Dynamical noise can enhance high-order statistical structure in complex systems

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Recent research has provided a wealth of evidence highlighting the pivotal role of high-order interdependencies in supporting the information-processing capabilities of distributed complex systems. These findings may suggest that high-order interdependencies constitute a powerful resource that is, however, challenging to harness and can be readily disrupted. In this paper we contest this perspective by demonstrating that high-order interdependencies can not only exhibit robustness to stochastic perturbations, but can in fact be enhanced by them. Using elementary cellular automata as a general testbed, our results unveil the capacity of dynamical noise to enhance the statistical regularities between agents and, intriguingly, even alter the prevailing character of their interdependencies. Furthermore, our results show that these effects are related to the high-order structure of the local rules, which affect the system's susceptibility to noise and characteristic times-scales. These results deepen our understanding of how high-order interdependencies may spontaneously emerge within distributed systems interacting with stochastic environments, thus providing an initial step towards elucidating their origin and function in complex systems like the human brain.

High-order interdependencies — such as synergy and redundancy — play important roles in a range of naturally occurring information-processing systems, including the human brain. These statistical structures rely in fine-tuned coordination of multiple parts, and hence one may wonder how is that nature manages to give rise to them in noisy scenarios. Here we show that high-order structures can be not only robust to noise but can even be enhanced by them, and that noise can modify their prevailing character. This demonstrates that high-order interactions can be manifested not only in the presence of noise, but also due to noise — providing an explanation of how they may spontaneously emerge in naturally occurring systems interacting with noisy environments.

I. INTRODUCTION

Complex systems are characterised by collective phenomena arising from the interactions between relatively simple elements.[1] Examples include the mesmerising synchronisation of fireflies and the murmuration of starlings,[2] the freezing of water and other phase transitions,[3] and the collective dynamics of decision-making[4] or political opinions.[5] Crucially, these collective phenomena often cannot be explained from the behaviour of the system's parts in isolation. Thus, the goal of complexity science is to identify common laws governing the emergence of collective behaviour, establishing universal links between local interactions and global phenomena.

Among the various approaches in the complexity scientist's toolbox, information theory has established itself as a promising framework to study distributed computation across complex systems throughout the physical, biological, and social sciences.[6–8] Within it, the mathematical framework of *partial information decomposition* (PID)[9,10] is being rapidly consolidated as a powerful approach to understand and quantify various aspects of distributed information processing. Crucially, PID provides a comprehensive taxonomy of *high-order interactions* — statistical structures describing a group of variables that cannot be written in terms of pairwise interdependencies. Specifically, PID distinguishes between qualitatively different types of high-order interactions,[11] including *redundancy* — information that can be extracted from multiple random variables — and *synergy* — information contained in a group of variables jointly but not in subsets of them separately.

A growing body of literature suggests an important role of high-order interactions supporting computation in complex systems. For example, alternating profiles of synergy and redundancy have been related to the different classes of elementary cellular automata,[12] while quantities related to synergy (specifically, integrated information[10]) have been associated with phase transitions and distributed computation in both discrete and continuous systems.[13] In the human brain, synergy between large-scale neural populations has been related to cognition and consciousness, while redundancy has been associated with sensory-motor processes.[14,15] In artificial neural networks, redundancy provides robustness to perturba-

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tion while synergy supports modality integration and flexible learning.[16] In the context of artistic expression, the balance of redundancy and synergy has also been related with various dimensions of musical depth and complexity.[17,18] Overall, a wealth of empirical evidence is uncovering a spectrum of computational advantages of high-order interactions, including links between redundancy and robustness, as well as between synergy and flexible information integration.

These findings implicitly portray synergy as a powerful yet delicate resource, that can provide flexibility and enhanced information-processing capabilities to a system, while at the same time being fragile and easily broken. This picture, however, leads to a natural question: how can this view be reconciled with the seeming prevalence of synergy across the natural world? Biological systems are usually subjected to noise, i.e. stochastic fluctuations that can perturb their dynamics in various ways. Moreover, noise is often considered an important element in shaping dynamics and information processing in the brain,[19] allowing the encoding of different stimulus features or improving the signal-to-noise ratio of sensory inputs.[20] Characterising the interplay between high-order interactions and noise is an important open question, which would deepen our understanding of how these structures may appear in physical systems.

