

PHASE TRANSITIONS IN FIELD THEORY
USING REAL-TIME TECHNIQUES

by

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For my parents

ABSTRACT

We look at the combination of Quantum Field Theory and Equilibrium Thermodynamics. We concentrate on the newer real-time formalism from a path-integral approach, following Niemi and Semenoff, which leads to the same Feynman rules as in the operator approach called Thermo-Field Dynamics. This is to be compared with the older Imaginary-Time Formalism.

We discuss several problems with the real-time formalism as found in the literature; a) physical interpretation of results, b) singularities in tadpole Feynman diagrams, c) extraction of information from vacuum diagrams, d) understanding the free-energy calculations in the literature, e) lack of analyticity at zero momentum in non-zero temperature Feynman diagrams. The latter problem also occurs in the Imaginary-Time Formalism.

By deriving what in the literature is an ad-hoc extra Feynman rule, we provide a solution for problems b), c), and d).

Applications of the real-time formalism to phase transitions are considered, and shown to introduce no additional problems over the imaginary-time methods. The $O(N)$ model is used to test out the techniques, and the lack of analyticity at zero momentum is shown to lead to difficulties.

A more interesting and realistic phase transition is found with chiral symmetry in QCD. Various approximation schemes are considered, and particular attention is given to possible thermal gluon effects. These are shown to be more complicated and important than the literature on QCD at finite temperature suggests, and they lead to significant changes in the calculation.

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PREFACE

The work presented in this thesis was carried out in the Theoretical Particle Physics group of the Department of Physics, Imperial College, London, between October 1984 and September 1987, under the supervision of Doctor Ray J. Rivers. Unless otherwise stated, the work is original and has not previously been submitted for a degree in this or any other university. The problems (b), (c), and (d) of chapter 2 and the whole of chapter 3 have been published in *Zeitschrift für Physik* C36, 153, (1987). The ideas and results of chapter 5 form the contents of a joint paper with Dr. R.J. Rivers, while it is intended that a more detailed version of chapter 5 together with the contents of chapter 4 will be published separately under my name alone.

NOTATION AND CONVENTIONS

We set $\hbar=c=k=1$ (the Planck constant, the speed of light, and the Boltzman constant). Note that the inverse temperature, β , is actually $\hbar/kT$.

We follow the conventions of Cheng and Li [CL/app.A], which are also those of [IZ, BD], and set

$$g^{\mu\nu} = \text{Diag}(+1; -1, -1, -1)$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = \gamma_5$$

We will often choose a convenient basis for the γ matrices where $(\gamma^0)^+ = \gamma^0$, $(\gamma^i)^+ = -\gamma^i$, and $(\gamma^5)^+ = \gamma^5$, e.g. the Dirac basis [CL, BZ, BD].

For Feynman rules we follow Cheng and Li [CL/app.B], so that a Lagrangian of the form $L[\phi] = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$ leads to vertices of the form $-i\lambda \cdot (2\pi)^4 \cdot \delta^4(\sum_i p_i^\mu)$ where p_i^μ are the four-momenta flowing into the vertices, and the lines are $i(k^2 - m^2 + i\epsilon)^{-1}$ propagators ($=i\Delta$).

Abbreviations

CS Chiral Symmetry, or Chirally Symmetric.
 CSB Chiral Symmetry Breaking.
 ITF Imaginary Time Formalism.
 NJL Nambu and Jona-Lasinio, [NJL1,2]
 PIRT Path Integral Real Time formalism.
 SD Schwinger-Dyson (as in their equations).
 TFD Thermo-Field Dynamics.
 u,d,s up, down, and strange quarks.

Indices

a,b,... = 1,2, (3,4) TFD/PIRT thermal indices (only in PIRT
 can the indices run to 3 and 4 but this
 sector decouples and is often ignored).
 $\alpha, \beta, \dots$ Spinor indices, and in chapter 4 only
 they label the fundamental representation
 of $O(N)$.
 A,B,... = 1,5 Label the scalar(=1) or pseudo-scalar
 (=5) modes in the NJL model.
 $\mu, \nu, \dots$ = 0,1,2,3 Lorentz space-time indices.
 X,Y,... = 1,2,...(N^2-1) Vector indices of $SU(N)$ in chapter 5,
 e.g. τ^X $SU(N)$.
 i,j,... = 1,2,... N_f Flavour indices in the NJL model.

Colour indices are usually suppressed.

Notation

The most common use of these symbols is given below, though they may be used for other purposes, in which case it is specified by the accompanying text.

[..] literature reference.

(..) equation number, unless preceded by 'fig.'.

C curves in the complex time plane.

k^μ internal loop 4-momenta.

p^μ external 4-momenta.

$p^\mu=0$ calculation to be done putting all external 4-momenta to zero before doing the loop integrations.

$p^\mu \rightarrow 0$ calculation to be done putting all external 4-momenta to zero after doing the loop integrations.

T Temperature

α QCD coupling constant, related to Lagrangian coupling constant, g , by $g^2=4\pi\alpha$

β Inverse temperature $=1/T$

μ Chemical potential.

CHAPTER 1

Introduction

1) INTRODUCTION.

In this thesis, we investigate aspects of high energy particle physics in hot and dense situations. High energy particle physics is the study of elementary particles and fundamental forces. This requires that both quantum and relativistic effects should be considered, leading to the use of relativistic Quantum Field Theory. By hot and dense situations, we mean that there are too many particles for Field Theory to deal with on the typical scale of fundamental interactions ($\sim 1\text{fm}$, $\sim 10^{-23}\text{sec}$). To deal with this aspect we use Thermodynamics, where we look at average behaviour, and characterise it by a few gross thermodynamic variables such as temperature T , and chemical potential μ . The combination of these two approaches is called 'Finite-Temperature Field Theory'.

There are several areas where we need to apply such techniques. In present models of the early universe, space-time and matter/energy were very dense, and to understand how the universe as we see it today was formed, we must take into account both quantum and thermodynamic effects. Heavy ion collisions can also create, for significant space-time volumes, regions of very high energy and particle density where Finite-Temperature Field Theory may be able to make actual testable predictions. Solid state and optical physics can also apply the techniques of Finite-Temperature Field Theory, though often in a non-relativistic situation.

Phase transitions, marking a sharp change in properties of a substance, are particularly interesting to study and

often lead to dramatic tests of a theory. Finite Temperature Field Theory can be used to study transitions where quantum effects are very important, e.g. deconfinement transition of QCD, GUT transition in the early universe.

In looking at problems where large numbers of particles are relevant, we simplify the problem by concentrating on a few particles of interest, usually the ones which are carefully measured in some 'experiment'. The remainder, to good approximation for large numbers of particles, are treated statistically, and their averaged properties are represented by a heat bath (reservoir) with the correct gross thermodynamic properties. This introduces a few extra thermodynamic variables into the problem, compared with the same 'experiment' at zero temperature, i.e. with no background heat bath of particles. From the point of view of relativity, an important aspect of high energy physics, the presence of the heat bath has one other important effect. That is, it is usually said that the presence of the heat bath leads to a loss of Lorentz invariance at finite temperature. But we could look at an individual interaction in our hot and dense particle soup, concentrating on one interaction for such a short time (much less than the time between interactions) that the background, the heat bath, was irrelevant. We could then describe this using the well-tested Lorentz invariant theories of zero temperature high-energy physics, so it seems puzzling that on larger time scales, when the heat bath is important, Lorentz invariance disappears. We have to be rather more careful at finite temperature than just introducing the obvious thermodynamic variables and saying Lorentz invariance

is not present. Lorentz invariance is still there, and it is the choice of the rest frame of the heat bath to observe events from that confuses the issue.

Consider a large box at equilibrium at some fixed temperature. Then the box will be filled by a thermal gas of particles at equilibrium (in Bose-Einstein or Fermi-Dirac distributions if approximately non-interacting). This then forms the reservoir or heat bath. The photons in this heat bath will have a black body distribution, and this distribution is the same everywhere in the box, and defines the temperature. Now suppose we put a particle physics experiment inside the heat bath, fig.(1.1), which is a small perturbation from thermal equilibrium. This could be as simple as a single propagating electron. This has four-momentum p^μ , $p^2=m^2$, measured from the rest frame of the heat bath, where m is the rest mass of the electron when viewed in the zero temperature vacuum.

Now let's start to move the experiment with respect to the heat bath at a 4-velocity u^μ ($u^2=1$), and 3-velocity $\underline{u}$, fig.(1.1.ii). What the experiment 'sees' is that photons in the heat bath coming towards it are blue shifted, and photons coming from behind are red shifted as expected from special relativistic Doppler shift formulae. What is not so obvious is that all the photons in any one direction are shifted in such a way that the radiation still has the shape of black-body radiation [We4] so that in any one direction there is a well defined temperature, always given by the usual formula relating T and the wavelength of maximum intensity, for

instance.

So our experiment 'sees' different temperatures in different directions. It isn't clear how the experiment responds to the well defined (if complicated) temperature distribution it now sees when moving at speed $u=|\underline{u}|$, but there seems no reason to suppose it responds in the same way as it did when it was at rest with respect to the heat bath. So an observer fixed with respect to the heat bath sees different physics when the experiment is moving at different speeds with respect to the heat bath.

To an outside observer moving at some arbitrary speed $v=|\underline{v}|$ with respect to the heat bath, it now appears that even though the experimental variable p^2 is fixed, and the thermodynamic variables T, μ unchanged, the physics may well change. The electron in the simple case may have different masses at different speeds with respect to the heat bath. Galilean invariance is still respected (direction of $\underline{u}$ is irrelevant), but the 'simplistic' observer concludes that because the results now depend also on, say p_0^2 , that Lorentz invariance doesn't hold. However, the observer always sees the same physics occurring in the experiment, whatever his speed v with respect to the heat bath is, and this is what Lorentz invariance really tells you. There is nothing wrong in finding different physics if all that changes is the speed u of the experiment with respect to the heat bath.

The correct way to think about this situation is that we now have a system that consists of the experiment and the heat bath. Thus we have the internal variables of the experiment ($\{p^\mu\}$ say), the thermodynamic characteristics of the heat bath

(T, μ etc), but we should also put in the 4-velocity of the heat bath with respect to the experiment, u^μ . Everything is independent of the velocity of the observer with respect to the system (experiment plus heat bath). It is the experiment with respect to heat bath 4-velocity u^μ that is often forgotten. It means there are more Lorentz scalars that can be made up than was the case at zero temperature, $\{p^\mu u_\mu\}$, and when Lorentz vectors, symmetric tensors etc. are needed, there are many more possibilities as we shall find when we look at fermion and gluon full propagators.

What is normally done is that the rest frame of the experiment is chosen to coincide with the rest frame of the heat bath (the frame where only one temperature T is observed in all directions), $u_\mu = (1; \vec{0})$. This means we have p^2 and p_0^2 appearing in our formulae, obscuring the fact that really we have $p_0^2 = p \cdot u$, and that Lorentz invariance is still present.

This extra four-vector or the apparent loss of Lorentz invariance is a general property of relativistic finite temperature calculations. We now turn to considering the various Finite Temperature Field Theory formalisms in more detail. We are interested in time ordered products of fields as at zero temperature but now we want to take a thermal average, so we look at

$$\text{Tr} \{ \exp(-\beta(H - \mu N)) \cdot T_c \phi(x_1) \dots \phi(x_2) \} \quad (1.1)$$

where H is the Hamiltonian, N some conserved number operator with associated chemical potential μ , T_c is path ordering

along some as yet arbitrary curve C in the complex time plane, and $\phi(x)$ is the field. The trace is taken over all states. One can quickly put a limitation on the curve C . Looking at the two point function $\Pi(t)$ (for $\mu=0$),

$$\begin{aligned}\Pi(t) &= \text{Tr}\{e^{-\beta H} T_C \phi(t) \phi(0)\} \\ &= \Pi^>(t) \cdot \theta_C(t) + \Pi^<(t) \cdot \theta_C(-t) \quad (1.2)\end{aligned}$$

where $t \in C$ and is extended to complex values, and we have set $\mu=0$ and dropped $\tilde{x}$ for simplicity. Working in a complete set of energy eigenstates of the Hamiltonian, $H|n\rangle = E_n|n\rangle$, inserting unity, and using $\phi(t) = e^{iHt} \phi(0) e^{-iHt}$, we have

$$\begin{aligned}\Pi^>(t) &= \Pi^<(-t) \\ &= \sum_{m,n} \langle n | \phi(0) | m \rangle^2 \exp(-iE_n t) \exp(iE_m(t+i\beta)) \quad (1.3)\end{aligned}$$

Assuming the exponential dominates and demanding analyticity in the time variable, we must have that curve C has an imaginary part which is constant or decreasing in size. We are also limited to $-\beta < \text{Im}(t) < \beta$.

We now look to use this in a path integral formalism, where the difference between the possible formalisms is very clear. For simplicity we look at a single scalar field with the generating functional for finite-temperature expectation values of the path ordered ϕ products given by

$$Z[j] = \text{Tr}[\exp\{-\beta H\} \cdot T_C \exp\{i \int_C dt \int d^3x j(x)\phi(x)\}] \quad (1.4)$$

Using a complete set of states in the Heisenberg picture, defined at some time τ , $\{|\phi, \tau\rangle\}$, to represent the trace

$$Z[j] = \int D\phi \langle \phi, \tau | \exp\{-\beta H\} \cdot T_C \exp\{i \int_C d^4x j(x)\phi(x)\} | \phi, \tau \rangle \quad (1.5)$$

$$= \int D\phi \langle \phi, \tau - i\beta | T_C \exp\{i \int_C d^4x j \cdot \phi\} | \phi, \tau \rangle \quad (1.6)$$

Now if we want to use standard path integral manipulations, we want C running from τ to $\tau - i\beta$. We already know it must always have constant or decreasing imaginary part. We can use standard manipulations to arrive at

$$Z[j] = \int_{\tau \in C} D\phi(\tau) \exp\{i \int_C d^4x (L(\phi) + j\phi)\} \quad (1.7)$$

with boundary condition $+$ ($-$) sign for bosonic (fermionic) fields

$$\phi(\tau) = \pm \phi(\tau - i\beta) \quad (1.8)$$

If we were using fermionic fields, they would be anti-periodic because of the minus sign associated with the fermionic trace. For the example of scalar theories, we have

$$L(\phi) = 1/2 \left(-\frac{\partial^2}{\partial \tau_C^2} + \frac{\partial^2}{\partial x^2} + m^2 \right) \phi - V(\phi) \quad (1.9)$$

where $\frac{\partial}{\partial t_c}$ is taken along the curve C in a positive sense.

This leads to

$$Z[j] = \exp\left\{-i \int_C V\left[-i \frac{\partial}{\partial j}\right]\right\} \cdot \exp\left\{-\frac{i}{2} \iint_C j \Delta_c j\right\} \quad (1.10)$$

where some of the measures have been dropped for convenience, but all time integrals are along C, all spatial integrals are over some volume V.

The periodic boundary conditions of (1.8) lead to the Kubo-Martin-Schwinger [Ku, MS] boundary condition on Δ_c

$$\Delta_c^>(t-i\beta, \underline{x}) = \Delta_c^<(t, \underline{x}) \quad (1.11)$$

where we have made the ansatz

$$\Delta_c(x-y) = \Delta_c^>(x-y) \cdot \theta_c(x^0-y^0) + \Delta_c^<(x-y) \cdot \theta_c(y^0-x^0) \quad (1.12)$$

When one chooses the solution with the correct causal structure, equivalent to applying Feynman boundary conditions at zero temperature, the unique solution for Δ_c , after Fourier transforming the space coordinates with respect to $\underline{k}$, is

$$\begin{aligned} \Delta_c(\tau, \omega) = -i(2\omega(1-\exp\{-\beta\omega\}))^{-1} \cdot [& (e^{-i\omega\tau} + e^{-\beta\omega+i\omega\tau}) \theta_c(\tau) \\ & + (e^{+i\omega\tau} + e^{-\beta\omega-i\omega\tau}) \theta_c(-\tau)] \quad (1.13) \end{aligned}$$

where $\omega^2 = \underline{k}^2 + m^2$. We see that we have a forward propagating particle and a thermal distribution of particles and

anti-particles at large positive times, and a backwards propagating anti-particle and a thermal background of particles and anti-particles at large negative times.

We now look at the choices for the time curve C . The simplest is shown in fig.(1.2). This runs straight from τ to $\tau-i\beta$, and so is called Imaginary-Time Formalism, ITF [Ma1, Fe, LvW/2.3]. With the (anti-)periodicity of (fermionic) bosonic fields, we get Fourier series rather than transforms. The energies are defined only at discrete imaginary values, $p_0=2\pi n/\beta$ for bosons ($=\pi(2n+1)/\beta$ for fermions), n an integer (the full Feynman rules are given in app.A). As we take the zero temperature limit, we recover the Euclidianised version of field theory. In fact at all temperatures, we have to analytically continue to get real times and energy values out of the functions calculated in this scheme, which are only defined at discrete values. To define a unique analytic continuation, we must also account for the positive and negative large time boundary conditions.

We could avoid the analytic continuation if we had real times present in the formalism already, and this can be achieved if we run the curve C out along the real time axis, and then return it to $\tau-i\beta$. The simplest way is to run it back parallel to the real axis as shown in fig.(1.3). We would then take $\tau=-T\rightarrow\infty$ (T here is a time coordinate in fig.(1.3)) at the end to get all the real times we need. Such a formalism we call Real-Time Path-Integral formalism, PIRT [NS1,2, LvW/2.4, and see KK1, Ko for correction of sign errors]. We could in fact run back and forth many times as in fig.(1.4)., but as [MNU2] show, the extra trips cancel out,

and we may as well stick to just the curve of fig.(1.3).

Choosing C to be the usual curve of fig.(1.3), we apply the Riemann-Lebesgue lemma [Ti] and the usual adiabatic switching on of sources

$$\lim_{\text{Re}(\tau) \rightarrow \pm\infty} j(\tau) \rightarrow 0 \quad (1.14)$$

to equations (1.10), (1.13). This produces the factorisation of Z into two parts as

$$\Delta_c(x_a - y_b) \rightarrow 0 \quad \text{if } x_a \in C_1, C_2 \quad y_b \in C_3, C_4 \quad (1.15)$$

$$\therefore Z[j] = Z_{12}[j] \cdot Z_{34}[j] \quad (1.16)$$

where

$$Z_{12}[j] = \exp\left\{\int_{C_1 C_2} V\left[\frac{-i\delta}{\delta j}\right]\right\} \cdot \exp\left\{-i/2 \iint_{C_1 C_2} j \Delta_c j\right\}$$

and similarly $Z_{34}[j]$. To get the Feynman rules, that if we were to set $\sigma=1/2$ in fig.(1.3) would coincide with the usual TFD/PIRT rules, we define, noting the comments of [KK1]

$$j_1(x) = j(x) , \quad j_2(t, \underline{x}) = -j(t-i\beta, \underline{x})$$

$$\Delta_{ab}(x-y) = \Delta_c(\text{Re}(x-y)) \quad x \in C_a \quad y \in C_b \quad a, b = 1, 2 \quad (1.17)$$

The Feynman rules based on this choice of signs for the PIRT sources and propagator are given in app.A. Care is needed as

some earlier works have significant incorrect signs, even for scalar theories, as [KK1] point out. Even more care is needed with fermion fields, where more sign errors exist [Ko]. To get the physical thermal Green functions we use

$$G(x_1, \dots, x_N) = Z^{-1} \prod_{m=1}^N \left(-i \frac{\partial}{\partial j_1(x_m)} \right) \cdot Z[j] \Big|_{j_1=j_2=0} \quad (1.18)$$

It is this functional differentiation with respect to just the type 1 sources, the sources with real time arguments, that lead to Feynman rule, rule 2 of appendix A, for extracting the physical part of Green functions.

For later use we now derive the free energy, which is the non-zero temperature generalisation of the effective potential. Up to an irrelevant constant [A1]

$$Z[j] = \exp\{-\beta F[j]\} \quad F = \text{Helmholtz free energy} \quad (1.19)$$

$$Z_{12}[j] = \exp\{iW_{12}[j]\}$$

W_{12} is the generating functional of the 1/2 TFD connected Green functions.

The effective action Γ for the 1/2 sector is defined as

$$\Gamma[\bar{\phi}] = W_{12}[j] - \int_C d^4x \, j \bar{\phi} \quad \bar{\phi} = \frac{\partial W_{12}}{\partial j} \quad (1.20)$$

The corresponding effective potential is for space-time independent sources and fields

$$V(\bar{\phi}_1, \bar{\phi}_2) = \Gamma[\bar{\phi}_1, \bar{\phi}_2] / \int_{-\infty}^{\infty} d^4x \quad (1.21)$$

Defining a similar Legendre transform for $F[j]$ when we take $j(x)$ to be a space-time constant J

$$V.\omega(\phi) = F[J] - \int_C J\phi \quad (1.22)$$

Then we get the desired relation between the obvious generalisation of the effective potential in PIRT based on the 1/2 parts of C only, that is $V(\bar{\phi}_1, \bar{\phi}_2)$, and with the real physical quantity, the free energy $\omega(\bar{\phi})$, associated with the whole of curve C

$$\beta V.\omega(\bar{\phi}) = iV(\bar{\phi}_1, \bar{\phi}_2).VT - \ln Z_{34} \quad (1.23)$$

So far we have left σ ($0 \leq \sigma \leq 1$) of fig.(1.3) arbitrary. Varying σ generates an equivalence class of theories [MNUMM, MNU2]. The choice $\sigma=0$ leads to an old non-equilibrium formalism of Keldysh [Ke, LvW/2.1.3]. However, $\sigma=1/2$ leads to a very neat formalism. It also leads to Feynman rules identical to those found in the quite different operator approach developed by Umezawa and others, which is known as Thermo-Field Dynamics, TFD, [TU, Ma2, UMT, LvW/2.5]. TFD is related to C^* algebra approaches to Quantum Statistical Mechanics, as Ojima has shown [Oj2, LvW/2.5]. It is possible in TFD to mimic the σ parameter of PIRT, and then it is very

clear that the choice $\sigma=1/2$ leads to great simplifications in the TFD formalism [MNU2, LvW/2.5.6].

TFD and PIRT are not identical. The fields used are subtly different though there is a known relation between them [LvW/2.5.6]. TFD has no equivalent of the vertical parts C_3/C_4 thrown away in PIRT, and it appears that TFD can be applied in models where the adiabatic condition (1.14) can not be used! [LvW/2.4.1], e.g. the Anderson model [MU].

We will always choose $\sigma=1/2$ and assume that C , the time curve, is given by fig.(1.3). We will either be just be using the Feynman rules of TFD/PIRT, or where we have to go deeper, in chapters 2 and 3, we will use the path integral approach, though many of the problems discussed need to be resolved in the operator approach of TFD as well.

There is also an old real-time formalism due to Dolan and Jackiw [DJ]. This has fatal singularities beyond the one-loop level, and can now be seen as an incomplete part of the TFD/PIRT matrix Feynman rules, where only the type one fields are kept.

Lastly, the problems in this thesis were approached with a view to using the real-time approaches of TFD/PIRT. The application to phase transition calculations of chapters 4 and 5 consist only of calculation of 1 loop, 1- or 2-point functions. At this level, there is little difficulty with the analytic continuation needed in ITF to get to real energies, e.g. by using (A.4). Beyond this level, calculations of quantities involving real energies or times in ITF, e.g. transport coefficients, propagating plasma modes in quark-gluon plasmas, require difficult analytic continuations

to be done, so this is where TFD/PIRT could be expected to be far superior, if all its problems were understood. TFD/PIRT also offer a much clearer physical interpretation than does ITF, and it is part of the power of physics over pure mathematics that general physical understanding can lead to important advances, guiding the mathematical analysis that then follows. In this respect TFD/PIRT seems a great improvement. Also very interesting is the recent development of the operator TFD approach to cover non-equilibrium situations. Many real problems involve non-equilibrium thermodynamics, e.g. the evolution of the scalar field in inflationary models of the early universe. We confine ourselves to equilibrium situations in all the following discussions.

CHAPTER 2

Problems with Finite-Temperature Field Theories

2) PROBLEMS WITH FINITE-TEMPERATURE FIELD THEORIES.

When one comes to use TFD/PIRT in calculations a number of problems appear. In this chapter, we shall discuss five of the most significant of them. Some are confined just to these newer real-time formalisms (problems a,b,c, & d), while another (problem e) is also found in the corresponding ITF calculations, and so would appear to be a deeper problem.

Problem a:- Physical interpretation of TFD/ITF results.

In all the literature on TFD/PIRT it is the type 1 field that is considered to represent the physical degrees of freedom, and in the canonical operator approach of TFD it is actually called the physical field. The type 2 or tilde field then takes care of the thermally excited degrees of freedom. This means that whenever we are calculating physical quantities using Feynman diagrams, all the external legs or vertices should be type 1. This is included as part of the TFD/PIRT Feynman rules stated in appendix (A). This identification is particularly clear in the path integral formulation. The general fields used in the path integral have time arguments lying on curve C of fig (1.3) for PIRT. Type 1 fields are then defined to be that part of the general time field with time arguments on C_1 , i.e. real, physical times. It is one of the great advantages of TFD/PIRT that it allows this direct link to the real physics (unlike say ITF,

where difficult analytic continuations are needed).

This interpretation of the type 1 fields as being the physical degrees of freedom has a simple counter-example, provided by the spectral representations of two-point functions and explicitly calculated to lowest order in simple theories by Kobes and Semenoff [KS1,2]. These functions are two by two matrices in the double field formalism of TFD/PIRT, and have a neat diagonalised form. This shows that for a two point function, say Σ , the apparently independent complex functions $\Sigma_{11}(=-\Sigma_{22}^*)$ and $\Sigma_{12}(=-\Sigma_{21}^*)$ (where we have used the 1 $\leftrightarrow$ 2 symmetry of TFD/PIRT), can actually be expressed in terms of one complex function of momentum and thermodynamic variables, $\bar{\Sigma}$, and factors from the known entries of the Bogoliubov transformation matrices $U_{B/F}$ of TFD (see app.B). For the case of a scalar two point 1PI function and with temperature as the only thermodynamic variable, we have the relations

$$\text{Re}\{\Sigma_{11}\} = \text{Re}\{\bar{\Sigma}\}$$

$$\text{Im}\{\Sigma_{11}\} = \text{Im}\{\bar{\Sigma}\} \cdot \epsilon(p_0) \coth(\beta p_0/2) \quad (2a.1)$$

where Σ 's are a function of the energy p_0 , 3-momentum $\mathbf{p}$, and inverse temperature, β . These results are confirmed in explicit one-loop calculations of scalar two point 1PI functions (scalar or fermion loop), as are similar relations for fermion functions [KS1,2]. However, we find that it is not Σ_{11} but $\bar{\Sigma}$ that controls the behaviour of the full

propagator Π , as we have from the spectral representation, say for the scalar case,

$$\underline{\underline{\Pi}} = \underline{\underline{U}}_B \cdot \underline{\underline{\Pi}} \cdot \underline{\underline{U}}_B, \quad \underline{\underline{\Pi}} = \begin{pmatrix} \bar{\Pi} & 0 \\ 0 & -\bar{\Pi} \end{pmatrix}, \quad \bar{\Pi} = (p^2 - m^2 + \bar{\Sigma})^{-1} \quad (2a.2)$$

The pole of the propagator is a physical quantity, and is controlled by $\bar{\Sigma}$, not Σ_{11} . From relations (2a.1) we see that if we were to use the usual TFD/PIRT interpretation that the type 1 fields contain the physics, we would get the real part of the position of the full propagator pole right. The imaginary part of the pole's position, which corresponds to the thermalising time for the field (time for small fluctuations away from equilibrium to come to thermodynamic equilibrium), is incorrect if we use Σ_{11} as the physical object. Further, Kobes and Semenoff [KS1,2] show that the ITF calculation agrees with $\bar{\Sigma}$ and not Σ_{11} .

Perhaps we should take the 'core' functions, here represented using a bar on top, of TFD to contain the full physics, not the Green functions with all external vertices or legs set to be type 1, (though for two-point functions, relations such as (2a.1) allow the core functions to be extracted from the pure type-1 functions).

The problem would then be what was wrong with the original interpretation in both TFD and PIRT that said the type 1 fields were unambiguously the physical degrees of freedom? Worse, how do we extract the physics from $N(>2)$ -point functions in TFD/PIRT (e.g. the e^-/e^+ /photon vertex of QED that gives us $(g-2)$) where there is no known

neat diagonal form of the functions, and where, because of the difficulty of analytic continuation of $N(>2)$ -point functions or multi-loop diagrams in ITF, we really need a real-time formalism like TFD/PIRT.

For the purposes of chapters 4 & 5, the calculations are concerned with masses at finite temperature, and so it is the real part of the position of the pole in a two point function that is used, and from the relations (2a.1), there is no real problem.

However, U. Heinz (private communication) has pointed out that in ITF the relation between a retarded propagator, and a propagator calculated with Feynman boundary conditions, is of the same form as (2a.1). Further investigation of the role of large time boundary conditions in TFD/PIRT and in the analytic continuation of ITF is needed, e.g. see [FW].

Problem b:- Singularities in TFD/PIRT calculations.

When we try to calculate Feynman diagrams (app.A) in the old real-time formalism of Dolan and Jackiw [DJ] or in TFD/PIRT, we see they both suffer widely from apparent singularities of the form

$$\int d^4k [\delta(k^2-m^2)]^N f(k) \quad N>2 \quad (2b.1)$$

where $f(k)$ is non-zero near $k^2=m^2$.

These arise from the products of propagators carrying the

same momentum in a Feynman integral (e.g. the structure fig.(2.1) appearing in a diagram), as the propagators in both formalisms contain delta functions. There are two distinct situations where this occurs.

The first situation is when a part, P , of a diagram, D , has the structure of fig.(2.1). This leads to part of the expression for the whole Feynman diagram, whatever the spin of the field is, of the form

$$D = (2\pi)^{-4} \int d^4k P_{ab}(k) R_{ab}(k)$$

$$P_{ab}(k^\mu) = \sum_{c,d=1,2} \{ \Delta_{ac}(k) \Sigma_{cd}(k) \Delta_{db}(k) \} \quad (2b.2)$$

where $R_{ab}(k)$ is the rest of the diagram, $\Sigma(k)$ is the scalar two point 1PI function and the indices are the TFD thermal ones. Though in practice the indices will usually be summed over and the momentum integrated over when the whole diagram is considered, we will consider the general case of P for arbitrary but fixed momentum k^μ and fixed indices a,b . From app.A, we see that each of the four terms in the sum in (2b.2) contains a singularity of type (2b.1). In the old real-time formalism, which only contains the type-one fields, we would only have the term $a=b=c=d=1$, and in this case the singularity is fatal. However, the additional matrix structure of TFD with the usual rule 1 leads to the four terms in (2b.2) rather than the single term of the old real time formalism. We can now see how these singularities cancel in TFD (see appendices B of [NS2, MOU] for details). We consider a scalar field for

simplicity. The 1PI two-point function, Σ , and the propagator, Δ , can be written in a diagonalised form (see app.B)

$$\underline{\underline{\Sigma}}(k) = \underline{\underline{\tau}} \cdot \underline{\underline{U}}_B(|k^0|) \cdot \underline{\underline{\Sigma}}(k) \cdot \underline{\underline{U}}_B(|k^0|) \cdot \underline{\underline{\tau}}$$

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \bar{\Sigma} & 0 \\ 0 & -\bar{\Sigma}^* \end{pmatrix} \quad (2b.3)$$

$$\underline{\underline{\Delta}} = \underline{\underline{U}}_B(|k^0|) \cdot \underline{\underline{\bar{\Delta}}} \cdot \underline{\underline{U}}_B(|k^0|) \quad (2b.4)$$

$$\underline{\underline{\Delta}} = \begin{pmatrix} \bar{\Delta} & 0 \\ 0 & -\bar{\Delta}^* \end{pmatrix} \quad \bar{\Delta} = (k^2 - m^2 + i\epsilon)^{-1}$$

where the scalar Bogoliubov matrix, U_B , is given in (B.1), and satisfies

$$\underline{\underline{U}}_B \cdot \underline{\underline{\tau}} \cdot \underline{\underline{U}}_B = \underline{\underline{\tau}} \quad \underline{\underline{\tau}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2b.5)$$

Combining these gives

$$\mathbb{P}_{ab} = [\underline{\underline{U}}_B \cdot \underline{\underline{\mathbb{P}}} \cdot \underline{\underline{U}}_B]_{ab} \quad \underline{\underline{\mathbb{P}}} = \begin{pmatrix} \bar{\mathbb{P}} & 0 \\ 0 & -\bar{\mathbb{P}}^* \end{pmatrix}$$

$$\mathbb{P} = (\bar{\Delta})^2 \bar{\Sigma}$$

$$= \frac{\partial}{\partial m^2} (\bar{\Delta} \cdot \bar{\Sigma}) \quad (2b.6)$$

The U_B are independent of m^2 so we get

$$P_{ab} = \frac{\partial}{\partial m^2} [\underline{\underline{\Delta \cdot \Sigma}}]_{ab} \quad (2b.7)$$

We can now do the m^2 differential after all the integrals

$$D = \frac{\partial}{\partial m^2} (2\pi)^{-4} \int d^4 k (\underline{\underline{\Delta \cdot \Sigma}})_{ab} R_{ab}(k) \quad (2b.8)$$

and comparing (2b.8) with (2b.2), we see that the singularity has cancelled when the four terms of (2b.2) were combined. For fermionic fields or higher order singularities, there are similar mass derivative formulae as [FMOU] shows.

So far so good. However, there is a problem in that the singularities (2b.1) can occur in a second way, and that is when we have the external legs of a diagram carrying zero momentum. If we are considering diagrams which have a propagator associated with each external leg, then because for massive theories the matrix propagator is diagonal at zero momentum, the vertex to which it is attached will be of the same type, 1 or 2, as the field associated with that leg. By rule 2 (of the TFD/PIRT Feynman rules set out in app.A), this will be fixed to be type 1 when we select the physical part of the function, just as a diagram with no propagators associated with external legs, and with zero external momentum flowing along such external legs, would have that external leg vertex fixed to be type 1 by rule 2 (app.A).

In either case, we have part, P, of a Feynman diagram, D, looking like fig.(2.2), giving a contribution of

$$D = (2\pi)^{-4} \int d^4k \, P_{ab} R_{ab}$$

$$P_{ab}(k) = \Delta_{ac}(k) \cdot \tau_{cd} \cdot \Delta_{db}(k) \Big|_{\text{Subject to rule 2}}$$

$$= \Delta_{a1} \cdot \Delta_{1b} \tag{2b.9}$$

where $R_{ab}(k)$ is the rest of the diagram. Here the τ matrix gives the sign factor between type-1 and -2 vertices in TFD/PIRT. If it were not for rule 2 (app.A) fixing the external vertex ($c,d=1$) in contradiction to the usual rule for summing over vertex types ($c,d=1,2$), (2b.9) would be a special case of (2b.6) with $\bar{\Sigma}=1$, and the singularity associated with zero momentum legs would also be eliminated. As it is, due to the existence of rule 2 to extract the physical part of a function, there is only one term in (2b.9) and the singularity remains.

