

**LOCAL BOUNDARY INTEGRAL EQUATION ANALYSIS
OF
STRESS AND POTENTIAL PROBLEMS**

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ABSTRACT

In this thesis, is presented a novel development of the current boundary integral equation method of two dimensional potential and elasticity theory governed by harmonic and biharmonic equations, respectively. This new technique is based on either Taylor or Fourier series expansions of the direct 'Boundary Integral Equations (BIEs)'. Zero-order integral equations (conventional BIEs) are repeatedly differentiated up to some order (hence, their names are higher-order BIEs) with respect to the position of a suitably located point within the solution domain. The main objective of this work is to devise a solution method which provides detailed information in a local area of particular interest (such as where stress concentrations and cracks occur) without the need to evaluate the solution over the rest of the domain. Consequently, the method is very economic in the use of computing resources, especially if three-dimensional problems are amenable to this new technique. The method is particularly useful in providing an efficient way of refining already existing approximate solutions obtained by other means.

Throughout, the solution domain is assumed to be isotropic and homogeneous and free of body forces (such as weight, charge, etc.). In the case of elastostatic problems, applied forces are such that they introduce small strains (i.e., they do not invalidate the small strain assumption of classical elastic theory) under the condition of either the plane strain or plane stress. Integrals are accurately evaluated by using Gaussian quadrature formulae. Elements with either quadratic or cubic variation of geometry are employed to interpolate Cartesian-global coordinates of those elements at the Gauss points. A computer program called 'LBIE2PE' has been developed capable of treating both harmonic and biharmonic problems. Numerical results from some test cases are provided to show the applicability of 'Local Boundary Integral Equation (LBIE)' technique to potential and stress problems.

Wherever necessary, classical theories are concisely explained and, for compactness, tensor notation is used with implied summation on repeated indices, and partial derivatives being denoted by comma-index notation.

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C H A P T E R 1

1. INTRODUCTION

A central position in mathematical physics is occupied by a class of problems known as 'Boundary Value Problems (BVP)', which are the mathematical models of many important practical problems in science and engineering. A differential equation (generally partial differential equation, if there are more than one independent variable due to multidimensionality, friction, air resistance, etc.) models the physics of the problem in the domain, R . To find a solution to this given differential equation is not so difficult (in fact, there can be found an infinity of solutions to the equations), but the major difficulty is to find that solution which fits the prescribed boundary values⁽²⁷⁾ (certain constraints or conditions on the boundary, S). Analytical solutions to BVPs may be possible to derive if the region, R , has a homogeneous structure, a simple geometry and relatively straightforward boundary conditions (i.e., not difficult to describe by simple mathematical functions), and if the governing differential equation is linear⁽⁹⁾. Otherwise, an approximate solution must be found. Usually, most of the 'real-world' problems cannot be solved by analytical methods, in which exact mathematical expressions of the functions are obtained in terms of independent variables. Approximate solutions have only become feasible with the advent of high-speed digital computers, and subsequently numerical analysis methods have been developed to be more accurate and versatile. Here, it is impossible to give a systematic development of the entire subject within the scope of this thesis, we shall therefore confine ourselves to the discussion of a newly-developed local BIE numerical analysis method for solving the BVPs of two-dimensional potential and elasticity theory. Nevertheless, a short background is given below for comparison purposes.

Numerical methods for solving BVPs can be classified by two major groups;

- i) domain type,
- ii) boundary type.

In the former, approximations are made throughout the region, R , whereas the latter implies that the approximations are made only on the boundary, S . Thus, it is evident that the second class, namely, boundary integral techniques, have the important advantage over the domain type numerical methods, such as finite element or finite difference ones, that the discretisation of the domain, except the boundary, is not necessary for the solution, which means the reduction of the dimensionality of the problem by one. The mesh design and data preparation become much easier processes, in contrast to domain type methods where the mesh design and data preparation can be very tedious and expensive. The user of the program does not have to take into consideration how well the mesh approximates the governing differential equation since there is no need for the equation to be approximated in the interior because each of the fundamental (singular) solutions satisfies it exactly in R , neither does he have to consider mesh spacing inside the domain. He need only know if the mesh is detailed enough along the boundary. Therefore, the user convenience is outstanding. Again, since no approximation is made in R , but only on the boundary, the accuracy of the results is potentially enhanced. Also, an outstanding computational advantage of BIE methods, as frequently claimed, is that they lead to much smaller numbers of simultaneous linear equations than those in domain methods for the same problem with the same accuracy⁽¹⁵⁾. Therefore, the computer storage requirements and program execution times, and hence the cost involved, are reduced. However, the coefficient matrix associated with the set of linear algebraic equations formed from the integral equations is not sparse and symmetric due to the fact that each singularity (fundamental solution) plays a role in every equation⁽⁹⁾. Generally, to overcome this, a banded matrix is obtained by dividing the problem domain under consideration into subregions, for each of which an integral equation is established⁽⁴²⁾. Although this may be an additional benefit, that different material properties can be allowed at each subregion, another problem arises that some integration over the subregions inside the domain must now be carried out. This means the loss of one of the advantages of the BIE methods. This is marked especially in that when the BIE methods are applied to non-linear problems in which the differential equations are no longer fully transformable to the boundary,

consequently wherever there is non-linear behaviour, then there is a need for internal integration over these regions⁽¹⁵⁾. However, if the whole body is homogeneous, or each distinct zone in the body is homogeneous, and if the body forces are to be included, or the governing differential equation is quasi-linear as in elasto-plasticity, then the internal subdivisions of the body do not cause any increase in the order of the final set of algebraic equations, since the BIEs are augmented by a volume integral involving interior subdivisions⁽²⁾. Also the cases where information is wanted at many internal points and large surface-to-volume ratios detract from the attractiveness of the BIE methods (e.g., in the case of shells, narrow thin strips, etc.⁽²⁾). Additionally, the advantage of BIE methods is less clear in plasticity. A boundary integral approach can be formulated due to the property of plasticity as a linear incremental theory. However, the need for the evaluation of a set of integral equations, numerically, for each increment makes it difficult in practice⁽⁴⁵⁾.

It should be noted there are many BIE formulations some having advantages over others for particular problems. However, they can be divided into three main categories as follows^(2,7,9).

The first alternative is the so-called indirect method which implies that the functions used to find the unknown boundary values have no direct physical meaning with respect to the boundary value problem. The integral equations are expressed in terms of those fictitious density functions which satisfy (or more correctly, are adjusted to satisfy) prescribed boundary conditions. Once they have been obtained from a numerical solution of the integral equations the values of the solution unknowns can be calculated from them by numerical integrations. This approach was developed for elasticity by Kupradze (1963) and has recently been applied to potential and elasticity problems by Banerjee, Butterfield, Hess, Jaswon, Massonnet, Oliviera, Symm, Tomplin, Watson, Rieder, Heise, Antes, Christiansen, Friemann and others⁽²⁾.

The second alternative is the semi-direct method in which the integral equations are formulated in terms of unknown functions which are related to stress functions in

elasticity or stream functions in potential theory. The solution of these terms is obtained from the integral equations. The required values, e.g., stresses, can then be determined by simple differentiation of the solution. This approach has been developed by Henry, Jaswon, Ponter, Rim and Symm⁽²⁾.

Finally, the third approach of the BIE methods is the direct formulation where the unknown functions in the integral equations are the actual physical variables, such as stresses and displacements. Hence, the solution of the integral equations gives the required values on the boundary directly, and those anywhere within the domain can be found from the boundary values by numerical integration. From the engineering point of view, Rizzo was the first one who applied the direct formulation to elasticity problems. Although it can be found in Kupradze's method of functional equations, it differs from Rizzo's approach by its solution procedure. This approach has been carried out by Cruse, Lachat, Shaw, Watson, and others⁽²⁾.

Although the BIE methods compared to other domain type numerical ones, such as finite element and finite difference, are superior for a great variety of practical problems in terms of labour in data preparation and computer execution time, it can still be expensive in some cases, especially for three-dimensional problems (if, e.g., a fine mesh is required using many elements). Furthermore, if the information is required only over a small part of the domain (such as, in a region where stress concentration occurs), then over the whole boundary of the domain evaluation of the solution will not be needed. This therefore opens up the possibility of minimizing the use of computer resources, and of getting more accurate results if a technique is developed such that attention can be focused on a particular area of interest in the problem domain. This is the objective of the work presented in this thesis.

In the following chapters a further development of the current BIE method, namely, the Local Boundary Integral Equation (LBIE) method is presented for two-dimensional potential and elasticity theory governed by harmonic and biharmonic equations, respectively. This new technique is essentially based on the work 'A Local

Boundary Integral Equation Method For Potential Problems' by Fenner & Watson (1988).

In Chapter 2, the mathematical basis of the boundary integral equation method of two-dimensional potential theory is briefly outlined and the integral equation for an internal point, say p , is derived. This integral equation is then differentiated repeatedly to obtain higher-order boundary integral equations which are used as a basis for the LBIE formulation. This is followed by the differentiation of kernel functions (the derivative forms of the fundamental (singular) solution to Laplace's equation) up to some order. These derivatives are the parts of the integrands of higher-order BIEs. A concise review of the two series, namely, Taylor and Fourier ones is also provided, since these series are the trial functions used to approximate the unknown boundary values, in the place of shape functions of conventional BIEs which are used to interpolate those unknown boundary values. In section 2.2 of the same chapter a full description of the numerical treatment of the higher-order BIEs is given. To assess the accuracy of the method presented here the results of some test problems are provided. It was useful to consider problems for which the exact solutions are known, to be able to compare the solutions with those exact ones. The potential functions considered involve polynomial, hyperbolic, and logarithmic variations within and over the boundary, S . The solution domains are such that their boundaries consist of straight-line segments and a mixture of curved and straight-line segments in order to demonstrate the ability of the method to deal with awkward regions.

In Chapter 3, the application of local boundary integral equation method to stress problems is studied. First, the basic equations of elasticity are reviewed, and the necessary tools for formulating the boundary integral equations, namely, reciprocal relationship (Betti's theorem) and fundamental solutions are given. After the derivation of the integral equation for an internal point, p , (Somigliana's identity) higher-order boundary integral equations of two-dimensional elastostatics are obtained by simple differentiation of it at the point p . The derivatives of the kernel functions in the integrands of the higher-order BIEs are given in appendix E since they are rather

lengthy to be inserted in the main text. In the numerical implementation of the LBIE method for stress problems, three possible approaches (formulations) are discussed. These are as follows;

- i) Airy's stress function,
- ii) Neuber-Papkovich representation,
- iii) Direct expansion of higher-order BIEs,

each having some possible advantages over the others.

In order to illustrate the applicability of the technique to stress problems, four test cases are considered. These are, namely, a square flat plate under simple tension, a simple cantilever beam with a shear force at the free end, a square flat plate subject to a uniform tensile stress with a small circular hole at its centre, and a cracked beam of rectangular cross-section simply supported at its ends. The latter two problems provide a severe test to the method since the stresses vary rapidly.

1.1. Historical Development

The modern BIE methods have their origins in classical potential theory. Until the development of modern digital computers they were considered to be different types of analytical methods and worked on by the famous 19th century mathematicians Green, Betti (1872), Somigliana (1885), and in the early 20th century by Fredholm (1905), Lauricella (1909), Kellog (1929), and more recently Courant and Hilbert (1953), Muskhelishvili (1954), Mikhlin (1957), Volterra (1959), Simirnov (1964), and Kupradze (1965)⁽³⁴⁾. However, with the advent of fast electronic digital computers solving the equations numerically has become possible. The first numerical treatment of potential theory was formulated by Jaswon (1963) and Symm (1963). Rizzo (1967) and Cruse (1969) applied the method to elastostatics. In their formulation, the boundary of the problem was represented by straight-line segments and the unknowns were assumed to be constant over each of these elements. Riccardella (1973) and Cruse (1974) improved on the earlier formulations by allowing the unknown functions to vary linearly over each boundary element. Lachat and Watson (1975) produced another

significant improvement by introducing the idea of using general curved boundary elements taken from the finite elements of Zienkiewicz and Ergatoudis. In their representation, curved boundary elements were allowed to vary quadratically, and unknown functions (displacements or tractions) to vary either linearly, quadratically, or cubically. A further development was made by Watson (1982) who suggested employing boundary elements with Hermitian cubic shape functions instead of those with more conventional polynomial forms.

To the above-mentioned references we can add many more names who published papers relevant to their more specialized topics.

C H A P T E R 2

2. TWO-DIMENSIONAL POTENTIAL THEORY

Integral equations provide a powerful means of attacking the BVPs of potential theory (and elasticity). Solving the integral equations analytically, for practical problems is generally speaking almost impossible. Some Russian mathematicians especially Muskhelishvili and his school gave some analytical solutions by utilizing complex functions. But not until 1960s was much attempt made to implement BIEs numerically. The reasons for this can be given as follows⁽²⁰⁾. Singular or weakly singular kernels involved in most integral equations were hampering the numerical procedures. Existence and uniqueness of a solution to any given integral equation was another difficulty of an analytical nature. Also, the effective numerical solution of an integral equation was not feasible before the advent of high-capacity and high-speed digital electronic computers.

In 1963, Jaswon and Symm, and Jaswon & Ponter presented the first numerical results for two-dimensional potential problems. They attacked the problems by representing the boundary by straight-line elements. Hess and Smith in 1967 calculated the potential-flow around arbitrary bodies, numerically. The report by Cruse (1977) was a good introduction to the topic. But his approach was essentially the same as Jaswon and Symm's. After the introduction of curved boundary elements by Lachat (1975) and Lachat and Watson in elasticity, potential theory was also formulated in terms of this higher-order parametric representation of geometry and unknowns. These can be followed in the books by Brebbia (1978), Brebbia & Walker (1980), and Banerjee and Butterfield (1981).

Here, we confine ourselves to two-dimensional potential problems. It is, however, clear that the method presented here can be extended to three-dimensional potential problems without any difficulty.

2.1. Formulation

Laplace's equation, which is harmonic, is fundamental for potential theory, and can be defined by $\nabla^2\phi=0$. It can be interpreted as a measure of the deviation of the value of ϕ at a given point from its average or equilibrium value⁽²⁷⁾. This interpretation is especially helpful in understanding the nature of the heat conduction equation. Since Laplace's equation is fundamental for potential theory, ϕ then represents the potential produced by gravitational masses or electrostatic charges or steady-state heat distributions.

In order to derive the formulation of an integral equation for such a problem, two properties are required;

- i) fundamental (singular) solution to the given differential equation,
- ii) an appropriate reciprocal relationship between two functions.

Without considering in detail the mathematical requirements the fundamental solution for Laplace's equation in two dimensions may be stated as (Fenner, 1986)

$$\psi(p,q) = \ln\left[\frac{1}{r(p,q)}\right], \quad r(p,q) \neq 0 \quad (2.1)$$

where $r(p,q)$ is the distance between two points p and q within the region of interest. The point ' q ' will be replaced with ' Q ' if it is on the boundary. Since the Laplace's equation we are dealing with is a second-order elliptic partial differential equation, then the functions to be used in the reciprocal relationship are continuous and possess continuous first partial derivatives. For such functions the divergence theorem

$$\int_R \text{div} \mathbf{F} \, dR = \int_S \mathbf{F} \cdot \mathbf{n} \, dS \quad (2.2)$$

can be applied to derive the following Green's second theorem

$$\int_R (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dR = \int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS \quad (2.3)$$

In equation (2.2) above $\mathbf{F}$ is a vector field and $\mathbf{n}$ is the direction of the local outward normal to the surface S , of the region R . In equation (2.3), ψ is the fundamental solution, and ϕ is an unknown function satisfying Laplace's equation everywhere. Here, the region R is an area and its boundary S a line.

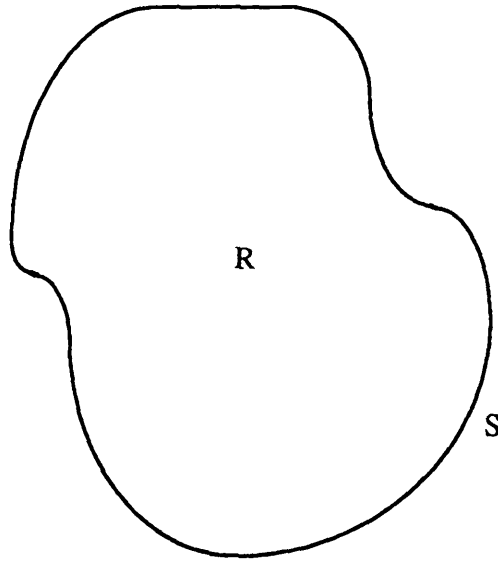


Figure 2.1: A two-dimensional solution domain, R , with the boundary, S .

2.1.1. The Integral Equation for an Internal Point p

Now, suppose that a point denoted as p serves as the origin for $r(p,q)$ in equation (2.1) in the region, R . $\psi(p,q)$, then, satisfies $\nabla^2 \psi = 0$ everywhere except at the origin, p . If a small circle of radius ϵ , surrounding p , is extracted from the region, R , Green's theorem may be applied to the region ' $R - R_\epsilon$ ' bounded by the circle S_ϵ and curve S by integrating along the boundaries in the directions as shown in Fig. 2.2.

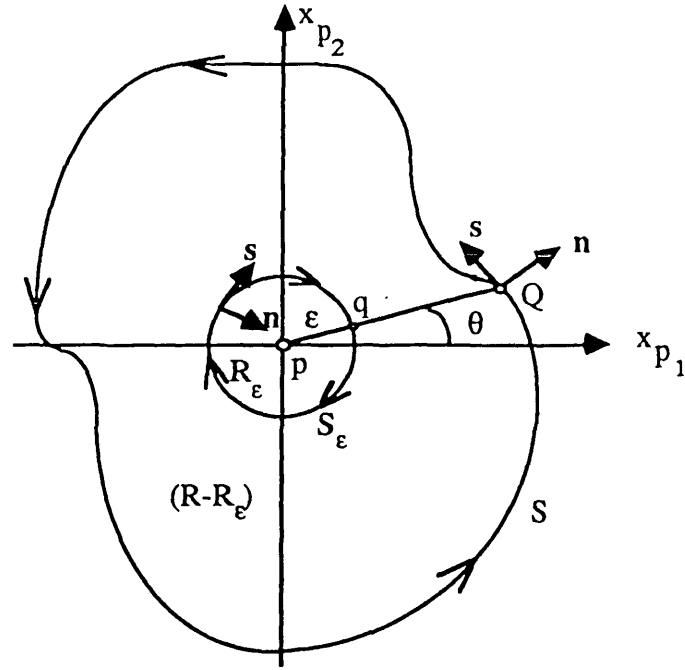


Figure 2.2: The solution domain with a small circular region of radius ϵ surrounding the point p with the circular boundary S_ϵ .

In Fig. 2.2 above, R_ϵ is the small circular region extracted from the solution domain, R .

Equation (2.3) can then be rewritten as

$$\int_{R-R_\epsilon} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dR = \int_{S+S_\epsilon} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS \quad (2.4)$$

Now, within $R-R_\epsilon$ $\nabla^2 \psi = \nabla^2 \phi = 0$ everywhere, therefore equation (2.4) reduces to

$$0 = \int_{S+S_\epsilon} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS \quad (2.5)$$

Note that on the surface S_ϵ

$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial \epsilon}, \quad \text{then} \quad \frac{\partial \psi}{\partial n} = -\frac{\partial \psi}{\partial \epsilon} = +\frac{1}{\epsilon} \quad (2.6)$$

also

$$dS = \epsilon d\theta \quad (2.7)$$

therefore, the first integral of equation (2.5) on the boundary S_ϵ in the limit as $q \rightarrow p$ (or $\epsilon \rightarrow 0$) becomes

$$\lim_{q \rightarrow p} \int_{S_\epsilon} \phi(p) \frac{1}{\epsilon} \epsilon d\theta = 2\pi\phi(p) \quad (2.8)$$

in the same manner, the second integral on S_ϵ as $q \rightarrow p$ is

$$\lim_{q \rightarrow p} \int_{S_\epsilon} \left(\frac{\partial \phi}{\partial n}\right)_p (-\ln \epsilon) \epsilon d\theta = 0 \quad (2.9)$$

Therefore, as the radius ϵ tends to zero, equation (2.5) yields

$$\int_S \left(\phi(Q) \frac{\partial \psi}{\partial n}(p, Q) - \psi(p, Q) \frac{\partial \phi}{\partial n}(Q) \right) dS(Q) + 2\pi\phi(p) = 0 \quad (2.10)$$

where p is an arbitrary point within the solution domain, R , and Q is the variable point on its boundary, S . This is an integral equation for an interior point p , which can be used to determine the value of the unknown function ϕ at p , provided the boundary values of ϕ and its normal derivative are known.

In a conventional BIE treatment equation (2.10) is developed into a form which involves integrations over only the boundary of the region of interest and allows these

boundary values to be determined. This boundary integral equation is given in equation (2.11) below, without considering mathematical details (Fenner, 1986),

$$\pi\phi(P) + \int_S \phi(Q) \frac{\partial\psi}{\partial n}(P, Q) dS(Q) = \int_S \psi(P, Q) \left(\frac{\partial\phi}{\partial n}\right)_Q dS(Q) \quad (2.11)$$

where P is now a boundary point.

On the other hand, in a local BIE method equation (2.10) is the one which is to be treated numerically in the present form to obtain the unknown boundary values. Since in a properly posed 'BVP' half of the boundary values are known, the other half can be determined from a set of linear equations by the use of polynomial shape functions of the type used in finite element techniques, in conventional BIE methods. However, in the local BIE technique the variations of the unknowns on the boundary are expressed by the series such as Taylor or Fourier ones. The variation of the surface geometry is defined by the same shape functions of intrinsic coordinates as in conventional BIE methods.

Note that in equation (2.10) the constant term 2π (or denoted as $C(p)$) is always valid even though we derived it from a circle of radius ϵ . In fact, it can also be found indirectly by applying a uniform potential over the entire boundary, then the normal derivative of potential must be zero. The same procedure applies to equation (2.11) for calculating the value ' π ', which can be attained only on a smooth part of the boundary. Hence, equation (2.10) and (2.11) both become

$$C(p) = - \int_S \frac{\partial\psi}{\partial n}(p, Q) dS(Q) \quad (2.12)$$

where p is replaced with P if it is the BIE (2.11).

2.1.2. Higher-Order BIEs

Let us recall the zero-order boundary integral equation (2.10)

$$\int_S \left(\phi(Q) \frac{\partial \psi}{\partial n}(p, Q) - \psi(p, Q) \frac{\partial \phi}{\partial n}(Q) \right) dS(Q) + 2\pi\phi(p) = 0 \quad (2.10)$$

This equation is expanded in series (Taylor and general four-term Fourier ones), which satisfy or are adjusted to satisfy the harmonic equation. Doing this, satisfactorily-accurate values of functions and their derivatives in a local area of interest can be obtained. Thus, zero-order integral equation (2.10) is differentiated repeatedly up to some order. Hence, they are called higher-order integral equations. Provided the point p is within the solution domain and not on the boundary, higher-order BIEs can be obtained by differentiating equation (2.10) with respect to directions x_{p1} and x_{p2} at p . Otherwise, that is if p is a point P on a part of the boundary which is not smooth, higher-order integrals do not converge to finite limits (Watson, 1982).

After differentiating equation (2.10), we have

$$2\pi\phi_{kl}(p) = - \int_S \phi(Q) A_{kl}(p, Q) dS(Q) + \int_S \left(\frac{\partial \phi}{\partial n} \right)_Q B_{kl}(p, Q) dS(Q) \quad (2.13)$$

where

$$\phi_{kl}(p) = \frac{\partial^{(k+l)} \phi(p)}{\partial x_{p1}^k \partial x_{p2}^l}, \quad (2.14)$$

and

$$B_{kl}(p,Q) = \frac{\partial^{(k+l)}}{\partial x_{p_1}^k \partial x_{p_2}^l} [\psi(p,Q)], \quad A_{kl}(p,Q) = \frac{\partial^{(k+l)}}{\partial x_{p_1}^k \partial x_{p_2}^l} \left[\frac{\partial \psi(p,Q)}{\partial n(Q)} \right] \quad (2.15)$$

Note that during the differentiation both $\phi(Q)$ and $(\partial\phi/\partial n)Q$ behave as constants, since they are independent of the position of p . The integrals in (2.13) always converge to finite limits as $\epsilon \rightarrow 0$ (see Fig. 2.2). Although equation (2.10) was directly differentiated to get equations (2.13), the previous limiting argument could have been used in order to obtain eqns (2.13) from eqn (2.5) as $\epsilon \rightarrow 0$. Equations (2.13) will be used as a basis for a local BIE technique. Also note that there are only two independent equations at each step of differentiation, since $\nabla^2 \psi = \nabla^2 \phi = 0$. In other words, while there is only one zero-order BIE ($k=l=0$ in eqns (2.13) which gives eqn (2.10)), there are two first-order equations, and two second-order equations, and so on. Table 2.1 shows the Pascal triangle form of the derivatives of ϕ , together with those chosen as the independent ones at each level.

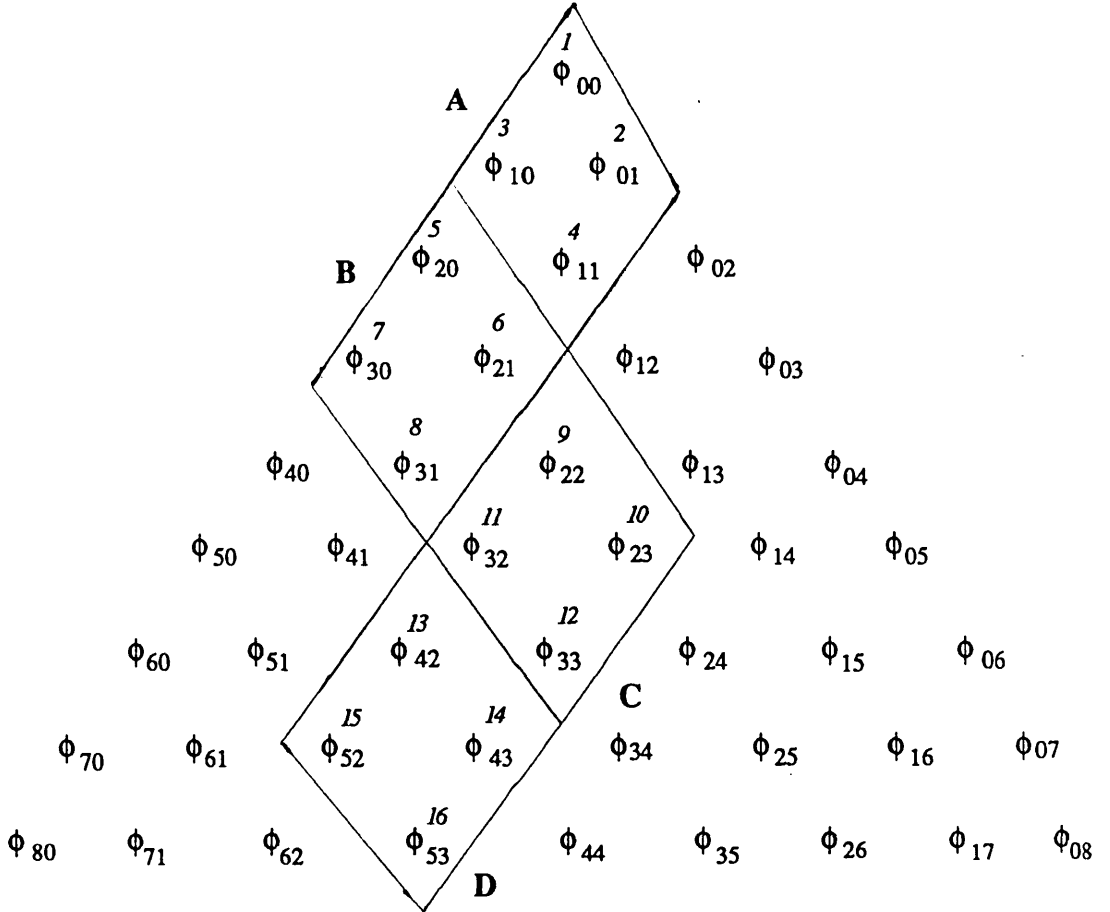


Table 2.1 : Pascal triangle of differentiation in two dimensions with submatrices A, B, C, and D which consist of the independent derivatives.

In Table 2.1 above subscripts mean differentiation with respect to coordinate directions x_{p1} and x_{p2} at p and those superscripts in italics show the independent derivatives in the order used to form a set of integral equations. As can be seen from Table 2.1, if n_d is the highest order of differentiation, then the total number of independent higher-order equations is

$$N = 2 n_d + 1 \quad (2.16)$$

2.1.3. Derivatives of Kernel Functions

The derivatives of the kernels in the integrands in equations (2.13) are required. One way of differentiating them is to make use of finite difference approximations. In order to get high accuracy in differencing the kernels Watson (1982) suggested multiplying the distance between field point and source point by 0.0002 to determine the distance between the two points at which the kernel functions are computed. He did this for two-dimensional elasticity problems, but there is no reason not to apply it here. However, finite difference approximation introduces an error additional to discretisation and numerical integration error, no matter how small it is. Since the analytical differentiation of kernel functions of potential theory is not so difficult, they are differentiated analytically here as follows in order to avoid the error incurred by differencing.

In Fig. (2.3) below, $|n_1|$ and $|n_2|$ denoted as n_1 and n_2 , respectively, are the magnitudes of the components of the unit outward vector normal to the boundary, S . From the figure, the magnitude of the displacement vector $|r(p,Q)| \equiv r(p,Q)$ is

$$r(p,Q) = (x_{p1} - x_{Q1}) \cos\theta + (x_{p2} - x_{Q2}) \sin\theta \quad (2.17)$$

note that in (2.17) the direction cosines are

$$\cos\theta = \frac{\partial r}{\partial x_{p_1}} = r_{,x_{p_1}} = \frac{(x_{p_1} - x_{Q_1})}{r} \quad (2.18)$$

and

$$\sin\theta = \frac{\partial r}{\partial x_{p_2}} = r_{,x_{p_2}} = \frac{(x_{p_2} - x_{Q_2})}{r} \quad (2.19)$$

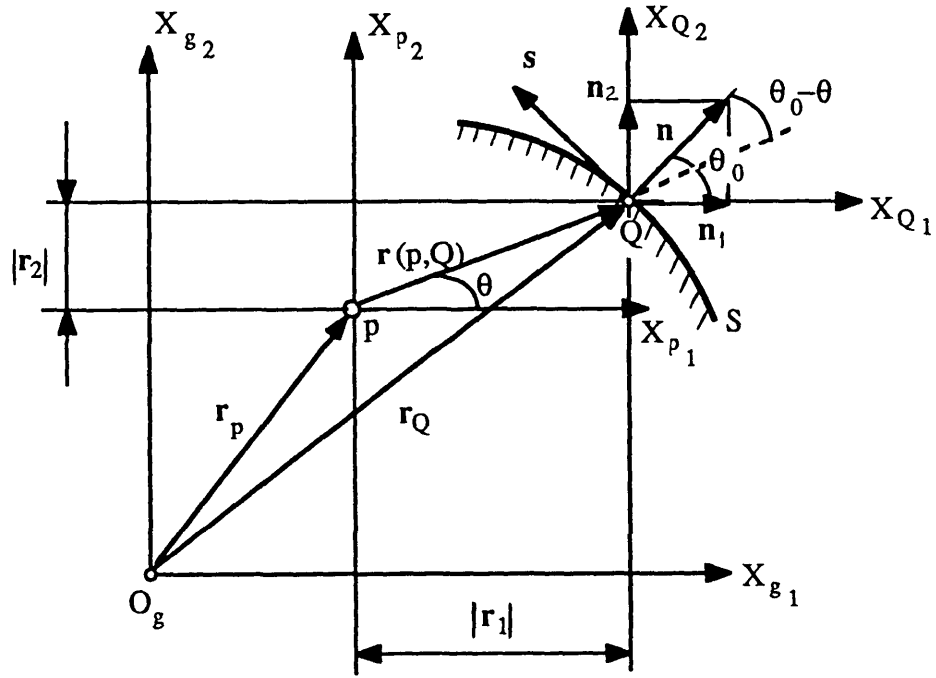


Figure 2.3 : A global reference coordinate system with a point Q on the boundary, S , and a point p internal to S .

by using the summation convention, r can be rewritten as

$$r = \{(x_{p_i} - x_{Q_i})^2\}^{1/2} \quad (2.20)$$

and since

$$\frac{\partial r}{\partial x_{Q_1}} = r_{,x_{Q_1}} = \frac{(x_{Q_1} - x_{P_1})}{r} = -r_{,x_{P_1}} \quad (2.21)$$

and

$$\frac{\partial r}{\partial x_{Q_2}} = r_{,x_{Q_2}} = \frac{(x_{Q_2} - x_{P_2})}{r} = -r_{,x_{P_2}} \quad (2.22)$$

then the normal derivative of r at Q is

$$\frac{\partial r(p, Q)}{\partial n(Q)} = \frac{\partial r}{\partial x_{Q_1}} n_1(Q) + \frac{\partial r}{\partial x_{Q_2}} n_2(Q) \quad (2.23)$$

or by the summation convention again,

$$\frac{\partial r(p, Q)}{\partial n(Q)} = \left(\frac{\partial r}{\partial x_i} \right)_Q n_i(Q) \quad (2.24)$$

Now, let us recall the first equation in (2.15)

$$B_{kl}(p, Q) = \frac{\partial^{(k+l)}}{\partial x_{P_1}^k \partial x_{P_2}^l} [\psi(p, Q)] \quad (2.15a)$$

where

$$\psi(p, q) = \ln \left[\frac{1}{r(p, Q)} \right], \quad (\text{see equation (2.1)})$$

Therefore* ,

$$B_{00}(p, Q) = \psi(p, Q) = -\ln[r(p, Q)] \quad (2.25)$$

$$B_{01}(p, Q) = -\frac{\partial}{\partial r} \{\ln r(p, Q)\} \frac{\partial r}{\partial y_p} = -\frac{1}{r} \frac{\partial r}{\partial y_p} = -\frac{(y_p - y_Q)}{r^2} \quad (2.26)$$

$$B_{10}(p, Q) = -\frac{\partial}{\partial r} \{\ln r(p, Q)\} \frac{\partial r}{\partial x_p} = -\frac{1}{r} \frac{\partial r}{\partial x_p} = -\frac{(x_p - x_Q)}{r^2} \quad (2.27)$$

$$B_{11}(p, Q) = 2 \frac{(x_p - x_Q)(y_p - y_Q)}{r^4} \quad (2.28)$$

$$B_{20}(p, Q) = 2 \frac{(x_p - x_Q)^2}{r^4} - \frac{1}{r^2} \quad (2.29)$$

$$B_{21}(p, Q) = 2 \frac{(y_p - y_Q)}{r^4} - 8 \frac{(x_p - x_Q)^2 (y_p - y_Q)}{r^6} \quad (2.30)$$

$$B_{30}(p, Q) = 6 \frac{(x_p - x_Q)}{r^4} - 8 \frac{(x_p - x_Q)^3}{r^6} \quad (2.31)$$

* From now on, for the sake of convenience, the first coordinate direction will be denoted as x and the second coordinate direction as y.

$$B_{31}(p,Q) = 48 \frac{(x_p - x_Q)^3 (y_p - y_Q)}{r^8} - 24 \frac{(x_p - x_Q) (y_p - y_Q)}{r^6} \quad (2.32)$$

$$B_{22}(p,Q) = 48 \frac{(x_p - x_Q)^2 (y_p - y_Q)^2}{r^8} - \frac{6}{r^4} \quad (2.33)$$

$$B_{23}(p,Q) = 24 \frac{(y_p - y_Q)}{r^6} + 96 \frac{(x_p - x_Q)^2 (y_p - y_Q)}{r^8} - 384 \frac{(x_p - x_Q)^2 (y_p - y_Q)^3}{r^{10}} \quad (2.34)$$

$$B_{32}(p,Q) = 24 \frac{(x_p - x_Q)}{r^6} + 96 \frac{(y_p - y_Q)^2 (x_p - x_Q)}{r^8} - 384 \frac{(y_p - y_Q)^2 (x_p - x_Q)^3}{r^{10}} \quad (2.35)$$

$$B_{33}(p,Q) = -720 \frac{(x_p - x_Q) (y_p - y_Q)}{r^8} + 3840 \frac{(x_p - x_Q)^3 (y_p - y_Q)^3}{r^{12}} \quad (2.36)$$

$$B_{42}(p,Q) = \frac{24}{r^6} - 144 \frac{(x_p - x_Q)^2}{r^8} + 96 \frac{(y_p - y_Q)^2}{r^8} - 1920 \frac{(x_p - x_Q)^2}{r^{10}} \\ (y_p - y_Q)^2 + 3840 \frac{(x_p - x_Q)^4 (y_p - y_Q)^2}{r^{12}} \quad (2.37)$$

$$\begin{aligned}
B_{43}(p,Q) = & -720 \frac{(y_p - y_Q)^8}{r^8} + 5760 \frac{(x_p - x_Q)^2 (y_p - y_Q)^{10}}{r^{10}} + 11520 \\
& \frac{(x_p - x_Q)^2 (y_p - y_Q)^3}{r^{12}} - 46080 \frac{(x_p - x_Q)^4 (y_p - y_Q)^3}{r^{14}}
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
B_{52}(p,Q) = & -432 \frac{(x_p - x_Q)^8}{r^8} + 1152 \frac{(x_p - x_Q)^3}{r^{10}} - 4608 \frac{(x_p - x_Q)}{r^{10}} \\
& (y_p - y_Q)^2 + 34560 \frac{(x_p - x_Q)^3 (y_p - y_Q)^2}{r^{12}} - 46080 \\
& \frac{(x_p - x_Q)^5 (y_p - y_Q)^2}{r^{14}}
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
B_{53}(p,Q) = & 17280 \frac{(x_p - x_Q) (y_p - y_Q)^{10}}{r^{10}} - 57600 \frac{(x_p - x_Q)^3 (y_p - y_Q)^{12}}{r^{12}} + \\
& 23040 \frac{(x_p - x_Q) (y_p - y_Q)^3}{r^{12}} - 322560 \frac{(x_p - x_Q)^3 (y_p - y_Q)^3}{r^{14}} \\
& + 645120 \frac{(x_p - x_Q)^5 (y_p - y_Q)^3}{r^{16}}
\end{aligned} \tag{2.40}$$

$$B_{44}(p,Q) = \frac{5040}{r^8} - 161280 \frac{(x_p - x_Q)^2 (y_p - y_Q)^2}{r^{12}} + 645120 \frac{(x_p - x_Q)^4 (y_p - y_Q)^4}{r^{16}} \quad (2.41)$$

$$B_{45}(p,Q) = -40320 \frac{(y_p - y_Q)}{r^{10}} - 322560 \frac{(x_p - x_Q)^2 (y_p - y_Q)}{r^{12}} + 1935360 \frac{(x_p - x_Q)^2 (y_p - y_Q)^3}{r^{14}} + 2580480 \frac{(x_p - x_Q)^4}{r^{16}} \\ (y_p - y_Q)^3 - 10321920 \frac{(x_p - x_Q)^4 (y_p - y_Q)^5}{r^{18}} \quad (2.42)$$

$$B_{54}(p,Q) = -40320 \frac{(x_p - x_Q)}{r^{10}} - 322560 \frac{(y_p - y_Q)^2 (x_p - x_Q)}{r^{12}} + 1935360 \frac{(y_p - y_Q)^2 (x_p - x_Q)^3}{r^{14}} + 2580480 \frac{(x_p - x_Q)^3}{r^{16}} \\ (y_p - y_Q)^4 - 10321920 \frac{(y_p - y_Q)^4 (x_p - x_Q)^5}{r^{18}} \quad (2.43)$$

Also by remembering the second equation in (2.15), that is,

$$A_{kl}(p,Q) = \frac{\partial^{(k+l)}}{\partial x_{p_1}^k \partial x_{p_2}^l} \left[\frac{\partial \psi(p,Q)}{\partial n(Q)} \right] \quad (2.15b)$$

Now,

$$\frac{\partial \psi(p, Q)}{\partial n(Q)} = \left(\frac{\partial \psi}{\partial x} \right)_Q n_1(Q) + \left(\frac{\partial \psi}{\partial y} \right)_Q n_2(Q) \quad (2.44)$$

where

$$\left(\frac{\partial \psi}{\partial x} \right)_Q = \frac{\partial \psi}{\partial x_Q} = -\frac{1}{r} \frac{\partial r}{\partial x_Q} = +\frac{x_p - x_Q}{r^2} = -B_{10}(p, Q) \quad (2.45)$$

and

$$\left(\frac{\partial \psi}{\partial y} \right)_Q = \frac{\partial \psi}{\partial y_Q} = -\frac{1}{r} \frac{\partial r}{\partial y_Q} = +\frac{y_p - y_Q}{r^2} = -B_{01}(p, Q) \quad (2.46)$$

hence equation (2.15b) for $k=l=0$ becomes

$$A_{00}(p, Q) = -B_{10}(p, Q) n_1(Q) - B_{01}(p, Q) n_2(Q) \quad (2.47)$$

then,

$$A_{01}(p, Q) = -B_{11}(p, Q) n_1(Q) + B_{20}(p, Q) n_2(Q) \quad (2.48)$$

$$A_{10}(p, Q) = -B_{20}(p, Q) n_1(Q) - B_{11}(p, Q) n_2(Q) \quad (2.49)$$

$$A_{11}(p,Q) = -B_{21}(p,Q) n_1(Q) + B_{30}(p,Q) n_2(Q) \quad (2.50)$$

$$A_{20}(p,Q) = -B_{30}(p,Q) n_1(Q) - B_{21}(p,Q) n_2(Q) \quad (2.51)$$

$$A_{21}(p,Q) = -B_{31}(p,Q) n_1(Q) - B_{22}(p,Q) n_2(Q) \quad (2.52)$$

$$A_{30}(p,Q) = +B_{22}(p,Q) n_1(Q) - B_{31}(p,Q) n_2(Q) \quad (2.53)$$

$$A_{31}(p,Q) = +B_{23}(p,Q) n_1(Q) - B_{32}(p,Q) n_2(Q) \quad (2.54)$$

$$A_{22}(p,Q) = -B_{32}(p,Q) n_1(Q) - B_{23}(p,Q) n_2(Q) \quad (2.55)$$

$$A_{23}(p,Q) = -B_{33}(p,Q) n_1(Q) + B_{42}(p,Q) n_2(Q) \quad (2.56)$$

$$A_{32}(p,Q) = -B_{42}(p,Q) n_1(Q) - B_{33}(p,Q) n_2(Q) \quad (2.57)$$

$$A_{33}(p,Q) = -B_{43}(p,Q) n_1(Q) + B_{52}(p,Q) n_2(Q) \quad (2.58)$$

$$A_{42}(p,Q) = -B_{52}(p,Q) n_1(Q) - B_{43}(p,Q) n_2(Q) \quad (2.59)$$

$$A_{43}(p,Q) = -B_{53}(p,Q) n_1(Q) - B_{44}(p,Q) n_2(Q) \quad (2.60)$$

$$A_{52}(p,Q) = +B_{44}(p,Q) n_1(Q) - B_{53}(p,Q) n_2(Q) \quad (2.61)$$

$$A_{53}(p,Q) = +B_{45}(p,Q) n_1(Q) - B_{54}(p,Q) n_2(Q) \quad (2.62)$$

Here, we differentiated the kernels up to the 8th order. As the order of differentiation increases, the derivatives of the kernel functions become more and more highly singular. Therefore, there is a limit to the order of differentiation (i.e., to the number of terms/equations). The accuracy in numerical integration of highly singular kernel derivatives is the criteria for this limit. This is to be discussed in sections (2.4) and (2.5).

2.1.4. Taylor Series

If a function ϕ is continuous in an interval $[x-a, y-b]$ and if it has subsequent partial derivatives of any order, the following equation can then be written to approximate the function, ϕ , about $x=a$ and $y=b$

$$\phi(x,y) = \phi(a,b) + D\phi + \frac{D^2\phi}{2!} + \dots + \frac{D^m\phi}{m!} \tag{2.63}$$

where the operator D is defined by

$$D = \left\{ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right\}_{\substack{x=a \\ y=b}} \tag{2.64}$$

Equation (2.63) provides very accurate polynomial approximations for a large class of functions which have derivatives of all orders (of order 'm', at least).

For convenience, Taylor series will be expanded at the origin of local coordinates, $x=y=0$, which is chosen at the point of particular interest. This gives the Maclaurin series for $\phi(x,y)$. Then, equation (2.63) and the operator D (eqn (2.64)) can be written as follows,

$$\phi(x,y) = \phi_0 + D\phi + \frac{D^2\phi}{2!} + \dots + \frac{D^m\phi}{m!} \quad (2.65)$$

where

$$D = \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\}_0 \quad (2.66)$$

Since $\nabla^2\phi=0$, not all the derivatives in (2.65) are independent. Defining those chosen as the independent ones in the order implied by Table 2.1 as the coefficients of Taylor series in (2.65), and denoting them as shown below

$$\alpha_1 = \phi_{00} , \quad \alpha_2 = \phi_{01} , \quad \alpha_3 = \phi_{10} , \dots, \text{and so on.} \quad (2.67)$$

we then have

$$\phi(x,y) = \alpha_1 + y \alpha_2 + x \alpha_3 + xy \alpha_4 + \dots + \left(\frac{x^3 y - xy^3}{6} \right) \alpha_8 + \dots \quad (2.68)$$

or representing the functions in eqn (2.68) as $f^1, f^2, \dots, f^m$, equation (2.68) becomes

$$\phi(x,y) = f^1 \alpha_1 + f^2 \alpha_2 + f^3 \alpha_3 + f^4 \alpha_4 + \dots + f^m \alpha_m \quad (2.69)$$

By simple differentiation of equation (2.69), the derivatives of the polynomial functions ($f^1, \dots, f^m$) required for obtaining the linear algebraic equations from the higher-order BIEs in eqns (2.13) can be determined. The functions and their first fifteen independent

Table 2.2 : The polynomial functions of the Taylor series, eqn (2.69), and their derivatives.

[illegible]

Table 2.2 : Continued.

m	12	13	14	15	16
$f^m_{,00} =$	$(20x^3y^3-6xy^5-6x^5y)/720$	$(15x^4y^2-15x^2y^4+y^6-x^6)/720$	$(35x^4y^3-7x^6y-21x^2y^5+y^7)/5040$	$(21x^5y^2-x^7-35x^3y^4+7xy^6)/5040$	$(56x^5y^3-8x^7y-56x^3y^5+8xy^7)/40320$
$f^m_{,01} =$	$(10x^3y^2-5xy^4-x^5)/120$	$(5x^4y-10x^2y^3+y^5)/120$	$(15x^4y^2-15x^2y^4+y^6-x^6)/720$	$(6x^5y-20x^3y^3+6xy^5)/720$	$(21x^5y^2-x^7-35x^3y^4+7xy^6)/5040$
$f^m_{,10} =$	$(10x^2y^3-5yx^4-y^5)/120$	$(10x^3y^2-5xy^4-x^5)/120$	$(20x^3y^3-6xy^5-6x^5y)/720$	$(15x^4y^2-15x^2y^4+y^6-x^6)/720$	$(35x^4y^3-7x^6y-21x^2y^5+y^7)/5040$
$f^m_{,11} =$	$(6x^2y^2-x^4-y^4)/24$	$(x^3y-xy^3)/6$	$(10x^3y^2-5xy^4-x^5)/120$	$(5x^4y-10x^2y^3+y^5)/120$	$(15x^4y^2-15x^2y^4+y^6-x^6)/720$
$f^m_{,20} =$	$(xy^3-yx^3)/6$	$(6x^2y^2-x^4-y^4)/24$	$(10x^2y^3-5yx^4-y^5)/120$	$(10x^3y^2-5xy^4-x^5)/120$	$(20x^3y^3-6xy^5-6x^5y)/720$
$f^m_{,21} =$	$(3xy^2-x^3)/6$	$(3x^2y-y^3)/6$	$(6x^2y^2-x^4-y^4)/24$	$(x^3y-xy^3)/6$	$(10x^3y^2-5xy^4-x^5)/120$
$f^m_{,30} =$	$(y^3-3yx^2)/6$	$(3xy^2-x^3)/6$	$(xy^3-yx^3)/6$	$(6x^2y^2-x^4-y^4)/24$	$(10x^2y^3-5yx^4-y^5)/120$
$f^m_{,31} =$	$(y^2-x^2)/2$	xy	$(3xy^2-x^3)/6$	$(3x^2y-y^3)/6$	$(6x^2y^2-x^4-y^4)/24$
$f^m_{,22} =$	xy	$(x^2-y^2)/2$	$(3x^2y-y^3)/6$	$(x^3-3xy^2)/6$	$(x^3y-xy^3)/6$
$f^m_{,23} =$	x	$-y$	$(x^2-y^2)/2$	$-xy$	$(x^3-3xy^2)/6$
$f^m_{,32} =$	y	x	xy	$(x^2-y^2)/2$	$(3x^2y-y^3)/6$
$f^m_{,33} =$	1	0	x	$-y$	$(x^2-y^2)/2$
$f^m_{,42} =$	0	1	y	x	xy
$f^m_{,43} =$	0	0	1	0	x
$f^m_{,52} =$	0	0	0	1	y
$f^m_{,53} =$	0	0	0	0	1

derivatives selected in an order based on the Pascal triangle of differentiation (see Table 2.1) are given in Table 2.2.

Now, in a typical BIE problem half of the boundary conditions are prescribed, and in a local BIE method non-prescribed values of the function are approximated by the series and they contribute to the coefficient matrix. If the unknown value is the gradient of the function it can then be determined from the equation

$$\left(\frac{\partial\phi}{\partial n}\right)_Q = \left(\frac{\partial\phi}{\partial x}\right)_Q n_1(Q) + \left(\frac{\partial\phi}{\partial y}\right)_Q n_2(Q) \quad (2.70)$$

where $n_1(Q)$ and $n_2(Q)$ are the components in the x and y directions of the unit outward normal vector at Q, and the derivatives of ϕ are found by differentiating eqn (2.69).

2.1.5. Fourier Series

The general four-term Fourier series with truncated terms, which satisfies harmonic equation $\nabla^2\phi=0$, is the second trial function given here as

$$\phi(x,y) = \sum_{m=1}^N [a_{m00} \cos(mfx) \cosh(mfy) + a_{m01} \cos(mfx) \sinh(mfy) + a_{m10} \sin(mfx) \cosh(mfy) + a_{m11} \sin(mfx) \sinh(mfy)] \quad (2.71)$$

where a_{m00} , a_{m01} , a_{m10} , and a_{m11} are constants and f is the frequency.

Now, for any piecewise continuous function of period p^* (i.e., if $\phi(x+p, y+p)^{**} = \phi(x,y)$) the Fourier series (2.71) will converge to $\phi(x,y)$ at all points of continuity, and to $1/2[\phi(x^+,y^+) + \phi(x^-,y^-)]$ at all points of discontinuity. Here, $\phi(x^+,y^+)$ and $\phi(x^-,y^-)$ are the limits approached from the right and left, respectively.

* p now stands for the period and should not be confused with interior point p.

** Here, the function ϕ is assumed to have the same periods in both directions.

The relation between the frequency f and the period p , for a whole range $(-P/2$ to $+P/2)$, is

$$f = \frac{2\pi}{p} \quad (2.72)$$

If the frequency f is taken as 1, which means, from (2.72), $p=2\pi$ (i.e., $-\pi < x, y < +\pi$), then the Fourier series representation (2.71) of the piecewise continuous function $\phi(x, y)$ can be written as⁽¹⁾

$$\phi(x, y) = \sum_{m=1}^N \left[a_{m00} \cos(mx) \cosh(my) + a_{m01} \cos(mx) \sinh(my) + a_{m10} \sin(mx) \cosh(my) + a_{m11} \sin(mx) \sinh(my) \right] \quad (2.73)$$

Even if the function $\phi(x, y)$ is not periodic of period p the Fourier series (2.73) would still represent it, if we are interested only in representing the function on $-\pi < x, y < +\pi$, since then it is unimportant to what values the series might converge outside this interval. Moreover, it should also be ensured that the first derivatives of ϕ are piecewise continuous in the interval and satisfy the Cauchy-Riemann conditions (see section 3.2.1.3) in order for the Fourier series representation to be differentiable at each point, since the function is differentiated repeatedly.

These conditions are certainly satisfied by choosing the local origin of coordinates x and y to be at the internal point p and by scaling the dimensions of the solution domain such that the maximum distance between any two points is unity :

$$\sqrt{x^2 + y^2} \leq 1$$

The value of the function at p and the first fifteen of its independent derivatives (selected in an order based on the Pascal triangle of derivatives in Table 2.1) are then given by Table 2.3 below,

$\phi_{00}(p) = + \sum_{m=1}^N a_{m00}$	$\phi_{20}(p) = - \sum_{m=1}^N m^2 a_{m00}$	$\phi_{22}(p) = - \sum_{m=1}^N m^4 a_{m00}$	$\phi_{42}(p) = + \sum_{m=1}^N m^6 a_{m00}$
$\phi_{01}(p) = + \sum_{m=1}^N m a_{m01}$	$\phi_{21}(p) = - \sum_{m=1}^N m^3 a_{m01}$	$\phi_{23}(p) = - \sum_{m=1}^N m^5 a_{m01}$	$\phi_{43}(p) = + \sum_{m=1}^N m^7 a_{m01}$
$\phi_{10}(p) = + \sum_{m=1}^N m a_{m10}$	$\phi_{30}(p) = - \sum_{m=1}^N m^3 a_{m10}$	$\phi_{32}(p) = - \sum_{m=1}^N m^5 a_{m10}$	$\phi_{52}(p) = + \sum_{m=1}^N m^7 a_{m10}$
$\phi_{11}(p) = + \sum_{m=1}^N m^2 a_{m11}$	$\phi_{31}(p) = - \sum_{m=1}^N m^4 a_{m11}$	$\phi_{33}(p) = - \sum_{m=1}^N m^6 a_{m11}$	$\phi_{53}(p) = + \sum_{m=1}^N m^8 a_{m11}$
A	B	C	D

Table 2.3 : The Fourier series representation of the unknown function $\phi(x,y)$ and its derivatives at p in the order implied by Table 2.1.

These expressions are used to define the terms on the left hand sides of equations (2.13) in terms of the Fourier constants. On the right hand sides, $\phi(Q)$ is defined in terms of the same constants by (2.73) (with x and y the coordinates of Q relative to the origin at p), $(\partial\phi/\partial n)_Q$ is similarly defined with the aid of (2.70). At any point on the boundary, however, either ϕ or its normal derivative (or a linear relationship between them) is known from the boundary conditions of the problem, so that the only one which is unknown has to be approximated by the Fourier series(2.73).

2.1.6. Transformation of Coordinates, Functions and their Derivatives

Series expansion is about the local origin of coordinates. Since, for the sake of convenience, the local origin need not be taken as the origin of global coordinates, then a transformation has to be made from global coordinates to local coordinates (and vice versa). Considering first the transformation formulae for coordinates one can see, using the geometric relations as indicated in Fig. 2.4 below, that

From the global to local coordinate system :

$$x_1 = (x_g - x_o) \cos\alpha + (y_g - y_o) \sin\alpha \quad (2.74)$$

$$y_1 = -(x_g - x_o) \sin\alpha + (y_g - y_o) \cos\alpha \quad (2.75)$$

From the local to global coordinate system :

$$x_g = x_o + x_1 \cos\alpha - y_1 \sin\alpha \quad (2.76)$$

$$y_g = y_o + x_1 \sin\alpha + y_1 \cos\alpha \quad (2.77)$$

where x_g, y_g are the cartesian coordinates of global coordinate system, x_1, y_1 are those of local coordinate system, and x_o, y_o are the global coordinates of the origin of the local coordinate system.

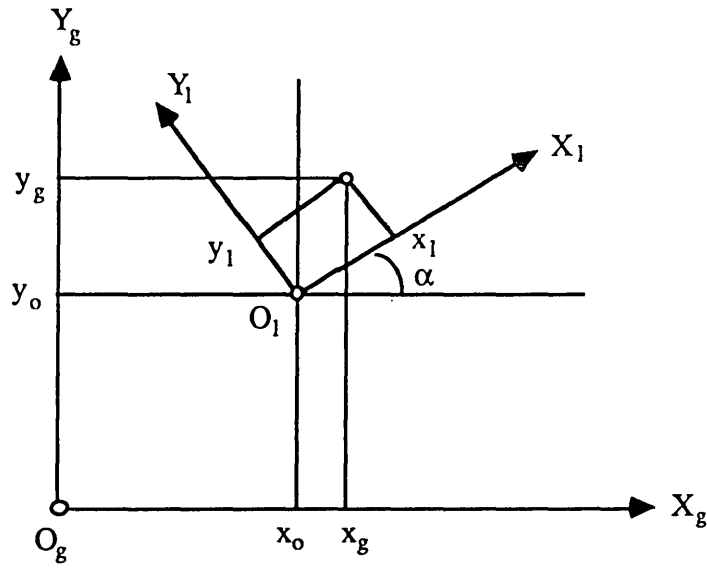


Figure 2.4 : Local and global coordinate system. α is the orientation of the local coordinate system with respect to the global coordinate system.

Transformation formulae for the derivatives of the function can be found with the aid of differential operators derived from Fig. 2.4 above. Thus,

$$\left(\frac{\partial \phi}{\partial x}\right)_1 = L_{x_1}\{\phi\}, \quad \left(\frac{\partial \phi}{\partial y}\right)_1 = L_{y_1}\{\phi\} \quad (2.78)$$

and

$$\left(\frac{\partial \phi}{\partial x}\right)_g = L_{x_g}\{\phi\}, \quad \left(\frac{\partial \phi}{\partial y}\right)_g = L_{y_g}\{\phi\} \quad (2.79)$$

where

$$\begin{aligned}
L_{x_g} &= \left\{ \frac{\partial}{\partial x_1} \cos\alpha - \frac{\partial}{\partial y_1} \sin\alpha \right\} \\
L_{y_g} &= \left\{ \frac{\partial}{\partial x_1} \sin\alpha + \frac{\partial}{\partial y_1} \cos\alpha \right\} \\
L_{x_1} &= \left\{ \frac{\partial}{\partial x_g} \cos\alpha + \frac{\partial}{\partial y_g} \sin\alpha \right\} \\
L_{y_1} &= \left\{ -\frac{\partial}{\partial x_g} \sin\alpha + \frac{\partial}{\partial y_g} \cos\alpha \right\}
\end{aligned} \tag{2.80}$$

Hence, higher-order operators are simply the operation of the operators on each other and it only remains to operate with them on the functions. They are given (up to the fifth-order) in appendix A.

2.2. Numerical Solution

Having formulated the BIE, the integrals can be performed either analytically (if possible) or numerically. Although numerical integration introduces some errors, in analytical integration the shape of the boundary of the solution domain should be assumed 'a priori' in a simple form, e.g., a sum of straight-line segments for the simplest representation. But even for such a simple case, analytical expressions of the integrands may be so complicated that analytical integration may turn out to be impossible, or rather involve a great amount of human effort in determining the integrals. Some solutions to singular integral equations for simple problems in plane elasticity were evaluated by using the complex variable method by Muskhelishvili (1954) and Mikhlin (1957). Since no general analytical method exists for integral equations to solve practical problems with complex geometries, numerical methods are employed to evaluate those integral equations.

In a conventional BIE method, the boundary of the solution domain is modelled by discretising it into elements. Each of these elements has a certain number of nodal points relevant to the order of shape functions associated with the element. Unknowns

over the boundary are expressed in terms of these functions. Approximating the surface and the unknown boundary values, in piecewise form, and defining them by suitable algebraic functions reduces the integral equations to a set of linear algebraic equations for unknowns at the nodal points of the elements. However, in the early numerical implementations of BIEs by Jaswon (1963), Symm (1963), and Jaswon & Ponter (1963) for potential theory, and by Rizzo (1967) and Cruse (1969) for elasticity, the introduced shape functions were only stepwise constant in order to express either the boundary unknowns or the boundary geometry. The surface was represented by straight-line elements, over each of which the unknowns were assumed to be constant. The integral equations were then written for the mid-point of each element or at the centroid of each triangular element, in two dimensions and three dimensions, respectively. The constant term $C(P)$ in equation (2.12) was then taken to be ' π ' for potential problems (see eqn (2.11)) and ' $1/2 \delta_{ij}$ ' for elasticity problems (see eqn (3.59)). Simpson's rule was employed to integrate the kernels by Jaswon&Ponter and Symm. Over these straight-line segments integration was performed analytically by Rizzo and Cruse to avoid the error produced by numerical integration due to rapid variation of the kernels as one argument approaches the other. This formulation was improved by Riccardella (1973) and Cruse (1974) by allowing the unknown functions to vary linearly over each element. The difference from the earlier formulation was that the integral equations were written for the end points of the elements (extremities in two dimensions, vertices in three dimensions), and unknowns were associated with these nodal points. But, if the source point P is not on a smooth part of the boundary (i.e., P is at an edge or a corner), then the constant of the free term, $C(P)$, is no longer ' $1/2 \delta_{ij}$ ' (for elasticity) and rigid body translation was considered (see equation (3.56)) by Cruse to compute this constant term implicitly.

This formulation was more efficient than the previous one. However, if the variation of the functions was rapid, then the required number of elements was still large compared to domain type methods. To overcome this problem Lachat (1975) and Lachat & Watson (1975) introduced the idea of using general curved boundary elements

defined by quadratic shape functions taken from the finite element formulation of Zienkiewicz and Ergatoudis. In their representation, the unknown functions were allowed to vary linearly, quadratically or cubically. Integrals were accurately evaluated by using Gaussian quadrature formulae. The coefficient of the free term of the integral equation was again computed considering rigid body translation. By the use of isoparametric formulations, it became more evident that BIE methods have the advantage over the domain type methods in accuracy. A further development was made by Watson (1982) who suggested employing boundary elements with Hermitian cubic shape functions instead of those with more conventional polynomial forms.

In the above formulations, the system of equations is scaled to obtain a numerically stable coefficient matrix, since the coefficients of the matrix are, in general, not of the same order of magnitude.

In the local BIE method, here, integrals are evaluated by using Gaussian quadrature formulae, as in the isoparametric BIE formulation of Lachat & Watson. Again, the boundary surface is discretised into elements with either quadratic or cubic variation to interpolate cartesian global coordinates of those elements at the Gauss points. However, the unknown functions are approximated by series (Taylor or Fourier) unlike the conventional BIE methods where they are expressed by the isoparametric shape functions of intrinsic coordinates. Consequently, the final set of linear algebraic equations is not formed from the integral equations written for the nodal points of the elements, instead it is obtained from the differentiation of the integral equations (higher-order BIEs). The coefficient matrix is scaled to ensure the numerical stability.

2.2.1. Approximate Series Solution

Now, assume that the unknown function $\phi(x,y)$ is represented by an appropriate series with a finite number of terms. Hence, this approximation of the function, ϕ , is written in the form of

$$\phi(x,y) = \sum_{m=1}^N \alpha_m f^m(x,y) \quad (2.81)$$

where the α_m are the constants to be determined from the solution of BIEs. Since the function $\phi(x,y)$ satisfies $\nabla^2\phi=0$, so should the functions f^m satisfy $\nabla^2f^m=0$, as shown in Table 2.2 and 2.3. In equation (2.81) the coordinates x and y are the cartesian coordinates of the point 'p' within the domain or those of the point 'Q' on the boundary. Therefore, equation (2.81) may explicitly be written as

$$\phi[p(x,y)] = \sum_{m=1}^N \alpha_m f^m[p(x,y)] \quad (2.82)$$

or

$$\phi[Q(x,y)] = \sum_{m=1}^N \alpha_m f^m[Q(x,y)] \quad (2.83)$$

and the normal derivative of ϕ on the boundary is

$$\left(\frac{\partial\phi}{\partial n}\right)_Q = \sum_{m=1}^N \alpha_m g^m(Q) \quad (2.84)$$

where

$$g^m(Q) = \left(\frac{\partial f^m}{\partial x}\right)_Q n_1(Q) + \left(\frac{\partial f^m}{\partial y}\right)_Q n_2(Q) \quad (2.85)$$

$n_1(Q)$ and $n_2(Q)$ being the components, in the coordinate directions x and y , of the unit outward normal vector $\mathbf{n}$ at Q .

Now, in a typical BIE problem, over every part of the boundary of the solution domain either $\phi(Q)$ (Dirichlet-Problem) or its normal gradient $(\partial\phi/\partial n)_Q$ (Neumann-Problem) is specified (or there may exist a linear relationship between them). The one which is not specified can be obtained by approximating with either equation (2.83) or (2.84). By substitution of equations (2.82), (2.83), and (2.84) in equations (2.13), we have

$$2\pi \sum_{m=1}^N \alpha_m f_{kl}^m(p) = \sum_{m=1}^N \alpha_m h_{kl}^m(Q) + b_{kl}(Q) \quad (2.86)$$

where, if $\phi(Q)$ is prescribed

$$h_{kl}^m(Q) = + \int_S g^m(Q) B_{kl}(p, Q) dS(Q) \quad (2.87)$$

$$b_{kl}(Q) = - \int_S \phi(Q) A_{kl}(p, Q) dS(Q) \quad (2.88)$$

and if $(\partial\phi/\partial n)_Q$ is prescribed

$$h_{kl}^m(Q) = - \int_S f^m(Q) A_{kl}(p, Q) dS(Q) \quad (2.89)$$

$$b_{kl}(Q) = + \int_S \left(\frac{\partial\phi}{\partial n}\right)_Q B_{kl}(p, Q) dS(Q) \quad (2.90)$$

equation (2.86) can therefore be rearranged as

$$\sum_{m=1}^N \alpha_m [2\pi f_{kl}^m(p) - h_{kl}^m(Q)] = b_{kl}(Q) \quad (2.91)$$

This is the final set of linear algebraic equations in the unknown α_m which can be found from the solution of it.

