

ASYMPTOTIC MODELLING OF
THERMOVISCOUS DISSIPATION IN
RESONANT ACOUSTIC STRUCTURES

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To my family, the Holleys, Russells and Gowns.

ABSTRACT

In this thesis, we employ asymptotic methods to systematically develop models for studying wave propagation and absorption in resonant acoustic structures. A novel aspect of our work is the inclusion from first principles of thermoviscous effects in multiple-scale media entailing complex geometric features. We initially focus on extraordinary transmission through narrow slits and slender holes embedded in rigid slabs, using scaling arguments and the method of matched asymptotic expansions to derive accurate approximations for field enhancements within the slits and holes, deviations of resonances from the standing-wave (Fabry–Pérot) frequencies expected in the absence of aperture effects, and sound absorption. Furthermore, we find that a well-known analogy between acoustic and electromagnetic extraordinary transmission can surprisingly be extended to include dissipative effects, in the case where these are associated with thin thermoviscous and electromagnetic-skin boundary layers, respectively in the acoustic and electromagnetic scenarios. Towards generalising the theory, we reformulate our asymptotic model for a hole resonator in terms of its spectral response, and extend to the case of multiple hole resonators via a multiple-scattering (Foldy-type) scheme. Armed with that extended theory, and a homogenisation approximation of it, we illuminate the acoustic properties of finite and infinite metasurface slabs formed of doubly periodic arrays of holes, covering extraordinary transmission, coherent perfect absorption (multi-channel critical coupling), lattice resonances of finite surfaces, and surface-wave effects.

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CHAPTER 1

Introduction

1.1 Background and motivation

Resonant structures in acoustics have received renewed attention in recent decades for their capacity to control sound waves. These structures exhibit local or global wave-sustaining dynamics, characterised by minimal energy loss over extended timescales. External forcing can excite these systems, manifesting distinct wave fields that are signatures of the structures and fluids involved. This excitation leads to concentrated energy amplification, facilitating coupling across the domain and with damping mechanisms. The emergence of resonances raises fundamental questions about the conditions required for excitation, sensitivity to frequency variations, and the spectral response of such systems. A precise understanding of these principles is essential for predicting and manipulating wave behaviour.

The emerging field of acoustic metamaterials, inspired by advancements in electromagnetics [107], leverages the collective dynamics of arrays made up of sub-wavelength resonators — structures with dimensions significantly smaller than the wavelengths at which they resonate [24, 80, 28]. These artificial metamaterials demonstrate unique properties not found in natural materials, profoundly influencing their interaction with sound. This has led to the development of proof-of-concept devices, such as acoustic superlenses and invisibility cloaks, with extraordinary capabilities, including collimation and focusing [19, 127], sub-wavelength imaging [55, 128], negative refraction [62], acoustic cloaking [29, 100, 129], and extraordinary acoustic transmission [49, 20].

The interaction between sound waves and structures represents a fundamental problem in acoustic theory. Elementary examples, including cylindrical and spherical shells, thin plates and columns, were extensively studied by Lord

Rayleigh in his seminal treatise *The Theory of Sound* in the nineteenth century [96]. The advent of metamaterials and metasurfaces has introduced new challenges for systematic modelling. These challenges stem from the interplay between resonant phenomena, complex geometries, and wave physics across disparate length scales. The surge of interest in this field has driven the development of new mathematical models to better understand the unique behaviours that emerge in these systems. These models utilise tools that include matched asymptotic expansions [60, 26, 14], operator methods [4, 3, 73, 70] and multiple-scale homogenisation [25, 104]. Still, for many of these models, a key limitation is their typical neglect of dissipation effects, which can destroy the extraordinary phenomena predicted by idealised scenarios. Moreover, dissipation is an integral aspect that features in the design of certain devices, as discussed below.

Dissipative effects in acoustic metamaterials are often associated with the presence of thermal and momentum boundary layers on rigid walls. In the context of sound propagation through air, it is commonly assumed that solid boundaries are perfectly rigid, given the substantial mismatch in acoustic impedances [69, 46]. This dynamic and thermal mismatch leads to the formation of thermodynamic boundary layers on the walls of any cavity confining airborne sound. Kirchhoff [65] provided the first comprehensive analysis of these effects in an infinitely long narrow tube. Subsequent studies have further explored these implications [122, 114, 110], including Lord Rayleigh’s specific examination of narrow slit-cavities [97]. Despite the detailed treatment of these effects in simple geometries, as previously highlighted, most analytical studies of acoustic metamaterials — particularly the more mathematically systematic ones — do not incorporate thermal and viscous (thermoviscous) effects.

This omission is frequently made to facilitate analysis and reduce the complexity of numerical simulations. This assumption is often rationalised by the significant disparity between the characteristic dimensions of the boundary layers and the resonator geometries — differing by more than two orders of magnitude in many cases [122, 64]. Nonetheless, the authors in [119, 85] demonstrate that the loss-free modelling of resonant structures can be unrealistic despite this disparity. Even when dissipation effects are included in the modelling, they are typically represented using heuristic methods and fitting parameters. One common approach begins with analysing dissipative sound propagation in well-defined geometries, such as infinite channels, where exact solutions are known. These solutions are then adapted to more complex configurations by introducing ‘end corrections’, which require case-specific empirical validation [106, 53, 89, 82, 56].

Building on the understanding of boundary-layer effects, considerable attention has been directed towards the development of metamaterials designed to enhance sound absorption. A prominent application of this technology is the integration of resonant chambers and perforated panels in combustion systems to reduce noise [82, 18]. A widely adopted approach for achieving optimal absorption in metamaterials involves the use of resonators tuned according to critical-coupling conditions [53, 126, 23, 37]. These conditions, which depend on the system's material properties and the geometry and configuration of resonators, describe a physical scenario in which the dissipation damping from internal losses is balanced by radiation damping from the energy emitted by the resonators. Notably, both theoretical and experimental investigations have demonstrated that certain metamaterials, when critically coupled, can achieve perfect absorption, absorbing all incoming sound at a designated frequency [125, 99, 98] (experimental results have observed absorption rates up to 99.7% of the incident acoustic power [125]). This concept extends to 'coherent perfect absorption', a two-port or more phenomenon where incoming waves are phase-matched and perfectly synchronised with the resonators, facilitating destructive interference and energy dissipation within the structure to achieve complete absorption [81, 120]. These phenomena further emphasise the need for models that incorporate thermoviscous effects.

1.2 Objectives and methodology

This thesis employs asymptotic methods to derive new systematic models for resonant acoustic structures, incorporating thermoviscous effects from first principles. In particular, we shall rely on the method of matched asymptotic expansions [27, 83], a method that systematically relates solutions across distinct asymptotic regimes, enabling smooth transitions between them. To model thermoviscous effects, we utilise the thermoviscous acoustic equations, which are fundamental to understanding sound propagation in the context of losses [90].

Our focus is on configurations where wave transmission is channelled via narrow slits and slender holes, which are widely used for sound manipulation and absorption and serve as fundamental models for metamaterials. We derive new formulae relevant to practical applications, including critical-coupling conditions for extraordinary transmission and coherent perfect absorption, thereby demonstrating the applicability of our results to real-world scenarios. Moreover, even though our attention is on the effects of thermoviscous dissipation, we also reveal new insights into lossless acoustics and explore interesting parallels with electromagnetic systems, all elucidated through asymptotic analysis.

The remaining sections of this introduction are organised as follows. In §1.3, we explore the acoustic properties of narrow slits and slender holes. In §1.4, we outline the fundamental equations and boundary conditions relevant to thermal and viscous acoustics.

1.3 Extraordinary transmission through narrow slits and slender holes

An elementary and fundamental example, which has been widely studied experimentally, theoretically and computationally, is sound transmission through a single slit in a rigid wall whose thickness is comparable to the wavelength and large compared to the slit width [20, 119]. When the system is absent of losses (lossless), the problem is mathematically analogous to transmission of transverse magnetic (TM)-polarised electromagnetic waves through a slit in a perfectly-conducting metallic wall [113, 124, 44, 112, 67, 15, 76, 73].

In these idealised settings, enhanced transmission, where waves propagate with greater efficiency than predicted, is attributed to excitation of standing waves in the slit, or so-called Fabry–Pérot resonances [10], at frequencies:

$$f_m = \frac{mc}{2l}, \quad (1.1)$$

where $m = 1, 2, \dots$ is a positive integer, c is the speed of sound (light), and l is the length of the slit. Close to these standing-wave frequencies, the transmission efficiency and slit-field magnitude see considerable enhancement; this phenomenon is known as extraordinary transmission. The term originated in electromagnetism [36, 93, 43] to describe enhanced transmission of wave energy through small apertures via excitation of surface plasmons, spoof plasmons and localised resonances, with similar effects identified in acoustics [20] and water waves [41, 39].