In this paper we argue against the misconception of synergy as fragile, showing that high-order interactions can be manifested not only in the presence of noise, but also *due to* noise. We illustrate our ideas with a comprehensive evaluation of the effect of noise on the high-order structures generated by elementary cellular automata (ECA), using the information-theoretic tools introduced in Refs.[17] and[21] to enable high-order analyses of relatively large systems. Our results reveal that high-order interactions — either of the synergistic or redundant type — can be enhanced by noise in the system's dynamics, in a way that depends on the system's structure and its behaviour in response to perturbations. Overall, the results in this study provide a first step towards understanding how high-order interactions may emerge in complex distributed systems subject to noise, including the human brain.

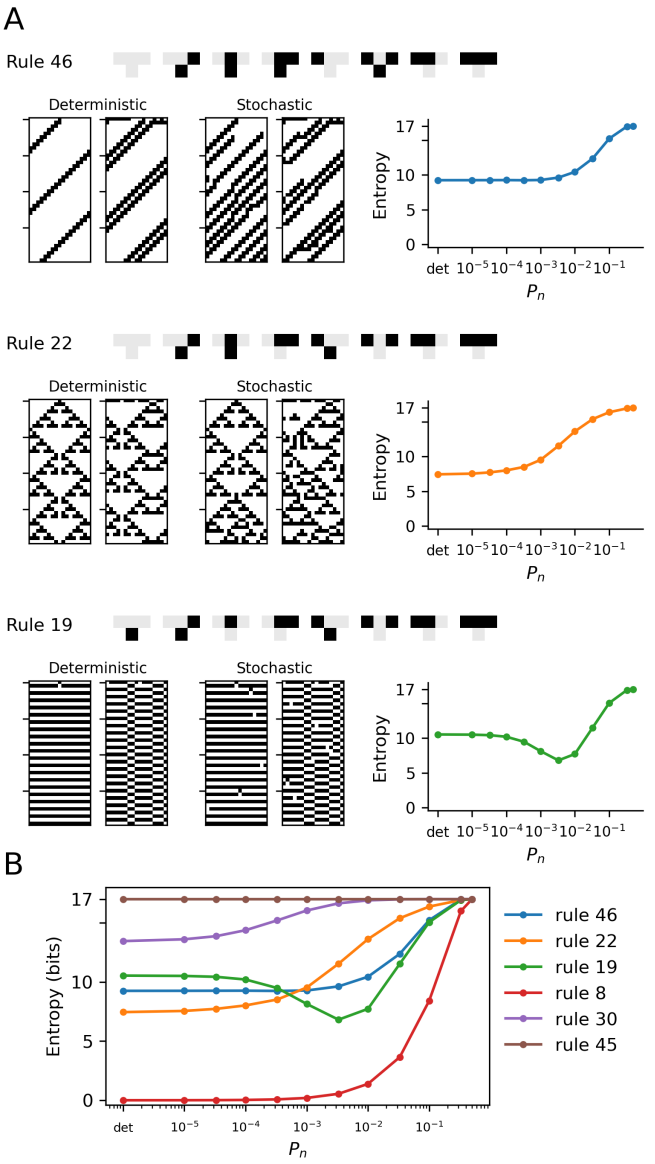


FIG. 1. **Entropy of elementary cellular automata in stochastic simulations.** **(A)** Realisations of rules 46, 22, and 19 are shown in the left: two trajectories without noise (deterministic), and two with probability $P_n = 0.01$ of flipping an agent at each step. The effect of the dynamical noise on the Shannon entropy of the resulting patterns is shown on the right. **(B)** Relationship between dynamical noise and Shannon entropy for other rules discussed in the main text.

II. RESULTS

A. Entropy of cellular automata and the effect of noise

The first step in our analysis was to investigate how noise impacts the diversity of the patterns resulting from the dynamics of elementary cellular automata (ECA). For this purpose, we studied the evolution of different ECA rules over a circular grid of 17 agents (or cells). We simulated the trajectories of all ECA rules for 800 steps considering all possible initial configurations, which allowed us to fully estimate the non-stationary joint probability distribution of agent patterns at each timestep after convergence (see Methods). Then, the resulting patterns were characterised in terms of their joint Shannon entropy, which quantifies the diversity of configurations explored by the system. Simulations were carried out under different levels of dynamical noise, which was characterised in terms of the probability P_n of flipping the resulting state of a given agent disobeying the corresponding ECA rule — with $P_n = 0$ being no noise, and $P_n = 1/2$ being maximal noise resulting on purely random dynamics. It is important to highlight that the noise implemented in these analyses is *dynamical noise*, i.e. perturbations to the transitions as the system's trajectory unfolds, in contrast to *observational noise*, i.e. the random flipping of values of agents when observing the system but without altering its trajectory. Observational noise has a much simpler (and less interesting) effect on entropy: it simply induces a monotonic increase of it for all rules (not shown).

