Coercive Inequalities and $U-$Bounds on Carnot Groups

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Esther Bou Dagher

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To my father's companion, Saint Estephan Nehme

Abstract

This thesis is concerned with coercive inequalities, such as the Logarithmic Sobolev and Poincaré inequalities, for generators defined as the sum of squares of degenerate and non-commuting vector fields of Hörmander type. In the setting of the Carnot group, we prove the Logarithmic Sobolev inequality for measures as a function of the Carnot-Carathéodory distance, and we show it implies the Talagrand transportation cost inequality. In the settings of the step-two Carnot group and the Anisotropic Heisenberg group, we prove the Poincaré and the $\phi-$Logarithmic Sobolev inequalities for measures as a function of the fundamental solution of the sub-Laplacian. We then extend our results to infinite dimensional Gibbs measures, with unbounded interaction potentials, on an infinite product of Carnot groups.

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Chapter 1

Introduction

1.1 A Brief History

Poincaré inequalities first arose in 1890 in the work of Henri Poincaré [113], where he used them to study partial differential equations of mathematical physics. In particular, he worked on the Fourier problem for the heat equation. He studied the Laplace problem, on a bounded open convex set $\Omega \in \mathbb{R}^n$ with Neumann boundary conditions, for a sequence of eigenvalues k_j and eigenfunctions u_j :

$$\Delta u_j + k_j u_j = 0; \;\; j \geq 1.$$

For smooth functions with mean zero, he used the following Poincaré inequality :

$$\kappa(\Omega) \int_\Omega f^2 dx \leq \int_\Omega |\nabla f|^2 dx,$$

to provide a lower bound on the first non-trivial eigenvalue $k_2 \geq \kappa(\Omega) > 0$, and to show that $k_j \to \infty$. We note that the Poincaré constant, $\kappa(\Omega)$, depends on the diameter of the domain Ω. This fact is especially important since it shows that the Poincaré inequality depends on the geometry of the space that we are working on, and this will, in turn, make the study of such inequalities in the setting of Carnot groups more difficult. Before we make this statement clearer in the context of the PhD thesis, we recap other useful application of the Poincaré inequality. In 1904, Adolf Hurwitz showed in [77] the connection between the Poincaré inequality and the isoperimetric inequality on the plane. In fact, if we consider a simple closed smooth curve in the plane, then the Poincaré inequality exactly translates to the isoperimetric inequality $L^2 \geq 4\pi A$, where L is the arc length of the curve, and A is the area enclosed by the curve. Another useful

interpretation of the Poincaré constant is that it is used as a parameter to control the convergence to equilibrium [50, 51]. Given a semigroup $(P_t)_{t\geq 0}$ with infinitesimal generator $\mathcal{L}$ solving the heat equation, under some conditions, we have the ergodic property that the semigroup converges to an invariant stationary measure as $t \to \infty$. The Poincaré inequality in this situation describes the L_2 convergence to equilibrium at an exponential rate. This phenomenon is applied in the study of Markov chains and Statistical mechanics.

A more powerful inequality, that implies the Poincaré inequality, is the dimension-free Logarithmic Sobolev inequality. This inequality has been useful in the infinite dimensional Wiener analysis, convergence to equilibrium of kinetic equations and Markov chains, measure concentration, information theory, and optimal transport [2, 90, 110, 127]. Moreover, Grigori Perelman used it in his study of the Ricci flow to solve the Millennium Prize Poincaré Conjecture [112].

The Poincaré and Logarithmic Sobolev inequalities belong to a wider class of functional inequalities called coercive inequalities. These are forcing inequalities which, when satisfied, "coerce" the generator and associated semigroup to behave in a certain way.

In this thesis, we will be concerned with proving and applying coercive inequalities in the setting of Carnot groups, whose tangent space at every point is spanned by a family of degenerate and non-commuting vector fields of Hörmander type.

1.2 The Logarithmic Sobolev inequality

The Logarithmic Sobolev inequality was first considered in [58] and states that

$$\int_\Omega f^2 \log\left(\frac{f^2}{\int_\Omega f^2 d\mu}\right) d\mu \leq C \int_\Omega |\nabla f|^2 d\mu, \tag{1.2.1}$$

where (Ω, μ) is a measure space, ∇ is the gradient, and $C \geq 0$.

We note that the important Poincaré inequality

$$\int_\Omega \left(f - \int_\Omega f d\mu\right)^2 d\mu \leq \tilde{C} \int_\Omega |\nabla f|^2 d\mu, \tag{1.2.2}$$

can be obtained from the Logarithmic Sobolev inequality by simply replacing f by $1 + \epsilon f$ in (1.2.1) and letting $\epsilon \to 0$.

In [69], L. Gross proved that the Logarithmic Sobolev inequality (1.2.1) holds when μ is the Gaussian measure on $\Omega = \mathbb{R}^n$, ∇ is the standard Euclidean gradient, and $C = 2$ is the optimal constant.

He also showed in [69] that the Logarithmic Sobolev inequality is equivalent to hypercontractiv-

ity[1] . This is a smoothing property for the semigroup $P_t = e^{t\mathcal{L}}$ generated by a given non-positive self-adjoint operator $\mathcal{L}$ on $L^2(\mu)$ such that

$$(-\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

L. Gross observed that the Logarithmic Sobolev inequality is uniform with respect to the dimension n. Hence, unlike the Sobolev inequality, it could be extended to infinite dimensions [69]. In fact, the Logarithmic Sobolev inequality enjoys the tensorisation property: if two measures μ_1 and μ_2 satisfy (1.2.1), then $\mu_1 \otimes \mu_2$ satisfies (1.2.1) with the same constant C. This property could be extended to the infinite dimensional space $\Omega^{\mathbb{Z}^d}$, where the product measure $\mu_\Lambda := \mu^{\otimes\Lambda}$ satisfies the inequality (1.2.1) with the same constant C for all $\Lambda \subset \mathbb{Z}^d$. A more interesting situation is when the infinite dimensional measure is not a product measure. For instance, we are concerned with Gibbs measures that have useful applications in studying models from quantum field theory and statistical mechanics (see Section 1.6 of this chapter for more details).

Moreover, R. Holley and D. Stroock showed in [75] that the Logarithmic Sobolev inequality is stable under bounded perturbations. This means that if a measure μ satisfies (1.2.1), then for a strictly positive, normalised, and bounded density ρ, the perturbed measure $\tilde{\mu}$ by ρ (defined as $d\tilde{\mu} = \rho d\mu$) satisfies the Logarithmic Sobolev inequality with constant $\tilde{C} = Ce^{\sup(\log\rho) - \inf(\log\rho)}$.

A natural question was to extend the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemannian manifolds. In [10], D. Bakry and M. Emery prove it under a Curvature-Dimension condition, which highlights a deep link between coercive inequalities and the geometry of the underlying space (see Section 1.4.1 and Section 1.4.2 of this chapter). In this more general situation, the hypercontractivity of the associated semigroup and the inequality's uniformness in the dimension of space still holds true. In my thesis, I work in the setting of Carnot groups, where the Curvature-Dimension condition fails because there is no bound on the curvature of space (the Laplacian is of Hörmander type and has some degeneracy, see Section 1.4.3 of this chapter). To bypass this geometric condition, an alternative method was developed in [72]. It states that in order to obtain coercive inequalities, it is sufficient to prove the so-called "$U-$Bound" inequalities, introduced in Section 1.4.5 of this chapter.

[1] $P_t = e^{t\mathcal{L}}$ is hypercontractive if for $q(t) \leq 1 + (q-1)e^{2t}$ and $q > 1$, the inequality $||P_t f||_{q(t)} \leq ||f||_q$ holds for all $f \in L^q(\mu)$.

1.2.1 The $q-$Logarithmic Sobolev inequality

A generalisation of the inequality (1.2.1) is the following $q-$Logarithmic Sobolev inequality for $q \in (1, 2]$

$$\int_\Omega |f|^q \log\left(\frac{|f|^q}{\int_\Omega |f|^q d\mu}\right) d\mu \leq C \int_\Omega |\nabla f|^q d\mu, \tag{1.2.3}$$

where (Ω, μ) is a measure space, ∇ is the gradient, and $C \geq 0$. When $q = 2$, we get back the inequality (1.2.1).

The inequality (1.2.3) was first obtained in [23] for probability measures on $\Omega = \mathbb{R}^n$ with log-concave densities of the form $d\mu = \frac{e^{-|x|^p}}{Z} dx$, where $Z = \int_{\mathbb{R}^n} e^{-|x|^p} dx$ is the normalisation constant. In [24], S. Bobkov and B. Zegarliński showed that the $q-$Logarithmic Sobolev inequality (1.2.3) with $q \in (1, 2)$ sharpens the inequality with $q = 2$. They showed, under some weak conditions and for $q \in (1, 2)$, the ultracontractivity[2] of the associated semigroup P_t. This contractivity property is stronger than the hypercontractivity of the semigroup obtained for $q = 2$.

Moreover, S. Bobkov and B. Zegarliński proved in [24] that under a $q-$Logarithmic Sobolev inequality (1.2.3) with $q \in (1, 2)$, one gets a much stronger decay of tails estimate than the $q = 2$ case. They showed that if the inequality $|\nabla f| \leq M$ holds $\mu-$almost everywhere for $M > 0$, then we have

$$\mu\left\{\left|f - \int_\Omega f d\mu\right| \geq s\right\} \leq 2 \exp\left\{-\frac{(q-1)^p}{M^p C^{p-1}} s^p\right\},$$

for $s > 0$ and p the finite index conjugate of q i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Notice that, for large s, this decay of tails inequality is tighter for $q \in (1, 2)$ than for $q = 2$.

1.2.2 Applications to the Isoperimetric inequality

The Logarithmic Sobolev inequality plays a key role in the study of isoperimetric problems, which are concerned with controlling the volume of a given set in terms of its surface area.

Given a metric space (Ω, d) with a probability measure μ, the problem is to estimate the largest isoperimetric profile $\mathcal{I}_\mu : [0, 1] \to \mathbb{R}^+$ such that

$$\mathcal{I}_\mu(\mu(A)) \leq \mu^+(A), \tag{1.2.4}$$

for all measurable sets A where $\mu^+(A)$[3] is the $\mu-$surface area of A.

In [13], D. Bakry and M. Ledoux proved that, in a space with Ricci curvature bounded from

[2] $P_t = e^{t\mathcal{L}}$ is ultracontractive if $||P_t f||_\infty \leq ||f||_p$ holds for all $t \geq 0$ and $p \in [1, \infty)$.

[3] $\mu^+(A) := \lim_{\epsilon \to 0} \frac{\mu(A^\epsilon) - \mu(A)}{\epsilon}$, where $A^\epsilon := \{x \in \Omega :$ there exists y $\in$ A such that $d(x, y) < \epsilon\}$.

below, the Logarithmic Sobolev inequality is equivalent to the following isoperimetric inequality

$$C\mathcal{U}(\mu(A)) \leq \mu^{+}(A), \tag{1.2.5}$$

where $\mathcal{U} = \Phi' \circ \Phi^{-1}$ and Φ is the distribution function of the normal distribution on $\mathbb{R}$. The inequality (1.2.5) is important since it is connected to the isoperimetric inequality (1.2.4). In fact, for the case where μ is the Gaussian measure, $\mathcal{I}_{\mu}(\mu(A)) = \mathcal{U}(\mu(A))$ (see [30, 125]).

Another isoperimetric inequality, called Cheeger's inequality

$$C\min\{\mu(\mathrm{A}), 1-\mu(\mathrm{A})\} \leq \mu^{+}(A), \tag{1.2.6}$$

implies the Poincaré inequality [45]. This inequality (1.2.6) is of interest since the isoperimetric profile $\mathcal{I}_{\mu}(\mu(A)) = \min\{\mu(\mathrm{A}), 1-\mu(\mathrm{A})\}$ if μ is the exponential measure on the real line [126].

Later on, many works [18, 21, 22, 24, 107, 117] investigated the role of functional inequalities in isoperimetry.

1.3 $\Phi-$Entropy inequality

A generalisation of the Logarithmic Sobolev inequality is the following $\Phi-$Entropy inequality, which was first introduced in [47],

$$\mathrm{Ent}_{\mu}^{\Phi}(f) := \int_{\Omega} \Phi(f) d\mu - \Phi\left(\int_{\Omega} f d\mu\right) \leq C \int_{\Omega} \Phi''(f) f(-\mathcal{L}f) d\mu, \tag{1.3.1}$$

where (Ω, μ) is a measure space, $\mathcal{L}$ is a non-positive self-adjoint Markov generator on $L^2(\mu)$, and Φ is a convex function.

In particular, if $\Phi : [0, \infty) \to \mathbb{R}$ is given by $\Phi(x) = x \log x$, then (1.3.1) is the Logarithmic Sobolev inequality. Moreover, if $\Phi(x) = x^2$, then (1.3.1) is the Poincaré inequality.

Under the additional conditions that $\Phi'' > 0$ and that $-\dfrac{1}{\Phi''}$ is convex, [40, 41] showed that, just like the Logarithmic Sobolev inequality, the $\Phi-$Entropy inequality (1.3.1) is tensorisable and is stable under bounded perturbations.

Many works [6, 8, 27, 28, 115, 78] have investigated the $\Phi-$Entropy inequality. In particular, the works [6, 27] have used the $\Phi-$Entropy inequality to study the convergence to equilibrium of Fokker-Planck-type equations.

We remark that the $\Phi-$Entropy inequality is not always homogeneous, and it is of interest to

study a similar homogeneous $F-$Sobolev inequality

$$\int_\Omega f^2 F(f^2) d\mu \leq C_1 \int_\Omega f(-\mathcal{L}f) d\mu + C_2, \tag{1.3.2}$$

where C_1 and C_2 are positive constants, $\int_\Omega f^2 d\mu = 1$, and $F : [0,\infty) \to [0,\infty)$ with $\lim_{x\to\infty} F(x) = \infty$.

Contractivity properties of $F-$Sobolev inequality (1.3.2), as well as its relation to the $\Phi-$Entropy inequality (1.3.1), were studied in [16].

The $F-$Sobolev inequality appears in many works [64, 131, 133, 134], and in particular in [131], where F. Y. Wang studied its implications to the spectral properties of the generator $-\mathcal{L}$. It is known that the Poincaré inequality (1.2.2) is equivalent to the fact that the operator $-\mathcal{L}$ has a gap at the bottom of its spectrum. Having the stronger $F-$Sobolev inequality (1.3.2), one can conclude additional information about the nature of the spectrum. In fact, F. Y. Wang showed in [131] that the $F-$Sobolev inequality (1.3.2) implies that the essential spectrum of $-\mathcal{L}$ is empty.

1.4 From Bakry-Émery Curvature-Dimension Condition to $U-$Bounds

1.4.1 Curvature-Dimension Condition

In [10], D. Bakry and M. Emery extended the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemannian manifolds under an important Curvature-Dimension condition. Before stating this sufficient condition (yet not necessary condition- see [130, 132]) to obtain the Logarithmic Sobolev inequality in this more general setting, we give a brief summary of the framework (which is described in more detail in [4, 5, 8, 12]):

Consider a measure space $(X, \mathcal{M}, \mu)$, and let $\mathcal{E}$ be a local and symmetric Dirichlet form with dense domain $D(\mathcal{E}) \subset L^2(X,\mu)$. Let $(P_t)_{t\geq 0}$ be the associated Markov semigroup on $L^2(X,\mu)$ with its generator $\mathcal{L}_\mathcal{E}$[4]. Let $\mathcal{A}$ be a suitable algebra dense in the domain of $\mathcal{L}_\mathcal{E}$. The Carré du champ operator[5] is a symmetric and bilinear form defined in the following way

$$\Gamma(f,g) := \frac{1}{2}\left(\mathcal{L}_\mathcal{E}(fg) - f\mathcal{L}_\mathcal{E} g - g\mathcal{L}_\mathcal{E} f\right), \quad f,g \in \mathcal{A}. \tag{1.4.1}$$

[4]An operator $\mathcal{L}_\mathcal{E}$ on $L^2(X,\mu)$ is the generator of a Markov semigroup $(P_t)_{t\geq 0}$ if $\mathcal{L}_\mathcal{E} f = \lim_{t\to 0} \frac{1}{t}(P_t - I)f$. Such operators are characterised by the Hille-Yosida Theorem (see Theorem 1.7 of [70]).

[5]The Carré du champ operator is linked to the Dirichlet form $\mathcal{E}$ by $\mathcal{E}(f,g) = \int_X \Gamma(f,g) d\mu, \quad \forall\, f,g \in \mathcal{A}$.

Define the symmetric and bilinear form Γ_2 as

$$\Gamma_2(f,g) := \frac{1}{2}\left(\mathcal{L}_{\mathcal{E}}\Gamma(f,g) - \Gamma(f,\mathcal{L}_{\mathcal{E}}g) - \Gamma(\mathcal{L}_{\mathcal{E}}f,g)\right), \quad f,g \in \mathcal{A}. \tag{1.4.2}$$

Notice that $\Gamma_2(f,g)$ is just an iteration of $\Gamma(f,g)$ in the sense that we apply Γ to every product of two functions in (1.4.1). The important Curvature-Dimension condition, denoted as $CD(\rho,\infty)$, states

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f), \tag{1.4.3}$$

where $\rho \in \mathbb{R}$ (see [5, 7, 118] for more information about $CD(\rho,\infty)$). Bakry and Emery proved in [10] that, for $\rho > 0$, the Curvature-Dimension condition (1.4.3) implies the symmetric measure μ satisfies the following generalised Logarithmic Sobolev inequality with constant $\frac{2}{\rho}$:

$$\int_X f^2 \log \frac{f^2}{\int_X f^2 d\mu} d\mu \leq \frac{2}{\rho}\int_X f(-\mathcal{L}_{\mathcal{E}}f)d\mu, \quad \forall\, f \in \mathcal{A}, \tag{1.4.4}$$

where it is additionally assumed that $\mathcal{L}_{\mathcal{E}}$ is a diffusion operator [6].

Moreover, they showed that under the same conditions, we equivalently have pointwise gradient estimates for the Markov semigroup (see [8, 12] for relevant applications):

$$\Gamma(P_t f, P_t f) \leq e^{-2\rho t} P_t(\Gamma(f,f)), \quad \forall\, f \in \mathcal{A}. \tag{1.4.5}$$

1.4.2 Link between the Logarithmic Sobolev inequality and the geometry of the underlying space

So far, this Curvature-Dimension condition seems abstract, but the following example considered in [91] will highlight a deep link between the Logarithmic Sobolev inequality (1.4.4), and similar inequalities, and the geometry of the underlying space. This is also of special importance in this thesis since we will be working in the setting of Carnot groups, where we will show that the Curvature-Dimension condition (1.4.3) fails because of the geometry of the space. Hence, one of the main purposes of the thesis is to bypass this condition and prove coercive inequalities despite having no bound on the curvature of space. The example in [91] goes as follows:

Let $(\mathbb{M}^d, \mathrm{g})$ be a smooth Riemannian manifold. Let $U : \mathbb{M}^d \to \mathbb{R}$ be a smooth potential, and

[6] $\mathcal{L}_{\mathcal{E}}$ is a diffusion operator means that $\mathcal{L}_{\mathcal{E}}\phi(f) = \phi'(f)\mathcal{L}_{\mathcal{E}}f + \phi''(f)\Gamma(f,f)$ for all $f \in \mathcal{A}$ and $\phi \in C_b^2(\mathbb{R})$.

consider the measure $\mu := e^{-U}\mathrm{Vol_g}$. Take $\mathcal{A} = C_c^\infty(\mathbb{M}^d)$, and for the standard gradient ∇, choose

$$\mathcal{E}(f,g) = \int_{\mathbb{M}^d} \langle \nabla f, \nabla g \rangle_{\mathrm{g}} d\mu, \quad \forall\, f, g \in \mathcal{A}.$$

Hence,

$$\Gamma(f,f) = |\nabla f|^2_{\mathrm{g}},$$

and $\mathcal{L}_\mathcal{E} = \Delta_{\mathrm{g}} - \langle \nabla U, \nabla \cdot \rangle_{\mathrm{g}}$, where Δ_{g} is the standard Laplace-Beltrami operator on $\mathbb{M}$. Using the Carré du champ operator Γ, the Riemannian distance $\mathrm{d_g}$ on $\mathbb{M}^d$ is recovered[7]. This indicates that Γ carries the metric information of $\mathbb{M}^d$. Moreover, for $f \in \mathcal{A}$,

$$\Gamma_2(f,f) = \frac{1}{2}\left(\mathcal{L}_\mathcal{E}\Gamma(f,f) - 2\Gamma(f, \mathcal{L}_\mathcal{E} f)\right)$$

$$= ||\mathrm{Hess_g} f||^2_{\mathrm{g}} + \mathrm{Ric_g}(\nabla f, \nabla f) + \langle \mathrm{Hess_g} U\ \nabla f, \nabla f \rangle_{\mathrm{g}}, \tag{1.4.6}$$

where the last equality is due to Bochner's formula. Provided that the quadratic form associated to the tensor $\mathrm{Ric_g} + \mathrm{Hess_g} U$ is bounded below by $\rho \mathrm{g} + \dfrac{1}{N-d}\nabla U \otimes \nabla U$ (where $\rho \in \mathbb{R}$, and $N > d$), the following $N-$Curvature-Dimension condition, $CD(\rho, N)$, holds true:

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{N}(\mathcal{L}_\mathcal{E} f)^2, \quad \forall\, f \in \mathcal{A}.$$

Note that in order to obtain the Logarithmic Sobolev inequality and the gradient estimates for the Markov semigroup, it is enough to consider the case $N = \infty$ (i.e. $CD(\rho, \infty)$ (1.4.3)). To satisfy this condition, it is required that the quadratic form associated to the tensor $\mathrm{Ric_g} + \mathrm{Hess_g} U$ is bounded below by $\rho \mathrm{g}$. The Logarithmic Sobolev inequality (1.4.4) , with constant $\dfrac{2}{\rho}$, in this case is

$$\int_{\mathbb{M}^d} f^2 \log \frac{f^2}{\int_{\mathbb{M}^d} f^2 d\mu} d\mu \leq \frac{2}{\rho} \int_{\mathbb{M}^d} |\nabla f|^2_{\mathrm{g}} d\mu, \quad \forall\, f \in \mathcal{A}.$$

The gradient estimates for the Markov semigroup (1.4.5) is now simply a commutation relation between the semigroup P_t and the gradient ∇ :

$$|\nabla P_t f| \leq e^{-\rho t} P_t |\nabla f| \quad \forall\, f \in \mathcal{A}.$$

Looking at the example described above, observe that in order for the Curvature-Dimension condition to be satisfied, some control over the curvature of space is needed.

[7]The Riemannian distance $\mathrm{d_g}$ on $\mathbb{M}^d$ is given by the formula $d_{\mathrm{g}}(x,y) = \sup\{\phi(y) - \phi(x) : \phi \in \mathcal{A},\ \Gamma(\phi,\phi) \leq 1\}$, for all $x, y \in \mathbb{M}^d$.

For instance, if $\mathbb{M}^d = \mathbb{R}^d$, the curvature of space is zero. Hence, to obtain the Logarithmic Sobolev inequality, notice from (1.4.6) that the only requirement for the Curvature-Dimension condition to hold is: the quadratic form associated to the tensor $\mathrm{Hess}_g U$ needs to be bounded from below by a positive constant. This is achieved when the measure μ is log-concave, and it encompasses the case of Gross's Gaussian measure on $\mathbb{R}^d$.

1.4.3 The Curvature-Dimension Condition fails for Carnot groups

In the setting of Carnot groups, there is no lower bound on the curvature of space since the Laplacian is of Hörmander type and has some degeneracy. To make this point clearer, we recap the considerations done in [9], where the authors start with the simplest example of the Carnot group: the Heisenberg group $\mathbb{H}$. It is a group isomorphic to $\mathbb{R}^3$, where the group operation is defined by

$$(x, y, z) \circ (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y) \right),$$

with $(x, y, z), (x', y', z') \in \mathbb{H}$. The left-invariant vector fields on the Heisenberg group are given by

$$X = \partial_x - \frac{y}{2}\partial_z$$

and

$$Y = \partial_y + \frac{x}{2}\partial_z.$$

The sub-gradient in this setting is defined as $\nabla_{\mathbb{H}} := (X, Y)$, and the self-adjoint operator for the Haar measure on $\mathbb{H}$ (equivalent to the Lebesgue measure on $\mathbb{R}^3$) is:

$$\mathcal{L} = X^2 + Y^2 = \partial_x^2 + \partial_y^2 + \frac{1}{4}(x^2 + y^2)\partial_z^2 + x\partial_{y,z}^2 - y\partial_{x,z}^2.$$

$\mathcal{L}$ is not elliptic since the Hessian associated to it is degenerate. However, $\mathcal{L}$ is hypoelliptic in the Hörmander sense, i.e. it satisfies Hörmander's condition: X, Y, and the commutator of X and Y, $[X, Y]$, span the tangent space of $\mathbb{H}$ at every point. In fact,

$$Z := [X, Y] = \partial_z, \quad \text{and} \quad [X, Z] = [Y, Z] = 0.$$

According to [121], if there is a uniqueness of solution in the space of square integrable functions for the following Cauchy problem

$$\begin{cases} \partial_t u = \mathcal{L}u \\ u_{|t=0} = f, \end{cases}$$

then the solution of the heat equation will be given by $u = P_t f$. Hypoellipticity means that this heat semigroup $(P_t)_{t\geq 0} = (e^{t\mathcal{L}})_{t\geq 0}$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^3$. In other words,

$$P_t f(x) = \int f(xy) p_t(y) dy,$$

where the heat kernel p_t is a smooth function. The computation of $\Gamma(f,f)$ (1.4.1) and $\Gamma_2(f,f)$ (1.4.2) is as follows:

$$\Gamma(f,f) = |\nabla_{\mathbb{H}} f|^2 = X(f)^2 + Y(f)^2,$$

and

$$\Gamma_2(f,f) = X^2(f)^2 + Y^2(f)^2 + \frac{1}{2}(XY+YX)(f)^2 + \frac{1}{2}(Zf)^2 + 2\Big(XZ(f)Y(f) - YZ(f)X(f)\Big).$$

The Curvature-Dimension condition (1.4.3) fails due the presence of $XZ(f)$ and $YZ(f)$ in the above equation. In other words, the Bakry-Emery Ricci tensor is $-\infty$ everywhere. In fact, the Ricci tensor associated to the elliptic operator $X^2 + Y^2 + \epsilon Z^2$ is

$$\begin{pmatrix} -\frac{1}{2\epsilon} & 0 & 0 \\ 0 & -\frac{1}{2\epsilon} & 0 \\ 0 & 0 & \frac{1}{2\epsilon^2} \end{pmatrix}.$$

To get a lower bound on Γ_2, a lower bound of the Ricci tensor associated to the hypoelliptic operator $\mathcal{L} = X^2 + Y^2$ is needed. Therefore, taking the limit $\epsilon \to 0$, observe that the lower bound of the Ricci tensor of $\mathcal{L}$ is $-\infty$ everywhere. In other words, there is no constant ρ such that $CD(\rho,\infty)$ holds true.

1.4.4 Important estimates, despite no Curvature-Dimension Condition on Carnot groups

Despite having the Curvature-Dimension condition fail in the setting of the Heisenberg group, other methods have been successful in proving gradient estimates and coercive inequalities. Using the group structure of the Heisenberg group and probabilistic methods from Malliavin calculus,

B. Driver and T. Melcher prove in [55] a commutation relation between the semigroup P_t and the sub-gradient $\nabla_{\mathbb{H}}$:

$$|\nabla_{\mathbb{H}} P_t f|^p(h) \leq C_p P_t |\nabla_{\mathbb{H}} f|^p(h), \tag{1.4.7}$$

for $h \in \mathbb{H}$, all smooth functions f, and $p > 1$ (notice that the important case $p = 1$ is not covered by this approach). Here, C_p is the smallest finite positive constant. T. Melcher extended this result [106] to the setting of the Carnot group $\mathbb{G}$:

$$|\nabla_{\mathbb{G}} P_t f|^p(h) \leq C_p P_t |\nabla_{\mathbb{G}} f|^p(h),$$

where now the sub-gradient is $\nabla_{\mathbb{G}} := (X_1, ..., X_k)$, such that $\{X_i\}_{i=1}^k$ are the generators of the Lie algebra of $\mathbb{G}$. The above gradient bounds directly imply a Poincaré inequality for P_t :

$$P_t(f^2) - (P_t f)^2 = 2 \int_0^t P_s(|\nabla_{\mathbb{G}} P_{t-s} f|^2) ds \leq 2t C_2 P_t |\nabla_{\mathbb{G}} f|^2,$$

as well as a reverse Poincaré inequality, which is useful for regularisation properties. A more important estimate is for the case when $p = 1$ in (1.4.7) since it would imply, in addition to the Poincaré inequality, Bobkov-type isoperimetric inequalities ([9, 13, 18]), Cheeger type inequalities ([9, 45, 101]), and the following Logarithmic Sobolev inequality for P_t :

$$P_t \left(f^2 \log \frac{f^2}{P_t f^2} \right) \leq C_1^2 t P_t |\nabla_{\mathbb{H}} f|^2.$$

We note that the approach by B. Driver and T. Melcher does not include this important case $p = 1$. However, H. Q. Li successfully proves that in [91] using a different approach based on heat kernel estimates. More precisely, the key ingredients he uses are the following necessary estimates on the heat kernel p_t for $t = 1$:

$$\frac{1}{L_1} \frac{e^{-\frac{d^2(x)}{4}}}{\sqrt{1 + |x| d(x)}} \leq p_1(x) \leq L_1 \frac{e^{-\frac{d^2(x)}{4}}}{\sqrt{1 + |x| d(x)}},$$

and

$$|\nabla_{\mathbb{H}} p_1(x)| \leq L_2 d(x) p_1(x),$$

for some constants $L_1 > 1$, $L_2 > 1$, $x \in \mathbb{H}$, and $d(x)$ the Carnot-Carathéodory distance, which is a natural sub-Riemannian distance function that will be defined in Section 1.7.1 of the Introduction. Using those estimates, the authors of [9, 80] simplified H. Q. Li's proof. Moreover, N. Eldredge extended the result for $p = 1$ to the setting of the more general Heisenberg-type groups [57]. His

proof also relied on those precise heat kernel estimates that he provided in this setting [56].

It is worth mentioning that in the setting of nilpotent Lie groups, heat kernel estimates were studied in [46, 65, 93, 95, 96, 97, 98, 129] to get a variety of coercive inequalities. We also note the following works [43, 44] that investigated coercive inequalities on Carnot groups.

1.4.5 The $U-$Bound Method

In [72], W. Hebisch and B. Zegarliński develop a method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space. Working on a general metric space equipped with non-commuting vector fields $\{X_1, \ldots, X_n\}$, their method is based on $U-$bounds, where U stands for the potential. Those are inequalities of the form:

$$\int f^q \mathcal{U}(d)\, d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu \tag{1.4.8}$$

where $d\mu = \dfrac{e^{-U(d)}}{Z} d\lambda$ is a probability measure, $U(d)$ and $\mathcal{U}(d)$ are functions having a suitable growth at infinity, Z is the normalisation constant, λ is a natural measure like the Lebesgue measure for instance (which is the Haar measure for nilpotent Lie groups), d is a metric related to the gradient $\nabla = (X_1, \ldots, X_n)$, and $q \in (1, \infty)$.

A detailed explanation of the main results of [72] is given in Section 2.1.2 of Chapter 2. The idea in broad terms is: to prove coercive inequalities like the Poincaré and the Logarithmic-Sobolev inequalities, it boils down to proving the sufficient $U-$bound inequality of the form (1.4.8). In this thesis, we prove the $U-$Bound inequality for measures as a function of the fundamental solution of the sub-Laplacian in the setting of step-two Carnot groups (Chapter 3) and the Anisotropic Heisenberg group (Chapter 4). The method that we develop in this thesis could be also adapted to other settings as well, such as the Filiform groups (considered in [42] and references therein).

1.5 Relation between the Logarithmic-Sobolev inequality and the Talagrand's Transportation Cost Inequality

In the setting of manifolds under D. Bakry and M. Emery's Curvature-Dimension condition, F. Otto and C. Villani showed in their celebrated paper [110] that the Logarithmic Sobolev inequality implies the Talagrand transportation cost inequality. Their proof relied on PDE's methods and Otto calculus in Wasserstein space [109]. The Talagrand transportation cost inequality was first introduced in 1996 by M. Talagrand [127]. His inequality bounds the optimal transport cost

in the sense of Monge-Kantorovich, $T_w(\mu,\nu)$, by the relative entropy functional, $\mathrm{Ent}_\nu\left(\frac{d\mu}{d\nu}\right) := \int \frac{d\mu}{d\nu}\log\left(\frac{d\mu}{d\nu}\right)d\nu$, in the following way:

$$T_w(\mu,\nu) \leq 2\mathrm{Ent}_\nu\left(\frac{d\mu}{d\nu}\right), \tag{1.5.1}$$

where μ a measure on $\mathbb{R}^N$ absolutely continuous with respect to the Gaussian measure ν, and $w(x,y) = \sum_{i=1}^{N}(x_i - y_i)^2$ is the cost of moving a unit mass from position x to position y. The transportation cost, $T_w(\mu,\nu)$, measures how much effort is required to transport a mass distributed according to μ to a mass distributed according to ν, relatively to the cost function w :

$$T_w(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^N\times\mathbb{R}^N} w(x,y)d\pi(x,y), \tag{1.5.2}$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{R}^N\times\mathbb{R}^N$ with μ the first marginal and ν the second marginal[8]. M. Talagrand showed how the Talagrand transportation cost inequality (4.1.6) implies the concentration of measure phenomenon, and his approach was motivated by the work of K. Marton [104, 105]. Later on, many works [5, 19, 20, 66, 67, 89, 90, 110] studied links between the Logarithmic Sobolev inequality, the Talagrand inequality, and the concentration of measure phenomenon, which is one of the main tools in probability theory, statistical mechanics, quantum information theory, stochastic dynamics, etc.

We are interested in showing that the Logarithmic Sobolev inequality implies the Talagrand inequality in the setting of Carnot groups. As mentioned in Section 1.4.3 of this chapter, D. Bakry and M. Emery's Curvature-Dimension condition does not hold, so we cannot apply the method by F. Otto and C. Villani [110].

In Chapter 2, in the setting of Carnot groups, we are first interested in proving the $q-$Logarithmic Sobolev inequality (see [23, 24] and Section 1.2.1 of the Introduction) for $q\in(1,\infty)$,

$$\int f^q \log\frac{f^q}{\int f^q d\mu}d\mu \leq c\int |\nabla f|^q d\mu, \tag{1.5.3}$$

for measures of the form $d\mu = \frac{e^{-U(d)}}{Z}d\lambda$. $U(d)$ is a function having a suitable growth at infinity, λ is a natural measure like the Lebesgue measure for instance, Z is the normalisation constant, and d is a metric related to the sub-gradient $\nabla = (X_1,\ldots,X_n)$, where $X_i's$ are the generators of the group's Lie algebra.

In [72], W. Hebisch and B. Zegarliński proved the $q-$Logarithmic Sobolev inequality for the

[8]For all bounded functions f and g on $\mathbb{R}^N$, $\int_{\mathbb{R}^N\times\mathbb{R}^N}(f(\omega)+g(\omega'))d\pi(\omega,\omega') = \int_{\mathbb{R}^N} f d\mu + \int_{\mathbb{R}^N} g d\nu$.

measure $d\mu = \frac{e^{-\alpha d^p}}{Z} d\lambda$, where p is the finite index conjugate of q and $\alpha > 0$. In Chapter 2, we enlarge the class of measures that satisfy the $q-$logarithmic Sobolev inequality to include measures of the form $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$, under some growth conditions for $U(d)$, and get examples such as $U(d) = (d+1)^p \log(d+1)$ and $U(d) = \sinh(d)$. The metric d can be thought of as a natural sub-Riemannian distance, like the Carnot-Carathéodory distance, introduced in Section 1.7.1 of this chapter. We note that those results do not apply for measures as a function of homogeneous norm "N", like the ones discussed in Chapter 3 and Chapter 4. In fact, it was proved in [72] that it is impossible to have a Logarithmic Sobolev inequality for measures as a function of homogeneous norms. Even though d and "N" are equivalent, their behavior around the "$z-$axis" are completely different, and so the approach to prove coercive inequalities is completely different. This point will be explained in more detail in Section 1.7.2 of the Introduction and in Section 2.1.1 of Chapter 2.

The goal is to apply the $q-$Logarithmic Sobolev inequality to get hypercontractivity[9] and to obtain the $p-$Talagrand inequality on (X, d, μ) with a constant K :

$$W_p(\mu, \nu)^p \leq \frac{1}{K} \mathrm{Ent}_\nu \left(\frac{d\mu}{d\nu} \right), \tag{1.5.4}$$

with p finite index conjugate of q, and μ is a probability measure absolutely continuous with respect to ν. (In Chapter 2, we prove that for $X = \mathbb{G}$, the Carnot group, and d the Carnot-Carathéodory distance.)

The $p-$Wasserstein distance (a generalisation of the above Talagrand's transportation cost (1.5.2)) between two probability measures on X is defined as

$$W_p(\mu, \nu)^p = \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x, y)^p d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the set of probability measures on $X \times X$ with μ the first marginal and ν the second marginal. In contrast with the definition (1.5.2) , the cost of moving a unit mass from position x to position y is now measured by $d(x, y)$. We note that for $p = 2$, (4.1.6) is a special case of (4.1.8).

The $p-$Talagrand inequality was introduced by Z.M. Balogh et al. in [15] who proved that it was implied by the $q-$Logarithmic Sobolev inequality for p the finite index conjugate of q; their work generalises the result by J. Lott and C. Villani [94] who proved the implication for the quadratic case $p = q = 2$. J. Lott and C. Villani [94] used the Hamilton-Jacobi infimum convolution operator under the assumption where the space (X, d, μ) supports local Poincaré inequality and the

[9] A semigroup, Q_t, is hypercontractive if $||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a$, for every bounded measurable function f on $\mathbb{R}^n$, $t \geq 0$, $a \in \mathbb{R}$, and $||.||_p$ the L^p-norm with respect to μ.

measure μ is a doubling measure [10]. This idea of using the Hamilton-Jacobi infimum convolution operator was first used by S. Bobkov et al. in [19] who explored the equivalence between the Logarithmic Sobolev inequality with a constant ρ in $\mathbb{R}^n$ and hypercontractivity of the quadratic Hamilton-Jacobi semigroup.

In the setting of the Heisenberg group $\mathbb{H}$, Z.M. Balogh et al. in [15] used W. Hebisch and B. Zegarliński's [72] $q-$Logarithmic Sobolev inequality for the measure $d\mu = \frac{e^{-\alpha d^p}}{Z} d\lambda$, where d is the Carnot-Carathéodory distance, to obtain the $p-$Talagrand inequality. To prove the above implication, Z.M. Balogh et al. observed that by Pansu's differentiability theorem [68], for Lipschitz continuous functions $f : \mathbb{H} \to \mathbb{R}$, the norm of the sub-Riemannian gradient $|\nabla f(x)| = \left(\sum_{i=1}^{N} |X_i f|^2 \right)^{\frac{1}{2}}$ used in [72] coincides with the metric sub-gradient in [15] for μ_p almost every x for which $|\nabla f(x)| > 0$.

In Chapter 2, we adapt Z.M. Balogh et al.'s Theorem 4.1 [15] to get the $p-$Talagrand inequality, and we adapt S. Bobkov et al.'s Theorem 2.1 in [19] to get hypercontractivity. However, instead, we use the Hamilton-Jacobi equation in the setting of Carnot groups by F. Dragoni [54]. This allows us to use the sub-gradient, and to show that, in the setting of Carnot groups, the Logarithmic Sobolev inequality for various probability measures implies the Talagrand inequality and hypercontractivity. We note that, in contrast with Z.M. Balogh et al.'s Theorem 4.1 [15], we do not require the probability measures to be doubling in the proofs.

1.6 Coercive Inequalities for Infinite-Dimensional Gibbs Measures

Studying the dynamics of particles on an infinitely large integer lattice has been a subject of interest since it models real-world situations such as phase transitions, neural networks, and spread of infection [92]. On those systems, we have dynamics and PDEs that we would like to study. Hence, it is essential to use some estimates i.e. coercive inequalities (like the Poincaré and Logarithmic Sobolev inequalities) that the Gibbs measures satisfy.

The studied system is an infinitely large integer lattice, and each point is a particle assigned with a random variable (see [25, 26, 60, 73, 74, 111, 139, 140, 141, 143, 144] for various setups). The configuration space considered is given by $\mathbf{M}^{\mathbb{Z}^d}$, where $\mathbf{M}$ is a manifold and $\mathbb{Z}^d$ is a $d-$dimensional integer lattice. We restrict the dynamics inside any compact setting $\Lambda \subset \mathbb{Z}^d$, with the potential, U_Λ^ω, depending on the phase, the interaction between neighbors, the boundary conditions $\omega \in \mathbf{M}^{\Lambda^c}$, and some parameters like the temperature and the magnetic field. The local Gibbs measure, E_Λ^ω,

[10] μ is doubling if the measure of any open ball is positive and finite and there exists a constant $c_d \geq 1$ such that for all $x \in X$ and $r > 0$, $\mu(B(x, 2r)) \leq c_d \mu(B(x, r))$.

associated to the potential on Λ, has a probability density that shows how likely the restricted subsystem in Λ will have an energy U^{ω}_{Λ}. Typically, it is the conditional expectation $E^{\omega}_{\Lambda} := \frac{1}{Z^{\omega}_{\Lambda}} e^{-U^{\omega}_{\Lambda}} dx_{\Lambda}$, where Z^{ω}_{Λ} is the normalisation constant. The local Gibbs measures represent the evolution of an interacting particle system converging to a global Gibbs state ν on Ω. The existence of the global Gibbs measure can be shown under some mild hypotheses and satisfies the Dobrushin-Landford-Ruelle (DLR) condition [52, 87]:

$$\nu E^{\omega}_{\Lambda} = \nu, \tag{1.6.1}$$

for all compact $\Lambda \subset \mathbb{Z}^d$.

M. Aizenmann and R. Holley in [3] studied the question of ergodicity of the system and the rate of convergence to equilibrium. In fact, they proved exponential rate of convergence in the uniform norm for stochastic Ising models with attractive interactions and under conditions which are stronger than Dobrushin's uniqueness condition for (1.6.1) in [52]. M. Aizenmann and R. Holley's conditions, however, were too strong to study other meaningful models, so another approach was needed. In order to deduce uniform ergodicity, this new approach boils down to proving the Logarithmic Sobolev inequality for the Gibbs measure. This method was first done by R. Holley and D. Stroock who used the Logarithmic Sobolev inequality to study the rate of convergence to equilibrium of Ising models with compact spin states ([75, 76]). In addition, D. Stroock and B. Zegarliński showed, for finite range lattice gases with a finite spin space, the equivalence of the Logarithmic Sobolev inequality to the Dobrushin-Shlosman mixing condition: which implies the uniqueness of the Gibbs state and its analytic dependence on the perturbation parameter [123]. Those works motivated to study the question of what classes of non-product Gibbs measures in infinite dimensions satisfy the Logarithmic Sobolev inequality and in which settings are they satisfied.

