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Turbulence Cascade in an Inhomogeneous Turbulent Flow

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I hereby certify that all work presented in this thesis is my own and that all other contributions have been appropriately acknowledged and referenced.

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To my grandfather, friend and forever an inspiration, Gilberto

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Abstract

The inhomogeneous, anisotropic turbulence downstream of a square prism is investigated by means of direct numerical simulations (DNS) and two-point statistics. As noted by Moffatt (2002) “it now seems that the intense preoccupation [...] with the problem of homogeneous isotropic turbulence was perhaps misguided” acknowledging there is now a revived interest in studying inhomogeneous turbulence.

The full description of the turbulence cascade requires a two-point analysis which revolves around the recently derived Kármán-Howarth-Monin-Hill equation (KMH). This equation is the inhomogeneous/anisotropic analogue to the so-called Kolmogorov equation (or Kármán-Howarth equation) used in Kolmogorov’s 1941 seminal papers (K41) which are the foundation to the most successful turbulence theory to date.

Particular focus is placed on the near wake region where the turbulence is anticipated to be highly inhomogeneous and anisotropic. Because DNS gives direct access to all velocity components and their derivatives, all terms of the KMH can be computed directly without resorting to any simplifications. Computation of the term associated with the non-linear inter-scale transfer of energy (Π) revealed that this rate is roughly constant over a range of scales which increases (within the bounds of our database) with distance to the wake generator, provided that the orientations of the pairs of points are averaged-out on the plane of the wake. This observation appears in tandem with a near $-5/3$ power law in the spectra of fluctuating velocities which deteriorates as the constancy of Π improves. The constant non-linear inter-scale transfer plays a major role in K41 and is required for deriving the $2/3$ -law (which is real space equivalent of the $-5/3$).

We extend our analysis to a triple decomposition where the organised motion associated with the vortex shedding is disentangled from the stochastic motions which do not display a distinct time signature. The imprint of the shedding-associated motion upon the stochastic component is observed to contribute to the small-scale anisotropy of the stochastic motion. Even though the dynamics of the shedding-associated motion differs drastically from that of the stochastic one, we find that both contributions are required in order to preserve the constant inter-scale transfer of energy. We further find that the inter-scale fluxes resulting from this decomposition display local (in scale-space) combinations of direct and inverse cascades. While the inter-scale fluxes associated with the coherent

motion can be explained on the basis of simple geometrical arguments, the stochastic motion shows a persistent inverse cascade at orientations normal to the centreline despite its energy appearing to be roughly isotropically distributed.

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Nomenclature

Acronyms

CFL	Courant-Friedrichs-Lewy number
$\mathcal{CV}$	Control volume in the context of FVM
$\partial\mathcal{CV}$	Surface bounding $\mathcal{CV}$
DNS	Direct Numerical Simulation
FVM	Finite Volume Method
HIT	Homogeneous Isotropic Turbulence
HPC	High Performance Computing
K41	Kolmogorov's theory of turbulence
KHMH	Kármán-Howarth-Monin-Hill equation (see eq. (3.1))
LES	Large Eddy Simulation
PDF	Probability Density Function
PIV	Particle Image Velocimetry
RANS	Reynolds-Averaged Navier-Stokes

Greek Letters

$\delta\tilde{q}^2$	Generalised second-order structure function associated with the coherent motions given by $\delta\tilde{q}^2 = \delta\tilde{u}_i\delta\tilde{u}_i$
δq^2	Generalised second-order structure function given by $\delta q^2 = \delta u_i\delta u_i$
$\delta q'^2$	Generalised second-order structure function associated with the stochastic motions given by $\delta q'^2 = \delta u'_i\delta u'_i$

$\delta \mathbf{u}, \delta u_i$	Two-point velocity difference vector $\delta \mathbf{u}$ with components δu_i
$\delta \mathbf{u}', \delta u'_i$	Two-point stochastic component of the velocity difference vector $\delta \mathbf{u}'$ with components $\delta u'_i$
$\delta \tilde{\mathbf{u}}, \delta \tilde{u}_i$	Two-point phase averaged (or coherent) component of the velocity difference vector $\delta \tilde{\mathbf{u}}$ with components $\delta \tilde{u}_i$
δu_r	Component of the two-point velocity difference vector $\delta \mathbf{u}$ in the direction $\mathbf{r}$
ϵ	Instantaneous turbulent dissipation $\nu \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}$
ε	Average turbulent dissipation $\langle \epsilon \rangle$
$\tilde{\varepsilon}$	Average dissipation due to the coherent motions
ε'	Average dissipation due to the stochastic motions
$\varepsilon_r, \varepsilon'_r, \tilde{\varepsilon}_r$	Two-point dissipation of the fluctuating $(\frac{\varepsilon^+ + \varepsilon^-}{2})$, stochastic $(\frac{(\varepsilon')^+ + (\varepsilon')^-}{2})$ and coherent $(\frac{\tilde{\varepsilon}^+ + \tilde{\varepsilon}^-}{2})$ motions, respectively
η	Kolmogorov length scale
ϕ	Phase angle
κ	One-dimensional wavenumber
λ	Taylor micro-scale given by $\mathcal{U} \sqrt{15 \frac{\nu}{\varepsilon}}$ in an isotropic context
λ_{ij}	Generalised Taylor micro-scale given by the curvature of the correlation ρ_{u_i} along x_j (see eq. (4.12))
ν	Kinematic viscosity of the fluid
Π	Non-linear inter-scale energy transfer in eq. (3.1)
Π'	Non-linear interscale transfer of the energy associated with stochastic motions by the stochastic motions appearing in eq. (4.14)
Π'_u	Non-linear interscale transfer of the energy associated with stochastic motions by the coherent motions appearing in eq. (4.14)
$\Pi_{u'}, \tilde{\Pi}_{\mathcal{P}_u}$	Non-linear interscale transfer of the total fluctuating energy by the stochastic motions resulting from $\Pi' + \tilde{\Pi}_{\mathcal{P}_u}$, the latter appearing in eq. (4.15)

$\tilde{\Pi}_{\tilde{u}}$	Non-linear interscale transfer of the energy associated with coherent motions by the coherent motions appearing in eq. (4.15)
$\Pi_U, \Pi'_U, \tilde{\Pi}_U$	Two-point linear inter-scale transfer of δq^2 , $\delta q'^2$ and $\delta \tilde{q}^2$, respectively
ψ	Arbitrary variable (not necessarily a scalar)
ρ_ψ	Two-point, two-time correlation for an arbitrary variable ψ (see eq. (3.7))
Θ_ψ	Integral time scale associated with the variable ψ (see eq. (4.10))
$\mathcal{T}'_{\tilde{u}}$	Non-linear physical-space transport of the energy associated with stochastic motions by the coherent motions appearing in eq. (4.14)
$\mathcal{T}'_{u'}$	Non-linear physical-space transport of the energy associated with stochastic motions by the stochastic motions appearing in eq. (4.14)
$\mathcal{T}_{u'}, \tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$	Non-linear physical-space transport of the total fluctuating energy by the stochastic motions resulting from $\mathcal{T}'_{u'} + \tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$, the latter appearing in eq. (4.15)
$\tilde{\mathcal{T}}_{\tilde{u}}$	Non-linear physical-space transport of the energy associated with coherent motions by the coherent motions appearing in eq. (4.15)

Operators

$\langle \psi \rangle$	Average of some variable ψ (not necessarily a scalar) used in general as the average in time and any homogeneous spatial direction (see eq. (2.21))
ψ^a	Orientation averaged value (see fig. 3.14) of some arbitrary variable ψ defined in scale space
$\{\psi\}$	Conditional average of some variable ψ based on the reference phase angle ϕ
$\mathcal{F}, \mathcal{F}^{-1}$	Direct and inverse Fourier transforms, respectively
$\mathcal{H}$	Hilbert transform (see eq. (4.5))
ψ^-	Variable ψ evaluated at $\mathbf{x}^- = \mathbf{X} - \frac{1}{2}\mathbf{r}$
ψ^+	Variable ψ evaluated at $\mathbf{x}^+ = \mathbf{X} + \frac{1}{2}\mathbf{r}$

Roman Letters

$\mathcal{A}, \mathcal{A}', \tilde{\mathcal{A}}$	Two-point advection by the mean flow of δq^2 , $\delta q'^2$ and $\delta \tilde{q}^2$, respectively
$\mathcal{A}_t, \mathcal{A}'_t, \tilde{\mathcal{A}}_t$	Unsteadiness (in time) of δq^2 , $\delta q'^2$ and $\delta \tilde{q}^2$, respectively

C_D	Drag coefficient
C_ε	Normalised dissipation coefficient given by $\varepsilon \frac{\mathcal{L}}{U^3}$
C_L	Lift coefficient
C_P	Mean pressure coefficient (see eq. (2.20))
d	Free-stream length scale, e.g. characteristic size of a bluff body
$\mathcal{D}_r, \mathcal{D}'_r, \tilde{\mathcal{D}}_r$	Two-point viscous diffusion in scale-space of δq^2 , $\delta q'^2$ and $\delta \tilde{q}^2$, respectively
$\mathcal{D}_x, \mathcal{D}'_x, \tilde{\mathcal{D}}_x$	Two-point viscous diffusion in physical-space of δq^2 , $\delta q'^2$ and $\delta \tilde{q}^2$, respectively
E_ψ	Power spectrum density of an arbitrary variable ψ
F_ψ	Flatness of the variable ψ (see eq. (4.8))
f_s	Vortex shedding frequency
k	Kinetic energy associated with the fluctuating velocity ($k = \frac{1}{2} u_i u_i$)
$\mathcal{L}$	Largest length scale of the turbulence, e.g. integral length scale
$\mathcal{L}_{ij}$	Integral length scale obtained by integrating ρ_{u_i} along x_j (see eq. (4.11))
$\mathcal{L}_{\parallel}$	Integral length scale computed from the integral of ρ_{u_1} along x_2
$\mathcal{L}_{\perp}$	Integral length scale computed from the integral of ρ_{u_1} along x_1
$\bar{p}$	Fluid pressure divided by density and $\bar{p} = P + p$
$\tilde{p}$	Phase averaged (or coherent) component of the fluctuating pressure given by $\tilde{p} = p - p'$
p'	Stochastic component of the fluctuating pressure given by $p' = p - \tilde{p}$
$\mathcal{P}_U, \mathcal{P}'_U, \tilde{\mathcal{P}}_U$	Two-point production by the mean flow of δq^2 , $\delta q'^2$ and $\delta \tilde{q}^2$, respectively
$\mathcal{P}'_u$	Two-point production of $\delta q'^2$ by the coherent motions
$\mathbf{r}, r_i$	Relative position between $\mathbf{x}^+$ and $\mathbf{x}^-$ representing a location in scale-space with coordinates r_i
r	Length scale given by the magnitude of $\mathbf{r}$
Re_∞	Reynolds number $Re_\infty = \frac{U_\infty d}{\nu}$

Re_λ	Local Reynolds number given by $Re_\lambda = \frac{u\lambda}{\nu}$
S_ψ	Skewness of some variable ψ (see eq. (4.7))
St	Strouhal number corresponding to $\frac{f_s d}{U_\infty}$
$\mathcal{T}_p, \mathcal{T}'_p, \tilde{\mathcal{T}}_p$	Two-point pressure-velocity correlation for the fluctuating, stochastic and coherent components, respectively, of the velocity and pressure
$\bar{\mathbf{u}}$	Fluid velocity with components $\bar{u}_i$ and $\bar{\mathbf{u}} = \mathbf{U} + \mathbf{u}$
$\mathbf{U}, U_i$	Mean velocity vector $\mathbf{U}$ with components U_i given by $\mathbf{U} = \bar{\mathbf{u}} - \mathbf{u}$
$\mathbf{u}, u_i$	Fluctuating velocity vector $\mathbf{u}$ with components u_i given by $\mathbf{u} = \bar{\mathbf{u}} - \mathbf{U}$
$\mathbf{u}', u'_i$	Stochastic component of the fluctuating velocity $\mathbf{u}'$ with components u'_i given by $\mathbf{u}' = \mathbf{u} - \tilde{\mathbf{u}}$
$\mathcal{U}$	Characteristic velocity scale of the turbulence, e.g. standard deviation of u_i
$\tilde{\mathbf{u}}, \tilde{u}_i$	Phase averaged (or coherent) component of the fluctuating velocity $\tilde{\mathbf{u}}$ with components $\tilde{u}_i$ given by $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}'$
U_∞	Free-stream reference velocity scale
$\mathbf{X}, X_i$	Mid-point between $\mathbf{x}^+$ and $\mathbf{x}^-$ representing a location in physical-space with coordinates X_i

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1 Introduction

Turbulence appears to arise ubiquitously in nature and has been a subject of interest and research by many people. The first recorded study of turbulence is usually attributed to Leonardo da Vinci in the 15th century (Gad-El-Hak, 2006; Macagno, 1986). Even as the equations governing the motion of fluids became available in the 19th century a complete understanding of turbulence is still lacking, as noted in the well-known anecdote (usually attributed to both Horace Lamb and Werner Heisenberg) comparing turbulence to quantum mechanics.

Most progress in describing turbulence has relied heavily on three assumptions that should hold in the statistical sense (Batchelor, 1953): that the turbulence is homogeneous, such that its properties are invariant under translations in space, that it is isotropic, such that its properties are invariant under rotations and that it is in equilibrium, such that the energy input is balanced by energy dissipated instantaneously (see §3.1 of Orszag, 1974). In fact, the Richardson-Kolmogorov cascade, a phenomenological description of turbulence which is arguably the most successful turbulence theory to date, was formulated precisely under the assumptions of homogeneity and isotropy. The predictions made by Kolmogorov, however, appear to hold in flows which do not fall under the three aforementioned assumptions, as noted by Kraichnan (1974): “Kolmogorov’s 1941 theory has achieved an embarrassment of success.”

It appears that, in an attempt to either support or disprove Kolmogorov’s theory, most fundamental studies of turbulence have focused in homogeneous (and usually also isotropic) turbulence. Such focus now appears to have been misguided (Davidson, Kaneda, et al., 2011), if not because turbulence in nature is more often than not inhomogeneous, but because by restricting research only to homogeneous flows we have neglected a part of the physics which may actually be essential to the description of turbulence.

We shall discuss Kolmogorov’s theory and its limitations in section 1.1 along with a brief discussion of its strengths and shortcomings. In section 1.2 we present a brief overview on the theory of planar wakes; even though the focus of the present work lies on the near wake, a discussion of Townsend’s contribution to the field is essential. Finally, an outline of this thesis is given in section 1.3.

1.1 An Overview of Turbulence Theory

“The single biggest problem of turbulence research is that we really have no exact solutions.” George (2017). George is referring to the (incompressible) Navier-Stokes equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \quad (1.1a)$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} \quad (1.1b)$$

where $\bar{u}_i$ are the components of $\bar{\mathbf{u}}$ the fluid velocity and $\bar{p}$ is the pressure (divided by density), at some time t and location $\mathbf{x}$ and ν is the kinematic viscosity of the fluid. If d and U_∞ are, respectively, representative length and velocity scales of the flow as a whole, then at high enough Reynolds number

$$Re_\infty = \frac{U_\infty d}{\nu} \quad (1.2)$$

the non-linear term of eq. (1.1b) dominates over the viscous one, i.e. inertial forces dominate over viscous ones, and the flow becomes turbulent.

While eq. (1.1) contains all the information necessary to describe the motions of fluids, the equations remain unclosed. As shown by Keller and Friedmann (1925), a hierarchy of equations arises from eq. (1.1b) as the non-linear term successively introduces higher order moments of $\bar{\mathbf{u}}$ as one multiplies eq. (1.1b) by the velocity. Without a solution for eq. (1.1), any success in modelling turbulence must involve some sort of closure for some moment of $\bar{\mathbf{u}}$ (Orszag, 1970). Furthermore, notice that the viscous term in eq. (1.1b) results in a loss of energy through viscous dissipation

$$\nu \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_j}$$

which keeps the total energy of the flow from piling up. Away from any boundaries, one can immediately foresee that the presence of turbulent motions gives rise to large gradients responsible for dissipating energy (Onsager, 1949; Tennekes and Lumley, 1972). We shall return to this (very) important property of the turbulence below.

The apparent randomness of the turbulence naturally calls for a statistical treatment (Batchelor, 1947; Kraichnan and S. Chen, 1989; Onsager, 1949; Townsend, 1976). Let us decompose the velocity into its mean $\mathbf{U}$ and fluctuating $\mathbf{u}$ components (and likewise for the pressures P and p) as introduced by O. Reynolds (1895). We use $\langle \cdot \rangle$ the process of averaging (e.g. ensemble averaging) which is able to commute with any derivative.

If one is interested in the mean velocity profile (see the discussion by Narasimha, 1990), its equation can be obtained by introducing $\mathbf{U} + \mathbf{u}$ in eq. (1.1b) and taking the average of the whole equation. However, the resulting equation (for the mean velocity) would require some knowledge of the turbulence due to the presence of the Reynolds' stresses $\langle u_i u_j \rangle$. While enormous effort has been devoted to modelling the Reynolds' stresses with varying degrees of success (see e.g. Pope, 2000; Tennekes and Lumley, 1972), this problem highlights the need of understanding turbulence. One way to circumvent this issue is to use of the equation for the turbulent kinetic energy $k = \frac{1}{2} u_i u_i$

$$\left\langle \frac{\partial k}{\partial t} \right\rangle + \left\langle u_i \frac{\partial k}{\partial x_i} \right\rangle = - \left\langle u_i \frac{\partial p}{\partial x_i} \right\rangle - \left\langle u_i u_j \frac{\partial U_i}{\partial x_j} \right\rangle + \left\langle \nu \frac{\partial^2 k}{\partial x_j \partial x_j} \right\rangle - \left\langle \nu \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle \quad (1.3)$$

which makes clear that the role of the Reynolds' stresses is to inject energy into the turbulence by interacting with the mean flow gradient (George, 2013).

For the simplest (possibly unrealistic) flow that is statistically stationary and homogeneous, eq. (1.3) reduces to a balance between the production of $\langle k \rangle$ at the large scales (over which $\mathbf{U}$ varies) given by $-\langle u_i u_j \frac{\partial U_i}{\partial x_j} \rangle$ and its dissipation

$$\varepsilon = \langle \epsilon \rangle = \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle \quad (1.4)$$

arising from the finest details of the turbulence, as described in the well known poem by Richardson (1922).

Thus, any hope of understanding turbulence must involve two ingredients: the mechanism responsible for transferring energy between scales and the turbulent dissipation.

1.1.1 Kolmogorov's Description of Turbulence

The first version of Kolmogorov's theory, introduced in Kolmogorov (1941a), was purely phenomenological and as it echoed the cascade idea proposed by Richardson (1922) it is often referred to as Richardson-Kolmogorov equilibrium cascade. Kolmogorov's theory is often referred to as K41 due to the publications (Kolmogorov, 1941a; Kolmogorov, 1941b; Kolmogorov, 1941c) where the theory was developed.

At the centre of this phenomenological description of turbulence lies the idea that there is a range of scales (called inertial range) in which turbulent eddies are so large that they are unaffected by viscosity, but still small enough that they do not feel the effects of the mean flow. This means that the dynamics of those eddies is governed solely by their non-linear interactions and the rate ε at which energy is dissipated. As a consequence, both the distribution of energy and its rate of transfer across scales have well-defined laws.

However, in Kolmogorov (1941c) it was successfully shown that the results obtained in Kolmogorov (1941a) could be derived directly from the Navier-Stokes equations by introducing the assumptions of local homogeneity, local isotropy and local stationarity (or equilibrium). These assumptions are local in scale-space, i.e. Kolmogorov assumes that inertial range eddies are homogeneous, isotropic and stationary regardless of the large scale details of the turbulence.

The success of K41 cannot be understated, in the words of Kraichnan (1974) “Kolmogorov’s 1941 theory has achieved an embarrassment of success. The $-5/3$ -spectrum has been found not only where it reasonably could be expected but also at Reynolds numbers too small for a distinct inertial range to exist and in boundary layers and shear flows where there are substantial departures from isotropy, and such strong effects from the mean shearing motion that the step-wise cascade appealed to by Kolmogorov is dubious.”

Kolmogorov’s work relies on describing turbulent eddies by means of structure functions, corresponding to the moments of the velocity differences (see Davidson, 2004)

$$\delta \mathbf{u}(\mathbf{X}, \mathbf{r}, t) = \mathbf{u}(\mathbf{X} + \frac{1}{2}\mathbf{r}, t) - \mathbf{u}(\mathbf{X} - \frac{1}{2}\mathbf{r}, t). \quad (1.5)$$

Due to the assumptions of homogeneity and isotropy, the moments of $\delta \mathbf{u}(\mathbf{X}, \mathbf{r}, t)$ are fully described by a single component of $\delta \mathbf{u}$ e.g. that aligned with $\mathbf{r}$,

$$\delta u_r(r, t) = (\mathbf{u}(\mathbf{x} + \frac{1}{2}\mathbf{r}, t) - \mathbf{u}(\mathbf{x} - \frac{1}{2}\mathbf{r}, t)) \cdot \frac{\mathbf{r}}{r} \quad (1.6)$$

as shown by von Kármán (1937) and von Kármán and Howarth (1938) in the context of two-point correlation functions (which are of course related to structure functions). In eq. (1.6) $r = \|\mathbf{r}\|$.

The second and third order moments of eq. (1.6) represent, roughly speaking, the energy contained in eddies of characteristic size r or less and the flux of energy across eddies of size r , respectively, as discussed in Davidson (2004), Davidson and Pearson (2005), Frisch (1995), and Townsend (1976).

The velocity difference defined in eq. (1.6) appears in the constitutive relation used in Kolmogorov (1941c)

$$\frac{\partial \langle \delta u_r^3 \rangle}{\partial r} + \frac{4}{r} \langle \delta u_r^3 \rangle = 6\nu \left(\frac{\partial^2 \langle \delta u_r^2 \rangle}{\partial r^2} + \frac{4}{r} \frac{\partial \langle \delta u_r^2 \rangle}{\partial r} \right) - 4\epsilon \quad (1.7)$$

which corresponds to the equation introduced by von Kármán and Howarth (1938) for the two-point correlations expressed in terms of eq. (1.6). Equation (1.7) is equivalent to the equation for the spectrum (written in Fourier space) derived by Lin (1949).

This equation relates the second and third order moments of δu_r , for homogeneous isotropic turbulence in equilibrium. The two main results of Kolmogorov's theory are that, for separations r within the inertial range, the second and third moments of δu_r are universal and follow

$$\langle \delta u_r^2 \rangle = C_K \varepsilon^{2/3} r^{2/3} \quad (1.8)$$

$$\langle \delta u_r^3 \rangle = -\frac{4}{5} \varepsilon r, \quad (1.9)$$

where C_K is a universal constant. Equation (1.9) follows from multiplying eq. (1.7) by r^4 and integrating it, then neglecting the viscous term; by further assuming the skewness of δu_r to be constant, eq. (1.8) is obtained.

Equation (1.7) thus provides the mathematical description of the Richardson-Kolmogorov equilibrium cascade (Kolmogorov, 1941a; Richardson, 1922). The distribution of energy among the different scales is given by eq. (1.8) and is related to the $-5/3$ slope of the power spectrum of the fluctuating velocity as shown in Obukhov (1941) and Obukhov and Yaglom (1953). Equation (1.9) can be interpreted as a flux of energy across eddies of size r and the fact that $\frac{\partial \langle \delta u_r^3 \rangle}{\partial r} = \text{const}$ means that rate of transfer of that energy is constant, i.e. there is no pile-up or depletion of energy due to non-linearities. Equation (1.9) is of particular relevance as it remains the only result in turbulence without adjustable constants that can be derived directly from the Navier-Stokes equations with some basic assumptions on the structure of the turbulence (Frisch, 1995). Furthermore, eq. (1.9) is ultimately related to the so called zeroth law of turbulence (Burattini et al., 2005; Pearson et al., 2004; Rollin et al., 2011) in which the normalised dissipation coefficient

$$C_\varepsilon = \varepsilon \frac{\mathcal{L}}{\mathcal{U}^3}, \quad (1.10)$$

is constant. In eq. (1.10) $\mathcal{L}$ and $\mathcal{U}$ are length and velocity scale of the largest motions in the turbulence (e.g. integral length scale and standard deviation of the fluctuating velocity, respectively). Equation (1.10) was firstly proposed by Taylor (1935) and also obtained by Kolmogorov (1941b).

The negative sign in eq. (1.9) indicates that the step-wise cascade occurs from large to small scales. Whilst this picture of the energy cascade certainly has its weaknesses (see George, 1992; Kraichnan, 1974; Kraichnan, 1991; Mazellier and Vassilicos, 2010; Shen and Warhaft, 2000; Vassilicos, 2015; Yakhot, 2006, and references therein), the fact remains that the $-5/3$ spectrum can be found in a variety of flows much broader than those

which satisfy the basic assumptions of K41 (see the extensive discussion in Kraichnan, 1974). Observing (1.9) is on the other hand less trivial due to experimental/numerical limitations, but there is evidence in support of both the scaling and the coefficient (see Hunt and Vassilicos, 1991).

Kolmogorov's students quickly recognised the shortcomings of eq. (1.7), in the sense that the equations could not possibly describe the turbulent motions in highly inhomogeneous anisotropic flows. Monin and Yaglom (1975) proposed a modification to eq. (1.7) that accounted for anisotropy in which, using

$$\delta q^2 = \delta u_i \delta u_i, \quad (1.11)$$

one obtains

$$\frac{\partial \langle \delta u_i \delta q^2 \rangle}{\partial r_i} = 2\nu \frac{\partial^2 \langle \delta q^2 \rangle}{\partial r_i \partial r_i} - 4\varepsilon. \quad (1.12)$$

This is to the so called Kármán-Howarth-Monin equation, as its derivation is largely attributed to Monin (Monin and Yaglom, 1975).

The assumptions of homogeneity and equilibrium are still present in eq. (1.12) through the absence of derivatives in physical-space $\mathbf{X}$ (recall eq. (1.5)) and the absence of a derivative in the direction over which the turbulence develops (time and/or advective terms) as noted by Lindborg (1999) (but see also eqs. (2.4) and (5.2) in Batchelor, 1947). While eqs. (1.8) and (1.9) could still be obtained by averaging over all orientations of $\mathbf{r}$, the question of inhomogeneous turbulence remained open.

To the author's knowledge, a first attempt at tackling the problem of inhomogeneous turbulence is given in Rotta (1951) who wrote the equation for the two-point correlation function without the assumption of homogeneity or isotropy. However, it was only after the work of Hill (1997) and Hill (2001) that the necessary tools to tackle the problem of inhomogeneous, anisotropic and unsteady turbulence became available, as will be discussed in chapter 3.

As a final remark, we should note that Kolmogorov's work consists of describing the turbulence in real-space. Even though the Fourier space was used in Obukhov (1941), who obtained the $-5/3$ spectrum and Lin (1949) who derived the equivalent of eq. (1.7), a description of inhomogeneous turbulence in Fourier space is not possible. Davidson (2004), Davidson and Pearson (2005), and Davidson and Krogstad (2008) provide interesting discussions (beyond the scope of this thesis) on the advantages and disadvantages of each representation.

1.1.2 Idealised vs. Real Turbulence

It is perhaps not surprising that the idealised turbulence under which Kolmogorov's theory is formulated has not much (if anything) to do with turbulence in nature. This is recognised in the opening paragraphs of Batchelor (1953). Nevertheless, the idea of homogeneous isotropic turbulence (HIT) was the only one for which a theory had been developed. As such, much effort has been invested in creating high Reynolds number flows that were as homogeneous and isotropic as possible (Batchelor, 1947; Moffatt, 2002). From Kolmogorov's assumptions, that of isotropy is perhaps the one that has received most attention since (see e.g. Elsinga and Marusic, 2016; Lindborg, 1996b). For example, George and Hussein (1991) proposed a theory for axisymmetric homogeneous turbulence.

Experimentally, HIT is usually achieved by passing a fluid through an array of bars or grid. Numerically, it is common practice to solve eq. (1.1) with the addition of a source term in eq. (1.1b) and use fully periodic boundary conditions. In general, the resulting velocity fields are approximately Gaussian and the resulting turbulence is, to some extent, homogeneous (Davidson, 2004; Frisch, 1995). The pursuit to generate the idealised HIT, however, is impaired on both accounts by *i*) experimental challenges in generating high Reynolds number homogeneous turbulence, *ii*) numerical difficulty in the resolution requirements imposed by high Re_∞ turbulence and *iii*) memory effects associated with how the turbulence is generated as a consequence of the proximity from the turbulence generator in experiments and the details of the forcing term in numerical simulations.

While we have already established that the $-5/3$ spectrum appears in a wide variety of flows (even if there is no reason to anticipate that it should), the verification of the dissipation scaling given by eq. (1.10), also a consequence of Kolmogorov's theory, has been less successful. In general, the inconsistency of eq. (1.10) is attributed to issues *i*), *ii*) and/or *iii*) above (see, e.g. Antonia and Pearson, 2000; Burattini et al., 2005; George, 2017; Sreenivasan, 1998). In recent years, however, it has become clear that the variability of C_ε between different sources is not only a consequence of experimental/numerical difficulties, but also the neglect of the non-stationary nature of the turbulence (see Vassilicos, 2015, and references therein).

As discussed in Vassilicos (2015), by measuring C_ε as the turbulence develops, it will first vary according to a non-equilibrium law until it becomes constant, at a flow-dependent value. The non-equilibrium dissipation law is given by

$$C_\varepsilon \propto \frac{Re_\infty^m}{Re_\lambda^n} \quad (1.13)$$

with $m \approx 1 \approx n$, where Re_∞ is a Reynolds number based on the initial conditions and Re_λ is the local Reynolds number. The latter is given by

$$Re_\lambda = \frac{\mathcal{U}\lambda}{\nu} \quad (1.14)$$

with $\mathcal{U}$ being a velocity scale of the turbulence (such as the standard deviation of some component of $\mathbf{u}$) and λ is the Taylor micro-scale introduced in Taylor (1935) which is related to the gradient of $\mathbf{u}$.

The implications of eq. (1.13) are discussed at length in Vassilicos (2015). We shall simply note two points: first, that as the turbulence decays, it becomes more and more out of equilibrium and second, that eq. (1.13) is most definitely present in regions of the flow where the $-5/3$ spectrum is observed over a broad range of wavenumbers. The former point means that the constant value reached by C_ε should not be anticipated on the arguments given by Kolmogorov; the latter means that whatever mechanism is responsible for transferring energy between scales, it is most certainly not that of Richardson-Kolmogorov's cascade.

It is important to note that non-equilibrium turbulence has been reported in a variety of flows other than grid turbulence, such as boundary layers, axisymmetric wakes, jets and periodic box turbulence (see e.g. Discetti et al., 2013; Goto and Vassilicos, 2016b; Nedić, Tavoularis, et al., 2017; Obligado et al., 2016); some of these flows are not only inhomogeneous, anisotropic and unsteady, but also contain strong coherent structures. In particular, Goto and Vassilicos (2016b) highlighted a link between the presence of coherent structures and eq. (1.13). Coherent structures, however they may be defined (see Cantwell, 1981), are definitely not accounted for in K41.

The extent in space over which eq. (1.13) has been observed (and, for that matter where the $-5/3$ spectrum is broadest) consists of several eddy turnover times and coincides with where both Re_λ and the turbulence intensity are the largest. Thus, the importance of the turbulence at such distances to the generator cannot be underestimated. These relatively short distances, in terms of the development of laboratory experiments, can nevertheless represent large distances in physical space: as noted by Vassilicos (2016), the non-equilibrium region downstream of a tree on a particularly gusty day could extend over the whole length of Hyde Park.

1.2 Wakes

A classical example of an inhomogeneous, anisotropic flow with strong coherent structures is that of the turbulent wake downstream of a bluff body. Wakes are found in various

engineering applications and are of great interest in structural dynamics among other fields in engineering (e.g. see remarks by von Kármán, 1963; Williamson, 1996). Notice that the grid turbulence achieved in laboratory experiments usually consists of turbulence arising from several interacting wakes (Roach, 1987; Roshko, 1952). In fact, as noted by Gomes-Fernandes (2016) these wakes, which are elementary parts of grid turbulence, are an ideal example of non-homogeneous anisotropic flows of relative simplicity under which the discussion in section 1.1 can be explored.

As noted by Bloor (1963) and Roshko (1952), it is important to distinguish between three ranges of Re_∞ : laminar flow without vortex shedding, laminar flow with vortex shedding and turbulent flow (see e.g. fig. 1 of Williamson, 1996). The actual values of Re_∞ that distinguish these types of motion are dependent on the geometry of the body.

At sufficiently large Re_∞ but still small enough such that turbulence does not ensue, the flow past a bluff body develops an array of alternating counter-rotating vortices moving downstream at a self-induced velocity (Bénard, 1908; Mallock, 1907; Mizota, 2000). The pattern that arises is usually referred to as a Kármán-street due to von Kármán’s contribution to the development of the stability theory of those vortices and how they influenced the drag experienced by the bluff body (von Kármán, 1911; von Kármán, 1912).

As the Re_∞ is further increased, turbulence develops in the wake; as noted by von Kármán (1963), “when the velocity is further increased, the periodical vortex shedding still persists, but the beautiful regular pattern no longer exists.” This observation is important, because planar wakes have been extensively investigated under Kolmogorov’s theory which does not accommodate for the presence of coherent structures.

The seminal work by Townsend on shear flows (including the planar wake) consisted of describing their nature sufficiently far downstream such that they became self-similar. While it is arguable if such a state exists (see Bearman, 1993; George, 2012; Wygnanski et al., 1986), we shall briefly describe Townsend’s work and its relation to Kolmogorov’s view of turbulence in section 1.2.1. In any case, self-similarity can only be expected to hold very far downstream; as reviewed in section 1.2.2, the flow closer to the wake generator exhibits a strong dependency on initial conditions but it is also where the turbulence is “richer” in the sense that it is of high intensity, strongly inhomogeneous and anisotropic.

1.2.1 Asymptotic Behaviour of Wakes

Chapter 6 of Townsend (1976) is dedicated to the evolution of free shear flows, obviously, we shall focus here on the planar wake. There is no intention in providing a detailed summary of Townsend’s work on free shear flows, instead, we wish only to highlight how

the Richardson-Kolmogorov equilibrium cascade impacts the determination of even the most basic flow properties, such as the mean velocity profile.

Townsend outlines a few assumptions on the structure of the flow, namely, that the turbulence should be locally homogeneous (in the sense of Kolmogorov, 1941a) and that the turbulence fluctuations are isotropic. The objective is to determine, far enough from the wake generator, what are the self-similar profiles of the mean velocity and turbulence intensities. Townsend motivates this goal by arguing that all flows should (at least in an asymptotic sense) tend to a state of moving equilibrium, which resonates very much with Kolmogorov’s assumption of steadiness/equilibrium, and implies an independence on initial conditions.

By making use of the boundary layer approximation and using the equations for the mean streamwise velocity and turbulent kinetic energy, Townsend used the self-similarity assumption to scale each term with the velocity deficit on the centreline u_0 , the turbulent kinetic energy on the centreline k_0 and the wake’s half-width ℓ_0 . To scale the dissipation appearing in eq. (1.3), Townsend made use of eq. (1.10) with $\mathcal{U} = \sqrt{k_0}$ and $\mathcal{L} = \ell_0$ and thus closed the equations. As explained in Vassilicos (2017), it is now known that this last step is the root cause of the incorrect predictions for the scalings of u_0 and ℓ_0 in axisymmetric wakes (see Nedić, Vassilicos, et al., 2013; Obligado et al., 2016).

As noted by George (2012), another symptomatic problem with Townsend’s approach is that it assumes an independence on the initial conditions, i.e. that the self-similar states of planar wakes are independent of the wake generator. One of the ways in which the initial conditions have been found to be relevant is through the presence of coherent structures (see Wygnanski et al., 1986) which retain their shape (determined by the wake generator) over large distances.

1.2.2 The Near Wake

As in Bearman (1993), we refer to the “near wake” as that region downstream of the bluff body where effects of initial conditions are present and thus the description given in section 1.2.1 certainly does not apply.

In the immediate vicinity of the wake the flow separates and gives rise to shear layers. At relatively low Re_∞ , these shear layers are stable and the turbulence arises from three-dimensional instabilities in the wake. As shown by Robichaux (1997) and Robichaux et al. (1999), at transitional Re_∞ different three-dimensional modes (which depend on the geometry of the wake generator) develop and give rise to stream-wise vorticity in braids that connect the coherent structures associated with the Kármán street.

As the Re_∞ increases, the shear layers themselves become unstable and break down into turbulence (see Lyn and Rodi, 1994; Williamson, 1996). In any case, i.e. regardless of Re_∞ (Cantwell, 1981), these shear layers roll up in the region downstream of the generator giving rise to coherent vortical structures. The presence of these coherent structures largely distorts the flow which is reflected in large turbulence intensities. As shown by Gerrard (1966), the location, with respect to the solid body, at which the standard deviation of the cross-stream velocity fluctuations peaks is an indicator of the region where the coherent structures are formed. As the structures depart, they exert both lift and drag forces on the body. When a vortex of positive circulation is formed, the circulation on the body is negative and *vice versa* leading to alternating positive and negative lift during the shedding cycle. On the other hand, the low-pressure at the core of the vortex creates a pressure difference between the upstream and downstream faces of the bluff body, causing a rise in drag twice per shedding cycle.

As a consequence of the intense “activity” in the vicinity of the wake generator, the mean velocity field is highly inhomogeneous. At sufficiently high Re_∞ , multiple saddle points develop in the mean streamlines (see, e.g. Lyn, Einav, et al., 1995). From a kinetic energy point of view, this strong inhomogeneity of the mean flow can be seen as having an effect of “feeding” the fluctuating motions (both the coherent structures and the turbulence) with energy, as the mean flow homogenises in the downstream direction less and less energy is transferred to the fluctuating motions which effectively decay with increasing downstream distance (Hussain, 1983). However, as noted by Cantwell (1981) even as the mean flow homogenises, the coherent structures still interact with the turbulence due to the local gradients introduced by those structures which, similarly to the mean flow production in eq. (1.3), generate turbulent kinetic energy.

Notice that even though there is no reason to expect the turbulence in this region to be described by eq. (1.7), there have been several reports of a $-5/3$ power law in the spectra of the fluctuating velocity (see e.g. Braza et al., 2006; Cantwell and Coles, 1983; Kravchenko and Moin, 2000; Lehmkuhl et al., 2013; Ma et al., 2000; Ong and Wallace, 1996; Trias et al., 2015; Wissink and Rodi, 2008, in the context of circular cylinders and square prisms). On the other hand, if the observations made by Goto and Vassilicos (2016b) with respect to the link between coherent structures and turbulent dissipation holds in this flow, one should expect $C_\varepsilon \neq \text{const}$. However, to the author’s knowledge, there have been no reports on the validity (or not) of $C_\varepsilon \neq \text{const}$ in the near wake. This is likely due to inadequacy of experimental techniques capable of measuring ε in flows with strong turbulence intensity and flow reversal, but also due to a difficulty in defining the velocity and length scales entering eq. (1.10).

1.3 Objectives and Thesis Outline

At the centre of the phenomenology of the Richardson-Kolmogorov equilibrium cascade lies the idea that the dynamics of eddies in the inertial range is governed solely by their non-linear interactions and the rate at which energy is dissipated ε . As a consequence, the distribution of energy and its rate of transfer among scales have well-defined laws, the later of which being necessarily equal to $-\varepsilon$ in the inertial range of length scales. Even though this theory is formulated under the assumptions of high Reynolds, homogeneity, isotropy and equilibrium, its predictions have been observed in a wider variety of flows (Kraichnan, 1974) Furthermore, there is growing evidence suggesting that the assumption of equilibrium is the most troublesome (see Vassilicos, 2015, and references therein).

Thus, this thesis focuses on the turbulence cascade in an inhomogeneous anisotropic flow which cannot be expected *a priori* to be in equilibrium: the wake generated by a square prism at $Re_\infty = 3900$. This choice follows from three reasons: it is a relatively simple set-up (from a numerical point of view) that leads to a rather complex flow, past studies have reported results predicted by Kolmogorov's theory (even if it cannot hold), and because an experimental investigation of the energy cascade that takes into account the periodic vortex shedding has been carried out (Thiesset, 2011).

The numerical methods used to perform the direct numerical simulations (DNS) are described in chapter 2. The code is validated against benchmark solutions to assess its accuracy and parallel performance. A series of low Re_∞ simulations of wakes past square prisms were performed as in that regime there is a considerable amount of numerical data to compare against. Finally, as a grid refinement study would prove to be too computationally costly, the main DNS at $Re_\infty = 3900$ is compared with experiments and other DNS.

In chapter 3 we study the energy cascade through the recently derived Kármán-Howarth-Monin-Hill equation (KMH), a generalised form of the Kármán-Howarth-Monin equation (see Hill, 2002a). In essence the KMH is a two-point energy equation which can be obtained directly from the Navier-Stokes equations without any of the aforementioned assumptions. Because our data are obtained from DNS, all terms in the KMH are accessible to us, including the pressure-velocity correlations and all components of the velocity-gradient tensor which are challenging (if not impossible) to estimate in experiments. To our knowledge, this is the first study in which the KMH is identically balanced (within numerical constraints) on a point-by-point and scale-by-scale fashion.

The study of the energy cascade is further extended in chapter 4 where the fluctuating component of the velocity and pressure fields are further decomposed to account for the

coherent motion. By studying versions of the KMH that distinguish between the two types of motion, we are able to precisely address how they interact with each other and with the mean flow. It is shown that some of the two-point properties of the coherent motions are a consequence of the particular geometrical shape of the coherent structures.

In chapter 5 we conclude with the main results presented throughout this thesis and suggest how they may be used in future studies of inhomogeneous, anisotropic, nonequilibrium turbulence.

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2 Numerical Methodology

Direct numerical simulations of the Navier-Stokes equations do not resort to any modelling assumptions and thus must resolve all scales of the flow. Particular care must be taken that the numerical discretisation is fine enough and has no impact on the solution. In this chapter we discuss the discretisation technique and the solver used to perform our DNS of the flow past a square prism.

In section 2.1 a review of the numerical schemes used is presented along with validation and assessment of the solver’s accuracy and parallel performance. In section 2.2 we compare results of simulations of square prisms at several Reynolds numbers with both experiments and other DNS, including a description of the set-up used to perform the main DNS (at $Re_\infty = 3900$).

2.1 Solver Description

The numerical simulations were performed using a cell-centred fully unstructured finite volume code called *Panta Rhei*. The PETSc library (Balay et al., 2016) is used solve the discretised equations in parallel.

The finite volume method (FVM) and spatial/time discretisations are described in section 2.1.1. As will be seen, special care must be taken in satisfying the equation for mass conservation eq. (1.1a); this is accomplished by means of an iterative procedure. The performance of the solver is illustrated in section 2.1.2 where the convergence (in terms of numerical error) and parallel scalability are discussed.

2.1.1 Finite Volume Discretisation and Numerical Integration

The goal of a discretisation method is to convert a set of (non-linear) partial differential equations into a system of linear algebraic equations while satisfying the necessary boundary conditions, in the form

$$A_{ij}y_j = b_i \tag{2.1}$$

which is solved to provide the values at discrete locations (e.g. cell centres) in the domain of interest. In eq. (2.1) A_{ij} is a coefficient matrix that depends on the mesh topology and

discretisation used, the vector y_j contains the unknown variable (e.g. any of the components of the velocity vector $\bar{\mathbf{u}}$) at the j -th node and b_i contains all known information (e.g. source terms, boundary conditions, etc.) at the i -th node. As will be discussed below, obtaining eq. (2.1) requires linearisation of the non-linear term of eq. (1.1b) (this allows usage of efficient iterative solvers for linear systems, as discussed in Ferziger and Perić, 2002).

One discretisation method to obtain eq. (2.1) from eq. (1.1) is the FVM. The main idea behind this approach is to reduce the order of the partial derivatives by taking advantage of Gauss' theorem (also known as divergence theorem)

$$\iiint_{\mathcal{CV}} \frac{\partial f_i}{\partial x_i} dV = \iint_{\partial \mathcal{CV}} f_i n_i dS \quad (2.2)$$

whereby the volume integral of the divergence of a flux $\mathbf{f}$ is converted to the surface integral of the fluxes $\mathbf{f} \cdot \mathbf{n}$ through the boundaries of a control-volume $\mathcal{CV}$. In eq. (2.2) $\partial \mathcal{CV}$ indicates the surface bounding $\mathcal{CV}$ and $\mathbf{n}$ is the outward-normal on that surface. This method is suitable for solving eq. (1.1b) as conservation is enforced by construction on the discrete level (Ferziger and Perić, 2002; Hirsch, 2007).

In eq. (2.1), the vector y_j represents a velocity component within the j -th control volume. The elements of matrix A_{ij} depend on the computational stencil, i.e. the different cells j used to approximate the derivatives around cell i . The term b_i contains not only the pressure term (from eq. (1.1b)) but also, as will be seen below, source terms due to the chosen discretisation and the imposed boundary conditions. After constructing the system given by eq. (2.1), the algebraic system is solved in parallel using routines implemented in the PETSc library (Balay et al., 2016).

2.1.1.1 Time Integration

The time integration is done using a second order accurate implicit method (backward differencing method). Assume that the solution is sought at the (discrete) time instant $t + 1$. The solution at different time steps are distinguished by their superscript, t represents the current time step and $t - 1$ the one before t (i.e. the solution is known at both t and $t - 1$).

As such, eq. (1.1b) is written for time step $t + 1$ and the time derivative term is estimated by Taylor expanding $\bar{\mathbf{u}}$ backwards in time about $t + 1$ which can be used to obtain

$$\left(\frac{\partial \bar{\mathbf{u}}}{\partial t} \right)^{t+1} = \frac{\frac{3}{2} \bar{\mathbf{u}}^{t+1} - 2 \bar{\mathbf{u}}^t + \frac{1}{2} \bar{\mathbf{u}}^{t-1}}{\Delta t} + \mathcal{O}(\Delta t^2), \quad (2.3)$$

In eq. (2.3) Δt is the time step and it can readily be seen that this estimate for $\frac{\partial \bar{\mathbf{u}}}{\partial t}$ is second order accurate. All the spatial terms of eq. (1.1b) are evaluated at time step $t + 1$. This type of time discretisation is referred to as implicit time discretisation and, for the same spatial discretisation, is expected to be more (numerically) stable than its explicit counterpart (see extensive discussion in Part 3 of Hirsch, 2007).

Since the solution is sought for time step $t + 1$, the time derivative is implemented in eq. (2.1) by including the term $-\frac{-2\bar{\mathbf{u}}^t + \frac{1}{2}\bar{\mathbf{u}}^{t-1}}{\Delta t}$ in b_i (i.e. as a source term) while the factor $\frac{3}{2}\frac{1}{\Delta t}$ contributes to A_{ii} , the diagonal coefficient of A_{ij} for the velocity $\bar{\mathbf{u}}$ at the i -th cell. As will become clear below, both these contributions to b_i and A_{ii} are also multiplied by the cell's volume.

2.1.1.2 Spatial Discretisation

Consider a domain $\mathcal{D}$ where eq. (1.1) is to be solved. We divide $\mathcal{D}$ into N cells (or control volumes) and focus on solving eq. (1.1b) for each $\mathcal{CV}$ as illustrated in fig. 2.1. For ease of illustration, a two-dimensional Cartesian domain is shown in fig. 2.1, however, the approach that follows also holds for three-dimensions and cells of arbitrary shape.

For those terms which cannot be expressed as divergences of fluxes, their integral over a $\mathcal{CV}$ is approximated by assuming its value to be constant within the $\mathcal{CV}$ and equal to its value at the centre of the $\mathcal{CV}$. Thus, for some arbitrary variable ψ one has

$$\iiint_{\mathcal{CV}} \psi dV = \iiint_{\mathcal{CV}} \left(\psi^{cv} + \Delta r_i \frac{\partial \psi^{cv}}{\partial x_i} + \Delta r_i \Delta r_j \frac{\partial^2 \psi^{cv}}{\partial x_i \partial x_j} + \dots \right) dV \approx \psi^{cv} V^{cv} + \mathcal{O}(\|\Delta \mathbf{x}\|^2), \quad (2.4)$$

where V^{cv} is the volume of $\mathcal{CV}$ and the vector $\Delta \mathbf{r}$ is the relative position to the centre of the control volume. As a consequence of this approximation, the integral will converge to the exact value with second order accuracy*.

One of the terms which cannot be written in terms of fluxes is the transient term $\frac{\partial \bar{u}_i}{\partial t}$; any source terms appearing in eq. (1.1b), also require a similar treatment. Using eq. (2.4), integration over the volume $\mathcal{CV}$ yields

$$\iiint_{\mathcal{CV}} \frac{\partial \bar{u}_i}{\partial t} dV \approx \frac{\partial \bar{u}_i^{cv}}{\partial t} V^{cv}. \quad (2.5)$$

* Notice that $\int_{\mathcal{CV}} \Delta r_i \frac{\partial \psi^{cv}}{\partial x_i} dV$ vanishes because a cell centred formulation is being used and as such $\int_{\mathcal{CV}} \Delta x_i dV = 0$ since Δr_i is the distance to the centre of $\mathcal{CV}$.

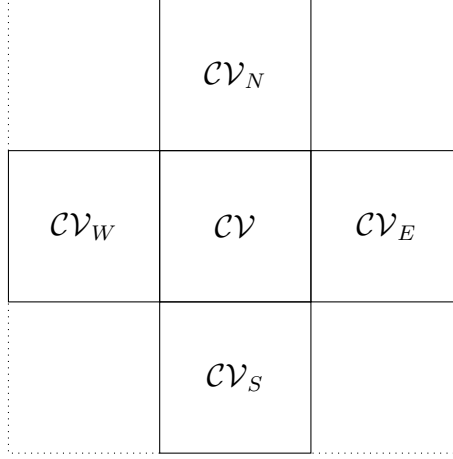


Figure 2.1: Illustration of a control-volume labelled as $\mathcal{CV}$ and its neighbours identified with subscripts which indicate their relative position to $\mathcal{CV}$ (i.e. N for north, E for east, S for south and W for west).

Similarly, whenever one requires a cell centre estimate for the gradient of some quantity ψ , it can be obtained by combining this property and eq. (2.2), such that

$$\frac{\partial \psi}{\partial x_i} V^{\mathcal{CV}} = \iint_{\partial \mathcal{CV}} \psi n_i dS. \quad (2.6)$$

Let us consider the non-linear term in eq. (1.1b). Making use of eq. (1.1a), the term $u_j \frac{\partial \bar{u}_i}{\partial x_j}$ can be written as the divergence of a flux, i.e. $\frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j}$, such that one has

$$\iiint_{\mathcal{CV}} \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} dV = \iint_{\partial \mathcal{CV}} \bar{u}_i \bar{u}_j n_j dS \quad (2.7)$$

as a direct consequence of eq. (2.2). The surface integral of the momentum flux components $\bar{u}_i \bar{u}_j n_j$ in $\partial \mathcal{CV}$ is required. As in the volume integral (recall eq. (2.4)), we assume the flux through $\partial \mathcal{CV}$ to be constant within that surface (trapezoidal rule). This procedure also converges to the exact integral with second order accuracy.

Because the discretisation employed here is cell-centred (i.e. the discrete variables are stored at the cell centres) the estimate for the fluxes at each face $\partial \mathcal{CV}$ must be obtained via some sort of interpolation between the cells sharing each face $\partial \mathcal{CV}$. This interpolation should be at least second order accurate, in order not to deteriorate the overall accuracy of the scheme. As explained in sections 5.6 and 7.5 of Ferziger and Perić (2002), implementation of higher order (third or higher) interpolation schemes would drastically increase the computational stencil (especially for unstructured grids) and would result in a matrix A_{ij} not being diagonally dominant which could pose problems to the algebraic

solver. Therefore, to efficiently obtain second order accuracy in the spatial discretisation, the deferred correction approach is used (as suggested in Ferziger and Perić, 2002). This approach consists in initially estimating the fluxes with first order accuracy and then improving that estimate iteratively by accounting for that error in the source term in eq. (2.1).

The first order estimate for the face value of the fluxes used here is the upwind interpolation. As the name suggests, the direction of the flux is used to infer the estimate of eq. (2.7). Notice that $\bar{\mathbf{u}}\bar{u}_jn_j$ can be interpreted as each component of $\bar{\mathbf{u}}$ being convected through $\partial\mathcal{CV}$ with the velocity $\bar{u}_jn_j$, we shall call $\bar{u}_jn_j$ the convection velocity. Thus, the non-linear term is linearised such that $\bar{\mathbf{u}}|_{\mathcal{CV}}$ is kept as an unknown, and the convection velocity $\bar{u}_jn_j$ is kept constant (when solving eq. (2.1)). Nevertheless, the convection velocity is estimated by linear interpolation such that, e.g. for the face $\partial\mathcal{CV}_E$ (separating $\mathcal{CV}$ from $\mathcal{CV}_E$ in fig. 2.1), one would have

$$\bar{\mathbf{u}}|_{\partial\mathcal{CV}_E} = \frac{\Delta_{EE}\bar{\mathbf{u}}_{\mathcal{CV}} + \Delta_E\bar{\mathbf{u}}_{\mathcal{CV}_E}}{\Delta_E + \Delta_{EE}} + \mathcal{O}(\Delta_E\Delta_{EE}), \quad (2.8)$$

where Δ_E is the distance between the centres of $\mathcal{CV}$ and $\mathcal{CV}_E$ in fig. 2.1 and Δ_{EE} would be the distance between $\mathcal{CV}_E$ and a cell to its eastern side. As such, variations in grid size should be smooth to allow $\Delta_E \approx \Delta_{EE}$ and have the truncation error in eq. (2.8) decrease approximately with the second power of grid spacing.

The upwind estimate for eq. (2.7) is then given by evaluating $\bar{\mathbf{u}}$ at the cell centre upstream of the face at which $\bar{u}_jn_j$ is interpolated, for example, using the nomenclature of fig. 2.1, one would have

$$\iint_{\partial\mathcal{CV}_E} \bar{u}_i\bar{u}_jn_j dS \approx \begin{cases} \bar{u}_i|_{\mathcal{CV}} (\bar{u}_jn_j)|_{\partial\mathcal{CV}_E} S_{\partial\mathcal{CV}_E}, & (\bar{u}_jn_j)|_{\partial\mathcal{CV}_E} \geq 0 \\ \bar{u}_i|_{\mathcal{CV}_E} (\bar{u}_jn_j)|_{\partial\mathcal{CV}_E} S_{\partial\mathcal{CV}_E}, & (\bar{u}_jn_j)|_{\partial\mathcal{CV}_E} < 0 \end{cases}, \quad (2.9)$$

and equivalently for the faces $\partial\mathcal{CV}_N$, $\partial\mathcal{CV}_W$ and $\partial\mathcal{CV}_S$. As such, the flux through each face of the control volume $\mathcal{CV}$ enters eq. (2.1) either as a diagonal term (when the flux is outwards from $\mathcal{CV}$) or off-diagonal (when the flux is inwards to $\mathcal{CV}$) in the matrix A_{ij} .

In order to raise the accuracy of the discretisation (as the upwind interpolation is over-dissipative, see e.g. Hirsch, 2007), the deferred approach, introduced by Khosla and Rubin (1974), is implemented within the same iterative procedure for obtaining the pressure (discussed below). At every iteration, the difference between the second order estimate obtained by linear interpolation between the cell centres neighbouring $\partial\mathcal{CV}$ (e.g. eq. (2.8)) and the upwind estimate (e.g. eq. (2.9)) is included explicitly in b_i in eq. (2.1) and the

first order error is iteratively removed from the solution which then becomes second order accurate.

Let us now turn our attention to the viscous term in eq. (1.1b). The starting point in discretising this term is the same as that for the non-linear term and thus applying eq. (2.7) to $\nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}$ results in

$$\iiint_{\mathcal{CV}} \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} dV = \iint_{\partial \mathcal{CV}} \nu \frac{\partial \bar{u}_i}{\partial x_j} n_j dS. \quad (2.10)$$

If the grid is orthogonal (i.e. the normals at $\partial \mathcal{CV}$ are aligned with a line connecting the centroid of the cells that share $\partial \mathcal{CV}$) then a second order estimate can be obtained via finite differences. By making use of Taylor expansion about the face centre (and in the direction normal to it, n) for the values of $\bar{u}_i$, one can obtain

$$\left. \frac{\partial \psi}{\partial n} \right|_f = \frac{\psi|_B - \psi|_A}{\Delta_{AB}} - \frac{\Delta_{Af}^2 - \Delta_{Bf}^2}{\Delta_{AB}} \left. \frac{\partial^2 \psi}{\partial n^2} \right|_f + \mathcal{O}\left(\frac{\Delta_{Af}^3 + \Delta_{Bf}^3}{\Delta_{AB}}\right) \quad (2.11)$$

where Δ_{Af} and Δ_{Bf} are the distances between the face centre and the centres of A and B and $\Delta_{AB} = \Delta_{Af} + \Delta_{Bf}$, and ψ is any component of $\bar{\mathbf{u}}$. If the grid is uniform, one has that $\Delta_{AB} = \frac{1}{2}\Delta_{Af} = \frac{1}{2}\Delta_{Bf}$ and eq. (2.11) becomes second order accurate; if that is not the case, the truncation error varies with $\frac{\Delta_{Af}^2 - \Delta_{Bf}^2}{\Delta_{AB}}$, this means that, when non-uniform grids are employed, the grid size should vary smoothly in order to keep $\frac{\Delta_{Af}^2 - \Delta_{Bf}^2}{\Delta_{AB}}$ small and minimise the error introduced by the linear interpolation.

If the grid is non-orthogonal, appropriate corrections must be made. Even though the grid used in our DNS was orthogonal, a brief description of the non-orthogonality correction is given here. The non-orthogonality correction is implemented via the deferred correction approach again, i.e., eq. (2.11) is used and then any necessary corrections are implemented as a source term; this is because implicit treatment of the correction would drastically increase the computational workload and the usage of eq. (2.11) prevents the occurrence of oscillatory behaviour (as mentioned above, but see also Ferziger and Perić, 2002; Muzaferija, 1994).

Corrections are made for the case when the face normal is not aligned with the vector connecting the cell centres (of the cells sharing $\partial \mathcal{CV}$) but not when the line connecting those cell centres does not intersect the face midpoint. While this issue could have been bypassed by directly interpolating the gradient (evaluated from eq. (2.6)) to the face centre (preserving second order accuracy), such procedure would not only involve a rather large stencil (since it would involve information on the cells that share $\partial \mathcal{CV}$ and all of their neighbours) but it would also be insensitive to oscillatory solutions (see the exten-

sive discussion in pp. 233-238 in Ferziger and Perić, 2002). Effectively, the direction $\mathbf{n}$ in eq. (2.11) for a non-orthogonal grid corresponds to the direction connecting the cell centres of those cells sharing $\partial\mathcal{CV}$. Thus, as suggested by Muzaferija (1994), a correction factor can be computed by interpolating the cell centre gradient (recall eq. (2.6)) to the face centre, projecting it both in the direction $\mathbf{n}$ and that of the actual face normal $\mathbf{n}_f$ and subtracting the difference to eq. (2.11).

2.1.1.3 Pressure Correction

Since we are dealing with an incompressible flow, the pressure can only be known up to an arbitrary constant (since only $\frac{\partial\bar{p}}{\partial x_i}$ enters eq. (1.1b)).

In order to extract the pressure an algorithm based on the PISO method (introduced by Issa, 1986) is used. The procedure consists of a predictor step and a series of corrector steps. In the predictor step, eq. (1.1b) is solved using the pressure from the previous time step, i.e. the central approximation of the pressure gradient is included in b_i of eq. (2.1). This results in a solution that satisfies eq. (1.1b) but may not satisfy eq. (1.1a), thus, necessary corrections are applied to the solution until $\bar{u}_i$ and $\bar{p}$ satisfies both eqs. (1.1a) and (1.1b) (i.e. eq. (1.1)).

In line with the notation used in Ferziger and Perić (2002), let us denote the value of $\bar{\mathbf{u}}$ at the predictor step with the superscript $*$ and denote with a tilde that part of the $\bar{\mathbf{u}}|_{\mathcal{CV}}$ given by eq. (2.1) that excludes both the contribution from the control volume $\mathcal{CV}$ and the contribution from the pressure gradient divided by the entry of A_{ij} for $i = j = \mathcal{CV}$, i.e.

$$\bar{u}_i|_{\mathcal{CV}}^{*t+1} = \hat{u}_i|_{\mathcal{CV}}^{*t+1} - \frac{1}{A_{\mathcal{CV}}} \frac{\partial\bar{p}}{\partial x_i} \Big|_{\mathcal{CV}}^t. \quad (2.12)$$

Now, we seek to obtain a pressure $\bar{p}^{*t}$ such that a second estimate of $\bar{u}_i|_{\mathcal{CV}}^{t+1}$ (let us denote it as $\bar{u}_i|_{\mathcal{CV}}^{**t+1}$) satisfies eq. (1.1a). Let us now take a perturbation on both the velocity $\bar{u}_i^{*t+1}$ and pressure $\bar{p}^t$, denoted respectively by $\bar{u}_i^{\parallel}$ and $\bar{p}^{\parallel}$, such that $\bar{u}_i|_{\mathcal{CV}}^{**t+1} = \bar{u}_i^{*t+1} + \bar{u}_i^{\parallel}$ and $\bar{p}^{*t} = \bar{p}^t + \bar{p}^{\parallel}$, the perturbations $\bar{u}_i^{\parallel}$ and $\bar{p}^{\parallel}$ must satisfy a relation similar to eq. (2.12), where $\hat{u}_i^{\parallel}|_{\mathcal{CV}}$ contains all the contributions from the cells neighbouring $\mathcal{CV}$ (notice that the right hand side of eq. (2.1) is not re-evaluated, hence $\hat{u}_i^{\parallel}|_{\mathcal{CV}}$ does not include any source terms). Thus, taking the divergence of $\bar{u}_i^{\parallel}|_{\mathcal{CV}}$ leads to a Poisson equation for the pressure perturbation

$$\left(\frac{\partial}{\partial x_i} \frac{1}{A_{\mathcal{CV}}} \frac{\partial\bar{p}^{\parallel}}{\partial x_i} \right) \Big|_{\mathcal{CV}} = \frac{\partial\hat{u}_i^{\parallel}}{\partial x_i} \Big|_{\mathcal{CV}} - \frac{\partial\bar{u}_i^{\parallel}}{\partial x_i} \Big|_{\mathcal{CV}} \quad (2.13)$$

which is solved using a conjugate gradient solver with multi-grid pre-conditioning available through the hypre library (Falgout, 2001). Notice, that the second term in the right

hand side of eq. (2.13) is equal to $-\frac{\partial \bar{u}_i}{\partial x_i}\bigg|_{\mathcal{CV}}^{*t+1}$ since $\frac{\partial \bar{u}_i}{\partial x_i}\bigg|_{\mathcal{CV}}^{**t+1} = 0$. The first term on the right hand side of eq. (2.13) will be ignored, because no estimate is available for it; this constitutes the basic assumption of the SIMPLE algorithm by Caretto et al. (1972). As argued in Ferziger and Perić (2002), this choice is hardly justifiable since $\frac{\partial \hat{u}_i}{\partial x_i}\bigg|_{\mathcal{CV}}$ is not necessarily zero at this stage. The value for $\bar{p}^{\parallel}$ is used to compute $\bar{p}^{*t}$ and $\bar{\mathbf{u}}^{**t+1}$ and take them as the correct values of velocity and pressure.

Issa (1986) proposed to repeat the procedure above for a second perturbation on $\bar{u}_i^{*t}$ and $\bar{p}^{*t}$, i.e. $\bar{u}_i^{\parallel\parallel}$ and $\bar{p}^{\parallel\parallel}$ which would result in a Poisson equation for $\bar{p}^{\parallel\parallel}$. However, the term $\frac{\partial \bar{u}_i}{\partial x_i}\bigg|_{\mathcal{CV}}^{**t+1}$ can now be approximated (rather than ignored). In fact, this step is repeated until both the divergence of the most recent velocity estimates and the difference between the two most recent velocity estimates fall below some threshold value. The Poisson equation for $\bar{p}^{\parallel\parallel}$ is similar to eq. (2.13) where the left hand side is now treated in the same way as the viscous term (recall eq. (2.10)).

Considering that some computational effort must be invested in improving the estimates of the velocity and pressure, the iterations within the PISO algorithm are done explicitly; this is achieved by extrapolating the flux $\bar{u}_j n_j$ through each cell face $\partial\mathcal{CV}$ from the values obtained in the previous two iterations. Furthermore, since the spatial discretisation employed is cell centred (both velocity $\bar{\mathbf{u}}$ and pressure $\bar{p}$ are defined at the cell centres), spurious pressure modes can arise in the pressure field; these spurious modes do not vanish with refinement of the grid (Hirsch, 2007) and thus appropriate care must be taken to prevent them.

The flux $\bar{u}_j n_j$ through each cell face $\partial\mathcal{CV}$ is then computed following the procedure introduced by Rhie and Chow (1983). Albeit the grids used here are orthogonal, it is worth mentioning that this method is similar to that described above for the viscous diffusion term, except that the correction is done iteratively, while for the viscous term it is added as a source term in eq. (2.1). Effectively, the pressure gradient at time step t is computed at each cell face $\partial\mathcal{CV}$ and cell centre, then, one can interpolate the cell centre estimates of $\frac{\partial \bar{p}}{\partial x_i}$ between neighbouring cells. The difference between both quantities is then added to the fluxes $u_j n_j$ at each face thus ensuring that the spurious modes are eliminated (see section 8.8 in Ferziger and Perić, 2002).

2.1.1.4 Boundary Conditions

In order to impose boundary conditions at a given cell face $\partial\mathcal{CV}$, a ghost cell (of zero thickness) is introduced such that it shares $\partial\mathcal{CV}$ with the cell $\mathcal{CV}$ (where the solution is affected by the boundary condition). The most basic boundary conditions (periodic) are treated directly by the algebraic solver. The treatment of all other boundary conditions

follows that of the corrections implemented above, i.e. the known quantities (at the ghost cells) are treated explicitly via the source term in eq. (2.1).

Let us consider first the inlet boundary condition where a Dirichlet condition is imposed for the velocity and a Neumann condition is imposed for the pressure, i.e. the faces at the inlet have constant velocity and zero pressure gradient. For example, if an inlet boundary condition is imposed on boundary $\partial\mathcal{CV}_W$ in fig. 2.1, one can include the value of $A_{ij}y_j$ (with i equal to $\mathcal{CV}$ and j equal to $\mathcal{CV}_W$) to the right hand side of eq. (2.1) using expressions analogous to eqs. (2.9) and (2.11) for the convective and diffusive terms, respectively.

The outlet boundary can be treated in a similar fashion. However, instead of a Dirichlet condition, the simplified advection equation

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_n \frac{\partial \bar{u}_i}{\partial x_n} = 0 \quad (2.14)$$

is used to determine the velocity at the outlet; n is the direction normal to the outlet. The solution of eq. (2.14) is obtained explicitly using Euler forward differences while upwind finite differences are used for the convective term. After the values of the velocity at the ghost cell are obtained, they can be accounted for in the same way as at the inlet.

The symmetry and wall boundary conditions are trivially implemented by noting that the former consists in a zero gradient at symmetry boundary and zero mass flux across the boundary and the latter corresponds to having zero velocity at the wall (i.e. nothing has to be added to the source term of eq. (2.1)). At the wall, a Neumann condition is used for the pressure (see discussion by Moin and J. Kim, 1980).

The last type of boundary condition considered is the so-called pressure boundary condition which consists in a Dirichlet condition for the pressure. The velocity at the ghost cell is extrapolated from inside the domain such that $\bar{\mathbf{u}}_{\text{ghost}} = \bar{\mathbf{u}}_{\mathcal{CV}} + (\bar{\mathbf{u}}_{\partial\mathcal{CV}} \cdot \mathbf{n} - \bar{\mathbf{u}}_{\mathcal{CV}} \cdot \mathbf{n}) \mathbf{n}$ where $\bar{\mathbf{u}}_{\partial\mathcal{CV}}$ is iteratively corrected following the procedure described above.

2.1.2 Error Assessment and Scalability

In order to assess the accuracy of the code and its scalability, several simulations were made of the classical Taylor-Green vortex case with $Re_\infty = 1600$ introduced by Taylor and Green (1937). This test case was chosen to assess the accuracy and scalability of the code (instead of the wake flow studied in the following chapters) as highly accurate benchmark solutions obtained by spectral methods (see van Rees et al., 2011) are available. Furthermore, the grid has uniform spacing and thus spatial derivatives are formally second order accurate.

	#cells				
Δt	32	64	128	256	512
$5 \cdot 10^{-3}$	×	×	×	×	×
$1 \cdot 10^{-2}$	×	×	×	×	×
$2 \cdot 10^{-2}$	×	×	×	—	—
$4 \cdot 10^{-2}$	—	×	—	—	—
$8 \cdot 10^{-2}$	×	—	—	—	—

Table 2.1: Combinations of grids (identified by #cells, the number of cells in each Cartesian direction) and sizes of time step (Δt) used for Taylor-Green vortex simulations marked with “×”. Entries marked with “—” indicate combinations not considered.

The initial condition used for the Taylor-Green vortex is given by

$$\begin{cases} u_1 = \sin(x_1) \cos(x_2) \cos(x_3) \\ u_2 = -\sin(x_1) \cos(x_2) \cos(x_3) \\ u_3 = 0 \\ p = \frac{1}{16} [\cos(2x_1) + \cos(2x_2)] [\cos(2x_3) + 2] \end{cases} \quad (2.15)$$

and the viscosity is set to $\frac{1}{1600}$ such that $Re_\infty = 1600$. The domain extent is given by $x_i \in [0, 2\pi]$ and periodicity is imposed at all boundaries. Given the absence of a mean-flow, no distinction shall be made between $\bar{\mathbf{u}}$ and $\mathbf{u}$ (and between $\bar{p}$ and p).

This flow is characterised by a continuous decay of kinetic energy as the large initial vortices break into smaller ones causing an initial build-up in the dissipation which rises, peaks and then decays. In order to assess the performance of the solver, a series of simulations were performed using the initial conditions given by eq. (2.15) and different grids, time steps and partitions as summarised in tables 2.1 and 2.2. The maximum Courant-Friedrichs-Lewy number (CFL) given by

$$CFL = \max(\bar{u}_i \frac{\Delta t}{\Delta x}) \quad (2.16)$$

was kept at 0.5 by decreasing the sizes of the time step Δt in proportion to the grid refinement (Δx is the typical cell size).

In fig. 2.2 we compare the numerical solution from the grids listed in table 2.1 using a time step Δt that results in a maximum CFL number of about 0.5 (the mean CFL number was always an order of magnitude smaller). Figure 2.2 shows how the resolution of the small scales (responsible for dissipating energy) improves with mesh refinement as the time evolution of the instantaneous dissipation ϵ integrated over the whole volume

#procs.	#cells				
	32	64	128	256	512
24	×	×	×	×	—
48	—	×	×	×	—
96	—	×	×	×	—
192	—	×	×	×	—
384	—	×	×	×	—
768	—	×	×	×	×
1536	—	×	×	×	×
3072	—	—	—	—	×

Table 2.2: Combinations of grids (identified by #cells, the number of cells in each Cartesian direction) and number of partitions (identified by #procs., the number of processors) used for Taylor-Green vortex simulations marked with “×”, entries marked with “—” indicate combinations of grid/number of partitions that were not considered.

appears to smoothly approach the reference profile from van Rees et al. (2011). On the other hand, the time evolution of kinetic energy, k , and its variation in time do not appear to be bounded by the solution given in van Rees et al. (2011), the overshoots observed are caused by dispersion errors which generate artificial fluctuations in the solution. In fact, the difference between $\frac{\partial k}{\partial t}$ and ϵ can only be due to numerical errors (see e.g. Warming and Hyett, 1974), as they are the only two terms appearing in the volume integral of the kinetic energy equation.

In fig. 2.3 we show how the mean error, computed as

$$\langle E \rangle = \frac{1}{20} \int_0^{20} \frac{|\psi - \psi_{\text{ref}}|}{\psi_{\text{ref}}} dt \quad (2.17)$$

(where ψ is k , ϵ or $\frac{\partial k}{\partial t}$) decreases as the grid is refined. The different reference solutions are distinguished below using the subscripts “spectral”, “#cells = 512³” and “ $\Delta t = 5 \cdot 10^{-3}$ ”, respectively, to distinguish between the spectral solution of van Rees et al. (2011), the finest grid ran here (512³ cells) using the smallest time-step used here ($\Delta t = 5 \cdot 10^{-3}$). When varying the time-step, the solution obtained for each grid at the smallest time step ($\Delta t = 5 \cdot 10^{-3}$) is also used as a reference.

Figure 2.3 shows the errors in volume integral of kinetic energy k , its rate of change in time $\frac{\partial k}{\partial t}$ and dissipation ϵ when the grid is refined and the time-step is kept constant. Notice that the error in enstrophy is not shown in fig. 2.3 given that

$$\iiint \epsilon dV = \iiint 2\nu \frac{1}{2} ||\omega|| dV \quad (2.18)$$

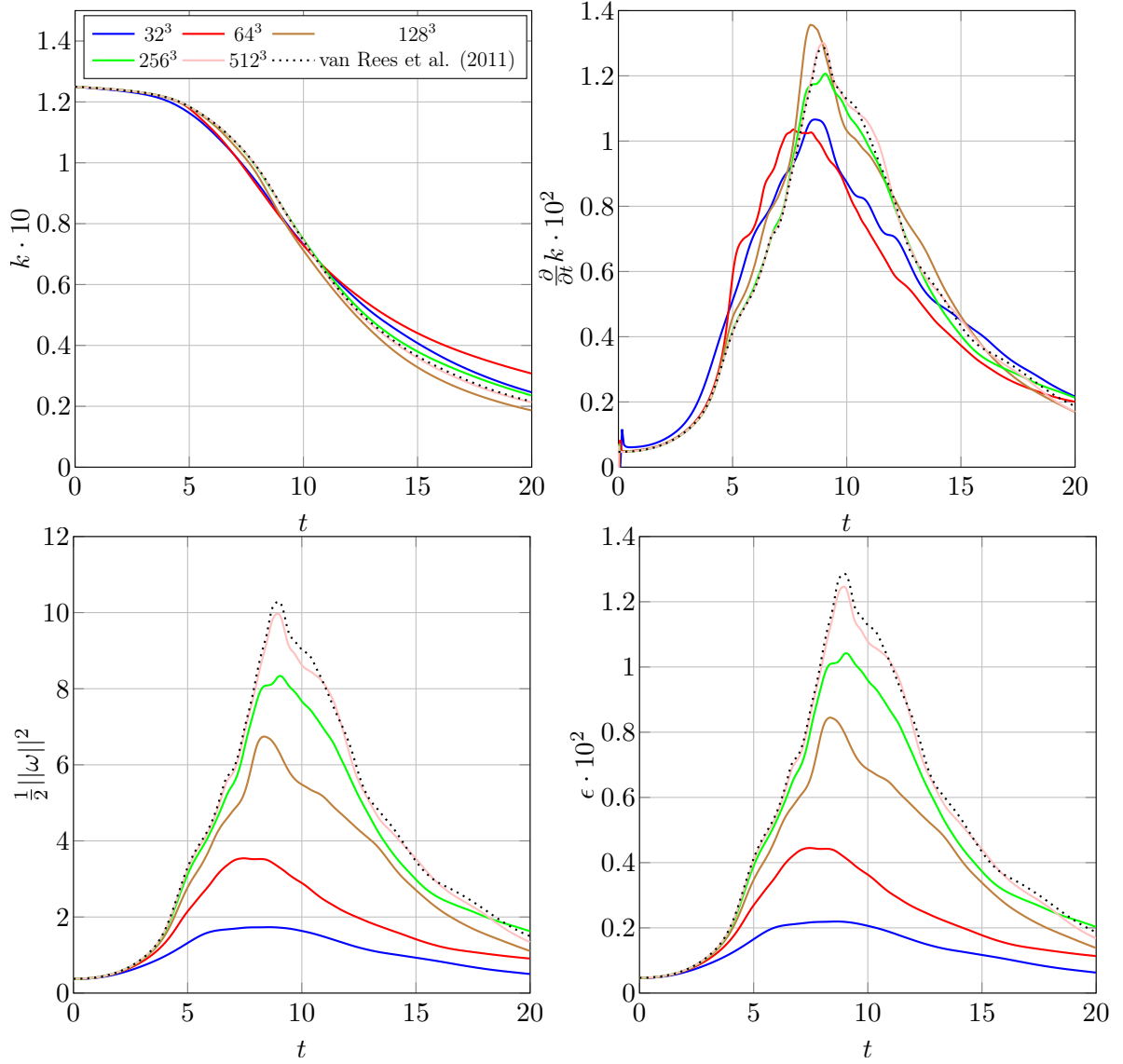


Figure 2.2: Comparison of the volume averaged k (top left), instantaneous dissipation $\frac{\partial k}{\partial t}$ (top right) and ϵ (bottom right), and enstrophy (bottom left) for a Taylor-Green vortex at $Re_\infty = 1600$ with different grids.

(see p. 20 in Frisch, 1995) and therefore both variables behave in the same way (as can be qualitatively assessed from fig. 2.2). It is clear from fig. 2.3 that as the grid is refined, the numerical solution approaches the reference solution of van Rees et al. (2011) obtained by highly accurate spectral methods. Furthermore, when comparing the different solutions to that obtained with the finest grid (512^3 cells) we see that the error converges to zero approximately following a quadratic power law (i.e. $\sim \mathcal{O}(\Delta x^2)$) which is in agreement with the discussion in section 2.1.1.

In fig. 2.4 we compare the solutions obtained with the different grids by varying the time step (recall table 2.1). The numerical solutions convergence nearly quadratically to the solutions obtained with the smallest time step since the error decreases nearly with a

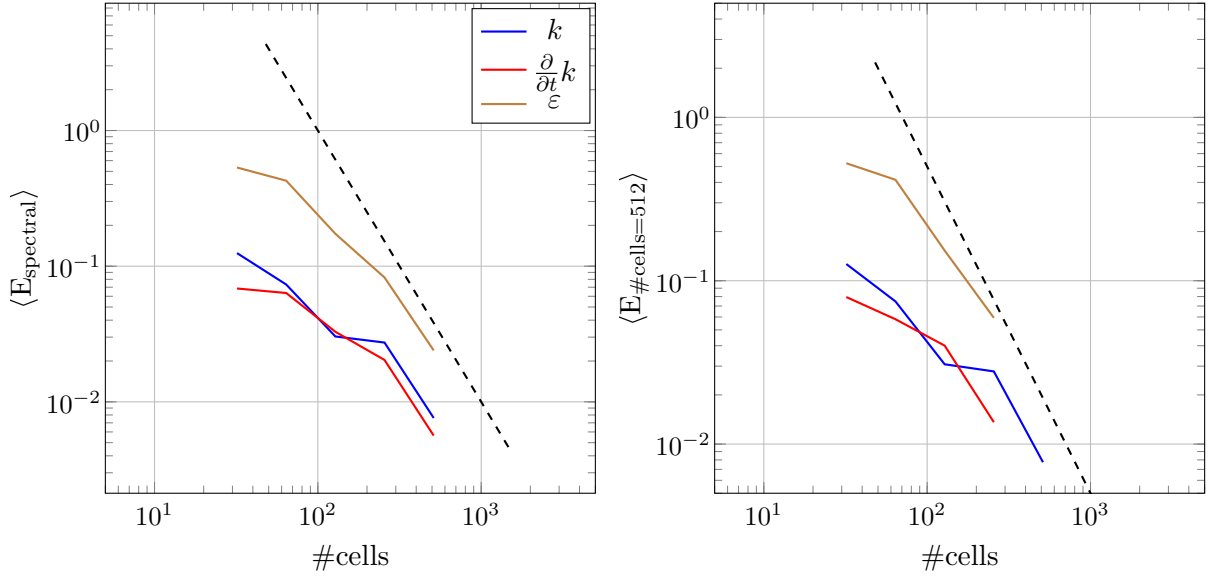


Figure 2.3: The errors of kinetic energy, time variation of kinetic energy and dissipation are computed for each grid (with the time step $\Delta t = 5 \cdot 10^{-3}$) comparing with the reference data of van Rees et al. (2011), on the left, and with the solution obtained with the finest grid, on the right. The dashed line indicates quadratic convergence.

second order power law as Δt decreases. Thus, fig. 2.4 highlights how the numerical errors introduced by the spatial and temporal discretisations are coupled and how appropriate refinement in both space and time is required to approach the exact solution.

