PHILOSOPHICAL TRANSACTIONS A

royalsocietypublishing.org/journal/rsta

Research

Cite this article: Erickson CW, Leung R, Stelle KS. 2022 Higgs effect without lunch. *Phil. Trans. R. Soc. A* **380**: 20210184.
https://doi.org/10.1098/rsta.2021.0184

Received: 13 January 2022
Accepted: 23 February 2022

One contribution of 12 to a theme issue 'The future of mathematical cosmology, Volume 2'.

Subject Areas:
gauge theory, mathematical physics, particle physics

Keywords:
dimensional reduction, effective field theory, gauge symmetry breaking

Author for correspondence:
K. S. Stelle
e-mail: k.stelle@imperial.ac.uk

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

Higgs effect without lunch

C. W. Erickson, Rahim Leung and K. S. Stelle

The Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2AZ, UK

RL, 0000-0002-0537-7486; KSS, 0000-0002-9779-995X

Reduction in effective space–time dimensionality can occur in field-theory models more general than the widely studied dimensional reductions based on technically consistent truncations. Situations where wave function factors depend non-trivially on coordinates transverse to the effective lower dimension can give rise to unusual patterns of gauge symmetry breaking. Leading-order gauge modes can be left massless, but naturally occurring Stueckelberg modes can couple importantly at quartic order and higher, thus generating a 'covert' pattern of gauge symmetry breaking. Such a situation is illustrated in a five-dimensional model of scalar electrodynamics in which one spatial dimension is taken to be an interval with Dirichlet/Robin boundary conditions on opposing ends. The Stueckelberg mode remains in the theory as a propagating scalar degree of freedom from a dimensionally reduced perspective, but it is not 'eaten' in a mass-generating mechanism. At leading order, it also makes no contribution to the conserved energy; for this reason, it may be called a (non-ghost) 'phantom'. This simple model illuminates a mechanism which also has been found in gravitational braneworld scenarios.

This article is part of the theme issue 'The future of mathematical cosmology, Volume 2'.

1. Vacuum structure and the Higgs mechanism

This article is about some of the unusual symmetry-breaking effects that can occur in systems whose low-energy effective dynamics is best described in a lowered space–time dimensionality.

The phenomenon of symmetry breaking triggered by a non-symmetric vacuum in a theory with a continuous

© 2022 The Authors. Published by the Royal Society under the terms of the Creative Commons Attribution License http://creativecommons.org/licenses/by/4.0/, which permits unrestricted use, provided the original author and source are credited.

THE ROYAL SOCIETY PUBLISHING

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

local symmetry is a cornerstone of the Standard Model's unification of the weak and electromagnetic interactions [1–4], as well as of candidate unifying extensions to include the strong interactions (e.g. [5]). In a now-traditional description of this phenomenon, a feast occurs: the symmetry orbit of vacuum states, which in a rigidly symmetric theory would give rise to Goldstone modes, gives rise in a locally symmetric gauge theory to masses for the gauge fields corresponding to the broken symmetry generators. The broken-generator gauge fields thus 'eat' the corresponding Goldstone modes, and become heavy.

Symmetry breaking can also occur in the context of a Kaluza–Klein dimensional reduction, either in a gravitational or a non-gravitational model. In a traditional reduction scenario with a compact extra space–time dimension or set of dimensions, fields are restricted to be independent of the coordinates parameterizing the extra dimensions. One speaks of a 'consistent truncation' in cases where such a restriction may be made fully compatible with the set of higher-dimensional field equations, so that general solutions to the restricted lower-dimensional theory can be 'lifted' (or, some say, 'oxidized') to solutions of the higher-dimensional theory. Such situations are mathematically interesting, and they have been the subject of considerable attention in the field-theory literature. A generalization of the traditional Kaluza–Klein construction allows for a limited form of dependence on the reduction space coordinates, while still preserving the truncation consistency—known as the Scherk–Schwarz mechanism [6]. In such a case, fields that always appear differentiated in the original higher-dimensional action can be allowed to have a linear dependence on a reduction coordinate, so their derivatives (or, e.g. for gauge fields, field strengths) remain constant over the reduction space.

Despite the seemingly virtuous terminology, 'consistent truncations' are not physically essential for dimensional reductions. What is more essential is that one be able to make sense of the effective lower-dimensional theory, at least within an appropriate range of length or energy scales. Interactions with higher modes which could simply be set to zero in a consistent truncation can, in a non-consistent case, give rise to small corrections to the leading-order effective theory, e.g. of a short-range Yukawa form for massive-field exchanges, or of higher-derivative form but suppressed by inverse powers of higher-mode masses [7].

Here, the focus will be on another type of symmetry breaking, in which the vacuum structure does not generate masses for gauge fields in the lower dimension, but in which local symmetry can be broken in a more surreptitious or 'covert' way—showing up mainly in the numerical values of interaction-term coefficients at higher orders in a perturbative expansion of the effective theory—typically at fourth order in the action, and higher [8].

A model where this can be seen [9] is in the effective theory for a Type IIA supergravity reduction from ten space–time dimensions to four, involving the non-compact space $\mathcal{H}^{(2,2)}$ [10]. Although not yielding a consistent truncation down to $D = 4$, an effective four-dimensional theory of supergravity nonetheless emerges thanks to the particular structure of the transverse-space differential equations in a separation-of-variables construction for solutions to the full $D = 10$ theory.

