

TIMBER HYPERBOLIC PARABOLOID SHELLS SUPPORTED
ON FLEXIBLE EDGE MEMBERS

By

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TO MY MOTHER

ABSTRACT

Existing theoretical and experimental investigations on thin shallow hyperbolic paraboloid shells are reviewed and discussed. The finite element method of analysis has also been reviewed and in particular the use of the triangular elements for the analysis of the hyperbolic paraboloid shell and the plate problems have been investigated. For defining the stiffness matrix for the triangular element under the actions of bending forces, a new polynomial function has been derived. This function has been used in extensive computations covering the plate and the hyperbolic paraboloid shell on fixed and simply supported boundary conditions.

In dealing with the hyperbolic paraboloid shell on flexible edge supports, the shell type of actions, where the membrane stresses are assumed present, has failed to predict the behaviour of the shell under the actions of lateral loading. Accordingly the shell analysis has been modified to exclude the membrane effect, retaining only the plate type of actions. This modification has been called the Curved Plate Theory. This theory has been found to predict the behaviour of the shell on flexible edge support.

Accounts are also given of two experiments carried out on a timber shell and a vybak models. Certain design faults have been pointed out in the experiment on the timber model which makes it difficult to give an adequate interpretation to the results of the investigation. The experiment on the vybak model confirmed that the shell transmits its load by the plate type of actions.

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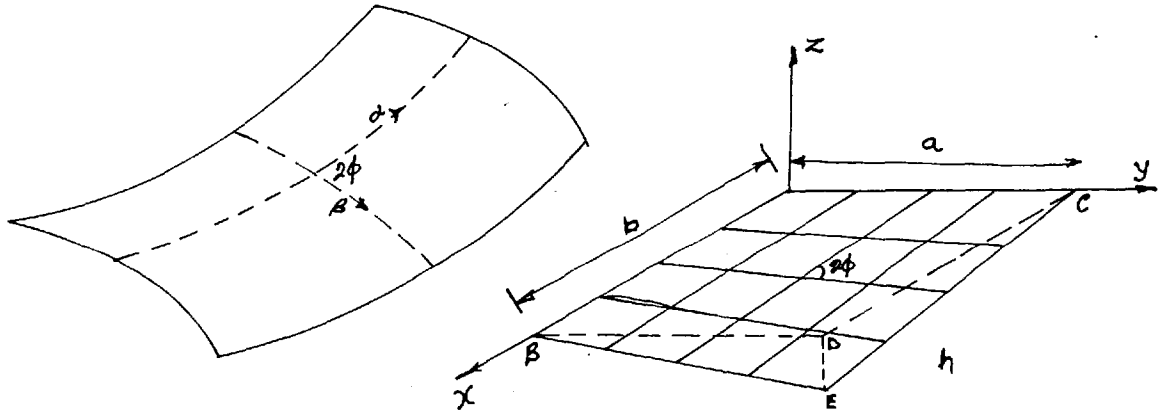
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CHAPTER 11.1 INTRODUCTION

In the last decade or two, hyperbolic paraboloid shells have gained a considerable amount of prominence among architects in roof construction for several reasons. Among these are, ease of construction associated with the straight line generators, aesthetically pleasing appearance, economy and the fact that from the design point of view, the membrane stresses are very simple to calculate. The hyperbolic paraboloid shell is also ideally suited to construction using timber laminates as the constructional material because of the natural advantages associated with ruled surfaces. Coupled with this is the extreme rapidity of construction and the possibility of factory produced components especially where small spans are involved. Figure 1.1 shows the basic units of the hyperbolic paraboloid shells with edges defined by the generating lines. Figure 1.2 shows the various combinations and permutations possible using the basic unit shown in figure 1.1b. Thus large areas can be covered by repetitive use of the basic unit.

Constructional problems associated with hyperbolic paraboloid shells are now largely solved, but the design procedure is far from satisfactory. It is possible to formulate the differential equations of equilibrium and compatibility using the shallow shell theory. The solution of the equations, taking into account the boundary conditions, by the classical approach, seems to present some difficulties. These arise from the mathematical properties of the equation which is essentially hyperbolic in nature. The solution also accepts only a limited form of boundary conditions and often the boundary conditions have to be made to suit the governing equation, a procedure which results in unrealistic boundary conditions.



(a) TRANSLATIONAL SURFACE

(b) RULED SURFACE

FIG 1.1

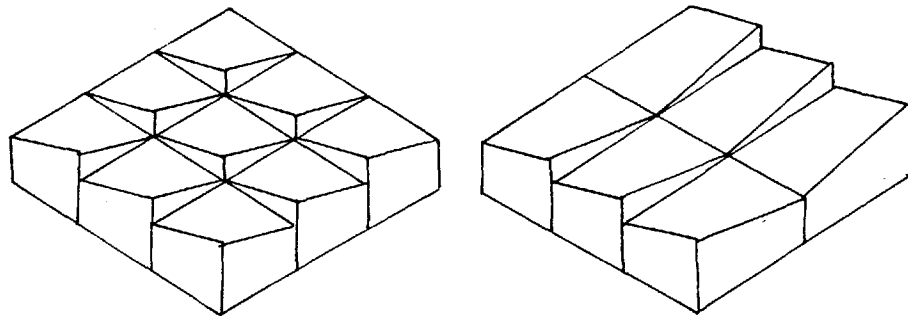


FIG 1.2 ARCHITECTURAL FORMS BUILT FROM THE
BASIC UNIT IN FIG 1.1b.

1.2 Geometry and Description of the Hyperbolic Paraboloid Shell

The hyperbolic paraboloid is a truly curved surface and comes mathematically into the general classification of ruled surfaces. Further subdivision within the general class is possible depending on the form of the ruling surfaces or generators. The generators may be curved fig. 1.1a in which case translational shell have been applied or straight fig. 1.1b when ruled surfaces are applied. These distinctions are however superficial. Physically, both surfaces are translational as well as ruled. If we consider a part of the surface bounded by the straight lines AB and CD of one set of generators fig. 1.1a and AC and BD of the other set, then the surface is developed if a straight line traverses one set but is kept parallel to the other set. On the other hand, the surface can be considered translational. For this purpose, we must realise that all planes parallel to the plane of symmetry are parabolae of opposite curvatures. The surface might therefore be conceived as that of the translation of one curve on the other. Moreover the curvatures of the parabolae being of opposite signs, the product of the curvatures known as the Gaussian curvature is negative, and as such, the hyperbolic paraboloid is part of the family of shells known as the shells of negative Gaussian curvature. For the purpose of this thesis, the above classification is adhered to and only hyperbolic paraboloids defined by straight line generators are treated.

If we denote by θ the angle between the plane of symmetry and the coordinate axis Ox say, then the general equation of the hyperbolic paraboloid of the ruled surface type is given by

$$Z = kxy \sin 2\theta \quad 1.1$$

where

$$k = 1/\sqrt{r_1 r_2}$$

and r_1, r_2 are radii of curvature of the parabolae. In the special case when $\phi = 45^\circ$, Ox and Oy are orthogonal, the intersecting curves are rectangular and the surface is then called a rectangular hyperbolic paraboloid. This is the conventional shape usually adopted in roof constructions.

If a parallelogram is described on two opposite generators for example AB and AC , fig. 1.1b, the fourth corner of the parallelogram being E , the line DE runs parallel to the axis Oz . The length DE is termed the linear distortion (v) of the hyper surface $ABCD$. If the projected area of the shell is Q on a plane perpendicular to the axis Oz , then the ratio v/Q is termed the specific distortion, and is equal to the quantity (k) in equation 1.1. This is the practical way of calculating (k). The rise of the shell is defined as the arithmetic mean of the difference of the heights of the low and high corners.

1.3 Review of Existing Theories

1.3.1 Membrane Theory

In practice the design of hyperbolic paraboloid shell has been based on the general assumption of a momentless state of stress in the shell. In this assumption only the inplane stresses are taken into consideration. The theory has therefore been appropriately termed the membrane theory. The governing equations are essentially those derived by Purcher⁽¹⁹³¹⁾ for shells of arbitrary geometry. In his derivation, the static equilibrium of the projection of the shell on a plane normal to the axial plane is considered, and the resulting differential equation for hyperbolic paraboloid shell results in the particularly simple expression

$$-\frac{2}{k} \frac{\partial^2 \phi}{\partial x \partial y} = L \quad 1.2$$

where ϕ is the Airy stress function with the usual meaning the theory of elasticity and

$$L = -Z + \frac{X_y}{k} + \frac{Y_x}{k} \quad 1.3$$

gives the load function.

Exact solutions of equation 1.2 have been derived by Tester⁽¹⁹⁴⁸⁾ and Reissner has generalised one of Tester's formulae for a constant load per unit area of the shell surface, rectangular in plan. Practical design methods with tables have been presented by Candela⁽¹⁹⁶⁰⁾, Parme⁽¹⁹⁵⁸⁾ and Portland Cement Association⁽¹⁹⁶⁰⁾.

Equation 1.2, though of a mathematically simple form however has some limitations when applied in practice. Reissner⁽¹⁹⁵⁵⁾ has discussed the various limitations of the simple membrane theory as applied to hyperbolic paraboloid, and Tottenham⁽¹⁹⁵⁷⁾ also on the same problem concludes that the stress function ϕ cannot in general be prescribed along all the edges, and therefore recommended that the effect of finite displacement should be taken into consideration to overcome the defects. We also note that the basic non-linear differential equations of equilibrium and compatibility for the membrane state are presented in the same publication. Flugge⁽¹⁹⁶⁰⁾, Flugge and Geyling⁽¹⁹⁵⁷⁾ have also shown in addition that discontinuities are developed in the internal stress system when the shell is partially loaded. The displacement consideration reveal further limitations of the membrane theory.

To illustrate briefly the problems encountered in the

membrane theory of hyperbolic paraboloid shell, the stress and deformation of the shell under a uniformly distributed load per unit area is considered. Setting $X = Y = 0$, we obtain

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{Zk}{2} = \frac{-qk}{2} \quad 1.4$$

where it has been assumed that $Z = -q$.

This equation has a general solution of the form

$$\phi = -\frac{1}{2} qkxy + f(y) + g(x) \quad 1.5$$

The stresses in the shell are then given by

$$N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \frac{1}{2} qk \quad \text{constant throughout the shell}$$

$$N_x = \frac{\partial^2 f(x)}{\partial x^2}, \quad \text{arbitrary} \quad 1.6$$

$$N_y = \frac{\partial^2 f(y)}{\partial y^2}, \quad \text{arbitrary}$$

In general, the forces N_x , N_y cannot be independently prescribed to satisfy the boundary condition. A trivial solution can be obtained if we put $N_x = N_y = 0$. Thus the shell is under a purely shearing state of stress. A rough estimate of the load transmission can be obtained if the shell is imagined to consist of a system of cables and suspension systems. A half of the load under a uniformly distributed load system can then be assumed to be carried by each system. Because all diagonal cross sections are parabolic and are generally part of the same parabola, the thrust in each cross-section is the same as the tension in the suspension system. The stress distribution in the shell can thus be described as one of tension along lines parallel to the low diagonal. The stress distribution thus results in shear stresses along the edge of the shell. Thus if

the edges are restrained by shear forces, then the shell should be capable of bearing loads. It is thus essential to have edge beams around the shell to provide the restraint.

The edge beam being under shear load, needs to be restrained in order to balance these forces. The restraints of the edge beam take several forms. The simplest type is the provision of a tie rod between the ~~low~~ corners and vertical supports at these corners. The more complicated types are buttresses to take the thrust resultants from the edge beams. These become heavy for large shells or shells with high rise.

Thus far we have used only the conditions of equilibrium of forces to determine the state of stress within the shell. To ensure however the conditions of compatibility of strain and displacements, we must introduce the displacement components u , v , w . The stress-strain relation and the strain-displacement relation will enable us to determine the displacements. The stress-strain relation has been derived by Reissner (1955) from a consideration of the minimum potential energy principle. In the publication, he showed that the purely membrane state does not allow the satisfaction of the symmetry condition.

The deformation of the shell comprises a shear deformation γ which if we assume the Poissons ratio ν to be zero without any loss of generality gives

$$\gamma = \frac{n_{xy}}{Gt} = -\frac{q}{Et_k} \quad 1.7$$

The shear strain is also given by the differential equation

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \text{Zero} \quad 1.8$$

If we equate the two expressions, we find a possible solution in the form

$$\begin{aligned} u &= 1/2\gamma y \\ v &= 1/2\gamma x \\ w &= 0 \end{aligned} \qquad 1.9$$

The shell deforms into a form of rhomboid, with the high corners moving out and the low corners moving in. To ensure a membrane state of stress therefore, the supporting members at the edge have to slide freely with the shell, an arrangement which is difficult to achieve in practice. Another mode of deformation is the following

$$u = v = 0, \quad w = \gamma/2k.$$

The possibility of this mode is present in principle, but in general the required displacement of the edge will not occur.

1.3.2 Bending Theory

As shown in section 1.31, the purely membrane state of stress is not possible as a result of lack of compatibility between shell edge and edge beam. Generally the shell would be accompanied by bending moment which in general is expected to be maximum towards the periphery of the shell.

The general theory of bending of shell has been in existence since the nineteenth century. The work of Aaron and Love are significant in this field. However most work in this field for doubly curved shells follow the theory developed by Marguere (1938) and later by Vlassov (1944). The complete theory yields two non-linear partial differential equations of the fourth order. No direct solution of these equations are known, and there has been a general simplification of the equations in order to achieve some solution.

The general assumptions have in many respects been similar to those used for deriving the differential equations for plate in bending. Plane sections are assumed plane before and after bending, hence stresses in the plane of the shell can be integrated across the thickness to give both the stress resultants and couples. Strains are also expressed in terms of the displacement of the middle surface, and thus the analysis of shell is reduced to the study of the deformation of the middle surface. The foregoing apply essentially to thin shells, in which category most shell roof structures and allied surfaces belong.

There are two general forms which the differential equations can take depending on the method of achieving the solution which the analyst has in mind. The two alternative forms are given below:

1. Marguere or Vlassov Equation

This comprises two partial differential equations expressed in terms of the stress function (F) and the transverse displacement (w). For shell whose middle surface is given by $z = kxy$, the resulting expression is

$$\begin{aligned} \nabla^4 F - 2Etk \frac{\partial^2 w}{\partial x \partial y} &= 0 \\ D \nabla^4 w + 2k \frac{\partial^2 F}{\partial x \partial y} &= -q \end{aligned} \quad 1.11$$

where $\nabla^4 = \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}^2$ and $D = Et^3/12(1 - \nu^2)$

and q is the uniform distributed load. The stresses and displacements are then in the form

$$\begin{aligned}
N_{xx} &= \frac{\partial^2 F}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \\
M_{xx} &= -D \left\{ \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right\} \quad \text{etc} \\
Q_x &= -D \frac{\partial}{\partial x} (\nabla^2 w) \quad \text{etc} \\
u &= \frac{1 - \nu^2}{D} \int \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} \right) dx \quad \text{etc}
\end{aligned} \tag{1.12}$$

A variation of the Marguere's equation is possible. Thus using the Vlassov's stress displacement function ϕ defined such that the first of equation (1.11) is identically satisfied thus

$$w = \nabla^4 \phi \quad F = 2Etk\phi^1 \tag{1.13}$$

the second reduces to a single eighth order equation

$$D \nabla^8 \phi + 4Etk^2 \phi^{.11} = -q \tag{1.14}$$

an equation which reminds of a similar equation in the theory of cylindrical shell.

2. Flugges Equation

By using the stress-strain and the strain-displacement relations, in the equilibrium equations, three simultaneous differential equations can be obtained in terms of the displacement u, v, w . The resulting expressions are

$$\begin{aligned}
(1 + \mu) \nabla_{,xy}^2 + 2u_{,yy} + (1 - \mu) \nabla_{,xx}^2 u - 2\pi(1 - \mu)w_{,x} &= 0 \\
(1 + \mu)u_{,xy} + 2\pi^2 \nabla_{,xx}^2 v + (1 - \mu) \nabla_{,yy}^2 v - 2\pi(1 - \mu)w_{,y} &= 0 \tag{1.15} \\
(2\pi w - \pi^2 u_{,x} - v_{,y}) + \Delta(w_{,yyyy} + 2\pi^2 w_{,xxyy} - \pi^2 w_{,xxxx}) &= Q
\end{aligned}$$

where u, v, w are displacements expressed in non-dimensional forms

$$u = u/a \quad v = v/b \quad w = wk$$

$$\pi = \pi/b, \quad \dot{\lambda} = t^2/12(1 - \mu)c^2\pi, \quad Q = \frac{-q(1 - \mu)a^2}{Etc}$$

and the subscripts after the comma indicate differentiation. The stresses are also expressed as in equation 1.12 in terms of the displacements. Equation 1.15 can be reduced to a single eighth order partial differential equation in any of the unknown by suitable transformations. The resulting expression is similar in form to equation 1.14.

Equations 1.11 and 1.15 have been employed by various investigators to analyse the hyperbolic paraboloid shell. Flugge and Conrad (1959) have used Vlassov's equation by expressing the load as a double Fourier series and then following a method similar to Ambartsumyan (1947) separated the plate solution from the shell solution, thus taking advantage of the rapid convergence of the remainder. Flugge and Conrad have also worked out examples of shells of positive and negative Gaussian Curvature under concentrated load. They infer, in particular, that the stiffness of the hyperbolic paraboloid is roughly equal to that of the flat plate. Levy type solutions have been used by Bouma (1959) and Apeland (1961). The later derived an influence coefficient method for a shell supported on two opposite boundaries, thus enabling the actual boundary conditions along the other two opposite edges to be applied. By plotting the real roots of the characteristic equation against certain shell parameters, it is shown that the edge disturbances in hyperbolic paraboloid shells are damped at a much slower rate than in shells of positive Gaussian Curvature.

1.3.3 Approximate Solutions

The set of partial differential equations 1.11 and 1.15 are to be solved in conjunction with the boundary conditions. It is then possible using relations given in equation 1.12 to obtain the displacement and stresses in the shell. A complete analytical solution of the eighth order partial differential equation cannot be obtained except in the case of the shell subject to pure bending - Maunder and Reissner (1957). Much attention has therefore been directed towards the approximate solution of the equations.

The Levy type solution has already been referred to in section 1.3.2. Some simplified solutions are based on the influence of the roots of the characteristic equations resulting from the Levy procedure. The Levy solution consists of assuming that the stress function ϕ and the load q can be expanded in series of the type

$$\begin{aligned} \phi &= \sum_{n=1}^{\infty} \phi_n e^{\gamma y x} \sin \gamma x \\ q &= \frac{4}{\pi} q_0 \sum_{n=1}^{\infty} \sin \gamma y \end{aligned} \quad 1.16$$

where $\gamma = n\pi/b$

which when applied to the eighth order differential equation 1.14 yields the characteristic equation

$$(p^2 - 1)^4 - \frac{1}{\beta^4} p^2 = 0 \quad 1.17$$

from which 4 real and 4 imaginary roots are obtained - Loof (1961)

$$\begin{aligned} p_{1-2} &= \pm \alpha_1 \\ p_{3-4} &= \pm \alpha_2 \end{aligned} \quad 1.18$$

$$p_{5-8} = \pm (\alpha_3 \pm i\mu_3)$$

Loof (1961) has presented the roots in a graphical form and showed

that the values of α_2 are small and can therefore be neglected. He thus arrived at a sixth order equation, requiring three boundary conditions on opposite edges and has shown that no generality is lost. He also presented a series of approximate formulae on the hyperbolic paraboloid shell supported on flexible edges. These formulae have been investigated by Batchelor (1963) who showed that results based on these formulae failed to be correlated by experimental results. By making assumptions similar to that of Loof, Bleich and Salvadori (1959) have reduced the equation 1.15 to one similar to that of a plate supported on an elastic foundation. In a similar manner, to Bleich and Salvadori, Vreedenburgh (1958/59) has reduced equation 1.14 to a fourth order one similar to that of the elastically supported beam.

Other approximate methods which have been used may be categorised under the heading of numerical procedures. Among these are finite differences, used by Das Gupta (1961) and the Variational technique using the Kantorovitch and Galerkin procedures. Cheetty (1961) has used the Kantorovitch procedure in conjunction with the Matrix Progression Technique developed by Tottenham (1962). In this method, a solution is sought in the form

$$\phi = \sum_{i=1}^n f_i(x) g_i(y) \quad 1.19$$

where $g_i(y)$ are the assumed functions which satisfy the boundary conditions in the y-direction and $f_i(x)$ are the functions to be determined. The initial set of differential equation is thus reduced to the form

$$F_i(\phi) = p \quad 1.20$$

where F_i is a set of ordinary differential operator and p are the external reactions. The Kantorovitch process is now used to minimise the residual in the form

$$\int [F_i(\phi) - p] g_i(y) dy = 0 \quad 1.21$$

The advantage of the method lies in its flexibility to allow the exact boundary conditions to be satisfied and any desired accuracy can be obtained by successive use and improvement of the functions assumed.

1.4 Review of Literature on Finite Element

The basic idea behind the finite element formulation of structural analysis has been known for a long time. The forerunner of the finite element analysis was the frame work analogy pioneered by Herenikoff⁽¹⁹⁴¹⁾. He conceived a structure as being represented by a series of interconnected frame work, whose elastic properties are suitably defined to correspond to that of the actual structure. The structure can then be solved by conventional analysis like the relaxation method and moment distribution. The accuracy of the Herenikoff's analogy clearly depends on the success with which the elastic properties can be defined. We shall call Herenikoff's idealisation as a one-dimensional element idealisation . The generalisation of the one dimensional element representation into two - and three dimensional idealisation was made possible by the advent of the electronic computer. Without this the solution is not possible because the number of simultaneous linear equations resulting is greater than that the normal conventional desk calculating machine can cope with.

1.4.1 Matrix formulation of the Finite Element Theory

Generalisation and unifications of the various structural theories have been the subject of the work of many investigators during the last two decades. Attempts have been made to summarise the past developments in structural analysis by means of concise mathematical statements, which are more convenient to write down

and also in such an easily processed form for machine computation. One such formulation is the matrix algebra approach.

The most comprehensive contribution has been made by research workers in the field of Aeronautical Engineering. This however is not surprising, because, they have always been faced with more complicated problems, especially as aircrafts become more complex and refined and solutions, like the series solution which are all embracing, are no longer adequate. Coupled with this, is the fact, that the limited use of the early electronic digital computing machines were confined mainly to the aircraft industry. The most general and most systematic formulation of the structural theory must necessarily be attributed to Argyris (1954/55) who expressed the various phases of the analysis in a standard sequence of matrix operations. Generalisation of the theory to make possible the two dimensional element representation of structure is due to Clough (1960) who defined a structure as being composed of an assemblage of a finite number of finite size elements that are interconnected at finite number of joints. It would appear that as a result of imposing continuity requirements at only these few joints, the flexibility of the idealised structure as compared with the actual structure will be excessively increased. However, by restricting the internal stresses to a few fundamental patterns that will approximate the actual stress distribution, the structure will be considerably stiffened. Although there is no rational procedure to determine precisely the relative and the resultant effects of each of these idealisations, it is natural to expect that the error resulting from the latter will diminish when the mesh size is refined. How fast that error will be reduced can only be tested by solving the given problem using different mesh approximations and observing the convergence of the results.

1.4.2 Properties of Element

The successful application of the finite element technique depends on the accuracy with which the approximating elements properties could be defined. Structures differ considerably in shape and so do elements. In fact there are infinite element types to analyse a given structure. But for shell type of structure which is considered in this thesis we are concerned mainly with elements which are capable of transmitting both membrane and bending stresses. Elements can roughly be categorised into displacement models and equilibrium models. Earlier models were based on the parametric representation of the displacement field (displacement model) or the stress field (equilibrium model). The unknown parameters are identifiable with either the nodal displacements (translation and rotation) for the displacement model or the nodal forces (moments and shears) for the ^{equilibrium} displacement field. The works of Pian (1964) and Clough (1958) respectively represent some very significant contribution in each field. We merely note here that the series representation is arbitrary. Theoretically the series must be infinite, but in practice it is finite and the way the infinite series is curtailed is left to the judgement of the author. The Argyris school have in recent publications - Argyris (Sept - Oct 1965) come out with a more rational approach towards defining element properties. The approach is based on what has appropriately been called the Natural Modes. These could be either of stresses or displacement. The significant point about using the natural mode approach is that the guess, as to which polynomial term to include or exclude from the infinite series, is taken out of the formulation.

In general an element could be an n-sided polygon having (n) nodes, but the general purpose element utilised in plate and shell

analysis is restricted to $n = 3$ - corresponding to the triangular element and $n = 4$ - corresponding to the quadrilateral element. The rectangular element which is a special case of the quadrilateral element has received a considerable attention than the triangular element although it has always been recognised that the triangular element has an intrinsic advantage over other elements by virtue of the fact that it has a great adaptability to meet the requirements of most structure and in particular curved and irregular boundaries. The formulation of the stiffness matrix for triangular element and rectangular elements under in-plane forces, seems to have been given exhaustive treatment in the following references: Turner et al (1956), Clough (1958). The bending stiffness formulation however has been confined solely to rectangular element (Zienkiewicz (1965) Melosh (1961), although recent publication by Clough et al (1964), Zienkiewicz (1966) show a trend towards the triangular element stiffness formulation.

1.4.3 Application of Finite Element to the Analysis of Shell Structure

It appears that the limited use of the finite element analysis to shell problem has been confined solely to the analysis of cylindrical shells and domes. Adini (1961) has used the triangular element to analyse a spherical shell while Zienkiewicz (1964) has used the rectangular elements to analyse an arch dam. Apart from these examples, the finite element has not found a general acceptance in the analysis of shell structure. There is no where in the literature where the hyperbolic paraboloid shell has been analysed using the finite element technique.

1.5 Review of Experimental Work

A considerable number of tests have been carried out mainly in reinforced mortar of hyperbolic paraboloid shells as an aid to design. Consequent on the nature of these tests no generalised conclusions can be drawn as to the behaviour of the shell in general. Much fewer tests have been carried out on Timber shell, and with the exception of one or two, all the tests have been carried out as aid to design. However some of the tests which have been reported have some significance for the problem discussed in this thesis and it is proposed to review them here.

Leicester and Massey (1960) have described tests on a $1/16$ scaled polyester resin model of a system of hyperbolic paraboloid shells consisting of six petals resting on six supports. They indicated the presence of quite large bending moments even in regions well removed from the boundaries of the shell. The same conclusion was reached by Flint (1961) and Rowe (1961). Lanletta (1961) reported tests carried out on a brick hyperbolic paraboloid shell supported on straight line generators. The tests were intended to study the statics of the shell under uniformly distributed loading. The author inferred that the shell behaviour differed considerably from that predicted by the membrane theory and approached that of a plate. Another important conclusion from the point of view of this thesis is that the test showed that the state of stress was not influenced to a great extent by the change in the rigidity of the edge beam. This is in support of the observation made in the course of the present investigation. We however observe that this is not the view held by Batchelor (1963).

The most comprehensive tests on shell with flexible edges

were carried out by Batchelor (1963). The shell was $8\frac{1}{2} \times 8\frac{1}{2} \times 3/4$ in with a rise of 14 in. The corners were all supported and tie rods were provided. Arrangements were also made to vary the edge beam stiffness. The model was of a reinforced concrete mortar. The result of the investigation was very interesting. We observe that the extensional or membrane theory failed to predict the behaviour of the model, neither could the bending theory. A systematic comparison was made between the shell behaviour and that of a plate of the same dimension but with fixed and simply supported boundary conditions. It was observed that experimental results lie within the solution obtained by treating the shell as a clamped or simply supported flat plate. The result also showed a great susceptibility towards the rigidity of the tie rod. With rigid tie rods, correlation with clamped plate solution was within 30%, whereas with flexible tie the behaviour was nearer to the simply supported plate. Batchelor concluded that the shell behaved more like a flat plate since the deflections in the middle region of the shell was reasonably predicted by regarding the structure as a flat plate.

A few tests however have been carried out using fixed boundary conditions by Chetty (1961), Brebbia (1966). The deflections and the stresses were observed to have been reasonably predicted by the bending theory of shell. No tests however have been reported on hyperbolic paraboloid shell on gable type or the simply supported kind of boundary conditions although the possibility always exists in theory.

1.6 The Elastic Properties and the Analysis of Timber Shells

The problems of the analysis of Timber Shell by the classical approach is necessarily complicated by its elastic properties. Timber is a variable material and displays high degree of heterogeneity. It may be represented as a cylindrically orthotropic material with the principal axes in the axial, radial and circumferential directions. When boards are cut from a log, these boards maintain the longitudinal axis, but the radial and the tangential directions are differently orientated, which condition makes the application of the theory of elasticity dubious. Radial and tangential elastic properties however are sufficiently similar to enable us to consider a piece of board as an orthotropic material, with the principal axes along and across the plane of the board.

Timber shell roofs are made up of layers of boards placed in different directions and nailed and or glued together. The number of layers and their arrangements in timber shell roofs vary very little. Shells with more than four layers are rare, and the orientation angles are normally 45° and 90° with the adjacent layers. Fortunately the thickness of the layers are invariably equal.

Because the cross-section of the shell is not homogeneous, the application of the theory of elasticity leads to a definition of several elastic constants - one set for plane stress or strain and the other set for bending. Assumptions in the shell theory also lead to coupling terms between extensional forces and bending moments. These problems have been discussed at great length by Dong (1964) and Reissner and Stavsky (1961). The general theory of the anisotropic shell have also been presented in the same publications. Comprehensive tests reported by Knipers and Loof (1962)

also show that tests on specimens of shell material do not follow the theoretical prediction of the elasticity theory. The use of average elastic properties based on experimental analysis is envisaged in this thesis.

1.7 Scope of Research

1. Investigation into an improved displacement function for the triangular finite element in bending.
2. Investigation of the stress and displacement distribution of the hyperbolic paraboloid shell whose edges are unsupported by the use of the Finite Element Technique.
3. Feasibility study of the model of a timber hyperbolic paraboloid shell using a table-sized model.

In cases 1 and 2 comparison are made between experimental results and those obtained from known approximate solution.

CHAPTER 2THEORY I2.1 GENERAL INTRODUCTION

The object of a structural analysis, is to be able to evaluate the response of a given structure to changes in its environment. The changes may come about as a result of the application of external forces, which may be of static or dynamic origin, or displacement, as for instance in the sinking of supports or changes in the thermal conditions around the structure. As the changes occur the structure is displaced to a new equilibrium position with consequent changes in the distribution of stresses throughout the body of the structure. The methods available for analysing any structure vary widely and depend on the complexity or simplicity of the structure. Basically, however, we are attempting in any method to satisfy the following two requirements simultaneously namely (a) equilibrium and (b) compatibility.

By satisfying the equilibrium condition, we are in effect establishing a balance between the external and the internal forces of the structure. If the statical conditions are insufficient to render the problem determinable, then we must consider in addition the question of kinematic or compatibility condition, which is the relationship between the displacement of the structure and the internally developed strains. Two sets of equations expressed by (a) forces i.e. the equilibrium relationships and (b) deformations i.e. the compatibility relationships are produced. The link up between (a) and (b) is provided by a third relationship which is quite distinct from the mechanics of the analysis. This is the relationship between the force and the displacement often identified with

the stress-strain relationship for the material of the structure and is thus a specific property of the particular structure. These three requirements are recognised as normal ingredients in a shell analysis program.

In this thesis one method of analysis, the Matrix Method of analysis, ^{as applied} ~~often referred to as~~ the Finite Element Method of analysis is presented. This method of analysis has been extensively developed and used in analysis of structures during the last decade in the field of plane stress and related problems. Extension of the method to include the effect of bending as in the problems of plates and shells is of recent origin. This thesis concerns itself with the particular application of the method to the analysis of hyperbolic paraboloid shells. It will be desirable here, however, to discuss the general scope and features of the method.

2.2 Definition of a Structure in the Finite Element Context

In the matrix algebra approach to structural analysis, a structure is defined as an assemblage of finite number of finite size elements, which may be of arbitrary size and geometry, each element is homogeneous and linearly elastic and the connections to each other are specifiable only in terms of some finite number of modes. By finite size element is meant elements which may be imagined to be mechanically sliced from the given structure. When the elements are assembled together in a given structure they must satisfy the following conditions:

- (1) Equilibrium: the internally developed forces, which appear as equivalent nodal forces must balance the

external forces on the nodes.

- (2) Compatibility: the elements must continue to fit together in their deformed states under the action of applied loads. In practice this is achieved only at the nodal points but the theory assumes that the element deformation has been described in such a way that connection along the sides of the element is achieved.

Satisfaction of these conditions is generally approximate. Customarily, one of these conditions is required explicitly, though not necessarily exactly, whereas the other follows implicitly according to the mechanics of the particular method being used. In the direct stiffness method, continuity of deformation is an explicit requirement, whereas the equilibrium requirement is implicit in the assembly procedure of the method.

2.3 Some Fundamental Energy Principles

The consistent formulation of a structural problem in the finite element concept is conveniently based on one of the various energy theorems or variational principles. As far as the finite element analysis is concerned the most widely used are the principles of variations of displacements and the principle of variation of stresses, also called the complementary energy principle. More general principles exist which in theory allow simultaneous approximations on displacements and stresses. ⁽³⁷⁾ Since these theorems are so fundamental to the finite element analysis, especially in defining the bounds to the solution, it is proposed to review here as briefly as possible, the most fundamental of the theorems namely the principle of the virtual work. In order to present the principle in its most generalised form, it is presented in the context of a three-dimen-

sional continuum.

Consider a three-dimensional body in space, whose position is fixed with respect to some prescribed axes which in this particular case has been chosen as the cartesian system of axes (fig. 2.1). The loading, the stresses and the strains are similarly defined in that system. For the equilibrium of the body in space we define the following force vectors:

- (i) Body forces/unit volume (fig. 2.2) $\bar{w}$ given by

$$\bar{w} = \{\bar{w}_x \quad \bar{w}_y \quad \bar{w}_z\} \quad 2.1$$

- (ii) Surface forces/unit surface area (fig. 2.5) φ given by

$$\varphi = \{\varphi_x \quad \varphi_y \quad \varphi_z\} \quad 2.2$$

- (iii) Internal stress vector σ given by

$$\sigma = \{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx}\} \quad 2.3$$

where the subscripts in equations 2.1, 2.2 and 2.3 have their usual meaning in the theory of elasticity and are in general functions of x , y and z .

For an elemental volume subjected to a variation of stresses (fig. 2.2), the equilibrium equations take the form

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \bar{w}_x = 0 \quad \text{etc} \quad 2.4$$

In matrix notation equation 2.4 can be written as

$$\begin{bmatrix} \frac{\partial}{\partial x} & . & . & \frac{\partial}{\partial y} & . & \frac{\partial}{\partial z} \\ . & \frac{\partial}{\partial y} & . & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & . \\ . & . & \frac{\partial}{\partial z} & . & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} + \begin{bmatrix} \bar{w}_x \\ \bar{w}_y \\ \bar{w}_z \end{bmatrix} = 0 \quad 2.4a$$

which can be represented more compactly as

$$D\sigma + \bar{w} = 0 \quad 2.4b$$

where D , is the rectangular matrix of differential operators.

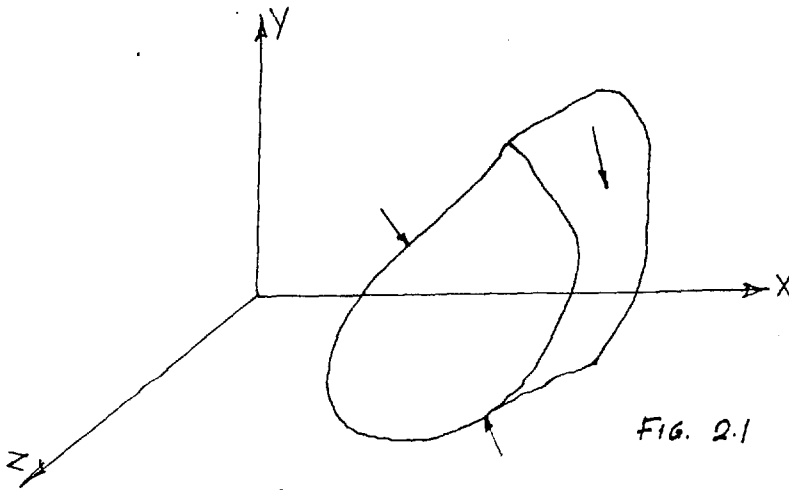


FIG. 2.1 A GENERAL 3-D BODY

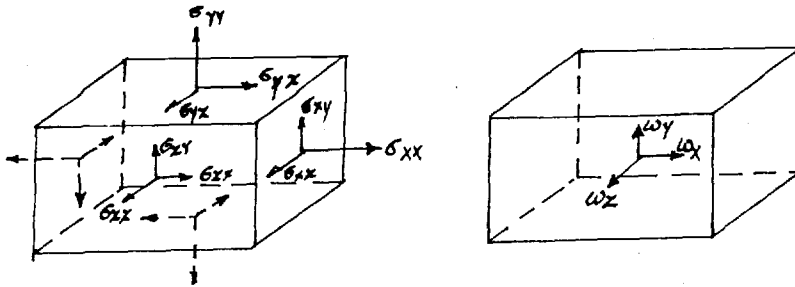


FIG. 2.2. CARTESIAN STRESSES AND BODY FORCES

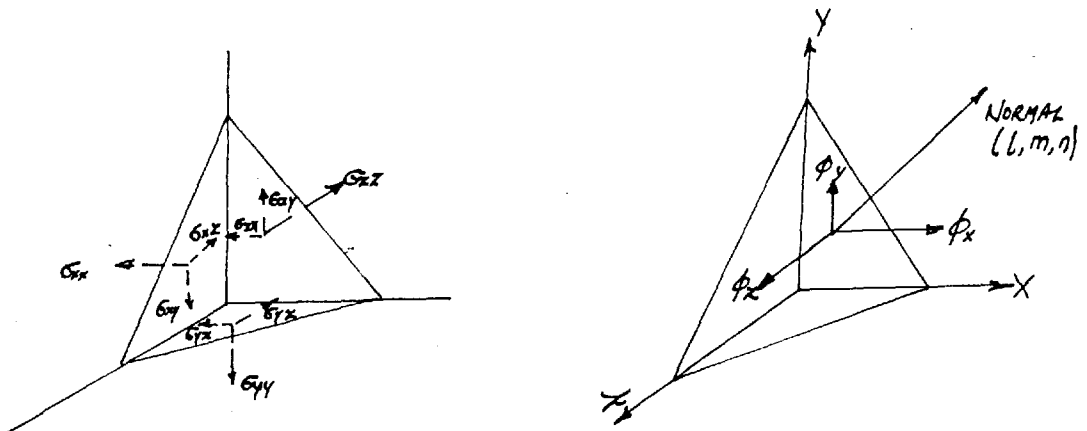


FIG. 2.3. STRESSES AND SURFACE FORCES.