Resonant transmission peaks are confined to extremely narrow frequency intervals. To accurately describe this phenomenon, it is necessary to determine the deviations of these peaks from the classical Fabry–Pérot resonances (1.1). In the absence of losses, the frequency shifts result from energy leakage through diffraction at the slit ends. The calculation of these shifts is complex, and comprehensive attempts in the literature are sparse. In [113], the author decomposes the wavefield into three distinct plane wave expansions – one valid internally in the slit and two in the external half-planes either side of the wall. By requiring continuity and

differentiability of the wavefield at the slit openings, they derive an infinite system of equations for the mode amplitudes inside the slit. Furthermore, by arbitrarily truncating this system to a small number of terms, a simple closed-form expression for the frequency shifts:

$$\left(\frac{\Delta f}{f_m}\right)_{\text{Ref. [113]}} = \frac{4h}{\pi} \left(\ln h + \ln(\pi m) - \frac{3}{2} \right), \quad (1.2)$$

where h is the slit's width-to-length ratio. While this approximation was thought to hold for small h , later experiments by other researchers in both acoustic and electromagnetic contexts revealed significant discrepancies with (1.2). These differences could not be fully explained by dissipative effects alone. The authors in [73] revisited the problem more than a decade later, using asymptotic analysis and layer-potential techniques to determine the corrected frequency shifts for $h \ll 1$:

$$\left(\frac{\Delta f}{f_m}\right)_{\text{Ref. [73]}} = \frac{4h}{\pi} \left(\ln h + \ln(2m) + \gamma_E + \ln(2\pi) + \frac{\pi}{\alpha} \right), \quad (1.3)$$

wherein $\gamma_E \approx 0.5772\dots$ is the Euler-Mascheroni constant, and α is a constant that was defined in the course of the analysis but was not provided explicitly. In Chapter 2, we use matched asymptotics to correct and extend existing literature results, aiming to provide a simpler and physically more insightful description of extraordinary transmission. Despite the rigorous nature of the models behind (1.3) and previous systematic works [59, 60, 61], their applicability is severely limited by neglecting thermoviscous effects. Experiments in acoustics have demonstrated that when slits are sufficiently narrow, thermoviscous effects can significantly diminish transmission peaks [119, 85] and cause a downward shift in resonance frequencies compared to predictions based on a lossless model, such as (1.3). Comparable effects have also been revealed in electromagnetics, where the finite conductivity of the plate induces dissipative effects [112].

So far, we have limited our discussion to a single slit within an infinite plate. However, acoustic transmission through slit arrays has also been analysed through both experimental and theoretical investigations [117, 74, 75]. Most of these studies consider cases where the distance between slits is on the order of the slit length l . In such cases, the slit array resonance frequencies match, to leading order, those of a single slit (1.1). However, the effects of multiple scattering are significant at the order of the leading frequency shift, which invalidates (1.3). As in the single slit scenario, transmission is enhanced close to resonance frequencies. Increasing dissipation decreases transmission peaks and transforms them into absorption peaks. In the literature, there is a scarcity of systematic studies accounting for

material losses in wave propagation through narrow slits. In Chapters 3 and 4, we derive an asymptotic model to study dissipation effects on acoustic wave transmission through narrow slits and explore its electromagnetic analogy.

The generalisation to three-dimensional subwavelength structures has been studied through physical experiments [49, 50] (and separately in electromagnetism [45, 78, 42, 16]) and shown to realise high field enhancement that can also be attributed to Fabry–Pérot type resonances (1.1). Nevertheless, far fewer analytical studies have systematically investigated acoustic transmission in a three-dimensional slender hole, defined as a circular hole perforated in an infinite slab of finite thickness, where the slab thickness is large compared to the radius of the hole [70, 40, 79]. In [70], the authors carry out an asymptotic analysis of a boundary-integral formulation of the problem to determine frequency shifts:

$$\left(\frac{\Delta f}{f_m} \right)_{\text{Ref. [70]}} = \frac{4h}{\alpha}, \quad (1.4)$$

where α is defined implicitly. The systematic inclusion of thermoviscous dissipation is even less prevalent than in its two-dimensional counterpart. In Chapters 5, 6 and 7, we extend our asymptotic theory to three-dimensions and focus on the effects of dissipative boundary layers on transmission.

This section highlights the significance of thermoviscous dissipation in fundamental problems that serve as the basis for acoustic metamaterials. Before proceeding with the analysis, we will introduce key definitions and the equations of thermoviscous acoustics.

1.4 Fundamentals of thermoviscous acoustics

Acoustic energy is converted into heat through complex thermoviscous processes. (Sound cannot perpetuate indefinitely within an enclosed environment, contrary to the beliefs held by the ancient Greeks [94].) Thermal losses occur when temperature gradients drive heat conduction within the transmission medium, converting sound energy into random thermal motion. Viscous losses arise from velocity gradients between adjacent fluid layers, generating shear stresses that transform organised wave energy into disordered kinetic energy. For low-amplitude waves, these effects are often negligible in the bulk medium but can become important near rigid boundaries or in the presence of small-scale features, ultimately contributing to the attenuation and dissipation of acoustic energy.

This section outlines the fundamental equations describing sound propagation, accounting for thermoviscous effects [65, 72]. We adopt the modern formulation of thermal and viscous acoustics, given in [90].

1.4.1 Governing equations

The flow of a compressible fluid is governed by the mass continuity equation

$$\frac{D\tilde{\rho}}{Dt} + \tilde{\rho} \nabla \cdot \tilde{\mathbf{u}} = 0, \quad (1.5)$$

and momentum equation

$$\tilde{\rho} \frac{D\tilde{\mathbf{u}}}{Dt} = \nabla \cdot \tilde{\boldsymbol{\sigma}}, \quad (1.6)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \quad (1.7)$$

represents the material derivative, t denotes time, and $\tilde{\rho}$, $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\sigma}}$ are the density, velocity and stress tensor fields, respectively. Throughout this section, the tilde $\sim$ notation denotes dimensional quantities.

Assuming the fluid is Newtonian, the stress tensor can be expressed as a sum of pressure and viscous contributions

$$\tilde{\boldsymbol{\sigma}} = -\tilde{p}\mathbf{I} + \tilde{\boldsymbol{\pi}}, \quad \tilde{\boldsymbol{\pi}} = \eta \left[\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^\dagger - \frac{2}{3}(\nabla \cdot \tilde{\mathbf{u}})\mathbf{I} \right], \quad (1.8)$$

where $\tilde{p}$ is the pressure field, $\mathbf{I}$ is the identity tensor, η is the constant dynamic viscosity of the fluid (neglecting bulk viscosity under Stokes' hypothesis [111]) and $\dagger$ denotes the transpose operator. Substituting (1.8) into (1.6) yields

$$\tilde{\rho} \frac{D\tilde{\mathbf{u}}}{Dt} = -\nabla \tilde{p} + \eta \left[\nabla^2 \tilde{\mathbf{u}} + \frac{1}{3} \nabla(\nabla \cdot \tilde{\mathbf{u}}) \right]. \quad (1.9)$$

Next, consider a time-dependent control volume $\mathcal{V}(t)$ that convects with the fluid flow. According to the first law of thermodynamics, the rate of change of total energy within this volume is

$$\frac{d}{dt} \int_{\mathcal{V}} \left(\frac{1}{2} \tilde{\rho} \tilde{u}^2 + \tilde{\rho} \tilde{e} \right) dV = \int_{\partial \mathcal{V}} \tilde{\mathbf{u}} \cdot \tilde{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} dA - \int_{\partial \mathcal{V}} \tilde{\mathbf{q}} \cdot \hat{\mathbf{n}} dA, \quad (1.10)$$

where dA and dV are differential area and volume elements, $\hat{\mathbf{n}}$ is the outward unit normal on the boundary $\partial \mathcal{V}$, and $\tilde{\rho} \tilde{u}^2/2$ and $\tilde{\rho} \tilde{e}$ are the respective kinetic and internal energy densities. The first term on the right-hand side of the energy

balance (1.10) represents the rate of work done by surface stresses, and the second term quantifies the rate of conductive heat transfer, where $\tilde{\mathbf{q}}$ denotes the heat flux density. Assuming that the heat flux obeys Fourier's law of heat conduction [68, 33],

$$\tilde{\mathbf{q}} = -\kappa \nabla \tilde{T}, \quad (1.11)$$

where κ is the thermal conductivity and $\tilde{T}$ is the temperature field.

