MAGNETOHYDRODYNAMIC STABILITY OF A TWO-FLUID PLASMA

by

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ABSTRACT

The assumption seems to have been accepted in the plasma physics community, that the single-fluid, magnetohydrodynamic (MHD) theory, is an adequate zeroth-order description of the magnetically confined, high-temperature plasma, and that this theory is able to describe a significantly large proportion of the experimentally observed properties of such plasmas. We believe this assumption to be incorrect, and we attempt to show in this thesis, that there is sufficient ground for questioning the pre-eminence of the MHD theory in magnetic fusion plasma analysis.

To this end we have examined the two-fluid model of an infinitely long, cylindrical plasma, suggested in /1/. This model differs from the so-called Ideal MHD model only in the retention of the ∇p_e and the $\underline{J} \times \underline{B}$ terms in the Ohm's law, and in the characterisation of the quasi-neutral plasma by separate ion and electron temperatures. We have critically analysed the work of /1/, and have found significant errors therein, some of which have resulted from the use of approximations (which we show to be unjustified), to simplify the two-fluid plasma problem. We have also developed a novel numerical method for solving one of the eigenvalue problems posed in /1/. Finally, we have used a numerical initial/boundary-value method to solve, without approximation, the full two-fluid equations. This method involves numerically integrating all the linearised equations forwards in time (using the numerical method of lines), until such time as the fastest exponentially-growing radial oscillation mode becomes dominant. We have used this technique to study three equilibria, corresponding to the z-current density profiles :

$$j_{oz} = j_0 \left(1 - \frac{r^2}{a^2}\right)^{\nu}$$

where $\nu = 1, 2, 4$ and a is the radius of the plasma.

The results that we have obtained indicate that the Ideal MHD theory significantly underestimates the linear growth-rates of all the radial oscillation modes examined, particularly the modes : m=3, n=2 ; m=2, n=1 ; and m=1, n=1 , on which some parametric studies were made. These results, coupled with other considerations which we present in the thesis, suggest that for any analytical (or numerical) plasma calculations to be relevant to experiment, such calculations would have to be done within some kind of two-fluid theoretical framework, although not necessarily this particular one.

CONTENTS

Abstract		2
Contents		3
Acknowledgements		5
Dedication		6
Introduction		8
Chapter 1		
1.1	The Basic Equations	23
1.2	Linearisation, Error I and Error II	27
1.3	Discussion of the Expression for v_{1r}	36
1.4	Discussion of the Further Approximations made to Equations (37) and (41) of /1/.	39
1.5	Numerical Solution of the 'Interior Case' Equation.	52
1.6	A Philosophical Objection to the Basic Coupled Equations of /1/.	66
1.7	Summary of Chapter 1	70
Chapter 2		
2.0	Background to the Method Used	71
2.1	The Equilibrium	78
2.2	The Equations to be Solved in the Initial-Value Treatment.	89

2.3	Boundary Conditions	100
2.3.1	Boundary Conditions at r=0 for m=0	100
2.3.2	Boundary Conditions at r=0 for m=1	101
2.3.3	Boundary Conditions at r=0 for m >1	102
2.3.4	Boundary Conditions at r=a.	103
2.4	Results Obtained From the Numerical Initial-Value Program.	105
2.4.1	The MHD Limit.	115
2.5	Summary of Chapter 2.	123
2.6	Conclusions	124

References 126

Appendixes :

Appendix A	On the 'Freezing-In' of Field Lines.	133
Appendix B	Derivation of the Expression for v_{1r}	137
Appendix C	The Mutual Cancelling of Error I and Error II.	139
Appendix D	Derivation of the 'Exterior Case' Equation.	142
Appendix E	Derivation of the 'Interior Case' Equation.	148
Appendix F	Determination of $\hat{A}(\omega, r_o)$ and hence of $\Lambda(\omega, r_o)$	153
Appendix G	Asymptotic Solutions of the Coupled Equations near r=0 .	158
Appendix H	Asymptotic Solutions of the Coupled Equations near $r=r_o$.	167
Appendix I	Numerical Initial-Value Integration Methods with Automatic Error Control.	176

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Dedication

I have a dream today.

I have a dream that one day
every valley shall be exalted,
every hill and mountain
shall be made low,
and the rough places will be made plain,
and the crooked places
will be made straight,
and the glory of the Lord
shall be revealed
and all flesh shall see it together.

This is our hope.
With this faith,
we will be able to hew out
of the mountain of despair
a stone of hope.

With this faith,
we will be able to transform
the jangling discords of our nation
into a beautiful symphony
of brotherhood.

With this faith,
we will be able to work together,
to pray together,
to struggle together,
to go to jail together,
knowing that we will be free one day.

Martin Luther King

For
the sons and daughters of
Sharpeville (21 March 1960)
and
Soweto (16 June 1976)
who demonstrate
that a man's dignity
cannot be taken from him
by force of arms.

INTRODUCTION

The simplest way to think about a plasma is as an ionised gas in which all,or a considerable number of atoms,have lost one or more of their free electrons.The strong electrostatic forces of attraction between the ions and electrons in such gases prevents substantial charge separation over distances that are greater than the Debye length,λ_D.That is to say that if scale lengths greater than λ_D are considered,the plasma will be found to be substantially neutral,/41/.With the concept of the Debye length as an aid, Tonks and Langmuir /2/,were able to provide a more formal definition of a plasma.Thus we may consider that an ionised gas will behave like a plasma if it is the case that $n\lambda_D^3 \gg 1$ and also $L \gg \lambda_D$, where L^3 is of the order of magnitude of the volume of the plasma, and n is the number density of ions (or electrons) in the plasma.This ensemble of ions and electrons that is called a plasma is interesting for many reasons.One of the most important of these is that if it can be confined in a dynamically stable manner,for sufficiently long periods of time (i.e. about 1 second),at high enough particle number densities (say, 10^{14} cm^{-3}),and if it also contains low mass-number ions (such as those of deuterium or tritium,or hydrogen),then at high enough temperatures (i.e. about 10^8 K),the conditions may be created under which thermonuclear fusion can take place,accompanied by the release of greater amounts of energy than was required to form the plasma.This controlled release of energy from thermonuclear fusion processes,is the goal of the vast bulk of contemporary research in the field of plasma physics. Accounts of the historical development of this subject are given in /11,15,16/.

Thus plasmas of thermonuclear fusion interest will probably be very dense,and because of the high temperatures at which they will have to be maintained,they will only be very weakly collisional.Another pertinent observation is that the charged particles that make up the plasma, generate electromagnetic fields through their elementary charges and currents.In order to evaluate these fields,and thus describe the future time-development of the plasma, it would be necessary to know the position and velocity of every particle that is a constituent of the plasma, at all times. The motions of the charges would also have to be followed in the sum of the fields that the charges themselves generate,and those that are externally imposed.This is not a realistic programme to follow because of the very large numbers of particles involved.It is because of this very large number of mutually

interacting particles in any plasma,and because of the difficulties encountered when any attempt is made to formulate a rigorous theory for a weakly collisional plasma,that attention has become focused on a number of approximate analytical models of plasmas.The most popular of these models is called the 'ideal' magnetohydrodynamic (MHD) model.This description of the plasma was developed by Lundqvist /103/.It assumes that the plasma is a single highly-collisional continuous fluid,of infinite electrical conductivity,whose mass is wholly associated with its ionic component,and which may only move,in bulk,with velocities that are much less than the speed of light,and may only have associated with it,oscillation frequencies that are much less than the electron plasma frequency /41/.This ideal MHD model was much used in the early days of magnetic confinement fusion plasma analysis /3,4,5/,because its relative simplicity allowed consideration of non- trivial plasma geometries,and because it was thought to provide a pessimistic appraisal of the plasma configuration under consideration. In the present day it seems to be considered,by experimental and theoretical plasma physicists alike,to be a good 'zeroth'-order description of the magnetically confined high-temperature plasma /6,7,8,9,10/,to which small modifications can be made to account for particular and specific effects. Formally,the MHD equations are a set of 8 nonlinear symmetric hyperbolic equations,and have been shown by Grad /104/ to include all the effects of standard fluid dynamics.Thus,although the MHD equations are themselves a simplification,they are still sufficiently complex to admit to solution only when relatively simple geometries,or a single dimension,are considered,and usually such solutions as are possible must be obtained numerically.A short review of the numerical methods that have been applied to the solution of the MHD equations has been given by Turkel /105/.

Apart from the single-fluid description of hot plasmas,there is also the two-fluid description,in which the presence of the electron fluid is explicitly acknowledged.There is also the 'guiding-centre' description (on which the so-called neo-classical theory /24/ is based),and the Boltzmann-Vlasov kinetic description /11/.However,we shall only concern ourselves in this thesis,with the single-fluid and two-fluid descriptions of hot plasmas.

A striking feature that we find when using the ideal MHD model to describe a plasma,is that the magnetic field is 'frozen' into the material of the plasma.(However see p.32 of /8/ or p.139 of /20/).In the words of Alfven /12,13,14/,

> 'Every motion(perpendicular to the field) of the liquid in relation to the lines of force is forbidden because it can give rise to infinite eddy currents.Thus the matter of the liquid is fastened to the lines of force.'

With the aid of the ideal MHD theory,it is possible to make general statements about the stability of such magnetic field configurations as might be proposed,for confining any particular hot plasma,in a given geometry.These statements originate from general global relations such as conservation equations and virial theorems /24/.

The concept of stability is very important in magnetic fusion plasma analysis,and for this reason we now proceed to define it.Thus,a particular equilibrium plasma/magnetic field configuration will be 'stable' if,when the plasma is subjected to a small perturbation,the amplitude of this perturbation is found to remain constant in time,or to decay in time.The equilibrium would be classed as 'unstable' if the amplitude of the perturbation were found to grow in time.On the other hand the plasma might remain in its new perturbed position,and in this case the equilibrium would be described as being 'marginally stable' .

Alfven /12,13/,found that because the magnetic field is tied (i.e. frozen) to the matter of the ideal MHD plasma,the magnetic field could behave like a stretched string,and,like such a string, may sustain wave-motion. He called such waves 'hydromagnetic waves',and if $\underline{B}$ is the strenght of the magnetic field,the velocity of these waves, V_a ,is given by

$$V_a^2 = \frac{B^2}{4\pi d}$$

where d is the mass density of the plasma.If L is a macroscopic length in the plasma,then the time taken by such a wave to travel a distance of this order is denoted by T_a .In such equilibrium plasma configurations as are found to be unstable using the ideal MHD analysis,the prediction of the analysis is that the plasma will survive for a time of the order of T_a after the onset of the instability,before being driven outward,and eventually destroyed by contact with the walls of the containment vessel.

The simplest plasma containment configuration is called the 'linear pinch'. It consists of a cylindrical column of gas along which a large current, $\underline{J}$ is passed. It is called a 'z-pinch' if this current is in the z-direction (i.e. along the axis of the plasma cylinder), and a ' e-pinch' , if the main current is in the e (or poloidal) direction. In either case the idea is that the inwardly directed $\underline{J} \times \underline{B}$ force, shall exactly balance the outward force due to the kinetic pressure of the gas, when the system is in equilibrium.

Having considered the simple 'pinch' containment device,we shall now briefly review the methods of producing plasmas with temperatures in the 10 keV range (i.e. hot enough to sustain thermonuclear reactions).The first method is called Joule or Ohmic heating.In this method what happens is that the electric field which is applied to the plasma accelerates the plasma electrons,thereby increasing their kinetic energies.These kinetic energies are then thermalised by means of collisions with the plasma ions, and as a consequence the temperature of the whole plasma is raised.The rate of Joule heating is given by

$$\underline{j}\cdot\underline{E} = \eta j^2 = \frac{E^2}{\eta}$$

where η is the plasma specific resistivity /44,51/. η is directly proportional to the electron-ion collision frequency which is in turn proportional to v^{-3},where v is the velocity of the electron in question. Thus,those few electrons in the 'tail' of the electron velocity distribution that have high velocities will make very few collisions.If the electric field is applied suddenly to the plasma,those of the high-velocity electrons that happen to be moving in the direction of $-\underline{E}$ will have their velocities increased even more.Their collision cross-section with the ions may then become so small that they are able to leave the plasma without making any collisions.Such electrons are called 'runaway' electrons,and constitute a significant energy loss mechanism for the plasma.They appear as an accelerated electron beam, detached from the main body of the distribution. Electrons can also be accelerated as a result of collisions with ions.As a result of this interaction the electrons emit electromagnetic radiation that is called 'bremsstrahlung'.The intensity of bremsstrahlung that is radiated from a unit volume of plasma is proportional to Z^2,where Z is the atomic number of the plasma ions.This form of radiative loss can thus become very important when the plasma contains significant quantities of high-Z impurities. Also the Kruskal-Shafranov criterion /8/ places a limit on how much current we can pass through the cylindrical plasma if we do not want it to become unstable.It is for these reasons that joule heating alone is unable to raise the plasma temperature to more than several keV.

Another important method of heating the plasma is called 'shock-wave' heating.What happens here is that the abruptly introduced large $\underline{J} \times \underline{B}$ force drives the plasma towards the centre of the plasma column,where the energy of the shock-wave is thermalised,and as a result,the plasma temperature is raised.

There are two ways in which the initialising large current of the plasma may be obtained,depending on the geometry of the plasma.If the plasma is a straight column,then the current can be obtained by the discharge of a

large capacitor through the gas column. On the other hand,if the plasma is toroidal,the current can be initiated by transformer action.Large primary coils are wound around the plasma in the poloidal direction. When current is passed through these primary coils,the plasma behaves as though it were the secondary winding of a transformer,and has a current induced in it.

Since ohmic and shock-wave heating alone are unlikely to be able to raise the plasma to thermonuclear temperatures,additional methods of heating the plasma are necessary.One such method is 'Neutral Beam Injection',in which the temperature of the plasma is raised by bombarding it with high energy neutral particles.Alternatively the plasma can be bombarded with radiation at the ion-cyclotron,or electron-cyclotron frequencies.These latter processes are known as Ion Cyclotron Resonance Heating (ICRH) and Electron Cyclotron Resonance Heating (ECRH).A more complete account of the heating processes that have been used to date for raising the temperatures of plasmas can be found in /11/,beginning at p.435 .

This thesis is concerned only with a cylindrical plasma system such as we have described,(although in our case the magnetic field is not wholly produced by the current flowing in the plasma,but is partially externally imposed),and the modes of instability to which such a system is susceptible. These instability modes are commonly denoted by the letter 'm',which may take all positive integer values.The four lowest instability modes for the cylindrical pinch are shown in figure 1.1. Note that m may also be zero. The growth of such ideal MHD instabilities is studied in an important paper

/17/,where it is found that the growth-rates depend not only on the average value of the equilibrium quantities,but also quite often sensitively,on the precise nature of the radial profiles.In /19/ the ideal MHD equilibrium of a plasma confined in a toroid is considered.Two fine expositions of ideal MHD theory are given in /18/ and /20/,and an exhaustive review of the subject is to be found in /21/.Now,in view of the fact that we shall later consider the drawbacks of the single-fluid theory,the following observation about the light in which the ideal MHD theory should be viewed, is pertinent.Thus it is probably the case that the ideal MHD equations (which are presented later in the thesis),are best thought of as describing a dissipationless,infinitely conducting fluid,obeying an adiabatic equation of state,and whose properties are sufficiently close to those of a real plasma to be of interest.

In a refinement to ideal MHD,called 'resistive' MHD,the plasma is allowed to have a small,but finite,electrical conductivity,and as a consequence the magnetic field lines are no longer frozen to the plasma,but may diffuse through it at a rate characterized by the resistive diffusion time /41/. According to the theory /11/,this resistive diffusion of the magnetic field with respect to the fluid,occurs at a significant rate only in a narrow region of space,surrounding a particular surface in the field which is known as the 'resonant surface' or 'singular surface',in the vicinity of which changes in the magnetic topology may occur.More fundamentally however,instabilities of an entirely different character to those of ideal MHD are found to arise /22/,called 'tearing','rippling' and 'gravitational interchange' instabilities.The short-wavelength rippling instability is found to be caused by local channelling of current through ripples in resistivity,(i.e. this instability is driven by gradients in the resistivity). If we consider a cartesian frame of reference,the gravitational resistive instability (or 'resistive g-mode'),occurs in the presence of a mass density gradient,and a y-directed body force (like gravity),when the main magnetic field is in the z-direction.The driving force is found to be destabilising if 'gravity' is in the same direction as the mass density gradient.This instability is the resistive equivalent of the ideal MHD 'interchange' instability /11/.The tearing instability is driven by the profile of equilibrium quantities far from the singular surface,and allows for the transfer of magnetic energy into plasma kinetic energy.In this long-wavelength instability,the current layer breaks up along the current flow lines.

The tearing instability is the resistive equivalent of the ideal MHD 'kink' instability,illustrated in figure 1·1. In the tearing instability the magnetic energy released by the change of topology,is converted into internal kinetic energy of the plasma,in the region around the resonant surface (also called the 'tearing layer').In contrast,the energy of the ideal MHD kink instability is converted into macroscopic motion of the plasma.

The essential feature of all these resistive instabilities is that they violate the topological restriction of the infinite conductivity analysis, (i.e. the preservation of the distinct magnetic surfaces).What essentially happens in the tearing instability,for instance,is that the magnetic field lines,which are no longer frozen to the plasma fluid,may break and reconnect to form 'magnetic islands',each of which has its own associated set of flux surfaces,distinct from the flux surfaces of the initial plasma/magnetic field configuration.

The acceptance of the single-fluid theory naturally depends on how well it is able to explain the experimentally observed properties of hot plasmas, and for this reason we shall now highlight some of the fluid-theoretical work which has been directed towards this end.

Firstly,we note that some workers have suggested that the tearing instability (sometimes called the tearing mode),is important in a wide variety of phenomena,ranging from the phenomenon of 'reconnection' in the magnetosphere,to the magnetic oscillations observed in the plasma confinement devices known as tokamaks /25,26/.The tokamak device is more fully described in /8/, but here it is worth pointing out that such machines are at the centre of most of the international research effort directed at achieving the goal of controlled thermonuclear fusion. To continue, we find that in /27,28/, experimental observation of the tearing mode is claimed by groups working on the toroidal plasma containment device known as the Reversed Field Pinch /29/. This same claim has been made for the linear pinch in /30,31,32,33/ , and for the 'hard-core' pinch configuration in /31,34,35/.Equally important however, in view

of the added impetus that it gave to theoretical plasma physics,was the experimental observation that tokamak plasma discharges which might be expected to continue for about 1ms or longer,are sometimes prone to terminate on a timescale of about 100μs.Such an occurrence is normally termed a 'major disruption',and since the whole process is thought to be due to some kind of instability,the phenomenon has been given the name 'disruptive instability'.Such attempts as have been made to describe the mechanics of this instability,have focused on a non-linear extension of the tearing mode theory /36/,involving the non-linear interaction of low m-number magnetic islands of different helicities.Now,remembering that plasmas of thermonuclear interest will only be weakly collisional,it is easy to realise that the classical theory of transport processes in a highly-collisional,low-temperature gas,such as is described in /44/,can have very little to say about transport processes in a fusion plasma. For this reason,a so-called neo-classical theory /24/,has been developed, with the aim of describing transport processes in a hot,toroidal plasma.

We now consider /37/,in which the view is expressed that the tearing instability might have two important effects in tokamaks.The first of these is that the magnetic fluctuations associated with the mode may lead to the disruption of the magnetic surfaces,and the second is that enhanced particle and energy transport may be the consequence.If these processes do indeed occur,we observe that they would necessarily be complicated by the presence of an electron temperature gradient,because this can destabilise the tearing mode (as is shown in /38/).In a process described in /38/, what is envisaged is that the non-linear development of the tearing instability relaxes the original temperature gradient,and that over the time in which the instability has taken place,the plasma will behave as though it had an enhanced thermal conductivity.Here,the use of the word 'enhanced' implies that the transport coefficient in question,has a value greater than the classically predicted value (or indeed the neo-classically predicted value).It must be pointed out however,that the neoclassical theory is not to be wholly believed.This is because the particle and energy transport found in actual tokamak experiments,is 50 times greater than the values predicted by the neo-classical theory.

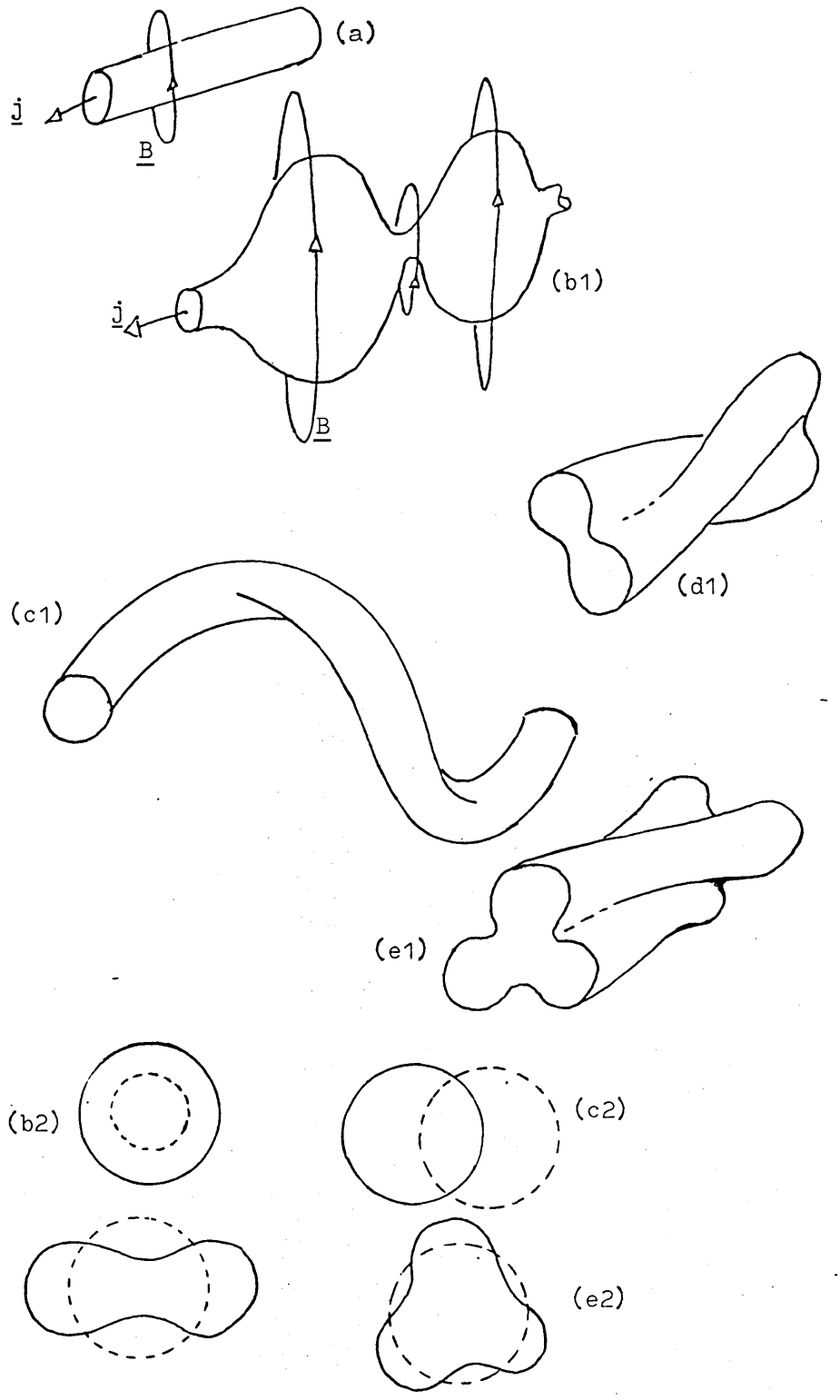


Figure (1.1) Lowest-order perturbations of a cylindrical plasma.

(a)	Unperturbed plasma, showing equilibrium current and magnetic field.
(b1), (b2)	m=0 perturbation
(c1), (c2)	m=1 perturbation
(d1), (d2)	m=2 perturbation
(e1), (e2)	m=3 perturbation

Reviews of the single-fluid hydromagnetic theories are available in /8,21,40/,and /43/ contains a particularly lucid description of resistive MHD.

Our short review has been presented in order to allow us to properly address ourselves to the question of how appropriate a description of high-temperature fusion plasmas,is afforded by the single-fluid theories.This is the purpose of the remainder of this introduction,and we begin by considering some of the shortfalls of these single-fluid theories.

Firstly,we note that the resistive MHD analysis breaks down in the limit of very high conductivity,becasue the 'region of discontinuity' /50/,over which the second derivative of the magnetic perturbation is large,becomes smaller than the particle gyro-radii. Such a situation,within the confines of a fluid theory is clearly inadmissible.Now,one of the most important differences between the ideal MHD theory and the resistive MHD theory,is that the ideal MHD analysis allows no electric field parallel to the equilibrium magnetic field,while the resistive MHD analysis does.This parallel electric field (produced by the slipping of magnetic field lines relative to the plasma in the small inner region of width Δ),produces a current layer of this same spatial width,by accelerating electrons along the local magnetic field.In the classical tearing mode theory and elsewhere /22,42,45,47,48/,it is assumed that $\Delta \gg r_L$, where $r_L = V_{th}/\omega_{ci}$ is the ion gyro-radius, and V_{th} and ω_{ci} are the ion thermal velocity and ion cyclotron frequency,respectively.However,as we have previously pointed out,high-temperature plasmas will be practically collisionless, and thus in view of the result of /49/ that $\Delta \ll r_L$ in the collisionless regime,the question arises as to whether tearing mode theory can have any relevance to fusion plasmas.We must also note that the tearing mode theory of /36/ ,fails to predict the time-dependent structure of the magnetic fluctuation amplitudes as observed during actual tokamak disruptions, and that the experimental observations of tearing instabilities claimed in /27,28/ ,are only in semi-quantitative agreement with the classical theory of /22/.

Further,as pointed out in /42,50/ ,it is not even necessary to have resistivity,in order to obtain tearing of the magnetic topology. All that is necessary is the inclusion of the electron inertia terms

($\frac{\partial \underline{j}}{\partial t}$ and $\underline{j}.\nabla\underline{j}$) in the generalised Ohm's Law /41/.This surely must be another source of doubt concerning the meaning of the standard single-fluid tearing mode theory.

Now,let us consider the resistive,single-fluid Ohm's law,which is expressed by the equation :

$$\underline{E} + \frac{1}{c}(\underline{V} \times \underline{B}) = \eta \underline{j} \qquad (1.1)$$

In this equation the various symbols have the following meanings :

$\underline{j}$	current density	
$\underline{E}$	Electric field	
$\underline{B}$	Magnetic field	
$\underline{V}$	Plasma fluid velocity	
c	Speed of light	
η	Plasma specific resistivity (see /51/)	(1.2)

It is easy to see that this equation cannot be correct for a hot tokamak plasma,because the anisotropy of the magnetic field will mean that there will be Hall currents present,flowing perpendicularly to $\underline{E}$ and $\underline{B}$,which may in principle and in general,be greater than that predicted by equation (1.1). It will be noticed that while the electric fields of equation (1.1) give rise to dissipation,the Hall electric field,does not.In the generalised Ohm's law,the Hall Effect manifests itself as the $\underline{j} \times \underline{B}$ term.It has been shown /1,52,53/,that this term and the ∇p_e term in the Ohm's law,can cause detachment of the ion fluid from the magnetic field,even in the absence of resistivity.The impact of this observation on the way in which we view results obtained from the tearing mode theory,is more easily appreciated when we realise that,as pointed out in /54/,the neglect of the Hall term is far more restrictive an approximation than the neglect of resistivity. Consequently,a theoretical account of experimental plasma behaviour that includes resistivity,but does not include the Hall term,must be regarded with extreme caution.

There are criticisms to be made of the ideal MHD theory too, in spite of the overtly approximate nature of this theory. To appreciate our argument it must be remembered that in the classical resistive MHD analysis, the ideal MHD equations apply everywhere,except in the

vicinity of the resonant surfaces.We shall also need to use the concept of the 'safety factor',defined in /8/,and normally denoted by the letter q. It is a very useful parameter for the description of the stability of given equilibrium plasma/magnetic field configurations,and is a function of r (in cylindrical coordinates).

Now,according to the ideal MHD theory of the 'kink' instability as presented in /82/,if $q(0) \gtrsim 1$, (where q(0) is the safety factor in the centre of the discharge),then for values of $q(a) \lesssim 2$, (where a is the radius of the cylindrical plasma,so that q(a) is the safety factor at the edge of the plasma),there is a very wide range of current profiles for which the plasma must be unstable to the m=2,n=1 kink mode (see figure 1.2).(Here,'n' is another number which may take integral values , that is used to describe the modes of instability of a cylindrical plasma). Yet,in spite of this prediction,some tokamaks have been made to operate in quite a stable manner,with q(a) close to or even less than 2.Furthermore, any attempt to explain this experimentally observed stability by an appeal to the linear theory will fail.This is because according to the linear theory,the instability zone narrows,and possibly disappears,only for current distributions that are so strongly peaked that $q(a)/q(0) \gtrsim 3.5$, /55/, and since $q(a) \lesssim 2$, this would require $q(0) \lesssim 0.5$. However,as pointed out in /56/,the realisation of the current profiles necessary to produce $q(0) \lesssim 0.5$ is inhibited by processes of internal disruption,which prevent noticeable increase of the current over a critical value at the edge of the plasma, and prevent q(0) from falling much below unity.What we have described therefore,is a serious failing of the MHD theory.Yet,skirting this problem others /57/,have suggested,mainly using results from non-linear numerical computation,that the tokamak of infinite major radius,may non-linearly evolve from an unstable cylindrical configuration,into a helically stable one.This attempt at explanation of the experimentally observed tokamak stability which we have just cited,also fails.This is because, as shown in /58/,the transition to such a helically stable equilibrium, is not possible.Finally,if we remember that within the single-fluid theory,magnetic fields are frozen into the plasma,we must ask,as /1/ also asks,how it is possible to obtain an 'anomalously' large cross-field diffusion of particles in tokamaks,without any observed anomaly in resistivity.

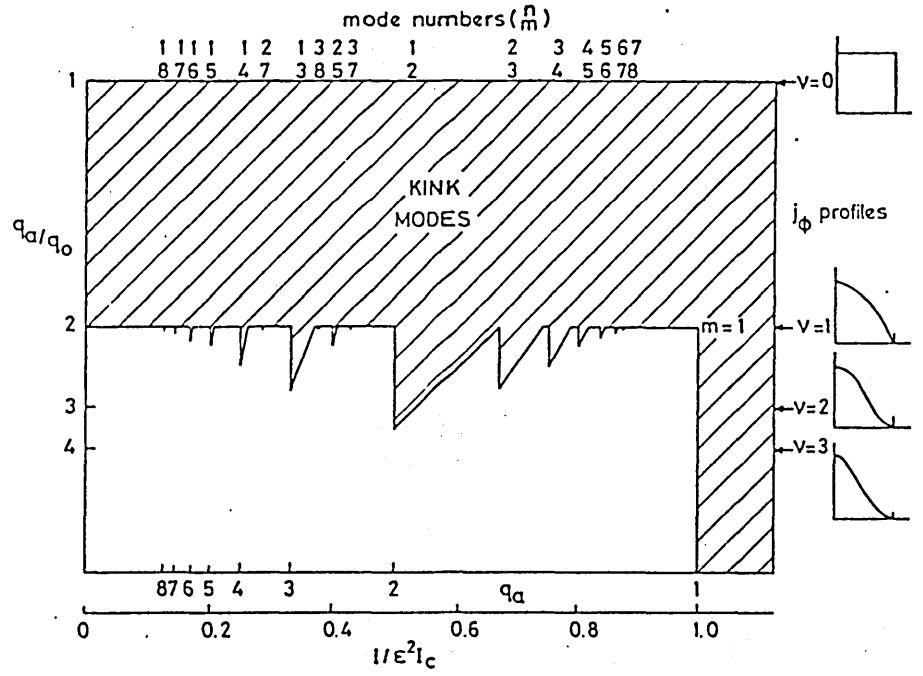


Figure 1.2 *Stability diagram for kink modes. The vertical axis measures the peaking of the current as given by q_a/q_0 (which is equal to $\nu+1$). The horizontal axis measures the total current I expressed as a fraction of $(a/R_0)^2 I_c$ where I_c is the current in the coil producing the toroidal magnetic field. This axis also gives q_a, which is inversely proportional to I. The mode numbers associated with these values of q_a are given at the top of the figure*

Note: This figure is taken from p.226 of /82/, and it is understood that

$$j_\phi(r) \equiv j_{oz}(r) .$$

The evidence that we have presented so far would seem to suggest that no serious case can be made (other than on the grounds of ease of formulation), for using single-fluid theory to try to describe confined plasmas at thermonuclear fusion temperatures. It would also seem to suggest that the single-fluid theory is far more incomplete a theory, than its ubiquity in the plasma physics literature would imply. This thesis provides further evidence in support of this viewpoint, by considering the two-fluid, dissipationless model used in /1/. The equations of the model are cast in a cylindrical geometry, linearised, and then Fourier analysed in the two ignorable space coordinates (θ and z). Then, after using the equation

$$\nabla \cdot \underline{B} = 0 \qquad (1.3)$$

to eliminate one of the components of $\underline{B}$, we are left with 8 equations for the 8 complex perturbation variables. This of course means that we are really dealing with 16 equations. These equations are initialised with an arbitrary initial perturbation, and then integrated forwards in time, until such time as the most rapidly exponentially-growing radial oscillation mode has become dominant. This means that our method can make no statements about the nature of the full eigenspectrum of the problem, since it only returns information about the eigenmode with the largest exponential growth-rate.

We note that this initial/boundary-value approach has been used successfully elsewhere /39,59,60,61/, where finite differencing in both space and time has been employed. We have however, used the numerical integration procedure known as the 'method of lines' (MOL), in which the spatial derivative $\frac{\partial}{\partial x_i}$ is replaced by finite differences, and the time derivative $\frac{\partial}{\partial t}$ is left continuous. If N spatial points are used to discretise the spatial domain, then a system of 16 x N coupled ordinary differential equations (ODEs) results, in place of the 16 coupled partial differential equations (PDEs), with which we started.

Herein lies the main advantage of our method, for our ODEs, like any other system of coupled ODEs, can be integrated, either implicitly or explicitly, by the multitude of sophisticated algorithms that have been developed for precisely this purpose. Examples of such algorithms which we have actually used are the Runge-Kutta-Fehlberg,

the Adams-Bashforth-Moulton and the Hindmarsh-Gear algorithms.These,and other algorithms that we have used are described in /65,69,72,77/.Thus,the use of the MOL allows us to make use of the facilities for automatic error-monitoring and automatic error-control,of these ODE integration schemes,as the integration proceeds.The same cannot be said of standard finite-difference replacements,such as the Cranck-Nicholson or Lax-Wendroff /61/ methods.Attempts have been made to implement variable-step finite-difference methods,but the step control mechanism tends to be crude and rather inefficient /97/.Much theoretical attention has been given to the MOL in the Russian literature /62/,but not elsewhere.

We have studied three plasma equilibria characterised the z-current density profiles:

$$j_{oz} = j_o\left(1 - \frac{r^2}{a^2}\right)^{\nu}$$

where ν takes the values 1,2, and 4. a is the radius of the plasma,and r is the radial coordinate of the cylindrical geometry.We have then uniformly altered our equilibrium quantities in order to bring our equations into the regimes in which the approximations of MHD hold,and as a result we have found that the single-fluid theory substantially underestimates the linear growth rates of such modes as are found to be unstable.

The motivation for this thesis was provided by the work of /1/,and so it was natural that we should criticize that work.We have found as a result,that serious errors have been made in the derivation that is given there of the two-fluid eigenvalue problem.We have also found,by using numerical and analytical considerations,that the approximations that have been made in /1/,in order to derive the eigenvalue problem,are not valid.

Finally,we have included as appendices,necessary algebraic manipulations which are too long to be included in the main text.

Chapter 1

1.1 The Basic Equations

The equations used in the fluid descriptions of plasmas are derivable from the Boltzmann-Vlasov 'moments' equations /68,69,106/,and it is not necessary to repeat such derivations here.Instead,for comparison with the two-fluid equation system with which we shall be concerned for the bulk of this thesis,we write down the equations used in the ideal MHD model.They are:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{v}) = 0 \qquad (1.1\text{-}1)$$

$$\rho\left(\frac{\partial}{\partial t} + \underline{v}.\nabla\right)\underline{v} = \frac{1}{c}(\underline{j} \wedge \underline{B}) - \nabla p \qquad (1.1\text{-}2)$$

$$\left(\frac{\partial}{\partial t} + \underline{v}.\nabla\right)\left[\frac{p}{\rho^{\gamma}}\right] = 0 \qquad (1.1\text{-}3)$$

$$\underline{\underline{E}} + \frac{1}{c}(\underline{v} \wedge \underline{B}) = 0 \qquad (1.1\text{-}4)$$

where

$$\rho = n m_i \qquad (1.1\text{-}5)$$

The symbols in these equations not already defined in equation (1.2), have the following meanings :

m_i	mass of each ion in the plasma fluid
n	number density of either ions or electrons
γ	ratio of specific heats
p	sum of the ionic and electronic fluid pressures

(1.1-6)

If we now supplement equations (1.1-1) to (1.1-4) with the two Maxwell equations :

$$\nabla \wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \qquad (1.1\text{-}7)$$

$$\nabla \wedge \underline{B} = \frac{4\pi}{c}\underline{j} \qquad (1.1\text{-}8)$$

we would then have the full prescription for the so-called ieal MHD model. If also, equation (1.1-4) is modified so that it reads :

$$\underline{E} + \frac{1}{c}(\underline{v} \wedge \underline{B}) = \eta \underline{j} \qquad (1.1\text{-}9)$$

then our system of equations would describe what is called the resistive MHD model. It will be noticed that all our equations so far have been written in Gaussian units.This system of units will be used throughout this thesis.It is also the system of units that has been used in the bulk of the published plasma physics literature.

Concerning the single-fluid equations that we have written down,it is seen that equation (1.1-1) expresses the continuity of the plasma fluid,and for this reason it is sometimes called the 'continuity equation'. Equation (1.1-2) expresses the conservation of the momentum of the fluid,and equation (1.1-3),the conservation of energy of the (assumed adiabatic) fluid. Finally,equation (1.1-4) is the perverted form of the generalised Ohm's law,which leads to the magnetic field being 'frozen' to the plasma fluid.If the plasma were to be assumed to be incompressible, or if we were only interested in such motions of the plasma as did not cause it to be compressed,then we could replace the energy equation above by :

$$\nabla \cdot \underline{v} = 0 \qquad (1.1\text{-}10)$$

The inclusion of this equation will substantially simplify any single-fluid (or two-fluid) analysis,since it will permit the elimination of one velocity component.

We now write down,for comparison,the equations which will be used in our two-fluid plasma study.They are :

$$\frac{\partial p_i}{\partial t} + (\underline{v}.\nabla)p_i = -\gamma p_i \nabla.\underline{v} \qquad (1.1\text{-}11)$$

$$\frac{\partial p_e}{\partial t} + (\underline{u}.\nabla)p_e = -\gamma p_e \nabla.\underline{u} \qquad (1.1\text{-}12)$$

$$\rho\left(\frac{\partial}{\partial t} + \underline{v}.\nabla\right)\underline{v} = en\left[\underline{E} + \frac{1}{c}(\underline{v}\wedge\underline{B})\right] - \nabla p_i \qquad (1.1\text{-}13)$$

$$0 = -en\left[\underline{E} + \frac{1}{c}(\underline{u}\wedge\underline{B})\right] - \nabla p_e \qquad (1.1\text{-}14)$$

$$\underline{j} = en(\underline{v} - \underline{u}) \qquad (1.1\text{-}15)$$

and substituting equation (1.1-15) into equation (1.1-14) we find that

$$\underline{E} + \frac{1}{c}(\underline{v}\wedge\underline{B}) = \frac{1}{enc}(\underline{j}\wedge\underline{B}) - \frac{1}{en}\nabla p_e \qquad (1.1\text{-}16)$$

The continuity equation is given by :

$$\frac{\partial n}{\partial t} + \nabla.(n\underline{v}) = 0 \qquad (1.1\text{-}17)$$

and as before,we supplement these equations with equation (1.1-7) and equation (1.1-8). We also note that equations (1.1-11) and (1.1-12) might also be written (if both fluids are assumed to be adiabatic),as follows :

$$n\left(\frac{\partial}{\partial t} + \underline{v}.\nabla\right)T_i + p_i(\gamma-1)\nabla.\underline{v} = 0 \qquad (1.1\text{-}18)$$

$$n\left(\frac{\partial}{\partial t} + \underline{u}.\nabla\right)T_e + p_e(\gamma-1)\nabla.\underline{u} = 0 \qquad (1.1\text{-}19)$$

where

$$p_i = nT_i \qquad (1.1\text{-}20)$$

and

$$p_e = nT_e \qquad (1.1\text{-}21)$$

We note immediately that in this model (as opposed to the single-fluid models), two fluid velocities ($\underline{u}$ and $\underline{v}$,where $\underline{u}$ is the electron fluid velocity),characterize the motion of the plasma in principle,although $\underline{u}$ can be eliminated from the system of equations.Other differences are evident in the Ohm's law (equation (1.1-16)),where it is seen that the ∇p_e and the $\underline{j} \times \underline{B}$ terms have been retained.We also note that there are different conservation of energy equations for the electrons and the ions (equation (1.1-11) and equation (1.1-12)),and that because we are assuming quasineutrality (i.e. that $n_i \simeq n_e$),we only need one continuity equation.We have chosen to use the ion continuity equation,as it is of simpler form.

In the next section we shall deal with the linearisation of our two-fluid equations.There,when linearising,we shall follow the common practice of taking the equilibrium ionic fluid velocity to be zero, i.e.

$$\underline{V}_o = 0 \qquad (1.1\text{-}21a)$$

For ideal MHD,this automatically implies that the equilibrium electric field, $\underline{E}_o = 0$, but within the two-fluid theory,this is not the case.We shall however,allow for an equilibrium flow of the electronic fluid,so that the equilibrium current flow will be given by

$$\underline{j}_o = - e n_o \underline{u}_o \qquad (1.1\text{-}22)$$

As is usual,a 'o' subscript will denote an equilibrium quantity,while a '1' subscript will denote a perturbed quantity.We also note that because of quasineutrality,it will be true that

$$\nabla \cdot \underline{j}_o = 0 \qquad (1.1\text{-}23)$$

Since we shall assume that

$$\underline{u}_o = [0 , u_{o\theta}(r) , u_{oz}(r)] \quad \text{and} \quad f_o = f_o(r) \qquad (1.1\text{-}24)$$

equations (1.1-22) and (1.1-23) will lead us to the conclusion that

$$\nabla \cdot \underline{u}_o = 0 \qquad (1.1\text{-}25)$$

where $f_o(r)$ is any equilibrium quantity.

1.2 Linearisation, Error I and Error II

To linearise the equations the variable f is written as $(f_o + f_1)$ in order to form the perturbed eqations, from which is subtracted the equilibrium equation (formed by replacing f by f_o). The equation that remains is the linearised equation, if no terms other than those which are first order in the perturbation variable f_1 , are retained. As a result of using this procedure, we find that the linearised equations for our two-fluid system of equations are :

$$n_o m_i \frac{\partial \underline{v}}{\partial t} = e n_o \left[\underline{E}_1 + \frac{\underline{v}_1 \wedge \underline{B}_o}{c} \right] + e n_1 \underline{E}_o - \nabla p_{1i} \qquad (1.2\text{-}1)$$

$$0 = -e n_o \left[\underline{E}_1 + \frac{\underline{u}_o \wedge \underline{B}_1}{c} + \frac{\underline{u}_1 \wedge \underline{B}_o}{c} \right] - e n_1 \underline{E}_o - \nabla p_{1e} - \frac{e n_1}{c} (\underline{u}_o \wedge \underline{B}_o) \qquad (1.2\text{-}2)$$

$$\frac{3}{2} n_o \frac{\partial T_{1i}}{\partial t} = -\frac{3}{2} n_o (\underline{v}_1 . \nabla) T_o - p_{oi} \nabla . \underline{v}_1 \qquad (1.2\text{-}3)$$

$$\frac{3}{2} n_o \frac{\partial T_{1e}}{\partial t} = -\frac{3}{2} n_o \left[(\underline{u}_o . \nabla) T_{1e} + (\underline{u}_1 . \nabla) T_{oe} \right] - p_{oe} \nabla . \underline{u}_1 \qquad (1.2\text{-}4)$$

$$\frac{\partial n_1}{\partial t} + n_o \nabla . \underline{v}_1 + (\underline{v}_1 . \nabla) n_o = 0 \qquad (1.2\text{-}5)$$

$$\nabla \wedge \underline{B}_1 = \frac{4\pi e}{c} \left[n_o (\underline{v}_1 - \underline{u}_1) - n_1 \underline{u}_o \right] \qquad (1.2\text{-}6)$$

$$\nabla \wedge \underline{E}_1 = -\frac{1}{c} \frac{\partial \underline{B}_1}{\partial t} \qquad (1.2\text{-}7)$$

Now, the balancing of magnetic pressure and kinetic pressure in equilibrium (obtained by adding together the equilibrium ion and electron momentum transfer equations), tells us that

$$\nabla p_o = \frac{1}{c} (\underline{j}_o \wedge \underline{B}_o) \qquad (1.2\text{-}8)$$

where

$$p_o = p_{oi} + p_{oe} \qquad (1.2\text{-}9)$$

Substituting from equation (1.1-22) for the equilibrium current density, into equation (1.2-8), we find that

$$\nabla p_0 = -\frac{e n_0}{c}(\underline{u}_0 \wedge \underline{B}_0) \qquad (1.2\text{-}10)$$

and when this equation is substituted into the Ohm's law equation (equation 1.2-2),that equation becomes

$$0 = -e n_0 \left[\underline{E}_1 + \frac{1}{c}(\underline{u}_0 \wedge \underline{B}_1) + \frac{1}{c}(\underline{u}_1 \wedge \underline{B}_0)\right] - e n_1 \underline{E}_0 - \nabla p_{1e} + \frac{n_1}{n_0}\nabla p_0 \qquad (1.2\text{-}11)$$

In /1/, the r-component of this equation in a cylindrical coordinate system,is given as

$$0 = -e n_0 \left[E_{1r} + \frac{u_{1\theta}B_{0z}}{c} + \frac{u_{0\theta}B_{1z}}{c} - \frac{u_{1z}B_{0\theta}}{c} - \frac{u_{0z}B_{1\theta}}{c}\right] - \frac{dp_{1e}}{dr} - e n_1 E_{0r} \qquad (1.2\text{-}12)$$

and this is obviously incorrect,since the last term of equation (1.2-11) has been omitted. We shall call this error in the work of /1/, Error I. It is a serious error,but there is one instance in which it is of no consequence,and this instance,along with the fact of the omission itself,must be stated explicitly,in order for the formulation to be considered not to be in error.We shall presently provide substantiation of this claim.