In [39], E. Carlen and D. Stroock were the first to apply Bakry-Emery Curvature-Dimension condition ([10]) to prove the Logarithmic Sobolev inequality for non-product and non-Gaussian measures in an infinite dimensional setting. They were able to obtain it on $\mathbf{M} = \mathbb{S}^d$, a unit sphere with dimension $d \geq 2$, at high temperatures. Afterwards, B. Zegarliński showed in [143], by using the structure of the Gibbs measures, a Sobolev inequality for short-range interactions on $\mathbf{M} = \mathbb{S}$. The theory was later extended to spin systems on compact spaces [99, 102, 103, 123, 124, 142] and non-compact systems with unbounded interactions [25, 26, 60, 73, 74, 111, 139, 140, 141, 144].

The primary interest in Chapter 5 of the thesis, is to prove the existence of coercive inequalities in infinite dimensions for different measures in the setting of a product space of nilpotent Lie groups, whose tangent space at every point is spanned by a family of degenerate and non-commuting vector

fields $\{X_i, i \in \mathcal{R}, 1 < |\mathcal{R}| < \infty\}$, where $|\mathcal{R}|$ is the cardinality of the index set $\mathcal{R}$. The standard Euclidean gradient and Laplacian are now replaced by the sub-gradient and sub-Laplacian which depend on $X_i's$, the generators of the corresponding Lie algebra. Before passing to an infinite dimensional setting, proving single-site coercive inequalities is needed. However, in the setting of nilpotent Lie groups, since the sub-Laplacian is of Hörmander type and has some degeneracy, D. Bakry and M. Emery's Curvature-Dimension condition in [10] will no longer hold true (see Section 1.4.3 of the Introduction for a more detailed explanation). In [72], a "$U-$Bound" method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space was developed. In Chapter 3 and Chapter 4, we prove the $U-$Bound inequalities for measures as a function of various homogeneous norms in the setting of step-two Carnot groups and the Anisotropic Heisenberg group. Those $U-$Bound inequalities turn out to play a key role in the passage to infinite dimensions for the settings we consider.

In [100], P. Ługiewicz and B. Zegarliński show, under some conditions, infinite dimensional Logarithmic Sobolev inequality in the case where $\mathbf{M}$ is a compact manifold without boundary equipped with the above X_i's. J. Inglis and I. Papageorgiou ([82]) considered the case where $\mathbf{M}$ is a non-compact spin space (mainly $\mathbf{M} = \mathbb{H}$, the Heisenberg group), and for at most quadratic interaction potentials. For interactions that are stronger than quadratic, see references [111, 81].

In Chapter 5 of the thesis, one aim is to show that certain infinite dimensional Gibbs measures with unbounded interaction potentials, greater than quadratic, as a function of homogeneous norms (such as the Kaplan norm for the Heisenberg group and the Anisotropic Heisenberg group) on an infinite product of Carnot groups satisfy the Poincaré inequality. So far, the passage to infinite dimensions in the setting of Nilpotent Lie groups required the condition $|\nabla d| \geq c$ outside the unit ball ([81, 82, 86]), which is not true for homogeneous norms. The known methods use the single site Poincaré inequality to pass to the global Poincaré inequality. To get results for measures as function of homogeneous norms, the key is to use the $U-$Bound inequalities (Chapter 3 and Chapter 4) to get a weak $U-$bound. Coupled with a weak single-site Poincaré inequality, we are able to pass to the infinite dimensional setting. Our result strongly generalises known results [81, 82, 86, 111] and provides a vast scope of applications in the setting of Carnot groups.

1.7 Setting

1.7.1 The Carnot Group and the Carnot-Carathéodory distance

Carnot groups are geodesic metric spaces that appear in many mathematical contexts like harmonic analysis in the study of hypoelliptic differential operators ([38, 121]) and in geometric measure theory (see extensive reference list in the survey paper [88]). The following series of definitions are from [29]:

Definition 1. We say that a Lie group on $\mathbb{R}^N$, $\mathbb{G} =(\mathbb{R}^N, \circ)$ is a (homogeneous) Carnot group if the following properties hold:

(C.1) $\mathbb{R}^N$ can be split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_r}$, and the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, ..., \lambda^r x^{(r)}), \qquad x^{(i)} \in \mathbb{R}^{N_i},$$

is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$. Then $(\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous Lie group on $\mathbb{R}^N$. Moreover, the following condition holds:

(C.2) If N_1 is as above, let $Z_1, ..., Z_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $Z_j(0) = \partial/\partial x_j|_0$ for $j = 1, ..., N_1$. Then

$$\text{rank}(\text{Lie}\{Z_1, ..., Z_{N_1}\}(x)) = N \qquad \forall x \in \mathbb{R}^N.$$

If (C.1) and (C.2) are satisfied, we shall say that the triple $G = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous Carnot group. We also say that $\mathbb{G}$ has step r and N_1 generators. The homogeneous dimension of $\mathbb{G}$ is denoted by $Q := \sum_{i=1}^{r} iN_i$. The vector fields $Z_1, ..., Z_{N_1}$ will be called the (Jacobian) generators of $\mathbb{G}$.

The importance of the differential operators $Z_1, ..., Z_{N_1}$ is that they allow us to define a differential calculus analogous to that on $\mathbb{R}^N$, while taking into account the group structure of $\mathbb{G}$. For instance, we may define a metric on $\mathbb{G}$ by considering horizontal curves whose derivative belongs to the sub-bundle spanned by $Z_1, ..., Z_{N_1}$:

Definition 2. We say that a Lipschitz curve $\gamma : [0, 1] \to \mathbb{G}$ is horizontal if there exist measurable functions $a_1, \ldots, a_{N_1} : [0, 1] \to \mathbb{R}$ such that

$$\gamma'(t) = \sum_{i=1}^{N_1} a_i(t) Z_i(\gamma(t))$$

for almost all $t \in [0,1]$ i.e. $\gamma'(t) \in \text{Span}\{Z_1(\gamma(t)), \ldots, Z_{N_1}(\gamma(t))\}$ almost everywhere. For such a horizontal curve γ, we define the length of γ to be

$$|\gamma| := \int_0^1 \left(\sum_{i=1}^{N_1} a_i^2(t) \right)^{\frac{1}{2}} dt.$$

Carnot groups portray fractal behavior since the only curves of finite length on such metric spaces are the horizontal curves. It turns out on $\mathbb{G}$ we can always connect two points $x, y \in \mathbb{G}$ by a horizontal curve using W. Chow's theorem (see Chapter 19 of [29]) . We may then define the Carnot-Carathéodory distance d as the minimum length of a curve which connects two points in unit time, and which can "only travel in the directions" of $Z_1, ..., Z_{N_1}$:

Definition 3. The Carnot-Carathéodory distance (alternatively called the control distance) between two points x and y is defined by

$$d(x,y) = \inf\{|\gamma| \text{ such that } \gamma : [0,1] \to \mathbb{G} \text{ is horizontal and } \gamma(0) = x,\ \gamma(1) = y\}.$$

The Carnot-Carathéodory distance endows $\mathbb{G}$ with geodesics that are different from the geodesics of $\mathbb{R}^N$. In particular, we consider the simplest example of the Carnot group: the Heisenberg group $\mathbb{H}$ which isomorphic to $\mathbb{R}^3$ (see the definition in Section 1.4.3). The geodesics of $\mathbb{R}^3$ are straight lines, yet the geodesics of $\mathbb{H}$ with respect to d are helices [71].

Moreover, the vector fields $Z_1, ..., Z_{N_1}$ allow us to define an analogue of the Euclidean gradient $\nabla_{euc} = (\partial_{x_1}, ..., \partial_{x_N})$ and the Euclidean Laplacian $\Delta_{euc} = \sum_{i=1}^{N} \partial_{x_i}^2$ in the following way:

Definition 4. The vector valued operator $\nabla := (Z_1, Z_2, ..., Z_{N_1})$ is called the sub-gradient on $\mathbb{G}$, and $\Delta = \sum_{i=1}^{N_1} Z_i^2$ is called the sub-Laplacian on $\mathbb{G}$.

It can be shown (proof found in [79]) that d is associated to the sub-gradient through:

$$|\nabla f(x)| = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Moreover, the metric d also induces a homogeneous norm on $\mathbb{G}$ by defining $d(x)$ as the distance between $x \in \mathbb{G}$ and the origin

$$d(x) \coloneqq d(x, 0).$$

In fact, d satisfies the triangle inequality defined with the group law i.e.

$$d(x \circ y) \leq d(x) + d(y),$$

and d *is* positively homogeneous with respect to the dilations in (C.1) i.e.

$$d(\delta_\lambda(x)) = \lambda d(x).$$

The Carnot Carathéodory distance also shares an important property with the Euclidean norm: it satisfies the eikonal equation away from the centre $\{(0, x^{(2)}, ..., x^{(r)}) : (x^{(2)}, ..., x^{(r)}) \in \mathbb{R}^{N_2} \times ... \times \mathbb{R}^{N_r}\} \subset \mathbb{G}$.

The metric d satisfies

$$|\nabla d(x^{(1)}, ..., x^{(r)})| = 1 \tag{1.7.1}$$

for all $x^{(1)} \neq 0$ (see [108] and Proposition 3.2.16 of [79]). Moreover, there exists a constant $K \in (0, \infty)$ such that

$$\Delta d \leq \frac{K}{d} \tag{1.7.2}$$

(see [24] and Proposition 3.2.17 of [79]).

The Carnot-Carathéodory distance thus seems to be the most natural choice of homogeneous norm on $\mathbb{G}$ that is analogous to the Euclidean norm. However, it differs from the Euclidean norm; for example, the isoperimetric problem in the Heisenberg group $\mathbb{H}$ is not optimised by $d-$balls [108].

Another natural homogeneous norm is the Kaplan norm, which is expressed as the fundamental solution of the sub-Laplacian raised to a certain exponent. This homogeneous norm will be introduced in the next section, and its difference from the Carnot-Carathéodory distance will be highlighted.

1.7.2 The Step-Two Carnot Group, the Heisenberg-type group, and the Kaplan Norm

The step-two Carnot group $\mathbb{G}$ (i.e. when $r = 2$ in the Definition 1 in Section 1.7.1) is a group isomorphic to $\mathbb{R}^{n+m}$ with the group law

$$(x,z)\circ(x',z') = \left(x_i + x'_i,\ z_j + z'_j + \frac{1}{2}\langle \Lambda^{(j)}x, x'\rangle\right)_{i=1,..,n;j=1,..,m}$$

for $x, x' \in \mathbb{R}^n, z, z' \in \mathbb{R}^m$, where the matrices $\Lambda^{(j)}$ are $n \times n$ skew-symmetric and linearly independent and $\langle .,. \rangle$ stands for the inner product on $\mathbb{R}^n$. One can verify that $(\mathbb{R}^{n+m}, \circ)$ is a Lie group whose identity is the origin and where the inverse is given by $(x,z)^{-1} = -(x,z)$.

The dilation

$$\delta_\lambda : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}, \qquad \delta_\lambda(x,z) = (\lambda x, \lambda^2 z)$$

is an automorphism of $(\mathbb{R}^{n+m}, \circ)$ for any $\lambda > 0$. Then, $\mathbb{G} = (\mathbb{R}^{n+m}, \circ, \delta_\lambda)$ is a homogeneous Lie group with homogeneous dimension $Q := n + 2m$.

The Jacobian matrix at $(0,0)$ of the left translation $\tau_{(x,z)}$ i.e the map

$$(x',z') \to \tau_{(x,z)}(x',z') := (x,z)\circ(x',z')$$

for fixed $(x,z) \in \mathbb{G}$, takes the following form

$$\mathcal{J}_{\tau_{(x,z)}}(0,0) = \left(\begin{array}{ccc|c} & \mathbb{I}_n & & 0_{n\times m} \\ \hline \frac{1}{2}\sum_{l=1}^{n}\Lambda^{(1)}_{1l}x_l & \cdots & \frac{1}{2}\sum_{l=1}^{n}\Lambda^{(1)}_{nl}x_l & \\ \vdots & \cdots & \vdots & \mathbb{I}_m \\ \frac{1}{2}\sum_{l=1}^{n}\Lambda^{(m)}_{1l}x_l & \cdots & \frac{1}{2}\sum_{l=1}^{n}\Lambda^{(m)}_{nl}x_l & \end{array}\right).$$

Then, the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$, is given by

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda^{(k)}_{il}x_l\frac{\partial}{\partial z_k} \qquad \text{and} \qquad Z_j = \frac{\partial}{\partial z_j},$$

for $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$.

Let $\nabla := (X_i)_{i=1,..,n}$ and $\Delta := \sum_{i=1,..,n} X_i^2$ denote the associated sub-gradient and sub-Laplacian, respectively.

The Heisenberg-type group $\tilde{\mathbb{H}}$ is an example of the step-two Carnot group $\mathbb{G}$. In fact, if we have

the two additional conditions

- $\Lambda^{(j)}$ are orthogonal matrices
- $\Lambda^{(i)}\Lambda^{(j)} = -\Lambda^{(j)}\Lambda^{(i)}$ for every $i, j \in \{1, .., m\}$ with $i \neq j$,

then we are in the setting of the Heisenberg-type group $\tilde{\mathbb{H}}$ isomorphic to $\mathbb{R}^{n+m}$. (Notice that if $n = 2$ and $m = 1$, we get back the Heisenberg group $\mathbb{H}$ introduced in Section 1.4.3.) The Kaplan distance was discovered on general Heisenberg-type groups in [84] by A. Kaplan, who extended the work of G.B. Folland [59] on the Heisenberg group. Namely, $\tilde{N} : \tilde{\mathbb{H}} \to [0, \infty)$ is given by

$$\tilde{N}(x, z) = (|x|^4 + 16|z|^2)^{\frac{1}{2}}, \text{ for } (x, z) \in \mathbb{R}^n \times \mathbb{R}^m,$$

where $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ and $|z| = \left(\sum_{i=1}^{m} z_i^2\right)^{\frac{1}{2}}$.
$\tilde{N}$ appears as a fundamental solution to the sub-Laplacian; in other words,

$$\Delta\tilde{N}(x, z)^{2-Q} = 0 \text{ in } \tilde{\mathbb{H}}\backslash\{0\}.$$

The Kaplan norm $\tilde{N}$ and the Carnot-Carathéodory distance d are topologically equivalent, i.e. there exists a constant $D > 0$ such that

$$D^{-1}d \leq \tilde{N} \leq Dd,$$

yet they exhibit different analytic behaviour. For instance, d is smooth away from the centre $\{(0, z) : z \in \mathbb{R}^m\} \subset \tilde{\mathbb{H}}$, while $\tilde{N}$ is smooth everywhere except at zero. Moreover, d satisfies the eikonal equation (1.7.1) $|\nabla d(x, z)| = 1$, whereas for N one has

$$|\nabla\tilde{N}(x, z)| = \frac{|x|}{\tilde{N}}, \tag{1.7.3}$$

and in particular $|\nabla\tilde{N}(x, z)|$ can be arbitrarily small. Moreover, since $\tilde{N}^{2-Q}$ is the fundamental solution to the sub-Laplacian, we have the following relation

$$\Delta\tilde{N}(x, z) = (Q - 1)\frac{|\nabla\tilde{N}(x, z)|^2}{\tilde{N}} = (Q - 1)\frac{|x|^2}{\tilde{N}^3}. \tag{1.7.4}$$

The proofs of these equations (1.7.3) and (1.7.4) can be found in Proposition 3.2.16. and Proposition 3.2.17 of [79], respectively.

In the setting of the more general step-two Carnot Group $\mathbb{G}$, we are motivated by the Kaplan

norm; hence, we consider a generalisation of the Kaplan norm:

$$N(x,z) := \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}},$$

with $a \in (0,\infty)$.

We have the following estimates of the sub-gradient and sub-Laplacian:

Lemma. *There exist constants* $A, C \in (0,\infty)$

$$A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\frac{|x|^2}{N^2} \tag{1.7.5}$$

and there exists a constant $B \in (0,\infty)$ *such that*

$$|\Delta N| \leq B\frac{|x|^2}{N^3} \tag{1.7.6}$$

The proof of this lemma can be found in Section 7.1 of the Appendix 7.

1.7.3 The Anisotropic Heisenberg group

The Carnot group $\mathbb{G}$ is said to be polarizable if N, where N^{2-Q} is the fundamental solution to the sub-Laplacian, is $\infty-$harmonic in $\mathbb{G}\backslash\{0\}$ [14], i.e.

$$\triangle_\infty N := \frac{1}{2}\langle \nabla\left(|\nabla N|^2\right), \nabla N\rangle = 0 \quad \text{ in } \mathbb{G}\backslash\{0\}. \tag{1.7.7}$$

The concept of polarizable Carnot groups was first introduced by Z. Balogh and J. Tyson in [14], where they used the $\infty-$harmonicity of N to create a procedure to construct polar coordinates. As a consequence, they showed that the fundamental solution of the $p-$sub-Laplacian can be expressed as the fundamental solution of the sub-Laplacian, proved capacity formulas, and produced sharp constants for the Moser-Trudinger inequality (which was established by L. Saloff-Coste in [120] in the setting of Carnot groups but without sharp constants). In settings related to polarizable Carnot groups, many authors showed Hardy-type inequalities as a function of N (see [48, 49, 62, 63, 85, 135]), proved Rellich-type inequalities (see [85, 138]), and studied Fuglede's $p-$module of system of measures [37].

However, we don't have a classification of polarizable Carnot groups: the only examples till now are Euclidean spaces and Heisenberg-type groups. This classification has been a longstanding open problem for two decades.

The concept of a polarizable Carnot group is a delicate one in the sense that under a small perturbation of the Lie algebra, the group is no longer polarizable. Z. Balogh and J. Tyson

provided in [14] the non-polarizable Anisotropic Heisenberg group on $\mathbb{R}^5$ as a counterexample with the following generators of the Lie algebra:

$$X_1 = \partial_{x_1} - \frac{\theta}{2}x_3\partial_t,\ X_2 = \partial_{x_2} - 2x_4\partial_t,\ X_3 = \partial_{x_3} + \frac{\theta}{2}x_1\partial_t, \text{ and } X_4 = \partial_{x_4} + 2x_2\partial_t, \tag{1.7.8}$$

where $\theta = 2$. To show (1.7.7) does not hold true for the Anisotropic Heisenberg group with $\theta = 2$, Z. Balogh and J. Tyson computed explicitly in [14] the fundamental solution of the sub-Laplacian using R. Beals, B. Gaveau, and P. Greiner's [17] explicit integral representation for the fundamental solution. Note that if $\theta = 4$, we have the polarizable Heisenberg group. In other words, just altering the constant θ in (1.7.8) changes the group from polarizable to non-polarizable. Moreover, T. Bieske [36] revisited this counterexample and proved that under a change of the inner product imposed on the vectors in the Lie algebra, which now requires the generators to be orthogonal instead of orthonormal, the Anisotropic Heisenberg group is turned into a group of Heisenberg-type i.e. it is now polarizable. Moreover, M. Gordina and L. Luo recently studied in [65] the Logarithmic Sobolev inequality with respect to a heat kernel measure in the setting of the Anisotropic Heisenberg group.

In Chapter 4, we extend the Anisotropic Heisenberg group from $\mathbb{R}^5$ ([14]) to $\mathbb{R}^{2n+1}$, compute explicitly the the fundamental solution of the sub-Laplacian, and prove coercive inequalities in that setting.

Recently, J. Tyson used our work in Chapter 4 to suggest the right quantitative form for the longstanding open problem of classification of polarizable step-two Carnot groups. More precisely, we quote the Conjecture 1.6 of his preprint [128], which relates the new notion of $H-$type stability to the classification of polarizable groups:

Conjecture 5 (J. Tyson [128])**.** *Let $\mathbb{G}$ be a step-two Carnot group with horizontal metric g_h. Denote by $\Sigma = \{g = \exp(x + t) \in \mathbb{G} : t = 0\}$. Let u be the fundamental solution for the sub-Laplacian $\mathcal{L}$ and let $N = u^{\frac{1}{2-Q}}$. Then,*

$$\sup_{g\in\sum} |N(g)(\mathcal{L}_\infty N)(g)| \simeq \delta(\mathbb{G})^2.$$

We note that $\delta(\mathbb{G})$ is the $H-$type deviation of $\mathbb{G}$, and if it is equal to zero, then we are in the setting of the Heisenberg-type group.

This conjecture, if true, would imply that all step-two polarizable Carnot groups are Heisenberg-type groups. J. Tyson proved that the results we obtained in Chapter 4 turned out to satisfy this conjecture.

In fact, we are one step closer to proving the longstanding conjecture of polarizable Carnot groups, and this motivates future work in this direction.

Chapter 2

The Logarithmic Sobolev and Talagrand Inequalities

This chapter is based on [35]. In the setting of Carnot groups, we prove the $q-$Logarithmic Sobolev inequality for probability measures as a function of the Carnot-Carathéodory distance. The main result in this chapter is the following $U-$bound theorem:

Theorem (8)**.** *Assume that outside the open unit ball* $B = \{d(x) < 1\}$, *the metric* d *satisfies the following:* $|\nabla d|$ *is bounded, say* $|\nabla d| \leq 1$, *and there exist finite positive constants* K *and* c_0 *such that*

$$\Delta d \leq K + U'(d)\left(|\nabla d|^2 - c_0\right).$$

(i) If $U'' \leq \beta U'$ *for some positive constant* β, *outside* B, *then for any* $q \in (1, \infty)$, *there exist constants* C_q, D_q, *independent of* f, *such that*

$$\int |f|^q |U'(d)|^q d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

(ii) If, in addition, $U \leq \gamma U'^q$ *for some positive constant* γ *and some* $q > 1$, *outside* B, *then*

$$\int |f|^q U(d)\, d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

Using W. Hebisch and B. Zegarliński's Theorem 7 ([72]), we obtain the $q-$Logarithmic Sobolev inequality for probability measures of the form

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda,$$

where Z is the normalisation constant, d is the Carnot-Carathéodory distance, and $U(d)$ satisfies (i) and (ii).

As an application, we use the Hamilton-Jacobi equation in the setting of Carnot groups to prove hypercontractivity and the $p-$Talagrand inequality:

Theorem (15)**.** *Let* $1 < q \leq 2$, *and* $p \geq 2$ *be its finite index conjugate, so that* $\frac{1}{p} + \frac{1}{q} = 1$.

If $(\mathbb{G}, d, \mu)$ *satisfies the* $q-$*Logarithmic Sobolev 2.1.6 inequality with constant* $c = (q-1)\left(\frac{q}{K}\right)^{q-1}$ *for some constant* $K > 0$, *then it also satisfies the* $p-$*Talagrand inequality with the same constant* K.

2.1 $q-$Logarithmic Sobolev Inequality

2.1.1 Overview

In the setting of the Carnot group $\mathbb{G}$, we would like to prove the $q-$Logarithmic Sobolev inequality

$$\int f^q \log \frac{f^q}{\int f^q d\mu} d\mu \leq c \int |\nabla f|^q d\mu, \tag{2.1.1}$$

where $q > 1$, ∇ is the sub-gradient as defined in Section 1.7.1 of the Introduction, and for a probability measure

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda,$$

where Z is the normalisation constant (i.e. $Z = \int e^{-U(d)} d\lambda$) and $U(d)$ is an increasing unbounded function of a certain metric d.

In Section 2.1.3 of this chapter, we will consider an important condition on the metric d that relates the norm of the sub-gradient $|\nabla d|$ to its sub-Laplacian Δd. This condition states: outside the open unit ball $B = \{d(x) < 1\}$, $|\nabla d|$ is bounded and there exist finite positive constants K and c_0 such that

$$\Delta d \leq K + U'(d)\left(|\nabla d|^2 - c_0\right). \tag{2.1.2}$$

This condition is satisfied by the Carnot-Carathéodory distance, defined in Section 1.7.1 of the Introduction. In fact, if d is the Carnot-Carathéodory distance, by (1.7.1),

$$|\nabla d(x^{(1)}, ..., x^{(r)})| = 1,$$

for all $(x^{(1)}, ..., x^{(r)}) \in \mathbb{G}$ such that $x^{(1)} \neq 0$. In addition, by (1.7.2), there exists a constant

$K \in (0, \infty)$ such that

$$\Delta d \leq \frac{K}{d}.$$

Hence, outside the unit ball, condition (2.1.2) is satisfied for the Carnot-Carathéodory distance.

However, this condition (2.1.2) fails for homogeneous norms, like the fundamental solution to the sub-Laplacian in the setting of the step-two Carnot group and the Anisotropic Heisenberg group, which are considered in Chapter 3 and Chapter 4. To illustrate, we consider the three-dimensional Heisenberg group $\mathbb{H}$, with the sub-gradient $\nabla := (X, Y)$, the sub-Laplacian $\Delta = X^2 + Y^2$, $X := \partial_x - \frac{y}{2}\partial_z$, and $Y = \partial_y + \frac{x}{2}\partial_z$.

The fundamental solution to the sub-Laplacian is given explicitly by $N(x, y, z)^{-2}$, where

$$N(x, y, z) = \left((x^2 + y^2)^2 + 16z^2\right)^{\frac{1}{4}},$$

i.e. $\Delta N(x, y, z)^{-2} = 0$ in $\mathbb{H}\backslash\{0\}$.

By a direct computation, we see that, for $(x, y, z) \in \mathbb{H}\backslash\{0\}$,

$$|\nabla N(x, y, z)| = \frac{(x^2 + y^2)^{\frac{1}{2}}}{N(x, y, z)},$$

and

$$\Delta N(x, y, z) = 3\frac{(x^2 + y^2)}{N(x, y, z)^3}.$$

Outside the open unit ball $B = \{N(x, y, z) < 1\}$, we see that the condition (2.1.2) fails around the "$z-$axis", i.e. when $x = y = 0$ and z is large.

This condition also fails for the same reason for the fundamental solution of the sub-Laplacian in the more general settings (step-two Carnot group and Anisotropic Heisenberg group) that we consider in Chapter 3 and Chapter 4.

We note that in the setting of Carnot groups, if we choose any homogeneous norm $\tilde{N}$ (for example like the ones considered in Chapter 3 and Chapter 4), it is equivalent to the Carnot-Carathéodory distance d. In other words, there exists a constant $D > 0$ such that

$$D^{-1}d \leq \tilde{N} \leq Dd.$$

However, there is a fundamental difference between the two metrics d and $\tilde{N}$: these distances are topologically the same, but they have different differential properties. In this Chapter, we will prove the Logarithmic Sobolev inequality (2.1.1) for measures as a function of the Carnot-Carathéodory distance d (and more general metrics that satisfy (2.1.2)). On the other hand,

W. Hebsich and B. Zegarliński prove in [72] that it is impossible for measures as a function of homogeneous norms $\tilde{N}$ to satisfy the Logarithmic Sobolev inequality (2.1.1). Instead, in Chapter 3 and Chapter 4, we will be interested in deriving a new method to prove the $\phi-$Logarithmic Sobolev inequality under some conditions on a concave function ϕ.

2.1.2 Coercive Inequalities through the U-Bound Method

As discussed in the Section 1.4.3 of the Introduction, D. Bakry and M. Emery's Curvature-Dimension condition [10] fails in the setting of Carnot groups. To bypass this geometric condition, a "$U-$Bound" method was developed in [72] that does not require a bound on the curvature of space. The goal of this section is to recap this method, which states that in order to obtain coercive inequalities (like the $q-$Poincaré and the $q-$Logarithmic Sobolev inequalities), it is sufficient to prove the "$U-$Bound" inequalities. We note that the results of this section are general in the sense that there is no condition on the metric. Hence, the Theorems in this section apply to measures as a function of the Carnot-Carathéodory distance (considered in this chapter) as well as measures as a function of the fundamental solution of the sub-Laplacian in the step-two Carnot group and the Anisotropic Heisenberg group (considered in Chapter 3 and Chapter 4).

Given the probability measure

$$d\mu = \frac{e^{-U}}{Z} d\lambda,$$

where U is an increasing unbounded function and Z is the normalisation constant. In broad terms, the general idea of this section is that a $U-$Bound of the form (2.1.3)

$$\int f^q \eta d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu$$

for η a suitable function "going to infinity in all directions" implies the $q-$Poincaré inequality (see Theorem 6 below).

If the inequality is satisfied for the specific

$$\eta = |\nabla U|^q + U,$$

then the $q-$Logarithmic Sobolev inequality is obtained (see Theorem 7 below).

Those theorems apply on general metric spaces equipped with non-commuting vector fields $\{X_1, X_2, ..., X_n\}$ satisfying Hörmander's condition, where the sub-gradient is $\nabla = (X_1, X_2, ..., X_n)$, and the sub-Laplacian is $\Delta = \sum_{i=1}^{n} X_i^2$. It is additionally assumed that the divergence of these vector fields with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}^n$ is zero. We are interested specifically in

the setting of the Carnot group $\mathbb{G}$ whose Lie algebra is generated by $\{X_1, X_2, ..., X_n\}$. The Carnot group setup is introduced in further detail in Section 1.7.1 of the Introduction. We state both theorems more precisely:

Let λ be a measure satisfying the $q-$Poincaré inequality for every ball $B_R = \{x : N(x) < R\}$, i.e. there exists a constant $C_R \in (0, \infty)$ such that

$$\frac{1}{|B_R|}\int_{B_R}\left|f - \frac{1}{|B_R|}\int_{B_R} f d\lambda\right|^q d\lambda \leq C_R \frac{1}{|B_R|}\int_{B_R} |\nabla f|^q d\lambda,$$

where $1 \leq q < \infty$.

Note that we have this Poincaré inequality on balls in the setting of Nilpotent lie groups thanks to J. Jerison's celebrated paper [83].

Theorem 6 (W. Hebisch, B. Zegarliński [72]). *Let μ be a probability measure on $\mathbb{R}^n$ which is absolutely continuous with respect to the measure λ and such that*

$$\int f^q \eta d\mu \leq C\int |\nabla f|^q d\mu + D\int f^q d\mu, \tag{2.1.3}$$

with some non-negative function η and some constants $C, D \in (0, \infty)$ independent of a function f. If for any $L \in (0, \infty)$ there is a constant A_L such that $\frac{1}{A_L} \leq \frac{d\mu}{d\lambda} \leq A_L$ on the set $\{\eta < L\}$ and, for some $R \in (0, \infty)$ (depending on L), we have $\{\eta < L\} \subset B_R$, then μ satisfies the $q-$Poincaré inequality

$$\int |f - \int f d\mu|^q d\mu \leq c\int |\nabla f|^q d\mu,$$

with some constant $c \in (0, \infty)$ independent of f.

To get the $q-$Poincaré inequality, notice that we have a freedom of choice of a "suitable" η. This fact will play an important role when proving coercive inequalities in Chapter 3 and Chapter 4.

Theorem 7 (W. Hebisch, B. Zegarliński [72]). *Suppose the following Sobolev inequality is satisfied*

$$\left(\int |f|^{q+\varepsilon} d\lambda\right)^{\frac{q}{q+\varepsilon}} \leq a\int |\nabla f|^q d\lambda + b\int |f|^q d\lambda,$$

and the following bound is true

$$\int |f|^q \left(|\nabla U|^q + U\right) d\mu \leq C'\int |\nabla f|^q d\mu + D'\int |f|^q d\mu. \tag{2.1.4}$$

Then, the following "defective" $q-$ Logarithmic Sobolev inequality is true

$$\int \left(f^q \log \frac{f^q}{\int f^q d\mu} \right) d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu. \tag{2.1.5}$$

Moreover, if $q \in (1,2]$ and the following $q-$Poincaré inequality holds $\int |f - \mu f|^q d\mu \leq \frac{1}{M} \int |\nabla f|^q d\mu$, then one has the $q-$Logarithmic Sobolev inequality

$$\int \left(f^q \log \frac{f^q}{\int f^q d\mu} \right) d\mu \leq c \int |\nabla f|^q d\mu \tag{2.1.6}$$

with some constant $c \in (0, \infty)$ independent of f.

The term "defective" $q-$Logarithmic Sobolev inequality is due to the presence of the additional term $\int |f|^q d\mu$ in (2.1.5). The $q-$Logarithmic Sobolev inequality (2.1.6) is obtained from the defective one (2.1.5) by a tightening procedure via Rothaus arguments (see [24]).

To get the $q-$Logarithmic Sobolev inequality, notice that the choice of η in the $U-$bound (2.1.4) is forced to be $|\nabla U|^q + U$.

2.1.3 Generalising the U-Bound Theorem for measures as a function of the Carnot-Carathéodory distance

We are interested in proving coercive inequalities for probability measures

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda, \tag{2.1.7}$$

where Z is the normalisation constant and $U(d)$ is an increasing unbounded function of a certain metric d.

We recall the condition on the metric (2.1.2) - which is satisfied by the Carnot-Carathéodory distance but not by the homogeneous norms in Chapter 3 and Chapter 4- mentioned in the Overview of this Chapter:

$$\Delta d \leq K + U'(d) \left(|\nabla d|^2 - c_0 \right).$$

This condition generalises the following metric condition, on the sub-gradient $|\nabla d|$ and the sub-Laplacian $|\Delta d|$, proposed by W. Hebisch and B. Zegarliński [72]: Outside the unit ball $B = \{d(x) < 1\}$,

$$\Delta d \leq K + \beta \varepsilon d^{p-1},$$

and

$$\frac{1}{\sigma^2} \leq |\nabla d| \leq 1,$$

where $\varepsilon \in \left(0, \frac{1}{\sigma}\right)$, $\beta \in (0, \infty)$, and $\sigma \in [1, \infty)$. Their condition arose from proving the measure

$$d\mu_p = \frac{e^{-d^p}}{Z} d\lambda$$

satisfies both $q-$Poincaré and $q-$Logarithmic Sobolev inequalities for q the finite index conjugate of p and $p > 1$. Notice that in this section, our work focuses on studying coercive inequalities for the more general measures of the form (2.1.7), and thus we have a more general condition of the form (2.1.2).

To get the $q-$Poincaré inequality from Theorem 6 of the previous section, W. Hebisch and B. Zegarliński proved a $U-$bound inequality (2.1.3) with a choice of $\eta = d^{p-1}$ (Theorem 2.1 of [72]). However, by Theorem 7 of the previous section, the $q-$Logarithmic Sobolev inequality requires the specific choice of $\eta = |\nabla U|^q + U$, where $U = d^p$ in W. Hebisch and B. Zegarliński's case. Computing and using the fact that $|\nabla d| \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\eta = |\nabla U|^q + U = |pd^{p-1}\nabla d|^q + d^p \leq (p^q + 1)d^p.$$

Therefore, to use Theorem 7 and prove the $q-$Logarithmic Sobolev inequality, W. Hebisch and B. Zegarliński proved an additional $U-$bound inequality with $\tilde{\eta} = (p^q + 1)d^p$ (Theorem 2.3 and Theorem 2.4 of [72]):

$$\int (p^q + 1)d^p f^q d\mu_p \leq C_q \int |\nabla f|^q d\mu_p + D_q \int |f|^q d\mu_p,$$

for q the finite index conjugate of p and $p > 1$.

In the following, we present a generalised version of Theorems 2.1, 2.3, and 2.4 of [72]. In place of the function $U(d) = d^p$, we let $U : [0, \infty) \to [0, \infty)$ be a twice differentiable and increasing function; and let $d\mu_U = \frac{e^{-U(d)}}{Z} d\lambda$ be a probability measure defined in terms of the function $U(d)$, where Z is the normalisation constant.

Theorem 8. *Assume that outside the open unit ball $B = \{d(x) < 1\}$, the metric d satisfies the following: $|\nabla d|$ is bounded, say $|\nabla d| \leq 1$, and there exist finite positive constants K and c_0 such that*

$$\Delta d \leq K + U'(d)\left(|\nabla d|^2 - c_0\right). \tag{2.1.8}$$

(i) If $U'' \leq \beta U'$ for some positive constant β, outside B, then for any $q \in (1, \infty)$, there exist constants C_q, D_q, independent of f, such that

$$\int |f|^q |U'(d)|^q d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

(ii) If, in addition, $U \leq \gamma U'^q$ for some positive constant γ and some $q > 1$, outside B, then

$$\int |f|^q U(d)\, d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

We note that, for d the Carnot-Carathéodory distance, by (1.7.1),

$$|\nabla d(x^{(1)}, ..., x^{(r)})| = 1,$$

for all $(x^{(1)}, ..., x^{(r)}) \in \mathbb{G}$ such that $x^{(1)} \neq 0$. In addition, by (1.7.2), there exists a constant $K \in (0, \infty)$ such that

$$\Delta d \leq \frac{K}{d}.$$

Hence, outside the unit ball, condition [72] is satisfied for the Carnot-Carathéodory distance.

The proof of (ii) is straightforward. In the following, we prove (i) where we adapt, with simplification, the general lines of proof in [72].

Proof. Using integration by parts and condition (2.1.8),

$$\int (\nabla d) \cdot \nabla \left(f e^{-U(d)} \right) d\lambda = - \int \Delta d \left(f e^{-U(d)} \right) d\lambda$$

$$\geq -K \int f e^{-U(d)} d\lambda - \int U'(d) |\nabla d|^2 f e^{-U(d)} d\lambda + c_0 \int U'(d) f e^{-U(d)} d\lambda. \tag{2.1.9}$$

Computing

$$\nabla \left(f e^{-U(d)} \right) = \nabla f e^{-U(d)} - U'(d) e^{-U(d)} \nabla d,$$

and taking the dot product with ∇d,

$$(\nabla d) \cdot \nabla \left(f e^{-U(d)} \right) = (\nabla f) \cdot (\nabla d) e^{-U(d)} - U'(d) f |\nabla d|^2 e^{-U(d)},$$

and replacing in (2.1.9) to get

$$c_0 \int U'(d) f e^{-U(d)} d\lambda \leq K \int f e^{-U(d)} d\lambda + \int |\nabla f| |\nabla d| e^{-U(d)} d\lambda.$$

Using the fact that $|\nabla d| \leq 1$ ($|\nabla d|$ bounded is enough), we get

$$\int U'(d)\, f d\mu_U \leq C \int f d\mu_U + D \int |\nabla f| d\mu_U. \tag{2.1.10}$$

To prove (i), we replace f in (2.1.10) by f^q to get

$$\int f^q \left(U'(d)\right)^q d\mu_U = \int \left[f^q U'(d)^{q-1}\right] U'(d)\, d\mu_U$$

$$\leq C \int \left[f^q U'(d)^{q-1}\right] d\mu_U + D \int \left|\nabla \left[f^q U'(d)^{q-1}\right]\right| d\mu_U$$

$$= C \int f f^{q-1} U'(d)^{q-1}\, d\mu_U + D \int q f^{q-1} |\nabla f| U'(d)^{q-1}\, d\mu_U$$

$$+ D \int (q-1)\, f f^{q-1} U'(d)^{q-2}\, U''(d)\, |\nabla d| d\mu_U. \tag{2.1.11}$$

Using Young's inequality with α, i.e. $ab \leq \alpha a^p + \dfrac{b^q}{q\,(\alpha p)^{q/p}}$ where p and q are conjugate finite indices in each of the three terms on the right-hand side of (2.1.11), in addition to using the conditions $|\nabla d| \leq 1$ and $U'' < \beta U'$ in the last term, we get

$$\int f^q U'(d)^q\, d\mu_U \leq \frac{C}{q\,(\alpha p)^{\frac{q}{p}}} \int f^q d\mu_U + C\alpha \int f^q U'(d)^q\, d\mu_U + \frac{D}{(\alpha p)^{\frac{q}{p}}} \int |\nabla f|^q d\mu_U$$

$$+ Dq\alpha \int f^q U'(d)^q\, d\mu_U + \frac{D\beta\,(q-1)}{q\,(\alpha p)^{\frac{q}{p}}} \int f^q d\mu_U + D\,(q-1)\,\alpha \int f^q U'(d)^q\, d\mu_U.$$

Choosing α small enough so that $C\alpha + Dq\alpha + D\,(q-1)\,\alpha \leq 1$, we get

$$\int |f|^q U'(d)^q\, d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

□

Corollary 9. *If the positive twice differentiable and increasing function $U(d)$ satisfies the two inequalities $U'' < \beta U'$ and $U \leq \gamma U'^q$, where β and γ are finite positive constants, then the measure*

$$d\mu_U = \frac{e^{-U(d)} d\lambda}{Z}$$

satisfies a $q-$Poincaré and a $q-$Logarithmic Sobolev inequality.

Proof. To get the Logarithmic Sobolev inequality, we verify inequality (2.1.4) from Theorem 7

$$|\nabla U(d)|^q + U(d) = |U'(d)\, \nabla d|^q + U(d) \leq (1+\gamma)\, U'^q(d)$$

Since $U''(d) < \beta U'(d)$ by assumption, we can use (i) of Theorem 8, and the result follows. □

Remark 10. In Corollary 9, it is enough to assume that the inequalities

$$U^{''} < \beta U^{'} \text{ and } U \leq \gamma U'^{q}$$

hold almost everywhere for β and γ finite positive constants.

Remark 11. To get Theorem 2.3 of [72], take $U(d) = d^p$, then

$$U'(d) = pd^{p-1} \text{ and } U''(d) = p(p-1)d^{p-2}.$$

Outside the open unit ball $B = \{d(x) < 1\}$,

$$\frac{U''(d)}{U'(d)} = \frac{p-1}{d} \leq p-1.$$

Therefore, (i) from Theorem 8 holds, and the statement of Theorem 2.3 of [72] is recovered.

Suppose q is the finite conjugate index of p, then,

$$\frac{U(d)}{U'^{q}(d)} = \frac{d^p}{p^q d^{(p-1)q}} = \frac{1}{p^q}.$$

Therefore, (ii) of Theorem 8 holds, and the statement of Theorem 2.4 of [72] is recovered.

Example 12. The $q-$Poincaré and a $q-$Logarithmic Sobolev inequality are satisfied for the measure

$$d\mu_U = \frac{e^{-(d+1)^p \log(d+1)} d\lambda}{Z}$$

for $q \geq \beta$, where β is the finite index conjugate to p.

Proof. Given $U(d) = (d+1)^p \log(d+1)$ .Then,

$$U'(d) = (d+1)^{p-1}(p\log(d+1)+1),$$

and

$$U''(d) = (d+1)^{p-2}(p(p-1)\log(d+1)+2p-1).$$

So, outside the open unit ball $B = \{d(x) < 1\}$,

$$\frac{U''(d)}{U'(d)} = \frac{(p(p-1)\log(d+1)+2p-1)}{d+1} \leq p(p-1) + \frac{2p-1}{d+1} \leq p^2 - \frac{1}{2}.$$

In addition,

$$\frac{U(d)}{U'^{q}(d)} = \frac{(d+1)^{p}\log(d+1)}{(d+1)^{(p-1)q}(p\log(d+1)+1)^{q}} \leq \frac{(d+1)^{p}\log(d+1)}{(d+1)^{(p-1)\beta}(p\log(d+1)+1)^{q}} \leq 1.$$

So, by Corollary 9, the $q-$Poincaré and a $q-$Logarithmic Sobolev inequality are satisfied for the measure $d\mu_U = \frac{e^{-(d+1)^{p}\log(d+1)}}{Z}d\lambda$ for $q \geq \beta$, where β is the finite index conjugate to p. □

Example 13. For $U(d) = \sinh(d)$, $U(d) = U''(d) \leq \cosh(d) = U'(d)$.

