On the singularity type of full mass currents in big cohomology classes

Tamás Darvas, Eleonora Di Nezza and Chinh H. Lu

Abstract

Let X be a compact Kähler manifold and $\{\theta\}$ be a big cohomology class. We prove several results about the singularity type of full mass currents, answering a number of open questions in the field. First, we show that the Lelong numbers and multiplier ideal sheaves of θ-plurisubharmonic functions with full mass are the same as those of a current with minimal singularities. Second, given another big and nef class $\{\eta\}$, we show the inclusion $\mathcal{E}(X,\eta)\cap \mathrm{PSH}(X,\theta)\subset \mathcal{E}(X,\theta)$. Third, we characterize big classes whose full mass currents are "additive". Our techniques make use of a characterization of full mass currents in terms of the envelope of their singularity type. As an essential ingredient we also develop the theory of weak geodesics in big cohomology classes. Numerous applications of our results to complex geometry are also given.

1. Introduction and main results

Since the fundamental work of Aubin [Aub] and Yau [Yau], the complex Monge-Ampère operator has found many important applications in differential geometry. In this vast area of research, pluripotential theory plays a crucial role, initiated by the seminal work of Bedford–Taylor [BT76, BT82, BT87] and Kołodziej [Kol98], to only mention a few.

Guedj and Zeriahi extended Bedford–Taylor theory to compact Kähler manifolds (X,ω) [GZ05, GZ07]. Their idea was to extend the definition of the complex Monge–Ampère operator to much larger sets of potentials, not only bounded ones. As a result, an adequate variational theory could be devised for global equations of complex Monge–Ampère type [BBGZ13, BBEGZ12] that has found many striking applications in Kähler geometry.

Additionally, the methods of [GZ07] have proven to be very robust, as they also apply in case of big cohomology classes that are non–Kähler, as explored in [BEGZ10]. Given a closed smooth $(1,1)$-form θ on X, we say that the class $\{\theta\}$ is *big*, if there exists a quasi–plurisubharmonic function u on X such that $\theta+dd^c u\geqslant \varepsilon\omega$ for some $\varepsilon>0$. Non–Kähler big classes arise naturally in constructions of algebraic geometry. Given a one point blowup of an arbitrary Kähler manifold, the simplest such example is given by the sum of the exceptional divisor class and a "sufficiently small" Kähler class.

When varying Kähler classes, one often has to study degenerate classes as well, and there has been a lot of work in trying to characterize the degenerate classes that admit special Kähler metrics [BBEGZ12, Dar16, SSY]. Recently, the solution to the complex Monge–Ampère equation in a big class has been used to show that on a projective manifold the cone of pseudoeffective

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classes is dual to the movable cone, solving an important open problem in complex algebraic geometry [WN16].

In most of the above mentioned works that study degenerate metrics finite energy pluripotential theory plays an important role. Partly motivated by this, and partly by a survey of open questions [DGZ15], we will investigate further the finite energy pluripotential theory of big cohomology classes. For a big class $\{\theta\}$, the class of full mass currents $\mathcal{E}(X,\theta)$ is of central interest, as in many ways it is the analog of the classical Sobolev spaces, given the role it plays in the variational study of complex Monge-Ampère equations (for the precise definition, see Section 2.1). Our first main result clarifies the local/global singular behavior of potentials in $\mathcal{E}(X,\theta)$ in various settings of geometric interest:

Theorem 1.1. *Let (X,ω) be a Kähler manifold. Assume that θ is a smooth closed $(1,1)$-form such that $\{\theta\}$ is big. Let V_θ be the envelope of θ. Then we have the following:*
(i) for any $\varphi \in \mathcal{E}(X,\theta)$ we have

$$\nu(\varphi,x) = \nu(V_\theta,x) \text{ and } \mathcal{I}(t\varphi,x) = \mathcal{I}(tV_\theta,x),\ \forall x \in X, t>0,$$

where $\nu(\varphi,x)$ is the Lelong number of φ at x, and $\mathcal{I}(t\varphi,x)$ is the germ of the multiplier ideal sheaf of $t\varphi$ at x.
(ii) If $\{\eta\}$ is a big and nef class, then

$$\mathcal{E}(X,\eta) \cap \mathrm{PSH}(X,\theta) \subset \mathcal{E}(X,\theta).$$

In particular, when $\theta=\omega$, this last inclusion gives that $\nu(\varphi,x)=0$ for any $x\in X, \varphi \in \mathcal{E}(X,\eta)$.

In the above statement V_θ is the "least singular" element of $\mathrm{PSH}(X,\theta)$ defined as

$$V_\theta := \{\varphi \quad \theta-\text{psh}, \ \varphi \leqslant 0 \text{ on X}\}.$$

For a precise definition of all other concepts in the above result we refer to Section 2.1. The statement of (ii) cannot hold in case $\{\eta\}$ is merely big. Indeed, if $\{\eta\}$ is big but not nef, the envelope V_η may have a positive Lelong number at some point $x\in X$, hence its complex Monge-Ampère measure can not have full mass with respect to a Kähler form $\omega \geqslant \eta$, as shown in [GZ07, Corollary 1.8]. Let us also emphasize that the very last statement of Theorem 1.1 also follows from the statement of (i) together with the fact that, when $\{\theta\}$ is big and nef, the Lelong numbers of V_θ are zero thanks to [Bou04, Propositions 3.2 and 3.6].

In the particular case when $\{\theta\}$ is semi-positive and big, Theorem 1.1 answers affirmatively an open question in [DGZ15, Question 36], saying that potentials in $\mathcal{E}(X,\theta)$ have zero Lelong numbers. A very specific instance of this was verified in [BBEGZ12, Theorem 1.1], using techniques from algebraic geometry.

Our arguments use the envelope construction originally due to Ross–Witt Nyström [RWN] that we recall now. For an upper semi-continuous function f on X, we let $P_\theta(f)$ be the largest θ-psh function lying below f, i.e. $P_\theta(f) = \sup\{u \mid \theta-\text{psh } u \leqslant f\}$. Given ψ,φ, two θ-psh functions, we define:

$$P_{[\theta,\psi]}(\varphi) := \Big\{ \lim_{C\to+\infty} P_\theta(\min(\psi+C,\varphi)) \Big\}^*. \tag{1}$$

In Section 2, we prove that whenever φ,ψ belong to $\mathcal{E}(X,\theta)$ then $P_\theta(\min(\varphi,\psi))$ also belongs to $\mathcal{E}(X,\theta)$. Coincidentally, with the help of this result we can settle a conjecture in [BEGZ10, Remark 2.16], regarding the convexity of finite energy classes associated to a big cohomology class (Corollary 2.12).

To familiarize the reader with the flavor of our arguments, we sketch the proof of the statement involving Lelong numbers in Theorem 1.1(i). If $\varphi \in \mathcal{E}(X,\theta)$ it follows from an approximation and balayage argument [BT82] that the non-pluripolar Monge–Ampère measure of $P_{[\theta,\varphi]}(0)$ vanishes on $\{P_{[\theta,\varphi]}(V_\theta) < V_\theta\}$ (See Proposition 2.14 c.f. [RWN, Theorem 4.1]). It thus follows from the domination principle that $P_{[\theta,\varphi]}(V_\theta) = V_\theta$. A local argument involving the pluricomplex Green function now implies that the Lelong numbers of φ are the same as those of V_θ (see (21)).

In the case when θ is Kähler a surprising characterization of the class $\mathcal{E}(X,\theta)$ in terms of the envelope construction of (1) was obtained in [Dar13, Theorem 3],[Dar14, Theorem 4]. Our next result, which is a vital ingredient in our proof of Theorem 1.1 (ii), shows that this characterization holds in the context of big classes as well :

Theorem 1.2. *Let $\{\theta\}$ be a big cohomology class and fix $\varphi \in \mathcal{E}(X,\theta)$. Then a function $\psi \in \mathrm{PSH}(X,\theta)$ belongs to $\mathcal{E}(X,\theta)$ if and only if $P_{[\theta,\psi]}(\varphi) = \varphi$.*

The implication "$\psi \in \mathcal{E}(X,\theta) \Longrightarrow P_{[\theta,\psi]}(\varphi) = \varphi$" follows essentially from the domination principle, by the arguments of [Dar13, RWN]. In order to prove the other implication, we introduce the seemingly unrelated notion of weak geodesics in big cohomology classes, mimicking Berndtsson's construction in the Kähler case [Bern, Section 2.2], and we prove that the Monge–Ampère energy I (sometimes called Aubin–Yau or Aubin–Mabuchi energy) is convex/linear along weak subgeodesics/geodesics (Theorems 3.8 and 3.12). Compared to the Kähler case, this is a very subtle issue and it serves as the key technical ingredient in the proof of Theorem 1.2.

When varying big classes, an important question is to understand how the class of full mass currents changes. Theorem 1.1(ii) already establishes a result in this direction in the particular case of big and nef classes. Paralleling this, as a consequence of Theorem 1.2, we can characterize the pairs of big classes that have "additive" full mass currents, greatly generalizing [DN15, Theorem B] in the process:

Theorem 1.3. *Let $\{\theta_1\},\{\theta_2\}$ be big classes on X. The following are equivalent:*
(i) $V_{\theta_1} + V_{\theta_2} \in \mathcal{E}(X,\theta_1+\theta_2)$.
(ii) For any $u \in \mathrm{PSH}(X,\theta_1), v \in \mathrm{PSH}(X,\theta_2)$ we have

$$u+v \in \mathcal{E}(X,\theta_1+\theta_2) \Longleftrightarrow u \in \mathcal{E}(X,\theta_1), v \in \mathcal{E}(X,\theta_2).$$

As it turns out, when $\{\theta_1\},\{\theta_2\}$ are big and nef, condition (i) in the above theorem is automatically satisfied (Corollary 4.2). This result also helps to partially confirm [BEGZ10, Conjecture 1.23] concerning log concavity of the non-pluripolar complex Monge-Ampère measure in the case of full mass currents of big and nef classes (see Corollary 5.3).

Organization of the paper. Section 2 mostly reviews background material on the pluripotential theory of big cohomology classes and we establish some preliminary results. In Section 3 we develop the theory of weak geodesics in big cohomology classes following Berndtsson's ideas, and then we prove Theorem 1.2 and Theorem 1.3. Theorem 1.1 will be proved in Section 4 while some other applications will be discussed in Section 5.

2. Pluripotential theory in big cohomology classes

2.1 Non-pluripolar Monge-Ampère measures

We recall basic facts concerning pluripotential theory of big cohomology classes. We borrow notation and terminology from [BEGZ10], and we also refer to this work for further details.

Let (X,ω) be a compact Kähler manifold of dimension n. We fix θ a smooth closed $(1,1)$-form on X such that $\{\theta\}$ is *big*, i.e., there exists $\psi \in \mathrm{PSH}(X,\theta)$ such that $\theta + dd^c\psi \geqslant \varepsilon\omega$ for some small constant $\varepsilon > 0$. Here, d and d^c are real differential operators defined as $d := \partial + \bar{\partial}$, $d^c := \frac{i}{2\pi}\left(\bar{\partial} - \partial\right)$. A function $\varphi : X \to \mathbb{R}\cup\{-\infty\}$ is called quasi-plurisubharmonic if it is locally written as the sum of a plurisubharmonic function and a smooth function. φ is called θ-plurisubharmonic (θ-*psh*) if it is quasi-psh such that $\theta + dd^c\varphi \geqslant 0$ in the sense of currents. We let $\mathrm{PSH}(X,\theta)$ denote the set of θ-psh functions which are not identically $-\infty$ (equivalently, it consists of θ-psh functions which are integrable on X).

A θ-psh function φ is said to have *analytic singularities* if there exists $c > 0$ such that locally on X,

$$\varphi = \frac{c}{2}\log\sum_{j=1}^{N}|f_j|^2 + u,$$

where u is smooth and $f_1,\dots,f_N$ are local holomorphic functions. The *ample locus* $\mathrm{Amp}(\{\theta\})$ of $\{\theta\}$ is the set of points $x \in X$ such that there exists a Kähler current $T \in \{\theta\}$ (satisfying $T \geqslant \varepsilon\omega$ for some $\varepsilon > 0$) with analytic singularities and smooth in a neighbourhood of x. The ample locus $\mathrm{Amp}(\{\theta\})$ is a Zariski open subset, and it is nonempty [Bou04].

Let $x \in X$. Fixing a holomorphic chart $x \in U \subset X$, the *Lelong number* $\nu(\varphi,x)$ of $\varphi \in \mathrm{PSH}(X,\theta)$ is defined as follows:

$$\nu(\varphi,x) = \sup\{\gamma \geqslant 0 \text{ s.t. } \varphi(z) \leqslant \gamma\log\|z-x\| + O(1) \text{ on } U\}. \tag{2}$$

One can also associate to φ a collection of *multiplier ideal sheafs* $\mathcal{I}(t\varphi),\ t \geqslant 0$, whose germs are defined by:

$$\mathcal{I}(t\varphi,x) = \{f \in \mathcal{O}_x \text{ s.t. } \int_V |f|e^{-t\varphi}\omega^n < \infty \text{ for some open set } x \in V \subset X\}.$$

If φ and φ' are two θ-psh functions on X, then φ' is said to be *less singular* than φ if they satisfy $\varphi \leqslant \varphi' + C$ for some $C \in \mathbb{R}$. A θ-psh function φ is said to have *minimal singularities* if it is less singular than any other θ-psh function. Such θ-psh functions with minimal singularities always exist, one can consider for example

$$V_\theta := \sup\{\varphi\ \theta\text{-psh}, \varphi \leqslant 0 \text{ on } X\}.$$

Trivially, a θ-psh function with minimal singularities has locally bounded potential in $\mathrm{Amp}(\{\theta\})$. It follows from Demailly's approximation theorem that V_θ is continuous in the ample locus $\mathrm{Amp}(\theta)$.

More generally, if f is a function on X, we define the Monge-Ampère envelope of f in the class $\mathrm{PSH}(X,\theta)$ by

$$P_\theta(f) := \left(\sup\{u \in \mathrm{PSH}(X,\theta) \mid u \leqslant f\}\right)^*,$$

with the convention that $\sup\emptyset = -\infty$. Observe that $P_\theta(f)$ is a θ-psh function on X if and only if there exists some $u \in \mathrm{PSH}(X,\theta)$ lying below f. Note also that $V_\theta = P_\theta(0)$.