2.2.2. Discretisation of Geometry

In the formulation presented here, the boundary is defined using the quadratic (three-point) and cubic (four-point) shape functions of intrinsic local coordinates.

The cartesian coordinates $x_i(\xi)$ of an arbitrary point of an element of the boundary are given in terms of the nodal coordinates x_i^c by

$$x_i(\xi) = N^c(\xi) x_i^c \quad (2.92)$$

where c ranges from 1 to 3 for a quadratic element, and 1 to 4 for a cubic element, and associated shape functions are^(6,13,49)

for quadratic variation :

$$\begin{aligned} N^1(\xi) &= \frac{1}{2} \xi (\xi - 1) \\ N^2(\xi) &= 1 - \xi^2 \\ N^3(\xi) &= \frac{1}{2} \xi (\xi + 1) \end{aligned} \quad (2.93)$$

for cubic variation :

$$\begin{aligned}
 N^1(\xi) &= \frac{1}{16} (1 - \xi) (-10 + 9(1 + \xi^2)) \\
 N^2(\xi) &= \frac{9}{16} (1 - \xi^2) (1 - 3\xi) \\
 N^3(\xi) &= \frac{9}{16} (1 - \xi^2) (1 + 3\xi) \\
 N^4(\xi) &= \frac{1}{16} (1 + \xi) (-10 + 9(1 + \xi^2))
 \end{aligned}
 \tag{2.94}$$

where ξ varies from -1 to $+1$.

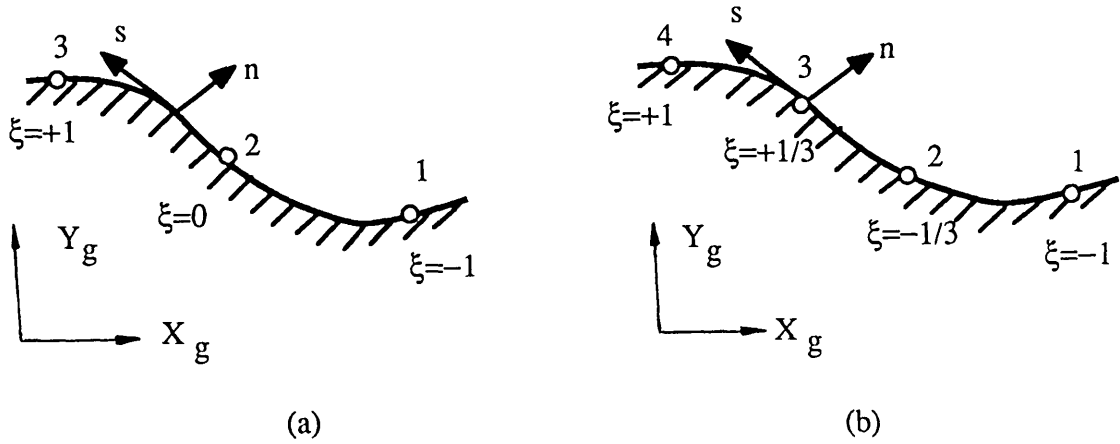


Figure 2.5 : Boundary element definition. a) A quadratic element,
b) a cubic element

Since the cartesian coordinates of an element are expressed in terms of the intrinsic local coordinates, there is then a need to transform from cartesian to curvilinear coordinates to be able to evaluate the integrals.

The components of the tangential vector, s , are proportional to the rates of change with the intrinsic coordinate, $\xi^{(16,34)}$.

$$s = \frac{dx}{d\xi} \mathbf{e}_1 + \frac{dy}{d\xi} \mathbf{e}_2 = s_1 \mathbf{e}_1 + s_2 \mathbf{e}_2 \quad (2.95)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors in the coordinate directions x and y , respectively.

Substitution of equation (2.92) in equation (2.95) yields

$$s_i = \frac{dx_i}{d\xi} = \frac{dN^c(\xi)}{d\xi} x_i^c \quad (2.96)$$

If $\mathbf{e}_3$ is the unit vector outward from the paper-plane (i.e., $\mathbf{e}_3$ is the normal to the plane formed by the normal vector, $\mathbf{n}$, and the tangential vector, s) the vector, $\mathbf{n}$, normal to the boundary is

$$\mathbf{n} = s \times \mathbf{e}_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{dx}{d\xi} & \frac{dy}{d\xi} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{dy}{d\xi} \mathbf{e}_1 - \frac{dx}{d\xi} \mathbf{e}_2 \quad (2.97)$$

$$= J_1 \mathbf{e}_1 + J_2 \mathbf{e}_2 \quad (2.98)$$

and the Jacobian of transformation from S to ξ is the modulus of equation (2.98),

$$J(\xi) = \frac{dS}{d\xi} = \sqrt{J_1^2 + J_2^2} \quad (2.99)$$

Hence, the components (equivalent to direction cosines) of the unit outward normal to the boundary are

$$n_i = \frac{J_i}{J(\xi)} \quad (2.100)$$

After defining the boundary geometry by using the quadratic and cubic shape functions of intrinsic local coordinates, both

$$\text{the integral } \int_S dS(Q) \quad \text{and the boundary point } Q(x,y)$$

can now be replaced with

$$\sum_{a=1}^q \int_{S_a} |J(\xi)| d\xi \quad \text{and} \quad Q(\xi) \quad ,\text{respectively,}$$

where a is the element being considered, and q is the total number of elements over the boundary. Then, equations (2.87), (2.88), (2.89), and (2.90) can be written as

If $\phi(Q)$ is prescribed,

$$h_{kl}^m(Q) = + \sum_{a=1}^q \int_{S_a=-1}^{+1} g^m(Q(\xi)) B_{kl}(p,Q(\xi)) |J(\xi)| d\xi \quad (2.101)$$

$$b_{kl}(Q) = - \sum_{a=1}^q \int_{S_a=-1}^{+1} \phi(Q(\xi)) A_{kl}(p, Q(\xi)) |J(\xi)| d\xi \quad (2.102)$$

If $(\partial\phi/\partial n)_Q$ is prescribed,

$$h_{kl}^m(Q) = - \sum_{a=1}^q \int_{S_a=-1}^{+1} f^m(Q(\xi)) A_{kl}(p, Q(\xi)) |J(\xi)| d\xi \quad (2.103)$$

$$b_{kl}(Q) = + \sum_{a=1}^q \int_{S_a=-1}^{+1} \left(\frac{\partial\phi}{\partial n} \right)_{Q(\xi)} B_{kl}(p, Q(\xi)) |J(\xi)| d\xi \quad (2.104)$$

In equations (2.102) and (2.104) note that the prescribed values $\phi(Q)$ and $(\partial\phi/\partial n)_Q$ over the element(s) were written as $\phi(Q(\xi))$ and $(\partial\phi/\partial n)_{Q(\xi)}$ to imply that they are not necessarily constant, and they may (are allowed to) vary from point to point over the element(s).

2.2.3. Numerical Integration Using Gaussian Quadrature

Since the integrations cannot generally be evaluated analytically (as mentioned before), a numerical integration technique must be adopted. The one which is used here to evaluate the integrals as accurately as possible is Gaussian quadrature.

Let us denote the integrands in equations (2.101), (2.102), (2.103), and (2.104) as $F(\xi)$. These integrals can then be expressed as

$$\int_{-1}^{+1} F(\xi) d\xi \quad (2.105)$$

which may be approximated by polynomial functions, and may be written in the form of

$$\int_{-1}^{+1} F(\xi) d\xi \cong \sum_{r=1}^S \omega_r F(\xi_r) \quad (2.106)$$

with weight factors ' ω_r '. In equation (2.106) the values of weight factors, ω_r , at the corresponding points of Gaussian quadrature abscissae, ξ_r , have been tabulated for a range of s . See, for example, Stroud and Secrest (1966).

Note that the source point and the field point never coincide over one element unlike the conventional BIE methods, and it is not necessary to use a Gaussian quadrature formulae with a logarithmic weight factor, $\log(1/\xi)$.

Therefore, the integrands always vary smoothly over the element(s), and never tend to infinity. Also, the existence of edges or sharp corners, where there is no unique tangent, along the boundary never causes any problem, because each element is considered separately and integrated one by one.

In order to obtain a numerically-stable coefficient matrix, the coordinates x and y are divided by the maximum dimension of the domain. The values obtained from the solution of the equations are of course rescaled accordingly.

2.3. Summary

The LBIE method developed herein has some common and different points from conventional BIE methods. It is therefore worthwhile to summarize the main points of the technique.

i) We begin with the zero-order integral equation (2.10) as the basis of the LBIE method.

ii) Equation (2.10) is then differentiated repeatedly at a point, p , in the x and y coordinate directions within the solution domain to obtain higher-order BIEs (see eqns (2.13)).

iii) In a typical 'BVP' at any point on the boundary either the potential function ϕ or its normal derivative (or a linear relationship between them) is known from the boundary conditions of the problem, so that only the one which is unknown, in eqns (2.13), has to be approximated. The variations of these unknowns on the boundary together with the potential and its subsequent derivatives at the internal point p , the terms on the left hand sides of eqns (2.13), are approximated by either Taylor series, eqn (2.69), or the Fourier series, eqn (2.73).

iv) The derivative forms of the kernel functions, parts of the integrands in eqns (2.13), are obtained by simple analytical differentiation (see section 2.1.3).

v) The integrations over the boundary are carried out by dividing the boundary into quadratic (three node) or cubic (four node) boundary elements, and applying gaussian quadrature to each one.

vi) The final set of linear algebraic equations is formed from the higher-order BIEs, eqns (2.13). It thus becomes a relatively small set of linear equations (16 equations are implied by Table 2.2 and 2.3) which are straightforward to solve by gaussian elimination.

vii) The coefficient matrix is scaled to overcome the possible numerical instability of a mixed dimensional matrix system.

viii) Having solved the linear equations the values of the coefficients of the series are known, and by rescaling them the required solution is obtained.

2.4. Test Cases

The accuracy of the numerical implementation of the LBIE method is assessed by reference to some test problems for which exact analytical solutions are known (for comparison purposes). For this reason a computer program called 'LBIE2PE' has been developed capable of evaluating both harmonic and biharmonic problems. The solution domains considered here are those of problems used previously to test the Taylor series

formulation of the Local BIE method by Fenner & Watson, namely, a square domain and a thick-walled cylinder. The latter provides an example of the use of curved boundary elements so that the ability of the technique to deal with awkward regions can be demonstrated. The potential functions considered involve the polynomial, hyperbolic, and logarithmic variations. The method is also used to locally refine the solution of a heat conduction problem. To avoid any interference from inaccuracies in the remote field solution, however, and to illustrate the capabilities of the method in such a local refinement procedure, the exact solution to the problem is applied outside the local area. Full ranges of solutions are given in Tables in appendices B, C, and D.

2.4.1. Square Domain

Figure 2.6 shows a square domain with sides of unit length, the local origin of expansion at the bottom left hand corner, the global coordinates of which are $x=y=1$.

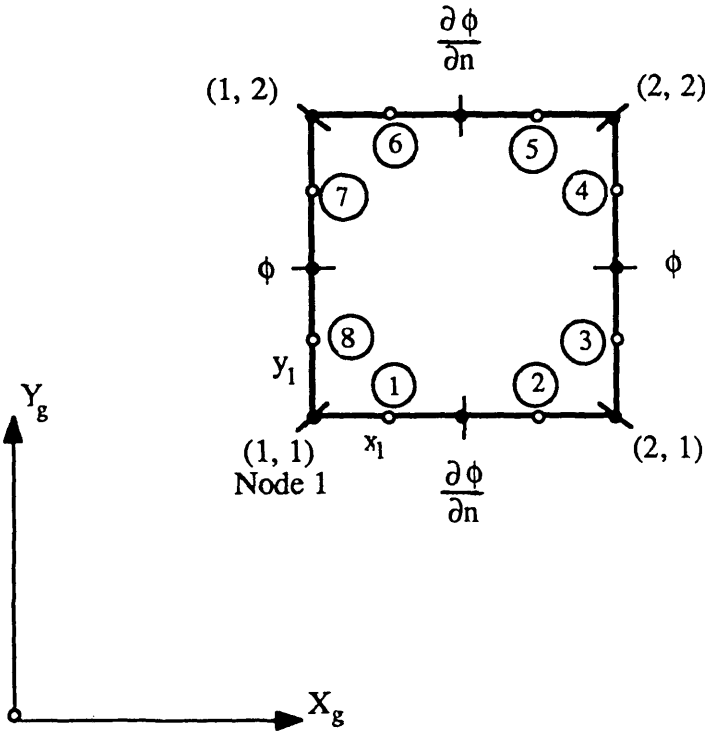


Figure 2.6. Square domain with mixed boundary conditions.

2.4.1.1. Linear Variation of Potential :

Let us first consider the boundary conditions which give rise to the following linear potential variation

$$\phi = 1 - x \quad \text{and} \quad \frac{\partial \phi}{\partial x} = -1 \quad (2.107)$$

with all other derivatives zero. The first solution scheme for this simple problem used eight quadratic boundary elements of equal length (two per side of the domain, see Fig. 2.6).

Tables 2.4 to 2.8 show the results obtained for the function ϕ and its first and second derivatives (i.e., the coefficients α_1 to α_5) using different numbers of terms and positions for internal point p in equations (2.13). Because of the simplicity of the problem one of the internal points was chosen at the mid point of the domain, and two others being progressively closer to the local point of expansion⁽¹⁷⁾, that is, (1.5, 1.5), (1.25, 1.25), and (1.1, 1.1)*.

Table 2.4 : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with different numbers of terms and the internal point at (1.5, 1.5)
using eight boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.5, 1.5)	5	+2.63x10 ⁻¹⁵	- 4.27x10 ⁻¹⁴	- 1.00000	+5.15x10 ⁻¹⁴	- 3.95x10 ⁻¹⁴
(1.5, 1.5)	7	- 2.12x10 ⁻¹⁴	+4.26x10 ⁻¹⁰	- 1.00000	- 1.25x10 ⁻⁰⁹	+1.25x10 ⁻⁰⁹
(1.5, 1.5)	11	- 9.34x10 ⁻¹¹	- 1.98x10 ⁻⁰⁹	- 1.00000	- 1.44x10 ⁻⁰⁹	- 1.59x10 ⁻⁰⁸
(1.5, 1.5)	15	- 1.59x10 ⁻¹⁰	+1.15x10 ⁻⁰⁹	- 1.00000	- 7.59x10 ⁻⁰⁶	+7.56x10 ⁻⁰⁶

* These are the coordinates with respect to the global coordinate system, see Fig. 2.6.

In Table 2.4 above with the internal point at (1.5, 1.5) and five terms in the series (Taylor) the results are perfect, being slightly less accurate as the number of terms retained in the series increases, though still in very good agreement with the exact values, confirming those results given by Fenner & Watson.

In more practical problems, where geometry and potential variations are more complex, the internal point should be taken closer to the local point of interest to obtain satisfactory results. With the internal point at (1.25, 1.25), however, the results deteriorate, while with the internal point at (1.1, 1.1) the results are far from accurate, as the number of terms increases. This is because the internal internal point is now too close to the boundary of the solution domain in relation to the size of the boundary elements used, and it is impossible to maintain adequate integration accuracy (even though ten gaussian quadrature points per element were used throughout*)⁽¹⁷⁾.

Table 2.5 : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with different numbers of terms and the internal point at (1.25, 1.25)
using eight boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.25, 1.25)	5	+1.01x10 ⁻⁰⁷	- 8.46x10 ⁻⁰⁶	- 1.00000	+1.53x10 ⁻⁰⁵	- 1.29x10 ⁻⁰⁵
(1.25, 1.25)	7	- 1.71x10 ⁻⁰⁵	- 9.24x10 ⁻⁰⁶	- 0.99981	- 1.12x10 ⁻⁰⁴	- 4.06x10 ⁻⁰⁴
(1.25, 1.25)	11	+1.20x10 ⁻⁰³	+6.42x10 ⁻⁰⁴	- 1.00078	- 4.96x10 ⁻⁰²	- 3.39x10 ⁻⁰²
(1.25, 1.25)	15	+8.22x10 ⁻⁰²	- 2.27x10 ⁻⁰¹	- 0.92014	- 1.17	- 3.30

* Unless stated, ten gaussian quadrature points per element are used for all the problems in this thesis.

Table 2.6 : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with different numbers of terms and the internal point at (1.1, 1.1)
using eight boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.1, 1.1)	5	$+1.03 \times 10^{-02}$	$- 6.01 \times 10^{-02}$	- 1.07523	$+6.59 \times 10^{-02}$	$- 2.23 \times 10^{-01}$
(1.1, 1.1)	7	$- 6.31 \times 10^{-02}$	$+6.81 \times 10^{-02}$	- 0.99542	$- 8.22 \times 10^{-01}$	+1.15
(1.1, 1.1)	11	$- 1.07 \times 10^{-02}$	$+5.55 \times 10^{-01}$	+0.13346	$- 7.42 \times 10^{-01}$	$- 2.47 \times 10^{-01}$
(1.1, 1.1)	15	$- 1.38 \times 10^{-03}$	$+9.30 \times 10^{-02}$	+0.02942	$+9.94 \times 10^{-03}$	$- 4.91 \times 10^{-01}$

As can be seen from Tables 2.4-2.6, the best results at all internal points are obtained when only five terms are used. See Table 2.7 below.

Table 2.7 : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using eight boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.5, 1.5)	5	$+2.63 \times 10^{-15}$	$- 4.27 \times 10^{-14}$	- 1.00000	$+5.15 \times 10^{-14}$	$- 3.95 \times 10^{-14}$
(1.25, 1.25)	5	$+1.01 \times 10^{-07}$	$- 8.46 \times 10^{-06}$	- 1.00000	$+1.53 \times 10^{-05}$	$- 1.29 \times 10^{-05}$
(1.1, 1.1)	5	$+1.03 \times 10^{-02}$	$- 6.01 \times 10^{-02}$	- 1.07523	$+6.59 \times 10^{-02}$	$- 2.23 \times 10^{-01}$

The values of the function and derivatives obtained using the Fourier series are also satisfactory (see Table 2.8 below), although they are not as good as those obtained using Taylor series⁽¹⁾. This is to be expected in view of the simplicity of the problem,

whose solution should be perfectly represented by a two term Taylor series⁽¹⁾. As the number of terms increases the results do improve, unlike with Taylor series, and it becomes necessary to retain relatively more terms to attain satisfactory accuracy (see Table B.2 in appendix B, for the full range of the results).

Table 2.8* : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using eight boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.5, 1.5)	15	+0.00002	- 0.00023	- 1.00013	+0.00253	- 0.00082
(1.25, 1.25)	9	+0.00060	- 0.00158	- 1.00027	- 0.00107	- 0.00859
(1.1, 1.1)	5	- 0.01140	+0.01803	- 0.90447	+0.09267	+0.00862

In order to improve the accuracy, particularly of the Fourier series method⁽¹⁾, the two elements adjacent to the local point of interest were each subdivided into ten elements, making 26 in all⁽¹⁷⁾. The ratio between the lengths of the successive subelements moving away from the local point of expansion was 1.5, which gave elements immediately adjacent to the local point of length 0.0044. The purpose of this element refinement was to permit much more accurate integrations, allowing the internal point to be placed much closer to the local point of interest. Table 9 and 10 show some of the results obtained from Taylor and the Fourier series with the internal points at (1.1, 1.1), (1.05, 1.05), and (1.01, 1.01) (see Table B.3 and B.4 in appendix B for the full range of the results).

* From now on, after five decimal places the results are rounded-off.

Table 2.9 : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with different numbers of terms and positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		0.	0.	- 1.	0.	0.
(1.1, 1.1)	5	+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.1, 1.1)	10	- 0.00000	+0.00000	- 1.00000	- 0.00000	+0.00000
(1.05, 1.05)		- 0.00000	+0.00000	- 1.00000	- 0.00000	- 0.00000
(1.01, 1.01)		+0.00000	+0.00002	- 0.99993	+0.00002	- 0.00000
(1.1, 1.1)	14	+0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		+0.00000	+0.02847	- 1.29386	+0.00260	+0.00016
(1.1, 1.1)	16	+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	- 0.00000
(1.01, 1.01)		+0.00000	- 0.05478	+0.004	+0.00001	+0.00006

Table 2.9 shows that the results obtained using Taylor series are still very good. There is, however, deterioration in accuracy with the internal point at (1.01, 1.01) as the number of terms in the series increases. With 14 terms the deterioration becomes significant, and particularly with 16 terms the results are bad, though the value of the potential function is still very good (at least up to five decimal places). This loss of accuracy is associated with the numerical integration of high-order kernel function derivatives⁽¹⁷⁾, which are increasingly singular as the number of terms in the series is increased.

Table 2.10 : Square domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with different numbers of terms and positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.1, 1.1)	8	+0.00000	- 0.00100	- 1.00105	+0.00050	+0.00265
(1.05, 1.05)		+0.00000	- 0.00077	- 1.00050	+0.00152	+0.00056
(1.01, 1.01)		+0.00000	- 0.00025	- 0.99908	+0.00106	+0.00041
(1.1, 1.1)	12	+0.00000	- 0.00003	- 0.99953	- 0.00038	- 0.00041
(1.05, 1.05)		- 0.00000	- 0.00012	- 0.99952	- 0.00022	- 0.00004
(1.01, 1.01)		- 0.00000	+0.00031	- 1.00511	- 0.00051	- 0.00004
(1.1, 1.1)	16	- 0.00000	- 0.00001	- 0.99999	- 0.00000	+0.00001
(1.05, 1.05)		+0.00000	- 0.00000	- 0.99998	+0.00000	- 0.00000
(1.01, 1.01)		+0.00000	- 0.04946	+0.005	+0.00002	+0.00006

Note that although the Fourier series method is still not as good as the Taylor series one, it gives much better accuracy than in Table 2.8, and as the number of terms increases it converges rapidly and the results with 16 terms become almost identical with those from Taylor series. The behaviour of the results from the Fourier series with the internal point at (1.01, 1.01) is like that of the results from Taylor series, as one may expect since the internal point is now too close to the boundary.

Based on the experience of this first problem, a number of higher-order potential functions can now be tested for the same square domain, with the same discretisation (i.e., 26 elements with local length ratio of 1.5 and smallest elements of length 0.0044).

2.4.1.2. Quadratic Variation of Potential :

The potential function to be applied over the domain is

$$\phi = x^2 - y^2 + 2xy \quad (2.108)$$

and its first and second order derivatives with respect to x and y coordinate directions are

$$\frac{\partial \phi}{\partial x} = 2x + 2y, \quad \frac{\partial \phi}{\partial y} = -2y + 2x \quad (2.109)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 2 \quad (2.110)$$

with all other derivatives zero. At x=y=1 (node 1), these become

$$\phi = 2 \quad (2.111)$$

$$\frac{\partial \phi}{\partial x} = 4, \quad \frac{\partial \phi}{\partial y} = 0 \quad (2.112)$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = 2, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 2 \quad (2.110)$$

Table 2.11 shows the optimum results obtained from Taylor series.

Table 2.11 : Square domain with quadratic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+2.	0.	+4.	+2.	+2.
(1.1, 1.1)	5	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)	5	+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)	5	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000

As can be seen from Table 2.11 above, the results obtained are excellent (although after twelve terms there is a slight loss of accuracy, see Table B.5 in appendix B). Five terms still give very good results because five-term Taylor series can perfectly represent second order polynomial functions (see the Pascal triangle of differentiation, Table 2.1). At the distance (1.01, 1.01), however, with the increment of the number of terms the results become unreliable as is the case in the first problem (see Table B.5 and B.6).

Table 2.12 shows the optimum results obtained from the Fourier series representation. Although Fourier series is not comparable with Taylor series with relatively small number of terms, the results, however, converge very well as the number of terms increases, and with 16 terms they are very good. The value of the potential function can be obtained even with smaller numbers of terms, and this is true for all the potential problems which follow (see Tables in appendices B, C, and D).

Table 2.12 : Square domain with quadratic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+2.	0.	+4.	+2.	+2.
(1.1, 1.1)	15	+2.00000	+0.00028	+3.99964	+2.00002	+1.99971
(1.05, 1.05)	16	+2.00000	+0.00007	+3.99877	+1.99993	+2.00000
(1.01, 1.01)	10	+2.00000	- 0.00048	+3.96953	+1.99822	+2.00126

2.4.1.3. Cubic Variation of Potential :

The potential function considered and its derivatives as follows

$$\phi = x^3 + 3x^2y - 3xy^2 - y^3 \quad (2.113)$$

and

$$\frac{\partial \phi}{\partial x} = 3x^2 + 6xy - 3y^2, \quad \frac{\partial \phi}{\partial y} = 3x^2 - 6xy - 3y^2 \quad (2.114)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 6x + 6y, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 6x - 6y \quad (2.115)$$

It is clear that the five-term Taylor series is now incapable of representing the function and it is necessary to retain at least 7 terms in the series. Table 2.13 shows the optimum results obtained from Taylor series and Table 2.14 those from Fourier series.

Table 2.13 : Square domain with cubic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	- 6.	+6.	0.	+12.
(1.1, 1.1)	7	- 0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.05, 1.05)	7	- 0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.01, 1.01)	7	+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000

Table 2.14 : Square domain with cubic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	- 6.	+6.	0.	+12.
(1.1, 1.1)	16	- 0.00000	- 5.99838	+6.00041	- 0.00053	+11.99783
(1.05, 1.05)	13	+0.00000	- 5.99855	+6.00188	+0.00339	+12.00053
(1.01, 1.01)	13	- 0.00000	- 6.00185	+6.03763	- 0.00233	+11.99993

2.4.1.4. Quartic Variation of Potential :

The potential function considered and its derivatives as follows

$$\phi = -x^4 + x^3y + 6x^2y^2 - xy^3 - y^4 \quad (2.116)$$

$$\frac{\partial \phi}{\partial x} = -4x^3 + 3x^2y + 12xy^2 - y^3 \quad (2.117)$$

$$\frac{\partial \phi}{\partial y} = -4y^3 + 3y^2x + 12yx^2 + x^3 \quad (2.118)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -12x^2 + 6xy + 12y^2, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 3x^2 + 24xy - 3y^2 \quad (2.119)$$

It is obvious that the nine-term Taylor series is able to reproduce the function. Table 2.15 and 2.16 show the optimum results obtained from Taylor and Fourier series, respectively.

Table 2.15 : Square domain with quartic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+4.	+6.	+10.	+24.	+6.
(1.1, 1.1)	9	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)	9	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.01, 1.01)	9	+4.00000	+5.99976	+ 9.99971	+23.99906	+5.99958

Table 2.16 : Square domain with quartic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+4.	+6.	+10.	+24.	+6.
(1.1, 1.1)	16	+4.00000	+5.99965	+10.00146	+23.99959	+6.00000
(1.05, 1.05)	16	+4.00000	+5.99988	+10.00035	+23.99995	+6.00002
(1.01, 1.01)	10	+4.00000	+5.99294	+10.05727	+23.99996	+6.00173

2.4.1.5. Quintic Variation of Potential :

The potential function considered and its derivatives as follows

$$\phi = -x^5 - 5x^4y + 10x^3y^2 + 10x^2y^3 - 5xy^4 - y^5 \quad (2.120)$$

$$\frac{\partial \phi}{\partial x} = -5x^4 - 20x^3y + 30x^2y^2 + 20xy^3 - 5y^4 \quad (2.121)$$

$$\frac{\partial \phi}{\partial y} = -5x^4 + 20x^3y + 30x^2y^2 - 20xy^3 - 5y^4 \quad (2.122)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -20x^3 - 60x^2y + 60xy^2 + 20y^3 \quad (2.123)$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -20x^3 + 60x^2y + 60xy^2 - 20y^3 \quad (2.124)$$

It is now necessary to employ at least 11 terms for a satisfactory Taylor series representation of the function. For the Fourier series again the larger the number of terms the better function representation is. Table 2.17 and 2.18 show the optimum results obtained.

Table 2.17 : Square domain with quintic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		+8.	+20.	+20.	+80.	0.
(1.1, 1.1)	11	+8.00000	+20.00000	+20.00000	+80.00000	+0.00000
(1.05, 1.05)	11	+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.01, 1.01)	10	+8.00000	+19.97971	+20.35543	+80.00894	- 0.00568

Table 2.18 : Square domain with quintic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		+8.	+20.	+20.	+80.	0.
(1.1, 1.1)	16	+8.00000	+19.97998	+20.00278	+80.00354	+0.02410
(1.05, 1.05)	16	+8.00000	+19.99574	+20.05714	+80.00291	+0.00040
(1.01, 1.01)	10	+8.00000	+19.92838	+21.75752	+80.01879	- 0.04852

2.4.1.6. Hyperbolic Variation of Potential :

The potential function considered and its derivatives are as follows

$$\phi = \text{Cos}x \text{ Cosh}y \quad (2.125)$$

$$\frac{\partial \phi}{\partial x} = -\text{Sin}x \text{ Cosh}y, \quad \frac{\partial \phi}{\partial y} = \text{Cos}x \text{ Sinh}y \quad (2.126)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\text{Cos}x \text{ Cosh}y, \quad \frac{\partial^2 \phi}{\partial x \partial y} = -\text{Sin}x \text{ Sinh}y \quad (2.127)$$

Table 2.19 and 2.20 show the optimum results obtained. Note that Fourier series representation is able to give slightly better results than the Taylor series one with relatively less numbers of terms since the potential function, eqn (2.125), is a part of the Fourier series (see eqn (2.73)).

Table 2.19 : Square domain with hyperbolic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.83373	+0.63496	- 1.29846	- 0.98890	- 0.83373
(1.1, 1.1)	16	+0.83373	+0.63497	- 1.29846	- 0.98890	- 0.83373
(1.05, 1.05)	16	+0.83373	+0.63496	- 1.29850	- 0.98890	- 0.83373
(1.01, 1.01)	10	+0.83373	+0.63426	- 1.29478	- 0.98934	- 0.83376

Table 2.20 : Square domain with hyperbolic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.83373	+0.63496	- 1.29846	- 0.98890	- 0.83373
(1.1, 1.1)	15	+0.83373	+0.63496	- 1.29846	- 0.98890	- 0.83372
(1.05, 1.05)	14	+0.83373	+0.63496	- 1.29844	- 0.98890	- 0.83373
(1.01, 1.01)	9	+0.83373	+0.63525	- 1.29869	- 0.98896	- 0.83394

2.4.1.7. Logarithmic Variation of Potential :

The potential function considered and its derivatives as follows

$$\phi = \ln r \quad (2.128)$$

where

$$r = \sqrt{x^2 + y^2}$$

$$\frac{\partial \phi}{\partial x} = \frac{x}{r^2}, \quad \frac{\partial \phi}{\partial y} = \frac{y}{r^2} \quad (2.129)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{r^2} - 2 \frac{x^2}{r^4}, \quad \frac{\partial^2 \phi}{\partial x \partial y} = -2 \frac{xy}{r^4} \quad (2.130)$$

Table 2.21 and 2.22 show the optimum results obtained. Both series converged well, although for logarithmic functions it is generally necessary to retain relatively many terms to be able to represent the variations well.

Table 2.21 : Square domain with logarithmic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.34657	+0.5	+0.5	- 0.5	0.
(1.1, 1.1)	15	+0.34657	+0.49992	+0.50000	- 0.49996	+0.00011
(1.05, 1.05)	12	+0.34657	+0.49996	+0.49994	- 0.49999	+0.00005
(1.01, 1.01)	10	+0.34657	+0.49955	+0.50309	- 0.50013	+0.00013

Table 2.22 : Square domain with logarithmic potential variation
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
with optimum numbers of terms and different positions for internal point
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.34657	+0.5	+0.5	- 0.5	0.
(1.1, 1.1)	16	+0.34657	+0.49999	+0.49992	- 0.49995	+0.00005
(1.05, 1.05)	16	+0.34657	+0.50000	+0.50010	- 0.49999	+0.00000
(1.01, 1.01)	10	+0.34657	+0.49965	+0.49898	- 0.49989	+0.00037

2.4.2. Thick-Walled Cylinder

Having solved several problems and obtained quite satisfactory results for a square domain we can now apply the same potential variations to a domain consisting of curved boundary segments, namely, a thick-walled cylinder in order to see whether or not the LBIE method is also applicable to problems with somewhat awkward regions.

Figure 2.7 shows one quadrant of the cross section of a thick-walled cylinder with inner and outer radii R_1 and R_2 , and prescribed values of potential, ϕ_1 and ϕ_2 , at the inner and outer surfaces. Due to symmetry only one quadrant is considered.

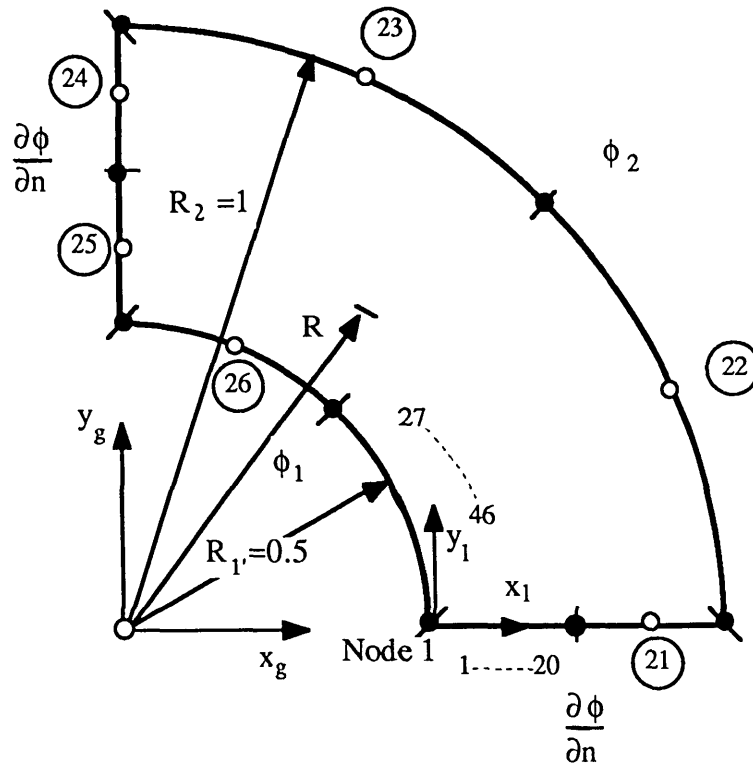


Figure 2.7 : One quadrant of a thick-walled cylinder with mixed boundary conditions.

Based on the experience of the first square domain problems the solution domain was discretised into 46 elements, 40 of which were in the vicinity of the local point of

interest which was selected to be at node 1 (see Fig. 2.7 above). The ratio between the length of successive elements in this region moving away from node 1 was 1.2, so that the elements immediately adjacent to this point of interest were of length 0.0013 along the x axis and 0.0021 along the inner surface of the cylinder⁽¹⁷⁾.

2.4.2.1. Linear Variation of Potential :

Let us recall the function and its derivative with respect to x direction

$$\phi = 1 - x \quad \text{and} \quad \frac{\partial \phi}{\partial x} = -1 \quad (2.107)$$

with all other derivatives zero. Table 2.23 shows the optimum results obtained using Taylor series for the function ϕ and its first and second derivatives (i.e., the parameters α_1 to α_5), using different numbers of terms and positions for internal point p in equations (2.13). Table 2.24 shows the optimum results obtained using the Fourier series. The results from both series are as good as those obtained for a square domain, and this is true for all the problems which follow (see Table C.1 and C.2 in appendix C for the full range of the results).

Table 2.23 : Cylinder domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.5	0.	- 1.	0.	0.
(0.52, 0.02)	5	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.51, 0.01)	5	+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.509, 0.009)	5	+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.508, 0.008)	5	+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000

Table 2.24 : Cylinder domain with linear potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.5	0.	- 1.	0.	0.
(0.52, 0.02)	15	+0.50000	+0.00000	- 1.00000	- 0.00000	- 0.00001
(0.51, 0.01)	13	+0.50000	- 0.00000	- 0.99998	+0.00001	- 0.00003
(0.509, 0.009)	13	+0.50000	- 0.00001	- 0.99998	+0.00001	- 0.00003
(0.508, 0.008)	11	+0.50000	- 0.00000	- 1.00001	+0.00002	+0.00002

2.4.2.2. Quadratic Variation of Potential :

Let us recall the potential function

$$\phi = x^2 - y^2 + 2xy \quad (2.108)$$

and its first and second order derivatives with respect to x and y coordinate directions

$$\frac{\partial \phi}{\partial x} = 2x + 2y, \quad \frac{\partial \phi}{\partial y} = -2y + 2x \quad (2.109)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 2 \quad (2.110)$$

with all other derivatives zero. At $x=0.5$ and $y=0$ (node 1), these become

$$\phi = 0.25 \quad (2.131)$$

$$\frac{\partial \phi}{\partial x} = 1, \quad \frac{\partial \phi}{\partial y} = 1 \quad (2.132)$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = 2, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 2 \quad (2.110)$$

Table 2.25 and 2.26 show the optimum results obtained from Taylor and Fourier series. Since five term Taylor series can perfectly represent second order polynomial functions the results obtained using five terms are perfect, like those obtained for the square domain (see section 2.4.1.2). Although the results from Fourier series is less accurate than those from Taylor series with relatively small numbers of terms, they, however, converge well as the number of terms increases, and with 13-16 terms they are almost identical with those from Taylor series.

Table 2.25 : Cylinder domain with quadratic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		+0.25	+1.	+1.	+2.	+2.
(0.52, 0.02)	5	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)	5	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)	5	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)	5	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000

Table 2.26 : Cylinder domain with quadratic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.25	+1.	+1.	+2.	+2.
(0.52, 0.02)	16	+0.25000	+1.00000	+1.00002	+2.00001	+1.99997
(0.51, 0.01)	13	+0.25000	+1.00000	+1.00002	+2.00001	+1.99996
(0.509, 0.009)	13	+0.25000	+1.00000	+1.00002	+2.00001	+1.99996
(0.508, 0.008)	14	+0.25000	+1.00000	+1.00000	+2.00001	+2.00001

2.4.2.3. Cubic Variation of Potential :

Let us recall the potential function

$$\phi = x^3 + 3x^2y - 3xy^2 - y^3 \quad (2.113)$$

and its first and second order derivatives with respect to x and y coordinate directions

$$\frac{\partial \phi}{\partial x} = 3x^2 + 6xy - 3y^2, \quad \frac{\partial \phi}{\partial y} = 3x^2 - 6xy - 3y^2 \quad (2.114)$$

$$\frac{\partial^2 \phi}{\partial x^2} = 6x + 6y, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 6x - 6y \quad (2.115)$$

Table 2.27 and 2.28 show the optimum results obtained from Taylor and Fourier series. The results from both series are very good. Note that seven terms in Taylor series are now required in order to be able to get satisfactory results.

Table 2.27 : Cylinder domain with cubic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.125	+0.75	+0.75	+3.	+3.
(0.52, 0.02)	7	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)	7	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)	7	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)	7	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000

Table 2.28 : Cylinder domain with cubic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.125	+0.75	+0.75	+3.	+3.
(0.52, 0.02)	16	+0.12500	+0.74999	+0.75003	+3.00002	+2.99994
(0.51, 0.01)	13	+0.12500	+0.75000	+0.74998	+2.99999	+3.00005
(0.509, 0.009)	13	+0.12500	+0.75000	+0.74997	+2.99999	+3.00007
(0.508, 0.008)	13	+0.12500	+0.75000	+0.74996	+2.99999	+3.00009

2.4.2.4. Quartic Variation of Potential :

Let us recall the potential function

$$\phi = -x^4 + x^3y + 6x^2y^2 - xy^3 - y^4 \quad (2.116)$$

and its first and second order derivatives with respect to x and y coordinate directions

$$\frac{\partial \phi}{\partial x} = -4x^3 + 3x^2y + 12xy^2 - y^3 \quad (2.117)$$

$$\frac{\partial \phi}{\partial y} = -4y^3 + 3y^2x + 12yx^2 + x^3 \quad (2.118)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -12x^2 + 6xy + 12y^2, \quad \frac{\partial^2 \phi}{\partial x \partial y} = 3x^2 + 24xy - 3y^2 \quad (2.119)$$

It is necessary to retain at least nine terms in Taylor series since less terms are not enough to reproduce the function. Table 2.29 and 2.30 show the optimum results obtained from Taylor and Fourier series. Both series give very good results.

Table 2.29 : Cylinder domain with quartic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.0625	+0.125	- 0.5	+0.75	- 3.
(0.52, 0.02)	9	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)	9	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.509, 0.009)	9	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.508, 0.008)	9	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000

Table 2.30 : Cylinder domain with quartic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.0625	+0.125	- 0.5	+0.75	- 3.
(0.52, 0.02)	14	- 0.06250	+0.12497	- 0.50001	+0.75007	- 2.99997
(0.51, 0.01)	14	- 0.06250	+0.12499	- 0.50000	+0.75001	- 3.00001
(0.509, 0.009)	14	- 0.06250	+0.12500	- 0.49999	+0.75000	- 3.00002
(0.508, 0.008)	14	- 0.06250	+0.12500	- 0.49994	+0.75001	- 3.00013

2.4.2.5. Quintic Variation of Potential :

Let us recall the potential function

$$\phi = -x^5 - 5x^4y + 10x^3y^2 + 10x^2y^3 - 5xy^4 - y^5 \quad (2.120)$$

and its first and second order derivatives with respect to x and y coordinate directions

$$\frac{\partial \phi}{\partial x} = -5x^4 - 20x^3y + 30x^2y^2 + 20xy^3 - 5y^4 \quad (2.121)$$

$$\frac{\partial \phi}{\partial y} = -5x^4 + 20x^3y + 30x^2y^2 - 20xy^3 - 5y^4 \quad (2.122)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -20x^3 - 60x^2y + 60xy^2 + 20y^3 \quad (2.123)$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -20x^3 + 60x^2y + 60xy^2 - 20y^3 \quad (2.124)$$

Table 2.31 and 2.32 show the optimum results obtained from Taylor and Fourier series.

Table 2.31 : Cylinder domain with quintic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	11	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)	11	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.509, 0.009)	11	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.508, 0.008)	11	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000

Table 2.32 : Cylinder domain with quintic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	16	- 0.03125	- 0.31234	- 0.31297	- 2.50033	- 2.49906
(0.51, 0.01)	16	- 0.03125	- 0.31239	- 0.31289	- 2.50021	- 2.49922
(0.509, 0.009)	16	- 0.03125	- 0.31289	- 0.31420	- 2.50042	- 2.49660
(0.508, 0.008)	13	- 0.03125	- 0.31254	- 0.31146	- 2.49981	- 2.50207

It is now obvious that eleven-term Taylor series is capable of representing the function. Although the Fourier series cannot produce the same accuracy the results are very much satisfactory.

2.4.2.6. Hyperbolic Variation of Potential :

Let us recall the potential function

$$\phi = \text{Cos}x \text{ Cosh}y \quad (2.125)$$

and its first and second order derivatives with respect to x and y coordinate directions

$$\frac{\partial \phi}{\partial x} = -\text{Sin}x \text{ Cosh}y, \quad \frac{\partial \phi}{\partial y} = \text{Cos}x \text{ Sinh}y \quad (2.126)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\text{Cos}x \text{ Cosh}y, \quad \frac{\partial^2 \phi}{\partial x \partial y} = -\text{Sin}x \text{ Sinh}y \quad (2.127)$$

Due to the fact that the hyperbolic function (2.125) is a part of the Fourier series representation it is not surprising that it can be reproduced with relatively less number of terms retained in the Fourier series, eqn (2.73). Table 2.33 and 2.34 show the optimum results obtained from Taylor and Fourier series. The results from both series are very good, and the same accuracy can be attained with almost the same numbers of terms in both series.

Table 2.33 : Cylinder domain with hyperbolic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.87758	0.	- 0.47943	0.	- 0.87758
(0.52, 0.02)	13	+0.87758	- 0.00000	- 0.47943	- 0.00000	- 0.87758
(0.51, 0.01)	13	+0.87758	- 0.00001	- 0.47943	+0.00002	- 0.87757
(0.509, 0.009)	13	+0.87758	- 0.00001	- 0.47943	+0.00002	- 0.87757
(0.508,0.008)	13	+0.87758	- 0.00000	- 0.47943	+0.00001	- 0.87757

Table 2.34 : Cylinder domain with hyperbolic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.87758	0.	- 0.47943	0.	- 0.87758
(0.52, 0.02)	13	+0.87758	- 0.00000	- 0.47943	+0.00000	- 0.87757
(0.51, 0.01)	13	+0.87758	- 0.00001	- 0.47944	+0.00002	- 0.87756
(0.509, 0.009)	13	+0.87758	- 0.00001	- 0.47944	+0.00002	- 0.87756
(0.508,0.008)	13	+0.87758	- 0.00000	- 0.47943	+0.00001	- 0.87757

2.4.2.7. Logarithmic Variation of Potential :

Let us recall the potential function

$$\phi = \ln r \quad (2.128)$$

where

$$r = \sqrt{x^2 + y^2}$$

and its first and second order derivatives with respect to x and y coordinate directions

$$\frac{\partial \phi}{\partial x} = \frac{x}{r^2}, \quad \frac{\partial \phi}{\partial y} = \frac{y}{r^2} \quad (2.129)$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{r^2} - 2 \frac{x^2}{r^4}, \quad \frac{\partial^2 \phi}{\partial x \partial y} = -2 \frac{xy}{r^4} \quad (2.130)$$

Table 2.35 and 2.36 show the optimum results obtained from Taylor and Fourier series. Since logarithmic variations are more complex than those polynomial and hyperbolic ones the accuracy attained from both series is comparably less, but satisfactory. Note that the results for the square domain are better than those for the cylinder because of somewhat awkward boundary. It is also of interest that the Fourier series is able to represent the function slightly better than the Taylor one, as is the case for the square domain but it can be more clearly seen here.

Table 2.35 : Cylinder domain with logarithmic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		- 0.69315	0.	+2.	0.	- 4.
(0.52, 0.02)	14	- 0.69315	+0.00298	+2.01818	+0.00132	- 4.03467
(0.51, 0.01)	15	- 0.69315	+0.00255	+2.02929	- 0.00481	- 4.05861
(0.509, 0.009)	15	- 0.69315	+0.00290	+1.97956	- 0.00544	- 3.95916
(0.508, 0.008)	14	- 0.69315	+0.00227	+2.04806	- 0.00448	- 4.09615

Table 2.36 : Cylinder domain with logarithmic potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
(0.52, 0.02)	14	- 0.69315	+0.00382	+2.03942	- 0.00296	- 4.07786
(0.51, 0.01)	15	- 0.69315	+0.00335	+2.02021	- 0.00639	- 4.04046
(0.509, 0.009)	15	- 0.69315	+0.00360	+1.99068	- 0.00690	- 3.98140
(0.508, 0.008)	14	- 0.69315	+0.00283	+2.05466	- 0.00565	- 4.10937

2.4.2.8. Radial Variation in Temperature :

Let us now apply the method to a practical problem where potential variation is more complicated. The steady state radial heat transfer at a rate Q per unit length of a cylinder from the inner surface to the outer surface is defined by⁽¹⁶⁾

$$Q = - 2\pi k \frac{d\phi}{dR} R \quad (2.133)$$

where k is the thermal conductivity of the material, R is the radial coordinate measured from the centre of the cylinder, and ϕ is the temperature through the wall having the values ϕ_1 at R_1 and ϕ_2 at R_2 (see Fig.2.7).

Integrating the equation above and applying the boundary conditions results in

$$\phi = \phi_1 + (\phi_2 - \phi_1) \frac{\ln(R/R_1)}{\ln(R_2/R_1)} \quad (2.134)$$

Hence, the first two nonzero derivatives of ϕ with respect to R are

$$\frac{d\phi}{dR} = \frac{(\phi_2 - \phi_1)}{R \ln(R_2/R_1)}, \quad \text{and} \quad \frac{d^2\phi}{dR^2} = \frac{(\phi_1 - \phi_2)}{R^2 \ln(R_2/R_1)} \quad (2.135)$$

with the values given as $R_1 = 0.5$, $R_2 = 1$, $\phi_1 = 6$, and $\phi_2 = 3$ we then have

$$\phi = 6 - 3 \frac{\ln(2R)}{\ln(2)}, \quad \frac{d\phi}{dR} = \frac{-3}{R \ln(2)}, \quad \frac{d^2\phi}{dR^2} = \frac{+3}{R^2 \ln(2)} \quad (2.136)$$

and at $R=R_1$ (node 1) they become

$$\phi = 6, \quad \frac{d\phi}{dR} = -8.65617, \quad \frac{d^2\phi}{dR^2} = +17.31234 \quad (2.137)$$

Table 2.37 and 2.38 show the optimum results obtained from both series (see Table C.15 and C.16 in appendix C for the full range of the results).

Table 2.37 : Cylinder domain with radial potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.52, 0.02)	14	+6.00000	- 0.01247	- 8.75397	- 0.00441	+17.50113
(0.51, 0.01)	15	+6.00000	- 0.01066	- 8.80508	+0.02019	+17.61029
(0.509, 0.009)	15	+6.00000	- 0.01228	- 8.57711	+0.02304	+17.15437
(0.508, 0.008)	14	+6.00000	- 0.00948	- 8.87669	+0.01883	+17.75354

Table 2.38 : Cylinder domain with radial potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
(0.52, 0.02)	14	+6.00000	- 0.01605	- 8.84254	+0.01369	+17.68125
(0.51, 0.01)	15	+6.00000	- 0.01411	- 8.76541	+0.02699	+17.53100
(0.509, 0.009)	15	+6.00000	- 0.01533	- 8.62453	+0.02939	+17.24924
(0.508, 0.008)	14	+6.00000	- 0.01192	- 8.90437	+0.02390	+17.80891

As can be seen from both Tables, when the number of terms retained in the series is fifteen the optimum point location is at about (0.509, 0.009), when the maximum error in the nonzero derivatives from Taylor series is about 0.9% and the maximum error in those from the Fourier one is about 0.4%. For all internal points the exact value of temperature (up to at least five decimal places) is obtained.

Based on the experience of the problems tested so far, it will be wise to increase the number of terms in order to improve the accuracy. But there should be a careful balance between the number of terms and the location of internal point. When the

number of terms is increased the internal point should be taken closer to the local point of interest. It is, however, necessary not to position the internal point so close to the local point in order to avoid the inaccuracies in numerical integration of high-order kernel function derivatives.

With the present mesh, when more than fifteen terms are used the accuracy of the results deteriorates. In order to use more terms it is necessary to select an internal point closer to the local point of interest on the boundary, which can only be done, in view of the foregoing discussion, if finer discretisation is used on the boundary in the region of this point*.

2.4.3. Local Refinement of a Solution

The examples considered so far show that in each case the LBIE method (with either series expansion) described here can be used to obtain a solution in its own right. It can also be used to locally refine the already existing approximate solutions obtained by other methods (analytical or numerical).

The same cylinder problem of a radial potential (temperature) variation with the same discretisation (46 elements with local length ratio of 1.2 and smallest elements of length 0.0013 and 0.0021 along the x direction and the inner surface of the cylinder, respectively) was solved. This time, a remote field function was applied outside the local area**. The exact analytical solution to the problem, eqn (2.134), was used as the remote field function to be able to focus on the capabilities of the method in the local solution procedure and to avoid any interference from inaccuracies in the remote field solution⁽¹⁷⁾.

The results obtained were perfectly in agreement with the exact solution. Both series converged rapidly when the number of terms was increased, and the results were

* There are two ways of refining the mesh. One is to use more elements in the local region of interest, and the other is to reduce the local area (a different length ratio may also be considered).

** By local area, we mean the area inside a circle of radius r , which is three times the distance between the internal point and the quadrature point on the boundary, although this is not necessarily precise depending on the type of the problem under consideration.

almost insensitive to changes in the position of the internal point. Table 2.39 and 2.40 show the optimum results obtained (see Table D.1 and D.2 in appendix D for the full range of the results).

Table 2.39 : Cylinder domain with radial potential variation
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest,
and remote field solution outside this region.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.55, 0.05)	15	+6.00000	- 0.00000	- 8.65613	- 0.00082	+17.31082
(0.54, 0.04)	15	+6.00000	+0.00000	- 8.65617	- 0.00016	+17.31227
(0.53, 0.03)	15	+6.00000	- 0.00000	- 8.65617	- 0.00004	+17.31231
(0.52, 0.02)	15	+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.51, 0.01)	15	+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)	13	+6.00000	+0.00000	- 8.65617	+0.00000	+17.31235
(0.508, 0.008)	14	+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234

Table 2.40 : Cylinder domain with radial potential variation
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
with optimum numbers of terms and different positions for internal point
using fine boundary elements in the region of interest,
and remote field solution outside this region.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.55, 0.05)	16	+6.00000	+0.00000	- 8.65617	- 0.00045	+17.31234
(0.54, 0.04)	16	+6.00000	+0.00000	- 8.65617	- 0.00014	+17.31234
(0.53, 0.03)	16	+6.00000	+0.00000	- 8.65617	- 0.00003	+17.31234
(0.52, 0.02)	15	+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.51, 0.01)	15	+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)	15	+6.00000	- 0.00000	- 8.65617	+0.00000	+17.31234
(0.508, 0.008)	13	+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234

2.5. Discussion

A local boundary integral equation method using series expansion techniques has been developed for two-dimensional potential problems governed by Laplace's equation. The method is essentially based on the work carried out by Fenner & Watson (1988) using the Taylor series expansion. Fourier and Taylor series have been re-examined and it has been found that both series methods produce very satisfactory results in regions of particular interest and they can be used in their own rights.

An important application for the LBIE technique is to problems where approximate solutions already exist and more detailed information is required at particular geometric features. It provides a way of locally refining approximate numerical solutions obtained by other means.

In order to successfully apply the method to problems of interest attention should be given to the choice of three parameters. They are as follows:

- i) number of terms retained in the series,
- ii) location of the internal point within the region of interest,
- iii) the ratio between the element lengths in the local area (particularly two elements immediately adjacent to the local point of interest) and the distance, from the internal point to these elements (more correctly, to the local point of interest).

There should be kept a careful balance between these three parameters. To be able to represent the potential function well the number of terms in the series should, in general, be large (the larger this number the better the function representation is). There is, however, a limit to this number. The limit is provided by the accuracy with which the highly-singular kernel functions can be integrated. It is observed that the results are quite satisfactory, when the numbers of terms are approximately 15-16. Although for relatively simple problems fewer can be used it is, however, much better to choose to retain 15-16 terms in the series, since in a practical problem not the function but only the

boundary conditions are known. Also, when the number of terms is increased the internal point chosen should be closer to the local point of interest. The ratio between the position of internal point and the element lengths adjacent to the local point of interest should, however, be kept in due proportion. Based on the experience of the test problems the accuracy is generally enhanced until the distance of the internal point from the local point on the boundary are approximately of the order of twice as the length of the local elements there (the two elements adjacent to the local point of interest). As the internal point moves closer to the local origin, the results (especially the derivatives) become unreliable, because the point is now too close to the boundary. The resulting integrals over the elements there will be difficult to evaluate numerically, because the high-order kernel functions within the integrands will have a large fluctuation across these elements. In such a case a finer discretisation of the local area becomes necessary.

Elements over the boundaries were defined by either quadratic (three point) or cubic (four point) shape functions for all the test problems considered in this chapter. In either case the results obtained were almost identical. Hence, the results obtained using cubic boundary elements are not given and in the next chapter only quadratic elements will be employed to represent boundary geometries.

Note that the number of algebraic equations does not increase with increasing number of elements in a mesh refinement, and is fixed for a given number of terms in the series, whereas in conventional BIE methods the number of equations increases as the number of nodes(elements) increases, which makes it an expensive process to refine the mesh⁽¹⁷⁾. Because of the small number of linear equations that have to be solved, the LBIE method is computationally very efficient compared to conventional numerical techniques.

All these make the LBIE method very attractive to use, particularly (due to the high level of accuracy) to locally refine existing approximate solutions, where detailed information in a small region of particular interest is needed.

It is therefore of interest to apply the method to some stress problems governed by biharmonic equations.

C H A P T E R 3

3. TWO-DIMENSIONAL ELASTICITY THEORY

Having formulated the local BIE method for problems of the harmonic type, the approach can now be extended to two-dimensional plane stress and plane strain biharmonic problems of elasticity theory.

Boundary integral (Somigliana's identity) equations of elastostatics are derived by the direct formulation, as was done for potential theory.

3.1. Formulation

In this section we will briefly discuss and give the fundamental equations of the classical theory of elasticity, which are necessary in formulating the LBIEs. The fundamental (singular) solutions are derived using two functions, namely, the Galerkin vector and Dirac delta function⁽¹²⁾. The reciprocal relationship is Betti's theorem. As in the case of potential theory, these two items (the fundamental solutions and reciprocal relationship) are required to establish an integral equation for a point, p , within the domain, R . The higher-order BIEs are then obtained by simple differentiation of that integral equation, repeatedly, at the internal point, p . The kernel functions (fundamental solutions) are differentiated in the same way, and they are given in appendix E. For compactness, wherever necessary, tensor notation will be used with implied summation on repeated indices, and partial derivatives being denoted by the comma-index notation.

3.1.1. Review of the Governing Equations of Elasticity

The partial differential equations of equilibrium in tensor form are given by^(18,43)

$$\sigma_{ij,j} + X_i = 0 \quad (3.1)$$

where X_i are the components of the body force vector in the x_i coordinate directions, and σ_{ij} is the stress tensor. The Hooke's law constitutive equation for a homogeneous

isotropic and linearly elastic material undergoing small strains in the absence of thermal strain effects is given by

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \quad (3.2)$$

or

$$E \epsilon_{ij} = (1 + \nu) \sigma_{ij} - \nu \delta_{ij} \sigma_{kk} \quad (3.3)$$

where ϵ_{ij} is the strain tensor and the Lamé elastic constants μ and λ are given in terms of Young's modulus, E , and Poisson's ratio, ν , by

$$\mu = \frac{E}{2(1 + \nu)}, \quad \text{and} \quad \lambda = \frac{2\mu\nu}{(1 - 2\nu)} \quad (3.4)$$

In equation (3.2) and (3.3) δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (3.5)$$

Since the strain-displacement relations are

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (3.6)$$

eqn (3.2) may then be expressed in terms of displacements as

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) \quad (3.7)$$

Substitution of eqn (3.7) to the stress equilibrium eqn (3.1) yields the well-known Navier's equations of equilibrium for linear elasticity for a homogeneous isotropic body in terms of displacements :

$$\mu u_{i,jj} + \left(\frac{\mu}{1-2\nu} \right) u_{j,ji} + X_i = 0 \quad (3.8)$$

This can be written in the form (see pp 154-156, Fung, 1965)

$$\mu \nabla^2 u_i + \left(\frac{\mu}{1-2\nu} \right) e_{,i} + X_i = 0 \quad (3.9)$$

where

$$e = u_{j,j} \quad (3.10)$$

and

$$\nabla^2 u_i = u_{i,jj} \quad (3.11)$$

The quantity e called dilatation is the divergence of the displacement vector u_i , and in expanded form,

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad (3.12)$$

which represents, in the infinitesimal strain theory, the change of area per unit area of the surface under strain.

If the body force vanishes, $X_i = 0$, then by taking the divergence of eqn. (3.8), we have

$$\mu u_{i,jji} + \left(\frac{\mu}{1-2\nu}\right) u_{j,jii} = 0 \quad (3.13)$$

and by interchanging these indices, since they are dummy indices, results in

$$u_{j,jii} = 0 \quad (3.14)$$

which can be written as

$$\nabla^2 e = 0 \quad (3.15)$$

Hence, the dilatation e is a harmonic function when the body force $X_i = 0$.

Now, operating on eqn. (3.9) with the laplacian ∇^2 , we obtain

$$\mu \nabla^4 u_i + \left(\frac{\mu}{1-2\nu}\right) \nabla^2 e_{,i} = 0 \quad (3.16)$$

what amounts to the same thing is

$$\nabla^4 u_i = 0 \quad (3.17)$$

Thus, the displacement components, u_i , are biharmonic. It follows that the strain components and the stress components, which are the linear combination of the first derivatives of the displacement components, are also biharmonic functions

$$\nabla^4 \sigma_{ij} = 0 \quad (3.18)$$

and

$$\nabla^4 \epsilon_{ij} = 0 \quad (3.19)$$

The solution to eqn (3.8) for a given problem should fit the prescribed boundary conditions of that problem.

The boundary conditions are such that over every part of the boundary either displacements

$$u_i(x) = g_i(x) \quad (3.20)$$

or tractions

$$t_i(x) = \sigma_{ij}(x) n_j(x) = g_i(x) \quad (3.21)$$

are specified (or there exists a linear relationship between them). The traction vector on the surface, S , in eqn (3.21) can be found from the substitution of Hooke's law (3.7) into eqn (3.21) as

$$t_i(x) = g_i(x) = \left\{ \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) \right\} n_j \quad (3.22)$$

The displacements and stresses are always continuous since the Navier's equations of elasticity, eqns (3.8), are elliptic for $-1 < \nu \leq 0.5$ (see pp 73-74 Cruse, 1977).

3.1.2. Derivation of Kernel Functions (the Fundamental Solutions)

The fundamental solution of Laplace's equation, eqn (2.1), depends only on the distance $r(p,q)$. It is clear that at infinity the fundamental solution is zero and when two points p and q coincide it becomes singular. This solution is produced by a unit source at an arbitrary point q subject to the condition that the solution at infinity is zero. Similarly, the fundamental solutions of equations (3.8) are the displacement fields due to unit concentrated forces at q in the cartesian coordinate directions, and singular at this point and zero at infinity. Because the unit force is a vector, pointing in some particular direction, the fundamental solution is not a function of r alone, and can be obtained by using a potential displacement function, G_i , called the Galerkin vector. It is assumed that the Galerkin vector has continuous derivatives through the fourth order. The basic equations (3.8) of elasticity, namely, Navier's equations are satisfied when⁽¹²⁾

$$2\mu u_i = 2(1 - \nu) G_{i,jj} - G_{j,ij} \quad (3.23)$$

and if the G_i satisfy the inhomogeneous biharmonic equations

$$G_{i,jjkk} = -\frac{X_i}{\mu} \quad (3.24)$$

If the body forces $X_i = 0$, eqns (3.24) are then biharmonic and their solutions are called biharmonic functions. Thus, to solve a problem in static equilibrium is to determine

these biharmonic functions. The solutions of (3.24) are closely related to solutions of Laplace's equations* (see p. 75 Cruse, 1977).

In order to find the fundamental solution to Navier's equations, the body forces are replaced by the Dirac delta function $D(p,q)$ and using eqns (3.24) to define new vector functions F_i such that⁽¹²⁾

$$\nabla^2 F_i = -\frac{1}{\mu} D(p,q) e_i \quad (3.25)$$

where the Dirac vector function $D(p,q) e_i$ replaces X_i in eqns (3.24). The singular solution for each component of eqns (3.24) in two dimensions is given by⁽¹²⁾

$$F_i = \frac{1}{2\mu\pi} e_i \ln\left[\frac{1}{r(p,q)}\right] \quad (3.26)$$

such that the appropriate Galerkin vector function in the absence of body forces may be given (see Remzi, 1981) by

$$G_i(p,q) = \frac{1}{8\mu\pi} \left[1 + \ln\left(\frac{1}{r}\right)\right] r^2 e_i \quad (3.27)$$

where e_i is a set of unit vectors in the x_i directions, and $r(p,q)$ is defined as the distance between field point q and the load point p with coordinates x and y . This function satisfies the conditions that the stresses vanish at infinity and are singular at origin, p .

Differentiation of equation (3.27) and substitution into eqn (3.23) yields the fundamental solution in terms of displacements as

* The relation between Neuber-Papkovich formulation (harmonic functions) and Galerkin vector (biharmonic functions) was pointed out by R.D. Mindlin, 1936 and 1953.

$$u_i(p,q) = \frac{1}{8\mu\pi(1-\nu)} \left\{ (3-4\nu) \delta_{ij} \ln\left(\frac{1}{r}\right) + r_{,i} r_{,j} \right\} e_j \quad (3.28)$$

and the fundamental solution in terms of tractions is obtained by differentiating (3.28) and substituting them into eqn (3.22) as

$$t_i(p,q) = \frac{-1}{4\pi(1-\nu)} \left(\frac{1}{r} \right) \left\{ \frac{\partial r}{\partial n} \left[(1-2\nu) \delta_{ij} + 2 r_{,i} r_{,j} \right] - (1-2\nu) [n_j r_{,i} - n_i r_{,j}] \right\} e_j \quad (3.29)$$

In eqns (3.28) and (3.29) all the implied differentiations of $r(p,q)$ are with respect to the coordinates of the field point, q (see section 2.1.3), that is to say :

$$r_{,i} = \frac{\partial r}{\partial x_{qi}} = \frac{1}{r} (x_{qi} - x_{pi}) = -\frac{\partial r}{\partial x_{pi}} \quad (3.30)$$

in eqn (3.29) the normal, n_i , is taken at the field point q , and the normal derivative is

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial x_{qi}} n_i(q) \quad (3.31)$$

In tensor notation, since

$$u_i(p,q) = U_{ji}(p,q) e_j \quad (3.32)$$

and

$$t_i(p,q) = T_{ji}(p,q) e_j \quad (3.33)$$

then,

$$U_{ij}(p,q) = \frac{1}{8\mu\pi(1-\nu)} \left\{ (3-4\nu) \delta_{ij} \ln\left(\frac{1}{r}\right) + r_{,i} r_{,j} \right\} \quad (3.34)$$

and

$$T_{ij}(p,q) = \frac{-1}{4\pi(1-\nu)} \left(\frac{1}{r} \right) \left\{ \frac{\partial r}{\partial n} \left[(1-2\nu) \delta_{ij} + 2 r_{,i} r_{,j} \right] - (1-2\nu) \left[n_j r_{,i} - n_i r_{,j} \right] \right\} \quad (3.35)$$

where the second index, j , refers to the direction of the concentrated point load at p and the first index, i , corresponds to the directions of the respective displacements and tractions at q .

