2 t} + a_3 e^{-i\omega_3 t}, \quad (30)$$

where a_k are arbitrary constants. If we form $(D + i\omega_2)(D + i\omega_3)x$, that is, $\ddot{x} + i(\omega_2 + \omega_3)\dot{x} - \omega_2\omega_3x$, we obtain

$$a_1 e^{-i\omega_1 t} = -\frac{(D + i\omega_2)(D + i\omega_3)x}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)} \quad (31)$$

in which the constants a_2 and a_3 do not appear. Similarly, we have

$$\begin{aligned} a_2 e^{-i\omega_2 t} &= -\frac{(D + i\omega_3)(D + i\omega_1)x}{(\omega_2 - \omega_3)(\omega_2 - \omega_1)}, \\ a_3 e^{-i\omega_3 t} &= -\frac{(D + i\omega_1)(D + i\omega_2)x}{(\omega_3 - \omega_1)(\omega_3 - \omega_2)}. \end{aligned} \quad (32)$$

So, assuming that the frequencies ω_k are all different, there are two distinct conserved quantities, namely

$$\begin{aligned} C_2 &= [\ddot{x} + i(\omega_2 + \omega_3)\dot{x} - \omega_2\omega_3x][\ddot{x} + i(\omega_1 + \omega_2)\dot{x} - \omega_1\omega_2x]^{-\omega_1/\omega_3}, \\ C_3 &= [\ddot{x} + i(\omega_2 + \omega_3)\dot{x} - \omega_2\omega_3x][\ddot{x} + i(\omega_1 + \omega_3)\dot{x} - \omega_1\omega_3x]^{-\omega_1/\omega_2}. \end{aligned} \quad (33)$$

These expressions and the equation of motion can be derived from the Hamiltonian

$$H = -i\omega_1 xp + \frac{b_2\omega_1}{\omega_1 - \omega_2} p^{1-\omega_2/\omega_1} + \frac{b_3\omega_1}{\omega_1 - \omega_3} p^{1-\omega_3/\omega_1}, \quad (34)$$

where b_2 and b_3 are arbitrary constants. Thus, $\dot{p} \equiv -\partial H/\partial x = i\omega_1 p$. This means that $p \propto e^{i\omega_1 t}$, so that $1/p$ is directly related to the combination in (31).

Then, from Hamilton's equation $\dot{x} \equiv \partial H/\partial p$ and from further differentiation with respect to t , we obtain

$$\begin{aligned} \dot{x} &= -i\omega_1 x + b_2 p^{-\omega_2/\omega_1} + b_3 p^{-\omega_3/\omega_1}, \\ \ddot{x} &= -i\omega_1 \dot{x} - i\omega_2 b_2 p^{-\omega_2/\omega_1} - i\omega_3 b_3 p^{-\omega_3/\omega_1}, \\ \dddot{x} &= -i\omega_1 \ddot{x} - \omega_2^2 b_2 p^{-\omega_2/\omega_1} - \omega_3^2 b_3 p^{-\omega_3/\omega_1}. \end{aligned} \quad (35)$$

This further differentiation is crucial because it is required to eliminate the constants b_2 and b_3 , and thus to obtain a differential equation that only contains $x(t)$ and is *independent of b_2 and b_3* . Combining these equations and performing some simplifying algebra, we obtain

$$\ddot{x} + i(\omega_1 + \omega_2 + \omega_3)\dot{x} - (\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1)x - i\omega_1\omega_2\omega_3x = 0. \quad (36)$$

We emphasize that the constants b_2 and b_3 have disappeared in this combination and we have reconstructed the equation of motion (26). These constants are reminiscent of Lagrange multipliers, but they are unlike Lagrange multipliers in that we do not vary the Hamiltonian with respect to them. Rather, we require that the equations of motion be *independent* of these constants.

Using only derivatives up to the second order, we can find expressions for $b_2 p^{-\omega_2/\omega_1}$ and $b_3 p^{-\omega_3/\omega_1}$: namely

$$\begin{aligned} b_2 p^{-\omega_2/\omega_1} &= i[\ddot{x} + i(\omega_1 + \omega_3)\dot{x} - \omega_1\omega_3 x]/(\omega_2 - \omega_3), \\ b_3 p^{-\omega_3/\omega_1} &= i[\ddot{x} + i(\omega_1 + \omega_2)\dot{x} - \omega_1\omega_2 x]/(\omega_3 - \omega_2), \end{aligned} \quad (37)$$

in which we recognize two of the quantities in square brackets that appear in (33). The third such quantity, namely $\ddot{x} + i(\omega_1 + \omega_2)\dot{x} - \omega_1\omega_2 x$, is closely related to H :

$$\begin{aligned} \ddot{x} + i(\omega_2 + \omega_3)\dot{x} - \omega_2\omega_3 x &= -(\omega_1 - \omega_2)(\omega_1 - \omega_3)x + ib_2(\omega_3 - \omega_1)p^{-\omega_2/\omega_1} + ib_3(\omega_2 - \omega_1)p^{-\omega_3/\omega_1} \\ &= i(\omega_1 - \omega_2)(\omega_3 - \omega_1)H/(\omega_1 p). \end{aligned}$$