Finally, in figs. 2.5 and 2.6 the weak and strong scaling of the code are shown, respectively, for the combinations of grid and partitions shown in table 2.2. Weak scaling considers how the execution time varies as the load per processor remains constant. Thus, the ideal weak scaling is achieved when the wall time per time step remains constant as the number of processors is increased. On the other hand, strong scaling examines how the computational time is reduced by partitioning a given problem into smaller subdomains. A good strong scaling is indicated by a linear decrease in the time consumed per time step when the number of processors is increased (for a given grid). These simulations were all ran on the Archer supercomputer's standard nodes, containing two 2.7 GHz, 12-core E5-2697 v2 (Ivy Bridge) series processors with 64 GB of memory available per node. The results are presented in terms of wall time per time step, i.e. how much physical time (in seconds) the computations require for a single time step. The speedup was not computed as running the simulations on a single processor would require very long simulations (and not enough memory would be available for the largest test cases).

As can be seen in fig. 2.5, the average time, t_{wall} , consumed per time step, Δt is approximately constant for workloads up to a few tens of thousand of cells per processor.

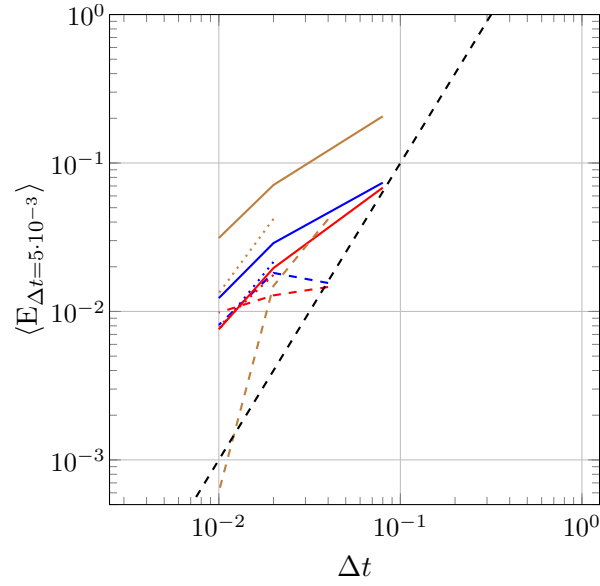


Figure 2.4: The errors of kinetic energy, time variation of kinetic energy and dissipation are computed for each grid (indicated by the subscript in the legend) comparing with the solution obtained with the smallest time step. The dashed line indicates quadratic convergence.

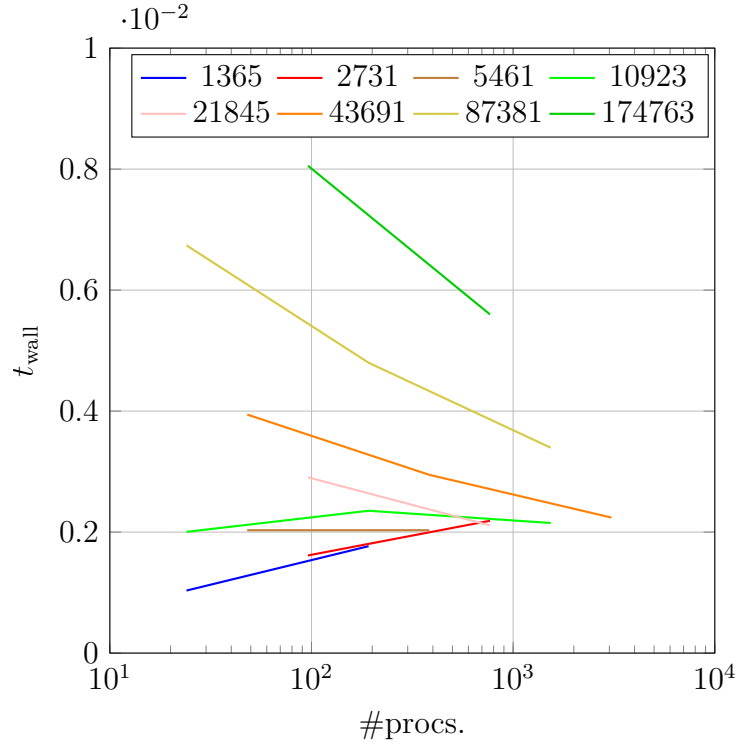


Figure 2.5: Weak scaling where t_{wall} is the total wall time (in seconds) per time step and the legend shows the workload density (i.e. $\# \text{cells} / \# \text{procs.}$, the number of cells per processor).

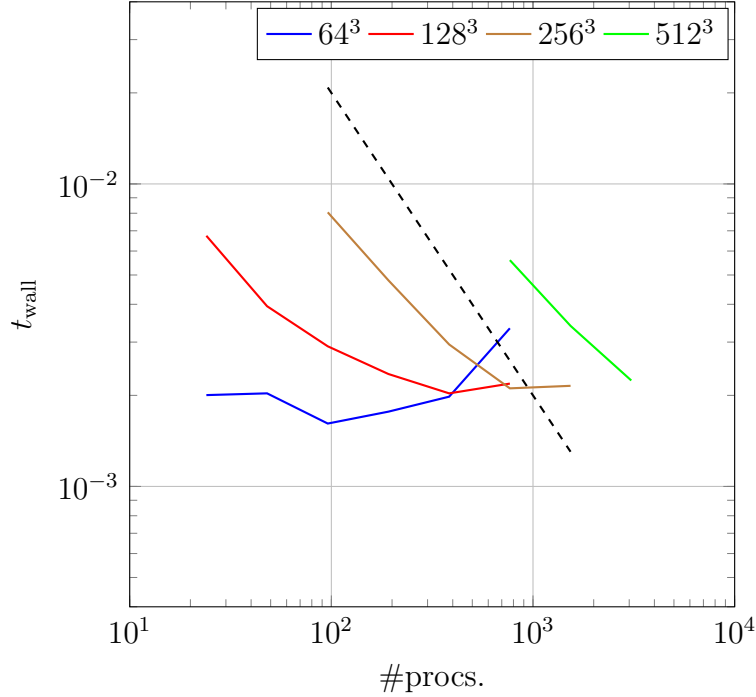


Figure 2.6: Strong scaling where t_{wall} is the total wall time (in seconds) per time step. The dashed line indicates linear scaling.

However, for larger loads, a decrease in the time consumed per time step is observed. This can be due to the fact that the portion of operations that benefit from partitioning may be larger for larger grids (see e.g. Barros and Kauranne, 1990). With regards to the strong scaling, fig. 2.6 shows that apart from the 64^3 case, significant reduction in the wall time per time step t_{wall} can be achieved by increasing the number of processors. Departures from the optimal (linear) scaling are observable only for the smaller grids using a large number of processors, as should be expected.

2.2 Numerical Set-up and Validation

In order to validate our results, we have compared several statistics with those reported in literature. In the present section we discuss statistics of the velocity, as well as the shedding frequency associated with the vortex shedding.

Many experimental results are available for the high Re_∞ regime, while for lower Re_∞ experiments are difficult and the majority of the results are computational. In section 2.2.1 we discuss results for low Reynolds numbers (up to $Re_\infty \sim 600$) for which a large amount of numerical results are available. In section 2.2.2 we present the validation for the main DNS with $Re_\infty = 3900$.

2.2.1 Simulations at Low/Moderate Re_∞

As discussed in section 1.2, at large enough Re_∞ a Kármán street develops downstream of the prism. For the square prism, this occurs for $Re_\infty \gtrsim 50$ (see Franke et al., 1990; Sohankar et al., 1998). For low Re_∞ , Sohankar et al. (1998) and Sohankar et al. (1999) proposed that the product of the Strouhal number $St = \frac{f_s d}{U_\infty}$, the vortex shedding frequency f_s normalised by free-stream variables, and the Reynolds number follows

$$Ro = St \cdot Re_\infty \approx 0.18Re - 3.7. \quad (2.19)$$

The product Ro is known as the Roshko number after the classical experiments of Roshko (1952) on circular cylinders.

A 3D simulation with $Re_\infty = 500$ was carried out using the same domain/grid used by Sohankar et al. (1999) for about 300 shedding cycles. Further (shorter) simulations for $50 \leq Re_\infty \leq 600$ were performed employing similar refinement in the vicinity of the prism but a larger cell density in the core of the wake. The domain dimensions used for the low/moderate Re_∞ simulation is sketched in fig. 2.7. A sensitivity to domain dimensions and boundary conditions was carried out by Sohankar et al. (1998).

As justified by Bearman (1993), a great deal of effort has been devoted in determining the properties of the vortex shedding and thus most of the data available in literature concerns St vs Re_∞ curves and instantaneous flow visualisations, especially at sub-critical Re_∞ where the flow is laminar. As Re_∞ increases, three-dimensional instabilities give rise to span-wise fluctuations and the wake is no longer two-dimensional; the first one of these instabilities occurs at $Re_\infty \approx 162$ (refer to Robichaux et al., 1999). In fig. 2.8 different values of St are shown for $0 < Re_\infty < 600$ both from two- and three-dimensional simulations and compared to experimental and DNS available in literature. Notice that despite the flow being fully three-dimensional at $Re_\infty \gtrsim 162$ some two-dimensional simulations are also available at higher Re_∞ . These Strouhal numbers were determined by finding the frequency at which the power spectrum of the lift coefficient was maximum.

The values of St obtained from the 3D simulations lie within the range of available experimental and DNS (3D) data showing how for $Re_\infty \gtrsim 200$ the shedding frequency decreases with increasing Re_∞ (the lowest Re_∞ experiments of Davis and Moore, 1982, are the only set of data that do not display this trend). At Re_∞ where the flow is laminar fig. 2.8 displays a good agreement between our 2D simulations and both experiments and simulations. In fact, the data seems to follow eq. (2.19) up to $Re_\infty = 150$ and fit reasonably well with the reference data up to $Re_\infty \approx 325$ where the flow is already three-dimensional. At higher Re_∞ , we report values of St roughly about half of those of Davis and Moore (1982). At such large Reynolds numbers the 2D simulations exhibit a chaotic

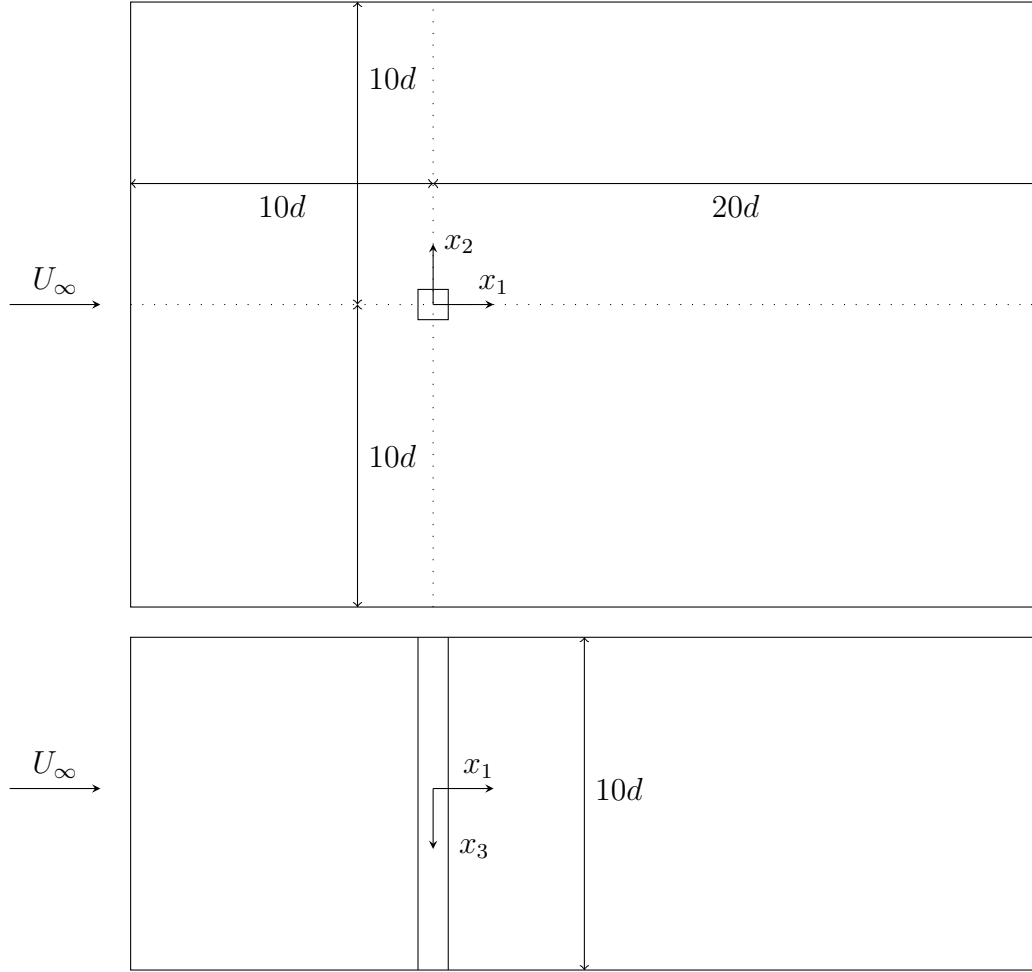


Figure 2.7: Domain dimensions for low Re_∞ simulations. Top: side view; bottom: top view. The origin of the coordinate system is at the centre of the square prism.

behaviour which consists in the shedding of an extra vortex pair which is not part of the Kármán street, we shall discuss this in more detail in section 2.2.1.1.

Along with St , the mean drag coefficient is also one of the quantities usually reported in studies of flows past bluff bodies. As shown in fig. 2.9 our results for the mean drag coefficient for all Re_∞ lie very close to the numerical results of Sohankar et al. (1999) and the experiments of Saha et al. (2000a) (as discussed by Graham, 1993, experimental force measurements at low Re_∞ are notoriously difficult). At low Re_∞ , these values appear to be consistently lower than those reported by Davis and Moore (1982), Franke et al. (1990), and Robichaux et al. (1999). However, not only are the discretisation methods employed different, there are also significant differences in blockage ratio and resolution between the set-ups of Davis and Moore (1982), Franke et al. (1990), and Robichaux et al. (1999)[†] and that of the simulations presented here as well as of Sohankar et al. (1999). In

[†] The numerical set-up used in Robichaux et al. (1999) is given in Robichaux (1997).

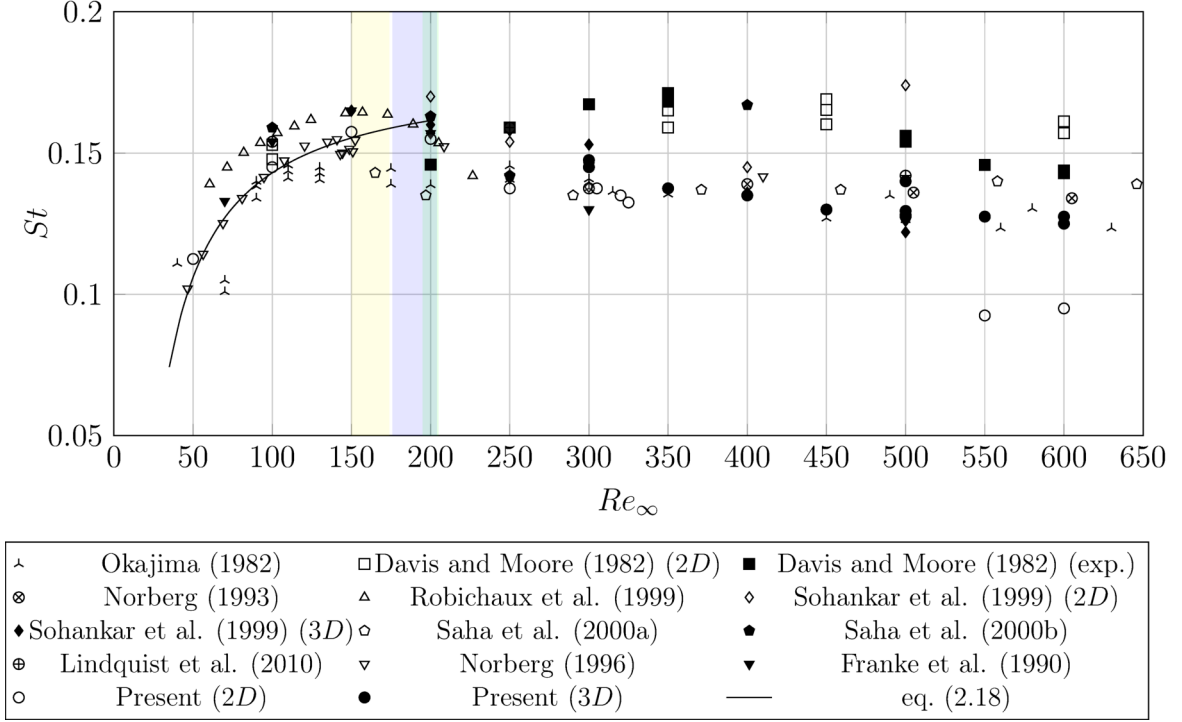


Figure 2.8: St versus Re_∞ from both experiments and DNS. Repeated symbols (for the same Re_∞) indicate different experimental/numerical conditions. The data cited as Norberg (1996) is available from Sohankar et al. (1999). The shaded areas indicate the ranges of Re_∞ predicted by Robichaux et al. (1999) where three-dimensional instabilities appear, the colours distinguish between the three unstable Floquet modes.

fact, the present set-up uses a pressure boundary condition at the top/bottom boundaries, which aims to further reduce blockage effects, and the number of cells employed was in general an order of magnitude larger than in those references.

Let us now consider the turbulent flow at $Re_\infty = 500$ which has been studied by Sohankar et al. (1999). Figure 2.10 shows a comparison of the mean stream-wise velocity and mean pressure coefficient

$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2} \quad (2.20)$$

along the geometrical centreline. The mean quantities (of any variable, e.g. ψ below) reported hereafter make use of the time and span-wise averages

$$\frac{1}{T_e - T_s} \int_{T_s}^{T_e} \psi dt \quad (2.21a)$$

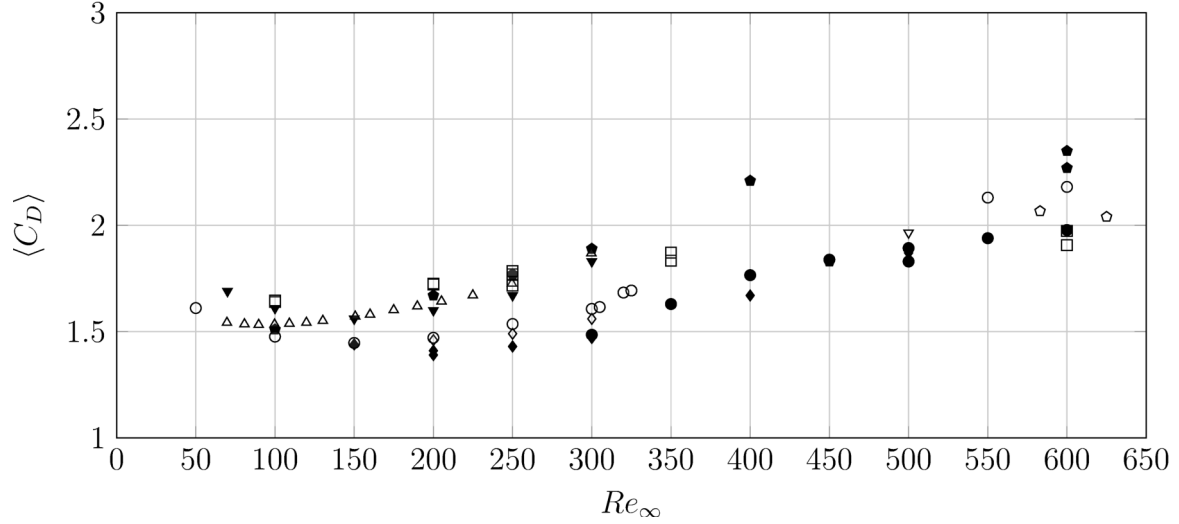


Figure 2.9: $\langle C_D \rangle$ versus Re_∞ from both experiments and DNS. Repeated symbols (for the same Re_∞) indicate different numerical conditions. See fig. 2.8 for legend.

$$\frac{1}{\pi} \int_0^\pi \psi dx_3, \quad (2.21b)$$

where T_s is the time after the flow becomes fully developed and T_e the time at which the simulation ended and x_3 is the span-wise direction where the mean flow is homogeneous (recall fig. 2.18). The procedure for obtaining this average (eq. (2.21)) is indicated through the operator $\langle \cdot \rangle$ such that, for example

$$\langle \psi \rangle = \frac{1}{T_e - T_s} \frac{1}{\pi} \int_{T_s}^{T_e} \int_0^\pi \psi dx_3 dt.$$

Given the lack of alternative sources to compare these results, the slight difference observed in fig. 2.10 between our results and those of Sohankar et al. (1999) prompted further investigation on possible causes for such difference. It was found that changing the top/bottom boundary condition from pressure to symmetry (as in Sohankar et al., 1999) only led to a slight decrease in St by about 10%.

Sohankar, 2014 reported that the averaging period used in Sohankar et al. (1999) was between 30 and 60 shedding cycles while the present results were averaged for almost 300 shedding cycles. Further inspection of the mean velocity field revealed that the reflection symmetry (about the centreline) of the mean velocity was not perfectly recovered after such long averaging period (see fig. 2.11). It was found that the statistical convergence in this range of Re_∞ is impaired by low frequency unsteadiness in the flow, reminiscent of the two-dimensional high Re_∞ flows, as discussed in the following section.

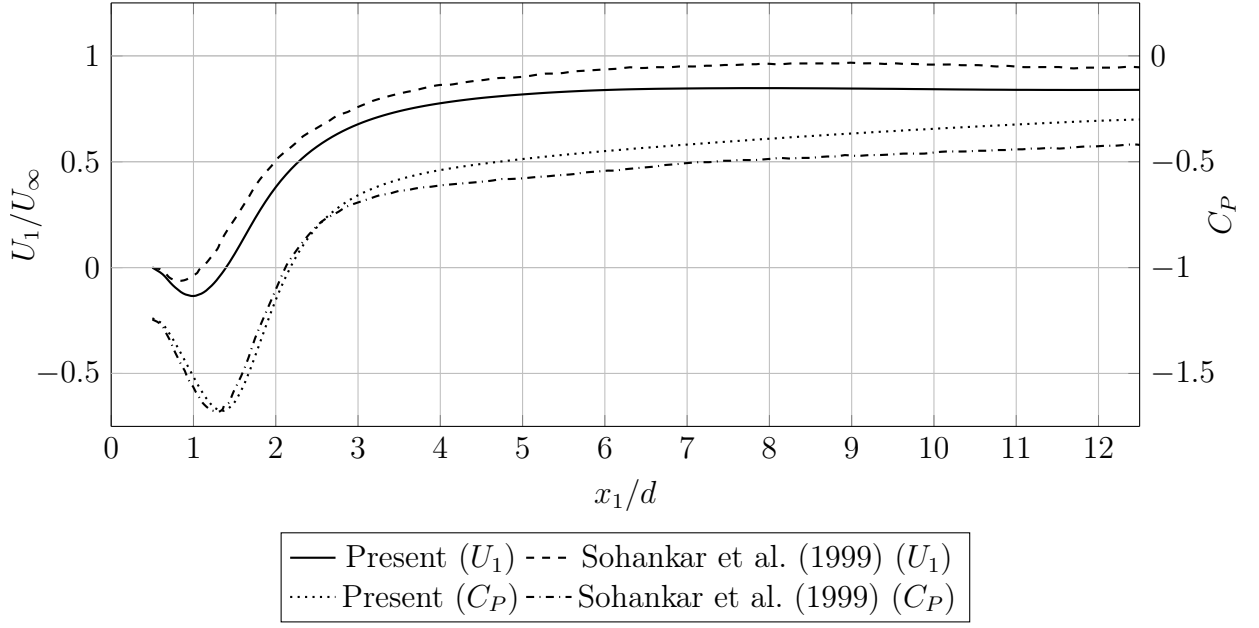


Figure 2.10: Mean stream-wise velocity component U_1 and mean pressure coefficient C_P along the geometric centreline for $Re_\infty = 500$.

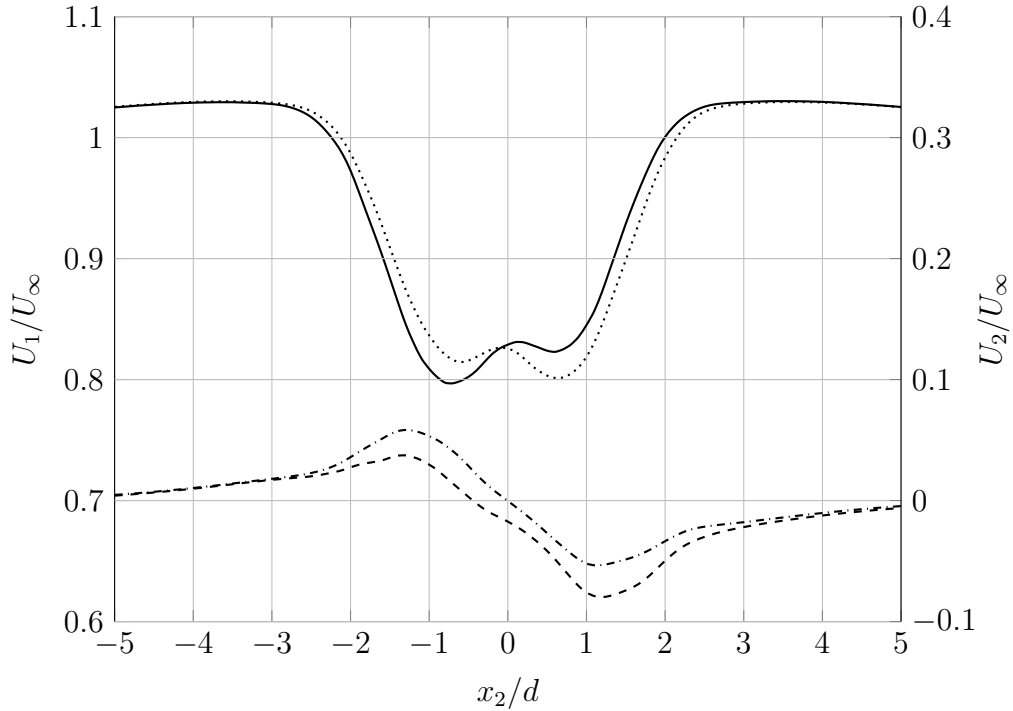
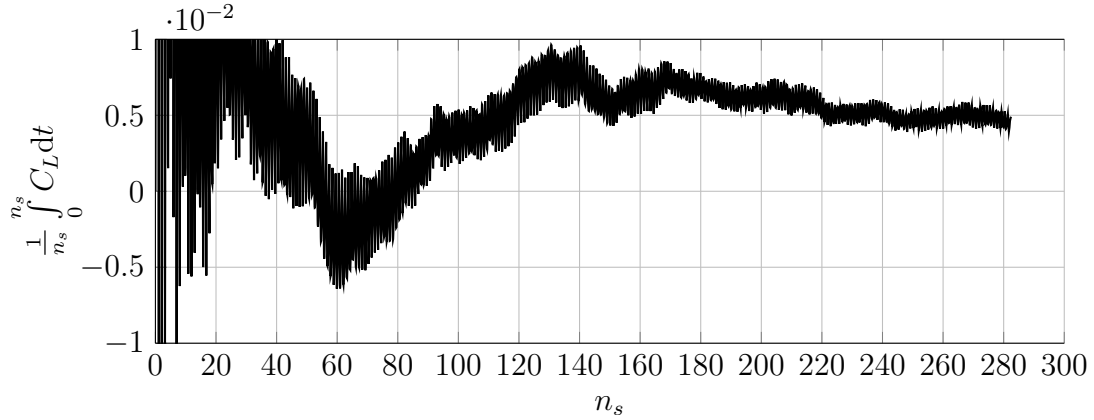


Figure 2.11: Profiles of U_1 (full and dotted lines) and U_2 (dashed and dash-dotted lines) along x_2/d at $x_1/d = 4$ for $Re_\infty = 500$. The full and dashed lines correspond to an averaging period of about 90 shedding periods, the dotted and dash-dotted lines correspond to an averaging period of about 240 shedding periods.


 Figure 2.12: Running average of C_L for $Re_\infty = 500$.

2.2.1.1 Low-Frequency Unsteadiness and Asymmetric States

As illustrated in fig. 2.11 when the averaging time was varied, it was observed that the mean velocity profile would slowly shift, such that the largest mean velocity deficit was observed either on the top or the bottom plane. This behaviour suggests that the wake experiences prolonged deflections either upward or downward (along the x_2 axis). This assumption is supported by fig. 2.12 which shows that the convergence of mean lift coefficient to zero is very slow. In fact, fig. 2.12 shows that there are extended periods of time over which the contribution of eq. (2.21) to $\langle C_L \rangle$ is either mostly positive or mostly negative (the same can be observed in fig. 5 of Trias et al., 2015, where $Re_\infty = 22 \times 10^3$).

Apart from the (conventional) vortex shedding which results in a symmetric mean flow, the behaviour illustrated in figs. 2.11 and 2.12 suggests the 3D wake exhibits two states in which the mean lift (over a few shedding cycles) is either positive or negative such that only the long term average is zero. In fact, if the shedding is perfectly symmetric (zero mean lift) the running average of C_L shown in fig. 2.12 should converge to zero at a rate given by t^{-1} . By examining periods over which this is not so, we find that there are three regimes over which the mean lift is approximately zero or ± 0.02 , i.e. symmetric and (two) asymmetric regimes, respectively. During the symmetric regime the mean drag coefficient is approximately 1.81 which is about 2% smaller than in the asymmetric regime. The three different regimes are clearly visible in fig. 2.13 which shows the history of C_L versus C_D curves over different periods of time associated with each regime. Notice how the coloured curves are reflections of each other while the black line shows no asymmetry about $C_L = 0$.

Non-zero mean lift coefficients have been reported by Robichaux et al. (1999), Saha et al. (2000b), Shiau et al. (1999), Sohankar et al. (1999), and Sohankar (2014) in 2D simulations of $Re_\infty \gtrsim 320$. Sohankar et al. (1999) and Sohankar (2014) observed that

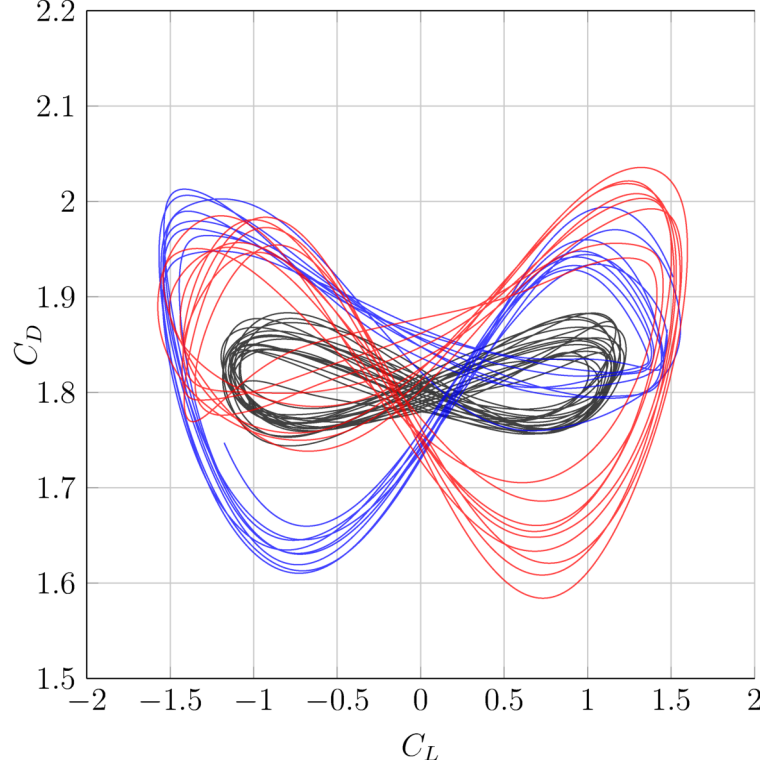


Figure 2.13: History of C_L versus C_D curves for two $Re_\infty = 500$ during three different regimes. The different colours are used to distinguish between the symmetric shedding with zero net lift (black), and asymmetric regimes with negative (blue) and positive (red) net lift.

at high Re_∞ the mean velocity profiles of the 2D simulations did not exhibit reflection symmetry about the centreline. Shiau et al. (1999) performed 2D simulations for $290 < Re_\infty < 330$ and showed that such (non-physical) flows undergo bifurcations where the mean lift coefficient is either positive or negative.

By imposing random small-amplitude fluctuations on an otherwise uniform flow as initial condition, different runs of 2D flows at high Re_∞ led to different states in which the mean lift coefficient was either positive or negative, but of equal magnitude for a given Re_∞ , as shown in fig. 2.14, and all other mean properties remained unchanged. It was found that the actual values of $\langle C_L \rangle$ depend strongly on the blockage ratio and boundary conditions at the top/bottom, for example, in Saha et al. (2000b) and Shiau et al. (1999), where they report values of $\langle C_L \rangle$ an order of magnitude smaller than in fig. 2.14, the blockage is of 10%, twice as large as in the present simulations and those of Sohankar et al. (1999) who report $\langle C_L \rangle = -0.04$ for $Re = 400$. The gap in Re_∞ shown in fig. 2.14 is a consequence of both the apparently chaotic nature of the solution, but also due to the fact that some simulations required longer run-times to reach steady state (and thus appear to be chaotic).

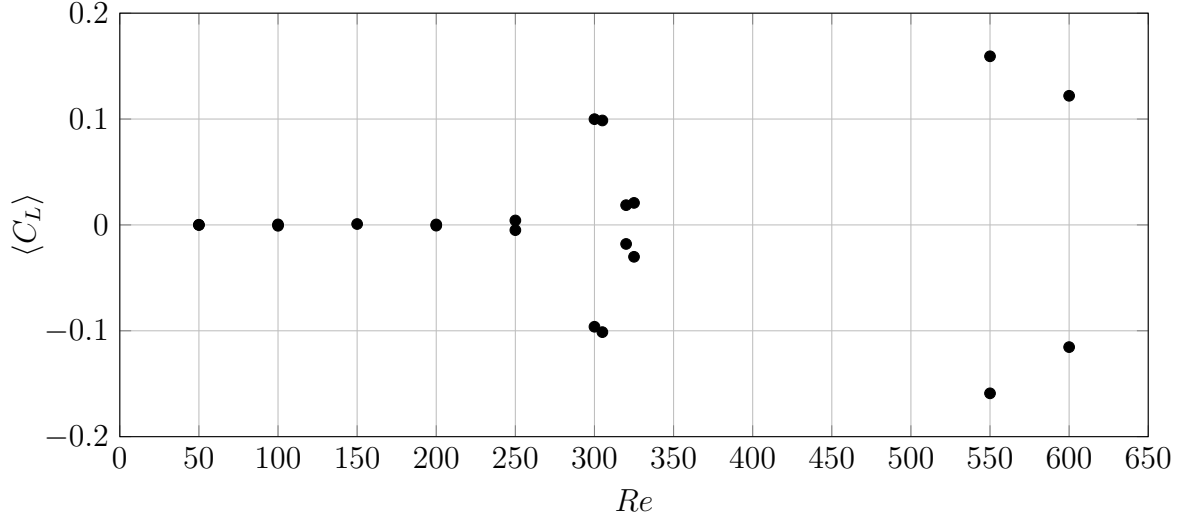


Figure 2.14: Mean lift coefficient $\langle C_L \rangle$ for 2D simulations with different initial conditions.

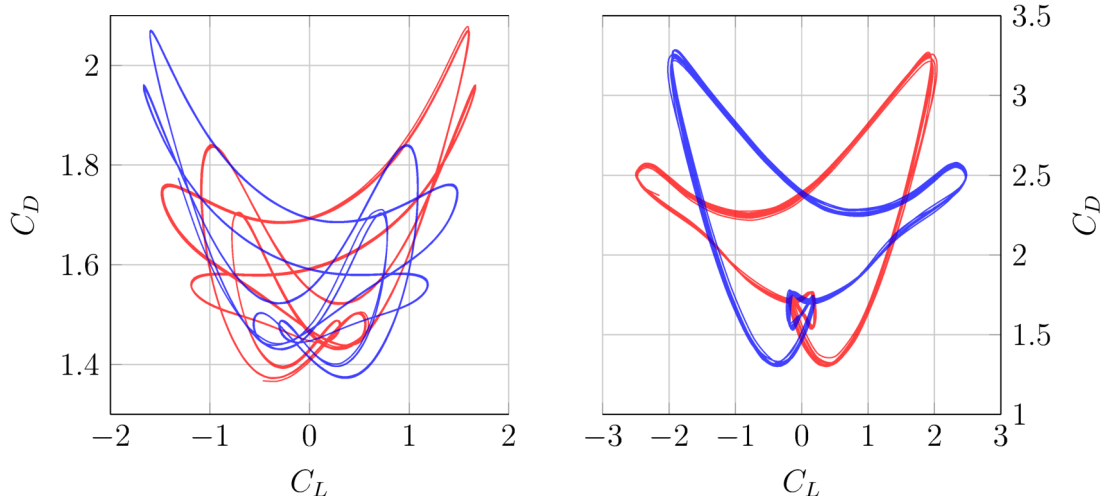


Figure 2.15: History of C_L versus C_D curves for two 2D simulations at $Re_\infty = 300$ (left) and $Re_\infty = 550$ (right). The different colours distinguish between separate simulations (the initial conditions were randomly perturbed).

As suggest by Shiau et al. (1999), at high Re_∞ the two distinct states can be recovered from each other by applying a reflection about x_2 . This is made clear in fig. 2.15 where it can be seen that the history of C_L and C_D of each state are mirrors of each other. Notice that at low Re_∞ the history of C_L versus C_D appears much more chaotic that at higher Re_∞ . This is a result of a vortex pair that does not “fit” in the Kármán street being shed from the prism at a frequency lower than that associated with the Kármán street. As illustrated in fig. 2.16, this vortex pair effectively distorts the spatial symmetry of the Kármán street as it travels through it.

The frequency at which the “anomalous” vortex pair is shed f_a was found to increase with Re_∞ until it becomes equal to f_s , the frequency associated with the Kármán street.

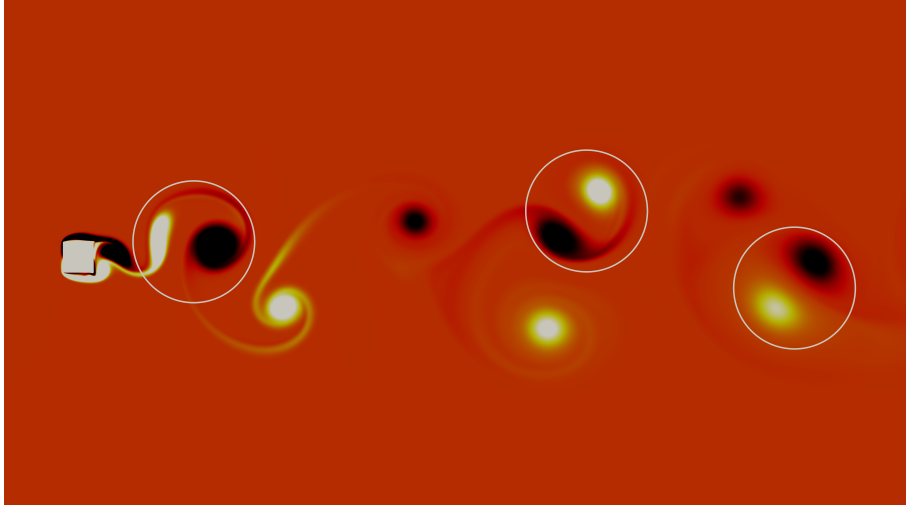


Figure 2.16: Iso-vorticity of a $Re_\infty = 500$ ($2D$) simulation. The domain used here was larger than in fig. 2.7. The gray circles indicate vortex pairs shed almost simultaneously which is not part of the developed Kármán street.

The spectra of C_L (and C_D) develops peaks at the frequencies f_s and f_a but also at $f_s - f_a$ and $f_s + f_a$ as well as multiple harmonics of these frequencies; it is for this reason that values reported here for St at $Re_\infty \gtrsim 500$ in $2D$ simulations are about half of those of reported by Davis and Moore (1982) (for their $2D$ simulations) since $f_s + f_a = 2f_s$. It is important to note that both the frequencies associated with the “anomalous” vortex pair as well as the values of mean lift observed in the $2D$ simulations appear to be highly sensitive to the blockage ratio.

Let us now return to the $3D$ simulation at $Re_\infty = 500$ which initiated this discussion. When we reported the different values of St in fig. 2.8, the spectrum of C_L was computed from the whole time series available. Instead, by computing the spectra of C_L within each of the three states observed in fig. 2.13 slight variations were observed in the shedding frequency: in the asymmetric shedding regime, the spectral peak occurred at a slightly smaller normalised frequency $St \approx 0.135$ than in the symmetric regime $St \approx 0.144$ (see fig. 2.17). This observation was further reinforced by the spectrum of C_D which, in the symmetric regime exhibits peaks at the normalised frequencies $St \approx 0.29$ and $St \approx 0.56$ while in the asymmetric regime the peaks are located at $St \approx 0.135$ and its harmonics as shown in fig. 2.17. Mahbub Alam et al. (2010) placed trip wires symmetrically on a circular cylinder and showed that for some cases the flow would transition between two states with slightly different shedding frequencies. Figure 2.17 thus shows how the spread of data observed in figs. 2.8 and 2.9 may not only be due to differences in experimental/numerical set-up but also due to the period of time over which statistics were collected.

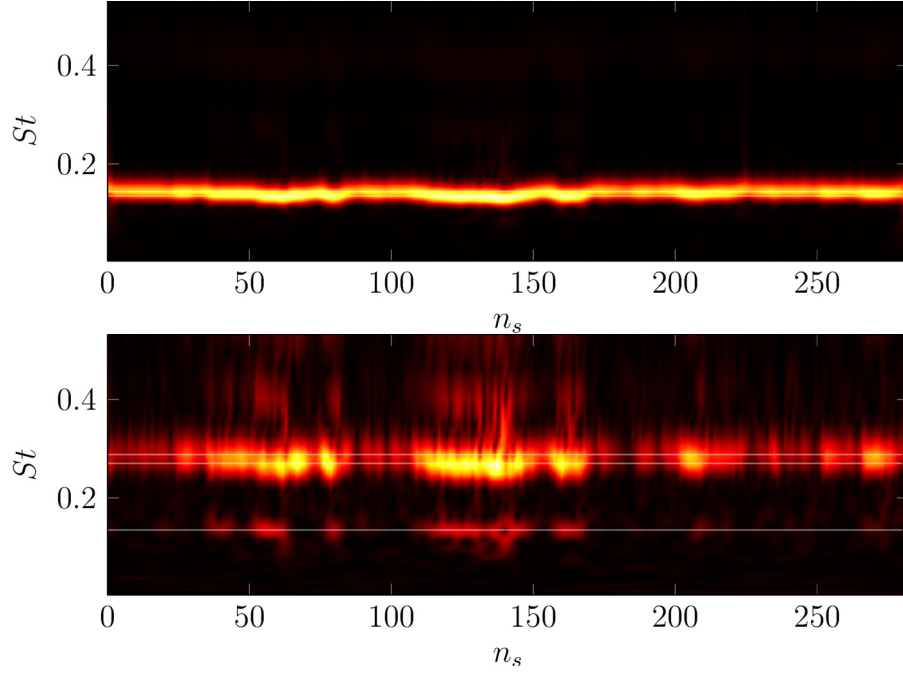


Figure 2.17: Spectrograms of C_L (top) and C_D (bottom) for $Re_\infty = 500$. The white lines indicate $St = 0.135$ and $St = 0.144$ (top) and $St = 0.135$, $St = 0.27$ and $St = 0.188$ (bottom).

Bi-stability in 3D wakes has been observed (experimentally) in the flow past circular cylinders at $Re_\infty \sim \mathcal{O}(10^5)$ (see Schewe, 1983, and references therein). Schewe (1986) showed that positive/negative mean lift states could be triggered upon tripping (momentarily) the boundary layer on the circular cylinder which effectively fixed the separation point. The present results for $Re_\infty = 500$ suggests that the transition between different modes of shedding occurs at a frequency which is at least one order of magnitude lower than the dominant shedding frequency (at least for this Re_∞). This appears to be related to the low-frequency unsteadiness observed in wakes of other geometries reported e.g. in Johnson et al. (2004), Lehmkuhl et al. (2013), and Najjar and Balachandar (1998).

2.2.2 DNS with $Re_\infty = 3900$

Performing a grid convergence study and sensitivity analysis of the domain size/boundary conditions would involve a steep increase in computational cost. As such, the results obtained with this configuration are compared below to experimental and DNS results with good agreement being observed between our data and that available in literature.

The fluid domain is sketched in fig. 2.18 along with its dimensions and the reference frame. At the inlet the velocity was set to U_∞ and at the outlet, the one dimensional advection equation is solved (convective boundary condition). The span-wise boundaries

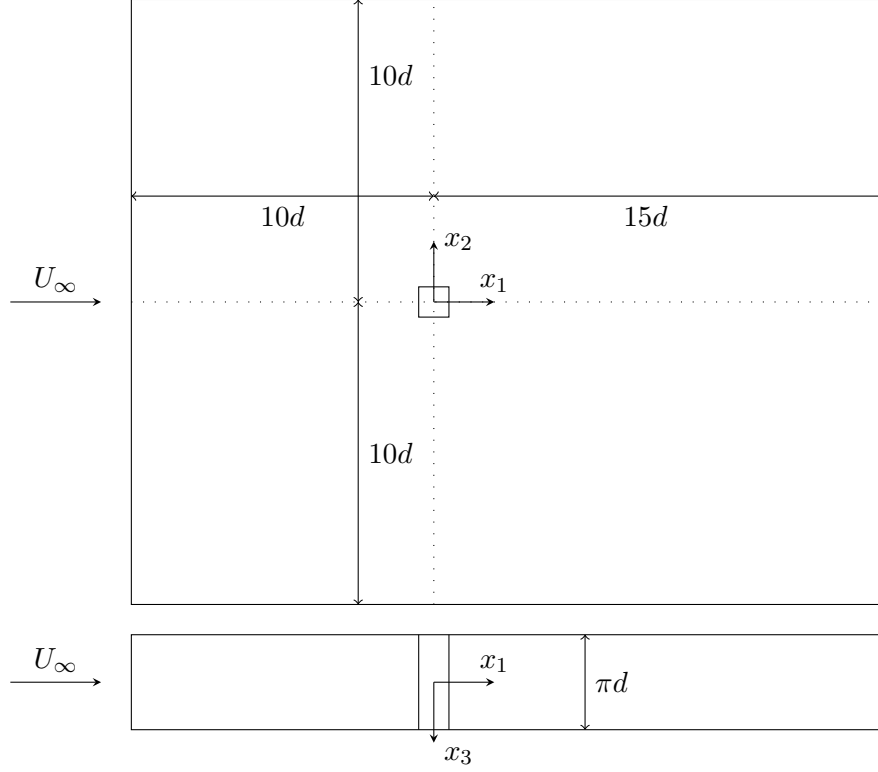


Figure 2.18: Domain dimensions for $Re_\infty = 3900$ DNS. Top: side view; bottom: top view. The origin of our coordinate system is at the centre of the square prism.

are treated as periodic while at the top and bottom the Dirichlet condition for pressure and Neumann condition for the velocity reduce blockage effects allowing the flow to be entrained across those boundaries.

This choice of domain dimensions was made based on existing DNS of bluff body wakes at similar Reynolds numbers (see e.g. Lehmkuhl et al., 2013; Sohankar et al., 1999). The inlet and outlet boundaries should be located far enough from the bluff body such that they have little effect on the solution, with regards to the inlet, experimental data of Durão et al. (1988) suggests that the mean stream-wise velocity is close the free-stream velocity at $x_1/d \approx 2$ but they also report non-negligible fluctuations in the cross-stream velocity component. The outlet was placed far enough downstream such that the computational cost remained amenable, the database generated from the DNS focuses on the flow $5d$ upstream of the outlet boundary. The confinement due to the top/bottom boundaries can lead to blockage effects that can heavily impact the solution (see e.g. Sahin and Owens, 2004), as seen from fig. 2.18, the blockage ratio (ratio between the characteristic size of the object and the height of the domain) in the present configuration is 5% which is considerably small, especially when compared to wind-tunnel experiments. Blockage effects are further minimised by imposing a pressure boundary condition at these boundaries

which allows fluid to be entrained across the boundaries (thus reducing confinement). The span-wise length of the domain used is the same as that used by Lehmkuhl et al. (2013) who studied the turbulent wake past a circular cylinder (at $Re_\infty = 3900$) who noticed no changes in first and second order statistics when the span-wise length was doubled. The span-wise length of πd is able to resolve the largest (unstable) structures in the wake (El Baroudi, 1960; Robichaux et al., 1999; Yamane et al., 1988)

The computational grid used had just under 40 million cells. The length of each side of the square prism in the (x_1, x_2) plane being d , the smallest cell size was $0.0015d$ and was at the corners of the prism. Over each edge 121 nodes were placed with most nodes being concentrated close to the corners. Near the inlet and top/bottom boundaries the cell sizes were set to $0.325d$ while near the outlet the cell size was $0.065d$. Appropriate care was placed in connecting the different grid sizes using stretching functions which allowed for a smooth variation of the cells' dimensions. The mesh was extruded in the span-wise direction generating 150 layers. In order to keep the computational requirements as low as possible, the mesh was stretched (in the x_1 and x_2 directions) allowing the finer cells to be located in the area around the prism. This ensured appropriate resolution of the thin separating shear layer. In the core of the wake the resolution was found to be at worse about 4 times the Kolmogorov length scale

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}}, \quad (2.22)$$

in fact, along the centreline the resolution varied between 3.8η and 2.9η at $x_1/d = 2$ and $x_1/d = 8$, respectively.

It was found that a maximum value of CFL of about 4 allowed the numerical solution to be stable and that led to an average of 0.2 across the whole domain. The largest values of the CFL were found in areas where the flow was still laminar. The resulting time step was $0.0025 \frac{U_\infty}{d}$ which corresponds to about 3000 time steps per shedding cycle. In the core of the wake, this value of the time step is smaller by at least one order of magnitude compared to the Kolmogorov time scale.

In fig. 2.19 we report values of the normalised shedding frequency, the Strouhal number, for three values of Re_∞ (100, 500 and 3900). In particular for the $Re_\infty = 3900$ case, the value of St was found to be 0.13. Following the extensive discussion in section 2.2.1.1 regarding low-frequency unsteadiness, it was found that at $Re_\infty = 3900$ the flow still exhibited multiple states identified by shedding cycles over which the mean lift is either close to zero or predominantly positive/negative; however, it was found that the transition between states occurred more often (i.e. the time spent in asymmetric states was much

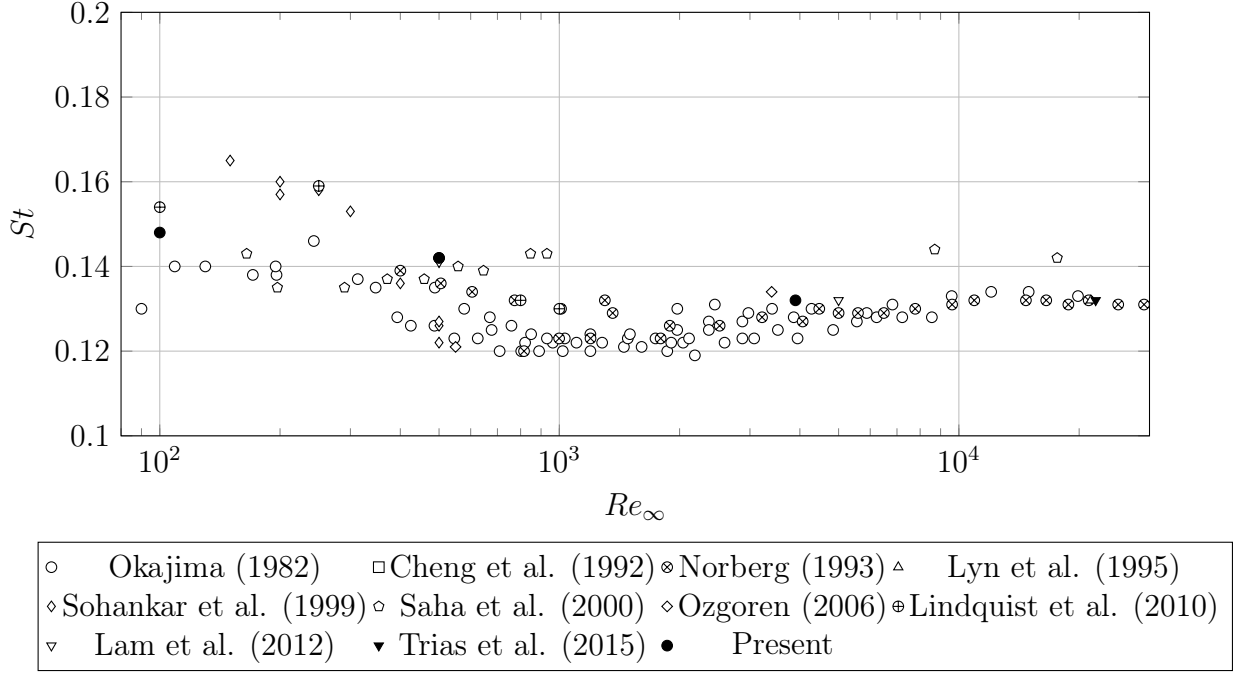


Figure 2.19: St versus Re_∞ from both experiments and DNS. Repeated symbols (for the same Re_∞) indicate different experimental/numerical conditions.

shorter) at a frequency close to 0.04 which is both larger than that observed at $Re_\infty = 500$ regime and smaller than the shedding frequency.

In figs. 2.20 to 2.23 we show the variation along the centreline of U_1 , $\sqrt{\langle u_1^2 \rangle}$, $\sqrt{\langle u_2^2 \rangle}$ and $\sqrt{\langle u_3^2 \rangle}$ which are the mean stream-wise velocity and the standard deviations of the stream-wise, cross-stream and spanwise velocity components, respectively. Apart from the DNS results of Arslan et al. (2012) and Trias et al. (2015) all remaining references refer to experimental results. The DNS of Arslan et al. (2012) employed a code developed for LES simulations, moreover, they report a resolution in terms of Kolmogorov length scales of about 7η . The DNS of Trias et al. (2015) on the other hand report a resolution of about 4η and even though they simulate a much larger Re_∞ flow, we find a very good agreement between our stream-wise profiles and theirs (database available in reference [36] of their paper).

Significant scatter in the statistics of the velocity is observed when comparing different references. In particular, the estimation of the maximum recirculation velocity and rear-stagnation point appears to be drastically different between experiments and DNS. This is in line with the results reported in Voke (1996) and Sohankar (2006) where LES simulations at $Re_\infty = 22000$ are compared with the reference data of Lyn, Einav, et al. (1995). On the other hand, significant differences are also observed in the recovery velocity: while the present results and those of Lyn, Einav, et al. (1995), Lee and G. Kim (2001b) and Trias et al. (2015) show a very slow increase of U_1/U_∞ along x_1/d from the

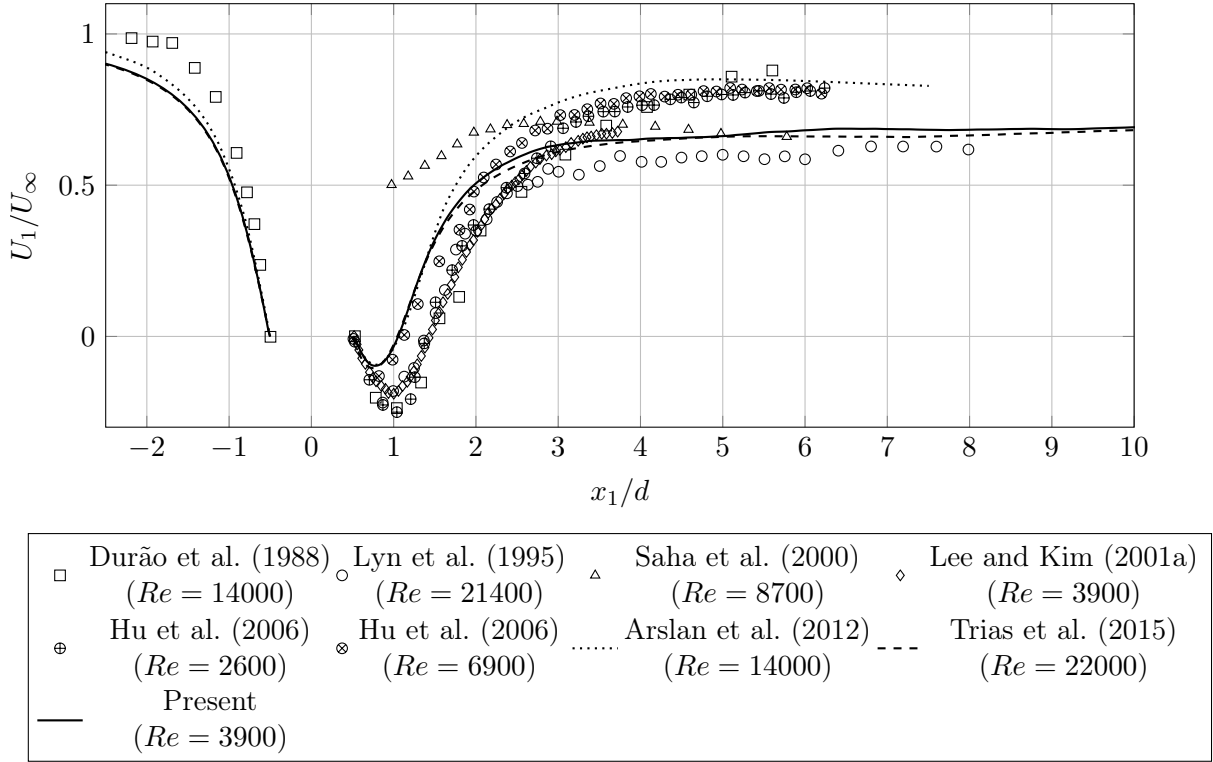


Figure 2.20: Profile of U_1 normalised by U_∞ along the geometrical centreline for $-2.5 < x_1/d < 10$. Experimental results from different references are indicated by symbols while the lines indicate numerical (DNS) results.

value of about 0.6 at $x_1/d \approx 3$ to around 0.7 at $x_1/d \approx 8$ others find the mean velocity to be over 20% higher. It is unlikely that these differences are solely due to wind tunnel blockage and free-stream turbulence. Lyn, Einav, et al. (1995), Lee and G. Kim (2001b) and Arslan et al. (2012) reported a blockage of 7% but Hu et al. (2006) and Trias et al. (2015) report a blockage of about 2% even though there are differences in their results. The free-stream turbulence on the other hand appears to have a more significant effect on U_1/U_∞ in the experiment of Durão et al. (1988) who report 6% free-stream turbulence; this explains the differences observed in the incoming profiles.

As seen in figs. 2.21 to 2.23, the disparity between the different references in the statistics of fluctuating velocities are even more significant than for U_1/U_∞ , particularly in the very near wake. The large values of u'_1 observed at $x_1/d < -0.5$ in the data of Durão et al. (1988) are due to their fairly high free-stream turbulence of 6%. While all references agree that the peak of $\sqrt{\langle u_1^2 \rangle}$ should occur just after $x_1 \approx d$, the actual value of $\sqrt{\langle u_1^2 \rangle}/U_\infty$ at its peak varies between 0.28 and 0.56. We find the peak value of $\sqrt{\langle u_1^2 \rangle}/U_\infty$ to be close to 0.41 which is in agreement with the results of Lyn, Einav, et al. (1995) and Trias et al. (2015) which were obtained at a larger Re_∞ .

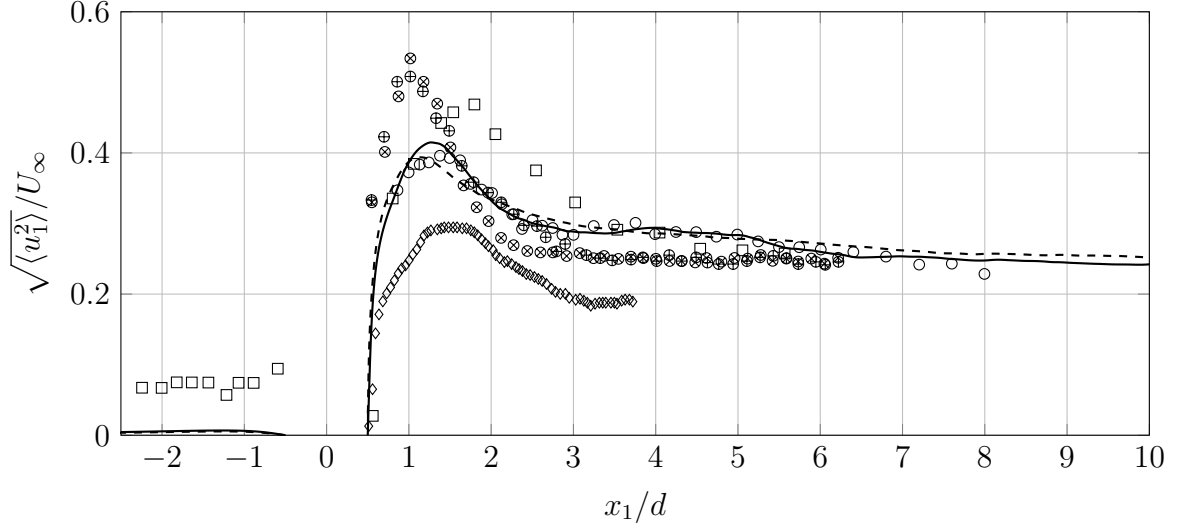


Figure 2.21: Profile of $\sqrt{\langle u_1^2 \rangle}$ normalised by U_∞ along the geometrical centreline for $-2.5 < x_1/d < 10$. Same legend as in fig. 2.20. Turbulent statistics of Lee and G. Kim (2001a) are shown in Lee and G. Kim (2001b).

The intensity of cross-stream fluctuations is mostly determined by the shedding. In fact, the distance from the prism to the location of the peak of $\sqrt{\langle u_2^2 \rangle}/U_\infty$ is used to define the vortex formation length. Both the present results and those of Trias et al. (2015) show a vortex formation length extending to $x_1/d \approx 1.5$, while the experiments of Durão et al. (1988), Lyn, Einav, et al. (1995) and Lee and G. Kim (2001b) place this further downstream, at $x_1/d \approx 1.9$. Apart from the PIV results of Lee and G. Kim (2001b), the peak value of $\sqrt{\langle u_2^2 \rangle}/U_\infty$ is reported to be close to 0.9 both in the present results and Lyn, Einav, et al. (1995) and Trias et al. (2015) and just over 0.8 in Durão et al. (1988). Differences are also observed in the decay of $\sqrt{\langle u_2^2 \rangle}/U_\infty$ with increasing stream-wise distance. Although Lee and G. Kim (2001b) did not measure sufficiently downstream, their results seem to agree with the present DNS and that of Trias et al. (2015) in a slower decrease of $\sqrt{\langle u_2^2 \rangle}/U_\infty$ with x_1/d in comparison to Durão et al. (1988) and Lyn, Einav, et al. (1995).

Fewer results are available for the spanwise fluctuating velocity. The hot-wire measurements of Hu et al. (2006) at two different Re_∞ show different evolutions of $\sqrt{\langle u_3^2 \rangle}/U_\infty$ with x_1/d , while the present DNS at $Re_\infty = 3900$ shows the same profile of $\sqrt{\langle u_3^2 \rangle}/U_\infty$ as in Trias et al. (2015) whose results refer to $Re_\infty = 22000$. In the DNS, the peak of $\sqrt{\langle u_3^2 \rangle}/U_\infty$ on the centreline appears to be at the same location as the mean stagnation point (where $U_1 = 0$).

Finally, we show in fig. 2.24 the contour of the normalised turbulent kinetic energy $\langle k \rangle/U_\infty^2$. The peak turbulent kinetic energy occurs at the centreline, due to the large contribution of $\sqrt{\langle u_2^2 \rangle}$ which also peaks at the centreline. However, it is possible to see

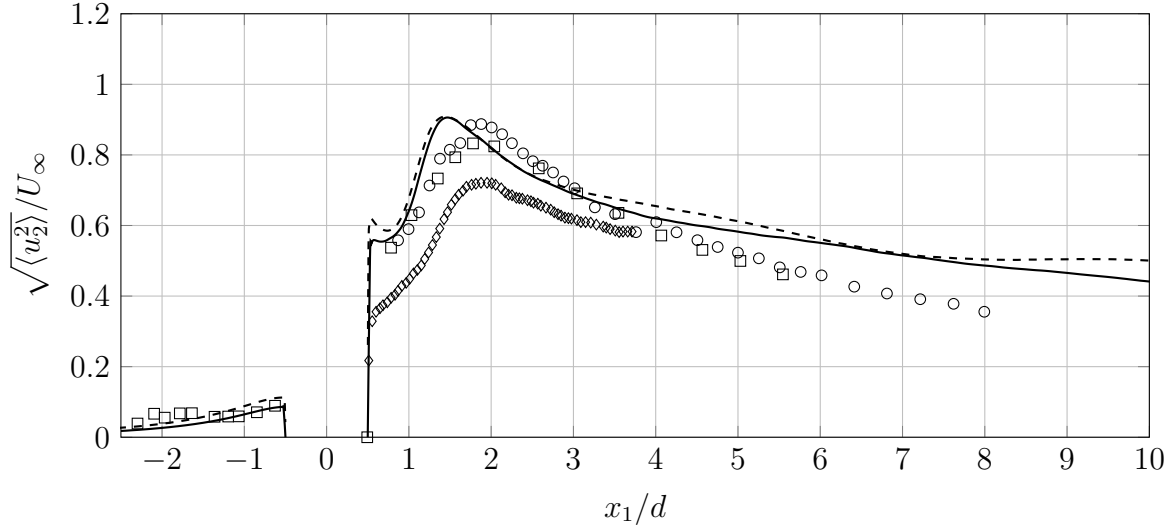


Figure 2.22: Profile of $\sqrt{\langle u_2^2 \rangle}$ normalised by U_∞ along the geometrical centreline for $-2.5 < x_1/d < 10$. Same legend as in fig. 2.20. Turbulent statistics of Lee and G. Kim (2001a) are shown in Lee and G. Kim (2001b).

another peak (of smaller magnitude) roughly at the stagnation point in between the two large recirculation bubbles at the top/bottom and back (see fig. 2.30). This is most likely due to the flapping of the shear layer as it corresponds also to a peak in $\sqrt{\langle u_1^2 \rangle} / U_\infty$.

The shedding frequency f_s was obtained by taking the power spectrum of the time signal of the lift coefficient C_L (shown in fig. 2.25). The alternating peaks and valleys correspond to the instances in time when each vortex departs from the vortex formation region. The drag coefficient C_D fluctuates with twice the shedding frequency (as seen in fig. 2.25) since each alternating vortex induces the same low pressure on the downstream face of the prism.

In figs. 2.26 to 2.28 we report, respectively, values of mean drag coefficient, root mean square of the lift coefficient, and ratio between the root mean square value of drag and lift coefficients, at different Re_∞ . Despite the little availability of statistics for the force coefficients at $Re_\infty \sim \mathcal{O}(10^3)$, we find our results to fall within the experimental and numerical data of the listed references. For the present Reynolds number of 3900, the fluctuating lift coefficient has a standard deviation of 1.52 and the drag coefficient has a mean value of 2.15 with a standard deviation of 0.2. Even though it appears that for the present Re_∞ the Strouhal number and mean drag coefficient are independent of Re_∞ , this appears not to be the case for the standard deviations of both C_L and C_D , as can be seen in figs. 2.27 and 2.28.

In fig. 2.29 the distribution of the mean span-wise averaged pressure coefficient (recall eq. (2.20)) over the surface of the square is shown and compared to experimental and DNS data. Good agreement with literature is found when comparing the obtained distribution

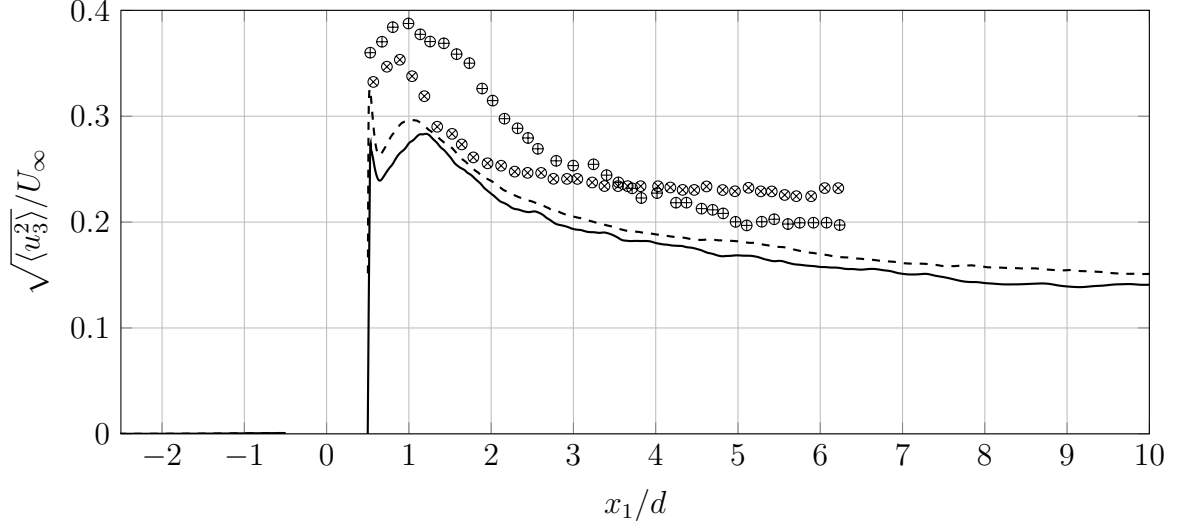


Figure 2.23: Profile of $\sqrt{\langle u_3^2 \rangle}$ normalised by U_∞ along the geometrical centreline for $-2.5 < x_1/d < 10$. Same legend as in fig. 2.20.

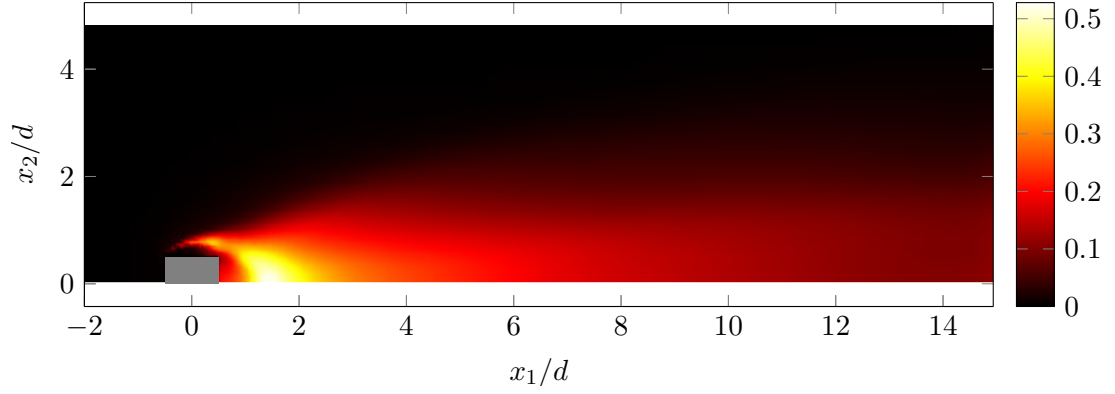


Figure 2.24: Contour (for positive x_2) of mean turbulent kinetic energy $\langle k \rangle$ normalised by U_∞^2 .

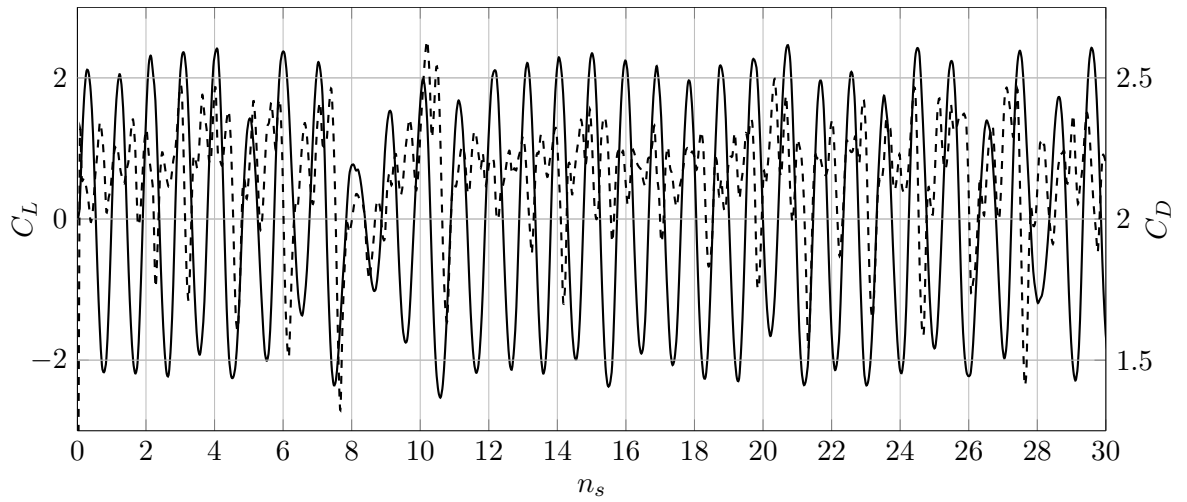


Figure 2.25: Signals of C_L (full line) and C_D (dashed line) versus number of shedding periods $n_s = t f_s$ for $Re_\infty = 3900$.

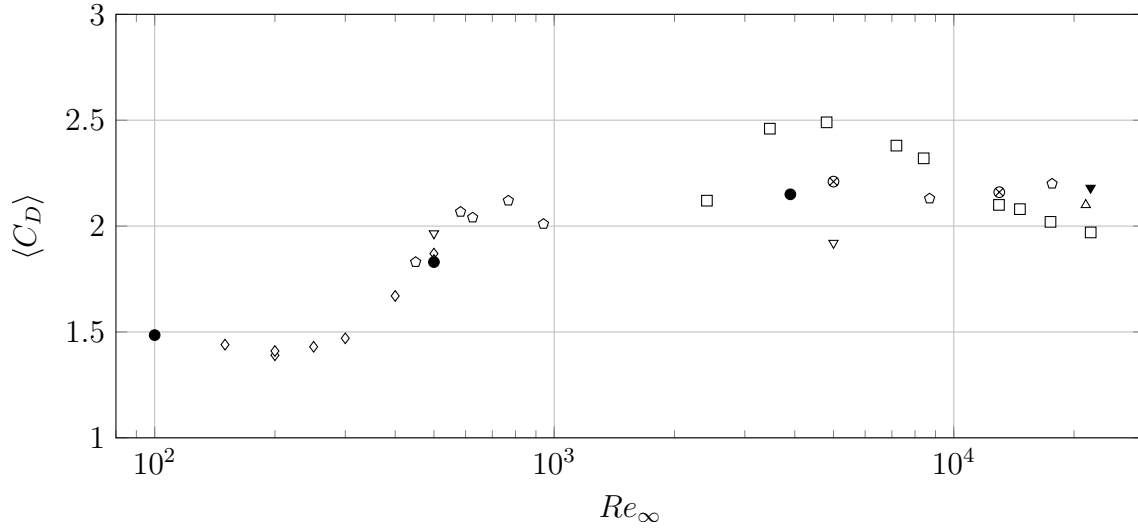


Figure 2.26: Mean drag coefficient at different Re_∞ . The symbols are the same as in fig. 2.19.

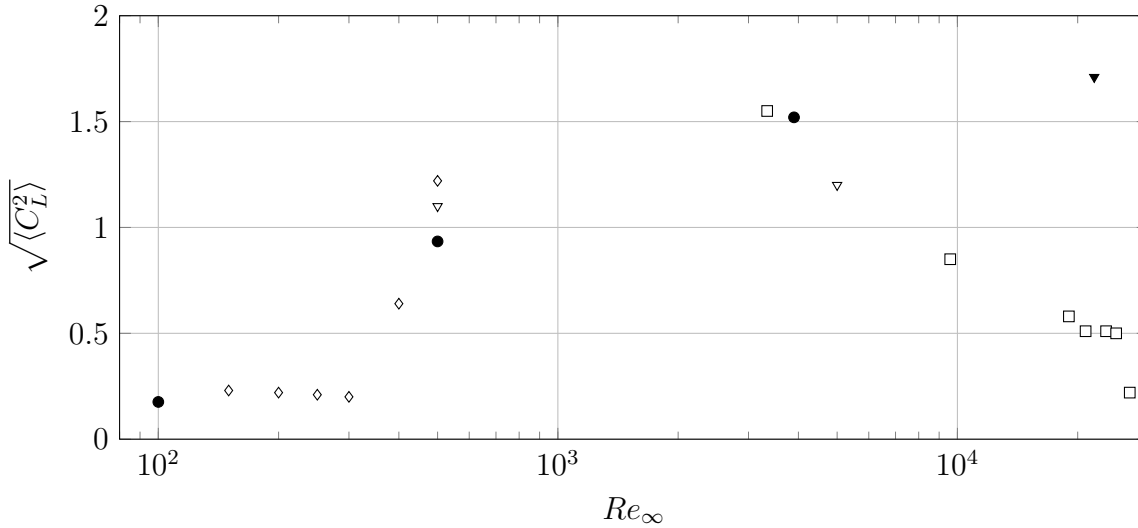


Figure 2.27: Root mean square of lift coefficient at different Re_∞ . The symbols are the same as in fig. 2.19.

of C_P along the surface. Despite the scatter in the experimental results, we find that in our DNS the distribution of C_p along the the back-face of the prism (CD in fig. 2.29) and on the top and bottom faces (BC and DA in fig. 2.29) is within the data of Bearman and Obasaju (1982), Norberg (1993), J. M. Chen and Liu (1999) and Trias et al. (2015).

