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Convergence of a Finite Element Approximation of Surfactant Spreading on a Thin Film in the Presence of van der Waals Forces

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We consider a fully practical finite element approximation of the following system of nonlinear degenerate parabolic equations

$$\tfrac{\partial u}{\partial t} + \tfrac{1}{2}\,\nabla.(u^2\,\nabla[\sigma(v)]) - \tfrac{1}{3}\,\nabla.(u^3\,\nabla w) = 0\,,$$
$$w = -c\,\Delta u - \delta\,u^{-\nu} + a\,u^{-3}\,,$$
$$\tfrac{\partial v}{\partial t} + \nabla.(u\,v\,\nabla[\sigma(v)]) - \rho\,\Delta v - \tfrac{1}{2}\,\nabla.(u^2\,v\,\nabla w) = 0\,.$$

The above models a surfactant driven thin film flow in the presence of both attractive, $a > 0$, and repulsive, $\delta > 0$ with $\nu > 3$, van der Waals forces; where u is the height of the film, v is the concentration of the insoluble surfactant monolayer and $\sigma(v) := 1 - v$ is the typical surface tension. Here $\rho \geq 0$ and $c > 0$ are the inverses of the surface Peclet number and the modified capillary number. In addition to showing stability bounds for our approximation, we prove convergence, and hence existence of a solution to this nonlinear degenerate parabolic system, (i) in one space dimension when $\rho > 0$; and, moreover, (ii) in two space dimensions if in addition $\nu \geq 7$. Furthermore, iterative schemes for solving the resulting nonlinear discrete system are discussed. Finally, some numerical experiments are presented.

Mathematics Subject Classification (2000): 65M60, 65M12, 35K55, 35K65, 35K35, 76A20, 76D08

1. Introduction

In the recent paper Barrett, Garcke & Nürnberg (2001), abbreviated to BGN throughout this paper, the authors proposed and analysed a fully practical finite element approximation of the following system of nonlinear degenerate parabolic equations

$$\tfrac{\partial u}{\partial t} + \tfrac{1}{2}\,\nabla.(u^2\,\nabla[\sigma(v)]) - \tfrac{1}{3}\,\nabla.(u^3\,\nabla w) = 0 \qquad \text{in } \Omega_T\,, \tag{1.1a}$$
$$w = -c\,\Delta u + \phi(u) \qquad \text{in } \Omega_T\,, \tag{1.1b}$$
$$\tfrac{\partial v}{\partial t} + \nabla.(u\,v\,\nabla[\sigma(v)]) - \rho\,\Delta v - \tfrac{1}{2}\,\nabla.(u^2\,v\,\nabla w) = 0 \qquad \text{in } \Omega_T\,; \tag{1.1c}$$

where $\Omega_T := \Omega \times (0,T]$ and Ω is a bounded domain in $\mathbb{R}^d$, $d = 1$ or 2. The above, derived using lubrication theory, models a surfactant driven thin film flow; where u is the height of the film, v is the concentration of the insoluble surfactant monolayer and w is the pressure (reduced if van der Waals forces are present; that is, $\phi \not\equiv 0$). In addition, $\sigma \in C^1(\mathbb{R}_{\geq 0})$ with

$$\sigma(s) \geq 0, \quad \sigma'(s) < 0 \qquad \forall\ s \in \mathbb{R}_{\geq 0} \tag{1.2}$$

is the constitutive equation of state relating the surface tension σ to v. We note that σ is a strictly monotonically decreasing function of v, which is natural to assume as the surfactant lowers surface tension. In the engineering literature it is often assumed that the surfactant concentration is dilute, i.e. $v \in [0,1]$, and that $\sigma : [0,1] \to [0,1]$; e.g.

$$\sigma(s) := (\beta+1)\,[1 + \theta(\beta)\,s]^{-3} - \beta, \quad \text{where} \quad \theta(\beta) := \left(\tfrac{\beta+1}{\beta}\right)^{\frac{1}{3}} - 1 \tag{1.3}$$

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and $\beta \in \mathbb{R}_{>0}$ relates to the activity of the surfactant; cf. Jensen & Grotberg (1992, p 262). For the van der Waals forces in (1.1b), we take the form suggested in Oron, Davis & Bankoff (1997); that is,

$$\phi(u) = \phi^+(u) + \phi^-(u), \qquad \phi^+(u) := -\delta\, u^{-\nu}, \quad \nu > 3\,, \qquad \phi^-(u) := a\, u^{-3}\,, \tag{1.4}$$

where $a \in \mathbb{R}_{\geq 0}$ is the scaled dimensionless Hamaker constant, $\delta \in \mathbb{R}_{\geq 0}$ represents the effect of repulsive van der Waals forces, and $a = 0 \Leftrightarrow \delta = 0$. Finally in (1.1a–c), $\rho \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}_{>0}$ are the inverses of the surface Peclet and the modified capillary numbers, respectively. Applications of the system (1.1a–c) range from the medical treatment of premature infants to industrial coating and drying processes, see BGN for further details and references.

As u and v can take on zero values (1.1a–c) is a degenerate parabolic system, which is fourth order in u. This degeneracy makes the analysis/numerical analysis of the system particularly difficult. As there is no maximum principle for parabolic equations of fourth order, a naive discretization does not guarantee the non-negativity of the approximation to u. If $\delta = 0$, following Barrett, Blowey & Garcke (1998), BGN imposed the non-negativity of the discrete approximation to u as a constraint; whereas if $\delta > 0$, the positivity of the approximation to u can be guaranteed for an appropriate discretization through the singularity in ϕ^+. In both cases, BGN were able to derive stability bounds in space dimensions $d = 1$ and 2. However, their main convergence result was restricted to $\rho > 0$ and one space dimension. The latter was due to the fact that the a priori bounds they derived only guarantee in one space dimension that the discrete approximation to u is uniformly bounded and equicontinuous, which was necessary to be able to pass to the limit in the discrete problem. For similar reasons, the results on related degenerate parabolic equations of fourth order in Bernis & Friedman (1990), Barrett, Blowey & Garcke (1998), Barrett, Blowey & Garcke (1999), Grün & Rumpf (2000), Barrett, Blowey & Garcke (2001) and Barrett & Blowey (2001) were restricted to one space dimension. However, recently Grün (2002) has proved convergence in two space dimensions of a finite element approximation to the thin film equation in the absence of surfactant, (1.1a–c) with $v \equiv 0$ and $u(\cdot, 0) > 0$. It is the aim of this paper to adapt the techniques in Grün (2002), and BGN, to propose a finite element approximation to (1.1a–c) and prove convergence in one space dimension if $\rho, a, \delta, u(\cdot, 0) > 0$ and, moreover, in two space dimensions if in addition $\nu \geq 7$.

The basic ingredients of our approach are two energy estimates for this surfactant driven flow. Firstly, as in BGN, we relate F to σ by the Gibbs identity

$$\sigma(v) = F(v) - v\, F'(v) \qquad \Rightarrow \qquad \sigma'(v) = -v\, F''(v)\,. \tag{1.5}$$

Knowing σ the above identity determines F up to a linear term. Furthermore, on noting (1.2), we have that F is convex and $F''(v) \to \infty$ as $v \to 0$. Assuming appropriate boundary conditions, which will be specified later on, one can derive the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\tfrac{c}{2}\, |\nabla u|^2 + \Phi(u) + F(v)\right] \mathrm{d}x + \int_\Omega \int_0^u |\partial_y \vec{\mathbf{V}}_H|^2 \,\mathrm{d}y\, \mathrm{d}x + \rho \int_\Omega F''(v)\, |\nabla v|^2 \,\mathrm{d}x = 0\,. \tag{1.6}$$

Here Φ is an antiderivative of ϕ, i.e. $\Phi' \equiv \phi$, and $\vec{\mathbf{V}}_H$ is the horizontal component of the velocity field. Due to the use of lubrication theory, see BGN for details, one has that

$$\int_0^u |\partial_y \vec{\mathbf{V}}_H|^2 \,\mathrm{d}y = u\,|\nabla[\sigma(v)]|^2 - u^2\,\nabla[\sigma(v)]\,.\,\nabla w + \tfrac{1}{3}\,u^3\,|\nabla w|^2$$
$$\geq \tfrac{1}{8}\,u\,|\nabla[\sigma(v)]|^2 + \tfrac{1}{21}\,u^3\,|\nabla w|^2\,, \tag{1.7}$$

where the inequality was obtained from applying the Young's inequality

$$|r\,s| \leq \tfrac{\gamma}{2}\,r^2 + \tfrac{1}{2\gamma}\,s^2 \quad \forall\ r,\ s \in \mathbb{R}, \quad \gamma \in \mathbb{R}_{>0}. \tag{1.8}$$

The energy estimate (1.6) will lead to important a priori estimates.

In order to obtain the second estimate, we define also a function $G \in C^\infty(\mathbb{R}_{>0})$ such that $u^3\,\nabla[G'(u)] = \nabla u$; that is

$$G''(s) = s^{-3} \quad \Rightarrow \quad G'(s) = -\tfrac{1}{2}\,s^{-2} \quad \Rightarrow \quad G(s) = \tfrac{1}{2}\,s^{-1}\,. \tag{1.9}$$

Testing (1.1a) with $G'(u)$ and (1.1b) with $-\Delta u$ formally yields, on assuming appropriate boundary conditions and noting (1.8), that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u)\,\mathrm{d}x + \tfrac{c}{3}\int_\Omega |\Delta u|^2\,\mathrm{d}x + \tfrac{1}{3}\int_\Omega (\phi^+)'(u)\,|\nabla u|^2\,\mathrm{d}x$$
$$= -\tfrac{1}{3}\int_\Omega (\phi^-)'(u)\,|\nabla u|^2\,\mathrm{d}x + \tfrac{1}{2}\int_\Omega G''(u)\,u^2\,\nabla[\sigma(v)]\,.\,\nabla u\,\mathrm{d}x$$
$$\leq C\left[\int_\Omega u\,|\nabla[\sigma(v)]|^2\,\mathrm{d}x + \int_\Omega [\,G''(u) - (\phi^-)'(u)\,]\,|\nabla u|^2\,\mathrm{d}x\right]. \tag{1.10}$$

This together with (1.4) and the bound

$$s^{-\alpha} \leq \gamma\,s^{-\beta} + C(\gamma,\alpha,\beta) \qquad \forall\ s,\ \gamma \in \mathbb{R}_{>0}, \quad \alpha \in (0,\beta), \quad \beta \in \mathbb{R}_{>0}\,, \tag{1.11}$$

for $\beta = \nu + 1$ and for both $\alpha = 3$ and 4, yields for $\delta > 0$ that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G(u)\,\mathrm{d}x + \tfrac{c}{3}\int_\Omega |\Delta u|^2\,\mathrm{d}x + \tfrac{1}{4}\int_\Omega (\phi^+)'(u)\,|\nabla u|^2\,\mathrm{d}x$$
$$\leq C\left[\int_\Omega u\,|\nabla[\sigma(v)]|^2\,\mathrm{d}x + \int_\Omega |\nabla u|^2\,\mathrm{d}x\right]. \tag{1.12}$$

From (1.12), (1.6) and (1.7) one can formally show that $u \in L^2(0,T;H^2(\Omega))$ if $u(\cdot,0) > 0$.

It is the goal of this paper to derive a finite element method that is consistent with the formal energy estimates (1.6) and (1.12). In one space dimension only the analogue of (1.6) is required to prove convergence, see BGN and Remark 3.2 below. As in Grün (2002) for the thin film equation, it is the discrete analogue of (1.12) which plays a crucial role in our convergence proof in the case of two space dimensions. We stress that it is only this bound that requires the presence of the repulsive van der Waals forces, $\delta > 0$, to control the surfactant term on the right hand side of (1.10). In the absence of surfactant, (1.12) holds with $(a, \delta > 0)$ and without $(a = \delta = 0)$ van der Waals forces.

To complete the system (1.1a–c), we will specify the boundary conditions

$$\tfrac{1}{2}\,u^2\,\tfrac{\partial[\sigma(v)]}{\partial\nu_{\partial\Omega}} - \tfrac{1}{3}\,u^3\,\tfrac{\partial w}{\partial\nu_{\partial\Omega}} = \tfrac{\partial u}{\partial\nu_{\partial\Omega}} = u\,v\,\tfrac{\partial[\sigma(v)]}{\partial\nu_{\partial\Omega}} - \tfrac{1}{2}\,u^2\,v\,\tfrac{\partial w}{\partial\nu_{\partial\Omega}} - \rho\,\tfrac{\partial v}{\partial\nu_{\partial\Omega}} = 0 \quad \text{on } \partial\Omega\,; \tag{1.13}$$

where $\nu_{\partial\Omega}$ is normal to $\partial\Omega$, the Lipschitz boundary of Ω. One can derive (1.13) on requiring a no penetration condition for the velocity $\vec{\mathbf{V}}_H$, a 90° angle condition for the film height u and a no flux condition for the surfactant concentration v to hold, see BGN for details. We remark that if either $u\,\rho > 0$ or $-u\,v\,\sigma'(v) > 0$ on $\partial\Omega$ then (1.13) is equivalent to $\frac{\partial u}{\partial \nu_{\partial\Omega}} = \frac{\partial w}{\partial \nu_{\partial\Omega}} = \frac{\partial v}{\partial \nu_{\partial\Omega}} = 0$ on $\partial\Omega$.

Altogether, in this paper we consider the following initial boundary value problem:
(P) Find functions u, w, $v : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\,\nabla.(u^2\,\nabla[\sigma(v)]) - \tfrac{1}{3}\,\nabla.(u^3\,\nabla\,w) = 0 \qquad \text{in } \Omega_T, \tag{1.14a}$$

$$w = -c\,\Delta u + \phi(u) \qquad \text{in } \Omega_T, \tag{1.14b}$$

$$\frac{\partial v}{\partial t} + \nabla.(u\,\lambda(v)\,\nabla[\sigma(v)]) - \rho\,\Delta v - \tfrac{1}{2}\,\nabla.(u^2\,\lambda(v)\,\nabla w) = 0 \quad \text{in } \Omega_T, \tag{1.14c}$$

$$u(x,0) = u^0(x) > 0, \qquad v(x,0) = v^0(x) \geq 0 \qquad \forall\, x \in \Omega, \tag{1.14d}$$

$$\begin{aligned}&\tfrac{1}{2}\,u^2\,\frac{\partial[\sigma(v)]}{\partial\nu_{\partial\Omega}} - \tfrac{1}{3}\,u^3\,\frac{\partial w}{\partial \nu_{\partial\Omega}} = \frac{\partial u}{\partial \nu_{\partial\Omega}} \\ &\qquad = u\,\lambda(v)\,\frac{\partial[\sigma(v)]}{\partial\nu_{\partial\Omega}} - \tfrac{1}{2}\,u^2\,\lambda(v)\,\frac{\partial w}{\partial\nu_{\partial\Omega}} - \rho\,\frac{\partial v}{\partial \nu_{\partial\Omega}} = 0 \qquad \text{on } \partial\Omega \times (0,T),\end{aligned} \tag{1.14e}$$

where $T > 0$ is a fixed positive time. In the above $c \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}_{\geq 0}$ are given constants, while $\sigma \in C^1(\mathbb{R}_{\geq 0})$ and $\phi : \mathbb{R}_{>0} \to \mathbb{R}$ are given by (1.2) and (1.4), with $a, \delta > 0$; u^0 and v^0 are given initial profiles. Throughout this paper, as in BGN, we restrict ourselves to the model constitutive equation of state

$$\sigma(v) := 1 - v\,, \tag{1.15}$$

the $\beta \to \infty$ limit of (1.3). However, the techniques in this paper do apply to a general σ satisfying (1.2), see Remark 2.2 below. As remarked previously, the physically relevant values of $v(x,t)$ lie in the interval $[0,1]$. Noting this, it is convenient for the analysis in this paper, as well as in BGN, to replace the terms $u^i\,v$, $i = 1 \to 2$, in (1.1c) by $u^i\,\lambda(v)$, where $\lambda : \mathbb{R} \to (-\infty, 1]$ is defined as

$$\lambda(s) := [s-1]_- + 1; \tag{1.16}$$

where

$$[s]_- := \min\{s, 0\} \qquad \text{and} \qquad [s]_+ := \max\{s,0\}. \tag{1.17}$$

This paper is organised as follows. In Section 2 we formulate a fully practical finite element approximation of the degenerate system (P) and derive discrete analogues of the energy estimates (1.6) and (1.12). In doing so, we adapt a technique introduced in Zhornitskaya & Bertozzi (2000) and Grün & Rumpf (2000) for deriving a discrete entropy bound for the thin film equation. In Section 3 we prove convergence, and hence existence of a solution to the system (P), in space dimensions $d = 1$ if $\rho, a, \delta > 0$; and $d = 2$ if in addition $\nu \geq 7$. In Section 4 we state an iterative scheme for solving the nonlinear discrete system at each time level and present some numerical computations in both one and two space dimensions.

Finally we note that there is very little work in the PDE literature on surfactant type problems. To our knowledge, the only relevant work on the degenerate system (P) is the following. Renardy (1996) shows that there exists a solution for T sufficiently small to a pure initial value problem similar to (P) for very smooth initial data. In the absence of van der Waals forces, Wieland (2002) establishes an existence result for (P) in one space dimension similar to that obtained in BGN.

Notation and Auxiliary Results

Let $D \subset \mathbb{R}^d$, $d = 1$ or 2, with a Lipschitz boundary ∂D if $d = 2$. We adopt the standard notation for Sobolev spaces, denoting the norm of $W^{m,q}(D)$ ($m \in \mathbb{N}$, $q \in [1, \infty]$) by $\|\cdot\|_{m,q,D}$ and the semi-norm by $|\cdot|_{m,q,D}$. We extend these norms and semi-norms in the natural way to the corresponding spaces of vector and matrix valued functions. For $q = 2$, $W^{m,2}(D)$ will be denoted by $H^m(D)$ with the associated norm and semi-norm written as, respectively, $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$. For notational convenience, we drop the domain subscript on the above norms and semi-norms in the case $D \equiv \Omega$. Throughout $(\cdot, \cdot)$ denotes the standard L^2 inner product over Ω, while q' denotes for any $q \in [1, \infty]$ the "dual exponent" such that $\frac{1}{q} + \frac{1}{q'} = 1$. In addition we define

$$\fint \eta := \frac{1}{\underline{m}(\Omega)} (\eta, 1) \qquad \forall\, \eta \in L^1(\Omega), \qquad \text{where } \underline{m}(\Omega) \text{ is the measure of } \Omega\,. \tag{1.18}$$

For later purposes, we recall the following well-known Sobolev interpolation results, e.g. see Adams & Fournier (1977): Let $q \in [1, \infty]$, $m \geq 1$,

$$r \in \begin{cases} [q, \infty] & \text{if } \; m - \frac{d}{q} > 0, \\ [q, \infty) & \text{if } \; m - \frac{d}{q} = 0, \\ [q, -\frac{d}{m-(d/q)}] & \text{if } \; m - \frac{d}{q} < 0, \end{cases}$$

and $\mu = \frac{d}{m}\,(\frac{1}{q} - \frac{1}{r})$. Then there is a constant C depending only on Ω, q, r, m such that for all $z \in W^{m,q}(\Omega)$ the inequality

$$|z|_{0,r} \leq C\, |z|_{0,q}^{1-\mu}\, \|z\|_{m,q}^{\mu} \tag{1.19}$$

holds. We recall also the following compactness results. Let X, Y and Z be Banach spaces with a compact embedding $X \overset{c}{\hookrightarrow} Y$ and a continuous embedding $Y \hookrightarrow Z$. Then the embeddings

$$\{\, \eta \in L^2(0,T;X) : \tfrac{\partial \eta}{\partial t} \in L^2(0,T;Z)\,\} \overset{c}{\hookrightarrow} L^2(0,T;Y) \tag{1.20a}$$

$$\text{and} \qquad \{\, \eta \in L^\infty(0,T;X) : \tfrac{\partial \eta}{\partial t} \in L^2(0,T;Z)\,\} \overset{c}{\hookrightarrow} C([0,T];Y) \tag{1.20b}$$

are compact and a generalised version of (1.20a), where the time derivative is replaced by a time translation, holds. That is, any bounded and closed subset E of $L^2(0,T;X)$ with

$$\lim_{\theta \to 0} \left\{ \sup_{\eta \in E} \|\eta(\cdot, \cdot + \theta) - \eta(\cdot, \cdot)\|_{L^2(0,T-\theta;Z)} \right\} = 0 \tag{1.20c}$$

is compact in $L^2(0,T;Y)$, see Simon (1987).

It is convenient to introduce the "inverse Laplacian" operator $\mathcal{G} : \mathcal{F} \to Z$ such that

$$(\nabla \mathcal{G} z, \nabla \eta) = \langle z, \eta \rangle_{q'} \quad \forall\, \eta \in W^{1,q'}(\Omega), \tag{1.21}$$

where $\mathcal{F} := \big\{ z \in (W^{1,q'}(\Omega))' : \langle z, 1 \rangle_{q'} = 0 \big\}$ and $Z := \{ z \in W^{1,q}(\Omega) : (z, 1) = 0 \}$. Here and throughout $\langle \cdot, \cdot \rangle_{q'}$ denotes the duality pairing between $(W^{1,q'}(\Omega))'$ and $W^{1,q'}(\Omega)$ for any $q \in (1,2]$. The well-posedness of $\mathcal{G}$ follows from the generalised Lax-Milgram theorem and the Poincaré inequality

$$|\eta|_{0,r} \leq C\,(\,|\eta|_{1,r} + |(\eta,1)|\,) \quad \forall\, \eta \in W^{1,r}(\Omega) \quad \text{and} \quad r \in [1,\infty]. \tag{1.22}$$

Throughout C denotes a generic constant independent of h, τ and ε; the mesh and temporal discretization parameters and the regularization parameter. In addition $C(a_1,\cdots,a_I)$ denotes a constant depending on the arguments $\{a_i\}_{i=1}^I$. Furthermore $\cdot_{(\star)}$ denotes an expression with or without the subscript $\star$, similarly with superscripts.

2. Finite Element Approximation

We consider the finite element approximation of (P) under the following assumptions on the mesh:

(A) Let Ω be a convex polygonal domain if $d=2$. Let $\{\mathcal{T}^h\}_{h>0}$ be a quasi-uniform family of partitionings of Ω into disjoint open simplices κ with $h_\kappa := \text{diam}(\kappa)$ and $h := \max_{\kappa\in\mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}^h}\overline{\kappa}$. In addition, it is assumed for $d=2$ that all simplices $\kappa \in \mathcal{T}^h$ are right-angled.

We note that the right-angled simplices assumption is not a severe constraint, as there exist adaptive finite element codes that satisfy this requirement, see e.g. Schmidt & Siebert (2000).

Associated with $\mathcal{T}^h$ is the finite element space

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \ \text{ is linear } \forall\, \kappa \in \mathcal{T}^h\} \subset H^1(\Omega). \tag{2.1}$$

We introduce also

$$S^h_{\geq 0} := \{\chi \in S^h : \chi \geq 0 \text{ in } \Omega\} \subset H^1_{\geq 0}(\Omega) := \{\eta \in H^1(\Omega) : \eta \geq 0 \ a.e. \text{ in } \Omega\}, \tag{2.2}$$

and similarly $S^h_{>0}$ and $H^1_{>0}(\Omega)$. Let J be the set of nodes of $\mathcal{T}^h$ and $\{p_j\}_{j\in J}$ the coordinates of these nodes. Let $\{\chi_j\}_{j\in J}$ be the standard basis functions for S^h; that is $\chi_j \in S^h_{\geq 0}$ and $\chi_j(p_i) = \delta_{ij}$ for all $i,j \in J$. We introduce $\pi^h : C(\overline{\Omega}) \to S^h$, the interpolation operator, such that $(\pi^h\eta)(p_j) = \eta(p_j)$ for all $j \in J$. A discrete semi-inner product on $C(\overline{\Omega})$ is then defined by

$$(\eta_1,\eta_2)^h := \int_\Omega \pi^h(\eta_1(x)\,\eta_2(x))\,\mathrm{d}x = \sum_{j\in J} m_j\,\eta_1(p_j)\,\eta_2(p_j), \tag{2.3}$$

where $m_j := (1,\chi_j) > 0$. The induced discrete semi-norm is then $|\eta|_h := [\,(\eta,\eta)^h\,]^{\frac{1}{2}}$, where $\eta \in C(\overline{\Omega})$. We introduce also the L^2 projection $Q^h : L^2(\Omega) \to S^h$ defined by

$$(Q^h\eta,\chi)^h = (\eta,\chi) \quad \forall\, \chi \in S^h. \tag{2.4}$$

On recalling (1.5) and (1.15), we then define a function F such that $v\,\nabla[F'(v)] = -\nabla[\sigma(v)]$; that is,

$$F''(s) = -s^{-1}\,\sigma'(s) = s^{-1} \quad \Rightarrow \quad F(s) = s\,(\ln s - 1) + 1\,. \tag{2.5}$$

For computational purposes, we replace $F \in C^\infty(\mathbb{R}_{>0})$ for any $\varepsilon \in (0,1)$ by the regularized function $F_\varepsilon : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$F_\varepsilon(s) := \begin{cases} \frac{s^2-\varepsilon^2}{2\,\varepsilon} + (\ln\varepsilon - 1)\,s + 1 & s \leq \varepsilon \\ s\,(\ln s - 1) + 1 & \varepsilon \leq s \leq 1 \\ \frac{1}{2}\,(s-1)^2 & 1 \leq s\,. \end{cases} \tag{2.6}$$

Hence $F_\varepsilon \in C^{2,1}(\mathbb{R})$ with the first two derivatives of F_ε given by

$$F_\varepsilon'(s) := \begin{cases} \varepsilon^{-1}\,s + \ln\varepsilon - 1 & s \leq \varepsilon \\ \ln s & \varepsilon \leq s \leq 1 \\ s-1 & 1 \leq s \end{cases} \quad \text{and} \quad F_\varepsilon''(s) := \begin{cases} \varepsilon^{-1} & s \leq \varepsilon \\ s^{-1} & \varepsilon \leq s \leq 1 \\ 1 & 1 \leq s\,, \end{cases} \tag{2.7}$$

respectively. For later purposes, we note that

$$F_\varepsilon(s) \geq \tfrac{s^2}{4} - \tfrac{1}{2} \quad \forall\ s \geq 0 \qquad \text{and} \qquad F_\varepsilon(s) \geq \tfrac{s^2}{2\,\varepsilon} \quad \forall\ s \leq 0\,; \tag{2.8}$$

see BGN.

