TYPICAL ORBITS OF QUADRATIC POLYNOMIALS WITH A NEUTRAL FIXED POINT: NON-BRJUNO TYPE

by

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Abstract. — We investigate the quantitative and analytic aspects of the near-parabolic renormalization scheme introduced by Inou and Shishikura in 2006. These provide techniques to study the dynamics of some holomorphic maps of the form $f(z) = e^{2\pi i\alpha}z + \mathcal{O}(z^2)$, including the quadratic polynomials $e^{2\pi i\alpha}z+z^2$, for some irrational values of α. The main results of the paper concern fine-scale features of the measure-theoretic attractors of these maps, and their dependence on the data. As a bi-product, we establish an optimal upper bound on the size of the maximal linearization domain in terms of the Siegel-Brjuno-Yoccoz series of α.

Résumé **(Orbites typiques des polynômes quadratiques avec un point fixe neutre: type non-Brjuno)**

On étudie les aspects quantitatifs et analytiques du procédé de renormalisation presque parabolique introduit par Inou et Shishikura en 2006. Ceci fournit des techniques pour étudier la dynamique de certaines applications holomorphes de la forme $f(z) = e^{2\pi i\alpha}z+\mathcal{O}(z^2)$, dont les polynômes quadratiques $e^{2\pi i\alpha}z+z^2$, pour certaines valeurs irrationnelles de α. Les principaux résultats de cet article concernent les propriétés à petite échelle des attracteurs au sens de la théorie de la mesure pour ces applications ainsi que de leur dépendance en fonction des données du problème. On obtient également une borne supérieure optimale sur la taille du domaine maximal de linéarisation en termes de la série de Brjuno-Siegel-Yoccoz de α.

1. Introduction

1.1. Neutral fixed points. — Let f be a holomorphic map of the form

$$f(z) = e^{2\pi \mathbf{i}\alpha}z + a_2z^2 + a_3z^3 + \dots,$$

defined on a neighborhood of $0 \in \mathbb{C}$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Asymptotically near 0, the orbits are governed by the rotation of angle α and are highly recurrent. Away from zero, the influence of non-linearity increases, eventually reaching the scale where the behavior is governed by the global topological structure of the map. For systems with unstable behavior near zero, the transition from local to global and back may occur infinitely often. This creates a delicate

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interplay among the arithmetic nature of α, the non-linearities of the large iterates of f, and the global covering structure of the iterates of f. In this paper we study this problem.

The ideal scenario is when the map f is conformally conjugate to the linear map $w \mapsto e^{2\pi \mathbf{i}\alpha} \cdot w$ on a neighborhood of 0. To discuss this further, let us denote the best rational approximatants of α with p_n/q_n, $n \geq 1$. By a landmark result of Siegel and Brjuno [**Sie42, Brj71**], if the series

$$\mathcal{B}(\alpha) = \sum_{n=1}^{\infty} q_n^{-1} \log q_{n+1}$$

is finite, then near 0 the map f is confomally conjugate to a linear map. When f is linearizable near 0, the maximal domain on which the conjugacy exists is called the *Siegel disk* of f. The geometry of the Siegel disks as well as the dynamics of f near their boundaries have been the subject of extensive studies over the last few decades. These involve a wide range of methods, with consequences often depending on the arithmetic nature of α. See for instance, [**Her85, McM98, GJ02, GŚ03, PZ04, ABC04, BC07, Yam08, Zak10, Zha11, Ché11**]. We note that for almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{B}(\alpha) < \infty$, while for generic choice of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{B}(\alpha) = \infty$.

On the other hand, by a celebrated result of Yoccoz [**Yoc95**], if $\mathcal{B}(\alpha) = \infty$, the polynomial

$$P_\alpha(z) := e^{2\pi \mathbf{i}\alpha} z + z^2$$

is not linearizable at zero. Although this optimality result has been further extended by similar ideas to special families of maps [**PM93, Gey01, Oku04**], it remains widely open in families of polynomials and rational maps. Also, due to a non dynamical step in those arguments, very little has been understood about the local dynamics of non-linearizable maps. In [**PM97**], Perez-Marco constructs non-trivial local invariant compact sets containing 0 for non-linearizable maps. But the necessary control on the geometry of these objects and the dynamics of the map on them has remained out of reach.

In 2006, Inou and Shishikura introduced a renormalization scheme that provides a powerful tool to study the dynamics of near parabolic maps, [**IS06**]. This involves an infinite-dimensional class of maps $\mathcal{F}$, and a nonlinear operator $\mathcal{R} : \mathcal{F} \to \mathcal{F}$, called near-parabolic renormalization. Every map in $\mathcal{F}$ is defined on a Jordan neighborhood of 0, has a neutral fixed point at 0, and a unique critical point of local degree two in its domain of definition. Given $f \in \mathcal{F}$, $\mathcal{R}(f)$ is defined as a sophisticated notion of the return map of f about 0 to a region in the domain of f, viewed in a certain canonically defined coordinate on that region. Precise definitions appear in Section 2.

In this paper we carry out a quantitative analysis of the near-parabolic renormalization scheme. This involves proving a number of foundational results on the combinatorial and analytic aspects of the scheme. In particular, we have slightly modified the definition of renormalization to make it suitable for applications.

Successive iterates of $\mathcal{R}$ at some $f \in \mathcal{F}$ produces a renormalization tower; a sequence of maps $\mathcal{R}^{\circ j}(f)$ which are related by the changes of coordinates. The general theme in theories of renormalization is that large iterates of f often break down into compositions of a small number of the changes of coordinates and the maps $\mathcal{R}^{\circ j}(f)$. However, due to the "semi-local"

nature of near-parabolic renormalization, there are a number of issues which require careful consideration.

The maps in $\mathcal{F}$ have a partial covering structure, involving a branched covering of local degree two. The change of coordinate in the definition of renormalization also has a partial covering structure with a branch point. We prove some detailed orbit relations on the renormalization tower, relating the combinatorial aspects of the orbits of f to the ones of $\mathcal{R}^{\circ j}(f)$, for $j \geq 1$.

The change of coordinates in the definition of renormalization involves transcendental mappings with highly distorting nature. Substantial part of the paper (Section 6) is devoted to proving uniform (distortion) estimates on these maps and their dependence on α. To this end we have introduced a new approach to compare the changes of coordinates to some model maps using quasi-conformal mappings.

The interplay between the arithmetic of α and the non-linearities of the iterates of f is manifested in geometric aspects of the renormalization tower. We present a systematic approach to employing the above combinatorial and analytic tools to study the dynamics of the maps in $\mathcal{F}$.

1.2. Statements of the results. — Define the set of irrational numbers

$$\mathrm{HT}_N := \{[0; a_1, a_2, \dots] \in \mathbb{R} \mid \forall i \geq 1, a_i \geq N\},$$

where $N \in \mathbb{N}$ and $[0; a_1, a_2, \dots] = 1/(a_1 + 1/(a_2 + 1/(\dots)))$ denotes the continued fraction expansion. For technical reasons, in this paper we require α to be in HT_N, for some fixed constant $N \in \mathbb{N}$. (1)

The class of maps $\mathcal{F}$ fibers over HT_N as

$$\mathcal{F} = \cup_{\alpha \in \mathrm{HT}_N} \mathcal{F}_\alpha, \quad \mathcal{F}_\alpha = \{P_\alpha\} \cup \mathcal{IS}_\alpha,$$

where $\mathcal{IS}_\alpha$ is the Inou-Shishikura class of maps defined precisely in Section 2.2. For $f \in \mathcal{IS}_\alpha$, $f'(0) = e^{2\pi \mathbf{i} \alpha}$. We note that for each $\alpha \in \mathrm{HT}_N$ there are polynomials and rational maps of arbitrarily large degree whose restriction to some neighborhood of 0 belong to $\mathcal{IS}_\alpha$. Also, $P_\alpha \notin \mathcal{IS}_\alpha$, but $\mathcal{R}(P_\alpha)$ is defined and belongs to $\mathcal{IS}_{1/\alpha} \subset \mathcal{F}$.

By classical results, the post-critical set of a holomorphic map provides key information about the dynamics of that map, in particular, its measurable dynamics. A map $f \in \mathcal{F}$ has a unique critical point in its (restricted) domain of definition, say cp_f. The *post-critical* set of f associated to cp_f is defined as

$$\mathcal{PC}(f) = \overline{\cup_{i=1}^{\infty} f^{\circ i}(\mathrm{cp}_f)}.$$

The main aim of this paper is to describe the geometry of the post-critical set and the iterates of the map near it. To this end, we build a decreasing nest of simply connected sets containing $\mathcal{PC}(f)$, denoted by Ω_0^n, $n \geq 0$. Each Ω_0^n is formed of about $q_{n+1} + q_n$ (topological) sectors landing at 0, which are ordered by the arithmetic of α, and are mapped to one another by the

1. This is also required in the near-parabolic renormalization scheme. However, it is conjectured that there exists a scheme with similar qualitative features for which $N = 1$. So, we hope that the arguments presented here will be eventually applied to all irrational rotation numbers.

map. Roughly speaking, the rotation element leads to a tangential action on each Ω_0^n, while the nonlinearity of the map results in a radial action on each Ω_0^n. The arithmetic of α and the non-linearities of the large iterates of f characterizes the relative geometry of each Ω_0^{n+1} in Ω_0^n, and the shapes of the sectors in each Ω_0^n. See Figure 5.

A large a_n in the expansion of α (or some p_{n+1}/q_{n+1} very close to α) results in q_n "relatively thick fjords" in $\Omega_0^n \setminus \Omega_0^{n+1}$. By a delicate analysis of the geometry of the renormalization tower we show that the Siegel-Brjuno-Yoccoz arithmetic condition corresponds to the tip of the fjords reaching 0 in the limit.

Theorem A. — *For all $\alpha \in \mathrm{HT}_N$ with $\mathcal{B}(\alpha) = \infty$ and every $f \in \mathcal{F}_\alpha$, $\mathcal{PC}(f) \setminus \{0\}$ is non-uniformly porous*[2]*. In particular, $\mathcal{PC}(f)$ has zero area.*

We establish a uniform contraction principle with respect to certain hyperbolic metrics on the renormalization tower. The map f permutes the sectors in each Ω_0^n according to the rotation of angle α. These are used to prove that the dynamics of f on $\mathcal{PC}(f)$ is highly recurrent, with the combinatorics of the returns given by the rotation of angle α.

Theorem B. — *There are constants M and $\mu < 1$ such that for every α in HT_N and every $f \in \mathcal{F}_\alpha$, on the set $\mathcal{PC}(f)$ we have*

$$|f^{\circ q_n}(z) - z| \leq M\mu^n.$$

By a general result, the orbit of almost every point in the Julia set accumulates on a subset of the post-critical set [**Sul83, Lyu83b**]. Thus, Theorem A allows us to obtain the following.

Corollary C. — *For all $\alpha \in \mathrm{HT}_N$ with $\mathcal{B}(\alpha) = \infty$, the orbit of Lebesgue almost every point in the Julia set of P_α is non-recurrent. In particular, there is no absolutely continuous invariant probability on the Julia set of P_α.*

Let $\Delta(f)$ denote the Siegel disk of f when f is linearizable at 0, and otherwise, let $\Delta(f) = \{0\}$. Using the uniform contraction principle along the renormalization tower we show the relation $\cap_n \Omega_0^n = \mathcal{PC}(f) \cup \Delta(f)$, see Proposition 5.10. This allows us to establish some topological properties of $\mathcal{PC}(f)$.

Theorem D. — *For all $\alpha \in \mathrm{HT}_N$ and all $f \in \mathcal{F}_\alpha$, $\mathcal{PC}(f)$ is a connected set.*

For small perturbations of α, the sets Ω_0^n, up to some finite level n, move continuously as a function of α. Using $\cap_n \Omega_0^n = \mathcal{PC}(f) \cup \Delta(f)$ we conclude a semi-continuity property of $\mathcal{PC}(f)$.

Theorem E. — *Let f_α, $\alpha \in [0,1]$, be a continuous family of maps such that for $\alpha \in \mathrm{HT}_N$ we have $f_\alpha \in \mathcal{F}_\alpha$. Then, for every $\alpha_0 \in \mathrm{HT}_N$ and every $\epsilon > 0$ there is $\delta > 0$ such that for every $\alpha \in \mathrm{HT}_N$ with $|\alpha_0 - \alpha| < \delta$, $\mathcal{PC}(f_\alpha) \cup \Delta(f_\alpha)$ is contained in ϵ-neighborhood of $\mathcal{PC}(f_{\alpha_0}) \cup \Delta(f_{\alpha_0})$.*

2. A set $E \subseteq \mathbb{C}$ is called non-uniformly porous, if there is $\lambda > 0$ satisfying the following property. For every $z \in E$ there is a sequence of real numbers $r_n \to 0$ such that each ball of radius r_n about z contains a ball of radius λr_n disjoint from E.

When $\mathcal{B}(\alpha_0) < \infty$, $\mathcal{PC}(f_\alpha)$ may not depend continuously on α at α_0, due to nearby non-linearizable maps. The above theorem states that the post-critical set of the perturbed map can only explode into the Siegel disk of the limiting map. On the other hand, when $\mathcal{B}(\alpha_0) = \infty$, $\Delta(f_{\alpha_0}) = \{0\} \subset \mathcal{PC}(f_{\alpha_0})$, and the above theorem boils down to the continuity of $\mathcal{PC}(f_\alpha)$ at α_0.

Theorem E plays a key role in constructions of examples based on successive small perturbations. A special case of the above theorem for P_α and when α_0 is of bounded type was proved earlier by Buff and Chéritat in [**BC12**]. That is a key step in their remarkable construction of quadratics P_α with positive area Julia sets. The flexibility of the arguments presented here allows one to perturb the parameter α away from the real line, and gain control on the post-critical sets of nearby maps. This forms an essential part of a recent construction of Feigenbaum quadratic polynomials with positive area Julia sets by Avila and Lyubich reported in [**AL15**].

Each sector in Ω_0^n, for $n \geq 1$, forms a "fundamental domain" for the dynamics of f. That is, the orbit of every point in $\mathcal{PC}(f)$ visits each such sector. When $f = P_\alpha$, the orbit of almost every point in the Julia set of f must visit all those sectors; see Proposition 3.6. We show that in each Ω_0^n there is a sector whose diameter is bounded by a uniform constant times $exp(-\sum_{i\leq n} q_i^{-1} \log q_{i+1})$.

Theorem F. — *Let $\alpha \in \mathrm{HT}_N$ with $\mathcal{B}(\alpha) = \infty$ and $f \in \mathcal{F}_\alpha$. Then, the orbit of every point in $\mathcal{PC}(f)$ visits every neighborhood of 0. In particular, there is no periodic point in $\mathcal{PC}(f)$ except 0.*

When $f = P_\alpha$, for Lebesgue almost every $z \in \mathbb{C}$, the orbit of z under P_α either tends to infinity, or visits every neighborhood of 0 infinitely often.

As the critical orbit may never enter the linearization domain, the size of the smallest sector in each Ω_0^n provides an upper bound on the size of the Siegel disk.

Theorem G. — *There exists $C > 0$ such that for every $\alpha \in \mathrm{HT}_N$ and every $f \in \mathcal{F}_\alpha$ we have*

$$\mathrm{d}(\partial\Delta(f), 0) \leq C \cdot e^{-\mathcal{B}(\alpha)},$$

where $\mathrm{d}(\partial\Delta(f), 0)$ denotes the distance from 0 to the boundary of $\Delta(f)$.

On the other hand, Yoccoz in [**Yoc95**] proves that there is a constant $C' > 0$ such that for normalized maps f that are defined and one-to-one on $B(0, 1)$, $\Delta(f)$ contains the ball of radius $C' \cdot e^{-\mathcal{B}(\alpha)}$ about 0. By an alternative (and beautiful) approach specific to the quadratic polynomials, Buff and Chéritat [**BC04**] had already established the bound in Theorem G for the quadratic polynomials P_α for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

The above theorem gives a direct proof of the optimality of the Siegel-Brjuno-Yoccoz arithmetic condition in $\mathcal{F}$. However, for $\alpha \in \mathrm{HT}_N$, there is a holomorphic motion of the orbit of the critical point over $\mathcal{IS}_\alpha$. Thus, the optimality of the arithmetic condition for P_α by Yoccoz, and the classical λ-lemma [**Lyu83a, MSS83**], may be used to derive the optimality of the arithmetic condition in $\mathcal{F}$.

There are points in $\mathcal{PC}(f)$ with dense orbits. When f is not linearizable at 0 (and even for some linearizable f) there is an abundance of non-trivial invariant sets in $\mathcal{PC}(f)$ in the form

of hedgehogs introduced by Perez-Marco. Do those sets have a non-trivial basin of attraction in the Julia set. This has been answered in [**Che13**] for all rotations in HT_N. That is, for Lebesgue almost every z in the Julia set of P_α, the set of accumulation points of the orbit of z under P_α is equal to $\mathcal{PC}(P_\alpha)$. This provides a complete description of the topological behavior of the typical orbits of P_α, modulo the topology of $\mathcal{PC}(P_\alpha)$. The topological description of $\mathcal{PC}(f)$ will appear in a forthcoming paper.

The analogue of Theorem A when $\mathcal{B}(\alpha) < \infty$ is proved in [**Che13**]. The reason for the distinction is that the study of the linearizable maps requires finer (distortion) estimates on the changes of coordinates that were not available at the time of writing this paper.

There has been recent advances on the dynamics of quadratic polynomials using the near-parabolic renormalization technique and the methods developed in this paper. The statistical behavior of the orbits of the maps $f \in \mathcal{F}$ is described in [**AC12**]. The 1/2-hölder continuity of a relation between the conformal radius of the Siegel disks and the Brjuno series is confirmed in [**CC15**]. It is also employed in [**CS15**] to prove the local connectivity of the Mandelbrot set on a Cantor set of parameters where the fine scale dynamics degenerates.

This paper is a step towards developing a theory based on near-parabolic renormalization in order to provide a comprehensive description of the dynamics of holomorphic maps with a neutral fixed point. One hopes to eventually build a unified language to treat problems of different nature associated with such maps.

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1.3. Frequently used notations. —

- $:=$ is used when a notation appears for the first time.
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the integer, rational, real, and complex numbers, respectively. $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.
- $\mathbf{i}$ denotes the imaginary unit complex number, and i is used as an integer index.
- $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$ denote the real part, the imaginary part, and the absolute value of a complex number z, respectively.
- $B(y, \delta) \subset \mathbb{C}$ denotes the ball of radius δ around y in the Euclidean metric, and $B_\delta(X) := \cup_{x \in X} B(x, \delta)$, for a given $X \subseteq \mathbb{C}$.
- $\operatorname{diam}(S)$ and $\operatorname{int}(S)$ denote the Euclidean diameter and the interior of a set $S \subset \mathbb{C}$.
- Given a map f, $f^{\circ n}$ denotes the n times composition of f with itself.
- $\operatorname{Dom} f$, $J(f)$, and $\mathcal{PC}(f)$ denote the domain of definition, the Julia set, and the post-critical set of a map f, respectively.
- Univalent map refers to a one-to-one holomorphic map.
- Given $g \colon \operatorname{Dom} g \to \mathbb{C}$, with only one critical point in its domain of definition, cp_g and cv_g denote the critical point and the critical value of g, respectively.
- For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.
- Unless otherwise stated, arg denotes the principal branch of argument with values in $(-\pi, \pi]$.

2. Inou-Shishikura class and near-parabolic renormalization

2.1. Preliminary definitions. — Let $f \colon U \subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map. Given $z \in U$, if $f(z) \in U$ we can define $f^{\circ 2}(z) := f \circ f(z)$. Similarly, if $f^{\circ 2}(z)$ also belongs to U, $f^{\circ 3}(z)$ is defined, and so on. The *orbit of* z, denoted by $\mathcal{O}(z)$, is the sequence, $z, f(z), f^{\circ 2}(z), \dots$, as long as it is defined. So it may be a finite or an infinite sequence. Given an infinite orbit $\mathcal{O}(z)$, we say that $\mathcal{O}(z)$ *eventually stays* in a given set $E \subset \hat{\mathbb{C}}$, if there exists an integer k such that $\mathcal{O}(f^{\circ k}(z))$ is contained in E.

The *Fatou set* of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as the largest open set $F(f) \subseteq \hat{\mathbb{C}}$ on which the sequence of iterates $\langle f^{\circ n} \rangle_{n=0,1,\dots}$ forms a pre-compact family in the compact-open topology. Its complement, $J(f)$, is the *Julia set* of f.

The *distortion* of f on U is defined as the supremum of $\log(|f'(z)/f'(w)|)$, for all z and w in U, in the spherical metric, (which may be finite or infinite). We frequently use the following distortion bounds due to Koebe and Grunsky, see [**Pom75**] or [**Dur83**, Theorem 3.6].

***Theorem 2.1* (Distortion Theorem).** — *Suppose that $f\colon B(0,1) \to \mathbb{C}$ is a univalent map with $f(0) = 0$, and $f'(0) = 1$. At every $z \in B(0,1)$ we have*

1) $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$,

2) $\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$,

3) $\frac{1-|z|}{1+|z|} \leq |zf'(z)/f(z)| \leq \frac{1+|z|}{1-|z|}$,

4) $|\arg(zf'(z)/f(z))| \leq \log \frac{1+|z|}{1-|z|}$.

This implies the 1/4-theorem: the image $f(B(0,1))$ contains $B(0,1/4)$.

Here we summarize the results of [**IS06**] in Theorems 2.2, 2.3 and 2.6, that we use in this paper. They follow from Theorem 2.1 and Main Theorems 1–3 in [**IS06**].

2.2. Inou-Shishikura class of maps. — Consider a map $h\colon \mathrm{Dom}\ h \to \mathbb{C}$, where $\mathrm{Dom}\ h \subseteq \mathbb{C}$ denotes the domain of definition (always assumed to be open) of h. Given a compact set $K \subset \mathrm{Dom}\ h$ and an $\varepsilon > 0$, a neighborhood of h is defined as

$$\mathcal{N}(h; K, \varepsilon) := \{g\colon \mathrm{Dom}\ g \to \mathbb{C} \mid K \subset \mathrm{Dom}\ g, \text{ and } \sup_{z \in K} |g(z) - h(z)| < \varepsilon\}.$$

By "the sequence $h_n : \mathrm{Dom}\ h_n \to \mathbb{C}$ converges to h" we mean that given an arbitrary neighborhood of h defined as above, h_n is contained in that neighborhood for large enough n. Note that the maps h_n are not necessarily defined on the same set.

Consider the cubic polynomial

$$P(z) := z(1+z)^2.$$

It has a *parabolic* fixed point at 0, that is, $P'(0) = 1$. Also, it has a critical point at $\mathrm{cp}_P := -1/3$ which is mapped to the critical value at $\mathrm{cv}_P := -4/27$, and another critical point at -1 which is mapped to 0. See Figure 1.

Consider the ellipse

$$E := \Big\{x + \mathbf{i}y \in \mathbb{C} \mid (\frac{x+0.18}{1.24})^2 + (\frac{y}{1.04})^2 \leq 1\Big\},$$

and let

$$U := g(\hat{\mathbb{C}} \setminus E), \text{ where } g(z) := \frac{-4z}{(1+z)^2}. \tag{1}$$

The domain U contains 0 and cp_P, but not the other critical point of P at -1.

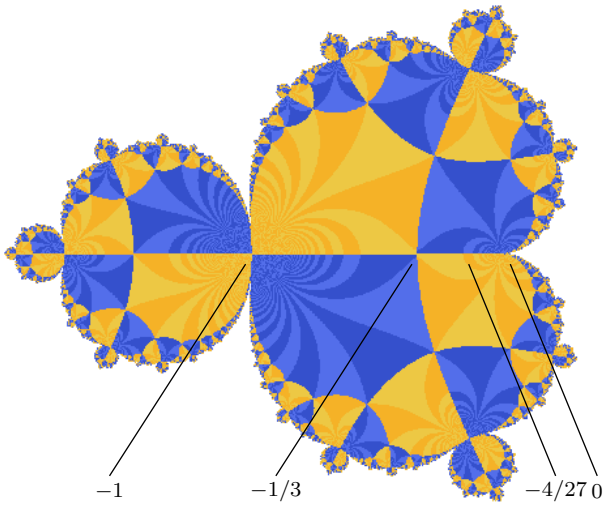


FIGURE 1. The dynamic plane of the polynomial $P(z) = z(1+z)^2$, and the special points -1, $-1/3$, $-4/27$, and 0. The shades of blue are mapped by the Fatou coordinate to strips of width one in the upper half plane, while the shades of yellow are mapped by the Fatou coordinate to strips of width one in the the lower half pane.

Following [**IS06**], we define the class of maps

$$\mathcal{IS}_0 := \left\{ f := P \circ \varphi^{-1} \colon U_f \to \mathbb{C} \;\middle|\; \varphi \colon U \to U_f \text{ is univalent, } \varphi(0) = 0,\ \varphi'(0) = 1 \right\}.$$

Every map in this class has a parabolic fixed point at 0 and a unique critical point at $\mathrm{cp}_f := \varphi(-1/3) \in U_f$.

The class $\mathcal{IS}_0$ corresponds to the class $\mathcal{F}_1$ (and also $\mathcal{F}_1^P$) in the notations of [**IS06**]. An extra condition on quasi-conformal extendibility of $\varphi^{-1} : U_f \to \mathbb{C}$ onto $\mathbb{C}$ is assumed in that paper. However, they have imposed this extra condition only to derive the hyperbolicity of the renormalization operator, which we shall introduce a moment. As we do not use the hyperbolicity of the renormalization operator in this paper, we have dropped the extra condition on the extendibility.

Theorem 2.2 **(Inou–Shishikura).** — *For all $h \in \mathcal{IS}_0$ there exist a domain $\mathcal{P}_h \subset U_h$ and a univalent map $\Phi_h \colon \mathcal{P}_h \to \mathbb{C}$ satisfying the following:*

(1) $\mathcal{P}_h$ is bounded by piece-wise analytic curves and is compactly contained in U_h. It contains cp_h and 0 on its boundary.
(2) $\Phi_h(\mathcal{P}_h) = \{\zeta \in \mathbb{C}; 0 < \mathrm{Re}\,\zeta\}$ and when $z \in \mathcal{P}_h \to 0$, $|\Phi_h(z)| \to +\infty$,
(3) $\Phi_h(h(z)) = \Phi_h(z) + 1$, for all $z \in \mathcal{P}_h$,
(4) the map Φ_h is unique once normalized by $\Phi_h(\mathrm{cp}_h) = 0$. Moreover, the normalized map Φ_h depends continuously on h.

The map $\Phi_h : \mathcal{P}_h \to \mathbb{C}$ in the above theorem is called the *Fatou coordinate* of h. The existence of such coordinate for the quadratic map $z \mapsto z + z^2$ was already known to Fatou, see for example [**Shi00**].

Given $\alpha \in \mathbb{R}$, let

$$\mathcal{IS}_\alpha := \{z \mapsto f(e^{2\pi \mathbf{i}\alpha} z) : e^{-2\pi\alpha\mathbf{i}} \cdot U_f \to \mathbb{C} \mid f \in \mathcal{IS}_0\}.$$

All maps in $\mathcal{IS}_\alpha$ have a critical value at $-4/27$. For the sake of simplicity of notations, we define and work with the quadratic family

$$Q_\alpha(z) := e^{2\pi\mathbf{i}\alpha} z + \frac{27}{16} e^{4\pi\alpha\mathbf{i}} z^2,$$

that enjoys the same normalization $\mathrm{cv}_{Q_\alpha} = -4/27$. Let us combine the two classes under the notation

$$\mathcal{QIS}_\alpha := \mathcal{IS}_\alpha \cup \{Q_\alpha\}.$$

The class $\cup_{\alpha\in\mathbb{R}} \mathcal{IS}_\alpha$ naturally embeds into the space of univalent maps on the unit disk with a neutral fixed point at 0. Hence, by the distortion theorem, it is a pre-compact class in the compact-open topology. Furthermore, it is an application of the area Theorem and the choice of P and U (see Main Theorem 1-a in [**IS06**] for details) that

$$\{|h''(0)|; h \in \mathcal{IS}_0\} \subset [2, 7]. \tag{2}$$

Any map $h = f_0(e^{2\pi\mathbf{i}\alpha}\cdot) \in \mathcal{IS}_\alpha$ has a fixed point at 0 with $h'(0) = e^{2\pi\mathbf{i}\alpha}$. Moreover, if α is small, h has another fixed point $\sigma_h \neq 0$ near 0 in U_h. The σ_h fixed point depends continuously on h and has asymptotic expansion $\sigma_h = -4\pi\alpha\mathbf{i}/f_0''(0) + o(\alpha)$, when h converges to f_0 in a fixed neighborhood of 0. Clearly $\sigma_h \to 0$ as $\alpha \to 0$.

Theorem 2.3 **(Inou–Shishikura)**. — *There exists a constant $r_1 > 0$ such that for every map $h\colon U_h \to \mathbb{C}$ in $\mathcal{QIS}_\alpha$ with $\alpha \in (0, r_1]$, there exist a domain $\mathcal{P}_h \subset U_h$ and a univalent map $\Phi_h\colon \mathcal{P}_h \to \mathbb{C}$ satisfying the following properties:*

(1) $\mathcal{P}_h$ is a simply connected region bounded by piece-wise analytic curves and is compactly contained in U_h. Also, it contains cp_h, 0, and σ_h on its boundary.

(2) we have

$$\Phi_h(\mathcal{P}_h) \supseteq \{\zeta \in \mathbb{C}; 0 < \mathrm{Re}\,\zeta \leq 1\},$$

with $\mathrm{Im}\,\Phi_h(z) \to +\infty$ as $z \in \mathcal{P}_h \to 0$, and $\mathrm{Im}\,\Phi_h(z) \to -\infty$ as $z \in \mathcal{P}_h \to \sigma_h$.

(3) Φ_h satisfies the Abel functional equation, that is,

$$\Phi_h(h(z)) = \Phi_h(z) + 1, \text{ whenever } z \text{ and } h(z) \text{ belong to } \mathcal{P}_h.$$

(4) Φ_h is unique once normalized by $\Phi_h(\mathrm{cp}_h) = 0$. Moreover, the normalized map Φ_h depends continuously on h.

In Section 6 we shall analyze the coordinates Φ_h introduced in the above theorem. In particular, we prove the following proposition in Section 6.7. It is frequently used in this paper. There is an alternative proof of this given in [**BC12**, Proposition 12].

Proposition 2.4. — *There exist a positive constant r_2, as well as integers $\boldsymbol{k}$ and $\hat{\boldsymbol{k}}$ such that for all $\alpha \in (0, r_2]$ and all $h \in \mathcal{QIS}_\alpha$, one may choose $\mathcal{P}_h$ such that the domain $\mathcal{P}_h$ and the map $\Phi_h \colon \mathcal{P}_h \to \mathbb{C}$ satisfy the following additional properties:*

(1) there exists a continuous branch of argument defined on $\mathcal{P}_h$ such that

$$\max_{z,z' \in \mathcal{P}_h} |\arg(z) - \arg(z')| \leq 2\pi\hat{\boldsymbol{k}};$$

(2) $\Phi_h(\mathcal{P}_h) = \{w \in \mathbb{C} \mid 0 < \mathrm{Re}(w) < \lfloor 1/\alpha \rfloor - \boldsymbol{k}\}$.

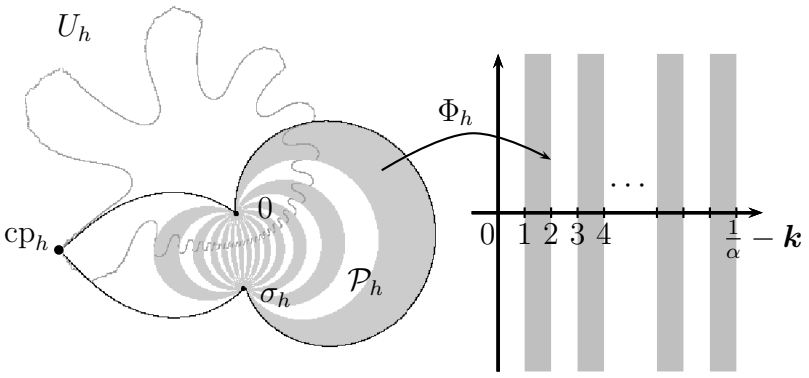


FIGURE 2. A perturbed Fatou coordinate Φ_h and its domain of definition $\mathcal{P}_h$. Similar colors are mapped on one another under Φ_h. The gray curve (amoeba) approximates the first few iterates of cp_h under h.

The map $\Phi_h : \mathcal{P}_h \to \mathbb{C}$ obtained in Theorem 2.3 is called the *perturbed Fatou coordinate* of h. In this paper, by the *perturbed Fatou coordinate* of h, or sometimes *Fatou coordinate* of h for short, we mean the coordinate that satisfies Proposition 2.4 or Theorem 2.2. See Figure 2.

2.3. Near-parabolic renormalization. — Let $h \colon U_h \to \mathbb{C}$ be in $\mathcal{QIS}_\alpha$, with $\alpha \in (0, r_2]$, where r_2 is the constant obtained in Proposition 2.4. Let $\Phi_h \colon \mathcal{P}_h \to \mathbb{C}$ denote the normalized Fatou coordinate of h. Define

$$\begin{aligned} \mathcal{C}_h &:= \{z \in \mathcal{P}_h : 1/2 \leq \mathrm{Re}(\Phi_h(z)) \leq 3/2\,,\ -2 < \mathrm{Im}\,\Phi_h(z) \leq 2\}, \\ \mathcal{C}_h^\sharp &:= \{z \in \mathcal{P}_h : 1/2 \leq \mathrm{Re}(\Phi_h(z)) \leq 3/2\,,\ 2 \leq \mathrm{Im}\,\Phi_h(z)\}. \end{aligned} \tag{3}$$

By definition, the critical value of h, cv_h, belongs to $\mathrm{int}\,(\mathcal{C}_h)$, and $0 \in \partial(\mathcal{C}_h^\sharp)$.

Assume for a moment that there exists a positive integer k_h, depending on h, with the following properties:

- For every integer $k \in \{1, 2, \ldots, k_h\}$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h^\sharp)$ which is compactly contained in Dom h and contains 0 on its boundary. We denote this component by $(\mathcal{C}_h^\sharp)^{-k}$.
- For every integer $k \in \{1, 2, \ldots, k_h\}$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h)$ which has non-empty intersection with $(\mathcal{C}_h^\sharp)^{-k}$, and is compactly contained in Dom h. This component is denoted by $\mathcal{C}_h^{-k}$.

– The sets $\mathcal{C}_h^{-k_h}$ and $(\mathcal{C}_h^\sharp)^{-k_h}$ are contained in

$$\{z \in \mathcal{P}_h \mid 0 < \operatorname{Re}\Phi_h(z) < \lfloor 1/\alpha \rfloor - \boldsymbol{k} - 1/2\}.$$

– The maps $h : \mathcal{C}_h^{-k} \to \mathcal{C}_h^{-k+1}$, for $2 \le k \le k_h$, and $h : (\mathcal{C}_h^\sharp)^{-k} \to (\mathcal{C}_h^\sharp)^{-k+1}$, for $1 \le k \le k_h$ are one-to-one onto. The map $h : \mathcal{C}_h^{-1} \to \mathcal{C}_h$ is a two-to-one branched covering.

Let k_h denote the smallest positive integer for which the above conditions hold, and define

$$S_h := \mathcal{C}_h^{-k_h} \cup (\mathcal{C}_h^\sharp)^{-k_h}.$$

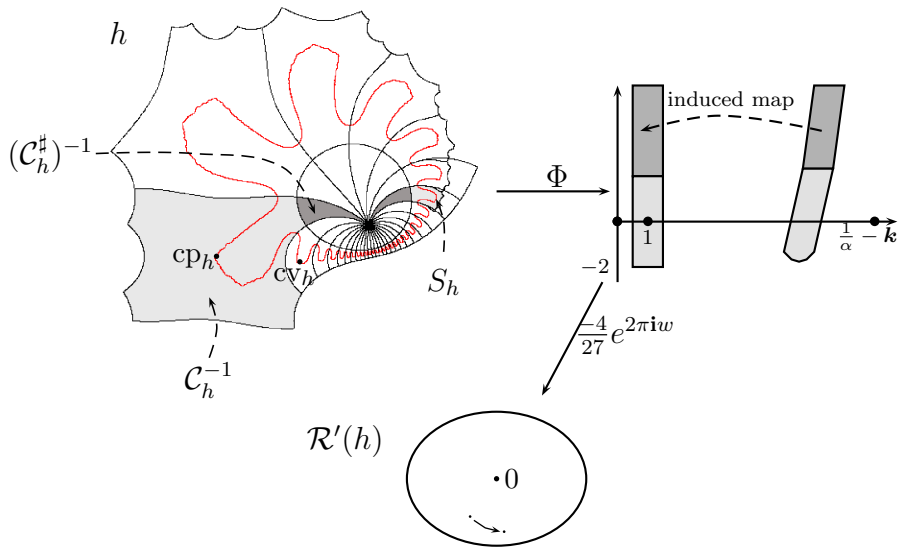


FIGURE 3. The sets $\mathcal{C}_h$, $\mathcal{C}_h^\sharp$,..., $\mathcal{C}_h^{-k_h}$, and $(\mathcal{C}_h^\sharp)^{-k_h}$. The "induced map" projects via $e^{2\pi \mathbf{i} w}$ to a well defined map $\mathcal{R}(h)$ on a neighborhood of 0.

Consider the map

$$\Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \to \mathbb{C}. \tag{4}$$

By the Abel functional equation, this map commutes with the translation by one, and hence projects via $z = \frac{-4}{27} e^{2\pi \mathbf{i} w}$ to a map $\mathcal{R}'(h)$ defined on a set punctured at zero. However, it extends across zero and has the form $z \mapsto e^{2\pi \frac{-1}{\alpha} \mathbf{i}} z + O(z^2)$ near there. See Figure 3.

The conjugate map $s \circ \mathcal{R}'(h) \circ s^{-1}$, where $s(z) := \bar{z}$ denotes the complex conjugation map, has the form $z \mapsto e^{2\pi \frac{1}{\alpha} \mathbf{i}} z + O(z^2)$ near 0. The map $\mathcal{R}(h) := s \circ \mathcal{R}'(h) \circ s^{-1}$, restricted to the interior of $s(\frac{-4}{27} e^{2\pi \mathbf{i}(\Phi_h(S_h))})$, is called the *near-parabolic renormalization* of h by Inou and Shishikura. We simply refer to it as the *renormalization* of h. One can see (Lemma 3.2) that one time iterating $\mathcal{R}(h)$ corresponds to several times iterating h, through the changes of coordinates. For some applications of closely related renormalizations (Douady-Ghys renormalization) one may refer to [**Dou87, Dou94, Yoc95, Shi98**] and the references therein.

It is a non-trivial task to control the shapes and the locations of the sets $\mathcal{C}_h^{-k}$ and $(\mathcal{C}_h^\sharp)^{-k}$ for a given map in $\mathcal{IS}_\alpha$. This is the key content of [**IS06**], which is carried out using a remarkable series of estimates on univalent mappings. In this paper we do not use the many statements proved on the geometry and locations of these sets in that paper, but need the following.

Proposition 2.5. — *There are positive constants $r_2' > 0$ and $C < 2\pi$ such that for every $\alpha \in (0, r_2']$ and every $h \in \mathcal{QIS}_\alpha$,*

$$\sup\{\arg z_1 - \arg z_2 \mid z_1, z_2 \in \mathcal{C}_h^{-1}\} \leq C,$$

for every continuous branch of argument defined on $\mathcal{C}_h^{-1}$.

Proof. — According to [**IS06**], for every $h \in \mathcal{QIS}_0$ the sets $\mathcal{C}_h^{-k}$ and $\mathcal{C}_h^{\sharp\,-k}$ are defined for all $k \geq 0$. That is, for large enough k these are contained in the repelling Fatou coordinate of the map h and then further pre-images are defined by the general properties of the Fatou coordinates. Comparing to their notations, $\mathcal{C}_h^{-1}$ is contained in the union

$$\psi_0(D_0) \cup \psi_0(D_0') \cup \psi_0(D_{-1}) \cup \psi_0(D_{-1}''),$$

where $\psi_0(z) = -4/z$. See Section 5.A–Outline of the proof. They prove in Proposition 5.7-(e) that the closure of the set $D_0 \cup D_0' \cup D_{-1} \cup D_{-1}''$ does not intersect the negative real axis. In particular, it follows that $\sup \arg z_1/z_2 < 2\pi$, for $z_1, z_2 \in \mathcal{C}_h^{-1}$, for each $h \in \mathcal{QIS}_0$. By the pre-compactness of the class of maps $\mathcal{QIS}_0$, there is a constant $C' < 2\pi$ such that the supremum is bounded from above by C' over all maps $h \in \mathcal{QIS}_0$. Then, by the continuous dependence of the Fatou coordinate on the map, there are $r_2' > 0$ and $C < 2\pi$ satisfying the conclusion of the proposition. □

The following theorem [**IS06**, Main theorem 3] states that the above definition of renormalization $\mathcal{R}$ can be carried out for certain perturbations of maps in $\mathcal{IS}_0$. In particular, this implies the existence of k_h satisfying the four properties listed in the definition of renormalization. There is also a detailed argument on this given in [**BC12**, Proposition 13] [3].

Define

$$V := P^{-1}(B(0, \frac{4}{27}e^{4\pi})) \setminus ((-\infty, -1] \cup \overline{B}) \tag{5}$$

where B is the component of $P^{-1}(B(0, \frac{4}{27}e^{-4\pi}))$ containing -1 (see Figure 4). By an explicit calculation (see [**IS06**, Proposition 5.2]) one can see that $\overline{U} \subset V$.

Theorem 2.6 **(Inou-Shishikura)**. — *There exist a constant $r_3 > 0$ such that if $h \in \mathcal{IS}_\alpha$ with $\alpha \in (0, r_3]$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}_{1/\alpha}$, that is, $\mathcal{R}(h)(z) :=$*

3. The sets $\mathcal{C}_h^{-k}$ and $(\mathcal{C}_h^\sharp)^{-k}$ defined here are (strictly) contained in the closure of the sets denoted by V^{-k} and W^{-k} in [**BC12**]. The set $\Phi_h(\mathcal{C}_h^{-k} \cup (\mathcal{C}_h^\sharp)^{-k})$ is contained in the closure of the union

$$D_{-k}^\sharp \cup D_{-k} \cup D_{-k}'' \cup D_{-k+1}' \cup D_{-k+1} \cup D_{-k+1}^\sharp$$

in the notation used in [**IS06**, Section 5.A].