Results show that while adding dynamical noise often resulted in an increase of entropy, a number of ECA rules displayed a remarkable decrease of entropy for intermediate noise levels (p_n between 10^{-3} and 10^{-2}). Hence, maximal noise always yielded — as expected — patterns of maximal entropy, but the path from deterministic to fully random dynamics was found not to always be monotonous. To illustrate this, Figure 1A shows three examples of rules with and without dynamical noise, together with the entropy measured at different values of P_{n}. This finding implies that, in some scenarios, adding noise results in a somewhat paradoxical reduction of the diversity of observed patterns. We refer to this as a 'biphasic' response to noise, where a small amount of noise first creates statistical structure, but further noise destroys it — which we tentatively interpret in terms of a loss of metastable configurations (see Discussion).

Additionally, our results also revealed that different ECA rules display different degrees of robustness against dynamical noise, exhibiting changes in their entropy at very different noise levels. For example, as illustrated by Figure 1B, some rules (e.g. rule 30) are affected by very low noise levels, while others are affected only when the noise level is high (e.g. rules 46 and 8).

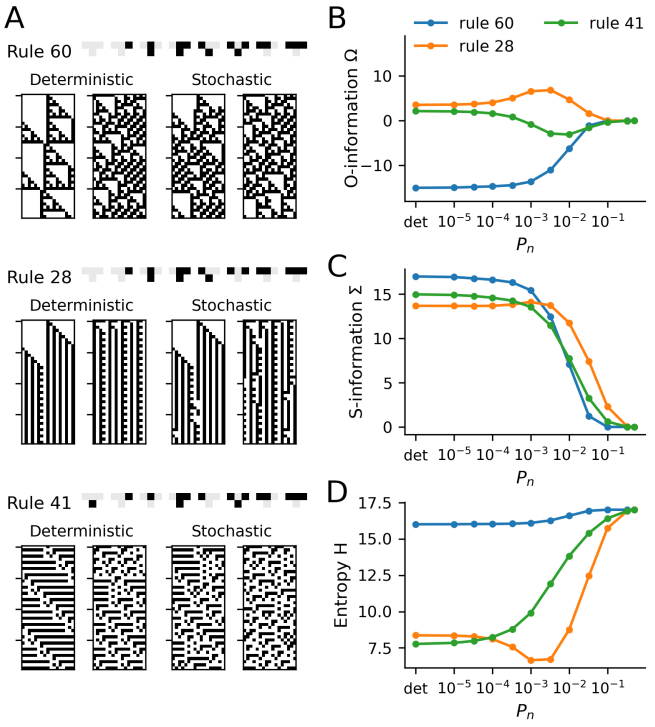


FIG. 2. **High-order interdependencies in stochastic automata. (A)** Rules 60, 28, and 41 shown schematically with two deterministic trajectories and two stochastic trajectories ($P_{\mathrm{n}} = 0.01$). **(B-D)** Effect of noise on the high-order structure induced by various rules as revealed by information-theoretic measures, including monotonic and biphasic behaviours.

B. Dominance of high-order interactions can be switched by noise

The second step in our analysis was to investigate how dynamical noise can affect different types of high-order interactions between the agents that are generate as a result of the rules. To do this, we calculated the O-information and S-information (denoted by Ω and Σ, respectively)[17] on the patterns exhibited by each of the ECA rules after 800 steps, as described in Methods. These two measures provide complementary accounts of the high-order structure of a system: Σ measures the total strength of the interdependencies in the system, while Ω measures whether these interdependencies are predominantly synergistic or redundant. In particular, a negative value of Ω indicates predominantly synergistic interactions, while a positive value indicates predominantly redundant interactions.[17]

Our analyses confirmed that in most ECAs, as one would expect, noise monotonically disrupts high-order structures, in the sense that both Ω and Σ monotonically decrease in absolute value. However, some rules exhibit an increase in the absolute magnitude of their O-information for intermediate levels of noise — more specifically, for levels of noise at which changes are visible but the full destruction of statistical interdependencies has not occurred. This increase can represent either a strengthening of the dominant tendency (e.g. a redundancy-dominated rule becoming even more dominanted by redundancy), or a change in the dominant tendency (e.g. a redundancy-dominated rule becoming synergy-dominated). Some examples are shown in Figure 2:

- Rule 60 exhibits a monotonically decreasing Ω and Σ, which is a signature of a progressive destruction of its highly synergistic structure (previously described in Ref.[12]).
- In contrast, Rule 28 is redundancy-dominated and exhibits an increase of its Ω, indicating that noise makes the dominance of redundancy stronger.
- Finally, Rule 41 is redundancy-dominated but its Ω becomes negative for certain values of P_{n}, indicating that for intermediate noise levels it switches to synergy-dominated.