When there is only one external leg (tadpole diagrams) by considering what the remaining part of the diagram is, we can show that the singularity is killed. Tadpole diagrams we can regard as being of the form (2b.2) where now $a=b$, the thermal label of the external vertex, and $a=b=1$ by (rule 2, app.A). Therefore tadpole diagrams, which must have zero momentum carried by the leg, have no singularity.

However, any more than one zero-momentum leg on a diagram will produce a singularity coming from parts of the diagram of the type fig.(2.2) when the Feynman rules as derived in the literature are used.

The simplest example is the bubble diagram, $-iB_{ab}(p^\mu, T)$, fig.(2.3), where $a, b=1, 2$ are the thermal indices. Following the usual rules we select the physical part of interest, setting $a=b=1$. Being so simple, the bubble only has singularities of the type (2b.1) when the external four-momentum is exactly zero, i.e. the sort that are not cancelled by the matrix structure of TFD/PIRT. We have,

$$B_{11} = B_0 + B_\beta + B_{\beta 2}$$

$$-iB_0 = (2\pi)^{-4} \int d^4k \Delta_0(k^\mu) \cdot \Delta_0(p^\mu + k^\mu)$$

$$-iB_\beta = 2 \cdot (2\pi)^{-4} \int d^4k \Delta_0(k^\mu) \cdot \Delta_\beta(p^\mu + k^\mu)$$

$$-iB_{\beta 2} = (2\pi)^{-4} \int d^4k \Delta_0(k^\mu) \cdot \Delta_\beta(k^\mu + p^\mu)$$

$$\Delta_0(k^\mu) = \frac{i}{k^2 - m^2 + i\epsilon} \quad \Delta_\beta(k) = \frac{2\pi\delta(k^2 - m^2)}{\exp(\beta|k_0|) - 1} \quad (2b.10)$$

Clearly for $p^\mu \neq 0$ the bubble, B_{11} , is well defined, but when we set p^μ exactly zero before we do the integrals, $B_{\beta 2}$ blows up as

$$B_{\beta 2} = i\delta(0) \cdot (\text{real constant}) \quad (2b.11)$$

It is just this sort of singularity associated with the presence of zero-momentum external legs that remains in

TFD/PIRT when the Feynman rules, as previously derived and as stated in app.A, are used.

An additional rule is needed to eliminate these singularities in this situation, which then leaves TFD/PIRT completely free of singular terms like (2b.1), and the form of the additional rule is known from the comparison of TFD/PIRT and ITF effective potential calculations.

Additional Rule:

In any diagram with external legs carrying zero momentum, vertices associated with such external legs are not to be fixed as type 1 as Rule 2 states, but varied between type 1 and 2 as usual. The only exception to this is if it would leave no vertex unfixed (i.e. there are no external legs carrying non-zero momentum) when we fix any one vertex in the diagram.

This is an ad-hoc rule [Oj1, Fu2,(pc)], with the main justification coming from the agreement obtained between TFD/PIRT and ITF calculations.

In the example of the bubble diagram, the ad-hoc rule tells us that the physical part, $B_{\text{phys}}(p^\mu, T)$, at zero momentum has an extra term, B_{12}

$$B_{\text{phys}}(p^\mu, T) = B_{11} + B_{12}$$

$$-iB_{12} = -(2\pi)^{-4} \int d^4k \left\{ \frac{\exp(\beta |k^0|/2) \cdot 2\pi \delta(k^2 - m^2)}{\exp(\beta |k^0|) - 1} \right\}^2 \quad (2b.12)$$

where the change of sign for a TFD type two vertex is included in the expression for B_{12} . This extra term now removes the singularity in the same way as the matrix structure of TFD/PIRT deals with singularities (2b.1) arising from internal parts of a diagram, as [FMOU] shows, to give

$$-iB_{\text{phys}}(p^\mu=0) = -iB_0 + \frac{i\partial}{\partial m^2} (2\pi)^{-4} \int d^4k \Delta_0(k^\mu) \cdot \Delta_\beta(k^\mu) \quad (2b.13)$$

This is finite and agrees with the ITF calculation.

This ad-hoc rule is necessary to eliminate such singularities in diagrams with no external leg carrying non-zero momentum, and it is sufficient to remove the singularities when we have large complicated diagrams with external legs carrying both zero and non zero momentum [FG1,2, NS2, MOU].

For calculations of tadpole diagrams [NS1,2, FGK], this additional rule coincides with the usual rule for extracting the physical part, Rule 2 (app.A), and there is no problem here. For calculations of diagrams involving more than one external leg carrying zero momentum, (which, in the case of effective potential calculation scheme of [CW], is most diagrams), the extra rule represents an important change from Rule 2.

Problem c:- Vacuum Diagram Difficulties in TFD/PIRT.

For vacuum diagrams we have a different problem, for with no external legs to fix, these diagrams give zero when the standard rules of TFD/PIRT given in section 2 are applied. This arises because of a symmetry between the physical type 1 fields and the type 2 fields associated with the heat bath's degrees of freedom. Niemi and Semenoff [NS1,2, LvW/2.4.44] show that this is a symmetry in the effective Lagrangian for the doubled fields of TFD.

$$L(\phi_1, \phi_2) = -L^*(\phi_2^*, \phi_1^*) \quad (2c.1)$$

This means that if we change all the vertex types in a diagram to the other type ($1 \leftrightarrow 2$) and add the new to the old diagram, then we get twice the imaginary part of the diagram [KS1,2]. For vacuum diagrams, which must be purely real, this tells us that if we use Rule 1 (app.A), we will sum over all possible vertex labellings, none are fixed by rule 2 (app.A), and we will get zero. However, if we were to fix any one vertex to be only type 1 (or 2) on any vacuum diagram (a departure from the usual rules), it would give us a non zero result. Further, calculations of the effective potential done using the vacuum method in TFD using this new rule [FGK, MNU] then give agreement with the usual ITF result. This has left us with an ad-hoc rule for vacuum diagrams. The wording of the ad-hoc rule needed to eliminate singularities in non vacuum diagrams is such that it covers the extra rule needed for vacuum diagrams as well. The derivation of this

additional ad-hoc rule for vacuum diagrams constitutes problem (c).

With regard to this problem, Matsumoto, Nakano and Umezawa [MNU] prove the ad-hoc rule needed for vacuum diagrams by using a small trick, but only in the context of an effective potential calculation, and only in the canonical TFD approach. Because the vacuum and tadpole methods are closely related, [MNU] also provides proof for the extra rule needed in the tadpole case, but there is no direct way of seeing how the summing all the 1PI zero external momenta diagrams method [CW] will work.

Problem d:- The Path Integral and the Effective Potential.

When we look at the non-zero temperature generalisation of the effective potential, the free energy, we find that the derivation in PIRT provided by Niemi and Semenoff [NS1,2] has a number of problems. This is in addition to the fact that we use vacuum or zero momentum diagrams to calculate the effective potential, and these diagrams are singular without the additional ad-hoc rule, as set out in problems b and c.

We can define the effective potential, $V(\phi_1, \phi_2)$, as the zero momentum term in a momentum expansion of the effective action. We then find that because the effective action is a generating functional for all the possible 1PI diagrams (which will include in TFD all possible vertex labellings), the effective potential, being the zero momentum external leg limit of these diagrams, will always be zero because of the 1-2 symmetry (2c.1) whatever calculational

scheme we use. This includes the zero temperature limit of TFD calculations! This definition must be incomplete, and yet the desired result is easily extracted from it, as the results (though not the reasoning) of Niemi and Semenoff [NS1,2] show. To extract the ITF result from the TFD/PIRT calculation, we need to fix just one vertex in each diagram of any of the methods, and this is what the extra ad-hoc rule used on the zero momentum diagrams involved is doing.

Further, when we consider the real physical object of interest at finite temperature, the Free Energy (which is usually taken to be identical to the zero-momentum term in a derivative expansion of the effective action), from the path integral viewpoint of Niemi and Semenoff, we find more problems. They show that the free energy density, ω , can be expressed as (see (1.23))

$$\omega(\phi_1, \phi_2) = -iV(\phi_1, \phi_2) \cdot T\beta^{-1} + (\beta V)^{-1} \cdot \ln(Z_{34}) \quad (2d.1)$$

where $V(\phi_1, \phi_2)$ is the effective potential defined as the zero momentum term in a derivative expansion of the TFD effective action, $V = \int d^3x$, T = length of C_1 and C_2 curves, and Z_{34} = part of partition function from C_3 and C_4 . C_a are parts of the PIRT time curve C (see fig.(1.3)).

It seems surprising from this that $V(\phi_1, \phi_2)$ can give any useful information, as it only gives imaginary contributions to the free energy. Also, since this $V(\phi_1, \phi_2)$ is exactly zero, it is the contribution from the vertical parts of the TFD contour, Z_{34} , that give the whole of the free energy. Yet

by fixing just one vertex (in contradiction with the usual TFD rules) in any of the effective potential calculation schemes, and so doing half of the calculations needed for $V(\phi_1, \phi_2)$, the ITF result is reproduced using only the 1 and 2 parts of the PIRT contour. So, understanding the free energy from a path integral viewpoint, i.e. equation (2d.1), is problem (d).

Using the approach of Matsumoto et al. [MNU] to the additional rules needed for vacuum diagram effective potential calculations, there is also no way of seeing how to explain (2d.1), why it is only the C_3 and C_4 parts of the contour that contribute to the free energy, and yet by fixing one vertex the correct result can be obtained.

The reason for the effective potential of Niemi and Semenoff being zero is seen from the approach of Matsumoto et al. It is zero because it is the energy density of the physical fields minus the energy density of the unphysical fields (i.e. the thermal degrees of freedom of the scalar field excited by the heat bath), which must be zero because TFD assumes thermal equilibrium. Ojima has also considered the problem [Oj1].

Problem e:- The Zero Momentum Limit.

The calculations in problems b,c, and d above, involve taking the limit of a space-time independent source before the integrations are done, i.e. setting $\{p^\mu\}=0$ on external legs of a diagram. This approach will be denoted by $p^\mu=0$. There is an obvious way of avoiding these problems in zero-momentum

diagrams. The alternative procedure, denoted by $p^\mu \rightarrow 0$, is that we calculate the diagram with non-zero external leg momenta, and then take the $\{p^\mu\} \rightarrow 0$ limit after all the loop momentum integrations have been performed.

However, we now see that this limit has its own problems for massive theories, independent of any physical context. At zero temperature with Lorentz invariance, the two methods give the same answer. At finite temperature, where we have lost Lorentz invariance, the two methods don't always agree, as pointed out by Fujimoto [Ful] (though this preprint contains calculational errors, it does contain the essence of this problem, and both Fujimoto [F(pc)] and Umezawa [Um(pc)] are aware of this problem). The problem can only occur where there is at least one variable external momentum; thus the problem only occurs for N-point functions, $N \geq 2$, and in particular does not affect the vacuum and tadpole diagram effective potential calculations. It would effect the approach to the effective potential where all the 1PI zero momentum graphs are summed.

We concentrate on the simplest case, the one loop, two point 1PI bubble diagram in a massive scalar theory. Because of the 'loss of Lorentz invariance at finite temperature' (see discussion in chapter 1), we find there are a series of limiting procedures $p^2 \rightarrow 0$, characterised by different values of $v = |\underline{v}|$, where

$$\underline{v} = \underline{p} / p^0 \quad (2e.1)$$

Only one value of v , $v=\infty$, gives the same result as the $p^\mu=0$ calculation. Generalising the calculations of Fujimoto [Fu1,(pc)], which correspond to $v=0$ and $v=\infty$, the other values of v give a whole range of results for the bubble, including some imaginary values. We define the bubble B , fig.(2.3), by

$$-iB_{11}(p^\mu) = \int d^4k [i\Delta(k^\mu) \cdot i\Delta(p^\mu+k^\mu)]_{11} \quad (2e.2)$$

where Δ is the TFD matrix propagator. Like the self energy function (app.B), this can be written as

$$-iB(p^\mu) = U_B(|p^0|) \cdot -i\bar{B}(p^\mu) \cdot U_B(|p^0|)$$

$$\bar{B} = \begin{pmatrix} \bar{B} & 0 \\ 0 & -\bar{B}^* \end{pmatrix} \quad (2e.3)$$

All the temperature dependence is contained in $B_\beta(p^\mu) = B(p^\mu;\beta) - B(p^\mu;\beta=\infty)$, as in (2b.10). The remainder of B is then the zero-temperature part, which doesn't suffer from problems of lack of analyticity at zero momentum.

Now consider $p^\mu=0$ and $p^\mu \rightarrow 0$ limits separately.

(i) $p^\mu=0$

Put the external leg momenta to zero before integrating. The ITF calculation is no problem, as $p^\mu=0$ is one of the allowed values $p^0=2\pi n/\beta$, n integer, though this would not be true if we were using fermions. The TFD calculation involves

the use of the ad-hoc additional rule of problem (b), which is derived in chapter 3. The result is the same in both cases,

$$B_{\beta}(\text{TFD}; p^{\mu}=0) = B(p^{\mu}=0) = F_2(\beta m) / 4\pi^2 \quad (2e.4)$$

with

$$F_2^{\pm}(b) = \int_1^{\infty} dx (\exp(bx) \pm 1)^{-1} \cdot (x^2 - 1)^{-1/2} \quad (2e.5)$$

There is no v dependence because we took $p^{\mu}=0$ before the integration, and the integrand of (2e.2) is analytic at zero external momentum at $T \neq 0$, unlike the final result! (equation (2e.8) vs. (2e.4)).

(ii) $p^{\mu} \rightarrow 0$

In this approach, we leave arbitrary momenta on external legs, do the loop integrations, and only then do we take $p \rightarrow 0$.

$$\text{i.e.} \quad B(p^{\mu} \rightarrow 0) = \lim_{p^{\mu} \rightarrow 0} (B(p^{\mu})) \quad (2e.6)$$

ITF involves an analytic continuation from the discrete set of imaginary values that p^0 is defined for. This is done using the usual method (see app.A). An advantage of TFD is that it avoids this complication. At this point problem (d) appears, in that TFD contains the ITF result, but has an extra term, $B_{\beta 2}$ of (2b.10).

$$B(\text{TFD}; p^\mu) = B(\text{ITF}; p^\mu) + \frac{i}{4\pi^2} \int d^4k \frac{\delta(k^2 - m^2)}{e^{-\beta|k_0|} - 1} \cdot \frac{\delta((p+k)^2 - m^2)}{e^{-i\beta|p_0+k_0|} - 1} \quad (2e.7)$$

The latter term in the TFD calculation comes from the product of Δ_{11} delta functions and corresponds to the pure imaginary difference between Σ_{11} and $\bar{\Sigma}$ ($B \in \Sigma$). We see that we can extract the ITF result, which is the same as $\bar{\Sigma}$ of TFD, from the Σ_{11} of the full TFD calculation by ignoring such extra terms, or by using the exact relation between $\bar{\Sigma}$ and Σ_{11} given in (2a.1), (see [KS1,2]). The result is

$$\bar{B}(\text{TFD}; p^\mu=0, v) = B(\text{ITF}; p^\mu \rightarrow 0; v) = G^-(v) / 4\pi^2 \quad (2e.8)$$

$$G^\pm(v) = \int_1^\infty dx \frac{1}{\exp(\beta x) \pm 1} \cdot \frac{(x^2 - 1)^{1/2}}{x^2 + v^2 \gamma^2} \quad \gamma^2 = 1/(1 - v^2) \quad (2e.9)$$

where we find that $G^\pm(v)$ has the properties

$$G^\pm(v=0) = \int_1^\infty dx \frac{1}{\exp(bx) \pm 1} \cdot \frac{(x^2 - 1)^{1/2}}{x^2} \quad (2e.10)$$

$$G^\pm(v=1) = 0 \quad (2e.11)$$

$$G^\pm(1 < v < \infty) \in \mathbb{C} \quad \text{see fig. (2.4), (2.5)} \quad (2e.12)$$

$$G^\pm(v=\infty) = F_2^\pm(b) \quad (b = \beta m) \quad (2e.13)$$

where v is defined in (2e.1). We now see that the result of the bubble integral is not analytic at zero external momentum, depending on the way we approach the zero parameterised by v . This in contrast with the analyticity of the integrand, seen in the lack of v dependence of the $p^\mu=0$ limit. In particular

$$\text{Re}\{B(p^\mu=0)\} = \text{Re}\{B(p^\mu \rightarrow 0; v=\infty)\} = F_2(\beta m)/4\pi^2 \quad (2e.14)$$

where B stands for an ITF or a TFD calculation. We see that the value $v=\infty$ is the only limiting procedure that gives the usual zero momentum result.

Since the bubble B is the simplest diagram where this problem occurs, it seems as if the lack of analyticity is a more general phenomenon of non-zero temperature field theory. Indeed, it appears that 3- and 4-leg one-loop diagrams have this problem $[Fu(pc)]$.

This lack of analyticity at zero external momentum has physical implications. The bubble, for instance, is the lowest order correction to Σ (2-point, 1PI function) in some ϕ^3 theory, and Σ can be related to the physical mass by looking at the full propagator Π , $\Pi=(p^2-m^2+\Sigma)^{-1}$. Suppose we were looking for a second order phase transition which would be signalled by the appearance of long-range forces, a massless field, say ϕ . This could be done by looking for a flattening of the effective potential, whose second derivative with respect to the ϕ is a mass defined at a momentum renormalisation point of $p^2=0$. We'll assume, that as in almost all of the literature, such calculations would be

performed by putting the external momenta to zero before doing any integration, $p^\mu=0$ approach, and thus would depend only on temperature T , and not on v . This mass we can call $m(\text{eff};T)$ and is defined by

$$\Sigma(p^2=0,T) = m^2(\text{eff};T) \quad (2e.15)$$

Alternatively a physical definition of mass $m(\text{phys};v,T)$ involves the momentum renormalisation point of $p^2=m^2(\text{phys};v,T)$ and is defined by

$$\Sigma(p^2=m^2(\text{phys};v,T),v,T) = m^2(\text{phys};v,T) \quad (2e.16)$$

In this second case, as we let the temperature T reach the second order phase transition, we naturally use a $p^2 \rightarrow 0$ approach to the zero momentum diagrams, but because of the lack of analyticity, we find we still have v dependence as we reach the critical temperature. Thus, looking at the solution to $m(\text{phys};v,T_c)=0$, tells us that the critical temperature T_c will be v dependent! So we get one clear answer for the transition temperature from an effective potential approach, but many answers from a physical mass definition. Thus we see that a lack of analyticity at zero momentum can have important physical implications.

A more general problem is illuminated in this discussion in that the effective potential, a Free energy, is intimately related to the zero momentum limit, as it is associated with space-time independent potentials. It can be thought of as the 'zeroth' order term in a derivative expansion of the

effective action. If the effective action is not analytic at zero momentum at non-zero temperature, what is the physical quantity?

The problem of the lack of analyticity at zero momentum in finite temperature field theory Feynman diagrams has only been discussed for the simplest 2-point function. Fujimoto [Fu(pc)] has found similar problems with one-loop 3- and 4-point scalar diagrams. However, the problems of interpretation of zero external momentum Feynman diagrams (do we use a $p^2=0$ approach, or $p^\mu \rightarrow 0$ when we must then interpret the extra variables c.f. v for 2-point functions?) have been completely ignored in the literature. The uncertainty in the physical meaning of a term, the effective potential, in an expansion of a functional, the effective action, about a point where it appears not to be analytic at non-zero temperature, is in direct contrast to the confidence placed in it in the literature.

CHAPTER 3

Zero-Momentum Diagrams

3) ZERO-MOMENTUM DIAGRAMS

Having stated some of the problems involved with finite temperature field theory as found in the literature, we now look at resolving some of these problems. In this chapter, we show it is possible to solve problems (b), (c), and (d) of chapter 2 using the path-integral approach to real-time formalisms. Specifically, the additional, though ad-hoc rule introduced in several references [FG1,2, FGK, MNU] and stated in chapter 2, is derived. This rule is needed to calculate $p^\mu=0$ diagrams and to obtain agreement with previous ITF calculations. This chapter derives this ad-hoc rule using the path integral rather than the TFD operator formalism, and uses it to understand the PIRT effective potential calculations, especially of Niemi and Semenoff, problem (d). However, why there should be a sudden discontinuity between $p^\mu=0$ and $p^\mu \rightarrow 0$ Feynman rules, and how we understand the lack of analyticity in non-zero temperature Feynman diagrams, problem (e), remains unsolved and much neglected.

In the discussion of problem (d), it was clear that in calculating the effective potential one must consider the whole of the time curve C , fig.(1.3), of PIRT (see (2d.1) and its derivation in chapter 1), and not just the C_1 and C_2 parts. We shall now take this idea further by applying it to individual vacuum and zero momentum external leg diagrams, independent of a particular physical context.

Consider a pure scalar theory with Lagrangian

$$L[\phi] = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - U(\phi) \quad (3.1)$$

The relevant generating functional is

$$Z[j] := \text{Tr} \{ \exp(-\beta H) \cdot T_C \exp(-\int_C dt \int d^3 \underline{x} j(\underline{x}, t) \phi(\underline{x}, t)) \} \quad (3.2)$$

$$Z[j] = \int_{\text{periodic}} D\phi \exp \left\{ -i \int_C dt \int d^3 \underline{x} (L[\phi] - j(\underline{x}) \phi(\underline{x})) \right\} \quad (3.3)$$

$$= \exp\{iW[j]\} \quad (3.4)$$

Standard manipulations show that $W[j]$ is the generating functional for the connected Green functions for a theory with Lagrangian L .

Now consider the space-time Feynman diagrams. The only new aspects at finite temperature are generalisations of the time integrals to run along the curve C , and a new propagator Δ_C . This satisfies the Klein-Gordon equation with time coordinate on C and periodic boundary conditions in the imaginary time direction. We also demand that a free theory give the Bose-Einstein occupation density. Take any one connected diagram, say D , fig.(3.1)

$$D = \prod_{m=1}^V \left(\int_C dt \int d^3 x \right) \cdot \prod_{\{n,p\}}^I (\Delta_C(x_n - x_p)) \cdot \prod_{r=1}^N j(x_r) \quad (3.5)$$

where diagram D has V vertices with coordinates $\{x^\mu\}$, I propagators between vertices n and p , and N sources.

Now do the integrals associated with any one vertex A, coordinates $\{x_a\}$, last.

$$D[j] = \int_C dt_a \int d^3 x_a (D_A[j; x_a]) \quad (3.6)$$

$$D_A[j; x_a] = \prod_{m=1}^{V-1} \left(\int_C dt_m \int d^3 x_m \right) \cdot \prod (\Delta_c(x_n - x_p)) \cdot \prod_{r=1}^N j(x_r) \quad (3.7)$$

D_A is a function of x_a , the coordinate of the fixed vertex A. Now shift all coordinates by a constant X^μ .

$$y_m^\mu = x_m^\mu + X^\mu \quad m = 1 \text{ to } V \quad (3.8)$$

The time integrals are now over a curve C shifted by X. However, the precise location of the curve is arbitrary; the results are independent of the value of τ in fig.(1.3). So we can run the new coordinates over the old curve, i.e.

$$\int_C dx^0 \equiv \int_{C-X^0} dx^0 = \int_C dy^0 \quad (3.9)$$

The propagator is also independent of such shifts

$$\Delta_c(x_1, x_2) = \Delta_c(x_1 - x_2) = \Delta_c(y_1 - y_2) \quad (3.10)$$

However, the sources are not necessarily independent of this shift. Only when we have space-time independent sources do we get D_A independent of vertex A's coordinates.

$$\begin{aligned} \therefore D_A[J; x_a] &= \prod_{m=1}^{V-1} \left(\int_C d^4 y_m \right) \cdot \prod^I (\Delta_C(y_n - y_p)) \cdot J^N = D_A[J; y_a] \quad (3.11) \\ &= D_A[J] \quad \text{say} \end{aligned}$$

where $j(x) = J$ is now a space-time independent source.

Therefore D_A is space-time independent, and we only need to calculate the Feynman diagrams which have one of their vertex coordinates fixed at any value, in order to calculate a diagram with space-time independent sources, corresponding to no external legs carrying non-zero momentum.

$$D[J] = D_A[J] \cdot \int_C d^4 x \quad (3.12)$$

This is clearly just the factoring out of the space-time volume element present when the sources are space-time independent.

What does this mean in terms of calculations with momentum space TFD Feynman rules? To see this, nominally associate D_A with a coordinate x_a confined to lie on just one portion of the curve C , say C_a , and reinstate the space-time integral for this variable but with time just running along C_a .

$$D_A = \frac{\int_{C_a} d^4 x_a (D_A(J; x_a))}{\int_{C_a} d^4 x} \quad a=1, 2, 3, \text{ or } 4. \quad (3.13)$$

Split all the other vertex coordinate integrals into a

sum of integrals over each of the four parts of C.

$$\int_C d^4 x_v = \sum_{v=1}^4 \int_{C_v} d^4 x_v \quad (3.14)$$

Take out the summations and leave these till last. The integrands now consist of coordinates confined to single parts of the C curve, allowing us to use the usual matrix representation of the propagator.

$$\Delta_c(x-y) = \Delta_{ab}(\text{Re}(x-y)) \quad x \in C_a, y \in C_b \quad (3.15)$$

It is the summations in (3.14) that lead to the usual TFD/PIRT sum of diagrams, with all vertices except external ones varying between type 1 and 2, as the 3-4 coordinates decouple from the 1-2 coordinates. When we consider general Green functions, we have taken functional derivatives of a generating functional with respect to the physical, i.e. type 1 sources only, (1.18), and this is what fixes the external legs to be type 1 in the usual Rule 2. However, here in calculating D_A , there has been no such differentiation, so all the vertices are summed over except vertex A, which we have restrained to lie on C_a , a fixed, in (3.14). So we have a diagram which has just one vertex of fixed type (but with its integral) and all the others varying over all possible types.

$$D = D_A \cdot \int_C d^4 x = \frac{\beta V}{\int_{C_a} d^4 x} \cdot D(\text{new rule}; a) \quad (3.16)$$

where the new rule is that the Feynman diagram, coordinate or momentum space, is to have one vertex of fixed type and all the others are to be varied.

The formula applies to any calculation of a connected or 1PI diagram with no non-zero momentum external legs, whatever the physical context. The manipulation of (3.12) into (3.13) was performed merely to aid the calculation of the diagram in TFD, and should not overshadow the key observation of this chapter, which is seen in comparing (3.6) with (3.12). While these manipulations work for any curve C , they are not always useful, e.g. ITF gives the simplest Feynman rules when curve C is not split up, making (3.16) trivial (no change to calculation), and showing why ITF presents no problems in such calculations. The new rule coincides with the ad-hoc rule, when we remember that the 1/2 and the 3/4 sectors decouple (see the propagator, app.A). As shown in problem (b), this ad-hoc rule is needed to eliminate the singularities in zero-momentum diagrams, and for extracting non-zero results from vacuum diagrams, and its derivation solves problems (b) and (c).

The formula also shows that the ends of the TFD contour C , which are usually discarded, are as important as the C_1 - C_2 parts, and that we could equally well have fixed the vertex to be type 3 or 4, and used the 3/4 propagators to calculate the diagram. It is only for ease of calculation that we use the C_1 and C_2 sector, and they all differ only by a factor of $\int_C d^4x$. That this is so is not surprising because the $p^0=0$ limit that we are looking at probes the whole of the time plane, as it corresponds to time independence. The formula

will also apply to calculations of connected diagrams with no external leg carrying non-zero energy components of their momentum four vectors, i.e. all $p^0=0$, as the space independence, $p=0$, corresponds to the universal factors of $\int d^3x=V$.

We now apply this approach to the problem with free energy calculations in PIRT, problem (d). Consider this rule for the vacuum and zero momentum connected diagrams within the specific context of the Helmholtz free energy calculation, using the definition via the partition function

$$Z[j] = \exp\{-\beta F[j]\} \quad (3.17)$$

Note that this partition function is the same as the generating functional of (3.2) up to an irrelevant constant that just shifts the energy origin and leaves the entropy unchanged [A1].

Looking at the free energy density, we have

$$f(J) \cdot \int d^3x = F[J] \quad (3.18)$$

$$f(j) = \frac{iW[J]}{\beta V} \quad (3.19)$$

W is just the generating functional for connected diagrams, i.e.

$$W = \sum_{\text{topologies}} D \quad (3.20)$$

and we can also fix one vertex, coordinates $\{x_A^\mu\}$, that is drop the usual $\int_C dt \int d^3x$, in each diagram D , so that

$$W_A := \sum_{\text{topologies}} D_A \quad (3.21)$$

The value of the diagrams, and hence of W_A , is independent of the value of the coordinates of the vertex, A , as (3.11) states. This makes the last vertex integral easy, as (3.12) states, and we have for W ,

$$\beta V.f(J) = i(VT - VT - i\beta\sigma V - i\beta(1-\sigma)V) W_A \quad (3.22)$$

This is to be compared with the formula for the free energy in Niemi and Semenoff's papers [NS1,2] and given in (1.23).

$$\text{c.f. } \beta\Omega = \beta V.\omega(\phi_1, \phi_2) = -iV(\phi_1, \phi_2) \cdot \int_{-\infty}^{\infty} d^4x - \ln(Z_{34}) \quad (2d.1)$$

where Ω is the Legendre transform of F with respect to J . We can then see that the effective potential of Niemi and Semenoff is

$$\begin{aligned} V(\phi_1, \phi_2) &= W_A \cdot (\int_{C_1} dt + \int_{C_2} dt) \cdot \int d^3x \\ &= W_A \cdot (VT - VT) = 0 \end{aligned} \quad (3.23)$$

$V(\phi_1, \phi_2)$ is the effective potential associated with C_1/C_2 , and Z_{34} the part of the partition function associated with C_3/C_4 ,

as defined from the path integral in (1.18) and [NS1,2]. It is clear that this effective potential is zero, just as the discussion in chapter 2.d suggested.

We can see by comparing (3.22) and (2d.1) that the two VT terms cancel and correspond to the most obvious effective potential, $V(\phi_1, \phi_2)$, being zero. This resolves part of problem (d) relating to the result (2d.1) of Niemi and Semenoff [NS1,2]. Now we see that indeed the free energy only gets a contribution from the Z_{34} part, which is usually thrown away.

Why does the effective potential of Niemi and Semenoff contain the correct answer, though? When the additional rule of chapter 2 is used in effective potential calculations, the ITF result was reproduced upto 2-loop level. It is now straightforward to understand this. From (3.22) and (2d.1) we have

$$\beta V.f = (V\beta\sigma + V\beta(1-\sigma)).W_A = \frac{-1}{\beta V} \ln(Z_{34}) \quad (3.24)$$

and so from (3.21)

$$f = \Sigma(D_A) \quad (3.25)$$

and we can now make use of our formula for calculating diagrams with one vertex fixed, to get

$$f = \frac{1}{V} \Sigma \frac{D(\text{new rule}; a)}{\int_{C_a} d^4 x} \quad (3.26)$$

where the new rule as before is to choose the time coordinate of one vertex to lie on C_a , and for this vertex only perform a time integration of just $\int_{C_a} dt$. We could choose these vertices to lie on C_3 or C_4 , as one would normally do if we wanted to calculate Z_{34} . Since in fact we are more interested in the C_1 and C_2 parts most of the time and our Feynman rules in PIRT are set up to do this, the most sensible version is to fix $a=1$ for all diagrams in the sum in (3.26). The new rule then coincides with the additional, and as far as the literature is concerned, ad-hoc rule stated in chapter 2. This is also the rule that has been used in the calculations of the free energy in TFD/PIRT to 2-loops, where the answer was compared with the ITF calculation [FG1,2, FGK, MNU].

We now understand how to calculate the free energy in PIRT (though not necessarily in the operator formalism of TFD) using the Feynman rules of PIRT with the new additional rule, and clear up the problems of the derivation of Niemi and Semenoff discussed in chapter 2, and problem (d) of chapter 2 completely resolved.

The Legendre transform from $W[J]$ to $\Omega(\Phi)$, the free energy or true effective potential at finite temperature, won't affect the application of the new rule to calculating 1PI rather than connected diagrams. As all the methods of calculating the free energy, C-W, vacuum, and tadpole, involve such diagrams as are covered by (3.16), the new rule is what should be applied to such calculations in TFD. It is exactly what has been applied in earlier calculations [FG1,2, FGK, MNU], to get agreement with ITF results and eliminate singularities.

CHAPTER 4

The $O(N)$ Scalar Model

4) THE O(N) SCALAR MODEL.

This model contains N scalar fields with a global O(N) symmetry and a quartic self interaction, as defined by the Lagrangian

$$L = 1/2(\partial_{\mu} \phi^{\alpha})(\partial^{\mu} \phi^{\alpha}) - (1/2)m^2 \phi^{\alpha} \phi^{\alpha} - (\lambda/8N)(\phi^{\alpha} \phi^{\alpha})^2 \quad (4.1)$$

where $\alpha=1,N$ and ϕ , m , and λ are all order 1, factors of N being made explicit. It is found that for certain parameter ranges there is a scalar O(N) invariant bound state at zero temperature. This bound state has a non-zero mass, not being associated with a Goldstone boson as are the pions in the models of massless QCD in chapter 5. As we heat up this model, we look for this mass going to zero to indicate that we have a second order transition, or for a sudden change in its value which may indicate a first order transition. Here it is the actual bound state mass that can be used to indicate phase transitions. The O(N) scalar model is renormalisable, but then this model has the problem that for one possible solution, the effective potential is unbounded from below as the expectation value of the scalar fields becomes large.

The O(N) scalar model has been considered in detail because of the ability to study bound states, probing the non-perturbative effects, by using a $1/N$ expansion just to leading order, even though the interesting phases appear to be metastable. The hope is then that the non-perturbative

mechanisms illuminated will be of wider importance and not crucially dependent on the approximation used to probe them in this model.

The model has been studied extensively at zero temperature [AKS, CJP, Ro, Sc]. More recently, it has been extended to finite temperature using the effective potential [FS1] and by using a variational approach [BM1,2]. They all used ITF and effectively worked exclusively at zero momentum. This is equivalent to looking at the Free Energy and using the second derivative as a mass, so that the mass used was defined using a zero-momentum renormalisation point. As external momenta are set to zero in these calculations, this approach misses an important problem in that finite temperature theories appear to be non-analytic, though finite, at zero momentum, problem (e) of chapter 2. To bring out this point, we use a physical definition of mass (momentum renormalisation point of $p^2=m^2$) in calculations with this model. This means that we look for $m^2 \rightarrow 0$ as a signal for a transition, and thus we look at $p^2 \rightarrow 0$ rather than $p^2=0$ limits of the Feynman diagrams involved (see problem (e), chapter 2).