We conclude that

$$C_2 = [i(\omega_1 - \omega_2)(\omega_3 - \omega_1)H/(\omega_1 p)]p[i(\omega_2 - \omega_3)b_3]^{-\omega_1/\omega_3} \propto H b_3^{-\omega_1/\omega_3}.$$

Similarly, we have

$$C_3 \propto H b_2^{-\omega_1/\omega_2}.$$

Thus C_2 and C_3 are constants of the motion because that they are both proportional to the Hamiltonian, with proportionality constants given by powers of b_2 and b_3 respectively.

To summarize, by eliminating the parameters b_2 and b_3 the evolution equation (26) can be derived from the unusual time-independent Hamiltonian (34) containing the single coordinate variable x and its conjugate momentum p . This Hamiltonian is a conserved quantity, which can be expressed as

$$H = i\omega_1 p \frac{\ddot{x} + i(\omega_2 + \omega_3)\dot{x} - \omega_2\omega_3 x}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)}. \quad (38)$$

The conserved quantities C_2 and C_3 are both proportional to H .

Before moving on, we must explain how a Hamiltonian with a single degree of freedom can give rise to a differential equation whose order is greater than two. The problem is as follows. Our Hamiltonian has the generic form

$$H = axp + f(p). \quad (39)$$

Therefore, the equations of motion are simply

$$\dot{x} = ax + g(p), \quad (40)$$

where $g(p) = f'(p)$, and

$$\dot{p} = -ap. \quad (41)$$

First, we solve (41)

$$p(t) = Ce^{-at}, \quad (42)$$

where C is an arbitrary constant. Next, we return to (40), which becomes

$$\dot{x} = ax + g(Ce^{-at})$$

after we eliminate p by using (42). This is a *first-order* equation. Thus, its solution has only *two* arbitrary constants:

$$x(t) = \phi(t, C, D). \quad (43)$$

We obtain the higher-order differential equation (36) by the sequence of differentiations in (35) that were required to eliminate the constants b_2 and b_3 . Of course, the solution to an n th-order equation can incorporate n pieces of data such as n initial conditions: $x(0)$, $\dot{x}(0)$, $\ddot{x}(0)$, $\dddot{x}(0)$, and so on. How is it possible to incorporate n pieces of data with only two arbitrary constants C and D ? There appear to be $n - 2$ missing arbitrary constants.

The answer is that the $n - 2$ pieces of initial data determine $n - 2$ parameters b_k multiplying each of the fractional powers of p in H . (One parameter can always be removed by a scaling.) We can incorporate the initial data into the Hamiltonian in the form of these parameters. These parameters specify an ensemble of Hamiltonians, all of which

give a unique n th-order field equation that is independent of these parameters and is capable of accepting n pieces of initial data.

For the triple-dot equation we can see from (37) that the ratio $b_2^{1/\omega_2}/b_3^{1/\omega_3}$ is related to the initial conditions. So, for the case of the third-order equation, the three arbitrary constants are C , D , and $b_2^{1/\omega_2}/b_3^{1/\omega_3}$. We emphasize that the Hamiltonian gives the higher-order equations of motion precisely because of the requirement that the parameters b_k drop out from the equation of motion. The parameters b_k in the Hamiltonian are crucial because they incorporate the initial data and are determined by the initial data. The nonzero parameter b_3 forces the evolution equation to be third order. Without b_3 the Hamiltonian does not know about the third frequency ω_3 . Indeed, if $b_3 = 0$, (34) reduces to (16) (with $b_2 = 1$). [This is consistent with (37) because setting $b_3 = 0$ there implies that $\ddot{x} + i(\omega_1 + \omega_2)\dot{x} - \omega_1\omega_2 x = 0$.]

Finally, one may ask what would happen if we followed the standard procedure for deriving the equations of motion for $x(t)$ from the Hamiltonian equations of motion $\dot{p} = -\partial H/\partial x$ and $\dot{x} = \partial H/\partial p$. This would mean solving the second equation for p in terms of x and $\dot{x}$ and then substituting back in the first to obtain a second-order equation for $x(t)$. In our case that would mean solving the first of equations (35) for p , which is not possible for general values of the parameters. However, it is instructive to see how this procedure works if we choose the parameters so that an explicit solution is possible. For example, if we choose $\omega_1 = 1$, $\omega_2 = -2$, $\omega_3 = 4$, $b_2 = 2$, and $b_3 = 1$, the equation can be solved to give $p^2 = -1 + \sqrt{\dot{x} + ix + 1}$. Substituting back into the equation $\dot{p} = ip$ gives, after some algebra, the nonlinear second-order equation

$$(\ddot{x} - 3i\dot{x} + 4x - 4i)^2 = -16(\dot{x} + ix + 1).$$

Further manipulation shows this to be equivalent to the constancy of C_2/C_3 .