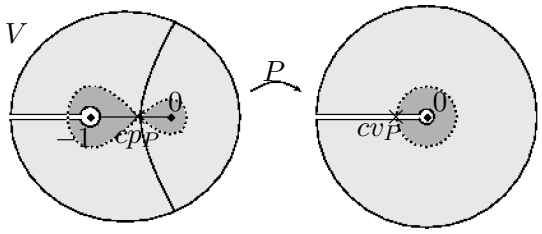


FIGURE 4. A schematic presentation of the polynomial P; its domain, and its range. Similar colors and line styles are mapped on one another.

$P \circ \psi^{-1}(e^{\frac{2\pi}{\alpha}\mathbf{i}} \cdot z)$ for a univalent map $\psi : U \to \mathbb{C}$. Moreover, ψ extends to a univalent map on V.

The same conclusion holds for the map $Q_\alpha(z) = e^{2\pi\mathbf{i}\alpha}z + \frac{27}{16}e^{4\pi\alpha\mathbf{i}}z^2$. That is, $\mathcal{R}(Q_\alpha)$ is well-defined and belongs to $\mathcal{IS}_{1/\alpha}$ provided $\alpha \in (0, r_3]$.

A uniform bound on k_h is established in Section 6.5 .

Proposition 2.7. — *There is $\boldsymbol{k''} \in \mathbb{N}$ such that for all $h \in \mathcal{IS}_\alpha$ with $\alpha \in (0, r_3]$, $k_h \leq \boldsymbol{k''}$.*

Let $[0; a_1, a_2, \dots]$ denote the continued fraction expansion of α as in the introduction. Define $\alpha_0 := \alpha$, and inductively for $i \geq 1$ define the sequence of real numbers $\alpha_i \in (0, 1)$ as

$$\alpha_i := 1/\alpha_{i-1} \pmod 1.$$

Then each α_i has expansion $[0; a_{i+1}, a_{i+2}, \dots]$. If we fix a constant $N \geq 1/r_3$, then $\alpha \in \mathrm{HT}_N$ implies that $\alpha_j \in (0, r_3]$, for $j = 0, 1, 2, \dots$. We use this constant N throughout the rest of this article.

Let $\alpha \in \mathrm{HT}_N$ and $f_0 \in \mathcal{QIS}_\alpha$. Then, using Theorem 2.6, we may inductively define the sequence of maps

$$f_{n+1} := \mathcal{R}(f_n) : U_{f_{n+1}} \to \mathbb{C}.$$

Let $U_n := U_{f_n}$ denote the domain of definition of f_n, for $n \geq 0$. Hence, for every n,

$$f_n : U_n \to \mathbb{C},\ f_n(0) = 0,\ f_n'(0) = e^{2\pi\alpha_n\mathbf{i}},\ \text{and } \mathrm{cv}_{f_n} = -4/27.$$

3. Dynamically defined neighborhoods of the post-critical set

Recall the constants $\boldsymbol{k}, \hat{\boldsymbol{k}}$ introduced in Proposition 2.4 and the constant N introduced at the end of the previous section.

Remark 3.1. — To slightly simplify the technical details of proofs, we assume that

$$N \geq \boldsymbol{k} + \hat{\boldsymbol{k}} + 2. \tag{6}$$

The reason to impose this is to make $\Phi_{f_n}(\mathcal{P}_{f_n})$ wide enough to contain a set defined later. However, one can avoid this condition by extending Φ_{f_n} and $\Phi_{f_n}^{-1}$ to larger domains, using the dynamics of f_n. We postpone this argument to Section 5.3.

3.1. Changes of coordinates, renormalization tower. — For $n \geq 0$, let $\Phi_n := \Phi_{f_n}$ denote the Fatou coordinate of $f_n \colon U_n \to \mathbb{C}$ defined on the set $\mathcal{P}_n := \mathcal{P}_{f_n}$. For our convenience we use the notation

$$\mathbb{E}\mathrm{xp}(\zeta) := \zeta \longmapsto \frac{-4}{27} s(e^{2\pi \mathbf{i}\zeta}) : \mathbb{C} \to \mathbb{C}^*, \text{ where } s(z) = \bar{z}.$$

By Proposition 2.4, Inequality (6), and that $\mathcal{P}_n$ is simply connected, there is an (anti-holomorphic) inverse branch

$$\eta_n : \mathcal{P}_n \to \Phi_{n-1}(\mathcal{P}_{n-1})$$

of $\mathbb{E}\mathrm{xp}$. There may be several choices for this map but we choose one of them (for each n) such that

$$\mathrm{Re}(\eta_n(\mathcal{P}_n)) \subset [0, \hat{\boldsymbol{k}} + 1] \tag{7}$$

holds, and fix this choice for the rest of this article. Now define

$$\psi_n := \Phi_{n-1}^{-1} \circ \eta_n \colon \mathcal{P}_n \to \mathcal{P}_{n-1}. \tag{8}$$

Each ψ_n extends continuously to $0 \in \partial\mathcal{P}_n$ by mapping it to 0.

For $n \geq 2$ we can form the compositions

$$\Psi_n := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_n \colon \mathcal{P}_n \to \mathcal{P}_0 \subset U_0.$$

For every $n \geq 0$, let $\mathcal{C}_n$ and $\mathcal{C}_n^\sharp$ denote the corresponding sets for f_n defined in (3) (i.e., replace h by f_n). Denote by k_n the smallest positive integer with

$$S_n^0 := \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \{z \in \mathcal{P}_n \mid 0 < \mathrm{Re}\, \Phi_n(z) < \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1/2\}.$$

By definition, the critical value of f_n is contained in $f_n^{\circ k_n}(S_n^0)$.

For every $n \geq 0$ and $i \geq 2$, define the sectors

$$S_n^1 := \psi_{n+1}(S_{n+1}^0) \subset \mathcal{P}_n,\ S_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n.$$

All these sectors contain 0 on their boundaries.

3.2. Orbit relations on the renormalization tower. —

Lemma 3.2. — *Let $z \in \mathcal{P}_n$ be a point with $w := \mathbb{E}\mathrm{xp} \circ \Phi_n(z) \in U_{n+1}$. There exists an integer ℓ_z with $1 \leq \ell_z \leq \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1 + k_n$, such that*

- *the finite orbit $z, f_n(z), f_n^{\circ 2}(z), \ldots, f_n^{\circ \ell_z}(z)$ is defined, $f_n^{\circ \ell_z}(z) \in \mathcal{C}_n \cup \mathcal{C}_n^\sharp$;*
- *$\mathbb{E}\mathrm{xp} \circ \Phi_n(f_n^{\circ \ell_z}(z)) = f_{n+1}(w)$;*
- *if in addition $w \in f_{n+1}(U_{n+1})$, then*

$$z, f_n(z), f_n^{\circ 2}(z), \ldots, f_n^{\circ \ell_z}(z) \in \bigcup_{i=0}^{k_n + \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2} f_n^{\circ i}(S_n^0).$$

Proof. — As $w \in \mathrm{Dom}\ f_{n+1}$, by the definition of renormalization $\mathcal{R}(f_n) = f_{n+1}$, there are $\zeta \in \Phi_n(S_n^0)$ and $\zeta' \in \Phi_n(\mathcal{C}_n \cup \mathcal{C}_n^\sharp)$, such that

$$\mathbb{E}\mathrm{xp}(\zeta) = w, \quad \mathbb{E}\mathrm{xp}(\zeta') = f_{n+1}(w), \text{ and } \quad \zeta' = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta).$$

Since $\mathbb{E}\mathrm{xp}(\Phi_n(z)) = w$, there exists an integer ℓ with

$$\Phi_n(z) + \ell = \zeta \text{ and } -k_n + 1 \leq \ell \leq \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1.$$

By the Abel functional equation for Φ_n, we have

$$\zeta' = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta) = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\Phi_n(z) + \ell) = \Phi_n \circ f_n^{\circ k_n + \ell}(z).$$

Letting $\ell_z := k_n + \ell$, we have

$$1 \leq \ell_z \leq k_n + \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1,\ f_n^{\circ \ell_z}(z) = \Phi_n^{-1}(\zeta') \in \mathcal{C}_n \cup \mathcal{C}_n^\sharp,$$
$$\mathbb{E}\mathrm{xp} \circ \Phi_n(f_n^{\circ \ell_z}(z)) = \mathbb{E}\mathrm{xp} \circ \Phi_n(\Phi_n^{-1}(\zeta')) = \mathbb{E}\mathrm{xp}(\zeta') = f_{n+1}(w).$$

This proves the first two parts.

For the last part, first note that by the assumption on w, $\operatorname{Im} \Phi_n(z) > -2$. Now, if $\ell > 0$, then

$$z, f_n(z), \dots, f_n^{\circ(\ell-1)}(z) \in \bigcup_{i=k_n-1}^{k_n + \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2} f_n^{\circ i}(S_n^0),\ f_n^{\circ \ell}(z), \dots, f_n^{\circ \ell_z}(z) \in \bigcup_{i=0}^{k_n} f_n^{\circ i}(S_n^0).$$

If $\ell \leq 0$, then

$$z, f_n(z), \dots, f_n^{\circ \ell_z}(z) \in \bigcup_{i=-\ell}^{k_n} f_n^{\circ i}(S_n^0).$$

□

Define

$$\mathcal{P}_n' := \{w \in \mathcal{P}_n \mid 0 < \operatorname{Re} \Phi_n(w) < \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1\}.$$

Lemma 3.3. — *For every $n \geq 1$ we have*

(1) for every $w \in \mathcal{P}_n'$, $f_{n-1}^{\circ \lfloor 1/\alpha_{n-1} \rfloor} \circ \psi_n(w) = \psi_n \circ f_n(w)$,

(2) for every $w \in S_n^0$, $f_{n-1}^{\circ (k_n \lfloor 1/\alpha_{n-1} \rfloor + 1)} \circ \psi_n(w) = \psi_n \circ f_n^{\circ k_n}(w)$.

This is summarized in the following two diagrams

$$\begin{array}{ccc} \mathcal{P}_{n-1} & \xrightarrow{f_{n-1}^{\circ \lfloor 1/\alpha_{n-1} \rfloor}} & \mathcal{P}_{n-1} \\ {\scriptstyle \psi_n}\uparrow & & \uparrow{\scriptstyle \psi_n} \\ \mathcal{P}_n' & \xrightarrow{f_n} & \mathcal{P}_n \end{array} \qquad \begin{array}{ccc} \mathcal{P}_{n-1} & \xrightarrow{f_{n-1}^{\circ k_n \lfloor 1/\alpha_{n-1} \rfloor + 1}} & \mathcal{P}_{n-1} \\ {\scriptstyle \psi_n}\uparrow & & \uparrow{\scriptstyle \psi_n} \\ S_n^0 & \xrightarrow{f_n^{\circ k_n}} & \mathcal{C}_n \cup \mathcal{C}_n^\sharp \end{array}$$

Proof. — Part (1): The proof is given in three steps.

Step 1: For every $w \in \mathcal{P}_n'$ there exists a positive integer m_w with

$$f_{n-1}^{\circ m_w} \circ \psi_n(w) = \psi_n \circ f_n(w).$$

By the definition of renormalization $\mathcal{R}f_{n-1} = f_n$, there are $\zeta \in \Phi_{n-1}(S^0_{n-1})$ and $\zeta' \in \Phi_{n-1}(\mathcal{C}_{n-1} \cup \mathcal{C}^\sharp_{n-1})$ as well as integers t_1 and t_2 with

$$\zeta' = \Phi_{n-1} \circ f_{n-1}^{\circ k_{n-1}} \circ \Phi_{n-1}^{-1}(\zeta), \zeta = \eta_n(w) + t_1, \zeta' = \eta_n(f_n(w)) + t_2$$
$$|t_i| \leq \lfloor 1/\alpha_{n-1} \rfloor - \boldsymbol{k}, \text{ for } i = 1, 2.$$

This implies that

$$\eta_n(f_n(w)) = \Phi_{n-1} \circ f_{n-1}^{\circ(k_{n-1}+t_1-t_2)} \circ \Phi_{n-1}^{-1}(\eta_n(w)).$$

Hence, $f_{n-1}^{\circ m_w} \circ \psi_n(w) = \psi_n \circ f_n(w)$, for $m_w = k_{n-1} + t_1 - t_2$.

Step 2: m_w is a constant independent of $w \in \mathcal{P}'_n$. We use the connectivity of $\mathcal{P}'_n$. For $j \in A := \{1, 2, \ldots, k_{n-1} + 2(\lfloor 1/\alpha_{n-1} \rfloor - \boldsymbol{k})\}$ set

$$X_j := \{w \in \mathcal{P}'_n \mid f_{n-1}^{\circ j}(\psi_n(w)) \text{ is defined and } f_{n-1}^{\circ j} \circ \psi_n(w) - \psi_n \circ f_n(w) = 0\}.$$

It follows from Step 1 that $\mathcal{P}'_n = \cup_{j \in A} X_j$. Let m be the smallest element of A such that $\operatorname{int}(X_m)$ is non-empty. We claim that $S := \cup_{j \in A, j \geq m} X_j$ is connected. Otherwise, $\mathcal{P}'_n \setminus S$ is an uncountable set contained in $\cup_{j=1}^{m-1} X_j$. This implies that at least one of $X_1, X_2, \ldots, X_{m-1}$, say X_i, is uncountable, and hence has an accumulation point in itself. As the set of points where $f_{n-1}^{\circ i}(\psi_n(w))$ is defined is open, and $f_{n-1}^{\circ i} \circ \psi_n - \psi_n \circ f_n$ is anti-holomorphic, $\operatorname{int}(X_i)$ must be non-empty. Therefore, S must be connected.

The anti-holomorphic map $f_{n-1}^{\circ m} \circ \psi_n - \psi_n \circ f_n$ is defined on the connected set S and is equal to 0 on an open subset of S. Hence, it must be 0 on all of S. Finally, since $\mathcal{P}'_n \setminus S$ is discrete, the equality holds on all of $\mathcal{P}'_n$.

Step 3: $m_w = \lfloor 1/\alpha_{n-1} \rfloor$.

By virtue of Step 2, it is enough to find the asymptotic value of m_w as $w \in \mathcal{P}'_n$ tends to 0. To prove this, we claim that for all continuous branches of arguments defined on $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$,

$$\arg(\psi_n(w_2)/\psi_n(w_1)) + \alpha_{n-1} \arg(w_2/w_1) \to 0 \pmod{2\pi},$$

as w_1 and w_2 tend to 0 within $\mathcal{P}_n$. Let us assume this for a moment. First note that since $f'_n(0) = e^{2\pi\alpha_n \mathbf{i}}$, when $w \to 0$ within $\mathcal{P}'_n$, we must have $\arg(f_n(w)/w) \to 2\pi\alpha_n \pmod{2\pi}$. Then, by the above statement, we must have $\arg(\psi_n(f_n(w))/\psi_n(w)) \to -2\pi\alpha_n\alpha_{n-1} \pmod{2\pi}$. However, since α_{n-1} is an irrational number, $\lfloor 1/\alpha_{n-1} \rfloor$ is the unique positive integer j for which $\arg(f_{n-1}^{\circ j}(w')/w') \to -2\pi\alpha_n\alpha_{n-1} \pmod{2\pi}$, as $w' \to 0$.

The proof of the above claim requires some features of the Fatou coordinate Φ_{n-1}, which are proved in Section 6. (The proofs in that section are independent of the rest of the paper, and in particular, do not depend on this proposition.) Below, we refer the reader to the statements needed to prove the above claim.

Th map Φ_{n-1}^{-1} is decomposed as $\Phi_{n-1}^{-1} = \tau_{n-1} \circ L_{n-1}^{-1}$, where L_{n-1}^{-1} is a univalent map defined on $\Phi_{n-1}(\mathcal{P}_{n-1})$, and τ_{n-1} is an explicit formula defined on $L_{n-1}^{-1}(\Phi_{n-1}(\mathcal{P}_{n-1}))$. See Equations (33) and (38). It is proved that L_{n-1}^{-1} converges to a translation by a constant near $+\mathbf{i}\infty$, see Equation (53). The covering formula, τ_{n-1} maps every straight vertical line going to $+\mathbf{i}\infty$ to a smooth curve landing at 0 at a well-defined angle. To show that these imply the above claim, let $w_1 \in \mathcal{P}_n$, $w_2 \in \mathcal{P}_n$, and consider the line segments $l_1 = w_1 \cdot (0, 1)$ and $l_2 = w_2 \cdot (0, 1)$ which

connect these two points to 0. The rays l_1 and l_2 are lifted through the covering map $\mathbb{E}\mathrm{xp}$ to two vertical lines whose real parts differ from each other by $|\arg(w_2/w_1)|/(2\pi)$. Then, L_{n-1}^{-1} maps these two vertical lines to two smooth curves which tend to $+\mathbf{i}\infty$, and the difference between their real parts tends to $|\arg(w_2/w_1)|/(2\pi)$. Then, the covering map τ_{n-1} maps these two curves to the two smooth curves $\psi_n(l_1)$ and $\psi_n(l_2)$ which land at 0 at well-defined angles. Moreover, the asymptotic angle between $\psi_n(l_1)$ and $\psi_n(l_2)$ at 0 is equal to $\alpha_{n-1}|\arg(w_2/w_1)|$, as evidences from the explicit formula for τ_{n-1}. Note that there is a change of signs due to the map $\mathbb{E}\mathrm{xp}$ being anti-holomorphic. One may refer to Section 6.4 for more details on the behavior of L_{n-1}.

Part (2): The above steps work to prove this part as well. In step 1, one needs to use Lemma 3.2 k_n times. In step 2, one only replaces $\mathcal{P}'_n$ by S^0_n, and uses connectivity of S^0_n. For the last step, one has $\arg(w/f_n^{\circ k_n}(w)) \to 2\pi(1-k_n\alpha_n) \pmod{2\pi}$, as $w \to 0$ in S^0_n. As in the previous case, $\arg(\psi_n(w)/\psi_n(f_n^{\circ k_n}(w))) \to -2\pi(1-k_n\alpha_n)\alpha_{n-1} \pmod{2\pi}$. This uniquely determines the number of iterates of f_{n-1} required to map $\psi_n(w)$ to $\psi_n(f_n^{\circ k_n}(w))$. □

Lemma 3.4. — *For every $n \geq 1$ we have*

(1) for every $w \in \mathcal{P}'_n$, $f_0^{\circ q_n} \circ \Psi_n(w) = \Psi_n \circ f_n(w)$,

(2) for every $w \in S^0_n$, $f_0^{\circ(k_n q_n + q_{n-1})} \circ \Psi_n(w) = \Psi_n \circ f_n^{\circ k_n}(w)$,

(3) similarly, for every $m < n$, $f_n \colon \mathcal{P}'_n \to \mathcal{P}_n$ and $f_n^{\circ k_n} : S^0_n \to (\mathcal{C}_n \cup \mathcal{C}_n^\sharp)$ are conjugate to some iterates of f_m on the set $\psi_{m+1} \circ \cdots \circ \psi_n(\mathcal{P}_n)$.

Parts (1) and (2) of the lemma are illustrated in the following diagrams

$$\begin{array}{ccc} \mathcal{P}_0 & \xrightarrow{f_0^{\circ q_n}} & \mathcal{P}_0 \\ {\scriptstyle\Psi_n}\uparrow & & \uparrow{\scriptstyle\Psi_n} \\ \mathcal{P}'_n & \xrightarrow{f_n} & \mathcal{P}_n \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{P}_0 & \xrightarrow{f_0^{\circ(k_n q_n + q_{n-1})}} & \mathcal{P}_0 \\ {\scriptstyle\Psi_n}\uparrow & & \uparrow{\scriptstyle\Psi_n} \\ S^0_n & \xrightarrow{f_n^{\circ k_n}} & \mathcal{C}_n \cup \mathcal{C}_n^\sharp \end{array}$$

Proof. — We give a proof for the first part in three steps. The other parts can be proved by the same arguments.

Step 1: For every $w \in \mathcal{P}'_n$ there exists a positive integer m_w with

$$f_0^{\circ m_w} \circ \Psi_n(w) = \Psi_n \circ f_n(w).$$

By Lemma 3.3, $\psi_n(w)$ is mapped to $\psi_n(f_n(w))$ under the iterate $f_{n-1}^{\circ\lfloor 1/\alpha_{n-1}\rfloor}$. The orbit

$$\psi_n(w), f_{n-1}(\psi_n(w)), \ldots, f_{n-1}^{\circ\lfloor 1/\alpha_{n-1}\rfloor}(\psi_n(w)) = \psi_n(f_n(w))$$

has a subset of the form

$$\psi_n(w), f_{n-1}(\psi_n(w)), \ldots, f_{n-1}^{\circ j}(\psi_n(w)) \\ , f_{n-1}^{\circ(j+k_{n-1})}(\psi_n(w)), f_{n-1}^{\circ(j+k_{n-1}+1)}(\psi_n(w)), \ldots, f_{n-1}^{\circ\lfloor 1/\alpha_{n-1}\rfloor}(\psi_n(w))$$

contained in $\mathcal{P}_{n-1}$, where $f_{n-1}^{\circ j}(\psi_n(w)) \in S^0_{n-1}$. Using Lemma 3.3 (with $n-1$) for each consecutive pair in the above list, one concludes that $\psi_{n-1}(\psi_n(w))$ is mapped to $\psi_{n-1}(\psi_n(f_n(w)))$

under some iterate of f_{n-2}. By an inverse inductive argument (at levels $n-2, n-3, \dots, 1$), one concludes the claim.

Step 2: m_w is a constant independent of $w \in \mathcal{P}'_n$.
The proof in Step 2 of the previous lemma works here as well. Indeed, as $f_0^{\circ j} \circ \Psi_n(w)$ is defined for all positive integers j and $w \in \mathcal{P}'_n$, the proof is slightly easier here.

Step 3: $m_w = q_n$.
Similar to the proof in the previous lemma, we use the property that for every j and $w_1, w_2 \in \mathcal{P}_j$, $\arg(\psi_j(w_2)/\psi_j(w_1)) + \alpha_{j-1} \arg(w_2/w_1) \to 0 \pmod{2\pi}$ as $w_1, w_2 \to 0$. This will be proved in Section 6 (see Equation (53)). Now, as $w \in \mathcal{P}'_n$ tends to 0, $\arg(f_n(w)/w)$ tends to $2\pi\alpha_n \pmod{2\pi}$. Hence,

$$\arg(\Psi_n(f_n(w))/\Psi_n(w)) \to (-1)^n 2\pi\alpha_0 \cdots \alpha_n \pmod{2\pi}.$$

On the other hand, q_n is the unique positive integer for which $\arg(f_0^{\circ q_n}(w')/w') \to (-1)^n 2\pi\alpha_0 \cdots \alpha_n \pmod{2\pi}$, as $w' \to 0$. □

3.3. A nest of neighborhoods of the post-critical set. — For $n \geq 0$, define the positive integers

$$b_n := k_n + \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2,$$

and consider the union

$$\Omega_n^0 := \cup_{i=0}^{b_n} f_n^{\circ i}(S_n^0) \cup \{0\}. \tag{9}$$

Using Lemma 3.4, we transfer the iterates in the above union to the dynamic plane of f_0 to obtain

$$\Omega_0^n := \cup_{i=0}^{q_n b_n + q_{n-1}} f_0^{\circ i}(S_0^n) \cup \big\{0\big\}.$$

The upper bound in the above union is obtained as follows. The first k_n iterates in (9) corresponds to $k_n q_n + q_{n-1}$ iterates on level 0 by Lemma 3.4-2. The remaining $\lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2$ iterates in (9) amounts to $q_n(\lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2)$ iterates by Lemma 3.4-1. The neighborhoods Ω_n^i, for $i \geq 1$, may be defined accordingly. Using Lemma 3.4, first choose the unique integer $l_{n,i}$ such that $f_{n+i}^{\circ k_{n+i} + \lfloor 1/\alpha_{n+i} \rfloor - \boldsymbol{k} - 2}$ on S_{n+i}^0 corresponds to $f_n^{\circ l_{n,i}}$ on S_n^i. Then, define

$$\Omega_n^i := \cup_{j=0}^{l_{n,i}} f_n^{\circ j}(S_n^i) \cup \{0\}.$$

See Figure 5.

Proposition 3.5. — *For every $f_0 \in \mathcal{QIS}_\alpha$, with $\alpha \in \mathrm{HT}_N$, and every $n \geq 0$,*

(1) Ω_0^{n+1} is compactly contained in the interior of Ω_0^n;
(2) $f_0 : \Omega_0^{n+1} \to \Omega_0^n$;
(3) $\mathcal{PC}(f_0)$ is contained in the interior of Ω_0^n. [4]

4. For further properties of this nest see Proposition 5.10.

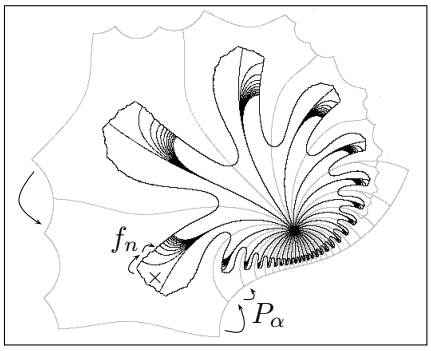


FIGURE 5. In the figure, Ω^0 is the union of the sectors with gray boundaries. The domain Ω^n is bounded by the black curve (amoeba) and some of the sectors in Ω^n are shown. The map f_n corresponds to the n-th renormalization of P_α. The critical point of P_α is denoted by "$\times$" here.

Proof. — Part (1): First we prove that $\Omega_0^{n+1} \subset \Omega_0^n$. To do this, it is enough to show that for every $z \in S_0^{n+1}$, there are a positive integer m, points $z_i \in S_0^n$, for $1 \leq i \leq m$, and non-negative integers t_i, for $0 \leq i \leq m$, satisfying the following properties:

(a) $f_0^{\circ t_0}(z_1) = z$,
(b) $f_0^{\circ t_j}(z_j) = z_{j+1}$, for all $j = 1, 2, \ldots, m-1$,
(c) $f_0^{\circ t_m}(z_m) = f_0^{\circ q_{n+1}b_{n+1}+q_n}(z)$,
(d) $t_j \leq q_n b_n + q_{n-1}$, for all $j = 0, 1, \ldots, m$.

To prove the above statement, fix an arbitrary $z \in S_0^{n+1}$. We define $m := b_{n+1}$. Let $\zeta := \Psi_{n+1}^{-1}(z) \in S_{n+1}^0$. Note that $f_{n+1}^{\circ m}(\zeta)$ is define. Also, $\Psi_n^{-1}(z)$ is defined and belongs to $\mathcal{P}_n$. First we show that there are points $\sigma_i \in S_n^0$, for $1 \leq i \leq m$, as well as positive integers ℓ_i, for $0 \leq i \leq m$, such that

(a') $f_n^{\circ \ell_0}(\sigma_1) = \Psi_n^{-1}(z)$,
(b') $f_n^{\circ \ell_j}(\sigma_j) = \sigma_{j+1}$, for all $j = 1, 2, \ldots, m-1$,
(c') $f_n^{\circ \ell_m}(\sigma_m) = \psi_{n+1}(f_{n+1}^{\circ m}(\zeta))$,
(d') $k_n \leq \ell_j \leq b_n$, for all $j = 0, 1, \ldots, m$.

By the definition of S_{n+1}^0, the iterates

$$\zeta, f_{n+1}(\zeta), f_{n+1}^{\circ 2}(\zeta), \ldots, f_{n+1}^{\circ m}(\zeta)$$

are defined and belong to $U_{n+1} \cap f_{n+1}(U_{n+1})$. We use Lemma 3.2 for each consecutive pair in the above orbit (in place of w and $f_n(w)$ in that lemma) to inductively introduce the sequence $\sigma_1, \sigma_2, \ldots, \sigma_m$ and $\ell_0, \ell_1, \ldots, \ell_m$ as follows.

Lemma 3.2 applied to $\xi_1 := \psi_{n+1}(\zeta)$ produces $\xi_2 \in \mathcal{P}_n$ and a positive $\ell \in \mathbb{Z}$ with

$$\mathbb{E}\exp \circ \Phi_n(\xi_2) = f_{n+1}(\zeta), \text{ and } f_n^{\circ \ell}(\xi_1) = \xi_2.$$

Now, there are $\sigma_1 \in S_n^0$ and $\ell_0 \in \mathbb{Z}$ with $k_n \leq \ell_0 \leq b_n$ such that $f_n^{\circ \ell_0}(\sigma_1) = \xi_1 = \Psi_n^{-1}(z)$.

As $\mathbb{E}\mathrm{xp}\circ\Phi_n(\xi_1) \in \mathrm{Dom}\ f_{n+1}$, we can choose a $\sigma_2 \in S_n^0$ with $\Phi_n(\xi_1) - \Phi_n(\sigma_2) \in \mathbb{Z}$. Let ℓ_1 denote the positive integer with $f_n^{\circ\ell_1}(\sigma_1) = \sigma_2$. The integer ℓ_1 satisfies (d'). That is because to go from σ_1 to σ_2, one needs at least k_n iterates to go from S_n^0 to $\mathcal{C}_n \cup \mathcal{C}_n^\sharp$ and then at most $\lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 2$ iterates to reach σ_2.

Repeating the above paragraph with σ_2 which satisfies $\mathbb{E}\mathrm{xp}\circ\Phi_n(\sigma_2) = f_{n+1}(\zeta)$, one obtains $\sigma_3 \in S_n^0$ and an integer ℓ_2 with $k_n \le \ell_2 \le b_n + 1$, such that $f_n^{\circ\ell_2}(\sigma_2) = \sigma_3$ and $\mathbb{E}\mathrm{xp}\circ\Phi_n(\sigma_3) = f_{n+1}^{\circ 2}(\zeta)$.

Repeating the above argument inductively, one obtains the sequence of pairs (σ_4, ℓ_3), (σ_5, ℓ_4), $\ldots$, (σ_{m+1}, ℓ_m) such that

$$\mathbb{E}\mathrm{xp}\circ\Phi_n(\sigma_{j+1}) = f_{n+1}^{\circ j}(\zeta),\ f_n^{\circ\ell_j}(\sigma_j) = \sigma_{j+1},\ \text{for } j = 3, 4, \ldots, m.$$

Finally, change ℓ_m to the positive integer ℓ with $f_n^{\circ\ell}(\sigma_m) = \psi_{n+1}(f_{n+1}^{\circ m}(\zeta))$. This introduces the points σ_j and integers ℓ_j satisfying (a')–(d').

Now define $z_j := \Psi_n(\sigma_j) \in S_0^n$, for $j = 1, 2, \ldots, m$. One can see that (a')–(d') implies (a)–(d), using Lemma 3.4. For example, we prove (a), (c), and the inequality for t_0 in (d).

$$\begin{aligned}
z &= \Psi_n(\Psi_n^{-1}(z)) \\
&= \Psi_n(f_n^{\circ\ell_0}(\sigma_1)) && \text{(by (a'))} \\
&= \Psi_n(f_n^{\circ(\ell_0-k_n)} \circ f_n^{\circ k_n}(\sigma_1)) \\
&= f_0^{\circ(\ell_0-k_n)q_n} \circ \Psi_n(f_n^{\circ k_n}(\sigma_1)) && \text{(by Lemma 3.4-1)} \\
&= f_0^{\circ(\ell_0-k_n)q_n} \circ f_0^{\circ(k_nq_n+q_{n-1})}(\Psi_n(\sigma_1)) && \text{(by Lemma 3.4-2)} \\
&= f_0^{\circ(\ell_0q_n+q_{n-1})}(z_1),
\end{aligned}$$

Let $t_0 := \ell_0 q_n + q_{n-1}$, and note that as $\ell_0 \le b_n$, t_0 satisfies the inequality in (d).

Similarly, (c) follows from the following equalities.

$$\begin{aligned}
\Psi_n(\psi_{n+1}(f_{n+1}^{\circ m}(\zeta))) & \\
&= \Psi_{n+1}(f_{n+1}^{\circ m}(\zeta)) \\
&= \Psi_{n+1}(f_{n+1}^{\circ(m-k_{n+1})} \circ f_{n+1}^{\circ k_{n+1}}(\zeta)) \\
&= f_0^{\circ(m-k_{n+1})q_{n+1}} \circ \Psi_{n+1}(f_{n+1}^{\circ k_{n+1}}(\zeta)) && \text{(by Lemma 3.4-1)} \\
&= f_0^{\circ(m-k_{n+1})q_{n+1}} \circ f_0^{\circ k_{n+1}q_{n+1}+q_n}(\Psi_{n+1}(\zeta)) && \text{(by Lemma 3.4-2)} \\
&= f_0^{\circ q_{n+1}b_{n+1}+q_n}(z),
\end{aligned}$$

and

$$\begin{aligned}\Psi_n(f_n^{\circ \ell_m}(\sigma_m)) &= \Psi_n(f_n^{\circ \ell_m - k_n} \circ f_n^{\circ k_n}(\sigma_m)) \\ &= f_0^{\circ(\ell_m - k_n)q_n} \circ \Psi_n(f_n^{\circ k_n}(\sigma_m)) && \text{(by Lemma 3.4-1)} \\ &= f_0^{\circ(\ell_m - k_n)q_n} \circ f_0^{\circ k_n q_n + q_{n-1}}(z_m) && \text{(by Lemma 3.4-2)} \\ &= f_0^{\circ t_m}(z_m). && \text{(with } t_m := \ell_m q_n + q_{n-1})\end{aligned}$$

It remains to show that $\partial\Omega_0^{n+1} \subset \text{int}\,(\Omega_0^n)$. First we claim that for all $n \geq 1$, $0 \in \text{int}\ \Omega_0^n$.

By the definition of the sectors, for every $n \geq 0$ there is $\varepsilon_n > 0$ such that for every $x_n \in B(0, \varepsilon_n)$, there is $x_n' \in S_n^0$, and a non-negative integer $s_n \leq b_n - 1$ with $f_n^{\circ s_n}(x_n') = x_n$. In particular, $B(0, \varepsilon_n) \subset \Omega_n^0$, $\forall n \geq 0$. Fix $n \geq 1$. For x_0 sufficiently close to zero we may obtain a sequence of points $x_j \in B(0, \varepsilon_j)$, $x_j' \in S_j^0$, non-negative integers $s_j \leq b_j - 1$ such that $f_j^{\circ s_j}(x_j') = x_j$, $\mathbb{E}\text{xp} \circ \Phi_j(x_j') = x_{j+1}$, for all $j = 0, 1, \ldots, n-1$. Now, by the definition of renormalization, $\Psi_n(x_n') \in S_0^n$ is mapped to x_0 under some iterate of f_0. To bound the number of iterates needed, let $N(s_0, s_1, \ldots, s_n)$ denote the resulting number of iterates of f_0 for given $s_0, s_1, \ldots, s_n$. By the upper bound on each s_j, we have

$$\begin{aligned}N(s_0, s_1, \ldots, s_n) \leq N(0, s_1 + 1, s_2, s_3, \ldots, s_n) \leq N(0, 0, s_2 + 1, s_3, s_4, \ldots, s_n) \\ \leq \cdots \leq N(0, 0, \ldots, 0, s_n + 1) = q_n b_n + q_{n-1}\end{aligned}$$

This implies that $x_0 \in \Omega_0^n$ and hence, finishes the proof of the claim.

Let $z' \neq 0$ belong to $\partial\Omega_0^{n+1}$. To show that $z' \in \text{int}\ \Omega_0^n$ we continue to use the notations of the earlier arguments. There exists $z \neq 0$ in ∂S_0^{n+1} with $f_0^{\circ t}(z) = z'$, for some non-negative $t \in \mathbb{Z}$. Hence, $\zeta = \Psi_{n+1}^{-1}(z)$ belongs to ∂S_{n+1}^0. On the other hand, the closure of S_{n+1}^0 is contained in $U_{n+1} \cap f_{n+1}(U_{n+1})$. But, for the point $\xi_1 = \psi_{n+1}(\zeta)$, σ_1 may belong to the boundary of S_n^0 (i.e. $\xi_1 \notin \text{int}\ S_n^0$). To rectify the problem, we slightly "thicken" the set S_n^0 on the left side. That is, there is an open set $\hat{S}_n^0$ such that, the closure of $\hat{S}_n^0$ intersects S_n^0, $f_n(\hat{S}_n^0) \subset \text{int}\ S_n^0$, $\mathbb{E}\text{xp} \circ \Phi_n(\hat{S}_n^0) \subset f_{n+1}(U_{n+1})$, and $\sigma_j \in \text{int}\,(\hat{S}_n^0 \cup S_n^0)$, for $j = 1, 2, \ldots, m$. Now, one uses the open mapping property of holomorphic and anti-holomorphic maps to see that $z_i \in \text{int}\,(S_0^n \cup \Psi_n(\hat{S}_n^0))$, for all i. Note that since $\hat{S}_n^0 \subset \Omega_n^0$ and $f_n(\hat{S}_n^0) \subset \text{int}\ S_n^0$, $f_0^{\circ j}(\Psi_n(\hat{S}_n^0))$ is defined and contained in $\text{int}\,(\Omega_0^n)$ for all j with $0 \leq j \leq q_n b_n + q_{n-1}$. By the open mapping property of f_0, this implies that those forward iterates of z_i are contained in $\text{int}\,(\Omega_0^n)$.

Part (2): Clearly, $f_0(0) = 0 \in \Omega_0^n$. Let z be an arbitrary point in $\Omega_0^{n+1} \setminus \{0\}$. By the previous part, $z \in \Omega_0^n$. If $z \in \Omega_0^n$ is not in the last sector $f_0^{\circ q_n b_n + q_{n-1}}(S_0^n)$, then $f_0(z)$ is defined and belongs to Ω_0^n, by definition.

Assume that $z \neq 0$ belongs to the last sector of the union Ω_0^n. By Lemma 3.4, the last sector of the union Ω_0^n is the image of the last sector in the union Ω_n^0 under the map Ψ_n. That is, $\Psi_n^{-1}(z)$ is defined and belongs to $f_n^{\circ b_n}(S_n^0) \subset \mathcal{P}_n$. On the other hand, we claim that $z \in \Omega_0^{n+1} \cap \Psi_n(\mathcal{P}_n)$ implies $\mathbb{E}\text{xp} \circ \Phi_n \circ \Psi_n^{-1}(z) \in \Omega_{n+1}^0 \subseteq \text{Dom}\,(f_{n+1})$. Assuming the claim for a moment, combining the two statements, we have $\Psi_n^{-1}(z) \in \cup_{l=0}^{k_n - 1} f_n^{\circ l}(S_n^0)$. By Lemma 3.4, this

implies that $z \in \cup_{l=0}^{k_n-1} f_0^{\circ(lq_n)}(S_0^n)$. Then, $f_0(z)$ is defined and belongs to Ω_0^n, by the definition of Ω_0^n.

Now we prove the claim. Recall the domain Ω_n^1. By Lemma 3.4, the iterates in the union Ω_0^{n+1} are obtained from the iterates of S_n^1 on level n to form Ω_n^1. In particular, the iterates within $\Omega_0^{n+1} \cap \Psi_n(\mathcal{P}_n)$ are obtained from the iterates of S_n^1 that lie in $\mathcal{P}_n$. Recall that by our choice of the branch of ψ_{n+1} in Section 3.1, S_n^1 is to the left of the last sector in Ω_n^0. Therefore, if $z \in \Omega_0^{n+1} \cap \Psi_n(\mathcal{P}_n)$ and also z belongs to the last sector in Ω_0^n, then

$$\Psi_n^{-1}(z) \in \bigcup_{l=0}^{b_{n+1}} \bigcup_{j=0}^{\lfloor 1/\alpha_n \rfloor - \boldsymbol{k}-2} f_n^{\circ \lfloor 1/\alpha_n \rfloor l+j}(S_n^1) \cap \mathcal{P}_n.$$

The set S_n^1, and all its consecutive iterates by f_n which lie on $\mathcal{P}_n$, project under $\mathbb{E}\mathrm{xp} \circ \Phi_n$ to the set S_{n+1}^0. By the definition of renormalization, see also Lemma 3.2, the iterate of S_n^1 that has returned to $\mathcal{P}_n$ after leaving $\mathcal{P}_n$, projects under $\mathbb{E}\mathrm{xp} \circ \Phi_n$ to the set $f_{n+1}(S_{n+1}^0)$. Repeating this argument, one concludes that the above union projects under $\mathbb{E}\mathrm{xp} \circ \Phi_n$ to the set Ω_{n+1}^0.

Part (3): Recall that for every $n \geq 1$, $f_n \colon S_n^0 \to f_n^{\circ k_n}(S_n^0)$ has a critical point. Thus, by Lemma 3.4-2, $f_0^{\circ(k_n q_n + q_{n-1})} \colon S_0^n \to \Psi_n(f_n^{\circ k_n}(S_n^0))$ must also have a critical point. Therefore, the critical point of f_0 belongs to Ω_0^n, for $n \geq 1$. On the other hand, by Part 2, f_0 can be iterated infinitely many times on $\cap_{n\geq 1}\Omega_0^n$, with values in this intersection. Now, the result follows from Part 1. □

By a lemma of Lyubich [**Lyu83b**], for a rational map $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, with $J(f) \neq \hat{\mathbb{C}}$, and any open set V containing the closure of the orbits of the critical values of f, the orbit of Lebesgue almost every $z \in J(f)$ eventually stays in V. Combined with Proposition 3.5, the orbit of almost every point in the Julia set of Q_α, $\alpha \in \mathrm{HT}_N$, eventually stays in every Ω_0^n.

Proposition 3.6. — *For every $\alpha \in \mathrm{HT}_N$ and every $f_0 \in \mathcal{QIS}_\alpha$ we have the following.*

(1) When $f_0 = Q_\alpha$, for every $n \geq 0$, every integer ℓ with $0 \leq \ell \leq q_n b_n + q_{n-1}$, and almost every $z \in J(Q_\alpha)$, $\mathcal{O}(z) \cap Q_\alpha^{\circ \ell}(S_0^n) \neq \emptyset$.

(2) For every $n \geq 0$, every integer ℓ with $0 \leq \ell \leq q_n b_n + q_{n-1}$, and every $z \in \mathcal{PC}(f_0)$, $\mathcal{O}(z) \cap f_0^{\circ \ell}(S_0^n) \neq \emptyset$.

Proof. — Evidently, it is enough to prove both statements for $\ell = 0$.

First we show that for every $n \geq 0$,

$$\left\{ z \in \mathbb{C} \mid \mathcal{O}(z) \cap \Omega_0^{n+2} \neq \emptyset \right\} \subseteq \left\{ z \in \mathbb{C} \mid \mathcal{O}(z) \cap (S_0^n \cup \{0\}) \neq \emptyset \right\}.$$

To see this, first note that if z is mapped to 0 by some iterate of f_0, we have $\mathcal{O}(z) \cap (S_0^n \cup \{0\}) \neq \emptyset$ and we are done. Below, we assume that the orbit of z does not land at 0.