By applying the Reynolds transport theorem [5], we obtain

$$\frac{d}{dt} \int_{\mathcal{V}} \left(\frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}}^2 + \tilde{\rho} \tilde{e} \right) dV = \int_{\mathcal{V}} \tilde{\rho} \frac{D}{Dt} \left(\frac{1}{2} \tilde{\mathbf{u}}^2 + \tilde{e} \right) dV. \quad (1.12)$$

Furthermore, applying the divergence theorem, results in

$$\int_{\partial \mathcal{V}} \tilde{\mathbf{u}} \cdot \tilde{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} dA = \int_{\mathcal{V}} \nabla \cdot (\tilde{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{u}}) dV, \quad \int_{\partial \mathcal{V}} \tilde{\mathbf{q}} \cdot \hat{\mathbf{n}} dA = \int_{\mathcal{V}} \nabla \cdot \tilde{\mathbf{q}} dV. \quad (1.13a, b)$$

Substituting these identities into equation (1.10) and recognising that the choice of control volume is arbitrary, we derive the local form of the energy balance

$$\tilde{\rho} \frac{D}{Dt} \left(\frac{1}{2} \tilde{\mathbf{u}}^2 + \tilde{e} \right) = \nabla \cdot (\tilde{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{u}}) - \nabla \cdot \tilde{\mathbf{q}}. \quad (1.14)$$

Equation (1.14) can be further simplified. We start by taking the dot product of $\tilde{\mathbf{u}}$ with (1.6), arriving at

$$\frac{\tilde{\rho}}{2} \frac{D \tilde{\mathbf{u}}^2}{Dt} = \nabla \cdot (\tilde{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{u}}) - \frac{1}{2\eta} \tilde{\boldsymbol{\sigma}} : \nabla \tilde{\mathbf{u}}. \quad (1.15)$$

Subtracting (1.15) from (1.14) isolates the internal energy component, resulting in

$$\tilde{\rho} \frac{D \tilde{e}}{Dt} = \tilde{\boldsymbol{\sigma}} : \nabla \tilde{\mathbf{u}} - \nabla \cdot \tilde{\mathbf{q}}. \quad (1.16)$$

Given the Newtonian assumption, equations (1.5) and (1.8) lead to the identity

$$\tilde{\boldsymbol{\sigma}} : \nabla \tilde{\mathbf{u}} = -\frac{\tilde{p}}{\tilde{\rho}} \frac{D \tilde{p}}{Dt} + \frac{1}{2\eta} \tilde{\boldsymbol{\pi}} : \tilde{\boldsymbol{\pi}}. \quad (1.17)$$

Substituting (1.17) and (1.11) into (1.16) gives

$$\tilde{\rho} \frac{D \tilde{e}}{Dt} + \frac{\tilde{p}}{\tilde{\rho}} \frac{D \tilde{p}}{Dt} = \frac{1}{2\eta} \tilde{\boldsymbol{\pi}} : \tilde{\boldsymbol{\pi}} + \kappa \nabla^2 \tilde{T}. \quad (1.18)$$

We use the fundamental thermodynamic relation

$$d\tilde{e} = \tilde{T}d\tilde{s} - \tilde{p}d\left(\frac{1}{\tilde{\rho}}\right), \quad (1.19)$$

which is valid for systems in local thermodynamic equilibrium, to derive an equation for the entropy density field $\tilde{s}$ [33, 66]. Hence, by taking the material derivative, we have

$$\tilde{T}\frac{D\tilde{s}}{Dt} = \frac{D\tilde{e}}{Dt} + \tilde{p}\frac{D}{Dt}\frac{1}{\tilde{\rho}}. \quad (1.20)$$

Thus, we can rewrite (1.18), using (1.20), as

$$\tilde{\rho}\tilde{T}\frac{D\tilde{s}}{Dt} = \frac{1}{2\eta}\tilde{\boldsymbol{\pi}}:\tilde{\boldsymbol{\pi}} + \kappa\nabla^2\tilde{T}. \quad (1.21)$$

This equation describes the evolution of entropy in the system, where the first and second terms on the right-hand side represent viscous dissipation and thermal conduction, respectively.

In closing the governing equations, we require two equations of state connecting the thermodynamic fields $\tilde{\rho}$, $\tilde{p}$, $\tilde{T}$, and $\tilde{s}$ (or equivalently, $\tilde{e}$). Generally, these equations are expressed as $\tilde{\rho} = \tilde{\rho}(\tilde{p}, \tilde{s})$ and $\tilde{T} = \tilde{T}(\tilde{p}, \tilde{s})$. In the case of an ideal gas [33, 66]

$$\tilde{\rho}(\tilde{p}, \tilde{s}) = \mathcal{H}\tilde{p}^{\frac{1}{\gamma}}e^{-\frac{\tilde{s}}{c_v}}, \quad \tilde{T}(\tilde{p}, \tilde{s}) = \frac{\tilde{p}}{\mathcal{R}\tilde{\rho}(\tilde{p}, \tilde{s})}, \quad (1.22a, b)$$

where $\mathcal{H}$ is a constant of proportionality, γ is the adiabatic index of air ($\gamma \approx 1.4$), c_v is the specific heat at constant volume, and $\mathcal{R}$ is the specific gas constant.

1.4.2 Boundary conditions at solid surfaces

At a stationary solid boundary, the fluid must satisfy the no-slip condition [5]

$$\tilde{\mathbf{u}} = \mathbf{0} \quad (1.23)$$

and temperature and heat flux continuity [68]

$$\tilde{T} = \tilde{T}_s, \quad \kappa\nabla\tilde{T} \cdot \hat{\mathbf{n}} = \kappa_s\nabla\tilde{T}_s \cdot \hat{\mathbf{n}}, \quad (1.24)$$

where κ_s and $\tilde{T}_s$ denote the thermal conductivity and temperature field of the solid, respectively. Since solids, particularly metals, typically have much higher thermal conductivity than fluids (the scenario of interest in this thesis, where $\kappa_s \gg \kappa$) and assuming the absence of heat sources, the solid's temperature remains

effectively uniform, equal to the equilibrium temperature $\tilde{T}_0$. Therefore,

$$\tilde{T} = \tilde{T}_0 \quad (1.25)$$

at the solid surface.

1.4.3 Linearisation and time-harmonic fields

The present formulation is generalised to apply to a range of scenarios concerning Newtonian fluids governed by Fourier-type heat conduction. In acoustics, our primary interest is in time-harmonic perturbations superimposed on an equilibrium state. To establish the fundamental equations for wave propagation, we define the fields as

$$\tilde{\rho} = \tilde{\rho}_0 + \tilde{\rho}', \quad \tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}', \quad \tilde{p} = \tilde{p}_0 + \tilde{p}', \quad \tilde{T} = \tilde{T}_0 + \tilde{T}', \quad \tilde{s} = \tilde{s}_0 + \tilde{s}', \quad (1.26a - e)$$

where we denote the equilibrium state by subscript 0 and perturbations by prime. The equilibrium state is assumed to be quiescent, i.e., $\tilde{\mathbf{u}}_0 = \mathbf{0}$. It is also homogeneous, meaning that $\tilde{\rho}_0$, $\tilde{p}_0$, $\tilde{T}_0$ and $\tilde{s}_0$ are uniform in space and constant in time.

Under the assumption of small perturbations, linearisation of (1.5), (1.9) and (1.21) yields the linearised equations for mass continuity

$$\frac{\partial \tilde{\rho}'}{\partial t} + \tilde{\rho}_0 \nabla \cdot \tilde{\mathbf{u}}' = 0, \quad (1.27)$$

momentum

$$\tilde{\rho}_0 \frac{\partial \tilde{\mathbf{u}}'}{\partial t} = -\nabla \tilde{p}' + \eta \left[\nabla^2 \tilde{\mathbf{u}}' + \frac{1}{3} \nabla (\nabla \cdot \tilde{\mathbf{u}}') \right], \quad (1.28)$$

and entropy

$$\tilde{\rho}_0 \tilde{T}_0 \frac{\partial \tilde{s}'}{\partial t} = \kappa \nabla^2 \tilde{T}'. \quad (1.29)$$

The linearised equations of state are expressed as

$$\tilde{\rho}' = \frac{1}{c^2} \tilde{p}' - \left(\frac{\tilde{\rho}_0 \beta \tilde{T}_0}{c_p} \right) \tilde{s}', \quad \tilde{T}' = \left(\frac{\beta \tilde{T}_0}{\tilde{\rho}_0 c_p} \right) \tilde{p}' + \left(\frac{\tilde{T}_0}{c_p} \right) \tilde{s}', \quad (1.30a, b)$$

where the speed of sound $c = (\partial \tilde{p}_0 / \partial \tilde{\rho}_0)_{\tilde{s}_0}$, the thermal expansion coefficient $\beta = \tilde{\rho}_0 [\partial (1/\tilde{\rho}_0) / \partial \tilde{T}_0]_{\tilde{p}_0}$, and c_p is the specific heat capacity. In certain instances, we can further simplify these equations; specifically, in the case of an ideal gas, equation (1.22) implies that $\beta \tilde{T}_0 = 1$. By utilising (1.30), it is possible to eliminate

