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Perturbation Dynamics of Turbulence

by

Wee Zhao, Chua Khoo

Department of Aeronautics, Imperial College London

*Submitted in partial fulfilment of the degree of Masters of Philosophy
in the Department of Aeronautics, Imperial College London*

Abstract

Turbulence encountered in engineering applications is commonly found with various forms of external disturbances and perturbations that affect the dynamics and statistics of turbulence. Their understanding provides valuable physical insights into controlling turbulence in a robust way. This thesis investigates the Navier-Stokes equations via a statistical state dynamics framework. Via the statistical state dynamics framework, the chaotic aspects of turbulence can be regularised, leading to a more quantitative behaviour for understanding turbulence. Central to this is the closure problem that arises when an averaging procedure is applied to the Navier-Stokes equations which is non-linear in nature. Although initially, the aim was to quantify the aspects of the Reynolds stress terms in terms of a linear operator, the methodology is deemed to be too computationally expensive. Following on, a transient growth analysis is done on an ensemble of realisation of the Navier-Stokes equations, effectively putting it in contrast with a linearised Navier-Stokes equations operator augmented with the Cess eddy viscosity model.

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Statement of Originality

This is to certify that this copy of my thesis embodies the results of my own course of study and research, and other aspects have been appropriately referenced.

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Chapter 1

Introduction

The incompressible Navier-Stokes equations are derived based on the conservation laws of mass and momentum that dictates the flow evolution of a Newtonian fluid. By adopting a three-dimensional Cartesian coordinates in space, denoted by $\mathbf{x} = [x, y, z]^T$, with $(\bullet)^T$ representing the vector / matrix transpose, with x denoting the stream-wise direction, y , the wall-normal direction, and z , the span-wise direction. Time is denoted by t . The evolution equations for the three-dimensional velocity vector field, $\mathbf{U}(\mathbf{x}, t) = [U, V, W]^T(\mathbf{x}, t)$, being the fluid velocity in the x , y , and z direction. The scalar pressure field is denoted by, $P(\mathbf{x}, t)$, with an external three-dimensional forcing input, $\mathbf{F}(\mathbf{x}, t)$, are written in non-dimensional form as

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} + \mathbf{F}, \quad (1.1a)$$

$$\nabla \cdot \mathbf{U} = 0. \quad (1.1b)$$

The no-slip boundary condition are enforced at the solid surface as the boundary condition, ensuring that $\mathbf{U}(\mathbf{x}, t) = \mathbf{0}$ at a solid boundary. The non-dimensional Reynolds number, Re , is defined as

$$Re = \frac{UL}{\nu},$$

where U and L are a relevant velocity and length-scale, respectively, with ν as the kinematic viscosity.

1.1 Wall-Bounded Shear Flows

The dynamics of the Navier-Stokes equations are rich and highly dependent on the boundary conditions that are involved in the set-up. In this literature review, the focus centres upon shear flows driven by a pressure gradient or wall movement while interacting with a wall boundary. Implicitly, the existence of the wall leads to the no-slip boundary condition being applied at the wall boundary. In other words, the fluid velocity at the wall remains at rest. Examples of wall-bounded shear flows are plane channel flow, plane Couette flow, pipe flow and the flat plate boundary layers.

1.1.1 Laminar Flow: Modal Analysis

For this section, we clarify the notation used to describe the three-dimensional velocity field of as

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t).$$

$\mathbf{U}(\mathbf{x})$ is defined as the time-independent laminar solution of the flow geometry. $\mathbf{u}(\mathbf{x}, t)$ will be the time-dependent three-dimensional velocity perturbation, and can be further written as

$$\mathbf{u}(\mathbf{x}, t) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}(\mathbf{x}, t),$$

where $\mathbf{x}$ represents the spatial coordinate vector in the flow field, while t represents time.

By linearising the Navier-Stokes equations about the laminar equilibrium point, the linearised Navier-Stokes equations is obtained as

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{U} &= -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned} \quad (1.2)$$

However, the existence of the pressure gradient term leads to inconvenience in expressing the state variables in a dynamical system of the form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{L} \mathbf{u}.$$

Based on the assumption of periodic boundary conditions in the stream-wise and span-wise flow domain, a resolution is provided by combining the equations derived by Orr (1907); Sommerfeld (1908); Squire (1933), effectively eliminating the pressure term in the linearised Navier-Stokes equations. The Orr-Sommerfeld-Squire (OSS) equations consist of two state variables for each point in space in the flow field; the wall-normal velocity, v , and the wall-normal vorticity, $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$. By substituting the modal ansatz,

$$\begin{aligned} v(\mathbf{x}, t) &= \tilde{v}(y) e^{\alpha x + \beta y - \omega t}, \\ \eta(\mathbf{x}, t) &= \tilde{\eta}(y) e^{\alpha x + \beta y - \omega t}, \end{aligned} \quad (1.3)$$

where α and β are the spatial wavenumbers ($\alpha = 2\pi n_x/L_x$ and $\beta = 2\pi n_z/L_z$), while ω is the temporal wavenumber; leads to the OSS eigenvalue problem, written as

$$\begin{aligned} \mathbf{L}_{\text{OSS}} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} &= \mathbf{M} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix}, \\ \begin{bmatrix} \mathcal{L}_{OS} & 0 \\ i\beta \frac{dU}{dy} & \mathcal{L}_{SQ} \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix} &= i\omega \begin{bmatrix} k^2 - \mathcal{D}^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ \tilde{\eta} \end{bmatrix}, \end{aligned} \quad (1.4)$$

$$\text{where } \mathcal{L}_{OS} = i\alpha U(k^2 - \mathcal{D}^2) + i\alpha \frac{d^2 U}{dy^2} \quad \text{and} \quad \mathcal{L}_{SQ} = i\alpha U + \frac{1}{Re}(k^2 - \mathcal{D}^2).$$

In eq.(1.4), $k^2 = \alpha^2 + \beta^2$ while $\mathcal{D}^2 = \frac{d^2}{dy^2}$. The equation is only applicable when the base flow is parallel to the stream-wise direction, that is $\mathbf{U}(\mathbf{x}, t) = [U(y), 0, 0]^\top$.

In general, since the formulation of the OSS equations, the earliest results to predict an unstable eigen-mode for the flat plate boundary layer is by Tollmien (1930) and Schlichting (1933). They calculated that the least stable eigenvalue ($\Im(\omega) = 0$), occurs at displacement thickness of $Re_{\text{crit}} = U_\infty \delta^* / \nu = 520$ (U_∞ being the upstream velocity magnitude and Re_{crit} being the critical Reynolds number that leads to a transition to turbulence), with the corresponding two dimensional coherent eigenvector being termed as the Tollmien-Schlichting wave (T-S wave) in literature. For plane channel flow, the critical value of instability predicted by Orszag (1971) is found to be $Re_{\text{crit}} = 5772.2$ for a wavenumber of $\alpha = 1.02$. However, experimental results reveals a contradiction, where transition to turbulent states are observed with Reynolds number lower than the predicted Re_{crit} through modal analysis (Davies & White, 1928; Patel & Head, 1969).

1.1.2 Laminar Flow: Non-Modal Analysis

For a linear operator, $\mathbf{L}$, transient amplification occurs due to the non-normal property of the linear operator, $\mathbf{L}\mathbf{L}^H \neq \mathbf{L}^H\mathbf{L}$, where $(\bullet)^H$ represent the complex conjugate transpose. The eigenvectors of the non-normal linear operator are highly non-orthogonal (nearly linearly independent) to each other, where varying decay rates of eigenvectors of opposing directions lead to a brief amplification of the perturbation from a linearly stable equilibrium before decaying. In the context of the Orr-Sommerfeld-Squire equations, the non-normality of the linear operator is reflected by the lift-up effect, represented by the $i\beta \frac{dU}{dy}$ term in $\mathbf{L}_{\text{OSS}}$. In Trefethen *et al.* (1993), stream-wise homogeneous perturbations with span-wise variation are known to induce the lift-up effect, leading to the non-normal transient amplification. These stream-wise homogeneous vortices, which are three dimensional perturbations, are known to be optimal (Reddy & Henningson, 1993). However, $\beta = 0$ does not necessarily imply that transient growth is absent from the $\mathbf{L}_{\text{OSS}}$ operator. When $\beta = 0$, two dimensional perturbations do permit transient growth via the Orr mechanism (Farrell, 1988), allowing the small amplitude two dimensional perturbations to evolve into a three dimensional finite amplitude perturbation. Here, we note that it is due to perturbation from these non-normal transient amplification that induces the sub-critical transition of the flow field from the laminar attractor to the turbulent attractor. Since the linearised operator is linearly stable for all Reynolds numbers for plane Couette flow and pipe flow, the transition to turbulence of the flow field does not involves the T-S waves that was predicted for boundary layers, which corresponds to the unstable eigen-mode from a linear stability analysis. Furthermore, transition to turbulence for plane Channel flow was observed prior to the predicted critical Reynolds number of $Re_{\text{crit}} = 5772$. Ideas from non-modal stability analysis (Schmid, 2007) are required to quantify the stability properties of the non-normal linear operator as the amplification involves the interactions between stable eigen-modes. The non-modal stability problem can be formulated in the time domain as an initial value problem from a perturbation or an optimal response problem due to external forcing, associated to the pseudo-spectra of the linear operator (Reddy *et al.*, 2006). The initial value problem corresponds to an optimisation problem of obtaining the largest energetic amplification from

an initial perturbation, $\delta \mathbf{u}(0)$, in a finite-time horizon, T^* , where

$$G(T^*) = \max_{\delta \mathbf{u}_0} \frac{\|\delta \mathbf{u}(T^*)\|^2}{\|\delta \mathbf{u}(0)\|^2} = \|e^{T^* \mathbf{L}}\|^2, \quad \text{where} \quad \|\delta \mathbf{u}(0)\|^2 = 1.$$

$\|(\bullet)\|$ refers to the L2-norm.

Alternatively, looking at the optimal response problem from an external harmonic forcing, $\tilde{\mathbf{f}}e^{i\omega_f t}$, on a linear operator is known as resolvent analysis in literature (Trefethen *et al.*, 1993). The dynamics from the linear operator will experience large energy amplification with response, $\tilde{\mathbf{g}}e^{i\omega_f t}$, at certain forcing frequencies, ω_f , determined by the resolvent operator's norm,

$$R(\omega_f) = \|(i\omega_f \mathbf{I} - \mathbf{L})^{-1}\|^2.$$

For a non-normal operator, the optimal forcing frequency are generally found to be *far apart* from the eigen-spectrum of the linear operator, and hence the phenomenon is termed pseudo-resonance.

Here, as discussed in Schmid & Brandt (2014), we touch upon the singular value decomposition and draw a connection towards the two optimisation problems above. For a matrix, $\mathbf{A} \in \mathbb{C}^{m \times n}$, the singular value decomposition would be

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^H.$$

$\mathbf{\Sigma} \in \mathbb{C}^{m \times n}$ is the diagonal matrix of the singular values. $\mathbf{U} \in \mathbb{C}^{m \times m}$, the unitary matrix of the left singular vectors, and, $\mathbf{V} \in \mathbb{C}^{n \times n}$, the unitary matrix of the right singular vectors.

For $\mathbf{A} = e^{t\mathbf{L}}$, the optimal perturbation, $\delta \mathbf{u}_0$, would correspond to the left singular vector of the largest singular value, while the corresponding right singular vector would describe the optimal response, $\delta \mathbf{u}(t^*)$ in the flow field at time t^* . The singular value itself would represent the value of maximum amplification.

For the case of the optimal response from a harmonic forcing, by allowing $\mathbf{A} = (i\omega_f \mathbf{I} - \mathbf{L})^{-1}$, the modes of the optimal forcing, $\tilde{\mathbf{f}}$, and the optimal response, $\tilde{\mathbf{g}}$, are the right and left singular vectors, respectively, that shares the largest singular vector. The singular vector itself would correspond to the resolvent norm, $R(\omega_f)$.

Through the lens of the singular value decomposition, the optimal response problem can be viewed as an input-output analysis, characterising the behaviour of specific forcing to its corresponding response.

1.1.3 Self-Sustaining Process of the Turbulent Boundary Layer

The dynamics of the turbulent boundary layer is characterised by the self-sustaining process (Waleffe *et al.*, 1993; Hamilton *et al.*, 1995). It can be broken into three distinct phases; streak formation, streak breakdown and vortex regeneration; and for the minimal flow unit of the plane Couette flow, the cycle has a period of approximately 80 - 100 time units. Streaks are near-wall coherent structures with alternating span-wise varying regions of high and low speed

stream-wise flows. Quasi stream-wise vortices, on the other hand, are velocity perturbations in v and w , inducing local stream-wise vorticity along the vortex.

Starting for stream-wise vortices, the general consensus in literature attributes the lift-up effect to streak generation. One side of the vortex advects the high stream-wise momentum of the laminar velocity profile from a region further away from the wall closer to the wall, creating a high stream-wise velocity streak closer to the wall, while on the other side of the vortex acts to propel the laminar velocity profile's low stream-wise momentum further from the wall, creating the low stream-wise velocity streak. The lift-up effect is a non-normal mechanism attributed to the non-zero cross diagonal term in the Orr-Sommerfeld-Squire linear operator. It holds the form

$$\mathbf{L}_{\text{lift-up}} \sim v \frac{\partial U_l(y)}{\partial y}.$$

Due to instability inherent in the streak structure or transient growth, the streaks would subsequently start to meander along the stream-wise direction. In doing so, the streak breakdown process starts. The meandering motion induces vortex stretching and non-linear vorticity interactions, regenerating the quasi stream-wise vortices, thus completing a cycle of the self-sustaining process.

1.2 Thesis Outline

At high Reynolds numbers, the turbulent solution to the Navier-Stokes equations exhibit chaos. Commonly termed as the butterfly effect, infinitesimal differences in initial conditions lead to diverging temporal trajectories when the initial conditions are propagated forward in time (Lorenz, 1963). When an initial condition is perturbed by a infinitesimal perturbation, the predictability of turbulence becomes challenging as the Lyapunov time scale decreases as Reynolds number increases (Ruelle, 1979; Mohan *et al.*, 2017).

Based on a suitable averaging procedure, the aims and objective of this thesis is to characterise the perturbation of turbulence in an *averaged* sense, leading to a linear operator that could encompass the effect of the Reynolds stresses.

1.3 Thesis Structure

The theory of statistical state dynamics is first introduced in §2, including the linearisation of its dynamics. Next, the introduction of the formulation of the ensemble mean operator is made in §3, elucidating a computational impracticality associated with the formulation. The method is then refined to investigate the transient amplification of the statistical state dynamics via a perturbation of the initial condition. The methodology is proven to be fruitful, where differing dynamics are revealed when compared to conventional understanding and a discussion is made. In the conclusion, a pertinent observation is made on the behaviour of statistical state dynamics when the linear operator possess an unstable eigenvalue, and subsequently, concluding remarks were made.

Chapter 2

Perturbation Dynamics of Turbulence

This chapter starts by discussing the idea of applying an averaging procedure to the Navier-Stokes equations, motivating the introduction of the statistical state dynamics framework. Expressing the temporal evolution of a dynamical system's statistics (ie: the turbulent mean and fluctuation covariance) using statistical state dynamics allows for these statistics to be amenable to theoretical analysis. The framework proved fruitful in elucidating the dynamics of barotropic (Farrell *et al.*, 2017b) and baroclinic beta-plane turbulence (Farrell & Ioannou, 2009) that does not permit an analogous analytical counterpart from analysing its dynamical realisation. The review on statistical state dynamics here would predominantly be based on Farrell *et al.* (2017a) and Farrel & Ioannou (2019).