3.1.3. Betti's Reciprocal Work Theorem

After deriving the fundamental solutions (3.34) and (3.35) due to unit point loads we can now establish the reciprocal relationship which is also required to form the integral equations. This is provided by Betti's theorem and may be stated in terms of point forces as⁽¹⁶⁾

$$\sum_j F_j d'_j = \sum_j F'_j d_j \quad (3.36)$$

where the d_j are displacements in the directions of the forces F_j , the point j ranging from 1 to n at which the forces act on the body, R . In another words, eqn (3.36) states that in

a linear elastic solid, the work done by a set of forces acting through the corresponding displacements produced by a second set of forces is equal to the work done by the second set of forces acting through the corresponding displacements produced by the first set of forces⁽¹⁸⁾. In eqn (3.36) body forces were ignored, but if required they can also be considered. By taking into consideration that a force F in reality acts over a finite area and assuming the applied forces act only on the boundary of the solution domain, eqn (3.36) can be expressed in terms of displacements u_i and tractions t_i , which are equivalent to the displacement vector d and force vector F , respectively, in the form of

$$\int_S t_i u'_i dS = \int_S t'_i u_i dS \quad (3.37)$$

where the integrations are carried out over the boundary, S . In eqn (3.37) the two independent sets, namely, unprimed displacements u_i and tractions t_i , and primed ones u'_i and t'_i form the two sets we require in order to establish the integral identity.

Eqn (3.37) is in fact a direct extension of the divergence theorem (see eqn (2.2)) and can be proved by it. Let us recall eqn (2.2)

$$\int_R \text{div} F dR = \int_S F \cdot n dS \quad (2.2)$$

and rewrite it in the form of

$$\int_R F_{i,i} dR = \int_S F_i n_i dS \quad (3.38)$$

where, as indicated before, F is a vector field with continuous first partial derivatives in the domain, R .

Again, consider two equilibrium states of the elastic body. One with displacements u_i due to the body forces X_i and surface tractions t_i , and the other with displacements u'_i due to the body forces X'_i and surface tractions t'_i . Now, we included the body forces (although not necessary here). Hence, eqn (3.37) can be written as

$$\int_S t_i u'_i dS + \int_R X_i u'_i dR = \int_S t'_i u_i dS + \int_R X'_i u_i dR \quad (3.39)$$

Consider the left-hand side of this equation :

Since

$$t_i = \sigma_{ij} n_j \quad (3.21)$$

and

$$\sigma_{ij,j} = -X_i \quad (3.1)$$

then

$$\int_S t_i u'_i dS + \int_R X_i u'_i dR = \int_S \sigma_{ij} n_j u'_i dS - \int_R \sigma_{ij,j} u'_i dR \quad (3.40)$$

and apply the divergence theorem (3.38) to the first integral on the right-hand side

$$\begin{aligned}
\int_S \sigma_{ij} n_j u'_i dS &= \int_R (\sigma_{ij} u'_{i,j})_{,j} dR \\
&= \int_R \sigma_{ij,j} u'_i dR + \int_R \sigma_{ij} u'_{i,j} dR
\end{aligned} \tag{3.41}$$

Substitution of eqn (3.41) into eqn (3.40) results in

$$\int_S t_i u'_i dS + \int_R X_i u'_i dR = \int_R \sigma_{ij} u'_{i,j} dR \tag{3.42}$$

and similarly performing the same operations as above on the right-hand side of eqn (3.39), we have

$$\int_S t'_i u_i dS + \int_R X'_i u_i dR = \int_R \sigma'_{ij} u_{i,j} dR \tag{3.43}$$

Now if we can show that

$$\int_R \sigma_{ij} u'_{i,j} dR = \int_R \sigma'_{ij} u_{i,j} dR \tag{3.44}$$

then the proof will be completed. Let us apply Hooke's law (3.2) on the right-hand side of (3.44)

$$\begin{aligned}
\int_R \sigma_{ij} u'_{i,j} dR &= \int_R (\lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}) u'_{i,j} dR \\
&= \int_R [\lambda u'_{i,i} u_{k,k} + \mu (u_{i,j} + u_{j,i}) u'_{i,j}] dR
\end{aligned}$$

$$= \int_R \left[\lambda u'_{k,k} u_{k,k} + \mu u_{i,j} u'_{i,j} + \mu u_{j,i} u'_{i,j} \right] dR \quad (3.45)$$

and since

$$\mu u_{j,i} u'_{i,j} = \mu u_{i,j} u'_{j,i} \quad (3.46)$$

then, the integral (3.45) is symmetric with respect to the primed and unprimed states. Therefore, eqn (3.44) and hence eqn (3.39) are true. And in the absence of body forces we have eqn (3.37). Since it is assumed that the solution domain is free of body forces, only eqn (3.37) will be treated.

3.1.4. The Integral Equation for an Internal Point p

In equation (3.37), let the unprimed set be the required solutions to the problem under consideration, and primed set be the fundamental solutions (3.34) and (3.35). Navier's equation (3.8), which is the governing partial differential equation, is satisfied by the fundamental solutions (3.34) and (3.35) everywhere except at the origin, p . If a small circle of radius ϵ , surrounding p , is extracted from the region, R , Betti's theorem may be applied to the region, $R-R_\epsilon$, bounded by the circle S_ϵ and the curve S by integrating along the boundaries in the directions as shown in Fig. 2.2. Equation (3.37) may then be rewritten as

$$\int_{S+S_\epsilon} t_i u'_i dS = \int_{S+S_\epsilon} t'_i u_i dS \quad (3.47)$$

unless the point p is on the boundary of the solution domain, the integrals over the boundary surface vary smoothly and present no difficulties, since there are no singularities. The same limiting argument as $\epsilon \rightarrow 0$ as in potential theory (see section

2.1.1) is used. Note that both the functions t_i and u_i are continuous in R , and take the values

$$t_i \equiv t_i(q) \quad \text{and} \quad u_i \equiv u_i(q) \quad \text{on } S_\epsilon \quad (3.48)$$

and as $q \rightarrow p$ (or $\epsilon \rightarrow 0$)

$$t_i \equiv t_i(p) \quad \text{and} \quad u_i \equiv u_i(p). \quad (3.49)$$

Hence, the first integral on S_ϵ in eqn (3.47) in the limit as $\epsilon \rightarrow 0$ is

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} t_i u_i' dS = \lim_{\epsilon \rightarrow 0} \left\{ t_i(p) \left[C_1 \ln\left(\frac{1}{\epsilon}\right) + C_2 \right] 2\pi\epsilon \right\} = 0 \quad (3.50)$$

since the displacement fundamental solution can be written as

$$u_i' = C_1 \ln\left(\frac{1}{\epsilon}\right) + C_2 \quad (3.51)$$

where C_1 and C_2 are constants, and $dS = \epsilon d\theta$.

The second integral on S_ϵ in eqn (3.47) may then be written as

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} t_i' u_i dS = u_i(p) \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} t_i' dS \quad (3.52)$$

but, since by definition the traction fundamental solution components t'_i are the components of concentrated point load of unity in each of the coordinate directions, then

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} t'_i dS = \delta_{ij} e_j \quad (3.53)$$

and eqn (3.52) becomes

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} t'_i u_i dS = u_i(p) \delta_{ij} e_j = u_j(p) e_j \quad (3.54)$$

Therefore, substitution of these results, eqns (3.50) and (3.54), into eqn (3.47) in the limit as $\epsilon \rightarrow 0$ yields

$$\int_S t_i u'_i dS = u_j(p) e_j + \int_S u_i t'_i dS \quad (3.55)$$

Let us recall the fundamental solutions (3.32) and (3.33)

$$u_i(p,q) \equiv u'_i(p,q) = U_{ji}(p,q) e_j \quad (3.32)$$

$$t_i(p,q) \equiv t'_i(p,q) = T_{ji}(p,q) e_j \quad (3.33)$$

Now, if Q is a point on the boundary of the solution domain eqn (3.55) (by interchanging the indices) can be expressed by

$$u_i(p) = \int_S t_j(Q) U_{ij}(p, Q) dS(Q) - \int_S u_j(Q) T_{ij}(p, Q) dS(Q) \quad (3.56)$$

This is the integral equation which is used in a local BIE treatment as the basis of the method, and it is known as Somigliana's identity for displacements. If the boundary values $u_j(Q)$ and $t_j(Q)$ are known, the value of u_i at p can be determined with the aid of eqn (3.56). In a conventional BIE treatment eqn (3.56) is developed into a form which involves integrations over only the boundary of the region of interest and allows those boundary values to be determined from the solution of it. It is given here without considering mathematical details (Remzi, 1981)

$$C_{ij}(P) u_j(P) = \int_S t_j(Q) U_{ij}(P, Q) dS(Q) - \int_S u_j(Q) T_{ij}(P, Q) dS(Q) \quad (3.57)$$

where P is now a boundary point, and $C_{ij}(P)$ is

$$C_{ij}(P) = \lim_{\epsilon \rightarrow 0} \left(- \int_{S_\epsilon} T_{ij}(P, Q) dS(Q) \right) \quad (3.58)$$

S_ϵ being the boundary of a region R_ϵ , which is the part of R contained within a circle of radius ϵ centered at P (see Fig. 3.1, below).

If the tangent plane at P is unique

$$C_{ij}(P) = \frac{1}{2} \delta_{ij} \quad (3.59)$$

and if the tangent plane is not unique, $C_{ij}(p)$ can be calculated indirectly by considering the rigid body translation (see section 2.1.1).

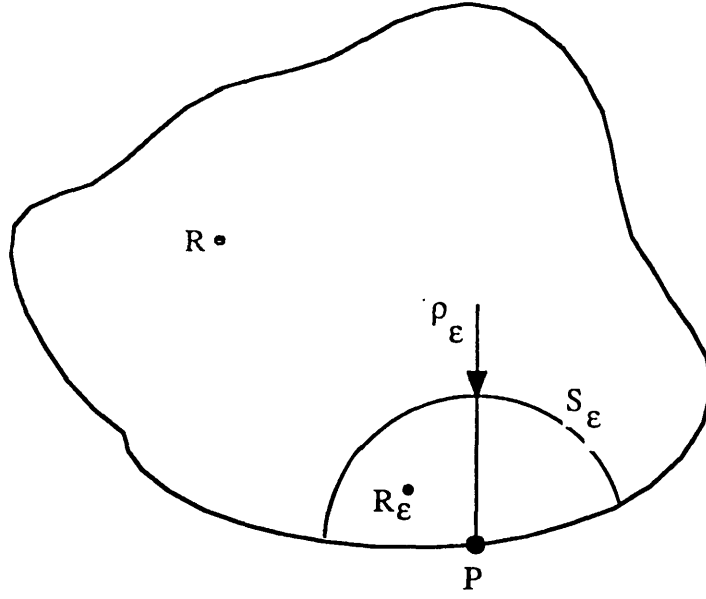


Figure 3.1 : The domain R_ϵ isolating P .

In order to obtain stresses within the solution domain, in conventional BIE methods, eqn (3.56) is differentiated at p and substituted into Hooke's law, eqn (3.7), which gives

$$\begin{aligned} \sigma_{ij}(p) = & \int_S t_k \left[\lambda \delta_{ij} U_{mk,m} + \mu (U_{ik,j} + U_{jk,i}) \right] dS - \\ & \int_S u_k \left[\lambda \delta_{ij} T_{mk,m} + \mu (T_{ik,j} + T_{jk,i}) \right] dS \end{aligned} \quad (3.60)$$

which may be written in the abbreviated form as

$$\sigma_{ij}(p) = \int_S t_k(Q) D_{kij}(p, Q) dS(Q) - \int_S u_k(Q) S_{kij}(p, Q) dS(Q) \quad (3.61)$$

where the third-order tensors D_{kij} and S_{kij} are given by

$$D_{kij}(p,Q) = \frac{1}{4\pi(1-\nu)} \left(\frac{1}{r}\right) \left\{ (1-2\nu) \left[\delta_{ki} r_{,j} + \delta_{kj} r_{,i} - \delta_{ij} r_{,k} \right] + 2 r_{,i} r_{,j} r_{,k} \right\} \quad (3.62)$$

and

$$S_{kij}(p,Q) = \frac{2\mu}{4\pi(1-\nu)} \left(\frac{1}{r^2}\right) \left\{ 2 \frac{\partial r}{\partial n} \left[(1-2\nu) \delta_{ij} r_{,k} + \nu (\delta_{ik} r_{,j} + \delta_{jk} r_{,i}) - 4 r_{,i} r_{,j} r_{,k} \right] + 2\nu (n_i r_{,j} r_{,k} + n_j r_{,i} r_{,k}) + (1-2\nu) \left[2 n_{,k} r_{,i} r_{,j} + n_j \delta_{ik} + n_i \delta_{jk} \right] - (1-4\nu) n_k \delta_{ij} \right\} \quad (3.63)$$

and the surface stresses may be evaluated by taking the limiting form of Somigliana's identity for stresses, eqn (3.61), as $p \rightarrow P$, being a boundary point, and applying the discretisation procedures. This gives high resolution of stresses, but is costly due to the need to carry out integrations (see Remzi, 1981).

Therefore, after the equations have been solved, the stresses at surface points may be obtained by substitution of the already calculated values of tangential strains and tractions into Hooke's law, eqn (3.7).

In an isoparametric formulation the tangential strains are found from the differentiation of the equation⁽³⁴⁾

$$u_i(\xi) = N^c(\xi) u_i^c \quad (3.64)$$

which, by using Hooke's law, gives the surface stresses in the local coordinate system and a simple transformation then yields the stresses in the global coordinate directions.

3.1.5. Higher-Order BIEs

Somigliana's identity for displacements, eqn (3.56), is differentiated with respect to the position of a suitably located point in the interior of the solution domain in order to obtain higher-order integral equations which are used in the series expansions, as was done in section 2.1.2 for potential theory. By adopting a different notation from the one used previously in section 2.1.2, for the sake of convenience, the differentiation of eqn (3.56) results in

$$u_{i,\partial}(p) = \int_S t_j(Q) U_{ij,\partial}(p,Q) dS(Q) - \int_S u_j(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.65)$$

where ∂ implies the partial differentiation of any order in the coordinate directions x and y at p , and can be expressed by

$$\partial = k \text{ } l \text{ } m \text{ } \dots\dots \quad (3.66)$$

where the indices $k, l, m, \dots = x, y$ (or $1, 2$) in two dimensions.

It is obvious that during the differentiation of eqn (3.56) $u_j(Q)$ and $t_j(Q)$ behave as constants since they are independent of the position of p (see section 2.1.2).

3.1.6. Derivatives of Kernel Functions

The derivatives of the kernels in the integrands in equations (3.65) are required. As mentioned in section 2.1.3, one way of differentiating is to make use of finite difference approximations. Since the expressions of the kernels, eqns. (3.34) and (3.35), are complicated and lengthy, unlike eqn. (2.1) of potential theory, finite difference approximations seem quite attractive. The other way is to differentiate them analytically. Analytical differentiation is rather tedious and time-consuming. The choice of course depends on the accuracy of high-order function differencing. In order to get

high accuracy in differencing the kernels Watson (1982) suggested, see section 2.1.3, multiplying the distance between field point and source point by 0.0002 to determine the distance between the two points at which the kernel functions are computed. Finite difference approximation, however, introduces an error additional to discretisation and numerical integration error, no matter how small it is. Also, in order to best assess the accuracy of the method, avoiding the error of numerical approximations as much as possible, analytical differentiation is performed. Since derivatives consist of very large expressions they are given in appendix E.

3.2. Numerical Solution

Detailed information about the solutions of conventional BIE methods, their historical development, and general implementation of the local BIE method numerically has been given in section 2.2, therefore no further explanation is given here. The same procedures necessarily apply to solve stress problems numerically. However, the formulation of LBIEs is now possible by three approaches, each having some advantages over each other. They are as follows.

3.2.1. Airy's Stress Function Approach

The differential equation of equilibrium in the absence of body forces is, in expanded form (see eqn (3.1)),

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0\end{aligned}\tag{3.1a}$$

The usual method of solving these equations is by introducing a new function ϕ , called stress function or Airy's stress function, which satisfies the above equations and

also the strain compatibility equation. It is evident that equations (3.1a) are satisfied if the stresses are expressed as

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \text{and} \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (3.67)$$

If these expressions are substituted into the strain compatibility equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0 \quad (3.68)$$

we see that stress function ϕ must satisfy the biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (3.69)$$

Thus, if the stress function, ϕ , can be found for a two-dimensional elastostatics problem such that it satisfies both the biharmonic equation $\nabla^4 \phi = 0$ and given boundary conditions, then the solution to the problem has been obtained.

3.2.1.1. Plane Strain Case :

If a body, the dimension of which is very large in the third coordinate direction z e.g. a long cylindrical or prismatic body, is loaded by forces which are perpendicular to the longitudinal elements and do not vary along the length, it may be assumed that all cross sections are in the same condition, and the displacements* are such that $w=0$ and hence u and v are functions of x and y only. Then, from the strain-displacement relations (3.6) we have

* The displacements in the coordinate directions x, y, z are denoted as u, v, w , respectively.

$$\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \quad (3.70)$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

and the *shear* stresses which act over the cross sections and at the fixed ends can be found from Hooke's law (3.2) and (3.3) as

$$\sigma_{xz} = \sigma_{yz} = 0, \quad \sigma_{xy} = 2\mu \epsilon_{xy} \quad (3.71)$$

and

$$\begin{aligned} E' \frac{\partial u}{\partial x} &= \sigma_{xx} - \nu' (\sigma_{yy} + \sigma_{zz}) \\ E' \frac{\partial v}{\partial y} &= \sigma_{yy} - \nu' (\sigma_{xx} + \sigma_{zz}) \end{aligned} \quad (3.72)$$

$$0 = \sigma_{zz} - \nu' (\sigma_{xx} + \sigma_{yy})$$

from the last equation of (3.72) finding σ_{zz} as

$$\sigma_{zz} = \nu' (\sigma_{xx} + \sigma_{yy}) \quad (3.73)$$

Thus, the plane strain problem reduces to the determination of σ_{xx} , σ_{yy} , and σ_{xy} as functions of x and y only.

Substitution of eqn (3.73) into the first two equations of (3.72) and dividing them by $(1 + \nu')$ results in

$$2 \mu \frac{\partial u}{\partial x} = (1 - \nu') \sigma_{xx} - \nu' \sigma_{yy} \quad (3.74)$$

$$2 \mu \frac{\partial v}{\partial y} = (1 - \nu') \sigma_{yy} - \nu' \sigma_{xx}$$

Let us write

$$P = \nabla^2 \phi \quad (3.75)$$

which is a harmonic function since $\nabla^2 P = \nabla^4 \phi = 0$ and represents $(\sigma_{xx} + \sigma_{yy})$, see compatibility equations (3.68). Now, if we insert the stress function, eqns (3.67), in equations (3.74), and use the definition (3.75), we then have

$$2 \mu \frac{\partial u}{\partial x} = (1 - \nu') P - \frac{\partial^2 \phi}{\partial x^2} \quad (3.76)$$

$$2 \mu \frac{\partial v}{\partial y} = (1 - \nu') P - \frac{\partial^2 \phi}{\partial y^2}$$

3.2.1.2. Plane Stress Case :

It is assumed here that the stress components σ_{xx} , σ_{yy} , and σ_{xy} are independent of the third coordinate direction z , i.e., they do not vary through the third dimension (e.g., the thickness of a plate). Although variation of stress does occur, this assumption is valid for sufficiently thin plates. Then,

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \quad (3.77)$$

and from Hooke's law (3.3)

$$\epsilon_{xz} = \epsilon_{yz} = 0, \quad \epsilon_{xy} = \frac{1}{2\mu} \sigma_{xy} \quad (3.78)$$

and

$$\begin{aligned} E \frac{\partial u}{\partial x} &= \sigma_{xx} - \nu \sigma_{yy} \\ E \frac{\partial v}{\partial y} &= \sigma_{yy} - \nu \sigma_{xx} \\ E \frac{\partial w}{\partial z} &= -\nu (\sigma_{xx} + \sigma_{yy}) \end{aligned} \quad (3.79)$$

Again, by introducing the stress function ϕ , in (3.79) and using (3.75) we have

$$\begin{aligned} E \frac{\partial u}{\partial x} &= P - (1 + \nu) \frac{\partial^2 \phi}{\partial x^2} \\ E \frac{\partial v}{\partial y} &= P - (1 + \nu) \frac{\partial^2 \phi}{\partial y^2} \end{aligned} \quad (3.80)$$

Now, comparing equations (3.76) with equations (3.80) we see that plane stress and plane strain cases can be represented by the same equation provided

$$v' = \frac{v}{1+v} \quad \text{and} \quad E' = E(1 - v'^2) \quad (3.81)$$

are substituted into equations (3.76) to get the plane stress case, equations (3.80).

3.2.1.3. Determination of Displacements :

The derivative forms of displacements (strains) were obtained above for both the plane strain and plane stress cases. But if the displacements are of interest to us, and they are indeed required in BIEs, we then must integrate the strains ϵ_{xx} in the x and ϵ_{yy} in the y coordinate directions to obtain the displacements.

Let us introduce an analytic function $f(z)$, of which P is the real part and Q imaginary (indeed, since P is harmonic it will have a conjugate harmonic function, Q , see pp 170-171 Timoshenko & Goodier, 1984). It is known that the integral (or derivative) of an analytic function is another analytic function, say $\phi(z)$.

Hence,

$$\phi(z) = \rho_x + i\rho_y = \frac{1}{4} \int f(z) dz = \frac{1}{4} \int (P + iQ) dz \quad (3.82)$$

and differentiating this, we also have

$$\frac{\partial \rho_x}{\partial x} + i \frac{\partial \rho_y}{\partial x} = \frac{\partial \phi(z)}{\partial x} = \frac{d\phi(z)}{dz} \frac{\partial z}{\partial x} \quad (3.83)$$

but

$$\frac{\partial z}{\partial x} = 1 \quad \text{since} \quad z = x + iy \quad (3.84)$$

then obviously

$$\frac{\partial \rho_x}{\partial x} + i \frac{\partial \rho_y}{\partial x} = \varphi'(z) = \frac{1}{4} (P + iQ) \quad (3.85)$$

by comparing the real and imaginary parts of both sides we obtain

$$\frac{\partial \rho_x}{\partial x} = \frac{1}{4} P \quad (3.86)$$

$$\frac{\partial \rho_y}{\partial x} = \frac{1}{4} Q$$

because the real and imaginary parts of an analytic function satisfy Cauchy-Riemann conditions (i.e., they are conjugate to each other), we also have

$$\frac{\partial \rho_x}{\partial x} = \frac{\partial \rho_y}{\partial y} = \frac{1}{4} P \quad (3.87)$$

and

$$\frac{\partial \rho_x}{\partial y} = -\frac{\partial \rho_y}{\partial x} = -\frac{1}{4} Q \quad (3.88)$$

By substituting these into equations (3.76) and integrating them, u and v are obtained as

$$2\mu u = -\frac{\partial \phi}{\partial x} + 4(1 - \nu') \rho_x \quad (3.89)$$

$$2\mu v = -\frac{\partial \phi}{\partial y} + 4(1 - \nu') \rho_y$$

where, from now on, ν' will be written as ν . Hence, in tensor notation

$$2\mu u_i = -\phi_{,i} + 4(1 - \nu) \rho_i \quad (3.90)$$

3.2.1.4. Determination of the Functions ρ_x and ρ_y :

In determining the displacements u and v in equations (3.90) we need to know the values of ρ_x and ρ_y . They can easily be found by noting the relation between P and ϕ , hence between ρ_i and ϕ . Once P and Q are known, then ρ_x and ρ_y can be determined with the aid of equations (3.87) and (3.88).

Now, let us recall the relation (3.75),

$$P = \nabla^2 \phi \quad (3.75)$$

and expand the stress function, ϕ , in series (either Taylor or Fourier ones) with a finite number of terms, which satisfy biharmonic equation (3.69). Equation (2.81) may then be recalled

$$\phi(x,y) = \sum_{m=1}^N \alpha_m f^m(x,y) \quad (2.81)$$

where α_m are the constants to be determined from the solution of BIEs, and since $\nabla^4\phi=0$, the functions f^m must now satisfy $\nabla^4f^m=0$. These functions from Taylor series are given in Table 3.1

It is easy to show that one may determine for any harmonic function P a function Q , conjugate to it, by

$$\frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x} \quad (3.91)$$

$$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$$

these are the Cauchy-Riemann conditions which relate the real and imaginary parts of an analytic function, say $f(z)$ (see equation (3.82)). It is straightforward to prove (3.91), and hence eqns. (3.87) and (3.88), by differentiating $f(z)$ with respect to the coordinate directions x and y as follows,

$$\frac{\partial f(z)}{\partial x} = \frac{df(z)}{dz} \frac{\partial z}{\partial x} = f'(z) \quad (3.92)$$

and

$$\frac{\partial f(z)}{\partial y} = \frac{df(z)}{dz} \frac{\partial z}{\partial y} = if'(z) \quad (3.93)$$

and since

$$\frac{\partial f(z)}{\partial x} = \frac{\partial P}{\partial x} + i \frac{\partial Q}{\partial x} \quad \text{and} \quad \frac{\partial f(z)}{\partial y} = \frac{\partial P}{\partial y} + i \frac{\partial Q}{\partial y} \quad (3.94)$$

Table 3.1 : The polynomial functions of Taylor series, eqn (2.69), and their derivatives, which satisfy the biharmonic equation (3.69).

m	1	2	3	4	5	6	7	8	9	10	11	12	13
$f^m_{,00} =$	1	x	y	$x^2/2$	xy	$y^2/2$	$x^3/6$	$x^2y/2$	$xy^2/2$	$y^3/6$	$(x^4 - 3x^2y^2)/24$	$x^3y/6$	$xy^3/6$
$f^m_{,10} =$	0	1	0	x	y	0	$x^2/2$	xy	$y^2/2$	0	$(4x^3 - 6xy^2)/24$	$x^2y/2$	$y^3/6$
$f^m_{,01} =$	0	0	1	0	x	y	0	$x^2/2$	xy	$y^2/2$	$-x^2y/4$	$x^3/6$	$xy^2/2$
$f^m_{,20} =$	0	0	0	1	0	0	x	y	0	0	$(12x^2 - 6y^2)/24$	xy	0
$f^m_{,11} =$	0	0	0	0	1	0	0	x	y	0	$-xy/2$	$x^2/2$	$y^2/2$
$f^m_{,02} =$	0	0	0	0	0	1	0	0	x	y	$-x^2/4$	0	xy
$f^m_{,30} =$	0	0	0	0	0	0	1	0	0	0	x	y	0
$f^m_{,21} =$	0	0	0	0	0	0	0	1	0	0	$-y/2$	x	0
$f^m_{,12} =$	0	0	0	0	0	0	0	0	1	0	$-x/2$	0	y
$f^m_{,03} =$	0	0	0	0	0	0	0	0	0	1	0	0	x
$f^m_{,40} =$	0	0	0	0	0	0	0	0	0	0	1	0	0
$f^m_{,31} =$	0	0	0	0	0	0	0	0	0	0	0	1	0
$f^m_{,13} =$	0	0	0	0	0	0	0	0	0	0	0	0	1
$f^m_{,04} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,50} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,41} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,14} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,05} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,60} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,51} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,15} =$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f^m_{,06} =$	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.1 : Continued.

m	14	15	16	17	18
$f_{,00}^m =$	$(y^4 - 3x^2y^2)/24$	$(x^5 - 5x^3y^2)/120$	$(x^4y - 5x^2y^3)/24$	$(y^4x - y^2x^3)/24$	$(y^5 - 5y^3x^2)/120$
$f_{,10}^m =$	$-xy^2/4$	$(x^4 - 3x^2y^2)/24$	$(2x^3y - xy^3)/12$	$(y^4 - 3x^2y^2)/24$	$-xy^3/12$
$f_{,01}^m =$	$(6y^3 - 6x^2y)/24$	$-yx^3/12$	$(x^4 - 3x^2y^2)/24$	$(2xy^3 - x^3y)/12$	$(y^4 - 3x^2y^2)/24$
$f_{,20}^m =$	$-x^2/4$	$(4x^3 - 6y^2x)/24$	$(6x^2y - y^3)/12$	$-xy^2/4$	$-y^3/12$
$f_{,11}^m =$	$-xy/2$	$-x^2y/4$	$(4x^3 - 6y^2x)/24$	$(2y^3 - 3x^2y)/12$	$-xy^2/4$
$f_{,02}^m =$	$(12y^2 - 6x^2)/24$	$-x^3/12$	$-x^2y/4$	$(6xy^2 - x^3)/12$	$(4y^3 - 6x^2y)/24$
$f_{,30}^m =$	0	$(2x^2 - y^2)/4$	xy	$-y^2/4$	0
$f_{,21}^m =$	$-y/2$	$-xy/2$	$(2x^2 - y^2)/4$	$-xy/2$	$-y^2/4$
$f_{,12}^m =$	$-x/2$	$-x^2/4$	$-xy/2$	$(2y^2 - x^2)/4$	$-xy/2$
$f_{,03}^m =$	y	0	$-x^2/4$	xy	$(2y^2 - x^2)/4$
$f_{,40}^m =$	0	x	y	0	0
$f_{,31}^m =$	0	$-y/2$	x	$-y/2$	0
$f_{,13}^m =$	0	0	$-x/2$	y	$-x/2$
$f_{,04}^m =$	1	0	0	x	y
$f_{,50}^m =$	0	1	0	0	0
$f_{,41}^m =$	0	0	1	0	0
$f_{,14}^m =$	0	0	0	1	0
$f_{,05}^m =$	0	0	0	0	1
$f_{,60}^m =$	0	0	0	0	0
$f_{,51}^m =$	0	0	0	0	0
$f_{,15}^m =$	0	0	0	0	0
$f_{,06}^m =$	0	0	0	0	0

Table 3.1 : Continued.

m	19	20	21	22
$f^m_{,00} =$	$(x^6 - 10x^4y^2 + 5x^2y^4)/720$	$(6x^5y - 10x^3y^3)/720$	$(6xy^5 - 10x^3y^3)/720$	$(y^6 - 10x^2y^4 + 5x^4y^2)/720$
$f^m_{,10} =$	$(6x^5 - 40x^3y^2 + 10xy^4)/720$	$(x^4y - x^2y^3)/24$	$(y^5 - 5x^2y^3)/120$	$(x^3y^2 - xy^4)/36$
$f^m_{,01} =$	$(x^2y^3 - yx^4)/36$	$(x^5 - 5x^3y^2)/120$	$(xy^4 - x^3y^2)/24$	$(6y^5 - 40x^2y^3 + 10yx^4)/720$
$f^m_{,20} =$	$(3x^4 - 12x^2y^2 + y^4)/72$	$(2yx^3 - y^3x)/12$	$-xy^3/12$	$(3x^2y^2 - y^4)/36$
$f^m_{,11} =$	$(xy^3 - 2x^3y)/18$	$(x^4 - 3x^2y^2)/24$	$(y^4 - 3x^2y^2)/24$	$(yx^3 - 2xy^3)/18$
$f^m_{,02} =$	$(3x^2y^2 - x^4)/36$	$-x^3y/12$	$(2xy^3 - x^3y)/12$	$(3y^4 - 12x^2y^2 + x^4)/72$
$f^m_{,30} =$	$(x^3 - 2xy^3)/6$	$(6x^2y - y^3)/12$	$-y^3/12$	$xy^2/6$
$f^m_{,21} =$	$(y^3 - 6yx^2)/18$	$(2x^3 - 3xy^2)/12$	$-xy^2/4$	$(6x^2y - 4y^3)/36$
$f^m_{,12} =$	$(6xy^2 - 4x^3)/36$	$-yx^2/4$	$(2y^3 - 3yx^2)/12$	$(x^3 - 6xy^2)/18$
$f^m_{,03} =$	$yx^2/6$	$-x^3/12$	$(6xy^2 - x^3)/12$	$(y^3 - 2yx^2)/6$
$f^m_{,40} =$	$(3x^2 - 2y^2)/6$	xy	0	$y^2/6$
$f^m_{,31} =$	$-4xy/6$	$(2x^2 - y^2)/4$	$-y^2/4$	$xy/3$
$f^m_{,13} =$	$xy/3$	$-x^2/4$	$(2y^2 - x^2)/4$	$-4xy/6$
$f^m_{,04} =$	$x^2/6$	0	xy	$(3y^2 - 2x^2)/6$
$f^m_{,50} =$	x	y	0	0
$f^m_{,41} =$	$-4y/6$	x	0	$y/3$
$f^m_{,14} =$	$x/3$	0	y	$-4x/6$
$f^m_{,05} =$	0	0	x	y
$f^m_{,60} =$	1	0	0	0
$f^m_{,51} =$	0	1	0	0
$f^m_{,15} =$	0	0	1	0
$f^m_{,06} =$	0	0	0	1

then, by comparing (3.92) with (3.93), we arrive at equations (3.91). If the same procedure is applied to $\varphi(z)$, see eqn. (3.82), we then obtain equations (3.87) and (3.88). In order to ensure the displacements u and v are single-valued in a simply connected domain, R , the functions ρ_x and ρ_y must be single-valued. Assume that the function $f(z)$ is single-valued and its differential is therefore exact

$$df(z) = df(x,y) = \frac{\partial f}{\partial x} dx + i \frac{\partial f}{\partial y} dy \quad (3.95)$$

where

$$\frac{\partial f}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y) \quad (3.96)$$

It is of interest that on the basis of Green's formula in two dimensions⁽³¹⁾

$$\int_S (M dx + N dy) = \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dR \quad (3.97)$$

the integral on the right-hand side vanishes for every part of the region R , if we have, throughout R ,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad (3.98)$$

which is the case here, when the differential, eqn (3.95), is exact. Note that if the first part of eqn (3.96) is differentiated with respect to direction y and the second part with respect to x we then arrive at eqn (3.98). Thus eqn (3.98) is the necessary and sufficient

condition for the existence of the function $f(x,y)$. In fact, if the condition (3.98) is satisfied the line integral on the left-hand side of eqn (3.97)

$$\int_S (M \, dx + N \, dy) \tag{3.97a}$$

does not depend on the path of integration, but on the starting and end points of this path. This can directly be shown as follows.

Consider any closed contour S within the region R as shown below.

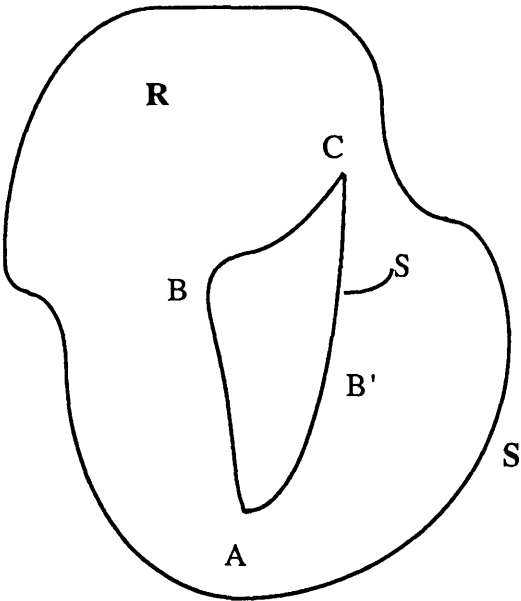


Figure 3.2 : Arbitrary paths of integration, within R.

The integral (3.97a) taken along S must then be equal to zero. One may write, by supposition,

$$\int_{ACB'A} (M dx + N dy) = 0 \quad (3.99)$$

or

$$\int_{ABC} df + \int_{CB'A} df = 0 \quad (3.100)$$

and by reversing the direction of integration in the second integral, that is to say, since

$$\int_{CB'A} df = - \int_{AB'C} df \quad (3.101)$$

then , one must have

$$\int_{ABC} df = \int_{AB'C} df \quad (3.102)$$

as was to be proved.

Therefore, if the integrand in (3.97a) is an exact differential the function $f(z)$ is then single valued. In fact, it follows from the representation (3.67) itself that $f(z)$ is always single-valued. Note that since the second derivatives of the biharmonic function ϕ are related to stresses by (3.67), the function $P = \nabla^2 \phi$ (eqn. 3.75) is necessarily single-valued (see p. 262, Sokolnikoff, 1956). If P is known its conjugate Q is determined by the relations (3.91) to within an arbitrary constant C , and hence $f(z) = P + iQ$, to within an imaginary constant C_i . If the region R is simply connected P , and hence $f(z)$, is single-valued. Furthermore, in a multiply connected region $f(z)$ is also

single-valued, if we guarantee that the integrals taken over each boundary of the solution domain is zero (see pp 697-707, Muskhelishvili, 1954). This can be managed by introducing some imaginary cuts to make the multiply connected region simply connected one. On the other words, the integration is carried out along the boundaries such that the solution domain always lies on the left of the direction of integration.

In the same way ρ_x and ρ_y , which are also harmonic (analytic) functions, are single-valued.

Hence, the displacements u_i should be continuous and single-valued.

In the light of what has been said above, the conditions (3.87), (3.88), and (3.91) can now be conveniently used to form Table 3.3, where the values of P , Q , ρ_x , and ρ_y from Taylor series are represented.

If the trial function used to approximate the stress function ϕ is Fourier series, then the sine Fourier series (in x) may be employed in the form of (see p. 53 Timoshenko & Goodier, 1984)

$$\phi(x,y) = \sum_{m=1}^N \sin(mx) [\alpha_{m1} \sinh(my) + \alpha_{m2} \cosh(my) + \alpha_{m3} y \sinh(my) + \alpha_{m4} y \cosh(my)] \quad (3.103)$$

where α_{m1} , α_{m2} , α_{m3} , and α_{m4} are the constants, and m is an integer.

Equation (3.103) satisfies the biharmonic eqn (3.69) $\nabla^4\phi=0$, and can be obtained by taking the function ϕ to be of

$$\phi(x,y) = \sin(mx) f(y) \quad (3.104)$$

Table 3.3 : The real functions P^m , ρ_x^m and their conjugates Q^m , ρ_y^m from the Taylor series, which satisfy the biharmonic function (3.69).

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$P^m =$	0	0	0	1	0	1	x	y	x	y	$(x^2 - y^2)/4$	xy	xy	$(y^2 - x^2)/4$	$(x^3 - 3xy^2)/12$
$Q^m =$	0	0	0	1	0	1	y	-x	y	-x	xy/2	$(y^2 - x^2)/2$	$(y^2 - x^2)/2$	-xy/2	$(3x^2y - y^3)/12$
$\rho_x^m =$	0	0	0	x/4	0	x/4	$(x^2 - y^2)/8$	xy/4	$(x^2 - y^2)/8$	xy/4	$(x^3 - 3xy^2)/48$	$(3x^2y - y^3)/24$	$(3x^2y - y^3)/24$	$(3xy^2 - x^3)/48$	$(x^4 - 6x^2y^2 + y^4)/192$
$\rho_y^m =$	0	0	0	y/4	0	y/4	xy/4	$(y^2 - x^2)/8$	xy/4	$(y^2 - x^2)/8$	$(3x^2y - y^3)/48$	$(3xy^2 - x^3)/24$	$(3xy^2 - x^3)/24$	$(y^3 - 3x^2y)/48$	$(x^3y - xy^3)/48$

m	16	17	18	19	20	21	22
$P^m =$	$(3x^2y - y^3)/12$	$(3xy^2 - x^3)/12$	$(y^3 - 3x^2y)/12$	$(x^4 - 6x^2y^2 + y^4)/72$	$(x^3y - xy^3)/12$	$(xy^3 - x^3y)/12$	$(x^4 - 6x^2y^2 + y^4)/72$
$Q^m =$	$(3xy^2 - x^3)/12$	$(y^3 - 3x^2y)/12$	$(x^3 - 3xy^2)/12$	$(x^3y - xy^3)/18$	$(6x^2y^2 - x^4 - y^4)/48$	$(x^4 + y^4 - 6x^2y^2)/48$	$(x^3y - xy^3)/18$
$\rho_x^m =$	$(x^3y - xy^3)/48$	$(6x^2y^2 - x^4 - y^4)/192$	$(xy^3 - x^3y)/48$	$(x^5 - 10x^3y^2 + 5xy^4)/1440$	$(5x^4y - 10x^2y^3 + y^5)/960$	$(10x^2y^3 - 5x^4y - y^5)/960$	$(x^5 - 10x^3y^2 + 5xy^4)/1440$
$\rho_y^m =$	$(6x^2y^2 - x^4 - y^4)/192$	$(xy^3 - x^3y)/48$	$(x^4 - 6x^2y^2 + y^4)/192$	$(5x^4y - 10x^2y^3 + y^5)/1440$	$(10x^3y^2 - 5xy^4 - x^5)/960$	$(x^5 - 10x^3y^2 + 5xy^4)/960$	$(5x^4y - 10x^2y^3 + y^5)/1440$

in which $f(y)$ is a function of y only. Substituting (3.104) into (3.69) we find the following equation for determining $f(y)$

$$m^4 f(y) - 2m^2 f''(y) + f^{IV}(y) = 0 \quad (3.105)$$

The general solution of this linear differential equation with constant coefficients is

$$f(y) = [\alpha_{m1} \sinh(my) + \alpha_{m2} \cosh(my) + \alpha_{m3} y \sinh(my) + \alpha_{m4} y \cosh(my)] \quad (3.106)$$

Hence, eqn (3.104) takes the form of (3.103) as m is a finite number ranging from 1 to N . The required derivatives can be obtained by differentiating (3.103) with respect to the coordinate directions x and y of the point p , or Q (see eqns (2.82) and (2.83)).

In order to determine the functions p_x and p_y in eqn (3.90), we need to find first the functions P and Q . Since $P = \nabla^2 \phi$, eqn (3.75), then

$$P(x,y) = \sum_{m=1}^N 2m \sin(mx) [\alpha_{m3} \cosh(my) + \alpha_{m4} \sinh(my)] \quad (3.107)$$

and its conjugate Q can be found with the aid of eqns (3.91) as

$$Q(x,y) = \sum_{m=1}^N 2m \cos(mx) [\alpha_{m3} \sinh(my) + \alpha_{m4} \cosh(my)] \quad (3.108)$$

Substituting eqns (3.107) and (3.108) into eqns (3.87) and (3.88), respectively, and carrying out the indicated operations results in

$$\rho_x(x,y) = - \sum_{m=1}^N \frac{1}{2} \cos(mx) [\alpha_{m3} \cosh(my) + \alpha_{m4} \sinh(my)] \quad (3.109)$$

and

$$\rho_y(x,y) = + \sum_{m=1}^N \frac{1}{2} \sin(mx) [\alpha_{m3} \sinh(my) + \alpha_{m4} \cosh(my)] \quad (3.110)$$

3.2.1.5. Approximate Series Solution :

In the light of what has been done in section 2.2.1, let us recall eqn (3.65)

$$u_{i,\partial}(p) = \int_S t_j(Q) U_{ij,\partial}(p,Q) dS(Q) - \int_S u_j(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.65)$$

where over every part of the boundary of the solution domain either the displacements $u_j(Q)$ or the tractions $t_j(Q)$ are prescribed (or there may exist a linear relationship between them). The one which is not specified can be obtained by approximating the stress function ϕ . If the tractions are not specified they can be determined with the aid of eqn (3.21), into which the stress function ϕ approximated with eqn (2.81) is substituted. Equation (3.21) may then be rewritten as

$$\begin{aligned}
t_x(Q) &= \left(\frac{\partial^2 \phi}{\partial y^2} \right)_Q n_1(Q) - \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_Q n_2(Q) \\
t_y(Q) &= - \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)_Q n_1(Q) + \left(\frac{\partial^2 \phi}{\partial x^2} \right)_Q n_2(Q)
\end{aligned}
\tag{3.111}$$

or in tensor notation

$$t_i(Q) = \left\{ \delta_{ij} \phi_{,\beta\beta} - \phi_{,ij} \right\}_Q n_j(Q) \tag{3.112}$$

If the displacements are not specified they can be obtained from eqns (3.90), where in addition to the stress function ϕ the two new functions dependent on ϕ , are used. These two functions (ρ_x and ρ_y) can be either approximated with eqns (3.109) and (3.110), if Fourier series is the trial function, or with the tabulated values given in Table 3.3, if the trial function is Taylor series.

Now, let us redefine eqns (3.90) and (3.112), respectively, as

$$u_i = \sum_{m=1}^N \alpha_m e_i^m \tag{3.113}$$

where

$$\begin{aligned}
e_x^m &= \frac{1}{2\mu} \left[-f_{,x}^m + 4(1-\nu) \rho_x^m \right] \\
e_y^m &= \frac{1}{2\mu} \left[-f_{,y}^m + 4(1-\nu) \rho_y^m \right]
\end{aligned}
\tag{3.14}$$

and

$$t_i = \sum_{m=1}^N \alpha_m E_i^m \quad (3.115)$$

where

$$\begin{aligned} E_x^m &= f_{,yy}^m n_1 - f_{,xy}^m n_2 \\ E_y^m &= -f_{,xy}^m n_1 + f_{,xx}^m n_2 \end{aligned} \quad (3.116)$$

eqn (3.65) may then be re-expressed by

$$\sum_{m=1}^N \alpha_m e_{i,\partial}^m(p) = \sum_{m=1}^N \alpha_m h_{i,\partial}^m(Q) + b_{i,\partial}(Q) \quad (3.117)$$

where, if the displacements $u_j(Q)$ are prescribed

$$h_{i,\partial}^m(Q) = + \int_S E_j^m(Q) U_{ij,\partial}(p,Q) dS(Q) \quad (3.118)$$

$$b_{i,\partial}(Q) = - \int_S u_j(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.119)$$

and, if the tractions $t_j(Q)$ are prescribed

$$h_{i,\partial}^m(Q) = - \int_S e_j^m(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.120)$$

$$b_{i,\partial}(Q) = + \int_S t_j(Q) U_{ij,\partial}(p,Q) dS(Q) \quad (3.121)$$

Equation (3.117) can therefore be rearranged as

$$\sum_{m=1}^N \alpha_m [e_{i,\partial}^m(p) - h_{i,\partial}^m(Q)] = b_{i,\partial}(Q) \quad (3.122)$$

This is the final set of linear algebraic equations in the unknowns α_m which can be found from the solution of it.

In order to have independent derivatives in finding the stress function ϕ the set of equations (3.122) is arranged in the order given below.

$$1) u = -\phi_{,x} c + d \rho_x$$

$$2) v = -\phi_{,y} c + d \rho_y$$

$$3) u_{,x} = -\phi_{,xx} c + d \rho_{x,x}$$

$$4) (u_{,y} + v_{,x})/2 = -\phi_{,xy} c$$

$$5) v_{,y} = -\phi_{,yy} c + d \rho_{y,y}$$

$$6) u_{,xx} = -\phi_{,xxx} c + d \rho_{x,xx}$$

$$7) (u_{,xy} + v_{,xx})/2 = -\phi_{,xxy} c$$

$$8) (u_{,yy} + v_{,xy})/2 = -\phi_{,xyy} c$$

$$9) v_{,yy} = -\phi_{,yyy} c + d \rho_{y,yy}$$

$$10) u_{,xxx} = -\phi_{,xxx} c + d \rho_{,xxx}$$

$$11) (u_{,xxy} + v_{,xxx})/2 = -\phi_{,xxy} c$$

$$12) (u_{,yyy} + v_{,xyy})/2 = -\phi_{,yyy} c$$

$$13) v_{,yyy} = -\phi_{,yyy} c + d \rho_{,yyy}$$

$$14) u_{,xxxx} = -\phi_{,xxxx} c + d \rho_{,xxxx}$$

$$15) (u_{,xxx} + v_{,xxx})/2 = -\phi_{,xxx} c$$

$$16) (u_{,yyy} + v_{,xyy})/2 = -\phi_{,yyy} c$$

$$17) v_{,yyy} = -\phi_{,yyy} c + d \rho_{,yyy}$$

$$18) u_{,xxxx} = -\phi_{,xxxx} c + d \rho_{,xxxx}$$

$$19) (u_{,xxx} + v_{,xxx})/2 = -\phi_{,xxx} c$$

$$20) (u_{,yyy} + v_{,xyy})/2 = -\phi_{,yyy} c$$

$$21) v_{,yyy} = -\phi_{,yyy} c + d \rho_{,yyy}$$

where

$$c = \frac{1}{2\mu} \quad \text{and} \quad d = \frac{2(1-\nu)}{\mu}$$

The boundary discretisation and numerical integration procedures are carried out exactly as was done for potential problems in sections (2.2.2) and (2.2.3).

3.2.2. Neuber-Papkovich Approach

The advantage of what is to be introduced here is that the determination of displacements can be achieved successfully without integration⁽³²⁾.

The starting point is the general solution of Navier's equations in terms of three arbitrary independent harmonic functions, namely, ϕ_0 , ϕ_1 , and ϕ_2 .

$$\begin{aligned} 2\mu u &= 4(1-\nu) \phi_1 - \frac{\partial}{\partial x} (x \phi_1 + y \phi_2 + \phi_0) \\ 2\mu v &= 4(1-\nu) \phi_2 - \frac{\partial}{\partial y} (x \phi_1 + y \phi_2 + \phi_0) \end{aligned} \quad (3.123)$$

It has been shown (Neuber, 1958) that any one of the functions above may be set equal to zero without affecting the generality of the solution. If we, here, set ϕ_0 equal to zero, then we have

$$\begin{aligned} 2\mu u &= (3-4\nu) \phi_1 - (x \phi_{1,x} + y \phi_{2,x}) \\ 2\mu v &= (3-4\nu) \phi_2 - (x \phi_{1,y} + y \phi_{2,y}) \end{aligned} \quad (3.124)$$

or more compactly

$$2\mu u_i = (3-4\nu) \phi_i - x_j \phi_{j,i} \quad (3.125)$$

where if the relations (3.81) are substituted the plane-stress case is obtained.

The stresses (and the tractions) can be found by differentiating eqns (3.125) and by substituting them into Hooke's law (3.7) as

$$\begin{aligned}
\sigma_{xx} &= 2\left((1-\nu)\frac{\partial\phi_1}{\partial x} + \nu\frac{\partial\phi_2}{\partial y}\right) - \left(x\frac{\partial^2\phi_1}{\partial x^2} + y\frac{\partial^2\phi_2}{\partial x^2}\right) \\
\sigma_{yy} &= 2\left((1-\nu)\frac{\partial\phi_2}{\partial y} + \nu\frac{\partial\phi_1}{\partial x}\right) - \left(x\frac{\partial^2\phi_1}{\partial x^2} + y\frac{\partial^2\phi_2}{\partial x^2}\right) \\
\sigma_{xy} &= (1-2\nu)\left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x}\right) - \left(x\frac{\partial^2\phi_1}{\partial x\partial y} + y\frac{\partial^2\phi_2}{\partial x\partial y}\right)
\end{aligned} \tag{3.126}$$

Substitution of these equations into (3.21) gives the components of the traction vector in the coordinate directions.

By noting the following relations

$$x_{\alpha} \phi_{\alpha, kk} = 0 \quad \text{since } \nabla^2 \phi_{\alpha} = 0$$

and

$$x_{\alpha, k} = \delta_{\alpha k} = \delta_{k \alpha}$$

equations (3.126) may compactly be rewritten as

$$\sigma_{ij} = 2\nu \delta_{ij} \phi_{k, k} + (1-2\nu) (\phi_{i, j} + \phi_{j, i}) - x_k \phi_{k, ij} \tag{3.127}$$

and the traction components are

$$t_i = 2\nu \phi_{k,k} n_i + (1 - 2\nu) (\phi_{i,j} + \phi_{j,i}) n_j - x_k \phi_{k,ij} n_j \quad (3.128)$$

3.2.2.1. Approximate Series Solution :

The displacement components are expressed by two independent harmonic functions, namely, ϕ_1 and ϕ_2 . These functions can now be determined by expanding them in series form with unknown constant terms,

$$\phi_1 = \sum_{m=1}^N \alpha_m f^m \quad (3.129)$$

and

$$\phi_2 = \sum_{m=1}^N \beta_m g^m \quad (3.130)$$

where the functions f^m and g^m could be taken as the same function, and $\nabla^2 f^m = \nabla^2 g^m = 0$.

Although the components of displacements satisfy the biharmonic equation (see eqn (3.17)), and at each step of differentiation there are only up to four independent terms, there are two independent terms at each stage of differentiation of harmonic functions (see section 2.1.2) which are used to express the displacement components. Therefore, to avoid possible duplication, a proper arrangement of the simultaneous equations must be made, as was done in the previous section.

Equations (3.125) and (3.128) can now be redefined as

$$u_i = \sum_{m=1}^N \alpha_{km} e_{ik}^m \quad (3.131)$$

which is in expanded form

$$u = \sum_{m=1}^N \alpha_{1m} e_{11}^m + \sum_{m=1}^N \alpha_{2m} e_{12}^m \quad (3.132)$$

$$v = \sum_{m=1}^N \alpha_{1m} e_{21}^m + \sum_{m=1}^N \alpha_{2m} e_{22}^m$$

where

$$\begin{aligned} e_{11}^m &= \frac{1}{2\mu} [(3-4\nu) f^m - x f_{,x}^m] \\ e_{12}^m &= -\frac{1}{2\mu} y f_{,x}^m \end{aligned} \quad (3.133)$$

$$e_{21}^m = -\frac{1}{2\mu} x f_{,y}^m$$

$$e_{22}^m = \frac{1}{2\mu} [(3-4\nu) f^m - y f_{,y}^m]$$

and

$$t_i = \sum_{m=1}^N \alpha_{km} E_{ik}^m \quad (3.134)$$

which is in expanded form

$$t_x = \sum_{m=1}^N \alpha_{1m} E_{11}^m + \sum_{m=1}^N \alpha_{2m} E_{12}^m \quad (3.135)$$

$$t_y = \sum_{m=1}^N \alpha_{1m} E_{21}^m + \sum_{m=1}^N \alpha_{2m} E_{22}^m$$

where

$$\begin{aligned} E_{11}^m &= [2(1-\nu) f_{,x}^m - x f_{,xx}^m] n_1 + [(1-2\nu) f_{,y}^m - x f_{,xy}^m] n_2 \\ E_{12}^m &= [2\nu f_{,y}^m - y f_{,yy}^m] n_1 + [(1-2\nu) f_{,x}^m - y f_{,xy}^m] n_2 \\ E_{21}^m &= [(1-2\nu) f_{,y}^m - x f_{,xy}^m] n_1 + [2\nu f_{,x}^m + x f_{,xx}^m] n_2 \\ E_{22}^m &= [(1-2\nu) f_{,x}^m - y f_{,xy}^m] n_1 + [2(1-\nu) f_{,y}^m + y f_{,yy}^m] n_2 \end{aligned} \quad (3.136)$$

Equation (3.65) may then be written as

$$\sum_{m=1}^N \alpha_{km} e_{ik,\partial}^m(p) = \sum_{m=1}^N \alpha_{km} h_{i,\partial}^m(Q) + b_{i,\partial}(Q) \quad (3.137)$$

where, if the displacements $u_j(Q)$ are prescribed

$$h_{i,\partial}^m(Q) = + \int_S E_{jk}^m(Q) U_{ij,\partial}(p,Q) dS(Q) \quad (3.138)$$

$$b_{i,\partial}(Q) = - \int_S u_j(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.139)$$

and, if the tractions $t_j(Q)$ are prescribed

$$h_{i,\partial}^m(Q) = - \int_S e_{jk}^m(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.140)$$

$$b_{i,\partial}(Q) = + \int_S t_j(Q) U_{ij,\partial}(p,Q) dS(Q) \quad (3.141)$$

Equation (3.137) can therefore be rearranged as

$$\sum_{m=1}^N \alpha_{km} [e_{ik,\partial}^m(p) - h_{i,\partial}^m(Q)] = b_{i,\partial}(Q) \quad (3.142)$$

This is the final set of linear algebraic equations in the unknowns α_{km} which can be found from the solution of it.

In order to avoid possible duplications of linear algebraic equations eqns (3.142) may be arranged in the order given below.

$$1) u = c [a \phi_1 - (x \phi_{1,x} + y \phi_{2,x})]$$

$$2) v = c [a \phi_2 - (x \phi_{1,y} + y \phi_{2,y})]$$

$$3) u_{,y} = c [a \phi_{1,y} - \phi_{2,x} - (x \phi_{1,xy} + y \phi_{2,xy})]$$

$$4) v_{,y} = c [(a-1) \phi_{2,y} + (x \phi_{1,xx} + y \phi_{2,xx})]$$

$$5) u_{,x} = c [(a-1) \phi_{1,x} - (x \phi_{1,xx} + y \phi_{2,xx})]$$

$$6) v_{,x} = c [a \phi_{2,x} - \phi_{1,y} - (x \phi_{1,xy} + y \phi_{2,xy})]$$

$$7) u_{,xy} = c [(a-1) \phi_{1,xy} - \phi_{2,xx} - (x \phi_{1,xxxy} + y \phi_{2,xxxy})]$$

$$8) v_{,xy} = c [(a-1) \phi_{2,xy} + \phi_{1,xx} + (x \phi_{1,xxx} + y \phi_{2,xxx})]$$

$$9) (u_{,xx} + u_{,yy})/2 = -c [\phi_{1,xx} + \phi_{2,xy}]$$

$$10) (v_{,xx} + v_{,yy})/2 = c [\phi_{2,xx} - \phi_{1,xx}]$$

$$11) (u_{,xxy} + u_{,xyy})/2 = 0.5 c [(a-2) \phi_{1,xxy} - (a-1) \phi_{1,xxx} - \phi_{2,xxx} \\ - 2 \phi_{2,xyy} - (x \phi_{1,xxxy} + y \phi_{2,xxxy}) - (x \phi_{1,xxxy} \\ + y \phi_{2,xxxy})]$$

$$12) (v_{,xxy} + v_{,xyy})/2 = 0.5 c [(a-1) \phi_{2,xxy} - (a-2) \phi_{2,xxx} + \phi_{1,xxy} \\ + 2 \phi_{1,xxx} - (x \phi_{1,xxxy} + y \phi_{2,xxxy}) + (x \phi_{1,xxxy} \\ + y \phi_{2,xxxy})]$$

$$13) (u_{,xxx} + u_{,yyy})/2 = 0.5 c [(a-3) \phi_{1,xxx} - a \phi_{1,xxy} + 3 \phi_{2,xxx} + \\ (x \phi_{1,xxxy} + y \phi_{2,xxxy}) + (x \phi_{1,xxxy} + y \phi_{2,xxxy})]$$

$$14) (v_{,xxx} + v_{,yyy})/2 = 0.5 c [a \phi_{2,xxx} - (a-3) \phi_{2,xxy} - 3 \phi_{1,xxy} - \\ (x \phi_{1,xxxy} + y \phi_{2,xxxy}) + (x \phi_{1,xxxy} + y \phi_{2,xxxy})]$$

$$15) (u_{,xxx} + u_{,yyy})/2 = -c [\phi_{1,xxx} + \phi_{2,xyy}]$$

$$16) (v_{,xxx} + v_{,yyy})/2 = c [\phi_{1,xxxy} + \phi_{2,xyyy}]$$

$$17) - (u_{,xxx} + u_{,yyy})/2 = u_{,xyy} = c [(a-2) \phi_{1,xxxy} - 2 \phi_{2,xxxy} \\ - (x \phi_{1,xxxyy} + y \phi_{2,xxxyy})]$$

$$18) - (v_{,xxx} + v_{,yyy})/2 = v_{,xyy} = c [(a-2) \phi_{2,xyyy} + 2 \phi_{1,xxxy} \\ - (x \phi_{1,xxxyy} + y \phi_{2,xxxyy})]$$

$$19) - (u_{,xxx} + u_{,yyy})/2 = u_{,xyy} = c [(a-2) \phi_{1,xyyy} - 3 \phi_{2,xxxyy} \\ - (x \phi_{1,xxxyyy} + y \phi_{2,xxxyyy})]$$

$$20) - (v_{,xxx} + v_{,yyy})/2 = v_{,xyy} = c [(a-3) \phi_{2,xyyy} + 2 \phi_{1,xxxy} \\ + (x \phi_{1,xxxyy} + y \phi_{2,xxxyy})]$$

$$21) - (u_{,xxx} + u_{,yyy})/2 = u_{,xyy} = c [(a-3) \phi_{1,xyyy} + 2 \phi_{2,xxxyy} \\ - (x \phi_{1,xxxyy} + y \phi_{2,xxxyy})]$$

$$22) - (v_{,xxx} + v_{,yyy})/2 = v_{,xyy} = c [(a-2) \phi_{2,xxxy} - 3 \phi_{1,xyyy} \\ - (x \phi_{1,xxxyy} + y \phi_{2,xxxyy})]$$

where

$$c = \frac{1}{2\mu} \quad \text{and}$$

$$a = 3 - 4\nu, \quad \text{if plane strain}$$

$$a = \frac{3 - \nu}{1 + \nu}, \quad \text{if plane stress}$$

3.2.3. Direct Expansion of the Integral Equations

The Somigliana identity for displacements for an internal point p and its subsequent derivatives with respect to the coordinate directions, eqns (3.65), can be expanded directly in series without the need to express displacements (and tractions) in terms of potential functions.

The components of the displacement vector are approximated as follows.

$$u = \sum_{m=1}^N \alpha_m f^m \quad (3.143)$$

and

$$v = \sum_{m=1}^N \beta_m f^m \quad (3.144)$$

where the functions f^m and g^m satisfy the biharmonic eqn (3.69) and can be represented by the same series, hence they are identical ($f^m \equiv g^m$).

Then, more compactly

$$u_i = \sum_{m=1}^N \alpha_{im} f^m \quad (3.145)$$

Differentiating (3.145) and substituting into Hooke's law (3.7) results in

$$\sigma_{ij} = \sum_{m=1}^N [\lambda \delta_{ij} \alpha_{km} f_{,k}^m + \mu \alpha_{im} f_{,j}^m + \mu \alpha_{jm} f_{,i}^m] \quad (3.146)$$

and using equation (3.21) the tractions are

$$t_i = \sum_{m=1}^N [\lambda n_i \alpha_{km} f_{,k}^m + \mu n_k \alpha_{im} f_{,k}^m + \mu n_k \alpha_{km} f_{,i}^m] \quad (3.147)$$

which may be redefined as

$$t_i = \sum_{m=1}^N \alpha_{km} G_{ik}^m \quad (3.148)$$

and in expanded form

$$t_x = \sum_{m=1}^N \alpha_{1m} G_{11}^m + \sum_{m=1}^N \alpha_{2m} G_{12}^m \quad (3.149)$$

$$t_y = \sum_{m=1}^N \alpha_{1m} G_{21}^m + \sum_{m=1}^N \alpha_{2m} G_{22}^m$$

where

$$\begin{aligned}
G_{11}^m &= (2\mu + \lambda) f_{,x}^m n_1 + \mu f_{,y}^m n_2 \\
G_{12}^m &= \lambda f_{,y}^m n_1 + \mu f_{,x}^m n_2 \\
G_{21}^m &= \mu f_{,y}^m n_1 + \lambda f_{,x}^m n_2 \\
G_{22}^m &= \mu f_{,x}^m n_1 + (\lambda + 2\mu) f_{,y}^m n_2
\end{aligned} \tag{3.150}$$

Equation (3.65) may then be written as

$$u_{i,\partial}(p) = \sum_{m=1}^N H_{i,\partial}^m(Q) + b_{i,\partial}(Q) \tag{3.151}$$

or by substitution of eqn (3.145) into (3.151) we have

$$\sum_{m=1}^N \alpha_{im} f_{,\partial}^m(p) = \sum_{m=1}^N H_{i,\partial}^m(Q) + b_{i,\partial}(Q) \tag{3.152}$$

where, if the displacements $u_j(Q)$ are prescribed

$$H_{i,\partial}^m(Q) = + \int_S \alpha_{km} G_{jk}^m(Q) U_{ij,\partial}(p,Q) dS(Q) \tag{3.153}$$

$$b_{i,\partial}(Q) = - \int_S u_j(Q) T_{ij,\partial}(p,Q) dS(Q) \tag{3.154}$$

and, if the tractions $t_j(Q)$ are prescribed

$$H_{i,\partial}^m(Q) = - \int_S \alpha_{jm} f^m(Q) T_{ij,\partial}(p,Q) dS(Q) \quad (3.155)$$

$$b_{i,\partial}(Q) = + \int_S t_j(Q) U_{ij,\partial}(p,Q) dS(Q) \quad (3.156)$$

Equation (3.152) can therefore be rearranged as

$$\sum_{m=1}^N [\alpha_{im} f_{,\partial}^m(p) - H_{i,\partial}^m(Q)] = b_{i,\partial}(Q) \quad (3.157)$$

This is the final set of linear algebraic equations in the unknowns α_{im} which can be found from the solution of it.

3.3. Summary

Although the LBIE method developed in Chapter 2 for potential theory essentially applies here there are now three alternative ways of formulating LBIEs for stress problems. Hence, the LBIE techniques for stress problems are summarized as follows.

i) We begin with the zero-order integral equation (3.56) as the basis of the LBIE methods.

ii) Equation (3.56) is then differentiated repeatedly at a point, p , in the x and y coordinate directions within the solution domain to obtain higher-order BIEs (see eqns (3.65)).

iii) In a typical 'BVP' of elasticity at any point on the boundary either displacements, eqn (3.20), or tractions, eqn (3.21), (or a linear combination of them) are known from the boundary conditions of the problem, so that only those which are

unknown, in eqns (3.65), have to be approximated. Note, however, that these unknowns (displacements or tractions) are now expressed in three different ways. They are, namely, Airy's stress function approach (ASFA), direct expansion approach (DEA), and Neuber-Papkovich approach (NPA). While the former two are biharmonic the latter one is harmonic in nature. Hence, depending on the type of formulation used, the variations of those unknowns on the boundary together with the displacements and their subsequent derivatives at the internal point p , the terms on the left hand sides of eqns (3.65), are approximated by either harmonic or biharmonic series (Taylor or Fourier ones).

iv) The derivative forms of the kernel functions, parts of the integrands in eqns (3.65), are obtained by simple analytical differentiation (see section 3.1.6).

v) The integrations over the boundary are carried out by dividing the boundary into quadratic (three node) boundary elements, and applying gaussian quadrature to each one.

vi) The final set of linear algebraic equations is formed from the higher-order BIEs, eqns (3.65). It thus becomes a relatively small set of linear equations, which is straightforward to solve by gaussian elimination. In order to avoid duplications in forming the set of equations when ASFA or NPA is used, however, the proper arrangements of the linear equations, eqns (3.65), should be made, as shown in sections (3.2.1.5) and (3.2.2.1).

vii) The coefficient matrix is scaled to overcome the possible numerical instability of a mixed dimensional matrix system.

viii) Having solved the linear equations the values of the coefficients of the series are known, and by rescaling them the required solution is obtained.