Additionally, for some rules (e.g. rule 28) we also found a biphasic change of S-information with intermediate levels of noise, showing an increase before decaying to zero (Figure 2C). This confirms that noise can effectively create high-order interdependencies — which in this case are predominantly redundant (Figure 2B). In contrast, rule 41 does not show a transient increase in S-information due to noise, despite having a biphasic behaviour of O-information (Figure 2B). Correspondingly, the change in sign of O-information may be interpreted as due to a destruction of redundant interactions rather than the creation of synergistic ones.

C. Relationship between information measures and susceptibility to noise

Next, we sought to determine how these different information-theoretic quantities are related to each other and

to the effect noise has on them. For this analysis, we excluded certain rules[22] that exhibited zero O-information and S-information at all noise levels. For the remaining 77 rules, we explored relationships between entropy, O-information, and S-information, and whether the effect of noise was monotonic or biphasic. After noticing that different rules are affected by very different levels of noise (some start being affected at $P_n = 10^{-4}$ while others only at $P_n = 10^{-2}$, see Figure 2C), we also evaluated the "half-noise" level for each measure — defined as the noise level that produces half of the total observed change, see Methods — and its relationships with information-theoretic quantities.

Our results revealed a negative correlation between the entropy and the O-information when both are measured in the absence of noise (Figure 3A). This means that synergy-dominated rules tend to exhibit a higher joint entropy (i.e. a larger repertoire of patterns in their final configuration), while redundancy-dominated rules tend to exhibit less diversity in their patterns. This is consistent with previous mathematical results showing that synergy tends to account for increasingly large fractions of the total entropy of a system.[23] Furthermore, results also show a negative correlation between the half-noise for entropy change and the S-information (Figure 3B), which implies that rules with a large amount of interdependencies are more sensitive to noise — i.e. they are affected by smaller values of P_n. Interestingly, neither entropy nor O-information showed a correlation with the entropy half-noise (Figure 3B).

Finally, results also showed that all the rules that exhibit an increase of Ω with noise (i.e. that become more redundancy-dominated with noise) also exhibit a biphasic change in entropy (i.e. they become less entropic with intermediate levels of noise), as shown in Figure 3C. This is consistent with the result shown in Figure 3A, where less entropy is associated with redundancy. Conversely, and with the exception of rules 72 and 104, almost all rules that show biphasic behaviour in entropy exhibit a positive change in O-information with noise, i.e. they become more redundancy-dominated with noise. Finally, we found that all the rules showing a negative drop in O-information (i.e. that transiently become more synergistic due to noise), do not show a decrease of entropy or an increase of S-information with noise. Overall, in agreement with our findings in the previous section, this suggests that the increase of the predominance of synergy caused by noise in some rules may not be accompanied by an increase of structure — and thus it may be due to a selective decrease of redundant interactions instead of the creation of synergistic ones.