In fig. 2.30 the mean streamlines are shown in the vicinity of the prism (only the top half of the domain is shown). The large recirculation bubbles (connected by a saddle point at $x_1/d \approx 0.49, x_2/d \approx 0.6$) can be clearly identified above the prism's top surface and just downstream of its backward side. In addition to these recirculation regions, two small separation bubbles exist on the top surface: an elongated one near the front-face

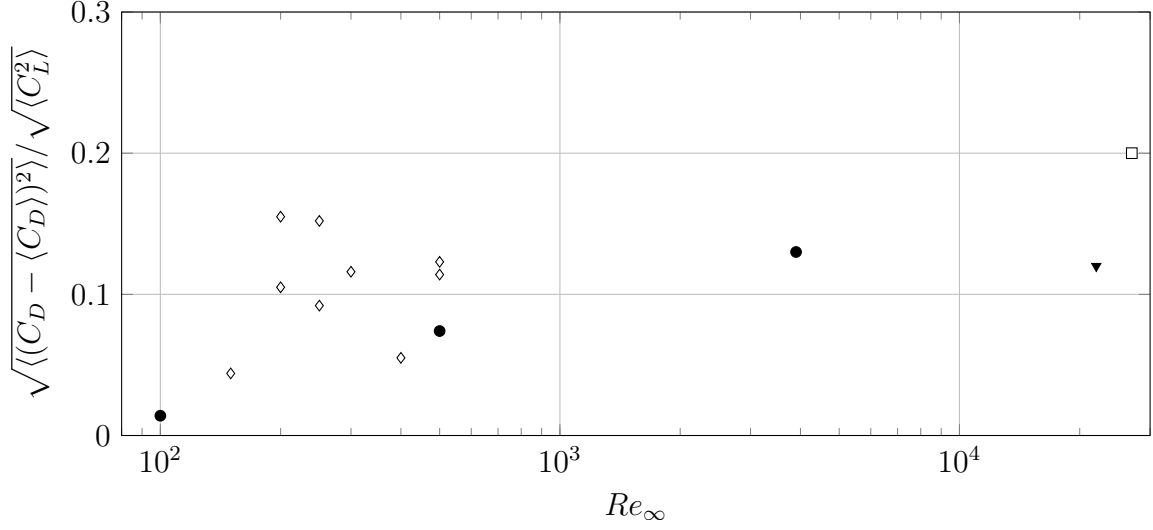


Figure 2.28: Ratio between the root mean square of drag and lift coefficients at different Re_∞ . The symbols are the same as in fig. 2.19.

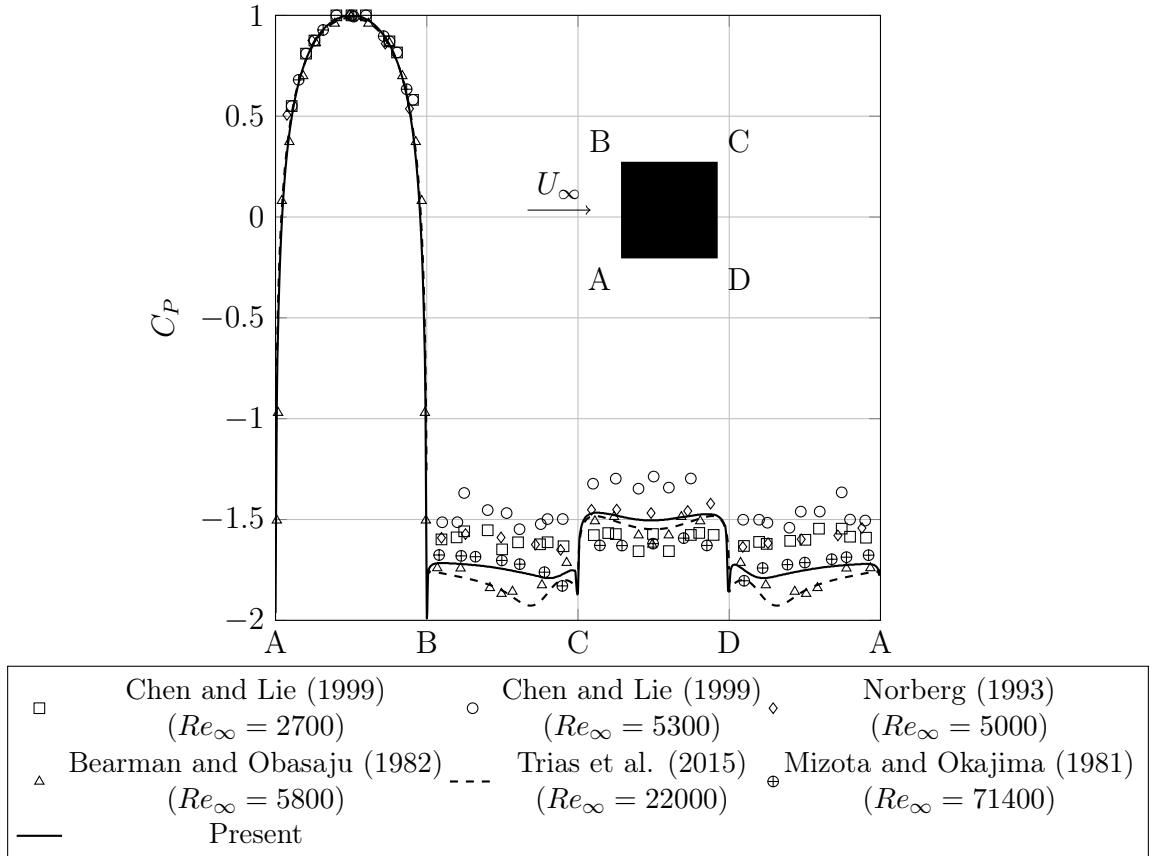


Figure 2.29: Distribution of the spanwise averaged mean pressure coefficient C_P over the sides of the prism. Experimental data are indicated by symbols, DNS by lines. The data from Norberg (1993) were taken from Sohankar et al. (1999). Bearman and Obasaju (1982) only measured C_P over the top half of the prism (from mid-point of AB to the midpoint of CD).

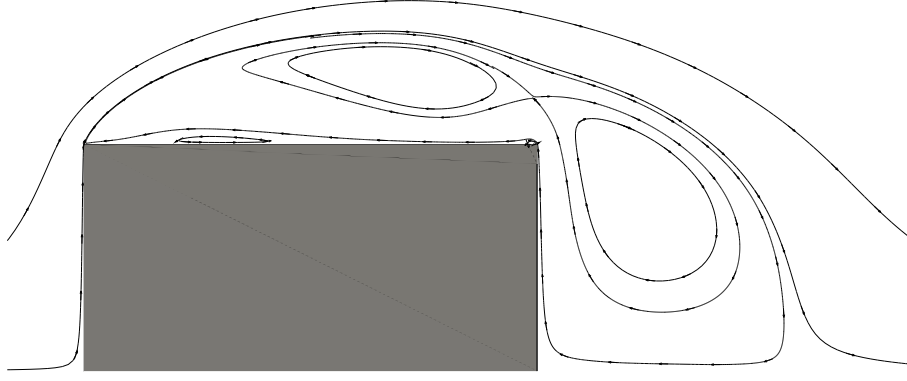


Figure 2.30: Mean streamlines in the vicinity of the prism. Only the top half of the domain is shown.

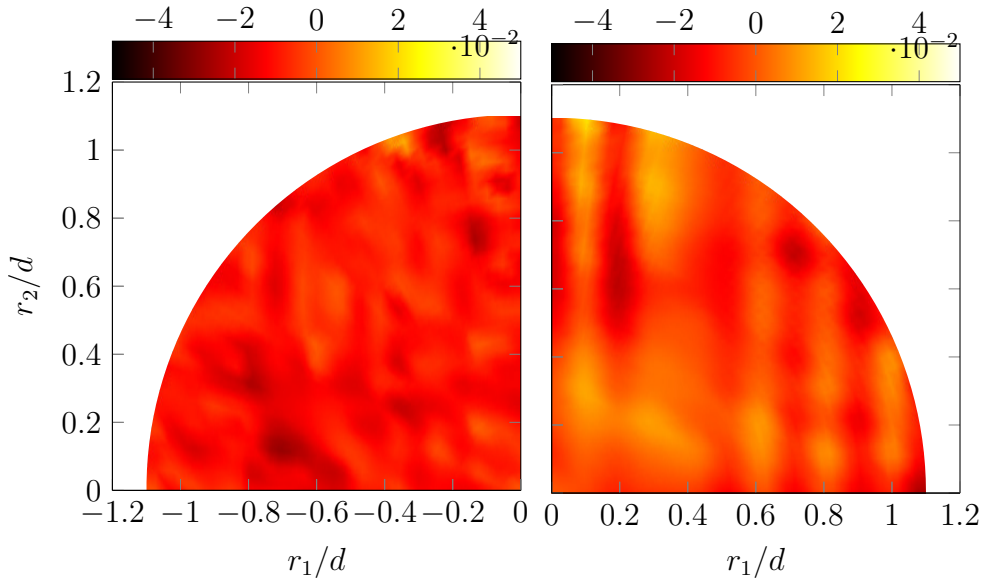


Figure 2.31: Balance of the KMH equation (sum of all terms in eq. (3.1)) normalised by the turbulent dissipation in scale space for $x_1/d = 2$ (left) and $x_1/d = 8$ (right) on the centreline.

and a much smaller one closer to the back-face. These separation bubbles have also been reported in Mizota and Okajima (1981), Sohankar (2006) and Trias et al. (2015) for flows at higher Reynolds numbers.

Finally, because the remainder of the thesis is concerned with budgets of turbulent kinetic energy in physical and scale space, it is important to assess to which extent the relevant equations are satisfied. The generalised two-point kinetic energy equation will be described in detail in chapter 3. As shown in fig. 2.31, the equation is balanced within $\pm 5\%$ of the value of the local dissipation, the residual was found to be even smaller when averaging over the orientations in scale space.

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3 The Turbulence Cascade in the Near Wake

Kolmogorov’s equilibrium cascade theory (Kolmogorov, 1941a; Kolmogorov, 1941b; Kolmogorov, 1941c) applies to locally homogeneous turbulence and is perhaps the one firm theoretical pillar underpinning much of our understanding of turbulent flows as well as much of one-point (e.g. RANS) and two-point (e.g. LES) modelling of such flows over the past 75 years. However, recent developments have shown that this theory does not apply in, at least, extended regions of many turbulent flows where the theory might have been expected to be valid given that local statistical homogeneity is present. Examples of such flows are unsteady periodic turbulence (Goto and Vassilicos, 2015; Goto and Vassilicos, 2016a), grid-generated decaying turbulence (Vassilicos, 2015), self-similar axisymmetric turbulent wakes (Castro, 2016; Obligado et al., 2016; Vassilicos, 2015, and references therein) and the outer region of turbulent boundary layers (Nedić, Tavoularis, et al., 2017). Even so, as Kraichnan (1974) already stated more than forty years ago, “Kolmogorov’s 1941 theory has achieved an embarrassment of success. The $-5/3$ -spectrum has been found not only where it reasonably could be expected but also at Reynolds numbers too small for a distinct inertial range to exist and in boundary layers and shear flows where there are substantial departures from isotropy, and such strong effects from the mean shearing motion that the step-wise cascade appealed to by Kolmogorov is dubious”.

Kraichnan was referring to the Kolmogorov-Obukhov prediction (Obukhov, 1941) that the turbulence energy spectrum is proportional to the $-5/3$ power of wavenumber in the inertial range. This particular prediction has indeed been an “embarrassment of success” because it is present even more widely than Kraichnan stated: it can indeed be found in relatively low Reynolds number flows, boundary layers, wakes, homogeneous shear flows, atmospheric turbulence and mixing layers (see e.g. Champagne et al., 1970; Klebanoff, 1955; Kravchenko and Moin, 2000; Ong and Wallace, 1996; Rogers and Moser, 1994; Ubroi and Freymuth, 1969; Wissink and Rodi, 2008); but it can also be found in those flow regions mentioned above where the Kolmogorov equilibrium is now known to be absent; and it can even be found in flow regions such as in the very inhomogeneous and

anisotropic very near field production region of grid-generated turbulence where it cannot be expected to apply from the very outset (see Gomes-Fernandes et al., 2015).

Even more fundamental than the $-5/3$ power law scaling of the energy spectrum is Kolmogorov’s prediction that, in the inertial range of scales, the interscale energy transfer rate Π is constant, i.e. independent of length-scale and viscosity ν , and equal to $-\varepsilon$ where ε is the turbulence dissipation rate per unit mass. There are various ways to derive the energy spectrum’s $-5/3$ power law shape in the inertial range (see, for example, the textbooks by Frisch, 1995; Lesieur, 2008; Mathieu and Scott, 2000; Pope, 2000; Tennekes and Lumley, 1972) but unlike the constancy of Π , no way is known which derives it directly from the Navier-Stokes equations without closure assumptions (Kraichnan, 1974; Leslie, 1973; McComb, 2014). A straightforward dimensional analysis is typically carried out which starts from the premise that the energy spectrum depends only on ε and wavenumber in the inertial range, a premise which can be justified by the result, itself obtained from Navier-Stokes, that $\Pi \approx -\varepsilon$ in the inertial range. If the energy spectrum is a one-dimensional spectrum (as typically measured in laboratory experiments) then an assumption of small-scale isotropy is also required (Frisch, 1995; Lesieur, 2008; Mathieu and Scott, 2000; Pope, 2000; Tennekes and Lumley, 1972).

While recent DNS (Goto and Vassilicos, 2016a) have shown that, in freely decaying periodic turbulence, Π is neither constant nor equal to $-\varepsilon$ even though a $-5/3$ power law scaling of the energy spectrum is clearly present, PIV measurements by Gomes-Fernandes et al. (2015) suggest that in the very near field of grid-generated turbulence where the turbulence is very inhomogeneous (even locally), anisotropic and building up, Π/ε may be approximately independent of scale over a significant range. Adding to Kraichnan’s “embarrassment of success”, Gomes-Fernandes et al. (2015) find that the energy spectrum has a very well-defined $-5/3$ power law dependence on frequency in this region, a result also obtained and discussed in Laizet, Vassilicos, and Cambon (2013), Laizet, Nedić, et al. (2015), and Melina et al. (2016).

The approach taken by Gomes-Fernandes et al. (2015) relies on the Kármán-Howarth-Monin-Hill (KMH) equation: this is the fully generalised form of the Kármán-Howarth or Kármán-Howarth-Monin equation (see Frisch, 1995) derived by Hill (2002a) which is valid under no assumptions whatsoever, i.e. irrespective of degrees of inhomogeneity, anisotropy and unsteadiness. The KMH equation is an energy balance for the energy associated with eddies of a certain size. It takes into account both interscale and interspace transfers as well as advection, turbulence dissipation and production. This equation is the only rigorous way to make statements concerning Π and ε in a highly inhomogeneous and anisotropic region of a turbulent flow such as the very near-field grid-generated turbulence studied by Gomes-Fernandes et al. (2015). These authors used a planar two-component

PIV to compute the various terms of this equation and therefore had to rely on a number of assumptions to obtain some of these terms. It is therefore essential to carry out a similar study in a well-resolved computational setting where all the terms of the KHM equation can be calculated without having to resort to any unjustified assumption.

In this chapter, without resorting to any assumption, we calculate every term of the KHM equation in a DNS of a turbulent planar wake generated by a square prism and study its near field up to about $10d$, where d is the side of the square. We study the near wake of a square prism because (i) it is numerically less demanding than a DNS of the near field of grid-generated turbulence and is nevertheless a significantly inhomogeneous and anisotropic flow; (ii) power law energy spectra have already been reported in the very near field of planar wakes (Braza et al., 2006; Ong and Wallace, 1996); and (iii) hot wire anemometry studies of highly reduced forms of the KHM equation have already been carried out in turbulent planar wakes by Thiesset, Danaila, and Antonia (2011), Thiesset, Danaila, Antonia, and T. Zhou (2011), Thiesset et al. (2013a) and Thiesset, Danaila, and Antonia (2014). This approach can give us the opportunity to both confirm and extend the results of Gomes-Fernandes et al. (2015) and Thiesset et al. (2013a). However we go beyond this indeed important and necessary confirmation and extension and offer some new insights concerning the various terms of the KHM equation and the processes they represent.

3.1 The Kármán-Howarth-Monin-Hill Equation

The most general forms of scale-by-scale energy budget for incompressible turbulent flows have been derived without making any assumption about the nature of the turbulence by Duchon and Robert (1999) without averaging and by Hill (1997), Hill (2001) and Hill (2002a) with averaging. Using the Reynolds decomposition (capital letters and $\langle \cdot \rangle$ indicate ensemble- and/or time-averaged quantities), the equation derived by Hill (1997), Hill (2001) and Hill (2002a) (which we refer to as KHM equation) takes the form (see also Danaila, Krawczynski, et al., 2012)

$$\begin{aligned} \frac{\partial \langle \delta q^2 \rangle}{\partial t} + \frac{\partial \frac{U_i^+ + U_i^-}{2} \langle \delta q^2 \rangle}{\partial X_i} + \frac{\partial \langle \delta u_i \delta q^2 \rangle}{\partial r_i} + \frac{\partial \delta U_i \langle \delta q^2 \rangle}{\partial r_i} = & -2 \langle \delta u_i \delta u_j \rangle \frac{\partial \delta U_j}{\partial r_i} - \\ & - \langle (u_i^+ + u_i^-) \delta u_j \rangle \frac{\partial \delta U_j}{\partial X_i} - \frac{\partial \langle \frac{u_i^+ + u_i^-}{2} \delta q^2 \rangle}{\partial X_i} - 2 \frac{\partial \langle \delta u_i \delta p \rangle}{\partial X_i} + \nu \frac{1}{2} \frac{\partial^2 \langle \delta q^2 \rangle}{\partial X_i \partial X_i} + \\ & + 2\nu \frac{\partial^2 \langle \delta q^2 \rangle}{\partial r_i \partial r_i} - 4\nu \left(\left\langle \frac{\partial \delta u_j}{\partial X_i} \frac{\partial \delta u_j}{\partial X_i} \right\rangle + \frac{1}{4} \left\langle \frac{\partial \delta u_j}{\partial r_i} \frac{\partial \delta u_j}{\partial r_i} \right\rangle \right) \quad (3.1) \end{aligned}$$

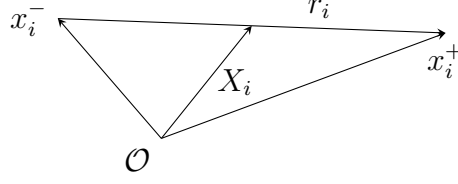


Figure 3.1: Illustration of the spatial position vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{x}^+ = (x_1^+, x_2^+, x_3^+)$ and $\mathbf{x}^- = (x_1^-, x_2^-, x_3^-)$ in a reference frame centred on $\mathcal{O}$, and of separation vector $\mathbf{r} = (r_1, r_2, r_3)$.

where $\delta q^2 = \delta u_i \delta u_i$ in terms of the fluctuating velocity differences $\delta u_i = u_i^+ - u_i^-$ (for components $i = 1, 2, 3$), $\delta U_i = U_i^+ - U_i^-$ where U_i is a mean flow velocity component, $\delta p = p^+ - p^-$ where p is fluctuating pressure, and the superscripts $+$ and $-$ distinguish quantities evaluated at $x^+ = X_i + r_i/2$ and $x^- = X_i - r_i/2$, respectively, as illustrated in fig. 3.1. Equation (3.1) is written in a six-dimensional reference frame X_i, r_i where coordinates X_i are associated with a location in physical space and the scale space is the space of all separations and orientations $\mathbf{r} = (r_1, r_2, r_3)$ between two-points (we refer to $r = |\mathbf{r}|$ as a scale). Throughout this chapter and in chapter 4, in the context of two-point statistics, we refer to isotropy as the independence of a given quantity on the orientation of the separation vector $\mathbf{r}$.

We follow Valente and Vassilicos (2015) and Gomes-Fernandes et al. (2015) in the way we identify the terms in (3.1) as

$$\mathcal{A}_t = -\mathcal{A} - \Pi - \Pi_U + \mathcal{P} + \mathcal{T}_u + \mathcal{T}_p + \mathcal{D}_x + \mathcal{D}_r - \varepsilon_r \quad (3.2)$$

where each term is associated with a physical process in the budget of $\langle \delta q^2 \rangle$:

- $\mathcal{A}_t = \frac{1}{4} \frac{\partial \langle \delta q^2 \rangle}{\partial t}$ represents the rate of change in time of $\langle \delta q^2 \rangle$ at a given physical point X_i and separation r_i . In this thesis statistics are collected in time as opposed to ensemble averages (recall eq. (2.21)) and therefore $\mathcal{A}_t$ vanishes.
- $\mathcal{A} = \frac{1}{4} \frac{\partial \frac{U_i^+ + U_i^-}{2} \langle \delta q^2 \rangle}{\partial X_i}$ is the advection term. This term represents the transport of $\langle \delta q^2 \rangle$ in physical space X_i by the mean flow. In fact, integrating $\mathcal{A}$ over a volume $\mathcal{V}_x$ in physical space and using Gauss' theorem yields $\iiint_{\mathcal{V}_x} \mathcal{A} dV = \oint_{\partial \mathcal{V}_x} \frac{U_i^+ + U_i^-}{2} \langle \delta q^2 \rangle n_i dS$ which is the integral of a flux through $\partial \mathcal{V}_x$, the boundary of $\mathcal{V}_x$. If this volume encompasses a set of mean streamlines, the flux integral is proportional to the difference of $\langle \delta q^2 \rangle$ between the downstream and upstream boundaries in the case of homogeneous turbulence, in which case $\mathcal{A}$ accounts for the decay of $\langle \delta q^2 \rangle$ in the direction of the mean flow (Hill, 2002b; Thiesset et al., 2013a).

- $\Pi = \frac{1}{4} \frac{\partial \langle \delta u_i \delta q^2 \rangle}{\partial r_i}$ is the non-linear inter-scale transfer rate and accounts for the effect of non-linear interactions in redistributing δq^2 within the r_i space and is given by the divergence in scale space of the flux $\langle \delta u_i \delta q^2 \rangle$. If one takes the integral of Π over a volume in scale space $\mathcal{V}_r$, then, similarly to the discussion above, one obtains the flux integral $\oint_{\partial \mathcal{V}_r} \langle \delta u_i \delta q^2 \rangle n_i dS$. If $\mathcal{V}_r$ is taken to be a sphere of radius ℓ , then $\iiint_{\mathcal{V}_r} \Pi dV$ is proportional to the orientation-averaged flux $\langle \delta \mathbf{u} \delta q^2 \rangle$ in scale space, corresponding to a length-scale equal to the radius of the sphere.
- $\Pi_U = \frac{1}{4} \frac{\partial \delta U_i \langle \delta q^2 \rangle}{\partial r_i}$ is the linear inter-scale transfer rate. Similarly to Π , this term accounts for transfer of δq^2 in scale space r_i , but instead of δu_i , it is the two-point difference of mean velocity δU_i that transports δq^2 in scale space.
- $\mathcal{P} = \frac{1}{4} \left[-2 \langle \delta u_i \delta u_j \rangle \frac{\partial \delta U_j}{\partial r_i} - \langle (u_i^+ + u_i^-) \delta u_j \rangle \frac{\partial \delta U_j}{\partial X_i} \right]$ can be associated with the production of $\langle \delta q^2 \rangle$ by mean flow gradients. Writing $4\mathcal{P}$ in terms of $x_i^\pm = X_i \pm r_i/2$ yields $-2 \langle u_i^+ u_j^+ \rangle \frac{\partial U_j^+}{\partial x_j^+} - 2 \langle u_i^- u_j^- \rangle \frac{\partial U_j^-}{\partial x_j^-} + 2 \langle u_i^- u_j^+ \rangle \frac{\partial U_j^+}{\partial x_j^+} + 2 \langle u_i^+ u_j^- \rangle \frac{\partial U_j^-}{\partial x_j^-}$ where the first two terms represent the production terms in the one-point turbulent kinetic energy equation and the last two represent combined actions of the mean flow gradients and the two-point correlation tensors $\langle u_i^\pm u_j^\mp \rangle$ (see e.g. Lindborg, 1996a, where such terms are related to the equation for $\langle \delta u_i \delta u_j \rangle$). Furthermore, if one writes the interscale energy budget for $\delta Q^2 = \delta U_i \delta U_i$, which is the mean-flow equivalent to (3.1), the term $\mathcal{P}$ appears as it does in (3.1) but with the opposite sign:

$$\begin{aligned}
 \frac{\partial \delta Q^2}{\partial t} + \frac{\partial \frac{U_i^+ + U_i^-}{2} \delta Q^2}{\partial X_i} + \frac{\partial \langle \delta u_i \delta u_j \delta U_j \rangle}{\partial r_i} + \frac{\partial \delta U_i \delta Q^2}{\partial r_i} &= 2 \langle \delta u_i \delta u_j \rangle \frac{\partial \delta U_j}{\partial r_i} + \\
 + \langle (u_i^+ + u_i^-) \delta u_j \rangle \frac{\partial \delta U_j}{\partial X_i} - \frac{\partial \langle \frac{u_i^+ + u_i^-}{2} \delta u_j \delta U_j \rangle}{\partial X_i} - 2 \frac{\partial \langle \delta U_i \delta P \rangle}{\partial X_i} + \nu \frac{1}{2} \frac{\partial^2 \delta Q^2}{\partial X_i \partial X_i} + \\
 + 2\nu \frac{\partial^2 \delta Q^2}{\partial r_i \partial r_i} - 4\nu \left(\langle \frac{\partial \delta U_j}{\partial X_i} \frac{\partial \delta U_j}{\partial X_i} \rangle + \frac{1}{4} \langle \frac{\partial \delta U_j}{\partial r_i} \frac{\partial \delta U_j}{\partial r_i} \rangle \right) \quad (3.3)
 \end{aligned}$$

It follows that the interscale energy budget for $\langle [(U_i^+ + u_i^+) - (U_i^- + u_i^-)][(U_i^+ + u_i^+) - (U_i^- + u_i^-)] \rangle$, which equals $\delta Q^2 + \langle \delta q^2 \rangle$, does not involve $\mathcal{P}$. This observation consolidates the interpretation of $\mathcal{P}$ as a production term. When $\mathcal{P}$ is positive/negative, energy is therefore lost/gained by δQ^2 and gained/lost by $\langle \delta q^2 \rangle$ at the same rate. This process is only significant at large enough values of r where δQ^2 is not negligible. Note that $\frac{\partial \langle \delta u_i \delta u_j \delta U_j \rangle}{\partial r_i}$ is the difference between the interscale energy transfer rate of the total two-point kinetic energy, $\frac{\partial \langle \delta u_i \frac{1}{2} (\delta U_j + \delta u_j) (\delta U_j + \delta u_j) \rangle}{\partial r_i}$, and the interscale energy transfer rate $2\Pi = \frac{\partial \langle \delta u_i \delta q^2 \rangle}{\partial r_i}$ of $\frac{1}{2} \delta q^2$. All the other terms in (3.3) can be inter-

preted in the same way as their analogues in (3.1), except for the term $\frac{\partial \langle \frac{u_i^+ + u_i^-}{2} \delta u_j \delta U_j \rangle}{\partial X_i}$ which we discuss in the next bullet point.

- $\bullet$ $\mathcal{T}_u = -\frac{1}{4} \frac{\partial \langle \frac{u_i^+ + u_i^-}{2} \delta q^2 \rangle}{\partial X_i}$ is the transport of δq^2 in physical space due to turbulent fluctuations. It features in (3.1) whereas the term $\frac{\partial \langle \frac{u_i^+ + u_i^-}{2} \delta u_j \delta U_j \rangle}{\partial X_i}$ which features in (3.3) is the difference between the turbulent transport of total two-point kinetic energy $\frac{1}{2}(\delta U_j + \delta u_j)(\delta U_j + \delta u_j)$ in physical space and $2\mathcal{T}_u$.
- $\bullet$ $\mathcal{T}_p = -\frac{1}{2} \frac{\partial \langle \delta u_i \delta p \rangle}{\partial X_i}$ is equal to -2 times the correlation between velocity differences and differences of pressure gradient.
- $\bullet$ $\mathcal{D}_x = \nu \frac{1}{8} \frac{\partial^2 \langle \delta q^2 \rangle}{\partial X_i \partial X_i}$ is the diffusion in physical space due to viscosity. This term is analogous to the diffusion term appearing in the single-point turbulent kinetic energy equation and thus its contribution to (3.1) is expected to be small.
- $\bullet$ $\mathcal{D}_r = \frac{1}{2} \nu \frac{\partial^2 \langle \delta q^2 \rangle}{\partial r_i \partial r_i}$ is the diffusion in scale space by viscosity. This term is equal to the dissipation ε when the two points coincide ($r = 0$) and can be shown (see Appendix B in Valente and Vassilicos, 2015) to be negligible for separations much larger than the Taylor micro-scale.
- $\bullet$ ε_r is the two-point average dissipation rate $\varepsilon_r = \frac{\varepsilon^+ + \varepsilon^-}{2}$ since, by expanding the gradients (see Appendix A) it equals $\frac{1}{2} \nu \left(\langle \frac{\partial u_j^+}{\partial x_i^+} \frac{\partial u_j^+}{\partial x_i^+} \rangle + \langle \frac{\partial u_j^-}{\partial x_i^-} \frac{\partial u_j^-}{\partial x_i^-} \rangle \right)$.

If a turbulent flow is locally homogeneous over length-scales r smaller than a certain inhomogeneity length-scale, then (3.3) reduces to the identity $0 = 0$ and (3.1) reduces to

$$\mathcal{A}_t + \mathcal{A} + \Pi \approx \mathcal{D}_r - \varepsilon \quad (3.4)$$

for such length-scales r . Given that $\mathcal{D}_r$ is negligible at length-scales r larger than the local Taylor length-scale, (see Appendix B of Valente and Vassilicos, 2015), one is left with

$$\mathcal{A}_t + \mathcal{A} + \Pi \approx -\varepsilon \quad (3.5)$$

in the intermediate range of r between the Taylor length-scale and the inhomogeneity length-scale (assuming we are considering a region of a turbulent flow where such a range exists). Kolmogorov's hypothesis of local equilibrium then leads to the equilibrium relation

$$\Pi \approx -\varepsilon \quad (3.6)$$

which is the central property of the Kolmogorov equilibrium cascade in locally homogeneous turbulence. A further step is needed to obtain Kolmogorov's $\langle \delta q^2 \rangle \sim \varepsilon^{2/3} r^{2/3}$ and $\langle \delta u_i^2 \rangle \sim \varepsilon^{2/3} r_i^{2/3}$ for $i = 1, 2, 3$ and the Kolmogorov-Obukhov $-5/3$ energy spectrum (Obukhov, 1941). This step requires additional hypotheses of small-scale isotropy and self-similarity, and proceeds either by dimensional analysis Kolmogorov (1941a) or by taking the skewness of the velocity differences to be constant Kolmogorov (1941c).

Under the assumptions of equilibrium, homogeneity and isotropy introduced in Kolmogorov (1941a) and Kolmogorov (1941c) it is possible to recover eq. (1.7) from eq. (3.1). The term $\mathcal{A}_t$ vanishes by assuming statistical stationarity while $\mathcal{A}$ vanishes under the assumption of equilibrium. Introduction of homogeneity removes Π_U , $\mathcal{P}$, $\mathcal{T}_u$ and $\mathcal{D}_x$ while $\mathcal{T}_p$ is zero in isotropic flows. The isotropy assumption can then be used to remove the dependence in the orientation (see the original derivation of eq. (1.7) for the correlation function in von Kármán and Howarth, 1938).

The implicit assumption of equilibrium done in K41 is reflected by the absence of a source term for the energy cascading down the scales. In an attempt to conciliate K41 with this implicit assumption of equilibrium it has been suggested to force the turbulence using a source term $\mathbf{f}(\mathbf{x}, t)$ in the Navier-Stokes equations (as done in Frisch, 1995, for the homogeneous case), requiring an additional term $2\delta u_i \delta f_i$ in eq. (3.1). Nie and Tanveer (1999) have shown that without the assumption of homogeneity and isotropy, such form of the Kármán-Howarth-Monin-Hill still leads to eqs. (1.8) and (1.9) provided that one averages over all space $\mathbf{X}$ and all orientations of $\mathbf{r}$.

Including an artificial forcing makes the interpretation of eq. (3.1) (at least in its homogeneous form) as a scale-by-scale energy budget more straightforward, as it allows the energy source (arising from $2\delta u_i \delta f_i$) to be placed explicitly and uniquely at the largest scales therefore retrieving Kolmogorov's relations for the inertial range (see Chapter 6 of Frisch, 1995). However, when describing naturally occurring flows, this forcing arises from the flow itself. Decomposing the flow into a mean and a fluctuating part reveals this effect (see description below eq. (3.1)). Effectively, one may consider these additional terms as a source term for the budget of $\langle \delta q^2 \rangle$ via $-\mathcal{A} - \Pi_U + \mathcal{P}$: while both Π_U and $\mathcal{P}$ vanish for an homogeneous mean flow $\mathcal{A}$ must be retained as $\langle \delta q^2 \rangle$ decays in the direction of $\mathbf{U}$, i.e. $\frac{\partial \delta q^2}{\partial x_{\parallel}} \neq 0$. In Hill (2002b) and Lindborg (1999) it is noted that the term $\mathcal{A}$ cannot be dropped as it should appear as a consequence of the material derivative of $\langle \delta q^2 \rangle$ while Danaila, Anselmetti, et al. (1999) stress that by Taylor's hypothesis this term can be related to the unsteady term emphasising its relation to the assumption of equilibrium.

The Kármán-Howarth-Monin-Hill has been applied to a few studies of turbulent flows. In Marati et al. (2004) eq. (3.1) is applied to a DNS of a channel flow. The authors make a precise description of the relevance of each term in the various regions of the channel.

Their approach is to address the flux terms in eq. (3.1) in terms of their wall-normal fluxes. They find that in different regions of the channel the terms in eq. (3.1) can act as either source or sink of energy. In particular, they observe that in the buffer region, some separations display an excess of gain of $\langle \delta q^2 \rangle$ in comparison with the dissipation which is counter acted by an inverse cascade mechanism. The importance of $\mathcal{T}_u$ is highlighted as acting as a “bridge” between the buffer layer and the outer region of the flow. In the log-layer, they recover the classical picture of equilibrium whereby $\mathcal{P}$ dominates at the large scales and then Π transfers the energy down the scales.

Equation (3.1) has also been studied in the context of wakes (see Thiesset, Danaila, and Antonia, 2011; Thiesset, Danaila, Antonia, and T. Zhou, 2011; Thiesset et al., 2013b, and references therein) as the coupling between the coherent motion and the turbulence is still not well understood (Cantwell and Coles, 1983). In Thiesset, Danaila, and Antonia (2011) and Thiesset, Danaila, Antonia, and T. Zhou (2011) the authors apply a triple decomposition to eq. (3.1) in order to assess the contribution of the coherent motion to the energy cascade. They find that such contribution is small but that it acts as a sink in eq. (3.1) counteracting the direct cascade associated with the turbulent motion. In Thiesset et al. (2013b) the same authors study planar wakes generated by different geometries. They find that over a wide range of scales, the budget of energy is satisfied by taking into account only $\mathcal{A}$, Π and ε in eq. (3.1). They also find that the scale at which $\mathcal{A}$ and Π are equal acts as normalisation length scale for the dissipation. It must be pointed out, however, that the measurements were taken much further downstream than the region investigated here. Furthermore, those studies have employed only hot-wires, which means that only separations aligned with the mean-flow could be investigated.

In Valente and Vassilicos (2015) the Kármán-Howarth-Monin-Hill is used to characterise the decay of turbulence generated by a grid. They show that terms such as $\mathcal{P}$ and $\mathcal{T}_u$ are more significant for eq. (3.1) at large scales and at the locations closest to the grid. As they investigate eq. (3.1) further downstream into the decay, they find (similarly to Thiesset et al., 2013b) that the budget is dominated by $\mathcal{A}$, Π and ε . Throughout the different locations investigated, they find that the peak value of Π scales with kinetic energy and integral length scale, regardless of the scaling of the ε . Furthermore, they stress that while these flows are clearly not in Richardson-Kolmogorov equilibrium (as $\Pi \neq \varepsilon$ at any scale) the power spectra of the fluctuating velocity exhibit a slope of $-5/3$ especially at locations closest to the generating grids.

In Gomes-Fernandes et al. (2015) a PIV study is carried out in turbulence generated by fractal grids. Such grids extended the region over which the turbulence is out of Richardson-Kolmogorov equilibrium but still exhibit the classical $-5/3$ slope of the power spectrum density (Gomes-Fernandes et al., 2014; Laizet and Vassilicos, 2011; Laizet, Vas-

silicos, and Cambon, 2013; Laizet, Nedić, et al., 2015; Mazellier and Vassilicos, 2010; Seoud and Vassilicos, 2007; Valente and Vassilicos, 2012). The region of the flow investigated in Gomes-Fernandes et al. (2015) is highly inhomogeneous and anisotropic, however, a clear $-5/3$ is observed in the power spectrum in the frequency domain. Analysis of the scale by scale budget of energy reveals that when averaging over orientations on the PIV plane (similar to the approach suggested in Nie and Tanveer, 1999) there is a range of scales over which $\Pi \approx \text{const}$ while the other terms of eq. (3.1) are non-negligible. In addition to this, they also find (in the same region) that both direct and inverse fluxes of energy coexist in scale space but that the orientation averaged flux occurs from large to small scales.

Due to experimental limitations which prevent explicit measurement and computation of many terms in (3.1), the laboratory studies of Thiesset, Danaila, and Antonia (2011), Thiesset, Danaila, Antonia, and T. Zhou (2011), Thiesset et al. (2013a), Thiesset, Danaila, and Antonia (2014), Valente and Vassilicos (2015) and Hearst and Lavoie (2014) have focused on (3.4), mostly in an intermediate region of planar wakes (at streamwise distances larger than $10d$ and typically $40d$ from the wake generator of size d). Gomes-Fernandes et al. (2015) attempted to measure and compute as many of the terms in (3.1) as possible from a planar two-component PIV in the very near field of a turbulence generated by a fractal square grid. Their results suggest that, upstream of the peak of turbulent kinetic energy where the turbulence is very inhomogeneous and anisotropic and therefore not decaying but rather building up, the inter-scale transfer rate Π is approximately constant within a sizeable range of scales.

A constant Π over a range of scales is what (3.6) would predict for a locally homogeneous turbulence. However, the measurements of Gomes-Fernandes et al. (2015) suggested that all the terms appearing in (3.1) have a non-negligible contribution, the exact opposite of what the Kolmogorov theory assumes for the purpose of deriving (3.6). Furthermore, Gomes-Fernandes et al. (2015) observed a combined forward and inverse cascade in scale space in the region of their measurements. Specifically their results suggested that the interscale energy flux $\langle \delta u_r \delta q^2 \rangle = \langle \delta u_i \delta q^2 \rangle \frac{r_i}{r}$ has different signs at different orientations. However, some of the terms in eq. (3.1) were inaccessible to Gomes-Fernandes et al. (2015) due to experimental limitations and others required assumptions to be computed from their data.

In this chapter we present DNS evidence which confirms and strengthens the conclusions of Gomes-Fernandes et al. (2015) and which permits us to go further and obtain new insights about (3.1) in the near wake of a square prism.

3.2 Spectra and Structure Functions

3.2.1 Energy Spectra in Frequency Domain and Taylor's Hypothesis

We start by looking at turbulence spectra in the region of the flow where we study the scale-by-scale energy budget (3.1) in the next section. The energy spectra E_{u_i} at several locations along the centreline are shown in figs. 3.2a, 3.3a and 3.3b for the streamwise, cross-stream and spanwise components of the fluctuating velocity, respectively. At these locations, the spectra display a power law which is close to $-5/3$ over a substantial range of frequencies at $x_1/d = 2$. As the distance x_1 to the prism increases, this power law deteriorates while the range of frequencies narrows. This is in spite of the flow becoming more homogeneous and undergoing an increase in local Reynolds' number.

In fig. 3.2b the $5/3$ compensated spectra of the streamwise fluctuations are shown in a linear-logarithmic plot to highlight that the spectra are close to a $-5/3$ power law over just under a decade of frequencies, particularly at $x_1 = 2d$ which is located straight after the peak of turbulent kinetic energy (as seen in fig. 2.24). As will become fully clear in the following section, this observation is in-line with the comment made by Kraichnan (1974) that such a power law is observed in regions of flows where Kolmogorov's theory is not expected to hold. Several other authors have reported such a power law in simulations and experiments of planar wakes (see e.g. Braza et al., 2006; Cantwell and Coles, 1983; Kravchenko and Moin, 2000; Lehmkuhl et al., 2013; Ma et al., 2000; Ong and Wallace, 1996; Trias et al., 2015; Wissink and Rodi, 2008) at comparable distances from the wake generator (on the centreline) and similar Reynolds' numbers.

Less reported by previous authors, however, is the fact that the range of frequencies over which a power law is observed appears to decrease with increasing downstream distance in agreement with a similar observation made by Laizet, Nedić, et al. (2015) in near-field grid-generated turbulence. It is also worth mentioning that the local Taylor length-based Reynolds number $Re_\lambda = \frac{\sqrt{\langle u_1^2 \rangle} \lambda}{\nu}$, where $\lambda = \sqrt{15 \frac{\nu}{\epsilon}} \sqrt{\langle u_1^2 \rangle}$, increases from about 120 at $x_1/d = 2$ to about 170 at $x_1/d = 10$. This is a first indication that these near $-5/3$ power laws do not have much to do with Kolmogorov theory where an increase in Reynolds number is expected to lead to a broader inertial range.

Due to the predominantly planar nature of the vortex shedding only the spectra of the streamwise and cross-stream components display pronounced peaks (figs. 3.2a and 3.3a) while this is not the case for the spectra of spanwise fluctuations (fig. 3.3b). For the streamwise component, the peak appears at $2f_s$ while for the cross-stream component the peak occurs at f_s . This is due to the alternating vortices shed from the top and

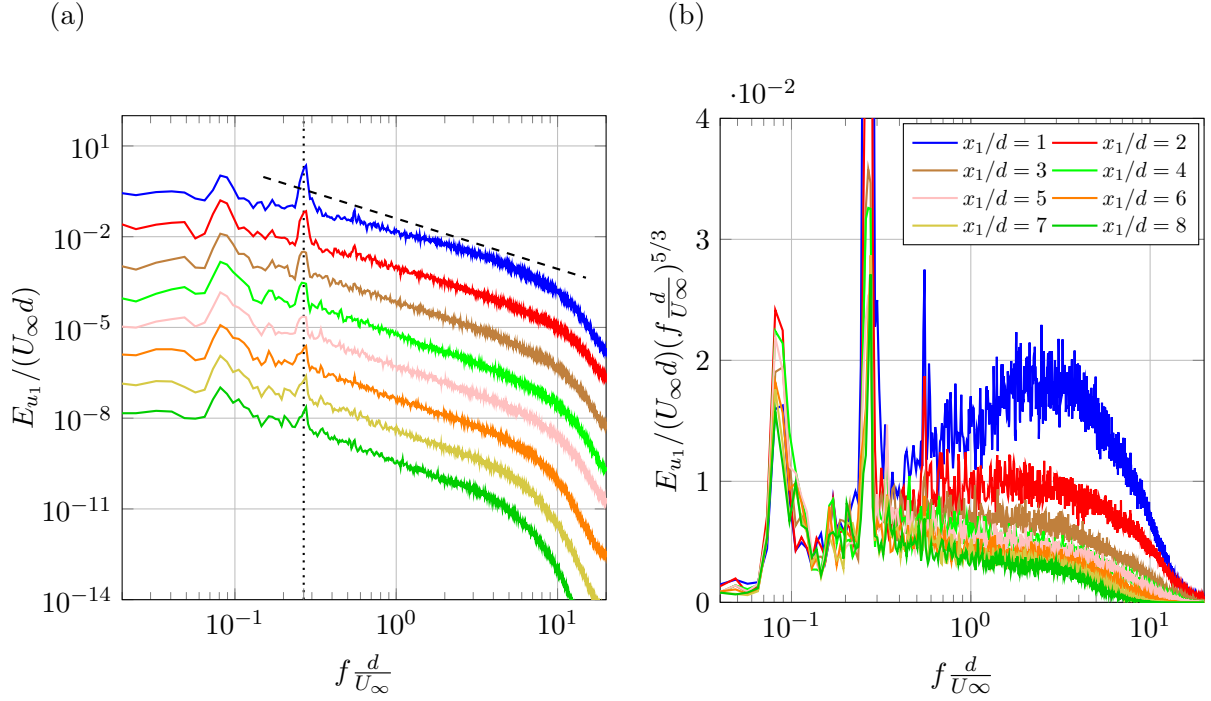


Figure 3.2: Power spectrum densities of streamwise fluctuating velocity at three locations on the centreline (different locations are offset by one decade) normalised by $U_\infty d$. In (a) the dashed line indicates a slope of $-5/3$ and the dotted line indicates $f = 2f_s$; in (b) compensated spectra are plotted in linear-logarithmic axes.

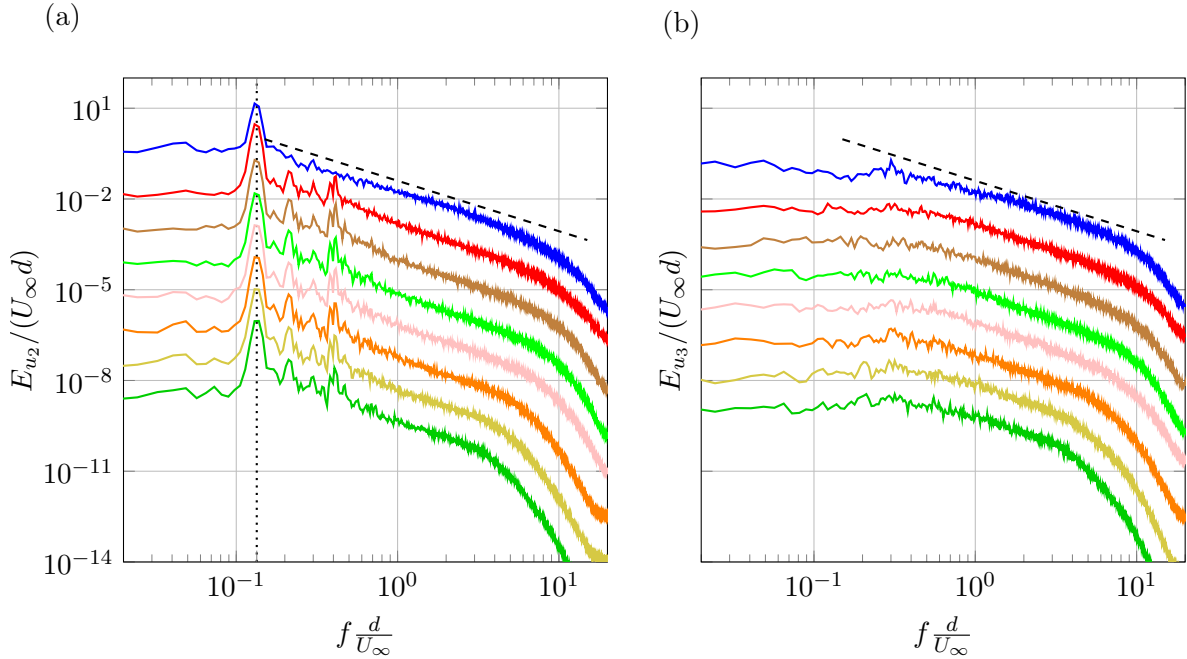


Figure 3.3: Power spectrum densities, normalised by $U_\infty d$, of the cross-stream and spanwise velocity components (a) and (b), respectively (different locations are offset by one decade). See fig. 3.2 for legend. The dashed line indicates a slope of $-5/3$. In (a) the dotted line indicates $f = f_s$.

bottom sides having spanwise vorticity of opposite signs, thus inducing on the centreline streamwise fluctuations in the same direction but cross stream fluctuations in opposite directions.

In fig. 3.2a it is possible to identify a peak which is not a harmonic of the shedding frequency, at $f \frac{d}{U_\infty} \approx 0.085$ while in fig. 3.3a there is a peak at $f \frac{d}{U_\infty} = f_s \frac{d}{U_\infty} + 0.085$. Such peak is most likely related to the bi-stable behaviour observed in section 2.2. This sort of low-frequency unsteadiness which has been reported in wakes generated by flat plates and other bluff bodies (see Johnson et al., 2004; Lehmkuhl et al., 2013; Najjar and Balachandar, 1998, and references therein) which are not yet well understood (and beyond the point of this thesis).

The $-5/3$ power law spectrum obtained by the Kolmogorov theory is for wavenumber spectra (Obukhov, 1941), not frequency spectra. The conversion from wavenumber to frequency required to interpret spectra obtained from velocity data collected over time at a particular point in space is usually based on Taylor's frozen turbulence hypothesis (introduced in Taylor, 1935). According to this hypothesis, high enough frequencies f are related to high enough wavenumbers κ by $\kappa = 2\pi f/U_c$ where U_c is a convection velocity, usually the mean flow velocity at the data collection point. Given that we find well-defined power laws in our frequency spectra with exponents more or less close to $-5/3$ it makes sense to test the validity of Taylor's hypothesis at those points where these frequency spectra were found.

To verify Taylor's hypothesis, we follow the approach of Laizet, Nedić, et al. (2015) and computed the correlation coefficient

$$\rho_{u_i}(x_j, \Delta x_j, \Delta t) = \frac{\langle u_i(x_j, t) u_i(x_j + \Delta x_j, t + \Delta t) \rangle}{\sqrt{\langle u_i(x_j, t)^2 \rangle \langle u_i(x_j + \Delta x_j, t + \Delta t)^2 \rangle}} \quad (3.7)$$

(no summation over repeated indices) with $i = 1$ and $j = 1$; the velocity component u_1 is of course also function of x_2 and x_3 in principle but the correlation is evaluated at $x_2 = 0 = x_3$. In fig. 3.4a, ρ_{u_1} is plotted for $x_1/d = 2$ and for five different values of the spatial offset Δx . Even at separations as large as $\Delta x/d = 1$ there is a strong correlation of the velocity signals. The peaks of the correlation ρ_{u_1} can be used to compute U_c by taking the ratio between the values of Δx and Δt where these peaks occur.

As shown in fig. 3.4c, there does seem to be a linear relationship between Δx_1 and Δt suggesting that U_c is well-defined, which supports the validity of the Taylor hypothesis. As is also clear in fig. 3.4c, U_c is slightly larger than the mean flow velocity on the centreline. For example, we have found the values $U_c/U_\infty \approx 0.67$ and $U_c/U_\infty \approx 0.83$ at the two centreline locations $x_1/d = 2$ and $x_1/d = 4$ respectively, which, as can be seen in fig. 2.20, are indeed larger than U_1 at the same locations. In fact, U_c appears to be close

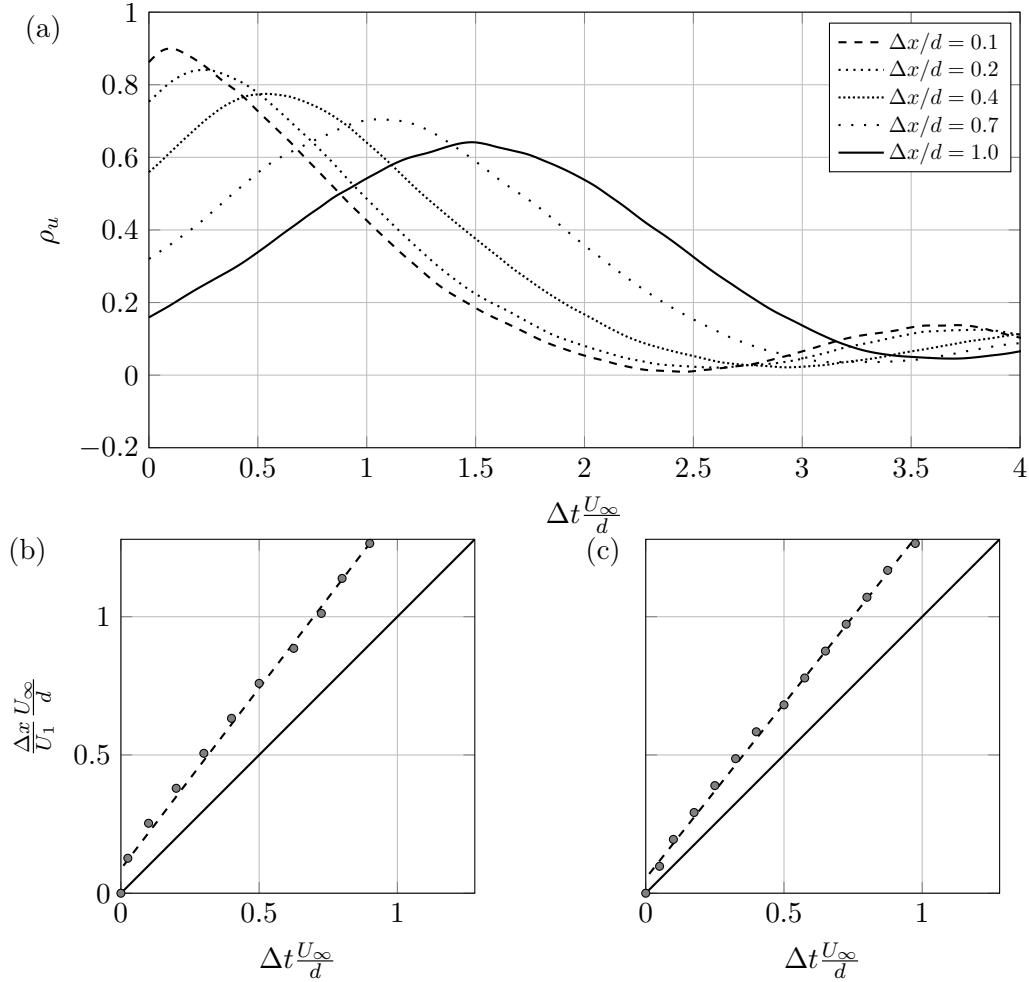


Figure 3.4: In (a) the correlation coefficient eq. (3.7) at $x_1/d = 2$ on the centreline is plotted for different separations Δx_1 . In (b) and (c) the symbols indicate the locations of the peaks of eq. (3.7) at both $x_1/d = 2$ (b) and $x_1/d = 4$ (c) on the centreline, the full lines indicate the slope $U_c/U_1 = 1$ and the dashed lines indicate linear fits to the data.

to the convection velocity of the large scale vortices as reported by Bloor and Gerrard (1966) and Y. Zhou and Antonia (1992). Similar conclusions can be reached for the components u_2 and u_3 of the fluctuating velocity in terms of a correlation coefficient defined as in eq. (3.7) but for u_2 and u_3 respectively. In fact we find $U_c(x_1, 0, 0) \approx 1.3U_1(x_1, 0, 0)$ for all three fluctuating velocity components in the range $x_1/d \geq 3$ and U_c/U_1 slightly higher but under or close to 1.4 at $x_1 = 2d$ on the centreline. While many other considerations regarding the conversion from frequency to wavenumber can arise (see Lumley, 1965; Wyngaard and Clifford, 1977), it is beyond this work to delve into such issues. The point of this limited space-time analysis has simply been to confer a little more substance to our observation of near $-5/3$ power law energy spectra by providing some evidence in favour of the Taylor frozen turbulence hypothesis which therefore suggests that the

power law frequency spectra that we have observed may be reflections of similar underlying wavenumber spectra. In the following subsection we briefly report on second order structure functions as they are Fourier analogues of wavenumber spectra.

3.2.2 Second Order Structure Functions and Corresponding Spectra

Our supporting evidence for the Taylor hypothesis is limited to spatial separations r_1 in the streamwise direction and cannot be expected to be valid in other directions. We therefore extract from our DNS data second order structure functions $\langle \delta u_i^2 \rangle$ for $i = 1, 2, 3$ as functions of $(r_1, 0, 0)$ at various locations X_1 on the centreline. Given the near $-5/3$ power law scalings observed in the frequency spectra, one might expect these second order structure functions to have a near $2/3$ power law dependence on r_1 in an intermediate range of scales. The power law range in the frequency spectra is, at most, about $0.5 < fd/U_\infty < 3$. Assuming that we can apply Taylor's hypothesis and therefore $r_1 f = U_c$, this power law range of frequencies translates to $\frac{1}{3}U_c/U_\infty < r_1/d < 2U_c/U_\infty$, i.e. typically around $0.25 < r_1/d < 1.5$. However, as seen in fig. 3.5, the structure functions $\langle \delta u_i^2 \rangle$ (for $i = 1, 2, 3$) do not have a near $2/3$ power law in this range. In fact, the approximate slopes of $\log(\langle \delta u_1^2 \rangle)$ and $\log(\langle \delta u_2^2 \rangle)$ vastly exceed $2/3$, while the $\log(\langle \delta u_3^2 \rangle)$ curve reaches it only tangentially in this range.

The striking difference between the planar structure functions $\langle \delta u_1^2 \rangle$ and $\langle \delta u_2^2 \rangle$ and the spanwise structure function $\langle \delta u_3^2 \rangle$ as well as the very steep growth of the planar structure functions compared to $r_1^{2/3}$ can be explained in terms of the vortex shedding which is clearly visible in figs. 3.2a and 3.3a but absent in fig. 3.3b. The lack of a clear $2/3$ power law in the r_1 dependence of $\langle \delta u_3^2 \rangle$ is consistent with the observation sometimes made (Frisch, 1995; Lindborg, 1999) that the range of wavenumbers (or frequencies) over which a well-defined $-5/3$ power law is observed is in general larger than the range of scales over which a $2/3$ power law for the second order structure is clearly present (see also Antonia, Smalley, et al., 2003). We now illustrate these two points in terms of a simple example.

We consider a model spectrum (see Pope, 2000) given by

$$E_u(\kappa) = E_0(\kappa)\kappa^{-\frac{5}{3}}E_\eta(\kappa) \quad (3.8)$$

where $E_0(\kappa)$ and $E_\eta(\kappa)$ characterise, respectively, the low and high wavenumber behaviour of the spectrum and are constant in the intermediate range of wavenumbers κ where $E_u(\kappa) \propto \kappa^{-\frac{5}{3}}$. We take functions $E_0(\kappa)$ and $E_\eta(\kappa)$

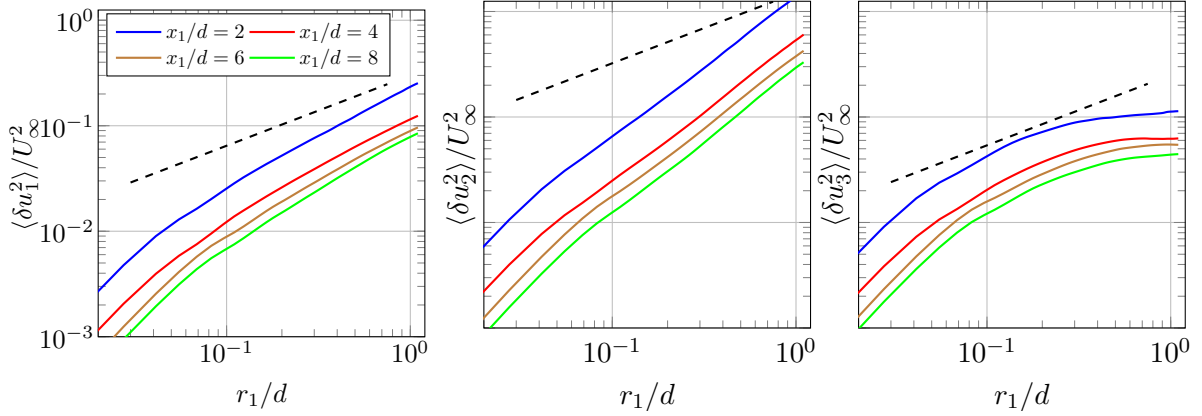


Figure 3.5: Second order structure functions of the three different velocity components along r_1 at three different locations on the centreline. The dashed line indicates a slope of $2/3$.

$$E_0(\kappa) = \left(\frac{\kappa/\kappa_0}{[(\kappa/\kappa_0)^2 + c_0]^{\frac{1}{2}}} \right)^{5/3} \quad (3.9a)$$

$$E_\eta(\kappa) = \exp(-c_\eta \kappa / \kappa_\eta) \quad (3.9b)$$

so that $E_u(\kappa) \propto \kappa^{-\frac{5}{3}}$ in the range $\kappa_0 \ll \kappa \ll \kappa_\eta$. We chose the constants c_0 and c_η and the inner and outer wavenumbers κ_0 and κ_η such that the spectrum takes the form shown in fig. 3.6. In the same figure we also plot another model spectrum obtained in the exact same way but with highly amplified energy around a wavenumber κ_s (both spectra integrate to the same total kinetic energy). We do so via

$$E_s(\kappa) = E(\kappa) \left\{ 1 + A \exp \left[\frac{-(\kappa - \kappa_s)^2}{2\Delta\kappa_s^2} \right] \right\} \quad (3.10)$$

where A is an amplification factor, κ_s is the wavenumber associated with the shedding and $\Delta\kappa_s$ represents the spread of the spectral peak. Equation (3.10) attempts to mimic the model introduced in Thiesse, Danaila, and Antonia (2014) in scale space which is effectively associated with a peak in spectral space. The $-5/3$ power law is well defined for over one decade of wavenumbers in both model spectra as made clear in the second plot of fig. 3.6 where the spectra have been compensated by $\kappa^{5/3}$.

We now calculate the second order structure functions corresponding to the two model spectra of fig. 3.6 from

$$\langle \delta u^2 \rangle = 2u'^2 [1 - \mathcal{F}^{-1}(E_u)] \quad (3.11)$$

where $\mathcal{F}^{-1}$ denotes an inverse Fourier transform. These structure functions are plotted in fig. 3.7 and one can make two clear observations. Firstly, there is no $2/3$ power law

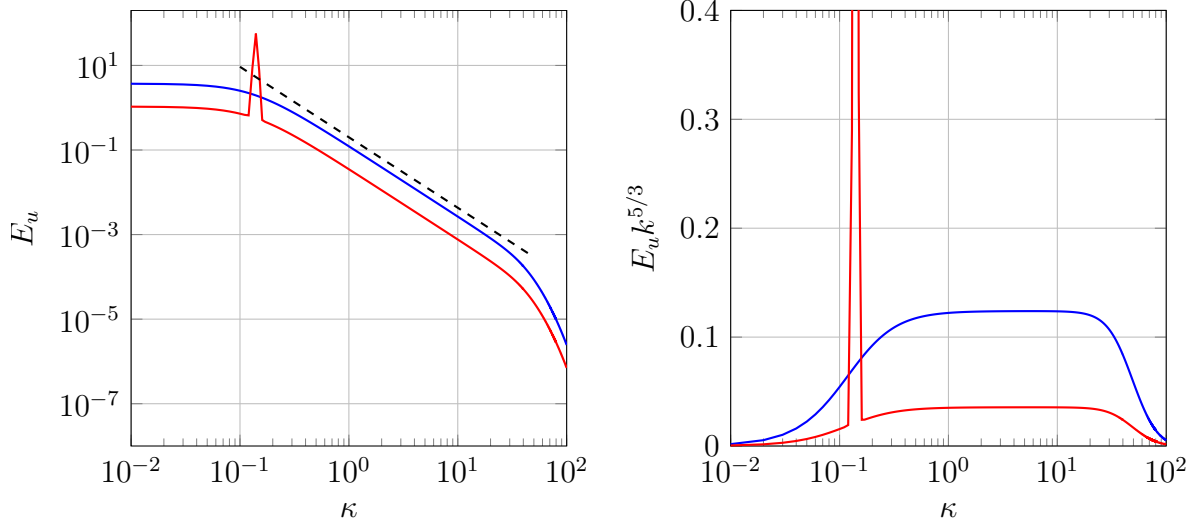


Figure 3.6: Models of a spectrum with a one decade inertial range. One of the spectra is amplified at a given wavenumber. On the right the spectra are compensated by $\kappa^{5/3}$.

even though there is a well-defined $-5/3$ power law in the spectrum from which these structure functions are derived. At best, $\langle \delta u^2 \rangle$ reaches $r^{2/3}$ only tangentially in the case of the model spectrum without the shedding peak. This is similar to our result plotted in fig. 3.5 for $\langle \delta u_3^2 \rangle$ which concerns spanwise fluctuating velocities unaffected by the planar vortex shedding. Secondly, the vortex shedding peak of one of the two model spectra has introduced a growth much steeper than a $2/3$ power law in the corresponding structure function. As noted by Thiesset, Danaila, and Antonia (2014), one should indeed expect the effect of the shedding on the second order structure function to be reflected by a sinusoid-like function; fig. 3.7 demonstrates that this sinusoid can, in fact, very significantly contaminate the power law range of the structure function even when the corresponding energy spectrum has a well-defined $\kappa^{-5/3}$ range over more than a decade. The resulting structure function is much steeper than a $2/3$ power law in the range of scales which corresponds to the $-5/3$ part of the spectrum. This is similar to our results plotted in fig. 3.5 for $\langle \delta u_1^2 \rangle$ and $\langle \delta u_2^2 \rangle$ which concerns planar fluctuating velocities which are affected by vortex shedding.

Inverting (3.11) and using it to obtain the spectra which correspond to the structure functions in fig. 3.5 does return $-5/3$ power law spectra over a range of wavenumbers comparable to the range of frequencies where figs. 3.2a, 3.3a and 3.3b exhibit $-5/3$ power laws (see figure fig. 3.8).

In conclusion, the evidence presented in this and the previous subsections supports the view that there is an underlying approximate $r_1^{2/3}$ dependence in all three second order structure functions on the centreline of our DNS but over a very short r_1 -range. In the

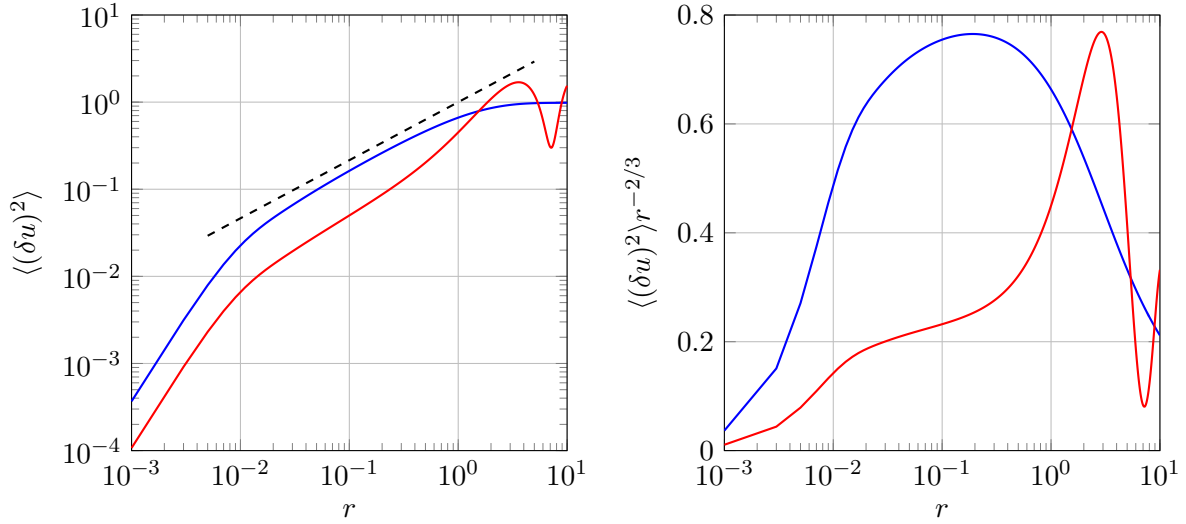


Figure 3.7: Second order structure functions obtained by the inverse Fourier transform of the spectra plotted in fig. 3.6. On the right the structure functions are compensated by $r^{-2/3}$.

case of the streamwise and cross-stream structure functions, this range is totally overshadowed by the vortex shedding signature. The energy spectrum amplifies the power law region and makes it appear as a clear near $-5/3$ power law over a wide range of frequencies/wavenumbers while at the same time disentangling it from the vortex shedding signature. However these near $-5/3$ power law spectra cannot be obviously explained by the K41: their range appears to reduce rather than increase with Reynolds number and, as we show in the following subsection, there is no isotropy in that range.

3.2.3 Distribution of Energy in Scale-space and Isotropy

In fig. 3.9 $\langle \delta q^2 \rangle$ is plotted for $r_3 = 0$ at two different locations in space. The distribution of $\langle \delta q^2 \rangle$ in scale-space shows that the energy is distributed anisotropically in the range $0.2 \leq r/d$ which includes most of the range $0.15 < r_1/d < 1.5$ where the $-5/3$ power laws are observed. Comparing fig. 3.9a with fig. 3.9b shows that there is an overall decrease of the energy and suggests some tendency towards isotropy with increasing downstream position.

The structure functions associated with each velocity component at $X_1/d = 2$ are shown in fig. 3.10. It is clear that the large values of $\langle \delta q^2 \rangle$ observed in fig. 3.9 are associated with the large contribution of $\langle \delta u_2^2 \rangle$ compared to the other two components. In fact, one could have anticipated that, at least for orientations aligned with the mean flow direction, the structure function associated with the vertical fluctuations would have a significant

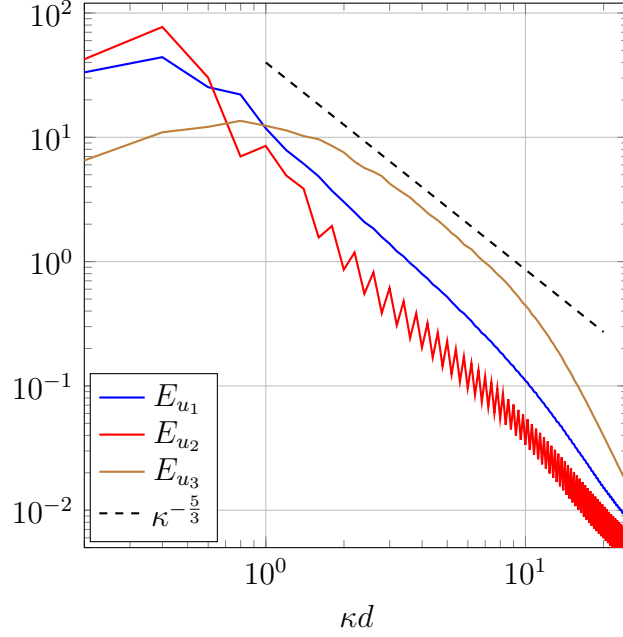


Figure 3.8: Energy spectra in wavenumber obtained by Fourier transform of the correlation functions of each velocity component at $x_1/d = 4$.

contribution to the total energy, as comparing fig. 2.22 with figs. 2.21 and 2.23 shows that it is that component which contributes the most to the turbulent kinetic energy.

The plots in fig. 3.10 show very distinct anisotropy except for $\langle \delta u_3^2 \rangle$ which is unaffected by the vortex shedding. As seen at the start of this section, the frequency spectra of u_1 and u_2 show a persistent peak at the frequency associated with the vortex shedding, indicating that the anisotropy observed in fig. 3.9 and fig. 3.10 is most likely due to the coherent motion as also suggested by Thiesset, Danaila, and Antonia (2013). However it is not easy with the tools used in this chapter to disentangle the shedding-induced anisotropy from an underlying potentially isotropic small-scale turbulence which may or may not be amenable to the Kolmogorov framework. We leave this issue for chapter 4 a triple decomposition of the velocity field which distinguishes between coherent and incoherent fluctuations is used. In the following section we proceed with the study of the various terms in the KMH equation which in fact brings further, and in fact more substantial, evidence of anisotropy.

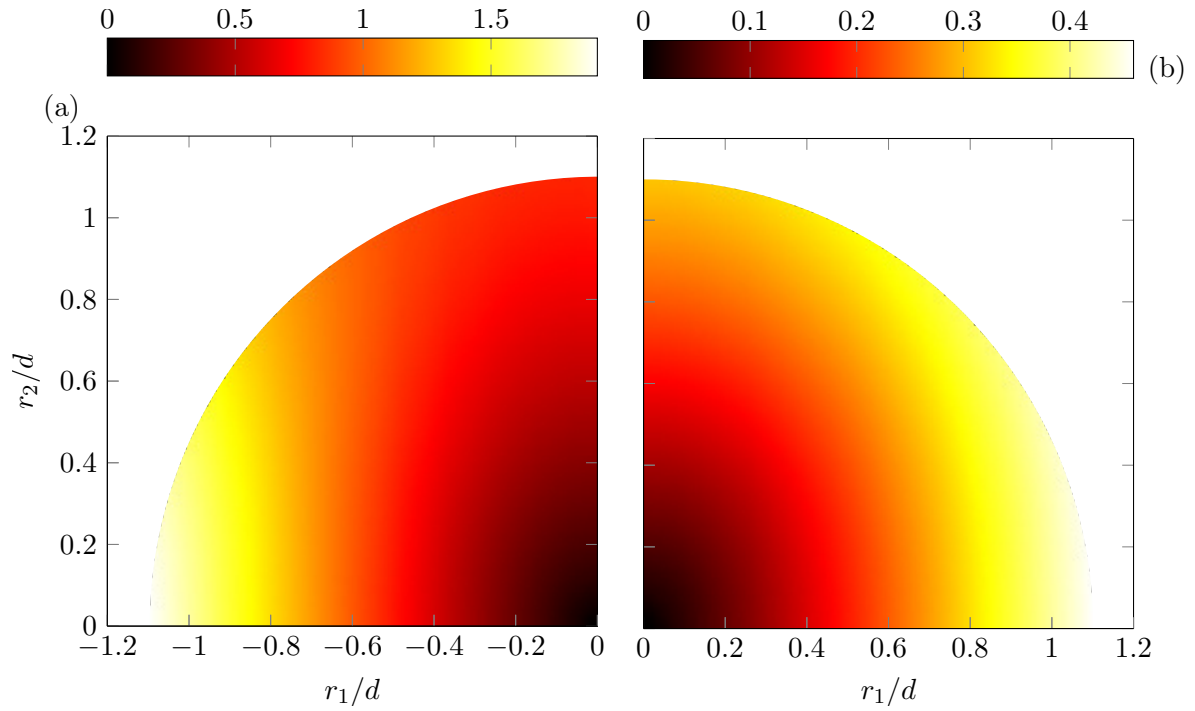


Figure 3.9: Distribution of $\langle \delta q^2 \rangle$ normalised by U_∞^2 in scale space on the geometrical centreline at $X_1/d = 2$ and $X_1/d = 8$ in (a) and (b), respectively.

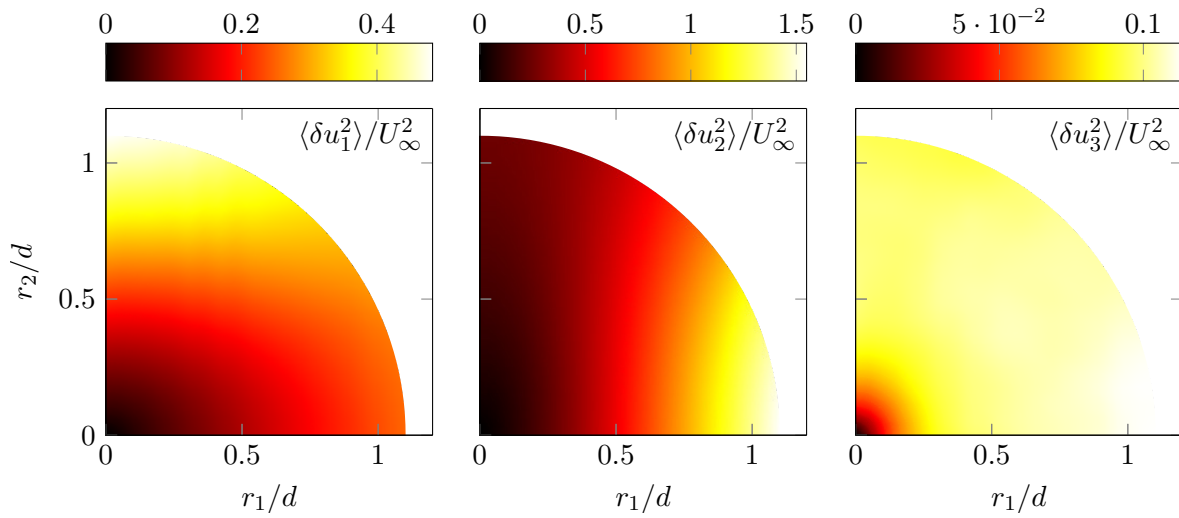


Figure 3.10: i -th component of the second order structure function $\langle \delta u_i^2 \rangle$ normalised by U_∞^2 at $X_1/d = 2$ on the centreline.

3.3 Scale-by-scale Budget of the Generalised Second Order Structure Function

3.3.1 Forward and Inverse Cascades by Non-linear Interactions

The non-linear inter-scale transfer rate Π is the divergence in scale space of the non-linear inter-scale flux vector $\langle \delta \mathbf{u} \delta q^2 \rangle$. The orientation of this flux is directly linked to the concept of energy cascade as it can characterise the direction where energy flows in scale space. In the context of the Richardson-Kolmogorov equilibrium cascade and the assumption of local isotropy, this flux has a radial component which points from large to small separations for any orientation.

From now on we concentrate our attention on four centreline locations, namely $x_1/d = 2, 4, 6, 8$, all far enough from the recirculation region and from the downstream end of our DNS domain, see figs. 2.18 and 2.20. In fig. 3.11 we plot the non-linear inter-scale flux in scale space at two centreline locations. The figures show the direction of the vector $\langle \delta \mathbf{u} \delta q^2 \rangle$ with its magnitude in the background. While the magnitude of $\langle \delta \mathbf{u} \delta q^2 \rangle$ is, in general, larger at large separations, its distribution in scale space is far from isotropic. Moreover, the orientation of $\langle \delta \mathbf{u} \delta q^2 \rangle$ varies drastically with the orientation of the scale vector $\mathbf{r}$, suggesting that $\langle \delta q^2 \rangle$ is, on average, redistributed in scale space, rather than solely transferred from large to small r .

Notice how, at $X_1 = 2$ the flux is directed outwards (from small to large r/d) at separations close to the $r_2 = 0$ axis. At a location further downstream ($X_1/d = 8$) the fluxes have rotated by 90 deg and the flux is now directed inwards (from large to small r/d) at separations close to the $r_2 = 0$ axis (in agreement with the measurements of Thiesse, Danaila, and Antonia (2014)) but outwards at separations close to the $r_1 = 0$ axis.

A necessary condition for a forward or inverse cascade is that the radial component

$$\langle \delta u_r \delta q^2 \rangle = \langle \delta u_i \delta q^2 \rangle r_i / r \quad (3.12)$$

of the flux $\langle \delta \mathbf{u} \delta q^2 \rangle$ is positive or negative.

While the sign of $\langle \delta u_r \delta q^2 \rangle$ indicates if the flux is from large to small (negative) or small to large (positive) scales, it is not enough to characterise the energy cascade as being direct or inverse. For the energy to be truly cascading from large to small scales it is necessary that negative values of $\langle \delta u_r \delta q^2 \rangle$ be associated with negative values of the radial component of the non-linear inter-scale transfer rate Π written in cylindrical coordinates (with the cylindrical axis in the spanwise direction) i.e.