We will also use a path integral approach [see Ri] with PIRT time curve (fig.(1.3)) being chosen, to see how the matrix structure works. The model was also studied using SD equations and large N approximation with TFD/PIRT, and then everything was repeated with using ITF. All methods appeared to give the same answers for the real parts of Green functions, including the masses of interest to us.

The starting point is the path-integral representation of the generating functional of thermal expectation values of

time-ordered products (which can be found in [FH] and for TFD/PIRT and finite-temperature formalisms in chapter 1 and appendix D, as well as in [NS1,2, La, LvW]).

$$Z[j^\alpha] = \int_{BC} \prod_{\alpha=1}^N [D\phi^\alpha] \exp\{i/h(\int_C dt \int_V d^3x [-(1/2)\phi^\alpha(\partial_0^2)\phi^\alpha - (\lambda_0/8N)(\phi^\alpha\phi^\alpha)^2 + j^\alpha\phi^\alpha])\} \quad (4.2)$$

where the boundary condition (BC) is the usual periodic one, $\phi^\alpha(t, \underline{x}) = \phi^\alpha(t-i\beta, \underline{x})$, V is the space volume to be taken to infinity, C is a curve in complex time space running from τ to $\tau-i\beta$ (τ is an arbitrary time), and we introduce N sources j^α . Different choices for C correspond to different finite-temperature formalisms [MNU2, LvW], with the one for TFD given by fig.(1.3) [NS1,2, LvW, MNU2].

The manipulations used from here for simple field theories to get the usual Feynman rules can be found in chapter 1 and [NS1,2, La, LvW], and the zero temperature manipulations of the $O(N)$ scalar model using this path integral approach can be found in [Ri].

We take the usual TFD curve C , and we can manipulate (4.2) into

$$Z[j^\alpha] = \int_{\alpha=1}^N \prod [D\phi_1^\alpha][D\phi_2^\alpha] \exp\{i(\int_{-\infty}^{\infty} dt \int d^3x [(1/2)\phi_a^\alpha \Delta_{ab}^{-1} \phi_b^\alpha - \theta_a(\lambda_0/8N)(\phi_a^\alpha\phi_a^\alpha)^2 + j_a^\alpha\phi_a^\alpha])\} \quad (4.3)$$

where $a, b=1, 2$ are the TFD/PIRT indices, $\theta_1=+1$, $\theta_2=-1$, Δ_{ab} is

the scalar TFD/PIRT propagator (app.A), and the time integral is now from $-\infty$ to $+\infty$ as with zero temperature field theory. All repeated indices are summed over, even if repeated more than once in any one term. This now looks like a zero temperature field theory but with double the fields we started with. All the temperature information is contained in the propagator and the double-field format, rather than in the boundary conditions and in the time paths, and this is the central result for path-integral TFD obtained by Niemi and Semenoff. The type one- and two-fields and sources are related to those of (4.3) by

$$\begin{aligned}\phi_1^\alpha(t, \underline{x}) &= \phi^\alpha(t, \underline{x}) \\ \phi_2^\alpha(t, \underline{x}) &= \phi^\alpha(t - i\beta/2, \underline{x}) \\ j_1^\alpha(t, \underline{x}) &= j^\alpha(t, \underline{x}) \\ j_2^\alpha(t, \underline{x}) &= -j^\alpha(t - i\beta/2, \underline{x})\end{aligned}\tag{4.4}$$

where t is real.

Note that there are some sign errors in [NS1,2] as pointed out by Kobes and Kowalski [KK], and the definitions given here give the Feynman rules for a scalar field of the form stated in app.A.

Now we use an auxiliary field, $\tilde{\chi}$, to leave a pure Gaussian integral in the ϕ^α fields, but to do this in this doubled field formalism we have to introduce a pair of such fields via a pair of functional Gaussians in the $\tilde{\chi}_1$ and $\tilde{\chi}_2$ fields, illustrating how all the fields seen at zero

temperature are doubled in the finite temperature formalism of TFD.

$$\text{constant} = \int D\tilde{\chi}_1 D\tilde{\chi}_2 \exp\{i \int d^4x (\tilde{\chi}_a - (\lambda/2N) \phi_a^\alpha \phi_a^\alpha)^2 (N/2\lambda_0) \theta_a\} \quad (4.5)$$

Dropping infinite constants, and from now on absorbing the space time integrals (now all from $-\infty$ to $+\infty$) in an implied matrix and vector notation, we get

$$Z[j^\alpha] = \int \prod_{\alpha=1}^N ([D\phi_1^\alpha][D\phi_2^\alpha]) D\tilde{\chi}_1 D\tilde{\chi}_2 \exp i \left\{ \frac{1}{2} \phi_a^\alpha \Delta_{ab} \phi_b^\alpha + (N/2\lambda_0) \theta_a \tilde{\chi}_a \tilde{\chi}_a \right. \\ \left. - \frac{1}{2} \theta_a \phi_a^\alpha \phi_a^\alpha \tilde{\chi}_a + \theta_a j_a^\alpha \phi_a^\alpha \right\} \quad (4.6)$$

The ϕ^α integrals are easily performed, and writing the resulting determinant as an exponential we get

$$Z[j_a^\alpha] = \int [D\tilde{\chi}_1][D\tilde{\chi}_2] \exp\{iW[j_a^\alpha, \tilde{\chi}_a]\} \quad (4.7)$$

$$W[j_a^\alpha, \tilde{\chi}_a] = N\{(2\lambda_0)^{-1} \tilde{\chi}_a \tilde{\chi}_a \theta_a + \frac{i}{2} \text{Tr}(\ln(\Delta_{ab}^{-1}(x-y; m^2 + \tilde{\chi}))) \\ - \frac{1}{2} j_a^\alpha \Delta_{ab} j_b^\alpha\} \quad (4.8)$$

where the trace is over both space-time and the thermal indices, and the propagator, Δ , is now for mass $m^2 + \tilde{\chi}$, and has no factors of i included, i.e. $\Delta^{-1} \sim p^2 - m^2 - \tilde{\chi}$. The last step is

to use the simplest version of the stationary phase approximation [MW] to produce the leading-order large N approximation, and we get

$$Z[j_a^\alpha] = \exp\{iW[j_a^\alpha, \tilde{\chi}_a]\} \Big|_{\text{subject to (4.10)}} \quad (4.9)$$

$$\frac{\partial W}{\partial \tilde{\chi}_a} = 0 \quad (4.10)$$

Note that the condition needed is just like the usual one encountered in field theories that the sources should be zero, and (4.10) is the form obtained in TFD. The auxiliary field can now seen in the non-local trace term to have picked up kinetic terms. It is useful to calculate the effective action, $\Gamma[\Phi_a^\alpha]$, where we use the definitions

$$\frac{\partial W}{\partial j_a^\alpha} = \Phi_a^\alpha = -\Delta_{ab} j_b^\alpha \quad (4.11)$$

$$\Gamma = W - \int_{C_1+C_2} j_a^\alpha \Phi_a^\alpha \quad (4.12)$$

to give us

$$\Gamma = (N/2\lambda) \tilde{\chi}_a \tilde{\chi}_a \theta_a + \frac{iN}{2} \text{Tr} \{ \ln [-\Delta_{ab}^{-1}(x-y)] \} + \frac{1}{2} \int \int \Phi_a^\alpha \Delta_{ab}(x-y) \Phi_b^\alpha \quad (4.13)$$

where the propagator $\Delta^{-1} = +m_0^2 + \tilde{\chi}$. Then the condition (4.10) is equivalent to

$$\frac{\partial \Gamma}{\partial \tilde{\chi}_a} = 0 \quad (4.14)$$

i.e.

$$(N/\lambda) \tilde{\chi}_a \theta_a - \frac{iN\theta}{2} \Delta_{aa}(0) - \frac{1}{2} \Phi_a^{\alpha} \Phi_a^{\alpha} = 0 \quad (4.15)$$

This gives us the self consistent equation for the mass, $\tilde{\chi}$, of the ϕ field. The other equation of interest is the inverse bound state propagator, as the auxiliary $\tilde{\chi}$ field becomes the bound state field, picking up kinetic terms from quantum corrections as seen through the non-locality of the trace term.

$$\begin{aligned} \frac{\partial^2 \Gamma}{\partial \tilde{\chi}_a(x) \partial \tilde{\chi}_b(y)} &= (N/\lambda) \tau_{ab} \delta(x-y) - \frac{iN\theta}{2} \Delta_{ab}(x-y) \Delta_{ba}(y-x) \theta_b \\ &= \Sigma_{ab}(x-y) \end{aligned} \quad (4.16)$$

where there is no sum over any index, and the Δ propagator has mass $m_0^2 + \tilde{\chi}$. Taking the Fourier transform, dropping an overall delta function, and using the diagonal form of the TFD matrices (app.B) gives

$$\underline{\Sigma}(p) = \underline{\tau} \cdot \underline{U} \cdot [(N/\lambda) \underline{\tau} + N \underline{B} \cdot \underline{\tau}] \cdot \underline{U} \cdot \underline{\tau} + \Phi \Delta^3 \Phi \quad (4.17)$$

so the matrix form of the full propagator has the expected

form, giving when $\Phi=0$

$$\underline{\underline{\Pi}} = \underline{\underline{U}} \cdot \underline{\underline{\Pi}} \cdot \underline{\underline{U}} \quad \underline{\underline{\Pi}} = \begin{pmatrix} \bar{\Pi} & 0 \\ 0 & -\bar{\Pi}^* \end{pmatrix}$$

$$\bar{\Pi}^{-1} = N(2/\lambda + \bar{B}) \quad (4.19)$$

where $-iB$ is the bubble diagram, fig.(2.3), with two identical scalar fields mass $m^2 + \chi$, and with no symmetry, vertex, or N factors included, so that

$$\begin{aligned} B_{ab}(p^\mu) &= i \int d^4k \Delta_{ab}(k) \Delta_{ab}(k+p) \\ &= [\underline{\underline{U}} \cdot \underline{\underline{B}} \cdot \underline{\underline{U}}]_{ab} \end{aligned} \quad (4.18)$$

Once again the behavior of the TFD matrix propagator is controlled by just one function $\bar{\Pi}$. The inverse propagator for the ϕ fields is also needed, and exactly the same thermal structure is found, where the poles of the matrix propagator depend only on one function $\bar{\Pi}_\phi$ similar to (4.18)

$$\bar{\Pi}_\phi^{-1}(p^\mu, T) = (p^2 - m_\phi^2 - \tilde{\chi}) \quad (4.20)$$

So we see that just as at zero temperature, $m_\phi^2 + \tilde{\chi}$ is the unrenormalised full quantum-corrected ϕ -field mass. The only temperature-dependence of the ϕ -field propagator poles comes in via the self-consistent mass, $\tilde{\chi}$, from equation (4.15).

If we believe in the effective potential/free energy at

non-zero temperature (see problem (e), chapter 2), we also require

$$\frac{\partial \Gamma}{\partial \bar{\phi}_a^\alpha} = 0 \quad (4.21)$$

so that we are at an extremum. This corresponds to tadpole diagrams where the external leg must by momentum conservation carry zero momentum, so problem (e) is avoided. This leads to the equation

$$(m_0^2 + \tilde{\chi}) \bar{\phi}_a^\alpha = 0 \quad (4.22)$$

Thus we have two distinct types of solution, $\bar{\phi}=0$ or $m_0^2 + \tilde{\chi}=0$. The latter gives a flat effective potential and is therefore uninteresting and usually neglected. It is in fact the new "endpoint" solution of [BM1,2] who check that it has a higher value of the effective potential than the metastable $\langle \phi \rangle=0$ solutions. This $m_0^2 + \tilde{\chi}=0$ solution does not satisfy the saddle-point constraint, however, and we might be tempting to drop this 'solution'. What has happened is that $\tilde{\chi}$ is no longer going to be a dynamical (bound state) field and we should not try to integrate over it in the path integral as we did above. What it now represents is the expectation value squared of ϕ^α , as we find we have $O(N) \rightarrow O(N-1)$ symmetry breaking. Working correctly, we end up with very similar equations but now we replace N by $N-1$ in the ϕ equations, as they now represent the $N-1$ massless Goldstone bosons present (as required by

Goldstone's theorem), and we don't get any of the $\tilde{\chi}$ field relations such as the gap equation (4.14), where $\tilde{\chi}$ is being treated like a dynamical field. This corresponds to the analysis in the variational method of [BM1,2] of their "new" endpoint solution.

Renormalisation is straight-forward, as finite temperature introduces no new divergences, so we renormalise at zero temperature, following Abbott et al. [AKS]. Before doing this, we redefine our $\tilde{\chi}_a$ fields by

$$\tilde{\chi}_a = \chi_a - m_0^2 \quad (4.23)$$

The field χ is now the constituent ϕ -field mass function. So far all our parameters and fields have been unrenormalised. If we now denote the unrenormalised quantities with a subscript 0, we can define renormalised parameters by

$$\frac{\chi}{g} = \frac{\chi_0}{g_0} + \frac{1}{2} \int d^4k \frac{i}{k^2 + i\epsilon} \quad (4.24)$$

$$\frac{1}{g} = \frac{1}{\lambda_0} + \frac{i}{2} \int d^4k \frac{i}{k^2 + i\epsilon} \cdot \frac{i}{k^2 - M^2 + i\epsilon} \quad (4.25)$$

where M is an arbitrary momentum renormalisation point.

The result of all this is that when we take a solution for the self consistent equation (4.15) for the χ_a 's we can take a $\chi_1 = \chi_2 = \chi$ solution, and this χ is the mass for the renormalised ϕ fields, as is seen by looking at the renormalised ϕ propagator. The bound state propagator has the form

$$\begin{aligned} \bar{\Pi}^{-1}(p^\mu, T) = & N(32\pi^2/g - 1 - \ln(\chi/M^2) + [2 - u \ln((u+1)/(u-1))]) \\ & + 16\pi^2 \bar{B}_\beta(p^\mu, T) \end{aligned} \quad (4.26)$$

where $u^2 = 1 - 4\chi/p^2$, and the only new term at finite temperature is the explicit temperature dependent part of the bubble, B_β (no symmetry or vertex factors, see (4.19)). Also note with reference to problem (a) of chapter 2, that we are looking at $\bar{\Pi}$ and not Π_{11} , but that there is no difference in the real part of these quantities.

Temperature also comes in via the self-consistent equation for the ϕ field mass, now χ , and from (4.15) and (4.23) we have

$$\chi/g = m^2/g + (\chi/32\pi^2) \ln(\chi/M^2) + \frac{1}{2} L_\beta^-(\chi, T) \quad (4.27)$$

where $L_\beta^-(\chi, T) = \int d^4k i \Delta_{11}(k; \chi)$ is the non-zero temperature part of the tadpole diagram for a scalar field of mass squared χ , excluding symmetry and vertex factors, and m and g are parameters of the theory. Note that this equation gives rise to a critical temperature above which no solution for χ exists, which means that the $\langle \phi \rangle = 0$ phase is not stable.

The mass of the bound state, which we'll denote by the function η , is found by looking for poles in the real part of the bound state propagator, and in this matrix formulation we can see from (4.18) that they come from zeros in $\bar{\Pi}^{-1}$ of the χ field

$$\text{Re}\{(32\pi^2/g) -1 -\ln(\chi/M^2) +[2-\text{uln}((u+1)/(u-1))]\}$$

$$+16\pi^2\bar{B}_\beta(\eta^2(v,T),v,T)\} =0 \quad (4.28)$$

where η is going to be a function of temperature, and because of the loss of Lorentz invariance, on $|\underline{y}| = |\underline{p}/p_0|$, p^μ being the four-momentum. Note that it is possible not to have bound states present, no solution to (4.28), even when there is a stable $\langle\phi\rangle=0$ phase, as [AKS] pointed out.

Now, for second order transitions, the point $\eta(v,T_c)=0$ is of interest, and this is easily calculated

$$32\pi^2/g -1 -\ln(\chi/M^2) +4G^-(v,T_c) =0 \quad (4.29)$$

where χ is temperature-dependent but v -independent, as (4.27) shows. Here, we have used the $p^2 \rightarrow 0$ limit in the fermion bubble calculation to get G^- of (2e.9), and the lack of analyticity in this finite-temperature calculation gives rise to a critical temperature, T_c , dependent on v . Previous authors [FS1, BM1,2] all used a $p^2=0$ approach, which agrees only in the $v \rightarrow \infty$ limit.

It is possible to eliminate the unphysical M renormalisation scale parameter, which won't appear in any physical quantity, as Abbott et al. did [AKS]. This makes the different solutions, depending on the values of the two parameters of the model, m and λ , much clearer.

We work with a ϕ -field self-consistent mass of $\rho(T)$ in

units of M'^2 , where

$$\rho := \chi(T) \cdot M'^{-2} \cdot \exp(-32\pi^2/g) = \chi/M'^2 \quad (4.30)$$

and we have the following equations to solve. First the gap equation which fixes ρ is given by

$$\rho \ln \rho = \rho_0 \ln \rho_0 - 8\rho F_1^-(b\rho^{1/2}) \quad (4.31)$$

where $b=\beta M'$, ρ_0 is the zero temperature value of ρ , and $F_1^-(\eta)$ is a standard function given by

$$F_1^+(\eta) = \int dx \cdot (\exp(\eta x \pm 1))^{-1} \cdot (x^2 - 1)^{1/2} \quad (4.32)$$

$$\rho_0 \ln \rho_0 = -32\pi^2 m^2 / (gM'^2) \quad (4.33)$$

The gap equation is sketched in fig.(4.1). We also have the bound state propagator $\Pi(p^2, v, T)$ (after suitable analytic continuation of the log in (4.28))

$$\begin{aligned} \Pi^{-1}(p^2, v, T) = (32\pi^2)^{-1} \{ -\ln \rho + 1 - z(2\tan^{-1}(z) - \pi) \} + \\ \frac{1}{2} \bar{B}_\beta(p^2, v, T; \chi) \end{aligned} \quad (4.34)$$

where $z^2 = (4\rho/(p^2/M'^2)) - 1$, and $\bar{B}_\beta$ is the explicit temperature dependent part of the bubble diagram, defined in (4.19). This equation is sketched in fig.(4.2). The $\text{Re}\{z(2\tan^{-1}(z) - \pi)\}$

function has a minimum of zero at $(p^2/M'^2)=4\rho$, rising smoothly on either side of this point. So if at zero temperature, $\rho=\rho_0>e$, then there are no bound states at $T=0$.

Note that the fundamental field's mass, ρ , doesn't depend on v , the parameter that represents the loss of Lorentz invariance for two point functions. This illustrates the simplicity of the model, as we would expect most parameters of a theory to exhibit the loss of Lorentz invariance, as in fact the poles of the bound state propagator in this model show. We now look at the phase structure and transitions of the $O(N)$ model at non-zero temperature, and compare with the zero-temperature structure given in [AKS] and the non-zero temperature results gained using different methods [BM1,2, FS1]. As [BM1] points out, the renormalisable $O(N)$ model corresponds to a bare coupling constant of $\lambda_0 \rightarrow 0$, and this can be seen in (4.25).

There are three solutions. The first is the uninteresting one which roughly corresponds to $\chi=0$ in equation (4.22) for the vanishing of the first derivative of the effective potential, though this solution doesn't satisfy the gap equation. This has a ϕ -independent effective potential and corresponds to $O(N) \rightarrow O(N-1)$ symmetry breaking. The χ parameter plays the role of the $\langle\phi^2\rangle$ parameter, and $\chi=0$ implies that there are $N-1$ massless ϕ fields modes (4.22), (4.23). This solution has a higher effective potential value than the others, while they exist. It corresponds to the so-called "new broken symmetry phase" solution of [BM1], corresponding to 'endpoint' solutions in their variational approach

Turning to the $O(N)$ symmetric solutions $\phi=0, \chi \neq 0$ allowed by the vanishing of the first derivative with respect to the ϕ field of the effective potential (4.21), there are at most two solutions corresponding to the two solutions of $\rho \ln \rho = \text{Number}$ which arises in the gap equation (4.15) (e.g. fig.(4.3)). The lower solution gives rise to tachyonic bound state poles while it exists, and corresponds to a local maximum in the effective potential. It led to confusion in the earlier zero-temperature $O(N)$ literature (see [AKS]). The upper $O(N)$ symmetric solution, while it exists, has the lowest value of the effective potential of all possible solutions, and no tachyons (i.e. is a local effective potential minimum), as is seen in [BM1].

To look at the phase structure, we consider heating up a theory with some choice of the free parameters (m^2/g) and (M'^2), though M' merely provides an overall mass scale. Suppose we find ourselves at zero temperature with the upper solution allowed by the gap equation such that $\rho_0 > e$, and then we find, quite consistently, that there are no bound state propagator poles, $1 - \ln \rho < 0$ and by (4.34), fig.(4.2), there are no solutions at zero temperature, i.e. no bound states. For example, we could have $\rho_0 = 3.3$, as used in fig.(4.3). This possibility was noted at zero temperature by Abbott et al. [AKS] but not mentioned at finite temperature in [BM1,2, FS].

Heating the system up, the value of ρ drops as fig.(4.1) and explicitly fig.(4.3) show, and β changes in (4.34), and we find that poles start to appear in the bound-state propagator corresponding to a weakly bound state and an associated

resonance at mass squared of $4\rho(T).M'^2$. However, the bound-state propagator is v -dependent, and we find that these poles first appear at different temperatures for different v , that is we can have bound states present at some velocities with respect to the heat bath, but not at others. This phase transition is not noted by the other non-zero temperature works [BM1,2, FS1]. This is a v -dependent critical temperature. Further heating reduces the bound states mass and raises the resonance's mass when calculated in units of χ , and the value of $\rho(\chi)$, the ϕ field mass, also drops.

At some temperature ($T=M'\sqrt{[3\rho_0\ln(\rho_0)/\pi^2]}$), when the gap equation first gives $\rho\ln\rho=0$, the lower ρ , $\langle\phi\rangle=0$ tachyonic solution appears, but this always has a higher effective potential. This solution can be seen in fig.(4.1.ii) and in fig.(4.3).

Eventually, the self-consistent upper ρ solution takes on the same value as the tachyonic lower ρ solution, fig.(4.1.iii). At any higher temperature there is no solution to the gap equation. If we were to look at the effective potential at this temperature, we would find that the minima and maxima corresponding to the $\langle\phi\rangle=0$ solutions coincide with each other and also with the flat $\chi=0$ solution, so that the effective potential mass, $\partial^2V/\partial\phi^2$, is also zero. Thus, we are getting a clear signal for a second order phase transition, and this is the one found by [BM1,2, FS]. Now let us consider the bound state at this transition. As we have already seen, the bound state has, in general, a v -dependent mass. Another way of looking for a phase transition would be to look at the physical, $p^2=m^2$, bound state mass, and to see at what

temperature it reaches zero. This would coincide with the effective potential definition of mass if the relevant Feynman diagrams were analytic at zero external momentum at non-zero temperature. In fact we find the v -dependence of the bound state mass doesn't disappear as the mass becomes zero at some v and T , and so though the mass may be zero at some temperature for one value of v , it isn't zero for other values. For this simple model, based on the bubble diagram, we find that the mass that reaches zero at the lowest temperature is the $v=\infty$ mass, which is the only one which coincides with the effective-potential and gap equation signals of this phase transition. Perhaps, then, it is the presence of any zero mass on-shell mode, even if at the unphysical infinite velocity, that triggers or signals the phase transition.

Different choices of the free parameters (m^2/g) and M'^2 can lead us to start further down this chain of behaviour, but nothing more than that.

There are four new points to be made beyond the conclusions of [BM1,2, FS1].

Firstly, the doubled field formalism of TFD/PIRT introduces no particular complications for this model where only two point functions were important and we could exploit their neat diagonalisation, whether we use a Schwinger-Dyson/diagrammatic approach, or a direct path-integral and saddle-point method, with or without an auxiliary field, all of which give the same answer, though not all reproduced here. It is clear that all fields and sources must be doubled, even auxiliary fields.

Secondly, in TFD/PIRT, it is $\bar{B}$, and not B_{11} , that controls the physics of the bound-state field as seen in the full propagator, (4.11), emphasising the comments made in problem (a) of chapter 2 that the usual interpretation of TFD/PIRT that the type 1 fields are the physical fields seems to be incorrect. Here we were interested only in the real part of the bubble for the bound state mass calculations, and there is no real problem in this simple case.

Thirdly, the phase structure is not quite as either [BM1,2] or [FS1] suggest. We do in fact have another phase transition, where the resonance and bound state first condense out. This follows on from the observation made in [AKS] that the $\langle\phi\rangle=0$ phase can be stable (gap equation has solutions) and yet the bound state propagator can have no poles. We also note that the "new broken symmetry phase" of [BM1] in their variational approach corresponds to the neglected and uninteresting $\langle\phi\rangle\neq 0$ solution noted in the zero temperature literature.

Fourthly, the problem of the lack of analyticity in Feynman diagrams at non-zero temperature and zero momentum, problem (e) of chapter 2, hits us in second order transitions as it leads to velocity-dependent transition temperatures when we look at the relevant physical mass that approaches zero at that transition. Here, we do have a clear signal when we use the v -independent, tadpole-diagram-based calculations for the gap equation and the $p^\mu=0$ (rather than the $p^\mu\rightarrow 0$) calculations which we used in the calculation of the effective potential.

However, it is not clear quite what the status is of the lowest-order term in a momentum expansion around zero of a quantity that is not analytic at zero momentum, the effective action. To resolve these mathematical problems, an understanding is needed of where the physics involved really lies at zero momentum. This problem, affecting both ITF and TFD/PIRT calculations, is almost totally neglected in the literature, and no reference to it is made in any published discussion of phase transitions.

CHAPTER 5

Chiral Symmetry and QCD

5) CHIRAL SYMMETRY AND QCD.

The purpose of this chapter is to look at the application of the techniques and problems of real time finite temperature field theory to the problem of the chiral phase transition in QCD.

The generally accepted theory of the strong interactions is QCD with its non-abelian $SU(3)$ gauge group of colour. It is the non-abelian colour symmetry that makes the analytic study of QCD, and its interesting non-perturbative aspects, the hadron and baryon properties, so much harder than, say, the $O(N)$ model of chapter 4. This is to be contrasted with the lattice approaches, where it is the fermionic nature of the quarks that causes fundamental problems. In trying to use an analytic approach to QCD, it is natural to start with the relatively easy quarks, and try to add on the difficult gluons later, hoping that we can, by treating the quarks fairly accurately but the gluons very simply, still arrive at a qualitatively and quantitatively accurate account of the quark and perhaps the bound state properties in QCD. This mimics the early pre-QCD approaches to the strong interactions of the '60's which concentrated on the quarks and their flavour symmetries. The use of flavour current algebras leads to some good predictions [CL/ch.5]. We hope that concentrating on the quarks and largely ignoring the gluons in finite temperature studies will yield successful results, as those earlier zero temperature studies managed to do.

These quark-orientated studies are based on the

observation that the 0^- meson octet is very light on the QCD scale, and could be regarded as approximate Goldstone bosons associated with the breaking of an approximate quark flavour symmetry among the u, d, and s quarks. In particular, there is the broken approximate symmetry of the u and d quarks, which is an especially good approximate symmetry, seen in the fact that the pions are remarkably light on QCD scales (manifested in the fact that high energy particle showers produce so many pions). The symmetry we are interested in is not due to the approximate equality of masses, $m_u = m_d (= m_s)$, which does have interesting consequences (isospin symmetries), but that on the QCD scale the u and d quarks, and to a lesser extent the s quark, are all approximately massless. The approximate symmetry (whose breaking is associated with the approximate Goldstone bosons, the 0^- octet) of the approximately massless quarks is chiral symmetry, under which a quark field transforms as

$$q_j \rightarrow \exp(i\alpha^a \lambda_{jk}^a \gamma_5) q_k$$

$$\bar{q}_j \rightarrow \bar{q}_k \exp(i\alpha^a \lambda_{jk}^{*a} \gamma_5) \quad (5.1)$$

where $a=1, \dots, (N_f^2-1)$, $j, k=1, \dots, N_f$, $\lambda^a \equiv SU(N_f)$, and the spinor indices are suppressed. We are using a basis for the γ matrices such that $\gamma_5 = \gamma_5^\dagger$. In the Lagrangian, the most important terms that don't respect this symmetry are mass terms, $m\bar{q}q$. The physical approximation that we are going to make first is that the bare quark masses are zero, and thus

that the pions (0^- octet) are the genuinely massless Goldstone bosons associated with the breaking of $SU_L(N_f) \times SU_R(N_f) \rightarrow SU(N_f)$, $N_f=2$ ($=3$) for u, d ($,s$) quarks. The fact that the u, d and to a greater extent the s quark masses are not zero but only small, accounts for the chiral symmetry being approximate and the Goldstone bosons acquiring small mass (smaller for the better $m_u=m_d=0$ approximation).

Chiral symmetry in the massless quark approximation provides a useful approach to calculating the properties of the strong interactions at non-zero temperature, just as it did at zero temperature. In particular, it also gives us a useful new handle on the phase structure of QCD. As with any broken symmetry, we expect this symmetry to be restored at high temperatures (c.f. ferromagnetism in real magnets). Thus, with exactly massless quarks, we can look for a chiral symmetry phase transition which is comparatively easy to locate in an analytic approach. The idea is that this chiral phase transition in the approximate theory of QCD and massless quarks, or its remnant in true QCD, is close to, if not exactly the same as, the interesting deconfinement transition of QCD, as lattice studies [FOOU, KKSU, Si, JL] and analytic studies [FM, YOPR, AD] have suggested. There are very obvious gross physical differences between confined and deconfined QCD phases. The reasonable assumption that a transition between these two phases exists, has implications for the physics of the early universe [We4/ch.15.11] and experiments on heavy ion collisions [JL/sec.5].

Thus it is of great physical interest to study the chiral phase transition in QCD, using the finite temperature

techniques discussed before. This approach entails concentrating on the quarks and their flavour symmetries rather than the gauge symmetry and the gluons in QCD. To try and get real physical results, we must try to probe the non-perturbative aspects of QCD, using some large N expansion or other consistent truncation of Schwinger-Dyson equations.

Finding chiral symmetry phase transitions.

How is chiral symmetry (CS) expressed in the full quantum theory? At the classical level, the chiral transformation (5.1) forbids the presence of quark masses. We will start with Lagrangians where any bare quark mass, m_0 , is put to zero for the case of exact chiral symmetry. At zero temperature, we can show that

$$\text{CS} \Leftrightarrow \{\gamma_5, \Sigma(p^\mu)\} = 0 \quad (5.2)$$

$$\text{CS} \Rightarrow m_{\text{physical}} = 0 \quad (5.3)$$

$$\text{CS} \Rightarrow \langle \bar{q}q \rangle = 0 \quad (5.4)$$

where Σ is the self mass, the two-point 1PI function, and m the mass for the fermion field q whose chiral symmetry is under investigation. Let's look at these relations at non-zero temperature.

We start from the generating functional Z for a theory with a fermion field ψ and source η

$$Z[\bar{\eta}, \eta] = \int_{B.C.} D\bar{\psi} D\psi \exp \left\{ i \int_C dt \int d^3 \underline{x} (L[\bar{\psi}, \psi] + \bar{\eta} \psi + \bar{\psi} \eta) \right\} \quad (5.5)$$

The Lagrangian L is invariant under chiral transformations $\psi \rightarrow \exp(i\alpha\gamma_5)\psi$. Thus, ignoring the possibility of non-invariant measure (anomalies) under chiral transformations, we have

$$Z[\bar{\eta}, \eta] = \int_{B.C.} D\bar{\psi} D\psi \exp \left\{ i \int_C dt \int d^3 \underline{x} (L[\bar{\psi}, \psi] + \bar{\eta}(1+i\alpha\gamma_5)\psi + \bar{\psi}(1+i\alpha\gamma_5)\eta) \right\} \quad (5.6)$$

If we demand that Z be invariant under CS transformations, $\partial Z / \partial \alpha = 0$, and then take $Z^{-1} \cdot \partial^2 / \partial \bar{\eta} \partial \eta$ of the resulting expression, setting the sources to zero gives $\{\gamma_5, \Pi\} = 0$, where Π is the full propagator, connected two-point function, for the chiral fermion. Thus we recover the zero temperature result (5.2). Note that all the effects of non-zero temperature are hidden in the time arguments ($t \in C$) and the boundary conditions of the fields and sources, and therefore in the Feynman rules involved in evaluating what are the same Feynman diagram expansions for Σ as at zero temperature. So the result (5.2) is true whatever the temperature, chemical potential etc. are and whatever formalism (ITF, TFD, PIRT etc.) we use. In PIRT, where without loss of information, we restrict our time arguments to lie on C_a , $a=1,2$, of Fig.(1.3), we have

$$\text{CS} \quad \Leftrightarrow \quad \{\gamma_5, \Sigma_{ab}(p^\mu)\} = 0 \quad (5.7)$$

where $a, b=1, 2$ are the thermal indices of PIRT. We can then use the diagonal structure of PIRT, (B.8), to arrive at

$$\text{CS} \quad \Leftrightarrow \quad \{\gamma_5, \bar{\Sigma}\} = 0 \quad (5.8)$$

Now let's make the ansatz for the spinor structure of Σ (or for $\bar{\Sigma}$ in the case of TFD/PIRT) that

$$\Sigma(p^\mu) = (Ap_0 + a)\gamma_0 - B\mathbf{p} \cdot \boldsymbol{\gamma} - M + m_0 \quad (5.9)$$

where A, a, B , and M are functions of p_0^2 , $\mathbf{p}^2$ and the thermodynamic variables, and m_0 is the bare fermion mass. The full propagator is then

$$\Pi(p^\mu) = (p \cdot \gamma - m_0 + \Sigma)^{-1} = \frac{((A+1)p_0 + a)\gamma_0 - (B+1)\mathbf{p} \cdot \boldsymbol{\gamma} + M}{((A+1)p_0 + a)^2 - (1+B)^2 \mathbf{p}^2 - M^2} \quad (5.10)$$

From (5.2) we find that

$$\text{CS} \quad \Leftrightarrow \quad M(p) = 0 \quad (5.11)$$

Thus, as Weldon [We1] has pointed out, the pole of the full propagator, the physical fermion mass m_{phys} , may not be at zero p^2 , even when we have quantum chiral symmetry. In fact the generalisation of (5.3) is

$$\text{CS} \Rightarrow m_{\text{phys}}=0 \quad \underline{\text{IFF}} \quad A=B \text{ and } a=0 \text{ at } p^2=m_{\text{phys}}^2 \quad (5.12)$$

If the chemical potential is zero, i.e. zero net fermion number, C symmetry ensures $a=0$. Lorentz symmetry at zero temperature ensures $A=B$. We then see that (5.12) contains the zero temperature and chemical potential result of (5.3). At general temperatures, however, a zero physical fermion mass is not required by quantum chiral symmetry in that fermion field!