When cast into the form of a Schrödinger equation, that transverse problem has a Pöschl–Teller integrable structure [11]. This Schrödinger problem generates a transverse-mode spectrum with a single zero-eigenvalue bound state, then a gap, and then a continuum of scattering states. The Schrödinger-problem eigenvalues translate to mass levels in the lower-dimensional effective theory. It is the existence of the mass gap that allows for a sensible interpretation as a lower-dimensional theory, at least for energies below the scale of the mass gap. The 'vacuum' solution in this sector of Type IIA supergravity is a lift to $D = 10$ of the vacuum solution of a six-dimensional model constructed in 1984 by Salam & Sezgin [12], a solution having the space–time structure of $D = 4$ Minkowski space–time times an S^2 two-sphere. An effective $D = 4$ supergravity theory then arises (at leading order) for massless fluctuations about this vacuum background, fluctuations incorporating the Pöschl–Teller zero mode in the transverse part of the wave function.

The happy feature of integrability for that transverse Schrödinger problem also allows for the explicit calculation of transverse-space integrals giving the coefficients of interaction terms involving the massless modes of the $D = 4$ effective theory. Then a funny thing happens: although

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

the $D=4$ effective theory has the massless spectrum of a supergravity theory, beginning in the action at quadratic order with a Fierz–Pauli spin-two Lagrangian, the interaction coefficients at fourth order and higher in fields turn out not to have the values one expects for a generally covariant theory in an order-by-order field expansion [13,14].

This 'funny numbers' phenomenon is the defining feature of covert symmetry breaking in an effective theory where the lower-dimensional gauge or gravitational fields remain otherwise massless: a Higgs effect without lunch. A full exploration of how this works in the Type IIA model of [9,10,12] becomes quite involved. However, simpler toy models of the phenomenon can be constructed by declaring non-standard boundary conditions for fields in the transverse space dimensions. This was done for a simple illustrative Maxwell-plus-scalar model in [15], which shows the essential features of such a covert symmetry breaking scenario, and to which we next turn. In reviewing [15], we will expand on and clarify a key issue surrounding the number of propagating degrees of freedom in the effective theory. In particular, we will shine a light on the existence of a 'phantom'—a *non*-ghostly scalar that only contributes to the energy at the interacting level.

2. Massless effective theory in a non-standard reduction: $D=5$ Dirichlet–Robin electrodynamics

(a) Maxwell action, boundary conditions and gauge symmetries

Instead of the technically rather involved Type IIA supergravity embedding of the Salam–Sezgin model, one can create an illustrative dimensional reduction model in which fluctuation fields need to have a non-trivial dependence on an extra 'transverse' coordinate. Since this will not generate a consistent reduction in the technical sense, the dimensional reduction will need to be understood in the sense of a low-energy effective theory where lower-dimensional dynamics is dominant.

Start with Maxwell theory in a five-dimensional space–time with structure

$$\mathcal{M}_5 = \mathbb{R}^{1,3} \times \mathbb{I}, \tag{2.1}$$

in which the intention will be to choose non-standard boundary conditions on fields at the $z=1$ end of the $\mathbb{I}=[0,\ 1]$ interval. The $D=5$ space–time metric is

$$ds_5{}^2 = \eta_{MN}\, \mathrm{d}X^M\, \mathrm{d}X^N = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + \mathrm{d}z^2, \tag{2.2}$$

denoting the reduction coordinate by z and the $d=4$ reduced space–time coordinates by x^μ.

The Maxwell gauge fields $A_\mu(x,z), A_z(x,z)$ will satisfy standard $D=5$ equations of motion and the A_μ components will be assigned standard Dirichlet boundary conditions at the $z=0$ end of the $\mathbb{I}=[0,\ 1]$ interval, but at the $z=1$ end they will be assigned Robin boundary conditions. The Dirichlet/Robin boundary conditions chosen for A_μ are accordingly

$$A_\mu(x,0)=0 \quad \text{and} \quad (\partial_z - 1)A_\mu(x,1)=0. \tag{2.3}$$

For the A_z component of the $D=5$ gauge field, it will prove to be necessary for its transverse-space derivative $\partial_z A_z$ to satisfy the boundary conditions (2.3). In the $d=4$ Minkowski subspace directions, all fields and their associated derivatives will be required to fall off as usual at spatial infinity.

The corresponding $D=5$ Maxwell action also needs to include a boundary term at the $z=1$ end

$$\begin{aligned} S[A_\mu, A_z] &= S_{\mathrm{Max}}[A_\mu, A_z] + S_{\mathrm{BT}}[A_\mu, A_z] \\ &= \int \mathrm{d}^4x \int_0^1 \mathrm{d}z \Big(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} F_{\mu z}F^{\mu z} \Big) + \frac{1}{2} \int \mathrm{d}^4x\, F_{\mu z}F^{\mu z}\Big|_{z=1}, \end{aligned} \tag{2.4}$$

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F_{\mu z} = \partial_\mu A_z - \partial_z A_\mu$. The $D = 5$ action (2.4) is invariant under standard U(1) gauge transformations with gauge parameter $\Lambda(x, z)$

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda \quad \text{and} \quad A_z \mapsto A_z + \partial_z \Lambda. \tag{2.5}$$

Varying the action (2.4) while taking care in imposing the transverse boundary conditions (2.3) and integrating by parts on the Minkowski boundary at infinity gives rise to a standard set of Maxwell equations

$$A_\mu : \quad (\Box_4 + \partial_z^2) A_\mu - \partial_\mu \partial^\nu A_\nu - \partial_\mu \partial_z A_z = 0 \tag{2.6}$$

and

$$A_z : \quad \Box_4 A_z - \partial_z \partial^\mu A_\mu = 0, \tag{2.7}$$

where $\Box_4 = \partial_\mu \partial^\mu$. Including the boundary term S_{BT} in the action (2.4) is necessary in order to allow the Robin condition for the field A_μ to be incorporated into a well-posed variational problem generating the field equations (2.6) and (2.7).