Related to statistical state dynamics framework, we discuss the triple decomposition of a turbulent velocity field, introduced by Hussain & Reynolds (1970). Triple decomposition decomposes a turbulent velocity field into three components; a time-independent component and two time-dependent components: the first, representing the effect on the velocity field from the forced dynamics; and the other, representing the natural fluctuations of turbulence. Based on the ergodic hypothesis, the time-independent temporal mean state is interpreted as the equilibrium of the ensemble mean.

Following on the introduction of an ensemble statistical state, the concept of an ensemble perturbation to this statistical state is defined based on velocity perturbations applied to each turbulent trajectory making up the ensemble. We introduce a general mathematical framework of introducing an ensemble of velocity perturbations. In contrast with conventional ideas about velocity perturbations on a turbulent trajectory, we see that the perturbation's ensemble statistical distribution affects the overall statistical evolution dynamics.

The idea that we introduce relates to applying an ensemble of perturbations on an ensemble of velocity fields. By interpreting the temporal mean time-independent component as an equilibrium of the statistical state, such as in Kashyap *et al.* (2022, 2024), an attempt is made to simplify the non-linear evolution ensemble statistical state equations.

2.1 Statistical State Dynamics

In literature, concept of a statistical state was introduced to the study of fluid mechanics by Hopf (1952). Hopf used the concept of cumulants to study the velocity field of homogeneous isotropic turbulence. Despite making limited progress due to the difficulty in obtaining a satisfactory closure to the cumulants' expansion, it is notable for its conceptual advancement.

For a state quantity, $(\bullet)(\mathbf{x}, t)$, we introduce the Reynolds averaging operator, $\langle(\bullet)\rangle_\Omega$, where Ω represents the sample space where the state quantity is drawn upon. In Hussain & Reynolds (1970), the sample space has been the temporal phase of a harmonic forcing, $\Omega = \phi$, hence the operator represents phase averaging. A streamwise spatial averaging procedure, $\Omega = x$, was adopted in the formulation of a statistical state dynamics of the Navier-Stokes equations, termed stochastic structural stability theory (S3T) in Farrell & Ioannou (2012). In general, the averaging procedure is required to satisfy the following Reynolds conditions:

$$\begin{aligned}\langle\alpha F + \beta G\rangle_\Omega &= \alpha\langle F\rangle_\Omega + \beta\langle G\rangle_\Omega \\ \langle\frac{\partial}{\partial t}F\rangle_\Omega &= \frac{\partial}{\partial t}\langle F\rangle_\Omega, \\ \langle\langle F\rangle_\Omega G\rangle_\Omega &= \langle F\rangle_\Omega\langle G\rangle_\Omega,\end{aligned}\tag{2.1}$$

where $F(\mathbf{x}, t)$ and $G(\mathbf{x}, t)$ are vector fields, while α and β are scalar constants. In this work, an ensemble averaging procedure is adopted, where an infinite number of realisations of a turbulent flow field trajectory is considered in the ensemble. The ensemble averaging procedure is defined as

$$\langle(\bullet)\rangle(\mathbf{x}, t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (\bullet)^{(i)},\tag{2.2}$$

where $(\bullet)^{(i)}$ denotes a state related to the i -th member of the ensemble.

By applying the Reynolds averaging operator, the states in eq.(1.1) can be decomposed into an ensemble mean component, $\langle(\bullet)\rangle(\mathbf{x}, t)$, and, a turbulent fluctuation component, $(\bullet)'(\mathbf{x}, t)$, written as

$$\begin{aligned}\mathbf{U}(\mathbf{x}, t) &= \langle\mathbf{U}\rangle(\mathbf{x}, t) + \mathbf{u}'(\mathbf{x}, t), \\ P(\mathbf{x}, t) &= \langle P\rangle(\mathbf{x}, t) + p'(\mathbf{x}, t), \\ \mathbf{F}(\mathbf{x}, t) &= \langle\mathbf{F}\rangle(\mathbf{x}, t) + \mathbf{f}'(\mathbf{x}, t).\end{aligned}\tag{2.3}$$

From the definitions above, it can be further inferred that $\langle\mathbf{u}'\rangle(\mathbf{x}, t) = \mathbf{0}$, $\langle p'\rangle(\mathbf{x}, t) = 0$, $\langle\mathbf{f}'\rangle(\mathbf{x}, t) = \mathbf{0}$.

For the ensemble mean statistical state evolution equations are

$$\frac{\partial}{\partial t}\langle\mathbf{U}\rangle + (\langle\mathbf{U}\rangle \cdot \nabla) \langle\mathbf{U}\rangle = -\nabla\langle P\rangle + \frac{1}{Re}\nabla^2\langle\mathbf{U}\rangle - \nabla \cdot \langle\mathbf{u}' \otimes \mathbf{u}'\rangle + \langle\mathbf{F}\rangle,\tag{2.4a}$$

$$\nabla \cdot \langle\mathbf{U}\rangle = 0,\tag{2.4b}$$

while the evolution equations for the turbulent fluctuations are

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u}' + (\langle \mathbf{U} \rangle \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \langle \mathbf{U} \rangle &= -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}' \\ &+ \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle - (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \mathbf{f}', \end{aligned} \quad (2.5a)$$

$$\nabla \cdot \mathbf{u}' = 0. \quad (2.5b)$$

From the perspective of statistical state dynamics, eq.(2.5) represents the dynamics of the fluctuating statistical states of each individual turbulent trajectory in the ensemble, where the $\nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle$ term represents the forcing for the maintenance of the ensemble mean statistical state that is representative property of the ensemble as a whole.

As a consequence of applying the Reynolds averaging operation on the Navier-Stokes equations, the Reynolds stress tensor, $\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$, is produced. Hence, the statistical state dynamics equations are under-determined and the turbulent fluctuating Reynolds stress term is required to be modelled. In the turbulence modelling community, this is known as the turbulence closure problem.

2.2 Triple Decomposition

In Hussain & Reynolds (1970), the introduction of the triple decomposition is motivated by an experimental setting where a vibrating ribbon is placed upstream generating a periodic forcing of period T_f onto a turbulent velocity field, $\mathbf{U}(\mathbf{x}, t)$. An identical decomposition is applied in this formulation, albeit the ensemble averaging procedure is adopted instead of the phase averaging procedure as in Hussain & Reynolds (1970). The triple decomposition splits the realisation of a turbulent trajectory sampled from an ensemble into

$$\mathbf{U}^{(i)}(\mathbf{x}, t) = \overline{\mathbf{U}}(\mathbf{x}) + \delta \langle \mathbf{U} \rangle(\mathbf{x}, t) + \mathbf{u}'^{(i)}(\mathbf{x}, t), \quad (2.6)$$

$\overline{\mathbf{U}}(\mathbf{x})$ is denoted as the temporal mean of the velocity field, $\delta \langle \mathbf{U} \rangle(\mathbf{x}, t)$ as the ensemble mean fluctuations, and, $\mathbf{u}'^{(i)}$ as the turbulent fluctuations of a realisation of turbulent trajectory. These quantities are mathematically defined as

$$\overline{\mathbf{U}}(\mathbf{x}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{U}^{(i)}(\mathbf{x}, t) dt \quad \text{and} \quad \langle \mathbf{U} \rangle(\mathbf{x}, t) = \overline{\mathbf{U}}(\mathbf{x}) + \delta \langle \mathbf{U} \rangle(\mathbf{x}, t).$$

The definitions also applies to the corresponding decomposition of the pressure field

$$P^{(i)}(\mathbf{x}, t) = \overline{P}(\mathbf{x}) + \delta \langle P \rangle(\mathbf{x}, t) + p'^{(i)}(\mathbf{x}, t). \quad (2.7)$$

By invoking ergodicity, for statistically stationary turbulent trajectories, the ensemble mean state can be interpreted to be equivalent to the temporal mean state of a single realisation of turbulent trajectory. It implies that the temporal mean state, eq.(2.8),

$$\left\langle \begin{bmatrix} \mathbf{U} \\ P \\ \mathbf{u}' \\ \mathbf{u}' \otimes \mathbf{u}' \end{bmatrix} \right\rangle_e (\mathbf{x}, t) = \overline{\begin{bmatrix} \mathbf{U} \\ P \\ 0 \\ \mathbf{u}' \otimes \mathbf{u}' \end{bmatrix}} (\mathbf{x}), \quad (2.8)$$

is an equilibrium state of the ensemble mean statistical state equations, eq.(2.4). For a wide range of probable velocity perturbations applied, due to the dissipative nature of the Navier-Stokes equations, the ensemble statistical state will eventually relaxes back to the temporal mean statistical equilibrium once the velocity perturbations is removed from the ensemble.

2.3 Perturbation Dynamics of Turbulence

2.3.1 Perturbations of an Ensemble

As a starting point for the modelling of the statistical state dynamics of the Navier-Stokes equations, the scope of perturbations are limited to a instantaneous velocity impulse introduced to each realisation of the ensemble members. It ensures that the natural temporal mean states sample from an ensemble are unmodified by a persistent perturbation or forcing, simplifying the temporal mean statistical state equation. Here, a perturbation state is represented as $\delta(\bullet)$, hence drawing a distinction between a realisation of velocity perturbation, $\delta\mathbf{U}$, the ensemble of velocity perturbations, denoted as $\langle\delta\mathbf{U}\rangle$, and, the perturbation of the ensemble mean state, denoted as $\delta\langle\mathbf{U}\rangle$.

For an ensemble of turbulent trajectories, perturbations to the trajectories is defined as an unique realisation of divergence-free velocity perturbation, $\delta\mathbf{U}^{(i)}$, applied to each member in the ensemble, $\mathbf{U}^{(i)}$. Adopting notation of ensembles, the ensemble of perturbations, denoted as

$$\delta\mathcal{U}_N = \{\delta\mathbf{U}^{(1)}, \delta\mathbf{U}^{(2)}, \dots, \delta\mathbf{U}^{(N)}\}, \quad (2.9)$$

applied to the ensemble, $\mathcal{U}_N$, is expressed as

$$\mathcal{U}_N \xRightarrow{\delta\mathcal{U}_N} {}^*\mathcal{U}_N, \quad (2.10)$$

where ${}^*(\bullet)$ denotes the perturbed state of $(\bullet)$. Essentially, the $\Rightarrow$ operation represents the i -th member of the ensemble, $\mathbf{U}^{(i)}$, is perturbed by the i -th member of the ensemble of velocity perturbations, $\delta\mathbf{U}^{(i)}$,

$${}^*\mathbf{U}^{(i)} = \mathbf{U}^{(i)} + \delta\mathbf{U}^{(i)}. \quad (2.11)$$

Hence, the perturbed ensemble can be denoted as

$${}^*\mathcal{U}_N = \{{}^*\mathbf{U}^{(1)}, {}^*\mathbf{U}^{(2)}, \dots, {}^*\mathbf{U}^{(N)}\}.$$

The effect of the ensemble under the defined ensemble of perturbations is inferred to be

$$\begin{aligned} {}^*\langle\mathbf{U}\rangle &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N \mathbf{U}^{(i)} + \delta\mathbf{U}^{(i)}, \\ &= \langle\mathbf{U}\rangle + \langle\delta\mathbf{U}\rangle, \\ &= \langle\mathbf{U}\rangle + \delta\langle\mathbf{U}\rangle, \end{aligned} \quad (2.12a)$$

where $\delta\langle\mathbf{U}\rangle = \langle\delta\mathbf{U}\rangle$, which states that the ensemble mean of the ensemble of perturbations, $\langle\delta\mathbf{U}\rangle$, will be equal to the ensemble mean state perturbation, $\delta\langle\mathbf{U}\rangle$, on the original ensemble system that the ensemble of perturbations is applied to.

Consequently, $\delta \mathbf{u}'^{(i)} = \delta \mathbf{U}^{(i)} - \langle \delta \mathbf{U} \rangle$ implies that $^* \mathbf{u}'^{(i)} = \mathbf{u}'^{(i)} + \delta \mathbf{u}'^{(i)}$, leading to

$$\begin{aligned} ^* \langle \mathbf{u}' \otimes \mathbf{u}' \rangle &= \langle (\mathbf{u}' + \delta \mathbf{u}') \otimes (\mathbf{u}' + \delta \mathbf{u}') \rangle, \\ &= \langle \mathbf{u}' \otimes \mathbf{u}' \rangle + \langle \delta \mathbf{u}' \otimes \delta \mathbf{u}' \rangle + \langle \delta \mathbf{u}' \otimes \mathbf{u}' \rangle + \langle \mathbf{u}' \otimes \delta \mathbf{u}' \rangle, \\ &= \langle \mathbf{u}' \otimes \mathbf{u}' \rangle + \delta \langle \mathbf{u}' \otimes \mathbf{u}' \rangle. \end{aligned} \quad (2.12b)$$

Equation (2.12) shows that the statistical state perturbations, $\delta \langle \mathbf{U} \rangle$ and $\delta \langle \mathbf{u}' \otimes \mathbf{u}' \rangle$, are dependent on the statistical distribution of the ensemble of velocity perturbations, $\langle \delta \mathbf{U} \rangle$ and $\delta \mathbf{u}'$, together with the initial distribution of the unperturbed turbulent fluctuations, $\mathbf{u}'$. Following on from these definitions of the statistical state perturbations, the dynamic evolution for the ensemble mean perturbation, $\delta \langle \mathbf{U} \rangle$, is described by

$$\begin{aligned} \frac{\partial}{\partial t} \delta \langle \mathbf{U} \rangle + (\delta \langle \mathbf{U} \rangle \cdot \nabla) \langle \mathbf{U} \rangle + (\langle \mathbf{U} \rangle \cdot \nabla) \delta \langle \mathbf{U} \rangle &= -\nabla \delta \langle P \rangle + \frac{1}{Re} \nabla^2 \delta \langle \mathbf{U} \rangle \\ &\quad - \nabla \cdot \delta \langle \mathbf{u}' \otimes \mathbf{u}' \rangle - (\delta \langle \mathbf{U} \rangle \cdot \nabla) \delta \langle \mathbf{U} \rangle, \end{aligned} \quad (2.13a)$$

$$\nabla \cdot \delta \langle \mathbf{U} \rangle = 0. \quad (2.13b)$$

The perturbation ensemble mean dynamics are coupled to the unperturbed ensemble mean equations and also indirectly affected by each realisation of the perturbed turbulent fluctuations through the evolving perturbed Reynolds stress tensor, $\delta \langle \mathbf{u}' \otimes \mathbf{u}' \rangle$.

2.3.2 Linearisation of the Statistical State Equations

Small Perturbation Approximation

When a sufficiently small amplitude perturbation, $\epsilon \delta \mathbf{U}$ (with $\epsilon \ll 1$), that possesses a prescribed ensemble statistical distribution as defined by eq.(2.12) is applied to the velocity field of each ensemble member, linearisation can be performed on the ensemble statistical state equations about this statistical equilibrium by applying a small perturbation expansion,

$$\left\langle \begin{bmatrix} \mathbf{U} \\ P \\ \mathbf{u}' \\ \mathbf{u}' \otimes \mathbf{u}' \end{bmatrix} \right\rangle (\mathbf{x}, t) = \overline{\begin{bmatrix} \mathbf{U} \\ P \\ 0 \\ \mathbf{u}' \otimes \mathbf{u}' \end{bmatrix}} (\mathbf{x}) + \epsilon \widetilde{\begin{bmatrix} \mathbf{u} \\ p \\ 0 \\ \mathbf{u}' \otimes \mathbf{u}' \end{bmatrix}} (\mathbf{x}, t). \quad (2.14)$$

The terms denoted $\widetilde{(\bullet)}$ are introduced to distinctly denote the perturbation states from a reference temporal mean statistical equilibrium state, $\langle (\bullet) \rangle_e = \overline{(\bullet)}$, and is conceptually equivalent to the more general case of the perturbation states from a non-equilibrium evolving reference trajectory, $\delta (\bullet)$, defined previously. The ensemble mean equilibrium state adheres to the time-invariant Navier-Stokes equations albeit augmented with the temporally averaged turbulent fluctuation Reynolds stress term, $\overline{\mathbf{u}' \otimes \mathbf{u}'}$,

$$(\overline{\mathbf{U}} \cdot \nabla) \overline{\mathbf{U}} + \nabla \cdot \overline{(\mathbf{u}' \otimes \mathbf{u}')} = -\nabla \overline{P} + \frac{1}{Re} \nabla^2 \overline{\mathbf{U}}, \quad (2.15a)$$

$$\nabla \cdot \overline{\mathbf{U}} = 0. \quad (2.15b)$$

Linearising the ensemble mean statistical state dynamics equations about the temporal mean equilibrium state leads to

$$\frac{\partial}{\partial t} \tilde{\mathbf{u}} + (\bar{\mathbf{U}} \cdot \nabla) \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla) \bar{\mathbf{U}} = -\nabla \tilde{p} + \frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} - \nabla \cdot (\widetilde{\mathbf{u}' \otimes \mathbf{u}'}), \quad (2.16a)$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0. \quad (2.16b)$$

2.4 Derivations

The derivation of the evolution equations of statistical state dynamics are as shown below.