Remembering that there is no radial current in equilibrium,the three components of Ampere's law (equation 1.2-6) are :

$$\frac{1}{r}\frac{\partial B_{1z}}{\partial \theta} - \frac{\partial B_{1\theta}}{\partial z} = \frac{4\pi e n_0}{c}(v_{1r} - u_{1r}) \qquad (1.2\text{-}13)$$

$$\frac{\partial B_{1r}}{\partial z} - \frac{\partial B_{1z}}{\partial r} = \frac{4\pi e n_0}{c}(v_{1\theta} - u_{1\theta}) - \frac{4\pi e n_1 u_{0\theta}}{c} \qquad (1.2\text{-}14)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_{1\theta}) - \frac{1}{r}\frac{\partial B_{1r}}{\partial \theta} = \frac{4\pi e n_0}{c}(v_{1z} - u_{1z}) - \frac{4\pi e n_1 u_{0z}}{c} \qquad (1.2\text{-}15)$$

For economy of notation, it will prove convenient to refer to equations in /1/, by the number of the equation in that paper, preceded by the letters TH.

We have found it to be the case that, for the purposes of deriving equation (TH37) and equation (TH41), which are of central importance to the eigenvalue formulation presented in /1/, there is only one instance in which Error I is inconsequential, and that is the instance in which the last term of equation (1.2-14) and the last term of equation (1.2-15) are omitted. (See Appendix C). There seems however, to be no obvious reason for these omissions, and for this reason we shall refer to them as Error II. We cannot tell from /1/ if Error II has been committed, since equations (1.2-14) and (1.2-15) are never written down in that paper. In Appendix C however, we have shown that this error must have been committed, and so we do not consider it any further.

In order to continue our criticism of /1/, it will prove convenient to follow the steps in the derivation of their basic eigenvalueproblem (i.e. the two coupled equations TH37 and TH41), rather closely. Thus, in order to permit Fourier analysis in the θ and z directions, and in time, we only consider perturbations of the form

$$f_1(r,\theta,z,t) = \hat{f}_{1mn\omega}(r)\exp\left(i\omega t + im\theta + \frac{inz}{R}\right) \qquad (1.2\text{-}16)$$

What we are trying to visualise is shown in figure (1.2-1). The hope is that the toroidal plasma of minor radius a, and major radius R, may be approximately represented by an infinitely long plasma cylinder of radius a, whose properties are repeated every $2\pi R$ cm in the z-direction. We note that for brevity in our linearised equations, we shall simply write f_1 for the amplitude function $\hat{f}_{1mn\omega}(r)$, and the dependence on m, n, ω, and r, will be understood. We shall also take the equilibrium magnetic field to have no radial component, so that we may write it as

$$\underline{B}_o = (0, B_{o\theta}, B_{oz}) \qquad (1.2\text{-}17)$$

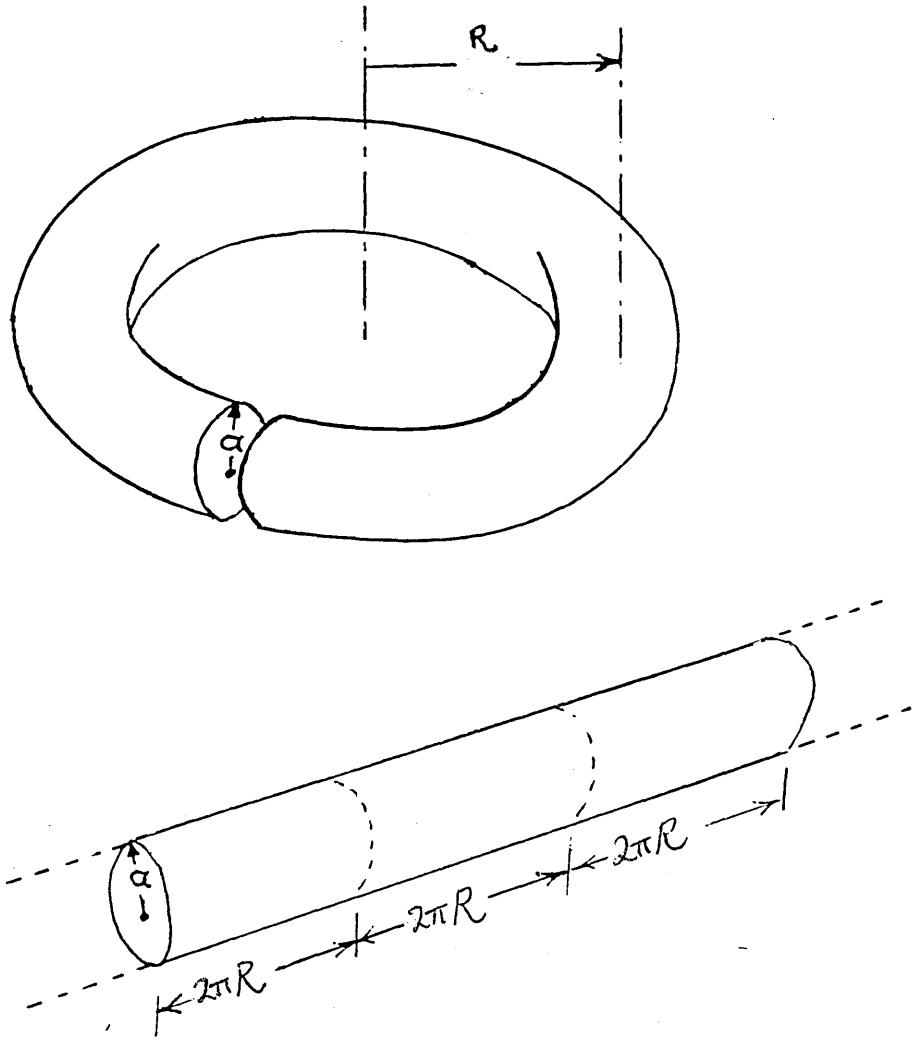


Figure 1.2-1 The approximation of a toroid as an infinite periodic cylinder.

Another equation which we may use, and which reduces the number of dependent variables by one, is

$$\nabla \cdot \underline{B}_1 = 0 \qquad (1.2\text{-}18)$$

i.e.

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_{1r}) + \frac{1}{r}\frac{\partial B_{1\theta}}{\partial \theta} + \frac{\partial B_{1z}}{\partial z} = 0 \qquad (1.2\text{-}19)$$

and it will be remembered that this equation is not an independent equation, but is forced to be true throughout the time-development of the plasma by Maxwell's equations, if it was true at the time when we began our consideration of the plasma. We now substitute equation (1.2-16) into our linearised equations, so that they take the form :

$$im_i\omega n_0 v_{1r} = en_0\left[E_{1r} + \frac{1}{c}(v_{1\theta}B_{0z} - v_{1z}B_{0\theta})\right] + en_1E_{0r} - \frac{dp_{1i}}{dr} \qquad (1.2\text{-}20)$$

$$im_i\omega n_0 v_{1\theta} = en_0\left[E_{1\theta} - \frac{v_{1r}B_{0z}}{c}\right] - \frac{im}{r}p_{1i} \qquad (1.2\text{-}21)$$

$$im_i\omega n_0 v_{1z} = en_0\left[E_{1z} + \frac{v_{1r}B_{0\theta}}{c}\right] - \frac{in}{R}p_{1i} \qquad (1.2\text{-}22)$$

These three equations are the three components of the ion-fluid momentum equation. The three components of the electron-fluid 'momentum' are :

$$0 = -en_0\left[E_{1r} + \frac{1}{c}(u_{0\theta}B_{1z} + u_{1\theta}B_{0z} - u_{0z}B_{1\theta} - u_{1z}B_{0\theta})\right] - en_1E_{0r} - \frac{dp_{1e}}{dr} + \frac{n_1}{n_0}\frac{dp_0}{dr} \qquad (1.2\text{-}23)$$

$$0 = -en_0\left[E_{1\theta} + \frac{1}{c}(u_{0z}B_{1r} - u_{1r}B_{0z})\right] - \frac{im}{r}p_{1e} \qquad (1.2\text{-}24)$$

$$0 = -en_0\left[E_{1z} + \frac{1}{c}(u_{1r}B_{0\theta} - u_{0\theta}B_{1r})\right] - \frac{in}{R}p_{1e} \qquad (1.2\text{-}25)$$

In the above equations we have used the fact that there is no radial current in equilibrium, i.e.

$$u_{or} = v_{or} = 0 \qquad (1.2\text{-}26)$$

The ion and electron energy equations are given respectively by :

$$\frac{3}{2} n_o \left[i\omega T_{1i} + V_{1r} \frac{dT_{oi}}{dr} \right] + n_o T_{oi} (\nabla \cdot \underline{V}_1) = 0 \qquad (1.2\text{-}27)$$

$$\frac{3}{2} n_o \left[i\omega T_{1e} + (\underline{u}_o \cdot \nabla) T_{1e} + u_{1r} \frac{dT_{oe}}{dr} \right] + n_o T_{oe} (\nabla \cdot \underline{u}_1) = 0 \qquad (1.2\text{-}28)$$

and the three components of Faraday's law are

$$\frac{im}{r} E_{1z} - \frac{in}{R} E_{1\theta} = - \frac{i\omega}{c} B_{1r} \qquad (1.2\text{-}29)$$

$$\frac{in E_{1r}}{R} - \frac{dE_{1z}}{dr} = - \frac{i\omega}{c} B_{1\theta} \qquad (1.2\text{-}30)$$

$$\frac{1}{r} \frac{d}{dr} (r E_{1\theta}) - \frac{im E_{1r}}{r} = - \frac{i\omega B_{1z}}{c} \qquad (1.2\text{-}30)$$

Finally, we write down the three components of Ampere's law

$$\frac{im}{r} B_{1z} - \frac{in}{R} B_{1\theta} = \frac{4\pi e n_o}{c} (V_{1r} - u_{1r}) \qquad (1.2\text{-}32)$$

$$\frac{in}{R} B_{1r} - \frac{dB_{1z}}{dr} = \frac{4\pi e n_o}{c} (V_{1\theta} - u_{1\theta}) - \underline{\frac{4\pi e n_1 u_{0\theta}}{c}} \qquad (1.2\text{-}33)$$

$$\frac{1}{r} \frac{d}{dr} (r B_{1\theta}) - \frac{im}{r} B_{1r} = \frac{4\pi e n_o}{c} (V_{1z} - u_{1z}) - \underline{\frac{4\pi e n_1 u_{oz}}{c}} \qquad (1.2\text{-}34)$$

The term whose omission in equation (1.2-23) constitutes Error I has been underlined, as have the terms in equations (1.2-33) and (1.2-34), whose omission constitutes Error II.

Equations (1.2-19) to (1.2-23) are the equations on which the eigenvalue formulation of /1/ is based. Now, in agreement with the notation of /1/, we define the following quantities :

$$
\begin{aligned}
T_0(r) &= T_{oe}(r) + T_{oi}(r) \\
P_o(r) &= n_0(r) T_0(r) \\
B_0^2(r) &= B_{o\theta}^2(r) + B_{oz}^2(r) \\
P_1(r) &= P_{1i}(r) + P_{1e}(r) \\
\psi(r) &= B_{1r}(r)/B_0(r) \\
k_{\parallel}(r) &= \frac{1}{B_0}(\underline{k}\cdot\underline{B}_0) = \frac{1}{B_0}\left[\frac{mB_{o\theta}}{r} + \frac{nB_{oz}}{R}\right] \\
\Omega_{mn} &= -\underline{k}\cdot\underline{u}_0 = -\left[\frac{mu_{o\theta}}{r} + \frac{nu_{oz}}{R}\right] \\
\Omega &= \omega - \Omega_{mn} = \omega + \underline{k}\cdot\underline{u}_0
\end{aligned}
\qquad (1.2\text{-}35)
$$

Let us call equations (1.2-19) to (1.2-34) the Basic Set of equations. We find that it is possible to perform simple eliminations among the Basic Set of equations in order to arrive at the result :

$$
u_{1r} = \frac{\Omega}{k_{\parallel}}\psi \qquad (1.2\text{-}36)
$$

which gives the radial perturbation velocity of the electron fluid.In Appendix A we have shown that this equation implies that the magnetic field is 'tied' to the electron fluid,as far as radial motion is concerned.This result has also been reported in /1/.The derivation of the equivalent result for the ion fluid is much more involved,but we have outlined the procedure in Appendix B.The result is

$$
V_{1r} = \left(1 + \frac{\omega}{\omega_{ci}}\cdot M\right)\frac{\omega\psi}{k_{\parallel}} + a_4 + \frac{1}{D}\cdot\frac{\omega}{\omega_{ci}}\left[a_1 + a_2 + a_3\right] \qquad (1.2\text{-}37)
$$

where

$$
a_1 = \frac{i^2 c T_0}{eB_0}\left(\frac{T_0'}{T_0}\Delta_P + \Delta_P'\right) \qquad (1.2\text{-}38)
$$

$$
a_2 = \frac{icT_0}{eB_0}\left[\frac{n_1 P_{oi}'}{n_0 P_0} - \frac{n_0' P_{1i}}{n_0 P_0}\right] \qquad (1.2\text{-}39)
$$

$$
a_3 = -\frac{1}{k^2 r}\left[\frac{nB_{o\theta}}{RB_0} - \frac{mB_{oz}}{rB_0}\right]\frac{d}{dr}\left[\frac{r\omega\psi}{k_{\parallel}}\right] \qquad (1.2\text{-}40)
$$

$$
a_4 = \frac{\omega^* P_0 \Delta_P}{D P_0'} \qquad (1.2\text{-}41)
$$

In /1/ we find that the terms a_1 and a_2 have been reported incorrectly as a_1^* and a_2^*, where

$$a_1^* = \frac{icT_o}{eB_o}\left[\frac{T_o'\Delta_p}{T_o} + \Delta_p'\right] \qquad (1.2-42)$$

$$a_2^* = \frac{icT_o}{eB_o}\left[\frac{n_1 p_{oi}'}{n_o p_o} - \frac{n_o'(p_{1i}+p_{1e})}{n_o p_o}\right] \qquad (1.2-43)$$

In equations (1.2-37) to (1.2-41), the primes denote differentiation with respect to the radial coordinate r, and the quantities involved which have not yet been defined, have the following meanings :

$$D = 1 - \frac{\omega^2}{\omega_{ci}^2}\left(1 + \frac{\omega_{pi}^2}{k^2c^2}\right)$$

$$k^2 = \frac{m^2}{r^2} + \frac{n^2}{R^2}$$

$$\omega_{ci} = \frac{eB_o}{m_i c} \qquad \text{(ion cyclotron frequency)}$$

$$\omega_{pi}^2 = \frac{4\pi e^2 n_o}{m_i} \qquad \text{(ion plasma frequency, squared)}$$

$$\omega^* = -\frac{cT_o p_o'}{eB_o^2 p_o}\left[\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right]$$

$$M = \frac{1}{D}\left[\frac{\omega}{\omega_{ci}} + \frac{2nB_{o\theta}}{B_o k^2 r R}\right]$$

$$\Delta_p = \frac{ip_1}{p_o} + \frac{p_o'\psi}{p_o k_{\parallel}} \qquad (1.2-44)$$

Notwithstanding the errors that have been made in /1/ in the derivation of equation (1.2-37), that equation shows that even in the absence of dissipation, the ion fluid may become detached from the magnetic field, and that in principle the nature of this detachment can be quite complicated. In order to reduce the complexity of the problem to more manageable levels, it was proposed in /1/ that only such low frequencies that

$$\frac{\omega}{\omega_{ci}} \ll 1 \qquad (1.2-44a)$$

be considered. Then arguing that for typical fusion tokamak parameters the quantities D and M are such that

$$D \sim 1 \quad , \quad M \ll 1 \qquad (1.2-44b)$$

the equation (1.2-37) can be written :

$$V_{ir} = \frac{\omega\psi}{k_{\parallel}} + a_4 \qquad (1.2-45)$$

where a_4 is given by equation (1.2-41). It will be observed that if this low-frequency version of equation (1.2-37) is correct, then both of the terms which have been reported incorrectly in /1/ will be insignificant. Also, since the typical value of ω_{ci} for fusion tokamaks will be about 10^8 s^{-1}, it is envisaged in /1/ that this theory will apply for frequencies below about 10^6 s^{-1}.

The coupled equations of the eigenvalue formulation of /1/ (i.e. equations TH37 and TH41), can be found, after much tedious algebra, to follow from the use of equation (1.2-45) for v_{1r}, in further eliminations within our Basic Set of equations. These two coupled equations are the cornerstone of the work presented in /1/, and the solution of these equations is the source of all the results presented in that paper. It is important to note then, that we believe that equation (1.2-45), which is crucial to the derivation of the coupled equations of /1/, is incorrect. We shall discuss our objections to this equation in the next section.

1.3 Discussion of the Expression for v_{1r} —

In /1/, only the case in which the resonant surface $r=r_o$ lies within the plasma volume, is considered. We shall now show that if such is the situation of the singular surface, then it is never legitimate to neglect the a_3 term in comparison to the a_4 term, in equation (1.2-37), as has been done in /1/, no matter how low the frequency is. In order to demonstrate this, let us assume, as is done in /1/, that $D \sim 1$. Then it will be true that the ratio of the terms in a_3 and a_4 in equation (1.2-37) is :

$$y \equiv \frac{\frac{\omega}{\omega_a} a_3}{a_4} \sim \frac{\left(\frac{\omega}{\omega_{ci}}\right)^3 \left(\frac{R}{r_L}\right)^2 \frac{d}{dr}\left[\frac{\psi}{k_{\parallel}}\right]}{\left[\frac{i p_1}{p_o} + \frac{p_o' \psi}{p_o k_{\parallel}}\right]} \qquad (1.3\text{-}1)$$

It is important to examine this ratio in the limit

$$r \rightarrow r_o \qquad (1.3\text{-}1a)$$

or equivalently

$$k_{\parallel} \rightarrow 0 \qquad (1.3\text{-}1b)$$

Let us also assume that

$$\psi(r_o) \neq 0 \qquad (1.3\text{-}1c)$$

It then follows that in the limit in (1.3-1b), the second term in the denominator of equation (1.3-1), dominates the first. If also we make the assumption that the radial derivatives of equilibrium quantities can be ordered as follows

$$\frac{d}{dr} \sim \frac{1}{a} \qquad (1.3\text{-}2)$$

where a is the radius of the plasma, then we can write

$$\frac{p_o'}{p_o} \sim \frac{1}{a} \qquad (1.3\text{-}3)$$

and it will follow that

$$y \sim \frac{a\left(\frac{\omega}{\omega_{ci}}\right)^3 \left(\frac{R}{r_L}\right)^2 \frac{d}{dr}\left[\frac{\psi}{k_{\parallel}}\right]}{\psi/k_{\parallel}} \qquad (1.3\text{-}4)$$

In equations (1.3-1) and (1.3-4), r_L is the ion cyclotron radius.

Now,let us assume that all the necessary eliminations for the derivation of equations (TH37) and (TH41),using equation (1.2-45),have been done. On examining these equations we would find that for each of them the point $r=r_o$ is a regular singular point /70/.This observation has interesting consequences for equation (1.2-37),which we will now consider.

Since r_o is a regular singular point of the equations,we may make the following single-term Frobenius expansion /70/,for the solution function Ψ , valid in the vicinity of $r=r_o$:

$$\psi(r-r_0) = C_1(r-r_0)^{\rho} + \cdots \qquad (1.3\text{-}4a)$$

where C_1 is an arbitrary constant and ρ is to be determined from the indicial equation /70/,but is not of specific interest here.Also,since $k_{\parallel}$ is a perfectly well-behaved function,we may Taylor-expand it in the vicinity of $r=r_o$ as follows :

$$k_{\parallel}(r-r_0) = \left[\frac{dk_{\parallel}}{dr}\right]_{r=r_0}\cdot(r-r_0) + \cdots \qquad (1.3\text{-}4b)$$

and so we find that

$$\frac{\Psi}{k_{\parallel}} \simeq C_2(r-r_0)^{\rho-1} \qquad (1.3\text{-}5)$$

where

$$C_2 = C_1/k_{\parallel}'(r_0) \qquad (1.3\text{-}5a)$$

is an arbitrary constant. Thus differentiating equation (1.3-5) with respect to r

$$\frac{d}{dr}\left[\frac{\Psi}{k_{\parallel}}\right] \simeq C_2(\rho-1)(r-r_0)^{\rho-2} \qquad (1.3\text{-}6)$$

and using equation (1.3-5) again along with equation (1.3-6) we find that :

$$\frac{\frac{d}{dr}\left[\frac{\Psi}{k_{\parallel}}\right]}{\left[\Psi/k_{\parallel}\right]} \simeq \frac{(\rho-1)}{(r-r_0)} \qquad (1.3\text{-}7)$$

It therefore follows that since we cannot expect that ρ is in general real and equal to unity,the ratio in equation (1.2-37) becomes infinite as $r \to r_o$. Consequently,the neglect of the a_3 term in comparison to the a_4 term is not justified.This means that the approximation made on equation (1.2-37) in order to derive the coupled equations,is therefore also not justified.

Let us try to state this result more simply. What we have found is that the approximations made on equation (1.2-37) lead to a pair of coupled differential equations for which $r=r_o$ is a regular singular point. However,we have also found that if $r=r_o$ is a regular singular point of the coupled equations,then the approximations made on equation (1.2-37) are invalid. The only sensible resolution of this inconsistency, if the Basic Set of equations is not inherently inconsistent,is that the approximation that has led to the inconsistency,is incorrect.Consequently, equation (1.2-45) is an incorrect rendering of equation (1.2-37).It also follows that equations (TH37) and (TH41),derived using equation (1.2-45), are equally incorrect,so that it is not clear that the solution of this pair of coupled equations,will constitute a sensible contribution toward the explanation of phenomena observed in experimental high-temperature plasmas.

We would thus expect equation (1.2-45) to actually look like

$$v_{ir} = \frac{\omega\psi}{k_\eta} + a_4 + \frac{1}{\mathcal{D}} \cdot \frac{\omega}{\omega_{ci}} a_3 \qquad (1.3\text{-}8)$$

so that the amount and nature of the 'slippage' between the ion fluid and the magnetic field,is far more complex than is indicated in /1/.

In the next section we shall begin with the coupled equations of /1/ as our starting point, in spite of the reservations that we have expressed as to the validity of those equations. The reason for this is chronological, and is simply that we attempted the computational solution of those equations as they stand, before we investigated their validity from an analytical standpoint. Thus we shall also consider in the next section, approximations that have been made in /1/ to the very complicated coupled equations, in order to permit some analytic treatment. As a result of these approximations, two simpler eigenvalue problems were derived in /1/, for one of which we have developed what we feel to be a novel numerical solution method.

1.4 Discussion of further approximations made to equations (TH37) and (TH41)

Although we have shown that the use of equation (1.2-45), describing the radial perturbed velocity of the ion fluid, is not legitimate, it is nevertheless true that the use of this equation in the manner prescribed in /1/, does lead to the coupled equations (TH37 and TH41). We will not reproduce the derivation of these two equations here, as it is very lengthy. We shall instead, simply state these equations as they are given in /1/, and use them as the starting point of our discussion.

Equation (TH37), which is essentially the continuity equation, reads :

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r\Psi}{k_{\parallel}}\right) = -\frac{1}{r}\frac{d}{dr}\left[\frac{r p_0 \omega^*}{p_0' \omega}\Delta_p\right] + \left[\frac{k_{\parallel}^2 V_{th}^2}{\omega^2} - \frac{3}{5} - \frac{p_0 n_0' \omega^*}{p_0' n_0 \omega}\left\{1 + \frac{3T_{0i}}{5T_0}\left(\frac{n_0 T_{0i}'}{n_0' T_{0i}} - \frac{2}{3}\right)\right\}\right]\Delta_p \qquad (1.4\text{-}1)$$

and equation (TH41), the radial component of the ion momentum, reads :

$$\frac{d}{dr}\left[\frac{B_0 k_{\parallel}}{4\pi k^2 r}\cdot\frac{d}{dr}(rB_0\Psi) + \Psi\left\{\frac{k_{\parallel}^2 p_0'}{k^2 k_{\parallel}} - \frac{j_{\parallel}}{ck^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\right\} - \frac{p_0 k_{\parallel}^2}{k^2}\Delta_p\right]$$

$$+ \frac{2B_{0\theta}}{4\pi r}\left[\frac{m^2}{k^2 r^2}\cdot\frac{d}{dr}(rB_0\Psi) + \frac{4\pi n}{k^2 R}\left\{\frac{j_{\parallel}}{c} + \frac{1}{B_0^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\frac{p_0'}{k_{\parallel}}\right\} - \frac{4\pi n p_0}{k^2 R B_0^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\Delta_p\right]$$

$$- \frac{k_{\parallel}B_0^2\Psi}{4\pi} + m_i n_0 \omega\left[\frac{\omega\Psi}{k_{\parallel}} + \frac{p_0\omega^*}{p_0'}\Delta_p\right] = 0 \qquad (1.4\text{-}2)$$

where

$$j_{\parallel} = \frac{1}{B_0}(\underline{j}_0 \cdot \underline{B}_0) \qquad \text{and} \qquad V_{th}^2 = T_0/m_i \qquad (1.4\text{-}2a)$$

and all other quantities are as previously defined in equations (1.2-35) and (1.2-44). Now, remembering that it is our intention to show that certain crucial approximations which are made in /1/ are illegitimate, it is plain that we must follow rather closely, the steps leading to those approximations. Thus, we make the following two transformations :

$$\xi = \Psi/k_{\parallel} \qquad (1.4\text{-}3a)$$

$$\zeta = \frac{p_0\omega^*}{p_0'}\Delta_p \qquad (1.4\text{-}3b)$$

Substituting equations (1.4-3a) and (1.4-3b) into equation (1.4-1) and multiplying by $-\omega$, we find that we can write equation (1.4-1) as follows

$$\frac{1}{r}\frac{d}{dr}(r\zeta) + \alpha(r)\zeta = -\frac{\omega}{r}\frac{d}{dr}[r\xi] \qquad (1.4\text{-}4)$$

where

$$\alpha(r) = \frac{n_o'}{n_o}\left[1 + \frac{3T_{oi}}{5T_o}\left(\frac{n_oT_{oi}'}{n_o'T_{oi}} - \frac{2}{3}\right) + \frac{3\omega p_o'}{5\omega^* p_o} - \frac{k_\parallel^2 v_{th}^2 p_o'}{\omega\omega^* p_o}\right] \qquad (1.4\text{-}5)$$

Equation (1.4-4) is also equation (TH43). However it has been given in /1/ with the incorrect sign on the right hand side (RHS) of the equation. We shall refer to this error as Error III. Also, we note that the point at which $k_\parallel = 0$, is a regular singular point of equation (1.4-1) and equation (1.4-2). That is to say that the point $r=r_o$ is a regular singular point of these equations.

A small rearrangement of equation (1.4-4) enables us to write it as

$$\frac{1}{r}\frac{d}{dr}\{r(\zeta+\omega\xi)\} = -\alpha\zeta \qquad (1.4\text{-}6)$$

so that

$$\int \frac{d}{dr}\{r(\zeta+\omega\xi)\}dr = -\int r\alpha\zeta\, dr \qquad (1.4\text{-}7)$$

Now, integrating both sides of this equation with respect to r, (which we change to the dummy variable s), we find that

$$r(\zeta+\omega\xi) = -\int s\alpha(s)\zeta(s)\, ds \qquad (1.4\text{-}7a)$$

or

$$\omega\xi = -\zeta - \frac{1}{r}\int s\,\alpha(s)\zeta(s)\, ds \qquad (1.4\text{-}7b)$$

That is to say

$$\omega\xi(r) = -\zeta\left[1 + A\{r,\omega,\zeta(r)\}\right] \qquad (1.4\text{-}8)$$

where

$$-A = \frac{1}{r\,\zeta(r)}\int_r^a s\,\zeta(s)\,\alpha(s)\, ds \qquad (1.4\text{-}9)$$

and boundary conditions

$$\xi(a) = \zeta(a) = 0 \qquad (1.4\text{-}9a)$$

corresponding to the presence of an infinitely conducting and rigid wall, in contact with the plasma, have been used.

Equation (1.4-9) is the basis of one of the major approximations made in /1/, and we shall comment on it at greater length presently. Meanwhile, substituting equations (1.4-3a) and (1.4-3b) into equation (1.4-2), allows us to obtain, after some algebra, equation (TH46), i.e.

$$\frac{d}{dr}\left\{\frac{rB_0^2k_{\parallel}^2}{4\pi k^2r^2}\cdot\frac{d}{dr}(r\xi)\right\} - \xi\left[\frac{B_0^2k_{\parallel}^2}{4\pi} - m_in_0\omega^2 + \frac{2mrk_{\parallel}B_0}{4\pi}\cdot\frac{d}{dr}\left(\frac{B_{0\theta}}{k^2r^3}\right) - \frac{2nB_{0\theta}}{ck^2rR}\left(\frac{mJ_{0\theta}}{r} + \frac{nJ_{0z}}{R}\right)\right]$$

$$= \frac{d}{dr}\left(\frac{k_{\parallel}^2p_0'\zeta}{k^2\omega^*}\right) - \left(\frac{2en_0nB_{0\theta}}{ck^2rR} + m_in_0\omega\right)\xi \qquad (1.4\text{-}10)$$

The eigenvalue problem that is posed in /1/, consists of the solution of the coupled equations (1.4-1) and (1.4-2), for the functions Ψ and Δ_p, after the imposition of suitable boundary conditions. The solution functions must also satisfy the restriction that they be everywhere square-integrable, in order that they shall be of physical interest. However, these coupled equations as they stand, are far too complicated to submit to standard analytical treatment. It is for this reason that it was felt in /1/ that some approximations were necessary. The first such approximation that was made, which would eventually facilitate the reduction of the two coupled equations to a single, second-order differential equation, was called the 'mean-value' approximation, and we shall now outline what was involved in this approximation. It was assumed that in equation (1.4-9), the integral

$$\int_r^a s\,\zeta(s)\alpha(s)\,ds \qquad (1.4\text{-}10a)$$

could be replaced by

$$\zeta(r)\int_{r_0}^a s\,\alpha(s)\,ds \qquad (1.4\text{-}10b)$$

We shall refer to this approximation as Approximation I. No real justification was presented in /1/ for this approximation, but it is possible to examine its validity from a heuristic standpoint. Thus, let us suppose that the form of the perturbation $\zeta(r)$, is that in figure (1.4-1a).

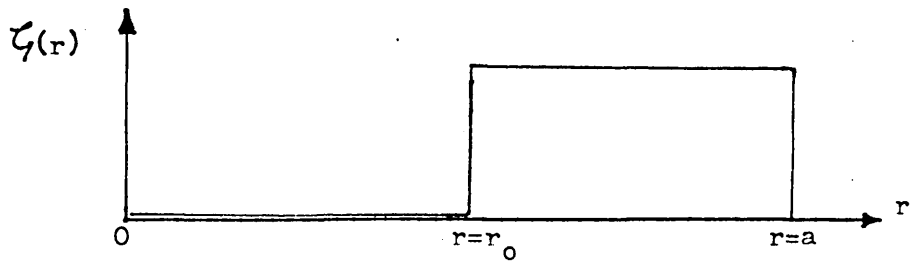


Figure 1.4-1a

It is evident that for this perturbation Approximation I is valid.We might loosely describe this perturbation as being slowly varying everywhere,except at $r=r_o$ and at $r=a$, but of course it is also a highly unphysical perturbation.Let us now consider the two more realistic perturbations in figure (1.4-1b).

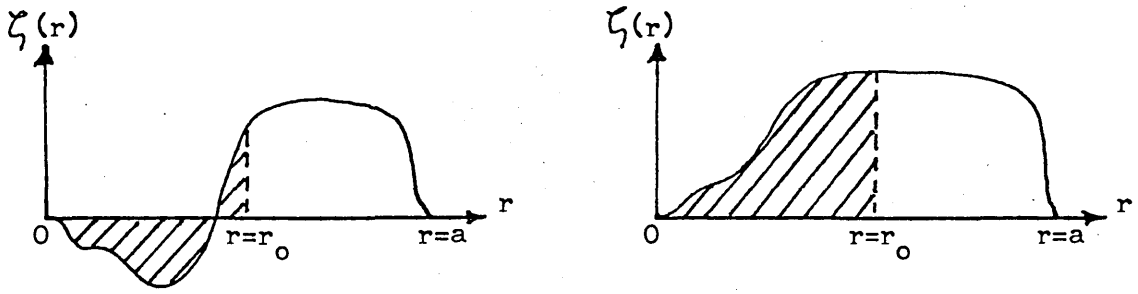


Figure 1.4-1b

Since these perturbations are nearly constant in the region $r_o \leq r \leq a$, Approximation I is not likely to lead to severe errors in this region. However,over the whole domain,its use can be expected to lead to quite substantial errors,as is shown by the shaded portion of figure (1.4-1b), whose contribution the integral of the approximation totally ignores. We would conclude then,that the approximation is not good for perturbations that only vary slowly with r,such as in figure (1.4-1b).

Finally,we consider the idealised rapidly varying perturbation in figure (1.4-1c),which is also meant to be very localised.Once again,the use of Approximation I leads to error of the order of magnitude indicated by the shaded portion of this figure.Quite evidently however,this error is not as large as for the non-localised and slowly-varying perturbations.

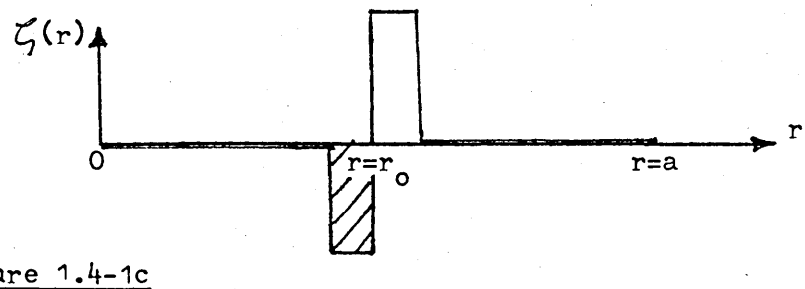


Figure 1.4-1c

It seems reasonable to expect therefore, that through the use of Approximation I, we may only expect to describe such eigenfunctions as are localised in space around $r=r_o$. If this is so, then it is only in the circumstance when all the eigenfunctions of this problem that exist are localised in space around $r=r_o$, that we may regard with any seriousness, any predictions as to the nature of the eigenvalue spectrum of this problem, that the use of this approximation may lead us to. This means that the use of this approximation is severely restrictive, and also highly suspect, because we cannot know a priori that there are no eigenfunctions of broad spatial extent, or equally, that all such eigenfunctions as exist will be localised in space around $r=r_o$. The critical nature of the influence which this approximation will exert on the eigenvalue spectrum which is found for this problem, has not been acknowledged in /1/.

Another approximation which is made in /1/ as part of the process of reducing the two coupled equations to a single equation, is that the nature of the eigenvalue spectrum to be found, and the properties of the solutions, are critically determined by the local behaviour of the coupled equations in the vicinity of the resonant surface, (cf. the discussion in section 1.3). Thus, the proposition is that the replacement of every equilibrium profile quantity $f_o(r)$, wherever it appears, by $f_o(r_o)$, will not significantly affect the nature of the eigenvalue spectrum of the problem. We shall call this approximation, Approximation II. It is easy to see that if the eigenfunctions are zero everywhere, except near the resonant point $r=r_o$, then it will only be near this point that we need bother solving the coupled equations. Thus Approximation II reduces to the assumption that all such eigenfunctions as exist, are localised around the resonant point. Once again, the justification of such an assumption, when the nature of the eigenfunctions cannot be known a priori, is not clear, unless we are being asked to accept that our two-fluid eigenfunctions will look like the ideal MHD eigenfunctions of /71/. This seems unreasonable however, since we are trying to find out if our two-fluid theory makes any different predictions to the single-fluid theory.

Thus, recognising the importance of the resonant radius, r_o, and using Approximation I, equation (1.4-9) can be written

$$-\hat{A} = \frac{1}{r_o}\int_{r_o}^{a} S\alpha(s)\,ds \qquad (1.4\text{-}11)$$

We can substitute this last equation into equation (1.4-8),which would then become :

$$\zeta(r) = \Lambda(\omega, r_0)\,\xi(r) \qquad (1.4\text{-}12)$$

where

$$\Lambda(\omega, r_0) = \frac{-\omega}{\{1 + \hat{A}(\omega, r_0)\}} \qquad (1.4\text{-}13)$$

Here we point out that the sign error which we have called Error III in /1/, has propagated and affected four subsequent equations in that paper,specifically equations (TH46),(TH43),(TH49) .All that is wrong with these equations is that the RHS of each is reported with the incorrect sign.We have given the correct versions of these equations in equations (1.4-8),(1.4-4), and (1.4-13) .

Having made the above approximation ,it was proposed in /1/ that the case only in which the resonant point lies within the plasma volume,be considered.With the aid of this and further approximations (which we shall detail in Appendix D and Appendix E) two further equations are derived in /1/,from equation (1.4-10),which purportedly describe the behaviour of the plasma in the 'interior' ($0 \leq r \leq r_o$) and 'exterior' ($r_o \leq r \leq a$) regions respectively.

The first of these two equations which we shall deal with is the 'exterior' equation,which we have rederived in Appendix D in order to make transparent the approximations involved.Notwithstanding the serious approximations that have been previously made in the formulation of /1/ (Approximations I and II), we consider that the further approximations that have been used in order to derive the exterior equation,are within themselves sufficient cause for doubting the accuracy of that equation. However,as is shown in Appendix D,if we define the new independent variable x,where

$$x = \frac{2|m|(r - r_o)}{r_o} \qquad (1.4\text{-}14)$$

and use the approximations detailed there,we can derive the following equation for $\xi(r)$:

$$\frac{d}{dx}\left(x^2 \frac{d\xi}{dx}\right) - \xi\left[\frac{x^2}{4} + \frac{x}{|m|} - (\lambda + \tfrac{1}{4})\right] = 0 \qquad (1.4\text{-}15)$$

where

$$\lambda + \tfrac{1}{4} =$$

$$\frac{4\pi m^2}{B_o^2 k_{||}^2 r_o^2}\left[m_i n_o \omega^2 - \frac{2B_{o\theta}^2}{mcB_{oz}}\left(\frac{mJ_{o\theta}}{r_o} + \frac{nJ_{oz}}{R}\right) + \Lambda(\omega, r_o)\left\{m_i n_o \omega - \frac{2en_o B_{o\theta}^2}{mcB_{oz}}\right\}\right] \qquad (1.4\text{-}16)$$

This equation has been solved analytically in /1/ where it is shown that only real positive λ values are admissible, and that the eigenfunctions corresponding to these λ values are not square-integrable near x=0, so that these eigenvalues form part of the continuous spectrum. One would conclude then, that no discrete solutions (corresponding to instability of the plasma) are possible in the exterior region.

There is a simple numerical experiment that can be performed on equation (1.4-15), which would lead one to suspect that all such eigenvalues as this equation might possess, are real and positive. This experiment, which we now present, is appealing because it does not involve any complicated mathematical analysis. Thus, using 3-point finite-differencing /62/, we may represent equation (1.4-15) at the ith mesh-point, as follows :

$$x_i^2\left[\xi_{i+1} - 2\xi_i + \xi_{i-1}\right]/h^2 + 2x_i\left[\xi_{i+1} - \xi_{i-1}\right]/2h - \left[\frac{x_i^2}{4} + \frac{x_i}{m}\right]\xi_i = -\lambda\xi_i$$

i.e.

$$-\left[\frac{x_i^2}{h^2} + \frac{x_i}{h}\right]\xi_{i+1} + \left[\frac{2x_i^2}{h^2} + \frac{x_i^2}{4} + \frac{x_i}{m}\right]\xi_i - \left[\frac{x_i^2}{h^2} - \frac{x_i}{h}\right]\xi_{i-1} = \lambda\xi_i \qquad (1.4\text{-}15a)$$

where

$$h = x_{i+1} - x_i \quad , \qquad i = [0, 1, 2, 3, \ldots, N]$$

If we also attach the boundary conditions

$$\xi_0 = \xi_{N+1} = 0 \qquad (1.4\text{-}15b)$$

and have N internal points in our space -mesh , our problem is reduced to that of solving N simultaneous equations for the ξ_i (i=1,2,3,...,N).. This simultaneous equation system can be written in matrix form as :

$$\underline{\underline{A}} \cdot \underline{\xi} = \lambda \underline{\xi} \qquad (1.4\text{-}15c)$$

and only has non-trivial solutions when

$$\det\left|\underline{\underline{A}} - \lambda \underline{\underline{I}}\right| = 0 \qquad (1.4\text{-}15d)$$

In equation (1.4-15c) $\underline{\underline{A}}$ is a square matrix,with all its elements, except those of the three leading diagonals,equal to zero. $\underline{\xi}$ is the column matrix which we may write

$$\underline{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_N \end{bmatrix} \qquad (1.4\text{-}15e)$$

and $\underline{\underline{I}}$ is the unit matrix,which like $\underline{\underline{A}}$ is of dimension N.
Equation (1.4-15d) is an Nth order polynomial in χ ,whose N roots are approximations to the eigenvalues which we seek.We have solved this equation on the computer, using the Numerical Algorithm Group (NAG)/83/ routine F02AGF,for various values of N,and have always found the eigenvalues to be an increasing,real,positive,sequence of numbers.
In figure (1.4-2) we have plotted the smallest of the numerically generated eigenvalues against N.This figure suggests that as $N \to \infty$,the smallest eigenvalue $\chi_{min} = \lambda_{min} + 1/4$,tends asymptotically to some positive value which is greater than 0·25 . Thus,this 'exact' method suggests that as the solution space becomes more continuous ($N \longrightarrow \infty$), only positive and real values of λ are obtainable.This is precisely what has been found in the analytic solution of equation (1.4-15) in /1/.Also,it should be noted that in equation (1.4-15a)

$$\chi = \lambda + 1/4 \qquad (1.4\text{-}15f)$$

In Appendix F we have derived,within the approximations made so far, the expression for $\Lambda(\omega,\tau_0)$, and we have found that this expression has been incorrectly given in /1/.This error is due to the sign error that we have called Error III.We are also able to demonstrate in Appendix F, the consequences of Error III on the results presented in /1/.

If then,equation (1.4-16) is to be believed,the expression for $\Lambda(\omega,\tau_0)$ may be substituted into it.The result is a quartic equation in ω . Therefore,since we know that λ may only take positive and real values, equation (1.4-16) then takes on the character of a 'dispersion equation', and we may substitute into it,these positive real λ values (from zero upwards) and solve it in order to find what values of the frequency

result.Thus,treating λ as a parameter running from zero upwards,we have been able to obtain the results shown in figure (1.4-3),in which we have labelled the four frequency branches ω_1 , ω_2 , ω_3 , ω_4. From this figure we can see that there are two high-frequency branches (ω_1 and ω_2 ,which are nearly coincident),and two much lower frequency branches (ω_3 and ω_4). Also,since we are only interested in frequencies such that $\frac{\omega}{\omega_{ci}} \ll 1$, we can see from figure (1.4-3) that we do not need to consider values of $\lambda \gtrsim 10$. It should also be noted that none of the four values of ω produced, for each positive λ used in equation (1.4-16),was complex (for the particular equilibrium profiles for which this equation was solved).This result shows that the case of λ positive,does not correspond,for these particular equilibrium profiles,to instability of the plasma in the 'exterior' region. This is in agreement with the result presented in /1/.

Using similar approximations to those already used in the derivation of the 'exterior' equation, the 'interior' equation,which it is purported in /1/ describes the behaviour of the plasma in the region $0 \leq r \leq r_o$, can be derived.Once again,in order to make transparent the approximations involved,this derivation has been carried out in Appendix E,where it is found that the equation reads :

$$\frac{d}{dr}\left[r(r-r_0)^2\frac{d\mathcal{Z}}{dr}\right] + \mathcal{Z}\left[\frac{r_0^2}{r}C - \frac{m^2}{r}(r-r_0)^2 - 2(r-r_0) + Dr\right] = 0 \qquad (1.4\text{-}17)$$

where

$$r\xi(r) = \mathcal{Z}(r) \qquad (1.4\text{-}18)$$

$$C = \frac{4\pi m_i n_o m^2\omega^2}{B_o^2 k_{\parallel}'^2 r_o^2}\left[1 + \frac{1}{\omega}\Lambda(\omega,r_o)\right] \qquad (1.4\text{-}19)$$

and

$$D = \frac{8\pi e n_o n}{c B_o q(o) R^2 k_{\parallel}'^2}\left[\Omega_{mn} + \Lambda(\omega,r_o)\right] \qquad (1.4\text{-}20)$$

The 'interior' equation is clearly more complex than the 'exterior' equation, with regard to the ease with which it will submit to analytic solution.This is partly because of the involvement of the eigenfrequency ω ,in coefficients that are dependent on r,and partly because the solution domain now includes two regular singular points (r=0 and $r=r_o$). The standard analytical approach would be to convert equation (1.4-17)

to Sturm-Liouville form,in order to be supported by the associated well-understood body of analytical theory /84/ behind such equations.We would find after doing this however (see equation A.18 of /1/),that the eigenfrequency was still involved in the coefficients of the equation in a non-standard way.

As a result of these difficulties,the analytical solution of equation (1.4-17) has not been attempted in /1/. Instead,another equation (which it is hoped in /1/ is sufficiently similar to the 'interior' equation to be of interest) is considered.This new equation has the desirable property that the eigenvalue appears in it in a standard way. We do not consider this modified equation any further, as we are not certain how closely related it is to the original interior case equation. Instead,we shall go on in the next section,to present a novel numerical method of solution of the interior equation as it stands.

Finally we note that in equation (1.4-20), q(0) is the safety factor at the origin, and Ω_{mn} is defined in equation (1.2-35).