Fix an arbitrary $z \in \mathbb{C}$ and an integer $t_1 \geq 0$ with $f_0^{\circ t_1}(z) \in \Omega_0^{n+2}$. Choose $t_2 \geq t_1$ with $f_0^{\circ t_2}(z)$ in the last sector $f_0^{\circ j}(S_0^{n+2})$, with $j = q_{n+2} b_{n+2} + q_{n+1}$. The last sector of Ω_{n+2}^0 is contained in $\mathcal{P}_{n+2}$, and is mapped to the sector $f_0^{\circ j}(S_0^{n+2})$. Thus, the point $\zeta := \Psi_{n+2}^{-1}(f_0^{\circ t_2}(z))$ is defined and belongs to $\mathcal{P}_{n+2}$. This implies that $f_{n+2}(\zeta)$ is defined. By Lemma 3.2, this implies that $\zeta' := \psi_{n+2}(\zeta)$ can be iterated at least two times under f_{n+1}. That is, ζ' and $f_{n+1}(\zeta')$

belong to U_{n+1}. Now, Lemma 3.2 applied to $\zeta'' := \psi_{n+1}(\zeta')$ implies that there is an orbit $\zeta'', f_n(\zeta''), \dots, f_n^{\circ\ell}(\zeta'')$, with $\mathbb{E}\mathrm{xp} \circ \Phi_n(f_n^{\circ\ell}(\zeta'')) = f_{n+1}(\zeta') \in U_{n+1}$. This implies that there exists a positive integer ℓ' with $f_n^{\circ\ell'}(\zeta'') \in S_n^0$. Now, using Lemma 3.4, $f_0^{\circ\ell''}(\Psi_n(\zeta'')) = f_0^{\circ\ell''}(z) \in S_0^n$, for some positive integer ℓ''. This finishes the proof of the above inclusion.

By Proposition 3.5, and the paragraph preceding Proposition 3.6, the orbit of almost every point in $J(f_0)$ visits Ω_0^{n+2}. Then, from the above equation we conclude part (1) of the proposition for $\ell = 0$.

By Proposition 3.5, $\mathcal{PC}(f_0)$ is contained in Ω_0^{n+2}, for every $n \geq 0$. Thus, from the above equation we also conclude part (2) of the proposition. (Indeed, the proof of Part 2 of the proposition is already present in the proof of Proposition 3.5.) □

4. Upper bound on the sizes of linearization domains

4.1. Approaches of the critical orbit to the fixed point. — In this section, we estimate the size of a sector (roughly the smallest one) in each union Ω_0^n in terms of a partial sum of the Brjuno series introduced in the Introduction. The main technical tool is stated in the next two propositions. They will be proved in Section 6.7, once we establish some estimates on the Fatou coordinates.

Let f be a map in $\mathcal{QIS}_\alpha$, with $\alpha \in (0, r_3]$. Recall the domain $\mathcal{P}_f$ defined in Proposition 2.3, the constant $\boldsymbol{k}$ in Proposition 2.4, as well as the sector S_f and the constant k_f defined in Section 2.3. Moreover, if the rotation of $\mathcal{R}(f)$ at 0 belongs to $(0, r_3]$, then $\mathcal{P}_{\mathcal{R}(f)}$, $\Phi_{\mathcal{R}(f)}$, and $\psi_{\mathcal{R}(f)}$ are also defined, where $\psi_{\mathcal{R}(f)} : \mathcal{P}_{\mathcal{R}(f)} \to \mathcal{P}_f$ is the change of coordinate defined in Section 3.1.

Proposition 4.1. — *There is $M_1 \geq 1$ such that for all $\alpha \in (0, r_3]$ and all $f \in \mathcal{QIS}_\alpha$, there exists $\eta(f)$ in the set $\{k_f, k_f + 1, \dots, \lfloor 1/(2\alpha) \rfloor + k_f\}$ such that*

$$\operatorname{diam}\left(f^{\circ\eta(f)}(S_f)\right) \leq M_1\alpha, \text{ and } f^{\circ\eta(f)}(S_f) \subseteq \mathcal{P}_f.$$

Recall the constant N defined at the end of Section 2.2. That is, $\alpha \in \mathrm{HT}_N$ guarantees that every $h \in \mathcal{QIS}_\alpha$ is infinitely near-parabolic renormalizable.

Proposition 4.2. — *There is $M_2 \geq 1$ such that for all $\alpha \in \mathrm{HT}_N$ and all $f \in \mathcal{QIS}_\alpha$, there exists $\kappa(f)$ in the set $\{0, 1, \dots, \lfloor 1/(2\alpha) \rfloor\}$ such that*

(1) $f^{\circ\kappa(f)} \circ \psi_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) \subseteq \mathcal{P}_f$,
(2) $\forall w \in \mathcal{P}_{\mathcal{R}(f)}$, $|f^{\circ\kappa(f)} \circ \psi_{\mathcal{R}(f)}(w)| \leq M_2\alpha|w|^\alpha$.

Assume $\alpha \in \mathrm{HT}_N$ and $f_0 \in \mathcal{QIS}_\alpha$. By Theorem 2.6, the sequence of renormalizations $f_n = \mathcal{R}^{\circ n}(f_0)$ and rotations α_n are defined for $n \geq 0$. In particular, we have the petals $\mathcal{P}_n$, the Fatou coordinates Φ_n, the lift maps $\psi_n : \mathcal{P}_n \to \mathcal{P}_{n-1}$, and the sectors S_n^i for each f_n. The latter are defined in Section 3.2. Applying the above Propositions to the maps f_n, we obtain the integers $\eta(n) = \eta(f_n)$ and $\kappa(n) = \kappa(f_n)$, for $n \geq 0$. Recall the integers b_n defined in Section 3.3.

Proposition 4.3. — *There is $M_3 \in \mathbb{R}$ such that for all $\alpha \in \mathrm{HT}_N$, all $f_0 \in \mathcal{QIS}_\alpha$, and all $m \geq 1$, there exist a non-negative integer $\nu(m) \leq q_m b_m + q_{m-1}$ with*

$$\operatorname{diam}(f_0^{\circ\nu(m)}(S_0^m)) \leq M_3 \cdot \alpha_0 \cdot \alpha_1^{\alpha_0} \cdot \alpha_2^{\alpha_0\alpha_1} \cdot \alpha_3^{\alpha_0\alpha_1\alpha_2} \dots \alpha_m^{\alpha_0\dots\alpha_{m-1}}.$$

Proof. — Let M be the maximum of the constants M_1 and M_2 obtain in the above two Propositions. Given $m \geq 1$, by Proposition 4.1,

$$\operatorname{diam}(f_m^{\circ\eta(m)}(S_m^0)) \leq M \cdot \alpha_m,\ f_m^{\circ\eta(m)}(S_m^0) \subset \mathcal{P}_m.$$

Using Proposition 4.2 with $n = m-1$ and $w \in f_m^{\circ\eta(m)}(S_m^0)$, we obtain

$$\begin{aligned}\operatorname{diam}(f_{m-1}^{\circ\kappa(m-1)} \circ \psi_m(f_m^{\circ\eta(m)}(S_m^0))) &\leq M \cdot \alpha_{m-1}(\operatorname{diam}(f_m^{\circ\eta(m)}(S_m^0)))^{\alpha_{m-1}} \\ &\leq M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}}.\end{aligned}$$

By Lemma 3.3, the above relation boils down to

$$\operatorname{diam}(f_{m-1}^{\circ\kappa(m-1)} \circ f_{m-1}^{\circ(\eta(m)\lfloor 1/\alpha_m\rfloor+1)}(\psi_m(S_m^0)) \leq M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}},$$

which is equivalent to

$$\operatorname{diam}(f_{m-1}^{\circ(\kappa(m-1)+\eta(m)\lfloor 1/\alpha_m\rfloor+1)}(S_{m-1}^1) \leq M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}}.$$

Again applying Proposition 4.2 with $n = m-2$, the last inequality implies that

$$\begin{aligned}&\operatorname{diam}(f_{m-2}^{\circ\kappa(m-2)} \circ \psi_{m-1}(f_{m-1}^{\circ(\kappa(m-1)+\eta(m)\lfloor 1/\alpha_m\rfloor+1)}(S_{m-1}^1)) \\ &\qquad\leq M \cdot \alpha_{m-2} \cdot \left(M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}}\right)^{\alpha_{m-2}},\end{aligned}$$

which, by Lemma 3.4, gives us

$$\begin{aligned}&\operatorname{diam}(f_{m-2}^{\circ(\kappa(m-2)+(\kappa(m-1)+\eta(m)\lfloor 1/\alpha_m\rfloor+1)\lfloor 1/\alpha_{m-1}\rfloor+1)}(\psi_{m-1}(S_{m-1}^1))) \\ &\qquad\leq M \cdot \alpha_{m-2} \cdot (M \cdot \alpha_{m-1} \cdot (M \cdot \alpha_m)^{\alpha_{m-1}})^{\alpha_{m-2}}.\end{aligned}$$

Inductively, repeating Proposition 4.2 with $m-3, m-4, \dots, 0$, one obtains

$$\begin{aligned}&\operatorname{diam}(f_0^{\circ\nu(m)}(S_0^m)) \\ &\quad\leq M \cdot \alpha_0 \cdot [M \cdot \alpha_1 [M \cdot \alpha_2 [\dots [M \cdot \alpha_m]^{\alpha_{m-1}}]^{\alpha_{m-2}} \dots]^{\alpha_1}]^{\alpha_0} \\ &\quad\leq M^{1+\alpha_0+\alpha_0\alpha_1+\dots+\alpha_0\alpha_1\cdots\alpha_{m-1}} \cdot \alpha_0 \cdot \alpha_1^{\alpha_0} \cdot \alpha_2^{\alpha_0\alpha_1} \cdot \alpha_3^{\alpha_0\alpha_1\alpha_2} \dots \alpha_m^{\alpha_0\dots\alpha_{m-1}} \\ &\quad\leq M^4 \cdot \alpha_0 \cdot \alpha_1^{\alpha_0} \cdot \alpha_2^{\alpha_0\alpha_1} \cdot \alpha_3^{\alpha_0\alpha_1\alpha_2} \dots \alpha_m^{\alpha_0\dots\alpha_{m-1}}\end{aligned}$$

for some integer $\nu(m)$. Here we have used that $\alpha_i\alpha_{i+1} \leq 1/2$, for $i \geq 0$. This finishes the proof of the estimate.

The bound on $\nu(m)$ follows from the upper bounds on $\eta(j)$ and $\kappa(j)$ (see the discussion on $N(s_0, s_1, \dots, s_n)$ in the proof of Proposition 3.5-1). □

Let $\beta_{-1} := 1$, and $\beta_n := \Pi_{j=0}^n \alpha_j$, for $n \geq 0$. Using elementary properties of continued fractions one can show that (see [**Yoc95**, Section 1.5] for further details)

$$\Big| \sum_{j=0}^{\infty} \beta_{j-1} \log \alpha_j^{-1} - \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \Big| \leq C \tag{10}$$

for some constant C independent of $\alpha_0 \in (0,1)$.

Proof of Theorem F. — The proof is immediate using Proposition 3.6, Proposition 4.3, and the uniform bound in Equation (10). □

Theorem 4.4. — *There exists a constant M such that for every $\alpha \in \mathrm{HT}_N$ and every $f \in \mathcal{QIS}_\alpha$, the conformal radius of the Siegel disk centered at 0 is bounded from above by $M \exp(-\sum_{n=0}^{\infty} q_n^{-1} \log q$*

Proof. — Recall that each $U_n = \mathrm{Dom}\ f_n$ contains a non-zero fixed point σ_{f_n}. By Lemma 3.2, this fixed point lifts to a periodic point of f_{n-1}, whose orbit crosses the set S^0_{n-1}. Then by the conjugacy relations in Lemma 3.4 this periodic point is sent by Ψ_{n-1} to a periodic point of f_0 whose orbit must cross $S^{n-1}_0 = \Psi_{n-1}(S^0_{n-1})$. Hence, every sector in the union Ω^{n-1}_0 contains at least a point of that cycle. Now the theorem follows from the 1/4-Theorem, Proposition 4.3, and Equation (10). □

Remark 4.5. — In [**AC12**] we prove a stronger version of Proposition 4.3, which is based on an infinitesimal estimate on the Fatou coordinates established in [**Che13**]. It is proved that given any neighborhood of the Siegel disk (or zero), as $n \to \infty$, the density of the number of sectors in Ω^n_0 which are contained in that neighborhood tends to one. Although not all sectors are necessarily contained in such neighborhoods, surprisingly, it is also proved in [**AC12**] that every neighborhood of the Siegel disk contains the orbit of infinitely many periodic points.

There is a large class of analytic maps of $\mathbb{C}$ or $\hat{\mathbb{C}}$ that have a restriction which belong to the Inou-Shishikura class. Thus the above results apply to these maps as well. Here is a simple example. Recall the domain U in (1). Let h be a rational map of the Riemann sphere that $h(0) = 0$, $h'(0) = 1$, and h is univalent on the connected component of $h^{-1}(U)$ containing 0. Then the map $h \cdot (1+h)^2$ belongs to $\mathcal{IS}_0$. Note that such maps may have arbitrarily large degrees. Pre-composing these maps with rotations of angle $\alpha \in \mathrm{HT}_N$, one has the bound on the conformal radius of their Siegel disk in the theorem, and in particular, the optimality of the Brjuno condition for their linearizability.

5. Measure and topology of the attractor

Let $\alpha \in \mathrm{HT}_N$ and $f_0 \in \mathcal{QIS}_\alpha$. By Theorem 2.6 in Section 2.3 the sequence of renormalizations $f_n = \mathcal{R}^{\circ n}(f_0)$, $n \geq 0$, are defined. One forms the domains Ω^n_0 and Ω^0_n for the map f_0, defined in Section 3. In this section we prove Theorem A. The plan is to show that $\cap^\infty_{n=0}\Omega^n_0$, which contains the post-critical set by Proposition 3.5, does not contain any Lebesgue density point. As the proof spans over several pages, we briefly outline the argument in the next paragraph.

In Subsection 5.2 we show that any point z_0 in $\cap_n \Omega^n_0$ can be mapped to arbitrarily deep levels of the renormalization planes using the changes of coordinates. Let z_n, for $n \geq 1$, denote the point obtained on level n in this process. In Proposition 5.3 we show that there are infinitely many levels n with $|z_n| \geq \alpha_n$. In Proposition 5.1, we state that if at some level we have $|z_n| \geq \alpha_n$, then there exists a ball of size comparable to its distance to z_n in the complement of $\mathcal{PC}(f_n)$. In Subsection 5.3 we define holomorphic maps g_i from an appropriate

subset V_i of the i-th renormalization level to a domain V_{i-1} on level $i-1$. The maps g_i, for $i = n, n-1, \dots, 1$ belong to a compact class of maps and $z_i \in V_i$ is mapped to $z_{i-1} \in V_{i-1}$ under g_i. In Lemma 5.5 we show that each g_i is uniformly contracting in the respective hyperbolic metrics, and in Lemma 5.6 we show that each g_i is univalent on a ball of definite hyperbolic size (independent of i and n) about z_i. The composition of these maps (from level n to level 0) sends the complimentary ball obtained in Proposition 5.1 to the dynamic plane of f_0. By uniform contraction of the maps g_i, after first few iterates the image of the ball shrinks and falls in the neighborhood of some z_j where g_j is univalent, and stays in the balls where the further maps g_i, for $i = j-1, j-2, \dots, 1$, are univalent. Then, we use compactness of the class of maps containing g_i and the distortion theorem to show that this composition provides us with a ball in the complement of $\mathcal{PC}(f_0)$ at a small scale near z_0.

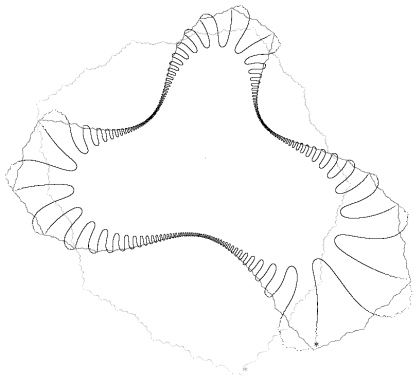


FIGURE 6. The three curves in different colors approximate the orbit of the critical points for different values of α. The light gray one is for $\alpha = [0; 3, 1, 1, 1, \dots]$, the gray one for $[0; 3, 50, 1, 1, 1, \dots]$, and the dark gray one for $[0; 3, 50, 10^5, 1, 1, 1, \dots]$.

5.1. Balls in the complement at deep levels. — Given $X \subseteq \mathbb{C}$, let $B_\delta(X) := \cup_{x \in X} B(x, \delta)$.

Proposition 5.1. — *For all $E \in \mathbb{R}$ there are positive constants δ_1, δ_2, and r^* satisfying the following. For every $\alpha \in (0, r_2]$, every $f \in \mathcal{QLS}_\alpha$, and every $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \le \frac{1}{2\pi}\log\alpha^{-1} + E$ and $\mathbb{E}\mathrm{xp}(\zeta) \in \Omega_0^0(f)$, there exists a curve $\gamma : [0,1] \to \mathbb{C}$, with $\gamma(0) = \zeta$, such that*

(1) $\mathbb{E}\mathrm{xp}\big(B_{\delta_1}\big(B(\gamma(1), r^*) \cup \gamma[0,1]\big)\big) \subseteq \mathrm{Dom}\ f \setminus \{0\}$,

(2) $\mathbb{E}\mathrm{xp}\big(B(\gamma(1), r^*)\big) \cap \Omega_0^0(f) = \emptyset$, $f\big(\mathbb{E}\mathrm{xp}(B(\gamma(1), r^*))\big) \cap \Omega_0^0(f) = \emptyset$,

(3) $\operatorname{diam}\operatorname{Re}\Big(B_{\delta_1}\big(B(\gamma(1), r^*) \cup \gamma[0,1]\big)\Big) \le 1 - \delta_1$,

(4) $\operatorname{mod}\ B_{\delta_1}\big(B(\gamma(1), r^*) \cup \gamma[0,1]\big) \setminus \big(B(\gamma(1), r^*) \cup \gamma[0,1]\big) \ge \delta_2$. [1]

The proof of the above proposition appears in Section 6.7. See Figure 9.

Recall the sets $\mathcal{C}_h$ (and $\mathcal{C}_n$ for f_n) introduced for the definition of renormalization.

1. mod denotes the conformal modulus of an annulus.

Lemma 5.2. — *There exists a real constant $\delta_3 < \min\{\delta_1, 1/8\}$ such that*
- *$\forall j \in \mathbb{Z}$, $\forall n \in \mathbb{N}$, $\mathbb{E}\mathrm{xp}(B(j, \delta_3)) \subset \mathrm{int}\,(\mathcal{C}_n) \subset \Omega_n^0$,*
- *$\forall n \in \mathbb{N}$, $\forall \xi \in \mathbb{C}$ with $\mathbb{E}\mathrm{xp}(\xi) \in \Omega_n^0$, we have $\mathbb{E}\mathrm{xp}(B(\xi, \delta_3)) \subset \mathrm{Dom}\ f_n$.*

Proof. — As each set $\mathcal{C}^{-i} \cup (\mathcal{C}^\sharp)^{-i}$, for $i = 0, 1, 2, \dots, k_n$, is compactly contained in Dom f_n, Ω_n^0 is compactly contained in Dom f_n. Therefore, it follows from continuous dependence of the Fatou coordinate on the map, the pre-compactness of $\mathcal{IS}_0$, and the uniform bound in Proposition 2.7 that there exists a real constant $\delta > 0$ such that

$$\forall n \geq 1, B(-4/27, \delta) \subset \mathcal{C}_n \text{ and } B_\delta(\Omega_n^0) \subset \mathrm{Dom}\ f_n. \tag{11}$$

The first inclusion implies the first part of the lemma and the second one implies the second part of the lemma. □

5.2. Going down the renormalization tower. — For every $n \geq 1$, let $\mathrm{Fil}(\Omega_n^0)$ denote the set obtained from adding the bounded components of $\mathbb{C} \setminus \Omega_n^0$ to Ω_n^0, if there is any. For $n \geq 1$ and $j = 0, 1, \dots, \lfloor \alpha_n^{-1} \rfloor - \boldsymbol{k} - 1$, let $I_{n,j}$ denote the closure of the connected component of

$$\mathrm{int}\,\big(\mathrm{Fil}(\Omega_n^0)\big) \cap \Phi_n^{-1}\{j + \tfrac{1}{2} + t\mathbf{i} : t \in \mathbb{R}\}$$

landing at 0. Each $I_{n,j}$ is a smooth curve in $\mathrm{Fil}(\Omega_n^0)$ that connects the boundary of Ω_n^0 to 0. For every such n and j, every closed loop (i.e. homeomorphic image of a circle) contained in $\Omega_n^0 \setminus I_{n,j}$ is contractible in $\mathbb{C} \setminus \{0\}$. This implies that there is a continuous inverse branch of $\mathbb{E}\mathrm{xp}$ defined on every $\Omega_n^0 \setminus I_{n,j}$.

By Proposition 2.4, Proposition 2.7, and the pre-compactness of $\cup_{\alpha \in (0, r_3]} \mathcal{IS}_\alpha$, there exists a positive integer $\boldsymbol{k}'$ such that

$$\forall n \geq 1 \text{ and } \forall j \text{ with } 0 \leq j < \tfrac{1}{\alpha_n} - \boldsymbol{k} - 1, \quad \sup_{z, z' \in \Omega_n^0 \setminus I_{n,j}} |\arg(z) - \arg(z')| \leq 2\pi \boldsymbol{k}', \tag{12}$$

for every continuous branch of argument defined on $\Omega_n^0 \setminus I_{n,j}$. To simplify the technical details of the proof in this section, we assume the following condition on the rotations

$$N \geq 2\boldsymbol{k}' + \boldsymbol{k} + \boldsymbol{k}'' + 1. \tag{13}$$

Fix an arbitrary point $z_0 \in \cap_{n=0}^\infty \Omega_0^n \setminus \{0\}$. We associate a sequence of quadruples

$$\langle (z_i, w_i, \zeta_i, \sigma(i)) \rangle_{i=0}^\infty \tag{14}$$

to z_0, where $z_i, w_i \in \mathrm{Dom}\ f_i$, $\zeta_i \in \Phi_i(\mathcal{P}_i)$, and $\sigma(i)$ is a non-negative integer. This sequence shall be the trace of z_0 while going down the renormalization tower, and will be used to transport the complementary balls on level n, introduced in Proposition 5.1, back to the dynamic plane of f_0. It is inductively defined as follows.

Define the sets

$$\mathscr{A}_n := \big\{ z \in \mathcal{P}_n \mid \mathrm{Re}\,\Phi_n(z) \in [\boldsymbol{k}' + 1/2, \lfloor 1/\alpha_n \rfloor - \boldsymbol{k}], \text{ or } \Phi_n(z) \in \cup_{j=1}^{\boldsymbol{k}'} B(j, \delta_3) \big\}$$
$$\mathscr{B}_n := \Omega_n^0 \setminus \mathscr{A}_n.$$

For $z_0 \in \mathscr{A}_0$, let $w_0 := z_0$, $\sigma(0) := 0$. For $z_0 \in \mathscr{B}_0$, let $w_0 \in S_0^0 \bigcap \cap_{n \geq 1} \Omega_0^n$ and positive integer $\sigma(0) < k_0 + \boldsymbol{k}'$ be such that $f_0^{\circ \sigma(0)}(w_0) = z_0$. In both cases, let $\zeta_0 := \Phi_0(w_0)$.

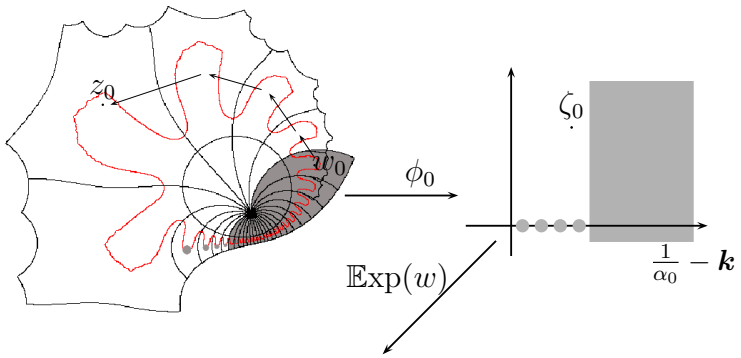


FIGURE 7. The two different colors correspond to the two different ways of going down the renormalization tower. The gray part corresponds to $\mathscr{A}$ and the rest to $\mathscr{B}$.

Define $z_1 := \mathbb{E}\mathrm{xp}(\zeta_0)$. Since $z_0 \in \Omega_0^1$, one can see that $z_1 \in \Omega_1^0$. Thus, we can repeat the above process to define the quadruple $(z_1, w_1, \zeta_1, \sigma(1))$, and so on.

In general, for every $l \geq 0$, we have

$$z_l = \mathbb{E}\mathrm{xp}(\zeta_{l-1}),\ z_l \in \Omega_l^0, f_l^{\circ\sigma(l)}(w_l) = z_l,\ \Phi_l(w_l) := \zeta_l, 0 \leq \sigma(l) < k_l + \boldsymbol{k}'. \tag{15}$$

Note that by the definition of this sequence and condition (13), for all $l \geq 0$, we have

$$\boldsymbol{k}' + 1/2 \leq \mathrm{Re}\,\zeta_l \leq \lfloor 1/\alpha_l \rfloor - \boldsymbol{k},\ \text{or}\ \zeta_l \in \cup_{j=1}^{\boldsymbol{k}'} B(j, \delta_3). \tag{16}$$

Proposition 5.3. — *Assume that $z_0 \in \cap_{n=0}^{\infty} \Omega_0^n \setminus \{0\}$ and α is a non-Brjuno number in HT_N. If $\langle \zeta_j \rangle_{j=0}^{\infty}$ is the sequence associated to z_0, there are arbitrarily large m with*

$$\mathrm{Im}\,\zeta_m \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}}. \tag{17}$$

To prove Proposition 5.3, we need the estimate in the following Proposition, which will be prove in Section 6.8.

Proposition 5.4. — *There exist positive constants D_1 and D_2 such that for all $n \geq 1$,*

$$\text{if } \mathrm{Im}\,\zeta_{n+1} \geq \frac{D_1}{\alpha_{n+1}},\ \text{then } \mathrm{Im}\,\zeta_{n+1} \leq \frac{1}{\alpha_{n+1}} \mathrm{Im}\,\zeta_n - \frac{1}{2\pi\alpha_{n+1}} \log \frac{1}{\alpha_{n+1}} + \frac{D_2}{\alpha_{n+1}}. \tag{18}$$

Proof of Proposition 5.3. — Assume in the contrary that there is $\ell \geq 1$, such that for all $m \geq l$, we have

$$\mathrm{Im}\,\zeta_m > \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}}. \tag{19}$$

Let $\beta_j := \Pi_{l=0}^{j} \alpha_l$, for $j \geq 0$. Note that since $\alpha_i \alpha_{i+1} \leq 1/2$, for all $i \geq 0$, we have

$$\sum_{i=0}^{\infty} \beta_i \leq (\beta_0 + \beta_2 + \beta_4 + \dots) + (\beta_1 + \beta_3 + \beta_5 + \dots) \leq 2(1 + 1/2 + 1/4 + \dots) = 4.$$

Evidently, one of the following two occurs.

(∗) There exists an integer $n_0 \geq \ell$ such that for every $j \geq n_0$, we have $\operatorname{Im} \zeta_j \geq \frac{D_1}{\alpha_j}$.
(∗∗) There are infinitely many integers j greater than or equal to ℓ with $\operatorname{Im} \zeta_j < \frac{D_1}{\alpha_j}$.

If (∗) holds, we can use Proposition 5.4 at every level $j \geq n_0$. Recursively using Equation (18), we obtain the following inequality for every $n > n_0$,

$$\begin{aligned}\operatorname{Im} \zeta_n \leq \frac{1}{\alpha_n \alpha_{n-1} \cdots \alpha_{n_0}} \operatorname{Im} \zeta_{n_0-1} - \frac{1}{2\pi \alpha_n \alpha_{n-1} \cdots \alpha_{n_0}} \log \frac{1}{\alpha_{n_0}} \\ - \frac{1}{2\pi \alpha_n \alpha_{n-1} \cdots \alpha_{n_0+1}} \log \frac{1}{\alpha_{n_0+1}} - \cdots - \frac{1}{2\pi \alpha_n} \log \frac{1}{\alpha_n} \\ + D_2 \left(\frac{1}{\alpha_n \alpha_{n-1} \cdots \alpha_{n_0}} + \frac{1}{\alpha_n \alpha_{n-1} \cdots \alpha_{n_0-1}} + \cdots + \frac{1}{\alpha_n} \right).\end{aligned}$$

By the contradiction hypothesis (19), in the above inequality we may replace $\operatorname{Im} \zeta_n$ by $\frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}}$, and then multiply both sides of it by $2\pi\beta_n$ to obtain

$$\sum_{j=n_0-1}^{n} \beta_j \log \frac{1}{\alpha_{j+1}} \leq 2\pi \beta_{n_0-1} \operatorname{Im} \zeta_{n_0-1} + 2\pi D_2 \left(\beta_{n_0-1} + \beta_{n_0} + \cdots + \beta_{n-1} \right)$$

As n was an arbitrary integer, the above inequality implies that

$$\sum_{j=n_0-1}^{+\infty} \beta_j \log \frac{1}{\alpha_{j+1}} \leq 2\pi \beta_{n_0-1} \operatorname{Im} \zeta_{n_0-1} + 2\pi D_2 \sum_{j=n_0-1}^{+\infty} \beta_j < +\infty.$$

By Equation (10), this contradicts α being non-Brjuno.

Now assume (∗∗) holds [5]. Let $n_1 < m_2 \leq n_2 < m_3 \leq n_3 < \cdots$ be an increasing sequence of positive integers with the following properties (the case that some $n_i = \infty$ is easier and follows from the following argument)

- for every integer j with $m_i \leq j \leq n_i$, we have $\operatorname{Im} \zeta_j < \frac{D_1}{\alpha_j}$
- for every integer j with $n_i < j < m_{i+1}$, we have $\operatorname{Im} \zeta_j \geq \frac{D_1}{\alpha_j}$.

As in case (∗), we may recursively use (18) for integers n with $n_i \leq n \leq m_{i+1} - 2$, and the contradiction hypothesis (19), to obtain the following inequality for every $i \geq 2$.

$$\sum_{j=n_i}^{m_{i+1}-1} \beta_j \log \frac{1}{\alpha_{j+1}} \leq 2\pi \beta_{n_i} \operatorname{Im} \zeta_{n_i} + 2\pi D_2 \left(\beta_{n_i} + \beta_{n_i+1} + \cdots + \beta_{m_{i+1}-2} \right).$$

5. In the proof of Proposition 5.11 we present an alternative argument for this case, where we prove a slightly weaker conclusion which is still enough for our purpose. More precisely, we show that if (∗∗) holds for some z, then there exists a constant E and infinitely many m with $\operatorname{Im} \zeta_m \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}} + E$.

Then, using $\operatorname{Im}\zeta_{n_i} < D_1/\alpha_{n_i}$,

$$\begin{aligned}\sum_{i=2}^{\infty}\sum_{j=n_i}^{m_{i+1}-1}\beta_j\log\frac{1}{\alpha_{j+1}} &\le \sum_{i=2}^{\infty}\Big(2\pi\beta_{n_i}\operatorname{Im}\zeta_{n_i}+2\pi D_2\sum_{j=n_i}^{m_{i+1}-2}\beta_j\Big)\\ &\le \sum_{i=2}^{\infty}\Big(2\pi D_1\beta_{n_i-1}+2\pi D_2\sum_{j=n_i}^{m_{i+1}-2}\beta_j\Big)\\ &\le (2\pi D_1+2\pi D_2)\sum_{i=0}^{\infty}\beta_i<+\infty.\end{aligned}$$

On the other hand, for all integers j with $m_i \le j \le n_i$, we have $\frac{1}{2\pi}\log\frac{1}{\alpha_{j+1}} < \operatorname{Im}\zeta_j < \frac{D_1}{\alpha_j}$, which implies that

$$\sum_{i=2}^{\infty}\sum_{j=m_i}^{n_i-1}\beta_j\log\frac{1}{\alpha_{j+1}} \le \sum_{i=2}^{\infty}\sum_{j=m_i}^{n_i-1}2\pi D_1\beta_{j-1} \le 2\pi D_1\sum_{i=0}^{\infty}\beta_i<+\infty.$$

Combining the above two inequalities, we obtain the contradiction

$$\sum_{j=m_2}^{\infty}\beta_j\log\frac{1}{\alpha_{j+1}}=\sum_{i=2}^{\infty}\Big(\sum_{j=m_i}^{n_i-1}\beta_j\log\frac{1}{\alpha_{j+1}}+\sum_{j=n_i}^{m_{i+1}-1}\beta_j\log\frac{1}{\alpha_{j+1}}\Big)<+\infty.$$

□

5.3. Going up the renormalization tower. — Assume $\operatorname{Im}\zeta_n \le \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}}$ holds for some positive integer n.

The argument presented in Section 5.3 is valid if $\operatorname{Im}\zeta_n \le \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}}+E$, for some constant E independent of n. But, to avoid unnecessary details we assume that $E=0$. See Proposition 5.8 and its proof for more details.

We may use Proposition 5.1 with $E=0$ and the map f_{n+1}, to obtain a curve γ_n and a ball $B(\gamma_n(1), r^*)$ that projects under $\mathbb{E}\mathrm{xp}$ into $\mathbb{C}\setminus\Omega_0^0(f_{n+1})$. Let us define the set

$$V_{n+1} := B_{\delta_1}\big(B(\gamma_n(1), r^*)\cup\gamma_n[0,1]\big).$$

We shall define domains $V_n, V_{n-1}, \dots, V_1$, a holomorphic map g_{n+1}, and anti-holomorphic maps $g_n, g_{n-1}, \dots, g_1$ as in diagram

$$V_{n+1}\xrightarrow{g_{n+1}}V_n\xrightarrow{g_n}V_{n-1}\xrightarrow{g_{n-1}}\cdots\xrightarrow{g_2}V_1\xrightarrow{g_1}V_0 := B_1(\Omega_0^0), \tag{20}$$

satisfying

- for all $i=1,2,\dots,n$, $V_i=\Omega_i^0\setminus I_{i,j(i)}$ for some $j(i)\in\{0,1,\dots,\lfloor 1/\alpha_i\rfloor-\boldsymbol{k}-1\}$;
- for all $i=1,2,\dots,n+1$, $g_i: V_i\to V_{i-1}$; for all $i=0,1,\dots,n$, $z_i\in V_i$;
- $g_{n+1}(\zeta_n)=z_n$; and for all $i=1,2,\dots,n$, $g_i(z_i)=z_{i-1}$.

We use an inverse inductive process to define the pairs (g_{i+1}, V_i), starting with $i=n$ and ending with $i=0$.

Base step $i = n$: Recall that $\zeta_n \in V_{n+1}$ satisfies (16). As $\operatorname{diam}(\operatorname{Re} V_{n+1}) \leq 1 - \delta_1$, and $\delta_3 < \delta_1$, there exists an integer $j \in \{0, 1\}$ such that

$$\operatorname{Re}(V_{n+1} - j) \subset (0, \alpha_n^{-1} - \boldsymbol{k}).$$

With this choice of j, we define $g_{n+1} : V_{n+1} \to \mathbb{C}$ as

$$g_{n+1}(\zeta) := f_n^{\circ(j+\sigma(n))}(\Phi_n^{-1}(\zeta - j)).$$

By Proposition 5.1-1, $\mathbb{E}\mathrm{xp}(V_{n+1} - j)$ is contained in Dom f_{n+1}. So, Lemma 3.2, combined with Equations (13) and (15), imply that $f_n^{\circ(j+\sigma(n))}$ is defined on $\Phi_n^{-1}(V_{n+1} - j)$. Indeed, we have $g_{n+1}(V_{n+1}) \subset \Omega_n^0$.

Since $g_{n+1}(V_{n+1})$ intersects at most $\sigma(n) + 1 \leq \boldsymbol{k}' + k_n$ of the curves $I_{n,j}$, there exists $j(n) \in \{0, 1, \ldots, \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1\}$ with $g_{n+1}(V_{n+1}) \cap I_{n,j(n)} = \emptyset$. We define $V_n := \Omega_n^0 \setminus I_{n,j(n)}$.

Finally, by the equivariance property of Φ_n,

$$g_{n+1}(\zeta_n) = f_n^{\circ(j+\sigma(n))}(\Phi_n^{-1}(\zeta_n - j)) = f_n^{\circ\sigma(n)}(w_n) = z_n.$$

Induction step: Assume that (g_{i+1}, V_i) is defined and we want to define (g_i, V_{i-1}). As every closed loop in V_i is contractible in $\mathbb{C} \setminus \{0\}$, there exists an inverse branch η_i of $\mathbb{E}\mathrm{xp}$ defined on V_i with $\eta_i(z_i) = \zeta_{i-1}$. Now we consider two separate cases.

$\mathscr{R}$ **:** $\operatorname{Re}(\eta_i(V_i)) \subset [1/2, \infty)$,
$\mathscr{L}$ **:** $\operatorname{Re}(\eta_i(V_i)) \cap (-\infty, 1/2) \neq \emptyset$.

If $\mathscr{R}$ occurs: Since $\zeta_{i-1} \in \eta_i(V_i)$ satisfies (16), and $\operatorname{diam}(\operatorname{Re} B_{\delta_3}(\eta_i(V_i))) \leq \boldsymbol{k}' + 1/4$ by Equation (12) and $\delta_3 < 1/8$, there exists an integer $j \in \{0, 1, \ldots, \boldsymbol{k}' + 1\}$ with

$$B_{\delta_3}(\eta_i(V_i)) - j \subset \{\xi \in \mathbb{C} : 3/8 \leq \operatorname{Re} \xi \leq \lfloor 1/\alpha_{i-1} \rfloor - \boldsymbol{k}\}. \tag{21}$$

By Lemma 5.2, $\mathbb{E}\mathrm{xp}(B_{\delta_3}(\eta_i(V_i))) \subset$ Dom f_i. Thus, Lemma 3.2 and Inequality (13) imply that $f_{i-1}^{\circ(j+\sigma(i-1))}$ is defined on $\Phi_{i-1}^{-1}(B_{\delta_3}(\eta_i(V_i)) - j)$. Define $\widetilde{g}_i$ on $B_{\delta_3}(\eta_i(V_i))$ as

$$\widetilde{g}_i(\zeta) := f_{i-1}^{\circ(j+\sigma(i-1))}(\Phi_{i-1}^{-1}(\zeta - j)), \tag{22}$$

and let

$$g_i := \widetilde{g}_i \circ \eta_i.$$

One can see that $\widetilde{g}_i(B_{\delta_3}(\eta_i(V_i)))$ intersects at most $\boldsymbol{k}' + k_{i-1} + 1$ of the curves $I_{i-1,j}$, for $j = 0, 1, \ldots, \lfloor 1/\alpha_{i-1} \rfloor - \boldsymbol{k} - 1$. Hence, by Equation (13), there is $j(i-1)$ in that set with $\widetilde{g}_i(B_{\delta_3}(\eta_i(V_i)) \cap I_{i-1,j(i-1)} = \emptyset$. Now, we define $V_{i-1} := \Omega_{i-1}^0 \setminus I_{i-1,j(i-1)}$ so that

$$\widetilde{g}_i(B_{\delta_3}(\eta_i(V_i))) \subset V_{i-1}. \tag{23}$$

Finally, by the equivariance property of Φ_{i-1}, we have

$$g_i(z_i) = f_{i-1}^{\circ(j+\sigma(i-1))}(\Phi_{i-1}^{-1}(\eta_i(z_i) - j)) = f_{i-1}^{\circ\sigma(i-1)}(w_{i-1}) = z_{i-1}.$$

If $\mathscr{L}$ occurs: Here, because $\operatorname{diam}(\operatorname{Re} \eta_i(V_i)) \leq \boldsymbol{k}'$ and $\zeta_{i-1} \in \eta_i(V_i)$ satisfies (16), we must have $\zeta_{i-1} \in B(j, \delta_3)$ for some j in $\{1, 2, \ldots, \boldsymbol{k}'\}$. Therefore, by Lemma 5.2 (and $\mathcal{C}_i \subset V_i$, $\delta_3 < 1/8$), $\eta_i(V_i) \supseteq B(j, \delta_3)$. This implies that

$$\eta_i(V_i) \cap B_{\delta_3}(\{0, -1, -2, \ldots, -\boldsymbol{k}'\}) = \emptyset,$$

or equivalently

$$B_{\delta_3}(\eta_i(V_i)) \cap \{0, -1, -2, \dots, -\boldsymbol{k}'\} = \emptyset. \tag{24}$$

Now, we extend $\Phi_{i-1} : \mathcal{P}_{i-1} \to \mathbb{C}$ over a larger domain, using the dynamics of f_{i-1}, so that a unique branch of Φ_{i-1}^{-1} is defined on $B_{\delta_3}(\eta_i(V_i))$.

Recall the sectors $\mathcal{C}_{i-1}^{-j} \cup (\mathcal{C}_{i-1}^{\sharp})^{-j}$, for $1 \le j \le k_{i-1}$, and $S_{i-1}^0 = \mathcal{C}_{i-1}^{-k_{i-1}} \cup (\mathcal{C}_{i-1}^{\sharp})^{-k_{i-1}}$ introduced for the definition of renormalization (of f_{i-1}). If $k_{i-1} < \boldsymbol{k}' + 1$, using (13), one can consider further pre-images for $j = k_{i-1} + 1, \dots, \boldsymbol{k}' + 1$ as

$$\mathcal{C}_{i-1}^{-j} := \Phi_{i-1}^{-1}(\Phi_{i-1}(\mathcal{C}_{i-1}^{-k_{i-1}}) - (j - k_{i-1})),$$
$$(\mathcal{C}_{i-1}^{\sharp})^{-j} := \Phi_{i-1}^{-1}(\Phi_{i-1}((\mathcal{C}_{i-1}^{\sharp})^{-k_{i-1}}) - (j - k_{i-1})).$$

Let $\mathcal{D}_{i-1} := \mathcal{C}_{i-1}^{-\boldsymbol{k}'-1} \cup (\mathcal{C}_{i-1}^{\sharp})^{-\boldsymbol{k}'-1}$, and observe that $f_{i-1}^{\circ(\boldsymbol{k}'+1)} : \mathcal{D}_{i-1} \to \mathcal{C}_{i-1} \cup \mathcal{C}_{i-1}^{\sharp}$. For $i \ge 1$, define the set

$$\mathcal{P}_{i-1}^{\natural} := \bigcup_{j=0}^{\boldsymbol{k}'} f_{i-1}^{\circ j}(\mathcal{D}_{i-1}).$$

Define $\Phi_{i-1}^{\natural} : \mathcal{P}_{i-1}^{\natural} \to \mathbb{C}$ as follows. For $z \in \mathcal{P}_{i-1}^{\natural}$, there is an integer j with $0 \le j \le \boldsymbol{k}' + 1$ and $f_{i-1}^{\circ j}(z) \in \mathcal{P}_{i-1}$. Let

$$\Phi_{i-1}^{\natural}(z) := \Phi_{i-1}(f_{i-1}^{\circ j}(z)) - j.$$

As Φ_{i-1} satisfies the Abel functional equation on $\mathcal{P}_{i-1}$, one can see that $\Phi_{i-1}^{\natural}$ is independent of the choice of j and hence, defines a holomorphic map on $\mathcal{P}_{i-1}^{\natural}$. The map $\Phi_{i-1}^{\natural}$ is not univalent. However, it still satisfies the Abel Functional equation on $\mathcal{P}_{i-1}^{\natural}$. Indeed, it has critical points at the critical point of f_{i-1} and its $\boldsymbol{k}'$ pre-images within $\mathcal{P}_{i-1}^{\natural}$. The $\boldsymbol{k}' + 1$ critical points of $\Phi_{i-1}^{\natural}$ are mapped to $0, -1, -2, \dots, -\boldsymbol{k}'$.

