

DEVELOPMENT AND APPLICATION OF LINEAR ESTIMATORS OF SCALE

by

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Abstract

The general distribution theory of order statistics and the asymptotic distribution of sample quantiles are discussed in chapter 1 in preparation for the development of large and small sample estimators of the standard deviation of a normal population considered in chapter 2. A simple large sample estimator is described and shown to be asymptotically unbiased and highly efficient relative to the root-mean-square estimator. For small samples the best linear estimator is discussed and its application extended to samples of greater than twenty observations using an approximation to the required coefficients. As an alternative to this a nearly-best linear estimator is developed and shown to be highly efficient and simply extended to larger samples and to censored samples. Simple approximation formulae are derived for the variances and efficiencies of these alternative estimators.

In chapter 3 test procedures are developed for both large and small samples in which the standard deviation is estimated using one of the linear estimators considered in chapter 2. Tables of percentage points of the distributions of these test procedures, which may be used as substitutes for Student's t-test, the χ^2 -test or the F-test for variances, are given for a range of sample sizes and comparisons are made between the powers of the standard tests and these alternatives.

The robustness of several linear estimators to departures from normality is considered in the last chapter. Exact results are given for a variety of non-normal distributions. Finally the results of a large scale simulation study of the robustness of the root-mean-square estimator and some of the linear estimators are summarized. The distributions used in this simulation study include some for which exact values are available, thus enabling comparisons between empirical and exact values to be made. Also considered are contaminated normal distributions with varying proportions of contamination. The linear estimators are shown to be more robust than the root-mean-square estimator as well as being highly efficient for precise normality.

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CHAPTER 1

INTRODUCTION AND GENERAL DISTRIBUTION

THEORY

1.1 Introduction

As is apparent from the increasing volume of published research on new theoretical aspects or practical applications of order statistics, interest in this branch of applied statistics is growing steadily. Industrialists, who already make extensive use of the range as a quick efficient estimator of the standard deviation for small sample quality control charts, are realizing the potentials of "short-cut" procedures based on relatively simple functions of the ordered observations. These short-cut procedures include not only quick efficient estimates of population parameters but simple tests of hypotheses. Although these methods are less efficient than the standard statistical techniques there are many situations where the cost and effort involved in obtaining the data is less than the cost of a complex analysis and thus, for a given fixed cost, greater efficiency may be obtained by collecting more information and using these non-optimal techniques.

Also there are many occasions when observations occur naturally in an ordered state, for example, the life of electronic components on a test bed or the breaking strength of strands of wool. In tests like these it is often necessary to stop the experiment before all the observations are available. This results in an ordered sample with some observations censored. Analysis of such samples may require complicated adaptations

of standard techniques but the theory and application of order statistics extends readily to include censored samples.

Currently there is also considerable interest in the problems of treating outlying observations. If outliers are rejected from the sample we have a situation similar to that above while, if the outliers are not rejected they may have a disastrous effect on any least-squares analysis to be performed. In either event therefore it would appear sensible to consider analysis using methods based on order statistics.

Order statistics, however, like the optimum statistical techniques are not distribution-free. Their sampling distributions depend on the form of the parent population and, for the normal distribution which is of most interest, the equations for the moments of these sampling distributions are extremely difficult to handle. Because of this the tables of coefficients required for the estimators of the parameters and the tables of percentage points of the distributions of test statistics are severely limited. In this thesis an attempt is made not only to develop new highly efficient estimators and test procedures which are more powerful than range-based tests, but also to extend existing tables by employing close approximations to the moments of these sampling distributions.

1.2 General distribution theory of order statistics

The following is a summary of the main results in the general theory of order statistics.

Suppose that a sample of n independent observations is taken from a population with distribution function $F(y)$ and continuous density function $f(y)$, and that the observations when ordered are given by

$$y_1 \leq y_2 \leq \dots \leq y_n ,$$

then y_i is called the i^{th} order statistic. Since $F(y)$ is continuous we may consider, with probability one, that the order statistics are of the form

$$y_1 < y_2 < \dots < y_n .$$

The distribution function of y_i , and hence its density function, may be obtained by finding the probability that y_i is less than or equal to some value, y say,

$$\text{i.e.} \quad P(y_i \leq y) = P(i \text{ or more of the observations are } \leq y)$$

$$= \sum_{j=i}^n \binom{n}{j} [F(y)]^j [1-F(y)]^{n-j} .$$

The density function of y_i may be found by differentiating with respect to y to give

$$\begin{aligned}
f_{y_i}(y) &= \sum_{j=i}^n \binom{n}{j} [F(y)]^{j-1} f(y) [1-F(y)]^{n-j} \\
&- \sum_{j=i}^n \binom{n}{j} (n-j) [F(y)]^j [1-F(y)]^{n-j-1} f(y) \\
&= nf(y) \left\{ \sum_{j=i}^n \binom{n-1}{j-1} [F(y)]^{j-1} [1-F(y)]^{n-j} \right. \\
&- \left. \sum_{j=i}^{n-1} \binom{n-1}{j} [F(y)]^j [1-F(y)]^{n-j-1} \right\}
\end{aligned}$$

All the terms in these two sums cancel except for the first term of the first sum leaving

$$f_{y_i}(y) = nf(y) \binom{n-1}{i-1} [F(y)]^{i-1} [1-F(y)]^{n-i}$$

as the density function of the i^{th} order statistic y_i .

J. Ogawa (1951) obtained the general form of the joint density function of the k order statistics $y_{n_1}, y_{n_2}, \dots, y_{n_k}$ where $n_1, n_2, \dots, n_k$ are k integers such that $1 \leq k \leq n$ and $1 \leq n_1 < n_2 < \dots < n_k \leq n$. This joint density function is

$$\begin{aligned}
\psi_k(y_{n_1}, \dots, y_{n_k}) &= \frac{n!}{\prod_{i=1}^{k+1} (n_i - n_{i-1} - 1)!} \prod_{i=1}^{k+1} [F(y_{n_i}) - F(y_{n_{i-1}})]^{n_i - n_{i-1} - 1} \\
&\quad \cdot \prod_{i=1}^k f(y_{n_i})
\end{aligned}$$

when $y_{n_1} < y_{n_2} < \dots < y_{n_k}$

= 0 otherwise,

where $n_0 = 0$, $n_{k+1} = n + 1$, $y_{n_0} = -\infty$ and $y_{n_{k+1}} = \infty$.

By suitable choice of the integers $n_1, n_2, \dots, n_k$ useful special cases of this joint density function may be obtained. For example:

(a) The joint density of the order statistics

$y_1, y_2, \dots, y_n$ is given by

$$\begin{aligned}\psi_n(y_1, \dots, y_n) &= n! f(y_1) f(y_2) \dots f(y_n) \\ &\quad \text{for } y_1 < y_2 < \dots < y_n. \\ &= 0 \text{ otherwise.}\end{aligned}$$

(b) The joint density function of the i^{th} and j^{th} order statistics for $i < j$ is

$$\begin{aligned}\psi_2(y_i, y_j) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(y_i)]^{i-1} \\ &\quad \cdot [F(y_j) - F(y_i)]^{j-i-1} [1 - F(y_j)]^{n-j} f(y_i) f(y_j) \\ &\quad \text{for } y_i \leq y_j \\ &= 0 \text{ otherwise.}\end{aligned}$$

(c) If $i = 1$ and $j = n$ in the above we have

$$\begin{aligned}\psi_2(y_1, y_n) &= n(n-1) [F(y_n) - F(y_1)]^{n-2} f(y_n) f(y_1), \\ &\quad \text{for } y_1 < y_n\end{aligned}$$

= 0 otherwise,

which may be used to find the distribution of the range $y_n - y_1$.

These density functions may be used to determine the moments and product moments of the order statistics. The general product moment of the order statistics is defined by

$$\begin{aligned}
 & E \left(y_{n_1}^{a_1} y_{n_2}^{a_2} \dots y_{n_k}^{a_k} \right) \\
 &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} y_{n_1}^{a_1} y_{n_2}^{a_2} \dots y_{n_k}^{a_k} \psi_k(y_{n_1}, y_{n_2}, \dots, y_{n_k}) \\
 &\quad \cdot dy_{n_1} dy_{n_2} \dots dy_{n_k} \\
 &= \frac{n!}{\prod_{i=1}^{k+1} (n_i - n_{i-1} - 1)!} \int_{-\infty < y_{n_1} < \dots < y_{n_k} < \infty} \prod_{i=1}^k y_{n_i}^{a_i} \\
 &\quad \prod_{i=1}^{k+1} [F(y_{n_i}) - F(y_{n_{i-1}})]^{n_i - n_{i-1} - 1} \\
 &\quad \cdot \prod_{i=1}^k f(y_{n_i}) \prod_{i=1}^k dy_{n_i} .
 \end{aligned}$$

Once again by choosing $n_1, n_2, \dots, n_k$ suitably special cases may be derived.

The expected value of the i^{th} order statistic y_i is

$$E(y_i) = \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} \xi [F(\xi)]^{i-1} [1 - F(\xi)]^{n-i} f(\xi) d\xi$$

The variance of the i^{th} order statistic y_i is given by

$$V(y_i) = E(y_i^2) - |E(y_i)|^2$$

where

$$E(y_i^2) = \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} \xi^2 [F(\xi)]^{i-1} [1 - F(\xi)]^{n-i} f(\xi) d\xi ,$$

and the covariance of the i^{th} and j^{th} order statistics
(with $i < j$) is

$$\text{Cov}(y_i, y_j) = E(y_i y_j) - E(y_i)E(y_j)$$

where

$$E(y_i y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_{-\infty}^{\infty} \int_{-\infty}^n \xi \eta [F(\xi)]^{i-1} [F(\eta) - F(\xi)]^{j-i-1} \\ \cdot [1 - F(\eta)]^{n-j} f(\xi) f(\eta) d\xi d\eta .$$

Thus the expected values and the variance-covariance matrix of the order statistics from any continuous distribution may be obtained by specifying the form of the population, i.e. $F(y)$, in the above equations. Evaluation of these integrals presents considerable difficulty except in the simpler cases of the uniform distribution on $(0,1)$ where $F(y) = y$ for $0 \leq y \leq 1$, and the exponential distribution with $F(y) = 1 - e^{-\lambda y}$ for $y \geq 0$.

The case where the observations come from a normal population has been studied extensively and a great deal of

literature is available. D. Teichroew (1956) computed the means and product moments of the order statistics for samples of up to size 20 and A.E. Sarhan and B.G. Greenberg (1956, 1958) calculated the corresponding variances and covariances using Teichroew's tables. These tables are fundamental to the consideration of the properties of linear estimators based on order statistics from a normal population and have been used repeatedly in the subsequent sections.

More recently several authors have considered other distribution functions in the above equations such as the logistic, gamma and right triangular distributions and have either produced tables of the means and variance-covariance matrix or derived simplified formulae or recurrence relations. These results make it possible not only to derive linear estimators suitable for these particular populations but also to compare the robustness of estimators proposed for a normal population. This subject is considered in detail in chapter 4.

1.3 Asymptotic distributions of quantiles

When considering the asymptotic distributions of order statistics we may have two types of limiting process. Firstly we may require the distribution of the i^{th} order statistic y_i in a sample of n as both i and n tend to infinity such that i/n remains constant. In this case we are considering the asymptotic distribution of a quantile of the sample. Secondly, if i remains fixed as n tends to infinity we are considering the asymptotic distribution of an extreme value. The term extreme is used since the cases of most interest are when i is close to 1 or n . These two limiting processes lead to radically different results. We consider here, for use in the next section, the asymptotic distributions of quantiles, the reader is referred to E.J. Gumbel (1958) for the theory of extreme values and applications. Most of the following results are due to F. Mosteller (1946).

Suppose that the density function of the population is $f(y)$ and that for any given number p such that $0 < p < 1$ the p -quantile, or 100p percentage point, is defined as the value y_p such that

$$p = \int_{-\infty}^{y_p} f(y) dy = F(y_p) .$$

For example $y_{0.5}$, $y_{0.25}$ and $y_{0.75}$ are the median and

the lower and upper quartiles, respectively, of the population. The p -quantile of the ordered sample $y_1, y_2, \dots, y_n$ is defined by

$$z_p = \begin{cases} y_{np} & \text{if } np \text{ is an integer} \\ y_{[np]+1} & \text{if } np \text{ is not an integer} \end{cases}$$

where $[np]$ denotes the largest integer not exceeding np .

The mode of the distribution of the i^{th} order statistic may be obtained by differentiating the density function

$$f_{y_i}(y) = \frac{n!}{(n-i)!(i-1)!} [F(y)]^{i-1} [1 - F(y)]^{n-i} f(y)$$

with respect to y (after taking logarithms) and equating to zero. This gives

$$\frac{(i-1)f(\tilde{y})}{F(\tilde{y})} - \frac{(n-i)f(\tilde{y})}{1 - F(\tilde{y})} + \frac{f'(\tilde{y})}{f(\tilde{y})} = 0$$

where $\tilde{y}$ is the mode.

For large n the term f'/f will in general be small compared to the other terms since $(i-1)$ and $(n-i)$ are both large. Thus as n tends to infinity such that $i = np$ we have

$$(np-1) f(\tilde{y})/F(\tilde{y}) = (n-np)f(\tilde{y})/[1 - F(\tilde{y})]$$

$$\text{i.e.} \quad F(\tilde{y}) = p.$$

Thus to order $1/n$ the mode of the distribution of the sample quantile is the population quantile.

$$\text{Suppose } F(y_i) = F(y_p) + \xi = p + \xi .$$

The random variable $F_i = F(y_i)$ is the i^{th} order statistic in a sample of n independent observations from a uniform distribution on $(0,1)$ and hence has density function proportional to $F_i^{i-1}(1 - F_i)^{n-i}$ which, using $F_i = p + \xi$, becomes

$$(p + \xi)^{i-1}(q-\xi)^{n-i}$$

where $q = 1 - p$.

Taking logs and expanding we have, except for constants,

$$(i - 1) \log \left(1 + \frac{\xi}{p}\right) + (n - i) \log \left(1 - \frac{\xi}{q}\right) .$$

For large n such that $i = np$ this becomes

$$np\left(\frac{\xi}{p} - \frac{1}{2} \frac{\xi^2}{p^2} + \dots\right) + nq\left(-\frac{\xi}{q} - \frac{1}{2} \frac{\xi^2}{q^2} - \dots\right)$$

$$\text{i.e. } -\frac{n\xi^2}{2pq} + o(\xi^3) .$$

Neglecting terms of order ξ^3 the density function of F_i is proportional to $\exp(-n\xi^2/2pq)$. The constant may be evaluated by integration to give

$$\frac{1}{\sqrt{2\pi}} \left(\frac{n}{pq}\right)^{\frac{1}{2}} \exp \left\{-\frac{n}{2pq} (F_i - p)^2\right\}$$

as the asymptotic density function of F_i . Thus F_i is asymptotically normal with mean p and variance pq/n .

The asymptotic distribution of the sample quantile may be found by expanding F_i in the neighbourhood of y_p as follows:

$$F_i = F(y_p) + (y_i - y_p)f(y_p) + \frac{(y_i - y_p)^2}{2} f'(y_p) + \dots$$

neglecting third and following terms we have

$$y_i = y_p + \frac{F_i - p}{f(y_p)}.$$

Thus when n tends to infinity such that $i = np$ the distribution of the sample p -quantile tends to a normal distribution with mean y_p and variance $pq/nf^2(y_p)$. In a similar way the asymptotic joint density function of two sample quantiles may be obtained. Suppose that $p_1 = \int_{-\infty}^{y_1} f(y)dy$ and $p_2 = \int_{-\infty}^{y_2} f(y)dy$ and that $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. The joint density function of the random variables $F(y_i)$ and $F(y_j)$, $i < j$, is proportional to

$$[F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1} [1 - F(y_j)]^{n-j}.$$

If $F(y_i) = p_1 + \xi_1$ and $F(y_j) = p_2 + \xi_2$, this becomes

$$(p_1 + \xi_1)^{i-1} (p_2 - p_1 + \xi_2 - \xi_1)^{j-i-1} (1 - p_2 - \xi_2)^{n-j}.$$

As before, for large n such that $i = np_1$ and $j = np_2$, taking logs, expanding and neglecting terms in ξ_1^3 , ξ_2^3 , etc., and higher we have that the joint density function of $F(y_i)$ and $F(y_j)$ is proportional to

$$\exp\left\{-\frac{n}{2(p_2-p_1)}\left(\frac{p_2}{p_1}\xi_1^2 - 2\xi_1\xi_2 + \frac{q_1}{q_2}\xi_2^2\right)\right\}.$$

Comparing this with the exponent of the bivariate normal distribution we have that $F(y_i)$ and $F(y_j)$ have a bivariate normal distribution with variances and covariances given by

$$\text{var}\{F(y_i)\} = \frac{(p_2 - p_1)}{n} \frac{p_1}{p_2} \frac{1}{(1-\rho^2)}$$

$$\text{var}\{F(y_j)\} = \frac{(p_2 - p_1)}{n} \frac{q_2}{q_1} \frac{1}{(1-\rho^2)}$$

$$\text{and } \text{cov}\{F(y_i), F(y_j)\} = \frac{(p_2 - p_1)}{n} \frac{\rho^2}{(1-\rho^2)}$$

where $\rho^2 = (p_1q_2)/(p_2q_1)$.

These become $\text{var}\{F(y_i)\} = p_1q_1/n$, $\text{var}\{F(y_j)\} = p_2q_2/n$ and $\text{cov}\{F(y_i), F(y_j)\} = p_1q_2/n$.

To order $1/n$ the p_1 and p_2 sample quantiles are asymptotically bivariate normally distributed with variances and covariances given by

$$\text{Var}(z_{p_1}) = p_1q_1/nf(y_{p_1})^2, \quad \text{Var}(z_{p_2}) = p_2q_2/nf(y_{p_2})^2$$

$$\text{Cov}(z_{p_1}, z_{p_2}) = p_1 q_2 / (f(y_{p_1}) f(y_{p_2})) .$$

The generalization of these results to k sample quantiles is as follows:

For k given real numbers for which

$$0 < p_1 < p_2 < \dots < p_k < 1 ,$$

let the p_i -quantile of the population be y_{p_i} ,

$$\text{i.e.} \quad p_i = \int_{-\infty}^{y_{p_i}} f(y) dy \quad i = 1, 2, \dots, k .$$

Then the joint distribution of the k sample quantiles

$z_{p_1}, z_{p_2}, \dots, z_{p_k}$ tends to a k -dimensional normal distribution

with means $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ and variance-covariance matrix

$$\frac{1}{n} \Delta = \frac{1}{n} \begin{bmatrix} \frac{p_1 q_1}{f_1^2} & \frac{p_1 q_2}{f_1 f_2} & \dots & \frac{p_1 q_k}{f_1 f_k} \\ \frac{p_1 q_2}{f_1 f_2} & \frac{p_2 q_2}{f_2^2} & \dots & \frac{p_2 q_k}{f_2 f_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_1 q_k}{f_1 f_k} & \frac{p_2 q_k}{f_2 f_k} & \dots & \frac{p_k q_k}{f_k^2} \end{bmatrix} .$$

where $f_i = f(y_{p_i})$ and $q_i = 1 - p_i$, $i = 1, 2, \dots, k$, as $n \rightarrow \infty$.

CHAPTER 2

DEVELOPMENT OF LINEAR ESTIMATORS FOR LARGE AND
SMALL SAMPLES

2.1 A family of proportion -p estimators for large samples

In this section we consider a family of estimators based on the difference between the means of a proportion p of the observations from each end of the ordered sample. It is shown below that for a wide range of values of p the asymptotic efficiency of the estimator is greater than 95 per cent. From this family of highly efficient estimators one having a particularly simple form is selected as a quick efficient estimator of the standard deviation. This estimator is given by one-third of the difference between the means of the largest one sixth and the smallest one-sixth of the observations.

Proportion -p estimators

Suppose $y_1 \leq y_2 \leq \dots \leq y_n$ is an ordered sample of n independent observations from a normal distribution with mean μ and variance σ^2 .

Let x_p be such that a proportion p of the standard normal distribution lies to the right of x_p

$$\text{i.e. } p = \int_{x_p}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du ,$$

$$\text{and let } f(x_p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_p^2}$$

$$\text{Suppose } S = \left\{ \sum_{i=n+1-r}^n y_i - \sum_{i=1}^r y_i \right\} / r$$

where $r = np$, rounded up to the nearest whole number if np is not an integer.

It is shown below that the proportion $-p$ estimator, σ_p^* , formed by multiplying S by $p/2f(x_p)$ is an asymptotically unbiased estimator of the standard deviation σ . The asymptotic efficiency of σ_p^* relative to the root mean square estimator $s = \sqrt{[\sum (y_i - \bar{y})^2 / (n - 1)]}$ is shown to be

$$[f(x_p)]^2 / \{p + [x_p p - f(x_p)][x_p - 2x_p p + 2f(x_p)]\}.$$

Asymptotic expectation and asymptotic efficiency of σ_p^*

The asymptotic properties of σ_p^* were derived using the results due to F. Mosteller, outlined in the previous section.

Let x_{p_i} be such that $p_i = \int_{-\infty}^{x_{p_i}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$.

Then y_i is asymptotically normally distributed with mean

$\sigma x_{p_i} + \mu$ and variance $\frac{p_i(1-p_i)}{nf(x_{p_i})^2} \sigma^2$, and the covariance of

y_i and $y_j (j > i)$ is

$$\frac{p_i(1-p_j)\sigma^2}{nf(x_{p_i})f(x_{p_j})}.$$

Let $\bar{y}_1 = \left(\sum_{i=1}^r y_i \right) / r$

$$\text{and } \bar{y}_2 = \left(\sum_{i=n-r+1}^n y_i \right) / r$$

$$\text{then } S = \bar{y}_2 - \bar{y}_1.$$

The asymptotic expectation of $\bar{y}_1$ is given by

$$\begin{aligned} E(\bar{y}_1) &= \lim_{n \rightarrow \infty} \frac{1}{r} \sum_{i=1}^r (\sigma x_{p_i} + \mu) / r \\ &= \mu + \sigma \lim_{n \rightarrow \infty} \frac{1}{r} \sum_{i=1}^r x_{p_i} \\ &= \mu + \sigma \frac{1}{p} \int_0^p x_{p_i} dp_i \\ &= \mu - \frac{\sigma f(x_p)}{p} \end{aligned}$$

Similarly the asymptotic expectation of $\bar{y}_2$ is

$$E(\bar{y}_2) = \mu + \frac{\sigma f(x_p)}{p}.$$

Thus the asymptotic expectation of σ_p^* is σ .

The asymptotic variance of $\bar{y}_1$ is

$$\begin{aligned} \text{Var}(\bar{y}_1) &= \sigma^2 \lim_{n \rightarrow \infty} \left\{ \frac{2}{r^2} \sum_{i=1}^r \sum_{j=1}^r \frac{p_i(1-p_j)}{nf(x_{p_i})f(x_{p_j})} \right. \\ &\quad \left. - \frac{1}{r^2} \sum_{i=1}^r \frac{p_i(1-p_i)}{nf(x_{p_i})^2} \right\} \\ &= \frac{\sigma^2}{np^2} \{ p + [x_p - px_p + f(x_p)] [px_p - f(x_p)] \} \quad (1) \end{aligned}$$

and the asymptotic covariance of $\bar{y}_1$ and $\bar{y}_2$ is

$$\begin{aligned} \text{Cov}(\bar{y}_1, \bar{y}_2) &= \sigma^2 \lim_{n \rightarrow \infty} \left\{ \frac{1}{r^2} \sum_{i=1}^r \sum_{j=n-r+1}^n \frac{p_i(1-p_i)}{nf(x_{p_i})f(x_{p_j})} \right\} \\ &= \frac{\sigma^2}{np^2} [f(x_p) - x_p p]^2 \end{aligned} \quad (2)$$

Equations (1) and (2), which have been given previously in a different form by R. Mead (1966), were used to find the asymptotic variance of σ_p^* .

This is given by

$$\text{Var}(\sigma_p^*) = \frac{\sigma^2}{2nf^2(x_p)} \{p + [x_p p - f(x_p)] [x_p - 2x_p p + 2f(x_p)]\}$$

The asymptotic efficiency of σ_p^* relative to s , i.e.

$\text{Var}(s)/\text{Var}(\sigma_p^*)$, is thus

$$f^2(x_p) / \{p + [x_p p - f(x_p)] [x_p - 2x_p p + 2f(x_p)]\}$$

Table 1 gives the asymptotic efficiencies of σ_p^* and the corresponding multiplying factors for several values of p in the range 0 to 0.45. Any proportion $-p$ estimator with p greater than 0.1 or less than 0.4 is about 90 per cent or more efficient. The maximum efficiency occurs when p is approximately $\frac{1}{5}$ but almost as efficient is σ_p^* for $p = \frac{1}{6}$ which has the attraction that the multiplying factor is very nearly $\frac{1}{3}$ (0.3335).

Table 1

Asymptotic efficiencies of σ_p^* relative to s and corresponding multiplying factors for several values of p .

p	Multiplying factor $p/2f(x_p)$	Asymptotic eff. of σ_p^*	p	Multiplying factor $p/2f(x_p)$	Asymptotic eff. of σ_p^*
0.025	0.2138	0.5415	0.250	0.3934	0.9626
0.050	0.2425	0.7205	0.275	0.4121	0.9558
0.075	0.2649	0.8218	0.300	0.4314	0.9468
0.100	0.2849	0.8838	0.325	0.4515	0.9364
0.125	0.3037	0.9232	0.350	0.4725	0.9252
0.150	0.3217	0.9473	0.375	0.4945	0.9138
0.175	0.3394	0.9604	0.400	0.5177	0.9028
0.200	0.3572	0.9662	0.425	0.5423	0.8927
0.225	0.3751	0.9665	0.450	0.5685	0.8842

The estimator $\sigma_{1/6}^*$ is thus very easy to remember and is simply applied to samples which are already partially ordered. It is asymptotically unbiased and has asymptotic efficiency 95.6 per cent.

Small sample bias and variance of the estimator $\sigma_{1/6}^*$

The tables of expected values of order statistics from a normal distribution, given by D. Teichroew (1956), were used to compute the expected values of $\sigma_{1/6}^*$ for sample sizes up to 20. These are compared in table 2 with the expected values of the root-mean-square estimator s , given by

$$E s = \sigma \{2/(n-1)\}^{\frac{1}{2}} \Gamma(\frac{n}{2}) / \Gamma(\frac{n-1}{2}) .$$

It is clear from table 2 that except for sample sizes under 20 the bias is unimportant being less than 10 per cent.

Table 2

$E\sigma_{1/6}^*/Es$ for sample sizes 2(1)20 .

n	$E\sigma_{1/6}^*/Es$	n	$E\sigma_{1/6}^*/Es$
2	0.471	11	0.905
3	0.637	12	0.936
4	0.745	13	0.835
5	0.825	14	0.864
6	0.888	15	0.889
7	0.733	16	0.913
8	0.787	17	0.935
9	0.831	18	0.955
10	0.871	19	0.880
		20	0.900

The variances and mean square errors for $\sigma_{1/6}^*$ and s are given in table 3 for sample sizes 6, 12 and 18. The mean square error is greatly affected by the bias when the sample size is small. For small samples it is preferable to use one of the alternative linear estimators to be discussed later.

Table 3

Variances and mean square errors of s and of $\sigma_{1/6}^*$ for sample sizes $n = 6, 12$ and 18 .

(Variances and mean square errors to be multiplied by σ^2).

n	Var(s)	M.S.E. of s	Var($\sigma_{1/6}^*$)	M.S.E. of $\sigma_{1/6}^*$	Eff = $\frac{\text{M.S.E. of } s}{\text{M.S.E. of } \sigma_{1/6}^*}$
6	0.0946	0.0969	0.0779	0.1020	0.950
12	0.0444	0.0449	0.0409	0.0482	0.932
18	0.0290	0.0292	0.0277	0.0312	0.936

2.2 Comparison with other large sample estimators

These proportion $-p$ estimators compare well with all other large sample estimators; they are simpler to use than others of equal efficiency and are more efficient than the simple estimators. The asymptotic theory of quantiles discussed in chapter 1 may be used to derive the properties of some of these alternative estimators. For example, F. Benson (1949) used Mostellers results to derive the variance of estimators of μ and σ based on a single pair of quantiles of order p and $1 - p$. These results had been derived earlier using a different method. The estimator of the standard deviation σ is given by

$$s' = a(z_{1-p} - z_p)$$

where a is an unbiassing constant and z_p is the sample quantile of order p , i.e. the expected value of z_p is given by $E(z_p) = y_p = \mu + x_p \sigma$. The unbiassing constant is therefore given by $1/2x_p$ and the variance of s' is

$$\begin{aligned}\text{Var}(s') &= \frac{\sigma^2}{4x_p^2 n} \left\{ \frac{p(1-p)}{f^2(x_p)} + \frac{p(1-p)}{f^2(x_p)} - \frac{2p^2}{f^2(x_p)} \right\} \\ &= \frac{\sigma^2 p(1-2p)}{2x_p^2 f^2(x_p) n}\end{aligned}$$

When $p = 0.25$ the estimator s' is the semi-interquartile range and has efficiency 36.8 per cent. The estimator of this form which is most efficient may be found by differentiating $\text{Var}(s')$ with respect to p and equating to zero. The resulting estimator has $p = 0.0692$ and corresponding x_p and $f(x_p)$ given by 1.482 and 0.13304 respectively. The efficiency of this estimator is about 65 per cent.

It is interesting to note that the "seven-eighths" rule for estimating σ given by M.H. Quenouille (1959) also employs the factor $\frac{1}{3}$. His estimator consists of determining one-third of the difference between the quantiles (x_p, x_{1-p}) with $p = \frac{1}{16}$. The efficiency of this estimator is 0.62. M.H. Quenouille also gives some alternatives which remain relatively simple yet have greater efficiency than the ones above. These are

$$s' = \frac{1}{4} \left(z_{\frac{15}{16}} + \frac{3}{4} z_{\frac{3}{4}} - \frac{3}{4} z_{\frac{1}{4}} - z_{\frac{1}{16}} \right)$$

with efficiency 73 per cent, and

$$s' = \frac{1}{6} \left(z_{\frac{32}{33}} + z_{\frac{6}{7}} - z_{\frac{1}{7}} - z_{\frac{1}{33}} \right)$$

with efficiency 81 per cent.

R. Mead (1966) considered several estimators based on dividing the ordered sample into equinumeric groups and forming

a linear combination of the means of the observations in each group. His estimators are

$$\begin{aligned}
 s_2' &= 0.62666 (\bar{y}_2 - \bar{y}_1) \\
 s_3' &= 0.45838 (\bar{y}_3 - \bar{y}_1) \\
 s_4' &= 0.36927 (\bar{y}_4 - \bar{y}_1) + 0.09431 (\bar{y}_3 - \bar{y}_2) \\
 s_5' &= 0.31213 (\bar{y}_5 - \bar{y}_1) + 0.11860 (\bar{y}_4 - \bar{y}_2) \\
 s_6' &= 0.27177 (\bar{y}_6 - \bar{y}_1) + 0.12373 (\bar{y}_5 - \bar{y}_2) + \\
 &\quad 0.03844 (\bar{y}_4 - \bar{y}_3)
 \end{aligned}$$

where s_k' is based on k groups and $\bar{y}_i$ is the mean of the observations in the i^{th} group.

Mead showed that the asymptotic efficiencies of these estimators are as given below:

k	2	3	4	5	6
Efficiency of s_k'	87.6	93.3	95.6	96.8	97.6

The estimator s_3' is equivalent to $\sigma_{1/3}^*$. The efficiency of $\sigma_{1/6}^*$ compares well with all these estimators and is simpler to compute.

2.3 Small sample estimators and the Generalized Least Squares Theorem

The earliest small sample estimator of the standard deviation of a normal distribution was the range and this is still used a great deal in quality control procedures. Although its distribution function and its main properties have been known for some time it must be regarded, except for very small sample sizes, as a rather poor estimator since its efficiency decreases very rapidly as the sample size increases.

The problem of finding the most efficient linear estimators of the parameters of a general distribution was considered by E.H. Lloyd (1952) using the theorem of Generalized Least Squares. This theorem states that if $\underline{y}$ is a vector of observations from a distribution with unknown parameter vector $\underline{\theta}$ such that the expected value of $\underline{y}$ is of the form

$$E(\underline{y}) = \underline{A}\underline{\theta},$$

where $\underline{A}$ is a matrix of known coefficients, and if the variance-covariance matrix of $\underline{y}$ is known up to a scale factor, i.e. $V(\underline{y}) = \underline{V}w^2$, then the least squares estimator of $\underline{\theta}$ is $\hat{\underline{\theta}}$ where

$$\hat{\underline{\theta}} = (\underline{A}'\underline{V}^{-1}\underline{A})^{-1}\underline{A}'\underline{V}^{-1}\underline{y}$$

and $V(\hat{\underline{\theta}}) = (\underline{A}'\underline{V}^{-1}\underline{A})^{-1}w^2$.

This estimator $\hat{\theta}$ is the minimum variance unbiased linear estimator of θ .

Lloyd pointed out that the conditions of this theorem are satisfied if $\underline{y}$ is the vector of ordered observations from a distribution with unknown mean μ and unknown variance σ^2 , since in this case

$$E(\underline{y}) = (\underline{1} : \underline{m}) \begin{pmatrix} \mu \\ \sigma \end{pmatrix},$$

where $\underline{1}$ is a column vector of ones and $\underline{m}$ is the vector of means of the ordered observations,

and $V(\underline{y}) = V\sigma^2$,

where V is the variance covariance matrix of the standardized order statistics. For any continuous distribution $\underline{m}$ and V may be found using the equations of chapter 1. Therefore, from the general formulae of the theorem, we have that the best linear estimators of μ and σ are

$$\hat{\mu} = -\underline{m}'\underline{\Gamma}\underline{y} \quad \text{and} \quad \hat{\sigma} = \underline{1}'\underline{\Gamma}\underline{y},$$

where $\underline{\Gamma} = V^{-1}(\underline{1}\underline{m}' - \underline{m}\underline{1}')V^{-1}/\Delta$

and $\Delta = (\underline{1}'V^{-1}\underline{1})(\underline{m}'V^{-1}\underline{m}) - (\underline{1}'V^{-1}\underline{m})^2$.

The variances and covariances of these estimators are

$$\text{Var}(\hat{\mu}) = (\underline{m}'V^{-1}\underline{m})\sigma^2/\Delta, \quad \text{Var}(\hat{\sigma}) = (\underline{1}'V^{-1}\underline{1})\sigma^2/\Delta$$

$$\text{and } \text{Cov}(\hat{\mu}, \hat{\sigma}) = -(\mathbf{l}'\mathbf{V}^{-1}\mathbf{m})\sigma^2/\Delta.$$

Provided $\mathbf{m}$ and $\mathbf{V}$ are available these equations may be used to determine best linear estimators of μ and σ for any continuous distribution, symmetric or unsymmetric. The vector $\mathbf{y}$ of ordered observations may be complete or incomplete. Of particular interest is the case where $\mathbf{y}$ is a censored sample, i.e. an ordered sample with some observations omitted from either or both ends.

The equations simplify considerably when we have a complete sample from a symmetric distribution since in this case $\mathbf{l}'\mathbf{V}^{-1}\mathbf{m} = 0$. The equations for the estimators and their variances and covariance take the form

$$\hat{\mu} = \sum y/n, \quad \hat{\sigma} = \mathbf{m}'\mathbf{V}^{-1}\mathbf{y}/\mathbf{m}'\mathbf{V}^{-1}\mathbf{m},$$

$$\text{Var}(\hat{\mu}) = \sigma^2/n, \quad \text{Cov}(\hat{\mu}, \hat{\sigma}) = 0, \quad \text{Var}(\hat{\sigma}) = \sigma^2/\mathbf{m}'\mathbf{V}^{-1}\mathbf{m}.$$

Exact values of the coefficients used to determine the best linear estimator of the standard deviation for a normal population were computed and tabulated by Sarhan and Greenberg (1956, 1958). They considered all forms of censoring from either end of the ordered sample and tabulated the coefficients $\mathbf{l}'\mathbf{V}^{-1}\mathbf{m}$ and the variance $\mathbf{l}'\mathbf{V}^{-1}\mathbf{l}/\Delta$ for samples of size $n \leq 20$ using Teichroew's tables for $\mathbf{m}$ and their own tables for the variance-covariance matrix $\mathbf{V}$. For complete samples these estimators have high efficiency, about 99 per cent, relative to the unbiased root-mean-square

estimator. However, the efficiency of estimators based on censored samples decreases very quickly as more observations are omitted.