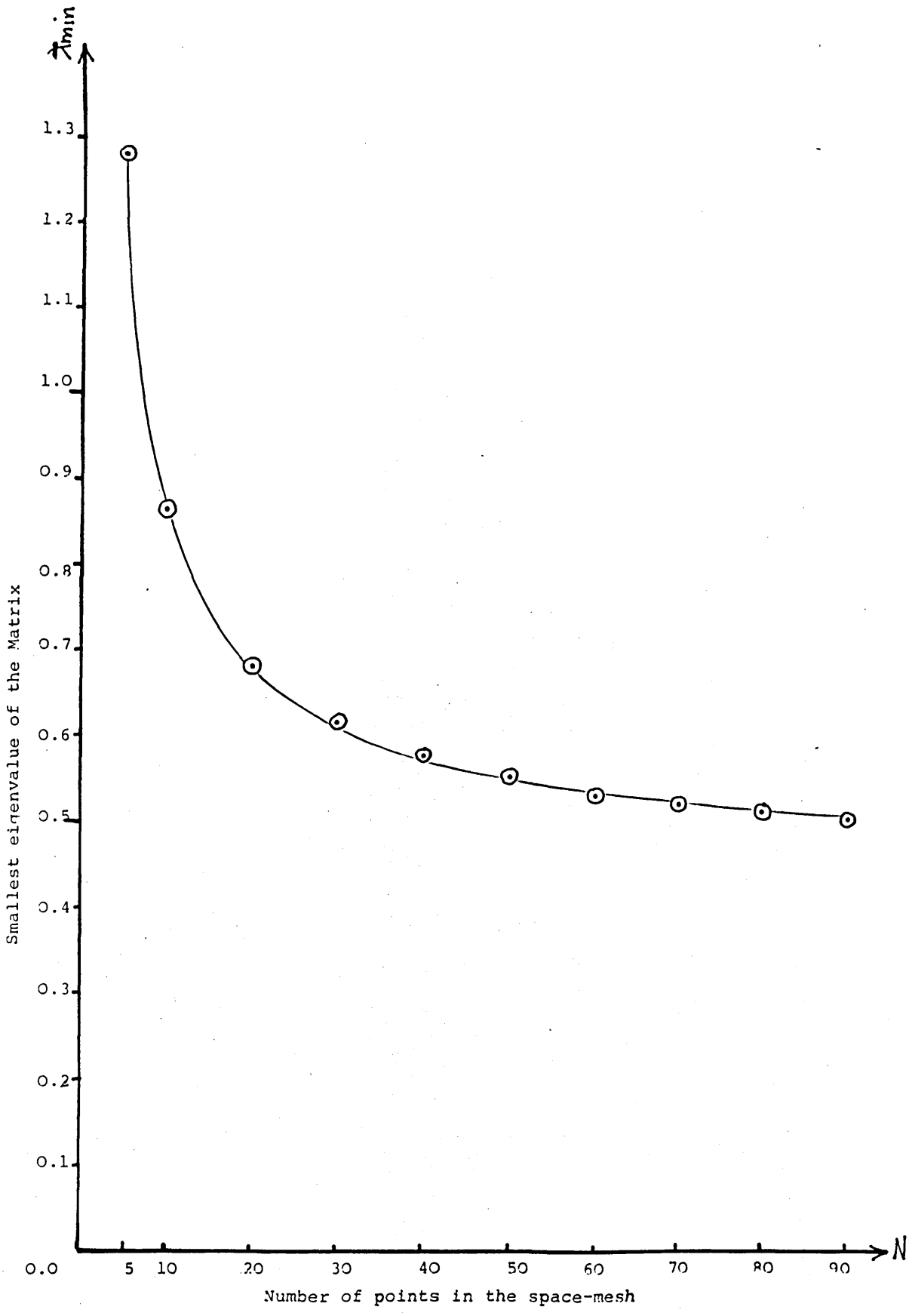


Figure (1.4-2)

Plot of λ_{min} v. N obtained by solving equation (1.4-15d), using Routine FC2AGF, of the Numerical Algorithms Group. /83/

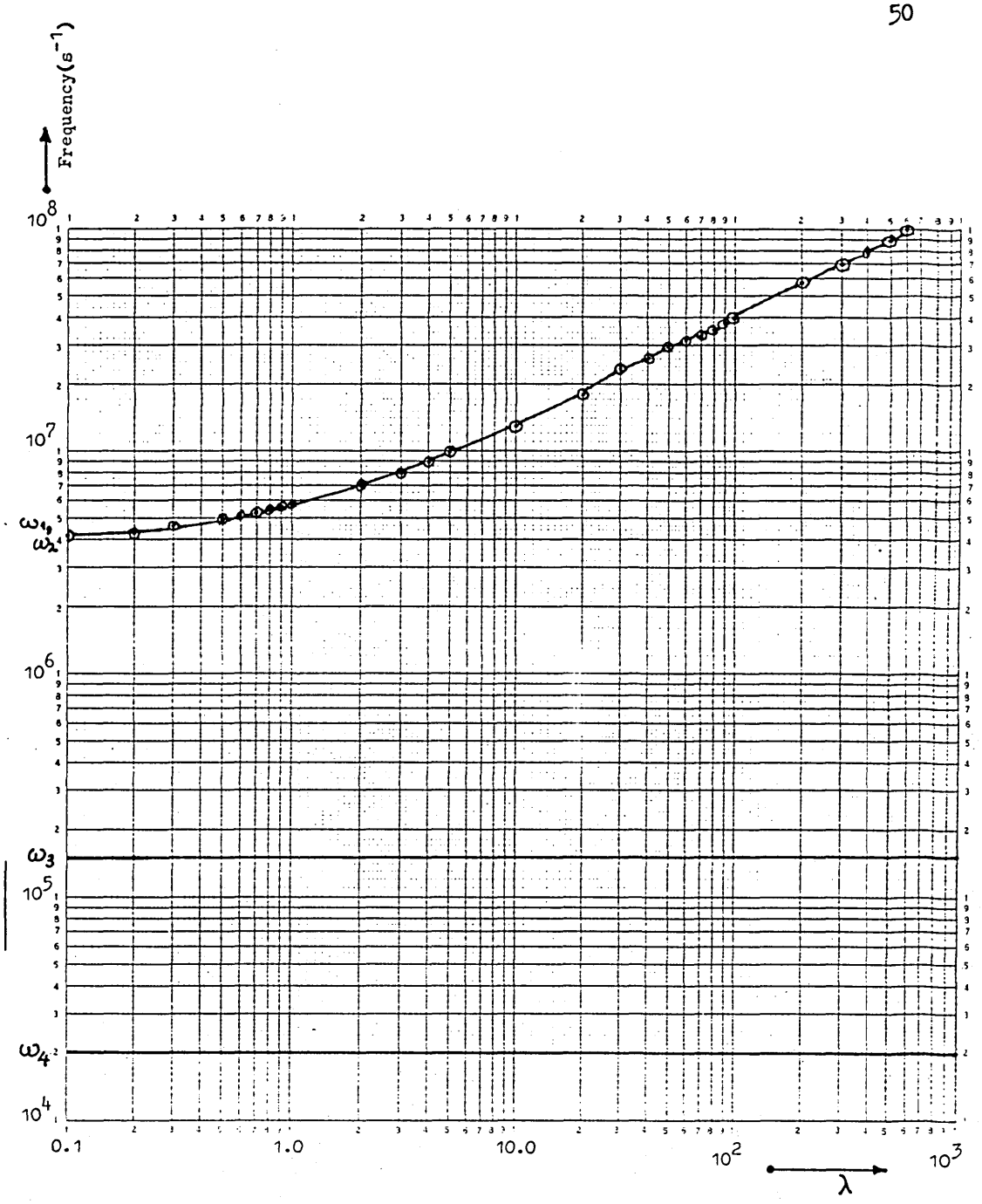


Figure (1.4-3) 'Dispersion relation' resulting from equation (1.4-16) showing the four frequency branches.

Table 1.4-1 Profiles used for the study of the approximate equations of /1/ .

number density : $n_0(r) = n_0(0)\, e^{-r^2/a^2}$

$$n_0(0) = 5 \times 10^{13}\ cm^{-3}$$

temperature : $T_{0i}(r) = T_{0e}(r) = T_0(0) e^{-r^2/2a^2}$

$$T_0(0) = 10^{-9}\ erg = 1\ keV$$

total current : $I_p = \int_0^a r\, j_{0z}(r)\, r\, 2\pi = 1.5 \times 10^{15}$ Stat Amp $\simeq 500$ kA

current density : $j_{0z}(r) = j_{0z}(0) \left[\frac{T_0(r)}{T_0(0)}\right] e^{-3r^2/4a^2}$

and by Ampere's law :

$$\frac{1}{r}\frac{d}{dr}\left[r B_{0\theta}(r)\right] = \frac{4\pi}{c} j_{0z}(r)$$

z component of magnetic field :

$$B_{0z} = \text{constant} = 3 \times 10^4\ \text{Gauss}.$$

1.5 Numerical solution of the 'interior' equation

In this section we shall consider the numerical solution of equation (1.4-17) using a method we believe to be novel.For ease of reference we repeat the equation here.It reads :

$$\frac{d}{dr}\left[r(r-r_0)^2\frac{dZ}{dr}\right] + Z\left[\frac{r_0^2}{r}C - (r-r_0)^2\frac{m^2}{r} - 2(r-r_0) + D r\right] = 0 \qquad (1.5\text{-}1)$$

where $C(\omega)$ and $D(\omega)$ are given by equations (1.4-19) and (1.4-20). As we have already noted elsewhere,equation (1.5-1) has regular singularities at r=0 and $r=r_o$,and since numerical integration cannot proceed through either singularity,we must obtain asymptotic expressions for the solution functions,valid in the vicinity of either singularity. These expressions are most easily obtained by examining equation (1.5-1),firstly in the limit $r \longrightarrow 0$, and then in the limit $r \longrightarrow r_o$. Following this procedure, it is evident that near r=0 equation (1.5-1) takes the form :

$$\frac{d}{dr}\left[r\frac{dZ}{dr}\right] + \frac{Z}{r}\left[C - m^2\right] = 0 \qquad (1.5\text{-}2)$$

which,when we perform the differentiation and multiply by r,becomes :

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + Z\left[C - m^2\right] = 0 \qquad (1.5\text{-}3)$$

We note that this equation is of Euler-Cauchy type,and so by making the transformation

$$r = e^x \qquad (1.5\text{-}4)$$

we can convert equation (1.5-3) to :

$$\frac{d^2Z}{dx^2} + Z\left[C - m^2\right] = 0 \qquad (1.5\text{-}5)$$

The characteristic equation of this equation is obtained by substituting into it the trial solution

$$Z = e^{\alpha x} \qquad (1.5\text{-}6)$$

which produces the result :

$$\alpha^2 + (C - m^2) = 0 \qquad (1.5\text{-}7)$$

The two values of α which solve equation (1.5-7) are :

$$\alpha_1 = i\,(C - m^2)^{\frac{1}{2}}$$
$$\alpha_2 = -i(C - m^2)^{\frac{1}{2}} \qquad (1.5\text{-}8)$$

Combinations of equation (1.5-4) and (1.5-6) then shows us that near the origin, the solution behaves like

$$Z = A_1 r^{\alpha_1} + \frac{A_2}{r^{\alpha_1}} \qquad (1.5\text{-}9)$$

α_1 will in general be complex, and so we can write :

$$\alpha_1 = Y_a + iY_b \qquad (1.5\text{-}10)$$

If for our choice of ω , $Y_a > 0$, then a bounded solution at r=0 will be possible only if A_2=0. On the other hand, if $Y_a < 0$, then a bound solution at r=0 will only be possible if A_1=0.

Similarly, in the vicinity of $r=r_o$, equation (1.5-1) will read :

$$\frac{d}{dr}\left[(r-r_o)^2 \frac{dZ}{dr}\right] + Z\left[C + D\right] = 0 \qquad (1.5\text{-}11)$$

If also, we make $x = r-r_o$ the new independent variable, then equation (1.5-11) becomes :

$$x^2 \frac{d^2 Z}{dx^2} + 2x \frac{dZ}{dx} + Z[C+D] = 0 \qquad (1.5\text{-}12)$$

which is also of Euler-Cauchy type. Thus, using the transformation in equation (1.5-4), we are able to produce an equation which has the following characteristic equation

$$\beta^2 + \beta + (C + D) = 0 \qquad (1.5\text{-}13)$$

This is the indicial equation associated with equation (1.5-11), (just as equation 1.5-7 is the indicial equation of equation 1.5-5). The two solutions of this equation are :

$$\beta_1 = -\tfrac{1}{2} + (0{\cdot}25 - C - D)^{\frac{1}{2}}$$
$$\beta_2 = -\tfrac{1}{2} - (0{\cdot}25 - C - D)^{\frac{1}{2}} \qquad (1.5\text{-}14)$$

It follows that in the vicinity of $r=r_o$ the solutions behave like :

$$Z = B_1 x^{\beta_1} + B_2 x^{\beta_2} \qquad (1.5\text{-}15)$$

We note from equation (1.5-14), that except in the special and unlikely

circumstance when $C(\omega) + D(\omega) = 1/4$, that if $Re(\beta_1) > 0$,then $Re(\beta_2) < 0$, and vice versa.We also note that if

$$|Re(\ 0{\cdot}25 - C - D\ )| \ < \ \tfrac{1}{2} \qquad (1.5\text{-}15a)$$

then no bounded solutions will exist in the vicinity of the resonant surface,for the particular frequency ω,and for the particular equilibrium profiles that we are using.

The numerical method that we propose to use in order to solve equation (1.5-1),relies on the fact that the frequency space in which the search for eigenvalues is to be carried out,is not infinite in extent.(By 'eigenvalues' we propose to mean those special values of ω ,which will in general be complex,for which an everywhere bounded solution of equation 1.5-1 is possible).For instance,since we know that the low-frequency approximation ($\omega \ll \omega_{ci}$) underlies all the work that has led up to this equation,there is already a natural upper bound to the range of frequencies that we are prepared to consider. In order to include the worst case,we can set this upper bound to be $\omega_{max}=\omega_{ci}$ .Secondly,it is clear that any instabilities which grow on a timescale of about 100ms, will not be consequential in determining the total time for which the plasma discharge persists.This is because the growth of such instabilities is so slow, that their overall effect could conceivably be very easily controlled by feedback devices peripheral to the plasma.The instabilities observed in experimental plasmas tend at any rate,to occur on a much faster timescale, so that the effect of such a slowly growing instability would probably never be seen anyway,since it would be swamped by that of the more rapidly growing instabilities.Arbitrarily then,we can state that frequencies less than $10\ s^{-1}$, are not interesting. Thus we have now defined a 'reduced frequency space' in which we will search for eigenvalues,and we have schematically represented this frequency space in figure (1.5-1).

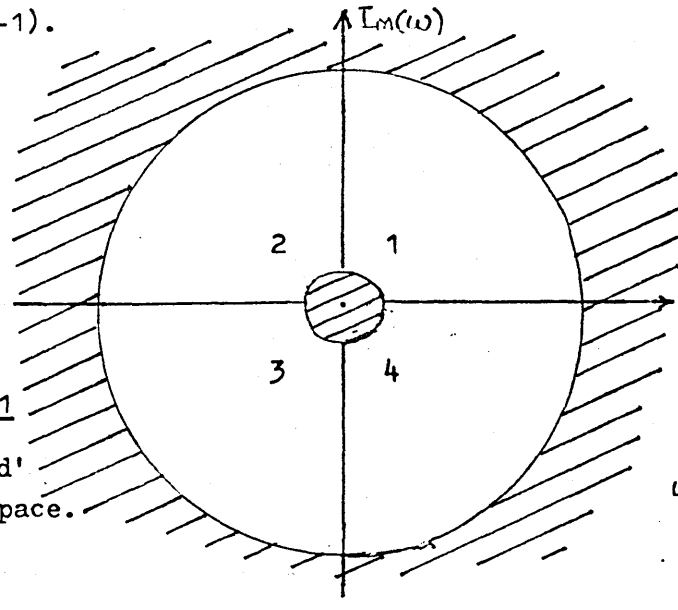


Figure 1.5-1

The 'Reduced' Frequency Space.

The four quadrants of the reduced frequency space in figure (1.5-1) have been numbered,and in this frequency space the complex frequency ω , is represented as

$$\omega = \omega_r + i.\omega_i \qquad (1.5\text{-}15b)$$

and,to reiterate

$$\omega_{max} = \omega_{ci}$$
$$\omega_{min} = 10 \qquad (1.5\text{-}15c)$$

Finally,we point out that the axes of the frequency space are given logarithmic (base 10) rather than linear divisions .

In equation (1.5-10) we defined

$$Y_a(\omega) = Re\left[i(C - m^2)^{\frac{1}{2}}\right] \qquad (1.5\text{-}16)$$

and from equation (1.5-14) we also choose to define

$$Z_a(\omega) = Re\left[(0.25 - C - D)^{\frac{1}{2}}\right] \qquad (1.5\text{-}17)$$

Now,let us consider the expression

$$y_1(\omega) = \Delta^{u(\omega) + iv(\omega)}$$
$$= \Delta^u . \exp(iv.\log\Delta) \qquad (1.5\text{-}18)$$

where Δ is a real number.This equation makes it plain that $y_1(\omega)$ is bound or unbound at $\Delta=0$, as $u(\omega)$ is greater or less than zero respectively,because the exponential factor always has unit modulus. Since u and v are also functions of r,we can obtain the gradient of the function y_1 by differentiating it with respect to r.Let the result of this differentiation be called y_2. Thus :

$$\frac{dy_1(\omega)}{dr} = y_2(\omega) \qquad (1.5\text{-}19)$$

In equation (1.5-18) suppose that $y_1(\omega)$ is the asymptotic value that we obtain,after moving a small distance Δ,away from either of the singularities of equation (1.5-1).Then,knowing y_1 at one singular point, it is possible (using the classical Runge-Kutta methods /72/ or the

the method of Nordsieck /78/) to numerically integrate equation (1.5-1), and to eventually arrive at some specified distance away from the other singular point. All that would be necessary for the integration to proceed would be a value of ω . Suppose that we start our numerical integration from the singularity near r=0, with some arbitrary value of ω, and that we continue integrating until we reach some target value of the independent variable, say $r=\lambda$. If this point is close enough to the $r=r_o$ singularity, then we will know (apart from an arbitrary constant) how the solution function and its gradient should behave at $r=\lambda$. Thus, at $r=\lambda$, we must make the numerical solution match the analytic solution (i.e. agree with $y_1(\omega)$ and $y_2(\omega)$) to any desired tolerance. However, for our first choice of ω the most likely occurrence is that our integrated solution y_1^* , y_2^* , will not match the actual solution y_1, y_2, to the desired degree of accuracy. We would then be free to begin the integration again with a new value of ω , and thereby obtain new values of y_1^*, y_2^* . In the literature of numerical analysis, this process is known as 'shooting' /85/. What is envisaged is illustrated in figure (1.5-2), where the second of three 'shots' , with frequencies ω_1 , ω_2 and ω_3 , has been successful.

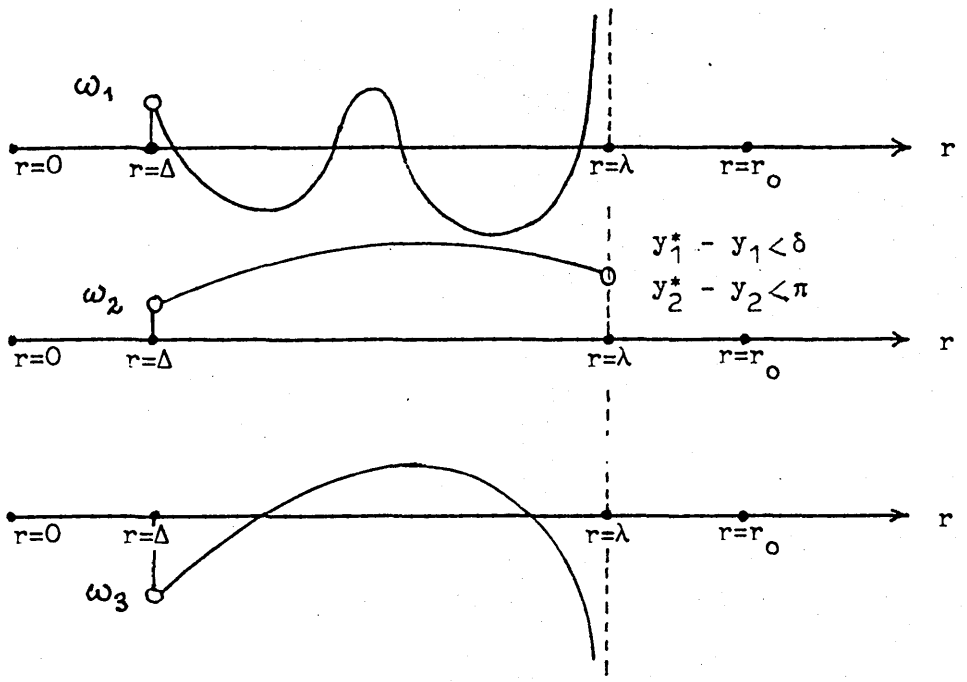


Figure 1.5-2 Illustrating the 'shooting' method

In this figure δ and π are the tolerances within which the solution function and its gradient must be found in order for a shot to be deemed successful.

As we have indicated in figure (1.5-2),a 'shot' will have been successful when the difference $y_1(\omega,\lambda) - y_1^*$ is less than some specified tolerance.The problem with the shooting method however, is that we cannot know,a priori ,what values of the frequency are likely to be successful.Thus, a method of choosing a region of frequency space in which the method of shooting is likely to produce results (or equivalently,deciding on a region in which it is almost certainly pointless to shoot),would be useful indeed. We will now outline how ,for this particular problem ,such a method may be obtained.

Suppose that we apply equation (1.5-18) in a region of the frequency space of figure (1.5-1) in which the functions $u(\omega)$ and $v(\omega)$ form flat surfaces.It is then clear that if for any value of ω taken from this flat region our shot fails,then it will also fail for all neighbouring values of ω within this flat region.That is to say that within this flat region the quantity

$$y_1(\omega,\lambda) - y_1^* = \epsilon(\omega) \qquad (1.5\text{-}19a)$$

will not vary much. It seems reasonable to conclude that solutions (and therefore discrete eigenvalues),are most likely to be found in regions where the functions $u(\omega)$ and $v(\omega)$ vary rapidly with ω (as illustrated in figure (1.5-3)),for in such regions the consequent large variations in $y_1(\omega)$ and $y_2(\omega)$ as ω is varied, will guarantee the generation of many different values of $\epsilon(\omega)$. In the instance when all these values of $\epsilon(\omega)$ do not all have the same sign,for a particular frequency range,then in that frequency range,there will exist at least one eigenvalue.In fact, if the equation under consideration does not suffer from some severe and unforseen pathology , we speculate that the form of the $u(\omega)$ and $v(\omega)$ surfaces which would correspond to an eigenvalue that is well separated from other eigenvalues of the equation,would be much as is shown in figure (1.5-3) . However,it is plain that since our frequency mesh is not infinitely fine,what figure (1.5-3) might correspond to (as suggested by figure (1.5-4)),is a group of closely spaced eigenvalues.

Thus if we make plots of $u(\omega)$ and $v(\omega)$ in the frequency space of figure (1.5-1) ,we may expect to be able to tell from these plots,roughly where in frequency space lie such eigenvalues as are to be found within our reduced frequency space.Iterative numerical methods,such as Newton's method,may then be applied to locate eigenvalues to within the required tolerance.

For the equilibrium profiles given in Table 1.4-1 ,the $Y_a(\omega)$ and $Z_a(\omega)$ surfaces are plotted in all four quadrants of the frequency space in figure (1.5-1).The plots of $Z_a(\omega)$ are given in figure (1.5-5), and combined plots of $Z_a(\omega)$ and $Y_a(\omega)$ are given in figure (1.5-6), in order that they may be directly compared.It turns out that a logarithmically divided space mesh of 54 x 54 is much finer than is necessary.In the plots of figure (1.5-5) and figure (1.5-6) we have only used 6 divisions on the ω_r axis,and 54 divisions on the ω_i axis.The reason for this,as can be seen from the plots themselves, is that the variation in height of the surfaces ,is far larger travelling along the ω_i direction,than along the ω_r direction. In figure (1.5-7) we have plotted the real and imaginary parts of the function $\Lambda(\omega,r_0)$ (labelled 'realam' and 'imlamd'),in all four quadrants of the reduced frequency space.It is clear from these plots that the pathology of the $Y_a(\omega)$ and $Z_a(\omega)$ plots,is entirely due to the pathology of the function $\Lambda(\omega,r_0)$,since Y_a and Z_a both depend on $\Lambda(\omega,r_0)$,through their dependence on $C(\omega)$ and $D(\omega)$(see equations(1.4-19) and (1.4-20)).In figure (1.5-8) we have used the full 54 x 54 capacity of our reduced frequency space ,in order to better appreciate the pathology of the $\Lambda(\omega,r_0)$ surface,since we shall presently have comment to make on this point.

Our heuristic investigations so far point to an important fact which seems totally to have gone unacknowledged in /1/.This fact is that the nature of the eigenvalue spectrum found in the 'interior' case,is critically dependent on the form of $\Lambda(\omega,r_0)$.However,the possibility of introducing the function $\Lambda(\omega,r_0)$ arose as a direct consequence of the application of Approximation I (see section 1.4). Consequently,the eigenvalue spectrum found in the 'interior' case problem (and also,by implication,the 'exterior' case),must be at least as much a function of Approximation I,as it is a function of the 'real' physical behaviour of the original two-fluid plasma.This is a very serious defect in the work of /1/,and is further evidence that

the results presented in /1/ should be treated with extreme caution.

The desirable behaviour of the surface function shown in figure (1.5-3) is found to occur only in the first and fourth quadrants of the reduced frequency space, for both $Y_a(\omega)$ and $Z_a(\omega)$. Using as our first guess any frequency in the region of pathology of $Z_a(\omega)$ in the first quadrant, and an iterated numerical integration procedure., the two eigenfunctions y_α and y_β in figure (1.5-9) ,were obtained. Both the real and the imaginary parts of each of these eigenfunctions is plotted. These eigenfunctions were obtained using the equilibrium profiles given in Table 1.4-1 ,and are for $m=3$ and $n=2$. It will also be noted that these eigenfunctions are not localised in space, and this would suggest, according to our discussion in section 1.4 ,that the approximations made in deriving equation (1.5-1) (i.e. Approximation I and Approximation II),are not valid. This is another highly unsatisfactory aspect of the formulation of /1/, and in view of it, we are not sure that any meaning can be attached to solutions of the interior case problem.

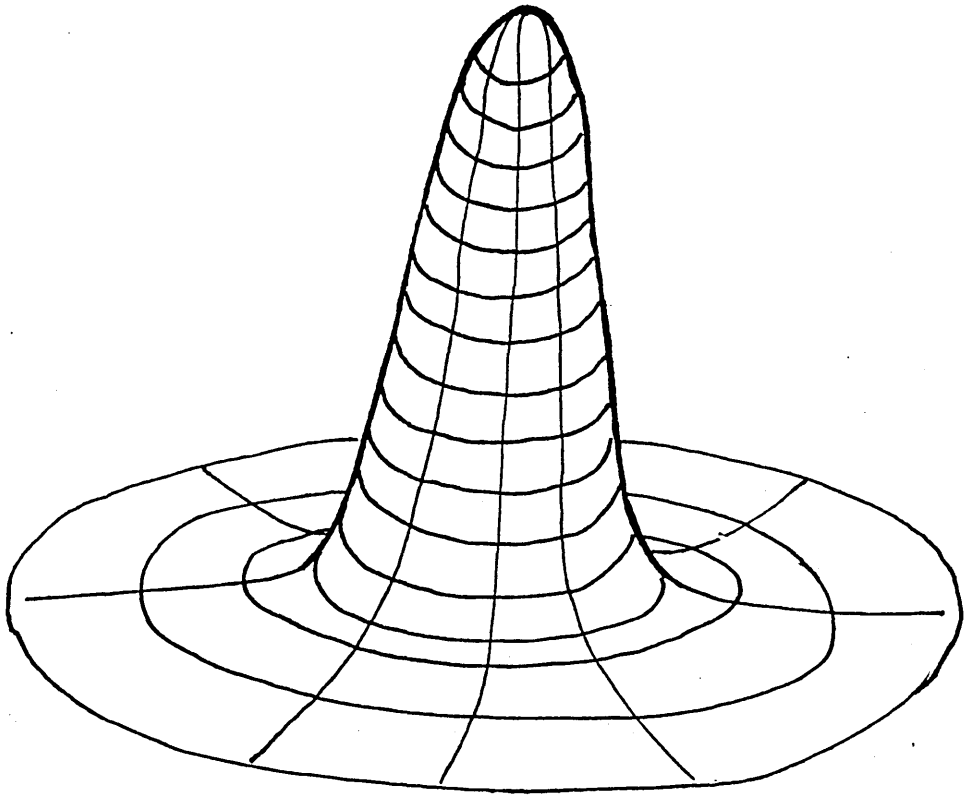


Figure 1.5-3 u(ω) behaviour for isolated eigenvalue

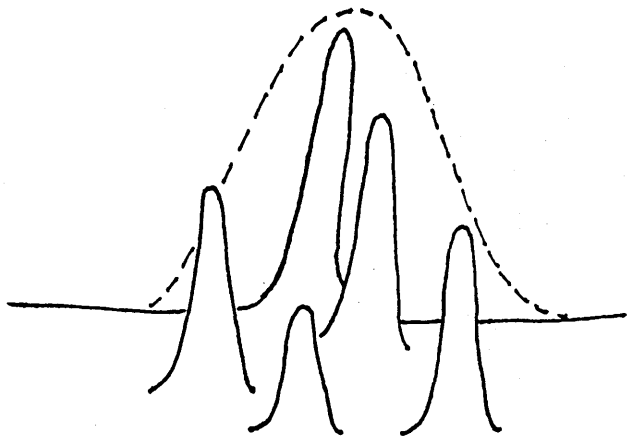


Figure 1.5-4 u(ω) behaviour for cluster of eigenvalues

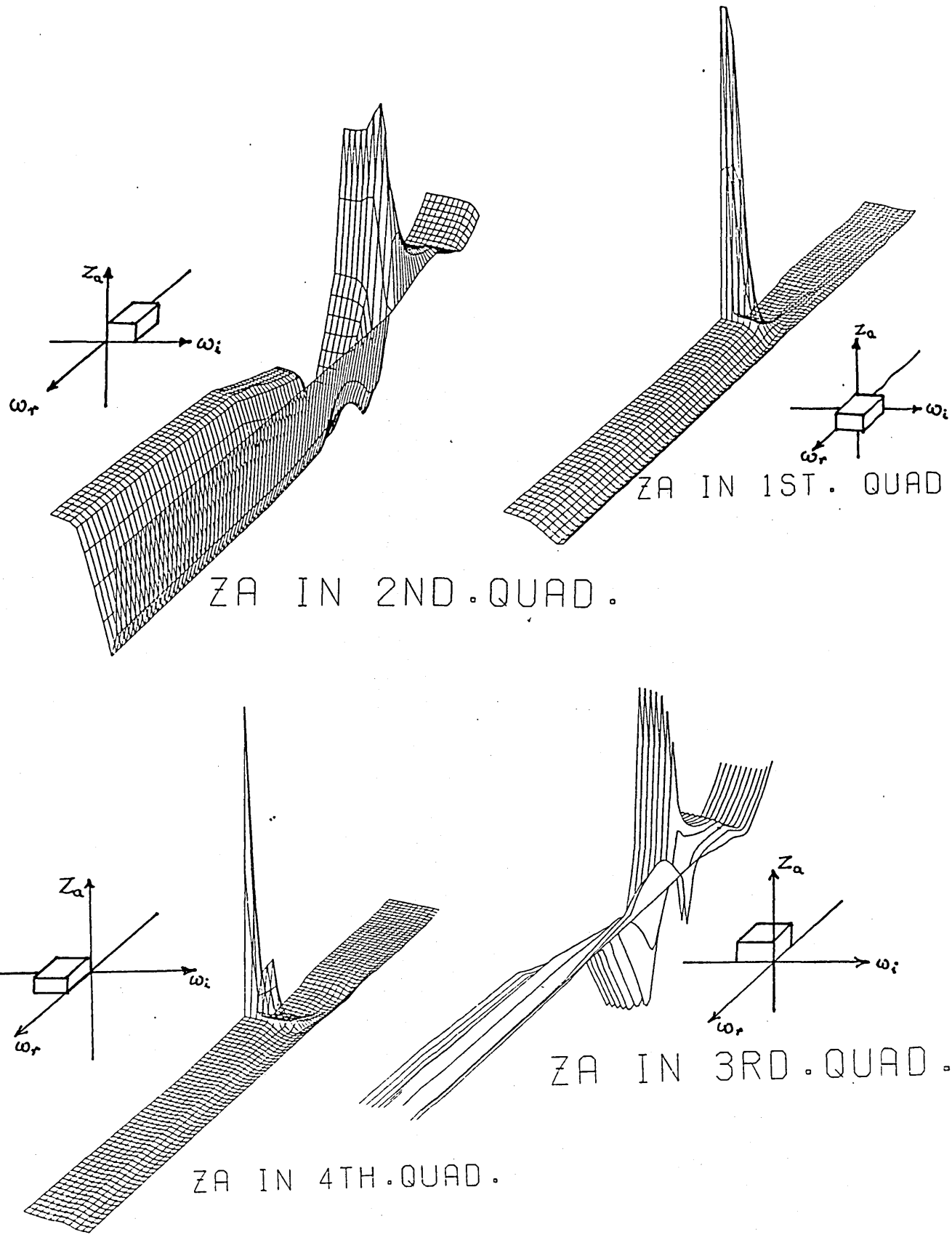


Figure 1.5-5 Plots of the $Z_a(\omega)$ surface in all four quadrants of frequency space.

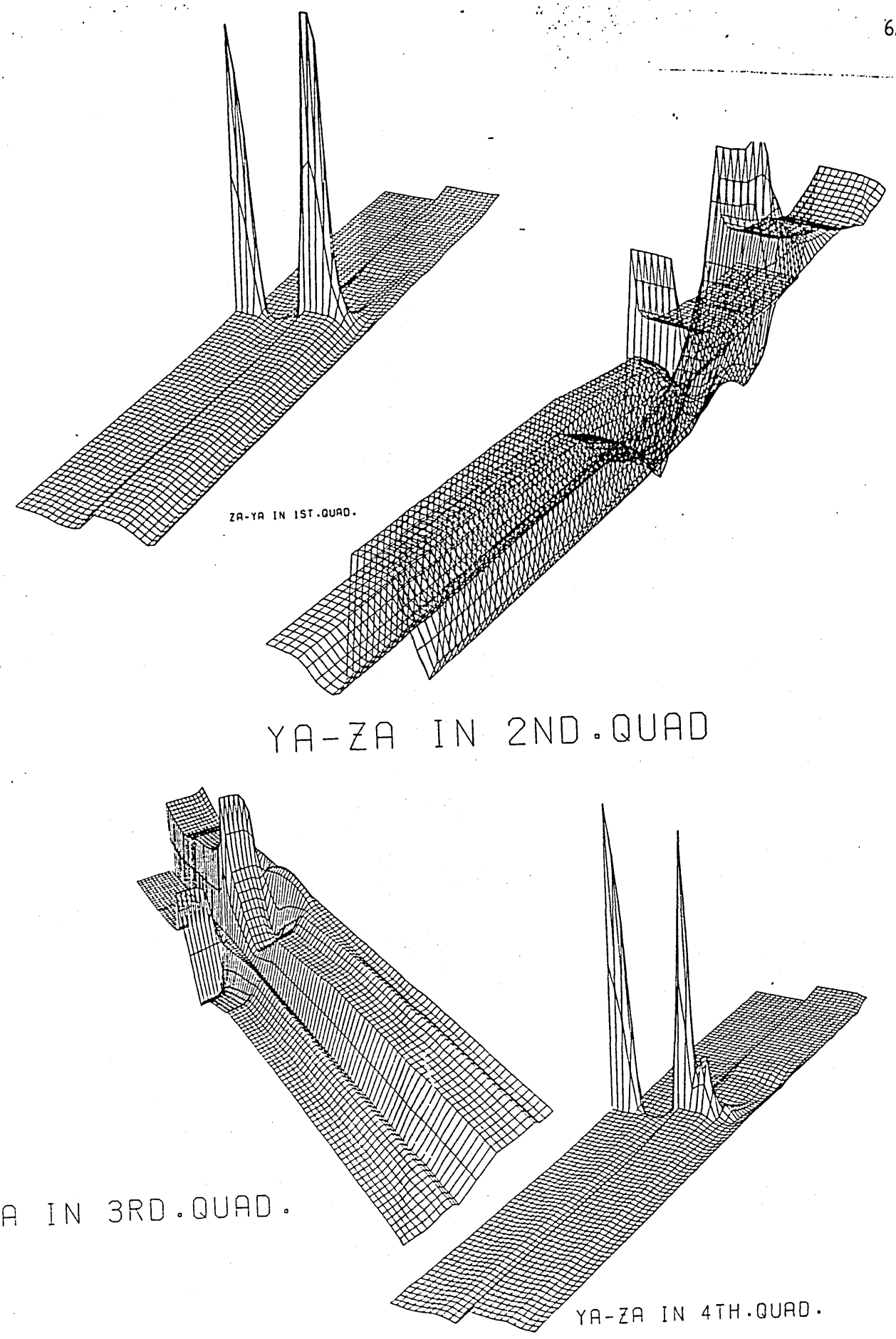


Figure 1.5-6 Plots of Y_a and Z_a in all four quadrants of frequency space.

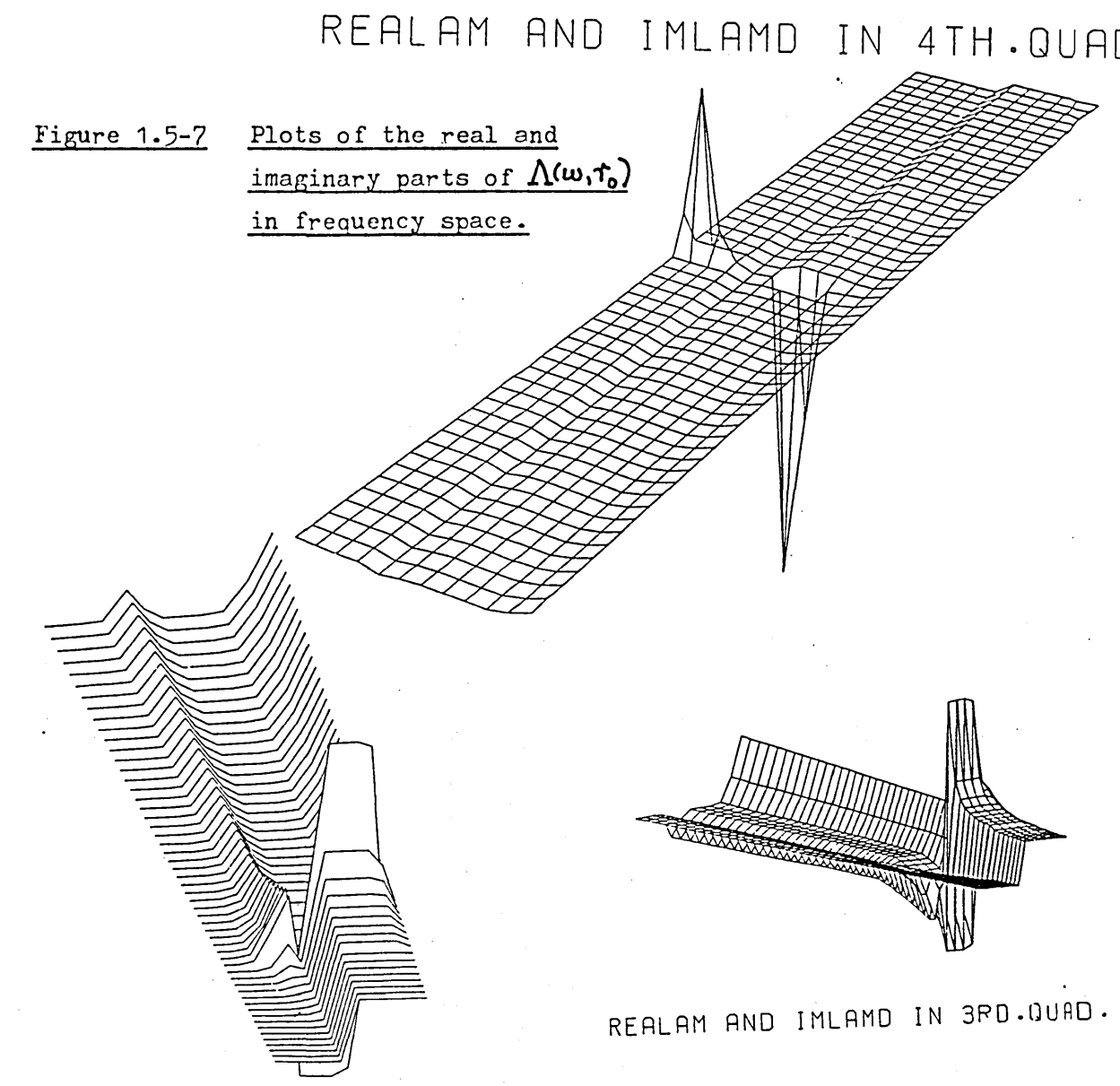


Figure 1.5-7 Plots of the real and imaginary parts of $\Lambda(\omega, t_0)$ in frequency space.

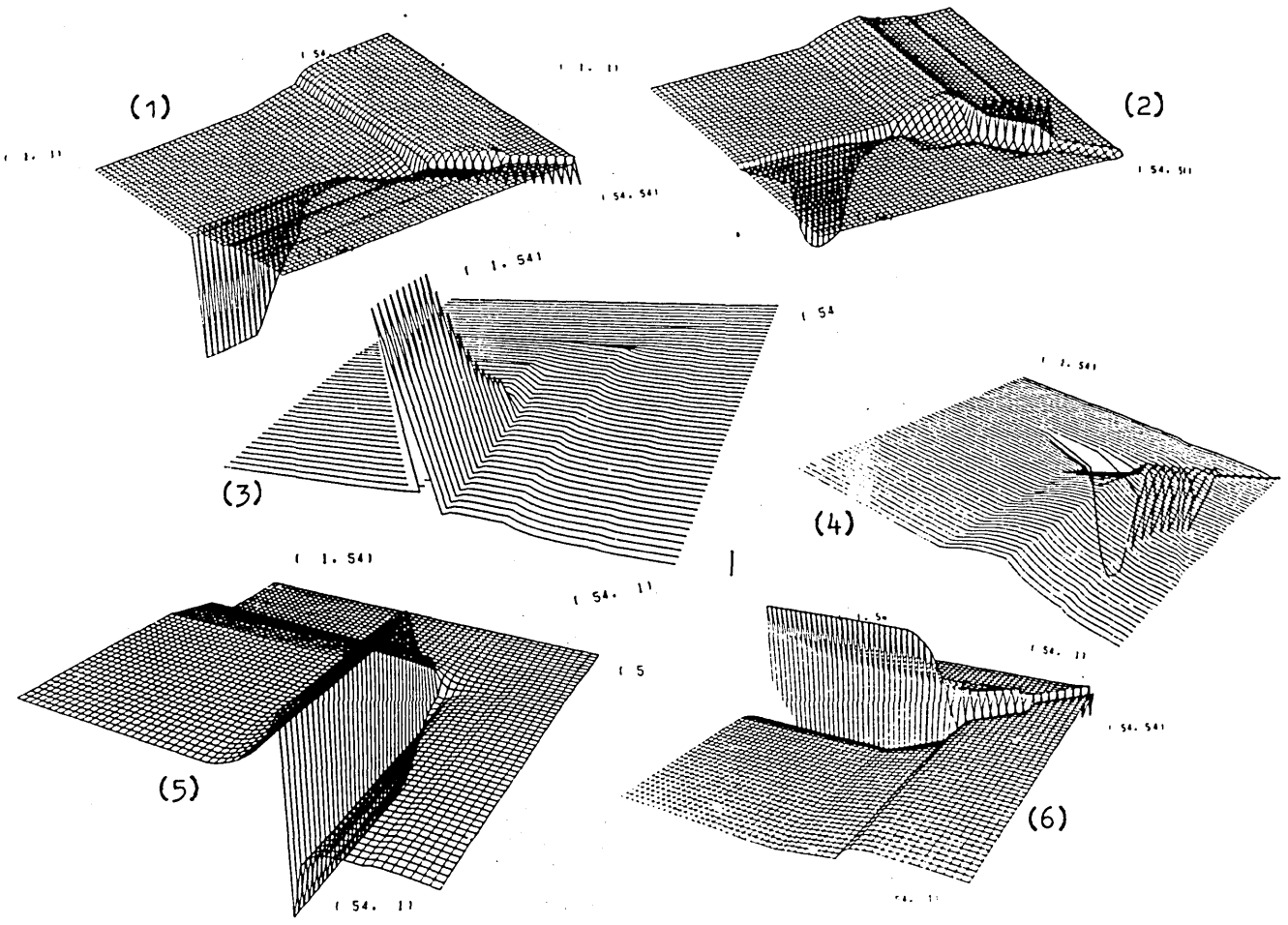


Figure 1.5-8 Real and imaginary parts of $\Lambda(\omega,r_0)$ in three quadrants of frequency space.

1.Real part of $\Lambda(\omega,r_0)$ in first quadrant.
2.Imaginary part of $\Lambda(\omega,r_0)$ in first quadrant.
3.Real part of $\Lambda(\omega,r_0)$ in fourth quadrant.
4.Imaginary part of $\Lambda(\omega,r_0)$ in fourth quadrant.
5.Real part of $\Lambda(\omega,r_0)$ in second quadrant.
6.Imaginary part of $\Lambda(\omega,r_0)$ in second quadrant.

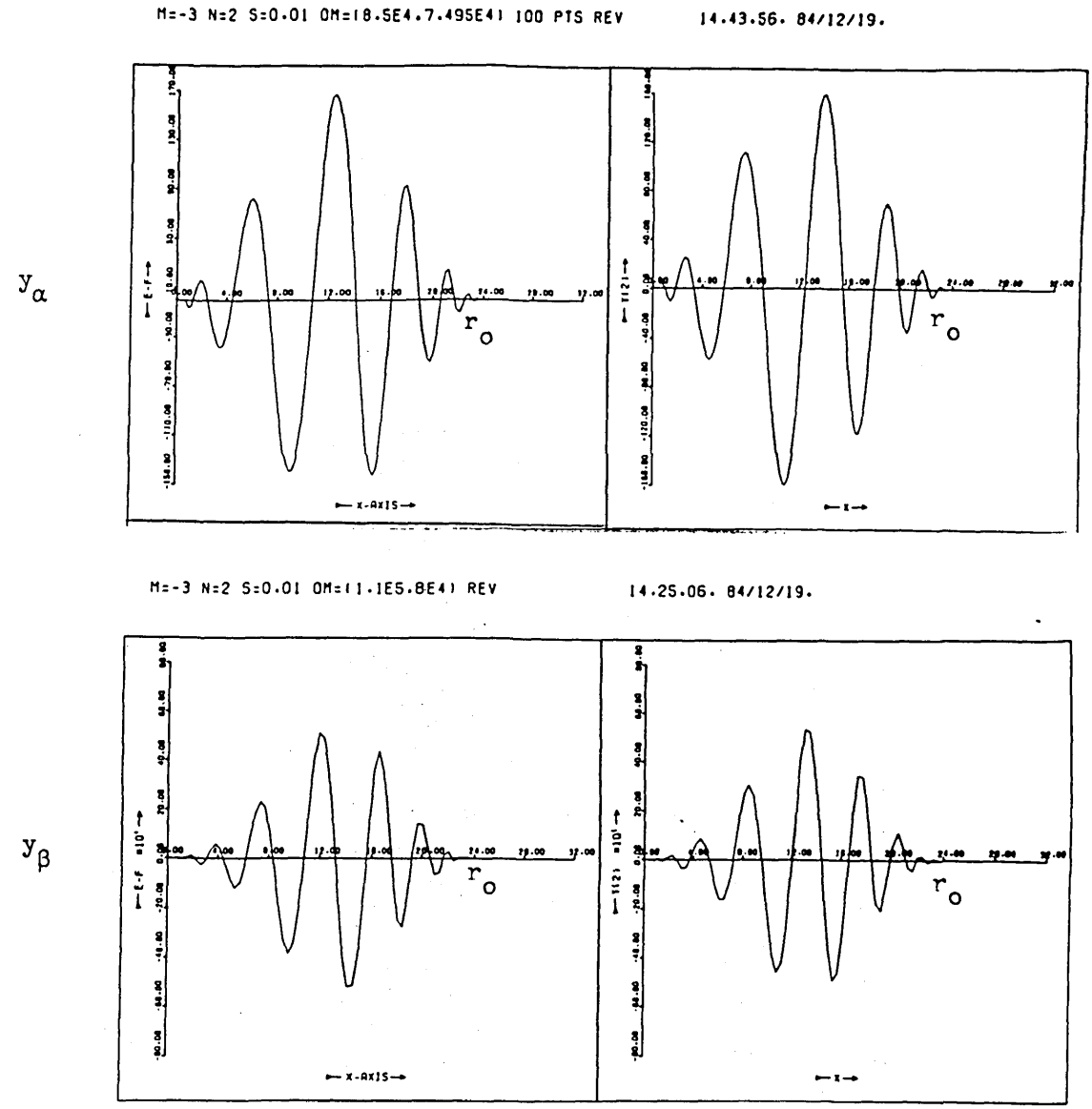


Figure 1.5-9 Two radial eigenfunctions (m=3 , n=2) of equation (1.5-1)

The resonant surface is located at r_o = 24·16 cm and the radius of the plasma is a = 30·0 cm . The integration has been carried to within λ = 0·01 r_L of each regular singular point (r=0 and r=r_o), and r_L = 0·1 cm .The direction of the integration was from r=r_o towards the origin.