Similarly to the approach in Zhornitskaya & Bertozzi (2000) and Grün & Rumpf (2000), we introduce $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ such that for all $z^h \in S^h$ and *a.e.* in Ω

$$\Lambda_\varepsilon(z^h) \text{ is symmetric and positive semi-definite,} \tag{2.9a}$$
$$\Lambda_\varepsilon(z^h)\,\nabla\pi^h[F_\varepsilon'(z^h)] = \nabla z^h. \tag{2.9b}$$

Firstly, we give the construction of Λ_ε in the simple case when $d = 1$. Given $z^h \in S^h$ and $\kappa \in \mathcal{T}^h$ having vertices p_j and p_k, we set

$$\Lambda_\varepsilon(z^h)\,|_\kappa := \begin{cases} \frac{z^h(p_k) - z^h(p_j)}{F_\varepsilon'(z^h(p_k)) - F_\varepsilon'(z^h(p_j))} = \frac{1}{F_\varepsilon''(z^h(\xi))} \text{ for some } \xi \in \kappa & \text{if } \ z^h(p_k) \neq z^h(p_j), \\ \frac{1}{F_\varepsilon''(z^h(p_k))} & \text{if } \ z^h(p_k) = z^h(p_j). \end{cases} \tag{2.10}$$

Clearly the piecewise constant construction in (2.10) satisfies the conditions (2.9a,b). Following Grün & Rumpf (2000) we extend the above construction to $d = 2$. Let $\{e_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that the j^{th} component of e_i is δ_{ij}, $i,\ j = 1 \to d$. Given non-zero constants α_i, $i = 1 \to d$; let $\widehat\kappa(\{\alpha_i\}_{i=1}^d)$ be the reference open simplex in $\mathbb{R}^d$ with vertices $\{\widehat p_i\}_{i=0}^d$, where $\widehat p_0$ is the origin and $\widehat p_i = \alpha_i\,e_i$, $i = 1 \to d$. Given a $\kappa \in \mathcal{T}^h$ with vertices $\{p_{j_i}\}_{i=0}^d$, such that p_{j_0} is the right-angled vertex, then there exists a rotation matrix R_κ and non-zero constants $\{\alpha_i\}_{i=1}^d$ such that the mapping $\mathcal{R}_\kappa : \widehat x \in \mathbb{R}^d \to p_{j_0} + R_\kappa \widehat x \in \mathbb{R}^d$ maps the vertex $\widehat p_i$ to p_{j_i}, $i = 0 \to d$, and hence $\widehat\kappa \equiv \widehat\kappa(\{\alpha_i\}_{i=1}^d)$ to κ. For any $z^h \in S^h$, we then set

$$\Lambda_\varepsilon(z^h)\,|_\kappa := R_\kappa\,\widehat\Lambda_\varepsilon(\widehat z^h)\,|_{\widehat\kappa}\,R_\kappa^T, \tag{2.11}$$

where $\widehat z^h(\widehat x) \equiv z^h(\mathcal{R}_\kappa \widehat x)$ for all $\widehat x \in \overline{\widehat\kappa}$ and $\widehat\Lambda_\varepsilon(\widehat z^h)\,|_{\widehat\kappa}$ is the $d \times d$ diagonal matrix with diagonal entries, $k = 1 \to d$,

$$[\widehat\Lambda_\varepsilon(\widehat z^h)\,|_{\widehat\kappa}]_{kk} := \begin{cases} \frac{\widehat z^h(\widehat p_k) - \widehat z^h(\widehat p_0)}{F_\varepsilon'(\widehat z^h(\widehat p_k)) - F_\varepsilon'(\widehat z^h(\widehat p_0))} \equiv \frac{z^h(p_{j_k}) - z^h(p_{j_0})}{F_\varepsilon'(z^h(p_{j_k})) - F_\varepsilon'(z^h(p_{j_0}))} = \frac{1}{F_\varepsilon''(z^h(\xi))} & \\ \qquad\qquad \text{for some } \xi \text{ between } p_{j_k} \text{ and } p_{j_0} & \text{if } \ z^h(p_{j_k}) \neq z^h(p_{j_0}), \\ \frac{1}{F_\varepsilon''(\widehat z^h(\widehat p_0))} \equiv \frac{1}{F_\varepsilon''(z^h(p_{j_0}))} & \text{if } \ z^h(p_{j_k}) = z^h(p_{j_0}). \end{cases} \tag{2.12}$$

As $R_\kappa^T \equiv R_\kappa^{-1}$, $\nabla z^h \equiv R_\kappa \widehat{\nabla} \widehat{z}^h$, where $x \equiv (x_1, \cdots, x_d)^T$, $\nabla \equiv (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d})^T$, $\widehat{x} \equiv (\widehat{x}_1, \cdots, \widehat{x}_d)^T$ and $\widehat{\nabla} \equiv (\frac{\partial}{\partial \widehat{x}_1}, \cdots, \frac{\partial}{\partial \widehat{x}_d})^T$, it easily follows that $\Lambda_\varepsilon(z^h)$ constructed in (2.11) and (2.12) satisfies (2.9a,b). It is this construction that requires the right angle constraint on the partitioning $\mathcal{T}^h$. Furthermore, we note from (2.7) that for all $\kappa \in \mathcal{T}^h$

$$
\begin{aligned}
\nabla z^h . \nabla \pi^h [F_\varepsilon'(z^h)] \,|_\kappa &\equiv R_\kappa \widehat{\nabla} \widehat{z}^h . R_\kappa \widehat{\nabla} \widehat{\pi}^h [F_\varepsilon'(\widehat{z}^h)] \,|_{\widehat{\kappa}} \\
&\equiv \widehat{\nabla} \widehat{z}^h . \widehat{\nabla} \widehat{\pi}^h [F_\varepsilon'(\widehat{z}^h)] \,|_{\widehat{\kappa}} \geq |\widehat{\nabla} \widehat{z}^h|^2 \,|_{\widehat{\kappa}} \equiv |\nabla z^h|^2 \,|_\kappa \\
\Longrightarrow (\nabla z^h, \nabla \pi^h [F_\varepsilon'(z^h)]) &\geq |z^h|_1^2 \qquad \forall\ z^h \in S^h\,,
\end{aligned}
\tag{2.13}
$$

where $(\widehat{\pi}^h \widehat{\eta})(\widehat{x}) \equiv (\pi^h \eta)(\mathcal{R}_\kappa \widehat{x})$ and $\widehat{\eta}(\widehat{x}) \equiv \eta(\mathcal{R}_\kappa \widehat{x})$ for all $\widehat{x} \in \overline{\widehat{\kappa}}$.

In a similar fashion we introduce $\Xi : S^h_{>0} \to [L^\infty(\Omega)]^{d \times d}$ such that for all $z^h \in S^h_{>0}$ and $a.e.$ in Ω

$$
\Xi(z^h) \text{ is symmetric and positive semi-definite,} \tag{2.14a}
$$
$$
[\Xi(z^h)]^3 \, \nabla \pi^h [G'(z^h)] = \nabla z^h. \tag{2.14b}
$$

We extend the construction (2.11) – (2.12) for Λ_ε to Ξ. In particular, in the case $d = 1$ we set for any $z^h \in S^h_{>0}$ and $\kappa \in \mathcal{T}^h$ having vertices p_j and p_k

$$
\Xi(z^h)\,|_\kappa := \begin{cases} \left(\frac{z^h(p_k) - z^h(p_j)}{G'(z^h(p_k)) - G'(z^h(p_j))} \right)^{\frac{1}{3}} = [G''(z^h(\xi))]^{-\frac{1}{3}} = z^h(\xi) & \\ \qquad\qquad\qquad\qquad \text{for some } \xi \in \kappa & \text{if} \quad z^h(p_k) \neq z^h(p_j), \\ [G''(z^h(p_k))]^{-\frac{1}{3}} = z^h(p_k) & \text{if} \quad z^h(p_k) = z^h(p_j). \end{cases} \tag{2.15}
$$

Throughout we make use of the fact that the matrices $\Xi(\chi)$ and $\Lambda_\varepsilon(z^h)$ commute for any $\chi \in S^h_{>0}$ and $z^h \in S^h$, see (2.11).

To define our approximation of (P) it is convenient to split Φ, recall (1.6), into its convex and concave parts. We have for given $a, \delta \in \mathbb{R}_{>0}$ and $\nu > 3$ that for all $s \in \mathbb{R}_{>0}$

$$
\Phi(s) = \Phi^+(s) + \Phi^-(s), \quad \text{where} \quad \Phi^+(s) := \tfrac{\delta}{\nu - 1}\, s^{1-\nu}, \quad \Phi^-(s) := -\tfrac{a}{2}\, s^{-2}. \tag{2.16}
$$

It holds, on recalling (1.4), that $\phi^+ \equiv (\Phi^+)'$ and $\phi^- \equiv (\Phi^-)'$. For future reference, we note that the following hold for all $r, s, \gamma \in \mathbb{R}_{>0}$ and $z^h \in S^h_{>0}$

$$
\Phi(s) \geq \Phi(\,(\tfrac{\delta}{a})^{\frac{1}{\nu-3}}) = -\tfrac{a\,(\nu-3)}{2\,(\nu-1)}\,(\tfrac{a}{\delta})^{\frac{2}{\nu-3}} \quad \text{and} \quad -\Phi^-(s) \leq \tfrac{a\,(\nu-3)}{2\,(\nu-1)}\,(\tfrac{2\,a}{\delta})^{\frac{2}{\nu-3}} + \tfrac{1}{2}\,\Phi^+(s)\,, \tag{2.17a}
$$
$$
(\nabla \pi^h [G'(z^h)], \nabla z^h) \leq \gamma\,(\nabla \pi^h [\phi^+(z^h)], \nabla z^h) + C(\gamma)\,|z^h|_1^2 \tag{2.17b}
$$

and if $\nu \geq 7$

$$
(\phi^-(r) - \phi^-(s))^2 \leq \tfrac{9\,a^2}{\delta\,\nu}\,(\phi^+(r) - \phi^+(s))\,(r - s) + 9\,a^2\,(r - s)^2\,. \tag{2.18}
$$

(2.17b) can be easily seen on considering the function $\psi : \mathbb{R}_{>0} \to \mathbb{R}$, where $\psi(r) := \gamma\,\phi^+(r) - G'(r) + C(\gamma)\,r$. Obviously, on noting (1.4), (1.9) and (1.11) with $\beta = \nu + 1$, ψ is monotonically increasing for $C(\gamma)$ sufficiently large. The result (2.18) is proved in Grün (2002, Lemma 4.4) for $\nu > 7$. However, the proof stated there is also valid for $\nu \geq 7$.

In addition to $\mathcal{T}^h$, let $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_n := t_n - t_{n-1}$, $n = 1 \to N$. We set $\tau := \max_{n=1\to N} \tau_n$. For any given $\varepsilon \in (0,1)$, we then consider the following fully practical finite element approximation of (P) with $\sigma(v)$ given by (1.15) and $\phi(u)$ given by (1.4):
$(\mathbf{P}_\varepsilon^{h,\tau})$ For $n \geq 1$ find $\{U_\varepsilon^n, W_\varepsilon^n, V_\varepsilon^n\} \in [S^h]^3$ such that for all $\chi \in S^h$

$$\left(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h + \tfrac{1}{3}\left([\Xi(U_\varepsilon^n)]^3 \nabla W_\varepsilon^n, \nabla\chi\right) = -\tfrac{1}{2}\left([\Xi(U_\varepsilon^n)]^{\frac{3}{2}} [\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}} \nabla V_\varepsilon^{n-1}, \nabla\chi\right), \quad (2.19a)$$

$$c\,(\nabla U_\varepsilon^n, \nabla\chi) + (\phi^+(U_\varepsilon^n) + \phi^-(U_\varepsilon^{n-1}), \chi)^h = (W_\varepsilon^n, \chi)^h, \quad (2.19b)$$

$$\left(\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h + \rho\,(\nabla V_\varepsilon^n, \nabla\chi) + (\Xi(U_\varepsilon^n)\, \Lambda_\varepsilon(V_\varepsilon^n)\, \nabla V_\varepsilon^n, \nabla\chi)$$
$$= -\tfrac{1}{2}\left([\Xi(U_\varepsilon^n)]^2\, \Lambda_\varepsilon(V_\varepsilon^n)\, \nabla W_\varepsilon^n, \nabla\chi\right); \quad (2.19c)$$

where $U_\varepsilon^0 \in S_{>0}^h$ and $V_\varepsilon^0 \in S^h$ are approximations of u^0 and v^0, respectively, e.g. $U_\varepsilon^0 \equiv \pi^h u^0$ or $Q^h u^0$ and similarly V_ε^0.

Setting $V_\varepsilon^n \equiv 0$, $n = 0 \to N$, (2.19a,b) collapses to the approximation of the thin film equation, (1.1a–c) with $v \equiv 0$, studied in Grün (2002). Note that the convex(concave) terms in Φ are approximated implicitly(explicitly) in (2.19b). Note furthermore that we approximate u^2 by $[\Xi(U_\varepsilon^n)]^{\frac{3}{2}} [\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}$ in (2.19a) and by $[\Xi(U_\varepsilon^n)]^2$ in (2.19c), respectively, in order for our discrete stability bounds to hold; cf. Theorem 2.4 below, and (2.63) – (2.66) in particular. As $U_\varepsilon^0 > 0$, one can ensure that $\Xi(U_\varepsilon^{n-1})$ and $\phi^-(U_\varepsilon^{n-1})$ are well-defined for $n \geq 1$; see Theorem 2.1 below.

REMARK 2.1 In BGN the following approximation to (P) was studied.
$(\mathbf{Q}_\varepsilon^{h,\tau})$ For $n \geq 1$ find $\{U_\varepsilon^n, W_\varepsilon^n, V_\varepsilon^n\} \in [S^h]^3$ such that for all $\chi \in S^h$

$$\left(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h + \tfrac{1}{3}\left(\pi^h[(U_\varepsilon^{n-1})^3]\, \nabla W_\varepsilon^n, \nabla\chi\right) = -\tfrac{1}{2}\left(\pi^h[(U_\varepsilon^{n-1})^2]\, \nabla V_\varepsilon^{n-1}, \nabla\chi\right), \quad (2.20a)$$

$$c\,(\nabla U_\varepsilon^n, \nabla\chi) + (\phi^+(U_\varepsilon^n) + \phi^-(U_\varepsilon^{n-1}), \chi)^h = (W_\varepsilon^n, \chi)^h, \quad (2.20b)$$

$$\left(\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h + \rho\,(\nabla V_\varepsilon^n, \nabla\chi) + (U_\varepsilon^n\, \Lambda_\varepsilon(V_\varepsilon^n)\, \nabla V_\varepsilon^n, \nabla\chi)$$
$$= -\tfrac{1}{2}\left(\pi^h[(U_\varepsilon^n)^{\frac{1}{2}}\, (U_\varepsilon^{n-1})^{\frac{3}{2}}]\, \Lambda_\varepsilon(V_\varepsilon^n)\, \nabla W_\varepsilon^n, \nabla\chi\right). \quad (2.20c)$$

Note that $(\mathrm{P}_\varepsilon^{h,\tau})$ in contrast to $(\mathrm{Q}_\varepsilon^{h,\tau})$ uses Ξ and approximates u in the term $u^3\,\nabla w$ at the new time level. Furthermore, the schemes vary in the way they approximate u^2 in the right hand side terms. We note that with imposing $U_\varepsilon^n \in S_{\geq 0}^h$ as a constraint and formulating (2.20b) as a variational inequality, $(\mathrm{Q}_\varepsilon^{h,\tau})$ can easily be adapted to the case $\delta = 0$ ($\phi^+ \equiv 0$). This makes it possible to prove convergence also in the absence of (repulsive) van der Waals forces. However, the convergence results for $(\mathrm{Q}_\varepsilon^{h,\tau})$ are restricted to the case $d = 1$, see BGN. In this paper, on the other hand, we will prove convergence in space dimensions $d = 1$ and 2 for $(\mathrm{P}_\varepsilon^{h,\tau})$. As this requires the presence of repulsive van der Waals forces, see Remark 3.2 below, we have to assume that $\inf u^0 > 0$. Finally we note that for $n \geq 1$ given $\{U_\varepsilon^{n-1}, V_\varepsilon^{n-1}\}$, while one can show that the solution U_ε^n of (2.20a) is unique; like Grün (2002) in the case $V_\varepsilon^{n-1} \equiv 0$, we are not able to establish this result for (2.19a).

REMARK 2.2 The restriction of σ to the linear case (1.15) is not crucial for the analysis in this paper. However, this choice simplifies our considerations and is also more practical. Different choices of σ, e.g. (1.3), can be incorporated into $(\mathrm{P}_\varepsilon^{h,\tau})$ by replacing (2.19a,c) with

$$\left(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h + \tfrac{1}{3}\left([\Xi(U_\varepsilon^n)]^3\,\nabla W_\varepsilon^n, \nabla\chi\right) = \tfrac{1}{2}\left([\Xi(U_\varepsilon^n)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}\,\nabla\pi^h[\sigma_\varepsilon(V_\varepsilon^{n-1})], \nabla\chi\right),$$

$$\left(\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h + \rho\,(\nabla V_\varepsilon^n, \nabla\chi) - \left(\Xi(U_\varepsilon^n)\,\widetilde{\Lambda}_\varepsilon(V_\varepsilon^n)\,\nabla\pi^h[\sigma_\varepsilon(V_\varepsilon^n)], \nabla\chi\right)$$
$$= -\tfrac{1}{2}\left([\Xi(U_\varepsilon^n)]^2\,\widetilde{\Lambda}_\varepsilon(V_\varepsilon^n)\,\nabla W_\varepsilon^n, \nabla\chi\right);$$

where $\widetilde{\Lambda}_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ is such that for all $z^h \in S^h$ and $a.e.$ in Ω

$$\widetilde{\Lambda}_\varepsilon(z^h) \text{ is symmetric and positive semi-definite,} \tag{2.22a}$$
$$\widetilde{\Lambda}_\varepsilon(z^h)\,\nabla\pi^h[\widetilde{F}_\varepsilon'(z^h)] = -\nabla\pi^h[\sigma_\varepsilon(z^h)]; \tag{2.22b}$$

and $\sigma_\varepsilon \in C^1(\mathbb{R})$ is a regularized version of σ defined by

$$\sigma_\varepsilon(s) := \begin{cases} \sigma(\varepsilon) + (s-\varepsilon)\,\sigma'(\varepsilon) & s \le \varepsilon \\ \sigma(s) & \varepsilon \le s \le M \\ \sigma(M) + (s-M)\,\sigma'(M) & M \le s\,. \end{cases} \tag{2.23}$$

Here it is implicitly assumed that $v(x,t) \in [0, M]$. Furthermore, $\widetilde{F}_\varepsilon \in C^2(\mathbb{R})$ in (2.22b) is such that

$$\widetilde{F}_\varepsilon''(s) := \begin{cases} -\sigma_\varepsilon'(\varepsilon)\,\varepsilon^{-1} & s \le \varepsilon \\ -\sigma_\varepsilon'(s)\,s^{-1} & \varepsilon \le s \le M \\ -\sigma_\varepsilon'(M)\,M^{-1} & M \le s\,. \end{cases} \tag{2.24}$$

We observe that $\widetilde{F}_\varepsilon$ approximates $F \in C^2((0, M])$ as in (1.5) and is convex. Similarly to (2.11) – (2.12), the construction of $\widetilde{\Lambda}_\varepsilon$ is straightforward and yields in the case $d = 1$ for any $z^h \in S^h$ and $\kappa \in \mathcal{T}^h$ having vertices p_j and p_k that

$$\widetilde{\Lambda}_\varepsilon(z^h)\,|_\kappa := \begin{cases} -\frac{\sigma_\varepsilon(z^h(p_k)) - \sigma_\varepsilon(z^h(p_j))}{\widetilde{F}_\varepsilon'(z^h(p_k)) - \widetilde{F}_\varepsilon'(z^h(p_j))} & \text{if } \ z^h(p_k) \neq z^h(p_j), \\[2mm] -\frac{\sigma_\varepsilon'(z^h(p_k))}{\widetilde{F}_\varepsilon''(z^h(p_k))} & \text{if } \ z^h(p_k) = z^h(p_j). \end{cases} \tag{2.25}$$

We note that $\widetilde{\Lambda}_\varepsilon$ approximates $\widetilde{\lambda}$, where $\widetilde{\lambda}(s) := [s - M]_- + M$ similarly to (1.16).

The convergence analysis for the linear case (1.15) presented in this paper can be adapted to more general σ, as above, but these details will be reported on elsewhere.

Below we recall some well-known results concerning S^h for any $\kappa \in \mathcal{T}^h$, $\chi, z^h \in S^h$, $m \in \{0, 1\}$, $p \in [1, \infty]$, $s \in [2, \infty]$ if $d = 1$ and $s \in (2, \infty]$ if $d = 2$:

$$|\chi|_{1,\kappa} \le C\,h_\kappa^{-1}\,|\chi|_{0,\kappa}\,; \tag{2.26}$$
$$|\chi|_{m,r,\kappa} \le C\,h_\kappa^{-d\,(\frac{1}{p}-\frac{1}{r})}\,|\chi|_{m,p,\kappa} \qquad \text{for any } r \in [p, \infty]\,; \tag{2.27}$$
$$\lim_{h\to 0}\|(I - \pi^h)\eta\|_{1,s} = 0 \qquad \forall\, \eta \in W^{1,s}(\Omega)\,; \tag{2.28}$$
$$|(I - \pi^h)\eta|_m \le C\,h^{2-m}\,|\eta|_2 \qquad \forall\, \eta \in H^2(\Omega)\,; \tag{2.29}$$
$$|(I - \pi^h)\eta|_{m,s} \le C\,h^{1-m}\,|\eta|_{1,s} \qquad \forall\, \eta \in W^{1,s}(\Omega)\,; \tag{2.30}$$
$$\textstyle\int_\kappa \chi^2\,\mathrm{d}x \le \int_\kappa \pi^h[\chi^2]\,\mathrm{d}x \le (d+2)\,\int_\kappa \chi^2\,\mathrm{d}x\,; \tag{2.31}$$

$$|(\chi, z^h) - (\chi, z^h)^h| \leq |(I - \pi^h)(\chi\, z^h)|_{0,1} \;\leq\; C\, h^{1+m}\, |\chi|_{m,p}\, |z^h|_{1,p'}\,. \tag{2.32}$$

It follows from (2.4) that

$$(Q^h\eta)(p_j) = \frac{(\eta, \chi_j)}{(1, \chi_j)} \quad \forall\ j \in J \quad \Longrightarrow \quad |Q^h\eta|_{0,\infty} \leq |\eta|_{0,\infty} \quad \forall\ \eta \in L^\infty(\Omega). \tag{2.33}$$

In addition, it holds for $m \in \{0, 1\}$ that

$$|(I - Q^h)\eta|_{m,r} \leq C\, h^{1-m}\, |\eta|_{1,r} \quad \forall\ \eta \in W^{1,r}(\Omega) \qquad \text{for any } r \in [2, \infty]. \tag{2.34}$$

On setting $Z^h := \{z^h \in S^h : (z^h, 1) = 0\}$, it is easily established that

$$|z^h|_{0,q} \leq C\, h^{-1}\, |\mathcal{G} z^h|_{1,q} \quad \forall\ z^h \in Z^h \qquad \text{for any } q \in (1, 2]. \tag{2.35}$$

On introducing the "discrete Laplacian" operator $\Delta^h : S^h \to Z^h$ such that

$$(\Delta^h z^h, \chi)^h = -(\nabla z^h, \nabla \chi) \qquad \forall\ \chi \in S^h; \tag{2.36}$$

it follows, as Ω is convex, that for all $z^h \in S^h$

$$|z^h|_{1,s} \leq C\, |\Delta^h z^h|_0\,, \quad \text{where } s = \infty \text{ if } d = 1,\ s \in (1, \infty) \text{ if } d = 2\,; \tag{2.37}$$

see for example Barrett, Langdon & Nürnberg (2002, Lemma 3.1). Finally, we note that the results (2.34), (2.35) and (2.37) above exploit the fact that we have a quasi-uniform family of partitionings $\{\mathcal{T}^h\}_{h>0}$.