3.4. Test Cases

The accuracy of the numerical implementation of the LBIE methods are assessed by reference to four test problems, namely, a square flat plate under uniform tension, a cantilever beam with shear force at the end, a square flat plate subject to a uniform tensile stress with a small circular hole at its centre, and a cracked beam of rectangular cross-section simply supported at its ends. Due to high stress variations the latter two problems, especially the last one, provide a severe test of, not only the LBIE methods developed here, but any numerical method of solution. The LBIE methods are also used to locally refine the solutions of the last two problems, because of having unsatisfactory results for them when the LBIE methods are employed on their own. The analytical solution to the stress concentration problem is applied as the remote field solution, whereas for the crack problem numerical results obtained from a conventional BIE program called BEETS* are applied outside the local area of interest. The full range of the solutions are given in Tables in appendices F, G, and H.

3.4.1. Square Flat Plate under Simple Tension

Let us consider a square flat plate with sides of length $2L$ loaded at the upper and lower edges by a uniform tensile stress $\sigma=1$ in the y coordinate direction. Due to the symmetry only one quadrant is modelled. Figure 3.3 shows the quadrant with sides of unit length, the local origin of expansion being at the bottom left hand corner, which is arbitrarily chosen to coincide with the global coordinate origin at node 1. For this very simple problem, with the experience based on the potential problems for a square domain (section 2.4.1), the mesh of eight quadratic boundary elements of equal length is used (two per side of the domain).

* BEETS was developed in 1987 by Dr. A. A. Becker, one of the staff in Mech. Eng. Dept., Imperial College.

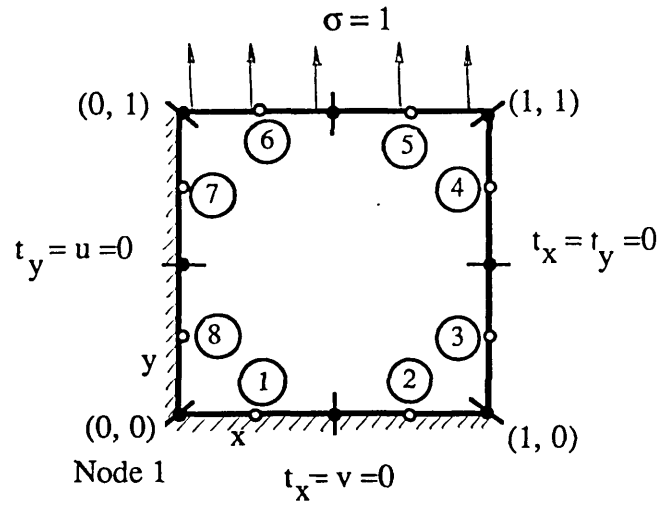


Figure 3.3 : One quadrant of a square flat plate under simple tension.

Poisson's ratio for the material of the plate is taken to be $\nu=0.3$, and elasticity modulus to be $E=1$. Plane strain conditions are assumed.

The analytical solution for this simple problem is given by⁽³⁾

$$u = -\sigma \nu (1 + \nu) \frac{x}{E} \quad (3.158)$$

$$v = +\sigma (1 - \nu^2) \frac{y}{E} \quad (3.159)$$

$$\sigma_{xx} = 0, \quad \sigma_{yy} = \sigma, \quad \sigma_{zz} = \nu \sigma \quad (3.160)$$

Tables 3.4 to 3.9 show the results obtained from Taylor series for the displacements u , v , and the stresses σ_{xx} , σ_{yy} , σ_{xy} using different numbers of terms and two positions for internal point p in equations (3.65). Due to the simplicity of the problem one of the internal points was chosen at the mid point of the domain, and the other being closer to the local point of interest, Node 1 (see section 2.4.1.1).

Table 3.4 shows the results obtained using the DEA. With the internal point at (0.5, 0.5) and five terms in the series (Taylor) the results are perfect, being progressively less accurate as the number of terms retained in the series increases, though still very close to the exact values. In fact, the three term series solution of the DEA could have been more accurate for this very simple problem, because all three stress components should be perfectly represented by a three term Taylor series, since the stress components are constants throughout the domain.

As the internal point moves to position (0.25, 0.25) the results start deteriorating, and as the numbers of terms increases with the internal point at (0.25, 0.25) the accuracy is reduced, like with the point at (0.5, 0.5).

Table 3.4 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
with different numbers of terms and the internal point at (0.5, 0.5)
using DEA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5)	5	- 3.76x10 ⁻¹⁰	- 3.87x10 ⁻¹⁰	+7.70x10 ⁻¹⁰	+2.11x10 ⁻¹⁰	+1.00000
(0.5, 0.5)	10	- 1.06x10 ⁻⁰⁹	+1.07x10 ⁻⁰⁹	+1.66x10 ⁻⁰⁹	- 5.12x10 ⁻¹⁰	+1.00000
(0.5, 0.5)	14	+1.42x10 ⁻⁰⁹	- 1.87x10 ⁻⁰⁹	- 2.39x10 ⁻⁰⁹	- 2.97x10 ⁻⁰⁹	+1.00000
(0.5, 0.5)	18	+5.37x10 ⁻⁰⁸	- 5.35x10 ⁻⁰⁸	- 1.83x10 ⁻⁰⁷	+1.78x10 ⁻⁰⁹	+1.00000

Table 3.5 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
with different numbers of terms and the internal point at (0.25, 0.25)
using DEA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.25, 0.25)	5	- 1.96x10 ⁻⁰⁶	+2.72x10 ⁻⁰⁵	- 5.99x10 ⁻⁰⁵	- 2.27x10 ⁻⁰⁵	+1.00000
(0.25, 0.25)	10	- 2.27x10 ⁻⁰⁴	+2.75x10 ⁻⁰⁴	+7.01x10 ⁻⁰⁴	- 6.66x10 ⁻⁰⁵	+0.99943
(0.25, 0.25)	14	+2.71x10 ⁻⁰³	- 2.66x10 ⁻⁰³	- 1.16x10 ⁻⁰²	+6.23x10 ⁻⁰⁴	+1.01048
(0.25, 0.25)	18	- 6.89x10 ⁻⁰³	- 1.32x10 ⁻⁰²	- 8.96x10 ⁻⁰²	- 1.34x10 ⁻⁰³	+0.86916

Tables 3.6 and 3.7 show the results obtained using NPA with internal points at (0.5, 0.5) and (0.25, 0.25) and using different numbers of terms.

Table 3.6 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
with different numbers of terms and the internal point at (0.5, 0.5)
using NPA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5)	5	- 3.15x10 ⁻¹⁰	+2.87x10 ⁻¹⁰	+5.55x10 ⁻¹⁰	+1.06x10 ⁻¹⁰	+1.00000
(0.5, 0.5)	7	- 1.06x10 ⁻⁰⁹	+1.07x10 ⁻⁰⁹	+1.66x10 ⁻⁰⁹	- 5.12x10 ⁻¹⁰	+1.00000
(0.5, 0.5)	9	+1.42x10 ⁻⁰⁹	- 1.87x10 ⁻⁰⁹	- 2.39x10 ⁻⁰⁹	- 2.97x10 ⁻⁰⁹	+1.00000
(0.5, 0.5)	11	+5.37x10 ⁻⁰⁸	- 5.35x10 ⁻⁰⁸	- 1.83x10 ⁻⁰⁷	+1.78x10 ⁻⁰⁹	+1.00000

Table 3.7 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
with different numbers of terms and the internal point at (0.25, 0.25)
using NPA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.25, 0.25)	5	-2.55×10^{-06}	-7.39×10^{-06}	-3.72×10^{-05}	-3.29×10^{-05}	+0.99999
(0.25, 0.25)	7	-2.27×10^{-04}	$+2.75 \times 10^{-04}$	$+7.01 \times 10^{-04}$	-6.66×10^{-05}	+0.99943
(0.25, 0.25)	9	$+2.71 \times 10^{-03}$	-2.66×10^{-03}	-1.16×10^{-02}	$+6.23 \times 10^{-04}$	+1.01048
(0.25, 0.25)	11	-6.89×10^{-03}	-1.32×10^{-02}	-8.96×10^{-02}	-1.34×10^{-03}	+0.86916

It is of interest that some of the results obtained using NPA are exactly the same as those obtained using DEA (compare Tables 3.4 and 3.5 with 3.6 and 3.7 above, or Tables F.1 with F.2 in appendix F). The results are identical, when the numbers of terms used are: 5, 7, 9, 11 (for NPA) and 6, 10, 14, 18 (for DEA), respectively (see Pascal triangles, Table 2.1 and Table 3.2).

Tables 3.8 and 3.9 show the results obtained using ASFA with internal points at (0.5, 0.5) and (0.25, 0.25) and using different numbers of terms.

Table 3.8 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
with different numbers of terms and the internal point at (0.5, 0.5)
using ASFA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5)	5	$+2.55 \times 10^{-11}$	-2.66×10^{-11}	-2.65×10^{-11}	-4.00×10^{-15}	+1.00000
(0.5, 0.5)	9	-4.76×10^{-10}	$+4.49 \times 10^{-10}$	-1.66×10^{-09}	$+1.07 \times 10^{-10}$	+1.00000
(0.5, 0.5)	13	-6.69×10^{-10}	$+6.82 \times 10^{-10}$	$+1.91 \times 10^{-08}$	-5.12×10^{-10}	+1.00000
(0.5, 0.5)	17	-4.61×10^{-10}	$+1.25 \times 10^{-11}$	-9.39×10^{-09}	-2.97×10^{-09}	+1.00000
(0.5, 0.5)	21	$+2.98 \times 10^{-08}$	-2.96×10^{-08}	-1.86×10^{-07}	$+1.78 \times 10^{-09}$	+1.00000

Table 3.9 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
with different numbers of terms and the internal point at (0.25, 0.25)
using ASFA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.25, 0.25)	5	- 1.37x10 ⁻⁰⁶	+1.54x10 ⁻⁰⁶	+2.34x10 ⁻⁰⁶	- 5.96x10 ⁻⁰⁷	+1.00000
(0.25, 0.25)	9	+9.70x10 ⁻⁰⁷	- 1.09x10 ⁻⁰⁵	- 1.47x10 ⁻⁰⁴	- 3.29x10 ⁻⁰⁵	+1.00010
(0.25, 0.25)	13	- 1.42x10 ⁻⁰⁴	+1.90x10 ⁻⁰⁴	+1.23x10 ⁻⁰³	- 6.66x10 ⁻⁰⁵	+0.99891
(0.25, 0.25)	17	+2.00x10 ⁻⁰³	- 1.94x10 ⁻⁰³	- 2.94x10 ⁻⁰²	+6.23x10 ⁻⁰⁴	+1.02826
(0.25, 0.25)	21	- 7.16x10 ⁻⁰³	- 1.29x10 ⁻⁰²	- 9.69x10 ⁻⁰²	- 1.34x10 ⁻⁰³	+0.87649

Comparison of these results with those obtained using the other two methods show that almost the same accuracy is achieved with ASFA. The difference comes from the error in integrating the two harmonic functions ρ_x and ρ_y , numerically, see section 3.5.

The cosine Fourier series in x in the form of (3.103) gave bad results, and a combination of cosine and sine Fourier series in x was necessary to obtain satisfactory results. For this reason the Fourier series (2.73), which consists of both sine and cosine terms in x, was used to solve the problem. Since this series, eqn (2.73), satisfies the harmonic equation $\nabla^2\phi=0$ it is obvious that it can be used only with NPA. The results obtained are quite satisfactory (see Table 3.10), although they are not as good as those obtained using Taylor series. As the number of terms increases the results do improve, unlike with Taylor series (see also section 2.4 in chapter 2).

Table 3.10* : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Fourier series (2.73)
with different numbers of terms and the internal point at (0.5, 0.5) and (0.25, 0.25)
using NPA and eight boundary elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	7	- 0.01449 +0.00129	+0.01410 - 0.00149	+0.03329 - 0.00820	+0.00184 - 0.00182	+0.96420 +1.01176
(0.5, 0.5) (0.25, 0.25)	9	- 0.00791 +0.00274	+0.00631 - 0.00243	+0.00301 - 0.00146	- 0.00639 +0.00033	+0.98073 +1.01617
(0.5, 0.5) (0.25, 0.25)	11	- 0.00027 - 0.00731	+0.00016 - 0.01412	+0.00205 - 0.10142	- 0.00167 - 0.00245	+1.00390 +0.85824

In order to improve the accuracy of the Fourier series method, the same refined mesh of the square domain in section 2.4.1 (i.e., 26 elements with local length ratio of 1.5 and smallest elements of length 0.0044 along both coordinate directions) was used so that the internal point could be placed much closer to the local point of interest. This gave some improvement, although the results are still not as accurate as those from Taylor series, see Table 3.11.

To improve the accuracy even further a much more refined mesh is necessary in order to allow the positions of internal points to move much closer to the local point.

* From now on, after five decimal places the results are rounded-off.

Table 3.11 : Square plate under simple tension
Displacements and stresses at node 1 (Fig. 3.3) from Fourier series (2.73)
with different numbers of terms and the internal point at (0.1, 0.1) and (0.05, 0.05)
using NPA and refined mesh in the region of interest.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
Exact :		0.	0.	0.	0.	1.
(0.1, 0.1) (0.05, 0.05)	7	- 0.00195 - 0.00023	+0.00023 +0.00010	- 0.05056 - 0.08556	- 0.00262 - 0.00087	+1.07776 +1.12406
(0.1, 0.1) (0.05, 0.05)	9	+0.00004 - 0.00001	- 0.00005 - 0.00001	- 0.00073 - 0.00081	+0.00024 - 0.00083	+1.00904 +0.99581
(0.1, 0.1) (0.05, 0.05)	11	+0.00000 - 0.00000	- 0.00000 - 0.00000	- 0.00020 - 0.00041	+0.00022 +0.00062	+0.99809 +0.99641

3.4.2. Cantilever Beam with a Shear Force at the End

Consider a rectangular beam having length L , width w , and depth d , built in a vertical wall at one end and bent by a shearing force S applied at the other end, as shown in Fig. 3.4.

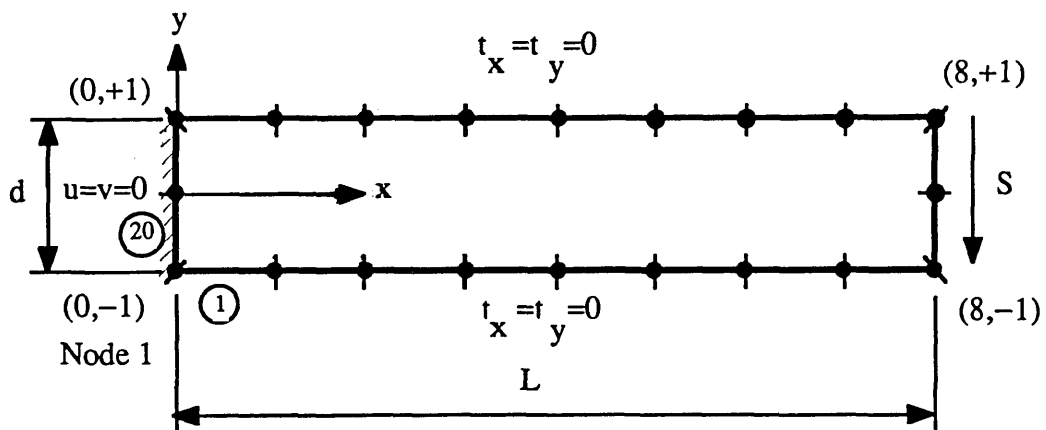


Figure 3.4 : Cantilever beam with a shear force at the end.

The analytical solution for stresses is given by⁽¹⁶⁾

$$\begin{aligned}\sigma_{yy} &= 0 \\ \sigma_{xx} &= \frac{My}{I}, \quad \sigma_{xy} = -\frac{S}{2I} \left[\left(\frac{d}{2} \right)^2 - y^2 \right]\end{aligned}\quad (3.161)$$

where $M=S(L-x)$ is the bending moment at a beam cross-section at a distance x from the built-in end and $I=\frac{2}{3}(d/2)^3$ is the second moment of the cross-section of the cantilever.

The analytical solution for displacements is given by⁽¹⁶⁾

$$u = \frac{S}{EI} \left[Lx - \frac{x^2}{2} \right] y + C_2 y + \frac{S(2+\nu)}{6EI} y^3 \quad (3.162)$$

$$v = -\frac{\nu S}{2EI} [L-x] y^2 + C_1 x + \frac{S}{EI} \left[\frac{x^3}{6} - \frac{Lx^2}{2} \right] \quad (3.163)$$

where

$$C_1 = -\frac{S(4+5\nu)}{6EI} \left(\frac{d}{2} \right)^2, \quad C_2 = -\frac{S(2+\nu)}{6EI} \left(\frac{d}{2} \right)^2 \quad (3.164)$$

The constants C_1 and C_2 given above are found by considering the conditions at the built-in end as

$$u=0 \quad \text{at} \quad x=0 \quad \text{and} \quad y=\pm d/2$$

In practice, the displacement conditions at the built-in end may be different, hence, eqns (3.162) and (3.163) do not provide exact solutions nor do the stress distributions (3.161). Although for comparatively long cantilevers at a considerable distance away from the support itself the solutions (3.161), (3.162), and (3.163), by means of Saint-Venant's principle, give satisfactory results we will here investigate the behaviour of the corner point at the built-in end (node 1, see Fig 3.4) and compare the results obtained from the LBIE methods with those obtained from the above analytical solutions.

For our purposes numerical values used are as follows:

$$L=8, \quad d=2, \quad w=1, \quad S=1, \quad \nu=0.3, \quad \text{and} \quad E=1.$$

and assuming a plane stress case, we have at node 1,

$$\sigma_{xx}=12, \quad \sigma_{xy}=0, \quad \sigma_{yy}=0$$

$$\text{and} \quad u=0, \quad v= - 1.4625$$

The boundary element mesh shown in Fig. 3.4 consists of twenty quadratic elements of equal size.

Tables (3.12) to (3.14) show the results obtained from the three techniques using different numbers of terms with the internal point at (4., 0.) and (2., 0.). For the full range of the results see Tables G.1, G.2, and G.3 in appendix G.

Table 3.12* : Cantilever beam loaded at the end with shearing stress
Displacements and stresses at node 1 (Fig. 3.4) from Taylor series
with optimum numbers of terms and with the internal point at (4., 0.) and (2.,0.)
using DEA and twenty quadratic elements

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic:	0.	- 1.4625	- 12.	0.	0.
(4, 0.)	10	+0.004	- 1.200	- 11.982	- 0.001	+0.001
(4, 0.)	14	+0.049	- 1.213	- 12.050	+0.000	+0.001
(4, 0.)	18	+0.092	- 1.215	- 11.983	+0.004	+0.003
(2, 0.)	10	+0.150	- 1.045	- 10.115	+0.013	+0.001
(2, 0.)	11	+0.186	- 1.270	- 12.174	+0.009	+0.004

Table 3.13 : Cantilever beam loaded at the end with shearing stress
Displacements and stresses at node 1 (Fig. 3.4) from Taylor series
with optimum numbers of terms and with the internal point at (4., 0.) and (2.,0.)
using NPA and twenty quadratic elements

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic:	0.	- 1.4625	- 12.	0.	0.
(4, 0.)	7	+0.004	- 1.200	- 11.982	- 0.001	+0.001
(4, 0.)	9	+0.049	- 1.213	- 12.050	+0.000	+0.001
(4, 0.)	11	+0.092	- 1.215	- 11.983	+0.004	+0.003
(2, 0.)	7	+0.150	- 1.045	- 10.115	+0.013	+0.001

* From now on, after three decimal places the results are rounded-off.

Table 3.14 : Cantilever beam loaded at the end with shearing stress
Displacements and stresses at node 1 (Fig. 3.4) from Taylor series
with optimum numbers of terms and with the internal point at (4., 0.) and (2.,0.)
using ASFA and twenty quadratic elements

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic:	0.	- 1.4625	- 12.	0.	0.
(4, 0.)	13	+1.	- 1.505	- 14.523	+0.130	- 0.055
(4, 0.)	17	+1.	- 1.436	- 14.504	+0.153	- 0.074
(2, 0.)	17	+0.468	- 1.445	- 11.430	- 0.083	+0.131

With the internal point at (4., 0.) using the third and higher order terms in Taylor series, the stresses obtained from DEA and NPA are in good agreement with the analytical results, whereas the results for the displacement in the y direction differ from that of the analytical solution by about 10%. The results from ASFA are, however, not encouraging, and moreover with some numbers of terms they are identical (see Table G.3 in appendix G).

With the present mesh the best results are obtained when the internal point is placed at (4., 0.), i.e., in the middle of the cantilever. As the internal point moves closer to the local point of interest, i.e., to (2., 0.), the results become inaccurate.

It is necessary to refine the mesh and use smaller elements in the local area in order to position the internal point much closer to the local point of interest. Hence, elements 1 and 20 were each subdivided into ten elements having the length ratio of 1.5. Although the mesh refinement gave just a little improvement (but not very much despite the increasing cost) in the results from ASFA the accuracy achieved with the other two methods (DEA and NPA) remained almost the same, hence none of these results are reported here.

With the experience based on the previous problem the Fourier series (2.73) was also applied as a trial function to solve this problem. The results obtained were very inaccurate. The reason for this is perhaps that the stress distributions are more complex than those in the previous simple problem, and the Fourier series in x only is not capable of representing the solution. Therefore, Fourier series in x and y should be superposed in order to be able to obtain satisfactory results (see p. 53, Timoshenko & Goodier, 1984). Since the results obtained using the Fourier series in the present form, eqn (2.73), did not give good results they are not given here.

If Tables (G.1) and (G.2) are compared it can be seen that the best results are obtained using the following numbers of terms: 7, 9, 11 (for NPA) and 10, 14, and 18 (for DEA), and other intermediate numbers of terms give slightly less accurate results, though they are still close to the values obtained for above numbers.

3.4.3. Stress Concentration at a Hole in a Square Plate under Tension

Consider that the square flat plate in section 3.4.1 has now a hole of radius a in the middle free of external forces as shown in Fig. 3.5. The existence of the hole produces stress distributions which have local peaks due to the discontinuity there.

Because of symmetry about both coordinate directions only one quadrant needs to be discretised. The ratio of the radius of the hole to the half width of the plate is taken to be $a/L=0.05$, and the other dimensions and the material properties being the same as given in the first problem (see section 3.4.1).

The first solution scheme for this problem used 26 quadratic boundary elements with the length ratio $R=1.4$ for the constrained elements along the x and y coordinate directions as shown in Fig. 3.5. Due to rapid stress variations attempts to apply the local solution techniques on their own were not successful. Hence, the LBIE methods (namely, DEA, NPA, and ASFA) were used to locally refine the solution of the problem.

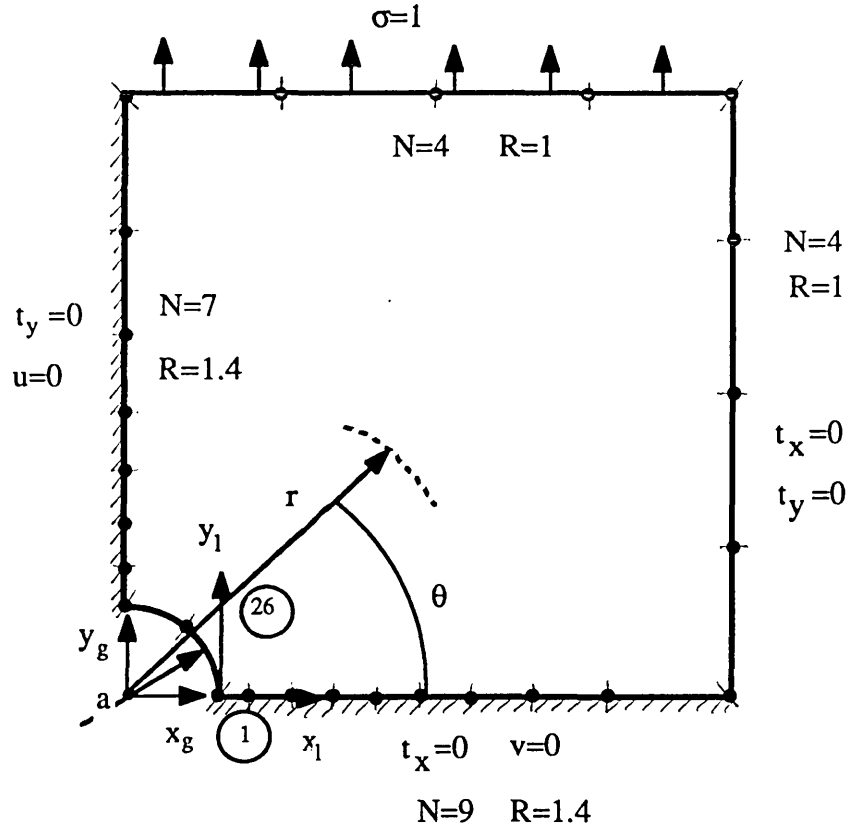


Figure 3.5 : One quadrant of a square flat plate subject to uniform tension with a hole in the middle. N stands for the number of elements and R for the length ratio.

For this reason, a mesh refinement was made in order to let the internal point p be positioned close to the local point of interest at node 1. The segment constrained in the y coordinate direction was divided into two parts, the first adjacent to the local point having the length of 0.05 and the second 0.9. Then, this first part and the element 26 were each subdivided into ten elements, making 44 elements in all. The ratio between the lengths of the successive subelements moving away from the local point of expansion was taken as 1.1. This gave elements immediately adjacent to the local point of length 0.003 along the x axis and 0.0025 along the surface of the hole. On the remainder of the boundary the same length ratios and numbers of elements of the mesh shown in Fig. 3.5 were used.

The analytical solution for an infinite plate, in polar coordinates, is given by⁽³⁷⁾

$$\sigma_{rr} = \frac{\sigma}{2} \left[\left(1 - \left(\frac{a}{r}\right)^2\right) - \left(1 - 4\left(\frac{a}{r}\right)^2 + 3\left(\frac{a}{r}\right)^4\right) \cos(2\theta) \right] \quad (3.165)$$

$$\sigma_{\theta\theta} = \frac{\sigma}{2} \left[\left(1 + \left(\frac{a}{r}\right)^2\right) + \left(1 + 3\left(\frac{a}{r}\right)^4\right) \cos(2\theta) \right] \quad (3.166)$$

$$\sigma_{r\theta} = \frac{\sigma}{2} \left[1 + 2\left(\frac{a}{r}\right)^2 - 3\left(\frac{a}{r}\right)^4 \right] \sin(2\theta) \quad (3.167)$$

and

$$u_r = \frac{\sigma}{8\mu r} \left[(\chi - 1)r^2 + 2a^2 - 2\left(a^2(\chi + 1) + r^2 - \frac{a^4}{2}\right) \cos(2\theta) \right] \quad (3.168)$$

$$u_\theta = \frac{\sigma}{4\mu r} \left[a^2(\chi - 1) + r^2 + \frac{a^4}{2} \right] \sin(2\theta) \quad (3.169)$$

where r and θ are the polar coordinates, a is the radius of the hole, σ is the applied stress (see Fig. 3.5), and

$$\chi = 3 - 4\nu, \quad \text{if plane strain}$$

$$\chi = \frac{3-\nu}{1+\nu}, \quad \text{if plane stress} \quad (3.170)$$

We note that the greatest local concentration of stress occurs at node 1 in the hoop direction, and has the value 3. This solution does, however, not represent the exact solution, because the plate has finite dimensions. A correction of the solution for a

finite width becomes necessary, and it is given as 3.008 for a plate with above dimensions (see Remzi, 1981). In the local refinement procedure the above analytical solution, eqns (3.165) to (3.169), was applied as the remote field function outside the local area.

Table 3.15 shows the optimum results obtained using DEA. The results are very good, when the numbers of terms used are between 10 to 18. The accuracy obtained for less than 10 terms is very poor (see Table H.1 in appendix H). It is also of interest that the best results are obtained when only 10, 14, and 18 terms are used, which was the case for the cantilever beam, and other intermediate numbers of terms give less accurate results. This is especially marked with the internal points at (0.06, 0.01) and (0.057, 0.007). When the internal point is positioned at (0.055, 0.005) and the number of terms used is 18 the highest accuracy is attained. This gives the stress concentration as 3.009, which is within 0.03% of the above quoted value. The other two components of the stress are also accurate up to the third decimal place. The values of the displacements are correct to three decimal places (see Table H.1 in appendix H for the full range of the results).

Table 3.15 : Plate with a hole in the middle subject to uniform stresses at the ends
Displacements and stresses at node 1 (Fig. 3.5) from Taylor series
using DEA and refined boundary elements in the region of interest
and remote field solution outside the local area
with optimum numbers of terms and different positions for internal point.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
Analytic: - 0.05			0.	0.	0.	3.008
(0.060, 0.010)	14	- 0.049	- 0.002	- 0.070	- 0.046	+2.988
(0.057, 0.007)	14	- 0.050	- 0.000	- 0.029	- 0.018	+2.995
(0.055, 0.005)	18	- 0.050	+0.000	- 0.001	- 0.000	+3.009
(0.054, 0.004)	18	- 0.050	+0.000	+0.002	- 0.001	+3.010
(0.053, 0.003)	14	- 0.050	- 0.000	- 0.004	- 0.002	+2.998
(0.0525, 0.0025)	10	- 0.050	+0.000	- 0.001	- 0.005	+3.013

Table 3.16 shows the optimum results obtained using NPA. Note they are identical with those in Table 3.15. This is to be expected after having obtained the results for the previous two problems.

Table 3.16 : Plate with a hole in the middle subject to uniform stresses at the ends
Displacements and stresses at node 1 (Fig. 3.5) from Taylor series
using NPA and refined boundary elements in the region of interest
and remote field solution outside the local area
with optimum numbers of terms and different positions for internal point.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic:	- 0.05	0.	0.	0.	3.008
(0.060, 0.010)	9	- 0.049	- 0.002	- 0.070	- 0.046	+2.988
(0.057, 0.007)	9	- 0.050	- 0.000	- 0.029	- 0.018	+2.995
(0.055, 0.005)	11	- 0.050	+0.000	- 0.001	- 0.000	+3.009
(0.054, 0.004)	11	- 0.050	+0.000	+0.002	- 0.001	+3.010
(0.053, 0.003)	9	- 0.050	- 0.000	- 0.004	- 0.002	+2.998
(0.0525, 0.0025)	7	- 0.050	+0.000	- 0.001	- 0.005	+3.013

Table 3.17 shows the results obtained for ASFA. Although some of the results are encouraging ASFA is still not as good as DEA and NPA, and moreover depending on the number of terms and locations of the internal point the results suddenly change, hence they are not reliable (see also Table H.3 in appendix H).

Table 3.17 : Plate with a hole in the middle subject to uniform stresses at the ends
Displacements and stresses at node 1 (Fig. 3.5) from Taylor series
using ASFA and refined boundary elements in the region of interest
and remote field solution outside the local area
with optimum numbers of terms and different positions for internal point.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic:	- 0.05	0.	0.	0.	3.008
(0.060, 0.010)	17	- 0.050	+0.014	+0.049	- 0.141	+2.989
(0.057, 0.007)	13	- 0.042	+0.016	+0.236	- 0.005	+2.980
(0.055, 0.005)	9	- 0.050	+0.001	+0.049	- 0.141	+2.979
(0.054, 0.004)	21	- 0.050	- 0.014	- 0.023	+0.342	+2.984
(0.053, 0.003)	21	- 0.048	- 0.006	- 0.325	+0.123	+2.937
(0.0525, 0.0025)	13	- 0.050	+0.007	+0.075	- 0.329	+2.994

3.4.4. Cracked Beam in Three-Point Bending

The last problem shows that the presence of a geometric discontinuity causes local stress concentrations. When this discontinuity is a sharp re-entrant corner (e.g., a notch), not smooth like the one in the previous case, the stress variations become even more severe around that notch, and as the notch becomes a crack (where the radius of the notch is now zero) the stresses at the crack tip give rise to infinity.

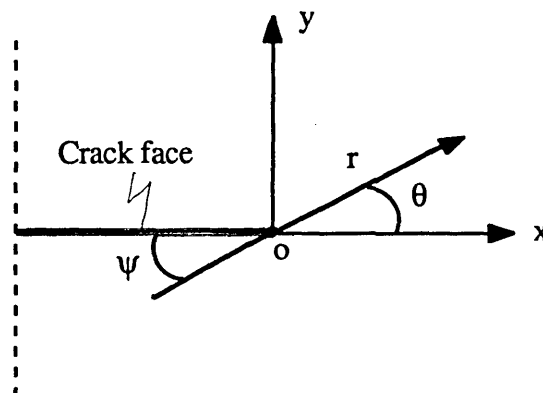


Figure 3.6 : A sharp crack with the coordinate system employed by Williams.

The severity of the stresses at the crack is determined by the stress intensity factor denoted as K_N , where $N=1, 2, 3$ indicates the three modes of fracture which can occur. Here, attention is confined only to mode I (K_I), sometimes called as the opening mode involving stresses which are symmetrical about $\theta=0$, since it is the most common type of fracture (for detailed information the reader is referred to fracture mechanics books, see also references 16, 23, and 34).

Fig. 3.7 shows a beam of rectangular cross section simply supported at its ends, having the dimensions of length L , width w , and depth d . At the lower side in the middle of the beam a crack with a length of a is introduced. On the opposite side a line load of magnitude S is applied.

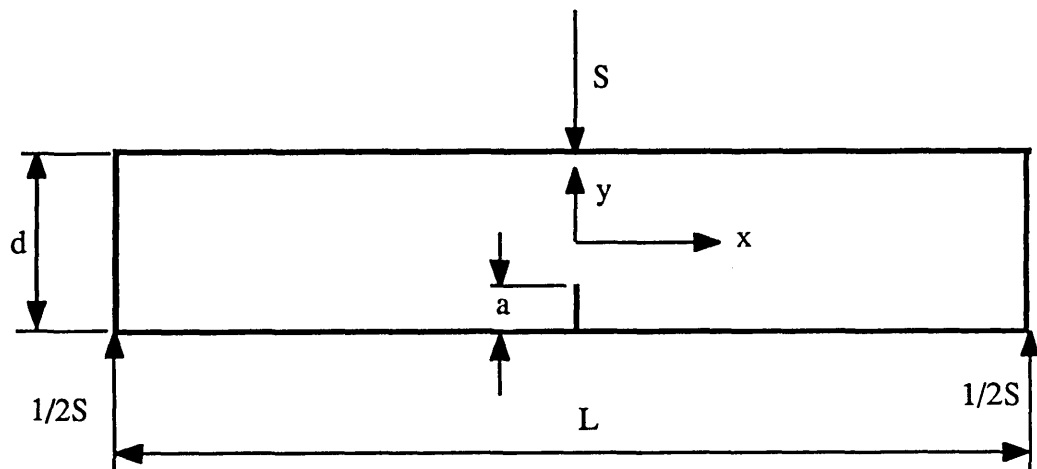


Figure 3.7 : Cracked beam subject to three-point bending.

For present purposes the numerical values and material properties are taken as

$$L=8, \quad d=1, \quad a=0.2, \quad w=1, \quad S=2, \quad E=1, \quad \nu=0.3$$

and a state of plane stress is assumed.

With the beam dimensions and load given above the quoted value for the stress intensity factor is 19.36, or in non-dimensional form⁽¹⁶⁾

$$\frac{K_I w d^{3/2}}{M} = 4.84$$

where the bending moment at the mid-section of the beam is $M=SL/4$

Only one half of the beam need be modelled, taking the advantage of symmetry with respect to the x coordinate direction. Since for a crack problem stress variations are very severe a fine mesh is used so that accurate integrations can be performed. Fig. 3.8 shows the mesh of quadratic line elements used to represent that half of the beam, together with the boundary conditions⁽¹⁶⁾ (58 elements in all, having the smallest elements immediately adjacent to the local point, which is now at the crack tip, of length 0.0094).

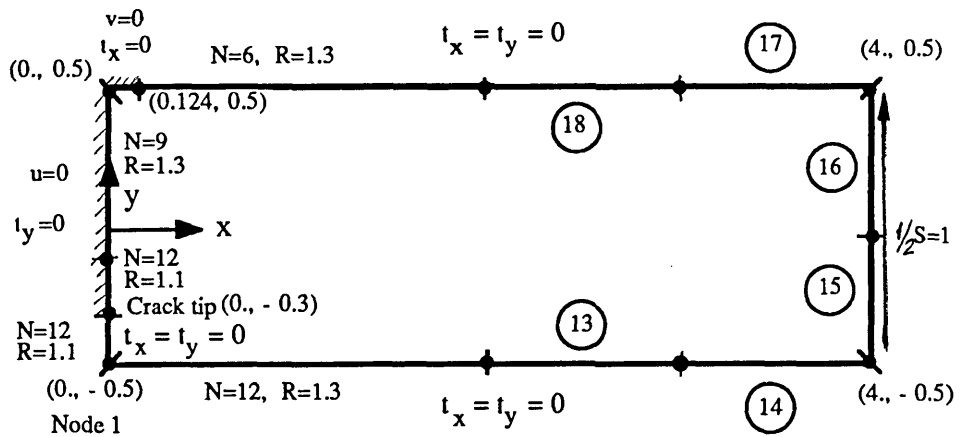


Figure 3.8 : Mesh of the one half of the cracked beam together with the boundary conditions. N stands for the numbers of elements and R for the length ratios of these elements.

Attempts to apply the LBIE methods* on their own to obtain reasonable values for the stress intensity factor were not successful, at all. This was to be expected after the experience based on the previous stress concentration problem solved. Therefore, the LBIE techniques (DEA and NPA) were also applied to locally refine the solution obtained by other means. This time, numerical values of displacements and stresses obtained from BEETS were used outside the local area as the remote field solution. The LBIE methods in the form already described were still not successful, unlike with the stress concentration problem. It was first thought that the reason for this was that the Taylor series was not capable of representing the rapid stress variations around the crack tip. Therefore, there was a need for a different type of series which could represent the solution well in the local area of interest, in the place of special crack tip elements of conventional BIE methods. The series considered was the polynomial one initially used by Williams. It is given by⁽⁴⁸⁾

$$\begin{aligned}\phi(r,\psi) = \sum_{m=1}^N r^{(m/2+1)} \left\{ \alpha_m \left[\sin\left(\frac{m}{2}-1\right)\psi - \frac{m-2}{m+2} \sin\left(\frac{m}{2}+1\right)\psi \right] \right. \\ \left. + \beta_m \left[\cos\left(\frac{m}{2}-1\right)\psi - \cos\left(\frac{m}{2}+1\right)\psi \right] \right\} \quad (3.171)\end{aligned}$$

which satisfies $\nabla^4\phi=0$, eqn (3.69). By writing this equation in terms of the bisector angle $\theta = \psi - \pi$, and retaining only the first terms, we have

$$\begin{aligned}\phi(r,\theta) = r^{3/2} \left[\alpha_1 \left(-\cos(\theta/2) - \frac{1}{3} \cos(3\theta/2) \right) + \right. \\ \left. \beta_1 \left(-\sin(\theta/2) - \sin(3\theta/2) \right) \right] \quad (3.172)\end{aligned}$$

* ASFA has now been abandoned, since it was, in general, unsuccessful.

where the first part in α_1 represents the symmetrical solution and the second part in β_1 the antisymmetrical solution⁽⁴⁸⁾. Since we are interested only in mode I the second part can be omitted.

The associated stresses, in polar coordinates, are then given by⁽⁴⁸⁾

$$\sigma_{rr}(r,\theta) = \frac{1}{4r^{1/2}} \alpha_1 [-5 \cos(\theta/2) + \cos(3\theta/2)] \quad (3.173)$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{1}{4r^{1/2}} \alpha_1 [-3 \cos(\theta/2) - \cos(3\theta/2)] \quad (3.174)$$

$$\sigma_{r\theta}(r,\theta) = \frac{1}{4r^{1/2}} \alpha_1 [-\sin(\theta/2) - \sin(3\theta/2)] \quad (3.175)$$

and the displacements by⁽⁴⁸⁾

$$u_r(r,\theta) = \frac{r^{1/2}}{4\mu} \alpha_1 [(2\chi - 1) \cos(\theta/2) - \cos(3\theta/2)] \quad (3.176)$$

$$u_\theta(r,\theta) = \frac{r^{1/2}}{4\mu} \alpha_1 [-(2\chi + 1) \sin(\theta/2) + \sin(3\theta/2)] \quad (3.177)$$

where χ is given by the relations (3.170).

Equations (3.176) and (3.177) both can be used with DEA. Note that the first parts of eqns (3.176) and (3.177) (i.e., $r^{1/2}\cos(\theta/2)$ and $r^{1/2}\sin(\theta/2)$) are harmonic, hence they can be used with NPA.

The same cracked beam with the same fine discretisation (58 elements with local length ratio of 1.1 and smallest elements of length 0.0094 immediately adjacent to the crack tip) was solved. This time, the size of the local area was gradually decreased starting from the crack face end at the corner to the crack tip. Outside the local area the same numerical solution from BEETS was again used as the remote field solution. Since the first terms in eqn (3.172), or eqns (3.173) to (3.177), dominates the solution in the vicinity of the crack tip it was expected to obtain more satisfactory results as the local area was reduced in size. Although there was a significant improvement in the results they were still not reasonably accurate, especially stresses.

Finally, the local area was reduced to zero, which meant that no series approximation made along the boundary except at the internal point, and the results were still far from satisfactory. Moreover, in all the results obtained for this crack problem it was observed that the results were very sensitive to changes in the position of the internal point and the numbers of terms used.

3.5. Discussion

In this chapter, the LBIE method developed for two-dimensional potential problems has now been extended to two-dimensional elasticity problems. Although the LBIE methods for stress problems essentially used the same principles as that for potential problems they are now formulated in three different ways. They are as follows;

- i) Airy's stress function approach (ASFA),
- ii) Neuber-Papkovich approach (NPA),
- iii) Direct expansion approach (DEA).

These three formulations use either harmonic functions or biharmonic ones. Accordingly, the functions in the series (Taylor, Fourier, or crack polynomial series) are chosen to satisfy either the harmonic equation or the biharmonic equation.

In the first simple problem, namely, a square flat plate under simple tension the three LBIE methods produced very accurate results, while for the second problem, of a cantilever beam, the ASFA was found to be unsuccessful. This was perhaps due to the error incurred by numerical integration of the conjugate functions ρ_x and ρ_y . Note that these two functions are determined from the harmonic functions P and Q using relations (3.87) and (3.88), and during their calculations the integration constants in the indefinite integrals were assumed to be zero. This is here a valid assumption since those constants vanish when the integrals become definite, in other words, when the integration limits are known. However, since the integrals are evaluated numerically the assumption of zero constants in the analytical expressions cease to be valid, and the additional error is produced. This is especially marked when stress variations are relatively complicated, and where there is a need for a fine mesh to obtain reasonable integration accuracy. The results of the stress concentration problem tend to confirm this explanation, because in this case the LBIE methods were used to locally refine the solution. The results from the ASFA were not as good as those from the DEA and NPA, both of which gave very accurate results. It is noticeable, however, that the ASFA now gave much better accuracy than that for the cantilever. This was because no approximation in ρ_x and ρ_y was made in the remote field, where the analytical solution was applied, and in the region of interest a fine mesh was used so that the integration accuracy was enhanced. Nevertheless, since the ASFA was, in general, comparatively unsatisfactory and unreliable it has been abandoned until a way can be found to determine the constants in ρ_x and ρ_y , indirectly (or may be directly). Note that the ASFA method would be the most efficient one, if it worked as well as the other two, because of using a smaller number of terms. In the other two, for a given number of terms in the series, the number of unknowns is doubled because the displacements (and the tractions) are expressed by two different undetermined coefficients (α_{km} , where $k=1, 2$ in two

dimensions and m is the number of terms): compare eqns (3.131), (3.134) for NPA and (3.145), (3.148) for DEA with eqns (3.113), (3.115) for ASFA.

Also, note that some of the results from the DEA and NPA for the problems solved here are identical. This is to be expected because these two formulations involve the analytical expressions of displacements (or stresses) in two different ways, and hence the functions (harmonic or biharmonic) used to represent the displacement field are obviously related (see the footnote on p. 99). In view of the foregoing discussion, the ASFA, which uses the biharmonic function ϕ and the harmonic ones ρ_i to represent the displacement field (see eqn (3.90)), is also expected to produce the same results as those from the other two (DEA and NPA). But it did not give the same results because of the existence of the integration constants in ρ_x and ρ_y , as mentioned before. The results from the DEA and NPA are identical only when the following numbers of terms are used: 6, 10, 14, 18 for DEA and 5, 7, 9, 11 for NPA, respectively (the corresponding numbers for ASFA would be 9, 13, 17, and 21, see Pascal triangles of differentiation, Table 2.1 and 3.2). Other intermediate numbers do not produce the same results because they do not provide corresponding analytical expressions. For this reason the above numbers are here referred to as the corresponding numbers of terms, and the remainder as intermediate numbers of terms. It is also of interest that in general, the best results are obtained for the corresponding numbers of terms. Hence, it is reasonable to retain only the corresponding numbers of terms in the series (Taylor) except for very simple problems where only few terms are enough to produce their analytical solutions. It is, therefore, clear that the NPA is superior to the DEA in that it uses a smaller number of terms for the same problem with the same accuracy, also in that it is easier to handle, since it uses harmonic functions, unlike the DEA which uses biharmonic functions.

Having obtained satisfactory results for the potential problems in chapter 2 with the Fourier series method, it was also considered for stress problems. For this reason, the sine Fourier series in x satisfying the biharmonic equation (3.69), eqn (3.103), was employed for the DEA (and ASFA) method, since they use biharmonic functions. Due

to inaccurate results, a combination of cosine and sine Fourier series in x became necessary. Since this form of Fourier series (eqn (2.73)) had already been developed in chapter 2 to solve the potential problems, it was used with the NPA. For the square plate under simple tension, quite satisfactory results were obtained. It did not, however, give accurate results for the cantilever where the stress distributions were more complicated. Therefore, a Fourier series in x and y should be superposed in order to obtain satisfactory results.

For the last problem, namely, the cracked beam in three-point bending, where the variations of stresses are very rapid and tend to infinity at the crack tip, Taylor series was also unable to represent the solution, even in the local refinement procedure. Hence, in the vicinity of the crack tip the polynomial stress function of Williams, eqn (3.171), with the first term retained was employed. With the DEA the displacements, eqns (3.176) and (3.177), derived from eqn (3.172) were used, while the harmonic parts of them were used with the NPA. Although the results dramatically improved when the number of terms was between 14-18, they were still not reasonably accurate, especially stresses. It was found that the results were very sensitive to changes in the position of the internal point and the number of terms used. Finally, to see whether the error was introduced by the series approximation on the boundary or the LBIEs were not able to produce the solution of rapid stress variations, no series approximation on the boundary was made, except at the internal point, and the results were still not satisfactory. There could be two reasons for this: i) The LBIE methods are not capable of representing the rapid stress variations due to highly ill-conditioned linear equations with increasing order of singularity in the higher-order BIEs. Note that with the experience based on the problems (either potential or stress) solved so far the number of terms should be increased as the function variations increase, and for a crack problem the larger this number the better the function representation is. ii) A human error was involved in coding the program during the modification to employ the crack tip series solution*.

* The program 'LBIE2PE' was checked twice, and no error was found.

The failure to obtain satisfactory results for the cracked beam does not, however, detract from the value of the LBIE methods presented here. They should be successfully used to locally refine the solutions of stress concentration problems, provided the stress variations are not as severe as at a sharp crack, and also other relatively simple problems can be solved very accurately with reduced costs.

In order to successfully apply the LBIE methods to problems of interest attention should be given to the choice of three parameters, as is the case for potential problems. Although these are mentioned at the end of chapter 2 in section 2.5 they are repeated here for completeness.

i) Number of terms retained in the series,

ii) location of the internal point within the region of interest,

iii) the ratio between the element lengths in the local area (particularly two elements immediately adjacent to the local point of interest) and the distance from the internal point to these elements (more correctly, to the local point of interest).

A careful balance should be kept between these three parameters. To be able to represent the stress variations well the number of terms in the series should, in general, be large (the larger this number the better the function representation is). There is, however, a limit to this number. The limit is provided by the accuracy with which the highly-singular kernel functions can be integrated (this can be best seen in the crack problem, as explained before). It is observed that the results are quite satisfactory when the numbers of terms are approximately 7-11 for NPA (or 10-18 for DEA). Although for relatively simple problems fewer can be used it is, however, much better to choose to retain 7-11 terms in the series, since in a practical problem not its analytical solution but only the boundary conditions are known. The results can be improved by positioning the internal point at distances from the local point of the same order as the length of the local elements there (the two elements adjacent to the local point of interest), provided the aforementioned three parameters are kept in due proportion. By observing these

three parameters it is possible to use more and more refined meshes in the local area to enhance the accuracy as required. This, however, does not increase the number of algebraic equations, unlike with conventional BIE methods. Therefore, the costs of the mesh refinement are much less compared to conventional BIE methods.

C H A P T E R 4

4. CONCLUSIONS

In this thesis, local boundary integral equation methods based on series expansions have been developed for both potential and stress problems. The series considered are Taylor and Fourier ones using either harmonic or biharmonic functions. The LBIE techniques presented here make it possible to determine the variables of interest in a small particular region of a solution domain, without the need to evaluate the solution over the rest of the boundary.

For potential problems, it has been found that both series methods produce very accurate results in regions of particular interest and they can be used in their own rights, whereas for stress problems the Fourier series method in the form described here cannot represent the solution, if the function variations are relatively complicated.

The major application for the technique is to problems where detailed information is required at particular geometric features of the domain for which approximate solutions already exist. This is because attention can be focused on a small pre-selected region, and therefore it provides a way of locally refining approximate solutions obtained by other means. Although the LBIE methods as described in chapter 3 did not represent the crack tip solutions it should not discredit them to be applicable to many stress problems having less severe variations like the stress concentration problem solved in section 3.4.3.

Because the number of linear equations to be solved is fixed for a given number of terms in the series and is very small, the technique is computationally very efficient compared to conventional numerical techniques, and consequently it is probably most beneficial for three-dimensional problems.

4.1. Further Suggestions

Although the LBIE methods developed here were, in general, successful they are not completely free of drawbacks. Firstly, in order to be able to apply the technique to problems of interest successfully a careful balance between the three parameters should be kept (see sections 2.5 and 3.5), otherwise the results obtained could be inaccurate. This makes it relatively difficult for an inexperienced user of the technique to apply. Secondly, when the variations of the variables of interest are very rapid, then the number of terms to be used should be large to represent the solution well, but this necessitates the use of highly singular kernel functions. If the variations are very severe, like those in the crack problem, it becomes impossible to maintain high accuracy in numerical integrations to obtain reasonable values for the unknowns. To overcome these problems, the zero and higher-order equations can be transformed to the boundary, as in the form described by J.O. Watson (1982), and expanded in the series about a point of interest on the boundary. Doing this, the need to consider the three parameters may be eliminated, and it may also be possible to use a relatively small number of terms, which implies relatively less singular kernel functions have to be employed. Watson did this using only up to (and including) the second order integral equations, but essentially in a conventional way using Hermitian cubic boundary elements (with additional singular shape functions at notch and crack tips), and he obtained quite satisfactory results.

Let us now recall the functions p_x and p_y in eqn (3.90). As we mentioned in the previous section, 3.5, the integration constants may be determined indirectly from certain boundary constraints (if possible), or they may be determined by comparing the known boundary condition values with the calculated values from the solution of integral equations. The difference between them is the relative error which has to be either added to or subtracted from eqn (3.90) and re-run the program again. This, of course, means that now the program should be run successively until the error is reduced to the required accuracy, which means an increase in computing costs (the possibility of this should also be investigated). Those constants may also be calculated

directly by the analytical integration outside the local area, e.g., by using constant elements there to simplify the procedure.

This analytical integration may be carried out not only for determining those integration constants but also it would be possible to perform the integrations analytically for the LBIE methods as a whole. The problem arose is that the boundary conditions in the remote field must now be written in the form of analytical expressions. This may be managed by a curve fitting process applied to the boundary conditions. The accuracy should be enhanced substantially. The displacement and traction kernel functions, eqns (3.34) and (3.35), can also be differentiated analytically, since their derivatives are lengthy and very tedious to derive manually. All these integrations and differentiations can be performed using a suitable symbolic programming language, e.g., ALTRAN*.

Finally, in the LBIE methods, besides Taylor and Fourier series other types of series can be employed provided they satisfy the required conditions. That is, they should satisfy either the harmonic or the biharmonic equation, and they should converge with finite number of terms. One example is the general stress function series (see p. 133, eqn (80), Timoshenko & Goodier, 1984). Also, a combination of Taylor and Fourier series could be used.

* ALTRAN stands for ALgebra TRANslator and can perform symbolic computations on algebraic data. It can be called within a FORTRAN program.

A P P E N D I C E S

A P P E N D I X A

Transformation Formulae

Transformation formulae for the derivatives of a function, say ϕ , can be found with the aid of differential operators given in section 2.1.6 as,

$$\left(\frac{\partial\phi}{\partial x}\right)_1 = L_{x_1}\{\phi\}, \quad \left(\frac{\partial\phi}{\partial y}\right)_1 = L_{y_1}\{\phi\} \quad (2.78)$$

and

$$\left(\frac{\partial\phi}{\partial x}\right)_g = L_{x_g}\{\phi\}, \quad \left(\frac{\partial\phi}{\partial y}\right)_g = L_{y_g}\{\phi\} \quad (2.79)$$

where

$$\begin{aligned} L_{x_g} &= \left\{ \frac{\partial}{\partial x_1} \cos\alpha - \frac{\partial}{\partial y_1} \sin\alpha \right\} \\ L_{y_g} &= \left\{ \frac{\partial}{\partial x_1} \sin\alpha + \frac{\partial}{\partial y_1} \cos\alpha \right\} \\ L_{x_1} &= \left\{ \frac{\partial}{\partial x_g} \cos\alpha + \frac{\partial}{\partial y_g} \sin\alpha \right\} \\ L_{y_1} &= \left\{ -\frac{\partial}{\partial x_g} \sin\alpha + \frac{\partial}{\partial y_g} \cos\alpha \right\} \end{aligned} \quad (2.80)$$

Hence, higher-order operators are simply the operation of the operators on each other and it only remains to operate with them on the functions. They are given (up to the fifth-order) as follows:

From the local to global coordinate system,

$$L_{x_1x_1} = \left\{ \frac{\partial^2}{\partial x^2} \cos^2 \alpha + 2 \frac{\partial^2}{\partial x \partial y} \cos \alpha \sin \alpha + \frac{\partial^2}{\partial y^2} \sin^2 \alpha \right\}_g \quad (A.1)$$

$$L_{x_1y_1} = \left\{ -\frac{\partial^2}{\partial x^2} \cos \alpha \sin \alpha + \frac{\partial^2}{\partial x \partial y} \cos 2\alpha + \frac{\partial^2}{\partial y^2} \cos \alpha \sin \alpha \right\}_g \quad (A.2)$$

$$L_{y_1y_1} = \left\{ \frac{\partial^2}{\partial x^2} \sin^2 \alpha - 2 \frac{\partial^2}{\partial x \partial y} \sin \alpha \cos \alpha + \frac{\partial^2}{\partial y^2} \cos^2 \alpha \right\}_g \quad (A.3)$$

$$L_{x_1x_1x_1} = \left\{ \frac{\partial^3}{\partial x^3} \cos^3 \alpha + 3 \frac{\partial^3}{\partial x^2 \partial y} \cos^2 \alpha \sin \alpha + 3 \frac{\partial^3}{\partial x \partial y^2} \cos \alpha \sin^2 \alpha + \frac{\partial^3}{\partial y^3} \sin^3 \alpha \right\}_g \quad (A.4)$$

$$L_{x_1x_1y_1} = \left\{ -\frac{\partial^3}{\partial x^3} \cos^2 \alpha \sin \alpha + \frac{\partial^3}{\partial x^2 \partial y} (\cos^3 \alpha - 2 \cos \alpha \sin^2 \alpha) + \frac{\partial^3}{\partial x \partial y^2} (2 \cos^2 \alpha \sin \alpha - \sin^3 \alpha) + \frac{\partial^3}{\partial y^3} \sin^2 \alpha \cos \alpha \right\}_g \quad (A.5)$$

$$L_{x_1y_1y_1} = \left\{ +\frac{\partial^3}{\partial x^3} \sin^2 \alpha \cos \alpha + \frac{\partial^3}{\partial x^2 \partial y} (\sin^3 \alpha - 2 \sin \alpha \cos^2 \alpha) - \frac{\partial^3}{\partial x \partial y^2} (2 \sin^2 \alpha \cos \alpha - \cos^3 \alpha) + \frac{\partial^3}{\partial y^3} \cos^2 \alpha \sin \alpha \right\}_g \quad (A.6)$$

$$\begin{aligned}
L_{y_1 y_1 y_1} = & \left\{ -\frac{\partial^3}{\partial x^3} \sin^3 \alpha + 3 \frac{\partial^3}{\partial x^2 \partial y} \cos \alpha \sin^2 \alpha - 3 \frac{\partial^3}{\partial x \partial y^2} \cos^2 \alpha \sin \alpha \right. \\
& \left. + \frac{\partial^3}{\partial y^3} \cos^3 \alpha \right\}_g
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
L_{x_1 x_1 x_1 x_1} = & \left\{ \frac{\partial^4}{\partial x^4} \cos^4 \alpha + 4 \frac{\partial^4}{\partial x^3 \partial y} \cos^3 \alpha \sin \alpha + 6 \frac{\partial^4}{\partial x^2 \partial y^2} \cos^2 \alpha \right. \\
& \left. \sin^2 \alpha + 4 \frac{\partial^4}{\partial x \partial y^3} \cos \alpha \sin^3 \alpha + \frac{\partial^4}{\partial y^4} \sin^4 \alpha \right\}_g
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
L_{x_1 x_1 x_1 y_1} = & \left\{ -\frac{\partial^4}{\partial x^4} \cos^3 \alpha \sin \alpha + \frac{\partial^4}{\partial x^3 \partial y} (\cos^4 \alpha - 3 \cos^2 \alpha \sin^2 \alpha) + \right. \\
& \frac{\partial^4}{\partial x^2 \partial y^2} (3 \cos^3 \alpha \sin \alpha - 3 \cos \alpha \sin^3 \alpha) + \frac{\partial^4}{\partial x \partial y^3} (3 \cos^2 \alpha \\
& \left. \sin^2 \alpha - \sin^4 \alpha) + \frac{\partial^4}{\partial y^4} \sin^3 \alpha \cos \alpha \right\}_g
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
L_{x_1 x_1 y_1 y_1} = & \left\{ \frac{\partial^4}{\partial x^4} \cos^2 \alpha \sin^2 \alpha + \frac{\partial^4}{\partial x^3 \partial y} 2 (\cos \alpha \sin^3 \alpha - \sin \alpha \cos^3 \alpha) \right. \\
& + \frac{\partial^4}{\partial x^2 \partial y^2} (\cos^4 \alpha - 4 \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha) + \frac{\partial^4}{\partial x \partial y^3} 2 \\
& \left. (\cos^3 \alpha \sin \alpha - \cos \alpha \sin^3 \alpha) + \frac{\partial^4}{\partial y^4} \cos^2 \alpha \sin^2 \alpha \right\}_g
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
L_{x_1 y_1 y_1 y_1} = & \left\{ -\frac{\partial^4}{\partial x^4} \cos \alpha \sin^3 \alpha + \frac{\partial^4}{\partial x^3 \partial y} (3 \cos^2 \alpha \sin^2 \alpha - \sin^4 \alpha) + \right. \\
& \frac{\partial^4}{\partial x^2 \partial y^2} 3 (\cos \alpha \sin^3 \alpha - \cos^3 \alpha \sin \alpha) + \frac{\partial^4}{\partial x \partial y^3} (\cos^4 \alpha \\
& \left. - 3 \cos^2 \alpha \sin^2 \alpha) + \frac{\partial^4}{\partial y^4} \cos^3 \alpha \sin \alpha \right\}_g \quad (A.11)
\end{aligned}$$

$$\begin{aligned}
L_{y_1 y_1 y_1 y_1} = & \left\{ \frac{\partial^4}{\partial x^4} \sin^4 \alpha - 4 \frac{\partial^4}{\partial x^3 \partial y} \sin^3 \alpha \cos \alpha + 6 \frac{\partial^4}{\partial x^2 \partial y^2} \cos^2 \alpha \right. \\
& \left. \sin^2 \alpha - 4 \frac{\partial^4}{\partial x \partial y^3} \sin \alpha \cos^3 \alpha + \frac{\partial^4}{\partial y^4} \cos^4 \alpha \right\}_g \quad (A.12)
\end{aligned}$$

$$\begin{aligned}
L_{x_1 x_1 x_1 x_1} = & \left\{ +\frac{\partial^5}{\partial x^5} \cos^5 \alpha + 5 \frac{\partial^5}{\partial x^4 \partial y} \cos^4 \alpha \sin \alpha + 10 \frac{\partial^5}{\partial x^3 \partial y^2} \right. \\
& \cos^3 \alpha \sin^2 \alpha + 10 \frac{\partial^5}{\partial x^2 \partial y^3} \cos^2 \alpha \sin^3 \alpha + 5 \frac{\partial^5}{\partial x \partial y^4} \\
& \left. \cos \alpha \sin^4 \alpha + \frac{\partial^5}{\partial y^5} \sin^5 \alpha \right\}_g \quad (A.13)
\end{aligned}$$

$$\begin{aligned}
L_{x_1 x_1 x_1 y_1} = & \left\{ -\frac{\partial^5}{\partial x^5} \cos^4 \alpha \sin \alpha + \frac{\partial^5}{\partial x^4 \partial y} (\cos^5 \alpha - 4 \cos^3 \alpha \sin^2 \alpha) \right. \\
& \left. + \frac{\partial^5}{\partial x^3 \partial y^2} (4 \cos^4 \alpha \sin \alpha - 6 \cos^2 \alpha \sin^3 \alpha) + \frac{\partial^5}{\partial x^2 \partial y^3} \right.
\end{aligned}$$

$$\begin{aligned}
& (6 \cos^3 \alpha \sin^2 \alpha - 4 \cos \alpha \sin^4 \alpha) + \frac{\partial^5}{\partial x \partial y^4} (4 \cos^2 \alpha \\
& \sin^3 \alpha - \sin^5 \alpha) + \frac{\partial^5}{\partial y^5} \cos \alpha \sin^4 \alpha \} _g \quad (A.14)
\end{aligned}$$

$$\begin{aligned}
L_{x_1 x_1 x_1 y_1 y_1} = & \left\{ \frac{\partial^5}{\partial x^5} \cos^3 \alpha \sin^2 \alpha + \frac{\partial^5}{\partial x^4 \partial y} (3 \cos^2 \alpha \sin^3 \alpha - 2 \cos^4 \alpha \right. \\
& \sin \alpha) + \frac{\partial^5}{\partial x^3 \partial y^2} (\cos^5 \alpha - 6 \cos^3 \alpha \sin^2 \alpha + 3 \cos \alpha \\
& \sin^4 \alpha) + \frac{\partial^5}{\partial x^2 \partial y^3} (\sin^5 \alpha - 6 \cos^2 \alpha \sin^3 \alpha + 3 \cos^4 \alpha \\
& \sin \alpha) + \frac{\partial^5}{\partial x \partial y^4} (3 \cos^3 \alpha \sin^2 \alpha - 2 \cos \alpha \sin^4 \alpha) + \\
& \left. \frac{\partial^5}{\partial y^5} \cos^2 \alpha \sin^3 \alpha \right\} _g \quad (A.15)
\end{aligned}$$

$$\begin{aligned}
L_{x_1 x_1 y_1 y_1 y_1} = & \left\{ -\frac{\partial^5}{\partial x^5} \cos^2 \alpha \sin^3 \alpha + \frac{\partial^5}{\partial x^4 \partial y} (3 \cos^3 \alpha \sin^2 \alpha - 2 \cos \alpha \right. \\
& \sin^4 \alpha) + \frac{\partial^5}{\partial x^3 \partial y^2} (6 \cos^2 \alpha \sin^3 \alpha - 3 \cos^4 \alpha \sin \alpha - \\
& \sin^5 \alpha) + \frac{\partial^5}{\partial x^2 \partial y^3} (\cos^5 \alpha - 6 \sin^2 \alpha \cos^3 \alpha + 3 \sin^4 \alpha \\
& \cos \alpha) - \frac{\partial^5}{\partial x \partial y^4} (3 \sin^3 \alpha \cos^2 \alpha - 2 \sin \alpha \cos^4 \alpha) + \\
& \left. \frac{\partial^5}{\partial y^5} \sin^2 \alpha \cos^3 \alpha \right\} _g
\end{aligned}$$

$$\frac{\partial^5}{\partial y^5} \sin^2 \alpha \cos^3 \alpha \}_g \quad (A.16)$$

$$\begin{aligned} L_{x_1 y_1 y_1 y_1 y_1} = & \left\{ \frac{\partial^5}{\partial x^5} \cos \alpha \sin^4 \alpha + \frac{\partial^5}{\partial x^4 \partial y} (\sin^5 \alpha - 4 \cos^2 \alpha \sin^3 \alpha) + \right. \\ & \frac{\partial^5}{\partial x^3 \partial y^2} (6 \cos^3 \alpha \sin^2 \alpha - 4 \cos \alpha \sin^4 \alpha) + \frac{\partial^5}{\partial x^2 \partial y^3} \\ & (6 \sin^3 \alpha \cos^2 \alpha - 4 \sin \alpha \cos^4 \alpha) - \frac{\partial^5}{\partial x \partial y^4} (4 \sin^2 \alpha \\ & \left. \cos^3 \alpha - \cos^5 \alpha) + \frac{\partial^5}{\partial y^5} \sin \alpha \cos^4 \alpha \right\}_g \end{aligned} \quad (A.17)$$

$$\begin{aligned} L_{y_1 y_1 y_1 y_1 y_1} = & \left\{ -\frac{\partial^5}{\partial x^5} \sin^5 \alpha + 5 \frac{\partial^5}{\partial x^4 \partial y} \sin^4 \alpha \cos \alpha - 10 \frac{\partial^5}{\partial x^3 \partial y^2} \sin^3 \alpha \right. \\ & \cos^2 \alpha + 10 \frac{\partial^5}{\partial x^2 \partial y^3} \sin^2 \alpha \cos^3 \alpha - 5 \frac{\partial^5}{\partial x \partial y^4} \sin \alpha \cos^4 \alpha \\ & \left. + \frac{\partial^5}{\partial y^5} \cos^5 \alpha \right\}_g \end{aligned} \quad (A.18)$$