On the surface, equilibrium relations of the following forms are satisfied;

$$\begin{aligned} \phi_x &= \sigma_{xx} \cos(n,x) + \sigma_{xy} \cos(n,y) + \sigma_{zx} \cos(n,z) \\ \phi_y &= \text{etc} \end{aligned} \quad 2.5$$

where $\cos(n,c)$ is the direction cosine between the outward normal and the (c) axis. Equation 2.5 can be put in the form of equation 2.4b if we make the substitution

$$\cos(n,c) = \frac{\partial n}{\partial c} \quad 2.5a$$

and equation 2.5 then assumes the form

$$Dn\sigma = \phi \quad 2.5b$$

Thus far in deriving the equilibrium relationships we have assumed no deformation of the element of the continuum and therefore the equilibrium conditions apply strictly to the undistorted system. This is the reason why there is a linear relationship between the stress system and the load system. One consequence of this is that we can have a superposition of equilibrium states. This is particularly significant in the derivation of element flexibility using the force method, in that we can build up a complicated stress system by combination of simpler equilibrium nodes.

To describe the deformation of the element we require the following vectors,

$$\begin{aligned} (\mathbf{w}) \quad & \text{The strain vector in the body of the element given by} \\ & = \{\epsilon_{xx} \epsilon_{yy} \epsilon_{zz} \epsilon_{xy} \epsilon_{yz} \epsilon_{zx}\} \end{aligned} \quad 2.6$$

$$\begin{aligned} (\mathbf{v}) \quad & \text{The cartesian displacement components} \\ \mu &= \{u \ v \ w\} \end{aligned} \quad 2.7$$

The compatibility condition is then expressed by

$$\epsilon = D^t \mu \quad 2.8$$

where the super-script indicates that the matrix D is transposed.

Again the strains are linear in the displacements and therefore we can superimpose compatible deformation states which give rise to a new state which is also compatible. This is an extremely useful idea when formulating the stiffness matrix of an element from pure kinematic arguments.

2.31 Principle of Virtual Work

If we consider a stress system $\bar{\sigma}$ which is arbitrary but is in equilibrium with w and ϕ , also an arbitrary strain system $\underline{\epsilon}$ which is kinematically consistent with μ , these two systems being independent of each other except in the context of correspondence then the principle of virtual work can be expressed by the scalar equation

$$\int_V \bar{\sigma} \underline{\epsilon} = \int_V w \bar{t} \mu \, dv + \int_S \phi \bar{t} u \, ds \quad 2.9$$

where the integration is carried through the volume v and over the surface s of the element. Equation 2.9 can be derived from equations 2.4b, 2.5b and 2.8 by the use of Greens transformation (25) Equation 2.9 can be interpreted loosely by saying that the work of the internal stresses and the corresponding strains is equal to the work of the external forces and the body forces and the corresponding external displacements. It is significant to observe the following:

- (1) $\bar{\sigma}$ is not necessarily the true system of stress but includes this special state. It must be in equilibrium with the external force system in other words it must be a statically equivalent stress system.
- (2) $\underline{\epsilon}$ is not necessarily the same as the true strain system but includes this special state. It must, however, be kinematically consistent with the external displacement.

In its broad application to structural problems, two corrolaries of the principles of virtual work are particularly important. They are

the principle of virtual forces and the principle of virtual displacement.

2.32 The Principle of Virtual Forces

Equation 2.9 is now cast into the more useful form suitable for engineering application in the form

$$\bar{S}^t \underline{v} = R^t r \quad 2.10$$

where $\bar{S}$ is the stress resultant replacing $\bar{\sigma}$ and $\underline{v}$ is the strain vector replacing $\underline{\epsilon}$, R is the external load vector replacing both w and ϕ , while r represents the external displacement replacing μ . To express the principle of virtual forces, 2.10 now assumes the form

$$\bar{S}^t v = R^t r \quad 2.11$$

where the bar below v has been dropped to indicate that it is the actual compatible strain, R is now arbitrary. If in particular we form a statically equivalent system expressed by

$$\bar{S} = \bar{b} R \quad 2.12$$

and substitute 2.12 in 2.11 we obtain

$$R^t \bar{b}^t v = R^t r \quad 2.13$$

and since R is arbitrary

$$r = \bar{b}^t v \quad 2.14$$

which is the mathematic expression for the unit load theorem. Suppose however that $S = S_e$ a self equilibrating stress system, 2.11 now assumes the form

$$S_e^t v = 0 \quad 2.15$$

which is a formal expression of the compatibility condition. Equations 2.14 and 2.15 form the basis of the force method of analysis.

2.33 Principle of Virtual Displacement

If S and R are in the actual system and $\underline{v}$ and r in the arbitrary system equation 2.11 assumes the form

$$\underline{v}^t S = r^t R \quad 2.16$$

which is the formal statement of the principle. If in particular from a kinematic analysis we obtain

$$\underline{v} = a r \quad 2.17$$

and substitute 2.17 in 2.16 we obtain

$$r^t a^t S = r^t R$$

Since r^t is arbitrary, we make it unity and thus obtain the unit displacement theorem, the resulting expression being given by

$$R = a^t S \quad 2.18$$

which is the equilibrium relationship established from pure kinematic argument.

2.4 Matrix Methods in the Finite Element Analysis

The methods used in the matrix analysis of structure can be classified into two categories namely, (1) The force method or the flexibility method and (2) The displacement method or the stiffness method. These methods are fairly well established in the analysis of frames and one dimensional structure in general. Extension of the methods to two and three dimensional problems is made possible by matrix notation, which enables groups of displacements or forces to be represented by single quantities, and so enables the general formulation of problems in all classes by the same matrix interpretative scheme. The flexibility and stiffness methods are interrelated to one another since one is the inverse of the other. The distinguishing factor is the order in which we satisfy the equilibrium and the compatibility conditions. The general formulation of structural problems in the flexibility and stiffness method is now presented in the light of the fundamental theorems presented in section 2.3.

2.41 The General Theory of the Force Method of Analysis

Equilibrium conditions alone are not sufficient in the general case to fix the stress matrix S for a given load R . But we can always find stress systems $\bar{S}$ which satisfy all the equilibrium conditions, such stress systems we define as being statically equivalent to the applied loading. Thus based on the undeformed geometry we can always write the equilibrium relations in the form

$$\bar{S} = \bar{b}R \quad 2.19$$

For a statically indeterminate structure, $\bar{b}$ is indeterminate. But if however we assign some redundancies X and form a statically equivalent system $b_o R$ we obtain

$$S = b_o R + b_i X \quad 2.20$$

where b_o - is a rectangular matrix of constants having m columns, whose i th column is a stress system in equilibrium with $R_i = 1$

b_i - is a rectangular matrix of constants having n columns, the i th column of which is a stress system in equilibrium with $X_i = 1$

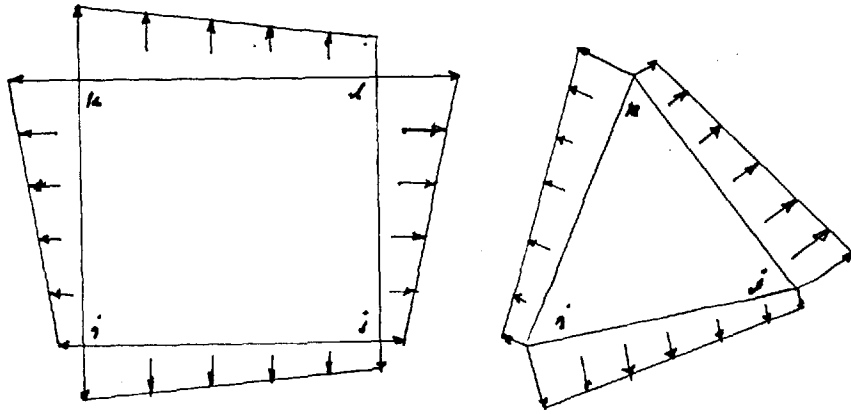
X - is the $(n \times 1)$ vector comprising the magnitudes of the self-equilibrating stress system more familiarly known in the literature as the redundancies.

Thus in the force method, equilibrium is first satisfied by assigning as many redundants as would make the structure determinate. We now denote the deformation of the structural members by the vector v and displacement of the structure by r . Using the virtual force or the complimentary energy principles, the compatibility condition is expressed by

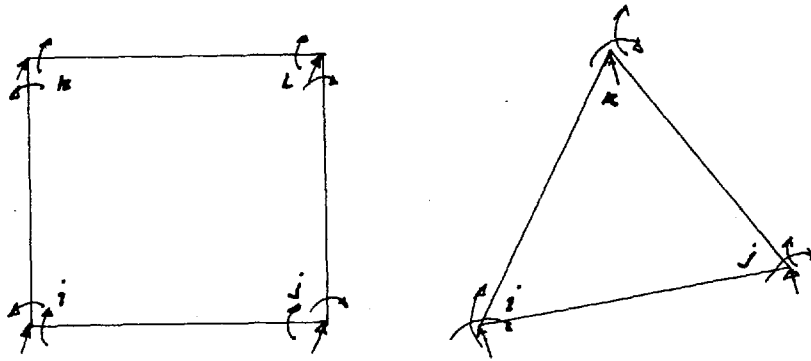
$$b_i^t v = 0 \quad 2.21$$

The deflection of the structure is now given by

$$r = b_o^t v \quad 2.22$$



(a) DISTRIBUTED MOMENTS IN ELEMENTS



(b) EQUIVALENT NODAL LOADS

FIG 2.4 DISTRIBUTED STRESSES AND EQUIVALENT NODAL LOADS
FOR FINITE ELEMENT ANALYSIS

If the stress strain law for the structural member is now given by a relation of the form

$$v = fS + H \quad 2.23$$

where f is the flexibility of the member and

H is a column of initial strains due, for instance, to self straining induced by incorrect fit, manufacturing error etc.

Substituting for v given by 2.23 in the compatibility equation 2.21 we obtain

$$b_1^t [fS + H] = 0 \quad 2.24$$

Now substitute for S from 2.20 we obtain

$$b_1^t [fb_oR + fb_1X + H] = 0 \quad 2.25$$

Expanding and rearranging equation 2.25 we obtain

$$X = - (b_1^t fb_1)^{-1} [b_1^t fb_oR + b_1^t H] \quad 2.26$$

Substituting for X in equation 2.20 we obtain

$$\begin{aligned} S &= b_oR - b_1 (b_1^t fb_1)^{-1} b_1^t fb_oR - b_1 (b_1^t fb_1)^{-1} b_1^t H \\ &= [b_o - b_1 (b_1^t fb_1)^{-1} b_1^t fb_o]R - b_1 (b_1^t fb_1)^{-1} b_1^t H \\ &= \bar{b}R - b_1 (b_1^t fb_1)^{-1} b_1^t H \end{aligned} \quad 2.27$$

$$\text{where } b = [b_o - b_1 (b_1^t fb_1)^{-1} b_1^t fb_o] \quad 2.28$$

The displacement of the structure can now be obtained from

$$\begin{aligned} r &= b o^t [f\bar{S} + H] \\ &= [b o^t fb_o - b o^t fb_1 (b_1^t fb_1)^{-1} b_1^t fb_o]R \\ &\quad + [b o^t - b o^t fb_1 (b_1^t fb_1)^{-1} b_1^t]H \\ &= FR + b^t H \end{aligned} \quad 2.29$$

$$\text{where } F = b o^t fb_o - b o^t fb_1 (b_1^t fb_1)^{-1} b_1^t fb_o \quad 2.30$$

F is the flexibility matrix of the structure.

2.42 The General Theory of the Displacement Method

A similar argument to the one used in section 2.42 shows that in general, for a set of displacements r , kinematic considerations

alone are not sufficient to determine the strains in the members of a structure. But we can always find a strain system v which is kinematically equivalent to r . If the r 's are small, then we can find linear relations of the form

$$v = ar \quad 2.31$$

Since the strain in the member is now uniquely defined, the stress follows from the stress-strain relation given in the general form

$$S = k_N v + J_N \quad 2.31$$

where k_N is the stiffness matrix and J_N is the initial stress conjugate to the initial strain H given before. If now we apply the virtual displacement theorem we obtain

$$S^t v = R^t r \quad 2.32$$

Substituting for v from 2.30 in 2.31 we obtain

$$S^t a r = R^t r \quad 2.33$$

and since r is arbitrary

$$R = a^t S \quad 2.34$$

which is the equilibrium relationship established from pure kinematic argument. If now we substitute for S in 2.34 we obtain

$$\begin{aligned} R &= a^t [k_N v + J_N] \\ &= a^t k_N a r + a^t J_N \\ &= K r + a^t J_N \end{aligned} \quad 2.35$$

$$\text{where } K = a^t k_N a \quad 2.36$$

K is the stiffness matrix for the structure.

2.5. Finite Element Analysis in the Displacement Method

In theory both the flexibility method and the stiffness method can be used to analyse any given structure. But for automatic computation, the stiffness matrix approach has proved so far to be more convenient than the flexibility approach. The short coming of the flexibility approach seems to be the difficulty of teaching the

machine to select a suitable redundant system which will result in a well condition flexibility matrix. Considerable amount of research is going on at present into the topological aspect with a view to overcoming this difficulty. Before going into the detailed consideration of the question of idealisation and the derivation of the individual element stiffness matrix it is convenient here to present in some details the sequence of matrix operations involved in the finite element analysis using the displacement theory.

For any structural member we seek the relationship between the forces (which could be static, dynamic or thermal) and the deformations of the member. In general a structural member may have 'n' assigned nodes or joints and both forces and displacements associated with these nodes. No loss of generality is involved if we assume that to each joint or node there is associated only one force and only one displacement. It is then always possible to establish a relationship between the system of forces acting at these joints and the corresponding displacements in the form

$$\begin{array}{rcl}
 S_1 & = & k_{11}v_1 + k_{12}v_2 + \dots + k_{1j}v_j + \dots + k_{1n}v_n \\
 S_2 & = & k_{21}v_1 + k_{22}v_2 + \dots + k_{2j}v_j + \dots + k_{2n}v_n \\
 & \vdots & \\
 S_n & = & k_{n1}v_1 + k_{n2}v_2 + \dots + k_{nj}v_j + \dots + k_{nn}v_n
 \end{array}
 \tag{2.37}$$

where S_i is the force acting at joint i

v_i is the displacement in the direction of the force R_i

k_{ij} are the stiffness influence coefficients.

In matrix form this system of linear equations takes the form

$$\begin{pmatrix} S_1 \\ S_2 \\ . \\ . \\ . \\ S_j \\ . \\ . \\ S_n \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & . & . & . & k_{1j} & . & . & . & k_{1n} \\ k_{21} & k_{22} & . & . & . & k_{2j} & . & . & . & k_{2n} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ k_{j1} & k_{j2} & . & . & . & k_{jj} & . & . & . & k_{jn} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ k_{n1} & k_{n2} & . & . & . & k_{nj} & . & . & . & k_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_j \\ . \\ . \\ v_n \end{pmatrix} \quad 2.37a$$

or more concisely as:

$$S = [k] \quad r \quad 2.37c$$

An observation of equation 2.37 shows that the geometrical configuration, the stress condition and the elastic properties should all be reflected in the stiffness matrix. Also if the material is homogeneous, isotropic and linearly elastic, then by Maxwell reciprocal theorem, the $[k]$ matrix is symmetric about the leading diagonal and positive definite. To conform with the finite element definition, the numerical value of n must be kept finite. Figure 2.4 shows two types of triangular elements with their boundary stress patterns, namely a triangular plate element subject to plane stress, and a triangular plate element subject to flexural stresses. The system of joint forces in fig 2.4b will be assumed to represent the resultant effect of distributed stress patterns shown in fig 2.4a.

By utilising suitable theories and techniques, and considering the various factors, such as the compatibility of the element with each other, it is possible to derive the stiffness relation in equation 2.37. Several stiffness matrices have been derived for plate elements in bending and we shall return to these later on.

obtain N equations of the form of 2.38 showing that the $[K]$ matrix is the 'unassembled' stiffness matrix for the structure. It is pertinent here to remark that the k_g 's need not be of the same size, this is equivalent to formally expressing the fact that a complex structure can be imagined to be composed of elements of different geometry and elastic properties.

When the element is a part of a complete structure its deformations are in various ways more constrained than when its free. The distortion of the element can be related to the actual displacement of the joints. This is accomplished by fixing all the nodes in the structure and displacing one node at a time. Only members adjacent to the displaced joints will be distorted and the components of this distortion may thus be determined by mere inspection. Carrying out this procedure at all the joints of the structure that will be displaced under the action of the external loads and assembling the results in one matrix equation will yield the following relationship

$$V = [a] r \quad 2.42$$

where r is given by the relation

$$r = \{r_1 \quad r_2 \quad r_3 \quad \dots \quad r_N\} \quad 2.43$$

and r is the displacement of joint i of the structure.

$[a]$ is a transformation matrix composed of constants relating the deformation of the elements $\{v\}$ to the structure displacement $\{r\}$. It is a linear transformation which implies that our displacements must be small and the principle of superposition of compatible deformation states applies.

The principle of virtual work can now be applied to obtain the relationship between the load and deformation of the structure. This follows from the fact that we know the strains in our structure

expressed by the nodal deformation matrix (V) and these strains are compatible with the structure's displacement expressed by the (r) vector. The resulting expression is

$$V^t S = r^t R \quad 2.44$$

But $V = ar$ and $S = KV$

Substituting for S and V in 2.44

$$r^t R = r^t a^t K ar \quad 2.45$$

Since this is an identity, we can equate the coefficients of r^t on each side of the equation to obtain

$$\begin{aligned} R &= a^t K ar \\ &= Kr \end{aligned} \quad 2.47$$

where $K = a^t K a$ is the stiffness matrix of the structure

R is the loading of the structure and its form in consonance with r defined by 2.43 is given by

$$R = \{ R_1 \ R_2 \ \dots \ R_j \ \dots \ R_N \} \quad 2.48$$

where R_i is the loading on the node i .

Equation 2.47 represents a set of simultaneous equations in N unknowns and must be solved to obtain the displacements. In its present form it may be singular, if we have included all the degrees of freedom including the rigid body freedoms and mechanisms which may have resulted from our idealisation. The rigid body freedoms and mechanisms must be eliminated. If this is done, the K matrix then becomes positive definite and possesses a unique inverse.

2.5 Direct Assembly of the K Matrix

It is extremely advantageous to include right from the beginning all possible degrees of freedom in the derivation of the stiffness matrix. The matrix $[a]$ will then be seen to contain elements which are zero, one or minus one. Because of the similarity of form to the Boolean algebra, this matrix has been named the Boolean

Matrix [26]. The boolean matrix operations are simple and will be illustrated under the section on computation.

Let the matrix [a] be partitioned in such a way that the transformation $a^t k a$ is possible, thus [a] takes the form

$$a = \{a_1 \ a_2 \ \dots \ a_g \ \dots \ a_N\} \quad 2.49$$

where N represents the total number of element. The stiffness matrix is thus seen to assume the form

$$K = \sum_{g=1}^N a_g^t k_g a_g \quad 2.50$$

the summation being taken over all the elements.

However a more obvious procedure of assembling the stiffness matrix can be obtained as follows. For an individual element, the generalised nodal force increment ΔS required to maintain a set of nodal displacement increment Δr is given by

$$\Delta S_g = k^e \Delta r \quad 2.51$$

Resultant nodal forces increments acting on the complete structure are

$$\Delta R = \sum_{g=1}^N \Delta S_g = K^e r \quad 2.52$$

where K the stiffness matrix of the complete structure is given by the summation

$$K = \sum k^e \quad 2.53$$

which provides an alternative basis for the matrix assembly. At each joint, the components of the stiffness matrices pertaining to the elements surrounding the nodes are summed directly to obtain the components of the overall stiffness matrix. Since any particular joint is only attached to a few elements surrounding it, this means that the summation is performed over a few elements at a time.

2.6 The Boundary Conditions

As indicated earlier on, it is extremely advantageous for logical and dynamic programming to include ab initio, all the degrees of freedom including the rigid body freedom, that may be associated with the element and structure. The final stiffness matrix is then singular and possesses no inverse since the determinant is zero. Both force boundary conditions and displacement boundary conditions can be imposed. For the displacement type boundary conditions the columns and rows pertaining to the specified boundary displacement is collapsed with consequent shrinkage of the stiffness matrix.

A more systematic approach however introduces the idea of partitioning. This is especially useful for solving mixed boundary value problems. Forces corresponding to a selected set of displacements are assigned and the remaining displacements are required to assume prescribed values. It is then required to determine the numerical values of the displacements corresponding to the prescribed forces and vice versa. Elements of the displacement and force vectors are rearranged as necessary and partitioned thus

$$\begin{aligned} r &= \{r_1, r_0\} \\ \text{and } R &= \{R_1, R_0\} \end{aligned} \quad 2.54$$

The subvectors R and r_0 are given and it is required to determine R_0 and r_1 . In conformity with equation 2.54 the stiffness matrix K is rearranged and partitioned thus

$$K = \begin{bmatrix} K_{11} & K_{10} \\ K_{01} & K_{00} \end{bmatrix} \quad 2.55$$

Then the force-deflection equations may be rearranged in the form of two matrix equations thus

$$\begin{aligned} R_1 &= K_{11}r_1 + K_{10}r_0 \\ R_0 &= K_{01}r_1 + K_{00}r_0 \end{aligned} \quad 2.56$$

from which follows by direct elimination and substitution

$$\begin{aligned} r_i &= [K_{ii}]^{-1} r_i - [K_{ii}]^{-1} [K_{io}] r_o \\ \text{and } R_o &= [K_{oi}] [K_{ii}]^{-1} R_i + [K_{oo} - K_{oi} K_{ii}^{-1} K_{io}] r_o \end{aligned} \quad 2.57$$

We also note the idea of the linear constraint matrix defined by

$$[M]r = 0 \quad 2.58$$

where the matrix M is essentially a rectangular matrix which the programmer specifies. The number m , of constraints assumed independent is equal to the number of rows of $[M]$. Its formulation is extremely simple since the elements are either unity or zero. The constraint conditions are to be used in eliminating a set of m displacements specified by the analyst. As before we rearrange and partition the matrix M in conformity with the first of equation 2.54, the elements of r_o being selected for elimination. Thus we have

$$[M_i : M_o] \begin{Bmatrix} r_i \\ r_o \end{Bmatrix} = 0 \quad 2.59$$

$$\text{or } M_i r_i + M_o r_o = 0 \quad 2.59a$$

Premultiplying the equation by M_o^{-1} we obtain

$$r_o = - M_o^{-1} M_i r_i \quad 2.60$$

and substituting for r_o in the first of equation 2.56, the force-displacement relations for the reduced system take the form

$$R = (K_{ii} - K_{io} M_o^{-1} M_i) r_i \quad 2.61$$

Equations 2.55 to 2.61 assume that the entire stiffness matrix is already generated prior to partitioning. A more systematic approach can however be obtained if the partitioning is carried out directly on the $[a]$ transformation matrix. The elements of the transformation matrix are rearranged and partitioned in conformity with equation 2.54 thus

$$\begin{aligned} v &= [a_i : a_o] \begin{Bmatrix} r_i \\ r_o \end{Bmatrix} \\ &= a r \end{aligned} \quad 2.62$$

$$\begin{aligned}
\text{But } K &= a^t \bar{k} a \\
&= \begin{bmatrix} a_1^t & a_0^t \\ a_1 & a_0 \end{bmatrix} \bar{k} \begin{bmatrix} a_1 & a_0 \end{bmatrix} \\
&= \begin{bmatrix} a_1^t k a_1 & a_1^t k a_0 \\ a_0^t k a_1 & a_0^t k a_0 \end{bmatrix} \\
&= \begin{bmatrix} K_{11} & K_{10} \\ K_{01} & K_{00} \end{bmatrix}
\end{aligned} \tag{2.63}$$

The procedure is thus seen to lead to a gradual build up of the various components of the K matrix in their required form to enable the analysis to proceed as indicated by equations 2.57 and 2.61.

2.70 Calculation of Stresses

The procedure outlined above enables us to determine the displacements by the solution of equation 2.47 thus

$$r = K^{-1}R \tag{2.64}$$

it being assumed that the boundary conditions etc have been imposed to enable the K matrix to be inverted. Substituting for r in equation 2.42 we obtain 'V' the element strains and similarly substituting for V in equation 2.40 we obtain the stress resultants or the equivalent nodal force. The result of the successive back substitution is

$$S = k a r = k a K^{-1} R \tag{2.65}$$

The theory thus far developed does not enable the individual element stresses to be obtained, this is embodied in the derivation of the stiffness matrix for the particular element. We merely note here that if σ denotes the element stress a relationship exists which is of the form

$$\sigma = Q S \tag{2.66}$$

where Q is the stress matrix, whose explicit form will be derived under the section on element stiffness matrix.

CHAPTER 3THEORY II3.1 Element Properties.3.1.1 Structural Idealisation

The formulation of any structural problem often involves a number of assumptions and idealisations. These assumptions may be in terms of stresses such as the idealisation of the stress-strain relations in the plastic stage of analysis, or of beams capable of carrying bending stresses only while shear stresses and shear deformations are neglected. Often the idealisation involves only the geometry. Thus a tapered beam could be approximated by a series of uniform segmented or stepped beams, or a curved beam by segmented straight beams. Experience often dictates what assumptions or idealisations are to be made in any particular problem and there are no hard and fast rules to be observed. The objective on the whole is to produce a mathematical model which is capable of reproducing to a great extent the behaviour of the actual structure. The quality of the analysis is to a large extent dependent on the extent to which the mathematical model simulates the behaviour of the actual structure.

A parallel idea is that of approximation. This manifests itself in several ways and often cannot be distinguished from idealisation. Thus in shell analysis, by neglecting the bending stresses, which are regarded as secondary, one obtains an approximate shell theory often called the membrane theory. Often approximations are made in the final stages of analysis as for instance in the Shorer

type of approximation, to obtain a simplified solution. Solution of the exact equations of the actual physical system by approximate mathematical expressions such as the finite differences is still another form of approximation.

In the finite element method of analysis, we are concerned with idealisations which, involve physical approximation to the geometry such as the idealisation of the elastic continua as assemblages of finite continua or finite elements. The central idea is 'cutting' of the structure rather than substituting for the structure an entirely different structure as for instance in the beam or lattice analogy for plate problem. Whereas in the methods of the substitute structure the elastic properties could to a large extent be modified, no such approximations are called for in the finite element method of analysis.

More severe approximations can however be made in the finite element method of analysis. Thus we may replace a curved surface by an assembly of flat elements. Such an approximation has a dual advantage. Firstly it makes possible a general formulation of a class of problems with similar geometry and stress pattern such as plates and shells of various shapes and curvature. Secondly, it ensures that one set of stiffness matrix shall be used, thus obviating the necessity of deriving as many stiffness matrices as there are elements. We merely add that the inverse idea is also admissible, such as representing a flat surface by segments of curved surfaces. Both ideas have been exhibited thus far in the analysis of folded plates and cylindrical shells.

In accordance with the definition of a structure in the finite element context it is first necessary to break up the structure into

suitable segments. The general shape of a two dimensional shell type of structure is shown in fig. 3.1a and fig. 3.1b shows its possible idealisation. In general the element is a polygon having N sides, but the numeric value of N must be kept finite both to simplify the procedure by which the stiffness matrix or flexibility matrix of the element could be derived and to conform with the finite element definition. Observation also shows that we are not restricted to any particular geometry within the finite element context. Thus we may use curved elements to follow the contours of a curved structure and flat elements to follow the contours of a flat structure.

3.1.2 Suitable Elements for Idealising Shell Surfaces

For chronological purpose shells are divided in this thesis into three categories namely:

1. Axially symmetric shells of revolution
2. Singly curved shells
3. Doubly curved shells

While we are not concerned in this thesis with the analysis of shell type 1, it is interesting to review a type of idealisation which has been employed by Lu et al (1963). The idealisation followed in that reference was that of the assemblies of conical frusta and a spherical cap as shown in fig. 3.2. It is also interesting to note that the same type of structure was analysed by Adini (1961) using the triangular elements. He took advantage of the symmetry of the entire structure, and analysed an eighth segment of the structure.

By far the most important elements for idealising shell types

2 and 3 are the rectangular elements and the triangular element. When the shell has a single curvature as in cylindrical shells the obvious choice is the rectangular element. With class 3 shells a distinction has to be made based on the Gaussian curvature. Thus for shells of positive Gaussian curvature like elliptic paraboloids, rectangular elements could be used, for negatively curved shells like hyperbolic paraboloid the choice is limited to triangular elements because of the necessity of following as far as possible the contours of the shell.

3.1.3 Element Geometry

In deriving the shell element stiffness we are faced with two distinct possibilities. We may form an idealised model in which the middle surface of the element coincides with the middle surface of the structure and the element is thus curved. The resulting element stiffness matrix is thus fully populated and both the extensional and flexural actions are coupled. The stiffness matrix in partitioned form may thus be represented by

$$k_s = \begin{pmatrix} k_r & k_c \\ k_c^t & k_d \end{pmatrix} \quad 3.1$$

where k_d is the membrane or extensional stiffness

k_r is the flexural or bending stiffness

k_c is a coupling submatrix.

The size of k_s will depend generally on the number of nodal points which has been used to define the geometry of the element. Considering a rectangular curved element having four nodal points the order of k_s is 24×24 . Moreover the matrix being defined with respect to a particular geometry becomes unsuitable for any other geometry and the versatility of the element is thus impaired.

The second possibility is the use of elements which do not coincide with the geometry of the shell except at the nodal point. Thus the curved sides of the element are replaced by straight lines. The element stiffness is thus independent of geometry within the element and is definable solely by the nodal geometry.

The stiffness matrix is of the form

$$k_s = \begin{pmatrix} k_d & 0 \\ 0 & k_r \end{pmatrix} \quad 3.2$$

The absence of the coupling submatrix is apparent. Although the order of the matrix is not reduced, the separate submatrices have been reduced from three components to two and what is more from the computational point of view can be generated and stored independently as

$$k_s = [k_d \quad k_r] \quad 3.3$$

with considerable saving in computer storage. Also a general purpose stiffness matrix results. This type of element has been adopted for analysing the hyperbolic paraboloid shell in this thesis.

3.2 General Theory of Derivation of Shell Element Stiffness

As pointed out earlier, the stiffness matrix for a shell element whose middle surface is flat can be built up from two independent stiffness matrices corresponding to the bending stiffness for the element and the in-plane stiffness for the element. Below the general theory is presented.

3.2.1 Natural and Cartesian Definition of Displacement and Forces; Elastic Properties

Two sets of displacements are required denoted here as natural and cartesian systems. 'Natural' and 'Cartesian' may be regarded

in what follows as being used in a rather loose sense in some sections to indicate 'local' and 'global' system of coordinates. The more rigid use will be explained later.

We start with the complete cartesian force and displacement vectors which are arranged in the sequence

$$P = \begin{Bmatrix} P_d & P_r \\ 18 \times 1 \end{Bmatrix} \quad p = \begin{Bmatrix} p_d & p_r \\ 18 \times 1 \end{Bmatrix} \quad 3.4$$

The suffices d and r stand for displacements (forces) and rotations or slopes (Moments). The component vectors P_d and P_r are next defined as

$$P_d = \begin{Bmatrix} P_1 & P_2 & P_3 \\ 9 \times 1 \end{Bmatrix} \quad P_r = \begin{Bmatrix} \Omega_1 & \Omega_2 & \Omega_3 \\ 9 \times 1 \end{Bmatrix} \quad 3.5$$

in which (see fig. 3.3)

$$P_i = \begin{Bmatrix} U_i & V_i & W_i \end{Bmatrix} \quad \Omega_i = \begin{Bmatrix} M_{xi} & M_{yi} & M_{zi} \end{Bmatrix} \quad 3.6$$

Similarly for the displacement vector

$$p_d = \begin{Bmatrix} p_1 & p_2 & p_3 \\ 9 \times 1 \end{Bmatrix} \quad p_r = \begin{Bmatrix} \bar{w}_1 & \bar{w}_2 & \bar{w}_3 \\ 9 \times 1 \end{Bmatrix} \quad 3.7$$

in which

$$p_i = \begin{Bmatrix} u_i & v_i & w_i \\ 3 \times 1 \end{Bmatrix} \quad \bar{w}_i = \begin{Bmatrix} \theta_i & \phi_i & \psi_i \\ 3 \times 1 \end{Bmatrix} \quad 3.8$$

Next natural forces and displacements are defined. The natural force and displacement vectors are arranged in the sequence

$$P_N = \begin{Bmatrix} P_{NM} & P_{NB} \end{Bmatrix} \quad p_N = \begin{Bmatrix} p_{NM} & p_{NB} \end{Bmatrix} \quad 3.9$$

where the subscript N has been used to indicate the natural system.

The order of the matrix vectors P_N and p_N is influenced by our definition of what we called the natural system. If we merely referred to the natural system as mere transformation to a convenient local coordinate system the P , P_N , p , p_N are of order 18×1 , assuming we include all the degrees of freedom in P_N and p_N . If on the other hand, we neglect the rigid body freedom as explained later and identify our P_N and p_N with straining forces and straining

modes of displacement, the orders of P_N and p_N are reduced to (9×1) . This^{fs} made up of

$$P_{NM} = \{P_\alpha \ P_\beta \ P_\gamma\}, P_{NB} = \{T_S \ T_A\} \quad 3.10$$

in which $P_\alpha \ P_\beta \ P_\gamma$ denote any three independent forces system which could be self equilibrating force system directed in some α, β, γ coordinate system which in ref. (27) has been taken as the direction of the sides. T_S and T_A are given by

$$T_S = \{T_{S\alpha} \ T_{S\beta} \ T_{S\gamma}\}; T_A = \{T_{A\alpha} \ T_{A\beta} \ T_{A\gamma}\} \quad 3.11$$

where T_{Si} and T_{Ai} are conveniently identified with the symmetric and antisymmetric nodal moments acting at the vertices of the side l_i , their vectorial direction being normal to the direction of l_i . Differentiation into symmetric and antisymmetric components is not essential to the theory.

Correspondingly, the natural displacement vectors p_{NM} and p_{NB} are described by

$$p_{NM} = \{p_\alpha \ p_\beta \ p_\gamma\}, p_{NB} = \{\bar{\phi}_S \ \bar{\phi}_A\} \quad 3.12$$

where

$$\bar{\phi}_S = \{\phi_{S\alpha} \ \phi_{S\beta} \ \phi_{S\gamma}\} \quad \bar{\phi}_A = \{\phi_{A\alpha} \ \phi_{A\beta} \ \phi_{A\gamma}\} \quad 3.13$$

Suffices M and N denote Membrane and bending respectively.

It is pertinent here to comment on the order of P_N and p_N . Ideally it should be $(18 - 6) \ 12 \times 1$ matrix and not a (9×1) matrix as given above. The reason for the reduced dimension is because we ignore the inplane rotational stiffness at the nodal point. As a result when investigating a flat portion of a shell we have to ignore a rotational degree of freedom preferably that associated with the axis nearest to the perpendicular of the plane in question.

We also require in what follows, the relation between the total natural strain and the natural component stress as defined by

the relation

$$\epsilon_N = E_N^{-1} \sigma_N \quad 3.14$$

where E_N is the 3×3 natural modulus of elasticity and may include arbitrary anisotropy. E_N is of the

$$E_N = E \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \quad 3.15$$

where E is the cartesian modulus.

3.2.2 Natural and Cartesian Stiffnesses

If the p_N and P_N are suitably defined then it is possible to write down by mere inspection the relationship between e_N and p_N as

$$e_N = a_N p_N \quad 3.16$$

where a is a square matrix expressing the linear relationship between e_N and p_N . Further more we can write

$$\begin{aligned} e_N &= [e_{NM} \quad e_{NB}] \\ p_N &= [p_{NM} \quad p_{NB}] \end{aligned} \quad 3.17$$

and since no coupling exists a_N is of the form

$$a_N = \begin{bmatrix} a_{NM} & 0 \\ 0 & a_{NB} \end{bmatrix} \quad 3.18$$

Using the principle of virtual work

$$\begin{aligned} p_N^t p_N &= \int_V e_N^t \sigma_N dV \\ &= p_N^t \int_V a_N^t E_N a_N p_N dV \end{aligned} \quad 3.19$$

$$\begin{aligned} \text{or} \quad p_N &= \left[\int_V a_N^t E_N a_N dV \right] p_N \\ &= K_{NS} p_N \end{aligned} \quad 3.20$$

where

$$\begin{aligned} K_{NS} &= \int_V \begin{bmatrix} a_{NM}^t & 0 \\ 0 & a_{NB} \end{bmatrix} E_N \begin{bmatrix} a_{NM} & 0 \\ 0 & a_{NB} \end{bmatrix} dV \\ &= \int \begin{bmatrix} a_{NM}^t E_N a_{NM} & a_{NB}^t E_N a_{NB} \end{bmatrix} dV \\ &= \begin{bmatrix} K_M & K_B \end{bmatrix} \end{aligned} \quad 3.21$$

Relation 3.21 confirms that the stiffness matrix of the shell segment can be built up of the membrane and binding matrices separately generated.