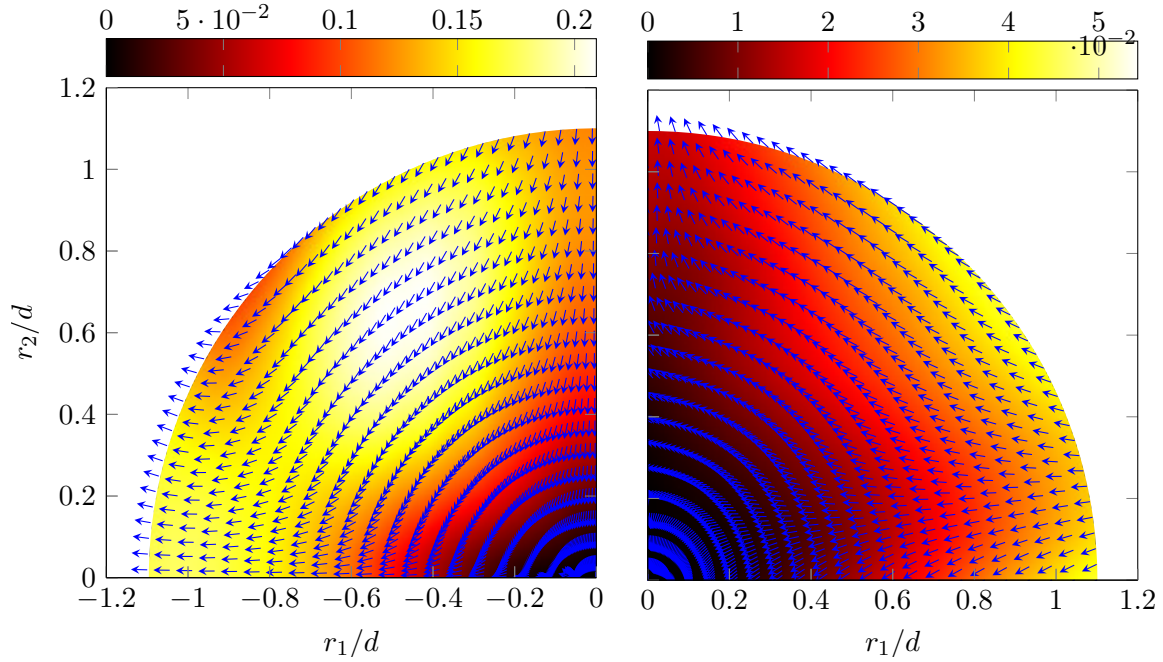


Figure 3.11: Inter-scale flux of energy due to non-linear interactions $\langle \delta \mathbf{u} \delta q^2 \rangle$ at $X_1/d = 2$ (left) and $X_1/d = 8$ (right) for $r_3 = 0$. The arrows indicate the orientation of the vector $\langle \delta \mathbf{u} \delta q^2 \rangle$ and the contour indicates the magnitude normalised by U_∞^3 .

$$\Pi_r = \frac{1}{4} \frac{\partial}{\partial r} \langle \delta u_r \delta q^2 \rangle + \frac{1}{4r} \langle \delta u_r \delta q^2 \rangle. \quad (3.13)$$

Conversely, for the energy to be cascading from small to large scales positive values of $\langle \delta u_r \delta q^2 \rangle$ must be associated with positive values of Π_r . In other words, the radial flux $\langle \delta u_r \delta q^2 \rangle$ is only responsible for an accumulation (depletion) of energy at a given separation vector $\mathbf{r}$ if the gradient given by eq. (3.13) is positive (negative) at that separation vector. Figures 3.12 and 3.13 map eqs. (3.12) and (3.13) in scale space at the same locations as fig. 3.11.

It is now clear that figs. 3.12 and 3.13 suggest a co-existence of forward and inverse non-linear cascades at both centreline locations $x_1/d = 2$ and $x_1/d = 8$. In agreement with fig. 3.11, the orientations where these two simultaneous cascades operate differ at these two locations. The cascade appears inverse in the streamwise and forward in the cross stream direction at $x_1/d = 2$ but forward in the streamwise and inverse in the cross stream direction at $x_1/d = 8$. It is important to note that these local, in scale space, inverse cascade behaviours coexist with energy spectra characterised by near $-5/3$ power laws, in particular at $x_1/d = 2$ where these power laws are closest to $-5/3$ and the inverse cascade behaviour is in the streamwise direction where the spectra are effectively evaluated. Such coexistence as the one at $x_1/d = 2$ was already observed in Gomes-Fernandes

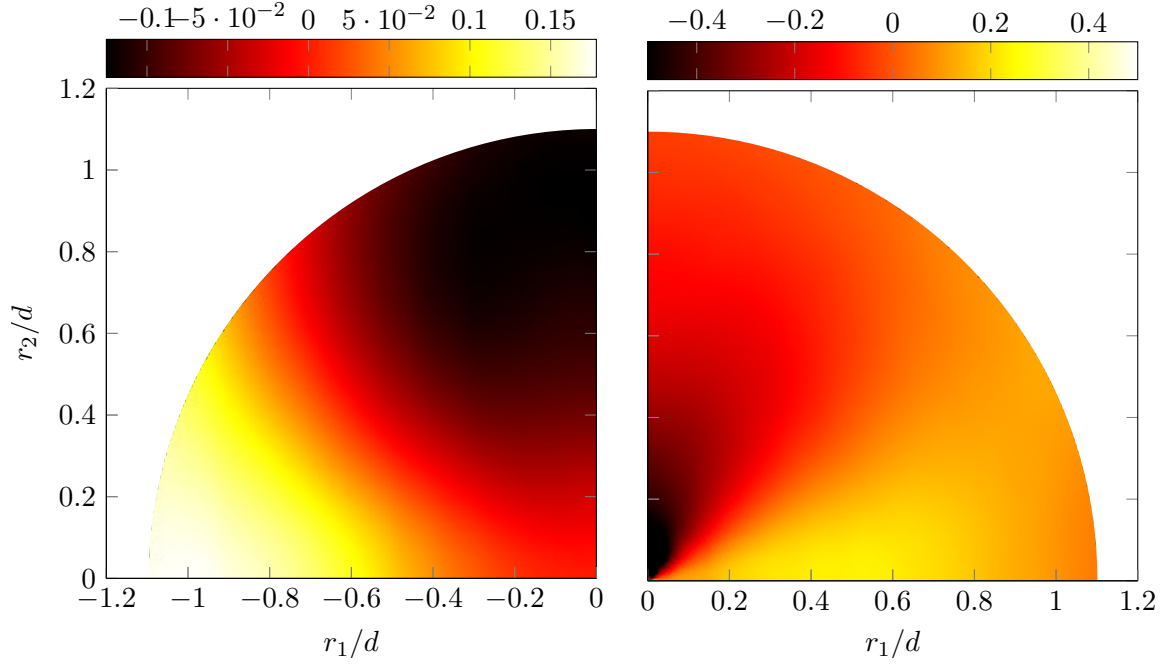


Figure 3.12: Radial contributions to the non-linear interscale flux (left) and transfer (right) of $\langle \delta q^2 \rangle$ at $X_1/d = 2$, eqs. (3.12) and (3.13) respectively.

et al. (2015) in the very near field of a different turbulent flow, while the experiments of Thiesset, Danaila, and Antonia (2014) did not allow capturing these phenomena because their measurements were carried out further downstream and because their measurement apparatus was limited to $r_2 = 0$.

Given the presence of simultaneous forward and inverse cascade behaviours along different scale space orientations, it is important to average over these orientation angles (as illustrated in fig. 3.14) and work out if the non-linear cascade is, on average, forward or inverse. The orientation averaged radial component of the non-linear inter-scale flux, i.e. $\langle \delta u_r \delta q^2 \rangle^a = \frac{1}{2\pi} \int_0^{2\pi} d\theta \langle \delta u_r \delta q^2 \rangle$ (where θ is the scale space orientation angle, i.e. $r_1/r = \cos \theta$, see fig. 3.14), is a function of r and is plotted in fig. 3.15 for centreline locations $X_1/d = 2, 4, 6, 8$. Apart from the location closest to the cylinder ($X_1/d = 2$) $\langle \delta u_r \delta q^2 \rangle^a$ remains negative over all scales investigated and even at $X_1/d = 2$ it remains negative over a wide range of scales, $r/d < 0.86$. This means that the non-linear cascade is forward on average on the centreline near field $2 \leq X_1/d \leq 8$ of a square prism's turbulent wake, and results from combined forward and inverse cascades in different directions as exemplified by figs. 3.12 and 3.13.

We distinguish between “forward/inverse cascades” and “forward/inverse cascade behaviours” because the possibility remains that turbulent flow inhomogeneities, in particular at the larger scales but *a priori* even potentially at some smaller scales, may have some contribution to the behaviour of the interscale flux vector $\langle \delta \mathbf{u} \delta q^2 \rangle$. Equally, inhomogene-

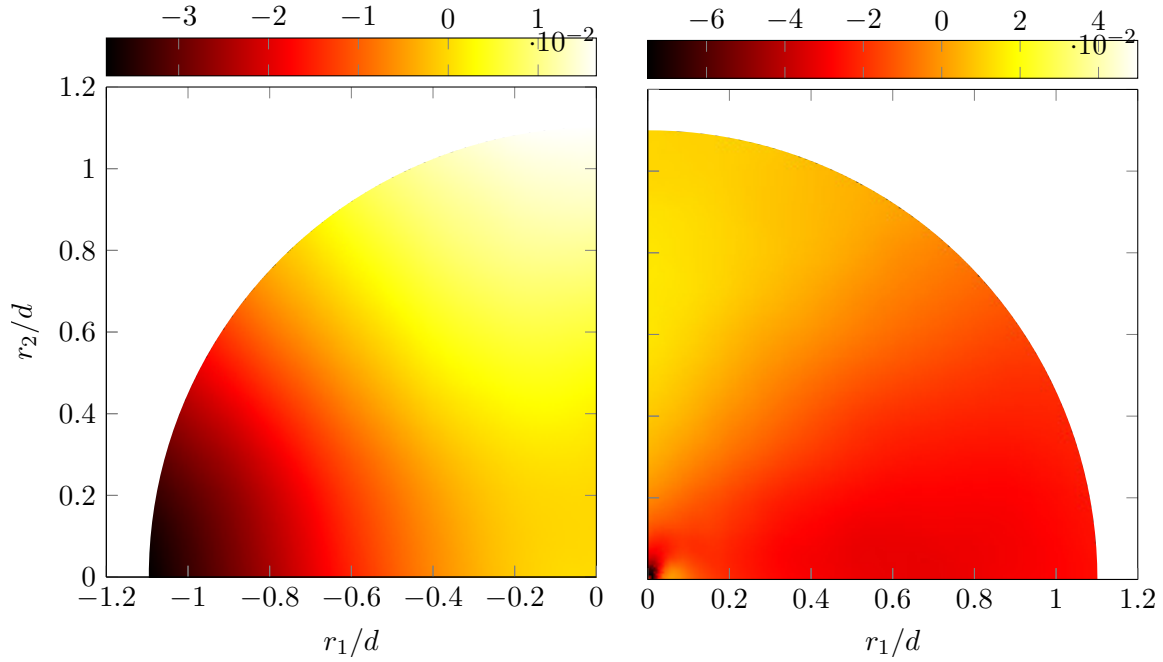


Figure 3.13: Radial contributions to the non-linear interscale flux (left) and transfer (right) of $\langle \delta q^2 \rangle$ at $X_1/d = 8$, eqs. (3.12) and (3.13) respectively.

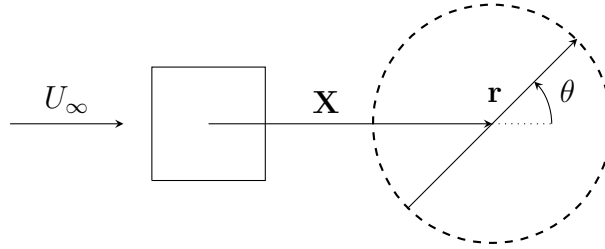


Figure 3.14: Illustration of the $r_3 = 0$ plane for $\mathbf{X}$ aligned with the centreline. The angle θ indicates the angle over which the orientations of $\mathbf{r}$ are averaged.

ity may be a priori and at least partly responsible for the predominantly negative values of $\langle \delta u_r \delta q^2 \rangle^a$ indicating a forward cascade on average. Untangling the inhomogeneity from the actual cascade contributions is a delicate and important question which will need to be carefully addressed in future works.

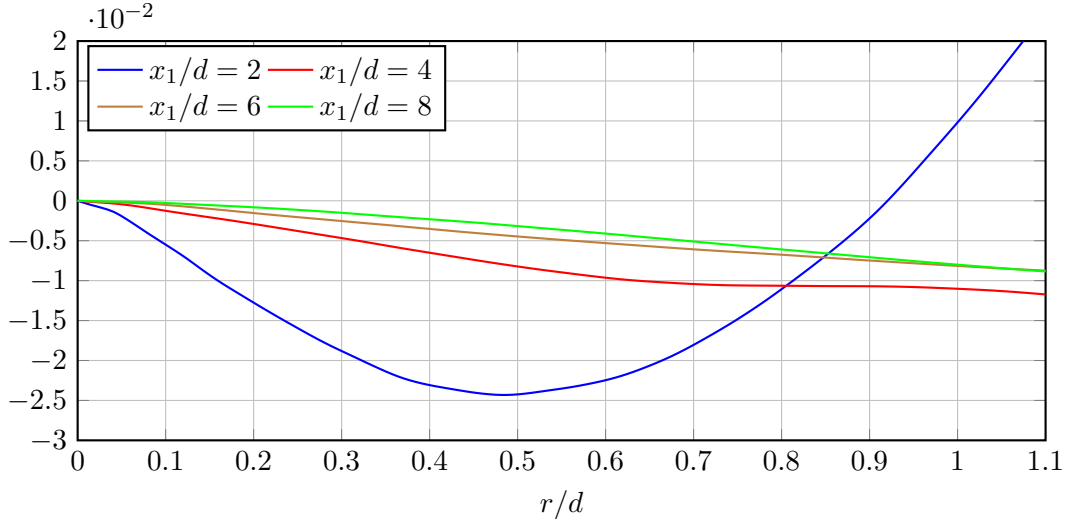


Figure 3.15: Orientation averaged radial component of the non-linear inter-scale flux $\langle \delta u_r \delta q^2 \rangle^a$ at four different locations along the centreline.

3.3.2 Contributions of the Different Terms in the KMH Equation

As the KMH equation clearly illustrates, the non-linear cascade does not happen in isolation unless the length-scales considered are so small that direct viscous diffusion effects matter and/or the turbulence is sufficiently inhomogeneous for at least one of the following, production $\mathcal{P}$, linear inter-scale transfer rate Π_U , spatial transport $\mathcal{T}_u$ and pressure term $\mathcal{T}_p$, to be significant. The type of inhomogeneity which causes the advection term $\mathcal{A}$ to be non-negligible when averaging is over time and $\mathcal{A}_t = 0$ by construction, is a reflection of the unsteady nature of the small-scale turbulence in the frame following a mean flow. In this subsection we examine all these additional processes in order to establish the energy exchange context in which the previous section's non-linear cascade occurs.

We first confirmed that the viscous terms $\mathcal{D}_x$ and $\mathcal{D}_r$ are indeed negligible in (3.1), except at very small scales (comparable to or smaller than the local Taylor microscale which is typically around $d/10$ at the centreline positions investigated) (see Appendix B in Valente and Vassilicos, 2015). We therefore now deal with Π and $\mathcal{T}_p$; $\mathcal{A}$ and $\mathcal{T}_u$; and $\mathcal{P}$ and Π_U .

In figs. 3.16 and 3.17 both Π and $\mathcal{T}_p$ are plotted in scale space at two different locations on the centreline ($X_1/d = 2$ and $X_1/d = 8$, respectively). In eq. (3.1) $\mathcal{T}_p$ is written as $-\frac{1}{2} \frac{\partial \langle \delta u_i \delta p \rangle}{\partial X_i}$; however, another possible formulation is $-\frac{1}{4} \frac{\partial \langle \delta u_i (p^+ + p^-) \rangle}{\partial r_i}$ which resembles the expression for Π with $-(p^+ + p^-)$ taking the place of δq^2 . At both locations ($X_1/d = 2$ and $X_1/d = 8$), Π acts simultaneously as a source ($\Pi < 0$) and a sink term ($\Pi > 0$) in eq. (3.1). However, $\mathcal{T}_p$ is initially (at $X_1/d = 2$) a sink term ($\mathcal{T}_p < 0$) at all orientations

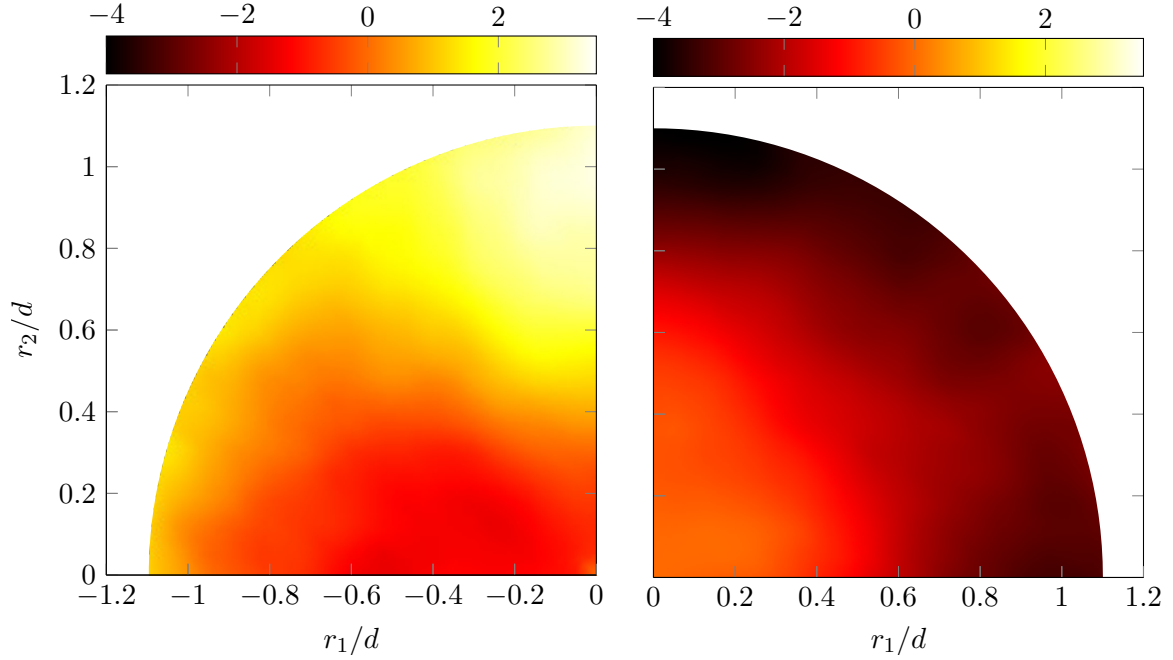


Figure 3.16: Non-linear interscale transfer Π (left) and pressure term $\mathcal{T}_p$ (right) normalised by ε_r at $X_1/d = 2$ on the centreline.

and amplitudes of $\mathbf{r}$ while further downstream (at $X_1/d = 8$), just like Π , it acts as both a source ($\mathcal{T}_p > 0$) and a sink term ($\mathcal{T}_p < 0$) in eq. (3.1).

It is important to distinguish Π from Π_r (see eq. (3.13)) shown in figs. 3.12 and 3.13. In section 3.3.1, Π_r is used to describe how δq^2 is exchanged from large to small scales (and vice versa). However, the term which actually appears in the budget of $\langle \delta q^2 \rangle$ is Π , the divergence in scale space of the non-linear interscale flux $\langle \delta \mathbf{u} \delta q^2 \rangle$. Figures 3.16 and 3.17 reveal that the non-linear interactions can act as a source of $\langle \delta q^2 \rangle$ independently from the orientation of $\langle \delta u_i \delta q^2 \rangle$, i.e., it is not necessary to have a forward energy flux ($\langle \delta u_r \delta q^2 \rangle < 0$) to observe $\Pi < 0$.

What is also, perhaps, surprising from figs. 3.16 and 3.17 is how $\mathcal{T}_p$ loses its somewhat isotropic distribution as X_1 increases. Moreover, it would appear from fig. 3.17 that there is some correlation at $X_1/d = 8$ between the distribution of $\mathcal{T}_p$ and Π in scale space, in agreement with recent results by Yasuda and Vassilicos (2017) for periodic turbulence. The fact that $\mathcal{T}_p$ remains a significant term in the KMH equation may not be fully unexpected, from the measurements of Gomes-Fernandes et al. (2015) for example, but it does not seem to have been previously reported. One may anticipate that $\mathcal{T}_p$ reflects the presence of the coherent vortices associated with the shedding which introduce strong pressure gradients in the flow.

Two other terms which also appear to correlate as downstream location increases are the terms responsible for transporting δq^2 in physical space: $\mathcal{A}$ and $\mathcal{T}_u$. In figs. 3.18

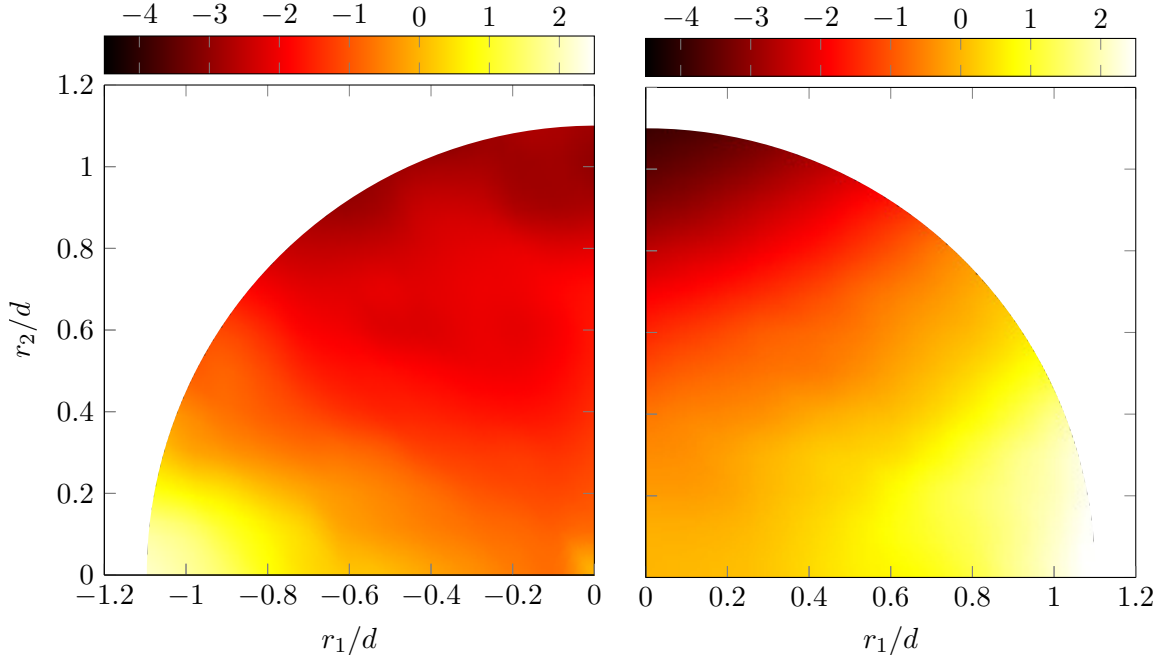


Figure 3.17: Non-linear interscale transfer Π (left) and pressure term $\mathcal{T}_p$ (right) normalised by ε_r at $X_1/d = 8$ on the centreline.

and 3.19 these two terms are shown side by side for $X_1/d = 2$ and $X_1/d = 8$ respectively. At all separation vectors $\mathbf{r}$ considered one can observe that $\mathcal{A} < 0$. This indicates that, regardless of the orientation and amplitude of $\mathbf{r}$, $\langle \delta q^2 \rangle$ decays in the direction of the mean flow. $\mathcal{T}_u$ also appears to be predominantly negative, especially at the location closest to the prism.

At $X_1/d = 2$ $\mathcal{A}$ appears to be roughly independent of the orientation of r_i suggesting that initially the decay of $\langle \delta q^2 \rangle$ depends only on $r = |\mathbf{r}|$. However, as X_1 increases and the production term weakens, $\mathcal{A}$ appears to tend to map in scale space in a way roughly similar to $\mathcal{T}_u$. This suggests the development with downstream distance of a sweeping mechanism (see Tsinober, 2009) whereby small scale turbulent eddies are transported by the larger turbulent eddies so that a tendency develops for $\mathcal{A}$ and $\mathcal{T}_u$ to more or less balance. The existence of a strong sweeping correlation between $\mathcal{A}$ and $\mathcal{T}_u$ is supported by the measurements of Gomes-Fernandes et al. (2015) which revealed an even stronger correlation between $\mathcal{A}$ and $\mathcal{T}_u$ than evidenced here (see fig. 22 of that paper) albeit in a different turbulent flow configuration, and by the recent DNS of periodic turbulence by Yasuda and Vassilicos (2017).

The two remaining terms to be considered in the KMH equation are $\mathcal{P}$ and Π_U which, unlike the previous terms which reflect inhomogeneity of the turbulence, are associated with inhomogeneity of the mean flow. One therefore expects their contribution

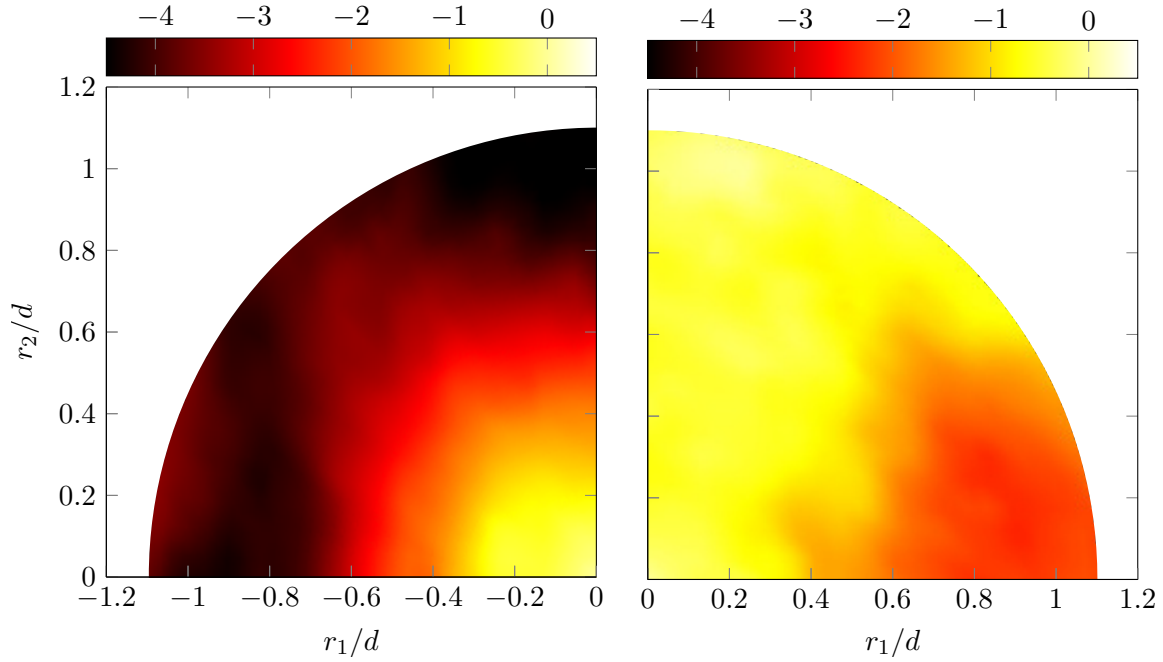


Figure 3.18: Advection $\mathcal{A}$ (left) and turbulent transport $\mathcal{T}_u$ (right) normalised by ε_r at $X_1/d = 2$ on the centreline.

in eq. (3.1) to decrease with increasing downstream location as is indeed observed by comparing fig. 3.20 and fig. 3.21.

Apart from the clear decrease in magnitude, it is also clear that the distribution of $\mathcal{P}$ undergoes a reorientation in scale space, perhaps similar to the one reported in fig. 3.11 for the interscale flux. $\mathcal{P}$ appears stratified along the r_1 axis at $X_1/d = 2$, presumably as a consequence of the strong mean flow gradients in the streamwise direction very close to the prism (see fig. 2.20). Further downstream, at $X_1/d = 8$, $\mathcal{P}$ appears stratified along the r_2 axis. The largest variations of mean velocity occur in the cross-stream direction at that position.

It may be interesting to contrast the scale space maps of $\mathcal{P}$ in fig. 3.20 and fig. 3.21 and those of the interscale flux in fig. 3.11. At both centreline locations, $X_1/d = 2$ and $X_1/d = 8$, the apparently inverse cascading interscale flux which points from small to large scales along r_1 at $X_1/d = 2$ and along r_2 at $X_1/d = 8$ also points from low to high $\mathcal{P}$ values. However, in the $X_1/d = 2$ case, it does clearly point from high to low $\mathcal{P}$ values along the r_2 axis where the cascade appears forward, which does appear natural. We must leave the explanation of the non-intuitive relation between inverse cascade and scale-space $\mathcal{P}$ map for a future study which may involve all three directions in scale space and a better representation of coherent structures in the scale by scale energy balance.

The scale space map of Π_U is mostly determined by the alignment of the mean velocity differences δU_i with the gradient (in scale space) of $\langle \delta q^2 \rangle$. Recall from fig. 3.9 that the

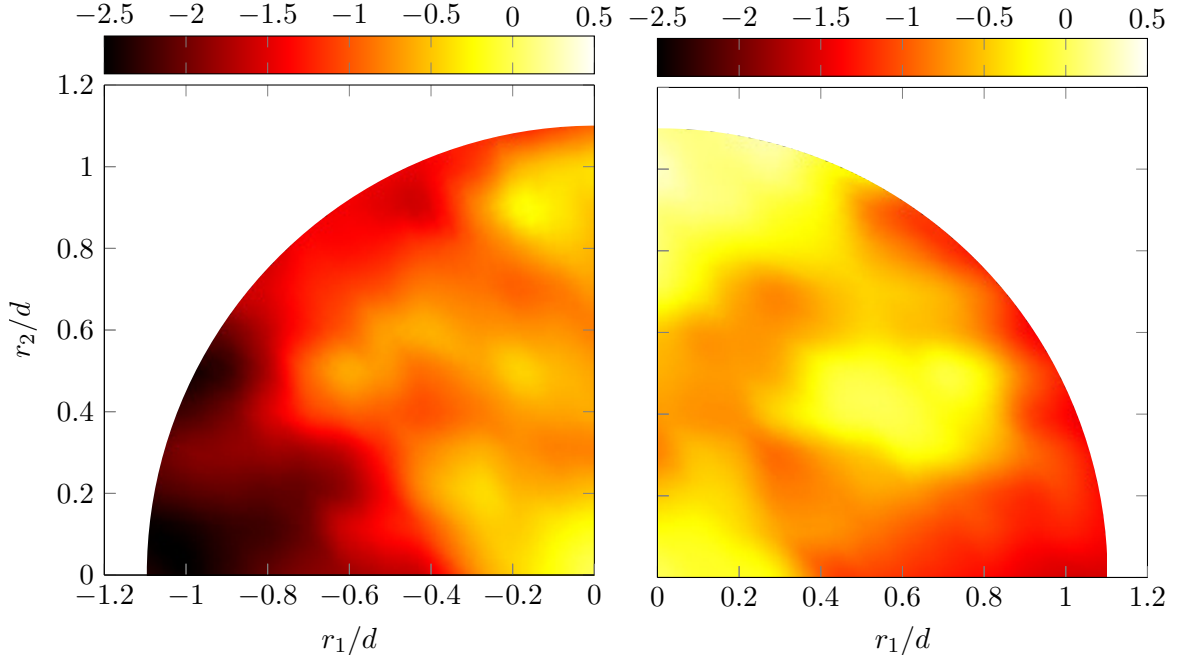


Figure 3.19: Advection $\mathcal{A}$ (left) and turbulent transport $\mathcal{T}_u$ (right) normalised by ε_r at $X_1/d = 8$ on the centreline.

distribution of $\langle \delta q^2 \rangle$ in scale-space does not undergo significant changes between the two locations investigated. On the other hand the components of δU_i can be expected to decrease significantly as the wake spreads and recovers towards U_∞ . In fact, δU_1 must remain positive for positive r_1 (as U_1 increases with x_1) and δU_2 must remain negative for positive r_2 due to the symmetry of the wake.

The results presented in this subsection illustrate how the present turbulent flow does not satisfy Kolmogorov's 1941 premises of equilibrium, local homogeneity and local isotropy even at separation length scales which correspond to the range where the energy spectra we presented have well-defined power laws with exponents close to $-5/3$. The non-linear cascade occurs in a context where various other processes affecting turbulent kinetic energy are active, including production, advection and turbulent transport in physical space, correlations between fluctuating velocity differences and differences of pressure gradients, and linear interscale transfer by the mean flow. In the following subsection, we show how these processes balance when we average over planar orientations, thereby reducing the orientation averaged KMH equation to $\Pi^a \approx -\varepsilon_r^a \approx -\varepsilon$ over a growing range of separations r for further downstream centreline positions X_1/d (the superscript a indicates an average over planar orientation angles as illustrated in fig. 3.14). We have ascertained that, at $X_1/d = 2$ and 8 on the centreline, ε_r^a and ε are very closely equal over the entire range $0 \leq r/d \leq 1.1$ of the figures discussed in the

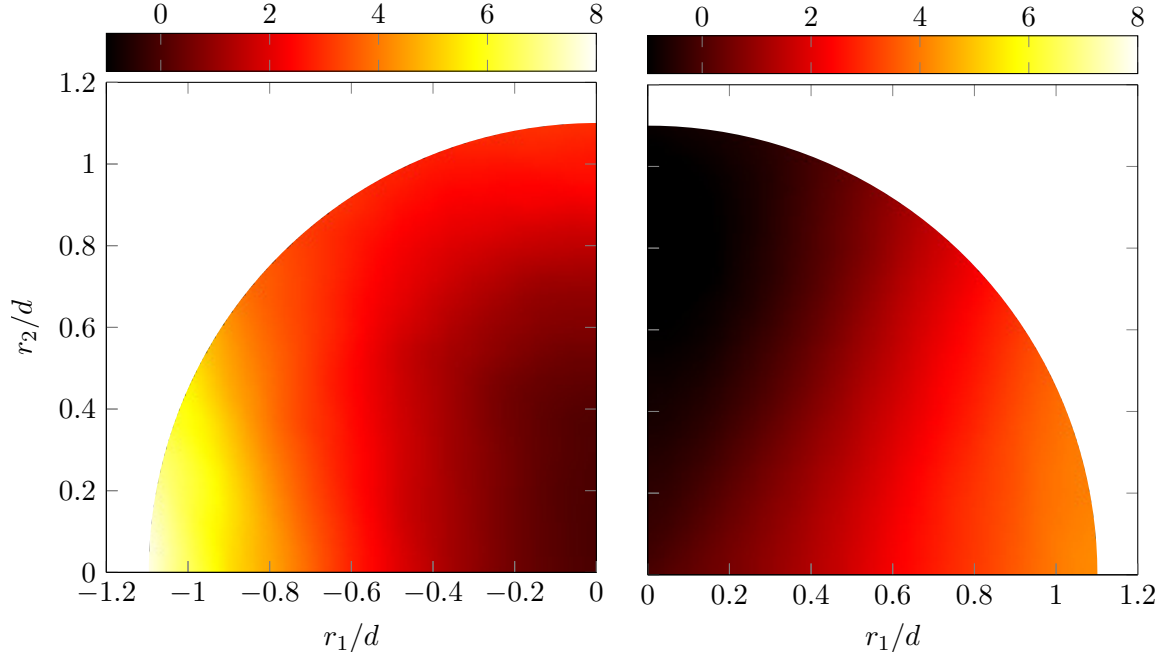


Figure 3.20: Production $\mathcal{P}$ (left) and linear interscale transfer Π_U (right) normalised by ε_r at $X_1/d = 2$ on the centreline.

following subsection, so that if the quantities plotted in these figures were normalised by ε then these figures would look effectively the same.

3.3.3 Orientation Averaged KMH Equation

In fig. 3.22 we plot the planar orientation averaged terms of eq. (3.1) normalised by ε_r^a . These plots are shown for two centreline positions $X_1/d = 2$ and $X_1/d = 8$. At $r = 0$ all terms vanish except ε_r^a and $\mathcal{D}_r^a$ which balance exactly as can actually be shown analytically. At both positions, the viscous diffusion terms are negligible except at separations r smaller than the Taylor length scale* which is about $0.06d$ at $x_1/d = 2$ and $0.1d$ at $x_1/d = 8$. None of the other terms is negligible at separations r larger than the Taylor length scale except Π_U at the larger centreline distance. In fact, at $X_1/d = 2$, all terms associated with inhomogeneity increase their contribution to eq. (3.1) as r increases.

As the distance to the prism is increased, the contribution of Π_u^a is considerably reduced (as already observed in fig. 3.21). It can also be seen in fig. 3.22 that even at the location furthest from the prism, the effect of the fluctuating pressure, included in $\mathcal{T}_p^a$, cannot be neglected, in particular at the largest separations. At the furthest of the two locations, it appears that the dependencies on r of $\mathcal{A}^a$ and $\mathcal{T}_u^a$ mirror each other with a tendency for one to partly balance the other. This behaviour is consistent with

* As will be seen in chapter 4 different definitions of Taylor length scales exist in anisotropic flows. The values reported here are the most conservative estimates we have obtained.

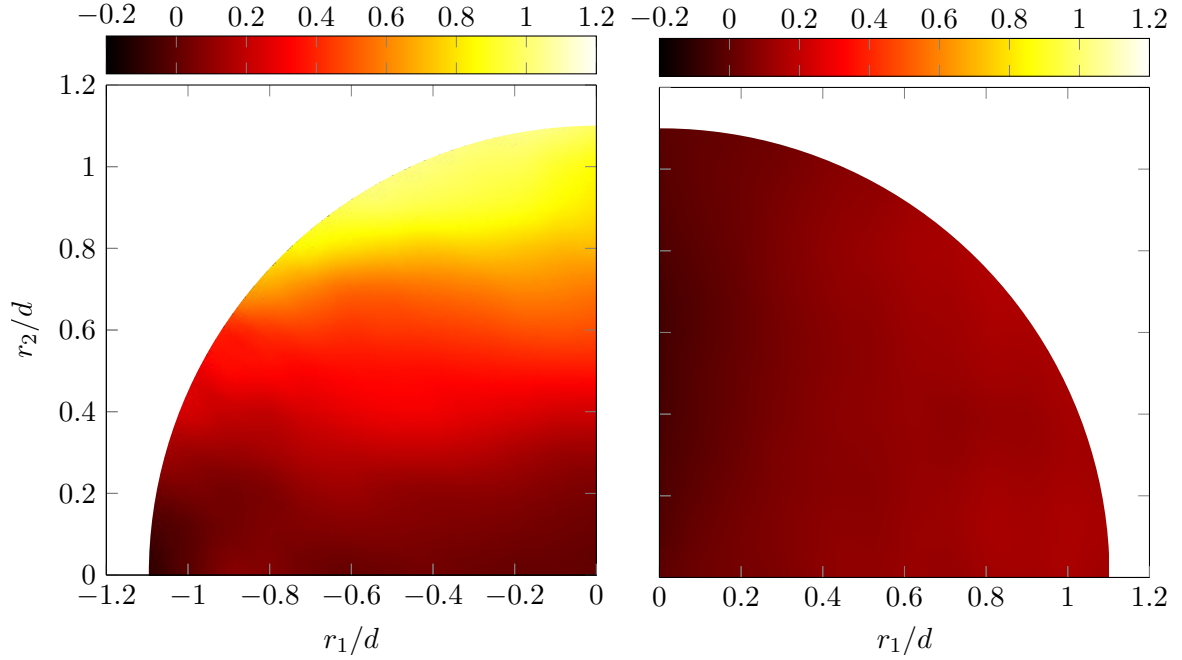


Figure 3.21: Production $\mathcal{P}$ (left) and linear interscale transfer Π_U (right) normalised by ε_r at $X_1/d = 8$ on the centreline.

the sweeping mechanism referred to in the previous subsection concerning $\mathcal{A}$ and $\mathcal{T}_u$ at $X_1/d = 8$.

Finally, at both locations, we find a range of separations r larger than the Taylor length scale where the orientation averaged KHMH equation reduces to the following couple of approximate balances

$$\Pi^a + \varepsilon_r^a \approx 0 \approx -\mathcal{A}^a + \mathcal{T}_u^a + \mathcal{T}_p^a - (\Pi_U^a - \mathcal{P}^a). \quad (3.14)$$

This range of separations where $\Pi^a \approx -\varepsilon_r^a \approx -\varepsilon$ increases with increasing X_1/d is clearly seen in fig. 3.23 where Π^a/ε_r^a is plotted at four different locations on the centreline ($X_1/d = 2, 4, 6, 8$). In fact, this range of separations covers one decade at $X_1/d = 4, 6, 8$ and $\Pi^a \approx -\varepsilon_r^a \approx -\varepsilon$ appears to increase towards 1 with increasing X_1/d in the range of separations where it is constant. The PIV measurements of Gomes-Fernandes et al. (2015) also found that Π^a is approximately constant over a range of separations r in the very near field of a different turbulent flow where the inhomogeneity terms of KHMH equation are also significant and in fact dominant. However their measurements indicated an apparently different value of $-\Pi^a/\varepsilon_r^a$, significantly larger than 1. It is not clear whether their different value of Π^a/ε_r^a is a result of the measurement limitations of their PIV and of the hypotheses that they were therefore forced to make to extract information on the KHMH equation from their measurements.

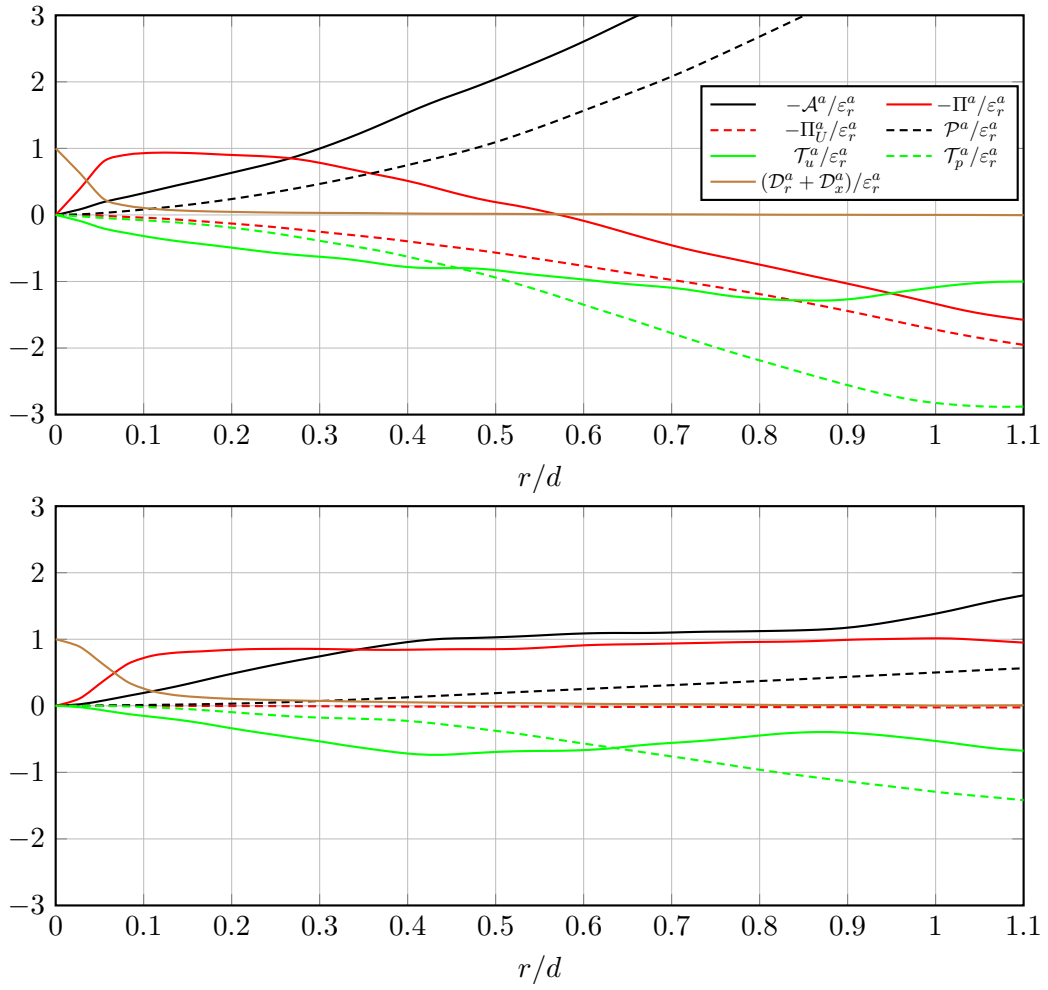


Figure 3.22: Orientation averaged terms of eq. (3.1) normalised by ε_r^a at $X_1/d = 2$ (top) and $X_1/d = 8$ (bottom).

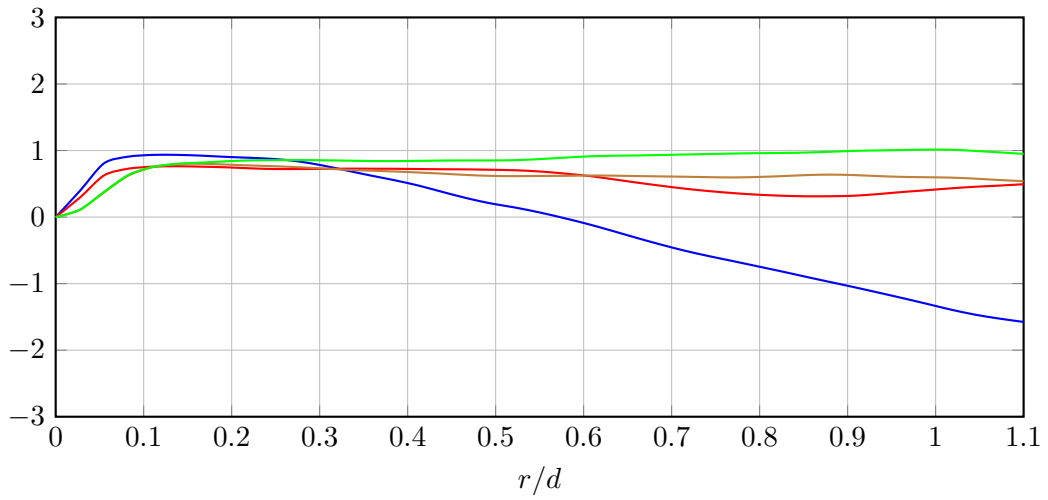


Figure 3.23: Orientation averaged non-linear inter-scale transfer Π^a normalised by $-\varepsilon_r^a$ at four different locations along the centreline.

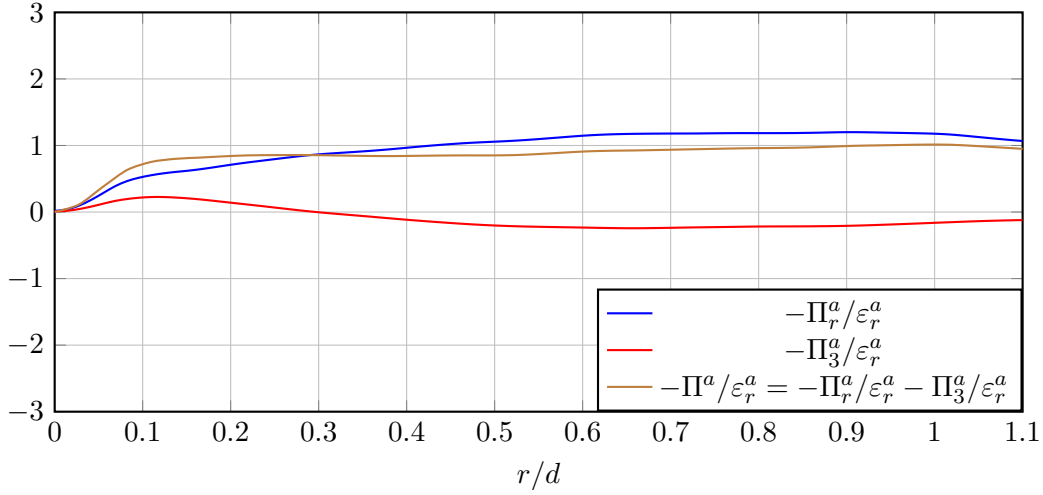


Figure 3.24: Orientation averaged in plane and out of plane components of Π at $X_1/d = 8$.

It is important to point out that the near constancy of Π^a requires the inclusion of both its in-plane (Π_r^a) and out-of-plane ($\Pi_z \equiv \Pi_3$) components (the angular contribution $\Pi_\theta^a = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r} \frac{\partial(\delta u_\theta \delta q^2)}{\partial \theta} d\theta$ is zero). This is illustrated in fig. 3.24 where these two components are plotted as functions of r . Albeit small, the contribution of Π_3^a is particularly relevant especially at the smaller separations, where both Π_r^a and Π_3^a are not constant but contribute to make $\Pi^a = \Pi_r^a + \Pi_3^a$ constant. It is likely that the inability of the experimental PIV of Gomes-Fernandes et al. (2015) to access all velocity field components may have affected their results.

As a final comment which echoes the last paragraph of section 3.3.1, the possibility remains that the inhomogeneity of the turbulence might be contributing to the constancy of Π^a .

3.4 Summary

The KMH equation offers the possibility to study energy exchanges in any turbulent flow irrespective of its particular inhomogeneous or anisotropic character and suggests that well-defined energy-related laws may exist and may perhaps even be the same for a range of such flows. Remarkably, some of these laws seem to be similar to those predicted by the Kolmogorov theory of homogeneous turbulence but their underlying causes can of course not be explained by this theory.

In this chapter we have investigated the turbulence in the very near wake of a square prism at $Re_\infty = 3900$. Our results reveal that the power spectrum densities of the fluctuating velocity exhibit a near $-5/3$ slope over a wide range of frequencies on the centreline

even though the flow is highly inhomogeneous and anisotropic. This slope is more evident at the location closest to the prism. We also find support for a Taylor frozen turbulence hypothesis. These spectra may therefore be the reflection of a Kolmogorov-like r_1 -dependence of second order structure functions, even if this dependence is masked by coherent structure signatures.

The local inhomogeneity and local anisotropy in the region of the flow that we have focused on is made amply manifest by our study of all the terms in the fully generalised scale-by-scale energy balance, the Kármán-Howarth-Monin-Hill equation (KMH). The energy exchange processes caused by inhomogeneity are all active and some of them, specifically advection, turbulent transport in physical space and the term representing the effects of fluctuating pressure, are, along with the non-linear interscale cascade, the dominant processes in play. The strong dependence of all inhomogeneity terms on the orientation of the two-point separation vector shows that a conclusive study of the KMH equation cannot be made by investigating pairs of points along a single orientation in a turbulent flow region such as the present one.

In spite of this extremely inhomogeneous and anisotropic setting and its related turbulent energy processes, the orientation averaged interscale transfer rate Π^a is approximately independent of separation r over a range of r which increases with increasing X_1/d from $X_1/d = 2$ to at least $X_1/d = 8$ on the centreline. Even more remarkably, this constant Π^a tends towards $-\varepsilon_r^a \approx -\varepsilon$, i.e $\Pi^a \approx -\varepsilon_r^a \approx -\varepsilon$, as X_1/d increases at least up to $X_1/d = 8$. The computational domain of our DNS is not large enough to allow us an insight on what happens at normalised distances X_1/d larger than 8.

Even in the majority of cases (different values of r and X_1) where the orientation averaged interscale transfer rate Π^a is negative and the non-linear cascade is therefore forward on average, this overall cascade consists of a mix of forward and inverse non-linear cascade behaviours in different orientations of the separation vector $\mathbf{r}$. These local, in scale space, inverse cascade behaviours coexist with energy spectra characterised by near $-5/3$ power laws. At $X_1/d = 2$ where energy spectral power laws are best defined with exponents closest to $-5/3$, the inverse cascade is in the streamwise direction where the spectra are effectively evaluated. Such coexistence between $-5/3$ spectral scalings and an inverse non-linear cascade behaviour was already observed in Gomes-Fernandes et al., 2015 in the very near field of a different fully three-dimensional turbulent flow.

At the furthest downstream distance from the prism that our DNS can reliably reach, an approximate alignment appears to form between the scale space maps of the pressure term $\mathcal{T}_p$ and Π on the one hand and between the advection $\mathcal{A}$ and the turbulent transport $\mathcal{T}_u$ on the other. The latter may suggest the appearance of a sweeping mechanism whereby the largest turbulent eddies transport small scale turbulence as it evolves.

Finally, our results reveal that the effects of the fluctuating pressure on the scale-by-scale energy balance cannot be neglected, at least in the near field of a turbulent wake.

4 The Effects of the Coherent Motions on the Inter-scale and Inter-space Energy Exchange

As observed in chapter 3, the presence of coherent structures (associated with the vortex shedding) can play an important role in the perceived distribution of energy in scale-space. These structures were identified as those responsible for the narrow band amplification of the frequency spectra of u_1 and u_2 and thus have a distinct time signature. While the model proposed by Thiesset, Danaila, and Antonia (2014) aims at describing their energy content in scale space, their actual influence on the combined inter-scale/inter-space energy exchange is still far from being well understood. We are faced with two possibilities: either i) the signature of the vortex shedding is passive insofar that the motion associated with it and that which is not are uncoupled or ii) the vortex shedding influences the dynamics of how turbulent kinetic energy is generated and exchanged in space and within scales.

Given eq. (3.14), which suggests a constant non-linear flux of energy from large to small scales in a range of scales bounded by a Taylor micro-scale from below and by the size of the large scale eddies from above, one may be inclined towards i) above. In fact, Batchelor (1953) argues that a Kolmogorov-like equilibrium could be achieved without the assumption of large Re_∞ provided that the scales at which the equilibrium occurs are independent of the energy-containing eddies. Effectively, the rate with which energy is fed into these scales must balance the rate with which it dissipates and thus $\Pi = -\varepsilon$ which in turn implies $C_\epsilon = \text{const}$ if, by virtue of the decoupling from the energy-containing eddies, Π at large scales is independent of inlet/initial conditions. The classical $-5/3$ spectrum then follows by appropriate dimensional analysis made possible by cascade equilibrium and independence from inlet/initial/boundary conditions (see pp. 148-152 in Batchelor, 1953). On the other hand, Goto and Vassilicos (2016b) have shown that for decaying turbulence in a periodic box, the turbulence does exhibit the $-5/3$ spectrum while it departs from equilibrium in the sense that $\Pi \neq -\varepsilon$. In fact, as their turbulence decays, this imbalance becomes more and more evident with the inequality between Π and $-\varepsilon$ be-

coming larger for a few eddy turnover times; at the same time, the dissipation coefficient eq. (1.10) follows Re_λ^{-1} which the authors show to be a consequence of the constant ratio between ε' , the rate of energy dissipation at the small scales, and $\tilde{\varepsilon}$, the rate of energy dissipated by large coherent scales, even if the latter is much smaller than the former.

In this chapter we explore the scale-by-scale energy dynamics in the near wake by extracting from the time signals of velocity and pressure a contribution associated with the vortex shedding. In section 4.1 we explain how this decomposition is carried out and characterise the flow in terms of the contributions arising from the phase and stochastic components of velocity and pressure; in section 4.2 we discuss how the coherent motions influence some of the statistics of the fluctuating velocity field, as well as their relationship with the non-equilibrium law for C_ε ; in section 4.3 we re-examine the two-point statistics in light of these contributions; in section 4.4 a model is proposed to explain the two-point statistics associated with the coherent motions and we conclude with a summary in section 4.5.

4.1 Triple Decomposition of the Flow

When the flow exhibits a dominant (periodic) pulsation, it is usual to decompose the velocity and pressure signals into their mean, phase and stochastic components (see Hussain and W. C. Reynolds, 1970; W. C. Reynolds and Hussain, 1972). Let $\bar{s}(t)$ be some signal that displays a dominant pulsation, then

$$\bar{s}(t) = S + \tilde{s}(t) + s'(t) \quad (4.1)$$

where S , $\tilde{s}$ and s' are the mean, phase and stochastic components of $\bar{s}$, respectively (notice that $s = \tilde{s} + s'$). The mean component S contains that part of $\bar{s}$ which is time independent, the phase component $\tilde{s}$ contains that part of $\bar{s}$ which fluctuates with a dominant frequency while the remainder, s' , represents the random-like behaviour of $\bar{s}$. How exactly one obtains $\tilde{s}$ and s' is discussed below in section 4.1.1.

The decomposition given by eq. (4.1) was introduced in Hussain and W. C. Reynolds (1970). In what follows, the operators $\langle \cdot \rangle$ and $\{ \cdot \}$ will be used to mean

$$\langle \bar{s} \rangle = \bar{s} - \tilde{s} - s' = S \quad (4.2a)$$

$$\{ \bar{s} \} = \bar{s} - s' = S + \tilde{s} \quad (4.2b)$$

i.e. the operators $\langle \cdot \rangle$ and $\{ \cdot \}$ filter out the fluctuating (all but the mean) and stochastic components, respectively*. As such, the following conditions (Hussain and W. C. Reynolds, 1970) hold:

$$\begin{cases} \{s'\} = 0, & \langle \tilde{s} \rangle = 0, & \langle s' \rangle = 0 \\ \langle S\psi \rangle = S \cdot \Psi, & \{ \tilde{s}\psi \} = \tilde{s} \cdot (\Psi + \tilde{\sigma}), & \{ S\psi \} = S \cdot (\psi + \tilde{\psi}) \end{cases}, \quad (4.3)$$

where ψ is some arbitrary variable.

We are now tasked with determining a suitable procedure for obtaining a reference phase, so that the phase averaging can be applied to the velocity and pressure fields. The approach taken here in obtaining the reference phase involves applying the Hilbert transform to a reference signal (the lift coefficient), as discussed in section 4.1.1. Some properties of the triply decomposed velocity and pressure fields are presented in section 4.1.2.

4.1.1 Phase Extraction

4.1.1.1 The Hilbert Transform

The Hilbert transform is a commonly used technique in engineering when information on the phase and envelope of a given signal is sought (Feldman, 2011; Wlezien and Way, 1979). Here, we are only interested in obtaining a reference phase in order to employ eq. (4.1). Then, the reference phase can be obtained by computing the angle of the complex signal

$$\bar{a}(t) = \bar{s}(t) - i\bar{h}(t), \quad (4.4)$$

where $\bar{s}(t)$ is a signal that exhibits some sort of periodicity in time (of which the phase ϕ is sought), t , and the function h equals the Hilbert transform $\mathcal{H}(\bar{s})$ of $\bar{s}(t)$ given by the convolution

$$\mathcal{H}(\bar{s}) = \int_{-\infty}^{+\infty} \bar{s}(\tau) \frac{1}{\pi(t - \tau)} d\tau, \quad (4.5)$$

where τ is a dummy variable (i.e. $\mathcal{H}(\bar{s})$ remains a function of time t). Equation (4.4) is also referred to as the analytic signal of $\bar{s}(t)$ (Poularikas, 1999).

We consider here, without loss of generality, that the signal $\bar{s}(t)$ has a mean value of zero (i.e. $S = 0$) as this simplifies the description that follows. If this were not the case, one could simply apply the same procedure to the signal $\bar{s} - S$. Due to this consideration, $s = \bar{s}$ and $a = \bar{a}$.

* Notice that $\langle \cdot \rangle$ retains the same meaning as introduced through eq. (2.21)

Let us now represent $a(t)$ from eq. (4.4) in exponential form as

$$a(t) = \alpha(t)e^{i\phi(t)} \quad (4.6)$$

where $\alpha(t) = \sqrt{s(t)^2 + h(t)^2}$ and $\phi(t) = \arctan[-h(t)/s(t)]$. The complex angle $\phi(t)$ then provides the instantaneous phase of $s(t)$ while the amplitude provides its instantaneous envelope through $\pm\alpha(t)$. While we will not make use of $\alpha(t)$ in the analysis that follows, this quantity is of great importance in other fields such structural mechanics (Feldman, 2011).

Notice that because eq. (4.5) is the convolution of $s(t)$ with $\frac{1}{\pi t}$ one has that, by the convolution theorem, the Hilbert transform of $s(t)$ equals $\mathcal{F}^{-1}[\mathcal{F}(s)\mathcal{F}(\frac{1}{\pi t})]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ indicate the direct and inverse Fourier transforms. Since $\mathcal{F}(\frac{1}{\pi t}) = -i \cdot \text{sgn}(2\pi f)$, where sgn is the sign function, $\mathcal{F}(\frac{1}{\pi t}) = -e^{\frac{\pi}{2}i}$ for positive frequencies f . Therefore, eq. (4.5) generates a function of opposite sign to $s(t)$ that is shifted (in phase) by $\frac{\pi}{2}$, hence the minus sign in eq. (4.4).

4.1.1.2 Reference Signal for the Phase Estimation

Several (experimental) techniques for obtaining reference signals from which to extract the phase ϕ are discussed in Wlezien and Way (1979). Examples of common techniques to obtain a reference signal are the use of a pressure tap on the cylinder (see e.g. Braza et al., 2006) or the fluctuating velocity signal either from within the turbulent flow (after appropriately filtering, as done in Thiesset, Danaila, and Antonia, 2014) or from the outside of the turbulent core (see e.g. Davies, 1976).

These approaches take a single signal to which eq. (4.5) is applied such that the phase angle can be inferred. Ideally, the phase would be estimated locally such that ϕ would be a function of $\mathbf{x}$. Yet, applying eq. (4.5) to every spatial location would be a very (computationally) demanding task. Instead, we have followed a similar approach to Thiesset, Danaila, and Antonia, 2014 where the phase is inferred from a single signal. However, instead of using u_2 as done in Thiesset, Danaila, and Antonia (2014) we have used the signal of the lift coefficient C_L . This allows us to post-process the whole domain and extract phase-averaged quantities in a single run, rather than having a reference phase for each position. Furthermore, as seen in fig. 2.25, the lift coefficient follows a near sinusoid (“clean” of any noise) due to the build-up and departure of the large scale vortices (from the vortex formation region)[†], such that no filtering is required prior to applying the Hilbert transform.

[†] Due to Kelvin’s circulation theorem (see e.g. da Costa Campos, 2010) as a vortex of positive (negative) circulation is formed, the circulation on the prism must be negative (positive).

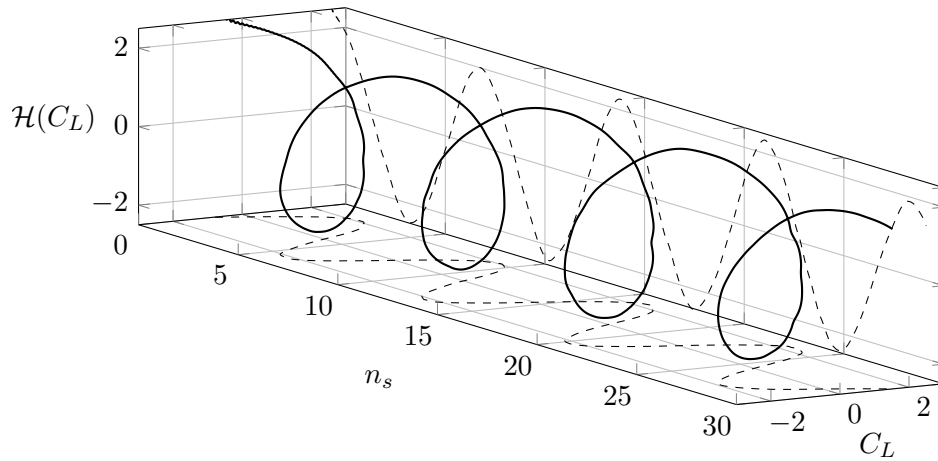


Figure 4.1: Illustration of the evolution of the analytic signal (full line) of C_L in time given by eq. (4.4). $n_s = tf_s$ indicates time normalised by the shedding frequency. The dashed lines, projected on the axes, represent the lift coefficient and its Hilbert transform.

Following the procedure described above (see eqs. (4.4) and (4.5)), the signal $\mathcal{H}(C_L)$ shown in fig. 4.1, was constructed using eq. (4.4) based on $s(t)$ being the lift coefficient $C_L(t)$, also shown in fig. 4.1. Using eq. (4.6), the phase angle for each time instant $\phi(t)$ was extracted. In order to obtain a phase average through eq. (4.2b), one must sample all time instants corresponding to the same value of $\phi(t)$, i.e. eq. (4.2b) is obtained by conditional average using the phase value as the condition.

Because the data is discrete, it becomes necessary to bin $\phi(t)$ in order to carry out the conditional averaging. The phase angle was binned into 32 groups and is shown in fig. 4.2 in comparison to ϕ without binning. A smaller bin size would have improved resolution along the phase but reduced the statistical convergence (as fewer time steps would correspond to each bin). Thus, each time instant is associated with a phase $\phi = -\pi + n\frac{2\phi}{32}$ with $0 < n < 31$. The 32 phase bins are then used to sample the data (velocity and pressure) throughout the domain. The resulting phase averaged lift and drag coefficients are shown in fig. 4.3 versus the phase angle ϕ , where $\phi = 0$ was chosen such that $\tilde{C}_L = 0$.

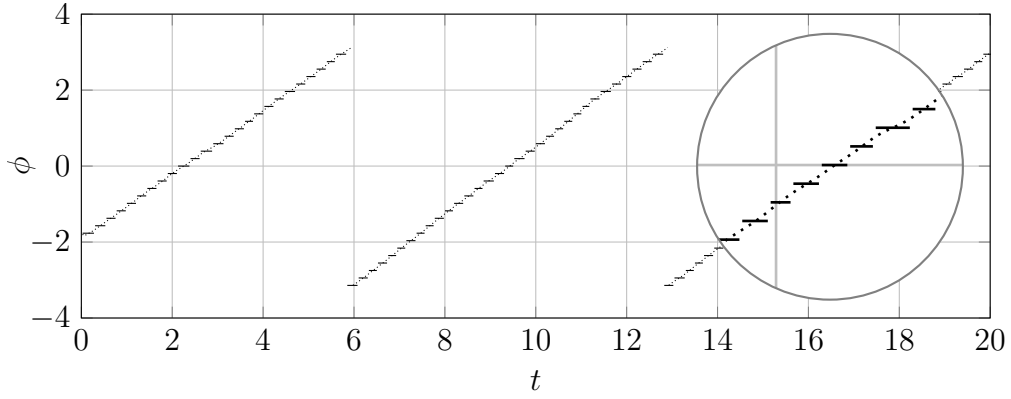


Figure 4.2: The phase angle ϕ along time t . The full and dotted lines indicate the phase angle after and before binning, respectively. The inset shows a magnified portion of the plot.

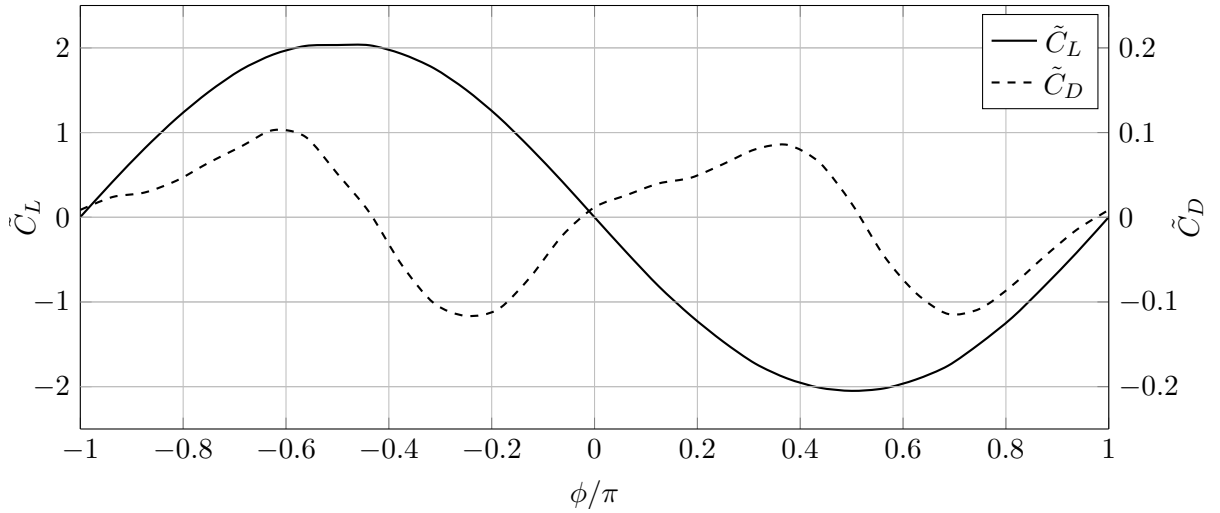


Figure 4.3: Evolution of $\tilde{C}_L$ and $\tilde{C}_D$ along the phase ϕ normalised by π .

4.1.2 Properties of the Triply Decomposed Solution

4.1.2.1 Phase Averaged Velocity and Pressure Fields

In addition to the phase selection (described above) used to compute the phase average, the phase averaging procedure also involves averaging in the spanwise direction as introduced in eq. (2.21b). In fact the phase components of velocity and pressure fields appear to have no significant dependence on the spanwise dimension (prior to applying eq. (2.21b)). For example, as illustrated in fig. 4.4 there are only slight variations of $||\tilde{\mathbf{u}}||$ in the direction of x_3 which are attributed to statistical convergence rather than spanwise inhomogeneity; the same was observed for other values of ϕ . Hence, hereafter, the phase averaging procedure $\{\cdot\}$ also includes averaging over the span.

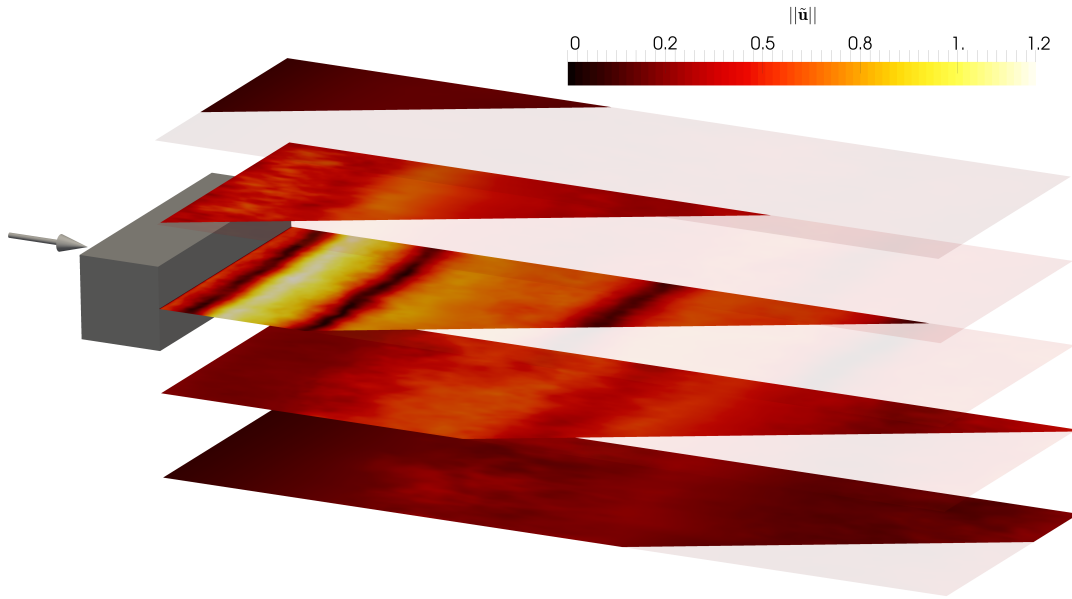


Figure 4.4: Contours of $||\tilde{\mathbf{u}}||(\phi = -\pi)$ normalised by U_∞ without spanwise averaging (eq. (2.21b)) on five planes normal to x_2 equally spaced between $x_2 = -2d$ and $x_2 = 2d$. The arrow indicates the direction of the free-stream velocity U_∞ .

The phase-averaged velocity and pressure fields are illustrated in figs. 4.5 and 4.6, respectively, for four different values of ϕ : 0 , $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ and $\frac{3}{4}\pi$. These phase angles were chosen such that half a period of the shedding process can be observed; the distributions of $\tilde{\mathbf{u}}$ and $\tilde{p}$ over the remaining part of the shedding cycle (not pictured here for brevity) can be inferred from figs. 4.5 and 4.6 since the cycle must follow reflection symmetry about the centreline over the full period.

The streamlines illustrated in figs. 4.5 and 4.6 show how the phase averaged velocity displays a structure similar to that of the Kármán vortex street (discussed in section 1.2) where the alternating vortices display opposite circulation, the positive ones travelling above the centreline and the negative ones below the centreline.

Figures 4.5 and 4.6 show how, at $\phi = \frac{\pi}{2} + n\pi$, the coherent structures are formed immediately downstream of the prism (within the so-called vortex formation region). At later stages of the cycle these structures are dislodged from the vortex formation region and travel downstream at a small distance from the centreline. Figure 4.5 suggests that the distance between successive vortices does not vary much (within the flow region investigated here). Furthermore, figs. 4.3 and 4.6 show how the drag coefficient builds up along the phase as the pressure downstream of the prism decreases while the vortex is being formed ($\phi = \frac{\pi}{2}$ in figs. 4.5 and 4.6), and quickly increases (reducing drag) as the vortex is released. A similar observation can be made with regards to the lift coefficient,

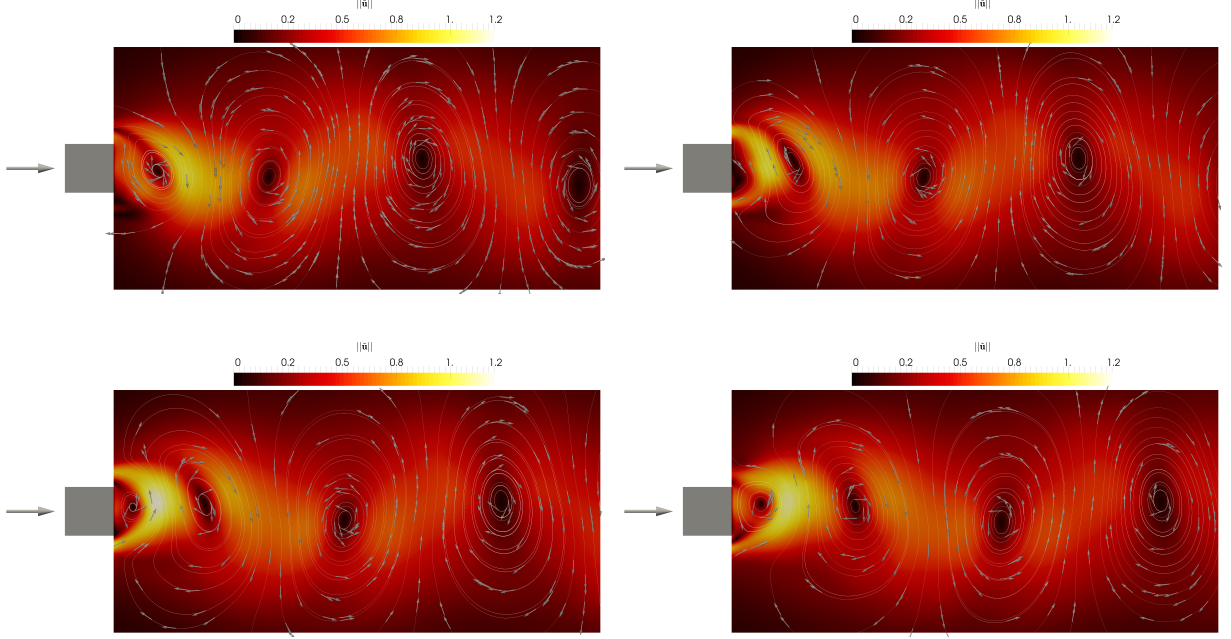


Figure 4.5: Contours of the magnitude of the phase-averaged velocity $\tilde{\mathbf{u}}$ (normalised by U_∞), the white lines indicate streamlines obtained from $\tilde{\mathbf{u}}$. The large arrow indicates the direction of the free-stream velocity U_∞ . Using the phase angles shown in fig. 4.3, on the top $\phi = 0$ (left) and $\phi = \frac{1}{4}\pi$ (right), on the bottom $\phi = \frac{1}{2}\pi$ (left) and $\phi = \frac{3}{4}\pi$ (right).

as seen in fig. 4.6. At $\phi = \frac{\pi}{2}$ a vortex of positive circulation is being formed and the pressure increases in the direction normal to the centreline, on the backward facing side of the prism. This would have been clearer had data been recorded over all faces of the prism, however that would have increased the storage requirements (recall that the mesh is refined in the vicinity of the prism).

As discussed in Hussain (1983), the streamlines are not necessarily a good indicator of presence of coherent structures; however, if lines of constant vorticity are overlaid on to the streamlines shown in fig. 4.5 one can clearly see that they both coincide with vortex like structures, as shown clearly in fig. 4.7. Lyn, Einav, et al. (1995) argue that, apart from the very near wake (which they call base region) where the coherent structures are formed, there is indeed a correspondence between iso-vorticity and streamlines in identifying coherent structures (at least within experimental error).

Notice that identifying the phase angle with respect to fig. 4.3 only provides information on the velocity and pressure fields in the immediate vicinity of the prism. In fact, in order to infer the flow structure in a given region of the domain from the phase angle one would require knowledge on the speed at which these structures propagate along the phase. Instead, in the sections that follow, $\phi = 0$ is taken to be the phase angle at which

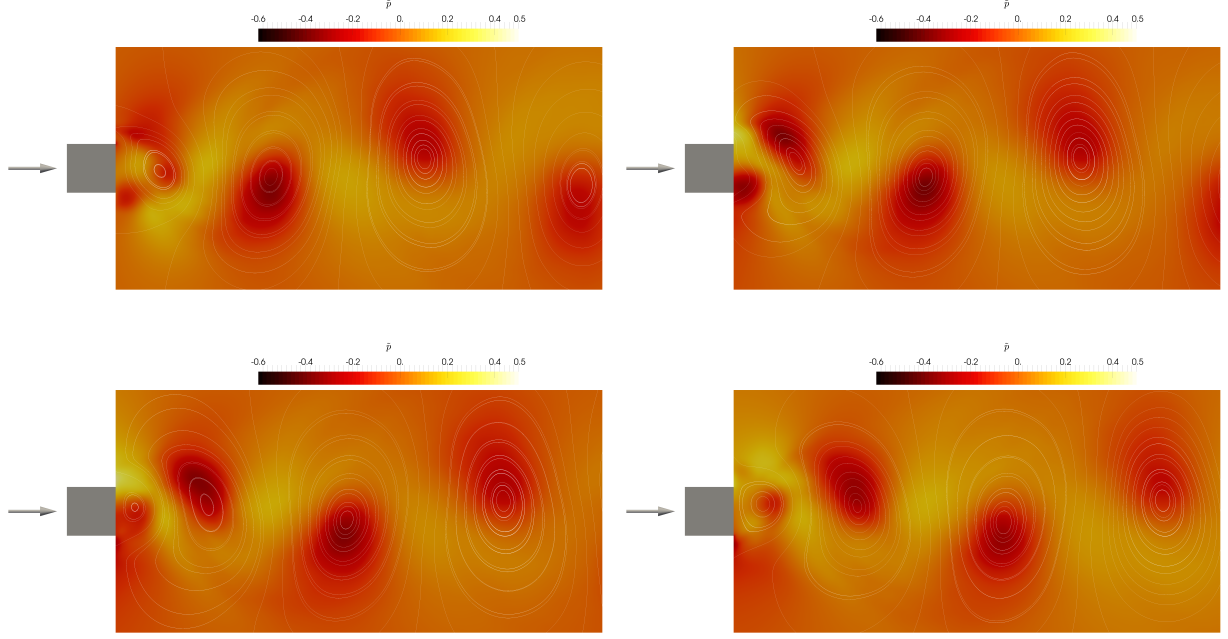


Figure 4.6: Contours of the magnitude of the phase-averaged pressure $\tilde{p}$ (normalised by $U_\infty^2 \rho_\infty$), the white lines indicate streamlines as in fig. 4.5. The large arrow indicates the direction of the free-stream velocity U_∞ . Using the phase angles shown in fig. 4.3, on the top $\phi = 0$ (left) and $\phi = \frac{1}{4}\pi$ (right), on the bottom $\phi = \frac{1}{2}\pi$ (left) and $\phi = \frac{3}{4}\pi$ (right).