Lemma 2.1 Let the assumptions (A) hold and let $\|\cdot\|$ denote the spectral norm on $\mathbb{R}^{d\times d}$. Then for any given $\varepsilon \in (0, 1)$ the function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ satisfies

$$\varepsilon\, \xi^T \xi \leq \xi^T \Lambda_\varepsilon(z^h) \xi \leq \xi^T \xi \qquad \forall\ \xi \in \mathbb{R}^d,\ z^h \in S^h \tag{2.38}$$

and is continuous. In particular it holds for all z_1^h, $z_2^h \in S^h$, $\kappa \in \mathcal{T}^h$ that

$$\begin{aligned} \|(\Lambda_\varepsilon(z_1^h) - \Lambda_\varepsilon(z_2^h))\,|_\kappa\,\| &= \|(\widehat{\Lambda}_\varepsilon(\widehat{z}_1^h) - \widehat{\Lambda}_\varepsilon(\widehat{z}_2^h))\,|_{\widehat{\kappa}}\,\| \\ &\leq \max_{s\in\mathbb{R}} F_\varepsilon''(s)\, \max_{s\in\mathbb{R}} [F_\varepsilon''(s)]^{-1} \max_{k=1\to d} \left[\, |z_1^h(p_{j_k}) - z_2^h(p_{j_k})| + |z_1^h(p_{j_0}) - z_2^h(p_{j_0})|\,\right] \\ &\leq \varepsilon^{-1} \max_{k=1\to d} \left[\, |z_1^h(p_{j_k}) - z_2^h(p_{j_k})| + |z_1^h(p_{j_0}) - z_2^h(p_{j_0})|\,\right]\,, \end{aligned} \tag{2.39}$$

where we have adopted the notation (2.11) and (2.12).

Proof. See the proof of Lemma 2.1 in BGN. □

We introduce also for any $\varepsilon \in (0, 1)$, on recalling (1.9), the regularized function $G_\varepsilon : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$G_\varepsilon(s) := \begin{cases} G(\varepsilon) + (s - \varepsilon)\, G'(\varepsilon) + \frac{(s-\varepsilon)^2}{2}\, G''(\varepsilon) & s \leq \varepsilon \\ G(s) & \varepsilon \leq s\,. \end{cases} \tag{2.40}$$

Hence $G_\varepsilon \in C^{2,1}(\mathbb{R})$ with the first two derivatives of G_ε given by

$$G_\varepsilon'(s) := \begin{cases} G'(\varepsilon) + (s - \varepsilon)\, G''(\varepsilon) & s \leq \varepsilon \\ G'(s) & \varepsilon \leq s \end{cases} \quad \text{and} \quad G_\varepsilon''(s) := \begin{cases} G''(\varepsilon) & s \leq \varepsilon \\ G''(s) & \varepsilon \leq s\,, \end{cases} \tag{2.41}$$

respectively. Furthermore we define $\Xi_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$, for any $\varepsilon \in (0, 1)$, such that for all $z^h \in S^h$ the analogues of (2.14a,b) with G replaced by G_ε hold.

LEMMA 2.2 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0,1)$ the functions $\Xi : S^h_{>0} \to [L^\infty(\Omega)]^{d\times d}$ and $\Xi_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ satisfy for all $\kappa \in \mathcal{T}^h$, $\beta \in (0,\infty)$

$$\min_{x\in\overline{\kappa}} [z^h(x)]^\beta\, \xi^T \xi \leq \xi^T\, [\Xi(z^h)]^\beta|_\kappa\, \xi \leq \max_{x\in\overline{\kappa}} [z^h(x)]^\beta\, \xi^T\xi \qquad \forall\, \xi \in \mathbb{R}^d,\ z^h \in S^h_{>0}; \quad (2.42a)$$

$$\varepsilon^\beta\, \xi^T\xi \leq \xi^T\, [\Xi_\varepsilon(z^h)]^\beta|_\kappa\, \xi \leq \max_{x\in\overline{\kappa}} [\max\{\varepsilon, z^h(x)\}]^\beta\, \xi^T\xi \qquad \forall\, \xi \in \mathbb{R}^d,\ z^h \in S^h. \quad (2.42b)$$

Moreover, Ξ_ε is continuous. In particular it holds for all $z^h_1,\, z^h_2 \in S^h_M := \{z^h \in S^h : |z^h| \leq M\}$, $\kappa \in \mathcal{T}^h$, $\beta \in (0,\infty)$ that

$$\begin{aligned}
&\|\,(\,[\Xi_\varepsilon(z^h_1)]^\beta - [\Xi_\varepsilon(z^h_2)]^\beta\,)\,|_\kappa\,\|^{\max\{\frac{3}{\beta},1\}} \\
&\quad \leq C(\beta, M)\, \max_{|s|\leq M} G''_\varepsilon(s)\, \max_{|s|\leq M} [G''_\varepsilon(s)]^{-1} \max_{k=1\to d} \left[\, |z^h_1(p_{j_k}) - z^h_2(p_{j_k})| + |z^h_1(p_{j_0}) - z^h_2(p_{j_0})|\,\right] \\
&\quad \leq C(\beta, M, \varepsilon) \max_{k=1\to d} \left[\, |z^h_1(p_{j_k}) - z^h_2(p_{j_k})| + |z^h_1(p_{j_0}) - z^h_2(p_{j_0})|\,\right]. \qquad (2.43)
\end{aligned}$$

Proof. The desired results (2.42a,b) follow from (1.9), (2.41) and the construction of $\Xi_{(\varepsilon)}$, cf. (2.15). The continuity of Ξ_ε follows similarly to the proof of Lemma 2.1. Adopting the notation (2.11) and (2.12) we have that

$$\begin{aligned}
\|\,(\,[\Xi_\varepsilon(z^h_1)]^\beta - [\Xi_\varepsilon(z^h_2)]^\beta\,)\,|_\kappa\,\| &= \left\| \left([\widehat{\Xi}_\varepsilon(\widehat{z}^h_1)]^\beta - [\widehat{\Xi}_\varepsilon(\widehat{z}^h_2)]^\beta \right) |_{\widehat{\kappa}} \right\| \\
&= \max_{k=1\to d} \left| \left[\left([\widehat{\Xi}_\varepsilon(\widehat{z}^h_1)]^\beta - [\widehat{\Xi}_\varepsilon(\widehat{z}^h_2)]^\beta \right) |_{\widehat{\kappa}} \right]_{kk} \right| \qquad (2.44a)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \left[\left([\widehat{\Xi}_\varepsilon(\widehat{z}^h_1)]^\beta - [\widehat{\Xi}_\varepsilon(\widehat{z}^h_2)]^\beta \right) |_{\widehat{\kappa}} \right]_{kk} \right| \\
&\qquad \leq \left| \left[\left([\widehat{\Xi}_\varepsilon(\widehat{z}^h_1)]^\beta - [\widehat{\Xi}_\varepsilon(\widehat{z}^h_{1,2})]^\beta \right) |_{\widehat{\kappa}} \right]_{kk} \right| + \left| \left[\left([\widehat{\Xi}_\varepsilon(\widehat{z}^h_{1,2})]^\beta - [\widehat{\Xi}_\varepsilon(\widehat{z}^h_2)]^\beta \right) |_{\widehat{\kappa}} \right]_{kk} \right| \\
&\qquad = \left| [G''_\varepsilon(\mu_1)]^{-\frac{\beta}{3}} - [G''_\varepsilon(\zeta)]^{-\frac{\beta}{3}} \right| + \left| [G''_\varepsilon(\zeta)]^{-\frac{\beta}{3}} - [G''_\varepsilon(\mu_2)]^{-\frac{\beta}{3}} \right|, \qquad (2.44b)
\end{aligned}$$

where $z^h_{1,2} \in S^h_M$ with $z^h_{1,2}(p_{j_0}) = z^h_2(p_{j_0})$ and $z^h_{1,2}(p_{j_k}) = z^h_1(p_{j_k})$, μ_i lies between $z^h_i(p_{j_0})$ and $z^h_i(p_{j_k})$, $i = 1 \to 2$, and ζ lies between $z^h_1(p_{j_k})$ and $z^h_2(p_{j_0})$.

Similarly to the proof of Lemma 2.1 one can easily show, on noting the Lipschitz and Hölder continuity of the function $[\cdot]^{\frac{\beta}{3}}$ for $\frac{\beta}{3} \geq 1$ and $\frac{\beta}{3} < 1$, respectively, that

$$\begin{aligned}
\left| [G''_\varepsilon(\mu_1)]^{-\frac{\beta}{3}} - [G''_\varepsilon(\zeta)]^{-\frac{\beta}{3}} \right| &\leq C(\beta, M) \left| [G''_\varepsilon(\mu_1)]^{-1} - [G''_\varepsilon(\zeta)]^{-1} \right|^{\min\{\frac{\beta}{3},1\}} \\
&\leq C(\beta, M) \left[\max_{|s|\leq M} G''_\varepsilon(s) \max_{|s|\leq M} [G''_\varepsilon(s)]^{-1}\, |z^h_1(p_{j_0}) - z^h_2(p_{j_0})| \right]^{\min\{\frac{\beta}{3},1\}} \\
&\leq C(\beta, M, \varepsilon)\, |z^h_1(p_{j_0}) - z^h_2(p_{j_0})|^{\min\{\frac{\beta}{3},1\}}. \qquad (2.45)
\end{aligned}$$

Combining (2.44a,b), (2.45) and a similar result for the second term on the right hand side of (2.44b) yields the desired result (2.43). ◻

We introduce for all $\varepsilon \in (0,1)$, $\lambda_\varepsilon : \mathbb{R} \to [\varepsilon, 1]$ defined, on recalling (1.16) and (1.17), by

$$\lambda_\varepsilon(s) := [\lambda(s) - \varepsilon]_+ + \varepsilon. \qquad (2.46)$$

LEMMA 2.3 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0,1)$ the functions $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ and $\Xi : S^h_{>0} \to [L^\infty(\Omega)]^{d\times d}$ are such that for all $\kappa \in \mathcal{T}^h$

$$\max_{x\in\kappa} \|\{\Lambda_\varepsilon(z^h) - \lambda_\varepsilon(z^h)\,\mathcal{I}\}(x)\| \leq h_\kappa\,|\nabla \lambda_\varepsilon(z^h)|_{0,\infty,\kappa} \leq h_\kappa\,|\nabla z^h|_\kappa\,| \qquad \forall\ z^h \in S^h, \quad (2.47a)$$

$$\max_{x\in\kappa} \|\{[\Xi(z^h)]^\beta - [z^h]^\beta\,\mathcal{I}\}(x)\| \leq \max\{\beta,1\}\,|z^h|_{0,\infty,\kappa}^{\max\{\beta-1,0\}}\,[\,h_\kappa\,|\nabla z^h|_\kappa\,|\,]^{\min\{\beta,1\}}$$
$$\forall\ \beta \in (0,\infty),\ z^h \in S^h_{>0}, \quad (2.47b)$$

where $\mathcal{I}$ is the $d\times d$ identity matrix.

Proof. Adopting the notation of (2.11), we have from (2.12), (2.7), (2.46), (1.16) and the Lipschitz continuity of λ_ε that

$$\begin{aligned}\max_{x\in\kappa} \|\{\Lambda_\varepsilon(z^h) - \lambda_\varepsilon(z^h)\,\mathcal{I}\}(x)\| &= \max_{\widehat{x}\in\widehat{\kappa}} \|\{\widehat{\Lambda}_\varepsilon(\widehat{z}^h) - \lambda_\varepsilon(\widehat{z}^h)\,\mathcal{I}\}(\widehat{x})\| \\ &= \max_{\widehat{x}\in\widehat{\kappa}}\ \max_{k=1\to d} |\,[\widehat{\Lambda}_\varepsilon(\widehat{z}^h)]_{kk} - \lambda_\varepsilon(\widehat{z}^h)(\widehat{x})\,| \leq h_\kappa\,|\widehat{\nabla}\lambda_\varepsilon(\widehat{z}^h)|_{0,\infty,\widehat{\kappa}} \\ &= h_\kappa\,|\nabla\lambda_\varepsilon(z^h)|_{0,\infty,\kappa} \leq h_\kappa\,|\nabla z^h|_\kappa\,|, \end{aligned} \quad (2.48)$$

where we have noted that $[\widehat{\Lambda}_\varepsilon(\widehat{z}^h)\,|_\kappa]_{kk} = \lambda_\varepsilon(\widehat{z}^h(\widehat{\xi}^{(k)})) \equiv \lambda_\varepsilon(z^h(\xi^{(k)}))$ with $\xi^{(k)} \equiv \mathcal{R}_\kappa\widehat{\xi}^{(k)} \in \overline{\kappa}$ for some point $\widehat{\xi}^{(k)} \in \overline{\widehat{\kappa}}$. Hence we obtain the desired result (2.47a). Similarly to (2.48) we have from (2.15), (1.9) and the Lipschitz and Hölder continuity of the function $[\cdot]^\beta$ for $\beta \geq 1$ and $\beta < 1$, respectively, that

$$\begin{aligned}\max_{x\in\kappa} \|\{[\Xi(z^h)]^\beta - [z^h]^\beta\,\mathcal{I}\}(x)\| &= \max_{\widehat{x}\in\widehat{\kappa}} \|\{[\widehat{\Xi}(\widehat{z}^h)]^\beta - [\widehat{z}^h]^\beta\,\mathcal{I}\}(\widehat{x})\| \\ = \max_{\widehat{x}\in\widehat{\kappa}}\ \max_{k=1\to d} |\,[\widehat{\Xi}(\widehat{z}^h)]^\beta_{kk} - [\widehat{z}^h(\widehat{x})]^\beta\,| &\leq \max\{\beta,1\}\,|z^h|_{0,\infty,\kappa}^{\max\{\beta-1,0\}}\,[\,h_\kappa\,|\nabla z^h|_\kappa\,|\,]^{\min\{\beta,1\}}\,,\end{aligned}$$

and hence the desired result (2.47b). □

In order to prove existence of a solution to $(\mathrm{P}^{h,\tau}_\varepsilon)$ we need to go through a regularization procedure, which is similar to that used for the logarithmic Cahn-Hilliard equation; see e.g. Barrett, Blowey & Garcke (1999) and Barrett & Blowey (2001). For this purpose we introduce for any $\mu \in \mathbb{R}_{>0}$, on recalling (2.16), the $C^{2,1}$ convex(concave) functions $\Phi^\pm_\mu : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\Phi^\pm_\mu(s) := \begin{cases} \Phi^\pm(\mu) + (s-\mu)\,\phi^\pm(\mu) + \frac{(s-\mu)^2}{2}\,(\phi^\pm)'(\mu) & s \leq \mu \\ \Phi^\pm(s) & \mu \leq s\,. \end{cases} \quad (2.49)$$

We set $\phi^\pm_\mu := (\Phi^\pm_\mu)'$, $\Phi_\mu := \Phi^+_\mu + \Phi^-_\mu$ and note that $\Phi^+(s) \geq \Phi^+_\mu(s) \geq 0 \geq \Phi^-_\mu(s) \geq \Phi^-(s)$ for all $s \in \mathbb{R}_{>0}$. Furthermore, Φ is strictly monotonically decreasing on $(0, [\frac{\delta}{a}]^{\frac{1}{\nu-3}})$ and Φ^{-1} is uniquely defined on $\mathbb{R}_{\geq 0}$. Moreover, we recall the regularized functions $G_\mu : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $\Xi_\mu : S^h \to [L^\infty(\Omega)]^{d\times d}$, see (2.40) and (2.14a,b).

THEOREM 2.1 Let $\phi(\cdot)$ satisfy (1.4) with $a, \delta > 0$. Let the assumptions (A) hold and $U^{n-1}_\varepsilon \in S^h_{>0}$, $V^{n-1}_\varepsilon \in S^h$. Then for all $\varepsilon \in (0,1)$ and for all $h,\ \tau_n > 0$ there exists a solution $\{U^n_\varepsilon, W^n_\varepsilon, V^n_\varepsilon\} \in S^h_{>0} \times S^h \times S^h$ to the n-th step of $(\mathrm{P}^{h,\tau}_\varepsilon)$ with $⨍ U^n_\varepsilon = ⨍ U^{n-1}_\varepsilon$ and $⨍ V^n_\varepsilon = ⨍ V^{n-1}_\varepsilon$.

Proof. Due to the singular nature of the nonlinearity $\phi^+(s)$ we have to go through a regularization procedure in order to prove existence of a solution to (2.19a,b). Hence we introduce the following regularized version for any fixed $\mathcal{T}^h$, $\varepsilon \in (0,1)$ and any $\mu > 0$: Find $\{U^n_{\varepsilon,\mu}, W^n_{\varepsilon,\mu}\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\left(\tfrac{U^n_{\varepsilon,\mu} - U^{n-1}_\varepsilon}{\tau_n}, \chi\right)^h + \tfrac{1}{3}\,([\Xi_\mu(U^n_{\varepsilon,\mu})]^3\,\nabla W^n_{\varepsilon,\mu}, \nabla\chi) = -\tfrac{1}{2}\,([\Xi_\mu(U^n_{\varepsilon,\mu})]^{\frac{3}{2}}\,[\Xi(U^{n-1}_\varepsilon)]^{\frac{1}{2}}\,\nabla V^{n-1}_\varepsilon, \nabla\chi), \tag{2.50a}$$

$$c\,(\nabla U^n_{\varepsilon,\mu}, \nabla\chi) + (\phi^+_\mu(U^n_{\varepsilon,\mu}) + \phi^-_\mu(U^{n-1}_\varepsilon), \chi)^h = (W^n_{\varepsilon,\mu}, \chi)^h. \tag{2.50b}$$

We now adapt the argument in Grün (2002, Lemma 3.2) to prove existence of a solution to (2.50a,b). Let $m := \fint U^{n-1}_\varepsilon$ and $D_\mu : Z^h \to Z^h$ be such that for any $\overline{U} \in Z^h$

$$\begin{aligned}(D_\mu(\overline{U}), \chi)^h = (\overline{U} - (I - \textstyle\fint)\,U^{n-1}_\varepsilon, \chi)^h + \tfrac{\tau_n}{3}\,([\Xi_\mu(\overline{U} + m)]^3\,\nabla\overline{W}, \nabla\chi)& \\ + \tfrac{\tau_n}{2}\,([\Xi_\mu(\overline{U} + m)]^{\frac{3}{2}}\,[\Xi(U^{n-1}_\varepsilon)]^{\frac{1}{2}}\,\nabla V^{n-1}_\varepsilon, \nabla\chi) \qquad \forall\,\chi \in S^h\,,& \qquad (2.51a)\\ \text{where} \qquad \overline{W} \equiv\; -c\,\Delta^h\overline{U} + \pi^h[\phi^+_\mu(\overline{U} + m) + \phi^-_\mu(U^{n-1}_\varepsilon)]\,.& \qquad (2.51b)\end{aligned}$$

Note that solving (2.50a,b) is equivalent to finding a $\overline{U} \in Z^h$ with $D_\mu(\overline{U}) = 0$, since given such a $\overline{U}$ then $\{U^n_{\varepsilon,\mu}, W^n_{\varepsilon,\mu}\} \equiv \{\overline{U} + m, -c\,\Delta^h\overline{U} + \pi^h[\phi^+_\mu(\overline{U} + m) + \phi^-_\mu(U^{n-1}_\varepsilon)]\}$ solves (2.50a,b).

Assume that for a given $R > 0$ there does not exist a $\overline{U} \in Z^h_R := \{z^h \in Z^h : |z^h|_1 \le R\}$ with $D_\mu(\overline{U}) = 0$. For any fixed $\mu > 0$ and $R > 0$, it follows immediately from (2.51a,b), (2.36), (2.26), (2.27), (2.43) and (2.49) that D_μ is continuous on Z^h_R. Hence we can define the continuous function $B_\mu : Z^h_R \to Z^h_R$, where $B_\mu(\overline{U}) := -\frac{R\,D_\mu(\overline{U})}{|D_\mu(\overline{U})|_1}$. As Z^h_R is a convex and compact subset of the finite dimensional space S^h, the *Brouwer fixed point theorem* (see e.g. Renardy & Rogers (1992)) asserts that there exists a $\overline{U}_R \in Z^h_R$ with $B_\mu(\overline{U}_R) = \overline{U}_R$. Moreover $|\overline{U}_R|_1 = R$. We will now prove a contradiction for R sufficiently large.

Let $\overline{W}_R \in S^h$ be defined by (2.51b) with $\overline{U}$ replaced by $\overline{U}_R$. Then (2.36), the monotonicity of ϕ^+_μ, (1.8), (2.31) and (1.22) yield that

$$\begin{aligned}(\overline{U}_R, \overline{W}_R)^h &= c\,|\overline{U}_R|_1^2 + (\overline{U}_R, \phi^+_\mu(\overline{U}_R + m) + \phi^-_\mu(U^{n-1}_\varepsilon))^h \\ &= c\,|\overline{U}_R|_1^2 + (\overline{U}_R + m - m, \phi^+_\mu(\overline{U}_R + m) - \phi^+_\mu(m))^h + (\overline{U}_R, \phi^+_\mu(m) + \phi^-_\mu(U^{n-1}_\varepsilon))^h \\ &\ge c\,|\overline{U}_R|_1^2 - \gamma\,|\overline{U}_R|_h^2 - C(\gamma, U^{n-1}_\varepsilon) \ge \tfrac{c}{2}\,|\overline{U}_R|_1^2 - C(\gamma, U^{n-1}_\varepsilon) = \tfrac{c}{2}\,R^2 - C(\gamma, U^{n-1}_\varepsilon) \\ &\qquad \forall\,\mu \in (0, m]\,, \qquad (2.52)\end{aligned}$$

for $\gamma > 0$ sufficiently small. Similarly to (2.52), and on noting the elementary identity

$$2\,r\,(r - s) = (r^2 - s^2) + (r - s)^2 \quad \forall\,r, s \in \mathbb{R}\,, \tag{2.53}$$

we obtain from (2.51a,b), (1.8) and the monotonicity of ϕ^+_μ that for all $\mu \in (0, \mu_1]$, where

$\mu_1 := \min_{x\in\overline{\Omega}} U_\varepsilon^{n-1}(x)$,

$$\begin{aligned}(D_\mu(\overline{U}_R),\overline{W}_R)^h &= (\overline{U}_R+m-U_\varepsilon^{n-1},\overline{W}_R)^h + \tfrac{\tau_n}{3}\,([\Xi_\mu(\overline{U}_R+m)]^3\,\nabla\overline{W}_R,\nabla\overline{W}_R)\\ &\qquad + \tfrac{\tau_n}{2}\,([\Xi_\mu(\overline{U}_R+m)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}\,\nabla V_\varepsilon^{n-1},\nabla\overline{W}_R)\\ &\geq \tfrac{c}{2}\,|\overline{U}_R|_1^2 - \tfrac{c}{2}\,|U_\varepsilon^{n-1}|_1^2 + (\overline{U}_R+m-U_\varepsilon^{n-1},\phi_\mu^+(\overline{U}_R+m)+\phi_\mu^-(U_\varepsilon^{n-1}))^h\\ &\qquad - \tfrac{\tau_n}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1},\nabla V_\varepsilon^{n-1})\\ &\geq \tfrac{c}{2}\,|\overline{U}_R|_1^2 - \tfrac{c}{2}\,|U_\varepsilon^{n-1}|_1^2 + (\overline{U}_R+m-U_\varepsilon^{n-1},\phi_\mu^+(U_\varepsilon^{n-1})+\phi_\mu^-(U_\varepsilon^{n-1}))^h\\ &\qquad - \tfrac{\tau_n}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1},\nabla V_\varepsilon^{n-1})\\ &\geq \tfrac{c}{4}\,|\overline{U}_R|_1^2 - C(\gamma,\tau_n,U_\varepsilon^{n-1},V_\varepsilon^{n-1}) = \tfrac{c}{4}\,R^2 - C(\gamma,\tau_n,U_\varepsilon^{n-1},V_\varepsilon^{n-1})\,,\end{aligned} \tag{2.54}$$

again for $\gamma > 0$ sufficiently small. Combining (2.52) and (2.54) yields for any $\mu \in (0,\mu_1]$ that

$$(\overline{U}_R,\overline{W}_R)^h > 0 \quad \text{and} \quad (D_\mu(\overline{U}_R),\overline{W}_R)^h > 0 \qquad \text{for } R \text{ sufficiently large}\,, \tag{2.55}$$

which clearly contradicts $(\overline{U}_R,\overline{W}_R)^h = (B_\mu(\overline{U}_R),\overline{W}_R)^h = -\frac{R}{|D_\mu(\overline{U}_R)|_1}\,(D_\mu(\overline{U}_R),\overline{W}_R)^h < 0$. Hence we have existence of a solution $\{U_{\varepsilon,\mu}^n, W_{\varepsilon,\mu}^n\}$ to (2.50a,b) for all $\mu \in (0,\mu_1]$.