To obtain the cartesian stiffness we require a transformation of the form

$$P_N = c_{NP} \quad \text{and} \quad P = c_N^t P_N \quad 3.22$$

we now apply the congruent transformation

$$k_S = c_N^t k_{NS} c_N \quad 3.23$$

to obtain the cartesian stiffness for the shell element. The subscript S has been used to denote shell. If P_N has been described in a local system which itself is cartesian, c_N can be identified with simple rotation of axes and c_N is given by the diagonal matrix

$$c_N = \begin{bmatrix} c & c & c \end{bmatrix} \quad 3.24$$

where

$$c = \begin{bmatrix} c_{\alpha x} & c_{\alpha y} & c_{\alpha z} \\ c_{\beta x} & c_{\beta y} & c_{\beta z} \\ c_{\gamma x} & c_{\gamma y} & c_{\gamma z} \end{bmatrix} \quad 3.25$$

and c_{ij} is the direction cosine relating axis i to j .

3.3 Displacement Functions

The theory presented in the last section shows that in order to establish the stiffness relationship, the strain in the element must be known or determinable in one form or another. It must also be expressible by equation 3.16. For linear elements such as beams, existing theories using the engineers theory of bending, slope deflection equations etc enable us to do this. But with two dimensional and three dimensional elements these theories are generally inadequate and we have to consider using the various energy theorems, such as Ritz principles, virtual work etc. If for instance a

displacement field is prescribed in a continuum by several unknown parameters then the theorem of least total potential energy can be used. The combination of the unknown parameters yielding a minimum of the energy represents the best possible approximation. Convergence of the result to the correct solution will be assured if by increasing infinitely the number of parameters the true distribution of displacement could be achieved.

In the usual approach based on Ritz method expansion of the displacement field in polynomial or Fourier series over the whole field is envisaged. Thus for a three dimensional continuum we have

$$\begin{aligned} u &= u_0(x, y, z) + \alpha_{ur} u_r(x, y, z) \\ v &= v_0(x, y, z) + \alpha_{vr} v_r(x, y, z) \\ w &= w_0(x, y, z) + \alpha_{wr} w_r(x, y, z) \end{aligned} \quad 3.26$$

where

u_0, v_0, w_0 satisfy the kinematic conditions where prescribed
 u_r, v_r, w_r are linearly independent functions which
 vanish at prescribed points

$\alpha_{ur}, \alpha_{vr}, \alpha_{wr}$ are unknown constants to be determined such that the
 total potential energy of the system is a minimum.

In the finite element approach, the parameters (α) of the displacement field are identified with the nodal displacement vector which is in the literature referred to as the generalised displacement. The nodal displacements prescribe the displacement field within the element uniquely.

According to Vebuek (1965) generalised displacements p are defined according to the following rules:

(a) Connection of the elements with one another across a common boundary is to be expressed by equating two by two the

generalised displacement pertaining to this boundary.

(b) The whole displacement field along this boundary must be determined uniquely by the generalised displacements pertaining to the boundary and conversely. In this manner, equating the generalised displacement secures the continuity of displacement along the boundary.

(c) The number of parameters (α) must be equal or superior to the number of required generalised displacements. If superior, convenient additional generalised displacements can be prescribed so that the matrices α and p can be connected by a non singular linear transformation $T = (T_{ij})$ such that

$$\alpha = Tp \text{ and } p = T^{-1}_{\alpha} \quad 3.27$$

3.3.1 Requirements of the Displacement Function

In view of the prominence given to the conditions which the displacement function must satisfy in recent publications it is proposed to review them here and comment on them.

(a) The displacement function must be continuous over the element ⁽³⁴⁾ and at no point of the structure should the strain be infinite otherwise the field is no longer integrable ⁽³⁷⁾. This is the basic requirements of the potential energy formulation and does not warrant further comment.

(b) It must be possible to represent the rigid body motions of an element otherwise the equilibrium condition of the element as a whole are falsified ⁽³⁴⁾. As shown in the theory in section 3.2.2 the rigid body displacements are not essential to the formulation of the natural stiffness matrix but is implicitly required in the formulation of the q_N matrix.

(c) It must be possible to represent states of constant stress otherwise as the mesh of elements is finely subdivided, there is no

guarantee that the stresses will converge towards a continuous function ⁽⁴¹⁾. This condition is no doubt reminiscent of the membrane analysis of triangular elements. As pointed out in ref ⁽²⁷⁾ it is not possible to assume that the bending stresses or moments are constant over the triangle in view of the necessity of transferring shear forces, which require varying bending moments.

(d) Where neighbouring element abut between nodes there must be no discontinuity of slope and displacement between adjacent elements ⁽⁴²⁾. While this condition is implicit in the theory it does not appear to be a necessary condition and appears very difficult to satisfy with elements in bending. Indeed the whole basis of analysis of shells using elements of planar middle surface would be invalidated if this condition were adhered to. This requirement appears to be the basis of partitioning of elements displacement functions into the so called 'conforming' and 'non-conforming' functions ⁽⁴¹⁾. The conforming functions are assumed to guarantee the continuity of displacements along the boundaries and continuity of normal slope across the boundaries while non-conforming functions are assumed to guarantee only the continuity of displacements, with the continuity of normal slope sacrificed.

3.3.2 Displacement Function Previously Derived

1. Non-Conforming Functions. Bending Elements

Earlier displacement functions have essentially been in the non-conforming category. The various functions have been based on the various interpolation formulae expressed by the scalar equation

$$w(x, y) = [1 \ x \ y \ x^2 \ xy \ y^2 \ x^2y \ xy^2 \ x^3 \ \dots] \alpha \quad 3.28$$

$$\text{or} \quad w = X^t \alpha \quad 3.28a$$

where the α 's are to be determined in terms of the generalised displacements. For elements in bending the generalised displacements at the nodes are taken as the normal displacement and the two cartesian slopes θ and ϕ . Some differentiation of the function is involved in order to obtain the slopes θ and ϕ and generally these functions are unsuitable for slope definition. For a rectangular element in bending, Melosh ⁽³⁴⁾ has improved this situation by using one of the osculating formulae in which the amplitude (α) of the displacement function w is expressed not only in terms of the nodal displacements but also in terms of the first derivative which is easily identified with θ and ϕ and so no further errors are introduced.

Furthermore in order to express the α uniquely in terms of the p as given by equation 3.27, the polynomials have been truncated quite arbitrarily and generally there has been loss of symmetry in the polynomial expressed by the loss of the higher terms in the general expansion of equation 3.28. Often by trial and error certain terms in the polynomials have been arbitrarily coupled as in the best formulation achieved thus far ⁽⁴³⁾ for triangular element given by

$$w = [1 \ x \ y \ x^2 \ y^2 \ (x^2y + xy^2) \ xy \ x^3 \ y^3] \begin{bmatrix} \alpha_1 \\ . \\ . \\ . \\ . \\ \alpha_q \end{bmatrix} \quad 3.29$$

2. Conforming Displacement Function

The task of deriving suitable displacement functions which satisfy 'conforming' definition has been difficult. These often involve specification of additional nodes at the middle of the side of the element boundary. At these points only is the continuity

of normal slope satisfied. In reference (41) a conforming displacement function for a triangular element is claimed to have been obtained. But we note that in the same reference, that the displacement function while ensuring convergence, the approximation was not always as good as that achievable with the simpler function discussed in the previous section. An example of the application of the various functions⁽⁴¹⁾ yielded a maximum of 3% for the non-conforming function and errors of up to 30% for the conforming function. This seems to show that the requirements of conformity seems to impose unnecessary additional constraint on the element which the simpler functions do not, and in terms of comparison of the result with the exact solution the simpler functions are vastly superior. Moreover the conforming solutions erred on the upper bound side of the solution while the non-conforming functions erred on the lower bound side.

Both types of function have been used in the analysis in this thesis and compared with a new function derived in this thesis.

3.3.3 Natural Modes of Deformation

After the initial trial and error efforts at deriving suitable deflecting pattern for elements in general, research effort has been directed towards finding a consistent and logical basis for deriving suitable deformation patterns. Notable in this field is the Argyris school. Their most important contribution is the introduction of what they have appropriately termed the natural modes of deformation.

An examination of equation 3.19 shows that as far as the energy of the system is concerned we should concern ourselves only with those modes of deformation which contribute to the strain

energy of the element. Thus we could neglect those displacement modes which are associated with the rigid body displacements. But we must have in accordance with requirements in section 3.31 a means of prescribing the rigid body modes since this is implicit in the assembly of the equilibrium equation of the entire structure as given by equation 3.23. The importance of neglecting the rigid body mode and deriving the stiffness matrix via the natural stiffness matrix does not seem to have been generally recognised.

To fix ideas, let us consider a triangular element under the action of inplane stress and inplane deformation. Clearly the degrees of freedom of the entire element are six, comprising the two cartesian displacements of each of the nodes. Thus the p vector in equation 3.23 is given by

$$p = \{u_x \ u_y \ u_z \ v_x \ v_y \ v_z\} \quad 3.30$$

But included in the p vectors are three rigid body modes. Thus the deformation or straining modes are three in number. Suppose however that instead of considering the degrees of freedom we have considered the force system P which correspond to these deformations. By corresponding forces we mean forces which do work on the displacement such that the relationship

$$W = P^t p$$

is satisfied. The six force system which correspond to the displacement system may be represented thus

$$P = \{U_x \ U_y \ U_z \ V_x \ V_y \ V_z\} \quad 3.31$$

Now considering the equilibrium condition, included in the P vector are three overall equilibrium conditions which consist of the equilibrium of the forces in the directions of x , and y and rotational equilibrium of the element about the normal. Therefore the statical indeterminacy of the triangular element is three which is also the kinematic indeterminacy. The entire nodal force system is

describable in terms of any three thus

$$P = \begin{pmatrix} P_N \\ P_O \end{pmatrix} = \begin{pmatrix} a_N \\ a_O \end{pmatrix} P_N \quad 3.32$$

where P_N are the redundant forces and P_O has been used to describe three overall equilibrium forces mentioned above. Since the P_N and P_O system are to be only statically equivalent to the P system, then we are free to choose the P_N and P_O system as we please. And in particular we may choose the P_N system as self-equilibrating system, and then the P_O system is wholly statically equivalent to the external system. With this background in mind we may choose the P_N system in a manner most natural to the geometry of the element. In ref ⁽²⁶⁾ the p_N system has been chosen as the elongations of the three sides of the triangle and the P_N system as flange loads in the tensile sense acting along the sides of the triangle as shown in figure 3.4.

One important conclusion is that these modes enable us to determine the stiffness relationship in a manner most natural to the element and any impositions about continuity etc if they exist, manifest themselves in a very obvious manner. Exactly the same procedure has been extended to problems of rectangular plate element in bending and under planar forces and startlingly accurate results have been achieved by this means.

Deriving a triangular element stiffness in bending based on the above consideration does not appear to have received any attention, and therefore attention has been focussed in that direction in this thesis.

3.4 Derivation of Stiffness Matrix for Triangular Element in Bending

3.4.1 Element Coordinate Systems

As pointed out earlier, the stiffness matrix for a shell element is derived in terms of what are referred to as a cartesian system and a natural system. Also pointed out is the crude sense in which the natural system has been used to define a local system of axis which could in fact be cartesian in itself. Fig. 3.5 shows the various coordinate systems which can be used to describe the geometry of the element. The local coordinate system can be seen in fig. 3.5(b) as being a translation and rotation of the cartesian axis. One axis of the latter is made to coincide with one of the sides of the element. The natural system shown in fig. 3.5(c) also referred to as the area coordinate is here denoted by a triangular system of axes. The coordinate of any point within the triangle is specified by the magnitude of the three perpendiculars drawn from the point to the sides of the triangle.

Consider an arbitrary triangle i, j, k , arbitrarily orientated in a cartesian coordinate system fig. 3.6. Then an arbitrary point (o) whose coordinates are x , and y can be equally described by a coordinate system which we may refer to as $\xi_i \xi_j \xi_k$, where $h_n \xi_n$ is the perpendicular distance from point(o) on to the side(l_n) of the triangle facing the node(n) and (h_n) is the perpendicular distance of the node(n) from the side(l_n).

By summation of the areas of the component triangles oij , ojk and oki , it is possible to present the relations between $\xi_i \xi_j \xi_k$ thus:

$$\Delta_{oij} = 1/2 \, l_k \, h_k \xi_k = \xi_k \Delta_{ijk} \quad 3.33$$

where Δ has been used to represent the area.

$$\text{Thus} \quad \xi_k = \frac{\Delta_{oij}}{\Delta_{ijk}} \quad 3.34$$

Similar expressions could be written for ξ_i, ξ_j .

$$\text{But } \Delta_{ijk} = \Delta_{oij} + \Delta_{ojk} + \Delta_{oik} = (\xi_i + \xi_j + \xi_k) \Delta_{ijk} \quad 3.35$$

$$\text{therefore} \quad \xi_i + \xi_j + \xi_k = 1 \quad 3.36$$

This relationship can be used to represent any of the ξ 's in terms of the other two.

To enable us to transform from one system to another we need the relationship between the ξ 's and x and y .

$$\begin{aligned} \text{Now } \Delta_{ijk} &= 1/2 \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \\ \Delta_{ojk} &= 1/2 \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \end{aligned} \quad 3.37$$

where the vertical bars have been used to indicate the determinantal evaluation of the terms within the brackets. Expanding equation 3.37 in terms of the first row and denoting Δ_{ijk} by the scalar D

$$\text{then } \xi_i = \frac{\Delta_{ojk}}{\Delta_{ijk}} = \left\langle \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} - x \begin{vmatrix} 1 & y_j \\ 1 & y_k \end{vmatrix} + y \begin{vmatrix} 1 & x_j \\ 1 & x_k \end{vmatrix} \right\rangle / D \quad 3.38$$

$$\xi_i = \alpha_i x + \beta_i y + \gamma_i \quad 3.39$$

where α_i and β_i have been used to denote the coefficient of x and y , and γ_i is the constant in expression 3.38 above.

$$\text{Similarly} \quad \xi_j = \alpha_j x + \beta_j y + \gamma_j \quad 3.39a$$

$$\text{and} \quad \xi_k = \alpha_k x + \beta_k y + \gamma_k \quad 3.39b$$

With respect to the geometry of the fig. 5.6(o) is a point midway between m and n , which we shall take as the moveable origin and it is a function of ξ_i . The coordinate of a point s moving between m and n is derived as follows:

Since s moves between m and n , one of its coordinates is specified by $h_i \xi_i$. If sl is the perpendicular distance to lj , then $sl = h_j \xi_j$ is the other coordinate. From the similarity of triangles snl and kjp we can write the relation

$$kp/jk = sl/sn \quad 3.40$$

$$\text{therefore} \quad sn = sl \cdot jk/kp = \xi_j h_j \cdot l_i/h_j = l_i \xi_j \quad 3.41$$

$$\begin{aligned} \text{But} \quad os = on - sn = z &= l_i(1 - \xi_i)/2 - l_i \xi_j \\ &= l_i(1 - \xi_i - 2\xi_j)/2 \end{aligned} \quad 3.42$$

$$\text{Thus} \quad z = l_i(1 - \xi_i - 2\xi_j)/2 \quad 3.43$$

This can be rendered non-dimensional by dividing through by l_i . Similar relationship can be obtained by cyclic substitution of i , j and k .

3.4.2 Derivation of the Equation of the Surface

The same argument used before in section 3.33 leads to the fact that for describing the natural straining modes for a triangular element in bending with three nodal points we required only six generalised displacements and three rigid body modes. Thus p_N is of the order (6×1) . Here we take as the generalised displacements, the generalised rotational vectors at the nodes whose directions are normal to the sides of the triangle and in the plane of the triangle. Thus the deflected surfaces are to be represented by polynomials whose amplitudes are the rotation of the corners in a direction parallel to the sides.

If two sides of the triangle are fixed with respect to the vertical displacement only and the third is free to deflect, then the deflected form of the surface can be derived as follows: The displacement of the free edge is expanded into a power series function as follows:

$$w [\xi=0] = a + b\bar{z} + c\bar{z}^2 + d\bar{z}^3 + \dots \quad 3.44$$

where $\bar{z}$ is the coordinate of a point moving along the edge. Since however we require a third degree polynomial, as indicated by four boundary conditions on the displacement curve, namely two displacements and two rotations of the ends, we terminate the series as shown above. The series can be separated into the symmetric function represented by the even powers of z thus

$$w(s) = a + c\bar{z}^2 \quad 3.44a$$

and an anti-symmetric function represented by the odd power function

$$w(a) = b\bar{z} + d\bar{z}^3 \quad 3.44b$$

Thus approximately we can represent the deflection of the edge and sections parallel to the edges by a symmetric parabola and an anti-symmetric function. For the general section, the bar above z is now dropped. The form of the deflection surfaces represented by the symmetric and the antisymmetric functions, is shown in fig. 3.7.

Again the slope is expanded as power series of ξ_i (say). Thus

$$\phi = a_0 + a_1(1 - \xi_i) + a_2(1 - \xi_i)^2 \quad 3.45$$

we again terminate this series in the second power basing our argument on the fact that the slope is the first differential of the displacement and must necessarily lag by one degree behind the displacement.

$\phi = a_0$ will be rejected as an admissible solution from purely physical argument since it implies that the slope at the corner i is finite and has the same magnitude as the maximum slope at the foot of the triangle. $\phi = a_0$ can indeed be identified with the rigid body displacement.

3.4.3 Symmetric Functions

The constants a and c in expression 3.34a can be found from the boundary conditions of the curve thus:

$$\text{at } z = \pm 1/2, \quad w = 0, \quad dw/dz = -\phi \quad 3.46$$

$$\text{From these conditions } c = -\phi/1 \text{ and } a = -c l^2/4 \quad 3.47$$

$$\text{and} \quad w = \phi l/4 [1 - (2z/l)^2] \quad 3.48$$

For the symmetric displacement we prescribe the linear variation

$$\phi = a_1(1 - \xi_i)$$

in equation 3.45. When $\xi_i = 0$, that is, at the edge ϕ becomes $\bar{\phi}_s$ and $a_1 = \bar{\phi}_s$. Then $\phi = \bar{\phi}_s(1 - \xi_i)$.

Substituting for ϕ , z/l , and $1 = l_i(1 - \xi_i)$

$$w = \frac{\bar{\phi}_s(1 - \xi_i)^2 l_i}{4} \left[1 - \frac{(1 - \xi_i - 2\xi_j)^2}{(1 - \xi_i)^2} \right] \quad 3.49$$

which after expansion reduces to

$$w(s) = \bar{\phi}_s l_i \xi_j \xi_k \quad 3.50$$

3.4.4 Antisymmetric Displacement

For the antisymmetric case we now take the remaining expression in the polynomial

$$w = bz + dz^3 \quad 3.51$$

The constants b , and d are determined as in the symmetric case, but now we use the parabolic slope distribution as given by

$$\phi = \bar{\phi}_a(1 - \xi_i)^2$$

The resulting expression simplifies to

$$w(a) = \bar{\phi}_a l_i \xi_j \xi_k (\xi_j - \xi_k) \quad 3.52$$

With one edge displaced and the other two supported, we can now write

$$w_i = (w_s + w_a)i$$

for the edge facing node i displaced.

$$\text{Thus } w_i = l_i [\bar{\phi}_{s_i} \xi_j \xi_k + \bar{\phi}_{a_i} \xi_j \xi_k (\xi_j - \xi_k)] \quad 3.53$$

By mere inspection we can write down the remaining expressions thus

$$w_j = l_j [\bar{\phi}_{s_j} \xi_k \xi_i + \bar{\phi}_{a_j} \xi_k \xi_i (\xi_k - \xi_i)] \quad 3.53a$$

$$w_k = l_k [\bar{\phi}_{s_k} \xi_i \xi_j + \bar{\phi}_{a_k} \xi_i \xi_j (\xi_i - \xi_j)] \quad 3.53b$$

and

$$w = w_i + w_j + w_k \quad 3.54$$

3.5 Computational Molecules

In contrast to the rectangular element, it is not possible or appears rather difficult to present an explicit formulation of the stiffness matrix for the triangular element in bending. However in what follows, the main algebraic matrix formulation is presented in a form which it would be assembled in a program for computer analysis.

The natural generalised displacement expressed by $\bar{\Phi}_S$ and $\bar{\Phi}_a$ are represented by the column matrix p_N thus:

$$p_N = \{\bar{\Phi}_{Si} \quad \bar{\Phi}_{Sj} \quad \bar{\Phi}_{Sk} \quad \bar{\Phi}_{ai} \quad \bar{\Phi}_{aj} \quad \bar{\Phi}_{ak}\} \quad 3.55$$

the rigid body displacements are represented by

$$p_o = \{\bar{\Phi}_{oi} \quad \bar{\Phi}_{oj} \quad \bar{\Phi}_{ok}\} \quad 3.56$$

we also require the diagonal matrix l given by

$$l = \begin{bmatrix} L & o \\ o & L \end{bmatrix} \quad 3.57$$

where $L = [l_i \quad l_j \quad l_k] \quad 3.58$

and l_n has been defined as the side length of the triangle opposite to node n of the triangle. If now we denote by X the row vector of the polynomial expansion in $\xi_i \xi_j$ up to the third power thus

$$X = [\xi_i \quad \xi_j \quad \xi_i \xi_j \quad \xi_i^2 \xi_j^2 \quad \xi_i^2 \xi_j \quad \xi_i \xi_j^2 \quad \xi_i^3 \xi_j^3] \quad 3.59$$

then we can write the set of displacements derived in the previous section using the equation 3.36 to eliminate ξ_k as

$$w_N = Xclp_N \quad 3.60$$

where c is a rectangular matrix of coefficients which has as many columns as there are generalised displacements p_N and as many rows as there are columns in X . It is obtained by expanding each of the displacement components obtained in the last section in power series involving only ξ_i and ξ_j and grouping the coefficients of X as column vectors in correspondence with the appropriate $\bar{\Phi}$ for conformal matrix multiplication. The c matrix is presented in table (3.1).

With respect to a rectangular coordinate system, the strain component $\ddot{\mu}_x$ is defined by

$$\ddot{\mu}_x = \begin{bmatrix} \frac{d^2 w}{dx^2} \\ \frac{d^2 w}{dy^2} \\ 2 \frac{d^2 w}{dx dy} \end{bmatrix} \quad 3.61$$

Similarly with respect to the natural triangular coordinate system defined in the previous section, we also define the functional $\ddot{\mu}_\xi$ given by

$$\ddot{\mu}_\xi = \begin{bmatrix} \frac{d^2 w}{d\xi_i^2} \\ \frac{d^2 w}{d\xi_j^2} \\ \frac{d^2 w}{d\xi_i d\xi_j} \end{bmatrix} \quad 3.62$$

By partial differentiation of the displacement function w_N with respect to x and y making use of the relation (3.49) between x , y and $\xi_i \xi_j \xi_k$ we obtain

$$\ddot{\mu}_x = [\alpha] \ddot{\mu}_\xi \quad 3.63$$

where the matrix α is given by

$$\alpha = \begin{bmatrix} \alpha_i^2 & \alpha_j^2 & 2\alpha_i \alpha_j \\ \beta_i^2 & \beta_j^2 & 2\beta_i \beta_j \\ 2(\alpha_i \beta_i) & 2(\alpha_j \beta_j) & 2(\alpha_i \beta_j + \alpha_j \beta_i) \end{bmatrix} \quad 3.64$$

If we represent by $\ddot{X}$ the rectangular matrix obtained by differentiating partially the row matrix X as indicated by $\ddot{\mu}$, then we can write

$$\mu_x = \alpha \ddot{X} c \quad 1 \quad P_N \quad 3.65$$

where the explicit form of $\ddot{\mu}$ is presented in table (3.2). The stress matrix σ_x is defined by

$$\sigma_x = \{m_x \quad m_y \quad m_{xy}\} \quad 3.66$$

$$\text{then } \sigma_x = D\mu_x = D\alpha \dot{x} c_1 p_N \quad 3.67$$

where the D matrix is the elastic relation defined before. To obtain the natural stiffness matrix we now use the principle of virtual work, giving

$$\begin{aligned} p_N^t p_N &= \int \ddot{\mu}_x^t \sigma_x dv \\ &= \int p_N^t c_1 \dot{x}^t \alpha^t D \alpha \dot{x} c_1 p_N dv \end{aligned} \quad 3.68$$

and since p_N^t is arbitrary we obtain

$$p_N = t_1^t c_1^t \left\{ \int_S \ddot{x}^t \alpha^t D \alpha \dot{x} dS \right\} c_1 p_N \quad 3.69$$

$$p_N = t_1^t c_1^t \bar{k} c_1 p_N \quad 3.70$$

$$\text{where } \bar{k} = \int_S \ddot{x}^t \alpha^t D \alpha \dot{x} dS = \int_S \ddot{x}^t D \dot{x} dS \quad 3.71$$

for a material with an isotropic elasticity, $\bar{k}$ has been evaluated and presented in table (3.3). In the integration involved we have made use of the relation

$$\int_S \xi_i^m \xi_j^n dS = \frac{2\Delta m! n!}{(m+n+2)!} \quad 3.72$$

where Δ denotes the area of the triangle and the natural stiffness matrix is given by

$$k_N = t_1^t c_1^t \bar{k} c_1 \quad 3.73$$

3.5.1 Stiffness Matrix in the Cartesian System.

To obtain the cartesian stiffness matrix k , we now include the rigid body displacement expressed by

$$w_R = \xi_i P_{oi} + \xi_j P_{oj} + \xi_k P_{ok} \quad 3.74$$

and the total displacement w is now expressed by

$$w = w_R + w_N \quad 3.75$$

If p now expresses the cartesian vector of generalised displacement given by

$$p = \{w \quad \theta \quad \phi\} \quad 3.76$$

$$\begin{aligned}
\text{where} \quad w &= \{w_i \quad w_j \quad w_k\} \\
e &= \{e_i \quad e_j \quad e_k\} \\
\phi &= \{\phi_i \quad \phi_j \quad \phi_k\}
\end{aligned} \tag{3.77}$$

and $e = -dw/dy$ $\phi = dw/dx$
 making a coordinate substitution, we obtain

$$w = [T_{oo}] p_o \tag{3.78}$$

and differentiating w with respect to x , and y and again making a coordinate substitution, we again obtain

$$\begin{Bmatrix} e \\ \phi \end{Bmatrix} = T_{on} p_o + T_{nn} p_n \tag{3.79}$$

T_{on} T_{nn} T_{oo} in the present analysis turns out to be a unit matrices of rank equal to 3.

Solving equation 3.78 we obtain

$$p_o = T_{oo}^{-1} w \tag{3.80}$$

substituting for p in equation 3.79 and rearranging we obtain

$$T_{nn} p_n = \begin{Bmatrix} e \\ 0 \end{Bmatrix} - T_{on} T_{oo}^{-1} w \tag{3.81}$$

$$= \begin{bmatrix} -T_{on} T_{oo}^{-1} & I_6 \\ (6 \times 3) & \end{bmatrix} p \tag{3.81a}$$

$$= A p \tag{3.81b}$$

The A matrix is presented in table (3.4)

Now solve for p_n we obtain

$$p_n = T_{nn}^{-1} A p = a_n p \tag{3.82}$$

k matrix is now obtained by the congruent transformation

$$k = a_n^t k_n a_n \tag{3.83}$$

[C] =

.	-1	.	.	1	.
-1	.	.	-1	.	.
.	1	.	.	-3	.
1	1	-1	2	-2	.
1	.	.	3	.	.
.	.	.	-1	3	1
.	.	.	-3	1	-1
.	.	.	.	2	.
.	.	.	-2	.	.

TABLE 3.1 COEFFICIENTS OF POLYNOMIAL EXPANSION OF
THE DISPLACEMENT FUNCTION.

[X] =

.	.	2	.	.	$2\xi_2$	.	$6\xi_1$	.
.	.	.	.	2	.	$2\xi_1$	.	$6\xi_2$
.	.	.	1	.	$2\xi_1$	$2\xi_2$	.	.

TABLE 3.2 X MATRIX IN EQUATION 3.65.

2Δ								
			$2D_{11}$					
			D_{13}	$D_{33}/2$				
			$2D_{12}$	D_{23}	$2D_{22}$			
			A	B	C	E		
			F	G	H	J	K	
			$2D_{11}$	D_{13}	$2D_{12}$	L	M	$3D_{11}$
			$2D_{12}$	D_{23}	$2D_{23}$	O	P	$3D_{12}/2$ $3D_{22}$

TABLE 3.3 COEFFICIENTS OF k_N MATRIX

$$A = 2D_{11}/3 + 2D_{13}$$

$$B = (D_{13} + D_{33})/3$$

$$C = 2(D_{12} + D_{23})/3$$

$$E = (D_{11} + D_{33} + D_{13})/3$$

$$F = 2(D_{12} + D_{13})/3$$

$$G = (D_{23} + D_{33})/3$$

$$H = 2(D_{22} + D_{23})/3$$

$$J = (D_{33} + D_{12} + 2D_{13} + 2D_{23})/6$$

$$K = (D_{22} + D_{33} + D_{23})/3$$

$$L = (D_{11} + 2D_{13})/2$$

$$M = (2D_{12} + D_{13})/2$$

$$O = (D_{12} + D_{23})/2$$

$$P = (2D_{22} + D_{23})/2$$

$$\begin{aligned}
D_{11} &= \alpha_1^4 + \beta_1^4 + 2v\alpha_1^2\beta_1^2 + 4\gamma\alpha_1^2\beta_1^2 \\
D_{12} &= \alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + v[\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2] + 4\gamma\alpha_1\alpha_2\beta_1\beta_2 \\
D_{13} &= 2\alpha_1^3\alpha_2 + 2\beta_1^3\beta_2 + 2v[\alpha_1^2\beta_1\beta_2 + \alpha_1\alpha_2\beta_1^2] \\
&\quad + 4\gamma\alpha_1\beta_1(\alpha_1\beta_2 + \alpha_2\beta_1) \\
D_{22} &= \alpha_2^4 + \beta_2^4 + 2v\alpha_2^2\beta_2^2 + 4\gamma\alpha_2^2\beta_2^2 \\
D_{23} &= 2\alpha_1\alpha_2^3 + 2\beta_1\beta_2^3 + 2[\alpha_2^3\beta_1\beta_2 + \alpha_1\alpha_2\beta_2^2] \\
&\quad + 4\gamma\alpha_2\beta_2[\alpha_1\beta_2 + \alpha_2\beta_1] \\
D_{33} &= 4\alpha_1^2\alpha_2^2 + 4\beta_1^2\beta_2^2 + 8v\alpha_1\alpha_2\beta_1\beta_2 \\
&\quad + 4\gamma(\alpha_1\beta_2 + \alpha_2\beta_1)(\alpha_1\beta_2 + \alpha_2\beta_1)
\end{aligned}$$

$$\gamma = (1 - v)/2$$

v = Poissons Ratio

TABLE 3.3a COEFFICIENT D_{mn} IN TABLE 3.3

$-\beta_1$	$-\beta_2$	$\beta_1 + \beta_2$	1	.	.	.	.	.
$-\beta_1$	$-\beta_2$	$\beta_1 + \beta_2$	.	1	.	.	.	.
$-\beta_1$	$-\beta_2$	$\beta_1 + \beta_2$	.	.	1	.	.	.
α_1	α_2	$-\alpha_1 - \alpha_2$	.	.	.	1	.	.
α_1	α_2	$-\alpha_1 - \alpha_2$	.	.	.	.	1	.
α_1	α_2	$-\alpha_1 - \alpha_2$	.	.	.	.	.	1

TABLE 3.4 COEFFICIENT OF THE MATRIX A.

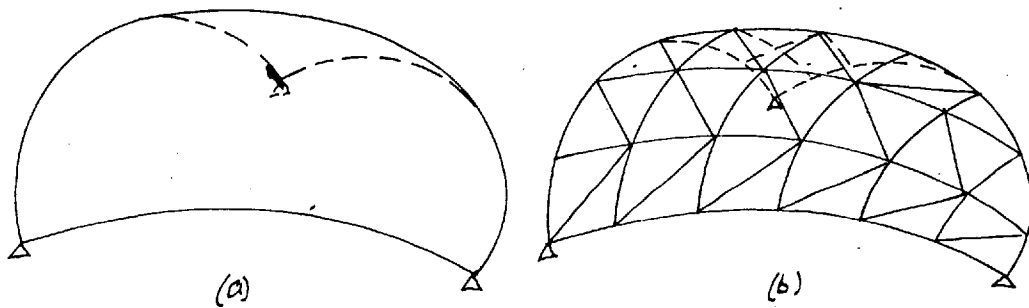


FIG. 3.1 GENERAL SHELL - TYPE STRUCTURE
AND POSSIBLE IDEALISATION

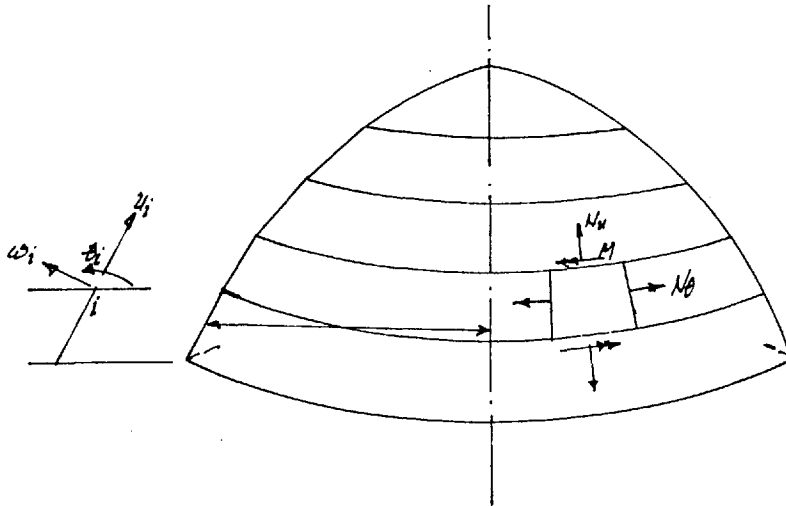


FIG 3.2 AXISYMMETRIC SHELL OF REVOLUTION AND
IDEALISATION BY CONICAL ELEMENTS.