2.4 Approximate coefficients for the best linear estimator of the standard deviation for complete normal samples

As mentioned previously, because of the computational difficulties involved in evaluating the required integrals these tables do not extend beyond samples of size 20. S.S. Shapiro and M.B. Wilk (1965) overcame this limitation when proposing a test for normality based on order statistics by using an approximation procedure. For use with their test they computed a table of coefficients which are multiples of the coefficients required for the best linear estimators of the standard deviation. The same approximation was employed in this section to extend Sarhan and Greenberg's table for samples up to size 50. The variances and efficiencies of the estimators are also given and a comparison between the exact and approximate values for small sample sizes is made.

Approximation Procedure

Let $\tilde{m}$ be the vector of expected values and $\tilde{V}$ the variance-covariance matrix of a complete ordered sample of size n from a standard normal distribution. Suppose $\tilde{y}$ is the vector of ordered observations. We have seen that the best linear estimator of the standard deviation σ is given by $\hat{\sigma} = (\tilde{m}'\tilde{V}^{-1}\tilde{y})/(\tilde{m}'\tilde{V}^{-1}\tilde{m})$ and its variance is $\sigma^2/(\tilde{m}'\tilde{V}^{-1}\tilde{m})$.

Using Shapiro and Wilk's notation let $\tilde{a}^{*'} = \tilde{m}'\tilde{V}^{-1}$,

$R^2 = \underline{m}' \underline{V}^{-1} \underline{m}$ and $C^2 = \underline{m}' \underline{V}^{-1} \underline{V}^{-1} \underline{m}$. Then $\hat{\sigma} = (\underline{a}^* \underline{y}) / R^2 = \underline{b}' \underline{y}$ where $\underline{b}'$ is the required vector of coefficients.

Shapiro and Wilk suggested the approximation $\hat{\underline{a}}^*$ for $\underline{a}^*$ where

$$\hat{a}_i^* = 2m_i, \quad i = 2, 3, \dots, n-1. \quad (1)$$

$$\text{and } -\hat{a}_1^* = \hat{a}_n^* = \begin{cases} [C^2 \Gamma(\frac{1}{2}n) / \{\sqrt{2} \Gamma(\frac{n+1}{2})\}]^{\frac{1}{2}}, & n \leq 20 \\ [C^2 \Gamma(\frac{n+1}{2}) / \{\sqrt{2} \Gamma(\frac{1}{2}n+1)\}]^{\frac{1}{2}}, & n > 20 \end{cases} \quad (2)$$

Exact values of R^2 and C^2 were computed for $n \leq 20$ and straight lines fitted by least squares. The resulting equations are

$$\begin{aligned} R^2 &= -2.411 + 1.981 n \\ C^2 &= -2.722 + 4.083 n \end{aligned} \quad (3)$$

In both cases the residual sum of squares is so small compared to the regression sum of squares that errors introduced by extrapolation using these equations for larger values of n will be small.

The expected values $\underline{m}$ for larger samples have been tabulated by H.L. Harter (1961) and were used to compute $\hat{\underline{a}}^*$, and hence $\hat{b}(=\hat{\underline{a}}^*/R^2)$, for all sample sizes up to 50.

The exact and approximate values of the coefficients are compared in table 4 for $n = 5, 10, 15$ and 20 .

Table 4

Exact and approximate coefficients $\{b_{n-i+1}\}$ for
 $n = 5, 10, 15$ and 20 .

i	n	5		10		15		20	
		Exact	Approx	Exact	Approx	Exact	Approx	Exact	Approx
1		.3724	.3848	.2044	.2020	.1444	.1436	.1128	.1136
2		.1352	.1321	.1172	.1151	.0927	.0914	.0765	.0757
3		.0000	.0000	.0763	.0754	.0699	.0694	.0611	.0608
4				.0436	.0432	.0526	.0524	.0497	.0495
5				.0142	.0141	.0379	.0378	.0402	.0401
6						.0247	.0246	.0318	.0317
7						.0122	.0121	.0241	.0241
8						.0000	.0000	.0169	.0169
9								.0101	.0101
10								.0033	.0033

It may be seen that as n increases the error in the approximation becomes very small, the largest error for samples of size 20 being about 1 per cent. The changed form of the approximation for $\hat{a}_n^*$ was suggested by Shapiro and Wilk because, when the effect of the approximate C^2 is eliminated, the exact value and the first approximation appeared to cross at $n = 19$.

Further comparisons between the exact values, these approximations and more complex approximations suggested by R.L. Plackett (1958) were considered by these authors to justify use of equations (1), (2) and (3) in their test for normality.

Variances and efficiencies of these best linear estimators

The variance of $\hat{\sigma}$, i.e. $\sigma^2 / (\underline{m}' \underline{V}^{-1} \underline{m})$ may be approximated by σ^2 / R^2 .

The efficiency of the estimator relative to the biased root-mean-square estimator s , may be approximated by

$$\begin{aligned} \text{Eff.} &= \frac{\text{Mean square error of } s}{(\sigma^2 / R^2)} \\ &= 2R^2(1 - E_s), \end{aligned}$$

where
$$E_s = \left(\frac{2}{n-1}\right)^{\frac{1}{2}} \Gamma(\frac{1}{2}n) / \Gamma\left(\frac{n-1}{2}\right).$$

The efficiency relative to the unbiased root-mean-square estimator s' given by $s' = s / E_s$, may be approximated by

$$\text{Eff}^* = R^2((Es)^{-2}-1)$$

The exact and approximate variances and efficiencies are compared in table 5.

Table of coefficients

The approximate coefficients $\{b_{n-i+1}\}$ for the best linear estimators for $20 < n \leq 50$ are listed with the corresponding variances and efficiencies in table 6. Exact values for $n \leq 20$ are included for completeness.

It is possible to obtain this table directly from the table of coefficients given by Shapiro and Wilk using

$$\hat{b}_i = a_i / \sum_{i=1}^n a_i m_i, \text{ where the } a_i \text{'s are their tabulated values,}$$

but Harter's table and equations (1), (2) and (3) were used to avoid accumulating rounding errors.

Although this approximation procedure might be extended beyond $n = 50$ the computations involved in determining the estimate from a sample become extensive and simpler estimators which are asymptotically highly efficient are more appropriate. For example the estimators based on means of groups of observations, discussed in section 2.1.

Table 5

Exact and approximate variances and efficiencies of the best linear estimator.

Sample size n	Variance		Efficiency ¹		Efficiency ^{*2}	
	Exact	Approx	Exact	Approx	Exact	Approx
2	.5708	.6448	.708	.627	1.000	.885
3	.2755	.2831	.826	.804	.992	.965
4	.1801	.1814	.874	.868	.989	.981
5	.1333	.1334	.900	.900	.988	.987
6	.1057	.1055	.917	.918	.988	.990
7	.0875	.0873	.929	.931	.989	.991
8	.0746	.0744	.938	.940	.989	.991
9	.0650	.0649	.944	.946	.989	.992
10	.0576	.0575	.949	.951	.990	.992
11	.0517	.0516	.954	.955	.990	.992
12	.0469	.0468	.957	.959	.990	.992
13	.0429	.0428	.960	.961	.991	.992
14	.0395	.0395	.964	.964	.992	.992
15	.0366	.0366	.966	.966	.992	.992
16	.0341	.0342	.967	.967	.992	.992
17	.0320	.0320	.968	.968	.992	.992
18	.0301	.0301	.969	.970	.992	.992
19	.0284	.0284	.971	.971	.992	.992
20	.0268	.0269	.973	.972	.993	.992

¹ Efficiency relative to biassed root-mean-square estimator.

² Efficiency^{*} relative to unbiased root-mean-square estimator.

TABLE 6

Coefficients for the best linear estimator of the standard deviation of a normal population.

For $n \leq 20$ the tabulated values are based on the exact variance-covariance matrix.

For $n > 20$ the values are obtained using the approximation formulae given.

Table 6

Coefficients $\{b_{n-i+1}\}$ for the best linear estimator with variances
and efficiencies.

$i \backslash n$	2	3	4	5	6	7	8	9	10
1	.8862	.5908	.4539	.3724	.3175	.2778	.2476	.2237	.2044
2		.0000	.1102	.1352	.1386	.1351	.1294	.1233	.1172
3				.0000	.0432	.0625	.0713	.0751	.0763
4						.0000	.0230	.0360	.0436
5								.0000	.0142
Variance	.5708	.2755	.1801	.1333	.1057	.0875	.0746	.0650	.0576
Efficiency ¹	.708	.826	.874	.900	.917	.929	.938	.944	.949
Efficiency ^{*2}	1.000	.992	.989	.988	.988	.989	.989	.989	.990

¹ Efficiency relative to biased root-mean-square estimator.

² Efficiency* relative to unbiased root-mean-square estimator.

Table 6 (continued)

Coefficients $\{b_{n-i+1}\}$ for the best linear estimator with variances and efficiencies.

$i \backslash n$	11	12	13	14	15	16	17	18	19	20
1	.1883	.1748	.1632	.1532	.1444	.1366	.1297	.1235	.1178	.1123
2	.1115	.1061	.1013	.0968	.0927	.0889	.0854	.0822	.0792	.0765
3	.0760	.0749	.0735	.0717	.0699	.0681	.0663	.0645	.0628	.0611
4	.0481	.0506	.0520	.0526	.0526	.0524	.0519	.0512	.0505	.0497
5	.0234	.0294	.0335	.0362	.0379	.0391	.0398	.0401	.0402	.0402
6	.0000	.0097	.0164	.0212	.0247	.0272	.0290	.0302	.0312	.0318
7			.0000	.0070	.0122	.0160	.0189	.0211	.0228	.0241
8					.0000	.0053	.0094	.0125	.0150	.0169
9							.0000	.0041	.0074	.0101
10									.0000	.0033
Variance	.0517	.0469	.0429	.0395	.0366	.0341	.0320	.0301	.0284	.0268
Efficiency ¹	.954	.957	.960	.964	.966	.967	.968	.969	.971	.973
Efficiency ^{*2}	.990	.990	.991	.992	.992	.992	.992	.992	.992	.993

¹ Efficiency relative to biased root-mean-square estimator.

² Efficiency* relative to unbiased root-mean-square estimator.

TABLE 6

APPROXIMATE COEFFICIENTS (b_{n-i+1}) FOR BEST LINEAR ESTIMATORS OF STANDARD DEVIATION (CONTINUED)

$n \backslash i$	21	22	23	24	25	26	27	28	29	30
1	0.1080	0.1041	0.1005	0.0972	0.0941	0.0913	0.0886	0.0862	0.0839	0.0817
2	0.0732	0.0708	0.0687	0.0666	0.0647	0.0629	0.0612	0.0596	0.0581	0.0567
3	0.0592	0.0577	0.0563	0.0549	0.0536	0.0524	0.0512	0.0500	0.0489	0.0479
4	0.0487	0.0478	0.0470	0.0461	0.0453	0.0445	0.0436	0.0429	0.0421	0.0413
5	0.0399	0.0396	0.0393	0.0389	0.0384	0.0380	0.0375	0.0370	0.0365	0.0360
6	0.0321	0.0324	0.0325	0.0325	0.0324	0.0323	0.0321	0.0319	0.0316	0.0314
7	0.0251	0.0258	0.0264	0.0268	0.0270	0.0272	0.0273	0.0273	0.0273	0.0272
8	0.0185	0.0197	0.0207	0.0214	0.0220	0.0225	0.0229	0.0231	0.0233	0.0235
9	0.0122	0.0139	0.0153	0.0164	0.0173	0.0181	0.0187	0.0192	0.0196	0.0199
10	0.0060	0.0083	0.0101	0.0116	0.0128	0.0139	0.0148	0.0155	0.0161	0.0166
11	0.0000	0.0027	0.0050	0.0069	0.0085	0.0098	0.0110	0.0119	0.0127	0.0134
12			0.0000	0.0023	0.0042	0.0059	0.0073	0.0084	0.0095	0.0103
13					0.0000	0.0019	0.0036	0.0050	0.0063	0.0073
14							0.0000	0.0017	0.0031	0.0044
15									0.0000	0.0015
VARIANCE	0.0255	0.0243	0.0232	0.0222	0.0212	0.0204	0.0196	0.0188	0.0182	0.0175
EFF. 1	0.9733	0.9741	0.9748	0.9755	0.9762	0.9767	0.9773	0.9778	0.9782	0.9787
EFF. 2	0.9917	0.9916	0.9916	0.9916	0.9915	0.9915	0.9915	0.9915	0.9914	0.9914

1. Efficiency relative to biased root mean square estimator.
2. Efficiency relative to unbiased root mean square estimator.

TABLE 6

APPROXIMATE COEFFICIENTS (b_{n-i+1}) FOR BEST LINEAR ESTIMATORS OF STANDARD DEVIATION (CONTINUED)

$i \backslash n$	31	32	33	34	35	36	37	38	39	40
1	0.0796	0.0777	0.0759	0.0741	0.0725	0.0709	0.0694	0.0680	0.0667	0.0654
2	0.0553	0.0540	0.0528	0.0516	0.0505	0.0494	0.0484	0.0475	0.0465	0.0456
3	0.0469	0.0459	0.0450	0.0441	0.0433	0.0424	0.0417	0.0409	0.0402	0.0395
4	0.0406	0.0399	0.0392	0.0386	0.0379	0.0373	0.0367	0.0361	0.0355	0.0350
5	0.0355	0.0350	0.0345	0.0340	0.0336	0.0331	0.0326	0.0322	0.0318	0.0313
6	0.0311	0.0308	0.0305	0.0301	0.0298	0.0295	0.0292	0.0288	0.0285	0.0282
7	0.0271	0.0270	0.0269	0.0267	0.0265	0.0263	0.0261	0.0259	0.0257	0.0254
8	0.0235	0.0236	0.0236	0.0235	0.0235	0.0234	0.0233	0.0232	0.0231	0.0229
9	0.0202	0.0204	0.0205	0.0206	0.0207	0.0207	0.0207	0.0207	0.0207	0.0206
10	0.0170	0.0174	0.0176	0.0179	0.0181	0.0182	0.0183	0.0184	0.0184	0.0185
11	0.0140	0.0145	0.0149	0.0153	0.0156	0.0158	0.0160	0.0162	0.0163	0.0164
12	0.0111	0.0117	0.0123	0.0128	0.0132	0.0135	0.0138	0.0141	0.0143	0.0145
13	0.0082	0.0090	0.0097	0.0103	0.0109	0.0113	0.0117	0.0121	0.0124	0.0126
14	0.0055	0.0064	0.0073	0.0080	0.0086	0.0092	0.0097	0.0101	0.0105	0.0109
15	0.0027	0.0038	0.0048	0.0057	0.0064	0.0071	0.0077	0.0082	0.0087	0.0091
16	0.0000	0.0013	0.0024	0.0034	0.0043	0.0050	0.0057	0.0064	0.0069	0.0074
17			0.0000	0.0011	0.0021	0.0030	0.0038	0.0045	0.0052	0.0057
18					0.0000	0.0010	0.0019	0.0027	0.0034	0.0041
19							0.0000	0.0009	0.0017	0.0024
20									0.0000	0.0008
VARIANCE	0.0169	0.0164	0.0159	0.0154	0.0149	0.0145	0.0141	0.0137	0.0134	0.0130
EFF. 1	0.9791	0.9794	0.9798	0.9801	0.9804	0.9807	0.9810	0.9813	0.9815	0.9817
EFF. 2	0.9914	0.9914	0.9913	0.9913	0.9913	0.9913	0.9913	0.9913	0.9912	0.9912

1. Efficiency relative to biased root mean square estimator.
2. Efficiency relative to unbiased root mean square estimator.

TABLE 6

APPROXIMATE COEFFICIENTS (b_{n-i+1}) FOR BEST LINEAR ESTIMATORS OF STANDARD DEVIATION (CONTINUED)

$i \backslash n$	41	42	43	44	45	46	47	48	49	50
1	0.0642	0.0630	0.0618	0.0608	0.0597	0.0587	0.0578	0.0568	0.0559	0.0551
2	0.0448	0.0440	0.0432	0.0424	0.0417	0.0410	0.0403	0.0396	0.0390	0.0384
3	0.0388	0.0382	0.0375	0.0369	0.0364	0.0358	0.0352	0.0347	0.0342	0.0337
4	0.0344	0.0339	0.0334	0.0329	0.0325	0.0320	0.0316	0.0311	0.0307	0.0303
5	0.0309	0.0303	0.0301	0.0297	0.0293	0.0290	0.0286	0.0282	0.0279	0.0275
6	0.0279	0.0276	0.0273	0.0270	0.0267	0.0264	0.0261	0.0258	0.0255	0.0252
7	0.0252	0.0250	0.0248	0.0245	0.0243	0.0241	0.0238	0.0236	0.0234	0.0232
8	0.0228	0.0226	0.0225	0.0223	0.0222	0.0220	0.0218	0.0217	0.0215	0.0213
9	0.0206	0.0205	0.0204	0.0203	0.0202	0.0201	0.0200	0.0199	0.0198	0.0196
10	0.0185	0.0185	0.0185	0.0184	0.0184	0.0183	0.0183	0.0182	0.0181	0.0181
11	0.0165	0.0166	0.0166	0.0167	0.0167	0.0167	0.0167	0.0167	0.0166	0.0166
12	0.0147	0.0148	0.0149	0.0150	0.0151	0.0151	0.0152	0.0152	0.0152	0.0152
13	0.0129	0.0131	0.0132	0.0134	0.0135	0.0136	0.0137	0.0138	0.0138	0.0139
14	0.0112	0.0114	0.0116	0.0119	0.0120	0.0122	0.0123	0.0124	0.0125	0.0126
15	0.0095	0.0098	0.0101	0.0104	0.0106	0.0108	0.0110	0.0111	0.0113	0.0114
16	0.0078	0.0082	0.0086	0.0089	0.0092	0.0094	0.0097	0.0099	0.0101	0.0102
17	0.0062	0.0067	0.0071	0.0075	0.0078	0.0081	0.0084	0.0086	0.0089	0.0091
18	0.0047	0.0052	0.0057	0.0061	0.0065	0.0068	0.0071	0.0074	0.0077	0.0079
19	0.0031	0.0037	0.0042	0.0047	0.0052	0.0056	0.0059	0.0063	0.0066	0.0068
20	0.0015	0.0022	0.0028	0.0034	0.0039	0.0043	0.0047	0.0051	0.0054	0.0058
21	0.0000	0.0007	0.0014	0.0020	0.0026	0.0031	0.0035	0.0039	0.0043	0.0047
22			0.0000	0.0007	0.0013	0.0018	0.0023	0.0028	0.0032	0.0036
23					0.0000	0.0006	0.0012	0.0017	0.0022	0.0026
24							0.0000	0.0006	0.0011	0.0016
25									0.0000	0.0005
VARIANCE	0.0127	0.0124	0.0121	0.0118	0.0115	0.0113	0.0110	0.0108	0.0106	0.0103
EFF. 1	0.9820	0.9822	0.9824	0.9826	0.9827	0.9829	0.9831	0.9832	0.9834	0.9835
EFF.* 2	0.9912	0.9912	0.9912	0.9912	0.9911	0.9911	0.9911	0.9911	0.9911	0.9911

1. Efficiency relative to biased root mean square estimator.
2. Efficiency relative to unbiased root mean square estimator.

2.5 Estimation of the standard deviation of a normal population from doubly censored samples using normal scores

In the preceding section Sarhan's and Greenberg's tables were extended beyond $n = 20$ by approximating the coefficients of the best linear estimator. An alternative approach is to develop nearly-best linear estimators whose coefficients may be derived exactly and to find approximation formulae for their variances.

In this section A.K. Gupta's estimator of the standard deviation of a normal population, depending only on the expected values of the normal order statistics, is considered and further results showing its high efficiency relative to the best linear estimator for all forms of censoring are given. Tables which facilitate computation of the estimator, based on a sample of size n , are provided for $2 \leq n \leq 50$ for complete samples and for $21 \leq n \leq 25$ for censored samples, thereby extending the tables which already exist for $n \leq 20$. Simple approximations for the variances of the estimators are also given.

Suppose that a sample of n observations is taken from a normal population with unknown mean μ and unknown variance σ^2 and that the sample is subjected to Type II censoring as defined by Gupta (1952). That is, the r_1 smallest and r_2 largest observations in the sample are not available or have

been discarded. Samples of this kind arise in life-testing procedures and when outliers have been rejected from the original sample. A review of rejection procedures has recently been given by Grubbs (1969).

We consider here an alternative estimator depending only on the normal scores, i.e. the expected values of the ordered observations. This estimator was originally proposed for singly censored samples by Gupta and later extended to doubly censored samples by Sarhan and Greenberg.

The normal scores have been evaluated and tabulated for a wide range of sample sizes by Harter (1961) and are given by both Pearson and Hartley (1966) and Fisher and Yates (1963) for all sample sizes up to 50. The coefficients required for the alternative estimator are thus readily available for all practical sample sizes.

The variance of the estimator involves the variance-covariance matrix. However, sufficiently accurate estimates of the variance may be obtained using the simple approximation procedures given below.

Derivation of the alternative estimator

Suppose that the observations remaining after censoring are ordered and denoted by

$$y_{r_1+1} \leq y_{r_1+2} \leq \dots \leq y_{n-r_2} \quad .$$

Let $\underline{y}$ be the vector of these ordered observations and let $\underline{x}$ be the vector of the standardized values

$$(y_i - \mu)/\sigma \quad (i = r_1 + 1, \dots, n - r_2);$$

$$\underline{m} = E(\underline{x}),$$

$$\underline{V} = E\{(\underline{x} - \underline{m})(\underline{x} - \underline{m})'\}$$

are the vector of normal scores and the variance-covariance matrix of the standardized ordered observations, respectively.

Gupta's alternative estimate is obtained from the equations in section 2.3 by replacing $\underline{V}^{-1}$ by the identity matrix, to give

$$\tilde{\sigma} = \underline{1}'(\underline{1} \underline{m}' - \underline{m} \underline{1}')\underline{y}/\tilde{\Delta},$$

where
$$\tilde{\Delta} = (\underline{1}'\underline{1})(\underline{m}'\underline{m}) - (\underline{1}'\underline{m})^2.$$

Also
$$\text{Var}(\tilde{\sigma}) = \underline{1}'(\underline{1} \underline{m}' - \underline{m} \underline{1}')\underline{V}(\underline{m} \underline{1}' - \underline{1} \underline{m}')\underline{1} \sigma^2/\tilde{\Delta}^2.$$

The efficiency of this alternative estimator relative to the most efficient linear estimator, i.e. $\text{Var}(\hat{\sigma})/\text{Var}(\tilde{\sigma})$, is given in Table 7 for small amounts of censoring for samples of size 20.

Table 7

Variance and percentage efficiency of $\tilde{\sigma}$ relative to $\hat{\sigma}$ for selected forms of censoring for $n = 20$

r_1	r_2	Var $\tilde{\sigma}$	efficiency	r_1	r_2	Var $\tilde{\sigma}$	efficiency
0	0	0.0269	99.91	1	3	0.0403	93.79
0	1	0.0298	97.55	1	4	0.0448	92.71
0	2	0.0332	95.11	2	2	0.0404	94.00
0	3	0.0368	93.09	2	3	0.0449	93.03
0	4	0.0409	91.40	2	4	0.0501	92.26
1	1	0.0328	96.69	3	3	0.0502	92.33
1	2	0.0364	95.12	3	4	0.0563	91.81

For a complete sample of size 20 the efficiency is 99.9 per cent and when at least one observation is censored from each end of the sample the efficiency is always greater than 90 per cent. The lowest efficiency is 85 per cent and occurs when the sample is heavily censored on one side only.

Although the efficiency of $\tilde{\sigma}$ is high for all forms of censoring it should be remembered that even in the most favourable cases censoring of any kind has a considerable effect on the variances of both $\tilde{\sigma}$ and $\hat{\sigma}$. From Table 7 it may be seen that for a sample of size 20 the variance of $\tilde{\sigma}$ when six observations have been censored is twice that of $\tilde{\sigma}$ based on the complete sample.

Complete samples

The equations giving the coefficients and variance of $\tilde{\sigma}$ simplify considerably when the sample is uncensored since in this case $1' m = 0$. The equations become

$$\tilde{\sigma} = \sum_{i=1}^n m_i y_i / \sum_{i=1}^n m_i^2$$

and
$$\text{Var}(\tilde{\sigma}) = \sum_{i=1}^n \sum_{j=1}^n m_i m_j v_{ij} / \left(\sum_{i=1}^n m_i^2 \right)^2,$$

where v_{ij} is the covariance of the i^{th} and j^{th} order statistics.

Calculation of $\tilde{\sigma}$ is further simplified if the subranges $(y_{n-i+1} - y_i)$, ($i = 1, \dots, [\frac{1}{2}n]$), are formed before multiplying by the normal scores. The estimator now takes the form

$$\tilde{\sigma} = \sum_{i=1}^{[\frac{1}{2}n]} m_i (y_{n-i+1} - y_i) / \sum_{i=1}^n m_i^2.$$

Fisher and Yates (1963) give values of $m_1^2 + \dots + m_n^2$ for $n \leq 50$ together with the table of normal scores. For convenience the reciprocals of $m_1^2 + \dots + m_n^2$ are given in Table 8.

Approximation procedure for the variance of $\tilde{\sigma}$ in complete samples

The reciprocal of the variance of $\tilde{\sigma}$ is very nearly a linear function of the sample size n . The linearity of this relationship may be summarized by the least squares equation

$$1/\text{Var}(\tilde{\sigma}) = 1.99n - 2.54.$$

Table 8

Reciprocals of the sums of squares of the normal scores for $2 \leq n \leq 50$.

n	0	10	20	30	40
1	-	0.1126	0.0536	0.0350	0.0260
2	1.5707	0.1015	0.0509	0.0339	0.0253
3	0.6981	0.0924	0.0485	0.0328	0.0247
4	0.4356	0.0848	0.0462	0.0317	0.0241
5	0.3130	0.0783	0.0442	0.0308	0.0236
6	0.2429	0.0727	0.0424	0.0298	0.0230
7	0.1979	0.0679	0.0407	0.0290	0.0225
8	0.1667	0.0636	0.0391	0.0282	0.0220
9	0.1438	0.0599	0.0376	0.0274	0.0215
10	0.1263	0.0566	0.0363	0.0267	0.0211

This equation for $n > 20$ leads to a very simple approximation namely $\text{Var}(\tilde{\sigma}) = \sigma^2 / (2n - \frac{5}{2})$ which is about equivalent to a root-mean-square estimator based on $(n - \frac{1}{4})$ degrees of freedom. This property of $\tilde{\sigma}$ together with a χ approximation to its distribution has been used to investigate the effect of using a linear estimator of σ in the usual t, χ^2 and F-tests for one and two-sample means and variances. Details of this work are given later in section 3.2.

It is of interest to compare $\tilde{\sigma}$ with the unbiased root-mean-square estimator s' defined by $s' = s/E(s)$ where $s^2 = \sum (y_i - \bar{y})^2 / (n-1)$ and $E(s) = \{2/(n-1)\}^{1/2} \Gamma(\frac{1}{2}n) / \Gamma(\frac{1}{2}(n-1))$. This estimator has variance $[\Gamma(\frac{1}{2}(n-1)) / \Gamma(\frac{1}{2}n)]^2 (n-1)/2 - 1$.

Table 9 shows $\text{Var}(\tilde{\sigma})$ and $\text{Var}(s')$ for a range of sample sizes. For $n \leq 20$ the values of $\text{Var}(\tilde{\sigma})$ were obtained using the variance-covariance matrix whereas for $n > 20$ the least squares approximation above was used.

Table 9

Comparison of the variances of the linear estimator $\tilde{\sigma}$ and the unbiased root-mean-square estimator s' .

n	Var $\tilde{\sigma}$	Var s'
5	0.13342	0.13177
10	0.05766	0.05701
15	0.03667	0.03633
20	0.02687	0.02665
25	0.02118	0.02105
30	0.01749	0.01739
35	0.01490	0.01481
40	0.01298	0.01290

Censored samples

The estimator for censored samples may more conveniently

be expressed in the form

$$\tilde{\sigma} = a \sum m_i y_i - b \sum y_i$$

where $a = (n-k) / \{ (n-k) \sum m_i^2 - (\sum m_i)^2 \}$

and $b = \sum m_i / \{ (n-1) \sum m_i^2 - (\sum m_i)^2 \}$.

Here $k = r_1 + r_2$ and the summations are over i where $i = r_1 + 1, \dots, n - r_2$.

Values of a and b for all forms of censoring for sample sizes $21 \leq n \leq 25$ are given in Table 10 thereby supplementing the tables for $n \leq 20$ in Sarhan and Greenberg. More extensive tables are available from the author.

The variance of $\tilde{\sigma}$ for censored samples

Sarhan and Greenberg point out that for given n and fixed size of sample left after censoring, i.e. $r_1 + r_2 = k$, a constant, the variance of the best linear estimator of σ is remarkably constant independently of the individual values of r_1 and r_2 . The same is true of the alternative estimator $\tilde{\sigma}$.

From the approximation formula for $\text{Var}(\tilde{\sigma})$ for complete samples we have that $u = (n - \frac{1}{4}) \text{Var}(\tilde{\sigma})$ is approximately independent of the sample size n . For censored samples with $n = 5, 10, 15$ and 20 the values of u averaged over all r_1 and r_2 such that $r_1 + r_2 = k$, were plotted against $k/(n-1)$. The resulting plots appear to converge fairly rapidly to the

TABLE 10. VALUES OF a AND b FOR ALL FORMS OF CENSORING
FOR SAMPLE SIZES $21 \leq n \leq 25$.

$n=21$

		r_2										
		0	1	2	3	4	5	6	7	8	9	
r_1	0	0.034 0.000	0.067 -0.006	0.080 -0.014	0.095 -0.024	0.111 -0.033	0.129 -0.050	0.150 -0.069	0.176 -0.092	0.207 -0.122	0.244 -0.162	0
	1	142.766 -8.449	0.087 0.000	0.107 -0.009	0.129 -0.020	0.156 -0.033	0.187 -0.054	0.225 -0.080	0.272 -0.114	0.331 -0.161	0.409 -0.225	1
	2	35.186 -4.184	138.773 -24.755	0.135 0.000	0.167 -0.012	0.206 -0.029	0.253 -0.052	0.313 -0.085	0.390 -0.131	0.491 -0.196	0.629 -0.291	2
	3	13.747 -2.470	33.682 -8.070	130.874 -39.291	0.212 0.000	0.267 -0.018	0.337 -0.045	0.428 -0.084	0.549 -0.143	0.717 -0.231	0.955 -0.367	3
	4	6.644 -1.608	12.941 -3.916	31.223 -11.364	119.351 -50.934	0.345 0.000	0.446 -0.029	0.584 -0.075	0.775 -0.147	1.051 -0.264	1.467 -0.459	4
	5	3.626 -1.112	6.137 -2.259	11.760 -5.062	27.879 -13.785	104.504 -58.590	0.595 0.000	0.804 -0.051	1.108 -0.138	1.572 -0.292	2.322 -0.571	5
	6	2.132 -0.798	3.274 -1.431	5.446 -2.726	10.239 -5.797	23.761 -15.071	86.931 -61.344	1.127 0.000	1.626 -0.100	2.438 -0.297	3.865 -0.703	6
	7	1.311 -0.586	1.871 -0.956	2.819 -1.624	4.590 -2.955	8.427 -6.018	19.029 -15.002	67.369 -58.453	2.476 0.000	3.985 -0.240	6.953 -0.835	7
	8	0.823 -0.433	1.107 -0.655	1.545 -1.019	2.271 -1.657	3.597 -2.890	6.394 -9.636	13.888 -13.405	46.825 -49.500	7.057 0.000	14.155 -0.844	8
	9	0.511 -0.315	0.652 -0.447	0.854 -0.646	1.156 -0.960	1.642 -1.492	2.498 -2.477	4.228 -4.576	8.632 -10.208	26.806 -34.768	35.691 0.000	9
	0	0.292 -0.214	0.352 -0.284	0.432 -0.382	0.542 -0.522	0.698 -0.732	0.933 -1.065	1.320 -1.642	2.040 -2.773	3.689 -5.514	9.637 -16.011	
		10	11	12	13	14	15	16	17	18	19	

[illegible]

1997, 1998, 1999, 2000, 2001, 2002, 2003, 2004, 2005, 2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022, 2023, 2024, 2025, 2026, 2027, 2028, 2029, 2030, 2031, 2032, 2033, 2034, 2035, 2036, 2037, 2038, 2039, 2040, 2041, 2042, 2043, 2044, 2045, 2046, 2047, 2048, 2049, 2050, 2051, 2052, 2053, 2054, 2055, 2056, 2057, 2058, 2059, 2060, 2061, 2062, 2063, 2064, 2065, 2066, 2067, 2068, 2069, 2070, 2071, 2072, 2073, 2074, 2075, 2076, 2077, 2078, 2079, 2080, 2081, 2082, 2083, 2084, 2085, 2086, 2087, 2088, 2089, 2090, 2091, 2092, 2093, 2094, 2095, 2096, 2097, 2098, 2099, 2100, 2101, 2102, 2103, 2104, 2105, 2106, 2107, 2108, 2109, 2110, 2111, 2112, 2113, 2114, 2115, 2116, 2117, 2118, 2119, 2120, 2121, 2122, 2123, 2124, 2125, 2126, 2127, 2128, 2129, 2130, 2131, 2132, 2133, 2134, 2135, 2136, 2137, 2138, 2139, 2140, 2141, 2142, 2143, 2144, 2145, 2146, 2147, 2148, 2149, 2150, 2151, 2152, 2153, 2154, 2155, 2156, 2157, 2158, 2159, 2160, 2161, 2162, 2163, 2164, 2165, 2166, 2167, 2168, 2169, 2170, 2171, 2172, 2173, 2174, 2175, 2176, 2177, 2178, 2179, 2180, 2181, 2182, 2183, 2184, 2185, 2186, 2187, 2188, 2189, 2190, 2191, 2192, 2193, 2194, 2195, 2196, 2197, 2198, 2199, 2200, 2201, 2202, 2203, 2204, 2205, 2206, 2207, 2208, 2209, 2210, 2211, 2212, 2213, 2214, 2215, 2216, 2217, 2218, 2219, 2220, 2221, 2222, 2223, 2224, 2225, 2226, 2227, 2228, 2229, 2230, 2231, 2232, 2233, 2234, 2235, 2236, 2237, 2238, 2239, 2240, 2241, 2242, 2243, 2244, 2245, 2246, 2247, 2248, 2249, 2250, 2251, 2252, 2253, 2254, 2255, 2256, 2257, 2258, 2259, 2260, 2261, 2262, 2263, 2264, 2265, 2266, 2267, 2268, 2269, 2270, 2271, 2272, 2273, 2274, 2275, 2276, 2277, 2278, 2279, 2280, 2281, 2282, 2283, 2284, 2285, 2286, 2287, 2288, 2289, 2290, 2291, 2292, 2293, 2294, 2295, 2296, 2297, 2298, 2299, 2300, 2301, 2302, 2303, 2304, 2305, 2306, 2307, 2308, 2309, 2310, 2311, 2312, 2313, 2314, 2315, 2316, 2317, 2318, 2319, 2320, 2321, 2322, 2323, 2324, 2325, 2326, 2327, 2328, 2329, 2330, 2331, 2332, 2333, 2334, 2335, 2336, 2337, 2338, 2339, 2340, 2341, 2342, 2343, 2344, 2345, 2346, 2347, 2348, 2349, 2350, 2351, 2352, 2353, 2354, 2355, 2356, 2357, 2358, 2359, 2360, 2361, 2362, 2363, 2364, 2365, 2366, 2367, 2368, 2369, 2370, 2371, 2372, 2373, 2374, 2375, 2376, 2377, 2378, 2379, 2380, 2381, 2382, 2383, 2384, 2385, 2386, 2387, 2388, 2389, 2390, 2391, 2392, 2393, 2394, 2395, 2396, 2397, 2398, 2399, 2400, 2401, 2402, 2403, 2404, 2405, 2406, 2407, 2408, 2409, 2410, 2411, 2412, 2413, 2414, 2415, 2416, 2417, 2418, 2419, 2420, 2421, 2422, 2423, 2424, 2425, 2426, 2427, 2428, 2429, 2430, 2431, 2432, 2433, 2434, 2435, 2436, 2437, 2438, 2439, 2440, 2441, 2442, 2443, 2444, 2445, 2446, 2447, 2448, 2449, 2450, 2451, 2452, 2453, 2454, 2455, 2456, 2457, 2458, 2459, 2460, 2461, 2462, 2463, 2464, 2465, 2466, 2467, 2468, 2469, 2470, 2471, 2472, 2473, 2474, 2475, 2476, 2477, 2478, 2479, 2480, 2481, 2482, 2483, 2484, 2485, 2486, 2487, 2488, 2489, 2490, 2491, 2492, 2493, 2494, 2495, 2496, 2497, 2498, 2499, 2500, 2501, 2502, 2503, 2504, 2505, 2506, 2507, 2508, 2509, 2510, 2511, 2512, 2513, 2514, 2515, 2516, 2517, 2518, 2519, 2520, 2521, 2522, 2523, 2524, 2525, 2526, 2527, 2528, 2529, 2530, 2531, 2532, 2533, 2534, 2535, 2536, 2537, 2538, 2539, 2540, 2541, 2542, 2543, 2544, 2545, 2546, 2547, 2548, 2549, 2550, 2551, 2552, 2553, 2554, 2555, 2556, 2557, 2558, 2559, 2560, 2561, 2562, 2563, 2564, 2565, 2566, 2567, 2568, 2569, 2570, 2571, 2572, 2573, 2574, 2575, 2576, 2577, 2578, 2579, 2580, 2581, 2582, 2583, 2584, 2585, 2586, 2587, 2588, 2589, 2590, 2591, 2592, 2593, 2594, 2595, 2596, 2597, 2598, 2599, 2600, 2601, 2602, 2603, 2604, 2605, 2606, 2607, 2608, 2609, 2610, 2611, 2612, 2613, 2614, 2615, 2616, 2617, 2618, 2619, 2620, 2621, 2622, 2623, 2624, 2625, 2626, 2627, 2628, 2629, 2630, 2631, 2632, 2633, 2634, 2635, 2636, 2637, 2638, 2639, 2640, 2641, 2642, 2643, 2644, 2645, 2646, 2647, 2648, 2649, 2650, 2651, 2652, 2653, 2654, 2655, 2656, 2657, 2658, 2659, 2660, 2661, 2662, 2663, 2664, 2665, 2666, 2667, 2668, 2669, 2670, 2671, 2672, 2673, 2674, 2675, 2676, 2677, 2678, 26