The map $\Phi_{i-1}^{\natural}$ is a natural extension of Φ_{i-1} to a multi-valued holomorphic map on $\mathcal{P}_{i-1}^{\natural} \cup \mathcal{P}_{i-1}$. However, the two maps

$$\Phi_{i-1}^{\natural} : \bigcup_{j=0}^{\boldsymbol{k}'} f_{i-1}^{\circ j}(\mathcal{D}_{i-1}) \to \mathbb{C}, \ \Phi_{i-1} : \bigcup_{j=\boldsymbol{k}'+1}^{\lfloor 1/\alpha_{i-1} \rfloor + \boldsymbol{k}' - \boldsymbol{k} - 1} f_{i-1}^{\circ j}(\mathcal{D}_{i-1}) \to \mathbb{C}$$

provide a well defined holomorphic map on every $\boldsymbol{k}' + 1$ consecutive sectors of the form $f_{i-1}^{j}(\mathcal{D}_{i-1})$. More precisely, for every l with $0 \le l < \lfloor 1/\alpha_{i-1} \rfloor - \boldsymbol{k}$,

$$\Phi_{i-1}^{\natural} \amalg_l \Phi_{i-1} : \bigcup_{j=0}^{\boldsymbol{k}'} f_{i-1}^{\circ(l+j)}(\mathcal{D}_{i-1}) \to \mathbb{C}$$

is defined as

$$\Phi_{i-1}^{\natural} \amalg_l \Phi_{i-1}(z) := \begin{cases} \Phi_{i-1}^{\natural}(z), & \text{if } z \in f_{i-1}^{\circ j}(\mathcal{D}_{i-1}) \text{ with } j < \boldsymbol{k}' + 1; \\ \Phi_{i-1}(z), & \text{if } z \in f_{i-1}^{\circ j}(\mathcal{D}_{i-1}) \text{ with } j \ge \boldsymbol{k}' + 1. \end{cases}$$

The set $B_{\delta_3}(\eta_i(V_i))$ has diameter strictly less than $\boldsymbol{k}' + 1$. Therefore, it can intersect at most $\boldsymbol{k}' + 1$ vertical strips of width one. In other words, $B_{\delta_3}(\eta_i(V_i))$ is contained in $\boldsymbol{k}' + 1$ consecutive

sets in the list

$$\Phi_{i-1}^{\natural}(\mathcal{D}_{i-1}), \Phi_{i-1}^{\natural}(f_{i-1}(\mathcal{D}_{i-1})), \dots, \Phi_{i-1}^{\natural}(f_{i-1}^{\circ \boldsymbol{k}'}(\mathcal{D}_{i-1})),$$
$$\Phi_{i-1}(f_{i-1}^{\circ(\boldsymbol{k}'+1)}(\mathcal{D}_{i-1})), \dots, \Phi_{i-1}(f_{i-1}^{\circ(2\boldsymbol{k}'+1)}(\mathcal{D}_{i-1})).$$

Thus, by the above argument about $\Phi_{i-1}^{\natural} \amalg_l \Phi_{i-1}$, and that every closed loop in $B_{\delta_3}(\eta_i(V_i))$ is contractible in the complement of the critical values of $\Phi_{i-1}^{\natural}$ by (24), there exists an inverse branch of this map defined on $B_{\delta_3}(\eta_i(V_i))$. We denote this map by $\widetilde{g}_i$ and let

$$g_i := \widetilde{g}_i \circ \eta_i : V_i \to \Omega_{i-1}^0.$$

One can similarly verify that $\widetilde{g}_i(B_{\delta_3}(\eta_i(V_i)))$ does not intersect some curve $I_{i-1,j(i-1)}$. We define $V_{i-1} := \Omega_{i-1}^0 \setminus I_{i-1,j(i-1)}$ and have $g_i : V_i \to V_{i-1}$. Indeed, we have shown

$$\widetilde{g}_i(B_{\delta_3}(\eta_i(V_i))) \subset V_{i-1}. \tag{25}$$

Here, $\sigma(i-1) = 0$, $\Phi_{i-1}(w_{i-1}) = \zeta_{i-1}$, and $w_{i-1} = z_{i-1}$. Hence $g_i(z_i) = z_{i-1}$. This finishes the definition of the domains and maps when $\mathscr{L}$ occurs. [6] [7]

5.4. Safe lifts. — Each domain V_j, for $j = n+1, n, \dots, 0$, is a hyperbolic Riemann surface. Let $\rho_i(z)|dz|$ be the complete metric of constant curvature -1 on V_i. The next two lemmas are natural consequences of the definition of the chain (20).

Lemma 5.5. — *For every $i \in \{1, 2, \dots n\}$, the map $g_i : (V_i, \rho_i) \to (V_{i-1}, \rho_{i-1})$ is uniformly contracting. More precisely, for every $z \in V_i$, we have*

$$\rho_{i-1}(g_i(z)) \cdot |g_i'(z)| \leq \delta_4 \cdot \rho_i(z),$$

with $\delta_4 := (2\boldsymbol{k}' + 1)/(2\boldsymbol{k}' + 1 + \delta_3)$.

Proof. — Let $\hat{\rho}_i(z)|dz|$ and $\tilde{\rho}_i(z)|dz|$ denote the Poincaré metric on the domains $\eta_i(V_i)$ and $B_{\delta_3}(\eta_i(V_i))$, respectively. By the definition of g_i and properties (23) and (25) we can decompose the map $g_i : (V_i, \rho_i) \to (V_{i-1}, \rho_{i-1})$ as follows:

$$(V_i, \rho_i) \xrightarrow{\eta_i} (\eta_i(V_i), \hat{\rho}_i) \xhookrightarrow{inc.} (B_{\delta_3}(\eta_i(V_i)), \tilde{\rho}_i) \xrightarrow{\tilde{g}_i} (V_{i-1}, \rho_{i-1}).$$

By Schwartz-Pick Lemma, the first map and the last map in the above chain are non-expanding.

To show that the inclusion map is uniformly contracting in the respective metrics, fix an arbitrary point ξ_0 in $\eta_i(V_i)$, and define

$$H(\xi) := \xi + (\xi - \xi_0)\frac{\delta_3}{(\xi - \xi_0 + 2\boldsymbol{k}' + 1)} : \eta_i(V_i) \to \mathbb{C}.$$

6. The chain of the domains and maps (20) defined here depends on the value of n. It is likely that the last parts of the chains defined for two different values of n are not identical.

7. For an alternative approach to going down and up the tower see [**Che13**, Section 3].

Since diam $\mathrm{Re}(\eta_i(V_i)) \le \boldsymbol{k}'$, we have $|\mathrm{Re}(\xi - \xi_0)| \le \boldsymbol{k}'$ for every $\xi \in \eta_i(V_i)$. This implies that $|\xi - \xi_0| < |\xi - \xi_0 + 2\boldsymbol{k}' + 1|$, and hence

$$|H(\xi) - \xi| = \delta_3 |\frac{\xi - \xi_0}{\xi - \xi_0 + 2\boldsymbol{k}' + 1}| < \delta_3.$$

So, $H(\xi)$ is a holomorphic map from $\eta_i(V_i)$ into $B_{\delta_3}(\eta_i(V_i))$. By Schwartz-Pick Lemma, H is non-expanding. In particular, at $H(\xi_0) = \xi_0$ we obtain

$$\tilde{\rho}_i(\xi_0)|H'(\xi_0)| = \tilde{\rho}_i(\xi_0)(1 + \frac{\delta_3}{2\boldsymbol{k}' + 1}) \le \hat{\rho}_i(\xi_0).$$

That is, $\tilde{\rho}_i(\xi_0) \le \delta_4 \cdot \hat{\rho}_i(\xi_0)$ with $\delta_4 = (2\boldsymbol{k}' + 1)/(2\boldsymbol{k}' + 1 + \delta_3)$. □

Lemma 5.6. — *There exists a constant $\delta_5 > 0$ independent of n such that*

(1) each $g_i : V_i \to V_{i-1}$, for $i = 1, 2, \ldots, n+1$, is either one-to-one or has only one simple critical point;

(2) each $g_i : V_i \to V_{i-1}$, for $i = 1, 2, \ldots, n$, is one-to-one on the hyperbolic ball

$$B_{\rho_i}(z_i, \delta_5) := \{z \in V_i \mid d_{\rho_i}(z, z_i) < \delta_5\}.$$

Proof. — Part (1): Each map g_i is a composition of at most four maps; η_i (this does not appear for g_{n+1}), a translation by an integer j, Φ_{i-1}^{-1}, and $f_{i-1}^{\circ(j+\sigma(i-1))}$. The first three maps are one-to-one. The map $f_{i-1}^{\circ(j+\sigma(i-1))}$ on $\Phi_{i-1}^{-1}(\eta_i(V_i) - j)$ is either one-to-one or has at most one simple critical point. To see this, first note that the relevant critical points of $f_{i-1}^{\circ(j+\sigma(i-1))}$ within Ω_{i-1}^0 are contained in $\cup_{l=0}^{j+\sigma(i-1)}\{f_{i-1}^{-l}(\mathrm{cp}_{f_{i-1}})\}$, which are all non-degenerate. If $\Phi_{i-1}^{-1}(\eta_i(V_i) - j)$ contains more than one element in the above list, by the equivariance property of Φ_{i-1}, there must be a pair of points $\xi, \xi + m$ (for some integer $m \neq 0$) in $\eta_i(V_i) - j$. As this set is contained in the lift of a simply connected domain under $\mathbb{E}\mathrm{xp}$, that is not possible.

Part (2): The proof is broken into four small steps.

Step 1. If g_i has a critical value, then it belongs to $\cup_{l=0}^{j+\sigma(i-1)-1}\{f_{i-1}^{\circ l}(-4/27)\}$, where j is the integer defined in case $\mathscr{R}$ of the inductive construction [8].

Looking back at the inductive process, the map g_i introduced in $\mathscr{L}$ is univalent, hence, we only need to look at maps considered in $\mathscr{R}$. By the definition (22), if g_i has a critical value, it must belong to the above set.

Let cv_{g_i} denote the critical value of g_i, if it exists.

Step 2. If $\mathrm{cv}_{g_i} = f_{i-1}^{\circ l}(-4/27) = \Phi_{i-1}^{-1}(l+1)$, for some l with $0 \le l \le \sigma(i-1) + j - 1$, then $z_{i-1} \notin \Phi_{i-1}^{-1}(B(l+1, \delta_3))$.

To see this, we refer to the definition of quadruples (14). If $z_{i-1} \in \mathscr{A}_{i-1}$, recall that $\sigma(i-1) = 0$ and $z_{i-1} = \Phi_{i-1}^{-1}(\zeta_{i-1})$. By the above step, and since $j \le \boldsymbol{k}' + 1$, we have $\mathrm{cv}_{g_i} \in \cup_{l=0}^{\boldsymbol{k}'}\{f_{i-1}^{\circ l}(-4/27)\}$. Now, if $\mathrm{Re}\,\zeta_{i-1} \ge \boldsymbol{k}' + 1/2$, then z_{i-1} can not belong to $\cup_{l=1}^{\boldsymbol{k}'}\Phi_{i-1}^{-1}(B(l, \delta_3))$. And if $\zeta_{i-1} \in \cup_{l=1}^{\boldsymbol{k}'}B(l, \delta_3)$, we have $j = 0$. By Step 1, g_i has no critical value.

When $z_{i-1} \in \mathscr{B}_{i-1}$, by definition, $z_{i-1} \notin \cup_{l=0}^{\boldsymbol{k}'}\Phi_{i-1}^{-1}(B(l, \delta_3))$.

8. When $\sigma(i-1) + j = 0$, we define the set to be empty.

Step 3. There exists a real constant $\delta > 0$ such that $B_{\rho_{i-1}}(z_{i-1}, \delta)$ is simply connected and does not contain cv_{g_i}.

By steps 1 and 2, it is enough to show that there exists a $\delta > 0$ such that for every $l \in \{1, 2, \dots, 2\boldsymbol{k}' + 1\}$ and every $i \leq n$, $B_{\rho_{i-1}}(\Phi_{i-1}^{-1}(l), \delta)$ is simply connected and is contained in $\Phi_{i-1}^{-1}(B(l, \delta_3))$. Recall that Φ_{i-1}^{-1} is univalent on $\{\xi \in \mathbb{C}; \mathrm{Re}\,\xi \in (0, \alpha_{i-1}^{-1} - \boldsymbol{k})\}$. As the balls $B(l, \delta_3)$ and the segments $\{s \cdot l + (1-s)(1/8 - 2\mathbf{i}), s \in (0,1)\}$, for $l = 1, 2, \dots, \boldsymbol{k}'$, are well contained in this strip, it follows from the distortion theorem that there are constants M_1 and M_2 such that

$$\Phi_{i-1}^{-1}(B(l, \delta_3)) \supset B(\Phi_{i-1}^{-1}(l), M_1 \cdot (\Phi_{i-1}^{-1})'(l)),$$
$$\forall z \in \Phi_{i-1}^{-1}(B(l, \delta_3)), d(z, \partial V_{i-1}) \leq M_2 \cdot (\Phi_{i-1}^{-1})'(l).$$

For the second line of the above equations we have used the Koebe distortion theorem on the segment $\{s \cdot l + (1-s)(1/8 - 2\mathbf{i}), s \in (0,1)\}$. As $\rho_{i-1}(\cdot)$ is comparable to $1/d(\cdot, \partial V_{i-1})$, one infers that $\Phi_{i-1}^{-1}(B(l, \delta_3))$ contains a round hyperbolic ball of radius uniformly bounded from below.

Step 4. part (2) of the lemma holds for $\delta_5 := \delta$.

By the contraction of g_i (Lemma 5.5), $g_i(B_{\rho_i}(z_i, \delta_5))$ is contained in the ball $B_{\rho_{i-1}}(z_{i-1}, \delta)$. As $B_{\rho_{i-1}}(z_{i-1}, \delta)$ is simply connected and does not contain any critical value of g_i, one can find a univalent inverse branch of g_i defined on this ball. Therefore, g_i is one-to-one on the ball $B_{\rho_i}(z_i, \delta_5)$. □

Let $\mathcal{G}_n$ denote the map

$$\mathcal{G}_n := g_1 \circ g_2 \circ \cdots \circ g_{n+1} : V_{n+1} \to \Omega_0^0.$$

Recall the curve γ_n obtained in Proposition 5.1 for the map f_{n+1}. We have $\gamma_n(0) = \zeta_n$. By the properties of the chain (20), $\mathcal{G}_n(\gamma_n(0)) = z_0$.

Lemma 5.7. — *For all $E \in \mathbb{R}$, there exists a constant $D_3 > 0$ such that for every $n \geq 1$ satisfying $\mathrm{Im}\,\zeta_n \leq (2\pi)^{-1} \log(1/\alpha_{n+1}) + E$, there exists $r_n \in (0, +\infty)$ such that*

(1) $\mathcal{G}_n(B(\gamma_n(1), r^*)) \cap \Omega_0^{n+1} = \emptyset$;

(2) $B(\mathcal{G}_n(\gamma_n(1)), r_n) \subset \mathcal{G}_n(B(\gamma_n(1), r^*))$, and $|\mathcal{G}_n(\gamma_n(1)) - z_0| \leq D_3 \cdot r_n$;

(3) $r_n \leq D_3 \cdot (\delta_4)^n$.

Proof. — Part (1): By Proposition 5.1-2, for every $\zeta \in B(\gamma_n(1), r^*)$ we have

$$\mathbb{E}\mathrm{xp}(\zeta) \notin \Omega_{n+1}^0, \text{ and } f_{n+1}(\mathbb{E}\mathrm{xp}(\zeta)) \notin \Omega_{n+1}^0.$$

We claim that this implies

$$g_{n+1}(\zeta) \notin \Omega_n^1, \text{ and } f_n(g_{n+1}(\zeta)) \notin \Omega_n^1.$$

It follows from the definition of the renormalization (see proof of Lemma 3.2) that since $\mathbb{E}\mathrm{xp}(\zeta) \notin \Omega_{n+1}^0$, then $\Phi_n^{-1}(\zeta) \notin \Omega_n^1$. Also from $f_{n+1}(\mathbb{E}\mathrm{xp}(\zeta)) \notin \Omega_{n+1}^0$, it follows that $f_n^{\circ j}(\Phi_n^{-1}(\zeta)) \notin \Omega_n^1$, for $j = 0, 1, 2, \dots, b_n + 1$. In particular, by (13), (15), and $j \leq \boldsymbol{k}' + 1$, $g_{n+1}(\zeta)$ and $f_n(g_{n+1}(\zeta))$ are not in Ω_n^1.

The same argument implies the following statement for every $i = n, n-1, \dots, 1$.

For all $w \in V_i$, if $w \notin \Omega_i^{n-i+1}$ and $f_i(w) \notin \Omega_i^{n-i+1}$,
then $g_i(w) \notin \Omega_{i-1}^{n-i+2}$ and $f_{i-1}(g_i(w)) \notin \Omega_{i-1}^{n-i+2}$

By an inductive argument, one infers from these that $\mathcal{G}_n(\zeta) \notin \Omega_0^{n+1}$.

Part (2): It follows from Proposition 5.1-4 that there exists a constant C such that $B(\gamma_n(1), r^*) \cup \gamma_n[0,1]$ has hyperbolic diameter (with respect to ρ_{n+1} in V_{n+1}) less than C. Let m denote the smallest non-negative integer with

$$C \cdot (\delta_4)^m \leq \delta_5/2.$$

Note that m is uniformly bounded from above independent of n. We decompose the map $\mathcal{G}_n$ into two maps as follows

$$\mathcal{G}_n^l := g_{n-m+1} \circ g_{n-m+2} \circ \cdots \circ g_{n+1} \text{ and } \mathcal{G}_n^u := g_1 \circ g_2 \circ \cdots \circ g_{n-m}.$$

By Lemma 5.5 and our choice of m, we have

$$\mathcal{G}_n^l(B(\gamma_n(1), r^*) \cup \gamma_n[0,1]) \subseteq B_{\rho_{n-m}}(z_{n-m}, \delta_5/2).$$

Since by Lemmas 5.5 and 5.6 each g_i, for $i = n-m, n-m-1, \dots, 1$, is univalent and uniformly contracting on $B_{\rho_i}(z_i, \delta_5)$, we conclude that $\mathcal{G}_n^u$ is univalent on $B_{\rho_{n-m}}(z_{n-m}, \delta_5)$. Thus, by the distortion theorem, $\mathcal{G}_n^u$ has bounded distortion on $\mathcal{G}_n^l(B(\gamma_n(1), r^*) \cup \gamma_n[0,1])$.

We claim that $\mathcal{G}_n^l$ belongs to a pre-compact class of maps. That is because it is a composition of m maps g_i, for $i = n+1, \dots, n-m+1$, where each of these maps is a composition of two maps as $g_i = \widetilde{g}_i \circ \eta_i$. The map η_i is univalent on V_i and, by the distortion theorem, has uniformly bounded distortion on sets of bounded hyperbolic diameter. The map $\widetilde{g}_i$ extends over the larger set $B_{\delta_3}(\eta_i(V_i))$, by (23) and (25). So, it belongs to a compact class. (Indeed, $f_i^{\circ(\sigma(i)+j)}$ is a uniformly bounded number (by Proposition 2.7, (15), and $j \leq \boldsymbol{k}' + 1$) of iterates of a map in the pre-compact class $\cup_{(0,r_3]} \mathcal{IS}_\alpha$).

Putting all these together, one infers that there exists a constant C' such that

$$\begin{aligned} |\mathcal{G}_n(\gamma_n(1)) - z_0| &= |\mathcal{G}_n(\gamma_n(1)) - \mathcal{G}_n(\gamma_n(0))| \\ &\leq C' \cdot \operatorname{diam}(\mathcal{G}_n(B(\gamma_n(1), r^*)). \end{aligned}$$

Also, $\mathcal{G}_n(B(\gamma_n(1), r^*))$ contains a round ball of Euclidean radius comparable to the diameter of $\mathcal{G}_n(B(\gamma_n(1), r^*))$.

Part (3): The domain $\mathcal{G}_n(B(\gamma_n(1), r^*))$ is contained in Ω_0^0 which is compactly contained in V_0. Thus, the Euclidean and the hyperbolic (with respect to ρ_0) metrics are comparable on Ω_0^0. Now, the uniform contraction with respect to the hyperbolic metric in Lemma 5.5 implies the claim. □

5.5. Semi-continuity of the post-critical set. —

Proposition 5.8. — *Let $\alpha \in \mathrm{HT}_N$ and $f_0 \in \mathcal{QIS}_\alpha$. Assume that for some non-zero z in $\cap_{n=0}^{\infty}\Omega_0^n(f_0)$ there exist a constant E and infinitely many distinct positive integers m for which*

$$\operatorname{Im}\zeta_n \leq \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}} + E. \tag{26}$$

Then $\cap_{n=0}^{\infty}\Omega_0^n(f_0)$ is non-uniformly porous at z.

Proof. — By the assumption on α, f_0 is infinitely near-parabolic renormalizable and hence we may define $f_n = \mathcal{R}^{\circ n} f_0$, $n \geq 0$. Let n_i, $i \geq 0$, be an increasing sequence of positive integers for which Inequality (26) holds. Applying Proposition 5.1 to the maps f_{n_i+1} there are curves γ_{n_i} and balls $B(\gamma_{n_i}(1), r^*)$ enjoying the properties in that proposition. The maps $\mathcal{G}_{n_i}$, by Lemma 5.7, provide us with a sequence of balls $B(\mathcal{G}_{n_i}(\gamma_{n_i}(1)), r_{n_i})$ satisfying

$$B(\mathcal{G}_{n_i}(\gamma_{n_i}(1)), r_{n_i}) \cap \Omega_0^{n_i+1} = \emptyset,\ |\mathcal{G}_{n_i}(\gamma_{n_i}(1)) - z_0| \leq D_3 \cdot r_{n_i},\ r_{n_i} \to 0.$$

This finishes the proof of the proposition. □

Proof of Theorem A. — Let $z_0 \in \mathcal{PC}(f_0) \setminus \{0\}$. By Proposition 3.5, $z_0 \in \cap_{n=0}^{\infty}\Omega_0^n \setminus \{0\}$. Thus, we can define the sequence of quadruples (14). Since α is a non-Brjuno number, Proposition 5.3 provides us with an strictly increasing sequence of integers n_i for which we have inequality (17) with $m = n_i$. By Proposition 5.8, $\mathcal{PC}(f_0)$ is non-uniformly porous at z_0. This implies that z_0 is not a Lebesgue density point of $\mathcal{PC}(f_0)$, and hence, by the Lebesgue density theorem, $\mathcal{PC}(f_0)$ must have zero area. Indeed, continuing the notations in the proof of Proposition 5.8, for $s_i := r_{n_i} + D_3 \cdot r_{n_i}$, we have

$$\frac{\operatorname{area}\left(B(z_0, s_i)\right) \cap \mathcal{PC}(f_0))}{\operatorname{area}\left(B(z_0, s_i)\right)} \leq \frac{\pi(s_i)^2 - \pi(r_{n_i})^2}{\pi(s_i)^2} \leq \frac{(D_3)^2 + 2D_3}{(D_3)^2 + 2D_3 + 1} < 1.$$ [9]

□

Proof of Corollary C. — By the argument before Proposition 3.6, the orbit of almost every point in the Julia set eventually stays in Ω_0^n, for $n \geq 0$. This implies that almost every point in the complement of Ω_0^n, for $n \geq 0$, is non-recurrent. As area Ω_0^n shrinks to zero, almost every point in the Julia set must be non-recurrent. The second part follows from the first part and Poincaré recurrence Theorem. □

Proposition 5.9. — *There exist $M > 0$ and $\mu < 1$ such that for all $\alpha \in \mathrm{HT}_N$, all $f \in \mathcal{QIS}_\alpha$, all $n \geq 1$, and all $z \in \Omega_0^{n+1}$ we have*

$$|f^{\circ q_n}(z) - z| \leq M \cdot \mu^n.$$

9. The proof does not imply that $\mathcal{PC}(f_0) \setminus \{0\}$ is porous (shallow), i.e. at every scale around a point in $\mathcal{PC}(f_0) \setminus \{0\}$ there is a disk of comparable radius in the complement of $\mathcal{PC}(f_0)$. Indeed, it seems that in Proposition 5.3, given any increasing sequence of positive integers $\langle n_i \rangle$ one can find a non-Brjuno α and $z \in \mathcal{PC}(P_\alpha)$ such that Inequality (17) holds only at levels n_i. Hence, the scales obtained in the above proof may shrink to zero arbitrarily fast.

In particular this holds on the post-critical set.

Proof. — The result basically follows from the uniform contraction in Lemma 5.5, but, since we are not concerned with the distortions of the maps here, one may go down the tower in a simpler fashion. We briefly outline the procedure here and leave further details to the reader.

Let $f_0 := f$, and $f_i := \mathcal{R}^{\circ i}(f)$, for $i \geq 1$. Given $z_0 \in \Omega_0^{n+1} \setminus \{0\}$, inductively define the sequence of points w_i, ζ_i, z_{i+1} and non-negative integers σ_i, for $i = 0, 1, \ldots, n$, according to the following rules. When $z_i \in \mathscr{A}_i' := \cup_{j=k_i+\boldsymbol{k}'}^{b_i} f_i^{\circ j}(S_i^0)$, then $w_i := z_i$, and $\sigma_i := 0$. When $z_i \in (\Omega_i^0 \setminus \mathscr{A}_i')$, one may choose $w_i \in (S_i^0 \cap \Omega_i^{n+1-i})$ and $\sigma_i \in \{1, 2, \ldots, k_i + \boldsymbol{k}' - 1\}$ so that $f_i^{\circ\sigma_i}(w_i) = z_i$. The existence of such w_i follows from the choice of the inverse branch ψ_{i+1} in (7) (that is, $\sup \operatorname{Re} \Phi_i \circ \psi_{i+1}(\mathcal{P}_{i+1}) \leq \inf \operatorname{Re} \Phi_i(S_i^0)$). However, w_i is not necessarily unique. In both cases, $\zeta_i := \Phi_i(w_i)$, $z_{i+1} := \mathbb{E}\mathrm{xp}(\zeta_i)$.

The last point $z_{n+1} \in \Omega_{n+1}^0$. By Lemma 5.2, we may choose a curve $\gamma_n : [0,1] \to \mathbb{C}$ with $\gamma_n(0) = \zeta_n$, $\gamma_n(1) = \zeta_n + 1$, and $\mathbb{E}\mathrm{xp}(B_{\delta_3}(\gamma_n)) \subset \operatorname{Dom} f_{n+1}$. By the distortion theorem, γ_n may be chosen to have uniformly bounded Euclidean length, independent of n. Define $V_{n+1} := B_{\delta_3}(\gamma_n \cup (\gamma_n - 1))$, and note that the hyperbolic distance between ζ_n and $\zeta_n + 1$ within V_{n+1} is uniformly bounded from above, independent of n.

We have $(V_{n+1} - 1) \subset \Phi_n(\operatorname{Dom} \Phi_n)$, and by Lemma 3.2 and Equation (13), f_n may be iterated $k_n + \boldsymbol{k}'$ times on $\Phi_n^{-1}(V_{n+1} - 1)$. Define $g_{n+1}(\zeta) := f_n^{\circ\sigma_n+1} \circ \Phi_n^{-1}(\zeta - 1)$ on V_{n+1}, and as in the previous argument, choose $j(n)$ in $\{0, 1, \ldots, \lfloor 1/\alpha_n \rfloor - \boldsymbol{k} - 1\}$ such that $g_{n+1}(V_{n+1}) \subset \Omega_n^0 \setminus I_{n,j(n)}$. Let $V_n := \Omega_n^0 \setminus I_{n,j(n)}$.

We have $\operatorname{Re} \zeta_i \in [\boldsymbol{k}' + 1/2, \lfloor 1/\alpha_i \rfloor - \boldsymbol{k} - 1/2]$, for $i = 0, 1, \ldots, n$. Repeating only case $\mathscr{R}$ of the construction in Section 5.3, one inductively defines the pairs (g_{i+1}, V_i), for $i = n-1, n-2, \ldots, 1$, such that $g_{i+1} := f_i^{\circ\sigma_i+j} \circ \Phi_i^{-1} \circ (\eta_{i+1} - j)$, for some $j \in \{0, 1, \ldots, \boldsymbol{k}'\}$ with $\operatorname{Re}(\eta_i(V_{i+1}) - j) \subset (0, \lfloor 1/\alpha_i \rfloor - \boldsymbol{k})$. Moreover, $j \in \{0, 1, \ldots, \lfloor 1/\alpha_i \rfloor - \boldsymbol{k} - 1\}$ is chosen so that $V_i = \Omega_i^0 \setminus I_{i,j(i)}$ contains $f_i^{\circ\sigma_i+j} \circ \Phi_i^{-1}(B_{\delta_3}(\eta_i(V_{i+1})) - j)$.

The composition of these maps, denoted by $\mathcal{G}_n$, satisfies $\mathcal{G}_n(\zeta_n) = z_0$. We claim that $\mathcal{G}_n(\zeta_n + 1) = f_0^{\circ q_n}(z_0)$. To see this, first note that $\Phi_n^{-1}(B(\zeta_n - 1, \delta_3)) \subset \mathcal{P}_n'$ and hence by Lemma 3.4, $\Psi_n \circ f_n \circ \Phi_n^{-1} = f_0^{\circ q_n} \circ \Psi_n \circ \Phi_n^{-1}$ on $B(\zeta_n - 1, \delta_3)$. On the other hand, by the definition of renormalization, one can see that $\mathcal{G}_n = f_0^{\circ s} \circ \Psi_n \circ \Phi_n^{-1}$ on V_{n+1}, for some non-negative integer s. The integer s is non-negative because of the choices of the branches of ψ_i in (7) (that is, $\operatorname{Re} \Phi_i(\psi_{i+1}(\mathcal{P}_{i+1}')) \leq \boldsymbol{k}' + 1$). Then, at every point $\xi \in B(\zeta_n - 1, \delta_3)$ we have

$$\begin{aligned} f_0^{\circ q_n}(\mathcal{G}_n(\xi)) = f_0^{\circ q_n} \circ f_0^{\circ s} \circ \Psi_n \circ \Phi_n^{-1}(\xi) &= f_0^{\circ s} \circ f_0^{\circ q_n} \circ \Psi_n \circ \Phi_n^{-1}(\xi) \\ &= f_0^{\circ s} \circ \Psi_n \circ f_n \circ \Phi_n^{-1}(\xi) = f_0^{\circ s} \circ \Psi_n \circ \Phi_n^{-1}(\xi + 1) = \mathcal{G}_n(\xi + 1). \end{aligned}$$

Since V_{n+1} is connected and the above equation holds on $B(\zeta_n - 1, \delta_3) \subset V_{n+1}$, it must hold on V_{n+1}. In particular, $\mathcal{G}_n(\zeta_n + 1) = f_0^{\circ q_n}(\mathcal{G}_n(\zeta_n))$. Now, by the uniform contraction of the maps g_i, one concludes the result. □

Recall that $\Delta(f)$ denotes the Siegel disk of $f \in \mathcal{QIS}_\alpha$ centered at 0, provided it exists. Theorem D is a special case of the following proposition.

Proposition 5.10. — *For all $\alpha \in \mathrm{HT}_N$ and all $f \in \mathcal{QIS}_\alpha$, the following properties hold.*

(1) If α is a non-Brjuno number, then $\mathcal{PC}(f) = \cap_{n=0}^{\infty} \Omega_0^n$.
(2) If α is a Brjuno number, then
  a) $\mathrm{int}\,(\cap_{n=0}^{\infty} \Omega_0^n) = \Delta(f)$,
  b) $\mathcal{PC}(f) = \cap_{n=0}^{\infty} \Omega_0^n \setminus \Delta(f)$, *in particular,* $\partial\Delta(f) \subseteq \mathcal{PC}(f)$.
(3) $\mathcal{PC}(f)$ is a connected set.

The connectivity of $\mathcal{PC}(P_\alpha)$ for irrational values of α follows from the main result of [**Chi08**], although it is not directly stated in that paper. Indeed, it is proved that $\mathcal{PC}(P_\alpha)$ is equal to the closure of a class of hedgehogs of P_α. By definition, each hedgehog of P_α is a connected set. See also [**BBCO10**] . However, proving the connectivity of the post-critical set in the greater generality of the class $\mathcal{IS}$ is particularly important when studying the applications of near-parabolic renormalization scheme. See for instance Lemma 4.5 in [**CC15**].

To prove the above proposition we need the next two Propositions, which shall be proved in Section 6.8.

Proposition 5.11. — *There exists $E \in \mathbb{R}$ such that for all Brjuno $\alpha \in \mathrm{HT}_N$, all f in $\mathcal{QIS}_\alpha$, and all z in $\cap_{n=0}^{\infty} \Omega_0^n \setminus \overline{\Delta(f)}$, there are infinitely many positive integers m with* $\mathrm{Im}\,\zeta_m \leq \frac{1}{2\pi} \log \alpha_{m+1}^{-1} + E$.

The set $\cap_{n=0}^{\infty} \Omega_0^n \setminus \overline{\Delta(f)}$ may be empty for some values of α, in which case the statement of the above lemma is void.

Proposition 5.12. — *For all $E \in \mathbb{R}$ there is $\delta_6 > 0$, such that for all $n \geq 1$, and all $\zeta \in \mathbb{E}\mathrm{xp}^{-1}(\Omega_{n+1}^0)$ with* $\mathrm{Im}\,\zeta \leq \frac{1}{2\pi} \log \alpha_{n+1}^{-1} + E$, *there is $\zeta' \in \mathbb{E}\mathrm{xp}^{-1} \circ \Phi_{n+1}^{-1}(\{1, 2, \ldots, \lfloor 1/(2\alpha_{n+1}) \rfloor\})$ with*

$$|\,\mathrm{Re}(\zeta' - \zeta)| \leq 1/2,\ d(\zeta', \zeta) \leq \delta_6.$$

Combining Proposition 5.8 and Proposition 5.11, we obtain the following corollary.

Corollary 5.13. — *Let α be a Brjuno number in HT_N and $f \in \mathcal{QIS}_\alpha$. Then, the set $\cap_{n=0}^{\infty} \Omega_0^n \setminus \overline{\Delta(f)}$ is non-uniformly porous. In particular, $\cap_{n=0}^{\infty} \Omega_0^n \setminus \Delta(f)$ has empty interior.*

Proof of Proposition 5.10. — Let $f_0 := f$ and $f_n := \mathcal{R}^n(f_0)$, for $n \geq 1$. Also, α_n denotes the asymptotic rotation of f_n at 0, for $n \geq 0$.

Part (1): Fix a point $z_0 \in \cap_{n=0}^{\infty} \Omega_0^n \setminus \{0\}$, and recall the sequence of quadruples $\langle (z_i, w_i, \zeta_i, \sigma_i) \rangle_{i=0}^{\infty}$ introduced in the proof of Proposition 5.9. Proposition 5.3 applies to the sequence ζ_i as well and provides an increasing sequence of positive integers n_i satisfying $\mathrm{Im}\,\zeta_{n_i} \leq \frac{1}{2\pi} \log \alpha_{n_i+1}^{-1}$. One uses Proposition 5.12 (with $E = 0$) to find

$$\zeta'_{n_i} \in \mathbb{E}\mathrm{xp}^{-1}(\Phi_{n_i+1}^{-1}(\{1, 2, \ldots, \lfloor 1/(2\alpha_{n_i+1}) \rfloor\}))$$

enjoying the properties in the proposition. The two points ζ_{n_i} and ζ'_{n_i} are mapped to the dynamic plane of f under the map $\mathcal{G}_{n_i}$ built in the proof of Proposition 5.9. Moreover, by

the uniform contraction of the changes of coordinates, see Lemma 5.5, $\mathcal{G}_{n_i}(\zeta'_{n_i})$ converges to $z_0 = \mathcal{G}_{n_i}(\zeta_{n_i})$, as n_i tends to infinity.

Elements of $\Phi^{-1}_{n_i+1}(\{1, 2, \dots, \lfloor 1/(2\alpha_{n_i+1})\rfloor\})$ belong to the orbit of the critical value of f_{n_i+1}, by the normalization of the coordinates Φ_{n_i}. Then, Ψ_{n_i} maps these elements into the orbit of the critical value of f_0, by the definition of renormalization. Moreover, $\mathcal{G}_{n_i} = f_0^{\circ s_{n_i}} \circ \Psi_{n_i}$, for some non-negative integer s_{n_i} (see the proof of Proposition 5.9). In particular, $\mathcal{G}_{n_i}(\zeta'_{n_i})$ belongs to the orbit of the critical value of f_0. Thus, $z_0 \in \mathcal{PC}(f)$.

Part (2)-a: Recall that by Proposition 3.5, the intersection $\cap_{n=0}^{\infty}\Omega_0^n$ is forward invariant under f_0, is compact, and is connected. Moreover, it contains 0 and the point cp_{f_0} outside of $\Delta(f_0)$. It follows that $\overline{\Delta(f_0)} \subseteq \cap_{n=0}^{\infty}\Omega_0^n$, and $\Delta(f_0) \subseteq \mathrm{int}\,(\cap_{n=0}^{\infty}\Omega_0^n)$.

On the other hand, by Corollary 5.13 a point in $\cap_{n=0}^{\infty}\Omega_0^n \setminus \overline{\Delta(f)}$ may not be in the interior of the set $\cap_{n=0}^{\infty}\Omega_0^n$. In other words, a point $z \in \mathrm{int}\,(\cap_{n=0}^{\infty}\Omega_0^n)$ either belongs to $\Delta(f_0)$ or belongs to $\partial\Delta(f_0)$. We claim that the latter may not occur, and hence $\mathrm{int}\,(\cap_{n=0}^{\infty}\Omega_0^n) \subseteq \Delta(f_0)$. Let U_0 be the connected component of $\mathrm{int}\,(\cap_{n=0}^{\infty}\Omega_0^n)$ containing 0. By the previous paragraph, U_0 contains $\Delta(f_0)$. If some z in $\mathrm{int}\,(\cap_{n=0}^{\infty}\Omega_0^n)$ belongs to $\partial\Delta(f_0)$, U_0 is strictly larger than $\Delta(f_0)$. Let $\hat{U}_0$ denote the filled-in set of U_0, and note that $f_0 : \hat{U}_0 \to \hat{U}_0$. Let $\psi : \hat{U}_0 \to \mathbb{D}$ denote the uniformization of $\hat{U}_0$ by the unit disk mapping 0 to 0. By the Schwarz lemma, $\psi \circ f_0 \circ \psi^{-1}$ is a rotation of $\mathbb{D}$. That is, f_0 is conjugate to a rotation on an strictly larger set than $\Delta(f_0)$, which contradicts the maximality of $\Delta(f_0)$.

Part (2)-b: As the orbit of cp_{f_0} is recurrent by Proposition 5.9, $\mathcal{PC}(f_0) \cap \Delta(f_0)$ is empty. By Proposition 3.5, we only need to show that $(\cap_{n=0}^{\infty}\Omega_0^n \setminus \Delta(f)) \subseteq \mathcal{PC}(f_0)$. Let $z \in \cap_{n=0}^{\infty}\Omega_0^n \setminus \Delta(f)$. By the previous part, $z \in \cap_{n=0}^{\infty}\Omega_0^n \setminus \mathrm{int}\,(\cap_{n=0}^{\infty}\Omega_0^n)$, and hence, there exists a sequence $z_i \in \partial\Omega_0^i$, for $i = 0, 1, \dots$, converging to z. We shall show that there exists a sequence w_i, for $i = 0, 1, \dots$, in the orbit of the critical point of f_0 with $d(z_i, w_i) \to 0$, as i tends to infinity. This proves that $z \in \mathcal{PC}(f_0)$.

Recall the sets $\mathcal{C}_n := \mathcal{C}_{f_n}$, $\mathcal{C}_n^{-1}, \dots, \mathcal{C}_n^{-k_n}$, and $\mathcal{C}_n^{\sharp} := \mathcal{C}_{f_n}^{\sharp}$, $(\mathcal{C}_n^{\sharp})^{-1}, \dots, (\mathcal{C}_n^{\sharp})^{-k_n}$ introduced in Section 3.1. To prove the above claim it is enough to show that for all $n \geq 1$ we have,

(a') $\partial\Omega_0^n \subset \bigcup_{j=0}^{q_n b_n + q_{n-1}} \overline{f_0^{\circ j}(\Psi_n(\mathcal{C}_n^{-k_n}))}$;

(b') $\forall j = 0, 1, \dots, q_n b_n + q_{n-1}$, $\mathcal{O}(\mathrm{cv}_{f_0}) \cap \overline{f_0^{\circ j}(\Psi_n(\mathcal{C}_n^{-k_n}))} \neq \emptyset$;

(c')
$$\lim_{n\to\infty} \sup \left\{ \mathrm{diam}\ \overline{f_0^{\circ j}(\Psi_n(\mathcal{C}_n^{-k_n}))} \;\middle|\; 0 \leq j \leq q_n b_n + q_{n-1} \right\} = 0.$$

Proof of (a'): Recall that by Theorem 2.6, $f_{n+1}(z) = P \circ \psi^{-1}(e^{2\pi\alpha_{n+1}\mathbf{i}} \cdot z)$, for some univalent map $\psi : U \to e^{-2\pi\alpha_{n+1}\mathbf{i}} \cdot \mathrm{Dom}\,(f_{n+1})$ with $\psi'(0) = 1$. The ellipse E defined in Section 2.2 is contained in $B(0, 2)$, and hence, U contains $B(0, 8/9)$. The 1/4-Theorem implies that $\psi(U)$ and Dom f_{n+1} contain $B(0, 2/9)$. Thus,

$$\{w : \mathrm{Im}\, w \geq 2\} \subset \mathbb{E}\mathrm{xp}^{-1}(\mathrm{Dom}\ f_{n+1}). \tag{27}$$

The definition of renormalization implies that

$$\mathbb{E}\mathrm{xp}(\Phi_n((\mathcal{C}_n^{\sharp})^{-k_n})) = f_{n+1}^{-1}(B(0, \tfrac{4}{27}e^{-4\pi})).$$

By an explicit estimate on the polynomial P and the distortion theorem applied to the map $z \mapsto \frac{8}{9}\psi(\frac{9}{8}\cdot z)$ one concludes that $f_{n+1}^{-1}(B(0,\frac{4}{27}e^{-4\pi})) \subset B(0,\frac{4}{27}e^{4\pi})$. Therefore, $\mathbb{E}\mathrm{xp}(\Phi_n((\mathcal{C}_n^\sharp)^{-k_n})) \subset B(0,\frac{4}{27}e^{4\pi})$, or in other words,

$$\Phi_n((\mathcal{C}_n^\sharp)^{-k_n}) \subset \{w \in \mathbb{C} \mid \operatorname{Im} w > -2\}. \tag{28}$$

Combining the inclusions (27) and (28), one infers that $\partial\Omega_n^0 \cap \cup_{j=0}^{b_n} f_n^{\circ j}((\mathcal{C}_n^\sharp)^{-k_n}) = \emptyset$. By Lemma 3.4, this implies that $\partial\Omega_0^n \cap \cup_{j=0}^{b_n q_n + q_{n-1}} f_0^{\circ j}(\Psi_n(\mathcal{C}_n^\sharp)^{-k_n}) = \emptyset$. By the definition of Ω_0^n, this finishes the proof of part (a').