From the global to local coordinate system,

$$L_{x_g x_g} = \left\{ \frac{\partial^2}{\partial x^2} \cos^2 \alpha - 2 \frac{\partial^2}{\partial x \partial y} \cos \alpha \sin \alpha + \frac{\partial^2}{\partial y^2} \sin^2 \alpha \right\}_l \quad (A.19)$$

$$L_{x_g y_g} = \left\{ + \frac{\partial^2}{\partial x^2} \cos \alpha \sin \alpha + \frac{\partial^2}{\partial x \partial y} \cos 2\alpha - \frac{\partial^2}{\partial y^2} \cos \alpha \sin \alpha \right\}_1 \quad (A.20)$$

$$L_{y_g y_g} = \left\{ \frac{\partial^2}{\partial x^2} \sin^2 \alpha + 2 \frac{\partial^2}{\partial x \partial y} \sin \alpha \cos \alpha + \frac{\partial^2}{\partial y^2} \cos^2 \alpha \right\}_1 \quad (A.21)$$

$$L_{x_g x_g x_g} = \left\{ \frac{\partial^3}{\partial x^3} \cos^3 \alpha - 3 \frac{\partial^3}{\partial x^2 \partial y} \cos^2 \alpha \sin \alpha + 3 \frac{\partial^3}{\partial x \partial y^2} \cos \alpha \sin^2 \alpha - \frac{\partial^3}{\partial y^3} \sin^3 \alpha \right\}_1 \quad (A.22)$$

$$L_{x_g x_g y_g} = \left\{ + \frac{\partial^3}{\partial x^3} \cos^2 \alpha \sin \alpha + \frac{\partial^3}{\partial x^2 \partial y} (\cos^3 \alpha - 2 \cos \alpha \sin^2 \alpha) - \frac{\partial^3}{\partial x \partial y^2} (2 \cos^2 \alpha \sin \alpha - \sin^3 \alpha) + \frac{\partial^3}{\partial y^3} \sin^2 \alpha \cos \alpha \right\}_1 \quad (A.23)$$

$$L_{x_g y_g y_g} = \left\{ + \frac{\partial^3}{\partial x^3} \sin^2 \alpha \cos \alpha - \frac{\partial^3}{\partial x^2 \partial y} (\sin^3 \alpha - 2 \sin \alpha \cos^2 \alpha) - \frac{\partial^3}{\partial x \partial y^2} (2 \sin^2 \alpha \cos \alpha - \cos^3 \alpha) - \frac{\partial^3}{\partial y^3} \cos^2 \alpha \sin \alpha \right\}_1 \quad (A.24)$$

$$\begin{aligned}
L_{y_g y_g y_g} = & \left\{ + \frac{\partial^3}{\partial x^3} \sin^3 \alpha + 3 \frac{\partial^3}{\partial x^2 \partial y} \cos \alpha \sin^2 \alpha + 3 \frac{\partial^3}{\partial x \partial y^2} \cos^2 \alpha \sin \alpha \right. \\
& \left. + \frac{\partial^3}{\partial y^3} \cos^3 \alpha \right\}_1
\end{aligned} \tag{A.25}$$

$$\begin{aligned}
L_{x_g x_g x_g x_g} = & \left\{ \frac{\partial^4}{\partial x^4} \cos^4 \alpha - 4 \frac{\partial^4}{\partial x^3 \partial y} \cos^3 \alpha \sin \alpha + 6 \frac{\partial^4}{\partial x^2 \partial y^2} \cos^2 \alpha \right. \\
& \left. \sin^2 \alpha - 4 \frac{\partial^4}{\partial x \partial y^3} \cos \alpha \sin^3 \alpha + \frac{\partial^4}{\partial y^4} \sin^4 \alpha \right\}_1
\end{aligned} \tag{A.26}$$

$$\begin{aligned}
L_{x_g x_g x_g y_g} = & \left\{ + \frac{\partial^4}{\partial x^4} \cos^3 \alpha \sin \alpha + \frac{\partial^4}{\partial x^3 \partial y} (\cos^4 \alpha - 3 \cos^2 \alpha \sin^2 \alpha) - \right. \\
& \frac{\partial^4}{\partial x^2 \partial y^2} (3 \cos^3 \alpha \sin \alpha - 3 \cos \alpha \sin^3 \alpha) + \frac{\partial^4}{\partial x \partial y^3} (3 \cos^2 \alpha \sin^2 \alpha - \sin^4 \alpha) \\
& \left. - \frac{\partial^4}{\partial y^4} \sin^3 \alpha \cos \alpha \right\}_1
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
L_{x_g x_g y_g y_g} = & \left\{ \frac{\partial^4}{\partial x^4} \cos^2 \alpha \sin^2 \alpha - \frac{\partial^4}{\partial x^3 \partial y} 2 (\cos \alpha \sin^3 \alpha - \sin \alpha \cos^3 \alpha) \right. \\
& + \frac{\partial^4}{\partial x^2 \partial y^2} (\cos^4 \alpha - 4 \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha) - \frac{\partial^4}{\partial x \partial y^3} 2 (\cos^3 \alpha \sin \alpha - \cos \alpha \sin^3 \alpha) \\
& \left. + \frac{\partial^4}{\partial y^4} \cos^2 \alpha \sin^2 \alpha \right\}_1
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
L_{x_g y_g y_g y_g} = & \left\{ + \frac{\partial^4}{\partial x^4} \text{Cos} \alpha \sin^3 \alpha + \frac{\partial^4}{\partial x^3 \partial y} (3 \text{Cos}^2 \alpha \sin^2 \alpha - \sin^4 \alpha) - \right. \\
& \frac{\partial^4}{\partial x^2 \partial y^2} 3 (\text{Cos} \alpha \sin^3 \alpha - \text{Cos}^3 \alpha \sin \alpha) + \frac{\partial^4}{\partial x \partial y^3} (\text{Cos}^4 \alpha \\
& \left. - 3 \text{Cos}^2 \alpha \sin^2 \alpha) - \frac{\partial^4}{\partial y^4} \text{Cos}^3 \alpha \sin \alpha \right\}_1 \quad (A.29)
\end{aligned}$$

$$\begin{aligned}
L_{y_g y_g y_g y_g} = & \left\{ \frac{\partial^4}{\partial x^4} \sin^4 \alpha + 4 \frac{\partial^4}{\partial x^3 \partial y} \sin^3 \alpha \text{Cos} \alpha + 6 \frac{\partial^4}{\partial x^2 \partial y^2} \text{Cos}^2 \alpha \right. \\
& \left. \sin^2 \alpha + 4 \frac{\partial^4}{\partial x \partial y^3} \sin \alpha \text{Cos}^3 \alpha + \frac{\partial^4}{\partial y^4} \text{Cos}^4 \alpha \right\}_1 \quad (A.30)
\end{aligned}$$

$$\begin{aligned}
L_{x_g x_g x_g x_g} = & \left\{ + \frac{\partial^5}{\partial x^5} \text{Cos}^5 \alpha - 5 \frac{\partial^5}{\partial x^4 \partial y} \text{Cos}^4 \alpha \sin \alpha + 10 \frac{\partial^5}{\partial x^3 \partial y^2} \right. \\
& \text{Cos}^3 \alpha \sin^2 \alpha - 10 \frac{\partial^5}{\partial x^2 \partial y^3} \text{Cos}^2 \alpha \sin^3 \alpha + 5 \frac{\partial^5}{\partial x \partial y^4} \\
& \left. \text{Cos} \alpha \sin^4 \alpha - \frac{\partial^5}{\partial y^5} \sin^5 \alpha \right\}_1 \quad (A.31)
\end{aligned}$$

$$\begin{aligned}
L_{x_g x_g x_g x_g y_g} = & \left\{ + \frac{\partial^5}{\partial x^5} \text{Cos}^4 \alpha \sin \alpha + \frac{\partial^5}{\partial x^4 \partial y} (\text{Cos}^5 \alpha - 4 \text{Cos}^3 \alpha \sin^2 \alpha) \right. \\
& \left. - \frac{\partial^5}{\partial x^3 \partial y^2} (4 \text{Cos}^4 \alpha \sin \alpha - 6 \text{Cos}^2 \alpha \sin^3 \alpha) + \frac{\partial^5}{\partial x^2 \partial y^3} \right.
\end{aligned}$$

$$\begin{aligned}
& (6 \cos^3 \alpha \sin^2 \alpha - 4 \cos \alpha \sin^4 \alpha) - \frac{\partial^5}{\partial x \partial y^4} (4 \cos^2 \alpha \\
& \sin^3 \alpha - \sin^5 \alpha) + \frac{\partial^5}{\partial y^5} \cos \alpha \sin^4 \alpha \} _1 \quad (A.32)
\end{aligned}$$

$$\begin{aligned}
L_{x_g x_g x_g y_g y_g} = & \left\{ \frac{\partial^5}{\partial x^5} \cos^3 \alpha \sin^2 \alpha - \frac{\partial^5}{\partial x^4 \partial y} (3 \cos^2 \alpha \sin^3 \alpha - 2 \cos^4 \alpha \right. \\
& \sin \alpha) + \frac{\partial^5}{\partial x^3 \partial y^2} (\cos^5 \alpha - 6 \cos^3 \alpha \sin^2 \alpha + 3 \cos \alpha \\
& \sin^4 \alpha) - \frac{\partial^5}{\partial x^2 \partial y^3} (\sin^5 \alpha - 6 \cos^2 \alpha \sin^3 \alpha + 3 \cos^4 \alpha \\
& \sin \alpha) + \frac{\partial^5}{\partial x \partial y^4} (3 \cos^3 \alpha \sin^2 \alpha - 2 \cos \alpha \sin^4 \alpha) - \\
& \left. \frac{\partial^5}{\partial y^5} \cos^2 \alpha \sin^3 \alpha \right\} _1 \quad (A.33)
\end{aligned}$$

$$\begin{aligned}
L_{x_g x_g y_g y_g y_g} = & \left\{ + \frac{\partial^5}{\partial x^5} \cos^2 \alpha \sin^3 \alpha + \frac{\partial^5}{\partial x^4 \partial y} (3 \cos^3 \alpha \sin^2 \alpha - 2 \cos \alpha \right. \\
& \sin^4 \alpha) - \frac{\partial^5}{\partial x^3 \partial y^2} (6 \cos^2 \alpha \sin^3 \alpha - 3 \cos^4 \alpha \sin \alpha - \\
& \sin^5 \alpha) + \frac{\partial^5}{\partial x^2 \partial y^3} (\cos^5 \alpha - 6 \sin^2 \alpha \cos^3 \alpha + 3 \sin^4 \alpha \\
& \cos \alpha) + \frac{\partial^5}{\partial x \partial y^4} (3 \sin^3 \alpha \cos^2 \alpha - 2 \sin \alpha \cos^4 \alpha) +
\end{aligned}$$

$$\frac{\partial^5}{\partial y^5} \sin^2 \alpha \cos^3 \alpha \}_1 \quad (A.34)$$

$$\begin{aligned} L_{x_g y_g y_g y_g y_g} = & \left\{ \frac{\partial^5}{\partial x^5} \cos \alpha \sin^4 \alpha - \frac{\partial^5}{\partial x^4 \partial y} (\sin^5 \alpha - 4 \cos^2 \alpha \sin^3 \alpha) + \right. \\ & \frac{\partial^5}{\partial x^3 \partial y^2} (6 \cos^3 \alpha \sin^2 \alpha - 4 \cos \alpha \sin^4 \alpha) - \frac{\partial^5}{\partial x^2 \partial y^3} \\ & (6 \sin^3 \alpha \cos^2 \alpha - 4 \sin \alpha \cos^4 \alpha) - \frac{\partial^5}{\partial x \partial y^4} (4 \sin^2 \alpha \\ & \left. \cos^3 \alpha - \cos^5 \alpha) - \frac{\partial^5}{\partial y^5} \sin \alpha \cos^4 \alpha \right\}_1 \quad (A.35) \end{aligned}$$

$$\begin{aligned} L_{y_g y_g y_g y_g y_g} = & \left\{ + \frac{\partial^5}{\partial x^5} \sin^5 \alpha + 5 \frac{\partial^5}{\partial x^4 \partial y} \sin^4 \alpha \cos \alpha + 10 \frac{\partial^5}{\partial x^3 \partial y^2} \sin^3 \alpha \right. \\ & \cos^2 \alpha + 10 \frac{\partial^5}{\partial x^2 \partial y^3} \sin^2 \alpha \cos^3 \alpha + 5 \frac{\partial^5}{\partial x \partial y^4} \sin \alpha \cos^4 \alpha \\ & \left. + \frac{\partial^5}{\partial y^5} \cos^5 \alpha \right\}_1 \quad (A.36) \end{aligned}$$

A P P E N D I X B

Numerical Results of the Potential Problems in a Square Domain

Table B.1 : Square domain with linear potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using eighth quadratic boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.5, 1.5)	5	+2.63x10 ⁻¹⁵	- 4.27x10 ⁻¹⁴	- 1.00000	+5.15x10 ⁻¹⁴	- 3.95x10 ⁻¹⁴
(1.25, 1.25)		+1.01x10 ⁻⁰⁷	- 8.46x10 ⁻⁰⁶	- 1.00000	+1.53x10 ⁻⁰⁵	- 1.29x10 ⁻⁰⁵
(1.1, 1.1)		+1.03x10 ⁻⁰²	- 6.01x10 ⁻⁰²	- 1.07523	+6.59x10 ⁻⁰²	- 2.23x10 ⁻⁰¹
(1.5, 1.5)	7	- 2.12x10 ⁻¹⁴	+4.26x10 ⁻¹⁰	- 1.00000	- 1.25x10 ⁻⁰⁹	+1.25x10 ⁻⁰⁹
(1.25, 1.25)		- 1.71x10 ⁻⁰⁵	- 9.24x10 ⁻⁰⁶	- 0.99981	- 1.12x10 ⁻⁰⁴	- 4.06x10 ⁻⁰⁴
(1.1, 1.1)		- 6.31x10 ⁻⁰²	+6.81x10 ⁻⁰²	- 0.99542	- 8.22x10 ⁻⁰¹	+1.15
(1.5, 1.5)	8	- 1.40x10 ⁻¹⁴	+6.25x10 ⁻¹⁰	- 1.00000	- 1.25x10 ⁻⁰⁹	+1.25x10 ⁻⁰⁹
(1.25, 1.25)		+2.06x10 ⁻⁰⁴	- 7.62x10 ⁻⁰⁴	- 1.00031	- 2.87x10 ⁻⁰⁴	- 7.05x10 ⁻⁰³
(1.1, 1.1)		- 7.72x10 ⁻⁰³	- 2.54x10 ⁻⁰²	- 0.68128	- 6.05x10 ⁻⁰¹	- 1.39
(1.5, 1.5)	9	+8.05x10 ⁻¹⁰	- 2.87x10 ⁻⁰⁹	- 1.00000	+1.91x10 ⁻⁰⁸	+1.03x10 ⁻⁰⁹
(1.25, 1.25)		- 9.34x10 ⁻⁰⁵	- 4.10x10 ⁻⁰⁵	- 0.99954	- 4.01x10 ⁻⁰³	- 2.36x10 ⁻⁰³
(1.1, 1.1)		- 1.08x10 ⁻⁰²	+4.89x10 ⁻⁰¹	+0.23752	+3.85x10 ⁻⁰¹	+2.06x10 ⁻⁰¹
(1.5, 1.5)	10	+8.05x10 ⁻¹⁰	- 2.87x10 ⁻⁰⁹	- 1.00000	+1.91x10 ⁻⁰⁸	+1.03x10 ⁻⁰⁹
(1.25, 1.25)		+2.56x10 ⁻⁰³	- 3.68x10 ⁻⁰³	- 1.00261	- 2.40x10 ⁻⁰²	- 3.60x10 ⁻⁰²
(1.1, 1.1)		- 8.32x10 ⁻⁰³	+2.97x10 ⁻⁰¹	+0.07003	- 5.31x10 ⁻⁰¹	+2.16x10 ⁻⁰¹
(1.5, 1.5)	11	- 9.34x10 ⁻¹¹	- 1.98x10 ⁻⁰⁹	- 1.00000	- 1.44x10 ⁻⁰⁹	- 1.59x10 ⁻⁰⁸
(1.25, 1.25)		+1.20x10 ⁻⁰³	+6.42x10 ⁻⁰⁴	- 1.00078	- 4.96x10 ⁻⁰²	- 3.39x10 ⁻⁰²
(1.1, 1.1)		- 1.07x10 ⁻⁰²	+5.55x10 ⁻⁰¹	+0.13346	- 7.42x10 ⁻⁰¹	- 2.47x10 ⁻⁰¹
(1.5, 1.5)	12	- 9.34x10 ⁻¹¹	- 1.98x10 ⁻⁰⁹	- 1.00000	- 1.44x10 ⁻⁰⁹	- 1.59x10 ⁻⁰⁸
(1.25, 1.25)		+5.43x10 ⁻⁰⁴	+1.25x10 ⁻⁰²	- 0.98744	- 1.88x10 ⁻⁰¹	- 3.57x10 ⁻⁰²
(1.1, 1.1)		- 1.79x10 ⁻⁰²	+6.72x10 ⁻⁰¹	+0.25368	- 3.28	- 5.72x10 ⁻⁰¹
(1.5, 1.5)	14	- 1.59x10 ⁻¹⁰	- 2.36x10 ⁻⁰⁹	- 1.00000	- 1.85x10 ⁻⁰⁹	- 2.10x10 ⁻⁰⁸
(1.25, 1.25)		+7.70x10 ⁻⁰²	- 2.01x10 ⁻⁰¹	- 0.91619	- 1.02	- 3.11
(1.1, 1.1)		- 3.32x10 ⁻⁰³	- 2.51x10 ⁻⁰¹	+0.06221	- 3.41x10 ⁻⁰¹	- 6.82x10 ⁻⁰¹
(1.5, 1.5)	15	- 1.59x10 ⁻¹⁰	+1.15x10 ⁻⁰⁹	- 1.00000	- 7.59x10 ⁻⁰⁶	+7.56x10 ⁻⁰⁶
(1.25, 1.25)		+8.22x10 ⁻⁰²	- 2.27x10 ⁻⁰¹	- 0.92014	- 1.17	- 3.30
(1.1, 1.1)		- 1.38x10 ⁻⁰³	+9.30x10 ⁻⁰²	+0.02942	+9.94x10 ⁻⁰³	- 4.91x10 ⁻⁰¹
(1.5, 1.5)	16	- 1.61x10 ⁻¹⁰	+1.15x10 ⁻⁰⁹	- 1.00000	- 7.59x10 ⁻⁰⁶	+7.56x10 ⁻⁰⁶
(1.25, 1.25)		+9.97x10 ⁻⁰²	+1.99x10 ⁻⁰¹	- 0.05560	- 4.71	- 3.40
(1.1, 1.1)		- 3.34x10 ⁻⁰³	+1.18x10 ⁻⁰¹	+0.00491	- 3.88x10 ⁻⁰¹	- 3.29x10 ⁻⁰¹

Table B.2 : Square domain with linear potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using eight quadratic boundary elements.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.5, 1.5)	5	- 5.27x10 ⁻⁰³	- 9.75x10 ⁻⁰²	- 0.93818	+1.06x10 ⁻⁰¹	- 2.36x10 ⁻⁰¹
(1.25, 1.25)		- 7.33x10 ⁻⁰³	- 1.26x10 ⁻⁰²	- 0.93199	+7.14x10 ⁻⁰²	- 1.14x10 ⁻⁰¹
(1.1, 1.1)		- 1.14x10 ⁻⁰²	+1.80x10 ⁻⁰²	- 0.90447	+9.27x10 ⁻⁰²	+8.62x10 ⁻⁰³
(1.5, 1.5)	7	- 2.83x10 ⁻⁰³	+2.01x10 ⁻⁰²	- 0.99252	- 7.66x10 ⁻⁰²	+4.84x10 ⁻⁰²
(1.25, 1.25)		- 6.71x10 ⁻⁰⁴	+1.47x10 ⁻⁰³	- 0.99756	- 1.02x10 ⁻⁰²	+1.41x10 ⁻⁰²
(1.1, 1.1)		- 4.51x10 ⁻⁰²	+6.29x10 ⁻⁰²	- 0.81730	- 7.26x10 ⁻⁰¹	+4.98x10 ⁻⁰¹
(1.5, 1.5)	8	- 2.83x10 ⁻⁰³	+2.01x10 ⁻⁰²	- 0.99252	- 7.66x10 ⁻⁰²	+4.84x10 ⁻⁰²
(1.25, 1.25)		- 5.49x10 ⁻⁰⁴	+1.06x10 ⁻⁰³	- 0.99782	- 1.04x10 ⁻⁰²	+1.05x10 ⁻⁰²
(1.1, 1.1)		- 7.78x10 ⁻⁰³	+2.92x10 ⁻⁰³	- 0.61720	- 6.05x10 ⁻⁰¹	- 1.25
(1.5, 1.5)	9	+1.65x10 ⁻⁰³	- 3.83x10 ⁻⁰³	- 1.00892	+4.27x10 ⁻⁰²	+1.23x10 ⁻⁰²
(1.25, 1.25)		+5.98x10 ⁻⁰⁴	- 1.58x10 ⁻⁰³	- 1.00027	- 1.07x10 ⁻⁰³	- 8.59x10 ⁻⁰³
(1.1, 1.1)		+7.75x10 ⁻⁰³	+2.97x10 ⁻⁰¹	+0.11456	- 1.77x10 ⁻⁰¹	+1.15x10 ⁻⁰¹
(1.5, 1.5)	10	+1.65x10 ⁻⁰³	- 3.83x10 ⁻⁰³	- 1.00892	+4.27x10 ⁻⁰²	+1.23x10 ⁻⁰²
(1.25, 1.25)		+2.94x10 ⁻⁰³	- 4.74x10 ⁻⁰³	- 1.00322	- 1.90x10 ⁻⁰²	- 3.84x10 ⁻⁰²
(1.1, 1.1)		- 7.92x10 ⁻⁰³	+2.82x10 ⁻⁰¹	+0.06727	- 5.18x10 ⁻⁰¹	+1.77x10 ⁻⁰¹
(1.5, 1.5)	11	+3.08x10 ⁻⁰⁴	- 8.18x10 ⁻⁰⁴	- 0.99765	- 8.58x10 ⁻⁰³	- 1.73x10 ⁻⁰²
(1.25, 1.25)		+1.58x10 ⁻⁰³	- 4.70x10 ⁻⁰⁴	- 1.00247	- 5.19x10 ⁻⁰²	- 4.14x10 ⁻⁰²
(1.1, 1.1)		- 9.06x10 ⁻⁰³	+4.80x10 ⁻⁰¹	+0.11989	- 7.15x10 ⁻⁰¹	- 3.84x10 ⁻⁰¹
(1.5, 1.5)	12	+3.08x10 ⁻⁰⁴	- 8.18x10 ⁻⁰⁴	- 0.99765	- 8.58x10 ⁻⁰³	- 1.73x10 ⁻⁰²
(1.25, 1.25)		- 7.28x10 ⁻⁰⁴	+1.68x10 ⁻⁰²	- 0.97793	- 1.80x10 ⁻⁰¹	+2.67x10 ⁻⁰⁵
(1.1, 1.1)		- 9.77x10 ⁻⁰³	+3.78x10 ⁻⁰¹	+0.14206	- 1.85	- 4.26x10 ⁻⁰¹
(1.5, 1.5)	14	+2.94x10 ⁻⁰⁵	+4.31x10 ⁻⁰⁴	- 1.00032	- 1.02x10 ⁻⁰³	+4.37x10 ⁻⁰³
(1.25, 1.25)		+7.63x10 ⁻⁰²	- 1.91x10 ⁻⁰¹	- 0.90537	- 1.09	- 3.06
(1.1, 1.1)		- 2.77x10 ⁻⁰³	+2.12x10 ⁻⁰¹	+0.05327	- 2.90x10 ⁻⁰¹	- 6.08x10 ⁻⁰¹
(1.5, 1.5)	15	+1.86x10 ⁻⁰⁵	- 2.30x10 ⁻⁰⁴	- 1.00013	+2.53x10 ⁻⁰³	- 8.16x10 ⁻⁰⁴
(1.25, 1.25)		+8.18x10 ⁻⁰²	- 2.16x10 ⁻⁰¹	- 0.91199	- 1.23	- 3.28
(1.1, 1.1)		- 8.16x10 ⁻⁰⁴	+5.37x10 ⁻⁰²	+0.02086	- 6.58x10 ⁻⁰²	- 3.58x10 ⁻⁰¹
(1.5, 1.5)	16	+1.86x10 ⁻⁰⁵	- 2.30x10 ⁻⁰⁴	- 1.00013	+2.53x10 ⁻⁰³	- 8.16x10 ⁻⁰⁴
(1.25, 1.25)		- 2.62x10 ⁻⁰¹	+1.16	+0.11105	+3.96	+1.48x10 ⁺⁰¹
(1.1, 1.1)		+7.26x10 ⁻⁰³	- 4.26x10 ⁻⁰¹	- 0.10831	- 4.36x10 ⁻⁰¹	+1.68

Table B.3 : Square domain with linear potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.1, 1.1)	5	+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.1, 1.1)	6	+0.00000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(1.05, 1.05)		+0.00000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(1.01, 1.01)		- 0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.1, 1.1)	7	+0.00000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	- 0.00000
(1.01, 1.01)		- 0.00000	+0.00000	- 1.00000	- 0.00000	- 0.00000
(1.1, 1.1)	8	- 0.00000	- 0.00000	- 1.00000	- 0.00000	+0.00000
(1.05, 1.05)		- 0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		- 0.00000	- 0.00000	- 1.00000	- 0.00000	+0.00000
(1.1, 1.1)	9	- 0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		- 0.00000	+0.00000	- 1.00000	- 0.00000	- 0.00000
(1.01, 1.01)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.1, 1.1)	10	- 0.00000	+0.00000	- 1.00000	- 0.00000	+0.00000
(1.05, 1.05)		- 0.00000	+0.00000	- 1.00000	- 0.00000	- 0.00000
(1.01, 1.01)		+0.00000	+0.00002	- 0.99993	+0.00002	- 0.00000
(1.1, 1.1)	11	- 0.00000	+0.00000	- 1.00000	- 0.00000	- 0.00000
(1.05, 1.05)		- 0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		- 0.00000	+0.00005	- 1.00076	- 0.00002	+0.00001
(1.1, 1.1)	12	- 0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		- 0.00000	+0.00037	- 1.00721	- 0.00027	- 0.00003
(1.1, 1.1)	13	- 0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		+0.00000	+0.00058	- 1.01184	+0.00020	- 0.00001
(1.1, 1.1)	14	+0.00000	- 0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		+0.00000	+0.02847	- 1.29386	+0.00260	+0.00016
(1.1, 1.1)	15	+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.01, 1.01)		+0.00000	- 0.02102	- 0.24	+0.00068	- 0.00021
(1.1, 1.1)	16	+0.00000	+0.00000	- 1.00000	+0.00000	+0.00000
(1.05, 1.05)		+0.00000	+0.00000	- 1.00000	+0.00000	- 0.00000
(1.01, 1.01)		+0.00000	- 0.05478	+0.004	+0.00001	+0.00006

Table B.4 : Square domain with linear potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	0.	- 1.	0.	0.
(1.1, 1.1)	5	- 0.01611	+0.05828	- 0.86507	+0.03455	+0.15852
(1.05, 1.05)		+0.00074	+0.01116	- 0.94914	- 0.00542	- 0.11121
(1.01, 1.01)		- 0.00007	+0.02166	- 0.94013	- 0.02984	- 0.06567
(1.1, 1.1)	6	- 0.00193	+0.01301	- 0.96831	+0.08301	+0.01041
(1.05, 1.05)		- 0.00024	+0.01585	- 0.96199	+0.03360	- 0.02697
(1.01, 1.01)		- 0.00001	+0.02212	- 0.93471	- 0.04112	- 0.08001
(1.1, 1.1)	7	+0.00008	- 0.00119	- 1.00045	- 0.00058	- 0.00214
(1.05, 1.05)		+0.00001	- 0.00076	- 0.99985	+0.00091	- 0.00056
(1.01, 1.01)		+0.00000	- 0.00016	- 0.99906	+0.00227	+0.00041
(1.1, 1.1)	8	+0.00000	- 0.00100	- 1.00105	+0.00050	+0.00265
(1.05, 1.05)		+0.00000	- 0.00077	- 1.00050	+0.00152	+0.00056
(1.01, 1.01)		+0.00000	- 0.00025	- 0.99908	+0.00106	+0.00041
(1.1, 1.1)	9	- 0.00004	+0.00005	- 0.99713	- 0.00007	+0.00365
(1.05, 1.05)		- 0.00000	- 0.00036	- 0.99808	- 0.00020	+0.00077
(1.01, 1.01)		+0.00000	- 0.00038	- 0.99805	- 0.00013	+0.00007
(1.1, 1.1)	10	+0.00000	- 0.00101	- 1.00093	+0.00151	+0.00219
(1.05, 1.05)		- 0.00000	- 0.00081	- 0.99998	+0.00093	+0.00180
(1.01, 1.01)		+0.00000	- 0.00008	- 0.99770	+0.00016	- 0.00003
(1.1, 1.1)	11	+0.00001	+0.00113	- 0.99829	- 0.00221	- 0.00398
(1.05, 1.05)		+0.00000	- 0.00016	- 0.99958	- 0.00002	+0.00014
(1.01, 1.01)		- 0.00000	+0.00030	- 1.00491	- 0.00049	- 0.00003
(1.1, 1.1)	12	+0.00000	- 0.00003	- 0.99953	- 0.00038	- 0.00041
(1.05, 1.05)		- 0.00000	- 0.00012	- 0.99952	- 0.00022	- 0.00004
(1.01, 1.01)		- 0.00000	+0.00031	- 1.00511	- 0.00051	- 0.00004
(1.1, 1.1)	13	- 0.00000	- 0.00000	- 1.00040	+0.00018	+0.00021
(1.05, 1.05)		- 0.00000	- 0.00012	- 0.99950	- 0.00023	- 0.00004
(1.01, 1.01)		+0.00000	+0.00066	- 1.01304	+0.00021	- 0.00001
(1.1, 1.1)	14	- 0.00000	+0.00004	- 0.99666	- 0.00194	- 0.00110
(1.05, 1.05)		+0.00000	+0.00003	- 1.00025	+0.00004	+0.00000
(1.01, 1.01)		+0.00000	+0.02177	- 1.18921	+0.00238	+0.00016
(1.1, 1.1)	15	- 0.00000	- 0.00000	- 0.99997	- 0.00001	- 0.00000
(1.05, 1.05)		- 0.00000	- 0.00002	- 0.99989	- 0.00001	- 0.00000
(1.01, 1.01)		- 0.00000	- 0.07941	+0.5	- 0.00163	+0.00059
(1.1, 1.1)	16	- 0.00000	- 0.00001	- 0.99999	- 0.00000	+0.00001
(1.05, 1.05)		+0.00000	- 0.00000	- 0.99998	+0.00000	- 0.00000
(1.01, 1.01)		+0.00000	- 0.04946	+0.005	+0.00002	+0.00006

Table B.5 : Square domain with quadratic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+2.	0.	+4.	+2.	+2.
(1.1, 1.1)	5	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.1, 1.1)	6	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.1, 1.1)	7	+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.1, 1.1)	8	+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.1, 1.1)	9	+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	- 0.00012	+3.99986	+1.99954	+1.99979
(1.1, 1.1)	10	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	- 0.00133	+3.99530	+1.99838	+2.00033
(1.1, 1.1)	11	+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	- 0.00511	+4.09063	+2.00234	+1.99854
(1.1, 1.1)	12	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.01, 1.01)		+2.00000	- 0.31933	+10.4	+2.25243	+2.03503
(1.1, 1.1)	13	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+4.00000	+1.99999	+2.00000
(1.01, 1.01)		+2.00000	- 0.43436	+13.	+1.99236	+2.02690
(1.1, 1.1)	14	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00002	+4.00006	+1.99997	+1.99999
(1.01, 1.01)		+1.99999	- 9.2	+102.	+1.23883	+1.97107
(1.1, 1.1)	15	+2.00000	+0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00003	+4.00008	+1.99997	+2.00001
(1.01, 1.01)		+2.00000	+5.	- 201.	+1.78460	+2.07649
(1.1, 1.1)	16	+2.00000	- 0.00000	+4.00000	+2.00000	+2.00000
(1.05, 1.05)		+2.00000	- 0.00000	+3.99989	+2.00000	+2.00001
(1.01, 1.01)		+2.00000	- 19.	- 30.	+1.30374	+2.26398

Table B.6 : Square domain with quadratic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+2.	0.	+4.	+2.	+2.
(1.1, 1.1)	5	+2.19105	- 0.62561	+2.75	+1.76758	- 0.5
(1.05, 1.05)		+1.99040	- 0.02109	+3.77776	+2.11942	+2.84607
(1.01, 1.01)		+2.00002	- 0.10966	+3.69230	+2.24961	+2.43716
(1.1, 1.1)	6	+2.02488	- 0.09502	+3.95967	+1.19953	+1.23421
(1.05, 1.05)		+2.00250	- 0.07905	+3.93654	+1.63733	+1.80534
(1.01, 1.01)		+2.00005	- 0.10549	+3.74197	+2.14608	+2.30558
(1.1, 1.1)	7	+2.00923	+0.01553	+4.20979	+1.85002	+1.33189
(1.05, 1.05)		+2.00091	+0.02553	+4.17482	+1.84308	+1.63914
(1.01, 1.01)		+2.00000	- 0.00743	+4.02526	+1.95504	+1.95153
(1.1, 1.1)	8	+1.99859	+0.04321	+4.12499	+2.00295	+2.00807
(1.05, 1.05)		+1.99985	+0.02487	+4.11046	+1.90321	+1.91267
(1.01, 1.01)		+2.00000	- 0.00924	+4.02491	+1.93108	+1.95968
(1.1, 1.1)	9	+2.00125	- 0.02101	+3.88517	+2.03798	+1.94674
(1.05, 1.05)		+2.00005	- 0.00576	+3.93040	+2.03173	+2.02099
(1.01, 1.01)		+2.00000	- 0.00190	+3.96790	+1.99691	+2.00170
(1.1, 1.1)	10	+1.99991	+0.01118	+4.00013	+1.99025	+1.99074
(1.05, 1.05)		+2.00000	+0.00732	+3.98510	+1.99908	+1.99116
(1.01, 1.01)		+2.00000	- 0.00048	+3.96953	+1.99822	+2.00126
(1.1, 1.1)	11	+1.99986	- 0.02843	+3.95124	+2.05885	+2.10470
(1.05, 1.05)		+2.00000	+0.00504	+3.98373	+2.00240	+1.99694
(1.01, 1.01)		+2.00000	- 0.02174	+4.37872	+2.03467	+2.00149
(1.1, 1.1)	12	+1.99996	+0.00122	+3.98298	+2.01252	+2.01388
(1.05, 1.05)		+2.00000	+0.00407	+3.98230	+2.00701	+2.00128
(1.01, 1.01)		+2.00001	- 0.26871	+8.8	+2.42359	+2.04034
(1.1, 1.1)	13	+2.00000	+0.00026	+4.00869	+1.99567	+1.99560
(1.05, 1.05)		+2.00000	- 0.00012	+4.00243	+1.99997	+1.99999
(1.01, 1.01)		+2.00000	- 0.48565	+14.	+1.98469	+2.02638
(1.1, 1.1)	14	+2.00004	- 0.00034	+3.94795	+2.03022	+2.01689
(1.05, 1.05)		+2.00000	- 0.00059	+4.00471	+1.99916	+1.99986
(1.01, 1.01)		+1.99999	- 7.1	+69.	+1.30599	+1.97162
(1.1, 1.1)	15	+2.00000	+0.00028	+3.99964	+2.00002	+1.99971
(1.05, 1.05)		+2.00000	+0.00291	+3.97936	+2.00294	+2.00007
(1.01, 1.01)		+2.00000	+21.	- 409.	+2.43980	+1.84955
(1.1, 1.1)	16	+2.00000	+0.00034	+4.00008	+1.99989	+1.99955
(1.05, 1.05)		+2.00000	+0.00007	+3.99877	+1.99993	+2.00000
(1.01, 1.01)		+2.00000	- 1.2	- 39.	+1.20879	+2.25011

Table B.7 : Square domain with cubic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	- 6.	+6.	0.	+12.
(1.1, 1.1)	5	+1.6	- 11.19	- 4.	- 0.35195	- 9.
(1.05, 1.05)		- 0.03839	- 5.99832	+4.98432	+1.2	+15.9
(1.01, 1.01)		- 0.00017	- 6.33431	+4.63109	+1.7	+14.48067
(1.1, 1.1)	6	+0.01453	- 6.13858	+5.53727	- 0.75389	+12.27787
(1.05, 1.05)		+0.00219	- 6.18857	+5.47165	- 0.29673	+12.48504
(1.01, 1.01)		+0.00014	- 6.27167	+5.18232	+0.51503	+13.00346
(1.1, 1.1)	7	- 0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.05, 1.05)		- 0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.01, 1.01)		+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.1, 1.1)	8	+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.05, 1.05)		+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.01, 1.01)		+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.1, 1.1)	9	+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.05, 1.05)		+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.01, 1.01)		+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.1, 1.1)	10	+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.05, 1.05)		+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.01, 1.01)		- 0.00000	- 6.00010	+5.99963	- 0.00009	+12.00004
(1.1, 1.1)	11	+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.05, 1.05)		+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.01, 1.01)		- 0.00000	- 6.00018	+6.00155	- 0.00001	+12.00000
(1.1, 1.1)	12	+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.05, 1.05)		+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.01, 1.01)		- 0.00000	- 6.00011	+6.00029	- 0.00006	+12.00000
(1.1, 1.1)	13	+0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.05, 1.05)		+0.00000	- 6.00000	+6.00000	+0.00000	+12.00000
(1.01, 1.01)		- 0.00000	- 6.00108	+6.02207	- 0.00225	+11.99993
(1.1, 1.1)	14	- 0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.05, 1.05)		- 0.00000	- 6.00001	+6.00002	- 0.00000	+12.00000
(1.01, 1.01)		+0.00000	- 5.94725	+5.47784	+0.00239	+12.00028
(1.1, 1.1)	15	- 0.00000	- 6.00000	+6.00000	- 0.00000	+12.00000
(1.05, 1.05)		- 0.00000	- 6.00001	+6.00002	- 0.00001	+12.00000
(1.01, 1.01)		- 0.00000	- 5.97114	+5.98965	+0.00146	+12.00010
(1.1, 1.1)	16	- 0.00000	- 6.00000	+6.00001	- 0.00000	+12.00000
(1.05, 1.05)		- 0.00000	- 6.00000	+6.00000	- 0.00001	+12.00000
(1.01, 1.01)		+0.00000	- 4.44	- 5.	+0.03183	+11.98826

Table B.8 : Square domain with cubic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	0.	- 6.	+6.	0.	+12.
(1.1, 1.1)	5	+0.90703	- 8.93817	+0.07	- 0.66096	+0.4
(1.05, 1.05)		- 0.05175	- 6.01069	+5.00385	+0.89901	+16.48181
(1.01, 1.01)		- 0.00009	- 6.45309	+4.56718	+1.5	+14.40578
(1.1, 1.1)	6	+0.10456	- 6.37593	+5.91027	- 3.4	+8.8
(1.05, 1.05)		+0.01019	- 6.30742	+5.81669	- 1.6	+11.15396
(1.01, 1.01)		+0.00020	- 6.41322	+5.04180	+0.46530	+13.14855
(1.1, 1.1)	7	+0.03855	- 5.90964	+6.96511	- 0.66093	+9.2
(1.05, 1.05)		+0.00357	- 5.87271	+6.80717	- 0.71370	+10.46307
(1.01, 1.01)		- 0.00000	- 6.03378	+6.13794	- 0.27392	+11.77860
(1.1, 1.1)	8	- 0.00687	- 5.79148	+6.60316	- 0.00822	+12.05112
(1.05, 1.05)		- 0.00074	- 5.87539	+6.54543	- 0.46917	+11.57550
(1.01, 1.01)		- 0.00001	- 6.03824	+6.13709	- 0.33276	+11.79861
(1.1, 1.1)	9	+0.00505	- 6.07841	+5.53159	+0.14833	+11.77708
(1.05, 1.05)		+0.00018	- 6.01654	+5.71567	+0.12308	+12.07463
(1.01, 1.01)		- 0.00000	- 6.00182	+5.85442	- 0.00637	+12.00699
(1.1, 1.1)	10	- 0.00045	- 5.94706	+6.00075	- 0.04651	+11.95664
(1.05, 1.05)		+0.00000	- 5.96542	+5.92953	- 0.00460	+11.95804
(1.01, 1.01)		- 0.00000	- 5.99556	+5.86163	- 0.00059	+12.00502
(1.1, 1.1)	11	- 0.00065	- 6.13835	+5.76467	+0.28479	+12.50696
(1.05, 1.05)		- 0.00000	- 5.97597	+5.92318	+0.01074	+11.98474
(1.01, 1.01)		+0.00000	- 6.03015	+6.52743	+0.05872	+12.00540
(1.1, 1.1)	12	- 0.00017	- 5.99424	+5.91891	+0.05963	+12.06561
(1.05, 1.05)		+0.00001	- 5.98071	+5.91620	+0.03338	+12.00600
(1.01, 1.01)		+0.00000	- 5.99727	+5.93412	+0.00694	+12.00022
(1.1, 1.1)	13	+0.00001	- 5.99876	+6.04023	- 0.01988	+11.97935
(1.05, 1.05)		+0.00000	- 5.99855	+6.00188	+0.00339	+12.00053
(1.01, 1.01)		- 0.00000	- 6.00185	+6.03763	- 0.00233	+11.99993
(1.1, 1.1)	14	+0.00020	- 6.00170	+5.74639	+0.14727	+12.08235
(1.05, 1.05)		- 0.00000	- 6.00280	+6.02252	- 0.00393	+11.99937
(1.01, 1.01)		+0.00000	- 5.96033	+5.69108	+0.00195	+12.00027
(1.1, 1.1)	15	+0.00000	- 5.99868	+5.99834	+0.00010	+11.99859
(1.05, 1.05)		+0.00000	- 5.98807	+5.91583	+0.01199	+12.00025
(1.01, 1.01)		- 0.00000	- 5.99653	+6.29214	+0.00052	+12.00043
(1.1, 1.1)	16	- 0.00000	- 5.99838	+6.00041	- 0.00053	+11.99783
(1.05, 1.05)		- 0.00000	- 5.99967	+5.99505	- 0.00029	+11.99996
(1.01, 1.01)		+0.00000	- 5.32723	- 4.	+0.03608	+11.98889

Table B.9 : Square domain with quartic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+4.	+6.	+10.	+24.	+6.
(1.1, 1.1)	5	+8.08	- 7.	- 15.	+27.66918	- 49.
(1.05, 1.05)		+3.86146	+7.12161	+9.73396	+30.00136	+16.
(1.01, 1.01)		+3.99775	+6.33840	+8.74274	+29.89478	+13.
(1.1, 1.1)	6	+3.93903	+6.40357	+11.15130	+26.63154	+6.07983
(1.05, 1.05)		+3.99305	+6.50463	+11.31435	+25.09047	+4.99925
(1.01, 1.01)		+3.99966	+6.71977	+12.09882	+22.88703	+3.51315
(1.1, 1.1)	7	+3.98713	+5.94487	+ 9.61961	+24.13606	+6.99961
(1.05, 1.05)		+3.99939	+5.95981	+ 9.78786	+24.23316	+6.40061
(1.01, 1.01)		+4.00002	+6.03605	+10.04097	+24.18321	+6.03858
(1.1, 1.1)	8	+4.00349	+5.89337	+ 9.72185	+23.92769	+5.97572
(1.05, 1.05)		+4.00035	+5.95622	+ 9.83006	+24.19190	+6.16844
(1.01, 1.01)		+4.00001	+6.03187	+10.03928	+24.12467	+6.05528
(1.1, 1.1)	9	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.01, 1.01)		+4.00000	+5.99976	+ 9.99971	+23.99906	+5.99958
(1.1, 1.1)	10	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.01, 1.01)		+4.00000	+5.99730	+ 9.99042	+23.99671	+6.00068
(1.1, 1.1)	11	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.01, 1.01)		+4.00000	+5.98955	+10.18576	+24.00481	+5.99701
(1.1, 1.1)	12	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.01, 1.01)		+4.00001	+5.35848	+23.	+24.50709	+6.07029
(1.1, 1.1)	13	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+6.00000	+10.00002	+23.99999	+6.00000
(1.01, 1.01)		+4.00000	+5.12772	+28.	+23.98537	+6.05398
(1.1, 1.1)	14	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+6.00000	+10.00013	+23.99994	+5.99999
(1.01, 1.01)		+3.99999	- 13.	+208.	+22.45429	+5.94055
(1.1, 1.1)	15	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+5.99994	+10.00017	+23.99995	+6.00002
(1.01, 1.01)		+4.00000	+16.	- 405.	+23.56208	+6.15452
(1.1, 1.1)	16	+4.00000	+6.00000	+10.00000	+24.00000	+6.00000
(1.05, 1.05)		+4.00000	+5.99999	+ 9.99978	+24.00000	+6.00002
(1.01, 1.01)		+3.99999	- 34.	- 55.	+22.57576	+6.53910

Table B.10 : Square domain with quartic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+4.	+6.	+10.	+24.	+6.
(1.1, 1.1)	5	+6.57694	- 2.	- 6.	+29.54010	- 27.
(1.05, 1.05)		+3.77506	+7.26780	+8.56077	+31.87078	+23.
(1.01, 1.01)		+3.99751	+5.85607	+7.00617	+32.21986	+16.
(1.1, 1.1)	6	+4.12120	+5.82110	+11.55747	+21.14504	- 0.80
(1.05, 1.05)		+4.01233	+6.13113	+11.67448	+22.41671	+2.41554
(1.01, 1.01)		+4.00000	+6.19848	+11.08284	+23.72288	+4.89270
(1.1, 1.1)	7	+4.10002	+5.97070	+11.89588	+22.02515	- 0.67
(1.05, 1.05)		+4.01095	+6.22134	+11.88002	+22.59419	+2.27218
(1.01, 1.01)		+4.00008	+6.04624	+10.64303	+24.01948	+5.44236
(1.1, 1.1)	8	+3.98792	+6.26237	+11.00240	+23.63633	+6.45763
(1.05, 1.05)		+3.99888	+6.21383	+11.14575	+23.28015	+5.39282
(1.01, 1.01)		+3.99998	+6.00744	+10.63561	+23.50671	+5.61673
(1.1, 1.1)	9	+3.98838	+6.25.133	+10.96118	+23.64235	+6.44709
(1.05, 1.05)		+3.99953	+6.11407	+10.55937	+23.69870	+5.74556
(1.01, 1.01)		+4.00000	+6.07045	+10.14647	+24.07150	+5.97731
(1.1, 1.1)	10	+4.00019	+5.96951	+ 9.95457	+24.06038	+6.06182
(1.05, 1.05)		+4.00000	+5.97903	+ 9.99446	+24.03595	+6.05353
(1.01, 1.01)		+4.00000	+5.99294	+10.05727	+23.99996	+6.00173
(1.1, 1.1)	11	+4.00030	+6.07617	+10.08621	+23.87564	+5.75495
(1.05, 1.05)		+4.00001	+5.99490	+10.00401	+24.01288	+6.01337
(1.01, 1.01)		+4.00000	+5.95836	+10.72270	+24.05923	+6.00211
(1.1, 1.1)	12	+4.00004	+5.99897	+10.00358	+23.99627	+5.99141
(1.05, 1.05)		+4.00000	+5.99816	+10.00882	+23.99730	+5.99874
(1.01, 1.01)		+4.00001	+5.45780	+19.8	+24.84750	+6.08085
(1.1, 1.1)	13	+4.00001	+5.99986	+ 9.97960	+24.01199	+6.00845
(1.05, 1.05)		+4.00001	+6.01451	+ 9.93031	+24.02478	+6.00376
(1.01, 1.01)		+4.00000	+5.02399	+29.6	+23.96992	+6.05295
(1.1, 1.1)	14	+3.99996	+6.00057	+10.04996	+23.97196	+5.98379
(1.05, 1.05)		+4.00000	+6.00261	+ 9.99948	+24.00026	+5.99988
(1.01, 1.01)		+3.99999	- 8.4	+141.	+22.59078	+5.94166
(1.1, 1.1)	15	+4.00000	+6.00001	+10.00399	+23.99881	+5.99907
(1.05, 1.05)		+4.00000	+6.00447	+ 9.96900	+24.00480	+6.00013
(1.01, 1.01)		+4.00001	+50.	- 829.	+24.89191	+5.69392
(1.1, 1.1)	16	+4.00000	+5.99965	+10.00146	+23.99959	+6.00000
(1.05, 1.05)		+4.00000	+5.99988	+10.00035	+23.99995	+6.00002
(1.01, 1.01)		+3.99999	+3.32212	- 73.	+22.38161	+6.51074

Table B.11 : Square domain with quintic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+8.	+20.	+20.	+80.	0.
(1.1, 1.1)	5	+12.01	+8.58	- 2.2	+105.44	- 50.
(1.05, 1.05)		+7.63638	+24.86801	+24.22408	+101.41	+23.
(1.01, 1.01)		+7.99056	+23.57735	+22.13273	+97.97245	+16.4
(1.1, 1.1)	6	+7.33064	+23.28684	+26.77823	+104.27	+12.
(1.05, 1.05)		+7.92930	+23.49467	+27.74193	+90.47384	- 1.6
(1.01, 1.01)		+7.99753	+24.96864	+34.37623	+72.40710	- 16.
(1.1, 1.1)	7	+7.72733	+19.50431	+14.14770	+83.68698	+20.
(1.05, 1.05)		+7.97754	+19.34691	+16.12067	+83.94710	+9.
(1.01, 1.01)		+7.99999	+20.23893	+20.14081	+81.37360	+1.1
(1.1, 1.1)	8	+8.04712	+18.49728	+16.14691	+79.61231	- 0.26820
(1.05, 1.05)		+8.00463	+19.24552	+17.31271	+82.78158	+2.6
(1.01, 1.01)		+8.00001	+20.26886	+20.15291	+81.79302	+0.97704
(1.1, 1.1)	9	+8.00775	+19.70094	+19.28683	+80.42856	+0.00595
(1.05, 1.05)		+8.00046	+19.77449	+19.36597	+80.46305	+0.52576
(1.01, 1.01)		+7.99999	+19.75892	+19.52437	+79.79819	+0.09241
(1.1, 1.1)	10	+8.00032	+19.87965	+19.89130	+80.17507	+0.26091
(1.05, 1.05)		+7.99996	+19.90394	+19.99340	+80.11057	+0.21665
(1.01, 1.01)		+8.00000	+19.97971	+20.35543	+80.00894	- 0.00568
(1.1, 1.1)	11	+8.00000	+20.00000	+20.00000	+80.00000	+0.00000
(1.05, 1.05)		+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.01, 1.01)		+8.00000	+19.97892	+20.37551	+80.00977	- 0.00606
(1.1, 1.1)	12	+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.05, 1.05)		+8.00000	+20.00000	+20.00000	+79.99999	- 0.00001
(1.01, 1.01)		+8.00002	+18.71413	+46.	+81.01644	+0.14081
(1.1, 1.1)	13	+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.05, 1.05)		+8.00000	+19.99999	+20.00005	+79.99997	- 0.00001
(1.01, 1.01)		+8.00000	+18.25225	+56.	+79.97217	+0.10817
(1.1, 1.1)	14	+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.05, 1.05)		+8.00000	+19.99994	+20.00024	+79.99989	- 0.00002
(1.01, 1.01)		+7.99997	- 18.	+418.	+76.88463	- 0.12058
(1.1, 1.1)	15	+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.05, 1.05)		+8.00000	+19.99988	+20.00032	+79.99990	- 0.00004
(1.01, 1.01)		+8.00000	+40.	- 817.	+79.11670	- 0.31055
(1.1, 1.1)	16	+8.00000	+20.00000	+20.00000	+80.00000	- 0.00000
(1.05, 1.05)		+8.00000	+19.99999	+19.99954	+80.00000	+0.00005
(1.01, 1.01)		+7.99998	- 60.	- 104.	+77.10910	+1.09334

Table B.12 : Square domain with quintic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+8.	+20.	+20.	+80.	0.
(1.1, 1.1)	5	+12.452	+7.	- 7.5	+10.83	- 52.
(1.05, 1.05)		+7.40340	+25.20544	+20.42943	+10.72759	+42.
(1.01, 1.01)		+7.98994	+22.20000	+16.80597	+10.51779	+26.
(1.1, 1.1)	6	+7.73250	+21.82975	+26.90462	+92.17745	- 2.
(1.05, 1.05)		+7.97538	+22.46522	+27.93585	+84.48439	- 7.
(1.01, 1.01)		+7.99863	+23.39608	+31.04610	+75.49724	- 12.
(1.1, 1.1)	7	+8.09952	+19.23739	+21.04018	+76.92592	- 5.
(1.05, 1.05)		+8.01399	+19.92991	+22.15913	+79.49626	- 3.
(1.01, 1.01)		+8.00025	+20.32269	+22.16763	+81.48473	- 0.59222
(1.1, 1.1)	8	+7.99542	+19.50821	+20.21055	+78.42197	+2.
(1.05, 1.05)		+8.00039	+19.92145	+21.33192	+80.26905	+0.58611
(1.01, 1.01)		+8.00001	+20.22971	+22.14984	+80.25581	- 0.17432
(1.1, 1.1)	9	+7.90577	+21.66614	+28.26955	+77.24461	+4.
(1.05, 1.05)		+7.99638	+20.53697	+24.95008	+77.68652	- 1.6
(1.01, 1.01)		+8.00001	+20.23482	+22.11016	+80.30163	- 0.14507
(1.1, 1.1)	10	+8.00545	+19.28787	+19.77481	+80.77232	+0.79378
(1.05, 1.05)		+7.99993	+19.52136	+20.70164	+80.22287	+0.72576
(1.01, 1.01)		+8.00000	+19.92838	+21.75752	+80.01879	- 0.04852
(1.1, 1.1)	11	+8.00799	+21.73612	+22.79632	+76.53212	- 6.
(1.05, 1.05)		+8.00003	+19.72588	+20.82480	+79.92547	+0.20824
(1.01, 1.01)		+7.99999	+20.22691	+16.01258	+79.50707	- 0.05179
(1.1, 1.1)	12	+8.00196	+19.93319	+20.86661	+79.34922	- 0.72777
(1.05, 1.05)		+8.99993	+19.78760	+20.91562	+79.63100	- 0.06846
(1.01, 1.01)		+8.00003	+18.88316	+40.	+81.62314	+0.15958
(1.1, 1.1)	13	+7.99992	+19.98561	+19.45762	+80.27258	+0.27395
(1.05, 1.05)		+8.00000	+20.01132	+19.84109	+80.00710	+0.00021
(1.01, 1.01)		+8.00000	+18.05213	+59.	+79.94193	+0.10613
(1.1, 1.1)	14	+7.99746	+20.02424	+23.31385	+78.07894	- 1.
(1.05, 1.05)		+8.00001	+20.03502	+19.72600	+80.04791	+0.00667
(1.01, 1.01)		+7.99997	- 9.	+284.	+77.16013	- 0.11835
(1.1, 1.1)	15	+7.99999	+19.98489	+20.03681	+79.99310	+0.01171
(1.05, 1.05)		+7.99998	+19.88380	+20.82162	+79.88440	- 0.00234
(1.01, 1.01)		+8.00001	+108.	- 1670.	+81.79598	+0.61744
(1.1, 1.1)	16	+8.00000	+19.97998	+20.00278	+80.00354	+0.02410
(1.05, 1.05)		+8.00000	+19.99574	+20.05714	+80.00291	+0.00040
(1.01, 1.01)		+7.99998	+13.71386	- 140.	+76.71511	+1.03581

Table B.13 : Square domain with hyperbolic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.83373	+0.63496	- 1.29846	- 0.98890	- 0.83373
(1.1, 1.1)	5	+0.96855	+0.18561	- 2.2	- 1.22905	- 2.6
(1.05, 1.05)		+0.83293	+0.58887	- 1.44496	- 1.03467	- 0.62950
(1.01, 1.01)		+0.83381	+0.56500	- 1.46354	- 0.95632	- 0.72072
(1.1, 1.1)	6	+0.84091	+0.58693	- 1.40897	- 1.26102	- 0.91675
(1.05, 1.05)		+0.83467	+0.58074	- 1.42414	- 1.09938	- 0.77664
(1.01, 1.01)		+0.83377	+0.55819	- 1.52347	- 0.83118	- 0.56011
(1.1, 1.1)	7	+0.83636	+0.63036	- 1.26397	- 1.02478	- 1.00382
(1.05, 1.05)		+0.83403	+0.63569	- 1.27018	- 1.01292	- 0.91798
(1.01, 1.01)		+0.83373	+0.63472	- 1.29314	- 0.97626	- 0.84278
(1.1, 1.1)	8	+0.83351	+0.63932	- 1.28176	- 0.98853	- 0.82568
(1.05, 1.05)		+0.83371	+0.63687	- 1.28414	- 0.99926	- 0.84117
(1.01, 1.01)		+0.83373	+0.63334	- 1.29370	- 0.99555	- 0.83728
(1.1, 1.1)	9	+0.83362	+0.63608	- 1.29021	- 0.99072	- 0.82641
(1.05, 1.05)		+0.83372	+0.63519	- 1.29068	- 0.99188	- 0.83469
(1.01, 1.01)		+0.83373	+0.63514	- 1.29147	- 0.98850	- 0.83415
(1.1, 1.1)	10	+0.83373	+0.63330	- 1.29961	- 0.98678	- 0.83038
(1.05, 1.05)		+0.83373	+0.63364	- 1.29819	- 0.98766	- 0.83098
(1.01, 1.01)		+0.83373	+0.63426	- 1.29478	- 0.98934	- 0.83376
(1.1, 1.1)	11	+0.83373	+0.63481	- 1.29824	- 0.98898	- 0.83366
(1.05, 1.05)		+0.83373	+0.63490	- 1.29811	- 0.98911	- 0.83382
(1.01, 1.01)		+0.83373	+0.63305	- 1.26417	- 0.98807	- 0.83433
(1.1, 1.1)	12	+0.83373	+0.63494	- 1.29816	- 0.98911	- 0.83397
(1.05, 1.05)		+0.83373	+0.63488	- 1.29810	- 0.98905	- 0.83376
(1.01, 1.01)		+0.83373	+0.50271	+1.4	- 0.88432	- 0.81920
(1.1, 1.1)	13	+0.83373	+0.63496	- 1.29842	- 0.98891	- 0.83375
(1.05, 1.05)		+0.83373	+0.63500	- 1.29860	- 0.98883	- 0.83372
(1.01, 1.01)		+0.83373	+0.45535	+2.4	- 0.99139	- 0.82255
(1.1, 1.1)	14	+0.83373	+0.63496	- 1.29840	- 0.98894	- 0.83375
(1.05, 1.05)		+0.83373	+0.63496	- 1.29842	- 0.98891	- 0.83373
(1.01, 1.01)		+0.83373	- 3.	+39.	- 1.3	- 0.84544
(1.1, 1.1)	15	+0.83373	+0.63497	- 1.29845	- 0.98890	- 0.83373
(1.05, 1.05)		+0.83373	+0.63495	- 1.29842	- 0.98891	- 0.83373
(1.01, 1.01)		+0.83373	+2.6	- 85.	- 1.07725	- 0.80234
(1.1, 1.1)	16	+0.83373	+0.63497	- 1.29846	- 0.98890	- 0.83373
(1.05, 1.05)		+0.83373	+0.63496	- 1.29850	- 0.98890	- 0.83373
(1.01, 1.01)		+0.83373	- 8.	- 12.	- 1.3	- 0.72231

Table B.14 : Square domain with hyperbolic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.83373	+0.63496	- 1.29846	- 0.98890	- 0.83373
(1.1, 1.1)	5	+0.84740	+0.57803	- 1.43526	- 1.06574	- 0.95649
(1.05, 1.05)		+0.83373	+0.61036	- 1.37074	- 1.01550	- 0.75553
(1.01, 1.01)		+0.83377	+0.60137	- 1.37676	- 0.97942	- 0.78933
(1.1, 1.1)	6	+0.83648	+0.61291	- 1.35571	- 1.10308	- 0.84235
(1.05, 1.05)		+0.83413	+0.60846	- 1.36553	- 1.03130	- 0.78964
(1.01, 1.01)		+0.83375	+0.59841	- 1.41200	- 0.90598	- 0.69599
(1.1, 1.1)	7	+0.83364	+0.63294	- 1.31041	- 0.98527	- 0.82466
(1.05, 1.05)		+0.83375	+0.63349	- 1.30851	- 0.98207	- 0.82942
(1.01, 1.01)		+0.83373	+0.63614	- 1.30301	- 0.97948	- 0.83221
(1.1, 1.1)	8	+0.83386	+0.63238	- 1.30871	- 0.98834	- 0.83823
(1.05, 1.05)		+0.83374	+0.63348	- 1.30876	- 0.98184	- 0.82837
(1.01, 1.01)		+0.83373	+0.63581	- 1.30307	- 0.98390	- 0.83071
(1.1, 1.1)	9	+0.83373	+0.63552	- 1.29700	- 0.99005	- 0.83524
(1.05, 1.05)		+0.83373	+0.63535	- 1.29781	- 0.98965	- 0.83496
(1.01, 1.01)		+0.83373	+0.63525	- 1.29869	- 0.98896	- 0.83394
(1.1, 1.1)	10	+0.83373	+0.63536	- 1.29758	- 0.98981	- 0.83545
(1.05, 1.05)		+0.83373	+0.63538	- 1.29766	- 0.98974	- 0.83503
(1.01, 1.01)		+0.83373	+0.63446	- 1.29959	- 0.98968	- 0.83369
(1.1, 1.1)	11	+0.83373	+0.63577	- 1.29707	- 0.99052	- 0.83664
(1.05, 1.05)		+0.83373	+0.63483	- 1.29800	- 0.98893	- 0.83363
(1.01, 1.01)		+0.83373	+0.62905	- 1.19533	- 0.98039	- 0.83363
(1.1, 1.1)	12	+0.83373	+0.63493	- 1.29796	- 0.98922	- 0.83409
(1.05, 1.05)		+0.83373	+0.63486	- 1.29795	- 0.98907	- 0.83376
(1.01, 1.01)		+0.83373	+0.52341	+0.7	- 0.81404	- 0.81702
(1.1, 1.1)	13	+0.83373	+0.63495	- 1.29849	- 0.98887	- 0.83371
(1.05, 1.05)		+0.83373	+0.63502	- 1.29876	- 0.98879	- 0.83371
(1.01, 1.01)		+0.83373	+0.43418	+2.7	- 0.99455	- 0.82276
(1.1, 1.1)	14	+0.83373	+0.63495	- 1.29878	- 0.98871	- 0.83361
(1.05, 1.05)		+0.83373	+0.63496	- 1.29844	- 0.98890	- 0.83373
(1.01, 1.01)		+0.83373	- 2.3	+25.	- 1.27	- 0.84521
(1.1, 1.1)	15	+0.83373	+0.63496	- 1.29846	- 0.98890	- 0.83372
(1.05, 1.05)		+0.83373	+0.63503	- 1.29898	- 0.98882	- 0.83372
(1.01, 1.01)		+0.83373	+9.5	- 170.	- 0.80940	- 0.89511
(1.1, 1.1)	16	+0.83373	+0.63495	- 1.29847	- 0.98889	- 0.83371
(1.05, 1.05)		+0.83373	+0.63496	- 1.29850	- 0.98890	- 0.83373
(1.01, 1.01)		+0.83373	- 0.1	- 16.	- 1.3	- 0.73