We will now show that, for μ sufficiently small, $\{U_{\varepsilon,\mu}^n, W_{\varepsilon,\mu}^n\}$ solves (2.19a,b). Choosing $\chi \equiv W_{\varepsilon,\mu}^n$ in (2.50a) and $\chi \equiv U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1}$ in (2.50b) yields that

$$\begin{aligned}&(U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1}, W_{\varepsilon,\mu}^n)^h + \tfrac{\tau_n}{3}\,([\Xi_\mu(U_{\varepsilon,\mu}^n)]^3\,\nabla W_{\varepsilon,\mu}^n,\nabla W_{\varepsilon,\mu}^n)\\ &\qquad\qquad = -\tfrac{\tau_n}{2}\,([\Xi_\mu(U_{\varepsilon,\mu}^n)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}\,\nabla V_\varepsilon^{n-1},\nabla W_{\varepsilon,\mu}^n)\,,\end{aligned} \tag{2.56a}$$

$$\begin{aligned}&c\,(\nabla U_{\varepsilon,\mu}^n,\nabla(U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1})) + (\phi_\mu^+(U_{\varepsilon,\mu}^n)+\phi_\mu^-(U_\varepsilon^{n-1}),U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1})^h\\ &\qquad\qquad = (W_{\varepsilon,\mu}^n, U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1})^h\,.\end{aligned} \tag{2.56b}$$

On noting (2.53), the convexity of Φ_μ^+ and the concavity of Φ_μ^-, it follows from (2.56b) that

$$\begin{aligned}&\tfrac{c}{2}\,|U_{\varepsilon,\mu}^n|_1^2 + \tfrac{c}{2}\,|U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1}|_1^2 + (\Phi_\mu(U_{\varepsilon,\mu}^n),1)^h\\ &\qquad\qquad \leq \tfrac{c}{2}\,|U_\varepsilon^{n-1}|_1^2 + (\Phi_\mu(U_\varepsilon^{n-1}),1)^h + (W_{\varepsilon,\mu}^n, U_{\varepsilon\,\mu}^n - U_\varepsilon^{n-1})^h\,.\end{aligned} \tag{2.57}$$

Combining (2.56a) and (2.57) yields for $\mu \in (0,\mu_1]$ that

$$\begin{aligned}&\tfrac{c}{2}\,|U_{\varepsilon,\mu}^n|_1^2 + (\Phi_\mu(U_{\varepsilon,\mu}^n),1)^h + \tfrac{\tau_n}{3}\,([\Xi_\mu(U_{\varepsilon,\mu}^n)]^3\,\nabla W_{\varepsilon,\mu}^n,\nabla W_{\varepsilon,\mu}^n) + \tfrac{c}{2}\,|U_{\varepsilon,\mu}^n - U_\varepsilon^{n-1}|_1^2\\ &\quad \leq \tfrac{c}{2}\,|U_\varepsilon^{n-1}|_1^2 + (\Phi(U_\varepsilon^{n-1}),1)^h - \tfrac{\tau_n}{2}\,([\Xi_\mu(U_{\varepsilon,\mu}^n)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}\,\nabla V_\varepsilon^{n-1},\nabla W_{\varepsilon,\mu}^n)\\ &\quad \leq C\,\left[|U_\varepsilon^{n-1}|_1^2 + (\Phi(U_\varepsilon^{n-1}),1)^h + \tfrac{\tau_n}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1},\nabla V_\varepsilon^{n-1})\right].\end{aligned} \tag{2.58}$$

The bound (2.58) yields the existence of constants C_i independent of $\mu \in (0,\mu_1]$ such that

$$(\Phi_\mu(U_{\varepsilon,\mu}^n),1)^h \leq C_1 \quad\Rightarrow\quad \Phi_\mu(U_{\varepsilon,\mu}^n(x_j)) \leq C_2\,h_{\min}^{-d} =: C_3 \quad \forall\ j \in J\,, \tag{2.59}$$

where $h_{\min} := \min_{\kappa\in\mathcal{T}^h} h_\kappa$. Combining (2.59), (2.49) and (2.16) yields for all $\mu \in (0,\mu_2]$, where $\mu_2 := \min\{\Phi^{-1}(C_3),\mu_1\}$, and for all $j \in J$ that

$$\Phi(U_{\varepsilon,\mu}^n(x_j)) = \Phi_\mu(U_{\varepsilon,\mu}^n(x_j)) \leq C_3 \quad\Rightarrow\quad U_{\varepsilon,\mu}^n(x_j) \geq \mu_2 > 0\,. \tag{2.60}$$

This yields that for all $\mu \in (0,\mu_2]$ a solution $\{U^n_{\varepsilon,\mu}, W^n_{\varepsilon,\mu}\}$ to (2.50a,b) solves (2.19a,b) as $\Xi_\mu(U^n_{\varepsilon,\mu}) \equiv \Xi(U^n_{\varepsilon,\mu})$, $\phi^+_\mu(U^n_{\varepsilon,\mu}) \equiv \phi^+(U^n_{\varepsilon,\mu})$ and $\phi^-_\mu(U^{n-1}_\varepsilon) \equiv \phi^-(U^{n-1}_\varepsilon)$. Hence we have existence of a solution $\{U^n_\varepsilon, W^n_\varepsilon\}$ to (2.19a,b).

Finally, existence of a solution V^n_ε to (2.19c) follows from a similar, but simpler, application of the *Brouwer fixed point theorem*; see the proof of Theorem 2.1 in BGN for details. □

LEMMA 2.4 Let the assumptions of Theorem 2.1 hold. Then for all $\varepsilon \in (0,1)$ and for all h, $\tau_n > 0$ a solution $\{U^n_\varepsilon, W^n_\varepsilon, V^n_\varepsilon\}$ to the n-th step of $(\mathrm{P}^{h,\tau}_\varepsilon)$ is such that

$$\begin{aligned}\mathcal{E}(U^n_\varepsilon, V^n_\varepsilon) + \tfrac{c}{2}\,|U^n_\varepsilon - U^{n-1}_\varepsilon|^2_1 + \tfrac12\,|V^n_\varepsilon - V^{n-1}_\varepsilon|^2_h + \rho\,\tau_n\,(\nabla V^n_\varepsilon, \nabla\pi^h[F'_\varepsilon(V^n_\varepsilon)])& \\ + \tfrac{\tau_n}{24}\,([\Xi(U^n_\varepsilon)]^3\,\nabla W^n_\varepsilon, \nabla W^n_\varepsilon) + \tfrac58\,\tau_n\,(\Xi(U^n_\varepsilon)\,\nabla V^n_\varepsilon, \nabla V^n_\varepsilon)& \\ \leq \mathcal{E}(U^{n-1}_\varepsilon, V^{n-1}_\varepsilon) + \tfrac{\tau_n}{2}\,(\Xi(U^{n-1}_\varepsilon)\,\nabla V^{n-1}_\varepsilon, \nabla V^{n-1}_\varepsilon)\,,& \end{aligned} \tag{2.61a}$$

where

$$\mathcal{E}(U^n_\varepsilon, V^n_\varepsilon) := \tfrac{c}{2}\,|U^n_\varepsilon|^2_1 + (\Phi(U^n_\varepsilon) + F_\varepsilon(V^n_\varepsilon), 1)^h\,. \tag{2.61b}$$

Furthermore if $\phi(\cdot)$ satisfies (1.4) with $\nu \geq 7$, then

$$\begin{aligned}(G(U^n_\varepsilon),1)^h + \tfrac{\tau_n}{4}\,(\nabla\pi^h[\phi^+(U^n_\varepsilon)], \nabla U^n_\varepsilon) + \tfrac{c}{3}\,\tau_n\,|\Delta^h U^n_\varepsilon|^2_h& \\ \leq (G(U^{n-1}_\varepsilon),1)^h + \tfrac{\tau_n}{8}(\nabla\pi^h[\phi^+(U^{n-1}_\varepsilon)], \nabla U^{n-1}_\varepsilon) + C\,\tau_n\,[\,|U^n_\varepsilon|^2_1 + |U^{n-1}_\varepsilon|^2_1\,]& \\ + \tfrac{\tau_n}{4}\,(\Xi(U^{n-1}_\varepsilon)\,\nabla V^{n-1}_\varepsilon, \nabla V^{n-1}_\varepsilon)\,.& \end{aligned} \tag{2.62}$$

Proof. Similarly to (2.58) we obtain, on choosing $\chi \equiv W^n_\varepsilon$ in (2.19a) and $\chi \equiv U^n_\varepsilon - U^{n-1}_\varepsilon$ in (2.19b), that

$$\begin{aligned}\tfrac{c}{2}\,|U^n_\varepsilon|^2_1 + (\Phi(U^n_\varepsilon),1)^h + \tfrac{\tau_n}{3}\,([\Xi(U^n_\varepsilon)]^3\,\nabla W^n_\varepsilon, \nabla W^n_\varepsilon) + \tfrac{c}{2}\,|U^n_\varepsilon - U^{n-1}_\varepsilon|^2_1& \\ \leq \tfrac{c}{2}\,|U^{n-1}_\varepsilon|^2_1 + (\Phi(U^{n-1}_\varepsilon),1)^h - \tfrac{\tau_n}{2}\,([\Xi(U^n_\varepsilon)]^{\frac32}\,[\Xi(U^{n-1}_\varepsilon)]^{\frac12}\,\nabla V^{n-1}_\varepsilon, \nabla W^n_\varepsilon)\,.& \end{aligned} \tag{2.63}$$

Choosing $\chi \equiv \pi^h[F'_\varepsilon(V^n_\varepsilon)]$ in (2.19c) and noting (2.9a,b) yields that

$$\begin{aligned}(V^n_\varepsilon - V^{n-1}_\varepsilon, F'_\varepsilon(V^n_\varepsilon))^h + \rho\,\tau_n\,(\nabla V^n_\varepsilon, \nabla\pi^h[F'_\varepsilon(V^n_\varepsilon)]) + \tau_n\,(\Xi(U^n_\varepsilon)\,\nabla V^n_\varepsilon, \nabla V^n_\varepsilon)& \\ = -\tfrac{\tau_n}{2}\,([\Xi(U^n_\varepsilon)]^2\,\nabla W^n_\varepsilon, \nabla V^n_\varepsilon)\,.& \end{aligned} \tag{2.64}$$

Now $F''_\varepsilon \geq 1$ implies that

$$(V^n_\varepsilon - V^{n-1}_\varepsilon, F'_\varepsilon(V^n_\varepsilon))^h \geq (F_\varepsilon(V^n_\varepsilon) - F_\varepsilon(V^{n-1}_\varepsilon), 1)^h + \tfrac12\,|V^{n-1}_\varepsilon - V^n_\varepsilon|^2_h\,. \tag{2.65}$$

Combining (2.63), (2.64), (2.65) and noting (1.8) and (2.61b) yields that

$$\begin{aligned}\mathcal{E}(U^n_\varepsilon, V^n_\varepsilon) + \tfrac{c}{2}\,|U^n_\varepsilon - U^{n-1}_\varepsilon|^2_1 + \tfrac12\,|V^n_\varepsilon - V^{n-1}_\varepsilon|^2_h + \rho\,\tau_n\,(\nabla V^n_\varepsilon, \nabla\pi^h[F'_\varepsilon(V^n_\varepsilon)])& \\ + \tfrac{\tau_n}{3}\,([\Xi(U^n_\varepsilon)]^3\,\nabla W^n_\varepsilon, \nabla W^n_\varepsilon) + \tau_n\,(\Xi(U^n_\varepsilon)\,\nabla V^n_\varepsilon, \nabla V^n_\varepsilon)& \\ \leq \mathcal{E}(U^{n-1}_\varepsilon, V^{n-1}_\varepsilon) - \tfrac{\tau_n}{2}\,([\Xi(U^n_\varepsilon)]^2\,\nabla W^n_\varepsilon, \nabla V^n_\varepsilon) - \tfrac{\tau_n}{2}\,([\Xi(U^n_\varepsilon)]^{\frac32}\,[\Xi(U^{n-1}_\varepsilon)]^{\frac12}\,\nabla W^n_\varepsilon, \nabla V^{n-1}_\varepsilon)& \\ \leq \mathcal{E}(U^{n-1}_\varepsilon, V^{n-1}_\varepsilon) + \tfrac{\zeta+\gamma}{4}\,\tau_n\,([\Xi(U^n_\varepsilon)]^3\,\nabla W^n_\varepsilon, \nabla W^n_\varepsilon)& \\ + \tfrac{\tau_n}{4\zeta}\,(\Xi(U^{n-1}_\varepsilon)\,\nabla V^{n-1}_\varepsilon, \nabla V^{n-1}_\varepsilon) + \tfrac{\tau_n}{4\gamma}\,(\Xi(U^n_\varepsilon)\,\nabla V^n_\varepsilon, \nabla V^n_\varepsilon),& \end{aligned} \tag{2.66}$$

for arbitrary ζ, $\gamma > 0$. Choosing $\zeta = \frac{1}{2}$ and $\gamma = \frac{2}{3}$ leads to the desired result (2.61a).

Furthermore, choosing $\chi \equiv \pi^h[G'(U_\varepsilon^n)]$ in (2.19a) and $\chi \equiv -\Delta^h U_\varepsilon^n$ in (2.19b) yields on noting (2.14b) and (2.36) that

$$\begin{aligned}\left(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}, G'(U_\varepsilon^n)\right)^h + \tfrac{1}{3}\,(\nabla W_\varepsilon^n, \nabla U_\varepsilon^n) &= -\tfrac{1}{2}\,([\Xi(U_\varepsilon^n)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}\,\nabla V_\varepsilon^{n-1}, \nabla\pi^h[G'(U_\varepsilon^n)]) \\ &\leq \tfrac{1}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1}, \nabla V_\varepsilon^{n-1}) + \tfrac{1}{4}\,(\nabla\pi^h[G'(U_\varepsilon^n)], \nabla U_\varepsilon^n)\,, \qquad (2.67a)\\ (\nabla W_\varepsilon^n, \nabla U_\varepsilon^n) &= c\,|\Delta^h U_\varepsilon^n|_h^2 + (\nabla\pi^h[\phi^+(U_\varepsilon^n) + \phi^-(U_\varepsilon^{n-1})], \nabla U_\varepsilon^n)\,. \qquad (2.67b)\end{aligned}$$

On noting the convexity of G it follows from (2.67a,b) that

$$\begin{aligned}&(G(U_\varepsilon^n) - G(U_\varepsilon^{n-1}), 1)^h + \tfrac{c}{3}\,\tau_n\,|\Delta^h U_\varepsilon^n|_h^2 + \tfrac{\tau_n}{3}\,(\nabla\pi^h[\phi^+(U_\varepsilon^n)], \nabla U_\varepsilon^n)\\ &\quad\leq \tfrac{\tau_n}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1}, \nabla V_\varepsilon^{n-1}) + \tfrac{\tau_n}{4}(\nabla\pi^h[G'(U_\varepsilon^n)], \nabla U_\varepsilon^n) - \tfrac{\tau_n}{3}(\nabla\pi^h[\phi^-(U_\varepsilon^{n-1})], \nabla U_\varepsilon^n)\,. \qquad (2.68)\end{aligned}$$

If $\nu \geq 7$, it follows from (2.68), (2.17b) and (2.18) that

$$\begin{aligned}&(G(U_\varepsilon^n) - G(U_\varepsilon^{n-1}), 1)^h + \tfrac{c}{3}\,\tau_n\,|\Delta^h U_\varepsilon^n|_h^2 + (\tfrac{1}{3} - \tfrac{\gamma}{4})\,\tau_n\,(\nabla\pi^h[\phi^+(U_\varepsilon^n)], \nabla U_\varepsilon^n)\\ &\quad\leq \tfrac{\tau_n}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1}, \nabla V_\varepsilon^{n-1}) + C(\gamma,\xi)\,\tau_n\,|U_\varepsilon^n|_1^2 + \tfrac{\xi}{6}\,\tau_n\,|\nabla\pi^h[\phi^-(U_\varepsilon^{n-1})]|_0^2\\ &\quad\leq \tfrac{\tau_n}{4}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1}, \nabla V_\varepsilon^{n-1}) + C(\gamma,\xi)\,\tau_n\,|U_\varepsilon^n|_1^2 + C\,\xi\,\tau_n\,|U_\varepsilon^{n-1}|_1^2\\ &\qquad + C\,\xi\,\tau_n\,(\nabla\pi^h[\phi^+(U_\varepsilon^{n-1})], \nabla U_\varepsilon^{n-1}) \qquad (2.69)\end{aligned}$$

for arbitrary ξ, $\gamma > 0$. The desired result (2.62) then follows on choosing ξ and γ appropriately. □

Remark 2.3 We note that (2.61a,b) and (2.62) are the discrete analogues of the formal energy estimates (1.6) and (1.12), respectively, on recalling (1.7) and (1.15).

Theorem 2.2 Let $\phi(\cdot)$ satisfy (1.4) with $a, \delta > 0$. Let the assumptions (A) hold and $U_\varepsilon^0 \in S_{>0}^h$, $V_\varepsilon^0 \in S^h$. Then for all $\varepsilon \in (0,1)$, $h > 0$ a solution $\{U_\varepsilon^n, W_\varepsilon^n, V_\varepsilon^n\}_{n=1}^N$ to $(\mathrm{P}_\varepsilon^{h,\tau})$ is such that $U_\varepsilon^n > 0$, $⨍ U_\varepsilon^n = ⨍ U_\varepsilon^0$ and $⨍ V_\varepsilon^n = ⨍ V_\varepsilon^0$, $n = 1 \to N$, and if $\tau_n \leq \frac{5}{4}\,\omega\,\tau_{n-1}$, $n = 2 \to N$ for an $\omega \in (0,1)$, then

$$\begin{aligned}&c \max_{n=1\to N} \|U_\varepsilon^n\|_1^2 + \max_{n=1\to N} (\Phi(U_\varepsilon^n), 1)^h + \max_{n=1\to N} (F_\varepsilon(V_\varepsilon^n), 1)^h\\ &\quad + \max_{n=1\to N} |V_\varepsilon^n|_0^2 + \varepsilon^{-1} \max_{n=1\to N} |\pi^h[V_\varepsilon^n]_-|_0^2\\ &\quad + c\sum_{n=1}^N \|U_\varepsilon^n - U_\varepsilon^{n-1}\|_1^2 + \sum_{n=1}^N |V_\varepsilon^n - V_\varepsilon^{n-1}|_0^2 + \rho\sum_{n=1}^N \tau_n(\nabla V_\varepsilon^n, \nabla\pi^h[F_\varepsilon'(V_\varepsilon^n)])\\ &\quad + \rho\sum_{n=1}^N \tau_n\|V_\varepsilon^n\|_1^2 + \sum_{n=1}^N \tau_n\,([\Xi(U_\varepsilon^n)]^3\,\nabla W_\varepsilon^n, \nabla W_\varepsilon^n) + (1-\omega)\sum_{n=1}^N \tau_n\,(\Xi(U_\varepsilon^n)\,\nabla V_\varepsilon^n, \nabla V_\varepsilon^n)\\ &\qquad \leq C\left[\,1 + \|U_\varepsilon^0\|_1^2 + (\Phi(U_\varepsilon^0) + F_\varepsilon(V_\varepsilon^0), 1)^h + (\Xi(U_\varepsilon^0)\,\nabla V_\varepsilon^0, \nabla V_\varepsilon^0)\,\right]. \qquad (2.70)\end{aligned}$$

In addition

$$\begin{aligned}&\sum_{n=1}^N \tau_n \left|\mathcal{G}[\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}]\right|_{1,q}^2 + \sum_{n=1}^N \tau_n \left|\mathcal{G}[\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\tau_n}]\right|_{1,q}^2\\ &\quad \leq C(\max_{n=0\to N} |[\Xi(U_\varepsilon^n)]^3|_{0,r})\left[\,1 + \|U_\varepsilon^0\|_1^2 + (\Phi(U_\varepsilon^0) + F_\varepsilon(V_\varepsilon^0), 1)^h + (\Xi(U_\varepsilon^0)\,\nabla V_\varepsilon^0, \nabla V_\varepsilon^0)\,\right], \qquad (2.71a)\end{aligned}$$

where $q = 2$ and $r = \infty$ if $d = 1$, $q \in (1, 2)$ and $r = \frac{2q}{2-q}$ if $d = 2$, and

$$|[\Xi(U_\varepsilon^n)]^\beta|_{0,s} \leq C \, \|U_\varepsilon^n\|_1^\beta \qquad \forall\, \beta \in (0, \infty), \quad \forall\, s \in \begin{cases} [1, \infty] & \text{if } \ d = 1, \\ [1, \infty) & \text{if } \ d = 2. \end{cases} \tag{2.71b}$$

Furthermore if $\phi(\cdot)$ satisfies (1.4) with $\nu \geq 7$, then

$$\begin{aligned} \max_{n=1\to N} (G(U_\varepsilon^n), 1)^h + c \sum_{n=1}^N \tau_n \, |\Delta^h U_\varepsilon^n|_h^2 + \sum_{n=1}^N \tau_n \, (\nabla \pi^h[\phi^+(U_\varepsilon^n)], \nabla U_\varepsilon^n) \\ \leq C \left[1 + \|U_\varepsilon^0\|_1^2 + (\Phi(U_\varepsilon^0) + F_\varepsilon(V_\varepsilon^0) + G(U_\varepsilon^0), 1)^h + (\nabla \pi^h[\phi^+(U_\varepsilon^0)], \nabla U_\varepsilon^0) \right. \\ \left. + (\Xi(U_\varepsilon^0) \, \nabla V_\varepsilon^0, \nabla V_\varepsilon^0) \right]. \end{aligned} \tag{2.72}$$

Proof. Summing (2.61a) from $n = 1 \to k$ and observing that $\tau_n \leq \frac{5}{4} \, \omega \, \tau_{n-1}$, $n = 2 \to k$, yields for any $k \leq N$ that

$$\begin{aligned} \mathcal{E}(U_\varepsilon^k, V_\varepsilon^k) + \tfrac{1}{2} \sum_{n=1}^k \left[c \, |U_\varepsilon^n - U_\varepsilon^{n-1}|_1^2 + |V_\varepsilon^n - V_\varepsilon^{n-1}|_h^2 \right] + \rho \sum_{n=1}^k \tau_n (\nabla V_\varepsilon^n, \nabla \pi^h[F_\varepsilon'(V_\varepsilon^n)]) \\ + \tfrac{1}{24} \sum_{n=1}^k \tau_n \, ([\Xi(U_\varepsilon^n)]^3 \, \nabla W_\varepsilon^n, \nabla W_\varepsilon^n) + \tfrac{5}{8} \, (1 - \omega) \sum_{n=1}^k \tau_n \, (\Xi(U_\varepsilon^n) \, \nabla V_\varepsilon^n, \nabla V_\varepsilon^n) \\ \leq \mathcal{E}(U_\varepsilon^0, V_\varepsilon^0) + \tfrac{\tau_1}{2} \, (\Xi(U_\varepsilon^0) \, \nabla V_\varepsilon^0, \nabla V_\varepsilon^0) \,. \end{aligned} \tag{2.73}$$

Therefore, on noting (2.8) and (2.17a), the bounds $1 \to 3$, $6 \to 8$ and $10 \to 11$ in (2.70) follow from (2.73), $U_\varepsilon^n - U_\varepsilon^{n-1} \in Z^h$, (1.22), (2.3) and (2.31). Combining the third bound in (2.70) and (2.8) yields the bounds $4 \to 5$ in (2.70). Bounds 4 and 8 in (2.70) yield, on noting (2.13), bound 9 in (2.70).