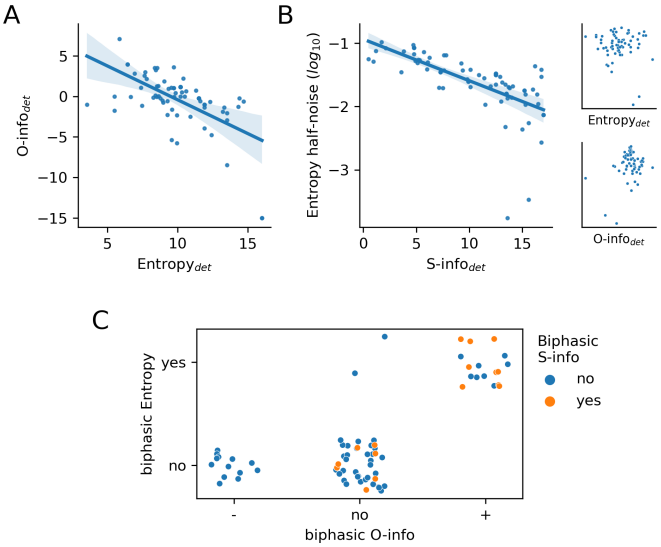


FIG. 3. **Correlations between information measures in stochastic automata. (A)** O-information plotted against Entropy for each rule, in the deterministic setting after 800 steps of rule evolution (Spearmann's $\rho = -0.55$, $p = 4 \times 10^{-7}$). **(B)** S-information in the deterministic setting versus the entropy "half-noise," defined as the value of P_n that causes half of the total shift in entropy ($\rho = -0.81$, $p = 10^{-17}$). Small plots show the absence of correlation between the entropy half-noise and entropy (top, $\rho = -0.08$, $p = 0.49$) or O-information (bottom, $\rho = -0.06$, $p = 0.6$). **C.** Rules classified according to the changes produced by noise on entropy (monotonic or biphasic) and O-information (negative biphasic, monotonic, or positive biphasic). The color of the dots represents the behavior of S-information (monotonic of biphasic). In all plots, rules with zero or maximal entropy were excluded, as they show no effects of noise on O-information or entropy, respectively.

D. Characteristic time and statistical interdependencies

We then sought to investigate how the convergence speed of various rules is related to the structure of the resulting statistical interdependencies. After starting from random initial conditions, some rules establish their steady states in just one or two steps (e.g. rule 60), while other rules take tens (e.g. rule 28) or hundreds (e.g. rule 110) of steps to reach stability (Figure 4A). To quantify these differences, we defined the "characteristic time" of each rule as the number of steps needed to reach a difference of less than 0.5 bits (or less than 2 standard deviations for periodic rules) from the steady state value (or mean) of their joint entropy.

We found that the characteristic time for both entropy and O-information have a negative correlation with the half-noise for entropy, as shown in Figure 4B. This implies that slower rules (i.e. the ones with a larger characteristic time) have a higher susceptibility to noise, and hence less noise is needed to increase their entropy. In addition, characteristic times for Entropy and O-information are highly correlated (Figure 4C, left), while there is a weak relationship between the characteristic times and and the half-noise for O-information (Figure 4C, right). Finally, the rules with negative biphasic effect of noise on O-information (made predominantly synergistic with noise) are significantly slower than those with a positive biphasic effect (Figure 4D).

Overall, these results show that the rules that become predominantly redundant with noise are rules that establish their interdependencies and entropy in few steps. Taking into account the results in Figure 4B, the rules that become redundant with noise are, at the same time, less susceptible to it.

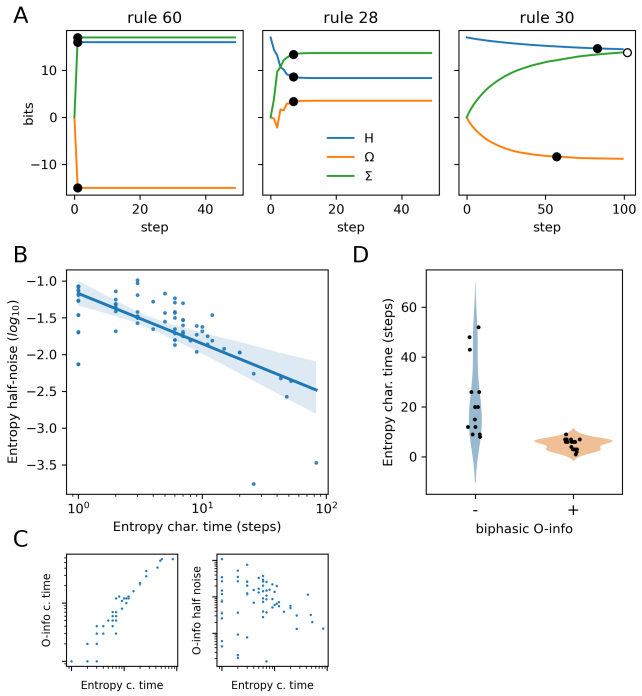


FIG. 4. **Time evolution of information measures. (A)** Evolution of Entropy (H), O-information (Ω) and S-information (Σ) in rules 60, 28, and 30 in the absence of noise. Black dots mark the timestep in which the measure is less than 0.5 bits away from its steady-state value, which we refer to as "characteristic time." For rule 30, the characteristic time for Σ is 108 (represented as an open circle). **(B)** Characteristic time and logarithm of half-noise for entropy are negatively correlated (Spearmann's $\rho = -0.66$, $p = 2.4 \times 10^{-10}$). **(C)** Left: positive correlation between the characteristic times of entropy and O-information. Right: absence of correlation between characteristic time of entropy and the half-noise for O-information. **(D)** Comparison of characteristic time for entropy for rules with either a negative or a positive biphasic O-information.