Now consider the $\langle \bar{q}q \rangle$ relation of (5.4). Using the full propagator (before the removal of the C_a $a=3,4$ parts in the case of PIRT) we have, from the definition of the thermal propagator as the thermal average of the time ordered fermion fields, that

$$\langle \bar{q}q \rangle = -\text{Tr} \left\{ \int_{\beta} d^4k \Pi(k) \right\} \quad (5.13)$$

where the trace has the usual minus sign for fermions and is over all indices except flavour, and the integral is as is used in the Feynman rules for whatever formalism we work in (e.g. an energy sum is implied in ITF). If we do the energy part we pick up only contributions from the poles of the fermion propagator. Doing the spinor trace leads to

$$\langle \bar{q}q \rangle = -N_f \int d^3 \underline{k} M(k^2=m_{\text{phys}}^2) \cdot (\text{rest of expression}) \quad (5.14)$$

Thus if we have chiral symmetry at non-zero temperature, (5.2)

holds for all momenta, including on shell, and this ensures that a zero quark condensate is still a signal of chiral symmetry and (5.4) holds at non-zero temperatures, chemical potentials etc.

If we want to use the two field formalism of TFD/PIRT we have to do a bit more. Firstly, the physical fields are meant to be the type 1 fields, so we want $\langle \bar{q}_a q_a \rangle$, $a=1$. Though we might be worried by problem (a) of chapter 2, we are essentially looking at tadpole diagrams, for which TFD and ITF agree. The only other point is that we consider the diagonalised version of the full propagator, the Π and Σ involved in the equations above become $\bar{\Pi}$ and $\bar{\Sigma}$ in the TFD/PIRT notation, and we have

$$\langle \bar{q}_a q_a \rangle = -\int d^4k \left[\underline{U}_F \cdot \text{Tr} \{ \bar{\Pi} \} \cdot \underline{U}_F \right]_{aa} \quad (5.15)$$

where $a=1$ for physical fields. We see that the trace doesn't change the thermal matrix structure. $\bar{\Pi}$ has the same form as (5.10) and we also end up with the equation (5.14) for TFD/PIRT as for the other formalisms, though we note that the M function comes from the same ansatz as (5.9) but now it is for the core TFD/PIRT function $\bar{\Sigma}$.

Thus when we are looking for signals for chiral phase transition, we should concentrate on $\text{Tr}(\Sigma)$, (5.2) or the quark condensate, the order parameter (5.4), and remember that at finite temperature there are the extra conditions of (5.12) needed when looking at the physical fermion mass as compared with the zero temperature result, (5.3).

So how do we look for a CS phase transition in view of the change in condition (5.3) to (5.12)? Suppose we have some scheme for calculating Σ for a field that is classically chirally invariant. We need to show that the preferred solution (maximum entropy, minimum thermodynamic potential) switches between $\text{Tr}(\Sigma(p^\mu))=0$ for all p^μ , and not zero, at some critical temperature, as (5.2) says this is the test for chiral invariance at the quantum level.

The CS and CSB (CS broken) solutions may both be valid metastable solutions (local potential minima) either side of the critical temperature, but which solution has the lowest absolute value of the thermodynamic potential decides which is the stable solution; see fig.(5.1).

What we expect for a second-order transition is for the CSB solution to merge into the CS solution in a smooth manner, i.e. $\text{Tr}(\Sigma) \rightarrow 0$ as $T \rightarrow T_c$ for the CSB solution where T_c is the transition temperature as in fig.(5.2). We could also look at the order parameter, $\langle \bar{q}q \rangle$ which would also show no discontinuity at the phase transition for a second order transition, related to $\text{Tr}\{\Sigma\}$ as (5.4) shows. There is another way of looking at second-order transitions and that is to look for the appearance of long-range correlations, long-range forces, or equivalently massless modes appearing at the critical temperature. Equation (5.3) suggests that the obvious candidate is the fermion mass. Many approaches at finite temperature [BMZ, BCS] use the zero temperature relation (5.3) and talk about $\text{Tr}\{\Sigma(p^2=m'^2)\}=m'$ as if it were the physical mass, though the earlier discussion following Weldon [We1] which led to (5.12) shows that this need not be

the case at non-zero temperature. They then try to use m' going to zero at some temperature as the signal for the transition, as in (5.3) or (5.2). We shall examine the validity of the assumption that $\text{Tr}\{\gamma^\mu \Sigma\}$ can be neglected in each of the approximation schemes used in this chapter. No detailed explanation is provided for the neglect of the $\text{Tr}\{\gamma^\mu \Sigma\}$ parts in the relevant literature [BMZ, BCS] as far as the invalidation of (5.3) is concerned.

A further problem with looking at the physical quark mass is the problem of the lack of analyticity at non-zero temperature and zero momentum diagrams as we would then be looking at $\Sigma(p^2=m_{\text{phys}}^2)$ and defining the critical temperature T_c as $\lim_{T \rightarrow T_c} \{m(T)\} \rightarrow 0$, as discussed in problem (e) of chapter 2.

The Quark Mass.

We are interested in using the quark self-mass Σ to locate the chiral phase transition in massless QCD. At zero temperature, one approach is to write down Schwinger-Dyson (SD) equations which then suggest possible truncations as analytically useful approximations. By use of self-consistent solutions, we then hope to have a useful probe of what will be non-perturbative effects, e.g. bound states. It is straightforward to generalise this approach to non-zero temperature. Firstly, ITF, TFD/PIRT etc. involve no alteration of the diagrammatic structure of the zero temperature diagrams, and so the SD equations, which could be regarded as diagrammatic identities, remain unchanged, all the

thermodynamic effects being hidden in the Feynman rules to be used. The important problem remaining is to extract the physics from the mathematics. It is all too easy to follow what is, at non-zero temperature, very similar maths to the old zero-temperature calculations and forget what physics we are trying to do. (A good example is given by the imaginary parts of masses, which are not so much related to decay rates of particles, but to thermalisation rates [We3], i.e. the time it takes for a small fluctuation away from thermal equilibrium to come to equilibrium).

We look at the SD equation for the quark self mass, Fig.(5.3). The whole emphasis is to relegate the gluons to a secondary status, so it is instructive to view the SD equation as in Fig.(5.4). This shows how we might expect to get a self consistent non-linear integral equation for Σ out of this approach, as we are left with all the complicated non-abelian gluon part in the kernel K, fig.(5.5). It is on the kernel K that the approximations/truncations are made.

Before looking at possible truncations of K, does the matrix structure of TFD introduce any particular difficulties over other methods? Again the diagram becomes a matrix expression, the two vertices of Σ having a thermal index in TFD/PIRT, and we have

$$i\Sigma_{ab}(p^\mu) \cdot \delta^4(p-p') = \int d^4k \ (-iT\gamma^\nu) \cdot i\Pi_{ac}(k^\mu) \delta^4((p+k)-(p'+k)) \cdot$$

$$K_{acb}^\nu(p+k, p'+k, k) \tag{5.16}$$

where only the thermal (a,b,c=1,2) and Lorentz indices are shown. We exploit the diagonalisation of two point functions (app.B) and we can see that the only unknown in Π is the core function $\bar{\Sigma}$. Now we come across part of the problem (a) of chapter 2. We now have to relate Σ_{ab} to $\bar{\Sigma}$, and unlike ITF we don't have a simple self-consistent integral equation as yet. However, the relation between what is usually considered to be the physical part Σ_{11} and the core function is straightforward.

$$\text{Re}\{\Sigma_{11}\} = \text{Re}\{\bar{\Sigma}\}$$

$$\text{Im}\{\Sigma_{11}(p^\nu, T)\} = \text{Im}\{\bar{\Sigma}(p^\nu, T)\} \cdot \varepsilon(p^0) \tanh((p^0 + \mu)\beta/2) \quad (5.17)$$

We will be interested in the real parts, the mass, rather than the imaginary parts, which are related to thermalising times [We3], so taking the real part of (5.16) leaves us with no extra complications and an integral equation for the important parameter $\text{Re}\{\bar{\Sigma}\}$.

It is interesting to note that when, as will usually be the case, the kernel is effectively replaced by a scalar type propagator, that the difference between Σ_{11} and $\bar{\Sigma}$ is the extra double thermal statistics term that appears in TFD but not in ITF (with the standard analytic continuation contained in the formula (A.4)), that is

$$\Sigma_{11} - \bar{\Sigma} \propto \int d^4k \frac{\delta(k^2 - m_q^2)}{\exp(\beta |k_0|) + 1} \cdot \frac{\delta(q^2)}{\exp(\beta |q_0|) - 1} \quad (5.18)$$

where $q^\mu = p^\mu + k^\mu$. So when this occurs, we will often drop such terms which obviously contribute to the imaginary part of Σ , while we want the real part.

Approximations to the quark/quark Scattering Kernel K.

The hardest work is in evaluating K. This is very similar to the electron-positron kernel of [BD/19.3]. We have to approximate the kernel K i.e. the gluonic part, if we are to make any progress.

The first approximation we might make is to replace the full quark-gluon vertex by just the bare vertex in fig.(5.3). This seems to be demanded when making a ladder approximation on the quark-quark scattering kernel that appears in fig.(5.4). The ladder approximation is to assume that the kernel K depends only on the momentum transferred between the quarks. All the schemes we look at in this chapter are ladder approximations. Unfortunately, it does look as if we are throwing away a large piece of the physics, the full vertex.

However, Adler and Davis [AD] show that this apparent replacement of the full by the bare vertex is actually consistent with keeping certain ladder vertex corrections beyond the bare vertex approximation, e.g. fig.(5.6), provided certain other approximations are made. To show this, [AD] ignore quark annihilation graphs (some sort of 'quenched' approximation that lattice calculations often use) and the non-commutivity of the colour matrices, so that they can use the Ward identities connecting the full quark-gluon vertex

with the full fermion propagator, in the equation, fig.(5.7), for the full vertex in terms of the quark-quark scattering kernel (see also [BD/19.2,19.3]. After making the ladder approximation on the quark-quark kernel, an equation of the form of fig.(5.4) is obtained, but where vertex corrections of the type in fig.(5.6) have been included.

The conclusion is that making the ladder approximation directly on the SD equation for the quark self-mass means that only the bare quark-gluon vertex is being considered. If we work in a quenched approximation as well, and ignore the non-commutivity of the colour matrices, then [AD] show that in fact we are consistently taking into account some vertex corrections. Only in section (d) of this chapter are quark-quark annihilation graphs, and the non-commutivity of the colour matrices vital. For the first three sections then we can consider that ladder corrections to the full vertex, e.g. fig.(5.6), have been included.

The problem is still too difficult, in that we must deal with the full gluon propagator in fig.(5.4). The non-abelian nature of the gluon prevents us from integrating out the gluon part in the equation for the quark's self mass (though by ignoring the quarks, progress might be made by looking at the SD equations for the gluon propagator and pure gluonic vertices, as has been done at zero temperature, e.g. [De]).

We shall consider the following further approximations, as shown in fig.(5.8). First we shall consider just ignoring the gluons altogether, approximating them by an instantaneous interaction. This gives us a $(\bar{q}q)^2$ interaction and we have a

Nambu & Jona-Lasinio type model [NJL1,2]. Secondly, we shall replace the gluon propagator by some 'sensible' potential, motivated perhaps by general physical observations or phenomenologically derived potentials. In the third section, we just try to replace the full gluon propagator by the zeroth order perturbative gluon propagator of (A.11). Lastly, in section (d), we try a gluon propagator, calculated perturbatively to one-loop.

a) Instantaneous Quark Interactions; the NJL model.

This is equivalent to taking the gluons to have infinite mass and zero range, and we end up with an instantaneous 4-fermion interaction which leads us to the NJL model [NJL1,2]. This is unrenormalisable, and so there is an ultraviolet cutoff, Λ , introduced. Though the approximation involved in the NJL model seems arbitrary and unphysical in some senses (no long range gluon effects at all, no non-abelian effects though we can model a strong coupling constant), the NJL model does emerge as the lowest order approximation in effective action studies of the QCD hadron spectrum [Ba], and so should not be dismissed out of hand. Considering the simplicity of the model, the zero temperature results are quite good [Be]. The model also gives us an analytic testing ground for the techniques to be used and problems likely to be encountered.

Taking the NJL model seriously has been suggested by Bernard at zero temperature [Be] and by Bernard et al. at non-zero temperatures [BMZ] (also see [HK1,2]). Bernard [Be] looked at the values of the pion decay constant and the quark condensate in the NJL model in order to fix the cutoff and produce a value for the quark mass. The paper [BMZ] generalising this work to finite temperature and density using ITF was received after this work was completed. We shall look at the use of TFD/PIRT, and how the problems of chapter 2 appear in the NJL model, neither of which was performed in [BMZ], and compare the common results while presenting some additional ones.

The effective Lagrangian we have for QCD in this

approximation, that is the NJL Lagrangian with isospin, is

$$L = \bar{\psi}_j (i\gamma^\mu \partial_\mu - m_0) \psi_j + \frac{g^2}{2} \left((\bar{\psi}_j \psi_j)^2 - \sum_X (\bar{\psi}_j \gamma_5 \sigma_{jk}^X \psi_k)^2 \right) \quad (5a.1)$$

where $X=1, \dots, (N^2-1)$ and the σ^X matrices are $SU(N)$ flavour matrices in a N_f dimensional representation, so the $j, k=1, \dots, N_f$. The spinor indices are suppressed and we have an overall $U(N_c)$ symmetry to represent the extra quark species due to colour without having any of the interesting properties of colour present in full QCD. The sign of the coupling constant ensures stable theories for $g^2 > 0$. For simplicity, the bare quark mass m_0 will be the same for all quarks, compatible with the good approximation that the u and d quarks have the same mass when we stick to 2 flavours. Only when the bare quark mass is zero, and when $\{\underline{\sigma}^X, \underline{\sigma}^Y\} = 2\delta^{XY} \underline{1}$ (i.e. for $N=2$, $SU(2)$ only) is the Lagrangian invariant under the chiral transformation

$$\psi \rightarrow \exp(i\gamma_5 \alpha^X \sigma^X) \psi \quad (5a.2)$$

We still need to use a further approximation to proceed, and that is to truncate the SD equations in a self-consistent manner at the 1 loop level. Writing down the SD equation for the quark self-mass Σ in this approximation reveals the gross simplicity of the NJL model to 1 loop, in that the self mass is momentum independent. The only contributions come from tadpole digrams, fig.(5a.1), as the approximation discards diagrams such as fig.(5a.2). We are still calculating an

infinite series of diagrams, and thus probing non-perturbative effects, but an explicit solution is now possible. A large N_c approximation, equivalent to the leading order of a mean-field approach, gives similar results, but in fig.(5a.1), diagrams (iii) and (iv) would then also be excluded.

Using the same approximation in TFD/PIRT we must deal with the new matrix structure of TFD/PIRT over ITF and zero temperature calculations. It is trivial to see that there are no off-diagonal terms in what is now a thermal matrix $\underline{\Sigma}$, $\Sigma_{12}=\Sigma_{21}=0$, as there is only one vertex in the diagrams in fig.(5a.1), which can not carry both a 1 and a 2 thermal label. Using the diagonalised version of Σ (see app.B), we see that we have for the NJL model in this approximation $\Sigma_{11}=-\Sigma_{22}^*=\bar{\Sigma}$, and so no physical interpretation problem (problem (a) of chapter 2). With no momentum dependence in Σ we also have no problems of lack of analyticity at zero momentum, problem (e) of chapter 2. This is indeed a simple model and approximation. Keeping the bare quark mass we have (see appendix C)

$$\bar{\Sigma} = (Ap_0+a)\gamma_0 - B\vec{p}\cdot\vec{\gamma} - M + m_0 \quad (5a.3)$$

then we have in the NJL model to one loop

$$\begin{aligned} \bar{\Sigma}(p^\mu) = & \frac{g^2}{2} \int d^4k \cdot \{ (-4M \cdot N_f \cdot N_c) - (0) + (((A+1)k_0+a)\gamma_0 - B\vec{k}\cdot\vec{\gamma} + M) \\ & - (-(N^2-1)((A+1)k_0+a)\gamma_0 - B\vec{k}\cdot\vec{\gamma} + M) \} \cdot iP_q \quad (5a.4) \end{aligned}$$

where the terms in the curly brackets come from fig.(5a.1), (i), (ii), (iii), and (iv) respectively, and P_q comes from the full fermion propagator with the gamma algebra removed

$$\underline{\underline{P}} = \underline{\underline{U}}_F \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}_F \quad \bar{P}^{-1} = ((A+1)k_0+a)^2 - (B+1)^2 \tilde{k}^2 - M^2$$

$$P_q = P_{11} = \frac{1}{\bar{P}^{-1} + i\epsilon} - \delta(\bar{P}^{-1}) \cdot f_{11}^+(k^0) \quad (5a.5)$$

where f_{11}^+ is the thermal factor given in (A.12). Because of the momentum independence of Σ , we find $A=B=0$, and taking $\text{Tr}(\Sigma \gamma^\mu)$ of (5a.4) shows that this self-consistent solution is possible. This is of great use in view of (5.12). Any CS transition will now be accompanied by the appearance of zero mass quarks. We also know from C symmetry that $a=0$ for zero chemical potential, and from rotation invariance that there is no $\underline{b} \cdot \underline{\gamma}$ term ($|\underline{b}|$ a function of p^2 , p_0^2 , and T, μ etc) allowed in Σ (see appendix C). We see this by looking at $\text{Tr}(\Sigma \gamma_0)$, where we first see that $A=B=0$ as mentioned before. Using this, we then have

$$a = \frac{g^2 N^2}{2} \int d^4 k (k_0 + a) \cdot i P_q(k^\mu; T, \mu) \quad (5a.6)$$

The interesting part is the M calculation, $\text{Tr}\{\Sigma\}$, for which

$$M = m_0 + \frac{g^2 C}{4} M \cdot L^+(M; T, \mu) \quad (5a.7)$$

$$C = 2N_f N_c - 1/2 + (N_f^2 - 1)/2 \quad (5a.8)$$

where $L^+(M; T, \mu) = \int d^4 k i P_q(k_\mu; M, T, \mu)$. The numerical coefficient,

C, agrees with those of [Be, BMZ, NJL2] when we set $N=2$ i.e. work in $SU(2)$, the only case where the NJL with isospin Lagrangian is invariant under the CS transformation of (5a.2).

Later, we shall use a large N_c approximation (mean-field) rather than the one-loop approximation of [Be, BMZ] to look at the bound states. The equation for M changes only by an effective scaling of g^2 , whose value is not used as input but is fixed by other, physical values anyway. We then find $a=O(N_c^{-1})$ and so it is dropped for all values of chemical potential (it is exactly zero for zero chemical potential). Again this does not matter as 'a' only appears as a shift in the value of the chemical potential in all other equations, and again it is this shifted value of chemical potential that is that is fixed by physical values (in particular the conserved quark number density).

As we have seen, the ambiguities of the different CS transition signals disappear in the NJL model where $A=B=0$, as do the physical interpretation and lack of zero momentum analyticity problems (problems (a) and (e) of chapter 2). This is all due to the simplicity of the model and approximation where Σ is based only on tadpole diagrams.

Unfortunately in these tadpole calculations there is a problem with non-zero chemical potential, μ . We have to evaluate expressions such as

$$\int d^4k. f(k_0-a, \vec{k}). \left(\frac{i}{(k_0-a)^2 - \omega^2 + i\epsilon} - 2\pi\delta((k_0-a)^2 - \omega^2). FD(k_0, a; T, \mu) \right)$$

(5a.9)

where M and a are parts of Σ , (5.9), $\omega^2 = \tilde{k}^2 + M^2$, and $f(x, y)$ is a well behaved function. In TFD/PIRT, the Fermi-Dirac thermal distribution factor, FD, is given by $FD(k_0, a; T, \mu) = n_{11}(k_0, x=0; T, \mu)$, where

$$n_{11}(k_0, x; T, \mu) = \frac{\theta(k_0 - x)}{e^{\phi} + 1} + \frac{\theta(k_0 - x)}{e^{-\phi} + 1} \quad \phi = \beta(k_0 - \mu) \quad (5a.10)$$

where $\omega^2 = \tilde{k}^2 + M^2$. However, in ITF we get the same expression (5a.9) as in TFD, but now the fermionic thermal factor is $FD(k_0, a; T, \mu) = n_{11}(k_0, x=a; T, \mu)$. The subtle change in the θ factors makes all the difference, and leads to different answers as (5a.9) stands. In the ITF approach of [BMZ], we find that the 'a' function only appears in the combination $(\mu - a)$ in any physical expression, and its only effect is to shift the chemical potential. We can remove 'a' by shifting the energy integral in the ITF expressions (5a.9), and then the μ used is the real physical μ to be fixed by the quark conserved number density. In TFD/PIRT, with its slightly different θ functions in the thermal factors, we can not do this as the expression (5a.9) stands. However, if we were to replace the $i\epsilon$ instruction in the propagator (and in the implicit regularisation for the δ functions) by the rule that, just as with the free propagator where the $i\epsilon$ came from, we still use an energy integration curve that runs above (below) the real axis for positive (negative) values of energy, then TFD agrees (after some manipulation) with ITF and the $\text{Tr}\{\gamma_0 \Sigma\}$ factor can be dropped, and the physical μ used throughout.

There is no reference to this problem in the literature and an explanation is needed. We can consider any result with chemical potential given here as an ITF result or TFD with this, at present, arbitrary prescription. We therefore drop the 'a' function of (5a.3) in the rest of the discussions.

Now to the phase transition. With $A=B=0$ in this model and approximation, the unambiguous signal for a CS transition is the appearance of a zero in $\text{Tr}\{\Sigma\}/4=M$, which is now the physical mass. For the quark condensate we have

$$\langle \bar{q}q \rangle = -4N_f \cdot M \cdot L^+(\mu, T) \quad (5a.11)$$

where $L^+ = \int d^4k P_q(k)$. The integral is not zero while there is a non zero solution for M , (5.25), i.e. at temperatures below the critical temperature, and so this signal is unambiguous. Evaluating (5.25) we have

$$M = m_0 + \frac{g^2 C \cdot M}{4} \left(\frac{M^2}{16\pi^2} \left(\frac{\Lambda^2}{M^2} - \ln\left(\frac{\Lambda^2}{M^2}\right) \right) - L_\beta^+(T, \mu) \right) \quad (5a.12)$$

where $L_\beta^+ = L^+(T, \mu) - L^+(T=0)$ and is given by

$$L_\beta^+ = \int d^4k \frac{\delta(k^2 - m^2)}{e^{\beta|k^0|} \pm 1} = \frac{m^2}{2\pi^2} F_1^\pm(\beta m) \quad (5a.13)$$

For exact chiral symmetry, $m_0=0$, there are two solutions to (5a.12), $M=0$ the CS solution, present at all temperatures, and a $M>0$ CSB solution which is only present at temperatures below a critical temperature and below a critical chemical

potential ($g^2 > 0$ assumed, needed for a stable solution). This second solution is the favoured solution while it exists and so we have a chiral phase transition from CSB at low temperatures and quark densities, to a CS solution at high temperatures and densities. Although [BMZ] found the same behaviour, they did not give the explicit expressions for the phase boundary, $T_c(\mu)$. They are given by

$$T_c(\mu=0) = \Lambda \left(\frac{3(1-G)}{(2\pi)} \right)^{1/2} \quad (5a.14)$$

where $G=64\pi^2(Cg^2\Lambda^2)^{-1}$ (note weak coupling means there is no CSB at any temperature), or in terms of the more physical zero-temperature dynamical quark mass $M_0=M(T=0)$

$$T_c(\mu=0) = M_0 \left(\frac{3}{2\pi^2} \ln\left(1+\frac{\Lambda^2}{M_0^2}\right) \right)^{1/2} \quad (5a.15)$$

and the ITF calculation (or the TFD calculation with the ad-hoc prescription) gives the explicit formula (not given by [BMZ]) for the phase boundary

$$T_c^2(\mu_c) + \frac{3\mu_c^2}{\pi^2} = T_c^2(\mu=0) \quad (5a.16)$$

We wish to calculate what is the physical chemical potential μ being used here (having removed the 'a' function of (5.9) from physical calculations). As usual in statistical mechanics, we look at the density of the conserved quantity for which μ is acting as the relevant Lagrange multiplier.

Here it is the quark density. This comes from the zero component of the conserved current $j^\mu = \bar{q}\gamma^\mu q$, and we get for the total (all flavours, all colours) quark density $\rho_q (=3 \times \text{baryon density})$

$$\begin{aligned}\rho_q &= -\text{Tr} \int d^4k \gamma^0 \Pi_{11}(k) \\ &= -N_f N_c \int dk \frac{k^2}{\pi^2} (n^+(k, \mu) - n^-(k, \mu))\end{aligned}$$

$$n^\pm(k, \mu) = (\exp(\beta(\omega \mp \mu)) + 1)^{-1} \quad \omega^2 = k^2 + M^2 \quad (5a.17)$$

For the critical chemical potential μ_c where $M=0$, we have

$$\rho_q = N_f N_c \int dk \frac{k^2}{\pi^2} \left(\frac{e^{\beta k} \cdot 2 \text{sh}(\beta \mu)}{e^{2\beta k} + 2e^{\beta k} \text{ch}(\beta \mu) + 1} \right) \quad (5a.18)$$

We can then follow the path of [Be] and [BMZ] who relate this NJL model with parameters Λ , M_0 (or equivalently g_0), and for explicit CSB, m_0 , to real strong interactions by getting the best fit of the following quantities in the model to their experimental values; the pion decay constant ($m_0 \neq 0$ needed), the quark condensate, and Lorentz invariance when a three-momentum ultra-violet cutoff is used at $T=0$.

We now look at the bound state spectrum, in particular the Goldstone bosons (the pions hopefully). A different path from the one followed by [Be, BMZ] was used to look at the bound state properties. Rather than looking at the pion decay constant, we choose to calculate the four-quark scattering vertex directly. To facilitate this, we now add the further

approximation of large N_c and working to lowest order (we assume the coupling g^2 to be of order N_c^{-1}). Though this means that in the diagrams for Σ , fig.(5a.1) (iii) and (iv) are now dropped, this is equivalent to a rescaling of the coupling g^2 in the equation for M , and $a=0$ for all values of chemical potential. Since g^2 is never used as input and a is not physical, the physical results presented so far are unchanged. What it does in looking at the bound states is that we can drop diagrams like fig.(5a.3), though diagrams like fig.(5a.4) were already excluded in the one loop approximation used for Σ . This enables us to get a direct hold on the bound-state masses by looking for poles in the quark-quark scattering function, now given by the sum of diagrams in fig.(5a.5). This is called the chain approximation by Lurié [Lu/ch.9], and is equivalent to the ladder approximation for the homogeneous Bethe-Salpeter wavefunction equation as used in [NJL1/(4.1)]. In this large N approximation, one particular class of diagrams is then summed, and as this includes some diagrams of arbitrarily large order in the coupling constant, it means we are probing the non-perturbative effects.

Searching for bound states in the NJL model, we look at the four-point fermion-fermion scattering function, as poles here will indicate bound states. Diagrammatically, we can draw the four-fermion vertices in a 'point-split' notation to indicate how the spinor indices are contracted, though there is no other meaning attached to this. Each vertex is either a $\mathbb{1}-\mathbb{1}$ or a $\gamma_5-\gamma_5$ vertex, so we can use labels $A, B, \dots = 1, 5$ to indicate this. The γ_5 vertices also have $\underline{\sigma}^X$ SU(N) matrices

associated with them. The rules of TFD also require a thermal index for each vertex and we will use $a, b, \dots = 1, 2$. A general vertex fig.(5a.6) will then represent

$$-ig_{AA} \cdot \Gamma_A^{\alpha\beta} \cdot \Gamma_A^{\gamma\delta} \cdot \tau_{aa} \cdot \rho_{Aij}^X \quad (5a.19)$$

where $g_{11}=g_1=-g_{55}=-g_5=g^2$, $g_{15}=g_{51}$, $\Gamma_1 = 1$ and $\Gamma_5 = \gamma_5$ with spinor indices $\alpha, \beta, \gamma, \delta$, and ρ for $A=1$ is δ_{ij} with the X index dropped and for $A=5$ is σ_{ij}^X . $\underline{\tau}$ is the thermal matrix of (B.6), and there is no sum over the repeated indices. We will largely suppress this index jungle which turns out to simplify a great deal, even with the new element in TFD/PIRT of the thermal indices.

For the large N approximation used in the NJL model, the four-point fermion-fermion scattering matrix, Π , only has contributions from Feynman diagrams consisting of a linear chain of bubbles, such as fig.(5a.5). The basic unit is the fermion bubble diagram, fig.(2.3). Suppressing all but thermal and the A, B scalar/pseudo-scalar indices

$$-iB_{ij}^{AB}(p^\mu) = (2\pi)^{-4} \cdot \int d^4k \text{Tr} \{ \Gamma_\rho^A \cdot S_{ij}(k) \cdot \Gamma_\rho^B \cdot S_{jk}(p+k) \} \quad (5a.20)$$

where $\underline{S}$ is the thermal fermion propagator matrix of (app.A), and the trace is over colour, flavour and spinor indices. As usual, no symmetry factors are included and only vertex factors needed for the trace are included, in particular the thermal $\underline{\tau}$ vertex matrices are excluded, but the $SU(N)$ σ

matrices are included with the γ^5 of the pseudo-scalar vertices. A simple sum of the chains gives an expression for Π of

$$i\Pi_{ab}^{AB,X}(p^\mu) = \sum_{n=0}^{\infty} \left[(i\underline{B}(p^\mu) \cdot \underline{\tau} - i\underline{g})^n \right]_{ab}^{AC,X} \cdot i\underline{g}^{CB} \quad (5a.21)$$

where we evaluate the matrix product over all indices. The trace in the bubble leaves only thermal $a, b=1, 2$, scalar/pseudo-scalar $A, B=1, 5$, and for the pseudo-scalar case X the vector $SU(N)$ index. This seems like a confusion of indices, but this is just to be complete and the structure is really very simple.

First, the $1-\gamma_5$ structure is trivial at zero temperature because

$$\text{Tr} \{ (p+m)\gamma_5 (p+k+m) \mathbf{1} \} = 0 \quad (5a.22)$$

so that in the $A, B, \dots=1, 5$ indices, the bubble B , and therefore the scattering matrix Π , are completely diagonal. The fermion propagators at finite temperature also have an identical gamma matrix structure (A.9, 10, 11, B.10), and so this result holds at all temperatures, i.e.

$$i\Pi_{jk}^{AB} = i\Pi_{jk}^A \cdot \delta^{AB} \quad (5a.23)$$

and we have only the pure scalar and the pure pseudo-scalar channels at all temperatures.

Then the vector $SU(N)$ indices X on the pseudo-scalar Π

diagonalise as $\text{Tr}\{\sigma^X, \sigma^Y\} = \delta^{XY} N_f$. Thus we have (N^2-1) identical pseudo-scalar isospin 1 (for $SU(2)$) modes, the pions, and a scalar isospin singlet mode, and so we can drop this vector index from now on.

The new element is the thermal matrix structure. Here we find that the bubble B , a spin-zero two-point function, has the same structure as the spectral representation of scalar two-point functions (B.8) (but with TFD/PIRT $\underline{\tau}$ factors at each end due to our definition of B , so we end up with an expression like (B.7)). This is not an obvious result, though proofs [NS2, MOU] of the cancellation of certain singular multiple delta function terms in two-point TFD functions, in any one set of diagrams of the same topology (just varying thermal vertex labeling), require that each such set of diagrams has the same thermal structure as the spectral representation. This is a necessary result for perturbative TFD to be a useful scheme, and all explicit calculations have had no problems in this regard. For the bubble diagram, Kobes and Semenoff have shown this explicitly [KS1,2], and the bubble does indeed have the thermal structure of (B.7) , that is

$$-i\underline{\underline{B}} = \underline{\underline{U}}_B \cdot -i\underline{\underline{B}} \cdot \underline{\underline{U}}_B$$

$$\underline{\underline{B}} = \begin{pmatrix} \bar{B}(p^\mu, T) & 0 \\ 0 & -\bar{B}^* \end{pmatrix} \quad (5a.24)$$

Thus there is only one complex function $\bar{B}$ that controls all the elements of $\underline{\underline{B}}$. The $\underline{\underline{U}}_B$ and the $\underline{\tau}$ combine nicely in the

chain of bubbles so we can use relation (B.5) together with (5a.21) and (5a.23) and we get, suppressing some indices

$$i\Pi_{ab}^A = \sum_{n=0}^{\infty} (-ig_A)^{n+1} \Gamma^A \Gamma^A \cdot [\underline{\tau} \cdot \underline{U} \cdot (-i\underline{B} \cdot \underline{\tau})^n \cdot \underline{\tau} \cdot \underline{U} \cdot \underline{\tau}]_{ab} \quad (5a.25)$$

Now $\underline{\underline{B}}$ and $\underline{\underline{\tau}}$ are diagonal matrices in all indices, so we can easily perform the sum and we get

$$i\Pi_{ab}^A = [\underline{\tau} \cdot \underline{U} \cdot i\underline{\underline{\Pi}}^A \cdot \underline{U} \cdot \underline{\tau}]_{ab}$$

$$\underline{\underline{\Pi}}^A = \begin{pmatrix} \underline{\underline{\Pi}}^A & 0 \\ 0 & -(\underline{\underline{\Pi}}^A)^* \end{pmatrix} \quad (5a.26)$$

$$i\underline{\underline{\Pi}}^A(p^\mu, T) = -ig_A \Gamma_A \Gamma_A [1 + g_A \underline{\underline{B}}^A(p^\mu, T)]^{-1} \quad (5a.27)$$

where there is no sum over the index A which denotes the scalar or pseudo-scalar cases. This is then the full TFD non-zero temperature generalisation of the zero temperature result which was of the same form as (5a.27)

Extracting the physics is not as straightforward as proposed. Usually in TFD it is the type-one fields that are considered to be the physical fields either in a canonical approach [UMT], or from a path integral viewpoint (where only the type-one fields have real time arguments) [NS2, MOU, LvW]. Given this, the physical Green functions are given by the Feynman diagrams whose external legs are all fixed to be type one. So the standard interpretation of TFD says the physical result in our case is the Π_{11}^A function of (5a.26).