Proof of (b'): By the definition of renormalization,

$$\mathbb{E}\mathrm{xp}(\Phi_n(\mathcal{C}_n^{-k_n})) = f_{n+1}^{-1}\big(B(0,\tfrac{4}{27}e^{4\pi}) \setminus B(0,\tfrac{4}{27}e^{-4\pi})\big).$$

We have $(P^{-1}(B(0,\frac{4}{27}e^{-4\pi})) \cap U) \subset B(0,\frac{8}{27}e^{-4\pi})$, and the distortion theorem applied to the map $z \mapsto \frac{8}{9}\psi(\frac{9}{8}\cdot z)$ implies that $\psi(B(0,\frac{8}{27}e^{-4\pi})) \subset B(0,1/10)$. Hence, $f_{n+1}^{-1}(B(0,\frac{4}{27}e^{-4\pi})) \subset B(0,1/10)$. On the other hand, by the previous part, we have $\psi(U) \supset B(0,2/9)$. Combining these, we have $-4/27 \in \mathbb{E}\mathrm{xp}(\Phi_n(\mathcal{C}_n^{-k_n}))$, or equivalently,

$$\Phi_n(\mathcal{C}_n^{-k_n}) \cap \{1,2,\dots,\lfloor 1/\alpha_n\rfloor - \boldsymbol{k} - 1\} \neq \emptyset.$$

This means that for $j = 0,1,\dots,b_n$, $f_n^{\circ j}(\mathcal{C}_n^{-k_n}) \cap \mathcal{O}(\mathrm{cv}_{f_n}) \neq \emptyset$. By the definition of renormalization, $\Psi_n(\mathcal{O}(\mathrm{cv}_{f_n}) \cap \mathcal{P}_n) \subset \mathcal{O}(\mathrm{cv}_{f_0})$. This finishes the proof of part (b').

Proof of (c'): Since ψ has univalent extension onto V, the distortion theorem implies that there exists a constant C, independent of n, such that Dom f_{n+1} is contained in $B(0,C)$. On the other hand, $P(B(0,\frac{4}{54}e^{-4\pi})) \subset B(0,\frac{4}{27}e^{-4\pi})$ and by 1/4-Theorem, $\psi(B(0,\frac{4}{54}e^{-4\pi})) \supseteq B(0,\frac{1}{54}e^{-4\pi})$. Combining these inclusions with the first equation in the proof of part (b'), we obtain $\mathbb{E}\mathrm{xp}(\Phi_n(\mathcal{C}_n^{-k_n})) \subset B(0,C) \setminus B(0,\frac{1}{54}e^{-4\pi})$. Thus,

$$\operatorname{Im}\Phi_n(\mathcal{C}_n^{-k_n}) \subseteq \big[\tfrac{1}{2\pi}\log(\tfrac{4}{27C}), 2 + \tfrac{1}{2\pi}\log 8)\big].$$

We also have

$$\operatorname{Re}\Phi_n(\mathcal{C}_n^{-k_n}) \subseteq [1/2, \lfloor 1/\alpha_n\rfloor - \boldsymbol{k} - 1/2], \text{ and } \operatorname{diam}\operatorname{Re}(\Phi_n(\mathcal{C}_n^{-k_n})) \leq \boldsymbol{k''},$$

by the choice of k_n, Proposition 2.7, and condition (13).

Let $V_{n+1} := B_{\delta_3}(\Phi_n(\mathcal{C}_n^{-k_n}))$, where δ_3 is the constant in Lemma 5.2. The above equations imply that $\Phi_n(\mathcal{C}_n^{-k_n})$ has uniformly bounded hyperbolic diameter in V_{n+1}, independent of n.

For every $n \geq 1$ and every $j = 0,1,\dots,b_nq_n + q_{n-1}$, there is a chain of maps as in (20) that maps the closure of $\mathcal{C}_n^{-k_n}$ to the closure of $f_0^{\circ j}(\Psi_n(\mathcal{C}_n^{-k_n}))$. That is, given j, there are non-negative integers $\sigma_i \in \{0,1,\dots,b_i\}$, for $i = 0,1,\dots,n$, such that j times iterating f_0 on the closure of $\Psi_n(\mathcal{C}_n^{-k_n})$ breaks down to σ_i times iterating f_i on level i, for $i = 0,1,\dots,n$, using the changes of coordinates. Then, one defines a chain of maps as in (20) so that each g_i is the composition of three maps η_i, Φ_i^{-1}, and $f_i^{\circ\sigma_i}$, where η_i is an appropriate inverse branch of $\mathbb{E}\mathrm{xp}$ and $\sigma_i \in \{0,1,\dots,b_i\}$. As we have used this argument several times before, here we leave further details to the reader. The uniform contraction in Lemma 5.5 implies that the supremum exponentially tends to 0.

Part (3): Each Ω_0^n, for $n \geq 0$, is a connected set. It is a finite union of connected sets (sectors) all containing 0. If α is not a Brjuno number, by the first part, $\mathcal{PC}(f_0)$ is the intersection of this nest of connected sets, and hence is connected. If α is a Brjuno number, each Ω_0^n, for $n \geq 0$, contains the full set $\Delta(f_0)$ in its interior. Thus each $\Omega_0^n \setminus \Delta(f_0)$, for $n \geq 0$, is a connected set. Therefore, their intersection, which is the post-critical set by Part 2-a, is a connected set. □

Proof of Theorem E. — Let f_α be a continuous family of maps in $\mathcal{QIS}_\alpha$. Define the notation $\Lambda(f_\alpha) = \mathcal{PC}(f_\alpha) \cup \Delta(f_\alpha)$. For values of α in HT_N, we may use the construction in Section 3.5 to define the nest of domains $\Omega_0^n(f_\alpha)$ for the map f_α. Denote $\Lambda(f_\alpha) = \mathcal{PC}(f_\alpha) \cup \Delta(f_\alpha)$. By Proposition 5.10, $\Lambda(f_\alpha) = \cap_{n=0}^{\infty} \Omega_0^n(f_\alpha)$.

Let us fix an arbitrary $\alpha_0 \in \mathrm{HT}_N$ and $\varepsilon > 0$. By the above paragraph, there is a positive integer n such that $\Omega_0^n(f_\alpha) \subseteq B(\Lambda(f_\alpha), \varepsilon/2)$.

Let I_n denote the set of rotation numbers $\alpha \in (0, r_3]$ such that the first $n+1$ entries of the continued fraction expansion of α are equal to the corresponding entries of α_0. The set I_n is a connected subset of the real line. By the hypothesis of the theorem, $\alpha_0 \in \mathrm{HT}_N$. Therefore, the first $n+1$ entries of every $\alpha \in I_n$ are bigger than or equal to N. Then, by Theorem 2.6, for all $\alpha \in I_n$, f_α is $n+1$ times near-parabolic renormalizable. In particular, the sets $\Omega_0^n(f_\alpha)$ are defined for all those α. Note that if f_α is $n+1$ times renormalizable, we may define $\Omega_0^n(f_\alpha)$.

Recall that the Fatou coordinate of a map, defined in Proposition 2.3, depends continuously on the map. Then, it follows from the definitions that for $\alpha \in (0, r_3]$, the closure of $\Omega_0^0(f_\alpha)$ depends continuously on α, with respect to the Hausdorff distance on compact subsets of $\mathbb{C}$. Moreover, for $\alpha \in I_n$, the maps $\mathcal{R}^{\circ i}(f_\alpha)$, for $0 \leq i \leq n$ are defined and depend continuously on α. This implies that there is a neighborhood $J_n \subseteq I_n$ of α_0 such that for all $\alpha \in J_n$ we have $\Omega_0^n(f_\alpha) \subseteq B(\Omega_0^n(f_\alpha), \varepsilon/2)$.

By the above paragraphs, for all $\alpha \in J_n$, we have $\Omega_0^n(f_\alpha) \subseteq B(\Lambda(f_{\alpha_0}), \varepsilon)$. For α in $J_n \cap \mathrm{HT}_N$, $\Lambda(f_\alpha) = \cap_{n=0}^{\infty} \Omega_0^n(f_\alpha) \subset \Omega_0^n(f_\alpha)$. In particular, we must have $\Lambda(f_\alpha) \subseteq B(\Lambda(f_{\alpha_0}), \varepsilon)$. As ε was chosen arbitrarily, this finishes the proof of the theorem. □

6. Perturbed Fatou coordinates

In this section we analyze the perturbed Fatou coordinates. Our approach incorporates ideas from quasi-conformal mappings and the generalized Cauchy integral formula, although quasi-conformal mappings do not directly appear here. After writing this paper, the methods presented here have been further developed in [**Che13**] to prove an optimal infinitesimal estimate on the perturbed Fatou coordinates. This method is also employed in [**CC15**] to prove some sharp estimates on the dependence of the Fatou coordinates on the linearity and non-linearity of the maps.

We shall work with the maps in the class

$$\mathcal{QIS}_\alpha := \mathcal{IS}_\alpha \cup \{Q_\alpha\},$$

where $\alpha \in \mathbb{R}$, and $\mathcal{IS}_\alpha$ as well as $\{Q_\alpha\}$ are the sets of maps defined in Section 2.2. The unique critical point of a map in $\mathcal{QIS}$ is used to normalized the Fatou coordinate, i.e. make it unique.

However, most of the arguments presented here may be applied to more general maps, provided there is a special point that can be used to make the Fatou coordinate unique.

6.1. Unwrapping the map, pre-Fatou coordinate. — Recall that an element of $\mathcal{IS}_\alpha$ is of the form

$$h(z) = P \circ \varphi^{-1}(e^{2\pi \mathbf{i}\alpha} \cdot z) \colon e^{-2\pi\alpha\mathbf{i}} \cdot \varphi(U) \to \mathbb{C},$$

where $\alpha \in \mathbb{R}$, as well as $\varphi : U \to \mathbb{C}$ is a univalent map with $\varphi(0) = 0$ and $\varphi'(0) = 1$.

Lemma 6.1. — *The domain U contains $B(0, 8/9)$. Every $h \in \mathcal{QIS}_\alpha$, with $\alpha \in \mathbb{R}$, is univalent on $B(0, 4/27)$, and $|\mathrm{cp}_h| \in [4/27, 4/3]$.*

Proof. — Recall from Section 2.2 that the ellipse E is contained in $B(0, 2)$. By a simple calculation, this implies that U contains $B(0, 8/9)$.

The polynomial P is univalent on $B(0, 1/3)$. That is because,

$$P(w_1) - P(w_2) = (w_1 - w_2)((1 + w_1 + w_2)^2 - w_1 w_2),$$

while for $w_1, w_2 \in B(0, 1/3)$, $\mathrm{Re}(1 + w_1 + w_2)^2 > 1/9$ and $-1/9 < \mathrm{Re}(w_1 w_2) < 1/9$. Hence, for w_1 and w_2 in $B(0, 1/3)$, $P(w_1) = P(w_2)$ only if $w_1 = w_2$.

The map $\psi(z) = \frac{3}{2}\varphi(\frac{2}{3} \cdot z)$ is defined and univalent on the disk $|z| < 1$. By the distortion Theorem 2.1, $\psi(B(0, 1/2))$ contains the ball of radius $(1/2)/(1 + 1/2)^2 = 2/9$. Moreover, $2/9 \le |\psi(-1/2)| \le 2$. This implies that $\varphi(B(0, 1/3)) \supset B(0, 4/27)$ and $|\mathrm{cp}_h| = |\varphi(-1/3)| \in [4/27, 4/3]$. In particular, h must be univalent on $B(0, 4/27)$.

The quadratic polynomial Q_α has a critical point at $-8e^{-2\pi\alpha\mathbf{i}}/27$. By a calculation similar to the above one for the polynomial P, Q_α is univalent on the ball $B(0, 8/27)$. □

Recall from Theorem 2.3 that every $h \in \mathcal{QIS}_\alpha$, with $\alpha \in (0, r_1]$, has a non-zero fixed point $\sigma_h \in \partial\mathcal{P}_h$. We will see in Lemma 6.4 that on some fixed neighborhood of 0 independent of h, 0 and σ_h are the only fixed points of h. Following [**Shi00**], we write

$$h(z) = z + z(z - \sigma_h)u_h(z),$$

with $u_h(z)$ a holomorphic function defined on Dom h. If 0 is a simple fixed point of h, i.e. $h'(0) \neq 1$, $u_h(0) \neq 0$. Differentiating this equation at 0, one obtains

$$\sigma_h = \frac{1 - e^{2\pi\mathbf{i}\alpha}}{u_h(0)}. \tag{29}$$

Let us define the constant

$$C_0 := \inf\{|u_h(0)| : h \in \cup_{\alpha\in(0,r_1]}\mathcal{IS}_\alpha\}. \tag{30}$$

Lemma 6.2. — *We have $C_0 > 0$.*

Proof. — For every $\alpha \in (0, r_1]$, and every h in the closure of the class of maps $\mathcal{IS}_\alpha$ (with respect to the uniform convergence on compact sets), h has two preferred fixed points at 0 and σ_h. Hence, for every such h, $u_h(z)$ is defined on Dom h. Moreover, since for all those h, $h'(0) = e^{2\pi\mathbf{i}\alpha} \neq 1$, 0 is a simple fixed point of h, and so $u_h(0) \neq 0$. This implies that for every $\alpha \in (0, r_1]$, $\inf\{|u_h(0)|; h \in \mathcal{IS}_\alpha\}$ is non-zero. On the other hand, when $\alpha = 0$, $\sigma_h = 0$

is a double fixed point of $h \in \mathcal{IS}_0$, and we may define $u_h(z) = (h(z) - z)/z^2$. In particular, $u_h(0) = h''(0)/2$. By Equation (2), $|h''(0)| \in [2, 7]$. The map $h \mapsto u_h(0)$ is defined on the closure of the set of maps $\cup_{\alpha\in[0,r_1]}\mathcal{IS}_\alpha$, and depends continuously on α and h. This finishes the proof of the lemma. □

For the map $h = Q_\alpha$, $\sigma_h = (1 - e^{2\pi\mathbf{i}\alpha})\frac{16}{27}e^{-4\pi\alpha\mathbf{i}}$. Define the constant

$$C_1 = \max\{\frac{2\pi}{C_0}, \frac{32\pi}{27}\}. \tag{31}$$

By Lemma 6.2 and Equation (29), for all $\alpha \in (0, r_1]$ and all $h \in \mathcal{QIS}_\alpha$,

$$|\sigma_h| \le C_1\alpha. \tag{32}$$

Define the map $\tau_h : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{0, \sigma_h\}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, by the formula

$$\tau_h(w) := \frac{\sigma_h}{1 - e^{-2\pi\mathbf{i}\alpha w}}. \tag{33}$$

This is a covering transformation from $\mathbb{C}$ onto $\hat{\mathbb{C}} \setminus \{0, \sigma_h\}$ with the deck transformation group generated by the map $w \mapsto w+1/\alpha$, that is, $\tau_h(w+1/\alpha) = \tau_h(w)$. Moreover, we have $\tau_h(w) \to 0$ as $\operatorname{Im} w \to +\infty$, and $\tau_h(w) \to \sigma_h$ as $\operatorname{Im} w \to -\infty$.

For $\alpha \in (0, r_1]$ and $R \in (0, +\infty)$, define the set

$$\Theta_\alpha(R) := \mathbb{C} \setminus \cup_{n\in\mathbb{Z}} B(n/\alpha, R).$$

Lemma 6.3. — *We have*

(1) for all $\delta > 0$, all α with $0 < \alpha \le \min\{r_1, \delta/(2C_1)\}$, and all $h \in \mathcal{QIS}_\alpha$,

$$\tau_h(\Theta_\alpha(C_1/(\pi\delta))) \subset B(0, \delta).$$

(2) for all $\alpha \in (0, r_1]$, all $h \in \mathcal{QIS}_\alpha$, all $r \in (0, 1/2]$, and all $w \in \Theta_\alpha(r/\alpha)$, we have

$$|\tau_h(w)| \le e^{2\pi}C_1\frac{\alpha}{r}e^{-2\pi\alpha\operatorname{Im} w}.$$

Proof. — Part (1) Fix $\delta > 0$ and let $\varepsilon = \min\{r_1, \delta/(2C_1)\}$ and $R = C_1/(\pi\delta)$. For all $\alpha \in (0, \varepsilon]$ and all $w \in \partial B(0, R)$, $|1 - e^{-2\pi\mathbf{i}\alpha w}| \ge 1 - e^{-2\pi\alpha R}$. Also, by elementary calculations, for all $\alpha \in [0, \varepsilon]$, $1 - e^{-2\pi\alpha R} - \pi\alpha R \ge 0$ (This holds at the end points 0 and ε as well as at the critical point $\delta \log 2/(2C_1)$). Thus, by Equation (32), and the periodicity of τ_h, for $w \in \partial\Theta_\alpha(0, R)$,

$$|\tau_h(w)| \le |\frac{\sigma_h}{1 - e^{-2\pi\alpha R}}| \le \frac{C_1\alpha}{\pi\alpha R} = \delta.$$

On the other hand, for every $\alpha \in (0, \varepsilon]$, and every $h \in \mathcal{QIS}_\alpha$, as $w \in \Theta_\alpha(R)$ tends to $+\mathbf{i}\infty$, $\tau_h(w) \to 0$, and as $w \to -\mathbf{i}\infty$, $\tau_h(w) \to \sigma_h$, where $|\sigma_h| \le C_1\alpha \le \delta/2$. Therefore, by the maximum principle, the inequality in part 1 holds on $\Theta_\alpha(R)$.

Part (2) For $w \in \partial\Theta_\alpha(r/\alpha)$, we have

$$|1 - e^{-2\pi\mathbf{i}\alpha w}| \ge 1 - e^{-2\pi r} \ge e^{-2\pi}re^{2\pi r} \ge e^{-2\pi}re^{2\pi\alpha\operatorname{Im} w}.$$

Also, as $\operatorname{Im} w \to \pm\infty$, $|1 - e^{-2\pi \mathbf{i}\alpha w}| \geq e^{-2\pi} r e^{2\pi\alpha \operatorname{Im} w}$ holds. By the maximum principle, for $w \in \Theta_\alpha(r/\alpha)$ we must have $|1 - e^{-2\pi\mathbf{i}\alpha w}| \geq e^{-2\pi} r e^{2\pi\alpha \operatorname{Im} w}$. Then, by Equation (32),

$$|\tau_h(w)| = \frac{|\sigma_h|}{|1 - e^{-2\pi\mathbf{i}\alpha w}|} \leq C_1 e^{2\pi} \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}.$$

□

One may lift h under τ_h to obtain a map F_h defined near $+\mathbf{i}\infty$ and $-\mathbf{i}\infty$. Any such lift satisfies

$$h \circ \tau_h(w) = \tau_h \circ F_h(w),\ F_h(w) + \alpha^{-1} = F_h(w + \alpha^{-1}), \tag{34}$$

wherever they are defined. In the next lemma we analyze the domain of definition of F_h and its asymptotic behaviors near $+\mathbf{i}\infty$ and $-\mathbf{i}\infty$. The plan is to study how the perturbed Fatou coordinate of h, Φ_h, compares with an appropriate inverse branch of τ_h.

6.2. Estimates on the lift F_h. —

Lemma 6.4. — *There are constants $r_2' > 0$, C_2, and C_3 such that for all $\alpha \in (0, r_2']$, and all $h \in \mathcal{QIS}_\alpha$, there exists a lift of h under τ_h, denoted by F_h such that*

(1) F_h is defined and univalent on $\Theta_\alpha(C_2)$;
(2) for all $w \in \Theta_\alpha(C_2)$ we have

$$|F_h(w) - (w+1)| < 1/4, \quad |F_h'(w) - 1| < 1/4;$$

(3) for all $r \in (0, 1/2]$ and all $w \in \Theta_\alpha(r/\alpha) \cap \Theta_\alpha(C_2)$ we have

$$|F_h(w) - (w+1)| < C_3 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}, \quad |F_h'(w) - 1| < C_3 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}.$$

Proof. — Part (1): By Lemma 6.1, for every $\alpha \in \mathbb{R}$, every map $h \in \mathcal{QIS}_\alpha$ is defined and is univalent on $B(0, 4/27)$. We apply Lemma 6.3-1 with $\delta = 4/27$ to obtain $\varepsilon_1 = \min\{r_1, 2/(27C_1)\}$ such that for all $\alpha \in (0, \varepsilon_1]$ and $h \in \mathcal{QIS}_\alpha$ we have $\tau_h(\Theta_\alpha(27C_1/(4\pi))) \subset B(0, 4/27)$. In particular, $\sigma_h = \lim_{\operatorname{Im} w \to -\infty} \tau_h(w)$ must be in $B(0, 4/27)$. On the other hand, as h is univalent on $B(0, 4/27)$, 0 and σ_h are the only pre-images of 0 and σ_h in $B(0, 4/27)$. This implies that any lift of h under τ_h is a well-defined, finite, and univalent function on $\Theta_\alpha(27C_1/(4\pi))$. There are many choices for this lift, but we may choose one with

$$\lim_{\operatorname{Im} w \to +\infty} (F_h(w) - w) = 1. \tag{35}$$

Since $\varepsilon_1 \leq 2\pi/(27C_1)$, $\Theta_\alpha(27C_1/(4\pi))$ is connected and the normalization of F_h near $+\mathbf{i}\infty$ uniquely determines the lift F_h on $\Theta_\alpha(27C_1/(4\pi))$. Although F_h may be defined beyond $\Theta_\alpha(27C_1/(4\pi))$, it may have singularities outside of $\Theta_\alpha(27C_1/(4\pi))$ when $h^{-1}(\sigma_h) \setminus \{\sigma_h\}$ is not empty.

Part (2): Using the first formula in Equation (34), one can see that F_h is given by the formal expression

$$\begin{aligned} F_h(w) &= w + \frac{1}{2\pi\alpha\mathbf{i}} \log\big(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}\big) \\ &= w + 1 + \frac{1}{2\pi\alpha\mathbf{i}} \log\big(e^{-2\pi\alpha\mathbf{i}}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)})\big), \text{ with } z = \tau_h(w). \end{aligned}$$

The branch of log on the second line of the above equations is determined by $-\pi < \operatorname{Im}\log(\cdot) < \pi$. This is consistent with our choice of the branch in Equation (35).

Recall that by Lemma 6.1 every $h \in \mathcal{QLS}_\alpha$ is defined on $B(0, 4/27)$. Moreover, by the distortion Theorem 2.1, $\cup_{\alpha\in[0,r_1]}\mathcal{QLS}_\alpha$ forms a pre-compact class of maps. By the continuous dependence of u_h on h, the restrictions of u_h to $B(0, 4/27)$ must also form a pre-compact class. This implies that there exists a positive constant $\delta_1 \le 4/27$ such that

$$\forall z \in B(0, \delta_1), \quad \Big|1 - \frac{u_h(z)}{(1 + z u_h(z))u_h(0)}\Big| < \frac{1}{4\pi}. \tag{36}$$

Using Lemma 6.3-1 with $\delta = \delta_1$ we obtain

$$\varepsilon_2 = \min\{r_1, \delta_1/(2C_1)\}$$

such that for all $w \in \Theta_\alpha(C_1/(\pi\delta_1))$, all $\alpha \in (0, \varepsilon_2]$ and all $h \in \mathcal{QLS}_\alpha$, one has $|z| = |\tau_h(w)| < \delta_1$. Replacing σ_h by the expression in (29) and using $|1 - e^{2\pi\mathbf{i}\alpha}| < 2\pi\alpha$, we obtain

$$\begin{aligned} \Big|e^{-2\pi\alpha\mathbf{i}}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)}) - 1\Big| &= \Big|1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)} - e^{2\pi\mathbf{i}\alpha}\Big| \\ &= \Big|(1 - e^{2\pi\mathbf{i}\alpha})(1 - \frac{u_h(z)}{(1 + z u_h(z))u_h(0)})\Big| \\ &< 2\pi\alpha \cdot \frac{1}{4\pi} < \frac{1}{2}. \end{aligned}$$

In particular, for all $z \in \tau_h(\Theta_\alpha(C_1/(\pi\delta_1)))$, in the expression for F_h the branch of log is well defined. Note that for all $x \in B(1, 1/2)$ we have $|\log x| \le 2|x - 1|$. Then, for all $w \in \Theta_\alpha(C_1/(\pi\delta_1))$, using the above inequality we get

$$\begin{aligned} |F_h(w) - (w + 1)| &= \Big|\frac{1}{2\pi\alpha\mathbf{i}} \log\Big(e^{-2\pi\alpha\mathbf{i}}(1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)})\Big)\Big| \\ &\le \frac{1}{2\pi\alpha} \cdot 2 \cdot (2\pi\alpha \cdot \frac{1}{4\pi}) < \frac{1}{4}. \end{aligned}$$

This finishes the proof of the first inequality in part 2. On the other hand, for w_0 in $\Theta_\alpha(C_1/(\pi\delta_1)+1)$, the map $F_h(w) - w - 1$ is defined on $B(w_0, 1)$ and its absolute value is bounded by $1/4$. By the Schwarz-Pick lemma, we must have $|F_h'(w_0) - 1| < 1/4$. We may define

$$C_2 := C_1/(\pi\delta_1) + 1.$$

Part (3): By Equation (36), for all $z \in \partial B(0,\delta_1)$ we have,

$$\Big|1-\frac{u_h(z)}{(1+zu_h(z))u_h(0)}\Big| < \frac{1}{4\pi} = \frac{1}{4\pi\delta_1}|z|.$$

As z tends to 0, the expression on the left hand of the above equation tends to 0. By the maximum principle, the above inequality must hold on $B(0,\delta_1)$. Therefore, for w in $\Theta_\alpha(r/\alpha)\cap \Theta_\alpha(C_1/(\pi\delta_1))$ we have

$$\begin{aligned}|F_h(w)-(w+1)| &= \Big|\frac{1}{2\pi\alpha\mathbf{i}}\log\Big(e^{-2\pi\alpha\mathbf{i}}(1-\frac{\sigma_h u_h(z)}{1+zu_h(z)})\Big)\Big| \\ &\le \frac{1}{2\pi\alpha}\cdot 2\cdot\Big|e^{-2\pi\alpha\mathbf{i}}(1-\frac{\sigma_h u_h(z)}{1+zu_h(z)})-1\Big| \\ &\le \frac{1}{2\pi\alpha}\cdot 2\cdot 2\pi\alpha\cdot\Big|1-\frac{u_h(z)}{(1+zu_h(z))u_h(0)}\Big| \le \frac{|z|}{2\pi\delta_1}.\end{aligned}$$

By the estimate in Lemma 6.3-2, we obtain the first inequality in part 3 with

$$C_3 = \frac{e^{2\pi}C_1}{2\pi\delta_1}.$$

The second inequality is similarly proved using the Schwarz-Pick lemma.

The polynomial $P(z)=z(1+z)^2$ belongs to $\mathcal{IS}_0$. This implies that $C_0 \le u_P(0) = 2$. Then,

$$\varepsilon_2 \le \frac{\delta_1}{2C_1} \le \frac{4}{27}\cdot\frac{1}{2}\cdot\min\Big\{\frac{C_0}{2\pi},\frac{27}{32\pi}\Big\} \le \frac{2}{27\pi}.$$

This guarantees that

$$2C_2+\frac{3}{2} = 2(\frac{C_1}{\pi\delta_1}+1)+\frac{3}{2} \le \frac{1}{\pi\varepsilon_2}+\frac{7}{2} \le \frac{1}{\varepsilon_2} \le \frac{1}{\alpha}. \tag{37}$$

In particular, for $\alpha \in (0,\varepsilon_2]$, the set $\Theta_\alpha(C_2)$ is connected. □

6.3. Fatou coordinate of F_h, L_h. — Let $h \in \mathcal{QIS}_\alpha$, $\alpha \in (0,r_1]$, with the perturbed Fatou coordinate $\Phi_h : \mathcal{P}_h \to \mathbb{C}$ introduced in Theorem 2.3. The set $\tau_h^{-1}(\mathcal{P}_h)$ is periodic of period $1/\alpha$ and is contained in $\mathbb{C}\setminus(\mathbb{Z}/\alpha)$. Each connected component of $\tau_h^{-1}(\mathcal{P}_h)$ is simply connected and is bounded by piece-wise analytic curves going from $-\mathbf{i}\infty$ to $+\mathbf{i}\infty$. Let $\widetilde{\mathcal{P}}_h$ denote the connected component of $\tau_h^{-1}(\mathcal{P}_h)$ separating 0 from $1/\alpha$. We have a univalent map

$$L_h := \Phi_h\circ\tau_h : \widetilde{\mathcal{P}}_h \to \mathbb{C}. \tag{38}$$

The map $h : \mathcal{P}_h \to \mathbb{C}\setminus\{0,\sigma_h\}$ lifts to a univalent map $F_h : \widetilde{\mathcal{P}}_h \to \mathbb{C}$ that agrees with the lift we chose in Lemma 6.4. There is a unique point $\mathrm{cp}_{F_h} \in \partial\widetilde{\mathcal{P}}_h$ with $\tau_h(\mathrm{cp}_{F_h}) = \mathrm{cp}_h$. As F_h is univalent on $\Theta_\alpha(C_2)$, cp_{F_h} must lie in $B(0,C_2)\cup B(1/\alpha,C_2)$. Because of orientation, cp_{F_h} must lie in $B(0,C_2)$.

By Theorem 2.3, L_h satisfies the following properties:

1) $L_h(\mathrm{cp}_{F_h}) = 0$;
2) as $\mathrm{Im}(w)\to+\infty$ in $\widetilde{\mathcal{P}}_h$, $\mathrm{Im}\,L_h(w)\to+\infty$, and as $\mathrm{Im}(w)\to-\infty$ in $\widetilde{\mathcal{P}}_h$, $\mathrm{Im}\,L_h(w)\to-\infty$;
3) $L_h(\widetilde{\mathcal{P}}_h) \supset \{w\in\mathbb{C} : 0<\mathrm{Re}(w)\le 1\}$;

4) If w and $F_h(w)$ belong to $\widetilde{\mathcal{P}}_h$, then

$$L_h(F_h(w)) = L_h(w) + 1. \tag{39}$$

We wish to estimate the map L_h on $\widetilde{\mathcal{P}}_h$. To this end, we need to extend L_h onto a larger domain containing $\widetilde{\mathcal{P}}_h$. This can be done using the above functional equation, carried out in the remaining of this section.

By Lemma 6.4-2, for every $w \in \Theta_\alpha(C_2)$,

$$|\arg(F_h(w) - w)| < \arcsin(1/4) < \arcsin((\sqrt{6} - \sqrt{2})/4) = \pi/12. \tag{40}$$

Let $a = C_2 e^{5\pi \mathbf{i}/12}$, and $\overline{a}$ denote the complex conjugate of a. Define the piece-wise analytic curve

$$\begin{aligned}\ell = \{w \in \mathbb{C} : \arg(w - a) = 11\pi/12\} \\ \cup \{w \in \mathbb{C} : \arg(w - \overline{a}) = -11\pi/12\} \\ \cup \{C_2 e^{2\pi \mathbf{i}\theta} : \theta \in [-5\pi/12, 5\pi/12]\}.\end{aligned}$$

Let ℓ' denote the curve $-\ell + 1/\alpha$. The union of ℓ and ℓ' decomposes $\mathbb{C}$ into three connected components. Let

$$A_1$$

denote the connected component of $\mathbb{C} \setminus (\ell \cup \ell')$ that contains $1/(2\alpha)$. The set A_1 is connected and simply connected.

The line ℓ intersects the vertical line $\operatorname{Re} w = 0$ at points denoted by b and $\overline{b}$, with $\operatorname{Im} b > \operatorname{Im} \overline{b}$. Let $\theta_0 := \arcsin(1/4)/2 + \pi/24 \in (\arcsin(1/4), \pi/12)$, and define

$$A := \{w \in \mathbb{C} \mid \arg(w - b) \in [\pi - \theta_0, 11\pi/12]\}\}.$$

Set

$$A_2 := A_1 \cup A \cup s(A) \cup (-A + 1/\alpha) \cup (-s(A) + 1/\alpha),$$

where $s(w)$ denotes the complex conjugate of $w \in \mathbb{C}$.

Each of the curves $L_h^{-1}(\mathbf{i}\mathbb{R})$ and $L_h^{-1}(1 + \mathbf{i}\mathbb{R})$ divide $\mathbb{C}$ into two connected components; say R for the right hand connected component of $\mathbb{C} \setminus L_h^{-1}(\mathbf{i}\mathbb{R})$ and L for the left hand connected component of $\mathbb{C} \setminus L_h^{-1}(1 + \mathbf{i}\mathbb{R})$. Also, let L' and R' denote the corresponding components of $\mathbb{C} \setminus A_2$. These are open subsets of $\mathbb{C}$. Define

$$A_3 := (R \cap L') \cup (L \cap R'), \text{ [10]}.$$

The real analytic curves bounding A_2 intersect the curves $L_h^{-1}(\mathbf{i}\mathbb{R})$ and $L_h^{-1}(1 + \mathbf{i}\mathbb{R})$ at most in a finite number of places. Thus, A_3 is a union of a finite number of simply connected domains.

Set

$$X := \operatorname{int}(A_2 \cup A_3).$$

The set X is connected, $\operatorname{cp}_{F_h} \in \partial X$, $\operatorname{cv}_{F_h} := F_h(\operatorname{cp}_{F_h}) \in X$, and $L_h^{-1}((0,1) + \mathbf{i}\mathbb{R}) \subset X$.

Lemma 6.5. — *For every $\alpha \in (0, r_2']$ and every $h \in \mathcal{QIS}_\alpha$ we have*

10. The set $L \cap R'$ is likely to be empty. We show in Lemma 6.10 that this is the case when α is small enough.

(1) X is simply connected and F_h is defined and is univalent on X;
(2) for all $w \in X$, there are integers $m_w < n_w$ such that for all integers j with $m_w < j < n_w$ we have $F_h^{\circ j}(w) \in X$, but for $j \in \{m_w, n_w\}$ we have $F_h^{\circ j}(w) \notin X$.
(3) for all $w \in X$, there exists a unique integer $j_w \in [m_w + 1, n_w - 1]$ with $F_h^{\circ j_w}(w) \in L_h^{-1}((0,1] + \mathbf{i}\mathbb{R})$;
(4) for all $w \in B(0, C_2) \cap X$, there is a positive integer l_w with $\operatorname{Re} F_h^{\circ l_w}(w) \geq C_2$. Moreover, l_w is uniformly bounded from above independent of α, h, and w.

Proof. — Part (1): As $A_2 \subset \Theta_\alpha(C_2)$, by Lemma 6.4, F_h is defined on A_2. The main issue we face here is to show that 0 and $1/\alpha$ do not belong to A_3.

Fix $h_0 \in \mathcal{IS}_0 \cup \{Q_0\}$ and for $\alpha \in (0, r_2']$ let $h_\alpha(z) := h_0(e^{2\pi \mathbf{i}\alpha} z)$ be an element of $\mathcal{QIS}_\alpha$. When $\alpha \to 0$, τ_{h_α} converges uniformly on compact sets to the map $\tau_{h_0}(w) := -2/(h_0''(0)w)$, and h_0 may be lifted under τ_{h_0} to a unique map F_{h_0}. Recall the petal $\mathcal{P}_{h_0}$, the Fatou coordinate Φ_{h_0}, and the relation $\Phi_{h_0}(\mathcal{P}_{h_0}) = (0, +\infty) + \mathbf{i}\mathbb{R}$ from Theorem 2.2. Under τ_{h_0}, $\mathcal{P}_{h_0}$ lifts to a set $\widetilde{\mathcal{P}}_{h_0}$ and Φ_{h_0} lifts to the univalent map $L_{h_0} : \widetilde{\mathcal{P}}_{h_0} \to \mathbb{C}$. Then, we have $L_{h_0}(\widetilde{\mathcal{P}}_{h_0}) = (0, +\infty) + \mathbf{i}\mathbb{R}$ and $L_{h_0}(F_{h_0}(w)) = L_{h_0}(w) + 1$, for all $w \in \widetilde{P}_{h_0}$. The map $L_{h_0}^{-1}$ extends onto $[0, +\infty) + \mathbf{i}\mathbb{R}$ and its image covers the right hand connected component of $\mathbb{C} \setminus L_{h_0}^{-1}(\mathbf{i}\mathbb{R})$. On the other hand, as $\mathcal{P}_{h_0}$ is compactly contained in Dom h_0, 0 does not belong to the closure of $\widetilde{\mathcal{P}}_{h_0}$. Thus, 0 must be in the left hand connected component of $L_{h_0}^{-1}(\mathbf{i}\mathbb{R})$. For all $\alpha \in (0, r_2']$, $0 \notin \tau_{h_\alpha}^{-1}(\text{Dom } h \setminus \{0, \sigma_{h_\alpha}\})$. Hence, by the continuous dependence of L_{h_α} on α, for all $\alpha \in (0, r_2']$, 0 must belong to the left hand connected component of $L_{h_\alpha}^{-1}(\mathbf{i}\mathbb{R})$. Similarly, this implies that for all $h \in \cup_{\alpha \in (0, r_2']} \mathcal{QIS}_\alpha$, $1/\alpha$ belongs to the right-hand side connected component of $\mathbb{C} \setminus L_h^{-1}(1 + \mathbf{i}\mathbb{R})$.

Recall the set $A_3 = (R \cap L') \cup (L \cap R')$. By the above paragraph, $0 \in L$ and $1/\alpha \in R$. Using the periodicity of F_h and the above argument, one may prove that $-n/\alpha \in L - n/\alpha$, for $n \in \mathbb{N}$, and $n/\alpha \in R + n/\alpha$, for $n \in \mathbb{N}$. Therefore, $-n/\alpha \in L$, for $n \in \mathbb{N} \cup \{0\}$ and $n/\alpha \in R$, for $n \in \mathbb{N}$. Clearly, $-n/\alpha \notin R'$, for $n \in \mathbb{N} \cup \{0\}$, and $n/\alpha \notin L'$, for $n \in \mathbb{N}$. Putting these together, we deduce that for all $n \in \mathbb{Z}$, $n/\alpha \notin A_3$.

Let $U_h = \text{Dom } h \subset \mathbb{C}$. The set $\hat{\mathbb{C}} \setminus U_h$ lifts under τ_h to countably many simply connected components each containing a unique n/α for some $n \in \mathbb{Z}$. Since every component of A_3 is a simply connected region whose boundary is contained in $\tau_h^{-1}(U_h)$, and A_3 avoids $\mathbb{Z}/\alpha$, $A_3 \subset \tau_h^{-1}(U_h)$. This implies that F_h is defined on A_3, except possibly on a discrete set of singularities. Such singularities might arise at τ_h^{-1} of $h^{-1}(\{0, \sigma_h\}) \setminus \{0, \sigma_h\}$, if such points lie in $\tau_h(A_3)$. Thus, so far we know that such singularities may not occur on ∂A_3, and hence may not occur on ∂X.

Recall the polynomial $P(z) = z(1+z)^2$ in the definition of the class $\mathcal{IS}_0$. There is $\varepsilon > 0$ such that $B(0, \varepsilon)$ is only covered once by the map $P : U \to \mathbb{C}$, that is, $P : P^{-1}(B(0, \varepsilon)) \cap U \to B(0, \varepsilon)$ is a homeomorphism. When α is small enough, σ_h belongs to $B(0, \varepsilon)$. Thus, 0 and σ_h must be the only pre-images of 0 and σ_h within U_h. Combining with the above paragraph, for small values of α, there is no singularity of F_h within X. On the other hand, ∂X has continuous dependence on α and h. The singularities also depend continuously on α and h, when they

exist, and do not hit ∂X by the above paragraph. Therefore, for all $\alpha \in (0, r_2']$ and $h \in \mathcal{QLS}_\alpha$, F_h has no singularity in X.

The set A_2 is simply connected and is bounded by a finite number of analytic curves. The set X is formed of attaching to A_2 a finite number of simply connected domains that share a connected and analytic boundary curve with A_2. This implies that X is simply connected. By the same argument $F_h(X)$ is also simply connected.

Let $u : X \to B(0,1)$ and $u' : F_h(X) \to B(0,1)$ be some uniformizations. The map $u' \circ F_h \circ u^{-1} : B(0,1) \to B(0,1)$ is a proper mapping. So, it must be a finite Blaschke product. But, F_h^{-1} is uniquely defined near $\pm\mathbf{i}\infty$. Thus, the blaschke product must be of degree one, or in other words, $F_h : X \to F_h(X)$ is a homeomorphism.