So, by Corollary 9, the $q-$Poincaré and $q-$Logarithmic Sobolev inequalities hold true for the measure $d\mu_U = \frac{e^{-\sinh(d)}}{Z}d\lambda$ for all $q \geq 1$.

2.2 $q-$Talagrand Inequality

2.2.1 Motivation

2.2.1.1 Origins

The Talagrand transportation cost inequality was first introduced in 1996 by M. Talagrand [127]. His inequality bounds the optimal transport cost in the sense of Monge-Kantorovich, $T_w(\mu,\nu)$, by the relative entropy functional, $\mathrm{Ent}_\nu\left(\frac{d\mu}{d\nu}\right) := \int \frac{d\mu}{d\nu}\log\left(\frac{d\mu}{d\nu}\right)d\nu$, in the following way:

$$T_w(\mu,\nu) \leq 2\mathrm{Ent}_\nu\left(\frac{d\mu}{d\nu}\right), \tag{2.2.1}$$

where μ a measure on $\mathbb{R}^N$ absolutely continuous with respect to the Gaussian measure ν, and $w(x,y) = \sum_{i=1}^{N}(x_i - y_i)^2$ is the cost of moving a unit mass from position x to position y. The transportation cost, $T_w(\mu,\nu)$, measures how much effort is required to transport a mass distributed according to μ to a mass distributed according to ν, relatively to the cost function w :

$$T_w(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^N\times\mathbb{R}^N} w(x,y)d\pi(x,y), \tag{2.2.2}$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with μ the first marginal and ν the second marginal[1].

To prove (2.2.1), Talagrand first considered the dimension $N = 1$, and the Gaussian measure in that case is $d\nu = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$. Consider $f := \frac{d\mu}{d\nu}$ to be a step function that does not take the value zero. To transport ν to μ, he defined the function ϕ by:

[1]For all bounded functions f and g on $\mathbb{R}^N$, $\int_{\mathbb{R}^N\times\mathbb{R}^N}(f(\omega)+g(\omega'))d\pi(\omega,\omega') = \int_{\mathbb{R}^N} f d\mu + \int_{\mathbb{R}^N} g d\nu$.

$$\nu((-\infty, x]) = \mu((-\infty, \phi(x)])$$

Meaning

$$\int_{-\infty}^{x} e^{-\frac{u^2}{2}} du = \int_{-\infty}^{\phi(x)} f(u) e^{-\frac{u^2}{2}} du \tag{2.2.3}$$

Then, μ is the image of ν under the map $x \to \phi(x)$. So, the image of π of ν on $\mathbb{R}^2$ under the map $x \to (x, \phi(x))$ has marginals ν and μ. Given a cost function $w(x, y) = (x - y)^2$,

$$T_w(\mu, \nu) \leq \int_{\mathbb{R}} (x - \phi(x))^2 d\nu(x) \leq 2 \int_{\mathbb{R}} f(x) \log f(x) d\nu(x) = 2\mathrm{Ent}_\nu \left(\frac{d\mu}{d\nu} \right).$$

The last inequality is proved by differentiating (2.2.3), using an integration by parts, and utilising the convexity of $\frac{x^2}{2}$. By induction over N, M. Talagrand proves that (2.2.1) holds in $\mathbb{R}^N$[127].

2.2.1.2 Relation to Concentration of Measure

M. Talagrand showed how the Talagrand transportation cost inequality (2.2.1) implies the concentration of measure phenomenon, and his approach was motivated by the work of K. Marton [104, 105].

In [127], Talagrand used for cost function the distance $d(x, y) = ||x - y||_2 = \left(\sum_{i=1}^{N} (x_i - y_i)^2 \right)^{\frac{1}{2}}$. Let $d\gamma$ be the Gaussian measure; it follows from Talagrand's transportation cost inequality (2.2.1) and Cauchy-Schwartz that:

$$T_d(\mu, \nu) \leq \sqrt{2 \int f \log f d\gamma}$$

Using the triangle inequality,

$$T_d(\mu, \nu) \leq T_d(\mu, \gamma) + T_d(\gamma, \nu).$$

Let $B \subset \mathbb{R}^n$ be a measurable set, and define for $t > 0$, $B_t = \{x \in \mathbb{R}^n; d(x, B) \leq t\}$. For the measures $d\mu = \frac{1_B}{\gamma(B)} d\gamma$, and $d\nu = \frac{1_{B_t^c}}{\gamma(B_t^c)} d\gamma$,

$$t = \inf\{||x - y||_2;\ x \in B, y \in B_t^c\} \leq \sqrt{2 \log \left(\frac{1}{\gamma(B)} \right)} + \sqrt{2 \log \left(\frac{1}{1 - \gamma(B_t)} \right)}.$$

Therefore, for $t \geq \sqrt{2\log(\frac{1}{\gamma(B)})}$,

$$1 - \gamma(B_t) \leq e^{-\frac{1}{2}\left(t - \sqrt{2\log(\frac{1}{\gamma(B)})}\right)^2}.$$

When $\gamma(B) = 1/2$, this is close to the best possible result given by the isoperimetric inequality $1 - \gamma(B_t) \leq e^{-\frac{t^2}{2}}$. The measure of B_t converges at at an exponential rate to 1 as t grows: this is a standard result in the concentration of measure in Gauss space. This concentration of measure phenomenon is one of the main tools in probability theory, statistical mechanics, quantum information theory, stochastic dynamics, etc., and many works [5, 19, 20, 66, 67, 89, 90, 110] studied its links with the Logarithmic Sobolev inequality and the Talagrand inequality.

2.2.1.3 Extension of the Talagrand inequality and its relation to the Logarithmic Sobolev inequality

In the setting of manifolds under D. Bakry and M. Emery's Curvature-Dimension condition, F. Otto and C. Villani showed in their celebrated paper [110] that the Logarithmic Sobolev inequality implies the Talagrand transportation cost inequality. Their proof relied on PDE's methods and Otto calculus in Wasserstein space [109].

We are interested in showing that the Logarithmic Sobolev inequality implies the Talagrand inequality in the setting of Carnot groups. As mentioned in Section 1.4.3 of the Introduction, D. Bakry and M. Emery's Curvature-Dimension condition does not hold, so we cannot apply the method by F. Otto and C. Villani [110].

In this chapter, we adapt Z.M. Balogh et al.'s Theorem 4.1 [15] to get the $p-$Talagrand inequality, and we adapt S. Bobkov et al.'s Theorem 2.1 in [19] to get hypercontractivity. However, instead, we use the Hamilton-Jacobi equation in the setting of Carnot groups by F. Dragoni [54]. This allows us to use the sub-gradient, and to show that, in the setting of Carnot groups, the Logarithmic Sobolev inequality for various probability measures implies the Talagrand inequality and hypercontractivity. We note that, in contrast with Z.M. Balogh et al.'s Theorem 4.1 [15], we do not require the probability measures to be doubling in the proofs. This allows us to obtain results for a wider class of probability measures that encompass the examples given in this Chapter, such as $U(d) = (d+1)^p \log(d+1)$ in Example 12 and $U(d) = \sinh(d)$ in Example 13.

2.2.2 The $q-$Logarithmic Sobolev inequality implies the $p-$Talagrand inequality in the setting of Carnot groups

Let $\mathbb{G}$ be a Carnot group whose Lie algebra is generated by $\{X_i\}_{i=1}^{N_1}$. Hence, the sub-gradient is given by $\nabla := (X_1, ..., X_{N_1})$ and the sub-Laplacian is given by $\Delta := \sum_{i=1}^{N_1} X_i^2$. Let d be the Carnot-Carathéodory distance. (This setup is described in more detail in Section 1.7.1 of the Introduction.)

The following Hamilton-Jacobi Theorem by F. Dragoni [54] is a key tool in proving that the Logarithmic Sobolev inequality implies the Talagrand inequality and Hypercontractivity in the setting of the Carnot group $\mathbb{G}$.

Theorem 14 (F. Dragoni [54]). *Let $\phi : [0,\infty) \to [0,\infty)$ be a continuous, non-decreasing, and convex function with $\phi(0) = 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ a lower semicontinuous function such that there exists $C > 0$:*

$$f(x) \geq -C(1 + d(x)).$$

Then a viscosity solution for the Hamilton-Jacobi problem in the Carnot Group $\mathbb{G}$

$$\begin{cases} u_t(x,t) + \phi(|\nabla u(x,t)|) = 0 & (x,t) \in \mathbb{G} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{G} \end{cases}$$

is given by the Hopf-Lax formula

$$Q_t f(x) := \inf_{y \in \mathbb{G}} \left\{ t\phi^* \left(\frac{d(x,y)}{t} \right) + f(y) \right\}$$

almost everywhere, where $d(x,y)$ is the Carnot-Carathéodory distance, ∇ is the sub-gradient, and ϕ^* is the Legendre transform of ϕ.

In this chapter, we will be considering $f(x)$ to be a continuous and bounded function ,and we will be choosing the function $\phi(s) = \frac{s^q}{q}$, where $1 < q \leq 2$.

Thus, the viscosity solution for the Hamilton-Jacobi problem in the Carnot group $\mathbb{G}$

$$\begin{cases} u_t(x,t) + \frac{|\nabla u(x,t)|^q}{q} = 0 & (x,t) \in \mathbb{G} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{G} \end{cases} \tag{2.2.4}$$

is given by the Hopf-Lax formula

$$Q_t f(x) = \inf_{y \in \mathbb{G}} \left\{ \frac{d(x,y)^p}{t^{p-1}} + f(y) \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

To prove that the $q-$Logarithmic Sobolev inequality implies the $p-$Talagrand inequality, we are inspired by the proof of Theorem 4.1 in [15]. However, instead of using the Hamilton-Jacobi equation as in that paper, we use the following Hamilton-Jacobi equation as in Theorem 4 in the paper by F. Dragoni [54] . In the setting of Carnot groups, the restriction for using a doubling measure μ is no longer needed.

Let d be the Carnot-Carathéodory distance and $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$ be a measure where Z is a normalisation constant, λ is the Lebesgue measure, and $U(d)$ satisfies the conditions of Theorem 8.

Theorem 15. *Let $1 < q \leq 2$, and $p \geq 2$ be its finite index conjugate, so that $\frac{1}{p} + \frac{1}{q} = 1$.*

If $(\mathbb{G}, d, \mu)$ satisfies the $q-$Logarithmic Sobolev 2.1.6 inequality with constant $c = (q-1)\left(\frac{q}{K}\right)^{q-1}$ for some constant $K > 0$, then it also satisfies the $p-$ Talagrand inequality with the same constant K.

For every continuous bounded function f where $Qf = Q_1 f$, we will be proving the dual formulation of the Talagrand inequality

$$\int_{\mathbb{G}} e^{KQf} d\mu \leq e^{K \int_{\mathbb{G}} f d\mu}, \tag{2.2.5}$$

which is shown equivalent to the $p-$Talagrand inequality (4.1.8) in [15].

Proof. For some $n \geq 1$, define

$$\phi(t) = \frac{1}{Kt^n} \log \left(\int_{\mathbb{G}} e^{kt^n Q_t f} d\mu \right).$$

Using the fact that f is bounded and the dominated convergence theorem, Balogh et al. prove in Theorem 4.1 of [15] that:

$$\lim_{t \to 0^+} \phi(t) = \int_{\mathbb{G}} f d\mu.$$

The main goal is to prove that $\phi(t)$ is non-increasing in t, which implies that $\phi(1) \leq \lim_{t \to 0^+} \phi(t)$. In other words, replacing:

$$\phi(1) = \frac{1}{K} \log \left(\int_{\mathbb{G}} e^{kQf} d\mu \right) \leq \lim_{t \to 0^+} \phi(t) = \int_{\mathbb{G}} f d\mu.$$

Rearranging the terms and exponentiating the last inequality, we get the dual Talagrand inequality (2.2.5). We now show that $\phi(t)$ is non-increasing. The two main ingredients to prove that are the

$q-$Logarithmic Sobolev inequality and the solution to the Hamilton-Jacobi problem (2.2.4). Fix $t \in (0,1]$. For $s > 0$, we have

$$\lim_{s\to 0^+} \frac{\phi(t+s)-\phi(t)}{s} = \lim_{s\to 0^+} \frac{1}{s}\left(\frac{1}{K(t+s)^n} - \frac{1}{Kt^n}\right) \log \int_{\mathbb{G}} e^{K(t+s)^n Q_{t+s}f} d\mu$$

$$+ \lim_{s\to 0^+} \frac{1}{Kt^n s}\left(\log \int_{\mathbb{G}} e^{K(t+s)^n Q_{t+s}f} d\mu - \log \int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu\right). \tag{2.2.6}$$

As $s \to 0^+$, the first term on the right-hand side of (2.2.6) converges to

$$-\frac{n}{Kt^{n+1}} \log\left(\int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu\right).$$

The limit of the second term of (2.2.6) is

$$\frac{1}{Kt^n} \frac{1}{\int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu} \lim_{s\to 0^+} \left[\frac{1}{s}\left(\int_{\mathbb{G}} e^{K(t+s)^n Q_{t+s}f} d\mu - \int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu\right)\right]$$

rewriting, we get

$$= \frac{1}{Kt^n} \frac{1}{\int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu} \lim_{s\to 0^+} \left[\int_{\mathbb{G}} \frac{e^{K(t+s)^n Q_{t+s}f} - e^{Kt^n Q_{t+s}f}}{s} d\mu + \int_{\mathbb{G}} \frac{e^{Kt^n Q_{t+s}f} - e^{Kt^n Q_t f}}{s} d\mu\right]$$

so, using the dominated convergence theorem for the first term, we get

$$= \frac{1}{Kt^n} \frac{\int_{\mathbb{G}} Knt^{n-1} Q_t f e^{Kt^n Q_t f} d\mu}{\int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu} + \frac{1}{Kt^n} \frac{1}{\int_{\mathbb{G}} e^{Kt^n Q_t f} d\mu} \lim_{s\to 0^+} \left[\int_{\mathbb{G}} \frac{e^{Kt^n Q_{t+s}f} - e^{Kt^n Q_t f}}{s} d\mu\right]. \tag{2.2.7}$$

In order to compute the limit in (2.2.7), we use the Hamilton-Jacobi equation in Theorem 5, which has the solution $Q_t f$, so

$$\frac{\partial Q_t f}{\partial_t} = -\frac{|\nabla Q_t f|^q}{q},$$

so,

$$\lim_{s\to 0^+} \left(\frac{e^{Kt^n Q_{t+s}f} - e^{Kt^n Q_t f}}{s}\right) = Kt^n e^{Kt^n Q_t f} \frac{\partial Q_t f}{\partial_t} = -Kt^n e^{Kt^n Q_t f} \frac{|\nabla Q_t f|^q}{q}. \tag{2.2.8}$$

Since $Q_{(.)}f(.)$ is Lipschitz on $\mathbb{G} \times \mathbb{R}_+$, $Q_{t+s}f = Q_t f + O(s)$ holds uniformly on $\mathbb{G}$. Since $Q_t f(x)$ is uniformly bounded in x, we can use the dominated convergence theorem and (2.2.8), to find the limit in (2.2.7):

$$\lim_{s\to 0^+} \left[\int_{\mathbb{G}} \frac{e^{Kt^n Q_{t+s}f} - e^{Kt^n Q_t f}}{s} d\mu\right] = -Kt^n \int_{\mathbb{G}} e^{Kt^n Q_t f} \frac{|\nabla Q_t f|^q}{q} d\mu.$$

In other words, equation (2.2.7) becomes:

$$\lim_{s\to 0^+}\frac{1}{Kt^n s}\left(\log\int_{\mathbb{G}} e^{K(t+s)^n Q_{t+s}f}d\mu-\log\int_{\mathbb{G}} e^{Kt^n Q_t f}d\mu\right)$$

$$=\frac{1}{Kt^{n+1}}\frac{1}{\int_{\mathbb{G}} e^{Kt^n Q_t f}d\mu}\left[\int_{\mathbb{G}} nKt^n Q_t f e^{Kt^n Q_t f}d\mu-\int_{\mathbb{G}} Kt^{n+1}\frac{|\nabla Q_t f|^q}{q}e^{Kt^n Q_t f}d\mu\right].$$

Replacing the last equation in (2.2.6), we get:

$$\lim_{s\to 0^+}\frac{\phi(t+s)-\phi(t)}{s}=\frac{1}{Kt^{n+1}}\frac{1}{\int_{\mathbb{G}} e^{Kt^n Q_t f}d\mu}$$

$$\times[-n\log\left(\int_{\mathbb{G}} e^{Kt^n Q_t f}d\mu\right)\int_{\mathbb{G}} e^{Kt^n Q_t f}d\mu+\int_{\mathbb{G}} nKt^n Q_t f e^{Kt^n Q_t f}d\mu$$

$$-\int_{\mathbb{G}} Kt^{n+1}\frac{|\nabla Q_t f|^q}{q}e^{Kt^n Q_t f}d\mu].$$

Let $g=e^{\frac{1}{q}Kt^n Q_t f}$, and $n=\dfrac{1}{q-1}$, we obtain:

$$\lim_{s\to 0^+}\frac{\phi(t+s)-\phi(t)}{s}=\frac{1}{Kt^{\frac{q}{q-1}}}\frac{1}{\int_{\mathbb{G}} g^q d\mu}$$

$$\times[-\frac{1}{q-1}\log\left(\int_{\mathbb{G}} g^q d\mu\right)\int_{\mathbb{G}} g^q d\mu+\frac{1}{q-1}\int_{\mathbb{G}} g^q\log(g^q)d\mu$$

$$-\int_{\mathbb{G}} Kt^{\frac{q}{q-1}}\frac{|\nabla Q_t f|^q}{q}g^q d\mu]. \tag{2.2.9}$$

Using the $q-$Logarithmic Sobolev inequality,

$$\mathrm{Ent}_\mu(g^q)=\int_{\mathbb{G}} g^q\log(g^q)d\mu-\log\left(\int_{\mathbb{G}} g^q d\mu\right)\int_{\mathbb{G}} g^q d\mu$$

$$\leq(q-1)\left(\frac{q}{K}\right)^{q-1}\int_{\mathbb{G}}|\nabla g|^q d\mu$$

in (2.2.9):

$$\lim_{s\to 0^+}\frac{\phi(t+s)-\phi(t)}{s}=\frac{1}{Kt^{\frac{q}{q-1}}}\frac{1}{\int_{\mathbb{G}} g^q d\mu}$$

$$\times\left[\left(\frac{q}{K}\right)^{q-1}\int_{\mathbb{G}}|\nabla g|^q d\mu-\int_{\mathbb{G}} Kt^{\frac{q}{q-1}}\frac{|\nabla Q_t f|^q}{q}g^q d\mu\right]$$

$$=\frac{1}{Kt^{\frac{q}{q-1}}}\frac{1}{\int_{\mathbb{G}} g^q d\mu}\left[\left(\frac{q}{K}\right)^{q-1}\int_{\mathbb{G}}|\nabla(e^{\frac{1}{q}Kt^n Q_t f})|^q d\mu-\int_{\mathbb{G}} Kt^{\frac{q}{q-1}}\frac{|\nabla Q_t f|^q}{q}g^q d\mu\right]=0.$$

Hence, $\phi(t)$ is non-increasing, and we obtain the dual Talagrand inequality . □

2.3 Hypercontractivty of Hamilton-Jacobi solutions

Denote $||.||_p$, $p \in \mathbb{R}$, the L^p-norms with respect to μ, and $||f||_0 = e^{\int \log|f| d\mu}$ whenever $\log|f|$ is $\mu-$integrable. Also, consider the Hamilton-Jacobi problem on a Carnot group $\mathbb{G}$ as in Theorem 14. The following Hypercontractivity theorem is inspired by Theorem 2.1 by Bobkov et al. in [19], but with the measure $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$ and the Euclidian gradient replaced by the sub-gradient ∇.

Theorem 16. *Assume we have the following 2-Logarithmic Sobolev inequality with the measure* $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$, *and in the setting of the Carnot group:*

$$\rho \mathrm{Ent}_\mu(f^2) = \rho \int_{\mathbb{G}} f^2 \log(f^2) d\mu - \rho \log\left(\int_{\mathbb{G}} f^2 d\mu\right) \int_{\mathbb{G}} f^2 d\mu \leq 2 \int_{\mathbb{G}} |\nabla f|^2 d\mu.$$

Then, for every bounded measurable function f *on* $\mathbb{G}$, *every* $t \geq 0$, *and every* $a \in \mathbb{R}$,

$$||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a.$$

The following proof adapts the proof of Theorem 2.1 done by Bobkov et al. in [19] in the Euclidean setting, but uses the Hamilton-Jacobi problem in Carnot groups (Theorem 14).

Proof. Let $F(t) = ||e^{Q_t f}||_{\lambda(t)}$, with $\lambda(t) = a + \rho t$, $t > 0$.

For all $t > 0$ and almost every x, the partial derivatives $\frac{\partial}{\partial_t} Q_t f(x)$ exist, and by the Hamilton-Jacobi problem (Theorem 5),

$$\frac{\partial}{\partial_t} Q_t f(x) = -\frac{1}{2} |\nabla Q_t f(x)|^2. \tag{2.3.1}$$

So, F is differentiable at every point $t > 0$ where $\lambda(t) \neq 0$. Computing,

$$\left(F(t)^{\lambda(t)}\right)' = \rho F(t)^{\lambda(t)} \log(F(t)) + \lambda(t) F(t)^{\lambda(t)-1} F'(t). \tag{2.3.2}$$

Also,

$$\left(F(t)^{\lambda(t)}\right)' = \frac{\partial}{\partial_t} \left(\int_{\mathbb{G}} e^{\lambda(t) Q_t f} d\mu\right)$$

$$= \int_{\mathbb{G}} \left(\rho Q_t f + \lambda(t) \frac{\partial}{\partial_t} Q_t f\right) e^{\lambda(t) Q_t f} d\mu$$

using (4.1.10),

$$= \frac{\rho}{\lambda(t)} \mathrm{Ent}_\mu(e^{\lambda(t) Q_t f}) + \frac{\rho}{\lambda(t)} \log\left(\int_{\mathbb{G}} e^{\lambda(t) Q_t f} d\mu\right) \int_{\mathbb{G}} e^{\lambda(t) Q_t f} d\mu - \frac{\lambda(t)}{2} \int_{\mathbb{G}} |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu$$

$$= \frac{\rho}{\lambda(t)} \mathrm{Ent}_\mu(e^{\lambda(t) Q_t f}) + \rho F(t)^{\lambda(t)} \log(F(t)) - \frac{\lambda(t)}{2} \int_{\mathbb{G}} |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu.$$

Combining this with (4.1.11), we get

$$\lambda(t)^2 F(t)^{\lambda(t)-1} F'(t) = \rho \mathrm{Ent}_\mu(e^{\frac{1}{2}\lambda(t)Q_t f})^2 - \frac{\lambda(t)^2}{2}\int_{\mathbb{G}} |\nabla Q_t f(x)|^2 e^{\lambda(t)Q_t f} d\mu$$

using the 2−Logarithmic Sobolev inequality,

$$\leq \int_{\mathbb{G}} |\nabla e^{\frac{\lambda(t)Q_t f}{2}}|^2 d\mu - \frac{\lambda(t)^2}{2}\int_{\mathbb{G}} |\nabla Q_t f(x)|^2 e^{\lambda(t)Q_t f} d\mu$$

$$= \frac{\lambda(t)^2}{4}\int_{\mathbb{G}} |\nabla Q_t f(x)|^2 e^{\lambda(t)Q_t f} d\mu - \frac{\lambda(t)^2}{2}\int_{\mathbb{G}} |\nabla Q_t f(x)|^2 e^{\lambda(t)Q_t f} d\mu$$

$$\leq 0,$$

for all $t > 0$. Thus, $F'(t) \leq 0$ for all $t > 0$. Since F is continuous, then it is non-increasing. Hence, $F(t) \leq F(0)$, for all $t > 0$.

$$||e^{Q_t f}||_{\lambda(t)} \leq ||e^{Q_0 f}||_{\lambda(0)}.$$

So,

$$||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a.$$

□

Chapter 3

Coercive Inequalities on Step-Two Carnot Groups

This chapter is based on [33]. In the setting of step-two Carnot groups, we prove the following $U-$Bound theorem:

Theorem (17)**.** *Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ with $a \in (0,\infty)$ and $g : [0,\infty) \to [0,\infty)$ be a differentiable increasing function such that $g''(N) \leq g'(N)^3 N^3$ on $\{N \geq 1\}$. Let*

$$d\mu = \frac{e^{-g(N)}}{Z} d\lambda$$

be a probability measure, and Z the normalisation constant. Then, for all locally Lipschitz functions f, the following $U-$bound

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \tag{3.0.1}$$

holds outside the unit ball $\{N < 1\}$ with C and D positive constants and $q \geq 2$.

We apply this $U-$bound to W. Hebisch and B. Zegarliński's Theorem 6 (introduced in Section 2.1.2 of Chapter 2) to get the $q-$Poincaré inequality for $q \geq 2$.

In [72], W. Hebisch and B. Zegarliński prove that it is impossible for measures as a function of homogeneous norms, like those considered in this chapter, to satisfy the Logarithmic Sobolev inequality (2.1.1). Instead, we prove in this chapter the following $\phi-$Logarithmic Sobolev inequality under some conditions on a concave function ϕ:

Theorem (24)**.** *Let U be a locally Lipschitz function on $\mathbb{R}^N$ which is bounded below such that $Z =$*

$\int e^{-U} d\lambda < \infty$, and $d\mu = \dfrac{e^{-U}}{Z} d\lambda$. Let $\phi : [0,\infty) \to \mathbb{R}^+$ be a non-negative, non-decreasing, concave function such that $\phi(0) > 0$, and $\phi'(0) > 0$. Assume the following classical Sobolev inequality is satisfied:

$$\left(\int |f|^{q+\epsilon} d\lambda \right)^{\frac{q}{q+\epsilon}} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda$$

for some a, $b \in [0,\infty)$, and $\epsilon > 0$. Moreover, if for some A, $B \in [0,\infty)$, we have:

$$\mu \left(|f|^q (\phi(U) + |\nabla U|^q) \right) \leq A\mu |\nabla f|^q + B\mu |f|^q,$$

Then, there exists constants C, $D \in [0,\infty)$ such that:

$$\mu \left(|f|^q \phi \left(\left| \log \frac{|f|^q}{\mu |f|^q} \right| \right) \right) \leq C\mu |\nabla f|^q + D\mu |f|^q,$$

for all locally Lipschitz functions f.

This $\phi-$Logarithmic Sobolev theorem we obtain is general in the sense that is is independent on the explicit Kaplan norm used. It will applied in Chapter 4 to obtain results for measures as a function of the fundamental solution to the sub-Laplacian in the setting of the Anisotropic Heisenberg group.

Moreover, we deduce from Theorem 24 and Theorem 17, a $\beta-$Logarithmic Sobolev inequality (Theorem 28). We also provide an interval where the $\beta-$Logarithmic Sobolev inequality cannot hold (Theorem 27).

3.1 Motivation

In Chapter 2, we were concerned with proving coercive inequalities for measures as a function of the Carnot-Carathéodory distance d. A suitable question to ask is whether we can replace the Carnot-Carathéodory distance by another natural homogeneous distance function: the Kaplan norm N. It is a homogeneous norm related to the fundamental solution of the sub-Laplacian in the setting of the Heisenberg-type group $\tilde{\mathbb{H}}$ isomorphic to $\mathbb{R}^{n+m}$. (See Section 1.7.2 of the Introduction for the definition of the Heisenberg-type group.) The Kaplan distance was discovered on general Heisenberg-type groups in [84] by A. Kaplan, who extended the work of G.B. Folland [59] on the Heisenberg group. Namely, $N : \tilde{\mathbb{H}} \to [0,\infty)$ is given by

$$N(x,z) = \left(|x|^4 + 16|z|^2 \right)^{\frac{1}{4}},$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, and $|z| = \left(\sum_{i=1}^{m} z_i^2\right)^{\frac{1}{2}}$. This Kaplan norm is related to the fundamental solution of the sub-Laplacian in the following way:

$$\Delta N(x,z)^{2-n-2m} = 0 \text{ in } \tilde{\mathbb{H}}\backslash\{0\}.$$

As mentioned in Section 2.1.1 of Chapter 2, there is a fundamental difference between the two metrics: the Carnot-Carathéodory distance d and the Kaplan norm N. Even though these distances are topologically the same i.e. they are equivalent in the sense that there exists a constant $D > 0$ such that

$$D^{-1}d \leq N \leq Dd,$$

they have different differential properties. In fact, in Chapter 2 we proved the Logarithmic Sobolev inequality for measures as a function of d, yet it is impossible for measures as a function of N [72].

In Chapter 4 of [79], J. Inglis worked in the setting of the $H-$ type group, equipped with a probability measure

$$d\nu_p := \frac{e^{-\alpha N^p}}{Z} d\lambda,$$

where $p \in (1,\infty)$, $\alpha > 0$, and Z is the normalisation constant.

His aim was to gain spectral information about the associated positive self-adjoint operator on $L^2(\nu_p)$:

$$\mathcal{T}_{\nu_p} = -\Delta + \alpha p N^{p-1} \nabla N \cdot \nabla.$$

Operators of this type, defined with the sub-gradient and the sub-Laplacian, are considered to be the sub-Riemannian analogue of the Schrödinger operators of the form $\mathcal{L} = -\Delta_{\text{euclidean}} + V$ on $\mathbb{R}^n$, where V is some potential. They are called Schrödinger operators since the initial value problem for the Schrödinger equation may be reduced to the investigation of the spectrum of these operators acting on a Hilbert space (see [114]).

J. Inglis proved in Theorem 4.5.2 of [79] that for $p \in (1,2)$, $\mathcal{T}_{\nu_p}$ does not even have a spectral gap, let alone an empty essential spectrum. However, he showed in Theorem 4.5.3 of [79] that for $p > 2$, $\mathcal{T}_{\nu_p}$ has a spectral gap. He also conjectured that when $p > 2$, $\mathcal{T}_{\nu_p}$ will also have a discrete spectrum.

In this chapter, we will work in the more general step-two Carnot group $\mathbb{G}$ equipped with the general probability measure

$$d\mu = \frac{e^{-g(N)}}{Z} d\lambda,$$

where $g(N)$ satisfies some growth conditions, and where we consider a more general $N :=$

$\left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ with $a > 0$. Note that when $a = 16$, N takes the form of the Kaplan norm in the Heisenberg-type group. Our work is motivated by gaining spectral information about the associated positive self-adjoint operator on $L^2(\mu)$:

$$\mathcal{T}_\mu = -\Delta + g'(N)\nabla N \cdot \nabla.$$

In other words, in this chapter, we will prove the Poincaré inequality under some conditions on $g(N)$. Our work generalises the results by J. Inglis in Chapter 4 of [79]. We note that our method is completely different from his approach: we proceed by proving a $U-$bound inequality (**??**). The importance of the $U-$bound proof that we developed is that its ideas could be adapted to other settings like in the setting of the Anisotropic Heisenberg group in Chapter 4. Also, so far, an extension to infinite dimensions for measures as a function of homogeneous norms has not been possible (due to restrictions in known methods not satisfied by the norms considered). However, using this $U-$bound inequality, we are able to provide a new method (discussed in Chapter 5) that overcomes those restrictions and allows for an extension to an infinite dimensional setting. Our result strongly generalises known results [81, 82, 86, 111] and provides a vast scope of applications in the setting of Carnot groups.

3.2 The U-Bound Inequality

Consider $\mathbb{G} \simeq \mathbb{R}^{n+m}$ a step-two Carnot group, whose Lie algebra is generated by $\{X_i\}_{i=1}^n$. Let $\nabla := (X_1, .., X_n)$ be the sub-gradient and $\Delta = \sum_{i=1}^{n} X_i^2$ be the sub-Laplacian. (See Section 1.7.2 of the Introduction for the definition of the step-two Carnot group.)

The $U-$bound Theorem is the following:

Theorem 17. *Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ with $a \in (0, \infty)$ as introduced above and $g : [0, \infty) \to [0, \infty)$ be a differentiable increasing function such that $g''(N) \leq g'(N)^3 N^3$ on $\{N \geq 1\}$. Let*

$$d\mu = \frac{e^{-g(N)}}{Z} d\lambda$$

be a probability measure, and Z the normalisation constant. Then, for all locally Lipschitz functions f, the following $U-$bound

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \tag{3.2.1}$$

holds outside the unit ball $\{N < 1\}$ with C and D positive constants and $q \geq 2$.

The goal is to utilise the $U-$bound inequality (3.2.1) to obtain the $q-$Poincaré inequality by using W. Hebisch and B. Zegarliński's Theorem 6 (introduced in Section 2.1.2 of Chapter 2).

Recall that Theorem 6 gave us freedom of choosing some η with the requirement that "η goes to infinity in all directions". In this case, the choice is

$$\eta = \frac{g'(N)}{N^2},$$

and the requirement would be $\eta \to \infty$ as $N \to \infty$. This point will be made clearer in the next section of this chapter, where we will provide examples of measures that satisfy the $q-$Poincaré inequality.

A natural question is: How did we choose this $\eta = \frac{g'(N)}{N^2}$ in our Theorem 17?

Before answering this question, we state the following technical lemma (the proof is in Section 7.1 of the Appendix 7):

Lemma 18. *There exist constants $A, C \in (0, \infty)$*

$$A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\frac{|x|^2}{N^2} \tag{3.2.2}$$

and there exists a constant $B \in (0, \infty)$ such that

$$|\Delta N| \leq B\frac{|x|^2}{N^3} \tag{3.2.3}$$

and

$$\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}. \tag{3.2.4}$$

This lemma provides bounds for the three quantities $|\nabla N|$, ΔN, and $x \cdot \nabla N$. Bounds of this form are vital when adapting our proof of the $U-$bound Theorem 17 to other settings, like the Anisotropic Heisenberg group setting (Chapter 4). Moreover, notice that we have the useful positivity of $x \cdot \nabla N \geq 0$ according to (3.2.4). In Chapter 4, we will see that $x \cdot \nabla N$ could be negative, and this generates a "bad term" that will, in turn, make the proof only work for a higher dimension $n > 5$.

The choice of η

We first consider the case $q = 2$. Integrating by parts,

$$\int (\nabla N) \cdot (\nabla f)\, e^{-g(N)} d\lambda = -\int \nabla \left(\nabla N e^{-g(N)}\right) f d\lambda$$

$$= -\int \Delta N f e^{-g(N)} d\lambda + \int |\nabla N|^2 f g'(N) e^{-g(N)} d\lambda.$$

Next, using (4.1.5) and (3.2.3) of the Technical Lemma 18,

$$A\int \frac{|x|^2}{N^2}g'(N)fe^{-g(N)}d\lambda - B\int \frac{|x|^2}{N^3}fe^{-g(N)}d\lambda \leq \int \nabla N\cdot\nabla f e^{-g(N)}d\lambda. \tag{3.2.5}$$

The first candidate for η is $\frac{|x|^2}{N^2}g'(N)$, which appears naturally in the above inequality. Notice that for $|x|=0$, we have $\eta=0$, yet the Kaplan norm $N=(a|z|^2)^{\frac{1}{4}}\to\infty$ as $|z|\to\infty$. This violates Theorem 6's requirement stating that "$\eta\to\infty$ in all directions", as this requirement fails around the $z-$ axis.

To obtain a suitable η, we replace f by $\frac{f^2}{|x|^2}$ in (3.2.5) to obtain:

$$\int f^2\left(A\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda \leq \int(\nabla N)\cdot\left(\nabla\left(\frac{f^2}{|x|^2}\right)\right)e^{-g(N)}d\lambda.$$

From the left-hand side of the above inequality, we now have the good candidate $\eta=\frac{g'(N)}{N^2}$. We are concerned with the right-hand side of the inequality, and we proceed with the following computations.

$$\begin{aligned}\int f^2\left(A\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda &\leq \int(\nabla N)\cdot\left(\nabla\left(\frac{f^2}{|x|^2}\right)\right)e^{-g(N)}d\lambda\\ &= \int(\nabla N)\cdot\left[2f\frac{\nabla f}{|x|^2}-\frac{2f^2\nabla|x|}{|x|^3}\right]e^{-g(N)}d\lambda\\ &= \int\frac{2f}{|x|^2}\nabla N\cdot\nabla f e^{-g(N)}d\lambda - 2\int f^2\frac{\nabla N\cdot x}{|x|^4}e^{-g(N)}d\lambda\end{aligned}$$

by (3.2.4) of the Technical Lemma 18,

$$\leq 2\int\frac{f}{|x|^2}|\nabla N||\nabla f|e^{-g(N)}d\lambda.$$

By (4.1.5) of the Technical Lemma 18,

$$\int f^2\left(A\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda \leq 2\sqrt{C}\int\frac{|f|}{N|x|}|\nabla f|e^{-g(N)}d\lambda.$$

We apply Cauchy's inequality with $\epsilon: ab\leq\frac{\epsilon a^2}{2}+\frac{b^2}{2\epsilon}$ with $a=\frac{|f|}{N|x|}e^{-\frac{g(N)}{2}}$ and $b=\sqrt{C}|\nabla f|e^{-\frac{g(N)}{2}}$, and obtain

$$\int f^2\left(A\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda \leq \epsilon\int\frac{|f|^2}{N^2|x|^2}e^{-g(N)}d\lambda+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda. \tag{3.2.6}$$

This inequality (3.2.6) resembles the required $U-$bound inequality (3.2.1), except for the different term $\epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda$. Due to the presence of the singularity at $|x|$, the first thing that comes to mind is to apply Hardy's inequality. We proceed by giving a few trials (that don't work) that will naturally lead to a proof that works. The reason we provide a description of the failed trials is to explain why we don't have a simpler proof and to justify why we use a complicated form of Hardy's inequality on an "artificial" set. The goal of the trials is also to provide a good intuition for the problem and to show how the geometry of the Carnot group comes into play.

Trial 1: Hardy's Inequality

The following Hardy's inequality is classical and could be found in [119] and references within:

Theorem 19 (Hardy's Inequality)**.** *Let $\mathbb{G} \simeq \mathbb{R}^{n+m}$ be a step-two Carnot group with sub-gradient $\nabla = (X_1, ..., X_n)$. Then, for $h \in C_0^\infty(\mathbb{R}^{n+m})$,*

$$\int \frac{h^2}{|x|^2} d\lambda \leq \frac{4}{(n-2)^2} \int |\nabla h|^2 d\lambda. \tag{3.2.7}$$

Proof. By a direct computation,

$$\int \frac{h^2}{|x|^2} d\lambda = \frac{1}{(n-2)} \int h^2 \nabla \cdot \left(\frac{x}{|x|^2} \right) d\lambda$$

integrating by parts,

$$= -\frac{2}{(n-2)} \int h \nabla h \left(\frac{x}{|x|^2} \right) d\lambda$$

$$\leq \frac{2}{(n-2)} \int \frac{|h|}{|x|} |\nabla h| d\lambda$$

using Cauchy's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = \frac{|h|}{|x|}$ and $b = \frac{2}{(n-2)} |\nabla h|$,

$$\leq \frac{1}{2} \int \frac{h^2}{|x|^2} d\lambda + \frac{2}{(n-2)^2} \int |\nabla h|^2 d\lambda.$$

Rearranging, we obtain Hardy's inequality. □

Notice that Hardy's inequality holds for the Haar measure $d\lambda$ (equivalent to the Lebesgue measure), while we are dealing with the measure $e^{-g(N)} d\lambda$. To account for that, we apply Hardy's inequality (3.2.7) with

$$h = \frac{f e^{\frac{-g(N)}{2}}}{N}.$$

Therefore, (3.2.6) becomes

$$\int f^2\left(A\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda \leq \epsilon\int\frac{\left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)^2}{|x|^2}d\lambda+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda$$

$$\leq \frac{4\epsilon}{(n-2)^2}\int\left|\nabla\frac{fe^{\frac{-g(N)}{2}}}{N}\right|^2d\lambda+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda$$

$$=\frac{\epsilon}{(n-2)^2}\int f^2\frac{g'(N)^2}{N^2}|\nabla N|^2e^{-g(N)}d\lambda+\text{remainder terms}+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda.$$

The problem here is that the term $\frac{\epsilon}{(n-2)^2}\int f^2\frac{g'(N)^2}{N^2}|\nabla N|^2e^{-g(N)}d\lambda$ cannot be absorbed into the left-hand side of the inequality due to the squared power on $g'(N)$. Hence, Trial 1 fails.

Trial 2: Introducing $F=\{(x,z)\in\mathbb{R}^{n+m}:|x|\sqrt{g'(N)}<1\}$

Set F is artificial: its choice is specifically made for the proof to go through. In fact, splitting the first integral on the right-hand side of (3.2.6) over F and F^c :

$$\int f^2\left(A\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda$$

$$\leq \epsilon\int_F\frac{|f|^2}{N^2|x|^2}e^{-g(N)}d\lambda+\epsilon\int_{F^c}\frac{|f|^2}{N^2|x|^2}e^{-g(N)}d\lambda+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda$$

$$\leq \epsilon\int_F\frac{|f|^2}{N^2|x|^2}e^{-g(N)}d\lambda+\epsilon\int_{F^c}\frac{g'(N)}{N^2}e^{-g(N)}d\lambda+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda$$

choosing ϵ small enough, we can absorb the integral over F^c into the left-hand side and get:

$$\int f^2\left((A-\epsilon)\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda$$

$$\leq \epsilon\int_F\frac{|f|^2}{N^2|x|^2}e^{-g(N)}d\lambda+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda. \tag{3.2.8}$$

Hence, the next step is to reprove Hardy's inequality taking into account the boundary term and the geometry of the step-two Carnot group. The following is a proof of the $U-$bound Theorem 17:

Hardy's inequality on $F_r=\{(x,z)\in\mathbb{R}^{n+m}:|x|\sqrt{g'(N)}<r\}$, where $1\leq r\leq 2$

Proving Hardy's inequality on the set F leads to a "loss of measure" at the boundary term. To bypass this issue, we prove Hardy's inequality on sets F_r. Then, integrating this boundary term with respect to r for $1\leq r\leq 2$ leads to recovering the full measure by the Coarea formula (see p.

468 of [29]).