As usual, we find problem (a) of chapter 2 appearing, in that here in TFD/PIRT, Π_{11} and $\bar{\Pi}$ are not identical. From (B.7), we see that they differ only in their imaginary parts, as given by

$$\text{Re}(\Pi_{11}^A) = \text{Re}(\bar{\Pi}^A)$$

$$\text{Im}\{\Pi_{11}^A(p^\mu)\} = \varepsilon(p_0)\coth(\beta p_0/2) \text{Im}\{\bar{\Pi}^A(p^\mu)\} \quad (5a.28)$$

The ITF calculation leads to an expression of the same form as (5a.27), but where the bubble function is calculated in ITF. Kobes and Semenoff have then shown explicitly that $\bar{B}(\text{TFD}) = B(\text{ITF})$, so that here we have $\Pi(\text{ITF}) = \bar{\Pi}(\text{TFD}) \neq \Pi_{11}(\text{TFD})$.

The main observation to be made is that there will be a significant difference between physical results of ITF and TFD/PIRT (with the standard interpretation), though this has only been shown explicitly for models based on bubble diagrams. Since TFD/PIRT matrix two-point functions are only controlled by one core function, and this for instance is why no extra counter-terms are needed in the renormalisation of the doubled field formalism of TFD/PIRT, it seems quite attractive that this should be the physical part, in contrast to the normal interpretation. Note that it is $\bar{B}$ and not B_{11} that appears in (5a.26), (5a.27). This has not, however, been demonstrated for anything beyond the two-point bubble diagram, and a more thorough investigation is needed.

For our purposes we shall only need to look at the real

part of Π , as it is poles in the real part that define the mass, and since there is no difference in the real parts of Π_{11} and $\bar{\Pi}$, we can ignore the problem of interpretation. Further, since the model is based only on the bubble diagram we can use the result of an explicit calculation by Kobes and Semenoff [KS1,2] that

$$\text{Re}(B_{11}(\text{TFD})) = \text{Re}(\bar{B}(\text{TFD})) = \text{Re}(B(\text{ITF})) \quad (5a.29)$$

This ensures that ITF and either interpretation of TFD will give the same temperature-dependent masses.

There is another point to be made. Since at non-zero temperature we lose Lorentz invariance, the heat bath defining a preferred frame, we get extra parameters in our calculations which can be written as a velocity $v=|\underline{v}|$ for each external four-momentum, say p^μ , where

$$\underline{v} = \underline{p}/p_0 \quad (5a.30)$$

We only have one p^μ in our two-point function which we use to define the mass of the bound state by looking for poles in the real part of the fermion-fermion scattering function. The definition of the mass $m(v,T)$ is

$$\text{Re}(\bar{\Pi}^{-1}(p^2=m^2(v,T),v,T)) = 0 \quad (5a.31)$$

There are three parameters that Π can depend upon, p , T and v , leading to a mass dependent on temperature and the velocity of

the bound state with respect to the heat bath, v . This latter dependence is not surprising, as an observer moving at speed v with respect to the heat bath, will see different temperatures in different directions, as Weinberg [We4] shows for the simplest case of black body radiation. This extra parameter v on top of the obvious temperature dependence has not been noted or exploited, and it will mean for instance that the renormalisation group at finite temperature at finite temperature is not a two-parameter group as stated [MNU4,5, MS2, FS1, FINY, FY] but a three-parameter group. Exactly the same phenomena occurs with the massless mode of Goldstone's boson, which can be massive at non-zero temperature although the relevant symmetry is still unbroken [MUYP, MNNU].

Now, at zero temperature, the results of the model for the bound state spectrum depend crucially on the pseudo-scalar bubble diagram having one part proportional to p^2 , and the scalar bubble diagram having a part proportional to $p^2 - 4m^2$, and both with a remaining term L^+ which is the integral we get from the single loop tadpole. This structure comes from the gamma algebra, which is unchanged in the finite-temperature Feynman rules. Thus, calculation of the bubble diagram at finite temperature gives exactly the same sort of structure.

$$-B^A(p^\mu) = N_f N_c \{ [4M^2 \pm (4M^2 - 2p^2)] A^+(p^\mu, T) \pm 4L^+(M, T) \} \quad (5a.32)$$

with

$$A^+(p^\mu, T) = (2\pi)^{-4} \int d^4k P_q(k^\mu) P_q(k^\mu + p^\mu) \quad (5a.33)$$

where P_q is given by (5a.5), and where we take the $+(-)$ sign for the $A=1$ scalar channel ($A=5$ pseudo-scalar channel). F_{11} is the full fermion propagator with the gamma algebra removed as in (5a.5), but with $A=B=0$ and the 'a' function removed.

If we take the non-chiral symmetric solution, $M(T) \neq 0$, which is the stable solution at zero temperature, then on substituting into (3.22), the fermion-fermion scattering matrix becomes (remembering $C \rightarrow 2N_f N_c$ in the large N_c approximation)

$$\begin{aligned} \text{Re}(\bar{\Pi}^1)^{-1} &\propto 2(p^2 - 4m^2) \text{Re}\{A^+(p^\mu, T)\} \cdot g^2 + 4m_0/M \\ \text{Re}(\bar{\Pi}^5)^{-1} &\propto -2p^2 \text{Re}\{A^+(p^\mu, T)\} \cdot g^2 + 4m_0/M \end{aligned} \quad (5a.34)$$

We have suppressed a vector $SU(N)$ index $X, Y=1, \dots, (N^2-1)$ for the pseudo-scalar case. There are no colour or flavour indices. So we have flavour and colour singlet bound states which are either part of a pseudo-scalar isospin 1 triplet, (for $SU(2)$) which are massless for exact CS, or a massive scalar isospin singlet.

Thus we see that in a large N_c solution, the NJL model does have a massless Goldstone boson when the Lagrangian had exact chiral symmetry, $m_0=0$, and we have the CSB $M \neq 0$ solution, and this is a pseudo-scalar particle corresponding to the pion. We also have a weakly-bound scalar particle. What is

more noticeable is that these results are exact, independent of the problems of the lack of analyticity at zero momentum, or the expectation at non-zero temperatures for all masses to get temperature corrections, and that at non-zero chemical potential particles and anti-particles will have different masses. That the Goldstone theorem seems to work at non-zero temperatures and chemical potentials doesn't seem to depend on the simplicity of the NJL model, for when we put in a bare quark mass, $m_0 \neq 0$, we now get v dependent, non-zero bound state masses, with differing particle and anti-particle masses if $\mu \neq 0$, just as we expect for the masses of general particles.

b) Using Quark/Quark Potentials.

It is possible to replace the qq kernel of fig.(5.4) by some chosen potential. This allows us great freedom to model the non-perturbative aspects of QCD gluon behaviour qualitatively, or to use phenomenologically inspired qq potentials. The most obvious property we want, missing from the other approaches, is quark confinement. Le Yaouanc et al. [YOPR], working at zero temperature, consider a range of confining potentials of the form $V(\underline{r}) \propto |\underline{r}|^\alpha$, $0 < \alpha < 3$ for 3 spatial dimensions, and find that the CS solution is unstable and that CS is always broken for these confining potentials. Adler and Davis [AD] used potentials to look at the infra-red and ultra-violet behavior of the quark mass-gap equation (the $\text{Tr}\{\Sigma\}$ equation) at $T=0$, and this was generalised to non-zero temperatures by Davis and Matheson [DM]. Because they are not working in the Landau gauge (see section (c) for why this gauge is special), the $\text{Tr}\{\gamma^\mu \Sigma\}$ functions are kept in the calculation. They found that for a simple confining potential of the form $V(\underline{r}) = \kappa |\underline{r}|$, and thus ignoring the expected non-confining behaviour of high momentum gluons, there was always a CSB solution unless the confinement was turned off 'by hand' at some critical temperature ($\kappa \rightarrow 0$).

The drawback of this approach is that there is no account taken of temperature effects in the potential, the gluon propagator and the vertex. This criticism applies also to the NJL model, which doesn't have the advantage of the potential approach in having confinement included! Even at the lowest

perturbative level, it is clear that the gluon propagator has temperature dependence of a well-defined nature (see app.(A)). It is clear that most quantities, masses, coupling constants etc. pick up temperature corrections and it is not easy to see how the potentials ought to be modified beyond the simplest argument used by [DM], where the simplest 'string' argument suggests a factor of $(1-T^2/T_c^2)$ for κ the string tension [AP]. The critical temperature is then an input variable whereas it is actually a complete unknown.

Atkinson, Johnson, and Koopmans [AJK], working at zero temperature use a potential which has the tensor structure of a gluon propagator, but with pole structure giving the gluons some finite mass. They add in a running coupling constant which depends on the gluon momentum only, which is consistent with the ladder approximation discussed here, and performs the ultra-violet regularisation. This approach seems far too simplistic in that either the gluon is massless with the zero temperature tensor structure, or we use the correct non-zero temperature tensor structure and distinguish the two quite different masses (corresponding to longitudinal and transverse modes in a three vector sense) that appear. There is no explanation given for the origin of the precise form used for either the propagator or the running coupling constant. The paper is interesting in that it suggests using the mathematics of integral equations, looking at the 'eigenvalues' of 'kernels' [MW/ch.11] in such equations.

The remaining approximations therefore concentrate on looking at well defined thermodynamic modifications to the kernel missing in the potential and NJL approaches. This

means sacrificing the modelling of confinement present in the potentials. One possible way of doing both, is to look at the SD equations for the gluon, as has been done at zero temperature by e.g. [AB, AJK, De] but this would be VERY difficult in itself (the answers differ dramatically even at zero temperature in the literature), and perhaps leave a difficult result to implement in the quark self-mass equation, fig.(5.3).

c) Exchange of a Single Free Gluon.

In this approach, the qq kernel of fig.(5.5) is approximated by just a single gluon perturbative propagator of (A.12), fig.(5.8.c). We see that we have thrown away any non-perturbative QCD effects, notably confinement, and we have $1/k^2$ rather than $1/k^4$ behavior. The great advantage is that we can easily include the correct temperature modifications to the kernel, here a free gluon propagator, in this approximation, unlike in the potential approach of section (b), as the finite temperature gluon propagator is well known. Thus we have

$$i\Sigma = \int d^4k (igT_X \gamma^\mu) i\Pi_{11}(k) (igT_X \gamma^\nu) \cdot D_{11\mu\nu}(p+k) \quad (5c.1)$$

where $g^2=4\pi\alpha$, the QCD coupling constant, and the T_X , $X=1,\dots,8$ are the SU(3) of colour matrices normalised such that $\text{Tr}(T_X, T_Y) = \frac{1}{2}\delta_{XY}$, so that $T_X = \lambda_X/2$ [CL/app.B], where λ are the Gell-Mann matrices [Ge/ch.VII]. Π is the full fermion propagator of (5.10), and we use the free gluon propagator which is given by [Oj2, MNU3, LvW/2.2.3]

$$(iD^{\mu\nu}(p^2))_{11} = -(g^{\mu\nu} + (\xi-1)\frac{p^\mu p^\nu}{p^2}) P_g \quad (5c.2)$$

where

$$P_g(p_\mu, T; \Pi) = \left(\frac{-i}{p^2 - \Pi + i\epsilon} - \frac{2\pi\delta(p^2 - \Pi)}{e^{\beta|p_0|} - 1} \right) \quad (5c.3)$$

where ξ is the gauge parameter, and we take the principal part of the p^{-2} in the $(\xi-1)$ term. We also in this section have $\Pi=0$, though there are non-zero gluon masses in section (d). This gives us

$$i\Sigma_{11} = -g^2 C \int d^4 k \gamma_\mu \left[(A(k)k_0 + a(k))\gamma_0 - B(k)\vec{k} \cdot \vec{\gamma} + M \right] \gamma_\nu \cdot$$

$$(g^{\mu\nu} + (\xi-1) \frac{q^\mu q^\nu}{q^2}) P_q P_g \quad (5c.4)$$

where $P_q = F_{11}$ is given by (5a.5) and is the full propagator with the gamma matrices removed. $C=4/3$ and is the Casimir eigenvalue of the quark's SU(3) representation, and P_g is given in (5c.3). Note that A,B and M relate to the ansatz, (5.3) for $\bar{\Sigma}$ and not to Σ_{11} . We must use the relation (5.17) between these two to get a self-consistent equation. While this is straightforward, it is an extra problem not found in the ITF calculation. The difference at the one loop level corresponds to the term in TFD containing two delta functions and two thermal statistical factors which is absent from ITF when we use the standard formula (A.4) for the ITF sums. This extra TFD/PIRT term only gives pure imaginary contributions to Σ , and as we are only interested in the real part of Σ , this extra term can be ignored, and ITF and TFD/PIRT agree.

This is still a very messy equation. In general at finite temperature, we have a set of six coupled non-linear real equations to solve for the three complex functions

$\text{Tr}\{\Sigma\}, \text{Tr}\{\Sigma\gamma_0\}$ and $\text{Tr}\{\Sigma\gamma\}$. We could try a nasty numerical solution, with all the problems of the loss of control of and the loss of 'feeling' for the solutions that this involves. Here, we try to probe the nature of the solutions by using further approximations to gain analytic solutions, while trying not to upset the basic physics we are interested in with our approximation.

We end up with an equation used by Bailin et al. [BCS], though they provide no justification for the equation they use. However, we can find an approach that leads to the justification of the equation used in [BCS] but only after a lot of work. This is based on the non-zero temperature generalisation of an approach used by Munczek [Mu] (also used in [Ak]) at zero temperature.

First we will consider the spinor structure of the quark self-mass, the A, B, and M functions of (5.9), in this model of QCD. We will work with $\mu=0$, zero net baryon density, and so have $\mu=0$, in view of the uncertainty with this aspect found in the NJL model of section (a). It is often said in the literature on CSB in massless QCD that there is no wavefunction renormalisation in the Landau gauge, e.g. [AD, DM, Ak]. However, there seems to be no reference in the literature to say that at all temperatures, $\text{Tr}\{\gamma^\mu\Sigma\}$ functions can be consistently removed for all momentum in the Landau gauge, but this is what appears to be assumed in many papers on CSB in QCD. We shall now examine to what extent these statements are true.

First of all, saying there is no wavefunction renormalisation does not ensure $A=B=0$ for all momentum, which

is what we are asking about, given the way we have defined Σ . Wavefunction renormalisation means that at some renormalisation point $\not{p}=\eta$ and say for zero temperature ($A=B$ then), we demand

$$\frac{\partial \Sigma}{\partial \not{p}} \Big|_{\not{p}=\eta} = 0 \quad (5c.5)$$

which, when we use $p^2=(\not{p})^2$, gives

$$\left\{ A + 2\not{p} \frac{\partial A(p^2)}{\partial (p^2)} - 2\not{p} \frac{\delta M(p^2)}{\delta (p^2)} \right\} \Big|_{\not{p}=\eta} = 0 \quad (5c.6)$$

Clearly, $A=0$ for all p^2 is not forced by wavefunction renormalisation as (5c.6) shows.

When we do the zero temperature calculation for Σ in a purely perturbative 1-loop calculation, we find that we do have both $A=0$ for all p^2 and no wavefunction renormalisation (i.e. (5.6) holds before any renormalisation) only in the Landau gauge $[J_0(p_c)]$. However, we are interested not in a pure perturbative calculation; the whole point of this chapter is to sample the non-perturbative effects to some extent by using self-consistent calculations for Σ . Atkinson and co-workers refer back to a paper by Dragovic, Mavlo and Filippov [DMF]. They have the same equation as us to examine, i.e. (5c.4), where the full vertex and full gluon propagator of fig.(5.3) have been replaced by their bare versions. Following their approach we look at this zero temperature calculation, and take $\text{Tr}\{\not{p}\Sigma(p^2)\}$ of (5c.4) and get

$$p^2 A(p^2) = \int d^4 k \frac{(A(k^2)+1)}{((1+A)^2 k^2 - M^2)} \cdot \frac{-i}{q^2} \cdot R \quad (5c.7)$$

where R is given by

$$\begin{aligned} R &= \frac{1}{4} \text{Tr} \{ \gamma_\mu k \gamma_\nu p \} \cdot (g^{\mu\nu} + (\xi-1) \frac{q^\mu q^\nu}{q^2}) \\ &= -2k \cdot p + \frac{(\xi-1)}{(p+k)^2} \cdot ((p^2+k^2)(k \cdot p) + 2k^2 p^2) \end{aligned} \quad (5c.8)$$

where $q^\mu = p^\mu + k^\mu$, and $p \cdot k = p^\mu k_\mu$. We Euclidianise, $p^0 \rightarrow p^4 = -ip^0$, and R keeps the same form in the Euclidian variables, but with an overall minus sign if we use $p \cdot k = p^4 k^4 + \underline{p} \cdot \underline{k}$. If we want to have $A(p^2)=0$ for all p^2 we want

$$0 = \int d^4 k \frac{1}{k^2 + M^2(k^2)} \cdot \frac{1}{(p+k)^2} \cdot R \quad (5c.9)$$

where everything is Euclidianised. To avoid making any assumptions about the behaviour of $M(k^2)$, we move to 4-dimensional spherical polars, writing [tHV]

$$\int d^4 k = \int_0^\infty dk \cdot k^3 \int_0^\pi d\theta \cdot \sin^2 \theta \int_0^\pi d\theta_2 \sin \theta_2 \int_0^{2\pi} d\phi \quad (5c.10)$$

where $k \cdot p = kp \cdot \cos \theta$, and $k = |k^j|$ now. We now require

$$0 = \int dk \cdot \frac{k^3}{k^2 + M^2(k^2)} \cdot \alpha(k, \xi)$$

$$\alpha(k, \xi) = \int d\theta \cdot \frac{\sin^2 \theta}{(p+k)^2} \cdot \left(3k \cdot p + \frac{2k^2 p^2 - 2(k \cdot p)^2}{(p+k)^2} + \xi R_\xi \right) \quad (5c.11)$$

where $R_\xi = \partial R / \partial \xi$. For the ξ independent part, the angular integration gives, where $x = \cos\theta$, $\zeta = (p^2 + k^2) / 2pk$,

$$\begin{aligned} \alpha(k, \xi=0) &\propto \int_{-1}^1 dx \left[\frac{3x(1-x^2)^{1/2}}{2(\zeta+x)} + \frac{(1-x^2)^{3/2}}{2(\zeta+x)^2} \right] \\ &= \int dx \left[\frac{3x(1-x^2)^{1/2}}{2(\zeta+x)} + \frac{(1-x^2)^{3/2}}{2} \cdot \frac{d}{dx} (-(\zeta+x)^{-1}) \right] \\ &= 0 \end{aligned} \tag{5c.12}$$

after integration by parts of the second term. So without any assumptions about the behaviour of $\text{Tr}\{\Sigma\}$, we have that the $A(p^2)$ function part of Σ , (5.9), can be consistently set to zero at zero temperature before any renormalisation, in these sort of approximations where the full gluon propagator and the full vertex are set to their bare versions in fig.(5.3). In particular, if we make $M(k^2)$ momentum independent, we recover the pure perturbative calculation mentioned above. The Landau gauge does indeed allow us to set the $\not{p}$ dependent parts of $\Sigma(p)$ to zero in these approximations at zero temperature.

Now the big question is what happens at non-zero temperatures? The literature using this approach at finite temperature, [BCS, Ak], gives no reference to justify why we can ignore $A(p^2)$ at finite temperatures. The crucial aspect of the above calculations is Lorentz invariance, but this is 'lost' at finite temperature, as discussed in chapter 1.

Firstly, this means that the mass function will now depend on k^2 and θ in the integral. This is very worrying because the explicit temperature-independent term in the finite-temperature expression for $A(p^2)$ (from (5c.1)) doesn't allow the simple proof given above to work. There is a potential problem with a non-zero $A(p^2)$ in that we might expect that it would be ultra-violet divergent (unlike in the zero-temperature formalism), and this would require wavefunction renormalisation. Finite-temperature, in this non-perturbative case, may introduce new ultra-violet divergences via the implicit temperature dependence of the quark mass. This is in contradiction to the usual perturbative proofs that this doesn't occur [MOU, NS2, LvW/2.4.4]. These can not be cancelled off by the additional terms, the explicitly temperature dependent terms that appear, which the thermal statistical factors ensure are ultra-violet finite (which is the real reason that perturbatively, finite temperature introduces no new ultra-violet divergences).

Secondly, these extra temperature-dependent terms have a totally different structure to them, and it is hard to see why they should be zero themselves.

We could look at ITF rather than ITF/PIRT, but exactly the same problems occur there, since the formalisms are so similar, as the use of (A.4) would show.

For the purposes of phase transitions, we only need a weaker condition, that $A(k^\mu)=B(k^\mu)=\text{constant}$ for all k^μ , when at the transition temperature, where $M=0$. We might be able to use the approximations that are necessary (even when we assume $A=B=0$) to gain further progress with the $M(p)$ equation, to

establish this result at least in a consistent manner within some further approximation. It seems very unlikely that the simple result we had at zero temperature (that in the Landau gauge and with a bare vertex and gluon propagator in fig.(5.3), we can self consistently set $A=B=0$ independent of the form of $\text{Tr}\{\Sigma(p^2)\}$), persists at non-zero temperature. This is a point that the literature using the Landau gauge and these approximations at non-zero temperature seems to ignore. With the need at infinitesimally small temperatures for wavefunction-renormalisation, the procedure even at zero temperature seems questionable.

For the approximation to be made here, used to gain progress with the calculation for M , it has not been possible to show that the A and B functions can be ignored in the calculation of the critical temperature. Like the rest of the literature following similar approaches [BCS, Ak] we shall have to just drop A and B as a further approximation, but unlike [BCS, Ak] we recognise what we are doing.

The additional approximation we are going to follow to make progress with the calculation of the quark mass is that put forward by Munczek [Mu], though it was used there at zero temperature. The approximation arises from the observation that one important contribution to the integral on the RHS of (5c.4) comes from the values of loop momentum where the gluons are on shell, $(p+k)^2=0$. If we push this further and say that the integral is dominated by the soft gluon pole, $|p^\mu+k^\mu|=0$, we are then looking at the behaviour of the quark with only long range perturbative gluon effects included. We expect perturbation theory to be good at short distances (high

momentum) because of asymptotic freedom. This approach is more self consistent than realistic in this respect in that we are using the long range properties of the perturbative gluon, but then we have neglected deconfinement anyway.

This approximation may seem arbitrary, but the real justification comes in the results produced at zero temperature by Munczek [Mu]. The behaviour of the running mass in this approximation is in good qualitative agreement with an expression obtained from renormalisation group calculations of Chang and Chang [CC]. The mass function, $M(p^2)$, has the same qualitative behavior as Peskin derives, as can be seen in comparing [Pe/fig.7] with [Mu/fig.1]. Munczek also uses his approximation in Bethe-Salpeter equations for the hadrons, and achieves reasonable values. This all gives confidence in this approximation.

What the approximation due to Munczek says, then, is that it makes little difference to the physics if we replace $\Sigma(k^\mu)$ inside the integral by its value at the soft gluon pole,

$$\Sigma(k_\mu) \rightarrow \Sigma(p_0 + m_g, p) \quad (5c.13)$$

where m_g is the mass of the gluon, here zero but later it will be non-zero. Under this assumption of the domination of soft gluon effects, we find that the Σ factors can be pulled outside the loop integral and we have effectively diagonalised the non-linear integral equations for Σ in momentum space.

Using the Munczek approximation on (5c.4) to get a guide to CSB, we arrive at the unrenormalised equation for the dynamical quark mass at p^μ , $M(p^\mu; T)$

$$M(p^\mu, T) = m_0 - 16\pi^2 G_0 \left(\int d^4 k \left\{ \frac{iM(p^\mu, T)}{(k^2 - M(p^\mu, T)^2 + i\epsilon)} \cdot \frac{i}{(p+k)^2} \right\} + B_\beta(k^\mu) \right) \quad (5c.14)$$

We note that all the M functions inside the integral have become independent of the loop integration. We also note that we have $1/q^2$ gluon behaviour rather than $1/q^4$ or whatever. We have here $G_0 = (1+\xi/3)\alpha_0/\pi$, and B_β is the explicitly temperature-dependent part of a fermion/massless scalar bubble diagram (excluding symmetry and vertex factors, and removing the gamma matrices), given by

$$B_\beta = \int d^4 k \left\{ - \frac{2\pi\delta(k^2 - M^2)}{(p+k)^2} \cdot f_{11}^\dagger(k_0) + \frac{2\pi\delta(k^2)}{(p+k)^2 - M^2 + i\epsilon} \cdot f_{11}^-(k_0) \right\} \quad (5c.15)$$

where $G = (1+\xi/3)\alpha/\pi$, $f_{11}^\pm$ is given in (A.12), and $M = M(p^\mu, T)$. We renormalise by redefining the two parameters of the theory G and m_0/G at $T=0$ using

$$G^{-1} = G_0^{-1} - \int d^4 k \frac{i}{(k^2 - \eta^2 + i\epsilon)^2} \cdot \frac{i}{(q+k)^2} \quad (5c.16)$$

$$\frac{m_0}{G_0} = \frac{m}{G}$$

where q^2 and η^2 are renormalisation-point parameters. This gives us after doing the explicit $T=0$ part

$$\frac{m}{G}(\eta^{-1} - M^{-1}) = \ln\left(\frac{\eta^2}{M^2}\right) + I(p^2, M^2) - I(q^2, \eta^2) - 16\pi^2 B_\beta(p_\mu, T) \quad (5c.17)$$

$$I(p^2, M^2) = \frac{(M^2 - p^2)}{p^2} \ln \frac{(M^2 - p^2)}{M^2} \quad (5c.18)$$

We now put the bare quark mass $m_0=0$ to get exact CS. This equation is nearly the same as used in [BCS], but in view of problem (e) of chapter 2, we extend the calculation to all values of v for a given p^2 , whereas in [BCS] only $v=0$ was examined. As $T \rightarrow T_c$, $p^2=M^2 \rightarrow 0$, the equation will depend on v , and we cannot just ignore this dependence as [BCS] do.

Putting $p^2=M^2$, for general sized M , we find that with zero gluon mass the term with the thermal gluon contributions is zero for all v ! The thermal properties of the gluon are irrelevant to the physical quark mass at all temperatures in this soft gluon approximation. Perhaps this is indicative that in a more realistic approach, thermal gluon effects are negligible c.f. the potential approach and NJL model. This is probably not really true, and here it is just a symptom of the simplifications made as section (d) will seem to indicate.

If we now use a physical renormalisation point, $p^2=\eta^2=M^2(T=0)=M_0^2$ so that all the parameters have their physical values, (where M_0 is the zero temperature dynamical quark mass (~ 300 MeV)), and take the $p^2=M^2 \rightarrow 0$ limit at zero chemical potential, we get

$$\frac{m(M_0^{-1}-M^{-1})}{G} = \ln\left(\frac{M_0^2}{M^2}\right) + \frac{2}{v\gamma M} \int_0^\infty d\omega \frac{1}{e^{\beta\omega}+1} \cdot \ln\left(\frac{k-v\gamma M}{k+v\gamma M}\right) \quad (5c.19)$$

where $\omega^2=k^2+M^2$, and $\gamma^2=(1-v^2)^{-1}$. So our physical mass $M(T)$ is velocity dependent and we were right to be careful. Putting

$m=0$ for exact CS and then looking for the signal of the CS transition in the disappearance of the $M \neq 0$ solution, we take the small mass M limit and get

$$\ln\left(\frac{M_0^2}{M^2}\right) - 4 \int_0^\infty d\omega \frac{1}{M} \frac{1}{e^{\beta\omega} + 1} \cdot \frac{1}{k} = 0 \quad (5c.20)$$

The most remarkable result is that the physical dynamical quark mass when small is v -independent in this approximation. We will now get a v -independent quark mass. By writing $2\ln(M_0^2/M^2)$ as $\int_0^\infty dk (\omega^{-1} - (k^2 + M_0^2)^{-1/2})$ we recover the form of the result in [BCS]. They do, however, take their ultraviolet cutoff Λ seriously; that is the ultra-violet cutoff is fixed at zero temperature by a choice of coupling constant, gauge parameter and the zero temperature quark mass. Because they have no temperature dependence in this cutoff (there is no explanation why this is so), this leads to differences with the renormalisation method used here, of the form

$$\int_\Lambda^\infty d\omega (e^{\beta\omega} + 1)^{-1} \cdot k^{-1} \quad (5c.21)$$

It is a well known fact that renormalisation is not changed by adding $T \neq 0$ terms: no new ultraviolet divergences are added [MOU, NS2, LvW/2.4.4]. If we believe in renormalisation, the approach in [BCS] is incorrect up to terms of order (5c.21). However, if we believe that there is some physics behind the use of a cutoff, and that physics is temperature-independent on the scales used here then we should use the cutoff

approach. The error is small unless the temperature scale, that is the critical temperature, is similar to the cutoff scale. In fact we find the cutoff in [BCS] is much larger than the critical temperature, so we need not worry about the unknown physics imposing the cutoff, and the cutoff approach and the renormalisation approach give very similar results.

Following my approach, we have

$$2\ln\left(\frac{M_0}{M}\right) - 4F_2^+(\beta M) = 0 \quad (5c.22)$$

where $F_2^+(b) = \int_1^\infty dx \{(\exp(bx) \pm 1) / (x^2 - 1)\}^{-1}$ is a standard function that crops up in one-loop finite-temperature work. Its limit as $b \rightarrow 0$ is

$$\lim_{b \rightarrow 0} \{F_2^+(b)\} = -\frac{1}{2}\ln(b) + \frac{1}{2}(\ln(\pi) - \gamma) - \frac{7}{4}\zeta'(-2)b^2 + O(b^3, b^3\ln(b)) \quad (5c.23)$$

where $\gamma=0.5772156649$ is Euler's constant, and ζ' is the derivative of the Riemann zeta function. As $M \rightarrow 0$, we find that the infrared divergence in the log term, which came from the explicitly zero-temperature term, is cancelled by a similar divergence in the finite-temperature addition, and we get the critical temperature to be given by

$$T_c = m_{\text{phys}}(T=0) \frac{e^\gamma}{\pi} \quad (5c.24)$$

Taking the zero temperature physical quark mass to be the same as the value used by [BCS], $M_0 \sim 315\text{MeV}$, we find that we get a

critical temperature of 180MeV. This is in close agreement ($\pm 2\%$) with the result of [BCS]. The difference arises from the different approaches to the ultraviolet cutoff.

In the approach used here, using the zero-temperature dynamical quark mass M_0 to fix the physics completely eliminates any direct dependence on the coupling constant, as (5c.24) makes clear. We seem to have possible CSB however weak the interaction is. This is in contradiction with the view of Peskin [Pe] who says CSB should be related to the strong coupling constant regime. However, with the input here being the physical zero temperature quark mass, there is no suggestion here how to calculate the parameter. Other work at zero temperature, e.g. Peskin [Pe], does suggest that to have such a large quark mass ($\sim 300\text{MeV}$) does require a large coupling constant ($\alpha \gtrsim 1$). In this finite temperature calculation, we have just hidden this aspect by the choice of the input parameter.

We should also note that there is no explicit gauge dependence (although it will appear in any calculation of the zero-temperature quark mass itself). We have, though, used the fact that by working in the Landau gauge, there is no wavefunction renormalisation, so that all the renormalised parameters used here are in fact the physical ones.

With the use of a 'serious' cutoff by Bailin et al. [BCS], they find some slight dependence on the coupling constant even having fixed M_0 , part of the small difference in results mentioned above. This is also due to the different approach taken to the ultra-violet divergences. The explicit gauge dependence of the [BCS] result might worry us, but we

see that they must have already assumed the Landau gauge in ignoring the wavefunction renormalisation and the A,B functions.

We can also look at the quark condensate, (5.13). Because we can do the energy integral first, we find that we get an overall factor of M evaluated at the quark propagator pole, (5.14). Since the zero in this has been used to define the transition, it is clear that the quark condensate, the order parameter, also signals a phase transition at the same temperature as $M(p^2=M^2)=0$, the definition we used above. This analysis is not performed in [BCS], who use a second approach, reminiscent of the NJL model when looking at the quark condensate, but this is totally incorrect and inconsistent, as a comparison with the discussion of the NJL model in section (a) shows. In particular, they have two different masses, one temperature dependent, one the zero temperature quark mass, appearing from a term which started with only one quark mass! ([BCS/(19b)]).

If we keep the Lagrangian quark mass m non-zero, we then have to solve the equation (5c.19) numerically. We note, though, that the Lagrangian quark mass m and the QCD coupling constant only appear in this equation in the combination m/α . This means that good approximate chiral symmetry can be obtained with large coupling constants.

To compare the results from this model with the NJL model of section 5a we use the same value for m , $\approx 5\text{MeV}$, as in [BMZ]. To relate it to the physical quantity in the model here, the zero temperature quark mass M_0 , we need to extract this from the results of [BMZ]. There, retrieving Lorentz invariance at

zero temperature having used a non-covariant cutoff (cut off only three-momentum as [BCS] in fact do), fixes Λ/M_0 , and then the parameters are fitted to give the best agreement with the experimental values of the pion decay constant, $f_\pi=93\text{MeV}$, and the sum-rule-derived value for the quark condensate, $(\langle\bar{q}q\rangle)^{1/3} = (250\pm 50)\text{MeV}$. This leads to two sets of values being used, which are (the second is given in parentheses) $\Lambda=700(900)\text{MeV}$, $f_\pi=100(134)\text{MeV}$, and $(\langle\bar{q}q\rangle)^{1/3}=270(296)\text{MeV}$, and therefore values of $M_0=350(463)\text{MeV}$, and $m/M_0=0.014(0.011)$. This is in contrast to the value for M_0 in [BCS] of 315MeV which with a $m=5\text{MeV}$ gives $m/M_0=0.016$. Munczek [Mu] looks at bound-state masses and fixes $M_0=490\text{MeV}$, $m=5.9\text{MeV}$, so that $m/M_0=0.012$.

In fact for these ranges of small m/M_0 , there is little variation in results. These are shown in fig.(5c.1) for typical $\alpha=0.75$. The results are quantitatively as expected. As the Lagrangian quark mass increases and CS becomes a worse symmetry, what is the CS transition at $m=0$ becomes weaker, and there is less and less sudden change in the dynamical quark mass, just as the NJL model shows [BMZ]. What is more interesting is that the 'critical' temperature that we can define remains fairly well defined for small Lagrangian quark mass and good CS, and more importantly it stays around the exact chiral symmetry value of $T/M_0=0.567$. This is a new result, and gives further confidence that chiral approximation to QCD gives reasonable results.

This analysis has therefore exposed the basis for the equation used by [BCS] to determine CS transition, and that is the approximation of Munczek in the Landau gauge. We have taken the analysis further in order to expose the v dependence

of the quark mass, M , but found it to disappear at the phase transition, leaving us with a unique critical temperature, contrary to the expectations of problem (e) of chapter 2. What physical principle, if any, is ensuring a unique temperature in this 'realistic' model that was absent in the $O(N)$ model is unclear.