Part (2): Let $\beta_l, \beta_r : \mathbb{R} \to \partial X$ be the piece-wise analytic curves bounding X, with β_l on the left side of β_r as well as $\operatorname{Im}\beta_l(t)$ and $\operatorname{Im}\beta_r(t)$ tending to $+\infty$ as $t \to +\infty$. There are four points $t_i \in \mathbb{R}$, for $i = 1, 2, 3, 4$, such that

$$\forall t \in (t_1, t_2), \operatorname{Im}\beta_l(t_1) < \operatorname{Im}\beta_l(t) < \operatorname{Im}\beta_l(t_2), \operatorname{Im}\beta_l(t_1) \le -C_2, \operatorname{Im}\beta_l(t_2) \ge C_2,$$
$$\forall t \in (t_3, t_4), \operatorname{Im}\beta_r(t_3) < \operatorname{Im}\beta_r(t) < \operatorname{Im}\beta_r(t_4), \operatorname{Im}\beta_r(t_3) \le -C_2, \operatorname{Im}\beta_r(t_4) \ge C_2.$$

Moreover, one may choose t_2 and t_4 arbitrarily large, as well as t_1 and t_3 arbitrarily small. Then adjust the curves β_r and β_l into simple curves $\hat\beta_r$ and $\hat\beta_l$ as follows.

$$\hat\beta_l(t) := \begin{cases} \beta_l(t_2) + (t - t_2)\mathbf{i} & \text{if } t \ge t_2 \\ \beta_l(t) & \text{if } t \in (t_1, t_2) \\ \beta_l(t_1) + (t - t_1)\mathbf{i} & \text{if } t \le t_1 \end{cases}, \hat\beta_r := \begin{cases} \beta_r(t_4) + (t - t_4)\mathbf{i} & \text{if } t \ge t_4 \\ \beta_r(t) & \text{if } t \in (t_3, t_4) \\ \beta_r(t_3) + (t - t_3)\mathbf{i} & \text{if } t \le t_3 \end{cases}$$

Let B denote the region bounded by the two curves $\hat\beta_l$ and $\hat\beta_r$. As $t_1, t_3 \to -\infty$ and $t_2, t_4 \to +\infty$, the corresponding sets B exhaust X. Note that F_h maps $\hat\beta_l$ into B and F_h^{-1} maps $\hat\beta_r$ into B. The domain $B' := B \cap F_h^{-1}(B)$ is simply connected and bounded by $\hat\beta_l$ and $F_h^{-1}(\hat\beta_r)$.

There is a harmonic function $\mathfrak{u} : B \to (0,1)$ such that $\mathfrak{u}(w) \to 0$ as $w \to \hat\beta_l$ and $\mathfrak{u}(w) \to 1$ as $w \to \hat\beta_r$. Near the upper end of B, $\mathfrak{u}(w)$ tends to a linear function of $\operatorname{Re} w$, that is, as $\operatorname{Im} w \to +\infty$, $\mathfrak{u}(w)$ tends to the function $(\operatorname{Re} w - \operatorname{Re}\hat\beta_l(t_2))/(\operatorname{Re}\hat\beta_r(t_4) - \operatorname{Re}\hat\beta_l(t_2))$. That is because, as $\operatorname{Im} w \to +\infty$, the probability of a Brownian motion in B that starts at height $\operatorname{Im} w$ to hit the height $\max\{\operatorname{Im}\hat\beta_r(t_4), \operatorname{Im}\hat\beta_l(t_2)\}$ tends to zero. Similarly, near the lower end of B, $\mathfrak{u}$ tends to a linear function of $\operatorname{Re} w$.

Consider the harmonic function $\mathfrak{u}_1 : B' \to \mathbb{R}$ defined as $\mathfrak{u}_1(w) := \mathfrak{u}(F_h(w)) - \mathfrak{u}(w)$. We claim that the infimum of $\mathfrak{u}_1$ on B' is strictly positive. By the maximum principle, we only need to show this on the boundary of B'. At $w \in \hat\beta_l$, $\mathfrak{u}_1(w) = \mathfrak{u}(F_h(w)) > 0$, and at $w \in F_h^{-1}(\hat\beta_r)$, $\mathfrak{u}_1(w) = 1 - \mathfrak{u}(w) > 0$. By the above paragraph, near the two ends of B, $\mathfrak{u}(w)$ tend to some linear functions of $\operatorname{Re} w$ and we have $|F_h(w) - w - 1| \le 1/4$. This implies that $\mathfrak{u}_1(w)$ is uniformly bounded away from 0 when $|\operatorname{Im} w|$ is large enough. This finishes the proof of the claim.

By the above paragraph, the forward orbit of every point in B eventually leaves B on the right hand of $\hat\beta_r$ and the backward orbit of every point in B eventually leaves B on the left

hand of $\hat{\beta}_l$. As the sets B exhaust X, we conclude the same statement about every orbit in X. In particular, any orbit in X must cross the closure of the region bounded by the two curves $L_h^{-1}(\mathbf{i}\mathbb{R})$ and $F_h(L_h^{-1}(\mathbf{i}\mathbb{R}))$.

Part (3): By the previous part, for every $w \in X$ both forward and backward orbit of w under F_h leave X. In particular, every such orbit must hit $L_h^{-1}((0,1]+\mathbf{i}\mathbb{R})$.

To see the uniqueness, assume that for some $j_1, j_2 \in \mathbb{Z}$ and $w \in X$, $F_h^{\circ j_1}(w)$ and $F_h^{\circ j_2}(w)$ belong to $L_h^{-1}((0,1]+\mathbf{i}\mathbb{R})$. Then $F_h^{\circ(j_2-j_1)}$ maps the point $F_h^{\circ j_1}(w)$ in $L_h^{-1}((0,1]+\mathbf{i}\mathbb{R})$ into $L_h^{-1}((0,1]+\mathbf{i}\mathbb{R})$. However, by Equation (39), F_h maps every point to the right of $L_h^{-1}(\mathbf{i}\mathbb{R})$ to a point to the right of $L_h^{-1}(1+\mathbf{i}\mathbb{R})$, and F_h^{-1} maps every point to the left of $L_h^{-1}(1+\mathbf{i}\mathbb{R})$ to a point to the left of $L_h^{-1}(\mathbf{i}\mathbb{R})$. Thus, we must have $j_1 = j_2$, which proves the uniqueness of j_w.

Part (4): Since the forward orbit of every point in X under F_h leaves the domain X on the right hand, the existence of l_w follows. The uniform bound on l_w is a result of the pre-compactness of the class $\cup_{\alpha\in[0,r_2']}\mathcal{QIS}_\alpha$. Indeed, as a sequence h_i tends to some map h (not necessarily in the class $\cup_{\alpha\in[0,r_2']}\mathcal{QIS}_\alpha$), the sequence of Fatou coordinates Φ_{h_i} converges uniformly on compact sets to some univalent map Φ_h that enjoys the same equivariance property on its domain of definition. Further details are left to the reader. □

Lemma 6.6. — *For all $\alpha \in (0, r_2']$ and all $h \in \mathcal{QIS}_\alpha$, L_h has a unique univalent extension onto X. In particular, when both w and $F_h(w)$ belong to X, $L_h(F_h(w)) = L_h(w) + 1$.*

Proof. — By Lemma 6.5, for all $w \in X$ there is a unique integer j_w with $F_h^{\circ j_w}(w)$ belongs to $L_h^{-1}((0,1]+\mathbf{i}\mathbb{R})$. Define $L_h(w) := L_h(F_h^{\circ j_w}(w)) - j_w$. Although j_w cannot be continuous in w, thanks to Equation (39) on $\widetilde{\mathcal{P}}_h$, this provides us with a well-defined holomorphic map on X.

To prove that L_h is one-to-one, assume that for some w_1 and w_2 in X, $L_h(w_1) = L_h(w_2)$. Choose $j \in \mathbb{Z}$ with $\operatorname{Re} L_h(w_1) = \operatorname{Re} L_h(w_2) \in (j, j+1]$. We must have $j_{w_1} = j_{w_2} = -j$ and the equation $L_h(w_i) = L_h(F_h^{\circ(-j)}(w_i)) + j$, for $i = 1, 2$. As L_h is univalent on $\widetilde{\mathcal{P}}_h \supset L_h^{-1}((0,1]+\mathbf{i}\mathbb{R})$ and F_h is univalent on X, then $w_1 = w_2$.

Since the holomorphic function $L_h(F_h(w)) - L_h(w) - 1$ is identically zero on $\widetilde{\mathcal{P}}_h$, by the uniqueness of the analytic continuation, it must be 0 on all of X. □

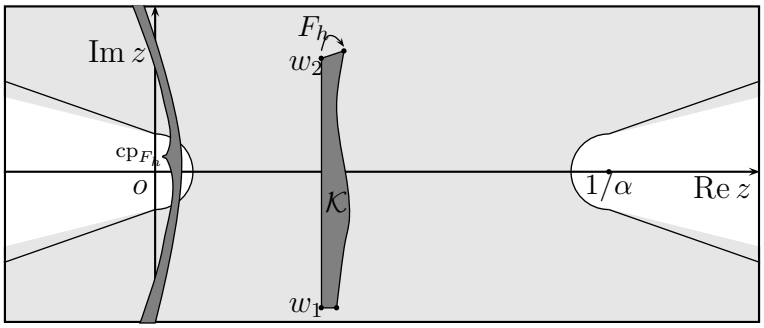


FIGURE 8. The light gray region shows the domain A_2, and the dark gray regions show the sets $\mathcal{K}$ and $L_h^{-1}([0,1]+\mathbf{i}\mathbb{R})$.

6.4. Estimates on L_h. — The univalent map $L_h : X \to \mathbb{C}$ provides us with two foliations on X that are the pre-images of the horizontal and vertical line segments in $L_h(X)$. In other words, the horizontal leaves are the solutions of the vector field $1/L_h'$, while the vertical ones are the integral curves of the vector field $\mathbf{i}/L_h'$. By Lemma 6.5, the horizontal leaves are invariant under F_h while the vertical leaves are mapped to one another under F_h. In this section we prove some geometric features of these foliations.

In the next lemma, the sets A_1 and A_2 refer to the sets defined in Section 6.3. The constants r_2' and C_3 are introduced in Lemma 6.4.

Lemma 6.7. — *There exists $C \in \mathbb{R}$ such that for all $\alpha \in (0, r_2']$ and all $h \in \mathcal{QIS}_\alpha$, the following hold.*

(1) For all w with $B(w,5) \subseteq A_2$, we have $|\arg(L_h'(w))| < \pi/3$ and $2/5 \le |L_h'(w)| \le 8/3$.

(2) For all $R \in [3.25, 1/(2\alpha)]$ and all w with $B(w,R) \subseteq A_2$,

$$|\arg(L_h'(w))| \le \frac{8C_3}{3R} e^{-2\pi\alpha \operatorname{Im} w} + \frac{40}{3R},$$

$$(1 - \frac{8C_3}{3R} e^{-2\pi\alpha \operatorname{Im} w})(1 - \frac{5}{R}) \le |L_h'(w)| \le (1 + \frac{8C_3}{3R} e^{-2\pi\alpha \operatorname{Im} w})(1 + \frac{40}{3R}).$$

(3) For all $w \in A_1$, $C^{-1} \le |L_h'(w)| \le C$.

(4) As $\operatorname{Im} w \to +\infty$ in A_1, $|L_h'(w) - 1| = O(1/\operatorname{Im} w + \alpha e^{-2\pi\alpha \operatorname{Im} w})$, with the constant in O independent of α and h.

Proof. — Part (1): Given w_0 with $B(w_0, 5) \subseteq A_2$, let $w_1 := 4(F_h(w_0) - w_0)/15$. By Lemma 6.4-2, $|F_h(w_0) - w_0| < 1 + 1/4$, $|w_1| < 1/3$, and $B(F_h(w_0), 15/4) \subset B(w_0, 5)$. Thus, L_h is defined and univalent on $B(F_h(w_0), 15/4)$. The function

$$\psi(w) = \Big(L_h(F_h(w_0)) - L_h(F_h(w_0) - \frac{15w}{4})\Big) \frac{4}{15 L_h'(F_h(w_0))},$$

is defined and univalent on $|w| < 1$ with $\psi(0) = 0$ and $\psi'(0) = 1$. Applying Theorem 2.1-4 to the above map at w_1 we obtain

$$|\arg(w_1 L_h'(w_0))| = |\arg(w_1 L_h'(w_0)\frac{15}{4})| = |\arg(w_1 \psi'(w_1)/\psi(w_1))|$$
$$\le \log \frac{1 + |w_1|}{1 - |w_1|} \le \log \frac{1 + 1/3}{1 - 1/3} < \frac{3}{4} < \frac{\pi}{4}.$$

Above, we have used $\log 2 = \int_1^2 1/x\, dx < 1/8(8/8 + 8/9 + 8/10 + \cdots + 8/15) < 3/4$.

By Equation (40), $|\arg(w_1)| < \pi/12$. Therefore, $|\arg(L_h'(w_0))| < \pi/12 + \pi/4 = \pi/3$.

Applying Theorem 2.1-3 to the function ψ we obtain

$$\frac{1}{2} \cdot \frac{4}{5} \le \frac{1 - 1/3}{1 + 1/3} \cdot |\frac{4}{15 w_1}| \le |L_h'(w_0)| \le |\frac{4}{15 w_1}| \cdot \frac{1 + 1/3}{1 - 1/3} \le \frac{4}{3} \cdot 2.$$

Above, we use $4/5 \le |4/(15w_1)| \le 4/3$ which is obtained from $|15w_1/4 - 1| \le 1/4$.

Part (2): Fix w_0 with $B(w_0, R) \subset A_2$. Let $R' := R - 1.25$ and $w_1 := (F_h(w_0) - w_0)/R'$. By Lemma 6.4-2, and a simple calculation,

$$|w_1| \le \frac{5}{4R'} \le \frac{5}{8}, \quad \frac{1}{R} \le \frac{1}{R'} \le \frac{2}{R}. \tag{41}$$

As $B(w_0, R) \subset A_2$, $d(w_0, \mathbb{Z}/\alpha) \ge R > R'$. Hence $w_0 \in \Theta_\alpha(R')$. Using Lemma 6.4-3 with $r = R'\alpha \in [0, 1/2]$, we have

$$|R'w_1 - 1| \le \frac{C_3}{R'} e^{-2\pi\alpha \operatorname{Im} w_0} \le \frac{2C_3}{R} e^{-2\pi\alpha \operatorname{Im} w_0}. \tag{42}$$

Then, using Equation (40), and the inequality $\mathrm{d} \arcsin x/\mathrm{d}x = 1/\sqrt{1-x^2} \le 4/3$ on $[0, 1/4]$, we get

$$|\arg(w_1)| = |\arg(R'w_1)| < \arcsin\big(\min\big\{\tfrac{2C_3}{R} e^{-2\pi\alpha \operatorname{Im} w_0}, \tfrac{1}{4}\big\}\big) \le \min\big\{\tfrac{4}{3} \cdot \tfrac{2C_3}{R} e^{-2\pi\alpha \operatorname{Im} w_0}, \tfrac{\pi}{12}\big\}.$$

By Lemma 6.4-2, $B(F_h(w_0), R') \subset B(w_0, R) \subset A_2$. Hence, L_h is defined and univalent on $B(F_h(w_0), R')$. The function

$$\psi(w) = \Big(L_h(F_h(w_0)) - L_h(F_h(w_0) - R'w)\Big) \frac{1}{R'L_h'(F_h(w_0))} \tag{43}$$

is defined and univalent on $|w| < 1$ with $\psi(0) = 0$ and $\psi'(0) = 1$. Using the distortion theorem 2.1-4 at w_1, we get

$$\begin{aligned} |\arg(w_1\psi'(w_1)/\psi(w_1))| &= |\arg(w_1 L_h'(w_0))| \\ &\le \log(\frac{1+|w_1|}{1-|w_1|}) = \log(1 + \frac{2|w_1|}{1-|w_1|}) \\ &\le \frac{2|w_1|}{1-|w_1|} \le \frac{16}{3}|w_1| \le \frac{16}{3} \cdot \frac{5}{4R'} \le \frac{40}{3R}. \end{aligned}$$

Therefore,

$$|\arg(L_h'(w_0))| \le |\arg(w_1)| + |\arg(w_1 L_h'(w_0))| \le \frac{8C_3}{3R} e^{-2\pi\alpha \operatorname{Im} w_0} + \frac{40}{3R}.$$

Applying Theorem 2.1-3 to the function ψ at w_1 we have

$$\frac{1-|w_1|}{1+|w_1|} \le \Big|\frac{w_1\psi'(w_1)}{\psi(w_1)}\Big| = |L_h'(w_0) w_1 R'| \le \frac{1+|w_1|}{1-|w_1|}$$

This implies

$$\begin{aligned} |L_h'(w_0)| &\le \Big|\frac{1}{R'w_1}\Big| \cdot \frac{1+|w_1|}{1-|w_1|} \\ &= \Big|\frac{R'w_1 + 1 - R'w_1}{R'w_1}\Big| \cdot \frac{1-|w_1|+2|w_1|}{1-|w_1|} \\ &\le (1 + \frac{|1-R'w_1|}{|R'w_1|})(1 + \frac{2|w_1|}{1-|w_1|}) \\ &\le (1 + \frac{4}{3}\frac{2C_3}{R} e^{-2\pi\alpha \operatorname{Im} w_0})(1 + \frac{40}{3R}). \end{aligned}$$

In the last inequality of the above estimates we have used the inequalities in Equations (42) and (41), as well as $|R'w_1| \geq 3/4$.

Similarly,

$$\begin{aligned} |L_h'(w_0)| &\geq \frac{1}{|R'w_1|} \cdot \frac{1-|w_1|}{1+|w_1|} \\ &\geq (1 - \frac{|1-R'w_1|}{|R'w_1|})(1 - \frac{2|w_1|}{1+|w_1|}) \\ &\geq (1 - \frac{4}{3}\frac{2C_3}{R} e^{-2\pi\alpha \operatorname{Im} w_0})(1 - \frac{5}{R}). \end{aligned}$$

Part (3): The set A_1 is simply connected and is bounded by two piece-wise analytic curves, say ℓ_1 and ℓ_2 with ℓ_1 on the left hand of ℓ_2. By Lemma 6.4-2, as well as Equations (40) and (37), $F_h(\ell_1)$ and $F_h^{-1}(\ell_2)$ are defined and belong to A_1. Indeed, there is $\delta > 0$ such that the Euclidean distances $d(F_h(\ell_1), \partial A_1)$ and $d(F_h^{-1}(\ell_2), \partial A_1)$ is at least δ.

Let $B = \{w \in A_1 : B(w,5) \subset A_1\}$, and let $\mathrm{d_{hyp}}$ denote the poincaré metric on A_1 (i.e. the complete metric of constant curvature -1). The set B is closed, but not necessarily connected. By the previous paragraph, there is $\delta' > 0$, depending only on δ, such that for every $w \in A_1$ there is $j_w \in \{-1, 0, 1\}$ such that $\mathrm{d_{hyp}}(F_h^{\circ j_w}(w), B) \leq \delta'$. Define $B' = \{w \in A_1 : \mathrm{d_{hyp}}(w, B) \leq \delta'\}$.

Let us fix an arbitrary w in A_1. To find uniform upper and lower bounds on $|L_h'(w)|$ we consider the three cases $w \in B$, $w \in B' \setminus B$, and $w \in A_1 \setminus B'$.

If $w \in B$, then by Part (1), $2/5 \leq |L_h'(w)| \leq 8/3$.

Assume that $w \in B' \setminus B$ (here $j_w = 0$). There is $w' \in B$ with $\mathrm{d_{hyp}}(w, w') \leq \delta'$, and a Riemann mapping $u : B(0,1) \to A_1$ with $u(0) = w'$. It follows that there is $\delta'' < 1$ depending only on δ', such that $|u^{-1}(w)| \leq \delta''$. By the distortion Theorem 2.1-1,

$$\frac{1-\delta''}{(1+\delta'')^3} \leq \frac{|(L_h \circ u)'(u^{-1}(w))|}{|(L_h \circ u)'(0)|} \leq \frac{1+\delta''}{(1-\delta'')^3},$$

and

$$\frac{1-\delta''}{(1+\delta'')^3} \leq \frac{|u'(u^{-1}(w))|}{|u'(0)|} \leq \frac{1+\delta''}{(1-\delta'')^3}.$$

These imply that

$$|L_h'(w)| \leq \frac{1+\delta''}{(1-\delta'')^3} \cdot \frac{(1+\delta'')^3}{1-\delta''} \cdot |L_h'(w')| \leq \frac{(1+\delta'')^4}{(1-\delta'')^4} \cdot \frac{8}{3}, \tag{44}$$

and

$$|L_h'(w)| \geq \frac{1-\delta''}{(1+\delta'')^3} \cdot \frac{(1-\delta'')^3}{1+\delta''} \cdot |L_h'(w')| \geq \frac{(1-\delta'')^4}{(1+\delta'')^4} \cdot \frac{2}{5}. \tag{45}$$

Assume that $w \in A_1 \setminus B'$. By the first paragraph, $w' = F_h^{\circ j_w}(w) \in B'$ for some $j_w \in \{+1, -1\}$. If $j_w = +1$, then

$$L_h'(w) = (L_h \circ F_h - 1)'(w) = L_h'(w') \cdot F_h'(w),$$

while the absolute value of the right hand of the above equation is bounded from above and away from 0 by Equations (44) and (45), and Lemma 6.4. If $j_w = -1$, then

$$L_h'(w) = (L_h \circ F_h^{-1} + 1)'(w) = L_h'(w') \cdot (F_h^{-1})'(w),$$

with the size of the right hand of the above equation uniformly bounded from above and away from 0 by Equations (44) and (45), and Lemma 6.4.

Part (4): Since there is a positive angle between the rays bounding the set A_1 and A_2, there is $\delta > 0$, independent of α and h, such that for $w_0 \in A_1$ with $\operatorname{Im} w_0 \geq 2C_2$, $B(w_0, \delta \operatorname{Im} w_0) \subset A_2$. The function

$$\psi(w) = \big(L_h(w_0) - L_h(w_0 - \delta \operatorname{Im} w_0 \cdot w)\big) \frac{1}{\delta \operatorname{Im} w_0 L_h'(w_0)}$$

is defined and univalent on $|w| < 1$, $\psi(0) = 0$, and $\psi'(0) = 1$. Let us define $w_1 = (w_0 - F_h(w_0))/(\delta \operatorname{Im} w_0)$. For $\operatorname{Im} w_0$ large enough, $|w_1| \leq 1/2$.

Applying Theorem 2.1-3 to ψ at w_1 we get

$$\frac{1}{|w_0 - F_h(w_0)|} \cdot \frac{1 - |w_1|}{1 + |w_1|} \leq |L_h'(F_h(w_0))| \leq \frac{1 + |w_1|}{1 - |w_1|} \cdot \frac{1}{|w_0 - F_h(w_0)|}$$

which implies

$$\big||L_h'(F_h(w_0))| - 1\big| = O(1/\operatorname{Im} w_0).$$

Applying Theorem 2.1-4 to ψ at w_1 we get

$$\begin{aligned} |\arg(L_h'(F_h(w_0)))| &\leq |\arg(L_h'(F_h(w_0))(w_1 \delta \operatorname{Im} w_0)) - \arg(w_1 \delta \operatorname{Im} w_0)| \\ &\leq \log \frac{1 + |w_1|}{1 - |w_1|} + |\arg(F_h(w_0) - w_0)| \\ &\leq O(1/\operatorname{Im} w_0) + O(\alpha e^{-2\pi\alpha \operatorname{Im} w_0}). \end{aligned}$$

In the last line of the above equations we have used $|F_h(w_0) - w_0 - 1| = O(\alpha e^{-2\pi\alpha \operatorname{Im} w_0})$ which is provided by Lemma 6.4 (with $r = 1/2$). Combining the above inequalities we conclude

$$\begin{aligned} |L_h'(F_h(w_0)) - 1| &\leq |\arg(L_h'(F_h(w_0)))| \cdot |L_h'(F_h(w_0))| + \big||L_h'(F_h(w_0))| - 1\big| \\ &= \big(O(1/\operatorname{Im} w_0) + O(\alpha e^{-2\pi\alpha \operatorname{Im} w_0})\big) + O(1/\operatorname{Im} w_0). \end{aligned}$$

Finally,

$$\begin{aligned} |L_h'(w_0) - 1| &= |L_h'(F_h(w_0)) \cdot F_h'(w_0) - 1| \\ &= |L_h'(F_h(w_0))(F_h'(w_0) - 1)| + |L_h'(F_h(w_0)) - 1| \\ &\leq C \cdot O(\alpha e^{-2\pi\alpha \operatorname{Im} w_0}) + O(1/\operatorname{Im} w_0) + O(\alpha e^{-2\pi\alpha \operatorname{Im} w_0}). \end{aligned}$$

The constant in all of O depend only on δ. □

We also need to control the geometry of the vertical leaves of the foliation in X. But, integrating the vector field $\mathbf{i}/L_h'$, using the estimates in Lemma 6.7, results in diverging integrals. We present an alternative approach to deal with this issue in the next proposition.

Proposition 6.8. — *For all $M' \in \mathbb{R}$, there is $M > 0$ such that for all α in $(0, r_2']$, all $h \in \mathcal{QIS}_\alpha$, and all $r \in (0, 1/2]$ the following holds. Let $w_1, w_2 \in A_1$ with*

- $\operatorname{Re} w_1 = \operatorname{Re} w_2$, *and* $\operatorname{Im} w_2 \geq \operatorname{Im} w_1 \geq M'/\alpha$,
- *for all* $t \in (0,1)$, $tw_1 + (1-t)w_2 \in \Theta_\alpha(r/\alpha) \cap A_1$.

Then,

(1) $|\operatorname{Re}(L_h(w_1) - L_h(w_2))| \leq M/r$,
(2) $|\operatorname{Im}(L_h(w_1) - L_h(w_2)) - \operatorname{Im}(w_1 - w_2)| \leq M/r$,
(3) As M' tends to $+\infty$, M tends to 0.

Proof. — For the simplicity of the notations let $t_i := \operatorname{Im} w_i$, $i = 1, 2$. Define

$$l := \{tw_1 + (1-t)w_2 : t \in [0,1]\}.$$

Let us assume that $F_h(l) \subset A_1$. By Lemma 6.4-2, the two curves l, $F_h(l)$, as well as the two line segments $w_1 + t(F_h(w_1) - w_1)$, $t \in [0,1]$, and $w_2 + t(F_h(w_2) - w_2)$, $t \in [0,1]$, cut $\mathbb{C}$ into two connected components. Denote the closure of the bounded one by $\mathcal{K}$ (see Figure 8). We have $\mathcal{K} \subset A_1$, and by Lemma 6.6, L_h is defined on $\mathcal{K}$.

Define the rectangle

$$\mathcal{D} := \{s + \mathbf{i}t : 0 \leq s \leq 1,\ t_1 \leq t \leq t_2\},$$

and the map $g : \mathcal{D} \to \mathcal{K}$ as

$$g(s + \mathbf{i}t) := (1-s)(\operatorname{Re}(w_1) + \mathbf{i}t) + sF_h(\operatorname{Re}(w_1) + \mathbf{i}t).$$

By the estimates in Lemma 6.4-2, g is uniformly close to a translation on $\mathcal{D}$. To prove the desired estimates in the proposition, we compare L_h to g^{-1} by analyzing the map $G := L_h \circ g : \mathcal{D} \mapsto \mathbb{C}$. Using the notations $\zeta = s + \mathbf{i}t$, $d\zeta = ds + \mathbf{i}dt$ and $d\bar{\zeta} = ds - \mathbf{i}dt$, by Green's Theorem, we have

$$\oint_{\partial\mathcal{D}} G(\zeta)\, d\zeta = \iint_{\mathcal{D}} -\frac{\partial G}{\partial \bar{\zeta}}(\zeta)\, d\zeta \wedge d\bar{\zeta}. \tag{46}$$

With notation $w = g(\zeta)$ and the Cauchy-Riemann equation $\partial L_h/\partial \bar{w} = 0$, the complex chain rule for G can be written as

$$\frac{\partial G}{\partial \bar{\zeta}} = (\frac{\partial L_h}{\partial w} \circ g)\frac{\partial g}{\partial \bar{\zeta}} + (\frac{\partial L_h}{\partial \bar{w}} \circ g)\frac{\partial \bar{g}}{\partial \bar{\zeta}} = (\frac{\partial L_h}{\partial w} \circ g)\frac{\partial g}{\partial \bar{\zeta}}.$$

A simple differentiation gives

$$\begin{aligned}\frac{\partial g}{\partial \bar{\zeta}}(s + \mathbf{i}t) &= \frac{1}{2}[\frac{\partial g}{\partial s} + \mathbf{i}\frac{\partial g}{\partial t}](s + \mathbf{i}t) \\ &= \frac{1}{2}[F_h(\operatorname{Re} w_1 + \mathbf{i}t) - (\operatorname{Re} w_1 + \mathbf{i}t) - 1 + s(1 - F_h'(\operatorname{Re} w_1 + \mathbf{i}t))].\end{aligned}$$

Since $l \subset \Theta_\alpha(r/\alpha)$, by Lemma 6.4-3,

$$|\frac{\partial g}{\partial \bar{\zeta}}(s + \mathbf{i}t)| \leq \frac{1}{2}|F_h(g(\mathbf{i}t)) - g(\mathbf{i}t) - 1| + \frac{1}{2}|F_h'(g(\mathbf{i}t)) - 1| \leq C_3\frac{\alpha}{r}e^{-2\pi\alpha t}.$$

Also recall that by Lemma 6.7, $|L_h'| \leq C$ on $\mathcal{K}$. Then, the right hand of Equation (46) may be controlled as follows:

$$\begin{aligned}
\Big| \iint_{\mathcal{D}} \frac{\partial G}{\partial \bar{\zeta}}(\zeta)\, d\zeta \wedge d\bar{\zeta} \Big| &\leq 2 \int_{t_1}^{t_2} \int_0^1 \Big| \frac{\partial G}{\partial \bar{\zeta}}(s + \mathbf{i}t) \Big|\, ds dt \\
&\leq 2 \sup_{w \in \mathcal{K}} |L_h'(w)| \int_{t_1}^{t_2} \int_0^1 C_3 \frac{\alpha}{r} e^{-2\pi\alpha t}\, ds dt \\
&\leq \frac{2CC_3\alpha}{r} \int_{M'/\alpha}^{\infty} e^{-2\pi\alpha t}\, dt \\
&\leq \frac{CC_3}{\pi r} e^{-2\pi M'}.
\end{aligned} \tag{47}$$

The left hand of Equation (46) may be written as

$$\begin{aligned}
\int_{t_1}^{t_2} G(\mathbf{i}t)\mathbf{i}\, dt + \int_0^1 G(s + \mathbf{i}t_2)\, ds \\
+ \int_{t_1}^{t_2} G(1 + \mathbf{i}(t_1 + t_2 - t))(-\mathbf{i})\, dt + \int_0^1 G(1 - s + \mathbf{i}t_1)(-1)\, ds.
\end{aligned}$$

By definition, for $t \in [t_1, t_2]$, we have $g(1 + \mathbf{i}t) = F_h(g(\mathbf{i}t))$. This implies that for all $t \in [t_1, t_2]$, $G(1 + \mathbf{i}t) = G(\mathbf{i}t) + 1$. Using this relation in the third integral of the above equation and then making a change of coordinate, the above sum is equal to

$$-\mathbf{i}(t_2 - t_1) + \int_0^1 G(s + \mathbf{i}t_2)\, ds + \int_0^1 -G(1 - s + \mathbf{i}t_1)\, ds.$$

Set

$$M_i := \sup_{s \in [0,1]} \big| L_h'(g(s + \mathbf{i}t_i)) - 1 \big|, i = 1, 2.$$

By Lemma 6.7-3, M_1 and M_2 are uniformly bounded from above independent of α and h. Moreover, by Lemma 6.4, $|\partial g/\partial s| \leq 5/4$ on $\mathcal{K}$ and $|\partial g/\partial s - 1| \leq C_3 \alpha e^{-2\pi M'}/r$ on $\mathcal{K}$. Hence,

$$\begin{aligned}
\Big| \int_0^1 G(s + \mathbf{i}t_2)\, ds - L_h(w_2) - 1/2 \Big| &= \Big| \int_0^1 G(s + \mathbf{i}t_2) - G(\mathbf{i}t_2) - s\, ds \Big| \\
&\leq \int_0^1 |G(s + \mathbf{i}t_2) - G(\mathbf{i}t_2) - s|\, ds \\
&\leq \sup_{s \in [0,1]} |G(s + \mathbf{i}t_2) - G(\mathbf{i}t_2) - s| \\
&\leq \sup_{s \in [0,1]} \Big| \frac{\partial G}{\partial s}(s + \mathbf{i}t_2) - 1 \Big| \\
&\leq \sup_{s \in [0,1]} \Big| \frac{\partial g}{\partial s}(s + \mathbf{i}t_2) \cdot L_h'(g(s + \mathbf{i}t_2)) - 1 \Big| \\
&\leq \frac{5}{4} \cdot M_2 + \frac{C_3\alpha}{r} e^{-2\pi M'}.
\end{aligned} \tag{48}$$

To get the fifth inequality in the above equation we have used the formula $AB-1=A(B-1)+A-1$. On the other hand,

$$\int_0^1 -G(1-s+\mathbf{i}t_1)\,ds + L_h(w_1) + 1/2 = \int_0^1 -G(1-s+\mathbf{i}t_1) + L_h(w_1) + (1-s)\,ds$$
$$= -\int_0^1 G(s+\mathbf{i}t_1) - L_h(w_1) - s\,ds.$$

Thus, the inequalities in Equation (48) may be repeated for the above integral to conclude

$$\Big|\int_0^1 -G(1-s+\mathbf{i}t_1)\,ds + L_h(w_1) + 1/2\Big| \le \frac{5}{4}\cdot M_1 + \frac{C_3\alpha}{r}e^{-2\pi M'}. \tag{49}$$

One infers the inequalities in Parts (1) and (2) of the proposition by considering the real part and the imaginary part of Equation (46) as well as using the bounds in Equations (48) and (49). For instance, for Part (2),

$$\begin{aligned}
&|\operatorname{Im}(w_2-w_1) - \operatorname{Im}(L_h(w_2)-L_h(w_1))| \\
&\qquad \le \Big|(t_2-t_1) - \operatorname{Im}\big(\int_0^1 G(s+\mathbf{i}t_2)\,ds\big) - \operatorname{Im}\big(\int_0^1 -G(1-s+\mathbf{i}t_1)\,ds\big)\Big| \\
&\qquad\quad + \Big|\operatorname{Im}\big(\int_0^1 G(s+\mathbf{i}t_2)\,ds\big) - \operatorname{Im}(L_h(w_2)) - 1/2\Big| \\
&\qquad\quad + \Big|\operatorname{Im}\big(\int_0^1 -G(1-s+\mathbf{i}t_1)\,ds\big) + \operatorname{Im}(L_h(w_1)) + 1/2\Big| \\
&\qquad \le \big(\frac{CC_3e^{-2\pi M'}}{\pi} + \frac{5M_2r}{4} + \frac{5M_1r}{4} + 2C_3\alpha e^{-2\pi M'}\big)\frac{1}{r}.
\end{aligned}$$

This finishes the proof of the inequalities in Parts (1) and (2) under the assumption $F_h(l)\subset A_1$ that we made at the beginning of the proof.

If $F_h^{-1}(l)\subset A_1$, one considers the bounded region $\mathcal{K}'$ enclosed by the curves l, $F_h^{-1}(l)$, $t\mapsto w_1+t(F_h^{-1}(w_1)-w_1)$, $t\in[0,1]$ and $t\mapsto w_2+t(F_h^{-1}(w_2)-w_2)$, $t\in[0,1]$. One may repeat the above calculations and estimates for the map $g:\mathcal{D}\to\mathcal{K}'$ defined as $g(s+\mathbf{i}t) := s(\operatorname{Re}(w_1)+\mathbf{i}t)+(1-s)F_h^{-1}(\operatorname{Re}(w_1)+\mathbf{i}t)$. This leads to the same estimates in Parts (1) and (2), under this condition.

There is also the possibility that for some w_1 and w_2 neither $F_h(l)\subset A_1$ nor $F_h^{-1}(l)\subset A_1$ holds. (This is when the two balls $B(0,C_2)$ and $B(1/\alpha,C_2)$ are close.) By the assumption on α in Equation (37), the balls $B(0,C_2)$ and $B(1/\alpha,C_2)$ are at least $5/4$ apart. Thus, there are $w_1'\in B(w_1,1)$ and $w_2'\in B(w_2,1)$ with $\operatorname{Re}w_1' = \operatorname{Re}w_2'$, the line segment l' connecting them remains in $\Theta_\alpha(r/(2\alpha))\cap A_1$, and either $F_h(l')\subset A_1$ or $F_h^{-1}(l')\subset A_1$. By the above argument we obtain the inequalities in the proposition for w_1' and w_2'. From that, one infers the inequalities in Parts (1) and (2) for w_i using the uniform bound on L_h' in Lemma 6.7-3, by making M large enough.

Part (3): For $M'\ge 1/2$, we may fix $r=1/2$ in the above arguments. Also, by Lemma 6.7-4, the constants M_1 and M_2 are $O(\alpha/M'+\alpha e^{-2\pi M'})$. In particular, M_1 and M_2 tend to 0 as M'

tends to $+\infty$. This implies that

$$M = \Big(\frac{CC_3 e^{-2\pi M'}}{\pi} + \frac{5M_2 r}{4} + \frac{5M_1 r}{4} + 2C_3\alpha e^{-2\pi M'}\Big)$$

tends to 0 as M' tends to $+\infty$. □

Lemma 6.9. — *The limit*

$$\ell_h := \lim_{\substack{\operatorname{Im} w\to+\infty \\ w\in A_1}} L_h(w) - w$$

exists and is finite.

Proof. — For w_1 and w_2 on the vertical line $\operatorname{Re} w = 1/(2\alpha)$ we may use Proposition 6.8 with $r = 1/2$ and large values of M' to conclude that $|(L_h(w_2) - L_h(w_1)) - (w_2 - w_1)|$ tends to 0 as $\operatorname{Im} w_1$ and $\operatorname{Im} w_2$ tend to $+\infty$. This implies that the function $L_h(w) - w$ satisfies the Cauchy criterion and hence its limit exists as $\operatorname{Im} w \to +\infty$ along $\operatorname{Re} w = 1/(2\alpha)$.

By Lemma 6.7-4, $|L_h'(w) - 1|$ tends to 0 as $\operatorname{Im} w \to +\infty$ in A_1. This implies that as $\operatorname{Im} w \to +\infty$ in the strip $\operatorname{Re} w \in [1/(2\alpha) - 1, 1/(2\alpha) + 1]$, the limit of $L_h(w) - w$ exists.

Using Lemma 6.4-2, for $w \in A_1$ there is an integer j with $|j| \le O(\operatorname{Im} w)$ such that $\operatorname{Re} F_h^{\circ j}(w) \in [1/(2\alpha) - 1, 1/(2\alpha) + 1]$. Moreover, by Lemma 6.4-3 with $r = 1/2$,

$$|F_h^{\circ j}(w) - j - w| \le j \cdot 2C_3\alpha e^{-2\pi\alpha \operatorname{Im} w} \le O((\operatorname{Im} w) \cdot e^{-2\pi\alpha \operatorname{Im} w}).$$

Therefore,

$$L_h(w) - w = L_h(F_h^{\circ j}(w)) - j - w = \big(L_h(F_h^{\circ j}(w)) - F_h^{\circ j}(w)\big) + \big(F_h^{\circ j}(w) - j - w\big).$$

Thus, $L_h(w) - w$ has a limit as $w \in A_1$ tends to $+\mathbf{i}\infty$. □

We shall give an upper bound on the size of ℓ_h in Corollary 6.20.

6.5. The image of L_h. — Recall that $L_h(X)$ is an open subset of $\mathbb{C}$ containing $(0, 1] + \mathbf{i}\mathbb{R}$. In this section we prove a lower bound on

$$x_h := \sup\{t \in (0, +\infty) \mid (0, t) + \mathbf{i}\mathbb{R} \subseteq L_h(X)\}. \tag{50}$$

Recall the constants r_2' and C_2 introduced in Lemma 6.4. Define the constant

$$r_2 := \min\{r_2', 1/(4C_2 + 20)\}. \tag{51}$$

For $\alpha \le r_2$, $2C_2 + 10$ and $\alpha^{-1} - 2C_2 - 10$ belong to X, and hence L_h is defined at these points. Let

$$s_1 = \operatorname{Re}(L_h(2C_2 + 10)), \quad s_2 = \operatorname{Re}(L_h(\alpha^{-1} - 2C_2 - 10)). \tag{52}$$

Lemma 6.10. — *For all $\alpha \in (0, r_2]$ and $h \in \mathcal{QLS}_\alpha$, we have*

(1) $1 < s_1 \le s_2 \le x_h$;
(2) $\forall w \in L_h^{-1}((s_1, s_2) + \mathbf{i}\mathbb{R})$, $B(w, 5) \subset A_1$;
(3) $\forall t \in \mathbb{R}, |\arg(L_h^{-1}(1 + \mathbf{i}t) - 2C_2 - 10)| \ge \pi/6$;
(4) $\forall t \in \mathbb{R}, |\arg(L_h^{-1}(x_h + \mathbf{i}t) - \alpha^{-1} + 2C_2 + 10)| \le 5\pi/6$;
(5) $\limsup_{|t|\to+\infty} |\arg(L_h^{-1}(\mathbf{i}t) - 2C_2 - 10)| \le 5\pi/6$;

(6) $\limsup_{|t|\to+\infty} |\arg(L_h^{-1}(x_h+\mathbf{i}t)-\alpha^{-1}+2C_2+10)| \geq \pi/6$.

Proof. — Let X' be the set of points $w \in \mathbb{C}$ such that

$$|\arg(w-2C_2-10)| \leq 5\pi/6, \text{ and } |\arg(w-1/\alpha+2C_2+10)| \geq \pi/6.$$

The condition $\alpha \leq r_2$ implies that X' is a connected set containing the points $2C_2+10$ and $\alpha^{-1}-2C_2-10$. Moreover, for $w \in X'$, $|w-i/\alpha| \geq C_2+5$, for all $i \in \mathbb{Z}$. In particular, for $w \in X'$, $B(w,5) \subset A_1 \subset A_2$. Then, by Lemma 6.7-1, for $w \in X'$, $\arg(L_h'(w))$ belongs to $(-\pi/3, \pi/3)$.

By the above paragraph, for every $w \in X'$, the argument of the tangent line to the vertical foliation passing through w at w belongs to $\pi/2+(-\pi/3,\pi/3) = (\pi/6, 5\pi/6)$, modulo π. In particular, $L_h^{-1}(s_1+\mathbf{i}\mathbb{R})$ and $L_h^{-1}(s_2+\mathbf{i}\mathbb{R})$ stay within $X' \subset X$, and spread from $-\mathbf{i}\infty$ to $+\mathbf{i}\infty$. This implies parts 1 and 2 of the lemma.

By the above paragraph, $L_h^{-1}(s_1+\mathbf{i}\mathbb{R})$ lies to the left of the curve $|\arg(w-2C_2-10)| = \pi/6$. Also, since vertical leaves are disjoint, the vertical leaf passing through cv_{F_h}, $L_h^{-1}(1+\mathbf{i}\mathbb{R})$, must lie to the left of $L_h^{-1}(s_1+\mathbf{i}\mathbb{R})$. This implies part 3 of the lemma. Similarly, $L_h^{-1}(s_2+\mathbf{i}\mathbb{R})$ lies to the right of the curve $|\arg(w-\alpha^{-1}+2C_2+10)| = 5\pi/6$, and $L_h^{-1}(x_h+\mathbf{i}\mathbb{R})$ lies to the right of the leaf $L_h^{-1}(s_2+\mathbf{i}\mathbb{R})$. This implies part 4 of the lemma.