In order for the boundary conditions (2.3) on A_μ to be gauge invariant, one requires the following restrictions on the form of the local gauge parameter Λ:

$$\Lambda(x, 0) = c_1 \quad \text{and} \quad (\partial_z - 1)\Lambda(x, 1) = c_2, \tag{2.8}$$

where c_1 and c_2 are constants. In order for the gauge parameter to vanish at asymptotic infinity, we require $c_1 = c_2 = 0$.

In addition to the local gauge symmetry (2.5), the Dirichlet/Robin boundary conditions (2.3) for A_μ allow an additional but restricted 'harmonic' gauge symmetry

$$A_\mu \mapsto A_\mu + \partial_\mu \Gamma \quad \text{and} \quad A_z \mapsto A_z, \tag{2.9}$$

where $\Box_4 \Gamma = 0$ and $\partial_z^2 \Gamma = 0$. Invariance of the boundary conditions on A_μ under this harmonic symmetry requires

$$\Gamma(x, 0) = c_3 \quad \text{and} \quad (\partial_z - 1)\Gamma(x, 1) = c_4, \tag{2.10}$$

where we will again take the constants c_3 and c_4 to vanish.

(b) Mode expansions

Now let us expand the $D = 5$ gauge fields in a mode expansion so that we can derive an effective theory for the leading-order $d = 4$ modes. For A_μ satisfying the boundary conditions (2.3), one needs an expansion basis satisfying the same boundary conditions. In order to solve equation (2.6) by separation of variables,

$$A_\mu(x, z) = \sum_{\ell=0}^{\infty} a_\mu^{(\ell)}(x) \xi_\ell(z), \tag{2.11}$$

one requires a complete set of basis functions $\xi_\ell, \ell = 0, 1, \ldots, \infty$, satisfying the transverse wave function problem

$$\xi_\ell''(z) = -\omega_\ell^2 \xi_\ell(z), \quad \xi_\ell(0) = 0 \quad \text{and} \quad \xi_\ell'(1) - \xi_\ell(1) = 0, \tag{2.12}$$

where $\xi_\ell'(z) = \partial_z \xi_\ell(z)$. The L_2 normalized solutions to this transverse problem are

$$\xi_0(z) = \sqrt{3} z \quad \text{and} \quad \xi_\ell(z) = \sqrt{2} \csc(\omega_\ell) \sin(\omega_\ell z), \quad \ell \in \{1, 2, \ldots\} \tag{2.13}$$

and

$$\tan \omega_\ell = \omega_\ell, \quad \omega_\ell > 0. \tag{2.14}$$

When it comes to A_z, the situation is somewhat different. No specific boundary conditions are required in deriving (2.7) from the variation of the action (2.4) because the only term containing δA_z on the boundaries of $\mathbb{I}$ already vanishes when the equations of motion are satisfied. One

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

learns the behaviour of A_z on the boundaries instead directly from compatibility with equation (2.6). Substituting the expansion (2.11) into (2.6), one obtains

$$\sum_{\ell=0}^{\infty}\left((\Box_d-\omega_\ell^2)a_\mu^{(\ell)}-\partial_\mu\partial^\nu a_\nu^{(\ell)}\right)\xi_\ell(z)-\partial_\mu\partial_z A_z=0, \tag{2.15}$$

where $\omega_0=0$. Accordingly, one sees that $\partial_z A_z$ must lie within the span of the $\xi_\ell(z)$ basis, i.e.

$$\partial_z A_z(x,z)=\sum_{\ell=0}^{\infty}b^{(\ell)}(x)\xi_\ell(z). \tag{2.16}$$

Integrating this equation and noting that for $\ell>0$ the indefinite integral of $\xi_\ell(z)$ is proportional to its derivative, one has the expansion

$$A_z(x,z)=h(x)\zeta(z)+\sum_{\ell=0}^{\infty}g^{(\ell)}(x)\xi_\ell'(z), \tag{2.17}$$

where it is convenient to choose $\zeta(z)=(3\sqrt{3}/10)-(\sqrt{3}/2z^2)$ satisfying $\zeta'(z)=-\xi_0(z)$, to complete the expansion of $\partial_z A_z$ in (2.16).

(c) Leading-order $d=4$ effective field theory: no lunch

Using the mode expansions (2.11) and (2.17) in the $D=5$ field equations (2.6) and (2.7), one obtains an equivalent formulation of the theory in terms of $d=4$ expansion fields. Our main interest here will be the structure of the leading-order theory. This may be done either by working directly with the $D=5$ action (2.4) and using orthonormality properties of §2(b) mode expansions to separate the leading-order terms or by making the mode expansions in the $D=5$ field equations (2.6) and (2.7), separating out the leading-order modes and then reconstructing the leading effective action. Either way, one obtains simply

$$I_{\rm Eff}=\int \mathrm{d}^4x\left(-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}+\partial^\mu h(\partial_\mu g-a_\mu)\right), \tag{2.18}$$

where $f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$ and $a_\mu(x)$ and $g(x)$ are just the $\ell=0$ modes in the expansions (2.11) and (2.17) with the $\ell=0$ index dropped. The higher $\ell\geq 1$ modes excluded from the leading-order action (2.18) are all massive, as in an ordinary Kaluza–Klein expansion. Thus, one expects (2.18) to accurately describe the leading gauge-field dynamics of the system at energies well below the mass of the lightest ($\ell=1$) such higher mode, even when interactions with sources are introduced.

What does the action (2.18) describe? A key point to note is that expanding in terms of the $\xi_i(z)$ and $\zeta(z)$ transverse wave function basis has not produced a leading-order $d=4$ system with a massive vector field. So this reduction has not generated a standard Higgs mechanism—the leading-order vector field $a_\mu(x)=a_\mu^0(x)$ remains massless. So there is no lunch-feast going on here.