1.6 A philosophical objection to the basic coupled equations of /1/.

As we have seen, the resonant surface $r=r_o$, and the origin , are regular singular points of the equations (1.4-1) and (1.4-2) . It follows that we shall require asymptotic expressions of the solution functions in the vicinity of either singularity, if we are to be able to solve these two coupled equations numerically. A procedure that we might follow is to consider the form of the coupled equations at a small distance from either singular point. Thus, suppose that we consider a distance r_1 away from the origin, and a distance r_2 away from the resonant surface. We now also define the dimensionless quantities :

$$x = \frac{r_1}{a} \qquad ; \qquad y = \frac{r_2}{a} \tag{1.6-1}$$

where a is the plasma radius, and we assume that since both $r=0$ and $r=r_o$, are regular singular points, it is reasonable to expect that the first term of a Frobenius expansion /70/, when substituted into the coupled equations, will produce an indicial equation obeyed by the coupled equation set. Thus, near $r=0$, we try the expansion

$$\psi = a_o x^{\rho} \qquad ; \qquad \Delta_p = b_o x^{\nu} \tag{1.6-2}$$

and find as a result (see Appendix G) that the powers are related as follows :

$$\rho = \nu - 1 \tag{1.6-3}$$

We also find in Appendix G that the quadratic indicial equation which results from the use of equation (1.6-2) has the two roots :

$$\rho_1 = -m - 1 \tag{1.6-4a}$$

$$\rho_2 = m - 1 \qquad (1.6\text{-}4b)$$

In the approximate theory of /1/ the equivalent of this equation is equation (1.5-8), which states that the indicial roots are complex numbers that depend on ω. However, the asymptotic expansion of the unapproximated coupled equations shows that the indicial roots are neither complex nor ω-dependent.

Similarly, in the vicinity of the resonant surface, we can attempt the single-term Frobenius expansions

$$\psi = a_o y^{\mu} \quad ; \qquad \Delta_p = b_o y^{\gamma} \qquad (1.6\text{-}5)$$

and we will be able to find (see Appendix H), that these indicies have the relationship :

$$\mu = \gamma + 1 \qquad (1.6\text{-}6)$$

Thus, provided we do not consider the uninteresting and very low frequencies characterized by $\omega \ll \omega^*$, then the quadratic indicial equation that we find has the two roots

$$\mu_1 = 0 \qquad (1.6\text{-}7a)$$

$$\mu_2 = \frac{2n}{a^2 r_o R k^2 k_{\parallel}'(r_o)} \qquad (1.6\text{-}7b)$$

Once again we see that the indicial roots are not complex or ω-dependent, in stark contradiction to the result of the approximate theory of /1/, which states that the behaviour of the indicial roots near $r=r_o$ is given by equation (1.5-14). Even if we do not make the approximation $\omega \ll \omega^*$, μ_1 and μ_2 would still only have very weak dependence on (see figure 1.6-1), and certainly not the strongly pathological dependence implied by equation (1.5-14), illustrated in the Y_a - Z_a plots of figure (1.5-6).

This failure of the approximate 'interior' formulation of /1/ to produce the correct asymptotic form of the solutions is significant. It indicates that the approximations which have led to the 'interior' equation (see Appendix H) are propably unreasonable, and consequently that any conclusions obtained through the use of this formulation of the two-fluid problem, may be unreliable.

We note that there is an important conclusion to be drawn from equations (1.6-7a) and (1.6-7b). From equation (1.6-6) we see that the values of γ corresponding to those reported in equation (1.6-7) are :

$$\gamma_1 = -1 \qquad (1.6\text{-}8a)$$

$$\gamma_2 = \frac{2n}{a^2 r_o R k^2 k_{\parallel}'(r_o)} - 1 \qquad (1.6\text{-}8b)$$

and from equation (F13) we know that

$$k_{\parallel}'(r_o) = \frac{m}{R}\left(\frac{1}{q}\right)'_{r_o} \qquad (1.6\text{-}8c)$$

Thus, it is clear that discrete bound solutions of both ψ and Δ_p (and therefore the possibility of instability), exists only if

$$\frac{2}{a^2 r_o k^2 \left(\frac{1}{q}\right)'_{r_o}} \cdot \frac{n}{m} > 1 \qquad (1.6\text{-}9)$$

If this condition is violated then the plasma is stable to all m-number perturbations, except possibly m=0, about which no statement is made. We bear in mind that the very low frequencies ($\omega \ll \omega^*$) are excluded for the validity of this criterion, and also that the whole analysis up to now has assumed that frequencies of interest are such that $\omega \ll \omega_{ci}$. We also note that this criterion has arisen directly from consideration of the asymptotic behaviour of the coupled equations of /1/, and point out that assessment of its value must be carried out in the light of the objections to those coupled equations which we have highlighted in section 1.3. As it stands however, the criterion does indicate that the higher m modes, are more stable than the lower m modes.

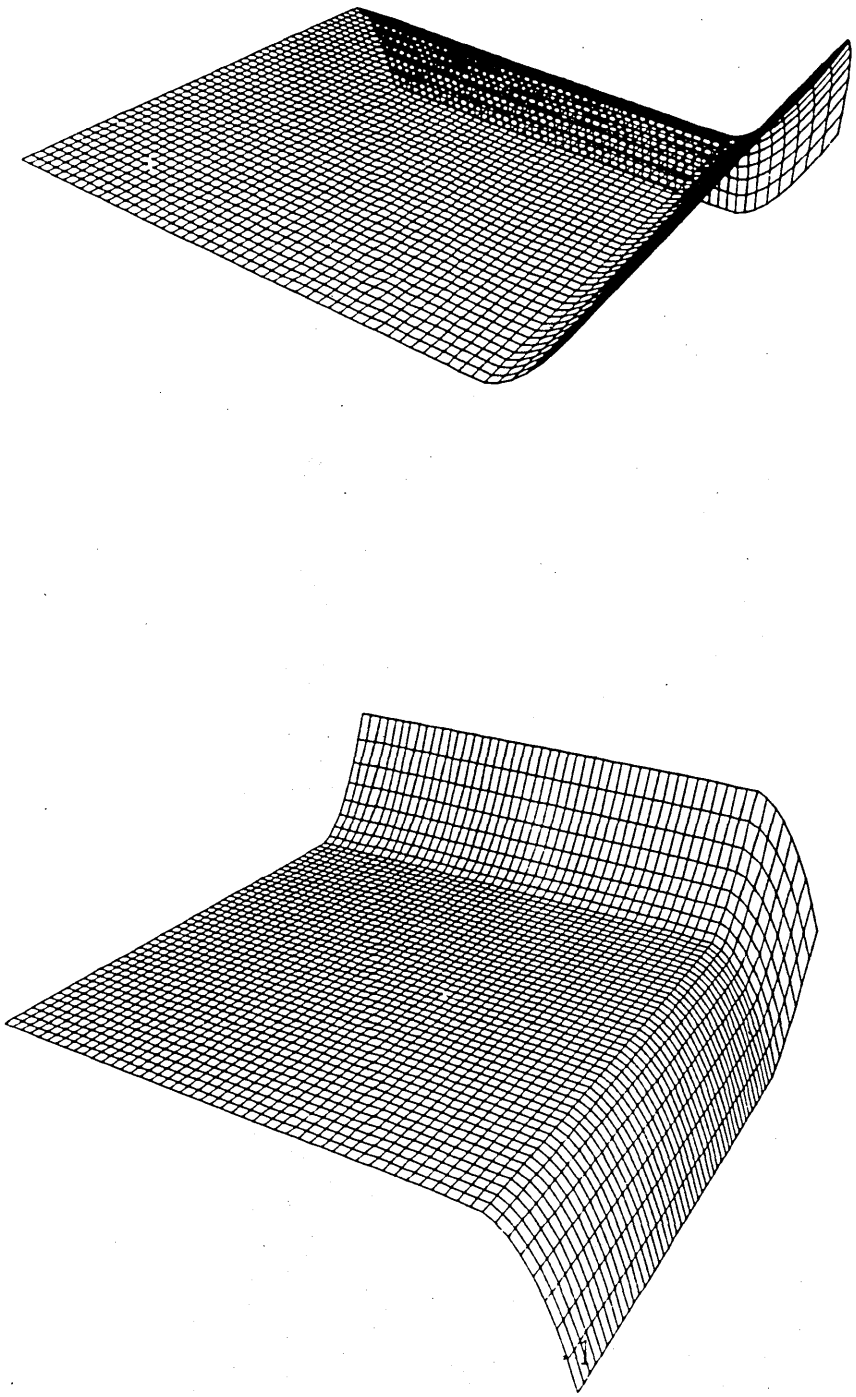


Figure 1.6-1 Behaviour of $Re(u_2)$ in the first quadrant of frequency space.

1.7 Summary of Chapter 1

In this chapter we have critically analysed the two-fluid theory presented in /1/, and have found,as a result,several errors in the formulation.We have also found an algebraic sign error that we showed to be consequential with regard to the conditions of stability predicted in /1/. Using heuristic and numerical and analytic considerations,we have shown two of the major approximations employed in the formulation of /1/,to be unjustified. It was also necessary to rederive all the approximate equations of /1/, in order to make transparent the nature of the approximations that were being employed there,and exactly how those approximations were being applied.We have in fact shown that there are grounds for regarding the results presented in /1/ with extreme caution.

Chapter 2

2.0 Background to the method used

In this chapter we shall consider a different approach to the solution of the two-fluid plasma equations presented in /1/, to that which we considered in chapter 1. There a normal-mode analysis was attempted and the time-derivative removed from the equations by the use of the following replacement :

$$\frac{\partial}{\partial t} \equiv i\omega$$

This formulation, from which in theory the whole spectrum of oscillations of the plasma can be derived, was found in spite of many approximations to have many associated difficulties. For instance (and in contrast to the ideal MHD situation) the eigenvalue problem that results from the normal-mode approach is not self-adjoint, so that in general such eigenvalues as exist will be complex. The linearised equations (equations 1·2-1 to 1·2-7) when Fourier analysed in time and space, were found to possess a singularity at the 'rational' magnetic surfaces, which are located wherever

$$\underline{k} \cdot \underline{B}_o = 0$$

These singularities also occur in ideal MHD theory (although, unlike in the two -fluid theory, their location is frequency dependent). At these singularities, the solutions to the ideal MHD differential equations are also singular. If resistivity is included, the ideal MHD solutions become regular at the rational surfaces. We shall now describe a method of solution of the linearised two-fluid equations, which avoids explicit consideration of the possible singular behaviour of the two-fluid solutions in the vicinity of the rational surfaces. In this method, the time-derivative is retained, and the procedure adopted is to integrate the linearised equations forwards in time, having initialised the problem with an arbitrary perturbation, until such time as the fastest exponentially growing radial oscillation, dominates all other oscillations that are present. (Only radial oscillations are of interest because, as in chapter 1, the linearised equations are first of all Fourier analysed in the

ignorable θ and z directions).Thus,through the use of this method, information will only be recovered about the behaviour of the fastest growing oscillation,and general statements about the nature of the full eigenspectrum,cannot be made.However,in order to avoid the difficulties encountered in /1/,this is probably not so great a price to pay.

The numerical integration method that we have used is called 'the method of lines' (MOL) /62,63,64,65,96/,and it involves the replacement of the spatial derivative operator $\frac{\partial}{\partial x_i}$,with finite differences,while the time derivative is left 'continuous' . Thus,if x is the spatial dimension which has been discretised into N_2 points, then a system of N_1 partial differential equations (PDEs), would be converted into a system of $(N_1 \times N_2)$ ordinary differential equations (ODEs). This system of ODEs can then be integrated numerically,using any one of the multitude of sophisticated algorithms written specifically for this purpose /65/.

In /62/ there is a very readable account of the theoretical foundations of the MOL,and also a wealth of references on the theoretical and practical aspects of the application of this method. V.I.Lebedev (see the reference at the bottom of p.596 of /62/) has shown that solutions obtained by the MOL,converge to the general solution of the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad (2.0\text{-}1)$$

in the rectangular box

$$0 \leq x \leq L \; ; \quad 0 \leq t \leq T \qquad (2.0\text{-}2)$$

provided that the initial and boundary conditions are homogeneous,and that the following inequalities are satisfied by the function $f(x,t)$:

$$\int_0^T \left\{ \int_0^L f^2 dx \right\}^{\frac{1}{2}} dt < C$$

$$\left[\int_0^L f^2 dx \right]_{t=0} < C$$

$$\int_0^T \left\{ \int_0^L \left(\frac{\partial f}{\partial x} \right)^2 dx \right\}^{\frac{1}{2}} dt < C. \qquad (2.0\text{-}3)$$

where C is a certain positive constant.It is clear that these inequalities are satisfied for $f(x,t)=0$,in which case equation (2.0-1) describes

undamped waves travelling in the positive and negative x-directions,with unit velocity.The homogeneous wave equation,which is what we now have, is purely hyperbolic.The numerical treatment of such equations is problematic in general,and special techniques must be employed in their numerical solution.Such techniques include the 'method of characteristics' /90/, the addition of an arbitrary 'artificial viscosity' /86/ to the equations, and also the complicated Flux Corrected Transport scheme /73/. In fact, according to Ames /90/,the method of characteristics is the natural numerical procedure for the integration of systems of hyperbolic partial differential equations.It is therefore important,for our purposes,to assess how well the MOL treat not only hyperbolic,but also parabolic equation systems.As an illustration,suppose that we are interested in applying the MOL to the solution of the one-dimensional heat equation :

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$

$$0 \leq x \leq 1$$

$$u(x,0) = 1.0$$

$$u(0,t) = 0.0$$

$$u(1,t) = 0.0 \qquad (2.0\text{-}4)$$

Suppose also that we use N_p=11 spatial points to characterise the region in which this equation applies.The boundary points are $x_o = 0$ and x_{10}= 1.0.At each of the internal spatial points we propose to write the heat diffusion equation as

$$\frac{du_i}{dt} = \left(u_{i+1} - 2u_i + u_{i-1}\right)/h^2 \qquad (i = 1, N-1) \qquad (2.0\text{-}5)$$

$$h = x_{i+1} - x_i$$

where we have taken c=1 ,and we have used three-point centred-differencing,for the second spatial derivative.As a result we are left with the following system of ODEs :

$$\frac{du_1}{dt} = -\frac{2u_1}{h^2} + \frac{u_2}{h^2}$$

$$\frac{du_2}{dt} = \frac{u_1}{h^2} - \frac{2u_2}{h^2} + \frac{u_3}{h^2}$$

$$\vdots$$

$$\frac{du_9}{dt} = \frac{u_8}{h^2} - \frac{2u_9}{h^2} \qquad (2.0\text{-}6)$$

Thus,the assumption of this method is that the time-development of the discrete dependent variables (u_i , i=1,9),at the fixed points in space (x_i ,i=1,9),or fixed lines in x-t space,fairly approximates the time-development of the original continuous dependent variable.A lucid exposition of the MOL treatment of parabolic PDEs is given in /96/.

The numerical integration software that we have used is that called FORSIM (FORmula SIMulation),written by Carver et al /77/.Communication with the FORSIM package is via a user-supplied subroutine which must be called UPDATE,in which the PDE system,along with its associated boundary and initial conditions,are defined.The FORSIM package has been used extensively in research laboratories around the world since it first appeared /94/,and the numerical routines that it contains have thus been well-tried and thoroughly tested over the intervening period of time.

Within FORSIM ,the system of PDEs is converted into a system of ODEs (in the manner of our example above),by the MOL algorithm of Sincovec and Madsen /63/.All that then remains is the problem of integrating forwards in time the new system of ODEs.This can be done using the Runge-Kutta or Adams integration routines present in FORSIM, or if the equation set is 'stiff' /95/,by using the Hindmarsh adaptation of C.W.Gear's integration routine /81/,also contained within the FORSIM package.Other than the seven different types of integration algorithms contained within it,FORSIM has many facilities which make it an attractive tool for the numerical solution of coupled ODE/PDE sets.Not least among these are its automatic starting /78,79,91,93/ and automatic error-control /72/ capabilities.In Appendix I we examine these two subjects,with reference to the Adams-Bashforth-Moulton integration routine.It is this routine that we have used for the bulk of the numerical results that shall be presented later in this chapter.

We shall now make several observations which must be borne in mind when using the MOL to solve a system of coupled PDEs.We should,for instance,note that the accuracy of a FORSIM (or indeed,of any MOL) solution,depends on the size of the spatial divisions,the order of the approximation of the spatial derivatives,the order of the time-integration algorithm and its step-size,the maximum truncation error allowed during the integration,and on the method of imposing the boundary conditions. All these topics are well discussed,with reference to FORSIM,in /87/..

It is shown in /87/ that the general FORSIM solution of equation (2.0-4) using pseudo-point boundary conditions and the Gear algorithm,is superior to the highly specialised ,purpose-built,implicit solution technique of von Rosenberg et al /89/ ,when heat loss at the boundary (i.e. a non-linear boundary condition) is allowed. This is quite surprising considering the generality of the FORSIM MOL solution method.We are also able to see,from results presented in /87/, that the MOL is far more efficient than the method of characteristics /90/,in solving systems of hyperbolic PDEs, such as the conservation equations of fluid flow. Now,let us also consider the results of Hocevar and Wineinger /88/, obtained from the numerical solution of the equations that arise in the analysis of loss-of-coolant accidents in nuclear reactors.Those results show the method of characteristics to be superior to the following methods :

(a) Explicit
(b) Six-point Implicit
(c) Modified Two-Step Lax-Wendroff

which were also tested in the equations used in the study of /88/.Thus, since it has been shown in /87/ that FORSIM (i.e. the MOL), exhibits far better accuracy than the method of characteristics,the implication is that the MOL is more accurate than the explicit,implicit, and Lax-Wendroff methods.Further analyses of the MOL treatment of hyperbolic equations are to be found in /74,75,76/.

We shall now consider two ways in which the accuracy of our numerical results could have been checked.Firstly,we could keep the spatial interval fixed,and increase the order of our difference approximations,or, secondly,we could reduce the spatial interval,keeping the order of our spatial difference approximations fixed.The idea would then be to continue each of these procedures independently,until tolerably small change is produced in the results.The former of these two methods is not infallible however,because the error in the solution will depend on the higher order derivatives, and for our particular equation system,these errors may increase rapidly with increasing order (see /91/).The control of this error would then prohibitively increase the execution time of our computations.Thus,suppose that a pathology is present,and that at some point the fourth derivative becomes infinite,while the

third derivative remains bounded.To approach too close to this point with an approximation of fourth order or higher is fraught with danger, whereas one can integrate past the pathology with a third order approximation,provided the integration step is reduced in the vicinity of the pathology.Further cautions against the indiscriminate use of higher order spatial differences can be found in /92/.

For the reasons outlined above then,we have used only three-point difference formulae,and have checked our results by reducing the space-mesh step size. We have found it sufficient to use only 20 spatial points, since the results produced using 40 spatial points differ from those obtained using 20 points,by less than 0.2 %.

It is also encouraging that identical results (to within 0.1%) are produced when our equations are integrated by each of the seven different algorithms present in FORSIM (except for the rudimentary two-step Euler method , whose results differ from those produced by the other more sophisticated integration routines by about 2.5 %). This suggests that the results that we have obtained are a property of our equation set,rather than a property of our numerical integration

There remain only a few more points to be made about our numerical procedures.For instance,since we have complex coefficients in our 8 Fourier analysed,linearised equations,there are really 16 equations,for 16 real variables,which must be integrated forwards in time.Thus,for the modest 20 spatial points that we have used,we are still involved in integrating forwards in time, 320 ODEs.Thus the program requires large amounts of storage in the central memory of the computer.It was found that for 45 spatial points,it was not possible to run the program interactively,and that for 50 spatial points,the single-user memory allocation was exceeded. All our programs were run on the CDC Cyber 175 and 855 computers of the Imperial College Computer Centre,where a FORTRAN 66 version of FORSIM is maintained.

Using the numerical method that we have just outlined,we are able to produce the eigenfunctions and growth-rates (imaginary parts of the complex eigenfrequency, ω),of the fastest growing radial oscillations, for the different m-modes and different equilibria that we have studied.

These equilibria are characterised by the z-current density profiles :

$$j_{oz} = j_0 \left[1 - \frac{r^2}{a^2}\right]^{\nu} \qquad (2.0\text{-}7)$$

where we have chosen to study the cases for which

$$\nu = 1 \, , 2 \, , 4 \qquad (2.0\text{-}8)$$

Using ideal MHD analysis we would find that all these equilibria are stable to the modes m=0 , m=1 , m=2 ,m=3 . We have found that the presence of the ∇p_e and the $\underline{J} \times \underline{B}$ terms in the Ohm's law, is sufficient to cause the m=1 , m=2 , and m=3 modes to become unstable.

In the next section we shall discuss in more detail,the above equilibria.

2.1 The Equilibrium

When the plasma is in equilibrium, we know (from adding together equation (1.1-13) and (1.1-14)), that the following equation holds :

$$\nabla p_0 = \frac{1}{c} \underline{j}_0 \wedge \underline{B}_0 \qquad (2.1\text{-}1)$$

This equation only has an r-component (in cylindrical coordinates), if the equilibrium magnetic field is given by :

$$\underline{B}_0 = (0 , B_{0\theta} , B_{0z}) \qquad (2.1\text{-}2)$$

Equation (2.1-1) then becomes the statement

$$\frac{dp_0}{dr} = \frac{1}{c} \left(j_{0\theta} B_{0z} - j_{0z} B_{0\theta} \right) \qquad (2.1\text{-}3)$$

and if we remember that the equilibrium Ampere's law

$$\nabla \wedge \underline{B}_0 = \frac{4\pi}{c} \underline{j}_0 \qquad (2.1\text{-}4)$$

will have in cylindrical coordinates, the two components :

$$j_{0\theta} = -\frac{c}{4\pi} \frac{dB_{0z}}{dr} \qquad (2.1\text{-}5)$$

$$j_{0z} = \frac{c}{4\pi r} \frac{d}{dr} (r B_{0\theta}) \qquad (2.1\text{-}6)$$

the equilibrium condition, equation (2.1-3), becomes

$$\frac{d}{dr} \left[p_0 + \left(\frac{B_{0\theta}^2 + B_{0z}^2}{8\pi} \right) \right] + \frac{B_{0\theta}^2}{4\pi r} = 0 \qquad (2.1\text{-}7)$$

From equation (2.1-7) we are able to see that once any two of the quantities $p_o(r)$, $B_{o\theta}(r)$ and $B_{oz}(r)$, are arbitrarily specified, then the remaining quantity is automatically defined apart from an arbitrary constant. It will be convenient to choose $p_o(r)$ to be one of the arbitrarily specified profiles, for since $p_o = n_o T_o$, it would then be easy to make sure that

$$\frac{n_o(0)}{n_o(a)} \gg 1 \qquad (2.1\text{-}7a)$$

This condition is necessary for the validity of one of the boundary conditions that shall later be applied to the equations.

In order to appreciate how equilibria corresponding to equation (2.1-7), and possibly describing actual plasma confinement experiments, can be set up, let us suppose that as well as defining $p_o(r)$, we also define $B_{o\theta}(r)$ as follows :

$$B_{o\theta}(r) = b_o r e^{-\nu r^2/a^2} \qquad (2.1\text{-}7b)$$

where b_o is an arbitrary constant with units of (magnetic field/length). With the aid of equation (2.1-6) we can then state that the z-current density profile is given by the expression :

$$j_{oz}(r) = \left(\frac{b_o c}{2\pi}\right) e^{-\nu r^2/a^2}\left[1 - \frac{\nu r^2}{a^2}\right] = j_o e^{-\nu r^2/a^2}\left[1 - \frac{\nu r^2}{a^2}\right]. \qquad (2.1\text{-}7c)$$

Now, suppose also that the equilibrium pressure profile that we have chosen to use is

$$P_o(r) = P_o(0)\, e^{-\epsilon\left(\frac{r}{a}\right)^{\alpha}} \qquad (2.1\text{-}7d)$$

then, from the equilibrium equation (equation 2.1-7)

$$\frac{d}{dr}\left(\frac{B_{oz}^2}{2}\right) = -\frac{B_{o\theta}^2}{r} - 4\pi \frac{dp_o}{dr} - \frac{d}{dr}\left(\frac{B_{o\theta}^2}{2}\right) \qquad (2.1\text{-}7e)$$

$B_{oz}^2(r)$ can be obtained by numerical integration of equation (2.1-7e) (i.e. treating it as an initial-value problem), provided $B_{oz}(0)$ is known. This is the input arbitrary constant of which we spoke above. This constant can be supplied by specifying the β value of the plasma equilibrium that we are trying to set up. We define this β in the following way :

$$\beta = \frac{p_o(0)}{p_o(0) + \frac{B_{oz}^2(0)}{4\pi}} \qquad (2.1\text{-}7f)$$

i.e.

$$p_o(0) = \frac{\beta\, B_{oz}^2(0)}{(1-\beta)4\pi} \qquad (2.1\text{-}7g)$$

In equation (2.1-7c)

$$j_o = \frac{cb_o}{2\pi} \qquad (2.1\text{-}7h)$$

and so we can write equation (2.1-7b) for $B_{o\theta}$ as :

$$B_{o\theta} = \frac{2\pi j_o}{c} \cdot r\, e^{-\nu r^2/a^2} \qquad (2.1\text{-}7i)$$

Equation (2.1-7e) can then be normalised by division by $B_{oz}^2(0)$. Once this is done, the normalised central pressure will be given by

$$\hat{p}_o(0) = \frac{\beta}{(1-\beta)} \qquad (2.1\text{-}7j)$$

and the normalised $B_{o\theta}$ field by

$$\hat{B}_{o\theta} = K \mu\, e^{-\nu\mu^2} \qquad (2.1\text{-}7k)$$

where

$$K = \frac{2\pi j_o a}{c B_{oz}(0)} \qquad (2.1\text{-}7l)$$

is a dimensionless number that is of order 0·2 for typical high-temperature tokamaks. In equation (2.1-7k) μ is the dimensionless independent variable, and is given by

$$\mu = \frac{r}{a} \qquad (2.1\text{-}7m)$$

The choice of the number K is determined by the ratio $(j_o/B_{oz}(0))$ in the experiment that we are trying to describe. The physical significance

of the number K is that from equation (2.1-7k)

$$\lim_{\mu \to 0} \left(\frac{\hat{B}_{o\theta}}{\mu} \right) = K \qquad (2.1\text{-}7m)$$

and this is obviously a quantity that varies from one experimental configuration to another.

In summary, the procedure of setting up an equilibrium is as follows:

1. Choose a pressure profile $\hat{p}_o(\mu) = \hat{p}_o(0).\tilde{\alpha}(\mu)$
2. Choose a β value (this defines $\hat{p}_o(0)$).
3. Choose values for K and $\tilde{\nu}$ (thereby defining $\hat{B}_{o\theta}$ through equation 2.1-7k).
4. Solve the normalised equilibrium equation as an initial-value problem

$$\frac{d(B_{oz}^2)}{d\mu} = -\frac{2B_{o\theta}^2}{\mu} - \frac{2\beta}{(1-\beta)} \cdot \frac{d\tilde{\alpha}}{d\mu} - \frac{d}{d\mu}(B_{o\theta}^2)$$

$$B_{oz}^2(0) = 1.0$$

thereby numerically determining $\hat{B}_{oz}(\mu)$.

The question of whether these equilibria are ideal MHD unstable is an important one to consider, and can be decided by the application of Suydam's criterion /3/. According to this criterion, for equilibria in the cylindrical configuration which we are considering to be stable, it is necessary that

$$S(r) = \frac{32\pi\, p'}{rB_{oz}^2} \left(\frac{\tilde{\mu}}{\tilde{\mu}'} \right)^2 + 1 \geqslant 0 \qquad (2.1\text{-}7n)$$

where the primes denote differentiation with respect to r, and

$$\tilde{\mu} = \frac{B_{o\theta}}{rB_{oz}} \qquad (2.1\text{-}7o)$$

The function S(r) is called the Suydam parameter, and while stability of the equilibrium is not guaranteed if $S(r) \geqslant 0$ throughout the plasma volume, instability of the plasma is guaranteed in regions where $S(r) < 0$. We must also point out that if the Suydam criterion is to be satisfied in the vicinity of the origin, then the central portion of the pressure profile generally needs to be very flat. To achieve this we have taken, for the sake of illustration, $\alpha = 4$, in equation (2.1-7d) .

Two sets of equilibrium profiles are shown in figure (2.1a) and figure (2.1b) for which $\tilde{\nu}=1$, $\epsilon=1$, $\kappa=0.2$ .For the first set of profiles $\beta=0.01$, and for the second set $\beta=0.1$,and we see that this difference in β is enough to make the second set of profiles Suydam-unstable over a large portion of the radius of the plasma. We would be able to shape the current profile (j_{oz}) even more by taking

$$\hat{B}_{o\theta}(u) = K\mu e^{-\tilde{\nu}\mu^{\delta}} \qquad (2.1-8)$$

for which the corresponding j_{oz} profile is

$$j_o(\mu) = j_o(1 - 0{\cdot}5\tilde{\nu}\delta\mu^{\delta}) e^{-\tilde{\nu}\mu^{\delta}} \qquad (2.1-9)$$

We have used the above equilibrium profiles for illustrative purposes only.The actual current density profiles that we have used in the studies that we have carried out are given by :

$$j_{oz}(\mu) = j_o(1 - \mu^2) \qquad (2.1-10)$$

This current profile has been used elsewhere /82/ for the study of free-boundary ideal MHD instabilities. The corresponding $B_{o\theta}$ magnetic field is given by

$$\hat{B}_{o\theta} = \frac{K}{(\nu+1)} \cdot \frac{1}{\mu} \cdot \left[1 - (1-\mu^2)^{\nu+1}\right] \qquad (2.1-11)$$

where

$$K = \frac{2\pi j_o a}{c B_a} \qquad (2.1-12)$$

and

$$B_a = B_{oz}(a) \qquad (2.1-13)$$

The number density and temperature profiles that we have chosen are :

$$n_o(\mu) = n_o(o) e^{-\epsilon\mu^2} \qquad (2.1-14)$$

$$T_o(\mu) = T_o(o) e^{-\epsilon(\Gamma-1)\mu^2} \qquad (2.1-15)$$

so that the pressure profile is given by :

$$P_o(\mu) = P_o(o) e^{-\epsilon\Gamma\mu^2} \qquad (2.1-16)$$

In equations (2.1-14) to (2.1-16), ϵ is an arbitrary positive and real number which adjusts the ratio $\frac{n_0(0)}{n_0(a)}$ of equation (2.1-7a). Γ is the ratio of specific heats. For integer $\nu \leq 4$, the analytic integration of the equilibrium equation (2.1-7), is just tolerably tedious.For example, for $\nu=1$ we obtain

$$B_{o\theta}(\mu) = K\mu\left(1 - \frac{\mu^2}{2}\right) \tag{2.1-17}$$

and

$$B_{oz}^2(\mu) = 1 + K^2(5 - 12\mu^2 + 9\mu^4 - 2\mu^6)/6 + \tilde{\beta}\, e^{-\epsilon\Gamma}\{1 - e^{+\epsilon\Gamma(1-\mu^2)}\}. \tag{2.1-18}$$

where $\tilde{\beta}$ has now been defined as

$$\tilde{\beta} = \frac{4\pi p_0(a)}{B_a^2} \tag{2.1-18a}$$

The 'safety factor' q(μ) is defined

$$q(\mu) = \frac{r B_{oz}}{R B_{o\theta}} = \epsilon_1 \mu B_{oz}(\mu)/B_{o\theta}(\mu) \tag{2.1-19}$$

where ϵ_1 is the 'inverse aspect ratio',and is defined by :

$$\epsilon_1 = \frac{a}{R} \tag{2.1-20}$$

It is relatively simple to show that the safety factor profile corresponding to the current density of equation (2.1-10) is (if B_{oz} varies by less than a few percent over the radius)

$$q(\mu) = \frac{\epsilon_1(\nu+1)\mu^2}{K\left[1 - (1-\mu^2)^{\nu+1}\right]} \tag{2.1-21}$$

From this it is easy to see that

$$q(0) = \epsilon_1/K \tag{2.1-22}$$

and also that

$$q(1) = \frac{\epsilon_1(\nu+1)}{K}. \tag{2.1-23}$$

From the previous two equations it therefore follows that

$$\frac{q(1)}{q(0)} \equiv \frac{q(a)}{q(0)} = \nu+1 \qquad (2.1\text{-}24)$$

We have illustrated the $j_{oz}(r), j_{o\theta}(r), B_{oz}(r), B_{o\theta}(r)$, and $q(r)$ profiles used in our numerical studies in figure (2.1-1) .

In the next section,we shall rewrite the equations of our two-fluid model (equations (1.2-1) to (1.2-7)),in a form suitable for our numerical initial-value approach.We shall also find that it is not necessary to solve explicitly for the electron fluid perturbation velocity $\underline{u}_1$,because this can be wholly expressed in terms of the other perturbation variables.Also,because $\nabla.\underline{B}_1 = 0$,only two components of the perturbed magnetic field (say B_{1r} and $B_{1\theta}$) are independent and need be solved for.Thus we have in total, 8 perturbed quantities to solve for,and these are :

B_{1r}	radial component of perturbed magnetic field
$B_{1\theta}$	θ component of perturbed magnetic field
v_{1r}	radial component of perturbed ion-fluid velocity
$v_{1\theta}$	θ component of perturbed ion-fluid velocity
v_{1z}	z component of perturbed ion-fluid velocity
p_{1i}	isotropic ion-fluid pressure perturbation
p_{1e}	isotropic electron-fluid pressure perturbation
n_1	number density perturbation.

(2.1-25)

It is to be noted also that we could reduce the number of variables that we have to solve for by one,by taking

$$\nabla.\underline{V}_1 = 0 \qquad (2.1\text{-}26)$$

in which case we would be assuming the plasma to be incompressible,but we have chosen not to do this.Our results therefore will apply to a fully compressible plasma.

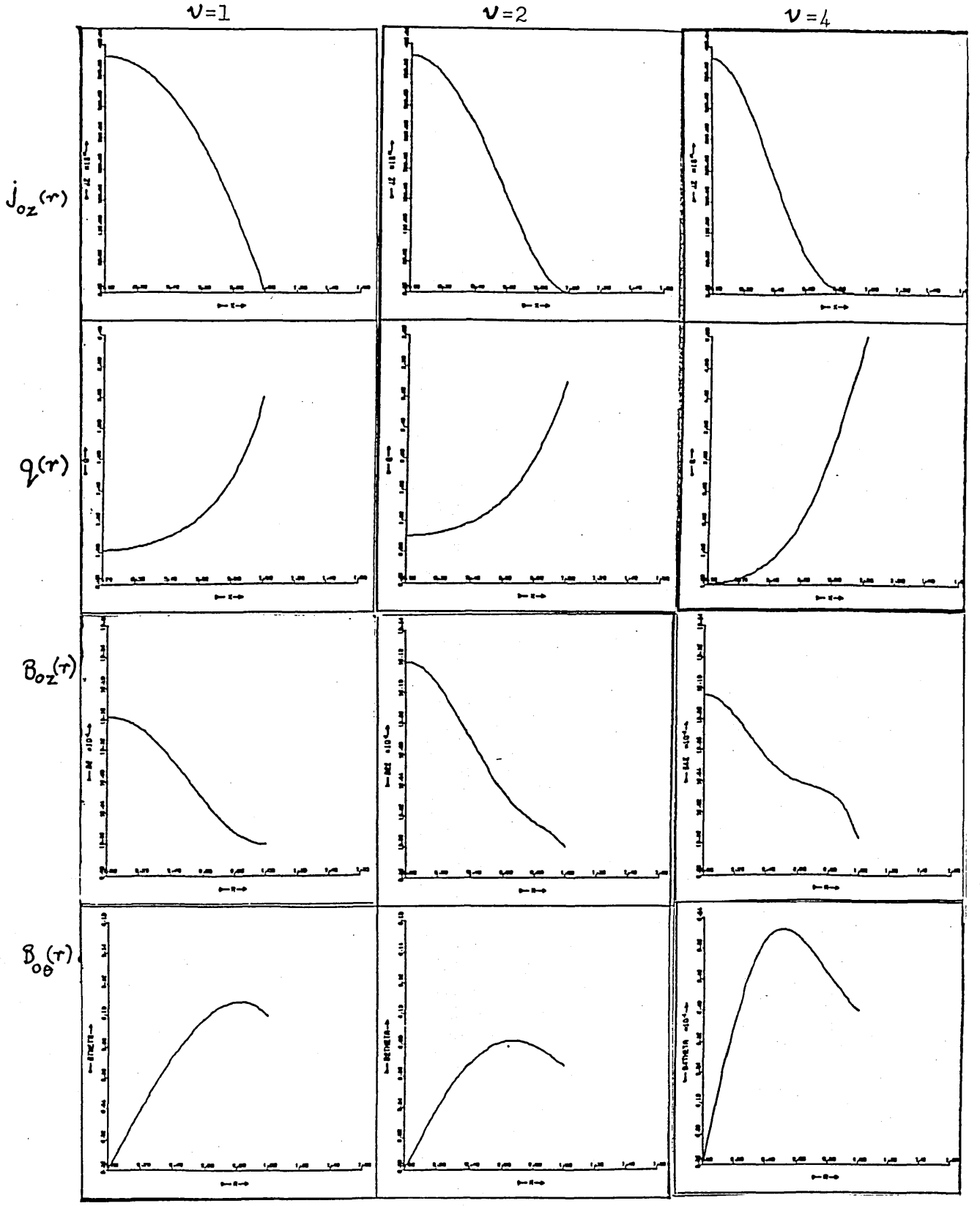


Figure 2.1-1 Equilibrium profiles for $j_{oz} = j_o \left[1 - \frac{r^2}{a^2}\right]^{\nu}$

$\tilde{\nu}=1$, $K=0.2$, $\epsilon=1$, $\epsilon_1=0.2$, $\beta=0.01$, $\delta=2$

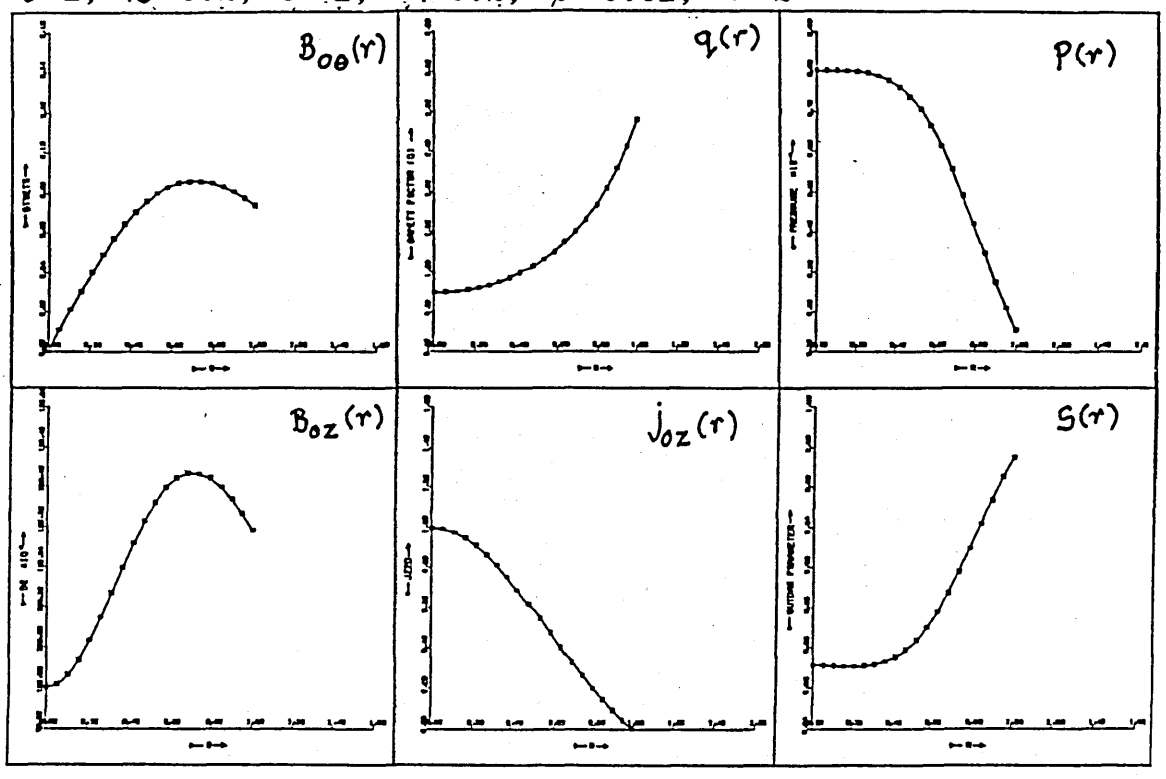


Figure 2.1a Equilibrium profiles for $j_o(\mu) = j_o[1 - 0.5\tilde{\nu}\delta\mu^\delta]$ with $\beta=0.01$. $p_o = p_o(0)\, e^{-\epsilon r\mu^4}$

$\tilde{\nu}=1$, $K=0.2$, $\epsilon=1$, $\epsilon_1=0.2$, $\beta=0.1$, $\delta=2.$

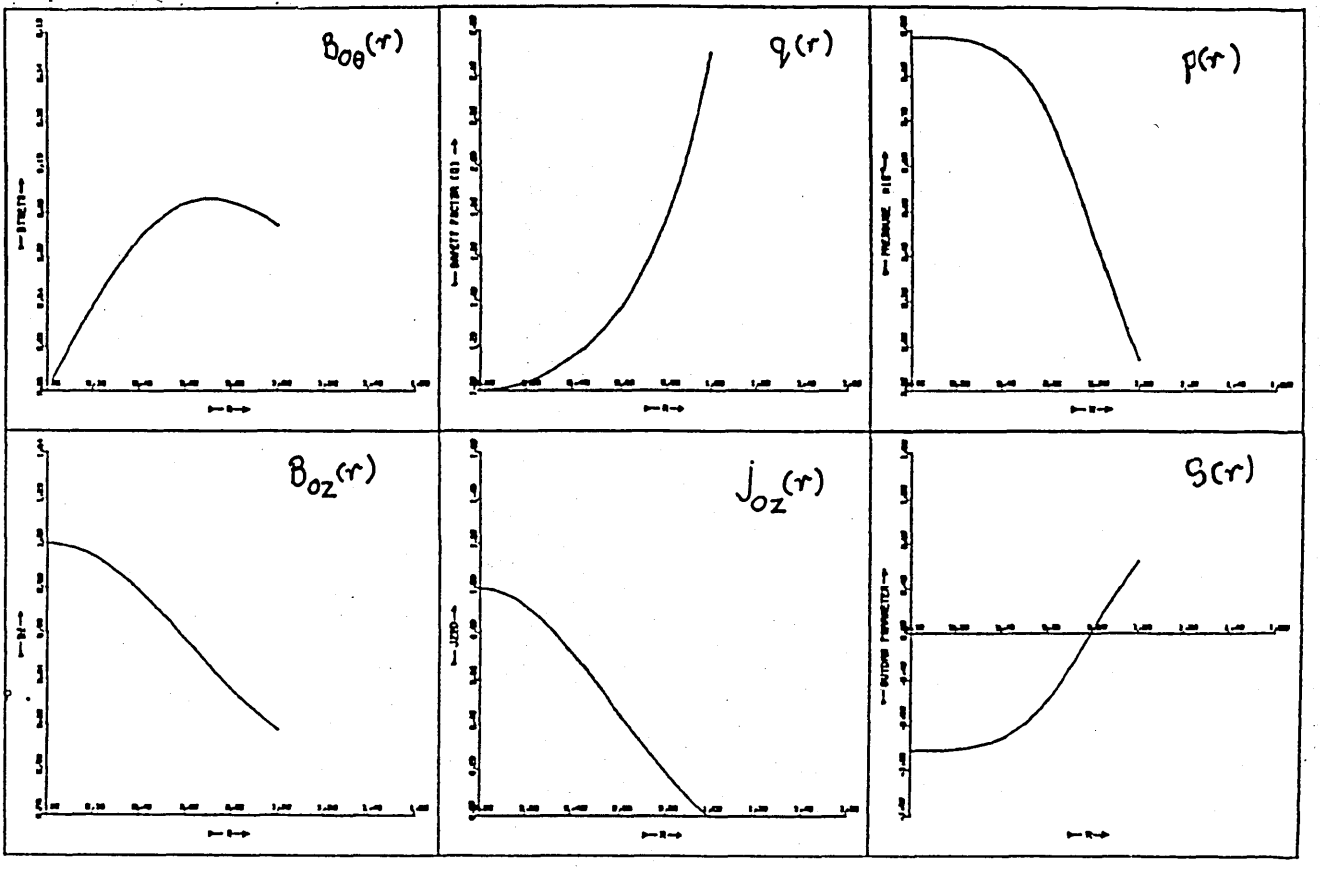


Figure 2.1b Equilibrium profiles for $j_0(\mu) = j_0[1 - 0.5\tilde{\nu}\delta\mu^\delta]$ with $\beta=0.1$ $\quad p_0 = p_0(0)e^{-\epsilon r\mu^4}$

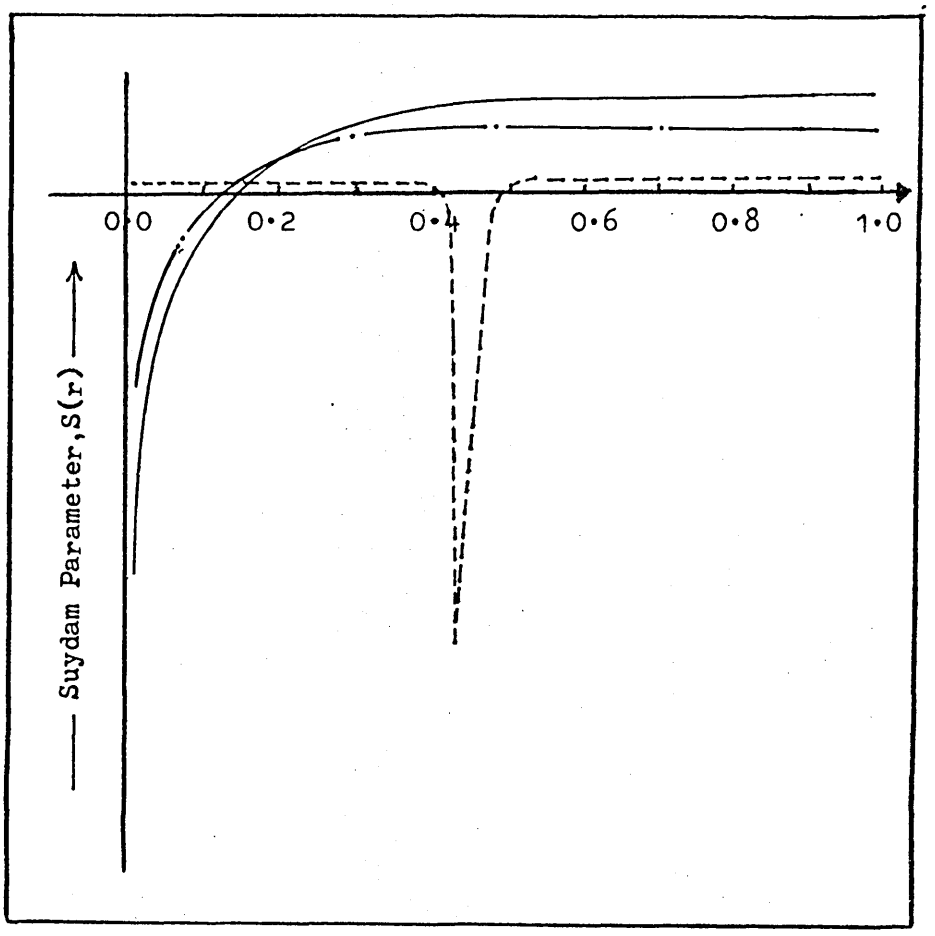


Figure 2.1-2 Suydam Parameter profile for the equilibria of figure (2.1-1)