Theorem 20 (Hardy's Inequality for F_r)**.** *Let $\mathbb{G} \simeq \mathbb{R}^{n+m}$ be a step-two Carnot group, whose Lie algebra is generated by $\{X_i\}_{i=1}^n$. Define $F_r = \{(x,z) \in \mathbb{R}^{n+m} : |x|\sqrt{g'(N)} < r\}$. Then,*

$$\int_{F_r} \frac{h^2}{|x|^2} d\lambda \leq \frac{4}{(n-2)^2} \int_{F_r} |\nabla h|^2 \, d\lambda + \frac{2}{n-2} \int_{\partial F_r} \frac{h^2}{|x|^2} \sum_{j=1}^{n} \frac{x_j \langle X_j I, \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \rangle}{\left| \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \right|} dH^{n+m-1},$$

where I is the identity map on $\mathbb{R}^{n+m}$ and ∇_{euc} is the Euclidean gradient on $\mathbb{R}^{n+m}$.

The following proof is in the same spirit of the classical Hardy's inequality (3.2.7) proof.

Proof. By a direct computation,

$$\int_{F_r} \frac{h^2}{|x|^2} d\lambda = \frac{1}{n-2} \int_{F_r} h^2 \nabla \cdot \left(\frac{x}{|x|^2} \right) d\lambda$$

integrating by parts,

$$= -\frac{2}{n-2} \int_{F_r} h \nabla h \left(\frac{x}{|x|^2} \right) d\lambda + \frac{1}{n-2} \int_{\partial F_r} \frac{h^2}{|x|^2} \sum_{j=1}^{n} \frac{x_j \langle X_j I, \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \rangle}{\left| \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \right|} dH^{n+m-1}$$

$$\leq \frac{2}{n-2} \int_{F_r} \frac{|h|}{|x|} |\nabla h| d\lambda + \frac{1}{n-2} \int_{\partial F_r} \frac{h^2}{|x|^2} \sum_{j=1}^{n} \frac{x_j \langle X_j I, \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \rangle}{\left| \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \right|} dH^{n+m-1}$$

using Cauchy's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = \frac{|h|}{|x|}$ and $b = \frac{2}{(n-2)} |\nabla h|$,

$$\leq \frac{1}{2} \int_{F_r} \frac{h^2}{|x|^2} d\lambda + \frac{2}{(n-2)^2} \int_{F_r} |\nabla h|^2 d\lambda + \frac{1}{n-2} \int_{\partial F_r} \frac{h^2}{|x|^2} \sum_{j=1}^{n} \frac{x_j \langle X_j I, \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \rangle}{\left| \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \right|} dH^{n+m-1}.$$

Rearranging, we get Hardy's inequality for F_r. □

Choosing

$$h = \frac{f e^{\frac{-g(N)}{2}}}{N}$$

in the last Theorem 20, and noting that $F := F_1 \subset F_r \subset F_2$, we have

$$\int_F \frac{f^2 e^{-g(N)}}{N^2 |x|^2} d\lambda$$

$$\leq \frac{4}{(n-2)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + \frac{2}{n-2}\int_{\partial F_r}\frac{f^2e^{-g(N)}}{N^2|x|^2}\sum_{j=1}^{n}\frac{x_j\langle X_jI, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle}{\left|\nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\right|}dH^{n+m-1}. \tag{3.2.9}$$

Recovering the full measure using the Coarea formula

Integrating (3.2.9) with respect to r from 1 to 2 :

$$\int_1^2\int_F \frac{f^2e^{-g(N)}}{N^2|x|^2}d\lambda dr$$

$$\leq \frac{4}{(n-2)^2}\int_1^2\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda dr$$

$$+\frac{2}{n-2}\int_1^2\int_{\partial F_r}\frac{f^2e^{-g(N)}}{N^2|x|^2}\sum_{j=1}^{n}\frac{x_j\langle X_jI, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle}{\left|\nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\right|}dH^{n+m-1}dr.$$

To recover the full measure in the boundary term, we use the Coarea formula (see p. 468 of [29]):

$$\int_F \frac{f^2e^{-g(N)}}{N^2|x|^2}d\lambda$$

$$\leq \frac{4}{(n-2)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda$$

$$+\frac{2}{n-2}\int_{\{1<|x|\sqrt{g'(N)}<2\}}\frac{f^2e^{-g(N)}}{N^2|x|^2}\sum_{j=1}^{n}x_j\langle X_jI, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle d\lambda. \tag{3.2.10}$$

Going back to Trial 2

Recall in Trial 2, we split the first integral on the right-hand side of (3.2.6) over F and F^c, and we obtained (3.2.8):

$$\int f^2\left((A-\epsilon)\frac{g'(N)}{N^2}-\frac{B}{N^3}\right)e^{-g(N)}d\lambda \leq \epsilon\int_F\frac{|f|^2}{N^2|x|^2}e^{-g(N)}d\lambda + \frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda$$

by (3.2.10),

$$\leq \frac{4\epsilon}{(n-2)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda$$

$$+\frac{2\epsilon}{n-2}\int_{\{1<|x|\sqrt{g'(N)}<2\}}\frac{f^2e^{-g(N)}}{N^2|x|^2}\sum_{j=1}^{n}x_j\langle X_jI, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle d\lambda$$

$$+\frac{C}{\epsilon}\int|\nabla f|^2e^{-g(N)}d\lambda. \tag{3.2.11}$$

The remainder of the proof is to use the conditions of Theorem 17, the technical Lemma 18, the domain of integrations, and the given fields X_j that are the generators of the Lie algebra of the step-two Carnot group $\mathbb{G}$. From the computation, we find a suitable ϵ which is required to satisfy the condition

$$\frac{10\epsilon}{n-2} + \epsilon < A.$$

Here, A is the constant that appears in the lower bound of $|\nabla N|$ of the technical Lemma 18, i.e. (4.1.5):

$$A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\frac{|x|^2}{N^2}.$$

For completeness, we give the full details of the computations below.

Computations

We will be computing explicitly the terms on the right-hand side of (3.2.11). To make things simpler, let

$$I_1 = \frac{4\epsilon}{(n-2)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda$$

and

$$I_2 = \frac{2\epsilon}{n-2}\int_{\{1<|x|\sqrt{g'(N)}<2\}}\frac{f^2e^{-g(N)}}{N^2|x|^2}\sum_{j=1}^{n}x_j\langle X_jI, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle d\lambda.$$

Computing I_1

$$I_1 = \frac{4\epsilon}{(n-2)^2}\int_{F_2}\left|\nabla f\frac{e^{-\frac{g(N)}{2}}}{N} - f\frac{g'(N)}{2}\frac{\nabla N e^{-\frac{g(N)}{2}}}{N} - \frac{f\nabla N e^{-\frac{g(N)}{2}}}{N^2}\right|^2 d\lambda$$

$$\leq \frac{16\epsilon}{(n-2)^2}\int_{F_2}\frac{|\nabla f|^2}{N^2}e^{-g(N)}d\lambda + \frac{4\epsilon}{(n-2)^2}\int_{F_2}f^2g'(N)^2\frac{|\nabla N|^2}{N^2}e^{-g(N)}d\lambda$$

$$+\frac{16\epsilon}{(n-2)^2}\int_{F_2}f^2\frac{|\nabla N|^2}{N^4}e^{-g(N)}d\lambda.$$

Using (4.1.5) of the technical Lemma 18 and taking into consideration that $N > 1$ and on F_2, $|\nabla N|^2 \leq \frac{C|x|^2}{N^2} \leq \frac{4C}{N^2g'(N)}$,

$$I_1 \leq \tilde{C}\int_{F_2}|\nabla f|^2e^{-g(N)}d\lambda + \tilde{D}\int_{F_2}f^2e^{-g(N)}d\lambda + \frac{16\epsilon C}{(n-2)^2}\int_{F_2}f^2\frac{g'(N)}{N^4}e^{-g(N)}d\lambda.$$

We do not worry about the third term in this inequality since it is dominated by $\int f^2\frac{g'(N)}{N^2}e^{-g(N)}d\lambda$ for $N > 1$.

Computing I_2

For e_i the standard Euclidean basis on $\mathbb{R}^{n+m}$,

$$X_j I \cdot e_i = \begin{cases} 0 & \text{for} \quad i \neq j \quad \textit{and} \quad i \leq n \\ 1 & \text{for} \quad i = j \quad \textit{and} \quad i \leq n \\ \frac{1}{2}\sum_{l=1}^{n} \Lambda_{jl}^{(i)} x_l & \text{for} \quad n+1 \leq i \leq n+m. \end{cases}$$

$$\nabla_{euc}\left(|x|\sqrt{g'(N)}\right) \cdot e_i = \begin{cases} \frac{x_i\sqrt{g'(N)}}{|x|} + \frac{|x|^3 g''(N) x_i}{2\sqrt{g'(N)}N^3} & \text{for} \quad i = j \quad \textit{and} \quad i \leq n \\ \frac{a|x|g''(N)z_i}{4\sqrt{g'(N)}N^3} & \text{for} \quad n+1 \leq i \leq n+m \end{cases}$$

Taking the dot product and summing,

$$\sum_{j=1}^{n} x_j \langle X_j I, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle$$

$$\begin{aligned} &= |x|\sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2\sqrt{g'(N)}N^3} + \sum_{j=1}^{n} x_j \sum_{i=n+1}^{n+m} \left(\frac{a|x|g''(N)}{8\sqrt{g'(N)}N^3}\right) z_i \sum_{l=1}^{n} \Lambda_{jl}^{(i)} x_l \\ &= |x|\sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2\sqrt{g'(N)}N^3} + \left(\frac{a|x|g''(N)}{8\sqrt{g'(N)}N^3}\right) \sum_{i=n+1}^{n+m} z_i \sum_{j=1}^{n}\sum_{l=1}^{n} \Lambda_{jl}^{(i)} x_l x_j \\ &= |x|\sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2\sqrt{g'(N)}N^3}, \end{aligned}$$

where $\sum_{j=1}^{n}\sum_{l=1}^{n} \Lambda_{jl}^{(i)} x_l x_j = 0$ since $\Lambda_{jl}^{(i)}$ is skew symmetric.

Therefore, replacing,

$$\begin{aligned} I_2 &= \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} x_j \langle X_j I, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\rangle d\lambda \\ &= \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2\sqrt{g'(N)}}{N^2|x|} e^{-g(N)} d\lambda + \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2|x|^3 g''(N)}{2N^5\sqrt{g'(N)}} e^{-g(N)} d\lambda. \end{aligned}$$

Using the fact that we are integrating over $\{1 < |x|\sqrt{g'(N)} < 2\}$,

$$I_2 \leq \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda + \frac{8\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g''(N)}{N^5 g'(N)^2} e^{-g(N)} d\lambda.$$

Using the condition of Theorem 17 that $g''(N) \leq g'(N)^3 N^3$, we get

$$I_2 \leq \frac{10\epsilon}{n-2}\int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda.$$

Combining I_1 and I_2

Replacing the estimates I_1 and I_2 in (3.2.11) we get

$$\int f^2 \left((A-\epsilon)\frac{g'(N)}{N^2} - \frac{B}{N^3}\right) e^{-g(N)} d\lambda \leq I_1 + I_2 + \frac{C}{\epsilon}\int |\nabla f|^2 e^{-g(N)} d\lambda$$

$$\leq \tilde{C}\int |\nabla f|^2 e^{-g(N)} d\lambda + \tilde{D}\int f^2 e^{-g(N)} d\lambda$$

$$+\frac{16\epsilon C}{(n-2)^2}\int f^2 \frac{g'(N)}{N^4} e^{-g(N)} d\lambda + \left(\frac{10\epsilon}{n-2}\right)\int \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda.$$

On $\{N > 1\}$, $\int f^2 \left(\frac{B}{N^3}\right) e^{-g(N)} d\lambda$ and $\frac{16\epsilon C}{(n-2)^2}\int f^2 \frac{g'(N)}{N^4} e^{-g(N)} d\lambda$ are of lower order. So, choosing $\left(\frac{10\epsilon}{n-2} + \epsilon\right) < A$ we get

$$\int f^2 \left(\frac{g'(N)}{N^2}\right) e^{-g(N)} d\lambda \leq C\int |\nabla f|^2 e^{-g(N)} d\lambda + D\int f^2 e^{-g(N)} d\lambda.$$

Hence, we have proved the $U-$ bound Theorem 17 for $q = 2$.

Extension of the U-Bound to $q > 2$

For $q > 2$, replacing $|f|$ by $|f|^{\frac{q}{2}}$,we get:

$$\int \frac{g'(N)}{N^2}|f|^q d\mu \leq C\int \left|\nabla |f|^{\frac{q}{2}}\right|^2 d\mu + D\int |f|^q d\mu. \tag{3.2.12}$$

Calculating,

$$\begin{aligned}\int \left|\nabla |f|^{\frac{q}{2}}\right|^2 d\mu &= \int \left|\frac{q}{2}|f|^{\frac{q-2}{2}} (\operatorname{sgn}(f))\nabla f\right|^2 d\mu \\ &\leq \int \frac{q^2}{4}|f|^{q-2}|\nabla f|^2 d\mu.\end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned}\int \left|\nabla |f|^{\frac{q}{2}}\right|^2 d\mu &\leq \int \frac{q^2}{4}|f|^{q-2}|\nabla f|^2 d\mu \leq \frac{q^2}{4}\left(\int |f|^q d\mu\right)^{\frac{q-2}{q}}\left(\int |\nabla f|^q d\mu\right)^{\frac{2}{q}} \\ &\leq \frac{q(q-2)}{4}\int |f|^q d\mu + \frac{q}{2}\int |\nabla f|^q d\mu.\end{aligned} \tag{3.2.13}$$

Where the last inequality uses $ab \leq \frac{a^{p'}}{p'} + \frac{b^{q'}}{q'}$, with $a = \left(\int |f|^q d\mu\right)^{\frac{q-2}{q}}$, $b = \left(\int |\nabla f|^q d\mu\right)^{\frac{2}{q}}$, and p' and q' conjugates. Choosing $p' = \frac{q}{q-2}$, we obtain $\frac{1}{q'} = 1 - \frac{q-2}{q}$, so $q' = \frac{q}{2}$. Using the inequalities (7.2.14) and (7.2.15), we get,

$$\begin{aligned}\int \frac{g'(N)}{N^2}|f|^q d\mu &\leq C\int \left|\nabla |f|^{\frac{q}{2}}\right|^2 d\mu + D\int |f|^q d\mu \\ &\leq C'\int |\nabla f|^q d\mu + D'\int |f|^q d\mu.\end{aligned}$$

3.3 Poincaré Inequality

We now have the $U-$Bound Theorem 17 at our disposal and are ready to prove the $q-$Poincaré inequality using W. Hebisch and B. Zegarliński's Theorem 6 (introduced in Section 2.1.2 of Chapter 2).

As discussed previously in this chapter, $\frac{g'(N)}{N^2}$ from the $U-$Bound Theorem 17 plays the role of η in W. Hebisch and B. Zegarliński's Theorem 6. We note that in [1], R.A Adams proved coercive inequalities on $\mathbb{R}^n$ for measures of the form $d\mu = \exp(-\psi(x))dx$, where $\psi(x)$ has a polynomial behaviour $\rho(x)$, an exponential behavior $\exp(\rho(x))$, or the composition of the exponential j times $\exp^{(j)}(\rho(x))$. This motivates the following corollaries in our setting:

Corollary 21. *The Poincaré inequality for $q \geq 2$ holds for the measure*

$$d\mu = \frac{\exp\left(-\cosh\left(N^k\right)\right)}{Z} d\lambda,$$

where λ is the Lebesgue measure, and $k \geq 1$ in the setting of the step-two Carnot group.

Proof. $g(N) = \cosh\left(N^k\right)$, so $g'(N) = kN^{k-1}\sinh\left(N^k\right)$, and

$$g''(N) = k(k-1)N^{k-2}\sinh(N^k) + k^2 N^{2k-2}\cosh(N^k).$$

First, on $\{N \geq 1\}$, $g''(N) \leq k^3 N^{3k}\sinh^3(N^k) = g'(N)^3 N^3$, so the condition of Theorem 17 is satisfied. Second,

$$\begin{aligned}\int \frac{g'(N)}{N^2} f^q d\mu &= \int f^q \left[kN^{k-3}\sinh\left(N^k\right)\right] d\mu \\ &= \int_{\{N<1\}} f^q \left[kN^{k-3}\frac{e^{N^k} - e^{-N^k}}{2}\right] d\mu + \int_{\{N\geq 1\}} f^q \left[kN^{k-3}\frac{e^{N^k} - e^{-N^k}}{2}\right] d\mu \\ &\leq \int_{\{N<1\}} f^q \left[k\frac{e - e^{-1}}{2}\right] d\mu + C' \int_{\{N\geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N\geq 1\}} |f|^q d\mu \\ &\leq C\int |\nabla f|^q d\mu + D\int |f|^q d\mu.\end{aligned}$$

Thus, the conditions of Theorem 6 are satisfied for $\eta = kN^{k-3}\sinh\left(N^k\right)$, and $k \geq 1$. So, the Poincaré inequality holds for $q \geq 2$. □

The following corollary was proven J. Inglis, Theorem 4.5.5 of [79] for $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$ in the setting of Heisenberg-type group. Here, we obtain a generalised version for $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ in the setting of the step-two Carnot group.

Corollary 22. *The Poincaré inequality for $q \geq 2$ holds for the measure*

$$d\mu = \frac{\exp\left(-N^k\right)}{Z}d\lambda,$$

where λ is the Lebesgue measure, and $k \geq 4$ in the setting of the step-two Carnot group.

Proof. Let $g(N) = N^k$, so $g^{'}(N) = kN^{k-1}$, and $g''(N) = k(k-1)N^{k-2}$. First, on $\{N \geq 1\}$, $g''(N) \leq k^3N^{3k} = g'(N)^3N^3$, so the condition of Theorem 17 is satisfied. Second,

$$\begin{aligned}\int \frac{g'(N)}{N^2}f^q d\mu &= \int f^q\left[kN^{k-3}\right]d\mu \\ &= \int_{\{N<1\}} f^q\left[kN^{k-3}\right]d\mu + \int_{\{N\geq 1\}} f^q\left[kN^{k-3}\right]d\mu \\ &\leq \int_{\{N<1\}} kf^q d\mu + C^{'}\int_{\{N\geq 1\}}|\nabla f|^q d\mu + D^{'}\int_{\{N\geq 1\}}|f|^q d\mu \\ &\leq C\int|\nabla f|^q d\mu + D\int|f|^q d\mu.\end{aligned}$$

Thus, the conditions of Theorem 6 are satisfied for $\eta = kN^{k-3}$, and $k \geq 4$. So, the Poincaré inequality holds for $q \geq 2$. □

The following corollary improves Corollary 22 in an interesting way. Namely, at a cost of a logarithmic factor, we now get the Poincaré inequality for polynomial growth of order $k \geq 3$.

Corollary 23. *The Poincaré inequality for $q \geq 2$ holds for the measure*

$$d\mu = \frac{\exp\left(-N^k\log\left(N+1\right)\right)}{Z}d\lambda,$$

where λ is the Lebesgue measure, and $k \geq 3$ in the setting of the step-two Carnot group.

Proof. Let $g(N) = N^k\log\left(N+1\right)$,so $g^{'}(N) = kN^{k-1}\log(N+1) + \dfrac{N^k}{N+1}$, and

$$g''(N) = k(k-1)N^{k-2}\log(N+1) + \frac{2kN^{k-1}}{N+1} - \frac{N^k}{(N+1)^2}.$$

First, on $\{N \geq 1\}$,

$$g''(N) \leq k^3 N^{3k} \log^3(N+1) + \frac{N^{3k+3}}{(N+1)^3} + \frac{3k^2 N^{3k+1} \log^2(N+1)}{N+1} + \frac{3kN^{3k+2} \log(N+1)}{(N+1)^2}$$

$$= N^3 g'(N)^3,$$

so the condition of Theorem 17 is satisfied. Second,

$$\int \frac{g'(N)}{N^2} f^q d\mu$$

$$\begin{aligned}
&= \int f^q \left[kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1} \right] d\mu \\
&= \int_{\{N<1\}} f^q \left[kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1} \right] d\mu + \int_{\{N\geq 1\}} f^q \left[kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1} \right] d\mu \\
&\leq \int_{\{N<1\}} (k \log(2) + 1) f^q d\mu + C' \int_{\{N\geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N\geq 1\}} |f|^q d\mu \\
&\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu.
\end{aligned}$$

Thus, the conditions of Theorem 6 are satisfied for $\eta = kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1}$, and $k \geq 3$. So, the Poincaré inequality holds for $q \geq 2$. □

3.4 $\phi-$Logarithmic Sobolev Inequality

After proving the $q-$Poincaré inequality for measures as a function of the homogeneous norm $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$, a natural question would be if one could obtain other coercive inequalities. J.Inglis et al.'s Theorem 2.1 in [80] proved that, for the measure $d\mu = \frac{e^{-U} d\lambda}{Z}$, provided we have the $U-$Bound

$$\mu(|f|(|U|^\beta + |\nabla U|)) \leq A\mu|\nabla f| + B\mu|f|,$$

one obtains

$$\mu\left(|f| \left| \log \frac{|f|}{\mu|f|} \right|^\beta \right) \leq C\mu|\nabla f| + B\mu|f|.$$

We will first extend their theorem, and then we will use the $U-$bound Theorem 17 to get more general coercive inequalities.

Theorem 24. *Let U be a locally Lipschitz function on $\mathbb{R}^N$ which is bounded below such that $Z = \int e^{-U} d\lambda < \infty$, and $d\mu = \frac{e^{-U}}{Z} d\lambda$. Let $\phi : [0, \infty) \to \mathbb{R}^+$ be a non-negative, non-decreasing, concave function such that $\phi(0) > 0$, and $\phi'(0) > 0$. Assume the following classical Sobolev inequality is*

satisfied:

$$\left(\int |f|^{q+\epsilon} d\lambda\right)^{\frac{q}{q+\epsilon}} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda$$

for some a, $b \in [0, \infty)$, *and* $\epsilon > 0$. *Moreover, if for some* A, $B \in [0, \infty)$, *we have:*

$$\mu\left(|f|^q(\phi(U) + |\nabla U|^q)\right) \leq A\mu|\nabla f|^q + B\mu|f|^q,$$

Then, there exists constants C, $D \in [0, \infty)$ *such that:*

$$\mu\left(|f|^q \phi\left(\left|\log \frac{|f|^q}{\mu|f|^q}\right|\right)\right) \leq C\mu|\nabla f|^q + D\mu|f|^q,$$

for all locally Lipschitz functions f.

Notice that in W. Hebsich and B. Zegarliński's Theorem 7 of Section 2.1.2, to obtain the Logarithmic Sobolev inequality, a forced choice of

$$\eta = U + |\nabla U|^q$$

is required. In this theorem, to get the $\phi-$Logarithmic Sobolev inequality, the specific choice of η is

$$\eta = \phi(U) + |\nabla U|^q.$$

Moreover, this theorem is general in the sense that is is independent on the explicit homogeneous norm used. It will applied in Chapter 4 to obtain results for measures as a function of the fundamental solution to the sub-Laplacian in the setting of the Anisotropic Heisenberg group.

Proof. Using the mean value theorem, there exists $0 < c < x$ such that $\phi(x) - \phi(0) = x\phi'(c)$. Since ϕ is concave, then ϕ' is non-increasing, so $\phi'(c) \leq \phi(0)$. Therefore, $\forall x \geq 0$,

$$\phi(x) \leq x\phi'(0) + \phi(0) \tag{3.4.1}$$

By the fundamental theorem of calculus,

$$\phi(y) - \phi(x) = \int_x^y \phi'(t)dt = \int_0^{y-x} \phi'(t+x)dt \leq \int_0^{y-x} \phi'(t)dt = \phi(y-x) - \phi(0)$$

Where the last inequality uses the fact that ϕ' is non-increasing. Using (3.4.1), we get, $\forall y \geq x$:

$$\phi(y) - \phi(x) \leq \phi'(0)(y-x). \tag{3.4.2}$$

Suppose first that $\int |f|^q = 1$, and let $E = \{x \in \mathbb{R}^{\mathbb{N}} : \ |\log |f|^q| > U\}$.

$$\int |f|^q \phi\left(|\log |f|^q|\right) d\mu$$

$$= \int_E |f|^q \phi\left(|\log |f|^q|\right) d\mu + \int_{E^c} |f|^q \phi\left(|\log |f|^q|\right) d\mu$$

$$= \int_E |f|^q \left[\phi\left(|\log |f|^q|\right) - \phi(U)\right] d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi\left(|\log |f|^q|\right) d\mu$$

$$\leq \phi'(0) \int_E |f|^q \left(|\log |f|^q| - U\right) d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi(U) d\mu,$$

where the last inequality uses (7.2.9) on E, and uses the fact that ϕ is non-decreasing on E^c, hence, $|\log |f|^q| < U$. Let $E_1 = \{\log |f|^q > U\}$, $E_2 = \{\log |f|^q < -U\}$, and $c = \int_{E_1} |f|^q e^{-U} d\lambda$.

$$\int |f|^q \phi\left(|\log |f|^q|\right) d\mu$$

$$\leq \frac{c\phi'(0)q}{\epsilon Z} \int_{E_1} \frac{\left(|f| e^{-\frac{U}{q}}\right)^q}{c} \log\left(|f| e^{-\frac{U}{q}}\right)^{\epsilon} d\lambda + \phi'(0) \int_{E_2} e^{-U} d\mu + \int |f|^q \phi(U) d\mu$$

Using Jensen's inequality,

$$\leq \frac{c\phi'(0)(q+\epsilon)}{Z\epsilon} \log\left(\int_{E_1} \frac{\left(|f| e^{-\frac{U}{q}}\right)^{q+\epsilon}}{c} d\lambda\right)^{\frac{q}{q+\epsilon}} + \phi'(0) \int_{E_2} 1 d\mu + \int |f|^q \phi(U) d\mu$$

$$\leq \frac{c^{\frac{\epsilon}{q+\epsilon}} \phi'(0)(q+\epsilon)}{Z\epsilon} \log\left(\int \left(|f| e^{-\frac{U}{q}}\right)^{q+\epsilon} d\lambda\right)^{\frac{q}{q+\epsilon}} + \phi'(0) Z + \int |f|^q \phi(U) d\mu$$

Using classical Sobolev inequality,

$$\leq a \int |f|^q d\mu + b \int \left|\nabla (f e^{-\frac{U}{q}})\right|^q d\lambda + \phi'(0) Z + \int |f|^q \phi(U) d\mu$$

$$= a + \phi'(0) Z + b \int \left|(\nabla f) e^{-\frac{U}{q}} - \frac{f}{q} (\nabla U) e^{-\frac{U}{q}}\right|^q d\lambda + \int |f|^q \phi(U) d\mu$$

$$\leq a + \phi'(0) Z + b 2^{q-1} Z \int |\nabla f|^q d\mu + \frac{Z b 2^{q-1}}{q^q} \int |f|^q |\nabla U|^q d\mu + \int |f|^q \phi(U) d\mu$$

$$\leq a + \phi'(0) Z + b 2^{q-1} Z \int |\nabla f|^q d\mu + \max\{\frac{Z b 2^{q-1}}{q^q}, 1\} \int |f|^q \left(|\nabla U|^q + \phi(U)\right) d\mu$$

Using the $U-$bound in the Theorem's condition

$$\leq A + B \int |\nabla f|^q d\mu$$

Finally, replace $|f|^q$ by $\frac{|f|^q}{\mu |f|^q}$ to get the desired inequality. $\square$

We note that, in the classical $\mathbb{R}^n$ setting, the following inequality is used for hypercontractivity in Orlicz spaces ([16]) and for tight Orlicz-Sobolev inequalities ([136]).

Corollary 25. *$\phi(x) = (1+x)^\beta$, for $\beta \in (0,1]$ is non-negative, non-decreasing, and concave function satisfying $\phi(0) = 1 > 0$, and $\phi'(0) = \beta > 0$. Therefore, Theorem 24 applies, and*

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) \leq \mu\left(|f|^q\left(1+\left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|\right)^\beta\right) \leq C\mu|\nabla f|^q + D\mu|f|^q.$$

Corollary 26. *Let $h^{(1)}(x) = \log(\alpha + x)$, where $\alpha > 1$. Define recursively*

$$h^{(n)}(x) = \log(\alpha + h^{(n-1)}(x)).$$

Then, for all $n \geq 1$, $h^{(n)}(x) = \phi(x)$ of Theorem 24 . Therefore, we obtain

$$\mu\left(|f|^q \log^{*(n)}\left(\frac{|f|^q}{\mu|f|^q}\right)\right) \leq \mu\left(|f|^q h^{(n)}\left(\frac{|f|^q}{\mu|f|^q}\right)\right) \leq C\mu|\nabla f|^q + D\mu|f|^q,$$

where $\log^{*(n)}$ is the positive part of $\log^{(n)}$.

Proof. The proof proceeds by induction. For $n = 1$, $h^{(1)}(x) = \log(\alpha + x)$, so $h^{(1)}(x)' = \dfrac{1}{\alpha + x}$, and $h^{(1)}(x)'' = \dfrac{-1}{(\alpha+x)^2}$. $h^{(1)}(0) = \log(\alpha) > 0$, and $h^{(1)}(0)' = \dfrac{1}{\alpha} > 0$. In addition, $h^{(1)}(x)$ is non-negative, non-decreasing, and concave; hence the conditions of Theorem 24 are satisfied.

Assume it is true for $n = k$, prove it is true for $n = k+1$: $h^{(k+1)}(x) = \log(\alpha + h^{(k)}(x))$, so $h^{(k+1)}(x)' = \dfrac{h^{(k)}(x)'}{\alpha + h^{(k)}(x)}$, and $h^{(k+1)}(x)'' = \dfrac{h^{(k)}(x)''}{\alpha + h^{(k)}(x)} - \dfrac{h^{(k)}(x)'}{(\alpha + h^{(k)}(x))^2}$. The result follows directly. □

Returning to the measure as a function of the homogeneous norm $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$,

$$d\mu = \frac{e^{-N^p}}{Z} d\lambda,$$

we will prove using Theorem 17 and Theorem 24, that the $\beta-$Logarithmic Sobolev inequality $(0 < \beta \leq 1)$ (Corollary 25) holds for $q \geq 2$, yet fails for $1 < q < \dfrac{2p\beta}{p-1}$. To start with, we will show why the $\beta-$Logarithmic Sobolev inequality fails for $1 < q < \dfrac{2p\beta}{p-1}$. The proof follows closely to Hebisch and Zegarliński's Theorem 6.3 of [72].

Theorem 27. *Let $\mathbb{G}$ be a stratified group, and N be a smooth homogenous norm on $\mathbb{G}$. For $\alpha > 0$, $p \geq 1$, let $d\mu = \dfrac{e^{-\alpha N^p}}{Z} d\lambda$, where Z is the normalisation constant. The measure μ satisfies no β-Logarithmic Sobolev inequality $(0 < \beta \leq 1)$ for $1 < q < \dfrac{2p\beta}{p-1}$.*

Proof. The proof is by contradiction. Let x_0 be such that $(\nabla N)(x_0) = 0$. For $t > 0$ put $r = t^{\frac{-p+1}{2}}$, and

$f = \max\left[\min\left(\frac{2 - d(x, tx_0)}{r}, 1\right), 0\right]$. On $B(tx_0, 2r) = \{x : d(x, tx_0) \leq 2r\}$, by homogeneity, by Lemma 6.3 of [72], and by the fact that $(\nabla N)(x_0) = 0$, we have $|N(x) - N(tx_0)| \leq c_1 r^2$, so $|N(x)^p - N(tx_0)^p| \leq c_2$. Consequently, the exponential factor in μ is comparable to a constant on the support of f. Also, $|\nabla f| \leq \frac{1}{r}$, and

$$\mu|f|^q \approx r^Q \left(e^{-\alpha N^p(tx_0)}\right) \tag{3.4.3}$$

$$\log(\mu|f|^q) \approx -t^p \tag{3.4.4}$$

$$\mu|\nabla f|^q \approx r^{-q} r^Q e^{-\alpha N^p(tx_0)} \tag{3.4.5}$$

Choose t large enough so that $r < \frac{2}{3}$. On $B(tx_0, 2r)$, $2 - d(x, tx_0) \geq 2 - 2r \geq r$. Thus,

$$f = \max\left[\min\left(\frac{2 - d(x, tx_0)}{r}, 1\right), 0\right] = 1,$$

and $\log |f|^q = 0$.

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) = \mu\left(|f|^q \left(|\log|f|^q - \log\mu|f|^q|^\beta\right)\right) = \mu\left(|f|^q \left(|\log\mu|f|^q|^\beta\right)\right)$$

by (7.2.11),

$$\approx \mu(|f|^q t^{p\beta})$$

by (7.2.10)

$$\approx t^{p\beta} r^Q e^{-\alpha N^p(tx_0)}.$$

Assuming we have $\beta-$Logarithmic Sobolev inequality, and using (7.2.12), we get:

$$t^{p\beta} r^Q e^{-\alpha N^p(tx_0)} \leq M r^{-q} r^Q e^{-\alpha N^p(tx_0)}$$

since $r = t^{\frac{-p+1}{2}}$,

$$t^{p\beta} \leq M t^{-q(\frac{-p+1}{2})}.$$

For t large enough, we get a contradiction when $p\beta > \frac{q(p-1)}{2}$ i.e. for $q < \frac{2p\beta}{p-1}$. So, the measure μ satisfies no $\beta-$Logarithmic Sobolev inequality for $1 < q < \frac{2p\beta}{p-1}$. □

Now we prove that for $q \geq 2$, $\beta-$Logarithmic Sobolev inequality holds true for $d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda$,

where $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ and $0 < \beta \leq \dfrac{p-3}{p}$.

Theorem 28. *Let $\mathbb{G}$ be an step-two Carnot group. Let the probability measure be $d\mu = \dfrac{e^{-g(N)}}{Z} d\lambda$, where Z is the normalisation constant and $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ with $a \in (0, \infty)$. Let $g : [0, \infty) \to [0, \infty)$ be a differentiable increasing function such that $g'(N)$ is increasing, $g(N) \leq \left(c\dfrac{g'(N)}{N^2}\right)^{\frac{1}{\beta}}$, and $g''(N) < dg'(N)^2$ on $\{N \geq 1\}$, for some constants c and d. Then*

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) \leq C\mu|f|^q + D\mu|\nabla f|^q,$$

for C and D positive constants and for $q \geq 2$.

Proof. On $\{N \geq 1\}$, $g''(N) < dg'(N)^2 \leq g'(N)^3 N^3$, so the condition of Theorem 17 is satisfied. Thus, on $\{N \geq 1\}$, we have the U-bound (3.2.1):

$$\mu\left(\frac{g'(N)}{N^2}|f|^q\right) \leq C\mu|\nabla f|^q + D\mu|f|^q.$$

By the condition $g(N) \leq \left(c\dfrac{g'(N)}{N^2}\right)^{\frac{1}{\beta}}$, we obtain $\phi(g(N)) = (1 + g(N))^\beta \leq \dfrac{g'(N)}{N^2}$ on $\{N \geq 1\}$. Hence, since $g(N)$ is increasing and using the U-bound,

$$\begin{aligned}
\mu\left(\phi(g(N))|f|^q\right) &\leq \int_{\{N\geq 1\}} \left(\frac{g'(N)}{N^2}|f|^q\right) d\mu + \int_{\{N<1\}} \phi(g(N))|f|^q d\mu \\
&\leq \int_{\{N\geq 1\}} \left(\frac{g'(N)}{N^2}|f|^q\right) d\mu + \int_{\{N<1\}} (1 + g(1))^\beta |f|^q d\mu \\
&\leq C\mu|\nabla f|^q + D\mu|f|^q.
\end{aligned} \tag{3.4.6}$$

In order to use Theorem 24, it remains to prove:

$$\mu\left(|f|^q|\nabla g(N)|^q\right) \leq C\mu|f|^q + D\mu|\nabla f|^q. \tag{3.4.7}$$

On $\{N < 1\}$, since $g'(N)$ is increasing and using (4.1.5),

$$\int_{\{N<1\}} \left(|f|^q|\nabla g(N)|^q\right) d\mu = \int_{\{N<1\}} \left(|f|^q|g'(N)\nabla N|^q\right) d\mu \leq C^{\frac{q}{2}} \int_{\{N<1\}} |f|^q|g'(1)|^q d\mu.$$

We now need to consider $\{N \geq 1\}$:

$$\int |f|^q \left(\nabla g(N) \cdot V - \nabla \cdot V\right) d\mu = \int \nabla|f|^q \cdot V d\mu \leq \frac{\epsilon}{p} \int |f|^q|V|^p d\mu + \frac{1}{\epsilon^{\frac{q}{p}}} q^{q-1} \int |\nabla f|^q d\mu, \tag{3.4.8}$$

where the last inequality uses $ab \leq \epsilon\dfrac{a^p}{p} + \dfrac{b^q}{\epsilon^{\frac{q}{p}} q}$, where $a = |f|^{q-1}|V|$, and $b = q|\nabla f|$.

Let $V = \nabla N \frac{|x|^{q-2}}{N^{q-2}} g'(N)^{q-1}$. Since $\nabla g(N) = g'(N)\nabla N$, then $\nabla g(N) \cdot V = |\nabla g(N)|^q$, which is the term on the left hand side of (4.2.12). Using the inequality (4.1.5) on the first term on the right hand side of (3.4.8) we get

$$\begin{aligned} \frac{\epsilon}{p}\int |f|^q |V|^p d\mu &= \frac{\epsilon}{p}\int |\nabla N|^p |f|^q \frac{|x|^{(q-2)p} g'(N)^q}{N^{(q-2)p}} d\mu \\ &\leq \frac{\epsilon C^{\frac{p}{2}}}{p}\int |f|^q \frac{|x|^q g'(N)^q}{N^q} d\mu \end{aligned}$$

which can subtracted from the left hand side of (3.4.8) since by choosing ϵ small enough and noting that using (4.1.5),

$$\nabla g(N) \cdot V = |\nabla N|^2 \frac{|x|^{q-2}}{N^{q-2}} g'(N)^q \geq A \frac{|x|^q}{N^q} g'(N)^q.$$

It remains to compute $\nabla \cdot V$. Using $|\Delta N| \leq B\frac{|x|^2}{N^3}$, (2.2.1), and $\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}$, (4.1.7),

$$\nabla \cdot V = \Delta N \frac{|x|^{q-2}}{N^{q-2}} g'(N)^{q-1} + (q-2)\frac{|x|^q g'(N)^{q-1}}{N^{q+1}}$$

$$-(q-2)\frac{|\nabla N|^2 |x|^{q-2} g'(N)^{q-1}}{N^{q-1}} + (q-1) g'(N)^{q-2} g''(N) \frac{|x|^{q-2}|\nabla N|^2}{N^{q-2}}.$$

$$|\nabla \cdot V| \leq B\frac{|x|^q}{N^{q+1}} g'(N)^{q-1} + (q-2)\frac{|x|^q g'(N)^{q-1}}{N^{q+1}} + (q-2)\frac{C|x|^q g'(N)^{q-1}}{N^{q+1}} + C(q-1) g'(N)^{q-2} g''(N)\frac{|x|^q}{N^q}.$$

All terms can be absorbed by the first term in (3.4.8). Using (4.2.10) and (4.2.12), the condition of Theorem 24, is satisfied, and we obtain $\beta-$Logarithmic Sobolev inequality:

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) \leq C\mu|f|^q + D\mu|\nabla f|^q$$

for C and D positive constants. □

Corollary 29. *Let $\mathbb{G}$ be a step-two Carnot group and $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ with $a \in (0,\infty)$. Let the probability measure be $d\mu = \frac{e^{-\beta N^p}}{Z} d\lambda$, where Z is the normalisation constant. Then, for $p \geq 4$ and $0 < \beta \leq \frac{p-3}{p}$,*

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) \leq C\mu|f|^q + D\mu|\nabla f|^q,$$

for C and D positive constants and for $q \geq 2$.

Chapter 4

Coercive Inequalities on Anisotropic Heisenberg Group

This chapter is based on [34]. We extend the Anisotropic Heisenberg group to higher dimensions from $\mathbb{H}_4(1,2)$ on $\mathbb{R}^5$ to $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ on $\mathbb{R}^{2n+1}$, where $\theta > 0$. Then, we compute the fundamental solution for the sub-Laplacian N^{2-Q}, where Q is the homogeneous dimension, using R. Beals, B. Gaveau, and P. Greiner's [17] explicit integral representation:

Theorem (30). *The fundamental solution for the group* $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ *(i.e.* $\Delta N^{2-Q} = \Delta N^{-2n} = 0$ *on* $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)\setminus\{0\}$*) is given by the following homogeneous norm*

$$N(x,t) = \frac{\left(B^2+t^2\right)^{\frac{1}{4n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2}-\frac{1}{4n}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{2}}}$$

where $A = \frac{\theta x_1^2}{2} + \frac{\theta x_{n+1}^2}{2} + \frac{\theta}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$ *and* $B = \frac{\theta x_1^2}{4} + \frac{\theta x_{n+1}^2}{4} + \frac{\theta}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$.

For general measures of the form

$$d\mu = \frac{e^{-g(N)}}{Z}d\lambda,$$

we prove the $U-$Bound theorem:

Theorem (31). *Let* N^{-2n} *be the fundamental solution in the setting of the Anisotropic Heisenberg group* $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ *with* $\theta > 0$ *and* $n > 5$. *Let* $g : [0,\infty) \to [0,\infty)$ *be a differentiable increasing*

function such that $g''(N) \leq g'(N)^2$ *on* $\{N \geq 1\}$. *Then, for* $q \geq 2$,

$$\int \frac{g'(N)}{N^2}|f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu$$

holds outside the unit ball $\{N < 1\}$ *with* C *and* D *positive constants independent of a function* f *for which the right hand side is well defined.*

We apply this $U-$bound to W. Hebisch and B. Zegarliński's Theorem 6 (introduced in Section 2.1.2 of Chapter 2) to get the $q-$Poincaré inequality for $q \geq 2$. We utilise this $U-$Bound to get a $\beta-$Logarithmic Sobolev inequality for $d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda$, for $p \geq 4$, $q \geq 2$, and $0 < \beta \leq \frac{p-3}{p}$ (Corollary 38) by using Theorem 24 (proved in Section 3.4 of Chapter 3).

4.1 Extension of the Anisotropic Heisenberg group and the Fundamental Solution of the sub-Laplacian

Extension of the Anisotropic Heisenberg group

As mentioned in Section 1.7.3 of the Introduction, the Carnot group $\mathbb{G}$ is said to be polarizable if N, where N^{2-Q} is the fundamental solution to the sub-Laplacian, is $\infty-$harmonic in $\mathbb{G}\backslash\{0\}$ [14], i.e.

$$\triangle_\infty N := \frac{1}{2}\langle \nabla \left(|\nabla N|^2\right), \nabla N\rangle = 0 \qquad \text{in } \mathbb{G}\backslash\{0\}. \tag{4.1.1}$$

The concept of polarizable Carnot groups was first introduced by Z. Balogh and J. Tyson in [14], where they used the $\infty-$harmonicity of N to create a procedure to construct polar coordinates. As a consequence, they showed that the fundamental solution of the $p-$sub-Laplacian can be expressed as the fundamental solution of the sub-Laplacian, proved capacity formulas, and produced sharp constants for the Moser-Trudinger inequality. In settings related to polarizable Carnot groups, many authors showed Hardy-type inequalities as a function of N (see [48, 49, 62, 63, 85, 135]), proved Rellich-type inequalities (see [85, 138]), and studied Fuglede's $p-$module of system of measures [37].