Table B.15 : Square domain with logarithmic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Taylor series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.34657	+0.5	+0.5	- 0.5	0.
(1.1, 1.1)	5	+0.40070	+0.33634	+0.19568	- 0.43944	- 0.76462
(1.05, 1.05)		+0.34472	+0.52115	+0.51944	- 0.41583	+0.10086
(1.01, 1.01)		+0.34653	+0.51386	+0.50828	- 0.42988	+0.06372
(1.1, 1.1)	6	+0.34613	+0.50794	+0.53430	- 0.45310	- 0.04536
(1.05, 1.05)		+0.34644	+0.51311	+0.54004	- 0.45310	- 0.04536
(1.01, 1.01)		+0.34656	+0.51997	+0.56199	- 0.54203	- 0.08021
(1.1, 1.1)	7	+0.34699	+0.49967	+0.50669	- 0.49808	- 0.02878
(1.05, 1.05)		+0.34660	+0.49889	+0.50021	- 0.50223	- 0.00817
(1.01, 1.01)		+0.34657	+0.49779	+0.49525	- 0.49999	+0.00169
(1.1, 1.1)	8	+0.34652	+0.50117	+0.50371	- 0.49200	+0.00107
(1.05, 1.05)		+0.34657	+0.49902	+0.49866	- 0.50071	+0.00037
(1.01, 1.01)		+0.34657	+0.49775	+0.49524	- 0.50058	+0.00186
(1.1, 1.1)	9	+0.34665	+0.49695	+0.49269	- 0.49867	+0.00011
(1.05, 1.05)		+0.34658	+0.49757	+0.49302	- 0.49434	+0.00597
(1.01, 1.01)		+0.34657	+0.49732	+0.49471	- 0.50225	+0.00112
(1.1, 1.1)	10	+0.34659	+0.49859	+0.49825	- 0.49720	+0.00245
(1.05, 1.05)		+0.34657	+0.49901	+0.50004	- 0.49829	+0.00251
(1.01, 1.01)		+0.34657	+0.49955	+0.50309	- 0.50013	+0.00013
(1.1, 1.1)	11	+0.34658	+0.50194	+0.50128	- 0.50207	- 0.00481
(1.05, 1.05)		+0.34657	+0.49965	+0.50008	- 0.49902	+0.00107
(1.01, 1.01)		+0.34657	+0.49881	+0.52164	- 0.49936	- 0.00022
(1.1, 1.1)	12	+0.34657	+0.49992	+0.50000	- 0.49991	+0.00022
(1.05, 1.05)		+0.34657	+0.49996	+0.49994	- 0.49999	+0.00005
(1.01, 1.01)		+0.34657	+0.44456	+1.6	- 0.45618	+0.00609
(1.1, 1.1)	13	+0.34657	+0.49989	+0.50023	- 0.50010	+0.00002
(1.05, 1.05)		+0.34657	+0.49981	+0.50058	- 0.50026	+0.00000
(1.01, 1.01)		+0.34657	+0.42465	+2.	- 0.50120	+0.00467
(1.1, 1.1)	14	+0.34657	+0.49990	+0.50032	- 0.50022	+0.00003
(1.05, 1.05)		+0.34657	+0.49998	+0.49989	- 0.49995	+0.00005
(1.01, 1.01)		+0.34657	- 1.1	+18.	- 0.63291	+0.00509
(1.1, 1.1)	15	+0.34657	+0.49992	+0.50000	- 0.49996	+0.00011
(1.05, 1.05)		+0.34657	+0.50003	+0.49984	- 0.49995	+0.00000
(1.01, 1.01)		+0.34657	+1.4	- 35.	- 0.53767	+0.01331
(1.1, 1.1)	16	+0.34657	+0.49993	+0.49996	- 0.49995	+0.00012
(1.05, 1.05)		+0.34657	+0.49998	+0.50015	- 0.50000	+0.00000
(1.01, 1.01)		+0.34657	- 3.	- 5.	- 0.62330	+0.04670

Table B.16 : Square domain with logarithmic potential variation.
Function and derivatives at node 1 (Fig. 2.6) from Fourier series
using refined boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.34657	+0.5	+0.5	- 0.5	0.
(1.1, 1.1)	5	+0.37638	+0.41414	+0.33420	- 0.39748	- 0.38653
(1.05, 1.05)		+0.34282	+0.52424	+0.48739	- 0.36832	+0.26927
(1.01, 1.01)		+0.34652	+0.50244	+0.46264	- 0.36969	+0.15688
(1.1, 1.1)	6	+0.34451	+0.49634	+0.50642	- 0.64323	+0.00170
(1.05, 1.05)		+0.34669	+0.50569	+0.53820	- 0.52260	- 0.06378
(1.01, 1.01)		+0.34657	+0.50896	+0.54033	- 0.53162	- 0.04891
(1.1, 1.1)	7	+0.34437	+0.49361	+0.49734	- 0.66894	+0.01423
(1.05, 1.05)		+0.34673	+0.50347	+0.53315	- 0.52696	- 0.06025
(1.01, 1.01)		+0.34657	+0.49736	+0.50683	- 0.50902	- 0.00704
(1.1, 1.1)	8	+0.34627	+0.50698	+0.52420	- 0.49642	+0.00895
(1.05, 1.05)		+0.34654	+0.50336	+0.52198	- 0.51652	- 0.01279
(1.01, 1.01)		+0.34657	+0.49715	+0.50679	- 0.51184	- 0.00608
(1.1, 1.1)	9	+0.34732	+0.50162	+0.54751	- 0.43557	- 0.20354
(1.05, 1.05)		+0.34658	+0.49738	+0.48684	- 0.49145	+0.00835
(1.01, 1.01)		+0.34657	+0.49835	+0.49748	- 0.50110	+0.00078
(1.1, 1.1)	10	+0.34656	+0.49980	+0.49657	- 0.49713	+0.00412
(1.05, 1.05)		+0.34657	+0.49953	+0.49585	- 0.49682	+0.00343
(1.01, 1.01)		+0.34657	+0.49965	+0.49898	- 0.49989	+0.00037
(1.1, 1.1)	11	+0.34656	+0.49941	+0.49610	- 0.49646	+0.00523
(1.05, 1.05)		+0.34657	+0.50070	+0.49656	- 0.49852	+0.00048
(1.01, 1.01)		+0.34657	+0.49526	+0.58359	- 0.49236	- 0.00042
(1.1, 1.1)	12	+0.34657	+0.50019	+0.49693	- 0.49767	+0.00285
(1.05, 1.05)		+0.34657	+0.50074	+0.49662	- 0.49872	+0.00029
(1.01, 1.01)		+0.34657	+0.45340	+1.3	- 0.42645	+0.00700
(1.1, 1.1)	13	+0.34670	+0.49340	+0.57125	- 0.43231	- 0.05886
(1.05, 1.05)		+0.34657	+0.49906	+0.50468	- 0.50155	- 0.00023
(1.01, 1.01)		+0.34657	+0.41580	+2.2	- 0.50252	- 0.00458
(1.1, 1.1)	14	+0.34658	+0.49989	+0.49167	- 0.49514	+0.00279
(1.05, 1.05)		+0.34657	+0.49993	+0.50049	- 0.50006	+0.00001
(1.01, 1.01)		+0.34657	- 0.7	+12.	- 0.62117	- 0.00499
(1.1, 1.1)	15	+0.34656	+0.51145	+0.75271	- 0.46418	- 0.06974
(1.05, 1.05)		+0.34657	+0.49972	+0.50200	- 0.50028	- 0.00000
(1.01, 1.01)		+0.34657	+0.43222	- 71.	- 0.42334	- 0.02629
(1.1, 1.1)	16	+0.34657	+0.49999	+0.49992	- 0.49995	+0.00005
(1.05, 1.05)		+0.34657	+0.50000	+0.50010	- 0.49999	+0.00000
(1.01, 1.01)		+0.34657	+0.22769	- 6.	- 0.64019	+0.04424

A P P E N D I X C

Numerical Results of the Potential Problems in a Thick-Walled Cylinder Domain

Table C.1 : Cylinder domain with linear potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.5	0.	- 1.	0.	0.
(0.52, 0.02)	5	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.51, 0.01)		+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.509, 0.009)		+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.52, 0.02)	6	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.51, 0.01)		+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.509, 0.009)		+0.50000	- 0.00000	- 1.00000	+0.00000	- 0.00000
(0.508, 0.008)		+0.50000	+0.00000	- 1.00000	+0.00000	- 0.00000
(0.52, 0.02)	7	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.51, 0.01)		+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.509, 0.009)		+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.508, 0.008)		+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.52, 0.02)	8	+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.51, 0.01)		+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.509, 0.009)		+0.50000	- 0.00000	- 1.00000	+0.00000	+0.00000
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.52, 0.02)	9	+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.51, 0.01)		+0.50000	- 0.00000	- 1.00000	- 0.00000	- 0.00000
(0.509, 0.009)		+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.508, 0.008)		+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.52, 0.02)	10	+0.50000	+0.00000	- 1.00000	- 0.00000	+0.00000
(0.51, 0.01)		+0.50000	- 0.00000	- 1.00000	+0.00000	+0.00000
(0.509, 0.009)		+0.50000	+0.00000	- 1.00000	- 0.00000	- 0.00000
(0.508, 0.008)		+0.50000	+0.00000	- 1.00000	- 0.00000	- 0.00000

Table C.1 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.5	0.	- 1.	0.	0.
(0.52, 0.02)	11	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.51, 0.01)		+0.50000	- 0.00000	- 1.00000	+0.00000	- 0.00000
(0.509, 0.009)		+0.50000	- 0.00000	- 1.00000	+0.00000	- 0.00000
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00000	+0.00000	- 0.00000
(0.52, 0.02)	12	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00000
(0.51, 0.01)		+0.50000	- 0.00001	- 1.00000	+0.00002	- 0.00001
(0.509, 0.009)		+0.50000	- 0.00001	- 1.00000	+0.00002	- 0.00001
(0.508, 0.008)		+0.50000	- 0.00001	- 1.00000	+0.00001	- 0.00000
(0.52, 0.02)	13	+0.50000	- 0.00000	- 1.00000	+0.00000	- 0.00000
(0.51, 0.01)		+0.50000	- 0.00000	- 1.00000	+0.00001	+0.00000
(0.509, 0.009)		+0.50000	- 0.00001	- 1.00000	+0.00001	+0.00001
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00000	+0.00001	+0.00000
(0.52, 0.02)	14	+0.50000	+0.00000	- 1.00000	+0.00000	+0.00001
(0.51, 0.01)		+0.50000	- 0.00001	- 0.99993	+0.00001	- 0.00015
(0.509, 0.009)		+0.50000	- 0.00001	- 0.99996	+0.00001	- 0.00007
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00018	+0.00001	+0.00037
(0.52, 0.02)	15	+0.50000	+0.00000	- 0.99999	- 0.00000	- 0.00002
(0.51, 0.01)		+0.50000	+0.00000	- 1.00128	- 0.00000	+0.00255
(0.509, 0.009)		+0.50000	+0.00000	- 1.00114	- 0.00000	+0.00227
(0.508, 0.008)		+0.50000	- 0.00025	- 0.97103	+0.00044	- 0.05794
(0.52, 0.02)	16	+0.50000	+0.00000	- 1.00000	- 0.00000	+0.00000
(0.51, 0.01)		+0.50000	- 0.00005	- 0.99999	+0.00009	- 0.00001
(0.509, 0.009)		+0.50000	+0.00003	- 1.00225	- 0.00006	+0.00450
(0.508, 0.008)		+0.50000	+0.00026	- 1.00757	- 0.00051	+0.01514

Table C.2 : Cylinder domain with linear potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.5	0.	- 1.	0.	0.
(0.52, 0.02)	5	+0.49623	+0.02290	- 0.75614	+0.09454	+0.86278
(0.51, 0.01)		+0.50057	- 0.01925	- 1.32882	- 0.27315	- 0.25120
(0.509, 0.009)		+0.50037	- 0.01688	- 1.29505	- 0.25722	- 0.16981
(0.508, 0.008)		+0.50024	- 0.01522	- 1.27092	- 0.24750	- 0.10742
(0.52, 0.02)	6	+0.49999	+0.00614	- 1.02502	+0.06641	+0.06163
(0.51, 0.01)		+0.50000	+0.00582	- 1.02892	+0.05596	+0.06969
(0.509, 0.009)		+0.50000	+0.00579	- 1.02915	+0.05514	+0.07051
(0.508, 0.008)		+0.50000	+0.00577	- 1.02928	+0.05449	+0.07131
(0.52, 0.02)	7	+0.50000	- 0.00143	- 0.99991	+0.00008	+0.00206
(0.51, 0.01)		+0.50000	- 0.00229	- 0.99737	- 0.00674	- 0.00562
(0.509, 0.009)		+0.50000	- 0.00254	- 0.99658	- 0.00864	- 0.00789
(0.508, 0.008)		+0.50000	- 0.00294	- 0.99522	- 0.01180	- 0.01169
(0.52, 0.02)	8	+0.50000	+0.00250	- 1.00049	+0.00794	+0.00753
(0.51, 0.01)		+0.50000	+0.00073	- 1.00028	+0.00415	+0.00478
(0.509, 0.009)		+0.50000	+0.00059	- 1.00031	+0.00384	+0.00458
(0.508, 0.008)		+0.50000	+0.00045	- 1.00038	+0.00354	+0.00444
(0.52, 0.02)	9	+0.50000	+0.00286	- 1.00067	+0.00903	+0.00876
(0.51, 0.01)		+0.50000	- 0.00038	- 0.99917	- 0.00086	- 0.00273
(0.509, 0.009)		+0.50000	- 0.00029	- 0.99926	- 0.00049	- 0.00221
(0.508, 0.008)		+0.50000	- 0.00024	- 0.99933	- 0.00024	- 0.00184
(0.52, 0.02)	10	+0.50000	- 0.00077	- 0.99933	- 0.00041	- 0.00212
(0.51, 0.01)		+0.50000	- 0.00046	- 0.99948	+0.00048	- 0.00094
(0.509, 0.009)		+0.50000	- 0.00049	- 0.99950	+0.00056	- 0.00085
(0.508, 0.008)		+0.50000	- 0.00055	- 0.99953	+0.00069	- 0.00073

Table C.2 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.5	0.	- 1.	0.	0.
(0.52, 0.02)	11	+0.50000	- 0.00001	- 0.99999	+0.00003	- 0.00000
(0.51, 0.01)		+0.50000	+0.00001	- 1.00002	+0.00001	+0.00004
(0.509, 0.009)		+0.50000	- 0.00000	- 1.00001	+0.00002	+0.00003
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00001	+0.00002	+0.00002
(0.52, 0.02)	12	+0.50000	- 0.00003	- 0.99996	+0.00000	- 0.00008
(0.51, 0.01)		+0.50000	+0.00004	- 1.00003	- 0.00005	+0.00006
(0.509, 0.009)		+0.50000	+0.00002	- 1.00001	- 0.00003	+0.00003
(0.508, 0.008)		+0.50000	+0.00002	- 1.00001	- 0.00003	+0.00001
(0.52, 0.02)	13	+0.50000	+0.00000	- 0.99999	+0.00000	- 0.00003
(0.51, 0.01)		+0.50000	- 0.00000	- 0.99998	+0.00001	- 0.00003
(0.509, 0.009)		+0.50000	- 0.00001	- 0.99998	+0.00001	- 0.00003
(0.508, 0.008)		+0.50000	- 0.00000	- 0.99998	+0.00001	- 0.00004
(0.52, 0.02)	14	+0.50000	- 0.00000	- 0.99996	- 0.00000	- 0.00009
(0.51, 0.01)		+0.50000	- 0.00001	- 0.99993	+0.00001	- 0.00015
(0.509, 0.009)		+0.50000	- 0.00001	- 0.99995	+0.00001	- 0.00010
(0.508, 0.008)		+0.50000	- 0.00000	- 1.00007	+0.00001	- 0.00015
(0.52, 0.02)	15	+0.50000	+0.00000	- 1.00000	- 0.00000	- 0.00001
(0.51, 0.01)		+0.50000	+0.00001	- 1.00129	- 0.00001	+0.00257
(0.509, 0.009)		+0.50000	+0.00000	- 1.00117	- 0.00001	+0.00234
(0.508, 0.008)		+0.50000	+0.00038	- 1.04124	- 0.00067	+0.08247
(0.52, 0.02)	16	+0.50000	+0.00000	- 1.00001	- 0.00000	+0.00001
(0.51, 0.01)		+0.50000	- 0.00005	- 0.99992	+0.00010	- 0.00016
(0.509, 0.009)		+0.50000	+0.00004	- 1.00222	- 0.00001	+0.00443
(0.508, 0.008)		+0.50000	+0.00018	- 1.00644	- 0.00034	+0.01287

Table C.3 : Cylinder domain with quadratic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.25	+1.	+1.	+2.	+2.
(0.52, 0.02)	5	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	6	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	7	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	8	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	9	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	10	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000

Table C.3 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.25	+1.	+1.	+2.	+2.
(0.52, 0.02)	11	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	12	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+0.99999	+1.00000	+2.00001	+2.00000
(0.509, 0.009)		+0.25000	+0.99999	+1.00000	+2.00001	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	13	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.509, 0.009)		+0.25000	+1.00000	+1.00000	+2.00001	+2.00000
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.52, 0.02)	14	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+1.00004	+2.00000	+1.99992
(0.509, 0.009)		+0.25000	+1.00000	+1.00002	+2.00001	+1.99996
(0.508, 0.008)		+0.25000	+1.00000	+0.99990	+2.00001	+2.00019
(0.52, 0.02)	15	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+1.00000	+0.99932	+2.00000	+2.00136
(0.509, 0.009)		+0.25000	+1.00000	+0.99936	+2.00000	+2.00127
(0.508, 0.008)		+0.25000	+0.99987	+1.01506	+2.00023	+1.96989
(0.52, 0.02)	16	+0.25000	+1.00000	+1.00000	+2.00000	+2.00000
(0.51, 0.01)		+0.25000	+0.99998	+1.00000	+2.00005	+1.99999
(0.509, 0.009)		+0.25000	+1.00002	+0.99881	+1.99997	+2.00237
(0.508, 0.008)		+0.25000	+1.00014	+0.99602	+1.99974	+2.00796

Table C.4 : Cylinder domain with quadratic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.25	+1.	+1.	+2.	+2.
(0.52, 0.02)	5	+0.24289	+1.03317	+1.44884	+2.17918	+3.64878
(0.51, 0.01)		+0.25106	+0.95304	+0.36648	+1.47605	+1.54793
(0.509, 0.009)		+0.25069	+0.95745	+0.42997	+1.50492	+1.70199
(0.508, 0.008)		+0.25044	+0.96050	+0.47521	+1.52200	+1.82026
(0.52, 0.02)	6	+0.24997	+1.00164	+0.94327	+2.12629	+2.14240
(0.51, 0.01)		+0.25000	+1.00039	+0.93304	+2.09778	+2.15413
(0.509, 0.009)		+0.25000	+1.00022	+0.93179	+2.09444	+2.15554
(0.508, 0.008)		+0.25000	+1.00005	+0.93050	+2.09101	+2.15700
(0.52, 0.02)	7	+0.24997	+0.98636	+0.99400	+1.99230	+2.02209
(0.51, 0.01)		+0.24999	+0.97791	+1.02050	+1.92401	+1.94540
(0.509, 0.009)		+0.24999	+0.97535	+1.02913	+1.90387	+1.92132
(0.508, 0.008)		+0.24999	+0.97105	+1.04391	+1.87031	+1.88069
(0.52, 0.02)	8	+0.25001	+1.01972	+0.98906	+2.05900	+2.06849
(0.51, 0.01)		+0.25000	+1.00689	+0.99262	+2.02832	+2.04494
(0.509, 0.009)		+0.25000	+1.00586	+0.99261	+2.02584	+2.04321
(0.508, 0.008)		+0.25000	+1.00489	+0.99238	+2.02355	+2.04183
(0.52, 0.02)	9	+0.25000	+0.99943	+0.99955	+1.99692	+1.99876
(0.51, 0.01)		+0.25000	+1.00057	+0.99894	+1.99986	+2.00234
(0.509, 0.009)		+0.25000	+1.00056	+0.99894	+1.99980	+2.00230
(0.508, 0.008)		+0.25000	+1.00056	+0.99894	+1.99973	+2.00226
(0.52, 0.02)	10	+0.25000	+1.00100	+0.99897	+2.00102	+2.00348
(0.51, 0.01)		+0.25000	+1.00058	+0.99900	+1.99957	+2.00196
(0.509, 0.009)		+0.25000	+1.00063	+0.99902	+1.99944	+2.00183
(0.508, 0.008)		+0.25000	+1.00073	+0.99905	+1.99922	+2.00166

Table C.4 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.25	+1.	+1.	+2.	+2.
(0.52, 0.02)	11	+0.25000	+0.99969	+1.00013	+2.00025	+1.99979
(0.51, 0.01)		+0.25000	+1.00024	+0.99941	+1.99992	+2.00122
(0.509, 0.009)		+0.25000	+1.00013	+0.99955	+1.99999	+2.00094
(0.508, 0.008)		+0.25000	+1.00007	+0.99963	+2.00004	+2.00076
(0.52, 0.02)	12	+0.25000	+0.99959	+1.00041	+2.00005	+1.99915
(0.51, 0.01)		+0.25000	+1.00079	+0.99925	+1.99902	+2.00151
(0.509, 0.009)		+0.25000	+1.00051	+0.99951	+1.99931	+2.00098
(0.508, 0.008)		+0.25000	+1.00037	+0.99965	+1.99946	+2.00071
(0.52, 0.02)	13	+0.25000	+1.00001	+1.00003	+1.99999	+1.99995
(0.51, 0.01)		+0.25000	+1.00000	+1.00002	+2.00001	+1.99996
(0.509, 0.009)		+0.25000	+1.00000	+1.00002	+2.00001	+1.99996
(0.508, 0.008)		+0.25000	+1.00000	+1.00002	+2.00001	+1.99996
(0.52, 0.02)	14	+0.25000	+1.00000	+1.00008	+1.99999	+1.99983
(0.51, 0.01)		+0.25000	+1.00000	+1.00008	+2.00000	+1.99984
(0.509, 0.009)		+0.25000	+1.00000	+1.00007	+2.00001	+1.99986
(0.508, 0.008)		+0.25000	+1.00000	+1.00000	+2.00001	+2.00001
(0.52, 0.02)	15	+0.25000	+1.00000	+0.99995	+2.00000	+2.00010
(0.51, 0.01)		+0.25000	+1.00000	+0.99927	+2.00000	+2.00146
(0.509, 0.009)		+0.25000	+1.00001	+0.99896	+1.99999	+2.00208
(0.508, 0.008)		+0.25000	+1.00028	+0.97037	+1.99952	+2.05926
(0.52, 0.02)	16	+0.25000	+1.00000	+1.00002	+2.00001	+1.99997
(0.51, 0.01)		+0.25000	+0.99997	+1.00007	+2.00006	+1.99985
(0.509, 0.009)		+0.25000	+1.00001	+0.99897	+1.99999	+2.00206
(0.508, 0.008)		+0.25000	+1.00013	+0.99616	+1.99976	+2.00769

Table C.5 : Cylinder domain with cubic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.125	+0.75	+0.75	+3.	+3.
(0.52, 0.02)	5	+0.15452	+0.57739	- 1.4	+2.70061	- 3.
(0.51, 0.01)		+0.11323	+1.24	+7.	+9.	+9.
(0.509, 0.009)		+0.11820	+1.14	+6.	+8.	+7.
(0.508, 0.008)		+0.12099	+1.07	+5.	+8.	+6.
(0.52, 0.02)	6	+0.12505	+0.68207	+0.98743	+2.34098	+2.41833
(0.51, 0.01)		+0.12497	+0.68530	+1.02	+2.44264	+2.34512
(0.509, 0.009)		+0.12497	+0.68550	+1.02	+2.44931	+2.33782
(0.508, 0.008)		+0.12497	+0.68558	+1.02	+2.45374	+2.33060
(0.52, 0.02)	7	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.52, 0.02)	8	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.52, 0.02)	9	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.52, 0.02)	10	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000

Table C.5 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.125	+0.75	+0.75	+3.	+3.
(0.52, 0.02)	11	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.52, 0.02)	12	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00001	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.52, 0.02)	13	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.509, 0.009)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.508, 0.008)		+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.52, 0.02)	14	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.75002	+3.00000	+2.99996
(0.509, 0.009)		+0.12500	+0.75000	+0.75001	+3.00000	+2.99998
(0.508, 0.008)		+0.12500	+0.75000	+0.74995	+3.00000	+3.00010
(0.52, 0.02)	15	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.75000	+0.74966	+3.00000	+3.00067
(0.509, 0.009)		+0.12500	+0.75000	+0.74969	+3.00000	+3.00063
(0.508, 0.008)		+0.12500	+0.74994	+0.75762	+3.00012	+2.98476
(0.52, 0.02)	16	+0.12500	+0.75000	+0.75000	+3.00000	+3.00000
(0.51, 0.01)		+0.12500	+0.74999	+0.75000	+3.00002	+3.00000
(0.509, 0.009)		+0.12500	+0.75001	+0.74940	+2.99999	+3.00121
(0.508, 0.008)		+0.12500	+0.75007	+0.74798	+2.99987	+3.00405

Table C.6 : Cylinder domain with cubic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.125	+0.75	+0.75	+3.	+3.
(0.52, 0.02)	5	+0.17264	+0.48799	- 3.	+2.36827	-7.
(0.51, 0.01)		+0.11783	+1.02	+5.	+7.	+7.
(0.509, 0.009)		+0.12032	+0.98779	+4.	+7.	+6.
(0.508, 0.008)		+0.12200	+0.96642	+4.	+7.	+5.
(0.52, 0.02)	6	+0.12498	+0.70017	+0.83087	+2.72430	+2.80834
(0.51, 0.01)		+0.12498	+0.69979	+0.84159	+2.74884	+2.76520
(0.509, 0.009)		+0.12498	+0.69962	+0.84075	+2.74814	+2.76138
(0.508, 0.008)		+0.12498	+0.69935	+0.83891	+2.74577	+2.75781
(0.52, 0.02)	7	+0.12497	+0.73114	+0.72811	+2.99573	+3.05206
(0.51, 0.01)		+0.12498	+0.71799	+0.77077	+2.88957	+2.93423
(0.509, 0.009)		+0.12498	+0.71411	+0.78405	+2.85917	+2.89783
(0.508, 0.008)		+0.12498	+0.70764	+0.80652	+2.80882	+2.83674
(0.52, 0.02)	8	+0.12502	+0.77502	+0.72161	+3.08347	+3.11309
(0.51, 0.01)		+0.12500	+0.75802	+0.73225	+3.03369	+3.07178
(0.509, 0.009)		+0.12500	+0.75694	+0.73279	+3.03037	+3.06892
(0.508, 0.008)		+0.12500	+0.75592	+0.73300	+3.02742	+3.06663
(0.52, 0.02)	9	+0.12496	+0.68514	+0.76811	+2.80840	+2.80414
(0.51, 0.01)		+0.12500	+0.75525	+0.73503	+3.02118	+3.05305
(0.509, 0.009)		+0.12500	+0.75343	+0.73698	+3.01313	+3.04184
(0.508, 0.008)		+0.12500	+0.75234	+0.73844	+3.00770	+3.03387
(0.52, 0.02)	10	+0.12500	+0.76155	+0.73975	+3.00752	+3.03346
(0.51, 0.01)		+0.12500	+0.75682	+0.74147	+2.99328	+3.01581
(0.509, 0.009)		+0.12500	+0.75727	+0.74174	+2.99197	+3.01446
(0.508, 0.008)		+0.12500	+0.75830	+0.74212	+2.98993	+3.01257

Table C.6 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.125	+0.75	+0.75	+3.	+3.
(0.52, 0.02)	11	+0.12500	+0.74908	+0.75077	+3.00021	+2.99842
(0.51, 0.01)		+0.12500	+0.75082	+0.74851	+2.99935	+3.00305
(0.509, 0.009)		+0.12500	+0.75053	+0.74888	+2.99955	+3.00229
(0.508, 0.008)		+0.12500	+0.75036	+0.74910	+2.99967	+3.00183
(0.52, 0.02)	12	+0.12500	+0.74901	+0.75095	+3.00008	+2.99798
(0.51, 0.01)		+0.12500	+0.75179	+0.74823	+2.99775	+3.00356
(0.509, 0.009)		+0.12500	+0.75116	+0.74883	+2.99841	+3.00236
(0.508, 0.008)		+0.12500	+0.75083	+0.74912	+2.99877	+3.00177
(0.52, 0.02)	13	+0.12500	+0.74997	+0.75009	+2.99995	+2.99981
(0.51, 0.01)		+0.12500	+0.75000	+0.74998	+2.99999	+3.00005
(0.509, 0.009)		+0.12500	+0.75000	+0.74997	+2.99999	+3.00007
(0.508, 0.008)		+0.12500	+0.75000	+0.74996	+2.99999	+3.00009
(0.52, 0.02)	14	+0.12500	+0.75001	+0.74976	+3.00000	+3.00049
(0.51, 0.01)		+0.12500	+0.75000	+0.74981	+2.99999	+3.00038
(0.509, 0.009)		+0.12500	+0.75000	+0.74981	+3.00000	+3.00039
(0.508, 0.008)		+0.12500	+0.75000	+0.74977	+2.99999	+3.00047
(0.52, 0.02)	15	+0.12500	+0.75001	+0.74984	+2.99999	+3.00032
(0.51, 0.01)		+0.12500	+0.75000	+0.74942	+2.99999	+3.00116
(0.509, 0.009)		+0.12500	+0.75000	+0.74885	+2.99998	+3.00231
(0.508, 0.008)		+0.12500	+0.75021	+0.72763	+2.99963	+3.04474
(0.52, 0.02)	16	+0.12500	+0.74999	+0.75003	+3.00002	+2.99994
(0.51, 0.01)		+0.12500	+0.74998	+0.75008	+3.00004	+2.99984
(0.509, 0.009)		+0.12500	+0.74999	+0.74964	+3.00003	+3.00073
(0.508, 0.008)		+0.12500	+0.75010	+0.74757	+2.99982	+3.00487

Table C.7 : Cylinder domain with quartic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		- 0.0625	+0.125	- 0.5	+0.75	- 3.
(0.52, 0.02)	5	- 0.05583	+0.18097	- 1.16	+1.28	- 4.01
(0.51, 0.01)		- 0.06525	+0.33552	+0.86	+2.80	- 1.09
(0.509, 0.009)		- 0.06407	+0.31101	+0.54	+2.56	- 1.59
(0.508, 0.008)		- 0.06341	+0.29620	+0.34	+2.42	- 1.90
(0.52, 0.02)	6	- 0.06245	+0.20446	- 0.63059	+1.20	- 2.72928
(0.51, 0.01)		- 0.06245	+0.20345	- 0.63767	+1.15	- 2.67067
(0.509, 0.009)		- 0.06246	+0.20366	- 0.63498	+1.15	- 2.66552
(0.508, 0.008)		- 0.06247	+0.20405	- 0.63063	+1.16	- 2.66059
(0.52, 0.02)	7	- 0.06242	+0.15638	- 0.46254	+0.73341	- 3.14099
(0.51, 0.01)		- 0.06247	+0.18108	- 0.54477	+0.95627	- 2.89712
(0.509, 0.009)		- 0.06247	+0.18777	- 0.56853	+1.02	- 2.82864
(0.508, 0.008)		- 0.06247	+0.19872	- 0.60834	+1.11	- 2.71594
(0.52, 0.02)	8	- 0.06257	+0.022	- 0.45378	+0.41	- 3.35702
(0.51, 0.01)		- 0.06251	+0.095	- 0.46823	+0.61200	- 3.20914
(0.509, 0.009)		- 0.06250	+0.10012	- 0.46783	+0.62538	- 3.20016
(0.508, 0.008)		- 0.06250	+0.10453	- 0.46642	+0.63749	- 3.19312
(0.52, 0.02)	9	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.509, 0.009)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.508, 0.008)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.52, 0.02)	10	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.509, 0.009)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.508, 0.008)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000

Table C.7 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.625	+0.125	- 0.5	+0.75	- 3.
(0.52, 0.02)	11	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.509, 0.009)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.508, 0.008)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.52, 0.02)	12	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.509, 0.009)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.508, 0.008)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.52, 0.02)	13	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.509, 0.009)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.508, 0.008)		- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.52, 0.02)	14	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.50001	+0.75000	- 2.99998
(0.509, 0.009)		- 0.06250	+0.12500	- 0.50001	+0.75000	- 2.99999
(0.508, 0.008)		- 0.06250	+0.12500	- 0.49998	+0.75000	- 3.00005
(0.52, 0.02)	15	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12500	- 0.49983	+0.75000	- 3.00035
(0.509, 0.009)		- 0.06250	+0.12500	- 0.49978	+0.75000	- 3.00044
(0.508, 0.008)		- 0.06250	+0.12503	- 0.50333	+0.74995	- 2.99334
(0.52, 0.02)	16	- 0.06250	+0.12500	- 0.50000	+0.75000	- 3.00000
(0.51, 0.01)		- 0.06250	+0.12501	- 0.50000	+0.74999	- 3.00000
(0.509, 0.009)		- 0.06250	+0.12500	- 0.49971	+0.75001	- 3.00058
(0.508, 0.008)		- 0.06250	+0.12497	- 0.49907	+0.75006	- 3.00187

Table C.8 : Cylinder domain with quartic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.0625	+0.125	- 0.5	+0.75	- 3.
(0.52, 0.02)	5	- 0.03357	+0.07	- 2.6	+0.83976	- 9.
(0.51, 0.01)		- 0.06682	+0.39405	+1.7	+3.6	- 0.34
(0.509, 0.009)		- 0.06530	+0.37567	+1.5	+3.5	- 0.96
(0.508, 0.008)		- 0.06428	+0.36283	+1.3	+3.4	- 1.44
(0.52, 0.02)	6	- 0.06241	+0.19855	- 0.56866	+1.05525	- 2.89092
(0.51, 0.01)		- 0.06245	+0.19928	- 0.56846	+1.04228	- 2.83605
(0.509, 0.009)		- 0.06246	+0.19966	- 0.56472	+1.04734	- 2.83191
(0.508, 0.008)		- 0.06247	+0.20019	- 0.55923	+1.05539	- 2.82830
(0.52, 0.02)	7	- 0.06239	+0.16386	- 0.45354	+0.75118	- 3.16394
(0.51, 0.01)		- 0.06245	+0.19724	- 0.56050	+1.02647	- 2.85504
(0.509, 0.009)		- 0.06246	+0.20682	- 0.59274	+1.10220	- 2.76449
(0.508, 0.008)		- 0.06246	+0.22270	- 0.64724	+1.22664	- 2.61389
(0.52, 0.02)	8	- 0.06255	+0.02255	- 0.43260	+0.46866	- 3.36048
(0.51, 0.01)		- 0.06251	+0.08372	- 0.45128	+0.61780	- 3.24506
(0.509, 0.009)		- 0.06251	+0.08877	- 0.45147	+0.63038	- 3.23601
(0.508, 0.008)		- 0.06250	+0.09360	- 0.45066	+0.64211	- 3.22859
(0.52, 0.02)	9	- 0.06248	+0.14318	- 0.49501	+0.83781	- 2.94584
(0.51, 0.01)		- 0.06250	+0.11327	- 0.48083	+0.75090	- 3.04580
(0.509, 0.009)		- 0.06250	+0.11380	- 0.48138	+0.75338	- 3.04280
(0.508, 0.008)		- 0.06250	+0.11418	- 0.48189	+0.75537	- 3.04042
(0.52, 0.02)	10	- 0.06250	+0.10362	- 0.48032	+0.73473	- 3.06456
(0.51, 0.01)		- 0.06250	+0.11266	- 0.48331	+0.76162	- 3.03149
(0.509, 0.009)		- 0.06250	+0.11187	- 0.48379	+0.76404	- 3.02899
(0.508, 0.008)		- 0.06250	+0.11001	- 0.48446	+0.76781	- 3.02551

Table C.8 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.0625	+0.125 - 0.5	+0.75 - 3.		
(0.52, 0.02)	11	- 0.06250	+0.12778	- 0.50166	+0.74888	- 2.99669
(0.51, 0.01)		- 0.06250	+0.12253	- 0.49487	+0.75166	- 3.01054
(0.509, 0.009)		- 0.06250	+0.12344	- 0.49604	+0.75104	- 3.00809
(0.508, 0.008)		- 0.06250	+0.12401	- 0.49677	+0.75063	- 3.00659
(0.52, 0.02)	12	- 0.06250	+0.12815	- 0.50269	+0.74962	- 2.99431
(0.51, 0.01)		- 0.06250	+0.11889	- 0.49383	+0.75761	- 3.01244
(0.509, 0.009)		- 0.06250	+0.12102	- 0.49584	+0.75543	- 3.00837
(0.508, 0.008)		- 0.06250	+0.12214	- 0.49685	+0.75422	- 3.00633
(0.52, 0.02)	13	- 0.06250	+0.12494	- 0.49979	+0.75004	- 3.00042
(0.51, 0.01)		- 0.06250	+0.12499	- 0.49974	+0.75001	- 3.00052
(0.509, 0.009)		- 0.06250	+0.12499	- 0.49974	+0.75000	- 3.00052
(0.508, 0.008)		- 0.06250	+0.12499	- 0.49974	+0.75000	- 3.00052
(0.52, 0.02)	14	- 0.06250	+0.12497	- 0.50001	+0.75007	- 2.99997
(0.51, 0.01)		- 0.06250	+0.12499	- 0.50000	+0.75001	- 3.00001
(0.509, 0.009)		- 0.06250	+0.12500	- 0.49999	+0.75000	- 3.00002
(0.508, 0.008)		- 0.06250	+0.12500	- 0.49994	+0.75001	- 3.00013
(0.52, 0.02)	15	- 0.06250	+0.12497	- 0.49948	+0.75002	- 3.00105
(0.51, 0.01)		- 0.06250	+0.12499	- 0.49891	+0.75003	- 3.00217
(0.509, 0.009)		- 0.06250	+0.12497	- 0.49687	+0.75005	- 3.00626
(0.508, 0.008)		- 0.06250	+0.12464	- 0.46144	+0.75064	- 3.07713
(0.52, 0.02)	16	- 0.06250	+0.12501	- 0.50015	+0.74991	- 2.99969
(0.51, 0.01)		- 0.06250	+0.12503	- 0.50010	+0.74994	- 2.99980
(0.509, 0.009)		- 0.06250	+0.12508	- 0.50045	+0.74985	- 2.99911
(0.508, 0.008)		- 0.06250	+0.12484	- 0.49705	+0.75031	- 3.00589

Table C.9 : Cylinder domain with quintic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	5	- 0.04328	- 0.10979	+1.19	- 3.19	- 1.09613
(0.51, 0.01)		- 0.02573	- 0.40068	- 2.62	- 6.0	- 6.7
(0.509, 0.009)		- 0.02799	- 0.35415	- 2.00	- 5.6	- 5.7
(0.508, 0.008)		- 0.02927	- 0.32585	- 1.62	- 5.3	- 5.1
(0.52, 0.02)	6	- 0.03083	- 0.15401	+0.19	- 3.03567	- 3.49826
(0.51, 0.01)		- 0.03115	- 0.14455	+0.29	- 2.81008	- 3.57830
(0.509, 0.009)		- 0.03117	- 0.14258	+0.31	- 2.77084	- 3.58936
(0.508, 0.008)		- 0.03118	- 0.14020	+0.33	- 2.72477	- 3.60136
(0.52, 0.02)	7	- 0.03089	- 0.066	- 0.11750	- 2.17708	- 2.74045
(0.51, 0.01)		- 0.03106	- 0.059	- 0.55366	- 1.1	- 1.5
(0.509, 0.009)		- 0.03106	+0.100	- 0.70488	- 0.7	- 1.1
(0.508, 0.008)		- 0.03106	+0.170	- 0.96426	- 0.1	- 0.4
(0.52, 0.02)	8	- 0.03141	- 0.52947	- 0.09	- 3.30329	- 3.48758
(0.51, 0.01)		- 0.03126	- 0.36820	- 0.17174	- 2.77190	- 3.07190
(0.509, 0.009)		- 0.03126	- 0.36012	- 0.17593	- 2.74284	- 3.04757
(0.508, 0.008)		- 0.03126	- 0.35265	- 0.17752	- 2.71751	- 3.02826
(0.52, 0.02)	9	- 0.03095	+0.19	- 0.41204	- 0.90	- 0.98
(0.51, 0.01)		- 0.03126	- 0.35248	- 0.18861	- 2.69863	- 2.96086
(0.509, 0.009)		- 0.03125	- 0.33612	- 0.20696	- 2.62264	- 2.85451
(0.508, 0.008)		- 0.03125	- 0.32629	- 0.22075	- 2.57269	- 2.77968
(0.52, 0.02)	10	- 0.03126	- 0.38842	- 0.24253	- 2.54553	- 2.72210
(0.51, 0.01)		- 0.03125	- 0.35861	- 0.25373	- 2.45318	- 2.60807
(0.509, 0.009)		- 0.03125	- 0.36100	- 0.25605	- 2.44529	- 2.59830
(0.508, 0.008)		- 0.03125	- 0.36657	- 0.25933	- 2.43351	- 2.58476

Table C.9 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	11	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.508, 0.008)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	12	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.508, 0.008)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	13	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.508, 0.008)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	14	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)		- 0.03125	- 0.31250	- 0.31251	- 2.50000	- 2.50000
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.508, 0.008)		- 0.03125	- 0.31250	- 0.31249	- 2.50000	- 2.50003
(0.52, 0.02)	15	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)		- 0.03125	- 0.31250	- 0.31241	- 2.50000	- 2.50018
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31241	- 2.50000	- 2.50017
(0.508, 0.008)		- 0.03125	- 0.31248	- 0.31447	- 2.50003	- 2.49605
(0.52, 0.02)	16	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.51, 0.01)		- 0.03125	- 0.31250	- 0.31250	- 2.50001	- 2.50000
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31234	- 2.50000	- 2.50031
(0.508, 0.008)		- 0.03125	- 0.31252	- 0.31198	- 2.49997	- 2.50104

Table C.10 : Cylinder domain with quintic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	5	- 0.03451	- 0.15593	+0.62	- 3.40	- 3.10
(0.51, 0.01)		- 0.03041	- 0.19476	+0.10	- 3.58	- 4.47
(0.509, 0.009)		- 0.03066	- 0.19149	+0.14	- 3.54	- 4.40
(0.508, 0.008)		- 0.03084	- 0.18893	+0.17	- 3.50	- 4.35
(0.52, 0.02)	6	- 0.03076	- 0.17262	+0.35	- 3.43	- 3.90
(0.51, 0.01)		- 0.03115	- 0.16175	+0.49	- 3.14	- 4.04
(0.509, 0.009)		- 0.03117	- 0.15970	+0.51	- 3.10	- 4.06
(0.508, 0.008)		- 0.03119	- 0.15730	+0.54	- 3.05	- 4.08
(0.52, 0.02)	7	- 0.03083	- 0.03	- 0.11226	- 2.19624	- 2.79624
(0.51, 0.01)		- 0.03103	- 0.12452	- 0.62435	- 0.93	- 1.38
(0.509, 0.009)		- 0.03103	+0.18	- 0.80025	- 0.53	- 0.91
(0.508, 0.008)		- 0.03103	+0.26	- 1.10	+1.42	- 0.09
(0.52, 0.02)	8	- 0.03138	- 0.50797	- 0.042	- 3.15	- 3.46
(0.51, 0.01)		- 0.03126	- 0.37969	- 0.13922	- 2.74448	- 3.12
(0.509, 0.009)		- 0.03126	- 0.37198	- 0.14500	- 2.71837	- 3.09
(0.508, 0.008)		- 0.03126	- 0.36470	- 0.14831	- 2.69525	- 3.07
(0.52, 0.02)	9	- 0.03087	+0.36	- 0.49072	- 0.49	- 0.48
(0.51, 0.01)		- 0.03126	- 0.37359	- 0.14532	- 2.71703	- 3.08
(0.509, 0.009)		- 0.03126	- 0.35467	- 0.16569	- 2.63333	- 2.96
(0.508, 0.008)		- 0.03125	- 0.34314	- 0.18103	- 2.57661	- 2.87548
(0.52, 0.02)	10	- 0.03127	- 0.44512	- 0.19199	- 2.58873	- 2.89038
(0.51, 0.01)		- 0.03125	- 0.39011	- 0.21281	- 2.42490	- 2.68510
(0.509, 0.009)		- 0.03125	- 0.39518	- 0.21600	- 2.40991	- 2.66938
(0.508, 0.008)		- 0.03125	- 0.40697	- 0.22043	- 2.38642	- 2.64747

Table C.10 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.03125	- 0.31250	- 0.31250	- 2.50000	- 2.50000
(0.52, 0.02)	11	- 0.03125	- 0.30232	- 0.31813	- 2.50508	- 2.48917
(0.51, 0.01)		- 0.03125	- 0.32072	- 0.29413	- 2.49500	- 2.53776
(0.509, 0.009)		- 0.03125	- 0.31750	- 0.29826	- 2.49717	- 2.52916
(0.508, 0.008)		- 0.03125	- 0.31553	- 0.30078	- 2.49859	- 2.52390
(0.52, 0.02)	12	- 0.03125	- 0.30021	- 0.32390	- 2.50092	- 2.47583
(0.51, 0.01)		- 0.03125	- 0.33474	- 0.29013	- 2.47207	- 2.54506
(0.509, 0.009)		- 0.03125	- 0.32687	- 0.29745	- 2.48017	- 2.53023
(0.508, 0.008)		- 0.03125	- 0.32276	- 0.30109	- 2.48469	- 2.52286
(0.52, 0.02)	13	- 0.03125	- 0.31211	- 0.31315	- 2.49936	- 2.49846
(0.51, 0.01)		- 0.03125	- 0.31248	- 0.31171	- 2.49982	- 2.50155
(0.509, 0.009)		- 0.03125	- 0.31250	- 0.31158	- 2.49983	- 2.50182
(0.508, 0.008)		- 0.03125	- 0.31254	- 0.31146	- 2.49981	- 2.50207
(0.52, 0.02)	14	- 0.03125	- 0.31267	- 0.30847	- 2.50005	- 2.50814
(0.51, 0.01)		- 0.03125	- 0.31257	- 0.30926	- 2.49988	- 2.50648
(0.509, 0.009)		- 0.03125	- 0.31256	- 0.30929	- 2.49989	- 2.50641
(0.508, 0.008)		- 0.03125	- 0.31256	- 0.30926	- 2.49988	- 2.50648
(0.52, 0.02)	15	- 0.03125	- 0.31265	- 0.31038	- 2.49988	- 2.50425
(0.51, 0.01)		- 0.03125	- 0.31258	- 0.30825	- 2.49987	- 2.50849
(0.509, 0.009)		- 0.03125	- 0.31261	- 0.30336	- 2.49980	- 2.51828
(0.508, 0.008)		- 0.03125	- 0.31306	- 0.25603	- 2.49901	- 2.61293
(0.52, 0.02)	16	- 0.03125	- 0.31234	- 0.31297	- 2.50033	- 2.49906
(0.51, 0.01)		- 0.03125	- 0.31239	- 0.31289	- 2.50021	- 2.49922
(0.509, 0.009)		- 0.03125	- 0.31289	- 0.31420	- 2.50042	- 2.49660
(0.508, 0.008)		- 0.03125	- 0.31276	- 0.30769	- 2.49949	- 2.50962

Table C.11 : Cylinder domain with hyperbolic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.87758	0.	- 0.47943	0.	- 0.87758
(0.52, 0.02)	5	+0.87768	- 0.00413	- 0.48152	- 0.01809	- 0.90863
(0.51, 0.01)		+0.87756	- 0.00211	- 0.45490	+0.00315	- 0.87265
(0.509, 0.009)		+0.87756	- 0.00242	- 0.45907	+0.00018	- 0.87905
(0.508, 0.008)		+0.87757	- 0.00261	- 0.46161	- 0.00158	- 0.88305
(0.52, 0.02)	6	+0.87758	- 0.00379	- 0.47377	- 0.01927	- 0.88989
(0.51, 0.01)		+0.87758	- 0.00375	- 0.47357	- 0.01732	- 0.89232
(0.509, 0.009)		+0.87758	- 0.00376	- 0.47370	- 0.01742	- 0.89253
(0.508, 0.008)		+0.87758	- 0.00378	- 0.47391	- 0.01767	- 0.89273
(0.52, 0.02)	7	+0.87759	- 0.00185	- 0.48054	- 0.00047	- 0.87330
(0.51, 0.01)		+0.87758	- 0.00298	- 0.47679	- 0.01064	- 0.88447
(0.509, 0.009)		+0.87758	- 0.00331	- 0.47562	- 0.01350	- 0.88782
(0.508, 0.008)		+0.87758	- 0.00385	- 0.47364	- 0.01822	- 0.89341
(0.52, 0.02)	8	+0.87759	+0.00410	- 0.48093	+0.01397	- 0.86372
(0.51, 0.01)		+0.87758	+0.00110	- 0.48043	+0.00573	- 0.86963
(0.509, 0.009)		+0.87758	+0.00090	- 0.48046	+0.00520	- 0.86998
(0.508, 0.008)		+0.87758	+0.00072	- 0.48052	+0.00471	- 0.87025
(0.52, 0.02)	9	+0.87758	+0.00081	- 0.47945	+0.00301	- 0.87515
(0.51, 0.01)		+0.87758	- 0.00019	- 0.47905	- 0.00025	- 0.87869
(0.509, 0.009)		+0.87758	- 0.00016	- 0.47908	- 0.00012	- 0.87852
(0.508, 0.008)		+0.87758	- 0.00014	- 0.47911	- 0.00003	- 0.87839
(0.52, 0.02)	10	+0.87758	- 0.00033	- 0.47912	- 0.00021	- 0.87856
(0.51, 0.01)		+0.87758	- 0.00020	- 0.47917	+0.00019	- 0.87806
(0.509, 0.009)		+0.87758	- 0.00021	- 0.47918	+0.00023	- 0.87802
(0.508, 0.008)		+0.87758	- 0.00023	- 0.47919	+0.00028	- 0.87796

Table C.11 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.87758	0.	- 0.47943	0.	- 0.87758
(0.52, 0.02)	11	+0.87758	+0.00001	- 0.47943	- 0.00001	- 0.87758
(0.51, 0.01)		+0.87758	- 0.00001	- 0.47941	- 0.00000	- 0.87761
(0.509, 0.009)		+0.87758	- 0.00000	- 0.47942	- 0.00000	- 0.87761
(0.508,0.008)		+0.87758	- 0.00000	- 0.47942	- 0.00000	- 0.87760
(0.52, 0.02)	12	+0.87758	+0.00001	- 0.47944	- 0.00000	- 0.87755
(0.51, 0.01)		+0.87758	- 0.00005	- 0.47939	+0.00007	- 0.87765
(0.509, 0.009)		+0.87758	- 0.00004	- 0.47941	+0.00006	- 0.87762
(0.508,0.008)		+0.87758	- 0.00002	- 0.47942	+0.00003	- 0.87760
(0.52, 0.02)	13	+0.87758	- 0.00000	- 0.47943	- 0.00000	- 0.87758
(0.51, 0.01)		+0.87758	- 0.00001	- 0.47943	+0.00002	- 0.87757
(0.509, 0.009)		+0.87758	- 0.00001	- 0.47943	+0.00002	- 0.87757
(0.508,0.008)		+0.87758	- 0.00000	- 0.47943	+0.00001	- 0.87757
(0.52, 0.02)	14	+0.87758	+0.00000	- 0.47944	+0.00000	- 0.87756
(0.51, 0.01)		+0.87758	- 0.00001	- 0.47930	+0.00001	- 0.87783
(0.509, 0.009)		+0.87758	- 0.00001	- 0.47936	+0.00002	- 0.87771
(0.508,0.008)		+0.87758	- 0.00000	- 0.47975	+0.00002	- 0.87693
(0.52, 0.02)	15	+0.87758	+0.00000	- 0.47941	- 0.00000	- 0.87761
(0.51, 0.01)		+0.87758	+0.00001	- 0.48168	- 0.00001	- 0.87307
(0.509, 0.009)		+0.87758	+0.00000	- 0.48150	- 0.00000	- 0.87343
(0.508,0.008)		+0.87758	- 0.00043	- 0.42899	+0.00077	- 0.97845
(0.52, 0.02)	16	+0.87758	+0.00000	- 0.47943	- 0.00000	- 0.87758
(0.51, 0.01)		+0.87758	- 0.00008	- 0.47942	+0.00016	- 0.87759
(0.509, 0.009)		+0.87758	+0.00006	- 0.48338	- 0.00010	- 0.86968
(0.508,0.008)		+0.87758	+0.00046	- 0.49278	- 0.00089	- 0.85088

Table C.12 : Cylinder domain with hyperbolic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.87758	0.	- 0.47943	0.	- 0.87758
(0.52, 0.02)	5	+0.87981	- 0.01327	- 0.62298	- 0.05522	- 1.39
(0.51, 0.01)		+0.87725	+0.01171	- 0.28468	+0.16300	- 0.72841
(0.509, 0.009)		+0.87736	+0.01032	- 0.30463	+0.15371	- 0.77650
(0.508, 0.008)		+0.87744	+0.00935	- 0.31888	+0.14809	- 0.81338
(0.52, 0.02)	6	+0.87759	- 0.00339	- 0.46448	- 0.03864	- 0.91437
(0.51, 0.01)		+0.87758	- 0.00314	- 0.46235	- 0.03197	- 0.91851
(0.509, 0.009)		+0.87758	- 0.00311	- 0.46222	- 0.03142	- 0.91893
(0.508, 0.008)		+0.87758	- 0.00309	- 0.46214	- 0.03095	- 0.91934
(0.52, 0.02)	7	+0.87759	+0.00103	- 0.47912	+0.00004	- 0.87964
(0.51, 0.01)		+0.87758	+0.00170	- 0.48116	+0.00054	- 0.87361
(0.509, 0.009)		+0.87758	+0.00189	- 0.48180	+0.00692	- 0.87181
(0.508, 0.008)		+0.87758	+0.00221	- 0.48288	+0.00942	- 0.86879
(0.52, 0.02)	8	+0.87758	- 0.00185	- 0.47870	- 0.00571	- 0.88364
(0.51, 0.01)		+0.87758	- 0.00060	- 0.47896	- 0.00285	- 0.88149
(0.509, 0.009)		+0.87758	- 0.00050	- 0.47894	- 0.00261	- 0.88134
(0.508, 0.008)		+0.87758	- 0.00040	- 0.47891	- 0.00240	- 0.88122
(0.52, 0.02)	9	+0.87758	- 0.00088	- 0.47920	- 0.00273	- 0.88029
(0.51, 0.01)		+0.87758	+0.00010	- 0.47965	+0.00028	- 0.87681
(0.509, 0.009)		+0.87758	+0.00007	- 0.47962	+0.00017	- 0.87697
(0.508, 0.008)		+0.87758	+0.00005	- 0.47960	+0.00009	- 0.87709
(0.52, 0.02)	10	+0.87758	+0.00021	- 0.47960	+0.00010	- 0.87704
(0.51, 0.01)		+0.87758	+0.00012	- 0.47955	- 0.00013	- 0.87736
(0.509, 0.009)		+0.87758	+0.00013	- 0.47955	- 0.00015	- 0.87738
(0.508, 0.008)		+0.87758	+0.00015	- 0.47954	- 0.00019	- 0.87742

Table C.12 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+0.87758	0.	- 0.47943	0.	- 0.87758
(0.52, 0.02)	11	+0.87758	+0.00002	- 0.47944	- 0.00001	- 0.87757
(0.51, 0.01)		+0.87758	- 0.00001	- 0.47940	+0.00000	- 0.87764
(0.509, 0.009)		+0.87758	- 0.00001	- 0.47940	+0.00000	- 0.87763
(0.508,0.008)		+0.87758	- 0.00000	- 0.47941	- 0.00000	- 0.87762
(0.52, 0.02)	12	+0.87758	+0.00002	- 0.47945	- 0.00000	- 0.87753
(0.51, 0.01)		+0.87758	- 0.00006	- 0.47938	+0.00008	- 0.87767
(0.509, 0.009)		+0.87758	- 0.00005	- 0.47940	+0.00008	- 0.87763
(0.508,0.008)		+0.87758	- 0.00003	- 0.47941	+0.00004	- 0.87761
(0.52, 0.02)	13	+0.87758	- 0.00000	- 0.47943	+0.00000	- 0.87757
(0.51, 0.01)		+0.87758	- 0.00001	- 0.47944	+0.00002	- 0.87756
(0.509, 0.009)		+0.87758	- 0.00001	- 0.47944	+0.00002	- 0.87756
(0.508,0.008)		+0.87758	- 0.00000	- 0.47943	+0.00001	- 0.87757
(0.52, 0.02)	14	+0.87758	+0.00000	- 0.47944	+0.00000	- 0.87755
(0.51, 0.01)		+0.87758	- 0.00001	- 0.47937	+0.00001	- 0.87770
(0.509, 0.009)		+0.87758	- 0.00001	- 0.47940	+0.00002	- 0.87763
(0.508,0.008)		+0.87758	- 0.00000	- 0.47962	+0.00001	- 0.87719
(0.52, 0.02)	15	+0.87758	+0.00000	- 0.47941	- 0.00000	- 0.87761
(0.51, 0.01)		+0.87758	+0.00001	- 0.48172	- 0.00001	- 0.87300
(0.509, 0.009)		+0.87758	+0.00001	- 0.48147	- 0.00001	- 0.87350
(0.508,0.008)		+0.87758	+0.00064	- 0.54842	- 0.00112	- 0.73960
(0.52, 0.02)	16	+0.87758	+0.00000	- 0.47944	- 0.00001	- 0.87755
(0.51, 0.01)		+0.87758	- 0.00009	- 0.47931	+0.00017	- 0.87781
(0.509, 0.009)		+0.87758	+0.00006	- 0.48336	- 0.00012	- 0.86972
(0.508,0.008)		+0.87758	+0.00030	- 0.49064	- 0.00058	- 0.85516

Table C.13 : Cylinder domain with logarithmic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.69315	0.	+2.	0.	- 4.
(0.52, 0.02)	5	- 0.68369	+0.05456	+1.58708	- 0.01270	- 6.36
(0.51, 0.01)		- 0.69636	+0.25327	+4.19	+2.0	- 2.77
(0.509, 0.009)		- 0.69493	+0.22293	+3.79	+1.8	- 3.40269
(0.508, 0.008)		- 0.69415	+0.20479	+3.55	+1.6	- 3.80089
(0.52, 0.02)	6	- 0.69263	+0.08631	+2.30025	- 0.12177	- 4.63045
(0.51, 0.01)		- 0.69303	+0.09585	+2.40900	+0.07966	- 4.65598
(0.509, 0.009)		- 0.69305	+0.09778	+2.42730	+0.11524	- 4.66014
(0.508, 0.008)		- 0.69307	+0.10012	+2.44901	+0.15784	- 4.66520
(0.52, 0.02)	7	- 0.69268	+0.17285	+1.99779	+0.71777	- 3.88944
(0.51, 0.01)		- 0.69296	+0.25343	+1.75465	+1.44	- 3.06112
(0.509, 0.009)		- 0.69297	+0.28698	+1.63610	+1.73	- 2.71788
(0.508, 0.008)		- 0.69297	+0.34549	+1.42218	+2.24	- 2.11571
(0.52, 0.02)	8	- 0.69320	- 0.28844	+2.02791	- 0.40196	- 4.63228
(0.51, 0.01)		- 0.69317	- 0.18030	+2.14210	- 0.30592	- 4.64079
(0.509, 0.009)		- 0.69316	- 0.16390	+2.15409	- 0.27264	- 4.62896
(0.508, 0.008)		- 0.69316	- 0.14812	+2.16587	- 0.23739	- 4.61619
(0.52, 0.02)	9	- 0.69389	- 1.37	+2.51301	- 4.	- 8.
(0.51, 0.01)		- 0.69313	+0.00254	+1.94587	+0.54660	- 3.34885
(0.509, 0.009)		- 0.69314	- 0.03950	+1.99327	+0.35034	- 3.62833
(0.508, 0.008)		- 0.69315	- 0.63834	+2.02764	+0.22573	- 3.82124
(0.52, 0.02)	10	- 0.69315	+0.02613	+2.10670	- 0.04485	- 4.19973
(0.51, 0.01)		- 0.69315	+0.01753	+2.10527	- 0.05417	- 4.21235
(0.509, 0.009)		- 0.69315	+0.01694	+2.10463	- 0.05197	- 4.20954
(0.508, 0.008)		- 0.69315	+0.01552	+2.10366	- 0.04847	- 4.20527

Table C.13 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.69315	0.	+2.	0.	- 4.
(0.52, 0.02)	11	- 0.69315	- 0.00755	+2.13775	- 0.06505	- 4.29828
(0.51, 0.01)		- 0.69315	+0.02910	+2.09052	- 0.06592	- 4.18521
(0.509, 0.009)		- 0.69315	+0.02764	+2.09218	- 0.06403	- 4.18786
(0.508, 0.008)		- 0.69315	+0.02766	+2.09171	- 0.06341	- 4.18623
(0.52, 0.02)	12	- 0.69314	+0.01911	+2.05866	- 0.01019	- 4.11685
(0.51, 0.01)		- 0.69315	- 0.04915	+2.12338	+0.05812	- 4.24776
(0.509, 0.009)		- 0.69315	- 0.03063	+2.10394	+0.03920	- 4.20839
(0.508, 0.008)		- 0.69315	- 0.02158	+2.09440	+0.02975	- 4.18907
(0.52, 0.02)	13	- 0.69315	- 0.00126	+2.08490	- 0.00835	- 4.17144
(0.51, 0.01)		- 0.69315	+0.00188	+2.07636	- 0.00550	- 4.15294
(0.509, 0.009)		- 0.69315	+0.00206	+2.07569	- 0.00540	- 4.15154
(0.508, 0.008)		- 0.69315	+0.00217	+2.07505	- 0.00524	- 4.15023
(0.52, 0.02)	14	- 0.69315	+0.00298	+2.01818	+0.00132	- 4.03467
(0.51, 0.01)		- 0.69315	+0.00245	+2.04392	- 0.00470	- 4.08787
(0.509, 0.009)		- 0.69315	+0.00238	+2.04588	- 0.00464	- 4.09180
(0.508, 0.008)		- 0.69315	+0.00227	+2.04806	- 0.00448	- 4.09615
(0.52, 0.02)	15	- 0.69315	+0.00283	+2.08155	- 0.00495	- 4.16316
(0.51, 0.01)		- 0.69315	+0.00255	+2.02929	- 0.00481	- 4.05861
(0.509, 0.009)		- 0.69315	+0.00290	+1.97956	- 0.00544	- 3.95916
(0.508, 0.008)		- 0.69315	+0.00579	+1.63300	- 0.01062	- 3.26606
(0.52, 0.02)	16	- 0.69315	+0.00179	+2.08765	- 0.00356	- 4.17541
(0.51, 0.01)		- 0.69315	- 0.00007	+2.09708	+0.00016	- 4.19418
(0.509, 0.009)		- 0.69315	- 0.00043	+2.10353	+0.00087	- 4.20706
(0.508, 0.008)		- 0.69315	- 0.00093	+2.11415	+0.00185	- 4.22829

Table C.14 : Cylinder domain with logarithmic potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.69315	0.	+2.	0.	- 4.
(0.52, 0.02)	5	- 0.63925	- 0.17863	- 1.32	- 0.96711	- 16.
(0.51, 0.01)		- 0.70091	+0.42103	+7.8	+4.3	- 0.7
(0.509, 0.009)		- 0.69816	+0.38788	+6.4	+4.1	- 1.8
(0.508, 0.008)		- 0.69632	+0.36489	+6.0	+4.0	- 2.7
(0.52, 0.02)	6	- 0.69254	+0.05868	+2.48222	- 0.56891	- 5.08
(0.51, 0.01)		- 0.69304	+0.07015	+2.62714	- 0.29375	- 5.18
(0.509, 0.009)		- 0.69306	+0.07221	+2.64948	- 0.25192	- 5.19
(0.508, 0.008)		- 0.69308	+0.07462	+2.67515	- 0.20409	- 5.20
(0.52, 0.02)	7	- 0.69262	+0.20443	+1.99859	+0.70858	- 3.94240
(0.51, 0.01)		- 0.69293	+0.31165	+1.68747	+1.57	- 2.93
(0.509, 0.009)		- 0.69294	+0.35425	+1.54590	+1.91	- 2.53
(0.508, 0.008)		- 0.69294	+0.42847	+1.29150	+2.49	- 1.83
(0.52, 0.02)	8	- 0.69317	- 0.27566	+2.06975	- 2.5	- 4.61012
(0.51, 0.01)		- 0.69317	- 0.19961	+2.17939	- 2.7	- 4.68962
(0.509, 0.009)		- 0.69316	- 0.18341	+2.18933	- 2.4	- 4.67998
(0.508, 0.008)		- 0.69316	- 0.16745	+2.19892	- 2.1	- 4.66825
(0.52, 0.02)	9	- 0.69389	- 1.49	+2.69991	- 4.	- 9.
(0.51, 0.01)		- 0.69313	- 0.02588	+2.00562	+0.51527	- 3.51799
(0.509, 0.009)		- 0.69314	- 0.06526	+2.04813	+0.34038	- 3.76802
(0.508, 0.008)		- 0.69315	- 0.08790	+2.07821	+0.22805	- 3.94102
(0.52, 0.02)	10	- 0.69315	+0.01029	+2.14171	- 0.05983	- 4.28341
(0.51, 0.01)		- 0.69315	+0.00593	+2.13564	- 0.04747	- 4.26912
(0.509, 0.009)		- 0.69315	+0.00410	+2.13426	- 0.04214	- 4.26317
(0.508, 0.008)		- 0.69315	- 0.00015	+2.13237	- 0.03340	- 4.25446