$\tilde{\rho}'$ and $\tilde{s}'$ from (1.27) and (1.29), yielding

$$\left(\frac{1}{c^2} - \frac{\beta^2 \tilde{T}_0}{c_p}\right) \frac{\partial \tilde{p}'}{\partial t} + \tilde{\rho}_0 \beta \frac{\partial \tilde{T}'}{\partial t} + \tilde{\rho}_0 \nabla \cdot \tilde{\mathbf{u}}' = 0, \quad (1.31)$$

and

$$\tilde{\rho}_0 c_p \frac{\partial \tilde{T}'}{\partial t} - \beta \tilde{T}_0 \frac{\partial \tilde{p}'}{\partial t} = \kappa \nabla^2 \tilde{T}'. \quad (1.32)$$

Consider next the time-harmonic perturbations, characterised by the form

$$\tilde{f}'(\tilde{\mathbf{x}}, t) = \text{Re} \left(\tilde{f}''(\tilde{\mathbf{x}}) e^{-i\omega t} \right), \quad (1.33)$$

where $\tilde{\mathbf{x}}$ denotes the position vector, ω the angular frequency and $\tilde{f}''$ the complex-valued phasor field corresponding to the perturbation field $\tilde{f}'$. Substituting the time-harmonic fields into (1.23), (1.25), (1.28), (1.31) and (1.32), we obtain

$$-i\omega \left(\frac{1}{c^2} - \frac{\beta^2 \tilde{T}_0}{c_p} \right) \tilde{p}'' - i\omega \tilde{\rho}_0 \beta \tilde{T}'' + \tilde{\rho}_0 \nabla \cdot \tilde{\mathbf{u}}'' = 0, \quad (1.34)$$

$$-i\tilde{\rho}_0 \omega \tilde{\mathbf{u}}'' = -\nabla \tilde{p}'' + \eta \left[\nabla^2 \tilde{\mathbf{u}}'' + \frac{1}{3} \nabla (\nabla \cdot \tilde{\mathbf{u}}'') \right], \quad (1.35)$$

and

$$-i\omega \tilde{\rho}_0 c_p \tilde{T}'' + i\omega \beta \tilde{T}_0 \tilde{p}'' = \kappa \nabla^2 \tilde{T}''; \quad (1.36)$$

with implied conditions

$$\tilde{\mathbf{u}}'' = 0, \quad \tilde{T}'' = 0 \quad (1.37a, b)$$

on a stationary solid boundary. The basic equations presented above constitute the foundational framework that will be utilised throughout this thesis.

1.4.4 Dissipation of acoustic energy

In acoustics, the acoustic energy is a key quantity, defined as

$$\tilde{E}_{\text{acoustic}} = \int \left(\frac{1}{2} \tilde{\rho}_0 \tilde{u}''^2 + \frac{\tilde{p}''^2}{2c^2 \tilde{\rho}_0} + \frac{\tilde{\rho}_0 \tilde{T}_0 \tilde{s}''^2}{2c_p} \right) dV, \quad (1.38)$$

which captures the energy corresponding to perturbation fields. In the absence of losses, the entropy density perturbation $\tilde{s}' = 0$ and the integrand of (1.38), which represents the energy density, is exactly the second-order term in the fluid's energy density expansion (1.10).

To demonstrate this, we use the thermodynamic relations $d\tilde{e} = -\tilde{p}d(1/\tilde{\rho})$ and $\tilde{p}' = c^2\tilde{\rho}'$, to derive an expression for the internal energy density expansion

$$\tilde{\rho}\tilde{e} \approx \frac{\tilde{p}_0\tilde{\rho}'}{\tilde{\rho}_0} + \frac{\tilde{p}'^2}{2\tilde{\rho}_0c^2}. \quad (1.39)$$

Accordingly, we arrive at the total energy balance

$$\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \tilde{\rho}\tilde{e} = \frac{\tilde{p}_0\tilde{\rho}'}{\tilde{\rho}_0} + \frac{1}{2}\tilde{\rho}_0\tilde{u}'^2 + \frac{\tilde{p}'^2}{2\tilde{\rho}_0c^2}, \quad (1.40)$$

where the term $\tilde{p}_0\tilde{\rho}'/\tilde{\rho}_0$ is unimportant in this context, as it cancels out in the energy equation owing to the conservation of mass. As a result, acoustic energy remains conserved in lossless scenarios.

However, when losses are accounted for, some of the acoustic energy is lost to thermoviscous effects. Precisely determining the rate of dissipation requires a second-order expansion of the governing equations to examine the energy balance, as detailed in [86]. Alternatively, the rate of dissipation can be calculated by appropriately manipulating the linearised equations. Although this approach may obscure the fundamental physics, it facilitates a more straightforward derivation of a conservation-type relation for the acoustic energy. This approach is prevalent in the acoustics literature, and thus, we will adopt this method in our analysis.

We proceed by combining the linearised equations,

$$\tilde{\mathbf{u}}' \cdot (1.28) + \frac{\tilde{p}'}{\tilde{\rho}_0} \times (1.27) + \tilde{\rho}_0\tilde{T}' \times (1.29). \quad (1.41)$$

After manipulation, this is equivalent to

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2}\tilde{\rho}_0\tilde{u}'^2 \right) + \frac{\tilde{p}'}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}'}{\partial t} + \tilde{\rho}_0\tilde{T}' \frac{\partial \tilde{s}'}{\partial t} = & -\nabla \cdot (\tilde{p}'\tilde{\mathbf{u}}') + \nabla \cdot (\tilde{\boldsymbol{\pi}}' \cdot \tilde{\mathbf{u}}') - \frac{1}{2\eta}\tilde{\boldsymbol{\pi}}' : \tilde{\boldsymbol{\pi}}' \\ & + \frac{\kappa}{\tilde{T}_0}\nabla \cdot (\tilde{T}'\nabla\tilde{T}') - \frac{\kappa}{\tilde{T}_0}|\nabla\tilde{T}'|^2. \end{aligned} \quad (1.42)$$

Applying the identities from (1.30), we obtain

$$\frac{\tilde{p}'}{\tilde{\rho}_0} \frac{\partial \tilde{\rho}'}{\partial t} + \tilde{\rho}_0\tilde{T}' \frac{\partial \tilde{s}'}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\tilde{p}'^2}{2c^2\tilde{\rho}_0} + \frac{\tilde{\rho}_0\tilde{T}_0\tilde{s}'^2}{2c_p} \right). \quad (1.43)$$

Substituting equation (1.43) into (1.42) yields

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\tilde{\rho}_0\tilde{u}'^2 + \frac{\tilde{p}'^2}{2c^2\tilde{\rho}_0} + \frac{\tilde{\rho}_0\tilde{T}_0\tilde{s}'^2}{2c_p} \right) + \nabla \cdot \tilde{\mathbf{J}}' = -\tilde{\delta}', \quad (1.44)$$

where

$$\tilde{\mathbf{j}}' = \tilde{\mathbf{p}}'\tilde{\mathbf{u}}' + \tilde{\mathbf{j}}'_T + \tilde{\mathbf{j}}'_v \quad (1.45)$$

denotes the acoustic energy flux vector, with

$$\tilde{\mathbf{j}}'_T = -\frac{\kappa}{\tilde{T}_0} \tilde{T}' \nabla \tilde{T}', \quad \tilde{\mathbf{j}}'_v = -\tilde{\boldsymbol{\pi}}' \cdot \tilde{\mathbf{u}}' \quad (1.46a, b)$$

representing the fluxes due to thermal and viscous effects, respectively. The term $\tilde{\mathcal{D}}' = \tilde{\mathcal{D}}'_T + \tilde{\mathcal{D}}'_v$ is the acoustic energy dissipated per unit volume and time, including both thermal and viscous contributions

$$\tilde{\mathcal{D}}'_T = \frac{\kappa}{\tilde{T}_0} |\nabla \tilde{T}'|^2, \quad \tilde{\mathcal{D}}'_v = \frac{1}{2\eta} \tilde{\boldsymbol{\pi}}' : \tilde{\boldsymbol{\pi}}'. \quad (1.47a, b)$$