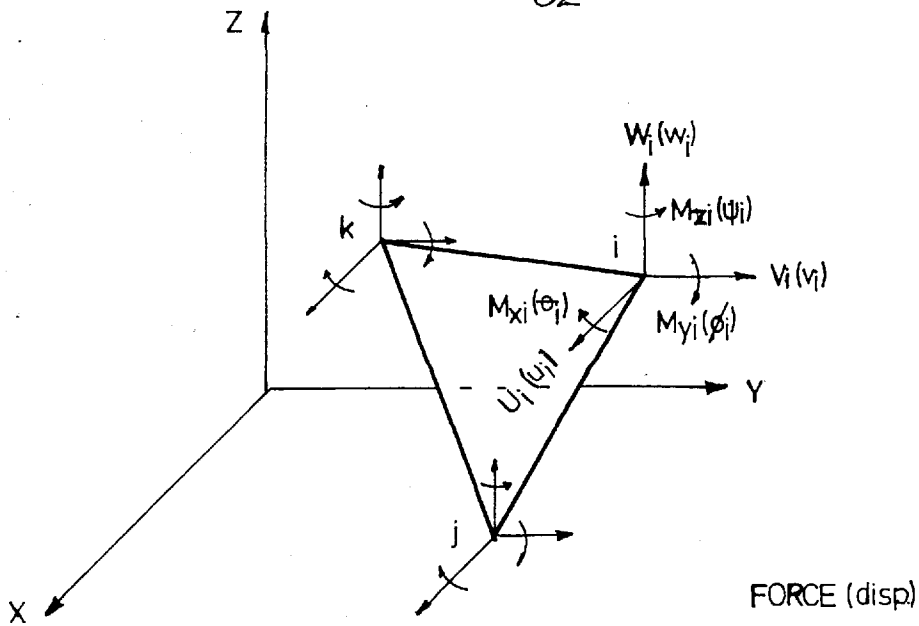


FIG 33 FORCE AND DISPLACEMENT COORDINATE

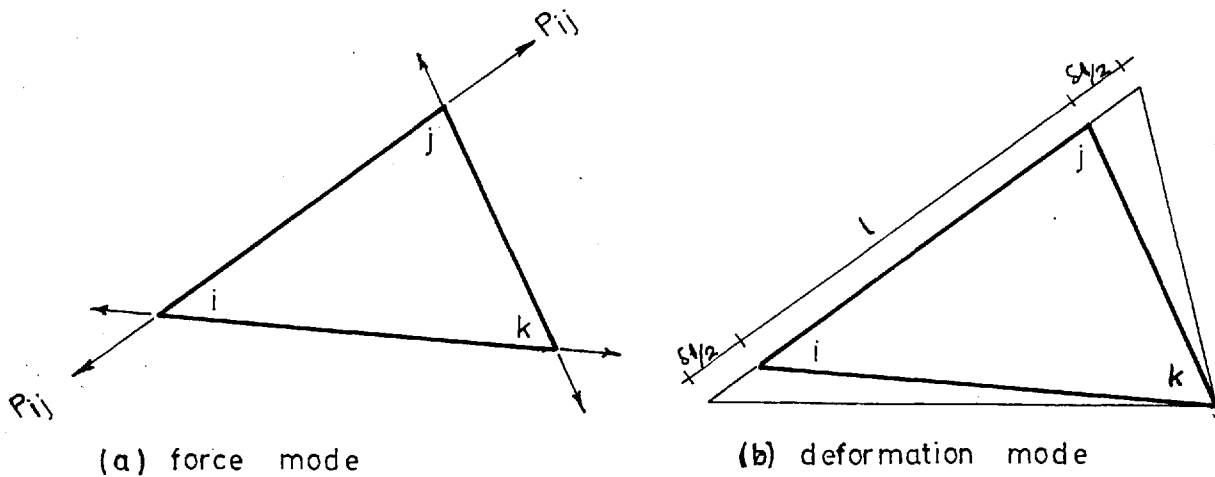
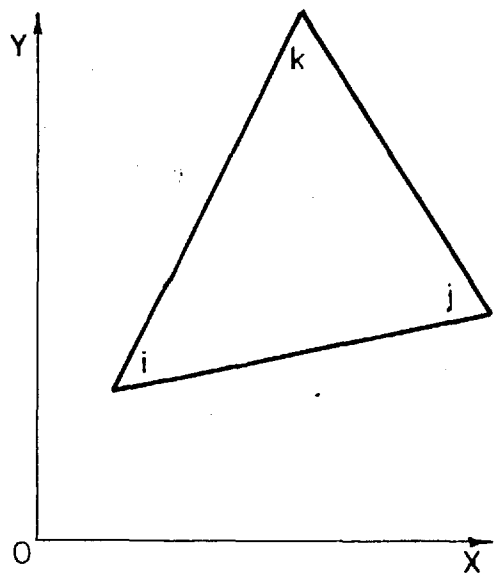
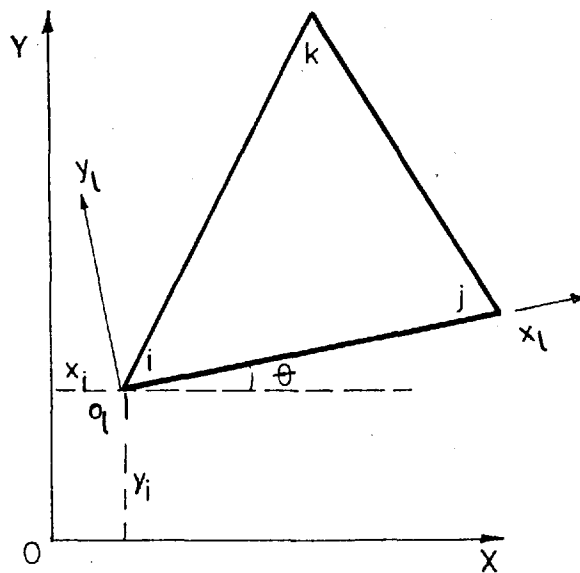


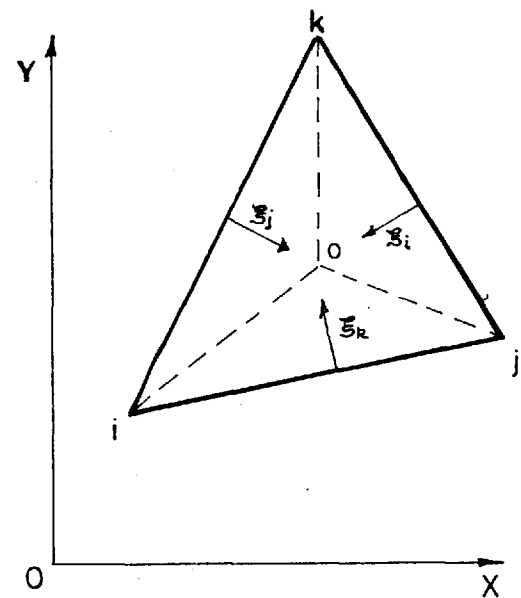
FIG3.4 NATURAL FORCES AND DEFORMATIONS



(a) cartesian coord.



(b) local coord.



(c) area coord.

$$g_i = \Delta_{ojk} / \Delta_{ijk}$$

FIG 3.5 COORDINATE SYSTEMS FOR THE TRIANGULAR ELEMENT.

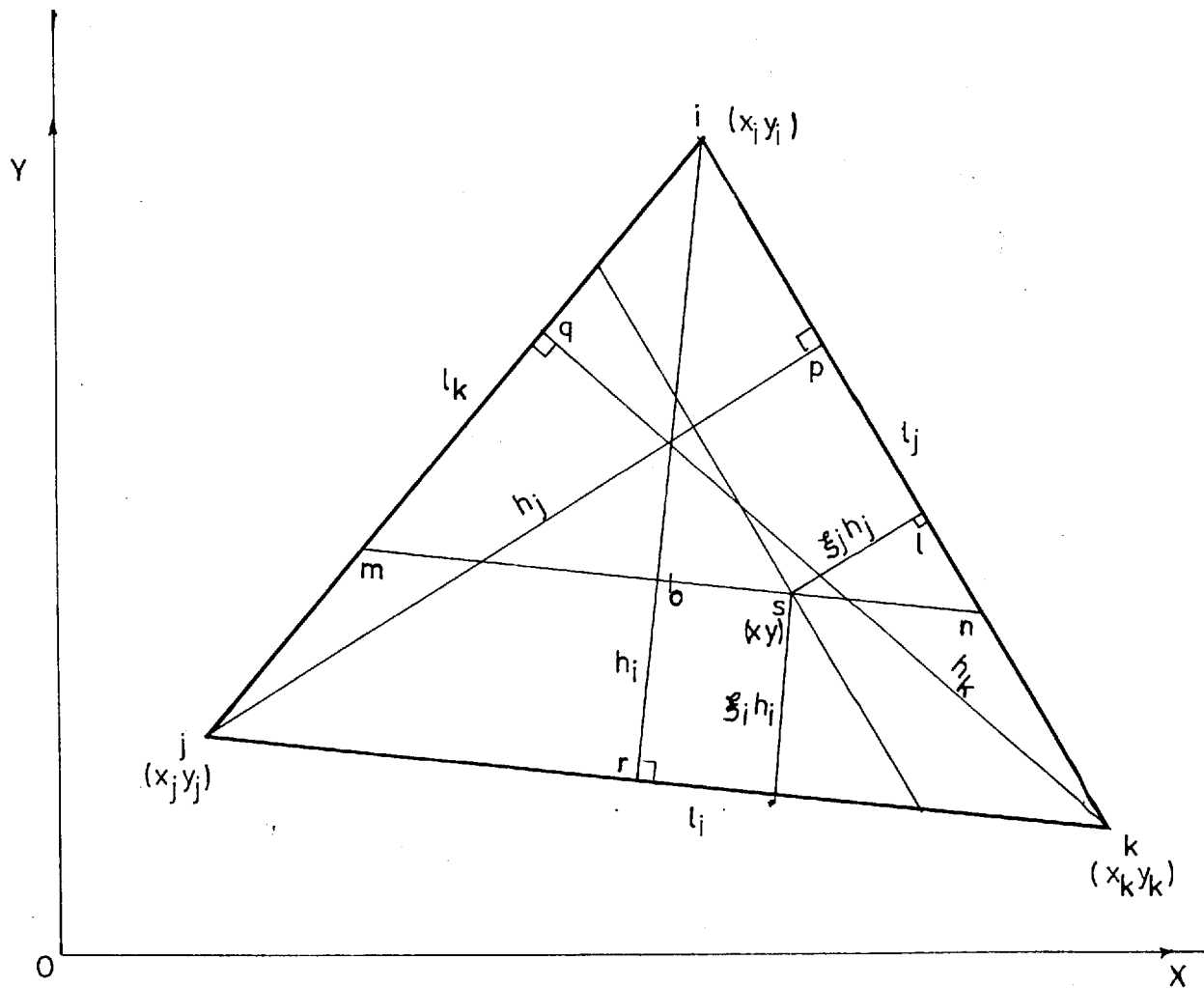


FIG 3-6 ELEMENT GEOMETRY.

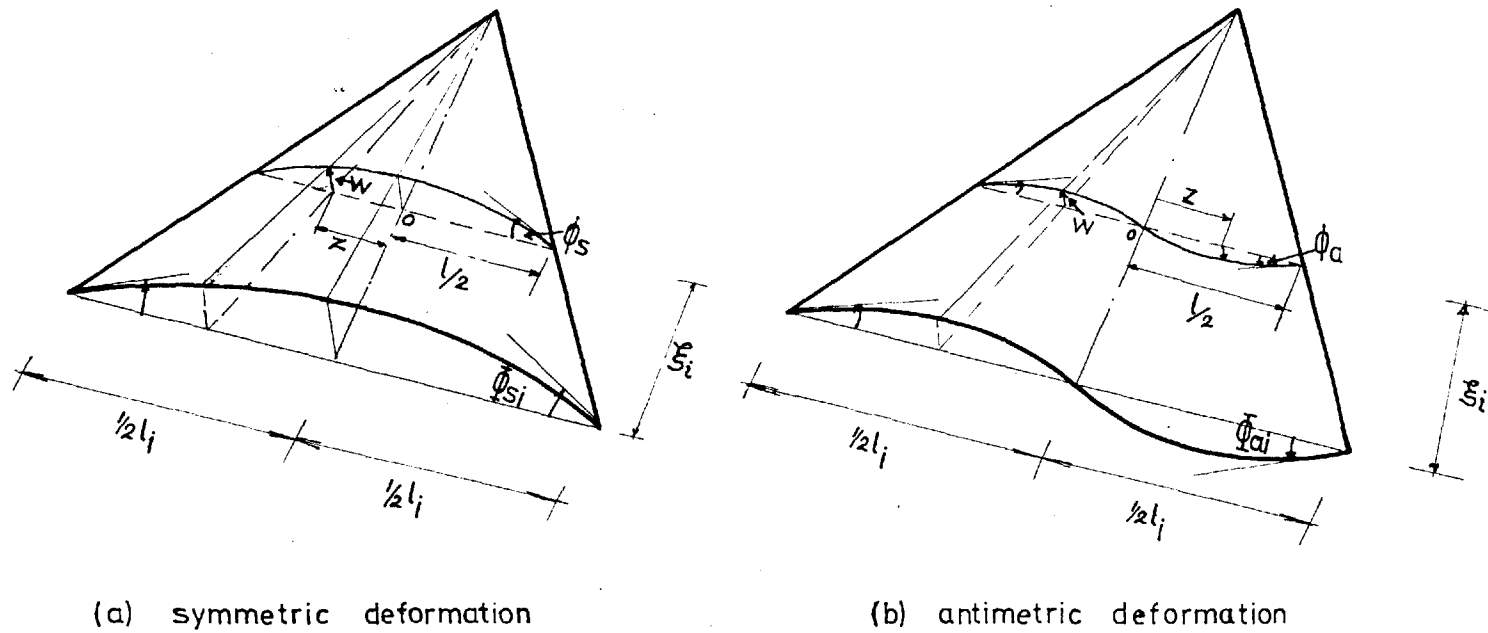


FIG 3.7 NATURAL GEOMETRY AND GENERALISED DISPLACEMENTS
FOR DERIVING A NEW FUNCTION FOR ELEMENT IN BENDING.

CHAPTER 4

COMPUTER PROGRAM

As shown in the section(3.2) on the derivation of the stiffness matrix for a shell element, both the inplane stiffness matrix and the bending stiffness matrix for the element are required. Quite a number of acceptable forms of the inplane stiffness matrix for the planar triangular element are available (26,29,32) and the one in reference (29) has been chosen for inclusion in the shell program. Of the bending stiffness matrices for the triangular element reviewed in chapter 3, two have been incorporated into two separate shell programs mainly for comparison with the one derived in this thesis. The details of these programs are now presented.

4.1 Local cartesian axes for shell element and the coordinate transformation matrix.

A subroutine has been written for the sequence of matrix operations derived in reference (26) to evaluate both the transformation matrix and the local axes for the shell element. In the programs, the side $i-j$ of the element, fig 3.5b, has been chosen as the local x -axis, while the node i has been chosen as the local origin of the element. Based on the global geometry of the nodes associated with the element, the program evaluates the direction cosines of the axes x , y and z of the local system. This is denoted here by c . If x_n denotes the coordinate vector $(x_n \ y_n \ z_n)$ the local axis coordinates of the nodes are computed from the formular

$$x_q = c[x_q - x_i]$$

where q takes the value of j and k respectively.

4.2 Assembly of the Stiffness Matrix for Shell Element

The inplane stiffness matrices and the bending stiffness matrices provide the relation between vectors which are defined as P_m p_m and P_b and p_b respectively. The subscripts m and b are used to denote membrane and bending respectively. These vectors have been defined in section 3.2. The relationships are as follows:

$$\begin{aligned} P_m &= \{ P_{mi} \ P_{mj} \ P_{mk} \} \\ p_m &= \{ p_{mi} \ p_{mj} \ p_{mk} \} \end{aligned} \quad 4.2$$

and

$$\begin{aligned} P_b &= \{ P_{bi} \ P_{bj} \ P_{bk} \} \\ p_b &= \{ p_{bi} \ p_{bj} \ p_{bk} \} \end{aligned} \quad 4.3$$

where $P_{mn} = \{ U_{mn} \ V_{mn} \}$ $p_{mn} = \{ u_{mn} \ v_{mn} \}$
and $P_{bn} = \{ M_{xn} \ M_{yn} \ W_n \}$ $p_{bn} = \{ e_n \ \phi_n \ w_n \}$

and n takes the value of i, j or k respectively.

Next we write two identification matrices M_m and M_b which are defined by the relationship:

$$P_m = M_m P_L \quad \text{and} \quad p_b = M_b P_L \quad 4.4$$

where p_L is of the same form as p in equation 3.30 but in the local system. The explicit form of M_m and M_b are as in table 4.1 and 4.2

The uncoupled stiffness matrix k in the local system is then obtained by the congruent transformation

$$\bar{k} = M_m^t k_m M_m + M_b k_b M_b \quad (18 \times 18) \quad 4.5$$

There is no arithmetical operation involved in equation (4.5).

The operation involves the sorting^{out} of the individual stiffness coefficients into the appropriate location in the 18×18 matrix. The operation could be performed as row and column transfers using the integer tabulation and ϕ table types of instructions, thus making a considerable saving in computer storage and time.

$$\begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix}_M = \begin{bmatrix} 1 & & & & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & & & & \\ & & & & & & 1 & & & & & & & & & & \\ & & & & & & & 1 & & & & & & & & & \\ & & & & & & & & & & & 1 & & & & & \\ & & & & & & & & & & & & 1 & & & & \end{bmatrix}$$

$$\begin{bmatrix} u_i \\ v_i \\ w_i \\ \phi_i \\ \phi_i \\ \psi_i \\ u_j \\ v_j \\ w_j \\ \phi_j \\ \phi_j \\ \psi_j \\ u_k \\ v_k \\ w_k \\ \phi_k \\ \phi_k \\ \psi_k \end{bmatrix}_S$$

TABLE 4.1 COEF OF M_m IN EQ. 4.4

ϕ_i			1													
ϕ_i				1												
w_i		1														
ϕ_j								1								
ϕ_j									1							
w_j							1									
ϕ_k													1			
ϕ_k														1		
w_k												1				

B

TABLE 4.2 COEF. OF M_b IN EQ. 4.4

u_i
v_i
w_i
ϕ_i
ϕ_i
ψ_i
u_j
v_j
w_j
ϕ_j
ϕ_j
ψ_j
u_k
v_k
w_k
ϕ_k
ϕ_k
ψ_k

S

Next we set up the super diagonal matrix C (18 x 18) thus

$$C = \begin{bmatrix} c & c & c & c & c & c \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \quad 18 \times 18 \quad 4.6$$

The global stiffness matrix is now obtained by the congruent transformation

$$k = C^t k_C C \quad 4.7$$

4.3 Assembly of the Stiffness Matrix for the entire Structure

As shown in section (2.51) the complete stiffness matrix is conveniently assembled by the use of the [a] matrix the derivation of which is now presented. The transpose of the [a] matrix which expresses the equilibrium equation between the external load and the nodal forces is used to obtain the [a] matrix.

For each element we redefine the element vector in the partitioned form

$$P_q = \{P_i \ P_j \ P_k\}_q \quad 4.8$$

where each subvector is of the order of 6 x 1 and

$$P_n = \{U_n \ V_n \ W_n \ M_{xn} \ M_{yn} \ M_{zn}\} \quad 4.9$$

where n refers to the nodal point (n) and takes the value i, j, k respectively. We now form a super vector P which is composed of the P vectors stacked one below another in their sequential form thus

$$P = [P_1 \ P_2 \ P_3 \ P_4 \ \dots\dots\dots P_N]^t \quad 6 \times 1 \quad 4.10$$

for a total number of N elements. Similarly we form a vector of external nodal loads on the structure R also in the sequence of nodal arrangement thus

$$R = [R_1 \ R_2 \ R_3 \ \dots\dots\dots R_Q]^t \quad 4.11$$

where each of the R_j 's is a 6 x 1 equivalent nodal loads composed of two inplane loads, three moments and one normal loads arranged in the same sequence as P_n thus

$$R_j = [U_j \ V_j \ W_j \ M_{xj} \ M_{yj} \ M_{zj}]^t \quad 4.12$$

$$\begin{bmatrix} \vdots \\ R_{i-1} \\ R_i \\ R_{i+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \begin{array}{c} \text{I} \\ \boxed{i \ j \ k} \end{array} & \begin{array}{c} \text{II} \\ \boxed{i \ j \ k} \end{array} & \begin{array}{c} \text{III} \\ \boxed{i \ j \ k} \end{array} & \begin{array}{c} \text{IV} \\ \boxed{i \ j \ k} \end{array} \\ \hline \boxed{0_6 \ I_6 \ 0_6} & \boxed{I_6 \ 0_6 \ 0_6} & \boxed{0_6 \ I_6 \ 0_6 \ 0_6 \ 0_6 \ I_6} & \\ \vdots & & & \end{bmatrix} \begin{bmatrix} \vdots \\ P_I \\ \cdot \\ P_{II} \\ \cdot \\ P_{III} \\ P_{IV} \\ \cdot \\ \vdots \end{bmatrix} \quad \text{--- 4.13}$$

$I_6 = 6 \times 6 \text{ unit matrix}$
 $0_6 = \text{null}$

16

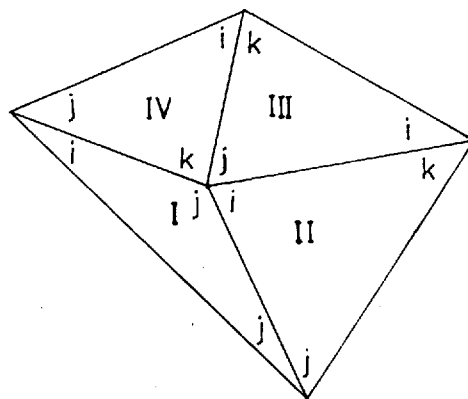


FIG 5.1

Then by considering the equilibrium of each node in turn for a total of Q nodes we find that the external load is equal to the sum of the element nodal forces meeting at the particular node. This can be represented by a matrix A which is in fact the required (a^t) matrix. We demonstrate this with respect to the simple structure shown in fig. (5.1). The equilibrium equation at node i is to be found.

The positioning of each of the unit matrix in equation 4.13 is seen to corresponding to the columns corresponding to the element nodal forces and in particular the corner identification symbols $i, j, \text{ or } k$. Within the subdivision represented by the vertical lines the unit matrix occurs only once. By similar reasoning the other rows of the a^t matrix are obtained. Since the rows of $[a]$ are sparsely populated we partition the a matrix as shown previously and in equation 4.13 by the two horizontal lines and merely record the positioning of the unit matrices by the integer ' and 0 table instructions. The rows of the entire matrix are now assembled by the algorithm

$$R_i = a_i^t k_i a_i \quad 4.14$$

The (a_i) sub matrix can be set up automatically in the computer as described. For a structure consisting of several elements which could run into hundreds, the amount of data input for the integer tabulation becomes enormous. Therefore the whole process has been turned here to the computer itself to compile. This is computed from the element nodal array. For each element there are three numbers corresponding to its vertices i, j, k . These numbers are presented as the input data in a sequential manner beginning with element 1 to element N . Also the number array is presented in the sequence i, j, k according to the tabulation below

<u>Nodal identification for element</u>				
<u>Node</u>	<u>No.</u>	<u>i = 1</u>	<u>j = 2</u>	<u>k = 3</u>
	1	.	.	.
	2	.	.	.
	.	.	.	.
	.	.	.	.
	N	.	.	.

The entire set of $3 \times N$ numbers is read in one go. For each node, starting with number 1, the set of $3N$ figures in the above table is scanned three at a time. If within any scan, a number occurs which is equal to the nodal number under the control, the element number in which it occurs is noted as well as whether its position is 1, 2 or 3 corresponding to i , j or k . Also of importance is a counter which registers the total number of elements meeting at the nodal point in question. This is useful in compiling equation 4.14, as well as finding the average stresses at the nodal point as will be explained later on.

Assembly of the Equations:

The program uses the data generated in the previous section according to the following algorithm. As a first step, the program assumes that the stiffness matrix for a particular element is partitioned by rows thus:

$$\begin{bmatrix} \overline{F_i} \\ \overline{F_j} \\ \overline{F_k} \end{bmatrix} = \begin{bmatrix} \overline{k_{ii}} & \overline{k_{ij}} & \overline{k_{ik}} \\ \overline{k_{ji}} & \overline{k_{jj}} & \overline{k_{jk}} \\ \overline{k_{ki}} & \overline{k_{kj}} & \overline{k_{kk}} \end{bmatrix} \begin{bmatrix} v_i \\ v_j \\ v_k \end{bmatrix} \quad \begin{array}{l} i^{\text{th}} \text{ row} \\ j^{\text{th}} \text{ row} \\ k^{\text{th}} \text{ row} \end{array} \quad 4.15$$

each submatrix k_{mn} being a 6×6 matrix. The row partitioning of the stiffness matrix assumes that the force vector is similarly partitioned but in the meanwhile the displacement vector is not. Then we set up a working space which has the dimension of $6 \times 6M$,

where M is the total number of element in the structure. Using the data generated in the previous section, we read the element number, bring the element stiffness to a working space where it is partitioned in accordance with [4.15]. Depending on whether the nodal identification number is 1, 2 or 3 we transfer the i^{th} , j^{th} of k^{th} row of the partitioned submatrix to the corresponding location in the $6 \times 6M$ working location. This is done for all elements directly attached to the node. The procedure described thus far correspond to forming the equation

$$R_i = \sum a^t E \quad \text{spaces} \quad 4.16$$

For the second stage, we now set up another $6 \times 6L$, where L is the maximum number of nodes. The matrix $6 \times 6M$ generated above is now transposed and partioned into 6×6 units. The process of addition of the stiffness matrix now takes place. Beginning with the first element, we add the component submatrices to location specified in the nodal connection data format.

4.4 Solution of the System of Equation

Since there are six unknown components of displacement per node to be determined the number of simultaneous equation to be solved grows enourmously as the number of nodes increases. This poses some problem as far as storage is concerned. However advantage can be taken of the sparse population of the matrix. Two scheems were tried.

4.41 Failure of the Gauss Sidel Iteration Process

The Gauss Sidel iterative scheem is based on the fact that we can write the equilibrium equation in the form

$$r_i = k_{ii}^{-1} \left[R_i - \sum_{l,i} k_{lj} r_j - \sum_{i+1,N} k_{ij} r_j \right] \quad 4.17$$

$$r_i = r_i^{(n+1)} - r_i^{(n)} \quad 4.18$$

$$\text{and } r_i^{(n+1)} = k_{ii}^{-1} \left[R_i - \sum k_{ij} r_j^{(n+1)} - \sum k_{ij} r_j^{(n)} \right] \quad 4.19$$

where n is the cycle of iteration. In theory the algorithm ought to work, but the result of the computation showed a divergence after only a few iterations, and the method was therefore abandoned.

4.42 Matrix Diagonalisation

The structure is divided by rows of nodes as shown in fig 5.2. All members having nodes wholly in row i or in both row i and $i + 1$ are said to be associated with row i . We now define the submatrix r_i such that each submatrix consists of node displacements for all nodes in the row. The submatrix R_i is similarly defined as the nodal loads in the row. We then see that for row-wise arrangement of nodes for a structure having q rows, K is of the form

$$K = \begin{bmatrix} K_{11} & K_{12} & & & \\ K_{21} & K_{22} & K_{23} & & \\ & K_{32} & K_{33} & K_{34} & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & K_{qq-1} & K_{qq} \end{bmatrix} \quad 4.20$$

By symmetry considerations

$$k_{ii} = K_{ii}^t \quad 4.21$$

$$K_{i,i-1} = K_{i-1,i}^t$$

A set of equations for which the matrix of coefficient is of the foregoing tridiagonal form may be solved relatively easily as shown by Wilson ⁽¹⁹⁵⁹⁾. We first write out equation (4.20) in the form of a set of simultaneous equations in the r_i 's thus:

$$\begin{aligned} K_{11}r_1 + K_{12}r_2 &= R_1 \\ K_{21}r_1 + K_{22}r_2 + K_{23}r_3 &= R_2 \end{aligned} \quad 4.22$$

$$K_{q-1,q-1}r_{q-1} + K_{q,q}r_q = R_q$$

The first equation is solved for r_1 . This result is then used to eliminate r_1 from the second equation which may then be used to solve for r_2 . This may then be used to eliminate r_2 from the third and so on. The sequence of operation is represented thus:

$$\begin{aligned} \text{Eqn. 1.} \quad r_1 &= K_{11}^{-1}[R_1 - K_{12}r_2] \\ &= R_1^{-1} - K_{11}^{-1}K_{12}r_2 = R_1^{-1} - Q_{12}r_2 \end{aligned} \quad 4.23$$

substitution of r_1 in equation 2 gives

$$K_{21}[R_1^{-1} - Q_{12}r_2] + K_{22}r_2 + K_{23}r_3 = R_2 \quad 4.24a$$

multiply, and rearrange we obtain

$$[K_{22} - K_{21}Q_{12}]r_2 = R_2 - K_{23}r_3 \quad 4.24b$$

$$\text{i.e.} \quad K_{22}^{-1}r_2 = R_2 - K_{23}r_3 \quad 4.24c$$

which is of the same form as equation 1 and solving for r gives

$$\begin{aligned} r_2 &= [K_{22}^{-1}]^{-1}[R_2 - K_{23}r_3] \\ &= R_2^{-1} - Q_{23}r_3 \end{aligned} \quad 4.24$$

and generally

$$r_j = R_j^{-1} - Q_{j,j+1}r_{j+1} \quad 4.25$$

If $j = q$, $r_{q+1} = 0$ and so we can make the backward substitution to obtain the values of the r 's. It will be observed that at any stage only Q_{ij} submatrices need be stored, thus leading to a greater saving in space compared to the method described by Paulling (1964). Machine size limits only the size of the submatrices Q_{ij} but not the total number of nodes.

4.5 Evaluation of Stresses in the Structure.

When the displacement of the structure have been obtained, the stresses in the elements are obtained by a backward substitution process. The strains in the elements are obtained by means of equation 3.16, and using the stress-strain relation given by equation 3.14 the stresses are determined. In general, stress jumps across the boundaries of the elements are to be expected. But by considering the equilibrium of a node under the actions of the forces contributed by each element and the external force system, it can be established that the stresses at a particular node in the structure can be obtained by taking the mean of the stresses at the corners of the elements meeting at the particular node.

97(a)

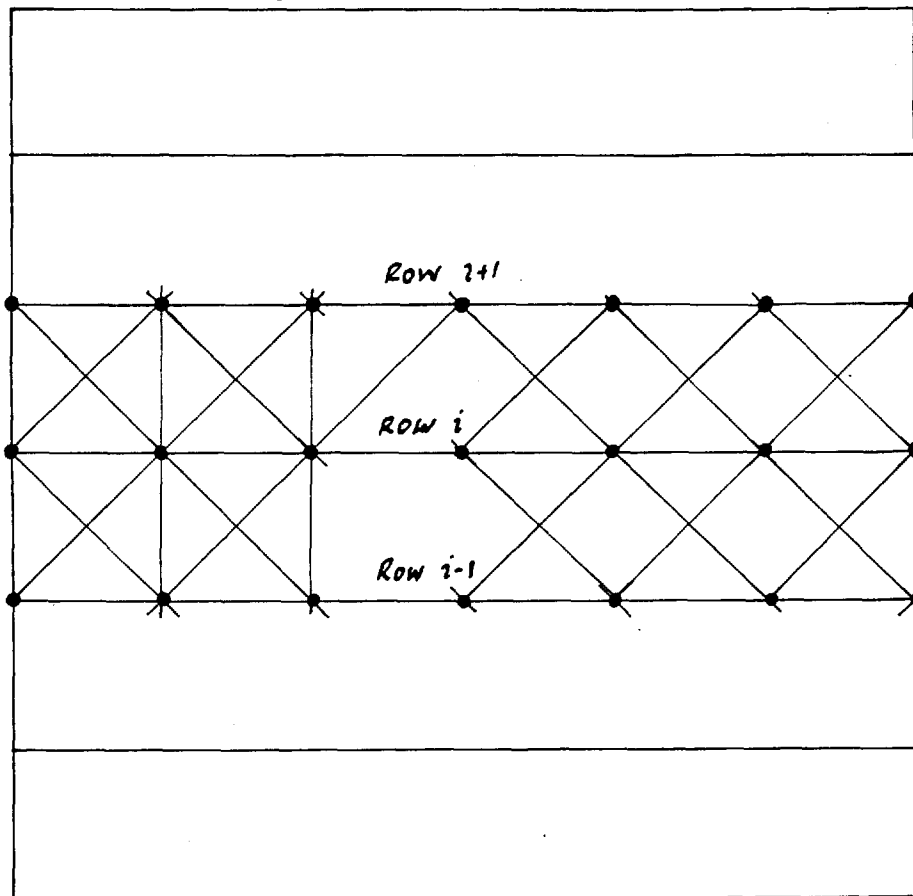


FIG 5.2 SUBDIVISION OF STRUCTURE INTO ROWS OF NODES
AND MEMBERS

CHAPTER 5

5. Tests on Timber and Vybak Models

5.1 Object of the Experiments

Experiments were conducted on hyperbolic paraboloid shells whose boundaries coincided with the straight line generators. Only the low corners of the timber shell were supported and a tie rod also was provided at these corners. All corners of the vybak model were fixed, with respect to vertical displacements and corner rotations.

The objects of the experiments were as follows:

1. To investigate the behavior of the shell under a uniformly distributed loading and flexible edge boundary conditions. The experimental results were to provide a basis for comparison for the theoretical solution based on the finite element analysis developed in chapter 3.
2. To investigate the suitability of a laboratory table sized model and the accuracy which may be achieved there from.
3. To study the suitability of balsa as a structural modeling material for timber hyperbolic paraboloid shell in particular and timber shells and platelike structures in general.

The objective described under '1' was primary, while those under '2' and '3' were secondary as far as the investigation carried out in this thesis is concerned.

5.2 Factor Governing the Choice and Scale and Material of the Model

Baron (1962) reported an exploratory test on a timber conoid shell roof, carried out at Southampton University. The object was to compare the behavior of the model with that of another model which was a half scale of the prototype. The material used for the laboratory model was balsa wood. The choice of balsa was influenced by the fact that it grows uniformly than any other species, thus ensuring some degree of control on the density and hence the strength properties. The apparent success of the model in predicting the behavior of the prototype in a rather qualitative manner prompted the choice of the material for the model of hypar shell planned in the present investigation.

A considerable number of experimental investigations have been carried out on models of timber shells as an aid to design. Some of these tests have been carried out on an eighth, quarter and half scale models of the prototype. The rise in labour and material cost has however diverted attention to the use of laboratory scaled models. One of such tests by Baron has been referred to above. Keresztesy (1964, 1965) has also reported a series of tests carried out on a model of timber hyperbolic paraboloid shell. This model was 5ft. square in plan and was planned to be a tenth scale model of a typical shell taken as 50ft. square in plan. It was felt that the use of very small scale sized models should be further investigated to determine their feasibility, and possible extension to more complex timber engineering structures. Thus taking the typical 50ft. square shell, a $1/25$ model was planned. This scale was applied to the membrane and thus led to the choice of $1/32$ in. thick veneers. The prototype would normally be nailed and or glued, but in the model, we are restricted to gluing the veneers together.

5.3 Description of the Structure

A former was constructed using plywood and the forming surface was shaped to the same curvature as the shell to be built. The former was 24 in. by 24 in. in plan and the rise of the high corners over the low corners was 4.8 in. Fig. 5.1 shows an isometric view of the former. To prevent the glue from percolating to the top of the former and so holding the finished shell down during the stripping off the former, strips of cellophane papers were laid on the former and held down by means of adhesive tape.

The veneers of balsa as delivered by Messrs Solarbo Ltd., were 24 in. long, 6 in. wide and $1/32$ in. thick. These had to be cut into strips $1/4$ in wide to conform to the model scale chosen. The cutting was done by means of a carving knife and a straight edge. This proved a particularly difficult operation since the knife tended to follow the grain of the wood. The grain was slightly inclined to the straight edge which was aligned to coincide with the edge of the sheet before starting to cut. The first layer of veneers was placed on the former in the direction of one of the straight line generators and held down by means of adhesive tape, while the next layer was spread with glue and laid on the next layer. The veneers were then pressed together and held down until the glue set. The second layer was orientated in the compression diagonal direction. The third layer was pointed in the direction of the other generator. A general view of the shell and veneer orientation is shown in fig. 5.2.

Two types of glue were tried for the membrane. The first was the traditional balsa cement normally used for gluing pieces of balsa together. The adhesion was found to be poor and large

quantities of glue spread had to be used especially as the cement was very poor at spreading. The second glue tried was the Tensol 7, which was found to be more suitable, and although rather slowly drying had a good spreading quality.

The edge beams were made from the same veneers as the membrane of the shell. Each beam was $1/4$ in. wide and $3/4$ in. deep. Eight lengths of 24 in. each were prepared. After the shell dried, the first set of four beams was placed around the top surface, and after drying, the shell was trimmed and taken off the former. The bottom set of edge beams was then fixed.

5.4 Corner Detail

The corner detail presented some problems. Since the material was very fragile and would easily be crushed if a bolted connection were used, reaction plates shown in fig. 5.3 were therefore glued on. The reaction plates 1 in. in length were obtained from $1/4$ in. mild steel angle. The glu-ing of the reaction plate has two disadvantages. First once glued on, it was very difficult to remove without damaging the shell should there be any need to rectify a mistake. Secondly it was difficult to align the plate prior to glue setting. The vertical reactions were designed to be transmitted to the frame by means of $1/8$ in diameter steel rods screw at one end to the corner plate and supported on the other by a ball bearing. The arrangement is shown in fig. 5.3a. This arrangement it was hoped would rectify any imperfections in the fixing of the reaction plate. The low corners were restrained by means of a tie rod made from $1/16$ in. diameter aluminium rod.

5.5 Loading and Instrumentation Arrangements

An air bag was used for applying the load on to the shell. The choice of the air bag was influenced by the need to apply a uniformly distributed load and to obtain an arrangement which permitted a continuous incremental loading to failure, with minimum of intervention with the test setup. The loading face of the bag, which was supplied by P. B. Cow Ltd., was shaped to the same surface as the shell and it was enclosed within a box fitted to the frame within which the shell was supported. The enclosing box is of the same shape and size as the former shown in fig. 5.1 except that it was completely hollow. The air bag had the internal dimension of the enclosing box. The air supply was obtained from an 80 psi supply using a reducing valve. The pressure of the air in the bag was measured by means of a water manometer connected to the supply line going to the bag.

Attention was confined to the measurement of displacements on the shell membrane and the edge beam. Dial gauges measuring up to 1/10,000 in. were placed according to the distribution shown in fig. 5.4a. 10mm Japanese strain gauges were placed on one edge beam both at the top and bottom according to the distribution shown in fig. 5.4b. No strain gauges were placed on the membrane because of the uncertainty envisaged in interpreting the stresses.

5.6 Testing of Specimen

Timber shells as constructed in practice are built up from veneer pieces which are either nailed or glued and in the majority of cases nailed and glued. It will thus be desirable that, in order to obtain a perfect model of the prototype, the means of connection should be reduced to the same scale as that of the model. Clearly

there is a physical limit to which this can be done. In the present investigation, the size of the model precluded the use of mechanical fastenings and hence the veneers had to be glued. The absence of any mechanical glue spreader and presser made it inevitable to avoid the liberal use of glue. It was therefore necessary to evaluate the influence of the glue stiffening on the shell membrane. A convenient model seemed to be the simple tension test.

5.6.1 Preparation of the Tension Specimen

The balsa as delivered by the manufacturers was already in prepared veneer form. Since there was no means of identifying which veneer came from the same part of the log from which the veneers were prepared, the matching of the specimens was not possible. The selection of the plies for the tension tests was therefore made on a random basis. The only degree of matching achieved was to ensure that a test specimen was built up from only one sheet of material. A jig and template was specially made. The template was used to cut the profile of the specimen fig. 5.5, thus ensuring similarly shaped specimens. The jig was used to align the holes in the bearing plates during the gluing process. Strains were measured by two displacement transducers fixed longitudinally at midspan one on each face. The gauge length used was 3 in.

5.7 Presentation of Results for the Timber Model

The results of the experiment on the timber model are presented in the following sequence:

- a. The readings obtained from the deflection measurement for the four loading cases are presented in fig. 5.14. The readings are recorded directly on the point where the deflections were

measured. Those points have been indicated by triangles. The numbers at the base of the triangle indicate the dial gauge numbering corresponding to that shown in fig. 5.4.

b. The strain measurements are presented in fig. 5.22 and fig. 5.23 as the differential of strains representing, a half of the difference of the top and bottom strains, and the mean of these strains respectively. From these graphs the bending stresses and the axial stress may be found.

c. The Young's moduli obtained from the tension test specimens are presented in a graphical form in fig. 5.20. A typical stress-strain relation is shown in fig. 5.18.

d. The plots of the displacements along the tension diagonal, the edge beam are presented in fig. 5.24 and fig. 5.25 respectively. The plot of the displacements at the centre of the shell corresponding to the various loading stages is presented in fig. 5.19.