Table C.14 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	- 0.69315	0.	+2.	0.	- 4.
(0.52, 0.02)	11	- 0.69315	- 0.01611	+2.16502	- 0.07530	- 4.35758
(0.51, 0.01)		- 0.69315	+0.03649	+2.09982	- 0.07835	- 4.20422
(0.509, 0.009)		- 0.69315	+0.03367	+2.10296	- 0.07535	- 4.20981
(0.508, 0.008)		- 0.69315	+0.03294	+2.10330	- 0.07399	- 4.20974
(0.52, 0.02)	12	- 0.69315	+0.01522	+2.07933	- 0.01362	- 4.15948
(0.51, 0.01)		- 0.69315	- 0.03676	+2.12072	+0.04149	- 4.24237
(0.509, 0.009)		- 0.69315	- 0.02409	+2.10795	+0.02933	- 4.21641
(0.508, 0.008)		- 0.69315	- 0.01747	+2.10109	+0.02295	- 4.20249
(0.52, 0.02)	13	- 0.69315	- 0.00282	+2.09561	- 0.01125	- 4.19380
(0.51, 0.01)		- 0.69315	+0.00193	+2.08321	- 0.00675	- 4.16675
(0.509, 0.009)		- 0.69315	+0.00227	+2.08204	- 0.00671	- 4.16433
(0.508, 0.008)		- 0.69315	+0.00251	+2.08084	- 0.00659	- 4.16186
(0.52, 0.02)	14	- 0.69315	+0.00382	+2.03942	- 0.00296	- 4.07786
(0.51, 0.01)		- 0.69315	+0.00308	+2.05248	- 0.00604	- 4.10500
(0.509, 0.009)		- 0.69315	+0.00302	+2.05329	- 0.00597	- 4.10664
(0.508, 0.008)		- 0.69315	+0.00283	+2.05466	- 0.00565	- 4.10937
(0.52, 0.02)	15	- 0.69315	+0.00355	+2.07770	- 0.00641	- 4.15547
(0.51, 0.01)		- 0.69315	+0.00335	+2.02021	- 0.00639	- 4.04046
(0.509, 0.009)		- 0.69315	+0.00360	+1.99068	- 0.00690	- 3.98140
(0.508, 0.008)		- 0.69315	- 0.00719	+3.13162	+0.01210	- 6.26
(0.52, 0.02)	16	- 0.69315	+0.00221	+2.08896	- 0.00446	- 4.17803
(0.51, 0.01)		- 0.69315	+0.00030	+2.09743	- 0.00059	- 4.19488
(0.509, 0.009)		- 0.69315	+0.00039	+2.09707	- 0.00078	- 4.19414
(0.508, 0.008)		- 0.69315	- 0.00135	+2.12508	+0.00267	- 4.25015

Table C.15 : Cylinder domain with radial potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
Exact :		+6.	0.	- 8.65617	0.	+17.31234
(0.52, 0.02)	5	+5.95917	- 0.23775	- 6.88372	+0.05059	+27.50
(0.51, 0.01)		+6.01386	- 1.1	- 18.	- 9.	+12.03
(0.509, 0.009)		+6.00770	- 0.96	- 16.	- 8.	+14.76
(0.508, 0.008)		+6.00431	- 0.89	- 15.	- 7.	+16.47438
(0.52, 0.02)	6	+5.99775	- 0.37477	- 9.96151	+0.52133	+20.05380
(0.51, 0.01)		+5.99949	- 0.41608	- 10.43250	- 0.35040	+20.16329
(0.509, 0.009)		+5.99958	- 0.42443	- 10.51181	- 0.50450	+20.18121
(0.508, 0.008)		+5.99966	- 0.43460	- 10.60588	- 0.68897	+20.20305
(0.52, 0.02)	7	+5.99799	- 0.74904	- 8.65332	- 3.1	+16.84888
(0.51, 0.01)		+5.99918	- 1.1	- 7.60	- 6.2	+13.26
(0.509, 0.009)		+5.99922	- 1.2	- 7.09	- 7.5	+11.78
(0.508, 0.008)		+5.99923	- 1.5	- 6.16	- 9.7	+ 9.17
(0.52, 0.02)	8	+6.00023	+1.3	- 8.78392	+1.8	+20.06978
(0.51, 0.01)		+6.00009	+0.78135	- 9.27975	+1.3	+20.10651
(0.509, 0.009)		+6.00006	+0.71020	- 9.33182	+1.2	+20.05524
(0.508, 0.008)		+6.00005	+0.64171	- 9.38299	+1.0	+19.99991
(0.52, 0.02)	9	+6.00320	+5.9	- 10.89	+17.	+36.
(0.51, 0.01)		+5.99993	- 0.01226	- 8.42799	- 2.4	+14.50
(0.509, 0.009)		+5.99997	+0.17028	- 8.63379	- 1.5	+15.71222
(0.508, 0.008)		+5.99999	+0.27592	- 8.78302	- 0.98	+16.54982
(0.52, 0.02)	10	+6.00000	- 0.11619	- 9.12602	+0.19204	+18.18866
(0.51, 0.01)		+6.00000	- 0.07734	- 9.12010	+0.23605	+18.24800
(0.509, 0.009)		+6.00000	- 0.07484	- 9.11739	+0.22674	+18.23615
(0.508, 0.008)		+6.00000	- 0.06877	- 9.11325	+0.21189	+18.21804

Table C.15 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.52, 0.02)	11	+6.00002	+0.03390	- 9.26433	+0.28204	+18.62773
(0.51, 0.01)		+6.00000	- 0.12736	- 9.05635	+0.28684	+18.13078
(0.509, 0.009)		+6.00000	- 0.12058	- 9.06416	+0.27833	+18.14346
(0.508,0.008)		+6.00000	- 0.12044	- 9.06245	+0.27542	+18.13706
(0.52, 0.02)	12	+5.99988	- 0.08180	- 8.92113	+0.04395	+17.84038
(0.51, 0.01)		+6.00000	+0.21138	- 9.19859	- 0.25006	+18.40153
(0.509, 0.009)		+6.00000	+0.13178	- 9.11509	- 0.16873	+18.23236
(0.508,0.008)		+6.00000	+0.09299	- 9.07409	- 0.12829	+18.14939
(0.52, 0.02)	13	+6.00000	+0.00530	- 9.03331	+0.03608	+18.07375
(0.51, 0.01)		+6.00000	- 0.00787	- 8.99660	+0.02325	+17.99417
(0.509, 0.009)		+6.00000	- 0.00864	- 8.99372	+0.02283	+17.98818
(0.508,0.008)		+6.00000	- 0.00906	- 8.99098	+0.02204	+17.98249
(0.52, 0.02)	14	+6.00000	- 0.01247	- 8.75397	- 0.00441	+17.50113
(0.51, 0.01)		+6.00000	- 0.01031	- 8.85734	+0.01978	+17.71484
(0.509, 0.009)		+6.00000	- 0.01002	- 8.86563	+0.01958	+17.73143
(0.508,0.008)		+6.00000	- 0.00948	- 8.87669	+0.01883	+17.75354
(0.52, 0.02)	15	+6.00000	- 0.01187	- 9.00909	+0.02083	+18.01846
(0.51, 0.01)		+6.00000	- 0.01066	- 8.80508	+0.02019	+17.61029
(0.509, 0.009)		+6.00000	- 0.01228	- 8.57711	+0.02304	+17.15437
(0.508,0.008)		+6.00000	- 0.02675	- 6.84	+0.04890	+13.69
(0.52, 0.02)	16	+6.00000	- 0.00759	- 9.03413	+0.01516	+18.06876
(0.51, 0.01)		+6.00000	- 0.00070	- 9.06294	+0.00129	+18.12594
(0.509, 0.009)		+6.00000	+0.00235	- 9.12160	- 0.00465	+18.24323
(0.508,0.008)		+6.00000	+0.00640	- 9.21547	- 0.01255	+18.43090

Table C.16 : Cylinder domain with radial potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.52, 0.02)	5	+5.75950	+0.81	+6.	+4.	+73.
(0.51, 0.01)		+6.03467	- 1.87	- 30.	- 19.	+ 3.
(0.509, 0.009)		+6.02239	- 1.72	- 28.	- 18.3	+ 8.
(0.508, 0.008)		+6.01419	- 1.62	- 27.	- 17.7	+12.
(0.52, 0.02)	6	+5.99736	- 0.24988	- 10.79158	+2.6	+22.15
(0.51, 0.01)		+5.99952	- 0.30039	- 11.42600	+1.4	+22.53
(0.509, 0.009)		+5.99962	- 0.30940	- 11.52344	+1.2	+22.58
(0.508, 0.008)		+5.99970	- 0.31994	- 11.63528	+1.0	+22.65
(0.52, 0.02)	7	+5.99769	- 0.89228	- 8.65986	- 3.1	+17.09234
(0.51, 0.01)		+5.99905	- 1.36	- 7.30	- 6.9	+12.68
(0.509, 0.009)		+5.99910	- 1.55	- 6.68	- 8.3	+10.94
(0.508, 0.008)		+5.99911	- 1.87	- 5.57	- 11.	+ 7.88
(0.52, 0.02)	8	+6.00012	+1.21	- 8.97101	+1.1	+20.01204
(0.51, 0.01)		+6.00009	+0.87	- 9.44519	+1.2	+20.34353
(0.509, 0.009)		+6.00006	+0.80	- 9.48839	+1.1	+20.30079
(0.508, 0.008)		+6.00005	+0.73	- 9.53020	+0.9	+20.24919
(0.52, 0.02)	9	+6.00322	+6.5	- 11.70375	+17.	+38.17
(0.51, 0.01)		+5.99994	+0.11100	- 8.68702	- 2.2	+15.23148
(0.509, 0.009)		+5.99998	+0.28203	- 8.87165	- 1.5	+16.31748
(0.508, 0.008)		+6.00000	+0.38039	- 9.00235	- 1.0	+17.06893
(0.52, 0.02)	10	+6.00001	- 0.04784	- 9.27758	+0.25698	+18.55086
(0.51, 0.01)		+6.00000	- 0.02719	- 9.25179	+0.20721	+18.49428
(0.509, 0.009)		+6.00000	- 0.01932	- 9.24590	+0.18437	+18.46882
(0.508, 0.008)		+6.00000	- 0.00106	- 9.23779	+0.14685	+18.43145

Table C.16 : Continued.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.52, 0.02)	11	+6.00003	+0.07104	- 9.38259	+0.32662	+18.88488
(0.51, 0.01)		+6.00000	- 0.15930	- 9.09696	+0.34068	+18.21375
(0.509, 0.009)		+6.00000	- 0.14668	- 9.11104	+0.32742	+18.23893
(0.508,0.008)		+6.00000	- 0.14333	- 9.11278	+0.32137	+18.23920
(0.52, 0.02)	12	+5.99999	- 0.06496	- 9.01053	+0.05884	+18.02484
(0.51, 0.01)		+6.00000	- 0.15803	- 9.18752	- 0.17848	+18.37901
(0.509, 0.009)		+6.00000	+0.10361	- 9.13266	- 0.12626	+18.26754
(0.508,0.008)		+6.00000	+0.07529	- 9.10321	- 0.09903	+18.20774
(0.52, 0.02)	13	+6.00000	+0.01200	- 9.08003	+0.04872	+18.17124
(0.51, 0.01)		+6.00000	- 0.00809	- 9.02641	+0.02865	+18.05429
(0.509, 0.009)		+6.00000	- 0.00956	- 9.02138	+0.02852	+18.04388
(0.508,0.008)		+6.00000	- 0.01056	- 9.01619	+0.02790	+18.03319
(0.52, 0.02)	14	+6.00000	- 0.01605	- 8.84254	+0.01369	+17.68125
(0.51, 0.01)		+6.00000	- 0.01304	- 8.89410	+0.02560	+17.78843
(0.509, 0.009)		+6.00000	- 0.01281	- 8.89754	+0.02535	+17.79531
(0.508,0.008)		+6.00000	- 0.01192	- 8.90437	+0.02390	+17.80891
(0.52, 0.02)	15	+6.00000	- 0.01501	- 8.99231	+0.02720	+17.98491
(0.51, 0.01)		+6.00000	- 0.01411	- 8.76541	+0.02699	+17.53100
(0.509, 0.009)		+6.00000	- 0.01533	- 8.62453	+0.02939	+17.24924
(0.508,0.008)		+6.00000	+0.03446	-13.89	- 0.05826	+27.78
(0.52, 0.02)	16	+6.00000	- 0.00944	- 9.03921	+0.01906	+18.07891
(0.51, 0.01)		+6.00000	- 0.00231	- 9.06418	+0.00456	+18.12842
(0.509, 0.009)		+6.00000	- 0.00117	- 9.09379	+0.00240	+18.18763
(0.508,0.008)		+6.00000	+0.00758	- 9.25387	- 0.01484	+18.50767

A P P E N D I X D

Numerical Results of a Local Refinement of the Solution of a Heat Conduction Problem

Table D.1 : Cylinder domain with radial potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Taylor series
using fine boundary elements in the region of interest,
and remote field solution outside this region.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.55, 0.05)	7	+6.00641	- 0.04838	- 8.58636	+0.41277	+15.71305
(0.54, 0.04)		+6.00368	- 0.03285	- 8.60251	+0.19450	+15.94252
(0.53, 0.03)		+6.00077	- 0.01062	- 8.64220	+0.21110	+16.78958
(0.52, 0.02)		+6.00018	- 0.00367	- 8.65107	+0.09478	+17.04773
(0.51, 0.01)		+6.00002	- 0.00071	- 8.65499	+0.01473	+17.20710
(0.509, 0.009)		+6.00001	- 0.00030	- 8.65584	+0.02544	+17.27160
(0.508, 0.008)		+6.00001	- 0.00037	- 8.65557	+0.00999	+17.24558
(0.55, 0.05)	8	+6.06479	- 0.49677	- 7.56023	- 4.33	- 10.44
(0.54, 0.04)		+6.02800	- 0.27528	- 8.04898	- 2.70	- 1.38
(0.53, 0.03)		+6.00117	- 0.01830	- 8.62575	+0.10773	+15.97391
(0.52, 0.02)		+6.00023	- 0.00549	- 8.64739	+0.06810	+16.73835
(0.51, 0.01)		+6.00002	- 0.00085	- 8.65477	+0.01774	+17.13648
(0.509, 0.009)		+6.00001	- 0.00038	- 8.65569	+0.02386	+17.23680
(0.508, 0.008)		+6.00001	- 0.00042	- 8.65550	+0.01296	+17.20569
(0.55, 0.05)	9	+6.00573	- 0.04244	- 8.55849	- 0.42799	+14.90629
(0.54, 0.04)		+6.00133	- 0.01269	- 8.62752	- 0.12554	+16.45982
(0.53, 0.03)		+6.00009	- 0.00127	- 8.65394	+0.01045	+17.22810
(0.52, 0.02)		+6.00001	- 0.00020	- 8.65585	+0.00510	+17.29566
(0.51, 0.01)		+6.00000	- 0.00001	- 8.65616	+0.00086	+17.31125
(0.509, 0.009)		+6.00000	- 0.00001	- 8.65616	+0.00063	+17.31158
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	+0.00046	+17.31187
(0.55, 0.05)	10	+6.00216	- 0.02061	- 8.62163	+0.08928	+16.29050
(0.54, 0.04)		+6.00040	- 0.00485	- 8.64876	+0.05601	+17.05207
(0.53, 0.03)		+6.00006	- 0.00098	- 8.65492	+0.02723	+17.25800
(0.52, 0.02)		+6.00001	- 0.00015	- 8.65602	+0.00910	+17.30321
(0.51, 0.01)		+6.00000	- 0.00001	- 8.65617	+0.00128	+17.31217
(0.509, 0.009)		+6.00000	- 0.00001	- 8.65617	+0.00094	+17.31196
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	+0.00067	+17.31232
(0.55, 0.05)	11	+5.99993	+0.00083	- 8.65830	+0.02687	+17.38810
(0.54, 0.04)		+5.99997	+0.00038	- 8.65710	+0.01272	+17.35221
(0.53, 0.03)		+6.00000	+0.00007	- 8.65635	+0.00322	+17.32295
(0.52, 0.02)		+6.00000	+0.00001	- 8.65620	+0.00068	+17.31485
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	+0.00006	+17.31257
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00002	+17.31243
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	+0.00002	+17.31243

Table D.1 : Continued

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.55, 0.05)	12	+5.99999	+0.00032	- 8.65671	+0.00452	+17.34267
(0.54, 0.04)		+5.99999	+0.00025	- 8.65671	+0.00603	+17.33868
(0.53, 0.03)		+6.00000	+0.00006	- 8.65629	+0.00174	+17.32049
(0.52, 0.02)		+6.00000	+0.00001	- 8.65620	+0.00057	+17.31469
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	+0.00007	+17.31258
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00002	+17.31244
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	+0.00003	+17.31244
(0.55, 0.05)	13	+5.99997	+0.00042	- 8.65710	+0.00808	+17.34935
(0.54, 0.04)		+5.99999	+0.00023	- 8.65668	+0.00564	+17.33653
(0.53, 0.03)		+6.00000	+0.00004	- 8.65625	+0.00106	+17.31726
(0.52, 0.02)		+6.00000	+0.00001	- 8.65619	+0.00037	+17.31391
(0.51, 0.01)		+6.00000	- 0.00000	- 8.65617	- 0.00001	+17.31231
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31235
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65619	- 0.00000	+17.31233
(0.55, 0.05)	14	+6.00001	- 0.00017	- 8.65562	- 0.01133	+17.29248
(0.54, 0.04)		+6.00001	- 0.00010	- 8.65586	- 0.00699	+17.29824
(0.53, 0.03)		+6.00001	- 0.00016	- 8.65572	- 0.01196	+17.28390
(0.52, 0.02)		+6.00000	+0.00001	- 8.65619	+0.00058	+17.31389
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	+0.00001	+17.31237
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31235
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234
(0.55, 0.05)	15	+6.00000	- 0.00001	- 8.65613	- 0.00082	+17.31082
(0.54, 0.04)		+6.00000	+0.00000	- 8.65617	- 0.00016	+17.31227
(0.53, 0.03)		+6.00000	- 0.00000	- 8.65617	- 0.00004	+17.31231
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31235
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	+0.00001	+17.31236
(0.55, 0.05)	16	+6.00000	+0.00011	- 8.65621	- 0.00305	+17.31812
(0.54, 0.04)		+6.00000	+0.00001	- 8.65618	- 0.00035	+17.31317
(0.53, 0.03)		+6.00000	+0.00000	- 8.65617	- 0.00005	+17.31246
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31235
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31235
(0.509, 0.009)		+6.00000	- 0.00000	- 8.65617	+0.00000	+17.31233
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31231

Table D.2 : Cylinder domain with radial potential variation.
Function and derivatives at node 1 (Fig. 2.7) from Fourier series
using fine boundary elements in the region of interest,
and remote field solution outside this region.

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.55, 0.05)	7	+6.00035	- 0.01760	- 8.66686	+0.88227	+17.12383
(0.54, 0.04)		+6.00015	- 0.00928	- 8.66238	+0.59839	+17.20831
(0.53, 0.03)		+6.00005	- 0.00401	- 8.65913	+0.35628	+17.26503
(0.52, 0.02)		+6.00001	- 0.00121	- 8.65716	+0.16737	+17.29724
(0.51, 0.01)		+6.00000	- 0.00015	- 8.65631	+0.04415	+17.31031
(0.509, 0.009)		+6.00000	- 0.00011	- 8.65627	+0.03595	+17.31085
(0.508, 0.008)		+6.00000	- 0.00008	- 8.65624	+0.02855	+17.31128
(0.55, 0.05)	8	+6.00035	- 0.01343	- 8.67105	- 0.88254	+17.37460
(0.54, 0.04)		+6.00015	- 0.00738	- 8.66428	+0.59849	+17.35092
(0.53, 0.03)		+6.00005	- 0.00335	- 8.65980	+0.35631	+17.33181
(0.52, 0.02)		+6.00001	- 0.00107	- 8.65730	+0.16737	+17.31918
(0.51, 0.01)		+6.00000	- 0.00014	- 8.65632	+0.04415	+17.31334
(0.509, 0.009)		+6.00000	- 0.00011	- 8.65628	+0.03595	+17.31308
(0.508, 0.008)		+6.00000	- 0.00008	- 8.65625	+0.02855	+17.31287
(0.55, 0.05)	9	+6.00003	- 0.00084	- 8.65851	+0.12887	+17.37680
(0.54, 0.04)		+6.00001	- 0.00031	- 8.65723	+0.06870	+17.35191
(0.53, 0.03)		+6.00000	- 0.00008	- 8.65654	+0.02999	+17.33216
(0.52, 0.02)		+6.00000	- 0.00001	- 8.65625	+0.00913	+17.31925
(0.51, 0.01)		+6.00000	- 0.00000	- 8.65618	+0.00116	+17.31335
(0.509, 0.009)		+6.00000	- 0.00000	- 8.65617	+0.00085	+17.31309
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	+0.00060	+17.31287
(0.55, 0.05)	10	+6.00002	+0.00012	- 8.65851	+0.09058	+17.41523
(0.54, 0.04)		+6.00001	+0.00005	- 8.65723	+0.05087	+17.36978
(0.53, 0.03)		+6.00000	+0.00002	- 8.65654	+0.02358	+17.33857
(0.52, 0.02)		+6.00000	+0.00000	- 8.65625	+0.00769	+17.32069
(0.51, 0.01)		+6.00000	+0.00000	- 8.65618	+0.00106	+17.31345
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00078	+17.31315
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	+0.00055	+17.31292
(0.55, 0.05)	11	+6.00000	+0.00012	- 8.65648	+0.00922	+17.33415
(0.54, 0.04)		+6.00000	+0.00005	- 8.65628	+0.00342	+17.32244
(0.53, 0.03)		+6.00000	+0.00002	- 8.65620	+0.00092	+17.31593
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	+0.00014	+17.31313
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	+0.00001	+17.31240
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31238
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31236

Table D.2 : Continued

Internal point	No. of terms	ϕ	ϕ,y	ϕ,x	ϕ,xy	ϕ,xx
	Exact :	+6.	0.	- 8.65617	0.	+17.31234
(0.55, 0.05)	12	+6.00000	+0.00022	- 8.65638	- 0.00091	+17.33416
(0.54, 0.04)		+6.00000	+0.00008	- 8.65625	- 0.00045	+17.32244
(0.53, 0.03)		+6.00000	+0.00002	- 8.65619	- 0.00015	+17.31593
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00003	+17.31313
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31240
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31238
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31236
(0.55, 0.05)	13	+6.00000	+0.00004	- 8.65619	- 0.00096	+17.31574
(0.54, 0.04)		+6.00000	+0.00001	- 8.65618	- 0.00047	+17.31355
(0.53, 0.03)		+6.00000	+0.00000	- 8.65617	- 0.00016	+17.31265
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00003	+17.31238
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234
(0.55, 0.05)	14	+6.00000	+0.00004	- 8.65617	- 0.00225	+17.31445
(0.54, 0.04)		+6.00000	+0.00001	- 8.65617	- 0.00087	+17.31315
(0.53, 0.03)		+6.00000	+0.00000	- 8.65617	- 0.00024	+17.31256
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00004	+17.31237
(0.51, 0.01)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	+0.00000	+17.31234
(0.55, 0.05)	15	+6.00000	+0.00001	- 8.65617	- 0.00045	+17.31264
(0.54, 0.04)		+6.00000	+0.00000	- 8.65617	- 0.00014	+17.31241
(0.53, 0.03)		+6.00000	+0.00000	- 8.65617	- 0.00003	+17.31235
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.51, 0.01)		+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)		+6.00000	- 0.00000	- 8.65617	+0.00000	+17.31234
(0.508, 0.008)		+6.00000	- 0.00000	- 8.65617	+0.00000	+17.31234
(0.55, 0.05)	16	+6.00000	+0.00000	- 8.65617	- 0.00045	+17.31234
(0.54, 0.04)		+6.00000	+0.00000	- 8.65617	- 0.00014	+17.31234
(0.53, 0.03)		+6.00000	+0.00000	- 8.65617	- 0.00003	+17.31234
(0.52, 0.02)		+6.00000	+0.00000	- 8.65617	- 0.00000	+17.31234
(0.51, 0.01)		+6.00000	- 0.00000	- 8.65617	- 0.00000	+17.31234
(0.509, 0.009)		+6.00000	- 0.00000	- 8.65617	+0.00000	+17.31234
(0.508, 0.008)		+6.00000	+0.00000	- 8.65617	+0.00000	+17.31235

A P P E N D I X E

Derivatives of the Displacement and Traction Kernels

The derivatives of the kernels U_{ij} and T_{ij} , eqns (3.34) and (3.35), can be obtained with the aid of the operations (3.30) and (3.31). Let us recall these equations,

$$U_{ij}(\rho, Q) = \frac{1}{8\sqrt{\mu}(1-\gamma)} \left\{ (3-4\gamma) \delta_{ij} \ln\left(\frac{1}{r}\right) + r_i r_j \right\} \quad (3.34)$$

and

$$\begin{aligned} T_{ij}(\rho, Q) = \frac{1}{4\sqrt{\mu}(1-\gamma)} \left(\frac{1}{r}\right) \left\{ \frac{\partial r}{\partial n} \left[(1-2\gamma) \delta_{ij} + 2r_i r_j \right] - \right. \\ \left. (1-2\gamma) \left[n_j r_i - n_i r_j \right] \right\} \end{aligned} \quad (3.35)$$

where

$$r_i = \frac{\partial r}{\partial x_{Qi}} = \frac{1}{r} (x_{Qi} - x_{\rho i}) = - \frac{\partial r}{\partial x_{\rho i}} \quad (3.30)$$

and

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial x_{Qi}} n_i(Q) \quad , \quad Q \text{ is a point on the boundary} \quad (3.31)$$

Then,

$$r_{ik} = \frac{1}{r} \left[\delta_{ik} - r_i r_k \right] \quad (E.1)$$

$$(r_i r_j)_{,k} = \frac{1}{r} \left[\delta_{ik} r_j + \delta_{jk} r_i - 2 r_i r_j r_k \right] \quad (E.2)$$

$$(r_i r_j r_k)_{,l} = \frac{1}{r} \left[\delta_{il} r_j r_k + \delta_{jl} r_i r_k + \delta_{kl} r_i r_j - 3 r_i r_j r_k r_l \right] \quad (E.3)$$

$$\begin{aligned} (r_i r_j r_k r_l)_{,m} = \frac{1}{r} \left[\delta_{im} r_j r_k r_l + \delta_{jm} r_i r_k r_l + \delta_{km} r_i r_j r_l + \right. \\ \left. \delta_{lm} r_i r_j r_k - 4 r_i r_j r_k r_l r_m \right] \end{aligned} \quad (E.4)$$

and so on.

Hence, using these identities we have (up to the fifth-order),

$$\begin{aligned} U_{ij,k}(\rho, Q) = \frac{1}{8\sqrt{\mu}(1-\gamma)} \left(\frac{1}{r}\right) \left\{ 2 r_i r_j r_k + (3-4\gamma) \delta_{ij} r_k - \delta_{ik} r_j \right. \\ \left. - \delta_{jk} r_i \right\} \end{aligned} \quad (E.5)$$

$$\begin{aligned} U_{ij,kl}(\rho, Q) = \frac{1}{8\sqrt{\mu}(1-\gamma)} \left(\frac{1}{r^2}\right) \left\{ 8 r_i r_j r_k r_l + 2(3-4\gamma) \delta_{ij} r_k r_l - \right. \\ \left. 2 \delta_{ik} r_j r_l - 2 \delta_{jk} r_i r_l - 2 \delta_{il} r_j r_k - \right. \end{aligned}$$

$$\left. \begin{aligned} & 2\delta_{ji}\Gamma_{ii}\Gamma_k - 2\delta_{kl}\Gamma_{ii}\Gamma_j - (3-4\gamma)\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} \\ & + \delta_{jk}\delta_{il} \end{aligned} \right\} \quad (E.6)$$

$$\begin{aligned} \mathbb{U}_{ij,klm}(\rho, \eta) = & \frac{1}{8\sqrt{\mu}(1-\gamma)} \left(\frac{1}{r^3} \right) \left\{ 48\Gamma_{ii}\Gamma_j\Gamma_k\Gamma_l\Gamma_m + 8(3-4\gamma)\delta_{ij}\Gamma_k\right. \\ & \Gamma_l\Gamma_m - 8\delta_{ik}\Gamma_j\Gamma_l\Gamma_m - 8\delta_{jk}\Gamma_{ii}\Gamma_l\Gamma_m - 8\delta_{il}\Gamma_j\Gamma_k\Gamma_m \\ & - 8\delta_{jl}\Gamma_{ii}\Gamma_k\Gamma_m - 8\delta_{kl}\Gamma_{ii}\Gamma_j\Gamma_m - 8\delta_{im}\Gamma_j\Gamma_k\Gamma_l - 8\delta_{jm} \\ & \Gamma_{ii}\Gamma_k\Gamma_l - 8\delta_{km}\Gamma_{ii}\Gamma_j\Gamma_l - 8\delta_{lm}\Gamma_{ii}\Gamma_j\Gamma_k - 2(3-4\gamma)\delta_{ij} \\ & \delta_{km}\Gamma_l - 2(3-4\gamma)\delta_{ij}\delta_{lm}\Gamma_k - 2(3-4\gamma)\delta_{ij}\delta_{kl}\Gamma_m + \\ & 2\delta_{ik}\delta_{jl}\Gamma_m + 2\delta_{jk}\delta_{il}\Gamma_m + 2\delta_{ik}\delta_{jm}\Gamma_l + 2\delta_{ik}\delta_{lm} \\ & \Gamma_j + 2\delta_{jk}\delta_{im}\Gamma_l + 2\delta_{jk}\delta_{lm}\Gamma_{ii} + 2\delta_{il}\delta_{jm}\Gamma_k + 2\delta_{il} \\ & \delta_{km}\Gamma_j + 2\delta_{jl}\delta_{im}\Gamma_k + 2\delta_{jl}\delta_{km}\Gamma_{ii} + 2\delta_{kl}\delta_{im}\Gamma_j \\ & \left. + 2\delta_{kl}\delta_{jm}\Gamma_{ii} \right\} \quad (E.7) \end{aligned}$$

$$\begin{aligned} \mathbb{U}_{ij,klmn}(\rho, \eta) = & \frac{1}{8\sqrt{\mu}(1-\gamma)} \left(\frac{1}{r^4} \right) \left\{ 384\Gamma_{ii}\Gamma_j\Gamma_k\Gamma_l\Gamma_m\Gamma_n + 48(3-4\gamma) \right. \\ & \delta_{ij}\Gamma_k\Gamma_l\Gamma_m\Gamma_n - 48\delta_{ik}\Gamma_j\Gamma_l\Gamma_m\Gamma_n - 48\delta_{jk}\Gamma_{ii}\Gamma_l\Gamma_m\Gamma_n \\ & \Gamma_n - 48\delta_{il}\Gamma_j\Gamma_k\Gamma_m\Gamma_n - 48\delta_{jl}\Gamma_{ii}\Gamma_k\Gamma_m\Gamma_n - 48 \\ & \delta_{kl}\Gamma_{ii}\Gamma_j\Gamma_m\Gamma_n - 48\delta_{im}\Gamma_j\Gamma_k\Gamma_l\Gamma_n - 48\delta_{jm}\Gamma_{ii}\Gamma_k \\ & \Gamma_l\Gamma_n - 48\delta_{km}\Gamma_{ii}\Gamma_j\Gamma_l\Gamma_n - 48\delta_{lm}\Gamma_{ii}\Gamma_j\Gamma_k\Gamma_n - \\ & 48\delta_{in}\Gamma_j\Gamma_k\Gamma_l\Gamma_m - 48\delta_{jn}\Gamma_{ii}\Gamma_k\Gamma_l\Gamma_m - 48\delta_{kn}\Gamma_{ii} \\ & \Gamma_j\Gamma_l\Gamma_m - 48\delta_{ln}\Gamma_{ii}\Gamma_j\Gamma_k\Gamma_m - 48\delta_{mn}\Gamma_{ii}\Gamma_j\Gamma_k\Gamma_l - \\ & 8(3-4\gamma)\delta_{ij}\delta_{km}\Gamma_l\Gamma_n - 8(3-4\gamma)\delta_{ij}\delta_{lm}\Gamma_k\Gamma_n - 8 \\ & (3-4\gamma)\delta_{ij}\delta_{kl}\Gamma_m\Gamma_n + 8\delta_{ik}\delta_{jl}\Gamma_m\Gamma_n + 8\delta_{jk}\delta_{il}\Gamma_m \\ & \Gamma_n + 8\delta_{ik}\delta_{jm}\Gamma_l\Gamma_n + 8\delta_{ik}\delta_{lm}\Gamma_j\Gamma_n + 8\delta_{jk}\delta_{im}\Gamma_l \\ & \Gamma_n + 8\delta_{jk}\delta_{lm}\Gamma_{ii}\Gamma_n + 8\delta_{il}\delta_{jm}\Gamma_k\Gamma_n + 8\delta_{il}\delta_{km}\Gamma_j \\ & \left. \right\} \end{aligned}$$

$$\begin{aligned}
& \Gamma_n + \mathcal{E} \delta_{jl} \delta_{im} \Gamma_k \Gamma_n + \mathcal{E} \delta_{jl} \delta_{km} \Gamma_i \Gamma_n + \mathcal{E} \delta_{kl} \delta_{im} \Gamma_j \\
& \Gamma_n + \mathcal{E} \delta_{kl} \delta_{jm} \Gamma_i \Gamma_n - \mathcal{E} (3-4\gamma) \delta_{ij} \delta_{km} \Gamma_l \Gamma_n - \mathcal{E} (3-4\gamma) \\
& \delta_{ij} \delta_{ln} \Gamma_k \Gamma_m - \mathcal{E} (3-4\gamma) \delta_{ij} \delta_{nm} \Gamma_k \Gamma_l + \mathcal{E} \delta_{ik} \delta_{jn} \Gamma_l \\
& \Gamma_m + \mathcal{E} \delta_{ik} \delta_{ln} \Gamma_j \Gamma_m + \mathcal{E} \delta_{ik} \delta_{nm} \Gamma_j \Gamma_l + \mathcal{E} \delta_{jk} \delta_{in} \Gamma_l \\
& \Gamma_m + \mathcal{E} \delta_{jk} \delta_{ln} \Gamma_i \Gamma_m + \mathcal{E} \delta_{jk} \delta_{nm} \Gamma_i \Gamma_l + \mathcal{E} \delta_{il} \delta_{jn} \Gamma_k \\
& \Gamma_m + \mathcal{E} \delta_{il} \delta_{km} \Gamma_j \Gamma_m + \mathcal{E} \delta_{il} \delta_{mn} \Gamma_j \Gamma_k + \mathcal{E} \delta_{jl} \delta_{in} \Gamma_k \\
& \Gamma_m + \mathcal{E} \delta_{jl} \delta_{kn} \Gamma_i \Gamma_m + \mathcal{E} \delta_{jl} \delta_{im} \Gamma_i \Gamma_k + \mathcal{E} \delta_{kl} \delta_{in} \Gamma_j \\
& \Gamma_m + \mathcal{E} \delta_{kl} \delta_{jn} \Gamma_i \Gamma_m + \mathcal{E} \delta_{kl} \delta_{nm} \Gamma_i \Gamma_j + \mathcal{E} \delta_{im} \delta_{jn} \Gamma_k \\
& \Gamma_l + \mathcal{E} \delta_{im} \delta_{kn} \Gamma_j \Gamma_l + \mathcal{E} \delta_{im} \delta_{ln} \Gamma_j \Gamma_k + \mathcal{E} \delta_{jm} \delta_{in} \Gamma_k \\
& \Gamma_l + \mathcal{E} \delta_{jm} \delta_{kn} \Gamma_i \Gamma_l + \mathcal{E} \delta_{jm} \delta_{ln} \Gamma_i \Gamma_k + \mathcal{E} \delta_{km} \delta_{in} \Gamma_j \\
& \Gamma_l + \mathcal{E} \delta_{km} \delta_{jn} \Gamma_i \Gamma_l + \mathcal{E} \delta_{km} \delta_{ln} \Gamma_i \Gamma_j + \mathcal{E} \delta_{ln} \delta_{im} \Gamma_j \\
& \Gamma_k + \mathcal{E} \delta_{ln} \delta_{jm} \Gamma_i \Gamma_k + \mathcal{E} \delta_{ln} \delta_{km} \Gamma_i \Gamma_j + 2(3-4\gamma) \delta_{ij} \\
& \delta_{km} \delta_{ln} + 2(3-4\gamma) \delta_{ij} \delta_{lm} \delta_{kn} + 2(3-4\gamma) \delta_{ij} \delta_{kl} \delta_{imn} \\
& - 2 \delta_{ik} \delta_{jl} \delta_{imn} - 2 \delta_{jk} \delta_{il} \delta_{imn} - 2 \delta_{ik} \delta_{jm} \delta_{in} - 2 \delta_{ik} \\
& \delta_{lm} \delta_{jn} - 2 \delta_{jk} \delta_{im} \delta_{ln} - 2 \delta_{jk} \delta_{lm} \delta_{in} - 2 \delta_{il} \delta_{jm} \\
& \delta_{kn} - 2 \delta_{il} \delta_{km} \delta_{jn} - 2 \delta_{jl} \delta_{im} \delta_{kn} - 2 \delta_{jl} \delta_{kn} \delta_{im} - \\
& 2 \delta_{kl} \delta_{im} \delta_{jn} - 2 \delta_{kl} \delta_{jm} \delta_{in} \} \quad (E.8)
\end{aligned}$$

$$\begin{aligned}
U_{ij,klmnop}(p,Q) = & \frac{1}{8\pi\mu(1-\gamma)} \left(\frac{1}{r^5} \right) \left\{ 3840 \Gamma_i \Gamma_j \Gamma_k \Gamma_l \Gamma_m \Gamma_n \Gamma_o + 384 \right. \\
& (3-4\gamma) \delta_{ij} \Gamma_k \Gamma_l \Gamma_m \Gamma_n \Gamma_o - 384 \delta_{ik} \Gamma_j \Gamma_l \Gamma_m \Gamma_n \Gamma_o - 384 \\
& \delta_{jk} \Gamma_i \Gamma_l \Gamma_m \Gamma_n \Gamma_o - 384 \delta_{il} \Gamma_j \Gamma_k \Gamma_m \Gamma_n \Gamma_o - 384 \delta_{jl} \Gamma_i \Gamma_k \Gamma_m \\
& \Gamma_n \Gamma_o - 384 \delta_{kl} \Gamma_i \Gamma_j \Gamma_m \Gamma_n \Gamma_o - 384 \delta_{im} \Gamma_j \Gamma_k \Gamma_l \Gamma_n \Gamma_o - 384 \\
& \delta_{jm} \Gamma_i \Gamma_k \Gamma_l \Gamma_n \Gamma_o - 384 \delta_{km} \Gamma_i \Gamma_j \Gamma_l \Gamma_n \Gamma_o - 384 \delta_{ln} \Gamma_i \Gamma_j \\
& \Gamma_k \Gamma_m \Gamma_o - 384 \delta_{in} \Gamma_j \Gamma_k \Gamma_l \Gamma_m \Gamma_o - 384 \delta_{jn} \Gamma_i \Gamma_k \Gamma_l \Gamma_m \Gamma_o - \\
& 384 \delta_{kn} \Gamma_i \Gamma_j \Gamma_l \Gamma_m \Gamma_o - 384 \delta_{ln} \Gamma_i \Gamma_j \Gamma_k \Gamma_m \Gamma_o - 384
\end{aligned}$$

[illegible]

[illegible]

$$\left. \delta_{ik} r_i = \epsilon \delta_{lm} \delta_{kn} \delta_l r_j^i - \epsilon \delta_{lm} \delta_{kn} \delta_j^i r_{li} \right\} \quad (E.9)$$

and by noting that

$$\left(\frac{\partial r}{\partial n} \right)_{,k} = \frac{1}{r} \left[n_k - r_k \frac{\partial r}{\partial n} \right] \quad (E.10)$$

Then,

$$\begin{aligned} \bar{T}_{ij,k}(p, q) = & \frac{1}{4\pi(1-\gamma)} \left(\frac{1}{r^2} \right) \left\{ \frac{\partial r}{\partial n} \left[\epsilon r_i r_j^i r_k + 2(1-2\gamma) \delta_j^i r_k - 2\delta_{ik} \right. \right. \\ & \left. \left. r_j - 2\delta_{jk} r_i \right] + m(1-2\gamma) \left[2 r_j r_k - \delta_{jk} \right] - n_j(1-2\gamma) \right. \\ & \left. \left[2 r_i r_k - \delta_{ik} \right] - n_k \left[2 r_i r_j + (1-2\gamma) \delta_j^i \right] \right\} \quad (E.11) \end{aligned}$$

$$\begin{aligned} \bar{T}_{ijkl}(p, q) = & \frac{1}{4\pi(1-\gamma)} \left(\frac{1}{r^3} \right) \left\{ \frac{\partial r}{\partial n} \left[4\epsilon r_i r_j^i r_k r_l + \epsilon(1-2\gamma) \delta_j^i r_k r_l - \right. \right. \\ & \left. \left. \epsilon \delta_{ik} r_j^i r_l - \epsilon \delta_{jk} r_i r_l - \epsilon \delta_{il} r_i r_k - \epsilon \delta_{il} r_j^i r_k - \epsilon \delta_{il} r_i r_l \right. \right. \\ & \left. \left. r_j - 2(1-2\gamma) \delta_j^i \delta_{il} + 2\delta_{ik} \delta_j^i + 2\delta_{jk} \delta_{il} \right] + m(1-2\gamma) \left[\epsilon \right. \right. \\ & \left. \left. r_i r_k r_l - 2\delta_{jk} r_i r_l - 2\delta_{jl} r_i r_k - 2\delta_{il} r_j^i r_l \right] - n_j(1-2\gamma) \left[\epsilon r_i \right. \right. \\ & \left. \left. r_k r_l - 2\delta_{ik} r_l - 2\delta_{il} r_k - 2\delta_{il} r_i \right] - n_k \left[\epsilon r_i r_j^i r_l + \right. \right. \\ & \left. \left. 2\delta_j^i(1-2\gamma) r_l - 2\delta_{il} r_j^i - 2\delta_{jl} r_i \right] - n_l \left[\epsilon r_i r_j^i r_k + \right. \right. \\ & \left. \left. 2(1-2\gamma) \delta_j^i r_k - 2\delta_{ik} r_j^i - 2\delta_{jk} r_i \right] \right\} \quad (E.12) \end{aligned}$$

$$T_{ij,klm}(p, q) = \frac{-1}{4i(1-i)} \left(\frac{1}{i^4} \right) \left\{ \frac{\partial}{\partial i} \left[384 r_i r_j r_k r_l r_m + 48(1-2i) \delta_j^i \right. \right.$$

$$\begin{aligned} & r_k r_l r_m - 48 \delta_i^k r_j r_l r_m - 48 \delta_j^k r_i r_l r_m - 48 \delta_l^k r_i r_j r_m - 48 \delta_j^l r_i r_k r_m \\ & - 48 \delta_i^l r_j r_k r_m - 48 \delta_k^l r_i r_j r_m - 48 \delta_m^i r_j r_k r_l - 48 \delta_j^m r_i r_k r_l - 48 \delta_k^m r_i r_j r_l \\ & - 48 \delta_l^m r_i r_j r_k - \epsilon(1-2i) \delta_j^i \delta_k^m r_l - \epsilon(1-2i) \delta_j^i \delta_m^k r_l + \epsilon \delta_i^k \delta_j^l r_m \\ & + \epsilon \delta_j^k \delta_i^l r_m + \epsilon \delta_i^k \delta_j^m r_l + \epsilon \delta_i^k \delta_l^m r_j + \epsilon \delta_j^k \delta_i^m r_l \\ & + \epsilon \delta_j^k \delta_l^m r_i + \epsilon \delta_j^l \delta_i^m r_k + \epsilon \delta_j^l \delta_k^m r_i + \epsilon \delta_i^l \delta_j^m r_k \\ & \left. + \epsilon \delta_i^l \delta_k^m r_j + \epsilon \delta_k^l \delta_i^m r_j + \epsilon \delta_k^l \delta_j^m r_i \right] + \end{aligned}$$

$$r_i(1-2i) \left[48 r_j r_k r_l r_m - \epsilon \delta_j^k r_l r_m - \epsilon \delta_j^l r_k r_m - \epsilon \right.$$

$$\left. \delta_k^l r_j r_m - \epsilon \delta_j^m r_k r_l - \epsilon \delta_k^m r_j r_l - \epsilon \delta_l^m r_j r_k + 2 \delta_j^k \delta_l^m \right. \\ \left. + 2 \delta_j^l \delta_k^m + 2 \delta_k^l \delta_j^m \right] - n_j(1-2i) \left[48 r_i r_k r_l r_m \right.$$

$$\left. - \epsilon \delta_i^k r_l r_m - \epsilon \delta_i^l r_k r_m - \epsilon \delta_k^l r_i r_m - \epsilon \delta_l^m r_i r_k - \epsilon \delta_k^m r_i r_l \right. \\ \left. - \epsilon \delta_l^m r_i r_k + 2 \delta_i^k \delta_l^m + 2 \delta_i^l \delta_k^m + 2 \delta_k^l \delta_i^m \right]$$

$$- n_k \left[48 r_i r_j r_l r_m + \epsilon(1-2i) \delta_j^i r_l r_m - \epsilon \delta_i^l r_j r_m - \epsilon \delta_j^l \right.$$

$$\left. r_i r_m - \epsilon \delta_l^m r_j r_i - \epsilon \delta_j^m r_i r_l - \epsilon \delta_l^m r_i r_j - 2(1-2i) \delta_j^i \delta_l^m \right. \\ \left. + 2 \delta_i^l \delta_j^m + 2 \delta_j^l \delta_i^m \right] - n_l \left[48 r_i r_j r_k r_m + \epsilon(1-2i) \right.$$

$$\left. \delta_j^i r_k r_m - \epsilon \delta_i^k r_j r_m - \epsilon \delta_j^k r_i r_m - \epsilon \delta_l^m r_j r_k - \epsilon \delta_j^m r_i r_k \right. \\ \left. - \epsilon \delta_k^m r_i r_j - 2(1-2i) \delta_j^i \delta_k^m + 2 \delta_i^k \delta_j^m + 2 \delta_j^k \delta_i^m \right] -$$

$$m \left[4\epsilon \Gamma_i \Gamma_j \Gamma_k \Gamma_l + 8(1-2\gamma) \delta_{ij} \Gamma_k \Gamma_l - \epsilon \delta_{ik} \Gamma_j \Gamma_l - \epsilon \delta_{jl} \Gamma_i \Gamma_k \right. \\ \left. - \epsilon \delta_{il} \Gamma_j \Gamma_k - \epsilon \delta_{jk} \Gamma_i \Gamma_l - 2(1-2\gamma) \delta_{ij} \delta_{kl} \right. \\ \left. + 2\delta_{ik} \delta_{jl} + 2\delta_{il} \delta_{jk} \right] \quad (E.13)$$

$$T_{ijklmn}(\rho, G) = \frac{+1}{4\pi(1-\gamma)} \left(\frac{1}{r^5} \right) \left\{ \frac{\partial}{\partial n} \left[384\epsilon \Gamma_i \Gamma_j \Gamma_k \Gamma_l \Gamma_m \Gamma_n + 384(1-2\gamma) \right. \right.$$

$$\begin{aligned} & \delta_{ij} \Gamma_k \Gamma_l \Gamma_m \Gamma_n - 384 \delta_{ik} \Gamma_j \Gamma_l \Gamma_m \Gamma_n - 384 \delta_{jk} \Gamma_i \Gamma_l \Gamma_m \Gamma_n \\ & - 384 \delta_{il} \Gamma_j \Gamma_k \Gamma_m \Gamma_n - 384 \delta_{jl} \Gamma_i \Gamma_k \Gamma_m \Gamma_n - 384 \delta_{kl} \Gamma_i \Gamma_j \Gamma_m \Gamma_n \\ & - 384 \delta_{im} \Gamma_j \Gamma_k \Gamma_l \Gamma_n - 384 \delta_{jm} \Gamma_i \Gamma_k \Gamma_l \Gamma_n - 384 \\ & \delta_{km} \Gamma_i \Gamma_j \Gamma_l \Gamma_n - 384 \delta_{lm} \Gamma_i \Gamma_j \Gamma_k \Gamma_n - 384 \delta_{in} \Gamma_j \Gamma_k \Gamma_l \Gamma_m \\ & - 384 \delta_{jn} \Gamma_i \Gamma_k \Gamma_l \Gamma_m - 384 \delta_{kn} \Gamma_i \Gamma_j \Gamma_l \Gamma_m - 384 \delta_{ln} \Gamma_i \Gamma_j \Gamma_k \Gamma_m \\ & - 384 \delta_{mn} \Gamma_i \Gamma_j \Gamma_k \Gamma_l - 48(1-2\gamma) \delta_{ij} \delta_{kl} \Gamma_m \Gamma_n - \\ & 48(1-2\gamma) \delta_{ij} \delta_{km} \Gamma_l \Gamma_n - 48(1-2\gamma) \delta_{ij} \delta_{ln} \Gamma_k \Gamma_m - 48(1-2\gamma) \\ & \delta_{ij} \delta_{kn} \Gamma_l \Gamma_m - 48(1-2\gamma) \delta_{ij} \delta_{ln} \Gamma_k \Gamma_m - 48(1-2\gamma) \delta_{ij} \delta_{mn} \\ & \Gamma_k \Gamma_l + 48 \delta_{ik} \delta_{jl} \Gamma_m \Gamma_n + 48 \delta_{jk} \delta_{il} \Gamma_m \Gamma_n + 48 \delta_{ik} \delta_{jm} \Gamma_l \Gamma_n \\ & + 48 \delta_{ik} \delta_{ln} \Gamma_j \Gamma_m + 48 \delta_{jk} \delta_{im} \Gamma_l \Gamma_n + 48 \delta_{jk} \delta_{ln} \Gamma_i \Gamma_m + \\ & 48 \delta_{il} \delta_{jm} \Gamma_k \Gamma_n + 48 \delta_{il} \delta_{kn} \Gamma_j \Gamma_m + 48 \delta_{il} \delta_{jn} \Gamma_k \Gamma_m + 48 \\ & \delta_{il} \delta_{kn} \Gamma_j \Gamma_m + 48 \delta_{kl} \delta_{im} \Gamma_j \Gamma_n + 48 \delta_{kl} \delta_{jn} \Gamma_i \Gamma_m + 48 \delta_{kl} \\ & \delta_{im} \Gamma_j \Gamma_n + 48 \delta_{kl} \delta_{jn} \Gamma_i \Gamma_m + 48 \delta_{kl} \delta_{mn} \Gamma_i \Gamma_j + 48 \delta_{im} \\ & \delta_{jn} \Gamma_k \Gamma_l + 48 \delta_{im} \delta_{kn} \Gamma_j \Gamma_l + 48 \delta_{im} \delta_{ln} \Gamma_j \Gamma_k + 48 \delta_{jm} \\ & \delta_{in} \Gamma_k \Gamma_l + 48 \delta_{jm} \delta_{kn} \Gamma_i \Gamma_l + 48 \delta_{jm} \delta_{ln} \Gamma_i \Gamma_k + 48 \delta_{km} \\ & \delta_{in} \Gamma_j \Gamma_l + 48 \delta_{km} \delta_{ln} \Gamma_i \Gamma_l + 48 \delta_{km} \delta_{ln} \Gamma_i \Gamma_k + 48 \delta_{km} \end{aligned}$$

$$\begin{aligned}
& \delta_{ik} \gamma_j \gamma_l + 4\epsilon \delta_{km} \delta_{jn} \gamma_l \gamma_i + 4\epsilon \delta_{km} \delta_{in} \gamma_l \gamma_j + 4\epsilon \delta_{im} \\
& \delta_{jn} \gamma_j \gamma_k + 4\epsilon \delta_{im} \delta_{jn} \gamma_l \gamma_k + 4\epsilon \delta_{im} \delta_{kn} \gamma_l \gamma_j + \epsilon(1-2\gamma) \\
& \delta_{ij} \delta_{kl} \delta_{mn} + \epsilon(1-2\gamma) \delta_{ij} \delta_{km} \delta_{ln} + \epsilon(1-2\gamma) \delta_{ij} \delta_{lm} \delta_{kn} \\
& - \epsilon \delta_{kl} \delta_{jm} \delta_{in} - \epsilon \delta_{jk} \delta_{il} \delta_{mn} - \epsilon \delta_{ik} \delta_{jm} \delta_{ln} - \epsilon \delta_{ik} \\
& \delta_{im} \delta_{jn} - \epsilon \delta_{jk} \delta_{im} \delta_{ln} - \epsilon \delta_{jk} \delta_{im} \delta_{in} - \epsilon \delta_{jl} \delta_{km} \delta_{in} - \\
& \epsilon \delta_{jl} \delta_{im} \delta_{kn} - \epsilon \delta_{il} \delta_{jm} \delta_{kn} - \epsilon \delta_{il} \delta_{km} \delta_{jn} - \epsilon \delta_{kl} \delta_{im} \\
& \delta_{jn} - \epsilon \delta_{kl} \delta_{jm} \delta_{in} \Big] + n(1-2\gamma) \Big[384 \gamma_l \gamma_j \gamma_k \gamma_m \gamma_n -
\end{aligned}$$

$$\begin{aligned}
& 4\epsilon \delta_{jk} \gamma_l \gamma_m \gamma_n - 4\epsilon \delta_{jl} \gamma_k \gamma_m \gamma_n - 4\epsilon \delta_{kl} \gamma_j \gamma_m \gamma_n - 4\epsilon \\
& \delta_{jm} \gamma_k \gamma_l \gamma_n - 4\epsilon \delta_{km} \gamma_j \gamma_l \gamma_n - 4\epsilon \delta_{lm} \gamma_j \gamma_k \gamma_n - 4\epsilon \delta_{jn} \\
& \gamma_k \gamma_l \gamma_m - 4\epsilon \delta_{kn} \gamma_j \gamma_l \gamma_m - 4\epsilon \delta_{ln} \gamma_j \gamma_k \gamma_m - 4\epsilon \delta_{mn} \gamma_j \\
& \gamma_k \gamma_l + \epsilon \delta_{jk} \delta_{lm} \gamma_n + \epsilon \delta_{jk} \delta_{ln} \gamma_m + \epsilon \delta_{jk} \delta_{mn} \gamma_l + \epsilon \delta_{jl} \\
& \delta_{km} \gamma_n + \epsilon \delta_{kl} \delta_{jm} \gamma_n + \epsilon \delta_{jl} \delta_{km} \gamma_m + \epsilon \delta_{jl} \delta_{mn} \gamma_k + \epsilon \delta_{kl} \delta_{jm} \\
& \gamma_m + \epsilon \delta_{kl} \delta_{mn} \gamma_j + \epsilon \delta_{jm} \delta_{kn} \gamma_l + \epsilon \delta_{jm} \delta_{ln} \gamma_k + \epsilon \delta_{km} \delta_{jn} \gamma_l \\
& + \epsilon \delta_{km} \delta_{ln} \gamma_j + \epsilon \delta_{lm} \delta_{jn} \gamma_k + \epsilon \delta_{lm} \delta_{kn} \gamma_j \Big] - n_j(1-2\gamma) \Big]
\end{aligned}$$

$$\begin{aligned}
& 384 \gamma_l \gamma_k \gamma_j \gamma_m \gamma_n - 4\epsilon \delta_{ik} \gamma_l \gamma_m \gamma_n - 4\epsilon \delta_{il} \gamma_k \gamma_m \gamma_n - 4\epsilon \\
& \delta_{kl} \gamma_l \gamma_m \gamma_n - 4\epsilon \delta_{im} \gamma_l \gamma_n \gamma_k - 4\epsilon \delta_{kn} \gamma_l \gamma_l \gamma_m - 4\epsilon \delta_{lm} \gamma_l \gamma_k \\
& \gamma_n - 4\epsilon \delta_{in} \gamma_k \gamma_l \gamma_m - 4\epsilon \delta_{kn} \gamma_l \gamma_l \gamma_m - 4\epsilon \delta_{ln} \gamma_l \gamma_k \gamma_m - \\
& 4\epsilon \delta_{mn} \gamma_l \gamma_k \gamma_l + \epsilon \delta_{ik} \delta_{lm} \gamma_n + \epsilon \delta_{il} \delta_{km} \gamma_m + \epsilon \delta_{kl} \delta_{im} \gamma_n + \\
& \epsilon \delta_{ik} \delta_{ln} \gamma_m + \epsilon \delta_{ik} \delta_{mn} \gamma_l + \epsilon \delta_{il} \delta_{km} \gamma_m + \epsilon \delta_{il} \delta_{mn} \gamma_k + \\
& \epsilon \delta_{kl} \delta_{im} \gamma_m + \epsilon \delta_{kl} \delta_{mn} \gamma_l + \epsilon \delta_{im} \delta_{kn} \gamma_l + \epsilon \delta_{im} \delta_{ln} \gamma_k + \\
& \epsilon \delta_{kn} \delta_{im} \gamma_l + \epsilon \delta_{kn} \delta_{ln} \gamma_i + \epsilon \delta_{lm} \delta_{in} \gamma_k + \epsilon \delta_{lm} \delta_{kn} \gamma_i \Big]
\end{aligned}$$

$$- n_k \Big[384 \gamma_l \gamma_j \gamma_l \gamma_m \gamma_n + 4\epsilon(1-2\gamma) \delta_{ij} \gamma_l \gamma_m \gamma_n - 4\epsilon \delta_{il} \gamma_j$$

$$\begin{aligned}
 & \Gamma_m \Gamma_n - 4E \delta_{ij}^l \Gamma_i \Gamma_m \Gamma_n - 4E \delta_{im} \Gamma_j^i \Gamma_n \Gamma_n - 4E \delta_{jn} \Gamma_i \Gamma_i \Gamma_l \\
 & \Gamma_n - 4E \delta_{ln} \Gamma_i \Gamma_j^i \Gamma_n - 4E \delta_{in} \Gamma_j^i \Gamma_i \Gamma_m - 4E \delta_{jn} \Gamma_i \Gamma_i \Gamma_m - \\
 & 4E \delta_{ln} \Gamma_i \Gamma_j^i \Gamma_m - 4E \delta_{mn} \Gamma_i \Gamma_j^i \Gamma_l - E(1-2\gamma) \delta_{ij}^l \delta_{lm} \Gamma_n - E \\
 & (1-2\gamma) \delta_{ij}^l \delta_{ln} \Gamma_m - E(1-2\gamma) \delta_{ij}^l \delta_{mn} \Gamma_l + E \delta_{il} \delta_{jm} \Gamma_n + E \delta_{jl}^i \\
 & \delta_{im} \Gamma_n + E \delta_{il} \delta_{jn} \Gamma_m + E \delta_{jl}^i \delta_{mn} \Gamma_j + E \delta_{ij}^l \delta_{ln} \Gamma_m + E \delta_{jl}^i \\
 & \delta_{mn} \Gamma_l + E \delta_{lm} \delta_{jn} \Gamma_l + E \delta_{im} \delta_{ln} \Gamma_j + E \delta_{jn} \delta_{in} \Gamma_l + E \delta_{jn} \\
 & \delta_{in} \Gamma_i + E \delta_{lm} \delta_{in} \Gamma_j + E \delta_{ln} \delta_{jn} \Gamma_i \Big] - n_i \int 3E4 \Gamma_i \Gamma_j^i \Gamma_k
 \end{aligned}$$

$$\begin{aligned}
 & \Gamma_m \Gamma_n + 4E(1-2\gamma) \delta_{ij}^l \Gamma_k \Gamma_m \Gamma_n - 4E \delta_{ik} \Gamma_j^i \Gamma_m \Gamma_n - 4E \delta_{jk} \\
 & \Gamma_i \Gamma_m \Gamma_n - 4E \delta_{im} \Gamma_j^i \Gamma_k \Gamma_n - 4E \delta_{jm} \Gamma_i \Gamma_k \Gamma_n - 4E \delta_{km} \Gamma_i \Gamma_j^i \\
 & \Gamma_n - 4E \delta_{in} \Gamma_j^i \Gamma_k \Gamma_m - 4E \delta_{jn} \Gamma_i \Gamma_k \Gamma_m - 4E \delta_{kn} \Gamma_i \Gamma_j^i \Gamma_m - \\
 & 4E \delta_{mn} \Gamma_i \Gamma_j^i \Gamma_k - E(1-2\gamma) \delta_{ij}^l \delta_{km} \Gamma_n - E(1-2\gamma) \delta_{ij}^l \delta_{mn} \Gamma_k - E \\
 & (1-2\gamma) \delta_{ij}^l \delta_{kn} \Gamma_m + E \delta_{ik} \delta_{jn} \Gamma_m + E \delta_{ik} \delta_{mn} \Gamma_j + E \delta_{jk} \delta_{in} \Gamma_m \\
 & + E \delta_{jk} \delta_{mn} \Gamma_i + E \delta_{im} \delta_{jn} \Gamma_k + E \delta_{im} \delta_{kn} \Gamma_j + E \delta_{ik} \delta_{jm} \Gamma_n \\
 & + E \delta_{jk} \delta_{im} \Gamma_n + E \delta_{jm} \delta_{in} \Gamma_k + E \delta_{jm} \delta_{kn} \Gamma_i + E \delta_{km} \delta_{in} \Gamma_j \\
 & + E \delta_{km} \delta_{jn} \Gamma_i \Big] - n_m \int 3E4 \Gamma_i \Gamma_j^i \Gamma_k \Gamma_l \Gamma_n + 4E(1-2\gamma) \delta_{ij}^l \Gamma_k
 \end{aligned}$$

$$\begin{aligned}
 & \Gamma_l \Gamma_n - 4E \delta_{ik} \Gamma_j^i \Gamma_l \Gamma_n - 4E \delta_{jk} \Gamma_i \Gamma_l \Gamma_n - 4E \delta_{jl}^i \Gamma_i \Gamma_k \Gamma_n - \\
 & 4E \delta_{il}^i \Gamma_j^i \Gamma_k \Gamma_n - 4E \delta_{kl} \Gamma_i \Gamma_j^i \Gamma_n - 4E \delta_{in} \Gamma_j^i \Gamma_k \Gamma_l - 4E \delta_{jn} \Gamma_i \\
 & \Gamma_k \Gamma_l - 4E \delta_{kn} \Gamma_i \Gamma_j^i \Gamma_l - 4E \delta_{ln} \Gamma_i \Gamma_j^i \Gamma_k - E(1-2\gamma) \delta_{ij}^l \delta_{kl} \Gamma_n \\
 & - E(1-2\gamma) \delta_{ij}^l \delta_{kn} \Gamma_l - E(1-2\gamma) \delta_{ij}^l \delta_{ln} \Gamma_k + E \delta_{ik} \delta_{jl}^i \Gamma_n + E \\
 & \delta_{jk} \delta_{il}^i \Gamma_n + E \delta_{ik} \delta_{jn} \Gamma_l + E \delta_{ik} \delta_{ln} \Gamma_j + E \delta_{jk} \delta_{in} \Gamma_l + E \delta_{jk} \\
 & \delta_{in} \Gamma_i + E \delta_{jl}^i \delta_{in} \Gamma_k + E \delta_{jl}^i \delta_{kn} \Gamma_l + E \delta_{il}^i \delta_{jn} \Gamma_k + E \delta_{il}^i \delta_{kn} \Gamma_j \\
 & + E \delta_{ki} \delta_{in} \Gamma_j + E \delta_{kl}^i \delta_{jn} \Gamma_l \Big] - n_n \int 3E4 \Gamma_i \Gamma_j^i \Gamma_k \Gamma_l \Gamma_m + 4E
 \end{aligned}$$

$$(1-2\gamma) \delta_{ij}^l \Gamma_k \Gamma_l \Gamma_m - 4E \delta_{ik} \Gamma_j^i \Gamma_l \Gamma_m - 4E \delta_{jk} \Gamma_i \Gamma_l \Gamma_m -$$