$\tilde{u}_2 = 0$ which identifies the instant at which a coherent structure passes by that location. If $\frac{\partial \tilde{u}_2}{\partial \phi} < 0$ then that vortex has negative circulation, as can be seen in figs. 4.5 and 4.7. Since the coherent structures travel downstream, their presence at a given location is felt at different times; this is illustrated in fig. 4.8 where $\tilde{u}_2$ is plotted at different locations on the centreline and compared to $\tilde{C}_L$ showing how the signals are out of phase between each other.

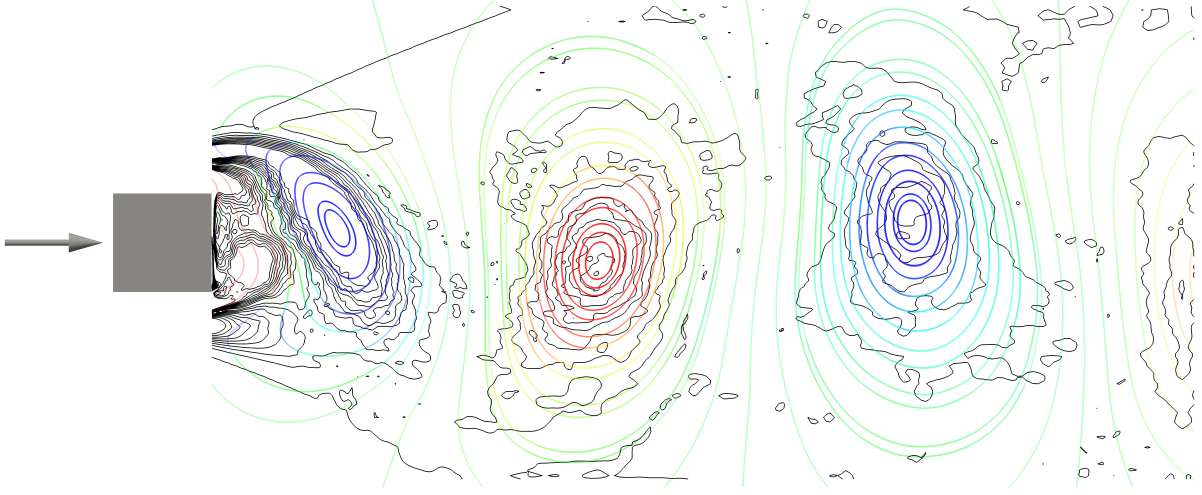


Figure 4.7: Comparison between iso-vorticity and streamlines of the phase-averaged velocity for $\phi = \frac{1}{4}\pi$. The iso-vorticity lines are shown in black while the streamlines are overlaid on top and coloured with the sign of the spanwise vorticity (blue is negative, red is positive). The arrow indicates the direction of the free-stream.

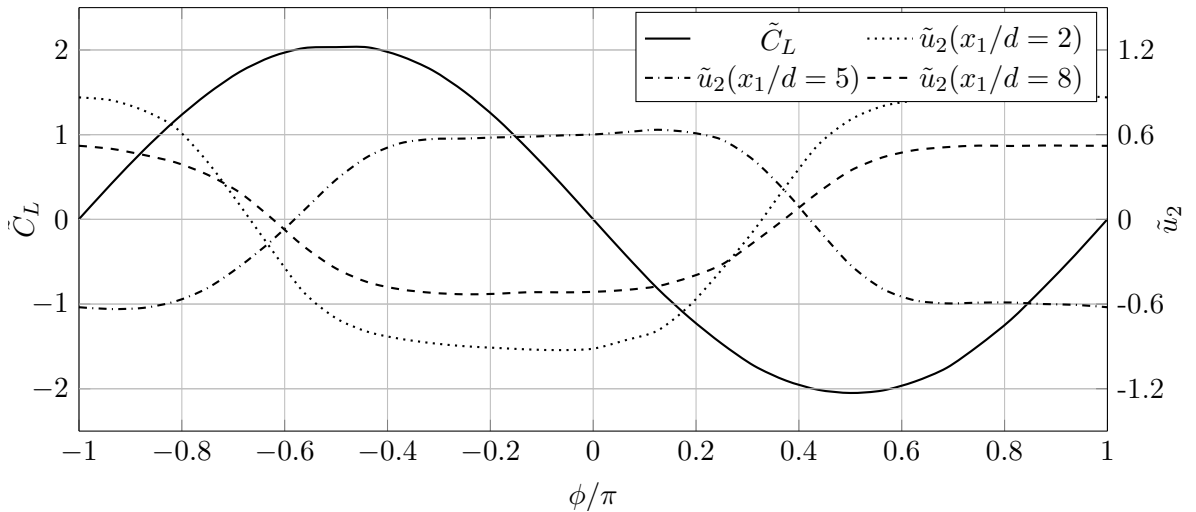


Figure 4.8: Evolution of the phase averaged cross-stream velocity at $x_1/d = 8$ on the centreline along the phase angle ϕ normalised by π . The lift coefficient from fig. 4.3 is included for comparison.

4.2 Manifestations of the Coherent Motion and Non-equilibrium

4.2.1 Gaussianity and Lack Thereof

The fact that turbulent fluctuating velocity differences are inherently non-Gaussian is well known as turbulent dissipation is a direct manifestation of this feature. This lack of Gaussianity is relevant in the context of turbulent eddies within the inertial and dissipative ranges as the skewness of velocity differences over separations within those ranges is related to the energy flux across scales and the energy dissipation (see Kolmogorov, 1941c; Townsend, 1947b).

Nevertheless, it is generally argued that the probability density functions (PDF) of the one-point fluctuating velocity should display near-Gaussian properties (see Batchelor, 1953; Davidson, 2004; Frisch, 1995; Pope, 2000; Tsinober, 2009, and other introductory books on turbulence).

That the PDF of the velocity fluctuations on the centreline of wakes follow a Gaussian, or near-Gaussian, distribution has been confirmed, as shown in Fabris (1979) and Townsend (1947a), for large distances from the wake generator (larger than those studied here). However, there is no reason to assume that such Gaussian distributions should also be observed in the near wake. In fact, Durão et al. (1988) show that the PDF of u_2 is far from Gaussian (exhibiting two peaks) but these authors also argue that the distribution of $u_2'^{\dagger}$ becomes approximately Gaussian at $x_1/d = 5$ on the centreline.

In fig. 4.9 we show how the streamwise velocity fluctuations do not display the near-Gaussian behaviour observed much further downstream by Fabris (1979) and Townsend (1947a). In fact, the PDFs of u_1 shown in fig. 4.9a remain far from Gaussian, displaying a gentle “bump” at negative (but small) values of u_1 at all locations investigated. These “bumps” for values of u_1 close to $-\sqrt{\langle u_1^2 \rangle}$ can be attributed to the effect of the coherent vortices, as they travel downstream and have their cores displaced from the centreline (recall fig. 4.7) a negative velocity component is introduced twice per shedding cycle. In fact, once the phase-component of the velocity is removed, the PDF loses these distinct “bumps”, as seen by contrasting fig. 4.9b with fig. 4.9a. Conversely, Baj et al. (2015) have observed a “bump” at positive values of u_1 (which also disappear once the coherent motion is filtered out); this is because their measurements are taken away from the centreline of the wakes (these authors study the wake generated by an array of rectangular bars) and thus the phenomenon described above has the opposite effect.

[†] In Durão et al. (1988) the “stochastic” component of the fluctuating velocity is obtained simply by band-rejecting around the shedding frequency.

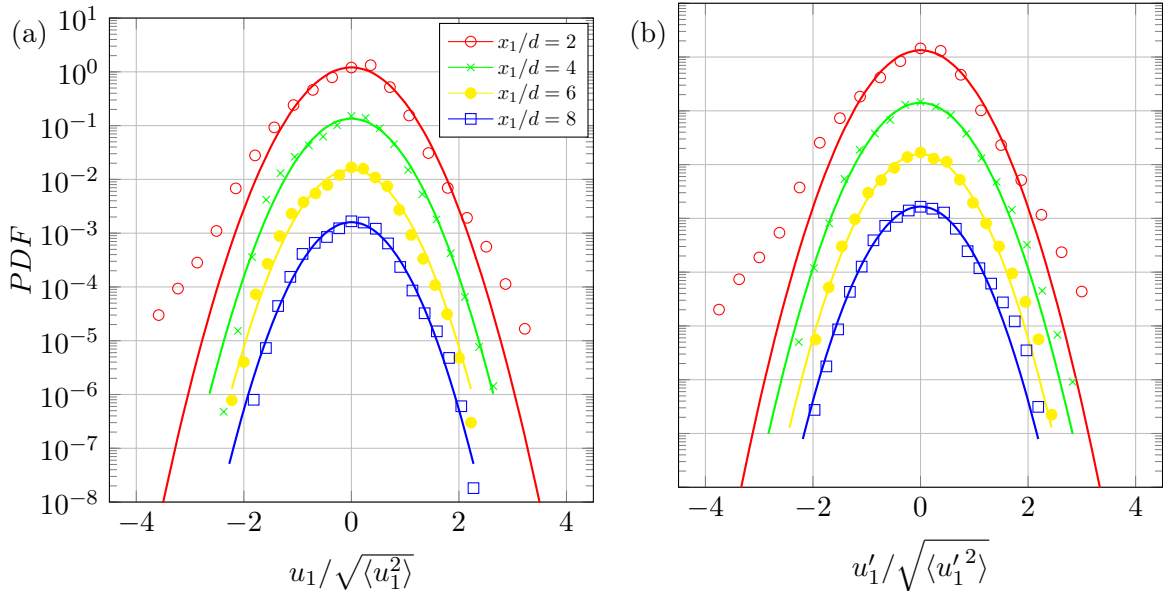


Figure 4.9: PDFs of $u_1 = \tilde{u}_1 + u'_1$ in (a) and u'_1 in (b) at $x_1/d = 2, 4, 6, 8$ on the geometric centreline. The PDFs are offset by a decade between consecutive locations. The full lines show a Gaussian fit to the data.

Furthermore, fig. 4.9b shows that the PDFs of u'_1 do approach Gaussian fits in the vicinity of the mean values, but develop wider “tails” at positive values of u'_1 similar to the decay region in grid turbulence as discussed in Maxey (1987). These positive tails appear to develop only after an initial region where the PDFs are strongly negatively skewed (at $x_1/d = 2$) and in fact, when comparing fig. 4.9a to fig. 4.9b, are seen to be due to the stochastic motion. This evolution from a strongly negatively skewed to slightly positively skewed PDF is made clearer in fig. 4.10 where the skewness, defined for some variable ψ ,

$$S_\psi = \frac{\langle \psi^3 \rangle}{\langle \psi^2 \rangle^{3/2}} \quad (4.7)$$

of the streamwise fluctuations is plotted along the centreline, both with and without the phase-component. Figure 4.10 also confirms that for $x_1/d \gtrsim 2$ the effect of the coherent motion is to reduce the values of S_{u_1} (since $S_{u'_1} > S_{u_1}$). Notice that this effect is opposite to that reported by Melina et al. (2016), but easily reconcilable given that their measurements were made away from the wake’s centreline and therefore the effect of the coherent motion must be opposite, i.e. the “bump” observed in fig. 4.9a would be located at positive values of u_1 in their flow. The error bars in fig. 4.10 (as well as figs. 4.11, 4.14 and 4.15 discussed below) have been calculated by appropriately accounting for propagation of uncertainty and then following the procedure introduced in Benedict and Gould (1996) to compute estimators for the variance of non-Gaussian distributions; the num-

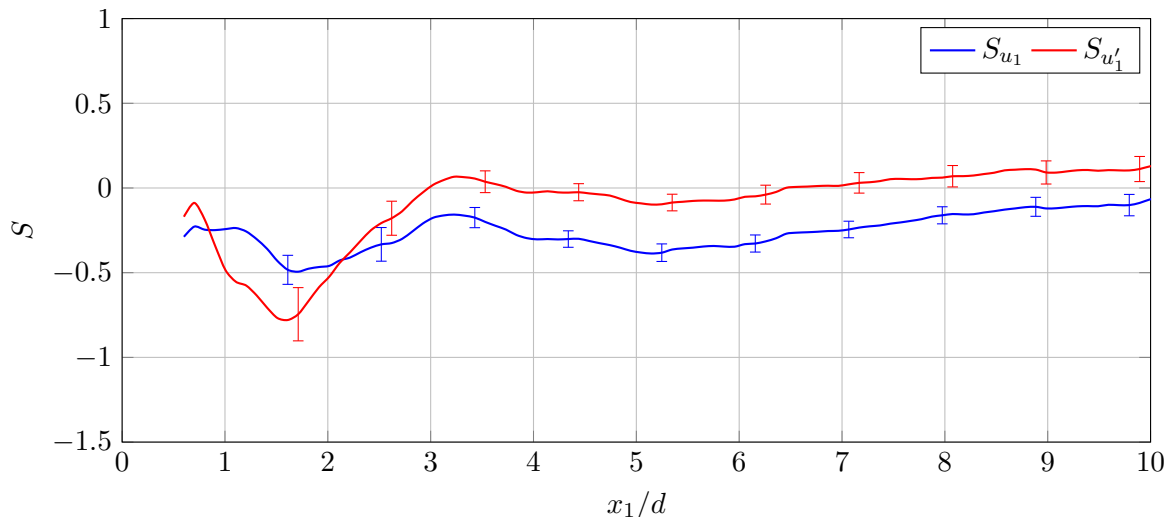


Figure 4.10: Skewness of u_1 and u'_1 along the geometrical centreline. The error bars are given by $\pm\sqrt{\frac{\sigma}{N_I}}$ where $\frac{\sigma}{N_I}$ is an estimator for the variance of the skewness given in Benedict and Gould (1996).

ber of independent samples was taken to be the number of integral time scales (at each x_1/d position) times the number of integral length scales obtained from the spanwise autocorrelation of u_3 (also at each x_1/d position).

In Ninni et al. (1999), Norberg (1998), and Ong and Wallace (1996) the authors study (experimentally) wakes generated by a circular cylinder. Ong and Wallace (1996) report positive skewness of u_1 along the centreline (for $3 \leq x_1/d \leq 10$), in contrast with the results of Ninni et al. (1999) where the skewness appears negative at $x_1/d = 11$ and $x_1/d = 14$. Even though the Re_∞ used in Ninni et al. (1999), which was $Re_\infty = 10000$, differed from the value of $Re_\infty = 3900$ used in Ong and Wallace (1996), the results of Norberg (1998) show that, for circular cylinders, after a positive peak in the vicinity of the vortex formation region, the skewness of u_1 becomes negative and then slowly grows (towards positive values) as x_1/d increases (the range of Re_∞ investigated by Norberg (1998) was $1500 \leq Re_\infty \leq 10000$).

Furthermore, as shown in fig. 1c in Norberg (1998), the value of the positive peak in the skewness of u_1 appears to decrease and shift upstream as the Reynolds number is increased. While this may not only explain the difference between the results obtained in Ninni et al. (1999) and Ong and Wallace (1996), it also suggests that the coherent structures generated by the square prism are “stronger” than those generated by a circular cylinder. As shown below, the local Reynolds number in the present flow is comparable to the values just above $Re_\lambda = 200$ reported in Ninni et al. (1999) at $x_1/d = 11$ and $x_1/d = 14$. This is in line with the arguments proposed by Lyn, Einav, et al. (1995) who compared the two wake generators and found substantial differences between the

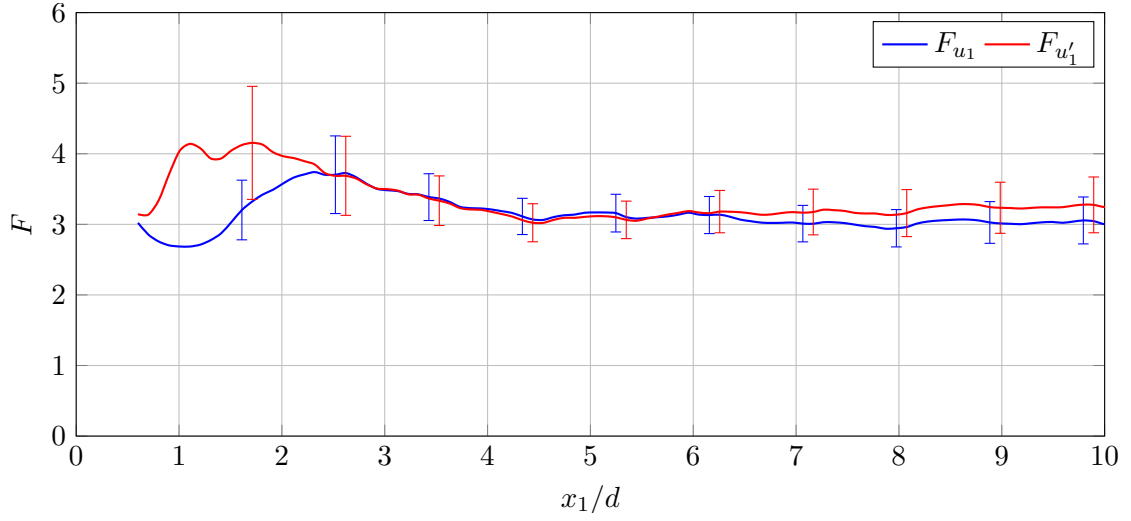


Figure 4.11: Flatness of u_1 and u'_1 along the geometrical centreline. The error bars are given by $\pm \sqrt{\frac{\sigma}{N_I}}$ where $\frac{\sigma}{N_I}$ is an estimator for the variance of the flatness given in Benedict and Gould (1996).

two flows in what they call the base region which is $x_1/d \lesssim 4$. Notice that it is precisely within that region that the profile of S_{u_1} shown in Norberg (1998) becomes positive.

As a final note on the statistics of u_1 , we show in fig. 4.11 the flatness (also known as kurtosis), defined for some variable ψ as

$$F_\psi = \frac{\langle \psi^4 \rangle}{\langle \psi^2 \rangle^2}, \quad (4.8)$$

of both u_1 and u'_1 . The error bars shown in fig. 4.11 were computed in the same way as those in fig. 4.10. Notice that, unlike the profiles of the skewness along the centreline, F_{u_1} and $F_{u'_1}$ do not differ much and are in fact fairly close to the value typical of Gaussian PDFs. This is in agreement with the observations of Ninmi et al. (1999), who report values of F_{u_1} close to 3 at $x_1/d = 11$ and $x_1/d = 14$, and not far from the results obtained by Ong and Wallace (1996) who report values of F_{u_1} decreasing from about 4 towards values close to 3 from $x_1/d = 3$ to $x_1/d = 10$. Despite the relatively large error bars in fig. 4.11, it may appear that $F_{u'_1}$ tends to slightly exceed the value of 3, i.e. the coherent motion may actually decrease (even if only slightly) the flatness of the streamwise fluctuations.

With regard to the cross-stream fluctuations, figs. 4.12a and 4.12b show that, apart from having zero skewness (due to the reflection symmetry), the PDFs of both u_2 and u'_2 are anything but Gaussian. The double peaked PDFs reported by Durão et al. (1988) are also observed here and shown in fig. 4.12a. Just as the gentle “bumps” observed in fig. 4.9a around $u_1/\sqrt{\langle u_1^2 \rangle} \approx -1$, the double peak feature is clearly related to the passage of the coherent vortices which induce strong cross-stream perturbations in the velocity

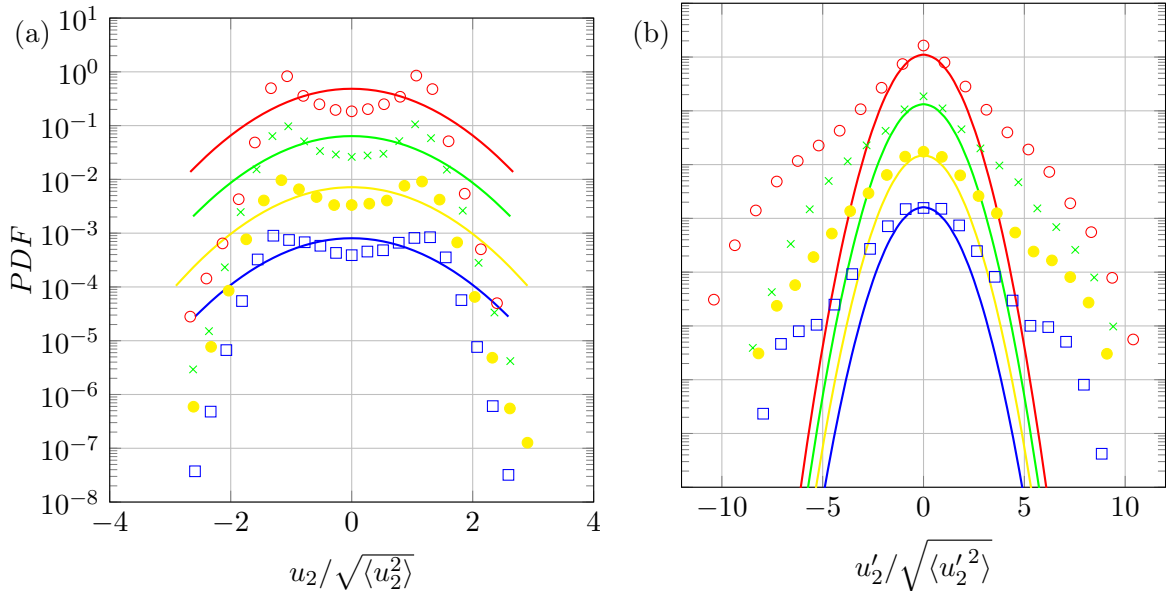


Figure 4.12: PDFs of $u_1 = \tilde{u}_2 + u'_2$ in (a) and u'_2 in (b) at $x_1/d = 2, 4, 6, 8$ on the geometric centreline. The PDFs are offset by a decade between consecutive locations. The full lines show a Gaussian fit to the data. See fig. 4.9a for legend.

of the fluid in-between vortices as discussed with regard to fig. 4.8. In fact, after the phase-component is extracted from u_2 , the PDFs display a single peak at the mean value and remain much flatter than the Gaussian fits especially for values of $|u'_2|$ larger than about $4\sqrt{\langle u_2'^2 \rangle}$. Durão et al. (1988) used a linear scale for the PDF of u'_2 and therefore did not report these tails..

It should be noted that the double peak in the PDF of u_2 is not observed by Ninni et al. (1999) in the wake of a circular cylinder, instead, they identify a wider spread of the PDF around the mean which may still be a signature of the same phenomena described above. Baj et al. (2015) reported a similar pattern to that of fig. 4.12a downstream of rectangular bars with their chord normal to the free-stream. Since one distinguishing feature between the present flow (and those of Baj et al., 2015; Durão et al., 1988) and the flow past a circular cylinder is the fixed separation point, one can perhaps reconcile fig. 4.12a with the results of Ninni et al. (1999) by arguing that the coherent motion is weaker in the case of bluff bodies without sharp edges. This would result in PDFs of u_2 that become flatter near the mean at locations closer to the wake generator.

Finally we turn our attention to the PDF of the spanwise fluctuations (shown in fig. 4.13). Unlike for the streamwise and cross-stream, there is no need to distinguish between u_3 and u'_3 since, as seen in fig. 4.4, the coherent motion occurs on the plane of the mean flow and thus $\{u_3\} = 0$. Figure 4.13 clearly shows that u_3 is non-Gaussian displaying flatter tails than the Gaussian fits, however, as for u_2 , the skewness of the PDFs

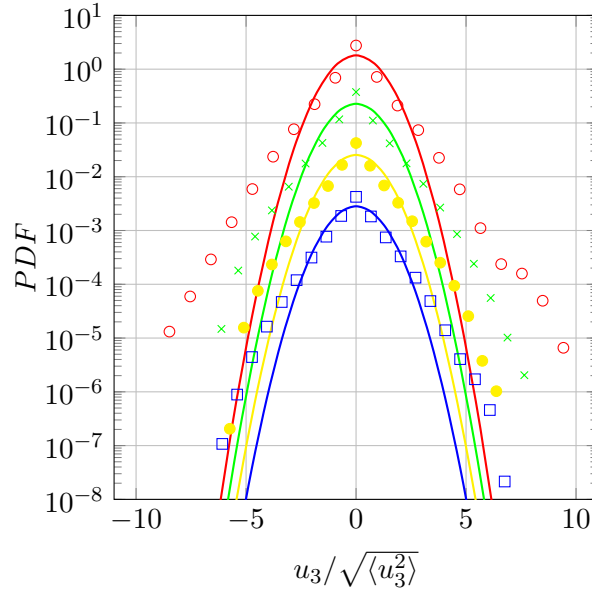


Figure 4.13: PDF of u_3 at $x_1/d = 2, 4, 6, 8$ on the geometric centreline. The PDFs are offset by a decade between consecutive locations. The full lines show a Gaussian fit to the data. See fig. 4.9a for legend.

of u_3 must be zero due to the reflection symmetry about the centreline. Apart from Ninni et al. (1999) there seems to be no other reports in literature on the PDFs of u_3 . Ninni et al. (1999) observe a slightly flatter PDF of u_3 with respect to a Gaussian fit (with flatness of about 3.2). Effectively, the distributions shown in fig. 4.13 highlight the relevance of the spanwise fluctuations in the dynamics of the near wake turbulence. While this velocity component is not directly affected by the shedding, the lack of Gaussianity must arise from the fact that neither u_1 nor u_2 are Gaussian, thus, the production of spanwise fluctuations appear to occur in a rather “violent” way, whereby strong fluctuations in u_3 are produced more often than if it were a purely random phenomenon.

As mentioned above, u_2 , u'_2 and u_3 must have zero skewness due to the reflection symmetry of the mean flow. While values of zero skewness are also a characteristic of Gaussian PDFs, figs. 4.12 and 4.13 show that this is far from being the case. In fact, unlike the PDFs of u_1 and u'_1 , the flatness of u'_2 and u_3 exceed by far the value of 3 (as seen in figs. 4.14 and 4.15) which is characteristic of Gaussian distributions. Furthermore, fig. 4.14 shows how the coherent motions actually decrease the values of F_{u_2} since, as seen in fig. 4.12a, the coherent motions contribute only to small excursions from the mean value (this effect is so pronounced that the error bars in fig. 4.14 for F_{u_2} are almost as small as the thickness of the lines).

Notice that the values of F_{u_2} and F_{u_3} reported here are somewhat different from those reported by Ninni et al. (1999) for a circular cylinder: they show $F_{u_2} \approx 2.5$ and $F_{u_3} \approx 3.2$

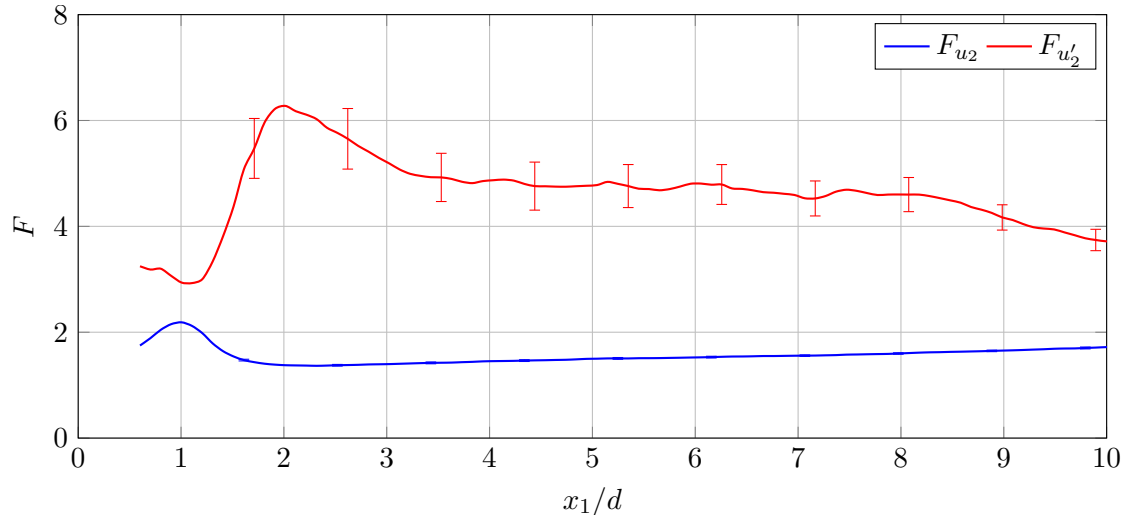


Figure 4.14: Flatness of u_2 and u'_2 along the geometrical centreline. See fig. 4.11 for details regarding the error bars.

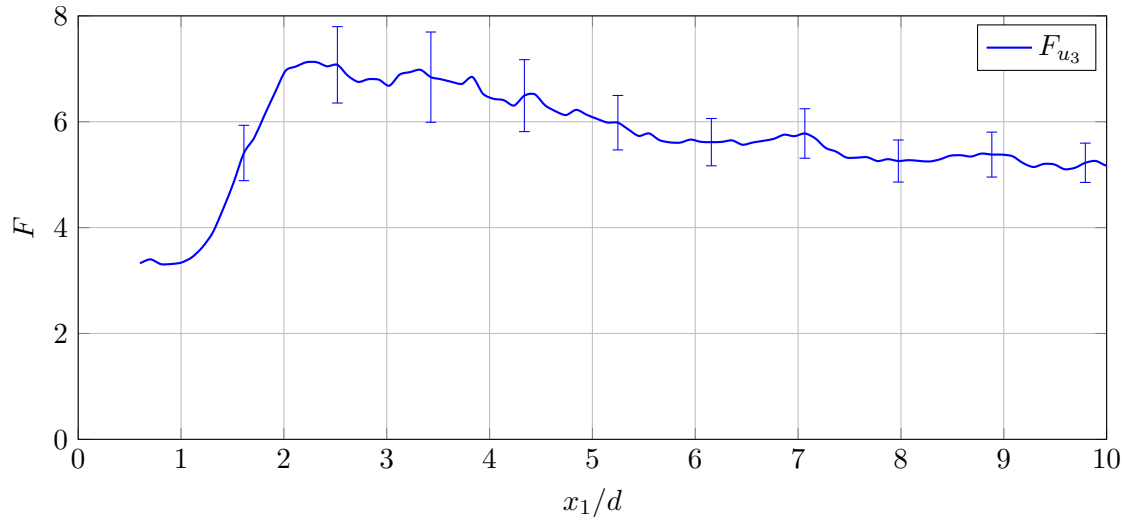


Figure 4.15: Flatness of u_3 along the geometrical centreline (recall that $u'_3 = u_3$). See fig. 4.11 for details regarding the error bars.

at $x_1/d = 11$ and $x_1/d = 14$. While fig. 4.12a suggests that F_{u_2} does increase with x_1/d (and thus may not be so far from the values reported in Ninni et al. (1999)), F_{u_3} remains fairly large and seems unlikely to decrease towards Gaussian levels. The differences between the present results and those of Ninni et al. (1999) appear to tie in with the argument presented in Lyn, Einav, et al. (1995) that the wake generated by a square prism displays larger velocity and length scales than that of the circular cylinder.

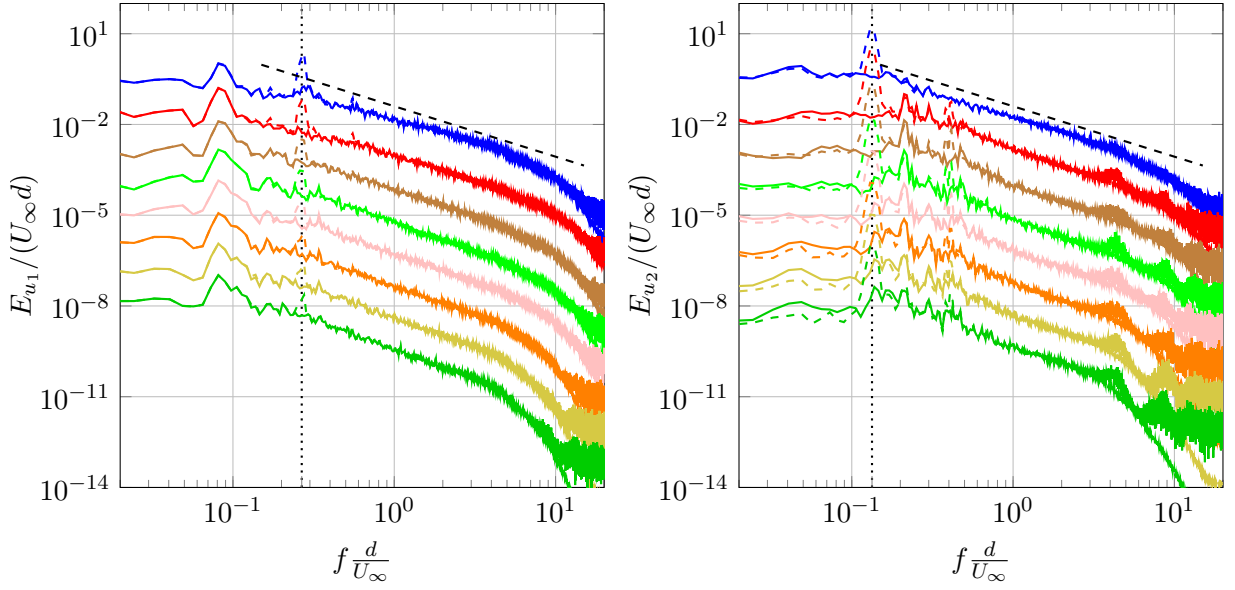


Figure 4.16: Power spectrum densities normalised by $U_\infty d$ of streamwise (left) and cross-stream (right) fluctuating velocities, before (dashed lines) and after (full lines) removing the phase component, between $x_1/d = 1$ and $x_1/d = 8$ (see fig. 3.2 for legend) offset by one decade. The dashed line indicates a slope of $-5/3$ and the dotted line indicates $f = 2f_s$ (left) and $f = f_s$ (right).

4.2.2 Kinetic Energy and Spectra

The purpose of the triple decomposition is to remove the time signature associated with the shedding from the fluctuating velocity and pressure. In the preceding section we have shown how removing the phase component of the velocity signals can drastically change their PDF. Given the already identified signature of the shedding on the spectra of u_1 and u_2 , one may expect this signature to vanish once the phase-averaging procedure is carried out. This can be seen in fig. 4.16 where the spectra of the fluctuating velocities u_1 and u_2 are compared to those of their stochastic counterparts: the velocity spectra remain mostly unchanged once the phase component is removed. The noticeable difference between the spectra computed for u'_i and $u_i = \tilde{u}_i + u'_i$ is the broadband peak around the shedding frequency.

Notice, however, that at large frequencies the spectra of u'_i are slightly disturbed (by comparison with figs. 3.2 and 3.3a). This happens because of the binning of the phase signal which effectively introduces artificial “steps” (jumps) in the time signal of u'_i . Indeed, if the number of bins is increased, this effect becomes less evident as shown in fig. 4.17. This, however, does not affect the results that follow as no time derivatives are computed and, as mentioned above, a smaller number of bins actually improves statistical convergence (since more samples of the stochastic signal are obtained for each bin of the phase-component).

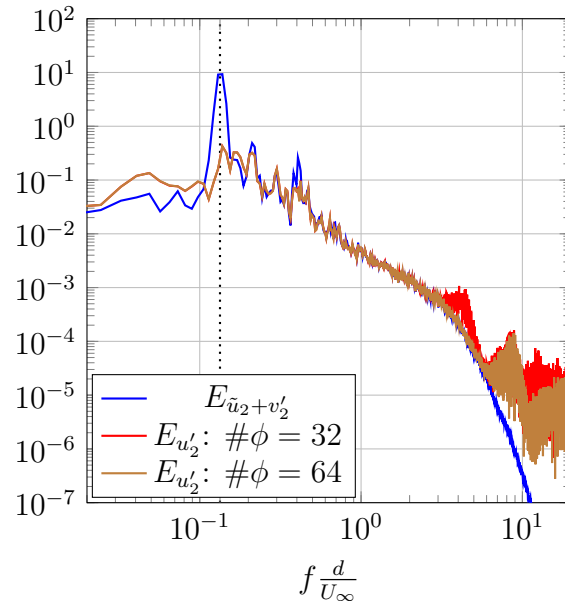


Figure 4.17: Comparison of spectra of the stochastic cross-stream velocity component when the number of bins (used for the phase averaging) is increased from 32 to 64.

The spectra in fig. 4.16 show how the phase averaging process removes from the fluctuating velocity the energy content associated with the band of frequencies neighbouring the shedding frequency f_s and its harmonics. This does not mean, however, that $E_{u'_1}(2f_s) = 0$ or $E_{u'_2}(f_s) = 0$. Since $\langle u_i^2 \rangle = \int_0^\infty E_{u_i} df$, one can obtain a quantitative measure of this difference by comparing $\langle \tilde{k} \rangle$, the mean kinetic energy associated with $\tilde{\mathbf{u}}$, to $\langle k' \rangle$, the mean kinetic energy associated with $\mathbf{u}'$. As can be seen in fig. 4.18, $\langle \tilde{k} \rangle$ is larger than $\langle k' \rangle$ up until $x_1/d \approx 10$, suggesting that the largest contribution to the total kinetic energy arises from the coherent motion, although $\langle k' \rangle$ is always of comparable order of magnitude.

One can reasonably assert that the phase averaged velocity is representative of the coherent structures created by the vortex shedding (Braza et al., 2006; Hussain, 1983; Lyn, Einav, et al., 1995) as $\tilde{\mathbf{u}}$ contains the characteristic time signature from the shedding (since that's how this quantity is defined) and its spatial distribution (recall fig. 4.5) resembles that of large scale vortices.

In Hussain (1983) and Hussain and Hayakawa (1987) it is argued that these coherent structures do not necessarily provide a large contribution to the turbulent kinetic energy. Of course, the regions of the flow considered in those references are much further downstream than the part of the flow studied here. Figure 4.18 makes it clear that these structures do contribute the most to the kinetic energy $\langle k \rangle$ in the very near wake and that this contribution ($\langle \tilde{k} \rangle$) decreases, in the direction of the mean flow, at a faster rate

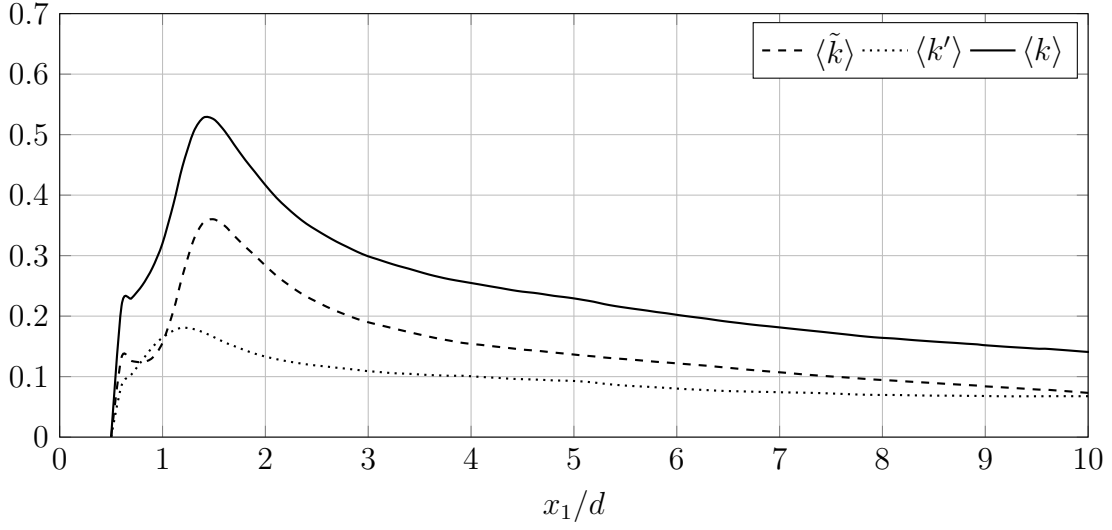


Figure 4.18: Profiles of kinetic energies $\langle \tilde{k} \rangle$ and $\langle k' \rangle$ computed from the phase and stochastic components, respectively, along the centreline. The total kinetic energy $\langle k \rangle = \langle \tilde{k} \rangle + \langle k' \rangle$ is also shown for comparison.

than the kinetic energy associated with the stochastic motion ($\langle k' \rangle$) in-line with Hussain (1983) and Hussain and Hayakawa (1987).

4.2.3 Dissipation and Nonequilibrium

Despite containing a large portion of the kinetic energy, the coherent structures appear to contribute very little to the total turbulent dissipation, as suggested by Hussain (1983) and Hussain and Hayakawa (1987). Figure 4.19 shows how the dissipation associated with the phase averaged velocity $\tilde{\varepsilon}$ and that associated with the stochastic motion ε' are strikingly different in magnitude. It can be seen that the largest contribution to the total dissipation ε arises from ε' , while $\tilde{\varepsilon}$ remains almost negligible throughout. In fact, fig. 4.19 clearly shows that the contribution of $\tilde{\varepsilon}$ to ε is almost negligible throughout the region investigated here.

While it is well known that $\tilde{\varepsilon} \ll \varepsilon'$, Goto and Vassilicos (2016a) have recently shown, in a DNS of decaying turbulence in a periodic box, that the ratio $\tilde{\varepsilon}/\varepsilon'$ remains approximately constant throughout a significant period of time during which the turbulence departs increasingly further from the equilibrium $\Pi = \text{const}$, i.e. $\mathcal{A}_t$ increases while Π decreases. Notice that since Goto and Vassilicos (2016a) study turbulence in a periodic box, the KMH equation reduces to $\mathcal{A}_t = -\Pi + \mathcal{D}_r - \varepsilon$, the real-space equivalent to the integrated Lin equation (Lin, 1949).

Figure 4.20 shows that, in the near wake region studied here the ratio $\varepsilon'/\tilde{\varepsilon}$ behaves in accordance with Goto and Vassilicos (2016a). During the decay of turbulence (recall from

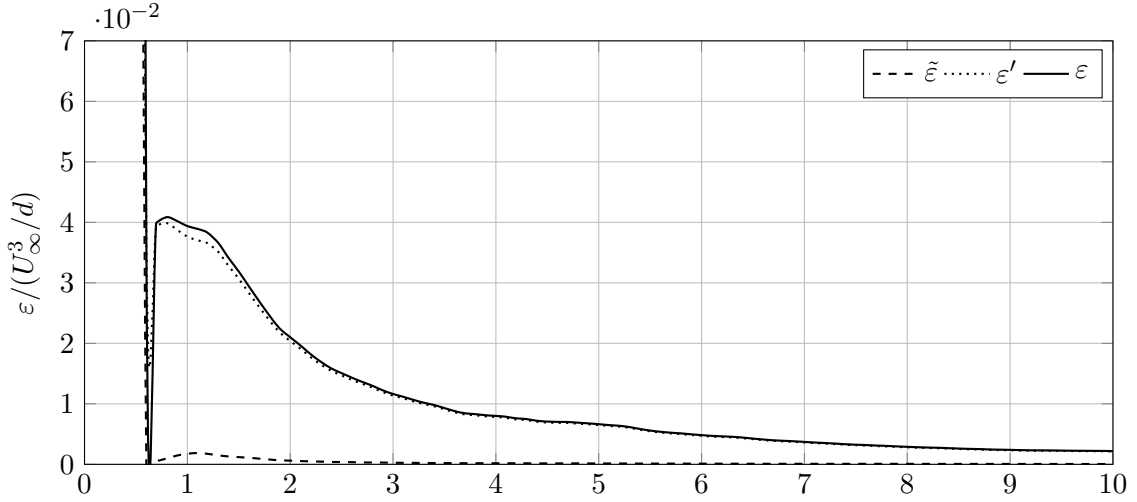


Figure 4.19: Profiles of the dissipations $\tilde{\varepsilon}$ and ε' computed from the phase and stochastic components, respectively, along the centreline. The total turbulent dissipation $\varepsilon = \tilde{\varepsilon} + \varepsilon'$ is also shown for comparison.

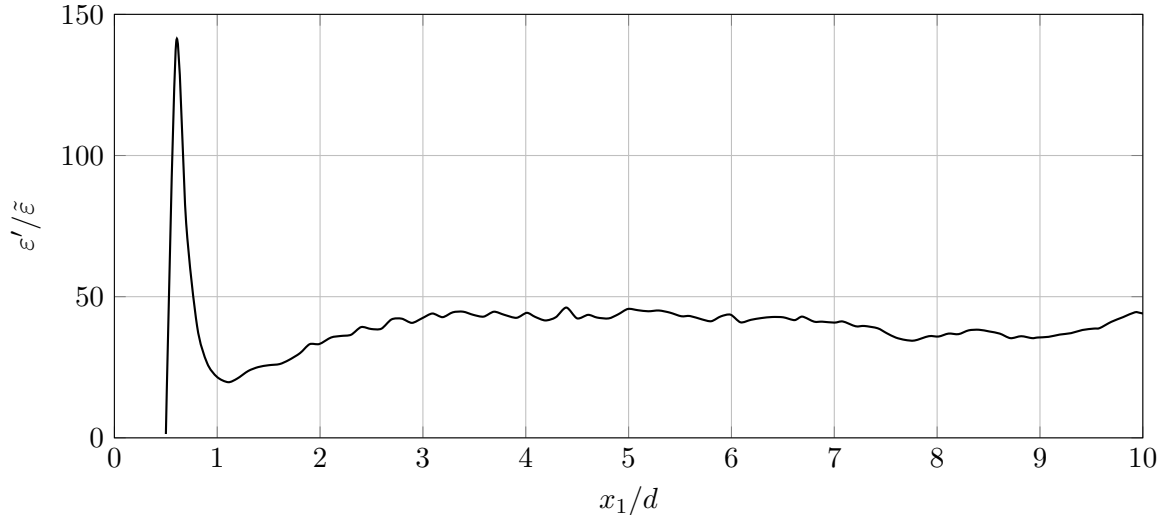


Figure 4.20: Ratio between the dissipation associated with the stochastic component ε' and that associated with the phase component $\tilde{\varepsilon}$ along the centreline.

fig. 4.18 that $\langle k \rangle$ decays in the streamwise direction for x_1/d larger than about 1.5) the ratio $\varepsilon'/\tilde{\varepsilon}$ remains about constant between $x_1/d \approx 3$ and $x_1/d = 10$, despite both terms being of significantly different magnitudes, as shown in fig. 4.19. This suggests that the stochastic motion is somehow coupled to the coherent motion.

It should be noted that the definition of dissipation by the coherent structures used in Goto and Vassilicos (2016a) is different from ours. Goto and Vassilicos (2016a) define $\tilde{\varepsilon} = 2\nu \int_0^{\kappa_c} \kappa^2 E_u dk$, where κ_c is some cut-off wavenumber characteristic of the large coherent eddies. This definition allowed those authors to show that $\tilde{\varepsilon}/\varepsilon' = \text{const}$ implies $C_\varepsilon \propto Re_\lambda^{-1}$. This dependence of C_ε on Re_λ is the same as the one in eq. (1.13) discussed

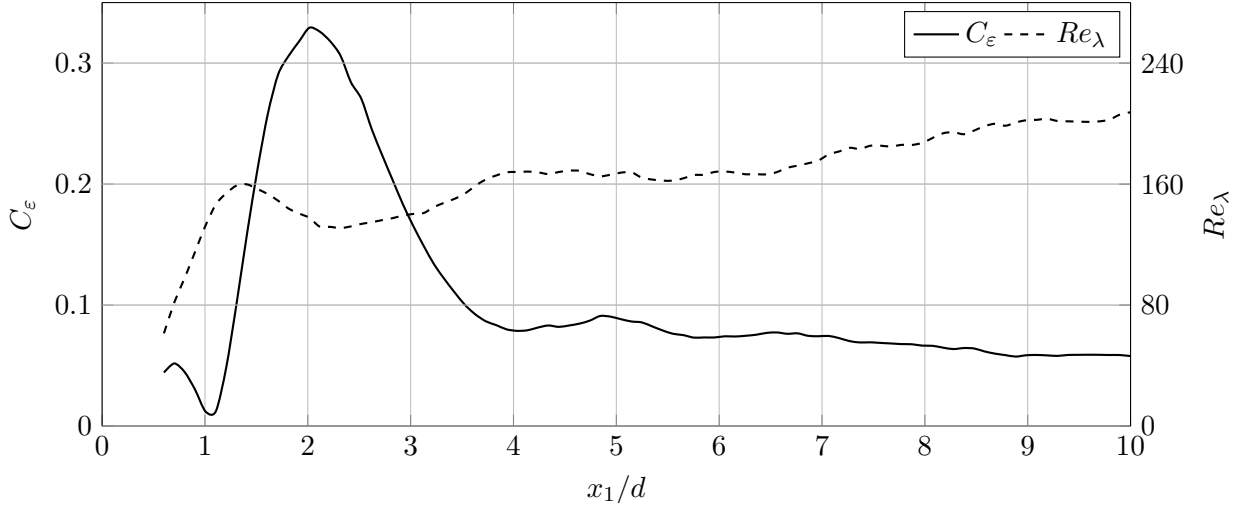


Figure 4.21: Evolution of C_ε and Re_λ along the centreline.

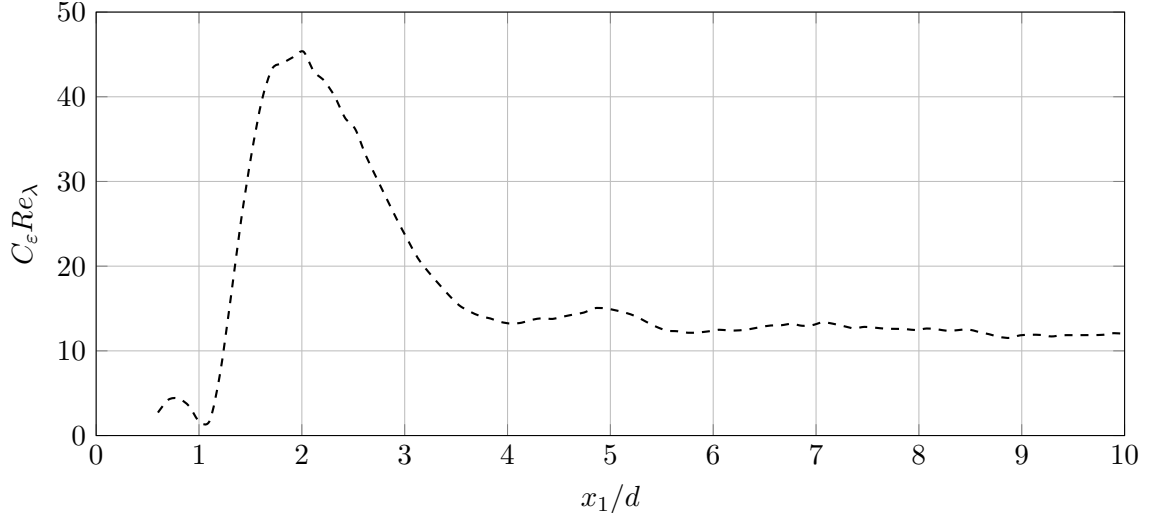


Figure 4.22: Evolution of $C_\varepsilon Re_\lambda$ along the centreline.

in section 1.1.2, the so-called nonequilibrium dissipation law (Vassilicos, 2015) which has been observed in a variety of flows (Discetti et al., 2013; Goto and Vassilicos, 2015; Goto and Vassilicos, 2016b; Mazellier and Vassilicos, 2008; Nedić, Tavoularis, et al., 2017; Seoud and Vassilicos, 2007). The evolution along the centreline of both the dissipation constant C_ε (see eq. (1.10)) and the local Reynolds number Re_λ (see eq. (1.14)) is shown in fig. 4.21. Notice that even though the turbulence decays along x_1 , the local Reynolds number increases with streamwise position and the dissipation “constant” decreases, so much so that, as shown in fig. 4.22, the product of both quantities remains approximately constant right after it peaks at about $x_1/d \approx 2$.

Before commenting on how C_ε and Re_λ in figs. 4.21 and 4.22 were computed, it is worth mentioning that mean deviation of $C_\varepsilon Re_\lambda$ from its mean value in the range

$3.5 \leq x_1/d \leq 10.5$ is about 4% (in fact, the difference between $C_\varepsilon Re_\lambda$ at $x_1/d = 3.5$ and $x_1/d = 10.5$ is about 1% of the mean value in that range), whereas the deviation of C_ε and Re_λ from their mean values in the same range is about 12%.

Given the anisotropy and inhomogeneity of this flow, some choices had to be made as to how to compute the velocity scale $\mathcal{U}$ and length scale $\mathcal{L}$ entering the definition of C_ε , as well as $\mathcal{U}$ and λ required to compute Re_λ . According to Taylor (1935), the integral length scale $\mathcal{L}$ at some location $\mathbf{x}$ would be given by the transverse integral length scale

$$\mathcal{L} = \mathcal{L}_\perp(\mathbf{x}) = \frac{1}{\sqrt{\langle u_1(\mathbf{x}, t)^2 \rangle}} \left\langle \int_{-\infty}^{\infty} u_1(\mathbf{x}, t) u_1(\mathbf{x} + \xi \mathbf{e}_2, t) d\xi \right\rangle, \quad (4.9)$$

where $\mathbf{e}_2$ is a unit-vector in the direction of x_2 . $\mathcal{L}_\perp(\mathbf{x})$ characterises the average size of the eddies passing by the location $\mathbf{x}$ as suggested in Taylor (1935). One can also define a streamwise integral length scale $\mathcal{L}_\parallel$ where the integral in eq. (4.9) is taken in the streamwise direction, a definition that is experimentally advantageous as it allows $\mathcal{L}_\parallel$ to be computed from single-point time series (using Taylor's frozen flow hypothesis, as discussed in Taylor, 1935). If the turbulence is isotropic, $\mathcal{L}_\perp$ and $\mathcal{L}_\parallel$ are related through $\mathcal{L}_\parallel = \frac{1}{2} \mathcal{L}_\perp$.

Computing $\mathcal{L}_\perp$ explicitly would require a large computational effort as the correlation may not cross zero (in practice, the bounds of the integral in eq. (4.9) are often taken as the zero crossings of the correlation function). Conversely, computation of the longitudinal integral length scale $\mathcal{L}_\parallel$ from correlations in space can only be done in a range of x_1/d that is far enough from the boundaries of the available data. An alternative is to compute $\mathcal{L}_\parallel$ through Taylor's hypothesis, where the integral time scale Θ is multiplied by the local mean velocity to obtain an integral length scale; to estimate Θ_{u_1} (the integral time scale computed from u_1) we have used the equivalence between the autocorrelation and the spectrum (see e.g. Pope, 2000)

$$\Theta_{u_1}(\mathbf{x}) = \frac{1}{\sqrt{\langle u_1(\mathbf{x}, t)^2 \rangle}} \left\langle \int_{-\infty}^{\infty} u_1(\mathbf{x}, t) u_1(\mathbf{x}, t + \tau) d\tau \right\rangle = \frac{E_{u_1}(f = 0)}{4 \langle u_1^2 \rangle} \quad (4.10)$$

as done in eq. (3.11). The value of $E_{u_1}(f = 0)$ was taken as the mean of the points lying on the plateau observed in fig. 3.2a. In fig. 4.23 the integral time scales computed through eq. (4.10) are shown along the centreline and it can be seen that the behaviour of the integral time scales associated with each velocity component do not vary in the same way along x_1/d . C_ε in figs. 4.21 and 4.22, where C_ε is shown to be proportional to Re_λ^{-1} in the range $3 \lesssim x_1/d < 10$, have been obtained by setting $\mathcal{L} = U_1 \Theta_{u_1}$ in the definition of C_ε .

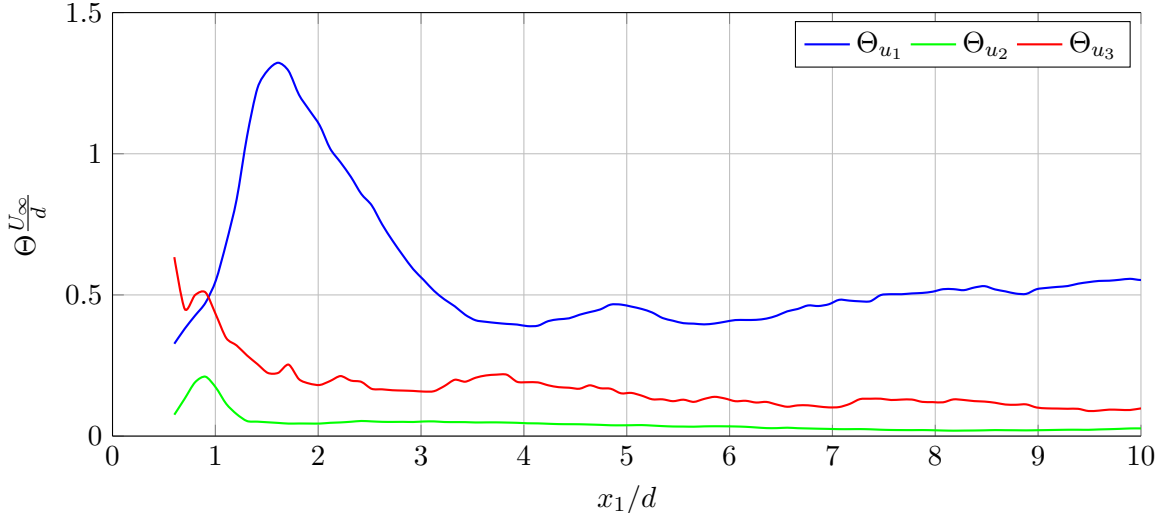


Figure 4.23: Integral time scale (given by eq. (4.10)) of each velocity component along the geometrical centreline.

As shown in fig. 4.23, only Θ_{u_1} increases along x_1 , therefore one cannot expect $C_\varepsilon \propto Re_\lambda^{-1}$ when either Θ_{u_2} or Θ_{u_3} are used to represent $\mathcal{L}$ since they decrease as x_1 increases.

The observation that $C_\varepsilon \propto Re_\lambda^{-1}$ on the centreline relies on the fact that the evolution of $U_1\Theta_{u_1}$ with x_1/d represents that of $\mathcal{L}_\parallel$ along x_1/d . While $\mathcal{L}_\parallel$ cannot be computed for all x_1/d where $U_1\Theta_{u_1}$ is available, we have used the autocorrelation function defined in eq. (3.7) to compute the integral length scales

$$\mathcal{L}_{ij} = \int_{-\infty}^{\infty} \rho_{u_i}(\xi \mathbf{e}_j) d\xi \quad (4.11)$$

associated with each velocity component i for $j = 1$ on the centreline for the range of x_1/d permitted by the boundaries of our database. The limits of the integral in eq. (4.11) were replaced by the zero crossing of the autocorrelation function on either side of $\xi = 0$ (notice that due to inhomogeneity the positive value of $\xi|_{\rho_{u_i}=0}$ is not necessarily equal in magnitude to its negative counterpart). Using eq. (4.11), one can readily see that $\mathcal{L}_\perp$ is equivalent to $\mathcal{L}_{12}$ and $\mathcal{L}_\parallel$ is equivalent to $\mathcal{L}_{11}$. In fig. 4.24 we show the different length scales normalised by $\mathcal{L}_{11}$, computed both from eq. (4.11) and from eq. (4.10) (multiplied by the local mean velocity).

Notice how in fig. 4.24 the range of x_1/d where data is available is much smaller than in fig. 4.23. This is because of the limited volume of data stored from the DNS which prevents us from computing the correlations at large separations. Despite the small range for which data is available, fig. 4.24 suggests that the ratio $U_1\Theta_{u_1}/\mathcal{L}_{11}$ is roughly constant, especially in the range where $C_\varepsilon \propto Re_\lambda^{-1}$ (recall fig. 4.21) where its average departure from

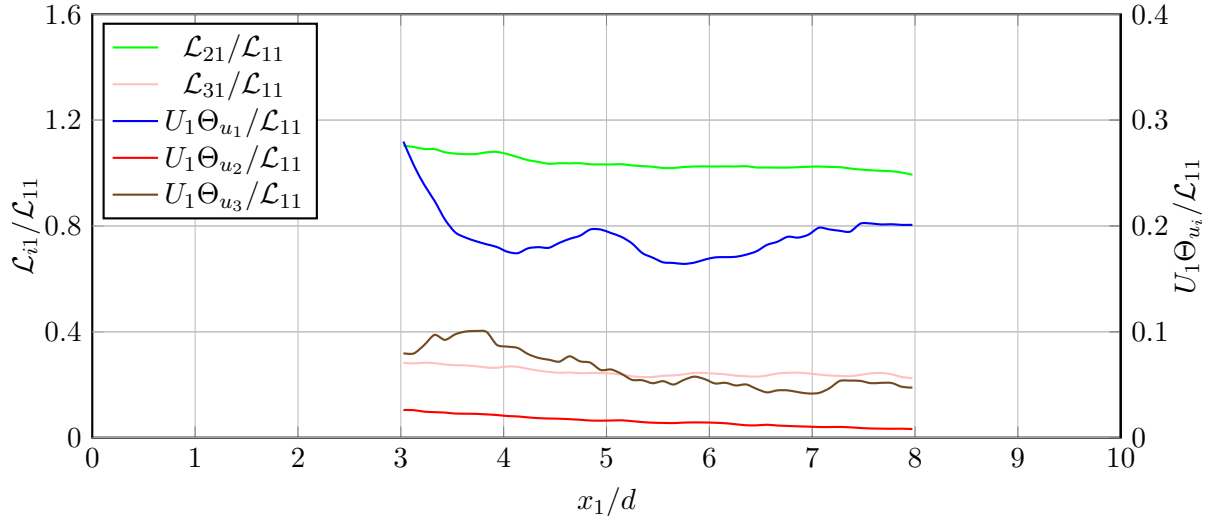


Figure 4.24: Integral length scale, both computed by integrating the spatial correlations up to the first zero crossing and from the integral time scale (fig. 4.23) of each velocity component along the geometrical centreline.

the mean value (in the range $3.5 < x_1/d < 8$) is about 5%; conversely, $U_1\Theta_{u_2}/\mathcal{L}_{11}$ and $U_1\Theta_{u_3}/\mathcal{L}_{11}$ decrease continuously in the range and depart from the mean value by more than 20%. Therefore, even though $U_1\Theta_{u_1}$ differs from $\mathcal{L}_{11}$ by a factor of a little more than 5, it appears to capture its growth with increasing x_1 .

Figure 4.24 shows that length scales $\mathcal{L}_{21}$ and $\mathcal{L}_{31}$ appear to follow the same evolution as $\mathcal{L}_{11}$ (their ratios vary by less than 4% of their mean value in the range $3.5 < x_1/d < 8$), meaning that, on the centreline, all integral length scales computed from the autocorrelation function of any velocity component in the direction of the mean-flow grow in the same manner. Thus, it appears that the choice of $\mathcal{L}_{i1}$ with $i = 1, 2, 3$ does not affect the observation that $C_\epsilon \propto Re_\lambda^{-1}$. Furthermore, $\mathcal{L}_{21} \approx \mathcal{L}_{11}$ means that $\mathcal{L}_{21}$ is twice as large as its isotropic value of $\frac{1}{2}\mathcal{L}_{11}$ (this is not surprising given that the coherent structures appear to be larger in the cross-stream direction as seen in fig. 4.7) and $\mathcal{L}_{31} \approx \frac{1}{4}\mathcal{L}_{11}$ is fairly close to the values obtained in grid generated turbulence by Valente and Vassilicos (2014).

With regard to the computation of the local Reynolds number Re_λ , the required parameters are a velocity scale $\mathcal{U}$ and a length scale λ , the Taylor micro-scale. As introduced in Taylor (1935), without summing over repeated indices,

$$\frac{1}{\lambda_{ij}^2} = \lim_{\xi \rightarrow 0} \left(\frac{1 - \langle u_i(\mathbf{x})u_i(\mathbf{x} + \xi\mathbf{e}_j) \rangle}{\xi^2} \right) \quad (4.12)$$

is the curvature of the autocorrelation of u_i along the direction x_j . For example, for $i = 1$ and $j = 2$, it can be seen that eq. (4.12) is related to the gradient of u_1 along x_2 since

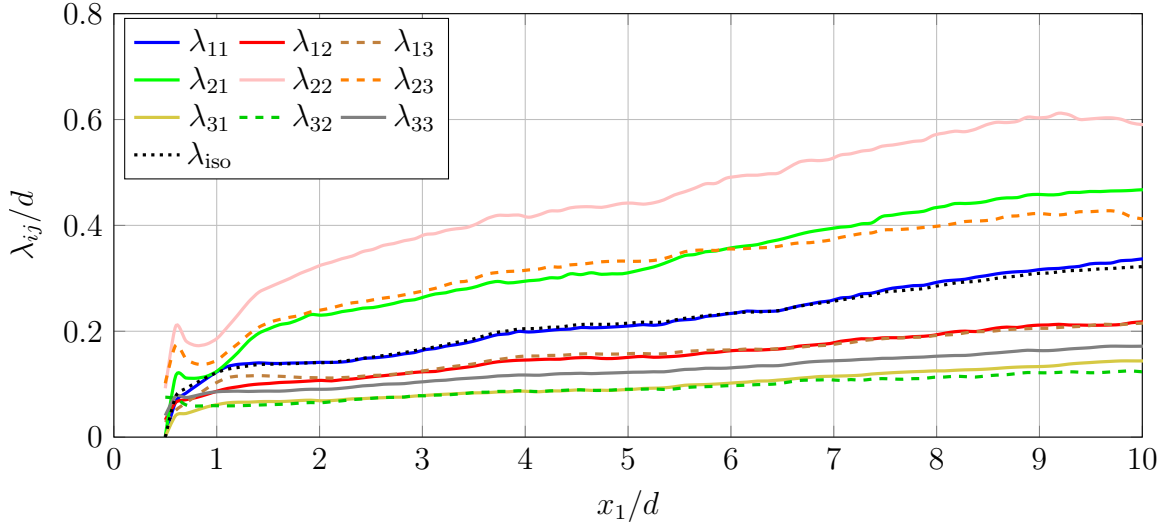


Figure 4.25: Different estimates for Taylor length scale computed through eq. (4.12). λ_{iso} is the isotropic estimate given by $\sqrt{\langle u_1^2 \rangle} \sqrt{15 \frac{\nu}{\varepsilon}}$.

the limit in eq. (4.12) is equal to $\frac{1}{2} \frac{\partial^2}{\partial \xi^2} \langle u_1(\mathbf{x}) u_1(\mathbf{x} + \xi \mathbf{e}_2) \rangle = -\frac{1}{4} \frac{\partial^2 u_1^2}{\partial x_2^2} + \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} \right)^2$ and, as discussed in Pope (2000) one has that $\frac{\partial^2 u_1^2}{\partial x_2^2} \ll \left(\frac{\partial u_1}{\partial x_2} \right)^2$.

For the case of isotropic turbulence, one has that the Taylor micro-scale λ equals $\mathcal{U} \sqrt{15 \frac{\nu}{\varepsilon}}$, where $\mathcal{U}$ is the root-mean square of any velocity component since, as shown by Taylor (1935), if the turbulence is isotropic λ_{ii} is the same for any i and equal to $\sqrt{2} \lambda_{ij}$ (with $i \neq j$). For simplicity, the isotropic definition is used to compute the Taylor micro-scale in fig. 4.21, specifically with $\mathcal{U} = \sqrt{\langle u_1^2 \rangle}$. Even though, as shown in fig. 4.25, the different scales given by eq. (4.12) are not isotropic (since $\lambda_{11} \neq \lambda_{22} \neq \lambda_{33}$ and $\lambda_{12} \approx \lambda_{13} \neq \lambda_{21} \approx \lambda_{23} \neq \lambda_{31} \approx \lambda_{32}$) all combinations of i and j used in eq. (4.12) consistently led to a profile of $C_\varepsilon Re_\lambda$ that remained constant in the range $3 \lesssim x_1/d < 10$ within (at worse) 5% of the average value in that range which is much smaller than the individual variation of C_ε . Figure 4.26 shows the ratios between the integral length scale and three estimates of λ , other than an offset between the profiles (consequence of difference values of each estimate) their evolution with x_1/d appears identical. As will be seen in section 4.3.1 axisymmetry appears to better represent the small scales than isotropy.

The velocity scale $\mathcal{U}$ used to compute Re_λ and C_ε was $\sqrt{\langle u_1^2 \rangle}$, for the sake of consistency with the fact that $\mathcal{L}$ was computed from the integral time scale of u_1 . Nevertheless, by taking $\mathcal{U}$ as any other velocity component ($\sqrt{\langle u_2^2 \rangle}$, $\sqrt{\langle u_3^2 \rangle}$, $\sqrt{\langle u_1'^2 \rangle}$, $\sqrt{\langle u_2'^2 \rangle}$ or $\sqrt{\langle u_3'^2 \rangle}$) the profiles of $C_\varepsilon Re_\lambda$ along x_1/d display the same plateau in the range $3 \lesssim x_1/d < 10$. In fact, the profiles of $C_\varepsilon Re_\lambda$ with $\mathcal{U}$ computed from any velocity component other than $\sqrt{\langle u_1^2 \rangle}$ varied, each, by less than 5% of their mean values in the range $3 \lesssim x_1/d < 10$.

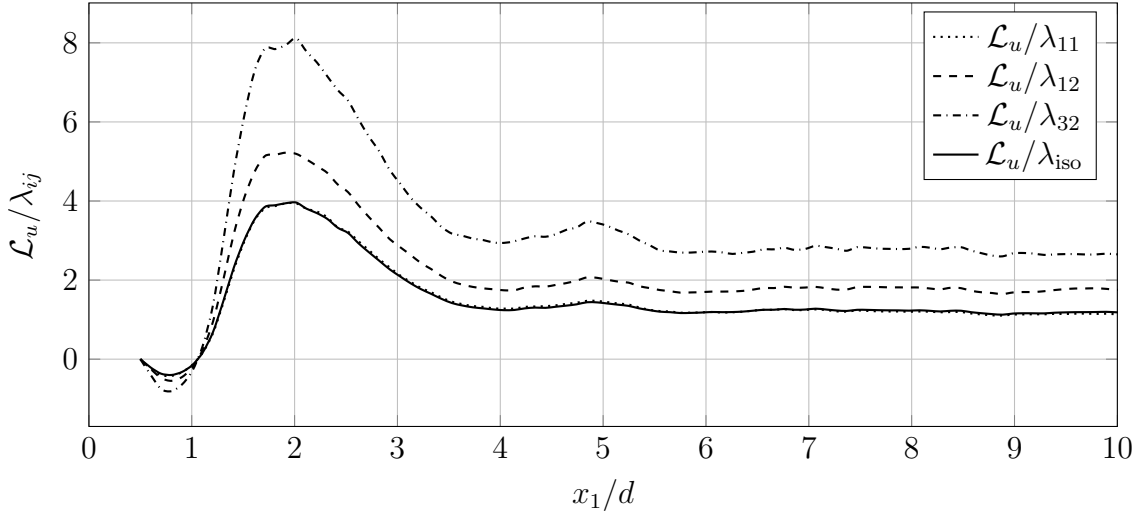


Figure 4.26: Ratio between integral length scale and some of the estimates of the Taylor micro-scale given by eq. (4.12), $\lambda_{\text{iso}} = \sqrt{\langle u_1^2 \rangle} \sqrt{15 \frac{\nu}{\varepsilon}}$.

Of the integral time scales shown in fig. 4.23, Θ_{u_1} was used to compute C_ε in fig. 4.21 as explained above. Notice that Θ_{u_1} can also be used to compute $\#_{\text{e.t.}}$ the number of eddy turnover times between two locations x_a and x_b , given by

$$\#_{\text{e.t.}} = \int_{x_a}^{x_b} U_1^{-1} \frac{\mathcal{U}}{\mathcal{L}} dx, \quad (4.13)$$

over which eq. (1.13) is observed as well as the number of eddy turnover times required for eq. (1.13) to establish itself. In eq. (4.13), the velocity and length scales of the large eddies were taken as $\sqrt{\langle u_1^2 \rangle}$ and $U_1 \Theta_{u_1}$, respectively. This results in $\#_{\text{e.t.}} \approx 1.8$ for $x_a/d = 2$ (where there is a small range of r/d where $\Pi^a \approx \text{const}$) and $x_b/d = 3.5$ (after which $C_\varepsilon \propto Re_\lambda^{-1}$) and $\#_{\text{e.t.}} \approx 8.8$ for $x_a/d = 3.5$ and the end of the domain investigated here, i.e. $x_b/d = 10$. This inspires confidence in the relevance of eq. (1.13), which, as seen in fig. 4.22 we were able to capture over a relatively small region in space that actually represents a large number of eddy turnover times.

4.3 Scale-by-scale Energy Budget - Revisited

When discussing the turbulence cascade in chapter 3 it was shown that Π^a , the rate at which energy is transferred between scales when averaged over orientations on the plane of the mean-flow (plane normal to x_3), was roughly constant over a wide range of scales, much like predicted by K41. However, the energy distribution over that same range of scales did not follow K41. Given that the dissipation of the stochastic component of the

velocity field seems in some way coupled or related to the dissipation of the coherent part of the velocity field, does the constancy of Π^a result from the stochastic or coherent components, or both?

Using eq. (4.1), one can separate the second order structure function into a stochastic and phase component, $\delta q'^2$ and $\delta \tilde{q}^2$ respectively. This distinction warrants eq. (3.1) to be re-derived in light of eqs. (4.1) and (4.3) as done at the end of Appendix A (but see also Thiesset, Danaila, and Antonia, 2014). The equations for $\langle \delta q'^2 \rangle$ and $\langle \delta \tilde{q}^2 \rangle$ are, respectively,

$$\begin{aligned} \left\langle \frac{\partial \delta q'^2}{\partial t} \right\rangle + \left\langle \frac{U_i^+ + U_i^-}{2} \frac{\partial \delta q'^2}{\partial X_i} \right\rangle + \left\langle \delta U_i \frac{\partial \delta q'^2}{\partial r_i} \right\rangle + \left\langle \delta \tilde{u}_i \frac{\partial \delta q'^2}{\partial r_i} \right\rangle + \left\langle \delta u'_i \frac{\partial \delta q'^2}{\partial r_i} \right\rangle = \\ - \left\langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \delta q'^2}{\partial X_i} \right\rangle - \left\langle \frac{u'_i{}^+ + u'_i{}^-}{2} \frac{\partial \delta q'^2}{\partial X_i} \right\rangle - \left\langle 2\delta u'_i \delta \frac{\partial p'}{\partial X_i} \right\rangle - \left\langle \delta u'_i (u'_j{}^+ + u'_j{}^-) \frac{\partial \delta U_i}{\partial X_j} \right\rangle - \\ \left\langle 2\delta u'_i \delta u'_j \frac{\partial \delta U_i}{\partial r_j} \right\rangle - \left\langle \delta u'_i (u'_j{}^+ + u'_j{}^-) \frac{\partial \delta \tilde{u}_i}{\partial X_j} \right\rangle - \left\langle 2\delta u'_i \delta u'_j \frac{\partial \delta \tilde{u}_i}{\partial r_j} \right\rangle + \\ \left\langle \nu \frac{1}{2} \frac{\partial^2 \delta q'^2}{\partial X_j \partial X_j} \right\rangle + \left\langle 2\nu \frac{\partial^2 \delta q'^2}{\partial r_j \partial r_j} \right\rangle - 4\nu \left(\left\langle \frac{\partial \delta u'_j}{\partial X_i} \frac{\partial \delta u'_j}{\partial X_i} \right\rangle + \frac{1}{4} \left\langle \frac{\partial \delta u'_j}{\partial r_i} \frac{\partial \delta u'_j}{\partial r_i} \right\rangle \right) \quad (4.14) \end{aligned}$$

and

$$\begin{aligned} \left\langle \frac{\partial \delta \tilde{q}^2}{\partial t} \right\rangle + \left\langle \frac{U_i^+ + U_i^-}{2} \frac{\partial \delta \tilde{q}^2}{\partial X_i} \right\rangle + \left\langle \delta U_i \frac{\partial \delta \tilde{q}^2}{\partial r_i} \right\rangle + \left\langle \delta \tilde{u}_i \frac{\partial \delta \tilde{q}^2}{\partial r_i} \right\rangle + \left\langle 2 \frac{\partial}{\partial r_i} (\delta u'_i \delta u'_j \delta \tilde{u}_j) \right\rangle = \\ - \left\langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \delta \tilde{q}^2}{\partial X_i} \right\rangle - \left\langle \frac{\partial}{\partial X_i} [(u'_i{}^+ + u'_i{}^-) \delta u'_j \delta \tilde{u}_j] \right\rangle - \left\langle 2\delta \tilde{u}_i \delta \frac{\partial \tilde{p}}{\partial X_i} \right\rangle - \left\langle \delta \tilde{u}_i (\tilde{u}_j{}^+ + \tilde{u}_j{}^-) \frac{\partial \delta U_i}{\partial X_j} \right\rangle - \\ \left\langle 2\delta \tilde{u}_i \delta \tilde{u}_j \frac{\partial \delta U_i}{\partial r_j} \right\rangle + \left\langle \delta u'_i (u'_j{}^+ + u'_j{}^-) \frac{\partial \delta \tilde{u}_i}{\partial X_j} \right\rangle + \left\langle 2\delta u'_i \delta u'_j \frac{\partial \delta \tilde{u}_i}{\partial r_j} \right\rangle \\ \left\langle \nu \frac{1}{2} \frac{\partial^2 \delta \tilde{q}^2}{\partial X_j \partial X_j} \right\rangle + \left\langle 2\nu \frac{\partial^2 \delta \tilde{q}^2}{\partial r_j \partial r_j} \right\rangle - 4\nu \left(\left\langle \frac{\partial \delta \tilde{u}_j}{\partial X_i} \frac{\partial \delta \tilde{u}_j}{\partial X_i} \right\rangle + \frac{1}{4} \left\langle \frac{\partial \delta \tilde{u}_j}{\partial r_i} \frac{\partial \delta \tilde{u}_j}{\partial r_i} \right\rangle \right). \quad (4.15) \end{aligned}$$

Evidently both eq. (4.14) and eq. (4.15) are rather similar to eq. (3.1) and therefore we make use of the same notation as in eq. (3.2) to identify the individual terms:

$$\mathcal{A}'_t + \mathcal{A}' + \Pi'_U + \Pi'_{\tilde{u}} + \Pi' = \mathcal{T}'_{\tilde{u}} + \mathcal{T}'_{u'} + \mathcal{T}'_{p'} + \mathcal{P}'_U + \mathcal{P}'_{\tilde{u}} + \mathcal{D}'_x + \mathcal{D}'_r - \varepsilon'_r \quad (4.16)$$

$$\tilde{\mathcal{A}}_t + \tilde{\mathcal{A}} + \tilde{\Pi}_U + \tilde{\Pi}_{\tilde{u}} + \tilde{\Pi}_{\mathcal{P}_{\tilde{u}}} = \tilde{\mathcal{T}}_{\tilde{u}} + \tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}} + \tilde{\mathcal{T}}_{\tilde{p}} + \tilde{\mathcal{P}}_U - \mathcal{P}'_{\tilde{u}} + \tilde{\mathcal{D}}_x + \tilde{\mathcal{D}}_r - \tilde{\varepsilon}_r \quad (4.17)$$

Notice that adding eq. (4.16) to eq. (4.17) results in eq. (3.2) by combining terms with apostrophe and tilde together and noticing that

$$\Pi = \Pi' + \Pi'_{\tilde{u}} + \tilde{\Pi}_{\tilde{u}} + \tilde{\Pi}_{\mathcal{P}_{\tilde{u}}} \quad (4.18)$$

and

$$\mathcal{T}_u = \mathcal{T}'_{u'} + \mathcal{T}'_{\tilde{u}} + \tilde{\mathcal{T}}_{\tilde{u}} + \tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}} \quad (4.19)$$

which are non-linear inter-scale and inter-space transfer terms in eq. (3.2).