58 r

TABLE 10. VALUES OF a AND b FOR ALL FORMS OF CENSORING FOR SAMPLE SIZES $21 \leq n \leq 25$.

n=23												r ₂	
0	1	2	3	4	5	6	7	8	9	10			
	0.048 0.000	0.060 -0.005	0.071 -0.011	0.082 -0.019	0.095 -0.028	0.109 -0.039	0.125 -0.053	0.144 -0.070	0.167 -0.091	0.193 -0.118	0.225 -0.152	0	
10	171.056 -9.248	0.076 0.000	0.092 -0.007	0.109 -0.016	0.129 -0.027	0.153 -0.041	0.180 -0.059	0.213 -0.083	0.253 -0.113	0.303 -0.154	0.366 -0.208	1	
9	42.258 -4.588	167.046 -27.202	0.114 0.000	0.138 -0.009	0.166 -0.022	0.200 -0.038	0.241 -0.061	0.292 -0.091	0.357 -0.131	0.439 -0.188	0.549 -0.266	2	
8	16.577 -2.716	40.755 -8.903	159.154 -43.545	0.171 0.000	0.210 -0.013	0.258 -0.032	0.318 -0.058	0.394 -0.095	0.493 -0.147	0.626 -0.222	0.808 -0.333	3	
7	8.061 -1.776	15.772 -4.343	38.287 -12.677	147.511 -57.215	0.265 0.000	0.330 -0.020	0.416 -0.050	0.528 -0.093	0.680 -0.159	0.892 -0.258	1.196 -0.413	4	
6	4.437 -1.235	7.555 -2.524	14.587 -5.697	34.914 -15.650	132.476 -67.235	0.423 0.000	0.546 -0.032	0.713 -0.082	0.948 -0.163	1.289 -0.293	1.807 -0.511	5	
5	2.642 -0.895	4.087 -1.616	6.862 -3.106	13.052 -6.676	30.729 -17.579	114.454 -72.685	0.724 0.000	0.976 -0.056	1.345 -0.152	1.911 -0.321	2.829 -0.631	6	
4	1.654 -0.665	2.384 -1.096	3.633 -1.883	6.000 -3.471	11.208 -7.182	25.856 -18.246	94.058 -72.807	1.364 0.000	1.963 -0.109	2.943 -0.326	4.674 -0.774	7	
3	1.068 -0.502	1.454 -0.769	2.060 -1.214	3.083 -2.007	4.990 -3.570	9.111 -7.131	20.462 -17.472	72.066 -67.041	2.982 0.000	4.794 -0.263	8.368 -0.916	8	
2	0.697 -0.380	0.905 -0.547	1.208 -0.806	1.677 -1.227	2.451 -1.957	3.862 -3.356	6.832 -6.450	14.767 -15.136	49.558 -55.208	8.472 0.000	16.984 -0.924	9	
1	0.448 -0.282	0.557 -0.386	0.706 -0.535	0.919 -0.758	1.238 -1.110	1.750 -1.699	2.650 -2.786	4.466 -5.087	9.079 -11.225	28.072 -37.838	42.764 0.000	10	
0	0.265 -0.195	0.314 -0.253	0.377 -0.330	0.461 -0.437	0.574 -0.589	0.736 -0.816	0.981 -1.175	1.383 -1.793	2.128 -2.999	3.834 -5.910	9.974 -17.009		
	11	12	13	14	15	16	17	18	19	20	21		
						r ₂							

TABLE 10 VALUES OF a AND b FOR ALL FORMS OF CENSORING
FOR SAMPLE SIZES $21 \leq n \leq 25$.

n=24

		r_2													
		0	1	2	3	4	5	6	7	8	9	10	11		
r_1	0	0.046 0.000	0.057 -0.005	0.067 -0.010	0.077 -0.017	0.088 -0.025	0.101 -0.035	0.116 -0.047	0.132 -0.062	0.151 -0.080	0.174 -0.102	0.201 -0.130	0.234 -0.166	0	
	10	186.663 -19.187	0.071 0.000	0.086 -0.008	0.101 -0.014	0.119 -0.024	0.139 -0.036	0.163 -0.052	0.191 -0.072	0.225 -0.097	0.266 -0.130	0.317 -0.174	0.382 -0.232	1	
	9	45.407 -7.102	178.673 -37.294	0.105 0.000	0.127 -0.008	0.151 -0.019	0.180 -0.034	0.215 -0.052	0.258 -0.077	0.311 -0.110	0.377 -0.155	0.463 -0.216	0.575 -0.302	2	
	8	17.733 -3.723	43.409 -11.401	168.831 -53.359	0.155 0.000	0.189 -0.012	0.229 -0.027	0.279 -0.049	0.341 -0.079	0.420 -0.121	0.524 -0.179	0.661 -0.263	0.851 -0.385	3	
	7	8.588 -2.273	16.735 -5.321	40.473 -15.056	155.417 -66.392	0.234 0.000	0.289 -0.017	0.359 -0.041	0.449 -0.076	0.567 -0.127	0.726 -0.202	0.947 -0.315	1.265 -0.490	4	
	6	4.709 -1.513	7.987 -2.996	15.368 -6.608	36.666 -17.824	138.704 -75.448	0.365 0.000	0.463 -0.026	0.593 -0.066	0.769 -0.127	1.016 -0.223	1.374 -0.373	1.918 -0.626	5	
	5	2.793 -1.062	4.306 -1.873	7.206 -3.536	13.664 -7.488	32.080 -19.477	119.167 -79.695	0.601 0.000	0.790 -0.043	1.057 -0.115	1.448 -0.235	2.046 -0.440	3.015 -0.808	6	
	4	1.742 -0.771	2.502 -1.246	3.802 -2.110	6.262 -3.843	11.667 -7.870	26.846 -19.815	97.422 -78.443	1.071 0.000	1.485 -0.080	2.123 -0.227	3.166 -0.505	5.002 -1.060	7	
	3	1.120 -0.570	1.521 -0.861	2.149 -1.342	3.208 -2.197	5.179 -3.873	9.433 -7.677	21.137 -18.680	74.279 -71.223	2.149 0.000	3.242 -0.172	5.181 -0.546	8.995 -1.417	8	
	2	0.728 -0.424	0.943 -0.604	1.256 -0.881	1.739 -1.329	2.536 -2.104	3.987 -3.584	7.038 -6.845	15.182 -15.976	50.846 -57.968	5.240 0.000	9.201 -0.481	18.341 -1.914	9	
	1	0.466 -0.310	0.578 -0.420	0.731 -0.578	0.949 -0.813	1.277 -1.182	1.801 -1.800	2.721 -2.936	4.578 -5.333	9.288 -11.714	28.666 -39.309	18.545 0.000	46.410 -2.411	10	
	0	0.274 -0.212	0.324 -0.272	0.389 -0.352	0.474 -0.463	0.589 -0.621	0.754 -0.856	1.003 -1.228	1.412 -1.866	2.169 -3.107	3.901 -6.099	10.132 -17.483	186.630 0.000	11	
		12	13	14	15	16	17	18	19	20	21	22			
		r_2													

TABLE 10. VALUES OF a AND b FOR ALL FORMS OF CENSORING
FOR SAMPLE SIZES $21 \leq n \leq 25$.

$n=25$

		r_2												
		0	1	2	3	4	5	6	7	8	9	10	11	
r_1	11	0.044 0.000	0.054 -0.004	0.063 -0.010	0.072 -0.016	0.083 -0.023	0.094 -0.032	0.107 -0.042	0.122 -0.053	0.139 -0.070	0.158 -0.089	0.181 -0.113	0.209 -0.142	0
	10	201.893 -10.047	0.067 0.000	0.080 -0.006	0.094 -0.013	0.110 -0.021	0.128 -0.032	0.149 -0.046	0.173 -0.063	0.201 -0.084	0.236 -0.112	0.278 -0.147	0.331 -0.193	1
	9	49.970 -4.990	197.897 -29.644	0.098 0.000	0.117 -0.007	0.138 -0.017	0.164 -0.029	0.194 -0.046	0.230 -0.067	0.274 -0.094	0.328 -0.130	0.396 -0.179	0.485 -0.245	2
	8	19.661 -2.960	48.460 -9.729	189.918 -47.740	0.142 0.000	0.171 -0.010	0.206 -0.024	0.247 -0.042	0.299 -0.067	0.363 -0.101	0.446 -0.147	0.553 -0.212	0.696 -0.304	3
	7	9.604 -1.942	18.856 -4.765	45.985 -13.969	178.268 -63.399	0.209 0.000	0.256 -0.014	0.314 -0.035	0.387 -0.064	0.481 -0.104	0.604 -0.163	0.769 -0.247	1.000 -0.374	4
	6	5.320 -1.357	9.098 -2.785	17.668 -6.319	42.595 -17.473	163.058 -75.655	0.319 0.000	0.398 -0.022	0.501 -0.054	0.637 -0.102	0.822 -0.174	1.081 -0.284	1.456 -0.458	5
	5	3.195 -0.989	4.972 -1.796	8.405 -3.476	16.124 -7.528	38.361 -20.010	144.739 -83.677	0.508 0.000	0.654 -0.035	0.854 -0.090	1.136 -0.178	1.547 -0.321	2.177 -0.563	6
	4	2.026 -0.742	2.939 -1.231	4.517 -2.132	7.538 -3.967	14.255 -8.300	33.384 -21.369	123.708 -86.654	0.865 0.000	1.163 -0.060	1.602 -0.165	2.279 -0.351	3.382 -0.691	7
	3	1.331 -0.567	1.828 -0.876	2.617 -1.397	3.966 -2.337	6.515 -4.214	12.108 -8.552	27.798 -21.367	100.656 -84.000	1.621 0.000	2.330 -0.119	3.495 -0.355	5.556 -0.844	8
	2	0.892 -0.437	1.171 -0.638	1.586 -0.952	2.235 -1.470	3.328 -2.385	5.361 -4.173	9.743 -8.215	21.787 -19.869	76.415 -75.338	3.532 0.000	5.675 -0.286	9.911 -0.996	9
	1	0.601 -0.337	0.759 -0.468	0.980 -0.660	1.302 -0.955	1.799 -1.429	2.618 -2.249	4.107 -3.808	7.236 -7.234	15.580 -16.798	52.083 -60.665	10.014 0.000	20.068 -1.004	10
	0	0.398 -0.255	0.484 -0.338	0.598 -0.454	0.755 -0.620	0.979 -0.867	1.314 -1.253	1.850 -1.899	2.790 -3.082	4.685 -5.574	9.490 -12.191	29.236 -40.741	50.473 0.000	11
		0.242 -0.180	0.283 -0.228	0.334 -0.290	0.400 -0.374	0.486 -0.489	0.604 -0.653	0.772 -0.896	1.025 -1.279	1.439 -1.936	2.208 -3.212	3.966 -6.282	10.283 -17.942	
		12	13	14	15	16	17	18	19	20	21	22	23	

limiting form given in Table 11. For $n > 20$ a sufficiently accurate approximation to the variance of $\tilde{\sigma}$ may be obtained by computing $\text{Var}(\tilde{\sigma}) = u\sigma^2/(n - 1\frac{1}{4})$ where u corresponds to $k/(n-1)$ and is found by linear interpolation in Table 11. An estimate of this variance is then obtained by replacing σ^2 by $\tilde{\sigma}^2$.

Table 11

Values of u for approximation to variance of $\tilde{\sigma}$ for censored samples.

$k/(n-1)$	u	$k/(n-1)$	u
.000	.500	.175	.710
.025	.530	.200	.750
.050	.555	.250	.825
.075	.580	.300	.915
.100	.610	.350	1.010
.125	.640	.400	1.130
.150	.675	.450	1.260

Table 12 compares the values obtained using this approximation formula with the exact variances for samples of size 20 subjected to small amounts of censoring.

Table 12

Comparison of approximate and exact variance of $\tilde{\sigma}$
for small amounts of censoring for $n = 20$.

k	k/(n-1)	u	Approx. variance	Exact variance		
				$r_1 = 0$	$r_1 = 1$	$r_1 = 2$
0	0	.500	.0267	.0269	-	-
1	.0526	.558	.0298	.0298	-	-
2	.1053	.616	.0329	.0332	.0328	-
3	.1579	.686	.0366	.0368	.0364	-
4	.2105	.765	.0408	.0409	.0403	.0404

For complete samples the inefficiency due to using this linear estimator instead of the root-mean-square estimator is more than overcome if one more observation is included in the sample.

For censored samples the linear estimator is about equivalent to a root-mean-square estimator based on a complete sample with $(n - \frac{1}{4})/2u$ degrees of freedom. For example, for $n = 20$ with two observations censored the effective sample size is $18.75/(2 \times 0.616) + 1 \approx 16$.

Summary of estimating procedure for censored samples

To compute the estimate of σ and its standard error given a censored sample one proceeds as follows:

(i) order the observations to obtain

$$y_{r_1+1} \leq y_{r_1+2} \leq \dots \leq y_{n-r_2};$$

(ii) compute $\sum_i m_i y_i$, $\sum_i y_i$

where the m_i ($i = r_1 + 1, \dots, n - r_2$) are the corresponding normal scores;

(iii) read the appropriate values of a and b from Table 10 and estimate σ using

$$\tilde{\sigma} = a \sum_i m_i y_i - b \sum_i y_i ;$$

(iv) compute $k/(n-1)$, where $k = r_1 + r_2$, and use Table 11 to find the corresponding value of u ;

(v) estimate the standard error of $\tilde{\sigma}$ using

$$\{u/(n - \frac{1}{4})\}^{\frac{1}{2}} \tilde{\sigma} .$$

CHAPTER 3

TESTS OF HYPOTHESES FOR MEANS AND VARIANCES

3.1 Large sample tests for means and variances using a linear estimator of the standard deviation

Several linear estimators of the standard deviation of a normal population have been shown to be highly efficient relative to the root-mean-square estimator and in most cases, except when a calculating machine or computer is at hand, simpler to calculate. However, their practical value is severely limited unless one is able to use them in tests of hypotheses concerning the means and variances of the populations from which the sample values are obtained. In this section four test procedures involving a particularly simple linear estimator are developed. This estimator was suggested in chapter 2 by the author as suitable for medium and large samples and shown to have an asymptotic efficiency of nearly 96 per cent. The test procedures considered may be used as alternatives to Student's t-tests for one and two-sample means, the χ^2 -test for a single variance or standard deviation and the F-test for comparing two sample variances. Tables of percentage points for the four test-statistics are given and in the final sections the powers of these alternative tests are compared with the powers of the standard tests.

A natural extension of the use of short-cut linear estimators of the standard deviation of a normal population is to introduce these estimators in tests of hypotheses. E. Lord

(1947 and 1950) produced tables of percentage points of test statistics suitable for small samples in which the range is used to estimate the standard deviation. The two sample-test for mean values in which the sample sizes are different was considered by P.G. Moore (1957) using a χ -approximation to the distribution of the standardized sum of the ranges of the two samples. R.F. Link (1950) and K.C.S. Pillai & A.R. Buenaventura (1961) considered the problem of using range-based F-tests for comparing two variances. H.L. Harter (1963) considerably extended earlier tables and produced percentage points of the distribution of the ratio of ranges determined from two independent samples from normal distributions with common variance for sample sizes up to 15. He also compared the power of this short-cut test procedure with that of the usual F-test. The reader is referred to the introduction to Biometrika Tables for Statisticians Volume I for illustrative examples and further details of these procedures and for the corresponding tables of percentage points. The efficiency of the range as an estimate of the standard deviation decreases so rapidly as the sample size increases that the test procedures outlined above are suitable for small samples only.

The theoretical difficulties involved in using the more efficient estimators in tests of hypotheses arise because the estimators suggested are linear functions of the ordered

sample values not of the independent observations themselves. The estimators are thus, at least for small samples, not normally distributed.

During the past few years there have been several attempts to prove the asymptotic normality of linear functions of ordered observations under more general conditions on the coefficients involved and with general underlying distributions. The distribution of the extreme value in a sample from a normal population has skewness and kurtosis coefficients which increase as the sample size increases. For this reason the magnitude of the weights given to the extreme observations is of major importance when considering the asymptotic behaviour of the linear function. S.M. Stigler (1969) gives a summary of results obtained to date and outlines the conditions imposed on the weights for the proofs of asymptotic normality to hold.

Our aim here is to derive test statistics that involve linear estimators and to consider their distributions, for practical sample sizes, under certain null hypotheses. For this problem it is not sufficient to establish asymptotic normality and to derive asymptotic means and variances for our estimators or test statistics. We need to investigate how quickly, i.e. for what range of sample sizes, a normal distribution is a sufficiently good approximation, and to derive approximation formulae for the parameters of this normal distribution which

will agree more closely with the exact values than do the asymptotic results.

It was decided in this section to restrict attention to consideration of one and two-sample tests for means and variances using a simple estimator of the standard deviation particularly suitable for medium and large samples. This estimator is considered in detail in section 2.1 and is obtained by calculating one-third of the difference between the means of the largest one-sixth and the smallest one-sixth of the observations.

There are several reasons for choosing this estimator rather than others which are equally as efficient. Firstly this estimator is easy to compute even for quite large samples since it is not necessary to order the observations completely. All that is required is that the sample be divided into three groups according to size in such a way that the extreme groups each contain about one-sixth of the observations. The sample values need not be ordered within these groups. The second reason for considering this estimator is that since all observations in the extreme groups are given equal weight it is conjectured that the distribution of the estimator converges to normality more quickly than does that of an estimator which assigns high weights to the extreme values.

The linear estimator, its mean and variance

Although the estimator may be calculated without

ordering the observations completely it is convenient to consider it as a linear function of the ordered observations for purposes of deriving its mean and variance.

Suppose $y_1 < y_2 < \dots < y_n$ is an ordered sample of n observations from a normal population with mean μ and variance σ^2 . The linear estimator of σ is defined by

$$\sigma^* = \frac{\sum_{i=n-r+1}^n y_i - \sum_{i=1}^r y_i}{3r}$$

where $r = n/6$, rounded up to the nearest whole number if $n/6$ is not an integer. This estimator is asymptotically unbiased and has asymptotic efficiency 96 per cent relative to the root-mean-square estimator.

Let $\underline{\delta}$ be the vector of coefficients assigned to the vector of ordered observations $\underline{y} = (y_1, y_2, \dots, y_n)'$.

Then δ_i is given by

$$\begin{aligned}\delta_i &= -\frac{1}{3r} \quad \text{for } 1 \leq i \leq r \\ &= 0 \quad \text{for } r+1 \leq i \leq n-r \\ &= \frac{1}{3r} \quad \text{for } n-r+1 \leq i \leq n,\end{aligned}$$

and the estimator is given by $\sigma^* = \underline{\delta}' \underline{y}$.

As before let $\underline{m}$ and $\underline{V}$ be the vector of normal scores and variance-covariance matrix of the standardized ordered observations respectively.

The mean and variance of σ^* are given by

$$E\sigma^* = \delta' \underline{E} y = \delta' \underline{m} \sigma$$

$$\text{and } \text{Var } \sigma^* = \delta' \underline{V} \delta \sigma^2.$$

Let $\delta' \underline{m}$ and $\delta' \underline{V} \delta$ for a sample of size n be represented by m_n and V_n respectively. The expected values, m_n , may be obtained for $n \leq 100$ and for selected values of n between 100 and 400 using the tables of normal scores given by H.L. Harter (1961). Determination of V_n , however, presents more of a problem as the variance-covariance matrix $\underline{V}$ is not available for samples larger than 20. Also since r is constant over sets of consecutive sample sizes, for example $r = 7$ for $n = 37, 38, 39, 40, 41$ and 42 , the variance behaves in a disjointed manner depending not only on n (or r) but on the position of n in these sets. Suppose k is defined in the following way

$$\begin{aligned} k &= n(\text{mod } 6) \quad \text{if } n \neq 6r \\ &= 6 \quad \quad \quad \text{if } n = 6r, \end{aligned}$$

then k takes the values 1 to 6 depending on the position of n in these sets of six consecutive sample sizes for which r is constant.

Exact values of the reciprocals of V_n for $n = 8, \dots, 20$ were computed and found to be almost exactly linearly related to

r and k . The least squares equation using these exact values is

$$\frac{1}{V_n} = 1.247 + 11.623r + 0.405k$$

with a regression mean square of 396.901 and residual mean square 0.0008. Table 13 gives the exact values of V_n and the approximate values obtained using this equation for $n = 8, \dots, 20$.

Table 13

Comparison of exact and approximate values of V_n for $n = 8, \dots, 20$.

n	r	k	Exact Variance	Approximate Variance
8	2	2	0.04379	0.04384
9	2	3	0.04313	0.04308
10	2	4	0.04240	0.04234
11	2	5	0.04166	0.04163
12	2	6	0.04094	0.04094
13	3	1	0.02935	0.02939
14	3	2	0.02903	0.02904
15	3	3	0.02870	0.02871
16	3	4	0.02837	0.02838
17	3	5	0.02804	0.02805
18	3	6	0.02772	0.02774
19	4	1	0.02192	0.02191
20	4	2	0.02173	0.02171

Approximation to the distribution of σ^*

Empirical sampling experiments involving 400 samples of

30 observations from a normal distribution were used to investigate the distribution of σ^* . The sampling moments and the skewness and kurtosis coefficients, γ_1 and γ_2 , of the distribution of σ^* were determined from these 400 estimates. The results of these experiments and the corresponding theoretical values and standard errors based on the assumption that the distribution of σ^* may be closely approximated by a normal distribution are shown below in table 14.

Table 14.

Comparison of observed and hypothesised characteristics of the distribution of σ^* for $n = 30$.

Characteristic	Empirical Value	Standard Error	Theoretical Value
Mean	0.95868	0.00660	0.96371
Variance	0.01639	0.00123	0.01686
Third moment	0.00044	0.00028	0.00000
Fourth moment	0.00081	0.00015	0.00085
Skewness γ_1	0.20712	0.12247	0.00000
Kurtosis γ_2	0.00838	0.24495	0.00000

It may be seen that both γ_1 and γ_2 are within two standard errors of the corresponding normal values, however, since the intention is to use this normal approximation to the

distribution of σ^* to derive percentage points of the distribution of test statistics it is essential that the approximation is good in the tails of the distribution as well as in the centre.

The tests of skewness and kurtosis using γ_1 and γ_2 and other tests such as Kolmogorov-Smirnov are not sufficiently sensitive to detect the slight non-normalities which are of importance. Thus we require a procedure for comparing empirical and theoretical distributions which is particularly sensitive to differences in their extremes. Such a procedure employing cumulative distribution functions is described by M.B. Wilk and R. Gnanadesikan (1968). Their method is a probability plotting procedure in which quantiles of one variable are plotted against quantiles of the other variable. Either or both of the cumulative distribution functions, which determine the quantiles, may be empirical. In our case one set of quantiles are obtained from the empirical sampling experiment and the other set are the corresponding quantiles from the standard normal distribution. This procedure is more convenient for computer application than plotting on arithmetic probability paper. If the variables are identically distributed the quantile-quantile plot (Q-Q plot) will be a straight line with slope 1 through the origin. If, however, one variable is a linear function of the other the plot will still

be linear but with possibly different intercept and gradient. To compare a theoretical distribution and an empirical distribution involving N observations the observations are ordered and the i^{th} ordered value is plotted against $F^{-1}[(i-\frac{1}{2})/N]$ for $i = 1, 2, \dots, N$, where F^{-1} is the inverse cumulative of the theoretical distribution. Figure 1 shows the Q-Q plot of 400 estimates of σ^* against the corresponding quantiles of the standard normal distribution. The linearity of this plot indicates that the normal approximation is reasonably good even in the tails of the distribution. This approximation is the basis of the theoretical results derived in the following sections.

The distribution of the ratio of two normal variables

The test-statistics to be considered in the following sections take the form of a ratio of two normal variables. The distributions of these statistics are derived using the following result due to R.C. Geary (1930). Let x and y be two normal variables with means zero, standard deviations α and β respectively and with correlation coefficient ρ . If $u = (b + y)/(a + x)$, where a and b are constants, the function

$$z = \frac{au - b}{\sqrt{(\alpha^2 u^2 - 2\rho\alpha\beta u + \beta^2)}}$$

has a standard normal distribution provided that $a + x$ is unlikely to assume negative values. This condition is amply

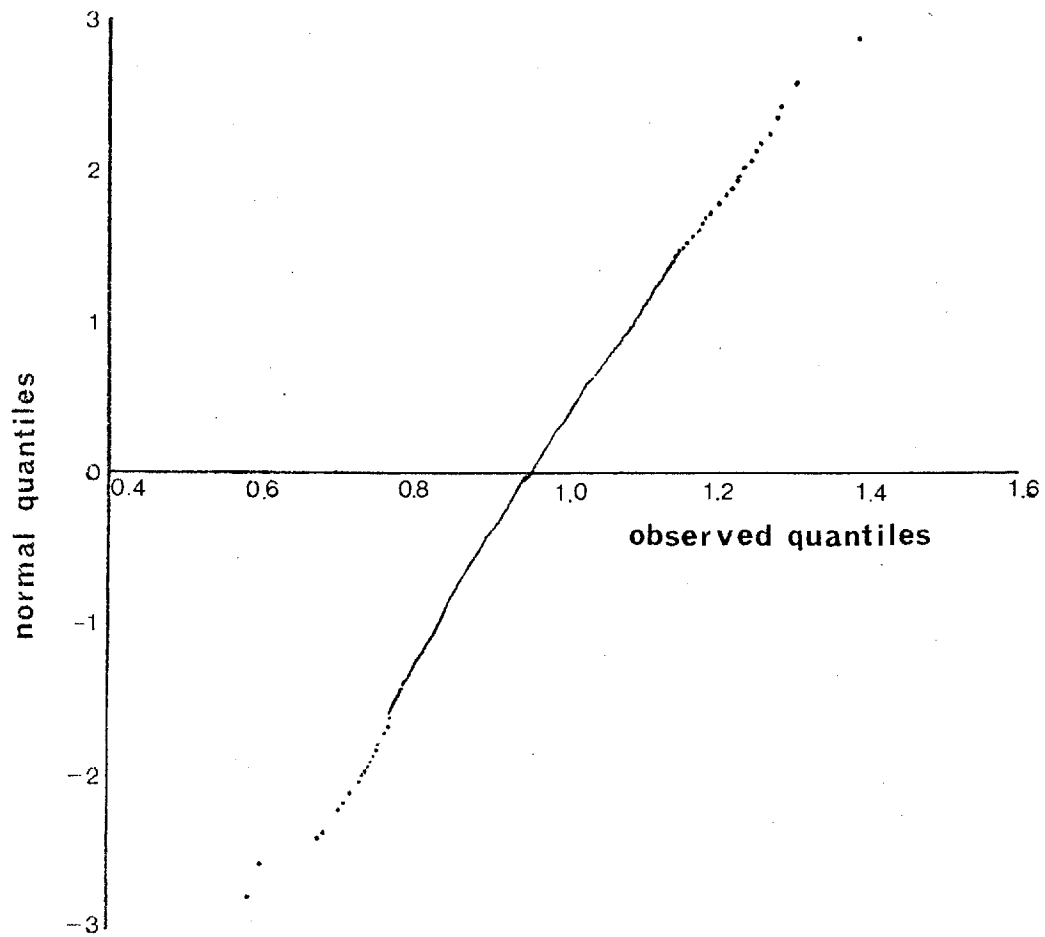


Figure 1. Q-Q plot of 400 estimates of σ^*

satisfied if $a \geq 3\alpha$.

Although it does not seem possible to derive a simple functional form for the distribution of u , specific percentage points may be obtained when $\rho = 0$ from the percentage points of the normal distribution.

One-sample test for mean value. (Alternative to Student's t-test)

Suppose the null hypothesis to be tested is that the mean of the normal population, from which we have a sample of size n , is μ_0 . The one-sample t-test consists of estimating σ^2 , the unknown variance of the normal population, by $s^2 = \sum (y_i - \bar{y})^2 / (n-1)$ and testing $(\bar{y} - \mu_0) / (s/\sqrt{n})$ using the t-distribution with $(n-1)$ degrees of freedom. If the standard deviation σ is estimated by the linear estimator σ^* instead of the root-mean-square estimator s the corresponding expression $u_1 = (\bar{y} - \mu_0) / (\sigma^* / \sqrt{n})$ does not have a t-distribution. This expression may, however, be rewritten in the form

$$u_1 = \frac{(\bar{y} - \mu_0) / (\sigma / \sqrt{n})}{(\sigma^* / \sigma)}.$$

Under the null hypothesis $(\bar{y} - \mu_0) / (\sigma / \sqrt{n})$ has a standard normal distribution and σ^* / σ is normal with mean m_n and variance V_n . The correlation coefficient ρ is zero since $\bar{y}$ and σ^* are orthogonal linear combinations of the observations. Since the condition $m_n \geq 3\sqrt{V_n}$ is amply satisfied for $n \geq 4$ we may use

Geary's theorem in which $a = m_n$, $b = 0$, $\rho = 0$, $\alpha^2 = V_n$ and $\beta^2 = 1$. Hence

$$z_1 = \frac{m_n u_1}{\sqrt{(V_n u_1^2 + 1)}}$$

has a standard normal distribution. The function z_1 is a monotonic increasing function of u_1 therefore the p -percent point of u_1 , i.e. $u_{1,p}$ may be obtained by solving the equation

$$z_p = \frac{m_n u_{1,p}}{\sqrt{(V_n u_{1,p}^2 + 1)}}$$

where z_p is the p -percent point of the standard normal distribution. This gives

$$u_{1,p} = [(m_n^2/z_p^2) - V_n]^{-\frac{1}{2}}$$

Values of $u_{1,p}$ for $p = 0.1, 0.05, 0.025, 0.02, 0.01, 0.005$ and 0.001 and for a range of sample sizes greater than 20 are given in table 15.

One-sample test for variance or standard deviation. (Alternative to chi-squared test).

The null hypothesis that $\sigma = \sigma_0$ may be tested using the normal approximation to the distribution of σ^* . Under the null hypothesis σ^* is normal with mean $m_n \sigma_0$ and variance $V_n \sigma_0^2$, i.e. $v_1 = \sigma^*/\sigma_0$ is normal mean m_n and variance V_n . The p -percent point of v_1 is given by $v_{1,p} = m_n + z_p \sqrt{V_n}$.

Upper and lower percentage points of v_1 are given in table 16.