In view of the problems with $\mu \neq 0$ in the TFD/PIRT version of the NJL model when compared with the results in ITF, no firm conclusion about this aspect in the context of the Munczek approximation has been reached.

d) Exchange of a Single Gluon, Calculated Perturbatively to
one loop.

One result of the analysis used above was that that there was no contribution being made to the result from the thermal properties of the gluon, so the kernel K again might as well not had any $T \neq 0$ corrections. This was one of the criticisms of the NJL model and potential approaches, of section (a) and (b), to the problem. At least in section (c), we have shown that rather than not knowing what the $T \neq 0$ corrections to the kernel of fig.(5.4) are, we have shown them to play no direct role. Another striking result was that the structure of the gluon propagator we used was only slightly altered; it was still a massless particle with the zero temperature tensor structure. First, we know that at non-zero temperature, masses often pick up temperature corrections. Also, what is often called the 'loss' of Lorentz invariance actually leads to the presence of another 4-vector in the problem, u^μ , the 4-velocity of the heat bath with respect to the observer. This means we actually have two new symmetric tensors available, so the tensor structure could change. All these effects first turn up at the one-loop level in the calculation of the gluon self mass. Thus it would be interesting to see how these more exotic finite temperature effects change the results we have obtained so far, so in this section we replace the kernel by a gluon propagator with one-loop corrections.

Unfortunately, doing this makes the whole approach unusable. No longer can we justify ignoring the A and B functions, and the Munczek approximation no longer simplifies

the situation enough to get a simple result such as (5c.24). We therefore have to approach the final 'results' as no more than sampling the effects, and not to be taken very seriously. The purpose of this section is mainly to present the application of known results on the gluon propagator at one loop to the problem of the CS transition in QCD, and to understand why nothing short of a major numerical study is needed to produce useful results (though using the outlines presented here, and standard approaches to non-linear integral equations, it is a feasible computing problem).

The first problem we encounter is in the overall tensor structure of the non-zero temperature gluon self mass $\Pi_{\mu\nu}$. It still satisfies the usual identity [LvW]

$$p^\mu \Pi_{\mu\nu}(p) = 0 \quad (5d.1)$$

This applies to the TFD function, either on any of the elements of the matrix propagator, or on the core function $\bar{\Pi}_{\mu\nu}$, since the diagonalising bosonic Bogoliubov matrices have no effect on the tensor structure. This result has been explicitly checked in TFD to 1 loop order in [FS2]. We have only four independent symmetric tensors to play with, $g_{\mu\nu}$ and the combinations $p_\mu p_\nu$, $u_\mu u_\nu$, and $p_\mu u_\nu + u_\mu p_\nu$. Following Weldon [We1] we define

$$P_{\mu\nu} = (g_{\mu\nu} - u_\mu u_\nu) + \frac{(p_\mu - p \cdot u u_\mu)(p_\nu - p \cdot u u_\nu)}{((p \cdot u)^2 - p^2)}$$

$$Q_{\mu\nu} = \frac{(p \cdot u p_\mu - p^2 u_\mu)(p \cdot u p_\nu - p^2 u_\nu)}{p^2((p \cdot u)^2 - p^2)} \quad (5d.2)$$

where P and Q satisfy (5d.1) and, suppressing some indices,

$$P \cdot P = P \quad Q \cdot Q = Q \quad P \cdot Q = Q \cdot P = 0 \quad (5d.3)$$

$$P_{\mu\nu} + Q_{\mu\nu} = (g_{\mu\nu} - p_\mu p_\nu / p^2) \quad (5d.4)$$

This gives us a self mass of the form

$$\Pi_{\mu\nu} = P_{\mu\nu} \Pi_T + Q_{\mu\nu} \Pi_L \quad (5d.5)$$

where Π_T and Π_L are functions of p^2 and $p \cdot u$. This leads to the full propagator (the core function for the TFD version)

$$-\bar{D}_{\mu\nu} = \frac{P_{\mu\nu}}{p^2 - \Pi_T + i\epsilon} + \frac{Q_{\mu\nu}}{p^2 - \Pi_L + i\epsilon} + \frac{\xi p_\mu p_\nu}{p^2 + i\epsilon} \cdot \frac{PP}{p^2} + c(p) \frac{C_{\mu\nu}}{p^2 + i\epsilon} \quad (5d.6)$$

where ξ is the gauge parameter (see free propagator, (A.11)). We now see that we have gluon masses appearing, $\sqrt{\Pi_{T/L}}$, corresponding to the two transverse and one longitudinal modes (in a 3 spatial dimensional sense rather than the 4-vector sense associated with equations like (5d.1)). The last term $c(p)C_{\mu\nu}$ contains the other independent symmetric vector not as yet used. It is zero in the Landau gauge [KK2] and has been found to be zero in all other approximations used [LvW/sec.3.3.2]. We shall ignore this last term. Thus the 1-1 TFD propagator is of the form

$$D_{11\mu\nu}(p) = -\sum_{x=1}^3 X_{\mu\nu}^x \left(\frac{i}{p^2 - \Pi_x + i\epsilon} + 2\pi\delta(p^2 - \Pi_x) \cdot f_{11}^+(p_0) \right)$$

$$X_{\mu\nu}^x = P_{\mu\nu}, Q_{\mu\nu}, \xi k_\mu k_\nu / k^2, \quad \Pi^x = \Pi_T, \Pi_L, 0$$

$$\text{for } x=1,2,3 \text{ respectively} \quad (5d.7)$$

where f_{11}^+ are given in (A.12). We calculate the Π functions from the self-mass function for the gluon, which to lowest order is the diagrams given in fig.(5d.1) and has been calculated up to a single last integral for general momentum in the Feynman gauge by Weldon [We1].

We now insert this into the equation for the quark self mass with the ladder approximation already used i.e. equation (5c.1). We end up with a form similar to the one we had in chapter 5, section c, but now the gauge-independent part of the propagator splits into two parts, with $\Pi_{L/T}$ masses coming in weighted 1:2, reflecting the number of longitudinal and transverse modes available. This gives us

$$M - m_0 = \frac{i}{4} \text{Tr} \{ \Sigma \} = g^2 C \int d^4 k \cdot M(k) \cdot P_q(k).$$

$$(2P_g(k+p, \Pi_T) + P_g(k+p, \Pi_L) + \xi P_g(k+p, \Pi=0)) \quad (5d.8)$$

where P_g was defined before in (5c.3), and $P_q = P_{11}$ of (5a.5) contains the a, A, and B functions of $\text{Tr} \{ \gamma^\mu \Sigma \}$ from (5.9).

Note how the gluon propagator contributions have a simple form

after the complicated look of (5d.7).

Now we apply the Munczek approximation to (5d.8), replacing all $\Sigma(k_\mu)$ inside the integral by $\Sigma(p_0 \pm \Pi_{\text{phys}}, \underline{p})$ where the Π_{phys} are the physical masses of the gluon modes, i.e. $\Pi_{\text{phys}}(\underline{p}) = \Pi(p^2 = \Pi_{\text{phys}})$. This now shows us one of the big problems of the Munczek approximation when we include temperature-induced gluon masses, as now the equation relates $M(p_\mu)$ to $M(p_0 + \Pi_{\text{phys}}, \underline{p})$. Surprisingly, we find that, to one-loop, the longitudinal gluon mode has a zero mass. The transverse gluon mode does have non-zero mass and so the Munczek approximation on this part means that the equation now needs solving at all p_0^2 , whatever we fix $\underline{p}$ to be, even when we only want the transition temperature. The same problem appears in the equations for A and B functions making simple analysis impossible for these functions.

We could try to forget the Munczek approximation and do the energy integral. Then all the functions A, B, M and the Π 's are only needed at the physical quark and gluon poles. Unfortunately, unlike at zero temperature, these are usually 3-momentum dependent. This is because without Lorentz invariance, the masses depend on p^2 and, say, p_0 , so the solution to $m_{\text{phys}} = m(p^2 = m, p^0)$ is $\underline{p}$ dependent. Even though the M and Π_L physical functions are going to be zero and hopefully v-independent at the transition temperature, the Π_T physical function is not zero and is 3-momentum dependent, so just as when we try the Munczek approximation, it is the transverse gluon mass that ruins everything.

It is useful to look at the properties of the gluon self-mass functions $\Pi_{L/T}$, for which we follow the results of Weldon

[We1]. This work was done with ITF but at least to 1 loop, TFD/PIRT seems to be in good agreement with ITF generally, so this is no problem. The expressions are also given for the Feynman gauge ($\xi=1$), rather than the Landau gauge ($\xi=0$) we prefer. It has been shown that the leading T^2 terms of the high temperature expressions are gauge invariant [Ue], though the poles of the gluon propagator are not physical and are generally gauge dependent. Weldon only gives exact expressions for the high temperature limit, but the cutoff found in the approach of [BCS] to the zero-loop approximation for the full gluon, suggests that momenta up to several times larger than the critical temperature are relevant. To extract some exact results is more difficult, as there is one last intractable integral left in the expression of [We1], (the thermal statistics factors almost always make calculation in terms of known functions difficult, an infinite sum of modified Bessel functions being the best one can do in some simple situations!). In the expressions of Weldon [We1] we put $p^2=0$ before doing the integrals but keep $\underline{p}$ arbitrary (so that the problem of non-zero momentum analyticity, problem (e) of chapter 2, is not relevant). This gives, exact to one loop, for all temperatures, and in the Feynman Gauge,

$$\Pi_L(p^2=0) = 0 \quad (5d.9)$$

$$\Pi_T(p^2=0) = (N_c + \frac{N_f}{2}) \frac{g^2}{6} T^2 = GT^2 \quad (5d.10)$$

where $g^2=4\pi\alpha$. This Feynman gauge calculation agrees with the axial gauge calculation [KK2/3.31,3.32] and the Coulomb gauge calculation [Sh]. It also agrees with the high temperature

limit [We1] which is known to be gauge-independent [Ue]. This limit is thought to be gauge-independent [Sh]. Equation (5d.9) also shows us that the physical longitudinal mode gluon mass, $\Pi(p^2=m_{\text{phys}}^2)=m_{\text{phys}}^2$, is zero, while the physical transverse mode mass is non-zero, and numerical investigations have confirmed this for a considerable range of momenta.

There seems to be no analytic way forward. Even extending the Munczek approximation to the gluon mass functions doesn't help because the physical transverse gluon mass is non-zero, and this upsets the $A=B=0$ approximation as well as everything else.

We might try to improve the work of Atkinson et al. [AJK], who at zero temperature have a massive gluon and a running coupling constant. If we used the correct non-zero temperature gluon propagator, distinguishing the longitudinal and transverse modes, and used renormalisation group calculations to get the running coupling constant [MNU4,5, MS2, FS1, FINY, FY], we could then systematically introduce the correct temperature corrections for these approximations to the full gluon propagator and full vertex of fig.(5.3). Whether the approach of [AJK] to look at the eigenvalues of integral equation kernels makes the calculations easier is doubtful.

Sticking to our approach, we therefore make the grossest assumptions to get a feel for the way the temperature-induced gluon masses may effect the solution. Even then, it is not as gross as the approximations made at zero temperature in [AJK]. For this purpose we ignore all momentum dependence of the fermion mass (self-consistent in that we are looking for the temperature at which the only solutions become of the form $M=0$

for all momenta). We also assume that the physical transverse mass is the same as the $\Pi_T(p^2=0, p)$ value, which is 3-momentum independent. Doing the k_0 integral, setting $|\underline{y}|=0$ for simplicity, and looking at the physical quark poles, $p^2=M^2$, we get

$$\begin{aligned}
 M(p^2=M^2) = m_0 + \sum_{\text{modes}} & - \int d^3 \underline{k} \, M \left\{ \frac{1}{2\omega} \cdot \frac{1}{2M(\omega+m)-\Pi_1(k+p)} \right\} \Bigg|_{k^2=M^2} \\
 & + \frac{1}{2\Omega} \cdot \frac{1}{\Pi_2(k)-2M\Omega} \Bigg|_{k^2=\Pi_2} \Bigg\} \\
 & + \int d^3 \underline{k} \, M \left\{ \frac{1}{e^{\beta\Omega}-1} \cdot \frac{1}{2\Omega} \cdot \frac{2\Pi_2}{\Pi_2^2(k)-4M^2\Omega^2} \right\} \Bigg|_{k^2=\Pi_2} \Bigg\} \\
 & - \int d^3 \underline{k} \, M \left\{ \frac{1}{e^{\beta\Omega}+1} \cdot \frac{1}{2\omega} \cdot \frac{2(2M^2-\Pi_1(k+p))}{(2M^2-\Pi_1(k+p))^2-4M^2\omega^2} \right\} \Bigg|_{k^2=M^2} \Bigg\}
 \end{aligned}
 \tag{5d.11}$$

where $\omega^2=\underline{k}^2+M^2$, $\Omega^2=\underline{k}^2+\Pi_2(k)$. There are two transverse modes, one longitudinal mode and a 'mode' containing the gauge dependent piece ($\propto \xi$ of (5d.12)) with $\Pi=0$ there. Π is the gluon self-mass function in each mode, where the subscript 1 indicates it is to be evaluated at the physical quark pole, and a subscript 2 means Π is evaluated at the gluon pole for that mode.

We have kept the momentum dependence of the gluon self mass functions so that we see the need to know the gluon mass at its 'physical' value and at the quark pole, the latter being at zero momentum for the transition calculation. Another significant point is how for a gluon mode with non-zero physical mass, the thermal statistical term of the

gluons, the term with a Bose-Einstein distribution in it, does make a direct contribution. This didn't happen in the case we had for the single, free gluon exchange, where the gluon was massless gluon, and the term with thermal gluon contributions dropped out. Thus, gluon modes with temperature induced non-zero masses, from the sea of real particles in the heat bath do interact with the long range forces at the second order phase transition (for the exact CS case), while the massless gluon modes do not. (This is the physical interpretation of the contribution of the term with bosonic thermal statistics). Proceeding we then get for small dynamical quark physical mass

$$0 = (1+\xi) \left(\ln((M_0)^2) \right)_{\overline{M}^2} + 2 \left(\ln((M_0)^2) - 1 \right)_{\overline{\Pi}_T} - (1+\xi) \left(\ln\left(\frac{T^2}{\overline{M}^2}\right) + 2(\ln(\pi) - \gamma) \right) - 2 \left(8F_1^-(\beta/\Pi_T) + \frac{8\pi^2 T^2}{12\Pi_T} \right) \quad (5d.12)$$

where $F_1^{\pm}(b) = \int dx [(x^2-1)^{1/2} \cdot (e^{\beta x} \pm 1)^{-1}]$. The massless longitudinal and 'gauge dependent' modes lead to $F_2^{\pm}$ functions whose small mass limit is given in (5c.23), γ being Euler's constant. The physical dynamical quark mass is $M=M(T)$, and $M_0=M(T=0)$. We see that there is no Π_L term as it is zero at one loop by (5d.9), so the longitudinal term is of the form found in section (c) (c.f. (5c.22), (5c.24)), where Bose-Einstein statistics play a role. The transverse mode comes in with a factor 2 over the longitudinal term and with bosonic statistics making a contribution via F_1^- . We also note that the transverse modes have no infrared-divergent

explicitly temperature-dependent terms, unlike the case with the massless gluon modes. The result is

$$T_c = M_0 \exp \left\{ -\frac{1}{2(2+(1+\xi))} \left(2 \ln G + 2 + 2(1+\xi)(\ln \pi - \gamma) + 2 \left(8F_1(\sqrt{G}) + \frac{2\pi^2}{3G} \right) \right) \right\} \quad (5d.13)$$

where $G = (N_c + N_f/2)g^2/6$ from the definition of Π_T in (5d.10). This result has been achieved by making several further approximations beyond the ladder approximation on the kernel of fig.(5.3) discussed in the first part of this chapter.

Firstly, we have ignored the effect of the momentum dependence of Σ . This is at least self-consistent in that we are looking for when the CSB solution, $M \neq 0$ for some momenta, smoothly becomes an $M=0$ solution for all momenta. It is as if we were doing a functional expansion of the quark propagator around $\Sigma=0$, which would be valid only at the critical temperature, and then only if we have $A=B=0$ for all momenta at the critical temperature.

The second assumption is that $A=B=0$ (the $\text{Tr}\{\gamma^\mu \Sigma\}$ functions, (5.9)). However, this is not even true at zero temperature in the Landau gauge if we have massive gluons according to [DMF]. We still might hope it was a good approximation in the Landau gauge, and try working with $\xi=0$.

Unfortunately, the calculation for Π , the gluon self-mass was done in the Feynman gauge, $\xi=1$. Though the high-temperature results are gauge independent [We1, Ue], the full function, $\Pi(p^\mu)$, is not. However, it is thought that at all temperatures, the $\Pi(p^2=0, p)$ values ought to be gauge independent [Sh]. The Feynman gauge results derived here,

(5d.9) and (5d.10), do in fact agree with axial gauge [KK2/3.31,3.32] and Coulomb gauge calculations [Sh]. We presume, therefore, that the variation in $\Pi(p^2=0, \underline{p})$, if any, is small in going from $\xi=1$ to $\xi=0$ (which the $A=B=0$ assumption seems to suggest we use), when taken with our dropping of any momentum dependence in Π .

Lastly, we are ignoring the momentum dependence of the gluon self-mass functions, otherwise we would have no hope of doing this analytically, since these functions for the transverse 'physical' mode appear to involve an intractable integral.

However, the purpose was to have a first attempt at incorporating the more general temperature effects such as massive gluons and the correct full tensor structure of the gluon propagator, which we have done. We now see what results all this has actually produced.

We note that the result has some explicit gauge dependent terms. Just like the single free gluon exchange model of section (c), there has been some implicit gauge fixing. The approximation $A=B=0$ works only in the Landau gauge, $\xi=0$, (if at all). The Π calculation was done by Weldon [We1] in the Feynman gauge, $\xi=1$, though the all-temperature results we use are the same as would be obtained in the gauge-independent high-temperature limit, so since we are already mauling Π in ignoring its momentum dependence, we will ignore any gauge dependence in Π . We hope for reasonable results when working with $\xi=0$, and show $\xi=1$ results for comparison. There is also some explicit coupling constant dependence.

The results are shown in fig.(5d.2). For a wide range of

values for α in the strong coupling regime, including what we think of as the physical values $\alpha \sim 0.75$, $0.5 \lesssim \alpha \lesssim 10$, there is significant CSB, shown by the high critical temperatures, $\sim 60\text{MeV}$. Here we use a value for the zero temperature quark mass of $\sim 300\text{MeV}$, compatible with [Be, BMZ, BCS]. The maximum seems to be at $\alpha = 1.05$ independent of the gauge. For weak coupling, $\alpha < 0.5$, we have feeble CSB, in that the critical temperature is low. This picture of CSB at strong coupling constant agrees with the view of Peskin [Pe].

It is interesting to compare this result with the critical temperatures of the NJL model approach of section (a) (independently calculated in [BMZ]), $T_c \sim 130\text{MeV}$ for $M_0 \sim 275\text{MeV}$ (M_0 fixed by other physical parameters), and with the single free gluon exchange model of section (c) (which can now be seen as the origin of the calculation of [BCS]), where $T_c \sim 180\text{MeV}$ for $M_0 \sim 300\text{MeV}$ (M_0 an input parameter). Since M_0 seems to be a reasonable scale, and to relate the approaches where M_0 may be given rather than fixed by other physical input values, we compare the ratio T_c/M_0 . This is about 0.55 for both the earlier approaches, but when we take into account the gluon masses and full non-zero temperature tensor structure of the gluon as here in section (d), we find $T_c/M_0 \sim 0.20$. It is a large shift, but it is difficult to see quite how significant this result is, given all the approximations used.

There are a few things we might do to improve the confidence in the actual number the approach of this section leads to. It should be simple enough to check that the $\Pi(p^2=0, p)$ values used, coming from the Feynman gauge expressions of [We1], are unchanged in the Landau gauge being used in the rest of the calculation. This value is the same in the Landau, axial [KK2], and Coulomb [Sh] gauges. More importantly, the role of wavefunction renormalisation, $A=B=0$ of (5.9), needs to be better understood, both here and in the simpler approximation of section (c). The work here is intended mainly as a pointer as to what the full effects of non-zero temperature are, especially as regards the gluons, and how they might alter the result.

In view of the problem with finite chemical potential encountered in the TFD/PIRT calculation for the NJL model, when compared with the ITF result of [BMZ], no attempt has been made to include the effect of non-zero baryon density.

CHAPTER 6

Conclusions

6) CONCLUSIONS.

The most serious problem that permeates this thesis is the lack of analyticity at zero momentum of finite temperature Feynman diagrams. This is the worst problem because it appears in both TFD/PIRT and ITF, and it has physical implications, and therefore seems a much deeper problem than the others in discussed in chapter 2. The zero energy limit is going to be associated with the large-time boundary conditions used in non-zero temperature formalisms. These boundary conditions are also relevant to other problems encountered. The difference between the physical results in TFD/PIRT and ITF (problem a of chapter 2) may well be related to the use of different large-time boundary conditions. Then there was the problem with the non-zero chemical potential in the non-perturbative TFD/PIRT calculation in the NJL model of section a of chapter 5. It seems as if by altering the $i\epsilon$ prescription in the propagators, that then the TFD/PIRT calculation can be made to agree with the ITF result of [BMZ]. This all seems to suggest that a more careful study of the role of the large-time boundary conditions in equilibrium finite-temperature field theory formalisms is necessary.

For the application to chiral symmetry in QCD at non-zero temperature, the main conclusion is that major improvements to the treatment of the gluons at non-zero temperature are needed. These lead to significant quantitative changes in the results (though due to the limitations of the approximation schemes used, these results could be doubted), and more

importantly, significantly different qualitative changes in the calculation. This is most clearly seen in section d of chapter 5, where the gluon propagator is seen to have a much more complicated tensor structure, and in the appearance of gluon modes with distinctly different properties, e.g. a massless longitudinal mode as opposed to the massive transverse modes found at the one-loop level. Unfortunately, while it appears this would introduce major changes in the calculation, (e.g. thermal gluonic effects only appear when there are massive gluonic modes as seem to be present at finite temperature), the calculations become very much more difficult. This is on top of the difficulties that the 'loss' of Lorentz invariance and the thermal distribution factors lead to when we compare with the zero-temperature calculation. The calculation in chapter 5 is a mere sketch because of these difficulties, but it does show the significant qualitative changes to the calculation, and that these can lead to significant quantitative effects, when the correct non-zero temperature effects are included for the gluons. This is in contrast to the existing literature, which concentrates on the quark thermal properties rather than those of the gluons.

APPENDICES

Appendix A:- Feynman Rules at Non-zero Temperature.

The diagrammatic structure of ITF and TFD/PIRT is unchanged from the zero-temperature case. The actual Feynman rules are unchanged unless explicitly stated below. In particular, four-momentum conserving delta functions, coupling constants, sign and i factors for vertices are present as in the zero temperature rules, and there are additional factors of -1 for fermion traces present. The conventions followed here coincide with those of [CL/app.B], so there can be found the zero-temperature versions of the rules given here.

Imaginary Time Formalism:ITF.

Work in the Euclidian version of the zero temperature theory, replacing the energies $\{k^0\}$ by their Euclidian versions $\{k^4\}$ ($=\{-ik^0\}$). Use all the zero temperature Euclidian propagators and vertex factors.

Replace ALL Euclidian energies $\{k^4\}$ by discrete valued variables $\{\omega_n\}$, a different integer variable n associated with each energy variable, where

$$\omega_n = (2n)\pi/\beta - \mu \quad (A.1)$$

$$\omega_n = (2n+1)\pi/\beta - \mu \quad (A.2)$$

where n is an integer, μ is the chemical potential, β the inverse temperature (see [LvW/sec.2.3.1]).

Replace all Euclidian energy integrals by sums over the

associated integer variable n

$$\int \frac{dk_4}{2\pi} \rightarrow \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \quad (\text{A.3})$$

If the result is a function of external energies, they will only be defined at integer imaginary Minkowski values. Analytic continuation must be performed to find physical real Minkowski energy values. This is not easy [LvW/sec.2.3.3, MNUMM]. The energy sums and analytic continuation of ITF can be done in a number of ways: one way which shows up the similarity with TFD expressions is to turn the sums over loop-energy variables in ITF calculations into integrals using (for the scalar, zero chemical potential case)

$$\begin{aligned} \sum_n \frac{1}{\beta} f(\omega_n) &= \left(\int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} + \int_{+\infty+i\varepsilon}^{-\infty+i\varepsilon} \right) dk_4 \frac{1}{2\pi i} \cdot f(k_4) \cdot \text{CT}((\beta k_4 + \mu)/2) \\ &= \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} f(k_4) + \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \frac{dk_4}{2\pi} (f(k_4) + f(-k_4)) \cdot f_{11}^{\bar{}}(ik_4) \end{aligned} \quad (\text{A.4})$$

where we have $\text{CT} = \cot(\tan)$, and use $f_{11}^{\dagger}(f_{11}^{\bar{}})$ for bosons (fermions) of (A.12). This should be compared with the 1-1 propagators of TFD, which are the old real-time formalism propagators of Dolan and Jackiw [DJ].

Thermo-Field Dynamics and Path Integral Real Time formalisms:
TFD/PIRT.

There are two approaches to these Feynman rules, a canonical approach operator approach, TFD [TU, Ma, UMT, LvW], and a particular version of older path integral techniques [NS1,2, LvW], detailed in chapter 1. The old real-time formalism of Dolan and Jackiw [DJ] appears as an incomplete version of these rules.

We have two types of every field we had at zero temperature, a type-one or physical field, and a type-two or tilde field. The latter is associated with the extra degrees of freedom present due to having a thermal reservoir as part of the whole system.

No mixing of these fields occurs at vertices, so we add a thermal label (a,b,...) which can take values 1,2 only, to each of the $T=0$ graph vertices, to indicate whether type 1 or 2 fields are interacting there. If it is type 2, then we have an additional factor of -1 for this vertex over the usual $T=0$ factors. This labelling is ignored for the purposes of calculating symmetry factors.

The derivations of Feynman rules in the literature then show the need for two extra Feynman rules for TFD/PIRT diagrams:-

Rule 1 states that we sum over all possible permutations of values for the vertex thermal labels.

Rule 2 states that if we want to extract the physical part of a function from a Feynman diagram calculation, since the type one fields are the physical fields, we set all external vertices or legs to be type 1, and this is in exception to rule 1.

The propagators in TFD do mix the type 1 and 2 fields in general. They are two by two matrices in thermal indices, the value of the thermal label on the vertex at each end of the propagator line in a Feynman diagram, selects the entry of the propagator matrix to be used. For instance, fig.(A.1) in TFD is proportional to

$$\theta_c \int d^4k \int d^4l \Delta_{ab}(k) \cdot \Delta_{ac}(p+k) \cdot \Delta_{cb}(l) \cdot \Delta_{cb}(p+k+l) \quad (A.5)$$

where Δ_{ab} are TFD/PIRT propagator matrices. We sum over $c=1,2$ by rule 1 but fix $a=b=1$ by rule 2, and $\theta_c=1,-1$ for $c=1,2$ for the additional minus sign for type 2 vertices.

Some care is needed with the forms of the TFD/PIRT propagators, as there were some small but significant errors in the early literature, as pointed out in [Ko, KK1], but we have

Scalar $\underline{\Delta}(k^\mu; T, \mu)$

$$i\Delta_{11} = \frac{i}{k^2+m^2+i\epsilon} + 2\pi\delta(k^2-m^2)f_{11}^-(k^0) \quad (A.6)$$

$$i\Delta_{12} = i\Delta_{21} = 2\pi\delta(k^2-m^2)f_{12}^-(k^0) \quad (A.7)$$

$$i\Delta_{22} = (i\Delta_{11})^* \quad (A.8)$$

Fermion $\underline{\underline{S}}(k^\mu; T, \mu)$

$$iS_{11} = \frac{i}{k-m+i\epsilon} - (\not{k}+m)2\pi\delta(k^2-m^2)f_{11}^\dagger(k^0) \quad (A.9)$$

$$iS_{12} = -iS_{21}e^{\beta\mu} = (\not{k}+m)2\pi\delta(k^2-m^2)f_{12}^\dagger(k^0) \quad (A.10)$$

$$iS_{22} = (iS_{11})^*$$

where no complex conjugation of the γ matrices is implied by the $*$.

Massless Vector Boson $\underline{\underline{D}}(k^\mu; T, \mu)$

$$iD_{\underline{\underline{\mu\nu}}} = -(g_{\mu\nu} + (\xi-1)\frac{k_\mu k_\nu}{k^2})i\underline{\underline{\Delta}} \quad (A.11)$$

Statistical Factors

$$f_{11}^{\pm}(k^0) = \frac{\theta(k^0)}{e^{x_{\pm}1}} + \frac{\theta(-k^0)}{e^{-x_{\pm}1}} \quad (\text{A.12})$$

$$f_{12}^{\pm}(k^0) = \left(\frac{\theta(k^0)e^{x/2}}{e^{x_{\pm}1}} + \frac{\theta(-k^0)e^{-x/2}}{e^{-x_{\pm}1}} \right) e^{\mu\beta} \quad (\text{A.13})$$

where $x=\beta(k^0-\mu)$.

A diagonalised form of these propagators is given in appendix B.

Appendix B:- Bogoliubov Matrices and Diagonalisation in
TFD/PIRT.

The matrix propagators and diagrams of TFD/PIRT often seem to have a neat diagonalised form, with Bogoliubov transformation matrices performing the diagonalisation. In the canonical operator approach of TFD to the real-time Feynman rules, a Bogoliubov transformation is central to the whole method, switching from a set of annihilation and creation operators that act on the zero temperature vacuum, to a set of operators built up from what will become a 'finite temperature' vacuum state, containing a condensate of particles in a thermal distribution, c.f. working in QCD with the usual operators but now using a vacuum with a quark condensate. A set of creation and annihilation operators and their vacuum state form a representation of the Hilbert space, and the Bogoliubov transformation relates different representations. In Quantum Field Theory, with an uncountably infinite number of possible representations for its space of all states (quantum mechanics has a countably infinite number of representations), different representations are generally unitarily inequivalent. TFD works by first doubling the zero temperature Hilbert space which contains the physical operators, while the new second copy is the unphysical tilde operators. Then it says that the physical vacuum is actually some state that forms the vacuum of another, unitarily inequivalent representation. The Bogoliubov transformation links the two representations, while the physics says we must use the operators of the old representation for physical

excitations, but that the physical vacuum is that of the other representation. The Bogoliubov transformation matrices are fixed by demanding that there is a thermal equilibrium distribution (Bose-Einstein or Fermi-Dirac) of real particles in the new thermal vacuum.

In the Path Integral approach, the Bogoliubov matrices only appear after the calculations have been done, and then they appear to be merely convenient matrices to write down useful diagonal versions of propagators etc.

The Bogoliubov transformation matrices that appear in the matrix formulation of TFD/PIRT are denoted $U_{B/F}$ for Bosonic/Fermionic statistics respectively, and are

$$\underline{U}_B(k^0) = \begin{pmatrix} \text{ch}(\theta) & e^{-\beta\mu} \text{sh}(\theta) \\ e^{\beta\mu} \text{sh}(\theta) & \text{ch}(\theta) \end{pmatrix} \quad (\text{B.1})$$

$$\text{sh}^2(\theta) = \frac{\theta(k^0)}{e^x - 1} + \frac{\theta(-k^0)}{e^{-x} - 1} = f_{11}^-(k^0) \quad (\text{B.2})$$

$$\underline{U}_F(k^0) = \begin{pmatrix} \cos(\theta) & -e^{-\beta\mu/2} \sin(\theta) \\ e^{\beta\mu/2} \sin(\theta) & \cos(\theta) \end{pmatrix} \quad (\text{B.3})$$

$$\sin^2(\theta) = \frac{\theta(k^0)}{e^x + 1} + \frac{\theta(-k^0)}{e^{-x} + 1} = f_{11}^+(k^0) \quad (\text{B.4})$$

where x and $f_{11}^\pm$ are found in (A.12). The U matrices have the property

$$\underline{\tau} \cdot \underline{U} \cdot \underline{\tau} \cdot \underline{U} = \underline{\mathbb{1}} \quad (\text{B.5})$$

where the τ matrix is

$$\underline{\underline{\tau}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{B.6})$$

The interesting thing is that the thermal 2-point functions, have Lehman-Källén (spectral) representations of the form, showing only the matrix structure,

$$\underline{\underline{\Pi}} = \underline{\underline{U}} \cdot \underline{\underline{\bar{\Pi}}} \cdot \underline{\underline{U}} \quad \underline{\underline{\bar{\Pi}}} = \begin{pmatrix} \bar{\Pi} & 0 \\ 0 & -\bar{\Pi}^* \end{pmatrix} \quad (\text{B.7})$$

$$\underline{\underline{\Sigma}} = \underline{\underline{\tau}} \cdot \underline{\underline{U}} \cdot \underline{\underline{\bar{\Sigma}}} \cdot \underline{\underline{U}} \cdot \underline{\underline{\tau}} \quad \underline{\underline{\bar{\Sigma}}} = \begin{pmatrix} \bar{\Sigma} & 0 \\ 0 & -\bar{\Sigma}^* \end{pmatrix} \quad (\text{B.8})$$

where Π and Σ are the connected and 1PI two-point functions respectively, and we distinguish between the diagonal matrices $\underline{\underline{\bar{\Pi}}}, \underline{\underline{\bar{\Sigma}}}$ and the functions $\bar{\Pi}, \bar{\Sigma}$. The U matrix is $U_{B/F}$ depending on the statistics of the field whose two point function it is. The important point is that the matrix structure of these functions is completely determined and known, since the Bogoliubov matrices are known, and there is only one unknown function $\bar{\Pi}$ or $\bar{\Sigma}$, called here 'core' functions and denoted by a bar on top and no thermal indices, containing most of the physics.

The free propagators are a special case of the functions above, and we have for a scalar $i\underline{\underline{\Delta}}$, spinor $i\underline{\underline{S}}$ (suppressing spin indices), and vector boson fields $i\underline{\underline{D}}_{\mu\nu}$ 'core' functions of the form

$$i\bar{\Delta} = \frac{i}{k^2 - m^2 + i\varepsilon} \quad (\text{B.9})$$

$$i\bar{S} = (\cancel{k} + m) \cdot i\bar{\Delta} \quad (\text{B.10})$$

$$i\bar{D}_{\mu\nu} = (g_{\mu\nu} + (\xi - 1)k_\mu k_\nu \cdot \frac{\text{PP}}{k^2}) \cdot i\bar{\Delta} \quad (\text{B.11})$$

where PP stands for principal part.