From (1.21), (2.4), (2.19a) and (2.34) we obtain for any $\eta \in W^{1,q'}(\Omega)$ that

$$\begin{aligned} (\nabla \mathcal{G}[\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}], \nabla \eta) &= (\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}, \eta) = (\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\tau_n}, Q^h \eta)^h \\ &= -(\tfrac{1}{3} \, [\Xi(U_\varepsilon^n)]^3 \nabla W_\varepsilon^n + \tfrac{1}{2} [\Xi(U_\varepsilon^n)]^{\frac{3}{2}} \, [\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}} \nabla V_\varepsilon^{n-1}, \nabla Q^h \eta) \\ &\leq C \, |[\Xi(U_\varepsilon^n)]^{\frac{3}{2}}|_{0,r} \left[|[\Xi(U_\varepsilon^n)]^{\frac{3}{2}} \, \nabla W_\varepsilon^n|_0 + |[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}} \, \nabla V_\varepsilon^{n-1}|_0 \right] \, |Q^h \eta|_{1,q'} \\ &\leq C \, |[\Xi(U_\varepsilon^n)]^{\frac{3}{2}}|_{0,r} \left[|[\Xi(U_\varepsilon^n)]^{\frac{3}{2}} \, \nabla W_\varepsilon^n|_0 + |[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}} \, \nabla V_\varepsilon^{n-1}|_0 \right] \, |\eta|_{1,q'}. \end{aligned} \tag{2.74}$$

Similarly to (2.74), from (1.21), (2.4), (2.19c), (2.38) and (2.34) we obtain that

$$\begin{aligned} &(\nabla \mathcal{G}[\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\tau_n}], \nabla \eta) = (\tfrac{V_\varepsilon^n - V_\varepsilon^{n-1}}{\tau_n}, Q^h \eta)^h \\ &= -\rho \, (\nabla V_\varepsilon^n, \nabla Q^h \eta) - (\Xi(U_\varepsilon^n) \, \Lambda_\varepsilon(V_\varepsilon^n) \, \nabla V_\varepsilon^n + \tfrac{1}{2} \, [\Xi(U_\varepsilon^n)]^2 \Lambda_\varepsilon(V_\varepsilon^n) \, \nabla W_\varepsilon^n, \nabla Q^h \eta) \\ &\leq C \left[\rho \, |\nabla V_\varepsilon^n|_0 + |[\Xi(U_\varepsilon^n)]^{\frac{1}{2}}|_{0,r} \left(|[\Xi(U_\varepsilon^n)]^{\frac{1}{2}} \, \nabla V_\varepsilon^n|_0 + |[\Xi(U_\varepsilon^n)]^{\frac{3}{2}} \, \nabla W_\varepsilon^n|_0 \right) \right] \, |\eta|_{1,q'}. \end{aligned} \tag{2.75}$$

Combining (2.74), (2.75), the assumptions on τ_n and the bounds $9 \to 11$ in (2.70) yields the bounds (2.71a).

The bound (2.42a) yields for any $\kappa \in \mathcal{T}^h$ with vertices $\{p_i\}_{i=0}^d$ that

$$\|\Xi(U_\varepsilon^n)\,|_\kappa\,\| \leq \max_{i=0\to d} |U_\varepsilon^n(p_i)| \quad \Rightarrow \quad \|[\Xi(U_\varepsilon^n)]^\beta\,|_\kappa\,\| \leq \max_{i=0\to d} |U_\varepsilon^n(p_i)|^\beta \quad \forall\, \beta \in (0,\infty)\,. \tag{2.76}$$

Now (2.71b), at first for $\beta\, s \geq 1$, follows from (2.76), (2.27) and (1.19). This preliminary result then immediately yields the desired result (2.71b) for all stated β and s.

Finally, summing (2.62) from $n = 1 \to k$ and observing that $\tau_n \leq \frac{5}{4}\,\omega\,\tau_{n-1}$, $n = 2 \to k$, yields for any $k \leq N$ that

$$\begin{aligned}(G(U_\varepsilon^k),1)^h &+ (\tfrac{1}{4} - \tfrac{5}{32}\,\omega) \sum_{n=1}^k \tau_n\,(\nabla \pi^h[\phi^+(U_\varepsilon^n)], \nabla U_\varepsilon^n) + \tfrac{c}{3} \sum_{n=1}^k \tau_n\,|\Delta^h U_\varepsilon^n|_h^2 \\ &\leq (G(U_\varepsilon^0),1)^h + C \sum_{n=1}^k \tau_n\,|U_\varepsilon^n|_1^2 + \tfrac{5}{16}\,\omega \sum_{n=2}^k \tau_{n-1}\,(\Xi(U_\varepsilon^{n-1})\,\nabla V_\varepsilon^{n-1}, \nabla V_\varepsilon^{n-1}) \\ &\quad + C\,\tau_1\,|U_\varepsilon^0|_1^2 + \tfrac{\tau_1}{8}\,(\nabla \pi^h[\phi^+(U_\varepsilon^0)], \nabla U_\varepsilon^0) + \tfrac{\tau_1}{4}\,(\Xi(U_\varepsilon^0)\,\nabla V_\varepsilon^0, \nabla V_\varepsilon^0)\,. \end{aligned} \tag{2.77}$$

The bounds (2.72) follow from (2.77) and the bounds 1 and 11 in (2.70). □

LEMMA 2.5 Let u^0, $v^0 \in H^1_{\geq 0}(\Omega)$, with $u^0 \in L^\infty(\Omega)$ and $u^0(x) \geq \zeta > 0$ for a.e. $x \in \Omega$, and the assumptions (A) hold. On choosing either $U_\varepsilon^0 \equiv Q^h u^0$ and $V_\varepsilon^0 \equiv Q^h v^0$, or $U_\varepsilon^0 \equiv \pi^h u^0$ and $V_\varepsilon^0 \equiv \pi^h v^0$ if either $d = 1$ or u^0, $v^0 \in W^{1,e}(\Omega)$ with $e > 2$; it follows that U_ε^0, $V_\varepsilon^0 \in S^h_{\geq 0}$ with $U_\varepsilon^0 \geq \zeta$ are such that for all $h > 0$

$$\|U_\varepsilon^0\|_1^2 + (\Phi(U_\varepsilon^0) + F_\varepsilon(V_\varepsilon^0) + G(U_\varepsilon^0),1)^h + (\nabla \pi^h[\phi^+(U_\varepsilon^0)], \nabla U_\varepsilon^0) + (\Xi(U_\varepsilon^0)\,\nabla V_\varepsilon^0, \nabla V_\varepsilon^0) \leq C. \tag{2.78}$$

Proof. The desired result (2.78) follows immediately from (2.33), (2.30), (2.34), (2.16), (2.6), (1.9), (1.4) and (2.42a). □

3. Convergence

Let

$$U_\varepsilon(t) := \tfrac{t-t_{n-1}}{\tau_n}\,U_\varepsilon^n + \tfrac{t_n-t}{\tau_n}\,U_\varepsilon^{n-1} \qquad t \in [t_{n-1}, t_n] \quad n \geq 1 \tag{3.1a}$$

and

$$U_\varepsilon^+(t) := U_\varepsilon^n, \qquad U_\varepsilon^-(t) := U_\varepsilon^{n-1} \qquad t \in (t_{n-1}, t_n] \quad n \geq 1. \tag{3.1b}$$

We note for future reference that

$$U_\varepsilon - U_\varepsilon^\pm = (t - t_n^\pm)\,\tfrac{\partial U_\varepsilon}{\partial t} \qquad t \in (t_{n-1}, t_n) \quad n \geq 1, \tag{3.2}$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$. We introduce also

$$\bar{\tau}(t) := \tau_n \qquad t \in (t_{n-1}, t_n] \quad n \geq 1. \tag{3.3}$$

Using (2.36) and the above notation, and introducing analogous notation for W_ε and V_ε, $(\mathrm{P}_\varepsilon^{h,\tau})$ can be restated as:

Find $\{U_\varepsilon,\, W_\varepsilon,\, V_\varepsilon\} \in [C([0,T];S^h)]^3$ such that for all $\chi \in L^2(0,T;S^h)$

$$\int_0^T \left[\left(\tfrac{\partial U_\varepsilon}{\partial t}, \chi\right)^h + \tfrac{1}{3}\left([\Xi(U_\varepsilon^+)]^3\, \nabla W_\varepsilon^+, \nabla\chi\right)\right] \mathrm{d}t = -\tfrac{1}{2}\int_0^T \left([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\, [\Xi(U_\varepsilon^-)]^{\frac{1}{2}}\, \nabla V_\varepsilon^-, \nabla\chi\right) \mathrm{d}t\,, \tag{3.4a}$$

$$\int_0^T \left[\left(\tfrac{\partial V_\varepsilon}{\partial t}, \chi\right)^h + \rho\,(\nabla V_\varepsilon^+, \nabla\chi) + (\Xi(U_\varepsilon^+)\, \Lambda_\varepsilon(V_\varepsilon^+)\, \nabla V_\varepsilon^+, \nabla\chi)\right] \mathrm{d}t$$
$$= -\tfrac{1}{2}\int_0^T \left([\Xi(U_\varepsilon^+)]^2\, \Lambda_\varepsilon(V_\varepsilon^+)\, \nabla W_\varepsilon^+, \nabla\chi\right) \mathrm{d}t\,; \tag{3.4b}$$

where

$$W_\varepsilon^+ \equiv -c\,\Delta^h U_\varepsilon^+ + \pi^h[\phi^+(U_\varepsilon^+) + \phi^-(U_\varepsilon^-)]. \tag{3.4c}$$

LEMMA 3.1 Let $\rho > 0$, $\phi(\cdot)$ satisfy (1.4) with $a, \delta > 0$, and $u^0 \in H^1(\Omega) \cap L^\infty(\Omega)$ with $u^0(x) \geq \zeta > 0$ for a.e. $x \in \Omega$ and $v^0 \in H^1_{\geq 0}(\Omega)$. Let $\{\mathcal{T}^h, U_\varepsilon^0, V_\varepsilon^0, \{\tau_n\}_{n=1}^N, \varepsilon\}_{h>0}$ be such that

(i) either $U_\varepsilon^0 \equiv Q^h u^0$, $V_\varepsilon^0 \equiv Q^h v^0$; or $U_\varepsilon^0 \equiv \pi^h u^0$, $V_\varepsilon^0 \equiv \pi^h v^0$ if either $d = 1$ or $u^0, v^0 \in W^{1,e}(\Omega)$ with $e > 2$;
(ii) Ω and $\{\mathcal{T}^h\}_{h>0}$ fulfil assumption (A), $\varepsilon \in (0,1)$ and $\tau_n \leq \frac{5}{4}\,\omega\,\tau_{n-1}$, $n = 2 \to N$, for an $\omega \in (0,1)$;
(iii) $\tau\, h^{-d\,(1-\frac{2}{p})} \to 0$ and $\varepsilon\, h^{-d\,(\frac{1}{2}-\frac{1}{p})} \to 0$ as $h \to 0$, where $p = 2$ if $d = 1$, $p > 2$ if $d = 2$.

Then there exists a subsequence of $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$, where $\{U_\varepsilon,\, W_\varepsilon,\, V_\varepsilon\}$ solve $(\mathrm{P}_\varepsilon^{h,\tau})$, and functions

$$u \in L^\infty(0,T; H^1_{\geq 0}(\Omega)) \cap H^1(0,T; (W^{1,q'}(\Omega))'), \tag{3.5a}$$
$$v \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1_{\geq 0}(\Omega)) \cap H^1(0,T; (W^{1,q'}(\Omega))') \tag{3.5b}$$
$$\text{such that} \quad \lambda(v) \in L^\infty(\Omega_T), \tag{3.5c}$$

with $u(\cdot,0) = u^0(\cdot)$ in Y_1, $v(\cdot,0) = v^0(\cdot)$ in Y_2, where $H^1(\Omega) \overset{c}{\hookrightarrow} Y_1$, $L^2(\Omega) \overset{c}{\hookrightarrow} Y_2$, and $⨍ u(\cdot,t) = ⨍ u^0 > 0$, $⨍ v(\cdot,t) = ⨍ v^0$ for *a.a.* $t \in (0,T)$, such that as $h \to 0$

$$\begin{aligned}
&U_\varepsilon,\ U_\varepsilon^\pm \to u && && \text{weak-}*\text{ in } L^\infty(0,T;H^1(\Omega)), && (3.6a)\\
&V_\varepsilon,\ V_\varepsilon^\pm \to v && \text{weak-}*\text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and} && \text{weakly in } L^2(0,T;H^1(\Omega)), && (3.6b)\\
&\mathcal{G}\tfrac{\partial U_\varepsilon}{\partial t} \to \mathcal{G}\tfrac{\partial u}{\partial t} && \text{and} \quad \mathcal{G}\tfrac{\partial V_\varepsilon}{\partial t} \to \mathcal{G}\tfrac{\partial v}{\partial t} && \text{weakly in } L^2(0,T;W^{1,q}(\Omega)), && (3.6c)\\
&U_\varepsilon,\ U_\varepsilon^\pm \to u && && \text{strongly in } L^2(0,T;L^s(\Omega)), && (3.7a)\\
&V_\varepsilon,\ V_\varepsilon^\pm \to v && && \text{strongly in } L^2(0,T;L^p(\Omega)), && (3.7b)\\
&[\Xi(U_\varepsilon^\pm)]^\beta \to u^\beta\,\mathcal{I}\,, && \text{for any } \beta \in (0,\infty), && \text{strongly in } L^2(0,T;L^s(\Omega)), && (3.8a)\\
&\Lambda_\varepsilon(V_\varepsilon^+) \to \lambda(v)\,\mathcal{I} && && \text{strongly in } L^2(0,T;L^p(\Omega)); && (3.8b)
\end{aligned}$$

where $s \in [2,\infty]$ and $q = 2$ if $d = 1$, $s \in [2,\infty)$ and $q \in (1,2)$ if $d = 2$.

Moreover, if $d = 1$ then u in addition to (3.5a) satisfies

$$u \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T) \quad \text{and} \quad u(x,t) > 0 \quad \forall\ (x,t) \in \overline{\Omega}_T \tag{3.9}$$

and there exists a subsequence of $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$ satisfying (3.6a–c), (3.7a,b), (3.8a,b) and as $h \to 0$

$$
\begin{aligned}
U_\varepsilon,\ U_\varepsilon^\pm &\to u && \text{uniformly on } \overline{\Omega}_T, && (3.10a)\\
\pi^h[\phi^\pm(U_\varepsilon^\pm)] &\to \phi^\pm(u) && \text{uniformly on } \overline{\Omega}_T, && (3.10b)\\
W_\varepsilon^+ &\to w := -c\,\Delta u + \phi(u) && \text{weakly in } L^2(0,T;H^1(\Omega)), && (3.10c)\\
[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+ &\to u^{\frac{3}{2}}\,\nabla w && \text{weakly in } L^2(\Omega_T). && (3.10d)
\end{aligned}
$$

Furthermore, if $d = 2$ and $\phi(\cdot)$ satisfies (1.4) with $\nu \geq 7$ then u in addition to (3.5a) satisfies

$$u \in L^2(0,T;H^2(\Omega)) \tag{3.11}$$

and there exists a subsequence of $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$ satisfying (3.6a–c), (3.7a,b), (3.8a,b) and as $h \to 0$

$$
\begin{aligned}
\Delta^h U_\varepsilon^+ &\to \Delta u && \text{weakly in } L^2(\Omega_T), && (3.12a)\\
U_\varepsilon,\ U_\varepsilon^\pm &\to u && \text{weakly in } L^2(0,T;W^{1,p}(\Omega)), && (3.12b)\\
U_\varepsilon &\to u && \text{strongly in } L^2(0,T;C^{0,\gamma}(\overline{\Omega})), \quad \text{for any } \gamma \in (0, 1-\tfrac{2}{p}). && (3.12c)
\end{aligned}
$$

Proof. Noting the definitions (3.1a,b) the bounds in (2.70) and (2.71a,b) together with (1.22), (2.78) and our assumption (i) imply that

$$
\begin{aligned}
&\|U_\varepsilon^{(\pm)}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|V_\varepsilon^{(\pm)}\|^2_{L^\infty(0,T;L^2(\Omega))} + \rho\,\|V_\varepsilon^{(\pm)}\|^2_{L^2(0,T;H^1(\Omega))}\\
&\quad + \varepsilon^{-1}\|\pi^h[V_\varepsilon^+]_-\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\bar{\tau}^{\frac{1}{2}}\tfrac{\partial U_\varepsilon}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} + \|\bar{\tau}^{\frac{1}{2}}\tfrac{\partial V_\varepsilon}{\partial t}\|^2_{L^2(\Omega_T)}\\
&\quad + \|\,[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|^2_{L^2(\Omega_T)} + \|\mathcal{G}\tfrac{\partial U_\varepsilon}{\partial t}\|_{L^2(0,T;W^{1,q}(\Omega))} + \|\mathcal{G}\tfrac{\partial V_\varepsilon}{\partial t}\|_{L^2(0,T;W^{1,q}(\Omega))} \leq C.
\end{aligned} \tag{3.13}
$$

Furthermore, we deduce from (3.2), (3.13) and (2.27) that

$$
\|U_\varepsilon - U_\varepsilon^\pm\|^2_{L^2(0,T;H^1(\Omega))} \leq \|\bar{\tau}\,\tfrac{\partial U_\varepsilon}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} \leq C\,\tau, \tag{3.14a}
$$

$$
\begin{aligned}
\|V_\varepsilon - V_\varepsilon^\pm\|^2_{L^2(0,T;L^p(\Omega))} &\leq C\,h^{-d\,(1-\frac{2}{p})}\,\|V_\varepsilon - V_\varepsilon^\pm\|^2_{L^2(\Omega_T)}\\
&\leq C\,h^{-d\,(1-\frac{2}{p})}\,\|\bar{\tau}\,\tfrac{\partial V_\varepsilon}{\partial t}\|^2_{L^2(\Omega_T)} \leq C\,h^{-d\,(1-\frac{2}{p})}\,\tau.
\end{aligned} \tag{3.14b}
$$

Hence on noting (3.13), (3.14a,b), $U_\varepsilon > 0$, (1.16), assumption (iii), and (1.20a) we can choose a subsequence $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$ such that the convergence results (3.5a–c), (3.6a–c) and (3.7a,b) hold. Then (3.5a,b) and Theorem 2.1 yield, on noting (1.20b), assumption (i), (2.30) and (2.34), that the subsequence satisfies the additional initial and integral conditions.

We now prove (3.8a). It holds for all $\beta \in (0,\infty)$ that

$$
\begin{aligned}
&\|u^\beta\,\mathcal{I} - [\Xi(U_\varepsilon^\pm)]^\beta\|_{L^2(0,T;L^s(\Omega))}\\
&\qquad \leq \|u^\beta - [U_\varepsilon^\pm]^\beta\|_{L^2(0,T;L^s(\Omega))} + \|\,[U_\varepsilon^\pm]^\beta\,\mathcal{I} - [\Xi(U_\varepsilon^\pm)]^\beta\|_{L^2(0,T;L^s(\Omega))}\,,
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
&\|u^\beta - [U_\varepsilon^\pm]^\beta\|_{L^2(0,T;L^s(\Omega))} \\
&\quad \leq C \begin{cases} \|u - U_\varepsilon^\pm\|^\beta_{L^2(0,T;L^s(\Omega))} & \text{if } \beta \in (0,1], \\ \beta \,\|\, [\max\{u, U_\varepsilon^\pm\}]^{\beta-1} (u - U_\varepsilon^\pm)\|_{L^2(0,T;L^s(\Omega))} & \text{if } \beta \geq 1 \end{cases} \\
&\quad \leq C \begin{cases} \|u - U_\varepsilon^\pm\|^\beta_{L^2(0,T;L^s(\Omega))} & \text{if } \beta \in (0,1], \\ \beta \left[\|u\|_{L^\infty(0,T;L^{\beta s}(\Omega))} + \|U_\varepsilon^\pm\|_{L^\infty(0,T;L^{\beta s}(\Omega))}\right]^{\beta-1} \|u - U_\varepsilon^\pm\|_{L^2(0,T;L^{\beta s}(\Omega))} & \\ & \text{if } \beta \geq 1 \end{cases}
\end{aligned}
\tag{3.16a}
$$

and, on noting (2.47b), (2.27) and (1.17),

$$
\begin{aligned}
&\| [U_\varepsilon^\pm]^\beta \mathcal{I} - [\Xi(U_\varepsilon^\pm)]^\beta\|_{L^2(0,T;L^s(\Omega))} \\
&\quad \leq C \begin{cases} h^{\beta - d\,[\frac{\beta}{2} - \frac{1}{s}]_+} \|\nabla U_\varepsilon^\pm\|^\beta_{L^2(\Omega_T)} & \text{if } \beta \in (0,1], \\ h^{1-d\,(\frac{1}{2} - \frac{1}{\beta s})}\, \beta \, \|U_\varepsilon^\pm\|^{\beta-1}_{L^\infty(0,T;L^{\beta s}(\Omega))} \|\nabla U_\varepsilon^\pm\|_{L^2(\Omega_T)} & \text{if } \beta \geq 1. \end{cases}
\end{aligned}
\tag{3.16b}
$$

Combining (3.15), (3.16a,b), (3.7a), (3.5a), (3.13) and (1.19) yields (3.8a).

Similarly for (3.8b), we have that

$$
\begin{aligned}
\|\lambda(v)\,\mathcal{I} - \Lambda_\varepsilon(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))} &\leq \|\lambda(v) - \lambda(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))} + \|\lambda(V_\varepsilon^+) - \lambda_\varepsilon(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))} \\
&\quad + \|\lambda_\varepsilon(V_\varepsilon^+)\,\mathcal{I} - \Lambda_\varepsilon(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))}.
\end{aligned}
\tag{3.17}
$$

Noting the global Lipschitz continuity of λ, (2.47a), (2.27) and (3.13), we have that

$$
\begin{aligned}
\|\lambda(v) - \lambda(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))} + \|\lambda_\varepsilon(V_\varepsilon^+)\,\mathcal{I} - \Lambda_\varepsilon(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))} \\
\leq \|v - V_\varepsilon^+\|_{L^2(0,T;L^p(\Omega))} + C\, h^{1-d\,(\frac{1}{2}-\frac{1}{p})} \|\nabla V_\varepsilon^+\|_{L^2(\Omega_T)} \\
\leq \|v - V_\varepsilon^+\|_{L^2(0,T;L^p(\Omega))} + C\, h^{1-d\,(\frac{1}{2}-\frac{1}{p})}\,.
\end{aligned}
\tag{3.18}
$$

It follows from (2.46), (1.16), (2.27) and (3.13) that

$$
\begin{aligned}
\|\lambda(V_\varepsilon^+) - \lambda_\varepsilon(V_\varepsilon^+)\|_{L^2(0,T;L^p(\Omega))} &\leq \|\varepsilon - [V_\varepsilon^+]_-\|_{L^2(0,T;L^p(\Omega))} \leq \|\varepsilon - \pi^h\,[V_\varepsilon^+]_-\|_{L^2(0,T;L^p(\Omega))} \\
&\leq C(T)\, h^{-d\,(\frac{1}{2}-\frac{1}{p})} \left[\, \varepsilon + \|\pi^h\,[V_\varepsilon^+]_-\|_{L^2(\Omega_T)}\right] \leq C(T)\,\varepsilon\, h^{-d\,(\frac{1}{2}-\frac{1}{p})}.
\end{aligned}
\tag{3.19}
$$

Combining (3.17), (3.18), (3.19) and noting (3.7b) and our assumption (iii) on ε yields the desired result (3.8b). Finally, we note that (2.38) and (3.8b) $\Rightarrow \lambda(v) \geq 0$ *a.e.* $\Rightarrow v \geq 0$ *a.e.* $\Rightarrow H^1_{\geq 0}(\Omega)$ in (3.5b).