Given $T_1 := \theta_1 + dd^c\varphi_1, ...,\ T_p := \theta_p + dd^c\varphi_p$ positive $(1,1)$-currents, where θ_j are closed smooth $(1,1)$-forms, following the construction of Bedford-Taylor [BT87] in the local setting, it has been shown in [BEGZ10] that the sequence of currents

$$\mathbf{1}_{\bigcap_j\{\varphi_j > V_{\theta_j}-k\}}(\theta_1 + dd^c\max(\varphi_1, V_{\theta_1}-k)) \wedge ... \wedge (\theta_p + dd^c\max(\varphi_p, V_{\theta_p}-k))$$

is non-decreasing in k and converges weakly to the so called *non-pluripolar product*

$$\langle T_1 \wedge \ldots \wedge T_p \rangle = \langle \theta_{\varphi_1} \wedge \ldots \wedge \theta_{\varphi_1} \rangle.$$

The resulting positive (p,p)-current does not charge pluripolar sets and it is *closed*. The particular case when $T_1 = \cdots = T_p$ will be important for us in the sequel. For a θ-psh function φ, the *non-pluripolar complex Monge-Ampère measure* of φ is

$$\theta_\varphi^n := \langle (\theta + dd^c\varphi)^n \rangle.$$

The volume of a big class $\{\theta\}$ is defined by

$$\mathrm{Vol}(\{\theta\}) := \int_X \theta_{V_\theta}^n.$$

Alternatively, by [BEGZ10, Theorem 1.16], in the above expression one can replace V_θ with any θ-psh function with minimal singularities. A θ-psh function φ is said to have *full Monge-Ampère mass* if

$$\int_X \theta_\varphi^n = \mathrm{Vol}(\{\theta\}),$$

and we then write $\varphi \in \mathcal{E}(X,\theta)$. Let us stress that since the non-pluripolar product does not charge pluripolar sets, for a general θ-psh function φ we only have $\mathrm{Vol}(\{\theta\}) \geqslant \int_X \theta_\varphi^n$.

By a *weight function*, we mean a smooth increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(0) = 0$ and $\chi(-\infty) = -\infty$. We say that $\varphi \in \mathrm{PSH}(X,\theta)$ has finite χ-energy if

$$E_\chi(\varphi) := \int_X (-\chi)(\varphi - V_\theta)\theta_\varphi^n < +\infty.$$

We denote by $\mathcal{E}_\chi(X,\theta)$ the set of full mass θ-psh potentials having finite χ-energy. If $\mathcal{W}^-$ denotes the set of weights χ that are convex on $\mathbb{R}^-$, then by [BEGZ10, Proposition 2.11] we have

$$\mathcal{E}(X,\theta) := \bigcup_{\chi \in \mathcal{W}^-} \mathcal{E}_\chi(X,\theta).$$

In the special case when $\chi = Id$ we simply denote the space $\mathcal{E}_\chi(X,\theta)$ by $\mathcal{E}^1(X,\theta)$. When $\varphi \in \mathrm{PSH}(X,\theta)$ has minimal singularities, the Monge-Ampère energy (in the class $\{\theta\}$) is defined by the formula

$$\mathrm{I}(\varphi) := \frac{1}{(n+1)\mathrm{Vol}(\theta)} \sum_{k=0}^{n} \int_X (\varphi - V_\theta)\langle \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k} \rangle.$$

For general $\varphi \in \mathrm{PSH}(X,\theta)$, using the monotonicity property of I, we have the following definition:

$$\mathrm{I}(\varphi) := \lim_{k\to\infty} \mathrm{I}(\max(\varphi, V_\theta - k)).$$

Though the above limit exists, it is possible that $\mathrm{I}(\varphi) = -\infty$. It was proved in [BEGZ10] that $\varphi \in \mathcal{E}^1(X,\theta)$ if and only if $\mathrm{I}(\varphi)$ is finite. Moreover, I is continuous under monotone and uniform convergence [BEGZ10, Proposition 2.10,Theorem 2.17].

2.2 Degenerate complex Monge-Ampère equation

We summarize recent results on the resolution of degenerate complex Monge-Ampère equations in big cohomology classes. The main source are [BEGZ10] and [BBGZ13]. Let μ be a non-pluripolar measure on X, i.e. a positive Borel measure that puts no mass on pluripolar sets. We want to

solve the complex Monge-Amère equation

$$\theta_\varphi^n = e^{\beta\varphi}\mu, \quad \varphi \in \mathcal{E}^1(X,\omega), \tag{3}$$

where $\beta > 0$ is a constant. As the treatment is the same for all β, we assume that $\beta = 1$. If μ is a smooth volume form (or it has bounded density with respect to ω^n) then one can use a fixed point argument [BEGZ10, Section 6.1] to solve the equation. For the general case we use the variational method of [BBGZ13] to show existence of a solution $\varphi \in \mathcal{E}^1(X,\theta)$. The proof that we give below is extracted from [BBGZ13], except for the argument of Theorem 2.2, which is inspired from [LN15]. The main point is to make it clear that the result is independent of [BD12].

2.2.1 *The variational method.* Let μ be a non pluripolar positive measure on X. For convenience we normalize θ so that its volume is 1.

Consider the following functional

$$F(\varphi) := \mathrm{I}(\varphi) - L_\mu(\varphi), \quad \varphi \in \mathrm{PSH}(X,\omega),$$

where $L_\mu(\varphi) := \int_X e^\varphi d\mu$. It follows from [BEGZ10] that I is upper semicontinuous with respect to L^1-convergence. Assume that φ_j is a sequence of θ-psh functions converging in $L^1(X,\omega^n)$ to $\varphi \in \mathrm{PSH}(X,\theta)$. Then by Hartogs lemma $\sup_X \varphi_j$ is uniformly bounded, hence the sequence e^{φ_j} stays in $\mathrm{PSH}(X,A\omega)$ for some positive constant A and e^{φ_j} converges to e^φ in $L^1(X,\omega^n)$. Since μ is non pluripolar it thus follows from an argument due to Cegrell [Ce98, Lemma 5.2] (see [BBGZ13, Theorem 3.10] or [GZ17, Lemma 11.5] for a proof in the compact case) that $L_\mu(\varphi_j) \to L_\mu(\varphi)$. This means L_μ is continuous with respect to the L^1-topology, hence F is upper semicontinuous on $\mathcal{E}^1(X,\theta)$.

Proposition 2.1. *There exists $\varphi \in \mathcal{E}^1(X,\theta)$ such that $F(\varphi) = \sup_{\psi\in\mathcal{E}^1(X,\theta)} F(\psi)$.*

Proof. Let (φ_j) be a sequence in $\mathcal{E}^1(X,\theta)$ such that $\lim_j F(\varphi_j) = \sup_{\mathcal{E}^1(X,\theta)} F > -\infty$. We claim that $\sup_X \varphi_j$ is uniformly bounded from above. Indeed, assume that it were not the case. Then by relabeling the sequence we can assume that $\sup_X \varphi_j$ increases to $+\infty$. By compactness property [GZ05, Proposition 2.7] it follows that the sequence $\psi_j := \varphi_j - \sup_X \varphi_j$ converges in $L^1(X,\omega^n)$ to some $\psi \in \mathrm{PSH}(X,\theta)$ such that $\sup_X \psi = 0$. In particular $\int_X e^\psi d\mu > 0$. It thus follows that

$$\int_X e^{\varphi_j} d\mu = e^{\sup_X \varphi_j} \int_X e^{\psi_j} d\mu \geqslant c e^{\sup_X \varphi_j}$$

for some positive constant c. Since $\mathrm{I}(\varphi_j) \leqslant \sup_X \varphi_j$, the above inequality gives that $F(\varphi_j)$ converges to $-\infty$, a contradiction. Thus $\sup_X \varphi_j$ is bounded from above as claimed. Since $F(\varphi_j) \leqslant \mathrm{I}(\varphi_j) \leqslant \sup_X \varphi_j$ it follows that $\mathrm{I}(\varphi_j)$ and hence $\sup_X \varphi_j$ is also bounded from below. It follows again from [GZ05, Proposition 2.7] that a subsequence of φ_j (still denoted by φ_j) converges in $L^1(X,\omega^n)$ to some $\varphi \in \mathrm{PSH}(X,\theta)$. Since I is upper semicontinuous it follows that $\varphi \in \mathcal{E}^1(X,\theta)$. Moreover, by continuity of L_μ it follows that $F(\varphi) \geqslant \sup_{\mathcal{E}^1(X,\theta)} F$ completing the proof. □

Next we prove that the maximizer obtained above is actually the solution to the complex Monge-Ampère equation (3). The proof relies on a differentiability property of the Monge-Ampère energy functional:

Theorem 2.2. *Fix $\varphi \in \mathcal{E}^1(X,\theta)$ and let χ be a continuous real valued function X. Set $\varphi_t = P_\theta(\varphi + t\chi)$, $t \in \mathbb{R}$. Then $t \mapsto I(\varphi_t)$ is differentiable and its derivative is given by*

$$\frac{d}{dt}\mathrm{I}(\varphi_t) = \int_X \chi \theta_{\varphi_t}^n, \ \forall t \in \mathbb{R}.$$

Note that $\varphi_t \geqslant \varphi - |t| \sup_X |\chi|$, hence $\varphi_t \in \mathcal{E}^1(X,\theta)$ for all $t \in \mathbb{R}$. This result was first proved in [BBGZ13, Lemma 4.2] using [BB08]. A simplification of the original argument has been given in [LN15], and we follow this approach here.

Proof. Let u be a continuous function on X and set $u_t := P_\theta(u+t\chi)$, $t \in \mathbb{R}$. Then for each $t \in \mathbb{R}$, u_t is a θ-plurisubharmonic function with minimal singularities. We claim that

$$\frac{d}{dt}\mathrm{I}(u_t) = \int_X \chi\theta_{u_t}^n, \ \forall t \in \mathbb{R}.$$

It suffices to prove the claim for $t = 0$. We only prove the equality for the right-derivative since the same argument can be applied to deal with the left derivative. We fix $t > 0$. It follows from the concavity of I [BEGZ10] that

$$\int_X (u_t - u_0)\theta_{u_t}^n \leqslant \mathrm{I}(u_t) - \mathrm{I}(u_0) \leqslant \int_X (u_t - u_0)\theta_{u_0}^n.$$

On the other hand, since $u + t\chi$ is continuous on X, it follows from a balayage argument (see [BT82]) that $\theta_{u_t}^n$ is supported on the contact set $\{u_t = u + t\chi\}$. It thus follows that

$$\int_X (u_t - u_0)\theta_{u_t}^n = \int_X (u + t\chi - u_0)\theta_{u_t}^n \geqslant t\int_X \chi\theta_{u_t}^n.$$

By a similar argument we also get

$$\int_X (u_t - u_0)\theta_{u_0}^n = \int_X (u_t - u)\theta_{u_0}^n \leqslant t\int_X \chi\theta_{u_0}^n.$$

We note that u_t converges uniformly to u_0 as $t \to 0^+$, hence by [BEGZ10] we have that $\theta_{u_t}^n$ converges weakly to $\theta_{u_0}^n$. Since χ is continuous, we can divide all of the above estimates with $t > 0$, and let $t \to 0$ to finish the proof of the claim.

Now, we come back to the proof of the theorem. We approximate φ from above by a sequence φ_j of continuous functions on X. For each j, we set $\varphi_{t,j} := P_\theta(\varphi_j + t\chi)$ and note that $\varphi_{t,j}$ decreases pointwise to φ_t as $j \to +\infty$. Since χ is continuous on X and $\varphi_{t,j}$ converges uniformly to $\varphi_{s,j}$ as $t \to s$ it follows from continuity of the complex Monge-Ampère operator together with our claim that the function $t \mapsto \mathrm{I}(\varphi_{t,j})$ is of class $\mathcal{C}^1$ on $\mathbb{R}$. We thus have that

$$\mathrm{I}(\varphi_{t,j}) - \mathrm{I}(\varphi_{0,j}) = \int_0^t \int_X \chi\theta_{\varphi_{s,j}}^n ds.$$

Letting $j \to +\infty$, and using the dominated convergence theorem we obtain

$$\mathrm{I}(\varphi_t) - \mathrm{I}(\varphi_0) = \int_0^t \int_X \chi\theta_{\varphi_s}^n ds.$$

By continuity of the Monge-Ampère operator the function $s \mapsto \int_X \chi\theta_{\varphi_s}^n$ is continuous on $\mathbb{R}$. Therefore, from the above equality we see that $t \mapsto \mathrm{I}(\varphi_t)$ is differentiable and its derivative is exactly as in the statement of the theorem. □

We are now ready to solve the equation (3).

Theorem 2.3. *Let μ be a non pluripolar positive measure on X. Then there exists a unique $\varphi \in \mathcal{E}^1(X,\theta)$ solving (3). Moreover, if $\mu = f\omega^n$ for some bounded nonnegative function f then φ has minimal singularities.*

Proof. As before we can assume that $\beta = 1$. It follows from Proposition 2.1 that there exists $\varphi \in \mathcal{E}^1(X,\theta)$ such that $F(\varphi) = \sup_{\mathcal{E}^1(X,\theta)} F$. Fix a continuous function χ and set

$$g(t) := I(\varphi_t) - \int_X e^{\varphi + t\chi} d\mu,\ t \in \mathbb{R},$$

where $\varphi_t := P_\theta(\varphi + t\chi) \in \mathcal{E}^1(X,\theta)$. Since $\varphi_t \leqslant \varphi + t\chi$ and since φ maximizes F on $\mathcal{E}^1(X,\theta)$ we have

$$g(0) = F(\varphi) \geqslant F(\varphi_t) \geqslant g(t),\ \forall t \in \mathbb{R}.$$

Thus g attains its maximum at 0. It follows from Theorem 2.2 and the dominated convergence theorem that g is differentiable at 0, hence $g'(0) = 0$ which means

$$\int_X \chi \theta_\varphi^n = \int_X \chi e^\varphi d\mu.$$

Since χ was chosen arbitrarily it follows that the equation (3) is satisfied in the weak sense of measure.

The uniqueness follows from Lemma 2.5 below. Finally if μ has bounded density then the right-hand side in (3) has bounded density (because φ is bounded from above) hence it follows from [BEGZ10, Thereom 4.1] that φ has minimal singularities. □

2.2.2 *The domination principle.* The following domination principle was proved in [BEGZ10, Corollary 2.5] for two θ-psh functions φ, ψ with φ having minimal singularities. The argument of Dinew [BL12] gives a generalization of this result to the case when $\varphi \in \mathcal{E}(X,\theta)$ does not necessarily have minimal singularities.

Proposition 2.4. *Let φ, ψ be θ-psh functions such that $\varphi \in \mathcal{E}(X,\theta)$. If $\psi \leqslant \varphi$ almost everywhere with respect to θ_φ^n then $\psi \leqslant \varphi$ everywhere.*

Proof. Fix $t > 0$. As $\{\varphi > \psi - t\}$ is plurifine open, it follows from locality of the non-pluripolar product with respect to the plurifine topology [BEGZ10, Proposition 1.4] that

$$\theta^n_{\max(\varphi,\psi-t)} \geqslant \mathbf{1}_{\{\varphi>\psi-t\}} \theta^n_{\max(\varphi,\psi-t)} = \mathbf{1}_{\{\varphi>\psi-t\}} \theta^n_\varphi = \theta^n_\varphi,$$

where in the last identity we used the assumption that $\theta^n_\varphi(\{\varphi < \psi\}) = 0$. As $\max(\varphi, \psi - t)$ also has full mass, the above inequality becomes equality. This together with the uniqueness theorem, proved in [BEGZ10, Theorem A] which is an adaptation of the original proof in the Kähler case due to Dinew [Dw09], gives that $\max(\varphi, \psi - t) = \varphi + C$, for some constant C which can be easily seen to be zero. Letting $t \to 0^+$ we obtain the desired result. □

Lemma 2.5. *Fix $\beta > 0$ and μ a non-pluripolar positive measure on X. Assume that $v \in \mathcal{E}(X,\theta)$ and $u \in \mathrm{PSH}(X,\theta)$ satisfy*

$$\theta_u^n \geqslant e^{\beta u} e^{-\phi}\mu\ ;\ \theta_v^n \leqslant e^{\beta v} e^{-\phi}\mu,$$

where ϕ is some Borel measurable function on X. Then $u \leqslant v$ on X.