Π' , $\Pi'_{\tilde{u}}$ and $\tilde{\Pi}_{\tilde{u}}$ can be interpreted as inter-scale transfer terms of either $\delta q'^2$ or $\delta \tilde{q}^2$. Π' represents the transfer of energy associated with the stochastic motions in scale space by the stochastic motions (i.e. transfer of $\delta q'^2$ in r_i by $\delta u'_i$), similarly, $\Pi'_{\tilde{u}}$ represents the transfer of the energy associated with the stochastic motions in scale space by the coherent motion (i.e. transfer of $\delta q'^2$ in r_i by $\delta \tilde{u}_i$). $\tilde{\Pi}_{\tilde{u}}$ represents the inter-scale transfer of the energy associated with the coherent motion by the coherent motion (i.e. transfer of $\delta \tilde{q}^2$ in r_i by $\delta \tilde{u}_i$). The term $\tilde{\Pi}_{\mathcal{P}_{\tilde{u}}}$ corresponds to the difference between the inter-scale transfer of the total fluctuating energy due to differences in stochastic velocity, $\langle \delta u'_i \frac{\partial(\delta \mathbf{u}' + \delta \tilde{\mathbf{u}})^2}{\partial r_i} \rangle$, and that of the energy associated with the stochastic motion $\langle \delta u'_i \frac{\partial \delta q'^2}{\partial r_i} \rangle$. Equivalently, $\mathcal{T}'_{u'}$, $\mathcal{T}'_{\tilde{u}}$, $\tilde{\mathcal{T}}_{\tilde{u}}$ represent transport in physical space, and $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ corresponds to the difference between the transport of the total fluctuating energy due to the two-point-average stochastic velocity, $\langle \frac{u'^+ + u'^-}{2} \frac{\partial(\delta \mathbf{u}' + \delta \tilde{\mathbf{u}})^2}{\partial X_i} \rangle$, and that of the energy associated with the stochastic motion $\langle \frac{u'^+ + u'^-}{2} \frac{\partial \delta q'^2}{\partial X_i} \rangle$. Thus, combining $\tilde{\Pi}_{\mathcal{P}_{\tilde{u}}}$ with Π' results in the net inter-scale transfer of the total fluctuating energy by the stochastic motion (i.e. transfer of δq^2 in $\mathbf{r}$ by $\delta \mathbf{u}'$) which we call $\Pi_{u'}$ such that eq. (4.18) can be written as

$$\Pi = \Pi_{u'} + \Pi'_{\tilde{u}} + \tilde{\Pi}_{\tilde{u}} \quad (4.20)$$

and equivalently, with regard to the transport in physical space, $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ and $\mathcal{T}'_{u'}$ are combined into $\mathcal{T}_{u'}$.

4.3.1 Anisotropy at the Small Scales

Large scale anisotropy can be expected in the planar wake (notice the rather different profiles of $\sqrt{\langle u_i^2 \rangle}$ for $i = 1, 2, 3$ shown in chapter 2) given the presence of the large coherent structures as already discussed. The scenario at smaller scales, however, may be different as such scales are usually believed to be isotropic (Taylor, 1935) as they may remain unaffected by large scale anisotropy. There is and has been considerable interest in verifying this assumption (see e.g. Biferale and Vergassola, 2001; Thiesset, Danaila, and Antonia, 2013).

As discussed in section 4.3, the triple decomposition allows us to distinguish between phase- and stochastic-component second order structure functions, $\langle \delta \tilde{q}^2 \rangle$ and $\langle \delta q'^2 \rangle$, respectively. Figures 4.27 and 4.28 show the distribution of these two quantities in scale-

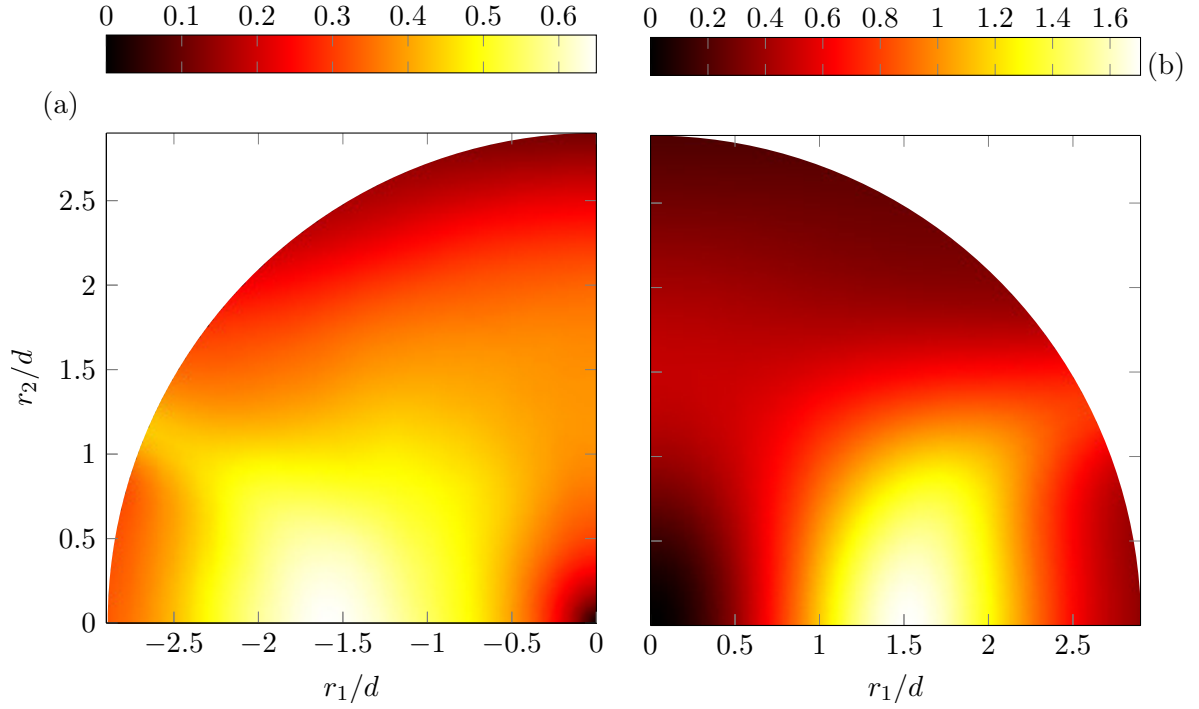


Figure 4.27: Distribution of $\langle \delta q'^2 \rangle$ and $\langle \delta \tilde{q}^2 \rangle$ normalised by U_∞^2 in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

space at the same locations investigated in chapter 3 ($X_1/d = 2$ and $X_1/d = 8$) where $\langle \delta q^2 \rangle$ was found to be anisotropically distributed in scale space.

Due to eq. (4.3) one has that $\langle \delta q^2 \rangle = \langle \delta \tilde{q}^2 \rangle + \langle \delta q'^2 \rangle$ and therefore combining figs. 4.27a and 4.27b results in fig. 3.9a (equivalently combining figs. 4.28a and 4.28b results in fig. 3.9b). The immediate observation one can make is that the anisotropy identified in figs. 3.9a and 3.9b is not uniquely due to the contribution of the phase component ($\langle \delta \tilde{q}^2 \rangle$) to the second order structure function as $\langle \delta q'^2 \rangle$ is also anisotropically distributed in scale space at both $X_1/d = 2$ and even at $X_1/d = 8$ if we consider scales larger than $\sim d$. Figures 4.27a, 4.27b, 4.28a and 4.28b also show how $\langle \delta q'^2 \rangle$ is much more evenly distributed in scale space, contrary to $\langle \delta \tilde{q}^2 \rangle$ which concentrates around a specific separations on the r_1 axis, related to the distance between the coherent structures (Thiesset, Danaïla, and Antonia, 2014).

As the distance to the prism increases figs. 4.27a and 4.28a clearly shows that there is a (progressively increasing) range of scales over which $\langle \delta q'^2 \rangle$ becomes independent of orientation, i.e. the anisotropy in the stochastic second order structure function is moved towards the larger scales. This is made clearer in fig. 4.29 where the range of r over which these quantities are plotted is reduced. It can be seen that at $X_1/d = 2$ $\langle \delta q'^2 \rangle$ becomes independent of the orientation of $\mathbf{r}$ for $r/d \lesssim 0.5$ while at $X_1/d = 8$ this increases to $r \lesssim$

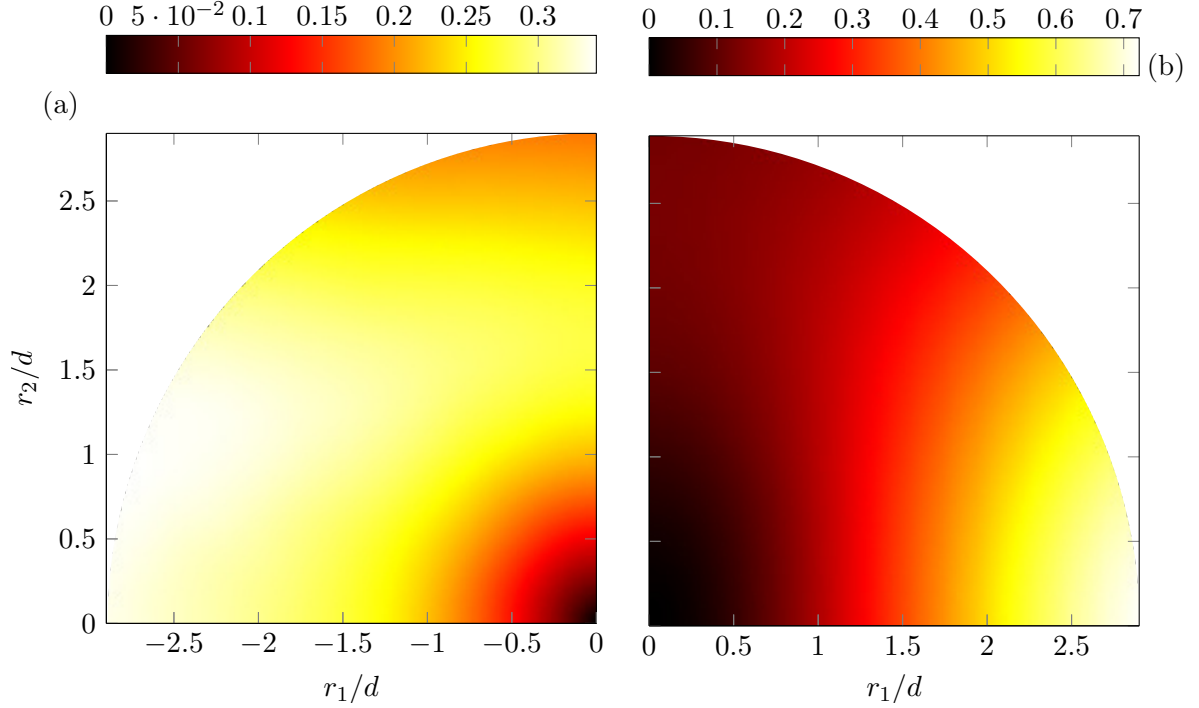


Figure 4.28: Distribution of $\langle \delta q'^2 \rangle$ and $\langle \delta \tilde{q}^2 \rangle$ normalised by U_∞^2 in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

1.5. Conversely, $\langle \delta \tilde{q}^2 \rangle$ remains stratified and very little energy content (from the coherent motion) is found at smaller length scales.

While Thiesset, Danaila, and Antonia (2013) and Thiesset, Danaila, and Antonia (2014) also report an increasing degree of anisotropy with length scale, they do not have access to orientations of $\mathbf{r}$ which are not parallel to r_1 . Here, we somehow reinforce their claim of small scale isotropy by showing that isotropy, in the sense of independence on the orientation of $\mathbf{r}$, of the stochastic second order structure function is present at small scales. As will be seen below, the term $\mathcal{P}'_{\tilde{u}}$ plays a major role in the scale-by-scale energy budget at the separations where $\langle \delta q'^2 \rangle$ becomes anisotropic[§].

Nevertheless, even though the present results appear to suggest the same as those of Thiesset, Danaila, and Antonia (2013) and Thiesset, Danaila, and Antonia (2014), it is important to distinguish what is meant by isotropy in both cases. Since Thiesset, Danaila, and Antonia (2013) and Thiesset, Danaila, and Antonia (2014) did not have access to separations with $r_2 \neq 0$, they could not verify independence of $\langle \delta q'^2 \rangle$ on the orientation of $\mathbf{r}$ and instead used the relation

[§] Thiesset, Danaila, and Antonia (2013) and Thiesset, Danaila, and Antonia (2014) propose that the isotropy is lost at the largest scales due to the presence of the coherent strain which actually contributes to $\mathcal{P}_{\tilde{u}}$.

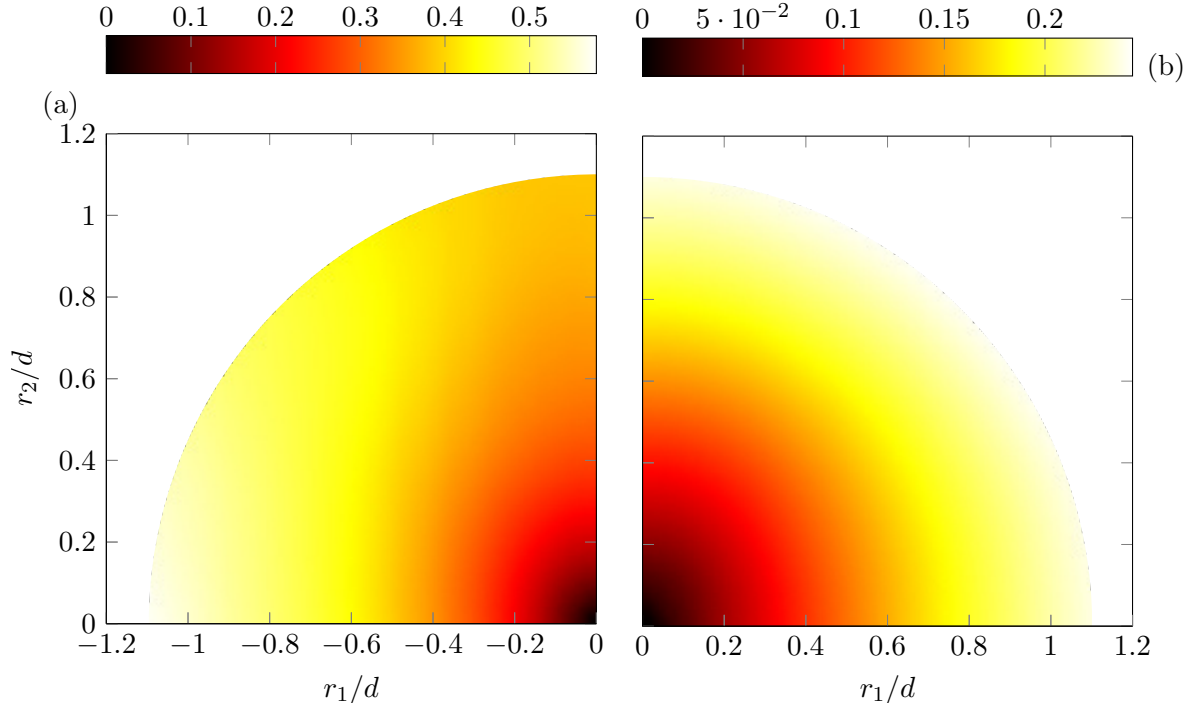


Figure 4.29: As figs. 4.27a and 4.28a, in (a) and (b), respectively, for separations $r < 1.1$.

$$\langle \delta u_{\perp}^{\prime 2} \rangle = \langle \delta u_i^{\prime 2} \rangle + \frac{1}{2} r_i \frac{\partial \langle \delta u_i^{\prime 2} \rangle}{\partial r_i}, \quad (4.21)$$

(with no implicit summation over repeated indices) where the symbol $\perp$ indicates any direction normal to r_i ; in Thiesset, Danaila, and Antonia (2013) and Thiesset, Danaila, and Antonia (2014) i is taken as 1. Thus, eq. (4.21) should hold for both $\langle \delta u_{\perp}^{\prime 2} \rangle = \langle \delta u_2^{\prime 2} \rangle$ and $\langle \delta u_{\perp}^{\prime 2} \rangle = \langle \delta u_3^{\prime 2} \rangle$ in the case of isotropic turbulence.

The variations of $\langle \delta \tilde{u}_i^2 \rangle$ and $\langle \delta u_i^{\prime 2} \rangle$ (for $i = 1, 2$) with r_1 are shown in fig. 4.30. The decomposition highlights the fact that the slopes of $\langle \delta u_1^{\prime 2} \rangle$ and $\langle \delta u_2^{\prime 2} \rangle$, observed in fig. 3.5, strongly exceed the $r_1^{2/3}$ power law due to the coherent motion. In fact, $\langle \delta \tilde{u}_1^2 \rangle$ and $\langle \delta \tilde{u}_2^2 \rangle$ deviate only slightly from an r_1^2 power law (the deviation is larger for $\langle \delta \tilde{u}_1^2 \rangle$), characteristic of smooth Taylor expandable fields. This is also consistent with the model proposed in Thiesset, Danaila, and Antonia (2014) where $\langle \delta \tilde{u}_2^2 \rangle \propto \sin^2(r_1 \frac{\pi}{\mathcal{L}_v})$ [¶].

After the coherent component is removed, $\langle \delta u_1^{\prime 2} \rangle$ displays a power law close to $r_1^{2/3}$ (even if only for a very small range of r/d) while $\langle \delta u_2^{\prime 2} \rangle$ displays a power law with exponent slightly larger than $2/3$. Since a $-5/3$ slope is observed in the spectra, deviations of the structure functions from $r_1^{2/3}$ may be due to two factors, inhomogeneity in space (e.g. decrease of the peak values of correlation coefficient) and the effect of the low frequency peak

[¶] In their model, Thiesset, Danaila, and Antonia (2014) define $\mathcal{L}_v$ as the distance between consecutive vortices. Notice that the first non-zero term in the Taylor expansion of $\sin^2(r_1 \frac{\pi}{\mathcal{L}_v})$ is $(r_1 \pi / \mathcal{L}_v)^2$.

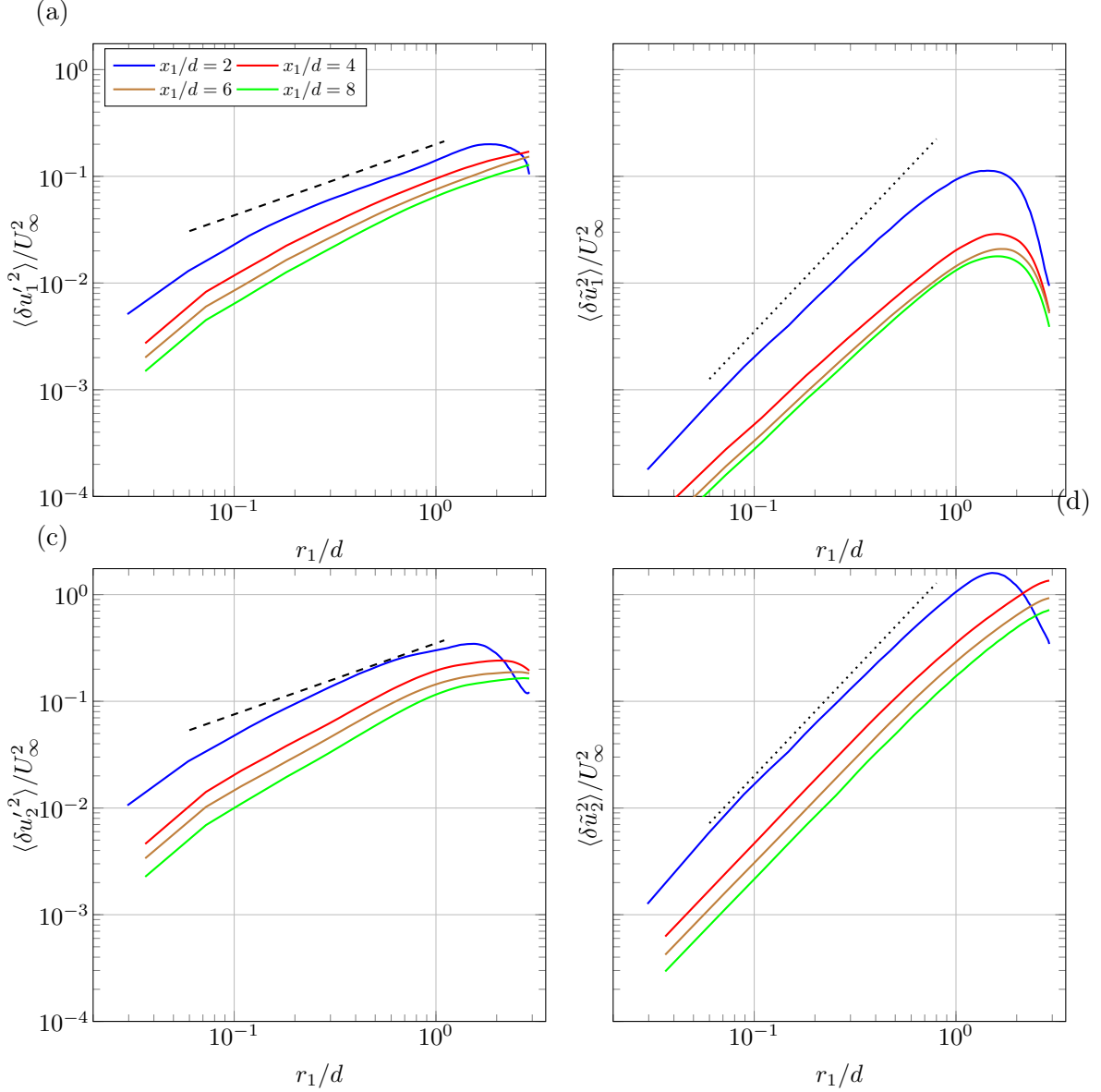


Figure 4.30: Profiles of the second order structure functions $\langle \delta u_i^2 \rangle$ along r_1/d at four locations on the centreline. The dashed lines indicate a power law of r_1^2 and the dotted lines indicate a power law of $r_1^{2/3}$.

observed in figs. 3.2a and 3.3a which, for the cross-stream component, appears at a higher frequency than for the streamwise component which would explain why the departure from $r_1^{2/3}$ is largest for the former than for the latter component.

More importantly, however, the isotropy defined by eq. (4.21) cannot possibly hold for relatively large values of r_1 given that, as seen in fig. 3.5, $\langle \delta u_3^2 \rangle$ does not display any power law for $r_1/d \gtrsim 0.1$ and eq. (4.21) effectively implies that one should have $\langle \delta u_2'^2 \rangle = \langle \delta u_3'^2 \rangle$ (recall that $\langle \delta u_3'^2 \rangle \equiv \langle \delta u_3^2 \rangle$). This is highlighted in fig. 4.31 where eq. (4.21) was computed both along r_1 and r_2 . Thus, it appears as if the different terms $\langle \delta u_i'^2 \rangle$, which

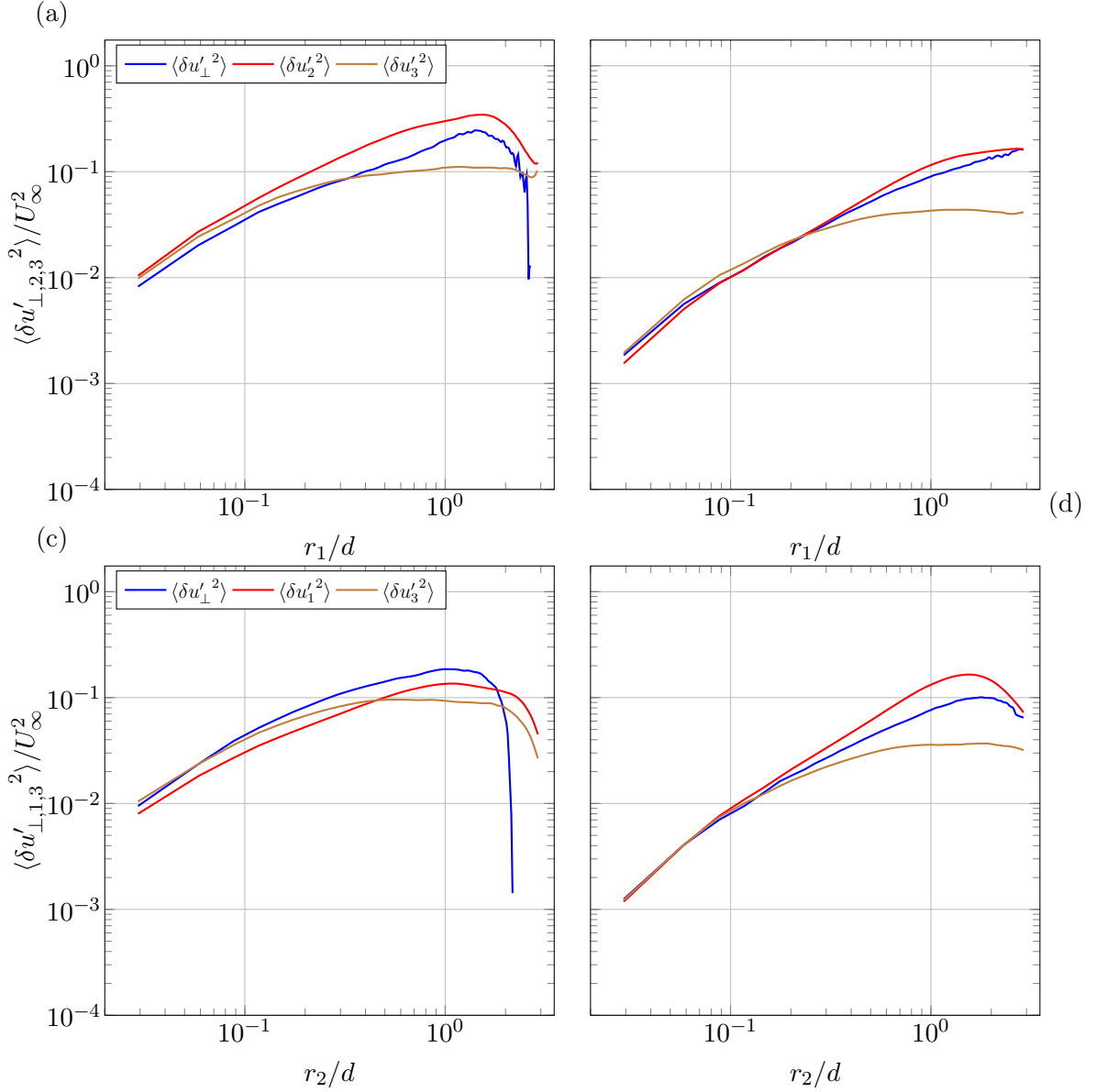


Figure 4.31: Assessment of eq. (4.21) for $i = 1$ (top) and $i = 2$ (bottom) at $X_1/d = 2$ (left) and $X_1/d = 8$ (right).

contribute to $\langle \delta q'^2 \rangle$, do not satisfy eq. (4.21) but somehow conspire in such a way that the total energy $\langle \delta q'^2 \rangle$ appears to be isotropically distributed among scales.

These observations beg the question if in fact, the smallest (dissipative) scales are indeed isotropic as often believed/assumed for high Reynolds number fully developed turbulence. While this assumption is often required in experiments due to the difficulty in accessing all components of the velocity gradient tensor $\langle \frac{\partial u_i}{\partial x_j} \rangle$, there is a wealth of evidence suggesting that small scales may not always be isotropic (see e.g. Browne et al., 1987; Champagne, 1978; Elsinga and Marusic, 2016; George and Hussein, 1991; Kraichnan, 1974; Mi and Antonia, 2010; Ninmi et al., 1999; Schenck and Jovanović, 2011; Shen and Warhaft, 2000; Warhaft, 2000, and references therein). For the case of planar wakes, there is some evi-

dence that small scales are axisymmetric rather than isotropic, as reported by Browne et al. (1987), George and Hussein (1991), Mi and Antonia (2010), and Schenck and Jovanović (2011).

In order to quantify the degree of small-scale isotropy, the parameters

$$\begin{aligned} K_1^i &= 2 \frac{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_2}{\partial x_1} \right)^2 \rangle}, & K_2^i &= 2 \frac{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_3}{\partial x_1} \right)^2 \rangle}, & K_3^i &= 2 \frac{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_1}{\partial x_2} \right)^2 \rangle}, \\ K_4^i &= 2 \frac{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_1}{\partial x_3} \right)^2 \rangle}, & K_5^i &= \frac{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_2}{\partial x_2} \right)^2 \rangle}, & K_6^i &= \frac{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_3}{\partial x_3} \right)^2 \rangle} \end{aligned} \quad (4.22)$$

introduced by Taylor (1935) must be equal to unity. Alternatively, small-scale axisymmetry can be assessed in the same way through

$$\begin{aligned} K_1^a &= \frac{\langle \left(\frac{\partial u_1}{\partial x_2} \right)^2 \rangle}{\langle \left(\frac{\partial u_1}{\partial x_3} \right)^2 \rangle}, & K_2^a &= \frac{\langle \left(\frac{\partial u_2}{\partial x_1} \right)^2 \rangle}{\langle \left(\frac{\partial u_3}{\partial x_1} \right)^2 \rangle}, & K_3^a &= \frac{\langle \left(\frac{\partial u_2}{\partial x_2} \right)^2 \rangle}{\langle \left(\frac{\partial u_3}{\partial x_3} \right)^2 \rangle}, \\ K_4^a &= \frac{\langle \left(\frac{\partial u_2}{\partial x_3} \right)^2 \rangle}{\langle \left(\frac{\partial u_3}{\partial x_2} \right)^2 \rangle}, & K_5^a &= 3 \frac{\langle \left(\frac{\partial u_2}{\partial x_2} \right)^2 \rangle}{\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \rangle + \langle \left(\frac{\partial u_2}{\partial x_3} \right)^2 \rangle} \end{aligned} \quad (4.23)$$

introduced by George and Hussein (1991). The small scales are said to be axisymmetric if the ratios in eq. (4.23) are all equal to 1. As noted by Laizet, Nedić, et al. (2015), the ratios eqs. (4.22) and (4.23) are very sensitive to grid resolution, which in the present case is always better than 4 times the Kolmogorov scale, similar to Laizet, Nedić, et al. (2015).

While Mi and Antonia (2010) and Schenck and Jovanović (2011) have examined eqs. (4.22) and (4.23) in planar wakes, they did not make use of a triple decomposition. Even though the contribution of the coherent motion to the dissipation, associated with gradients of $\mathbf{u}$, appears to be rather small (in comparison to that of the stochastic motion) as seen in fig. 4.20, it seems pertinent to ask whether the presence of the coherent motion influences the approach (or departure) from either small-scale isotropy or axisymmetry.

Figure 4.32 shows a comparison between eq. (4.22) and eq. (4.23) computed from the total fluctuating velocity (i.e. $\tilde{u}_i + u'_i$) along the centreline. Notice that even though the values obtained for eq. (4.22) are reasonably close to unity (for example, in Laizet, Nedić, et al., 2015, the bounds of ± 0.2 are used to justify small-scale isotropy), the values obtained for eq. (4.23) are much closer to one. While Mi and Antonia (2010) deem that their planar wake generated by a circular cylinder becomes axisymmetric only after $x_1/d = 40$,

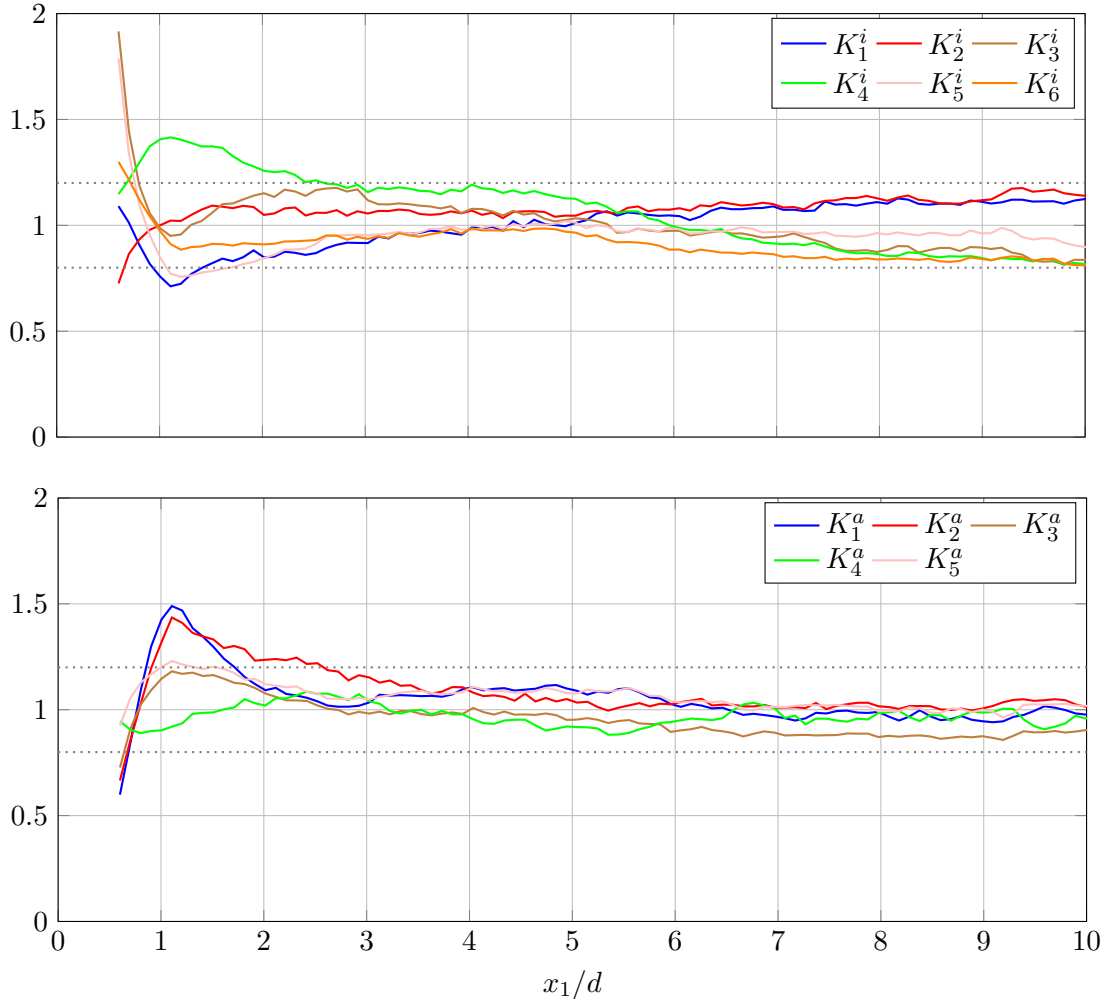


Figure 4.32: Small-scale isotropy (top) and axisymmetry (bottom) of the total fluctuating velocity. The different indicators are computed from eqs. (4.22) and (4.23).

their results have shown that in general values of the ratios in eq. (4.23) lie much closer to unity than the ratios in eq. (4.22), even at their closest measurement station.

We now turn our attention to fig. 4.33 where the ratios in eqs. (4.22) and (4.23) are computed for the stochastic component of the velocity (i.e. the gradients in eqs. (4.22) and (4.23) are now gradients of u'_i instead of u_i). Figure 4.33 suggests that the departure from isotropy is even more significant once the coherent component of the velocity is removed as the ratios given by eq. (4.22) appear to be further from 1 than in Figure 4.32. Conversely, the axisymmetry indicators given by eq. (4.23) remain within the bounds of 1 ± 0.2 . Thus, we can conclude that the coherent motion has practically no contribution to the observation of small-scale axisymmetry.

The fact that the individual second order structure functions $\langle \delta u_i'^2 \rangle$ do not satisfy eq. (4.21) thus appears to be due to the fact that neither the small scales (as shown in figs. 4.32 and 4.33) nor the large scales (recall fig. 4.24) are isotropic. It remains curious,

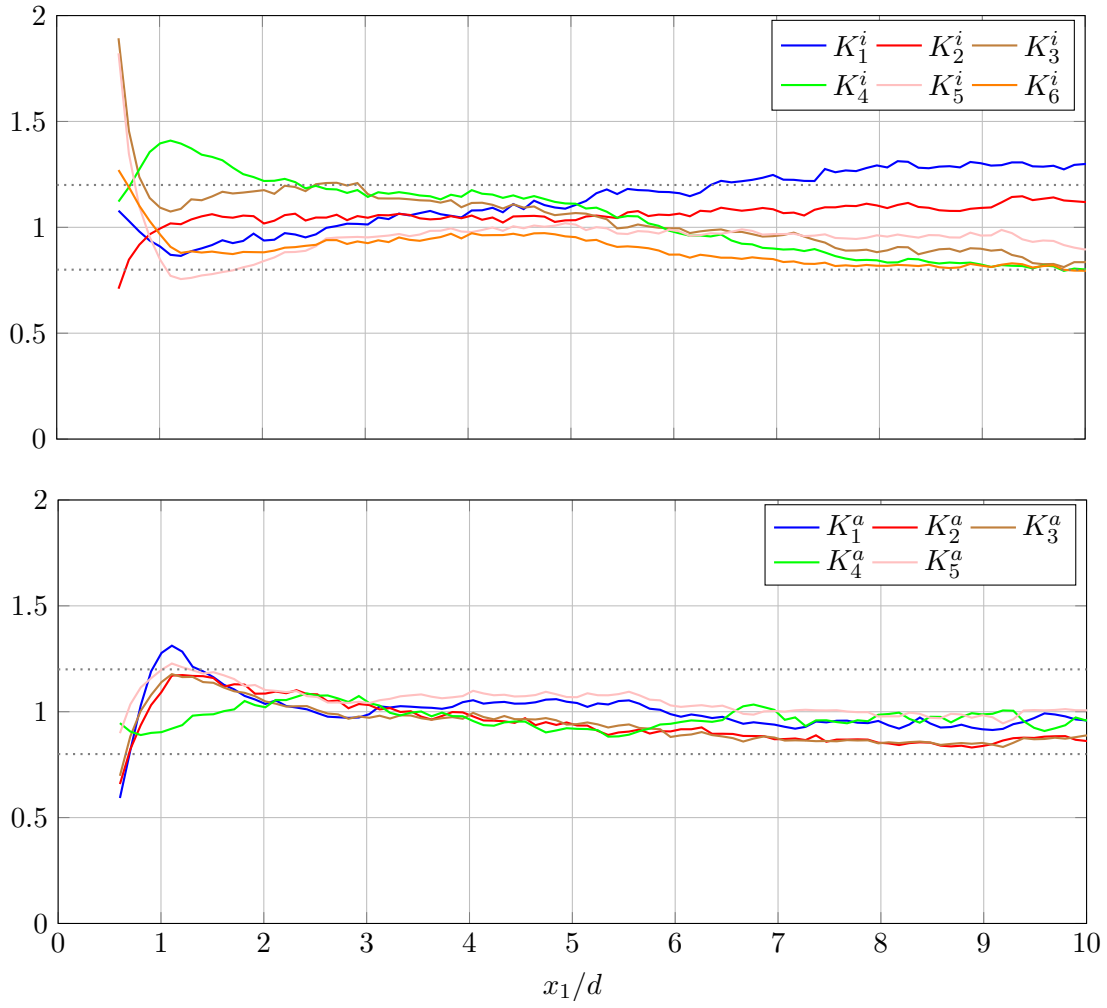


Figure 4.33: Small-scale isotropy (top) and axisymmetry (bottom) of the stochastic component of the fluctuating velocity. The different indicators are computed from eqs. (4.22) and (4.23).

however, that $\langle \delta q'^2 \rangle$ appears to be isotropically distributed in scale space even though $\langle \delta q'^2 \rangle$ results from adding the individual structure functions $\langle \delta u_i'^2 \rangle$.

4.3.2 Effect of the Coherent motion on the Inter-scale Energy Transfer and Scale-space Fluxes

In chapter 3 it was shown how the average non-linear inter-scale transfer of δq^2 is roughly constant when the orientations of $\mathbf{r}$ are averaged out in the $r_3 = 0$ plane, despite its distribution in scale space being far from so. In fact, this was observed even at $x_1/d = 2$ (albeit for a very small range of separations) where the coherent motion contributes a large portion of the total fluctuating kinetic energy (recall fig. 4.18). Furthermore, we have observed in section 4.2, that the non-equilibrium dissipation law and the related proportionality between $\tilde{\varepsilon}$ and ε' appear to be triggered just after $x_1/d = 2$. In this subsection we determine how the behaviour observed in chapter 3 depends on contributions arising from the coherent and stochastic motions individually, and we discuss how it can be consistent with $\tilde{\varepsilon} \propto \varepsilon'$ and the non-equilibrium dissipation scaling.

4.3.2.1 Constant Non-linear Inter-scale Transfer as a Combined Effect

By introducing the triple decomposition, the total non-linear inter-scale transfer (examined in chapter 3) is decomposed into three terms given by eq. (4.20). They are $\Pi_{u'}$, the non-linear inter-scale transfer of δq^2 by the stochastic motions, $\Pi'_{\tilde{u}}$ the non-linear inter-scale transfer of $\langle \delta q'^2 \rangle$ by the coherent motion and $\tilde{\Pi}_{\tilde{u}}$ the non-linear inter-scale transfer of $\langle \delta \tilde{q}^2 \rangle$ by the coherent motion.

These terms are shown in fig. 4.34, after averaging out the orientations of $\mathbf{r}$ in the plane of the wake (see fig. 3.14). One immediately observes that $\tilde{\Pi}_{\tilde{u}}^a \approx 0$ over the same range of r/d for which Π^a/ε_r^a (also shown in fig. 4.34) appears to be constant. This means that the observation that Π^a/ε_r^a appears constant must be due to terms $\Pi_{u'}$ and $\Pi'_{\tilde{u}}$, i.e. eq. (4.20) reduces to

$$\Pi^a \approx \Pi_{u'}^a + \Pi'_{\tilde{u}}^a \quad (4.24)$$

in the range where Π^a is approximately constant and close to ε^a , i.e. $0.1 < r/d < 0.3$ at $X_1/d = 2$ and $0.2 < r/d < 1$ at $X_1/d = 8$. It is clear that $\Pi^a \approx \text{const}$ is not just due to the stochastic motions, as Π^a is not just equal to Π'^a which results only from interactions of the stochastic motions (recall eq. (4.18)). This means that the coherent structures play a significant role in the near-constancy of Π^a . This role is dual: on the one hand, the stochastic fluctuations transfer, on average, both stochastic and coherent fluctuating energy from large to small scales (by means of $\Pi_{u'}^a$); on the other hand, the coherent fluctuations transfer, on average, stochastic energy from small to large scales (by means of $\Pi'_{\tilde{u}}$). This latter contribution is the smallest of the two appearing in eq. (4.24), but it does have a noticeable correcting action.

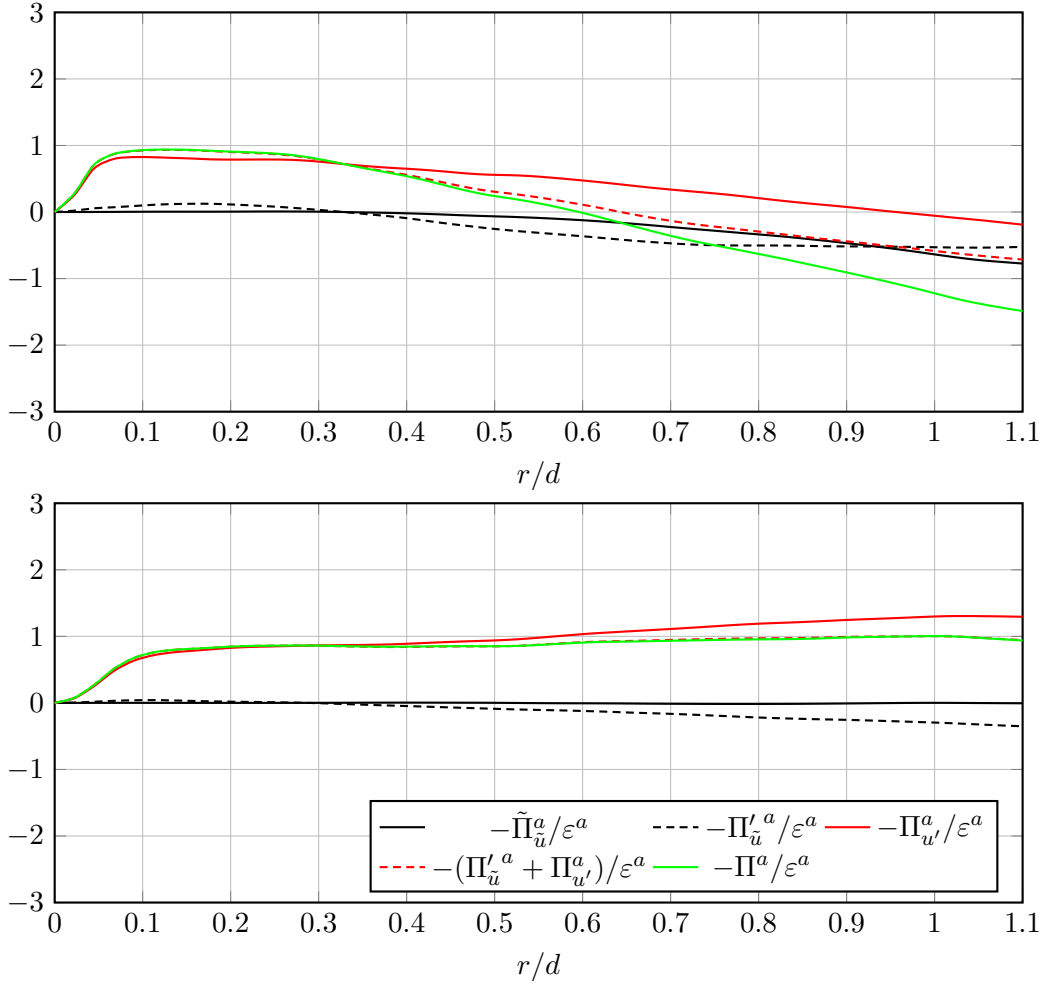


Figure 4.34: Orientation averaged non-linear inter-scale transfer terms from eqs. (4.14) and (4.15) normalised by ε_r^a at $X_1/d = 2$ (top) and $X_1/d = 8$ (bottom).

One can also see from fig. 4.34 that the non-linear inter-scale transfer of $\delta q'^2$ due to the coherent motions (Π_u') appears to transfer energy from small to large separations at scales $r/d \gtrsim 0.3$. This is confirmed by observing the positive sign of $\langle \delta \tilde{u}_r \delta q'^2 \rangle^a$ in fig. 4.35. This means that the only contribution to the direct non-linear cascade arises from the stochastic motions through $\langle \delta u'_r \delta q^2 \rangle^a$ which, as shown in fig. 4.35, remains negative for all separations investigated here. Notice that effectively, it is mainly $\Pi_{u'}$ that causes the value of Π^a/ε_r^a to be approximately constant and close to 1. The small contribution of Π_u' helps improve this constancy slightly and becomes more significant at sufficiently large separations. Hence, in the same range where $\Pi^a \approx \text{const}$, we can write

$$\Pi_{u'}^a + \Pi_u'^a + \varepsilon_r^a \approx 0 \quad (4.25)$$

which re-writes eq. (3.14) with more information.

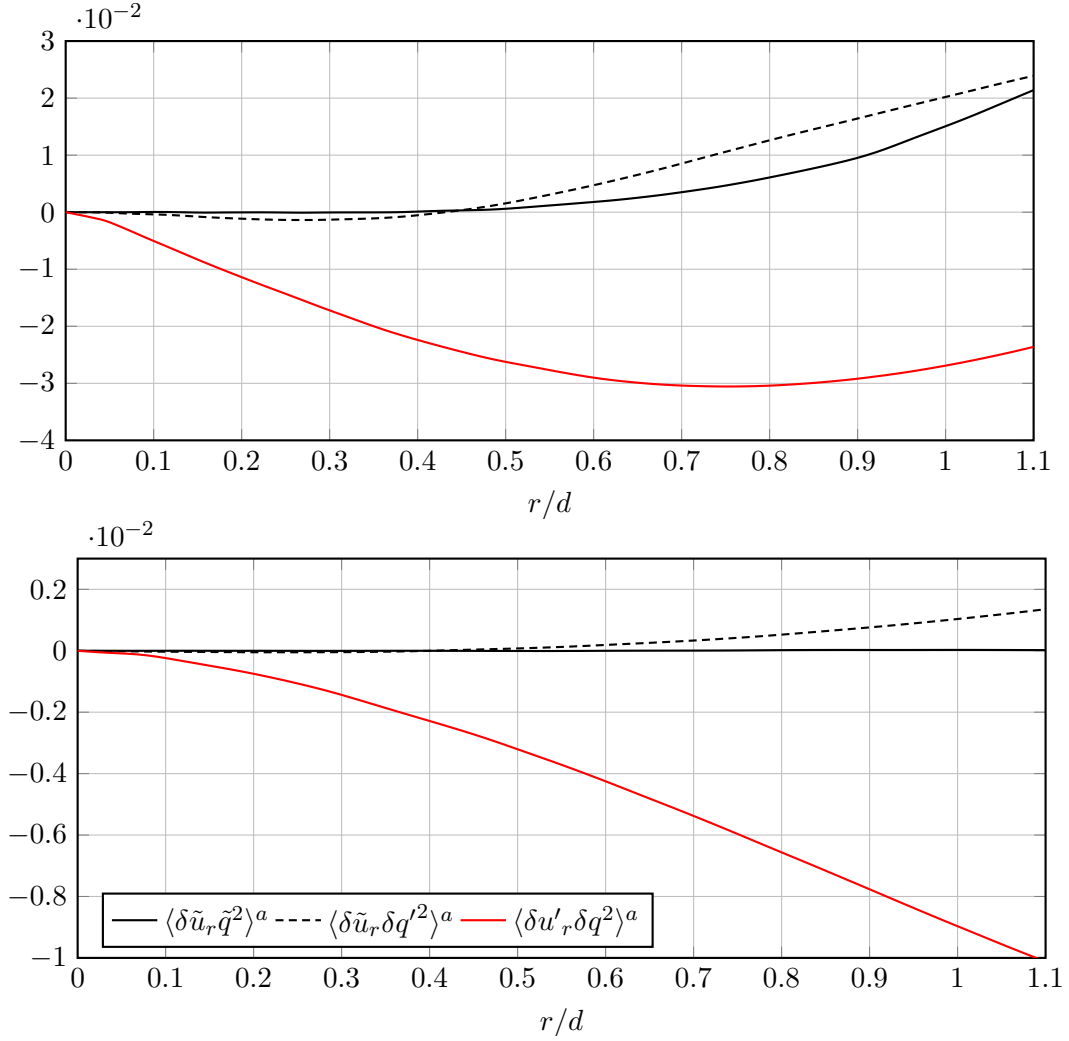


Figure 4.35: Orientation averaged non-linear inter-scale radial fluxes terms at $X_1/d = 2$ (top) and $X_1/d = 8$ (bottom).

The relation $\Pi^a \approx -\varepsilon^a$ suggests some kind of cascade equilibrium which appears at odds with the presence of the non-equilibrium scaling $C_\varepsilon \propto Re_\lambda^{-1}$ over the same range of streamwise distances x_1/d . A similar cascade equilibrium is the main consequence of the Kolmogorov (1941a) and Kolmogorov (1941c) theory of statistically homogeneous and approximately stationary small-scale turbulence and it implies $C_\varepsilon = \text{const}$. These two conclusions are reached in the Kolmogorov cascade theory by, (i) dropping the advective time derivative term and all non-homogeneity terms in the Kármán-Howarth-Monin-Hill equation and then (ii) assuming that the large-scale non-linear inter-scale transfer is directly determined by the large scale eddies, i.e. by their kinetic energy and size, and is therefore independent of initial/inlet/boundary conditions. The Kármán-Howarth-Monin-Hill equation (integrated over all directions if one wants to avoid using isotropy assumptions) then implies (i) a balance between the non-linear inter-scale transfer rate and the turbu-

lence dissipation over a range of scales from the large to the smallest turbulent eddies and (ii) $C_\epsilon = \text{const}$ as a result of the large-scale scaling of the large-scale non-linear inter-scale transfer rate and its balance with dissipation.

In the present context, the turbulence is neither homogeneous nor stationary at all scales larger than the Taylor micro-scale and the Kolmogorov theory is therefore inapplicable. Furthermore, there is no reason to expect the large-scale non-linear inter-scale transfer, even when averaged over orientations in the $r_3 = 0$ plane, to be independent of inlet/initial/boundary conditions. What is remarkable, is that the non-homogeneity and the advective terms balance in the orientation averaged KMH equation along the centreline in the near-field region we are focusing on even though these terms are in no way individually negligible. This leaves us with $\Pi^a \approx -\epsilon^a$ over a range of scales bounded from below by the Taylor micro-scale and from above by a scale between $d/3$ and d but without the constraint that Π^a has to be independent of inlet/initial/boundary conditions. We therefore cannot use $\Pi^a \approx -\epsilon^a$ to derive the scaling of ϵ^a as one would do in the context of the Kolmogorov theory. However, the proportionality between $\tilde{\epsilon}$ and ϵ' suggests that $C_\epsilon \propto Re_\lambda^{-1}$ as argued by Goto and Vassilicos (2016b) which, in turn, suggests that $\Pi^a \propto Re_\lambda^{-1}$ in the present near-field centreline context.

Note the following two points. Firstly, as we have shown in this subsection, the balance $\Pi^a \approx \epsilon^a$ is achieved with a substantial contribution to Π^a from the coherent turbulent fluctuations, which fully justifies the fact that Π^a depends on inlet conditions. Secondly, the cascade in the flow and flow region we are studying is fully non-stationary and out of equilibrium. The balance between Π^a and ϵ^a is actually a non-equilibrium balance achieved by a cancellation between the non-stationary and the non-homogeneous terms in the KMH equation. Such a balance cannot be achieved in the decaying periodic turbulence of Goto and Vassilicos (2016b) where there is no inhomogeneity to balance the non-stationarity in the KMH equation. Hence, Goto and Vassilicos (2016b) find $C_\epsilon \propto Re_\lambda^{-1}$ with an accompanying relation similar to $\tilde{\epsilon} \propto \epsilon'$ but no balance between non-linear inter-scale transfer and dissipation.

4.3.2.2 Fluxes of $\langle \delta q'^2 \rangle$ and $\langle \delta \tilde{q}^2 \rangle$ in Scale-Space

In chapter 3 we have shown that, in the same region where $\Pi^a \approx -\epsilon^a$, the orientation averaged non-linear inter-scale flux reflects a direct cascade, i.e. energy is transported from large to small scales. However, we have shown in section 3.3.1 that this is a result of combined direct and inverse cascades occurring at particular orientations $\mathbf{r}$ and that the orientations at which the inverse cascade occurs changes from r_1 at $X_1/d = 2$ to r_2 at $X_1/d = 8$. We have seen in section 4.3.2.1 that within that range, the direct cascade (when orientations are averaged) is dominated by the stochastic motions and thus we

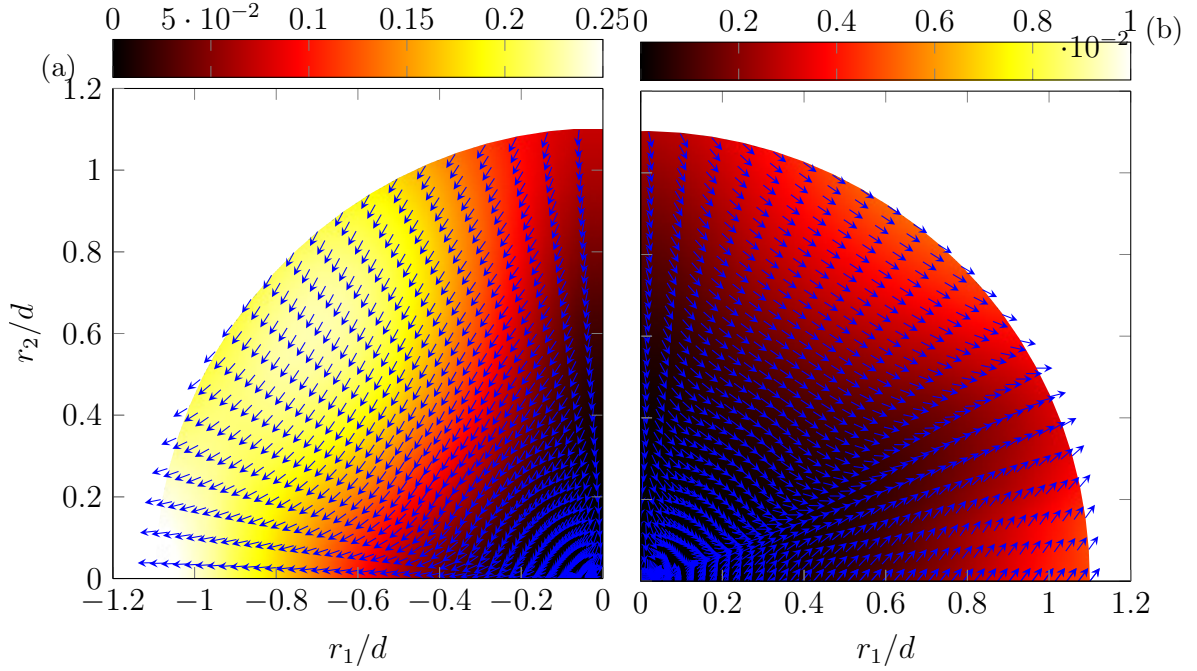


Figure 4.36: Distribution of $\langle \delta \tilde{u}_i \delta \tilde{q}^2 \rangle$ in scale space at $X_1/d = 2$ and $X_1/d = 8$ on the centreline.

must now determine if the inverse cascade (identified in section 3.3.1) is a consequence of the presence of the coherent structures or if it is also present in the stochastic motion.

We have shown in section 4.3.1 that the anisotropy observed in $\langle \delta q^2 \rangle$ is dominated by the contribution of the coherent structures ($\langle \delta \tilde{q}^2 \rangle$). However, we have seen in the same section that despite $\langle \delta q^2 \rangle$ appearing to be independent of the orientations of $\mathbf{r}$ (for $r_3 = 0$), the individual structure functions $\langle \delta u_i^2 \rangle$ are not isotropically distributed in the inertial range. This distinction is important since, if the stochastic motion were truly fully isotropic in the inertial range, the inter-scale flux $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ should always be oriented from large to small r .

We start by considering fig. 4.36 which shows the distribution of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$, the inter-scale flux of $\delta \tilde{q}^2$ due to the coherent motion, in scale space at $X_1/d = 2$ and $X_1/d = 8$. Recall from fig. 4.35 that, when orientations are averaged out, the contribution of this term to the total flux is negligible. Figure 4.36 shows that as the distance to the prism is increased (from $X_1/d = 2$ to $X_1/d = 8$) the orientations of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ do not change dramatically but its magnitude does. In fact, as will be seen below, the magnitude of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ at $X_1/d = 8$ is the smallest of all contributions to the inter-scale flux. An argument attempting to explain the orientations of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ is given in section 4.4.

The other flux associated with the coherent motion is $\langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle$, shown in fig. 4.37. The orientations of $\langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle$ near the r_1 and r_2 axes are similar to those of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$, i.e. the flux is outward near the r_1 axis and inward near the r_2 axis. However, a distinct critical

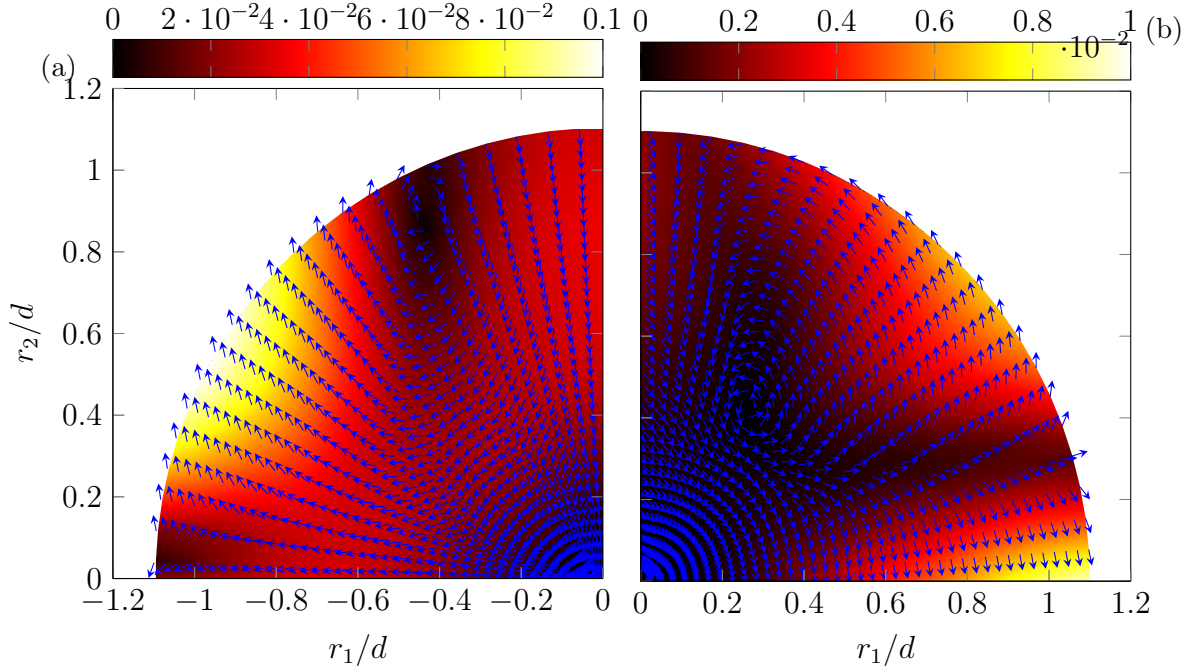


Figure 4.37: Distribution of $\langle \delta \tilde{u}_i \delta q'^2 \rangle$ in scale space at $X_1/d = 2$ and $X_1/d = 8$ on the centreline.

point in scale-space can be observed in the distribution of $\langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle$. This suggests that the coherent structures redistribute the energy associated with the stochastic motions at preferential orientations, possibly as a consequence of the anisotropy of the coherent structures themselves.

Let us now consider the inter-scale fluxes associated with velocity differences of the stochastic motions. In section 4.3.2.1 we have considered $\langle \delta u'_r \delta q'^2 \rangle$, the flux of the total energy due to the stochastic motion. However, we can express this flux as two different contributions (recall the description above eq. (4.20)), $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ and $\langle 2\delta \mathbf{u}' (\delta \mathbf{u}' \cdot \delta \tilde{\mathbf{u}}) \rangle$. The former represents the inter-scale flux of the energy associated with the stochastic motion by the stochastic motion itself, the latter reflects the effect of the stochastic motion in the inter-scale flux of the total fluctuating energy, i.e. $\langle 2\delta \mathbf{u}' (\delta \mathbf{u}' \cdot \delta \tilde{\mathbf{u}}) \rangle$ is the difference between $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ and $\langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle$. The distributions of both $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ and $\langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle - \langle \delta \mathbf{u}' \delta q'^2 \rangle$ in scale-space are shown in fig. 4.38 and fig. 4.39, respectively, for $X_1/d = 2$ and $X_1/d = 8$.

Unlike the fluxes associated with the coherent motions, the distributions of both $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ and $\langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle - \langle \delta \mathbf{u}' \delta q'^2 \rangle$ in scale-space undergo a significant change between $X_1/d = 2$ and $X_1/d = 8$. In particular, fig. 4.38 shows that $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ has the behaviour of $\langle \delta \mathbf{u} \delta q'^2 \rangle$ observed in section 3.3.1 (recall fig. 3.11) where the direction at which the inverse cascade occurs shifts from being aligned with r_1 at $X_1/d = 2$ to being aligned with r_2 at $X_1/d = 8$. The flux $\langle \delta \mathbf{u}' \delta q'^2 \rangle - \langle \delta \tilde{\mathbf{u}} \delta q'^2 \rangle$ (which enters the equation for $\delta \tilde{q}^2$

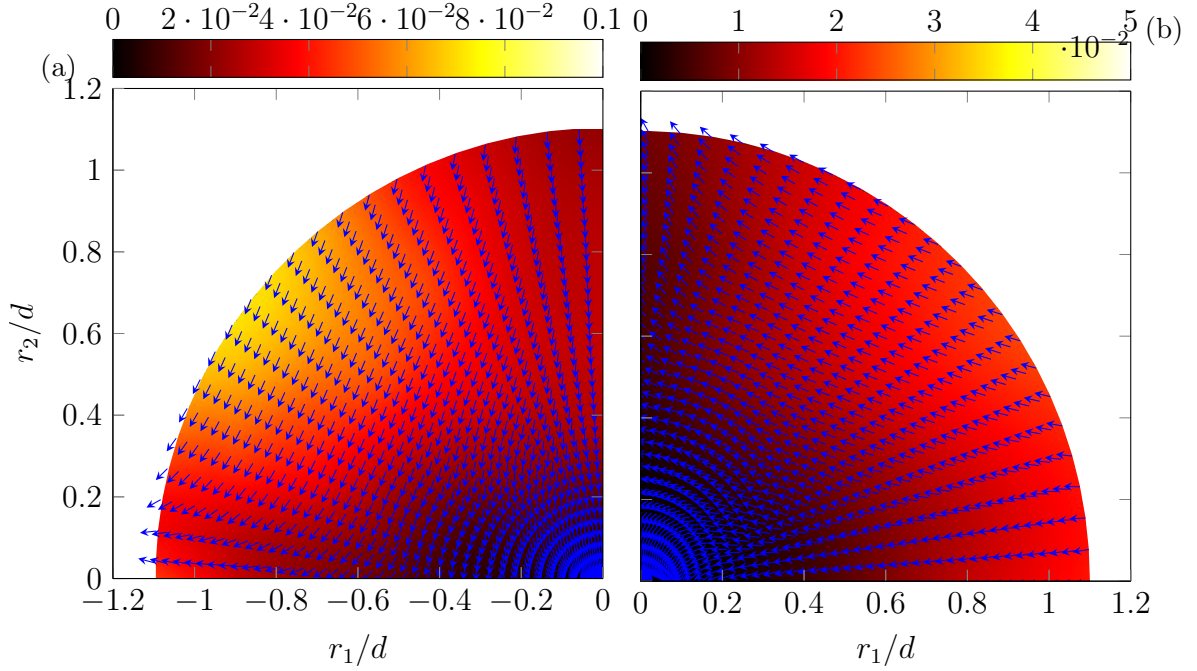


Figure 4.38: Distribution of $\langle \delta u'_i \delta q'^2 \rangle$ in scale space at $X_1/d = 2$ and $X_1/d = 8$ on the centreline.

correlates with $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ at $X_1/d = 8$ but differs from it at $X_1/d = 2$ where the fluxes point mostly inward at all orientations investigated.

We are thus led to conclude from figs. 4.36 to 4.39 that in the very near wake, the inverse cascade that occurs along r_1 is a combined effect of both the stochastic flux of energy associated with the stochastic motions and the fluxes associated with the coherent motion. As the distance to the prism increases, the fluxes associated with the coherent motions become less significant (in magnitude) when compared to $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ which in turn induces an inverse cascade along separations aligned with r_2 .

At the present moment we are unable to explain why the stochastic motion sustains (in a statistical sense) an inverse cascade as *a priori* one might expect the stochastic component of the velocity to behave closer to homogeneous isotropic turbulence, especially at the smaller scales. We have shown in section 4.3.1 how despite the energy associated with the stochastic motion appearing to be isotropically distributed, the turbulence is far from so. Furthermore, as will be seen in section 4.3.3, the different mechanisms involved in injecting and extracting energy from the stochastic motion (recall eq. (4.14)) are not isotropically distributed in scale-space. It may therefore be that the local (in scale space) direct and inverse cascades are required to evenly distribute energy in scale space, i.e. in order for the energy associated with the stochastic motion to be eventually isotropically distributed in scale-space, it must be anisotropically exchanged between different scales.

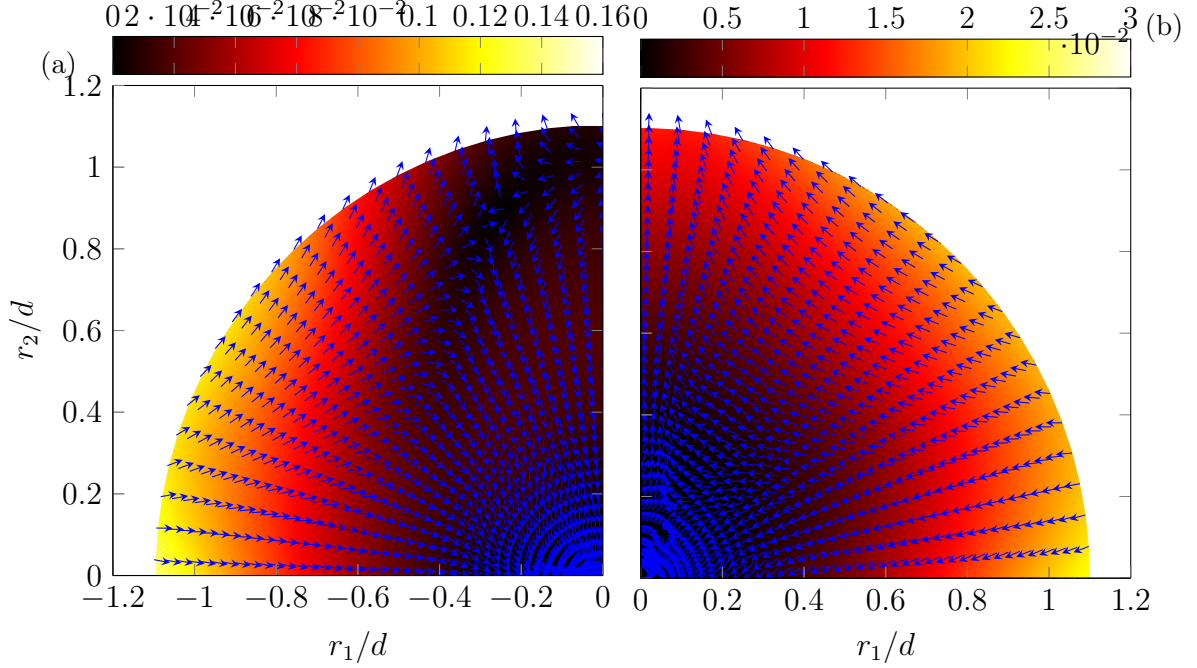


Figure 4.39: Distribution of $\langle \delta \mathbf{u}' \delta q^2 \rangle - \langle \delta \mathbf{u}' \delta q'^2 \rangle$ in scale space at $X_1/d = 2$ and $X_1/d = 8$ on the centreline.

The analysis of figs. 4.36 to 4.39 in this section warrants a short comment on statistical convergence. By the reflection symmetry of the wake, the vector fields shown in these figures should have a zero component along r_2 for separations along the r_1 axis and vice versa. The slight deviations from this symmetry are observed mainly for $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ and $\langle \delta \mathbf{u}' \delta q^2 \rangle - \langle \delta \mathbf{u}' \delta q'^2 \rangle$ at $X_1/d = 8$, however, both terms are significantly smaller than the remaining fluxes and thus the conclusions drawn above remain unaffected.

4.3.3 Budget of $\langle \delta q'^2 \rangle$

Given the main role played by the stochastic motion in the non-linear inter-scale energy transfer, and effectively the observation that $\Pi^a \approx -\varepsilon^a$, we now turn our attention to how the energy (associated with the stochastic motion) is generated and exchanged by the remaining mechanisms that appear in eq. (4.14). The different terms of eq. (4.14) are shown in fig. 4.40 for $X_1/d = 2$ and $X_1/d = 8$ when the orientations of $\mathbf{r}$ are averaged out (in the $r_3 = 0$ plane).

In analysing eq. (4.14) and eq. (4.15), we have seen that the link between mean-flow and stochastic motions and between coherent motions and stochastic motions is established by the production terms $\mathcal{P}'_U$ and $\mathcal{P}'_{\tilde{u}}$, respectively. All remaining terms in eqs. (4.14) and (4.15) (other than the dissipation) are only able to transfer energy either in scale-space or physical space. As made clear by fig. 4.40, the mean-flow inhomogeneity has very

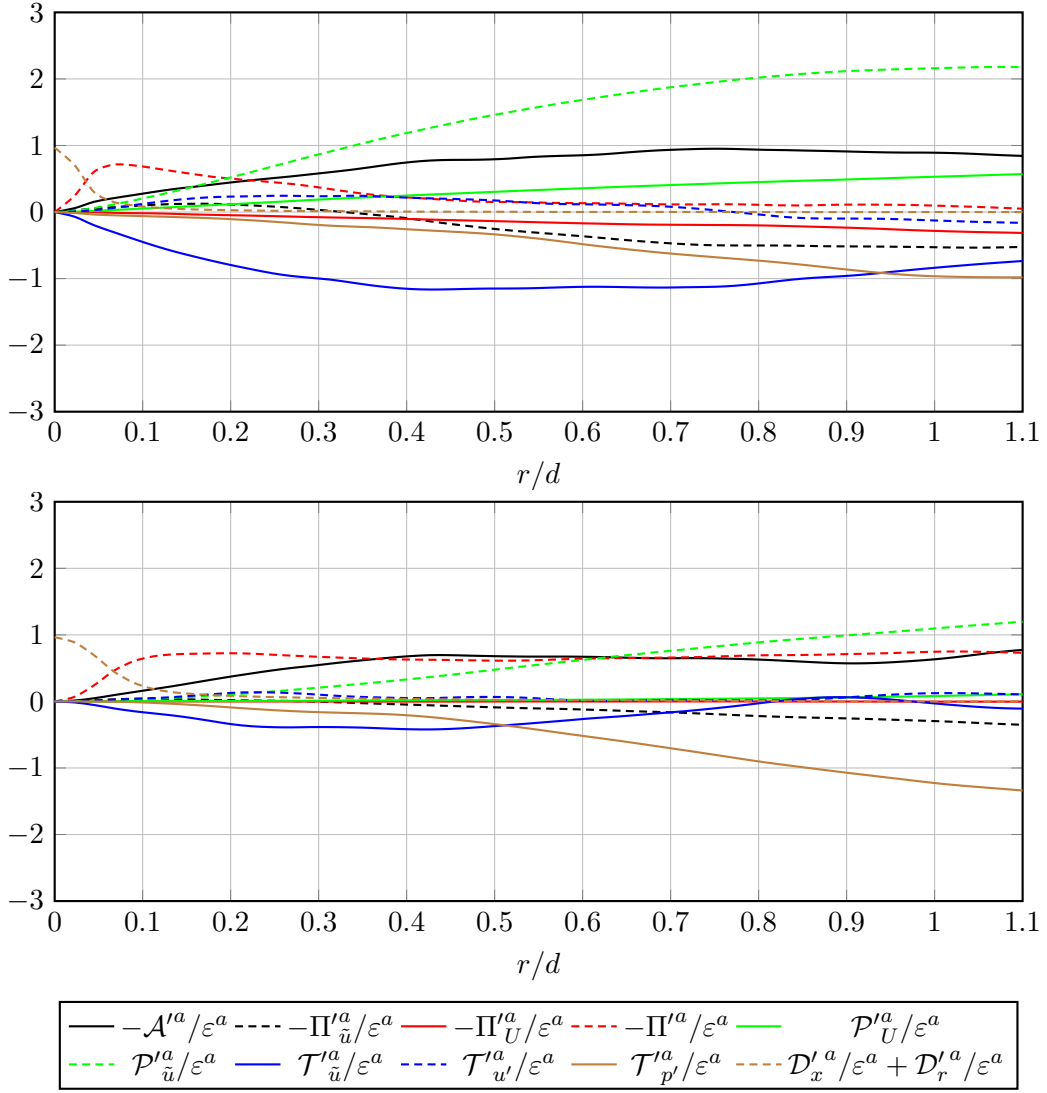


Figure 4.40: Orientation averaged terms of eq. (4.14) normalised by ε_r^a at $X_1/d = 2$ (top) and $X_1/d = 8$ (bottom) on the geometric centreline.

little impact on the production of $\langle \delta q'^2 \rangle$. Instead, it is the inhomogeneity of the coherent motions that is responsible for producing $\langle \delta q'^2 \rangle$ as reflected by the large contribution of $\mathcal{P}'_{\tilde{u}}{}^a$.

Figure 4.40 also illustrates how the advection term ($\mathcal{A}'^a$) acts as a source of $\langle \delta q'^2 \rangle$ in eq. (4.14). This term reflects the unsteadiness of the stochastic motions in the direction of the mean-flow and the fact that it is non-negligible effectively means that the underlying assumption of steadiness taken in Kolmogorov (1941a) is inapplicable to the stochastic motions in the present case. For small enough separations ($r/d \lesssim 0.2$ at $X_1/d = 2$ and $r/d \lesssim 0.4$ at $X_1/d = 8$) the advection term appears to be somewhat balanced by the transport due to the coherent motions ($\mathcal{T}'_{\tilde{u}}{}^a$). This reflects the sweeping mechanism identified in chapter 3, however, we are now able to determine that the transport by the

stochastic motion ($\mathcal{T}'^a_u$) plays a negligible role in transporting energy in physical space when compared to the transport of $\delta q'^2$ by the coherent motion. Thus, it appears that, in the context of the orientation averaged statistics, the eddies associated with the stochastic motion are swept by the large coherent structures (by means of $\mathcal{T}'^a_u$), i.e. the energy extracted by the sweeping due to the coherent structures is mostly replenished by that advected by the mean flow.