2.4.1 Dynamics of Ensemble Mean State

The evolution equation of the ensemble mean statistics are derived by applying the Reynolds averaging operator to the Navier-Stokes equations, eq.(1.1),

$$\left\langle \frac{\partial}{\partial t} \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right\rangle = \left\langle -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} + \mathbf{F} \right\rangle, \\ \langle \nabla \cdot \mathbf{U} \rangle = 0.$$

Applying the linearity property of the Reynolds condition, eq.(2.1), we have

$$\left\langle \frac{\partial}{\partial t} \mathbf{U} \right\rangle + \langle (\mathbf{U} \cdot \nabla) \mathbf{U} \rangle = \langle -\nabla P \rangle + \left\langle \frac{1}{Re} \nabla^2 \mathbf{U} \right\rangle + \langle \mathbf{F} \rangle, \\ \langle \nabla \cdot \mathbf{U} \rangle = 0,$$

and, since differential operators and the Reynolds averaging operator are commutative, it leads to

$$\frac{\partial}{\partial t} \langle \mathbf{U} \rangle + \langle (\mathbf{U} \cdot \nabla) \mathbf{U} \rangle = -\nabla \langle P \rangle + \frac{1}{Re} \nabla^2 \langle \mathbf{U} \rangle + \langle \mathbf{F} \rangle, \\ \nabla \cdot \langle \mathbf{U} \rangle = 0.$$

The non-linear advection term in the Navier-Stokes operator consists of numerous $\langle U_i U_j \rangle$ operations. Taking the Reynolds decomposition, eq.(2.3), and expanding it leads to

$$\begin{aligned} \langle U_i U_j \rangle &= \langle [\langle U_i \rangle + u'_i] [\langle U_j \rangle + u'_j] \rangle \\ &= \langle \langle U_i \rangle \langle U_j \rangle + u'_i u'_j + \langle U_i \rangle u'_j + \langle U_j \rangle u'_i \rangle \\ &= \langle \langle U_i \rangle \langle U_j \rangle \rangle + \langle u'_i u'_j \rangle + \langle \langle U_i \rangle u'_j \rangle + \langle \langle U_j \rangle u'_i \rangle \\ &= \langle U_i \rangle \langle U_j \rangle + \langle u'_i u'_j \rangle + \langle U_i \rangle \langle u'_j \rangle + \langle U_j \rangle \langle u'_i \rangle. \end{aligned}$$

Since $\langle u' \rangle = 0$,

$$\langle U_i U_j \rangle = \langle U_i \rangle \langle U_j \rangle + \langle u'_i u'_j \rangle. \quad (2.17)$$

Applying eq.(2.17) into the non-linear advective operator converts the mean state equation into

$$\frac{\partial}{\partial t} \langle \mathbf{U} \rangle + (\langle \mathbf{U} \rangle \cdot \nabla) \langle \mathbf{U} \rangle + \langle (\mathbf{u}' \cdot \nabla) \mathbf{u}' \rangle = -\nabla \langle P \rangle + \frac{1}{Re} \nabla^2 \langle \mathbf{U} \rangle + \langle \mathbf{F} \rangle, \\ \nabla \cdot \langle \mathbf{U} \rangle = 0.$$

Finally, the turbulent fluctuations advection operator, $\langle (\mathbf{u}' \cdot \nabla) \mathbf{u}' \rangle$, can be expressed in its divergence form as

$$(\mathbf{u}' \cdot \nabla) \mathbf{u}' = \frac{\partial}{\partial x} \begin{bmatrix} u'u' \\ u'v' \\ u'w' \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} u'v' \\ v'v' \\ v'w' \end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix} u'w' \\ v'w' \\ w'w' \end{bmatrix} = \nabla \cdot (\mathbf{u}' \otimes \mathbf{u}'),$$

leading to

$$\begin{aligned} \frac{\partial}{\partial t} \langle \mathbf{U} \rangle + \langle \langle \mathbf{U} \rangle \cdot \nabla \rangle \langle \mathbf{U} \rangle &= -\nabla \langle P \rangle + \frac{1}{Re} \nabla^2 \langle \mathbf{U} \rangle - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle + \langle \mathbf{F} \rangle, \\ \nabla \cdot \langle \mathbf{U} \rangle &= 0, \end{aligned}$$

which is as seen in eq.(2.4).

2.4.2 Dynamics of Turbulent Fluctuations

The turbulent fluctuations state is derived from the Navier-Stokes equations and the ensemble mean state evolution equations through the following expression:

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u}' &= \frac{\partial}{\partial t} \mathbf{U} - \frac{\partial}{\partial t} \langle \mathbf{U} \rangle, \\ \nabla \cdot \mathbf{u}' &= \nabla \cdot \mathbf{U} - \nabla \cdot \langle \mathbf{U} \rangle. \end{aligned}$$

Substituting the respective evolution equations and expressing the operands of the involved linear differential operators into the turbulent fluctuations coordinates, we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u}' + (\mathbf{U} \cdot \nabla) \mathbf{U} - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle &= -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}' + \langle \langle \mathbf{U} \rangle \cdot \nabla \rangle \langle \mathbf{U} \rangle + \mathbf{f}', \\ \nabla \cdot \mathbf{u}' &= 0. \end{aligned}$$

The non-linear advection term, $(\mathbf{U} \cdot \nabla) \mathbf{U}$, in the turbulent fluctuations momentum equations can be further be written as

$$(\mathbf{U} \cdot \nabla) \mathbf{U} = \langle \langle \mathbf{U} \rangle \cdot \nabla \rangle \langle \mathbf{U} \rangle + (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \langle \langle \mathbf{U} \rangle \cdot \nabla \rangle \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \langle \mathbf{U} \rangle,$$

leading to

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{u}' + \langle \langle \mathbf{U} \rangle \cdot \nabla \rangle \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \langle \mathbf{U} \rangle &= -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}' \\ &\quad + \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle - (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \mathbf{f}', \\ \nabla \cdot \mathbf{u}' &= 0, \end{aligned}$$

which is as seen in eq.(2.5).

2.4.3 Linearised Statistical States Dynamics

Starting from the ensemble mean equation of statistical state dynamics, eq.(2.4), and, applying a small perturbation expansion about the time-invariant equilibrium mean state, we have

$$\begin{aligned} \frac{\partial}{\partial t} [\bar{\mathbf{U}} + \epsilon \tilde{\mathbf{u}}] + \nabla \cdot \left[\overline{(\mathbf{u}' \otimes \mathbf{u}')} + \epsilon \widetilde{(\mathbf{u}' \otimes \mathbf{u}')} \right] &= -\nabla [\bar{P} + \epsilon \tilde{p}] + \frac{1}{Re} \nabla^2 (\bar{\mathbf{U}} + \epsilon \tilde{\mathbf{u}}) \\ &\quad - ([\bar{\mathbf{U}} + \epsilon \tilde{\mathbf{u}}] \cdot \nabla) [\bar{\mathbf{U}} + \epsilon \tilde{\mathbf{u}}], \\ \nabla \cdot [\bar{\mathbf{U}} + \epsilon \tilde{\mathbf{u}}] &= 0. \end{aligned}$$

Decomposing the equations above based on their order of magnitude and omitting terms of $\mathcal{O}(\epsilon^2)$, we have

$\mathcal{O}(1)$ equations:

$$\begin{aligned}(\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{U}} &= -\nabla \bar{P} + \frac{1}{Re} \nabla^2 \bar{\mathbf{U}} - \nabla \cdot \overline{(\mathbf{u}' \otimes \mathbf{u}')}, \\ \nabla \cdot \bar{\mathbf{U}} &= 0,\end{aligned}$$

and, $\mathcal{O}(\epsilon)$ equations:

$$\begin{aligned}\frac{\partial}{\partial t} \delta \tilde{\mathbf{u}} + (\bar{\mathbf{U}} \cdot \nabla) \delta \tilde{\mathbf{u}} + (\delta \tilde{\mathbf{u}} \cdot \nabla) \bar{\mathbf{U}} &= -\nabla \delta \tilde{p} + \frac{1}{Re} \nabla^2 \delta \tilde{\mathbf{u}} - \nabla \cdot \delta \widetilde{(\mathbf{u}' \otimes \mathbf{u}')}, \\ \nabla \cdot \delta \tilde{\mathbf{u}} &= 0,\end{aligned}$$

as shown in eq.(2.15) and eq.(2.16).

Chapter 3

Ensemble Mean Operator Inference

In this chapter, a data-driven mathematical formulation is introduced to solve the inverse problem to identify the an *ideal* linear operator for the linearised ensemble mean dynamics, eq. (2.16). In Reynolds-Averaged Navier-Stokes equations, the Reynolds stress terms are modelled through the use of the ensemble mean statistical states via the eddy-viscosity hypothesis, which is as proposed here.

A cursory analysis of the inverse problem yields an computational expensive procedure which is deemed infeasible. Following on, a data-driven approach as in Iyer *et al.* (2019) is used to identify the relevant transient growth seen in the turbulent boundary layers (Pujals *et al.*, 2009; Hwang & Cossu, 2010a).

3.1 Problem Statement

The statistical states of the ensemble, $\mathcal{U}_N$, is required to be statistically stationary, implying that $\mathcal{U}_N$ possesses statistical equilibrium, eq.(3.1),

$$\langle \mathbf{U} \rangle = \overline{\mathbf{U}} + \Delta_i, \quad (3.1a)$$

$$\langle \mathbf{u}' \rangle = \mathbf{0} + \Delta_j, \quad (3.1b)$$

$$\langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \overline{\mathbf{u}' \otimes \mathbf{u}'} + \Delta_k. \quad (3.1c)$$

Since only finite turbulent realisations exist in $\mathcal{U}_N$ as $N \neq \infty$, the numerical errors, Δ , exist and are required to be numerically small, $\Delta \ll 1$. The ensemble mean equations, eq.(2.15), introduced in §2, are adopted for the data-driven formulation here, written as

$$\begin{aligned} \frac{\partial}{\partial t} {}^* \langle \mathbf{U} \rangle + ({}^* \langle \mathbf{U} \rangle \cdot \nabla) {}^* \langle \mathbf{U} \rangle + \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle &= -\nabla {}^* \langle P \rangle + \frac{1}{Re} \nabla^2 {}^* \langle \mathbf{U} \rangle, \\ \nabla \cdot {}^* \langle \mathbf{U} \rangle &= 0. \end{aligned}$$

The ensemble perturbations on the initial conditions which is the temporal mean equilibrium state are furthermore constrained to be small,

$${}^* \mathbf{U}^{(i)} = \mathbf{U}^{(i)} + \delta \mathbf{U} \quad \text{with} \quad \delta \mathbf{U} \sim \mathcal{O}(\epsilon \langle \mathbf{U} \rangle),$$

allowing the ensemble mean equations to be further simplified to a linear operator as in eq.(2.16). The linearised approximation is written as

$$\frac{\partial}{\partial t} \tilde{\mathbf{u}} = -(\bar{\mathbf{U}} \cdot \nabla) \tilde{\mathbf{u}} - (\tilde{\mathbf{u}} \cdot \nabla) \bar{\mathbf{U}} - \nabla \cdot (\widetilde{\mathbf{u}' \otimes \mathbf{u}'}) + \frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} - \nabla \tilde{p}.$$

By adopting the linear eddy-viscosity hypothesis, the Reynolds stress terms are approximated by $\widetilde{(\mathbf{u}' \otimes \mathbf{u}')} \approx \nu_{\mathbf{T}} (\nabla \tilde{\mathbf{u}} + \nabla \tilde{\mathbf{u}}^{\top})$.

As periodic boundary conditions exist in the streamwise and spanwise directions, these directions are homogeneous in space. Hence, the linear operator are decoupled into individual block diagonal components by each wavenumber pair, $\mathbf{k} = \left(k_x = \frac{2\pi n_x}{L_x}, k_z = \frac{2\pi n_z}{L_z} \right)^{\top}$. The linearised perturbation ensemble mean dynamics is approximated by a linear operator, $\mathcal{L}[\tilde{\mathbf{u}}]$. The numerical realisation of the linear operator for wavenumber $\mathbf{k}$, is denoted, $\mathbf{L}(\mathbf{k})$, with dimension $\mathbb{C}^{3N_y \times 3N_y}$, where N_y is the dimension of the grid discretisation in the wall-normal direction.

The central assumption for this data-driven formulation for operator inference lies in the fact that in statistical state dynamics equations, any perturbation in initial conditions applied to the ensemble will eventually lead to the ensemble mean state relaxing back to the temporal mean state. Hence, the linear subspace that is generated by the linearisation of the stable manifold on the equilibrium point in state space will possess only stable eigenvalues with its corresponding set of eigenvectors, $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{3N_y}]$. $\mathbf{\Lambda}$ will be denoted for the diagonal matrix of eigenvalues, while $\mathbf{v}_i$ is the i -th eigenvector with a corresponding eigenvalue, λ_i . The linear response from an initial velocity perturbation is dictated by

$$\tilde{\mathbf{u}}(t) = e^{t\mathbf{L}(\mathbf{k})} \tilde{\mathbf{u}}(0).$$

With an existing set of orthonormal basis functions, $\Phi(\mathbf{k})$, the response vector, $\tilde{\mathbf{u}}_i \in \mathbb{C}^{3N_y \times 1}$, at time τ , is denoted as

$$\tilde{\mathbf{u}}_i = e^{\tau\mathbf{L}(\mathbf{k})} \phi_i,$$

where the basis function, ϕ_i , are also termed the velocity perturbation vector. To proceed with the operator inference, perturbations to the steady-state temporal mean ensemble is applied with the set of orthonormal basis functions, Φ . The velocity response matrix, $\tilde{\mathbf{U}}_R(\tau)$, is defined as $\tilde{\mathbf{U}}_R(\tau) = [\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \dots, \tilde{\mathbf{u}}_{3N_y}] (\tau)$, leading to

$$\tilde{\mathbf{U}}_R(\tau) = e^{\tau\mathbf{L}(\mathbf{k})} \Phi. \quad (3.4)$$

With the subsequent steps,

$$\begin{aligned} \tilde{\mathbf{U}}_R(\tau) &= e^{\tau\mathbf{L}(\mathbf{k})} \Phi, \\ \tilde{\mathbf{U}}_R(\tau) \Phi^{\mathbf{H}} &= e^{\tau\mathbf{L}(\mathbf{k})} \Phi \Phi^{\mathbf{H}}, \\ \tilde{\mathbf{U}}_R(\tau) \Phi^{\mathbf{H}} &= e^{\tau\mathbf{L}(\mathbf{k})} \mathbf{I}, \end{aligned}$$

where the unitary property, $\mathbf{I} = \Phi^{\mathbf{H}} \Phi = \Phi \Phi^{\mathbf{H}}$, is applied. The solution to the operator inference inverse problem is stated as

$$\mathbf{L}(\mathbf{k}) = \frac{1}{\tau} \log \left[\tilde{\mathbf{U}}_R(\tau) \Phi^{\mathbf{H}} \right]. \quad (3.5)$$

A drawback is that equation for direct operator inversion, eq.(3.5), is computationally expensive, as $3N_y$ sets of ensemble simulations are required to invert the response matrix to identify the linear operator.