—·—·—·— $\nu = 1$

———— $\nu = 2$

- - - - - - - - - $\nu = 4$

2.2 The equations to be solved in the initial-value treatment

The equations that we shall use in our numerical initial-value treatment of the two-fluid problem follow directly (after much tedious algebra),from equations (1.2-1) to (1.2-7) supplemented by

$$\nabla\cdot\underline{B}_1 = 0 \qquad (2.2\text{-}0)$$

The equations are :

$$n_0 m_i \frac{\partial v_{1r}}{\partial t} = \frac{iR}{4\pi n}\left[\frac{n}{R}(\underline{k}.\underline{B}_0) - \frac{d}{dr}\left(\frac{B_{0z}}{r}\right)\right]B_{1r} - \frac{iR}{4\pi n}\left[\frac{1}{r}\frac{d}{dr}(rB_{0z})\right]\frac{\partial B_{1r}}{\partial r}$$
$$-\frac{iR}{4\pi n}\left[B_{0z}\right]\frac{\partial^2 B_{1r}}{\partial r^2} + \frac{R}{4\pi n}\left[m\frac{d}{dr}\left(\frac{B_{0z}}{r}\right) - \frac{n}{R}\frac{dB_{0\theta}}{dr} - \frac{2nB_{0\theta}}{rR}\right]B_{1\theta}$$
$$+\frac{R}{4\pi n}\left[\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right]\frac{\partial B_{1\theta}}{\partial r} - \frac{\partial p_1}{\partial r} \qquad (2.2\text{-}1)$$

$$n_0 m_i \frac{\partial v_{1\theta}}{\partial t} = \frac{R}{4\pi n}\left[\frac{1}{r}\left\{\underline{k}.\underline{B}_0 + \frac{nr}{R}\frac{dB_{0\theta}}{dr}\right\}\right]B_{1r} + \frac{R}{4\pi n}\left[\frac{mB_{0z}}{r}\right]\frac{\partial B_{1r}}{\partial r}$$
$$+\frac{R}{4\pi n}\left[ik^2 B_{0z}\right]B_{1\theta} - \frac{im}{r}p_1 \qquad (2.2\text{-}2)$$

$$n_0 m_i \frac{\partial v_{1z}}{\partial t} = \frac{R}{4\pi n}\left[\frac{1}{r}\left(\frac{nr}{R}\frac{dB_{0z}}{dr} - \frac{mB_{0\theta}}{r}\right)\right]B_{1r} - \frac{R}{4\pi n}\left[\frac{mB_{0\theta}}{r}\right]\frac{\partial B_{1r}}{\partial r}$$
$$-\frac{R}{4\pi n}\left[ik^2 B_{0\theta}\right]B_{1\theta} - \frac{in}{R}p_1 \qquad (2.2\text{-}3)$$

$$\frac{\partial B_{1r}}{\partial t} = \frac{c}{4\pi e n_0}\left[i\left\{ \frac{n}{R}\left(\frac{dB_{0\theta}}{dr} + \frac{B_{0\theta}}{r}\right) - \frac{m}{r}\frac{dB_{0z}}{dr} + \frac{mR}{nr^2}(\underline{k}\cdot\underline{B}_0)\right\}\right] B_{1r}$$

$$\frac{c}{4\pi e n_0}\left[\frac{imR}{nr}(\underline{k}\cdot\underline{B}_0)\right]\frac{\partial B_{1r}}{\partial r} - \frac{c}{4\pi e n_0}\left[\frac{k^2R}{n}(\underline{k}\cdot\underline{B}_0)\right] B_{1\theta}$$

$$+ i(\underline{k}\cdot\underline{B}_0)\, V_{1r} \qquad (2.2\text{-}4)$$

$$\frac{\partial B_{1\theta}}{\partial t} = \frac{c}{4\pi e n_0}\left[\frac{dB_{0z}}{dr}\left(\frac{1}{\Gamma p_{0e}}\cdot\frac{dp_{0e}}{dr} - \frac{1}{r}\right) + \frac{d^2B_{0z}^2}{dr^2} + \frac{R}{n}(\underline{k}\cdot\underline{B}_0)\cdot\left(\frac{n^2}{R^2}+\frac{1}{r^2}\right)\right.$$

$$\left.\frac{mR}{nr}\left\{\frac{B_{0\theta}}{r\Gamma p_{0e}}\cdot\frac{dp_{0e}}{dr} - \frac{d}{dr}\left(\frac{B_{0\theta}}{r}\right)\right\}\right] B_{1r}$$

$$+ \frac{c}{4\pi e n_0}\left[\frac{R}{n}\left\{\frac{mB_{0\theta}}{\Gamma p_{0e}}\cdot\frac{dp_{0e}}{dr} - m\frac{d}{dr}\left(\frac{B_{0\theta}}{r}\right) + \frac{(\underline{k}\cdot\underline{B}_0)}{r}\right]\frac{\partial B_{1r}}{\partial r}\right.$$

$$+ \frac{c}{4\pi e n_0}\cdot\left[\frac{R}{n}(\underline{k}\cdot\underline{B}_0)\right]\frac{\partial^2 B_{1r}}{\partial r^2}$$

$$+ \frac{c}{4\pi e n_0}\left[\frac{iRk^2}{n}\left\{\frac{B_{0\theta}}{\Gamma B_e}\cdot\frac{dp_{0e}}{dr} - r\frac{d}{dr}\left(\frac{B_{0\theta}}{r}\right)\right\} - \frac{iR}{n}\cdot\frac{m^2}{r^2}(\underline{k}\cdot\underline{B}_0)\right.$$

$$\left.-i\left\{\frac{n}{R}\left(\frac{dB_{0\theta}}{dr} + \frac{B_{0\theta}}{r}\right) - \frac{m}{r}\frac{dB_{0z}}{dr}\right\}\right] B_{1\theta}$$

$$-\left[\frac{dB_{0\theta}}{dr}\right] V_{1r} - [B_{0\theta}]\frac{\partial V_{1r}}{\partial r} + \frac{in}{R}(B_{0z}V_{1\theta} - B_{0\theta}V_{1z})$$

$$-\left[\frac{icn}{en_0^2 R}\cdot\frac{dp_{0i}}{dr}\right] n_1 + \left[\frac{icn}{en_0^2 R}\left(\frac{n_0}{\Gamma p_0}\right)\frac{dp}{dr}\right] p_{1e}$$

$$(2.2\text{-}5)$$

$$\frac{\partial p_{1i}}{\partial t} = -\left[\frac{dp_{oi}}{dr} + \frac{\Gamma p_{oi}}{r}\right] v_{1r} - \left[\Gamma p_{oi}\right]\frac{\partial v_{1r}}{\partial r} - \left[\frac{im\Gamma p_{oi}}{r}\right] v_{1\theta} - \left[\frac{in\Gamma p_{oi}}{R}\right] v_{1z} \tag{2.2-6}$$

$$\frac{\partial p_{1e}}{\partial t} = -\left[\frac{dp_{oe}}{dr} + \frac{\Gamma p_{oe}}{r}\right] v_{1r} - \left[\Gamma p_{oe}\right]\frac{\partial v_{1r}}{\partial r} - \left[\frac{im\Gamma p_{oe}}{r}\right] v_{1\theta} - \left[\frac{in\Gamma p_{oe}}{R}\right] v_{1z}$$

$$- \left[\frac{i\Gamma p_{oe}.c}{4\pi e n_o^2}\left\{\frac{n}{R}\left(\frac{dB_{o\theta}}{dr} + \frac{B_{o\theta}}{r}\right) - \frac{m}{r}\frac{dB_{oz}}{dr}\right\}\right] n_1$$

$$+ \left[\frac{ic}{4\pi e n_o}\left\{\frac{n}{R}\left(\frac{dB_{o\theta}}{dr} + \frac{B_{o\theta}}{r}\right) - \frac{m}{r}\frac{dB_{oz}}{dr}\right\}\right] p_{1e} \tag{2.2-7}$$

$$\frac{\partial n_1}{\partial t} = -\left[\frac{n_o}{r} + \frac{dn_o}{dr}\right] v_{1r} - \left[n_o\right]\frac{\partial v_{1r}}{\partial r} - \left[\frac{in_o m}{r}\right] v_{1\theta} - \left[\frac{in_o n}{R}\right] v_{1z} . \tag{2.2-8}$$

In the above equations, the symbols not yet defined in this chapter have the following meanings :

n_o	equilibrium number density of ions or electrons	
m_i	ionic mass	
$p_1 = p_{1i} + p_{1e}$	total perturbed pressure	
$k^2 = \frac{m^2}{r^2} + \frac{n^2}{R^2}$		
$\underline{k}.\underline{B}_o = \frac{mB_{o\theta}}{r} + \frac{nB_{oz}}{R}$		
e	electronic charge	
Γ	ratio of specific heats	(2.2-9)

where

$$V_A^2 = \frac{B_a^2}{4\pi \langle n_o \rangle m_i}$$ Alfven velocity,

$$\omega_{ci} = \frac{eB_a}{m_i c}$$ Ion cyclotron frequency,

$$V_{th}^2 = \frac{\langle T_o \rangle}{m_i}$$ Ion thermal velocity,

$$T_A = \frac{a}{V_A}$$ Alfven wave transit time,

$$r_L = \frac{V_{th}}{\omega_{ci}}$$ Ion Larmor radius,

$$\omega_{pi}^2 = \frac{4\pi e^2 \langle n_o \rangle}{m_i}$$ Ion plasma frequency,

$$B_a = B_{oz}(a)$$ z-magnetic field applied at plasma boundary.

$$\langle f_o \rangle \equiv f_o(a) \,.$$ (2.2-10b)

It will also be useful to define the following quantities :

$$\underline{k}^* = \frac{1}{a}\left(\frac{m}{\mu}\hat{\theta} + n\epsilon_1 \hat{z}\right)$$

$$k_{\parallel}^* = \frac{1}{a}\left(\frac{m B_{o\theta}^*}{\mu} + n\epsilon_1 B_{oz}^*\right)$$

$$k^{*2} = \frac{1}{a^2}\left(\frac{m^2}{\mu^2} + n^2\epsilon_1^2\right)$$ (2.2-10c)

where $\hat{\theta}$ and $\hat{z}$ are unit vectors in the θ and z directions respectively.

Equations (2.2-1) to (2.2-8) are most meaningful when cast in dimensionless form.In order to do this we define the following dimensionless quantities :

$$\mu = \frac{r}{a}$$

$$\tau = \frac{t}{T_A}$$

$$\underline{v}_1^* = \underline{V}_1 / V_A$$

$$\underline{B}_1^* = \underline{B}_1 / B_a$$

$$p_1^* = P_1 / \langle p_o \rangle$$

$$n_1^* = n_1 / \langle n_o \rangle$$

$$n_o^* = n_o / \langle n_o \rangle$$

$$p_o^* = P_o / \langle p_o \rangle$$

$$\underline{B}_o^* = \underline{B}_o / B_a$$

$$\epsilon_1 = a / R$$

$$\epsilon_2 = a / r_L$$

$$\epsilon_3 = \omega_{ci} T_A = \epsilon_2 \beta^{1/2} = \frac{\omega_{pi} a}{c} = \frac{a}{\lambda_c}$$

$$\beta = \frac{V_{th}^2}{V_A^2} = \frac{\langle p_o \rangle}{(B_a^2 / 4\pi)}$$

$$\Gamma = C_p / C_v \quad (= 5/3 \text{ , for an ideal monatomic gas.})$$

λ_c = ion collisionless skin depth = c/ω_{pi} (2.2-10a)

We now find that if we use the definitions in equations (2.2-10a) to (2.2-10c), then equations (2.2-1) to (2.2-8) can be written in dimensionless form (if all the asterisks are understood, and therefore omitted), as follows :

$$\frac{\partial B_{1r}}{\partial \tau} = a_1(iB_{1r}) + a_2\left(i\frac{\partial B_{1r}}{\partial \mu}\right) + a_3 B_{1\theta} + a_4(iV_{1r}) \tag{2.2-11a}$$

$$\frac{\partial B_{1\theta}}{\partial \tau} = b_0\frac{\partial^2 B_{1r}}{\partial \mu^2} + b_1\frac{\partial B_{1r}}{\partial \mu} + b_2 B_{1r} + b_3(iB_{1\theta}) + b_4 V_{1r} + b_5\frac{\partial V_{1r}}{\partial \mu} + b_6(iV_{1\theta}) + b_7(iV_{1z}) + b_8(i\eta_1) + b_9(ip_{1e}) \tag{2.2-11b}$$

$$\frac{\partial V_{1r}}{\partial \tau} = c_0\left(i\frac{\partial^2 B_{1r}}{\partial \mu^2}\right) + c_1\left(i\frac{\partial B_{1r}}{\partial \mu}\right) + c_2(iB_{1r}) + c_3 B_{1\theta} + c_4\frac{\partial B_{1\theta}}{\partial \mu} + c_5\frac{\partial p_{1i}}{\partial \mu} + c_5\frac{\partial p_{1e}}{\partial \mu} \tag{2.2-11c}$$

$$\frac{\partial V_{1\theta}}{\partial \tau} = d_1 B_{1r} + d_2\frac{\partial B_{1r}}{\partial \mu} + d_3(iB_{1\theta}) + d_4(ip_{1i}) + d_4(ip_{1e}) \tag{2.2-11d}$$

$$\frac{\partial V_{1z}}{\partial \tau} = e_1 B_{1r} + e_2\frac{\partial B_{1r}}{\partial \mu} + e_3(iB_{1\theta}) + e_4(ip_{1i}) + e_4(ip_{1e}) \tag{2.2-11e}$$

$$\frac{\partial p_{1i}}{\partial \tau} = f_1 V_{1r} + f_2 \frac{\partial V_{1r}}{\partial \mu} + f_3 (i V_{1\theta}) + f_4 (i V_{1z}) \qquad (2.2\text{-}11f)$$

$$\frac{\partial p_{1e}}{\partial \tau} = f_1 V_{1r} + f_2 \frac{\partial V_{1r}}{\partial \mu} + f_3 (i V_{1\theta}) + f_4 (i V_{1z}) + f_5 (i n_1) + f_6 (i p_{1e}). \qquad (2.2\text{-}11g)$$

$$\frac{\partial n_1}{\partial \tau} = h_1 V_{1r} + h_2 \frac{\partial V_{1r}}{\partial \mu} + h_3 (i V_{1\theta}) + h_4 (i V_{1z}). \qquad (2.2\text{-}11h)$$

In equations (2.2-11a) to (2.2-11h), the subscripted coefficient quantities have the following definitions :

$$a_1 = \frac{1}{\epsilon_3} \left[n\epsilon_1 \left(\frac{\partial B_{0\theta}}{\partial \mu} + \frac{B_{0\theta}}{\mu} \right) - \frac{m}{\mu} \frac{\partial B_{0z}}{\partial \mu} + \frac{m}{n\epsilon_1 \mu^2} \cdot k_{\parallel} \right] \qquad (2.2\text{-}12a)$$

$$a_2 = \frac{m k_{\parallel}}{n \epsilon_1 \epsilon_3 \mu} \qquad (2.2\text{-}12b)$$

$$a_3 = -\frac{k^2 k_{\parallel}}{n \epsilon_1 \epsilon_3} \qquad (2.2\text{-}12c)$$

$$a_4 = k_{\parallel} \qquad (2.2\text{-}12d)$$

$$b_0 = \frac{-k_{\parallel}}{n\epsilon_1\epsilon_3} \tag{2.2-13a}$$

$$b_1 = \frac{1}{n\epsilon_1\epsilon_3}\left[\frac{m}{\mu\Gamma\rho_0}\frac{\partial\rho_0}{\partial\mu}B_{0\theta} + \frac{m}{\mu}\left(\frac{B_{0\theta}}{\mu} - \frac{\partial B_{0\theta}}{\partial\mu}\right) + \frac{k_{\parallel}}{\mu}\right] \tag{2.2-13b}$$

$$b_2 = \frac{1}{n\epsilon_1\epsilon_3}\left[n\epsilon_1\left(\frac{1}{\Gamma\rho_0}\cdot\frac{\partial\rho_0}{\partial\mu} - \frac{1}{\mu}\right)\frac{\partial B_{0z}}{\partial\mu} + \frac{\partial^2 B_{0z}}{\partial\mu^2} + k_{\parallel}\left(n^2\epsilon_1^2 + \frac{1}{\mu^2}\right) + \frac{m}{n\mu^2}\left(\frac{1}{\Gamma\rho_0}\cdot\frac{\partial\rho_0}{\partial\mu}B_{0\theta} + \frac{B_{0\theta}}{\mu} - \frac{\partial B_{0\theta}}{\partial\mu}\right)\right] \tag{2.2-13c}$$

$$b_3 = \frac{1}{n\epsilon_1\epsilon_3}\left[k^2\left(\frac{1}{\Gamma\rho_0}\cdot\frac{\partial\rho_0}{\partial\mu}\cdot B_{0\theta} + \frac{B_{0\theta}}{\mu} - \frac{\partial B_{0\theta}}{\partial\mu}\right) - \frac{mk_{\parallel}}{\mu^2} - n\epsilon_1\left\{n\epsilon_1\left(\frac{\partial B_{0\theta}}{\partial\mu} + \frac{B_{0\theta}}{\mu}\right) - \frac{m}{\mu}\frac{\partial B_{0z}}{\partial\mu}\right\}\right] \tag{2.2-13d}$$

$$b_4 = -\frac{\partial B_{0\theta}}{\partial\mu} \tag{2.2-13e}$$

$$b_5 = -B_{0\theta} \tag{2.2-13f}$$

$$b_6 = n\epsilon_1 B_{0z} \tag{2.2-13g}$$

$$b_7 = -n\epsilon_1 B_{0\theta} \tag{2.2-13h}$$

$$b_8 = \frac{n\epsilon_1\beta\rho_0}{2\epsilon_3 n_0^2} \tag{2.2-13i}$$

$$b_9 = \frac{n\epsilon_1\beta}{\Gamma\epsilon_3 n_0^2}\cdot\frac{\partial\rho_0}{\partial\mu} \tag{2.2-13j}$$

$$c_0 = -\frac{B_{0z}}{n\epsilon_1} \tag{2.2-14a}$$

$$c_1 = \frac{-1}{n\epsilon_1}\left[\frac{B_{0z}}{\mu} + d\frac{B_{0z}}{d\mu}\right] \tag{2.2-14b}$$

$$c_2 = \frac{1}{n\epsilon_1}\left[n\epsilon_1 k_{\parallel} - \frac{1}{\mu}\frac{dB_{0z}}{d\mu} + \frac{B_{0z}}{\mu^2}\right] \tag{2.2-14c}$$

$$c_3 = \frac{1}{n\epsilon_1}\left[\frac{m}{\mu}\left(\frac{dB_{0z}}{d\mu} - \frac{B_{0z}}{\mu}\right) - n\epsilon_1\frac{dB_{0\theta}}{d\mu} - \frac{2n\epsilon_1 B_{0\theta}}{\mu}\right] \tag{2.2-14d}$$

$$c_4 = \frac{1}{n\epsilon_1}\left[\frac{mB_{0z}}{\mu} - n\epsilon_1 B_{0\theta}\right]. \tag{2.2-14e}$$

$$c_5 = -\beta \tag{2.2-14f}$$

$$d_1 = \frac{1}{n\epsilon_1\mu}\left[\left(\frac{mB_{0z}}{\mu} + n\epsilon_1 B_{0\theta}\right) + n\epsilon_1\mu\frac{dB_{0\theta}}{d\mu}\right] \tag{2.2-15a}$$

$$d_2 = \frac{mB_{0z}}{n\epsilon_1\mu} \tag{2.2-15b}$$

$$d_3 = \frac{k^2 B_{0z}}{n\epsilon_1} \tag{2.2-15c}$$

$$d_4 = -\frac{m\beta}{\mu} \tag{2.2-15d}$$

$$e_1 = \frac{1}{n\epsilon_1\mu}\left[n\epsilon_1\mu\frac{dB_{0z}}{d\mu} - \frac{mB_{0\theta}}{\mu}\right] \qquad (2.2\text{-}16a)$$

$$e_2 = -\frac{mB_{0\theta}}{n\epsilon_1\mu} \qquad (2.2\text{-}16b)$$

$$e_3 = \frac{k^2 B_{0\theta}}{n\epsilon_1} \qquad (2.2\text{-}16c)$$

$$e_4 = -n\epsilon_1\beta \qquad (2.2\text{-}16d)$$

$$f_1 = -\frac{1}{2}\left[\frac{dp_0}{d\mu} + \frac{\Gamma p_0}{\mu}\right] \qquad (2.2\text{-}17a)$$

$$f_2 = -\frac{\Gamma p_0}{2} \qquad (2.2\text{-}17b)$$

$$f_3 = -\frac{m\Gamma p_0}{2\mu} \qquad (2.2\text{-}17c)$$

$$f_4 = -\frac{n\epsilon_1\Gamma p_0}{2} \qquad (2.2\text{-}17d)$$

$$f_5 = -\frac{\Gamma p_0}{2}\cdot f_6 \qquad (2.2\text{-}17e)$$

$$f_6 = \frac{1}{\epsilon_3}\left[n\epsilon_1\left(\frac{\partial B_{0\theta}}{\partial\mu} + \frac{B_{0\theta}}{\mu}\right) - m\frac{\partial B_{0z}}{\partial\mu}\right]. \qquad (2.2\text{-}17f)$$

$$h_1 = -\left(\frac{n_0}{\mu} + \frac{\partial n_0}{\partial\mu}\right) \qquad (2.2\text{-}18a)$$

$$h_2 = -n_0 \qquad (2.2\text{-}18b)$$

$$h_3 = -\frac{mn_0}{\mu} \qquad (2.2\text{-}18c)$$

$$h_4 = -n\epsilon_1 n_0 \qquad (2.2\text{-}18d)$$

We point out that for simplicity in equations (2.2-12a) to (2.2-18d) we have taken

$$P_{oi} = P_{oe} = \frac{P_o}{2} \tag{2.2-19}$$

If we now remember that each of the perturbed variables in equations (2.2-11a) to (2.2-11h) is a complex variable,and write down from these equations,the equations for the time-variation of the real and imaginary part of each perturbed variable,the set of 16 coupled partial differential equations which we have used in our numerical study,would then be obtained.We shall henceforth refer to this set of equations as the Numerical Set of equations,and in the next section we shall discuss the boundary conditions which the perturbed variables must satisfy, if it is assumed that the plasma is bounded by a perfectly conducting rigid wall.

2.3 Boundary conditions

It is the case that before the equation set that we have called the Numerical Set (see section 2.2),can be integrated,the appropriate boundary conditions must be applied to each perturbed variable,at each of the two limits of the integration (i.e. r=0 and r=a).Let us first consider the origin,where it transpires that three sets of boundary conditions must be imposed,corresponding to the cases :

$$m=0 \quad , \quad m=1 \quad , \quad \text{and} \quad m>1. \tag{2.3-1}$$

We shall consider these three cases in the following three subsections.

2.3.1 Boundary conditions at r=0 for m=0

The important feature of the m=0 mode,is that all the perturbed variables are independent of the poloidal angle θ .Thus,it is clear that the scalar perturbation variables, p_{1i}, p_{1e}, n_1 , and also the z-component of the perturbed ion fluid velocity, v_{1z} , obey the following conditions ar r=0 :

$$\left.\frac{\partial p_{1i}}{\partial r}\right|_{r=0} = \left.\frac{\partial p_{1e}}{\partial r}\right|_{r=0} = \left.\frac{\partial n_1}{\partial r}\right|_{r=0} = \left.\frac{\partial v_{1z}}{\partial r}\right|_{r=0} = 0. \tag{2.3.1-1}$$

It is equally clear that if we allow any of the perturbed variables v_{1r} ,$v_{1\theta}$,B_{1r} , or $B_{1\theta}$,to be non-zero as $r \longrightarrow 0$,then the θ - independence of the variable would ensure that it was a multiple-valued function at r=0.Therefore,in order to avoid this,we must insist that these variables obey the following conditions at r=0 :

$$V_{1r}(0) = V_{1\theta}(0) = B_{1r}(0) = B_{1\theta}(0) = 0 \tag{2.3.1-2}$$

2.3.2 Boundary conditions at r=0 for m=1.

In order to derive the boundary conditions which the perturbed variables obey in this case, we consider the normal cross-section of the plasma illustrated in figure (2.3.2-1). If f_1 is a perturbed variable, then, for the m=1 mode, the e-variation of f_1 is such that it takes a complete poloidal transversal around the axis, before f_1 returns to its starting value. In figure (2.3.2-1) the points A and B are diametrically opposed, and we suppose that f_A is the value of f_1 at A, and that f_B is the value of f_1 at B. The e-variation of the perturbed variables corresponding to the m=1 mode now makes it clear that if

$$f_A = f_o \qquad (2.3.2\text{-}1a)$$

then

$$f_B = -f_o \qquad (2.3.2\text{-}1b)$$

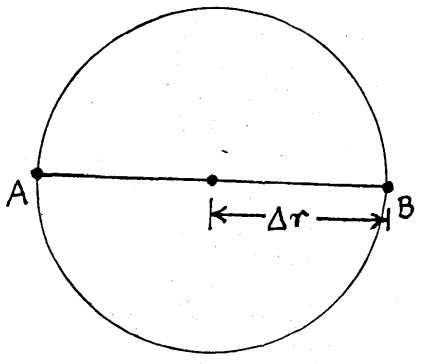


Figure (2.3.2-1)

Thus, if the perturbed variables of interest are p_{1i}, p_{1e}, n_1 or v_{1z}, then in order to prevent their being multiple-valued at r=0 as $\Delta r \longrightarrow 0$, we would have to insist that

$$p_{1i}(0)=p_{1e}(0)=n_1(0)=v_{1z}(0) = 0 \qquad (2.3.2\text{-}2)$$

On the other hand, for the perturbed variables v_{1r}, v_{1e}, B_{1r}, and B_{1e}, as well as the satisfaction of equations (2.3.2-1a) and (2.3.2-1b), there are conventions which must be adhered to.

For instance, suppose we make the convention that v_{1r} is a positive number, when it characterises perturbed radial motion of the plasma which is directed at the origin. Then, according to equations (2.3.2-1a) and (2.3.2-1b), f_A and f_B are as illustrated in figure (2.3.2-2). From this diagram it is clear that in the limit as $\Delta r \longrightarrow 0$, then $\frac{\partial v_{1r}}{\partial r}\Big|_{r=0} = 0$. This same argument applies to the perturbed variables B_{1r}, $v_{1\theta}$ and $B_{1\theta}$, if, for the latter two, we make the convention that clockwise motion around the origin as we look at the page, is positive. Thus, we have deduced that for the m=1 mode, the following boundary conditions apply:

$$\frac{\partial v_{1r}}{\partial r}\Big|_{r=0} = \frac{\partial v_{1\theta}}{\partial r}\Big|_{r=0} = \frac{\partial B_{1r}}{\partial r}\Big|_{r=0} = \frac{\partial B_{1\theta}}{\partial r}\Big|_{r=0} = 0 \qquad (2.3.2\text{-}3)$$

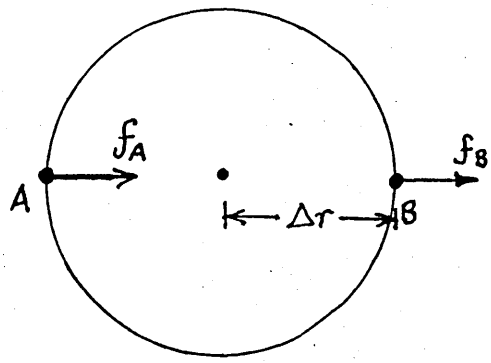


Figure (2.3.2-2)

2.3.3 Boundary conditions at r=0 for m>1.

In this case we find that each perturbed variable now goes through two or more cycles in the values which it may take, for every complete circuit around the origin. Thus, in order to prevent each perturbed variable from being multiple-valued at r=0, we must insist that

$$B_{1r}(0) = B_{1\theta}(0) = V_{1r}(0) = V_{1\theta}(0) = V_{1z}(0) = p_{1i}(0) = p_{1e}(0) = n_1(0) = 0 \qquad (2.3.3\text{-}1)$$

2.3.4 Boundary Conditions at r=a

If we assume that the wall of the vessel which contains the plasma is in contact with the plasma,and is both rigid and infinitely conducting, then there can be no loss of magnetic flux,or plasma,at the boundary. Thus we can write

$$v_{1r}(a) = 0 \qquad (2.3.4\text{-}1)$$

$$B_{1r}(a) = 0 \qquad (2.3.4\text{-}2)$$

The θ-component of the linearised ion momentum equation is

$$n_0 m_i \frac{\partial V_{1\theta}}{\partial t} = e n_0 \left[E_{1\theta} - \frac{V_{1r} B_{0z}}{c}\right] - \frac{im}{r} p_{1i} \qquad (2.3.4\text{-}3)$$

and since we know that at the perfectly conducting wall no electric field can exist,we can write

$$E_{1\theta}(a) = 0 \qquad (2.3.4\text{-}4)$$

Substituting equation (2.3.4-1) and equation (2.3.4-4) into equation (2.3.4-3),we find that the following relation holds

$$n_0 m_i \frac{\partial V_{1\theta}}{\partial t}(a) = -\frac{im}{a} p_{1i}(a) \qquad (2.3.4\text{-}5)$$

However,we know from equation (2.2-2) that we can also write

$$n_0 m_i \frac{\partial V_{1\theta}}{\partial t} = \left[\frac{R}{4\pi n r}\left\{\left(\frac{m B_{0\theta}}{r} + \frac{n B_{0z}}{R}\right) + \frac{nr}{R}\frac{d}{dr}\left(\frac{B_{0\theta}}{r}\right)\right\}\right] B_{1r}$$
$$+ \left[\frac{R}{4\pi n}\cdot\frac{m B_{0z}}{r}\right]\frac{\partial B_{1r}}{\partial r} + \left[\frac{i\left(\frac{m^2}{r^2}+\frac{n^2}{R^2}\right) R B_{0z}}{4\pi n}\right] B_{1\theta} - \frac{im}{r} p_{1i} \qquad (2.3.4\text{-}6)$$

If we then use the result in equation (2.3.4-5) in the above equation, and also the fact that $B_{1r}(a) = 0$,we are able to deduce the result

$$B_{1\theta}(a,t) = \frac{im}{a\left(\frac{m^2}{a^2}+\frac{n^2}{R^2}\right)}\cdot\left(\frac{\partial B_{1r}}{\partial r}\right)_{r=a} \qquad (2.3.4\text{-}7)$$

It is stated in /60/ (although no derivation is given),that provided

$$\underline{k}.\underline{B}_o \neq 0 \quad , \text{ at } r=a \qquad (2.3.4\text{-}8)$$

and

$$\frac{n_o(a)}{n_o(0)} \longrightarrow 0 \; ; \quad \frac{p_o(a)}{p_o(0)} \longrightarrow 0 \; ; \text{ and } \quad \frac{T_o(a)}{T_o(0)} \longrightarrow 0 \qquad (2.3.4\text{-}9)$$

the remaining boundary conditions at r=a are

$$P_{1i}(a,t) = P_{1e}(a,t) = n_1(a,t) = 0$$

$$V_{1\theta}(a,t) = \frac{ia}{m^2+\frac{n^2}{R^2}a^2}\cdot\left(\frac{\partial V_{1r}}{\partial r}\right)_{r=a}\cdot\left[m - \frac{aB_{o\theta}}{B_o^2}.\{\underline{k}.\underline{B}_o(r=a)\}\right]$$

$$V_{1z}(a,t) = \frac{ia^2}{m^2+\frac{n^2a^2}{R^2}}\cdot\left(\frac{\partial V_{1r}}{\partial r}\right)_{r=a}\cdot\left[\frac{n}{R} - \frac{B_{oz}}{B_o^2}\{\underline{k}.\underline{B}_o(r=a)\}\right] \qquad (2.3.4\text{-}10)$$

We note that in spite of the conditions for validity of these boundary conditions (equations 2.3.4-8 and 2.3.4-9), in /39,60/ they have been used for the magnetic field equilibrium known as the Bessel Function Model, a 'force-free' plasma equilibrium model which is characterised by the equation $\nabla p = 0$.

We have both used the boundary conditions above, and also left the boundary conditions for $B_{1\theta}$, $v_{1\theta}$, and v_{1z} 'free' (i.e. simply left them with their integrated values at the boundaries,and applied no constraint),and we have found that the results produced in the two cases (i.e. the eigenfrequencies determined),are not significantly different.

2.4 Results obtained from the numerical initial-value program.

In this section we shall present some of the results that we have obtained from the numerical solution,as an initial-value problem,of equations (2.2-1) to (2.2-8),subject to boundary conditions (see section 2.3) which we have already discussed.Our procedure was to initialise the equations with an arbitrary perturbation,and then to integrate them forwards in time,until the fastest-growing radial oscillation mode dominates.What this means is that if the fastest-growing radial oscillation mode has associated with it the complex eigenfrequency

$$\omega = \alpha + i\gamma \qquad (2.4\text{-}1)$$

then,after sufficient time,the amplitude of this mode will be much greater than the amplitudes of all the other modes present,and only this mode will be seen on the space-mesh.Also,we have found that the following initial perturbation

$$u_1(x) = \exp\left[-C\,.\,(x-0{\cdot}5)^2\right].\sin(3n\pi x^2) \qquad (2.4\text{-}2)$$

where n is the toroidal mode number (see equation 1.2-16),$x=\frac{r}{a}$ and C may take values between 50 and 125,starts off the integration very efficiently.

It is assumed in this linear analysis,that the time-dependence of the perturbation mode that becomes dominant is given by

$$u_1(x,t) = u_1(x)\exp(-i\omega t) \qquad (2.4\text{-}3)$$

and so it follows from equation (2.4-1) that γ is the exponential growth-rate of the mode in question.We have found it convenient to measure time in units of an average poloidal Alfven wave transit-time τ_A where

$$\tau_A = a\,.\,\frac{\sqrt{4\pi n_0 m_i}}{\bar{B}_{o\theta}} \qquad (2.4\text{-}4)$$

and $\overline{B}_{oe}$ is the average value of B_{oe} over the radius of the plasma column.

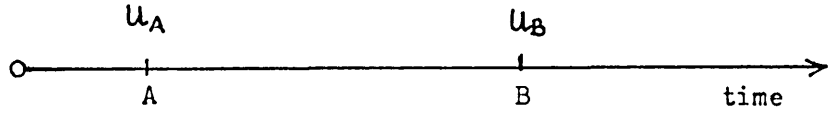


Figure 2.4-1

We will now show how the complex frequency associated with the perturbed variable u ,at a given spatial point, can be calculated from the knowledge of u at two adjacent times (t=A and t=B), at that given spatial point.Thus,suppose that at time t=A the value of u is u_A , and that at time t=B ,the value of u is u_B .We can then write

$$u_A = u_r(A) + i\, u_i(A) \qquad (2.4\text{-}5)$$

$$u_B = u_r(B) + i\, u_i(B) \qquad (2.4\text{-}6)$$

and it will follow from equation (2.4-3) that the exponential growth-rate is given by

$$\gamma = \frac{1}{(B-A)} \cdot \log\left\{\frac{|u_B|}{|u_A|}\right\} \qquad (2.4\text{-}7)$$

where

$$|u| = (u_r^2 + u_i^2)^{1/2} \qquad (2.4\text{-}8)$$

The derivation of an expression for α in equation (2.4-1) is slightly longer.To obtain this expression we note that if Z is an arbitrary real constant,we can write

$$u_A = Z.\exp(\gamma A).\left[\cos(\alpha A) - i \sin(\alpha A)\right] \qquad (2.4\text{-}9)$$

and

$$u_B = Z.\exp(\gamma B).\left[\cos(\alpha B) - i\sin(\alpha B)\right] \qquad (2.4\text{-}10)$$

Now, suppose that

$$B - A = \Delta \qquad (2.4\text{-}11)$$

it would then follow that

$$\frac{u_i(B)}{u_r(B)} = -\tan(\alpha\Delta + \alpha A) = \frac{\tan(\alpha A) + \tan(\alpha\Delta)}{\tan(\alpha A)\tan(\alpha\Delta) - 1} \qquad (2.4\text{-}12)$$

and since

$$\frac{u_i(A)}{u_r(A)} = -\tan(\alpha A) \qquad (2.4\text{-}13)$$

we can find, after a little algebra, that

$$\alpha = \frac{1}{\Delta}\tan^{-1}.\left\{\frac{u_r(B)u_i(A) - u_r(A)u_i(B)}{u_r(A)u_r(B) + u_i(A)u_i(B)}\right\} \qquad (2.4\text{-}14)$$

Once the fastest-growing mode , with eigenfrequency

$$\omega = \alpha + i\gamma$$

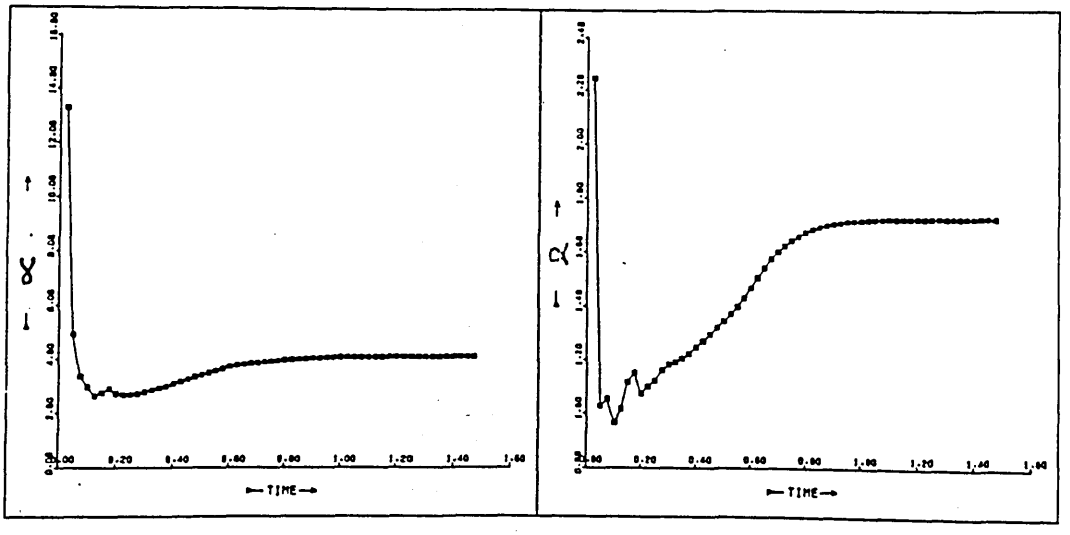
has become dominant, each of the 8 dependent perturbed variables must be found to exhibit this eigenfrequency at each spatial point in the domain. In our computations we have been content to ensure that the eigenvalue has converged to the same value (to within 0.01%), in a block of 8 adjacent space cells in the centre of the spatial domain. Figure (2.4-2) illustrates the emergence of the frequency of the fastest-growing m=3, n=2 mode , for the variables v_{1r} and $v_{1\theta}$ at the median space point, and also the emergence of the fastest-growing m=2, n=1 mode, for the variable B_{1r}, also at the median space point. In figure (2.4-3) we have illustrated all the eigenfunctions of the fastest-growing m=3, n=2 mode, for the equilibrium with $\nu = .4$ $K = 0{\cdot}2$. We point out here that increasing K increases the growth-rates (γ), that are found.

Some of the eigenfunctions for the equilibrium with $\nu=1$, $K=0.2$ for the m=3,n=2 mode, are shown in figure (2.4-3a),and it will be immediately noticed that they are of broader spatial extent,and more oscillatory than the eigenfunctions in figure (2.4-3) for the $\nu=4$ equilibrium. Figure (2.4-3b) illustrates some of the eigenfunctions of the fastest growing m=1,n=1 mode, for the $\nu=4$, $K=0.3$ equilibrium. In figure (2.4-4a) and (2.4-4b) we have illustrated the whole set of eigenfunctions of the m=4,n=3 mode , for the equilibrium with $\nu=2$ and $K=0.2$. Our numerical investigations have also shown that for the equilibrium parameters which we have chosen (such as number density, temperature , z-magnetic field and β), the growth rates for the $\nu=2$ and the $\nu=4$ equilibria,are nearly identical for all modes (although $\gamma_{\nu=2} > \gamma_{\nu=4}$), and that the growth rates for these equilibria are substantially greater than the growth rates for the $\nu=1$ equilibrium. The implication of this is that the equilibria with more sharply peaked current profiles, are more unstable than the equilibrium with parabolic current profile. In fact we have found that for our particular equilibrium parameters

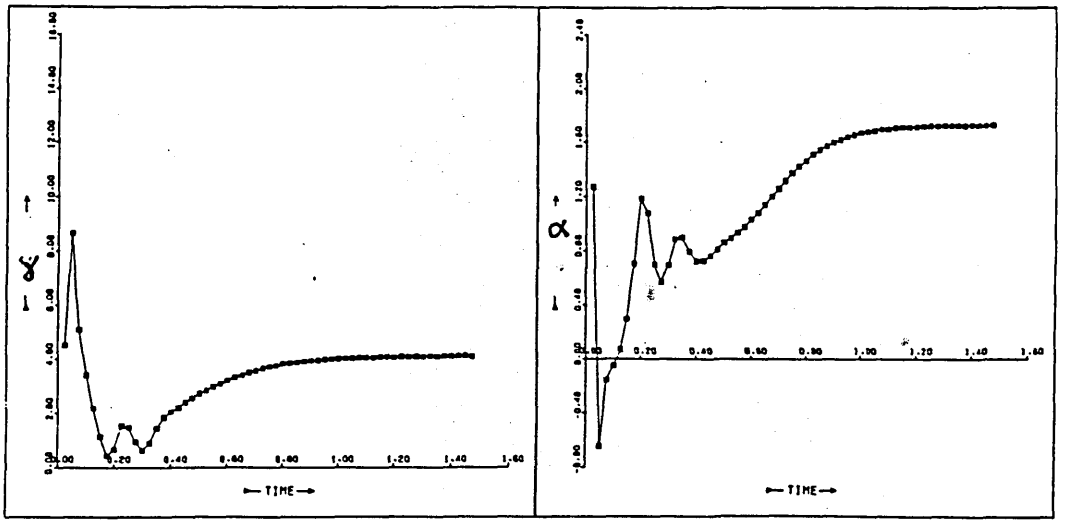
$$\gamma_{\nu=2,4} \simeq 1.5\,\gamma_{\nu=1} \qquad (2.4\text{-}15)$$

The numerical factor here is dependent on what numbers are chosen for such quantities as $n_o(0)$,$T_o(0)$ and β , and this will be made apparent in the next section.

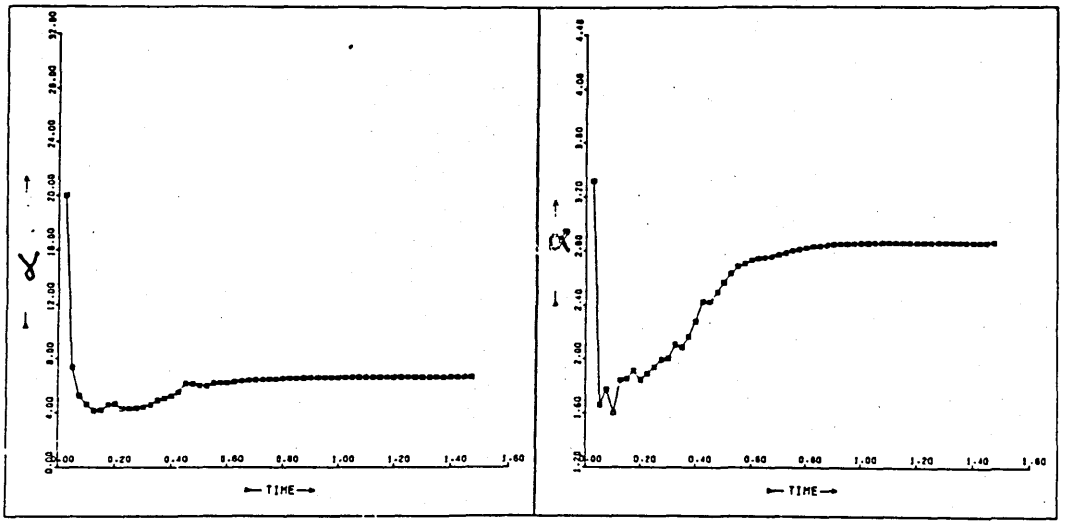
We have found that the eigenfunctions that we have obtained using this two-fluid model,tend to be localised in space (except for the case of the $\nu=1$ parabolic current profile equilibrium). However,although these eigenfunctions are often localised near the resonant surface, $x=x_o$, they are not (in contrast to results obtained using single-fluid theory /39,60/) always so.This may be an indication of the dissimilar properties of the resonant surface in single-fluid and two-fluid theory, (see also /107/).