TABLE 15. PERCENTAGE POINTS OF DENSITY FUNCTION OF U ,
FOR ONE SAMPLE TEST FOR MEANS

n	p						
	0.1000	0.0500	0.0250	0.0200	0.0100	0.0050	0.0010
21	1.4458	1.8832	2.2808	2.4031	2.7712	3.1274	3.9385
22	1.4173	1.8448	2.2325	2.3515	2.7093	3.0544	3.8359
23	1.3914	1.8100	2.1888	2.3049	2.6534	2.9887	3.7443
24	1.3678	1.7782	2.1490	2.2625	2.6027	2.9292	3.6620
25	1.4557	1.8911	2.2835	2.4034	2.7621	3.1055	3.8716
26	1.4311	1.8583	2.2427	2.3600	2.7105	3.0454	3.7898
27	1.4085	1.8281	2.2051	2.3201	2.6632	2.9904	3.7154
28	1.3876	1.8003	2.1705	2.2833	2.6197	2.9399	3.6473
29	1.3682	1.7744	2.1385	2.2493	2.5794	2.8932	3.5847
30	1.3501	1.7504	2.1087	2.2177	2.5420	2.8500	3.5269
31	1.4214	1.8419	2.2178	2.3320	2.6716	2.9933	3.6983
32	1.4027	1.8171	2.1872	2.2995	2.6333	2.9493	3.6401
33	1.3851	1.7939	2.1585	2.2691	2.5976	2.9082	3.5859
34	1.3687	1.7721	2.1316	2.2407	2.5642	2.8698	3.5353
35	1.3532	1.7515	2.1064	2.2139	2.5328	2.8338	3.4881
36	1.3386	1.7322	2.0826	2.1887	2.5033	2.7999	3.4437
37	1.3985	1.8092	2.1744	2.2850	2.6124	2.9206	3.5884
38	1.3834	1.7893	2.1500	2.2591	2.5821	2.8860	3.5434
39	1.3691	1.7704	2.1269	2.2346	2.5535	2.8534	3.5011
40	1.3555	1.7526	2.1050	2.2115	2.5265	2.8225	3.4611
41	1.3426	1.7356	2.0842	2.1895	2.5008	2.7932	3.4232
42	1.3303	1.7194	2.0644	2.1686	2.4765	2.7654	3.3873
43	1.3821	1.7859	2.1436	2.2516	2.5706	2.8697	3.5124
44	1.3694	1.7693	2.1233	2.2302	2.5456	2.8412	3.4759
45	1.3574	1.7534	2.1040	2.2097	2.5219	2.8142	3.4412
46	1.3458	1.7383	2.0855	2.1902	2.4992	2.7884	3.4081
47	1.3348	1.7238	2.0678	2.1715	2.4775	2.7637	3.3766
48	1.3242	1.7100	2.0509	2.1537	2.4567	2.7402	3.3465
49	1.3697	1.7684	2.1206	2.2267	2.5395	2.8319	3.4566
50	1.3589	1.7542	2.1033	2.2084	2.5183	2.8078	3.4260
51	1.3484	1.7405	2.0867	2.1909	2.4980	2.7847	3.3966
52	1.3384	1.7274	2.0707	2.1740	2.4784	2.7626	3.3685
53	1.3288	1.7148	2.0553	2.1578	2.4597	2.7413	3.3415
54	1.3195	1.7026	2.0406	2.1422	2.4416	2.7209	3.3156
55	1.3601	1.7548	2.1028	2.2075	2.5155	2.8028	3.4139
56	1.3506	1.7424	2.0877	2.1915	2.4971	2.7819	3.3876
57	1.3414	1.7304	2.0731	2.1761	2.4793	2.7618	3.3622
58	1.3326	1.7188	2.0590	2.1613	2.4622	2.7424	3.3377
59	1.3240	1.7076	2.0455	2.1470	2.4457	2.7237	3.3142
60	1.3157	1.6968	2.0324	2.1332	2.4297	2.7057	3.2915
70	1.3239	1.7061	2.0419	2.1426	2.4383	2.7129	3.2930
80	1.3300	1.7131	2.0490	2.1497	2.4449	2.7185	3.2945
90	1.3043	1.6796	2.0081	2.1064	2.3946	2.6613	3.2215
100	1.3113	1.6878	2.0173	2.1158	2.4044	2.6711	3.2302
150	1.2956	1.6660	1.9891	2.0855	2.3674	2.6270	3.1681
200	1.3014	1.6726	1.9960	2.0924	2.3739	2.6327	3.1707
250	1.2939	1.6625	1.9834	2.0790	2.3579	2.6142	3.1459
300	1.2896	1.6568	1.9761	2.0712	2.3486	2.6033	3.1312
350	1.2932	1.6611	1.9810	2.0763	2.3540	2.6088	3.1367
400	1.2896	1.6563	1.9750	2.0699	2.3465	2.6003	3.1255
∞	1.2816	1.6449	1.9600	2.0537	2.3263	2.5758	3.0902

TABLE 16. PERCENTAGE POINTS OF DENSITY FUNCTION OF V,
FOR ONE SAMPLE TEST FOR STANDARD DEVIATIONS

n	p							
	0.9950	0.9900	0.9750	0.9500	0.0500	0.0250	0.0100	0.0050
21	0.5283	0.5649	0.6186	0.6649	1.1475	1.1937	1.2475	1.2841
22	0.5472	0.5836	0.6371	0.6832	1.1637	1.2098	1.2633	1.2997
23	0.5651	0.6014	0.6547	0.7005	1.1790	1.2248	1.2781	1.3144
24	0.5821	0.6183	0.6713	0.7170	1.1934	1.2391	1.2921	1.3283
25	0.5562	0.5892	0.6376	0.6792	1.1139	1.1555	1.2039	1.2369
26	0.5721	0.6050	0.6532	0.6947	1.1279	1.1694	1.2176	1.2504
27	0.5873	0.6201	0.6681	0.7095	1.1411	1.1825	1.2305	1.2633
28	0.6019	0.6345	0.6824	0.7236	1.1538	1.1950	1.2428	1.2755
29	0.6158	0.6483	0.6961	0.7371	1.1658	1.2069	1.2546	1.2871
30	0.6292	0.6616	0.7092	0.7501	1.1773	1.2182	1.2658	1.2982
31	0.6044	0.6345	0.6786	0.7166	1.1129	1.1509	1.1950	1.2251
32	0.6171	0.6471	0.6911	0.7290	1.1241	1.1620	1.2060	1.2360
33	0.6294	0.6593	0.7031	0.7409	1.1349	1.1726	1.2165	1.2464
34	0.6412	0.6710	0.7147	0.7524	1.1453	1.1829	1.2266	1.2564
35	0.6526	0.6823	0.7259	0.7634	1.1552	1.1927	1.2363	1.2661
36	0.6636	0.6932	0.7367	0.7741	1.1648	1.2022	1.2457	1.2753
37	0.6604	0.6682	0.7090	0.7442	1.1108	1.1459	1.1867	1.2145
38	0.6510	0.6787	0.7194	0.7545	1.1202	1.1552	1.1959	1.2236
39	0.6612	0.6889	0.7295	0.7644	1.1292	1.1642	1.2048	1.2324
40	0.6711	0.6987	0.7392	0.7741	1.1380	1.1728	1.2134	1.2410
41	0.6808	0.7083	0.7487	0.7835	1.1465	1.1812	1.2217	1.2492
42	0.6901	0.7176	0.7579	0.7926	1.1547	1.1894	1.2297	1.2571
43	0.6685	0.6945	0.7327	0.7655	1.1082	1.1410	1.1792	1.2052
44	0.6775	0.7035	0.7416	0.7743	1.1163	1.1490	1.1871	1.2130
45	0.6863	0.7122	0.7502	0.7829	1.1241	1.1568	1.1948	1.2207
46	0.6949	0.7207	0.7586	0.7912	1.1317	1.1643	1.2022	1.2281
47	0.7032	0.7290	0.7668	0.7994	1.1391	1.1716	1.2095	1.2352
48	0.7113	0.7371	0.7748	0.8073	1.1463	1.1788	1.2165	1.2422
49	0.6912	0.7157	0.7517	0.7826	1.1055	1.1365	1.1724	1.1969
50	0.6991	0.7235	0.7594	0.7903	1.1126	1.1435	1.1794	1.2038
51	0.7068	0.7312	0.7670	0.7978	1.1195	1.1503	1.1861	1.2105
52	0.7143	0.7386	0.7744	0.8051	1.1262	1.1570	1.1927	1.2171
53	0.7216	0.7459	0.7816	0.8123	1.1328	1.1635	1.1991	1.2235
54	0.7288	0.7531	0.7887	0.8193	1.1392	1.1698	1.2054	1.2297
55	0.7100	0.7332	0.7673	0.7967	1.1029	1.1323	1.1664	1.1896
56	0.7170	0.7402	0.7742	0.8035	1.1092	1.1385	1.1725	1.1957
57	0.7238	0.7470	0.7810	0.8102	1.1154	1.1446	1.1786	1.2017
58	0.7305	0.7536	0.7875	0.8167	1.1214	1.1506	1.1845	1.2076
59	0.7371	0.7601	0.7940	0.8231	1.1273	1.1564	1.1903	1.2133
60	0.7435	0.7665	0.8003	0.8294	1.1330	1.1621	1.1959	1.2189
70	0.7563	0.7774	0.8084	0.8350	1.1132	1.1399	1.1708	1.1919
80	0.7666	0.7862	0.8150	0.8397	1.0980	1.1227	1.1515	1.1710
90	0.7928	0.8116	0.8393	0.8630	1.1114	1.1351	1.1628	1.1816
100	0.7985	0.8162	0.8423	0.8646	1.0985	1.1209	1.1469	1.1646
150	0.8412	0.8559	0.8773	0.8957	1.0883	1.1068	1.1282	1.1428
200	0.8573	0.8698	0.8883	0.9041	1.0697	1.0856	1.1040	1.1166
250	0.8756	0.8869	0.9035	0.9178	1.0666	1.0809	1.0974	1.1087
300	0.8884	0.8988	0.9140	0.9270	1.0633	1.0764	1.0916	1.1019
350	0.8938	0.9034	0.9174	0.9294	1.0551	1.0671	1.0811	1.0906
400	0.9026	0.9115	0.9247	0.9359	1.0538	1.0651	1.0782	1.0872

Two-sample test for mean values. (Alternative to two-sample Student's t-test).

Suppose that two samples of size n_1 and n_2 are taken from normal distributions with common variance σ^2 and means μ_1 and μ_2 respectively, and that it is required to test the null hypothesis that the difference between the means is equal to some value d . Let the sample means be $\bar{y}_1$ and $\bar{y}_2$ and let σ_1^* and σ_2^* denote the linear estimates of σ derived from the two samples. The first step is to form a weighted estimate σ^* of σ from the two linear estimates σ_1^* and σ_2^* using

$$\sigma^* = \frac{n_1 \sigma_1^* + n_2 \sigma_2^*}{n_1 + n_2}$$

The test-statistic is now defined as

$$u_2 = (\bar{y}_1 - \bar{y}_2 - d)/\sigma^*,$$

which may be rewritten in the form

$$u_2 = \frac{(\bar{y}_1 - \bar{y}_2 - d)/\{\sigma^2(1/n_1 + 1/n_2)\}^{\frac{1}{2}}}{(n_1 \sigma_1^{*2} + n_2 \sigma_2^{*2})/[(n_1 + n_2)\{\sigma^2(1/n_1 + 1/n_2)\}^{\frac{1}{2}}]}$$

This is the ratio of two uncorrelated normal variables, the numerator is a standard normal variable while the denominator is normal with mean

$$(n_1 n_2)^{\frac{1}{2}} (n_1 m_{n_1} + n_2 m_{n_2})/(n_1 + n_2)^{3/2}$$

and variance

$$n_1 n_2 (n_1^2 v_{n_1} + n_2^2 v_{n_2}) / (n_1 + n_2)^3 .$$

Provided the condition

$$n_1 m_{n_1} + n_2 m_{n_2} \geq 3(n_1^2 v_{n_1} + n_2^2 v_{n_2})^{\frac{1}{2}}$$

holds Geary's theorem may be used again. The above condition is satisfied for $n_1, n_2 \geq 4$ so we have that

$$z_2 = \frac{(n_1 n_2)^{\frac{1}{2}} (n_1 m_{n_1} + n_2 m_{n_2}) u_2 (n_1 + n_2)^{-3/2}}{[n_1 n_2 (n_1^2 v_{n_1} + n_2^2 v_{n_2}) u_2^2 (n_1 + n_2)^{-3} + 1]^{\frac{1}{2}}}$$

has a standard normal distribution.

The percentage points of u_2 may be obtained from the corresponding percentage points of the normal distribution using the equation

$$u_{2,p} = [n_1 n_2 / (n_1 + n_2)^3]^{-\frac{1}{2}} [(n_1 m_{n_1} + n_2 m_{n_2})^2 z_p^{-2} - (n_1^2 v_{n_1} + n_2^2 v_{n_2})]^{-\frac{1}{2}}$$

The 5 per cent and 2½ per cent points of u_2 for $21 \leq n_1$, $n_2 \leq 50$ are given in table 17.

Two-sample test for variances. (Alternative to the F-test).

Suppose that two samples of size n_1 and n_2 are taken from normal distributions with variances σ_1^2 and σ_2^2

TABLE 17 0.025 POINTS FOR TEST FOR DIFFERENCE BETWEEN TWO MEANS

$n_1 \backslash n_2$	21	22	23	24	25	26	27	28	29	30
21	0.685	0.670	0.656	0.644	0.659	0.647	0.636	0.625	0.615	0.606
22	0.670	0.656	0.643	0.630	0.645	0.633	0.622	0.612	0.602	0.593
23	0.656	0.643	0.630	0.618	0.631	0.620	0.610	0.600	0.590	0.581
24	0.644	0.630	0.618	0.606	0.619	0.608	0.598	0.588	0.579	0.570
25	0.659	0.645	0.631	0.619	0.632	0.620	0.609	0.599	0.589	0.580
26	0.647	0.633	0.620	0.608	0.620	0.609	0.598	0.588	0.579	0.570
27	0.636	0.622	0.610	0.598	0.609	0.598	0.588	0.578	0.569	0.560
28	0.625	0.612	0.600	0.588	0.599	0.588	0.578	0.569	0.560	0.551
29	0.615	0.602	0.590	0.579	0.589	0.579	0.569	0.560	0.551	0.543
30	0.606	0.593	0.581	0.570	0.580	0.570	0.560	0.551	0.543	0.535
31	0.620	0.607	0.594	0.583	0.592	0.581	0.571	0.562	0.553	0.544
32	0.611	0.598	0.586	0.575	0.584	0.573	0.563	0.554	0.545	0.536
33	0.603	0.590	0.578	0.567	0.575	0.565	0.555	0.546	0.537	0.529
34	0.595	0.582	0.570	0.559	0.567	0.557	0.548	0.539	0.530	0.522
35	0.587	0.575	0.563	0.552	0.560	0.550	0.541	0.532	0.524	0.516
36	0.580	0.567	0.556	0.546	0.553	0.543	0.534	0.525	0.517	0.509
37	0.593	0.580	0.568	0.557	0.564	0.554	0.544	0.535	0.527	0.518
38	0.586	0.573	0.562	0.551	0.557	0.547	0.538	0.529	0.520	0.512
39	0.579	0.567	0.555	0.544	0.551	0.541	0.532	0.523	0.514	0.507
40	0.572	0.560	0.549	0.538	0.544	0.535	0.526	0.517	0.509	0.501
41	0.566	0.554	0.543	0.533	0.538	0.529	0.520	0.511	0.503	0.496
42	0.560	0.548	0.537	0.527	0.532	0.523	0.514	0.506	0.498	0.490
43	0.572	0.560	0.549	0.538	0.543	0.533	0.524	0.515	0.507	0.499
44	0.566	0.554	0.543	0.532	0.537	0.528	0.519	0.510	0.502	0.494
45	0.561	0.549	0.538	0.527	0.532	0.522	0.513	0.505	0.497	0.489
46	0.555	0.544	0.533	0.522	0.527	0.517	0.508	0.500	0.492	0.485
47	0.550	0.539	0.528	0.517	0.521	0.512	0.504	0.495	0.488	0.480
48	0.545	0.534	0.523	0.513	0.517	0.508	0.499	0.491	0.483	0.476
49	0.556	0.544	0.533	0.523	0.526	0.517	0.508	0.500	0.492	0.484
50	0.551	0.540	0.529	0.518	0.522	0.512	0.504	0.495	0.487	0.480

TABLE 17 0.025 POINTS FOR TEST FOR DIFFERENCE BETWEEN TWO MEANS

$n_1 \backslash n_2$	31	32	33	34	35	36	37	38	39	40
21	0.620	0.611	0.603	0.595	0.587	0.580	0.593	0.586	0.579	0.572
22	0.607	0.598	0.590	0.582	0.575	0.567	0.580	0.573	0.567	0.560
23	0.594	0.586	0.578	0.570	0.563	0.556	0.568	0.562	0.555	0.549
24	0.583	0.575	0.567	0.559	0.552	0.546	0.557	0.551	0.544	0.538
25	0.592	0.584	0.575	0.567	0.560	0.553	0.564	0.557	0.551	0.544
26	0.581	0.573	0.565	0.557	0.550	0.543	0.554	0.547	0.541	0.535
27	0.571	0.563	0.555	0.548	0.541	0.534	0.544	0.538	0.532	0.526
28	0.562	0.554	0.546	0.539	0.532	0.525	0.535	0.529	0.523	0.517
29	0.553	0.545	0.537	0.530	0.524	0.517	0.527	0.520	0.514	0.509
30	0.544	0.536	0.529	0.522	0.516	0.509	0.518	0.512	0.507	0.501
31	0.554	0.545	0.538	0.530	0.523	0.517	0.526	0.519	0.513	0.507
32	0.545	0.538	0.530	0.523	0.516	0.509	0.518	0.512	0.506	0.500
33	0.538	0.530	0.523	0.516	0.509	0.503	0.511	0.505	0.499	0.493
34	0.530	0.523	0.516	0.509	0.502	0.496	0.504	0.498	0.492	0.487
35	0.523	0.516	0.509	0.502	0.496	0.490	0.497	0.491	0.486	0.480
36	0.517	0.509	0.503	0.496	0.490	0.484	0.491	0.485	0.480	0.474
37	0.526	0.518	0.511	0.504	0.497	0.491	0.498	0.492	0.486	0.481
38	0.519	0.512	0.505	0.498	0.491	0.485	0.492	0.486	0.481	0.475
39	0.513	0.506	0.499	0.492	0.486	0.480	0.486	0.481	0.475	0.470
40	0.507	0.500	0.493	0.487	0.480	0.474	0.481	0.475	0.470	0.464
41	0.502	0.495	0.488	0.481	0.475	0.469	0.476	0.470	0.465	0.459
42	0.496	0.489	0.483	0.476	0.470	0.464	0.470	0.465	0.460	0.454
43	0.505	0.498	0.491	0.484	0.478	0.471	0.477	0.472	0.466	0.461
44	0.500	0.492	0.486	0.479	0.473	0.467	0.473	0.467	0.461	0.456
45	0.495	0.488	0.481	0.474	0.468	0.462	0.468	0.462	0.457	0.452
46	0.490	0.483	0.476	0.470	0.464	0.458	0.463	0.458	0.452	0.447
47	0.485	0.478	0.472	0.465	0.459	0.454	0.459	0.453	0.448	0.443
48	0.480	0.474	0.467	0.461	0.455	0.449	0.454	0.449	0.444	0.439
49	0.489	0.481	0.475	0.468	0.462	0.456	0.461	0.456	0.450	0.445
50	0.484	0.477	0.471	0.464	0.458	0.452	0.457	0.451	0.446	0.441

TABLE 17 0.025 POINTS FOR TEST FOR DIFFERENCE BETWEEN TWO MEANS

$n_1 \backslash n_2$	41	42	43	44	45	46	47	48	49	50
21	0.566	0.560	0.572	0.566	0.561	0.555	0.550	0.545	0.556	0.551
22	0.554	0.548	0.560	0.554	0.549	0.544	0.539	0.534	0.544	0.540
23	0.543	0.537	0.549	0.543	0.538	0.533	0.528	0.523	0.533	0.529
24	0.533	0.527	0.538	0.532	0.527	0.522	0.517	0.513	0.523	0.518
25	0.538	0.532	0.543	0.537	0.532	0.527	0.521	0.517	0.526	0.522
26	0.529	0.523	0.533	0.528	0.522	0.517	0.512	0.508	0.517	0.512
27	0.520	0.514	0.524	0.519	0.513	0.508	0.504	0.499	0.508	0.504
28	0.511	0.506	0.515	0.510	0.505	0.500	0.495	0.491	0.500	0.495
29	0.503	0.498	0.507	0.502	0.497	0.492	0.488	0.483	0.492	0.487
30	0.496	0.490	0.499	0.494	0.489	0.485	0.480	0.476	0.484	0.480
31	0.502	0.496	0.505	0.500	0.495	0.490	0.485	0.480	0.489	0.484
32	0.495	0.489	0.498	0.492	0.488	0.483	0.478	0.474	0.481	0.477
33	0.488	0.483	0.491	0.486	0.481	0.476	0.472	0.467	0.475	0.471
34	0.481	0.476	0.484	0.479	0.474	0.470	0.465	0.461	0.468	0.464
35	0.475	0.470	0.478	0.473	0.468	0.464	0.459	0.455	0.462	0.458
36	0.469	0.464	0.471	0.467	0.462	0.458	0.454	0.449	0.456	0.452
37	0.476	0.470	0.477	0.473	0.468	0.463	0.459	0.454	0.461	0.457
38	0.470	0.465	0.472	0.467	0.462	0.458	0.453	0.449	0.456	0.451
39	0.465	0.460	0.466	0.461	0.457	0.452	0.448	0.444	0.450	0.446
40	0.459	0.454	0.461	0.456	0.452	0.447	0.443	0.439	0.445	0.441
41	0.454	0.450	0.456	0.451	0.447	0.442	0.438	0.434	0.440	0.436
42	0.450	0.445	0.451	0.446	0.442	0.438	0.434	0.430	0.435	0.431
43	0.456	0.451	0.457	0.452	0.447	0.443	0.439	0.435	0.440	0.436
44	0.451	0.446	0.452	0.447	0.443	0.438	0.434	0.430	0.436	0.432
45	0.447	0.442	0.447	0.443	0.438	0.434	0.430	0.426	0.431	0.427
46	0.442	0.438	0.443	0.438	0.434	0.430	0.426	0.422	0.427	0.423
47	0.438	0.434	0.439	0.434	0.430	0.426	0.422	0.418	0.423	0.419
48	0.434	0.430	0.435	0.430	0.426	0.422	0.418	0.414	0.419	0.415
49	0.440	0.435	0.440	0.436	0.431	0.427	0.423	0.419	0.424	0.420
50	0.436	0.431	0.436	0.432	0.427	0.423	0.419	0.415	0.420	0.416

TABLE 17 0.050 POINTS FOR TEST FOR DIFFERENCE BETWEEN TWO MEANS

$n_1 \backslash n_2$	21	22	23	24	25	26	27	28	29	30
21	0.570	0.558	0.547	0.536	0.549	0.539	0.530	0.521	0.513	0.505
22	0.558	0.546	0.535	0.525	0.537	0.528	0.519	0.510	0.502	0.495
23	0.547	0.535	0.525	0.515	0.526	0.517	0.508	0.500	0.492	0.485
24	0.536	0.525	0.515	0.505	0.516	0.507	0.499	0.491	0.483	0.476
25	0.549	0.537	0.526	0.516	0.527	0.517	0.508	0.500	0.492	0.484
26	0.539	0.528	0.517	0.507	0.517	0.508	0.499	0.491	0.483	0.475
27	0.530	0.519	0.508	0.499	0.508	0.499	0.491	0.482	0.475	0.468
28	0.521	0.510	0.500	0.491	0.500	0.491	0.482	0.475	0.467	0.460
29	0.513	0.502	0.492	0.483	0.492	0.483	0.475	0.467	0.460	0.453
30	0.505	0.495	0.485	0.476	0.484	0.475	0.468	0.460	0.453	0.446
31	0.518	0.506	0.496	0.486	0.494	0.485	0.477	0.469	0.461	0.454
32	0.510	0.499	0.489	0.479	0.487	0.478	0.470	0.462	0.455	0.448
33	0.503	0.492	0.482	0.473	0.480	0.472	0.464	0.456	0.449	0.442
34	0.496	0.486	0.476	0.467	0.474	0.465	0.457	0.450	0.443	0.436
35	0.490	0.480	0.470	0.461	0.467	0.459	0.451	0.444	0.437	0.431
36	0.484	0.474	0.464	0.455	0.461	0.453	0.446	0.439	0.432	0.425
37	0.495	0.484	0.474	0.465	0.471	0.463	0.455	0.447	0.440	0.433
38	0.489	0.479	0.469	0.460	0.465	0.457	0.449	0.442	0.435	0.428
39	0.483	0.473	0.464	0.455	0.460	0.452	0.444	0.437	0.430	0.423
40	0.478	0.468	0.458	0.450	0.455	0.447	0.439	0.432	0.425	0.419
41	0.473	0.463	0.454	0.445	0.449	0.442	0.434	0.427	0.420	0.414
42	0.468	0.458	0.449	0.440	0.445	0.437	0.430	0.423	0.416	0.410
43	0.478	0.468	0.458	0.449	0.454	0.445	0.438	0.430	0.424	0.417
44	0.473	0.463	0.454	0.445	0.449	0.441	0.433	0.426	0.419	0.413
45	0.468	0.458	0.449	0.440	0.444	0.436	0.429	0.422	0.415	0.409
46	0.464	0.454	0.445	0.436	0.440	0.432	0.425	0.418	0.411	0.405
47	0.460	0.450	0.441	0.432	0.436	0.428	0.421	0.414	0.407	0.401
48	0.455	0.446	0.437	0.428	0.432	0.424	0.417	0.410	0.404	0.398
49	0.465	0.455	0.446	0.437	0.440	0.432	0.425	0.418	0.411	0.404
50	0.461	0.451	0.442	0.433	0.436	0.428	0.421	0.414	0.407	0.401

TABLE 17 0.050 POINTS FOR TEST FOR DIFFERENCE BETWEEN TWO MEANS

$n_1 \backslash n_2$	31	32	33	34	35	36	37	38	39	40
21	0.518	0.510	0.503	0.496	0.490	0.484	0.495	0.489	0.483	0.478
22	0.506	0.499	0.492	0.486	0.480	0.474	0.484	0.479	0.473	0.468
23	0.496	0.489	0.482	0.476	0.470	0.464	0.474	0.469	0.464	0.458
24	0.486	0.479	0.473	0.467	0.461	0.455	0.465	0.460	0.455	0.450
25	0.494	0.487	0.480	0.474	0.467	0.461	0.471	0.465	0.460	0.455
26	0.485	0.478	0.472	0.465	0.459	0.453	0.463	0.457	0.452	0.447
27	0.477	0.470	0.464	0.457	0.451	0.446	0.455	0.449	0.444	0.439
28	0.469	0.462	0.456	0.450	0.444	0.439	0.447	0.442	0.437	0.432
29	0.461	0.455	0.449	0.443	0.437	0.432	0.440	0.435	0.430	0.425
30	0.454	0.448	0.442	0.436	0.431	0.425	0.433	0.428	0.423	0.419
31	0.462	0.455	0.449	0.443	0.437	0.432	0.439	0.434	0.429	0.424
32	0.455	0.449	0.443	0.437	0.431	0.426	0.433	0.428	0.423	0.418
33	0.449	0.443	0.437	0.431	0.425	0.420	0.427	0.422	0.417	0.412
34	0.443	0.437	0.431	0.425	0.420	0.414	0.421	0.416	0.411	0.407
35	0.437	0.431	0.425	0.420	0.414	0.409	0.416	0.411	0.406	0.401
36	0.432	0.426	0.420	0.414	0.409	0.404	0.410	0.405	0.401	0.396
37	0.439	0.433	0.427	0.421	0.416	0.410	0.416	0.411	0.407	0.402
38	0.434	0.428	0.422	0.416	0.411	0.405	0.411	0.406	0.402	0.397
39	0.429	0.423	0.417	0.411	0.406	0.401	0.407	0.402	0.397	0.393
40	0.424	0.418	0.412	0.407	0.401	0.396	0.402	0.397	0.393	0.388
41	0.419	0.413	0.408	0.402	0.397	0.392	0.397	0.393	0.388	0.384
42	0.415	0.409	0.403	0.398	0.393	0.388	0.393	0.389	0.384	0.380
43	0.422	0.416	0.410	0.405	0.399	0.394	0.399	0.394	0.390	0.385
44	0.418	0.412	0.406	0.400	0.395	0.390	0.395	0.390	0.386	0.381
45	0.413	0.408	0.402	0.396	0.391	0.386	0.391	0.386	0.382	0.378
46	0.409	0.404	0.398	0.393	0.388	0.383	0.387	0.383	0.378	0.374
47	0.405	0.400	0.394	0.389	0.384	0.379	0.384	0.379	0.375	0.370
48	0.402	0.396	0.391	0.385	0.380	0.376	0.380	0.375	0.371	0.367
49	0.408	0.403	0.397	0.392	0.386	0.381	0.386	0.381	0.377	0.372
50	0.405	0.399	0.393	0.388	0.383	0.378	0.382	0.378	0.373	0.369

TABLE 17 0.050 POINTS FOR TEST FOR DIFFERENCE BETWEEN TWO MEANS

$n_1 \backslash n_2$	41	42	43	44	45	46	47	48	49	50
21	0.473	0.468	0.478	0.473	0.468	0.464	0.460	0.455	0.465	0.461
22	0.463	0.458	0.468	0.463	0.458	0.454	0.450	0.446	0.455	0.451
23	0.454	0.449	0.458	0.454	0.449	0.445	0.441	0.437	0.446	0.442
24	0.445	0.440	0.449	0.445	0.440	0.436	0.432	0.428	0.437	0.433
25	0.449	0.445	0.454	0.449	0.444	0.440	0.436	0.432	0.440	0.436
26	0.442	0.437	0.445	0.441	0.436	0.432	0.428	0.424	0.432	0.428
27	0.434	0.430	0.438	0.433	0.429	0.425	0.421	0.417	0.425	0.421
28	0.427	0.423	0.430	0.426	0.422	0.418	0.414	0.410	0.418	0.414
29	0.420	0.416	0.424	0.419	0.415	0.411	0.407	0.404	0.411	0.407
30	0.414	0.410	0.417	0.413	0.409	0.405	0.401	0.398	0.404	0.401
31	0.419	0.415	0.422	0.418	0.413	0.409	0.405	0.402	0.408	0.405
32	0.413	0.409	0.416	0.412	0.408	0.404	0.400	0.396	0.403	0.399
33	0.408	0.403	0.410	0.406	0.402	0.398	0.394	0.391	0.397	0.393
34	0.402	0.398	0.405	0.400	0.396	0.393	0.389	0.385	0.392	0.388
35	0.397	0.393	0.399	0.395	0.391	0.388	0.384	0.380	0.386	0.383
36	0.392	0.388	0.394	0.390	0.386	0.383	0.379	0.376	0.381	0.378
37	0.397	0.393	0.399	0.395	0.391	0.387	0.384	0.380	0.386	0.382
38	0.393	0.389	0.394	0.390	0.386	0.383	0.379	0.375	0.381	0.378
39	0.388	0.384	0.390	0.386	0.382	0.378	0.375	0.371	0.377	0.373
40	0.384	0.380	0.385	0.381	0.378	0.374	0.370	0.367	0.372	0.369
41	0.380	0.376	0.381	0.377	0.373	0.370	0.366	0.363	0.368	0.365
42	0.376	0.372	0.377	0.373	0.369	0.366	0.363	0.359	0.364	0.361
43	0.381	0.377	0.382	0.378	0.374	0.370	0.367	0.363	0.368	0.365
44	0.377	0.373	0.378	0.374	0.370	0.367	0.363	0.360	0.364	0.361
45	0.373	0.369	0.374	0.370	0.367	0.363	0.360	0.356	0.361	0.358
46	0.370	0.366	0.370	0.367	0.363	0.360	0.356	0.353	0.357	0.354
47	0.366	0.363	0.367	0.363	0.360	0.356	0.353	0.350	0.354	0.351
48	0.363	0.359	0.363	0.360	0.356	0.353	0.350	0.346	0.350	0.347
49	0.368	0.364	0.368	0.364	0.361	0.357	0.354	0.350	0.355	0.351
50	0.365	0.361	0.365	0.361	0.358	0.354	0.351	0.347	0.351	0.348

respectively and that it is required to test the null hypothesis that $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Let σ_1^* and σ_2^* denote the linear estimates of σ_1 and σ_2 . Under the null hypothesis σ_1^* and σ_2^* are approximately normally distributed with means $m_{n_1}\sigma$ and $m_{n_2}\sigma$ and variances $V_{n_1}\sigma^2$ and $V_{n_2}\sigma^2$ respectively. The test-statistic $v_2 = \sigma_1^*/\sigma_2^*$ is thus the ratio of two normal variables and, since the condition $m_{n_2} > 3V_{n_2}^{1/2}$ is satisfied for $n_2 \geq 4$, Geary's theorem may be employed with $a = m_{n_2}$, $b = m_{n_1}$, $\rho = 0$, $\alpha^2 = V_{n_2}$ and $\beta^2 = V_{n_1}$.

Hence we have that

$$z_4 = (m_{n_2} v_2 - m_{n_1}) / (V_{n_2} v_2^2 + V_{n_1})^{1/2}$$

has a standard normal distribution. The percentage points of v_2 may be obtained from those of the standard normal distribution since z_4 is a monotonic increasing function of v_2 . The equation giving the upper p-percent point of v_2 in terms of z_p is

$$v_{2,p} = \frac{m_{n_1} m_{n_2} + z_p [m_{n_1}^2 V_{n_2} + m_{n_2}^2 V_{n_1} - z_p^2 V_{n_1} V_{n_2}]^{1/2}}{(m_{n_2}^2 - z_p^2 V_{n_2})}$$

The 5 and 2½ percentage points of v_2 for $21 \leq n \leq 50$ are given in table 18. Lower percentage points may be obtained by entering the table with n_1 and n_2 reversed and taking the reciprocal of the tabulated value.

TABLE 18 0.025 POINTS FOR TEST FOR DIFFERENCE BETWEEN VARIANCES

$n_1 \backslash n_2$	21	22	23	24	25	26	27	28	29	30
21	1.598	1.555	1.517	1.482	1.565	1.531	1.499	1.470	1.444	1.419
22	1.623	1.580	1.541	1.505	1.589	1.554	1.522	1.493	1.466	1.441
23	1.647	1.603	1.563	1.527	1.612	1.576	1.544	1.514	1.487	1.461
24	1.670	1.624	1.584	1.548	1.634	1.597	1.564	1.534	1.506	1.481
25	1.561	1.519	1.481	1.447	1.527	1.493	1.462	1.434	1.408	1.384
26	1.583	1.540	1.502	1.467	1.548	1.514	1.482	1.454	1.427	1.403
27	1.604	1.560	1.521	1.486	1.568	1.533	1.501	1.472	1.445	1.421
28	1.624	1.579	1.540	1.504	1.587	1.552	1.520	1.490	1.463	1.438
29	1.643	1.598	1.558	1.521	1.606	1.570	1.537	1.507	1.479	1.454
30	1.661	1.615	1.575	1.538	1.623	1.587	1.553	1.523	1.495	1.469
31	1.572	1.529	1.491	1.456	1.536	1.502	1.470	1.442	1.415	1.391
32	1.590	1.546	1.507	1.472	1.553	1.518	1.487	1.457	1.431	1.406
33	1.607	1.563	1.523	1.488	1.570	1.534	1.502	1.473	1.446	1.421
34	1.623	1.579	1.539	1.503	1.585	1.550	1.517	1.487	1.460	1.435
35	1.639	1.594	1.554	1.517	1.601	1.565	1.532	1.501	1.474	1.448
36	1.654	1.609	1.568	1.531	1.615	1.579	1.546	1.515	1.487	1.461
37	1.580	1.536	1.497	1.462	1.542	1.507	1.476	1.446	1.420	1.395
38	1.595	1.551	1.511	1.476	1.557	1.521	1.489	1.460	1.433	1.408
39	1.609	1.564	1.525	1.489	1.570	1.535	1.502	1.473	1.445	1.420
40	1.623	1.578	1.538	1.502	1.584	1.548	1.515	1.485	1.458	1.432
41	1.636	1.591	1.550	1.514	1.597	1.561	1.528	1.497	1.469	1.444
42	1.649	1.604	1.563	1.526	1.609	1.573	1.540	1.509	1.481	1.455
43	1.585	1.541	1.502	1.466	1.546	1.511	1.479	1.450	1.423	1.398
44	1.598	1.553	1.514	1.478	1.559	1.523	1.491	1.461	1.434	1.409
45	1.610	1.565	1.525	1.489	1.571	1.535	1.502	1.472	1.445	1.420
46	1.622	1.577	1.537	1.500	1.582	1.546	1.513	1.483	1.456	1.430
47	1.634	1.589	1.548	1.511	1.594	1.557	1.524	1.494	1.466	1.440
48	1.646	1.600	1.559	1.522	1.605	1.568	1.535	1.504	1.476	1.450
49	1.589	1.545	1.505	1.469	1.549	1.514	1.482	1.452	1.425	1.400
50	1.600	1.556	1.516	1.480	1.560	1.525	1.492	1.462	1.435	1.410

TABLE 18 0.025 POINTS FOR TEST FOR DIFFERENCE BETWEEN VARIANCES

$n_1 \backslash n_2$	31	32	33	34	35	36	37	38	39	40
21	1.487	1.462	1.439	1.418	1.398	1.379	1.436	1.417	1.399	1.382
22	1.509	1.484	1.461	1.439	1.419	1.400	1.457	1.438	1.420	1.403
23	1.531	1.505	1.481	1.459	1.439	1.419	1.478	1.458	1.440	1.422
24	1.551	1.525	1.501	1.479	1.458	1.438	1.497	1.477	1.458	1.441
25	1.449	1.425	1.403	1.382	1.362	1.344	1.398	1.380	1.362	1.346
26	1.469	1.444	1.421	1.400	1.380	1.362	1.417	1.398	1.380	1.364
27	1.487	1.462	1.439	1.418	1.398	1.379	1.435	1.416	1.398	1.381
28	1.505	1.480	1.456	1.435	1.414	1.395	1.452	1.432	1.414	1.397
29	1.522	1.497	1.473	1.451	1.430	1.410	1.468	1.448	1.430	1.412
30	1.538	1.513	1.488	1.466	1.445	1.425	1.483	1.463	1.445	1.427
31	1.456	1.431	1.409	1.387	1.367	1.349	1.404	1.385	1.367	1.350
32	1.472	1.447	1.424	1.402	1.382	1.363	1.419	1.400	1.382	1.365
33	1.487	1.462	1.439	1.417	1.396	1.377	1.433	1.414	1.396	1.379
34	1.502	1.476	1.453	1.431	1.410	1.391	1.447	1.428	1.409	1.392
35	1.516	1.490	1.466	1.444	1.423	1.404	1.461	1.441	1.422	1.405
36	1.530	1.504	1.479	1.457	1.436	1.416	1.474	1.454	1.435	1.417
37	1.460	1.435	1.412	1.391	1.371	1.352	1.406	1.387	1.369	1.353
38	1.473	1.448	1.425	1.403	1.383	1.364	1.419	1.400	1.382	1.365
39	1.486	1.461	1.438	1.416	1.395	1.376	1.431	1.412	1.394	1.377
40	1.499	1.473	1.450	1.427	1.407	1.387	1.443	1.424	1.405	1.388
41	1.511	1.485	1.461	1.439	1.418	1.398	1.455	1.435	1.417	1.399
42	1.523	1.497	1.473	1.450	1.429	1.409	1.466	1.446	1.427	1.410
43	1.463	1.438	1.415	1.393	1.373	1.354	1.408	1.389	1.371	1.354
44	1.474	1.449	1.426	1.404	1.383	1.364	1.419	1.400	1.382	1.364
45	1.486	1.460	1.436	1.414	1.394	1.374	1.430	1.410	1.392	1.375
46	1.496	1.471	1.447	1.425	1.404	1.384	1.440	1.421	1.402	1.385
47	1.507	1.481	1.457	1.435	1.414	1.394	1.450	1.431	1.412	1.394
48	1.517	1.491	1.467	1.445	1.424	1.404	1.460	1.440	1.421	1.404
49	1.465	1.440	1.416	1.394	1.374	1.355	1.409	1.390	1.372	1.355
50	1.475	1.450	1.426	1.404	1.383	1.364	1.419	1.400	1.381	1.364