Proof. It follows from the comparison principle (see [BEGZ10, Corollary 2.3]) that

$$\int_{\{v<u\}} e^{\beta u - \phi} d\mu \leqslant \int_{\{v<u\}} \theta_u^n \leqslant \int_{\{v<u\}} \theta_v^n \leqslant \int_{\{v<u\}} e^{\beta v - \phi} d\mu \leqslant \int_{\{v<u\}} e^{\beta u - \phi} d\mu.$$

Thus, the inequalities above become equalities and we have in particular that

$$\int_{\{v<u\}} (e^{\beta u} - e^{\beta v}) e^{-\phi} d\mu = 0.$$

Since ϕ is bounded from above on X it follows that $\int_{\{v<u\}}(e^{\beta u}-e^{\beta v})d\mu = 0$. We then deduce that $\mu(\{-\infty < v < u\}) = 0$. Since μ is non-pluripolar it follows that $\mu(\{v < u\}) = 0$. Consequently $\theta_v^n(\{v < u\}) = 0$, and the domination principle (Proposition 2.4) gives that $u \leqslant v$ on X. □

2.3 Regularity of quasi-psh envelopes

By a deep result of Berman and Demailly [BD12], the envelope V_θ has locally bounded Laplacian in the ample locus $\mathrm{Amp}(\{\theta\})$ and its complex Monge-Ampère measure satisfies

$$\theta_{V_\theta}^n = \mathbf{1}_{\{V_\theta=0\}}\theta^n. \tag{4}$$

In the case when $\{\theta\}$ is integral, this result has been obtained by Berman [Ber08] using different methods. In [Ber13], Berman introduced a new approximation scheme of the envelope by solving a suitably chosen family of complex Monge-Ampère equations. In the case when the class $\{\theta\}$ is big and nef, using this method, he gave a PDE proof of the regularity of V_θ. Given the almost $\mathcal{C}^{1,1}$-regularity of V_θ it is classical to obtain (4). In the general case of a big class Berman's method gives the following weaker version of (4) :

Theorem 2.6. *Suppose (X,ω) is Kähler. Let θ be a smooth $(1,1)$-form such that $\{\theta\}$ is big. Then the envelope V_θ satisfies*

$$\theta_{V_\theta}^n \leqslant \mathbf{1}_{\{V_\theta=0\}}\theta^n.$$

We will use Theorem 2.6 in the construction of the rooftop envelope of two functions in $\mathcal{E}(X,\theta)$ (see Theorem 2.10). In the case when $\{\theta\}$ is Kähler a detailed study of this envelope has been done in [Dar14, DR] which has found many interesting applications in Kähler geometry.

For a proof of Theorem 2.6 we refer to [Ber13] (see the paragraph "on the proof" at page 6 and Lemma 2.12 in version 3 of [Ber13]). An alternative proof using viscosity theory was recently given in [GLZ, Proposition 5.2].

2.4 Comparison of capacities

Given a big class $\{\theta\}$, recall that the θ-capacity of a set $E \subset X$ is defined as follows (see [BEGZ10, Section 4.1] for further details):

$$\mathrm{Cap}_\theta(E) := \sup\left\{\int_E \theta_u^n \mid u \in \mathrm{PSH}(X,\theta) \quad V_\theta - 1 \leqslant u \leqslant V_\theta\right\}.$$

The global extremal θ-psh function of E is defined as the usc regularization of

$$V_{\theta,E} := \sup\{u \in \mathrm{PSH}(X,\theta) \mid u \leqslant 0 \text{ on } E\}.$$

By the definition of $V_{\theta,E}$ it follows that $V_{\theta,E}^* = V_{\theta,F}^*$ if $E = F \cup P$ for some pluripolar set P. To see this it suffices to observe that any pluripolar set is contained in the $-\infty$ locus of some $\varphi \in \mathrm{PSH}(X,\theta)$. The latter standard fact can be quickly explained as follows. If E is pluripolar then by [GZ05, Theorem 7.2] there exists $u \in \mathrm{PSH}(X,\omega)$ such that $u(x) = -\infty$ for all $x \in E$. As $\{\theta\}$ is big, there exists $\psi \in \mathrm{PSH}(X,\theta)$ such that $\theta + dd^c\psi \geqslant \varepsilon\omega$ for some small constant $\varepsilon > 0$. It is clear that the function $\varphi := \psi + \varepsilon u$ belongs to $\mathrm{PSH}(X,\theta)$ and takes value $-\infty$ on E as we claimed.

Denote by $T_\theta(E) := \exp(-M_{\theta,E}) := \exp(-\sup_X V_{\theta,E}^\star)$ the *Alexander–Taylor capacity* of E. We recall the following useful relation between the θ-capacity and the Alexander–Taylor capacity (see [DN15, Proposition 5.4] and [BEGZ10, Lemma 4.2]):

Proposition 2.7. *There exists $A > 0$ such that for all Borel subsets $E \subset X$,*

$$\exp\left[-\frac{A}{\mathrm{Cap}_\theta(E)}\right] \leqslant T_\theta(E) \leqslant e \cdot \exp\left[-\left(\frac{\mathrm{Vol}(\{\theta\})}{\mathrm{Cap}_\theta(E)}\right)^{\frac{1}{n}}\right].$$

The second inequality was proved in [BEGZ10] while the first one was proved in [DN15, Proposition 5.4] using the main result of [BD12]. But, what was actually used in the proof of [DN15, Proposition 5.4] is the inequality in Theorem 2.6. Thus Proposition 2.7 is independent of [BD12].

Using Proposition 2.7 and a sharp analysis in the ample locus we get the following comparison between two θ-capacities. Let us emphasize that this result significantly extends [DN15, Theorem 5.6] where an extra assumption is required (see [DN15, Definition 4.2]).

Theorem 2.8. *Suppose $\{\theta_1\}$ and $\{\theta_2\}$ are big. Then there exists $C = C(\theta_1, \theta_2) > 0$ such that*

$$C^{-1}\mathrm{Cap}^n_{\theta_1}(E) \leqslant \mathrm{Cap}_{\theta_2}(E) \leqslant C\mathrm{Cap}^{1/n}_{\theta_1}(E),$$

for all Borel subsets $E \subset X$.

Proof. Since $\{\theta_2\}$ is big we can find $\psi \in \mathrm{PSH}(X, \theta_2)$ that is smooth in $\mathrm{Amp}(\{\theta_2\})$, ψ has analytic singularities such that $\theta_2 + dd^c\psi \geqslant \varepsilon\omega \geqslant \varepsilon\theta_1$ for some $\varepsilon > 0$ ([Bou04]). Normalize ψ by $\sup_X \psi = 0$ and denote by $U = \{\psi > -1\}$ which is a nonempty open subset of X, hence $M_{\theta_1,U} = \sup_X V_{\theta_1,U} < +\infty$. Now, the function $u = \psi + \varepsilon V^*_{\theta_1,E}$ is θ_2-psh and satisfies $u \leqslant 0$ on E (modulo a pluripolar set). Thus by definition we have $u \leqslant V^*_{\theta_2,E}$. It follows that

$$M_{\theta_2,E} \geqslant \sup_U u \geqslant \varepsilon \sup_U V^*_{\theta_1,E} - 1.$$

On the other hand $V^*_{\theta_1,E} - \sup_U V^*_{\theta_1,E}$ is θ_1-psh and takes nonpositive values on U, hence $V^*_{\theta_1,E} - \sup_U V^*_{\theta_1,E} \leqslant V_{\theta_1,U}$. This together with the above inequality yields

$$M_{\theta_2,E} \geqslant \varepsilon(M_{\theta_1,E} - M_{\theta_1,U}) - 1,$$

Giving that, for some $C > 0$ fixed we have $T_{\theta_2}(E) \leqslant CT_{\theta_1}(E)^\varepsilon$. An elementary calculation using the double estimate of Proposition 2.7 finishes the proof. □

From this comparison of capacities and standard arguments in pluripotential theory we immediately get the following convergence result.

Corollary 2.9. *Assume that $\{\theta\}$ is big and $\{\varphi^i_j\}_j, i = 1, ..., n$ are sequences of θ-psh functions with minimal singularities that converge decreasingly (or uniformly) to $\varphi^i, i = 1, ..., n$ (also with minimal singularities). If f_j is a sequence of uniformly bounded quasi-continuous functions converging monotonically to f (also quasi-continuous) then*

$$\int_X f_j\theta_{\varphi^1_j} \wedge ... \wedge \theta_{\varphi^n_j} \to \int_X f\theta_{\varphi^1} \wedge ... \wedge \theta_{\varphi^n}.$$

Proof. Let ω be a Kähler form on X. By the definition of quasi-continuity, for any $\varepsilon > 0$ one can find an open set $U \subset X$ such that all f_j are continuous in $X \setminus U$ and $\mathrm{Cap}_\omega(U) \leqslant \varepsilon$. Using Theorem 2.8, a standard argument now gives that

$$\int_U \theta_{\varphi^1_j} \wedge ... \wedge \theta_{\varphi^n_j} \leqslant C\varepsilon^{1/n}, \quad \int_U \theta_{\varphi^1} \wedge ... \wedge \theta_{\varphi^n} \leqslant C\varepsilon^{1/n}. \tag{5}$$

Now we can use Tietze's theorem to extend each $f_j|_{X\setminus U}, f|_{X\setminus U}$ to a continuous functions $\tilde{f}_j, \tilde{f}$ on X whose L^∞ norm is controlled. It follows from [BEGZ10, Theorem 2.17] that for j_0 fixed we

have

$$\int_X \tilde{f}_{j_0}\theta_{\varphi_j^1}\wedge ...\wedge\theta_{\varphi_j^n}\to\int_X \tilde{f}_{j_0}\theta_{\varphi^1}\wedge ...\wedge\theta_{\varphi^n},\quad \int_X \tilde{f}\theta_{\varphi_j^1}\wedge ...\wedge\theta_{\varphi_j^n}\to\int_X \tilde{f}\theta_{\varphi^1}\wedge ...\wedge\theta_{\varphi^n}.$$

Using (5) and the uniform boundedness of $f_j, \tilde{f}_j, f, \tilde{f}$, we can subsequently write that

$$\int_X f_{j_0}\theta_{\varphi_j^1}\wedge ...\wedge\theta_{\varphi_j^n}\to\int_X f_{j_0}\theta_{\varphi^1}\wedge ...\wedge\theta_{\varphi^n},\quad \int_X f\theta_{\varphi_j^1}\wedge ...\wedge\theta_{\varphi_j^n}\to\int_X f\theta_{\varphi^1}\wedge ...\wedge\theta_{\varphi^n}.$$

Finally, using the above and the monotonicity of f_j we can write that

$$\lim_{j_0\to\infty}\int_X f_{j_0}\theta_{\varphi^1}\wedge ...\wedge\theta_{\varphi^n}\geqslant\lim_{j\to\infty}\int_X f_j\theta_{\varphi_j^1}\wedge ...\wedge\theta_{\varphi_j^n}\geqslant\int_X f\theta_{\varphi^1}\wedge ...\wedge\theta_{\varphi^n}.$$

After invoking the dominated convergence theorem, the proof is finished. □

2.5 The operator $P(\varphi,\psi)$

Consider X a compact Kähler manifold and $\{\theta\}$ a big cohomology class. Given an usc function f on X, it is natural to ask whether there exists $u\in\mathrm{PSH}(X,\theta)$ lying below f . We will pay particular attention to the case when $f=\min(\varphi,\psi)$, where $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, in which case more can be said. Indeed, when $\{\theta\}$ is Kähler and $\varphi,\psi\in\mathcal{E}(X,\theta)$, it was shown in [Dar14] that $P_\theta(\varphi,\psi)\in\mathcal{E}(X,\theta)$. The analogue of this result holds in the big case as well:

Theorem 2.10. *Let $\chi\in\mathcal{W}^-$, i.e., χ is convex increasing with $\chi(0)=0$ and $\chi(-\infty)=-\infty$. If $\varphi,\psi\in\mathcal{E}_\chi(X,\theta)$, then $P_\theta(\varphi,\psi):=P_\theta(\min(\varphi,\psi))\in\mathcal{E}_\chi(X,\theta)$. In particular, if $\varphi,\psi\in\mathcal{E}(X,\theta)$, then $P_\theta(\varphi,\psi)\in\mathcal{E}(X,\theta)$, thus $P_{[\theta,\psi]}(\varphi)\in\mathcal{E}(X,\theta)$.*

Proof. Without loss of generality we can assume that $\varphi,\psi\leqslant 0$. Let $\varphi_j:=\max(\varphi,V_\theta-j)$, $\psi_j:=\max(\psi,V_\theta-j)$ be the canonical approximants. For each $j>0$, it follows from Lemma 2.11 below that there exists a unique $u_j\in\mathrm{PSH}(X,\theta)$ with minimal singularities such that

$$\theta_{u_j}^n=e^{u_j-\varphi_j}\theta_{\varphi_j}^n+e^{u_j-\psi_j}\theta_{\psi_j}^n. \tag{6}$$

Additionally, it follows from Lemma 2.5 (see also the proof of the uniqueness part in Lemma 2.11) that $u_j\leqslant\min(\varphi_j,\psi_j)$. Consequently $u_j\leqslant P_\theta(\varphi_j,\psi_j)$.

Next we claim that

$$\inf_j\int_X\chi(u_j-V_\theta)\theta_{u_j}^n>-\infty.$$

To prove the claim, in view of (6) it suffices to prove that

$$\inf_j\int_X\chi(u_j-V_\theta)e^{u_j-\varphi_j}\theta_{\varphi_j}^n>-\infty. \tag{7}$$

By convexity of χ it follows that $\chi(t+s)\geqslant\chi(t)+\chi(s)$, and also $\chi(t)e^t\geqslant -C$, for all $t,s\leqslant 0$ for some $C>0$. Thus to prove (7) it suffices to check that

$$\inf_j\int_X\chi(\varphi_j-V_\theta)e^{u_j-\varphi_j}\theta_{\varphi_j}^n>-\infty.$$

But this holds since $u_j\leqslant\varphi_j$ and $\varphi\in\mathcal{E}_\chi(X,\theta)$. Thus the claim is proved.