The next question is: what are the degrees of freedom described by the effective action (2.18)? The answer involves the U(1) gauge symmetry (2.5) and the harmonic symmetry (2.9). For the leading-order modes, the U(1) gauge transformations are, expanding $\Lambda(x,z)$ also in the $\xi_i(z)$ basis with $\lambda(x)=\lambda^0(x)$,

$$\delta a_\mu(x)=\partial_\mu\lambda(x),\quad \delta g(x)=\lambda(x)\quad\text{and}\quad \delta h(x)=0. \tag{2.19}$$

Note the role of the $g(x)$ field in (2.18): it accompanies the otherwise non-gauge-invariant $\partial^\mu h a_\mu$ term to make the U(1) gauge-invariant combination $\partial^\mu h(\partial_\mu g-a_\mu)$. This is a place where reduction with a non-standard transverse wave function zero mode has had an effect: the $g(x)$ field is behaving like a Stueckelberg field [16–18] repairing an otherwise non-gauge-invariant term. Unlike the original Stueckelberg implementation of such a field in a vector field mass term, however, the $g(x)$ field in (2.18) does not participate in the generation of an effective-theory mass.

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc. A 380: 20210184

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

To analyse the dynamical radiative degrees of freedom of the system, we employ Fourier analysis on the source-free field equations following from (2.18):

$$\partial^\mu(\partial_\mu a_\nu - \partial_\nu a_\mu) - \partial_\nu h = 0 \tag{2.20}$$

$$\Box_4 g - \partial^\mu a_\mu = 0 \tag{2.21}$$

and

$$\Box_4 h = 0. \tag{2.22}$$

Fourier transforming the fields

$$a_\mu(x) = \int \mathrm{d}^4p \exp(-\mathrm{i}p_\nu x^\nu)a_\mu(p), \quad g(x) = \int \mathrm{d}^4p \exp(-\mathrm{i}p_\nu x^\nu)g(p)$$

and

$$h(x) = \int \mathrm{d}^4p \exp(-\mathrm{i}p_\nu x^\nu)h(p), \tag{2.23}$$

gives the momentum-space equations

$$p^2 a_\nu - p_\nu p^\mu a_\mu - \mathrm{i}p_\nu h = 0 \tag{2.24}$$

$$p^2 g - \mathrm{i}p^\mu a_\mu = 0 \tag{2.25}$$

and

$$p^2 h = 0. \tag{2.26}$$

Now decompose $a_\mu(p) = \tilde{a}_\mu(p) + p_\mu\kappa(p)$ where at every point p^μ in momentum space the vectors $\tilde{a}_\mu(p)$ and p_μ are taken to be linearly independent. Inserting this decomposition into equations (2.24) and (2.25), one obtains

$$p^2\tilde{a}_\nu - p_\nu(p^\mu\tilde{a}_\mu + \mathrm{i}h) = 0 \tag{2.27}$$

and

$$p^2 g - \mathrm{i}p^\mu\tilde{a}_\mu - \mathrm{i}p^2\kappa = 0. \tag{2.28}$$

Linear independence of the vectors $\tilde{a}_\nu(p)$ and p_ν at the point p^μ in momentum space in equations (2.27), 2.31) and 2.26) then implies

$$p^2\tilde{a}_\nu = 0 \tag{2.29}$$

$$p^\mu\tilde{a}_\mu + \mathrm{i}h = 0 \tag{2.30}$$

$$p^2 g - \mathrm{i}p^\mu\tilde{a}_\mu - \mathrm{i}p^2\kappa = 0 \tag{2.31}$$

and

$$p^2 h = 0. \tag{2.32}$$

From equation (2.29), one learns that the support of $\tilde{a}_\nu(p)$ is only on the lightcone, i.e. $\mathrm{supp}(\tilde{a}_\nu(p)) = \{p_\mu | p^2 = 0\}$. Similarly, from (2.32), one also learns that $\mathrm{supp}(h(p)) = \{p_\mu | p^2 = 0\}$. Considering equation (2.31) on the $p^2 = 0$ lightcone subspace then implies $(p^\mu\tilde{a}_\mu)|_{p^2=0} = 0$, and consequently $p^\mu\tilde{a}_\mu(p) = 0$ throughout momentum space. However, equation (2.30) then implies $h(p) = 0$ except for $p^\mu = 0$ (remembering the p_ν factor in (2.27)). Consequently, one has it that $h(x) =$ constant in $d = 4$ Minkowski space, and is thus non-dynamical.

The remaining dynamical content of the theory resides in equations (2.29) and (2.31) after setting $p_\nu h(p) = 0$

$$p^2\tilde{a}_\nu = 0 \tag{2.33}$$

and

$$p^2 g - \mathrm{i}p^2\kappa = 0. \tag{2.34}$$

These are still U(1) gauge invariant. In momentum space, one has $\delta a_\mu(p) = -\mathrm{i}p_\mu\lambda$ so $\tilde{a}_\mu(p)$ does not transform, but κ does, with $\delta\kappa = -\mathrm{i}\lambda$. The transformation of the Stueckelberg field $g(p)$ is $\delta g(p) = \lambda(p)$. Consequently, the combination $g - \mathrm{i}\kappa$ is gauge invariant and so equations (2.33) and (2.34) are independently U(1) gauge invariant.