TABLE 18 0.025 POINTS FOR TEST FOR DIFFERENCE BETWEEN VARIANCES

$n_1 \backslash n_2$	41	42	43	44	45	46	47	48	49	50
21	1.366	1.351	1.400	1.385	1.370	1.356	1.343	1.331	1.374	1.361
22	1.387	1.371	1.421	1.405	1.390	1.376	1.363	1.350	1.394	1.381
23	1.406	1.390	1.440	1.425	1.410	1.395	1.382	1.369	1.413	1.399
24	1.424	1.408	1.459	1.443	1.428	1.413	1.399	1.386	1.431	1.417
25	1.330	1.315	1.363	1.348	1.333	1.320	1.307	1.295	1.336	1.324
26	1.348	1.333	1.381	1.365	1.351	1.337	1.324	1.312	1.354	1.341
27	1.365	1.349	1.398	1.382	1.368	1.354	1.340	1.328	1.370	1.357
28	1.380	1.365	1.414	1.398	1.384	1.369	1.356	1.343	1.386	1.373
29	1.396	1.380	1.430	1.414	1.399	1.384	1.371	1.358	1.401	1.388
30	1.410	1.395	1.445	1.428	1.413	1.399	1.385	1.372	1.416	1.402
31	1.334	1.319	1.367	1.351	1.337	1.323	1.310	1.298	1.339	1.326
32	1.349	1.333	1.381	1.366	1.351	1.337	1.324	1.311	1.353	1.340
33	1.362	1.347	1.395	1.380	1.365	1.351	1.337	1.325	1.367	1.354
34	1.376	1.360	1.409	1.393	1.378	1.364	1.350	1.337	1.380	1.367
35	1.388	1.373	1.422	1.406	1.391	1.376	1.363	1.350	1.393	1.379
36	1.400	1.385	1.434	1.418	1.403	1.388	1.375	1.361	1.405	1.391
37	1.337	1.321	1.369	1.353	1.339	1.325	1.312	1.299	1.340	1.328
38	1.349	1.333	1.381	1.365	1.351	1.337	1.323	1.311	1.352	1.339
39	1.360	1.345	1.393	1.377	1.362	1.348	1.335	1.322	1.364	1.351
40	1.371	1.356	1.404	1.388	1.373	1.359	1.346	1.333	1.375	1.362
41	1.382	1.367	1.415	1.399	1.384	1.370	1.356	1.343	1.386	1.373
42	1.393	1.377	1.426	1.410	1.395	1.380	1.366	1.353	1.396	1.383
43	1.338	1.323	1.370	1.354	1.340	1.326	1.312	1.300	1.341	1.328
44	1.348	1.333	1.380	1.365	1.350	1.336	1.322	1.310	1.351	1.338
45	1.358	1.343	1.390	1.375	1.360	1.346	1.332	1.319	1.361	1.348
46	1.368	1.352	1.401	1.385	1.370	1.355	1.342	1.329	1.371	1.358
47	1.378	1.362	1.410	1.394	1.379	1.365	1.351	1.338	1.380	1.367
48	1.387	1.371	1.420	1.404	1.388	1.374	1.360	1.347	1.390	1.376
49	1.339	1.323	1.370	1.355	1.340	1.326	1.312	1.300	1.341	1.328
50	1.348	1.332	1.379	1.364	1.349	1.335	1.321	1.309	1.350	1.337

TABLE 18 0.050 POINTS FOR TEST FOR DIFFERENCE BETWEEN VARIANCES

$n_1 \backslash n_2$	21	22	23	24	25	26	27	28	29	30
21	1.474	1.438	1.404	1.375	1.454	1.424	1.397	1.371	1.348	1.327
22	1.498	1.461	1.427	1.396	1.477	1.447	1.419	1.393	1.369	1.347
23	1.520	1.482	1.448	1.417	1.499	1.468	1.439	1.413	1.389	1.367
24	1.542	1.503	1.468	1.437	1.520	1.488	1.459	1.433	1.408	1.385
25	1.442	1.406	1.373	1.344	1.421	1.392	1.365	1.340	1.317	1.295
26	1.463	1.426	1.393	1.363	1.441	1.411	1.384	1.358	1.335	1.313
27	1.482	1.445	1.411	1.381	1.460	1.430	1.402	1.376	1.352	1.330
28	1.501	1.463	1.429	1.398	1.478	1.447	1.419	1.393	1.369	1.347
29	1.518	1.480	1.445	1.414	1.496	1.464	1.435	1.409	1.385	1.362
30	1.535	1.497	1.462	1.430	1.512	1.480	1.451	1.425	1.400	1.377
31	1.454	1.417	1.384	1.354	1.432	1.402	1.374	1.349	1.325	1.304
32	1.471	1.433	1.400	1.369	1.448	1.417	1.389	1.364	1.340	1.318
33	1.487	1.449	1.415	1.384	1.463	1.433	1.404	1.378	1.355	1.332
34	1.502	1.464	1.429	1.398	1.478	1.447	1.419	1.392	1.368	1.346
35	1.517	1.478	1.443	1.412	1.493	1.461	1.432	1.406	1.381	1.359
36	1.531	1.492	1.457	1.425	1.507	1.475	1.446	1.419	1.394	1.371
37	1.462	1.425	1.391	1.361	1.439	1.408	1.380	1.355	1.331	1.309
38	1.476	1.438	1.404	1.374	1.452	1.422	1.393	1.368	1.344	1.322
39	1.490	1.452	1.417	1.386	1.465	1.434	1.406	1.380	1.356	1.333
40	1.503	1.464	1.430	1.398	1.478	1.447	1.418	1.392	1.367	1.345
41	1.515	1.477	1.442	1.410	1.490	1.459	1.430	1.403	1.379	1.356
42	1.528	1.488	1.453	1.421	1.502	1.470	1.441	1.414	1.390	1.367
43	1.468	1.430	1.396	1.366	1.444	1.413	1.385	1.359	1.335	1.313
44	1.480	1.442	1.408	1.377	1.455	1.424	1.396	1.370	1.346	1.324
45	1.492	1.453	1.419	1.388	1.467	1.436	1.407	1.381	1.356	1.334
46	1.503	1.464	1.430	1.398	1.478	1.446	1.417	1.391	1.367	1.344
47	1.514	1.475	1.440	1.408	1.489	1.457	1.428	1.401	1.376	1.354
48	1.525	1.486	1.450	1.418	1.499	1.467	1.438	1.411	1.386	1.363
49	1.473	1.435	1.400	1.369	1.447	1.416	1.388	1.362	1.338	1.316
50	1.483	1.445	1.410	1.379	1.458	1.426	1.398	1.372	1.348	1.325

TABLE 18 0.050 POINTS FOR TEST FOR DIFFERENCE BETWEEN VARIANCES

$n_1 \backslash n_2$	31	32	33	34	35	36	37	38	39	40
21	1.391	1.369	1.349	1.330	1.312	1.295	1.350	1.333	1.317	1.301
22	1.413	1.391	1.370	1.351	1.332	1.315	1.370	1.353	1.337	1.321
23	1.433	1.411	1.390	1.370	1.352	1.334	1.390	1.373	1.356	1.340
24	1.453	1.430	1.408	1.388	1.370	1.352	1.409	1.391	1.374	1.358
25	1.358	1.337	1.317	1.298	1.281	1.264	1.317	1.300	1.284	1.270
26	1.377	1.355	1.335	1.316	1.298	1.281	1.335	1.318	1.302	1.287
27	1.395	1.373	1.352	1.333	1.315	1.298	1.352	1.335	1.319	1.303
28	1.412	1.389	1.369	1.349	1.331	1.314	1.368	1.351	1.334	1.319
29	1.428	1.406	1.384	1.365	1.346	1.329	1.384	1.366	1.350	1.334
30	1.444	1.421	1.399	1.379	1.361	1.343	1.399	1.381	1.364	1.348
31	1.367	1.345	1.325	1.306	1.288	1.271	1.324	1.307	1.291	1.276
32	1.382	1.360	1.339	1.320	1.302	1.285	1.339	1.322	1.305	1.290
33	1.397	1.374	1.354	1.334	1.316	1.299	1.353	1.335	1.319	1.304
34	1.411	1.388	1.367	1.347	1.329	1.312	1.366	1.349	1.332	1.316
35	1.424	1.401	1.380	1.360	1.342	1.324	1.379	1.361	1.345	1.329
36	1.437	1.414	1.393	1.373	1.354	1.336	1.392	1.374	1.357	1.341
37	1.372	1.350	1.330	1.311	1.293	1.276	1.329	1.312	1.295	1.280
38	1.385	1.363	1.342	1.323	1.305	1.288	1.341	1.324	1.307	1.292
39	1.398	1.375	1.354	1.335	1.316	1.299	1.353	1.335	1.319	1.303
40	1.410	1.387	1.366	1.346	1.327	1.310	1.364	1.347	1.330	1.314
41	1.421	1.398	1.377	1.357	1.338	1.321	1.375	1.358	1.341	1.325
42	1.432	1.409	1.388	1.368	1.349	1.331	1.386	1.368	1.351	1.335
43	1.376	1.354	1.333	1.314	1.296	1.279	1.332	1.314	1.298	1.283
44	1.387	1.365	1.344	1.324	1.306	1.289	1.342	1.325	1.308	1.293
45	1.398	1.375	1.354	1.335	1.316	1.299	1.353	1.335	1.318	1.303
46	1.408	1.386	1.364	1.345	1.326	1.309	1.363	1.345	1.328	1.312
47	1.419	1.396	1.374	1.354	1.335	1.318	1.372	1.354	1.338	1.322
48	1.429	1.405	1.384	1.364	1.345	1.327	1.382	1.364	1.347	1.331
49	1.379	1.357	1.336	1.316	1.298	1.281	1.334	1.316	1.300	1.285
50	1.389	1.366	1.345	1.326	1.307	1.290	1.343	1.326	1.309	1.294

TABLE 18 0.050 POINTS FOR TEST FOR DIFFERENCE BETWEEN VARIANCES

$n_1 \backslash n_2$	41	42	43	44	45	46	47	48	49	50
21	1.287	1.274	1.320	1.307	1.293	1.281	1.269	1.257	1.298	1.287
22	1.307	1.293	1.341	1.326	1.313	1.300	1.288	1.276	1.318	1.306
23	1.325	1.311	1.360	1.345	1.332	1.319	1.306	1.294	1.337	1.325
24	1.343	1.329	1.378	1.363	1.349	1.336	1.323	1.312	1.354	1.342
25	1.255	1.242	1.288	1.274	1.261	1.249	1.237	1.226	1.266	1.254
26	1.272	1.259	1.305	1.291	1.278	1.266	1.254	1.242	1.283	1.271
27	1.289	1.275	1.322	1.308	1.294	1.282	1.269	1.258	1.299	1.287
28	1.304	1.290	1.337	1.323	1.310	1.297	1.285	1.273	1.314	1.302
29	1.319	1.305	1.353	1.338	1.325	1.312	1.299	1.287	1.329	1.317
30	1.333	1.319	1.367	1.353	1.339	1.326	1.313	1.301	1.343	1.331
31	1.262	1.248	1.294	1.280	1.267	1.254	1.243	1.231	1.271	1.260
32	1.276	1.262	1.308	1.294	1.281	1.268	1.256	1.244	1.285	1.273
33	1.289	1.275	1.322	1.307	1.294	1.281	1.269	1.257	1.298	1.286
34	1.302	1.288	1.335	1.320	1.307	1.294	1.281	1.270	1.311	1.299
35	1.314	1.300	1.347	1.333	1.319	1.306	1.293	1.282	1.323	1.311
36	1.326	1.312	1.359	1.345	1.331	1.318	1.305	1.293	1.335	1.323
37	1.266	1.252	1.298	1.284	1.270	1.258	1.246	1.234	1.274	1.263
38	1.277	1.263	1.309	1.295	1.282	1.269	1.257	1.246	1.286	1.274
39	1.289	1.275	1.321	1.307	1.293	1.280	1.268	1.256	1.297	1.285
40	1.299	1.285	1.332	1.318	1.304	1.291	1.279	1.267	1.308	1.296
41	1.310	1.296	1.343	1.328	1.315	1.302	1.289	1.277	1.318	1.306
42	1.320	1.306	1.353	1.339	1.325	1.312	1.299	1.287	1.329	1.316
43	1.268	1.254	1.300	1.286	1.273	1.260	1.248	1.236	1.276	1.264
44	1.278	1.264	1.310	1.296	1.283	1.270	1.258	1.246	1.286	1.274
45	1.288	1.274	1.320	1.306	1.292	1.279	1.267	1.255	1.296	1.284
46	1.297	1.283	1.330	1.316	1.302	1.289	1.276	1.265	1.305	1.293
47	1.307	1.292	1.339	1.325	1.311	1.298	1.285	1.273	1.315	1.302
48	1.316	1.301	1.349	1.334	1.320	1.307	1.294	1.282	1.324	1.311
49	1.270	1.256	1.302	1.288	1.274	1.261	1.249	1.238	1.277	1.266
50	1.279	1.265	1.311	1.296	1.283	1.270	1.258	1.246	1.286	1.274

Further comments on use of normal approximation

The results obtained in the preceeding sections are based on the assumption that the distribution of σ^* may be closely approximated by a suitable normal distribution. In order to investigate further the validity of this assumption sampling experiments were carried out and Q-Q plots produced for the empirical distribution functions of z_1, z_3 and z_4 . In each of these cases the theoretical distribution is the standard normal so that the Q-Q plots of the empirical quantiles against the normal quantiles should be straight lines with unit slope passing through the origin. The results of the experiments are shown in figures 2, 3 and 4 from which it may be seen that the theoretical representation is reasonable over almost the entire range of the variables and that errors in the percentage points tabulated are likely to be small even for the 98 per cent points and beyond.

Summary of test procedures

In this summary the notation used to describe samples of observations from normal population is as follows:

n = sample size

μ = population mean

σ = population standard deviation

$\bar{y}$ = sample mean

σ^* = linear estimate of σ

An obvious subscript notation is used in the two-sample situation.

a) One-sample test for mean value.

Null hypothesis: $\mu = \mu_0$.

Compute $u_1 = (\bar{y} - \mu_0)/(\sigma^*/\sqrt{n})$ and consult table 15.

b) One-sample test for standard deviation.

Null hypothesis: $\sigma = \sigma_0$.

Compute $v_1 = \sigma^*/\sigma_0$ and consult table 16.

c) Two-sample test for mean values.

Null hypothesis: $\mu_1 - \mu_2 = d$ (d may be zero).

Compute $u_2 = (\bar{y}_1 - \bar{y}_2 - d)/\sigma^*$ where

$$\sigma^* = (n_1\sigma_1^* + n_2\sigma_2^*)/(n_1 + n_2)$$

and enter table 17 with the appropriate values of n_1 and n_2 .

d) Two-sample test for standard-deviations.

Null hypothesis: $\sigma_1 = \sigma_2$.

Compute $v_2 = \sigma_1^*/\sigma_2^*$ and enter table 18 with the row corresponding to n_1 and the column corresponding to n_2 .

For lower percent points of v_2 enter the table with n_1 and n_2 reversed and take the reciprocal of the tabulated value.

Powers of the one and two-sample tests for means compared with Student's t-tests.

a) One-sample tests.

Table 15 gives percentage points of u_1 under the null hypothesis $H_0: \mu = \mu_0$. In order to determine how much power is lost by using this substitute for the standard t-test

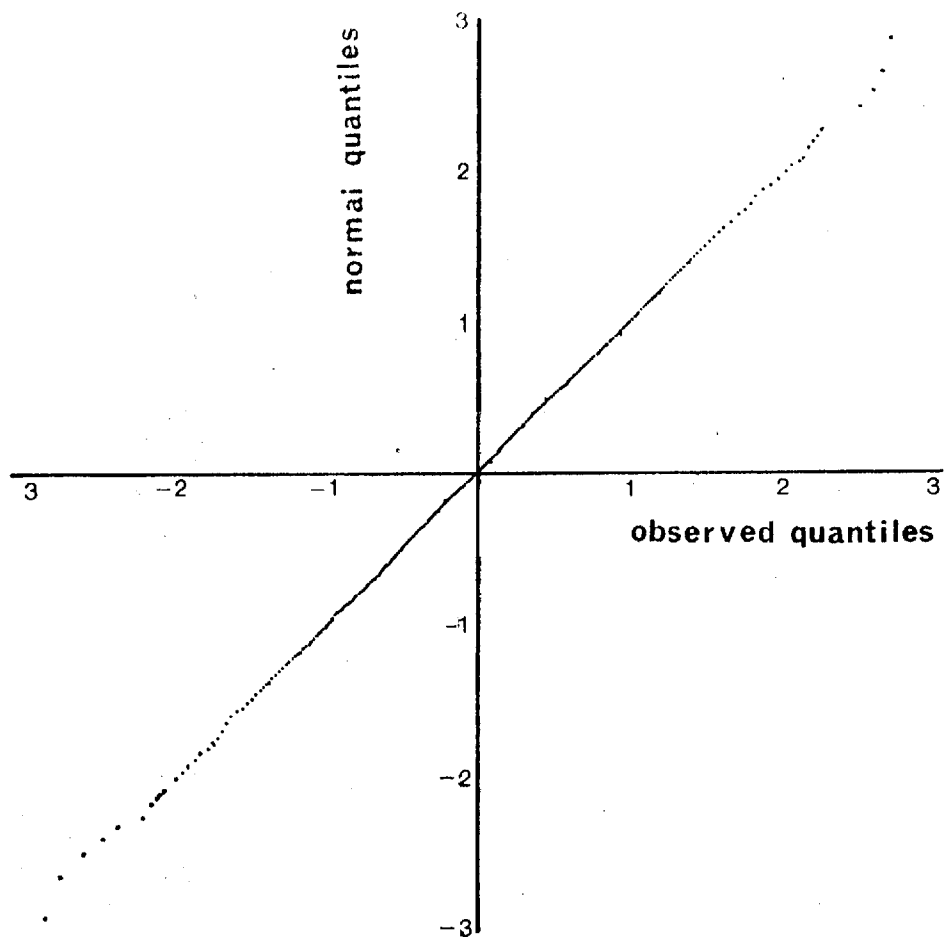


Figure 2. Q-Q plot of 500 estimates of z_i

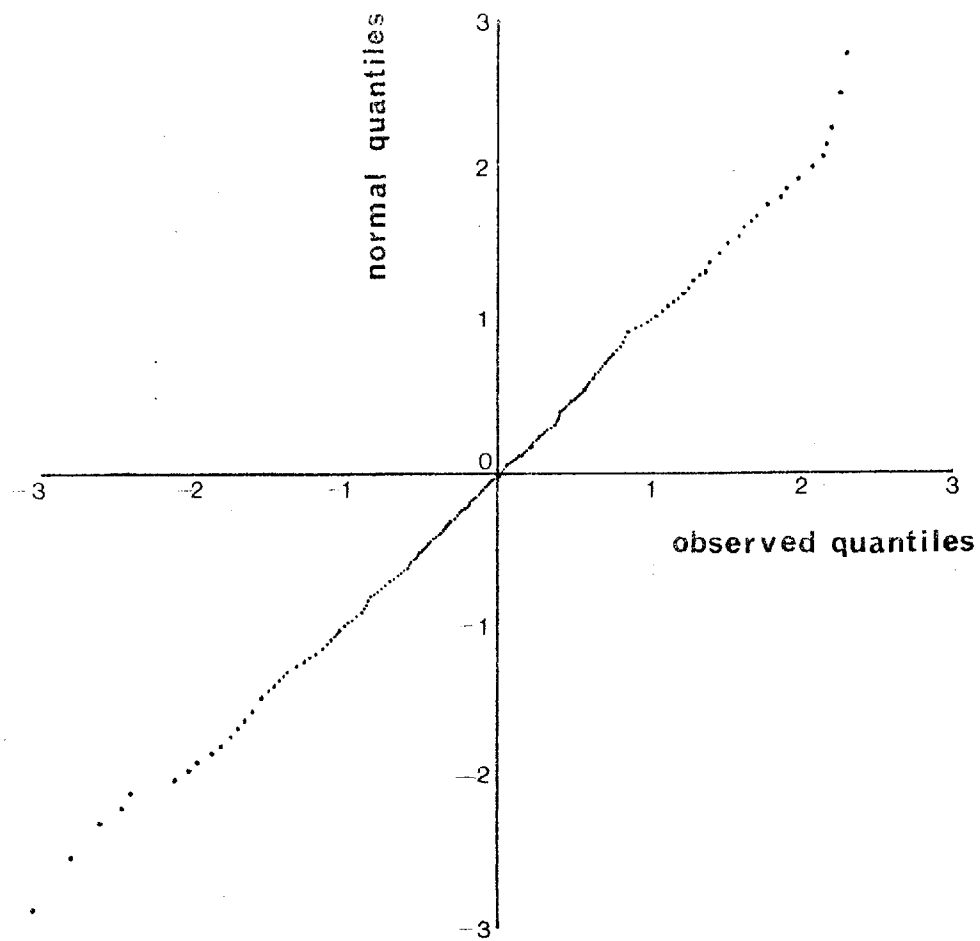


Figure 3. Q-Q plot of 250 estimates of z_3

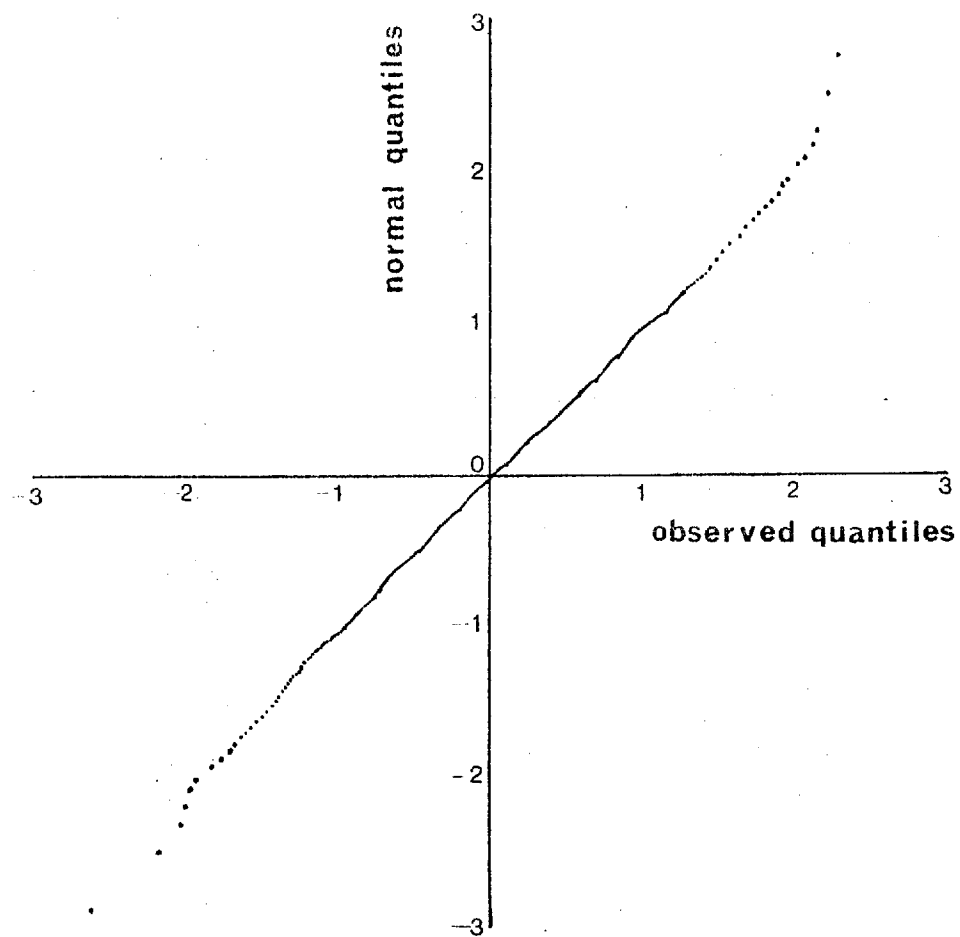


Figure 4. Q-Q plot of 250 estimates of z_4

it was decided to consider the powers of the two tests against the alternative hypothesis $H_1 : \mu = \mu_0 + \epsilon\sigma$ for several values of ϵ using a two-tail test with significance level α . Suppose that $u_{\alpha/2}$ is the 100 $(1 - \alpha/2)$ percent point of u_1 then the critical region for the u_1 -test is given by $|u_1| \geq u_{\alpha/2}$.

Under the alternative hypothesis the test statistic $u_1 = (\bar{y} - \mu_0)/(\sigma^*/\sqrt{n})$ is the ratio of two uncorrelated normal variables with means $\sqrt{n}\epsilon$ and m_n and variances 1 and V_n respectively. Thus

$$z_1 = (m_n u_1 - \sqrt{n}\epsilon)/(V_n u_1^2 + 1)^{\frac{1}{2}}$$

has a standard normal distribution. The power of the u_1 test is therefore given by

$$\begin{aligned} \text{Power} &= P(|u_1| \geq u_{\alpha/2}) \\ &= P\{(-m_n u_{\alpha/2} - \sqrt{n}\epsilon)/(V_n u_{\alpha/2}^2 + 1)^{\frac{1}{2}} \leq z_1 \leq (m_n u_{\alpha/2} - \sqrt{n}\epsilon)/(V_n u_{\alpha/2}^2 + 1)^{\frac{1}{2}}\} \\ &= 1 + F\{(-m_n u_{\alpha/2} - \sqrt{n}\epsilon)/(V_n u_{\alpha/2}^2 + 1)^{\frac{1}{2}}\} \\ &\quad - F\{(m_n u_{\alpha/2} - \sqrt{n}\epsilon)/(V_n u_{\alpha/2}^2 + 1)^{\frac{1}{2}}\} \end{aligned}$$

where F is the cumulative distribution function of the standard normal distribution.

The corresponding t-test statistic is

$t = (\bar{y} - \mu_0) / (s / \sqrt{n})$ which may be expressed as the ratio of a normal variable with mean $\sqrt{n} \epsilon$ and variance 1 to the square-root of a χ^2 -variable with $n-1$ degrees of freedom divided by $n-1$. Therefore, under H_1 , t is distributed as a non-central t with $n-1$ degrees of freedom and non-centrality parameter $\sqrt{n} \epsilon$. The power of the t-test against H_1 is given by

$$\text{Power} = 1 - T(-t_{\alpha/2, n-1}, \sqrt{n} \epsilon) + T(t_{\alpha/2, n-1}, \sqrt{n} \epsilon) \text{ where}$$

$T(t, \delta)$ is the cumulative distribution function of the non-central t and $t_{\alpha/2, n-1}$ is the $100(1 - \alpha/2)$ percent point of the t -distribution with $n-1$ degrees of freedom.

The powers of the u_1 -test and the t -test were computed for a range of values of n and ϵ using a 5 per cent level of significance; the results are compared in table 19. It may be seen from these results that the loss in power resulting from using this alternative to the t -test is negligible.

b) Two-sample tests.

The null hypothesis for the u_2 -test for the means of two populations is $H_0 : \mu_1 - \mu_2 = d$. Suppose the alternative hypothesis is $H_1 : \mu_1 - \mu_2 = d + \epsilon \sigma$, then $u_2 = (\bar{y}_1 - \bar{y}_2 - d) / \sigma^*$ where $\sigma^* = (n_1 \sigma_1^{*2} + n_2 \sigma_2^{*2}) / (n_1 + n_2)$ is the ratio of a normal variable with mean $\{n_1 n_2 / (n_1 + n_2)\}^{1/2} \epsilon$ and variance 1 to a normal variable with mean $g = (n_1 n_2)^{1/2} (n_1^m n_1 + n_2^m n_2) / (n_1 + n_2)^{3/2}$ and

$$\text{variance } h = (n_1 n_2) (n_1^2 V_{n_1} + n_2^2 V_{n_2}) / (n_1 + n_2)^3 .$$

Hence $z_3 = \{g - (n_1 n_2)^{\frac{1}{2}} (n_1 + n_2)^{-\frac{1}{2}} \epsilon\} / (h + 1)^{\frac{1}{2}}$ has a standard normal distribution.

As in the one-sample case the powers of the u_2 -test for given n_1, n_2, ϵ and significance level α may be obtained from the cumulative normal distribution tables and the corresponding powers of the t-test from the non-central t distribution with $n_1 + n_2 - 2$ degrees of freedom and non-centrality parameter $(n_1 n_2)^{\frac{1}{2}} (n_1 + n_2)^{-\frac{1}{2}} \epsilon$. The powers of the two tests, using a two-sided test with $\alpha = 0.05$, are compared in table 20. As with the one-sample case use of this alternative test involves only slight loss of power.

Power of one and two-sample tests for variances or standard deviations

a) One-sample test compared with the χ^2 -test.

The null hypothesis for the one-sample test for a variance is $H_0: \sigma = \sigma_0$. The standard procedure for testing this null hypothesis is to compute $(n-1)s^2/\sigma_0^2$ and test as a χ^2 -variable with $n-1$ degrees of freedom. The powers of this χ^2 -test and the v_1 -test against the alternative hypothesis $H_1: \sigma = k\sigma_0$ are compared below using a one-tail test with significance level α . For the sample sizes under consideration both sets of powers may be computed using the normal distribution tables.

Table 19

Comparison of u_1 -test and t-test.

n	Values of ϵ										
21	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
	0.050	0.072	0.140	0.256	0.412	0.584	0.741	0.860	0.935	0.974	0.991
	0.050	0.072	0.141	0.258	0.415	0.587	0.744	0.862	0.936	0.975	0.992
25	0.050	0.077	0.160	0.300	0.482	0.668	0.819	0.918	0.969	0.991	0.998
	0.050	0.077	0.160	0.302	0.484	0.670	0.821	0.919	0.970	0.991	0.998
29	0.050	0.081	0.179	0.343	0.546	0.737	0.875	0.953	0.986	0.997	0.999
	0.050	0.082	0.180	0.345	0.548	0.739	0.877	0.953	0.986	0.997	0.999
31	0.050	0.084	0.190	0.365	0.576	0.767	0.898	0.964	0.990	0.998	1.000
	0.050	0.084	0.190	0.366	0.578	0.769	0.898	0.965	0.991	0.998	1.000
41	0.050	0.096	0.239	0.465	0.704	0.877	0.963	0.992	0.999	1.000	1.000
	0.050	0.096	0.240	0.466	0.705	0.878	0.963	0.992	0.999	1.000	1.000

Upper entries are powers of u_1 , lower entries are powers of t.

Table 20

Comparison of powers of u_2 -test and t-test.

n_1	n_2	Values of ϵ										
21	41	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.8	0.90	1.00
		0.050	0.066	0.113	0.196	0.311	0.449	0.594	0.727	0.834	0.909	0.956
		0.050	0.066	0.114	0.196	0.311	0.450	0.595	0.728	0.835	0.910	0.956
31	31	0.050	0.067	0.121	0.213	0.340	0.490	0.641	0.773	0.872	0.936	0.972
		0.050	0.067	0.121	0.214	0.341	0.491	0.642	0.774	0.873	0.937	0.972
25	35	0.050	0.066	0.116	0.203	0.323	0.466	0.614	0.747	0.851	0.922	0.963
		0.050	0.067	0.117	0.204	0.324	0.468	0.616	0.749	0.852	0.922	0.964

Upper entries are powers of u_2 , lower entries are powers of t.

Under the alternative hypothesis the test statistic $v_1 = \sigma^*/\sigma_o$ is normally distributed with mean $m_n k$ and variance $V_n k^2$. If $v_{1,\alpha}$ is the 100(1- α) percentage point of v_1 as given in table 4 the power of the v_1 -test is

$$\text{Power} = P(v_1 \geq v_{1,\alpha}) = P\{z_2 \geq (v_{1,\alpha} - m_n k)/(V_n k^2)^{1/2}\}$$

where z_2 is standard normal.

However $v_{1,\alpha}$ is computed using $v_{1,\alpha} = m_n + z_\alpha V_n^{1/2}$ where z_α is the 100(1- α) percentage point of the normal distribution, therefore the power is also given by

$$\begin{aligned} \text{Power} &= P\{z_2 \geq (m_n + z_\alpha V_n^{1/2} - m_n k)/(V_n k^2)^{1/2}\} \\ &= 1 - F\{m_n(1-k)/(V_n k^2)^{1/2} + z_\alpha/k\} \end{aligned}$$

The power of the χ^2 -test against H_1 is given by

$\text{Power} = P\{(n-1)s^2/\sigma_o^2 \geq \chi_{\alpha, n-1}^2\}$ where $\chi_{\alpha, n-1}^2$ is the 100(1- α) percentage point of the χ^2 distribution with $n-1$ degrees of freedom. This probability may be found using the Wilson-Hilferty cubic approximation formula which is expressed as

$$\text{Power} = F[-\{4.5(n-1)\}^{1/2}\{(\chi_{\alpha, n-1}^2/k^2(n-1))^{1/3} + 2/(9(n-1)) - 1\}]$$

The powers of the v_1 -test and the χ^2 -test using a 5 percent level of significance are shown in table 21. The differences in power

Table 21

Comparison of powers of v_1 -test and χ^2 -test.

n	Values of k										
	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
21	0.050	0.175	0.366	0.564	0.722	0.832	0.901	0.942	0.966	0.980	0.988
	0.050	0.167	0.351	0.549	0.716	0.833	0.907	0.949	0.973	0.986	0.993
24	0.050	0.185	0.393	0.601	0.761	0.865	0.926	0.960	0.978	0.988	0.993
	0.050	0.178	0.380	0.593	0.761	0.871	0.934	0.968	0.984	0.993	0.996
28	0.050	0.200	0.431	0.652	0.810	0.903	0.952	0.977	0.989	0.994	0.997
	0.050	0.197	0.418	0.645	0.811	0.909	0.959	0.982	0.992	0.997	0.999
30	0.050	0.208	0.447	0.673	0.828	0.916	0.960	0.982	0.991	0.996	0.998
	0.050	0.199	0.437	0.699	0.832	0.924	0.968	0.987	0.995	0.998	0.999
41	0.050	0.242	0.535	0.774	0.907	0.966	0.988	0.996	0.999	0.999	1.000
	0.050	0.235	0.528	0.776	0.914	0.972	0.991	0.998	0.999	1.000	1.000

Upper entries are powers of v_1 , lower entries are

powers of χ^2

are greater than those for the tests for the means, however, some of this difference is due to the approximation formula above. The results of table 21 indicate that the loss in power is small.

b) Two-sample test compared with the F-test.

The v_2 -test is proposed as an alternative to the F-test in which $F = s_1^2/s_2^2$ is tested under the null hypothesis $H_0: \sigma_1 = \sigma_2 = \sigma$ as an F-variable with n_1-1 and n_2-1 degrees of freedom. The powers of these two tests are compared in table 22 against the alternative hypothesis $\sigma_1 = k\sigma_2 = \sigma$. If this alternative hypothesis is true $v_2 = \sigma_1^*/\sigma_2^*$ is the ratio of a normal variable with mean $m_{n_1}\sigma$ and variance $V_{n_1}\sigma^2$ and a normal variable with mean $m_{n_2}\sigma/k$ and variance $V_{n_2}\sigma^2/k^2$. From Geary's theorem we have that

$$z_4 = (m_{n_2} v_2 / k - m_{n_1}) / (V_{n_2} v_2^2 / k^2 + V_{n_1})^{\frac{1}{2}}$$

is a standard normal variable. Therefore the power of the v_2 -test using a one-tailed test with significance level α is given by

$$\text{Power} = 1 - F\{(m_{n_2} v_{2,\alpha} - m_{n_1} k) / (V_{n_2} v_{2,\alpha}^2 + V_{n_1} k^2)^{\frac{1}{2}}\}$$

where $v_{2,\alpha}$ is the $100(1-\alpha)$ percentage point of v_2 under H_0 .

The power of the corresponding F-test may be computed from the Incomplete Beta-Function as

$$\text{Power} = I_x \left[(n_2 - 1)/2, (n_1 - 1)/2 \right]$$

where $x = (n_2 - 1) / \{ (n_2 - 1) + (n_1 - 1) F_\alpha / k^2 \}$,

F_α is the $100(1-\alpha)$ percentage point of the F distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, and $I_x[a, b]$ is the Incomplete Beta-Function. (Biometrika Tables for Statisticians, Vol. 1, p. 33). From table 22 it may be seen that the loss in power is greatest for small sample sizes and small values of k , however, even in these cases the loss is small.