However, we don't have a classification of polarizable Carnot groups: the only examples till now are Euclidean spaces and Heisenberg-type groups. This classification has been a longstanding open problem for two decades.

The concept of a polarizable Carnot group is a delicate one in the sense that under a small perturbation of the Lie algebra, the group is no longer polarizable. Z. Balogh and J. Tyson provided

in [14] the non-polarizable Anisotropic Heisenberg group $\mathbb{H}_4(1,2)$ on $\mathbb{R}^5$ with the dilation

$$\delta_\lambda(x_1,x_2,x_3,x_4,t)=(\lambda x_1,\lambda x_2,\lambda x_3,\lambda x_4,\lambda^2 t)$$

and the composition law

$$(x_1,x_2,x_3,x_4,t)\circ(\eta_1,\eta_2,\eta_3,\eta_4,\tau)$$

$$=(x_1+\eta_1,x_2+\eta_2,x_3+\eta_3,x_4+\eta_4,t+\tau+x_1\eta_3-x_3\eta_1+2x_2\eta_4-2\eta_2x_4),$$

where $(x_1,x_2,x_3,x_4,t)\in\mathbb{R}^5$ and $(\eta_1,\eta_2,\eta_3,\eta_4,\tau)\in\mathbb{R}^5$.

$\mathbb{H}_4(1,2)$ is a homogeneous Carnot group of step two with generators

$$X_1=\partial_{x_1}-x_3\partial_t,\ X_2=\partial_{x_2}-2x_4\partial_t,\ X_3=\partial_{x_3}+x_1\partial_t,\ \text{and}\ \ X_4=\partial_{x_4}+2x_2\partial_t.$$

In this setting, the sub-gradient is $\nabla=(X_1,X_2,X_3,X_4)$, the sub-Laplacian is $\Delta=X_1^2+X_2^2+X_3^2+X_4^2$, and the homogeneous dimension is $Q=6$.

We note that just altering X_1 to "$X_1=\partial_{x_1}-2x_3\partial_t$" and X_3 to "$X_3=\partial_{x_3}+2x_1\partial_t$", makes the group a polarizable Heisenberg group. Hence, we observe that polarizability is affected by just a small perturbation of the Lie algebra.

In this chapter, we extend the Anisotropic Heisenberg group from $\mathbb{H}_4(1,2)$ on $\mathbb{R}^5$ to $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ on $\mathbb{R}^{2n+1}$, where $\theta>0$. Let the dilation be

$$\delta_\lambda(x_1,x_2,...,x_{2n},t)=(\lambda x_1,\lambda x_2,...,\lambda x_{2n},\lambda^2 t)$$

and let the composition law be

$$(x_1,x_2,...,x_{2n},t)\circ(\eta_1,\eta_2,...,\eta_{2n},\tau)$$

$$=\left(x_1+\eta_1,x_2+\eta_2,...,x_{2n}+\eta_{2n},t+\tau+\frac{\theta x_1\eta_{n+1}}{2}-\frac{\theta x_{n+1}\eta_1}{2}+\theta\sum_{j=2}^{n}(x_j\eta_{j+n}-\eta_j x_{j+n})\right),$$

where $(x_1,x_2,...,x_{2n},t)\in\mathbb{R}^{2n+1}$ and $(\eta_1,\eta_2,...,\eta_{2n},\tau)\in\mathbb{R}^{2n+1}$.

$\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ is a homogeneous Carnot group of step two with generators

$$X_j=\begin{cases}\partial_{x_1}-\dfrac{\theta x_{n+1}}{2}\partial_t & j=1\\ \partial_{x_{n+1}}+\dfrac{\theta x_1}{2}\partial_t & j=n+1\\ \partial_{x_j}-\theta x_{j+n}\partial_t & j=2,3,..,n\\ \partial_{x_j}+\theta x_{j-n}\partial_t & j=n+2,n+3,..,2n.\end{cases} \tag{4.1.2}$$

In this setting, the sub-gradient is $\nabla=(X_1,X_2,...,X_{2n})$ and the sub-Laplacian is $\Delta=\sum_{i=1}^{2n}X_i^2$, and the homogeneous dimension is $Q=2n+2$.

We note that for $n=2$ and $\theta=2$, we get back the setting considered by Z. Balogh and J. Tyson in [14], as introduced above.

The generalisation of the Anisotropic Heisenberg group that we consider, in addition to the results we obtain in this chapter, were used by J. Tyson [128] to formulate the suitable Conjecture 5 (stated in Section 1.7.3 of the Introduction). His conjecture relates the new notion of $H-$type stability to the classification of polarizable groups, and if true, it would imply that all step-two polarizable Carnot groups are Heisenberg-type groups.

The Fundamental Solution of the sub-Laplacian

Z. Balogh and J. Tyson showed in Example 6.6 of [14] that the fundamental solution of the sub-Laplacian for the group $\mathbb{H}_4(1,2)$ (i.e. $\Delta N^{2-Q}=\Delta N^{-4}=0$ on $\mathbb{H}_4(1,2)\setminus\{0\}$) is given by the following homogeneous norm

$$N(x,t)=\frac{\left(B^2+t^2\right)^{\frac{1}{8}}\left(A-B+\sqrt{B^2+t^2}\right)^{\frac{3}{8}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{8}}}, \tag{4.1.3}$$

for $(x,t)=(x_1,x_2,x_3,x_4,t)\in\mathbb{R}^5$, $A=x_1^2+x_2^2+x_3^2+x_4^2$, and $B=\frac{x_1^2}{2}+\frac{x_3^2}{2}+x_2^2+x_4^2$.

We note that Z. Balogh and J. Tyson showed in [14] that this homogeneous norm (4.1.3) does not satisfy the polarizability condition (4.1.1).

In the setting of the higher dimensional Anisotropic Heisenberg group $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ introduced above, we compute explicitly the fundamental solution of the sub-Laplacian:

Theorem 30. *The fundamental solution for the group* $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$

(i.e. $\Delta N^{2-Q} = \Delta N^{-2n} = 0$ *on* $\mathbb{H}_{2n}\left(\frac{\theta}{2}, \theta\right)\backslash\{0\}$*) is given by the following homogeneous norm*

$$N(x,t) = \frac{\left(B^2+t^2\right)^{\frac{1}{4n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2}-\frac{1}{4n}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{2}}} \tag{4.1.4}$$

where $A = \frac{\theta x_1^2}{2} + \frac{\theta x_{n+1}^2}{2} + \frac{\theta}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$ *and* $B = \frac{\theta x_1^2}{4} + \frac{\theta x_{n+1}^2}{4} + \frac{\theta}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$.

We note that for $n = 2$ and $\theta = 2$, and by using the identity

$$\left(AB+t^2+A\sqrt{B^2+t^2}\right) = \left(B+\sqrt{B^2+t^2}\right)\left(A-B+\sqrt{B^2+t^2}\right),$$

we get back the fundamental solution (4.1.3) as calculated in [14] .

Proof of Theorem 30

To compute explicitly the fundamental solution, we use R. Beals, B. Gaveau, and P. Greiner's Theorem 2 of [17], which states:

For $\mathbb{G}$ a step-two Carnot group, the fundamental solution for the sub-laplacian $\Delta = \sum_{j=1}^{2n} X_j^2$ on $\mathbb{G}$ with singularity at 0 is:

$$u(x,t) = \frac{\Gamma(\frac{Q}{2}-1)}{2(2\pi)^{\frac{Q}{2}}}\int_{\mathbb{R}} \frac{V(\tau)}{f(x,t,\tau)^{\frac{Q}{2}-1}} d\tau, \tag{4.1.5}$$

for $(x,t) \in \mathbb{G}$, $x \neq 0$. $Q = 2n+2$ is the homogeneous dimension, $V : \mathbb{R} \to \mathbb{R}$ is such that

$$V(\tau) = \prod_{j=1}^{2n} w_j(\tau)^{\frac{1}{2}} \operatorname{csch}(w_j(\tau))^{\frac{1}{2}},$$

and $f : \mathbb{G}\times\mathbb{R} \to \mathbb{C}$ is such that

$$f(x,t,\tau) = \frac{1}{2}\sum_{j=1}^{2n} w_j(\tau)\coth(w_j(\tau))|x\cdot e_j(\tau)|^2 - \sqrt{-1}t\tau.$$

In the case of the Anisotropic Heisenberg group $\mathbb{H}_{2n}\left(\frac{\theta}{2}, \theta\right)$, $w_j(\tau)$ are the eigenvalues and $e_j(\tau)$ are the corresponding normalised eigenvectors of the $2n \times 2n$ matrix $\Omega(\tau) = -\sqrt{-1}\tau M$, where

$$M_{kl} = \begin{cases} -\frac{\theta}{2} & k = 1, \quad l = n+1 \\ \frac{\theta}{2} & k = n+1, \quad l = 1 \\ -\theta & k = 2, 3, ..., n, \quad l = k+n \\ \theta & k = n+2, n+3, ..., 2n, \quad l = k-n \\ 0 & \text{otherwise} \end{cases}$$

The eigenvalues are: $w_1(\tau) = -\frac{\theta\tau}{2}$, $w_{n+1}(\tau) = \frac{\theta\tau}{2}$, $w_j(\tau) = -\theta\tau$, for $j = 2, 3, .., n$, and $w_j(\tau) = \theta\tau$, for $j = n+2, n+3, ..., 2n$. The corresponding normalised eigenvectors are:

$$(e_j(\tau))_i = \begin{cases} -\frac{\sqrt{-1}}{\sqrt{2}} & i = j \\ \frac{1}{\sqrt{2}} & i = j+n \\ 0 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, ..., n$, and

$$(e_j(\tau))_i = \begin{cases} \frac{\sqrt{-1}}{\sqrt{2}} & i = j-n \\ \frac{1}{\sqrt{2}} & i = j \\ 0 & \text{otherwise,} \end{cases}$$

for $j = n+1, n+2, ..., 2n$. Thus,

$$V(\tau) = \prod_{j=1}^{2n} w_j(\tau)^{\frac{1}{2}} \operatorname{csch}(w_j(\tau))^{\frac{1}{2}}$$

$$= \left(\frac{(\theta\tau)^2}{4} \operatorname{csch}\left(\frac{\theta\tau}{2}\right)^2 (\theta\tau)^{2n-2} \operatorname{csch}(\theta\tau)^{2n-2} \right)^{\frac{1}{2}} = \frac{\theta^n \tau^n}{2} \operatorname{csch}\left(\frac{\theta\tau}{2}\right) \operatorname{csch}(\theta\tau)^{n-1}, \qquad (4.1.6)$$

$$
\begin{aligned}
f(x,t,\tau) &= \frac{1}{2}\sum_{j=1}^{2n} w_j(\tau)\coth(w_j(\tau))|x\cdot e_j(\tau)|^2 - \sqrt{-1}t\tau \\
&= \frac{1}{2}w_1(\tau)\coth(w_1(\tau))|x\cdot e_1(\tau)|^2 + \frac{1}{2}w_{n+1}(\tau)\coth(w_{n+1}(\tau))|x\cdot e_{n+1}(\tau)|^2 \\
&+\tfrac{1}{2}\textstyle\sum_{j=2,j\neq n+1}^{2n} w_j(\tau)\coth(w_j(\tau))|x\cdot e_j(\tau)|^2 - \sqrt{-1}t\tau \\
&= -\frac{\theta\tau}{4}\coth\left(-\frac{\theta\tau}{2}\right)\left|-\frac{\sqrt{-1}}{\sqrt{2}}x_1 + \frac{x_{n+1}}{\sqrt{2}}\right|^2 + \frac{\theta\tau}{4}\coth\left(\frac{\theta\tau}{2}\right)\left|\frac{\sqrt{-1}}{\sqrt{2}}x_1 + \frac{x_{n+1}}{\sqrt{2}}\right|^2 \\
&+\frac{1}{2}\sum_{j=2}^{n} -\theta\tau\coth(-\theta\tau)\left|-\frac{\sqrt{-1}}{\sqrt{2}}x_j + \frac{x_{j+n}}{\sqrt{2}}\right|^2 \\
&+\tfrac{1}{2}\textstyle\sum_{j=n+2}^{2n}\theta\tau\coth(\theta\tau)\left|\frac{\sqrt{-1}}{\sqrt{2}}x_j + \frac{x_{j+n}}{\sqrt{2}}\right|^2 - \sqrt{-1}t\tau \\
&= \frac{\tau}{2}\coth\left(\frac{\theta\tau}{2}\right)\left(\frac{\theta x_1^2}{2} + \frac{\theta x_{n+1}^2}{2}\right) + \frac{\tau}{2}\coth(\theta\tau)\left(\sum_{j=2,j\neq n+1}^{2n}\theta x_j^2\right) - \sqrt{-1}t\tau.
\end{aligned}
$$

Letting $A = \dfrac{\theta x_1^2}{2} + \dfrac{\theta x_{n+1}^2}{2} + \dfrac{\theta}{2}\displaystyle\sum_{j=2,j\neq n+1}^{2n} x_j^2$ and $B = \dfrac{\theta x_1^2}{4} + \dfrac{\theta x_{n+1}^2}{4} + \dfrac{\theta}{2}\displaystyle\sum_{j=2,j\neq n+1}^{2n} x_j^2$, we obtain:

$$
f(x,t,\tau) = \tau\coth\left(\frac{\theta\tau}{2}\right)(A-B) + \tau\coth(\theta\tau)(2B-A) - \sqrt{-1}t\tau. \tag{4.1.7}
$$

Replacing (4.1.6) and (4.1.7) in (4.1.5), we get:

$$
\begin{aligned}
u(x,t) &= \frac{\Gamma(n)}{2(2\pi)^{n+1}}\int_{\mathbb{R}}\frac{V(\tau)}{f(x,t,\tau)^n}d\tau \\
&= \frac{\Gamma(n)}{2(2\pi)^{n+1}}\int_{\mathbb{R}}\frac{\frac{\theta^n\tau^n}{2}\operatorname{csch}(\frac{\theta\tau}{2})\operatorname{csch}(\theta\tau)^{n-1}}{\left(\tau\coth\left(\frac{\theta\tau}{2}\right)(A-B) + \tau\coth(\theta\tau)(2B-A) - \sqrt{-1}t\tau\right)^n}d\tau \\
&= \frac{\Gamma(n)}{(2\pi)^{n+1}}\int_0^{\infty}\operatorname{Re}\left\{\frac{\frac{\theta^n\tau^n}{2}\operatorname{csch}(\frac{\theta\tau}{2})\operatorname{csch}(\theta\tau)^{n-1}}{\left(\tau\coth\left(\frac{\theta\tau}{2}\right)(A-B) + \tau\coth(\theta\tau)(2B-A) - \sqrt{-1}t\tau\right)^n}\right\}d\tau \\
&= \frac{\theta^n\Gamma(n)}{2(2\pi)^{n+1}}\int_0^{\infty}\operatorname{Re}\left\{\frac{\operatorname{csch}(\frac{\theta\tau}{2})\operatorname{csch}(\theta\tau)^{n-1}\coth\left(\frac{\theta\tau}{2}\right)^n}{\coth\left(\frac{\theta\tau}{2}\right)^n\left(\coth\left(\frac{\theta\tau}{2}\right)(A-B) + \coth(\theta\tau)(2B-A) - \sqrt{-1}t\right)^n}\right\}d\tau
\end{aligned}
$$

using the trigonometric identity $\coth(\theta\tau) = \dfrac{1}{2\coth\left(\frac{\theta\tau}{2}\right)} + \dfrac{\coth\left(\frac{\theta\tau}{2}\right)}{2}$,

$$= \frac{\theta^n\Gamma(n)}{2(2\pi)^{n+1}}\int_0^\infty \mathrm{Re}\left\{\frac{\mathrm{csch}(\frac{\theta\tau}{2})\mathrm{csch}(\theta\tau)^{n-1}\coth\left(\frac{\theta\tau}{2}\right)^n}{\left(\frac{A}{2}\coth\left(\frac{\theta\tau}{2}\right)^2 + B - \frac{A}{2} - \sqrt{-1}t\ \coth\left(\frac{\theta\tau}{2}\right)\right)^n}\right\}d\tau$$

using the trigonometric identities $\mathrm{csch}(\theta\tau) = \dfrac{(\mathrm{csch}\left(\frac{\theta\tau}{2}\right))^2}{2\coth\left(\frac{\theta\tau}{2}\right)}$ and $\coth\left(\frac{\theta\tau}{2}\right)^2 = 1 + \mathrm{csch}\left(\frac{\theta\tau}{2}\right)^2$,

$$= \frac{\theta^n\Gamma(n)}{(2\pi)^{n+1}}\int_0^\infty \mathrm{Re}\left\{\frac{\mathrm{csch}(\frac{\theta\tau}{2})\mathrm{csch}\left(\frac{\theta\tau}{2}\right)^{2n-2}\coth\left(\frac{\theta\tau}{2}\right)}{\left(A\mathrm{csch}\left(\frac{\theta\tau}{2}\right)^2 + 2B - 2\sqrt{-1}t\ \sqrt{1+\mathrm{csch}\left(\frac{\theta\tau}{2}\right)^2}\right)^n}\right\}d\tau$$

letting $u = \mathrm{csch}\left(\dfrac{\theta\tau}{2}\right)$, we have $du = -\dfrac{\theta}{2}\coth\left(\dfrac{\theta\tau}{2}\right)\mathrm{csch}\left(\dfrac{\theta\tau}{2}\right)d\tau$

$$= \frac{\theta^{n-1}\Gamma(n)}{\pi(2\pi)^n}\mathrm{Re}\left\{\int_0^\infty \frac{u^{2n-2}}{\left(Au^2+2B-2\sqrt{-1}t\ \sqrt{1+u^2}\right)^n}du\right\}$$

$$= \frac{\theta^{n-1}\Gamma(n)}{\pi(2\pi)^n}\mathrm{Re}\left\{\frac{(-1)^{n-2}}{(n-1)!}\frac{\partial^{n-2}}{\partial A^{n-2}}\int_0^\infty \frac{u^2}{\left(Au^2+2B-2\sqrt{-1}t\ \sqrt{1+u^2}\right)^2}du\right\}$$

$$= \frac{(-1)^{n-2}\theta^{n-1}}{\pi(2\pi)^n}\frac{d^{n-2}}{dA^{n-2}}\mathrm{Re}\left\{\int_0^\infty \frac{u^2}{\left(Au^2+2B-2\sqrt{-1}t\ \sqrt{1+u^2}\right)^2}du\right\}. \tag{4.1.8}$$

The next step is to compute

$$I(A,B,t) = \mathrm{Re}\left\{\int_0^\infty \frac{u^2}{\left(Au^2+2B-2\sqrt{-1}t\ \sqrt{1+u^2}\right)^2}du\right\}$$

$$= \int_0^\infty \frac{(Au^2+2B)^2u^2 - 4t^2(1+u^2)u^2}{\left((Au^2+2B)^2+4t^2(1+u^2)\right)^2}du.$$

Write $I(A,B,t) = I_1(A,B,t) - I_2(A,B,t)$, where

$$I_1(A,B,t) = \int_0^\infty \frac{u^2}{(Au^2+2B)^2+4t^2(1+u^2)}du$$

and

$$I_2(A,B,t) = \int_0^\infty \frac{8t^2(1+u^2)u^2}{\left((Au^2+2B)^2+4t^2(1+u^2)\right)^2}du.$$

Remark that

$$I_2(A,B,t) = -t\partial_t I_1(A,B,t),$$

so

$$I(A,B,t) = I_1(A,B,t) + t\partial_t I_1(A,B,t). \tag{4.1.9}$$

We now compute $I_1(A,B,t) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(Ax^2+2B)^2 + 4t^2(1+x^2)}dx$ using complex theory. Write denominator in the integrand as a polynomial

$$p(z) = (Az^2+2B)^2 + 4t^2(1+z^2) = A^2z^4 + (4AB+4t^2)z^2 + 4B^2 + 4t^2. \tag{4.1.10}$$

$p(z)$ has four complex roots. If z is a root, then $-z$ is a root. Since $p(z)$ has real coefficients, then $\bar{z}$ is a root, and $-\bar{z}$ is the fourth root. Let α be a root in the upper half plane, so $-\bar{\alpha}$ is also in the upper half plane. We thus have

$$p(z) = A^2(z-\alpha)(z+\alpha)(z-\bar{\alpha})(z+\bar{\alpha}) = A^2(z^4 - (\alpha^2+\bar{\alpha}^2)z^2 + |\alpha|^4).$$

By identification with (4.1.10), we get that $\alpha^2 + \bar{\alpha}^2 = -\frac{4AB+4t^2}{A^2}$ and $|\alpha|^4 = \frac{4B^2+4t^2}{A^2}$. From this, we can calculate $\mathrm{Im}(\alpha) = \frac{\alpha - \bar{\alpha}}{2i}$. In fact,

$$\mathrm{Im}(\alpha)^2 = -\frac{\alpha^2 + \bar{\alpha}^2 - 2|\alpha|^2}{4} = \frac{AB + t^2 + A\sqrt{B^2+t^2}}{A^2}. \tag{4.1.11}$$

Now, we go back to computing the integral: Let $f(z) = \frac{z^2}{(Az^2+2B)^2 + 4t^2(1+z^2)}$, by the residue theorem:

$$I_1(A,B,t) = \pi i \left(\mathrm{Res}(f,\alpha) + \mathrm{Res}(f,-\bar{\alpha})\right). \tag{4.1.12}$$

$$\mathrm{Res}(f,\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}(z-\alpha)\frac{z^2}{A^2(z-\alpha)(z+\alpha)(z-\bar{\alpha})(z+\bar{\alpha})} = \frac{\alpha}{2A^2(\alpha^2-\bar{\alpha}^2)}$$

and

$$\mathrm{Res}(f,-\bar{\alpha}) = \lim_{z\to-\bar{\alpha}}(z+\bar{\alpha})f(z) = \lim_{z\to-\bar{\alpha}}(z+\bar{\alpha})\frac{z^2}{A^2(z-\alpha)(z+\alpha)(z-\bar{\alpha})(z+\bar{\alpha})} = \frac{\bar{\alpha}}{2A^2(\alpha^2-\bar{\alpha}^2)}.$$

Replacing in (4.1.12), we get:

$$I_1(A,B,t) = \pi i \left(\mathrm{Res}(f,\alpha) + \mathrm{Res}(f,-\bar{\alpha})\right)$$

$$= \pi i \left(\frac{\alpha}{2A^2(\alpha^2-\bar{\alpha}^2)} + \frac{\bar{\alpha}}{2A^2(\alpha^2-\bar{\alpha}^2)}\right)$$

$$= \frac{2\pi i}{4A^2(\alpha - \bar{\alpha})} = \frac{\pi}{4A^2 \mathrm{Im}(\alpha)}$$

using (4.1.11)

$$= \frac{\pi}{4A\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{1}{2}}}.$$

We can now compute $I(A, B, t)$ using (4.1.9):

$$I(A,B,t) = I_1(A,B,t) + t\partial_t I_1(A,B,t)$$

$$= \frac{\pi}{4A\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{1}{2}}} + t\partial_t \frac{\pi}{4A\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{1}{2}}}$$

$$= \frac{\pi}{4A\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{1}{2}}} - \frac{\pi(2t^2\sqrt{B^2+t^2} + At^2)}{8A\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{3}{2}}}$$

$$= \pi \frac{2\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right) - 2t^2\sqrt{B^2+t^2} - At^2}{8A\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{3}{2}}}$$

$$= \pi \frac{2B\sqrt{B^2+t^2} + 2B^2 + t^2}{8\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{3}{2}}}$$

$$= \frac{\pi\left(B + \sqrt{B^2+t^2}\right)^2}{8\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{3}{2}}}.$$

Using (4.1.8),

$$u(x,t) = \frac{(-1)^{n-2}\theta^{n-1}}{\pi(2\pi)^n} \frac{d^{n-2}}{dA^{n-2}} I(A,B,t)$$

$$= \frac{(-1)^{n-2}\theta^{n-1}}{(2\pi)^n} \frac{d^{n-2}}{dA^{n-2}} \left(\frac{\left(B + \sqrt{B^2+t^2}\right)^2}{8\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{3}{2}}} \right)$$

$$= \frac{(-1)^{n-2}(-1)^{n-2}\theta^{n-1}\prod_{k=3}^{n}(2k-3)}{(2\pi)^n 2^{n-2}} \left(\frac{\left(B + \sqrt{B^2+t^2}\right)^{2+n-2}}{8\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{\frac{3}{2}+n-2}} \right)$$

$$= \frac{\theta^{n-1}\prod_{k=3}^{n}(2k-3)}{(2\pi)^n 2^{n+1}} \left(\frac{\left(B + \sqrt{B^2+t^2}\right)^n}{\sqrt{B^2+t^2}\left(AB + t^2 + A\sqrt{B^2+t^2}\right)^{n-\frac{1}{2}}} \right).$$

The fundamental solution is given up to a constant multiple:

$$N(x,t) = u(x,t)^{\frac{1}{2-Q}} = u(x,t)^{-\frac{1}{2n}}$$

using the last equation,

$$N(x,t) = \frac{\left(B^2+t^2\right)^{\frac{1}{4n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2}-\frac{1}{4n}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{2}}} \tag{4.1.13}$$

where $A = \frac{\theta x_1^2}{2} + \frac{\theta x_{n+1}^2}{2} + \frac{\theta}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$ and $B = \frac{\theta x_1^2}{4} + \frac{\theta x_{n+1}^2}{4} + \frac{\theta}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$.

4.2 U-Bound Inequality

In the setting of the Anisotropic Heisenberg group $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$, the goal is to utilise the following $U-$bound inequality (4.2.1) to obtain the $q-$Poincaré inequality for measures of the general form

$$d\mu = \frac{e^{-g(N)}}{Z}d\lambda,$$

where N^{-2n} is the fundamental solution of the sub-Laplacian, as computed in Theorem 30.

Similarly to Chapter 3, the plan is to utilise the $U-$bound inequality (4.2.1) to obtain the $q-$Poincaré inequality via W. Hebisch and B. Zegarliński's Theorem 6 (introduced in Section 2.1.2 of Chapter 2).

The $U-$bound Theorem is the following:

Theorem 31. *Let N^{-2n} be the fundamental solution in the setting of the Anisotropic Heisenberg group $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ with $\theta > 0$ and $n > 5$. Let $g : [0,\infty) \to [0,\infty)$ be a differentiable increasing function such that $g''(N) \leq g'(N)^2$ on $\{N \geq 1\}$. Let*

$$d\mu = \frac{e^{-g(N)}}{Z}d\lambda$$

be a probability measure and Z the normalisation constant. Then, for $q \geq 2$,

$$\int \frac{g'(N)}{N^2}|f|^q d\mu \leq C\int |\nabla f|^q d\mu + D\int |f|^q d\mu \tag{4.2.1}$$

holds outside the unit ball $\{N < 1\}$ with C and D positive constants independent of a function f for which the right hand side is well defined.

Recall that W. Hebisch and B. Zegarliński's Theorem 6 gave us freedom of choosing some η with the requirement that "η goes to infinity in all directions". In this case, for similar reasons as discussed in Chapter 3, we will show that the choice is

$$\eta = \frac{g'(N)}{N^2},$$

and the requirement would be $\eta \to \infty$ as $N \to \infty$.

The plan is to adapt the $U-$Bound proof introduced in Section 3.2 of Chapter 3 and to portray the differences between both settings. In particular, note that one difference is that the dimension in this theorem is required to be $n > 5$, for reasons that will be discussed in this section.

As in Section 3.2 of Chapter 3, we require the following technical lemma, which is proved in Section 7.2 of the Appendix 7:

Lemma 32. *The fundamental solution of the sub-Laplacian N^{-2n} on $\mathbb{H}_{2n}(\frac{\theta}{2}, \theta)$ satisfies*

$$x \cdot \nabla N \geq -\frac{\theta |x|^2}{4nN}, \tag{4.2.2}$$

$$|\nabla N|^2 \geq \frac{\theta^2 |x|^2}{2^{5+\frac{2}{n}} N^2}, \tag{4.2.3}$$

$$|\nabla N|^2 \leq \frac{\theta^2 (2n+1)^2 |x|^2}{2^3 n^2 N^2}, \tag{4.2.4}$$

where $|x| = \left(\sum_{j=1}^{2n} x_j^2 \right)^{\frac{1}{2}}$ is the Euclidean norm.

This lemma provides bounds for the two quantities $|\nabla N|$ and $x \cdot \nabla N$. In contrast with the technical Lemma 18 in Section 3.2 of Chapter 3, the bounds of this lemma are more precise in terms of the dimension n.

Recall that in the technical Lemma 18 in Section 3.2 of Chapter 3, in addition to bounds on $|\nabla N|$ and $x \cdot \nabla N$, a bound on a third quantity ΔN is needed. In the case of the Anisotropic Heisenberg group setting, since N^{-2n} is the fundamental solution of the sub-Laplacian, then ΔN is related to $|\nabla N|$ by:

$$\Delta N = (2n+1) \frac{|\nabla N|^2}{N}, \tag{4.2.5}$$

In fact, for any fundamental solution $\tilde{N}^{2-Q}$ to the sub-Laplacian, we have the relation

$$\Delta \tilde{N} = (Q-1) \frac{|\nabla \tilde{N}|^2}{\tilde{N}},$$

where Q is the homogeneous dimension of the group (see [29] for more details). Hence, we can deduce, from this technical lemma, bounds on ΔN that are necessary in the proof of the $U-$bound Theorem.

Moreover, notice that Technical Lemma 18 had the useful property $x \cdot \nabla N \geq 0$. Here, according to (4.2.2), the quantity $x \cdot \nabla N$ could be negative: we will show that this will lead to an additional "bad term" that we did not encounter in the proof of Theorem 17. This additional "bad term" is the reason behind the requirement that $n > 5$ in the $U-$bound Theorem 31.

Proof of Theorem 31

We proceed as in Theorem 17 (Section 3.2 of Chapter 3), while highlighting the differences between the two settings. As in Theorem 17, we first prove the result for $q = 2$.

The choice of η

Using integration by parts,

$$\int (\nabla N)\cdot(\nabla f)\, e^{-g(N)} d\lambda = -\int \nabla \left(\nabla N e^{-g(N)}\right) f d\lambda = -\int \Delta N f e^{-g(N)} d\lambda + \int |\nabla N|^2 f g^{'}(N)\, e^{-g(N)} d\lambda.$$

Since N^{-2n} is the fundamental solution, we relate ΔN to $|\nabla N|$ by (4.2.5). Hence,

$$\int |\nabla N|^2 f \left[g'(N) - \frac{(Q-1)}{N}\right] e^{-g(N)} d\lambda = \int (\nabla N)\cdot(\nabla f)\, e^{-g(N)} d\lambda, \tag{4.2.6}$$

As discussed in the proof of Theorem 17, we replace f by $\dfrac{f^2}{|x|^2}$ in order to obtain a suitable η that "goes to infinity in all directions" (i.e. we avoid the problem around the "z-axis").

Replacing f by $\dfrac{f^2}{|x|^2}$ and using (4.2.3) of the Technical Lemma 32, the left-hand side of (4.2.6) becomes:

$$\begin{aligned}\int |\nabla N|^2 \frac{f^2}{|x|^2} \left[g'(N) - \frac{(Q-1)}{N}\right] e^{-g(N)} d\lambda &\geq \frac{\theta^2}{2^{5+\frac{2}{n}}} \int f^2 \left[\frac{g'(N)}{N^2} - \frac{(Q-1)}{N^3}\right] e^{-g(N)} d\lambda \\ &\geq \frac{\theta^2}{2^{5+\frac{2}{n}}} \int f^2 \left[\frac{g'(N)}{N^2} - (Q-1)\right] e^{-g(N)} d\lambda. \end{aligned} \tag{4.2.7}$$

Where the last inequality is true since $N > 1$. Just like Theorem 17, the good candidate $\eta = \dfrac{g'(N)}{N}$ is obtained. For the right-hand side of (4.2.6),

$$\int (\nabla N)\cdot\left(\nabla\left(\frac{f^2}{|x|^2}\right)\right) e^{-g(N)} d\lambda = \int (\nabla N)\cdot\left[2f\frac{\nabla f}{|x|^2} - \frac{2f^2\nabla|x|}{|x|^3}\right] e^{-g(N)} d\lambda$$

$$= \int (\nabla N) \cdot \left[2f \frac{\nabla f}{|x|^2} - \frac{2f^2 x}{|x|^4} \right] e^{-g(N)} d\lambda. \tag{4.2.8}$$

Using the bound on $\nabla N \cdot x$ from (4.2.2) of the Technical Lemma 32, we get:

$$\int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda = \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int f^2 \frac{\nabla N \cdot x}{|x|^4} e^{-g(N)} d\lambda$$

$$\leq \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda + \frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$$

$$\leq 2 \int \frac{f}{|x|^2} |\nabla N| |\nabla f| e^{-g(N)} d\lambda + \frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda.$$

Notice that since $\nabla N \cdot x$ could be negative, we have the additional "bad term" $\frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$ that we did not encounter in the proof of Theorem 17. This additional "bad term" is the reason behind the requirement that $n > 5$. In fact, bounding this term by the left-hand side of the inequality will be our main concern in this proof. Using the bound on $|\nabla N|$ from (4.2.4) of the Technical Lemma 32, we get

$$\int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda$$

$$\leq \frac{(2n+1)\theta}{2^{\frac{1}{2}} n} \int \frac{|f|}{N|x|} |\nabla f| e^{-g(N)} d\lambda + \frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$$

applying Cauchy's inequality with $\beta : ab \leq \beta a^2 + \frac{b^2}{4\beta}$ with

$$a = \frac{|f|}{N|x|} e^{-\frac{g(N)}{2}}$$

and

$$b = \frac{(2n+1)}{2^{\frac{1}{2}} n} |\nabla f| e^{-\frac{g(N)}{2}},$$

$$\leq \beta \int \frac{f^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda + \frac{(2n+1)^2 \theta^2}{8n^2 \beta} \int |\nabla f|^2 e^{-g(N)} d\lambda.$$

Combining the last inequality with (4.2.7), we get

$$\frac{\theta^2}{2^{5+\frac{2}{n}}} \int f^2 \left[\frac{g'(N)}{N^2} - (Q-1) \right] e^{-g(N)} d\lambda \leq \beta \int \frac{f^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda \tag{4.2.9}$$

$$+ \frac{(2n+1)^2 \theta^2}{8n^2 \beta} \int |\nabla f|^2 e^{-g(N)} d\lambda.$$

This inequality (4.2.9) resembles the required $U-$bound inequality (4.2.1), except for the dif-

ferent terms $\beta \int \frac{f^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$. We notice that β could be chosen arbitrarily small, and since $N > 1$, we will only be concerned with the term $\frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$.

For n large enough, to be determined later, the goal is to bound $\frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$ by the right hand side of (4.2.9) by $\frac{\theta^2}{2^{5+\frac{2}{n}}} \int f^2 \frac{g'(N)}{N^2} e^{-g(N)} d\lambda$.

Due to the presence of the singularity at $|x|$, the plan is to apply Hardy's inequality, as in the proof of Theorem 17. In Section 3.2 of Chapter 3, we provided a few unsuccessful trials that led naturally to a proof of Theorem 17. By the same reasoning, for α a parameter to be determined later, we introduce the "artificial" set

$$F := \{(x, z) \in \mathbb{R}^{2n+1} : \alpha \frac{|x|^2 g'(N)}{N} < 1\},$$

specifically made for the proof to go through. Notice that in Theorem 17, $F = \{(x, z) \in \mathbb{R}^{n+m} : |x|^2 g'(N) < 1\}$. This is because, unlike this theorem, we were dealing with the term $\epsilon \int \frac{f^2}{N^2|x|^2} e^{-g(N)} d\lambda$, with a quadratic N in the denominator and a free choice of ϵ. We plan to proceed as in Theorem 17: proving Hardy's inequality over the set $F_r := \{(x, z) \in \mathbb{R}^{2n+1} : \alpha \frac{|x|^2 g'(N)}{N} < r\}$, where $1 \leq r \leq 2$ taking into account the boundary term, and then using the Coarea formula (see p. 468 of [29]) to recover the full measure.

Introducing $F := \{(x, z) \in \mathbb{R}^{2n+1} : \alpha \frac{|x|^2 g'(N)}{N} < 1\}$

Splitting the following integral on F and F^c :

$$\frac{\theta}{2n} \int \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda = \frac{\theta}{2n} \int_F \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda + \frac{\theta}{2n} \int_{F^c} \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda$$

$$\leq \frac{\theta}{2n} \int_F \frac{f^2}{N|x|^2} e^{-g(N)} d\lambda + \frac{\alpha\theta}{2n} \int_{F^c} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda. \tag{4.2.10}$$

where (4.2.10) is true since $F^c = \{(x, z) : \frac{1}{|x|^2} \leq \alpha \frac{g'(N)}{N}\}$. Notice that for a suitable α to be chosen later, the second term of (4.2.10) can be absorbed by the left-hand side of (4.2.9). The aim now is to estimate the first term of (4.2.10).

Hardy's inequality on $F_r = \left\{(x, z) \in \mathbb{R}^{2n+1} : \alpha \frac{|x|^2 g'(N)}{N} < r\right\}$, where $1 \leq r \leq 2$

As mentioned in the proof of Theorem 17, proving Hardy's inequality on the set F leads to a "loss of measure" at the boundary term. To bypass this issue, we prove Hardy's inequality on sets F_r. Then, integrating this boundary term with respect to r for $1 \leq r \leq 2$ leads to recovering the full

measure by the Coarea formula (see p. 468 of [29]).

Theorem 33 (Hardy's Inequality for F_r)**.** *Let $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ with $\theta > 0$ Anisotropic Heisenberg group, whose Lie algebra is generated by $\{X_i\}_{i=1}^{2n}$ as defined in (4.1.2). Define $F_r = \{(x,z) \in \mathbb{R}^{2n+1} : \alpha\frac{|x|^2 g'(N)}{N} < r\}$. Then,*

$$\int_{F_r} \frac{h^2}{|x|^2} d\lambda \leq \frac{1}{(n-1)^2}\int_{F_r} |\nabla h|^2 \, d\lambda + \frac{1}{n-1}\int_{\partial F_r} \frac{h^2}{|x|^2}\sum_{j=1}^{2n} \frac{x_j \langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle}{\left|\nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\right|} dH^{2n},$$

where I is the identity map on $\mathbb{R}^{2n+1}$ and ∇_{euc} is the Euclidean gradient on $\mathbb{R}^{2n+1}$.

This follows our proof of Theorem 20; the difference between the two settings is the set F_r we are considering.