E. Relating high-order interdependencies to local interactions within the rules

Finally, we sought to investigate how statistical interdependencies may arise from the local interactions determined by the ECA rules. To this end, we characterised the local interactions of each rule in two ways: First, we calculated the Redundancy-Synergy index (RSI, see Methods) over each rule's three inputs bits and the output bit. The RSI quantifies the balance between synergy and redundancy in a directed — i.e. from inputs to output — manner, which is ideal to study the local effects of three neighbouring agents determining one single output agent in the next timepoint.[24] Second, the dependency between the inputs and the outputs was decomposed into constituent logical operations. Specifically, for each rule we calculated the number of binary operations of the XOR and AND/OR types, as well as unary operations of the IF/NOT type. An example is shown in Figure 5, which presents operations of the three types. While most rules exhibit only one or two types, a few rules (6, 18, 33, 40, 72, 130, 132) show the three of them.

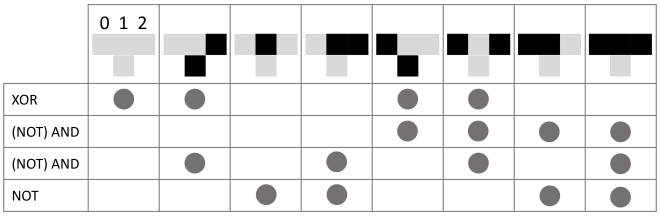


FIG. 5. **Logical operations found in a rule.** Rule 18 performs the following rules: a XOR operation over the bits 0 and 2 when the bit 1 is off, a not-AND (NOR) operation over bits 1 and 2 when bit 0 is on, a NOR operation over bits 0 and 1 when bit 2 is on, and a NOT operation over bit 1.

We studied the relationship between RSI, the type of operations at the local level, O-information, and S-information. This analysis revealed several connections between local and collective patterns of the high-order interactions. In particular, the O-information was found to be negatively correlated with RSI — which was expected as the RSI is (by construction) positive for synergistic interactions and negative for redundant ones. Moreover, the O-information was found to be negatively correlated with the number of XOR operations, meaning that as more of these operations are present, the rules are more synergistic. Additionally, local synergy dominance (i.e. a positive RSI) was found to be lead to higher values of S-information. The S-information was also found to be negatively correlated with the unary operations IF and NOT, meaning that unary operations make rules have less interdependencies in general. In contrast, the joint entropy was found not to be correlated to any of the local measures tested. These results are summarised in Table I.

We also investigated the relationship between RSI, local operations, the half-noise (resistance to perturbations), and characteristic time of the O-information and S-information. Results show that the entropy half-noise was negatively correlated with RSI, as well as the number of XOR operations, which suggests that local synergy makes a rule more susceptible to perturbations. In contrast, the presence of unary IF/NOT operations increase the resistance to noise, making the rule more robust. Furthermore, the same operations that reduce the half-noise were found to increase the characteristic times for entropy, O-information and S-information. Overall, rules with more synergy in the local interactions are more susceptible to noise and have longer characteristic times, while rules with unary operations are more resistant and have shorter timescales.

To assess potential combined relationships between the local rule patterns and the global information measures, we performed further analyses using Generalized Linear Models (GLM). Results largely confirm what was found using simple linear correlations: O-information is negatively correlated with the number of XORs; S-information is negatively correlated with IF/NOTs and positively with RSI; and the half-noise is higher when there are more IF/NOTs and less XORs (see Table II).

Measure	XOR	OR/AND	IF/NOT	RSI
Entropy	× (p=0.91)	× (p=0.08)	× (p=0.83)	× (p=0.97)
O-information	**-0.38** (p=8e-4)	× (p=0.3)	× (p=0.51)	**-0.27** (p=0.019)
S-information	× (p=0.09)	× (p=0.22)	**-0.45** (p=6.8e-5)	**0.35** (p=0.003)
Entropy half-noise	**-0.43** (p=1.4e-4)	× (p=0.34)	**0.38** (p=8.4e-4)	**-0.44** (p=7.6e-5)
O-info half-noise	× (p=0.29)	× (p=0.67)	× (p=0.38)	× (p=0.94)
S-info half-noise	**-0.38** (p=8.2e-4)	× (p=0.6)	× (p=0.32)	× (p=0.029)
Entropy char. time	0.44 (p=7.9e-5)	× (p=0.16)	-0.53 (p=1.3e-6)	0.51 (p=3.2e-6)
O-info char. time	0.42 (p=2.1e-4)	× (p=0.25)	-0.47 (p=2e-5)	0.49 (p=1e-5)
S-info char. time	0.39 (p=6.5e-4)	× (p=0.12)	-0.53 (p=1.3e-6)	0.49 (p=1e-5)