Finally, notice that fig. 4.40 suggests that the pressure term $\mathcal{T}'^a_p$ appears to play a significant role in the budget of $\langle \delta q'^2 \rangle$ especially at the largest scales ($r/d \gtrsim 0.5$). This pressure term is responsible for re-distributing energy both in scale- and physical-space due to anisotropy of the stochastic motions.

It is important to bear in mind that the correlations observed in fig. 4.40 are made for the orientation averaged quantities of eq. (4.14) and should not be expected to hold on a scale-by-scale basis given the already discussed anisotropy of the turbulence. We shall now examine the scale-space distribution of some of the terms entering eq. (4.14).

4.3.3.1 Mean-Flow Advection and Transport in Physical Space by the Coherent Motions

Contrasting $\mathcal{A}'$ with $\mathcal{T}'_u$ on a scale by scale basis, as done in figs. 4.41 and 4.42, reveals that the sweeping mechanism postulated above does not hold as clearly. At $X_1/d = 2$, the mean flow advection $\mathcal{A}'$ appears almost isotropically distributed in scale space (much like in fig. 4.52a) over a wide range of separations whereas $\mathcal{T}'_u$ achieves a rather different distribution along r_1 in comparison to r_2 . In fact, the significant contribution to $\mathcal{T}'^a_u$ seems to arise from the vicinity of the r_1 axis where $|\mathcal{T}'_u|$ is the largest and represents a sink in the equation for $\langle \delta q'^2 \rangle$. This effect is lost at the downstream location ($X_1/d = 8$) as seen in fig. 4.42b. Curiously, at $X_1/d = 8$ the term $\mathcal{T}'_u$ appears to become rather insensitive to the orientations of $\mathbf{r}$ while $\mathcal{A}'$ sees its largest magnitude concentrated near the r_1 axis (see fig. 4.42a).

Even though both $\mathcal{A}'$ and $\mathcal{T}'_u$ represent a transfer of $\langle \delta q'^2 \rangle$ in physical space, the advection term is directly linked to the decay of $\langle \delta q'^2 \rangle$ in the direction of the mean flow, whereas $\mathcal{T}'_u$ represents a dynamic transfer of $\langle \delta q'^2 \rangle$ (given that it is a third order statistic). In fact, given eq. (4.3) $\mathcal{T}'_u$ can be rewritten as

$$\mathcal{T}'_u = -\frac{1}{4} \left\langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \delta q'^2}{\partial X_i} \right\rangle = -\frac{1}{4} \left\langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \{\delta q'^2\}}{\partial X_i} \right\rangle.$$

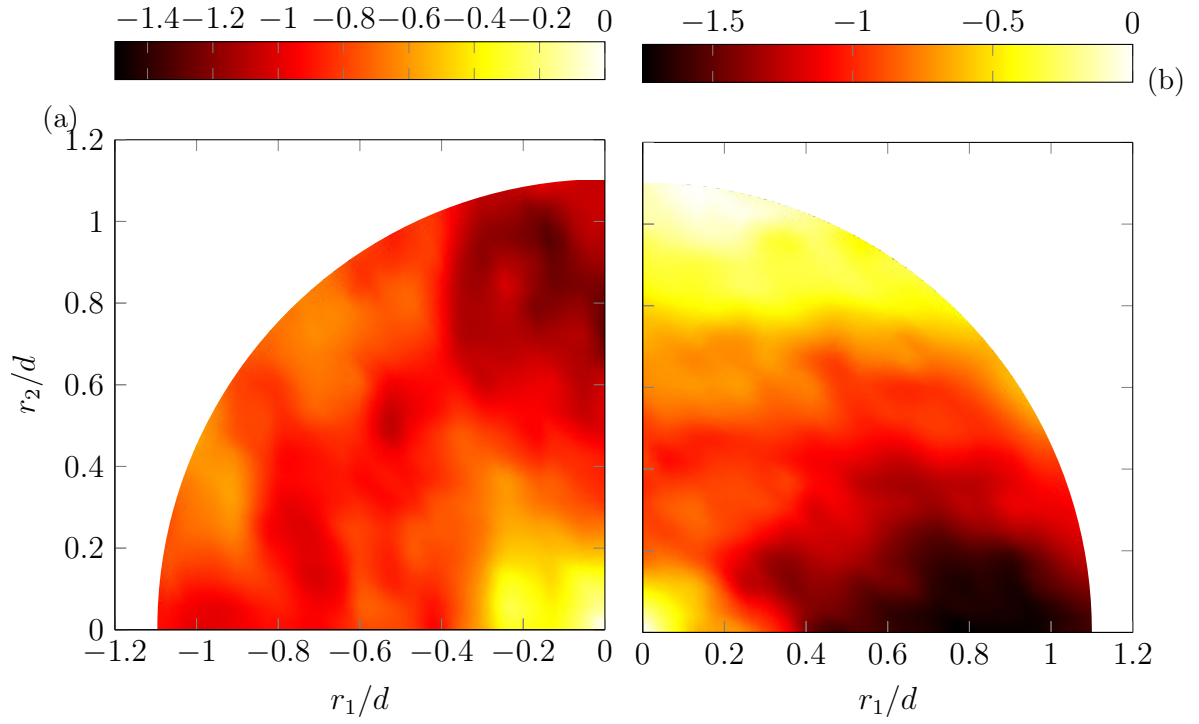


Figure 4.41: Distribution of $\mathcal{A}'$ and $\mathcal{T}'_u$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

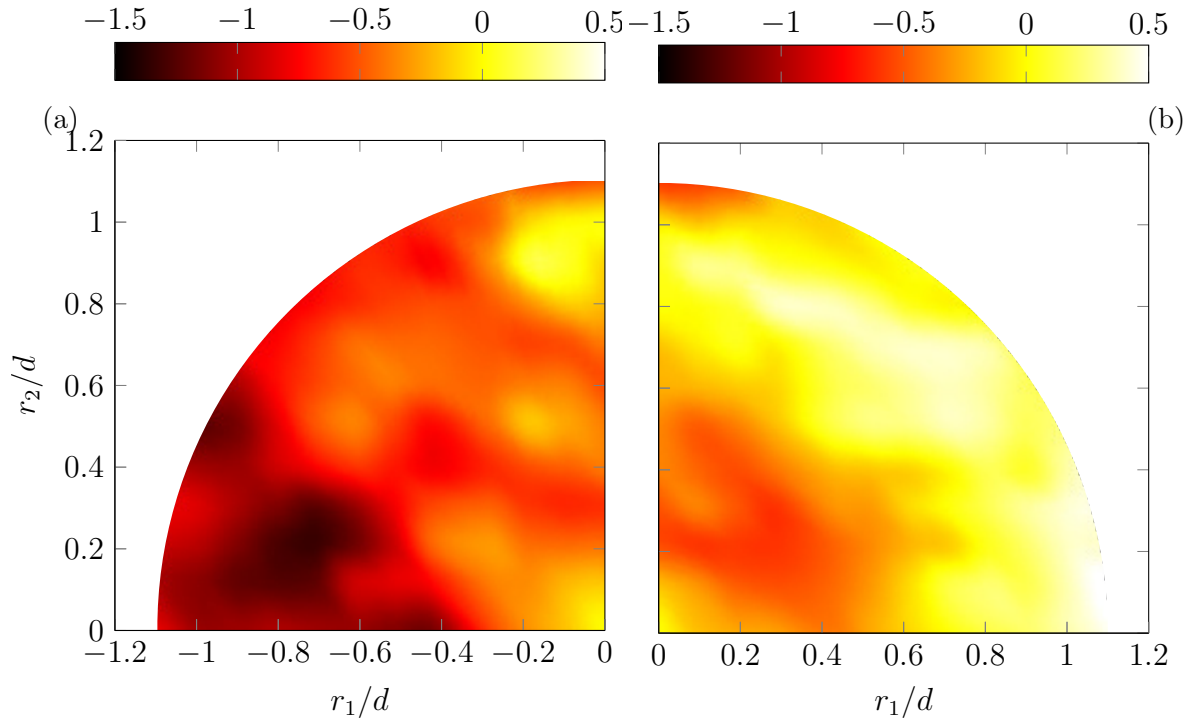


Figure 4.42: Distribution of $\mathcal{A}'$ and $\mathcal{T}'_u$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

This form of $\mathcal{T}'_u$ allows us to interpret the transfer of $\langle \delta q'^2 \rangle$ in physical space by the coherent motion as the average alignment between the gradient of $\{\delta q'^2\}$ in X_i and $\frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2}$, the two-point average of the velocity field associated with the coherent motion.

Let us consider a whole period of the shedding. Given how close to the centreline the coherent structures are observed, the aforementioned alignment will be small when the coherent structure is centred at a given x_i (in the two-point sense, i.e. $\frac{x_i^+ + x_i^-}{2}$), since the points diametrically opposed on the vortex will have approximately the same velocity (and thus $\tilde{u}_i^+ + \tilde{u}_i^-$ will be small). Conversely, when the mid-point between successive vortices corresponds to $\frac{x_i^+ + x_i^-}{2}$, $\frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2}$ is the largest and will be mostly aligned with x_2 . Thus, $\mathcal{T}'_u$ can be seen as the coherent vortices transporting $\langle \delta q'^2 \rangle$ (at a particular scale) mostly towards or away from the centreline, for $\mathcal{T}'_u < 0$ and $\mathcal{T}'_u > 0$ respectively.

The fact that $\mathcal{T}'_u$ tends to become more isotropically distributed may be related to the fact that the coherent structures undergo a strong distortion in the region immediately downstream of the prism, as observed in fig. 4.5. On the other hand, it is perhaps surprising that $\mathcal{A}'$ appears to be so independent of $\mathbf{r}$ at $X_1/d = 2$ and not so much at $X_1/d = 8$, given that at the location further upstream the mean flow is rather inhomogeneous but also due to the apparent tendency of $\langle \delta q'^2 \rangle$ to become independent of the orientation of $\mathbf{r}$ (in the $r_3 = 0$ plane) with increasing downstream position (recall fig. 4.29).

4.3.3.2 Production by the Mean and Phase-Averaged Velocity Components

It was seen in fig. 4.40 that the production due to the inhomogeneity of the coherent motions was much larger than that associated with mean-flow inhomogeneity where the orientations of $\mathbf{r}$ were averaged out. In figs. 4.43 and 4.44 the distributions of both $\mathcal{P}'_U$ and $\mathcal{P}'_u$ in scale-space are shown for $X_1/d = 2$ and $X_1/d = 8$.

Notice that the conclusion drawn when the orientations were averaged still holds, i.e. $\mathcal{P}'_U$ is indeed negligible in contributing as a source of energy of the stochastic motions. Furthermore, notice how the largest values of $\mathcal{P}'_u$ are concentrated along the r_1 axis. At $X_1/d = 8$, this coincides with the direction along which the non-linear inter-scale fluxes occur in the direct sense (from large to small scales), as seen in section 4.3.2.2. While this is not the case at $X_1/d = 2$, both the pressure term (discussed below in figs. 4.47 and 4.48) and the transport due to the coherent motions (seen in figs. 4.41 and 4.42) act as sink terms along the r_1 axis at that location, and then as source terms further downstream. One can thus argue that the reason why the fluxes appear to be oriented in the unconventional way (from small scales towards scales at which production is large) is a consequence that the energy injected by the production term is removed by other mechanisms other than the inter-scale transfer. Between the two locations investigated, it appears that, at separations close to the r_1 axis, as the different mechanisms capable

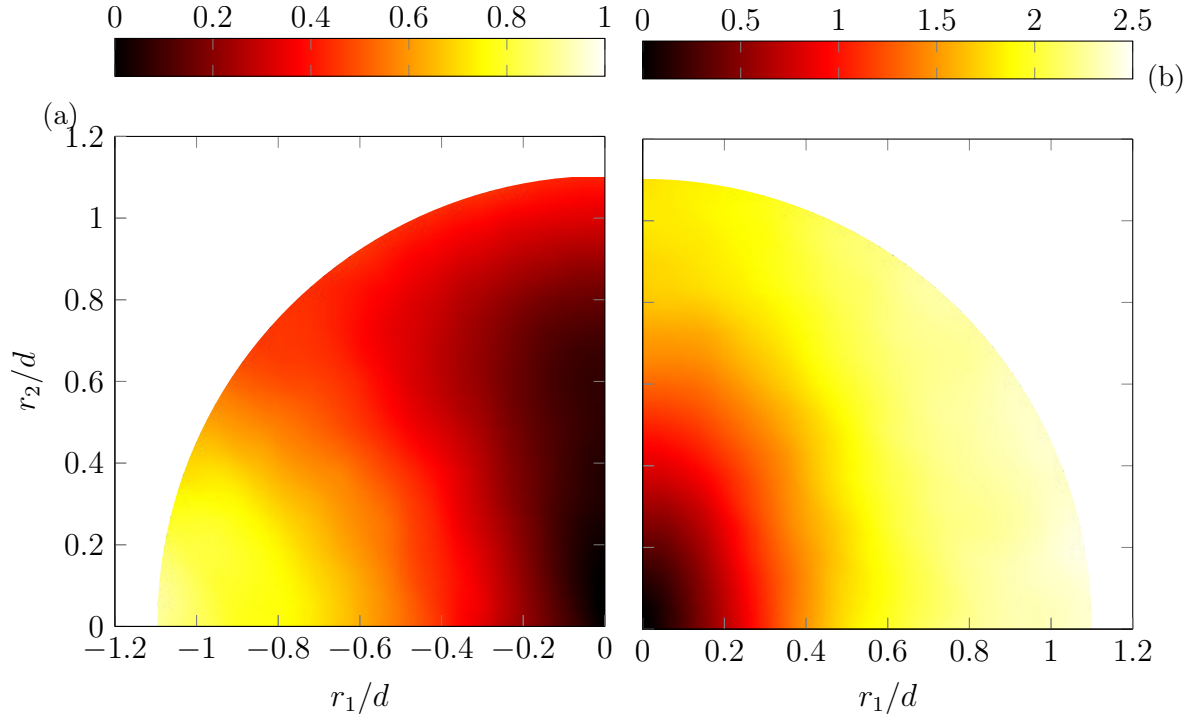


Figure 4.43: Distribution of $\mathcal{P}'_U$ and $\mathcal{P}'_{\tilde{u}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

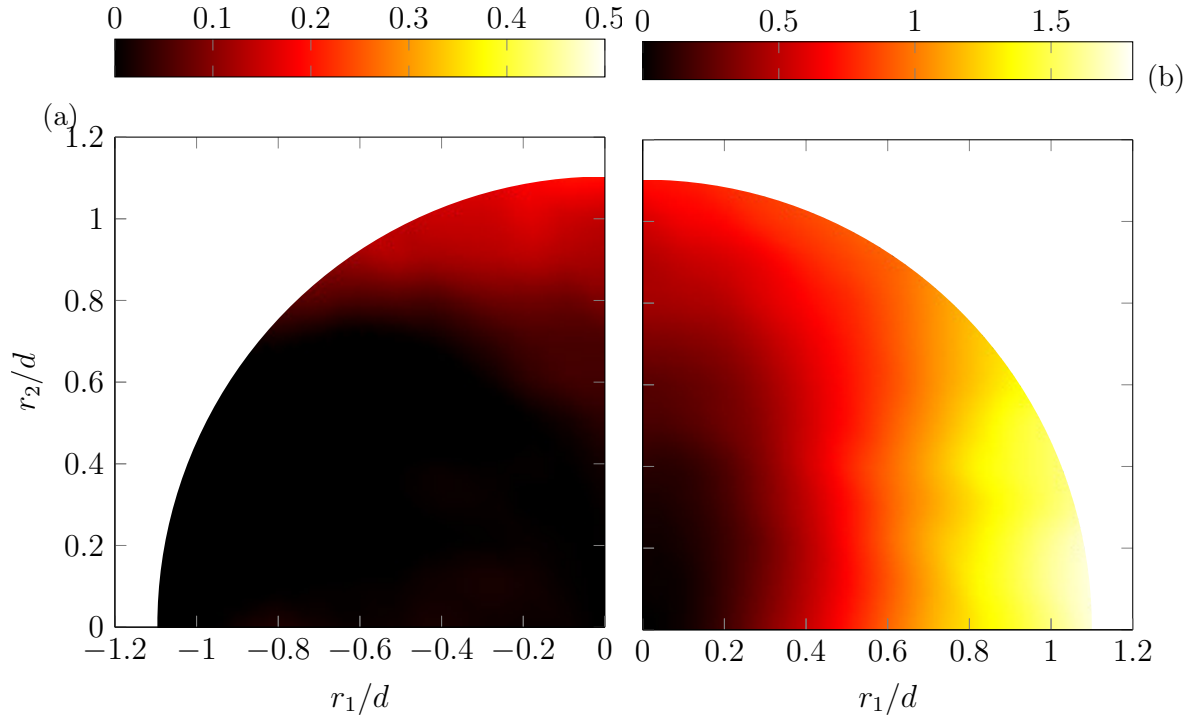


Figure 4.44: Distribution of $\mathcal{P}'_U$ and $\mathcal{P}'_{\tilde{u}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

of redistributing the energy injected through $\mathcal{P}'_{\tilde{u}}$ become more and more insignificant, the non-linear inter-scale transfer adapts and transfers the energy from where $\mathcal{P}'_{\tilde{u}}$ is large towards the small scales. We can further postulate that, as $\mathcal{P}'_{\tilde{u}}$ is larger than ε' , the remainder of the energy must be redistributed towards other orientations thus causing the outward fluxes along r_2 discussed in section 4.3.2.2.

4.3.3.3 Inter-Scale Transfer by Differences of Mean and Phase-Averaged Velocities

While both the mean flow and the coherent motions contribute largely to the transfers in physical space, their influence on the inter-scale transfer was seen to be of much smaller magnitude when averaging over orientations^{||}. When looking at the distributions of $\Pi'_{\tilde{u}}$ and $\Pi''_{\tilde{u}}$ in scale space (see figs. 4.45 and 4.46) the linear inter-scale transfer of $\langle \delta q'^2 \rangle$ is clearly negligible at most orientations. The inter-scale transfer by the coherent motion, on the other hand, despite its small magnitude when averaging out the orientations of $\mathbf{r}$, is comparable to ε and has a distinct distribution in scale space where $\Pi'_{\tilde{u}} > 0$ for separations aligned with the free-stream direction and $\Pi'_{\tilde{u}} < 0$ for separations aligned with the cross-stream direction. In fact, at $X_1/d = 8$ it seems like the magnitude of $\Pi'_{\tilde{u}}$ increases by some factor independent of the orientation of $\mathbf{r}$ in such a way that the orientation averaged $\Pi'^a_{\tilde{u}}$ remains small in comparison to $\Pi^a_{u'}$ (recall fig. 4.34).

4.3.3.4 Non-Linear Inter-Scale Transfer and Pressure Correlation

Let us now turn our attention to the only two terms in eq. (4.14) that depend only on the stochastic motions, Π' and $\mathcal{T}'_{p'}$. In figs. 4.47 to 4.49 the non-linear inter-scale transfer of $\langle \delta q'^2 \rangle$, the pressure correlation term and the turbulent transport are plotted at $X_1/d = 2$ and $X_1/d = 8$.

In particular, the distributions of Π' and $\mathcal{T}'_{p'}$ in scale space behave very much in the same way as Π and $\mathcal{T}_p$ as seen in section 3.3.2. In particular, notice that between the two locations $X_1/d = 2$ and $X_1/d = 8$ the distribution of Π' undergoes a rotation of $\pi/2$ in scale space, very much like Π in section 3.3.2. This reorientation of Π' correlates with the reorientation of the fluxes $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ seen in section 4.3.2.2. As discussed above, one possible explanation for this may be that the non-linear inter-scale transfer reacts to the changes imposed by the other terms in the KMH equation.

In the very near wake, the pressure term appears somewhat independent of the orientation of $\mathbf{r}$, as seen in fig. 4.47b, but perhaps more interestingly, its magnitude is much

^{||} While $\Pi'^a_{\tilde{u}}$ may be small, its contribution is still relevant in retrieving the K41-like equilibrium $\Pi^a \approx -\varepsilon^a$, as discussed in section 4.3.2.1

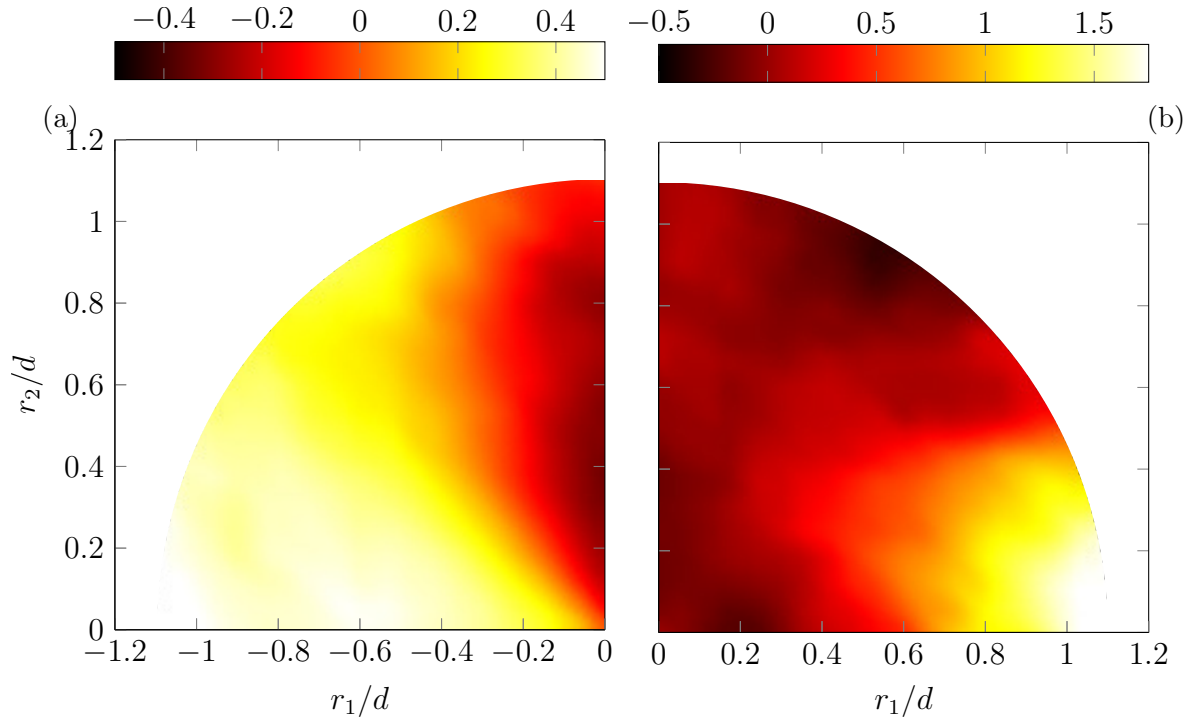


Figure 4.45: Distribution of Π'_U and Π'_u normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

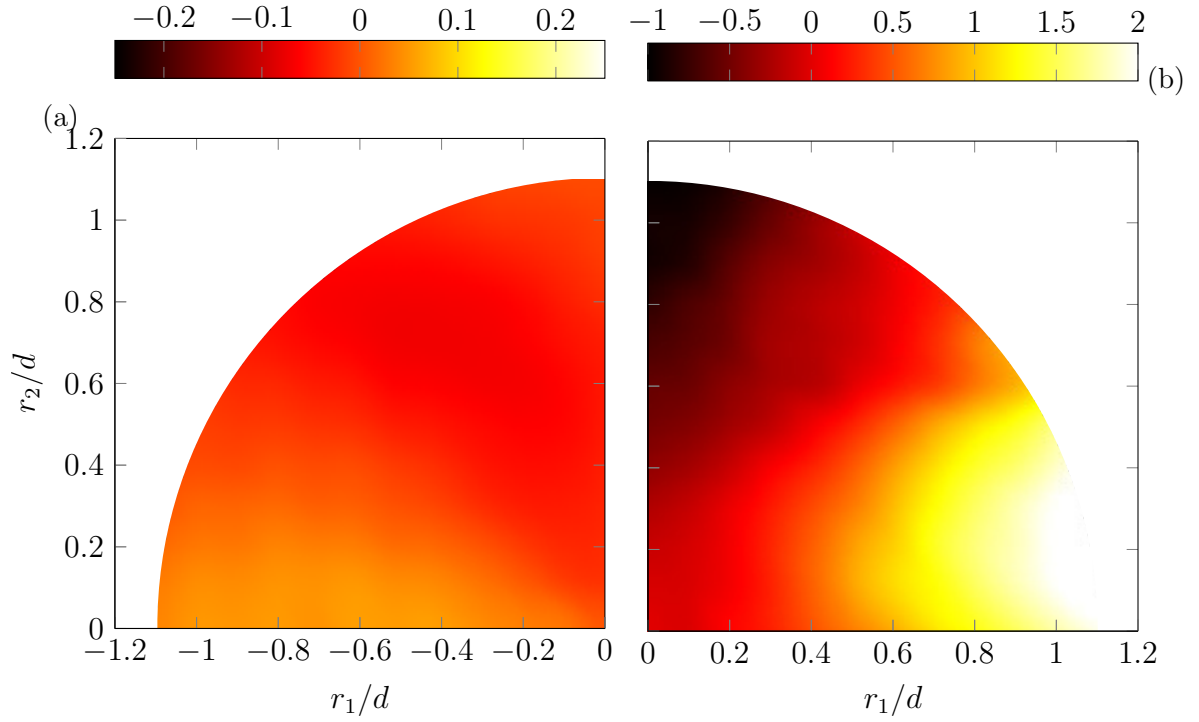


Figure 4.46: Distribution of Π'_U and Π'_u normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

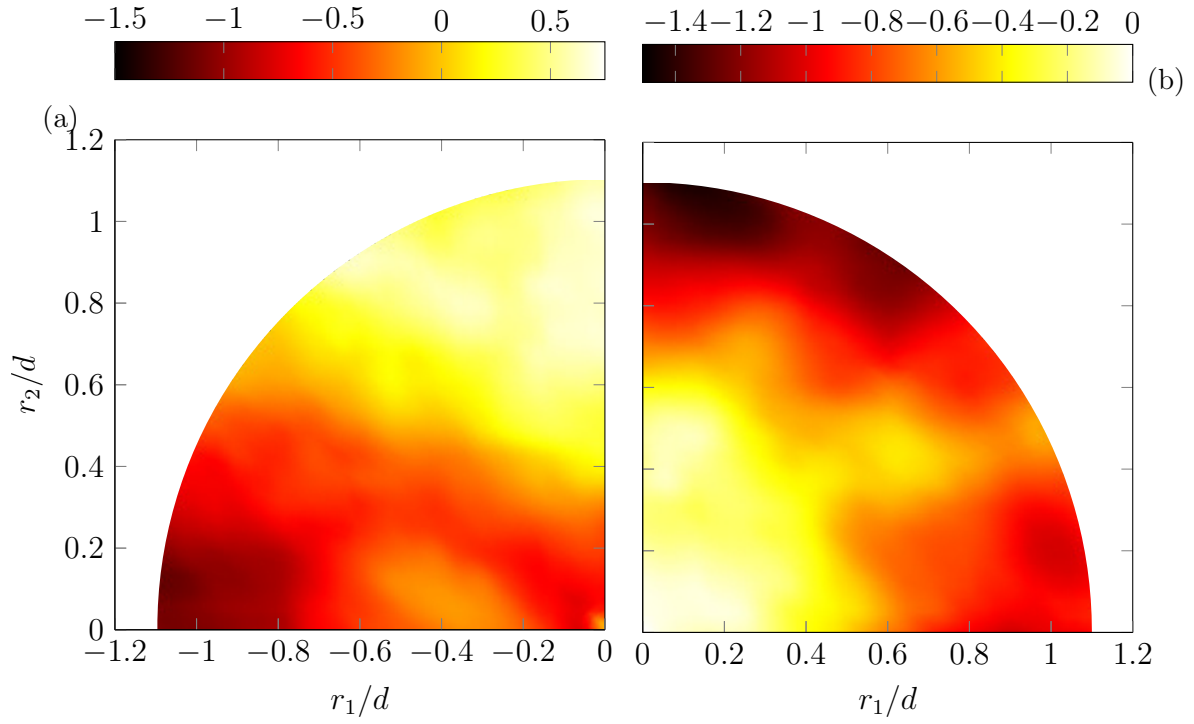


Figure 4.47: Distribution of Π' and $\mathcal{T}'_{p'}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

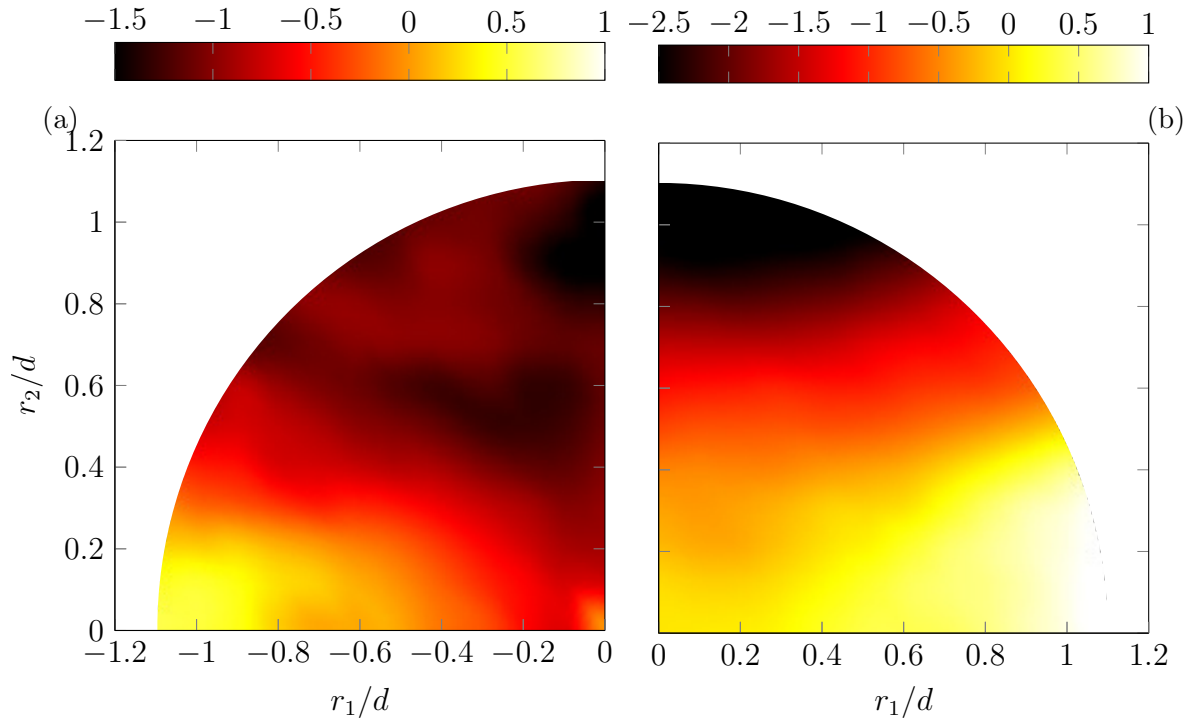


Figure 4.48: Distribution of Π' and $\mathcal{T}'_{p'}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

smaller (in comparison to ε_r) than that of Π' . Further downstream (at $X_1/d = 8$), the correlation between Π and $\mathcal{T}_p$ observed in fig. 3.17 appears to also develop between Π' and $\mathcal{T}'_p$ at $X_1/d = 8$ (see fig. 4.48). Yasuda and Vassilicos (2017) have made similar observations in periodic box turbulence where no coherent structures are present, suggesting that this correlation between the pressure term and the non-linear inter-scale transfer is not specific to the type of turbulence investigated here.

In general, the term $\mathcal{T}'_p$ would be negligible for isotropic turbulence (Lindborg, 1996b). The present results suggest that $\mathcal{T}'_p$ is more sensitive to anisotropy in the inter-scale transfer, rather than the anisotropy of the energy distribution $\langle \delta q'^2 \rangle$, which was seen to appear to become roughly isotropic as the distance to the prism increased.

4.3.3.5 Transport by the Stochastic Motions

Finally, we consider the distribution of the turbulent transport in physical space of $\langle \delta q'^2 \rangle$ by the stochastic motion itself. Figure 4.49 shows the distributions of $\mathcal{T}'_{u'}$ in scale space at the same two locations considered so far. This term was found to be mostly negligible in the scale by scale energy budget when orientations were averaged. In fact, its magnitude, despite not as small as that of e.g. Π'_U , seems to be always smaller than ε_r . At $X_1/d = 2$, $\mathcal{T}'_{u'}$ acts as a source of energy for separations aligned with streamwise direction, but this appears to be lost further downstream (at $X_1/d = 8$), where the positive values of $\mathcal{T}'_{u'}$ appear concentrated at an angle with the r_1 axis but also at large separations close to the r_2 axis. Notice that the stochastic transport in physical space appears to counter-act the effect of $\mathcal{T}'_u$, in particular in the very near wake where $\mathcal{T}'_u < 0$ mostly near the r_1 axis (recall fig. 4.41b).

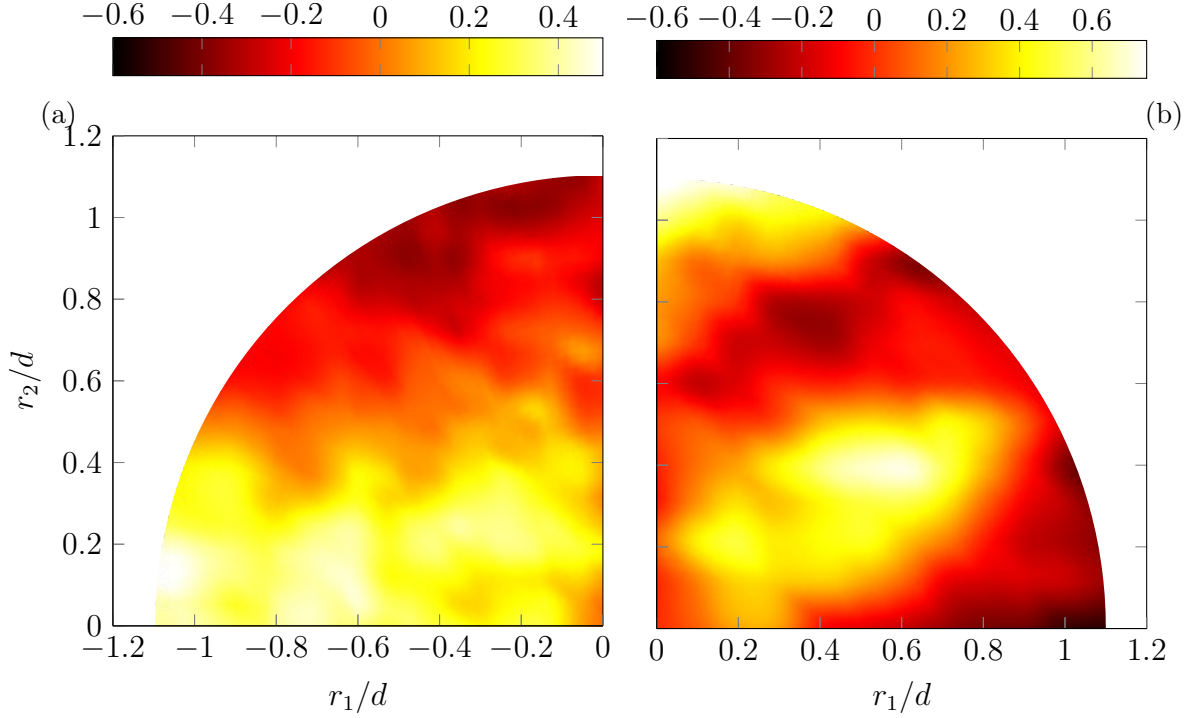


Figure 4.49: Distribution of $\mathcal{T}'_w$ normalised by ε_r in scale space on the geometrical centre-line at $X_1/d = 2$ in (a) and at $X_1/d = 8$ in (b).

4.3.4 Budget of $\langle \delta \tilde{q}^2 \rangle$

In the previous section, we have seen how the coherent motion is the main mechanism by which energy associated with the stochastic motions is produced. As discussed in section 4.1.2, the kinetic energy associated with the phase-averaged velocity undergoes a strong depletion downstream of the location where the coherent vortices are created (recall fig. 4.18). In fact, as the coherent structures are formed, they represent the largest portion of the total kinetic energy, but $\tilde{k}$ quickly becomes comparable in magnitude to k' . Figure 4.18 also shows how the dissipation associated with the coherent motion is practically negligible and it is thus reasonable to assume that $\tilde{\varepsilon}$ cannot be the driving factor in depleting $\tilde{k}$. Thus, it becomes relevant to characterise the budget of $\langle \delta \tilde{q}^2 \rangle$ in order to understand what mechanisms are responsible for producing and extracting energy from the coherent motion.

As noted by Hussain (1983), the decay of $\tilde{k}$ is caused by the dissipation $\tilde{\varepsilon}$ and the interaction with both the mean flow and the stochastic motion, via the respective production terms, equivalent to $\tilde{\mathcal{P}}_U$ and $\mathcal{P}'_{\tilde{u}}$ in eq. (4.17). However, eq. (4.17) allows us to determine at which scales the interactions between the mean, phase and stochastic components occur. The different terms of eq. (4.17) are shown in figs. 4.50 and 4.51 when the orientations of $\mathbf{r}$ are averaged out in the $r_3 = 0$ plane at both $X_1/d = 2$ and $X_1/d = 8$, respectively.

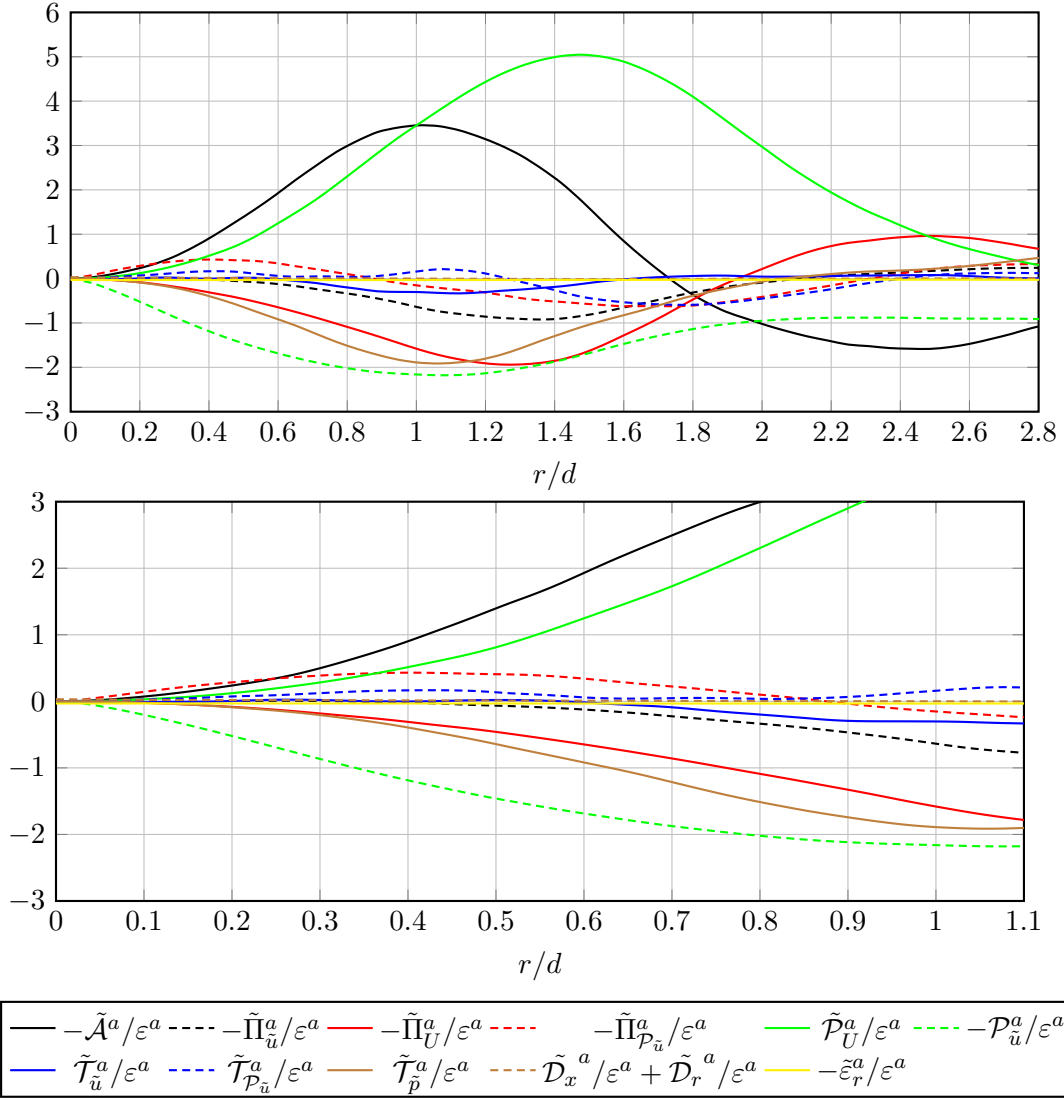


Figure 4.50: Orientation averaged terms of eq. (4.15) normalised by ε_r^a at $X_1/d = 2$. On the bottom the range of r/d is reduced.

Apart from $\tilde{\Pi}_{\mathcal{P}_u}^a$ (recall that $\tilde{\Pi}_{\mathcal{P}_u}^a$ is equal to $\Pi' - \Pi'_{u'}$), the major contributions to eq. (4.17) at both locations arise from the advection $\tilde{\mathcal{A}}^a$ and the production by the mean flow $\tilde{P}_U^a$ acting as sources along with $\mathcal{P}_u^a$ acting as a sink. Notice that close to the prism (at $X_1/d = 2$) and for very large separations ($r/d \gtrsim 1.7$) $\tilde{\mathcal{A}}^a$ actually changes sign. Furthermore, at $X_1/d = 2$ the linear transfer by the mean velocity $\tilde{\Pi}_U^a$ and pressure term $\tilde{T}_{\tilde{p}}^a$ also contribute to eq. (4.17) as sink terms, potentially redistributing both in scale- and physical-space such that the energy injected by the mean flow production can be lost to the stochastic motions through $\mathcal{P}_u^a$. As the distance to the prism increases, the contributions of $\tilde{\Pi}_U^a$ and $\tilde{T}_{\tilde{p}}^a$ as well as those of the remaining terms in eq. (4.17) become individually negligible but contribute together to balance out the residual of $\tilde{\mathcal{A}}^a + \tilde{\Pi}_{\mathcal{P}_u}^a - \tilde{P}_U^a + \mathcal{P}_u^a$.

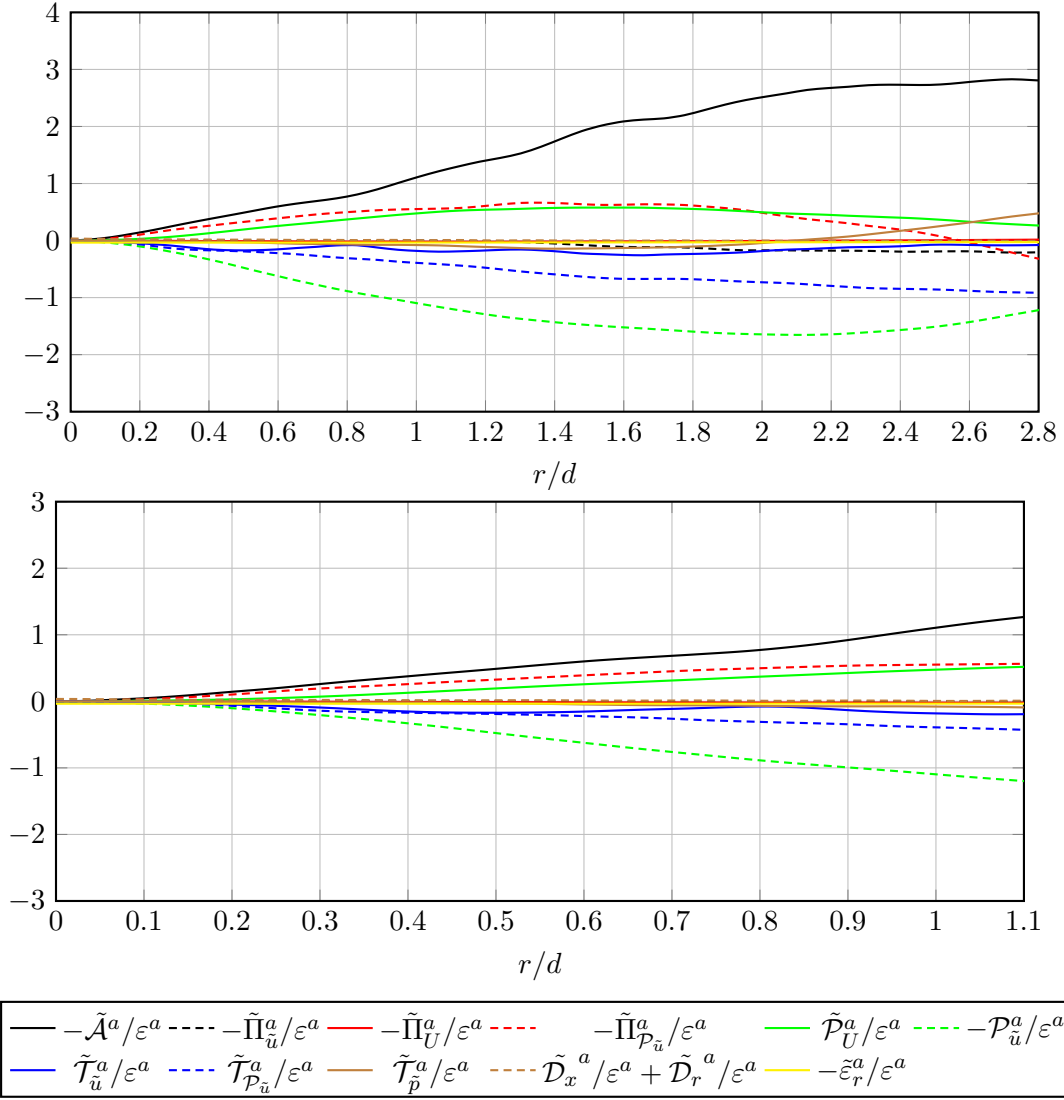


Figure 4.51: Orientation averaged terms of eq. (4.15) normalised by ε_r^a at $X_1/d = 8$. On the bottom the range of r/d is reduced.

Thus, figs. 4.50 and 4.51 suggest that very close to the prism ($X_1/d = 2$) $\langle \delta \tilde{q}^2 \rangle$ is produced by the mean flow inhomogeneity, the pressure and linear transfer terms redistribute the energy towards smaller scales where $\mathcal{P}'_u$ effectively feeds energy to the stochastic motions. Simultaneously, a large portion of the energy produced by the mean flow at very large scales is advected downstream through $\tilde{\mathcal{A}}$. Further downstream from the prism (at $X_1/d = 8$), the mean flow replenishes $\langle \delta \tilde{q}^2 \rangle$ to some extent via the production term, but this effect becomes less significant as the mean flow is much more homogeneous than at $X_1/d = 2$ (this also causes $\tilde{\Pi}_U$ to become smaller). Nevertheless, the production of $\langle \delta q'^2 \rangle$ is still the main sink of energy associated with the coherent motion which appears to be balanced by the energy arriving by the mean flow advection.

The budget shown in figs. 4.50 and 4.51 warrants a comment on the statistical convergence, which would be represented by the term $\tilde{\mathcal{A}}_t$ in eq. (4.17). Since the time derivatives were not stored at run-time, $\tilde{\mathcal{A}}_t$ must be inferred as the residual of eq. (4.17), this residual was found to always be an order of magnitude smaller than ε_r^a at small separations ($r/d < 1$) which is negligible in comparison to the remaining terms.

Having determined that only a few terms play an important role in the budget of $\langle \delta \tilde{q}^2 \rangle$ in the context of orientation averaged statistics, we turn now our attention to how these terms are distributed in scale space. As noted in chapter 3, while averaging out the orientations of $\mathbf{r}$ may highlight interesting properties of the scale-by-scale energy budget, the contribution of each individual term to such budget may differ drastically when its distribution in scale space is considered. Apart from a few terms in eq. (4.17), our focus will be on separations $r/d < 1$ for two main reasons: because statistical convergence deteriorates at larger separations but more importantly because it is at these separations where the non-linear inter-scale transfer attains the balance expressed by eq. (4.25).

4.3.4.1 Mean-Flow Advection and Linear Inter-Scale Transfer

In fig. 4.52 the advection by the mean flow and linear inter-scale transfer of $\langle \delta \tilde{q}^2 \rangle$ are shown at $X_1/d = 2$ on the $r_3 = 0$ plane. The distribution of $\tilde{\mathcal{A}}$ is roughly isotropic, similarly to $\mathcal{A}'$ seen in the previous section, but in fact represents the largest contribution to $\mathcal{A}$ shown in fig. 3.18 (recall that $\mathcal{A}$ is equal to $\tilde{\mathcal{A}} + \mathcal{A}'$). Conversely, the distribution of $\tilde{\Pi}_U$ in scale space shows the effect of mean flow inhomogeneity in re-distributing $\langle \tilde{q}^2 \rangle$ in scale space (as a sink for separations near the r_1 axis and a source for separations near the r_2 axis). Recall from figs. 4.5 and 4.7 that the cross section of the coherent structures appears to change from a circular shape within the formation region to a more elliptical one further downstream. The same terms are plotted in fig. 4.53 further downstream (at $X_1/d = 8$). The linear transfer term becomes practically negligible at most orientations given that the mean flow tends to become more homogeneous as the distance to the prism increases. The advection term on the other hand decreases in magnitude and is no longer isotropically distributed; fig. 4.53 shows that more energy is fed to $\langle \delta \tilde{q}^2 \rangle$ by $\tilde{\mathcal{A}}$ at orientations aligned with r_1 than at other orientations.

4.3.4.2 Inter-Scale Transfer by the Coherent and Stochastic Motions

The two remaining terms associated with inter-scale transfer, $\tilde{\Pi}_{\tilde{u}}$ and $\tilde{\Pi}_{\mathcal{P}_{\tilde{u}}}$, are shown in figs. 4.54 and 4.55 for $X_1/d = 2$ and $X_1/d = 8$, respectively. Even though $\tilde{\Pi}_{\tilde{u}}^a$ was seen to be negligible when averaging out the orientations of $\mathbf{r}$ (see section 4.3.2) its local (in scale space) contribution to eq. (4.15) appears to be of the order of ε_r at $X_1/d = 2$, its

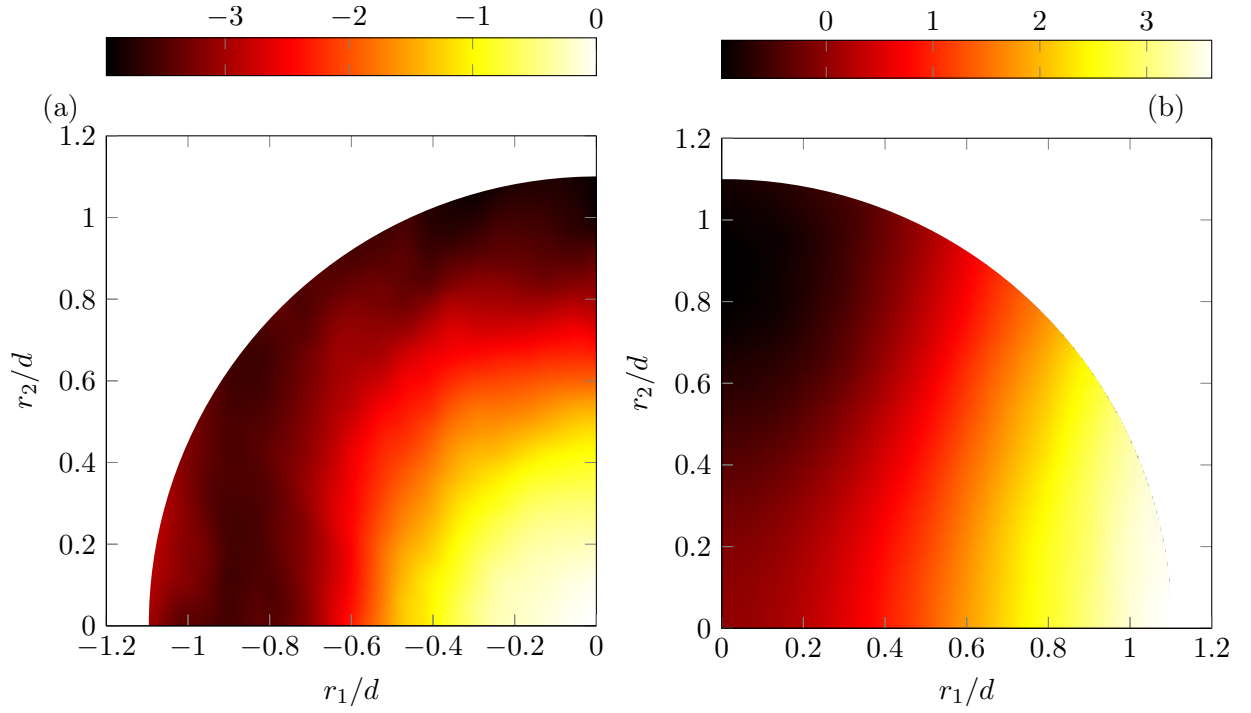


Figure 4.52: Distribution of $\tilde{\mathcal{A}}$ and $\tilde{\Pi}_U$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

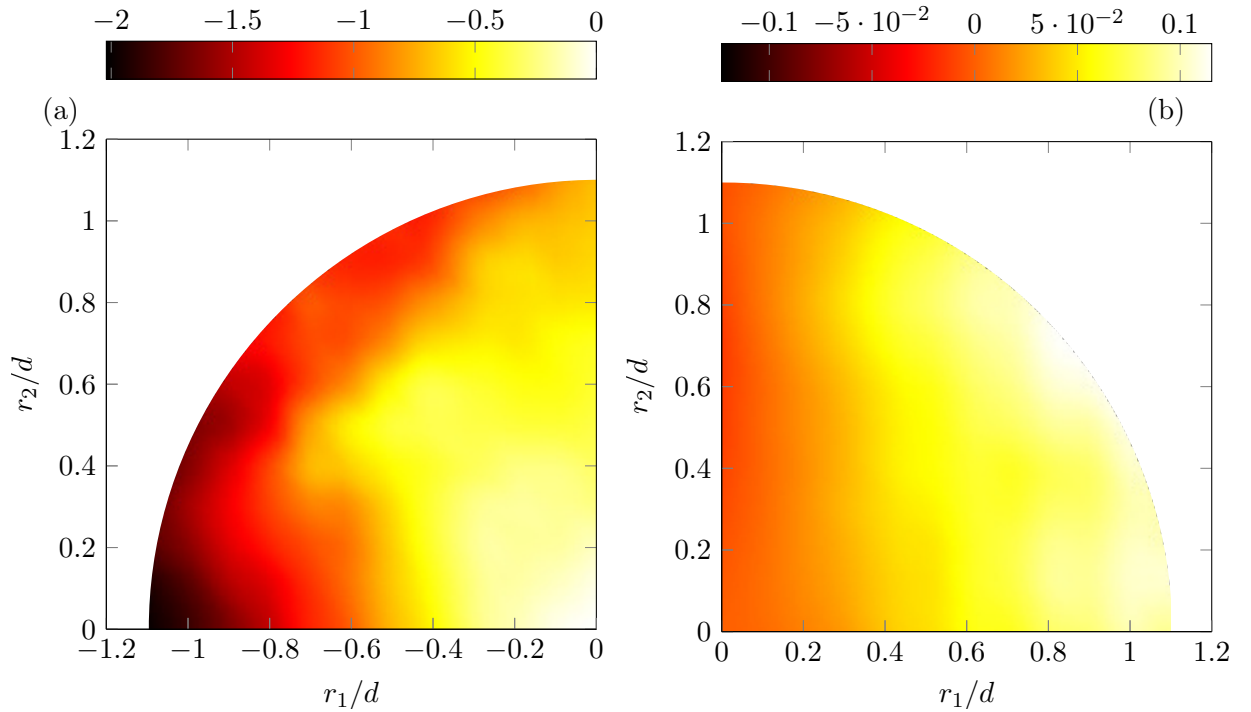


Figure 4.53: Distribution of $\tilde{\mathcal{A}}$ and $\tilde{\Pi}_U$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

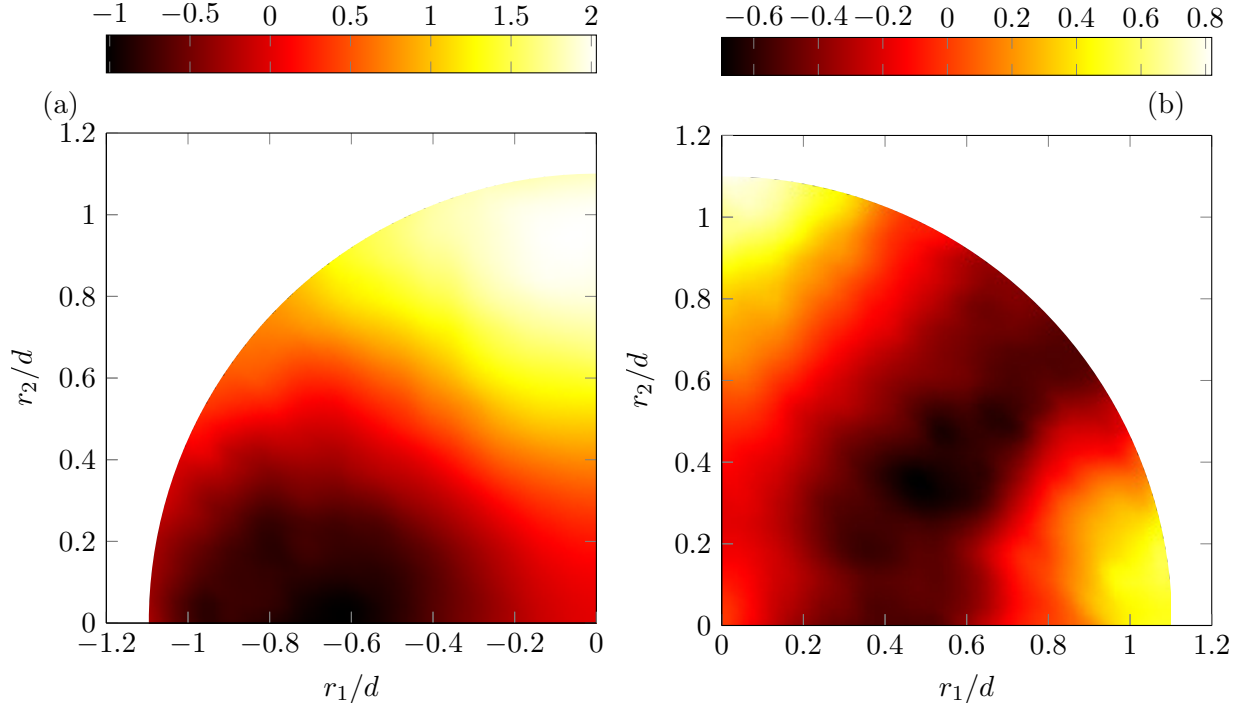


Figure 4.54: Distribution of $\tilde{\Pi}_{\bar{u}}$ and $\tilde{\Pi}_{\mathcal{P}_{\bar{u}}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

magnitude decreases slightly at $X_1/d = 8$ but the orientations at which $\tilde{\Pi}_{\bar{u}}$ acts as a source or sink remains the same. It is not unreasonable to assume that $\tilde{\Pi}_{\bar{u}}$ reflects the deformation undergone by the coherent structures, the fact that $\tilde{\Pi}_{\bar{u}}^a$ is approximately 0 means that whatever energy is lost at a particular separation (by means of the non-linear interactions of the coherent motions) is fed at some other orientation of $\mathbf{r}$ with the same magnitude, i.e. there is only a redistribution of energy.

With regard to the mixed term $\tilde{\Pi}_{\mathcal{P}_{\bar{u}}}$, recall that this term reflects the inter-scale transfer of the total energy by the stochastic motion as noted above eq. (4.20). While it is hard to provide a physical interpretation of $\tilde{\Pi}_{\mathcal{P}_{\bar{u}}}$ on its own, this term reflects how the stochastic motions interact with the coherent motions. Thus, we just comment on the peculiar distributions of $\tilde{\Pi}_{\mathcal{P}_{\bar{u}}}$ at $X_1/d = 2$ and $X_1/d = 8$ (shown in figs. 4.54 and 4.55) where the nature of $\tilde{\Pi}_{\mathcal{P}_{\bar{u}}}$ shifts from being a sink to a source at specific orientations at $X_1/d = 2$ but then at $X_1/d = 8$ becomes a sink of $\langle \delta \tilde{q}^2 \rangle$ at practically all orientations.

4.3.4.3 Pressure Correlations and Production by the Mean-Flow

Figures 4.56 and 4.57 shows how $\tilde{T}_{\bar{p}}$ and $\tilde{\mathcal{P}}_U$ are distributed in scale space at $X_1/d = 2$ and $X_1/d = 8$. Both terms undergo significant changes as the distance to the prism is increased. In particular, $\tilde{T}_{\bar{p}}$ acts as a sink at all orientations but its largest values appear to be especially concentrated at large r close to the r_1 and r_2 axis while at $X_1/d = 8$ it

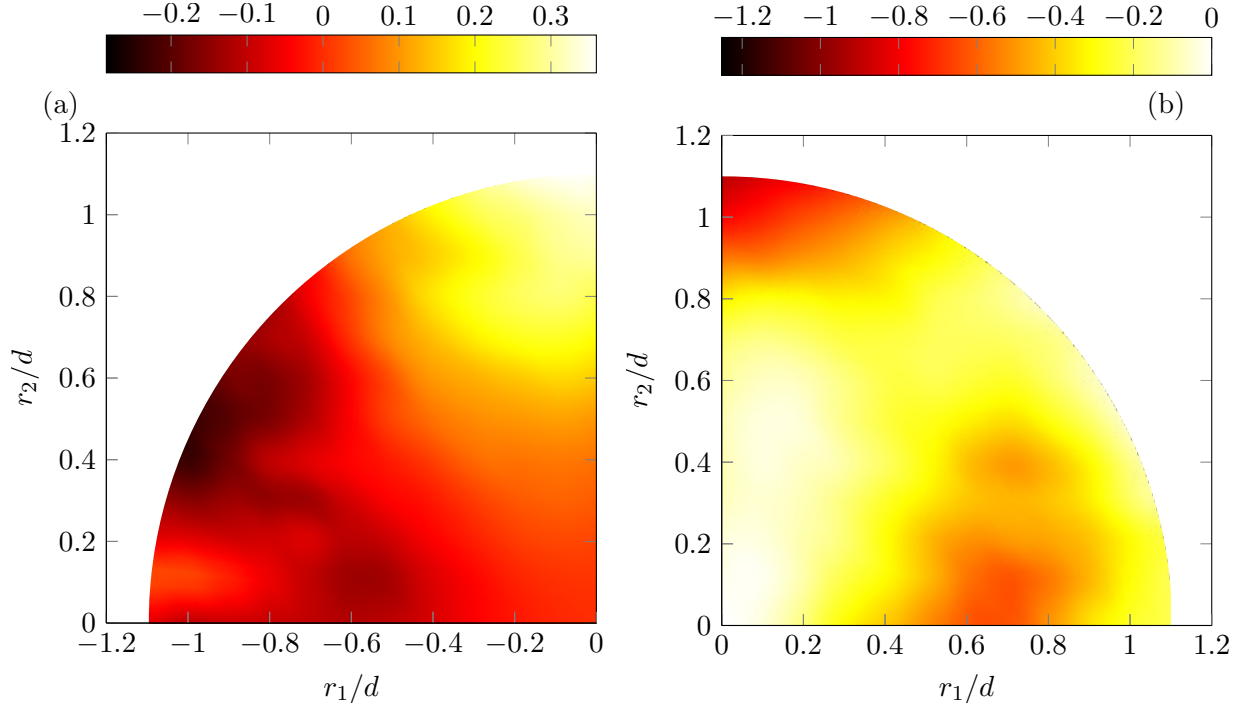


Figure 4.55: Distribution of $\tilde{\Pi}_{\tilde{u}}$ and $\tilde{\Pi}_{\mathcal{P}_{\tilde{u}}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

acts as a source of energy near the r_1 axis. Notice that the distribution of $\tilde{\mathcal{T}}_{\tilde{p}}$ in scale space correlates with that of $\mathcal{T}'_p$ shown in figs. 4.47b and 4.48b. While a physical interpretation of the pressure terms is not obvious, this correlation raises the possibility that the cascades of energy associated with the coherent and stochastic motions are able to exchange information via the pressure correlation term.

Let us now focus on the other term shown in figs. 4.56 and 4.57. The production of $\langle \delta \tilde{q}^2 \rangle$ by the mean flow, $\tilde{\mathcal{P}}_U$ has an almost identical distribution as $\mathcal{P}$ (recall figs. 3.20 and 3.21) at both locations; this suggests that the production of $\langle \delta q^2 \rangle$ by the mean flow occurs mostly due to the interaction between the mean flow and the coherent motions, rather than between the mean flow and the stochastic motions. As noted in section 3.3.2, the large concentration of $\tilde{\mathcal{P}}_U$ near the r_1 axis can be attributed to the large gradient of U_1 in the streamwise direction.

4.3.4.4 Inter-Space Transport by the Coherent and Stochastic Motions

Finally, the transport terms $\tilde{\mathcal{T}}_{\tilde{u}}$ and $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ (recall that $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ is the physical space analogue of $\tilde{\Pi}_{\mathcal{P}_{\tilde{u}}}$) are shown in figs. 4.58 and 4.59 for $X_1/d = 2$ and $X_1/d = 8$. These figures show that both $\tilde{\mathcal{T}}_{\tilde{u}}$ and $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ are relatively small in absolute value when compared to the terms discussed above. Furthermore, we can see that these terms are much smaller than $\mathcal{T}'_{\tilde{u}}$ which was shown in the previous section to play a significant role in the budget of $\langle \delta q'^2 \rangle$.

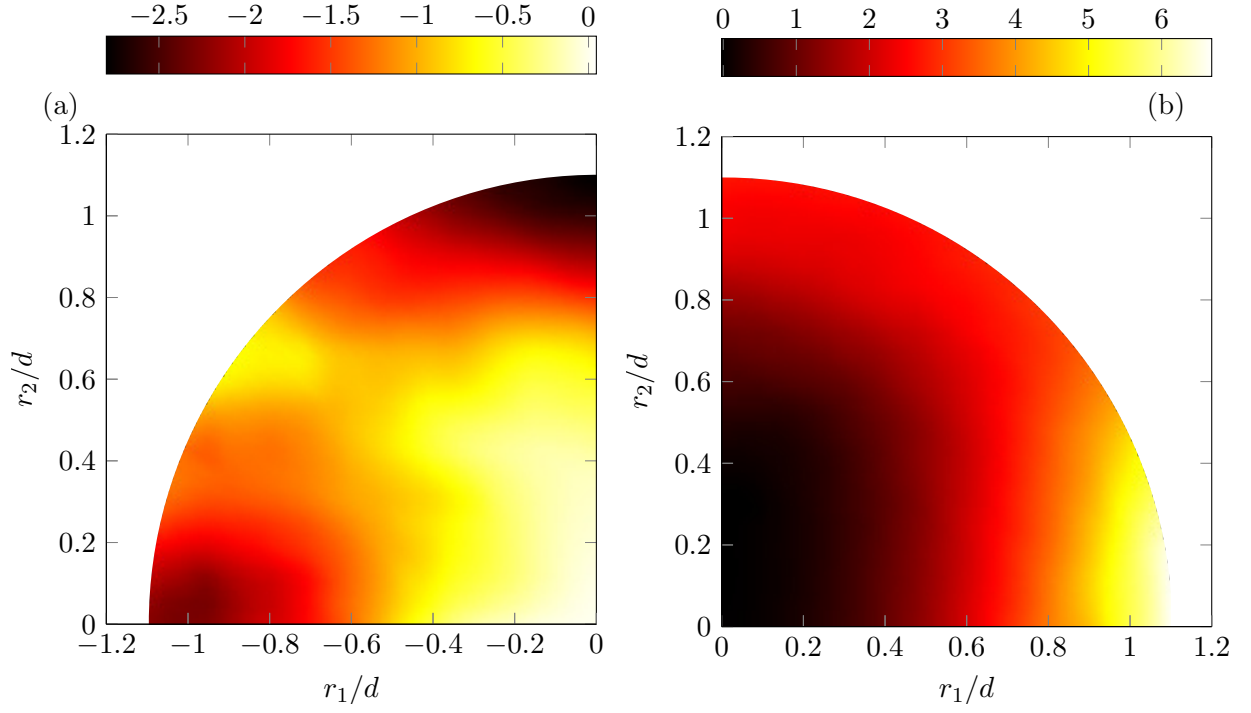


Figure 4.56: Distribution of $\tilde{\mathcal{T}}_{\tilde{p}}$ and $\tilde{\mathcal{P}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

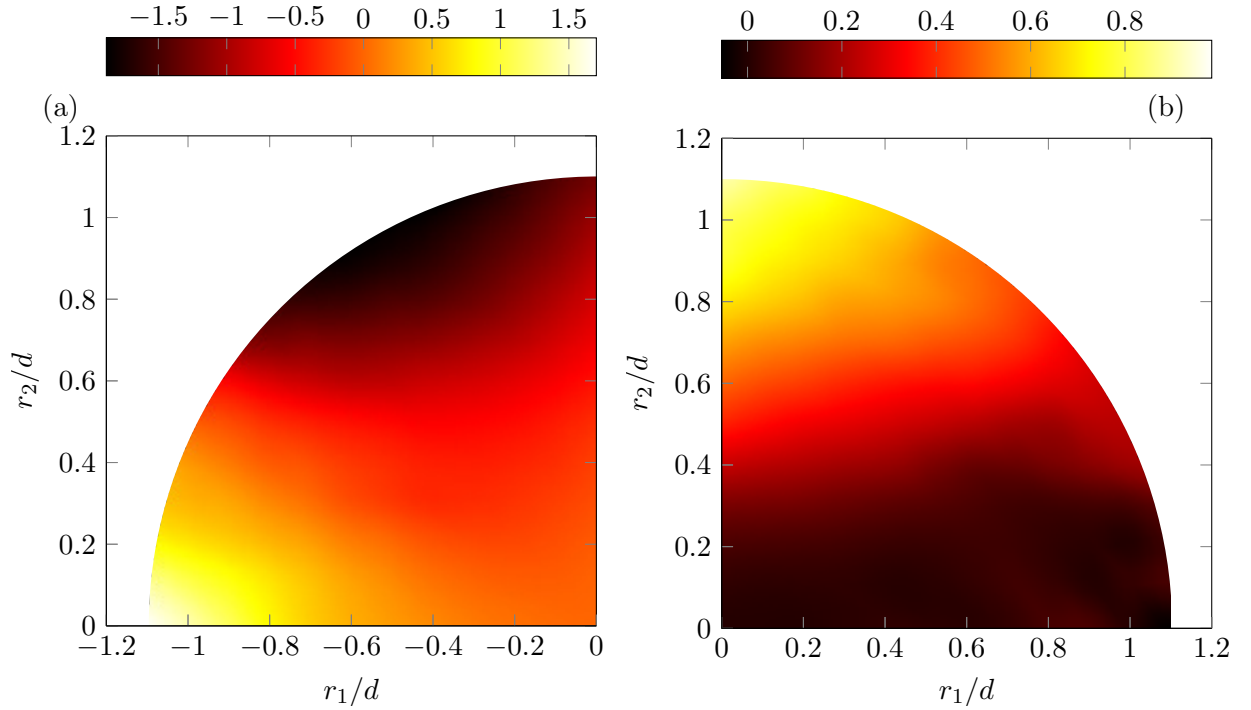


Figure 4.57: Distribution of $\tilde{\mathcal{T}}_{\tilde{p}}$ and $\tilde{\mathcal{P}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

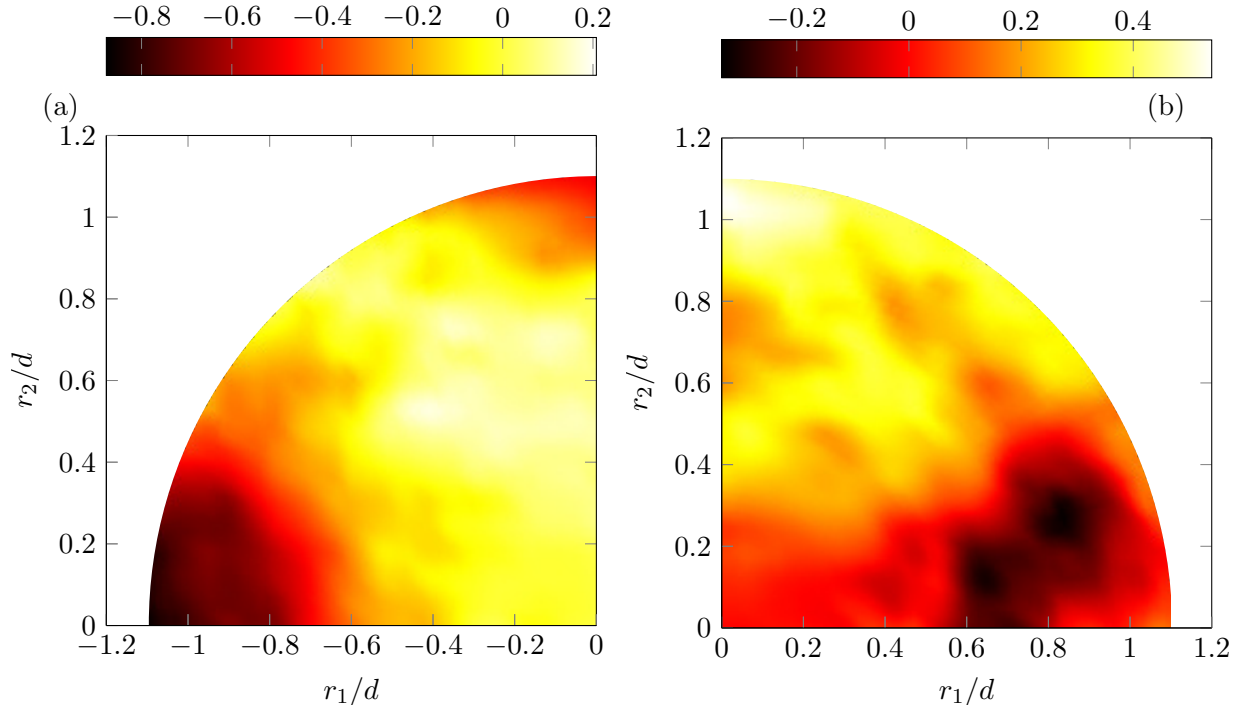


Figure 4.58: Distribution of $\tilde{\mathcal{T}}_{\tilde{u}}$ and $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 2$ in (a) and (b), respectively.

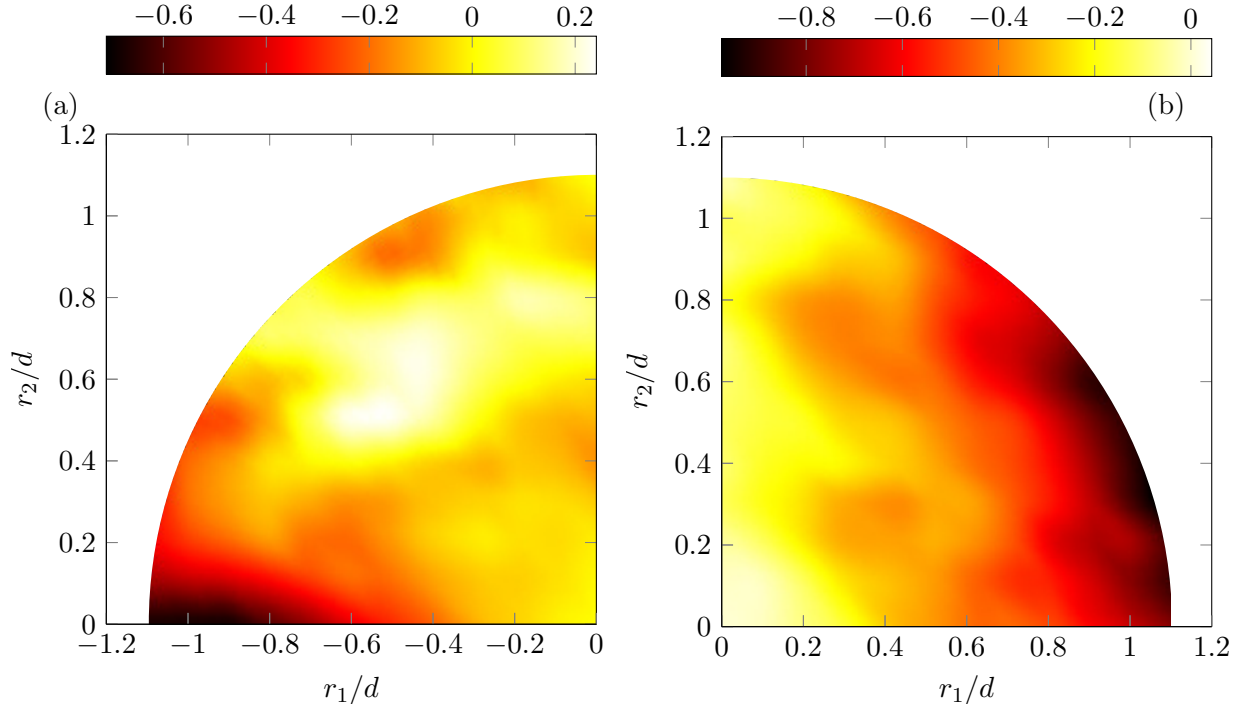


Figure 4.59: Distribution of $\tilde{\mathcal{T}}_{\tilde{u}}$ and $\tilde{\mathcal{T}}_{\mathcal{P}_{\tilde{u}}}$ normalised by ε_r in scale space on the geometrical centreline at $X_1/d = 8$ in (a) and (b), respectively.

4.4 A Model for the Phase-component that Reproduces its Two-point Dynamics

As emphasised by Thiesset, Danaïla, and Antonia (2014), modelling the phase-averaged properties of the flow is important not just because it represents a large portion of the fluctuating kinetic energy but also due to its ubiquitous presence in flows past bluff bodies. A first step in deriving such a model is given by those same authors under the form

$$\delta\tilde{u}_2^2/\langle\tilde{u}_2^2\rangle = 8\sin^2(\phi)\sin^2\left(\pi\frac{r_1}{\mathcal{L}_v}\right) \quad (4.26)$$

where $\mathcal{L}_v$ is the distance between successive coherent vortices. The model allows one to compute the second order structure function $\delta\tilde{u}_2^2$ as a function of phase angle and separation in the streamwise direction. The fact that eq. (4.26) can only be used along r_1 relates to the fact that Thiesset, Danaïla, and Antonia (2014) carried hot-wire measurements and were therefore unable to investigate other orientations of $\mathbf{r}$. Furthermore, they acknowledge that the accuracy of eq. (4.26) deteriorates with decreasing distance to the wake generator and therefore we should not expect our data to provide a reasonable fit.