5.8 Evaluation of the Timber Model set up

There were some design faults which affected the behavior of the shell in a manner which was not easy to evaluate. These faults are now set out below.

a. Bag Loading

The original intention in the design of the test set up was to load the model from the top. It was found however that the bag could not be properly fixed into its box container. The bag tended to slump on to the model causing the shell to deflect in an unsymmetry manner. An unsatisfactory remedy was made by turning the test rig

upside down so that the model rested on the bag. This in turn produced a secondary effect. The bag slumped into its container and the reaction supports had to be given some initial pre-tension to bring the model surface into contact with the bag surface. This had the effect of giving the shell an initial loading which is not directly due to applied loading. This can be observed in fig. 5.19, where it would be observed that for zero deflection, the plot of the displacement does not pass through the origin. It would also appear that since the hyperbolic paraboloid is not a developable surface, the fabrication of the bag can at best be approximate to the true shape. Under full loading however one might expect the bag to stretch to accommodate the surface of the shell. The provision of an edge beam also makes the use of a bag loading unsuitable especially in a shell of this dimension because of the constructional difficulties involved.

b. Support Condition

The support of the shell also proved unsatisfactory. The four corners of the shell were originally supported. But during the loading stage, the high corners were rendered ineffective by the corners moving in the direction rather than against the reaction supports. The lateral stability thus depended entirely on the support provided by rubbing against the air bag. It was therefore left to the low corners to carry the loads. The low corners deflected excessively as can be seen in fig. 5.21. The scale of the model did not permit the measurement of the diagonal extension of the tie rod.

5. 8.1 Evaluation of the Test Results

The Youngs moduli curves presented in fig. 5.20, show that superimposed on the variability of material, there is a definite evidence that the glue has a considerable effect on the tensile specimen.

Fig. 5.21 shows the plot of the displacement along the edge beam. The actual displacements have been plotted to show in particular the deflection of the low corner. If however the deflection values as shown in fig. 5.14 are compared for the low corners and the high corners, there is definite lack of symmetry and the reduction of the displacement of the edge beam shown in fig. 5.21 cannot be reduced realistically to exclude the effect of the reaction displacement. However this reduction has been done in fig. 5.25 based on the dial gauge 1 only. The displacement trend can be observed from this figure. The differential of the strains and the mean of the strains on the edge beam are also shown in fig. 5.22 and fig. 5.23. However to ensure that the shell still behaved linearly, the displacement at the centre of the shell has been plotted for different loading conditions and shown in fig. 5.19. From this graph it can be observed that the shell still behaved linearly.

It was felt that there were too many parameters involved in the test as carried out to enable a proper correlation to be made and hence a decision^{was} made to carry out an experiment on a Vybak model but drawing on the experiences gained on the timber model.

5.9 Details of Construction of the Vybak Model

The material chosen was a type of plastic under the trade name of VYBAK. It was chosen because it has a low softening temperature of 90°C and could be easily formed to any shape. In contrast to perspex, it does not creep very much at room temperature. Approximately 2 ft. square of the material, which was $1/4$ in. thick, was placed in the steam bath and softened. The softened vybak was then placed over the former fig. 5.1, weighted down and allowed to cool. When cooled, the model took the shape of the former. A certain amount of local irregularity was observed on one of the edges, but this was soon cured by reheating and reforming as described above. One quarter inch wide pieces of the same material for the edge beams had meanwhile been cut and glued by means of Tensol cement which had been recommended by the manufacturer. The joint was found to be structurally unsatisfactory, and other types of glue were tried with little success. A bolted connection was therefore chosen both for the edge beam and the corners. The edge beams consisted of two $1/4$ in. by $1/4$ in. laminations symmetrically placed at the top and bottom surface of the edges of the shell. The edge beams were attached to the shell by $1/8$ in. diameter studs at 2 in. intervals along the edges. Fig. 5.6 shows the edge beams in position and also the details of connection.

5.10 Corner Details

The corners were secured to the test frame by means of a rigid bolted connection shown in fig. 5.6. Steel plate pieces, pointed in the diagonal directions, were rigidly fixed to the four corners of the test frame. The high corner plates were on the same level, but on different level corresponding to the shell rise, from the

lower corner plates. These plates were provided with $1/2$ in. wide horizontal slots to allow for free diagonal horizontal movement of the shell if this is desired. A Y-piece plate was provided at each corner and formed in such a way that the V-prongs of the piece follow the profile of the edge beams and was attached to the shell edge beams by means of eight $1/4$ in. diameter bolts. The tail piece was provided with two vertical slots, to ensure that before bolting to the piece on the frame, the shell was not constrained in the vertical direction. All the corners of the shell were fixed. A general view of the shell mounted on the test frame is shown in fig. 5.7.

5.1.1 Loading and Instrumentation Arrangement

The model was loaded by means of lead shot which were measured into cans suspended from the model. $1/16$ in diameter holes were drilled through the membrane on grid points at 2 in. interval as shown in fig. 5.13. There were a total of 121 such holes. The cans were suspended by means of threads which were looped, passed through the holes and tied into a knot at the top of the shell. Thin pieces of wire rods fig. 5. 7 prevented the knots coming through the holes when the shell was loaded. The loop in the thread engaged the hook on the spring attached to the can. When the cans were not loading the shell, they were supported on a wooden platform fixed to a screw jack. The platform could be raised or lowered by means of the screw jack. The loading arrangement is shown in fig. 5.8.

Strain gauges were fixed on the model both at the top and bottom in accordance with the distribution shown in fig. 5.10. Two types of strain gauges were used. They were the 5mm rosettes and crosses. The crosses were used on the diagonals. The numbering of the strain gauges for the purposes of interpreting readings

obtained from the Solartron are shown in fig. 5.11 and fig. 5.12. For the deflection measurement, dial gauges reading up to 1/10,000 in. were used. The distribution of the dial gauges are as shown in fig. 5.13.

5.1.2 Deflection and Strain Measurements

The model size did not permit an extensive measurement of the shell at a single loading of the shell. This was due to the difficulty of arranging the dial gauges side by side on the 2 in. grid interval. Hence the deflection measurements were carried out a little at a time. Vertical displacements were measured along the sections marked A-A, B-B, C-C, J-J and K-K as shown in fig. 5.14 for a load of 2316/ sq. ft. For the same loading, the readings of all the strain gauges were recorded. The deflections of the centre point of the shell were also measured for two other loadings of 14.4 and 32.4 lb/sq. ft.

5.1.3 Presentation of the Results of tests on Vybak Model

The presentation of the displacement readings is similar to that used for the Timber Model and is shown in fig. 5.15. Here however the numbering of the dial gauges is related to what has been described as the possible position of the gauge. The positions which are twelve in all are located as the mid point of the loading points. The directions at which the displacements have been recorded are indicated. The displacements for these indicated directions are plotted as shown in fig. 5. 26 - 5.30 inclusive.

The presentation of the strain recordings is similar to that of the displacements. Fig. 5.16 and 5.17 show the results of the

reduced strains for all the gauges at the top and bottom respectively. Fig. 5.32, 5.34 inclusive show the plots of the strains at the indicated positions shown in fig. 5.16 and fig. 5.17.

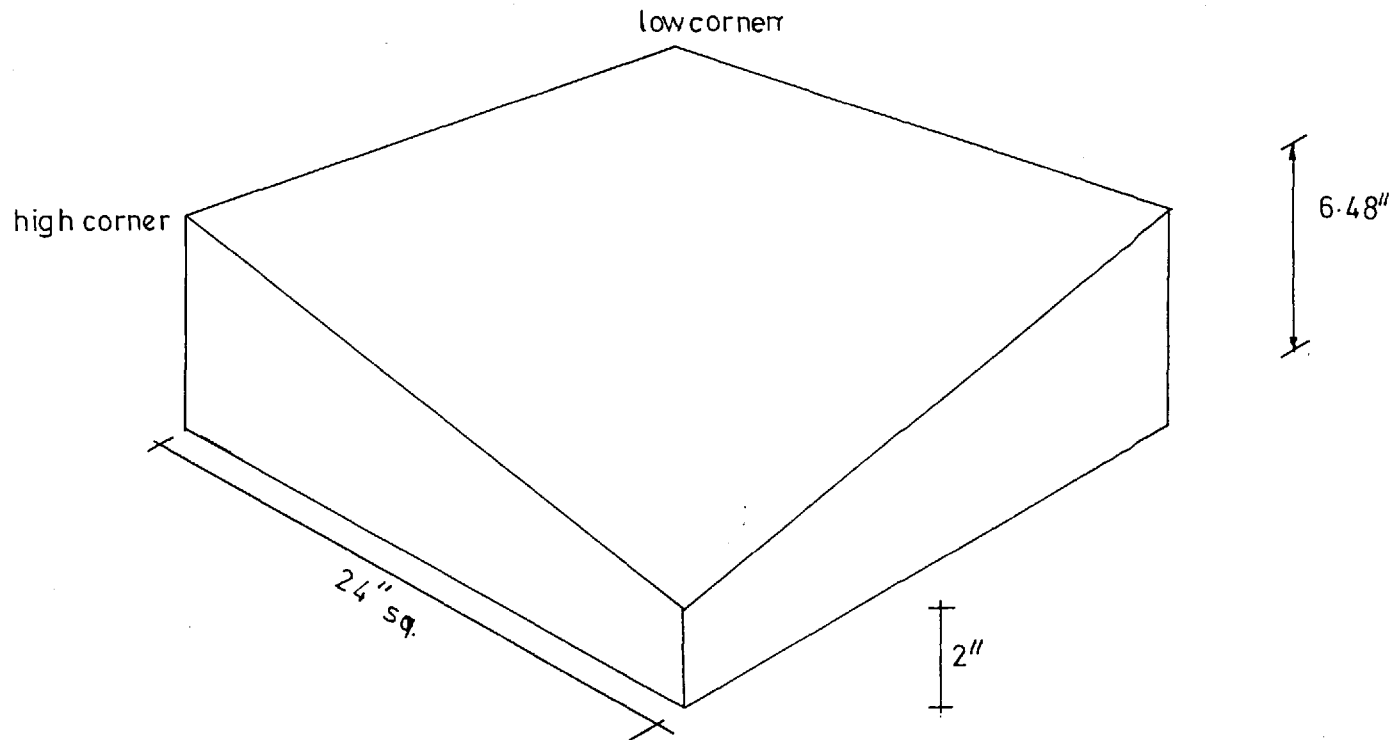


FIG 5.1 ISOMETRIC VIEW OF THE FORMER

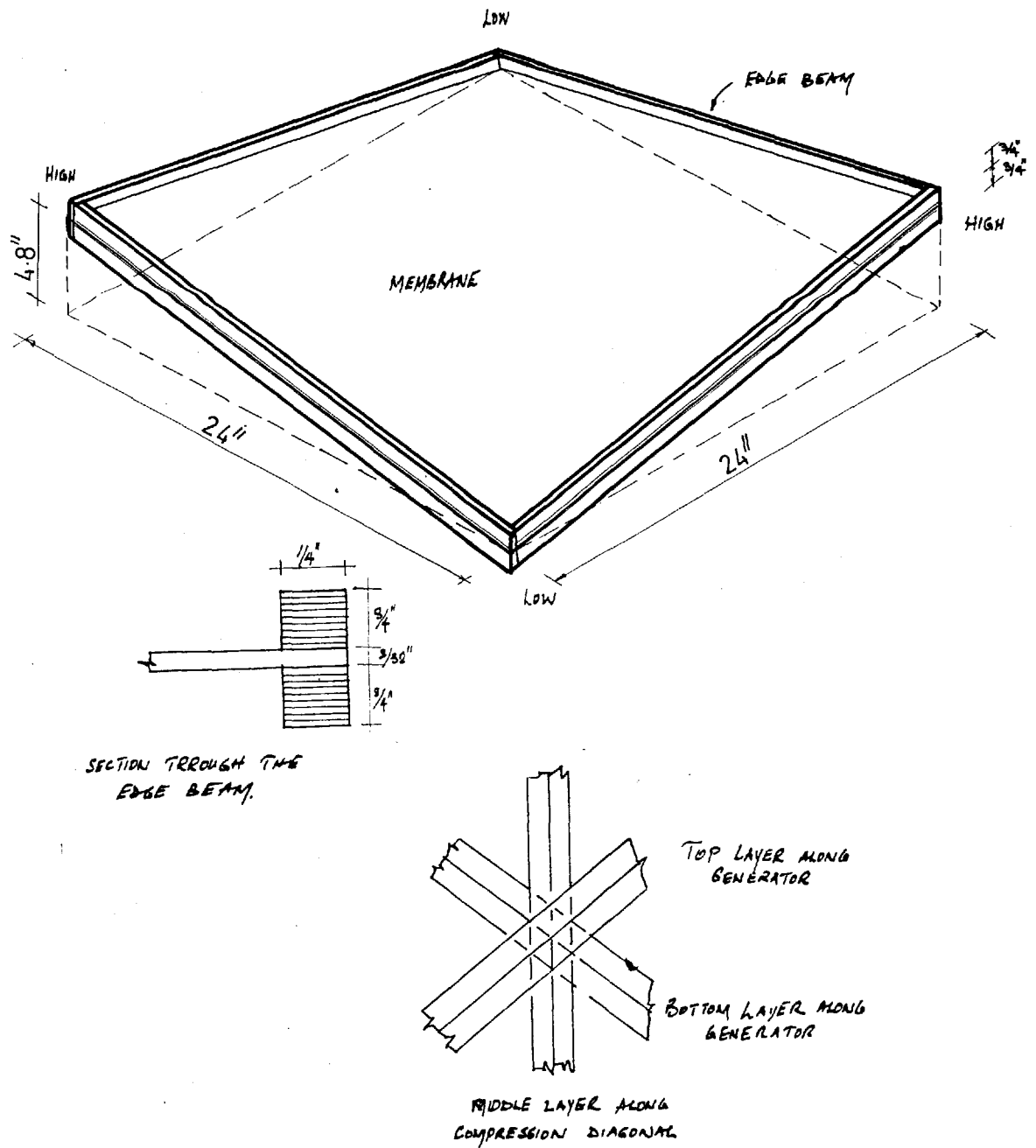
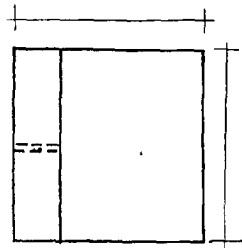
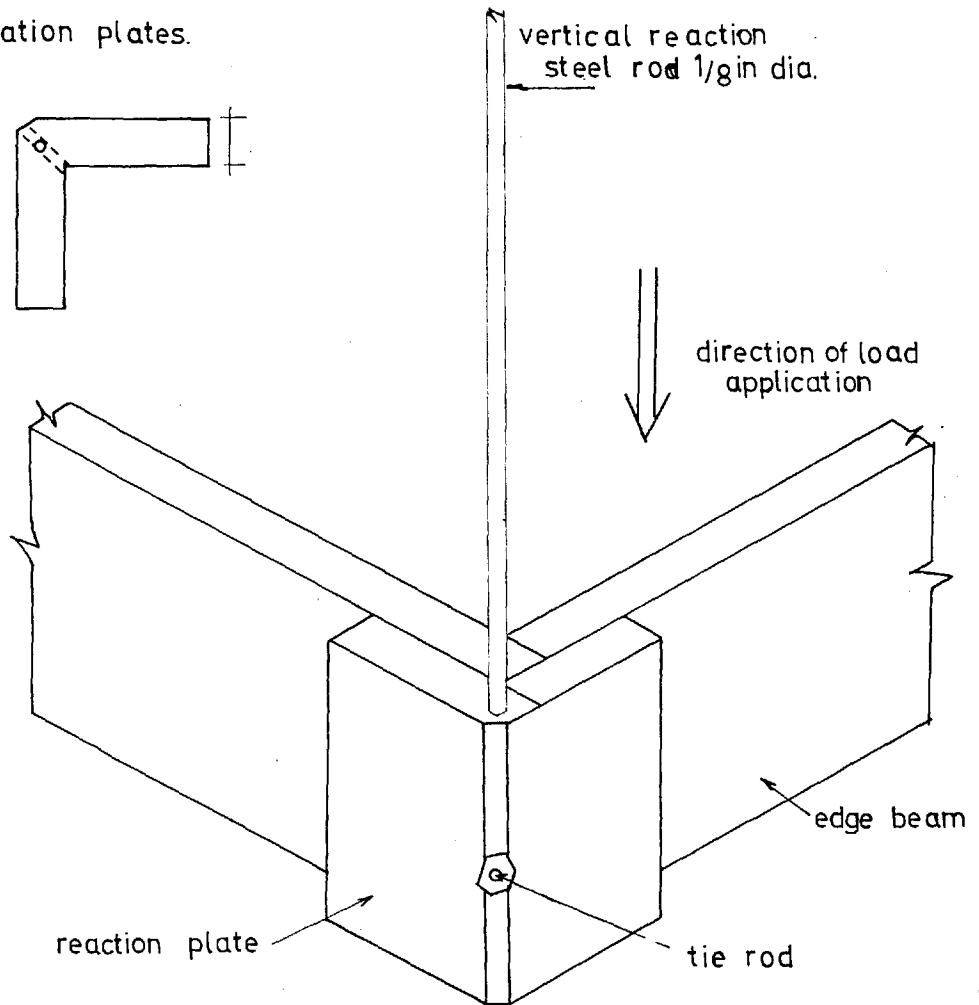
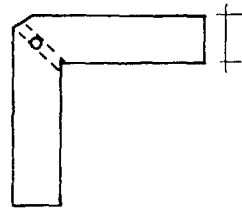


FIG. 5.2 ISOMETRIC VIEW OF SHELL AND ORIENTATION OF VENEERS

TIMBER MODEL



(a) reaction plates.



(b) corner detail

FIG 5.3 CORNER-DETAIL TIMBER MODEL

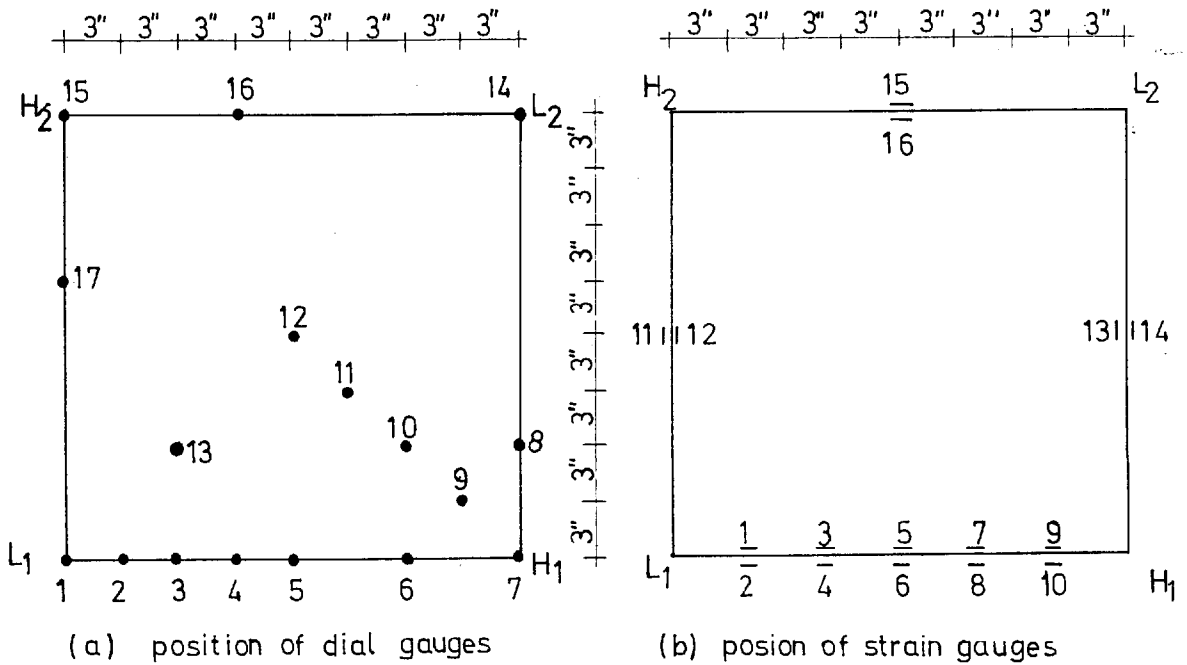


FIG 5-4 LOCATION OF DISPLACEMENT AND STRAIN GAUGES.
TIMBER MODEL

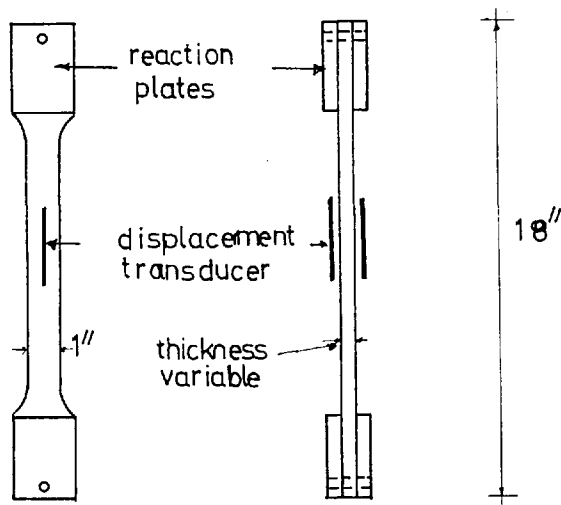


FIG 5-5 TENSION TEST SPECIMEN
TIMBER AND VYBAK MODELS.

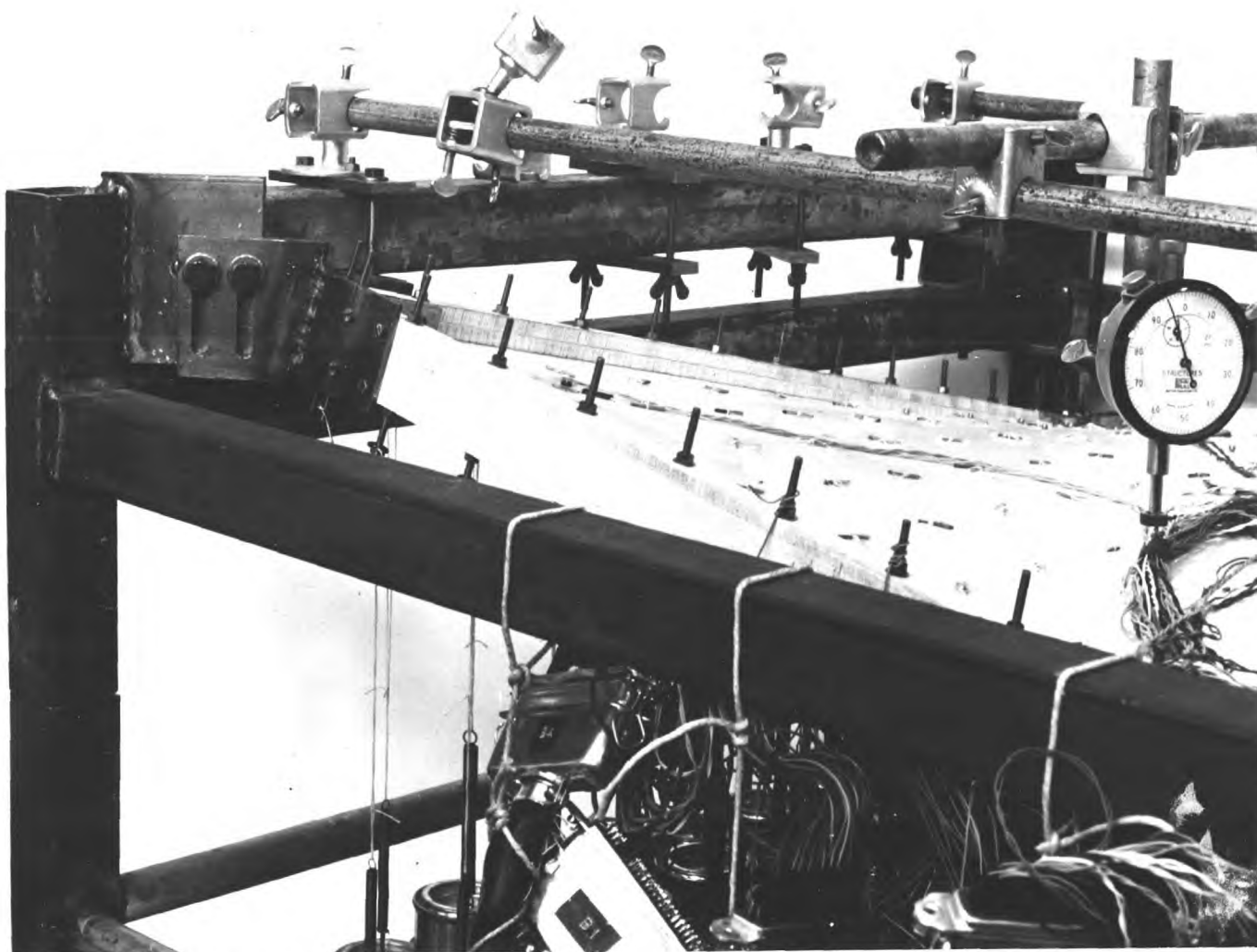


FIG 5.6 CORNER AND EDGE BEAM DETAILS VYBAK MODEL

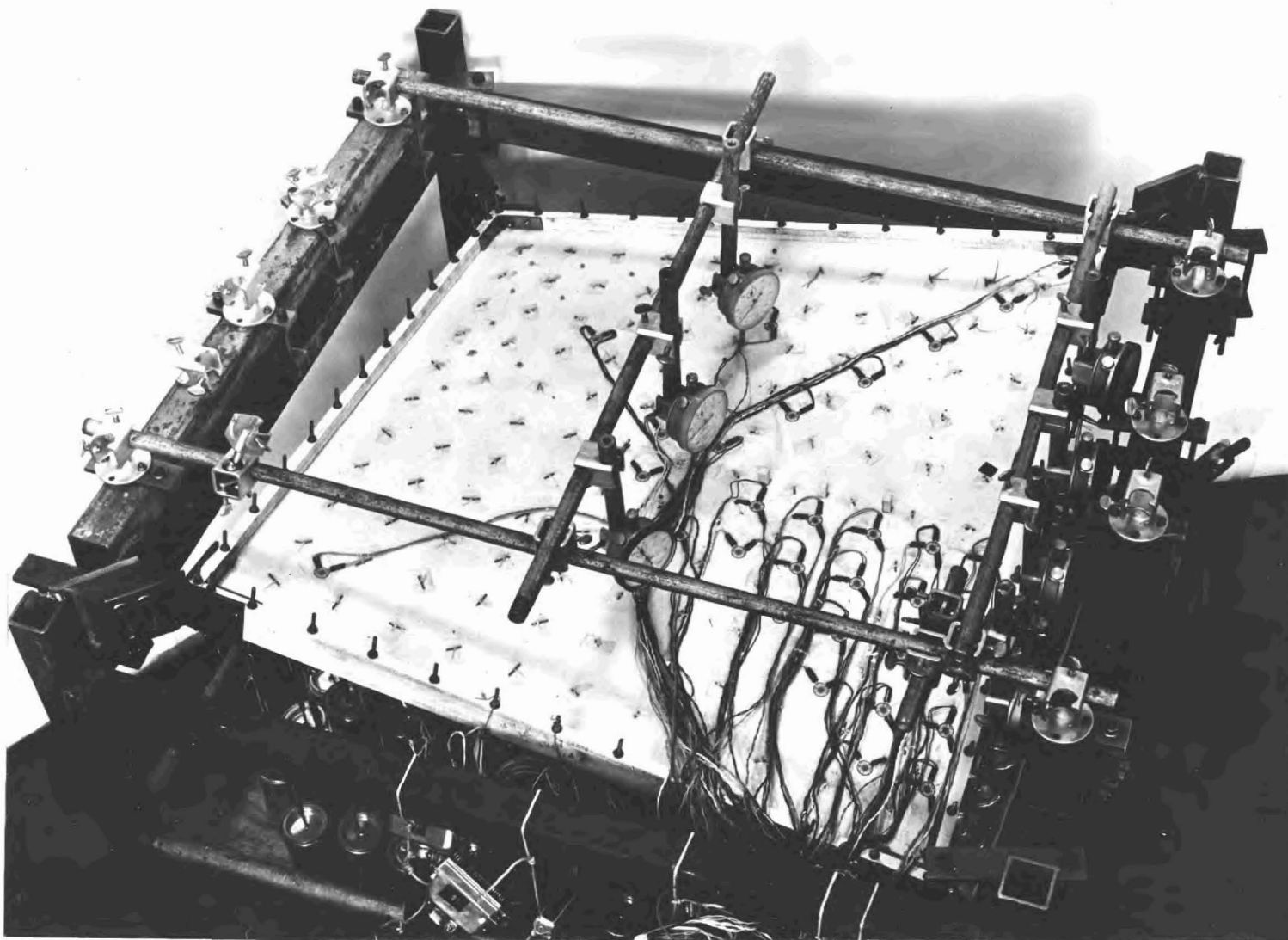


FIG 5.7 LOADING POINTS AND STRAIN GAUGE DISTRIBUTION ON VYBAK MODEL



FIG. 5.8 LOADING ARRANGEMENT AND TEST FRAME
VYBAK MODEL.