Hence, the problem of a linear operator inference is relegated to a subsequent chapter, while a study of the transient growth effect of perturbations on an ensemble is carried out, which is in comparison to the transient growth analysis of turbulent flows done using a proxy linearised model as in Hwang & Cossu (2010*a,b*).

3.2 Refined Problem Statement

The plane channel flow is the main flow configuration investigated in this and subsequent chapters. It is a three-dimensional domain with the stream-wise direction denoted by x , the span-wise direction denoted by z . Two parallel walls exist two spatial units apart in y , with y being known as the wall-normal direction. The channel is driven by a pressure gradient in the stream-wise direction with the mass flow rate being kept constant across the simulation time. Additionally, the stream-wise and span-wise directions are periodic. We express the space, time and velocity field with non-dimensional units. Hence, the dynamics of the flow field are dictated by three parameters; two spatial length scales defining the stream-wise and span-wise domain, L_x and L_z respectively, and the Reynolds number, Re_h ,

$$Re_h = \frac{U_l h}{\nu}.$$

h denotes the wall-normal half height, and, ν , the kinematic viscosity, with U_l the laminar centerline velocity.

To *accurately* identify the dynamics of the ensemble state velocity perturbation, $\tilde{\mathbf{u}}$, an ensemble simulation is carried out where each ensemble simulation is derived by applying the ensemble mean operation to a set of N turbulent trajectory realisations. N is chosen to be large, tending to infinity.

For each ensemble simulation, an unique velocity perturbation is applied to the Fourier velocity component, $\hat{\tilde{\mathbf{u}}}$, of the targeted plane Fourier mode. With the energy norm defined to be

$$A = \frac{1}{2h} \int_0^{2h} \hat{\tilde{\mathbf{u}}}^H(y) \cdot \hat{\tilde{\mathbf{u}}}(y) dy, \quad (3.6)$$

a perturbation with amplitude, A , is applied to the ensemble members to generate the transient amplification dynamics. The methodology for obtaining the unique velocity perturbation is generated by solving the optimisation problem for the optimal initial condition as in Hwang & Cossu (2010*a*),

$$\max_{\hat{\tilde{\mathbf{u}}}(t_0)} G(t) = \frac{\|\hat{\tilde{\mathbf{u}}}(t)\|^2}{\|\hat{\tilde{\mathbf{u}}}(t_0)\|^2} \quad \text{where} \quad \|\hat{\tilde{\mathbf{u}}}\|^2 = A, \quad (3.7)$$

for the linearised Navier-Stokes operator augmented with a temporal mean flow fitted eddy viscosity,

$$\nu = \left[1 - \frac{y^+}{Re_\tau} \right] \left(\frac{d\bar{U}^+}{dy^+} \right)^{-1} \quad (3.8)$$

The model is termed the LNS_ν model in subsequent sections.

Once converted into a eigenvalue problem, the symmetric nature of the energy norm operator leads to a set of perturbation eigenvectors that are orthogonal for each target time-horizon, t^* . The normalised leading 6 perturbation eigenvectors based on the optimal gain, $G(t^*)$, is taken to the the unique velocity perturbations and used to generate the transient growth dynamics. They are respectively termed $\delta\langle\hat{\mathbf{U}}\rangle_{1-6}$ in subsequent sections, ranked in descending order based on its predicted optimal gain via the LNS_ν model.

3.2.1 Identification of Small Perturbations Maximal Amplification

The finite-sized ensemble of turbulent trajectories, $\mathcal{U}_{200}$, is perturbed with a small coherent velocity perturbation, $\delta\langle\hat{\mathbf{U}}\rangle$, as an initial condition. The ensemble flowfield is subsequently sampled in snapshots with a time interval Δt , where $\hat{\mathbf{u}}_i(t_j)$ represents the velocity state vector at time t_j from a perturbation eigenvector of $\delta\hat{\mathbf{U}}_i$.

Since the perturbation eigenvectors are derived from the linearised Navier-Stokes operator augmented with the data-fitted eddy viscosity, it would not lead to the optimal gain corresponding from an ensemble of turbulent flowfields. By simulating the response based on a set of orthogonal perturbation eigenvectors, an eigenvalue problem can be formulated to identify initial conditions that lead to a larger gain based on the superposition of the perturbation eigenvectors.

For a target time, t_j , due to the superposition property of a linear system, the ensemble velocity response at t_j is

$$\tilde{\mathbf{u}}(t_j) = \begin{bmatrix} \tilde{\mathbf{u}}_1(t_j) \\ \tilde{\mathbf{u}}_2(t_j) \\ \tilde{\mathbf{u}}_3(t_j) \\ \vdots \end{bmatrix} \mathbf{a} = \mathbf{X}_{\tilde{\mathbf{u}}}(t_j) \mathbf{a}. \quad (3.9)$$

Thus, an optimisation problem can be formulated as

$$\max_{\mathbf{a}} G(t_j) = \frac{\|\tilde{\mathbf{u}}(t_j)\|^2}{\|\tilde{\mathbf{u}}(t_0)\|^2} \quad \text{where} \quad \|\tilde{\mathbf{u}}(t_0)\|^2 = 1, \quad (3.10)$$

leading to an eigenvalue problem of the form

$$\mathbf{X}_{\tilde{\mathbf{u}}}^H(t_j) \mathbf{W} \mathbf{X}_{\tilde{\mathbf{u}}}(t_j) \mathbf{a} = \lambda \mathbf{X}_{\tilde{\mathbf{u}}}^H(t_0) \mathbf{W} \mathbf{X}_{\tilde{\mathbf{u}}}(t_0) \mathbf{a}. \quad (3.11)$$

3.3 Numerical Methods

The methodology is tested on plane channel flow, investigated is at $Re_\tau = 100$ (where $Re_\tau = \frac{u_\tau h}{\nu}$), with a domain of $(L_x, L_z) = (12.6h, 4.2h)$. With the simulation employing the Fourier-

Galerkin method for the periodic directions and central difference for the wall-normal directions, a mesh discretisation of $(N_x, N_y, N_z) = (64, 66, 64)$ is opted. To progress the simulation forward in time, an implicit-explicit time-stepping method of Runge-Kutta-Wray 3 and Crank-Nicholson is implemented. The temporal mean equilibrium profile generated by the direct numerical simulation is validated against the results of Bewley *et al.* (2001) with an ensemble of $\mathcal{U}_{200}$. Subsequently, the linear transient amplification dynamics is sampled from the same ensemble size of $\mathcal{U}_{200}$. The initial ensemble of 200 members is sampled from a single statistically stationary turbulent trajectory at $Re_\tau = 100$, spaced $\Delta t = 100 \frac{h}{U_l}$ apart, where U_l is the laminar centerline velocity at channel half height.

Since plane channel flow is homogeneous in the streamwise and spanwise directions, $\widehat{\mathbf{U}}(y; n_x, n_z)$ is defined as the Fourier velocity component for the respective plane Fourier mode with wavenumber (n_x, n_z) and expected to possess an linear operator described in terms of its own Fourier velocity component and decoupled from other Fourier velocity components of a different wavenumber. The wavenumber investigated in this study are selected to be from wavenumbers $(n_x, n_z) = (0, 2), (0, 4), (2, 2), (4, 4)$ as they describes the streamwise streaks and its meandering streak instability mode, with corresponding $(k_x h, k_z h) \simeq (0, 3), (0, 6), (1, 3), (2, 6)$. The inner length scale for $(k_x h, k_z h) \simeq (0, 6), (2, 6)$ fourier modes are $(\lambda_x^+, \lambda_z^+) \simeq (\infty, 100), (300, 100)$ respectively.

3.3.1 Linearity Validation

This data-driven linearisation methodology requires the $\delta\langle\widehat{\mathbf{U}}\rangle$ perturbation amplitude to be small for linearity to hold, while being large enough to have a good signal to noise ratio. A parametric study is carried out to validate the amplitude applied onto the $(k_x h, k_z h) \simeq (0, 3)$ mode for an ensemble perturbation of $\delta\widehat{\mathbf{U}}_1$. As seen by the gains, $G(t)$, in fig 3.1, the range of amplitudes lead to a consistent linear dynamics, especially in the amplification regime before the peak ($0 \frac{h}{U_l} < t < 15 \frac{h}{U_l}$). Subsequently, an amplitude of $A = 0.01$ is used as it ensure a good signal to noise ratio from the background turbulence.

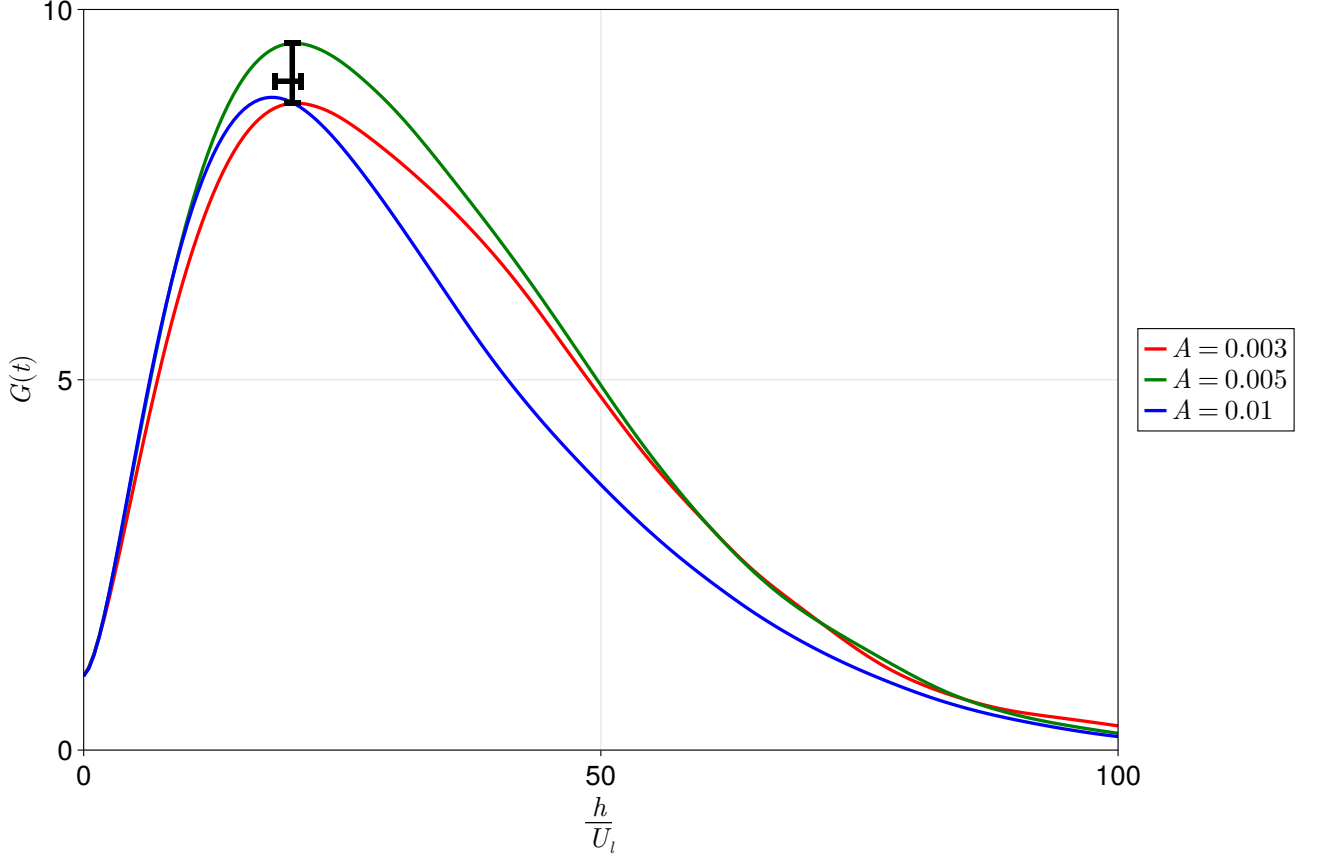


Figure 3.1: Transient gain dynamics for a range of ensemble perturbation amplitudes for $(k_x h, k_z h) \simeq (0, 3)$.

3.4 Evolution of Transient Amplification

The temporal evolution of the transient amplification dynamics for the perturbed wavenumbers are shown in this section, with the total gain, $G(t)$, decomposed into its stream-wise, wall-normal and span-wise energetic components, $G_{\hat{u}}(t)$, $G_{\hat{v}}(t)$ and $G_{\hat{w}}(t)$, respectively, and defined as

$$G_{\hat{u}}(t) = \frac{\|\hat{u}(t)\|^2}{\|\hat{\mathbf{u}}(t_0)\|^2} \quad G_{\hat{v}}(t) = \frac{\|\hat{v}(t)\|^2}{\|\hat{\mathbf{u}}(t_0)\|^2} \quad G_{\hat{w}}(t) = \frac{\|\hat{w}(t)\|^2}{\|\hat{\mathbf{u}}(t_0)\|^2}. \quad (3.12)$$

For the sake of clarity, the amplification response of the ensemble direct numerical simulation (eDNS) from the primary 4 perturbation eigenvectors are shown, in comparison to the LNS_v amplification response.

For the $(k_x h, k_z h) \simeq (0, 3)$ Fourier mode in fig 3.2, although the $\delta\langle\hat{\mathbf{U}}\rangle_1$ mode exhibit good correspondence between the data-driven eDNS and LNS_v model, the dominant dynamics of the lift-up effect are notably different as seen from the transient evolution of the wall-normal and span-wise velocities. It is observed that the wall-normal and span-wise velocities exhibit a slower energetic decay for the LNS_v model. Another interest note is on the evolution of the $\delta\langle\hat{\mathbf{U}}\rangle_3$ eigenvector, where a secondary transient amplification is observed for the eDNS model. To validate that the reason of the transient amplification is due to the non-normal nature of

the linear operator, a separate numerical experiment is done where the flowfield at $1.5 \frac{tu_\tau}{h}$ is extracted from the $\mathcal{U}_{200}$ ensemble and then insert to the LNS_ν model as an initial condition. A similar transient amplification is observed, validating the hypothesis that the secondary amplification is due to non-normal dynamics.