So, in the cases where the standard procedure can be followed in practice, the resulting nonlinear second-order equation is equivalent to an equation for a constant of the motion (which of course depends on the parameters b_2 and/or b_3).

IV. GENERALIZATION TO n TH ORDER

It is straightforward to generalize to the case of an arbitrary n th-order linear constant-coefficient evolution equation

$$\left[\prod_{r=1}^n (D + i\omega_r) \right] x(t) = 0, \quad (44)$$

whose general solution is

$$x(t) = \sum_{r=1}^n a_r e^{-i\omega_r t}. \quad (45)$$

For simplicity, we assume first that the frequencies ω_r are all distinct; at the end of this section we explain what happens if some of the frequencies are degenerate.

Corresponding to (32) and (33), we have

$$e^{-i\omega_s t} \propto \left[\prod_{r \neq s} (D + i\omega_r) \right] x(t). \quad (46)$$

Thus, the quantity

$$Q_s \equiv \left\{ \left[\prod_{r \neq s} (D + i\omega_r) \right] x(t) \right\}^{1/\omega_s} \quad (47)$$

is proportional to e^{-it} for all s . Hence, the $n - 1$ independent ratios Q_s/Q_1 ($s > 1$) are all conserved. Any other conserved quantities can be expressed in terms of these ratios.

The equation of motion and the conserved quantities can be derived from the Hamiltonian

$$H = -i\omega_1 x p + \sum_{r \neq 1}^n \frac{b_r \omega_1 p^{1-\omega_r/\omega_1}}{\omega_1 - \omega_r}, \quad (48)$$

which is the n th order generalization of (34) for the cubic case. In this expression the $n-1$ coefficients b_r are arbitrary, and must be eliminated to give the n th-order equation of motion. Note that in constructing the Hamiltonian H there is nothing special about the subscript “1” and it may be replaced by the subscript “ s ” ($1 < s \leq n$).

Degenerate frequencies

Until now, we have assumed that the frequencies ω_r are all distinct. However, if some of the frequencies are degenerate, there is a simple way to construct the appropriate Hamiltonian: If the frequencies ω_1 and ω_2 are equal, we make the replacement

$$\frac{\omega_1}{\omega_1 - \omega_2} p^{1-\omega_2/\omega_1} \longrightarrow \log(p). \quad (49)$$

(In making this replacement we are shifting the Hamiltonian by an infinite constant.) Thus, for $\omega_1 = \omega_2$ the Hamiltonian H_1 in (16) reduces to

$$H_1 = -i\omega_1 xp + \log p. \quad (50)$$

Hamilton’s equations for this Hamiltonian immediately simplify to (21) with $\omega_1 = \omega_2$. Similarly, for the case $\omega_1 = \omega_2$ the Hamiltonian (34) reduces to

$$H = -i\omega_1 xp + b_2 \log p + \frac{b_3 \omega_1}{\omega_1 - \omega_3} p^{1-\omega_3/\omega_1} \quad (51)$$

and Hamilton’s equations for this Hamiltonian readily simplify to (36) with $\omega_1 = \omega_2$.

Also, if the frequencies are triply degenerate, $\omega_1 = \omega_2 = \omega_3 = \omega$, the Hamiltonian in (34) is replaced by

$$H = -i\omega xp + b \log p + \frac{1}{2}c(\log p)^2, \quad (52)$$

where b and c are two parameters that are determined by the initial data. Once again, Hamilton’s equations for this Hamiltonian combine to give (36) with $\omega_1 = \omega_2 = \omega_3 = \omega$.

V. QUANTIZATION

The obvious question to be addressed next is whether it is possible to use the Hamiltonians that we have constructed to quantize classical systems that obey linear constant-coefficient evolution equations. Let us begin by discussing the simple case of the quantum harmonic oscillator (QHO), whose Hamiltonian H_1 is given in (25).