For $d = 1$ the first and eighth bound, for $q = 2$, in (3.13) imply that the $C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T)$ norm of U_ε is bounded independently of h, τ, ε and T, see for example Lemma 3.1 in BGN and Barrett, Blowey & Garcke (1998, Theorem 2.2). Therefore by the Arzelà-Ascoli theorem there exists a subsequence $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$ satisfying (3.6a–c), (3.7a,b), (3.8a,b) and such that

$$
U_\varepsilon,\ U_\varepsilon^\pm \to u \in C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\overline{\Omega}_T) \qquad \text{uniformly on } \overline{\Omega}_T \text{ as } h \to 0. \tag{3.20}
$$

Hence the desired result (3.10a). The bounds (2.70) and (2.78), on noting (3.1b), imply that

$$\max_{t\in[0,T]} \int_\Omega \pi^h[\Phi(U_\varepsilon^+)](x,t)\,\mathrm{d}x \leq C. \tag{3.21}$$

From the uniform Hölder continuity of U_ε^+ in space and (3.21) it follows that there exists $u_{min} \in \mathbb{R}_{>0}$ independent of h, τ and ε such that for all $t \in [0,T]$

$$U_\varepsilon^+(x,t) \geq u_{min} \quad \forall\, x \in \overline{\Omega}, \tag{3.22}$$

see e.g. Grün & Rumpf (2001, Corollary 5.3). Combining (3.22) and (3.20) yields the desired result (3.9).

Combining (3.5a), (3.9), (3.10a), the continuity of $\phi^\pm$ and $(\phi^\pm)'$ on $\mathbb{R}_{>0}$, (1.19) and (2.28) yields for all $t \in [0,T]$ that

$$\begin{aligned}
&\|\pi^h[\phi^\pm(U_\varepsilon^\pm)](\cdot,t) - \phi^\pm(u(\cdot,t))\|_{0,\infty} \\
&\quad \leq \|\pi^h[\phi^\pm(U_\varepsilon^\pm) - \phi^\pm(u)](\cdot,t)\|_{0,\infty} + \|[(I-\pi^h)\,\phi^\pm(u)](\cdot,t)\|_{0,\infty} \\
&\quad \leq \|\phi^\pm(U_\varepsilon^\pm(\cdot,t)) - \phi^\pm(u(\cdot,t))\|_{0,\infty} + \|[(I-\pi^h)\,\phi^\pm(u)](\cdot,t)\|_1 \to 0 \quad \text{as } h \to 0,
\end{aligned} \tag{3.23}$$

the desired result (3.10b). Furthermore, it follows from (3.13), (2.42a) and (3.10a) that there exists an h_0 such that

$$[\tfrac{1}{2}\min_{(x,t)\in\Omega_T} u(x,t)]^3\, \|\nabla W_\varepsilon^+\|^2_{L^2(\Omega_T)} \leq \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\, \nabla W_\varepsilon^+\|^2_{L^2(\Omega_T)} \leq C \quad \forall\, h \leq h_0\,. \tag{3.24}$$

Now choosing $\chi \equiv 1$ in (2.19b) yields, on noting (3.9), (3.10b) and (3.24), that

$$\|W_\varepsilon^+\|_{L^2(0,T;H^1(\Omega))} \leq C \qquad \text{and hence} \quad c\,\|\Delta^h U_\varepsilon^+\|_{L^2(\Omega_T)} \leq C\,; \tag{3.25}$$

on recalling (3.4c). Moreover, it follows from (3.25), (2.36), (2.30), (2.32), (3.13) and (3.6a) that for any $\eta \in L^2(0,T;H^1(\Omega))$

$$\begin{aligned}
\int_0^T (\Delta^h U_\varepsilon^+, \eta)\,\mathrm{d}t &= \int_0^T (\Delta^h U_\varepsilon^+, (I-\pi^h)\eta)\,\mathrm{d}t + \int_0^T \left[(\Delta^h U_\varepsilon^+, \pi^h\,\eta) - (\Delta^h U_\varepsilon^+, \pi^h\,\eta)^h\right]\mathrm{d}t \\
&\quad + \int_0^T (\nabla U_\varepsilon^+, \nabla(I-\pi^h)\eta)\,\mathrm{d}t - \int_0^T (\nabla U_\varepsilon^+, \nabla\eta)\,\mathrm{d}t \\
&\to -\int_0^T (\nabla u, \nabla \eta)\,\mathrm{d}t \qquad \text{as } h \to 0\,.
\end{aligned} \tag{3.26}$$

Combining (3.25), (3.10b) and (3.26) yields, on possibly extracting a further subsequence, (3.10c). Furthermore, we have on noting (3.13), (3.8a) and (3.10c) that (3.10d) holds.

For the remainder of the proof, let $d = 2$. It follows from (2.72), (2.78), (2.3), (2.31) and (3.1a,b) that

$$\|\Delta^h U_\varepsilon^+\|_{L^2(\Omega_T)} \leq C\,. \tag{3.27}$$

Similarly to (3.26) it follows from (3.27), (2.36), (2.30), (2.32), (3.13) and (3.6a) that for any $\eta \in L^2(0,T;W^{1,p}(\Omega))$

$$\int_0^T (\Delta^h U_\varepsilon^+, \eta)\,\mathrm{d}t \to -\int_0^T (\nabla u, \nabla\eta)\,\mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.28}$$

Combining (3.27), (3.28) and the denseness of $L^2(0,T;W^{1,p}(\Omega))$ in $L^2(\Omega_T)$ yields (3.12a) and, in particular, $\Delta u \in L^2(\Omega_T)$. This together with elliptic regularity, as Ω is convex polygonal, and (3.5a) proves (3.11). Moreover, we obtain from (2.27), (2.78) and our assumptions (iii) on τ that

$$\tau_1 \, \|U^0_\varepsilon\|^2_{1,p} \leq C \, \tau \, h^{-d\,(1-\frac{2}{p})} \, \|U^0_\varepsilon\|^2_1 \leq C \, . \tag{3.29}$$

Combining (3.29), (3.27) and (2.37) yields that

$$\|U^{(\pm)}_\varepsilon\|_{L^2(0,T;W^{1,p}(\Omega))} \leq C, \tag{3.30}$$

and hence (3.12b) holds on extracting a further subsequence. Finally, (3.12c) follows from (3.12b), (3.6c), (1.20a) and the compact embedding $W^{1,p}(\Omega) \stackrel{c}{\hookrightarrow} C^{0,\gamma}(\overline{\Omega})$. □

Remark 3.1 We note that in the case $d = 1$ we proved (3.10a–d) without the help of the entropy bound (2.72). For $d = 2$, however, we need (2.72) to hold and require the additional assumption $\nu \geq 7$. Furthermore, for $d = 2$, we need the following lemma in order to prove our main result.

Lemma 3.2 Let $d = 2$ and all the assumptions of Lemma 3.1 hold. If in addition $\tau_n = \tau$, $n = 1 \to N$, then

$$\int_0^{T-\theta} |U^\pm_\varepsilon(t+\theta) - U^\pm_\varepsilon(t)|^2_0 \, \mathrm{d}t \leq C \, \theta \qquad \forall \, \theta \in (0,T). \tag{3.31}$$

Furthermore, it holds that the subsequence of $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$ in Lemma 3.1 is such that as $h \to 0$

$$U^\pm_\varepsilon \to u \qquad \text{strongly in } L^2(0,T;C^{0,\gamma}(\overline{\Omega})), \quad \text{for any } \gamma \in (0, 1-\tfrac{2}{p}), \tag{3.32a}$$

and for $a.a.\ t \in (0,T)$

$$u(\cdot,t) \in C^{0,\gamma}(\overline{\Omega}) \quad \text{with} \quad u(x,t) \geq \zeta(t) > 0 \qquad \forall \, x \in \overline{\Omega} \, . \tag{3.32b}$$

Moreover, on extracting a further subsequence, it holds as $h \to 0$ that for $a.a.\ t \in (0,T)$

$$\pi^h[\phi^\pm(U^\pm_\varepsilon)](\cdot,t) \to \phi^\pm(u(\cdot,t)) \qquad \text{strongly in } C(\overline{\Omega}), \tag{3.33a}$$

$$W^+_\varepsilon(\cdot,t) \to w(\cdot,t) \equiv -c\,\Delta u(\cdot,t) + \phi(u(\cdot,t)) \qquad \text{weakly in } H^1(\Omega) \tag{3.33b}$$

and

$$[\Xi(U^+_\varepsilon)]^{\frac{3}{2}} \, \nabla W^+_\varepsilon \to u^{\frac{3}{2}} \, \nabla w \qquad \text{weakly in } L^2(\Omega_T). \tag{3.33c}$$

Proof. The proof is similar to the proofs in Grün (2002, Lemmas 7.1 and 8.8). It follows from (2.19a) for $m = 0 \to N - l$, $l \in \{1, \ldots, N\}$ fixed, that

$$\sum_{n=m+1}^{m+l} \tau_n \left(\tfrac{U^n_\varepsilon - U^{n-1}_\varepsilon}{\tau_n}, U^{m+l}_\varepsilon - U^m_\varepsilon \right)^h = -\tfrac{1}{3} \sum_{n=m+1}^{m+l} \tau_n \left([\Xi(U^n_\varepsilon)]^3 \, \nabla W^n_\varepsilon, \nabla (U^{m+l}_\varepsilon - U^m_\varepsilon) \right)$$
$$- \tfrac{1}{2} \sum_{n=m+1}^{m+l} \tau_n \left([\Xi(U^n_\varepsilon)]^{\frac{3}{2}} \, [\Xi(U^{n-1}_\varepsilon)]^{\frac{1}{2}} \, \nabla V^{n-1}_\varepsilon, \nabla (U^{m+l}_\varepsilon - U^m_\varepsilon) \right) . \tag{3.34}$$

Similarly to (2.74), we obtain from (3.34), (2.71b), (3.1a,b) and (3.13) for $r = \frac{2p}{p-2}$ that

$$
\begin{aligned}
|U_\varepsilon^{m+l} - U_\varepsilon^m|_h^2 &\leq C \sum_{n=m+1}^{m+l} \tau_n \, |[\Xi(U_\varepsilon^n)]^{\frac{3}{2}}|_{0,r} \, a_n \, |U_\varepsilon^{m+l} - U_\varepsilon^m|_{1,p} \\
&\leq C \sum_{k=1}^{l} \tau_{m+k} \, a_{m+k} \, |U_\varepsilon^{m+l} - U_\varepsilon^m|_{1,p},
\end{aligned} \tag{3.35}
$$

where $a_n := |[\Xi(U_\varepsilon^n)]^{\frac{3}{2}} \nabla W_\varepsilon^n|_0 + |[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}} \nabla V_\varepsilon^{n-1}|_0$. Summing (3.35) for $m = 0 \to N - l$ and using the uniform time step assumption yields on noting (2.70), (2.78) and (3.30) that

$$
\begin{aligned}
\sum_{m=0}^{N-l} \tau \, |U_\varepsilon^{m+l} - U_\varepsilon^m|_h^2 &\leq C \sum_{k=1}^{l} \tau \sum_{m=0}^{N-l} \tau \, a_{m+k} \, |U_\varepsilon^{m+l} - U_\varepsilon^m|_{1,p} \\
&\leq C \sum_{k=1}^{l} \tau \left(\sum_{m=0}^{N-l} \tau \, a_{m+k}^2 \right)^{\frac{1}{2}} \left(\sum_{m=0}^{N-l} \tau \, |U_\varepsilon^{m+l} - U_\varepsilon^m|_{1,p}^2 \right)^{\frac{1}{2}} \leq C \, l \, \tau.
\end{aligned} \tag{3.36}
$$

Combining (3.36), (2.3), (2.31) and (3.1b) yields (3.31) for $\theta = l\,\tau$. For arbitrary $\theta \in (0,T)$ with $\theta = \mu\,\tau$, $\mu \in (0,N)$, we argue as follows. Let $l = \lfloor \mu \rfloor := \max\{\ell \in \mathbb{N} : \ell \leq \mu\}$, $\vartheta = \mu - \lfloor \mu \rfloor \in [0,1)$ and $m \in \{0, \ldots, N-l\}$ be such that $t \in (m\,\tau, (m+1)\,\tau]$. Hence

$$
U_\varepsilon^{\pm}(t + \mu\,\tau) = \begin{cases} U_\varepsilon^{\pm}(t + l\,\tau) & \text{if } \; t \in (m\,\tau, m\,\tau + (1-\vartheta)\,\tau], \\ U_\varepsilon^{\pm}(t + (l+1)\,\tau) & \text{if } \; t \in (m\,\tau + (1-\vartheta)\,\tau, (m+1)\,\tau] \end{cases} \tag{3.37}
$$

and we obtain on noting (3.36) that

$$
\begin{aligned}
&\int_0^{T-\mu\,\tau} |U_\varepsilon^{\pm}(t + \mu\,\tau) - U_\varepsilon^{\pm}(t)|_h^2 \, \mathrm{d}t \\
&\qquad \leq \tau\,(1-\vartheta) \sum_{m=0}^{N-l} |U_\varepsilon^{m+l} - U_\varepsilon^m|_h^2 + \tau\,\vartheta \sum_{m=0}^{N-l-1} |U_\varepsilon^{m+l+1} - U_\varepsilon^m|_h^2 \\
&\qquad \leq C\,[(1-\vartheta)\,l + \vartheta\,(l+1)]\,\tau = C\,\mu\,\tau.
\end{aligned} \tag{3.38}
$$

Combining (3.38), (2.3) and (2.31) yields (3.31) for all $\theta \in (0,T)$. It follows from (3.12b) and (3.31), on noting (1.20c) and the compact embedding $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} C^{0,\gamma}(\overline{\Omega})$, that (3.32a) holds.

The strong convergence (3.32a) yields on extracting a further subsequence for *a.a.* $t \in (0,T)$ that

$$
U_\varepsilon^{\pm}(\cdot,t) \to u(\cdot,t) \quad \text{strongly in } C^{0,\gamma}(\overline{\Omega}) \qquad \text{and} \qquad u(x,t) \geq \zeta(t) > 0 \quad \forall\, x \in \overline{\Omega}, \tag{3.39}
$$

where the latter result comes from applying (3.21)–(3.22) for *a.a.* $t \in (0,T)$. Hence we have the desired result (3.32b). Combining (3.39), the continuity of $\phi^{\pm}$ and $(\phi^{\pm})'$ on $\mathbb{R}_{>0}$, (2.28) and (3.23), with $\|\cdot\|_1$ replaced by $\|\cdot\|_{1,p}$ on using (1.19) with $d = 2$ and (3.12b), we obtain the desired result (3.33a) for *a.a.* $t \in (0,T)$.

On noting (3.27) we have for $a.a.$ $t \in (0,T)$ that

$$|\Delta^h U_\varepsilon^+(\cdot,t)|_0 \leq C(t)\,. \tag{3.40}$$

Similarly to (3.28) this yields, on noting (3.39), for $a.a.$ $t \in (0,T)$ that as $h \to 0$

$$\Delta^h U_\varepsilon^+(\cdot,t) \to \Delta u(\cdot,t) \qquad \text{weakly in } L^2(\Omega). \tag{3.41}$$

Combining (3.4c), (3.41) and (3.33a) yields for $a.a.$ $t \in (0,T)$ that as $h \to 0$

$$\begin{aligned} W_\varepsilon^+(\cdot,t) &\equiv -c\,\Delta^h U_\varepsilon^+(\cdot,t) + \pi^h[\phi^+(U_\varepsilon^+) + \phi^-(U_\varepsilon^-)](\cdot,t) \\ &\qquad \to -c\,\Delta u(\cdot,t) + \phi(u(\cdot,t)) \qquad \text{weakly in } L^2(\Omega). \end{aligned} \tag{3.42}$$

Furthermore, it follows from (3.13), (2.42a) and (3.39) for $a.a.$ $t \in (0,T)$ that there exists an $h_0(t)$ such that

$$[\tfrac{1}{2}\,\zeta(t)]^3\,|\nabla W_\varepsilon^+(\cdot,t)|_0^2 \leq |[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}(\cdot,t)\,\nabla W_\varepsilon^+(\cdot,t)|_0^2 \leq C(t) \qquad \forall\, h \leq h_0(t)\,. \tag{3.43}$$

Combining (3.42) and (3.43) yields (3.33b) on extracting a further subsequence. Moreover, we have on noting (3.13) that there exists a limit $\underline{z} \in L^2(\Omega_T)$ such that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$\int_0^T ([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+, \nabla\eta)\,\mathrm{d}t \;\to\; \int_0^T (\underline{z}, \nabla\eta)\,\mathrm{d}t \quad \text{as } h \to 0. \tag{3.44}$$

It follows from (3.33b) and (3.8a) that $\underline{z}(\cdot,t) \equiv u^{\frac{3}{2}}(\cdot,t)\,\nabla w(\cdot,t)$ for $a.a.$ $t \in (0,T)$ and hence that (3.33c) holds. □

THEOREM 3.1 Let all the assumptions of Lemma 3.1 and in addition if $d=2$ those of Lemma 3.2 hold. Then there exists a subsequence of $\{U_\varepsilon, W_\varepsilon, V_\varepsilon\}_h$, where $\{U_\varepsilon,\, W_\varepsilon,\, V_\varepsilon\}$ solve $(\mathrm{P}_\varepsilon^{h,\tau})$, and functions $\{u,v\}$ satisfying (3.5a–c) and either (3.9) if $d=1$ or (3.11) and (3.32b) if $d=2$. In addition, as $h \to 0$ the following hold: (3.6a–c), (3.7a,b), (3.8a,b) and either (3.10a–d) if $d=1$ or (3.12b–c), (3.32a), (3.33a,b), for $a.a.$ $t \in (0,T)$, and (3.33c) if $d=2$. Furthermore, we have that u and v fulfil $u(\cdot,0) = u^0(\cdot)$ in Y_1, $v(\cdot,0) = v^0(\cdot)$ in Y_2, where $H^1(\Omega) \stackrel{c}{\hookrightarrow} Y_1$, $L^2(\Omega) \stackrel{c}{\hookrightarrow} Y_2$, and satisfy for all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$, with $q'=2$ if $d=1$ and $q' \in (2,\infty)$ if $d=2$,

$$\int_0^T \langle \tfrac{\partial u}{\partial t}, \eta\rangle_{q'}\,\mathrm{d}t + \tfrac{1}{3}\int_{\Omega_T} u^3\,\nabla w\,.\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\int_{\Omega_T} u^2\,\nabla v\,.\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t = 0\,, \tag{3.45a}$$

$$\int_0^T \langle \tfrac{\partial v}{\partial t}, \eta\rangle_{q'}\,\mathrm{d}t + \int_{\Omega_T} [\,\rho\,\nabla v\,.\,\nabla\eta + u\,\lambda(v)\,\nabla v\,.\,\nabla\eta\,]\,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\int_{\Omega_T} u^2\,\lambda(v)\,\nabla w\,.\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t = 0\,; \tag{3.45b}$$

where $w(\cdot,t) \equiv -c\,\Delta u(\cdot,t) + \phi(u(\cdot,t))$ for $a.a.$ $t \in (0,T)$.

Proof. For any $\eta \in L^2(0,T;W^{1,q'}(\Omega))$ and $\widetilde{\eta} \in H^1(0,T;W^{1,\infty}(\Omega))$ we choose $\chi \equiv \pi^h\eta$

in (3.4a,b) and now analyse the subsequent terms. Firstly (2.32), (2.35), (1.19) in time, (3.13) and (2.30) yield for $Z \equiv U_\varepsilon$ and V_ε, respectively, that

$$\begin{aligned}
&\left| \int_0^T \left[\left(\tfrac{\partial Z}{\partial t}, \pi^h \eta\right)^h - \left(\tfrac{\partial Z}{\partial t}, \pi^h \eta\right) \right] \mathrm{d}t \right| \\
&\quad \leq \left| \int_0^T \left[\left(\tfrac{\partial Z}{\partial t}, \pi^h[\eta - \widetilde{\eta}]\right)^h - \left(\tfrac{\partial Z}{\partial t}, \pi^h[\eta - \widetilde{\eta}]\right) \right] \mathrm{d}t \right| \\
&\qquad + \left| - \int_0^T \left(Z, \tfrac{\partial(\pi^h \widetilde{\eta})}{\partial t} \right)^h \mathrm{d}t + (Z(\cdot,T), \pi^h \widetilde{\eta}(\cdot,T))^h - (Z(\cdot,0), \pi^h \widetilde{\eta}(\cdot,0))^h \right. \\
&\qquad\qquad \left. + \int_0^T \left(Z, \tfrac{\partial(\pi^h \widetilde{\eta})}{\partial t} \right) \mathrm{d}t - (Z(\cdot,T), \pi^h \widetilde{\eta}(\cdot,T)) + (Z(\cdot,0), \pi^h \widetilde{\eta}(\cdot,0)) \right| \\
&\quad \leq C\, \|\mathcal{G} \tfrac{\partial Z}{\partial t}\|_{L^2(0,T;W^{1,q}(\Omega))}\, \|\pi^h[\eta - \widetilde{\eta}]\,\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\qquad + C\, h\, \|Z\|_{L^\infty(0,T;L^2(\Omega))}\, \|\pi^h \widetilde{\eta}\|_{H^1(0,T;H^1(\Omega))} \\
&\quad \leq C\, \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} + C\, h\, \|\widetilde{\eta}\|_{H^1(0,T;W^{1,q'}(\Omega))}\,. \qquad (3.46)
\end{aligned}$$

Furthermore, it follows from (1.21) and (3.13) that

$$\begin{aligned}
\left| \int_0^T \left(\tfrac{\partial Z}{\partial t}, (I - \pi^h)\eta\right) \mathrm{d}t \right| &\leq C\, \|\mathcal{G} \tfrac{\partial Z}{\partial t}\|_{L^2(0,T;W^{1,q}(\Omega))}\, \|(I - \pi^h)\eta\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\leq C\, \|(I - \pi^h)\eta\|_{L^2(0,T;W^{1,q'}(\Omega))}. \qquad (3.47)
\end{aligned}$$

Combining (3.46), the denseness of $H^1(0,T;W^{1,\infty}(\Omega))$ in $L^2(0,T;W^{1,q'}(\Omega))$, (3.47), (2.28), (1.21) and (3.6c) yields for $z \equiv u$ and v, respectively, and for all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T \left(\tfrac{\partial Z}{\partial t}, \pi^h \eta\right)^h \mathrm{d}t \to \int_0^T \langle \tfrac{\partial z}{\partial t}, \eta \rangle_{q'}\, \mathrm{d}t \quad \text{as } h \to 0. \qquad (3.48)$$

In view of (2.38), (3.1b), (2.71b) and (3.13), as $\rho > 0$, we deduce with r as defined in (2.71a) that

$$\begin{aligned}
&\left| \int_0^T ([\Xi(U_\varepsilon^+)]^2\, \Lambda_\varepsilon(V_\varepsilon^+)\, \nabla W_\varepsilon^+, \nabla (I - \pi^h)\eta)\, \mathrm{d}t \right| \\
&\quad \leq \|[\Xi(U_\varepsilon^+)]^2\, \nabla W_\varepsilon^+\|_{L^2(0,T;L^q(\Omega))}\, \|(I - \pi^h)\eta\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\quad \leq C\, \|[\Xi(U_\varepsilon^+)]^{\frac{1}{2}}\|_{L^\infty(0,T;L^r(\Omega))}\, \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\, \nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\, \|(I - \pi^h)\eta\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\quad \leq C\, \|(I - \pi^h)\eta\|_{L^2(0,T;W^{1,q'}(\Omega))} \qquad \forall\, \eta \in L^2(0,T;W^{1,q'}(\Omega)), \qquad (3.49\text{a})
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left| \int_0^T ([\Xi(U_\varepsilon^+)]^3\, \nabla W_\varepsilon^+, \nabla (I - \pi^h)\eta)\, \mathrm{d}t \right| + \left| \int_0^T (\Xi(U_\varepsilon^+)\, \Lambda_\varepsilon(V_\varepsilon^+)\, \nabla V_\varepsilon^+, \nabla (I - \pi^h)\eta)\, \mathrm{d}t \right| \\
&\quad + \left| \int_0^T ([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\, [\Xi(U_\varepsilon^-)]^{\frac{1}{2}}\, \nabla V_\varepsilon^-, \nabla (I - \pi^h)\eta)\, \mathrm{d}t \right| + \left| \int_0^T (\nabla V_\varepsilon^+, \nabla (I - \pi^h)\eta)\, \mathrm{d}t \right| \\
&\qquad \leq C\, \|(I - \pi^h)\eta\|_{L^2(0,T;W^{1,q'}(\Omega))} \qquad \forall\, \eta \in L^2(0,T;W^{1,q'}(\Omega)). \qquad (3.49\text{b})
\end{aligned}$$