Integrating (1.44) over a fixed volume $\mathcal{V}$ and using the divergence theorem, one has

$$\frac{d\tilde{E}_{\text{acoustic}}}{dt} + \int_{\partial\mathcal{V}} \tilde{\mathbf{j}}' \cdot \hat{\mathbf{n}} dA = -\tilde{\mathcal{D}}'. \quad (1.48)$$

In time-harmonic problems, the energy balance is typically averaged over a period of $2\pi/\omega$. Consequently, the first term in equation (1.48) vanishes, as the energy injection into the volume is periodic. Additionally, leveraging the identity

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \tilde{f}' \tilde{g}' dt = \frac{1}{2} \text{Re} \left[\tilde{f}'' (\tilde{g}'')^* \right], \quad (1.49)$$

where $\tilde{f}', \tilde{g}'$ are given by (1.33) and the asterisk denotes the complex conjugate, we derive the energy-dissipation relation

$$\int_{\partial\mathcal{V}} \tilde{\mathbf{j}}'' \cdot \hat{\mathbf{n}} dA = -\tilde{\mathcal{D}}'', \quad (1.50)$$

with

$$\tilde{\mathbf{j}}'' = \frac{1}{2} \text{Re} \left(\tilde{\mathbf{p}}'' \tilde{\mathbf{u}}''^* - \frac{\kappa}{\tilde{T}_0} \tilde{T}'' \nabla \tilde{T}''^* - \tilde{\boldsymbol{\pi}}'' \cdot \tilde{\mathbf{u}}''^* \right), \quad (1.51)$$

and $\tilde{\mathcal{D}}'' = \tilde{\mathcal{D}}''_T + \tilde{\mathcal{D}}''_v$, where

$$\tilde{\mathcal{D}}''_T = \frac{\kappa}{2\tilde{T}_0} \int |\nabla \tilde{T}''|^2 dV, \quad \tilde{\mathcal{D}}''_v = \frac{1}{4\eta} \int \tilde{\boldsymbol{\pi}}'' : \tilde{\boldsymbol{\pi}}''^* dV. \quad (1.52a, b)$$

1.4.5 Lossless (ideal) acoustics

In many acoustic resonators, dissipation effects are often weak. Thus, the time-harmonic fields are governed, over most of the spatial domain, by the adiabatic

and inviscid limits of the equations (1.34)–(1.36),

$$-i\frac{\omega}{c^2}\tilde{p}'' + \tilde{\rho}_0\nabla \cdot \tilde{\mathbf{u}}'' = 0, \quad -i\tilde{\rho}_0\omega\tilde{\mathbf{u}}'' = -\nabla\tilde{p}'', \quad \tilde{\rho}_0c_p\tilde{T}'' - \beta\tilde{T}_0\tilde{p}'' = 0. \quad (1.53a - c)$$

The Helmholtz equation for the pressure field can be derived by combining the first two equations,

$$\nabla^2\tilde{p}'' + \frac{\omega^2}{c^2}\tilde{p}'' = 0. \quad (1.54)$$

This serves as the fundamental equation governing lossless acoustics. After determining the pressure, the temperature field follows from the third equation in (1.53), reflecting the adiabatic nature of the problem.

By omitting thermoviscous effects, the complexity of the partial differential equations (1.34)–(1.36) is reduced. Consequently, the boundary conditions (1.37) are no longer applicable for the simplified problem. Rather, on a stationary solid boundary, they must be replaced with the impermeability condition

$$\hat{\mathbf{n}} \cdot \nabla\tilde{p}'' = 0 \quad (\hat{\mathbf{n}} \cdot \tilde{\mathbf{u}}'' = 0). \quad (1.55)$$

Moreover, in unbounded domains, solutions to the Helmholtz equation (1.54) propagate to infinity without attenuation. Therefore, in scattering problems, the far-field of $\tilde{p}''$ is generally decomposed into an incident field $\tilde{p}''^{(i)}$ and a scattered field $\tilde{p}''^{(s)}$, such that $\tilde{p}'' = \tilde{p}''^{(i)} + \tilde{p}''^{(s)}$. The incident field is typically specified, and the reflected or scattered field must satisfy an outward propagation condition at infinity, commonly the Sommerfeld radiation condition [109],

$$\left(\frac{\partial}{\partial\tilde{r}} - i\frac{\omega}{c}\right)\tilde{p}''^{(s)} = o\left(\frac{1}{\tilde{r}^{\frac{d+1}{2}}}\right) \quad \text{as } \tilde{r} \rightarrow \infty, \quad (1.56)$$

where d corresponds to the spatial dimensionality, and $\tilde{r}$ denotes the radial distance from the scatterer.

1.4.6 Thermal and viscous boundary layers

As discussed in §1.4.5, in the absence of losses, the pressure field $\tilde{p}''$ fails to satisfy the original boundary conditions prescribed at solid surfaces (1.37). This necessitates the presence of thermoviscous boundary layers, where dissipation effects become significant and these boundary conditions are met.

The viscous boundary layer thickness can be estimated by balancing the terms $i\tilde{\rho}_0\omega\tilde{\mathbf{u}}''$ and $\eta\nabla^2\tilde{\mathbf{u}}''$ in equation (1.35), yielding

$$l_v = \sqrt{\frac{\eta}{\tilde{\rho}_0\omega}}. \quad (1.57)$$

Similarly, the thermal boundary layer thickness is determined by balancing the terms $i\omega\tilde{\rho}_0c_p\tilde{T}''$ and $\kappa\nabla^2\tilde{T}''$ in equation (1.36), giving

$$l_T = \sqrt{\frac{\kappa}{\tilde{\rho}_0\omega c_p}}. \quad (1.58)$$

The ratio of the viscous to thermal boundary layer thicknesses is

$$\frac{l_v}{l_T} = \sqrt{\text{Pr}}, \quad (1.59)$$

where $\text{Pr} = \eta c_p / \kappa$ is the Prandtl number. For air, $\text{Pr} \approx 0.71$, indicating that the two length scales are comparable.

Thermoviscous boundary layers generally dominate the dissipation integrals (1.52). When dissipation effects are weak, these boundary layers are thin compared to the bulk fluid region, where losses are negligible. However, the acoustic fields within these boundary layers vary over much shorter length scales, which amplifies the magnitude of the integrands in (1.52). Since the dissipation integrals depend quadratically on the gradients of the fields, the contributions from the boundary layers outweigh those from the bulk.

1.5 Publications

The research presented in this thesis is, in part, based on the author's publications:

1. **Extraordinary transmission through a narrow slit** [48]
J. R. Holley and O. Schnitzer, *Wave Motion*, **91**, 102381, 2019
2. **Boundary-layer effects on electromagnetic and acoustic extraordinary transmission through narrow slits** [12]
R. Brandão, J. R. Holley and O. Schnitzer, *Proceedings of the Royal Society A*, **476**, 20200444, 2020

CHAPTER 2

Extraordinary transmission through narrow slits

2.1 Introduction

In this chapter, we systematically develop an approximate theory of acoustic extraordinary transmission through narrow slits, neglecting losses. As discussed in Chapter 1, the idealised scenario has an exact electromagnetic analogy. Takakura [113] was the first to put forward an approximate theory of ideal transmission through a narrow slit (in the context of the electromagnetic problem), based on an *ad hoc* truncation of an exact mode-matching scheme. Takakura’s key result, a simple closed-form approximation for the deviations of the transmission peaks from the naive standing-wave (Fabry–Pérot) resonance frequencies, was found to agree poorly with electromagnetic experiments [112]; the discrepancy could not be solely attributed to material losses or a skin effect, as it persisted even for slits wide enough for the metal to be considered perfectly conducting (still narrow relative to the wall thickness). (Earlier, less accurate, experiments [124] seemed to show good agreement with Takakura’s approximation.) More recent attempts to derive rigorous approximations for ideal transmission through a single narrow slit [60, 73] have not provided explicit formulae or physical insight into the discrepancy.

Our theory is based on an asymptotic analysis of the ideal transmission problem (described in the language of acoustics) in the pertinent limit where the slit width is small compared to the wall thickness, the latter being comparable to the wavelength. We demonstrate that our results, which include asymptotic formulae for the transmission efficiency, field enhancement and frequency shifts, are in excellent agreement with numerical and experimental data; moreover, we expose the logarithmic error rendering Takakura’s formula inaccurate, and point

out the common [93, 43, 20] but flawed physical assumption underlying that approximation.

The notably simple form of our theory follows from the manner in which we treat the disparate length scales and distinguished frequency regimes in the problem. Namely, rather than attempting to find a single approximation for the wave field that is uniformly valid everywhere and for all frequencies, we use the method of matched asymptotic expansions [115, 47, 26] to systematically decompose the physical domain into asymptotically overlapping regions (representing the slit, the external regions and transition regions near the apertures) and separately consider off- and near-resonance frequencies; our analysis is thereby distinct from recent asymptotic analyses of related transmission problems [38, 101, 39, 91].