3.4.1 Fourier mode: $(k_x h, k_z h) \simeq (0, 3)$

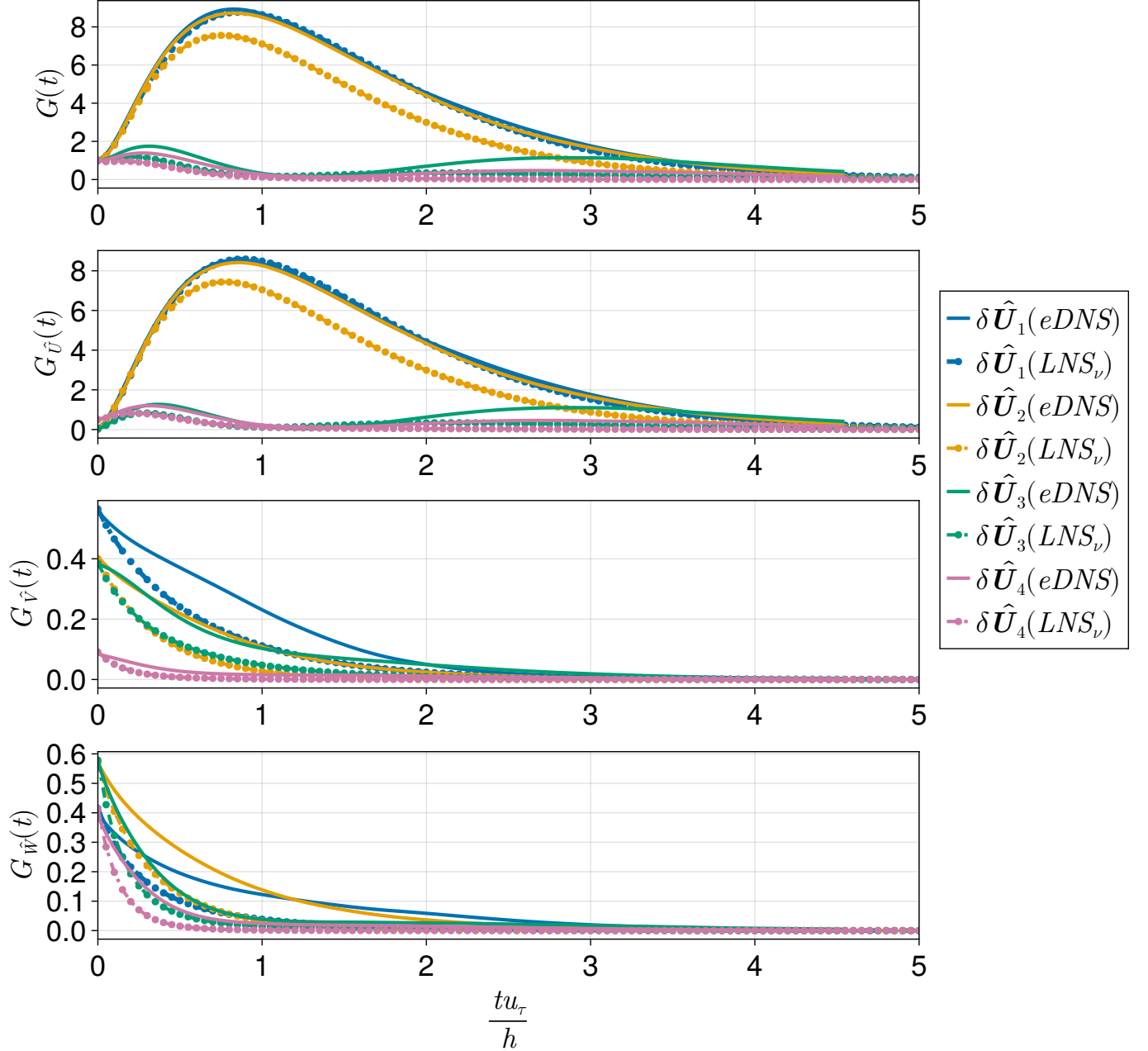


Figure 3.2: Transient gain dynamics for a range of ensemble perturbation eigenvectors for $(k_x h, k_z h) \simeq (0, 3)$.

3.4.2 Fourier mode: $(k_x h, k_z h) \simeq (0, 6)$

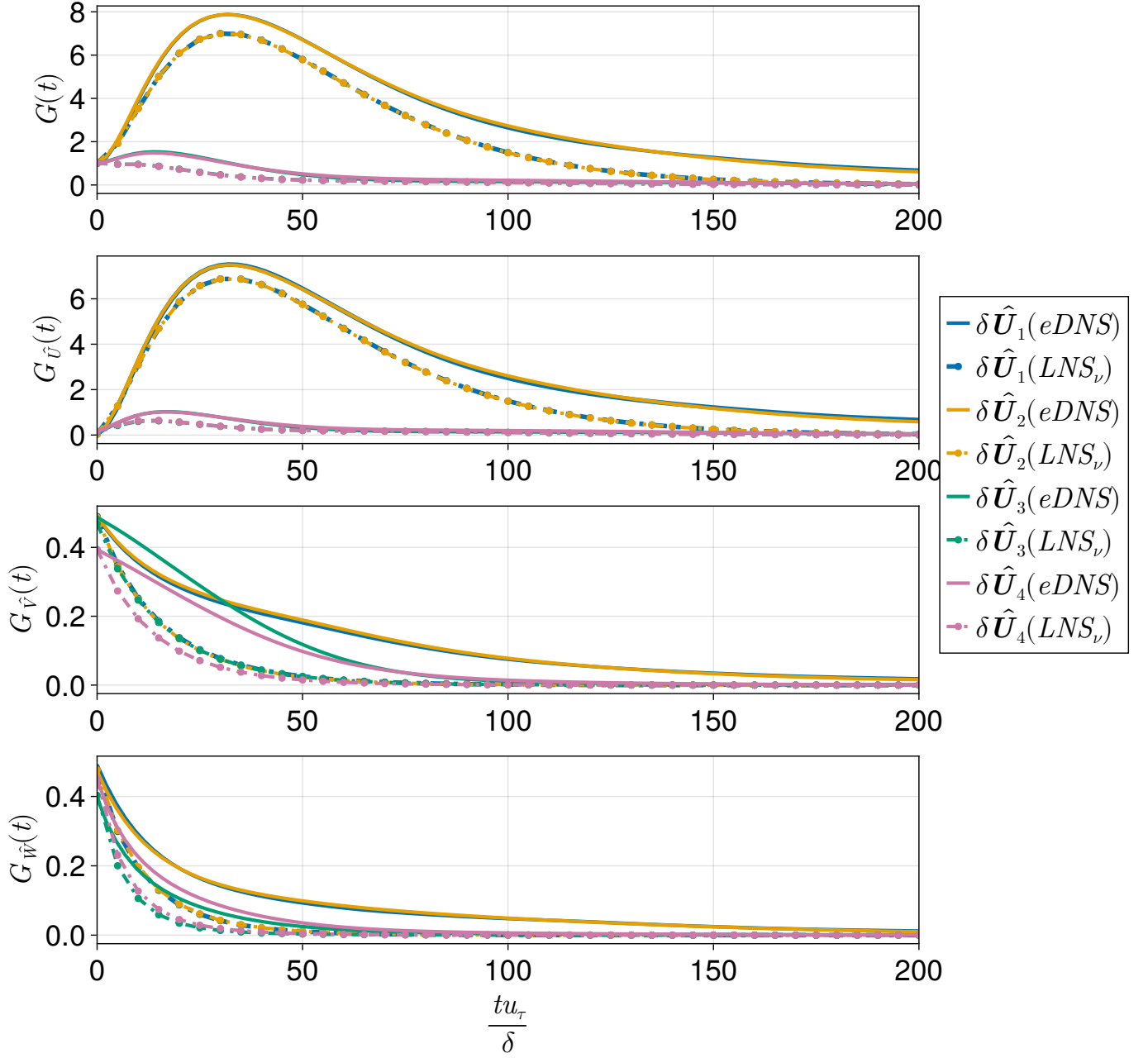


Figure 3.3: Transient gain dynamics for a range of ensemble perturbation amplitudes for $(k_x h, k_z h) \simeq (0, 6)$.

For the $(k_x h, k_z h) \simeq (0, 6)$ Fourier mode in fig 3.3, it is observed that the LNS $_\nu$ model generally exhibits a weaker amplification when compared to the eDNS model. The dominant dynamics of the lift-up effect are notably different as seen from the transient evolution of the wall-normal and span-wise velocities, and the wall-normal and span-wise velocities exhibit a slower energetic decay.

3.4.3 Fourier mode: $(k_x h, k_z h) \simeq (1, 3)$

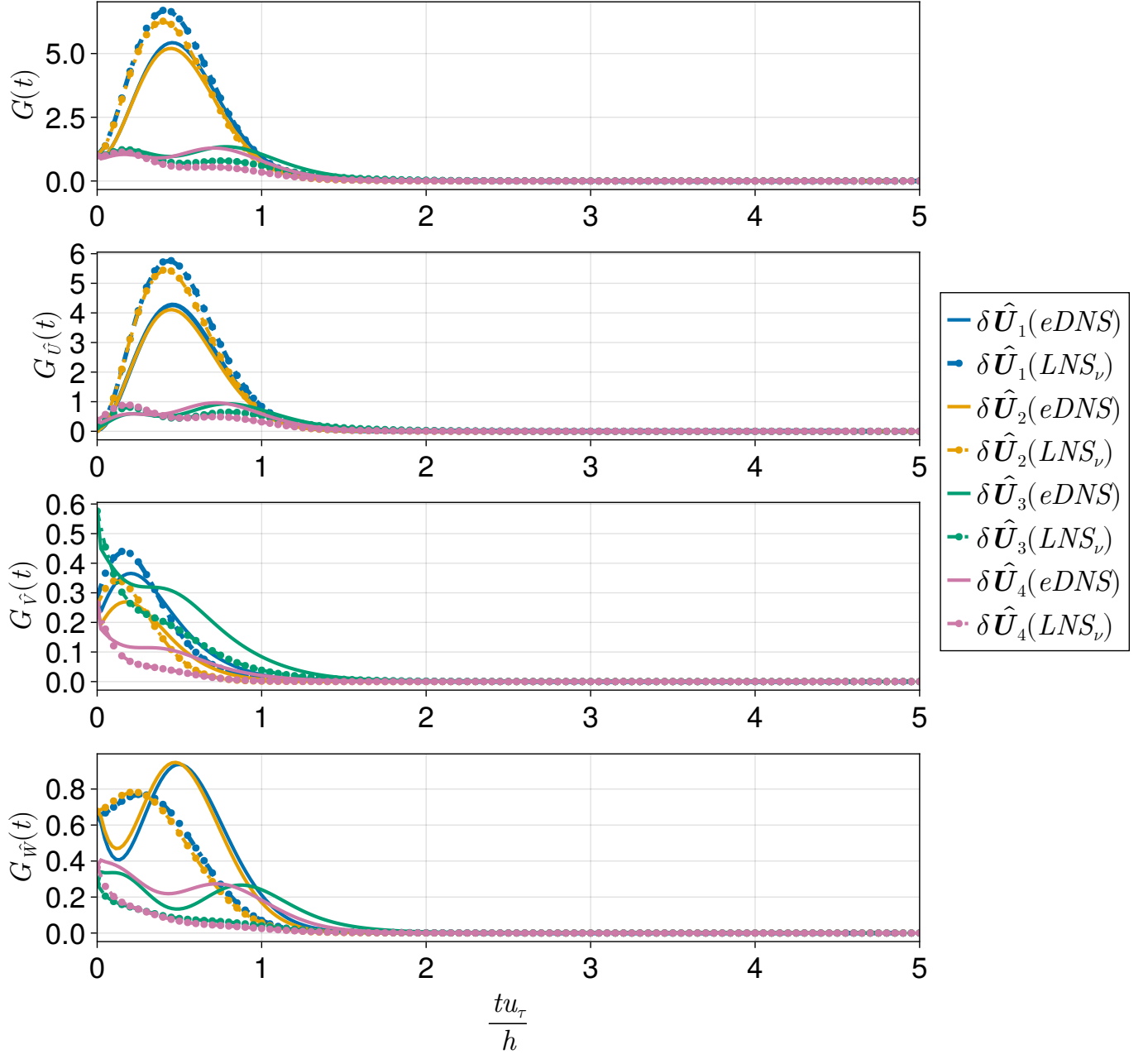


Figure 3.4: Transient gain dynamics for a range of ensemble perturbation amplitudes for $(k_x h, k_z h) \simeq (1, 3)$.

3.4.4 Fourier mode: $(k_x h, k_z h) \simeq (2, 6)$

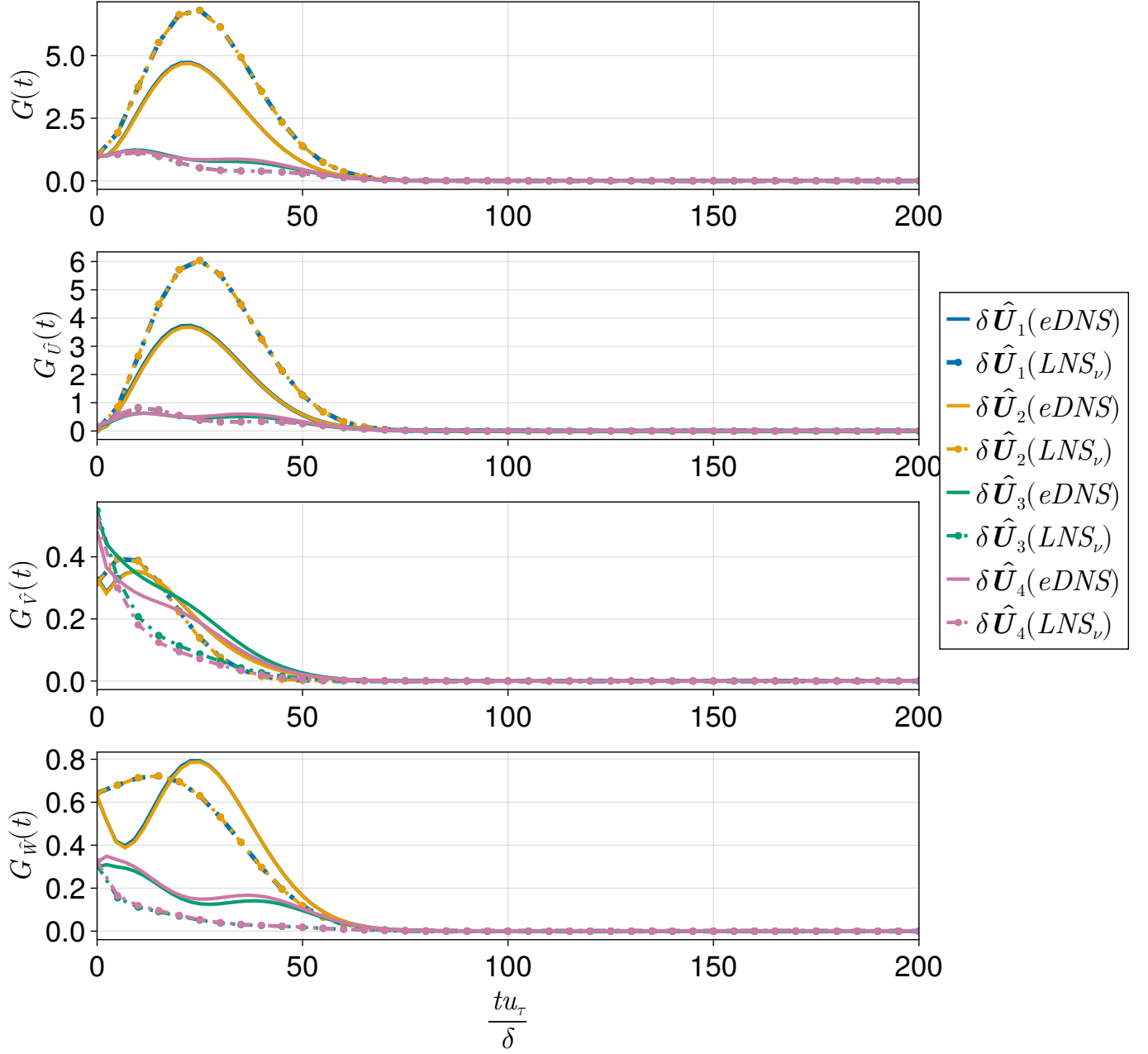


Figure 3.5: Transient gain dynamics for a range of ensemble perturbation amplitudes for $(k_x h, k_z h) \simeq (2, 6)$.

For the streak instability Fourier modes $[(k_x h, k_z h) \simeq (1, 3)]$ in fig. 3.4 and $(k_x h, k_z h) \simeq (2, 6)$ in fig. 3.5], in general, the eDNS dynamics exhibit a smaller magnitude of transient amplification when compared to the LNS_ν model dynamics. Traces of the Orr mechanism in play can be clearly for the primary two eigenvector perturbations as seen from the huge amplification in the stream-wise energetics, albeit with a smaller magnitude for the eDNS cases. Interestingly, the two primary eigenvector perturbations lead to a decay and then amplification on the span-wise

energetics that peaked higher than the initial condition, which is starkly different to the LNS_ν model dynamics.