Appendix C:- Fermion Self-Mass Function at Non-Zero
Temperature

The self-mass of fermions at finite-temperature and density has quite a different form because of the 'loss' of various symmetries, in particular Lorentz symmetry and particle/anti-particle symmetry. We choose p^0 and $\underline{p}$ as our variables rather than p^2 and $p \cdot u$ where u^μ is the heat bath's velocity with respect to the observer. Considering the spinor structure of Σ , we make the ansatz that

$$\Sigma(p^\mu) = (Ap_0 + a)\gamma_0 - B\underline{p} \cdot \underline{\gamma} - M + m_0 \quad (5c.1)$$

where A, a, B , and M are functions of p_0 , $\underline{p}^2$, and the thermodynamic variables, and m_0 is the bare fermion mass. We note that for zero fermion number density, particle/anti-particle symmetry is restored, so that $\Sigma(p^0) = -\Sigma(-p^0)$, and thus $a=0$.

The full propagator is, with this ansatz for Σ ,

$$\Pi(p^\mu) = (p \cdot \gamma - m_0 + \Sigma)^{-1} = \frac{((A+1)p_0 + a)\gamma - (B+1)\underline{p} \cdot \underline{\gamma} + M}{((A+1)p_0 + a)^2 - (1+B)^2 \underline{p}^2 - M^2} \quad (5c.2)$$

Appendix D:- Path Integral Real Time Formalism.

Consider a scalar field theory, with the generating functional for finite temperature expectation values of the path ordered ϕ products given by

$$Z[j] = \text{Tr}[\exp\{-\beta H\} \cdot T_c \exp\{i \int_C dt \int d^3x j(x)\phi(x)\}] \quad (D.1)$$

Using a complete set of states at some time τ , $\{|\phi, \tau\rangle\}$, to represent the trace

$$Z[j] = \int D\phi \langle \phi, \tau | \exp\{-\beta H\} \cdot T_c \exp\{i \int_C d^4x j(x)\phi(x)\} | \phi, \tau \rangle \quad (D.2)$$

$$= \int D\phi \langle \phi, \tau - i\beta | T_c \exp\{i \int_C d^4x j \cdot \phi\} | \phi, \tau \rangle \quad (D.3)$$

Now if we consider C running from τ to $\tau - i\beta$, we can use standard manipulations to arrive at

$$Z[j] = \int_{\tau \in C} D\phi(\tau) \exp\{i \int_C d^4x (L(\phi) + j\phi)\} \quad (D.4)$$

with boundary condition

$$\phi(\tau) = \phi(\tau - i\beta) \quad (D.5)$$

For the scalar theories we are interested in we have

$$L(\phi) = 1/2 \left(-\frac{\partial^2}{\partial t_c^2} + \frac{\partial^2}{\partial x^2} + m^2 \right) \phi - V(\phi) \quad (D.6)$$

where $\frac{\partial}{\partial t_c}$ is taken along the curve C. This leads to

$$Z[j] = \exp\left\{-i \int_C V\left[-i \frac{\partial}{\partial j}\right]\right\} \cdot \exp\left\{-\frac{i}{2} \iint_C j \Delta_c j\right\} \quad (D.7)$$

where some of the measures have been dropped for convenience.

The periodic boundary conditions of (A.5) lead to the Kubo-Martin-Schwinger [Ku, MS] boundary condition on Δ_c

$$\Delta_c^>(x-y) = \Delta_c^<(x-y) \quad (D.8)$$

where we have made the ansatz

$$\Delta_c(x-y) = \Delta_c^<(x-y) \cdot \theta(x^0 - y^0) + \Delta_c^>(x-y) \cdot \theta(y^0 - x^0) \quad (D.9)$$

When one chooses the solution with the correct causal structure, equivalent to applying Feynman boundary conditions at zero temperature, the unique solution for Δ_c , after Fourier transforming the space coordinates with respect to $\underline{k}$, is

$$\begin{aligned} \Delta_c(\tau, \omega) = -i(2\omega(1 - \exp\{-\beta\omega\}))^{-1} \cdot [& (e^{-i\omega\tau} + e^{-\beta\omega + i\omega\tau}) \theta_c(\tau) \\ & + (e^{+i\omega\tau} + e^{-\beta\omega - i\omega\tau}) \theta_c(-\tau)] \quad (D.10) \end{aligned}$$

where $\omega^2 = \underline{k}^2 + m^2$

Now choose C to be the usual curve of fig.(1.3).

Applying the Riemann-Lebesgue lemma [Ti] and also applying the usual adiabatic switching on of sources

$$\lim_{\text{Re}(\tau) \rightarrow \pm\infty} j(\tau) \rightarrow 0 \quad (\text{D.11})$$

produces the factorisation of Z into two parts as

$$\Delta_c(x_a - y_b) \rightarrow 0 \quad \text{if } x_a \in C_1, C_2 \quad y_b \in C_3, C_4 \quad (\text{D.12})$$

$$Z[j] = Z_{12}[j] \cdot Z_{34}[j] \quad (\text{D.13})$$

where

$$Z_{12}[j] = \exp\left\{\int_{C_1 C_2} V\left[\frac{-i\delta}{\delta j}\right]\right\} \cdot \exp\left\{-i/2 \iint_{C_1 C_2} j \Delta_c j\right\}$$

and similarly $Z_{34}[j]$. To get the usual TFD rules we define, noting the comments of [KK1],

$$j_1(x) = j(x) \quad , \quad j_2(t, \tilde{x}) = -j(t - i\beta, \tilde{x})$$

$$\Delta_{ab}(x-y) = \Delta_c(\text{Re}(x-y)) \quad x \in C_a \quad y \in C_b \quad a, b = 1, 2 \quad (\text{D.14})$$

To get the physical thermal Green functions we use

$$G(x_1, \dots, x_N) = Z^{-1} \prod_{m=1}^N \left(\frac{-i\delta}{\delta j_1(x_m)} \right) \cdot Z[j] \Big|_{j_1=j_2=0} \quad (\text{D.15})$$

It is this functional differentiation with respect to just the type 1 sources, the sources with real time arguments, that leads to rule 2 of appendix A for extracting the physical part of Green functions. Up to an irrelevant constant [A1]

$$Z[j] = \exp\{-\beta F[j]\} \quad F = \text{Helmholtz free energy} \quad (\text{D.16})$$

$$Z_{12}[j] = \exp\{iW_{12}[j]\}$$

W_{12} is the generating functional of the 1/2 TFD connected Greens functions.

The effective action Γ for the 1/2 sector is defined as

$$\Gamma[\bar{\phi}] = W_{12}[j] - \int_C d^4x \, j\bar{\phi} \quad \bar{\phi} = \frac{\partial W_{12}}{\partial j} \quad (\text{D.17})$$

The corresponding effective potential is for space-time independent sources and fields

$$V(\bar{\phi}_1, \bar{\phi}_2) = \Gamma[\bar{\phi}_1, \bar{\phi}_2] / \int_{-\infty}^{\infty} d^4x \quad (\text{D.18})$$

Defining a similar Legendre transform for $F[j]$ when we take $j(x)$ to be a space-time constant J

$$V.\omega(\phi) = F[J] - \int_C J\phi \quad (\text{D.19})$$

Then we get the desired relation between the obvious generalisation of the effective potential in PIRT based on the

1/2 parts of C only that is $V(\bar{\phi}_1, \bar{\phi}_2)$, and the real physical quantity, the free energy $\omega(\bar{\phi})$, associated with the whole of curve C

$$\beta V \cdot \omega(\bar{\phi}) = iV(\bar{\phi}_1, \bar{\phi}_2) \cdot VT - \ln Z_{34} \quad (D.20)$$

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REFERENCES

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FIGURES

FIGURE CAPTIONS

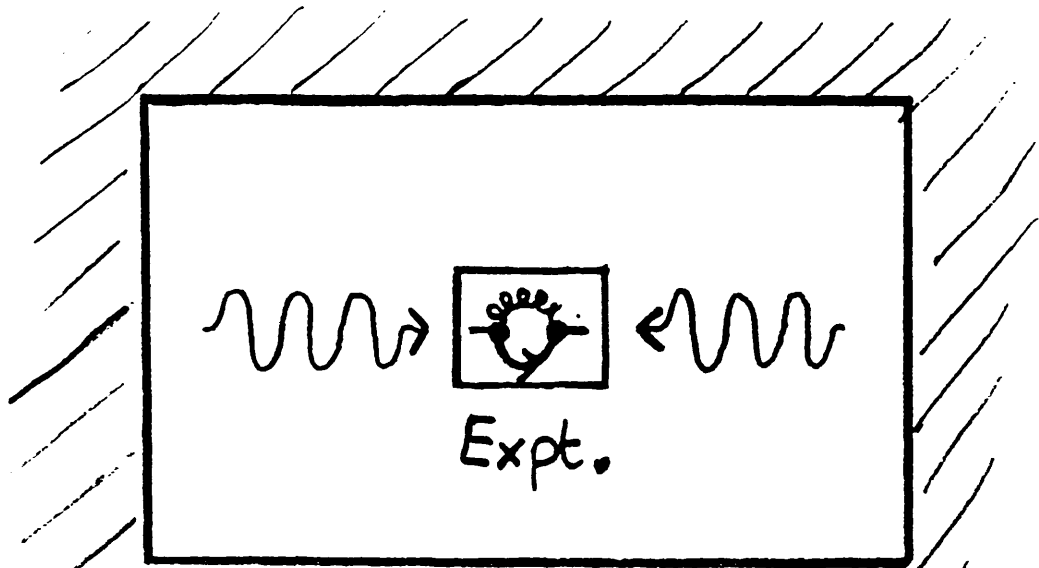
- 1.1 Diagram to illustrate the loss of Lorentz invariance at non-zero temperatures; what an experiment in a heat bath of photons sees when either (i) stationary or (ii) moving with respect to the heat bath.
- 1.2 Curve C in the complex time plane for ITF.
- 1.3 Curve C in the complex time plane for PIRT formalism.
At the end take $\tau \rightarrow -\infty$, $T \rightarrow \infty$. For the usual Feynman rules of TFD/PIRT take $\sigma=1/2$.
- 1.4 Possible alternative type of curve C in the complex time plane that would also lead to a real time formalism.
- 2.1 Diagrammatic structure that leads to apparent singularities in TFD/PIRT, but which eventually cancel.
- 2.2 Diagrammatic structure that leads to apparent singularities in TFD/PIRT which do not eventually cancel when using the Feynman rules of the literature (as in app.A).
- 2.3 Bubble diagram, $-iB_{ab}(p^\mu; T, \mu, \dots)$. Thermal indices, $a, b=1, 2$ are shown.

- 2.4 Analytic structure of the the bubble diagram in the complex energy plane for scalar field, mass m
- 2.5 Diagram to show when the bubble diagram has an imaginary contribution.
- 3.1 A general diagram D , with V vertices of coordinates $\{x^\mu\}$, I propagators, and N sources.
- 4.1 Diagram to illustrate the behavior of the gap equation in the $O(N)$ model.
- 4.2 Diagram to illustrate the behavior of the equation for the bound state masses in the $O(N)$ model.
- 4.3 Solutions for ρ , the constituent field mass of the scalar $O(N)$ model in the phase with $O(N)$ symmetry, for $\rho_0 = \rho(T) = 3.3$, when there are no bound states at zero temperature.
- 5.1 Behaviour of the Free Energy in a first order transition as the order parameter is varied.
- 5.2 Behaviour of the Free Energy in a second order transition as the order parameter is varied.
- 5.3 SD equation for the quark self-mass $\Sigma(p^\mu, T, \mu, \dots)$

- 5.4 SD equation for the quark self-mass $\Sigma(p^\mu, T, \mu, \dots)$ from the point of view of the quarks, illustrating the self-consistent approach.
- 5.5 The quark/quark scattering kernel $K_{acd}^\nu(p^\mu, p'^\mu, k^\mu; T, \mu, \dots)$ to be approximated in the calculation for Σ .
- 5.6 Diagram shown by [AD] to be included in the approximate equation for Σ , when a ladder approximation in a quenched approximation on the SD equation for the vertex function plus Ward identities are used. Diagram is not included when a ladder approximation is applied direct to the SD equation for Σ .
- 5.7 The quark/quark scattering kernel upon which [AD] make their ladder approximation.
- 5.8 The various further approximations to be made on the Kernel K of fig.(5.5):-
- a) Instantaneous quark interaction
 - b) Quark potential, supplied from beyond this calculation
 - c) Exchange of a single free gluon
 - d) Exchange of a gluon, calculated perturbatively to 1-loop.
- 5a.1 Diagrams included in 1-loop approximation for Σ in the NJL model, (iii), (iv) being excluded when working to leading order in $1/N_c$ expansion.

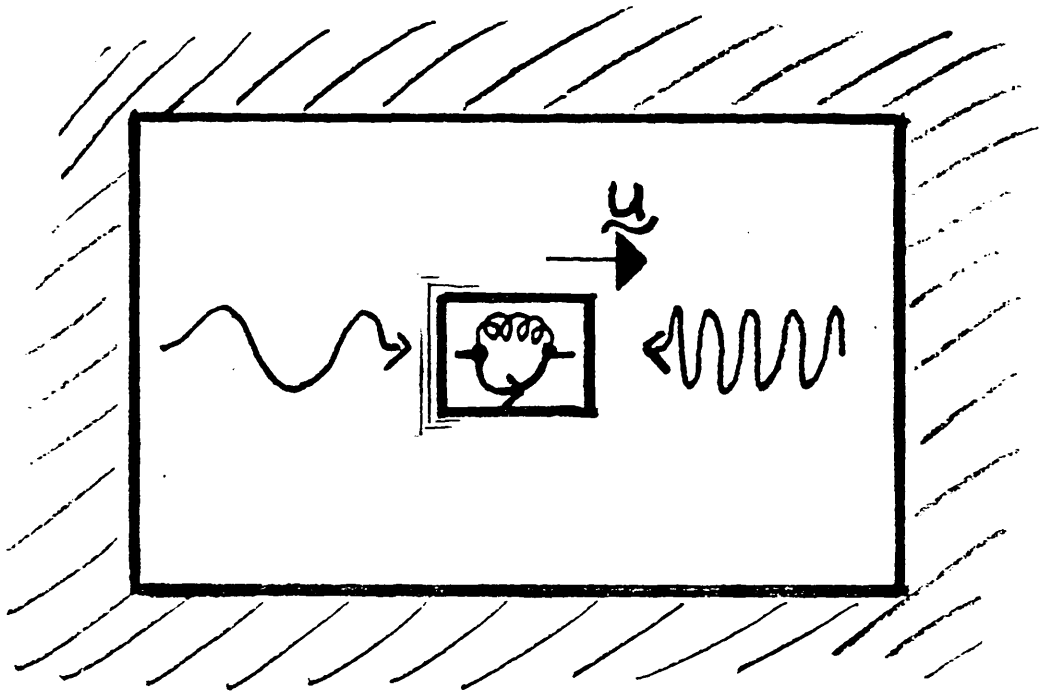
- 5a.2 Type of diagram excluded in expression for Σ either in a 1-loop or a large N_c approximation in the NJL model.
- 5a.3 Diagrams included in the quark/quark scattering function in the NJL model if working to 1-loop but excluded in a large N_c approximation.
- 5a.4 Diagrams not included in the quark/quark scattering function of the NJL model in either a 1-loop or a large N_c approximation.
- 5a.5 The quark/quark scattering function in the large N_c approximation to the NJL model.
- 5a.6 A general vertex in the NJL model, with indices; $a=1,2$ (thermal), $A=1,5$ (scalar/pseudo-scalar index), colour, flavour, spin, and for the pseudo-scalar case, a vector flavour index, $X=1,\dots,(N_f^2-1)$.
- 5c.1 Plots of dynamical 'physical' quark mass against temperature for quarks at rest with respect to the heat bath ($\vec{y}=0$), in units of the zero temperature quark mass, M_0 .
- 5d.1 Diagrams involved in the 1-loop gluon self-mass calculation (the quarks in the fermion loop are taken to be massless).

5d.2 Plot of critical temperature against coupling constant in the 'reasonable' gauges, $0 \leq \xi \leq 1$, for the approximation of the gluon propagator by a 1-loop calculation of section (d) of chapter 5.



(i)

HEAT BATH (BLACK BODY)



(ii)

Fig. 1.1

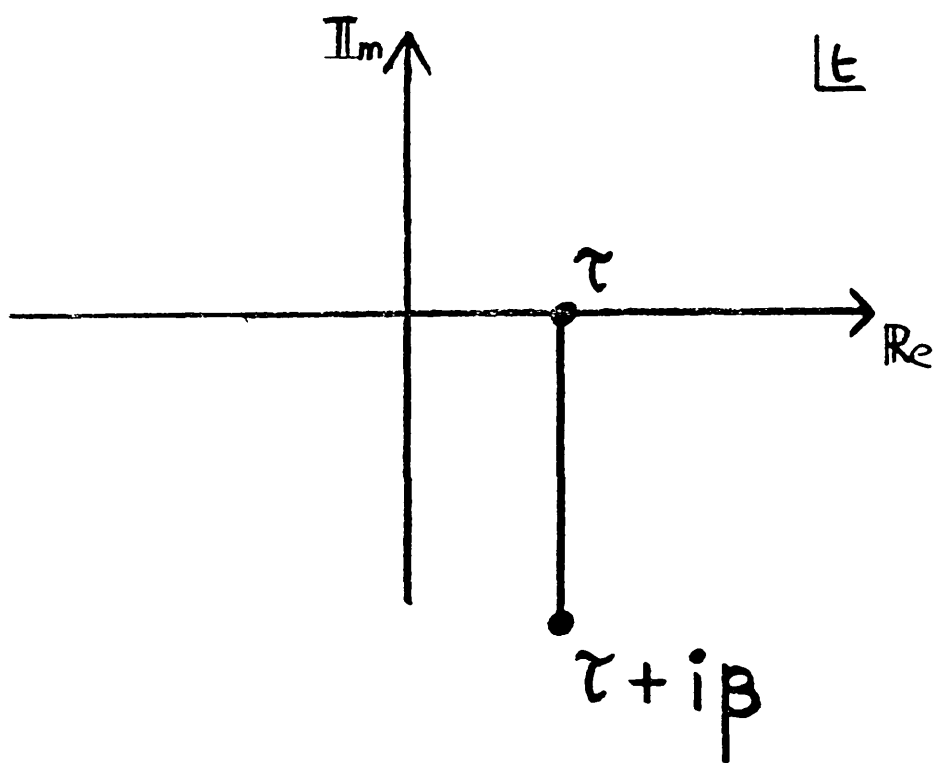


Fig. (1.2)

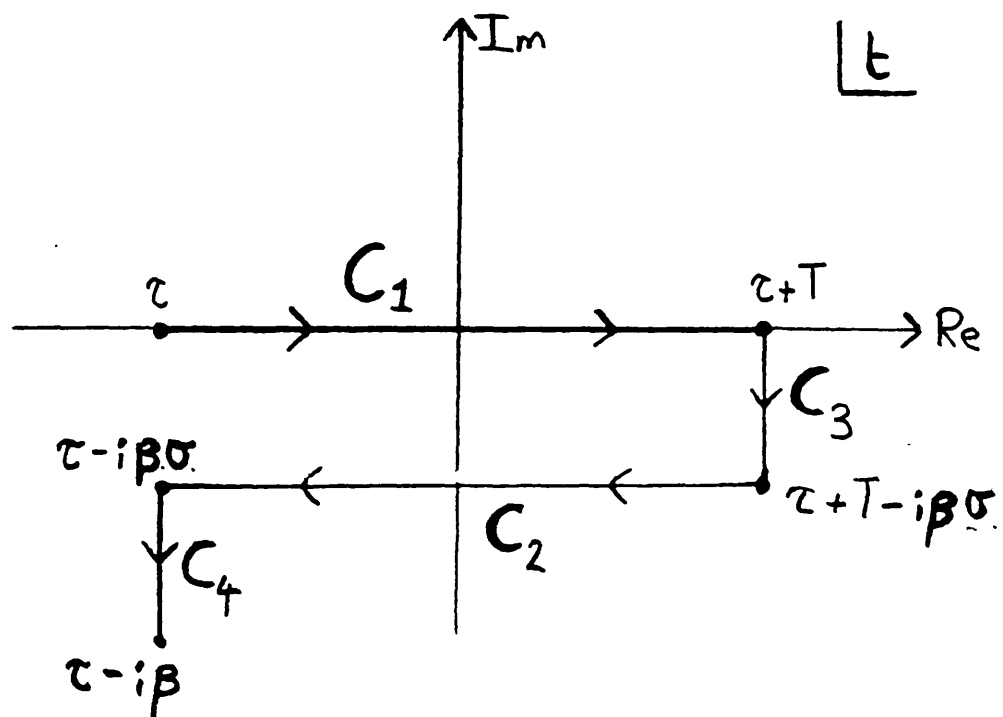


Fig.(1.3).

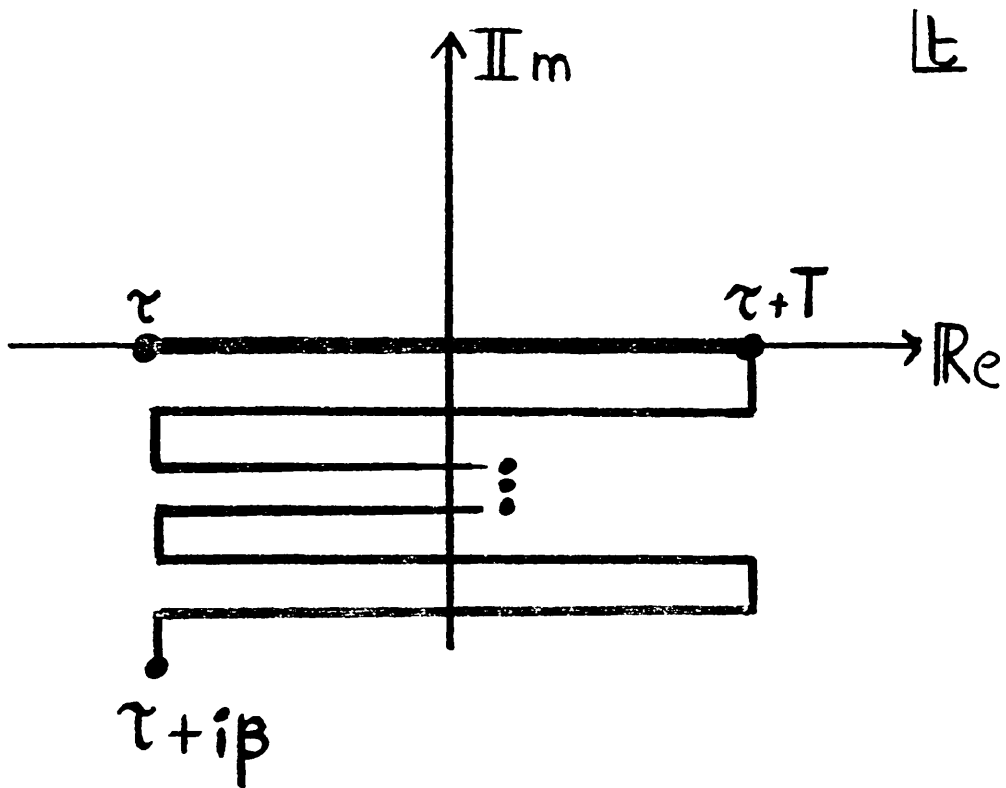


Fig. (1.4).

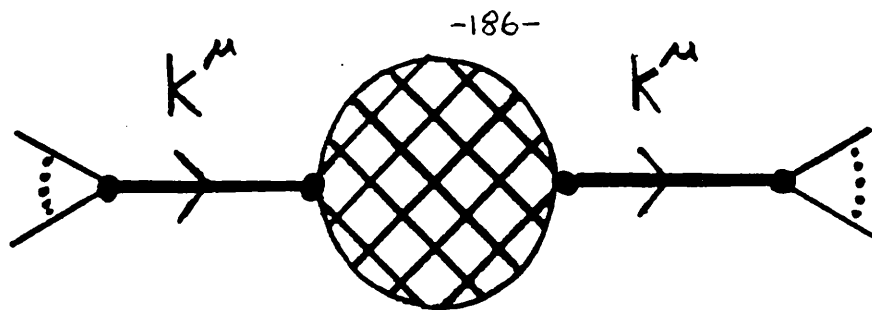


Fig.(2.1).

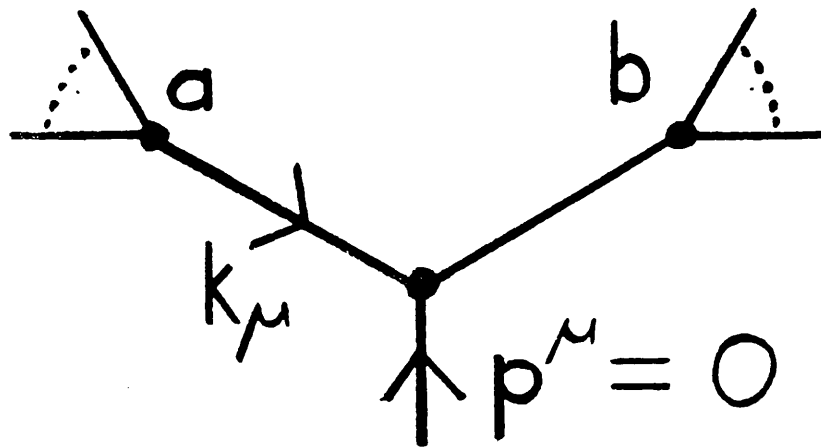


Fig. (2.2)

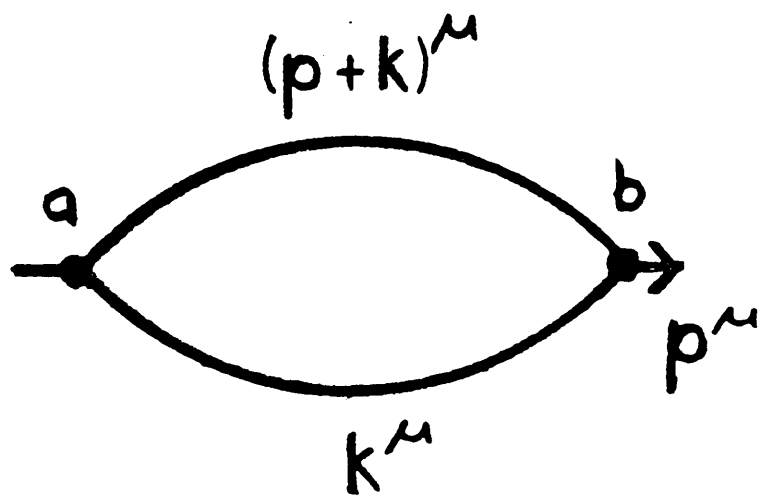


Fig. (2.3).

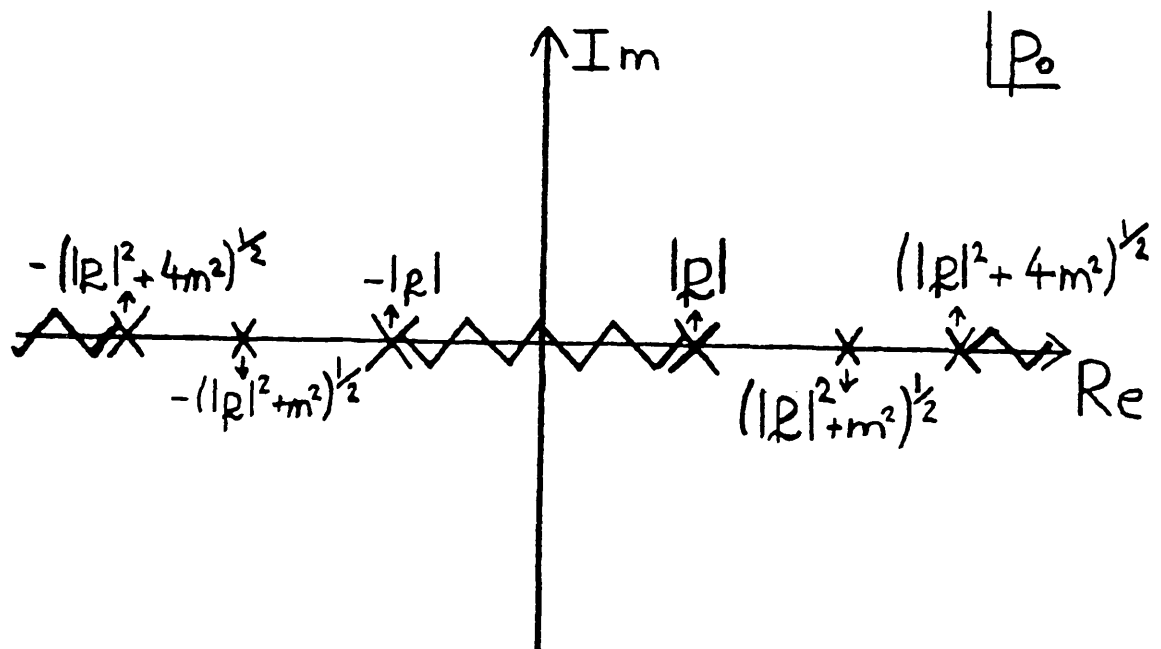


Fig. (2.4).

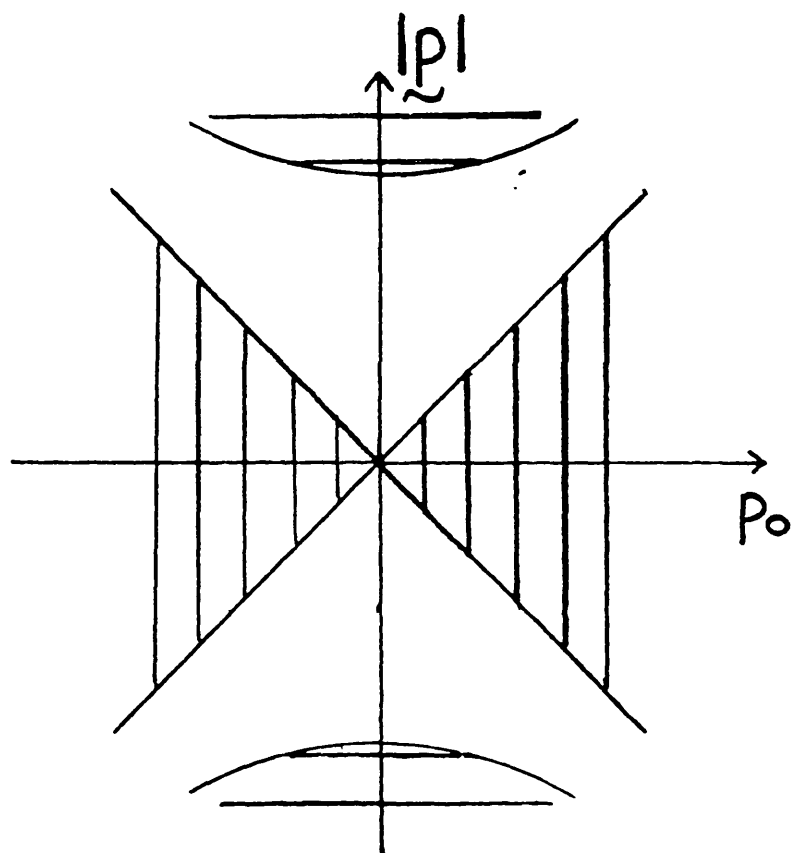


Fig.(2.5).

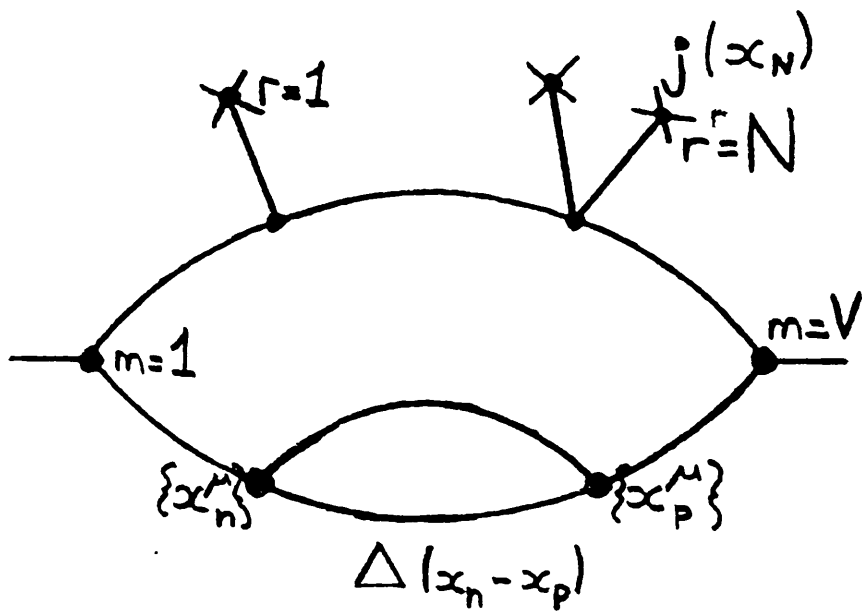
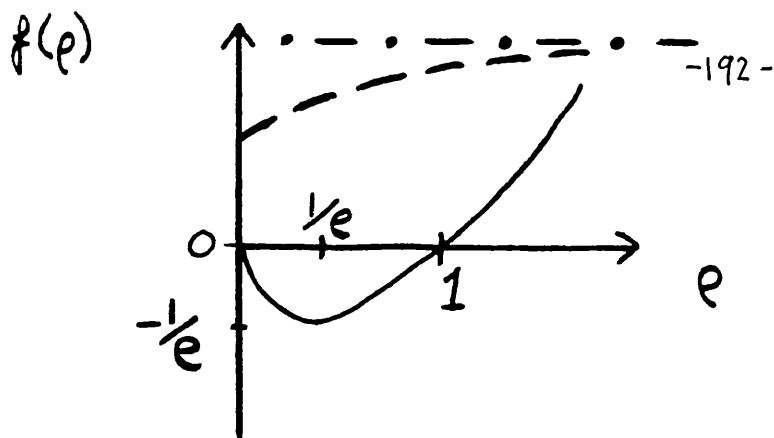
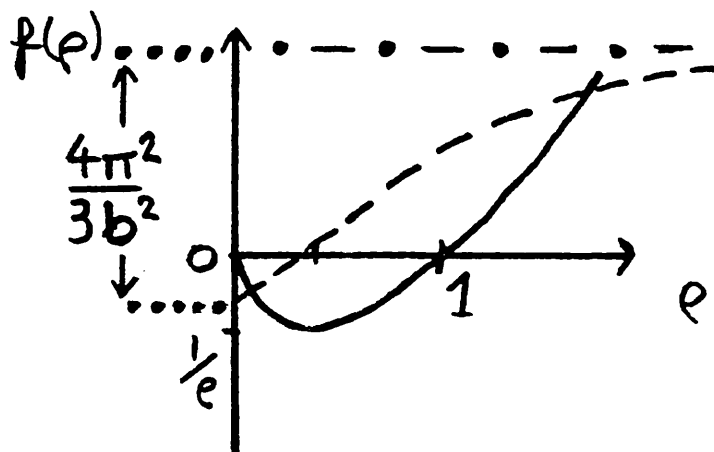


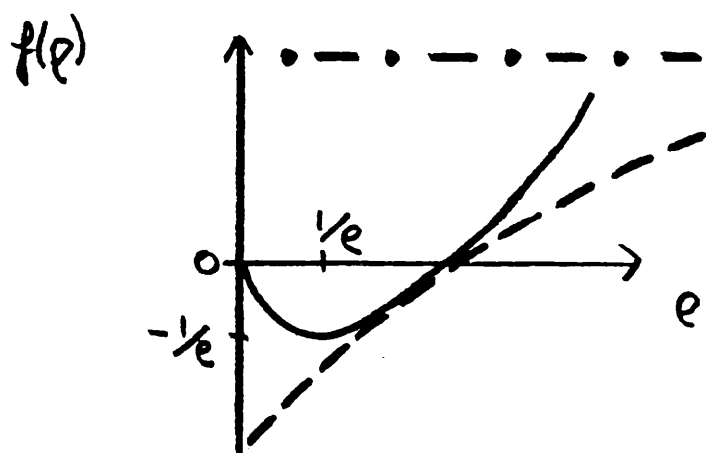
Fig.(3.1).