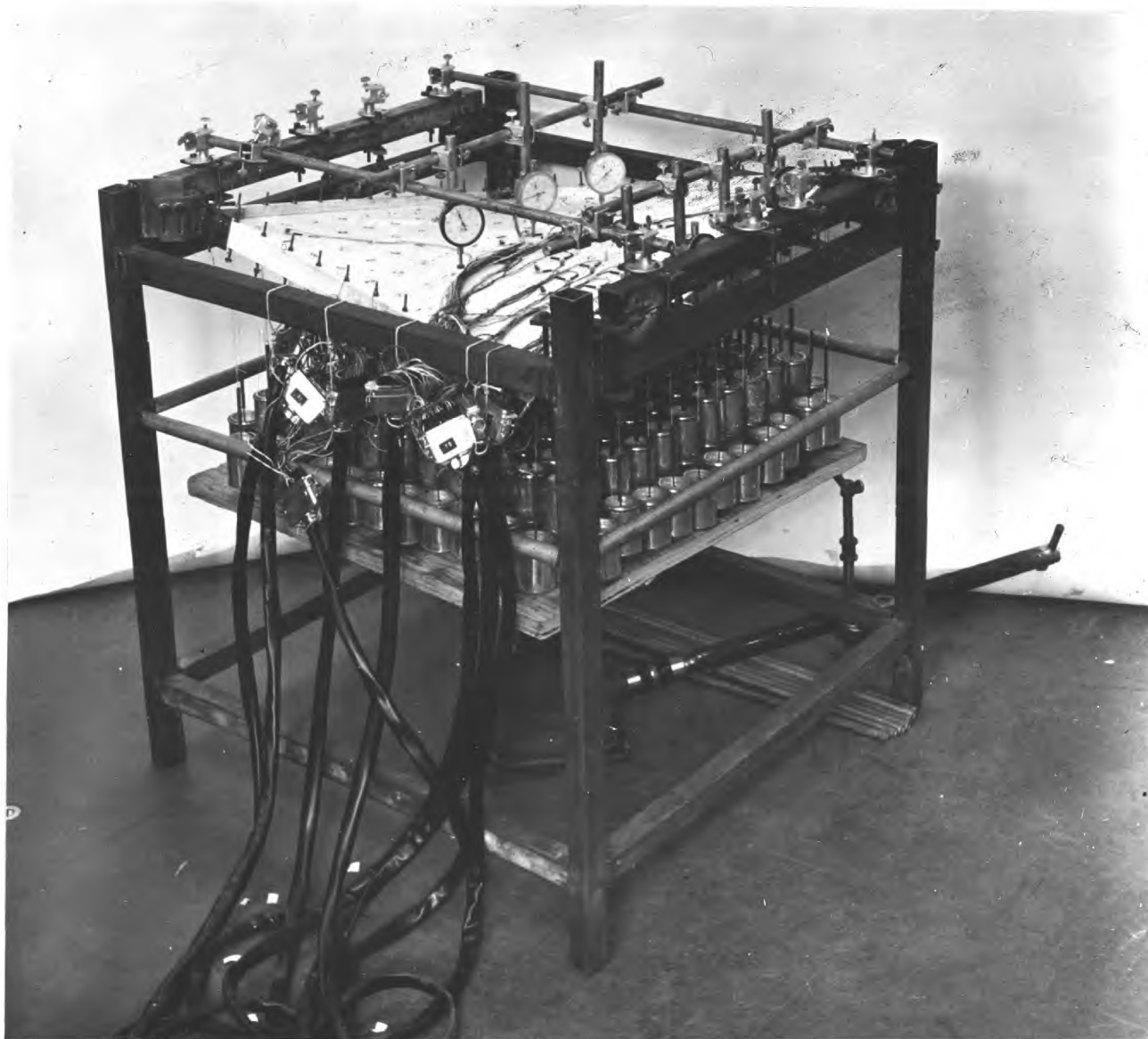


FIG 5.9 GENERAL VIEW OF TEST SET UP VYBAK MODEL

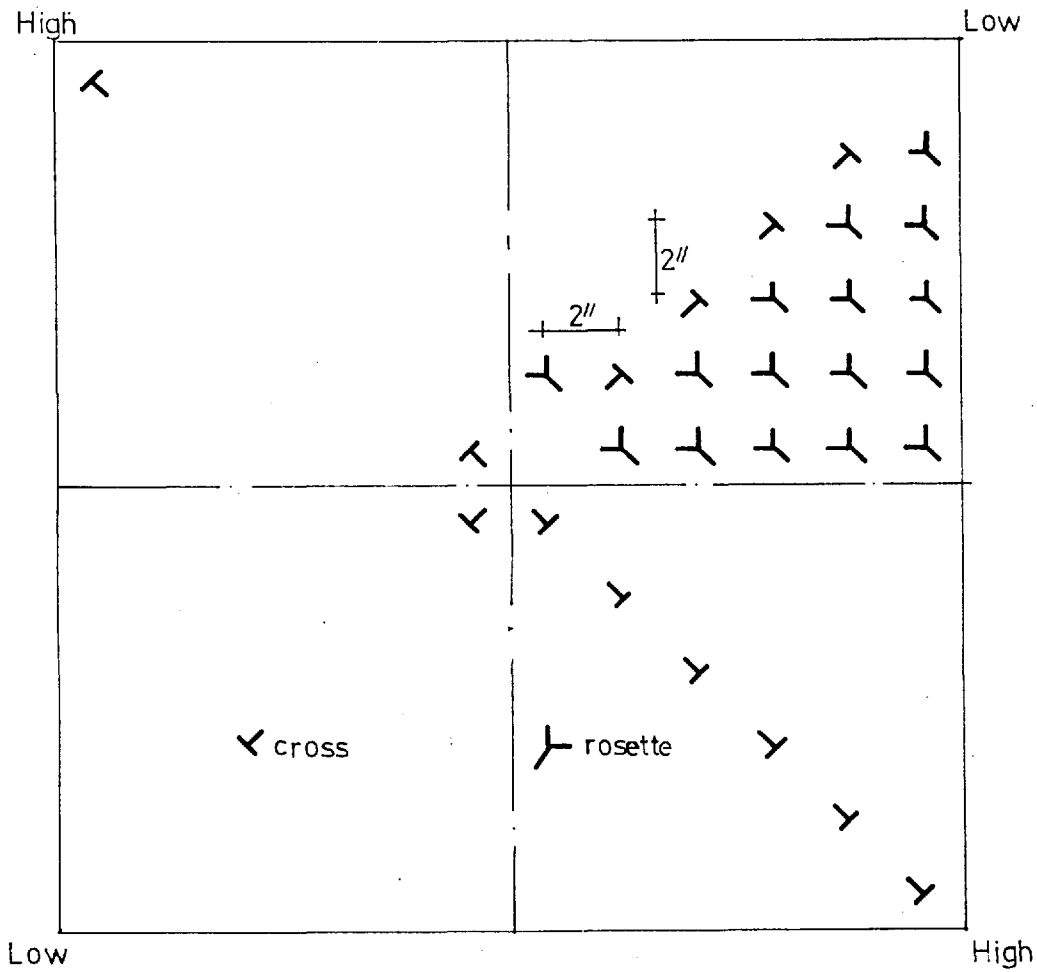


FIG 5.10 LOCATION OF STRAIN GAUGES
VYBAK MODEL

high

low

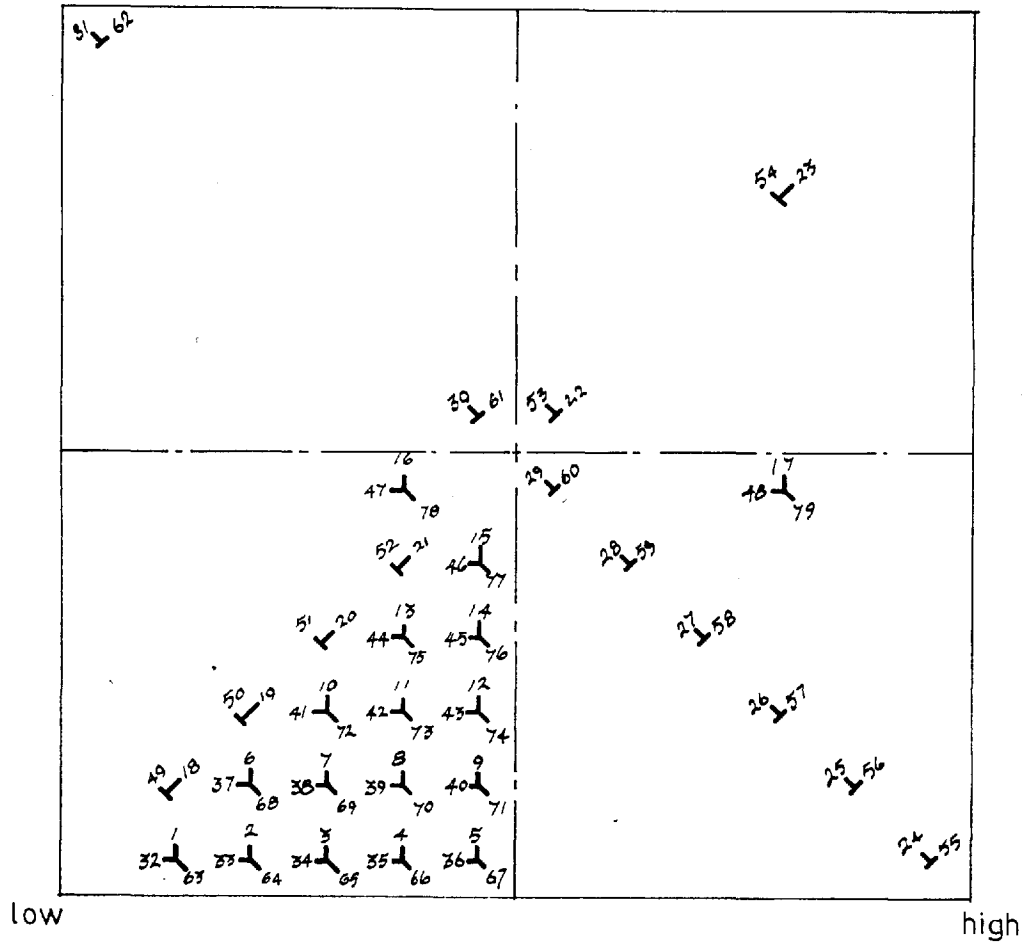


FIG 5.11 CHANNEL NO. FOR COMPONENT GAUGES
TOP SURFACE VYBAK MODEL

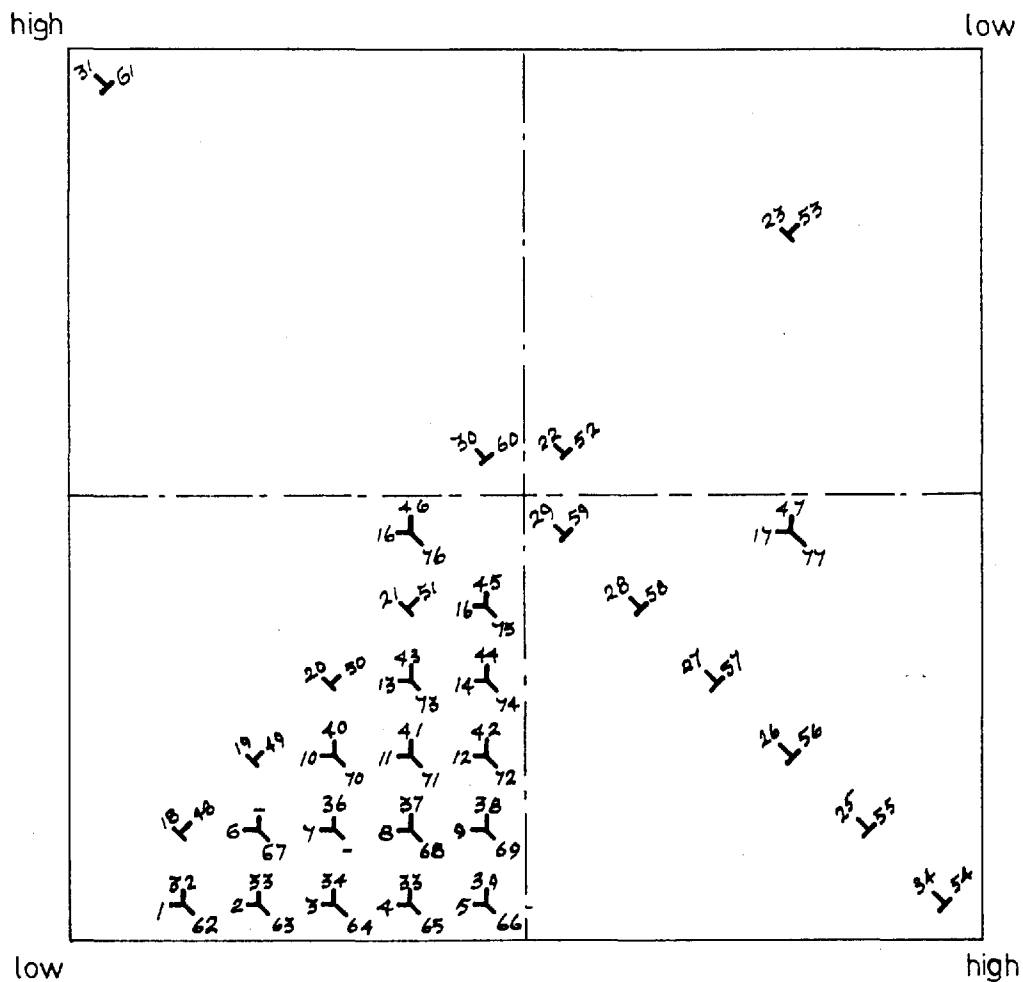


FIG 5.12 CHANNEL NO. FOR COMPONENT GAUGES
BOTTOM SURFACE VYBAK MODEL

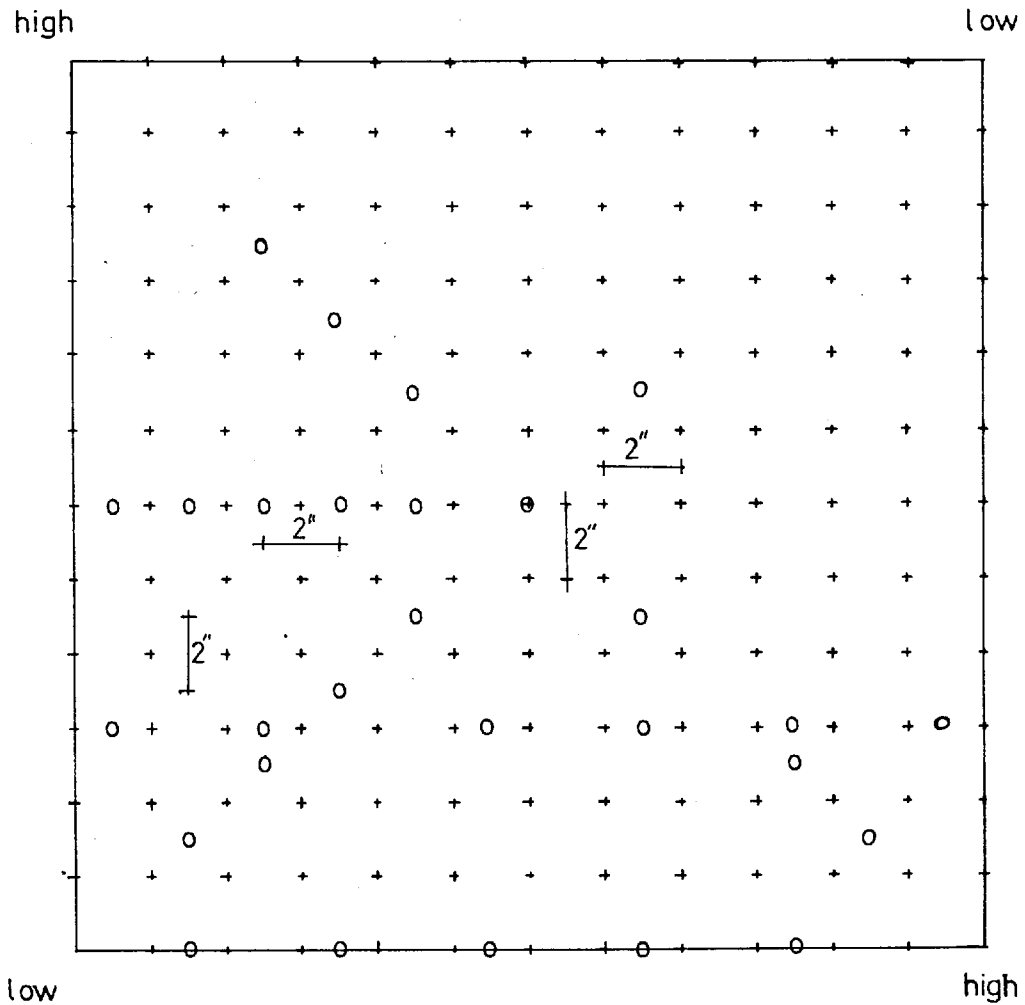


FIG 5.13 POSITION OF DIAL GAUGES (o) AND LOADING POINTS (+)

VYBAK MODEL

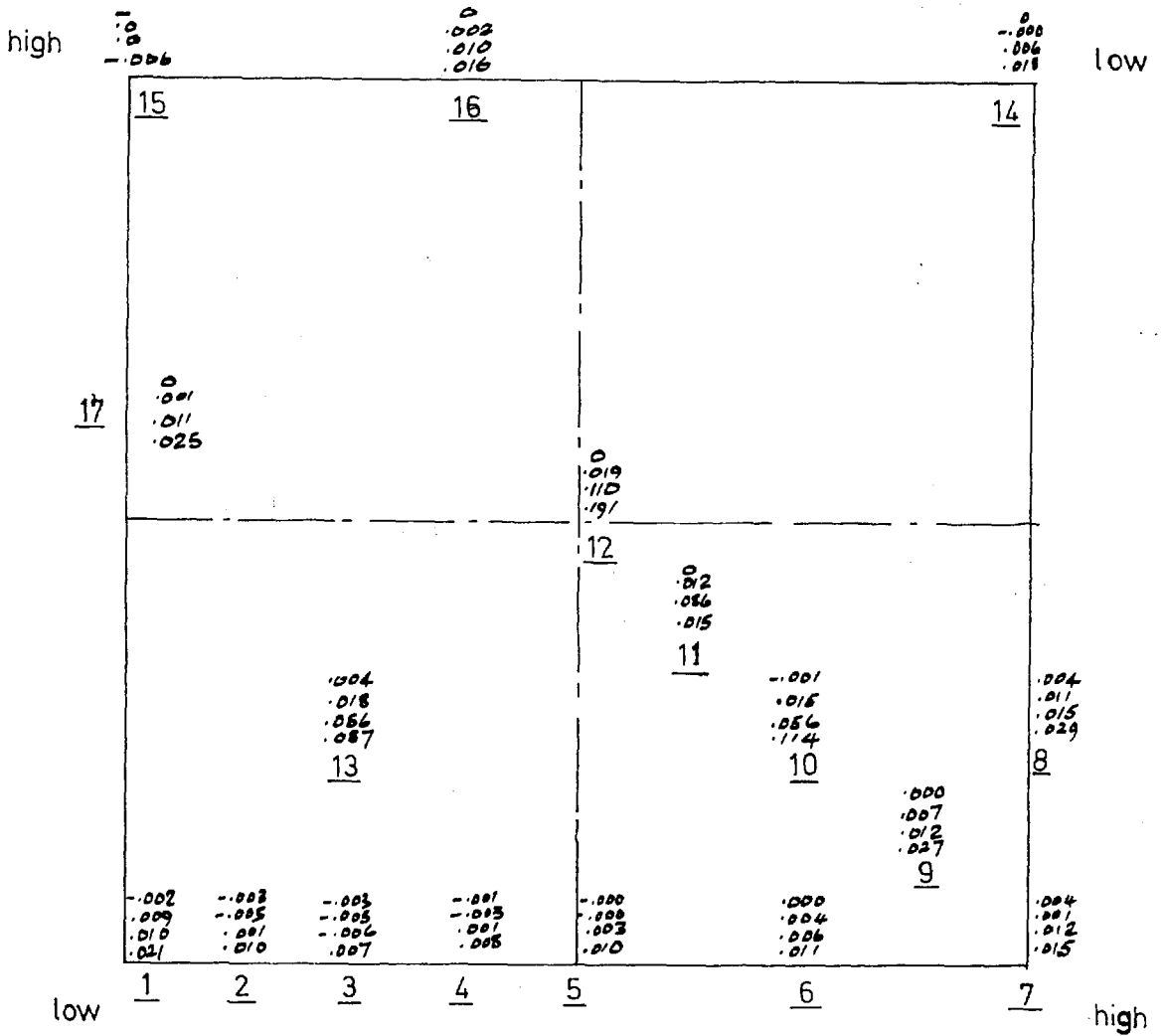


FIG 514 READINGS OF DISPLACEMENTS
TIMBER MODEL

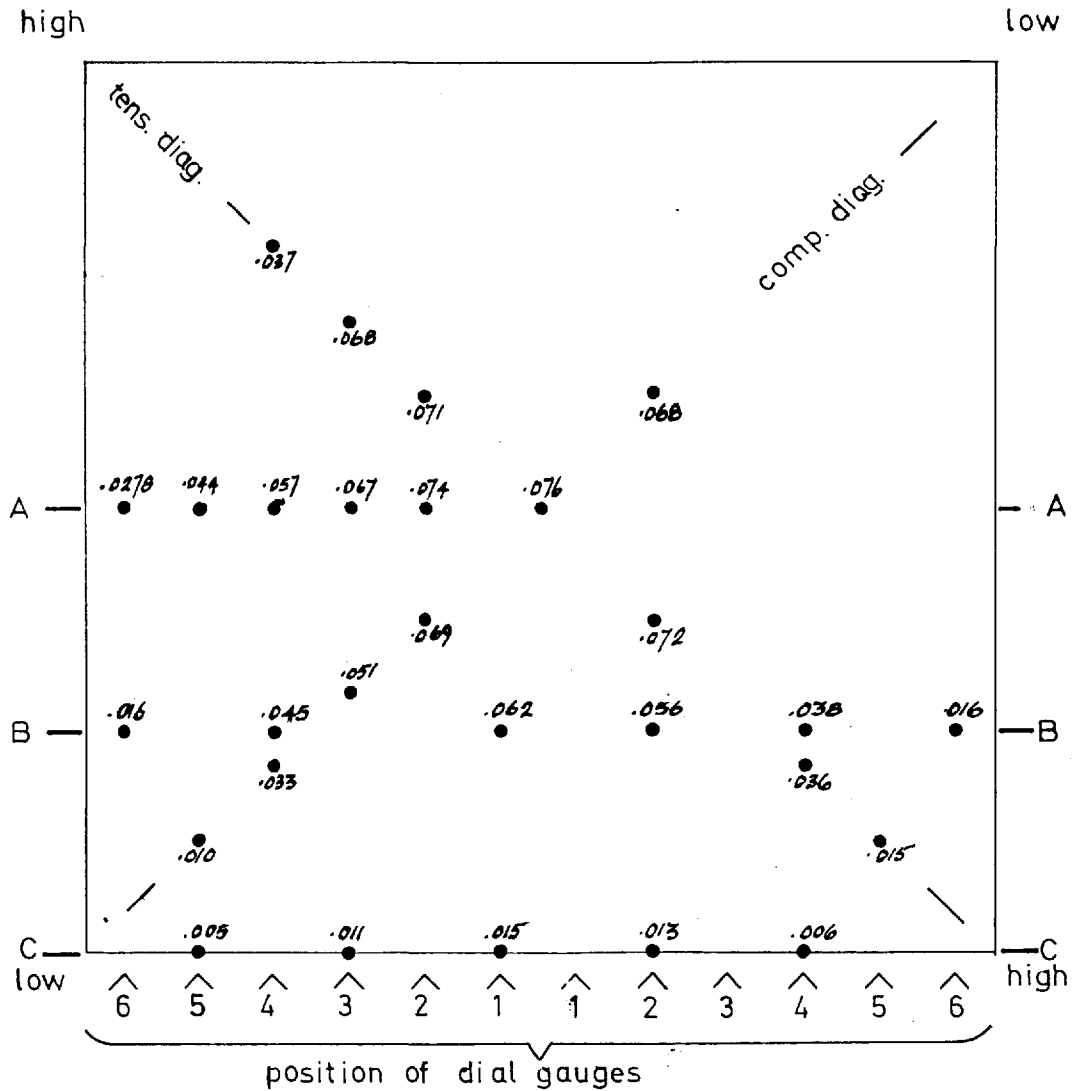


FIG 5-15 READINGS OF DISPLACEMENTS
VYBAK MODEL

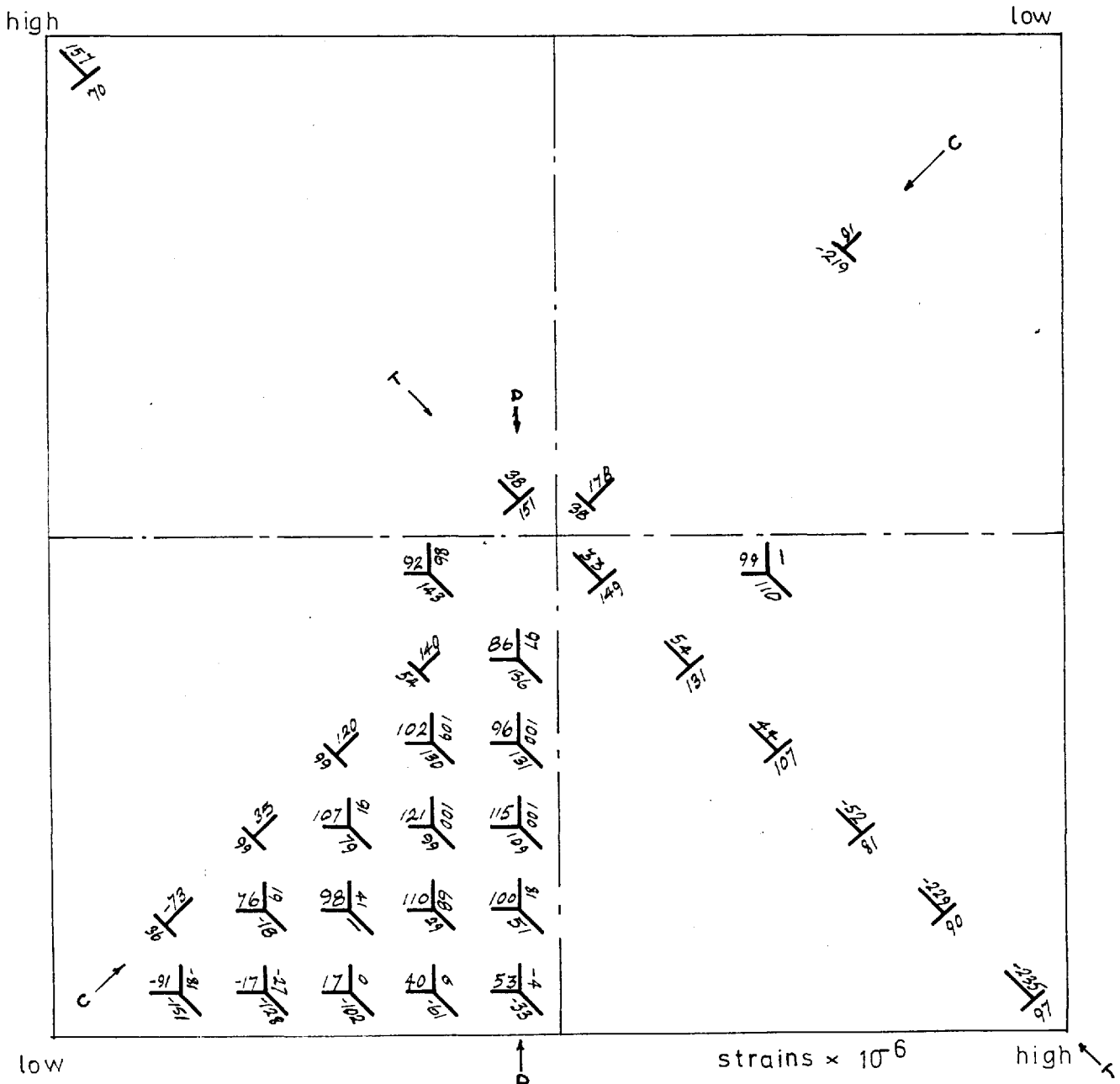


FIG 5.16 REDUCE STRAINS FOR TOP GAUGES VYBAK MODEL

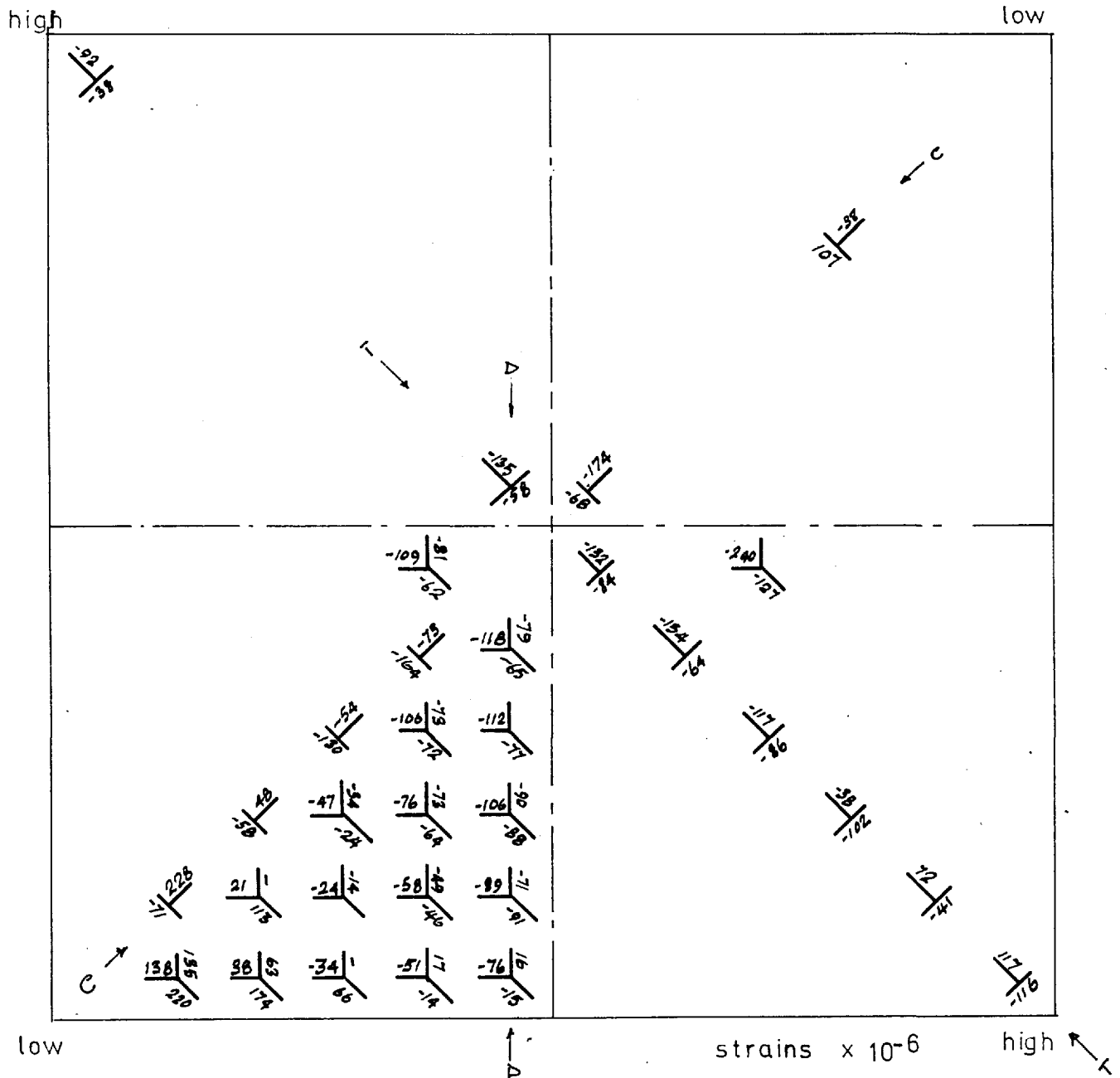


FIG 5-17 REDUCED STRAINS FOR BOTTOM GAUGES WYBAK MODEL

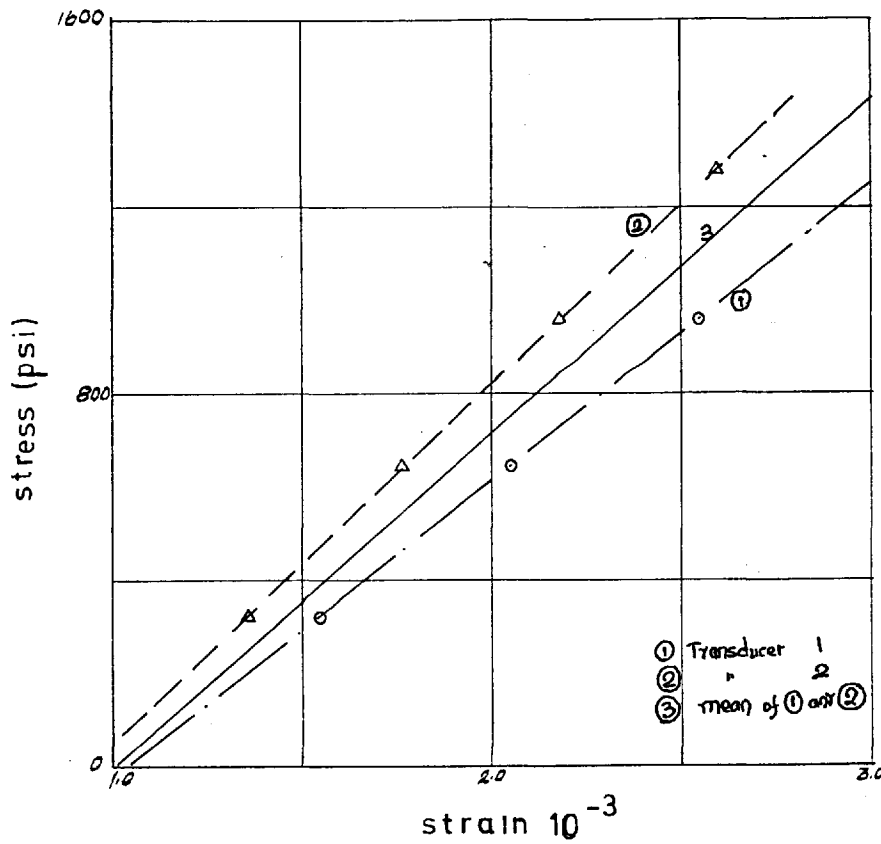


FIG 5.18 TYPICAL STRESS-STRAIN RELATION 3 ply Balsa SPE.

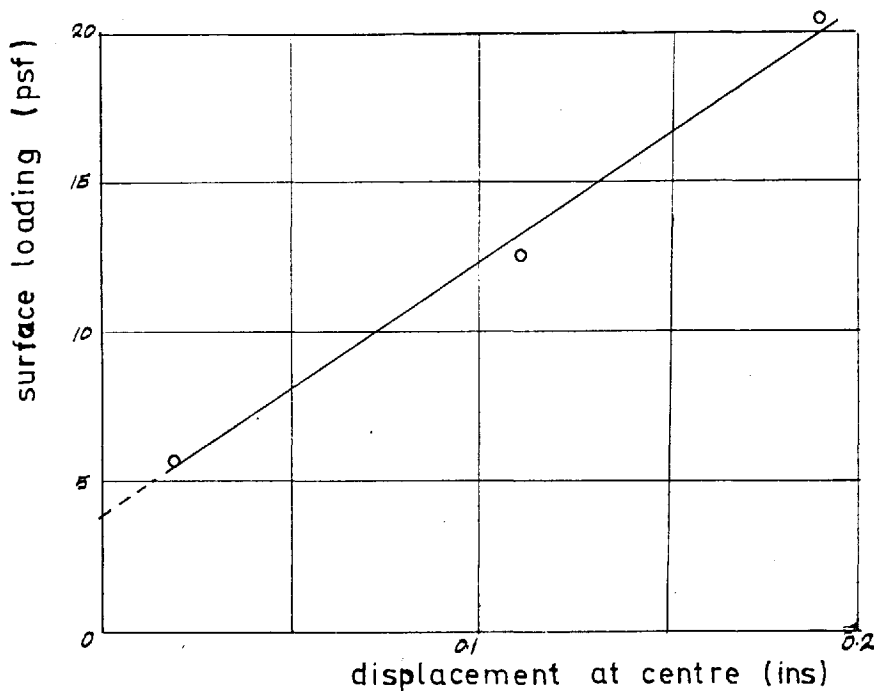


FIG 5.19 LOAD- DEFLECTION RELATION FOR CENTER OF TIMBER SHELL.

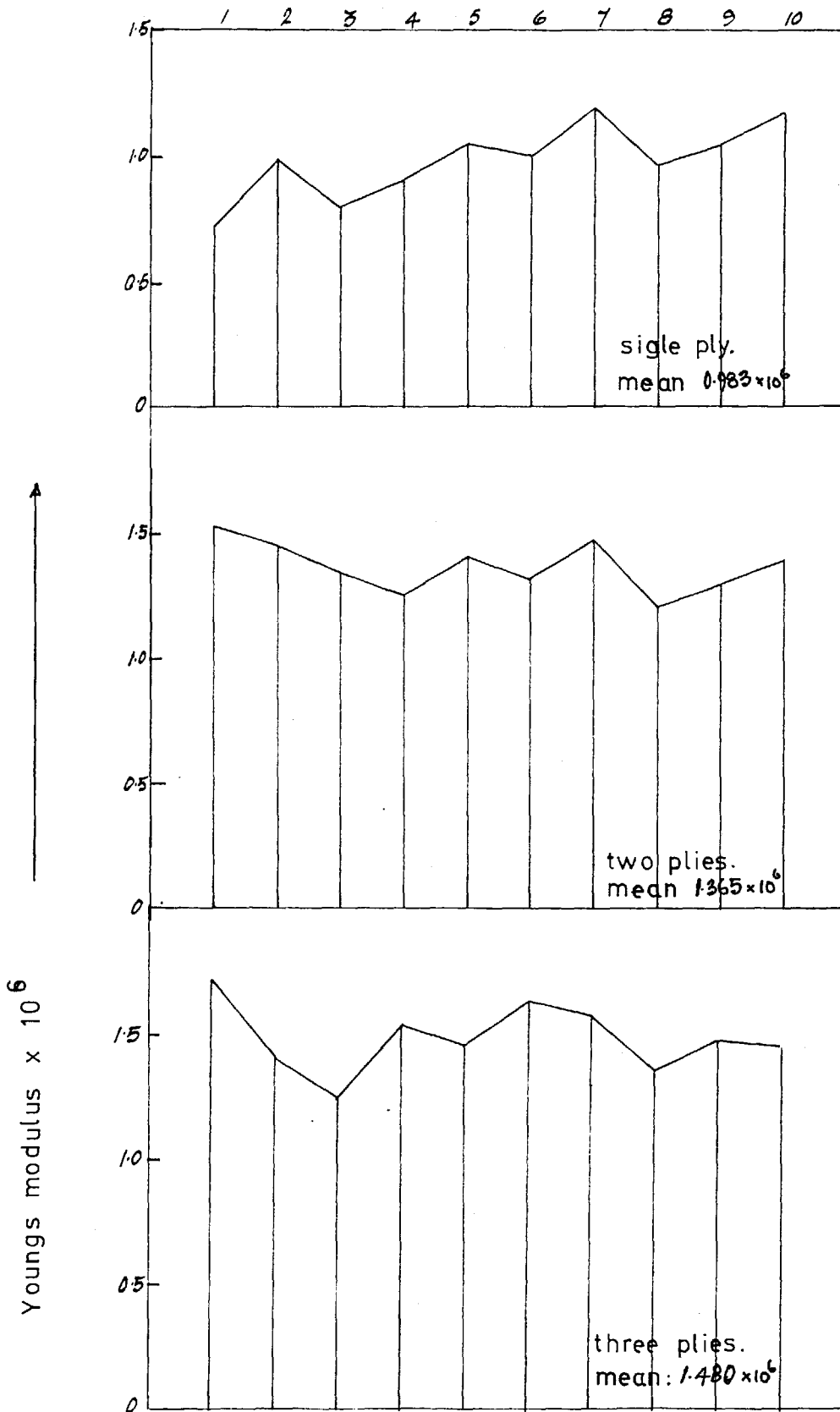


FIG 5.20 VARIATION OF YOUNGS MODULUS FOR 1,2,3 PLIES. BALSA.

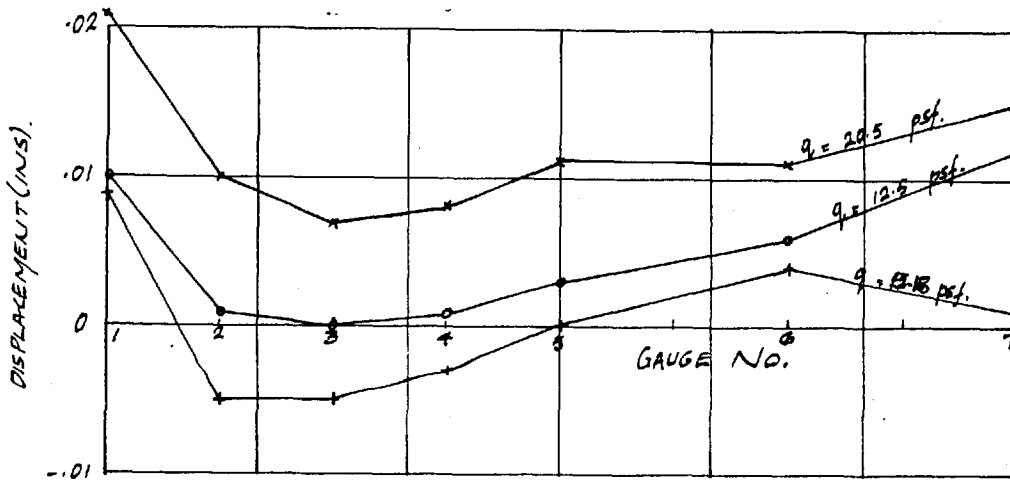


FIG 5.21 DISPLACEMENT ALONG THE EDGE BEAM
TIMBER MODEL

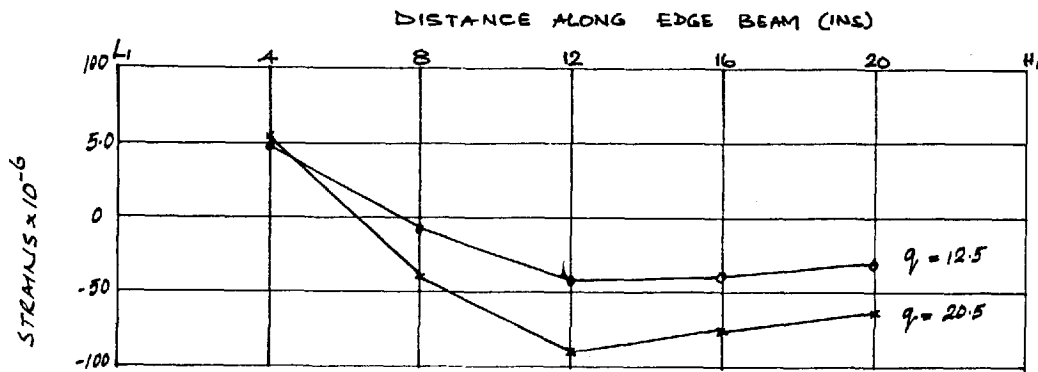


FIG 5.22 DIFFERENTIAL OF STRAINS ON EDGE BEAM
TIMBER MODEL

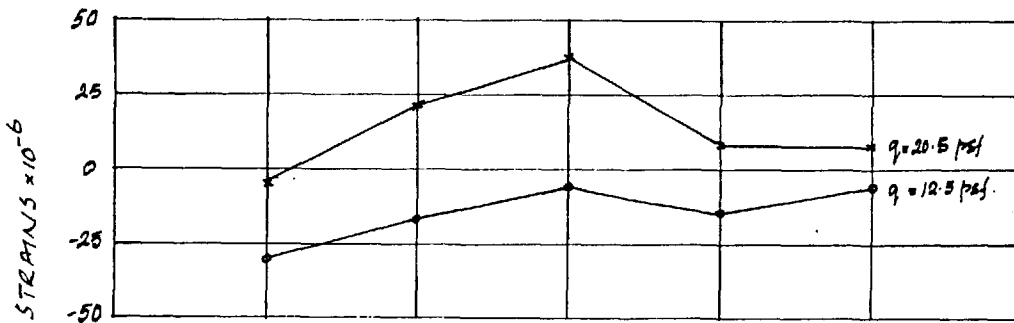


FIG 5.23 MEAN OF THE STRAINS ON EDGE BEAM
TIMBER MODEL

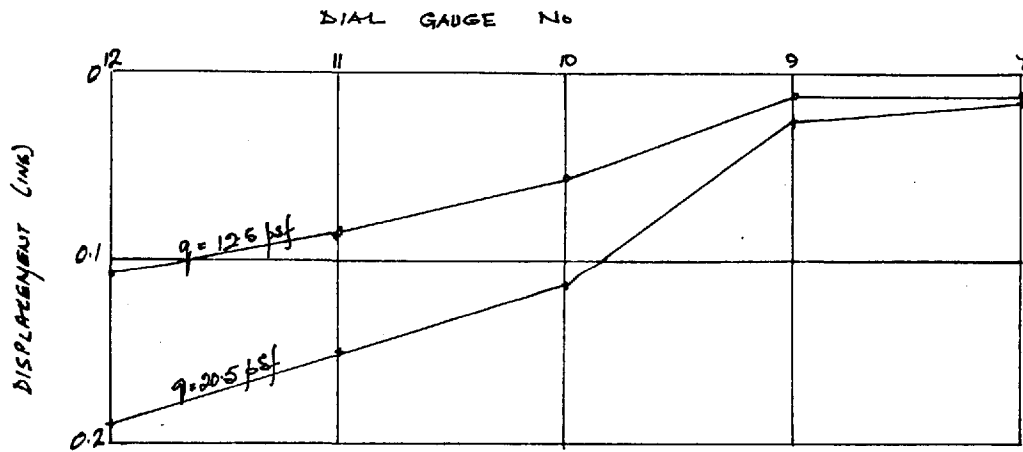


FIG 5.24 DISPLACEMENT ALONG THE TENSION DIAGONAL

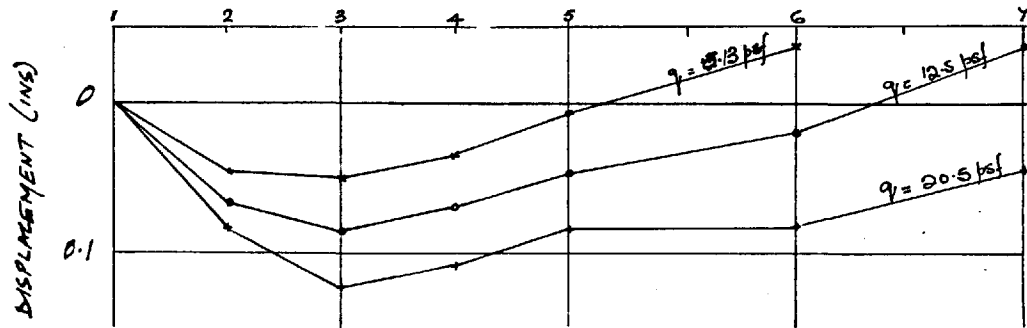


FIG 5.25 EDGE BEAM DISPLACEMENT

TIMBER MODEL

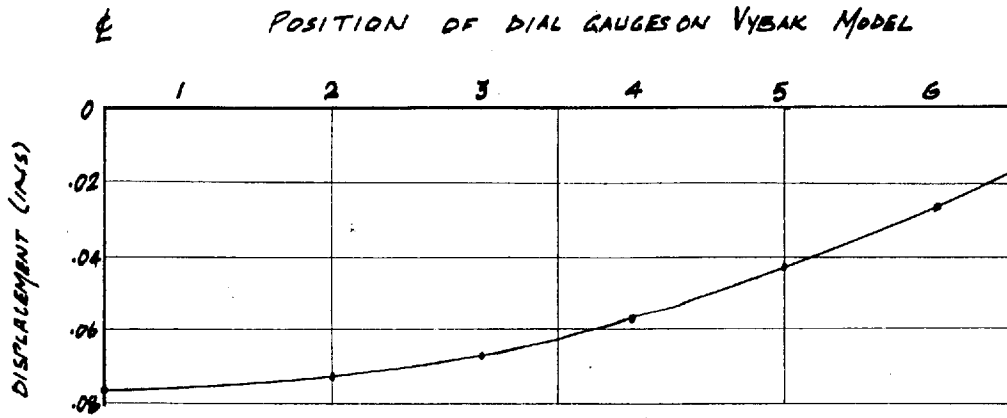


FIG 5.26 DISPLACEMENT ALONG LINE A-A FIG 5.15.