(a)



(b)



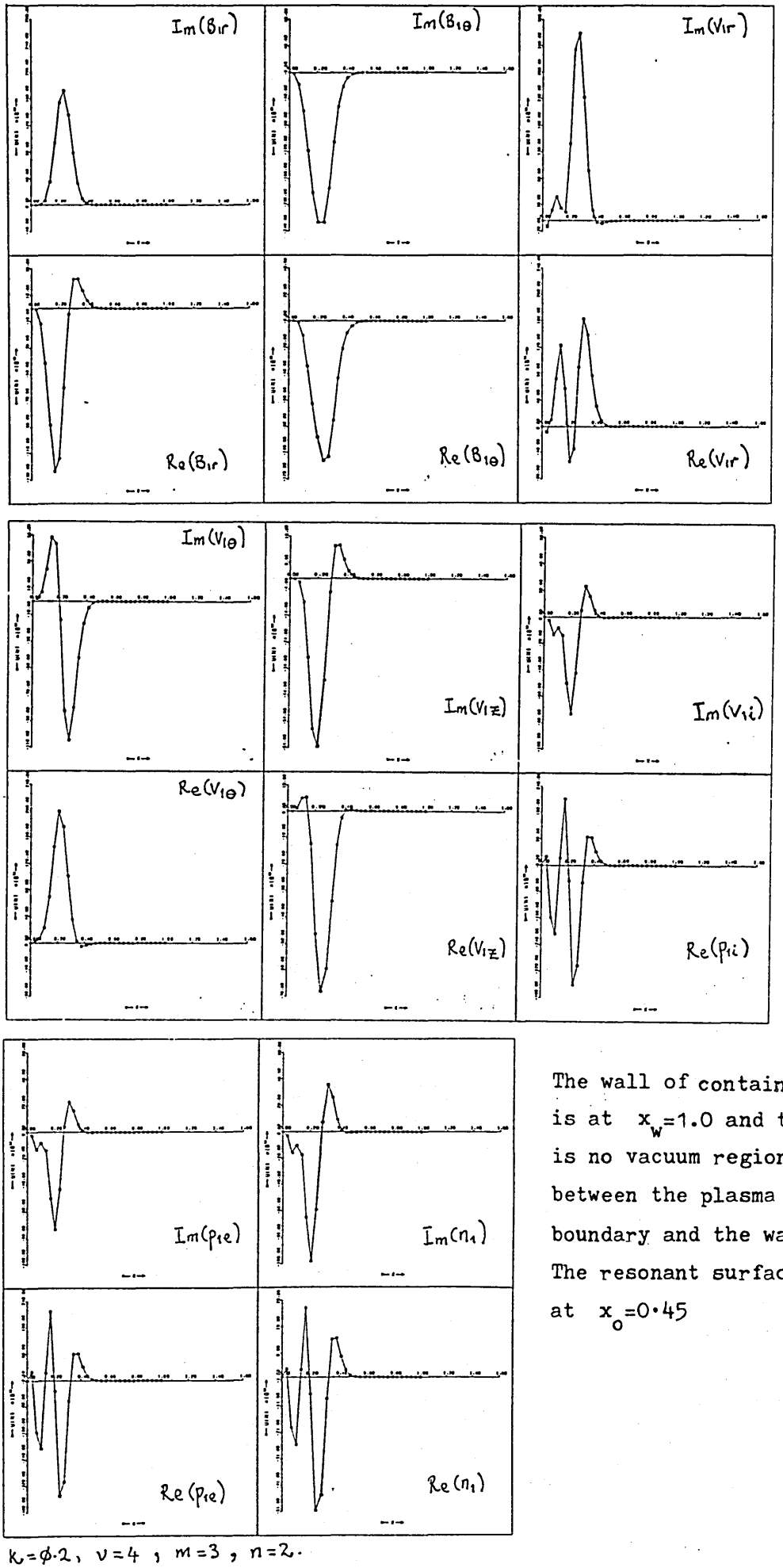
(c)

Figure 2.4-2 (a) Emergence of the fastest-growing m=3,n=2 mode for v_{1r} at the median space point.

(b) Emergence of the fastest-growing m=3,n=2 mode for $v_{1\theta}$ at the median space point.

(c) Emergence of the fastest-growing m=2,n=1 mode for B_{1r} at the median space point.

Time is measured in units of Alfven transit times, τ_A .



The wall of containment is at $x_w=1.0$ and there is no vacuum region between the plasma boundary and the wall. The resonant surface is at $x_o=0.45$

$k_z=0.2$, $\nu=4$, $m=3$, $n=2$.

Figure 2.4-3 Eigenfunctions of the fastest-growing m=3 , n=2 mode.

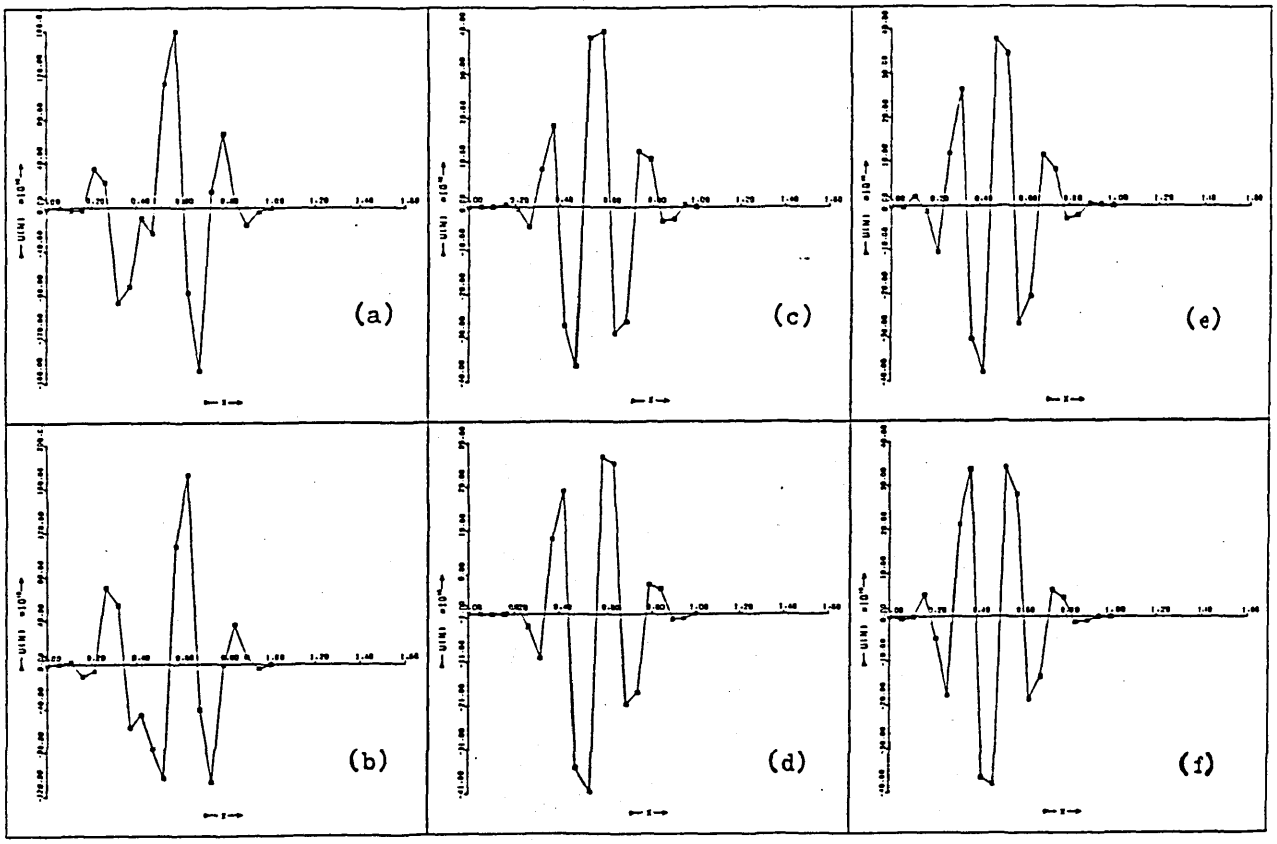


Figure 2.4-3a Some eigenfunctions of the fastest-growing m=3, n=2 mode, for the $\nu=1$ equilibrium. ($\kappa=0{\cdot}2$)

(a) Imaginary part of B_{1r}
(b) Real part of B_{1r}
(c) Imaginary part of $B_{1\theta}$
(d) Real part of $B_{1\theta}$
(e) Imaginary part of v_{1r}
(f) Real part of v_{1r}

The resonant radius is at $x_o=0{\cdot}816$

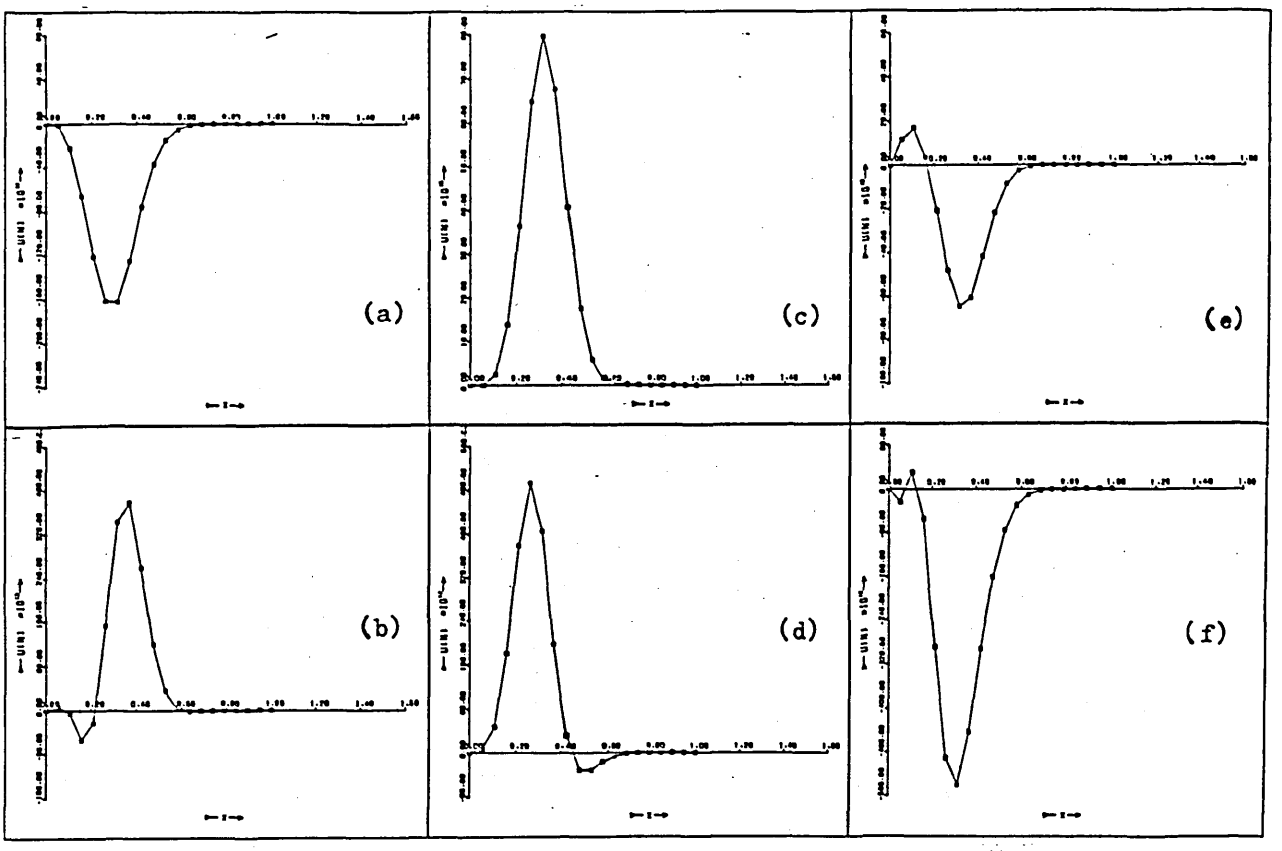


Figure 2.4-3b Some eigenfunctions of the fastest growing m=1, n=1 mode for the ν =4 equilibrium. (κ =0·3)

(a) Imaginary part of B_{1r}
(b) Real part of B_{1r}
(c) Imaginary part of $B_{1\theta}$
(d) Real part of $B_{1\theta}$
(e) Imaginary part of v_{1r}
(f) Real part of v_{1r}

The resonant radius is at x_o=0·45

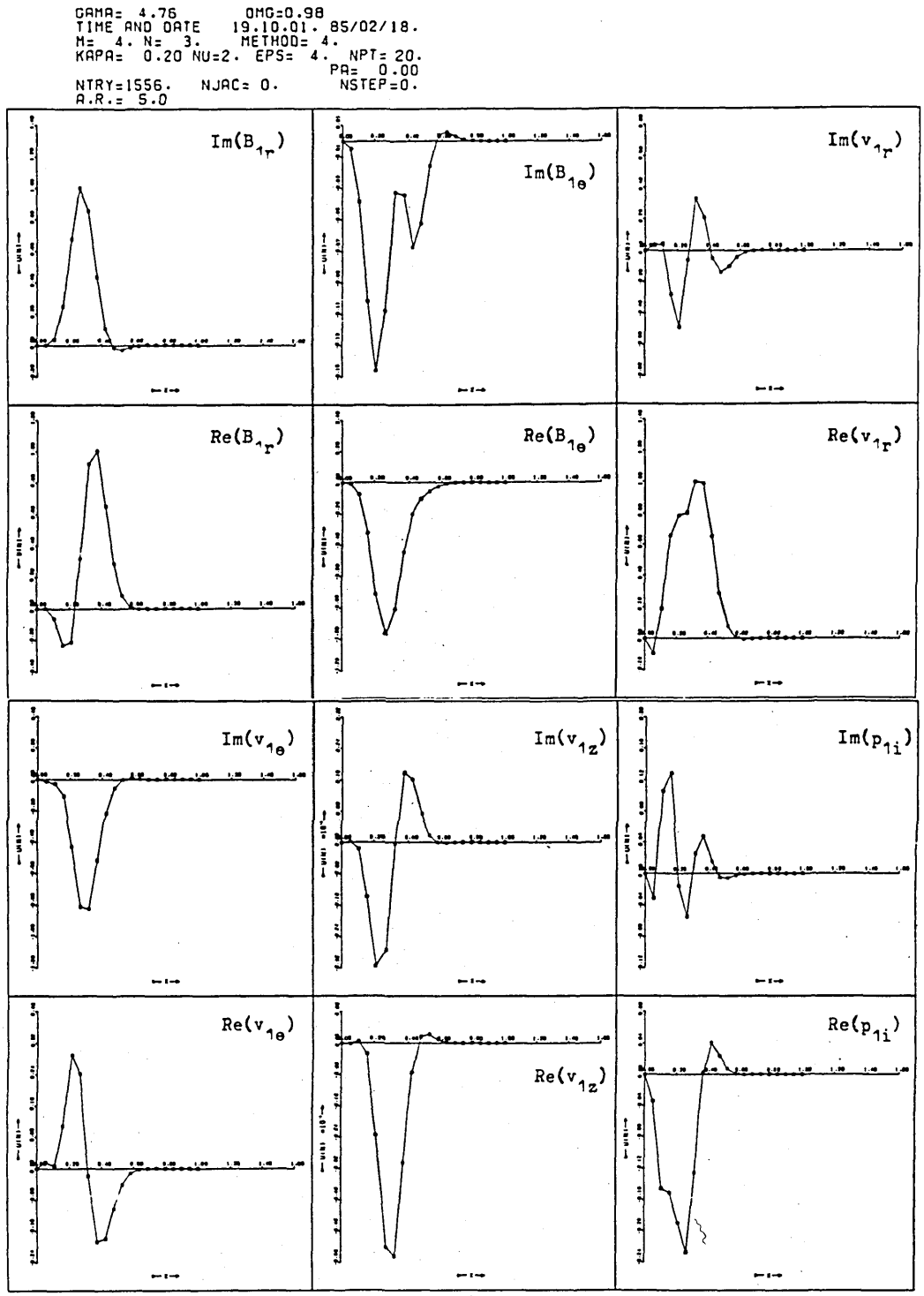


Figure 2.4-4a Eigenfunctions of the fastest-growing m=4, n=3 mode, for the ν=2, κ=0.2 equilibrium. The resonant surface is at x_0 = 0.525

The remaining eigenfunctions of this mode will be found in figure 2.4-4b .

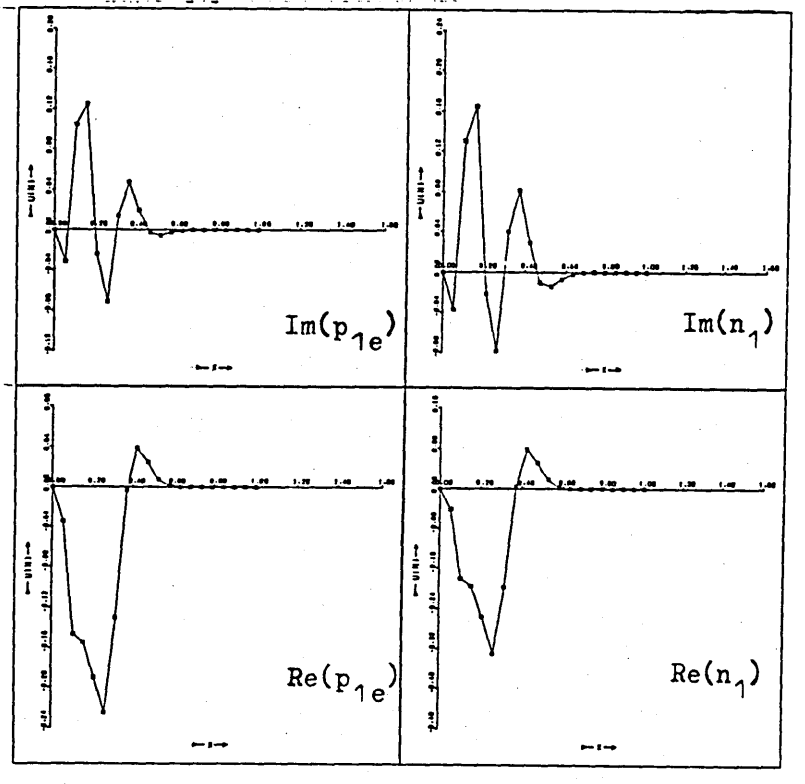


Figure 2.4-4b Eigenfunctions of the fastest-growing m=4, n=3 mode, for the $\nu=2$, $K_c=0{\cdot}2$ equilibrium. The resonant surface is at $x_0 = 0{\cdot}525$

2.4.1 The MHD Limit

It will prove to be interesting to examine the behaviour of our equations in the MHD regime, because the difference in the predictions made by the single-fluid theory and those made by our two-fluid theory, would then become apparent. The ideal MHD Ohm's law is

$$\underline{E} + \frac{1}{c}\left(\underline{v} \times \underline{B}\right) = 0 \qquad (2.4.1\text{-}0a)$$

and the Ohm's law used in our two-fluid theory is

$$\underline{E} + \frac{1}{c}\left(\underline{v} \times \underline{B}\right) = \frac{1}{enc}\left(\underline{j} \times \underline{B}\right) + \frac{1}{en}\nabla p_e \qquad (2.4.1\text{-}0b)$$

Thus, in order to obtain the MHD limit we must find out under what circumstances the terms on the RHS of (2.4.1-0b) are negligible in in comparison to those on the left. We can proceed by linearising equation (2.4.1-0b)

$$en_0\left[\underline{E}_1 + \frac{\underline{v}_1 \wedge \underline{B}_0}{c} + \frac{\underline{v}_0 \wedge \underline{B}_1}{c}\right] + en_1\left[\underline{E}_0 + \frac{\underline{v}_0 \wedge \underline{B}_0}{c}\right]$$

$$= \frac{(\underline{J}_0 \wedge \underline{B}_1)}{c} + \frac{(\underline{J}_1 \wedge \underline{B}_0)}{c} - \nabla p_{1e}$$

(Note: $\underline{v}_0 = 0$). (2.4.1-0c)

The terms on the RHS of this equation are all of the same order. To see this we take the equilibrium equation

$$\nabla p_0 = \frac{\underline{j} \times \underline{B}}{c} \qquad (2.4.1\text{-}0d)$$

and linearise it to obtain the result

$$\nabla p_1 = \frac{1}{c}(\underline{j}_0 \wedge \underline{B}_1) + \frac{1}{c}(\underline{j}_1 \wedge \underline{B}_0). \qquad (2.4.1\text{-}0e)$$

Since we are interested in phenomena occuring on the MHD timescale, it will be possible for us to order the time derivative as follows :

$$\frac{\partial}{\partial t} \equiv \frac{V_A}{a} = \frac{1}{\tau_A} \qquad (2.4.1\text{-}0f)$$

where V_A is the Alfven speed, and a is any macroscopic length, such as the radius of the plasma. In order to see when the Hall term is negligible, it will therefore be sufficient to consider the ratio of the first term on the RHS of (2.4.1-0c) to the first term on the LHS of the same equation, i.e. we are interested in the ratio

$$\epsilon_3 = \frac{en_o E_1}{\left(\frac{j_o B_1}{c}\right)} \qquad (2.4.1\text{-}0g)$$

From Faraday's law

$$\nabla \wedge \underline{E}_1 = -\frac{1}{c}\frac{\partial \underline{B}_1}{\partial t}$$

and thus equation (2.4.1-0f) will allow us to say that

$$E_1 \sim \frac{V_A B_1}{c} \qquad (2.4.1\text{-}0h)$$

Ampere's law states that

$$\nabla \wedge \underline{B}_o = \frac{4\pi}{c}\underline{j}_o$$

which means that we will be able to order j_o as :

$$j_o = \frac{cB_o}{4\pi a} \qquad (2.4.1\text{-}0i)$$

Substituting (2.4.1-0h) and (2.4.1-0i) into (2.4.1-0g), we find that

$$\epsilon_3 \doteqdot \frac{en_o\left(\frac{V_A B_1}{c}\right)}{\left(\frac{cB_o}{4\pi a}\right)\cdot\left(\frac{B_1}{c}\right)}$$

$$= \frac{a}{c}\left(\sqrt{\frac{4\pi e^2 n_o}{m_i}}\right) = \frac{a}{\lambda_c} \qquad (2.4.1\text{-}0j)$$

where λ_c is the ion collisionless skin depth, and is given by

$$\lambda_c = c/\omega_{pi} \qquad (2.4.1\text{-}2)$$

and ω_{pi} is the ion plasma frequency.

Thus, the terms on the LHS of equation (2.4.1-0c), are dominant over the terms on the RHS in the limit in which $\epsilon_3 \to \infty$ .i.e. in order to obtain the MHD limit in our equations we must let

$$\epsilon_3 \to \infty \qquad (2.4.1\text{-}3)$$

For the equilibria that we have studied, we have taken $n_0(0) = 5{\cdot}0 \times 10^{13} cm^{-3}$. This means that

$$\omega_{pi} = 0{\cdot}93 \times 10^{10} \ s^{-1}$$

so that

$$\lambda_c = \frac{c}{\omega_{pi}} = 3.22 \cdot cm$$

and thus

$$\epsilon_3 = \frac{a}{\lambda_c} = \frac{30}{3{\cdot}22} = 9.3105 \qquad (2.4.1\text{-}4)$$

We can call this value of ϵ_3 , ϵ_3^0 , to indicate that it is the value of ϵ_3 in the equilibrium that we have chosen. We have varied ϵ_3 as follows :

$$\epsilon_3 = \epsilon_3^0 \times 10^p \qquad (2.4.1\text{-}5)$$

in order to investigate continuously, the results of allowing different amounts of the Hall and ∇p_e terms to be present in our equations. Both these terms are removed, and the MHD limit recovered, as $p \to \infty$. In this limit, the two pressure equations become identical (so that they are in reality only a single equation). The equations for

$$\frac{\partial V_1}{\partial \tau} \ , \ \frac{\partial n_1}{\partial \tau}$$

are not affected. This is because the present two-fluid model has used the same ion-fluid momentum equation, and the same continuity equation as is used in ideal MHD. The affected equations now read

$$\frac{\partial B_{1r}}{\partial \tau} = ik_{\parallel} B_{1r} \qquad (2.4.1\text{-}6)$$

$$\frac{\partial B_{1\theta}}{\partial \tau} = \left[-\frac{dB_{0\theta}}{d\mu}\right] V_{1r} - B_{0\theta} \frac{\partial V_{1r}}{\partial \mu} + in\epsilon_1 \left[B_{0z} V_{1\theta} - B_{0\theta} V_{1z}\right] \qquad (2.4.1\text{-}7)$$

$$\frac{\partial \rho_1}{\partial \tau} = -\frac{1}{2}\left(\frac{d\rho_0}{d\mu} + \Gamma\rho_0\right)v_{1r} - \frac{\Gamma\rho_0}{2}\frac{\partial v_{1r}}{\partial \mu} - \frac{\Gamma\rho_0}{2}\left[\frac{imv_{1\theta}}{\mu} + in\epsilon_1 v_{1z}\right] \qquad (2.4.1\text{-}8)$$

It can be seen from p.152 of /11/ that the above three equations can be derived by the use of the ideal MHD model. Figure (2.4-4) illustrates the effect of the limiting process in equation (2.4.1-5) , on the growth rates of the m=3,n=2 ; m=2,n=1; and . . m=1,n=1 modes. This figure shows that for the tokamak reactor parameters that we have used :

$$B_o = 3{\cdot}0 \times 10^4 \text{ gauss}$$

$$T_o = 1 \text{ keV}$$

$$n_o = 5{\cdot}0 \times 10^{13} \text{ cm}^{-3}$$

$$\beta = 0{\cdot}01 \qquad (2.4.1\text{-}9)$$

the general effect of the extra terms in our two-fluid Ohm's law, over and above those in the ideal MHD Ohm's law, is strongly destabilising. We have run our program in the ideal MHD limit, for all the three equilibria, for as long as 50 Alfven transit times , and have not obtained any converged growth rates. We are tempted, in view of the fact that our equilibria only violate Suydam's criterion in very small regions of space,to conclude that the Suydam-stability of the plasma in the bulk of the spatial domain,may mean that if on the whole our equilibria are ideal MHD-unstable,this instability may be manifested as a very small growth rate,for which longer integration times than we have allowed, may be required.It may also be that the regions that are Suydam-stable, compensate for the regions that are Suydam-unstable,and that as a result,the plasma is globally stable. In the full two-fluid runs that we have done only times of the order of $0{\cdot}5\tau_A$ are normally required for the emergence of the fastest-growing mode. It will thus be appreciated that the $50\tau_A$ run times that we have allowed in our MHD runs,are very long indeed. In figure (2.4-5) we have illustrated the frequency .v. time plots for the m=2,n=1 mode, for three values of ϵ_3 ,the last of which corresponds to the MHD limit,$(\epsilon_3 \to \infty)$. The reduction of the growth-rate as ϵ_3 is increased is evident. We can also see from figure (2.4-4) that for each mode,there is a value of ϵ_3 for which the Hall and ∇p_e destabilisation is a maximum.

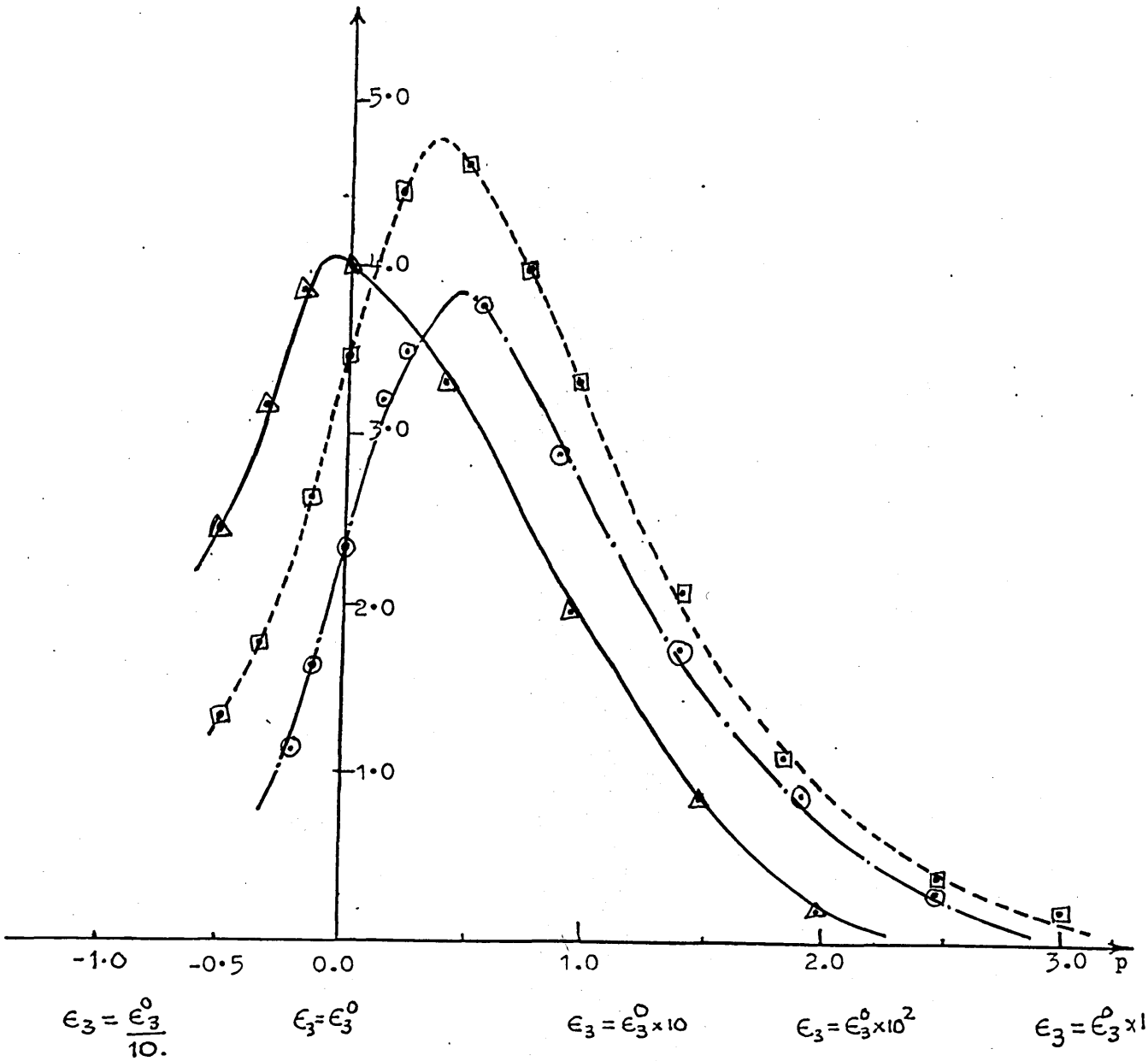


Figure 2.4-4 The behaviour of the growth-rates of three radial modes as the parameter ϵ_3 is varied. The MHD limit occurs at large values of p.

——— m=1,n=1

- - - - - - - - m=2,n=1

——·—— m=3,n=2

$$\epsilon_3^0 = \left(\frac{a}{\lambda_c}\right)^0 = 9.31$$

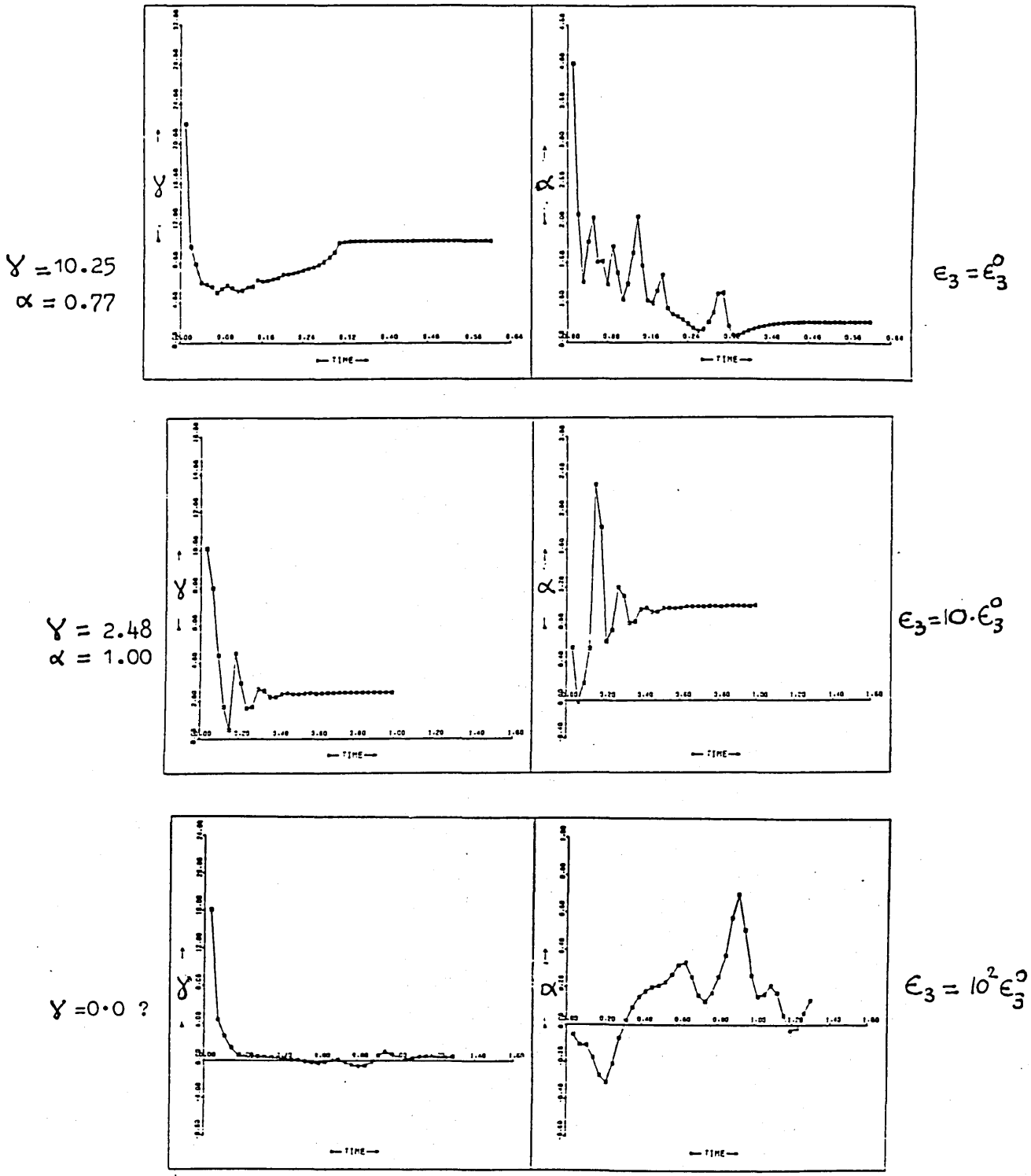


Figure 2.4-5 Sequence of frequency .v. time plots for the m=2, n=1 mode in the ν=2, κ=0·2 equilibrium, showing the effect of increasing ϵ_3 .

Time is in units of Alfven transit time, τ_A .

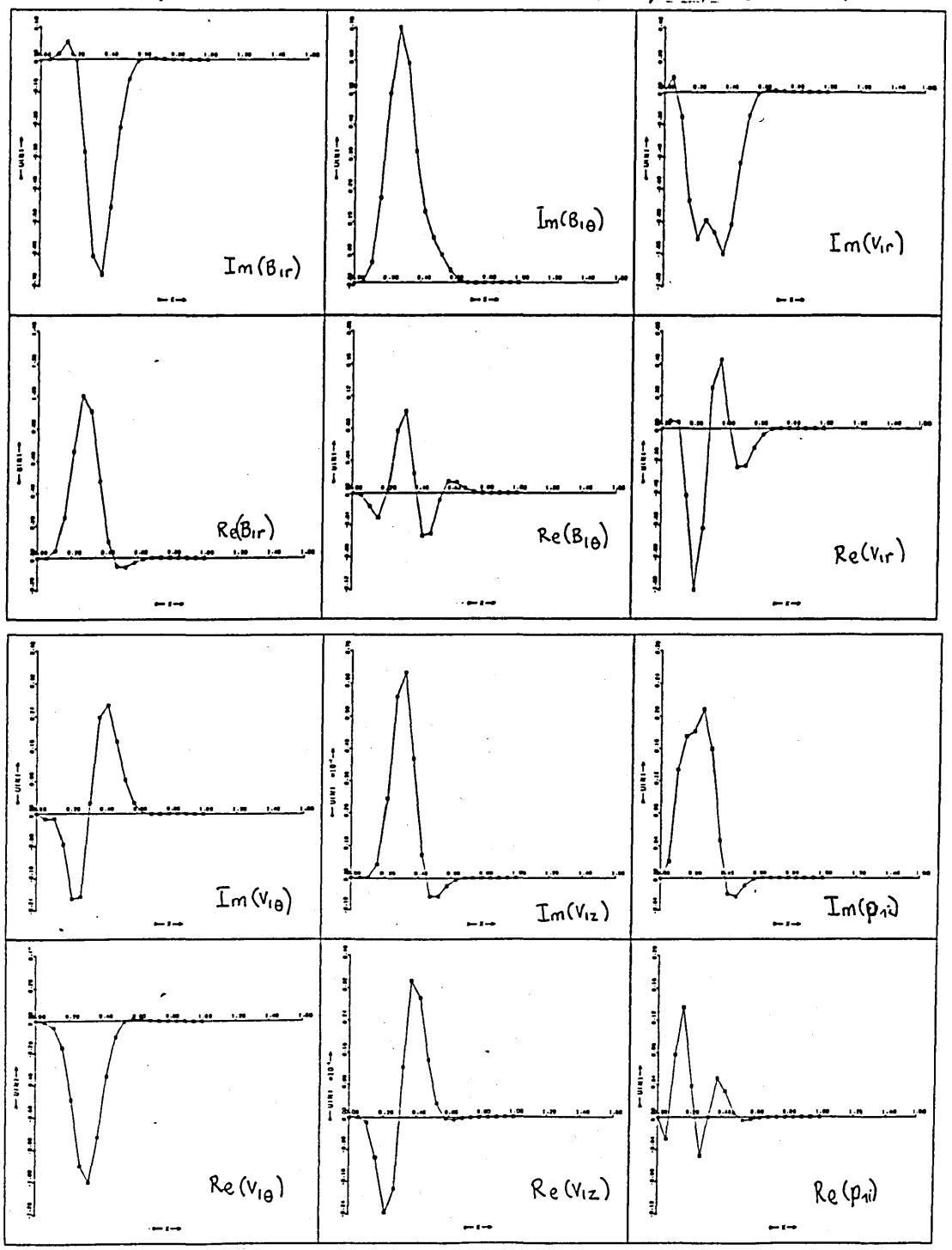


Figure 2.4-6 6 eigenfunctions of the m=5,n=3 mode for the $\nu=2$ equilibrium. The resonant surface is at $x_0=0{\cdot}747$

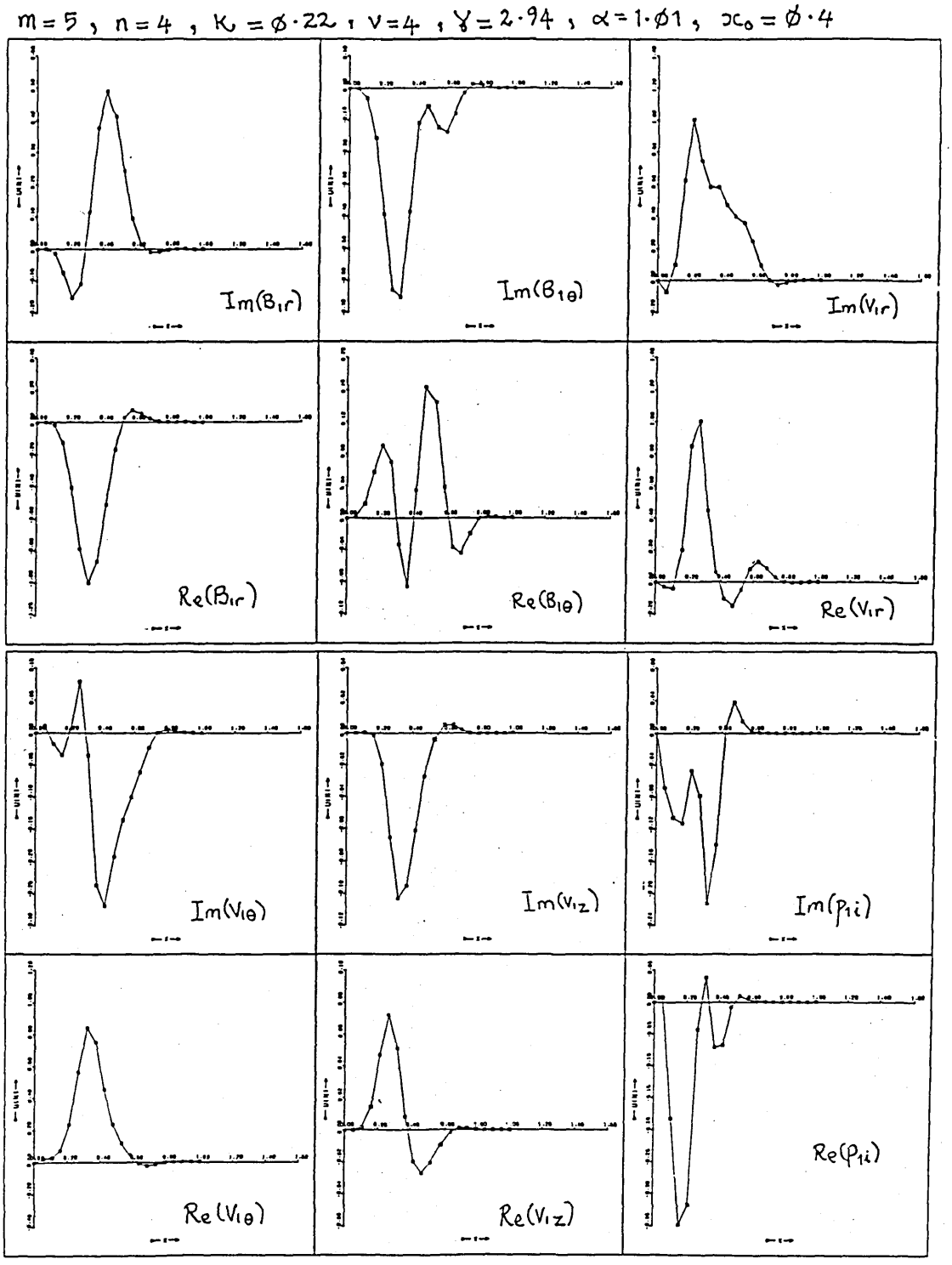


Figure 2.4-7 6 eigenfunctions of the m=5,n=4 mode for the ν=2 equilibrium.The resonant surface x_o= 0·4

2.5 Summary of Chapter 2

In this chapter we have used a numerical initial/boundary-value approach to solve the linearised two-fluid equations,under ideal MHD boundary conditions derived by Robinson /60/.The equations were initialised with an arbitrary perturbation,and then integrated forwards in time using the numerical method of lines, until such time as the fastest-growing radial oscillation mode should become dominant. We have studied several low m-number perturbations, and have found that the sharper current profile equilibria (ν =2,4) are more unstable than the broad (ν=1) equilibrium,for all modes (i.e. have larger associated growth-rates). We have also found that the eigenfunctions produced in the ν=2,4 equilibria, are much more localised than are those for the ν=1 equilibrium. Also,although these eigenfunctions are found often to be localised near the resonant surface, or actually around the resonant surface (see figure 2.4-6 and figure 2.4-7),for some radial modes, for other radial oscillation modes,the relationship between the resonant surface and the region of localisation of the eigenfunctions,is not apparent. Finally, as a result of a limiting process in which the MHD equations are recovered, we are able to state that the effect of the Hall and ∇p_e terms in the two-fluid Ohm's law (equation 1.1-16), on all equilibria that we have studied, is destabilising. That is to say, that the linear growth-rates obtained using this two-fluid theory, are significantly greater than would be obtained using single-fluid theory alone.

2.6 Conclusions

In this thesis we have considered the two-fluid model of an infinitely conducting cylindrical plasma used in /1/, with a view to providing evidence to support the proposition that single-fluid theory (resistive or ideal) cannot properly describe magnetically confined plasmas at temperatures high enough for fusion to occur. We have also followed through the analysis presented in /1/, a paper having the same broad aims as the present work, and although we are not in agreement with some aspects of the detailed analysis presented there, two general conclusions of that paper seem beyond doubt. Firstly, it has been seen that even in the absence of resistivity, the magnetic field is not 'frozen' to the ion fluid (in contrast to the ideal MHD theory, which holds that the whole plasma fluid and the magnetic field are frozen together). We have shown that this decoupling of the magnetic field and the ion fluid is more complex than is indicated in /1/. It is also the case that unless we are dealing with a situation in which $\nabla T \times \nabla n = 0$, then even the electron fluid is not frozen to the magnetic field. The detachment of the ion fluid from the magnetic field will occur throughout the volume of the plasma, and not just in the vicinity of the resonant surface (as is the case in resistive MHD), and so, given the central importance of magnetic decoupling to the tearing-mode theory /22,36,37/, which has often be used in describing high-temperature tokamak plasmas, it seems unreasonable to leave out of the Ohm's law, terms (like the ∇p_e term), which can not only cause decoupling of the magnetic field from the ion fluid, but are in any case (for plasma parameters such as those of JET), about three orders of magnitude greater than the resistive term /1/. That is to say that from the point of view of magnitude alone, when high-temperature tokamaks are being considered, the Hall and ∇p_e terms should be included before the resistive term. Secondly, we have seen that the MHD equations can be recovered from the two-fluid equations in the limit $\frac{V_A}{a\omega_{ci}} = \frac{\lambda_c}{a} \longrightarrow 0$. However, since it is the case that for high-temperature tokamaks such as JET $\frac{\lambda_c}{a} \sim 1$, it is clear that the single-fluid description of such plasmas is not legitimate.

The results of our numerical treatment of the linearised two-fluid equations also support the conclusion that the single-fluid description of high-temperature tokamak plasmas is not correct, since they indicate that the MHD theory would significantly underestimate the growth-rates

of such modes as are found to be unstable in the equilibria that we have studied.

It is well known that in ideal MHD the resonant point is a point at which the linearised plasma equations and their solutions are singular, and also that the location of the singularity is frequency dependent.It is likely that the same is not true of the equations of the present two-fluid model. This is seen by considering the two-fluid, single temperature theory of /107/, for which the resonant point is also a point at which the equations and their solutions are singular, but the location of the singularity is not frequency dependent. Thus,if, as we have argued, a two-fluid theory is necessary in order to more properly describe high-temperature tokamak plasmas, it is not clear that the present single-fluid descriptions of the behaviour of the plasma in the vicinity of the resonant point /3,5,22/, will have any relevance to tokamaks.

In conclusion, it seems clear to us that it will always be necessary, if the fluid description of tokamak plasmas at fusion temperatures is the aim, to use two-fluid, rather than single-fluid theory.

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Appendix A On the 'freezing-in' of field lines.

It is well-known that the ideal MHD Ohm's law

$$\underline{E} + \frac{1}{c}(\underline{v} \wedge \underline{B}) = 0 \tag{A1}$$

coupled with Faraday's law

$$\nabla \wedge \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \tag{A2}$$

leads to the result that magnetic field lines are 'frozen' into the material of the plasma.We shall now derive a single equation which embodies this result.

Taking the curl of (A1) and substituting (A2) into the result gives :

$$\nabla \wedge (\underline{v} \wedge \underline{B}) = \frac{\partial \underline{B}}{\partial t} \tag{A3}$$

The LHS of this equation can be expanded by a vector identity (remembering that $\nabla \cdot \underline{B} = 0$),to yield :

$$-\underline{B}(\nabla \cdot \underline{v}) + (\underline{B} \cdot \nabla)\underline{v} - (\underline{v} \cdot \nabla)\underline{B} = \frac{\partial \underline{B}}{\partial t} \tag{A3a}$$

i.e.

$$-\underline{B}(\nabla \cdot \underline{v}) + (\underline{B} \cdot \nabla)\underline{v} = \frac{d\underline{B}}{dt} \tag{A4}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{v} \cdot \nabla \tag{A5}$$

We can then use the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{v}) = 0 \tag{A5a}$$

to find that we can write

$$\nabla\cdot\underline{V} = -\frac{1}{n}\left[\frac{\partial n}{\partial t} + (\underline{V}\cdot\nabla)n\right] = -\frac{1}{n}\frac{dn}{dt} \qquad (A6)$$

Then substituting from (A6) for $\nabla\cdot\underline{V}$ into (A4),we find that

$$(\underline{B}\cdot\nabla)\underline{V} = \frac{d\underline{B}}{dt} - \frac{\underline{B}}{n}\frac{dn}{dt} \qquad (A6a)$$

i.e.

$$\left(\frac{\underline{B}}{n}\cdot\nabla\right)\underline{V} = \frac{d}{dt}\left(\frac{\underline{B}}{n}\right) \qquad (A7)$$

This is the equation that embodies the freezing-in law,and if we can show that the electron fluid motion in the radial direction,obeys this equation,we shall have shown that as far as radial motion is concerned,the electron fluid is frozen to the magnetic field.