One can fix the U(1) gauge symmetry by picking any convenient gauge. One common example is Coulomb gauge, $\partial_i a_i(x) = 0$, becoming $p_i a_i = 0$ in momentum space. Setting $p_0 h = 0$ in the $\nu = 0$ component of (2.24) then implies $p_i p_i a_0 = 0$, or in position space $\nabla^2 a_0 = 0$. Requiring a falloff

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

condition at spatial infinity, $a_0 \to 0$ as $|x^i| \to \infty$, then implies $a_0 = 0$. Consequently, there remain only the standard two dynamical degrees of freedom in the a_μ Maxwell field. The momentum-space Lorentz condition is also obtained: $p^\mu a_\mu = -p_o a_0 + p_i a_i = 0$.

The remaining degree of freedom for the system (2.18) is then seen in equation (2.25): $p^2 g(p) = 0$. Thus, in Coulomb gauge, there is an additional candidate scalar degree of freedom residing in the Stueckelberg field. This would be entirely expected in a standard Kaluza–Klein reduction where dependence on the z reduction coordinate is simply suppressed, in which case the A_z component of the $D = 5$ Maxwell field becomes a massless scalar in $d = 4$. In the present Dirichlet/Robin system, however, the remaining g field has a more tenuous existence. As we have seen in (2.9), there is one more transformation which we have not yet exploited: $D = 5$ pure Maxwell theory with Dirichlet/Robin boundary conditions on the transverse interval $\mathbb{I}$ also has the harmonic symmetry. By combining the harmonic transformation (2.9) with a U(1) gauge transformation (2.5), one can trade the $\delta a_\mu = \partial\gamma$ harmonic symmetry transformation of a_μ for a transformation $\delta g = -\gamma$, leaving a_μ unchanged. The harmonic condition on γ corresponds precisely to the support set of g, so one can use the reformulated harmonic transformation to send $g \to 0$.

One can summarize the degree-of-freedom count for the reduced effective-theory system described by the action (2.18) as follows: there is no massive lunch, and even the remaining scalar g has a kind of 'phantom' (but not ghost!) existence—barely a crumb on the side. Only the two $d = 4$ Maxwell degrees of freedom are unambiguously dynamical. We will investigate this structure further by looking at the Hamiltonian, in order to see what impact the presence of g would have on the conserved energy.

(d) Effective-theory energy

From the leading-order effective action (2.18), one passes to a canonical form of the action in the standard way, defining a canonical momentum $\pi_a(x) = \delta I/\delta\dot{\psi}_a$ conjugate to each field ψ_a in (2.18). This yields

$$\pi_i = \dot{a}_i - \partial_i a_0 \tag{2.35}$$

$$\pi_0 = 0 \tag{2.36}$$

$$\pi_h = a_0 - \dot{g} \tag{2.37}$$

and

$$\pi_g = -\dot{h}, \tag{2.38}$$

where $\dot{\psi} = \partial\psi/\partial t$. The corresponding canonical action

$$I_{\text{canonical}} = \int \mathrm{d}t\, \mathrm{d}^3x (\pi_i \dot{a}_i + \pi_g \dot{g} + \pi_h \dot{h} - \mathcal{H}) \tag{2.39}$$

has the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\pi_i\pi_i + \frac{1}{4}f_{ij}f_{ij} - a_0\partial_i\pi_i + \pi_g(a_0 - \pi_h) + \partial_i h(a_i - \partial_i g). \tag{2.40}$$

One may verify that Hamilton's equations following from variation of the canonical action (2.39) are a first-order system that is fully equivalent to the Lagrangian field equations (2.20)–(2.22).

The system's total energy is the value of $E = \int \mathrm{d}^3x\mathcal{H}$ for solutions to the field equations, as discussed above in §2(c). From the Coulomb-gauge analysis there, one has $h =$ constant, so from (2.38), one also has $\pi_g = 0$. Moreover, the field a_0 has no conjugate momentum and its variation yields the constraint $\partial_i\pi_i = 0$ just as in standard $d = 4$ Maxwell theory. Consequently, for solutions to the effective-theory field equations arising from the action (2.18), one has the conserved total energy

$$E = \int \mathrm{d}^3x \left(\frac{1}{2}\pi_i\pi_i + \frac{1}{4}f_{ij}f_{ij}\right), \tag{2.41}$$

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

which is simply equal to the standard, positive semidefinite, energy of Maxwell theory alone. Although allowed by the field equations, the g field does not contribute to the total energy. Clearly, the fact that g may be removed by the residual harmonic symmetry is the reason for its non-appearance in the total energy (2.41). At least in the source-free theory we have considered so far, the g field may be considered redundant.

3. Covert symmetry breaking

(a) Scalar electrodynamics

In §2, we have seen that the unambiguous leading-order dynamics of the $D=5$ system (2.4) is just $d=4$ Maxwell theory. The non-standard Dirichlet/Robin boundary conditions on the interval $\mathbb{I}$ do give rise to the h and g fields in the leading-order effective action (2.18), but the analysis of §2 shows that h has no dynamics while g may be considered redundant, giving no contribution to the total energy. Following [15], we now extend the model by U(1) gauge-covariantly coupling the system (2.4) also to a $D=5$ complex scalar field Φ

$$\begin{aligned} S[A_\mu, A_z, \Phi, \overline{\Phi}] &= S_{\mathrm{SQED}}[A_\mu, A_z, \Phi, \overline{\Phi}] + S_{\mathrm{BT}}[A_\mu, A_z] \\ &= \int \mathrm{d}^d x \int_0^1 \mathrm{d}z \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu z} F^{\mu z} - \overline{(D_\mu \Phi)} D^\mu \Phi \Big) \\ &\quad + \frac{1}{2} \int \mathrm{d}^d x \, F_{\mu z} F^{\mu z} \Big|_{z=1}, \end{aligned} \tag{3.1}$$

where $D_M \Phi = \partial_M \Phi - \mathrm{i}e A_M \Phi$, $M = \mu, z$, and where e is the charge of the complex scalar field Φ. For simplicity, one may take the $D=5$ scalar field Φ to satisfy Dirichlet/Dirichlet boundary conditions on the interval $\mathbb{I}$.