Proof. By a direct computation,

$$\int_{F_r} \frac{h^2}{|x|^2} d\lambda = \frac{1}{2n-2}\int_{F_r} h^2 \nabla \cdot \left(\frac{x}{|x|^2}\right) d\lambda$$

integrating by parts,

$$= -\frac{1}{n-1}\int_{F_r} h\nabla h\left(\frac{x}{|x|^2}\right) d\lambda + \frac{1}{2n-2}\int_{\partial F_r} \frac{h^2}{|x|^2}\sum_{j=1}^{2n} \frac{x_j \langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle}{\left|\nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\right|} dH^{2n}$$

$$\leq \frac{1}{n-1}\int_{F_r} \frac{|h|}{|x|}|\nabla h| d\lambda + \frac{1}{2n-2}\int_{\partial F_r} \frac{h^2}{|x|^2}\sum_{j=1}^{2n} \frac{x_j \langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle}{\left|\nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\right|} dH^{2n}$$

using Cauchy's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ with $a = \frac{|h|}{|x|}$ and $b = \frac{1}{(n-1)}|\nabla h|$,

$$\leq \frac{1}{2}\int_{F_r} \frac{h^2}{|x|^2} d\lambda + \frac{1}{2(n-1)^2}\int_{F_r} |\nabla h|^2 d\lambda + \frac{1}{2n-2}\int_{\partial F_r} \frac{h^2}{|x|^2}\sum_{j=1}^{2n} \frac{x_j \langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle}{\left|\nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\right|} dH^{2n}.$$

Rearranging, we get Hardy's inequality for F_r. □

Choosing

$$h = \frac{f e^{\frac{-g(N)}{2}}}{N^{\frac{1}{2}}}$$

in the last Theorem 33, and noting that $F := F_1 \subset F_r \subset F_2$, we have

$$\left(\frac{\theta}{2n}\right)\int_F \frac{f^2 e^{-g(N)}}{N|x|^2} d\lambda$$

$$\leq \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N^{\frac{1}{2}}}\right)\right|^2 d\lambda$$

$$+\left(\frac{\theta}{2n}\right)\frac{1}{n-1}\int_{\partial F_r}\frac{f^2e^{-g(N)}}{N|x|^2}\sum_{j=1}^{2n}\frac{x_j\langle X_jI,\nabla_{euc}\left(\alpha\frac{|x|^2g'(N)}{N}\right)\rangle}{\left|\nabla_{euc}\left(\alpha\frac{|x|^2g'(N)}{N}\right)\right|}dH^{2n}. \tag{4.2.11}$$

Recovering the full measure using the Coarea formula

Integrating both sides of the inequality (4.2.11) from $r=1$ to $r=2$, we get:

$$\left(\frac{\theta}{2n}\right)\int_1^2\int_{F_1}\frac{|fe^{\frac{-g(N)}{2}}|^2}{N|x|^2}d\lambda dr$$

$$\leq \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_1^2\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N^{\frac{1}{2}}}\right)\right|^2 d\lambda dr$$

$$+\left(\frac{\theta}{2n}\right)\frac{1}{n-1}\int_1^2\int_{\partial F_r}\frac{f^2e^{-g(N)}}{N|x|^2}\sum_{j=1}^{2n}\frac{x_j\langle X_jI,\nabla_{euc}\left(\alpha\frac{|x|^2g'(N)}{N}\right)\rangle}{\left|\nabla_{euc}\left(\alpha\frac{|x|^2g'(N)}{N}\right)\right|}dH^{2n}dr$$

To recover the full measure in the boundary term, we use the Coarea formula:

$$\left(\frac{\theta}{2n}\right)\int_{F_1}\frac{|fe^{\frac{-g(N)}{2}}|^2}{N|x|^2}d\lambda \leq \frac{\theta}{2n(n-1)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N^{\frac{1}{2}}}\right)\right|^2 d\lambda$$

$$+\frac{\theta}{2n(n-1)}\int_{\{1<\alpha\frac{|x|^2g'(N)}{N}<2\}}\frac{f^2e^{-g(N)}}{N|x|^2}\sum_{j=1}^{2n}x_j\langle X_jI,\nabla_{euc}\left(\alpha\frac{|x|^2g'(N)}{N}\right)\rangle d\lambda. \tag{4.2.12}$$

Computations

It remains to compute the right hand side of (4.2.12): Define the first term to be

$$A := \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N^{\frac{1}{2}}}\right)\right|^2 d\lambda,$$

and the second term to be

$$B := \frac{\theta}{2n(n-1)}\int_{\{1<\alpha\frac{|x|^2g'(N)}{N}<2\}}\frac{f^2e^{-g(N)}}{N|x|^2}\sum_{j=1}^{2n}x_j\langle X_jI,\nabla_{euc}\left(\alpha\frac{|x|^2g'(N)}{N}\right)\rangle d\lambda.$$

Computing A

$$A = \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_{F_2}\left|\nabla\left(\frac{fe^{\frac{-g(N)}{2}}}{N^{\frac{1}{2}}}\right)\right|^2 d\lambda$$

$$= \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_{F_2}\left|\nabla f\frac{e^{-\frac{g(N)}{2}}}{N^{\frac{1}{2}}} - f\frac{g'(N)}{2}\frac{\nabla N e^{-\frac{g(N)}{2}}}{N^{\frac{1}{2}}} - \frac{1}{2}\frac{f\nabla N e^{-\frac{g(N)}{2}}}{N^{\frac{3}{2}}}\right|^2 d\lambda$$

$$= \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_{F_2}\frac{|\nabla f|^2}{N}e^{-g(N)}d\lambda + \left(\frac{\theta}{8n}\right)\frac{1}{(n-1)^2}\int_{F_2} f^2 g'(N)^2\frac{|\nabla N|^2}{N}e^{-g(N)}d\lambda$$

$$+\left(\frac{\theta}{8n}\right)\frac{1}{(n-1)^2}\int_{F_2} f^2\frac{|\nabla N|^2}{N^3}e^{-g(N)}d\lambda - \frac{\theta}{2n(n-1)^2}\int_{F_2} f\nabla f\cdot\nabla N\frac{g'(N)}{N}e^{-g(N)}d\lambda$$

$$-\frac{\theta}{2n(n-1)^2}\int_{F_2} f\nabla f\cdot\nabla N\frac{1}{N^2}e^{-g(N)}d\lambda + \frac{\theta}{4n(n-1)^2}\int_{F_2} f^2 g'(N)\frac{|\nabla N|^2}{N^2}e^{-g(N)}d\lambda.$$

Using (4.2.4),

$$A \le \left(\frac{\theta}{2n}\right)\frac{1}{(n-1)^2}\int_{F_2}\frac{|\nabla f|^2}{N}e^{-g(N)}d\lambda + \frac{(2n+1)^2\theta^3}{2^6n^3(n-1)^2}\int_{F_2} f^2 g'(N)^2\frac{|x|^2}{N^3}e^{-g(N)}d\lambda$$

$$+\frac{(2n+1)^2\theta^3}{2^6n^3(n-1)^2}\int_{F_2} f^2\frac{|x|^2}{N^5}e^{-g(N)}d\lambda + \frac{(2n+1)\theta^2}{2^{\frac{5}{2}}n^2(n-1)^2}\int_{F_2}|f||\nabla f|\frac{g'(N)|x|}{N^2}e^{-g(N)}d\lambda$$

$$+\frac{(2n+1)\theta^2}{2^{\frac{5}{2}}n^2(n-1)^2}\int_{F_2}|f||\nabla f|\frac{|x|}{N^3}e^{-g(N)}d\lambda + \frac{(2n+1)^2\theta^3}{2^5n^3(n-1)^2}\int_{F_2} f^2 g'(N)\frac{|x|^2}{N^4}e^{-g(N)}d\lambda.$$

Using the fact that $F_2 = \left\{\alpha\frac{|x|^2 g'(N)}{N} < 2\right\}$ and that $|x| \le C_n N$,

$$A \le \left(\frac{1}{2n}\right)\frac{\theta}{(n-1)^2}\int_{F_2}\frac{|\nabla f|^2}{N}e^{-g(N)}d\lambda + \frac{(2n+1)^2\theta^3}{\alpha 2^5n^3(n-1)^2}\int_{F_2} f^2\frac{g'(N)}{N^2}e^{-g(N)}d\lambda$$

$$+\frac{(2n+1)^2\theta^3C_n^2}{2^6n^3(n-1)^2}\int_{F_2} f^2\frac{1}{N^3}e^{-g(N)}d\lambda + \frac{(2n+1)\theta^2C_n}{2^{\frac{5}{2}}n^2(n-1)^2}\int_{F_2}|f||\nabla f|\frac{g'(N)}{N}e^{-g(N)}d\lambda$$

$$+\frac{(2n+1)\theta^2C_n}{2^{\frac{5}{2}}n^2(n-1)^2}\int_{F_2}|f||\nabla f|\frac{1}{N^2}e^{-g(N)}d\lambda + \frac{(2n+1)^2\theta^3}{\alpha 2^4n^3(n-1)^2}\int_{F_2} f^2\frac{1}{N^3}e^{-g(N)}d\lambda.$$

Using Cauchy's inequality with $\gamma : ab \le \gamma a^2 + \frac{b^2}{4\gamma}$ on

$$\frac{(2n+1)\theta^2C_n}{2^{\frac{5}{2}}n^2(n-1)^2}\int_{F_2}|f||\nabla f|\frac{g'(N)}{N}e^{-g(N)}d\lambda$$

with

$a = \frac{|f|g'(N)}{N}$ and $b = \frac{(2n+1)\theta^2C_n}{2^{\frac{5}{2}}n^2(n-1)^2}|\nabla f|$ and Cauchy's inequality with γ on

$$\frac{(2n+1)\theta^2C_n}{2^{\frac{5}{2}}n^2(n-1)^2}\int_{F_2}|f||\nabla f|\frac{1}{N^2}e^{-g(N)}d\lambda$$

with $a = \frac{|f|}{N^2}$ and $b = \frac{(2n+1)\theta^2 C_n}{2^{\frac{5}{2}} n^2 (n-1)^2} |\nabla f|$ in addition to $N > 1$,

$$A \leq \left(\left(\frac{\theta}{2n} \right) \frac{1}{(n-1)^2} + \frac{\theta^4 C_n^2 (2n+1)^2}{\gamma 2^6 n^4 (n-1)^4} \right) \int_{F_2} |\nabla f|^2 e^{-g(N)} d\lambda$$

$$+ \left(\frac{\theta^3 C_n^2 (2n+1)^2}{2^6 n^3 (n-1)^2} + \frac{\theta^3 (2n+1)^2}{\alpha 2^4 n^3 (n-1)^2} + \gamma \right) \int_{F_2} f^2 e^{-g(N)} d\lambda$$

$$+ \left(\frac{\theta^3 (2n+1)^2}{\alpha 2^5 n^3 (n-1)^2} + \gamma \right) \int_{F_2} f^2 \frac{g'(N)}{N^2} e^{-g(N)} d\lambda.$$

Hence,

$$A \leq \left(\frac{\theta^3 (2n+1)^2}{\alpha 2^5 n^3 (n-1)^2} + \gamma \right) \int_{F_2} f^2 \frac{g'(N)}{N^2} e^{-g(N)} d\lambda + C \int_{F_2} |\nabla f|^2 e^{-g(N)} d\lambda + D \int_{F_2} f^2 e^{-g(N)} d\lambda. \tag{4.2.13}$$

Computing B

For the second term of (4.2.12),

$$B = \left(\frac{\theta}{2n} \right) \frac{1}{n-1} \int_{\{1 < \alpha \frac{|x|^2 g'(N)}{N} < 2\}} \frac{f^2 e^{-g(N)}}{N|x|^2} \sum_{j=1}^{2n} x_j \langle X_j I, \nabla_{euc} \left(\alpha \frac{|x|^2 g'(N)}{N} \right) \rangle d\lambda.$$

On the Anisotropic Heisenberg group, for e_i the standard Euclidean basis on $\mathbb{R}^{2n+1}$,

$$X_j I \cdot e_i = \begin{cases} 0 & \text{for} \quad i \neq j \quad \text{and} \quad i \leq 2n \\ 1 & \text{for} \quad i = j \quad \text{and} \quad i \leq 2n \\ \sum_{l=1}^{2n} \Lambda_{jl} x_l & \text{for} \quad i = 2n+1, \end{cases}$$

where

$$\Lambda_{jl} = \begin{cases} -\frac{\theta}{2} & j = 1, \quad l = n+1 \\ \frac{\theta}{2} & j = n+1, \quad l = 1 \\ -\theta & j = 2, 3, ..., n, \quad l = k+n \\ \theta & j = n+2, n+3, ..., 2n, \quad l = k-n \\ 0 & \text{otherwise} \end{cases}$$

Also,

$$\nabla_{euc} \left(\alpha \frac{|x|^2 g'(N)}{N} \right) \cdot e_i$$

$$= \begin{cases} 2\alpha\dfrac{x_i g'(N)}{N} + \dfrac{\alpha|x|^2 g''(N)\partial_{x_i}N}{N} - \dfrac{\alpha|x|^2 g'(N)\partial_{x_i}N}{N^2} & \text{for } i=j \text{ and } i \le 2n \\ \dfrac{\alpha|x|^2 g''(N)\partial_t N}{N} - \dfrac{\alpha|x|^2 g'(N)\partial_t N}{N^2} & \text{for } i = 2n+1. \end{cases}$$

Taking the dot product and summing,

$$\sum_{j=1}^{2n} x_j \langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle$$

$$= 2\alpha\frac{|x|^2 g'(N)}{N} + \sum_{j=1}^{2n} x_j \frac{\alpha|x|^2 g''(N)\partial_{x_j}N}{N} - \sum_{j=1}^{2n} x_j \frac{\alpha|x|^2 g'(N)\partial_{x_j}N}{N^2}$$

$$+ \sum_{j=1}^{2n} x_j \left(\frac{\alpha|x|^2 g''(N)\partial_t N}{N} - \frac{\alpha|x|^2 g'(N)\partial_t N}{N^2}\right)\sum_{l=1}^{2n}\Lambda_{jl}x_l$$

$$= 2\alpha\frac{|x|^2 g'(N)}{N} + \frac{\alpha|x|^2 g''(N)x\cdot\nabla N}{N} - \frac{\alpha|x|^2 g'(N)x\cdot\nabla N}{N^2}$$

$$+ \left(\frac{\alpha|x|^2 g''(N)\partial_t N}{N} - \frac{\alpha|x|^2 g'(N)\partial_t N}{N^2}\right)\sum_{j=1}^{2n}\sum_{l=1}^{2n}\Lambda_{jl}x_l x_j$$

$$= 2\alpha\frac{|x|^2 g'(N)}{N} + \frac{\alpha|x|^2 g''(N)x\cdot\nabla N}{N} - \frac{\alpha|x|^2 g'(N)x\cdot\nabla N}{N^2},$$

where $\sum_{j=1}^{2n}\sum_{l=1}^{2n}\Lambda_{jl}x_l x_j = 0$ since Λ is skew symmetric. Hence, using (4.2.4),

$$\left|\sum_{j=1}^{2n} x_j \langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle\right| \le 2\alpha\frac{|x|^2 g'(N)}{N} + \frac{\alpha|x|^3 g''(N)|\nabla N|}{N} + \frac{\alpha|x|^3 g'(N)|\nabla N|}{N^2}$$

$$\le 2\alpha\frac{|x|^2 g'(N)}{N} + \frac{\alpha\theta(2n+1)|x|^4 g''(N)}{2^{\frac{3}{2}}nN^2} + \frac{\alpha\theta(2n+1)|x|^4 g'(N)}{2^{\frac{3}{2}}nN^3}.$$

Therefore, replacing,

$$B = \left(\frac{\theta}{2n}\right)\frac{1}{n-1}\int_{\{1<\alpha\frac{|x|^2 g'(N)}{N}<2\}} \frac{f^2 e^{-g(N)}}{N|x|^2}\sum_{j=1}^{2n} x_j\langle X_j I, \nabla_{euc}\left(\alpha\frac{|x|^2 g'(N)}{N}\right)\rangle d\lambda$$

$$\le \left(\frac{\theta}{n}\right)\frac{\alpha}{n-1}\int_{\{1<\alpha\frac{|x|^2 g'(N)}{N}<2\}} \frac{f^2 g'(N)}{N^2}e^{-g(N)}d\lambda$$

$$+ \frac{\theta^2\alpha(2n+1)}{2^{\frac{5}{2}}n^2(n-1)}\int_{\{1<\alpha\frac{|x|^2 g'(N)}{N}<2\}} \frac{f^2|x|^2 g''(N)}{N^3}e^{-g(N)}d\lambda$$

$$+ \frac{\theta^2\alpha(2n+1)}{2^{\frac{5}{2}}n^2(n-1)}\int_{\{1<\alpha\frac{|x|^2 g'(N)}{N}<2\}} \frac{f^2|x|^2 g'(N)}{N^4}e^{-g(N)}d\lambda.$$

Using the fact that we are integrating over $\{1 < \alpha\frac{|x|^2 g'(N)}{N} < 2\}$,

$$B \leq \left(\frac{\theta}{n}\right)\frac{\alpha}{n-1}\int_{\{1<\alpha\frac{|x|^2g'(N)}{N}<2\}}\frac{f^2g'(N)}{N^2}e^{-g(N)}d\lambda$$

$$+\frac{(2n+1)\theta^2}{2^{\frac{3}{2}}n^2(n-1)}\int_{\{1<\alpha\frac{|x|^2g'(N)}{N}<2\}}\frac{f^2g''(N)}{N^2g'(N)}e^{-g(N)}d\lambda$$

$$+\frac{\theta^2(2n+1)}{2^{\frac{3}{2}}n^2(n-1)}\int_{\{1<\alpha\frac{|x|^2g'(N)}{N}<2\}}\frac{f^2}{N^3}e^{-g(N)}d\lambda. \tag{4.2.14}$$

Using the condition of the theorem that $g''(N) \leq g'(N)^2$ on $\{N \geq 1\}$, we bound the second term on the right hand side of (4.2.14).

Combining A and B

Now we go back to (4.2.9):

$$\frac{\theta^2}{2^{5+\frac{2}{n}}}\int f^2\left[\frac{g'(N)}{N^2}-(Q-1)\right]e^{-g(N)}d\lambda$$

$$\leq \beta\int\frac{f^2}{N^2|x|^2}e^{-g(N)}d\lambda+\frac{\theta}{2n}\int\frac{f^2}{N|x|^2}e^{-g(N)}d\lambda+\frac{\theta^2(2n+1)^2}{8n^2\beta}\int|\nabla f|^2e^{-g(N)}d\lambda.$$

From (4.2.10) we have that

$$\frac{\theta}{2n}\int\frac{f^2}{N|x|^2}e^{-g(N)}d\lambda \leq \frac{\theta}{2n}\int_F\frac{f^2}{N|x|^2}e^{-g(N)}d\lambda+\frac{\alpha\theta}{2n}\int_{F^c}\frac{f^2g'(N)}{N^2}e^{-g(N)}d\lambda.$$

From (4.2.13) and (4.2.14) we have that

$$\frac{\theta}{2n}\int_F\frac{f^2}{N|x|^2}e^{-g(N)}d\lambda$$

$$\leq\left(\frac{(2n+1)^2\theta^3}{\alpha 2^5n^3(n-1)^2}+\gamma+\frac{\alpha\theta}{n(n-1)}\right)\int_{F_2}f^2\frac{g'(N)}{N^2}e^{-g(N)}d\lambda$$

$$+C\int_{F_2}|\nabla f|^2e^{-g(N)}d\lambda+D\int_{F_2}f^2e^{-g(N)}d\lambda.$$

We combine the last three inequalities and repeat the same procedure to $\beta\int\frac{f^2}{N^2|x|^2}e^{-g(N)}d\lambda$ to get:

$$\left(\frac{\theta^2}{2^{5+\frac{2}{n}}}-\frac{\alpha\theta}{2n}-\frac{(2n+1)^2\theta^3}{\alpha 2^5n^3(n-1)^2}-\frac{\alpha\theta}{n(n-1)}-\gamma-\beta\right)\int f^2\frac{g'(N)}{N^2}e^{-g(N)}d\lambda \tag{4.2.15}$$

$$\leq C\int_{F_2}|\nabla f|^2e^{-g(N)}d\lambda+D\int_{F_2}f^2e^{-g(N)}d\lambda.$$

Choosing the dimension n

To get the $U-$Bound, we need the left hand side of (4.2.15) to be positive i.e. we need to find suitable α, n, γ, and β. Since γ and β can be chosen to be arbitrarily small, we need to find solutions to the following inequality

$$\frac{\theta}{2^{5+\frac{2}{n}}} - \frac{\alpha}{2n} - \frac{(2n+1)^2\theta^2}{\alpha 2^5 n^3 (n-1)^2} - \frac{\alpha}{n(n-1)} > 0. \tag{4.2.16}$$

First, we determine α : Let $f(\alpha) = \frac{\theta}{2^{5+\frac{2}{n}}} - \frac{\alpha}{2n} - \frac{(2n+1)^2\theta^2}{\alpha 2^5 n^3 (n-1)^2} - \frac{\alpha}{n(n-1)}$.

$$f'(\alpha) = -\frac{1}{2n} + \frac{(2n+1)^2\theta^2}{\alpha^2 2^5 n^3 (n-1)^2} - \frac{1}{n(n-1)}.$$

Solving for $f'(\alpha) = 0$,

$$\alpha = \frac{\theta(2n+1)}{2^2 n (n^2-1)^{\frac{1}{2}}}.$$

Replacing α in (4.2.16),

$$\frac{\theta}{2^{5+\frac{2}{n}}} - \frac{\theta(2n+1)}{2^3 n^2 (n^2-1)^{\frac{1}{2}}} - \frac{\theta(n^2-1)^{\frac{1}{2}}(2n+1)}{2^3 n^2 (n-1)^2} - \frac{\theta(2n+1)}{2^2 n^2 (n-1)(n^2-1)^{\frac{1}{2}}} > 0$$

$$1 > 2^{3+\frac{2}{n}} \left(\frac{2n+1}{n^2}\right) \left(\frac{(n+1)^{\frac{1}{2}}}{(n-1)^{\frac{3}{2}}}\right).$$

This holds true if we choose $n > 5$. Hence, we get:

$$\int f^2 \left(\frac{g'(N)}{N^2}\right) e^{-g(N)} d\lambda \leq C \int |\nabla f|^2 e^{-g(N)} d\lambda + D \int f^2 e^{-g(N)} d\lambda,$$

and the $U-$bound Theorem 31 is proved for $q = 2$. For $q > 2$, the proof follows as Theorem 17.

4.3 $q-$Poincaré Inequality

We now have the $U-$Bound Theorem 31 at our disposal and are ready to prove the $q-$Poincaré inequality using W. Hebisch and B. Zegarliński's Theorem 6 (introduced in Section 2.1.2 of Chapter 2).

As discussed previously in this chapter, $\frac{g'(N)}{N^2}$ from the $U-$Bound Theorem 31 plays the role of η in W. Hebisch and B. Zegarliński's Theorem 6. We note that in [1], R.A Adams proved coercive inequalities on $\mathbb{R}^n$ for measures of the form $d\mu = \exp(-\psi(x))dx$, where $\psi(x)$ has a polynomial behaviour $\rho(x)$, an exponential behavior $\exp(\rho(x))$, or the composition of the exponential j times

$\exp^{(j)}(\rho(x))$. This motivates the following corollaries in our setting:

Corollary 34. *The Poincaré inequality for $q \geq 2$ holds for the measure*

$$d\mu = \frac{\exp\left(-\cosh\left(N^k\right)\right)}{Z} d\lambda,$$

where λ is the Lebesgue measure, and $k \geq 1$ in the setting of the anisotropic Heisenberg group $\mathbb{R}^{2n+1}$ with $n > 5$.

Proof. $g(N) = \cosh\left(N^k\right)$, so $g'(N) = kN^{k-1}\sinh\left(N^k\right)$, and

$$g''(N) = k(k-1)N^{k-2}\sinh(N^k) + k^2N^{2k-2}\cosh(N^k).$$

First

$$g''(N) \leq k^2N^{2k-2}\sinh^2(N^k) = g'(N)^2,$$

on the set $\{N > \frac{3}{2}\}$, so the condition of Theorem 31 is satisfied. Secondly,

$$\int \frac{g'(N)}{N^2} f^q d\mu = \int f^q \left[kN^{k-3}\sinh\left(N^k\right)\right] d\mu$$

$$= \int_{\left\{N<\frac{3}{2}\right\}} f^q \left[kN^{k-3}\frac{e^{N^k} - e^{-N^k}}{2}\right] d\mu + \int_{\left\{N\geq\frac{3}{2}\right\}} f^q \left[kN^{k-3}\frac{e^{N^k} - e^{-N^k}}{2}\right] d\mu$$

$$\leq \int_{\left\{N<\frac{3}{2}\right\}} f^q \left[\left(\frac{3}{2}\right)^{k-3} k\frac{e^{\left(\frac{3}{2}\right)^k} - e^{-\left(\frac{3}{2}\right)^k}}{2}\right] d\mu + C' \int_{\{N\geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N\geq 1\}} |f|^q d\mu$$

$$\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu.$$

Thus, the conditions of Theorem 6 are satisfied for $\eta = kN^{k-3}\sinh\left(N^k\right)$, and $k \geq 1$. So, the Poincaré inequality holds for $q \geq 2$. □

Corollary 35. *The Poincaré inequality for $q \geq 1$ holds for the measure*

$$d\mu = \frac{\exp\left(-N^k\right)}{Z} d\lambda,$$

where λ is the Lebesgue measure, and $k \geq 4$ in the setting of the anisotropic Heisenberg group $\mathbb{R}^{2n+1}$ with $n > 5$.

Proof. Let $g(N) = N^k$, so $g'(N) = kN^{k-1}$, and $g''(N) = k(k-1)N^{k-2}$. First

$$g''(N) \leq k^2N^{2k-2} = g'(N)^2,$$

so the condition of Theorem 31 is satisfied. Second,

$$\int \frac{g'(N)}{N^2} f^q d\mu = \int f^q \left[kN^{k-3}\right] d\mu = \int_{\{N<1\}} f^q \left[kN^{k-3}\right] d\mu + \int_{\{N\geq 1\}} f^q \left[kN^{k-3}\right] d\mu$$

$$\leq \int_{\{N<1\}} k f^q d\mu + C' \int_{\{N\geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N\geq 1\}} |f|^q d\mu$$

$$\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu.$$

Thus, the conditions of Theorem 6 are satisfied for $\eta = kN^{k-3}$, and $k \geq 4$. So, the Poincaré inequality holds for $q \geq 2$. □

The following corollary improves Corollary 35 in an interesting way. Namely, at a cost of a logarithmic factor, we now get the Poincaré inequality for polynomial growth of order $k \geq 3$.

Corollary 36. *The Poincaré inequality for $q \geq 2$ holds for the measure*

$$d\mu = \frac{\exp\left(-N^k \log(N+1)\right)}{Z} d\lambda,$$

where λ is the Lebesgue measure, and $k \geq 3$ in the setting of the anisotropic Heisenberg group $\mathbb{R}^{2n+1}$ with $n > 5$.

Proof. Let $g(N) = N^k \log(N+1)$, so $g'(N) = kN^{k-1}\log(N+1) + \frac{N^k}{N+1}$, and

$$g''(N) = k(k-1)N^{k-2}\log(N+1) + \frac{2kN^{k-1}}{N+1} - \frac{N^k}{(N+1)^2}.$$

First $g''(N) \leq k^2 N^{2k-2}(\log(N+1))^2 + \frac{N^{2k}}{(N+1)^2} + 2kN^{k-1}\log(N+1)\frac{N^{k-1}}{N+1} = g'(N)^2$, so the condition of Theorem 31 is satisfied. Secondly,

$$\int \frac{g'(N)}{N^2} f^q d\mu = \int f^q \left[kN^{k-3}\log(N+1) + \frac{N^{k-2}}{N+1}\right] d\mu$$

$$= \int_{\{N<1\}} f^q \left[kN^{k-3}\log(N+1) + \frac{N^{k-2}}{N+1}\right] d\mu + \int_{\{N\geq 1\}} f^q \left[kN^{k-3}\log(N+1) + \frac{N^{k-2}}{N+1}\right] d\mu$$

$$\leq \int_{\{N<1\}} (k\log(2)+1) f^q d\mu + C' \int_{\{N\geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N\geq 1\}} |f|^q d\mu$$

$$\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu.$$

Thus, the conditions of Theorem 6 are satisfied for $\eta = kN^{k-3}\log(N+1) + \frac{N^{k-2}}{N+1}$, and $k \geq 3$. So, the Poincaré inequality holds for $q \geq 2$. □

4.4 $\beta-$Logarithmic Sobolev Inequality

Theorem 27 in Section 3.4 of Chapter 3 proves that for

$$d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda,$$

where $\alpha > 0$, $p \geq 1$, N is a smooth homogeneous norm, and Z is the normalisation constant, the measure μ satisfies no β-logarithmic Sobolev inequality ($0 < \beta \leq 1$) for $1 < q < \frac{2p\beta}{p-1}$.

However for $p \geq 4$, $0 < \beta \leq \frac{p-3}{p}$, and $q \geq 2$, we will show that μ satisfies $\beta-$Logarithmic Sobolev inequality. To get the $\beta-$Logarithmic Sobolev inequality, we use the $\phi-$Logarithmic Sobolev Theorem 24, proved in Section 3.4 of Chapter 3, combined with the $U-$Bound Theorem 31, proved in this chapter. We note that, in the classical $\mathbb{R}^n$ setting, this kind of inequality is used for hypercontractivity in Orlicz spaces ([16]) and for tight Orlicz-Sobolev inequalities ([136]).

Theorem 37. *Let $\mathbb{R}^{2n+1}$ be the Anisotropic Heisenberg group with $n > 5$. Let the probability measure be $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$, where Z is the normalisation constant and N^{-2n} the fundamental solution. Let $g : [0, \infty) \to [0, \infty)$ be a differentiable increasing function such that $g'(N)$ is increasing, $g(N) \leq \left(c\frac{g'(N)}{N^2} \right)^{\frac{1}{\beta}}$, and $g''(N) < dg'(N)^2$ on $\{N \geq 1\}$, for some constants c and d. Then*

$$\mu\left(|f|^q \left| \log\left(\frac{|f|^q}{\mu|f|^q} \right) \right|^\beta \right) \leq C\mu|f|^q + D\mu|\nabla f|^q,$$

for C and D positive constants and for $q \geq 2$.

The proof follows closely that of Theorem 28 proved in Section 3.4 of Chapter 3, with some modification.

Proof. Choose $\phi(x) = (1 + x)^\beta$, which satisfies the conditions of the $\phi-$Logarithmic Sobolev Theorem 24 in Section 3.4 of Chapter 3. On $\{N \geq 1\}$, $g''(N) < dg'(N)^2 \leq g'(N)^3 N^3$, so the condition of the $U-$Bound Theorem 31 is satisfied. Thus, on $\{N \geq 1\}$, we have the $U-$bound (4.2.1):

$$\mu\left(\frac{g'(N)}{N^2} |f|^q \right) \leq C\mu|\nabla f|^q + D\mu|f|^q.$$

By the condition $g(N) \leq \left(c\frac{g'(N)}{N^2} \right)^{\frac{1}{\beta}}$, we obtain $\phi(g(N)) = (1 + g(N))^\beta \leq \frac{g'(N)}{N^2}$ on $\{N \geq 1\}$.

Hence, since $g(N)$ is increasing and using the $U-$bound,

$$\begin{aligned}\mu\left(\phi(g(N))|f|^q\right) &\leq \int_{\{N\geq 1\}}\left(\frac{g'(N)}{N^2}|f|^q\right)d\mu + \int_{\{N<1\}}\phi(g(N))|f|^q d\mu \\ &\leq \int_{\{N\geq 1\}}\left(\frac{g'(N)}{N^2}|f|^q\right)d\mu + \int_{\{N<1\}}(1+g(1))^\beta|f|^q d\mu \qquad (4.4.1)\\ &\leq C\mu|\nabla f|^q + D\mu|f|^q.\end{aligned}$$

In order to use the $\phi-$Logarithmic Sobolev Theorem 24, it remains to prove:

$$\mu\left(|f|^q|\nabla g(N)|^q\right) \leq C\mu|f|^q + D\mu|\nabla f|^q. \qquad (4.4.2)$$

On $\{N<1\}$, since $g'(N)$ is increasing and using (4.2.4),

$$\int_{\{N<1\}}\left(|f|^q|\nabla g(N)|^q\right)d\mu = \int_{\{N<1\}}\left(|f|^q|g'(N)\nabla N|^q\right)d\mu \leq \frac{\theta^q(2n+1)^q}{2^{\frac{3q}{2}}n^q}\int_{\{N<1\}}|f|^q|g'(1)|^q d\mu.$$

We now need to consider $\{N\geq 1\}$:

$$\int|f|^q\left(\nabla g(N)\cdot V - \nabla\cdot V\right)d\mu = \int\nabla|f|^q\cdot V d\mu \leq \frac{\epsilon}{p}\int|f|^q|V|^p d\mu + \frac{1}{\epsilon^{\frac{q}{p}}}q^{q-1}\int|\nabla f|^q d\mu, \quad (4.4.3)$$

where the last inequality uses $ab \leq \epsilon\frac{a^p}{p} + \frac{b^q}{\epsilon^{\frac{q}{p}}q}$, where $a = |f|^{q-1}|V|$, and $b = q|\nabla f|$. Let

$$V = \nabla N\frac{|x|^{q-2}}{N^{q-2}}g'(N)^{q-1}.$$

Since $\nabla g(N) = g'(N)\nabla N$, then $\nabla g(N)\cdot V = |\nabla g(N)|^q$, which is the term on the left hand side of (4.4.2). Using the inequality (4.2.4) on the first term on the right hand side of (4.4.3) we get

$$\begin{aligned}\frac{\epsilon}{p}\int|f|^q|V|^p d\mu &= \frac{\epsilon}{p}\int|\nabla N|^p|f|^q\frac{|x|^{(q-2)p}g'(N)^q}{N^{(q-2)p}}d\mu \\ &\leq \frac{\epsilon}{p}\frac{\theta^p(2n+1)^p}{2^{\frac{3p}{2}}n^p}\int|f|^q\frac{|x|^q g'(N)^q}{N^q}d\mu\end{aligned}$$

which can subtracted from the left hand side of (4.4.3) since by choosing ϵ small enough and noting

that using (4.2.3),

$$\nabla g(N)\cdot V = |\nabla N|^2 \frac{|x|^{q-2}}{N^{q-2}} g'(N)^q \geq \frac{\theta^2}{2^{5+\frac{2}{n}}} \frac{|x|^q}{N^q} g'(N)^q.$$

It remains to compute $\nabla \cdot V$. Using the fact that N^{-2n} is the fundamental solution,

$$\triangle N = \frac{(Q-1)|\nabla N|^2}{N},$$

$Q = 2n+2$ the homogeneous dimension and $|\nabla N|^2 \leq \dfrac{\theta^2(2n+1)^2|x|^2}{2^3 n^2 N^2}$ (4.2.4),

$$\nabla \cdot V = \Delta N \frac{|x|^{q-2}}{N^{q-2}} g'(N)^{q-1} + (q-2)\frac{|x|^{q-4} g'(N)^{q-1} x\cdot \nabla N}{N^{q-2}} - (q-2)\frac{|\nabla N|^2 |x|^{q-2} g'(N)^{q-1}}{N^{q-1}}$$

$$+(q-1) g'(N)^{q-2} g''(N) \frac{|x|^{q-2}|\nabla N|^2}{N^{q-2}}.$$

$$|\nabla \cdot V| \leq \frac{(Q-1)\theta^2(2n+1)^2}{2^3 n^2} \frac{|x|^q}{N^{q+1}} g'(N)^{q-1} + (q-2)\frac{\theta(2n+1)|x|^{q-2} g'(N)^{q-1}}{2^{\frac{3}{2}} n N^{q-1}}$$

$$+(q-2)\frac{\theta^2(2n+1)^2}{2^3 n^2}\frac{|x|^q g'(N)^{q-1}}{N^{q+1}} + (q-1)\frac{\theta^2(2n+1)^2}{2^3 n^2} g'(N)^{q-2} g''(N)\frac{|x|^q}{N^q}.$$

All terms can be absorbed by the first term in (4.4.3). Using (4.4.1) and (4.4.2), the condition of the $\phi-$Logarithmic Sobolev Theorem 24 is satisfied, and we obtain $\beta-$Logarithmic Sobolev inequality:

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) \leq C\mu|f|^q + D\mu|\nabla f|^q$$

for C and D positive constants independent of f. □

As a corollary, we obtain the following result:

Corollary 38. *Let $\mathbb{R}^{2n+1}$ be the Anisotropic Heisenberg group with $n > 5$ and N the homogeneous norm corresponding to the fundamental solution of the sub-Laplacian equation. Let the probability measure be $d\mu = \dfrac{e^{-\beta N^p}}{Z} d\lambda$, where Z is the normalisation constant. Then, for $p \geq 4$ and $0 < \beta \leq \dfrac{p-3}{p}$,*

$$\mu\left(|f|^q \left|\log\left(\frac{|f|^q}{\mu|f|^q}\right)\right|^\beta\right) \leq C\mu|f|^q + D\mu|\nabla f|^q,$$

for C and D positive constants and for $q \geq 2$.

Chapter 5

Poincaré inequality in Infinite Dimensions

This Chapter is based on [31]. We formulate general sufficient conditions for the existence of a Poincaré inequality for Gibbs measures on the infinite dimensional Carnot Group $\mathbb{G}^{\mathbb{Z}^d}$ in terms of weak U-bounds and weak single-site Poincaré inequalities. We then formulate general sufficient conditions on the associated local specification in terms of the phase and interaction so that these conditions hold and give examples of measures constructed using homogeneous norms (like the fundamental solution to the sub-Laplacian) and generalising previous results for the Carnot-Carathéodory distance.

For N that is the generalisation of the fundamental solution to the sub-Laplacian in the Heisenberg group (in the setting of step-two Carnot groups, Chapter (3)) or the fundamental solution to the sub-Laplacian in the Anisotropic Heisenberg group (in the setting of the Anisotropic Heisenberg group, Chapter (4)), we have the following result:

Theorem (45)**.** *Let $\alpha > 0$ and $p > 3$. Define a local specification on $\mathbb{G}^{\mathbb{Z}^{\mathbb{D}}}$ by the following potential:*

$$U_\Lambda^\omega(x_\Lambda) = \alpha \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda} \beta_{ij} Q(x_i, x_j) + \sum_{i\in\Lambda,\, j\notin\Lambda} \beta_{ij} Q(x_i, \omega_j).$$

Suppose that:

(1) *ϕ is a semi-bounded generalised polynomial in N of order p with leading term αN^p,*

(2) *$|\beta_{ij}| \leq \beta$ for all $i, j \in \mathbb{Z}^D$, and*

(3) *$Q(x_i, x_j) = \sum_{\ell=1}^k c^\ell \varrho^\ell(x_i)^{\alpha_\ell} \delta^\ell(x_j)^{\beta_\ell}$ is a finite sum of $k \in \mathbb{Z}_{\geq 1}$ products of powers of homogeneous norms $(\varrho^\ell, \delta^\ell)_{\ell=1,\cdots,k}$ with bounded sub-gradient and for $c^\ell \in \mathbb{R}$, $\alpha^\ell, \beta^\ell \in \{0\} \cup [1, \infty)$*

and $0 \leq \alpha^\ell + \beta^\ell \leq p$.

Then for all sufficiently small $\beta > 0$*, there exists a unique Gibbs measure* ν *which satisfies the* 2*-Poincaré inequality*

$$\nu|f - \nu f|^2 \leq C_{SG}\nu|\nabla f|^2.$$

We note that the key is to have a $U-$bound theorem that holds in the corresponding setting, i.e. Theorem 17 of Chapter (3) and Theorem 31 of Chapter (4).

5.1 Setup

The goal of this chapter is to extend the following Poincaré inequality to an infinite dimensional setting:

$$\int_\Omega |f - \int_\Omega f d\nu|^q d\nu \leq C \int_\Omega |\nabla f|^q d\nu, \tag{5.1.1}$$

where Ω is the Carnot group. We are interested in $\Omega = \mathbb{G}$, the step-two Carnot group, where ν is a probability measure as function of the homogeneous norm considered in Chapter 3. Moreover, we will also consider $\Omega = \mathbb{H}_{2n}\left(\frac{\theta}{2}, \theta\right)$ with $\theta > 0$, the generalised Anisotropic Heisenberg group, where ν is a probability measure as function of the fundamental solution of the sub-Laplacian considered in Chapter 4.

So far, the passage to infinite dimensions in the setting of Nilpotent Lie groups required the condition $|\nabla N| \geq c$ outside the unit ball ([82, 86]), which is not true for homogeneous norms considered in Chapter 3 and Chapter 4. In fact, the trouble arises from the "$z-$axis", where $|\nabla N| = 0$ outside the unit ball (see Subsection 2.1.1 of Chapter 2 for more details). However, using the $U-$bound inequality, we are able to provide a new method that overcomes those restrictions and allows for an extension to an infinite dimensional setting. The following is a brief setup and statement of the Poincaré inequality theorem in an infinite dimensional setting:

We are interested in the Poincaré inequality of the form (5.1.1) where the configuration space $\Omega = \mathbb{G}^{\mathbb{Z}^d}$ (or $\Omega = \mathbb{H}_{2n}\left(\frac{\theta}{2}, \theta\right)^{\mathbb{Z}^d}$) is the product of infinitely many copies of the the step-two Carnot group (or the generalised Anisotropic Heisenberg group) indexed by the $d-$dimensional integer lattice $\mathbb{Z}^d$.

This resembles many particle systems in condensed matter physics that we see in ionic compounds and crystals for example. On those systems, we have dynamics and PDEs that we would like to study. Hence, it is essential to use some estimates i.e. coercive inequalities (like the Poincaré inequality) that the Gibbs measures satisfy.

From the point of view of statistical mechanics, each lattice point $i \in \mathbb{Z}^d$ represents a particle

with unbounded spin taking values in $\mathbb{G}$ (or $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$), while a point $\omega \in \Omega$ represents a spin configuration, i.e. a description of the spin of every particle in the system. We write the following setup for the step-two Carnot group $\mathbb{G}$, but we note that it could be replaced by the generalised Anisotropic Heisenberg group $\mathbb{H}_{2n}\left(\frac{\theta}{2},\theta\right)$ for $\theta > 0$.

The measures ν on Ω that we consider are constructed in the following way. If $\Lambda \subset \mathbb{Z}^d$ is a compact subset and $\omega \in \Omega$, we define the local Gibbs measure on $\mathbb{G}^\Lambda \subset \Omega$ by

$$\mathbb{E}^\omega_\Lambda(dx_\Lambda) = \frac{1}{Z^\omega_\Lambda} e^{-U^\omega_\Lambda} dx_\Lambda$$

where dx_Λ is the Lebesgue measure on $\mathbb{G}^\Lambda$ and Z^ω_Λ is a normalisation factor, and where the potential U^ω_Λ is

$$U^\omega_\Lambda(x_\Lambda) \coloneqq \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda} \beta_{ij} V(x_i,x_j) + \sum_{i\in\Lambda,\, j\notin\Lambda} \beta_{ij} V(x_i,\omega_j). \tag{5.1.2}$$

Here $\phi \in C(\mathbb{G},\, \mathbb{R})$, $V \in C^1(\mathbb{G}\times\mathbb{G},\mathbb{R})$, and $|\beta_{ij}| < \beta$ for some $\beta > 0$ and for each pair $i,j \in \mathbb{Z}^d$. That is, each site $i \in \Lambda$ contributes both a phase part $\phi(x_i)$ and an interaction part $\sum_{j\in\mathbb{Z}^d} V(x_i,x_j)$ where $x_j = \omega_j$ whenever $j \notin \Lambda$.

We call the family $(\mathbb{E}^\omega_\Lambda)^{\omega\in\Omega}_{\Lambda\subset\mathbb{Z}^d}$ a local specification, and we denote the integral of a function $f : \mathbb{G}^\Lambda \to \mathbb{R}$ against the measure by

$$\mathbb{E}^\omega_\Lambda f = \int_{\mathbb{G}^\Lambda} f d\mathbb{E}^\omega_\Lambda. \tag{5.1.3}$$

The functions f appearing in (5.1.1) belong to $W^{1,q}(\nu) \coloneqq W^{1,q}(\Omega,\nu)$ where the L^q-norm of the sub-gradient is defined by

$$|\nabla_\Lambda f|^q \coloneqq \sum_{i\in\Lambda} |\nabla_i f|^q$$

for all subsets $\Lambda \subset \mathbb{Z}^d$, and where $\nabla_i f(\omega) = \nabla f^\omega_i(\omega_i) \coloneqq \nabla_{\{i\}} f^\omega_{\{i\}}(\omega_i)$ and $|\nabla_i f|^q$ is the $q-$th power of the L^2-norm of $\nabla_i f$.

With respect to the family $(\mathbb{E}^\omega_\Lambda)^{\omega\in\Omega}_{\Lambda\subset\mathbb{Z}^d}$, a measure ν on Ω is called a global Gibbs measure for this local specification if it satisfies the Dobrushin-Landford-Ruelle (DLR) equation

$$\nu\mathbb{E}_\Lambda = \nu, \tag{5.1.4}$$

for all $\Lambda \subset \mathbb{Z}^d$. This construction, which follows the work of Dobrushin [52], Lanford, and Ruelle [87], means that the measure ν is constructed by first providing a description of the behaviour of ν on finite volume subsystems $\Lambda \subset \mathbb{Z}^d$. The existence of such a measure follows Dobrushin's

criterion [53, Theorem 1] for spin systems on arbitrary metric spaces. Moreover, the proof of the main theorem will go on to show that the Gibbs measure ν is unique, which is related to the physical phenomenon of phase transition [61].

In this chapter, we state our results that are concerned with finding conditions on the phase ϕ and interaction V so that the Gibbs measure ν exists, is unique, and satisfies the q-Poincaré inequality (5.1.1). The complete details and proofs of the results are found in [31].

5.2 Passage to Infinite Dimensions

As mentioned in the previous sections, the works [82, 86] investigated coercive inequalities for the global Gibbs measure ν as a function of the Carnot-Carathéodory distance (Introduced in Section 1.7.1 of the Introduction and Chapter 2). Our main aim is to consider Gibbs measures ν as function of the homogeneous norm considered in Chapter 3 and in Chapter 4, where the condition $|\nabla N| \geq c$ outside the unit ball is not satisfied.

In this section, we bypass this condition and strongly generalise the results of [81, 82, 86, 111]. We show that it suffices that a so-called weak U-bound (that is achieved for instance using the $U-$bound inequalities proved in Chapter 3 and in Chapter 4) and a weak q-Poincaré inequality hold. The term "weak" signifies that the right-hand side of the $U-$Bound inequalities (Theorem 17 in Chapter 3 and Theorem 31 in Chapter 4) contains defective terms involving the sub-gradients of f at other sites. It turns out however that this is sufficient, provided that the defective terms decay sufficiently fast as one moves further and further away from the given site $i \in \mathbb{Z}^d$. Similarly, the right-hand side of the weak $q-$Poincaré inequality contains defective terms at other sites. Thus, our result could be interpreted as saying that we expect each particle to interact with one another provided this interaction decays quickly.