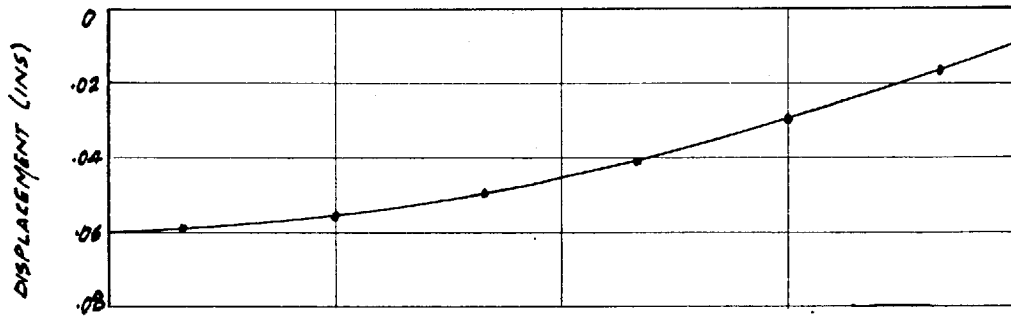


FIG 5.27 DISPLACEMENT ALONG LINE B-B FIG 5.15.

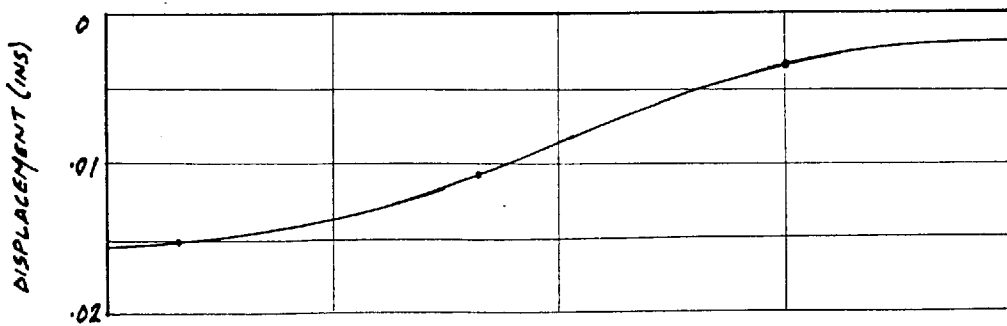
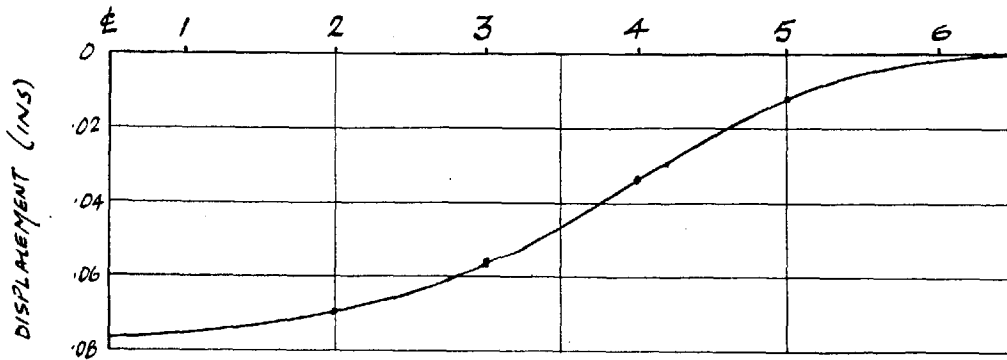
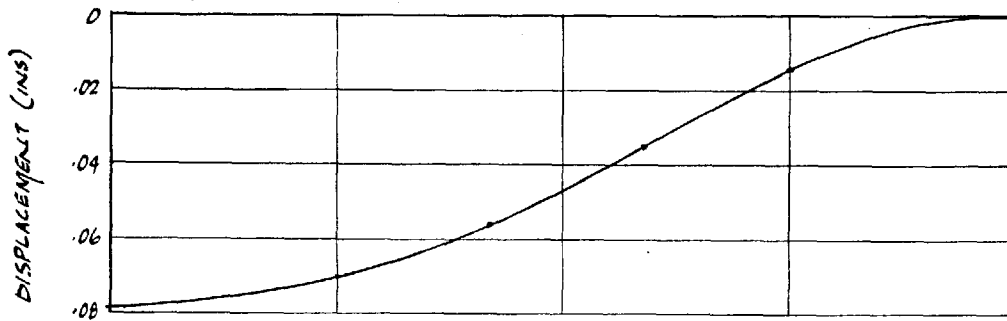
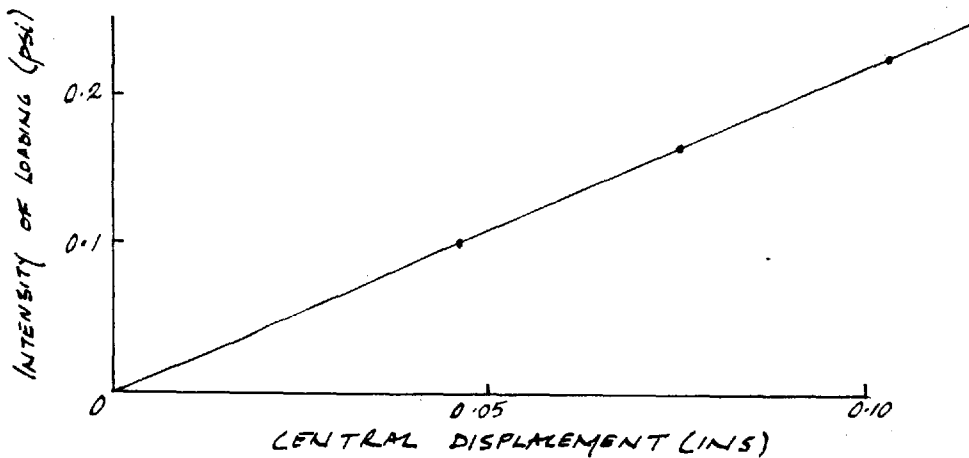


FIG 5.28 DISPLACEMENT ALONG LINE C-C FIG 5.15

POSITION OF GAUGES ON THE VYBAK MODEL

FIG 5.29 DISPLACEMENT ALONG THE COMPRESSION DIAGONALFIG 5.30 DISPLACEMENT ALONG THE TENSION DIAGONALFIG 5.31 LOAD - DEFLECTION CURVE FOR VYBAK MODEL.

STRAIN GAUGE POSITION ON VYBAK MODEL

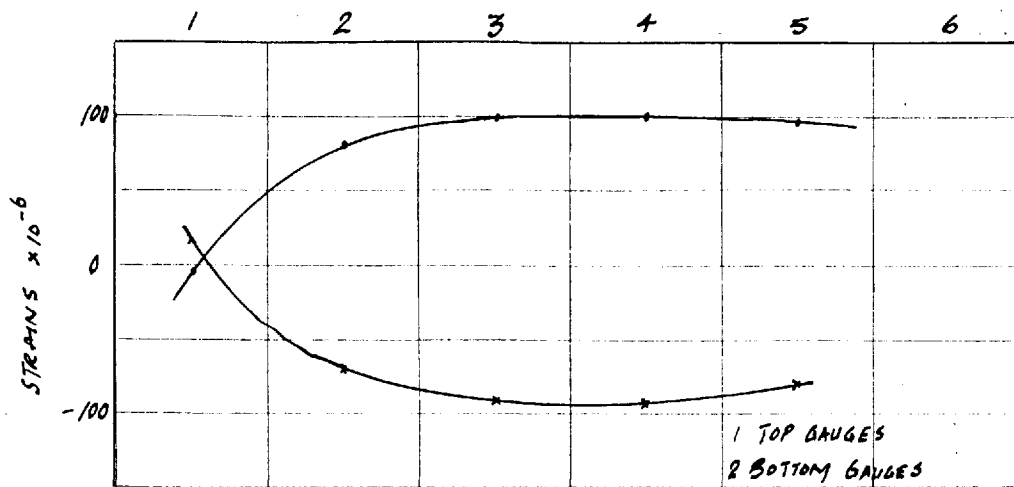


FIG 5.32 STRAIN DISTRIBUTION ALONG LINE D-D

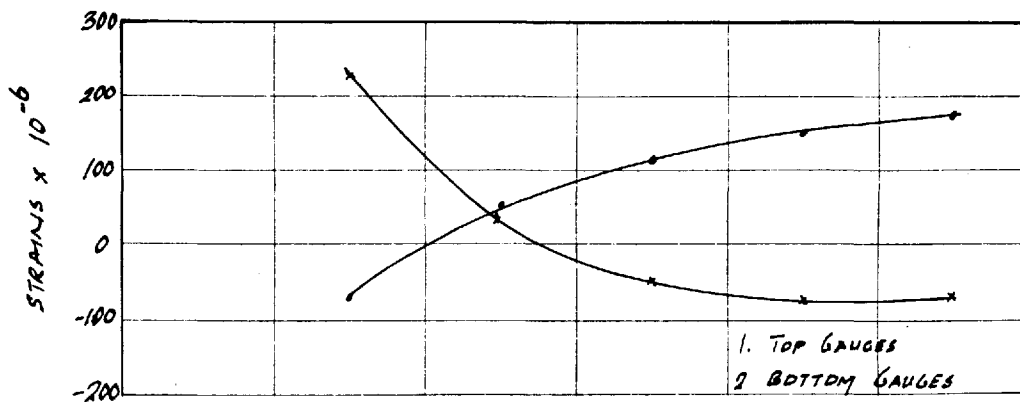


FIG 5.33 STRAIN DISTRIBUTION ALONG LINE C-C

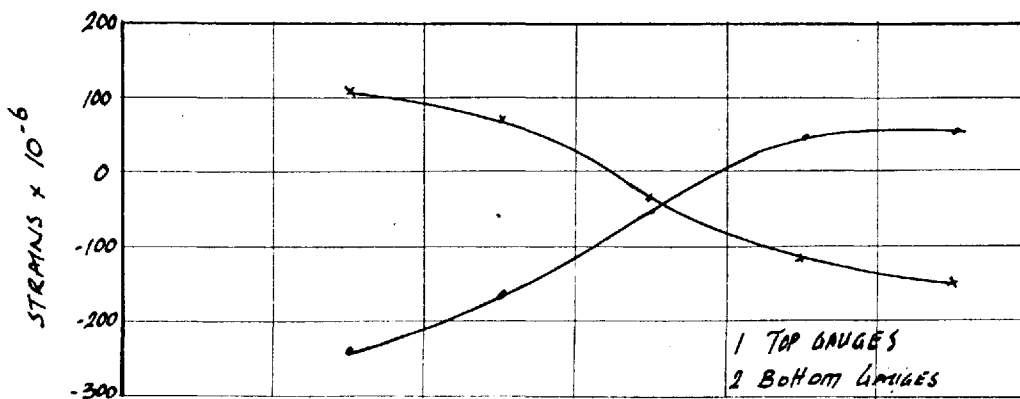


FIG 5.34 STRAIN DISTRIBUTION ALONG LINE T-T

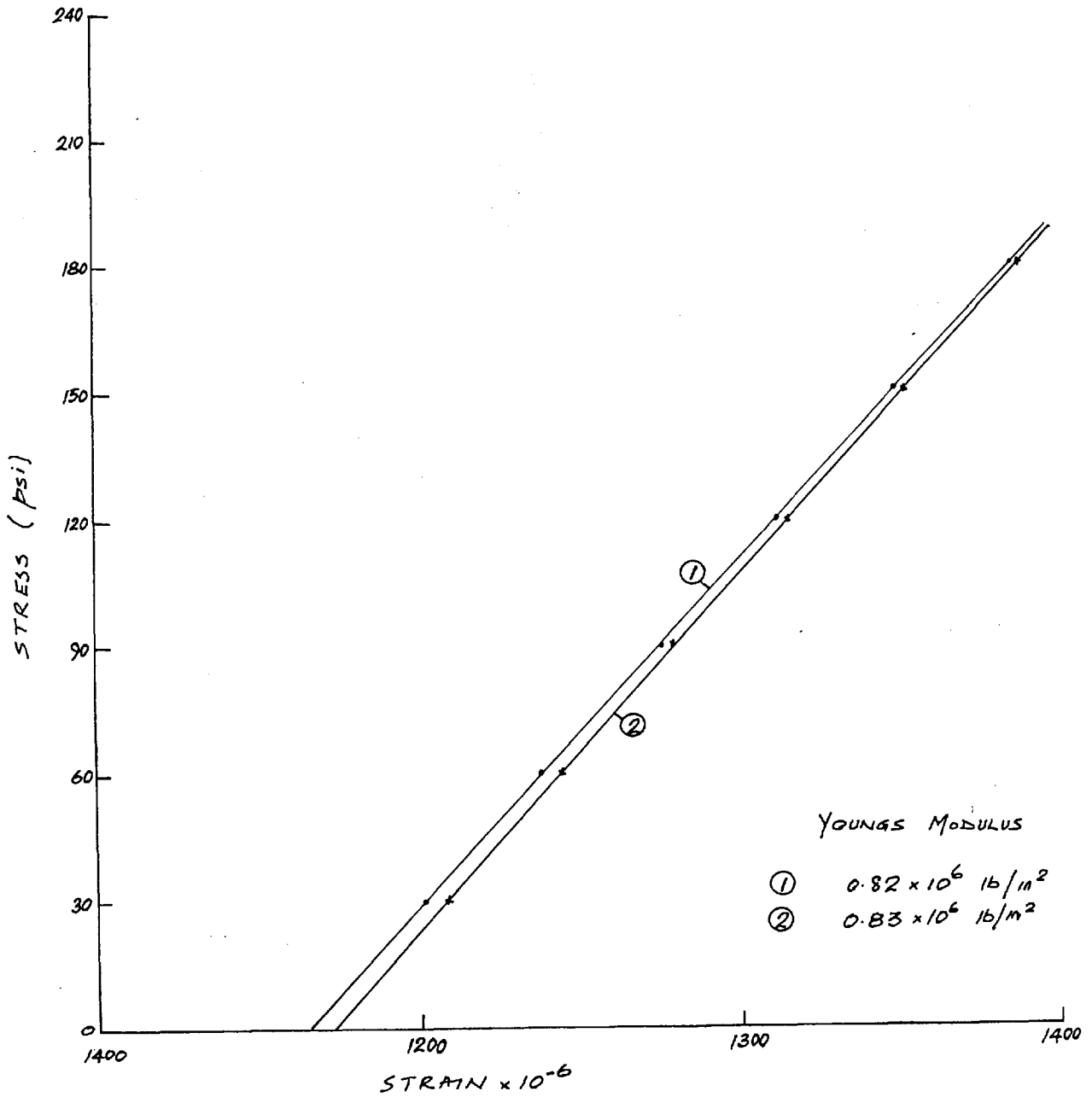


FIG 5.35 STRESS-STRAIN RELATIONSHIP FOR VYBAK.

CHAPTER 6DISCUSSION AND CONCLUSION6.1 Discussion of the Results of the Theoretical Computations

A considerable amount of effort was directed towards establishing that the theory as programmed for the computer could analyse some problems for which solutions are known. This was necessary to gain some confidence, in its application, as to the correctness of the formulation and the approximations involved. More important was the verification of the correctness and convergence of the formulation of the new function for triangular element in bending which has been derived in section 3.4. To this end, the problems of the plate under lateral loading and with various boundary conditions have been analysed. This is particularly relevant, because many research workers who have developed similar functions for triangular and rectangular elements in bending have chosen these particular problems as the test case to verify their formulations and the convergence. The problems of the hyperbolic paraboloid shells have also been analysed. The solutions obtained have been compared with the various theoretical solutions available. Where experimental data are available, these results have been correlated by means of the finite element solution. The details of these solution are now discussed under the head of the plate problems and the hyperbolic paraboloid shell problems.

6.2 The Plate Problems

Of the few functions available for triangular element in bending, two have been chosen for comparison along with the function

which has been derived in section 3.4 These functions are identified as Clough and Zienkiewicz functions respectively and the function derived in section 3.4 is referred to as Segun. The test problem has been chosen as a square isotropic plate with all the edges fixed. A further check is obtained by comparison with an exact solution. By exact is meant a solution based on an infinite series representation taking as many terms as possible with little changes in the solution after some terms. This solution is taken from the work of Timoshenko (1948). Fig. 6.2 shows the plot obtained from the various solutions. The solutions based on the finite element functions and carried out using a mesh division of 16×16 are shown in fig. 6.2. The Clough's function and Segun's function can be observed to give solutions which converge to the Timoshenko's solution from below. The Zienkiewicz function can be observed to give solutions which is above the Timoshenko solution. Zienkiewicz (1965) has however shown that the solution obtained using an 8×8 mesh absolutely converged to the Timoshenko's solution. If this 8×8 mesh solution is considered along with the 16×16 mesh solution obtained here, it would thus seem that the convergence is away from the exact solution.

The convergence of Segun's function is further exhibited in table 6.1. Here the results of the computation, using 8×8 , 12×12 , 16×16 and 20×20 mesh division, for the central deflection of the clamped plate using this function are presented. The accuracy of the function compared with the other functions is also clearly exhibited in this table. For the 16×16 mesh division, the Clough's function gave an error of $+13\%$, the Zienkiewicz function -10% , while Segun's function gave an error of $+2\%$.

AUTHOR	MESH	CENTRAL DEFL.	% of Exact
SEGUN	8 x 8	1.300	103
	12 x 12	1.288	102.5
	16 x 16	1.283	102
	20 x 20	1.279	101.5
CLOUGH	8 x 8	1.520	120
	16 x 16	1.420	113
ZIENKIEWICZ	16 x 16	1.130	90
TIMOSHENKO	EXACT'	1.27	100
MULTIPLIER $q L^4/D \times 10^{-3}$			

TABLE 6.1 Comparison of solutions for a clamped square plate.

Fig. 6.1 shows the comparison of solutions for a simply supported square plate. The solid curve shows the result of a 16 x 16 mesh finite difference solution. Because the curves practically coincide, the finite element solution using Segun's function and a 12 x 12 mesh division has been shown as dots. The value of the exact solution is 4.06. The finite element solution gave 4.1 an error of 1 %.

Fig. 6.3 shows the result of the computation for the plate on only four corner supports. This problem has also been discussed by Timoshenko. The exact solution has been given as 25.7 and the finite element solution for a 20 x 20 mesh is 26 again showing an error of +1 %.

The diverse nature of the problems investigated on plates and the accuracy which has been achieved demonstrates the capability of the finite element using the triangular element stiffness formulation. In particular, it can be observed that the triangular element stiffness in bending based on the function derived in section 3.4 yielded consistently more accurate and rapidly converging results compared with the other formulations discussed.

6.3 Hyperbolic Paraboloid Shell Problems

6.31 Solutions for Shell with Fixed and Simply Supported Boundaries

Of the various solutions available, the most abundant are for the shell with the boundaries fixed. The reason is simple. For analytical solutions of the Rayleigh - Ritz type, deflection modes have to be assumed and this can be achieved far more easily for the fixed boundary condition, than for any other type of boundary condition. Even the simple support type of boundary condition seems to present some problem in defining suitable functions. Problems originating from extrapolations at the boundary and corner points also make it easier for the fixed boundary conditions to be tackled more easily by the finite difference method. On the experimental front, it appears that more reports have been presented on the fixed shell than on the simply supported shell. This appears to be due to the fact that the fixed shell is more easily set up than the simply supported shell. The practical shell however is the flexible edge type. This has been investigated exhaustively by means of models, but no theoretical solution is available.

Since no exact solution is available for the shell with the fixed boundaries, the finite element solution for this problem has been compared with two approximate solutions. The first solution

is based on the work of Chetty (1962) who used a versatile method based on the Kantarovitch approximation. Basically this is a series solution. He also compared his solution with another series solution presented by Loof (1961) and the results of the experiment performed by him on an araldite model. The second solution is based on the work of Brebbia (1966) who used the finite difference method to solve equation 1.15. He also compared his solution with the experimental results on a model, he tested. For the solution of the simply supported shell, comparison has been made with the solution obtained by Michael (1966). He used a beam vibration function in the Gerlerkin approximation process.

These examples were chosen with two purposes in view. First to ensure that the finite element method could be applied to the analysis of hyperbolic paraboloid. Secondly, it was necessary to have contrasting examples to demonstrate the convergence trend of the solution, so as to give an indication of the element size necessary to assure convergence for varying shell dimension.

6.32 Comparison of Solutions for the Hyperbolic Paraboloid Shell with the fixed and simply supported Boundaries

For convenience of making cross references, the comparisons made below has been designated as problem 1, 2 and 3. Problem 1 deals with the comparison with the work of Chetty, problem 2 that of Brebbia, and problem 3 that of Michael (1966).

The data for problem 1 are as follows: length 12 in., rise 1.304 in. and thickness $1/4$ in. Fig. 6.5 shows the comparison of the deflections along the centre line of the shell. We shall not concern ourself with the experimental result which had been

dealt with at great length in the reference from which it has been detained. Presented along the finite element solution are results of two other solutions referred to as Loof Solution and K. G - a shortened form for Kantarovitch-Garlerkin method. The finite element is based on a shell program incorporating Segun's function. The mesh size used in this particular problem was 8×8 . The excellent comparison between the solutions can be observed.

Fig. 6.4 shows the plot of the displacements for shell type of problem 2. The data for this problem are as follows: length 100 cm, rise 10 cm and thickness 0.8 cm. We note that the 8×8 mesh which was quite adequate for problem 1 is no longer accurate enough for problem 2. Thus we had to employ successively 12×12 and 16×16 mesh subdivisions. In the figure comparison has been made with a series solution using eight harmonics, and we have also presented a solution using Clough's function. It can be observed that the solutions are of the same order of magnitude, but towards the centre of the shell and in general where boundary conditions have been applied, the finite element solutions show some divergence from the series solution. But within the shell agreement is reasonably good. Again we observe that Segun's function appears to give for the same mesh subdivision a better agreement at the centre than Clough's function.

The data for problem 3 was taken as that in problem 2. Fig. 6.6 shows the comparison between the finite element solutions and the series solution using the beam vibration function. 8×8 mesh and 16×16 mesh divisions have been used in this problem so as to exhibit the convergence trend. The large discrepancy at the centre using the 8×8 mesh can be observed, and the resulting improvement using a 16×16 mesh subdivision can also be observed. Within the

shell however, the discrepancies are not so pronounced. It can be observed that as the mesh division proceeds, the finite element solution can be seen to be approaching the series solution.

6.4 Hyperbolic Paraboloid Shell on Flexible Edge Supports

6. 41 Application of the Shell Analysis

In continuation of the procedure which has been followed thus far, it would be ideal to compare the result of the finite element solution with that of a series solution or solution by any other approximate method, for the shell on flexible edge supports. No solution of this problem has been found in the literature thus far. Hence the computational results are compared with the experimental results. This experiment has been reported in section 5.9. In order to eliminate the effect of the edge beam stiffness, the comparison has been made, as a preliminary step, for a shell with zero edge beam stiffness. This case being expected to be the limiting case in comparison with the identical plate problem discussed in 6.2. The corners of the shell were assumed restrained at the four corners with respect to the vertical displacements and at the lower corners with respect to horizontal movements. These conditions represent the least constraints to ensure stability. Fig. 6.7 shows the plot of the displacement along the central line of the shell. 8 x 8 and 16 x 16 mesh divisions have been used in the computation. We immediately observe the lifting of the centre of the shell. This lifting of the central portion of the shell has been associated with lack of convergence and has been previously noted in the discussion on problem 2 in section 6.2. The subdivision into 16 x 16 mesh, which had in the previous calculation on problem 2 been sufficient to give improved result which can be compared with other solutions

and the experimental result, does not seem to be adequate. But more disturbing is the fact that when the theoretical values are compared with the experimental values, both results do not agree and are not of the same order of magnitude. The experimental result is shown on this graph but to a scale ten times that of the theoretical curves' scale. Table 6.2 below gives an idea of the order of magnitudes involved. The displacement has been compared at A - the centre of the shell, B - a point midway between the centre of the shell and the edge and C - the edge point on the central line.

MESH	DISPL. A	% of EXPT.	DISPL. B	% of EXPT.	DISPL. C	% of EXPT.
8 x 8	.0032	4.26	.0058	9.0	.0038	22.4
16 x 16	.0046	6.14	.0065	10.3	.0050	29.6
Expt	.0760	100	.0640	100	.0170	100

TABLE 6.2. Comparison of experimental and theoretical results using the shell theory.

Although the convergence problem seems to have obscured the results, our experiences on the results of problems 1-3 would suggest however that convergence alone cannot account for the discrepancy in both results. Table 6.2 also shows some convergence trend although not as fast as would be desirable to approach the experimental result. It was therefore necessary to examine the state of stress in the shell. Hence the design of the Vybak model.

6.42 Stresses in the Vybak Model

The strains measured in the Vybak model have been shown in

fig. 5.32 - 34 inclusive. The actual strains recorded by the top and bottom gauges have been plotted. Also plotted are the mean of the strains and the differentials of the strains from which the direct stresses and the bending stresses can be obtained. Within the limits of the error in reading of the digital strain recorder which is of the order of ± 5 microstrains, hardly any direct strains could be observed in the model except along the tension and compression diagonals. Even here, we observe that the direct strains are negligible and could be considered as secondary. One would therefore justifiably be led to conclude that the shell transmits most of its load by pure bending and that its behaviour approximated more towards the plate rather than the shell actions.

6.43 Modification of the Shell Analysis to exclude the Membrane Effect

With the experimental results indicating that the shell behaved more like a plate rather than a shell, it therefore was necessary to modify the mathematical model to take into account this fact. From the finite element point of view, we have ascribed more degrees of freedom in the mathematical model than the physical model possessed. In the theoretical formulation of the shell theory, we have assumed that the strain energy of the shell is of the form

$$U = \int_V \epsilon^T N^t \sigma N dV$$

where $\epsilon^T N$ and ϵN have been defined by equation 3.17. $\epsilon^T N$ and ϵN were defined as being composed of flexural and extensional components of strains and stresses. If the extensional components are null, the theory is not invalidated. If therefore a null vector is substituted for the extensional strains in equation 3.21 and the corresponding displacements u and v are suppressed, the

resulting transformation gives the stiffness matrix for an element in bending. The geometric transformation matrix $[a]$ then ensures that the structure is analysed as a curved plate. The resulting theory has been captioned the Curved Plate Theory. In essence it is the analysis of a plate but with the coordinate transformation matrix $[c]$ taking other values than unity.

6.44 Correlation of the Test Result by the Curved Plate Theory

Fig. 6.8 shows the comparison between the experimental result and the theoretical result based on the curved plate theory. Segun's function has been used in the curved plate analysis and solutions have been obtained for 16 x 16 mesh and 20 x 20 mesh divisions. The edge beam effect has also been included in the computation. The results show a better agreement than those obtained when shell type of actions are assumed.

One peculiarity of the solution is however worth noting. When the computation was carried out with the edge beam effect included the computer result for the deflection at the centre of the shell was 500.253. When the computation was repeated without the edge beam effect, the result was 500.376. When the results were reduced to obtain the actual displacement in inches, the differences cancel out in the number of significant digits retained. In fact the solution presented in fig. 6.8 also represents the result of the shell without an edge beam. It was therefore felt that the computation should be carried out for the case in which the edges remain undeflected representing the case of the shell with an edge beam of infinite stiffness. The result is plotted on the same graph and captioned curved plate S.S (indicating simply supported). If the values of the displacements at the centre of the shell are compared,

the reduction in the displacement based on the zero edge stiffness case is of the order of 3.6 %. Generally within the boundary of the central portion of the shell, the shell is insensitive to the presence or absence of the edge beam. Towards the edges however the displacement is affected by the presence of the edge beam. This effect seems to have been borne out by tests conducted by Batchelor (1963) on a concrete model. An extract from his thesis is presented in fig. 6.10. It will be observed that the effect of replacing an edge beam made of channel section having a sectional moment of inertia of 1.78in^4 to a box section having a sectional moment of inertia of 3.56in^4 and a greater torsional rigidity has not produced significant changes in the distribution of displacements in the shell. It would thus appear that the rigidity of the shell is more important in determining the stiffness of the shell. Test will however be required to show that this still holds for a large span thickness ratio.

6.45 Comparison of the curved plate solution and the flat plate solution

In fig. 6.8 the computation using the shell data for a simply supported plate has been plotted. Comparison between the simply supported plate solution and the curved plate analysis shows that the shell stiffness is roughly of the same order as that of a flat plate. This fact has also been pointed out by Batchelor (1963). The results of his investigation on a concrete model with boundary conditions similar to the one conducted in this thesis also failed to be correlated by the shell theory and he therefore compared the behaviour of the shell with that of flat plate under fixed and simply supported conditions. When he used a flexible tie rod his results compared with a clamped plate solution, while with

a more rigid tie rod, the results compared with that of a plate which is simply supported. His broad conclusion that the shell behaviour could be approximated by the clamped plate behaviour must be open to questions. Clearly the results must be governed by the shell rise. If the warping constant is reduced to zero, the shell tends to the plate on four corner support to which a solution has been given in section 6.1. Thus with very low shell rise one might expect the result to tend towards the solution of the plate on four corner support. The little information available cannot make a generalised conclusion to be drawn between the solutions. However one can say that the shell does not behave as a plate under identical boundary conditions.

6.4.6 Curved Plate Analysis and the Shell Analysis for the Shell with Simply Supported Boundary Conditions

The result for the simply supported condition obtained from the curved plate analysis raises an interesting problem. We observe from considerations of problems 1 and 2 that, when the edges of the shell were fixed, the assumption of the presence of membrane stresses and the consequent use of the shell theory, enabled the test results to be correlated. Also based on the shell theory, we have assumed the presence of the membrane stresses in problem 3, leading to the displacement pattern shown in fig. 6.6 and again this solution compared very well with the series solution. However for problem 3, if we ignore the presence of the membrane stresses, we obtain the result of the curved plate S.S. shown in fig. 6.8. These results are shown again in fig. 6.9. It will thus appear that there are two possible configurations for the shell which is simply supported along the boundary depending on which assumptions on stresses is used. Which of the configuration the shell will assume can only be decided by experimental analysis.

6.5 GENERAL REMARKS

6.5.1 Mesh Size and Model Size

As already pointed out in the discussion on the results of problem 1, the 8 x 8 mesh division of the structure was sufficient to assure the convergence to the result obtained by other methods. The model size was 12 in x 12 in thus giving the leading dimension of the triangular elements as 3 in. When the model size was 50 cm x 50 cm for problem 2, we find that the 8 x 8 mesh division giving the element leading dimension of 6 cm was not sufficient to assure convergence. The 16 x 16 mesh division giving the leading element dimension of 3.125 cm was required. Even for this we observe from the result that convergence has not fully materialised. It would thus appear that as the numerical dimension of the shell becomes very large we will require more divisions of the structure to obtain convergence. However more subdivision into finer mesh entail greater number of nodes and hence very large systems of equations to be solved. Most of the cost of computing is however accounted for by the computing time. In the present computation, it was found that with the program organisation adopted, the computation time varies roughly as the cube of the mesh division as shown in fig. 6.11. It thus appears therefore that the finite element idealisation of shell in its present form is not an economic proposition for the everyday use compared with the other form of analysis of the problems considered.

6.5.2 Lifting of the Shell Centre

The cause of the lifting of the centre of the shell, observed in problems 2 and 3 and also on the problem of the shell on flexible support is not very easy to trace. The computations carried

out on the shell with the flexible edge by assuming the shell type behaviour and the curved plate action seem to suggest that it is associated with the membrane stresses. The portion can be taken as the slowly converging portion of the analysis. As shown in the computations on problems 2 and 3 its effect is minimised by division into finer mesh. The worsening of this effect in the computation on the shell with the flexible edge support however suggests that the boundary conditions could also have some part to play. This effect is totally absent on the plate problems and the curved plate analysis.

6.5.3. Bending moments in plate and shell.

The results of the computation of the bending moments are presented in fig. 6.12-6.15 inclusive. In general, for the plate problems, the bending moments are reasonably predicted by the finite element method using the triangular elements. Errors in the bending moments are maximum towards the edges and to a lesser extent at lines of symmetry. The results for the clamped plate and the simply supported plates are shown in fig 6.14 and 6.15 respectively. The same remarks apply to the problem of shell fixed along the boundary. The results, compared with other solutions and experimental results, have been presented for Problems 1 and 2 and shown in fig 6.15 and 6.14 respectively.

6.6 CONCLUSIONS

The various problems discussed in section 6.1 - 6.4 have demonstrated the ability of the finite element analysis to tackle the problems of bending of plates and shells under the fixed and simple support conditions. In particular the use of the triangular elements have been demonstrated as capable of producing the same degree of accuracy as solutions based on the series solution especially for plate problems. For shell type of problems, the finite element solution has been shown to suffer some convergence trouble which can be improved by division into fine mesh. A new algebraic function for defining the stiffness properties of the triangular element in bending has been developed. This function captioned Segun's function has been shown to give a consistently more accurate solution than Clough's and Zienkiewicz's functions for the analysis of plate problems. Its convergence has also been shown to be better than the other functions.

The shell theory has been shown to fail to predict the behaviour of the hyperbolic paraboloid shell on flexible edge supports. This failure has been explained by the fact that the shell does not develop membrane stresses as assumed in the shell theory for this particular boundary condition. An experiment conducted on a Vybak model established that the shell transmits its load by means of bending stresses which are associated with plate type of actions. A modified shell theory which neglects membrane actions has been advanced to explain the behaviour of the hyperbolic paraboloid shell on flexible edge support. This theory captioned the Curved plate theory has been shown to predict the shell behaviour under the flexible boundary conditions.

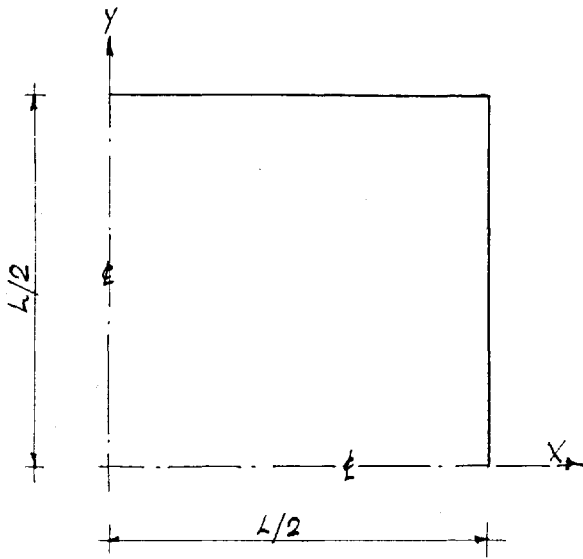
The large number equations which result from analysis based on shell type action, coupled with the convergence problem makes the finite element analysis less attractive than the other methods for the fixed and simple support condition.