3.5 *Optimal* Transient Amplification

With the dynamical trajectories of the statistical state from 6 of the perturbation eigenvector computed, the eigenvalue problem in eq.(3.11) can be solved for each Fourier mode.

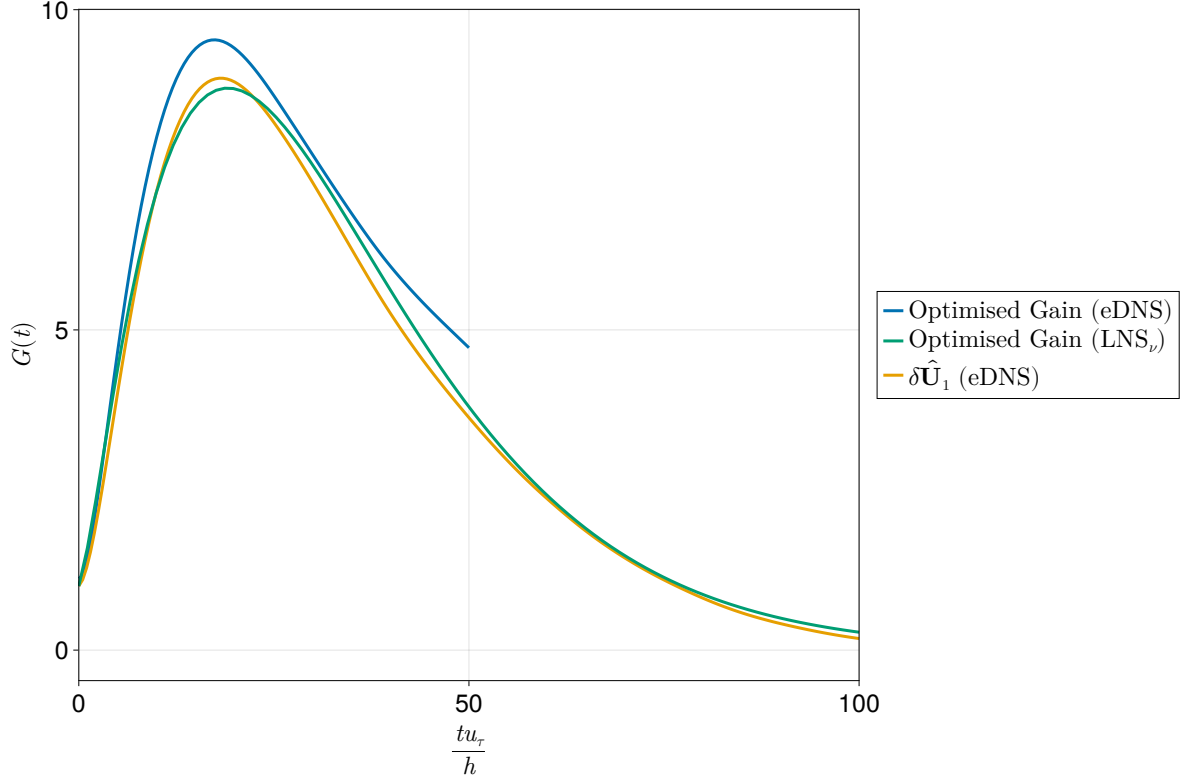


Figure 3.6: Transient gain dynamics for the *optimal* transient gain for $(k_x h, k_z h) \simeq (0, 3)$.

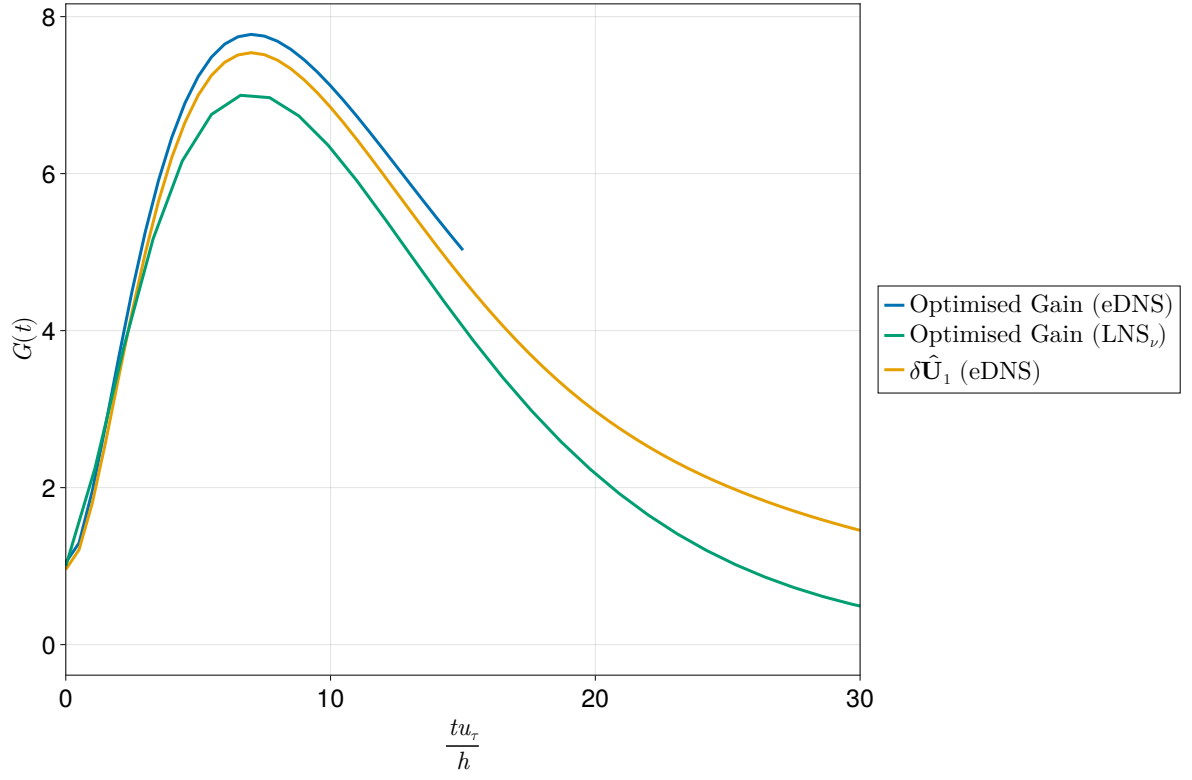


Figure 3.7: Transient gain dynamics for the *optimal* transient gain for $(k_x h, k_z h) \simeq (0, 6)$.

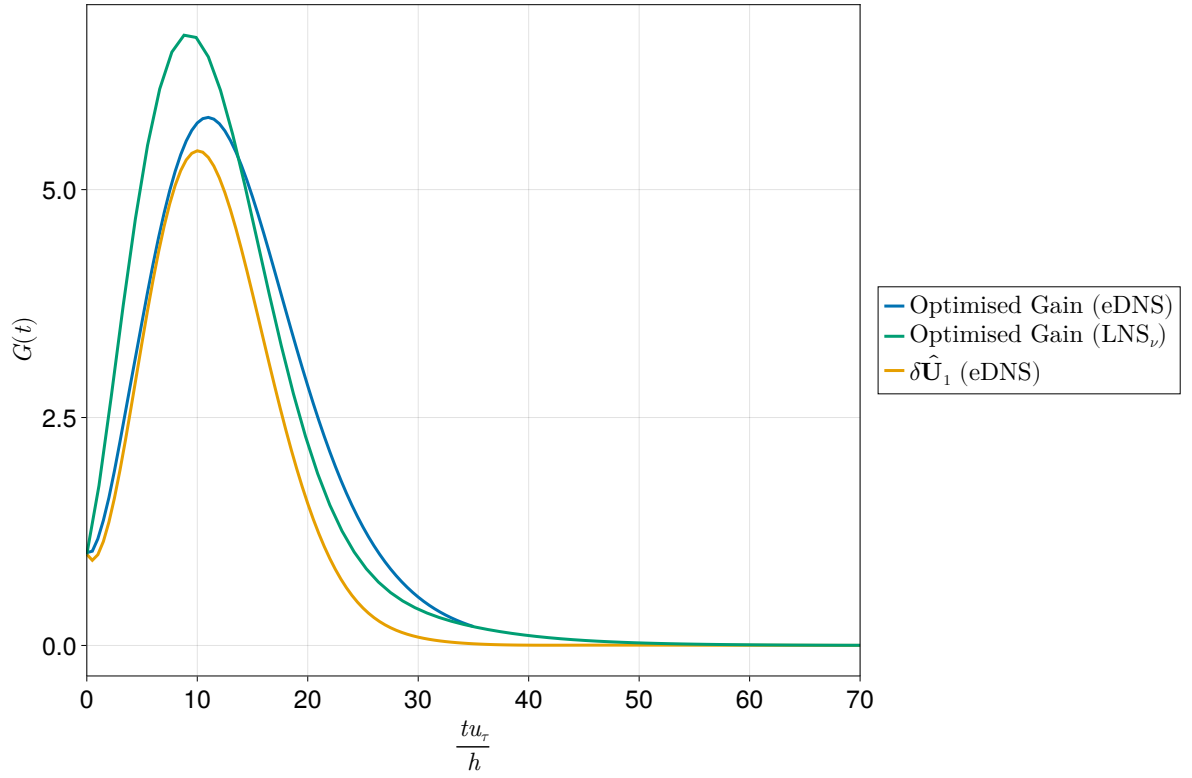


Figure 3.8: Transient gain dynamics for the *optimal* transient gain for $(k_x h, k_z h) \simeq (1, 3)$.

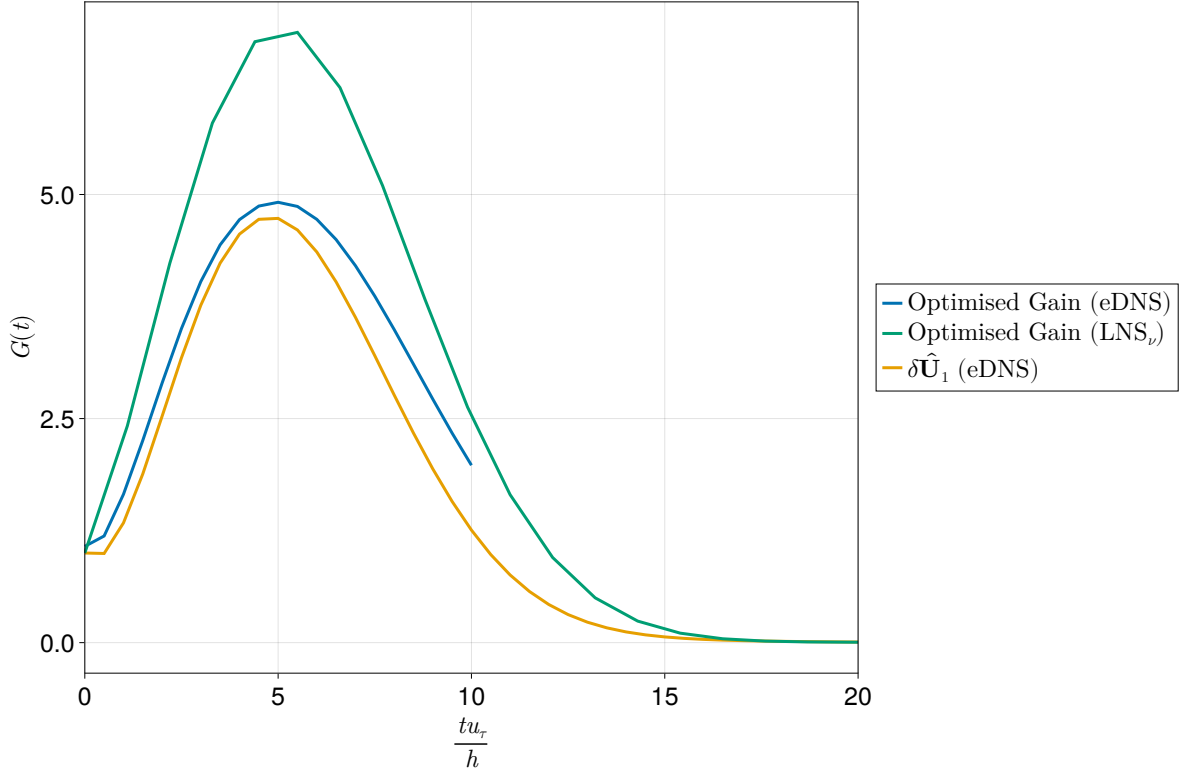


Figure 3.9: Transient gain dynamics for the *optimal* transient gain for $(k_x h, k_z h) \simeq (2, 6)$.

In general the optimised transient gain dynamics for eDNS, in fig. 3.6, fig. 3.7, fig. 3.8, and fig. 3.9, all found a larger optimal gain when compared to the transient gain dynamics from a perturbation of the primary perturbation eigenvector ($\delta \hat{\mathbf{U}}_1$) of an eDNS simulation. For the stream-wise homogeneous case, the optimised gain of an eDNS is found to be higher than an LNS_ν model. However, it is not the case for the streak-instability Fourier modes, where the optimised gain of an eDNS did not exceed the peak of the LNS_ν model.

Another observation is that the peaks of the optimised eDNS transient gains does not necessarily coincide with the time span of the $\delta \hat{\mathbf{U}}_1$ perturbation eigenvector, as seen in fig. 3.6 and fig. 3.8. The observation elucidate a deeper fact about the optimisation procedure in eq.(3.11). It hinges on the set of perturbation eigenvectors that was predicted to be optimal based on the eigenvalue optimisation procedure carried out with the LNS_ν model, eq.(3.7). It is expected to be computationally expensive to test the set of optimal perturbation eigenvectors for all ranges of time periods. This is admittedly the drawback of the current optimisation procedure and hence it is called *optimal* in a narrow sense, as it is based on only the set of perturbation eigenvectors predicted to have the largest transient gains based on the LNS_ν model.

3.6 Flow Structure of *Optimal* Transient Amplification

Following on from §3.5, the flow structure of the transient amplification's peak gain is illustrated and compared with the peak in the LNS_ν model. For the $(k_x h, k_z h) \simeq (0, 3)$ and $(k_x h, k_z h) \simeq (0, 6)$ Fourier modes, the peaks are illustrated with the wall-normal - span-wise

view, as it is expected that the lift-up effect is dominant, while the For the $(k_x h, k_z h) \simeq (1, 3)$ and $(k_x h, k_z h) \simeq (2, 6)$ Fourier modes are illustrated with the wall-normal - stream-wise view, to bring out features of the Orr mechanism.