Noting (3.49b), (2.28), and (3.6b), we have for all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T (\nabla V_\varepsilon^+, \nabla(\pi^h \eta)\,)\,\mathrm{d}t \to \int_0^T (\nabla v, \nabla \eta)\,\mathrm{d}t \quad \text{as } h \to 0. \tag{3.50}$$

It also follows from (3.13), (3.1b), (2.71b), (3.5a,c), (1.19) and (2.38) with r defined as in (2.71a) and s and p as in Lemma 3.1 that

$$\begin{aligned}
&\left| \int_0^T (\,([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^2\,\mathcal{I})\,\nabla V_\varepsilon^-, \nabla \eta)\,\mathrm{d}t \right| \\
&\quad \leq \left| \int_0^T (\,([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^2\,\mathcal{I})\,\nabla V_\varepsilon^-, \nabla \widetilde{\eta})\,\mathrm{d}t \right| \\
&\qquad + \left| \int_0^T (\,([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^2\,\mathcal{I})\,\nabla V_\varepsilon^-, \nabla (\eta - \widetilde{\eta}))\,\mathrm{d}t \right| \\
&\quad \leq \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^2\,\mathcal{I}\|_{L^2(\Omega_T)}\,\|V_\varepsilon^-\|_{L^2(0,T;H^1(\Omega))}\,\|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\qquad + \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^2\,\mathcal{I}\|_{L^\infty(0,T;L^r(\Omega))}\,\|V_\varepsilon^-\|_{L^2(0,T;H^1(\Omega))}\,\|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\quad \leq C \Big[\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,([\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^{\frac{1}{2}}\,\mathcal{I})\|_{L^2(\Omega_T)} + \|u^{\frac{1}{2}}\,([\Xi(U_\varepsilon^+)]^{\frac{3}{2}} - u^{\frac{3}{2}}\,\mathcal{I})\|_{L^2(\Omega_T)} \\
&\qquad\qquad + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \Big] \\
&\quad \leq C \Big[\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\|_{L^\infty(0,T;L^{2s}(\Omega))}\,\|[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^{\frac{1}{2}}\,\mathcal{I}\|_{L^2(0,T;L^{2s'}(\Omega))} \\
&\qquad + \|u^{\frac{1}{2}}\|_{L^\infty(0,T;L^{2s}(\Omega))}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}} - u^{\frac{3}{2}}\,\mathcal{I}\|_{L^2(0,T;L^{2s'}(\Omega))} + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \Big] \\
&\quad \leq C \Big[\|[\Xi(U_\varepsilon^-)]^{\frac{1}{2}} - u^{\frac{1}{2}}\,\mathcal{I}\|_{L^2(0,T;L^{2s'}(\Omega))} + \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}} - u^{\frac{3}{2}}\,\mathcal{I}\|_{L^2(0,T;L^{2s'}(\Omega))} \\
&\qquad\qquad + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \Big]
\end{aligned} \tag{3.51a}$$

and

$$\begin{aligned}
&\left| \int_0^T (\,[\,\Xi(U_\varepsilon^+)\,\Lambda_\varepsilon(V_\varepsilon^+) - u\,\lambda(v)\,\mathcal{I}\,]\,\nabla V_\varepsilon^+, \nabla \eta)\,\mathrm{d}t \right| \\
&\quad \leq \left| \int_0^T (\,[\,\Xi(U_\varepsilon^+)\,\Lambda_\varepsilon(V_\varepsilon^+) - u\,\lambda(v)\,\mathcal{I}\,]\,\nabla V_\varepsilon^+, \nabla \widetilde{\eta})\,\mathrm{d}t \right| \\
&\qquad + \left| \int_0^T (\,[\,\Xi(U_\varepsilon^+)\,\Lambda_\varepsilon(V_\varepsilon^+) - u\,\lambda(v)\,\mathcal{I}\,]\,\nabla V_\varepsilon^+, \nabla (\eta - \widetilde{\eta}))\,\mathrm{d}t \right| \\
&\quad \leq \|\Xi(U_\varepsilon^+)\,\Lambda_\varepsilon(V_\varepsilon^+) - u\,\lambda(v)\,\mathcal{I}\|_{L^2(\Omega_T)}\,\|V_\varepsilon^+\|_{L^2(0,T;H^1(\Omega))}\,\|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\qquad + \|\Xi(U_\varepsilon^+)\,\Lambda_\varepsilon(V_\varepsilon^+) - u\,\lambda(v)\,\mathcal{I}\|_{L^\infty(0,T;L^r(\Omega))}\,\|V_\varepsilon^+\|_{L^2(0,T;H^1(\Omega))}\,\|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\quad \leq C \Big[\|\Lambda_\varepsilon(V_\varepsilon^+)\,(\Xi(U_\varepsilon^+) - u\,\mathcal{I})\|_{L^2(\Omega_T)} + \|u\,(\Lambda_\varepsilon(V_\varepsilon^+) - \lambda(v)\,\mathcal{I})\|_{L^2(\Omega_T)} \\
&\qquad\qquad + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \Big] \\
&\quad \leq C \Big[\|\Xi(U_\varepsilon^+) - u\,\mathcal{I}\|_{L^2(\Omega_T)} + \|\Lambda_\varepsilon(V_\varepsilon^+) - \lambda(v)\,\mathcal{I}\|_{L^2(0,T;L^p(\Omega))} + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \Big].
\end{aligned} \tag{3.51b}$$

Noting that $H^1(0,T;W^{1,\infty}(\Omega))$ is dense in $L^2(0,T;W^{1,q'}(\Omega))$, (3.49b), (2.28), (3.51a,b), (3.8a,b), (3.5a,c), (1.19) and (3.6b), we have for all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T ([\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^-)]^{\frac{1}{2}}\,\nabla V_\varepsilon^-, \nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (u^2\,\nabla v, \nabla\eta)\,\mathrm{d}t \qquad \text{as } h \to 0, \quad (3.52\text{a})$$

$$\int_0^T (\Xi(U_\varepsilon^+)\,\Lambda_\varepsilon(V_\varepsilon^+)\,\nabla V_\varepsilon^+, \nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (u\,\lambda(v)\,\nabla v, \nabla\eta)\,\mathrm{d}t \quad \text{as } h \to 0. \quad (3.52\text{b})$$

Similarly to (3.51a,b), it follows from (3.13), (3.1b), (2.71b), (3.5a,c), (2.38) and (1.19) that

$$\begin{aligned}
&\left|\int_0^T (\,([\Xi(U_\varepsilon^+)]^3 - u^{\frac{3}{2}}\,[\Xi(U_\varepsilon^+)]^{\frac{3}{2}})\,\nabla W_\varepsilon^+, \nabla\eta)\,\mathrm{d}t\right| \\
&\quad \leq \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}} - u^{\frac{3}{2}}\,\mathcal{I}\|_{L^2(\Omega_T)}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\,\|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\qquad + \|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}} - u^{\frac{3}{2}}\,\mathcal{I}\|_{L^\infty(0,T;L^r(\Omega))}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\,\|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\quad \leq C\left[\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}} - u^{\frac{3}{2}}\,\mathcal{I}\|_{L^2(\Omega_T)} + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))}\right]
\end{aligned} \quad (3.53\text{a})$$

and

$$\begin{aligned}
&\left|\int_0^T (\,([\Xi(U_\varepsilon^+)]^2\,\Lambda_\varepsilon(V_\varepsilon^+) - u^{\frac{1}{2}}\,[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\lambda(v))\,\nabla W_\varepsilon^+, \nabla\eta)\,\mathrm{d}t\right| \\
&\leq \left|\int_0^T (\,([\Xi(U_\varepsilon^+)]^2\,\Lambda_\varepsilon(V_\varepsilon^+) - u^{\frac{1}{2}}\,[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\Lambda_\varepsilon(V_\varepsilon^+))\,\nabla W_\varepsilon^+, \nabla\eta)\,\mathrm{d}t\right| \\
&\qquad + \left|\int_0^T (\,([u^{\frac{1}{2}}\,[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\Lambda_\varepsilon(V_\varepsilon^+) - u^{\frac{1}{2}}\,[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\lambda(v))\,\nabla W_\varepsilon^+, \nabla\eta)\,\mathrm{d}t\right| \\
&\leq \|[\Xi(U_\varepsilon^+)]^{\frac{1}{2}} - u^{\frac{1}{2}}\,\mathcal{I}\|_{L^2(\Omega_T)}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\,\|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\quad + \|[\Xi(U_\varepsilon^+)]^{\frac{1}{2}} - u^{\frac{1}{2}}\,\mathcal{I}\|_{L^\infty(0,T;L^r(\Omega))}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\,\|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\quad + \|u^{\frac{1}{2}}\,[\Lambda_\varepsilon(V_\varepsilon^+) - \lambda(v)\,\mathcal{I}]\,\|_{L^2(\Omega_T)}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\,\|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\quad + \|u^{\frac{1}{2}}\,[\Lambda_\varepsilon(V_\varepsilon^+) - \lambda(v)\,\mathcal{I}]\,\|_{L^\infty(0,T;L^r(\Omega))}\,\|[\Xi(U_\varepsilon^+)]^{\frac{3}{2}}\,\nabla W_\varepsilon^+\|_{L^2(\Omega_T)}\,\|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))} \\
&\leq C\left[\|[\Xi(U_\varepsilon^+)]^{\frac{1}{2}} - u^{\frac{1}{2}}\,\mathcal{I}\|_{L^2(\Omega_T)} + \|\Lambda_\varepsilon(V_\varepsilon^+) - \lambda(v)\,\mathcal{I}\|_{L^2(0,T;L^p(\Omega))} + \|\eta - \widetilde{\eta}\|_{L^2(0,T;W^{1,q'}(\Omega))}\right].
\end{aligned} \quad (3.53\text{b})$$

Noting that $H^1(0,T;W^{1,\infty}(\Omega))$ is dense in $L^2(0,T;W^{1,q'}(\Omega))$, (3.49a,b), (2.28), (3.53a,b), (3.8a,b), (3.5a,c), (1.19), (3.10d) and (3.33c), we have for all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T ([\Xi(U_\varepsilon^+)]^3\,\nabla W_\varepsilon^+, \nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (u^3\,\nabla w, \nabla\eta)\,\mathrm{d}t \qquad \text{as } h \to 0, \quad (3.54\text{a})$$

$$\int_0^T ([\Xi(U_\varepsilon^+)]^2\,\Lambda_\varepsilon(V_\varepsilon^+)\,\nabla W_\varepsilon^+, \nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (u^2\,\lambda(v)\,\nabla w, \nabla\eta)\,\mathrm{d}t \quad \text{as } h \to 0. \quad (3.54\text{b})$$

Combining (3.4a–c), (3.48), (3.50), (3.52a,b), (3.54a,b), (3.10c) and (3.33b) yields that the functions $\{u,\,w,\,v\}$ satisfy (3.5a–c) and (3.45a,b), as well as the results of Lemma 3.1 and Lemma 3.2. □

Remark 3.2 As indicated in Remark 3.1, for $d = 1$ we proved our main convergence result for $(\mathrm{P}_{\varepsilon}^{h,\tau})$ with $\delta > 0$ without the help of the entropy bound (2.72). Therefore for $d = 1$ it appears possible to prove convergence of a modified version of $(\mathrm{P}_{\varepsilon}^{h,\tau})$ to a solution of (P) also in the absence van der Waals forces ($a = \delta = 0$), and hence with $u^0 \geq 0$. Similarly to Grün & Rumpf (2000) and Grün (2002) for the thin film equation, (1.1a–c) with $v \equiv 0$ and $a = \delta = 0$, one would replace Ξ by Ξ_ε in (2.19a–c) to obtain an approximation of (P), which we will denote as $(\mathrm{P}_{\varepsilon,\varepsilon}^{h,\tau})$. Clearly the stability bounds (2.70) and (2.71a,b) remain true with Ξ replaced by Ξ_ε. These bounds give rise to the convergence results (3.6a–c) , (3.7a,b) and (3.8b). For the above thin film equation one still has an analogue of the entropy bound (2.72) if $u^0 \geq \zeta > 0 \Rightarrow G(u^0)$ is non-singular. From this one can show that the resulting approximation $U_\varepsilon \geq \frac{1}{2}\varepsilon$ for h sufficiently small; see Grün & Rumpf (2000) for details. However, with either $u^0 \geq 0$ or surfactant present one does does not have an analogue of the entropy bound (2.72). Hence one has no control on the non-negativity of the resulting approximation. Nevertheless, one can still prove strong convergence in $L^2(0,T;L^\infty(\Omega)$ of $[\Xi_\varepsilon(U_\varepsilon^\pm)]^\beta \to [u]_+^\beta \mathcal{I}$ for any $\beta \in (0,\infty)$. Similarly to the techniques in BGN, on introducing $\{u > 0\} := \{(x,t) \in \Omega_T \,:\, u(x,t) > 0\}$, one would then be able to show that the limit u satisfies the weak solution concept

$$\int_0^T \langle \tfrac{\partial u}{\partial t}, \eta\rangle_2 \,\mathrm{d}t + \tfrac{1}{3}\int_{\{u>0\}} u^3\, \nabla(\Delta\, u)\,.\,\nabla\eta \,\mathrm{d}x\,\mathrm{d}t + \tfrac{1}{2}\int_{\{u>0\}} u^2\,\nabla v\,.\,\nabla\eta\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad (3.55)$$

for all $\eta \in L^2(0,T;H^1(\Omega))$, with a similar modification for the v equation. The non-negativity of the solution u could then be recovered by testing (3.55) with $[u]_-$, similar to the procedure described in Yin (1992) for the thin film equation. However, $(\mathrm{P}_{\varepsilon,\varepsilon}^{h,\tau})$ offers no advantages over the approximation (2.20a–c) described in BGN. Moreover, the solution $U_\varepsilon^n(\cdot)$ can now become negative in many disconnected regions where $u(\cdot,t_n) = 0$, which makes the location of the film boundary more difficult.

4. Numerical Results

Before presenting some numerical results in both one and two space dimensions, we briefly state algorithms for solving the resulting system of algebraic equations for $\{U_\varepsilon^n, W_\varepsilon^n, V_\varepsilon^n\}$ arising at each time level from the approximation $(\mathrm{P}_{\varepsilon}^{h,\tau})$. As (2.19a,b) in $(\mathrm{P}_{\varepsilon}^{h,\tau})$ are independent of V_ε^n, we first solve these to obtain $\{U_\varepsilon^n, W_\varepsilon^n\}$; then solve (2.19c) for V_ε^n. We use the following iterative approach to solve (2.19a,b) for $\{U_\varepsilon^n, W_\varepsilon^n\}$: Given $U_\varepsilon^{n,0} \in S^h_{>0}$, for $k \geq 1$ find $\{U_\varepsilon^{n,k}, W_\varepsilon^{n,k}\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\begin{aligned}\left(\tfrac{U_\varepsilon^{n,k}-U_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h &+ \tfrac{1}{3}\,([\Xi(U_\varepsilon^{n,k-1})]^3\,\nabla W_\varepsilon^{n,k}, \nabla\chi)\\ &= -\tfrac{1}{2}\,([\Xi(U_\varepsilon^{n,k-1})]^{\frac{3}{2}}\,[\Xi(U_\varepsilon^{n-1})]^{\frac{1}{2}}\,\nabla V_\varepsilon^{n-1}, \nabla\chi), \qquad (4.1a)\\ c\,(\nabla U_\varepsilon^{n,k}, \nabla\chi) + (\phi^+(U_\varepsilon^{n,k}) &+ \phi^-(U_\varepsilon^{n-1}), \chi)^h = (W_\varepsilon^{n,k}, \chi)^h. \qquad (4.1b)\end{aligned}$$

Then, having obtained $\{U_\varepsilon^n, W_\varepsilon^n\}$, we solve (2.19c) for V_ε^n using the following iterative approach: Given $V_\varepsilon^{n,0} \in S^h$, for $k \geq 1$ find $V_\varepsilon^{n,k} \in S^h$ such that

$$\begin{aligned}\left(\tfrac{V_\varepsilon^{n,k}-V_\varepsilon^{n-1}}{\tau_n}, \chi\right)^h &+ \rho\,(\nabla V_\varepsilon^{n,k}, \nabla\chi) + (\Xi(U_\varepsilon^n)\,\Lambda_\varepsilon(V_\varepsilon^{n,k-1})\,\nabla V_\varepsilon^{n,k}, \nabla\chi)\\ &= -\tfrac{1}{2}\,([\Xi(U_\varepsilon^n)]^2\,\Lambda_\varepsilon(V_\varepsilon^{n,k-1})\,\nabla W_\varepsilon^n, \nabla\chi) \qquad \forall\, \chi \in S^h. \quad (4.2)\end{aligned}$$

(4.1a,b) and (4.2) are the natural extensions of the iterative procedure proposed in Grün & Rumpf (2000) for solving a finite element approximation of the thin film equation. An analogue of (4.2) is used in BGN to solve (2.20c), see Remark 2.1. As $U_\varepsilon^{n,k-1} > 0$, it is easily established on noting (2.42a) that there exists a unique solution $\{U_\varepsilon^{n,k}, W_\varepsilon^{n,k}\} \in S^h_{>0} \times S^h$ to (4.1a,b). As (4.2) is linear, existence of $V_\varepsilon^{n,k}$ follows from uniqueness; and this is easily established on noting $\rho \geq 0$, (2.9a) and (2.14a). Hence the iterations (4.1a,b) and (4.2) are well defined.

For the iterative algorithms (4.1a,b) and (4.2) we set, for $n \geq 1$, $\{U_\varepsilon^{n,0}, V_\varepsilon^{n,0}\} \equiv \{U_\varepsilon^{n-1}, V_\varepsilon^{n-1}\}$ and adopted the stopping criteria

$$|U_\varepsilon^{n,k} - U_\varepsilon^{n,k-1}|_{0,\infty} < tol \quad \text{and} \quad |V_\varepsilon^{n,k} - V_\varepsilon^{n,k-1}|_{0,\infty} < tol, \tag{4.3}$$

respectively, with $tol = 10^{-8}$. Furthermore, we set $\{U_\varepsilon^n, W_\varepsilon^n, V_\varepsilon^n\} \equiv \{U_\varepsilon^{n,k}, W_\varepsilon^{n,k}, V_\varepsilon^{n,k}\}$ for (4.1a,b) and (4.2).

Remark 4.1 The nonlinear system (4.1a,b) can be solved using an inexact Newton's method, applying a BiCGSTAB algorithm at each Newton iteration. The linear system (4.2), on the other hand, can be solved efficiently using a conjugate gradient algorithm. Although we are unable to show convergence of the iterations (4.1a,b) and (4.2) for $\{U_\varepsilon^n,\, W_\varepsilon\}$ and V_ε^n, respectively, we observed good convergence properties in practice.

4.1 Numerical Results for $d = 1$

Firstly, we present numerical experiments in one space dimension. Throughout we chose a uniform partitioning of $\Omega = (-L, L)$, where $L \geq 1$, with mesh points $p_j = -L + (j-1)h$, $j = 1 \to 2^{10} + 1$, where $h = 2^{-9}\, L$. In addition we chose uniform time steps $\tau_n = \tau = 10^{-4}$ and throughout set the regularization parameter $\varepsilon = 10^{-5}$. For the initial profiles u^0 and v^0, we chose

$$u^0(x) = \bar{u}^0 + \sum_{m=1}^{M} \tfrac{\alpha_m}{M} \cos(\tfrac{m\pi x}{L}) \quad \text{and} \quad v^0(x) = \tfrac{v^0_{\max}}{2}\left[1 - \tanh(10\,|x| - 5)\right], \tag{4.4}$$

where $\bar{u}^0 > 0$, $v^0_{\max} \geq 0$ and random $\alpha_m \in [-\alpha, \alpha]$, $m = 1 \to M$, $\alpha \geq 0$. (4.4) with $\alpha = 0$ and $v^0_{\max} > 0$ or $v^0_{\max} = 0$ resemble a uniform liquid film of height $\bar{u}^0$ with and without surfactant on top of it, respectively. In the absence of surfactant a uniform film is a steady state. (4.4) with $\alpha > 0$ represents a small random perturbation to the uniform film. Throughout the data for $(\mathrm{P}_\varepsilon^{h,\tau})$ will be such that the assumptions of our main convergence result, Theorem 3.1, hold. In particular we will choose $U_\varepsilon^0 \equiv \pi^h u^0$ and $V_\varepsilon^0 \equiv \pi^h v^0$ as the discrete initial data on noting that u^0, $v^0 \in W^{1,\infty}(\Omega)$.

We investigated the effect of van der Waals forces and surfactant, respectively, on a slightly perturbed uniform liquid film. We set $L = 4$, $c = 5 \times 10^{-5}$, $\rho = 2 \times 10^{-3}$, $\nu = 4$, $\bar{u}^0 = 1$, $M = 20$ and varied the remaining parameters α, a, δ and $v^0_{\max}$. In Figure 1 we plot $U_\varepsilon(x, T)$ for $\alpha = 0.02$, $a = 2 \times 10^{-3}$, $\delta = 10^{-5}$, $v^0_{\max} = 0$ and $T = 140$. Note that this is a "numerical steady state" for $(\mathrm{P}_\varepsilon^{h,\tau})$, i.e. (4.3) is satisfied for $k = 1$. Note furthermore that $\min_{x\in\overline{\Omega}} U_\varepsilon(x, T) = \arg\min_{s\in\mathbb{R}_{>0}} \Phi(s) = (\frac{\delta}{a})^{\frac{1}{\nu-3}} = 5 \times 10^{-3}$, recall (2.17a). From the plot of the energy

$$E(t) := \tfrac{t-t_{n-1}}{\tau_n}\, \mathcal{E}(U_\varepsilon^n, V_\varepsilon^n) + \tfrac{t_n - t}{\tau_n}\, \mathcal{E}(U_\varepsilon^{n-1}, V_\varepsilon^{n-1}) \qquad t \in [t_{n-1}, t_n] \quad n \geq 1, \tag{4.5}$$

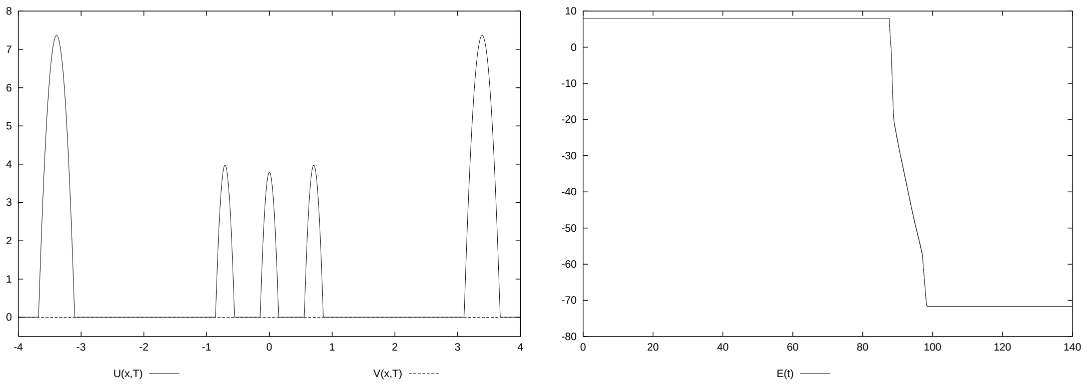

FIG. 1. $U_\varepsilon(x,T)$ and $V_\varepsilon(x,T)$ plotted against x at $T = 140$ and $E(t)$ for $t \in [0,T]$.

recall (2.61b), in Figure 1 one can clearly see that the main evolution of the film takes place in a comparatively short period of time and that the solution $\{U_\varepsilon(\cdot,T),\, V_\varepsilon(\cdot,T)\}$ attained a “numerical steady state”. In Figure 2, using a different vertical scale, we give some more details on the evolution of the solution U_ε. In the presence of attractive van der Waals forces and no surfactant, it is easily established using linear stability analysis that the uniform liquid film, $u^0 \equiv \bar{u}^0$, is unstable if

$$a > \tfrac{1}{3}\,(\bar{u}^0)^4 \left[\nu\,\delta\,(\bar{u}^0)^{-(\nu+1)} + c\,(\tfrac{\pi}{L})^2\right]. \tag{4.6}$$

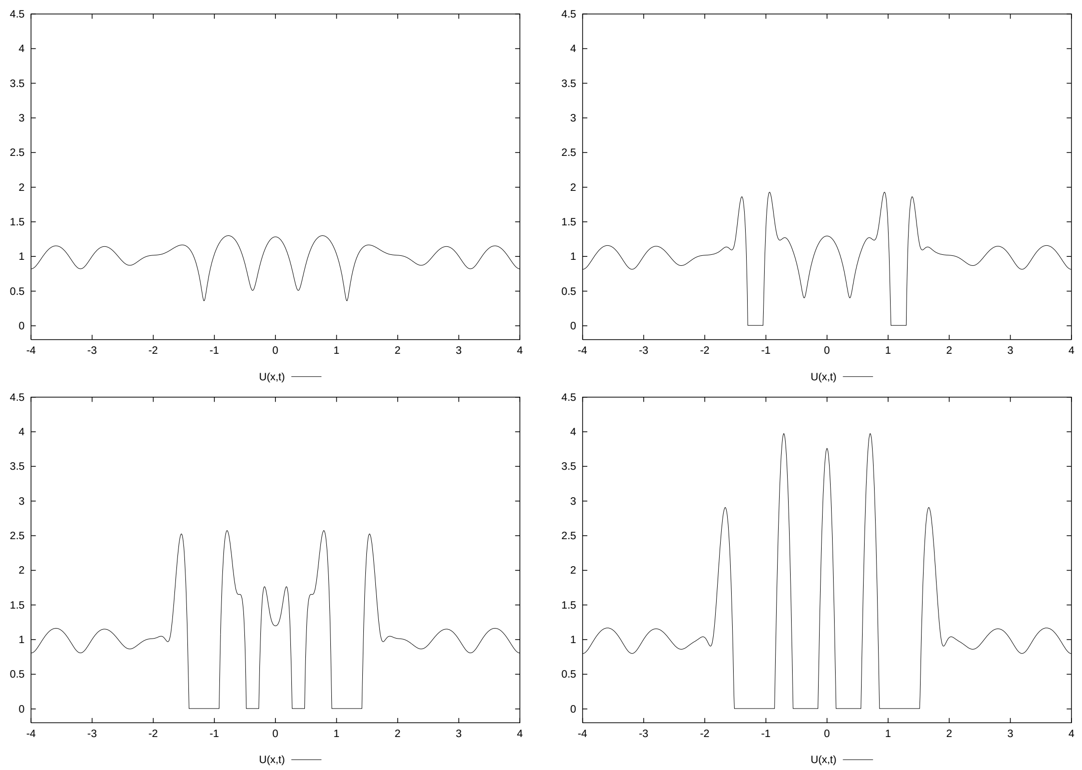

FIG. 2. $U_\varepsilon(x,t)$ plotted against x at $t = 87.5, 88, 88.5, 89$.