By the second paragraph, $L_h^{-1}(s_1+\mathbf{i}\mathbb{R})$ lies to the right of the curve $|\arg(w-2C_2-10)| = 5\pi/6$. By the uniform bound on $|F_h(w)-w-1|$ in Lemma 6.4, $L_h^{-1}(\mathbf{i}\mathbb{R})$ lies within uniformly bounded distance from $L_h^{-1}(s_1+\mathbf{i}\mathbb{R})$. This implies part 5 of the lemma. Similarly, part 6 of the lemma follows from comparing $L_h^{-1}(x_h+\mathbf{i}\mathbb{R})$ to the curve $L_h^{-1}(s_2+\mathbf{i}\mathbb{R})$. □

By Lemma 6.10-(5)-(6), the top end of $L_h^{-1}((0,x_h)+\mathbf{i}\mathbb{R})$ is contained in A_1. Therefore, by Lemma 6.9, we have

$$\lim_{\substack{\mathrm{Im}\, w\to+\infty \\ \mathrm{Re}\, w\in(0,x_h)}} L_h^{-1}(w)-w = -\ell_h. \tag{53}$$

Lemma 6.11. — *For all $\varepsilon>0$ there is M_ε such that for all $\alpha \in (0,r_2]$, all $h \in \mathcal{QIS}_\alpha$, and all $\zeta \in ([0,x_h]+\mathbf{i}\mathbb{R}) \setminus B(0,\varepsilon)$, we have*

$$M_\varepsilon^{-1} \leq |(L_h^{-1})'(\zeta)| \leq M_\varepsilon.$$

Proof. — By Lemma 6.10-3, $A_3 \cap B(1/\alpha, C_2) = \emptyset$ (in the definition of A_3, $L \cap R' = \emptyset$). Hence, by Lemma 6.4-1, on $X \setminus B(0,C_2)$, we have $|F_h'-1| < 1/4$.

By the pre-compactness of the class $\cup_{\alpha\in[0,r_2]}\mathcal{QIS}_\alpha$ and the continuous dependence of L_h on h, there is $\varepsilon'>0$ such that $L_h(X \cap B(\mathrm{cp}_{F_h},\varepsilon')) \subset B(0,\varepsilon)$. For the same reason, $|F_h'|$ is uniformly bounded from above and away from zero on $(B(0,C_2)\cap X)\setminus B(0,\varepsilon')$.

By Lemmas 6.10-2 and 6.7-3, when $s_1 \leq \mathrm{Re}\,\zeta \leq s_2$, $C^{-1} \leq |(L_h^{-1})'(\zeta)| \leq C$, where C is a uniform constant.

On the other hand, by Lemmas 6.4-2, 6.5-4, and 6.6, s_1 and $x_h - s_2$ are uniformly bounded from above. Thus, for ζ with $0 \leq \mathrm{Re}\,\zeta \leq x_h$, there is $j_\zeta \in \mathbb{Z}$, with $|j_\zeta|$ uniformly bounded from above, such that $s_1 \leq \mathrm{Re}(\zeta - j_\zeta) \leq s_2$. Then, the desired bounds on $|(L_h^{-1})'(\zeta)|$ follow from the

functional equation $L_h \circ F_h = L_h + 1$ in Lemma 6.6 and the above upper and lower bounds on $|F_h'|$. □

Define the sets

$$B_0 := \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta \in [0,1]\},\ B_1 := \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta \in [x_h - 1, x_h]\}.$$

Lemma 6.12. — *For all $\alpha \in (0, r_2]$, all $h \in \mathcal{QIS}_\alpha$, and all $w \in L_h^{-1}(B_0) + 1/\alpha$ with $|\operatorname{Im} w| \geq 3C_2 + 5$, there is $l_w \in \mathbb{Z}$ such that $F_h^{\circ l_w}(w)$ is defined and belongs to $L_h^{-1}(B_1)$. Moreover,*

(1) if $l_w \geq 0$, then for $0 \leq j \leq l_w$ we have $F_h^{\circ j}(w) \in X$;
(2) if $l_w < 0$, for $l_w \leq j \leq 0$, we have $F_h^{\circ j}(w) \in X$;
(3) for every constant $c \geq 5$ and for every w with $3C_2 + 5 \leq |\operatorname{Im} w| \leq 3C_2 + c$, $|l_w|$ is bounded from above by a constant depending only on c (independent of w, α, and h);
(4) as $\operatorname{Im} w \to \pm\infty$, we have $\operatorname{Im} F_h^{\circ l_w}(w) \to \pm\infty$.

Proof. — By Lemma 6.10-3, $L_h^{-1}(B_0) + 1/\alpha$ lies to the left of the curve $|\arg(w - 2C_2 - 10 - 1/\alpha)| = \pi/6$, and $L_h^{-1}(B_1)$ lies to the right-hand of $|\arg(w + 2C_2 + 10 - 1/\alpha)| = 5\pi/6$. The curve $|\arg(w - 2C_2 - 10 - 1/\alpha)| = \pi/6$ crosses the left-hand boundary of A_1 at two points with imaginary parts $\pm 2C_2/(\sqrt{6} - \sqrt{2}) + C_2 + 5$. Note that $2C_2/(\sqrt{6} - \sqrt{2}) + C_2 + 5 \leq 3C_2 + 5$. Thus, the intersection of $L_h^{-1}(B_0) + 1/\alpha$ and $\{w : |\operatorname{Im} w| \geq 3C_2 + 5\}$ is contained in A_1.

By Lemma 6.5, the forward orbit and the backward orbit of every point in A_1 eventually leave X. Combining with the above paragraph, the backward or the forward orbit of any $w \in L_h^{-1}(B_0) + 1/\alpha$ with $|\operatorname{Im} w| \geq 3C_2 + 5$ must cross $L_h^{-1}(B_1)$. Moreover, the uniform estimate in Lemma 6.4-2 shows that for $w \in L_h^{-1}(B_0) + 1/\alpha$ with $3C_2 + 5 \leq |\operatorname{Im} w| \leq 3C_2 + c$, $|l_w|$ is uniformly bounded from above by a constant depending only on C_2.

Part (4) of the lemma follows from the upper bound on $|\arg(F_h(w) - w)| \leq \pi/12$ in Equation (40) and the above argument. □

For ζ near the top and bottom ends of B_0 let $w = L_h^{-1}(\zeta) + 1/\alpha$ and $m_\zeta = l_w$ be the integer defined in Lemma 6.12. For some of those ζ, there may be more than one choice for m_ζ, in which case, one may choose either one. Then, consider the map

$$T_h(\zeta) := L_h(F_h^{\circ m_\zeta}(L_h^{-1}(\zeta) + 1/\alpha)), \tag{54}$$

near the two ends of B_0, with values in B_1.

Lemma 6.13. — *There is $\eta > 1$ such that for all $\alpha \in (0, r_2]$ and all $h \in \mathcal{QIS}_\alpha$, the map*

$$T_h : \{\zeta \in B_0 : |\operatorname{Im}\zeta| \geq \eta\}/\mathbb{Z} \to B_1/\mathbb{Z}$$

is defined and univalent. Moreover,

(1) $\operatorname{Im} T_h(\zeta) \to \pm\infty$, as $\operatorname{Im}\zeta \to \pm\infty$;
(2) for $\eta \leq |\operatorname{Im}\zeta| \leq \eta + 1$, $|m_\zeta|$ is uniformly bounded from above independent of α, ζ, and h.

Proof. — By the pre-compactness of the class $\mathcal{QIS}_\alpha$ (see the proof of Lemma 6.11) there is $\eta > 1$, independent of α and h, such that for all $\zeta \in B_0$ with $|\operatorname{Im}\zeta| \geq \eta$, $|\operatorname{Im} L_h^{-1}(\zeta)| \geq 3C_2 + 5$. Combining this with Lemma 6.12, we conclude that T_h is defined above the height η and below

the height $-\eta$. Moreover, by Equation (34) and Lemma 6.6, $T_h(\zeta+1) = T_h(\zeta)+1$ when $\operatorname{Re}\zeta = 0$. Therefore, T_h projects to a well-defined map from $\{\zeta \in B_0 : |\operatorname{Im}\zeta| \geq \eta\}/\mathbb{Z}$ to $B_1/\mathbb{Z}$. Also, as F_h and L_h are univalent on X, T_h must be univalent.

We have $\operatorname{Im} L_h(\zeta) \to \pm\infty$ when $\operatorname{Im}\zeta \to \pm\infty$ within B_0, and $\operatorname{Im} F_h^{\circ l_w}(w) \to \pm\infty$ as $\operatorname{Im} w \to \pm\infty$ by Lemma 6.12-4. This implies the asymptotic behavior of T_h in part 1. Part (2) of the lemma follows from Lemma 6.12-3 and the uniform bound on $|L_h'|$ in Lemma 6.11. □

Remark 6.14. — The map T_h projects under $z \mapsto (-4/27)e^{2\pi \mathbf{i} z}$ to the inverse of $\mathcal{R}'(h)$ restricted to the ball $B(0, e^{-2\pi\eta})$, see Section 2.3. The reason for considering this inverse here is that *a priori* we do not know how large Dom T_h^{-1} is, that is, the values of L_h^{-1} on B_1.

Recall the covering map τ_h and the relation $L_h = \Phi_h \circ \tau_h$ in Equations (33) and (38). Define

$$y_h := \sup\{t \in (0, x_h) \mid \tau_h \text{ is univalent on } L_h^{-1}((0,t)+\mathbf{i}\mathbb{R})\}. \tag{55}$$

We have $y_h \geq 1$. That is because, Φ_h and L_h are univalent maps, and $\Phi_h(\mathcal{P}_h) \supset (0,1]+\mathbf{i}\mathbb{R}$.

Proposition 6.15. — *There is $k > 0$ such that for all $\alpha \in (0, r_2]$ and all $h \in \mathcal{QLS}_\alpha$, we have*

(1) $\alpha^{-1} - k \leq y_h \leq \alpha^{-1}$;
(2) $x_h \leq \alpha^{-1} + k$.

Proof. — First we claim that

$$b_h' = \sup\{|m_\zeta| : \zeta \in B_0, |\operatorname{Im}\zeta| \geq \eta\}$$

is uniformly bounded from above independent of α and h. That is because, by Lemma 6.13, T_h projects under $e^{2\pi\mathbf{i}z}$ and $e^{-2\pi\mathbf{i}z}$ to univalent maps on $B(0, e^{-2\pi\eta})$, denoted by $\hat{T}_{h,t}$ and $\hat{T}_{h,b}$ respectively, satisfying $\hat{T}_{h,t}(0) = \hat{T}_{h,b}(0) = 0$. By the distortion Theorem 2.1, the image of any ray $\{re^{\mathbf{i}\theta} : r \in (0, e^{-2\pi(\eta+1)})\}$, for fixed $\theta \in [0, 2\pi)$, under $\hat{T}_{h,t}$ and $\hat{T}_{h,b}$ have uniformly bounded spirals about 0. (Indeed, using a sharp distortion theorem on the argument obtained from Loewner theory (see Thm 3.5 in [**Dur83**]) the total spiral is bounded by $\log((1+e^{-2\pi(\eta+1)})/(1-e^{-2\pi(\eta+1)})) \leq 2e^{-2\pi(\eta+1)} \ll 2\pi$. But we don't need this exact value here.) In terms of the lift map T_h and the integers m_ζ, this means that $|m_\zeta - m_{\zeta'}|$ is uniformly bounded from above for ζ and ζ' in B_0 with $|\operatorname{Im}\zeta| \geq |\operatorname{Im}\zeta'| \geq \eta+1$. On the other hand, when $|\operatorname{Im}\zeta| \in [\eta, \eta+1]$, $|m_\zeta|$ is uniformly bounded from above, independent of α and h, by Lemma 6.13-2. This finishes the proof of the claim.

Recall that τ_h is periodic of period $1/\alpha$. If $L_h^{-1}(B_0) + 1/\alpha$ lies to the right of $L_h^{-1}(B_1)$, then τ_h is univalent on $L_h^{-1}((0, x_h)+\mathbf{i}\mathbb{R})$ and we have $y_h = x_h$. Otherwise, as we show below, y_h is obtained from subtracting a uniformly bounded value from x_h.

By Lemmas 6.10 and 6.11, $\inf \operatorname{Re} L_h^{-1}(\mathbf{i}[-\eta,\eta])$ is uniformly bounded from below, and $\sup \operatorname{Re} L_h^{-1}(x_h + \mathbf{i}[-\eta,\eta]) - 1/\alpha$ is uniformly bounded from above, both independent of α and h. Then, Lemma 6.4-2 implies that there is a positive integer j, uniformly bounded from above, such that

$$\sup \operatorname{Re}\left(F_h^{-j}(L_h^{-1}(x_h + \mathbf{i}[-\eta,\eta]))\right) < \sup \operatorname{Re}\left((L_h^{-1}(\mathbf{i}[-\eta,\eta]) + 1/\alpha)\right).$$

By the first paragraph,

$$F_h^{-b'_h-2}(L_h^{-1}(x_h+\mathbf{i}[\eta,+\infty])\cap(L_h^{-1}(\mathbf{i}\mathbb{R})+1/\alpha)=\emptyset$$
$$F_h^{-b'_h-2}(L_h^{-1}(x_h+\mathbf{i}(-\infty,-\eta])\cap(L_h^{-1}(\mathbf{i}\mathbb{R})+1/\alpha)=\emptyset.$$

Therefore, there is $b_h>0$, uniformly bounded from above independent of α and h, such that $L_h^{-1}(\mathbf{i}\mathbb{R})+1/\alpha$ lies to the right of $L_h^{-1}(x_h-b_h+\mathbf{i}\mathbb{R})$. This implies that, τ_h is univalent on $L_h^{-1}((0,x_h-b_h)+\mathbf{i}\mathbb{R})$. That is, $y_h\geq x_h-b_h$. Note that $x_h-b_h\geq 1$, since $L_h^{-1}(\mathbf{i}\mathbb{R})+1/\alpha$ lies to the right of $L_h^{-1}(1+\mathbf{i}\mathbb{R})$.

By Equation (53), L_h^{-1} tends to a translation near the top end of $(0,x_h)+\mathbf{i}\mathbb{R}$, and τ_h is periodic of period $1/\alpha$. Hence, $y_h\leq 1/\alpha$. For the same reason, near the top end of B_0, $|m_\zeta-(x_h-1/\alpha)|\leq 2$. Hence, $x_h\leq 1/\alpha+2+b'_h$ and $y_h\geq x_h-b_h\geq 1/\alpha+m_\zeta-2-b_h\geq 1/\alpha-2-b_h$, where b_h is uniformly bounded from above. □

The argument in the proof of Proposition 6.15 through studying T_h has a key consequence stated in the next proposition.

Proposition 6.16. — *There is $C'>0$ such that for all $\alpha\in(0,r_2]$ and all $h\in\mathcal{QIS}_\alpha$, we have*

$$|L_h^{-1}(x_h)-1/\alpha|\leq C'.$$

Proof. — Let $\eta\geq 1$ be the constant introduced in Lemma 6.13. For $\zeta\in B_0$ with $\operatorname{Im}\zeta\geq\eta$, define $E_h(\zeta):=L_h\circ F_h^{-n_\zeta}\circ F_h^{\circ m_\zeta}(L_h^{-1}(\zeta)+1/\alpha)$, where m_ζ is the integer in Equation (54) and n_ζ is the number of backward iterates required to map $F_h^{m_\zeta}(L_h^{-1}(\zeta)+1/\alpha)$ into $L_h^{-1}(B_0)$. The integer n_ζ exists because of Lemma 6.5. By Lemma 6.13, E_h is defined above the height η, and its values belong to B_0.

By Equations (35) and (53), $\operatorname{Im}E_h(\zeta)-\operatorname{Im}\zeta\to 0$ when $\operatorname{Im}\zeta\to+\infty$. Also, by Equation (34) and Lemma 6.6, $E_h(\zeta+1)=E_h(\zeta)+1$, when $\operatorname{Re}\zeta=0$. Hence, E_h projects under $e^{2\pi\mathbf{i}\zeta}$ to a well defined univalent map $\widetilde{E}_h:B(0,e^{-2\pi\eta})\to\mathbb{C}$ satisfying $\widetilde{E}_h(0)=0$ and $|\widetilde{E}'_h(0)|=1$. By the distortion Theorem 2.1, $\widetilde{E}_h(B(0,e^{-2\pi\eta}))$ contains the ball $B(0,e^{-2\pi\eta}/4)$. Then, applying the distortion theorem to the map $\widetilde{E}_h^{-1}:B(0,e^{-2\pi\eta}/4)\to\mathbb{C}$ we conclude that $|\widetilde{E}_h^{-1}(e^{-4\pi\eta})|$ is uniformly bounded from above and away from 0. This implies that $|\operatorname{Im}E_h^{-1}(2\eta\mathbf{i})|$ must be uniformly bounded from above and below.

On the other hand, by the pre-compactness of $\mathcal{QIS}_\alpha$, on any given compact subset of B_0, $|\operatorname{Im}L_h^{-1}(\zeta)-\operatorname{Im}\zeta|$ is uniformly bounded from above. Combining with the above paragraph, we conclude that $|\operatorname{Im}L_h^{-1}\circ E_h^{-1}(2\eta\mathbf{i})|\leq C_1$, for some constant independent of α and h.

Now we have,

$$\begin{aligned}
|\operatorname{Im} L_h^{-1}(x_h)| &\le |\operatorname{Im} L_h^{-1}(n_\zeta)| + 2M_1 \\
&\le |\operatorname{Im} L_h^{-1}(n_\zeta + 2\eta\mathbf{i})| + M_1 2\eta + 2M_1 \\
&= |\operatorname{Im} F_h^{\circ n_\zeta}(L_h^{-1}(2\eta\mathbf{i}))| + 2M_1(\eta+1) \\
&\le |\operatorname{Im} F_h^{\circ(-m_\zeta+n_\zeta)}(L_h^{-1}(2\eta\mathbf{i}))| + |m_\zeta|\frac{1}{4} + 2M_1(\eta+1) \\
&\le |\operatorname{Im}\big(F_h^{\circ(-m_\zeta+n_\zeta)}(L_h^{-1}(2\eta\mathbf{i})) - \frac{1}{\alpha}\big)| + b_h'\frac{1}{4} + 2M_1(\eta+1) \\
&\le |\operatorname{Im} L_h^{-1} \circ E_h^{-1}(2\eta\mathbf{i})| + \frac{b_h'}{4} + 2M_1(\eta+1) \\
&\le C_1 + \frac{b_h'}{4} + 2M_1(\eta+1).
\end{aligned}$$

In the first line of the above equation we have used $n_\zeta \in (x_h - 2, x_h)$ and Lemma 6.11 with $\varepsilon = 1$. In the second line we have used Lemma 6.11 with $\varepsilon = 1$. In the third line we have used the functional equation (39). In the fourth line we have used $|F_h(w) - w - 1| \le 1/4$ from Lemma 6.4. In the fifth line we have used $|m_\zeta| \le b_h'$, where b_h' is the constant in the proof of Proposition 6.15, which is uniformly bounded from above.

On the other hand, by the definition of x_h, $L_h^{-1}(x_h + \mathbf{i}\mathbb{R})$ touches the right hand boundary of X. Combining with the upper bound on $|\arg L_h^{-1}(x_h + \mathbf{i}\mathbb{R}) - \pi/2|$ in Lemma 6.10, we conclude that $|\operatorname{Re} L_h^{-1}(x_h) - 1/\alpha|$ is also uniformly bounded from above. This finishes the proof of the proposition. □

6.6. A uniform bound on $|L_h^{-1}(\zeta) - \zeta|$. —

Proposition 6.17. — *There exists a constant C_4 such that for every α in $(0, r_2]$, every h in $\mathcal{QIS}_\alpha$, and every $t \in (0, x_h)$,*

$$|L_h^{-1}(t) - t| \le C_4 \min\{\log(2+t), \log(2 + x_h - t)\}.$$

Proof. — Within this proof all the constants $D_1, D_2, D_3, \ldots$ are assumed to be independent of α and h.

Let us define

$$x_h' = \sup\{t \ge 0 \mid (0,t) \subset L_h(X)\}.$$

By definition, $x_h' \ge x_h$. However, by Lemma 6.4-2 and Proposition 6.16, there is a constant D_1 such that $x_h' \le x_h + D_1$ and $L_h^{-1}(x_h') \in B(1/\alpha, D_1)$.

Let l_1 denote the vertical line $\operatorname{Re} w = C_2 + 5$, l_2 denote the vertical line $\operatorname{Re} w = 1/(2\alpha)$, and l_3 denote the vertical line $\operatorname{Re} w = 1/\alpha - C_2 - 5$. By Lemma 6.6, the closure of the curve $L_h^{-1}(0, x_h')$ connects the left hand boundary of X to the right hand boundary of X. Let $t_1 \in (0, x_h')$ be the smallest element with $L_h^{-1}(t_1) \in l_1$ and let $t_3 \in (0, x_h')$ be the largest element with $L_h^{-1}(t_3) \in l_3$. By Lemma 6.4-2 and the above paragraph, there is a constant D_2 such that $t_1 \le D_2$, $x_h' - t_3 \le D_2$. For the same reason, there is also a constant D_3 such that

$$|\operatorname{Im} L_h^{-1}(t)| \le D_3, \forall t \in [0, t_1] \cup [t_3, x_h'].$$

So far we have shown that the inequality in the proposition holds for $t \in [0, t_1] \cup [t_3, x_h]$. Below we deal with values of $t \in [t_1, t_3]$.

For $w \in X$ lying between l_1 and l_3 we have $B(w, 5) \subset A_1 \subset A_2$. Then by Lemma 6.7-1, for $t \in (t_1, t_3)$, $L_h^{-1}(t)$ lies between the lines l_1 and l_3, and we have $|\arg(L_h'(L_h^{-1}(t)))| \leq \pi/3$ as well as $2/5 \leq |L_h'(L_h^{-1}(t))| \leq 8/3$. It follows that there is a unique $t_2 \in (t_1, t_3)$ such that $L_h^{-1}(t_2) \in l_2$, and

$$\begin{aligned} \operatorname{Re} L_h^{-1}(t) &\geq \operatorname{Re} L_h^{-1}(t_1) + \frac{2}{5}\cos(\frac{\pi}{3})(t - t_1) \geq C_2 + 5 + \frac{1}{5}(t - t_1), \forall t \in (t_1, t_2), \\ \operatorname{Re} L_h^{-1}(t) &\leq \operatorname{Re} L_h^{-1}(t_3) - \frac{2}{5}\cos(\frac{\pi}{3})(t_3 - t) \leq \frac{1}{\alpha} - C_2 - 5 - \frac{1}{5}(t_3 - t), \forall t \in (t_2, t_3), \\ |\operatorname{Im} L_h^{-1}(t)| &\leq |\operatorname{Im} L_h^{-1}(t_1)| + \frac{8}{3}\sin(\frac{\pi}{3})(t_3 - t_1) \leq D_3 + \frac{4}{\sqrt{3}}(t_3 - t_1), \forall t \in (t_1, t_3). \end{aligned} \tag{56}$$

In particular, by Proposition 6.15, $|\alpha \operatorname{Im} L_h^{-1}(t)| \leq \alpha(D_3 + 1/\alpha + k + D_1) \leq D_3 + 1 + k + D_1$. Let $D_4 = D_3 + 1 + k + D_1$.

On the other hand,

$$\begin{aligned} d(L_h^{-1}(t), \mathbb{Z}/\alpha) = d(L_h^{-1}(t), 0) \geq \operatorname{Re} L_h^{-1}(t), \forall t \in (t_1, t_2), \\ d(L_h^{-1}(t), \mathbb{Z}/\alpha) = d(L_h^{-1}(t), 1/\alpha) \geq 1/\alpha - \operatorname{Re} L_h^{-1}(t), \forall t \in (t_2, t_3). \end{aligned} \tag{57}$$

Therefore,

$$\begin{aligned} B(L_h^{-1}(t), \operatorname{Re} L_h^{-1}(t) - C_2) \subset A_1, \forall t \in (t_1, t_2), \\ B(L_h^{-1}(t), \frac{1}{\alpha} - \operatorname{Re} L_h^{-1}(t) - C_2) \subset A_1, \forall t \in (t_2, t_3). \end{aligned}$$

Now, it follows from Lemma 6.7-2 and Equations (56) and (57) that there is a constant D_5, such that

$$\begin{aligned} |\operatorname{Re} L_h'(L_h^{-1}(t)) - 1| \leq \frac{D_5}{5 + (t - t_1)/5}, \forall t \in (t_1, t_2) \\ |\operatorname{Re} L_h'(L_h^{-1}(t)) - 1| \leq \frac{D_5}{5 + (t_3 - t)/5}, \forall t \in (t_2, t_3) \end{aligned} \tag{58}$$

and

$$\begin{aligned} |\operatorname{Im} L_h'(L_h^{-1}(t))| \leq \frac{D_5}{5 + (t - t_1)/5}, \forall t \in (t_1, t_2) \\ |\operatorname{Im} L_h'(L_h^{-1}(t))| \leq \frac{D_5}{5 + (t_3 - t)/5}, \forall t \in (t_2, t_3). \end{aligned} \tag{59}$$

Integrating the inequalities in (58) we conclude that there is a constant D_6 such that for $t \in (t_1, t_2)$

$$|\operatorname{Re} L_h^{-1}(t) - t| \leq |\operatorname{Re} L_h^{-1}(t_1) - t_1| + \int_{t_1}^{t} \frac{D_5}{5 + (t - t_1)/5}\, dt \leq D_6 + D_6 \log(1 + (t - t_1)),$$

and for $t \in (t_2, t_3)$

$$|\operatorname{Re} L_h^{-1}(t) - t| \leq |\operatorname{Re} L_h^{-1}(t_3) - t_3| + \int_{t}^{t_3} \frac{D_5}{5 + (t - t_1)/5}\, dt \leq D_6 + D_6 \log(1 + (t_3 - t)).$$

Similarly, integrating the inequalities in (59) we obtain

$$|\operatorname{Im} L_h^{-1}(t)| \le D_3 + D_6 \log(1 + (t - t_1)), \forall t \in (t_1, t_2),$$
$$|\operatorname{Im} L_h^{-1}(t)| \le D_3 + D_6 \log(1 + (t_3 - t)), \forall t \in (t_2, t_3).$$

These imply the desired inequality in the proposition for $t \in [t_1, t_3]$, with a constant C_4 depending only on D_3 and D_6. □

For $z \in \mathbb{C} \setminus \{0\}$ there is an inverse branch of the covering map $\mathbb{E}\mathrm{xp}$ defined on a neighborhood of z. The derivative at z of any such inverse branch is well-defined and is independent of the choice of the lift and the neighborhood. We let $(\mathbb{E}\mathrm{xp}^{-1})'(z)$ denote this complex number. As a corollary of Proposition 6.17 we obtain the following estimate on the derivative of Φ_h, which is convenient to write in terms of $(\mathbb{E}\mathrm{xp}^{-1})'$. Recall the constant k defined in Proposition 6.15.

Proposition 6.18. — *There exists a constant C_5 such that for every α in $(0, r_2]$, every h in $\mathcal{QLS}_\alpha$, and every t with $1 \le t \le \min\{1/(2\alpha), x_h\}$, we have*

$$\frac{1}{C_4 t} \le (\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_h^{-1})'(t) \le \frac{C_4}{t}.$$

Proof. — By Proposition 6.15, $x_h \ge \alpha^{-1} - k$, for some uniform constant k. For large values of α, x_h may be less than $1/(2\alpha)$. The condition $1 \le t \le \min\{1/(2\alpha), x_h\}$ guarantees that $\Phi_h^{-1}(t)$ is defined.

We assume that the constants D_i, $i = 1, 2, 3, \ldots$, within this proof, are independent of α and h.

Using Lemma 6.11 with $\varepsilon = 1$, there is D_1 such that for all $t \in [1, x_h]$, $D_1^{-1} \le |(L_h^{-1})'(t)| \le D_1$. Therefore, it is enough to show that for some constant D_2 we have

$$\frac{1}{D_2 t} \le (\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h)'(L_h^{-1}(t)) \le \frac{D_2}{t}. \tag{60}$$

There are essentially two arguments to prove the above bounds. To introduce these, recall the constant C_4 in Proposition 6.17. There is $t_0 \ge 1$ such that for $t \ge t_0$ we have $C_4 \log(2+t) \le t/2$.

First we consider the case $\alpha \ge \min\{1/(5k), 1/(2t_0)\}$. By definition, for $t \in [1, x_h]$, $L_h^{-1}(t) \in X$, and for nonzero integers n, $X \cap B(n/\alpha, C_2) = \emptyset$. On the other hand, by Lemma 6.11 with $\varepsilon = 1/2$, the image of the strip $1/2 \le \operatorname{Re}\zeta \le 1$ under L_h^{-1} uniformly separates $L_h^{-1}(t)$ from 0. That is, there is a constant D_3 such that for all $t \in [1, x_h]$, $|L_h^{-1}(t)| \ge D_3$. Therefore, for $t \in [1, x_h]$, $L_h^{-1}(t)$ is uniformly away from the set $\mathbb{Z}/\alpha$. On the other hand, since $x_h \le \alpha^{-1} + k \le 6k$ is uniformly bounded from above, by the pre-compactness of the class of maps h, for $t \in [1, x_h]$, $|L_h^{-1}(t)|$ is uniformly bounded from above independent of t, α, and h. These imply that $|\tau_h(L_h^{-1}(t))|$ is uniformly bounded from above and away from 0, by constants independent of t, α, and h.

By the above paragraph, and explicit estimates of the formula for $\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h$, there is a constant D_4 such that $D_4^{-1} \le |(\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h)'(L_h^{-1}(t))| \le D_4$. In this case, t is uniformly bounded away from 0 and from above by $1 \le t \le x_h \le \alpha^{-1} + k \le 6k$. Therefore, one may adjust the constant D_4 to some uniform constant D_2 so that Equation (60) holds for $t \in [1, x_h]$.

Now assume that $\alpha \leq \min\{1/(5k), 1/(2t_0)\}$. Here, by Proposition 6.15, $1/(2\alpha) \leq x_h$, and $t_0 \leq 1/(2\alpha)$. On the uniformly bounded subset $1 \leq t \leq t_0$, the above argument may be repeated to conclude that $t \cdot |(\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h)'(L_h^{-1}(t))|$ is uniformly bounded from above and away from 0 on this interval. It remains to consider $t \in [t_0, 1/(2\alpha)]$.

For $t \in [t_0, 1/(2\alpha)]$ define the set

$$O_t := \{\xi \in \mathbb{C} : |\operatorname{Im} \xi| \leq t/2, |\operatorname{Re} \xi - t| \leq t/2\}.$$

Using the uniform bound $|\sigma_h| \leq C_1\alpha$ in Equation (32), and some explicit estimates of $\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h$, there exists a constant D_5 such that

$$\frac{1}{D_5 t} \leq |(\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h)'(t)| \leq \frac{D_5}{t}.$$

For $1 \leq t \leq 1/(2\alpha)$, the modulus of the annulus $\Phi_h(\mathcal{P}_h) \setminus O_t$ is uniformly bounded away from zero, by a constant independent of t, α, and h. Then, one infers from the above bounds and the distortion Theorem 2.1 that there is a constant D_6 such that for all $\xi \in O_t$,

$$\frac{1}{D_6 t} \leq |(\mathbb{E}\mathrm{xp}^{-1} \circ \tau_h)'(\xi)| \leq \frac{D_6}{t}.$$

By Proposition 6.17 and the choice of t_0, for $t_0 \leq t \leq 1/(2\alpha)$, $L_h^{-1}(t) \in O_t$. Thus, we obtain the uniform bounds in Equation (60) for $t \in [t_0, 1/(2\alpha)]$. □

Proposition 6.19. — *For every $M' \in \mathbb{R}$, there is $M > 0$, such that for all $\alpha \in (0, r_2]$, all $h \in \mathcal{QIS}_\alpha$, and all $\zeta \in [0, x_h] + \mathbf{i}[M', +\infty)$ we have*

$$|L_h^{-1}(\zeta) - \zeta| \leq M \log(1 + 1/\alpha).$$

Proof. — All the constants $D_1, D_2, D_3, \ldots$ within this proof are assumed to be independent of α and h. Without loss of generality we may assume that $M' \leq 0$.

Recall the numbers s_1 and s_2 introduced in Equation (52), and define

$$Y_1 = [0, x_h] + \mathbf{i}[M', +\infty), \quad Y_2 = [s_1, s_2] + \mathbf{i}[M', +\infty).$$

By Lemmas 6.4-2, 6.5-4, and 6.6, there is a constant D_1 such that $s_1 \leq D_1$ and $x_h - s_2 \leq D_1$. In particular, by Lemma 6.4-2, and Equation (39), it is enough to prove the uniform upper bound in the proposition for values of $\zeta \in Y_2$. On the other hand, by the maximum principle, it is enough to prove the inequality on the boundary of Y_2.

Assume that $\zeta \in \partial Y_2$ with $\operatorname{Re} \zeta = s_1$. By Lemma 6.10-(2), $B(L_h^{-1}(\zeta), 5) \subset A_1 \subset A_2$. Then, by Lemma 6.7-(1), $|\arg L_h'(L_h^{-1}(\zeta))| \leq \pi/3$ and $2/5 \leq |L_h'(L_h^{-1}(\zeta))| \leq 8/3$. Hence, the function $\operatorname{Im} \zeta \mapsto \operatorname{Im} L_h^{-1}(\zeta)$ is strictly monotone, and there is a constant D_2 such that

$$\frac{8}{3} \operatorname{Im} \zeta + D_2 \geq \operatorname{Im} L_h^{-1}(\zeta) \geq \frac{2}{5} \sin \frac{\pi}{3} \cdot (\operatorname{Im} \zeta) - D_2.$$

This implies that there are $t_1 \leq t_2$, with $|t_1|$ and $\alpha|t_2|$ uniformly bounded from above, such that $\operatorname{Im} L_h^{-1}(s_1 + \mathbf{i}t_1) \geq 3.25$ and $\operatorname{Im} L_h^{-1}(s_1 + \mathbf{i}t_2) \geq 1/(2\alpha)$. By Lemma 6.7-(2), there is a constant D_3 such that for all $t \in [t_1, t_2]$, $|(L_h^{-1})'(s_1 + \mathbf{i}t) - 1| \leq D_3/t$. Integrating this inequality we conclude that there is a constant D_4 such that for $\operatorname{Im} \zeta \in [t_1, t_2]$, $|L_h^{-1}(\zeta) - \zeta|$ is bounded from above by a uniform constant times $\log \alpha^{-1} + D_4$.

On the other hand, by Proposition 6.8 (with $M' = 1/2$ and $r = 1/2$), the map L_h is uniformly close to a translation on vertical lines with imaginary part bigger than $1/(2\alpha)$, with an error bounded by $M/r = 2M$. It follows from Lemma 6.11 that for $t \geq t_2$, the function $t \mapsto L_h^{-1}(s_1 + \mathbf{i}t)$ is uniformly close to a translation. Combining this with the above paragraph, we conclude that there is a constant D_5 such that $|L_h^{-1}(\zeta) - \zeta| \leq D_5 \log(1 + 1/\alpha)$, for ζ with $\operatorname{Re}\zeta = s_1$ and $\operatorname{Im}\zeta \geq t_1$.

By an identical argument, one can show that there are t_1', with $|t_1'|$ uniformly bounded from above, and a constant D_6 such that $|L_h^{-1}(\zeta) - \zeta| \leq D_6 \log(1 + 1/\alpha)$, for ζ with $\operatorname{Re}\zeta = s_2$ and $\operatorname{Im}\zeta \geq t_1'$.

Let $D_7 = \max\{t_1, t_2\}$ and assume that $\zeta \in \partial Y_2$ with $\operatorname{Im}\zeta \leq D_7$. Recall that $s_1 > 1$ by Lemma 6.10-(1). Let M_1 be the constant obtained from applying Lemma 6.11 with $\varepsilon = 1$. Then, by virtue of Propositions 6.15 and 6.17, we have

$$\begin{aligned} |L_h^{-1}(\zeta) - \zeta| &\leq |L_h^{-1}(\zeta) - L_h^{-1}(\operatorname{Re}\zeta)| + |L_h^{-1}(\operatorname{Re}\zeta) - \operatorname{Re}\zeta| + |\operatorname{Re}\zeta - \zeta| \\ &\leq M_1(D_7 - M') + C_4 \log(2 + \alpha^{-1} + k) + (D_7 - M'). \end{aligned}$$

This implies the desired inequality, with a uniform bound, on the boundary of Y_2 with imaginary part less than D_7.

By Equation (53), $|L_h^{-1}(\zeta) - \zeta|$ tends to a constant near the top end of Y_2. The limit is independent of ζ, and, by the above paragraphs, it is bounded from above by a uniform constant times $\log(1 + \alpha^{-1})$ along $\operatorname{Re}\zeta = s_1$. Then, $|L_h^{-1}(\zeta) - \zeta|$ is bounded by a uniform constant times $\log(1 + \alpha^{-1})$ near the top end of Y_2. Combining with the above paragraphs, the desired inequality with a uniform bound is proved on the boundary of Y_2. This finishes the proof of the proposition. □

The above proposition gives an upper bound on the asymptotic translation of L_h^{-1} in Equation (53), which we state below for reference purposes.

Corollary 6.20. — *There is $C > 0$ such that for all $\alpha \in (0, r_2]$ and all $h \in \mathcal{QIS}_\alpha$, we have*

$$|\ell_h| \leq C \log(1 + \alpha^{-1}).$$

Remark 6.21. — When α tends to zero, $h(z) := P \circ \phi^{-1}(e^{2\pi\mathbf{i}\alpha} \cdot z)$ tends to a map h_0 with a parabolic fixed point at 0. Then, L_h tends to some univalent map L_{h_0} which is the lift of the attracting Fatou coordinate of h_0 under the change of coordinate $w = -2/(h''(0)z)$. It is known that L_{h_0} has asymptotic expansion $w + a \log w + c + o(1)$ near $+\mathbf{i}\infty$, for some constants a and c, for instance see [**Shi00**, Prop. 2.2.1]. So, it seems the logarithmic bound in Propositions 6.17 and 6.19 is necessary. The main point used in this paper is that since $\alpha \log(1 + 1/\alpha) \to 0$ as $\alpha \to 0$, the logarithmic error is absorbed in the formula of τ_h as the rotation numbers degenerate along the renormalization tower.

6.7. Geometry of the petals. — Recall that $L_h = \Phi_h \circ \tau_h$, where τ_h is given by the formula (33), L_h and its domain of definition X are defined in Section 6.3, and Φ_h is the Fatou coordinate defined in Theorem 2.3. In this section we employ the estimates on L_h established

in Sections 6.4, 6.5, and 6.6, as well as some explicit estimations of τ_h to prove the geometric properties of the maps Φ_h, as well as their domain and range, stated in Propositions 2.4, 2.7, 4.1, and 5.1.

Proof of Proposition 2.4. — Let r_2 be the constant defined in Equation (51). Let us fix $h \in \mathcal{QIS}_\alpha$, $\alpha \in (0, r_2]$. By the definitions of the constants $x_h \geq y_h$ in Equations (50) and (55), the map L_h^{-1} is defined on $(0, y_h) + \mathbf{i}\mathbb{R}$, and τ_h is univalent on $L_h^{-1}((0, y_h) + \mathbf{i}\mathbb{R})$. Moreover, by Lemma 6.6, L_h^{-1} is univalent on $(0, y_h) + \mathbf{i}\mathbb{R}$. We define

$$\mathcal{P}_h = \tau_h \circ L_h^{-1}((0, y_h) + \mathbf{i}\mathbb{R}), \Phi_h = L_h \circ \tau_h^{-1} : \mathcal{P}_h \to \mathbb{C}.$$

By Proposition 6.15, $y_h \geq \alpha^{-1} - k$, where k is a constant independent of α and h. This implies part 2 of the proposition by defining $\boldsymbol{k} = k$. Below we prove part 1 of the proposition.

Since $\mathcal{P}_h$ is simply connected and $0 \notin \mathcal{P}_h$, there is a continuous branch of argument defined on $\mathcal{P}_h$.

The map L_h has a holomorphic extension onto a neighborhood of $L_h^{-1}(\mathbf{i}\mathbb{R})$ using the functional equation $L_h(F_h(\zeta)) = L_h(\zeta) + 1$. Then, $L_h^{-1}(\mathbf{i}\mathbb{R})$ intersects the real axis at least once and at most at a finite number of points. There is $t_1 \in \mathbb{R}$ such that $L_h^{-1}(t_1\mathbf{i})$ is the closest point to 0 among all such intersections. That is, $L_h^{-1}(t_1\mathbf{i}) \in \mathbb{R}$. Moreover, there is $t_2 > t_1$ such that for $t \geq t_2$, $L_h^{-1}(t\mathbf{i}) \notin B(0, C_2)$, where C_2 is the constant in Lemma 6.4. By the normalization of L_h, $L_h^{-1}(0) = \mathrm{cp}_{F_h} \in B(0, C_2)$, where cp_{F_h} denotes the critical point of F_h. It follows from the pre-compactness of the class of maps h that $|t_1|$ and t_2 are uniformly bounded from above independent of h and α.

Let us define the curve γ_1 as the interval $(0, L_h^{-1}(t_1\mathbf{i})] \subset \mathbb{R}$, the curve γ_2 as $L_h^{-1}(t\mathbf{i})$ for $t \in [t_1, t_2]$, and the curve γ_3 as $L_h^{-1}(t\mathbf{i})$ for $t \in [t_2, \infty)$. We denote the union of these curves by γ. Since $L_h^{-1}((0, y_h) + \mathbf{i}\mathbb{R}) \subset X$ and $X \cap B(n/\alpha, C_2) = \emptyset$, for integers $n \neq 0$, $\tau_h(\gamma)$ is a simple curve in $\mathbb{C} \setminus \mathcal{P}_h$. Moreover, $\tau_h(\gamma)$ belongs to $\mathbb{C} \setminus \{0\}$ and tends to 0 and infinity at its two ends. To prove the existence of a constant $\hat{\boldsymbol{k}}$ for part 1 of the proposition, it is enough to show that for any continuous branch of argument defined on $\tau_h(\gamma)$, $\sup |\arg w - \arg w'|$, for $w, w' \in \tau_h(\gamma)$, is uniformly bounded from above by a constant independent of h and α.

The curve γ_1 is mapped to a half-line under τ_h tending to infinity in $\mathbb{C}$. Hence, $\sup |\arg w - \arg w'|$, for $w, w' \in \tau_h(\gamma_1)$ is bounded by π. On the other hand, for every h, $\sup |\arg w - \arg w'|$, for $w, w' \in \tau_h(\gamma_2)$ is finite. Hence, by the pre-compactness of the class of maps h, there is a uniform upper bound on these numbers independent of h and α.