This chapter is based on paper [48].

2.2 Formulation of the problem

Consider acoustic transmission of a normally incident plane wave (angular frequency ω , wave speed c) through a slit of width $2lh$ in a rigid wall of thickness l . We suppress the time dependence $e^{-i\omega t}$ in the usual way and adopt a dimensionless formulation where lengths are normalised by l and φ denotes the velocity potential normalised by the amplitude of the incident wave. Figure 2.1 shows a dimensionless schematic of the problem and defines the Cartesian coordinates (x, y) .

In the fluid domain, φ satisfies the Helmholtz equation

$$\nabla^2 \varphi + \Omega^2 \varphi = 0, \quad (2.1)$$

where we define the dimensionless frequency

$$\Omega = \frac{\omega l}{c}. \quad (2.2)$$

On the walls, impermeability implies that the normal derivative vanishes,

$$\frac{\partial \varphi}{\partial n} = 0. \quad (2.3)$$

In addition, the scattered field $\varphi - \varphi^{(i)}$, where $\varphi^{(i)} = e^{i\Omega y}$ is the incident wave, must propagate outwards at large distances from the slit. (In the electromagnetic analogy mentioned in the introduction, the same dimensionless problem holds with φ replaced by the scaled out-of-plane magnetic-field component.)

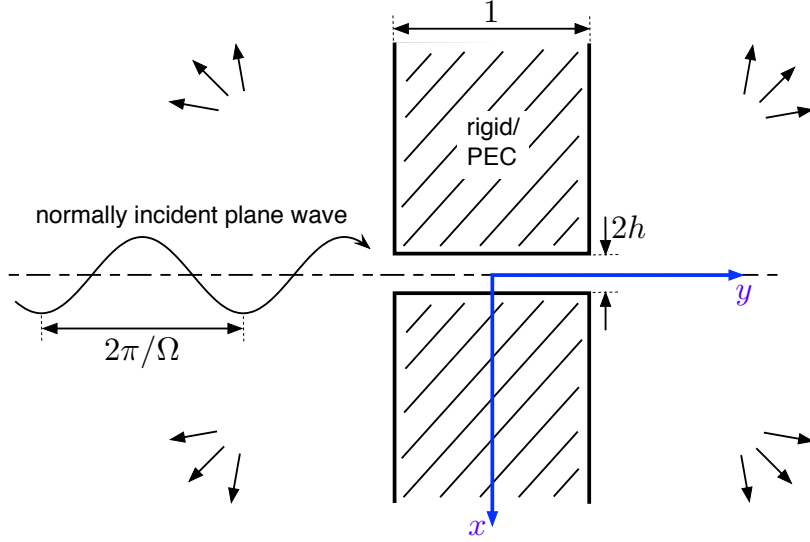


Figure 2.1: Schematic of the dimensionless transmission problem.

In what follows, we shall be interested in the resonant peaks of the transmission efficiency τ , defined as the ratio of the time-averaged acoustic (or electromagnetic) power transmitted through the slit to that transmitted through the same cross section in the absence of a wall:

$$\tau \equiv \frac{1}{2h\Omega} \int_{-h}^h \text{Im} \left(\varphi^* \frac{\partial \varphi}{\partial y} \right) dx, \quad (2.4)$$

where the asterisk denotes complex conjugation and the integrand is evaluated for fixed y such that $|y| < 1/2$.

2.3 Analysis off-resonance

Let us first, naively, consider the narrow-slit limit $h \ll 1$, holding Ω fixed. The slit openings reduce to points, as depicted in figure 2.2(a). Assuming that the slit potential is comparable in magnitude to the incident wave, as we shall later verify, diffraction from the slit is negligible in the domain left of the wall; the potential there, say φ^- , is accordingly

$$\varphi^- \approx e^{i\Omega y} + R e^{-i\Omega y}, \quad (2.5)$$

where the reflection coefficient $R = e^{-i\Omega}$ follows from (2.3) on $y = -1/2$ and $\approx$ denotes a relative asymptotic error of algebraic order, i.e., scaling with some positive power of h . Right of the wall, there is no incident nor reflected wave, only weak diffraction from the slit; the potential there, φ^+ , is accordingly algebraically small.

The above approximations are outer expansions, namely with position held fixed. We next derive an inner approximation for the wave field inside the slit by considering the limit $h \rightarrow 0$ with the stretched transverse coordinate $X = x/h$ fixed. Defining the slit field $\varphi(x, y) = \Phi(X, y)$, for $|y| < 1/2$, it is straightforward to show from a regular perturbation (cf. §3.3.4 in Chapter 3) of (2.1) and (2.3) that $\Phi \approx U(y)$, where $U(y)$ satisfies

$$\frac{d^2 U}{dy^2} + \Omega^2 U = 0 \quad (2.6)$$

with solution (see figure 2.2(b))

$$U(y) = A \cos(\Omega y) + B \sin(\Omega y). \quad (2.7)$$

To determine the constants A and B , we asymptotically match the slit field to the outer domains; noting that (2.5) tends to a limiting value as $y \rightarrow -1/2$ and that φ^+ is asymptotically small, we find

$$A = e^{-i\Omega/2} \sec \frac{\Omega}{2}, \quad B = -e^{-i\Omega/2} \csc \frac{\Omega}{2}. \quad (2.8)$$

Based on this solution, the dimensionless flux density in the y direction, $\partial\varphi/\partial y$, is $O(1)$ in the slit; at the right end of the slit, the corresponding net flux is $\approx 2hU'(1/2)$. This flux is approximately conserved on the small scale of the aperture, implying that the outer potential φ^+ is given by a transmitted cylindrical wave

$$\varphi^+ \approx hQ\mathcal{H}_0(\Omega r^+), \quad (2.9)$$

where $\mathcal{H}_0$ is the zeroth-order Hankel function, $r^\pm = \sqrt{x^2 + (y \mp 1/2)^2}$ and

$$Q = 2ie^{-i\Omega/2}\Omega \csc \Omega. \quad (2.10)$$

(Higher-order fundamental solutions must be discarded as their algebraic singularity contradicts the order of magnitude of the slit potential.) In deriving (2.10), we compared the known flux at the aperture to the limiting value using the local asymptotic relation

$$\mathcal{H}_0(t) \approx \frac{2i}{\pi} \left(\ln \frac{t}{2} + \gamma_E \right) + 1 \quad \text{as } t \rightarrow 0, \quad (2.11)$$

wherein the algebraic error is in t and $\gamma_E = 0.5772\dots$ is the Euler–Mascheroni constant [1].

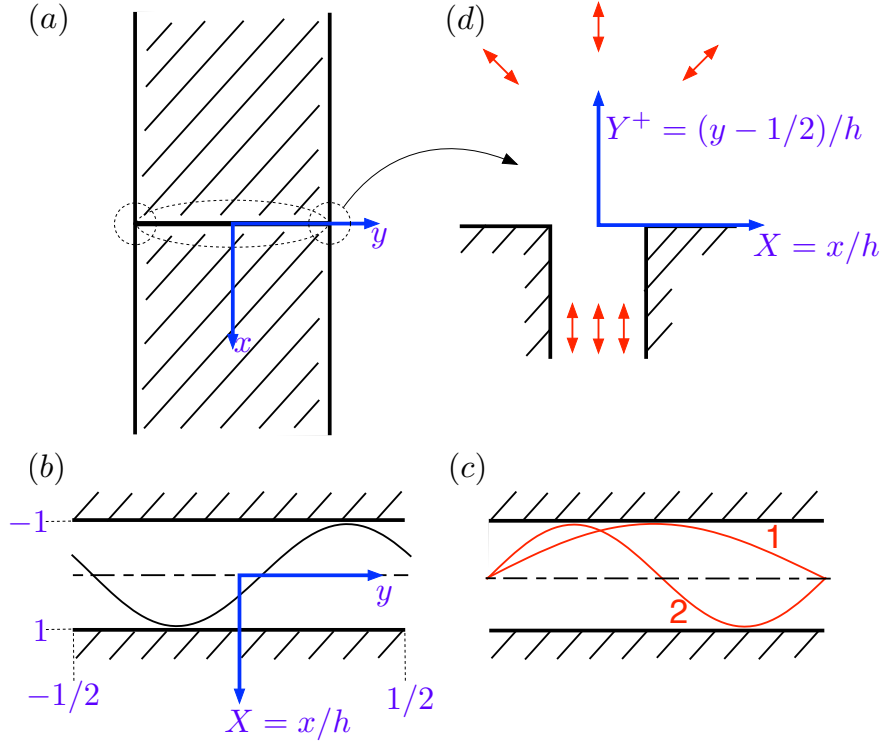


Figure 2.2: Asymptotic outer regions (a) and inner slit region (b). Near resonances, the slit field is approximately a standing-wave (Fabry–Pérot) mode enhanced by an $\mathcal{O}(h^{-1})$ factor (c) with the transition regions (d) close to the apertures playing a dominant role.