For the electron fluid,(A7) reads :

$$\left(\frac{\underline{B}}{n}\cdot\nabla\right)\underline{u} = \frac{d_e}{dt}\left(\frac{\underline{B}}{n}\right) \quad ; \quad \frac{d_e}{dt} \equiv \frac{\partial}{\partial t} + \underline{u}\cdot\nabla \qquad (A8)$$

and on expansion and multiplication by n^2 this becomes

$$n(\underline{B}\cdot\nabla)\underline{u} = n\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\underline{B} - \underline{B}\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)n \qquad (A9)$$

When linearised,this equation will read :

$$n_0(\underline{B}_0\cdot\nabla)\underline{u}_1 + n_0(\underline{B}_1\cdot\nabla)\underline{u}_0 + \underline{n}_1(\underline{B}_0\cdot\nabla)\underline{u}_0 =$$

$$n_0\left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)\underline{B}_1 + n_0(\underline{u}_1\cdot\nabla)\underline{B}_0 + n_1(\underline{u}_0\cdot\nabla)\underline{B}_0$$

$$-B_0\left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)n_1 - \underline{B}_0(\underline{u}_1\cdot\nabla)n_1 - \underline{B}_1(\underline{u}_0\cdot\nabla)n_0 \qquad (A10)$$

From texts on vector analysis we know that in cylindrical polar coordinates, the radial component of the vector $(\underline{A}\cdot\nabla)\underline{B}$ is given by :

$$\left[(\underline{A}.\nabla)\underline{B}\right]_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r} \qquad (A11)$$

If we then use this expression, and the form for the perturbed quantities given in equation (1·2-16), and bear in mind that $u_{or}=v_{or}=0$, it will be possible to find that the radial component of (A10) reads :

$$in_o B_o k_{\parallel} u_{1r} = in_o(\omega + \underline{k}\cdot\underline{u}_o)B_{1r} \qquad (A12)$$

i.e.

$$u_{1r} = \frac{(\omega + \underline{k}\cdot\underline{u}_o)}{k_{\parallel}} \cdot \frac{B_{1r}}{B_o} \qquad (A13)$$

or, in the notation of section 1·2 :

$$u_{1r} = \frac{\Omega}{k_{\parallel}}.\psi \qquad (A14)$$

which is also equation (1·2-36). In /1/ it was stated that this equation implied that the magnetic field was frozen to the electron fluid as far as radial motion was concerned . However, before we can claim to have shown this to be the case, we must show either that (A8) follows from the two-fluid Ohm's Law that we have used, or at the very least, we must show under what (if any) conditions, (A8) would follow from our two-fluid Ohm's Law. The two-fluid Ohm's Law that we have used is :

$$\underline{E} + \frac{1}{c}(\underline{v} \times \underline{B}) = \frac{1}{enc}(\underline{j} \times \underline{B}) - \frac{\nabla p_e}{en} \qquad (A15)$$

If we take the curl of this equation, the result would be :

$$\begin{aligned} &-\frac{1}{c}\frac{\partial \underline{B}}{\partial t} + \frac{1}{c}\left[\underline{v}(\nabla\cdot\underline{B}) + (\underline{B}\cdot\nabla)\underline{v} - \underline{B}(\nabla\cdot\underline{v}) - (\underline{v}\cdot\nabla)\underline{B}\right] = \\ &+\frac{1}{enc}\left[\underline{j}(\nabla\cdot\underline{B}) - \underline{B}(\nabla\cdot\underline{j}) + (\underline{B}\cdot\nabla)\underline{j} - (\underline{j}\cdot\nabla)\underline{B}\right] \qquad (A16) \\ &+\frac{1}{ec}\cdot\nabla\left(\frac{1}{n}\right)\wedge(\underline{j}\wedge\underline{B}) - \frac{1}{e}\nabla\left(\frac{1}{n}\right)\wedge\nabla p_e \end{aligned}$$

In (A16) we have used (A2) to substitute for $\nabla\wedge\underline{E}$. From the quasi-neutrality assumption we know that $\nabla\cdot\underline{j} = 0$, and also of course, $\nabla\cdot\underline{B} = 0$. If we also consider a plasma of constant density n, then it will be true that $\nabla\left(\frac{1}{n}\right) = 0$. For this circumstance, (A16) would read :

$$-\frac{\partial \underline{B}}{\partial t} + (\underline{B}\cdot\nabla)\underline{V} - \underline{B}(\nabla\cdot\underline{V}) - (\underline{V}\cdot\nabla)\underline{B} =$$

$$\underline{B}\cdot\nabla\left(\frac{\underline{j}}{en}\right) - \left(\frac{\underline{j}}{en}\cdot\nabla\right)\underline{B} \qquad \text{(A17)}$$

i.e.

$$\frac{\partial \underline{B}}{\partial t} + \left\{\left[\underline{V} - \frac{\underline{j}}{en}\right],\nabla\right\}\underline{B} - \left\{\underline{B}\cdot\nabla\right\}\left(\underline{V} - \frac{\underline{j}}{ne}\right) + B(\nabla\cdot\underline{V}) = 0 \qquad \text{(A18)}$$

However, since we are assuming that the plasma is quasineutral, and that it has a constant number density n, the following relations will hold :

$$\underline{u} = \underline{V} - \frac{\underline{j}}{en} \quad , \quad \nabla\cdot\underline{j} = 0 \quad \therefore \quad \nabla\cdot\underline{u} = \nabla\cdot\underline{j}$$

where $\underline{u}$ is the velocity of the electronic fluid .It therefore follows that we can write (A18) as :

$$\frac{\partial \underline{B}}{\partial t} + (\underline{u}\cdot\nabla)\underline{B} - (\underline{B}\cdot\nabla)\underline{u} + \underline{B}(\nabla\cdot\underline{u}) = 0 \qquad \text{(A19)}$$

This is equation (A3a), with $\underline{u}$ written in place of $\underline{v}$. Thus, we can state that equation (A14) only implies that the radial motion of the electronic fluid is tied to the magnetic field, if the plasma in question is assumed to have a constant number density.

Appendix B Derivation of the expression for v_{1r}

We saw in equation (1.2-36) that the radial perturbed velocity of the electron fluid is given by :

$$u_{1r} = \frac{\Omega}{k_{\parallel}} \psi \tag{B1}$$

In order to derive the equivalent expression for the ions, we shall need expressions for the three components of the perturbed electric field. Substituting (B1) into the θ and z components of the Ohm's law (equations (1.2-24) and (1.2-25)), allows us to write

$$E_{1\theta} = \left(\frac{\omega B_{0z}}{k_{\parallel} c}\right)\psi - \frac{m}{e n_0 r}\left[\frac{p_0'}{k_{\parallel}} \cdot \psi + i p_{1e}\right] \tag{B2}$$

$$E_{1z} = -\left(\frac{\omega B_{0\theta}}{k_{\parallel} c}\right)\psi - \frac{n}{e n_0 R}\left[\frac{p_0'}{k_{\parallel}} \psi + i p_{1e}\right] \tag{B3}$$

In order to determine an expression for E_{1r} , we form the following linear combination of the θ and z components of Faraday's law :

$\frac{n}{R}$ x equation (1.2-30) $-$ $\frac{m}{r}$ x equation (1.2-31)

and then we use the radial component of Ampere's law, equation (1.2-32) to substitute for the expression :

$$\frac{imB_{1z}}{r} - \frac{in}{R} B_{1\theta}$$

on the RHS of the resulting equation. As a result we are able to write for E_{1r} the following expression :

$$E_{1r} = -\frac{i}{k^2}\left[\frac{n}{R} E_{1z}' + \frac{m}{r} E_{1\theta}' + \frac{m}{r^2} E_{1\theta} + \frac{4\pi e n_0 \omega}{c^2}(v_{1r} - u_{1r})\right] \tag{B4}$$

where the primes denote differentiation with respect to r.

Then, substituting (B2) and (B3) into (B4), we find that

$$E_{1r} = -\frac{i}{k^2}\left[k^2\left(\frac{i\rho_{1e}}{en_0} + \frac{p_0'\psi}{en_0 k_{\parallel}}\right)' + \frac{2}{r}\cdot B_{0\theta}\cdot\frac{n\omega}{Rc}\cdot\frac{\psi}{k_{\parallel}} - \frac{\omega}{rc}\left(\frac{nB_{0\theta}}{R} - \frac{mB_{0z}}{r}\right)\right.$$

$$\left. + \frac{4\pi e n_0 \omega}{c^2}\left(v_{1r} - \frac{\omega\psi}{k_{\parallel}}\right)\right] \qquad \text{(B5)}$$

If we also substitute (B2) and (B3) into the θ and z components of the ion momentum (equations (1.2-21) and (1.2-22)) respectively , we find that :

$$v_{1\theta} = \frac{1}{im_i\omega n_0}\left[-\frac{m}{r}\left(i\rho_1 + \frac{p_0\psi}{k_{\parallel}}\right) - \frac{en_0 B_{0z}}{c}\left(v_{1r} - \frac{\omega\psi}{k_{\parallel}}\right)\right] \qquad \text{(B6)}$$

and

$$v_{1z} = \frac{1}{im_i\omega n_0}\left[\frac{-n}{R}\left(i\rho_1 + \frac{p_0\psi}{k_{\parallel}}\right) + \frac{en_0 B_{0\theta}}{c}\left(v_{1r} - \frac{\omega\psi}{k_{\parallel}}\right)\right] \qquad \text{(B7)}$$

The expression for v_{1r} is obtained by substituting the above expressions for E_{1r} , $v_{1\theta}$, and v_{1z} , into the radial component of the ion momentum equation (equation (1.2-20)). Substantial and involved algebraic manipulation is required, before the final form of the expression emerges. Such manipulation, because of its length, is not worth producing, even in an appendix, and we simply state the result :

$$v_{1r} = \left(1 + M\frac{\omega}{\omega_{ci}}\right)\frac{\omega\psi}{k_{\parallel}} + \frac{\omega^* p_0}{\mathcal{D}p_0'}\Delta_p$$

$$+ \frac{1}{\mathcal{D}}\left(\frac{\omega}{\omega_{ci}}\right)\left[-\frac{cT_0}{eB_0}\left(\frac{T_0'\Delta_p}{T_0} + \Delta_p'\right) + \frac{cT_0}{eB_0}\left(\frac{n_1 p_{0i}'}{n_0 p_0} - \frac{n_0' p_{1i}}{n_0 p_0}\right)\right.$$

$$\left. - \frac{1}{k^2 r B_0}\left(\frac{nB_{0\theta}}{R} - \frac{mB_{0z}}{r}\right)\frac{d}{dr}\left(\frac{r\omega\psi}{k_{\parallel}}\right)\right] \qquad \text{(B8)}$$

The quantities appearing in this equation are defined in equations (1.2-35) and (1.2-44)

Appendix C The mutual cancelling of Error I and Error II

The radial component of the linearised ion momentum equation is given by equation (1.2-20), and the equivalent equation for the electron fluid by equation (1.2-23) Respectively these two equations are

$$im_i\omega n_0 v_{1r} = en_0\left[E_{1r} + \tfrac{1}{c}(v_{1\theta}B_{0z} - v_{1z}B_{0\theta})\right] + en_1E_{0r} - \frac{dp_{1i}}{dr} \qquad (C1)$$

$$0 = -en_0\left[E_{1r} + \tfrac{1}{c}(u_{0\theta}B_{1z} + u_{1\theta}B_{0z} - u_{0z}B_{1\theta} - u_{1z}B_{0\theta})\right] - en_1E_{0r} - \frac{dp_{1e}}{dr} + \underline{\frac{n_1}{n_0}\frac{dp_0}{dr}} \qquad (C2)$$

①

The term whose omission in /1/ we have called Error I, is underlined. Adding (C1) and (C2) we find that

$$im_i\omega n_0 v_{1r} = \frac{en_0}{c}\left[B_{0z}(v_{1\theta} - u_{1\theta}) - B_{0\theta}(v_{1z} - u_{1z}) + (u_{0z}B_{1\theta} - u_{0\theta}B_{1z})\right] - \frac{dp_{1i}}{dr} + \frac{n_1}{n_0}\frac{dp_0}{dr} \qquad (C3)$$

We can eliminate $(v_{1\theta} - u_{1\theta})$ and $(v_{1z} - u_{1z})$ from this equation by using the θ and z components of the linearised form of Ampere's Law (equations (1.2-33) and (1.2-34)) These equations are repeated here for ease of reference

$$(v_{1\theta} - u_{1\theta}) = \frac{c}{4\pi en_0}\left(\frac{inB_{1r}}{R} - \frac{dB_{1z}}{dr}\right) + \underline{\frac{n_1u_{0\theta}}{n_0}} \qquad (C4)$$

②

$$(v_{1z} - u_{1z}) = \frac{c}{4\pi en_0}\left[\frac{1}{r}\frac{d}{dr}(rB_{1\theta}) - \frac{imB_{1r}}{r}\right] + \underline{\frac{n_1u_{0z}}{n_0}} \qquad (C5)$$

②

and we have underlined the terms whose omission would constitute Error III .

Substituting (C4) and (C5) into (C3) we then find that

$$
\begin{aligned}
i M_i \omega n_0 v_{1r} = \frac{e n_0}{c}\Big[& \frac{c B_{0z}}{4\pi e n_0}\left(\frac{in B_{1r}}{R} - \frac{dB_{1z}}{dr}\right) + \frac{n_1 u_{0\theta} B_{0z}}{n_0} \quad (2) \\
& - \frac{c B_{0\theta}}{4\pi e n_0}\left\{\frac{1}{r}\frac{d}{dr}(r B_{1\theta}) - \frac{im B_{1r}}{r}\right\} - \frac{n_1 u_{0z} B_{0\theta}}{n_0} \quad (2) \\
& + (u_{0z} B_{1\theta} - u_{0\theta} B_{1z})\Big] - \frac{dp_1}{dr} + \frac{n_1}{n_0}\frac{dp_0}{dr} \quad (1) \\
= \Big[& \frac{B_{0z}}{4\pi}\left(\frac{in B_{1r}}{R} - \frac{dB_{1z}}{dr}\right) - \frac{B_{0\theta}}{4\pi}\left\{\frac{1}{r}\frac{d}{dr}(r B_{1\theta}) - \frac{im B_{1r}}{r}\right\} + \frac{e n_0}{c}(u_{0z} B_{1\theta} - u_{0\theta} B_{1z})\Big] \\
& - \frac{dp_1}{dr} + \frac{e n_1}{c n_0}\left(n_0 u_{0\theta} B_{0z} - n_0 u_{0z} B_{0\theta}\right) + \frac{n_1}{n_0}\cdot\frac{dp_0}{dr} \quad (1) \\
& \qquad (2)
\end{aligned}
\tag{C6}
$$

The condition for the equilibrium of pressure forces and magnetic forces is :

$$\nabla p = \frac{\underline{J} \wedge \underline{B}}{c} \tag{C7}$$

which only has a radial component in equilibrium. This component is given by :

$$
\begin{aligned}
\frac{dp_0}{dr} &= \frac{1}{c}(J_{0\theta} B_{0z} - J_{0z} B_{0\theta}) \\
&= -\frac{e}{c}\left(n_0 u_{0\theta} B_{0z} - n_0 u_{0z} B_{0\theta}\right)
\end{aligned}
\tag{C8}
$$

where we have used the fact that the equilibrium current is given by :

$$\underline{J}_0 = -e n_0 \underline{u}_0 \tag{C9}$$

We are now able to see that by virtue of (C8), the last two terms in (C6) will cancel, and so we obtain finally :

$$
\begin{aligned}
i M_i \omega n_0 v_{1r} = \Big[& \frac{B_{0z}}{4\pi}\left(\frac{in B_{1r}}{R} - \frac{dB_{1z}}{dr}\right) - \frac{B_{0\theta}}{4\pi}\left\{\frac{1}{r}\frac{d}{dr}(r B_{1\theta}) - \frac{im B_{1r}}{r}\right\} \\
& - \frac{e n_0}{c}(u_{0z} B_{1\theta} - u_{0\theta} B_{1z}) - \frac{dp_1}{dr}
\end{aligned}
\tag{C10}
$$

We saw in the thesis that Error I had been committed in /1/ . If we note that equation (C10) is the direct precursor of equation (40) of /1/ ,we may fairly deduce that Error II has also been committed in /1/.As we have pointed out in the thesis,the cancellation of these errors (of omission),is not sufficient justification for their commission.At the very least,such omissions should be explicitly acknowledged.

Finally,we note that in equation (C6) th terms associated with Error I have been marked (1),and those associated with Error II have been marked (2) .

Appendix D Derivation of the 'exterior case' equation

In this appendix we shall derive the equation (A.1) of /1/ , which it is claimed in /1/ ,describes the behaviour of the two-fluid plasma in the region $r_o \leq r \leq a$,in order that we might more fully appreciate the approximations involved in this equation,and thus be in a position to assess its validity.

In the thesis we saw how,by the use of the transformations in equation (1.4-3),the basic coupled equations of the formulation of /1/ ,(equations (1.4-1) and (1.4-2)),were converted into equations (1.4-4) and (1.4-10),respectively.We shall repeat equation (1.4-4) here for convenience :

$$\frac{1}{r}\frac{d}{dr}(r\zeta) + \alpha(r)\zeta(r) = -\frac{\omega}{r}\frac{d}{dr}(r\xi) \qquad (D1)$$

We have also seen how,using the approximations discussed in section 1.4, this equation led to the new approximate equation :

$$\zeta(r) = \Lambda(\omega, r_o)\xi(r) \qquad (D2)$$

where the object $\Lambda(\omega, r_o)$ is defined in equation (1.4-13). Then,by substituting equation (D2) into equation (1.4-10),the two coupled equations in and $\xi(r)$,are reduced to a single equation in $\xi(r)$, which reads :

$$\underset{(1)}{\frac{d}{dr}\left[\frac{B_o^2 k_{11}^2}{4\pi k^2 r}\frac{d}{dr}(r\xi)\right]} - \xi\left[\underset{(2)}{\frac{B_o^2 k_{11}^2}{4\pi}} - \underset{(3)}{m_i n_o \omega^2} + \underset{(4)}{\frac{2mrk_{11}B_o}{4\pi}\frac{d}{dr}\left(\frac{B_{o\theta}}{k^2 r^3}\right)} - \underset{(5)}{\frac{2nB_{o\theta}}{ck^2 rR}\left(\frac{mJ_{o\theta}}{r} + \frac{nJ_{oz}}{R}\right)}\right] =$$

$$-\Lambda(\omega, r_o)\xi\left[\underset{(6)}{\frac{2en_o nB_{o\theta}}{ck^2 rR}} + \underset{(7)}{m_i n_o \omega}\right]. \qquad (D3)$$

We also follow /1/ and make the expansion

$$k_{\parallel}(r) = k'_{\parallel}(r_0).(r-r_0) \tag{D4}$$

and also define the new independent variable x,where

$$x = \frac{2|m|(r-r_0)}{r_0} \tag{D5}$$

so that

$$\frac{dr}{dx} = \frac{r_0}{2|m|} \quad \text{or} \quad \frac{dx}{dr} = \frac{2|m|}{r_0} \tag{D6}$$

and therefore

$$\frac{d}{dx} \equiv \left(\frac{2|m|}{r_0}\right)\frac{d}{dr} \tag{D7}$$

We therefore find that we can write equation (D4) as

$$k_{\parallel}(x) = \left.\frac{dk_{\parallel}}{dx}\right|_{x=0} . x \tag{D8}$$

As part of the approximation process of /1/ it must also be envisaged that the resonant surface, $r=r_0$,is not close to the origin,and that such replacements as

$$\frac{d}{dr}(r\xi) \equiv \frac{d}{dr}(r_0\xi) = r_0\frac{d\xi}{dr} \tag{D9}$$

which will be used,are not fundamentally serious distortions of the truth.In fact ,it is in general assumed that the replacement

$$f_0(r) \equiv f_0(r_0) \tag{D10}$$

is not a serious distortion,where $f_0(r)$,is any equilibrium profile quantity,and also that we can write

$$k^2 \simeq \frac{m^2}{r^2} \simeq \frac{m^2}{r_0^2} \tag{D11}$$

The z-component of the magnetic field is also not assumed to vary with r, so that since

$$k_{\parallel}(r) = \frac{1}{B_0}\left(\frac{mB_{0\theta}}{r} + \frac{nB_{0z}}{R}\right) \tag{D12}$$

we shall have

$$\frac{dk_{\parallel}}{dr} = \frac{d}{dr}\left(\frac{mB_{0\theta}}{rB_0}\right) \tag{D13}$$

Now, since it is the case that at the resonant surface $k_{\parallel}=0$, we are in a position to write

$$\frac{B_{0\theta}(r_0)}{B_{0z}(r_0)} = -\frac{nr_0}{mR} \tag{D14}$$

It is also the case that ordering the radial derivatives of equilibrium quantities as follows

$$\frac{d}{dr} \sim \frac{1}{r_0} \tag{D15}$$

is thought in /1/, not to be seriously consequential (in terms of perversion of the 'true' nature of whatever results underlie the problem). We may now substitute (D14) into (D13) to find

$$\left.\frac{dk_{\parallel}}{dr}\right|_{r=r_0} \sim \frac{n}{Rr_0} \tag{D16}$$

and using this result in (D8) we are able to find that

$$\left.\frac{dk_{\parallel}}{dx}\right|_{x=0} \sim \frac{n}{2mR} \tag{D17}.$$

Equation (D9) shows why it is hoped in /1/ that this treatment will characterise the outer region ($r_0 \leq r \leq a$), for such replacements can certainly not be true in the 'interior' domain ($0 \leq r \leq r_0$) , which includes the origin. Thus , it is not at all clear that the

equation at which we shall arrive, will indeed describe the behaviour of the plasma in the region $r_o \leq r \leq a$, if the 'outerness' of this equation derives solely from equation (D9). Also, the ordering in equation (D15) is likely to wildly incorrect if r_o lies close to the origin.

Now, with the aid of the approximations that we have just outlined, the terms of equation (D3) may be considered individually, and made to read as follows :

$$\text{Term 1} = \frac{d}{dr}\left\{\frac{B_o^2 k_{||}^2}{4\pi k^2 r}\cdot\frac{d}{dr}(r\xi)\right\} = \frac{B_o^2 k_{||}'^2}{\pi}\frac{d}{dx}\left(x^2\frac{d\xi}{dx}\right)$$

$$\text{Term 2} = -\xi\left(\frac{B_o^2 k_{||}^2}{4\pi}\right) = -\frac{B_o^2 k_{||}'^2}{4\pi}(x^2\xi)$$

$$\text{Term 3} = +m_i n_o \omega^2 \xi$$

$$\text{Term 4} = -\frac{k_{||}B_o \cdot 2mr}{4\pi}\cdot\frac{d}{dr}\left(\frac{B_{o\theta}}{k^2 r^3}\right)\xi = -\frac{B_o^2 k_{||}'^2}{\pi m}(x\xi)$$

$$\text{Term 5} = \frac{2nB_{o\theta}}{ck^2 rR}\left(\frac{mJ_{o\theta}}{r}+\frac{nJ_{oz}}{R}\right)\xi = \frac{2nB_{o\theta}(r_o)}{ck^2 r_o R}\left[\frac{mJ_{o\theta}(r_o)}{r_o}+\frac{nJ_{oz}(r_o)}{R}\right]$$

$$\text{Term 6} = -\frac{2en_o B_{o\theta}\Lambda(\omega,r_o)\xi}{ck^2 rR} = -\frac{2en_o nB_{o\theta}(r_o)\Lambda(\omega,r_o)\xi}{ck^2 r_o R}$$

$$\text{Term 7} = -m_i n_o \omega \Lambda(\omega,r_o)\xi$$

(D18)

It is to be noted that in the above identifications, the prime denotes differentiation with respect to x. Finally, collecting together the terms in (D18), we are able to write equation (D3) as :

$$\frac{d}{dx}\left(x^2\frac{d\xi}{dx}\right) - \xi\left[\frac{x^2}{4} + \frac{x}{m} - (\lambda+1/4)\right] = 0 \qquad \text{(D19)}$$

where

$$\lambda + 1/4 = \frac{\pi}{B_0^2 k_{11}'^2}\left[m_i n_0 \omega^2 + \frac{2nB_{0\theta}}{ck^2 r_0^2 R}\left(\frac{mJ_{0\theta}}{r_0} + \frac{nJ_{0z}}{R}\right)\right.$$

$$\left. + \Delta(\omega, r_0)\left(\frac{2en_0 nB_{0\theta}}{ck^2 R r_0} + m_i n_0 \omega\right)\right] \qquad \text{(D20)}$$

However, using equations (D11) and (D14) we find that

$$\frac{2nB_{0\theta}}{ck^2 r_0 R} = -\frac{2B_{0\theta}^2}{mcB_{0z}} \qquad \text{(D21)}$$

and using equation (D7) we see that

$$k_{11}'^2 = \frac{r_0^2}{4m^2}\left(\frac{dk_{11}}{dr}\right)^2 \qquad \text{(D22)}$$

Then using results (D21) and (D22), we are finally able to write equation (D20) as

$$\lambda + 1/4 = \frac{4\pi m^2}{r_0^2 B_0^2 \left(\frac{dk_{11}}{dr}\right)^2}\left[m_i n_0 \omega^2 - \frac{2B_{0\theta}^2}{mcB_{0z}}\left(\frac{mJ_{0\theta}}{r_0} + \frac{nJ_{0z}}{R}\right)\right.$$

$$\left. + \Delta(\omega, r_0)\left(\frac{-2en_0 B_{0\theta}^2}{mcB_{0z}} + m_i n_0 \omega\right)\right] \qquad \text{(D23)}$$

Equation (D19) is to be solved between the limits x=0 and $x=x_w$,the value of x at which the infinitely conducting,rigid plasma wall,is located.It will be noticed that this equation has a regular singular point at x=0 (corresponding to $r=r_o$),and it is clear that any solutions obtained,in order to be physically acceptable,must be square-integrable near x=0 . In /1/ ,this equation is solved analytically in terms of confluent hypergeometric functions ,/70/. It is found there that the only acceptable values of λ are real and positive ,and that the solutions obtained are not square-integrable near x=0,and do not therefore form part of the discrete spectrum. It is thus concluded in /1/ that only continuum modes are possible in this exterior region,corresponding to stability of the plasma.

It is not clear to us however,in view of the large number of approximations that have been necessary in order to derive this equation (and not only Approximations I and II, which are detailed in the thesis , but also the further approximations which we have highlighted in this appendix),that the conclusions that are drawn from the solution of this equation,carry a great deal of weight.

Appendix E Derivation of the internal case equation

In this appendix we shall derive the approximate equation of /1/ which is supposed to describe the behaviour of the plasma in the region $0 \leq r \leq r_0$ .As in Appendix D ,this is done so that we might more fully appreciate the approximations involved in this equation, and also become aware of the source of its specifically 'interior' quality.

As in Appendix D ,we are required to use the ordering

$$\frac{d}{dr} \sim \frac{1}{r_0} \tag{E1}$$

and the expansion

$$k_{\parallel}(r) = k_{\parallel}'(r_0).(r-r_0) \tag{E2}$$

We also define the quantity

$$\underline{k} \cdot \underline{J}_0 = \frac{m j_{0\theta}(r_0)}{r_0} + \frac{n J_{0z}(r_0)}{R} \tag{E3}$$

and in the spirit of Appendix D we make use of the approximation

$$k^2 \simeq \frac{m^2}{r^2} \tag{E4}$$

We have also already seen in Appendix D,how the two basic coupled equations of the formulation of /1/ ,are reduced to a single equation (equation (D3)).We shall use this equation as the starting point of our discussion,after substituting into it the new independent variable

Z, where

$$r\xi = Z \tag{E5}$$

If we then substitute this equation into equation (D3), we obtain an equation whose terms devolve as follows :

$$\text{Term 1} = \frac{B_0^2 k_{11}'^2}{4\pi m^2} \frac{d}{dr}\left[(r-r_0)^2 r \frac{dZ}{dr}\right]$$

$$\text{Term 2} = -\frac{B_0^2 k_{11}'^2}{4\pi} \cdot \frac{(r-r_0)^2}{r} \cdot Z$$

$$\text{Term 3} = \frac{m_i n_0 \omega^2 Z}{r}$$

$$\text{Term 4} = -\frac{2mrB_0 k_{11}'(r-r_0)}{4\pi m^2} \cdot \left[\frac{d}{dr}\left(\frac{B_{0\theta}}{r}\right)\right] \cdot \left(\frac{Z}{r}\right)$$

$$= -2(B_0 k_{11}')^2 (r-r_0) Z / 4\pi m^2$$

$$\text{Term 5} = \frac{2nrB_{0\theta}(\underline{k}\cdot\underline{J}_0)Z}{cm^2 R r}$$

$$\text{Term 6} = \frac{-2\Lambda(\omega, r_0) e n_0 n r}{cm^2 R} \cdot \left(\frac{Z}{r}\right)$$

$$\text{Term 7} = -m_i n_0 \omega \Lambda(\omega, r_0) \cdot \left(\frac{Z}{r}\right)$$

(E6)

Collecting together all these terms, and multiplying the result by $\left(\frac{4\pi m^2}{B_0^2 k_{11}'^2}\right)$ we are able to obtain the new equation :

$$\frac{d}{dr}\left[r(r-r_0)^2\frac{d}{dr}Z\right] - \frac{m^2(r-r_0)}{r}Z - 2(r-r_0)Z$$

$$+ \frac{4\pi m^2 m_i n_0 \omega^2}{B_0^2 k_{11}'^2}\left[1+\frac{\Lambda(\omega,r_0)}{\omega}\right]\frac{Z}{r} + \frac{4\pi.2en_0 n\Lambda(\omega,r_0)B_{0\theta}Z}{B_0^2 k_{11}'^2 cR}$$

$$+ \frac{4\pi.2nB_{0\theta}(\underline{k}.\underline{J}_0)Z}{B_0^2 k_{11}'^2 cRr} = 0 \qquad \text{(E7)}$$

In determining Term 4 we have used the fact that since

$$k_{11} = \frac{1}{B_0}\left(\frac{mB_{0\theta}}{r} + \frac{nB_{0z}}{R}\right) \qquad \text{(E7a)}$$

it follows that within our approximations

$$\frac{d}{dr}\left(\frac{B_{0\theta}}{r}\right) = \frac{B_0 k_{11}'}{m} \qquad \text{(E7b)}$$

Also, since

$$\dot{\underline{J}}_0 = -en_0\underline{u}_0 \qquad \text{(E7c)}$$

it follows that

$$\underline{k}.\dot{\underline{J}}_0 = -en_0(\underline{k}.\underline{u}_0) \qquad \text{(E7d)}$$

and therefore that

$$\underline{k}.\dot{\underline{J}}_0 = en_0\Omega_{mn} \qquad \text{(E7e)}$$

where equation (1.2-35) has been used to substitute for $(\underline{k}.\underline{u}_0)$. If we now define

$$C = \frac{4\pi m^2 m_i n_0 \omega^2}{B_0^2 k_{11}'^2 r_0^2}\left[1 + \frac{\Lambda(\omega,r_0)}{\omega}\right] \qquad \text{(E8)}$$

we will be able to write equation (E7) as :

$$\frac{d}{dr}\left[r(r-r_0)^2\frac{dZ}{dr}\right] - \frac{m^2(r-r_0)^2}{r}Z - 2(r-r_0)Z$$

$$+\left(\frac{Cr_0^2}{r}\right)Z + \frac{8\pi e n_0 n B_{o\theta}}{B_o^2 k_{11}'^2 cR}\left[\Lambda(\omega, r_0) + \Omega_{mn}\right] \qquad \text{(E9)}$$

The safety factor q(r) is given by

$$q(r) = \frac{rB_o}{RB_{o\theta}(r)} \qquad \text{(E10)}$$

Thus it follows that

$$\frac{B_{o\theta}(r)}{r} = \frac{B_o}{Rq(r)} \qquad \text{(E11)}$$

and the limit of this as $r \longrightarrow 0$ is :

$$\lim_{r \to 0} \frac{B_{o\theta}(r)}{r} = \frac{B_o}{q(0)R} \qquad \text{(E12)}$$

Therefore,if this 'inner region' ($0 \leq r \leq r_o$) ,is so localised around the origin that the variation of $B_{o\theta}(r)$ with r is substantially linear ,(i.e. if r_o is close enough to the origin),then we can write :

$$B_{o\theta}(r) = \left[\frac{B_o}{Rq(0)}\right].r \qquad \text{(E13)}$$

With this approximation,the last term of equation (E9) can be written

$$D.r \qquad \text{(E14)}$$

where

$$D = \frac{8\pi e n_0 n\left\{\Lambda(\omega, r_0) + \Omega_{mn}\right\}}{B_o k_{11}'^2 cR^2 q(0)} \qquad \text{(E15)}$$

We can therefore now write equation (E9) as

$$\frac{d}{dr}\left[r(r-r_0)^2\frac{dZ}{dr}\right] + Z\left[\frac{r_0^2}{r}C - \frac{m^2(r-r_0)^2}{r} - 2(r-r_0) + Dr\right] = 0 \qquad \text{(E16)}$$

and this is the 'interior' case equation described in /1/, which, if the claim of /1/ is to be believed, will describe the behaviour of our two-fluid plasma in the region $0 \leq r \leq r_0$.

From our derivation we can see that (E4) and (E13) are the only possible sources of the purportedly specifically 'interior' nature of this equation. As we have pointed out however, these approximations become more true as the singular surface approaches the origin, and the important ordering in (E1), less true. It is clear, for instance, that the approximations (E4) and (E13), are not true if r_0 is situated near the plasma boundary (at r = a), especially if the plasma does not have a large aspect ratio*.

The arguments that we have presented in this appendix and in Appendix D, have shown that the process of dividing the plasma into 'interior' and 'exterior' regions is dubious. They have also shown that the equations derived in /1/, and intended to characterize the behaviour of the plasma in those regions, even given the overtly approximate nature of those equations, are of questionable legitimacy.

*aspect ratio = $\frac{R}{a}$

Appendix F Determination of $\hat{A}(\omega, r_0)$ and hence of $\Delta(\omega, r_0)$.

In equation (1.4-11) we saw that

$$-\hat{A}(\omega,r_0) = \frac{1}{r_0}\int_{r_0}^{a}\left[\frac{n_0'}{n_0}\left\{1+\frac{3T_{0i}}{5T_0}\left(\frac{nT_{0i}'}{n_0'T_{0i}}-\frac{2}{3}\right)\right\} + \frac{3\omega p_0'}{5\omega^* p_0} - \frac{k_{\parallel}^2 V_{th}^2 p_0'}{\omega\omega^* p_0}\right] s\,ds. \qquad (F1)$$

(1) (2) (3)

and from equation (1.2-44) we can write

$$\frac{p_0'}{p_0\omega^*} = \frac{eB_0^2}{cT_0\left(\frac{nB_{0\theta}}{R}-\frac{mB_{0z}}{r}\right)} \qquad (F2)$$

Thus if we use the approximation that $\frac{nB_{0\theta}}{R} \ll \frac{mB_{0z}}{r}$ as we have done in Appendix D and Appendix E ,and if we also assume that $B_{0z} \simeq B_0 \simeq$ constant ,then we can write

$$\frac{p_0'}{p_0\omega^*} \simeq -\frac{eB_0 r}{cT_0 m} \qquad (F3)$$

so that the second term of (F1) can then be written

$$\text{Term 2} = -\frac{3\omega}{5r_0}\int_{r_0}^{a}\frac{eB_0 s}{cT_0 m}\cdot s\,ds = -\frac{\omega e B_0 r_0^2}{5mcT_0}\left[\left(\frac{a}{r_0}\right)^3-1\right]$$

$$= -\omega\left(\frac{eB_0 r_0^2}{5mcT_0}\right)\Delta_2 \qquad (F4)$$

where Δ_2 is given by

$$\Delta_2 = \left[\left(\frac{a}{r_0}\right)^3-1\right] \qquad (F5)$$

We may also integrate the third term of (F1)

$$\text{Term } 3 = -\frac{1}{r_0}\int_{r_0}^{a} \frac{k_{\parallel}^2 v_{th}^2}{\omega}\left(\frac{P_0'}{P_0 \omega^*}\right) s\,ds$$

$$= \frac{1}{r_0}\left(\frac{eB_0}{cmT_0}\right)\int_{r_0}^{a} \frac{k_{\parallel}^2 v_{th}^2}{\omega} s^2 ds \qquad \text{(F6)}$$

If we now use the equation (D4) for $k_{\parallel}(r)$, suggested in /1/, in the above result, we find that

$$\text{Term } 3 = \frac{k_{\parallel}'^2 v_{th}^2}{r_0 \omega} \cdot \frac{eB_0}{cmT_0} \int_{r_0}^{a} s^2 (s - r_0)^2 ds$$

$$= \frac{C_3}{r_0} \int_{r_0}^{a} s^2 (s - r_0)^2 ds \qquad \text{(F7)}$$

so that by making the transformation

$$t = s/r_0 \qquad \text{(F8)}$$

we may derive the result

$$\text{Term } 3 = C_3 r_0^4 \int_{1}^{a/r_0} t^2 (t-1)^2 dt \qquad \text{(F9)}$$

where

$$C_3 = \frac{k_{\parallel}'^2 v_{th}^2}{\omega} \cdot \frac{eB_0}{cmT_0} = \left(\frac{\omega_{ci}}{\omega}\right)\frac{k_{\parallel}'^2}{m} \qquad \text{(F10)}$$

in which the relations

$$v_{th} = \sqrt{\frac{T_0}{m_i}} \quad , \quad \omega_{ci} = \frac{eB_0}{m_i c} \qquad \text{(F11)}$$

for the ion thermal velocity and the ion cyclotron frequency respectively, have been used.

If q(r) is the safety factor, defined by

$$q(r) = \frac{r\, B_o(r)}{R\, B_{o\theta}(r)} \qquad \text{(F12)}$$

it is easy to show that

$$k_{\parallel}'(r_0) = \frac{m}{R}\left(\frac{1}{q}\right)'_{r_0} \qquad \text{(F13)}$$

and through substituting this equation into equation (F1) we can find that C_3 is given by

$$C_3 = \frac{\omega_{ci}}{\omega} \cdot \frac{m}{R^2}\left(\frac{1}{q}\right)'^2 \qquad \text{(F14)}$$

If then we make the identification

$$\Delta_3 = \int_1^{a/r_0} t^2(t-1)^2 dt \qquad \text{(F15)}$$

we find that

$$\text{Term 3} = C_3 r_0^4 \Delta_3 = \frac{1}{\omega}\left[\frac{\omega_{ci} m r_0^4 \Delta_3}{R^2}\left(\frac{1}{q}\right)'^2\right] \qquad \text{(F16)}$$

Now, let us also make the identification

$$C_2 = \left[\frac{n_0'}{n_0}\left\{1 + \frac{3\bar{T}_{oi}}{5\bar{T}_o}\left(\frac{n_0 \bar{T}_{oi}'}{n_0' \bar{T}_{oi}} - \frac{2}{3}\right)\right\}\right]_{r_0} \qquad \text{(F17)}$$

We then find that in equation (F1), the first term has the value

$$\text{Term 1} = \frac{C_2}{r_0}\int_{r_0}^{a} s\, ds = \frac{C_2}{r_0}\left(\frac{a^2 - r_0^2}{2}\right)$$

$$= \Delta_1 \quad , \text{ say.} \qquad \text{(F18)}$$

However, according to equation (1.4-13)

$$\Delta(\omega, r_0) = \frac{-\omega}{1 + \hat{A}(\omega, r_0)} \qquad \text{(F19)}$$

and so from equations (F4), (F16) and (F18) we see that

$$1 + \hat{A}(\omega, r_0) = (1 + \text{Term 1}) + \text{Term 2} + \text{Term 3}$$

$$= K_1 + \omega K_2 + \frac{K_3}{\omega} \qquad \text{(F20)}$$

where

$$K_1 = 1 + \Delta_1 \qquad \text{(F21)}$$

$$K_2 = -\left(\frac{eB_0 r_0^2}{5mcT_0}\right)\Delta_2 \qquad \text{(F22)}$$

$$K_3 = \left[m\omega_{ci} r_0^2 \left(\frac{r_0^2}{R^2}\right)\left(\frac{1}{q}\right)'^2\right]\Delta_3 \qquad \text{(F23)}$$

Equation (F19) is reported in /1/ with the incorrect sign as equation (49). When equation (F19) is substituted into equation (D23) in order to derive the quartic dispersion relation, this sign error naturally feeds into the coefficients of ω. One effect of this that is significant is that equation (TH60) is only correct if there is a negative sign associated with Δ_3. Equation (TH60) is derived by considering the low-frequency limit of the quartic dispersion relation (equation TH66), in the situation in which $|u^2 + \Sigma| \ll 1$, and states that :

$$f^2 = \left[\frac{\frac{1}{2}\left(\frac{r_0}{R}\right)^2 r_0^2 \left(\frac{1}{q}\right)'^2}{1 + r_0^2 q^2 \left(\frac{1}{q}\right)'^2} \cdot \Delta_3 \cdot \left(\frac{\omega_A}{\omega_H}\right)^2\right](u^2 + \Sigma) \qquad \text{(F24)}$$

where f is a frequency normalised with respect to the eigenfrequency ω , u^2 is the continuum eigenvalue (see section 1.4) ,and all other quantities on the RHS of equation (F24) are real,and defined in /1/.

In /1/ it is claimed that equation (F24) implies instability if

$$u^2 + \Sigma < 0 \qquad \text{(F25)}$$

However,the sign error in Δ_3 means that this conclusion is incorrect,and that equation (F24) actually implies instability if

$$u^2 + \Sigma > 0 \qquad \text{(F26)}$$

Now,for the equilibrium profiles which we used in order to obtain the dispersion graph in figure (1.4-3), and which are presented in Table 1.4-1, the value of Σ that we obtained was

$$\Sigma = \phi.244 \qquad \text{(F27)}$$

Thus equation (F26) is stating that for those profiles,we would expect instability (i.e. the discrete spectrum) to exist for

$$u^2 > -\phi.244 \quad \text{or} \quad \lambda > -\phi.244 \qquad \text{(F28)}$$

This region of λ includes all positive values of λ ,so that what we have here as a direct consequence of equation (F26),is a contradiction of the results obtained in /1/ from analytic solution of equation (1.4-15).The only possible conclusion that we can draw from this,is that the approximations made in /1/ in order to reduce the two coupled equations to a single equation,are so severe that they render the subsequent formulation internally inconsistent. This inconsistency has not been noted in /1/ because of the sign error (Error III) ,which they have made.

Appendix G Asymptotic solutions of the coupled equations near r=0.

In this appendix we shall determine the behaviour of the solution functions of the basic coupled equations of /1/ (i.e. equations (1.4-1) and (1.4-2)),in the vicinity of the origin,by assuming that they will obey the following single term expansions :

$$\psi = a_0 x^{\rho} \quad ; \quad \Delta_p = b_0 x^{\nu} \tag{G1}$$

since the origin is a regular singular point of each of the coupled equations.In the above equation if $r=r_1$ is the small distance away from the origin at which the expansion is to apply,then x is defined by

$$x = r_1/a \tag{G1a}$$

We will now write down an expression for the value of the parallel component of the propagation vector ($K_{\parallel}(r)$) at the origin.In order to do this we identify as K_0 the following limit

$$K_0 = \lim_{r \to 0} \left(\frac{B_{0\theta}}{r} \right). \tag{G1b}$$

and since we know,from the z-component of the Ampere law in cylindrical coordinates,that

$$\frac{1}{r} \frac{d}{dr} (r B_{0\theta}) = \frac{4\pi}{c} j_{0z} \tag{G1c}$$

we can also write

$$K_0 = \frac{4\pi j_{\parallel}(0)}{2c} \tag{G1d}$$

or,in terms of the safety factor,which is defined by

$$q(r) = \frac{r B_{0z}}{R B_{0\theta}} \tag{G1e}$$

we can write :

$$K = \frac{B_{0z}}{R q(0)} \tag{G1f}$$

We are now in a position to write that

$$k_{\parallel}(0) = \lim_{r \to 0} \frac{1}{B_0}\left(\frac{m B_{0\theta}}{r} + \frac{n B_{0z}}{R}\right) = \frac{1}{B_0}\left(mK + \frac{n B_{0z}}{R}\right) \tag{G2}$$

and if we define the dimensionless parameter

$$\epsilon_1 = a/R \tag{G2a}$$

we can write

$$k_{\parallel}(0) = \frac{1}{a}\hat{k}_{\parallel}(0) \tag{G2b}$$

where

$$\hat{k}_{\parallel}(0) = \frac{1}{B_0}\left(maK + n\epsilon_1 B_{0z}\right) \tag{G2c}$$

We will also find it convenient to define the following quantity, a number which describes the ratio of the equilibrium kinetic plasma pressure, to the equilibrium magnetic pressure due to the toroidal magnetic field

$$\beta = P_0 \Big/ \left(\frac{B_0^2}{4\pi}\right) \tag{G2d}$$

It is also important to realise that since $B_{o\theta}$ is zero on axis, B_{oz} on axis is identified with the total magnetic field, B_o. For ease of reference, we will now write down here again the two coupled equations basic to the work of /1/ (i.e. equations (1.4-1) and (1.4-2)).