Given the relevance of modelling the phase-averaged statistics noted above and the limitations of eq. (4.26), there is interest in investigating a model for these statistics valid for different orientations of $\mathbf{r}$. We propose that expressing the phase averaged velocity field by means of elliptical viscous vortices tilted with respect to the free-stream direction yield (in a qualitative sense) similar distributions of $\langle\delta\tilde{q}^2\rangle$ and $\langle\delta\tilde{\mathbf{u}}\delta\tilde{q}^2\rangle$ in scale space as those observed in section 4.3 and explains the orientations of the energy fluxes associated with the coherent motions (recall fig. 4.36). In fact, it may also help interpret the outward energy fluxes occurring at separations aligned with the free-stream direction found in Gomes-Fernandes et al. (2015) given that the peaks in their spectra suggest that coherent structures may be present.

4.4.1 Representing the Phase-averaged Velocity Field

As seen in section 1.2, there are similarities between the Kármán vortex street (present only for sub-critical Re_∞) and the velocity field obtained at high Re_∞ . In fact, as noted in von Kármán (1963): “When the velocity is further increased, the periodical vortex shedding still persists, but the beautiful regular pattern no longer exists”. One may thus expect that the point vortex model used in von Kármán (1911) and von Kármán (1912) to describe the Kármán street cannot be expected to hold in a viscous fluid at high Re_∞ . For this reason, Domm (1954) made use of the Oseen vortex model in the context of stability

theory for vortex streets in viscous flows. Introduced by Oseen (1915), the azimuthal velocity of the vortex is given by

$$||\mathbf{u}|| = \frac{\Gamma}{2\pi r} \left[1 - \exp\left(-\frac{r^2}{4r_v^2}\right) \right] \quad (4.27)$$

where Γ is the circulation and r_v is a viscous core radius (which may be a function of time), such that for $r \gg r_v$ eq. (4.27) leads to a potential vortex.

The success (or lack thereof) of the approach taken by Domm (1954) in the context of stability of the vortex street is discussed at length in Wille (1960). Here, however, we are concerned with representing $\tilde{u}$ as close as possible which is why eq. (4.27) is used as a starting point (instead of a potential vortex model). Let us superimpose the velocity fields associated with an infinite array of such vortices such that

$$u_1 = \sum_{v=-\infty}^{\infty} \frac{\mu_v}{\sqrt{1 + \left(\frac{r_{v1}}{r_{v2}}\right)^2}} \text{sgn}(r_{v2}) \quad (4.28a)$$

$$u_2 = \sum_{v=-\infty}^{\infty} \frac{-\mu_v}{\sqrt{1 + \left(\frac{r_{v2}}{r_{v1}}\right)^2}} \text{sgn}(r_{v1}) \quad (4.28b)$$

$$\mu_v = (-1)^{v+1} \frac{\Gamma}{2\pi} \frac{1}{r_v} \left[1 - \exp\left(-\frac{r_v^2}{4r_v^2}\right) \right], \quad \mathbf{r}_v = \mathbf{x} - \xi_v, \quad r_v = ||\mathbf{r}_v||, \quad (4.28c)$$

where the subscript v is used to distinguish successive vortices and each vortex is centred at ξ_v such that $\xi_{(v+2)1} - \xi_{v1} = b$ is the distance between vortex cores of equal circulation and $\xi_{v2} = a \cdot \text{sgn}(\Gamma)$, where a is the distance of the vortex core to the centreline along x_2 .

Notice that if r_v is constant, the velocity field generated by eq. (4.28) represents a frozen array of vortices travelling along x_1 . Thus, taking statistics based on this velocity field requires taking averages along x_1 , i.e. translations in x_1 correspond translations along the the phase ϕ .

Unfortunately, one can immediately see that, by construction, eq. (4.28) cannot lead to the direct and inverse fluxes associated with the coherent motions observed in section 4.3.2.2. Consider, for example, a pair of points lying on the x_1 axis: as one integrates over a period of the phase, the presence of vortices of alternating circulation leads to the consecutive increment and cancellation of any contributions to $\delta\tilde{u}_1\delta\tilde{q}^2$. This is a consequence of the translation symmetry of eq. (4.28) with respect to the phase given that translations along x_1 are identical to translations along $-x_1$ when the sign of Γ in eq. (4.28c) is changed. Notice that even though the phase averaged lift coefficient, appears to satisfy this symmetry, the phase averaged drag coefficient does not, as seen clearly in

fig. 4.3. As we will see, the introduction of an asymmetry along the phase gives rise to the locally direct and inverse fluxes in $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$, similar to what was observed in section 4.3.2.2.

Inspection of fig. 4.7 hints to one possible way of introducing an asymmetry to eq. (4.28): the coherent structures depicted in that figure appear to have a near-elliptical shape with their major axis lying at a small angle with respect to x_2 . For structures with positive circulation, this angle is negative (in the x_1, x_2 plane) and equivalently for structures of negative circulation. One possible way to construct such a velocity field is to prescribe two different values of r_ν in eq. (4.27), say $r_{\nu M}$ and $r_{\nu m}$ for the major and minor axes, respectively such that an elliptical velocity distribution is obtained. Let us define the coordinates r_1^* and r_2^* along the major and minor axes of an ellipse, we can re-write eq. (4.27) as

$$\|\mathbf{u}\| = \frac{\Gamma}{2\pi} \sqrt{\frac{1}{\left(\frac{r_1^*}{r_{\nu M}}\right)^2} + \left(\frac{r_2^*}{r_{\nu m}}\right)^2} \frac{1}{r_{\nu M} r_{\nu m}} \left[1 - \exp\left(-\frac{r_1^{*2}}{4r_{\nu M}^2} - \frac{r_2^{*2}}{4r_{\nu m}^2}\right) \right]. \quad (4.29)$$

Notice that eq. (4.29) reduces to eq. (4.27) when $r_{\nu M}$ and $r_{\nu m}$ are equal. Equation (4.29) is a crude attempt to reproduce the biaxially strained vortices obtained (in a different context) by Moffatt et al. (1994). As in eq. (4.28), we can now construct a velocity field by superposing the velocities associated with each vortex

$$u_1 = \sum_{v=-\infty}^{\infty} u_{v1}^* \cos(-\alpha_v) - u_{v2}^* \sin(-\alpha_v), \quad \alpha_v = (-1)^{v+1} \alpha \quad (4.30a)$$

$$u_2 = \sum_{v=-\infty}^{\infty} u_{v1}^* \sin(-\alpha_v) + u_{v2}^* \cos(-\alpha_v) \quad (4.30b)$$

where α is the angle between x_1 and the major axis of the vortices with positive circulation, u_{v1}^*, u_{v2}^* are the velocity components of the v -th vortex given by

$$u_{v1}^* = \frac{\mu_v}{\sqrt{1 + \left(\frac{r_{v1}^*}{r_{\nu M}} \frac{r_{\nu m}^2}{r_{\nu M}^2}\right)^2}} \text{sgn}(r_{v2}^*) \quad (4.31a)$$

$$u_{v2}^* = \frac{-\mu_v}{\sqrt{1 + \left(\frac{r_{v2}^*}{r_{\nu m}} \frac{r_{\nu M}^2}{r_{\nu m}^2}\right)^2}} \text{sgn}(r_{v1}^*) \quad (4.31b)$$

$$\mu_v = (-1)^{v+1} \frac{\Gamma}{2\pi} \sqrt{\frac{1}{\left(\frac{r_{v1}^*}{r_{\nu M}}\right)^2} + \left(\frac{r_{v2}^*}{r_{\nu m}}\right)^2} \frac{1}{r_{\nu M} r_{\nu m}} \left[1 - \exp\left(-\frac{r_{v1}^{*2}}{4r_{\nu M}^2} - \frac{r_{v2}^{*2}}{4r_{\nu m}^2}\right) \right] \quad (4.31c)$$

and the coordinates r_{v1}^*, r_{v2}^* are computed through

$$r_{v1}^* = (x_1 - \xi_{v1}) \cos(\alpha_v) - (x_2 - \xi_{v2}) \sin(\alpha_v) \quad (4.32a)$$

$$r_{v2}^* = (x_1 - \xi_{v1}) \sin(\alpha_v) + (x_2 - \xi_{v2}) \cos(\alpha_v). \quad (4.32b)$$

The parameters that appear in eq. (4.30) were tuned empirically such that the velocity field obtained (after removing the mean velocity) is qualitatively similar to $\tilde{\mathbf{u}}$ as shown in fig. 4.60 in the vicinity of $x_1/d = 8$. The values used here were $a = 0.4$, $b = 6$, $\Gamma = \pi \frac{3}{2}$, $r_{\nu M} = b \frac{1}{10}$, $r_{\nu m} = r_{\nu M} \frac{3}{4}$ and $\alpha = 0.9 \frac{\pi}{2}$. The choice of representing the velocity field in the vicinity of $x_1/d = 8$ is justified by the fact that the model cannot represent the spatial inhomogeneity of the phase averaged velocity field (as we have assumed a frozen array of vortices travelling along x_1). This inhomogeneity is particularly significant at shorter distances to the prism decreases (see fig. 4.60). Effectively this means that we cannot expect to represent any behaviour that does not satisfy the equivalence between the phase angle (or time) and streamwise position, such as the deformation undergone by the coherent structures. Nevertheless, if we were interested in representing $\tilde{\mathbf{u}}$ closer to the prism, the radii $r_{\nu m}$ and $r_{\nu M}$ would have to be smaller and α would have to be larger.

Let us further identify the role of the different parameters in the model by comparing the velocity signals along the phase, both from the DNS and from eq. (4.30). As shown in fig. 4.61, as consequence of the reflection symmetry about the centreline, the cross-stream component of $\tilde{\mathbf{u}}$ displays a sine-like evolution with respect to ϕ . The streamwise component, also shown in fig. 4.61, displays broader peaks than valleys, this can be interpreted as a consequence of the coherent structures not being perfectly circular. This is shown in fig. 4.61 where $\tilde{u}_1$ was computed from eq. (4.30) by setting $r_{\nu m} = r_{\nu M}$ (circle superscript) and $r_{\nu m} = r_{\nu M} \frac{3}{4}$ with $\alpha = \frac{\pi}{2}$ (ellipse superscript). Setting α slightly smaller than $\frac{\pi}{2}$ causes the peaks of $\tilde{u}_1$ to become slightly left-side lobed.

Nevertheless, as will be shown in the following section, the local (in scale space) direct and inverse fluxes observed in section 4.3.2.2 for the phase averaged velocity can be explained solely by the fact that $r_{\nu m} \neq r_{\nu M}$ and α not being multiple of $\frac{\pi}{2}$. We can speculate that any other (more elaborate) vortex model capable of representing the asymmetry/anisotropy inherent to the structures observed in figs. 4.7 and 4.60 would lead to the same conclusions.

4.4.2 Two-point Properties of the Model

As a first step, it is important to assess to what extent the present approach complies with the model developed by Thiesset, Danaila, and Antonia (2014) for $\langle \delta \tilde{u}_2^2 \rangle$. As shown in fig. 4.62, the second-order structure function of the cross stream phase averaged velocity component obtained from eq. (4.30) is practically indistinguishable from that predicted

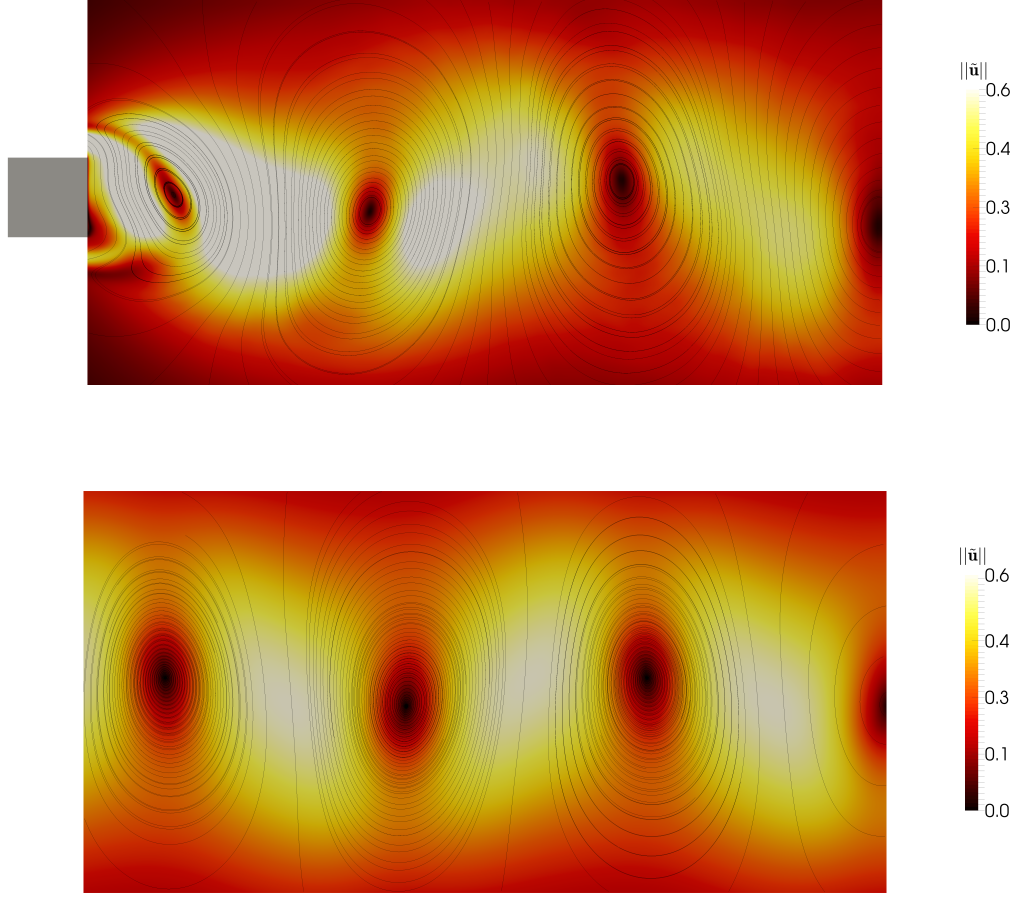


Figure 4.60: Phase averaged velocity field from the DNS at an arbitrary phase (top) and the velocity field given by eq. (4.31) with $a = 0.2$, $b = 6$, $\Gamma = \pi \frac{3}{2}$, $r_{\nu M} = b \frac{1}{10}$, $r_{\nu m} = r_{\nu M} \frac{3}{4}$ and $\alpha = 0.9 \frac{\pi}{2}$ (bottom). The black lines indicate streamlines placed at arbitrary locations. The DNS contour extends from $x_1/d = 0.5$ to $x_1/d = 10.5$ and from $x_2/d = -2.5$ to $x_2/d = 2.5$.

by eq. (4.26). Both estimates, however, appear to equally over-predict the profile of $\langle \delta \tilde{u}_2^2 \rangle / \langle \tilde{u}_2^2 \rangle$ along r_1 obtained from the DNS ($\langle \tilde{u}_2^2 \rangle$ was taken at the mid-point $X_1/d = 8$). This difference is most likely due to spatial inhomogeneity of the DNS data given the decay of $\langle \tilde{u}_2^2 \rangle$ in the streamwise direction but also because the distance between consecutive vortices (that appears in eqs. (4.26) and (4.30)) may be ill-defined if the coherent structures are convected at a non-uniform velocity along x_1 (see e.g. Y. Zhou and Antonia, 1992).

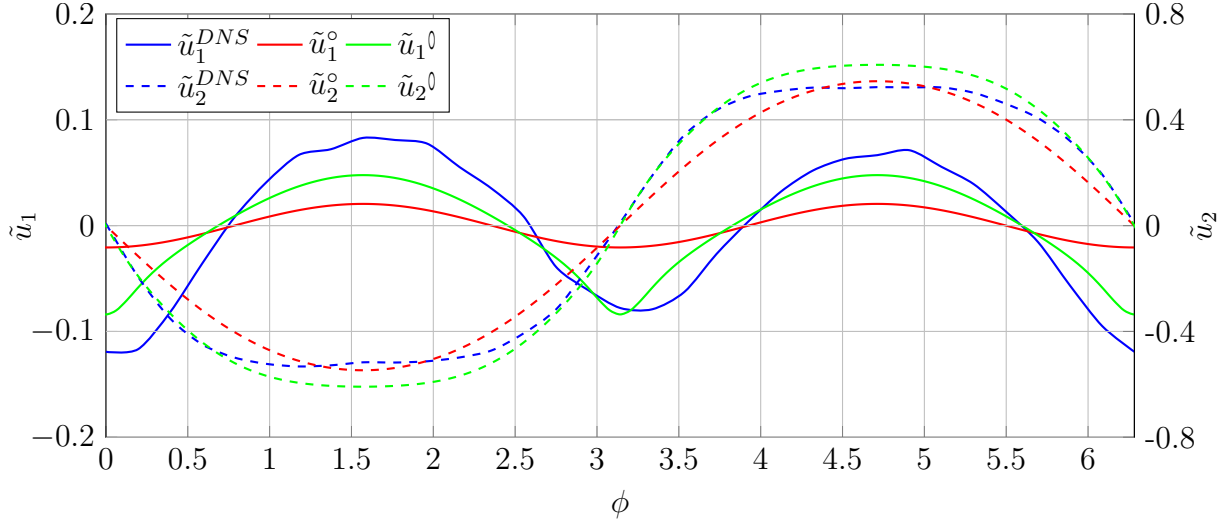


Figure 4.61: Signals of $\tilde{u}_1$ and $\tilde{u}_2$ along the phase ϕ . The superscript *DNS* indicates data obtained from the DNS at $x_1/d = 8$, the circle superscript indicates data generated from eq. (4.31) with $r_{\nu m} = r_{\nu M}$ and the ellipse superscript indicates data generated from eq. (4.31) with $r_{\nu m} = r_{\nu M} \frac{3}{4}$ (and $\alpha = 0$).

The distribution of $\langle \delta \tilde{q}^2 \rangle$ in scale-space computed from eq. (4.30), shown in fig. 4.63, also compares favourably with that obtained from the DNS (recall fig. 4.28b). The peak of $\langle \delta \tilde{q}^2 \rangle$ falls on the r_1 axis at separations close to $b/2$, as noted by Thiesset, Danaila, and Antonia (2014). Naturally, changing some of the parameters in eq. (4.30) leads to slightly different distributions of $\langle \delta \tilde{q}^2 \rangle$ in scale space, for example, the peak value of $\langle \delta \tilde{q}^2 \rangle$ becomes smaller as $r_{\nu m}$ approaches $r_{\nu M}$, as shown in fig. 4.63.

While eq. (4.30) was found to always lead to distributions of $\langle \delta \tilde{q}^2 \rangle$ in scale space not strikingly different than those obtained from the DNS, the third order structure functions $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ were found to be sensitive to both the ratio between $r_{\nu m}$ and $r_{\nu M}$ and, more importantly, the angle α . In fig. 4.64 the distribution of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ in scale space, computed from eq. (4.30), is compared to that obtained from the DNS**. In addition to the DNS data, three different instances of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ computed from eq. (4.30) are shown in fig. 4.64, one using the same parameters as those used in fig. 4.60, another where $\alpha = 0.99 \frac{\pi}{2}$ (instead of $\alpha = 0.9 \frac{\pi}{2}$) and one where $\alpha = 1.01 \frac{\pi}{2}$. The later configuration (fig. 4.64d), albeit not observed in our DNS data, represents a (hypothetical) case where elliptical-shaped coherent structures have their major axis at an angle larger than $\frac{\pi}{2}$ with respect to x_2 and illustrates how it is both a combination of the shape and orientation of the coherent

** In section 4.3.2.2 we have noted that the statistical convergence deteriorates at large separations and as the distance to the prism increases (see also Appendix A in Alves Portela et al., 2017), this is especially true for odd order statistics. For the sake of comparing the distributions of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ obtained from the DNS with the distributions of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ obtained from eq. (4.30) (which does not suffer from lack of statistical convergence) we have enforced the reflection symmetry such that $\langle \delta \tilde{u}_1 \delta \tilde{q}^2 \rangle(r_1, r_2) = -\langle \delta \tilde{u}_1 \delta \tilde{q}^2 \rangle(-r_1, r_2)$ and $\langle \delta \tilde{u}_2 \delta \tilde{q}^2 \rangle(r_1, r_2) = -\langle \delta \tilde{u}_2 \delta \tilde{q}^2 \rangle(r_1, -r_2)$.

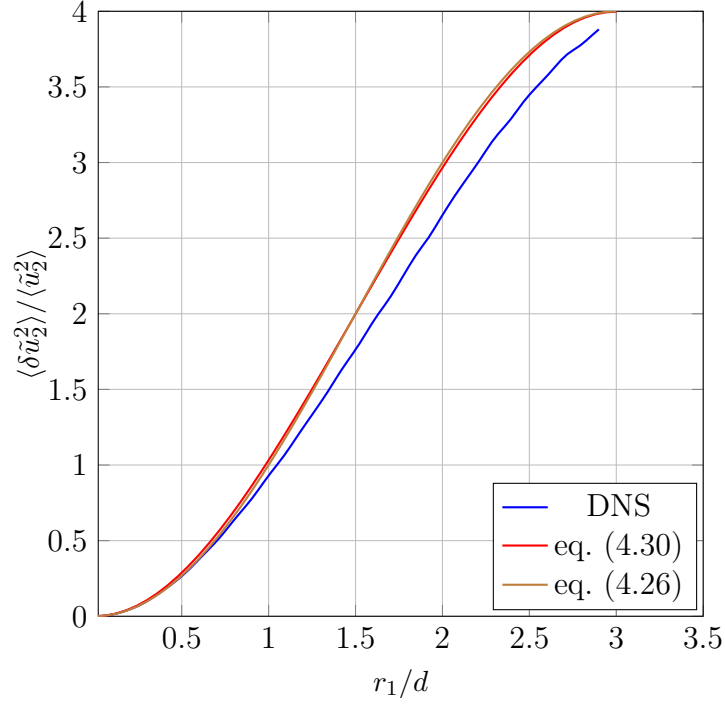


Figure 4.62: $\langle \delta \tilde{u}_2^2 \rangle / \langle \tilde{u}_2^2 \rangle$ computed from the DNS (at $X_1/d = 8$), from the infinite array of elliptical vortices (eq. (4.30)) and eq. (4.26) introduced in Thiesset, Danaila, and Antonia (2014).

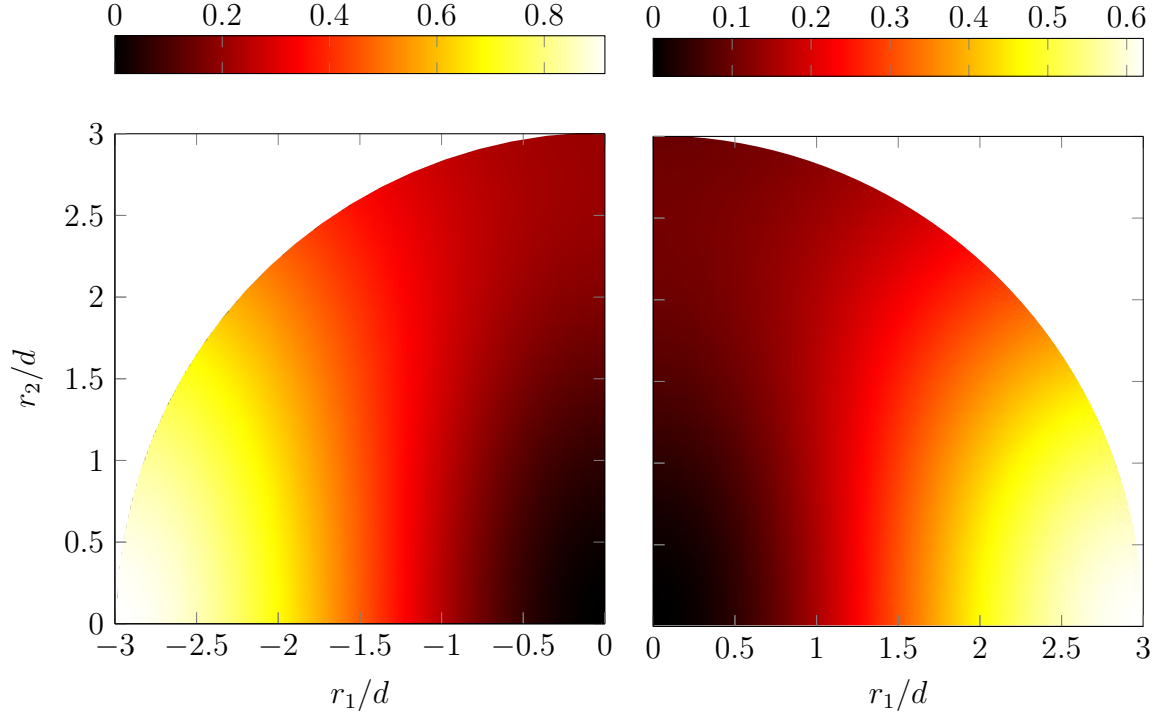


Figure 4.63: Generalised second-order structures functions $\langle \delta \tilde{q}^2 \rangle$ obtained through eq. (4.30). On the left, $r_{\nu m} = \frac{3}{4}r_{\nu M}$ and on the right $r_{\nu m} = r_{\nu M}$.

structures that leads to the appearance of local (in scale-space) direct and inverse fluxes associated with the coherent motions.

Inspection of fig. 4.64a and figs. 4.64b and 4.64c reveals that the fluxes $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ are directed from small to large separations at orientation close to the r_1 axis while the reverse is observed at orientations close to the r_2 axis. In fact, once the angle α is increased above the value of $\frac{\pi}{2}$ (even if only slightly), as shown in fig. 4.64d the orientations of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ are reversed. Furthermore, notice that a critical point can be identified in all cases at relatively large separations, in fig. 4.64a the critical point is observed at $|r_1| \approx 1.75d$, $|r_2| \approx 2.25d$ while in figs. 4.64b to 4.64d is observed at $|r_1| \approx 1.5d$, $|r_2| \approx 1.5d$. In fact, the location of this critical point in scale space was found to vary as the ratio $r_{\nu M}/r_{\nu m}$ was changed. Finally, notice that the magnitude of $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ decreases as α approaches $\frac{\pi}{2}$, as seen by comparing fig. 4.64b with fig. 4.64c.

Despite the empirical model used here to represent the phase averaged velocity field being relatively crude, it has proven useful in interpreting the two-point properties of the phase averaged velocity discussed in previous sections. In particular, the concentration of $\langle \delta \tilde{q}^2 \rangle$ along the r_1 axis is associated with the alternating vortices travelling approximately along the x_1 axis and thus introducing large variations in $\tilde{u}_2$ in regions in-between vortices (recall fig. 4.61); this observation is consistent with the measurements of Thiesset, Danaila, and Antonia (2014) and in fact leads to similar predictions as the model of $\langle \delta \tilde{u}_2^2 \rangle / \langle \tilde{u}_2^2 \rangle$ developed by those authors. While the actual shape of the coherent structures was found that have little influence on the energy distribution in scale-space, their distortion along a preferential axis leads to locally direct and inverse fluxes in scale space when the axes are not aligned with the coordinate system $\mathbf{x}$. This model, based on representing the phase averaged velocity by the superposition of ellipse-like structures, shows eccentricity and misalignment of the coherent structures (introduced in the vortex formation region) acts as a memory of initial conditions: as the distance to the prism increases, these structures become less eccentric and/or more aligned with the coordinate system $\mathbf{x}$ thus reducing their imprint in the total inter-scale energy flux.

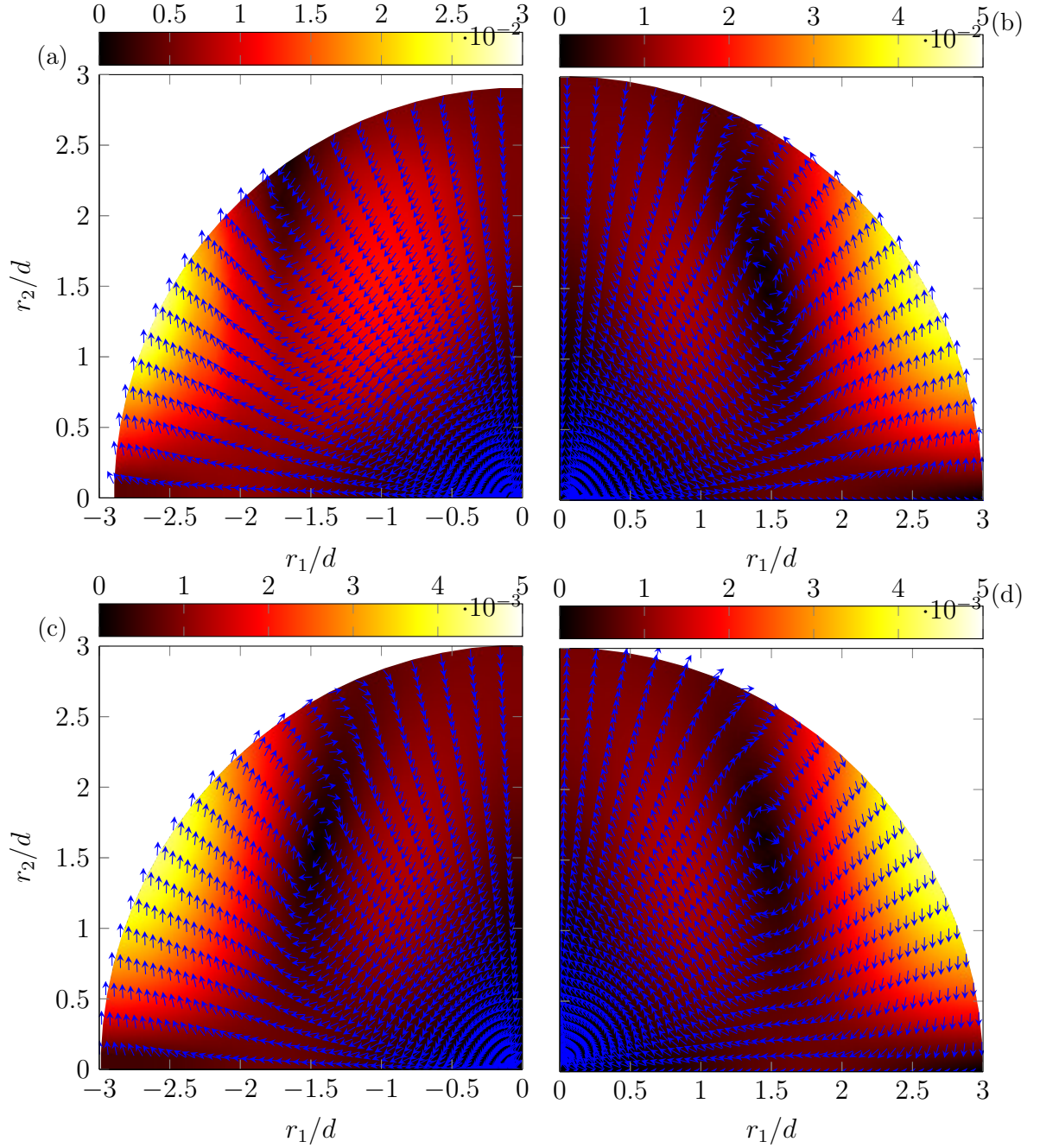


Figure 4.64: Third order structure functions $\langle \delta \tilde{u} \delta \tilde{q}^2 \rangle$ in scale space, the arrows indicate the orientation and the contour indicates the magnitude. In (a) the vectors correspond to data extracted from the DNS at $X_1/d = 8$ (recall ** on page 177). In (b), (c) and (d) the vectors are obtained from data generated through eq. (4.30), where the parameters in (a) are the same as those used in fig. 4.60, in (c) the angle α is $0.99\pi/2$ and in (d) it is $1.01\pi/2$.

4.5 Summary

By conditionally sampling the velocity and pressure fields in the wake generated by a square prism (as introduced in the classical work of Hussain and W. C. Reynolds (1970)) those fluctuating fields were decomposed into two components, a phase averaged component whose time signature follows the vortex shedding and a stochastic component which can be interpreted as the turbulent fluctuations which are superimposed onto the organised motion associated with the vortex shedding.

Similarly to the periodic box turbulence of Goto and Vassilicos (2016a), it was found that, along the centreline of the wake, the dissipation associated with the coherent motions is practically negligible with the bulk of the turbulent dissipation arising from the stochastic motions ($\tilde{\varepsilon} \ll \varepsilon'$). Nevertheless, the ratio between both components of the turbulent dissipation remains approximately constant over the same range where C_ε is proportional to Re_λ^{-1} (Goto and Vassilicos, 2016a, provide an argument for how $\tilde{\varepsilon}/\varepsilon'$ implies $C_\varepsilon \propto Re_\lambda^{-1}$). The later claim is explored extensively in order to assure it is not an artefact of the choice of estimates for the velocity and length scales used.

At first, this appears at odds with the observation that $\Pi^a \approx \text{const.}$ Kolmogorov (1941c) uses the assumptions of homogeneity and steadiness of the turbulence to obtain $\Pi = \text{const}$ from which $C_\varepsilon = \text{const}$ follows, under the same aforementioned assumptions. On the other hand, the homogeneous turbulence examined by Goto and Vassilicos (2016a) is unsteady and thus equilibrium, in the sense of $\Pi = \text{const}$, is not possible. The turbulence in the near wake (examined here), however, is both inhomogeneous and unsteady. While the energy transferred between scales appears constant (when averaging over orientations, i.e. $\Pi^a = \text{const}$) there is also a redistribution of energy, both in scale- and physical-space, by mechanisms other than the non-linear inter-scale transfer. These mechanisms represent the effect of inhomogeneity and contain information regarding the initial and boundary conditions, thus preventing ε from scaling with $\mathcal{U}^3/\mathcal{L}$.

The triple decomposition highlights the pivotal role of the stochastic motions in transferring energy down the scales. In fact, the self-interaction of the coherent motions plays a negligible role in the inter-scale energy transfer. The near-constancy of Π^a is retrieved by accounting for the total fluctuating energy transferred between scales by the stochastic motions and a small corrective contribution associated with the inter-scale transfer of energy associated with the stochastic motions by the coherent motions. The later contribution, albeit small, appears to resist the energy transfer in the direct sense since $\Pi_u'^a > 0$ at large enough scales. The re-orientation of the inter-scale energy fluxes observed in chapter 3 was found to be a consequence of the combined reorientation of $\langle \delta \mathbf{u}' \delta q'^2 \rangle$ and a decrease in the contribution of the fluxes associated with the coherent motions.

Examination of the individual budgets of $\langle \delta \tilde{q}^2 \rangle$ and $\langle \delta q'^2 \rangle$ reveals that the stochastic motions remain inhomogeneous and anisotropic throughout the range of scales over which $\Pi^a \approx \text{const.}$ The coherent motions interact with the stochastic motions in two special ways (other than Π'_u), the production and turbulent transport mechanisms. The production $\mathcal{P}'_u$ was found to be the main cause of loss of energy associated with the coherent motions which can only be replenished by the mean-flow inhomogeneity, through $\tilde{\mathcal{P}}_U$. The turbulent transport of stochastic energy by the coherent motions, when averaged over the orientations of $\mathbf{r}$ in the $r_3 = 0$ plane, follows closely the profile of $-\mathcal{A}^a$, the advection by the mean flow; this suggests that, in the context of orientation averaged statistics, the energy associated with the stochastic motions arriving due to the advection by the mean flow is swept away by the coherent motions.

Finally, we show how representing the coherent structures by ellipse-shaped viscous vortices allows one to explain the two-point properties of the coherent motions. Specifically, the concentration of $\langle \delta \tilde{q}^2 \rangle$ at large r on the r_1 axis is caused by the alternating circulation of the coherent structures. More interesting, however, is that the inter-scale energy flux arising from the self-interaction of the coherent motions can be seen as a consequence of the geometrical shape assumed by the coherent structures. As the cross section of these structures becomes less eccentric, or alternatively as the axis that define their near-elliptical shape become aligned with the cross- and streamwise directions, the magnitude of $\langle \delta \mathbf{u} \delta \tilde{q}^2 \rangle$ decreases, but the orientations at which the $\langle \delta \mathbf{u} \delta \tilde{q}^2 \rangle$ represent direct and inverse fluxes remains the same, so far as the quadrants where the ellipse's axes lie in remains the same.

The memory of the initial conditions is thus reflected not only on the non-equilibrium dissipation law observed along the centreline, but also in the local inter-scale fluxes arising from the coherent motions which Thiesset, Danaila, and Antonia (2014) reports to play a non-negligible role at distances as large as $20d$ from the wake generator.

5 Conclusion

All turbulent flows in nature and engineering are to some degree inhomogeneous, anisotropic and unsteady and the past 75 years must now be taken as clear evidence that the focus on homogeneous turbulence has been contrived and with arguably limited fruition. In this thesis we have carried out a highly resolved numerical simulation of the turbulent wake generated by a square prism in light of the objectives outlined in section 1.3.

The results shown in chapter 3 reveal that the power spectrum densities of the fluctuating velocity develop a near $-5/3$ power law in the frequency domain, immediately after the recirculation region. These Kolmogorov-like power laws are more evident in Fourier space than in scale space (as noted by Davidson and Pearson, 2005; Frisch, 1995). This happens for two reasons: because structure functions (defined in scale space) are contaminated by convolution of spectral peaks across all scales as illustrated by Fourier transforming a simple model spectrum and because the ranges over which power-laws are observed are wider in Fourier space than in real space. In chapter 4 we confirm that the second-order structure functions associated with the stream-wise velocity component of the stochastic motions exhibit near $2/3$ power laws over a narrow range of scales.

Moreover, the development of near $-5/3$ power laws appears in tandem with combined direct and inverse energy fluxes associated with the non-linear interactions. We find that in the very near wake, the distribution of these fluxes in scale space is quite similar to that found in the production region of grid generated turbulence by Gomes-Fernandes et al. (2015). At $x_1/d = 2$, the closest location to the prism investigated here, the direct flux dominates at separations normal to the centreline and is inverse along it. As the distance to the prism increases, the non-linear inter-scale fluxes see their distribution in scale space re-orient such that the direct fluxes dominate at separations lying on the centreline while inverse fluxes are observed at separations normal to it. While the effect of inhomogeneity cannot be discarded, the combined direct and inverse fluxes appear to be a direct consequence of the anisotropy of the turbulence.

The anisotropy of the turbulence is also clearly visible in the scale space distribution of all the terms appearing in the KMH equation (the generalised two-point energy budget equation). In the region of the flow where the spectra display power laws close to $-5/3$ the

terms of the KHH equation resulting from inhomogeneity are non-negligible throughout the different scales. The combined direct and inverse non-linear energy cascades and the effect of inhomogeneity mean that Kolmogorov's theory cannot be applied to obtain spectra with power laws of $-5/3$. However, once the orientations of the pairs of points are averaged on the plane of the wake, the combined effect of inhomogeneity and unsteadiness becomes negligible altogether. This observation is directly linked to the striking result that, in the same orientation-averaged sense, the energy flux occurs from large to small scales and at an approximately constant rate (close to $-\varepsilon$) the very ingredients that constitute the basis to Kolmogorov's theory. This is reminiscent of the prediction by Nie and Tanveer (1999) but with the important distinction that there appears to be no need to average over all solid angles (in scale-space) and over all physical-space.

Despite the different terms in the KHH conspiring together to have a negligible net effect on the energy budget, their individual contributions to the budget itself are in general comparable in magnitude to the turbulent dissipation. In particular, the advection term $\mathcal{A}$ appears to play an important role in the energy budget. As noted by Vassilicos (2015) this term represents the unsteadiness of the turbulence in the direction of the mean flow. A non-negligible $\mathcal{A}$ thus means that the turbulence is out of equilibrium.

By decomposing the flow field into its phase and stochastic components, we were able to determine that the dissipation associated with the coherent motions $\tilde{\varepsilon}$, despite being small in magnitude, remained proportional to the dissipation associated with the stochastic motions ε' . Goto and Vassilicos (2016a) had previously shown, in periodic box turbulence, that a constant ratio $\tilde{\varepsilon}/\varepsilon'$ led to the non-equilibrium scaling of the turbulent dissipation introduced in Seoud and Vassilicos (2007) where $C_\varepsilon \propto Re_\lambda^{-1}$. The same scaling was obtained along the centreline of the wake, in the same region where the range of scales over which $\Pi^a = \text{const}$ increases and over which $\tilde{\varepsilon}/\varepsilon'$ is constant. The non-classical behaviour of the turbulence is further evidenced by the fact that the stochastic motions themselves are strongly non-Gaussian (which is not the case in the work of Goto and Vassilicos, 2016a).

Thus this constitutes the first observation that $\Pi^a \approx \text{const}$ and $C_\varepsilon \propto Re_\lambda^{-1}$ can coexist in the same region of the flow. This may initially appear paradoxical as $\Pi \approx \text{const}$ is used in Kolmogorov (1941c) to determine that $C_\varepsilon = \text{const}$. However, Kolmogorov neglects the fact that the effects of inhomogeneity (and memory of initial conditions) can transpire through the remaining terms in the KHH equation. Even though their net effect is negligible (given that $\Pi^a \approx \text{const}$), they can still re-distribute energy both in scale- and in physical-space.

The distinction between phase and stochastic components of the flow revealed that the anisotropic distribution of energy in scale space is mostly due to the coherent mo-

tions. The latter, however, were found to have an underlying anisotropy as the structure functions for each velocity component of the stochastic motions are found to not satisfy the isotropy relation (see e.g. von Kármán, 1937). This is even more clear in the third order statistics where we find that the energy cascade from large to small scales (in the orientation averaged sense) is dominated by the stochastic motions. In fact, it is the inter-scale energy transfer due to the stochastic motions $\Pi_{u'}$ that causes Π^a to be negative and close to $-\varepsilon$. The coherent motions on the other hand only contribute to $\Pi^a \approx \text{const}$ as a small correction due to the inter-scale transfer of the energy associated with the stochastic motions.

Interestingly, we find that the non-linear fluxes associated with the self-interaction of the coherent motions $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$ are determined by their anisotropic nature. As these structures travel downstream, the orientations at which the direct/inverse fluxes (associated with $\langle \delta \tilde{\mathbf{u}} \delta \tilde{q}^2 \rangle$) dominate remain the same while their magnitude decreases. This means that the reorientation of the total non-linear inter-scale fluxes must be caused by a reorientation of the fluxes associated with the stochastic motions, which we indeed observe. The fact that the stochastic motions are able to sustain (in a statistical sense) inverse cascades at directions normal to the centreline must be seen as a consequence of the underlying anisotropy which we find to persist even at the small scales. In fact the smallest scales appear to be better described by the local axisymmetry introduced by George and Hussein (1991).

The discussion presented in this thesis has strong implications in how we should steer turbulence research and turbulence modelling. The fact that inhomogeneous, anisotropic and unsteady turbulence displays some features *à la* Kolmogorov suggests that there must be some underlying mechanism of the turbulence, other than equilibrium, local homogeneity and local isotropy, that triggers the rate of non-linear energy transfer to become constant in some average sense which appears to be less strict than that suggested by Nie and Tanveer (1999). The fact that this constant is close to the turbulent dissipation and results from combined direct and inverse energy cascades testifies to the importance of studying inhomogeneous turbulence.

Some of the important research questions raised by our results are as follows:

- First, and foremost, it is important to determine the Re dependence of $\Pi^a = \text{const}$. Such a study must result from combined DNS and experiments. In particular, our results suggest that the full velocity gradient tensor is required to obtain $\Pi^a = \text{const}$, such information can only be obtained experimentally via tomographic-PIV while DNS are restricted (in terms of Re) by computational costs.

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- While we have found that the pressure term is indeed non-negligible in the KHH equation (as postulated by Gomes-Fernandes et al., 2015), the apparent correlation in scale space between $\mathcal{T}_p$ and Π (also found in box turbulence by Yasuda and Vassilicos, 2017) still lacks a physical justification.
 - LES models have a hard time predicting flows such as that studied here (see e.g. Bosch and Rodi, 1998; Voke, 1996). In part, this difficulty may be associated with the anisotropic nature of the inter-scale energy transfer. While eddy-viscosity-based models can be tuned to dynamically change the direction of the energy fluxes, they cannot account for the multiple direct/inverse cascades observed in scale-space. It would be interesting to address the two-point properties of these models and others such as implicit-LES where the tuning of directional cascades may be more intuitive. In particular, whatever anisotropy is introduced into these models would need to, on average, be consistent with the growing evidence that $\Pi^a = \text{const}$ in regions of intense turbulence.
 - Do the combined direct/inverse cascades, especially those associated with the stochastic motions, persist further downstream in the wake? If so, it will be useful to carry a combined analysis of fluxes in both physical and scale space. As shown by Cimarelli et al. (2013) when locally inverse cascades occur in wall turbulence, the energy at those scales is transported in physical space to locations where the scale-space flux becomes direct (from large to small scales).
 - The important role of the coherent structures is found in the observation that $C_\varepsilon \propto Re_\lambda^{-1}$ (as explained by Goto and Vassilicos, 2016b). In the present flow such structures were defined on the basis of their time signature. It would be interesting to investigate if other types of structures (with less pronounced spectral signatures) also lead to $C_\varepsilon \propto Re_\lambda^{-1}$ and, perhaps more importantly, if they are also required to sustain $\Pi^a = \text{const}$.
 - Finally, it would be very interesting to study the probability distributions of the different terms in the KHH equation, especially that of Π , in order to determine if the observations made with regard to $\Pi^a = \text{const}$ and the directionality of the cascades are a result of some intermittent behaviour or if they are stable features of the turbulence.

Of course, “questions are only interesting if they lead to insightful answers, otherwise they remain literally ‘interesting’, i.e. ‘inter’ ‘este’ = ‘between’ ‘existences’, i.e. in limbo...

nowhere...” Vassilicos (2014). The issues raised in the list above suggest that the stalemate reached in turbulence research may soon be circumvented by a revived interest in realistic (as opposed to idealised) turbulence.

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Appendices

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A Derivation of the Generalised Two-Point Energy Equations

The Kármán-Howarth-Monin-Hill (KMH) equation is a two-point equation derived directly from the Navier-Stokes equations and as such it is exact in the sense that no approximations are required to obtain it (Hill, 2001). In fact, the classical Kármán-Howarth and Kármán-Howarth-Monin equations (see von Kármán and Howarth, 1938; Monin and Yaglom, 1975) can be obtained directly from the KMH equation after introducing the necessary assumptions of local isotropy and (or) local homogeneity.

In order to obtain the KMH equation, let us write the incompressible Navier-Stokes equations (eq. (1.1)) in two reference frames, x_i^+ and x_i^- as in eq. (A.1), where the superscripts indicate the reference frame in which the velocity and pressure are evaluated*. The two reference frames share the same origin but are independent of each other, i.e. changes in either x_i^+ or x_i^- are independent of the x_i^- or x_i^+ , respectively.

$$\begin{cases} \frac{\partial u_i^+}{\partial x_i^+} = 0 \\ \frac{\partial u_i^+}{\partial t} + u_j^+ \frac{\partial u_i^+}{\partial x_j^+} = -\frac{\partial p^+}{\partial x_i^+} + \nu \frac{\partial}{\partial x_j^+} \frac{\partial u_i^+}{\partial x_j^+} \end{cases} \quad (\text{A.1a})$$

$$\begin{cases} \frac{\partial u_i^-}{\partial x_i^-} = 0 \\ \frac{\partial u_i^-}{\partial t} + u_j^- \frac{\partial u_i^-}{\partial x_j^-} = -\frac{\partial p^-}{\partial x_i^-} + \nu \frac{\partial}{\partial x_j^-} \frac{\partial u_i^-}{\partial x_j^-} \end{cases} \quad (\text{A.1b})$$

We shall use eq. (A.1) to derive an equation for the energy associated with the two-point velocity differences δu_i , where the symbol δ preceding some quantity ϕ is defined through

$$\phi^+ = \phi(x_j^+, t) \quad \phi^- = \phi(x_j^-, t) \quad \delta\phi = \phi^+ - \phi^-. \quad (\text{A.2})$$

The differences of any quantity ϕ (obtained through eq. (A.2)) are defined in a six-dimensional reference frame associated with the pair x_i^+, x_i^- . Let us define a new reference frames x_i, r_i obtained from x_i^+, x_i^- through

* Notice that time “runs” simultaneously in both reference frames.

$$r_i = x_i^+ - x_i^- \quad x_i = \frac{1}{2} (x_i^+ + x_i^-), \quad (\text{A.3})$$

where r_i represents a scale (a distance between two points) and x_i represents a position in the laboratory reference frame (the mid-point between x_i^+ and x_i^-).

Because points in x_i^+ and x_i^- are independent from each other, the derivatives of any function of the former with respect to the latter (and vice versa) are zero, i.e., $\frac{\partial f(x_i^+)}{\partial x_j^-} = 0$, $\frac{\partial f(x_i^-)}{\partial x_j^+} = 0$. This allows us to write the Jacobians associated with the transformation eq. (A.3) as

$$J_{ij}(r_k, x_k^+) = \delta_{ij} \quad J_{ij}(r_k, x_k^-) = -\delta_{ij} \quad J_{ij}(x_k, x_k^+) = \frac{1}{2}\delta_{ij} \quad J_{ij}(x_k, x_k^-) = \frac{1}{2}\delta_{ij} \quad (\text{A.4})$$

and realise that, as a consequence[†]

$$\frac{\partial(\phi^+ + \phi^-)}{\partial x_i} = 2 \frac{\partial(\phi^+ - \phi^-)}{\partial r_i} \quad (\text{A.5})$$

holds.

Let us now make use of a Reynolds decomposition where capitalised letters indicate mean quantities and lower-case indicate fluctuations are the mean. Applying this decomposition to the momentum equations in eq. (A.1) yields

$$\begin{aligned} \frac{\partial}{\partial t} (U_i^+ + u_i^+) + (U_j^+ + u_j^+) \frac{\partial}{\partial x_j^+} (U_i^+ + u_i^+) = \\ - \frac{\partial}{\partial x_i^+} (P^+ + p^+) + \nu \frac{\partial}{\partial x_j^+} \frac{\partial}{\partial x_j^+} (U_i^+ + u_i^+) \end{aligned} \quad (\text{A.6a})$$

$$\begin{aligned} \frac{\partial}{\partial t} (U_i^- + u_i^-) + (U_j^- + u_j^-) \frac{\partial}{\partial x_j^-} (U_i^- + u_i^-) = \\ - \frac{\partial}{\partial x_i^-} (P^- + p^-) + \nu \frac{\partial}{\partial x_j^-} \frac{\partial}{\partial x_j^-} (U_i^- + u_i^-) \end{aligned} \quad (\text{A.6b})$$

while one can easily show that, as a consequence of eqs. (A.4) and (A.5), the mass conservation equations in eq. (A.1) hold for δu_i (and δU_i) with either r_i or x_i in place of either x_i^+ or x_i^- .

Taking the difference of (A.6a) and (A.6b) results in the expression below.

[†] More precisely, notice that since variables defined at either point are independent of the other (point) we have that $\frac{\partial \phi^+}{\partial x^+} = \frac{\partial(\phi^+ \pm \phi^-)}{\partial x^+}$ and $\frac{\partial \phi^-}{\partial x^-} = \frac{\partial(\phi^+ \pm \phi^-)}{\partial x^-}$. This implies that $\frac{\partial(\phi^+ \pm \phi^-)}{\partial x^+} = \frac{\partial(\phi^+ \mp \phi^-)}{\partial x^+}$ (similarly for x^-) by expanding the derivatives on each side with respect to x_i and r_i through (A.4) one obtains the relation given by (A.5).

$$\begin{aligned}
\frac{\partial}{\partial t} (\delta U_i + \delta u_i) + (U_j^+ + u_j^+) \frac{\partial}{\partial x_j^+} (U_i^+ + u_i^+) - (U_j^- + u_j^-) \frac{\partial}{\partial x_j^-} (U_i^- + u_i^-) = \\
- \left(\frac{\partial}{\partial x_j^+} (P^+ + p^+) - \frac{\partial}{\partial x_j^-} (P^- + p^-) \right) + \\
\nu \frac{\partial}{\partial x_j^+} \frac{\partial}{\partial x_j^+} (U_i^+ + u_i^+) - \nu \frac{\partial}{\partial x_j^-} \frac{\partial}{\partial x_j^-} (U_i^- + u_i^-) \quad (\text{A.7})
\end{aligned}$$

Now let us multiply eq. (A.7) by $2\delta u_i$ and take the ensemble average (denoted by the $\langle \cdot \rangle$), eq. (A.7) becomes

$$\begin{aligned}
\langle 2\delta u_i \frac{\partial \delta u_i}{\partial t} \rangle + \langle 2\delta u_i U_j^+ \frac{\partial u_i^+}{\partial x_j^+} \rangle + \langle 2\delta u_i u_j^+ \frac{\partial U_i^+}{\partial x_j^+} \rangle + \langle 2\delta u_i u_j^+ \frac{\partial u_i^+}{\partial x_j^+} \rangle \\
- \langle 2\delta u_i U_j^- \frac{\partial u_i^-}{\partial x_j^-} \rangle - \langle 2\delta u_i u_j^- \frac{\partial U_i^-}{\partial x_j^-} \rangle - \langle 2\delta u_i u_j^- \frac{\partial u_i^-}{\partial x_j^-} \rangle = \\
- \left(\langle 2\delta u_i \frac{\partial p^+}{\partial x_i^+} \rangle - \langle 2\delta u_i \frac{\partial p^-}{\partial x_i^-} \rangle \right) + \langle 2\delta u_i \nu \frac{\partial}{\partial x_j^+} \frac{\partial}{\partial x_j^+} u_i^+ \rangle - \langle 2\delta u_i \nu \frac{\partial}{\partial x_j^-} \frac{\partial}{\partial x_j^-} u_i^- \rangle. \quad (\text{A.8})
\end{aligned}$$

Given (A.4) we have that $\frac{\partial}{\partial x_i^+} = \frac{1}{2} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial r_i}$ and $\frac{\partial}{\partial x_i^-} = \frac{1}{2} \frac{\partial}{\partial x_i} - \frac{\partial}{\partial r_i}$.

We now consider the pairs of terms in the equation above which take a similar form (i.e. the corresponding terms where $+$ is replaced by $-$). We treat all terms on the left hand side of eq. (A.8) (except the time derivative) in the following way:

1. For gradients in x^+ subtract u^- (or U^-) from the numerator; for gradients in x^- subtract u^+ (or U^+);
2. Apply the properties $2\phi^+ = \phi^+ + \phi^- + \delta\phi$ and $2\phi^- = \phi^+ + \phi^- - \delta\phi$ (these are a direct consequence of (A.2)) to the velocities with repeating index of the gradient by which they are multiplied;
3. Expand the gradients in x^+ and x^- to gradients in x and r using (A.4).

$$\begin{aligned}
& \langle 2\delta u_i U_j^+ \frac{\partial u_i^+}{\partial x_j^+} \rangle - \langle 2\delta u_i U_j^- \frac{\partial u_i^-}{\partial x_j^-} \rangle = \langle 2\delta u_i U_j^+ \frac{\partial \delta u_i}{\partial x_j^+} \rangle + \langle 2\delta u_i U_j^- \frac{\partial \delta u_i}{\partial x_j^-} \rangle = \\
& = \langle \delta u_i (U_j^+ + U_j^- + \delta U_j) \left(\frac{1}{2} \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_i}{\partial r_j} \right) \rangle + \langle \delta u_i (U_j^+ + U_j^- - \delta U_j) \left(\frac{1}{2} \frac{\partial \delta u_i}{\partial x_j} - \frac{\partial \delta u_i}{\partial r_j} \right) \rangle = \\
& = \langle \delta u_i (U_j^+ + U_j^-) \frac{\partial \delta u_i}{\partial x_j} \rangle + 2\langle \delta u_i \delta U_j \frac{\partial \delta u_i}{\partial r_j} \rangle = \\
& = \frac{U_j^+ + U_j^-}{2} \langle \frac{\partial \delta u_i \delta u_i}{\partial x_j} \rangle + \langle \delta U_j \frac{\partial \delta u_i \delta u_i}{\partial r_j} \rangle
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
& \langle 2\delta u_i u_j^+ \frac{\partial U_i^+}{\partial x_j^+} \rangle - \langle 2\delta u_i u_j^- \frac{\partial U_i^-}{\partial x_j^-} \rangle = \langle 2\delta u_i u_j^+ \frac{\partial \delta U_i}{\partial x_j^+} \rangle + \langle 2\delta u_i u_j^- \frac{\partial \delta U_i}{\partial x_j^-} \rangle = \\
& = \langle \delta u_i (u_j^+ + u_j^- + \delta u_j) \left(\frac{1}{2} \frac{\partial \delta U_i}{\partial x_j} + \frac{\partial \delta U_i}{\partial r_j} \right) \rangle + \langle \delta u_i (u_j^+ + u_j^- - \delta u_j) \left(\frac{1}{2} \frac{\partial \delta U_i}{\partial x_j} - \frac{\partial \delta U_i}{\partial r_j} \right) \rangle = \\
& = \langle (u_j^+ + u_j^-) \delta u_i \rangle \frac{\partial \delta U_i}{\partial x_j} + 2\langle \delta u_j \delta u_i \rangle \frac{\partial \delta U_i}{\partial r_j}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
& \langle 2\delta u_i u_j^+ \frac{\partial u_i^+}{\partial x_j^+} \rangle - \langle 2\delta u_i u_j^- \frac{\partial u_i^-}{\partial x_j^-} \rangle = \langle 2\delta u_i u_j^+ \frac{\partial \delta u_i}{\partial x_j^+} \rangle + \langle 2\delta u_i u_j^- \frac{\partial \delta u_i}{\partial x_j^-} \rangle = \\
& = \langle \delta u_i (u_j^+ + u_j^- + \delta u_j) \left(\frac{1}{2} \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_i}{\partial r_j} \right) \rangle + \langle \delta u_i (u_j^+ + u_j^- - \delta u_j) \left(\frac{1}{2} \frac{\partial \delta u_i}{\partial x_j} - \frac{\partial \delta u_i}{\partial r_j} \right) \rangle = \\
& = \langle \delta u_i (u_j^+ + u_j^-) \frac{\partial \delta u_i}{\partial x_j} \rangle + 2\langle \delta u_i \delta u_j \frac{\partial \delta u_i}{\partial r_j} \rangle = \\
& = \langle \frac{u_j^+ + u_j^-}{2} \frac{\partial \delta u_i \delta u_i}{\partial x_j} \rangle + \langle \delta u_j \frac{\partial \delta u_i \delta u_i}{\partial r_j} \rangle
\end{aligned} \tag{A.11}$$

The pressure can be tackled using some of the steps above leading to the simplification below.

$$\begin{aligned}
& \langle 2\delta u_i \frac{\partial p^+}{\partial x_i^+} \rangle - \langle 2\delta u_i \frac{\partial p^-}{\partial x_i^-} \rangle = 2\langle \delta u_i \frac{\partial \delta p}{\partial x_i^+} \rangle + 2\langle \delta u_i \frac{\partial \delta p}{\partial x_i^-} \rangle = \\
& = 2\langle \delta u_i \left(\frac{1}{2} \frac{\partial \delta p}{\partial x_i} + \frac{\partial \delta p}{\partial r_i} \right) \rangle + 2\langle \delta u_i \left(\frac{1}{2} \frac{\partial \delta p}{\partial x_i} - \frac{\partial \delta p}{\partial r_i} \right) \rangle = \\
& = 2\langle \delta u_i \frac{\partial \delta p}{\partial x_i} \rangle
\end{aligned} \tag{A.12}$$

Notice that one can re-write eq. (A.12) as $2\langle \delta u_i \delta \frac{\partial p}{\partial x_i} \rangle$, i.e. the correlation between the velocity difference and the difference of pressure gradients between x_i^+ and x_i^- .

Finally we may deal with the viscous terms. The procedure requires the same steps used for the remaining terms in addition to $\psi \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_i} \phi = \frac{\partial}{\partial \xi_i} \left(\psi \frac{\partial}{\partial \xi_i} \phi \right) - \left(\frac{\partial}{\partial \xi_i} \psi \right) \left(\frac{\partial}{\partial \xi_i} \phi \right)$ (where ψ and ϕ are any two variables defined in the same reference frame ξ). The viscous term in A.8 is thus simplified as shown below.

$$\begin{aligned}
 \langle 2\delta u_i \nu \frac{\partial^2 u_i^+}{\partial x_j^+ \partial x_j^+} \rangle - \langle 2\delta u_i \nu \frac{\partial^2 u_i^-}{\partial x_j^- \partial x_j^-} \rangle &= 2\nu \langle \delta u_i \frac{\partial^2 u_i}{\partial x_j^+ \partial x_j^+} \rangle + 2\nu \langle \delta u_i \frac{\partial^2 u_i}{\partial x_j^- \partial x_j^-} \rangle = \\
 &= 2\nu \left\langle \frac{\partial}{\partial x_j^+} \left(\delta u_i \frac{\partial \delta u_i}{\partial x_j^+} \right) \right\rangle - 2\nu \left\langle \frac{\partial \delta u_i}{\partial x_j^+} \frac{\partial \delta u_i}{\partial x_j^+} \right\rangle + 2\nu \left\langle \frac{\partial}{\partial x_j^-} \left(\delta u_i \frac{\partial \delta u_i}{\partial x_j^-} \right) \right\rangle - \\
 &2\nu \left\langle \frac{\partial \delta u_i}{\partial x_j^-} \frac{\partial \delta u_i}{\partial x_j^-} \right\rangle = \nu \left\langle \frac{\partial^2 \delta u_i \delta u_i}{\partial x_j^+ \partial x_j^+} \right\rangle + \nu \left\langle \frac{\partial^2 \delta u_i \delta u_i}{\partial x_j^- \partial x_j^-} \right\rangle - 2\nu \left(\left\langle \frac{\partial \delta u_i}{\partial x_j^+} \frac{\partial \delta u_i}{\partial x_j^+} \right\rangle + \left\langle \frac{\partial \delta u_i}{\partial x_j^-} \frac{\partial \delta u_i}{\partial x_j^-} \right\rangle \right) = \\
 &= \nu \left\langle \left(\frac{1}{2} \frac{\partial}{\partial x_j} + \frac{\partial}{\partial r_j} \right) \left(\frac{1}{2} \frac{\partial}{\partial x_j} + \frac{\partial}{\partial r_j} \right) \delta u_i \delta u_i \right\rangle + \\
 &\nu \left\langle \left(\frac{1}{2} \frac{\partial}{\partial x_j} + \frac{\partial}{\partial r_j} \right) \left(\frac{1}{2} \frac{\partial}{\partial x_j} - \frac{\partial}{\partial r_j} \right) \delta u_i \delta u_i \right\rangle - 2\nu \left(\left\langle \frac{\partial \delta u_i}{\partial x_j^+} \frac{\partial \delta u_i}{\partial x_j^+} \right\rangle + \left\langle \frac{\partial \delta u_i}{\partial x_j^-} \frac{\partial \delta u_i}{\partial x_j^-} \right\rangle \right) = \\
 &= \nu \frac{1}{2} \frac{\partial^2 \langle \delta u_i \delta u_i \rangle}{\partial x_j \partial x_j} + 2\nu \frac{\partial^2 \langle \delta u_i \delta u_i \rangle}{\partial r_j \partial r_j} - 2\nu \left(\left\langle \frac{\partial \delta u_i}{\partial x_j^+} \frac{\partial \delta u_i}{\partial x_j^+} \right\rangle + \left\langle \frac{\partial \delta u_i}{\partial x_j^-} \frac{\partial \delta u_i}{\partial x_j^-} \right\rangle \right) \tag{A.13}
 \end{aligned}$$

The last two terms in eq. (A.13) can be re-written as gradients of the single-point velocities, since the contributions of u_i^- in the gradient with respect to x_j^+ are zero, and vice-versa for u_i^+ and x_j^- , thus results in the averaged between the dissipation in x_i^+ and x_i^- .

Now let us call $\delta q^2 = \delta u_i \delta u_i$ the two-point energy. Collecting eqs. (A.9) to (A.13) and rearranging some terms allows eq. (A.8) to be written as

$$\begin{aligned}
 \frac{\partial \langle \delta q^2 \rangle}{\partial t} + \frac{U_j^+ + U_j^-}{2} \left\langle \frac{\partial \delta q^2}{\partial x_j} \right\rangle + \langle \delta u_j \frac{\partial \delta q^2}{\partial r_j} \rangle + \delta U_j \left\langle \frac{\partial \delta q^2}{\partial r_j} \right\rangle = \\
 - 2 \langle \delta u_j \delta u_i \rangle \frac{\partial \delta U_i}{\partial r_j} - \langle (u_j^+ + u_j^-) \delta u_i \rangle \frac{\partial \delta U_i}{\partial x_j} - \left\langle \frac{\partial^{u_j^+ + u_j^-}}{\partial x_j} \delta q^2 \right\rangle - 2 \langle \delta u_i \frac{\partial \delta p}{\partial x_i} \rangle + \\
 \nu \frac{1}{2} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \langle \delta q^2 \rangle + 2\nu \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_j} \langle \delta q^2 \rangle - 2\nu \left[\left\langle \left(\frac{\partial \delta u_i}{\partial x_j^+} \right) \left(\frac{\partial \delta u_i}{\partial x_j^+} \right) \right\rangle + \left\langle \left(\frac{\partial \delta u_i}{\partial x_j^-} \right) \left(\frac{\partial \delta u_i}{\partial x_j^-} \right) \right\rangle \right]. \tag{A.14}
 \end{aligned}$$

The discussion of the different terms can be found in chapter 3. Notice that eq. (A.14) is a particular case of the equation for the tensor $\langle \delta u_i \delta u_k \rangle$ ($\langle \delta q^2 \rangle$ it is in fact the sum

of the diagonal of that tensor) as discussed by Hill (2001), Hill (2002a), and Lindborg (1996b)

Let us now consider the case where a triple decomposition of the fluctuating velocity field is used. As described in chapter 4, the velocity and pressure fields are now decomposed into their mean, phase component (denoted with a tilde) and stochastic component (denoted with an apostrophe). A two-point energy equation can be obtained for $\langle \delta \tilde{q}^2 \rangle$, the second order structure function associated with the coherent component, and $\langle \delta q'^2 \rangle$, the second order structure function associated with the stochastic component.

After making use of the triple decomposition, eq. (A.7) can to obtain equations for $\langle \delta \tilde{q}^2 \rangle$ or $\langle \delta q'^2 \rangle$. One has

$$\begin{aligned} & \langle 2\delta \tilde{u}_i \frac{\partial \delta \tilde{u}_i + \delta u'_i}{\partial t} \rangle + \langle 2\delta \tilde{u}_i U_j^+ \frac{\partial \tilde{u}_i^+ + u'^+_i}{\partial x_j^+} \rangle + \langle 2\delta \tilde{u}_i (\tilde{u}_j^+ + u'^+_j) \frac{\partial U_i^+}{\partial x_j^+} \rangle + \langle 2\delta \tilde{u}_i (\tilde{u}_j^+ + u'^+_j) \frac{\partial \tilde{u}_i^+ + u'^+_i}{\partial x_j^+} \rangle \\ & - \langle 2\delta \tilde{u}_i U_j^- \frac{\partial \tilde{u}_i^- + u'^-_i}{\partial x_j^-} \rangle - \langle 2\delta \tilde{u}_i (\tilde{u}_j^- + u'^-_j) \frac{\partial U_i^-}{\partial x_j^-} \rangle - \langle 2\delta \tilde{u}_i (\tilde{u}_j^- + u'^-_j) \frac{\partial \tilde{u}_i^- + u'^-_i}{\partial x_j^-} \rangle = \\ & - \left(\langle 2\delta \tilde{u}_i \frac{\partial \tilde{p}^+ + p'^+}{\partial x_i^+} \rangle - \langle 2\delta \tilde{u}_i \frac{\partial \tilde{p}^- + p'^-}{\partial x_i^-} \rangle \right) + \langle 2\delta \tilde{u}_i \nu \frac{\partial^2 \tilde{u}_i^+ + u'^+_i}{\partial x_j^+ \partial x_j^+} \rangle - \langle 2\delta \tilde{u}_i \nu \frac{\partial^2 \tilde{u}_i^- + u'^-_i}{\partial x_j^- \partial x_j^-} \rangle \end{aligned} \quad (\text{A.15})$$

and

$$\begin{aligned} & \langle 2\delta u'_i \frac{\partial \delta u'_i + \delta u'_i}{\partial t} \rangle + \langle 2\delta u'_i U_j^+ \frac{\partial \tilde{u}_i^+ + u'^+_i}{\partial x_j^+} \rangle + \langle 2\delta u'_i (\tilde{u}_j^+ + u'^+_j) \frac{\partial U_i^+}{\partial x_j^+} \rangle + \langle 2\delta u'_i (\tilde{u}_j^+ + u'^+_j) \frac{\partial \tilde{u}_i^+ + u'^+_i}{\partial x_j^+} \rangle \\ & - \langle 2\delta u'_i U_j^- \frac{\partial \tilde{u}_i^- + u'^-_i}{\partial x_j^-} \rangle - \langle 2\delta u'_i (\tilde{u}_j^- + u'^-_j) \frac{\partial U_i^-}{\partial x_j^-} \rangle - \langle 2\delta u'_i (\tilde{u}_j^- + u'^-_j) \frac{\partial \tilde{u}_i^- + u'^-_i}{\partial x_j^-} \rangle = \\ & - \left(\langle 2\delta u'_i \frac{\partial \tilde{p}^+ + p'^+}{\partial x_i^+} \rangle - \langle 2\delta u'_i \frac{\partial \tilde{p}^- + p'^-}{\partial x_i^-} \rangle \right) + \langle 2\delta u'_i \nu \frac{\partial^2 \tilde{u}_i^+ + u'^+_i}{\partial x_j^+ \partial x_j^+} \rangle - \langle 2\delta u'_i \nu \frac{\partial^2 \tilde{u}_i^- + u'^-_i}{\partial x_j^- \partial x_j^-} \rangle, \end{aligned} \quad (\text{A.16})$$

where eq. (A.15) results from the time average of product between eq. (A.7) and $\langle 2\delta \tilde{u}_i^2 \rangle$ while eq. (A.16) results from the time average of the product between eq. (A.7) and $\langle 2\delta u'^2_i \rangle$.

In order to write eqs. (A.15) and (A.16) as equations for $\langle \delta \tilde{q}^2 \rangle$ or $\langle \delta q'^2 \rangle$, one can follow the same steps used to obtain eq. (A.14), by making use of the fact that the time average $\langle \cdot \rangle$ is the same as the time average of the phase average $\langle \{ \cdot \} \rangle$ together with the properties given by eq. (4.3). The resulting equations are

$$\begin{aligned}
 & \langle \frac{\partial \delta \tilde{q}^2}{\partial t} \rangle + \langle \frac{U_i^+ + U_i^-}{2} \frac{\partial \delta \tilde{q}^2}{\partial x_i} \rangle + \langle \delta U_i \frac{\partial \delta \tilde{q}^2}{\partial r_i} \rangle + \langle \delta \tilde{u}_i \frac{\partial \delta \tilde{q}^2}{\partial r_i} \rangle + \langle 2\delta u'_i \delta \tilde{u}_j \frac{\partial}{\partial r_i} \delta u'_j \rangle = \\
 & - \langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \delta \tilde{q}^2}{\partial x_i} \rangle + \langle [(u_i'^+ + u_i'^-) \delta \tilde{u}_j] \frac{\partial}{\partial x_i} \delta u'_j \rangle - \langle 2\delta \tilde{u}_i \delta \frac{\partial \tilde{p}}{\partial x_i} \rangle - \langle \delta u'_i (u_j'^+ + u_j'^-) \frac{\partial \delta U_i}{\partial x_j} \rangle - \\
 & \quad \langle \delta u'_i \delta u'_j \frac{\partial \delta U_i}{\partial r_j} \rangle + \langle \delta u'_i (u_j'^+ + u_j'^-) \frac{\partial \delta \tilde{u}_i}{\partial x_j} \rangle + \langle \delta u'_i \delta u'_j \frac{\partial \delta \tilde{u}_i}{\partial r_j} \rangle \\
 & \quad \langle \nu \frac{1}{2} \frac{\partial^2 \delta \tilde{q}^2}{\partial x_j \partial x_j} \rangle + \langle 2\nu \frac{\partial^2 \delta \tilde{q}^2}{\partial r_j \partial r_j} \rangle - 4\nu \left(\langle \frac{\partial \delta \tilde{u}_j}{\partial x_i} \frac{\partial \delta \tilde{u}_j}{\partial x_i} \rangle + \frac{1}{4} \langle \frac{\partial \delta \tilde{u}_j}{\partial r_i} \frac{\partial \delta \tilde{u}_j}{\partial r_i} \rangle \right) \quad (\text{A.17})
 \end{aligned}$$

for $\langle \tilde{q}^2 \rangle$ and

$$\begin{aligned}
 & \langle \frac{\partial \delta q'^2}{\partial t} \rangle + \langle \frac{U_i^+ + U_i^-}{2} \frac{\partial \delta q'^2}{\partial x_i} \rangle + \langle \delta U_i \frac{\partial \delta q'^2}{\partial r_i} \rangle + \langle \delta \tilde{u}_i \frac{\partial \delta q'^2}{\partial r_i} \rangle + \langle \delta u'_i \frac{\partial \delta q'^2}{\partial r_i} \rangle = \\
 & - \langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \delta q'^2}{\partial x_i} \rangle - \langle \frac{u_i'^+ + u_i'^-}{2} \frac{\partial \delta q'^2}{\partial x_i} \rangle - \langle 2\delta u'_i \delta \frac{\partial p'}{\partial x_i} \rangle - \langle \delta u'_i (u_j'^+ + u_j'^-) \frac{\partial \delta U_i}{\partial x_j} \rangle - \\
 & \quad \langle \delta u'_i \delta u'_j \frac{\partial \delta U_i}{\partial r_j} \rangle - \langle \delta u'_i (u_j'^+ + u_j'^-) \frac{\partial \delta \tilde{u}_i}{\partial x_j} \rangle - \langle \delta u'_i \delta u'_j \frac{\partial \delta \tilde{u}_i}{\partial r_j} \rangle + \\
 & \quad \langle \nu \frac{1}{2} \frac{\partial^2 \delta q'^2}{\partial x_j \partial x_j} \rangle + \langle \nu 2 \frac{\partial^2 \delta q'^2}{\partial r_j \partial r_j} \rangle - 4\nu \left(\langle \frac{\partial \delta u'_j}{\partial x_i} \frac{\partial \delta u'_j}{\partial x_i} \rangle + \frac{1}{4} \langle \frac{\partial \delta u'_j}{\partial r_i} \frac{\partial \delta u'_j}{\partial r_i} \rangle \right) \quad (\text{A.18})
 \end{aligned}$$

for $\langle \delta q'^2 \rangle$. Notice that in eq. (A.18) the terms containing $\tilde{u}_i$ have a similar form to those containing U_i . The terms in eq. (A.17) all have a similar form to those in eq. (A.14) with the addition of the two terms

$$\langle 2\delta u'_i \delta \tilde{u}_j \frac{\partial}{\partial r_i} \delta u'_j \rangle \quad (\text{A.19})$$

and

$$\langle [(u_i'^+ + u_i'^-) \delta \tilde{u}_j] \frac{\partial}{\partial x_i} \delta u'_j \rangle. \quad (\text{A.20})$$

These can be re-written as

$$\langle 2\delta u'_i \delta u'_j \frac{\partial}{\partial r_i} \delta \tilde{u}_j \rangle - \langle 2 \frac{\partial}{\partial r_i} (\delta u'_i \delta \tilde{u}_j \delta u'_j) \rangle \quad (\text{A.21})$$

and

$$\langle [(u_i'^+ + u_i'^-) \delta \tilde{u}_j] \frac{\partial}{\partial x_i} \delta u'_j \rangle - \langle \frac{\partial}{\partial x_i} [(u_i'^+ + u_i'^-) \delta \tilde{u}_j \delta u'_j] \rangle, \quad (\text{A.22})$$

respectively, where the first terms in both eq. (A.21) and eq. (A.22) also appear in eq. (A.18) but with a different sign (as shown in chapter 4, these terms are associated with the production of $\delta q'^2$ by the coherent motion). The remaining terms in eqs. (A.21) and (A.22) are associated with the transfer (in scale-space) and transport (in physical space) of the total fluctuating energy δq^2 by the stochastic motions (see chapter 4). Thus, the final form of eq. (A.17) is

$$\begin{aligned}
& \langle \frac{\partial \delta \tilde{q}^2}{\partial t} \rangle + \langle \frac{U_i^+ + U_i^-}{2} \frac{\partial \delta \tilde{q}^2}{\partial x_i} \rangle + \langle \delta U_i \frac{\partial \delta \tilde{q}^2}{\partial r_i} \rangle + \langle \delta \tilde{u}_i \frac{\partial \delta \tilde{q}^2}{\partial r_i} \rangle + \langle 2 \frac{\partial}{\partial r_i} (\delta u'_i \delta u'_j \delta \tilde{u}_j) \rangle = \\
& - \langle \frac{\tilde{u}_i^+ + \tilde{u}_i^-}{2} \frac{\partial \delta \tilde{q}^2}{\partial x_i} \rangle - \langle \frac{\partial}{\partial x_i} [(u_i'^+ + u_i'^-) \delta u'_j \delta \tilde{u}_j] \rangle - \langle 2 \delta \tilde{u}_i \delta \frac{\partial \tilde{p}}{\partial x_i} \rangle - \langle \delta \tilde{u}_i (\tilde{u}_j^+ + \tilde{u}_j^-) \frac{\partial \delta U_i}{\partial x_j} \rangle - \\
& \quad \langle 2 \delta \tilde{u}_i \delta \tilde{u}_j \frac{\partial \delta U_i}{\partial r_j} \rangle + \langle \delta u'_i (u_j'^+ + u_j'^-) \frac{\partial \delta \tilde{u}_i}{\partial x_j} \rangle + \langle 2 \delta u'_i \delta u'_j \frac{\partial \delta \tilde{u}_i}{\partial r_j} \rangle \\
& \quad \langle \nu \frac{1}{2} \frac{\partial^2 \delta \tilde{q}^2}{\partial x_j \partial x_j} \rangle + \langle 2\nu \frac{\partial^2 \delta \tilde{q}^2}{\partial r_j \partial r_j} \rangle - 4\nu \left(\langle \frac{\partial \delta \tilde{u}_j}{\partial x_i} \frac{\partial \delta \tilde{u}_j}{\partial x_i} \rangle + \frac{1}{4} \langle \frac{\partial \delta \tilde{u}_j}{\partial r_i} \frac{\partial \delta \tilde{u}_j}{\partial r_i} \rangle \right). \quad (\text{A.23})
\end{aligned}$$

The terms in eqs. (A.14), (A.18) and (A.23) can be computed directly by storing the velocity and pressure differences as six-dimensional arrays and then applying any numerical method (such as finite differences) to estimate the derivatives in each dimension. This approach, however, may introduce some errors of numerical nature. This issue cannot be circumvented in experiments. If the data available is obtained from some DNS, then eqs. (A.14), (A.18) and (A.23) can be balanced as accurately as permitted by the numerical methods used in the DNS. In other words, the different terms that appear in eqs. (A.14), (A.18) and (A.23) should be expanded in terms of derivatives in $\mathbf{x}^+$ and $\mathbf{x}^-$ which can then be estimated with the same schemes used in the DNS. This is particularly important if the numerical schemes are not energy conservative (Felten and Lund, 2006).