Theorem 39. *Let $q \in (1, 2]$, and let ν be a Gibbs measure for the local specification $(\mathbb{E}_\Lambda^\omega)_{\Lambda \subset \mathbb{Z}^d}^{\omega \in \Omega}$ given in (5.1.2). Let $f \in W^{1,q}(\nu)$ and suppose the following two conditions hold for $\beta > 0$ sufficiently small:*

(**A1**) *for any $i \in \mathbb{Z}^d$ and $j \in \mathbb{Z}^d$ with $||j - i||_{L^1} = 1$, one has the weak single-site $U-$bound*

$$\nu\left(|f|^q |\nabla_j V(x_i, x_j)|^q\right) \leq A\nu|f|^q + A \sum_{k \in \mathbb{Z}^d} M_{ki} \nu |\nabla_k f|^q,$$

(**A2**) *and, for any $i \in \mathbb{Z}^d$, one has the weak single-site $q-$Poincaré inequality*

$$\nu \mathbb{E}_i^\omega |f - \mathbb{E}_i^\omega f|^q \leq A \sum_{k \in \mathbb{Z}^d} M_{ki} \nu |\nabla_k f|^q$$

for some constant $A > 0$ and some symmetric matrix $(M_{ij})_{i,j\in\mathbb{Z}^d}$ such that

(**B1**) *$A = \mathcal{O}(\beta^{-1/2})$ as $\beta \to 0^+$, and*

(**B2**) *$M_{ij} \geq 0$ for all $i, j \in \mathbb{Z}^d$ and satisfies $\sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} M_{ij} < \infty$.*

Then for all $\beta > 0$ sufficiently small the following global q-Poincaré inequality holds

$$\nu|f - \nu f|^q \leq C_{SG}\nu|\nabla f|^q$$

with some constant $C_{SG} > 0$ possibly depending on β, for all $f \in W^{1,q}(\nu)$.

The idea is that the summability condition (**B2**) implies that if the interaction matrix M_{ij} decays sufficiently fast at infinity, then the defective terms can be dispensed of.

To prove this theorem, we need some lemmata. The first is sweeping out inequalities - these were introduced in [70] in order to prove infinite dimensional Logarithmic Sobolev inequalities. Throughout the remainder of this section we fix $q \in (1, 2]$ and assume the conditions of Theorem 39 hold. Moreover, we use the notation $j \sim i$ to mean j is a neighbour of site i, i.e. j is adjacent to i in the sense $||j - i|| := ||j - i||_{L^1} = 1$, and we use the notation $\{\sim i\}$ to denote the set of neighbors of i, i.e. the $2d$ points $j \in \mathbb{Z}^d$ with $||j - i|| = 1$. Lastly, whenever we use Landau notation, we implicitly understand it with respect to the limit $\beta \to 0^+$; we will need to track the decay (or growth) of some constants and we will typically reserve uppercase letters such as K for large constants that are $\mathcal{O}(1)$ and lowercase letters such as c for small constants that are $o(1)$.

The first lemma is a single-site sweeping out inequality:

Lemma 40. *For all $i, j \in \mathbb{Z}^d$, the following single-site sweeping out inequality holds*

$$\nu|\nabla_j(\mathbb{E}_i^\omega f)|^q \leq K_1\nu|\nabla_j f|^q + c_1 \sum_{k\in\mathbb{Z}^d} N_{ki}\nu|\nabla_k f|^q \tag{5.2.1}$$

with some constants $K_1 = \mathcal{O}(1)$ and $c_1 = o(1)$ independent of i and f.

This lemma is proved using the DLR equation (5.1.4), Jensen's inequality, and the conditions of Theorem 39. Moreover, we use this lemma inductively to prove a global sweeping out inequality on a chosen subsystem having infinite volume. The first step is to form a finite partition of $\mathbb{Z}^d$ consisting of infinite volume subsystems $(\Gamma_i)_{i=0,1,\cdots,N}$ with the property that no interaction exists within each Γ_i which, in our case, translates to each pair of points $i, j \in \Gamma_i$ satisfying $||i - j|| > 1$ whenever $i \neq j$. The partition chosen here will be consist of $N = 2^d$ translations of the cube $\Gamma_0 = (2\mathbb{Z})^d$ translating 0 to another point in $[0, 1]^d \subset \mathbb{Z}^d$.

We fix in the sequel our choice of partition $(\Gamma_i)_{i=0,1,\cdots,N-1}$, and we begin by proving global

sweeping out inequalities for Γ_i; the self-similarity of the Γ_i will mean the proof follows for all if it follows for one.

Lemma 41. *For any $\Lambda \subset \mathbb{Z}^d$ and $i \in \{0, 1, \cdots, N-1\}$, the sub-lattice sweeping out inequality*

$$\nu|\nabla_\Lambda(\mathbb{E}_{\Gamma_n} f)|^q \leq K_2 \nu|\nabla_\Lambda f|^q + c_2 \nu|\nabla_{\Lambda^c} f|^q \tag{5.2.2}$$

holds for some constants $K_2 = \mathcal{O}(1)$ and $c_2 = o(1)$ independently of i and Λ.

As mentioned earlier, it is enough to we prove the case $i = 0$. First, rewrite the gradient into its single-site form

$$\nu|\nabla_\Lambda(\mathbb{E}_{\Gamma_0} f)|^q = \sum_{j \in \Lambda} \nu|\nabla_j(\mathbb{E}_{\Gamma_0} f)|^q. \tag{5.2.3}$$

Let $i \in \Gamma_0$ and $j \in \Lambda$. Since Γ_0 has no self-interaction, the measure $\mathbb{E}_{\Gamma_0}$ can be realised as the product of the single-site measures $\mathbb{E}_{\Gamma_0} = \mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots$ where $(i_m)_{m\geq 1}$ enumerates Γ_0. This reduces the problem of studying $\nabla_j \mathbb{E}_{\Gamma_0}$ to the problem of studying $\nabla_j \mathbb{E}_i$ for $i \in \Gamma_0$. Hence, to prove this lemma, we can use an induction argument on Lemma (40).

The next problem is to use the sweeping out inequality on the sub-lattice to transfer the weak single-site $q-$Poincaré inequality to a weak sub-lattice $q-$Poincaré inequality; in particular we want to show the weak $q-$Poincaré inequality holds for each of the N sub-lattices Γ_i.

Lemma 42. *Let $i = 0, 1, \cdots, N-1$. Then the following weak sub-lattice $q-$Poincaré inequality*

$$\nu\mathbb{E}^\omega_{\Gamma_i}|f - \mathbb{E}^\omega_{\Gamma_i} f|^q \leq C_0 \nu|\nabla f|^q \tag{5.2.4}$$

holds for some constant $C_0 > 0$ independently of i such that $C_0 = \mathcal{O}(\beta^{-1/2})$.

The proof of this lemma constitutes of using the DLR equation (5.1.4), a telescoping argument, the sweeping out inequality of Lemma (5.2.2), and the conditions of Theorem 39. Note that this is still a weak $q-$Poincaré inequality as the right-hand side contains $|\nabla f|^q$ instead of $|\nabla_{\Gamma_i} f|^q$. However, unlike the weak $q-$Poincaré inequality of **(A2)** the interaction matrix M_{ki} is absent. This reflects the fact the sub-lattices infiltrate the lattice so that the distance between an arbitrary point and the sub-lattice is bounded above.

Having obtained a weak $q-$Poincaré inequality for the sub-lattices, we give an argument to show this implies the (strong) $q-$Poincaré inequality on the entire lattice.

Lemma 43. *Let $\mathcal{P} \coloneqq \mathbb{E}_{\Gamma_{N-1}} \cdots \mathbb{E}_{\Gamma_0}\mathbb{E}_{\Gamma_0}$. Then $\mathcal{P}$ satisfies the following four conditions:*

(**1**) *$\mathcal{P}$ satisfies the DLR equation*

$$\nu f = \nu(\mathcal{P} f).$$

(**2**) *There exists a constant* C_1 *such that*

$$\nu|f|^q - \nu|\mathcal{P}f|^q \leq C_1\nu|\nabla f|^q,$$

(**3**) *There exists* $\xi \in (0,1)$ *such that for any* $m \in \mathbb{Z}_{\geq 0}$ *one has*

$$\nu|\nabla(\mathcal{P}^m f)|^q \leq \xi^m \nu|\nabla f|^q.$$

(**4**) *The following limit*

$$\lim_{m\to\infty} \mathcal{P}^m f = \nu f$$

holds ν*-almost surely.*

Proof. (**1**) follows immediately from the DLR equation ((5.1.4)). To prove (**2**), denote by $f_{-1} = f$ and $f_n = \mathbb{E}_{\Gamma_n} \cdots \mathbb{E}_{\Gamma_0} f$, for $n = 0, 1, \cdots, N-1$. By applying the sub-quadratic inequality

$|a|^q - |b|^q \leq \mathrm{sgn}(b) q|b|^{q-1}(a-b) + 2|a-b|^q$,

for $a, b \in \mathbb{R}$,and the weak sub-lattice q-Poincaré inequality (5.2.4) we obtain

$$\begin{aligned}
\nu|f|^q - \nu|\mathcal{P}f|^q &= \sum_{n=0}^{N-1} \nu\mathbb{E}_{\Gamma_n}\left(|f_{n-1}|^q - |\mathbb{E}_{\Gamma_n} f_{n-1}|^q\right) \\
&\leq 2\sum_{n=0}^{N-1} \nu\mathbb{E}_{\Gamma_n}|f_{n-1} - \mathbb{E}_{\Gamma_n} f_{n-1}|^q \\
&\leq 2C_0 \sum_{n=0}^{N-1} \nu|\nabla f_{n-1}|^q \\
&= 2C_0\nu|\nabla f|^q + 2C_0 \sum_{n=0}^{N-2} \nu|\nabla_{(\Gamma_n)^c}(\mathbb{E}_{\Gamma_n} f_{n-1})|^q
\end{aligned}$$

since $f_n = \mathbb{E}_{\Gamma_n} f_{n-1}$ depends only on coordinates in $(\Gamma_n)^c$. The first term requires no further estimation, while we can apply Lemma 41 inductively on the second term to obtain

$$\nu|\nabla_{(\Gamma_n)^c}(\mathbb{E}_{\Gamma_n} f_{n-1})|^q \leq K_2^{n+1}\nu|\nabla f|^q$$

which gives

$$\nu|f|^q - \nu|\mathcal{P}f|^q \leq C_1\nu|\nabla f|^q$$

which is exactly **(2)** with $C_1 = 2C_0 \sum_{n=0}^{N-1} K_2^n \leq 2K_2^N C_0$.

To prove **(3)**, we define $\Gamma^{(m)} = \cup_{n=0}^m \Gamma_n$ and apply Lemma 41 again to obtain

$$\nu|\nabla_{\Gamma^{(m)}} f_m|^q = \nu|\nabla_{\Gamma^{(m-1)}} f_m|^q \leq K_2\nu|\nabla_{\Gamma^{(m-1)}} f_{m-1}|^q + c_2\nu|\nabla f_{m-1}|^q.$$

Applying Lemma 41 as in the proof for **(2)**, we obtain

$$\nu|\nabla_{\Gamma^{(m)}} f_m|^q \leq K_2 \nu|\nabla_{\Gamma^{(m-1)}} f_{m-1}|^q + c_2 K_2^m \nu|\nabla f|^q. \tag{5.2.5}$$

This equation can be inductively applied on the first term, i.e. applying (5.2.5) on $\nu|\nabla_{\Gamma^{(m-1)}} f_{m-1}|^q$ and starting from $m = N-1$ we find

$$\nu|\nabla(\mathcal{P}f)|^q \leq K_2^N \nu|\nabla_{\Gamma_0} f_0|^q + c_2 K_2^N \nu|\nabla f|^q = c_2 K_2^N \nu|\nabla f|^q$$

since $\nabla_{\Gamma_0} f_0 = \nabla_{\Gamma_0} \mathbb{E}_{\Gamma_0} f = 0$. Since $K_2 = \mathcal{O}(1)$ and $c_2 = o(1)$, we obtain **(3)** at $m = 1$ with $\xi = c_2 K_2^N$ with β small enough; the general case for $m \in \mathbb{Z}_{\geq 1}$ follows by replacing f by $\mathcal{P}f$ and proceeding by straightforward induction.

To prove the final statement **(4)**, following the arguments of **(2)** we first bound the q-variance with respect to $\mathcal{P}$ by

$$\nu|f - \mathcal{P}f|^q \leq N^N \sum_{n=0}^{N-1} \nu \mathbb{E}_{\Gamma_n} |f_{n-1} - \mathbb{E}_{\Gamma_n} f_{n-1}|^q \leq N^N C_1 \nu|\nabla f|^q$$

using the Hölder inequality $(\sum_{i=1}^N |x_i|)^q \leq N^{q-1} \sum_{i=1}^N |x_i|^q$ for nonnegative (x_i) on the telescoping sum $f - \mathcal{P}f = \sum_{n=0}^{N-1} f_{n-1} - f_n$. For $m \in \mathbb{Z}_{\geq 1}$, replacing f by $\mathcal{P}^m f$ in the above estimate we obtain

$$\nu|\mathcal{P}^m f - \mathcal{P}^{m+1} f|^q \leq N^N C_1 \nu|\nabla(\mathcal{P}^m f)|^q \leq N^N C_1 \xi^m \nu|\nabla f|^q.$$

Since $\xi < 1$, the Borel-Cantelli lemma implies $\mathcal{P}^m$ converges ν-almost surely to some function g, and similarly $|\nabla(\mathcal{P}^m f)|$ converges ν-almost surely to zero by **(3)**, so that g is constant ν-almost everywhere. Note the decay of C_1 with respect to β does not matter since β is sufficiently small, but fixed. Assuming that $f \in W^{1,p}(\nu)$ is bounded, by dominated convergence we have

$$\lim_{m\to\infty} \mathcal{P}^m f = g = \nu(g) = \lim_{m\to\infty} \nu(\mathcal{P}^m f) = \nu(f)$$

with the last equality obtained by **(1)**. The general case of possibly unbounded $f \in W^{1,p}(\nu)$ is obtained by density. □

We can now prove the main theorem of this section.

Proof of Theorem 39. Denote by $f^m = \mathcal{P}^m f$ for $m \in \mathbb{Z}_{\geq 0}$. By applying **(1)**, **(2)**, and **(3)** of

Lemma 43 in succession,

$$\begin{aligned}\nu|f-\nu f|^q &= \sum_{n=0}^{N-1}\left(\nu|f_n-\nu f|^q-\nu|\mathcal{P}(f_k-\nu f)|^q\right)+\nu|f_N-\nu f|^q\\ &\leq C_1\sum_{n=0}^{N-1}\nu|\nabla f_n|^q+\nu|f_N-\nu f|^q\\ &\leq C_1\left(\sum_{n=0}^{N-1}\xi^n\right)\nu|\nabla f|^q+\nu|f_n-\nu f|^q\\ &\leq \frac{JC_1}{1-\xi}\nu|\nabla f|^q+\nu|f_N-\nu f|^q.\end{aligned}$$

Taking $N\to\infty$, the remainder term satisfies $\nu|f_N-\nu f|^q\to 0$ by **(4)** which gives the global $q-$Poincaré inequality

$$\nu|f-\nu f|^q\leq C_{SG}\nu|\nabla f|^q$$

with $C_{SG}=C_1/(1-\xi)$. □

5.3 Sufficient Conditions on the Phase and Interaction

Having now given general conditions - the existence of a weak single-site $U-$bound and a weak single-site $q-$Poincaré inequality - which guarantee the existence of a global $q-$Poincaré inequality, we now consider conditions on the phase $\phi\in C(\mathbb{G})$ and interaction $V\in C^1(\mathbb{G}\times\mathbb{G})$ appearing in (5.1.2) such that the conditions of Theorem 39 hold. The first will be a (strong) single-site U-bound without interaction in terms of a general function η, and the second will place conditions on the interaction V in terms of η.

Theorem 44. *Let $q\in(1,2]$, and let ν be a Gibbs measure for the local specification $(\mathbb{E}_\Lambda^\omega)_{\Lambda\in\mathbb{Z}^d}^{\omega\in\Omega}$ given in (5.1.2). Let $f\in W^{1,q}(\nu)$ and suppose the following two conditions hold on ϕ and V for sufficiently small $\beta>0$:*

(**C1**) *There exists a constant $B>0$ and a function $\eta\in C(\mathbb{G};\mathbb{R}_{\geq 0})$ diverging to infinity all directions such that*

$$\int|g|^q\eta(h)e^{-\phi(h)}dh\leq B\int\left(|\nabla g|^q+|g|^q\right)e^{-\phi(h)}dh$$

for all $g\in C(\mathbb{G})$ for which the left-hand side is finite, and

(**C2**) *the interaction $V\in C(\mathbb{G}\times\mathbb{G})$ satisfies the gradient bound*

$$|\nabla_{h_1}V(h_1,h_2)|^q+|\nabla_{h_2}V(h_1,h_2)|^q\leq B(1+\eta(h_1)+\eta(h_2)).$$

Then for all sufficiently small $\beta > 0$ the global q-Poincaré

$$\nu|f - \nu f|^q \leq C_{SG}\nu|\nabla f|^q$$

holds for all $f \in W^{1,q}(\nu)$ satisfying $\sup_{i\in\mathbb{Z}^d} \nu(\eta(x_i)|f|^q) < \infty$ with some constant $C_{SG} > 0$.

The inequality in condition (**C1**) is exactly the $U-$bound inequality we proved in Theorem (17) of Chapter (3) in the setting of the step-two Carnot group and Theorem (31) of Chapter (4) in the setting of the Anisotropic Heisenberg group with

$$\eta = \frac{g'(N)}{N^2},$$

and the phase

$$\phi = g(N),$$

where N is the generalisation of the fundamental solution to the sub-Laplacian in the Heisenberg group (Chapter (3)) or the fundamental solution to the sub-Laplacian in the Anisotropic Heisenberg group (Chapter (4)).

In addition, if the interaction $V \in C(\mathbb{G} \times \mathbb{G})$ satisfies the condition (**C2**), then Theorem (44) implies that using the $U-$bound inequality, we are able to extend the $q-$Poincaré inequality to an infinite dimensional setting.

Note that although this setup allows for more general $U-$bounds of the form (**C1**), the generic case occurs when the phase, the interaction, and the $U-$bound are all powers of a homogeneous norm. For instance, in the case of the Carnot-Carathéodory distance d, [82] proves the q-logarithmic Sobolev inequality with $\phi = d^p$, $V(x, y) = (d(x) \pm d(y))^2$, and $\eta = d^p$, in the case $p \geq 2$.

The generic choice in our settings is

$$g(N) = \alpha N^p.$$

For this choice,

$$\eta = \alpha p N^{p-3},$$

and we have proved in Theorem (17) of Chapter (3) and Theorem (31) of Chapter (4) the condition (**C1**):

$$\int |g|^q N^{p-3} e^{-\alpha N^p} dh \leq B \int \left(|\nabla g|^q + |g|^q\right) e^{-\alpha N^p} dh,$$

for any $\alpha > 0$, $p, q \geq 2$, and g supported away from the unit N-ball $\{N < 1\}$. We deduce from

Theorem (44) a generalised theorem for potentials of this standard form:

Theorem 45. *Let $\alpha > 0$ and $p > 3$. Define a local specification on $\mathbb{G}^{\mathbb{Z}^{\mathbb{D}}}$ by the following potential:*

$$U_\Lambda^\omega(x_\Lambda) = \alpha \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda} \beta_{ij} Q(x_i, x_j) + \sum_{i\in\Lambda,\, j\notin\Lambda} \beta_{ij} Q(x_i, \omega_j).$$

Suppose that:

(1) *ϕ is a semi-bounded generalised polynomial in N of order p with leading term αN^p,*

(2) *$|\beta_{ij}| \leq \beta$ for all $i, j \in \mathbb{Z}^D$, and*

(3) *$Q(x_i, x_j) = \sum_{\ell=1}^k c^\ell \varrho^\ell(x_i)^{\alpha_\ell} \delta^\ell(x_j)^{\beta_\ell}$ is a finite sum of $k \in \mathbb{Z}_{\geq 1}$ products of powers of homogeneous norms $(\varrho^\ell, \delta^\ell)_{\ell=1,\cdots,k}$ with bounded sub-gradient and for $c^\ell \in \mathbb{R}$, $\alpha^\ell, \beta^\ell \in \{0\} \cup [1, \infty)$ and $0 \leq \alpha^\ell + \beta^\ell \leq p$.*

Then for all sufficiently small $\beta > 0$, there exists a unique Gibbs measure ν which satisfies the 2-Poincaré inequality

$$\nu|f - \nu f|^2 \leq C_{SG} \nu|\nabla f|^2.$$

Note that in this theorem N could be:

1. The generalisation of the fundamental solution to the sub-Laplacian in the Heisenberg group (in the setting of step-two Carnot groups, Chapter (3))

2. The fundamental solution to the sub-Laplacian in the Anisotropic Heisenberg group (in the setting of the Anisotropic Heisenberg group, Chapter (4))

3. Replaced by d, the Carnot-Carathéodory distance (in the setting of Carnot groups, Chapter (2)).

In other words, N should satisfy the $U-$bound inequality (**C1**).

Idea of proof for Theorem (44)

The main problem in the proof is to show that (**C1**) and (**C2**) imply the existence of a weak q-Poincaré inequality. Since $|\nabla_j V|^q$ is controlled by η, the weak $U-$bound of (**A1**) follows immediately, and the difficulty lies with (**A2**) . In Theorem (6), Hebisch and Zegarlinski showed that in the finite dimensional case that one can pass from a $U-$bound to a q-Poincaré inequality [72], and in [82, Lemma 4.3] it was shown in some class of models that one can pass from a strong single-site $U-$bound to the strong q-Poincaré inequality. Here we show the statement remains true with "weak" replacing "strong".

The first step in establishing the weak $U-$bound for η is to prove a weak $U-$bound for η_i that is defective in the sense the right-hand side contains copies of the weak $U-$bound for neighbouring η_j. We will then approach in the spirit of the proof in Theorem (39) where we prove that inductively inserting the $U-$bound into these defective copies "converges" appropriately. It will also be noted as a corollary that the proof of this weak $U-$bound will give the existence of the Gibbs measure by Dobrushin's criterion [53, Theorem 1].

The following lemma is needed:

Lemma 46. *For all $i \in \mathbb{Z}^d$, the following weak single-site $U-$bound*

$$\nu(\eta_i|f|^q) \leq K_2 \left(\nu|f|^q + \sum_{k\in\mathbb{Z}^d} c_2^{||k-i||} \nu|\nabla_k f|^q \right) \tag{5.3.1}$$

holds for some constants $K_2 = \mathcal{O}(1)$ and $c_2 = o(1)$.

This lemma is proved by an induction on the inequality

$$\mathbb{E}_i^\omega(\eta_i|f|^q) \leq K_1 \ (\mathbb{E}_i^\omega|\nabla_i f|^q + \mathbb{E}_i^\omega|f|^q) + c_1 \sum_{j\neq i} |\beta_{ij}|^q \ \mathbb{E}_i^\omega(\eta_j|f|^q),$$

which is, in turn, easily proved by applying the conditions of Theorem (44).

To verify the conditions of Theorem 39 hold (and hence prove the $q-$Poincaré inequality), the key ingredient is Lemma (46) - it is straightforward to see that the weak $U-$bound is immediate, and more difficult to see that the weak $U-$bound implies the weak $q-$Poincaré inequality of (**A2**). The proof is an adaptation of [82, Lemma 4.3], but in the weak case. The basic idea is that one can split the left-hand side of (**A2**) into two parts, one where the boundary conditions are bounded and the other where they are not. It turns out that the latter is simpler to handle as the $U-$bound can be inserted.

Chapter 6

Summary and Open Problems

Although the question of obtaining coercive inequalities- such as the Poincaré or the Logarithmic Sobolev inequalities- for probability measures on a metric measure space has been a subject of numerous works (i.e. to study PDEs of mathematical physics, isoperimetric inequalities, convergence to equilibrium of systems in Statistical mechanics, and to solve the Millennium Prize Poincaré Conjecture), the literature on this topic in the setup of Carnot groups is scarce. My PhD thesis is centered around proving and applying such coercive inequalities for various probability measures in the setting of non-compact Carnot groups. The simplest example of the Carnot group is the Heisenberg group, which plays an important role in several branches of mathematics, such as representation theory, PDEs, number theory, several complex variables and quantum mechanics.

We are interested in the following Logarithmic Sobolev inequality (which also implies the Poincaré inequality):

$$\int f^2 \log\left(\frac{f^2}{\int f^2 d\mu}\right) d\mu \leq 2\int |\nabla f|^2 d\mu,$$

where ∇ is the gradient and μ is a probability measure.

In [69], L. Gross proved that the Logarithmic Sobolev inequality is equivalent to hypercontractivity for the semigroup $P_t = e^{t\mathcal{L}}$ generated by a given operator $\mathcal{L}$. Moreover, he showed that the Logarithmic Sobolev inequality, unlike the Sobolev inequality, could be extended to infinite dimensions. This has important applications in studying models from quantum field theory and statistical mechanics.

L. Gross worked with the Gaussian measure on $\mathbb{R}^n$, and a natural question was to extend the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemannian

manifolds. In [10], D. Bakry and M. Emery prove it under a Curvature-Dimension condition, which highlights a deep link between coercive inequalities and the geometry of the underlying space. In my thesis, I worked in the setting of Carnot groups, where the Curvature-Dimension condition fails because there is no bound on the curvature of space (the Laplacian is of Hörmander type and has some degeneracy). To bypass this geometric condition, a method was developed in [72], which states that in order to obtain coercive inequalities, it is sufficient to prove the so-called "$U-$Bound" inequalities.

In Chapter 2, in the setting of the general Carnot group, we proved the Logarithmic Sobolev inequality for measures of the form $d\mu = \frac{e^{-g(d)}}{Z} d\lambda$, where d is the Carnot-Carathéodory distance. Notice that $g(d)$ is a function of d in contrast with the homogeneous norms N that were considered in Chapter 3 and Chapter 4. Even though d and N are equivalent, their behavior around the "$z-$axis" are completely different, and so the approach to prove coercive inequalities is completely different.

Moreover, F. Otto and C. Villani showed in their celebrated paper [110] that in the setting of manifolds, under D. Bakry and M. Emery's Curvature-Dimension condition, the Logarithmic Sobolev inequality implies the Talagrand transportation cost inequality. However, the Curvature-Dimension condition fails in the setting of Carnot groups. We used a different approach to prove that the Logarithmic Sobolev inequality implies the Talagrand inequality and hypercontractivity in our setting. The key in our proof is to use the Hamilton-Jacobi equation in the setting of Carnot groups done by F. Dragoni [54].

In [110], F. Otto and C. Villani obtained the HWI inequality, which is an important functional inequality and unifies the Talagrand inequality and Logarithmic Sobolev inequality. It relates the entropy (H) with respect to some reference measure, the Wasserstein distance (W) and the Fisher information (I). One direction of work is to prove the HWI inequality in the setting of the general Carnot group for measures as a function of the Carnot-Carathéodory distance.

In Chapter 3, we considered the setting of the step-two Carnot group, $\mathbb{G}$, and we let X_i, $1 \leq i \leq n$ be the Jacobian basis of its Lie algebra (here n corresponds to the dimension of the first layer of stratification). We proved coercive inequalities for general measures of the form $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$, where λ is the Haar measure, Z is the normalisation constant, g is a differentiable increasing function, and N is a generalisation of the Heisenberg Kaplan norm. The interest behind choosing such N is that it generalises the unique fundamental solution of the sub-Laplacian in the Heisenberg group. Under some growth conditions on g, we proved the $U-$Bound inequality outside the unit

$N-$ball:

$$\int_{\mathbb{G}} \frac{g'(N)}{N^2}|f|^q d\mu \leq C\int_{\mathbb{G}} |\nabla f|^q d\mu + D\int_{\mathbb{G}} |f|^q d\mu,$$

for the sub-gradient $\nabla := (X_1, X_2, ..., X_n)$, C and D positive constants, and $q \geq 2$. From there, we deduced the Poincaré inequality. We also derived a new method to prove the $\phi-$Logarithmic Sobolev inequality under some conditions on a concave function ϕ (we note that it is impossible to have a Logarithmic Sobolev inequality in this specific setting [72]).

The importance of the $U-$bound proof that we developed is that its ideas could be adapted to other settings as well like in the setting of the Anisotropic Heisenberg group.

Moreover, so far, an extension to infinite dimensions for measures as a function of homogeneous norms has not been possible due to restrictions in known methods not satisfied by the norms considered. However, using this $U-$bound inequality, we were able to provide a new method that overcomes those restrictions and allows for an extension to an infinite dimensional setting (Chapter 5).

In [24], Bobkov and Zegarliński showed that for $q \in (1, 2)$, the $q-$Logarithmic Sobolev inequality is better than the Logarithmic Sobolev inequality. They showed that one gets a stronger decay of tail estimates and the corresponding semigroup is ultracontractive (see Section 1.2.1 of the Introduction for more details). Hence, it is of interest to study stronger coercive inequalities with $q \in (1, 2)$. In Chapter 3 and Chapter 4, we proved the $U-$bound inequality for $q \geq 2$; however, the case where we have the stronger inequality for $q \in (1, 2)$ is still an open problem.

Moreover, the goal of a current ongoing project [32] is to prove higher order coercive inequalities, like the Poincaré and Logarithmic Sobolev inequalities, for various probability measures. Our aim is to extend Adams' results in [1] and Rosen's results in [116] for higher order inequalities to the setting of Carnot groups.

In Chapter 4, we proved coercive inequalities in "non-polarizable" higher-dimensional anisotropic Heisenberg group. We extended the anisotropic Heisenberg groups to higher dimensions, computed explicitly the fundamental solution $\tilde{N}$ of the sub-Laplacian, and proved Poincaré and $\phi-$Logarithmic Sobolev inequalities for measures as a function of $\tilde{N}$. The results we obtained were used recently in a preprint by J. Tyson [128] to formulate a suitable conjecture relating the new notion of $H-$type stability to the classification of polarizable groups.

Those results for the Anisotropic Heisenberg group provided a suitable quantitative form for the longstanding conjecture about the classification of polarizable groups: this motivates future work in proving this open problem.

Moreover, Balogh and Tyson considered in [14] the Anisotropic Heisenberg group for $n = 2$. In Chapter 4, we considered the dimension $n > 5$, where we proved coercive inequalities on the Higher-dimensional Anisotropic Heisenberg group. In fact, the proof of the $U-$bound Theorem 31 required the dimension $n > 5$. Proving coercive inequalities for $n < 5$ are still not achieved in this method, and this is a possible direction for future work.

In Chapter 5, we showed that certain infinite dimensional Gibbs measures with unbounded interaction potentials as a function of homogeneous norms (such as homogeneous norms introduced in Chapter 3 and Chapter 4) on an infinite product of Carnot groups satisfy the Poincaré inequality. This is used to study spin systems and statistical mechanics, and allows us to gain a better understanding of phase transition. We note that in our new method, the $U-$Bound inequality is an essential tool to pass to the infinite dimensional setting. Our work strongly generalised known results [81, 82, 86, 111] and provided a vast scope of applications to study different models in the setting of Carnot groups.
Moreover, in [42], $q-$Poincaré inequalities were proved for a homogeneous norm N on Filiform groups, that generalise Engel-type groups. However, the condition $|\nabla N| \geq c$ outside the unit ball is not satisfied; therefore, the extension to infinite dimensions is not possible using known methods [82, 86]. A direction of future work is to apply the strategy of the thesis in the setting of Filiform groups i.e proving the $U-$bound inequality as in Theorem 17, and then extending the Poincaré inequality to infinite dimensions using the results of this Chapter.

In conclusion, this thesis highlights the importance of the $U-$bound inequality as a key to prove coercive inequalities on Carnot groups. The $U-$bound inequality was also an essential tool to extend the Poincaré inequality for measures as a function of homogeneous norms to infinite dimensions. Our results have laid a foundation stone for research on coercive inequalities on Carnot groups, and our novel methods have paved the way for promising future research in other non-commutative settings.

Chapter 7

Appendix

In this appendix, we give proofs to the technical lemmas that provide bounds for the three quantities $|\nabla N|$, ΔN, and $x \cdot \nabla N$, which are necessary in the proofs of the $U-$bound theorems in Chapter 3 and Chapter 4.

7.1 Proof of Technical Lemma 18

We recap the following Technical Lemma 18, which is used in the $U-$bound Theorem 17 of Chapter 3:

Lemma (EBD and B. Zegarliński [33])**.** *There exist constants* $A, C \in (0, \infty)$

$$A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\frac{|x|^2}{N^2} \tag{7.1.1}$$

and there exists a constant $B \in (0, \infty)$ *such that*

$$|\Delta N| \leq B\frac{|x|^2}{N^3} \tag{7.1.2}$$

and

$$\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}. \tag{7.1.3}$$

Proof. We first compute $\nabla N = (X_i N)_{i=1,\ldots,n}$.

$$X_i N = N^{-3}\left(|x|^2 x_i + \frac{a}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)} x_l z_k\right).$$

Therefore,

$$\begin{aligned}|\nabla N|^2 &= N^{-6}\left(|x|^6+\frac{a}{2}\sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}|x|^2x_ix_lz_k+\sum_{i=1}^{n}\frac{a^2}{16}\sum_{k,k'=1}^{m}\sum_{l,l'=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')}x_lx_{l'}z_kz_{k'}\right)\\ &= N^{-6}\left(|x|^6+\frac{a^2}{16}\sum_{i=1}^{n}\sum_{k,k'=1}^{m}\sum_{l,l'=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')}x_lx_{l'}z_kz_{k'}\right)\\ &= \frac{|x|^2}{N^2}N^{-4}\left(|x|^4+\frac{a^2}{16}\sum_{i=1}^{n}\sum_{k,k'=1}^{m}\sum_{l,l'=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')}\frac{x_l}{|x|}\frac{x_{l'}}{|x|}z_kz_{k'}\right),\end{aligned} \tag{7.1.4}$$

where we used that for each skew-symmetric matrix $\Lambda^{(k)}$, all $k\in\{1,...,m\}$, we have that

$$\sum_{l=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}x_lx_i=0.$$

From (7.1.4), that with some constants $A,C\in(0,\infty)$, we have

$$A\frac{|x|^2}{N^2}\leq|\nabla N|^2\leq C\frac{|x|^2}{N^2}.$$

By choosing $a\in(0,\infty)$ sufficiently small, we can ensure that $C\leq 1$. We note that using antisymmetry of matrices $\Lambda_{il}^{(k)}$ we get

$$\frac{x}{|x|}\cdot\nabla N=\sum_{i=1}^{n}\frac{x_i}{|x|}N^{-3}\left(|x|^2x_i+\frac{a}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}x_lz_k\right)$$

Next we compute

$$\begin{aligned}X_i^2N &= \left(\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}x_l\frac{\partial}{\partial z_k}\right)\left(N^{-3}\left(|x|^2x_i+\frac{a}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}x_lz_k\right)\right)\\ &= -3\left(N^{-7}\left(|x|^2x_i+\frac{a}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}x_lz_k\right)^2\right)\\ &+\left(N^{-3}\left(|x|^2+2x_i^2+\frac{a}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}\delta_{il}z_k\right)\right)\\ &+\left(N^{-3}\left(\frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\Lambda_{il}^{(k)}x_l\frac{a}{4}\sum_{k'=1}^{m}\sum_{l'=1}^{n}\Lambda_{il'}^{(k')}x_{l'}\delta_{kk'}\right)\right).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\Delta N &= \sum_{i=1}^{n} X_i^2 N \\ &= -3\left(N^{-7}\left(|x|^6 + 2\frac{a}{4}\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{l=1}^{n}\Lambda_{il}^{(k)}|x|^2 x_i x_l z_k\right)\right. \\ &\left. + \frac{a^2}{16}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{k'=1}^{m}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')} x_{l'} x_l z_k z_{k'}\right) \\ &+ \left(N^{-3}\left((n+2)|x|^2 + \frac{a}{4}\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{l=1}^{n}\Lambda_{il}^{(k)}\delta_{il} z_k\right)\right) \\ &+ \left(N^{-3}\left(\frac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)} x_l \frac{a}{4}\sum_{k'=1}^{m}\sum_{l'=1}^{n}\Lambda_{il'}^{(k')} x_{l'}\delta_{kk'}\right)\right).\end{aligned}$$

which after simplifications yields

$$\begin{aligned}\Delta N &= \sum_{i=1}^{n} X_i^2 N \\ &= -3\left(N^{-7}\left(|x|^6 + \frac{a^2}{16}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{k'=1}^{m}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')} x_{l'} x_l z_k z_{k'}\right)\right) \\ &+ \left(N^{-3}\left((n+2)|x|^2\right)\right) \\ &+ \left(N^{-3}\left(\frac{a}{8}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k)} x_{l'} x_l\right)\right).\end{aligned}$$

Thus we get

$$\begin{aligned}\Delta N &= \frac{|x|^2}{N^3}\left[-3\left(N^{-4}\left(|x|^4 + \frac{a^2}{16}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{k'=1}^{m}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')}\frac{x_l}{|x|}\frac{x_{l'}}{|x|} z_k z_{k'}\right)\right)\right. \\ &\left. + \left(n+2+\tfrac{a}{8}\textstyle\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k)}\frac{x_l}{|x|}\frac{x_{l'}}{|x|}\right)\right].\end{aligned}$$

which can be represented as follows

$$\begin{aligned}\Delta N &= (n-1)\frac{|x|^2}{N^3} \\ &+ \frac{|x|^2}{N^3}\left[-3\left(N^{-4}\left(-a|z|^2 + \frac{a^2}{16}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{k'=1}^{m}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k')}\frac{x_l}{|x|}\frac{x_{l'}}{|x|} z_k z_{k'}\right)\right)\right. \\ &\left. \frac{a}{8}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{l'=1}^{n}\sum_{i=1}^{n}\Lambda_{il}^{(k)}\Lambda_{il'}^{(k)}\frac{x_l}{|x|}\frac{x_{l'}}{|x|}\right].\end{aligned}$$

Hence, there exists a constant $B \in (0,\infty)$ such that

$$|\Delta N| \leq B\frac{|x|^2}{N^3}$$

Remark: If $a > 0$ is small, $\Delta N \geq 0$. For large a, in some directions ΔN can be negative. □

7.2 Proof of Technical Lemma 32

Next, we recap the following Technical Lemma 32, which is used in the $U-$bound Theorem 31 of Chapter 4:

Lemma (EBD and B. Zegarliński [34]). *The fundamental solution of the sub-Laplacian N^{-2n} on $\mathbb{H}_{2n}(\frac{\theta}{2},\theta)$ satisfies*

$$x\cdot\nabla N\geq-\frac{\theta|x|^2}{4nN}, \tag{7.2.1}$$

$$|\nabla N|^2\geq\frac{\theta^2|x|^2}{2^{5+\frac{2}{n}}N^2}, \tag{7.2.2}$$

$$|\nabla N|^2\leq\frac{\theta^2(2n+1)^2|x|^2}{2^3n^2N^2}, \tag{7.2.3}$$

Lemma 47. *where $|x|=\left(\sum_{j=1}^{2n}x_j^2\right)^{\frac{1}{2}}$ is the Euclidean norm.*

Proof. In the setting of the anisotropic Heisenberg group $\mathbb{R}^{2n+1}$, as shown in Theorem 30, we have the fundamental solution

$$N=\left(\frac{\left(B^2+t^2\right)^{\frac{1}{2n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{1-\frac{1}{2n}}}{B+\sqrt{B^2+t^2}}\right)^{\frac{1}{2}},$$

where $A=\frac{\theta x_1^2}{2}+\frac{\theta x_{n+1}^2}{2}+\frac{\theta}{2}\sum_{j=2,j\neq n+1}^{2n}x_j^2$ and $B=\frac{\theta x_1^2}{4}+\frac{\theta x_{n+1}^2}{4}+\frac{\theta}{2}\sum_{j=2,j\neq n+1}^{2n}x_j^2$.

We first calculate $\partial_{x_j}N$ and ∂_tN.