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r\psi}{k_{\parallel}}\right) = -\frac{1}{r}\frac{d}{dr}\left(\frac{r p_o \omega^*}{p_o' \omega}\cdot \Delta_p\right)$$

$$+\left[\frac{k_{\parallel}^2 V_{th}^2}{\omega^2} - \frac{3}{5} - \frac{p_o n_o' \omega^*}{p_o' n_o \omega}\left\{1 + \frac{3\bar{T}_{oi}}{5\bar{T}_o}\left(\frac{\bar{n}_o'\bar{T}_{oi}'}{n_o'T_{oi}} - \frac{2}{3}\right)\right\}\right]\Delta_p \qquad (G3)$$

$$\frac{d}{dr}\left[\frac{B_o k_{\parallel}}{4\pi k^2 r}\cdot\frac{d}{dr}(rB_o\psi) - \psi\left\{\frac{k_{\parallel}^2 p_o'}{k^2 k_{\parallel}} - \frac{j_{\parallel}}{ck^2}\left(\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right)\right\} - \frac{p_o k_{\parallel}^2}{k^2}\Delta_p\right]$$

$$+\frac{2B_{o\theta}}{4\pi r}\left[\frac{m^2}{k^2r^2}\frac{d}{dr}(rB_o\psi) + \frac{4\pi n}{k^2 R}\left\{\frac{j_{\parallel}}{c} + \frac{1}{B_o^2}\left(\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right)\frac{p_o'}{k_{\parallel}}\right\}\psi\right.$$

$$\left. - \frac{4\pi n p_o}{k^2 R B_o^2}\left(\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right)\Delta_p\right] - \frac{k_{\parallel}B_o^2\psi}{4\pi} + m_i n_o \omega\left(\frac{\omega\psi}{k_{\parallel}} + \frac{p_o\omega^*\Delta_p}{p_o'}\right) = 0 \qquad (G4)$$

We shall begin by considering the form of the terms of equation (G4) near $r=0$, recognising that near the origin we can write $k^2 = \frac{m^2}{r^2}$.

$$\text{Term 1} = \frac{d}{dr}\left\{\frac{B_o k_{\parallel}}{4\pi k^2}\cdot\frac{1}{r}\cdot\frac{d}{dr}(rB_o\psi)\right\}$$

$$= \left(\frac{B_o^2}{4\pi m^2 a}\right)\cdot \hat{k}_{\parallel}(o)\cdot\frac{d}{dx}\left\{x\frac{d}{dx}(x\psi)\right\} \qquad (G4a)$$

$$\text{Term 2} = \frac{d}{dr}\left[\left\{\frac{k_{\parallel}p_0'}{k^2} - \frac{j_{\parallel}}{ck^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\right\}\psi\right]$$

$$= -\frac{j_{\parallel}(0)B_0}{mc}\cdot\frac{d}{dx}(x\psi)$$

$$= -\frac{2m\epsilon_1}{q(0)}\left(\frac{B_0^2}{4\pi a m^2}\right)\cdot\frac{d}{dx}(x\psi) \qquad \text{(G4b)}$$

Note that here we have used the fact that the pressure gradient is zero on axis, and have neglected the last term in Term 2 because $B_{0\theta} \to 0$ as $r \to 0$. Also, as in Term 1 and subsequently, we assume that the equilibrium profiles (except $B_{0\theta}$) are constant over the small distance $0 \leq r \leq r_1$, so that equilibrium quantities are in general replaced by a constant equal to their value at $r=0$.

$$\text{Term 3} = \frac{d}{dr}\left\{-\frac{p_0 k_{\parallel}^2 \Delta_p}{k^2}\right\}$$

$$= -\frac{\hat{k}_{\parallel}^2(0)\,p_0(0)}{am^2}\frac{d}{dx}\left(x^2\Delta_p\right) \qquad \text{(G4c)}$$

$$\text{Term 4} = -\frac{B_0^2 k_{\parallel}\psi}{4\pi}$$

$$= -\frac{B_0^2\hat{k}_{\parallel}(0)\psi}{4\pi a} \qquad \text{(G4d)}$$

$$\text{Term 5} = \frac{2}{4\pi}\left(\frac{B_{0\theta}}{r}\right)\left[\frac{m}{k^2r^2}\cdot\frac{d}{dr}(rB_0\psi)\right]$$

$$= 2\left(\frac{B_0^2}{4\pi am}\right)\frac{\epsilon_1}{q(0)}\cdot\frac{d}{dx}(x\psi) \qquad \text{(G4e)}$$

$$\text{Term 6} = \frac{2}{4\pi}\left(\frac{B_{o\theta}}{r}\right)\frac{4\pi n}{k^2 R}\left\{\frac{j_{\parallel}}{c} - \frac{1}{B_o^2}\left(\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right)\frac{P_o'}{k_{\parallel}}\right\}\psi$$

$$= \left(\frac{2\epsilon_1^2 n B_{o\theta} x c^2}{m^2 c q(o)}\right) j_{\parallel}(o).\psi$$

$$= \left(\frac{B_o^2}{4\pi a m}\right)\left(\frac{4\epsilon_1^3 n}{q(o)^2}\right) x^2\psi \qquad \text{(G4f)}$$

$$\text{Term 7} = -\frac{2}{4\pi}\left(\frac{B_{o\theta}}{r}\right) P_o \cdot \frac{4\pi n}{k^2 R B_o^2}\left(\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right)$$

$$= -\left[\frac{2P_o(o)}{a\, q(o)\, m}\right].\epsilon_1^2.(x\Delta_p) \qquad \text{(G4g)}$$

$$\text{Term 8} = \frac{m_i n_o \omega^2}{k_{\parallel}}\psi$$

$$= \left(\frac{a m_i n_o(o)\omega^2}{\hat{k}_{\parallel}(o)}\right)\psi \qquad \text{(G4h)}$$

$$\text{Term 9} = m_i n_o \omega\left(\frac{P_o \omega^*}{P_o'}\right)\Delta_p = -m_i n_o \omega \Delta_p \cdot \frac{cT_o}{eB_o^2}\left(\frac{mB_{oz}}{r} - \frac{nB_{o\theta}}{R}\right)$$

$$= -n_o T_o \omega\, \Delta_p \left(\frac{m_i c}{eB_o}\right).\frac{m}{r}$$

$$= -\frac{m P_o(o)}{a}\left(\frac{\omega}{\omega_{ci}}\right).\left(\frac{\Delta_p}{x}\right) \qquad \text{(G4i)}$$

Thus we may state that near the origin equation (G4) looks like

$$\hat{k}_{\parallel}(o)\frac{d}{dx}\left[x\frac{d}{dx}(x\dot{\psi})\right] - \frac{2m\epsilon_1}{q(o)}\frac{d}{dx}(x\psi) - \hat{k}_{\parallel}^2(o)\beta\frac{d}{dx}(x^2\Delta_p) - m^2\hat{k}_{\parallel}(o)\psi$$

$$+\frac{2m\epsilon_1}{q(o)}\frac{d}{dx}(x\psi) + \frac{4mn\epsilon_1^3}{q(o)^2}x^2\psi - \frac{2m\epsilon_1^2}{q(o)}\beta.x\Delta_p$$

$$+\frac{m^2}{\hat{k}_{\parallel}(o)}\left(\frac{a\omega}{V_A}\right)^2\psi - m^3\left(\frac{\omega}{\omega_{ci}}\right)\beta.\frac{\Delta_p}{x} = 0 \qquad \text{(G5)}$$

after dividing by $\left(B_0^2/4\pi m^2 a\right)$. In equation (G5) V_A is the Alfven velocity,which is given by :

$$V_A = \frac{B_0}{(4\pi n_0 m_i)^{1/2}} \qquad \text{(G5a)}$$

It will also be noticed in equation (G5) that the second and fifth terms cancel.Substitution of equation (G1) into equation (G5) then gives :

$$\hat{k}_{\parallel}(0)\, a_0 (\rho+1)^2 x^{\rho} - \hat{k}_{\parallel}^2(0)\, b_0 (\nu+2)\beta\, x^{\nu+1}$$

$$- m^2 \hat{k}_{\parallel}(0)\, a_0 x^{\rho} + \frac{4mn\epsilon_1^3 a_0 x^{\rho+2}}{q(0)^2} - \frac{2\epsilon_1^2 m\beta b_0 x^{\nu+1}}{q(0)}$$

$$+ \frac{m^2}{k_{\parallel}(0)}\left(\frac{a\omega}{V_A}\right)^2 a_0 x^{\rho} - m^3\beta\left(\frac{\omega}{\omega_{ci}}\right) b_0 x^{\nu-1} = 0 \qquad \text{(G6)}$$

In this equation the most significant terms are those in and . Keeping only these we produce the equation :

$$a_0 \hat{k}_{\parallel}(0)\left[(\rho+1)^2 - m^2\right] x^{\rho} + \frac{m^2}{\hat{k}_{\parallel}(0)}\left(\frac{a\omega}{V_A}\right)^2 a_0 x^{\rho}$$

$$- m^3\beta\left(\frac{\omega}{\omega_{ci}}\right) b_0 x^{\nu-1} = 0 \qquad \text{(G7)}$$

We shall only be able to recover information about the asymptotic behaviour of both the functions ψ and Δ_p ,if the powers of x in this equation are equal.So we find that :

$$\rho = \nu - 1 \qquad \text{(G8)}$$

We shall now move on to consider the behaviour of equation (G3) near the origin, and again we do this by considering the individual terms of this equation. Thus we have

$$\text{Term 1} = \frac{1}{r}\cdot\frac{d}{dr}\left(\frac{r\psi}{k_{11}}\right)$$

$$= \frac{1}{\hat{k}_{11}(0)}\cdot\frac{1}{x}\cdot\frac{d}{dx}(x\psi) \qquad \text{(G3a)}$$

$$\text{Term 2} = -\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{P_0\omega^*}{\omega P_0'}\right)\Delta_P\right]$$

$$= -\frac{1}{r}\frac{d}{dr}\left[r\left\{-\frac{cT_0}{eB_0^2}\left(\frac{mB_{0z}}{r}-\frac{nB_{0\theta}}{R}\right)\right\}\frac{\Delta_P}{\omega}\right]$$

$$= m\left(\frac{r_L}{a}\right)^2\left(\frac{\omega_{ci}}{\omega}\right)\cdot\frac{1}{x}\cdot\frac{d\Delta_P}{dx} \qquad \text{(G3b)}$$

Now, if we assume that the two quantities

$$\left(\frac{n_0T_{oi}'}{n_0'T_{oi}}\right) \quad \text{and}$$

$$\left(\frac{P_0n_0'\omega^*}{P_0'n_0\omega}\right) \qquad \text{(G3c)}$$

are bounded at r=0, and we also write

$$\frac{k_{11}^2V_{th}^2}{\omega^2} = \left(\frac{r_L}{a}\right)^2\left(\frac{\omega_{ci}}{\omega}\right)^2\hat{k}_{11}(0) \qquad \text{(G3d)}$$

it will follow that equation (G3) can be written in the vicinity of the origin as :

$$\frac{1}{\hat{k}_{\parallel}(0)} \cdot \frac{1}{x} \frac{d}{dx}(x\psi) = m\left(\frac{r_L}{a}\right)^2 \left(\frac{\omega_{ci}}{\omega}\right) \frac{1}{x} \frac{d\Delta_p}{dx}$$

$$+ \text{(constant)} . \Delta_p \qquad \text{(G9)}$$

Substituting equation (G1) into this equation and keeping only the most significant terms ,we arrive at the equation :

$$\frac{a_0}{\hat{k}_{\parallel}(0)} \cdot (\rho+1)\, x^{\rho-1} = \left[m\left(\frac{r_L}{a}\right)^2 \frac{\omega_{ci}}{\omega}\right] \nu b_0 x^{\nu-2} \qquad \text{(G10)}$$

and if this equation is not to be trivial, the powers of x must be equal.Thus we find that

$$\rho - 1 = \nu - 2 \qquad \text{(G11)}$$

in agreement with equation (G8) .

Now,using equation (G11) in equation (G10) we find that

$$a_0 = \left[m\left(\frac{r_L}{a}\right)^2 \left(\frac{\omega_{ci}}{\omega}\right) \hat{k}_{\parallel}(0)\right] b_0 \qquad \text{(G12)}$$

We can then use this result to substitute for a_0 in the second term of equation (G7),in order to find that

$$\frac{m^2}{\hat{k}_{\parallel}(0)} \left(\frac{a\omega}{V_A}\right)^2 a_0 = m^3 \left(\frac{\omega}{\omega_{ci}}\right) \beta b_0 \qquad \text{(G13)}$$

and because of this result we realise that the last two terms of equation (G7) cancel,so that that equation now reads :

$$a_0 \hat{k}_{\parallel}(0) \left[(\rho+1)^2 - m^2\right] x^{\rho} = 0 \qquad \text{(G14)}$$

This means that the indicial equation obeyed by the coupled equations (equations (G3) and (G4)), is

$$(\rho+1)^2 - m^2 = 0 \qquad \text{(G15)}$$

and the two roots of this equation are :

$$\rho_1 = m-1$$

$$\rho_2 = -m-1 \qquad \text{(G16).}$$

The implication of this result is that no bound solutions exist at the origin for m=0 ,and that since the indicial roots have an integer difference (see /70/), one of the solutions near the origin is logarithmically unbounded for $|m| \geq 1$,while the other is bounded.

Appendix H Asymptotic solutions of the coupled equations near $r=r_o$

In this appendix we shall determine, using the method employed in Appendix G, the asymptotic behaviour of the solution functions to the basic coupled equations of /1/, in the vicinity of the resonant surface, $r=r_o$. We shall move a very small distance r_2 away from the resonant surface, and we shall define the dimensionless independent variable as

$$y = \frac{r_2}{a} = \frac{(r-r_o)}{a} \qquad \text{(H1)}$$

and since we know that $r=r_o$ is a regular singular point of the coupled equations, we shall assume that in the vicinity of the resonant surface the coupled equation solutions behave like

$$\psi = a_o y^{\mu} \quad ; \quad \Delta_p = b_o y^{\gamma} \qquad \text{(H1a)}$$

The equilibrium profile that we have to treat with care at this singularity is $k_{\parallel}(r)$. We know that

$$k_{\parallel}(r_o) = 0 \qquad \text{(H1b)}$$

and since $k_{\parallel}(r)$ is a perfectly well-behaved function we may Taylor expand it as follows

$$k_{\parallel}(r-r_o) = (r-r_o) \cdot \left.\frac{dk_{\parallel}}{dr}\right|_{r=r_o} \qquad \text{(H1c)}$$

We will then assume that all other equilibrium profile quantities do not vary much from their values at $r=r_o$, in the small distance r_2

that we shall move away from the resonant surface.We also define the dimensionless quantities :

$$\epsilon_2 = r_0/a \quad ; \epsilon_1 = a/R \tag{H1d}$$

so that we can write $k^2(r_0)$ as

$$k^2(r_0) = \hat{k}^2(0)/a^2 \tag{H1e}$$

where

$$\hat{k}^2(0) = \left(\frac{m^2}{\epsilon_2^2} + n^2\epsilon_1^2\right) \tag{H1f}$$

We also note that for economy of notation we shall write :

$$k_{\parallel}(y) = \frac{y}{a} \cdot k_{\parallel}^{(1)} \tag{H1g}$$

where

$$k_{\parallel}^{(1)} = \left.\frac{dk_{\parallel}(y)}{dy}\right|_{y=0} \tag{H1h}$$

We will now move on to consider the terms of the coupled equations (equations (G3) and (G4)) individually,as we did in Appendix G,starting with equation (G4).

$$\text{Term 1} = \frac{d}{dr}\left\{\frac{B_0 k_{\parallel}}{4\pi k^2} \cdot \frac{1}{r} \cdot \frac{d}{dr}(rB_0\psi)\right\}$$

$$= \left(\frac{B_0^2}{4\pi a}\right) \cdot k_{\parallel}^{(1)} \cdot \frac{d}{dy}\left\{y\frac{d\psi}{dy}\right\} \tag{H2}$$

$$\text{Term 2} = \frac{d}{dr}\left[\left\{\frac{k_{11}P_0'}{k^2} - \frac{j_{11}}{ck^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\right\}\psi\right]$$

$$= -\frac{\epsilon_2}{ma}\cdot P_0^{(1)}\cdot\frac{d\psi}{dy} \qquad \text{(H2a)}$$

$$\text{Term 3} = -\frac{d}{dr}\left[\frac{P_0 k_{11}^2}{k^2}\cdot\Delta_P\right]$$

$$= -\frac{P_0(r_0)}{a}\cdot\frac{k_{11}^{(1)^2}}{\hat{k}^2}\cdot\frac{d}{dy}\left[y^2\Delta_P\right] \qquad \text{(H2b)}$$

$$\text{Term 4} = -\frac{B_0^2 k_{11}\psi}{4\pi} = -\frac{B_0^2}{4\pi a}\cdot k_{11}^{(1)}\cdot(y\psi) \qquad \text{(H2c)}$$

$$\text{Term 5} = \frac{2}{4\pi}\left(\frac{B_{0\theta}}{r}\right)\frac{m}{k^2r^2}\frac{d}{dr}(rB_0\psi) = -\left(\frac{B_0^2}{4\pi a}\right)\cdot\frac{2n\epsilon_1}{\epsilon_2\hat{k}^2}\cdot\frac{d\psi}{dy} \qquad \text{(H2d)}$$

$$\text{Term 6} = \frac{2}{4\pi}\left(\frac{B_{0\theta}}{r}\right)\frac{4\pi n}{k^2R}\left[\frac{j_{11}}{c} + \frac{1}{B_0^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\frac{P_0'}{k_{11}}\right]\psi$$

$$= -\frac{2n^2\epsilon_1^2\epsilon_2}{m^2 k_{11}^{(1)}}\cdot\left[\frac{P_0^{(1)}}{a}\right]\cdot\frac{\psi}{y} \qquad \text{(H2e)}$$

$$\text{Term 7} = -\frac{2}{4\pi}\left(\frac{B_{0\theta}}{r}\right)\frac{4\pi n P_0}{k^2RB_0^2}\left(\frac{mB_{0z}}{r} - \frac{nB_{0\theta}}{R}\right)\Delta_P$$

$$= \frac{2an^2\epsilon_2 P_0(r_0)\epsilon_1^2}{m^2 a}\cdot\Delta_P \qquad \text{(H2f)}$$

$$\text{Term 8} = \frac{m_i n_0 \omega^2 \psi}{k_{\parallel}} = \frac{m_i n_0(r_0)\omega^2 a}{k_{\parallel}^{(1)}}\left(\frac{\psi}{y}\right) \tag{H2g}$$

$$\text{Term 9} = m_i n_0 \omega\left(\frac{P_0 \omega^*}{P_0'}\right) = a\, m_i \omega n_0(r_0)\left[\frac{P_0(r_0)\omega^*}{P_0^{(1)}}\right]\Delta_P \tag{H2h}$$

It is now clear that near the resonant point equation (G4) looks like :

$$Z k_{\parallel}^{(1)} \frac{d}{dy}\left\{y \frac{d\psi}{dy}\right\} - \frac{\epsilon_2 P_0^{(1)} d\psi}{ma\, dy} - \left[\left(\frac{P_0}{a}\right).\frac{k_{\parallel}^{(1)^2}}{\hat{k}^2}\right]\frac{d}{dy}(y^2\Delta_P)$$

$$- Z k_{\parallel}^{(1)} y\psi - \left\{\frac{Z.2n\epsilon_1}{\epsilon_2 \hat{k}^2}\right\}\frac{d\psi}{dy} - \left(\frac{2n^2\epsilon_1^2\epsilon_2}{m^2 k_{\parallel}^{(1)}}.\frac{P_0^{(1)}}{a}\right)\frac{\psi}{y}$$

$$+\left(\frac{2n^2\epsilon_1^2\epsilon_2 P_0}{m^2 a}\right)\Delta_P + \left(\frac{m_i n_0 \omega^2 a}{k_{\parallel}^{(1)}}\right)\frac{\psi}{y} + \left(\frac{a m_i n_0 P_0 \omega^* \omega}{P_0^{(1)}}\right)\Delta_P = 0 \tag{H3}$$

where we have put

$$Z = \frac{B_0^2}{4\pi a} \tag{H4}$$

Now if we substitute equation (H1a) into equation (H3) and keep only the most significant terms in x, we find that :

$$Z k_{\parallel}^{(1)} a_0 \mu^2 y^{\mu-1} - \epsilon_2 P_0^{(1)} a_0 \mu y^{\mu-1} - \frac{2nZ\epsilon_1 a_0 \mu y^{\mu-1}}{\epsilon_2 \hat{k}^2}$$

$$- \frac{2n^2\epsilon_1^2\epsilon_2 P_0^{(1)} a_0 y^{\mu-1}}{m^2 k_{\parallel}^{(1)} a} + \frac{2n^2\epsilon_1^2\epsilon_2 P_0 b_0 y^{\gamma}}{m^2 a}$$

$$+ \frac{m_i n_0 a \omega^2 a_0 y^{\mu-1}}{k_{\parallel}^{(1)}} + \frac{a m_i n_0 P_0 \omega^* \omega\, b_0 y^{\gamma}}{P_0^{(1)}} = 0 \tag{H5}$$

As in Appendix G, in order for this equation to return information about the asymptotic behaviour of both the solution functions ψ and Δ_p ,we require that the powers of y shall be equal.Thus :

$$\gamma = \mu - 1 \qquad \text{(H6)}$$

We now turn our attention to the second of the two coupled equations (equation (G3)).For this equation

$$\text{Term 1} = \frac{1}{r}\frac{d}{dr}\left(\frac{r\psi}{k_{\parallel}}\right) = \frac{1}{k_{\parallel}^{(1)}}\frac{d}{dy}\left(\frac{\psi}{y}\right) \qquad \text{(H7a)}$$

$$\text{Term 2} = -\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{p_0\omega^*}{\omega p_0'}\right)\Delta_p\right] = -\frac{p_0\omega^*}{p'\omega}\left(\frac{d\Delta_p}{dy}\right) \qquad \text{(H8a)}$$

$$\text{Term 3} = \text{constant} \times \Delta_p \qquad \text{(H8b)}$$

Substituting the expansions in equation (H1a) for ψ and Δ_p into the above equations we find that equation (G3) implies that

$$\frac{1}{k_{\parallel}^{(1)}} \cdot a_0(\mu-1)y^{\mu-2} = \frac{p_0\omega^*}{p_0^{(1)}\omega}\gamma b_0 y^{\gamma-1} \qquad \text{(H9)}$$

where we have kept only the most significant terms in y . If this equation is not to be trivial we must have equality of the powers of y. Thus :

$$\gamma - 1 = \mu - 2 \qquad \text{(H10)}$$

which is in agreement with equation (H6).Thus,using equation (H10) in equation (H9), we are able to find that

$$a_0 = -k_{\parallel}^{(1)} \cdot \frac{p_0 \omega^*}{p_0^{(1)} \omega} \cdot b_0 \qquad \text{(H11)}$$

and substituting this equation into equation (H5), thereby eliminating a_0 ,we find that the last two terms of equation (H5) cancel,.We also find that the fourth term in equation (H5) is equal to

$$\frac{2n^2 \epsilon_1^2 \epsilon_2}{m^2 a} \cdot \left(\frac{p_0 \omega^*}{\omega} \right) b_0 \qquad \text{(H12)}$$

With the help of this fact we can write equation (H5) as follows :

$$\left(k_{\parallel}^{(1)} \cdot Z \right) \mu^2 - \left(\frac{\epsilon_2 p_0^{(1)}}{ma} \right) \mu - \left(\frac{2n \epsilon_1 Z}{\epsilon_2 \hat{k}^2} \right) \mu$$

$$+ \frac{2n^2 \epsilon_1^2 \epsilon_2 p_0}{m^2 a} \left(\frac{\omega^*}{\omega} + 1 \right) = 0 \qquad \text{(H13)}$$

or,in terms of r-derivatives :

$$a k_{\parallel}' \left(\frac{B_0^2}{4\pi} \right) \mu^2 - \left[\frac{p_0' r_0}{ma} + \frac{2n \left(\frac{B_0^2}{4\pi} \right)}{aR k^2 r_0} \right] \mu$$

$$+ \frac{2n^2 r_0 p_0}{m^2 R^2} \left(1 + \frac{\omega^*}{\omega} \right) = 0 \qquad \text{(H14)}$$

If it is the case that we are dealing with a low-β plasma ,then the second term in the square brackets in equation (H14) is much greater than the first,and so within this approximation we may write equation (H14) as

$$a k_{\parallel}' \left(\frac{B_0^2}{4\pi} \right) \mu^2 - \left[\frac{2n \left(\frac{B_0^2}{4\pi} \right)}{aR k^2 r_0} \right] \mu + \frac{2n^2 r_0 p_0}{m^2 R^2} \left(1 + \frac{\omega^*}{\omega} \right) = 0 \qquad \text{(H15)}$$

Now,if we assume that the following relations are true

$$B_{0\theta} = 0.1 B_{0z}$$

$$r_0 \sim a$$

$$\frac{dB_{0\theta}}{dr} \sim \frac{B_{0\theta}}{r_0}$$

$$R = 5a$$

$$B_{0z} \sim \text{constant} \qquad \text{(H16)}$$

then it will be true that

$$\frac{\left(\frac{2n^2 r_0 p_0}{m^2 R^2}\right)}{ak_{\parallel}'\left(\frac{B_0^2}{4\pi}\right)} \sim 0.08\beta\frac{n^2}{m^3} \qquad \text{(H17)}$$

so that unless $\omega \ll \omega^*$,it will be true that the last term of equation (H15) will be very much less than the first.This very low frequency range is not interesting,if we are concerned with instabilities that will grow rapidly enough to destroy the confinement of the plasma before useful amounts of fusion energy can be derived from it.Thus we can write equation (H15) as :

$$ak_{\parallel}'\left(\frac{B_0^2}{4\pi}\right)\mu^2 - \left[\frac{2n\,(B_0^2/4\pi)}{aRk^2 r_0}\right]\mu = 0 \qquad \text{(H18)}$$

The two roots of this indicial equation are :

$$\mu_1 = 0$$

$$\mu_2 = \frac{2n}{a^2 R r_0 k^2 k_{\parallel}'} \qquad \text{(H19)}$$

Our arguments have made it clear that the indicial roots of equation (H15) have only very weak dependence on ω for $\omega \gtrsim \omega^*$.

The solution of equation (H15) is plotted in frequency space as figure (1.6-1).

If for completeness we decide to examine the case $\omega \to 0$, we will find that the two roots of the indicial equation (equation (H15)), are nearly equal, and are given by

$$\mu_{1,2} = \pm \left\{ \frac{-c_1}{c_2 k_{11}'(r_0)\omega} \right\}^{1/2} \qquad \text{(H20)}$$

where

$$c_1 = \frac{2n^2 r_0 \rho_0(r_0) \omega^*(r_0)}{m^2 R^2}$$

$$c_2 = \frac{a B_0^2(r_0)}{4\pi} \qquad \text{(H21)}$$

These indicies are either purely real or purely imaginary, as $k_{11}'(r_0)$ is < 0 or > 0, respectively. In the first case, since the indices are two real, non-integer numbers, that are not equal, one of them corresponds to a bound solution at $r=r_0$, and the other corresponds to an unbound solution. In the second case, since both indicies are purely complex, they both imply bound solutions in ψ that oscillate infinitely rapidly in space as $r \to r_0$. However, because of equation (H10) both the corresponding solutions in Δ_p would be unbound. We should point out here that in the first case discussed above, what is true of ψ is also true of Δ_p, because the indicies are very large real numbers (i.e. much greater than unity). Thus, according to this model, as we approach the case of marginal stability ($\omega \to 0$) the possibility of the plasma being unstable will exist only if

$$k_{11}'(r_0) < 0 \qquad \text{(H22)}$$

In equation (H21) it must be borne in mind (see equation (1.2-44)) that ω^* is assumed to be a positive number, because it is assumed that in general (p_0'/ρ_0) will be a negative number.

Appendix I Numerical initial-value integration methods with automatic error-control

The most rudimentary numerical method of integrating the equation :

$$\frac{dy}{dx} = f(x,y) \qquad (I1)$$

subject to the initial condition

$$y(x_o) = y_o \qquad (I2)$$

is the Euler method.If the spatial domain is divided into discrete ordinates x_i ($x_{i+1} - x_i = h$, $i=1,2,3,.....$),then the Euler method uses the statement

$$\hat{y}_{i+1} = y_i + h.f(x_i,y_i) \qquad (I3)$$

as an approximation for the exact expression

$$y(x_{i+1}) = y_i + \int_{x_i}^{x_{i+1}} f[x,y(x)]\, dx \qquad (I4)$$

The meanings of equations (I3) and (I4) are illustrated in figure (I.1).

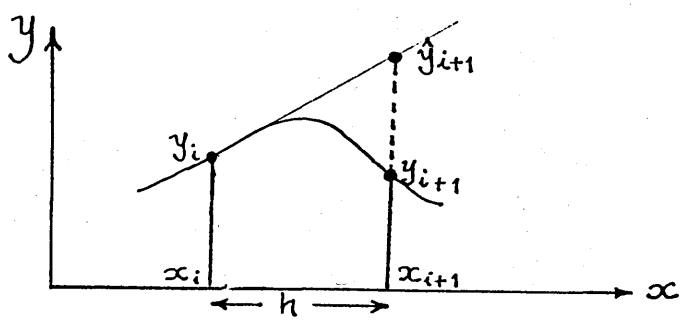


Figure I.1 A step of Euler's method.

Euler's method has a truncation error of only $O(h^2)$. (The truncation error is the error which has been introduced into the computation by the discretisation of the continuous space dimension,into bins of finite width, h.) A method that has truncation error $O(h^3)$ is Heun's method, which,in the present notation can be written :

$$\hat{y}_{i+1} = y_i + \frac{h}{2}.\left[f(x_i,y_i) + f(x_{i+1},y_{i+1})\right] \qquad (I5)$$

The problem here is that y_{i+1} is not known. We may therefore use Euler's method to predict its value, thereby obtaining the value $y_{i+1(p)}$. With this estimate we might reasonably hope to produce a better value for y_{i+1}, i.e.

$$y_{i+1(c)} = y_i + \tfrac{1}{2}.h.\left[f(x_i,y_i) + f(x_{i+1},y_{i+1(p)})\right] \qquad (I6)$$

Thus, Euler's method has been used as a <u>predictor</u> for y_{i+1}, and Heun's method as a <u>corrector</u> for y_{i+1}.

If we use the Euler predictor and the Heun corrector, then the truncation error in the Heun corrector is

$$\varepsilon_{i+1(c)} = \left(\frac{h^3}{12}\right)y_i''' + \frac{h}{2}.K_{i+1}.\varepsilon_{i+1(p)} + O(h^4) \qquad (I7)$$

($\varepsilon_{i+1(p)}$: predictor error)

where the contribution of the Euler predictor has been indicated. In equation (I7), K_{i+1} is given by

$$K_{i+1} = \frac{\partial f}{\partial y}\left(x_{i+1}, \zeta_{i+1}\right) \qquad (I8)$$

where

$$y_{i+1(p)} \leq \zeta_{i+1} \leq y_{i+1} \qquad (I9)$$

Thus, if we can find a predictor with truncation error $O(h^3)$, the error in the Heun corrector, due to the predictor error, could be made $O(h^4)$. Such a predictor is the <u>modified Euler's method</u>, whose expression is :

$$y_{i+1(p)} = y_{i-1} + 2h\,y_i' \qquad (I10)$$

If we use this predictor, the truncation error in the Heun corrector will be :

$$\varepsilon_{i+1(c)} = \frac{h^3}{12} y_i''' + O(h^4) \qquad (I11)$$

It is shown in /72/ that the truncation error of the modified Euler's method is

$$\varepsilon_{i+1(p)} \equiv y_{i+1(p)} - y_{i+1} = -\frac{h^3}{3} y_i''' + O(h^4) \qquad (I12)$$

and in the same way we can write the truncation error in the Heun corrector as

$$\varepsilon_{i+1(c)} \equiv y_{i+1(c)} - y_{i+1} = \frac{h^3}{12} y_i''' + O(h^4) \qquad (I13)$$

From these last two equations we are able to see that

$$y_{i+1(c)} - y_{i+1(p)} = \frac{5h^3}{12} y_i''' + O(h^4) \qquad (I14)$$

If we then substitute equation (I11) into equation (I14) we will find that

$$\frac{1}{5}\left[y_{i+1(c)} - y_{i+1(p)} \right] = \varepsilon_{i+1(c)} + O(h^4) \qquad (I15)$$

Since both the predictor and corrector have truncation error $O(h^3)$, it will be possible for us, through a process known as mopping-up, to produce from equations (I12) and (I13), an approximation for y_{i+1} (which we shall call $y_{i+1(m)}$), that has truncation error $O(h^4)$. Thus

$$\frac{4}{5} \cdot \text{equation (I13)} + \frac{1}{5} \cdot \text{equation (I12)} = \frac{1}{5}\left(4 y_{i+1(c)} + y_{i+1(p)}\right)$$

$$= y_{i+1(m)} + O(h^4) \qquad (I16)$$

We do not know after mopping-up, what exactly the truncation error in $y_{i+1(m)}$ is, but we do know that it is in general less than the truncation error of the corrector. Thus equation (I15) is a conservative estimate of the error in $y_{i+1(m)}$. This proceedure of computing an error

estimate from values computed in the method, rather than by evaluating formulas not closely related computationally to the solution process, is called automatic error estimation. It allows significant reduction in the computational time spent on error-control, since only quantities which have to be computed anyway in the computational process, are used. Notice that mop-up, and thus automatic error control, are only possible because the predictor and the corrector have the same order truncation error.

For the results presented in chapter 2, the so-called Adams-Bashforth predictor

$$y_{i+1(p)} = y_i + h.[55y_i' - 59y_{i-1}' + 37y_{i-2}' - 9y_{i-3}']/24 \qquad (I17)$$

and the so-called Adams-Moulton corrector

$$y_{i+1(c)} = y_i + \left(\frac{h}{24}\right).[9y_{i+1}' + 19y_i' - 5y_{i-1}' + y_{i-2}'] \qquad (I18)$$

have been used. The truncation error in the Adams-Bashforth predictor is

$$y_{i+1(p)} - y_{i+1} \equiv \varepsilon_{i+1(p)} = -\left(\frac{251}{720}\right)h^5 y_i^{(5)} + O(h^6) \qquad (I19)$$

and that of the Adams-Moulton corrector is

$$y_{i+1(c)} - y_{i+1} \equiv \varepsilon_{i+1(c)} = \left(\frac{19}{720}\right)h^5 y_i^{(5)} + O(h^6) \qquad (I20)$$

Since the predictor and corrector errors are of opposite sign (as in our previous example), then for small enough h, the predictor and corrector values will straddle the correct value. If we subtract equation (I19) from equation (I20), we obtain

$$\begin{aligned} y_{i+1(c)} - y_{i+1(p)} &= \left(\frac{270}{720}\right)h^5 y_i^{(5)} + O(h^6) \\ &= \left[\frac{19}{720}.h^5 y_i^{(5)}\right]\frac{270}{19} + O(h^6) \end{aligned} \qquad (I21)$$

so that our automatic error estimate is

$$\varepsilon_{i+1(c)} \simeq \frac{19}{270}\left[y_{i+1(c)} - y_{i+1(p)}\right] \qquad \text{(I22)}$$

We can then mop-up to make the truncation error $O(h^6)$, through use of the formula

$$y_{i+1(m)} = \left[251\, y_{i+1(c)} + 19\, y_{i+1(p)}\right]/270 \qquad \text{(I23)}$$

Now, suppose that the truncation error comitted per step of length h, does not vary much as the integration proceeds from $x=x_o$ to $x=x_N$. Suppose also that over this range we are prepared to tolerate a total error TOL. We would then be able to compute an upper bound on the permissible truncation error per integration step, i.e.

$$\left|\frac{\varepsilon_{step}}{h}\right| < \frac{TOL}{|x_N - x_o|} \qquad \text{(I24)}$$

We can now present the overall solution algorithm :

1. Compute the upper bound on the allowable truncation error per integration step (C in equation (I24)).

2. Find the correct step length h, in the following way :

   (a) Guess a value $h=h_1$

   (b) Do three steps of a Runge-Kutta integration with $h=h_1$

   (c) Apply the Adams method for 1 step with $h=h_1$, and compute the automatic error estimate, E. (i.e. the estimate of ε_{step}). Since the actual error is $O(h_1^6)$,

   $|\varepsilon_{step}/h_1|$ is $O(h_1^5)$, i.e. $\frac{E}{h_1} = Kh_1^5$

   where K is some constant. We remember that we are looking for a step h such that

   $$C = Kh^5 \qquad \text{i.e.} \qquad C = \left(\frac{E}{h_1^6}\right).h^5$$

Therefore we find

$$h = \left(\frac{Ch_1^6}{E}\right)^{1/5}$$

3. Using the h determined in (2), do 3 Runge-Kutta steps to compute $\hat{y}_1, \hat{y}_2, \hat{y}_3$, and set the space index to $i=3$.

4. Using the present h, do 1 step of the Adams method to compute $\hat{y}_{i+1}$, and compute the automatic error estimate E.

5. If $|E| > C|h|$ go to (6).
If $|E| < C|h|/32$ go to (7).

6. Halve h. Assuming that $\hat{y}_{i+1}$ is the value just computed and that the subscripts are in terms of the old h value, use the Runge-Kutta method with the new h value starting at x_i to compute $\hat{y}_{i+1/2}$, and the Runge-Kutta method starting at x_{i-1} to compute $\hat{y}_{i-1/2}$. Increment i and go to (4) with the new h value. i.e. start from x_{i+1} to compute the next $\hat{y}$ value.

7. Double h. Increment i and go to (4). To allow this interval doubling, at all steps we must save not only $\hat{y}_i, \hat{y}'_i, \hat{y}_{i-1}, \hat{y}'_{i-1}, \hat{y}'_{i-2}$ and the new value $\hat{y}_{i+1}$, but also $\hat{y}'_{i-3}, \hat{y}_{i-3}, \hat{y}'_{i-4}, \hat{y}_{i-4}, \hat{y}'_{i-5}, \hat{y}_{i-5}$.

We may fail to arrive exactly at x_N by this method. In this case we can use the Runge-Kutta method to compute the last fractional step.

Stability Considerations

In the numerical solution of an ODE a sequence of approximations y_n to the true solution $y(x_n)$ is generated. Roughly speaking, the stability of a numerical method refers to the behaviour of the difference or error, $y_n - y(x_n)$, as n becomes large. There are broadly, two categories of errors which are incurred during the integration process. One of these, which we have already mentioned above, is the truncation error, and this is the error introduced by the approximation of the differential equation by a difference equation. In the other category are the errors due to the deviation of the numerical solution from the exact theoretical solution

of the difference equation.Included in this class are round-off errors, and errors which are incurred if the difference equation is implicit, and not solved exactly at each step. An additional source of error results from the use of an auxiliary method (as is the case for the Adams-Bashforth-Moulton method) to develop 'starting' values for the difference equation. Now,in order to proceed,we must define the important concept of consistency. Thus,the difference approximation to the ODE is consistent if the sequence of numbers y_n , converges uniformly to a continuous function $y(x)$,as n becomes very large (or the step $h \to 0$). We can also say that the difference approximation to the ODE is consistent if it is true that

$$\frac{\text{truncation error}}{h} \longrightarrow 0 \quad , \quad \text{as} \quad h \longrightarrow 0 \quad .$$

We can now use Lax's equivalence theorem /98/ to give formal meaning to the concept of stability. This theorem states that :

> " Given a properly posed linear initial-value problem and a finite difference approximation to it that satisfies the consistency condition,stability is the necessary and sufficient condition for convergence. "

Now,suppose that we wish to integrate the ODE

$$y' = \lambda y \quad , \qquad y_o = 1 \tag{I25}$$

Use of the Euler method would lead us to the expression :

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda)y_n \tag{I26}$$

Now,assuming that the error E_n at the step n,leads to error E_{n+1} ,at the step n+1 ,then the errors at these two adjacent levels are related by the equation :

$$E_{n+1} = (1 + h\lambda)E_n \tag{I27}$$

In order to solve this equation,we substitute into it the trial solution :

$$E_n = \mu^n \tag{I28}$$

and obtain as a result,the equation :

$$\mu^{n+1} = (1+h\lambda)\mu^n \tag{I29}$$

which is called the 'characteristic equation' of equation (I27) . The solution of this characteristic equation is

$$\mu = (1 + h\lambda) \tag{I30}$$

It follows from the above equations that we can write

$$E_n = (1 + h\lambda)^n \tag{I31}$$

and it is clear from this equation that E_n is bounded as $n \to \infty$, only if

$$|1 + h\lambda| < 1 \tag{I32}$$

We are not however,normally interested in integrating only a single ODE,but rather a system of coupled ODEs. We can write such a system in matrix notation as follows :

$$\underline{y}' = \underline{\underline{A}}\,\underline{y} \qquad , \quad \underline{y}(0) = \underline{y}_o \tag{I33}$$

where the elements of the matrix $\underline{\underline{A}}$ are denoted by a_{ij} and are assumed constant.The formal solution of equation (I33) is

$$\underline{y}(x) = \underline{y}_o \exp(\underline{\underline{A}}\, x) \tag{I34}$$

and it is clear that if λ_{max} is the largest eigenvalue of the matrix $\underline{\underline{A}}$, the characteristic equation of equation (I34) will enable us to state that the condition for the stability of equation (I34), if the Euler method is used,is :

$$|1 + h\lambda_{max}| < 1 \tag{I35}$$

According to equation (I34),the exact solution will change over a step h, by the factor $\exp(\underline{\underline{A}}h)$, i.e.

$$y(x_{n+1}) = y(x_n) \exp(\underline{\underline{A}}h) \tag{I36}$$

so that any single-step method which can be expressed as

$$\underline{y}_{n+1} = \underline{\underline{M}}(\underline{\underline{A}}h).\,\underline{y}_n \tag{I37}$$

has properties which are essentially determined by how well $\underline{\underline{M}}(\underline{\underline{Ah}})$ approximates $\exp(\underline{\underline{Ah}})$. It is shown in /99/ that for the single-step method of equation (I37), the largest eigenvalue of $\underline{\underline{A}}$ still determines the choice of h for the stable integration of equation (I33).

Our main interest however, is to determine the stability of numerical methods for the integration of the nonlinear ODE

$$\underline{y}' = \underline{f}(x,\underline{y}) \qquad (I38)$$

Given this equation, the way to proceed at any step n, would be to form a Taylor series about $(\underline{y}_n, x_n)$, truncated after the first-order terms,

$$\underline{y}' = \underline{f}_n + [\underline{\underline{f}}_{\underline{y}}]_n (\underline{y} - \underline{y}_n) + O(h^2) \qquad (I39)$$

giving a linear non-homogeneous ODE at each step. The eigenvalues of equation (I39) are the eigenvalues of the Jacobian matrix $[\underline{\underline{f}}_{\underline{y}}]_n$, and will vary from step to step.

Suppose that the largest eigenvalue of the Jacobian matrix at the point n is $(\lambda_{max})_n$. Stable integration of equation (I38) could then conceivably be achieved in three ways :

1. For each $(\lambda_{max})_n$ choose the largest permissible h (say h_n), consistent with stability, or
2. Find the largest of the set of numbers $(\lambda_{max})_n$, and use this to determine a single constant h, which will then be used for the integration of equation (I38) throughout the domain of interest.
3. Use the method of (2) not over the whole domain, but over fixed blocks of steps.

The above procedures are reasonable because it has been shown in /92/ and /100/, that the stability characteristics of equation (I39) are very similar to those of equation (I38).

In the solution algorithm presented earlier in this appendix, it was indicated that the size of the first step taken could be guessed, but we need to know how to make a reasonable guess. For the fourth-order Adams predictor-corrector (P-C) formulas (equations (I17) and (I18)), typical

characteristic equations are (see p.184 of /99/) :

$$\mu^4 - \left(1 + \frac{55}{24}h\lambda\right)\mu^3 + \frac{59}{24}h\lambda\cdot\mu^2 - \frac{37}{24}h\lambda\cdot\mu + \frac{9}{24}h\lambda = 0 \qquad \text{(I40)}$$

$$\left(1 - \frac{9}{24}h\lambda\right)\mu^3 - \left(1 + \frac{19}{24}h\lambda\right)\mu^2 + \frac{5}{24}h\lambda\cdot\mu - \frac{h\lambda}{24} = 0 \qquad \text{(I41)}$$

$$\mu^4 - \left(1 + \frac{7}{6}h\lambda + \frac{55h^2\lambda^2}{64}\right)\mu^3 + \left(\frac{5h\lambda}{24} + \frac{59h^2\lambda^2}{64}\right)\mu^2$$

$$- \left(\frac{h\lambda}{24} + \frac{37}{64}h^2\lambda^2\right)\mu + \frac{9h^2\lambda^2}{64} = 0 \qquad \text{(I42)}$$

$$\mu^5 - \left(1 + \frac{7h\lambda}{6} + h^2\lambda^2\right)\mu^4 + \left(\frac{5h\lambda}{24} + \frac{95h^2\lambda^2}{64}\right)\mu^3$$

$$- \left(\frac{h\lambda}{24} + \frac{91h^2\lambda^2}{64}\right)\mu^2 + \frac{45h^2\lambda^2}{64}\cdot\mu - \frac{9h^2\lambda^2}{64} = 0 \qquad \text{(I43)}$$

corresponding respectively to the P,C,PECE and $PE(CE)^2$ algorithms. What is meant by PECE is the following :

1. Using the predicted value $y_{n+1(P)}$, a derivative evaluation is performed to obtain $y'_{n+1} = f(x_{n+1}, y_{n+1(P)})$. 2. With this new derivative the corrector is used to obtain a new estimate of the solution $y_{n+1(C)}$. This value is then used in another derivative evaluation.

For the $PE(CE)^2$ algorithm the above procedure is followed by another application of the corrector equation, and then by another derivative evaluation. In figure (I.2) we have illustrated the stability diagram in the complex $h\lambda$-plane , for the Adams-Moulton method in the PECE mode, as given by Krogh /101/. It is seen that this algorithm is stable if

$$\mathrm{Re}(h\lambda) < (-\lambda h)_{calc} \quad , \quad \text{where} \quad (\lambda h)_{calc} \sim 1.25 \qquad \text{(I44)}$$

Thus, in order to determine the upper bound h_{max}, on the step size, we would first of all linearise our system of nonlinear ODEs, in order to determine the Jacobian matrix. The largest negative eigenvalue of this matrix can then be calculated (for instance by the 'power method',/72/). A bound on h_{max} then follows from

$$h_{max} \leqslant \frac{(-\lambda h)_{calc}}{|\lambda_{max}|} \qquad (I45)$$

and all computations must operate within this bound. It would then be reasonable to choose our first integration step conservatively, for instance, such that

$$h_{first} = \frac{h_{max}}{1.1} \qquad (I46)$$

Within this framework, our computation is then adjusted for accuracy, as is done in the solution algorithm presented earlier.

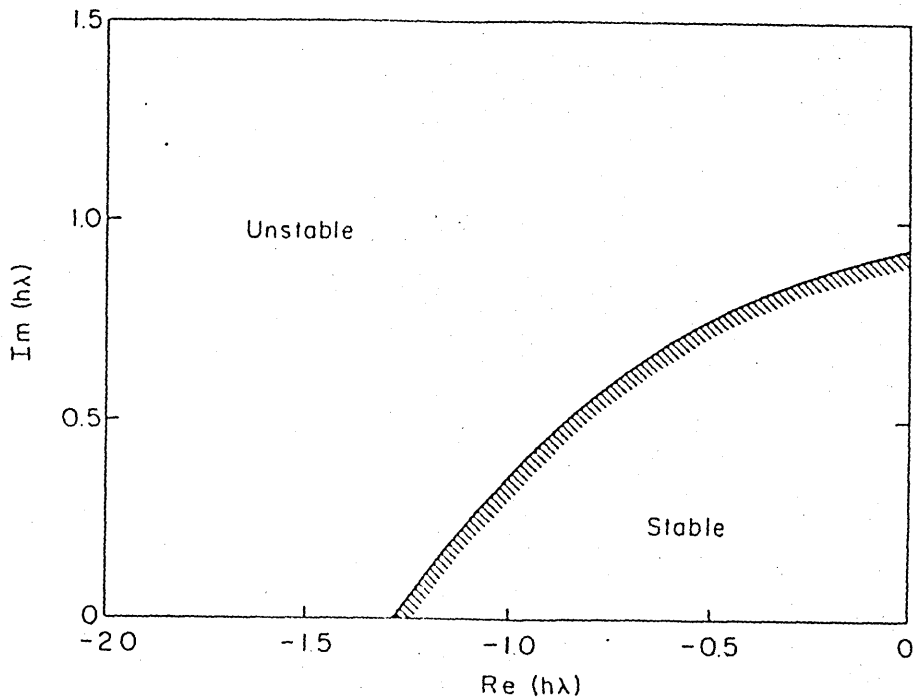


Figure I.2 Region of stability in complex $h\lambda$ plane for fourth-order Adams P–C combination. [Adapted and Reprinted from F. T. Krogh, *Journal of the ACM*, Volume 13, pp. 374–385; copyright © 1966, Association for Computing Machinery, Inc.]

The reader who finds our argument from equation (I25) onwards hard to follow, is referred to Crane and Klopfenstein /102/. There it is shown that the investigation of the stability of the stability of a numerical method of solving equation (I38)

$$\underline{y}' = \underline{f}(x, \underline{y})$$

can be reduced to the study of the stability of that particular numerical method when applied to equation (I25)

$$y' = \lambda y$$

for different values of λ. The values of λ to be considered are the eigenvalues of the Jacobian matrix

$$\underline{\underline{f}}_{\underline{y}} = \frac{\partial \underline{f}}{\partial \underline{y}} \qquad \text{(I47)}$$

which may in general be complex.