One may expand Φ in a transverse basis appropriate to the Dirichlet/Dirichlet boundary conditions,

$$\Phi(x,z) = \sum_{n=1}^{\infty} \phi^{(n)}(x) \theta_n(z), \tag{3.2}$$

where $\{\theta_n(z) = \sqrt{2}\sin(m_n z)\}$ with $n = 1, 2, \ldots$ and $m_n = n\pi$.

Under the U(1) gauge symmetry (2.5), the $\phi^{(n)}$ complex scalar modes transform in a way that mixes the various n levels

$$\phi^{(n)} \mapsto \sum_{m=1}^{\infty} \exp\left(\mathrm{i}e\lambda^{(\ell)} I_\ell\right)^{nm} \phi^{(m)}, \tag{3.3}$$

where the matrix $(I_i)^{nm}$ is

$$(I_\ell)^{nm} = I_\ell^{nm} = \int_0^1 \mathrm{d}z \, \xi_\ell(z) \theta_n(z) \theta_m(z). \tag{3.4}$$

One may expand the system (3.1) into $d=4$ modes either by substituting the expansions of A_μ, A_z and Φ into the higher-dimensional equations of motion or by inserting these expansions into the higher-dimensional action in order to obtain an action for the $d=4$ formulation of the system. The two procedures give equivalent results. Keeping all expansion modes for A_μ, A_z and Φ, the U(1) gauge symmetry is maintained to all orders, albeit in a rather complicated way.

Now consider just the leading-order effective theory, keeping just the $\ell=0$ massless gauge sector as described in §2 and the lightest $n=1$ complex scalar mode. The main point of this illustrative $D=5$ scalar electrodynamics model now comes into focus: it is simple enough that the various mode integrals governing expansions beyond the free theory (i.e. for action terms cubic and higher in fields) can straightforwardly be done. In addition to the matrix I_ℓ^{nm} given in

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

(3.4), one needs also

$$I^{nm}_{k\ell} = \int_0^1 \mathrm{d}z\, \xi_k(z)\xi_\ell(z)\theta_n(z)\theta_m(z). \tag{3.5}$$

Noting that terms $\Phi\partial_\mu\overline{\Phi}$ and $\overline{\Phi}\Phi A_\mu$ obey Dirichlet/Robin conditions, and thus can be expanded in the $\{\xi_\ell(z)\}$ basis, one needs the product rules

$$\theta_n(z)\theta_m(z) = I^{nm}_\ell \xi_\ell(z) \quad \text{and} \quad \theta_n(z)\theta_m(z)\xi_k(z) = I^{nm}_{k\ell}\xi_\ell(z). \tag{3.6}$$

The covariant derivative operator for the scalar field is then

$$D^{nm}_\mu = \delta_{nm}\partial_\mu - \mathrm{i}eI^{nm}_\ell a^{(\ell)}_\mu. \tag{3.7}$$

Keeping just the leading $\ell = 0$ gauge modes and the $n = 1$ modes of the scalar, writing simply $\phi = \phi^1$ and keeping terms up to cubic order in the interacting theory, one has

$$\begin{aligned} I_{\mathrm{Eff}} = \int \mathrm{d}^4x \Big(&-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \partial^\mu h(\partial_\mu g - a_\mu) - \partial_\mu\overline{\phi}\partial^\mu\phi \\ &- \pi^2\overline{\phi}\phi + \mathrm{i}eI^{11}_0 a^\mu(\overline{\phi}\partial_\mu\phi - \partial_\mu\overline{\phi}\phi)\Big), \end{aligned} \tag{3.8}$$

where $I^{11}_0 = \sqrt{3}/2$. One can accordingly identify the $d = 4$ effective-theory charge for the leading gauge-scalar coupling to be $q_{\mathrm{Eff}} = \sqrt{3}/2e$ in terms of the charge e of the $D = 5$ theory.

(b) An unanticipated seagull coefficient

Continuing on to quartic order in fields, one encounters a key peculiarity of this scalar electrodynamics construction with non-standard boundary conditions. As always in a model with a U(1) gauge field coupled to a complex scalar field, one expects to have $\int \mathrm{d}^4x a_\mu a^\mu\overline{\phi}\phi$ 'seagull' terms occurring at quartic order.[1] From the effective-theory charge q_{Eff} identified at cubic order, one would expect the gauge-field-coupled scalar kinetic term to become $-\int \mathrm{d}^4x\overline{(D_\mu\phi)}D^\mu\phi$ where

$$D_\mu = \partial_\mu - \mathrm{i}q_{Eff}a_\mu \tag{3.9}$$

is the usual gauge covariant derivative. Were that the case, the quartic-order $\int \mathrm{d}^4x a_\mu a^\mu\overline{\phi}\phi$ seagull term coefficient would be just q^2_{Eff}. That is not what arises from the transverse-space integrals, however. Instead, what one finds involves $I^{11}_{00} = 1 - (3/2\pi^2) \neq (I^{11}_0)^2$. This is the phenomenon of covert symmetry breaking: no mass generation occurs as in a standard Higgs mechanism, but symmetry breaking in higher couplings, starting at quartic order in fields, does.

Of course, if one keeps all gauge and scalar modes, the full $D = 5$ theory is retained and no symmetry breaking occurs. But this $D = 5$ U(1) symmetry mixes the various gauge and scalar field mode levels in a complicated way. Covert symmetry breaking is intrinsically a phenomenon related to a low-energy approximation, in which the massive higher modes are not independently excited, but are integrated out using the leading-order results of their field equations.