For $j=1$ and $j=n+1$:

$$\partial_{x_j}N=\frac{\theta x_j}{4nN}\left(\frac{B\left(B^2+t^2\right)^{\frac{1}{2n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta x_j(2n-1)}{4nN}\left(\frac{\left(B^2+t^2\right)^{\frac{1}{2n}}\left(\frac{A}{2}\sqrt{B^2+t^2}+B\sqrt{B^2+t^2}+B^2+t^2+\frac{AB}{2}\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}\left(B^2+t^2\right)^{\frac{1}{2}}}\right)$$

$$-\frac{\theta x_j}{4N}\left(\frac{\left(B^2+t^2\right)^{\frac{1}{2n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)}{(B+\sqrt{B^2+t^2})(B^2+t^2)^{\frac{1}{2}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$=\frac{x_j\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{\frac{1}{2}B^2A+(B-\frac{A}{2})t^2+\frac{1}{2}\sqrt{B^2+t^2}AB}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{x_j\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{(n-1)(B^2+t^2)\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{x_j\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{nB\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right) \tag{7.2.4}$$

Notice that

$$x_j\partial_{x_j}N\geq 0 \qquad\qquad for\ j=1,n+1 \tag{7.2.5}$$

To get a bound from above for $|\partial_{x_j}N|$, since $B\leq\sqrt{B^2+t^2}$, $B-\frac{A}{2}\leq\frac{A}{2}$, and $B^2+t^2\leq AB+t^2+A\sqrt{B^2+t^2}$,

$$|\partial_{x_j}N|\leq\frac{\theta|x_j|}{4nN}\left(\frac{\frac{1}{2}B^2A+(\frac{A}{2})t^2+\frac{1}{2}\sqrt{B^2+t^2}AB+(2n-1)(B^2+t^2)\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)}\right)$$

since $A\leq 2B$, so

$$\frac{1}{2}B^2A+(\frac{A}{2})t^2+\frac{1}{2}\sqrt{B^2+t^2}AB$$

$$\leq B\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)\leq(B^2+t^2)\left(B+\sqrt{B^2+t^2}\right),$$

$$|\partial_{x_j}N|\leq\frac{\theta|x_j|}{2N} \qquad\qquad \text{for } j=1,n+1. \tag{7.2.6}$$

Now we calculate $\partial_{x_j}N$ for $j=2,..,n,n+2,..,2n$:

$$\partial_{x_j}N=\frac{\theta x_j}{4nN}\left(\frac{2B\left(B^2+t^2\right)^{\frac{1}{2n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta x_j(2n-1)}{4nN}\left(\frac{\left(B^2+t^2\right)^{\frac{1}{2n}}\left(A\sqrt{B^2+t^2}+B\sqrt{B^2+t^2}+B^2+t^2+AB\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}(B^2+t^2)^{\frac{1}{2}}}\right)$$

$$-\frac{\theta x_j}{4N}\left(\frac{2\left(B^2+t^2\right)^{\frac{1}{2n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)}{(B+\sqrt{B^2+t^2})(B^2+t^2)^{\frac{1}{2}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$=\frac{\theta x_j\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{AB\sqrt{B^2+t^2}+AB^2+(2B-A)t^2}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta x_j\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{(2n-1)B\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$-\frac{\theta x_j\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)(B^2+t^2)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right) \tag{7.2.7}$$

Notice that

$$x_j\partial_{x_j}N \geq \frac{\theta x_j^2\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{-t^2}{\left(B+\sqrt{B^2+t^2}\right)\sqrt{B^2+t^2}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $t^2 \leq \left(B+\sqrt{B^2+t^2}\right)\sqrt{B^2+t^2}$ and $B^2+t^2 \leq AB+t^2+A\sqrt{B^2+t^2}$,

$$x_j\partial_{x_j}N \geq -\frac{\theta x_j^2}{4nN}. \tag{7.2.8}$$

We will now be able to get a lower bound for $x\cdot\nabla N$. In fact,

$$x\cdot\nabla N$$

$$= x_1\left(\partial_{x_1}N - \frac{\theta x_{n+1}}{2}\partial_t N\right) + x_{n+1}\left(\partial_{x_{n+1}}N + \frac{\theta x_1}{2}\partial_t N\right)$$

$$+\sum_{j=2}^{n} x_j\left(\partial_{x_j}N - \theta x_{j+n}\partial_t N\right) + \sum_{j=n+2}^{2n} x_j\left(\partial_{x_j}N + \theta x_{j-n}\partial_t N\right)$$

$$= x_1\partial_{x_1}N + x_{n+1}\partial_{x_{n+1}}N + \sum_{j=2, j\neq n+1}^{2n} x_j\partial_{x_j}N$$

using (7.2.5) and (7.2.8),

$$\geq -\sum_{j=2, j\neq n+1}^{2n}\frac{\theta x_j^2}{4nN}$$

$$\geq -\sum_{j=1}^{2n}\frac{\theta x_j^2}{4nN}.$$

Using the above defined Euclidean norm $|x| = \left(\sum_{j=1}^{2n} x_j^2\right)^{\frac{1}{2}}$,

$$x\cdot\nabla N \geq -\frac{\theta|x|^2}{4nN}.$$

It remains to bound $|\nabla N|$.

$$|\nabla N|^2 = \left(\partial_{x_1}N - \frac{\theta x_{n+1}}{2}\partial_t N\right)^2 + \left(\partial_{x_{n+1}}N + \frac{\theta x_1}{2}\partial_t N\right)^2$$

$$+\sum_{j=2}^{n}\left(\partial_{x_j}N - \theta x_{j+n}\partial_t N\right)^2 + \sum_{j=n+2}^{2n}\left(\partial_{x_j}N + \theta x_{j-n}\partial_t N\right)^2$$

$$= \sum_{j=1}^{2n} (\partial_{x_j} N)^2 + \frac{\theta^2 x_1^2}{4} (\partial_t N)^2 + \frac{\theta^2 x_{n+1}^2}{4} (\partial_t N)^2 + \sum_{j=2, j\neq n+1}^{2n} \theta^2 x_j^2 (\partial_t N)^2. \tag{7.2.9}$$

To bound (7.2.9) from below, we use the fact that for each $j = 2, 3, ..., n, n+2, .., 2n$,

$$|\partial_{x_j} N|^2 + \theta^2 x_j^2 |\partial_t N|^2 \geq \frac{1}{2} \left(|\partial_{x_j} N| + \theta |x_j| |\partial_t N| \right)^2,$$

in addition to that we have for $j = 1, n+1$,

$$|\partial_{x_j} N|^2 + \frac{\theta^2 x_j^2}{4} |\partial_t N|^2 \geq \frac{1}{2} \left(|\partial_{x_j} N| + \frac{\theta |x_j|}{2} |\partial_t N| \right)^2$$

so,

$$|\nabla N|^2 \geq \frac{1}{2} \left(|\partial_{x_1} N| + \frac{\theta |x_1|}{2} |\partial_t N| \right)^2$$

$$+ \frac{1}{2} \left(|\partial_{x_{n+1}} N| + \frac{\theta |x_{n+1}|}{2} |\partial_t N| \right)^2 + \frac{1}{2} \sum_{j=2, j\neq n+1}^{2n} \left(|\partial_{x_j} N| + \theta |x_j| |\partial_t N| \right)^2. \tag{7.2.10}$$

Now we proceed to calculate $\partial_t N$:

$$\partial_t N = \frac{1}{2N} \left(\frac{t}{2n} \frac{2 \left(B^2 + t^2\right)^{\frac{1}{2n}} \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)}{\left(B + \sqrt{B^2 + t^2}\right) \left(B^2 + t^2\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right)$$

$$+ \frac{1}{2N} \left(t \left(\frac{2n-1}{2n} \right) \frac{\left(B^2 + t^2\right)^{\frac{1}{2n}} \left(2\sqrt{B^2 + t^2} + A\right)}{\left(B + \sqrt{B^2 + t^2}\right) \left(\sqrt{B^2 + t^2}\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right)$$

$$- \frac{t}{4N} \left(\frac{2 \left(B^2 + t^2\right)^{\frac{1}{2n}} \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)}{\left(B + \sqrt{B^2 + t^2}\right)^2 \left(\sqrt{B^2 + t^2}\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right)$$

$$= \frac{t \left(B^2 + t^2\right)^{\frac{1}{2n}}}{4nN} \left(\frac{2 \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)}{\left(B + \sqrt{B^2 + t^2}\right) \left(B^2 + t^2\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right)$$

$$+ \frac{t \left(B^2 + t^2\right)^{\frac{1}{2n}}}{4nN} \left(\frac{(2n-1)\sqrt{B^2 + t^2} \left(2\sqrt{B^2 + t^2} + A\right)}{\left(B + \sqrt{B^2 + t^2}\right) \left(B^2 + t^2\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right)$$

$$- \frac{t \left(B^2 + t^2\right)^{\frac{1}{2n}}}{4nN} \left(\frac{2n \frac{\sqrt{B^2 + t^2} \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)}{\left(B + \sqrt{B^2 + t^2}\right)}}{\left(B + \sqrt{B^2 + t^2}\right) \left(B^2 + t^2\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right)$$

$$= \frac{t \left(B^2 + t^2\right)^{\frac{1}{2n}}}{4nN} \left(\frac{2B(A - B) + A\sqrt{B^2 + t^2} + 2n \frac{2B^3 + 2t^2 B + 2B^2 \sqrt{B^2 + t^2} + t^2 \sqrt{B^2 + t^2}}{\left(B + \sqrt{B^2 + t^2}\right)}}{\left(B + \sqrt{B^2 + t^2}\right) \left(B^2 + t^2\right) \left(AB + t^2 + A\sqrt{B^2 + t^2}\right)^{\frac{1}{2n}}} \right). \tag{7.2.11}$$

For $j = 2, .., n, n+2, .., 2n$, we use the reverse triangle inequality on (7.2.7) in addition to (7.2.11),

$$|\partial_{x_j} N| + \theta|x_j||\partial_t N| \geq \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}$$

$$\cdot\left(\frac{AB\sqrt{B^2+t^2}+AB^2+(2B-A)t^2+(2n-1)B\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)-t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{2B(A-B)|t|+A|t|\sqrt{B^2+t^2}+2n|t|\frac{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$=\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{AB\sqrt{B^2+t^2}+A|t|\sqrt{B^2+t^2}+AB^2}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{2AB|t|-At^2-B^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{-t^2\sqrt{B^2+t^2}-2B^2|t|-B^3+Bt^2}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{n\left(2B^2\sqrt{B^2+t^2}+2B|t|\sqrt{B^2+t^2}+2B^2|t|+2B^3+2Bt^2+2|t|^3\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $A|t|\sqrt{B^2+t^2} \geq At^2$, $AB\sqrt{B^2+t^2} \geq B^2\sqrt{B^2+t^2}$, $AB^2 \geq B^3$, and $2AB|t| \geq 2B^2|t|$,

$$\geq \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{-t^2\sqrt{B^2+t^2}+Bt^2}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{2n\left(B^2\sqrt{B^2+t^2}+B|t|\sqrt{B^2+t^2}+B^2t+B^3+t^2(B+|t|)\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $B+|t| \geq \sqrt{B^2+t^2}$,

$$\geq \frac{\theta|x_j|(2n-1)\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{B\sqrt{B^2+t^2}(B+|t|)+B^2(|t|+B)+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $B+|t| \geq \sqrt{B^2+t^2}$,

$$\geq \frac{\theta|x_j|(2n-1)\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{B^2+t^2}{AB+t^2+A\sqrt{B^2+t^2}}\right)^{\frac{1}{2n}}.$$

Since $B^2+t^2 \geq \frac{1}{4}\left(AB+t^2+A\sqrt{B^2+t^2}\right)$, then,

$$|\partial_{x_j}N| + \theta|x_j||\partial_t N| \geq (2n-1)\frac{\theta|x_j|}{2^{\frac{1}{n}+2}nN}. \tag{7.2.12}$$

For $j=1, n+1$, using the calculations (7.2.4) and (7.2.11),

$$|\partial_{x_j}N| + \frac{\theta|x_j|}{2}|\partial_t N|$$

$$= \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{\frac{1}{2}B^2A+(B-\frac{A}{2})t^2+\frac{1}{2}\sqrt{B^2+t^2}AB}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{(n-1)(B^2+t^2)\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{nB\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{B(A-B)|t|+\frac{1}{2}A|t|\sqrt{B^2+t^2}+n|t|\frac{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$= \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{\frac{1}{2}B^2A+(B-\frac{A}{2})t^2+\frac{1}{2}\sqrt{B^2+t^2}AB}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{(n-1)(B^2+t^2)\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{nB\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{B(A-B)|t|+\frac{1}{2}A|t|\sqrt{B^2+t^2}+n|t|\frac{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$= \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}$$

$$\cdot\left(\frac{\frac{1}{2}AB\sqrt{B^2+t^2}+\frac{1}{2}A|t|\sqrt{B^2+t^2}+\frac{AB^2}{2}+AB|t|-\frac{At^2}{2}-B^2\sqrt{B^2+t^2}-t^2\sqrt{B^2+t^2}-B^2|t|-B^3}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{n\left(2B^2\sqrt{B^2+t^2}+B|t|\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}+B^2|t|+2B^3+2Bt^2+|t|^3\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $\frac{1}{2}A|t|\sqrt{B^2+t^2} \geq \frac{At^2}{2}$, $AB|t| \geq B^2|t|$, and $\frac{1}{2}AB\sqrt{B^2+t^2} + \frac{AB^2}{2} \geq B^3$,

$$\geq \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{-B^2\sqrt{B^2+t^2}-t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{n\left(2B^2\sqrt{B^2+t^2}+B|t|\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}+B^2|t|+2B^3+2Bt^2+|t|^3\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$=\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{(2n-1)B^2\sqrt{B^2+t^2}+(n-1)t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$+\frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{n\left(B|t|\sqrt{B^2+t^2}+(B^2+t^2)(B+|t|+B)\right)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

$$\geq \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4N}\left(\frac{(B^2+t^2)(B+|t|+B)}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $|t|+B \geq \sqrt{B^2+t^2}$,

$$\geq \frac{\theta|x_j|}{4N}\left(\frac{B^2+t^2}{AB+t^2+A\sqrt{B^2+t^2}}\right)^{\frac{1}{2n}}$$

Since $B^2+t^2 \geq \frac{1}{4}\left(AB+t^2+A\sqrt{B^2+t^2}\right)$, then for $j=1, n+1$,

$$|\partial_{x_j}N| + \frac{\theta|x_j|}{2}|\partial_t N| \geq \frac{\theta|x_j|}{2^{2+\frac{1}{n}}N}. \tag{7.2.13}$$

Replacing (7.2.12) and (7.2.13) in (7.2.10) we get a lower bound for $|\nabla N|^2$:

$$|\nabla N|^2 \geq \frac{1}{2}\left(|\partial_{x_1}N| + \frac{\theta|x_1|}{2}|\partial_t N|\right)^2$$

$$+\frac{1}{2}\left(|\partial_{x_{n+1}}N| + \frac{\theta|x_{n+1}|}{2}|\partial_t N|\right)^2 + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n}\left(|\partial_{x_j}N| + \theta|x_j||\partial_t N|\right)^2$$

$$\geq \frac{\theta^2|x_1|^2}{2^{5+\frac{2}{n}}N^2} + \frac{\theta^2|x_{n+1}|^2}{2^{5+\frac{2}{n}}N^2} + \frac{\theta^2(2n-1)^2}{2^{5+\frac{2}{n}}n^2N^2}\sum_{j=2, j\neq n+1}^{2n}|x_j|^2.$$

Hence, using the defined Euclidean norm $|x| = \left(\sum_{j=1}^{2n}x_j^2\right)^{\frac{1}{2}}$,

$$|\nabla N|^2 \geq \frac{\theta^2|x|^2}{2^{5+\frac{2}{n}}N^2}.$$

The last thing to do in this subsection is to obtain an upper bound for $|\nabla N|^2$. By (7.2.9),

$$|\nabla N|^2 = \sum_{j=1}^{2n}(\partial_{x_j}N)^2 + \frac{\theta^2 x_1^2}{4}(\partial_t N)^2 + \frac{\theta^2 x_{n+1}^2}{4}(\partial_t N)^2 + \sum_{j=2, j\neq n+1}^{2n}\theta^2 x_j^2(\partial_t N)^2$$

$$\leq \sum_{j=1}^{2n}(\partial_{x_j}N)^2 + \theta^2|x|^2(\partial_t N)^2. \tag{7.2.14}$$

For $j = 2, .., n, n+2, ..., 2n$, from (7.2.7) and the triangle inequality,

$$|\partial_{x_j}N| = \frac{\theta|x_j|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}$$

$$\cdot\left(\frac{AB\sqrt{B^2+t^2}+AB^2+(2B-A)t^2+(2n-1)B\sqrt{B^2+t^2}\left(B+\sqrt{B^2+t^2}\right)+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $\left(B^2+t^2\right) \leq \left(AB+t^2+A\sqrt{B^2+t^2}\right)$,

$$\leq \frac{\theta|x_j|}{4nN}\left(\frac{(2n-1)B^3+(2n-1)B^2\sqrt{B^2+t^2}+(2n+1)Bt^2}{B^3+B^2\sqrt{B^2+t^2}+Bt^2+t^2\sqrt{B^2+t^2}}\right)$$

$$+\frac{\theta|x_j|}{4nN}\left(\frac{AB\sqrt{B^2+t^2}+AB^2-At^2+t^2\sqrt{B^2+t^2}}{B^3+B^2\sqrt{B^2+t^2}+Bt^2+t^2\sqrt{B^2+t^2}}\right)$$

since $B \leq A \leq 2B$,

$$\leq \frac{\theta|x_j|}{4nN}\left(\frac{(2n+1)B^3+(2n+1)B^2\sqrt{B^2+t^2}+(2n)Bt^2+t^2\sqrt{B^2+t^2}}{B^3+B^2\sqrt{B^2+t^2}+Bt^2+t^2\sqrt{B^2+t^2}}\right).$$

So,

$$|\partial_{x_j}N| \leq \frac{(2n+1)\theta|x_j|}{4nN}. \tag{7.2.15}$$

From (7.2.11),

$$|\partial_t N| = \frac{|t|\left(B^2+t^2\right)^{\frac{1}{2n}}}{4nN}\left(\frac{2B(A-B)+A\sqrt{B^2+t^2}+2n\frac{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2n}}}\right)$$

since $\left(B^2+t^2\right) \leq \left(AB+t^2+A\sqrt{B^2+t^2}\right)$,

$$\leq \frac{|t|}{4nN}\left(\frac{2B(A-B)+A\sqrt{B^2+t^2}+2n\frac{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}{\left(B+\sqrt{B^2+t^2}\right)}}{\left(B+\sqrt{B^2+t^2}\right)\left(B^2+t^2\right)}\right)$$

since $|t| \leq \sqrt{B^2+t^2}$,

$$\leq \frac{1}{4nN}\left(\frac{\left(B+\sqrt{B^2+t^2}\right)\left(2B(A-B)+A\sqrt{B^2+t^2}\right)}{\left(B+\sqrt{B^2+t^2}\right)^2\sqrt{B^2+t^2}}\right)$$

$$+\frac{1}{4nN}\left(\frac{2n(2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2})}{\left(B+\sqrt{B^2+t^2}\right)^2\sqrt{B^2+t^2}}\right)$$

$$=\frac{1}{4nN}\left(\frac{3B^2A-2B^3+3AB\sqrt{B^2+t^2}-2B^2\sqrt{B^2+t^2}+At^2}{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}\right)$$

$$+\frac{1}{4nN}\left(\frac{2n\left(2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}\right)}{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}\right)$$

since $A \leq \frac{B}{2}$

$$\leq \frac{1}{4nN}\left(\frac{-\frac{1}{2}B^3-\frac{1}{2}B^2\sqrt{B^2+t^2}+At^2+2n\left(2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}\right)}{2B^3+2t^2B+2B^2\sqrt{B^2+t^2}+t^2\sqrt{B^2+t^2}}\right).$$

Hence,

$$|\partial_t N| \leq \frac{(2n+1)}{4nN}. \tag{7.2.16}$$

Inserting (7.2.6), (7.2.15), and (7.2.16) in (7.2.14), we get:

$$|\nabla N|^2 \leq \sum_{j=1}^{2n}(\partial_{x_j}N)^2+\theta^2|x|^2(\partial_t N)^2$$

$$\leq \frac{\theta^2|x_1|^2}{4N^2}+\frac{\theta^2|x_{n+1}|^2}{4N^2}+\sum_{j=2,j\neq n+1}^{2n}\frac{(2n+1)^2\theta^2|x_j|^2}{2^4n^2N^2}+\frac{(2n+1)^2\theta^2}{2^4n^2N^2}|x|^2.$$

So,

$$|\nabla N|^2 \leq \frac{(2n+1)^2\theta^2|x|^2}{2^3n^2N^2}.$$

□

Bibliography

[1] R.A. Adams. *General logarithmic Sobolev inequalities and Orlicz embeddings.* J. Func. Anal. 34, 292– 303 (1979). https://doi.org/10.1016/0022-1236(79)90036-3

[2] S. Aida and D. Stroock. *Moment estimates derived from Poincaré and logarithmic Sobolev inequalities.* Mathematical Research Letters. 1994 Jan;1(1):75-86.

[3] M. Aizenmann and R. Holley. *Rapid convergence to equilibrium of stochastic Ising models in the Dobrushin-Shlosman regime.* In H. Kesten, editor, Percolation Theory and Ergodic Theory of Infinite Particle Systems, pages 1–11. Springer, 1987.

[4] L. Ambrosio, N. Gigli, G. Savaré. *Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bounds.* The Annals of Probability, 43(1) 339-404 February 2015. https://doi.org/10.1214/14-AOP907

[5] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev arithmiques.* Number 10 in Panoramas et Synthèses. Soc. Math. France, Paris, 2000.

[6] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. *On convex Sobolev in- equalities and the rate of convergence to equillibrium for Fokker-Planck type equations.* Comm. Partial Differential Equations, 26(1-2):43–100, 2001.

[7] D. Bakry. *L'hypercontractivité et son utilisation en théorie des semigroups.* Pages 1–114. Number 1581 in Lecture Notes in Math. Springer, 1994.

[8] D. Bakry. *Functional inequalities for Markov semigroups.* In Probability Measures on Groups: Recent Directions and Trends 91–147. Tata Inst. Fund. Res., Mumbai. MR2213477, 2006.

[9] D. Bakry, F. Baudoin, M. Bonnefont, and D. Chafaï. *On gradient bounds for the heat kernel on the Heisenberg group.* J. Funct. Anal., 255:1905–1938, 2008.

[10] D. Bakry and M. Emery. *Diffusions hypercontractives.* Séminaire de probabilités, XIX, 1983/84, vol. 1123, Springer, Berlin, 1985, pp. 177–206.

[11] D. Bakry and M. Emery. *Hypercontractivité de semigroupes de diffusion.* C. R. Acad. Sci. Paris S´er. I Math., 299(15):775 – 778, 1984.

[12] D. Bakry and M. Ledoux. *A logarithmic Sobolev form of the Li-Yau parabolic inequality.* Rev. Mat. Iberoam. 22 683–702. MR2294794, 2006.

[13] D. Bakry, M. Ledoux. *Lévy–Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator.* Invent. Math. 123 (2) (1996) 259–281.

[14] Z. Balogh and J. Tyson. *Polar Coordinates in Carnot Groups.* Math Z. 2002, 241:4, 697–730.

[15] Z.M. Balogh, A. Engulatov, L. Hunzinker, O.E. Maasalo. *Functional Inequalities and Hamilton-Jacobi Equations in Geodesic Spaces.* Springer Science+Business Media B.V. 2011

[16] F. Barthe, P. Cattiaux, C. Roberto. *Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and application to isoperimetry.* Rev. Mat. Iberoamericana 22, 993–1067 (2006)

[17] R. Beals, B. Gaveau, and P. Greiner. *The Green function of model step two hypoelliptic operators and the analysis of certain tangential Cauchy Riemann complexes.* Adv. Math. 121 (1996), 288–345.

[18] S.G. Bobkov. *A functional form of the isoperimetric inequality for the Gaussian measure.* J. Funct. Anal. 135 (1996) 39–49.

[19] S. Bobkov, I. Gentil, and M. Ledoux. *Hypercontractivity of Hamilton–Jacobi equations.* J. Math. Pures Appl. 80 (2001) 669–696.

[20] S.G. Bobkov and F. Götze. *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities.* J. Funct. Anal. 163 (1999) 1–28.

[21] S. Bobkov and C. Houdré. *Isoperimetric constants for product probability measures.* Ann. Probab., 25(1):184–205, 1997.

[22] S. Bobkov and C. Houdré. *Some connections between isoperimetric and Sobolev-type inequalities.* Mem. Amer. Math. Soc., 129(616), 1997.

[23] S. Bobkov and M. Ledoux. *From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities.* Geom. Funct. Anal., 10(5):1028–1052, 2000.

[24] S. Bobkov and B. Zegarliński. *Entropy bounds and isoperimetry.* Mem. Amer. Math.Soc., 176(829), 2005.

[25] T. Bodineau and B. Helffer. The log-Sobolev inequality for unbounded spin systems. J. Funct. Anal., 166:168–178, 1999.

[26] T. Bodineau and B. Helffer. *Correlations, spectral gap and log-Sobolev inequalities for unbounded spin systems.* In Differential Equations and Mathematical Physics (Birmingham, AL, 1999), number 16 in AMS/IP Stud. Adv. Math., pages 51–66. Amer. Math. Soc., 2000.

[27] F. Bolley and I. Gentil. *Phi-entropy inequalities and Fokker-Planck equations.* Progress in Analysis and Its Applications, World Sci. Publ., Hackensack, NJ, 2010, 463–469

[28] F. Bolley and I. Gentil. *Phi-entropy inequalities for diffusion semigroups.* J. Math. Pures Appl., 93(9), 2010, 449–473.

[29] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie Groups and Potential Theory for their Sub-Laplacians.* Springer Monographs in Mathematics. Springer, 2007.

[30] C. Borell. *The Brunn-Minkowski inequality in Gauss space.* Invent. Math., 30:207– 216, 1975.

[31] E. Bou Dagher, Y. Qiu, B. Zegarliński, and M. Zhang. *Spectral Gap Inequalities on Nilpotent Lie Groups in Infinite Dimensions.* https://arxiv.org/abs/2211.14188

[32] E. Bou Dagher, Y. Wang, and B. Zegarliński. Higher Order Coercive Inequalities on Carnot Groups. Project in progress.

[33] E. Bou Dagher and B. Zegarliński. *Coercive Inequalities and U-Bounds on Step-Two Carnot Groups.* Potential Anal (2021). https://doi.org/10.1007/s11118-021-09979-0

[34] E. Bou Dagher and B. Zegarliński. *Coercive inequalities in higher-dimensional anisotropic heisenberg group.* Anal. Math. Phys. 12, 3 (2022). https://doi.org/10.1007/s13324-021-00609-x

[35] E. Bou Dagher. *Note on the $q-$Logarithmic Sobolev Inequality and the $p-$Talagrand Inequality on Carnot Groups.* Communications in Contemporary Mathematics. https://doi.org/10.1142/S0219199722500705

[36] T. Bieske. *On the Lie Algebra of polarizable Carnot groups.* Anal. Math. Phys. 10, 80 (2020).

[37] M. Brakalova, I. Markina and A. Vasil'ev. *Modules of systems of measures on polarizable Carnot groups.* Ark. Mat., 54 (2016), 371-401.

[38] L. Capogna, D. Danielli, S.D. Pauls, and J. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* Progress in Mathematics, 259. Birkhäuser Verlag, Basel, 2007. xvi+223 pp. ISBN: 978-3- 7643-8132-5.

[39] E. Carlen and D. Stroock. *An application of the Bakry-Emery criterion to infinite dimensional diffusions.* In Séminaire de Probabilités, XX, number 1204 in Lecture Notes in Math., pages 341–348. Springer, 1986.

[40] D. Chafaï. *Entropies, convexity and functional inequalities: on Phi-entropies and Phi-Sobolev inequalities.* J. Math. Kyoto Univ., 44(2):325–363, 2004.

[41] D. Chafaï. *Binomial-Poisson entropic inequalities and the $M/M/\infty$ queue.* ESAIM Probab. Stat. 10, 10:317–339, 2006.

[42] M. Chatzakou, S. Federico, and B. Zegarlinski. *q-Poincaré inequalities on Carnot Groups with filiform type Lie algebra.* arXiv:2007.04689 [math.FA]

[43] M. Chatzakou, A. Kassymov, and M. Ruzhansky. *Logarithmic Hardy-Rellich inequalities on Lie groups.* arXiv:2107.04874 [math.AP]

[44] M. Chatzakou, A. Kassymov, and M. Ruzhansky. *Logarithmic Sobolev inequalities on Lie groups.* arXiv:2106.15652 [math.AP]

[45] J. Cheeger. *A lower bound for the smallest eigenvalue of the Laplacian.* Problems in Analysis, papers dedicated to Salomon Bochner, 1969, Princeton Univ. Press, Princeton, NJ, 1970, pp. 195–199.

[46] W.S. Cohn, G. Lu, and P. Wang. *Sub-elliptic global high order Poincaré inequalities in stratified Lie groups and applications.* (English summary) J. Funct. Anal. 249 (2007), no. 2, 393–424.

[47] I. Csiszár. *A class of measures of informativity of observation channels.* I. Period. Math. Hungar., 2(191-213), 1972.

[48] L. D'Ambrosio. *Hardy type inequalities related to degenerate elliptic differential operators.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 4(3):451–486, 2005.

[49] D. Danielli, N. Garofalo, and N.C. Phuc. *Hardy–Sobolev type inequalities with sharp constants in Carnot–Carathéodory spaces.* Potential Anal., 34:223–242, 2011.

[50] P. Diaconis and L. Miclo. *On quantitative convergence to quasi-stationarity.* InAnnales de la Faculté des sciences de Toulouse: Mathématiques 2015 (Vol. 24, No. 4, pp. 973-1016).

[51] P. Diaconis and D. Stroock. *Geometric bounds for eigenvalues of Markov chains.* The annals of applied probability (1991): 36-61.

[52] R. Dobrushin. *The description of Random Fields by Means of Conditional Probabilities and Conditions of its Regularity.* Theor. Prob. its Appl. 13, 197-224 (1968).

[53] R. Dobrushin. *Prescribing a system of random variables by conditional distributions.* Theory of Probability & Its Applications 15, 3 (1970), pp. 458-486.

[54] F. Dragoni. *Metric Hopf–Lax formula with semicontinuous data.* Discrete Contin. Dyn. Syst. 17(4), 713–729 (2007).

[55] B. Driver and T. Melcher. *Hypoelliptic heat kernel inequalities on the Heisenberg group.* J. Funct. Anal., 221:340–365, 2005.

[56] N. Eldredge. *Precise estimates for the subelliptic heat kernel on H-type groups.* J. Math. Pures Appl., 92:52–85, 2009.

[57] N. Eldredge. *Gradient estimates for the subelliptic heat kernel on H-type groups.* J. Funct. Anal., 258(2):504–533, 2010.

[58] P. Federbush. *Partially alternative derivation of a result of Nelson.* J. Math. Phys., 10:50–52, 1969.

[59] G. B. Folland. *Subelliptic estimates and function spaces on nilpotent Lie groups.* Ark. Mat., 13:161–207, 1975.

[60] I. Gentil and C. Roberto. *Spectral Gaps for Spin Systems: Some Non-convex Phase Examples.* J. Funct. Anal., 180:66-84, 2001.

[61] H. O. Georgii. *Gibbs measures and phase transitions* (de Gruyter, 2011).

[62] J.A. Goldstein and I. Kombe. *The Hardy inequality and nonlinear parabolic equations on Carnot groups.* Nonlinear Anal., 69(12):4643– 4653, 2008.

[63] J.A. Goldstein, I. Kombe, and A. Yener. *A unified approach to weighted Hardy type inequalities on Carnot groups.* Discrete Contin. Dyn. Syst., 37(4):2009–2021, 2017.

[64] F. Z. Gong and F. Y. Wang. *Functional inequalities for uniformly integrable semigroups and application to essential spectrums.* Forum Mathematicum, 14:293–313, 2002.

[65] M. Gordina and L Luo. L*ogarithmic Sobolev inequalities on non-isotropic Heisenberg groups.* J. Funct. Anal., 283(2):109500, 2022. https://doi.org/10.1016/j.jfa.2022.109500

[66] N. Gozlan. *A characterization of dimension free concentration in terms of transportation inequalities.* Ann. Probab. 37(6), 2480–2498 (2009)

[67] N. Gozlan, C. Roberto, and P.M. Samson. *From concentration to logarithmic Sobolev and Poincaré inequalities.* J. Funct. Anal. 260(5), 1491–1522 (2010).

[68] M. Gromov. *Carnot–Carathéodory spaces seen from within.* In: Sub-Riemannian Geometry, Progr. Math., vol. 144, pp. 79–323. Birkhäuser, Basel (1996).

[69] L. Gross. *Logarithmic Sobolev inequalities.* Amer. J. Math., 97:1061–1083, 1975.

[70] A. Guionnet and B. Zegarliński. *"Lectures on logarithmic Sobolev inequalities".* In: S´eminaire de probabilit´es XXXVI. Springer, pp. 1–134 (2003).

[71] P. Hajłasz and S. Zimmerman. *Geodesics in the Heisenberg group.* Analysis and Geometry in Metric Spaces 3, 1 (2015).

[72] W. Hebisch and B. Zegarliński. *Coercive inequalities on metric measure spaces.* J. Funct. Anal., 258:814–851, 2010.

[73] B. Helffer. Remarks on decay of correlations and Witten Laplacians III. *Application to logarithmic Sobolev inequalities.* Ann. Inst. H. Poincaré Probab. Statist., 35:483– 508, 1999.

[74] B. Helffer. *Semiclassical Analysis, Witten Laplacians, and Statistical Mechanics.* Number 1 in Partial Differential Equations and Applications. World Scientific, 2002.

[75] R. Holley and D. Stroock. *Logarithmic Sobolev inequalities and stochastic Ising models.* J. Stat. Phys., 46(5-6):1159–1194, 1987.

[76] R. Holley and D. Stroock. *Uniform and L2 convergence in one dimensional stochastic Ising models.* Comm. Math. Phys., 123:85–93, 1989.

[77] A. Hurwitz. *Sur le probléme des isopérimétres.* C. R. Acad. Sci. Paris 132 (1901), 401–403; Math. Werke, 1. Bd., Birkhäuser, 1932, pp. 490-491.

[78] Y. Z. Hu. *A unified approach to several inequalities for Gaussian and diffusion measures.* In Séminaire de Probabilite´s, XXXIV, volume 1729 of Lecture Notes in Math., pages 329–335. Springer, 200.

[79] J. Inglis. *Coercive Inequalities for Generators of Hörmander Type.* Doctor of Philosophy of the University of London and the Diploma of Imperial College, Department of Mathematics Imperial College, 2010.

[80] J. Inglis, V. Kontis, and B. Zegarlinski. *From U-bounds to isoperimetry with applications to H-type groups.* J. Funct. Anal. 260, 76–116 (2011)

[81] J. Inglis and I. Papageorgiou. *Log-Sobolev inequalities for infinite-dimensional Gibbs measures with non-quadratic interactions.* Markov Processes Relat. Fields 25, 879-897 (2019).

[82] J. Inglis and I. Papageorgiou. *Logarithmic Sobolev Inequalities for Infinite Dimensional Hörmander Type Generators on the Heisenberg Group.* Potential Anal., 31, 79-102 (2009).

[83] J. Jerison. *The Poincaré inequality for vector fields satisfying Hörmander's condition.* Duke Math. J. 53 (1986), no. 2, 503-523.

[84] A. Kaplan. *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms.* Trans. Amer. Math. Soc., 258(1):147–153, 1980.

[85] I. Kombe. *Sharp weighted Rellich and uncertainty principle inequalities on Carnot groups.* Commun. Appl. Anal., 14(2):251–271, 2010.

[86] T. Konstantopoulos and I. Papageorgiou. *The Log-Sobolev Inequality for Spin Systems of Higher Order Interactions.* arXiv:1906.11980 .

[87] O.E. Lanford, D. Ruelle. *Observables at infinity and states with short range correlations in statistical mechanics.* Comm. Math. Phys. 13:194–215, 1969.

[88] E. Le Donne. *A Primer on Carnot Groups: Homogenous Groups, Carnot-Carathéodory Spaces, and Regularity of Their Isometries.* Anal. Geom. Metr. Spaces 2017; 5:116–137.

[89] M. Ledoux. *Concentration of measure and logarithmic Sobolev inequalities.* in: Séminaire de Probabilités XXXIII, in: Lecture Notes in Mathematics, vol. 1709, Springer-Verlag, Berlin, 1999, pp. 120–216.

[90] M. Ledoux. *The Concentration of Measure Phenomenon.* American Mathematical Society, Providence, 2001.

[91] H. Q. Li. *Estimation optimale du gradient du semigroupe de la chaleur sur le groupe de Heisenberg.* J. Funct. Anal., 236:369–394, 2006.

[92] T. Liggett. *Interacting Particle Systems.* Springer, 2004.

[93] X. Li, C.Z. Lu, and H.L. Tang. *Poincaré inequalities for vector fields satisfying Hörmander's condition in variable exponent Sobolev spaces.* Acta Math. Sin. (Engl. Ser.) 31 (2015), no. 7, 1067–1085.

[94] J. Lott and C. Villani. *Hamilton–Jacobi semigroup on length spaces and applications.* J. Math. Pures Appl. (9) 88(3), 219–229 (2007).

[95] G. Lu. *Local and global interpolation inequalities on the Folland-Stein Sobolev spaces and polynomials on stratified groups.* (English summary) Math. Res. Lett. 4 (1997), no. 6, 777–790.

[96] G. Lu. *Polynomials, higher order Sobolev extension theorems and interpolation inequalities on weighted Folland-Stein spaces on stratified groups.* (English summary) Acta Math. Sin.(Engl. Ser.) 16 (2000), no. 3, 405–444.

[97] G. Lu and R.L. Wheeden. H*igh order representation formulas and embedding theorems on stratified groups and generalizations.* Studia Math. 142 (2000), no. 2, 101–133. (Reviewer: G. B. Folland).

[98] G. Lu and R.L. Wheeden. *Simultaneous representation and approximation formula and high-order Sobolev embedding theorems on stratified groups.* (English summary) Constr. Approx. 20 (2004), no. 4, 647–668.

[99] S. L. Lu and H. T. Yau. *Spectral gap and logarithmic Sobolev inequalities for Kawasaki and Glauber dynamics.* Comm. Math. Phys., 156:399–433, 1993.

[100] P. Ługiewicz and B. Zegarliński. *Coercive inequalities for Hörmander type generators in infinite dimensions.* J. Funct. Anal. 247(2):438–476, 2007.

[101] P. Maheux, L. Saloff-Coste. *Analyse sur les boules d'un opérateur sous-elliptique.* Math. Ann. 303 (1995) 713–740.

[102] F. Martinelli and E. Olivieri. *Approach to equilibrium of Glauber dynamics in the one phase region I. The attactive case.* Comm. Math. Phys., 161:447–486, 1994.

[103] F. Martinelli and E. Olivieri. *Approach to equilibrium of Glauber dynamics in the one phase region II. The general case.* Comm. Math. Phys., 161:487–514, 1994.

[104] K. Marton. *A simple proof of the blowing-up lemma.* IEEE Trans. Inform. Theory 32, 445–446 (1986).

[105] K. Marton. *Bounding d-distance by informational divergence: a method to prove measure concentration.* Ann. Probab. 24, 857–866 (1996).

[106] T. Melcher. *Hypoelliptic heat kernel inequalities on Lie groups.* PhD thesis, University of California, San Diego, 2004.

[107] E. Milman. *On the role of convexity in functional and isoperimetric inequalities.* Proc. London Math. Soc., 99(3):32–66, 2009.

[108] R. Monti. *Some properties of Carnot-Carathéodory balls in the Heisenberg group.* Rend. Mat. Acc. Lincei, 11:155–167, 2000.

[109] F. Otto. *The geometry of dissipative evolution equations: the porous medium equation.* Comm. Partial Differential Equations 26, 101–174 (2001).

[110] F. Otto and C. Villani. *Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality.* J. Funct. Anal. 173, 361–400 (2000).

[111] I. Papageorgiou. *The Logarithmic Sobolev Inequality in Infinite dimensions for Unbounded Spin Systems on the Lattice with non Quadratic Interactions.* Markov Processes Relat. Fields 16, 447-484 (2010).

[112] G. Perelman. *The entropy formula for the Ricci flow and its geometric applications.* arXiv preprint math/0211159 (2002).

[113] H. Poincaré. *Sur les Equations aux Derivees Partielles de la Physique Mathematique.* American Journal of Mathematics, Vol. 12, No. 3, 1890, pp. 211-294. doi:10.2307/2369620

[114] M. Reed and B. Simon. *IV: Analysis of Operators. Methods of Modern Mathematical Physics.* Academic Press, 1978.

[115] C. Roberto and B. Zegarliński. *Orlicz-Sobolev inequalities for sub-Gaussian measures and ergodicity of Markov semigroups.* J. Funct. Anal., 243(1):28–66, 2007.

[116] J. Rosen. *Sobolev Inequalities For Weight Spaces and Supercontractivity.* Transactions of the American Mathematical Society Volume 222, 197

[117] O. Rothaus. *Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities.* J. Funct. Anal., 64:296–313, 1985.

[118] O. Rothaus. *Hypercontractivity and the Bakry-Emery criterion for compact Lie groups.* J. Funct. Anal., 65(3):358–367, 1986.

[119] M. Ruzhansky and D. Suragan. *Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups.* Adv. Math. 317 (2017), 799–822.

[120] L. Saloff-Coste. *Théorèmes de Sobolev et inégalités de Trudinger sur certain groupes de Lie.* C. R. Acad. Sci. Paris 306 (1988), 305–308.

[121] E. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, New Jersey, 1993.

[122] D. Stroock and B. Zegarliński. *The logarithmic Sobolev inequality for continuous spin systems on a lattice.* J. Funct. Anal., 104:299–326, 1992.

[123] D. Stroock and B. Zegarliński. *The logarithmic Sobolev inequality for discrete spin systems on a lattice.* Comm. Math. Phys., 149:175–193, 1992.

[124] D. Stroock and B. Zegarliński. *On the ergodic properties of Glauber dynamics.* J. Stat. Phys., 81:1007–1019, 1995.

[125] V. N. Sudakov and B. S. Tsirel'son. *Extremal properties of half-spaces for spherically invariant measures.* J. Sov. Math., 9:9–18, 1978.

[126] M. Talagrand. *A new isoperimetric inequality and the concentration of measure phenomenon.* In Geometric Aspects of Functional Analysis, number 1469 in Lecture Notes in Math., pages 94–124. Springer-Verlag, 1991.

[127] M. Talagrand. *Transportation cost for Gaussian and other product measures.* Geom. Funct. Anal. 6, 587–600 (1996).

[128] J. Tyson. *Stability theorems for H-type Carnot groups.* Preprint 2022. arXiv:2208.04925 [math.DG]

[129] N.T. Varopoulos, L. Saloff-Coste and T. Coulhon. *Analysis and Geometry on Groups.* Cambridge Tracts in Mathematics, 100, Cambridge University Press, Cambridge (1992).

[130] F. Y. Wang. *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds.* Probab. Theory Related Fields, 109:417–424, 1997.

[131] F. Y. Wang. *Functional inequalities for empty essential spectrum.* J. Funct. Anal., 170:219–245, 2000.

[132] F. Y. Wang. *Logarithmic Sobolev inequalities: conditions and counterexamples.* J. Operator Theory, 46:183–197, 2001.

[133] F. Y. Wang. *Functional inequalities and spectrum estimates: the infinite measure case.* J. Funct. Anal., 194:288–310, 2002.

[134] F. Y. Wang. *Super and weak Poincare´ inequalities for hypoelliptic operators.* Acta Math. Appl. Sin. Engl. Ser., 25(4):617–630, 2009.

[135] J. Wang and P. Niu. *Sharp weighted Hardy type inequalities and Hardy–Sobolev type inequalities on polarizable Carnot groups.* C. R. Math. Acad. Sci. Paris Ser. I, 346:1231–1234, 2008.

[136] Y. Wang and B. Zegarlinski. *Higher Order Coercive Inequalities.* Potential Anal (2021). https://doi.org/10.1007/s11118-021-09940-1

[137] Y. Xiao. *An improved Hardy type inequality on Heisenberg group.* J. of Inequalities and Applications, 2011:38, 2011.

[138] Q. Yang. *Best constants in the Hardy-Rellich type inequalities on the Heisenberg group.* J. Math. Anal. Appl., 342, 423-431 (2008).

[139] N. Yosida. *The log-Sobolev inequality for weakly coupled lattice fields.* Probab. Theory Related Fields, 115:1–40, 1999.

[140] N. Yosida. *Application of log-Sobolev inequality to the stochastic dynamics of unbounded spin systems on the lattice.* J. Funct. Anal., 173:74–102, 2000.

[141] N. Yosida. *The equivalence of the log-Sobolev inequality and a mixing condition for unbounded spin systems on a lattice.* Ann. Inst. H. Poincaré Probab. Statist., 37(2):223–243, 2001.

[142] B. Zegarliński. *Log-Sobolev inequalities for infinite one-dimensional lattice systems.* Comm. Math. Phys., 133(147-162), 1990.

[143] B. Zegarliński. *On log-Sobolev inequalities for infinite lattice systems.* Lett. Math. Phys., 20(3):173–182, 1990.

[144] B. Zegarliński. *The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice.* Comm. Math. Phys., 175:401–432, 1996.