Our fixed-frequency approximation gives no pronounced enhancement, *viz.* $\tau = \mathcal{O}(1)$, except near the Fabry–Pérot frequencies

$$\bar{\Omega} = m\pi, \quad m = 1, 2, \dots, \quad (2.12)$$

where (2.8) and (2.10) are singular; as $\Omega \rightarrow \bar{\Omega}$ the slit field $U(y)$ diverges in amplitude and approaches a standing-wave solution, vanishing at $y = \pm 1/2$. The singular nature of the fixed-frequency limit is evident on physical grounds: as diffraction from the slit ends is neglected (and material losses have been discarded from the outset), the limiting standing waves constitute resonant modes. For $\Omega - \bar{\Omega} = \mathcal{O}(h)$, however, our present approximation suggests that the amplitude of the slit field is $\mathcal{O}(1/h)$ and the diffracted waves are $\mathcal{O}(1)$, comparable to the incident and reflected waves. Our asymptotic analysis therefore needs to be modified in this regime.

2.4 Analysis near resonance

In light of the above, we now define

$$\frac{\Omega - \bar{\Omega}}{\bar{\Omega}} \equiv 2h\Omega' \quad (2.13)$$

and in what follows consider the near-resonance limit $h \rightarrow 0$ with Ω' (and m) fixed. In this modified limit we anticipate the diffracted waves in the outer regions to be $O(1)$, i.e.,

$$\varphi^- \approx e^{i\bar{\Omega}y} + e^{-i\bar{\Omega}(1+y)} + Q^- \mathcal{H}_0(\bar{\Omega}r^-) \quad (2.14)$$

and

$$\varphi^+ \approx Q^+ \mathcal{H}_0(\bar{\Omega}r^+), \quad (2.15)$$

where the coefficients $Q^\pm$ remain to be determined. The slit potential is accordingly amplified,

$$\Phi(X, y) \approx h^{-1}\mathcal{U}(y), \quad (2.16)$$

where $\mathcal{U}(y)$ satisfies equation (2.6) with $\bar{\Omega}$ replacing Ω , whereas the relatively smaller magnitude of the outer potentials (2.14) and (2.15) implies the boundary conditions $\mathcal{U}(\pm 1/2) = 0$. The resulting homogeneous problem for $\mathcal{U}(y)$, by construction, has as non-trivial solutions

$$\mathcal{U}(y) = \mathcal{A} \times \begin{Bmatrix} \cos(\bar{\Omega}y) \\ \sin(\bar{\Omega}y) \end{Bmatrix}, \quad (2.17)$$

where $\mathcal{A}$ is a prefactor and henceforth the upper element of an array corresponds to odd m (even standing wave) and the lower to even m (odd standing wave).

The analysis of the slit region can be readily extended one further algebraic order by writing

$$\Phi(X, y) \approx h^{-1}\mathcal{U}(y) + \mathcal{V}(y), \quad (2.18)$$

where $\mathcal{V}(y)$ satisfies

$$\frac{d^2\mathcal{V}}{dy^2} + \bar{\Omega}^2\mathcal{V} = -4\Omega'\bar{\Omega}^2\mathcal{U}. \quad (2.19)$$

(It is straightforward to verify that the slit potential deviates from a unidimensional profile only at higher algebraic order.) Multiplying equation (2.19) by $\mathcal{U}^*$ and subtracting the conjugate of equation (2.6) (with $\bar{\Omega}$ replacing Ω) multiplied by $\mathcal{V}$,

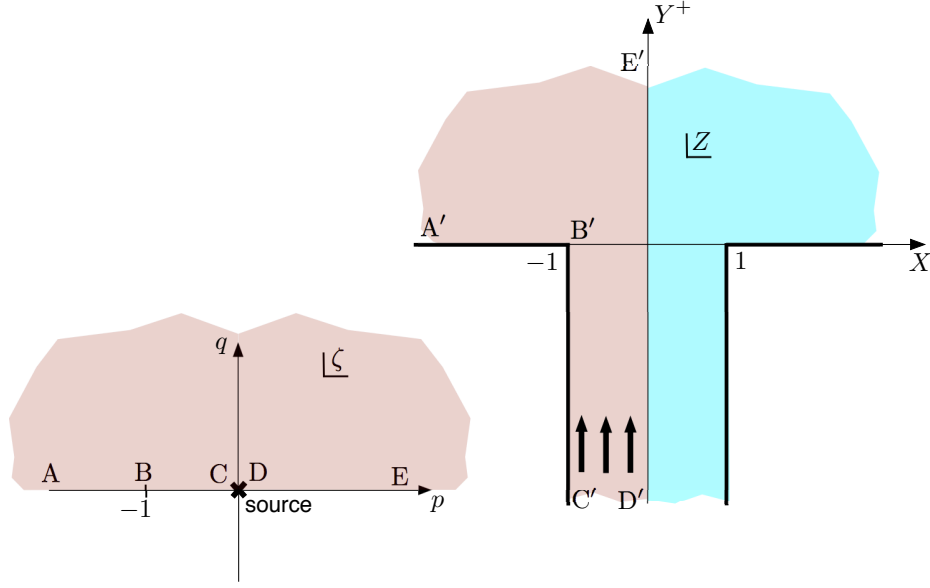


Figure 2.3: Conformal mapping of the left half ($X < 0$) of the transition region into the upper half of an auxiliary ζ plane.

followed by integration along the slit using $\mathcal{U}(\pm 1/2) = 0$, gives

$$\left[\mathcal{V} \frac{d\mathcal{U}^*}{dy} \right]_{-1/2}^{1/2} = 4\Omega' \bar{\Omega}^2 \int_{-1/2}^{1/2} |\mathcal{U}|^2 dy. \quad (2.20)$$

Substituting (2.17) we find

$$\mathcal{A} = \frac{i^m}{2\Omega' \bar{\Omega}} \times \left\{ \begin{array}{c} i\mathcal{V}(1/2) + i\mathcal{V}(-1/2) \\ \mathcal{V}(1/2) - \mathcal{V}(-1/2) \end{array} \right\}. \quad (2.21)$$

As in the off-resonance analysis, the leading fluxes at the slit ends can be matched with the corresponding diffraction terms in the outer regions; we thereby find

$$Q^\pm = i^m \bar{\Omega} \mathcal{A} \times \left\{ \begin{array}{c} 1 \\ \mp i \end{array} \right\}. \quad (2.22)$$

Unlike in the off-resonance analysis, however, the slit and outer potentials cannot be directly matched, since the leading-order outer potentials are singular as $r^\pm \rightarrow 0$. It is therefore necessary to consider transition regions near the apertures defined by intermediate limits with $X = x/h$ and $Y^\pm = \pm(y \mp 1/2)/h$ fixed (we also define $R^\pm = r^\pm/h$); under these scalings, the slit appears infinite and the boundaries are $Y^\pm = 0$ for $|X| > 1$ and $X = \pm 1$ for $Y^\pm < 0$ (see figure 2.2(d)). It is evident from the form of the slit and outer expansions that the potential is $O(1)$ in the transitions regions. We therefore write $\varphi(x, y) \approx T^\pm(X, Y^\pm)$, where $T^\pm$ satisfy

Laplace's equation [cf. (2.1)]

$$\frac{\partial^2 T^\pm}{\partial X^2} + \frac{\partial^2 T^\pm}{\partial Y^{\pm 2}} = 0, \quad (2.23)$$

the Neumann boundary condition

$$\frac{\partial T^\pm}{\partial N} = 0, \quad (2.24)$$

where N denotes $Y^\pm$ for the external walls and X for the inner slit walls, as well as asymptotic matching with the outer and slit regions. In the off-resonance limit, the $O(1)$ outer and slit potentials were regular hence $T^\pm$ were constants; in the present near-resonance limit, however, matching with the outer expansions (2.14) and (2.15) gives

$$T^\pm \sim iQ^\pm \frac{2}{\pi} \ln R^\pm + Q^\pm \left[1 + \frac{2i}{\pi} \left(\ln \frac{\bar{\Omega}h}{2} + \gamma_E \right) \right] + (1 \mp 1)e^{-i\bar{\Omega}/2} + o(1) \quad \text{as } R^\pm \rightarrow \infty, \quad (2.25)$$

while matching with the slit expansion (2.18) gives

$$T^\pm \sim iQ^\pm Y^\pm + (\mathcal{V})_{\pm 1/2} + o(1) \quad \text{as } Y^\pm \rightarrow -\infty, \quad (2.26)$$

where the leading term in (2.26) is written in terms of $Q^\pm$ using (2.22). In the acoustics interpretation, the above transition-region problems represent incompressible potential flows through the subwavelength apertures.