TABLE I. **Linear correlation coefficients between information-based measures and local properties of rules.** XOR, OR/AND, and IF/NOT are the number of operations compose each rule, as depicted in Figure 5. RSI is the Redundancy-Synergy Index between the inputs and the rule output. Coefficients are not shown when the associated p-value is higher than 0.02. Rules with null S-information or maximal entropy were excluded from this analysis.

Measure	GLM p-val	XOR	OR, AND	IF, NOT
Entropy	0.001	× (p=0.342)	× (p=0.0314)	**-1.77** (p=1.8e-4)
O-information	0.0041	**-1.55** (p=6.5e-4)	× (p=0.652)	× (p=0.43)
S-information	4.13e-7	× (p=0.826)	× (p=0.592)	**-3.23** (p=1.7e-7)
Entropy half-noise	2.1e-6	**-0.191** (p=1.8e-3)	× (p=0.272)	**0.159** (p=0.006)
O-info half-noise	0.644	× (p=0.391)	× (p=0.632)	× (p=0.722)
S-info half-noise	0.004	**-0.266** (p=4e-4)	× (p=0.227)	× (p=0.501)

TABLE II. **Generalized Linear Model fit of information-based measures and local binary or unary operations performed by each rule.** The table lists the p-value of the global fit and the coefficients (with p-values) associated to each predictor.

III. DISCUSSION

This paper investigates the role of noise in the emergence of high-order statistical interdependencies in complex systems, focusing on the case of elementary cellular automata. Our results reveal that intermediate levels of noise not only preserve, but can even enhance the statistical structure built by certain rules — as evidenced by a reduction of the joint Shannon entropy of the observed patterns and corresponding increase of the S-information. This case of "antifragility" (i.e. a system developing stronger structures under external perturbations) can be tentatively interpreted as a system with unstable attractors or transient metastable configurations. Interestingly, all rules that decrease their entropy and/or increase their S-information with intermediate noise belong to Classes 1 or 2 according to Wolfram's classification[25] (i.e., they converge either to a uniform state or to a repetitive state). We speculate that some of these states may be disrupted by perturbations, translating into a more restricted repertoire of patterns exhibited by the system.

Additionally, our results show that dynamical noise is capable of altering the balance between high-order statistical structures of a system in highly non-trivial ways. In our simulations, we found systems where noise either enhanced the dominant high-order trend (redundant or synergistic), or changed the dominant trend from one to another. By comparing the effect of noise on the overall strength of high-order interdependencies (S-information) and their dominant tendency (O-information), we observed that when structure is created by intermediate levels of noise (i.e. a biphasic S-information profile takes place) then it is of the redundant type (i.e. a biphasic O-information profile in the positive direction is observed), and not synergistic. Future work may investigate other systems to confirm if synergistic structures can be enhanced via dynamical noise or not. For example, there is a rich literature that investigates the effect of noise on continuous dynamical systems (e.g. physical oscillators[26–28] or neural systems[29–31]), and future work may interrogate such systems with the information-theoretic tools employed here.

Our results reveal that systems with patterns that have more statistical structure (as evidenced by higher S-information) are affected by lower levels of noise. In other words, a high amount of interdependencies — regardless of whether they are of the synergistic or redundant type — makes a rule more susceptible to the either constructive or destructive effects of noise. An explanation for this phenomenon arises from the analysis of the temporal dynamics of the informational measures: rules with higher S-information need in general more steps to establish its informational structure at the beginning of the simulation. Thus, it is reasonable to expect that it may also take longer to rebuild it when the structure is disrupted by noise.

Finally, we found correspondences between the global high-order structures and local operations within the rules themselves. Our results showed that rules with a synergy-dominated input-output relationship (i.e. positive RSI) tend to generate more synergy-dominated global patterns (i.e. negative O-information), are typically more susceptible to noise, and require a longer time to settle. On the contrary, rules with

redundancy-dominated input-output relationships or consisting of more unary operations tend to exhibit more robustness to noise and settle faster. These findings provide some clues related to why it is easier for dynamical noise to enhance redundant, rather than synergistic structures.