$$\begin{aligned}
(1-23) \delta_{ij} & \left(\delta_{kl} \delta_{mn} \tau_{ic} + \delta_{km} \delta_{ln} \tau_{io} + \delta_{im} \delta_{kn} \tau_{io} + \delta_{kl} \right. \\
& \delta_{no} \tau_{in} + \delta_{kl} \delta_{no} \tau_{im} + \delta_{km} \delta_{io} \tau_{in} + \delta_{kn} \delta_{no} \tau_{il} + \delta_{im} \delta_{ko} \\
& \tau_{in} + \delta_{im} \delta_{no} \tau_{ik} + \delta_{kn} \delta_{io} \tau_{im} + \delta_{kn} \delta_{no} \tau_{il} + \delta_{in} \delta_{ko} \tau_{im} + \\
& \delta_{in} \delta_{no} \tau_{ik} + \delta_{km} \delta_{ko} \tau_{il} + \delta_{kn} \delta_{io} \tau_{ik} \left. \right) - 4 \varepsilon \left(\delta_{ik} \delta_{jl} \delta_{mn} \right. \\
& \tau_{ic} + \delta_{jk} \delta_{il} \delta_{mn} \tau_{ic} + \delta_{ik} \delta_{jm} \delta_{ln} \tau_{ic} + \delta_{ik} \delta_{lm} \delta_{jn} \tau_{ic} + \delta_{jk} \\
& \delta_{im} \delta_{ln} \tau_{ic} + \delta_{jk} \delta_{lm} \delta_{in} \tau_{ic} + \delta_{jl} \delta_{im} \delta_{kn} \tau_{ic} + \delta_{jl} \delta_{km} \delta_{in} \\
& \tau_{ic} + \delta_{il} \delta_{jm} \delta_{kn} \tau_{ic} + \delta_{il} \delta_{km} \delta_{jn} \tau_{ic} + \delta_{kl} \delta_{im} \delta_{jn} \tau_{ic} + \delta_{kl} \\
& \delta_{jm} \delta_{in} \tau_{ic} + \delta_{ik} \delta_{jl} \delta_{no} \tau_{im} + \delta_{ik} \delta_{jl} \delta_{no} \tau_{im} + \delta_{jk} \delta_{il} \\
& \delta_{no} \tau_{im} + \delta_{jk} \delta_{il} \delta_{no} \tau_{im} + \delta_{ik} \delta_{jm} \delta_{lo} \tau_{in} + \delta_{ik} \delta_{jm} \delta_{no} \tau_{il} \\
& + \delta_{ik} \delta_{lm} \delta_{jn} \tau_{in} + \delta_{ik} \delta_{lm} \delta_{no} \tau_{ij} + \delta_{jk} \delta_{im} \delta_{lo} \tau_{in} + \delta_{jk} \\
& \delta_{im} \delta_{no} \tau_{il} + \delta_{jk} \delta_{im} \delta_{lo} \tau_{in} + \delta_{jk} \delta_{lm} \delta_{no} \tau_{il} + \delta_{jl} \delta_{im} \delta_{ko} \tau_{in} \\
& + \delta_{jl} \delta_{im} \delta_{no} \tau_{ik} + \delta_{jl} \delta_{km} \delta_{io} \tau_{in} + \delta_{jl} \delta_{km} \delta_{no} \tau_{il} + \delta_{il} \delta_{jm} \\
& \delta_{ko} \tau_{in} + \delta_{il} \delta_{jm} \delta_{no} \tau_{ik} + \delta_{il} \delta_{kn} \delta_{jn} \tau_{in} + \delta_{il} \delta_{kn} \delta_{no} \tau_{ij} + \\
& \delta_{kl} \delta_{im} \delta_{jn} \tau_{in} + \delta_{kl} \delta_{im} \delta_{no} \tau_{ij} + \delta_{kl} \delta_{jm} \delta_{io} \tau_{in} + \delta_{kl} \delta_{jm} \\
& \delta_{no} \tau_{il} + \delta_{kl} \delta_{jn} \delta_{lo} \tau_{im} + \delta_{kl} \delta_{jn} \delta_{no} \tau_{il} + \delta_{ik} \delta_{ln} \delta_{jn} \tau_{in} + \\
& \delta_{ik} \delta_{ln} \delta_{no} \tau_{ij} + \delta_{ik} \delta_{mn} \delta_{jo} \tau_{il} + \delta_{ik} \delta_{mn} \delta_{lo} \tau_{ij} + \delta_{jk} \delta_{in} \\
& \delta_{io} \tau_{im} + \delta_{jk} \delta_{in} \delta_{no} \tau_{il} + \delta_{jk} \delta_{ln} \delta_{io} \tau_{im} + \delta_{jk} \delta_{im} \delta_{no} \tau_{il} \\
& + \delta_{jk} \delta_{mn} \delta_{io} \tau_{il} + \delta_{jk} \delta_{mn} \delta_{lo} \tau_{il} + \delta_{jl} \delta_{in} \delta_{ko} \tau_{im} + \delta_{jl} \delta_{in} \\
& \delta_{no} \tau_{ik} + \delta_{jl} \delta_{kn} \delta_{io} \tau_{im} + \delta_{jl} \delta_{kn} \delta_{no} \tau_{il} + \delta_{jl} \delta_{mn} \delta_{io} \tau_{ik} \\
& + \delta_{jl} \delta_{mn} \delta_{ko} \tau_{il} + \delta_{il} \delta_{jn} \delta_{ko} \tau_{im} + \delta_{il} \delta_{jn} \delta_{no} \tau_{ik} + \delta_{il} \\
& \delta_{kn} \delta_{jo} \tau_{im} + \delta_{il} \delta_{kn} \delta_{no} \tau_{ij} + \delta_{il} \delta_{mn} \delta_{jo} \tau_{ik} + \delta_{il} \delta_{mn} \delta_{ko} \\
& \tau_{ij} + \delta_{kl} \delta_{in} \delta_{jo} \tau_{im} + \delta_{kl} \delta_{in} \delta_{no} \tau_{ij} + \delta_{kl} \delta_{jn} \delta_{io} \tau_{im} + \\
& \delta_{kl} \delta_{jn} \delta_{no} \tau_{il} + \delta_{kl} \delta_{mn} \delta_{jo} \tau_{il} + \delta_{kl} \delta_{mn} \delta_{lo} \tau_{il} + \delta_{im} \\
& \delta_{jn} \delta_{ko} \tau_{il} + \delta_{im} \delta_{jn} \delta_{lo} \tau_{ik} + \delta_{im} \delta_{kn} \delta_{jo} \tau_{il} + \delta_{im} \delta_{kn} \delta_{lo} \\
& \tau_{ij} + \delta_{im} \delta_{ln} \delta_{jo} \tau_{ik} + \delta_{im} \delta_{ln} \delta_{ko} \tau_{ij} + \delta_{jm} \delta_{in} \delta_{ko} \tau_{il} + \delta_{jm} \\
& \delta_{in} \delta_{lo} \tau_{ik} + \delta_{jm} \delta_{kn} \delta_{io} \tau_{il} + \delta_{jm} \delta_{kn} \delta_{lo} \tau_{il} + \delta_{jm} \delta_{ln} \delta_{io}
\end{aligned}$$

$$\begin{aligned} & r_k + \delta_{jn} \delta_{li} \delta_{ko} r_{li} + \delta_{kn} \delta_{in} \delta_{jo} r_{le} + \delta_{km} \delta_{in} \delta_{lo} r_{lj} + \\ & \delta_{kn} \delta_{jn} \delta_{io} r_{li} + \delta_{km} \delta_{jn} \delta_{lo} r_{li} + \delta_{km} \delta_{in} \delta_{io} r_{lj} + \delta_{kn} \delta_{in} \delta_{lo} \\ & \delta_{jo} r_{li} + \delta_{lm} \delta_{in} \delta_{jo} r_{lk} + \delta_{lm} \delta_{in} \delta_{ko} r_{lj} + \delta_{lm} \delta_{jn} \delta_{io} \\ & r_{lk} + \delta_{lm} \delta_{jn} \delta_{ko} r_{li} + \delta_{lm} \delta_{kn} \delta_{io} r_{lj} + \delta_{lm} \delta_{kn} \delta_{jo} r_{li} \Big] \end{aligned}$$

$$+ n_i (1-2\gamma) \Big[3840 r_{ij} r_{lk} r_{li} r_{lm} r_{ln} r_{lo} - 384 (\delta_{jk} r_{li} r_{lm} r_{ln}$$

$$\begin{aligned} & r_{lo} + \delta_{jl} r_{ik} r_{lm} r_{ln} r_{lo} + \delta_{kl} r_{ij} r_{lm} r_{ln} r_{lo} + \delta_{jm} r_{lk} r_{li} r_{ln} r_{lo} \\ & + \delta_{km} r_{ij} r_{li} r_{ln} r_{lo} + \delta_{lm} r_{ij} r_{lk} r_{ln} r_{lo} + \delta_{jn} r_{lk} r_{li} r_{lm} r_{lo} + \\ & \delta_{kn} r_{ij} r_{li} r_{lm} r_{lo} + \delta_{ln} r_{ij} r_{lk} r_{lm} r_{lo} + \delta_{mn} r_{ij} r_{lk} r_{li} r_{lo} + \delta_{jo} \\ & r_{lk} r_{li} r_{lm} r_{ln} + \delta_{ko} r_{ij} r_{li} r_{lm} r_{ln} + \delta_{lo} r_{ij} r_{lk} r_{lm} r_{ln} + \delta_{mo} \\ & r_{ij} r_{lk} r_{li} r_{ln} + \delta_{no} r_{ij} r_{lk} r_{li} r_{lm} \Big) + 4\mathcal{E} (\delta_{jk} \delta_{lm} r_{ln} r_{lo} + \delta_{jk} \\ & \delta_{ln} r_{lm} r_{lo} + \delta_{jk} \delta_{mn} r_{li} r_{lo} + \delta_{jl} \delta_{km} r_{ln} r_{lo} + \delta_{kl} \delta_{jm} r_{ln} r_{lo} \\ & + \delta_{jl} \delta_{kn} r_{lm} r_{lo} + \delta_{jl} \delta_{mn} r_{lk} r_{lo} + \delta_{kl} \delta_{jn} r_{lm} r_{lo} + \delta_{kl} \delta_{mn} \\ & r_{ij} r_{lo} + \delta_{jm} \delta_{kn} r_{li} r_{lo} + \delta_{jm} \delta_{ln} r_{lk} r_{lo} + \delta_{km} \delta_{jn} r_{li} r_{lo} + \delta_{km} \\ & \delta_{ln} r_{ij} r_{lo} + \delta_{ln} \delta_{jn} r_{lk} r_{lo} + \delta_{lm} \delta_{kn} r_{ij} r_{lo} + \delta_{jk} \delta_{lo} r_{lm} r_{ln} + \\ & \delta_{jk} \delta_{mo} r_{li} r_{ln} + \delta_{jl} \delta_{ko} r_{lm} r_{ln} + \delta_{jk} \delta_{no} r_{lm} r_{li} + \delta_{jl} \delta_{mo} r_{lk} r_{ln} \\ & + \delta_{jl} \delta_{no} r_{lk} r_{lm} + \delta_{kl} \delta_{jo} r_{lm} r_{ln} + \delta_{kl} \delta_{mo} r_{ij} r_{ln} + \delta_{kl} \delta_{no} \\ & r_{ij} r_{lm} + \delta_{jm} \delta_{ko} r_{li} r_{ln} + \delta_{jm} \delta_{lo} r_{lk} r_{ln} + \delta_{jm} \delta_{no} r_{lk} r_{li} + \delta_{km} \\ & \delta_{jo} r_{li} r_{ln} + \delta_{km} \delta_{lo} r_{ij} r_{ln} + \delta_{km} \delta_{no} r_{ij} r_{li} + \delta_{lm} \delta_{jo} r_{lk} r_{ln} \\ & + \delta_{lm} \delta_{ko} r_{ij} r_{ln} + \delta_{lm} \delta_{no} r_{ij} r_{lk} + \delta_{jn} \delta_{ko} r_{li} r_{ln} + \delta_{jn} \delta_{lo} \\ & r_{lk} r_{ln} + \delta_{jn} \delta_{mo} r_{lk} r_{li} + \delta_{kn} \delta_{jo} r_{li} r_{lm} + \delta_{kn} \delta_{lo} r_{ij} r_{lm} + \\ & \delta_{kn} \delta_{mo} r_{ij} r_{li} + \delta_{ln} \delta_{jo} r_{lk} r_{lm} + \delta_{ln} \delta_{ko} r_{ij} r_{lm} + \delta_{ln} \delta_{mo} r_{ij} \\ & r_{lk} + \delta_{mn} \delta_{jo} r_{lk} r_{li} + \delta_{mn} \delta_{ko} r_{ij} r_{li} + \delta_{mn} \delta_{lo} r_{ij} r_{lk} \Big) - \\ & \mathcal{E} (\delta_{jk} \delta_{ln} \delta_{no} + \delta_{jk} \delta_{ln} \delta_{mo} + \delta_{jk} \delta_{mn} \delta_{lo} + \delta_{jl} \delta_{km} \\ & \delta_{no} + \delta_{kl} \delta_{jm} \delta_{no} + \delta_{jl} \delta_{kn} \delta_{mo} + \delta_{jl} \delta_{mn} \delta_{ko} + \delta_{kl} \delta_{jn} \delta_{mo} \end{aligned}$$

$$+ Skl Simo Sjo + Sjm Skno Sls + Sjm Sln Sks + Skm Sjn Sls + Skm Sln Sjo + Sln Sjn Sks + Sln Skn Sjo) \Big] - n_i(1-2i) \Big]$$

$$3840 \text{ } \Gamma_i \Gamma_k \Gamma_l \Gamma_m \Gamma_n \Gamma_o - 384 (Sik \Gamma_l \Gamma_m \Gamma_n \Gamma_o + Sil \Gamma_k \Gamma_m \Gamma_n \Gamma_o + Skl \Gamma_l \Gamma_m \Gamma_n \Gamma_o + Sim \Gamma_l \Gamma_n \Gamma_k \Gamma_o + Skm \Gamma_l \Gamma_l \Gamma_l \Gamma_o + Sln \Gamma_l \Gamma_k \Gamma_n \Gamma_o + Sin \Gamma_k \Gamma_l \Gamma_m \Gamma_o + Skn \Gamma_l \Gamma_l \Gamma_l \Gamma_o + Sln \Gamma_l \Gamma_k \Gamma_l \Gamma_o + Smn \Gamma_l \Gamma_k \Gamma_l \Gamma_o + Sio \Gamma_k \Gamma_l \Gamma_m \Gamma_n + Sko \Gamma_l \Gamma_l \Gamma_l \Gamma_m + Sls \Gamma_l \Gamma_k \Gamma_m \Gamma_n + Smo \Gamma_l \Gamma_k \Gamma_l \Gamma_n + Snc \Gamma_l \Gamma_k \Gamma_l \Gamma_m) + 48 (Sik Sim \Gamma_n \Gamma_o + Sil Skm \Gamma_n \Gamma_o + Skl Sim \Gamma_n \Gamma_o + Sik Sln \Gamma_m \Gamma_o + Sik Smn \Gamma_l \Gamma_o + Sil Skn \Gamma_m \Gamma_o + Sil Smn \Gamma_k \Gamma_o + Skl Sin \Gamma_m \Gamma_o + Skl Smo \Gamma_l \Gamma_o + Sin Skn \Gamma_l \Gamma_o + Sim Sln \Gamma_k \Gamma_o + Skm Sin \Gamma_l \Gamma_o + Skm Sln \Gamma_l \Gamma_o + Sim Sin \Gamma_k \Gamma_o + Sln Skn \Gamma_l \Gamma_o + Sik Sls \Gamma_m \Gamma_n + Sik Smo \Gamma_l \Gamma_n + Sik Smn \Gamma_l \Gamma_m + Sil Sks \Gamma_m \Gamma_n + Sil Sln \Gamma_k \Gamma_m + Sil Snc \Gamma_k \Gamma_l + Skl Sls \Gamma_m \Gamma_n + Skl Smo \Gamma_l \Gamma_m + Snc Skl \Gamma_l \Gamma_m + Sin Sls \Gamma_o \Gamma_k + Sim Snc \Gamma_k \Gamma_l + Sim Sks \Gamma_l \Gamma_n + Skm Sls \Gamma_l \Gamma_n + Skm Sls \Gamma_l \Gamma_l + Sln Sls \Gamma_k \Gamma_l + Sln Sln \Gamma_l \Gamma_k + Sin Sln \Gamma_k \Gamma_l + Smn Sks \Gamma_l \Gamma_l + Smn Sls \Gamma_l \Gamma_k) - \epsilon (Sik Sim Snc + Sil Skm Snc + Skl Sim Snc + Sik Sln Snc + Sik Smn Sls + Sil Skn Snc + Sil Smn Sks + Skl Sin Snc + Skl Smn Sio + Sim Sks Sls + Sim Sln Sks + Skm Sin Sls + Skm Sln Sio + Sln Sin Sks + Sln Sln Sio) \Big] - n_k \Big[3840 \text{ } \Gamma_i \Gamma_j \Gamma_l$$

$$\begin{aligned}
& \delta_{mc} \tau_{ij} \tau_{klm} + \delta_{no} \tau_{ij} \tau_{klm}) - 4\epsilon(1-2\gamma) \delta_{ij} (\delta_{km} \\
& \tau_{nlc} + \delta_{mn} \tau_{klc} + \delta_{kn} \tau_{mlc} + \delta_{ko} \tau_{mln} + \delta_{mo} \tau_{kl} \\
& \tau_{ln} + \delta_{no} \tau_{klm}) + 4\delta (\delta_{ik} \delta_{jn} \tau_{mlc} + \delta_{ik} \delta_{mn} \tau_{jlc} + \\
& \delta_{jk} \delta_{in} \tau_{mlc} + \delta_{jk} \delta_{mn} \tau_{lhc} + \delta_{im} \delta_{jn} \tau_{klc} + \delta_{im} \delta_{kn} \\
& \tau_{jlc} + \delta_{ik} \delta_{jm} \tau_{nlc} + \delta_{jk} \delta_{im} \tau_{nlc} + \delta_{jm} \delta_{in} \tau_{klc} + \\
& \delta_{jm} \delta_{kn} \tau_{lhc} + \delta_{km} \delta_{in} \tau_{jlc} + \delta_{km} \delta_{jn} \tau_{lhc} + \delta_{ik} \delta_{jc} \\
& \tau_{lmn} + \delta_{ik} \delta_{mc} \tau_{jln} + \delta_{ik} \delta_{no} \tau_{jlm} + \delta_{jk} \delta_{ic} \tau_{lmn} \\
& \delta_{jk} \delta_{mc} \tau_{lin} + \delta_{jk} \delta_{nc} \tau_{ilm} + \delta_{im} \delta_{jc} \tau_{klm} + \delta_{im} \delta_{ko} \\
& \tau_{jln} + \delta_{im} \delta_{no} \tau_{jlk} + \delta_{jm} \delta_{io} \tau_{klm} + \delta_{jm} \delta_{ko} \tau_{lin} \\
& + \delta_{jm} \delta_{nc} \tau_{lik} + \delta_{km} \delta_{ic} \tau_{jln} + \delta_{km} \delta_{jc} \tau_{lin} + \delta_{km} \\
& \delta_{nc} \tau_{ilj} + \delta_{in} \delta_{jc} \tau_{klm} + \delta_{in} \delta_{ko} \tau_{jlm} + \delta_{in} \delta_{mo} \tau_{jlk} \\
& + \delta_{jn} \delta_{ic} \tau_{klm} + \delta_{jn} \delta_{ko} \tau_{lin} + \delta_{jn} \delta_{mc} \tau_{lik} + \delta_{kn} \delta_{ic} \\
& \tau_{jlm} + \delta_{kn} \delta_{jc} \tau_{lin} + \delta_{kn} \delta_{mo} \tau_{ilj} + \delta_{mn} \delta_{ic} \tau_{j} \\
& \tau_{lk} + \delta_{mn} \delta_{jc} \tau_{lik} + \delta_{mn} \delta_{ko} \tau_{ilj}) + \epsilon(1-2\gamma) \delta_{ij} (\delta_{km} \\
& \delta_{nc} + \delta_{im} \delta_{ko} + \delta_{kn} \delta_{mc}) - 8 (\delta_{ik} \delta_{jn} \delta_{mc} + \delta_{ik} \delta_{mn} \\
& \delta_{jc} + \delta_{jk} \delta_{in} \delta_{mc} + \delta_{jk} \delta_{mn} \delta_{io} + \delta_{im} \delta_{jn} \delta_{ko} + \delta_{im} \delta_{kn} \\
& \delta_{jc} + \delta_{ik} \delta_{jm} \delta_{nc} + \delta_{jk} \delta_{im} \delta_{nc} + \delta_{jm} \delta_{in} \delta_{ko} + \delta_{jm} \\
& \delta_{kn} \delta_{ic} + \delta_{km} \delta_{in} \delta_{jo} + \delta_{km} \delta_{jn} \delta_{io})] - nm \left[\right.
\end{aligned}$$

$$\begin{aligned}
& 3\delta_{40} \tau_{ij} \tau_{klm} \tau_{lnl} + 3\delta_{44} (1-2\gamma) \delta_{ij} \tau_{klm} \tau_{lnl} \\
& - 3\delta_{44} (\delta_{ik} \tau_{jln} \tau_{mlc} + \delta_{jk} \tau_{ilm} \tau_{mlc} + \delta_{jl} \tau_{lik} \\
& \tau_{mlc} + \delta_{il} \tau_{jlk} \tau_{mlc} + \delta_{kl} \tau_{ilj} \tau_{mlc} + \delta_{in} \tau_{j} \\
& \tau_{lk} \tau_{mlc} + \delta_{jn} \tau_{lik} \tau_{mlc} + \delta_{kn} \tau_{ilil} \tau_{jlc} + \delta_{ln} \tau_{il} \\
& \tau_{jlk} \tau_{mlc} + \delta_{io} \tau_{j} \tau_{klm} \tau_{lnl} + \delta_{jo} \tau_{lik} \tau_{lnl} + \delta_{ko} \tau_{il} \\
& \tau_{jlk} \tau_{mlc} + \delta_{lc} \tau_{ilj} \tau_{klm} + \delta_{nc} \tau_{ilj} \tau_{klm}) - 4\epsilon \\
& (1-2\gamma) \delta_{ij} (\delta_{kl} \tau_{mlc} + \delta_{kn} \tau_{lhc} + \delta_{ln} \tau_{klc} + \delta_{ko} \tau_{lc}
\end{aligned}$$

$$\begin{aligned} & \left(\begin{aligned} & \tau_{in} + \delta_{ik} \tau_{ik} \tau_{in} + \delta_{no} \tau_{ik} \tau_{ic} \Big) + 4E \Big(\delta_{ik} \delta_{jk} \tau_{in} \tau_{ic} \\ & + \delta_{jk} \delta_{il} \tau_{in} \tau_{io} + \delta_{ik} \delta_{jn} \tau_{il} \tau_{ic} + \delta_{ik} \delta_{ln} \tau_{ij} \tau_{ic} + \delta_{jk} \\ & \delta_{in} \tau_{il} \tau_{io} + \delta_{jk} \delta_{in} \tau_{il} \tau_{ic} + \delta_{jl} \delta_{in} \tau_{ik} \tau_{io} + \delta_{jl} \delta_{kn} \tau_{il} \\ & \tau_{ic} + \delta_{il} \delta_{jn} \tau_{ik} \tau_{ic} + \delta_{il} \delta_{kn} \tau_{ij} \tau_{ic} + \delta_{kl} \delta_{in} \tau_{ij} \tau_{ic} + \\ & \delta_{kl} \delta_{jn} \tau_{il} \tau_{io} + \delta_{ik} \delta_{jo} \tau_{il} \tau_{in} + \delta_{ik} \delta_{lo} \tau_{ij} \tau_{in} + \delta_{ik} \\ & \delta_{no} \tau_{ij} \tau_{ic} + \delta_{jk} \delta_{ic} \tau_{il} \tau_{in} + \delta_{jk} \delta_{ic} \tau_{il} \tau_{in} + \delta_{jk} \\ & \delta_{no} \tau_{il} \tau_{ic} + \delta_{jl} \delta_{io} \tau_{ik} \tau_{in} + \delta_{jl} \delta_{ko} \tau_{il} \tau_{in} + \delta_{jl} \delta_{no} \\ & \tau_{ij} \tau_{ic} + \delta_{il} \delta_{jo} \tau_{ik} \tau_{in} + \delta_{il} \delta_{ko} \tau_{ij} \tau_{in} + \delta_{il} \delta_{no} \tau_{ij} \tau_{ic} \\ & + \delta_{kl} \delta_{io} \tau_{ij} \tau_{in} + \delta_{kl} \delta_{jo} \tau_{il} \tau_{in} + \delta_{kl} \delta_{no} \tau_{il} \tau_{ij} + \delta_{in} \\ & \delta_{jo} \tau_{ik} \tau_{ic} + \delta_{in} \delta_{ko} \tau_{ij} \tau_{il} + \delta_{in} \delta_{io} \tau_{ij} \tau_{ic} + \delta_{jn} \delta_{io} \tau_{ik} \\ & \tau_{ic} + \delta_{jn} \delta_{ko} \tau_{il} \tau_{ic} + \delta_{jn} \delta_{io} \tau_{il} \tau_{ic} + \delta_{kn} \delta_{io} \tau_{ij} \tau_{ic} + \delta_{kn} \\ & \delta_{jo} \tau_{il} \tau_{ic} + \delta_{kn} \delta_{ic} \tau_{il} \tau_{ij} + \delta_{ln} \delta_{ic} \tau_{ij} \tau_{ic} + \delta_{in} \delta_{jo} \tau_{il} \tau_{ic} \\ & + \delta_{ln} \delta_{ko} \tau_{il} \tau_{ij} \Big) + E(1-2i) \delta_{ij} \Big(\delta_{kl} \delta_{no} + \delta_{kn} \delta_{io} + \\ & \delta_{ln} \delta_{ko} \Big) - E \Big(\delta_{ik} \delta_{jl} \delta_{no} + \delta_{jk} \delta_{il} \delta_{no} + \delta_{ik} \delta_{jn} \\ & \delta_{lo} + \delta_{ik} \delta_{ln} \delta_{jo} + \delta_{jk} \delta_{in} \delta_{lo} + \delta_{jk} \delta_{ln} \delta_{io} + \delta_{jl} \\ & \delta_{in} \delta_{ko} + \delta_{jl} \delta_{kn} \delta_{io} + \delta_{il} \delta_{jn} \delta_{ko} + \delta_{il} \delta_{kn} \delta_{jo} + \\ & \delta_{kl} \delta_{in} \delta_{jo} + \delta_{kl} \delta_{jn} \delta_{io} \Big) \Big] - m \int 3840 \tau_{il} \tau_{ij} \tau_{ik} \tau_{ic} \tau_{in} \end{aligned} \right. \end{aligned}$$

$$\begin{aligned} & \text{tic} + 3E4(1-2V) \delta_{ij} \text{tiklilmlic} - 3E4(\delta_{ik} \epsilon_{jl} \text{tillimlic} \\ & + \delta_{jk} \text{tillililmlic} + \delta_{jl} \text{tillikilmlic} + \delta_{il} \epsilon_{jk} \text{tillimlic} + \\ & \delta_{kl} \text{tillijlilmlic} + \delta_{im} \epsilon_{jk} \text{tillilic} + \delta_{jm} \text{tilliklilmlic} + \\ & \delta_{km} \text{tillijlilic} + \delta_{lm} \text{tillijlilklic} + \delta_{ic} \epsilon_{jk} \text{tillilim} + \\ & \delta_{jc} \text{tilliklilm} + \delta_{kc} \text{tillijlilm} + \delta_{lc} \text{tillijlilkim} + \delta_{nc} \\ & \text{tillijlilklic}) - 4E(1-2V) \delta_{ij} (\delta_{kl} \text{tillilic} + \delta_{km} \text{tillilic} + \\ & \delta_{lm} \text{tilliklic} + \delta_{kc} \text{tillilim} + \delta_{lc} \text{tillikim} + \delta_{nc} \text{tilliklic}) + 4E \\ & (-\delta_{ik} \delta_{jl} \text{tillilic} + \delta_{jk} \delta_{il} \text{tillilic} + \delta_{ik} \delta_{jm} \text{tillilic} + \delta_{ik} \end{aligned}$$

$$\begin{aligned}
& \delta_{im} \tau_{j'ic} + \delta_{jk} \delta_{im} \tau_{ilic} + \delta_{jk} \delta_{im} \tau_{ilic} + \delta_{jl} \delta_{im} \\
& \tau_{ikic} + \delta_{jl} \delta_{km} \tau_{ilic} + \delta_{il} \delta_{jm} \tau_{ikic} + \delta_{il} \delta_{km} \tau_{j'ic} + \\
& \delta_{kl} \delta_{im} \tau_{j'ic} + \delta_{kl} \delta_{jm} \tau_{ilic} + \delta_{ik} \delta_{jc} \tau_{ilim} + \delta_{ik} \delta_{lo} \\
& \tau_{j'ium} + \delta_{ik} \delta_{mo} \tau_{j'ic} + \delta_{jk} \delta_{ic} \tau_{ilim} + \delta_{jk} \delta_{lo} \tau_{ilim} \\
& + \delta_{jk} \delta_{mo} \tau_{ilic} + \delta_{jl} \delta_{ic} \tau_{ikim} + \delta_{jl} \delta_{ko} \tau_{ilim} + \\
& \delta_{jl} \delta_{mo} \tau_{ikic} + \delta_{il} \delta_{jc} \tau_{ikim} + \delta_{il} \delta_{ko} \tau_{j'ium} + \delta_{il} \\
& \delta_{mo} \tau_{j'ic} + \delta_{kl} \delta_{ic} \tau_{j'ium} + \delta_{kl} \delta_{jc} \tau_{ilim} + \delta_{kl} \delta_{mo} \\
& \tau_{ilic} + \delta_{im} \delta_{jc} \tau_{ikic} + \delta_{im} \delta_{ko} \tau_{j'ic} + \delta_{im} \delta_{lo} \tau_{j'ic} \\
& + \delta_{jm} \delta_{ic} \tau_{ikic} + \delta_{jm} \delta_{ko} \tau_{ilic} + \delta_{jm} \delta_{lo} \tau_{ikic} + \delta_{km} \\
& \delta_{ic} \tau_{j'ic} + \delta_{km} \delta_{jc} \tau_{ilic} + \delta_{km} \delta_{lo} \tau_{ilic} + \delta_{lm} \delta_{ic} \tau_{j'ic} \\
& \tau_{ik} + \delta_{lm} \delta_{jc} \tau_{ilic} + \delta_{lm} \delta_{ko} \tau_{ilic}) + 8(1-2i) \delta_{ij} (\\
& \delta_{kl} \delta_{mo} + \delta_{km} \delta_{ic} + \delta_{lm} \delta_{ko}) - 8(\delta_{ik} \delta_{jl} \delta_{mo} + \\
& \delta_{jk} \delta_{il} \delta_{mo} + \delta_{ik} \delta_{jm} \delta_{lo} + \delta_{ik} \delta_{im} \delta_{jc} + \delta_{jk} \delta_{im} \delta_{ic} \\
& + \delta_{jk} \delta_{im} \delta_{ic} + \delta_{jl} \delta_{im} \delta_{ko} + \delta_{jl} \delta_{km} \delta_{ic} + \delta_{il} \delta_{jm} \\
& \delta_{ko} + \delta_{il} \delta_{km} \delta_{jc} + \delta_{kl} \delta_{im} \delta_{jc} + \delta_{kl} \delta_{im} \delta_{jc} + \delta_{kl} \delta_{jm} \\
& \delta_{ic})] - 176 \left[3840 \tau_{ilic} \tau_{ikic} \tau_{ilim} + 384(1-2i) \right.
\end{aligned}$$

$$\begin{aligned}
& \delta_{ij} \tau_{ik} \tau_{ilim} - 384(\delta_{ik} \tau_{j'ic} \tau_{ilim} + \delta_{jk} \tau_{ilic} \tau_{ilim} \\
& \tau_{ilic} + \delta_{jl} \tau_{ilic} \tau_{ilim} + \delta_{il} \tau_{j'ic} \tau_{ilim} + \delta_{kl} \tau_{ilic} \tau_{ilim} \\
& + \delta_{im} \tau_{j'ic} \tau_{ilim} + \delta_{jm} \tau_{ilic} \tau_{ilim} + \delta_{km} \tau_{ilic} \tau_{ilim} \\
& + \delta_{lm} \tau_{j'ic} \tau_{ilic} + \delta_{im} \tau_{j'ic} \tau_{ilim} + \delta_{jm} \tau_{ilic} \tau_{ilim} \\
& + \delta_{km} \tau_{ilic} \tau_{ilim} + \delta_{lm} \tau_{j'ic} \tau_{ilic} + \delta_{im} \tau_{j'ic} \tau_{ilic}) \\
& - 48(1-2i) \delta_{ij} (\delta_{kl} \tau_{ilim} + \delta_{km} \tau_{ilim} + \delta_{lm} \tau_{ikic} + \\
& \delta_{ko} \tau_{ilim} + \delta_{lo} \tau_{ikic} + \delta_{mo} \tau_{ikic}) + 48(\delta_{ik} \delta_{jl} \tau_{ilim} \\
& \tau_{ilic} + \delta_{jk} \delta_{il} \tau_{ilim} + \delta_{ik} \delta_{jm} \tau_{ilim} + \delta_{ik} \delta_{im} \tau_{j'ic} + \delta_{jk} \\
& \delta_{im} \tau_{ilim} + \delta_{jk} \delta_{im} \tau_{ilim} + \delta_{jl} \delta_{im} \tau_{ikic} + \delta_{jl} \delta_{km}
\end{aligned}$$

$$\begin{aligned}
& \tau_{il} \tau_{in} + \delta_{il} \delta_{jn} \tau_{k \tau_{in}} + \delta_{il} \delta_{km} \tau_{j \tau_{in}} + \delta_{kl} \delta_{im} \tau_{j \tau_{in}} + \\
& \delta_{kl} \delta_{jn} \tau_{i \tau_{in}} + \delta_{ik} \delta_{jn} \tau_{l \tau_{in}} + \delta_{ik} \delta_{ln} \tau_{j \tau_{in}} + \delta_{ik} \delta_{mn} \\
& \tau_{j \tau_{il}} + \delta_{jk} \delta_{im} \tau_{l \tau_{in}} + \delta_{jk} \delta_{ln} \tau_{i \tau_{in}} + \delta_{jk} \delta_{mn} \tau_{i \tau_{il}} + \\
& \delta_{jl} \delta_{im} \tau_{k \tau_{in}} + \delta_{jl} \delta_{kn} \tau_{i \tau_{in}} + \delta_{jl} \delta_{mn} \tau_{i \tau_{lk}} + \delta_{il} \\
& \delta_{jn} \tau_{k \tau_{in}} + \delta_{il} \delta_{kn} \tau_{j \tau_{in}} + \delta_{il} \delta_{mn} \tau_{j \tau_{lk}} + \delta_{kl} \delta_{in} \tau_{j \tau_{in}} \\
& \tau_{in} + \delta_{kl} \delta_{jn} \tau_{i \tau_{in}} + \delta_{kl} \delta_{mn} \tau_{i \tau_{lj}} + \delta_{im} \delta_{jn} \tau_{k \tau_{il}} + \delta_{im} \\
& \delta_{kn} \tau_{j \tau_{il}} + \delta_{im} \delta_{ln} \tau_{j \tau_{lk}} + \delta_{jm} \delta_{in} \tau_{k \tau_{il}} + \delta_{jm} \delta_{kn} \tau_{i \tau_{il}} \\
& \tau_{il} + \delta_{jm} \delta_{ln} \tau_{i \tau_{lk}} + \delta_{kn} \delta_{in} \tau_{j \tau_{il}} + \delta_{kn} \delta_{jn} \tau_{i \tau_{il}} + \\
& \delta_{kn} \delta_{ln} \tau_{i \tau_{lj}} + \delta_{lm} \delta_{in} \tau_{j \tau_{lk}} + \delta_{lm} \delta_{jn} \tau_{i \tau_{lk}} + \delta_{ln} \delta_{kn} \\
& \tau_{i \tau_{lj}} + \delta (1-2\nu) \delta_{ij} (\delta_{kl} \delta_{mn} + \delta_{kn} \delta_{ln} + \delta_{im} \delta_{kn}) \\
& - \delta (\delta_{ik} \delta_{jl} \delta_{mn} + \delta_{jk} \delta_{il} \delta_{mn} + \delta_{ik} \delta_{jm} \delta_{ln} + \delta_{ik} \delta_{lm} \\
& \delta_{jn} + \delta_{jk} \delta_{im} \delta_{ln} + \delta_{jk} \delta_{lm} \delta_{in} + \delta_{jl} \delta_{im} \delta_{kn} + \delta_{jl} \delta_{kn} \\
& \delta_{in} + \delta_{il} \delta_{jm} \delta_{kn} + \delta_{il} \delta_{km} \delta_{jn} + \delta_{kl} \delta_{im} \delta_{jn} + \delta_{kl} \delta_{jn} \\
& \delta_{in}) \Bigg\} \quad (E.15)
\end{aligned}$$

Note that the derivatives of the traction kernels can also be obtained by differentiating the following equation (obtained from eqns (3.22), (3.32) and (3.33)) and substituting the derivatives of the displacement kernels into it,

$$T_{ij}(p, Q) = \lambda U_{ik,k} n_j + \mu (U_{ij,k} + U_{ki,j}) n_k \quad (E.16)$$

A P P E N D I X F

Numerical Results of the Square Plate Problem

Table F.1 : Square plate under uniform tension.
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
using direct expansion approach and eight quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	5	- 3.76x10-10 - 1.96x10-06	- 3.87x10-10 +2.72x10-05	+7.70x10-10 - 5.99x10-05	+2.11x10-10 - 2.27x10-05	+1.00000 +1.00000
(0.5, 0.5) (0.25, 0.25)	6	- 3.15x10-10 - 2.55x10-06	+2.87x10-10 - 7.39x10-06	+5.55x10-10 - 3.72x10-05	+1.06x10-10 - 3.29x10-05	+1.00000 +0.99999
(0.5, 0.5) (0.25, 0.25)	7	- 1.04x10-09 - 5.67x10-05	+2.91x10-10 - 1.18x10-04	+1.20x10-09 - 5.37x10-04	+1.82x10-10 - 1.23x10-04	+1.00000 +1.00013
(0.5, 0.5) (0.25, 0.25)	8	- 5.59x10-10 +3.68x10-03	+1.26x10-09 - 9.53x10-04	+8.93x10-10 +3.73x10-03	- 7.78x10-10 +1.48x10-03	+1.00000 +1.02648
(0.5, 0.5) (0.25, 0.25)	9	- 1.04x10-09 - 2.13x10-04	+1.02x10-09 +7.59x10-04	+1.70x10-09 +2.19x10-03	- 5.48x10-10 - 2.57x10-04	+1.00000 +0.99741
(0.5, 0.5) (0.25, 0.25)	10	- 1.06x10-09 - 2.27x10-04	+1.07x10-09 +2.75x10-04	+1.66x10-09 +7.01x10-04	- 5.12x10-10 - 6.66x10-05	+1.00000 +0.99943
(0.5, 0.5) (0.25, 0.25)	11	- 3.11x10-09 +3.15x10-03	- 2.45x10-09 - 4.25x10-03	+8.64x10-09 - 1.17x10-03	+2.27x10-09 +2.92x10-03	+1.00000 +1.00306

Table F.1 : Continued

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	12	- 1.53x10-09 +3.82x10-03	- 2.22x10-09 - 4.92x10-03	+3.38x10-09 - 6.99x10-05	+1.38x10-09 +5.14x10-03	+1.00000 +1.00502
(0.5, 0.5) (0.25, 0.25)	13	- 4.50x10-10 +2.14x10-03	- 1.97x10-09 - 2.06x10-03	+4.12x10-09 - 6.07x10-03	- 1.15x10-09 +3.49x10-04	+1.00000 +1.00870
(0.5, 0.5) (0.25, 0.25)	14	+1.42x10-09 +2.71x10-03	- 1.87x10-09 - 2.66x10-03	- 2.39x10-09 - 1.16x10-02	- 2.97x10-09 +6.23x10-04	+1.00000 +1.01048
(0.5, 0.5) (0.25, 0.25)	15	+1.72x10-08 - 1.02x10-01	+4.71x10-09 - 1.04x10-01	- 1.19x10-07 - 1.21	- 1.12x10-07 - 2.02x10-01	+1.00000 +0.36978
(0.5, 0.5) (0.25, 0.25)	16	+3.06x10-08 +3.57x10-03	- 2.95x10-08 - 4.59x10-04	- 1.02x10-07 +3.18x10-02	- 1.31x10-07 - 4.78x10-03	+1.00000 +0.98042
(0.5, 0.5) (0.25, 0.25)	17	+1.92x10-07 +1.25x10-02	+1.20x10-07 +2.15x10-02	- 7.58x10-07 +2.69x10-01	- 1.01x10-07 +1.49x10-02	+1.00000 +1.16357
(0.5, 0.5) (0.25, 0.25)	18	+5.37x10-08 - 6.89x10-03	- 5.35x10-08 - 1.32x10-02	- 1.83x10-07 - 8.96x10-02	+1.78x10-09 - 1.34x10-03	+1.00000 +0.86916

Table F.2 : Square plate under uniform tension.
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
using Neuber-Papkovich approach and eight quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	5	- 3.15x10-10 - 2.55x10-06	+2.87x10-10 - 7.39x10-06	+5.55x10-10 - 3.72x10-05	+1.06x10-10 - 3.29x10-05	+1.00000 +0.99999
(0.5, 0.5) (0.25, 0.25)	6	+4.48x10-11 - 9.20x10-06	+1.64x10-10 +4.33x10-04	- 1.19x10-09 +2.33x10-03	+3.92x10-10 - 1.23x10-03	+1.00000 +1.00115
(0.5, 0.5) (0.25, 0.25)	7	- 1.06x10-09 - 2.27x10-04	+1.07x10-09 +2.75x10-04	+1.66x10-09 +7.01x10-04	- 5.12x10-10 - 6.66x10-05	+1.00000 +0.99943
(0.5, 0.5) (0.25, 0.25)	8	+4.51x10-09 +6.11x10-04	- 5.75x10-09 - 3.97x10-04	- 1.26x10-08 - 1.99x10-03	- 9.07x10-10 +2.78x10-05	+1.00000 +1.00271
(0.5, 0.5) (0.25, 0.25)	9	+1.42x10-09 +2.71x10-03	- 1.87x10-09 - 2.66x10-03	- 2.39x10-09 - 1.16x10-02	- 2.97x10-09 +6.23x10-04	+1.00000 +1.01048
(0.5, 0.5) (0.25, 0.25)	10	+7.11x10-08 - 2.23x10-03	- 2.43x10-08 - 9.59x10-04	- 2.89x10-07 +5.37x10-02	+2.34x10-07 +2.67x10-02	+1.00000 +0.96029
(0.5, 0.5) (0.25, 0.25)	11	+5.37x10-08 - 6.89x10-03	- 5.35x10-08 - 1.32x10-02	- 1.83x10-07 - 8.96x10-02	+1.78x10-09 - 1.34x10-03	+1.00000 +0.86916

Table F.3 : Square plate under uniform tension.
Displacements and stresses at node 1 (Fig. 3.3) from Taylor series
using Airy's stress function approach and eight quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	5	+2.55x10-11 - 1.37x10-06	- 2.66x10-11 +1.54x10-06	- 2.65x10-11 +2.34x10-06	- 4.00x10-15 - 5.96x10-07	+1.00000 +1.00000
(0.5, 0.5) (0.25, 0.25)	6	+2.48x10-10 +8.36x10-06	- 3.65x10-11 - 9.75x10-07	- 4.49x10-10 - 5.45x10-05	- 6.51x10-12 +3.23x10-06	+1.00000 +1.00046
(0.5, 0.5) (0.25, 0.25)	7	- 5.73x10-11 +1.09x10-05	+2.45x10-10 - 3.41x10-06	- 5.39x10-11 - 5.97x10-05	+2.91x10-10 - 5.05x10-08	+1.00000 +1.00053
(0.5, 0.5) (0.25, 0.25)	8	- 1.92x10-10 - 6.30x10-06	+2.40x10-10 - 1.73x10-06	+4.16x10-11 +2.79x10-05	+1.58x10-10 - 2.82x10-05	+1.00000 +0.99996
(0.5, 0.5) (0.25, 0.25)	9	- 4.76x10-10 +9.70x10-07	+4.49x10-10 - 1.09x10-05	- 1.66x10-09 - 1.47x10-04	+1.07x10-10 - 3.29x10-05	+1.00000 +1.00010
(0.5, 0.5) (0.25, 0.25)	10	- 1.58x10-09 - 2.90x10-06	+1.61x10-09 +3.34x10-05	- 1.04x10-08 +2.37x10-04	- 1.30x10-09 - 6.74x10-05	+1.00000 +0.99987
(0.5, 0.5) (0.25, 0.25)	11	- 6.16x10-10 - 3.69x10-04	+5.43x10-11 +3.68x10-04	- 5.30x10-09 +4.33x10-03	- 2.76x10-09 - 4.56x10-04	+1.00000 +0.99664
(0.5, 0.5) (0.25, 0.25)	12	+6.78x10-11 - 8.31x10-03	- 3.36x10-11 +5.76x10-03	- 9.31x10-09 +7.00x10-02	- 9.87x10-10 +2.03x10-03	+1.00000 +0.92082
(0.5, 0.5) (0.25, 0.25)	13	- 6.69x10-10 - 1.42x10-04	+6.82x10-10 +1.90x10-04	+1.91x10-08 +1.23x10-03	- 5.12x10-10 - 6.66x10-05	+1.00000 +0.99891

Table F.3 : Continued

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	14	- 1.74x10-09 +1.54x10-04	+1.12x10-09 - 1.40x10-03	+8.80x10-08 - 5.73x10-03	- 1.20x10-08 - 3.06x10-03	+1.00000 +1.00505
(0.5, 0.5) (0.25, 0.25)	15	- 1.69x10-10 +4.52x10-04	- 5.51x10-10 - 2.24x10-03	+1.99x10-09 - 1.82x10-02	+2.66x10-10 - 2.09x10-03	+1.00000 +1.01361
(0.5, 0.5) (0.25, 0.25)	16	+4.10x10-11 +1.81x10-03	- 2.22x10-10 - 1.90x10-03	- 7.58x10-09 - 2.92x10-02	+1.79x10-09 +8.36x10-05	+1.00000 +1.02844
(0.5, 0.5) (0.25, 0.25)	17	- 4.61x10-10 +2.00x10-03	+1.25x10-11 - 1.94x10-03	- 9.39x10-09 - 2.94x10-02	- 2.97x10-09 +6.23x10-04	+1.00000 +1.02826
(0.5, 0.5) (0.25, 0.25)	18	+1.20x10-08 +4.15x10-03	- 2.73x10-08 - 1.67x10-03	- 1.71x10-07 +1.17x10-02	- 9.50x10-08 - 3.65x10-03	+1.00000 +1.00503
(0.5, 0.5) (0.25, 0.25)	19	+1.37x10-08 +6.50x10-03	- 2.77x10-08 +1.23x10-02	- 1.43x10-07 +1.84x10-01	- 1.05x10-07 +1.03x10-02	+1.00000 +1.03489
(0.5, 0.5) (0.25, 0.25)	20	+1.41x10-08 +1.02x10-01	- 3.03x10-08 +9.26x10-02	- 9.74x10-08 +1.55	- 8.65x10-08 +2.16x10-01	+1.00000 +1.95062
(0.5, 0.5) (0.25, 0.25)	21	+2.98x10-08 - 7.16x10-03	- 2.96x10-08 - 1.29x10-02	- 1.86x10-07 - 9.69x10-02	+1.78x10-09 - 1.34x10-03	+1.00000 +0.87649

Table F.4 : Square plate under uniform tension.
Displacements and stresses at node 1 (Fig. 3.3) from Fourier series (2.73)
using Neuber-Papkovich approach and eight quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.5, 0.5) (0.25, 0.25)	7	- 0.01449 +0.00129	+0.01410 - 0.00149	+0.03329 - 0.00820	+0.00184 - 0.00182	+0.96420 +1.01176
(0.5, 0.5) (0.25, 0.25)	9	- 0.00791 +0.00274	+0.00631 - 0.00243	+0.00301 - 0.00146	- 0.00639 +0.00033	+0.98073 +1.01617
(0.5, 0.5) (0.25, 0.25)	11	- 0.00027 - 0.00731	+0.00016 - 0.01412	+0.00205 - 0.10142	- 0.00167 - 0.00245	+1.00390 +0.85824

Table F.5 : Square plate under uniform tension.
Displacements and stresses at node 1 (Fig. 3.3) from Fourier series (2.73)
using Neuber-Papkovich approach and refined mesh in the region of interest.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Exact :	0.	0.	0.	0.	1.
(0.1, 0.1) (0.05, 0.05)	7	- 0.00195 - 0.00023	+0.00023 +0.00010	- 0.05056 - 0.08556	- 0.00262 - 0.00087	+1.07776 +1.12406
(0.1, 0.1) (0.05, 0.05)	9	+0.00004 - 0.00001	- 0.00005 - 0.00001	- 0.00073 - 0.00081	+0.00024 - 0.00083	+1.00904 +0.99581
(0.1, 0.1) (0.05, 0.05)	11	+0.00000 - 0.00000	- 0.00000 - 0.00000	- 0.00020 - 0.00041	+0.00022 +0.00062	+0.99809 +0.99641

A P P E N D I X G

Numerical Results of the Cantilever Beam Problem

Table G.1 : Cantilever beam loaded at the end with shearing stress.
Displacements and stresses at node 1 (Fig. 3.4) from Taylor series
using direct expansion approach and twenty quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic :	0.	- 1.4625	- 12.	0.	0.
(4., 0.) (2., 0.)	7	+10. +0.205	- 3.7 +1.346	+12. +15.	+0.322 +0.163	+0.007 - 0.020
(4., 0.) (2., 0.)	8	+10. +1.	- 1.882 - 0.106	- 10.550 - 4.4	- 0.285 - 0.344	- 2.5 - 0.679
(4., 0.) (2., 0.)	9	+1.6 +0.785	- 0.914 - 0.954	- 7.804 - 9.792	- 0.451 - 0.473	- 0.022 - 0.036
(4., 0.) (2., 0.)	10	+0.004 +0.150	- 1.200 - 1.045	- 11.982 - 10.115	- 0.001 +0.013	+0.001 +0.001
(4., 0.) (2., 0.)	11	+0.017 +0.186	- 1.186 - 1.270	- 11.945 - 12.174	- 0.001 +0.009	+0.002 +0.004
(4., 0.) (2., 0.)	12	+0.090 - 6.	- 1.208 +9.	- 12.046 +94.	- 0.001 +0.645	- 0.015 +1.
(4., 0.) (2., 0.)	13	+0.379 - 0.461	- 1.145 +0.05	- 11.892 + 2.	+0.032 - 0.024	- 0.164 +0.385
(4., 0.) (2., 0.)	14	+0.049 +2.	- 1.213 - 7.	- 12.050 - 68.	+0.000 +0.086	+0.001 +0.176
(4., 0.) (2., 0.)	15	+0.057 +0.570	- 1.205 - 1.968	- 12.078 - 18.436	- 0.001 +0.039	+0.004 +0.019
(4., 0.) (2., 0.)	16	+0.130 - 0.073	- 1.207 - 0.005	- 12.011 - 0.4	+0.001 +0.028	- 0.017 - 0.001
(4., 0.) (2., 0.)	17	+0.097 - 0.073	- 1.213 +0.1	- 11.975 + 0.4	+0.001 +0.013	+0.002 - 0.040
(4., 0.) (2., 0.)	18	+0.092 - 0.044	- 1.215 +0.09	- 11.983 + 0.1	+0.004 - 0.003	+0.003 - 0.049

Table G.2 : Cantilever beam loaded at the end with shearing stress.
Displacements and stresses at node 1 (Fig. 3.4) from Taylor series
using Neuber-Papkovich approach and twenty quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic :	0.	- 1.4625	- 12.	0.	0.
(4., 0.) (2., 0.)	5	+16. +0.190	- 3.9 +1.315	+14. +14.	+0.347 +0.158	- 0.009 - 0.019
(4., 0.) (2., 0.)	6	- 65. - 0.092	- 8. +1.	- 20.3 +10.	- 0.564 +0.156	+5. - 0.158
(4., 0.) (2., 0.)	7	+0.004 +0.150	- 1.200 - 1.045	- 11.982 - 10.115	- 0.001 +0.013	+0.001 +0.001
(4., 0.) (2., 0.)	8	- 0.091 +0.044	- 1.275 - 0.268	- 12.167 - 3.025	+0.004 +0.024	- 0.004 - 0.013
(4., 0.) (2., 0.)	9	+0.049 +2.	- 1.213 - 7.	- 12.050 - 68.	+0.000 +0.086	+0.001 +0.176
(4., 0.) (2., 0.)	10	+1.030 - 0.023	- 0.698 - 0.976	- 13.320 - 2.229	- 0.074 +0.024	+0.094 - 0.029
(4., 0.) (2., 0.)	11	+0.092 - 0.044	- 1.215 +0.09	- 11.983 + 0.1	+0.004 - 0.003	+0.003 - 0.049

Table G.3 : Cantilever beam loaded at the end with shearing stress.
Displacements and stresses at node 1 (Fig. 3.4) from Taylor series
using Airy's stress function approach and twenty quadratic elements.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic :	0.	- 1.4625	- 12.	0.	0.
(4., 0.)	5	- 0.083	- 8.	- 0.000	+0.064	- 0.000
(2., 0.)		- 0.036	- 1.549	- 0.000	+0.027	- 0.000
(4., 0.)	6	- 0.083	- 8.	- 0.000	+0.064	- 0.000
(2., 0.)		- 0.036	- 1.549	+0.000	+0.027	+0.000
(4., 0.)	7	- 2.	- 49.	- 0.000	+2.	- 0.273
(2., 0.)		- 0.453	- 7.	+0.000	+0.348	- 0.073
(4., 0.)	8	- 2.	- 49.	- 0.000	+2.	- 0.273
(2., 0.)		- 0.453	- 7.	+0.000	+0.348	- 0.073
(4., 0.)	9	- 0.020	- 2.279	+5.	+0.015	+0.014
(2., 0.)		- 0.060	+0.580	+7.	+0.047	- 0.001
(4., 0.)	10	- 0.020	- 2.279	+5.	+0.015	+0.014
(2., 0.)		- 0.060	+0.580	+7.	+0.047	- 0.001
(4., 0.)	11	- 1.036	- 5.7	+11.	+0.823	- 0.375
(2., 0.)		- 0.140	+0.9	+8.	+0.117	- 0.089
(4., 0.)	12	+1.	- 1.505	- 14.523	+0.130	- 0.055
(2., 0.)		+0.341	+2.	+9.	+0.590	- 0.444
(4., 0.)	13	+1.	- 1.505	- 14.523	+0.130	- 0.055
(2., 0.)		+0.341	+2.	+9.	+0.590	- 0.444

Table G.3 : Continued.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic :	0.	- 1.4625	- 12.	0.	0.
(4., 0.) (2., 0.)	14	+1. +0.341	- 1.505 +2.	- 14.523 +9.	+0.130 +0.590	- 0.055 - 0.444
(4., 0.) (2., 0.)	15	+1. - 1.326	- 2.271 +0.256	- 17.662 +4.	+0.258 - 0.069	- 0.130 +0.053
(4., 0.) (2., 0.)	16	+1. - 1.326	- 2.271 +0.256	- 17.662 +4.	+0.258 - 0.069	- 0.130 +0.053
(4., 0.) (2., 0.)	17	+1. +0.468	- 1.436 - 1.445	- 14.504 - 11.430	+0.153 - 0.083	- 0.074 +0.131
(4., 0.) (2., 0.)	18	+1. +0.468	- 1.436 - 1.445	- 14.504 - 11.430	+0.153 - 0.083	- 0.074 +0.131
(4., 0.) (2., 0.)	19	+0.371 - 0.176	- 10. +0.393	- 23. +5.	+3. - 0.055	- 2. +0.032
(4., 0.) (2., 0.)	20	+2. - 0.049	- 1.699 +0.3	- 33. +13.	+0.317 - 0.009	- 0.426 - 0.852
(4., 0.) (2., 0.)	21	+2. - 0.049	- 1.699 +0.3	- 33. +13.	+0.317 - 0.009	- 0.426 - 0.852

A P P E N D I X H

Numerical Results of the Stress Concentration Problem

Table H.1 : Plate with a hole in the middle subject to uniform stresses at the ends.

Displacements and stresses at node 1 (Fig. 3.5) from Taylor series using direct expansion approach and fine boundary elements in the region of interest, and remote field solution outside the local area.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
Analytic : - 0.05		0.	0.	0.	0.	+3.008
(0.060, 0.010)	7	- 0.054	+0.006	+0.203	- 0.363	+2.795
(0.057, 0.007)		- 0.053	+0.004	+0.100	- 0.275	+2.951
(0.055, 0.005)		- 0.051	+0.001	+0.023	- 0.162	+3.094
(0.054, 0.004)		- 0.052	+0.002	+0.047	- 0.150	+3.252
(0.053, 0.003)		- 0.051	+0.002	+0.041	- 0.128	+3.235
(0.0525, 0.0025)		- 0.050	+0.000	+0.002	- 0.062	+3.064
(0.060, 0.010)	8	- 0.040	- 0.011	- 0.020	- 0.135	+2.316
(0.057, 0.007)		- 0.044	- 0.006	- 0.024	- 0.071	+2.605
(0.055, 0.005)		- 0.048	- 0.002	- 0.020	- 0.041	+2.832
(0.054, 0.004)		- 0.048	- 0.002	- 0.005	- 0.018	+2.818
(0.053, 0.003)		- 0.049	- 0.001	- 0.003	- 0.010	+2.904
(0.0525, 0.0025)		- 0.050	- 0.001	- 0.005	- 0.010	+2.957
(0.060, 0.010)	9	- 0.021	- 0.024	- 0.537	- 0.475	+0.869
(0.057, 0.007)		- 0.111	+0.048	+1.764	+0.718	+9.550
(0.055, 0.005)		- 0.041	- 0.007	- 0.334	- 0.185	+1.839
(0.054, 0.004)		- 0.052	+0.014	+0.101	+0.021	+3.424
(0.053, 0.003)		- 0.051	+0.001	+0.077	+0.015	+3.340
(0.0525, 0.0025)		- 0.053	+0.003	+0.265	+0.097	+4.052
(0.060, 0.010)	10	- 0.046	- 0.004	- 0.043	- 0.182	+2.742
(0.057, 0.007)		- 0.050	- 0.000	- 0.014	- 0.074	+2.974
(0.055, 0.005)		- 0.050	+0.000	- 0.005	- 0.028	+3.016
(0.054, 0.004)		- 0.050	+0.000	+0.001	- 0.014	+3.029
(0.053, 0.003)		- 0.050	+0.000	+0.000	- 0.008	+3.022
(0.0525, 0.0025)		- 0.050	+0.000	- 0.001	- 0.005	+3.013

Table H.1 : Continued.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
Analytic : - 0.05			0.	0.	0.	+3.008
(0.060, 0.010)	11	- 0.045	- 0.005	- 0.071	- 0.143	+2.622
(0.057, 0.007)		- 0.048	- 0.002	- 0.050	- 0.061	+2.838
(0.055, 0.005)		- 0.050	- 0.000	- 0.022	- 0.023	+2.946
(0.054, 0.004)		- 0.050	- 0.000	- 0.026	- 0.014	+2.933
(0.053, 0.003)		- 0.050	- 0.000	- 0.015	- 0.008	+2.969
(0.0525, 0.0025)		- 0.050	- 0.000	- 0.011	- 0.006	+2.987
(0.060, 0.010)	12	- 0.051	+0.002	- 0.046	- 0.119	+3.142
(0.057, 0.007)		- 0.051	+0.001	- 0.028	- 0.061	+3.132
(0.055, 0.005)		- 0.050	+0.000	- 0.014	- 0.022	+3.063
(0.054, 0.004)		- 0.050	+0.000	- 0.011	- 0.016	+3.047
(0.053, 0.003)		- 0.050	+0.000	- 0.006	- 0.008	+3.031
(0.0525, 0.0025)		- 0.050	+0.000	- 0.005	- 0.005	+3.015
(0.060, 0.010)	13	- 0.051	+0.002	- 0.017	- 0.082	+3.067
(0.057, 0.007)		- 0.051	+0.001	- 0.015	- 0.037	+3.113
(0.055, 0.005)		- 0.050	+0.000	- 0.011	- 0.010	+3.027
(0.054, 0.004)		- 0.051	+0.000	- 0.003	- 0.009	+3.194
(0.053, 0.003)		- 0.050	+0.000	- 0.006	- 0.005	+3.066
(0.0525, 0.0025)		- 0.050	+0.000	- 0.004	- 0.002	+3.015
(0.060, 0.010)	14	- 0.049	- 0.002	- 0.070	- 0.046	+2.988
(0.057, 0.007)		- 0.050	- 0.000	- 0.029	- 0.018	+2.995
(0.055, 0.005)		- 0.050	- 0.000	- 0.016	- 0.007	+2.996
(0.054, 0.004)		- 0.050	- 0.000	- 0.008	- 0.004	+2.996
(0.053, 0.003)		- 0.050	- 0.000	- 0.004	- 0.002	+2.998
(0.0525, 0.0025)		- 0.050	- 0.000	- 0.004	- 0.001	+2.998

Table H.1 : Continued.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
	Analytic : - 0.05		0.	0.	0.	+3.008
(0.060, 0.010)		- 0.052	+0.003	+0.066	- 0.099	+3.461
(0.057, 0.007)		- 0.050	+0.001	+0.011	- 0.023	+3.142
(0.055, 0.005)		- 0.050	+0.000	- 0.000	- 0.006	+3.034
(0.054, 0.004)	15	- 0.050	+0.000	- 0.003	- 0.003	+3.027
(0.053, 0.003)		- 0.050	+0.000	- 0.003	- 0.001	+3.020
(0.0525, 0.0025)		- 0.050	+0.000	- 0.004	- 0.001	+2.996
(0.060, 0.010)		- 0.049	- 0.000	- 0.029	- 0.041	+3.027
(0.057, 0.007)		- 0.050	- 0.000	- 0.016	- 0.008	+2.994
(0.055, 0.005)		- 0.050	- 0.000	- 0.010	- 0.003	+2.995
(0.054, 0.004)	16	- 0.050	- 0.000	- 0.007	- 0.000	+2.981
(0.053, 0.003)		- 0.050	- 0.000	- 0.004	- 0.000	+2.992
(0.0525, 0.0025)		- 0.050	- 0.000	- 0.004	- 0.001	+2.995
(0.060, 0.010)		- 0.054	+0.007	- 0.032	- 0.113	+3.673
(0.057, 0.007)		- 0.050	+0.000	- 0.009	- 0.007	+3.046
(0.055, 0.005)		- 0.050	- 0.000	- 0.005	- 0.001	+2.996
(0.054, 0.004)	17	- 0.050	+0.000	+0.004	- 0.001	+3.046
(0.053, 0.003)		- 0.050	+0.000	+0.001	- 0.004	+3.040
(0.0525, 0.0025)		- 0.050	- 0.000	- 0.004	- 0.001	+2.992
(0.060, 0.010)		- 0.052	+0.002	+0.039	- 0.006	+3.245
(0.057, 0.007)		- 0.050	+0.000	+0.009	+0.002	+3.058
(0.055, 0.005)		- 0.050	+0.000	- 0.001	- 0.000	+3.009
(0.054, 0.004)	18	- 0.050	+0.000	+0.002	- 0.001	+3.010
(0.053, 0.003)		- 0.050	+0.000	- 0.003	- 0.001	+2.998
(0.0525, 0.0025)		- 0.050	+0.000	- 0.004	- 0.001	+2.998

Table H.2 : Plate with a hole in the middle subject to uniform stresses at the ends.
Displacements and stresses at node 1 (Fig. 3.5) from Taylor series
using Neuber-Papkovich approach and fine boundary elements in the region of interest,
and remote field solution outside the local area.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
Analytic : - 0.05		0.	0.	0.	0.	+3.008
(0.060, 0.010)	7	- 0.046	- 0.004	- 0.043	- 0.182	+2.742
(0.057, 0.007)		- 0.050	- 0.000	- 0.014	- 0.074	+2.974
(0.055, 0.005)		- 0.050	+0.000	- 0.005	- 0.028	+3.016
(0.054, 0.004)		- 0.050	+0.000	+0.001	- 0.014	+3.029
(0.053, 0.003)		- 0.050	+0.000	+0.000	- 0.008	+3.022
(0.0525, 0.0025)		- 0.050	+0.000	- 0.001	- 0.005	+3.013
(0.060, 0.010)	9	- 0.049	- 0.002	- 0.070	- 0.046	+2.988
(0.057, 0.007)		- 0.050	- 0.000	- 0.029	- 0.018	+2.995
(0.055, 0.005)		- 0.050	- 0.000	- 0.016	- 0.007	+2.996
(0.054, 0.004)		- 0.050	- 0.000	- 0.008	- 0.004	+2.996
(0.053, 0.003)		- 0.050	- 0.000	- 0.004	- 0.002	+2.998
(0.0525, 0.0025)		- 0.050	- 0.000	- 0.004	- 0.001	+2.998
(0.060, 0.010)	11	- 0.052	+0.002	+0.039	- 0.006	+3.245
(0.057, 0.007)		- 0.050	+0.000	+0.009	+0.002	+3.058
(0.055, 0.005)		- 0.050	+0.000	- 0.001	- 0.000	+3.009
(0.054, 0.004)		- 0.050	+0.000	+0.002	- 0.001	+3.010
(0.053, 0.003)		- 0.050	+0.000	- 0.003	- 0.001	+2.998
(0.0525, 0.0025)		- 0.050	+0.000	- 0.004	- 0.001	+2.998

Table H.3 : Plate with a hole in the middle subject to uniform stresses at the ends.
Displacements and stresses at node 1 (Fig. 3.5) from Taylor series
using Airy's stress function approach and fine boundary elements in the region of interest
and remote field solution outside the local area.

Internal point	No. of terms	u	v	σ_{xx}	σ_{xy}	σ_{yy}
Analytic : - 0.05		0.	0.	0.	0.	+3.008
(0.060, 0.010)	9	- 0.035	+0.076	+0.534	+0.163	+2.796
(0.057, 0.007)		- 0.041	+0.009	+0.111	- 0.056	+2.834
(0.055, 0.005)		- 0.050	+0.001	+0.049	- 0.141	+2.979
(0.054, 0.004)		- 0.038	- 0.005	- 0.498	- 0.305	+3.071
(0.053, 0.003)		- 0.040	- 0.004	- 0.505	- 0.338	+3.329
(0.0525, 0.0025)		- 0.050	+0.001	+0.013	- 0.050	+3.059
(0.060, 0.010)	13	- 0.047	+0.043	+0.323	+0.027	+2.983
(0.057, 0.007)		- 0.042	+0.016	+0.236	- 0.005	+2.980
(0.055, 0.005)		- 0.050	- 0.001	+0.024	- 0.033	+2.858
(0.054, 0.004)		- 0.038	- 0.006	- 0.488	- 0.239	+2.966
(0.053, 0.003)		- 0.041	- 0.005	- 0.491	- 0.243	+3.153
(0.0525, 0.0025)		- 0.050	- 0.000	+0.005	- 0.006	+2.997
(0.060, 0.010)	17	- 0.050	+0.014	+0.049	- 0.141	+2.989
(0.057, 0.007)		- 0.048	+0.001	+0.130	- 0.044	+2.954
(0.055, 0.005)		- 0.050	- 0.007	- 0.101	+0.095	+2.755
(0.054, 0.004)		- 0.050	- 0.011	- 0.433	+0.466	+2.567
(0.053, 0.003)		- 0.048	- 0.007	- 0.519	+0.414	+2.928
(0.0525, 0.0025)		- 0.050	+0.001	+0.049	- 0.141	+2.979
(0.060, 0.010)	21	- 0.051	+0.099	+0.108	- 0.023	+3.564
(0.057, 0.007)		- 0.049	+0.076	+0.152	- 0.033	+3.143
(0.055, 0.005)		- 0.050	- 0.007	- 0.091	+0.100	+3.056
(0.054, 0.004)		- 0.050	- 0.014	- 0.023	+0.342	+2.984
(0.053, 0.003)		- 0.048	- 0.006	- 0.325	+0.123	+2.937
(0.0525, 0.0025)		- 0.050	+0.007	+0.075	- 0.329	+2.994

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