One can predict the most unstable mode $\cos(\frac{m^*\pi x}{L})$, where

$$\frac{m^*\pi}{L} \approx (\tfrac{1}{\bar{u}^0})^2 \, (\tfrac{3\,a-\nu\,\delta\,(\bar{u}^0)^{3-\nu}}{2\,c})^{\frac{1}{2}} \approx 7.7 \iff m^* \approx 9.8 \tag{4.7}$$

for the parameters in Figure 2. This predicted behaviour can clearly be seen in the early stages of the evolution, cf. Figure 2.

In the following we report on the evolutions of U_ε and V_ε up to time $T = 8$ with the choices $\alpha = 0$, $a = \delta = 0$, $v^0_{\max} = 1$ and $\alpha = 0$, $a = 2 \times 10^{-3}$, $\delta = 10^{-5}$, $v^0_{\max} = 1$. In Figure 4 we plot $U_\varepsilon(x, t_n)$ and $V_\varepsilon(x, t_n)$ for $t_n = 0.1, 6.0, 7.0, 7.3, T = 8.0$ for the two respective evolutions, while the energy (4.5) is given in Figure 3. The above models the evolution of a uniform liquid film with a blob of surfactant on top of it without and with van der Waals forces being present, respectively. We remark that the same experiments for a slightly perturbed liquid film, i.e. $\alpha > 0$, yielded virtually the same results as shown in Figure 4. This indicates that as long as the film thickness is bounded away from zero, the effect of surface tension gradients dominates the impact of the van der Waals forces. Once the film thickness reaches a certain threshold, however, the film tries to rupture due to the attractive van der Waals forces, ϕ^-, in the effected regions, see Figure 4b. Although the film height becomes extremely thin, it can never actually rupture ($U_\varepsilon = 0$) due to the presence of the repulsive van der Waals forces, ϕ^+. In fact, as in Figure 1, it holds that $\min_{x\in\overline{\Omega}} U_\varepsilon(x, T) = \arg\min_{s\in\mathbb{R}_{>0}} \Phi(s) = 5 \times 10^{-3}$. We would also like to mention that we repeated the experiment with the parameters for Figure 4b on a very fine mesh. As we obtained virtually identical results, we are satisfied that the oscillations shown in Figure 4b are not due to mesh effects. In fact the instabilities are in agreement with linear stability analysis for the thin film equation in the absence of surfactant, see above. Finally, we recall Remark 3.2, where we state that the modified approximation, $(\mathrm{P}^{h,\tau}_{\varepsilon,\varepsilon})$, converges in the absence of van der Waals forces. Moreover, for the evolution shown in Figure 4a $U_\varepsilon >> \varepsilon$ and so the discretization $(\mathrm{P}^{h,\tau}_{\varepsilon,\varepsilon})$ is equivalent to $(\mathrm{P}^{h,\tau}_{\varepsilon})$.

4.2 Numerical Results for $d = 2$

Finally, we present a numerical experiment in two space dimensions with $\Omega = (-L, L) \times (-L, L)$. We took a uniform mesh of squares of length $h = \frac{2L}{256}$, each of which was divided into two triangles by its north east diagonal. We chose the following parameters for $(\mathrm{P}^{h,\tau}_{\varepsilon})$:

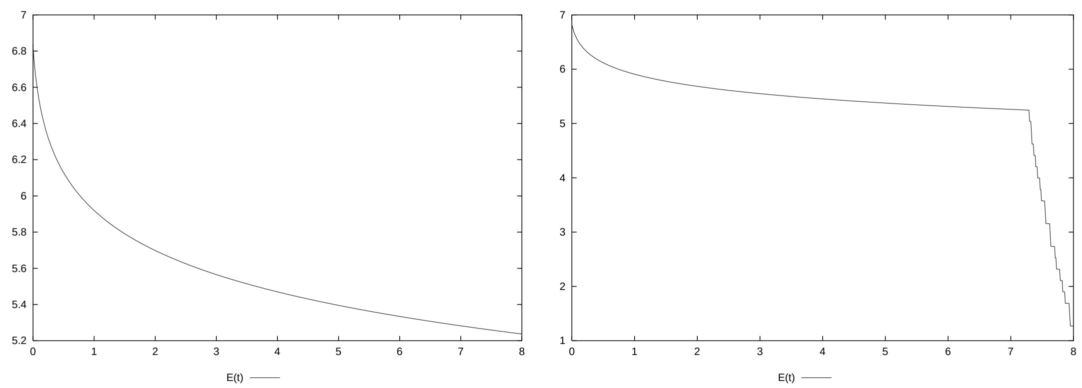

FIG. 3. $E(t)$ for $t \in [0, T]$ in the absence and presence of van der Waals forces.

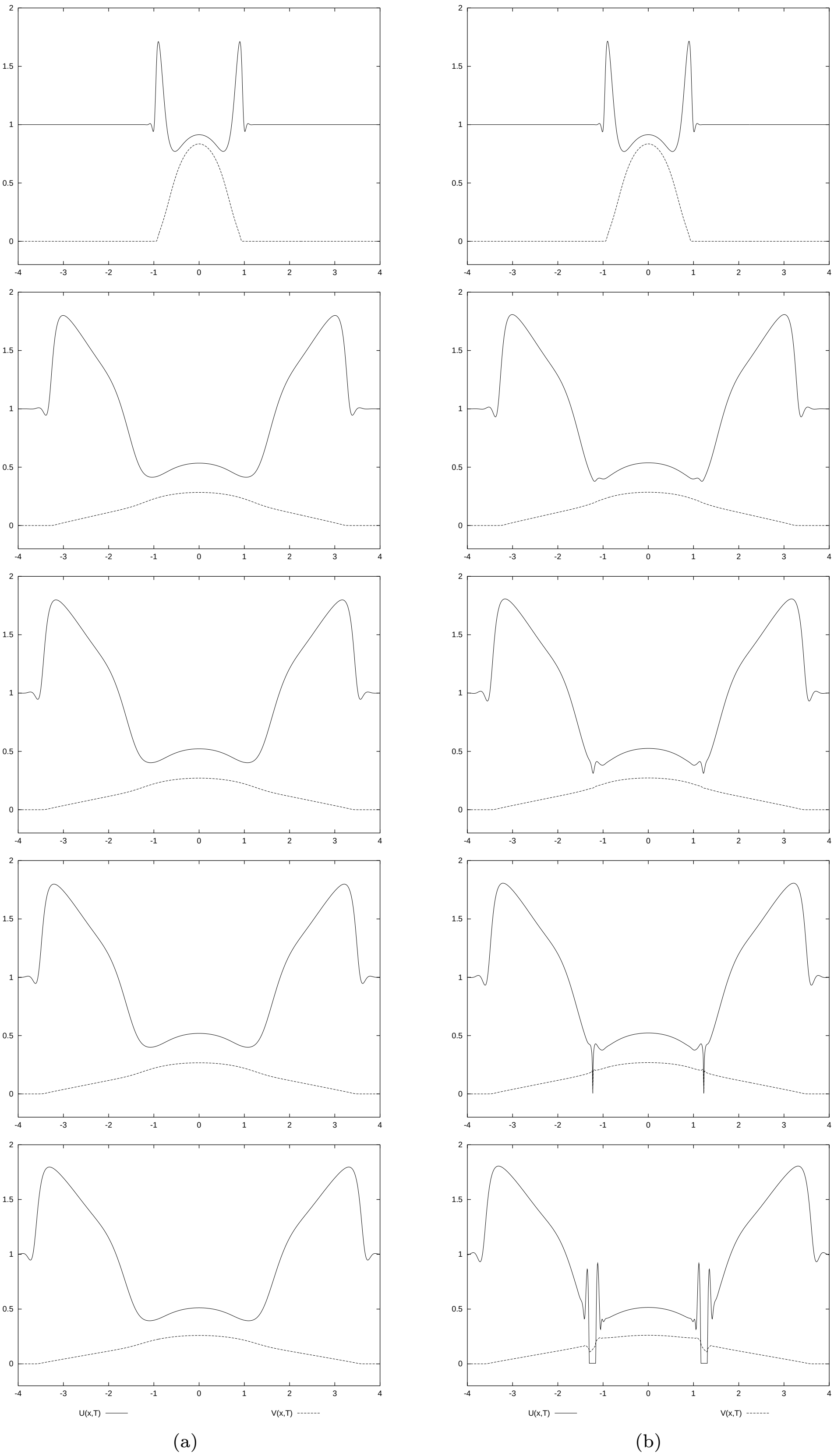

FIG. 4. $U_\varepsilon(x, t_n)$ and $V_\varepsilon(x, t_n)$ plotted against x at times $t_n = 0.1, 6.0, 7.0, 7.3, 8.0$; (a) without and (b) with van der Waals forces being present.

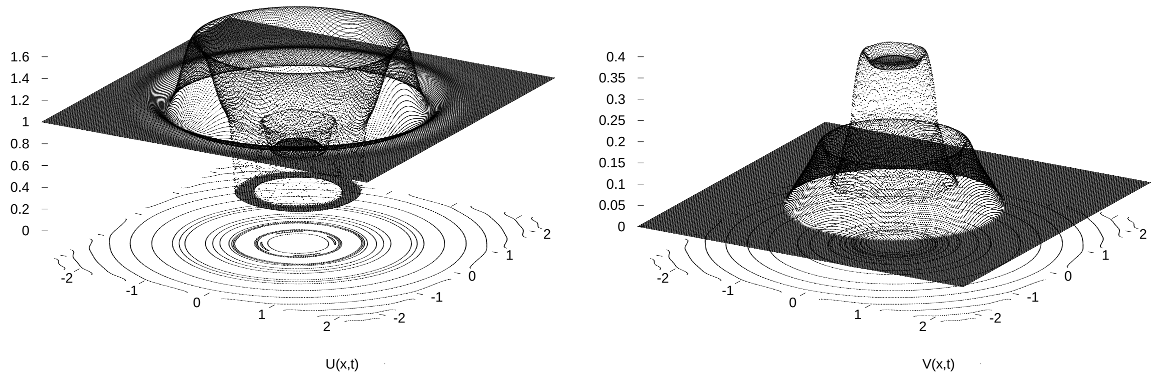

Fig. 5. $U_\varepsilon(x,T)$ and $V_\varepsilon(x,T)$ plotted against x for $T=1$.

$L=2.5$, $a=2\times10^{-2}$, $\delta=10^{-5}$, $c=10^{-3}$, $\rho=2\times10^{-3}$, $\nu=7$, $\tau_n=\tau=10^{-3}$, $\varepsilon=10^{-5}$ and for the initial profiles chose (4.4) with $\bar{u}^0=1$, $\alpha=0$ and $v^0_{\max}=1$. We set $U^0_\varepsilon\equiv\pi^h u^0$ and $V^0_\varepsilon\equiv\pi^h v^0$. In Figure 5 we plot $U_\varepsilon(x,T)$ and $V_\varepsilon(x,T)$ for $T=1$, while a contour plot of $U_\varepsilon(x,T)$ along with a plot of $U_\varepsilon(x,T)\,|_{x_2=0}$ is given in Figure 6. Furthermore, the results for an experiment with exactly the same data as above, except $a=\delta=0$, i.e. no van der Waals forces, can be seen in Figure 8. Qualitatively we can observe the same behaviour as in our experiments for $d=1$, cf. Figure 4. However, we note that here c is chosen larger than before and for our experiment with van der Waals forces present it holds that $\min_{x\in\overline{\Omega}} U_\varepsilon(x,T)\approx\arg\min_{s\in\mathbb{R}_{>0}}\Phi(s)\approx0.15$ due to the different choice of the parameters a, δ and ν. For smaller values of c and $\frac{\delta}{a}$ highly oscillatory solutions are observed, in line with the one dimensional argument (4.7) in the absence of surfactant. To approximate these accurately one needs to use highly adaptive meshes. We will report on such numerical experiments elsewhere. Finally, we give some more details on the evolution of U_ε for our last two experiments in Figure 7, where we plot $U_\varepsilon(x,t_n)$ for $t_n=0.01,0.1,0.5,0.7$ and 0.8.

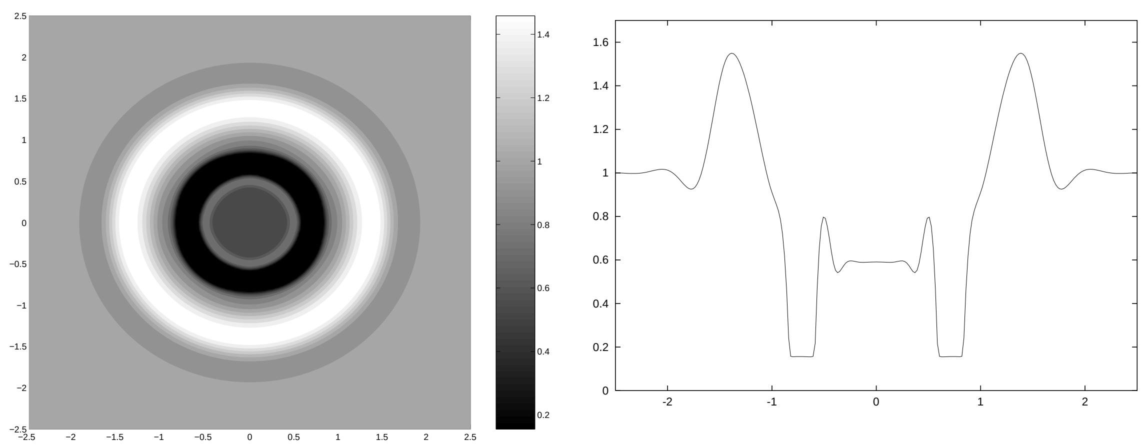

Fig. 6. Contour plot of $U_\varepsilon(x,T)$ and $U_\varepsilon(x,T)\,|_{x_2=0}$ plotted against x_1 for $T=1$.

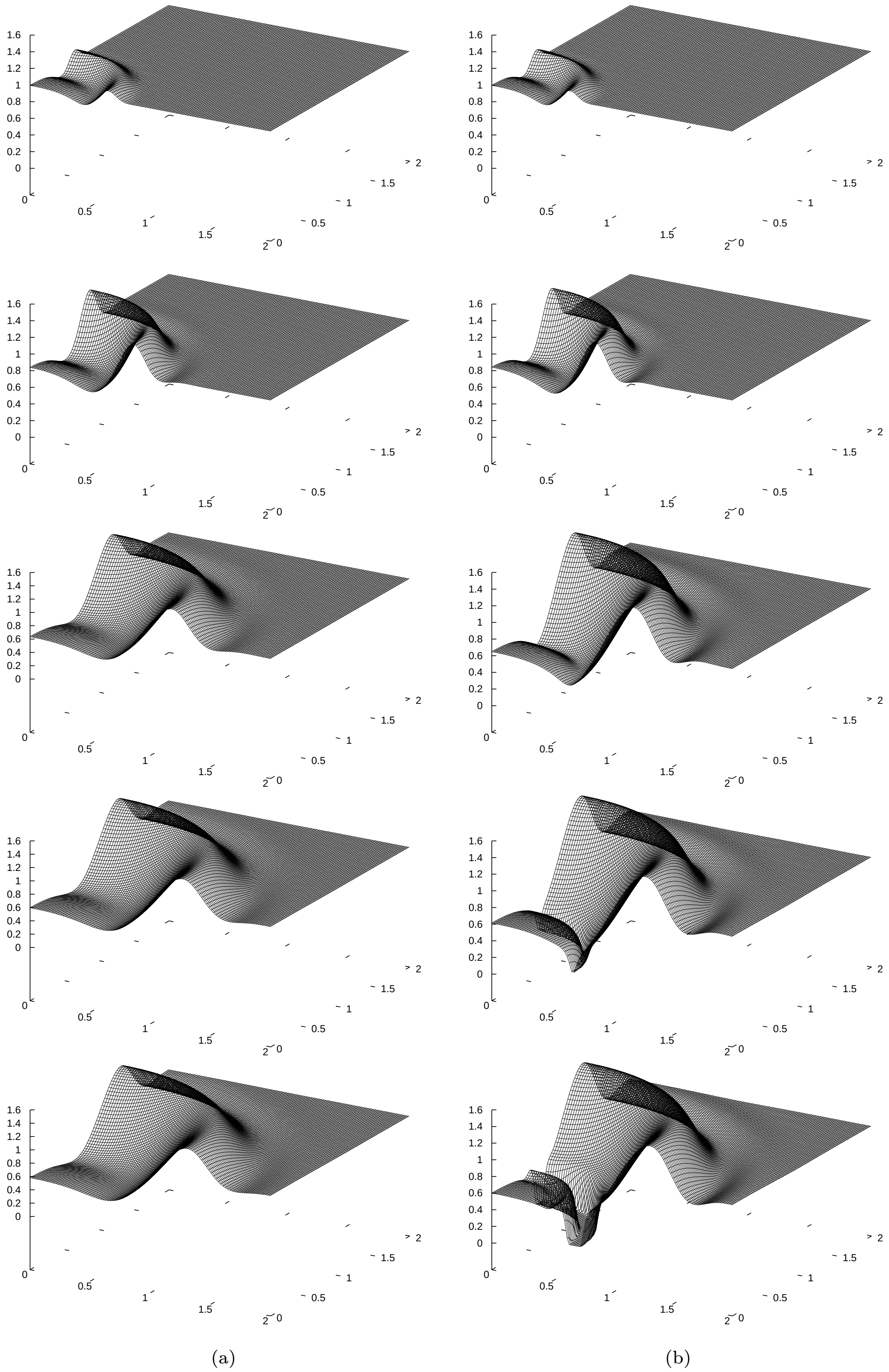

Fig. 7. $U_\varepsilon(x, t_n)$ plotted against x at times $t_n = 0.01, 0.1, 0.5, 0.7, 0.8$; (a) without and (b) with van der Waals forces being present.

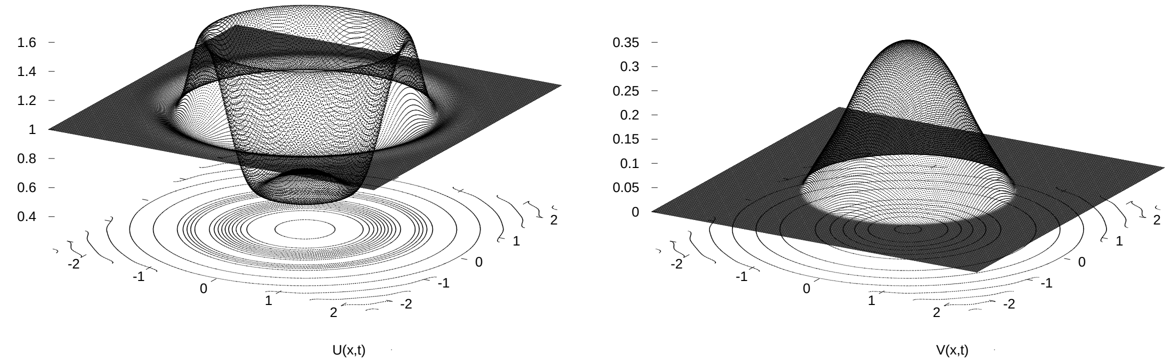

Fig. 8. $U_\varepsilon(x,T)$ and $V_\varepsilon(x,T)$ plotted against x for $T=1$.

References

Adams, R.A. & Fournier, J. (1977) Cone conditions and properties of Sobolev spaces. *J. Math. Anal. Appl.*, **61**, 713–734.

Barrett, J.W. & Blowey, J.F. (2001) Finite element approximation of a degenerate Allen-Cahn/Cahn-Hilliard system. *SIAM J. Numer. Anal.*, **39**, 1598–1624.

Barrett, J.W., Blowey, J.F. & Garcke, H. (1998) Finite element approximation of a fourth order nonlinear degenerate parabolic equation. *Numer. Math.*, **80**, 525–556.

Barrett, J.W., Blowey, J.F. & Garcke, H. (1999) Finite element approximation of the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Numer. Anal.*, **37**, 286–318.

Barrett, J.W., Blowey, J.F. & Garcke, H. (2001) On fully practical finite element approximations of degenerate Cahn-Hilliard systems. *M^2AN*, **35**, 713–748.

Barrett, J.W., Garcke, H. & Nürnberg, R. (2001) Finite element approximation of surfactant spreading on a thin film. Preprint 744, *SFB256*, University Bonn; (submitted for publication).

Barrett, J.W., Langdon, S. & Nürnberg, R. (2002) Finite element approximation of a sixth order nonlinear degenerate parabolic equation. (submitted for publication).

Bernis, F. & Friedman, A. (1990) Higher order nonlinear degenerate parabolic equations. *J. Diff. Eqns.*, **83**, 179–206.

Grün, G. (2002) On the convergence of entropy consistent schemes for lubrication type equations in multiple space dimensions. Preprint 682, *SFB256*, University of Bonn; and *Math. Comp.*, (to appear).

Grün, G. & Rumpf, M. (2000) Nonnegativity preserving numerical schemes for the thin film equation. *Numer. Math.*, **87**, 113–152.

Grün, G. & Rumpf, M. (2001) Simulation of singularities and instabilities in thin film flow. *Euro. J. Appl. Math.*, **12**, 293–320.

Jensen, O.E. & Grotberg, J.B. (1992) Insoluble surfactant spreading on a thin viscous film: shock evolution and film rupture. *J. Fluid Mech.*, **240**, 259–288.

ORON, A., DAVIS, S.H. & BANKOFF, S.G. (1997) Long-scale evolution of thin liquid films. *Rev. of Mod. Phys.*, **69**, 931–980.

RENARDY, M. (1996) A singularly perturbed problem related to surfactant spreading on thin films. *Nonlinear Anal.*, **27**, 287–296.

RENARDY, M. & ROGERS, R.C. (1992) *An Introduction to Partial Differential Equations.* New York: Springer.

SCHMIDT, A. & SIEBERT, K.G. (2001) Albert – software for scientific computations and applications. *Acta Math. Univ. Comenian.*, **70**, 105–122.

SIMON, J. (1987) Compact sets in the space $L^p(0,T;B)$. *Ann. Math. Pura. Appl.*, **146**, 65–96.

WIELAND, S. (2002) *Ph.D. thesis*, Bonn, (in preparation).

YIN, J. (1992) On the existence of nonnegative continuous solutions of the Cahn-Hilliard equation. *J. Diff. Eqns.*, **97**, 310–327.

ZHORNITSKAYA, L. & BERTOZZI, A.L. (2000) Positivity preserving numerical schemes for lubrication-type equations. *SIAM J. Numer. Anal.*, **37**, 523–555.