Since $\chi(-\infty)=-\infty$, the claim implies that $\sup_X u_j$ is uniformly bounded. It thus follows from [BEGZ10, Proposition 2.19] that some subsequence of u_j converges in $L^1(X,\omega^n)$ to some $u\in\mathcal{E}_\chi(X,\theta)$. Since $u_j\leqslant P_\theta(\varphi_j,\psi_j)$ it follows that $u\leqslant P_\theta(\min(\varphi,\psi))$. Now, it follows from [BEGZ10, Proposition 2.14] that $P_\theta(\varphi,\psi)\in\mathcal{E}(X,\theta)$ and so is $P_{[\theta,\psi]}(\varphi)$. □

Lemma 2.11. *Assume that $u, v \in \mathrm{PSH}(X,\theta)$ with minimal singularities. Then there exists a unique $\varphi \in \mathrm{PSH}(X,\theta)$ with minimal singularities such that*

$$\theta_\varphi^n = e^{\varphi-u}\theta_u^n + e^{\varphi-v}\theta_v^n.$$

Proof. The uniqueness follows from the Lemma 2.5. Indeed, assume that $\psi \in \mathcal{E}(X,\theta)$ is another solution, i.e.

$$\theta_\psi^n = e^{\psi-u}\theta_u^n + e^{\psi-v}\theta_v^n.$$

Set $\mu = e^u\theta_v^n + e^v\theta_u^n$ and $\phi := u+v$. Then we can write $\theta_\varphi^n = e^{\varphi-\phi}\mu$ and $\theta_\psi^n = e^{\psi-\phi}\mu$. It thus follows from Lemma 2.5 that $\varphi = \psi$.

To prove the existence we approximate u by $u_j := \max(u,-j)$ and note that these are bounded functions. Observe that, for each j,

$$\mu_j := e^{-u_j}\theta_u^n + e^{-v_j}\theta_v^n$$

is a non-pluripolar positive measure on X. For each $j>0$ it follows from Theorem 2.3 that there exists $\varphi_j \in \mathrm{PSH}(X,\theta)$ with minimal singularities such that

$$\theta_{\varphi_j}^n = e^{\varphi_j}\mu_j.$$

Let $C>0$ be a constant such that $\sup_X |u-v| \leqslant 2C$. This is possible because u and v have minimal singularities. The function $\phi := \frac{u+v}{2} - C - n\log 2$ is θ-psh with minimal singularities and it satisfies

$$\theta_\phi^n \geqslant e^\phi \mu_j.$$

It thus follows from Lemma 2.5 that $\varphi_j \geqslant \phi$ for all j. It also follows from Lemma 2.5 that φ_j is decreasing in j, the pointwise limit $\varphi := \lim_{j\to+\infty}\varphi_j$ has minimal singularities. By continuity of the Monge-Ampère operator (see [BEGZ10]) it follows that φ is the solution we are looking for. □

As a simple consequence of the above result, we can settle the conjecture of [BEGZ10, Remark 2.16] about the convexity of the classes $\mathcal{E}_\chi(X,\theta)$. A similar result in the Kähler case was obtained in [Dar14].

Corollary 2.12. *Suppose $\chi \in \mathcal{W}^-$. Then $\mathcal{E}_\chi(X,\theta)$ is convex.*

Proof. Given $u_0, u_1 \in \mathcal{E}_\chi(X,\theta)$ it follows that $P_\theta(u_0,u_1) \leqslant tu_0 + (1-t)u_1,\ t\in[0,1]$. By the above result $P_\theta(u_0,u_1) \in \mathcal{E}_\chi(X,\theta)$. Now [BEGZ10, Proposition 2.14] implies that $tu_0+(1-t)u_1 \in \mathcal{E}_\chi(X,\theta)$. □

If f is smooth, it follows from a balayage argument that $\theta^n_{P_\theta(f)}$ is concentrated on the contact set $\{P_\theta(f)=f\}$. Using the capacity theory developed in the previous subsection, the result also holds for more general functions f:

Proposition 2.13. *Assume that $\{\theta\}$ is big, $P_\theta(f) \neq -\infty$ and f is quasi-continuous and usc on X. Then $\theta^n_{P_\theta(f)}$ does not charge $\{P_\theta(f)<f\}$.*

Proof. Without loss of generality we can assume that $f \leqslant 0$. If f is smooth then the result follows from a balayage argument (or directly from Theorem 2.6). To treat the general case, we approximate f from above (by semicontinuity) by a sequence of smooth functions (f_j). We can also assume that $f_j \leqslant 0$. Set $\varphi_j := P_\theta(f_j)$, $\varphi := P_\theta(f)$ and note that $\varphi_j \searrow \varphi$. For each $j\in\mathbb{N}$ the

measure $\theta^n_{\varphi_j}$ vanishes in the set $\{\varphi_j < f_j\}$. Now, we want to pass to the limit as $j \to +\infty$. We first fix $k, l \in \mathbb{N}$ and set

$$U_{k,l} = \{\varphi_k < f\} \cap \{\varphi > V_\theta - l\}.$$

For any $j > k$, note that on $U_{k,l}$ we have $\varphi_j \geqslant \varphi > V_\theta - l$ and $\{\varphi_k < f\} \subset \{\varphi_j < f_j\}$. It thus follows from definition of the non-pluripolar product (see [BEGZ10]) that for any $j > k$, the measure $\theta^n_{\max(\varphi_j, V_\theta - l)} = \theta^n_{\varphi_j}$ vanishes on $U_{k,l}$. By assumption f is quasi-continuous, hence $U_{k,l}$ is quasi open. More precisely, for any fixed $\varepsilon > 0$ there exists an open set V_ε such that the set $G_\varepsilon := (V_\varepsilon \setminus U_{k,l} \cup U_{k,l} \setminus V_\varepsilon)$ satisfies $\mathrm{Cap}_\omega(G_\varepsilon) \leqslant \varepsilon$. Observe that for fixed l all the functions $\max(\varphi_j, V_\theta - l)$ are sandwiched between $V_\theta - l$ and V_θ. It then follows that

$$\sup_{j\in\mathbb{N}} \int_{G_\varepsilon} \theta^n_{\max(\varphi_j, V_\theta - l)} \leqslant A\mathrm{Cap}_\theta(G_\varepsilon) \leqslant A'\mathrm{Cap}^{1/n}_\omega(G_\varepsilon) \leqslant A'\varepsilon^{1/n},$$

where the last inequality follows from the comparison of capacities in Theorem 2.8. Consequently, $\sup_{j\in\mathbb{N}} \int_{V_\varepsilon} \theta^n_{\max(\varphi_j, V_\theta - l)} \leqslant A'\varepsilon^{1/n}$, and the continuity of the Monge-Ampère operator allows to take the limit, and we ultimately obtain:

$$\int_{U_{k,l}} \theta^n_{\max(\varphi, V_\theta - l)} \leqslant C\varepsilon^{1/n},$$

for some positive constant C independent of ε (but dependent on l). Now letting $\varepsilon \to 0$ we see that $\theta^n_{\max(\varphi, V_\theta - l)}$ vanishes in $U_{k,l}$. Letting $l \to +\infty$, and using the definition of the non-pluripolar product, we see that θ^n_φ vanishes in $\{\varphi_k < f\}$. Now, letting $k \to +\infty$ we obtain the result. □

With the previous result in hands, we can prove that the non-pluripolar Monge-Ampère measure of $P_{[\theta,\varphi]}(\psi)$ is supported on the contact set $\{P_{[\theta,\varphi]}(\psi) = \psi\}$. The latter was already proved in [RWN, Lemma 4.9] when $\{\theta\}$ is a Kähler class and ψ is continuous.

Proposition 2.14. *Let $\{\theta\}$ be a big class. Assume that $\varphi, \psi, P(\varphi, \psi) \in \mathrm{PSH}(X, \theta)$. Then $\theta^n_{P_{[\theta,\varphi]}(\psi)}$ does not charge $\{P_{[\theta,\varphi]}(\psi) < \psi\}$.*

Proof. For each $t > 0$, since $\min(\varphi + t, \psi)$ is usc and quasi-continuous, it follows from Proposition 2.13 that $\theta^n_{\psi_t}$ vanishes on $\{\psi_t < \min(\varphi + t, \psi)\}$, where $\psi_t := P_\theta(\min(\varphi + t, \psi))$. Because $\{\psi_t < \psi < \varphi + t\} \subset \{\psi_t < \min(\varphi + t, \psi)\}$ it thus follows that

$$\int_{\{\psi_t < \psi < \varphi + t\}} \theta^n_{\psi_t} = 0.$$

Fix $s > 0, j > 0$ and set $v := P_{[\theta,\varphi]}(\psi) = (\lim_{t\to+\infty} \psi_t)^*$, $\psi_{t,j} := \max(\psi_t, V_\theta - j)$, $v_j := \max(v, V_\theta - j)$. It is clear that $\psi_{t,j} \nearrow v_j$ almost everywhere as $t \nearrow \infty$. By the plurifine property of the non-pluripolar Monge-Ampère measure it follows that

$$\int_{\{V_\theta - j < \psi_t < \psi < \varphi + t\}} \theta^n_{\psi_{t,j}} = 0.$$

For $t > s$, we have $\psi_s \leqslant \psi_t \leqslant v \leqslant \psi$. Consequently, $\{V_\theta - j < \psi_s \leqslant v < \psi < \varphi + s\} \subset \{V_\theta - j < \psi_t < \psi < \varphi + t\}$ and we have

$$\int_{\{V_\theta - j < \psi_s \leqslant v < \psi < \varphi + s\}} \theta^n_{\psi_{t,j}} = 0.$$

Now, using the same trick as in the proof of Proposition 2.13 we let $t \to +\infty$ and arrive at

$$\int_{\{V_\theta - j < \psi_s \leqslant v < \psi < \varphi + s\}} \theta^n_{v_j} = 0.$$

Letting $s \to +\infty$, then $j \to +\infty$, and noting that θ_v^n is a non-pluripolar measure, we can conclude that θ_v^n vanishes on $\{v < \psi\}$. This concludes the proof. □

Theorem 2.15. *Assume that $\psi, \varphi \in \mathcal{E}(X,\theta)$. Then $P_{[\theta,\varphi]}(\psi) = \psi$.*

Proof. It follows from Theorem 2.10 that $P_{[\theta,\varphi]}(\psi) \in \mathcal{E}(X,\theta)$. The conclusion then follows from Proposition 2.14 and the domination principle (Proposition 2.4). □

3. Weak geodesics in big cohomology classes

3.1 Berndtsson's construction

We introduce a notion of weak geodesics in big cohomology classes mimicking Berndtsson's construction in [Bern, Section 2.2]. Fix θ a smooth closed $(1,1)$-form such that $\{\theta\}$ is big and also fix $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$ with minimal singularities. A *subgeodesic* of φ_0, φ_1 is a curve $[0,1] \ni t \mapsto u_t \in \mathrm{PSH}(X,\theta)$ such that

(i) For each $t \in [0,1]$, the function u_t has minimal singularities,

(ii) $u_{0,1} = \lim_{t\to 0,1} u_t \leqslant \varphi_{0,1}$ pointwise,

(iii) The complexification $X \times D \ni (x,z) \mapsto U(x,z) := u_{\log|z|}(x)$ is $\pi^*\theta$-psh on $X \times D$, where $D := \{z \in \mathbb{C} \mid 1 < |z| < e\}$ is the annulus in $\mathbb{C}$ and $\pi : X \times D \to X$ is the trivial projection.

Let us mention that a curve $(\alpha,\beta) \in t \to u_t \in \mathrm{PSH}(X,\theta)$ is a (general) subgeodesic if it satisfies only the appropriate version of (iii) above. In this case α, β may even take $\pm\infty$ as a value. We will not make a difference between the curve $t \to u_t$ and its complexification U.

The *weak geodesic* $[0,1] \ni t \to \varphi_t \in \mathrm{PSH}(X,\theta)$ with minimal singularities joining φ_0 to φ_1 is defined as the envelope of all subgeodesics, i.e.,

$$\varphi_l(x) := \sup\{u_l(x), \text{ where } t \to u_t \text{ is a subgeodesic of } \varphi_0, \varphi_1\}, \quad l \in [0,1], x \in X. \tag{8}$$

Lemma 3.1. *Let $t \to \varphi_t$ be the weak geodesic joining $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$ with minimal singularities, constructed as above. Then there exists $C = C(\varphi_0,\varphi_1) > 0$ such that*

$$|\varphi_t - \varphi_{t'}| \leqslant C|t - t'|, \quad t, t' \in [0,1]. \tag{9}$$

Additionally, for the complexification $\Phi(x,z) := \varphi_{\log|z|}(x)$ we have

$$(\pi^*\theta + dd^c\Phi)^{n+1} = 0 \text{ on } \mathrm{Amp}(\{\theta\}) \times D, \tag{10}$$

where equality is understood in the weak sense of measures.

Proof. The proof is essentially the same as in [Bern, Section 2.2], so we only point out the ideas. Consider the following (barrier) subgeodesic of φ_0, φ_1:

$$u_t(x) := \max(\varphi_0 - Ct, \varphi_1 + C(t-1)),$$

where C is a positive constant such that $\varphi_1 - C \leqslant \varphi_0 \leqslant \varphi_1 + C$. It is clear that $[0,1] \ni t \to u_t \in \mathrm{PSH}(X,\theta)$ is a subgeodesic of φ_0, φ_1, and by t-convexity of $t \to \varphi_t$ we can write $u_t \leqslant \varphi_t \leqslant (1-t)\varphi_0 + t\varphi_1$, hence the conclusion about Lipschitz continuity of $t \to \varphi_t$ follows.

The proof of (10) follows from a standard balayage argument and we refer the interested reader to [Bern, Section 2.2], to see how the ideas from there generalize to our setting. □

Next we prove a version of the comparison principle:

Proposition 3.2. *Assume that $u, v \in \mathrm{PSH}(X \times D, \pi^*\theta)$ satisfies $V_\theta - C \leqslant u_s, v_s, s \in D$ for some $C > 0$. If $\liminf_{(x,z)\to X\times\partial D}(u - v) \geqslant 0$ then*

$$\int_{\{u<v\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c v)^{n+1} \leqslant \int_{\{u<v\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c u)^{n+1}.$$

Proof. Fix $\varepsilon > 0$, $\delta > 0$. As θ is big we can find $\psi \in \mathrm{PSH}(X, \theta)$, $\sup_X \psi = 0$, with analytic singularities such that $X \setminus \mathrm{Amp}(\{\theta\}) = \{\psi = -\infty\}$ and $\psi \leqslant u_s, v_s,\ s \in D$, such that $\theta + dd^c\psi$ dominates a Kähler form. Consider

$$u_\varepsilon := \max(u, v_\varepsilon),\ v_\varepsilon := (1-\delta)v + \delta\lambda\psi - \varepsilon,$$

for some constant $\lambda > 1$. If $\lambda - 1$ is small enough we have

$$\theta + dd^c\lambda\psi = \lambda(\theta + dd^c\psi) + (1-\lambda)\theta \geqslant \lambda\omega + (1-\lambda)\theta \geqslant 0,$$

where ω is a Kähler form on X. Thus v_ε is $\pi^*\theta$-psh on $X \times D$. Observe that for some open relatively compact $\Omega' \Subset \mathrm{Amp}(\{\theta\})$ (Ω' depends on λ), $K \Subset D$, we have $u_\varepsilon \equiv u$ in a neighbourhood containing $(X \setminus \Omega') \times (D \setminus K)$. It follows that $\int_Y (\pi^*\theta + dd^c u)^{n+1} = \int_Y (\pi^*\theta + dd^c u_\varepsilon)^{n+1}$, where $Y := \Omega' \times K$. Indeed, for any test function $0 \leqslant \chi \in \mathcal{C}^\infty(Y)$ which is identically 1 in an open neighborhood U inside Y such that $\{u < u_\varepsilon\} \subset U$ we have

$$\int_Y \chi[(\pi^*\theta + dd^c u)^{n+1} - (\pi^*\theta + dd^c u_\varepsilon)^{n+1}] = \int_Y (u - u_\varepsilon) dd^c\chi \wedge T = 0,$$

where T is a positive (n,n)-current.