Concluding Remarks

From the comparisons of power given in tables 19-22 it is apparent that the loss in power resulting from using these tests as substitutes for the t -, χ^2 - and F-tests is small and more than compensated for by the ease of calculation of the estimates of standard deviation. If observations are easily obtained the slight loss in power may be overcome by including one extra observation in each sample.

The approximations involved in determining the percentage points for u_1 , u_2 , v_1 and v_2 are good for $n \geq 30$ and reasonable for $30 \geq n > 20$. These tests are not suitable for

TABLE 22 COMPARISON OF POWERS OF v_2 -TEST AND F-TEST

		VALUES OF k									
n_1	n_2	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8
24	24	0.050	0.202	0.452	0.691	0.851	0.934	0.972	0.988	0.995	0.998
		0.050	0.213	0.474	0.715	0.869	0.946	0.980	0.993	0.997	0.999
24	30	0.050	0.226	0.504	0.745	0.888	0.954	0.982	0.993	0.997	0.999
		0.050	0.234	0.520	0.762	0.901	0.964	0.987	0.996	0.999	1.000
30	24	0.050	0.209	0.476	0.726	0.881	0.955	0.984	0.994	0.998	0.999
		0.050	0.222	0.501	0.750	0.898	0.964	0.988	0.997	0.999	1.000
30	30	0.050	0.236	0.535	0.783	0.917	0.972	0.991	0.997	0.999	1.000
		0.050	0.248	0.556	0.803	0.930	0.978	0.994	0.998	1.000	1.000
24	40	0.050	0.254	0.561	0.798	0.919	0.970	0.989	0.996	0.998	0.999
		0.050	0.259	0.571	0.810	0.930	0.977	0.993	0.998	0.999	1.000
30	40	0.050	0.270	0.601	0.839	0.946	0.984	0.995	0.998	1.000	1.000
		0.050	0.276	0.614	0.851	0.955	0.988	0.997	0.999	1.000	1.000
40	24	0.050	0.210	0.505	0.765	0.913	0.973	0.992	0.998	0.999	1.000
		0.050	0.236	0.538	0.792	0.928	0.979	0.995	0.999	1.000	1.000
40	40	0.050	0.290	0.651	0.884	0.970	0.993	0.999	1.000	1.000	1.000
		0.050	0.300	0.667	0.896	0.976	0.995	0.999	1.000	1.000	1.000
24	50	0.050	0.274	0.599	0.827	0.935	0.977	0.992	0.997	0.999	0.999
		0.050	0.277	0.606	0.838	0.945	0.983	0.995	0.999	1.000	1.000
50	50	0.050	0.341	0.741	0.939	0.990	0.999	1.000	1.000	1.000	1.000
		0.050	0.350	0.755	0.947	0.992	0.999	1.000	1.000	1.000	1.000

Upper entries are powers of v_2 , lower entries are powers of F.

samples smaller than this, however similar tests suitable for small sample sizes which use different linear estimators and more appropriate approximations to their distributions are considered in the next section.

3.2 Small sample tests for means and variances using linear estimators of the standard deviation

One and two-sample tests for means and variances are considered in which the standard deviations of the normal populations are estimated by one of several linear estimators. The distributions of the squares of the linear estimators are shown to be closely approximated by Gamma distributions with parameters related to the variances of the estimators. On the basis of these Gamma distributions it is shown that hypotheses concerning the means and variances of the populations may be tested using the usual t , χ^2 and F -tests. The only differences in the standard one and two-sample test procedures being that the root-mean-square estimator is replaced by the linear estimator and the t , χ^2 and F -tables are entered with non-integer degrees of freedom.

The linear estimators and their variances

Three linear estimators were considered as alternatives to the root-mean-square estimator. These are the best linear unbiased estimator $\hat{\sigma}$. Dixon's estimator $\sigma(D)$ and Gupta's normal scores estimator $\tilde{\sigma}$.

The best linear estimator of the standard deviation σ of a normal population is

$$\hat{\sigma} = (\tilde{m}'\tilde{V}^{-1}\tilde{y})/(\tilde{m}'\tilde{V}^{-1}\tilde{m})$$

with variance given by

$$\text{var } \hat{\sigma} = \sigma^2 / (\underline{m}' \underline{V}^{-1} \underline{m}) .$$

As discussed in 2.4 Shapiro S.S. and Wilk M.B. (1965) used this estimator as the basis of an analysis of variance test for normality and showed that $\underline{m}' \underline{V}^{-1} \underline{m}$ is almost perfectly linearly related to the sample size n . The least-squares equation is

$$\sigma^2 / \text{var } \hat{\sigma} = \underline{m}' \underline{V}^{-1} \underline{m} = 1.981n - 2.411$$

with a regression mean square of 1725.7 and a residual mean square of 0.0016 .

Dixon's linear estimator $\sigma(D)$.

The main disadvantage of using the most efficient linear estimator $\hat{\sigma}$ is that the sum of products of the coefficients $\underline{m}' \underline{V}^{-1} / (\underline{m}' \underline{V}^{-1} \underline{m})$ with the ordered observations has to be computed. This computational difficulty may be overcome using an estimator suggested by Dixon W.J. (1957) who considered the use of subranges, $y_{n-i+1} - y_i = w_{(i)}$, in forming simple estimates of the population standard deviation. He calculated the unbiased estimate of the type $\sigma(D) = k \sum w_{(i)}$, where k is an unbiassing constant and the summation is over the subset of all $w_{(i)}$ which gives minimum variance, for $n \leq 20$. The efficiency of $\sigma(D)$ relative to $\hat{\sigma}$ is nowhere less than 96.6 per cent.

The variance of $\sigma(D)$ may be obtained from the variance-covariance matrix $\underline{V}$ and, as with $\hat{\sigma}$, the reciprocal of its variance may be approximated using the least-squares equation

$$\sigma^2/\text{var } \sigma(D) = 1.967n - 2.640 .$$

This equation was obtained by the author using exact values of $1/\text{var } \sigma(D)$ for $5 \leq n \leq 20$. The regression mean square is 1315.445 with residual mean square 0.0026.

Gupta's normal scores estimate $\tilde{\sigma}$.

The third linear estimator considered is the normal scores estimator discussed in section 2.5. The complete-sample estimator is given by

$$\tilde{\sigma} = \underline{\tilde{m}}'\underline{\tilde{y}}/(\underline{\tilde{m}}'\underline{\tilde{m}})$$

and is obtained by replacing the variance-covariance matrix $\underline{V}$ in the expression for $\hat{\sigma}$ by the identity matrix. The efficiency of $\tilde{\sigma}$ relative to $\hat{\sigma}$ is 99.9 per cent for $n = 5, 10, 15$ and 20 . The variance of $\tilde{\sigma}$ is given by $(\underline{\tilde{m}}'\underline{\tilde{V}}\underline{\tilde{m}})\sigma^2/(\underline{\tilde{m}}'\underline{\tilde{m}})^2$ and the least squares equation for $\sigma^2/\text{var } \tilde{\sigma}$ based on exact values for $n = 5, 10, 15$ and 20 is

$$\sigma^2/\text{var } \tilde{\sigma} = 1.988n - 2.538 .$$

The advantage of this last estimator is that the

coefficients are functions of the normal scores only and are thus readily available for a wide range of values of n .

Fisher's and Yates' Statistical Tables contain the vectors $\tilde{m}$ and values of $\tilde{m}'\tilde{m}$ for all sample sizes up to 50. The variance of $\tilde{\sigma}$ may conveniently be approximated by $\text{var } \tilde{\sigma} = \sigma^2 / (2n - \frac{5}{2})$.

Distribution of the linear estimators.

Student's t -distribution and the F -distribution are derived using the fact that, if the sample comes from a normal distribution with variance σ^2 , then $(n-1)s^2/\sigma^2$ has a χ^2 distribution with $(n-1)$ degrees of freedom. Although it is difficult to derive the exact distributions of the linear estimators $\hat{\sigma}$, $\sigma(D)$ or $\tilde{\sigma}$ it seems reasonable to expect them to behave in a manner similar to s . Empirical sampling experiments were conducted using 500 samples of 10 observations using each of the linear estimators, similar results were obtained for all three. Suppose σ' represents any one of $\hat{\sigma}$, $\sigma(D)$ or $\tilde{\sigma}$. It was conjectured that $z = \sigma'^2/\sigma^2$ behaves as a Gamma variable with parameters α and p , i.e. the density function of z is

$$f(z) = z^{p-1} \exp(-z/\alpha) / \{\Gamma(p)\alpha^p\}, \quad z \geq 0.$$

If this conjecture is true the mean and variance of σ'^2/σ^2 may

be used to derive the parameters α and p . Since the estimators are unbiased

$$E(\sigma'/\sigma) = 1$$

and
$$\text{Var}(\sigma'/\sigma) = v \text{ (say).}$$

where v is about 0.1 for $n = 6$ decreasing to about 0.03 for $n = 20$. The mean and variance of $(\sigma'/\sigma)^2$ are given by

$$E(\sigma'/\sigma)^2 = v + 1$$

and
$$\text{Var}(\sigma'/\sigma)^2 \approx 4 \text{Var}(\sigma'/\sigma) = 4v.$$

The moment generating function of the Gamma distribution with parameters α and p is $(1 - \alpha t)^{-p}$ from which we may obtain the mean and variance of this distribution as

$$\text{mean} = \alpha p$$

and
$$\text{variance} = \alpha^2 p.$$

Equating moments we have $\alpha p = 1 + v$ and $\alpha^2 p = 4v$. To the first order in v the solution of these equations is $\alpha = p^{-1} = 4v$.

In order to test the hypothesis that $(\sigma'/\sigma)^2$ has a Gamma distribution with parameters p^{-1} and p where $p = (4v)^{-1}$ we require a procedure which is sensitive to departures in the tails of this hypothesized Gamma distribution. Such a procedure is the probability plotting procedure suggested by Wilk M.B. and Gnanadesikan R. (1968) and described in the previous section.

In the present situation we wish to compare the empirical distribution function obtained from 500 estimates of $(\sigma'/\sigma)^2$ with the theoretical cumulative of the Gamma distribution. The quantile-quantile plot (Q-Q plot) using N observations is obtained by ordering the N observations and plotting these ordered values against $F^{-1}[(i - \frac{1}{2})/N]$ for $i = 1, 2, \dots, N$, where F^{-1} is the inverse cumulative of the appropriate Gamma distribution. Values of F^{-1} may be derived from Tables of the Incomplete Γ -Function, Pearson K. et. al. (1922).

Figure 5 shows the Q-Q plot obtained by plotting 500 ordered values of $(\hat{\sigma}/\sigma)^2$ for $n = 10$ against the corresponding quantiles of the Gamma distribution with $p = 4.35$. The linearity of the plot indicates that the Gamma distribution is a good approximation over almost the entire range of the variable at least as far as the 99 per cent point which is about 2.5 on the quantile scales. Similar plots were obtained for the other linear estimators.

The remainder of this section deals with test-procedures in which the distribution of the square of the linear estimator is assumed to be Gamma with parameter $p = (4v)^{-1}$.
One and two-sample tests for means and variances.

Suppose that we have a sample of n independent observations from a normal population with mean μ and variance σ^2 . Let $\bar{y}$, σ' and s^2 be the sample mean, the linear

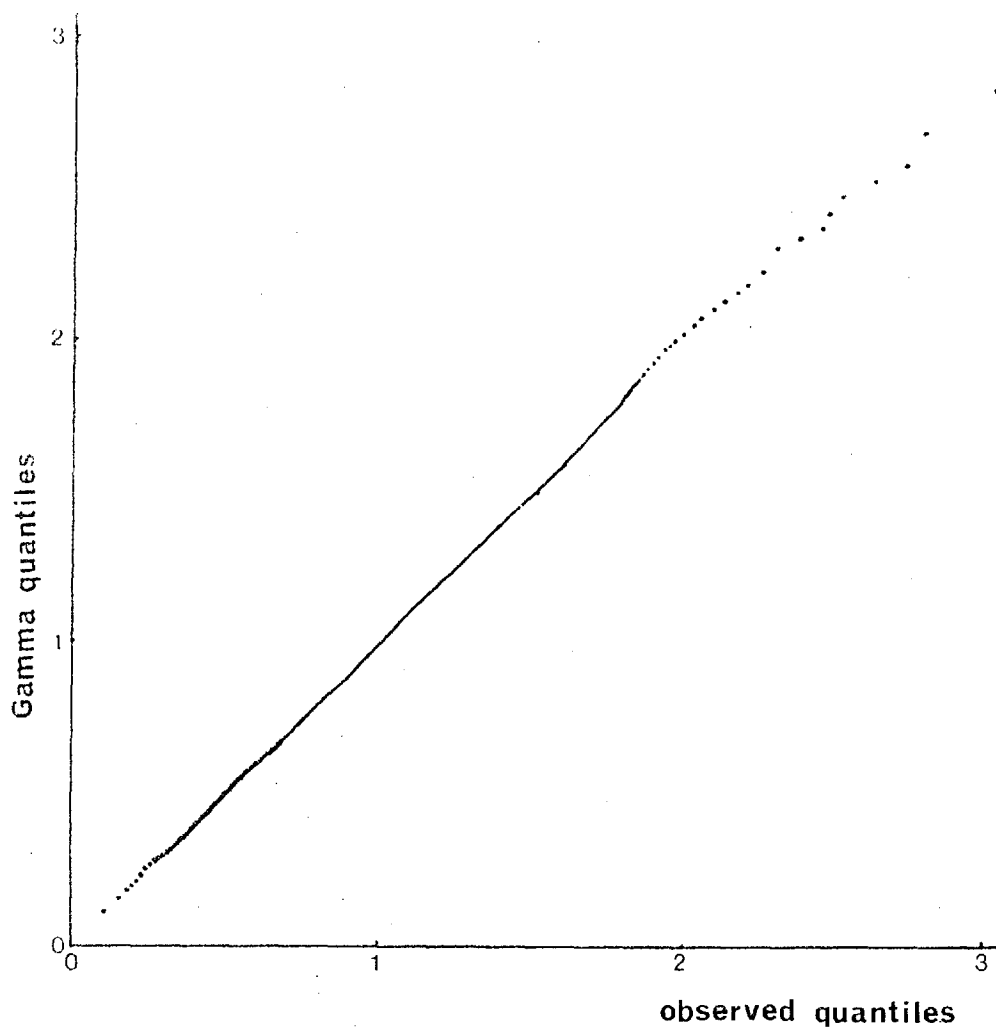


Figure 5. Q-Q plot of 500 observed values of $(\hat{\sigma}/\sigma)^2$ against Gamma quantiles with $p = 4.35$ for $n = 10$.

estimator of σ and the mean-square estimator of σ^2 respectively. An obvious subscript notation will be used in the two-sample situation.

Consider first the one-sample test for a mean value.

The standard procedure to test the null hypothesis $H_0: \mu = \mu_0$ is to test $t = (\bar{y} - \mu_0)/(s/\sqrt{n})$ using Student's t-distribution with $(n-1)$ degrees of freedom. The corresponding test-statistic with s replaced by σ' is $t' = (\bar{y} - \mu_0)/(\sigma'/\sqrt{n})$ which may be expressed as $t' = \{(\bar{y} - \mu_0)/(\sigma/\sqrt{n})\}/(\sigma'/\sigma)$, i.e. $t' = u/\omega^{1/2}$ where u has a standard normal distribution and ω has a Gamma distribution with parameters p^{-1} and p . Since σ' is invariant under a change of origin and the original observations are normally distributed we also have that u and ω are independent. A simple proof of this is given by H.O. Hartley (1950). The distribution of t' may be obtained by transforming the joint density function of u and ω and then integrating out the unwanted variable. The resulting density function of t' is

$$g(t') = \Gamma(p + \frac{1}{2})(1 + t'^2/2p)^{-(p+\frac{1}{2})}/\{(2p\pi)^{\frac{1}{2}}\Gamma(p)\}$$

which is Student's t-distribution with $2p$ degrees of freedom.

Since $p = (4v)^{-1}$, where v is the variance of σ' , the degrees of freedom for the test, i.e. $(2v)^{-1}$, will be non-integer.

For Gupta's estimate, $\tilde{\sigma}$, a good approximation to v^{-1} is

$(2n - \frac{5}{2})$ giving $(n - 1\frac{1}{4})$ as the degrees of freedom for the substitute t-test. It is clear from the similarity in the approximation formulae for the variances of the three linear estimators considered, that the percentage points of t' may be obtained by interpolation in the existing tables of Student's t-distribution using $(n - 1\frac{1}{4})$ as the degrees of freedom irrespective of which linear estimator has been used instead of s .

For the two-sample test for mean values it may be shown in a similar way that, if the variances of the two populations may be assumed equal, the null hypothesis that $\mu_1 - \mu_2 = d$, (d may be zero), may be tested using

$$t' = (\bar{y}_1 - \bar{y}_2 - d) / \{\sigma'^2 (1/n_1 + 1/n_2)\}^{\frac{1}{2}}$$

$$\text{where } \sigma'^2 = \{(n_1 - 1\frac{1}{4})\sigma_1'^2 + (n_2 - 1\frac{1}{4})\sigma_2'^2\} / (n_1 + n_2 - 2\frac{1}{2})$$

and σ_1' , σ_2' are the linear estimators of σ for the two samples. Under the null hypothesis the test-statistic t' has a t-distribution with $(n_1 + n_2 - 2\frac{1}{2})$ degrees of freedom.

Suppose we wish to test the null hypothesis that $\sigma^2 = \sigma_o^2$ using a single sample of size n . The standard procedure is to test $\chi^2 = (n-1)s^2/\sigma_o^2$ as a variable having a χ^2 -distribution with $(n-1)$ degrees of freedom. Using the assumption that σ'^2/σ^2 is approximately Gamma with parameters

p^{-1} and p where $p = (4v)^{-1}$ we have that $2p\sigma'^2/\sigma_o^2$ has a Gamma distribution with parameters 2 and p under the null hypothesis. In other words $2p\sigma'^2/\sigma_o^2$ has a χ^2 -distribution with $2p$ degrees of freedom. But $2p = (2v)^{-1} \approx (n-1\frac{1}{4})$. Therefore under the null hypothesis $(n-1\frac{1}{4})\sigma'^2/\sigma_o^2$ has a χ^2 -distribution with $(n-1\frac{1}{4})$ degrees of freedom.

The two-sample test for variances may be considered in a similar way. The null hypothesis that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ may be tested using $F' = \sigma_o'^2/\sigma_2'^2$ and interpolating in the tables of the F-distribution, since $\sigma_1'^2/\sigma_2'^2 = (\sigma_1'^2/\sigma^2)/(\sigma_2'^2/\sigma^2)$ which under the null hypothesis is the ratio of two χ^2 -variables each divided by its degrees of freedom. Thus F' has an F-distribution with $(n_1 - 1\frac{1}{4})$ and $(n_2 - 1\frac{1}{4})$ degrees of freedom.

CHAPTER 4

COMPARISON OF ROBUSTNESS OF LINEAR ESTIMATORS

4.1 General comments on robustness of estimators.

It has been known for many years that even slight non-normality in the population being sampled has a considerable effect on the sampling distribution of the sample variance. J.W. Tukey (1960) surveys earlier work on sampling from contaminated distributions and shows that "If contamination is a real possibility (and when is it not?) neither mean nor variance is likely to be a wisely chosen basis for making estimates from a large sample". Tukey considers a family of contaminated distributions formed by adding a proportion λ of a normal distribution with mean zero and standard deviation 3 to a proportion $1-\lambda$ of a standard normal distribution. Asymptotic results are derived which show that even though the sample variance is a sufficient statistic for precise normality other estimators are to be preferred on grounds of both efficiency and relative freedom from bias when even a very small proportion of contamination is present. In fact, for large samples, the variance of the sample variance exceeds that of the mean deviation when λ exceeds 0.008. One would need samples of several thousand observations to detect contamination of this magnitude. It seems sensible therefore to consider using alternative efficient linear estimators which are not affected to the same degree by contami-

nation. For most of the estimators discussed earlier there is certainly little to lose and possibly much to be gained by using them rather than the sample standard deviation.

In the following sections we consider the effect of non-normality on estimators of the standard deviation. Firstly, exact values are derived for several linear estimators based on small samples from a variety of non-normal distributions. Finally a summary of the results of a large scale simulation study is given in which the sample standard deviation is compared with some of the linear estimators.

4.2 Some exact results for a variety of distributions.

All the estimators considered in this section are of the form

$$u = \sum_{i=1}^n a_i y_i$$

where $y_1 < y_2 < \dots < y_n$ are the ordered observations and a_i , $i = 1, 2, \dots, n$ are the multiplying coefficients based on the assumption that the observations come from a normal population. If, however, the observations are from some other distribution, with mean μ and variance σ^2 , the mean of the estimator u is given by

$$\begin{aligned} E(u) &= \sum_{i=1}^n a_i E(y_i) \\ &= \sum_{i=1}^n a_i \{ \sigma E(x_i) + \mu \} , \end{aligned}$$

where the x_i are the standardized ordered observations. Thus

$$E(u) = \sum_{i=1}^n a_i \sigma m_i + \mu \sum_{i=1}^n a_i$$

where m_i is the expected value of the i^{th} order statistic from the standardized distribution, and since $\sum_{i=1}^n a_i = 0$ for an unbiased estimator, we have

$$E(u) = \sigma \sum_{i=1}^n a_i m_i .$$

The variance of u is given by

$$\text{Var}(u) = \sigma^2 \sum_{i=1}^n \sum_{j=1}^n a_i a_j v_{ij}$$

where v_{ij} is the covariance of the i^{th} and j^{th} order statistics from the standardized distribution.

Thus, to consider the behaviour of these linear unbiased estimators it is necessary to know the expectations $\bar{m}$ and the variance-covariance matrix $\bar{V}$ for the non-normal distributions.

Although the equations given in chapter 1 for the means, variances and covariances of the order statistics simplify to convenient expressions for both the uniform and the negative exponential distributions, for most other distributions they lead at best to multiple sums or recurrence relationships.

Published tables were used for the right triangular distribution and the logistic distribution but computer programs were written for the uniform, negative exponential and the gamma distribution with shape parameter 2 and 3. Although tables are available for the negative exponential distribution the necessary values were computed as a part check on the gamma distribution program. The published tables contain values for small samples only, therefore, although the values of $\bar{m}$ and $\bar{V}$ are available for the computed distributions for larger samples, the comparisons below are given for $n \leq 10$ only.

The estimators considered.

Depending on the sample size four or five linear estimators were considered, these are:

- 1) the best linear estimator $\hat{\sigma}$,
 - 2) Gupta's normal scores estimator $\tilde{\sigma}$,
 - 3) Downton's estimator with polynomial coefficients
(or Gini's mean difference) $\sigma(G)$,
 - 4) Dixon's short-cut estimator $\sigma(D)$ (see section 3.2),
- and
- 5) Range (when different from Dixon's) $\sigma(R)$.

F. Downton (1966) was interested in formulating an estimator which is mathematically tractable not only in its coefficients but in the computation of the standard error of the estimates obtained. His method produced the linear estimator with polynomial coefficients of the form

$$\sigma(G) = \sqrt{\pi} \sum_{i=1}^n (2i - n - 1) y_i / \{n(n - 1)\} ,$$

which, as pointed out by H.A. David (1968), is an alternative form of Gini's mean difference (Nair 1936). Downton showed that the variance of $\sigma(G)$ may be written explicitly as

$$n \text{ Var}\{\sigma(G)\} = \sigma^2 \{0.5113 + 0.6303/(n - 1)\} .$$

F.C. Barnett, K. Mullen and J.G. Saw (1967) discuss use of $\sigma(G)$ in a test for a mean value as an alternative to

Student's t-test and show, using a normal approximation to the distribution of the usual test statistic with s replaced by $\sigma(G)$, that this test is almost as powerful as Student's t-test.

Comparison of estimators for samples from a normal distribution.

As the first step in this comparison of the behaviour of the estimators, the means, variances and efficiencies were obtained for the normal distribution. The results are summarized in table 23 in which the efficiency is relative to the best linear estimator $\hat{\sigma}$.

Table 23

Efficiencies of the linear estimators for the normal distribution.

Sample size Estimator	5	6	7	8	9	10
Gupta $\hat{\sigma}$	99.93	99.91	99.90	99.86	99.91	99.92
Downton $\sigma(G)$	99.67	99.52	99.38	99.21	99.18	99.11
Dixon $\sigma(D)$	96.61	96.85	97.79	98.05	97.90	97.45
Range $\sigma(R)$	96.61	94.38	92.16	89.99	87.85	85.89

Except for the range all of these estimators are highly efficient relative to the best linear estimator which, as we have seen, is itself over 99 per cent efficient relative

to the unbiased root-mean-square estimator. Thus there is little to lose by using any one of them when we know that the observations are normally distributed.

Comparison of the estimators for non-normal distributions.

a) Uniform distribution on (0,1) .

For the uniform distribution the expected values and variance-covariance matrix elements are most conveniently expressed in terms of the uniform distribution on (0,1) . If y_i is the i^{th} order statistic from this distribution we have that

$$E(y_i) = \frac{i}{n+1}$$

and
$$\text{Cov}(y_i, y_j) = \frac{i(n-j+1)}{(n+1)^2(n+2)}, \quad i \leq j .$$

These values were computed for $5 \leq n \leq 10$ and used to evaluate the means, variances, mean-square-errors and efficiencies of the estimators. Finally these values were scaled so that the uniform distribution had unit standard deviation. The results are given in table 24a where, once again, efficiencies are relative to the best linear estimator. As expected the range is by far the most efficient, however, this estimator has considerable bias which increases as the sample size increases. The estimators which give low weight to the extreme values have low efficiency for this distribution.

Table 24a Uniform.

n	Estimator	Mean	Variance	M.S.E.	Eff.
5	BEST LINEAR	1.01615	0.07895	0.07921	100.00
	GUPTA	1.01947	0.08107	0.08145	97.25
	DOWNTON	1.02330	0.08375	0.08429	93.97
	DIXON (RANGE)	0.99288	0.07040	0.07045	112.43
	BEST LINEAR	1.01265	0.05740	0.05765	100.00
6	GUPTA	1.01720	0.05965	0.05995	96.17
	DOWNTON	1.02315	0.06277	0.06330	91.08
	DIXON	1.03670	0.07144	0.07280	79.20
	RANGE	0.97625	0.04246	0.04802	120.06
	BEST LINEAR	1.00986	0.04451	0.04461	100.00
7	GUPTA	1.01531	0.04652	0.04676	95.40
	DOWNTON	1.02324	0.04982	0.05036	88.58
	DIXON	1.02622	0.05136	0.05205	85.70
	RANGE	0.96075	0.03413	0.03567	125.04
	BEST LINEAR	1.00737	0.03598	0.03604	100.00
8	GUPTA	1.01365	0.03782	0.03800	94.83
	DOWNTON	1.02362	0.04113	0.04169	86.44
	DIXON	1.01478	0.03857	0.03879	92.90
	RANGE	0.94625	0.02575	0.02864	125.82
	BEST LINEAR	1.00520	0.03009	0.03012	100.00
9	GUPTA	1.01162	0.03173	0.03186	94.53
	DOWNTON	1.02321	0.03501	0.03554	84.74
	DIXON	1.00291	0.02997	0.02998	100.47
	RANGE	0.93307	0.01986	0.02434	123.74
	BEST LINEAR	1.00350	0.02560	0.02561	100.00
10	GUPTA	1.01019	0.02705	0.02716	94.30
	DOWNTON	1.02320	0.03025	0.03080	83.15
	DIXON	0.99150	0.02369	0.02376	107.76
	RANGE	0.92005	0.01571	0.02196	116.61
	BEST LINEAR	1.00350	0.02560	0.02561	100.00

b) Negative exponential distribution

The density function of the negative exponential distribution is

$$f(y) = e^{-y/\sigma}/\sigma, \quad 0 \leq y.$$

The expected value of the standardized order statistic y_i in a sample of size n from this distribution is

$$E(y_i) = \sum_{k=1}^i \frac{1}{n-k+1}$$

and the variances and covariances are

$$\text{Var}(y_i) = \text{Cov}(y_i, y_j) = \sum_{k=1}^i \frac{1}{(n-k+1)^2} \quad \text{for } i < j.$$

Table 24b gives the properties of the linear estimators based on samples of size $5 \leq n \leq 10$ from this distribution. Because the exponential distribution has large positive skewness the situation is the reverse of that for the uniform distribution. The estimator based on range now becomes increasingly poor as the sample size increases. There is little to choose between the best linear and Gupta's estimators but since less weight is given to the extreme observations Downton's estimation has greatest relative efficiency which is increasing with sample size. Dixon's estimation behaves in a disjointed manner depending on the form of the estimator. When the estimator is based on four

Table 24b Exponential.

n	Estimator	Mean	Variance	M.S.E.	Eff.
5	BEST LINEAR	0.88850	0.24730	0.25282	100.00
	GUPTA	0.88742	0.23801	0.25068	100.85
	DOWNTON	0.88622	0.23562	0.24856	101.71
	DIXON (RANGE)	0.89571	0.26315	0.27403	92.26
6	BEST LINEAR	0.88949	0.19754	0.20976	100.00
	GUPTA	0.88802	0.19493	0.20747	101.10
	DOWNTON	0.88615	0.19196	0.20492	102.36
	DIXON	0.88171	0.18756	0.20155	104.07
	RANGE	0.90000	0.22790	0.23770	88.24
7	BEST LINEAR	0.89041	0.16799	0.18000	100.00
	GUPTA	0.88867	0.16537	0.17777	101.26
	DOWNTON	0.88617	0.16206	0.17501	102.85
	DIXON	0.88477	0.16189	0.17517	102.76
	RANGE	0.90597	0.20395	0.21279	84.59
8	BEST LINEAR	0.89124	0.14629	0.15812	100.00
	GUPTA	0.88962	0.14379	0.15602	101.35
	DOWNTON	0.88666	0.14031	0.15321	103.21
	DIXON	0.88823	0.14413	0.15662	100.96
	RANGE	0.91063	0.18647	0.19446	81.31
9	BEST LINEAR	0.89194	0.12964	0.14132	100.00
	GUPTA	0.88974	0.12720	0.13936	101.41
	DOWNTON	0.88614	0.12360	0.13657	103.48
	DIXON	0.89147	0.13099	0.14277	98.99
	RANGE	0.91512	0.17317	0.18037	78.35
10	BEST LINEAR	0.89271	0.11650	0.12801	100.00
	GUPTA	0.89033	0.11419	0.12621	101.42
	DOWNTON	0.88622	0.11053	0.12348	103.67
	DIXON	0.89482	0.12092	0.13198	96.99
	RANGE	0.91925	0.16258	0.16910	75.70

observations, i.e. for $6 \leq n \leq 10$ the efficiency decreases from 104 to 97 per cent. The estimator changes for $n = 11$ and its efficiency is again above 100 per cent and will gradually decrease until the estimator changes again at $n = 17$ and so on.

c) Logistic distribution

The density function of the logistic distribution is

$$f(y) = e^{-y} (1 + e^{-y})^{-2} \quad -\infty < y < \infty .$$

This is a symmetric distribution with mean zero and variance $\pi^2/3$. It is very similar in shape to a normal distribution except in the extreme tails. A. Birnbaum and J. Dudman (1963), R.L. Plackett (1958) and S.S. Gupta & B.K. Shah (1965) have studied the problem of evaluating the expectations and variance-covariance matrix of the order statistics from a logistic distribution and B.K. Shah (1966) has tabulated the values of V for the standardized distribution for samples of size $n \leq 10$.

Gupta and Shah derived their results by way of the moment generating function of the k^{th} order statistic, which they expanded as a complicated power series in t containing Bernoulli numbers and Sterling numbers of the first kind. From this expansion these authors obtained exact expressions for the moments of the k^{th} order statistic which they tabulated for $n \leq 10$. This table was used for the vector $\underline{m}$ in the comparisons below. In order to find the covariances Gupta and Shah considered

the joint moment generating function of the i^{th} and j^{th} order statistics which they expanded and differentiated to give an expression for the bivariate moments. Later Shah (1965) simplified the expansion of the joint moment generating function so that it contained only one summation and derived from it several inter-related recurrence relations in terms of the sample size, and the covariances. These recurrence relations may be used together with earlier results by H. Ruben (1956) and Z. Govindarajulu (1963) to build up the variance-covariance matrix for increasing sample size. Shah's table for $n \leq 10$ was used to compare the properties of the linear estimators which are summarized in table 24c.