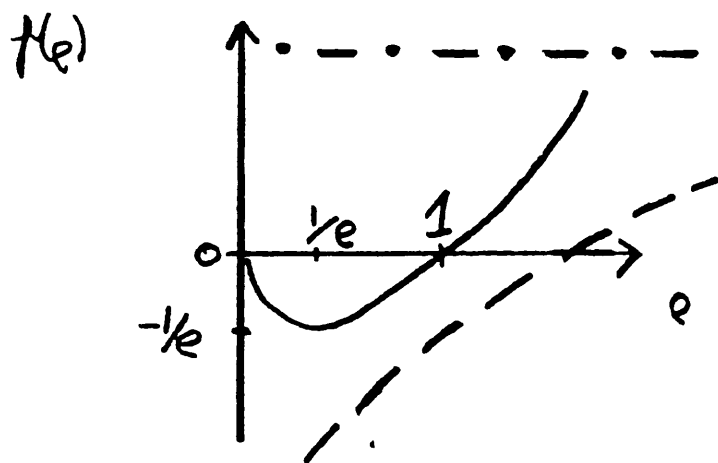
(i)



(ii)



(iii)



(iv)

····· $f(\rho) = \rho_0 \ln \rho_0$

— $f(\rho) = \rho \ln \rho$

--- $f(\rho) = \rho_0 \ln \rho_0 - 8\rho F_1(b\sqrt{\rho})$

Fig. (4.1).

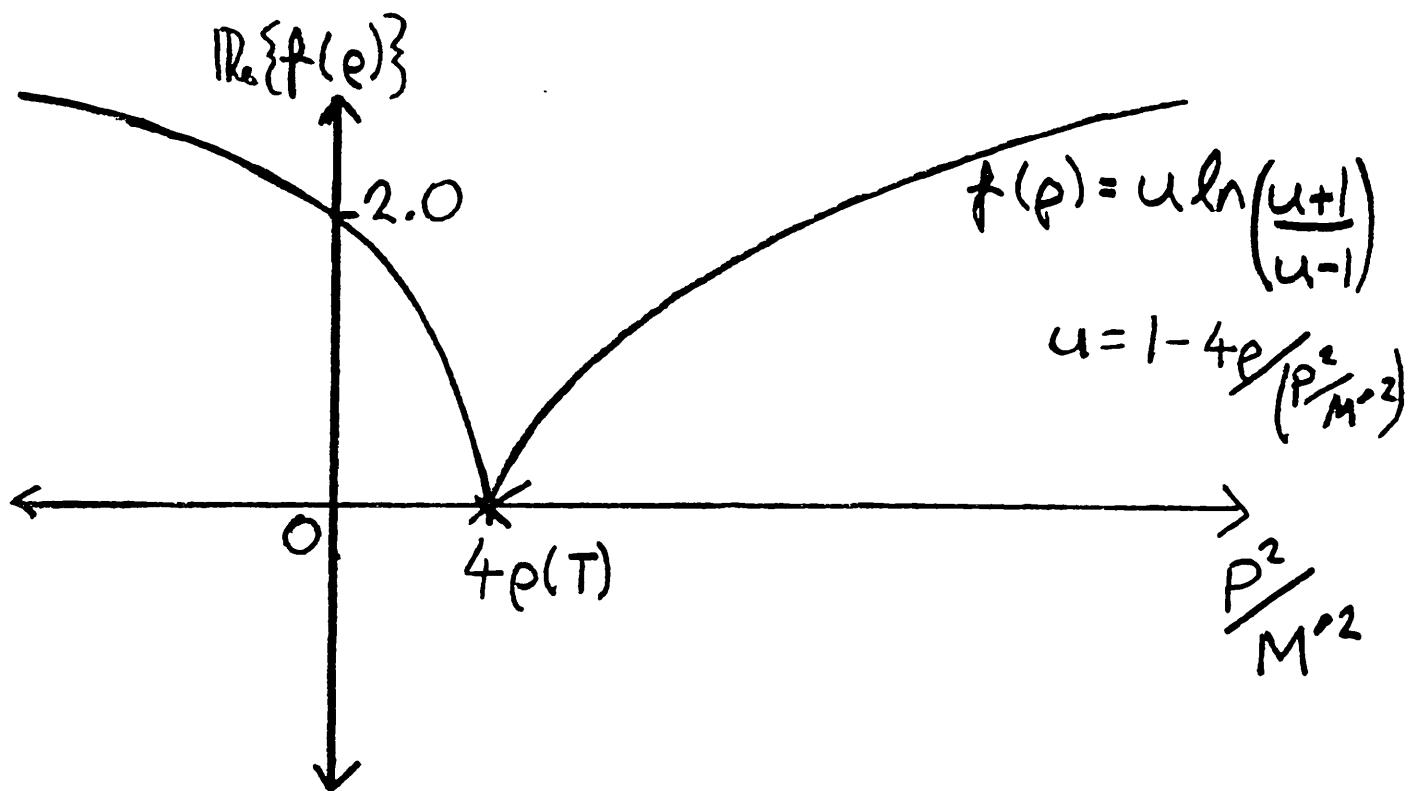
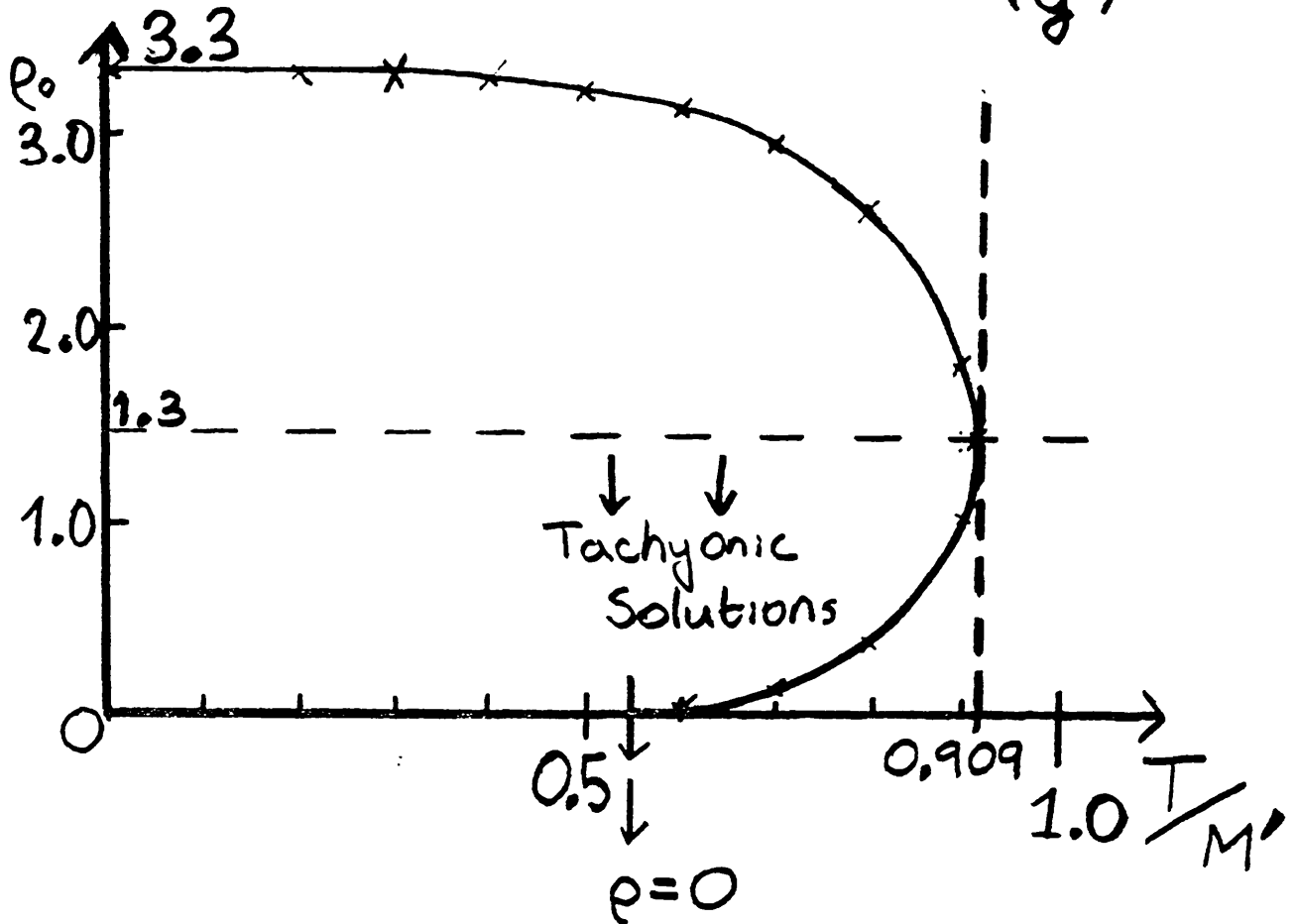


Fig. (4.2).

$$\rho(T) \quad \rho_0 \ln \rho_0 = -32\pi^2 \left(\frac{m^2}{g}\right) / M'^2 = 4.1$$



when

$$\rho_0 \ln \rho_0 = \frac{4\pi^2 T^2}{3 M'^2}$$

Fig. (4.3).

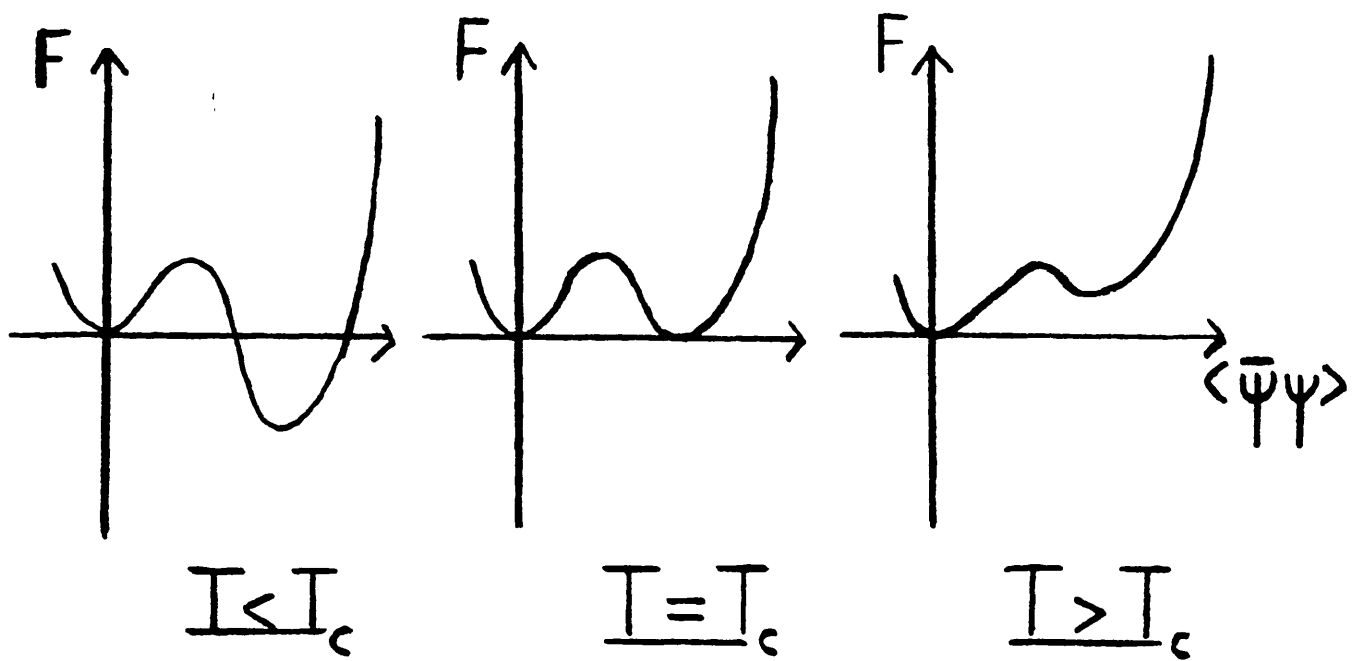


Fig. (5.1).

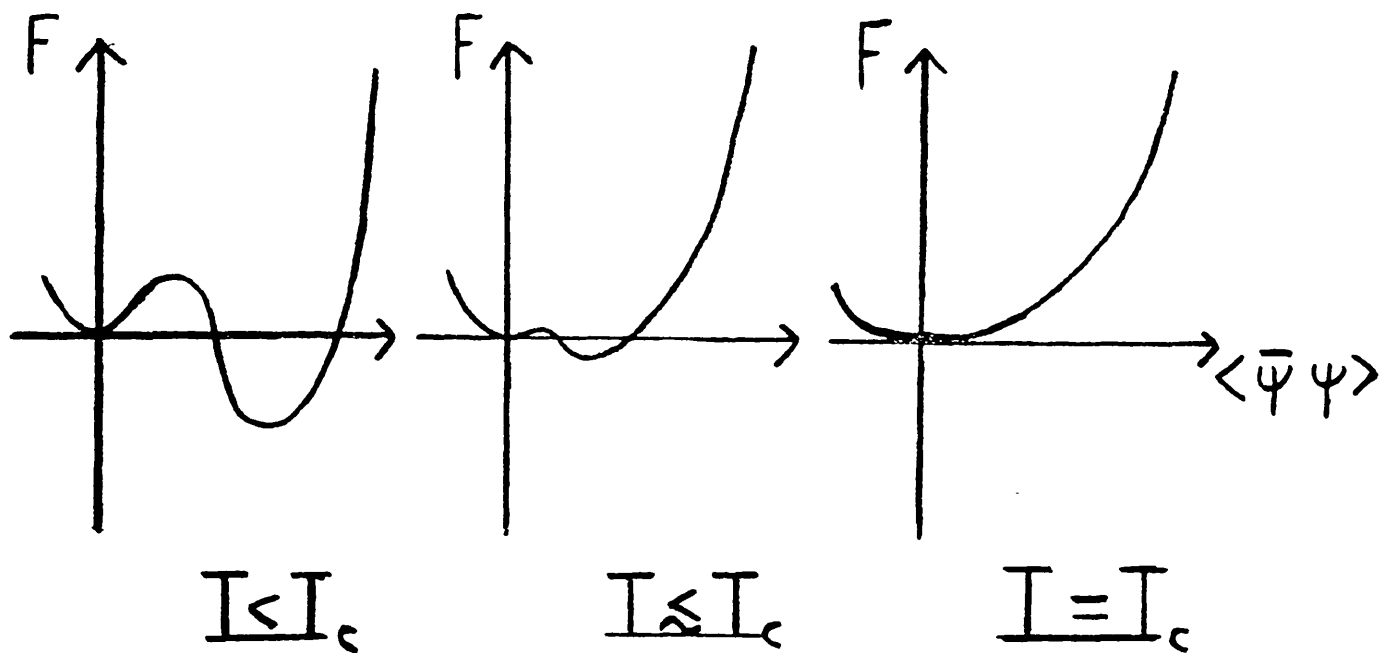


Fig. (5.2).

$$i\Sigma =$$
$$=$$

The diagram illustrates the self-energy $i\Sigma$ for a fermion. It consists of a horizontal fermion line with an incoming momentum k^μ and an outgoing momentum p^μ . The line contains two vertices: a shaded circle and a cross-hatched circle. A loop is formed by two wavy lines (representing photons) connecting these two vertices. The top vertex of the loop is also shaded. The momentum of the top vertex is labeled $(p+k)^\mu$.

Fig. (5.3).

$$i\Sigma = \text{[Cross-hatched circle with incoming and outgoing arrows]}_{p^\mu}$$
$$= \text{[Loop diagram with a cross-hatched top vertex, a circle labeled K at the bottom, and external momenta p^\mu and k^\mu]}_{p^\mu}$$

Fig. (5.4).

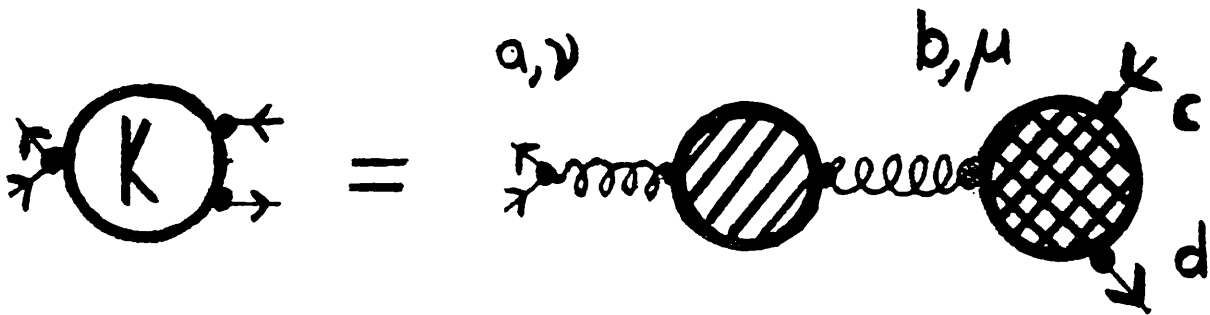


Fig. (5.5).

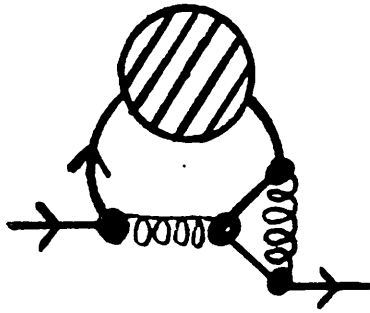


Fig. (5.6).

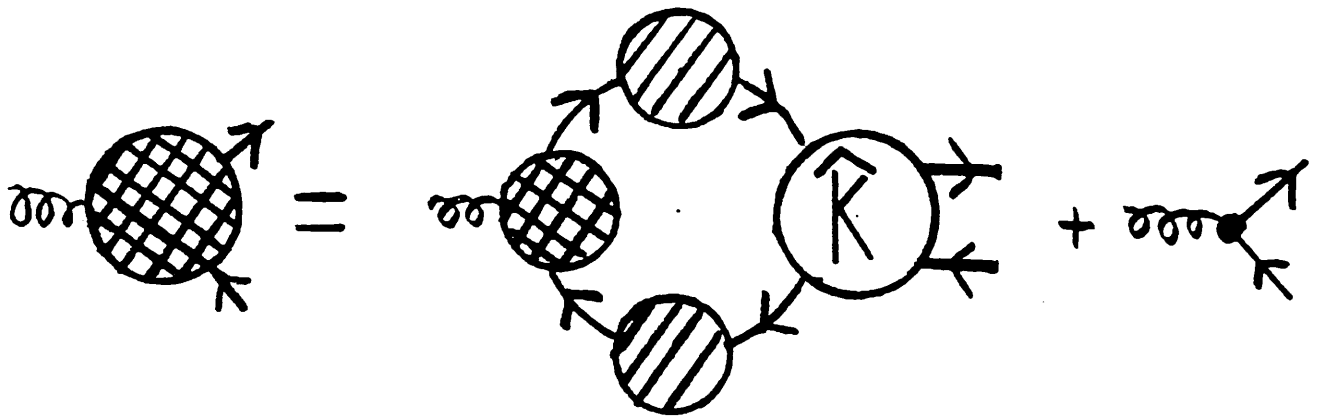


Fig.(5.7).

$$\text{Diagram (K)} = \text{Diagram (a)} \quad \underline{\text{(a)}}$$

$$= \text{Diagram (b)} \quad \underline{\text{(b)}}$$

$$= \text{Diagram (c)} \quad \underline{\text{(c)}}$$

$$= \text{Diagram (d)} \quad \underline{\text{(d)}}$$

Fig. (5.8).

$$i\Sigma = \text{[Cross-hatched circle with external lines]}$$

$$= \begin{array}{c} \text{[Diagram (i)]} \quad \text{(i)} \\ \text{[Diagram (ii)]} \quad \text{(ii)} \\ \text{[Diagram (iii)]} \quad \text{(iii)} \\ \text{[Diagram (iv)]} \quad \text{(iv)} \end{array} +$$

1 5 1 5

Fig. (5a.1).

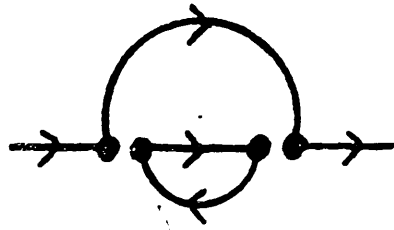


Fig. (5a.2).

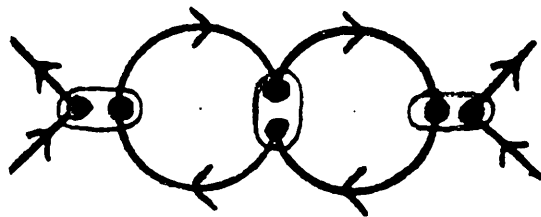
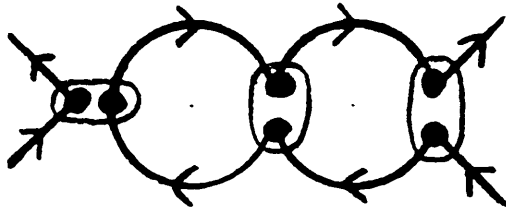


Fig. (5a.3).

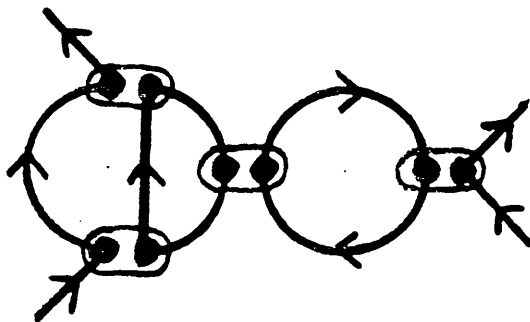


Fig. (5a.4).

$$\begin{aligned}
 i\Pi(p^\mu) = & \text{diagram 1} + \text{diagram 2} + \\
 & \dots + \text{diagram 3} \\
 & \xrightarrow{p^\mu} \\
 & + \dots
 \end{aligned}$$

The diagrams represent Feynman diagrams for the polarization function $\Pi(p^\mu)$. Diagram 1 is a tadpole diagram with two external lines and a loop. Diagram 2 is a bubble diagram with two external lines and two loops. Diagram 3 is a chain of three bubble diagrams with two external lines. The arrow labeled p^μ indicates the momentum flow through the chain of bubbles.

Fig. (5a.5).

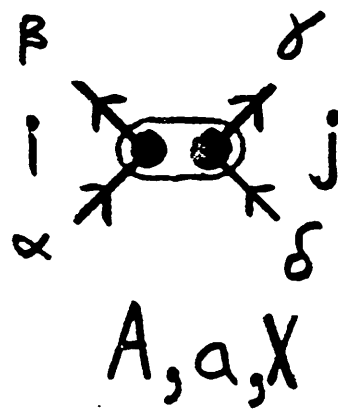
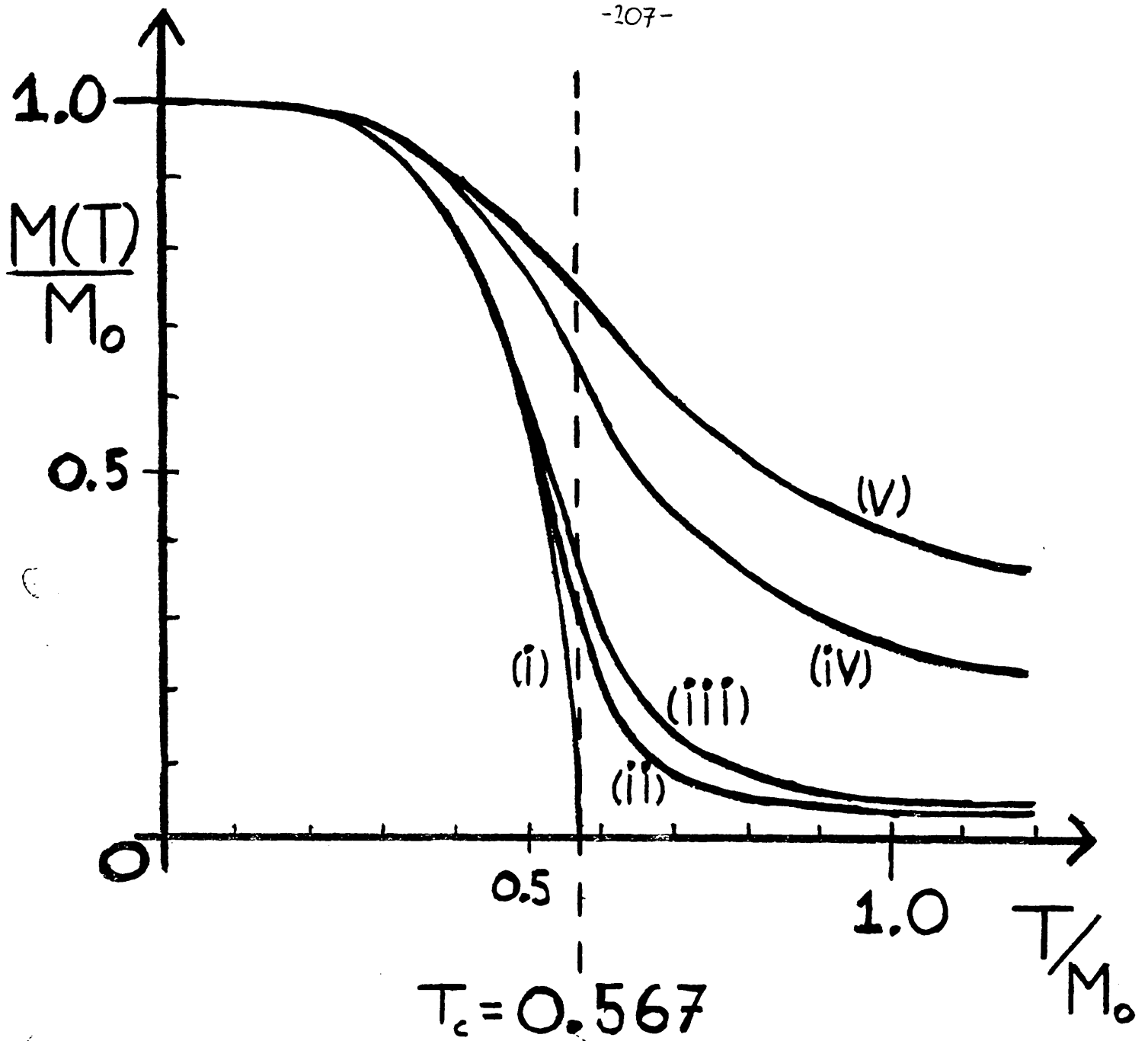


Fig. (5a.6).



$m =$	(i)	0.0
	(ii)	0.01
	(iii)	0.015
	(iv)	0.10
	(v)	0.20

$\alpha =$ (i) ANY VALUE
(ii), (iii), (iv), (v)
0.75

Fig. (5c.1).

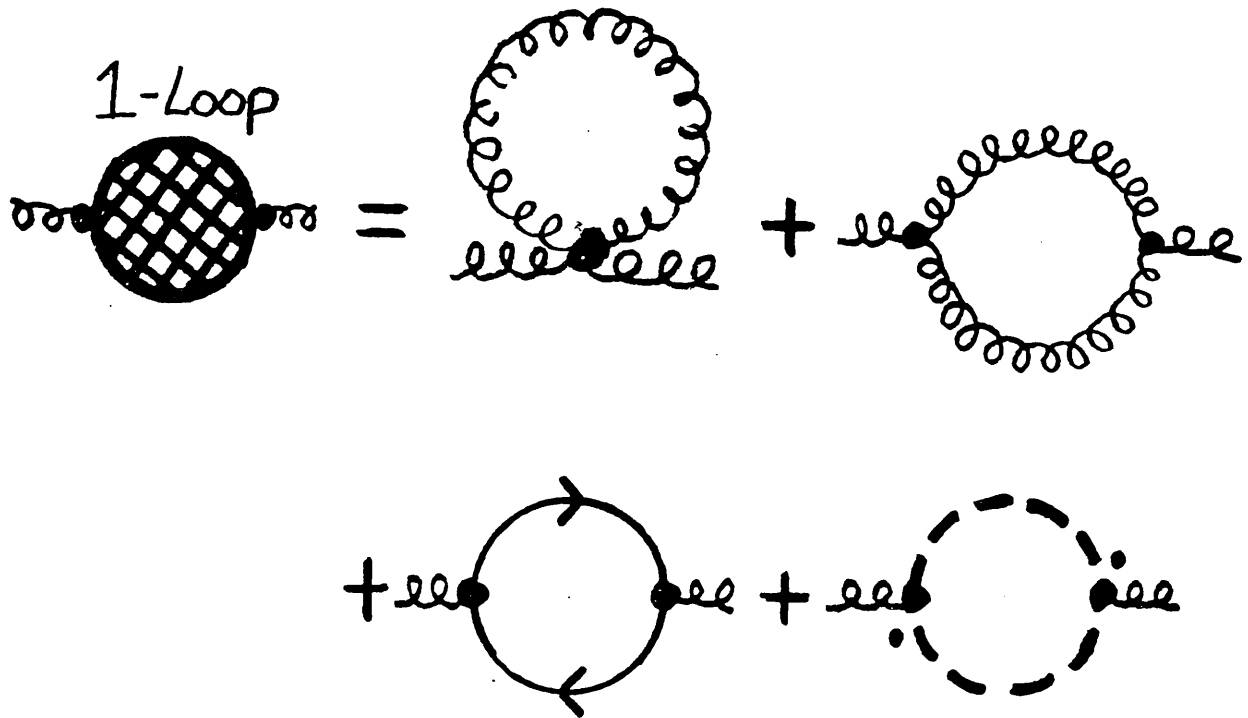


Fig. (5d.1).

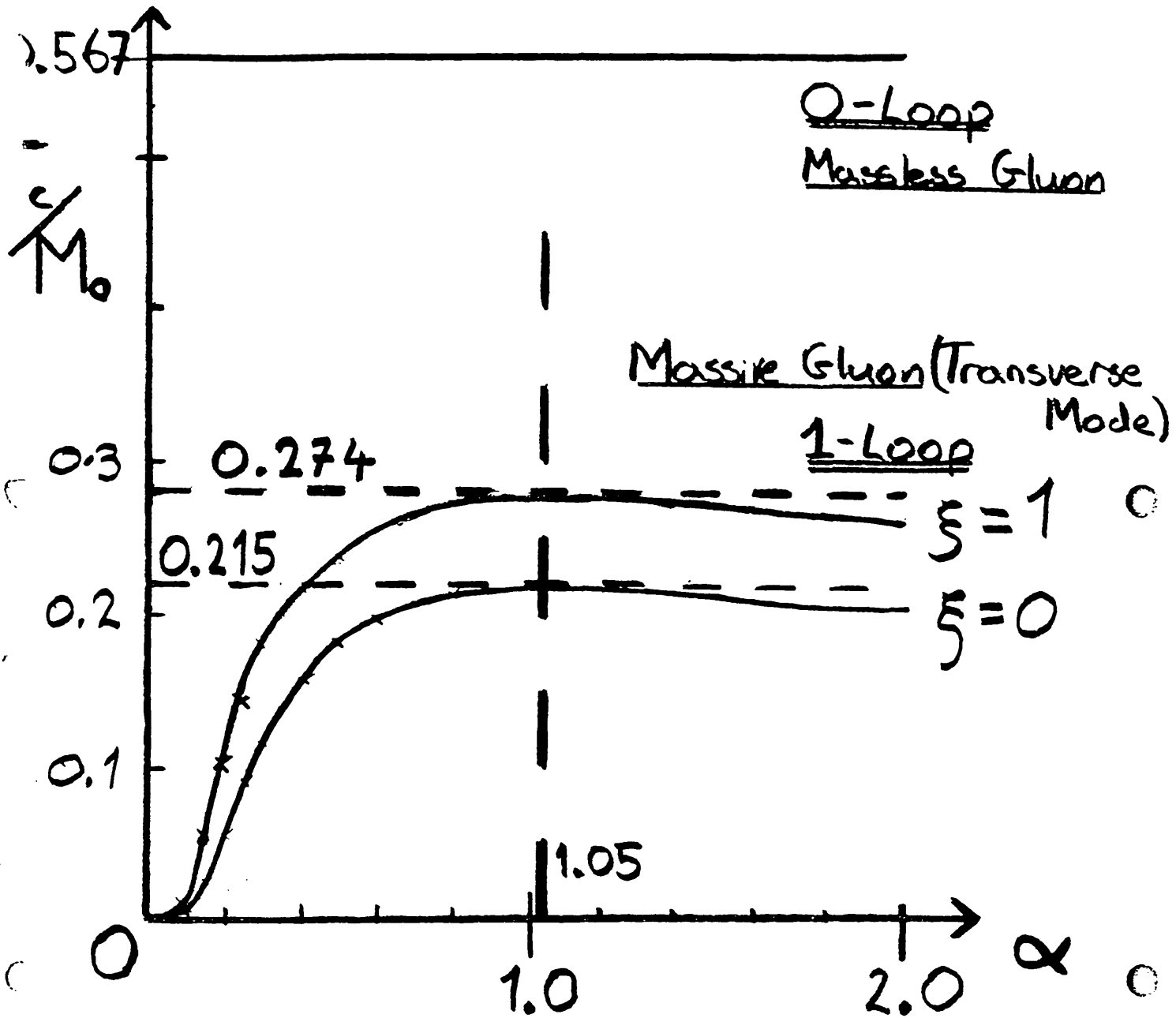


Fig. (5d.2).