Recall that for locally bounded $\pi^*\theta$-psh functions φ, ψ, the maximum principle for the complex Monge-Ampère operator yields

$$\mathbf{1}_{\{\varphi<\psi\}}(\pi^*\theta + dd^c \max(\varphi, \psi))^{n+1} = \mathbf{1}_{\{\varphi<\psi\}}(\pi^*\theta + dd^c\psi)^{n+1}.$$

Using the above facts we can write

$$\begin{aligned}
\int_{\{u<v_\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c v_\varepsilon)^{n+1} &= \int_{\{u<v_\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c u_\varepsilon)^{n+1} \\
&= \int_Y (\pi^*\theta + dd^c u_\varepsilon)^{n+1} - \int_{\{u\geqslant v_\varepsilon\}\cap Y} (\pi^*\theta + dd^c u_\varepsilon)^{n+1} \\
&\leqslant \int_Y (\pi^*\theta + dd^c u)^{n+1} - \int_{\{u>v_\varepsilon\}\cap Y} (\pi^*\theta + dd^c u_\varepsilon)^{n+1} \\
&= \int_Y (\pi^*\theta + dd^c u)^{n+1} - \int_{\{u>v_\varepsilon\}\cap Y} (\pi^*\theta + dd^c u)^{n+1} \\
&\leqslant \int_{\{u<(1-\delta)v+\delta\lambda\psi-\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c u)^{n+1}.
\end{aligned}$$

The left-hand side in the above estimate can be further estimated in the following way:

$$\int_{\{u<v_\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c v_\varepsilon)^{n+1} \geqslant (1-\delta)^n \int_{\{u<v_\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c v)^{n+1}.$$

Letting $\delta, \varepsilon \to 0$, the proof is finished. □

Proposition 3.3. *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X, \theta)$ with minimal singularities and $[0,1] \ni t \to u_t$ be a subgeodesic of φ_0, φ_1 with minimal singularities satisfying (9) and $u_{0,1} = \varphi_{0,1}$. If the complexification satisfies $(\pi^*\theta + dd^c U)^{n+1} = 0$ on $\mathrm{Amp}(\{\theta\}) \times D$ then $t \to u_t$ is the weak geodesic connecting φ_0 to φ_1.*

Proof. Let $t \to \varphi_t$ be the weak geodesic connecting φ_0, φ_1. It suffices to show that $\Phi \leqslant U$. Fix ρ a smooth function in D such that $\rho = 0$ on the boundary and $dd^c\rho = \sqrt{-1}dz \wedge d\bar{z}$. Fix also $\tau \in \mathrm{PSH}(X, \theta)$ with minimal singularities such that $(\theta + dd^c\tau)^n = c\omega^n$, for some positive normalization constant c and some fixed Kähler form ω on X. Observe that such a potential exists thanks to [BEGZ10, Theorem 4.1]. We normalize τ so that $\tau \leqslant \min(\varphi_0, \varphi_1)$. Applying the comparison principle (Proposition 3.2) for $\Phi_\varepsilon := (1-\varepsilon)\Phi + \varepsilon(\rho + \tau)$ and U we get

$$\int_{\{U<\Phi_\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^c\Phi_\varepsilon)^{n+1} \leqslant \int_{\{U<\Phi_\varepsilon\}\cap \mathrm{Amp}(\{\theta\})\times D} (\pi^*\theta + dd^cU)^{n+1} = 0.$$

By expanding $(\pi^*\theta + dd^c\Phi_\varepsilon)^{n+1} \geqslant \varepsilon^n(\theta + dd^c\tau)^n \wedge dd^c\rho = \varepsilon^n c\omega^n \wedge dd^c\rho > 0$, the inequality above gives that $U \geqslant \Phi_\varepsilon$ almost everywhere, hence everywhere in $X \times D$. Letting $\varepsilon \to 0$ we get the desired result. □

In the Kähler case, geodesics joining smooth potentials are $C^{1,\bar{1}}$-smooth and the Monge–Ampère energy is linear along these geodesics. When θ is also nef, by adapting the proof of Chen [Che00] (see [Bou12]), we expect that a similar regularity result still holds in the ample locus of $\{\theta\}$.

Recall that for potentials with minimal singularities the Monge–Ampère energy is defined by the following expression:

$$\mathrm{I}(u) := \frac{1}{(n+1)\mathrm{Vol}(\theta)} \sum_{j=0}^{n} \int_X (u - V_\theta)\theta_u^j \wedge \theta_{V_\theta}^{n-j}.$$

We proceed now to show that, as perhaps expected, the Monge–Ampère energy is linear/convex along geodesics/subgeodesics with minimal singularities. However to argue this, we will need to use a careful mollification argument for subgeodesics in the time variable that will be detailed in the next subsection.

3.2 Approximation of subgeodesics

Unless specified otherwise, assume for this subsection that $(0,1) \ni t \to \varphi_t \in \mathrm{PSH}(X, \theta)$ is a subgeodesic with minimal singularities such that $|\varphi_t - \varphi_{t'}| \leqslant C|t - t'|$, for some positive constant C. Consider a smoothing kernel $\chi : \mathbb{R} \to [0,1]$ supported in $[-1,1]$ with $\int_{\mathbb{R}} \chi(t)dt = 1$ and $\chi(t) = \chi(-t)$. We then set $\chi_\varepsilon(t) := \frac{1}{\varepsilon}\chi(t/\varepsilon)$, so that the support of χ_ε is $[-\varepsilon, \varepsilon]$ and $\int_{\mathbb{R}} \chi_\varepsilon(t)dt = 1$. For each $\varepsilon > 0$ we consider

$$\varphi_{\varepsilon,t}(x) := \int_0^1 \varphi_s(x)\chi_\varepsilon(t-s)ds, \tag{11}$$

which is well defined for $t \in (\varepsilon, 1-\varepsilon)$.

Lemma 3.4. *For each $\varepsilon > 0$, $(\varepsilon, 1-\varepsilon) \ni t \to \varphi_{\varepsilon,t} \in \mathrm{PSH}(X, \theta)$ is a t-Lipschitz subgeodesic with minimal singularities.*

Proof. The fact that $t \to \varphi_{\varepsilon,t}$ is a subgeodesic is trivial. Because $t \to \varphi_t$ is t-Lipschitz, it follows that $\varphi_{\varepsilon,t}$ has minimial singularities for all $t \in [\varepsilon, 1-\varepsilon]$. Working directly with (11) we obtain $|\varphi_{\varepsilon,t} - \varphi_{\varepsilon,t'}| \leqslant C|t - t'|$. □

The next observation is simple but will be crucial for our approximation procedure.

Lemma 3.5. *There exists $A_\varepsilon, B_\varepsilon > 0$ (that may blow up as $\varepsilon \to 0$) such that $\dot{\varphi}_{\varepsilon,t}/A_\varepsilon$, $\ddot{\varphi}_{\varepsilon,t}/B_\varepsilon$ can be written as a difference of θ-psh functions with minimal singularities $t \in (\varepsilon, 1-\varepsilon)$.*

Proof. Write $\chi'_\varepsilon(t) = \rho^+_\varepsilon(t) - \rho^-_\varepsilon(t)$ where $\rho^+_\varepsilon, \rho^-_\varepsilon$ are the positive and negative parts of $\chi'_\varepsilon(t)$. They are clearly bounded (but the bound blows up as $\varepsilon \to 0$). Now, let

$$u^\pm_{\varepsilon,t}(x) := \int_0^1 \varphi_s(x)\rho^\pm_\varepsilon(t-s)ds.$$

It follows that

$$dd^c u^\pm_{\varepsilon,t} = \int_0^1 (dd^c\varphi_s)\rho^\pm_\varepsilon(t-s)ds \geqslant -A_\varepsilon\theta,$$

where $A_\varepsilon = \int_{-\varepsilon}^\varepsilon \rho^+_\varepsilon = \int_{-\varepsilon}^\varepsilon \rho^-_\varepsilon$. Observe indeed that $\int_{-\varepsilon}^\varepsilon \rho^+_\varepsilon - \rho^-_\varepsilon = \int_{-\varepsilon}^\varepsilon \chi'_\varepsilon = 0$. To see that $u^\pm_{\varepsilon,t}/A_\varepsilon$ has minimal singularities we note that $|\varphi_t - \varphi_{t'}| \leqslant C|t-t'|$, hence

$$u^\pm_{\varepsilon,t}(x) := \int_{-\varepsilon}^\varepsilon \varphi_{t-s}(x)\rho^\pm_\varepsilon(s)ds \leqslant \int_{-\varepsilon}^\varepsilon \varphi_{t'-s}(x)\rho^\pm_\varepsilon(s)ds + CA_\varepsilon|t-t'|.$$

For $\ddot\varphi_{\varepsilon,t}$, a similar argument works with the choice $B_\varepsilon := \int_{-\varepsilon}^\varepsilon \max(\chi''_\varepsilon(t), 0)dt$. □

Lemma 3.6. *The Monge-Ampère energy is twice differentiable along $t \to \varphi_{\varepsilon,t}$ and the derivatives are given by*

$$\frac{d}{dt}\mathrm{I}(\varphi_{\varepsilon,t}) = \int_X \dot\varphi_{\varepsilon,t}\theta^n_{\varphi_{\varepsilon,t}},\ \frac{d^2}{dt^2}\mathrm{I}(\varphi_{\varepsilon,t}) = \int_X \ddot\varphi_{\varepsilon,t}\theta^n_{\varphi_{\varepsilon,t}} - n\int_X d\dot\varphi_{\varepsilon,t}\wedge d^c\dot\varphi_{\varepsilon,t}\wedge\theta^{n-1}_{\varphi_t}. \tag{12}$$

Proof. For notational convenience we remove the subscript ε. By basic properties of the I functional (see [BEGZ10, BB10]) we have, for $t \in (\varepsilon, 1-\varepsilon), s > 0$,

$$\int_X (\varphi_{t+s} - \varphi_t)\theta^n_{\varphi_{t+s}} \leqslant \mathrm{I}(\varphi_{t+s}) - \mathrm{I}(\varphi_t) \leqslant \int_X (\varphi_{t+s} - \varphi_t)\theta^n_{\varphi_t}.$$

Dividing by $s > 0$ the right-hand side then converges (as $s \to 0$) to $\int_X \dot\varphi_t\theta^n_{\varphi_t}$. The left-hand side can be estimated by

$$\frac{1}{s}\int_X (\varphi_{t+s} - \varphi_t)\theta^n_{\varphi_{t+s}} \geqslant \int_X \dot\varphi_t\theta^n_{\varphi_{t+s}}.$$

As $\dot\varphi_t$ is bounded and quasi-continuous (with respect to the Monge-Ampère capacity Cap_ω), Corollary 2.9 allows to pass to the limit as $s \to 0$ and the differentiability of $\mathrm{I}(\varphi_t)$ follows.

To prove the formula for the second derivative we fix $t \in (\varepsilon, 1-\varepsilon)$ and $s > 0$ small enough and prove that

$$\lim_{s\to 0^+} d(s) := \lim_{s\to 0^+} \frac{1}{s}\left(\int_X \dot\varphi_{t+s}\theta^n_{\varphi_{t+s}} - \dot\varphi_t\theta^n_{\varphi_t}\right)$$

equals to the right-hand side of (12). The same formula for the left limit will then follow automatically, as we can always "reverse" the time direction of a subgeodesic. Setting $T_s := \sum_{j=1}^{n-1}\theta^j_{\varphi_t}\wedge\theta^{n-1-j}_{\varphi_{t+s}}$, we can write

$$d(s) = \frac{1}{s}\left(\int_X \dot\varphi_{t+s}dd^c(\varphi_{t+s} - \varphi_t)\wedge T_s + \int_X (\dot\varphi_{t+s} - \dot\varphi_t)\theta^n_{\varphi_t}\right).$$

Thanks to Theorem 1.14 in [BEGZ10] we can integrate by parts in the first term and obtain

$$d(s) = \frac{1}{s}\left(\int_X (\varphi_{t+s} - \varphi_t)dd^c\dot\varphi_{t+s}\wedge T_s + \int_X (\dot\varphi_{t+s} - \dot\varphi_t)\theta^n_{\varphi_t}\right). \tag{13}$$

As $s \to 0$, $(\varphi_{t+s} - \varphi_t)/s$ decreases to $\dot\varphi_t$, all of them being quasi-continuous and uniformly bounded. On the other hand, by Lemma 3.5 we can write $\dot\varphi_{t+s}/A_\varepsilon = u^+_{t+s} - u^-_{t+s}$, where $u^\pm_{t+s}$ are θ-psh functions with minimal singularities that converge uniformly to $u^\pm_t$ as $s \to 0$. By using

Corollary 2.9, the first term in the right-hand side of (13) converges to $n\int_X \dot\varphi_t dd^c\dot\varphi_t\wedge\theta_{\varphi_t}^{n-1}$. Moreover, using dominated convergence for the second term we obtain

$$\lim_{s\to 0^+}\frac{1}{s}\int_X(\dot\varphi_{t+s}-\dot\varphi_t)\theta_{\varphi_t}^n=\int_X\ddot\varphi_t\theta_{\varphi_t}^n.$$

Observe in fact that $\ddot\varphi_t$ is uniformly bounded from above as it can be written as the difference of two qpsh functions. Finally, an integration by parts gives

$$\lim_{s\to 0^+} d(s)=-n\int_X d\dot\varphi_t\wedge d^c\dot\varphi_t\wedge\theta_{\varphi_t}^{n-1}+\int_X\ddot\varphi_t\theta_{\varphi_t}^n.$$

□

Corollary 3.7. *The Monge-Ampère energy is convex along $t\to\varphi_{\varepsilon,t}$.*

Proof. We again drop the ε subscript and fix $t\in(\varepsilon,1-\varepsilon)$ for the duration of the proof. In view of Lemma 3.6 it is enough to prove that

$$-nd\dot\varphi_t\wedge d^c\dot\varphi_t\wedge\theta_{\varphi_t}^{n-1}+\ddot\varphi_t\theta_{\varphi_t}^n\geqslant 0 \tag{14}$$

in the weak sense of measures in $\mathrm{Amp}(\{\theta\})$. This property is local, hence we can work in relatively compact open subset K of $\mathrm{Amp}(\{\theta\})$, and approximate $\varphi_t(x)$ using a convolution in the space variable x. As we show now, since $\dot\varphi_t$, $\ddot\varphi_t$ are bounded in K and can be written as the difference of two quasi-psh functions with minimal singularities, the convergence of the appropriate approximating measures to the left-hand side in (14) follows from standard pluripotential theory. Indeed, fix a coordinate ball $B\subset K$. We will show that (14) holds in B in the weak sense of measures.