Downton's estimator is least affected by the larger tails of the distribution, however, Gupta's estimator also has increasing efficiency as n increases and has smaller bias than Downton's. As would be expected the efficiency of the range decreases even more rapidly for observations from the logistic distribution than from the normal distribution.

d) Right-triangular distribution

The right-triangular distribution is an unsymmetrical distribution with density function

$$\begin{aligned} f(y) &= \{(y-\mu)/\sigma + 2\sqrt{2}\}/9\sigma & \mu-2\sqrt{2}\sigma \leq y \leq \mu+\sqrt{2}\sigma \\ &= 0 & \text{otherwise.} \end{aligned}$$

Table 24c Logistic.

n	Estimator	Mean	Variance	M.S.E.	Eff.
5	BEST LINEAR	0.97971	0.16567	0.16608	100.00
	GUPTA	0.97852	0.16444	0.16490	100.71
	DOWNTON	0.97729	0.16330	0.16382	101.38
	DIXON (RANGE)	0.98766	0.17048	0.17064	92.45
	BEST LINEAR	0.98083	0.13425	0.13461	100.00
6	GUPTA	0.97928	0.13280	0.13332	100.97
	DOWNTON	0.97717	0.13146	0.13198	102.00
	DIXON	0.97223	0.13013	0.13090	102.83
	RANGE	0.90352	0.15266	0.15270	88.16
	BEST LINEAR	0.98187	0.11305	0.11338	100.00
7	GUPTA	0.97995	0.11168	0.11208	101.16
	DOWNTON	0.97718	0.11006	0.11059	102.53
	DIXON	0.97563	0.11044	0.11103	102.11
	RANGE	0.90965	0.13484	0.13484	84.09
	BEST LINEAR	0.98270	0.09769	0.09799	100.00
8	GUPTA	0.98070	0.09638	0.09675	101.28
	DOWNTON	0.97761	0.09468	0.09519	102.94
	DIXON	0.97039	0.09697	0.09740	100.61
	RANGE	1.00408	0.12203	0.12205	80.29
	BEST LINEAR	0.98377	0.08608	0.08635	100.00
9	GUPTA	0.98137	0.08479	0.08514	101.42
	DOWNTON	0.97766	0.08302	0.08353	103.38
	DIXON	0.98204	0.08720	0.08749	98.69
	RANGE	1.00902	0.11241	0.11250	76.76
	BEST LINEAR	0.98435	0.07700	0.07725	100.00
10	GUPTA	0.98173	0.07577	0.07610	101.50
	DOWNTON	0.97710	0.07396	0.07448	103.71
	DIXON	0.98667	0.07980	0.07998	96.59
	RANGE	1.01361	0.10488	0.10507	73.52
	BEST LINEAR	0.98435	0.07700	0.07725	100.00

For the standardized distribution R. Downton (1954) has shown that the vector $\underline{m}$ and matrix $\underline{V}$ are given by

$$m_i = \{6n^{(n-i+1)} 2^{n-i+1} / (2n+1)^{(n-i+1)-4}\} / \sqrt{2} ,$$

$$v_{ii} = 18\{i/(n+1) - [n^{(n-i+1)} 2^{n-i+1} / (2n+1)^{(n-i+1)}]^2\} ,$$

$$v_{ij} = (j-1)^{(j-i)} 2^{j-i} v_{jj} / (2j-1)^{(j-i)} , \quad i < j$$

where $r^{(s)} = r(r-1) \dots (r-s+1)$

and $(ar+b)^{(s)} = (ar+b)(a[r-1]+b) \dots (a[r-s+1]+b) .$

These expressions were used by Downton to tabulate $\underline{m}$ and $\underline{V}$ for $n \leq 10$. The results given in table 24d are based on Downton's tabulated values. Although the range is the best of these linear estimators due to the negative kurtosis of this distribution, it is interesting to note that the normal best linear and Gupta's estimator are about equivalent and compare reasonably well with the best linear estimator for the right-triangular scale parameter whose coefficients and variances are also tabulated by Downton (1954) for $n \leq 10$. The efficiency of the normal best linear to the right-triangular best linear decreases from 96 per cent from $n = 5$ to 86 per cent for $n = 10$. The bias is almost negligible thus for small samples, even from this skew platykurtic distribution, these linear estimators

Table 24d Right triangular.

n	Estimator	Mean	Variance	M.S.E.	Eff.
5	BEST LINEAR	0.00835	0.10053	0.10053	100.00
	GUPTA	1.00034	0.11108	0.11108	98.61
	DOWNTON	1.00264	0.11310	0.11310	96.84
	DIXON (RANGE)	0.98447	0.10479	0.10503	104.28
6	BEST LINEAR	0.00631	0.08389	0.08391	100.00
	GUPTA	0.99903	0.08553	0.08553	98.11
	DOWNTON	1.00250	0.08796	0.08797	95.38
	DIXON	1.01082	0.09545	0.09557	87.79
7	RANGE	0.97450	0.07823	0.07888	106.37
	BEST LINEAR	0.00461	0.06755	0.06758	100.00
	GUPTA	0.99780	0.06914	0.06915	97.73
	DOWNTON	1.00267	0.07180	0.07181	94.11
8	DIXON	1.00476	0.07396	0.07398	91.35
	RANGE	0.96503	0.06140	0.06262	107.91
	BEST LINEAR	0.00301	0.05633	0.05638	100.00
	GUPTA	0.99684	0.05785	0.05786	97.44
9	DOWNTON	1.00290	0.06062	0.06063	92.99
	DIXON	0.99802	0.05948	0.05948	94.78
	RANGE	0.95607	0.04098	0.05191	108.61
	BEST LINEAR	0.00171	0.04818	0.04825	100.00
10	GUPTA	0.99558	0.04956	0.04958	97.32
	DOWNTON	1.00255	0.05235	0.05236	92.16
	DIXON	0.99107	0.04918	0.04926	97.96
	RANGE	0.96795	0.04179	0.04650	108.44
10	BEST LINEAR	0.00077	0.04204	0.04212	100.00
	GUPTA	0.99672	0.04330	0.04333	97.22
	DOWNTON	1.00264	0.04608	0.04608	91.41
	DIXON	0.98460	0.04158	0.04182	100.72
10	RANGE	0.94037	0.03565	0.03921	107.43

are reasonably efficient estimators of the standard deviation.

e) Gamma distribution

The density function of the Gamma distribution with parameter r is

$$f(y) = e^{-y} y^{r-1} / \Gamma(r) \quad 0 \leq y.$$

S.S. Gupta (1960) used the relationship between this gamma distribution and the exponential distribution to express the cumulative distribution function of y in the form of a partial sum of probabilities in a Poisson distribution as follows

$$F(y) = \sum_{j=r}^{\infty} \frac{e^{-y} y^j}{j!}.$$

Substituting for $F(y)$ in the equations in Chapter 1 gives, as the i^{th} moment of the k^{th} order statistic in a sample of size n ,

$$\mu_i(k, n) = \frac{n!}{(k-1)!(n-k)!} \int_0^{\infty} \left[1 - \sum_{j=0}^{r-1} \frac{e^{-y} y^j}{j!} \right]^{k-1} \left[\sum_{j=0}^{r-1} \frac{e^{-y} y^j}{j!} \right]^{n-k} \frac{e^{-y} y^{r+i-1}}{\Gamma(r)} dy.$$

Let $D(y) = \sum_{j=0}^{r-1} \frac{y^j}{j!}$, then since

$$[1 - e^{-y} D(y)]^{k-1} = \sum_{p=0}^{k-1} \binom{k-1}{p} (-1)^p e^{-yp} D(y)^p,$$

$$\mu_i(k, n) = \frac{n!}{(k-1)!(n-k)! \Gamma(r)} \sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} \int_0^{\infty} D(y)^{n-k+p} e^{-y(n-k+p+1)} y^{r+i-1} dy.$$

$$= \frac{n!}{(k-1)!(n-k)!\Gamma(r)} \sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} \sum_{m=0}^{(r-1)(n-k+p)} a_m(r, n-k+p)$$

$$\cdot \frac{\Gamma(r+i+m)}{(n-k+p+1)^{r+i+m}} \text{ where } D(y)^p = \sum_{m=0}^{(r-1)p} a_m(r, p) y^m.$$

When $k = 1$ we have

$$\mu_i(1, n) = \frac{n}{\Gamma(r)} \sum_{m=0}^{(r-1)(n-1)} a_m(r, n-1) \frac{\Gamma(r+i+m)}{n^{r+i+m}}.$$

$$\text{Thus } \mu_i(k, n) = \frac{n!}{(k-1)(n-k)!} \sum_{p=0}^{k-1} (-1)^p \binom{k-1}{p} \frac{\mu_i(1, n-k+p+1)}{(n-k+p+1)}$$

may be used to derive the moments of the k^{th} order statistics from those of the first order statistic. The coefficients

$a_m(r, p)$ may be derived by the recurrence relation

$$a_m(r, p) = a_m(r, p-1) + a_{m-1}(r, p-1) + \frac{1}{2!} a_{m-2}(r, p-1) +$$

$$\dots + \frac{1}{(r-1)!} a_{m-r+1}(r, p-1)$$

$$\text{and } a_m(r, 1) = 1/m! \quad m = 0, 1, \dots, (r-1).$$

Suppose that ξ and η are the ℓ^{th} and m^{th} order statistics from this gamma distribution, then we have

$$E(\xi\eta) = C \int_0^\infty \int_0^\infty \xi \eta F^{\ell-1}(\xi) [F(\eta) - F(\xi)]^{m-\ell-1} [1-F(\eta)]^{n-m} f(\xi) f(\eta) d\xi d\eta$$

$$\text{where } C = n! / \{(\ell-1)!(m-\ell-1)!(n-m)!\}.$$

which becomes after considerable algebraic manipulation

$$E(\xi_n) = \frac{C}{[\Gamma(r)]^2} \sum_{\alpha, \beta, t, p, j} (-1)^{\alpha+\beta} \binom{\ell-1}{\alpha} \binom{m-\ell-1}{\beta} \\ \cdot a_p(r, q-1) a_t(r, s-1) \frac{\Gamma(t+r+1) \Gamma(p+r+j+1)}{s^{t+r-j+1} j! (s+q)^{p+r+j+1}}$$

where $q = \alpha + m - \ell - \beta$, $s = n - m + \beta + 1$ and the summation is over α, β, t, p and j where

$$0 \leq \alpha \leq \ell-1$$

$$0 \leq \beta \leq m-\ell-1$$

$$0 \leq t \leq (r-1)(s-1)$$

$$0 \leq p \leq (r-1)(q-1)$$

$$0 \leq j \leq t+r.$$

Although Gupta tabulated the first four moments of the k^{th} order statistic for $n \leq 10$ and the percentage points of its distribution for several values of r the covariances were not tabulated. A computer program was written to evaluate the means, variances and covariances using the equations outlined above. Various checks were built into the program and wherever possible the computed values were checked against Gupta's tables. The variance-covariance matrix for $r = 1$ was checked against the exponential distribution matrix and no discrepancies, even for samples of size 20, were observed. This program was used to

derive $\bar{m}$ and $\bar{V}$ for the gamma distribution with parameter $r = 2$ and 3 . The results were then scaled so that the distributions had unit variance. The properties of the linear estimators are summarized in Tables 24e and 24f for these two distributions.

In both cases Downton's estimator is most efficient although there is little to choose between the best linear, Gupta's or Downton's. As the parameter r increases the efficiency and the bias for Downton's, Gupta's and Dixon's estimators decrease for n fixed. However, the bias of the range decreases and its efficiency increases with increasing r . This is to be expected since, apart from a scale factor these distributions are equivalent to chi-squared distributions with 4 and 6 degrees of freedom respectively. The properties of the estimators will converge to their normal equivalents as r increases.

Comparison of the different estimators.

Figure 6 shows the expected values of the five estimators for the different distributions plotted against the sample size n . All the estimators are biased for these non-normal distributions. Except for the range the expected values of the estimators decrease as the kurtosis of the distribution increases. The Graphs for the estimators which use all the observations are very similar. Downton's form of Gini's mean

Table 24e Gamma r = 2.

n	Estimator	Mean	Variance	M.S.E.	Eff.
5	BEST LINEAR	0.94104	0.19105	0.19452	100.00
	GUPTA	0.94053	0.19082	0.19336	100.60
	DOWNTON	0.93990	0.18871	0.19232	101.15
	DIXON (RANGE)	0.94433	0.20536	0.20846	93.31
6	BEST LINEAR	0.94150	0.15521	0.15863	100.00
	GUPTA	0.94080	0.15385	0.15736	100.81
	DOWNTON	0.93994	0.15247	0.15608	101.64
	DIXON	0.93780	0.15140	0.15534	102.11
	RANGE	0.94670	0.17431	0.17714	89.55
7	BEST LINEAR	0.94102	0.13087	0.13424	100.00
	GUPTA	0.94112	0.12951	0.13298	100.95
	DOWNTON	0.93997	0.12796	0.13157	102.03
	DIXON	0.93920	0.12852	0.13220	101.54
	RANGE	0.94909	0.15350	0.15609	86.00
8	BEST LINEAR	0.94222	0.11321	0.11654	100.00
	GUPTA	0.94152	0.11192	0.11534	101.04
	DOWNTON	0.94022	0.11030	0.11387	102.34
	DIXON	0.94088	0.11274	0.11624	100.26
	RANGE	0.95114	0.13851	0.14090	82.71
9	BEST LINEAR	0.94250	0.09081	0.10310	100.00
	GUPTA	0.94153	0.09853	0.10195	101.13
	DOWNTON	0.93980	0.09684	0.10045	102.64
	DIXON	0.94224	0.10116	0.10450	98.66
	RANGE	0.95322	0.12721	0.12939	79.68
10	BEST LINEAR	0.94306	0.08032	0.09256	100.00
	GUPTA	0.94185	0.08810	0.09148	101.18
	DOWNTON	0.93990	0.08638	0.08998	102.87
	DIXON	0.94376	0.09237	0.09553	96.89
	RANGE	0.95513	0.11831	0.12033	76.93

Table 24f Gamma $r=3$

n	Estimator	Mean	Variance	M.S.E.	Eff.
5	BEST LINEAR	0.96006	0.17271	0.17431	100.00
	GUPTA	0.95972	0.17191	0.17353	100.45
	DOWNTON	0.95936	0.17126	0.17291	100.81
	DIXON (RANGE)	0.96215	0.18396	0.18540	94.02
6	BEST LINEAR	0.96032	0.13947	0.14104	100.00
	GUPTA	0.95986	0.13857	0.14019	100.61
	DOWNTON	0.95930	0.13776	0.13942	101.16
	DIXON	0.95798	0.13802	0.13979	100.90
	RANGE	0.96376	0.15450	0.15581	90.52
7	BEST LINEAR	0.96060	0.11708	0.11863	100.00
	GUPTA	0.96009	0.11619	0.11778	100.73
	DOWNTON	0.95936	0.11527	0.11692	101.46
	DIXON	0.95889	0.11611	0.11780	100.71
	RANGE	0.96521	0.13489	0.13610	87.17
8	BEST LINEAR	0.96070	0.10095	0.10249	100.00
	GUPTA	0.96036	0.10010	0.10168	100.80
	DOWNTON	0.95958	0.09915	0.10079	101.68
	DIXON	0.95992	0.10110	0.10270	99.79
	RANGE	0.96666	0.12084	0.12197	84.03
9	BEST LINEAR	0.96102	0.08875	0.09027	100.00
	GUPTA	0.96030	0.08790	0.08948	100.89
	DOWNTON	0.95926	0.08690	0.08856	101.94
	DIXON	0.96073	0.09012	0.09166	98.49
	RANGE	0.96785	0.11030	0.11134	81.08
10	BEST LINEAR	0.96137	0.07924	0.08073	100.00
	GUPTA	0.96055	0.07843	0.07998	100.94
	DOWNTON	0.95936	0.07740	0.07905	102.13
	DIXON	0.96170	0.08180	0.08326	96.96
	RANGE	0.96907	0.10204	0.10300	78.39

difference has the interesting property that the expected value is independent of n . This property, also indicated by A.E. Sarhan (1954) as one of this estimator's main advantages, enables an unbiased estimator of the standard deviation for any continuous distribution to be expressed as

$$\sigma(G) = C \sum_{i=1}^n (2i - n - 1)y_i / \{n(n - 1)\}$$

where C is an unbiassing constant, independent of n , depending on the form of the distribution.

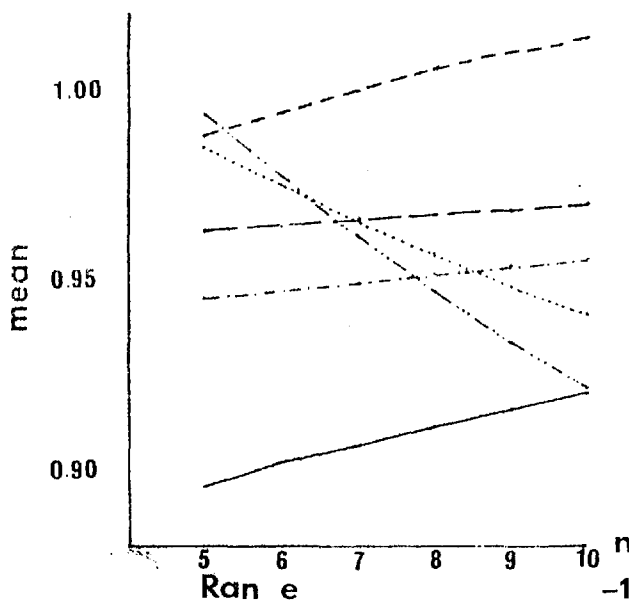
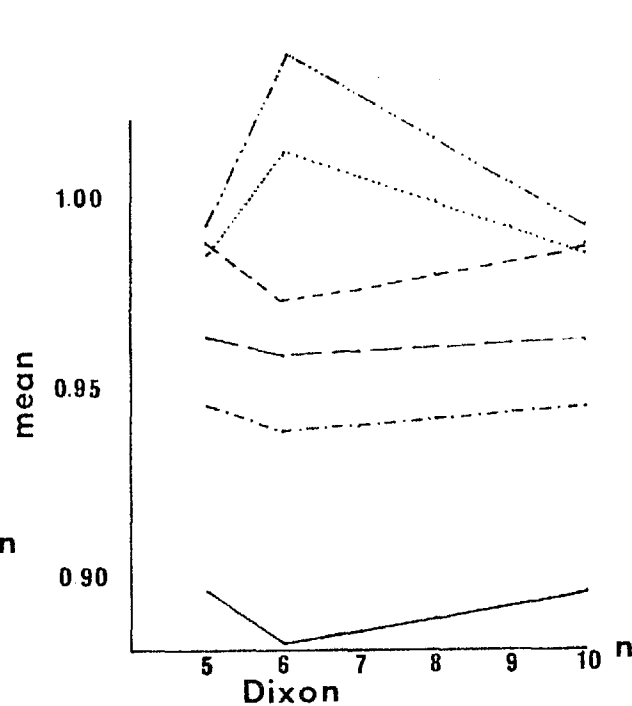
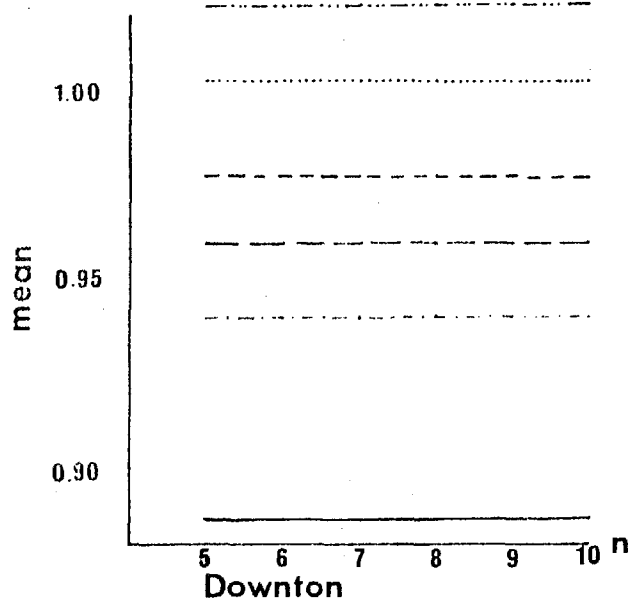
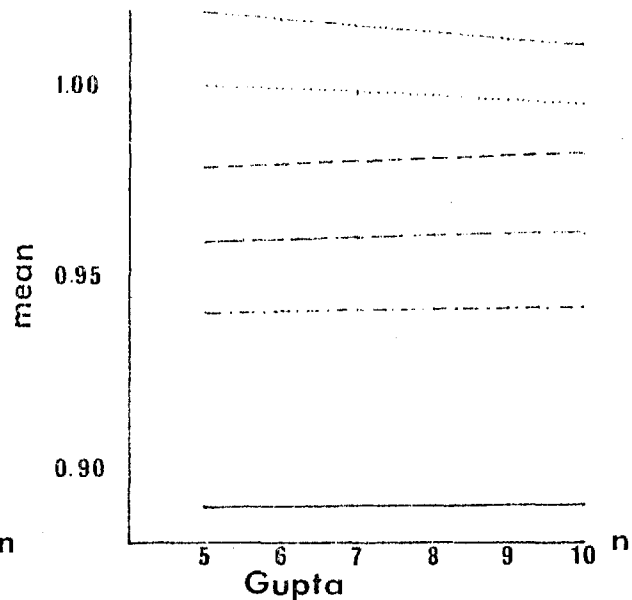
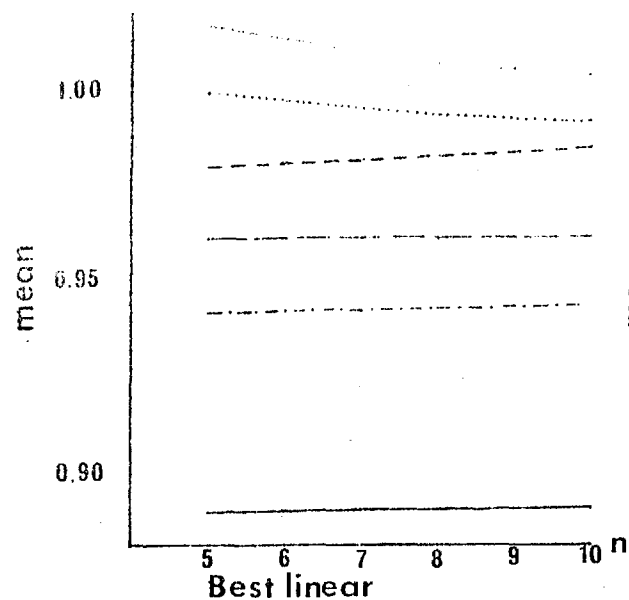
This poses several questions worth further consideration;

- (i) Is C simply related to γ_1 and γ_2 , the coefficients of skewness and kurtosis?
- (ii) How does the variance of this estimator compare with the best linear estimator for non-normal distributions?
- and (iii) How robust is this estimator to departures from the particular non-normal distribution?

For both the exponential and the right triangular distributions this linear estimator is approximately 80 per cent efficient relative to the corresponding best linear estimator for a sample of size 10. Also we have seen that for a normal sample with $n = 10$ the relative efficiency is over 99 per cent.

We may also ask

- (iv) For what range of non-normal distributions does



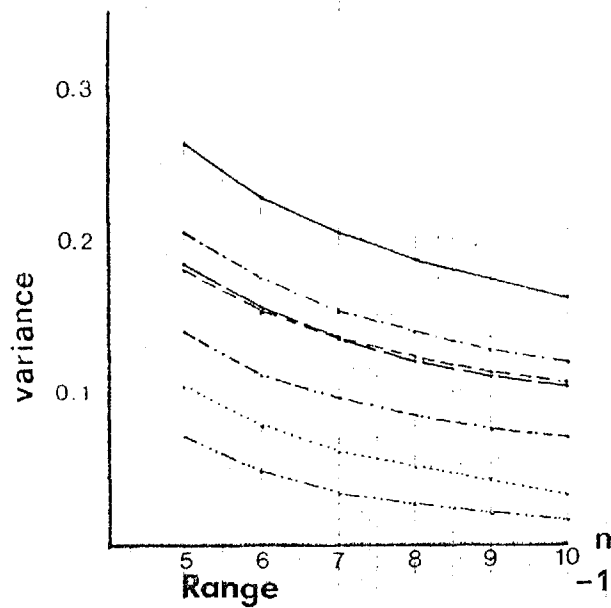
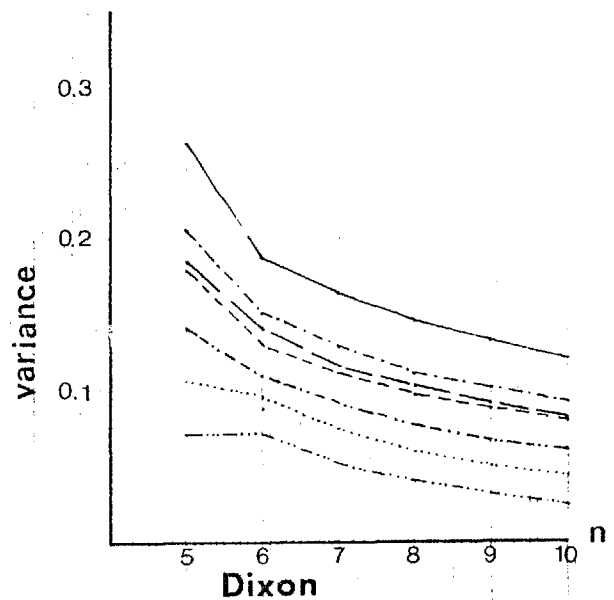
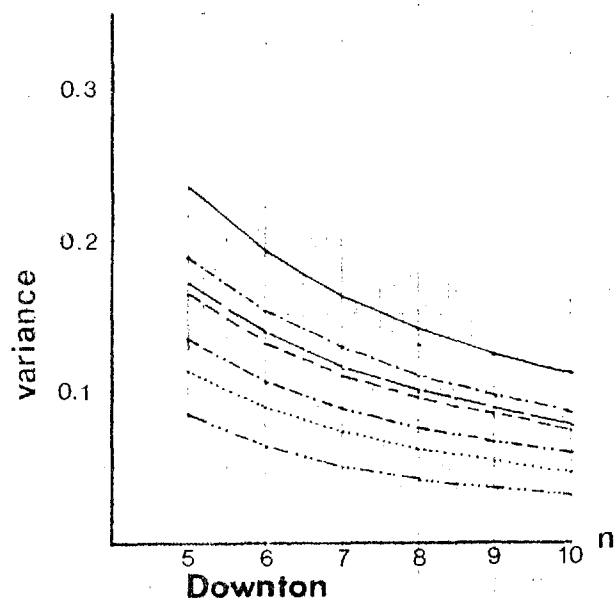
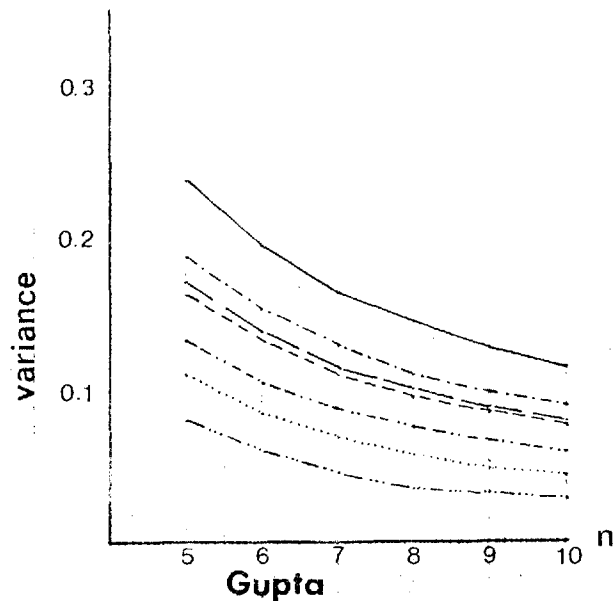
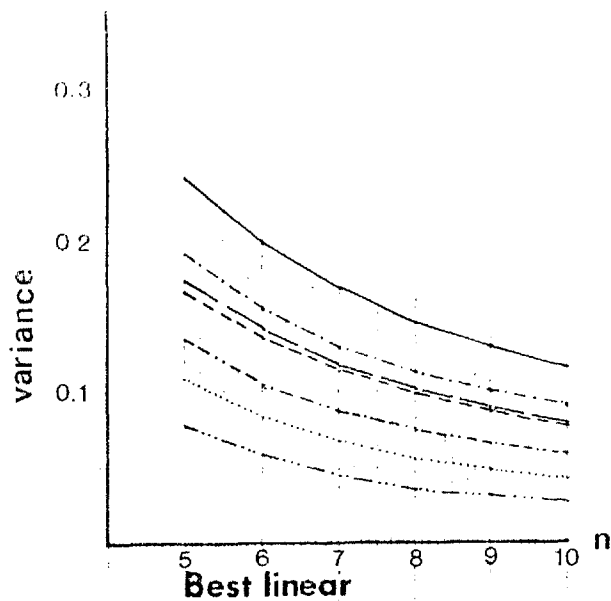
Uniform —————
 Right triangular
 Logistic - - - - -
 Gamma $r=3$ ————
 Gamma $r=2$ ————
 Exponential ————

Figure 6. Comparison of bias.

this estimator have relative efficiency greater than some value, say 95 per cent?

The answers to some of these questions may be contained in Downton's work or in earlier papers on Gini's mean difference. U.S. Nair (1936) derived formulae for the mean and variance of Gini's mean difference for the normal distribution and for the uniform and exponential distributions. Z.A. Lomnicki (1952) used a simpler method to derive the same equations which are given in M.G. Kendall and A. Stuart's, *Advanced Theory of Statistics*, Vol. 1. It is hoped that some of these points will be considered in more detail elsewhere.

Figure 7 shows the variances of each of the estimators for all the distributions considered including the normal. These diagrams emphasize the similarity between the best linear estimator, Gupta's normal scores estimator and Downton's estimator. For platykurtic distributions the variances are less than the corresponding normal variances, whereas for leptokurtic distributions the variances are greater. The spread of the lines in the range diagram indicates that for leptokurtic distributions the range has low efficiency relative to the other estimators and for platykurtic distributions the range has high efficiency. Dixon's estimator is very much better than the range for $n \geq 6$ and for $n = 6, 7$ or 8 is not a great deal worse than the other three.



Normal	—
Uniform	·····
Right triangular	- - - -
Logistic	- · - ·
Gamma $r=3$	— — — —
Gamma $r=2$	- - - -
Exponential	—

Figure 7. Comparison of variances.

These theoretical comparisons confirm several expected features of the behaviour of these linear estimators. Firstly the range is not robust to leptokurtic departures from normality, secondly Dixon's estimator, giving equal weight to several extreme values, is not affected as much as the range and has the advantage that its form is simple and convenient, and thirdly the other estimators, although more complicated to use, are least affected and about equivalent to each other.

The important question of how these estimators compare with the root-mean-square estimator for samples from non-normal distributions is considered in the following section.

4.3 A simulation study

Theoretical evaluation of the properties of the root-mean-square estimators for samples from non-normal distributions is difficult even for some of the standard distributions considered in the preceding section. Since we are also interested in comparing the behaviour of this estimator for small samples from contaminated normal distributions with the large sample results derived by Tukey (1960), it was decided to carry out a large scale simulation study of the robustness of s and of a selection of alternative estimators using small samples from several non-normal distributions including contaminated normal distributions with varying proportions of contamination.

Using the transformations given below 1000 sets of n random deviates were generated from each non-normal distribution. Each sample of n was used to estimate the standard deviation of the distribution for each of the estimators. The 1000 sets of estimates were then used to determine the means and variances of their distributions.

The estimators considered.

In view of the large amount of computing time required for these comparisons it was decided to restrict attention to those linear estimators which behaved differently in the preceding section and to leave out Gupta's estimator and Downton's estimator

in favour of the best linear. For leptokurtic distributions conclusions about the best linear estimator will also hold for Gupta's and Downton's estimators. Apart from including the biassed and unbiased root-mean-square estimators it was decided to include the mean deviation about the mean for comparison with Tukey's asymptotic results. The estimators considered, based on a sample of n observations such that $y_1 < y_2 < \dots < y_n$, were as follows:

- a) the root-mean-square estimator

$$s = \{ \sum (y_i - \bar{y})^2 / (n - 1) \}^{1/2} ;$$

- b) the unbiased root-mean-square estimator

$$s' = \{ (n - 1)/2 \}^{1/2} \Gamma(\frac{n - 1}{2}) s / \Gamma(\frac{n}{2}) ;$$

- c) the best linear estimator $\hat{\sigma}$;

- d) the mean deviation about the mean

$$m = b \sum |y_i - \bar{y}| / n ,$$

where b is an unbiassing constant depending on n ;

- e) Dixon's estimator ;

- and f) the range.

Non-normal distributions.

The program written for these experiments produced either a random uniform deviate x in the interval $(0,1)$ using the Lehmer congruence method, see K.D. Tocher (1963), or a random normal deviate z using G.B. Wetherill's (1965) transformation of the uniform deviate x . The remaining non-normal random deviates were obtained by applying the appropriate transformation to either x or z . The standardized density functions and the transformations employed are listed below:

i) negative exponential distribution:

$$\text{density function: } f(y) = e^{-(y+1)}, \quad -1 \leq y.$$

$$\text{transformation : } y = -\log(1-x) - 1.$$

ii) double-sided exponential distribution:

$$\text{density function: } f(y) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|y|), \quad -\infty < y < \infty$$

$$\text{transformation : } y = \frac{1}{\sqrt{2}} \log 2x \quad \text{if } 0 \leq x \leq \frac{1}{2}$$

$$y = -\frac{1}{\sqrt{2}} \log 2x \quad \text{if } \frac{1}{2} < x \leq 1.$$

iii) right-triangular distribution:

$$\text{density function: } f(y) = (2\sqrt{2} - y)/9, \quad -\sqrt{2} \leq y \leq 2\sqrt{2}$$

$$\text{transformation : } y = \sqrt{2} (3|x_1 - x_2| - 1)$$

where x_1 and x_2 are independent random uniform deviates on $(0,1)$;

iv) symmetric triangular distribution:

$$\text{density function: } f(y) = (\sqrt{6} - |y|)/6, \quad -\sqrt{6} \leq y \leq \sqrt{6}$$

$$\text{transformation: } y = \sqrt{6} (x_1 + x_2 - 1) ;$$

v) lognormal distribution:

$$\text{density function: } f(y) = \exp \left[-\frac{1}{2} \{ \log(y+a)+b \}^2 \right] / \{ \sqrt{2\pi}(y+a) \}$$

$$-a \leq y ;$$

$$\text{where } a = (e-1)^{-\frac{1}{2}} = 0.76287 \text{ and } b = 0.5 - \log a = 0.771,$$

$$\begin{aligned} \text{transformation: } y &= (e^z - e^{\frac{1}{2}}) / (e^2 - e)^{\frac{1}{2}} \\ &= 0.46271e^z - 0.76287, \end{aligned}$$

for further details see M.H. Quenouille (1959);

vi) the standardized uniform distribution:

$$\text{density function: } f(y) = 1/2\sqrt{3}, \quad -\sqrt{3} \leq y \leq \sqrt{3},$$

$$\text{transformation: } y = 2\sqrt{3} (x - \frac{1}{2}) ;$$

vii) contaminated normal distributions:

These distributions were obtained by adding a proportion λ of a normal distribution with mean μ_2 and variance σ_2^2 to a proportion $(1-\lambda)$ of a standard normal distribution and standardizing the resulting distribution so that its mean is zero and its variance one.

The density function is

$$\begin{aligned} f(y) &= \left[(1-\lambda) (2\pi)^{-\frac{1}{2}} \{ \exp -\frac{1}{2} (y\sigma + \lambda\mu_2)^2 \} + \right. \\ &\quad \left. \lambda (2\pi\sigma_2^2)^{-\frac{1}{2}} \exp \{ -\frac{1}{2} (y\sigma - \mu_2 + \mu_2\lambda)^2 / \sigma_2^2 \} \right] \sigma \end{aligned}$$

$$\text{where } \sigma^2 = 1-\lambda + \lambda\sigma_2^2 + \lambda(1-\lambda)\mu_2^2 .$$

The transformation used to generate random deviates from this distribution is

$$\begin{aligned} y &= (z - \lambda\mu_2)/\sigma & \text{if } x \geq \lambda \\ y &= \{\sigma_2 z + \mu_2(1-\lambda)\}/\sigma & \text{if } x < \lambda . \end{aligned}$$

The third and fourth cumulants of the standardized contaminated normal distribution are given by

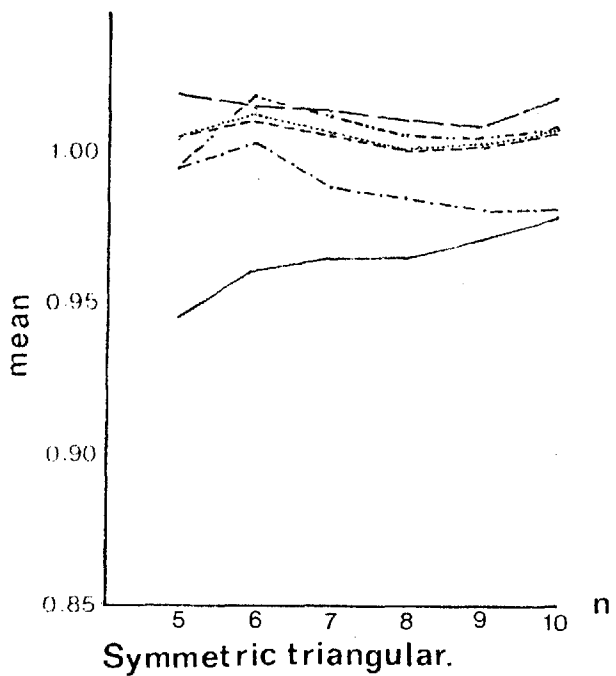
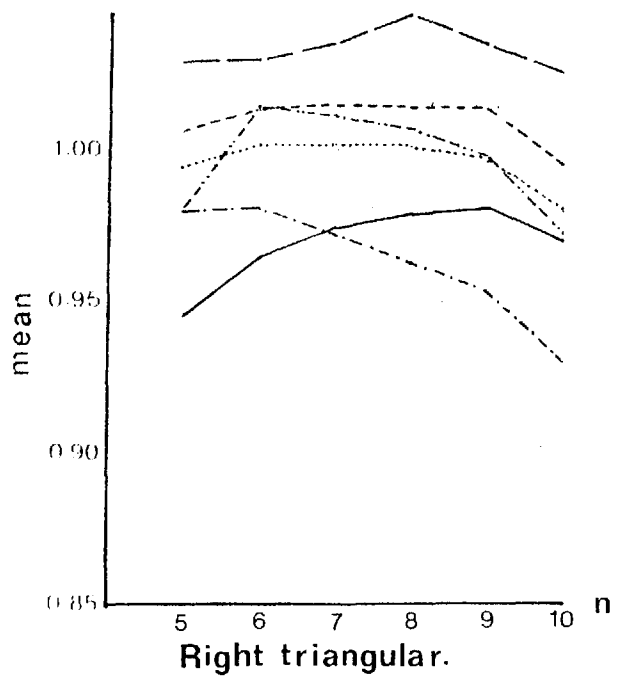
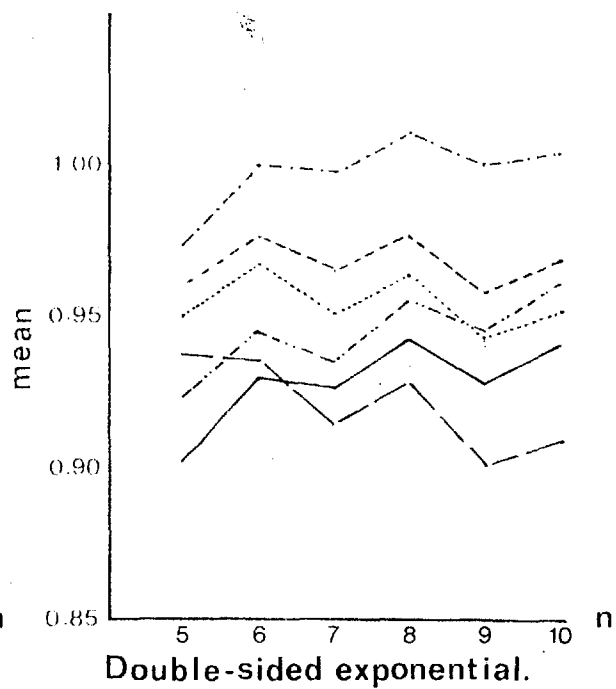
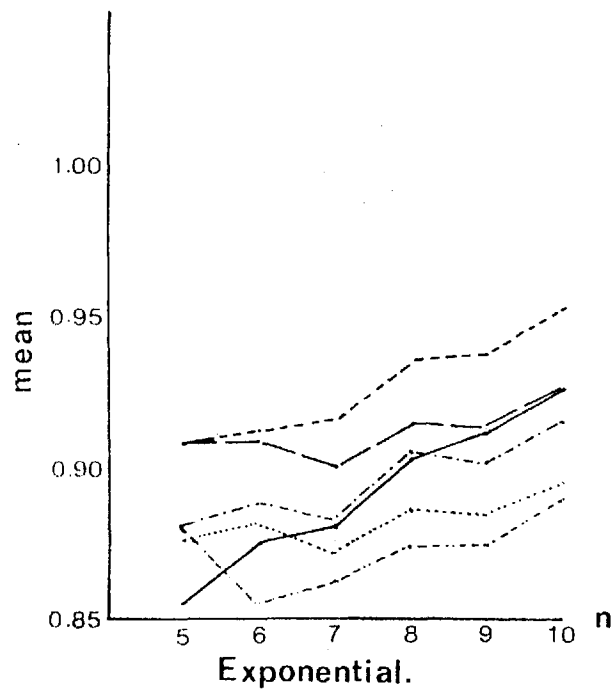
$$k_3 = (1-\lambda)\lambda\mu_2\{(1-2\lambda)\mu_2^2 + 3(\sigma_2^2-1)\}/\sigma^3 ,$$

$$\begin{aligned} \text{and } k_4 &= [3(1-\lambda)+\lambda\sigma_2^4)+(1-\lambda)\lambda\mu_2^2\{6(\lambda+\sigma_2^2-\lambda\sigma_2^2) \\ &\quad +\mu_2^2(1-3\lambda+3\lambda^2)\}]/\sigma^4-3 . \end{aligned}$$

Two such contaminated distributions were considered. Firstly the symmetric distribution with $\lambda = 0.05$, $\mu_2 = 0$ and $\sigma_2 = 3$ and secondly the skew distribution with $\lambda = 0.05$, $\mu_2 = 1$ and $\sigma_2 = 3$. Finally the properties of the estimators were estimated using samples of size 10 from contaminated normals with $\mu_2 = 0$, $\sigma_2 = 3$ and proportion of contamination λ increasing from 0 to 0.9.