Using Proposition 6.19 with $M' = 0$, there is $M > 0$ such that for $t \geq t_2$, $|L_h^{-1}(t\mathbf{i}) - t\mathbf{i}| \leq M \log(1 + 1/\alpha)$. Therefore, γ_3 is contained in the set

$$[-M \log(1 + 1/\alpha), M \log(1 + 1/\alpha)] + \mathbf{i}[-M \log(1 + 1/\alpha), +\infty).$$

Recall that τ_h is periodic of period $1/\alpha$, and it maps every vertical line to an arc of a circle connecting 0 to σ_h (Each such arc segment spirals at most by π about 0). Since $\alpha \log(1 + \alpha^{-1})$ is uniformly bounded from above, it follows that $\sup |\arg w - \arg w'|$, for $w, w' \in \tau_h(\gamma_3)$, is uniformly bounded from above by a constant depending only on M. □

Proof of Proposition 2.7. — Fix h with $h'(0) = e^{2\pi \mathbf{i}\alpha}$ and $\alpha \in (0, r_3]$. By Equation (53), each curve $\tau_h \circ L_h^{-1}(t + \mathbf{i}\mathbb{R})$, $t \in [0, 1/\alpha - \boldsymbol{k}]$, approaches zero with a well-defined tangent at 0. This implies that if $w \in \mathcal{C}_h \cup \mathcal{C}_h^\sharp$ is close enough to zero, there exists a unique inverse orbit $w, h^{-1}(w), \dots, h^{-j}(w)$ staying near zero such that j is the smallest positive integer with $h^{-j}(w) \in \mathcal{P}_h$. Comparing with the rotation of angle α, one can see that for $|w|$ small enough $\boldsymbol{k} + 1 \le j \le \boldsymbol{k} + 2$.

By Theorem 2.6, $\mathcal{R}(h)$ is of the form $z \mapsto P \circ \psi^{-1}(e^{2\pi \mathbf{i}/\alpha} \cdot z)$, where $\psi : U \to \mathbb{C}$ is a univalent mapping that has a univalent extension onto the larger domain V which contains the closure of U. By the distortion Theorem 2.1, $\mathcal{R}(h)$ must be uniformly close to a rotation on U, with a constant independent of h and α. Moreover, we also conclude that the pre-image (under $\mathcal{R}(h)$) of any ray in $P(U)$ landing at 0 must have uniformly bounded spiral about zero. Thus, any lift of $\mathcal{R}(h)$ under $\mathbb{E}\mathrm{xp}$ must be uniformly close to some translation, with the bound independent of h and α. This implies that $|k_h - j|$, for any j as above, is uniformly bounded from above. In particular, $k_h \le |k_h - j| + j$ is uniformly bounded from above independent of h and α. □

Proof of Proposition 4.1. — Recall that for $\alpha \le r_3$ the sector S_f is defined (see Theorem 2.6). Let M denote the constant produced by Proposition 6.19 applied with $M' = -2$. We consider two cases separately.

Recall the constant k from Proposition 6.15. The first case is to assume that α is small enough such that

$$\alpha \le 1/(3 + 2k) \text{ and } 3/2 + M \log(1 + 1/\alpha) \le 1/(4\alpha).$$

Define the set $A := [\lfloor 1/(2\alpha) \rfloor + 1/2, \lfloor 1/(2\alpha) \rfloor + 3/2] + \mathbf{i}[-2, +\infty)$. By definitions, $A = \Phi_f \circ f^{\circ(k_f + \lfloor 1/(2\alpha) \rfloor)}(S_f)$, and by the above condition on α, A is contained in $[0, y_h] + \mathbf{i}[-2/\alpha, \infty)$. By Proposition 6.19 with $M' = -2$, for all $\zeta \in A$ we have

$$\operatorname{Im} L_h^{-1}(\zeta) \ge -2 - M \log(1 + 1/\alpha),\ |\operatorname{Re} L_h^{-1}(\zeta) - 1/(2\alpha)| \le 3/2 + M \log(1 + 1/\alpha).$$

Now, using the second condition on α, the uniform bound $|\sigma_f| \le C_1 \alpha$ in Equation (32), and an explicit calculation of the formula for τ_f, there is a constant M_1' (depending only on M) such that

$$\operatorname{diam}(f^{\circ(k_f + \lfloor 1/2\alpha \rfloor)}(S_f)) = \operatorname{diam}(\Phi_f^{-1}(A)) = \operatorname{diam}(\tau_f \circ L_f^{-1}(A)) \le M_1' \cdot \alpha.$$

Here, when estimating τ_f, one uses that $\alpha \log(1 + 1/\alpha)$ is uniformly bounded from above on $(0, 1)$.

The second case is to assume that α is larger than some constant, but still less than r_3. Recall the constant s_1 defined in Equation (52), and define $A := [\lfloor s - 1/2 \rfloor, \lfloor s + 1/2 \rfloor] + \mathbf{i}[-2, +\infty)$. By Lemmas 6.10-2 and 6.7-1, for all $t \in \mathbb{R}$, $|\arg L_h'(L_h^{-1}(s_1 + \mathbf{i}t))| \le \pi/3$. Then, by Lemma 6.4, $L_h^{-1}(A)$ must be contained within $5/4$ of the set $|\arg(w - 2C_2 - 10) - \pi/2| \le \pi/3$. Also, as in the previous case, A lies above the line $\operatorname{Im} w = -2 - M \log(1 + 1/\alpha)$. Then, an explicit calculation on τ_f shows that for all $\zeta \in A$, $|\tau_f \circ L_f^{-1}(\zeta)| \le M_1''$, for some constant M_1'', independent of α and f. Here we only use that $|\sigma_f|$ is uniformly bounded from above independent of α and f (Indeed, $|\sigma_f| \le 4/27$ is assumed for $\alpha \le r_2'$ in the proof of Lemma 6.4). Because α is bounded from below in this case, one can adjust the constant M_1'' such that $|\tau_f \circ L_f^{-1}(\zeta)| \le M_1'' \alpha$ holds. □

Proof of Proposition 5.1. — There are two arguments; one for values of α near 0 and one for values of α away from zero. First we present the former case, where we impose a number of upper bounds on α along the way so that the proof works. The second case is based on a pre-compactness argument, and is presented at the end of this proof.

All the constants $D_1, D_2, D_3, \dots$ introduced within this proof are assumed to be independent of α and $f \in \mathcal{QIS}_\alpha$.

Consider the line segment

$$\vartheta(t) := t - (2 + t/2)\mathbf{i}, \text{ for } t \in [2, 1/(2\alpha)].$$

Let $\hat{\eta}_f : \mathcal{P}_f \to \mathbb{C}$ be an arbitrary inverse branch of the covering map $\mathbb{E}\mathrm{xp}$, and define

$$\chi_f := \hat{\eta}_f \circ \Phi_f^{-1} : \Phi_f(\mathcal{P}_f) \to \mathbb{C}. \tag{61}$$

We shall fixed the choice of the branch of $\hat{\eta}_f$ in a moment, but until then, all statements involving $\hat{\eta}_f$ are independent of the choice of the branch.

For

$$\alpha \le \frac{1}{4\boldsymbol{k}}, \tag{62}$$

by Proposition 6.15, the image of ϑ is contained in the domain of χ_f.

Sublemma 6.22. — *There is $D_1 > 0$ such that*

$$|\operatorname{Im} \chi_f(\vartheta(2))| \le D_1, \quad \operatorname{Im} \chi_f(\vartheta(\frac{1}{2\alpha})) \ge \frac{1}{2\pi} \log \frac{1}{\alpha} - D_1.$$

Proof. — There is a topological annulus A in $\Phi_f(\mathcal{P}_f)$ which separates the pair of points 1 and $\vartheta(2)$ from a neighborhood of $+\mathbf{i}\infty$ and its modulus is uniformly bounded away from 0, independent of α and f. Recall that $\chi_f(1) \in \mathbb{Z}$. The univalent map χ_f lifts A to an annulus of the same modulus in $(\mathbb{C} \setminus \mathbb{Z}) \cup \{\chi_f(1)\}$, which encloses the pair of points $\chi_f(1)$ and $\chi_f(\vartheta(2))$. As the modulus of this annulus is uniformly bounded away from 0, $|\chi_f(1) - \chi_f(\vartheta(2))|$ must be uniformly bounded from above. This implies the first inequality in the sublemma.

By Proposition 6.19 (with $M' = 0$) and an explicit estimate on τ_f, there is a constant D_2 such that $|\Phi_f^{-1}(1/(2\alpha))| \le D_2\alpha$. Therefore,

$$\operatorname{Im} \chi_f(\frac{1}{2\alpha})) \ge \frac{1}{2\pi} \log \frac{1}{\alpha} - \frac{1}{2\pi} \log \frac{27D_2}{4}.$$

Let γ denote the line segment connecting $1/(2\alpha)$ to $\vartheta(1/(2\alpha))$. Then, by Equation (62), the modulus of the annulus $\Phi_f(\mathcal{P}_f) \setminus \gamma$ is uniformly bounded away from 0 by a constant independent of α and f. By Proposition 6.18 and the distortion Theorem 2.1, $|\chi_f'|$ on γ is bounded from above by a uniform constant times α. This implies that the length of the curve $\chi_f(\gamma)$ is uniformly bounded from above. Combining this with the above inequality we conclude the second inequality in the sublemma. □

Sublemma 6.23. — *There is $D_3 > 0$ such that for all $t \in [2, 1/(2\alpha)]$ we have*

$$\frac{1}{D_3 t} \le |\chi_f'(\vartheta(t))| \le \frac{D_3}{t}.$$

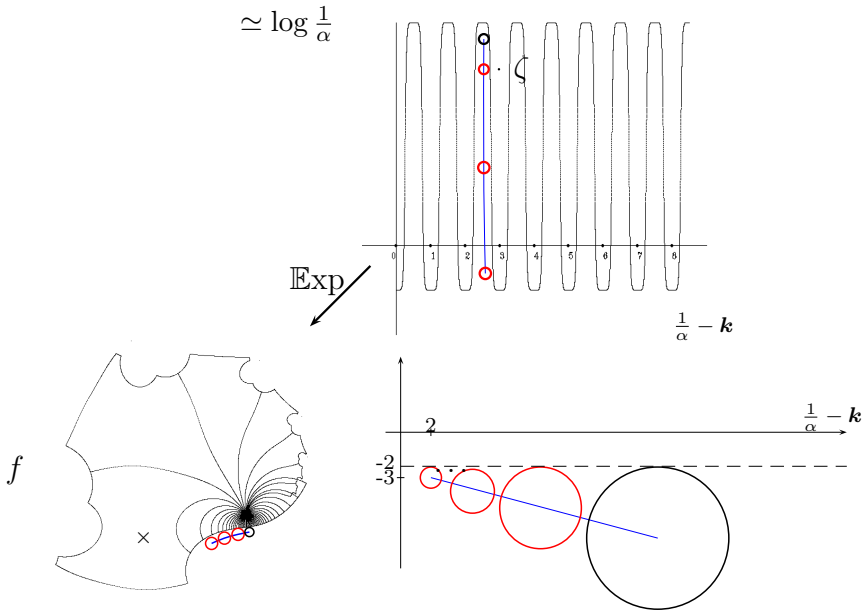


FIGURE 9. The figure shows a cartoon of the lift of the sectors under $\mathbb{E}\mathrm{xp}$, and the balls in their complement.

Proof. — First note that by Equation (62) and Proposition 6.15, $1/(2\alpha) \leq x_h$. For each t in $[2, 1/(2\alpha)]$, let γ_t denote the line segment connecting t to $\vartheta(t)$. For α satisfying Equation (62), the modulus of the annulus $\Phi_f(\mathcal{P}_f)\backslash\gamma_t$ is uniformly bounded away from 0. Thus, by the uniform bounds in Proposition 6.18 and the distortion Theorem 2.1, we obtain the uniform bound in the sublemma. □

When

$$\alpha \leq \frac{1}{4(\boldsymbol{k}'' + \boldsymbol{k} + 2)}, \tag{63}$$

for every $t \in [2, 1/(2\alpha)]$, we have

$$\begin{aligned} B(\vartheta(t), t/2) &\subset \{w \in \mathbb{C} : 1 \leq \mathrm{Re}(w) \leq \alpha^{-1} - \boldsymbol{k} - \boldsymbol{k}'' - 2, \mathrm{Im}\, w \leq -2\}, \\ B(\vartheta(t), t/2) + 1 &\subset \{w \in \mathbb{C} : 1 \leq \mathrm{Re}(w) \leq \alpha^{-1} - \boldsymbol{k} - \boldsymbol{k}'' - 1, \mathrm{Im}\, w \leq -2\}. \end{aligned} \tag{64}$$

In particular, χ_f is defined and univalent on $B(\vartheta(t), t/2)$. By Sublemma 6.23 and the Koebe 1/4-theorem,

$$B\big(\chi_f(\vartheta(t)), \frac{1}{8D_3}\big) \subseteq \chi_f\big(B(\vartheta(t), t/2)\big). \tag{65}$$

Let us define

$$D_4 := \min\{1/(8D_3), 1/4\}.$$

Let $\zeta_0 \in \mathbb{C}$ be an arbitrary point that satisfies the hypothesis of the proposition. We have

$$\operatorname{Im}\zeta_0 \leq \frac{1}{2\pi}\log\frac{1}{\alpha} + E. \tag{66}$$

Let us assume that ζ_0 also satisfies

$$\operatorname{Im}\zeta_0 \geq 1. \tag{67}$$

(We shall deal with $\operatorname{Im}\zeta_0 \leq 1$ in a moment.)

By Sublemma 6.22, $\operatorname{Im}\chi_f(\vartheta(2)) \leq D_1$ and $\operatorname{Im}\chi_f(\vartheta(1/(2\alpha))) \geq (2\pi)^{-1}\log\alpha^{-1} - D_1$. If we assume that

$$\alpha \leq e^{-4\pi D_1 - 2\pi}, \tag{68}$$

then

$$\frac{1}{2\pi}\log\frac{1}{\alpha} - D_1 \geq D_1 + 1.$$

Then, it follows from Equations (66) and (67) that there exists $t' \in [2, 1/(2\alpha)] = \operatorname{Dom}\vartheta$, such that

$$\operatorname{Im}(\chi_f(\vartheta(t'))) \geq 1, \quad -D_1 + 1 \leq \operatorname{Im}\zeta_0 - \operatorname{Im}\chi_f(\vartheta(t')) \leq D_1 + E. \tag{69}$$

Note that $(\Phi_f \circ \mathbb{E}\mathrm{xp})^{-1}(\vartheta(t'))$ forms a 1-periodic set of points. Then, there is a choice of the branch of $\hat{\eta}_f$ such that

$$|\operatorname{Re}\zeta_0 - \operatorname{Re}\chi_f(\vartheta(t'))| \leq 1/2. \tag{70}$$

From here on we shall fix this choice of the inverse branch χ_f. See Figure 9.

Let us define the curve

$$\gamma(s) = (1-s)\zeta_0 + s\chi_f(\vartheta(t')), s \in [0,1].$$

Fix an arbitrary $\delta_1 \in (0, 1/16)$.

Since $D_4 \leq 1/4$, by Equations (67) and (69) the set $B(\gamma(1), D_4) \cup \gamma[0,1]$ is contained above the line $\operatorname{Im}\zeta = 3/4$. On the other hand, by Lemma 6.1, $\operatorname{Dom} f \supset B(0, 8/9)$, and hence,

$$\mathbb{E}\mathrm{xp}\left(\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \geq 0\}\right) \subset \operatorname{Dom} f \setminus \{0\}.$$

In particular,

$$\mathbb{E}\mathrm{xp}\left(B_{\delta_1}\big(B(\gamma(1), D_4) \cup \gamma[0,1]\big)\right) \subseteq \operatorname{Dom} f \setminus \{0\}.$$

This proves Part (1) of the proposition.

As $D_4 \leq 1/4$, by Equation (70),

$$\operatorname{diam}\left(\operatorname{Re}(B(\gamma(1), D_4) \cup \gamma[0,1])\right) \leq 1/2 + 1/4 = 3/4,$$

which implies Part (3) of the proposition.

As γ is a straight line segment of uniformly bounded length, one may choose a uniform δ_2 for Part (4) of the proposition.

Recall that by Proposition 2.7, $k_f \leq \boldsymbol{k}''$. This implies that

$$\mathcal{P}_f \cap \cup_{j=0}^{k_f - 1} f^{\circ j}(S_f) \subset \Phi_f^{-1}\left(\{w \in \mathbb{C} : \operatorname{Re} w \in [0, 1/2] \cup (\alpha^{-1} - \boldsymbol{k} - \boldsymbol{k}'' - 1, \alpha^{-1} - \boldsymbol{k})\}\right).$$

On the other hand,

$$\mathcal{P}_f \cap \cup_{j=k_f}^{k_f+\lfloor 1/\alpha \rfloor - \boldsymbol{k}-1} f^{\circ j}(S_f) \subset \Phi_f^{-1}\big(\{w \in \mathbb{C} : \operatorname{Im} w \geq -2\}\big).$$

Hence, by Equation (64),

$$\Omega_0^0(f) \cap \Phi_f^{-1}(B(\vartheta(t'), t'/2)) = \emptyset, \quad \Omega_0^0(f) \cap f(\Phi_f^{-1}(B(\vartheta(t'), t'/2))) = \emptyset.$$

On the other hand, by Equation (65),

$$\mathbb{E}\mathrm{xp}(B(\gamma(1), D_4)) \subset \mathbb{E}\mathrm{xp}(\chi_f(B(\vartheta(t'), t'/2))) = \Phi_f^{-1}(B(\vartheta(t'), t'/2))).$$

Hence,

$$\mathbb{E}\mathrm{xp}(B(\gamma(1), D_4)) \cap \Omega_0^0(f) = \emptyset, \quad f(\mathbb{E}\mathrm{xp}(B(\gamma(1), D_4))) \cap \Omega_0^0(f) = \emptyset,$$

as desired in Part (2) of the proposition.

This finishes the proof of the proposition when α satisfies Equations (62), (63), and (68), as well as ζ_0 satisfies Equation (67). Below we consider the remaining case.

By the assumption $\operatorname{Im} \zeta_0 \leq \frac{1}{2\pi} \log \frac{1}{\alpha} + E$, if any of the conditions in Equations (62), (63), (68), and (67) does not hold, there is a uniform constant $D_5 \geq 1$ such that

$$\operatorname{Im} \zeta_0 \leq D_5.$$

Below we prove the position for such points ζ_0 (while assuming that $\alpha \in (0, r_2]$).

Recall the sector S_f defined in Section 2.3. Let us denote the connected component of $f^{-1}(S_f)$ which lies in $\mathcal{P}_f$ by S'_f. In other words, $S'_f = \Phi_f^{-1}(\Phi_f(S_f) - 1)$. The set S'_f might be contained in $\Omega_0^0(f)$, but this does not make any difference in the argument we present below.

There is a constant $D_7 > 0$, independent of α and f, such that $\Omega_0^0(f) \cup S'_f \subset B(0, D_7)$. To see this, first note that there is an integer $n_f \geq 0$, uniformly bounded from above, such that $L_f^{-1}(\{\zeta \in \mathbb{C} \mid n_f + 1/2 \leq \operatorname{Re} \zeta \leq x_f\})$ is contained in $\Theta_\alpha(C_2)$, where C_2 is the constant in Lemma 6.4. Then,

$$\bigcup_{i=k_f+n_f}^{k_f+\lfloor 1/\alpha \rfloor - \boldsymbol{k}-2} f^{\circ i}(S_f) \subseteq \Phi_f^{-1}(\{\zeta \in \mathbb{C} \mid n_f + 1/2 \leq \operatorname{Re} \zeta \leq x_f\}) \subset \tau_f(\Theta_\alpha(C_2)).$$

The diameter of $\tau_f(\Theta_\alpha(C_2))$ is uniformly bounded from above, independent of α and f. On the other hand, for $f \in \mathcal{QLS}_0$ the sets $f^{\circ i}(S_f)$, for $i \geq 0$, are defined and compactly contained in Dom f. Indeed, according to [**IS06**], for every $f \in \mathcal{QLS}_0$, S_f and all its forward and backward iterates are defined and contained in Dom f. Moreover, $f^{\circ i}(S_f)$, for $i \geq 0$, tend to 0 in the attracting direction, and $f^{\circ i}(S_f)$ within the repelling Fatou coordinate, for $i \leq 0$, tend to 0 in the repelling direction. For each $\alpha \in [0, r_2]$ and $f \in \mathcal{QLS}_\alpha$, the diameter of each $f^{\circ i}(S_f)$, for $0 \leq i \leq k_f + n_f$, (let $n_f = \infty$ when $\alpha = 0$), are finite. Similarly, the diameter of each S'_f is also finite. Therefore, by the pre-compactness of the class of maps $\cup_{\alpha \in [0, r_2]} \mathcal{QLS}_\alpha$ and the continuous dependence of the Fatou coordinates on the maps, the diameters of these sets are uniformly bounded from above, independent of α and f. This proves the existence of the uniform constant D_7.

By the pre-compactness of the class of maps $\cup_{\alpha\in[0,r_2]}\mathcal{QIS}_\alpha$, there is a constant $\delta > 0$ such that

$$B_\delta(\Omega_0^0(f) \cup S'_f) \subset \text{Dom } f.$$

See Equation (11) for further details.

Let $\mathbb{E}\text{xp}(\zeta_0) = z$. By the hypothesis of the proposition we have $z \in \Omega_0^0(f)$, and hence by the above paragraph, $|z| \le D_7$. Moreover, because $\text{Im}\,\zeta_0 \le D_5$ here, $|z| \ge 4e^{-2\pi D_5}/27$ is uniformly bounded from below.

Consider the smallest $r \ge 1$ such that $rz \in \partial(\Omega_0^0(f) \cup S'_f)$. That is, $r'z \in \Omega_0^0(f) \cup S'_f$ for all $r' \in [1, r]$. Let $z' = rz$.

Recall the set $\mathcal{C}_f^{-1}$ defined in Section 2.3. We are looking for a small ball near z' that is outside $\Omega_0^0(f)$ and is mapped outside $\Omega_0^0(f)$ by f. However, points outside but near $\partial\mathcal{C}_f^{-1}$ may be mapped into $\Omega_0^0(f)$, due to the branched covering $f : \mathcal{C}_f^{-1} \to \mathcal{C}_f$. Also, if the topological interior of the set $S'_f \setminus \Omega_0^0(f)$ is not empty, points within this set are mapped into $S_f \subset \Omega_0^0(f)$. But, these are the only ways in which this issue occurs. Indeed, it follows from the definition of the sets $f^{\circ i}(S_f)$, $0 \le i \le k_f + \lfloor 1/\alpha\rfloor - \boldsymbol{k} - 2$, and the way they are mapped to one another, that a given point $z_0 \in \partial\Omega_0^0(f)$ is mapped to a point on $\partial\Omega_0^0(f)$, unless either

(a) $z_0 \in \partial\mathcal{C}_f^{-1}$ and $\Phi_f(f(z_0)) \in (1/2 + \mathbf{i}[-2, 2]) \cup ([1/2, 3/2] + 2\mathbf{i}) \cup (3/2 + \mathbf{i}[-2, 2])$; or
(b) both z_0 and $f(z_0)$ belong to the boundary of S_f.

Moreover, case (b) only occurs if z_0 belongs to the common boundaries of S_f and S'_f. We may avoid case (b) by assuming that z' is on the boundary of $\Omega_0^0(f) \cup S'_f$. While due to the issue arising in case (a) we need to analyze two separate cases, which are presented below.

Let us first assume that $z' \notin \partial\mathcal{C}_f$. There is a round ball $B(z'', \delta') \subset B_\delta(\Omega_0^0(f)) \subset \text{Dom } f$ such that $|z' - z''| \le \delta/2$, $B(z'', \delta') \cap \Omega_0^0(f) = \emptyset$, and $f(B(z'', \delta')) \cap \Omega_0^0(f) = \emptyset$. Indeed, by the pre-compactness of the class $\mathcal{QIS}_\alpha$, and the continuous dependence of the sets $\mathcal{C}_f^{-i}$ on f, we may assume that δ' is uniformly bounded from below, independent of α and f.

Define the curve γ' as the union of the line segment connecting z to z' and the line segment connecting z' to z''. By definition, $\sup_{w,w'\in\gamma'} |\arg(w/w')| < \pi$. Moreover, γ' is contained in $B(0, D_7 + \delta) \setminus \{0\}$, and is uniformly away from 0.

Define the curve γ as the lift of γ' under $\mathbb{E}\text{xp}$ which starts at ζ_0. Since $|z''|$ is uniformly bounded from above, $|(\mathbb{E}\text{xp}^{-1})'(z'')|$ is uniformly bounded away from 0. Combining with a bounded distortion argument (Theorem 2.1), we conclude that the lift of $B(z'', \delta')$ under $\mathbb{E}\text{xp}$ contains a round ball about $\mathbb{E}\text{xp}^{-1}(z'')$ whose size is uniformly bounded away from 0, say by r^*. If necessary, we reduce the size of the ball to make it less than $1/4$. These readily imply Parts (1) and (2) of the proposition for some $\delta_1 > 0$ uniformly away from 0. For Part (3) of the proposition we note that

$$\text{diam Re}(B_{\delta_1}(B(\gamma(1), r^*) \cup \gamma[0, 1])) \le 2\delta_1 + \frac{\pi}{2\pi} + r^*,$$

where the term $\pi/(2\pi)$ comes form the bound on the spiral of γ' in the above paragraphs. Then, to make the above diameter less than $1 - \delta_1$, we need to choose $\delta_1 \le 3^{-1}(1/2 - r^*)$.

Finally since the diameter of γ is uniformly bounded from above, the modulus in Part (4) is uniformly bounded from above. So we may choose a uniform δ_2 in this case. This finishes the proof of the proposition in the case $z' \notin \partial\mathcal{C}_f$.

In the remaining case $z' \in \partial\mathcal{C}_f$ we only need to slightly modify the above argument. That is, by Proposition 2.5, there is a smooth curve $\gamma' \subset \Omega_0^0(f) \setminus B(0, 4e^{-2\pi D_5}/27)$ connecting z to $z' \in \partial\Omega_0^0(f) \setminus \mathcal{C}_f$ such that the length of γ' is uniformly bounded from above and $\sup_{w,w'\in\gamma'} |\arg(w/w')| \leq C < 2\pi$. Now by the above argument, there is a ball $B(z'', \delta') \subset B_\delta(\Omega_0^0(f))$, whose size is uniformly bounded from below, $B(z'', \delta') \cap \Omega_0^0(f) = \emptyset$, and $f(B(z'', \delta')) \cap \Omega_0^0(f) = \emptyset$.

Define the curve γ'' as the union of the curve γ' and the line segment connecting z' to z''. As in the above paragraphs, lifting $\gamma'' \cup B(z'', \delta')$ under $\mathbb{E}\mathrm{xp}$, we obtain the curve γ as well as a round ball whose size is uniformly bounded away from 0. Here, one must choose $r^* \leq (1 - C/(2\pi))/2$ and $\delta_1 \leq (1 - C/(2\pi) - r^*)/3$. □

6.8. Metric properties of the orbits in the renormalization tower. —

Proof of Proposition 4.2. — By the pre-compactness of the class of maps $\cup_{\alpha\in[0,r_2]} \mathcal{QIS}_\alpha$, and the continuous dependence of the normalized Fatou coordinate on the map, the diameter of $\mathcal{P}_{\mathcal{R}(f)}$ is uniformly bounded from above. In particular, the absolute value of $w \in \mathcal{P}_{\mathcal{R}(f)}$ is uniformly bounded from above.

Let $\eta_{\mathcal{R}(f)} : \mathcal{P}_{\mathcal{R}(f)} \to \Phi_f(\mathcal{P}_f)$ be an inverse branch of $\mathbb{E}\mathrm{xp}$ that satisfies $\mathrm{Re}(\eta_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}f})) \subset [0, \hat{\boldsymbol{k}} + 1]$ as in Equation (7). Let $\zeta := \eta_{\mathcal{R}(f)}(w)$. By the above paragraph, $\mathrm{Im}\,\zeta = \frac{-1}{2\pi} \log \frac{27|w|}{4}$ must be bounded from below by a uniform constant, say M'. Let M be the constant produced by Proposition 6.19 for M'.

To prove the proposition, we consider two cases; small values of α and large values of α. Recall the constant $\boldsymbol{k}$ and $\hat{\boldsymbol{k}}$ from Proposition 2.4. First we assume that α is small enough so that

$$\alpha \leq 1/(2\boldsymbol{k} + 2\hat{\boldsymbol{k}} + 2), \text{ and } 1/(4\alpha) \geq \hat{\boldsymbol{k}} + 2 + M \log(1 + 1/\alpha).$$

In this case we set $\kappa(f) := \lfloor 1/(2\alpha) \rfloor$. By the first condition on α above, we have

$$0 < \mathrm{Re}(\eta_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}f})) + \kappa(f) \leq \hat{\boldsymbol{k}} + 1 + 1/(2\alpha) \leq 1/\alpha - \boldsymbol{k}.$$

In particular, combining with Proposition 2.4, we conclude that $\eta_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) + \kappa(f) \subset \Phi_f(\mathcal{P}_f)$. This implies that

$$f^{\circ\kappa(f)} \circ \psi_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) = f^{\circ\kappa(f)} \circ \Phi_f^{-1} \circ \eta_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) = \Phi_f^{-1}\big(\eta_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) + \kappa(f)\big) \subset \mathcal{P}_f.$$

This proves part (1) of the proposition in this case.

On the other hand, one can see that by the above conditions on α, $\zeta + \kappa(f)$ belongs to the set $[0, 1/\alpha - \boldsymbol{k}] + \mathbf{i}[M', +\infty)$. Then, by Proposition 6.19, we have

$$|L_f^{-1}(\zeta + \kappa(f)) - (\zeta + \kappa(f))| \leq M \log(1 + 1/\alpha).$$

Hence,

$$\operatorname{Im} L_f^{-1}(\zeta+\kappa(f)) \geq \operatorname{Im}\zeta - M\log(1+1/\alpha), \tag{71}$$

$$|\operatorname{Re} L_f^{-1}(\zeta+\kappa(f)) - 1/(2\alpha)| \leq \hat{\boldsymbol{k}} + 2 + M\log(1+1/\alpha) \leq 1/(4\alpha). \tag{72}$$

Using Lemma 6.3-2 (with $r = 1/4$) and the above inequalities, there is a constant M_2 depending only on C_1 and M, such that

$$\begin{aligned}
|f^{\circ\kappa(f)}(\psi_{\mathcal{R}(f)}(w))| &= |\Phi_f^{-1}(\eta_{\mathcal{R}f}(w)+\kappa(f))| \\
&= |\tau_f \circ L_f^{-1}(\eta_{\mathcal{R}f}(w)+\kappa(f))| \\
&= |\tau_f(L_f^{-1}(\zeta+\kappa(f)))| \\
&\leq 4C_1 e^{2\pi}\alpha e^{-2\pi\alpha \operatorname{Im} L_f^{-1}(\zeta+\kappa(f))} \\
&\leq 4C_1 e^{2\pi}\alpha e^{-2\pi\alpha(\operatorname{Im}\zeta - M\log(1+1/\alpha))} \leq M_2\cdot\alpha|w|^\alpha.
\end{aligned}$$

This proves Part (2) of the proposition for small values of α.

Now we consider larger values of α that do not satisfy the above conditions. Here we set $\kappa(f) = 0$. Then, $f^{\circ\kappa(f)} \circ \psi_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) = \psi_{\mathcal{R}(f)}(\mathcal{P}_{\mathcal{R}(f)}) \subset \mathcal{P}_f$, by Equation (8). This proves Part (1) of the proposition in this case.

Since $\zeta \in [0, 1/\alpha - \boldsymbol{k}] + \mathbf{i}[M', +\infty)$, as in the above argument, we must have $\operatorname{Im} L_f^{-1}(\zeta) \geq \operatorname{Im}\zeta - M\log(1+1/\alpha)$. (Here we do not need the bound in (72).) Then, by an elementary estimate on τ_h, there is a uniform constant M_2 such that $|\tau_f(L_f^{-1}(\zeta))| \leq M_2\cdot|w|^\alpha$. That is, $|\psi_{\mathcal{R}(f)}(w)| \leq M_2|w|^\alpha$. However, since α is bounded from below here, one may adjust M_2 to accommodate the parameter α in the formula. This finishes the proof of Part (2) in this case. □

Proof of Proposition 5.4. — By Proposition 6.19, with $M' = 0$, we find a constant M (independent of n) such that for all $\zeta \in [0, x_h] + \mathbf{i}[0, +\infty)$ we have

$$\operatorname{Im} L_{n+1}^{-1}(\zeta) \geq \operatorname{Im}\zeta - M\log(1+1/\alpha_{n+1}).$$

Choose $D_1 > 0$ such that for all $\alpha \in (0,1)$, we have

$$\frac{D_1}{\alpha} - M\log(1+\frac{1}{\alpha}) \geq \frac{1}{4\alpha}.$$

If $\operatorname{Im}\zeta_{n+1} \geq D_1/\alpha_{n+1}$, the above equations guarantee that

$$\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}) \geq \frac{1}{4\alpha_{n+1}}.$$

This implies that $L_{n+1}^{-1}(\zeta_{n+1}) \in \Theta_{\alpha_{n+1}}(\frac{1}{4\alpha_{n+1}})$. By Lemma 6.3-2, with $r = 1/4$,

$$\begin{aligned}
|\tau_{n+1}(L_{n+1}^{-1}(\zeta_{n+1}))| &\leq 4C_1 e^{2\pi}\alpha_{n+1} e^{-2\pi\alpha_{n+1}(\operatorname{Im}\zeta_{n+1} - M\log(1+1/\alpha_{n+1}))} \\
&\leq C\alpha_{n+1} e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}},
\end{aligned}$$

for some constant C that depends only on C_1 and M. Here we have used that $\alpha_{n+1}\log(1 + 1/\alpha_{n+1})$ is uniformly bounded from above independent of $\alpha_{n+1} \in (0,1)$. Recall that $\Phi_{n+1}(w_{n+1}) = \zeta_{n+1}$, and $\Phi_{n+1}^{-1} = \tau_{n+1} \circ L_{n+1}^{-1}$. Hence, we have shown that $|w_{n+1}| \leq C\alpha_{n+1} e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}}$.

By Proposition 2.7 and Equation (15), w_{n+1} is mapped to z_{n+1} in a uniformly bounded number of iterates of f_{n+1}. The map f_{n+1} is of the form

$$z \mapsto P \circ \phi_{n+1}^{-1}(e^{2\pi \mathbf{i}\alpha_{n+1}} \cdot z) : (e^{-2\pi \mathbf{i}\alpha_{n+1}} \cdot \phi_{n+1}(U)) \to \mathbb{C},$$

with $|\phi'_{n+1}(0)| = 1$ and ϕ_{n+1} has univalent extension over the larger domain V (see Theorem 2.6). This implies that, there exists a uniform constant C' such that $|z_{n+1}| \leq C'|w_{n+1}|$.

Recall that $\mathbb{E}\mathrm{xp}(\zeta_n) = z_{n+1}$. Combining the above two paragraphs, we have

$$\frac{4}{27}e^{-2\pi \operatorname{Im}\zeta_n} = |\frac{-4}{27}e^{-2\pi \mathbf{i}\overline{\zeta_n}}| = |z_{n+1}| \leq C'C\alpha_{n+1}e^{-2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1}}.$$

Multiplying the above equation by $27/4$, and then taking log, we obtain

$$2\pi\alpha_{n+1}\operatorname{Im}\zeta_{n+1} \leq \log(\frac{27CC'}{4}) + \log\alpha_{n+1} + 2\pi\operatorname{Im}\zeta_n.$$

Then dividing through by $2\pi\alpha_{n+1}$ we obtain,

$$\operatorname{Im}\zeta_{n+1} \leq \frac{1}{2\pi\alpha_{n+1}}\log(\frac{27CC'}{4}) + \frac{1}{2\pi\alpha_{n+1}}\log\alpha_{n+1} + \frac{1}{\alpha_{n+1}}\operatorname{Im}\zeta_n.$$

This is the desired inequality in the proposition when we define the constant $D_2 = \frac{1}{2\pi}\log(\frac{27CC'}{4})$. □

Proof of Proposition 5.11. — First we prove that for every $D > 0$ there exists $E > 0$ such that if $\operatorname{Im}\zeta_{n+1} \leq D/\alpha_{n+1}$ then $\operatorname{Im}\zeta_n \leq \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}} + E$.

The map f_{n+2} has the form

$$z \mapsto P \circ \phi_{n+2}^{-1}(e^{2\pi\alpha_{n+2}\mathbf{i}} \cdot z) : (e^{-2\pi\alpha_{n+2}\mathbf{i}} \cdot \phi_{n+2}(U)) \to \mathbb{C},$$

with $|\phi'_{n+2}(0)| = 1$. Recall that by Theorem 2.6 the map ϕ_{n+2} has univalent extension onto the larger domain V. By the distortion theorem 2.1, this implies that $\mathrm{Dom}\, f_{n+2} = \phi_{n+2}(U) \cdot e^{-2\pi\alpha_{n+2}\mathbf{i}}$ has a uniformly bounded diameter in $\mathbb{C}$. Then, as $\mathbb{E}\mathrm{xp}(\zeta_{n+1}) \in \mathrm{Dom}\, f_{n+2}$, $\operatorname{Im}\zeta_{n+1}$ must be uniformly bounded from below by a constant M' independent of n. Using Proposition 6.19, with M', we obtain a constant M, independent of n, such that $\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}) \leq \operatorname{Im}\zeta_{n+1} + M\log(1 + 1/\alpha_{n+1})$. For points ζ_{n+1} with $\operatorname{Im}\zeta_{n+1} \leq D/\alpha_{n+1}$, we obtain

$$\operatorname{Im} L_{n+1}^{-1}(\zeta_{n+1}) \leq D/\alpha_{n+1} + M\log(1 + 1/\alpha_{n+1}).$$

By an explicit estimate on the covering map $\tau_{f_{n+1}}$ given by the formula in Equation (33), there is a constant C independent of n such that

$$|w_{n+1}| = |\tau_{f_{n+1}}(L_{n+1}^{-1}(\zeta_{n+1}))| \geq C\alpha_{n+1}.$$

The point w_{n+1} is mapped to z_{n+1} by a uniformly bounded number of iterates of f_{n+1}. Moreover, if z_{n+1} is close to 0, then w_{n+1} must be also close to 0. That is due to the covering structure of P on U, which covers a neighborhood of 0 only once. These imply that there is a constant C' independent of n such that $|z_{n+1}| \geq C'\alpha_{n+1}$. Then, as ζ_n is mapped to z_{n+1} by $\mathbb{E}\mathrm{xp}$, we obtain

$$\operatorname{Im}\zeta_n \leq \frac{-1}{2\pi}\log(\frac{27C'}{4}) + \frac{1}{2\pi}\log\frac{1}{\alpha_{n+1}}.$$

This finishes the proof of the claim by introducing $E = \frac{-1}{2\pi}\log(\frac{27C'}{4})$.

It is proved in Lemma 3.11 in [**Che13**] that there exists a constant $D > 0$ such that given any $z \in \cap_{n=0}^{\infty} \Omega_0^n \setminus \overline{\Delta(f)}$ there are infinitely many integers m with $\operatorname{Im} \zeta_m \leq D/\alpha_m$. (Indeed, the statement of the lemma in that paper concerns $z \in \mathcal{PC}(f) \setminus \overline{\Delta(f)}$, however, the proof is written for $\cap_{n=0}^{\infty} \Omega_0^n \setminus \overline{\Delta(f)}$.) Combining this with the statement in the first paragraph, we conclude that there are infinitely many levels m with $\operatorname{Im} \zeta_{m-1} \leq \frac{1}{2\pi} \log \frac{1}{\alpha_m} + E$. □

Proof of Proposition 5.12. — Fix $\zeta \in \mathbb{E}\mathrm{xp}^{-1}(\Omega_{n+1}^0)$ satisfying $\operatorname{Im} \zeta \leq \frac{1}{2\pi} \log \alpha_{n+1}^{-1} + E$. Since Ω_{n+1}^0 has a uniformly bounded diameter, independent of n, there is a uniform constant C_1 such that $\operatorname{Im} \zeta \geq C_1$. See, for instance, the proof of Proposition 5.11 for further details.

First assume that α_{n+1} is small enough so that $2 \leq 1/(2\alpha_{n+1}) \leq 1/\alpha_{n+1} - \boldsymbol{k}$. For an inverse branch of $\mathbb{E}\mathrm{xp}$, denoted by $\hat{\eta}_{n+1}$, we may consider the continuous curve $\Upsilon_{n+1}(t) := \hat{\eta}_{n+1} \circ \Phi_{n+1}^{-1}(t)$, for $1 \leq t \leq \frac{1}{2\alpha_{n+1}}$. By the same argument as in the proof of Sublemma 6.22, there exists a uniform constant C_2 such that for every choice of $\hat{\eta}_{n+1}$, we have

$$\frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}} - C_2 \leq \operatorname{Im} \Upsilon_{n+1}(\frac{1}{2\alpha_{n+1}}).$$

Moreover,

$$\operatorname{Im} \Upsilon_{n+1}(1) = \operatorname{Im} \hat{\eta}_{n+1} \circ \Phi_{n+1}^{-1}(1) = \operatorname{Im} \hat{\eta}_{n+1}(-4/27) = 0.$$

By Proposition 6.15, $\alpha_{n+1}^{-1} - \boldsymbol{k} \leq y_h \leq x_h$. Thus, by our assumption in the above paragraph, $1/(2\alpha_{n+1}) \leq x_h$. Then, Proposition 6.18 guarantees that for $1 \leq t \leq \frac{1}{2\alpha_{n+1}}$, $|\Upsilon'_{n+1}(t)| \leq C_4$. Hence, for every integer $i \in [2, \frac{1}{2\alpha_{n+1}}]$, the Euclidean distance $d(\Upsilon_{n+1}(i-1), \Upsilon_{n+1}(i))$ is at most C_4. It follows that there exists a choice of the inverse branch $\hat{\eta}_{n+1}$ and an integer $i \in [1, \frac{1}{2\alpha_{n+1}}]$ such that $\zeta' := \Upsilon_{n+1}(i)$ satisfies the desired inequalities in the proposition.

Now let us assume that α_{n+1} is bounded from below, that is, $1/(2\alpha_{n+1}) > 1/\alpha_{n+1} - \boldsymbol{k}$. (We are still assuming that $\alpha_{n+1} \leq r_2$, so there may not be any such α_{n+1}.) Then, by the hypothesis of the proposition, $\operatorname{Im} \zeta$ is uniformly bounded from above. As in the second paragraph, it is also uniformly bounded from below. Then, there is an element ζ' in the set $\mathbb{E}\mathrm{xp}^{-1} \circ \Phi_{n+1}^{-1}(1) = \mathbb{Z}$ that satisfies the desired inequalities in the proposition. □

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