We can assume existence of a smooth local potential $\tau\in C^\infty(B)$ such that $\theta=dd^c\tau$. Let $\tilde\rho_\delta(x)$ be a smoothing kernel in $\mathbb{C}^n$ and consider

$$\tilde\varphi_t^\delta(x):=\int_B(\tau(x-\zeta)+\varphi_t(x-\zeta))\tilde\rho_\delta(\zeta)dV(\zeta)-\tau(x).$$

Since the complexification of $\tilde\varphi_t^\delta$ is smooth and $\pi^*\theta$-psh in $B_\delta\times D_\varepsilon$, it follows that (14) holds for $\tilde\varphi_t^\delta$ and $\tilde\varphi_t^\delta\searrow\varphi_t$.

By Lemma 3.5 we can write $\dot\varphi_t:=u_+-u_-$, where u_+,u_- are bounded quasi psh functions on B_δ. Then the corresponding smooth quasi psh functions $v_\pm^\delta$, defined by

$$v_\pm^\delta(x):=\int_0^1\int_B(\tau(x-\zeta)+\varphi_s(x-\zeta))\tilde\rho_\delta(\zeta)\,\rho_\varepsilon^\pm(t-s)ds\,dV(\zeta),$$

converge decreasingly to $u_\pm$ and we have $\dot{\tilde\varphi}_t^\delta=v_+^\delta-v_-^\delta$. A similar thing is true for the second derivatives $\ddot\varphi_t,\ddot{\tilde\varphi}_t^\delta$ as well. As all the functions involved are quasi psh and bounded on B_δ, by Bedford–Taylor theory, positivity in (14) is preserved as we take the limit $\delta\to 0$, and (14) holds restricted to B_δ, finishing the proof. □

Theorem 3.8. *Assume that $\{\theta\}$ is big and $(0,1)\ni t\to\varphi_t\in\mathrm{PSH}(X,\theta)$ is a subgeodesic with minimal singularities. Then the Monge-Ampère energy* I *is convex along $t\to\varphi_t$.*

Proof. Fix $\varepsilon>0$ for the moment. As it suffices to show convexity of $t\to\mathrm{I}(\varphi_t)$ on $(\varepsilon,1-\varepsilon)$, and $t\to\varphi_t$ has minimal singularity potentials, without loss of generality we can assume that $t\to\varphi_t$ is Lipschitz in t and let $C>0$ be such that $|\varphi_t-\varphi_{t'}|\leqslant C|t-t'|$. Let $\varphi_{\varepsilon,t}$ be the subgeodesics approximating $t\to\varphi_t$ constructed above. Since $\mathrm{I}(\varphi_{\varepsilon,t})\to\mathrm{I}(\varphi_t)$ as $\varepsilon\to 0$ [BB10, Proposition 4.3], it suffices to prove the convexity of $t\to\mathrm{I}(\varphi_{\varepsilon,t})$. But this follows from Corollary 3.7. □

3.3 Linearity of the Monge-Ampère energy along geodesics

The regularization technique in the Subsection 3.2 can also be used to prove linearity of Monge–Ampère energy along geodesics in big classes. Although this result is not essentially used in this paper (in the proof of the important direction in Lemma 3.15 one only needs Theorem 3.8), we think it will be useful in the future.

Assume θ is a smooth closed $(1,1)$-form whose cohomology class is big. Fix φ_0, φ_1 two θ-psh functions on X with minimal singularities. The complex plane $\mathbb{C}$ is now viewed as a piece of $\mathbb{CP}^1$ (so that $\mathbb{CP}^1 = \mathbb{C} \cup H_\infty$) equipped with the Fubini-Study metric ω_{FS}. Accordingly, the product $M := X \times \mathbb{CP}^1$ is equipped with a smooth $(1,1)$-form $\Theta := \pi_1^*\theta + \pi_2^*\omega_{FS}$. We use the following change of coordinates between $\mathbb{CP}^1$ and $\mathbb{C}$:

$$\mathbb{CP}^1 \setminus H_\infty \ni [z_1 : z_2] \mapsto \frac{z_1}{z_2} \in \mathbb{C},$$

where $H_\infty := \{z_2 = 0\}$.

Lemma 3.9. *One has $V_\Theta(x,z) := V_\theta(x)$ for all $z \in \mathbb{CP}^1$.*

Proof. By definition

$$V_\Theta(x,z) := \sup\{U(x,z) \quad \Theta - \text{psh} \ : \ U \leqslant 0 \text{ on } M\}.$$

Clearly $V_\Theta(x,z) \geqslant V_\theta(x)$ since $V_\theta(x)$ is a candidate in the envelope. Moreover, we observe that for each $z \in \mathbb{CP}^1$, $V_\Theta(x,z)$ is a θ-psh function and $V_\Theta(x,z) \leqslant 0$ on X, thus $V_\Theta(x,z) \leqslant V_\theta(x)$. Hence the conclusion. □

Lemma 3.10. *Let $F \in \mathcal{C}^\infty(M,\mathbb{R})$ be a smooth function on M which is S^1-invariant when restricted to $X \times \mathbb{C}$. Denote by $\Phi := P_\Theta(F)$ the Monge-Ampère envelope on M of F with respect to Θ. Then the function $\mathbb{CP}^1 \ni z \mapsto G(z) := \mathrm{I}(\Phi(\cdot, z))$ satisfies*

$$\omega_{FS} + dd_z^c G = (\pi_2)_\star(\Theta + dd_{x,z}^c \Phi)^{n+1}, \tag{15}$$

in the sense of currents.

Proof. As H_∞ is pluripolar in $\mathbb{CP}^1$ it suffices to prove (15) in $\mathbb{C}$. As F is S^1-invariant it follows that $\Phi(x,z)$ is also S^1-invariant in z. Using convolution as in Subsection 3.2 we denote by Φ_ε the approximants, i.e.

$$\Phi_\varepsilon(x,z) := \int_{\mathbb{C}} \Phi(x,\zeta)\chi_\varepsilon(|z-\zeta|^2)dV(\zeta),$$

where χ_ε is a family of smoothing kernels. For each $\varepsilon > 0$, Φ_ε is Θ_ε-psh on M where $\Theta_\varepsilon := \pi_1^*\theta + \gamma_\varepsilon \pi_2^*\omega_{FS}$, with $\gamma_\varepsilon > 1$ decreasing to 1. Indeed,

$$dd_{x,z}^c \Phi \star \chi_\varepsilon = dd^c \Phi \star \chi_\varepsilon \geqslant (-\theta - \omega_{FS}) \star \chi_\varepsilon \geqslant -\theta - \omega_{FS} \star \chi_\varepsilon.$$

Moreover, since ω_{FS} is smooth on $\mathbb{CP}^1$, we have

$$|\omega_{FS} - \omega_{FS} \star \chi_\varepsilon| \leqslant \int_{\mathbb{C}} |\omega_{FS}(z) - \omega_{FS}(z-\zeta)|\chi_\varepsilon(|\zeta|^2)dV(\zeta) \leqslant C \int_{\mathbb{C}} |\zeta|\, \chi_\varepsilon(|z'|^2)dV(z') \leqslant C\varepsilon.$$

This means that we can find γ_ε decreasing to 1 such that $\omega_{FS} \star \chi_\varepsilon \leqslant \gamma_\varepsilon \omega_{FS}$. It can be shown in the same way as in Subsection 3.2 that $G_\varepsilon(z) := \mathrm{I}(\Phi_\varepsilon(\cdot, z))$ is smooth and its complex Hessian is given by

$$\gamma_\varepsilon \omega_{FS} + dd_z^c G_\varepsilon(z) = (\pi_2)_\star(\Theta_\varepsilon + dd_{x,z}^c \Phi_\varepsilon)^{n+1}. \tag{16}$$

To prove this we first explain how to compute all the second order partial derivatives of G_ε. Fix $\xi \in \mathbb{C} = \mathbb{R}^2$ a unit vector and denote by $\partial_\xi f(z) := \lim_{h \to 0^+}(f(z+h\xi) - f(z))/h$ the derivative of

f in the direction ξ. Note that

$$\frac{1}{h}\left[\chi_\varepsilon(|z+h\xi|^2)-\chi_\varepsilon(|z|^2)\right]\to\partial_\xi(\chi_\varepsilon(|z|^2))$$

as $h\to 0$ uniformly in z. It follows that

$$\frac{1}{h}\left[\Phi_\varepsilon(x,z+h\xi)-\Phi_\varepsilon(x,z)\right]\to\partial_\xi\Phi_\varepsilon(z)$$

as $h\to 0$ uniformly in x,z. By the same arguments as in Subsection 3.2 the first and second partial derivatives of $z\mapsto\Phi_\varepsilon(x,z)$ can be written as $C_\varepsilon(u^+-u^-)$ where $u^\pm$ are θ-psh functions with minimal singularities. They are thus uniformly bounded by a constant C'_ε (which blows up as $\varepsilon\to 0$). Thus the same arguments as in Subsection 3.2 show that $z\mapsto G_\varepsilon(z)$ is twice differentiable (even smooth). Set $g_\varepsilon=\gamma_\varepsilon\log(1+|z|^2)$, where $g=\log(1+|z|^2)$ is the potential of ω_{FS} in $\mathbb{C}$. The second derivative $\partial_z\partial_{\bar z}$ of $G_\varepsilon+g_\varepsilon$ is given by

$$\partial_z\partial_{\bar z}(G_\varepsilon+g_\varepsilon)=\int_X\partial_z\partial_{\bar z}(\Phi_\varepsilon+g_\varepsilon)(\theta+dd^c_x\Phi_\varepsilon)^n-nd_x\partial_z(\Phi_\varepsilon+g_\varepsilon)\wedge d^c_x\partial_{\bar z}(\Phi_\varepsilon+g_\varepsilon)\wedge(\theta+dd^c_x\Phi_\varepsilon)^{n-1}.$$

Let $H(z)$ denote the integrand in the right-hand side which is a positive measure on X. Indeed, since it is a local property we can argue locally and use an approximation technique as in the proof of Corollary 3.7. Moreover, we infer that $\frac{\sqrt{-1}}{\pi}H(z)dz\wedge d\bar z$ is the Monge-Ampère measure of $\Phi_\varepsilon+g_\varepsilon$ with respect to the form $\pi_1^*\theta$, i.e. $(\pi_1^*\theta+dd^c(\Phi_\varepsilon+g_\varepsilon))^{n+1}$. This together with

$$\gamma_\varepsilon\omega_{FS}+dd^c_zG_\varepsilon(z)=dd^c_z(g_\varepsilon+G_\varepsilon)=dd^c_z\mathrm{I}(\Phi_\varepsilon(\cdot,z)+g_\varepsilon(z))$$

justify the formula (16).

Fix $\chi:\mathbb{C}\to\mathbb{R}$ a smooth function with compact support in $\mathbb{C}$. We want to prove that

$$\int_{\mathbb{C}}\chi(\omega_{FS}+dd^cG)=\int_M\chi(\Theta+dd^c\Phi)^{n+1}.$$

The above formula is true for the approximants G_ε and Θ_ε as we discussed above. Now we explain how we can insure the convergence when we take the limit in (16) as $\varepsilon\to 0$. To deal with the left-hand side we prove that G_ε converge pointwise to G. For fixed $z\in\mathbb{C}$ we can find constants c_ε converging to 0 such that $\Phi_\varepsilon+c_\varepsilon$ decreases to Φ. Then $G_\varepsilon(z)$ converges to $G(z)$ by basic properties of the I functional [BB10, Proposition 4.3]. As G_ε is uniformly bounded independent of ε, the convergence of the current dd^cG_ε follows. Now we treat the convergence of the right-hand side of (16). Fix a Kähler form ω_M on M and $\delta>0$. Let U be an open neighbourhood of the pluripolar set $E=(X\setminus\mathrm{Amp}(\{\theta\}))\times H_\infty$ such that $\mathrm{Cap}_{\omega_M}(U)<\delta$. Note that Theorem 2.8 gives that

$$\mathrm{Cap}_{\Theta_\varepsilon}(U)\leqslant C\mathrm{Cap}_{\omega_M}(U)^{1/n}=O(\delta^{1/n}),$$

where C is independent of ε. And, by Bedford-Taylor theory [BT82] the Monge-Ampère measure $(\Theta_\varepsilon+dd^c\Phi_\varepsilon)^{n+1}$ converges to $(\Theta+dd^c\Phi)^{n+1}$ in $M\setminus U$. Letting $\varepsilon\to 0$ and then $\delta\to 0$ we arrive at the conclusion. □

We move forward to proving the linearity of the I functional. Consider (M,Θ) as above and let $\rho:D\to\mathbb{R}$ be a smooth potential of ω_{FS} in $D\subset\mathbb{CP}^1$ with zero boundary values.

Lemma 3.11. *Assume that F is a smooth function on M which is S^1-invariant in $X\times\mathbb{C}$ in the variable z. Let φ_t be the geodesic connecting $\varphi_0=P_\theta(F(\cdot,1))$ to $\varphi_1=P_\theta(F(\cdot,e))$. Let Φ be the envelope on M of F with respect to Θ. If $F(x,z)+\rho(z)>\varphi_{\log|z|}(x)$ in $X\times D$ then $z\mapsto\mathrm{I}(\Phi(\cdot,z)+\rho(z))$ is harmonic in D.*

Proof. When restricted to $X \times D$, we have $\Phi + \rho$ is $\pi_1^*\theta$-psh and has boundary values $\varphi_{0,1}$, thus by definition, $\Phi \leqslant \varphi_t - \rho < F$. It follows from [BT82] that $(\Theta + dd^c\Phi)^{n+1} = 0$ in $X \times D$, which in turn implies that $\mathrm{I}(\Phi(\cdot, z) + \rho(z))$ is harmonic in D thanks to Lemma 3.10. □

Theorem 3.12. *The* I *energy is linear along weak geodesics with minimal singularities.*

Proof. Fix f_0, f_1 two smooth functions in X and denote by $\varphi_i = P_\theta(f_i), i = 0, 1$ the envelopes of f_0, f_1 respectively. Let φ_t be the geodesic connecting φ_0, φ_1. Observe also that by approximating any two given potentials with minimal singularities with a sequence of smooth functions, it suffices to prove linearity of I along φ_t.