Comparison of the estimators

The means and mean square errors of the estimators for each of the non-normal distributions are shown in figures 8a, 8b and 9a, 9b respectively. Before discussing these results in detail



Biassed root-mean-square
 Unbiased
 Best linear
 Mean deviation
 Range
 Dixon

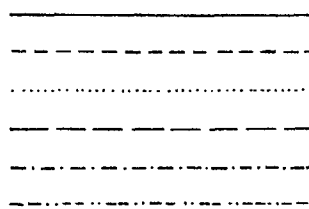
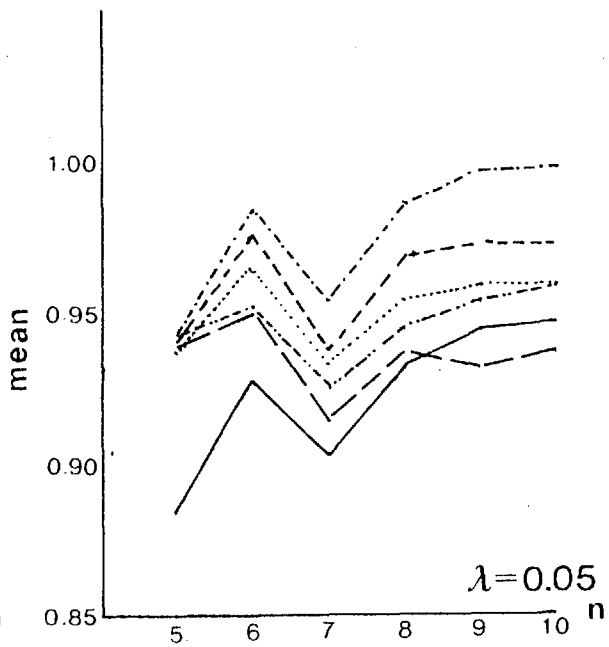
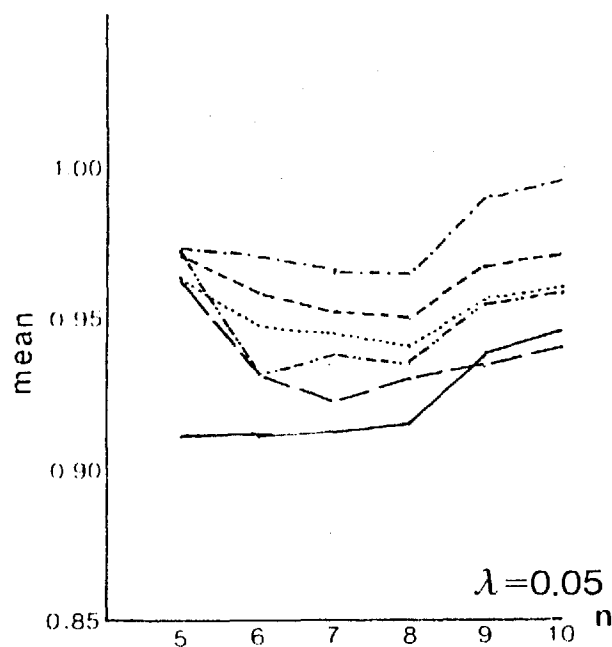
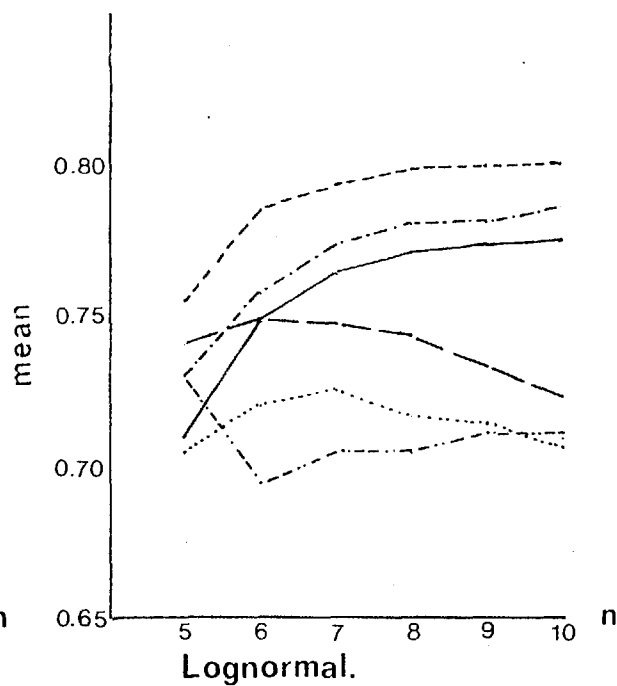
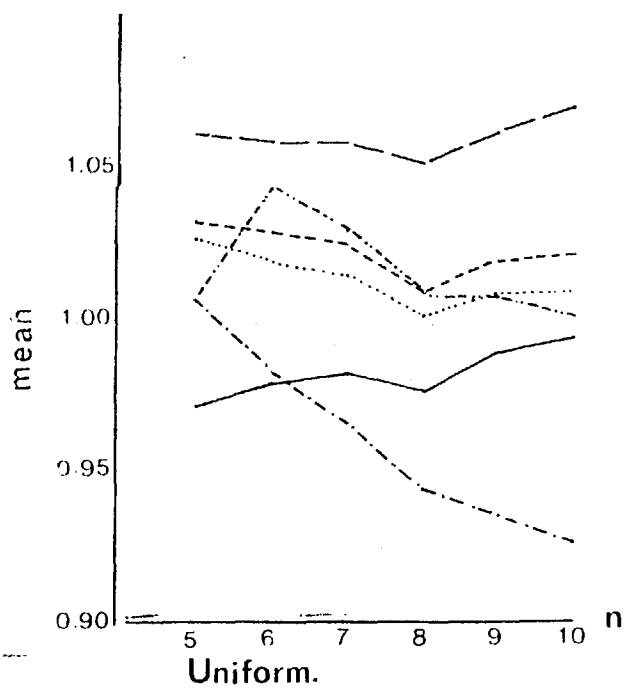


Figure 8a. Comparison of bias.



Contaminated normal (0,3).

Contaminated normal (1,3).

Biased root-mean-square

Unbiased

Best linear

Mean deviation

Range

Dixon

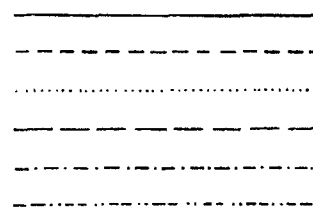
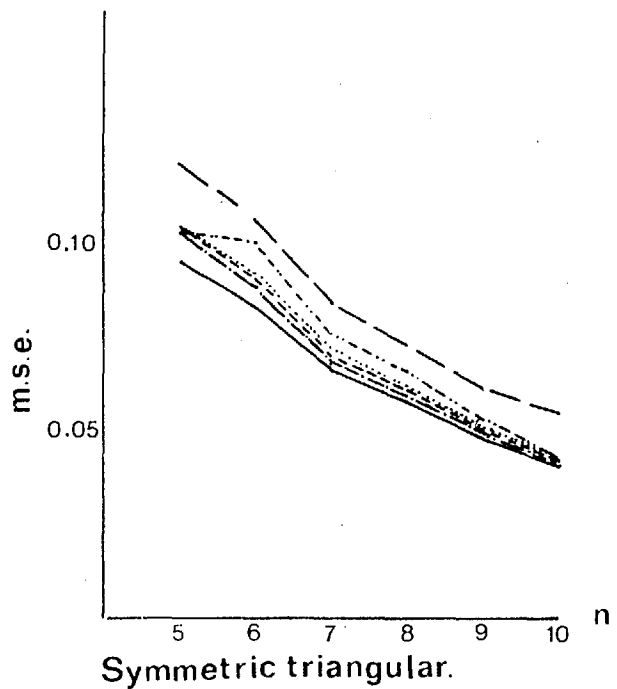
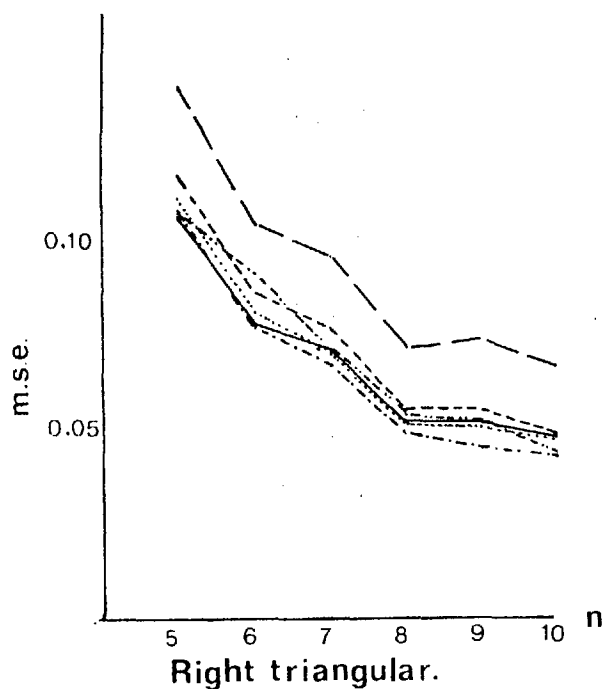
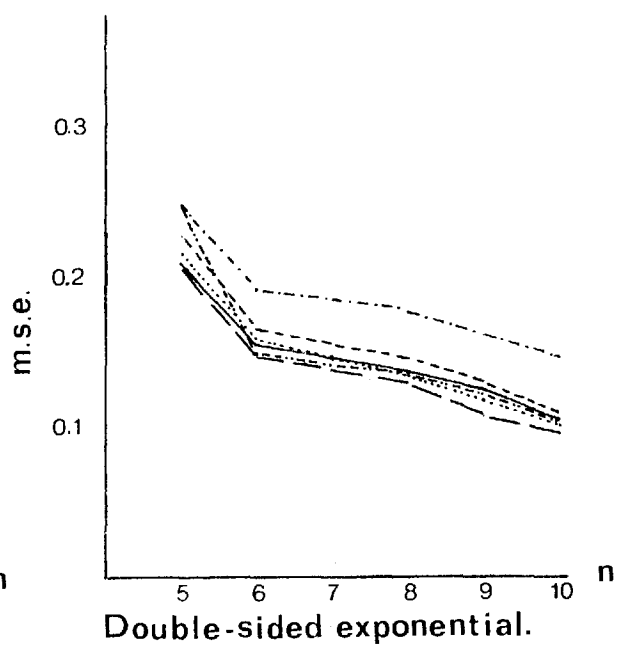
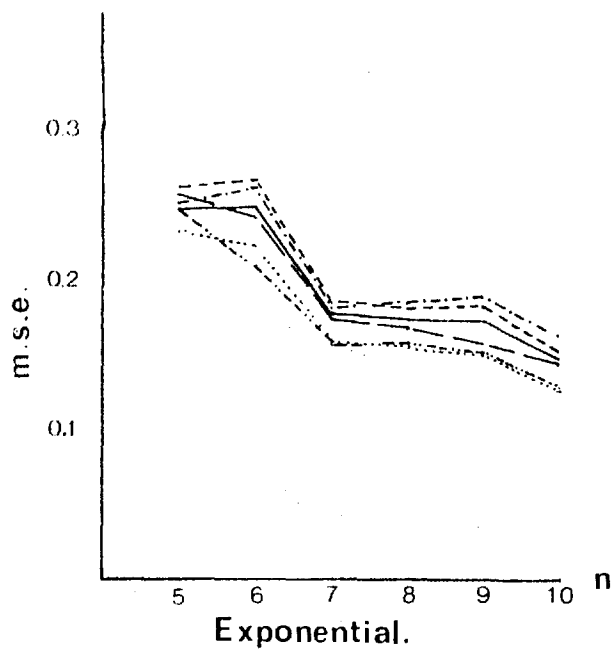


Figure 8b. Comparison of bias.



Biassed root-mean-square
 Unbiased
 Best linear
 Mean deviation
 Range
 Dixon

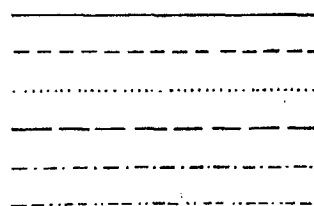
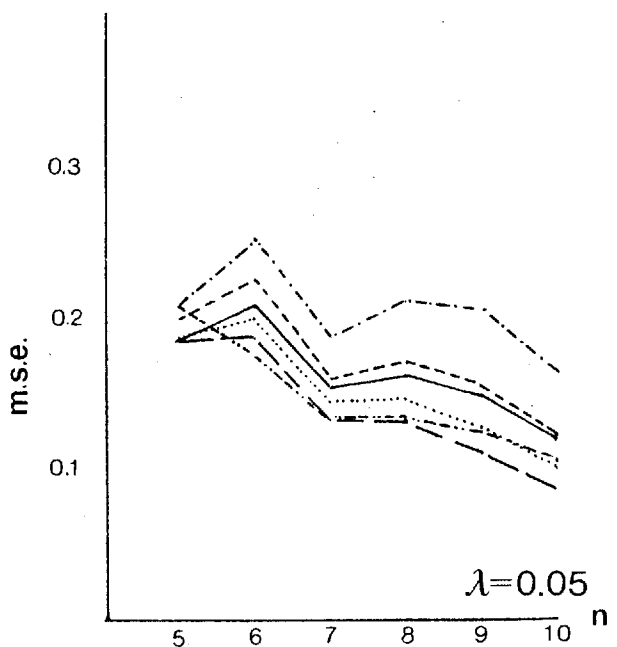
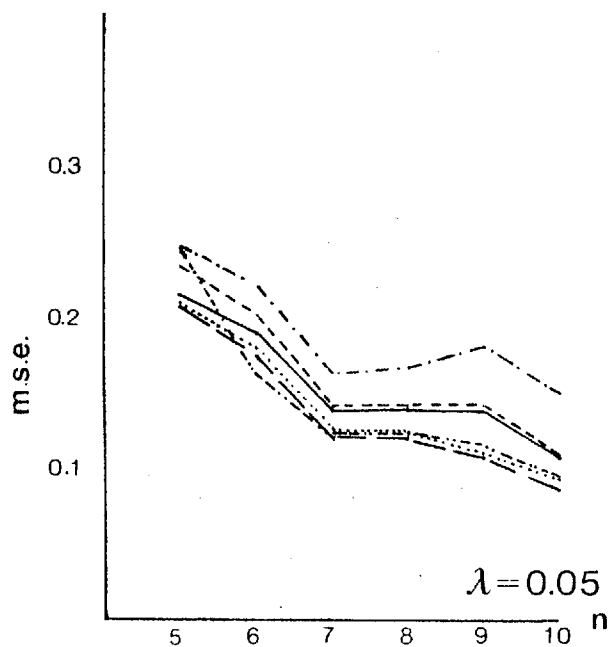
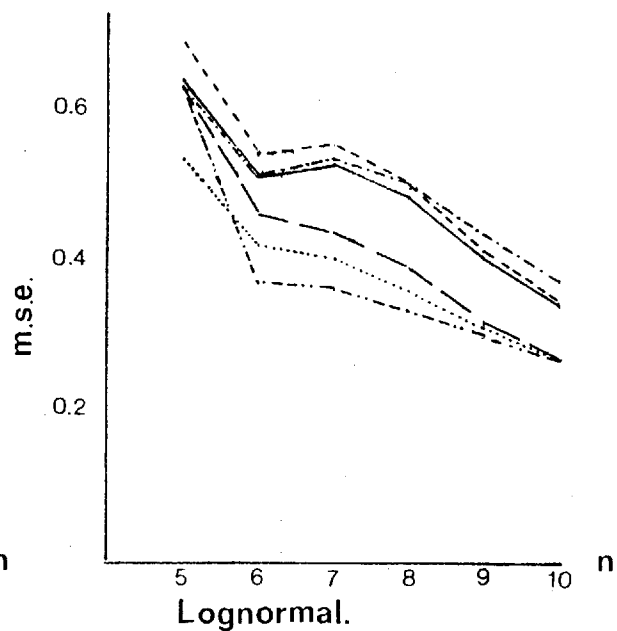
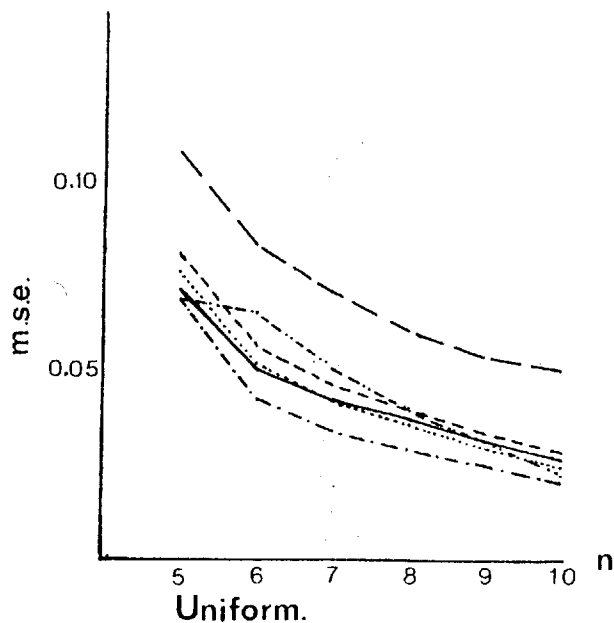


Figure 9a. Comparison of mean square errors.



Contaminated normal (0.3).

Contaminated normal (1,3).

Biassed root-mean-square

Unbiased

Best linear

Mean deviation

Range

Dixon

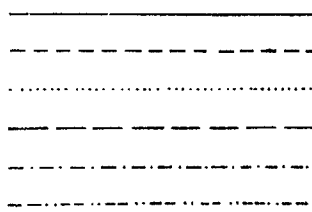


Figure9b. Comparison of mean square errors.

it is interesting to compare these empirical values with the exact results given in the preceding section. Comparisons may be made for nine cases. The three estimators, best linear, Dixon and range, may be taken with each of the uniform, exponential and right triangular distributions. Apart from some sampling fluctuations the results are in close agreement for both means and mean square errors for all these situations. In view of this close agreement and the fact that the estimators are highly correlated, since they were obtained from the same sets of 1000 observations for each sample size for each distribution, valid comparisons may be made between the estimators.

From figures 8a and 8b in which the biases of the estimators are compared we may see that, for the platykurtic distributions, the range and biased root-mean-square estimators have largest biases and that there is little to choose between the best linear, Dixon's and the unbiased root-mean-square. The mean deviation has largest mean value for these distributions. As may have been expected the situation is almost reversed for the positively skewed lognormal and exponential distributions for which the best linear and Dixon's estimators have greatest bias. The two contaminated distributions and the double-sided exponential distribution have similar patterns of bias. The range is most affected by the inclusion of observations from the tails of the distribution and has largest mean value for all three distributions. As the sample size

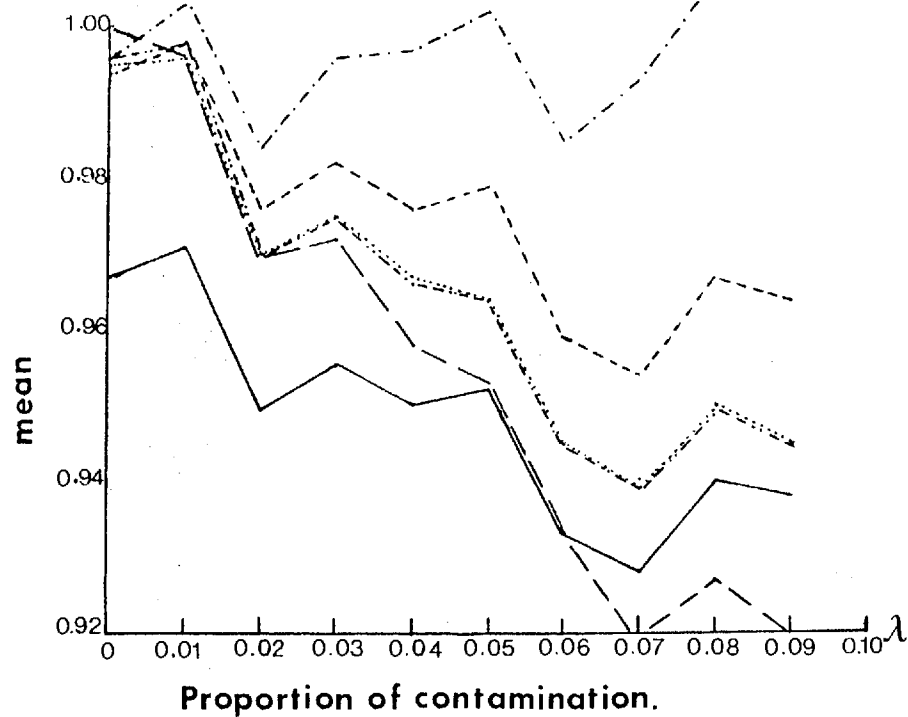
increases, and therefore a greater number of the samples contain large values, the means of the root-mean-square estimators increase more rapidly than the mean of the mean deviation, in fact for all the leptokurtic distributions considered the lines for the mean deviation and the biased root-mean-square estimator cross for some value of $n \leq 10$.

Of greater interest are the comparison of the mean-square-errors of the estimators shown in figures 9a and 9b. These diagrams illustrate that the mean-square-error patterns fall into two main types; those for leptokurtic parent distributions and those for platykurtic distributions. For the platykurtic distributions, i.e. the uniform, right triangular and the symmetric triangular, the mean deviation is by far the most inefficient of the estimators considered. For the uniform distribution with $n = 10$ the mean deviation is only about 50 per cent as efficient as its nearest competitor. For all three distributions the range has high efficiency but there is not much difference between any of the remaining estimators. However, for the leptokurtic distributions the situation is radically different. The range is now the most inefficient estimator with the mean-deviation, Dixon's estimator and the best-linear having approximately equal but much lower mean-square-errors. For the contaminated distributions the mean deviation has greatest efficiency with the best linear and Dixon's estimator in

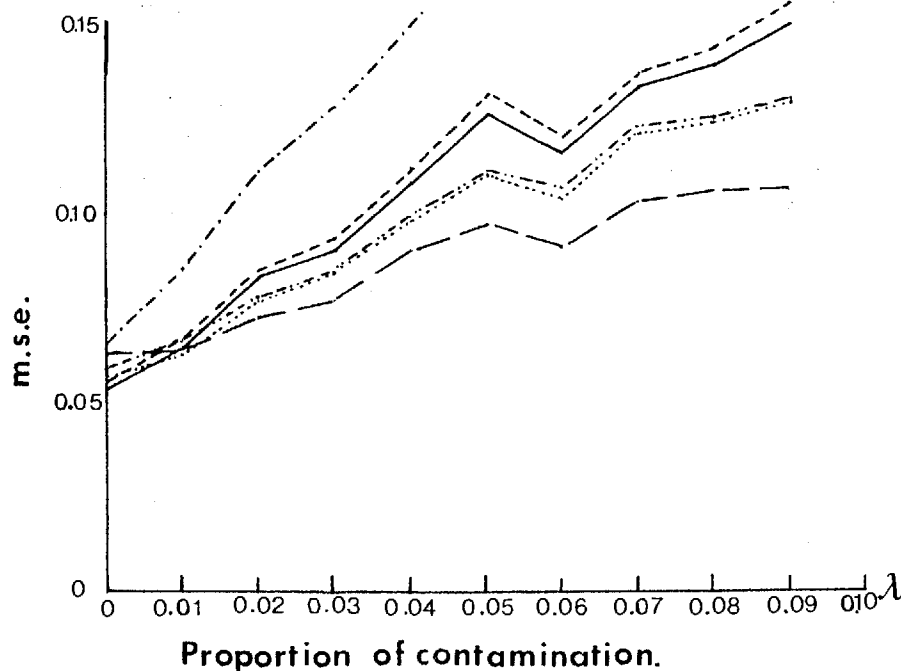
about equal second position. Both root-mean-square estimators are inefficient by comparison with these three alternatives.

Figure 10 shows the means and mean-square-errors of the estimators based on samples of 10 observations from contaminated normal distributions with $\mu_2 = 0$, $\sigma_2 = 3$ and increasing proportion of contamination λ . From 10a we have that the mean deviation and biased root-mean-square estimator have large bias with the mean of the mean deviation decreasing more rapidly than that of any other estimator. There is little to choose between the best linear and Dixon's estimator which, although biased for $\lambda \neq 0$ or 1, have smaller bias for a range of proportions of contamination than the biased root-mean-square estimator based on a sample from a precise normal distribution. The range is the only estimator to remain comparatively unbiased over the λ values considered. However, from figure 10b we see that the range is highly variable having an efficiency, for $\lambda \approx 0.09$, about 50 per cent relative to the next most variable which is the unbiased root-mean-square estimator. In spite of its large bias the mean-square-error of the mean deviation crosses those of the other estimators, which are more efficient for the normal distribution, for $\lambda \approx 0.01$. The best linear estimator and Dixon's estimator become more efficient than either of the root-mean-square estimators at about the same value of λ . These results for $n = 10$ are consistent with the asymptotic results

(a)



(b)



Biassed root-mean-square

Unbiased

Best linear

Mean deviation

Range

Dixon

Figure 10. Means and mean square errors for samples of 10 from a contaminated normal distribution (0, 3).

derived by Tukey and it may be conjectured that his conclusions about the inadvisability of using the root-mean-square estimator also hold for small samples.

Which estimator to use?

The exact and empirical results of the preceding sections show that on considerations of robustness alone these alternative estimators deserve more widespread application than they enjoy at present. However there is still the problem of deciding which estimator to use.

We may eliminate the root-mean-square estimators from our list since it is impossible in practice to detect non-normality when present at the level which results in these estimators being no longer optimal. Although the mean-deviation had smallest mean-square-error for the contaminated distributions for $\lambda \geq 0.015$, it has several disadvantages which outweigh this advantage from both a robustness and a mathematical point of view. Use of the mean-deviation does not give protection against platykurtic departures from normality, being consistently poorer than any other estimator considered for all sample sizes for the three platykurtic distributions studied. This drawback, together with the mathematical difficulties associated with employing the mean deviation in tests of hypotheses serve to eliminate this estimator also. This leaves the five linear estimators, range, best-linear, Dixon, Downton and Gupta. For small sample sizes i.e. $n \leq 5$,

the range is efficient and fairly robust. Tables of percentage points of statistics using the range are easily accessible, for references see the introduction to section 3.1. For sample sizes $6 \leq n \leq 20$ any one of the remaining four estimators may be used. Dixon's is easiest to calculate and Downton's is most robust to leptokurtic non-normality. The variances of all four estimators are available exactly for $n \leq 20$. The best-linear and Dixon's estimator are not available for $n > 20$. This leave a choice between Gupta's normal scores estimator and Downton's form of Gini's mean difference. For complete samples Downton's is to be preferred both from a robustness point of view and since the variance of the estimate is given exactly. However the normal scores estimate is almost as robust as Downton's and its variance may be conveniently approximated using the results of section 2.5.

Gupta's estimator has the advantage over Downton's that it may be extended for use with censored samples, see section 2.5. The following table summarizes the above conclusions.

Sample size	Estimator	Normal variance
Complete $n \leq 5$	Range	Exact
" $6 \leq n \leq 20$	Best linear Dixon Gupta Downton	All exact
" $20 \leq n$	Downton Gupta	Exact Approx. section 2.5
" large n	Prescott	Approx. section 3.1
Incomplete $n \leq 20$	Best linear Gupta	Exact
" $n > 20$	Gupta	Approx. section 2.5

The list of references following this section is not meant to be exhaustive and is far from complete containing only those cited in the preceding chapters. This list is meant to act as a starting point only to the available literature on the subject and as an indication of the volume of research being carried out on order-statistics and their applications and of the increasing importance of linear estimators in practical statistics.

REFERENCES

1. Barnett, F.C., Mullen, K. & Saw, J.G. (1967). Linear estimates of a population scale parameter. Biometrika 54 551-4
2. Benson, F. (1949). A note on the estimation of mean and standard deviation from quantiles. J. Roy. Statist. Soc. 11. 91.
3. Birnbaum, A. & Dudman, J. (1963). Logistic order statistics Ann. Math. Statist. 34. 658-663.
4. David, H.A. (1968). Gini's mean difference rediscovered. Biometrika 55, 573-5.
5. Dixon, W.J. (1957). Estimation of the mean and standard deviation of a normal population. Ann. Math. Statist., 28, 806-9.
6. Downton, F. (1954). Least squares estimates using ordered observations. Ann. Math. Statist. 25, 303-316.
7. Downton, F. (1966). Linear estimates with polynomial coefficients. Biometrika 53, 129-141.
8. Fisher, R.A. & Yates, F. (1938-63). Statistical Tables for Biological, Agricultural and Medical Research. Oliver & Boyd.
9. Geary, R.C. (1930). The frequency distribution of the quotient of two normal variates. J. Roy. Statist. Soc. 93, 442-446.
10. Govindarajulu, Z (1963). On moments of order statistics and quasi-ranges from normal populations. Ann. Math. Statist. 34, 633-651.
11. Grubbs, F.E. (1969). Procedures for detecting outlying observations in samples. Technometrics 11, 1-22.
12. Gumbel, E.J. (1958). Statistics of Extremes, Columbia University Press, New York.
13. Gupta, A.K. (1952). Estimation of the mean and standard deviation of a normal population from a censored sample. Biometrika 39, 260-273.

14. Gupta, S.S. (1968). Order statistics from the gamma distribution. Technometrics 2, 243-262.
15. Gupta, S.S. & Shah, B.K. (1965). Exact moments and percentage points of the order statistics and the distribution of the range from the logistic distribution. Ann. Math. Statist. 36, 907-920.
16. Harter, H.L. (1961). Expected values of normal order statistics. Biometrika 48, 151-165
17. Harter, H.L. (1963). Percentage points of the ratio of two ranges and power of the associated test. Biometrika 50, 187-194
18. Hartley, H.O. (1950). The use of range in analysis of variance. Biometrika 37, 271-280.
19. Kendall, M.G. & Stuart, A. (1963). Advanced theory of statistics. Vol. 1. Charles Griffin, London.
20. Link, R.F. (1950). The sampling distribution of the ratio of two ranges from independent samples. Ann. Math. Statist., 21, 112-116.
21. Lloyd, E.H. (1952). Least-squares estimation of location and scale parameters using order statistics. Biometrika, 39, 88-95.
22. Lomnicki, Z.A. (1952). The standard error of Gini's mean difference. Ann. Math. Statist. 13, 91.
23. Lord, E. (1947). The use of range in place of standard deviation in the t-test. Biometrika, 34, 41-67.
24. Lord, E. (1950). Power of the modified t-test (u-test) based on range. Biometrika, 37, 64-77.
25. Mead, R. (1966). A quick method of estimating the standard deviation. Biometrika 53, 559-564.
26. Moore, P.G. (1957). The two-sample t-test based on range. Biometrika 44, 482-489.
27. Mosteller, F. (1946). On some useful "inefficient" statistics. Ann. Math. Statist. Vol. 17. 377-408.

28. Nair, U.S. (1936). The standard error of Gini's mean difference. Biometrika 23, 428-434.
29. Ogawa, J. (1951). Contributions to the theory of systematic statistics I. Osaka. Math. J., 3 175-213.
30. Pearson, E.S. and Hartley, H.O. (1956). Biometrika Tables for Statisticians. Vol. 1. Cambridge University Press, England.
31. Pearson, K. (1957). Tables of the Incomplete Gamma Function. Cambridge University Press, England.
32. Pillai, K.C.S. and Buenaventura, A.R. (1961). Upper percentage points of a substitute F-ratio using ranges. Biometrika, 48, 195-6.
33. Plackett, R.L. (1958). Linear estimation from censored data. Ann. Math. Statist. 29, 131-42.
34. Prescott, P. (1968). A simple method of estimating dispersion from normal samples. J.R. Statist. Soc. C. 17, 70-74.
35. Prescott, P. (1970). Estimation of the standard deviation of a normal population from doubly censored samples using normal scores. Biometrika, 57, in press.
36. Quenouille, M.H. (1959). Rapid statistical calculations. Griffin, London.
37. Quenouille, M.H. (1959). Tables of 1000 standardized random deviates from certain non-normal distributions. Biometrika 46, 178-204.
38. Ruben, H. (1956). On the moments of the range and product moments of extreme order statistics in normal samples. Biometrika 43, 458-460.
39. Sarhan, A.E. (1954). Estimation of the mean and standard deviation by order statistics. Ann. Math Statist. 25, 317-28.
40. Sarhan, A.E. and Greenberg, B.G. (1956). Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Part I. The normal distribution up to samples of size 10. Ann. Math. Statist., 27, 427-451.

41. Sarhan, A.E. and Greenberg, B.G. (1958). Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Part II. Ann. Math. Statist. 29, 79-105.
42. Sarhan, A.E. and Greenberg, B.G. (1962). Contributions to order statistics, Wiley.
43. Shah, B.K. (1966). On the bivariate moments of order statistics from a logistic distribution. Ann. Math. Statist. 37, 1002-1010.
44. Shapiro, S.S. and Wilk, M.B. (1965). An analysis of variance test for normality (complete samples). Biometrika 52, 591-611.
45. Stigler, S.M. (1969). Linear functions of order statistics. Ann. Math. Statist. 40, 770-788.
46. Teichroew, D. (1956). Tables of expected values of order statistics and products of order statistics for samples of size 20 and less from the normal distribution Ann. Math. Statist. Vol. 27, 410-426.
47. Tocher, K.D. (1963). The art of simulation. English Universities Press, London.
48. Tukey, J.W. (1960). A survey of sampling from contaminated distributions. Contributions to Probability and Statistics. Edited by Olkin, I. et al. Stanford University Press, California.
49. Wetherill, G.B. (1965). An approximation to the inverse normal function suitable for the generation of random normal deviates on electronic computers. J. Roy. Statist. Soc. Series C. 14, 201-205.
50. Wilk, M.B. and Gnanadesikan, R. (1968). Probability plotting methods for the analysis of data. Biometrika 55, 1-17.
51. Wilson, E.B. and Hilferty, M.M. (1931). The distribution of chi-squared. Proc. Nat. Acad. Sci. 17, 684.