That such a phenomenon can occur starting at quartic order may be expected on general grounds, as well as the fact that when it does occur, it cannot be fully remedied by field redefinitions [8]. What can be altered by field redefinitions, however, is the presentation of the phenomenon. The leading-order effective theory is obtained by integrating out all higher massive modes. For the gauge-field sector, that process is unambiguous when one restricts attention to the leading $\ell = 0$ modes. For the complex scalar, however, some rearrangement is possible. Integrating out massive modes at leading order basically involves discarding derivatives of such a mode in its field equation, but keeping the mass term in what then becomes an algebraic equation at leading order, solving for the mode in question. Field redefinitions involving the Stueckelberg

[1]These terms are called 'seagulls' as they generate interactions between the particles corresponding to the gauge field and scalar that look diagrammatically like a seagull.

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

modes $g^{(\ell)}$ change the way the $D=5$ U(1) symmetry acts on the various modes. If one defines [15]

$$\varphi^{(n)} = \exp(\mathrm{i}e g^{(\ell)} I^{nn}_{\ell}) \exp(-\mathrm{i}e g^{(k)} I_k)^{nm} \phi^{(m)}, \tag{3.10}$$

then instead of the $D=5$ U(1) transformations mixing between n modes of the scalar field, the $\varphi^{(n)}$ transform diagonally and canonically,

$$\varphi^{(n)} \mapsto \exp(\mathrm{i}e\lambda^{(\ell)} I^{nn}_{\ell}) \varphi^{(n)}, \tag{3.11}$$

noting that $\exp(\mathrm{i}e\lambda^{(\ell)} I^{nn}_{\ell})$ is a phase and not a matrix.

Integrating out the $n>1$ massive modes and restricting to the $\ell=0$ gauge modes and the $\varphi=\varphi^{(1)}$ scalar field, the result through cubic terms remains the same as in (3.8), but when keeping terms up to quartic order, one has

$$\begin{aligned} \tilde{I}_{\mathrm{Eff}} = \int \mathrm{d}^4x \Big(& -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \overline{(D_\mu \varphi)} D^\mu \varphi - \pi^2 \overline{\varphi}\varphi + \partial^\mu h \big(\partial_\mu g - a_\mu\big) \\ & - q^2_{\mathrm{Eff}} \tilde{I} (a_\mu - \partial_\mu g)(a^\mu - \partial^\mu g) \overline{\varphi}\varphi - q^2_{\mathrm{Eff}} \widetilde{X} h^2 \overline{\varphi}\varphi \Big), \end{aligned} \tag{3.12}$$

where D_μ is the usual covariant derivative (3.9) and

$$\tilde{I} = (I^{11}_0)^{-2} (I^{11}_{00} - (I^{11}_0)^2) = \frac{1}{3} - \tfrac{2}{\pi^2}, \tag{3.13}$$

while $\widetilde{X}$ is a more complicated but explicitly calculable positive coefficient. In the effective-action form (3.12), one sees both the preservation of the U(1) symmetry by Stueckelberg fields but also its breaking. The top line of (3.12) has the normal $d=4$ gauge-covariant coupling of a_μ to the complex scalar field φ, while the second line shows the difference seagull vertex $(a_\mu - \partial_\mu g)(a^\mu - \partial^\mu g)\overline{\varphi}\varphi$ with a coefficient involving the $\tilde{I}$ combination which gives the deviation from the standard seagull coefficient. Since this structure involves the U(1) gauge invariant combination $(a^\mu - \partial^\mu g)$, it makes the preservation of the $D=5$ U(1) symmetry manifest, but since this is only achieved via the presence of the Stueckelberg field g, it also makes the covert symmetry breaking phenomenon manifest.

The alternative between the original $\phi^{(n)}$ expansion and the transformed $\varphi^{(n)}$ expansion may be likened to the alternative between the gauge basis and the mass basis in the CKM mechanism [19,20] in the Standard Model. In the $\phi^{(n)}$ basis, there are no terms directly mixing the $a^{(\ell)}_\mu$ gauge modes and the $g^{(\ell)}$ Stueckelberg modes, but the transformations of the $\phi^{(n)}$ scalar modes mix between different n levels. In the $\varphi^{(n)}$ basis, on the other hand, the scalar modes transform diagonally and canonically but then there are terms involving products of the $a^{(\ell)}_\mu$ and the $g^{(\ell)}$ Stueckelberg modes together with scalar modes, as one sees in (3.12).

4. Outlook

The key issues that we have considered in this article concern the low-energy effective-theory dynamics of a higher-dimensional theory which has a natural effective-theory interpretation in a lower dimension, but in which transverse wave function modes have a non-trivial dependence on a transverse coordinate. Such effective reductions can occur in a wide variety of situations, including the Type IIA supergravity lift of the Salam–Sezgin model as considered in [9] as well as in the simpler Dirichlet/Robin scalar electrodynamics considered here and in [15]. A take-home message is that there are interesting wider varieties of symmetry breaking mechanisms than the standard BEHGHK mechanism. Moreover, reductions in effective space–time dimensionality can occur without requiring technically consistent dimensional reductions.

The $D=5$ Dirichlet/Robin scalar electrodynamics model illustrated here has leading-order field dynamics in which massless Maxwell theory is unambiguously part of the $d=4$ effective theory, but there is also the g scalar mode, which, however, makes no contribution to the conserved energy at leading order and may be considered redundant at that level. The full story of that mode in the interacting theory remains to be clarified, as well as the roles of analogous modes

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024

in the IIA supergravity/Salam–Sezgin gravitational model. Another question going beyond what has been considered here is what happens when the Dirichlet/Robin scalar electrodynamics system generates a localized source in five dimensions for the gauge sector. One may expect a transition between near-field $D = 5$ behaviour near the source and far-field $d = 4$ behaviour away from it. Such a near-field/far-field transition is found when a delta-function source localized in the higher dimension is coupled to the Type IIA/Salam–Sezgin model [21]; one might expect similar behaviour in the $D = 5$ scalar electrodynamics case.