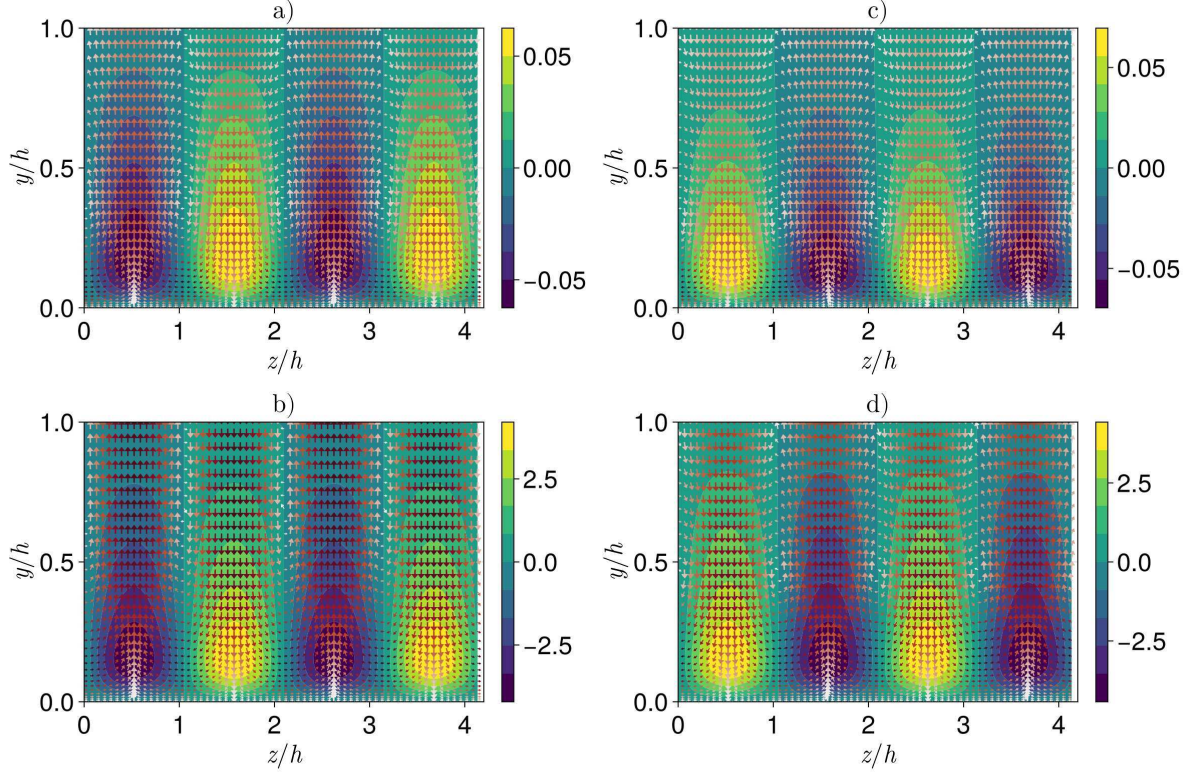


Figure 3.10: Flow contour for the *optimal* transient gain for $(k_x h, k_z h) \simeq (0, 3)$. The top row, (a, c), illustrates the initial condition of the flow field, while the bottom row, (b, d), illustrates the transient peak. The left column (a, b), are from the LNS_ν model, while the right column, (c, d), are from the eDNS simulation.

In general, by observing the stream-wise contours from the initial conditions in fig. 3.10, it is observed that the stream-wise velocity field for the eDNS simulation exacerbates a stronger lift-up effect when compared to the LNS_ν model. It suggests that vortices that fuel the lift-up effect is stronger for the eDNS case.

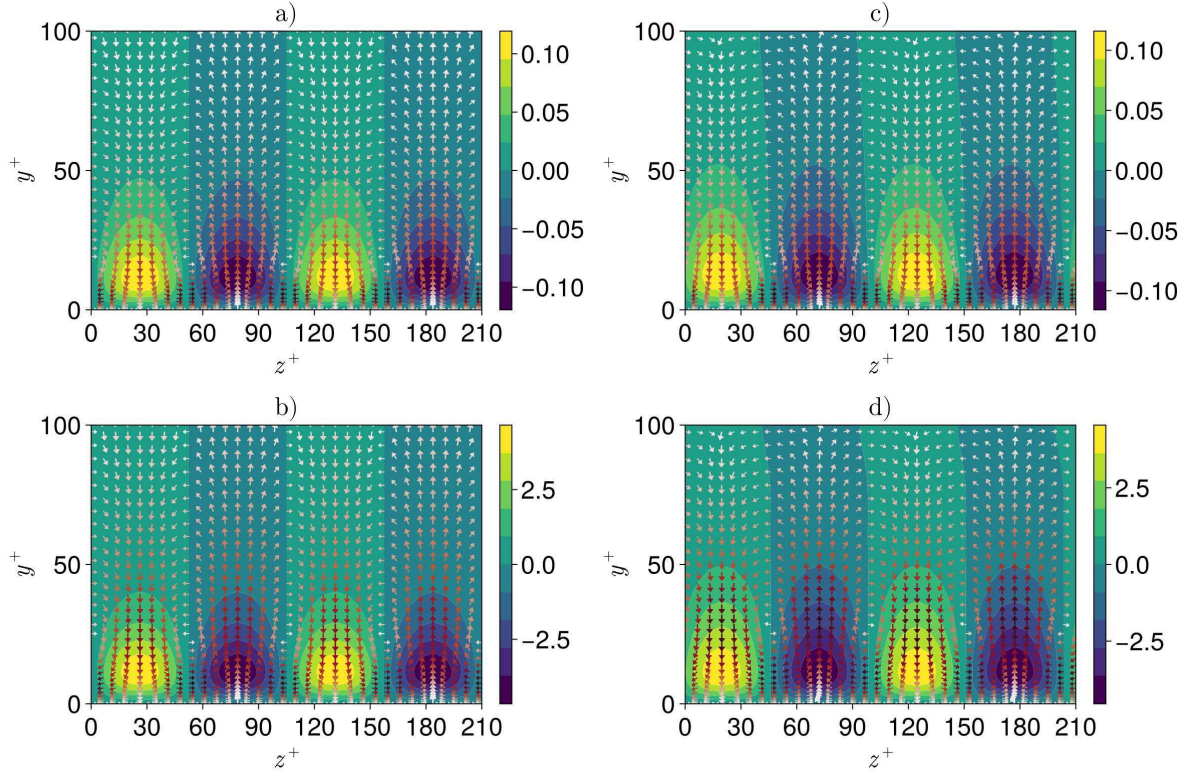


Figure 3.11: Flow contour for the *optimal* transient gain for $(k_x h, k_z h) \simeq (0, 6)$. The top row, (a, c), illustrates the initial condition of the flow field, while the bottom row, (b, d), illustrates the transient peak. The left column (a, b), are from the LNS_ν model, while the right column, (c, d), are from the eDNS simulation.

For the $(k_x h, k_z h) \simeq (0, 6)$ Fourier mode in fig. 3.11, it is observed that the stream-wise streaks are more elongated in the wall-normal direction when comparing the eDNS case to the LNS_ν model, suggesting that the eDNS simulation prefers a larger streak structure for triggering an optimal lift-up effect. Comparing the peaks, the streak structure in the eDNS simulation is comparatively larger than the LNS_ν model.

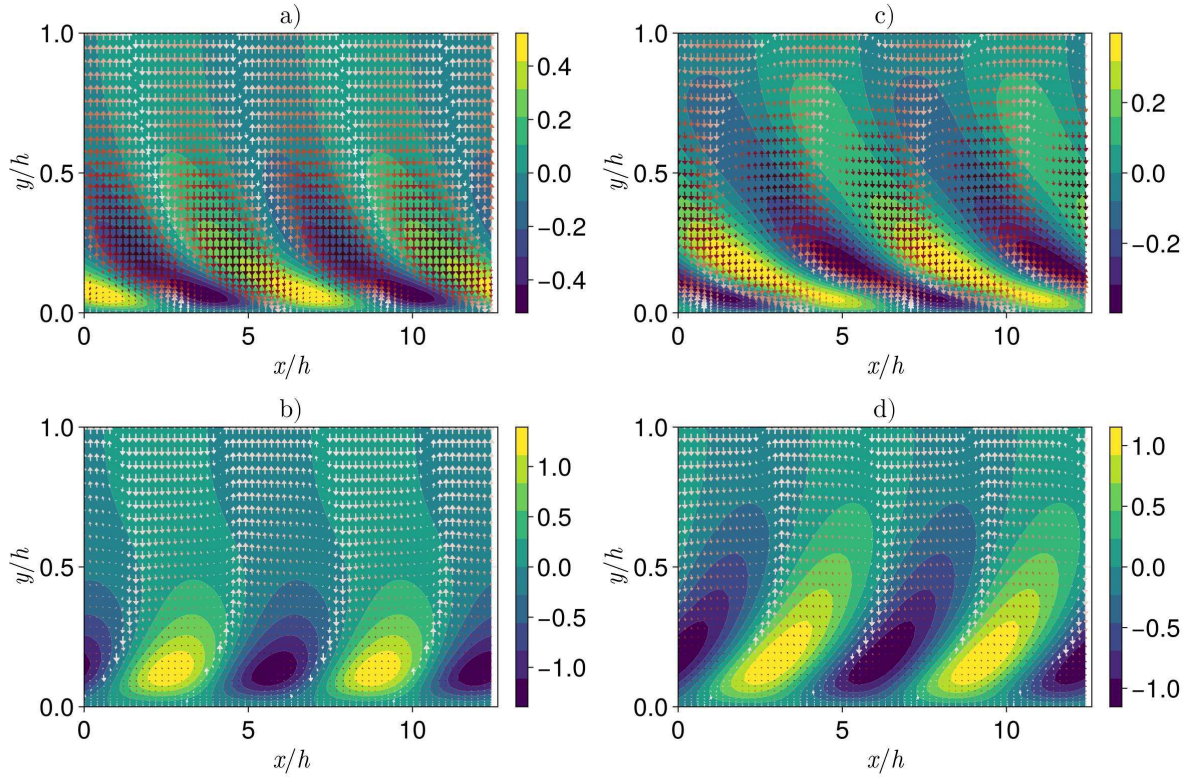


Figure 3.12: Flow contour for the *optimal* transient gain for $(k_x h, k_z h) \simeq (1, 3)$. The top row, (a, c), illustrates the initial condition of the flow field, while the bottom row, (b, d), illustrates the transient peak. The left column (a, b), are from the LNS_v model, while the right column, (c, d), are from the eDNS simulation.

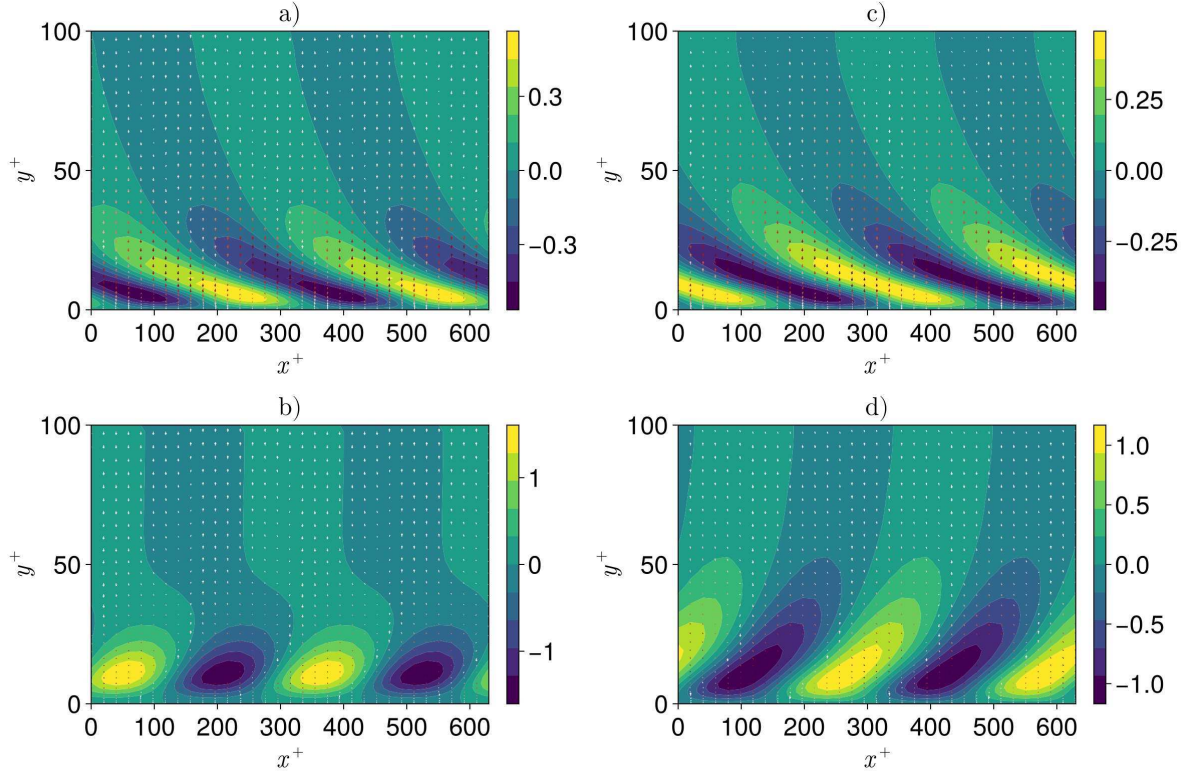


Figure 3.13: Flow contour for the *optimal* transient gain for $(k_x h, k_z h) \simeq (2, 6)$. The top row, (a, c), illustrates the initial condition of the flow field, while the bottom row, (b, d), illustrates the transient peak. The left column (a, b), are from the LNS_ν model, while the right column, (c, d), are from the eDNS simulation.

In fig. 3.12 and fig. 3.13, it is observed that the eDNS simulations have a preference for fingers more stretched in the wall-normal direction, covering a larger portion of the flow domain, which is more effectively in triggering the Orr mechanism. By doing so, the Orr mechanism manages to spread more effectively upwards as seen in the contours illustrating the peak gains. However, as shown in the previous section, (fig. 3.8 and fig. 3.9), the Orr mechanism has effectively a weaker effect in the eDNS simulation as the peak gains are lower than the LNS_ν model.

3.7 Reynolds Stresses Compared to Eddy Viscosity

Here, using data from the ensemble velocity fields, a construction of the Reynolds stresses, $\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$, is carried out. To ensure an equivalent comparison, the ensemble flow field's peak gain is taken as the reference field for each Fourier mode, and the rate of extracted energy of the divergence of the Reynolds stresses, $\langle \mathbf{U} \rangle \cdot [\nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle]$ is compared to the rate of extracted energy from the Cess's eddy viscosity model, $\langle \mathbf{U} \rangle \cdot [\nabla \cdot \boldsymbol{\nu}_t (\nabla \langle \mathbf{U} \rangle + \nabla \langle \mathbf{U} \rangle^T)]$, resulting in three components of the extracted energy. Since Fourier mode velocity fields are represented as complex numbers, only the magnitude of the extracted energy is shown as $\|C_u\|$, $\|C_v\|$, $\|C_w\|$ components. They represent the stream-wise, wall-normal and span-wise component respectively.