Overall, the presented results provide a first step towards a deeper understanding of the role of noise in the generation of high-order statistical interdependencies, which may lead to an explanation of their pervasiveness in the natural world. We hope our findings may inspire future efforts to explore the structure-enhancing potential of noise in biological, technological, or social complex systems.

IV. METHODS

A. Cellular automata simulations

Each elementary cellular automata rule was run for 800 steps, the number of steps chosen after checking for convergence of results on most rules. Noise was introduced via a probability that governed how likely an agent would disobey the rule, i.e. do the opposite that the rule dictated. A probability of 0.5 is equivalent to agents being completely random. For the deterministic simulations, simulations were ran for the whole set of 2^{17} possible initial conditions; this number was increased to 2^{21} for simulations considering noise, as each initial condition was repeated 8 times to allow a more thorough sampling of the possible trajectories. For some selected rules (those with longer characteristic times) the results were confirmed with longer simulation of 1500 steps, without finding noticeable differences. All simulations were carried out using custom-made code in Python.

Among the possible 256 possible elementary rules (which involve the input of three agents), we ignored those that are equivalent up to colour inversion or mirroring transformation, as well as the trivial rule 0. Thus, only 87 non-equivalent rules were analysed.[32]

B. Calculating information-theoretic metrics

Following Ref.[17] and others, the total correlation (TC) and dual total correlation (DTC) can be defined as

$$\mathrm{TC}(\boldsymbol{X}) = \sum_{j=1}^{n} H(X_j) - H(\boldsymbol{X}),$$
$$\mathrm{DTC}(\boldsymbol{X}) = \sum_{j=1}^{n} H\left(\boldsymbol{X}^{-j}\right) - (n-1)H(\boldsymbol{X}^{n}),$$

where we use $\boldsymbol{X} = (X_1,\ldots,X_n)$ and $\boldsymbol{X}^{-j} = (X_1,\ldots,X_{j-1},X_{j+1},\ldots X_n)$ as short-hand notations, and $H(\cdot)$ is the Shannon entropy. Then, the O-information (Ω) and S-information (Σ) can then be defined as

$$\Omega(\boldsymbol{X}) = \mathrm{TC}(\boldsymbol{X}) - \mathrm{DTC}(\boldsymbol{X}),$$
$$\Sigma(\boldsymbol{X}) = \mathrm{TC}(\boldsymbol{X}) + \mathrm{DTC}(\boldsymbol{X}).$$

Following Ref.[21] (see also Ref.[11]), we calculated the Redundancy-Synergy index (RSI) between the three inputs of the rule $\boldsymbol{S} = (S_1, S_2, S_3)$ and its output Y as

$$\mathrm{RSI}(\boldsymbol{S};Y) = I(\boldsymbol{S};Y) - \sum_{i=1}^{3} I(S_i;Y)$$

where $I(S_i;Y) = H(S_i) - H(S_i|Y)$ is the mutual information between variables Y and S_i.

When computing these quantities, the O-information and S-information were calculated with respect to the corresponding empirical distribution obtained by initialising the systems in a state of maximal entropy (i.e. each agent independent and maximally random), as described in Section IV A. The redundancy-synergy index was calculated on a maximum entropy distribution over the inputs. We used the Plugin estimator for deterministic simulations (as in those cases the exact joint distribution could be computed), and the NSB estimator[33] for simulations with noise. All information-theoretic metrics were calculated using the `ndd` package (pypi.org/project/ndd).

C. Half-noise calculation

The *half-noise* for each rule and statistical measure is defined as the level of noise that causes half of the total observed change on that metric. To determine the half-noise, the aggregated effect of noise on the metric is estimated by calculating the cumulative sum of the absolute value of the changes of the metric with respect to changes in P_n — i.e., the integral of the absolute value of the derivative of the metric with respect to the noise. In this way, potentially biphasic curves are converted into monophasic. Then, the amount of noise corresponding to the midpoint between the initial and final value was determined by a linear spline interpolation.

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AUTHOR CONTRIBUTIONS STATEMENT

All the authors conceptualised the research and designed the methodology. P.O. ran the simulations and performed the numerical analysis. All the authors analysed the results, drafted, reviewed, and approved the manuscript.

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