Models that come close to gauge-symmetric models but which display gauge-symmetry breaking without generating masses for the leading-order gauge modes can have subleading corrections which could be of interest from a variety of different perspectives. One such might be whether these corrections can serve as hints of hitherto unrecognized higher-dimensional space–time structure.

Data accessibility. This article has no additional data.

Authors' contributions. C.W.E.: investigation; R.L.: investigation; K.S.S.: investigation.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

Funding. The work of K.S.S. was supported in part by the STFC under Consolidated grant nos. ST/P000762/1 and ST/T000791/1 and the work of C.W.E. was supported by the United States Department of Veterans Affairs under the Post 9/11 GI Bill.

Acknowledgements. We are grateful to Carl Bender, Stanley Deser, Jonathan Halliwell, Alexander Harrold, Jean-Luc Lehners and Massimo Porrati for helpful discussions.

References

1. Englert F, Brout R. 1964 Broken symmetry and the mass of gauge vector mesons. *Phys. Rev. Lett.* **13**, 321–323. (doi:10.1103/PhysRevLett.13.321)
2. Higgs PW. 1964 Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.* **13**, 508–509. (doi:10.1103/PhysRevLett.13.508)
3. Guralnik GS, Hagen CR, Kibble TWB. 1964 Global conservation laws and massless particles. *Phys. Rev. Lett.* **13**, 585–587. (doi:10.1103/PhysRevLett.13.585)
4. Kibble TWB. 1967 Symmetry breaking in non-Abelian gauge theories. *Phys. Rev.* **155**, 1554–1561. (doi:10.1103/PhysRev.155.1554)
5. Georgi H, Glashow SL. 1974 Unity of all elementary particle forces. *Phys. Rev. Lett.* **32**, 438–441. (doi:10.1103/PhysRevLett.32.438)
6. Scherk J, Schwarz JH. 1979 How to get masses from extra dimensions. *Nucl. Phys. B* **153**, 61–88. (doi:10.1016/0550-3213(79)90592-3)
7. Duff MJ, Ferrara S, Pope CN, Stelle KS. 1990 Massive Kaluza-Klein modes and effective theories of superstring moduli. *Nucl. Phys. B* **333**, 783–814. (doi:10.1016/0550-3213(90)90139-5)
8. Deser S, Stelle KS. 2019 Field redefinition's help in constructing non-Abelian gauge theories. *Phys. Lett. B* **798**, 135007. (doi:10.1016/j.physletb.2019.135007)
9. Crampton B, Pope CN, Stelle KS. 2014 Braneworld localisation in hyperbolic spacetime. *JHEP* **2014**, 35. (doi:10.1007/JHEP12(2014)035)
10. Cvetic M, Gibbons GW, Pope CN. 2004 A string and *M* theory origin for the Salam-Sezgin model. *Nucl. Phys. B* **677**, 164–180. (doi:10.1016/j.nuclphysb.2003.10.016)
11. Pöschl G, Teller E. 1933 Bemerkungen zur quantenmechanik des anharmonischen oszillators. *Z. Phys.* **83**, 143–151. (doi:10.1007/BF01331132)
12. Salam A, Sezgin E. 1984 Chiral compactification on Minkowski $\times S^2$ of $N = 2$ Einstein-Maxwell supergravity in six-dimensions. *Phys. Lett. B* **147**, 47–51. (doi:10.1016/0370-2693(84)90589-6)
13. Arnowitt RL, Deser S, Misner CW. 2008 The dynamics of general relativity. *Gen. Rel. Grav.* **40**, 1997–2027. (doi:10.1007/s10714-008-0661-1)
14. Weinberg S. 1965 Photons and gravitons in perturbation theory: derivation of Maxwell's and Einstein's equations. *Phys. Rev.* **138**, B988–B1002. (doi:10.1103/PhysRev.138.B988)
15. Erickson CW, Harrold AD, Leung R, Stelle KS. 2020 Covert symmetry breaking. *JHEP* **2020**, 157. (doi:10.1007/JHEP10(2020)157)

16. Stueckelberg ECG. 1938 Die Wechselwirkungs Kraefte in der Elektrodynamik und in der Feldtheorie der Kernkraefte. *Helv. Phys. Acta* **11**, 225–244.
17. Stueckelberg ECG. 1938 Die Wechselwirkungs Kraefte in der Elektrodynamik und in der Feldtheorie der Kernkraefte. *Helv. Phys. Acta* **11**, 299–312.
18. Stueckelberg ECG. 1930 Die Wechselwirkungs Kraefte in der Elektrodynamik und in der Feldtheorie der Kernkraefte. *Helv. Phys. Acta* **11**, 312–328.
19. Cabibbo N. 1963 Unitary symmetry and leptonic decays. *Phys. Rev. Lett.* **10**, 531–533. (doi:10.1103/PhysRevLett.10.531)
20. Kobayashi M, Maskawa T. 1973 CP violation in the renormalizable theory of weak interaction. *Prog. Theor. Phys.* **49**, 652–657. (doi:10.1143/PTP.49.652)
21. Erickson CW, Leung R, Stelle KS. 2022 Taxonomy of brane gravity localisations. *JHEP* **2022**, 130. (doi:10.1007/JHEP01(2022)130)

Downloaded from https://royalsocietypublishing.org/ on 18 December 2024