Let F be a function on M which is S^1-invariant in $X \times \mathbb{C}$ in the variable z and such that $F = f_{0,1}$ on $X \times \partial D$ and $+\infty$ elsewhere. Consider a sequence $(F_j)_j$ of smooth functions $F_j \uparrow F$, which are also S^1-invariant in $X \times \mathbb{C}$ in the variable z and such that $F_j + \rho > \varphi_{\log|z|}(x)$ in $X \times D$. Let Φ_j be the envelope on M of F_j with respect to Θ. Then $(\Theta + dd^c\Phi_j)^{n+1}$ is supported on $\{\Phi_j = F_j\}$. As $X \times \partial D$ is non-pluripolar in M it follows that Φ_j is uniformly bounded from above. Thus Φ_j converges increasingly (almost everywhere) to Φ a Θ-psh function with minimal singularities. We claim that

$$\int_{\{\Phi < P_\Theta(F)\}} (\Theta + dd^c\Phi)^{n+1} = 0. \tag{17}$$

Too see this we observe that the Monge-Ampère measure of Φ is concentrated on $X \times \partial D$ and that$\{\Phi < P_\Theta(F)\} \subset X \times (\mathbb{CP}^1 \setminus \partial D)$ since $\Phi = P_\Theta(F) = \varphi_{0,1}$ on $X \times \partial D$. Indeed, on any open relatively compact subset K of $X \times (\mathbb{CP}^1 \setminus \partial D)$ one has that $\Phi_j < F_j$ for j large enough (since F_j increases to $+\infty$ uniformly in K and Φ_j is uniformly bounded from above). By the continuity property of the complex Monge-Ampère operator we get that $(\Theta + dd^c\Phi)^{n+1}(K) = 0$. It follows from (17) and the domination principle [BEGZ10, Corollary 2.5] that $\Phi = P_\Theta(F)$.

Now, we claim that $\Phi + \rho = \varphi_{\log|z|}$ in $X \times D$. Indeed, consider

$$\begin{cases} U_0(x,z) := \varphi_0(x) + A(\log(|z|^2 + 3) - \log(|z|^2 + 1) - \log 2); \\ U_1(x,z) := \varphi_1(x) + A(\log|z|^2 - \log(|z|^2 + 1) + \log(e^2 + 1) - 2). \end{cases}$$

For $A > 0$ big enough $U := \max(U_0, U_1) = \varphi_{0,1}$ on ∂D and it is Θ_A-psh, where $\theta_A := \pi_1^*\theta + A\pi_2^*\omega_{FS}$. So we can apply our previous analysis with this $(1,1)$-form Θ_A instead of Θ. By definition of envelope we have $\Phi \geqslant U$ and in particular $\Phi \geqslant \varphi_{0,1}$ on $X \times \partial D$. Moreover, for each $z \in \partial D$, $\Phi(\cdot, z)$ is θ-psh and dominated by $F = f_{0,1}$. It then follows that $\Phi \leqslant \varphi_{0,1}$ on $X \times \partial D$, giving the equality on the boundary. Furthermore, it follows from the proof of Lemma 3.11 and the continuity of the Monge-Ampère operator that the Monge-Ampère measure $(\pi_1^*\theta + dd^c\rho + dd^c\Phi)^{n+1}$ vanishes in $X \times D$. Proposition 3.3 thus insures that $\Phi + \rho$ is the unique weak geodesic with minimal singularities joining φ_0, φ_1. Hence the claim.

Now, thanks to Lemma 3.11 and [BB10, Proposition 4.3], we know that $\mathrm{I}(\Phi(\cdot, z) + \rho(z)) = \mathrm{I}(\varphi_{\log|z|})$ is harmonic in D (and S^1-invariant!). Hence, I is linear along φ_t, with $t = \log|z|$. This is what we wanted. □

3.4 Geodesic rays and the proof of Theorem 1.2

Given $\varphi \geqslant \psi$ two θ-psh functions such that φ has minimal singularities, we define the weak geodesic ray attached to φ, ψ in the following way (see [Dar13] for the Kähler case). For fixed $l > 0$, we denote by $[0, l] \ni t \to u_t^l \in \mathrm{PSH}(X, \theta)$ the weak geodesic segment joining φ and $\max(\varphi - l, \psi)$. The same argument as in [Dar13, Lemma 4.2] shows that u^l forms an increasing

family of weak geodesics and we can then define the limit subgeodesic ray:

$$v(\varphi,\psi)_t := \left(\lim_{l\to+\infty} u^l_t\right)^*,\ t\in[0,+\infty). \tag{18}$$

Lemma 3.13. *Assume that θ is big. The subgeodesic ray $[0,\infty)\ni t\to v(\varphi,\psi)_t\in \mathrm{PSH}(X,\theta)$ is a weak geodesic ray.*

Proof. As all complexifications $U^l\in\mathrm{PSH}(X\times D_l,\pi^*\theta)$ satisfy the appropriate complex Monge–Ampère equation on the domains $\mathrm{Amp}(\{\theta\})\times D_l$ and are locally bounded there, it follows from continuity property of the complex Monge–Ampère operator that the complexification of $t\to v(\varphi,\psi)_t$ satisfies the homogeneous Monge-Ampère equation on $\mathrm{Amp}(\{\theta\})\times D_\infty$ as well. Since all the curves $t\to u^l_t$ are uniformly t-Lipschitz continuous, so is their limit $t\to v(\varphi,\psi)_t$, hence Proposition 3.3 gives that $t\to v(\varphi,\psi)_t$ must be a weak geodesic ray, i.e., for any closed interval $I\subset[0,\infty)$, the restriction $I\ni t\to v(\varphi,\psi)_t\in\mathrm{PSH}(X,\theta)$ is the weak geodesic joining the potentials corresponding to the endpoints of I. □

Now we introduce an invariant of ψ. It is clear that $[0,\infty)\ni t\to\max(\varphi-t,\psi)$ is a subgeodesic ray with minimal singularities. Thus $t\to \mathrm{I}(\max(\varphi-t,\psi))$ is convex by Theorem 3.8 and decreasing by [BEGZ10, Proposition 2.8]. This implies that the following limit is well defined:

$$c_\psi := \lim_{t\to\infty}\frac{\mathrm{I}(\max(\varphi-t,\psi))-\mathrm{I}(\varphi)}{t}\leqslant 0.$$

Recall the cocycle formula of [BB10, Corollary 3.2]:

$$\mathrm{I}(u)-\mathrm{I}(v)=\frac{1}{\mathrm{Vol}(\theta)(n+1)}\sum_{j=0}^n\int_X(u-v)\theta_u^j\wedge\theta_v^{n-j}, \tag{19}$$

where $u,v\in\mathrm{PSH}(X,\theta)$ have minimal singularities. From this formula it follows that c_ψ is independent of the choice of potential with minimal singularities φ satisfying $\varphi\geqslant\psi$. Finally, the following result gives an attractive characterization of potentials ψ for which $c_\psi=0$.

Proposition 3.14. *Given $\psi\in\mathrm{PSH}(X,\theta)$, we have $c_\psi=0$ if and only if $\psi\in\mathcal{E}(X,\theta)$.*

Proof. The proof is an adaptation of the arguments of [Dar13, Theorem 2.5] to our more general setting. As c_ψ only depends on ψ it is enough to work with the special subgeodesic ray $t\to\psi_t:=\max(V_\theta-t,\psi)$. The cocycle formula 19 implies that $c_\psi=c_{\psi+c}$, thus we can assume that $\psi\leqslant V_\theta\leqslant 0$. By [BBGZ13, Proposition 2.8]

$$\int_X(\psi_t-V_\theta)\,\theta^n_{\psi_t}\leqslant \mathrm{I}(\psi_t)\leqslant(n+1)^{-1}\int_X(\psi_t-V_\theta)\,\theta^n_{\psi_t}$$

so our claim is equivalent to showing that

$$\psi\in\mathcal{E}(X,\theta)\Longleftrightarrow t^{-1}\int_X(\psi_t-V_\theta)\theta^n_{\psi_t}\to 0.$$

Fix $s\in(0,1)$. Note that on $\{\psi>V_\theta-t\}$, the two measures $\theta^n_{\psi_t}$ and θ^n_ψ coincide. Additionally, using that $X=\{\psi\leqslant V_\theta-t\}\cup\{V_\theta-t<\psi\leqslant V_\theta-st\}\cup\{V_\theta-st<\psi\}$ we can estimate the right-hand side above as follows:

$$0\geqslant\int_X\frac{\psi_t-V_\theta}{t}\theta^n_{\psi_t}\geqslant-\int_{\{\psi\leqslant V_\theta-t\}}\theta^n_{\psi_t}-\int_{\{\psi\leqslant V_\theta-ts\}}\theta^n_\psi-s\mathrm{Vol}(\theta).$$

By [GZ07, Lemma 1.2], $\psi\in\mathcal{E}(X,\theta)$ if and only if $\int_{\{\psi\leqslant V_\theta-t\}}\theta^n_{\psi_t}\to 0$ as $t\to+\infty$. Hence, the above estimates give the conclusion. □

Lemma 3.15. *Suppose $\psi \in \mathrm{PSH}(X,\theta)$ satisfies $\psi \leqslant V_\theta$. Let $t \to v(V_\theta,\psi)$ be the geodesic ray constructed in* (18). *Then*

$$\mathrm{I}(v(V_\theta,\psi)_t) = tc_\psi. \tag{20}$$

In particular, $t \to v(V_\theta,\psi)_t$ is constant if and only if $\psi \in \mathcal{E}(X,\theta)$.

Proof. We go back to the construction of $t \to v(V_\theta,\psi)_t$ in (18). By Theorem 3.12, for each l fixed, the function $t \to \mathrm{I}(u_t^l)$ is linear hence we can write:

$$\mathrm{I}(u_t^l) = \mathrm{I}(V_\theta) + \frac{t}{l}(\mathrm{I}(\max(\psi, V_\theta - l)) - \mathrm{I}(V_\theta)) = \frac{t}{l}(\mathrm{I}(\max(\psi, V_\theta - l)) - \mathrm{I}(V_\theta)).$$

Letting $l \to +\infty$, by [BB10, Proposition 3.3] we obtain (20). If $t \to v(V_\theta,\psi)_t$ is constant equal to V_θ, it follows that $c_\psi = 0$, thus by Proposition 3.14 we get $\psi \in \mathcal{E}(X,\theta)$. Conversely, if $\psi \in \mathcal{E}(X,\theta)$ then Proposition 3.14 yields that $c_\psi = 0$, hence I is constant along $v(V_\theta,\psi)$, thus $v(V_\theta,\psi)$ is constant. □

Remark 3.16. To show that if $t \to v(V_\theta,\psi)_t$ is constant then $\psi \in \mathcal{E}(X,\theta)$, we only need to prove the estimate $\mathrm{I}(v(V_\theta,\psi)_t) \leqslant tc_\psi$. This last inequality is a simple consequence of the convexity of the Monge–Ampère energy (Theorem 3.8) and the construction of the ray $t \to v(V_\theta,\psi)_t$ (18).

Lemma 3.17. *Given a weak geodesic ray $[0,+\infty) \ni t \to \phi_t \in \mathrm{PSH}(X,\theta)$, its Legendre transform $\mathbb{R} \ni \tau \to \phi_\tau^* = \inf_{t\in[0,+\infty)}(\phi_t - t\tau)$ satisfies*

$$\phi_\tau^* = P_\theta(\phi_\tau^* + C, \phi_0),\ \tau \in \mathbb{R}, C > 0.$$

In particular, $P_{[\theta,\phi_\tau^*]}(\phi_0) = \phi_\tau^*$.

Proof. One can repeat the argument in [Dar13, Theorem 5.3]. Fix $\tau \in \mathbb{R}$. The fact that ϕ_τ^* is θ-psh follows from Kiselman's minimum principle [Kis78]. Suppose that $\phi_\tau^* \neq -\infty$ and fix $C > 0$. Since $\phi_\tau^* \leqslant \phi_0$, it results that $P_\theta(\phi_\tau^* + C, \phi_0) \geqslant \phi_\tau^*$. Hence we only have to argue that:

$$P_\theta(\phi_\tau^* + C, \phi_0) \leqslant \phi_\tau^*.$$

Let $[0,1] \ni t \to g_t^l, h_t \in \mathrm{PSH}(X,\theta),\ l \geqslant 0$, be the weak geodesic segments defined by the formulas:

$$g_t^l = \phi_{tl} - tl\tau,$$

$$h_t = P_\theta(\phi_\tau^* + C, \phi_0) - Ct.$$

Then we have $h_0 \leqslant \phi_0 = \lim_{t\to 0} g_t^l = g_0^l$ and $h_1 \leqslant \phi_\tau^* \leqslant g_1^l$ for any $l \geqslant 0$. Hence, by definition of weak geodesics (8) we have

$$h_t \leqslant g_t^l,\ t \in [0,1], l \geqslant 0.$$

Taking the infimum in the above estimate over $l \in [0,+\infty)$ and then taking the supremum over $t \in [0,1]$, we obtain:

$$P_\theta(\phi_\tau^* + C, \phi_0) \leqslant \phi_\tau^*.$$

Letting $C \to +\infty$ we obtain the last statement of the proposition. □

Proof of Theorem 1.2

Theorem 1.2 is a consequence of the following result.

Theorem 3.18. *Let θ be a smooth closed $(1,1)$-form whose cohomology class is big. For any $\psi \in \mathrm{PSH}(X,\theta)$, the following are equivalent*
(i) $\psi \in \mathcal{E}(X,\theta)$;

(ii) $P_{[\theta,\psi]}(\varphi) \in \mathcal{E}(X,\theta)$ *for all* $\varphi \in \mathcal{E}(X,\theta)$*;*
(iii) $P_{[\theta,\psi]}(\varphi) = \varphi$ *for all* $\varphi \in \mathcal{E}(X,\theta)$*;*
(iv) $P_{[\theta,\psi]}(V_\theta) \in \mathcal{E}(X,\theta)$*;*
(v) $P_{[\theta,\psi]}(V_\theta) = V_\theta$*.*

Proof. We can assume without loss of generality that $\psi \leqslant 0$. The implication $(i) \Longrightarrow (iii)$ follows from Theorem 2.15 while the implications $(iii) \Longrightarrow (ii) \Longrightarrow (iv)$ and $(iii) \Longrightarrow (v)$ are trivial. The implication $(iv) \Longrightarrow (v)$ simply follows from Proposition 2.14 and the domination principle (Proposition 2.4).

We now prove that $(v) \Longrightarrow (i)$. Suppose that $P_{[\theta,\psi]}(V_\theta) = V_\theta$. From the construction of the ray $t \to v(V_\theta,\psi)_t$ in (18) it automatically follows that $v(V_\theta,\psi)_t \geqslant \psi$. This trivially gives $v_0^* = \inf_{t\in[0,\infty)} v(V_\theta,\psi)_t \geqslant \psi$. By definition of envelope we have $V_\theta \geqslant P_{[\theta,v_0^*]}(V_\theta) \geqslant P_{[\theta,\psi]}(V_\theta) = V_\theta$. Combining this with Lemma 3.17 we obtain that $v_0^* = P_{[\theta,v_0^*]}(V_\theta) = V_\theta$. As the ray $t \to v(V_\theta,\psi)_t$ is decreasing in t, this automatically gives that $V_\theta = v(V_\theta,\psi)_0 \geqslant v(V_\theta,\psi)_t \geqslant v_0^* = V_\theta$, hence $t \to v(V_\theta,\psi)_t$ is constant. Invoking Lemma 3.15 we obtain that $\psi \in \mathcal{E}(X,\theta)$. □

Theorem 1.3 follows directly from Theorem 1.2.