6.7 Suggestions for further Research

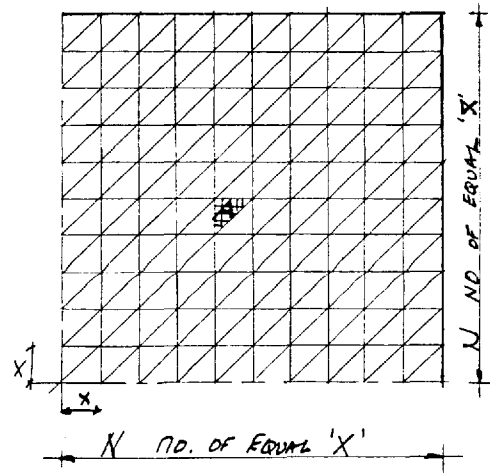
The test reported in Chapter 5 on the Vybak model has shown that the hyperbolic paraboloid shell tended to behave like a curved plate. Based on this test, the curved plate theory has been proposed. It will therefore be useful to investigate the extent to which this behaviour still holds when several parameters of the shell are varied, particularly the rise/span and the span/thickness ratios. The curved plate analysis has also demonstrated the relative unimportance of the edge beam stiffness for the test data obtained in the present investigation. Since the greater part of the cost for this type of structure is taken up by the provision for the edge beam it will be useful to determine the maximum ratio of shell/beam stiffness beyond which the beam stiffness does not contribute significantly to the stiffness of the shell. The tie rod influence could also be simultaneously studied in the same program.

A test of the shell on a simple type of support will also be useful to verify the findings of the curved plate and shell theory discussed in section 6.3.6. This could be used to simplify the analysis and design of the hyperbolic paraboloid shell by separation of the edge influences.

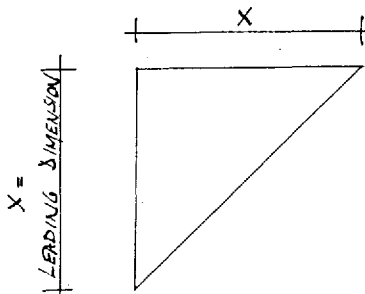
More investigation into the basic properties of glued veneers of balsa and relation to full scale specimens would appear to be necessary if the results of model studies on timber shell were to be used to interpret the behaviour of the prototype.



(a) COORDINATE SYSTEM
SHELL AND PLATE PROBLEMS



(b) $2N \times 2N$ MESH DIVISION



(c) TYPICAL ELEMENT 'A'

FIG 6 COORDINATE SYSTEM AND MESH DIVISIONS
FOR PLATE AND SHELL PROBLEMS

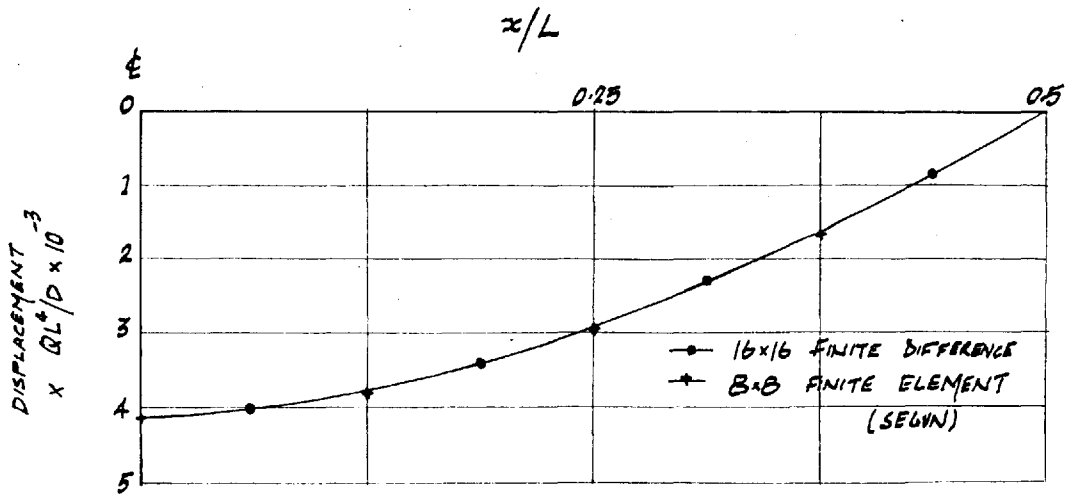


FIG 6.1 SIMPLY SUPPORTED SQUARE PLATE UNDER U.D.L. (Q)

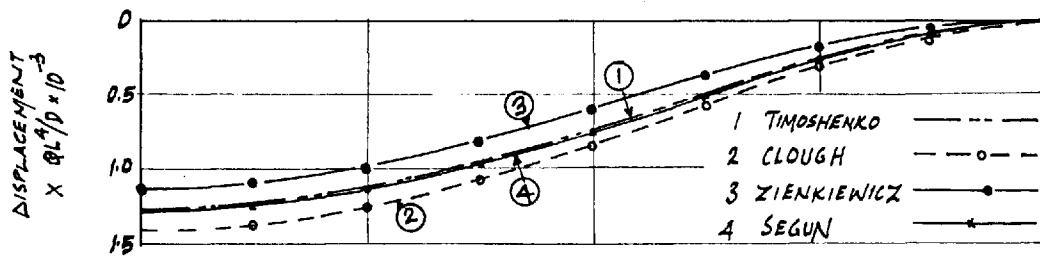


FIG 6.2 SQUARE PLATE WITH CLAMPED EDGES UNDER U.D.L. (Q)
COMPARISON OF RESULTS USING VARIOUS FUNCTIONS FOR TRIANGULAR
ELEMENT AND 16x16 MESH DIVISIONS OF THE PLATE

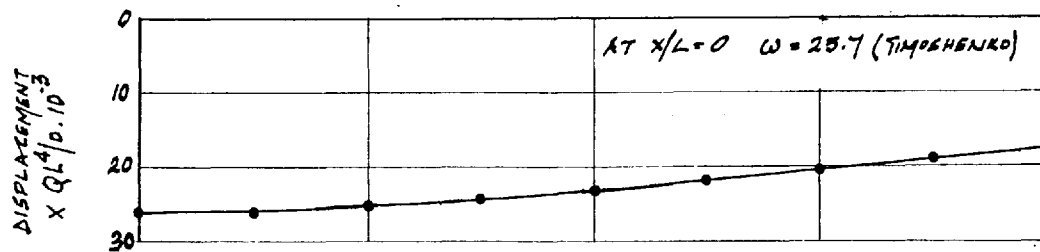


FIG 6.3 SQUARE PLATE ON FOUR CORNER SUPPORTS
FINITE ELEMENT SOLUTION USING TRIANGULAR ELEMENTS (SEGUN)

DISTANCE ALONG SHELL CENTRE - LINE

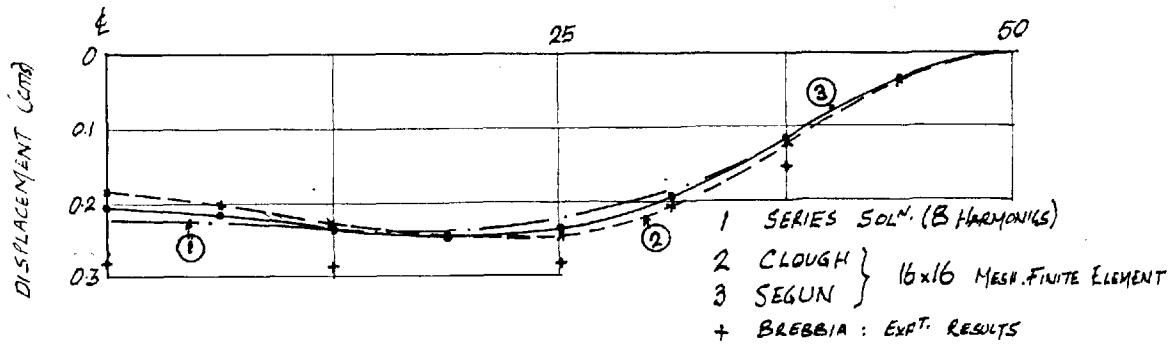


FIG 6.4 HYPERBOLIC PARABOLOID SHELL WITH FIXED EDGES
COMPARISON OF SOLUTIONS ON CENTRE LINE PROBLEM 2

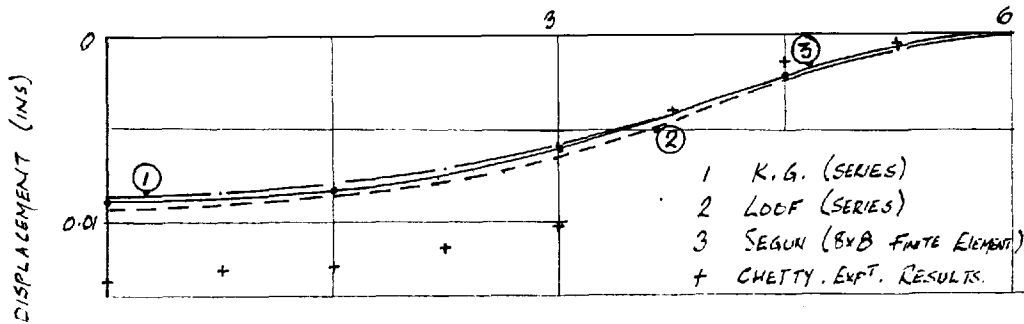


FIG 6.5 SHELL WITH FIXED EDGES PROBLEM 1
COMPARISON OF SOLUTIONS ON CENTRE LINE

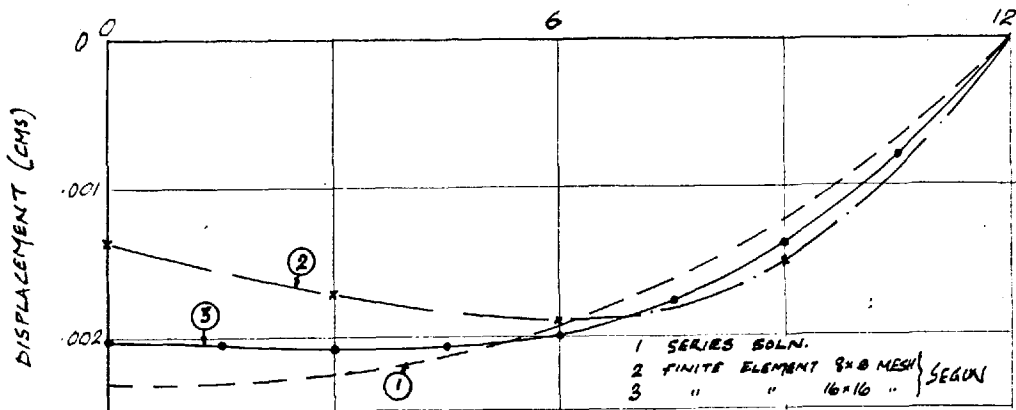


FIG 6.6 SHELL WITH GABLE SUPPORT PROBLEM 3
COMPARISON OF SOLUTION ON CENTRE - LINE

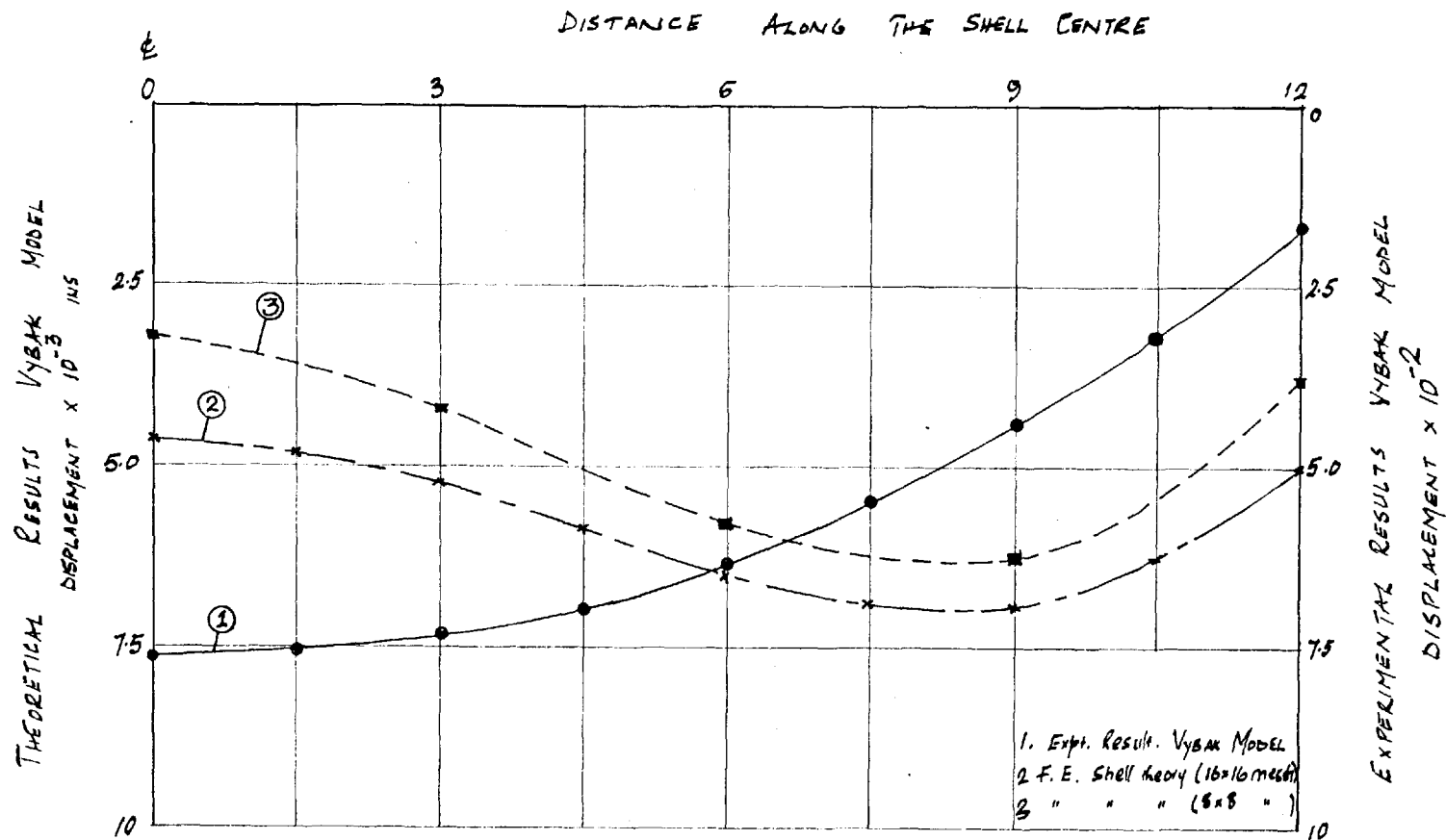


FIG. 6.7 HYPERBOLIC PARABOLOID SHELL ON FLEXIBLE SUPPORT.

COMPARISON OF FINITE ELEMENT SHELL SOLUTION WITH THE EXPERIMENTAL RESULT.

DISPLACEMENT ALONG LINE A-A.

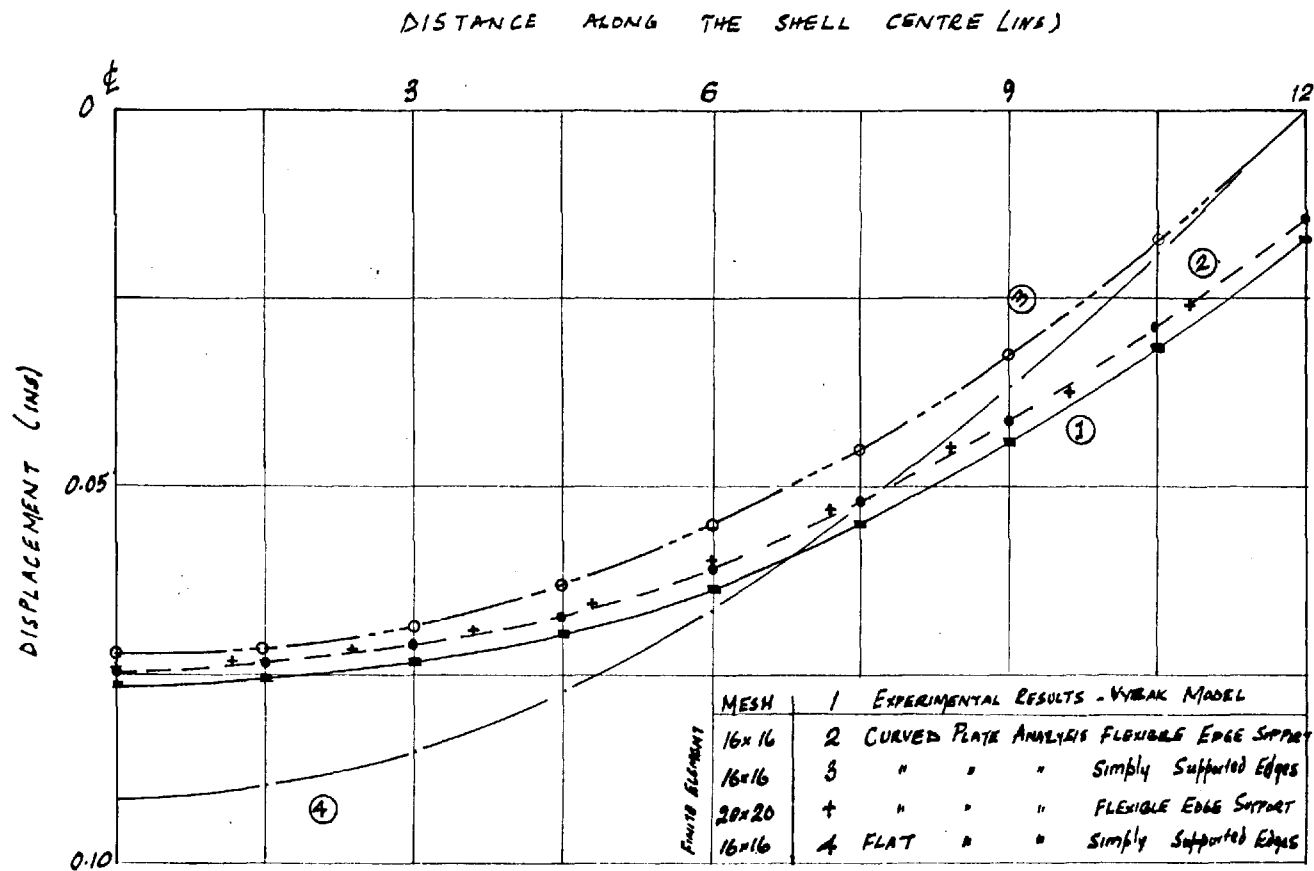


FIG 6.8 HYPERBOLIC PARABOLOID SHELL ON FLEXIBLE EDGE SUPPORTS.
COMPARISON OF EXPERIMENTAL RESULTS AND THEORETICAL RESULTS
BASED ON THE CURVED PLATE IDEALISATION.

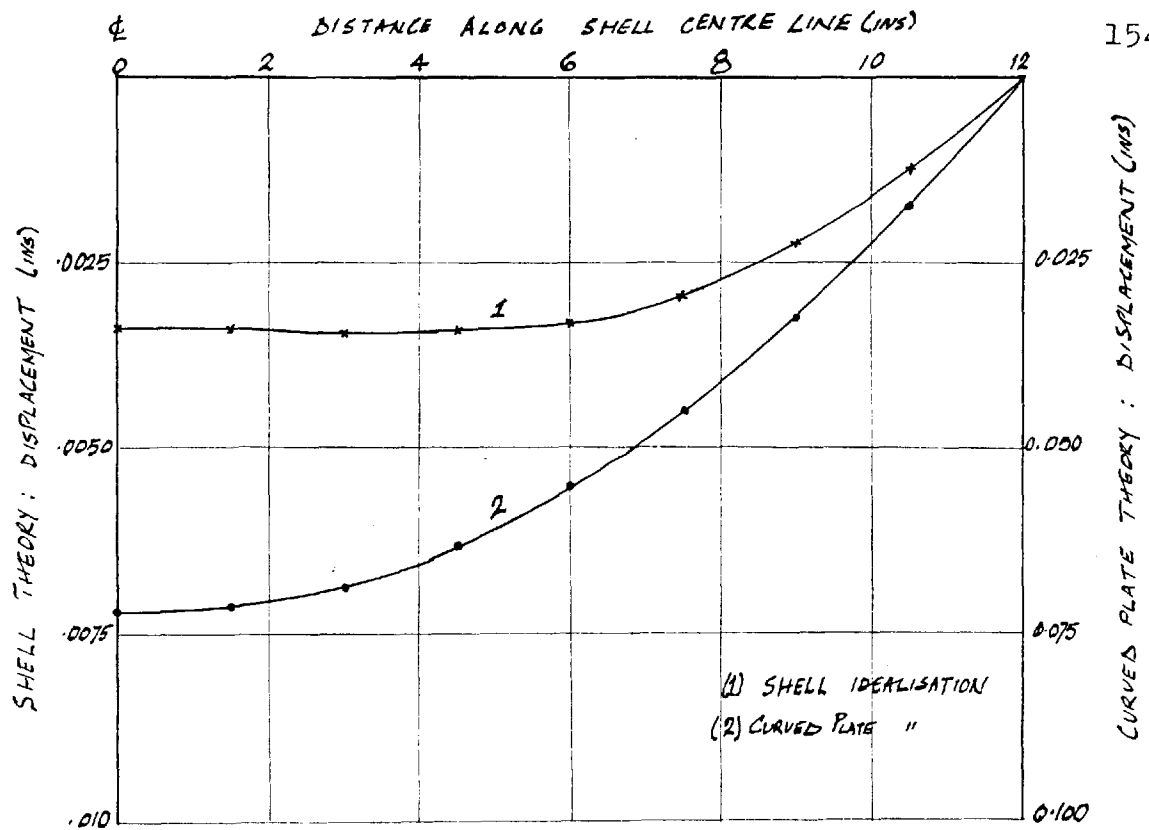


FIG. 6.9 HYPERBOLIC PARABOLOID SHELL SIMPLY SUPPORTED ALONG THE BOUNDARY, COMPARISON OF FINITE ELEMENT SOLUTIONS BASED ON SHELL AND CURVED PLATE IDEALISATIONS.

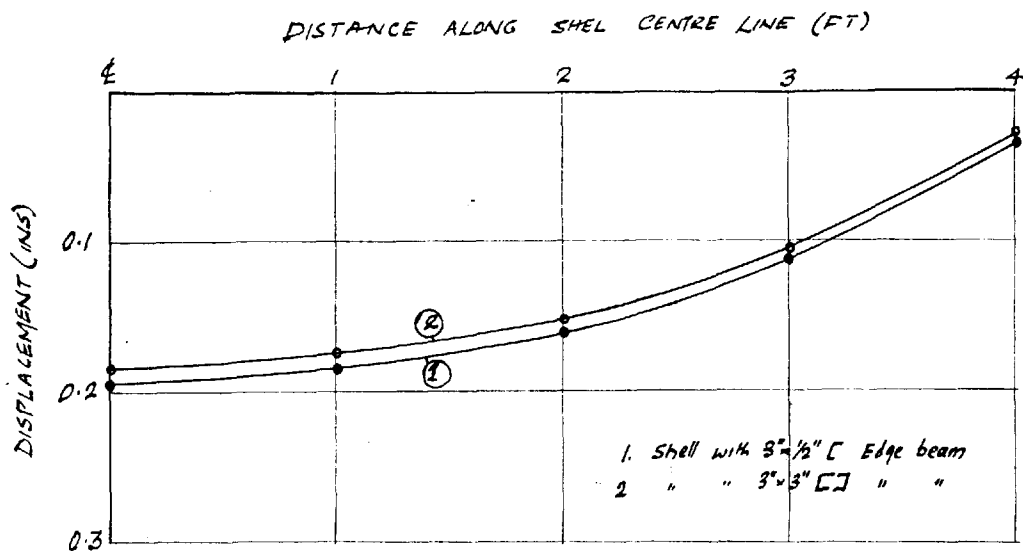


FIG 6.10 EFFECT OF EDGE BEAM STIFFNESS ON SHELL DEFORMATION (BATCHELOR 1963)

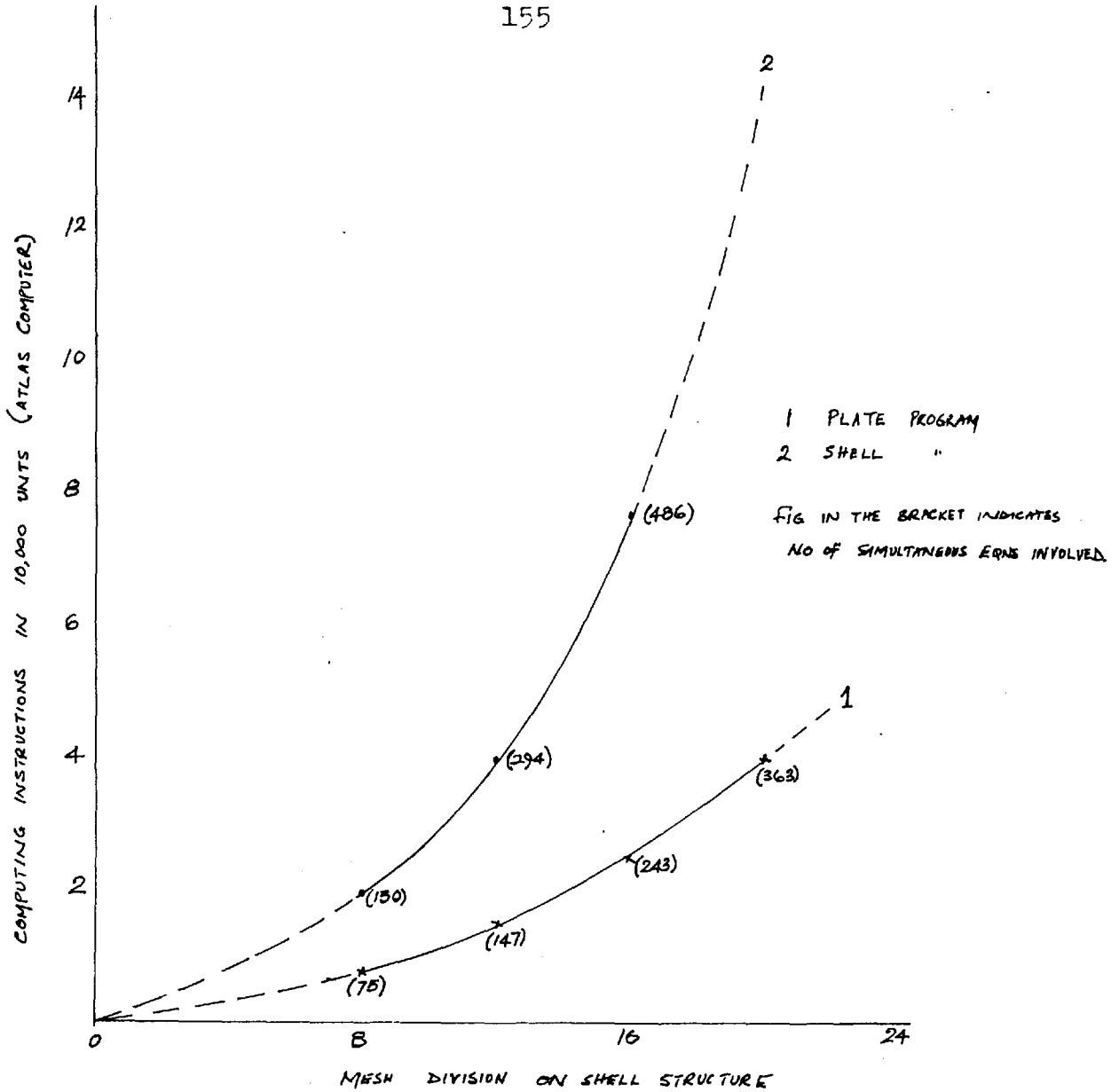


FIG. 6.11 COMPUTING INSTRUCTIONS (COST) AGAINST MESH DIVISIONS
FOR FINITE ELEMENT ANALYSIS

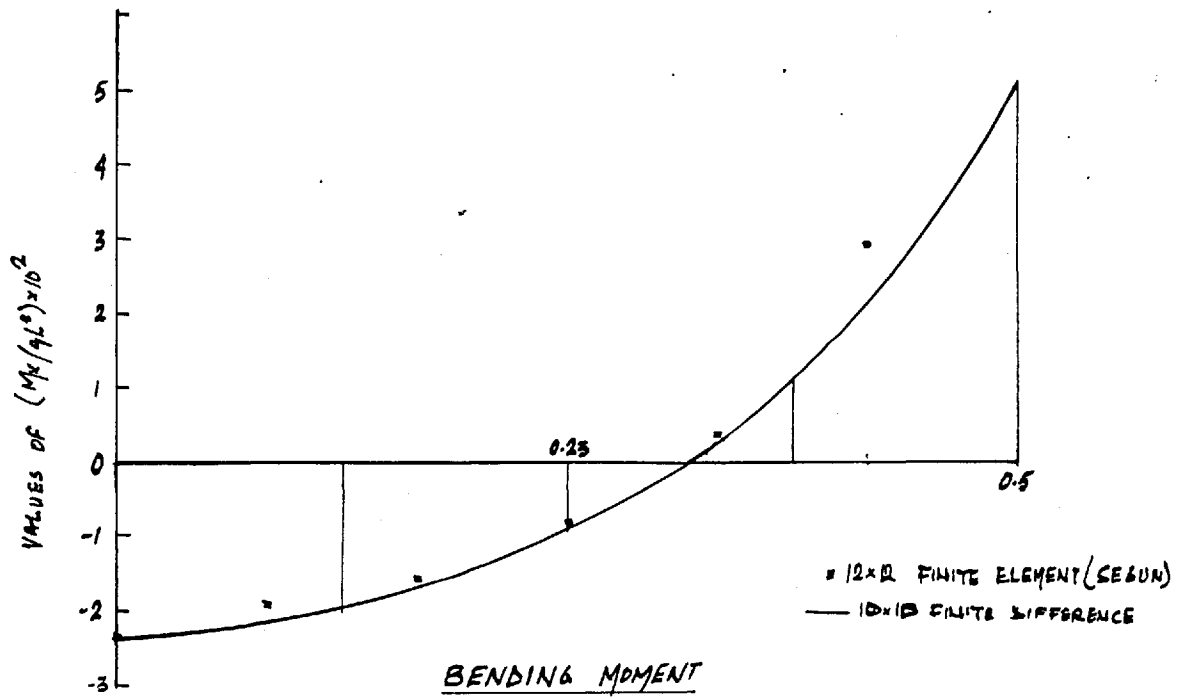


FIG. 6.12 CLAMPED SQUARE PLATE UNDER UNIFORM LOAD q

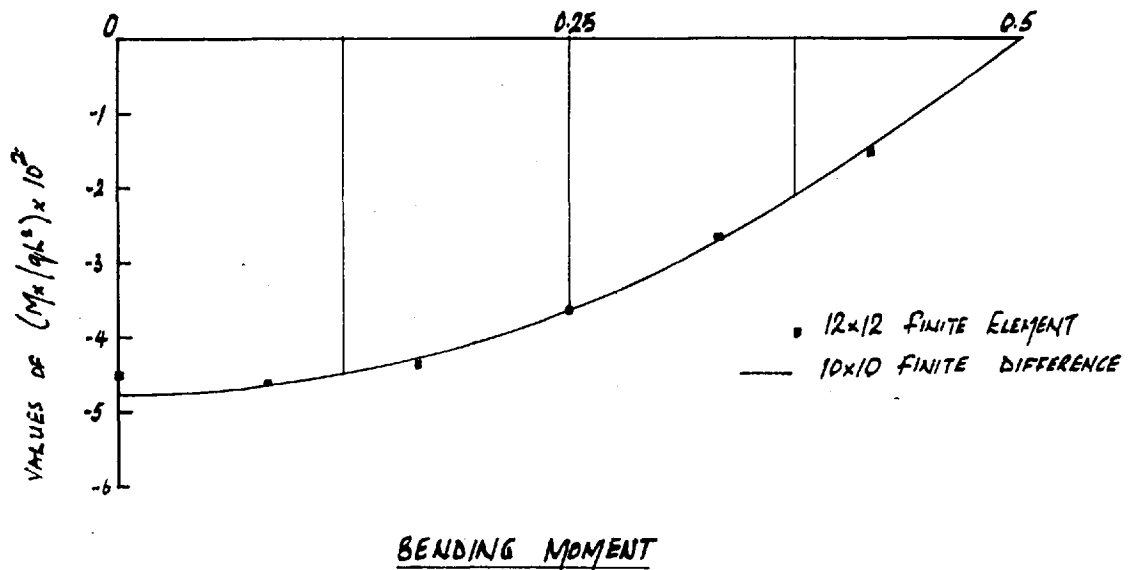


FIG. 6.13 SIMPLY SUPPORTED SQUARE PLATE UNDER UNIFORM LOAD q

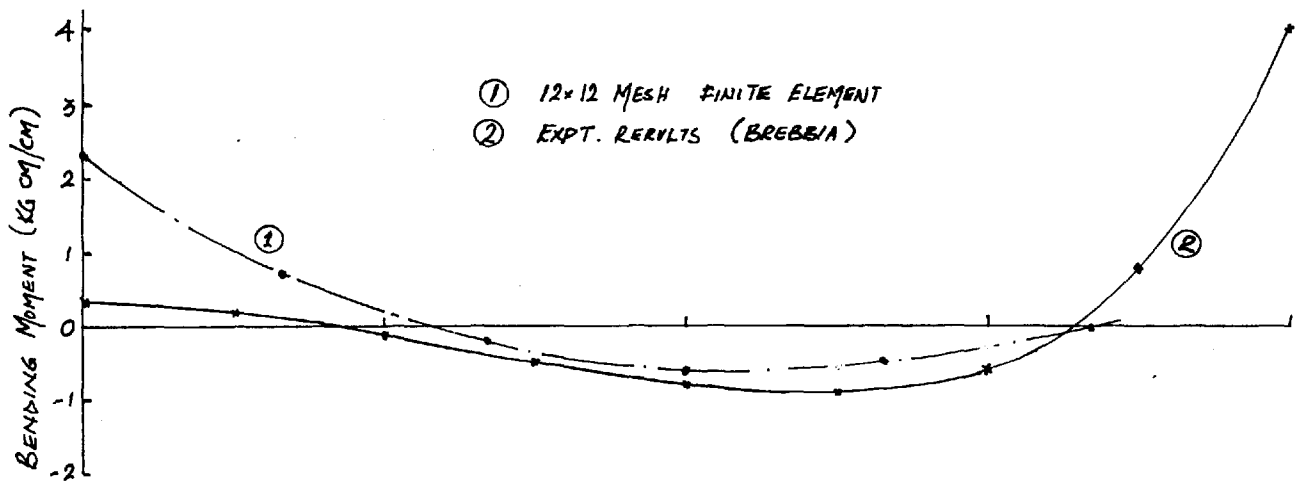


FIG 6.14 COMPARISON OF BENDING MOMENT AT CENTRE OF SHELL

PROBLEM 2

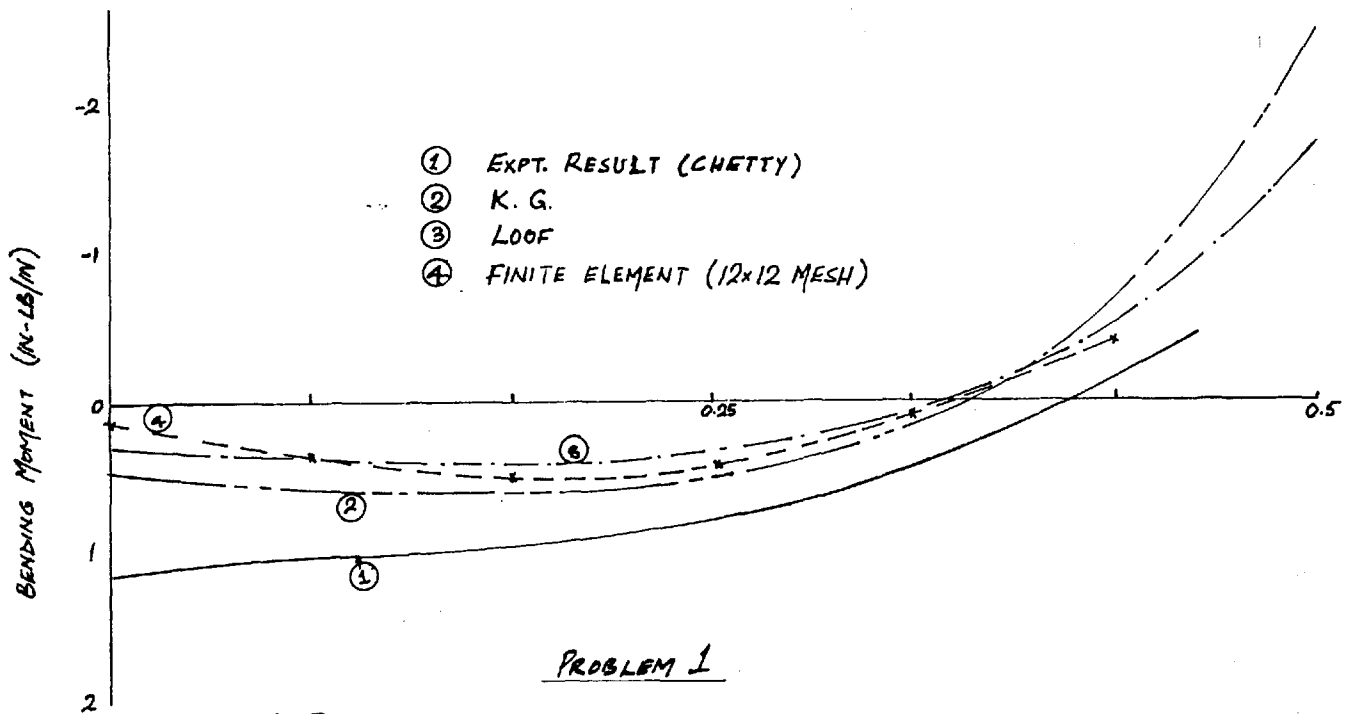


FIG 6.15 COMPARISON OF BENDING MOMENT AT CENTRE OF SHELL

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