One possibility is to quantize the Hamiltonian in p -space by setting $x = id/dp$. Then by shifting E by ω we see that the time-independent Schrödinger eigenvalue equation is

$$H_1 \tilde{\psi}(p) = \left(\omega p \frac{d}{dp} + \frac{1}{2} p^2 \right) \tilde{\psi}(p) = E \tilde{\psi}(p), \quad (53)$$

whose solution is

$$\tilde{\psi}(p) \propto p^{E/\omega} e^{-p^2/(4\omega)}. \quad (54)$$

In this way of doing things we derive the quantization condition by demanding that $\tilde{\psi}$ be a well defined, nonsingular function, which requires that $E = n\omega$, where n is a nonnegative integer [9]. However, these “momentum-space” eigenfunctions are problematical because p has no clear physical interpretation as a momentum. The p -space eigenfunctions are certainly not orthonormal in any simple sense because they do not solve a Sturm-Liouville boundary-value problem [10].

However, we can calculate the corresponding x -space eigenfunctions by Fourier transform using the formula [11]

$$H_n(z) = \frac{(-i)^n}{2\sqrt{\pi}} e^{z^2} \int_{-\infty}^{\infty} dp e^{ipz} p^n e^{-p^2}. \quad (55)$$

We find that

$$\psi_n(x) \propto e^{-\frac{1}{2}\omega^2 x^2} \varphi_n(x), \quad (56)$$

where $\varphi_n(x)$ is the n th eigenfunction of the QHO. This is consistent with our remark above that H_1 is related to the standard QHO Hamiltonian by the transformation $p \rightarrow p - i\omega x$. This transformation is achieved at the operator level

by the similarity transformation $p \rightarrow e^{-\omega^2 x^2/2} p e^{\omega^2 x^2/2}$ [12]. Because of this additional factor, our eigenfunctions are orthonormal with respect to the metric $\eta = e^{\omega^2 x^2}$. As an alternative approach, we can cast (25) in x -space as

$$H_1 = -\frac{1}{2} \frac{d^2}{dx^2} - \omega^2 \left(1 + x \frac{d}{dx} \right), \quad (57)$$

from which we can obtain the $\psi_n(x)$ directly.

To summarize, the quantized version of (25) corresponds to a transformed version of the QHO, where the x -space eigenfunctions are simply related to the standard eigenfunctions and are orthonormal with respect to an additional weight function. The p -space eigenfunctions can be written down but their interpretation is not obvious (the operator p corresponds to the conventional raising operator $a^\dagger$) and are not orthogonal in any simple way. Moreover, p , represented as $-id/dx$, is not Hermitian because the overlap integral between two wave functions has to be calculated with the inclusion of the metric $\eta(x)$, so integration by parts is no longer simply a matter of a minus sign. Instead, p is *pseudo-Hermitian* [13, 14] with respect to η , i.e. $p^\dagger = \eta p \eta^{-1}$. In p space the weight function $e^{\omega^2 x^2}$ becomes the highly nonlocal operator $e^{-\omega^2 d^2/dp^2}$.

If we generalize to the damped harmonic oscillator, we can still find a solution $\tilde{\psi}(p)$ to the time-independent Schrödinger equation, namely [3]

$$\tilde{\psi}(p) \propto p^{E/\omega_1} \exp \left[-\frac{\omega_1}{(\omega_1 - \omega_2)^2} p^{1-\omega_1/\omega_2} \right], \quad (58)$$

but even if we take $E = n\omega_1$ to make the prefactor nonsingular, we are still left with a nonintegral, and in general complex, power of p in the exponential. (See, also, the comments in Ref. [1].) Thus, in addition to the previously discussed problems with $\tilde{\psi}(p)$, we would now have to consider it to be a function in a cut plane. Moreover, there is no simple formula like (55) whereby one could obtain the x -space eigenfunctions. Furthermore, if we cast the equation in x -space we obtain

$$H_1 = \frac{1}{1 - \omega_2/\omega_1} \left\{ -i \frac{d}{dx} \left[\left(-i \frac{d}{dx} \right)^{-\omega_2/\omega_1} - i(\omega_1 - \omega_2)x \right] \right\}, \quad (59)$$

in which the difficulty associated with a fractional derivative is manifest.

Evidently, quantizing Hamiltonians of the form in (39) is nontrivial. The problem of quantizing the cubic equation describing the back-reaction force on a charged particle was solved in Ref. [8]. However, the system that was actually quantized was a *pair* of *coupled* cubic equations in the *unbroken* $\mathcal{PT}$ -symmetric region. Thus, it may be that the most effective approach for quantizing a Hamiltonian of the form (39) is to introduce a large number of additional degrees of freedom.

VI. SUMMARY

We have shown that any n th-order linear constant-coefficient evolution equation can be derived from a nonconventional but simple Hamiltonian of the form (39). Remarkably, this Hamiltonian has only one degree of freedom, that is, one pair of dynamical variables (x, p) . Furthermore, we have shown that for such a system there are $n - 1$ independent constants of the motion and we have constructed these conserved quantities in terms of $x(t)$ and its time derivatives. However, we find that it is not easy to formulate a general procedure to quantize the system described by the Hamiltonian, and this remains an extremely interesting but difficult open problem.

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