Theorem 3.19. *Let* $\{\theta_1\},\{\theta_2\}$ *be big classes. The following are equivalent:*
(i) $V_{\theta_1} + V_{\theta_2} \in \mathcal{E}(X,\theta_1+\theta_2)$*;*
(ii) For any $u \in \mathrm{PSH}(X,\theta_1), v \in \mathrm{PSH}(X,\theta_2)$ *we have*

$$u+v \in \mathcal{E}(X,\theta_1+\theta_2) \Longleftrightarrow u \in \mathcal{E}(X,\theta_1), v \in \mathcal{E}(X,\theta_2).$$

Proof. Since $V_{\theta_j} \in \mathcal{E}(X,\theta_j), j=1,2$, the implication $(ii) \Longrightarrow (i)$ is trivial. Assume (i) holds. The implication $(\Longrightarrow)$ in (ii) follows from [DN15, Theorem B]. Assume that $\varphi_j \in \mathcal{E}(X,\theta_j), j=1,2$. We want to show that $\varphi := \varphi_1+\varphi_2 \in \mathcal{E}(X,\theta_1+\theta_2)$. By assumption that (i) holds we get that $V_{\theta_1}+V_{\theta_2} \in \mathcal{E}(X,\theta_1+\theta_2)$. Hence, by definition of envelopes we can write

$$P_{[\theta_1+\theta_2,\varphi]}(V_{\theta_1}+V_{\theta_2}) \geqslant P_{[\theta_1,u]}(V_{\theta_1}) + P_{[\theta_2,v]}(V_{\theta_2}) = V_{\theta_1}+V_{\theta_2}.$$

Ultimately, it follows from Theorem 1.2 that $\varphi \in \mathcal{E}(X,\theta_1+\theta_2)$. □

4. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 and discuss some immediate consequences.

Proof of Theorem 1.1. We first argue the equality of Lelong numbers in (i). If $\varphi \in \mathcal{E}(X,\theta)$ then, as follows from Theorem 2.15, $P_{[\theta,\varphi]}(V_\theta) = V_\theta$. For any fixed $x \in X$ one trivially has $\nu(\varphi,x) \geqslant \nu(V_\theta,x)$. Assume, for the sake of seeking a contradiction, that $\nu(\varphi,x) > \nu(V_\theta,x)$. Fix a holomorphic coordinate around x so that we identify x with $0 \in \mathbb{B} \subset \mathbb{C}^n$ where B is the unit ball in $\mathbb{C}^n$. By definition of the Lelong numbers (2) we have

$$\varphi(z) \leqslant \gamma \log\|z\| + O(1),$$

where $\gamma = \nu(\varphi,x) > 0$. Let g be a smooth local potential for θ in $\mathbb{B}$ and observe that if $\psi \in \mathrm{PSH}(X,\theta)$ then $g+\psi$ is psh in $\mathbb{B}$. Furtheremore, w.l.o.g. we can assume that $g+\varphi, g+V_\theta \leqslant 0$ in $\mathbb{B}$. By the definition of the envelope we have the following inequality

$$V_\theta + g = P_{[\theta,\varphi]}(V_\theta) + g \leqslant \sup\{v \in \mathrm{PSH}(\mathbb{B}) \mid v \leqslant 0\,,\ v \leqslant \gamma\log\|z\| + O(1)\},$$

in $\mathbb{B}$. The right-hand side is the pluricomplex Green function $G_{\mathbb{B}}(z,0)$ of $\mathbb{B}$ with a logarithmic pole at 0 of order γ. By [Kli91, Proposition 6.1] we have that

$$G_{\mathbb{B}}(z,0) = \gamma \log \|z\| + O(1). \tag{21}$$

But this contradicts with the assumption that $\nu(V_\theta, x) < \gamma$.

Now we argue the equality of multiplier ideal sheafs in (i). This will be an application of Theorem 1.2 and the resolution of the strong openess conjecture of Guan and Zhou [GuZh], in the specific form provided by Lempert [Lemp]. Indeed, from Theorem 1.2 it follows that

$$P_\theta(\varphi + c, V_\theta)(x) \nearrow V_\theta(x) \text{ as } c \to \infty, \text{ for a.e. } x \in X. \tag{22}$$

As $\varphi \leqslant V_\theta + c'$ for some $c' \in \mathbb{R}$, we note that φ and $P_\theta(\varphi + c, V_\theta)$ have the same singularity type for any $c \in \mathbb{R}$, ultimately giving $\mathcal{I}(tP_\theta(\varphi + c, V_\theta), x) = \mathcal{I}(t\varphi, x),\ x \in X, t \geqslant 0$.

Finally, (22) and [Lemp, Theorem 1.1] imply that $\mathcal{I}(tP_\theta(\varphi + c, V_\theta), x) = \mathcal{I}(tV_\theta, x)$ for large enough c, proving that $\mathcal{I}(t\varphi, x) = \mathcal{I}(tV_\theta, x)$.

Now we turn to part (ii). Fix ω a Kähler form on X. We can suppose that $\theta, \eta \leqslant \tilde{\omega} := \eta + \omega$ and $\tilde{\omega}$ is Kähler. Assume that $\varphi \in \mathcal{E}(X, \eta) \cap \mathrm{PSH}(X, \theta)$. By Theorem 1.2 we get that $P_{[\eta,\varphi]}(V_\eta) = V_\eta$. This implies $P_{[\tilde{\omega},\varphi]}(V_\eta) = V_\eta$ since $P_{[\eta,\varphi]}(V_\eta) \leqslant P_{[\tilde{\omega},\varphi]}(V_\eta) \leqslant V_\eta$.

Furthermore, we claim that $V_\eta \in \mathcal{E}(X, \tilde{\omega})$, i.e., $\int_X \tilde{\omega}^n_{V_\eta} = \mathrm{Vol}(\tilde{\omega})$. Indeed, as θ is nef, expanding the sum of Kähler classes $(\eta + (1+\varepsilon)\omega)^n$ gives

$$\mathrm{Vol}(\{\eta + (1+\varepsilon)\omega\})^n = \sum_{k=0}^{n} \binom{n}{k} \{\eta + \varepsilon\omega\}^k \cdot \{\omega\}^{n-k}.$$

It follows from the comments after [BEGZ10, Definition 1.17] and [BFJ09, Proposition 2.9] that the left-hand side converges to $\mathrm{Vol}(\tilde{\omega})$ while the right-hand side converges to $\sum_{k=0}^{n} \binom{n}{k} \{\eta\}^k \cdot \{\omega\}^{n-k}$, ultimately giving

$$\mathrm{Vol}(\{\tilde{\omega}\}) = \mathrm{Vol}(\{\eta + \omega\}) = \sum_{k=0}^{n} \binom{n}{k} \{\eta\}^k \cdot \{\omega\}^{n-k}.$$

On the other hand, by multilinearity of the non-pluripolar product we get

$$\int_X \tilde{\omega}^n_{V_\eta} = \int_X (\eta + \omega + dd^c V_\eta)^n = \sum_{k=0}^{n} \binom{n}{k} \int_X (\eta + dd^c V_\eta)^k \wedge \omega^{n-k},$$

and moreover $\{(\eta + dd^c V_\eta)^k\} = \{\eta\}^k$ for each $0 \leqslant k \leqslant n-1$ thanks to [BEGZ10, Definition 1.17], proving the claim.

Given that $P_{[\tilde{\omega},\varphi]}(V_\eta) = V_\eta$ and $V_\eta \in \mathcal{E}(X, \tilde{\omega})$, we can use [Dar14, Theorem 4] to conclude that $\varphi \in \mathcal{E}(X, \tilde{\omega})$. Because $\theta \leqslant \tilde{\omega}$ and $\varphi \in \mathrm{PSH}(X, \theta)$, we get $\varphi \in \mathcal{E}(X, \theta)$, as follows from [DN15, Theorem B]. □

Remark 4.1. Observe that Theorem 1.1 (ii) is in general false for classes $\{\eta\}$ that are big but not nef. Indeed, if $\{\eta\}$ is only big, it may happen that V_η has a non-zero Lelong number at some point, and then [GZ07, Corollary 2.18] would give us that V_η does not have full mass with respect to any Kähler class $\{\theta\}$ satisfying $\eta \leqslant \theta$, contradicting $\mathcal{E}(X, \eta) \cap \mathrm{PSH}(X, \theta) \subset \mathcal{E}(X, \theta)$.

As a direct consequence we obtain the following additivity property of the set of full mass currents of big and nef cohomology classes, effectively proving that condition (i) in Theorem 1.3 is automatically satisfied.

Corollary 4.2. *Let $\{\theta_1\},\{\theta_2\}$ be big and nef classes. Then for any $\varphi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X,\theta_2)$ we have*

$$\varphi_1+\varphi_2 \in \mathcal{E}(X,\theta_1+\theta_2) \Longleftrightarrow \varphi_1 \in \mathcal{E}(X,\theta_1), \varphi_2 \in \mathcal{E}(X,\theta_2).$$

Proof. Fix a Kähler form ω such that $\theta_j \leqslant \omega$, $j=1,2$. It follows from part (ii) of Theorem 1.1 that $\varphi_j \in \mathcal{E}(X,\omega), \forall j=1,2$. By the convexity of $\mathcal{E}(X,\omega)$ proved in [GZ07, Proposition 1.6] it follows that $\varphi_1+\varphi_2$ belongs to $\mathcal{E}(X,2\omega)$. Now, [DN15, Theorem B] gives that $\varphi_1+\varphi_2 \in \mathcal{E}(X,\theta_1+\theta_2)$, hence the result follows. □

5. Further Applications

5.1 Invariance of finite energy classes

The following result says that finite energy classes are invariant under bimeromorphic maps as soon as the volume is preserved. The result was recently obtained in [DNFT]. As an application of Theorem 1.1 we give a slightly different proof of the "baby case", i.e. when f is a blow-up along a smooth center.

Proposition 5.1. *Let $\pi : X \to Y$ be a blow up with smooth center $\mathcal{Z}$ between Kähler manifolds and E be the exceptional divisor. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big class. Then the following conditions are equivalent:*

(i) $\mathrm{Vol}(\alpha)=\mathrm{Vol}(\pi_\star\alpha)$;

(ii) given a positive $(1,1)$-current T in $\pi_\star\alpha$, then $S=\pi^\star T+\gamma[E]$, where γ is a cohomological factor, is a positive $(1,1)$-current on X.

(iii) $\pi_\star(\mathcal{E}(X,\theta))=\mathcal{E}(Y,f_\star\theta)$,

(iv) $\pi_\star(\mathcal{E}_\chi(X,\theta))=\mathcal{E}_\chi(Y,f_\star\theta)$ *for any weight χ.*

Proof. Recall that, given a smooth representative θ of the class α, it follows from $\partial\bar\partial$-lemma that any positive $(1,1)$-current can be written as $T=\theta+dd^c\varphi$ where the global potential φ is a θ-psh function, i.e. $\theta+dd^c\varphi \geqslant 0$. The implications $(iii) \Rightarrow (i)$ and $(iv) \Rightarrow (i)$ are trivial while the fact that (ii) implies (i),(iii) and (iv) are [DN15, Proposition 3.3]. We want to prove $(i) \Rightarrow (ii)$. it suffices to show that for any positive $(1,1)$-current T we have $\gamma \geqslant -\nu(T,\mathcal{Z})$. Let $S_{\min}$ be a positive current with minimal singularities in α, then it writes as

$$S_{\min}=\pi^\star T_Y+\gamma[E]$$

where $\gamma \geqslant -\nu(T_Y,\mathcal{Z})$. It easy to check that $T_Y \in \mathcal{E}(Y,\pi_\star\alpha)$. Indeed,

$$\mathrm{Vol}(\pi_\star\alpha)=\mathrm{Vol}(\alpha)=\int_X \langle S_{\min}^n\rangle = \int_Y \langle T_Y^n\rangle.$$

Thus, it follows from Theorem 1.1 that $\nu(T_Y,y)=\nu(T_{\min},y)$ for any $y \in Y$ and for any $T_{\min}$ current with minimal singularities on Y. Hence $\nu(T_{\min},\mathcal{Z})+\gamma \geqslant 0$. Furthermore any positive current $T \in \pi_\star\alpha$ is such that $\nu(T_{\min},\mathcal{Z}) \leqslant \nu(T,\mathcal{Z})$, thus $\nu(T,\mathcal{Z})+\gamma \geqslant 0$. □

5.2 Log concavity of non-pluripolar product

It was conjectured in [BEGZ10, Conjecture 1.23] that

$$\int_X \langle T_1\wedge ...\wedge T_n\rangle \geqslant \left(\int_X \langle T_1^n\rangle\right)^{1/n} ... \left(\int_X \langle T_n^n\rangle\right)^{1/n}, \tag{23}$$

for all positive currents $T_1, ..., T_n$. The result holds for currents with analytic singularities as mentioned in [BEGZ10]. In this subsection we confirm this conjecture in the case of full mass currents in big and nef classes.

With the help of Corollary 4.2, we can make obvious adjustments in the proof of [BEGZ10, Corollary 2.15] to get the following result:

Proposition 5.2. *Let α_i, $i = 1, \cdots, n$ be big and nef cohomology classes and let $T_i \in \mathcal{E}(X, \alpha_i)$. Then we have*

$$\int_X \langle T_1 \wedge \cdots \wedge T_n \rangle = \int_X \langle T_{1,\min} \wedge \cdots \wedge T_{n,min} \rangle.$$

Using this we can prove the log concavity of full mass currents in the big and nef case:

Corollary 5.3. *If $T_j, j = 1, \cdots n$, are full mass currents in big and nef cohomology classes then* (23) *holds.*

Proof. Let μ denote the non-pluripolar measure $\mu := \langle T_1 \wedge ... \wedge T_n \rangle$ and let $\lambda_j, j = 1, \cdots, n$ be positive constants such that

$$\lambda_j \mu(X) = \int_X \langle T_j^n \rangle. \tag{24}$$

For each j, using [BEGZ10, Theorem A] there exists a positive full mass current $S_j \in \{T_j\}$ such that

$$\langle S_j^n \rangle = \lambda_j \mu.$$

By [BEGZ10, Proposition 1.11] we have that

$$\langle S_1 \wedge ... \wedge S_n \rangle \geqslant (\lambda_1 \cdots \lambda_n)^{1/n} \mu.$$

Proposition 5.2 gives that $\int_X \langle S_1 \wedge ... \wedge S_n \rangle = \int_X \langle T_1 \wedge ... \wedge T_n \rangle$. Hence after integrating the above inequality, due to (24), the result follows. □

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Tamás Darvas tdarvas@math.umd.edu
University of Maryland

Eleonora Di Nezza e.di-nezza@imperial.ac.uk
Imperial College London

Chinh H. Lu chinh.lu@sns.it, hoang-chinh.lu@u-psud.fr
Scuola Normale Superiore, Pisa, Italy, Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France