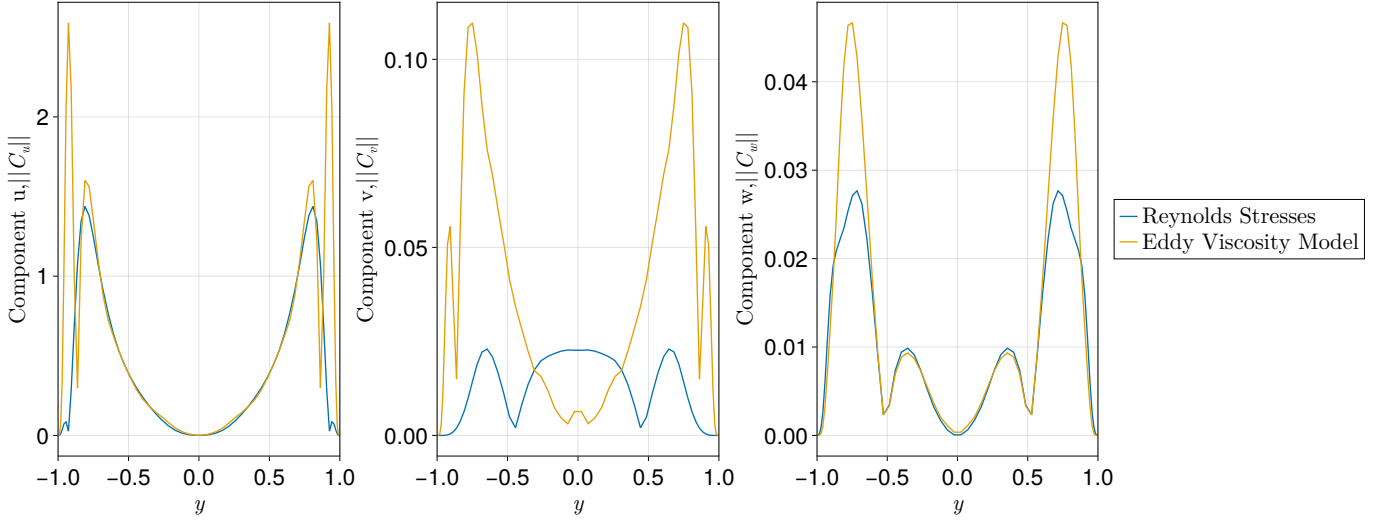


Figure 3.14: Energetic comparison of Reynolds Stresses and Cess eddy viscosity model for $(k_x h, k_z h) \simeq (0, 3)$

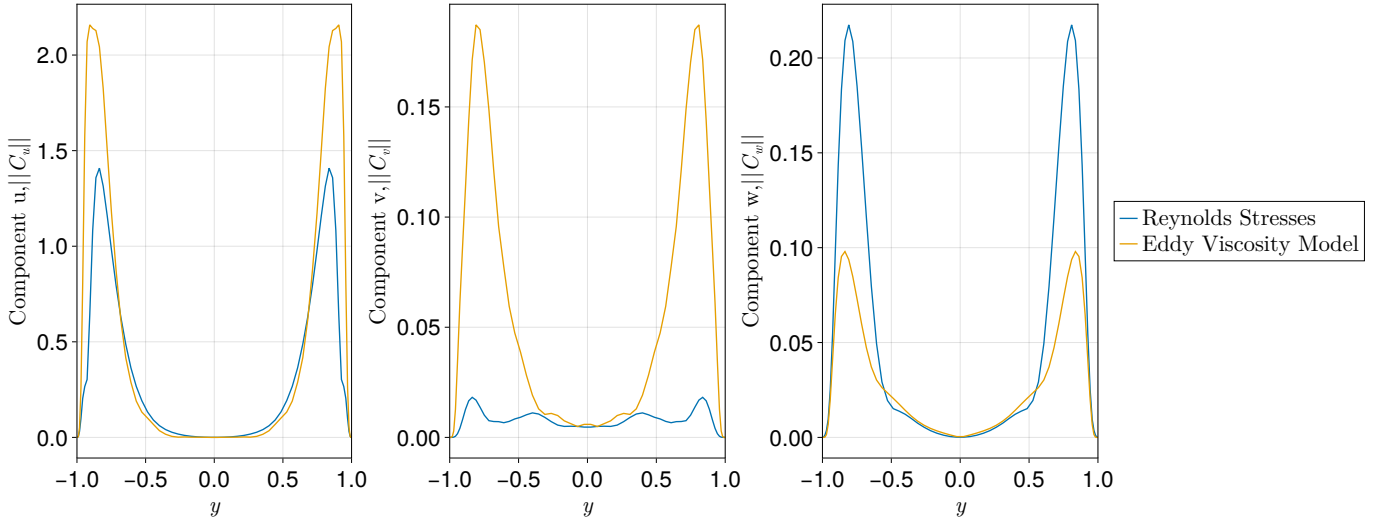


Figure 3.15: Energetic comparison of Reynolds Stresses and Cess eddy viscosity model for $(k_x h, k_z h) \simeq (1, 3)$

In general, the eddy viscosity model exhibits weak agreement when compared to the ground truth Reynolds stresses, especially near the wall regions, $-1 \leq y \leq -0.5$, and, $0.5 \leq y \leq 1$. For the $(k_x h, k_z h) \simeq (0, 3)$ Fourier mode, the eddy viscosity model shows good agreement to the effect of Reynolds stresses for the stream-wise and span-wise energetic extraction over the $-0.5 \leq y \leq 0.5$ region. A similar observation can be said for the $(k_x h, k_z h) \simeq (1, 3)$ Fourier wave too. However, for both cases, the wall-normal energetic extraction of the eddy viscosity model is deemed to be inaccurate. Another interesting fact can be seen from the centreline energetic extraction, where the stream-wise and span-wise extractions based on the Reynolds stresses are essentially zero for both Fourier modes. However, it is not the case for the wall-normal components.

Chapter 4

Conclusion

4.1 Summary

In this thesis, the statistical state dynamics of the Navier-Stokes equations on the geometry of the plane channel flow is investigated, where the Reynolds stresses have an effect of augmenting the dynamics of the statistical states which is non-trivial and differs significantly from an eddy viscosity model. Initial theoretical investigations to obtain a descriptive linear operator encompassing the Reynolds stress dynamics suggests that the number of transient growth numerical experiments would need to scale linearly with the degrees of freedom in the wall normal direction for each Fourier modes. The approach is then forgone. Instead, transient growth numerical experiments were applied to identify the differing dynamics from a LNS_ν operator formulated as in Hwang & Cossu (2010a).

Although the optimisation procedure in eq.(3.11) leads to a higher energetic peak in transient growth by combining the various perturbation transient growth response in a linear superposition, the methodology is not conclusive as the possible perturbation vector is vast due to multiple sets of orthonormal eigenvectors by selecting differing optimal time horizon in the transient gain eigenvalue problem based on the LNS_ν model in eq.(3.7).

The differing discrepancies in the comparison of eDNS dynamics and the LNS_ν model (with the Fourier modes which have a strong lift-up mechanism having a higher peak versus streak-instability Fourier modes having a lower peak when compared to LNS_ν model), suggests that there's no one magnitude of eddy viscosity that can encompass all cases of flow dynamics in the flow and should be varied accordingly to the Fourier modes. In future, these type of investigations can lead to a ground truth reference for the modelling of the Reynolds stresses.

4.2 Outlook

In statistical state dynamics, it is assumed that the statistical states will return to the temporal mean equilibrium statistical state once the effect of a perturbation is removed. This is enforced by the fact that the linearisation of the statistical state of the dynamical system possesses stable eigenvalues, especially for the mean statistical state of the turbulent channel flow. However, this is not the case for all statistical dynamical system.

Here, the state equations of the Lorenz system is introduced as

$$\partial_t \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\sigma x + \sigma y \\ \rho x - y - xz \\ -\beta z + xy \end{bmatrix}, \quad (4.1)$$

while the equations for the ensemble mean dynamics is noted as

$$\partial_t \left\langle \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\rangle = \begin{bmatrix} -\sigma \langle x \rangle + \sigma \langle y \rangle \\ \rho \langle x \rangle - \langle y \rangle - \langle x \rangle \langle z \rangle \\ -\beta \langle z \rangle + \langle x \rangle \langle y \rangle \end{bmatrix} + \begin{bmatrix} 0 \\ -\langle x' z' \rangle \\ \langle x' y' \rangle \end{bmatrix}, \quad (4.2)$$

with the turbulent fluctuations described by the following equations,

$$\partial_t \begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} -\sigma x' + \sigma y' \\ \rho x' - y' - x' z' \\ -\beta z' + x' y' \end{bmatrix} + \begin{bmatrix} 0 \\ \langle x' z' \rangle \\ -\langle x' y' \rangle \end{bmatrix} + \begin{bmatrix} 0 \\ -\langle z \rangle x' - \langle x \rangle z' \\ \langle y \rangle x' + \langle x \rangle y' \end{bmatrix}. \quad (4.3)$$

For the typical Lorenz system parameter of $\sigma = 10.0$, $\rho = 28.0$ and $\beta = \frac{8}{3}$, the statistical mean, by sampling 5×10^6 instances of the Lorenz system at $\Delta t = 0.1$ apart, is computed to be $\langle x, y, z \rangle^T = [-0.002, -0.002, 23.55]^T$, with eigenvalues $\lambda = -13.5, -2.67, 2.55$ respectively. Although, it is obvious that the statistical state will revert back to the temporal mean statistical equilibrium after a perturbation ends, it highlights strongly that the effect is solely due to the *Reynolds stress* terms,

$$\begin{bmatrix} 0 \\ -\langle x' z' \rangle \\ \langle x' y' \rangle \end{bmatrix}. \quad (4.4)$$

Furthermore, with each instance of the turbulent fluctuating state equations coupled to the ensemble mean states via

$$\begin{bmatrix} 0 \\ -\langle z \rangle x' - \langle x \rangle z' \\ \langle y \rangle x' + \langle x \rangle y' \end{bmatrix}.$$

It suggests that the statistical state dynamics equations are deeply coupled between the ensemble mean states and the individual turbulent fluctuation states. It raises the question that if it is a practical venture to still model the ensemble mean dynamics solely by augmenting the linearised Navier-Stokes equations with an eddy viscosity model as an eddy viscosity model equations are only based on the ensemble mean states.

References

- BEWLEY, THOMAS R., MOIN, PARVIZ & TEMAM, ROGER 2001 DNS-based predictive control of turbulence: an optimal benchmark for feedback algorithms. *Journal of Fluid Mechanics* **447**, 179–225, publisher: Cambridge University Press.
- DAVIES, S. J. & WHITE, C. M. 1928 An experimental study of the flow of water in pipes of rectangular section. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **119** (781), 92–107, publisher: The Royal Society.
- FARREL, BRIAN F. & IOANNOU, PETROS J. 2019 Statistical State Dynamics: A New Perspective on Turbulence in Shear Flow. In *Zonal Jets: Phenomenology, Genesis, and Physics*, pp. 380–400. Cambridge University Press.
- FARRELL, BRIAN F. 1988 Optimal excitation of perturbations in viscous shear flow. *The Physics of Fluids* **31** (8), 2093–2102, publisher: AIP Publishing.
- FARRELL, B. F., GAYME, D. F. & IOANNOU, P. J. 2017*a* A statistical state dynamics approach to wall turbulence. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* Publisher: The Royal Society Publishing.
- FARRELL, BRIAN F. & IOANNOU, PETROS J. 2009 A Theory of Baroclinic Turbulence. *Journal of the Atmospheric Sciences* **66** (8), 2444–2454, publisher: American Meteorological Society Section: Journal of the Atmospheric Sciences.
- FARRELL, BRIAN F. & IOANNOU, PETROS J. 2012 Dynamics of streamwise rolls and streaks in turbulent wall-bounded shear flow. *Journal of Fluid Mechanics* **708**, 149–196, publisher: Cambridge University Press.
- FARRELL, BRIAN F., IOANNOU, PETROS J. & NIKOLAIDIS, MARIOS-ANDREAS 2017*b* Instability of the roll-streak structure induced by background turbulence in pretransitional Couette flow. *Physical Review Fluids* **2** (3), 034607, publisher: American Physical Society.
- HAMILTON, JAMES M., KIM, JOHN & WALEFFE, FABIAN 1995 Regeneration mechanisms of near-wall turbulence structures. *Journal of Fluid Mechanics* **287**, 317–348, publisher: Cambridge University Press.
- HOPF, EBERHARD 1952 Statistical Hydromechanics and Functional Calculus. *Indiana University Mathematics Journal* **1** (1), 87–123.
- HUSSAIN, A. K. M. F. & REYNOLDS, W. C. 1970 The mechanics of an organized wave in turbulent shear flow. *Journal of Fluid Mechanics* **41** (2), 241–258, publisher: Cambridge University Press.

- HWANG, YONGYUN & COSSU, CARLO 2010a Amplification of coherent streaks in the turbulent Couette flow: an input–output analysis at low Reynolds number. *Journal of Fluid Mechanics* **643**, 333–348.
- HWANG, YONGYUN & COSSU, CARLO 2010b Linear non-normal energy amplification of harmonic and stochastic forcing in the turbulent channel flow. *Journal of Fluid Mechanics* **664**, 51–73, publisher: Cambridge University Press.
- IYER, A. S., WITHERDEN, F. D., CHERNYSHENKO, S. I. & VINCENT, P. E. 2019 Identifying eigenmodes of averaged small-amplitude perturbations to turbulent channel flow. *Journal of Fluid Mechanics* **875**, 758–780, publisher: Cambridge University Press.
- KASHYAP, PAVAN V., DUGUET, YOHANN & DAUCHOT, OLIVIER 2022 Linear Instability of Turbulent Channel Flow. *Physical Review Letters* **129** (24), 244501, publisher: American Physical Society.
- KASHYAP, P. V., DUGUET, Y. & DAUCHOT, O. 2024 Linear stability of turbulent channel flow with one-point closure. *Physical Review Fluids* **9** (6), 063906, publisher: American Physical Society.
- LORENZ, EDWARD N. 1963 Deterministic Nonperiodic Flow Section: Journal of the Atmospheric Sciences.
- MOHAN, PRAKASH, FITZSIMMONS, NICHOLAS & MOSER, ROBERT D. 2017 Scaling of Lyapunov exponents in homogeneous isotropic turbulence. *Physical Review Fluids* **2** (11), 114606, publisher: American Physical Society.
- ORR, W. M. F 1907 The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. Part I: A perfect liquid. Part II: A viscous liquid. *Proceedings of the Royal Irish Academy* **29** (9), 68.
- ORSZAG, STEVEN A. 1971 Accurate solution of the Orr–Sommerfeld stability equation. *Journal of Fluid Mechanics* **50** (4), 689–703, publisher: Cambridge University Press.
- PATEL, V. C. & HEAD, M. R. 1969 Some observations on skin friction and velocity profiles in fully developed pipe and channel flows | Journal of Fluid Mechanics. *Cambridge Core* .
- PUJALS, GREGORY, GARCÍA-VILLALBA, MANUEL, COSSU, CARLO & DEPARDON, SEBASTIEN 2009 A note on optimal transient growth in turbulent channel flows. *Physics of Fluids* **21** (1), publisher: AIP Publishing.
- REDDY, SATISH C. & HENNINGSON, DAN S. 1993 Energy growth in viscous channel flows. *Journal of Fluid Mechanics* **252**, 209–238, publisher: Cambridge University Press.
- REDDY, SATISH C., SCHMID, PETER J. & HENNINGSON, DAN S. 2006 Pseudospectra of the Orr–Sommerfeld Operator. *SIAM Journal on Applied Mathematics* Publisher: Society for Industrial and Applied Mathematics.
- RUELLE, DAVID 1979 Microscopic fluctuations and turbulence. *Physics Letters A* **72** (2), 81–82, publisher: North-Holland.

- SCHLICHTING, HERMAN 1933 Berechnung der Anfachung kleiner Störungen bei der Plattenströmung **13**, 50.
- SCHMID, PETER J. 2007 Nonmodal Stability Theory. *Annual Review of Fluid Mechanics* **39** (1), 129–162, eprint: <https://doi.org/10.1146/annurev.fluid.38.050304.092139>.
- SCHMID, PETER J. & BRANDT, LUCA 2014 Analysis of Fluid Systems: Stability, Receptivity, Sensitivity. *Applied Mechanics Reviews* **66** (2), 024803.
- SOMMERFELD, A. 1908 Ein Beitrag zur hydrodynamischen Erklärung der turbulenten Flüssigkeitsbewegungen. In *Congr. Internat. dei Mat. III*, , vol. 4, pp. 116–124.
- SQUIRE, HERBERT BRIAN 1933 On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **142** (847), 621–628, publisher: The Royal Society.
- TOLLMIEH, W. 1930 Über die Entstehung der Turbulenz. In *Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete*, pp. 18–21. Springer, Berlin, Heidelberg.
- TREFETHEN, LLOYD N., TREFETHEN, ANNE E., REDDY, SATISH C. & DRISCOLL, TOBIN A. 1993 Hydrodynamic Stability Without Eigenvalues. *Science* **261** (5121), 578–584.
- WALEFFE, FABIAN, KIM, JOHN & HAMILTON, JAMES M. 1993 On the Origin of Streaks in Turbulent Shear Flows. In *Turbulent Shear Flows 8*, pp. 37–49. Springer, Berlin, Heidelberg.