The Laplace problem for T^+ (similarly T^-) is uniquely solvable, up to an arbitrary additive constant, with just the leading terms in the far-field conditions (2.25) and (2.26) prescribed; that unique solution therefore determines the differences between the $O(1)$ constant terms in those conditions. To treat both Laplace problems simultaneously we note that $T^\pm/iQ^\pm$ solve a similar canonical problem where the difference

$$\beta \equiv \lim_{Y^\pm \rightarrow -\infty} \left(\frac{T^\pm}{iQ^\pm} - Y^\pm \right) - \lim_{R^\pm \rightarrow \infty} \left(\frac{T^\pm}{iQ^\pm} - \frac{2}{\pi} \ln R^\pm \right) \quad (2.27)$$

is a pure number, which depends only on the geometry of the aperture. In the present geometry, we obtain β by defining the complex variable $Z = X + iY^+$ and considering the conformal mapping [22]

$$Z = \frac{2i}{\pi}(1 + \zeta)^{1/2} + \frac{i}{\pi} \log \frac{(1 + \zeta)^{1/2} - 1}{(1 + \zeta)^{1/2} + 1} \quad (2.28)$$

from the upper half-plane of an auxiliary complex variable $\zeta = p + iq$ to the left half ($X < 0$) of the $X - Y^+$ domain (see figure 2.3). The form of the solution is evident in terms of the auxiliary variable [101]:

$$\frac{T^+}{iQ^+} = \frac{1}{\pi} \ln |\zeta| + \text{const.} \quad (2.29)$$

Using (2.29) and the limits of (2.28) as $\zeta \rightarrow 0$ and $\zeta \rightarrow \infty$, definition (2.27) yields

$$\beta = \frac{2}{\pi} \left(\ln \frac{4}{\pi} - 1 \right). \quad (2.30)$$

With β determined, the complex amplitude of the slit wave is found from (2.21), (2.22) and (2.25)–(2.27) as

$$\mathcal{A} = \frac{1/\bar{\Omega}}{\Omega' - \frac{2}{\pi} \left(\ln \frac{2\bar{\Omega}h}{\pi} + \gamma_E - 1 \right) + i} \times \begin{Bmatrix} i \\ -1 \end{Bmatrix}. \quad (2.31)$$

From this main result we find the transmission efficiency,

$$\tau \approx h^{-1} \bar{\Omega} |\mathcal{A}|^2, \quad (2.32)$$

and field enhancement in the slit, $|\mathcal{A}|/h$, in terms of the amplitude

$$|\mathcal{A}|^2 = \frac{(\pi m)^{-2}}{1 + \left[\Omega' - \frac{2}{\pi} (\ln(2mh) + \gamma_E - 1) \right]^2}, \quad (2.33)$$

a Lorentzian with a peak magnitude decreasing with m ; the frequency deviations Δf , of the resonant peaks from the Fabry–Pérot frequencies

$$f_m = \frac{mc}{2l}, \quad (2.34)$$

are provided as

$$\frac{\Delta f}{f_m} \approx \frac{4h}{\pi} (\ln h + \ln(2m) + \gamma_E - 1). \quad (2.35)$$

2.5 Comparison with numerics and previous theoretical results

It is illuminating to compare our asymptotic theory with numerical calculations, experimental data and existing approximations. To begin with, we have validated

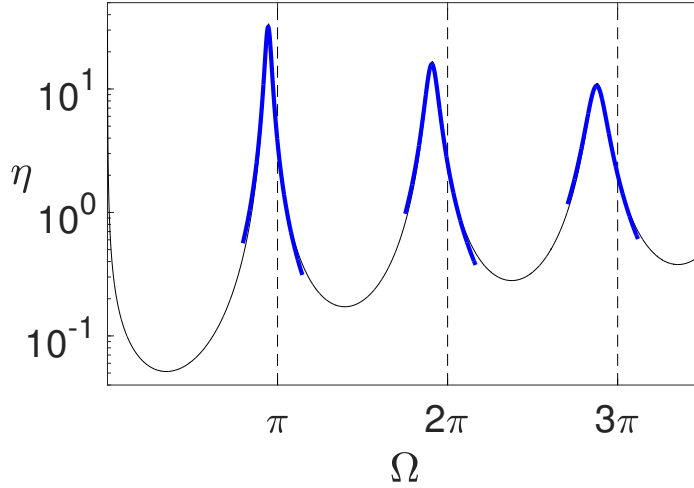


Figure 2.4: Numerical (black line) and near-resonance asymptotic (thick blue lines) transmission efficiency, for $h = 0.01$, plotted as a function of the normalized frequency.

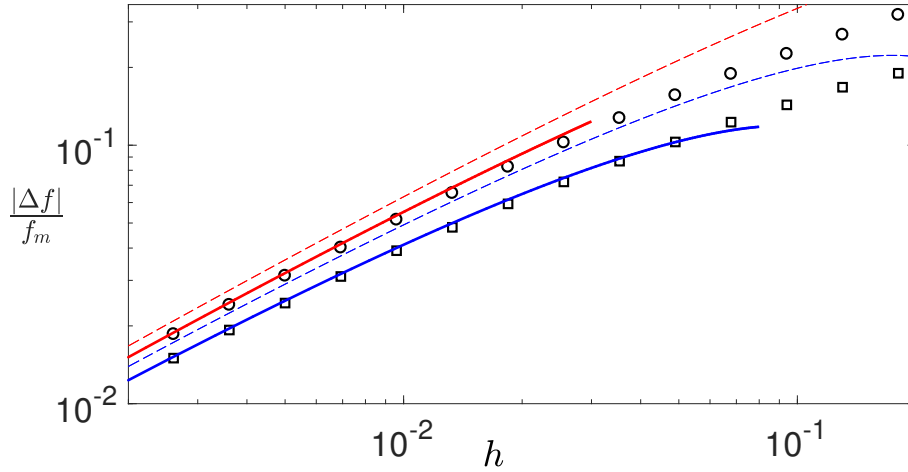


Figure 2.5: Relative frequency shifts from Fabry-Pérot values (numerics: symbols; asymptotics: solid lines) as a function of h for modes $m = 1$ (circles) and $m = 3$ (squares). The dashed lines depict Takakura's approximation (2.36).

our theory against a mode-matching scheme solving the ideal transmission problem exactly (see Appendix A for details); in particular, figures 2.4 and 2.5 present excellent agreement between our on-resonance asymptotic predictions for τ and Δf [cf. (2.32)–(2.35)] and the corresponding numerical values. In figure 2.5 we have added a dashed line depicting Takakura's prediction for the frequency shifts [113]:

$$\left(\frac{\Delta f}{f_m}\right)_{\text{Ref. [113]}} = \frac{4h}{\pi} \left(\ln h + \ln(\pi m) - \frac{3}{2} \right) \quad (2.36)$$

(ignoring terms of quadratic order in h). Takakura's approximation is seen to be in poor agreement with the numerical data. Note that (2.35) and (2.36) disagree at $O(h)$ which is practically comparable to the leading $O(h \ln h)$ term; our analysis suggests that the logarithmic error in Takakura's approximation is a result of disregarding the details of the wave field close to the apertures, which are seen to be important specifically near resonance. (The leading $O(h \ln h)$ term was also found by Lin *et al.* [73] using a different approach involving layer-potential techniques. Their expression for the $O(h)$ term, however, includes a constant which was not calculated.)

It is also constructive to interpret our results in terms of a corrected slit length $l + \Delta l$ [64, 51]. Following Ward *et al.* [119], we define the latter based upon the Fabry–Pérot model (2.34):

$$f_m + \Delta f = \frac{mc}{2(l + \Delta l)}, \quad (2.37)$$

where $f_m + \Delta f$ is the m th resonance frequency and we take the end correction Δl to account for both apertures. A comparison of (2.37) with our asymptotic prediction (2.35) gives

$$\frac{\Delta l}{l} \approx \frac{4h}{\pi} \left(\ln \frac{1}{2hm} + 1 - \gamma_E \right). \quad (2.38)$$

Notably, the end correction extracted from our asymptotic theory depends upon the mode number m ; accordingly, it does not constitute an intrinsic property of the aperture geometry (except to a leading logarithmic approximation). This unconventional feature can be traced to the weak singularity of the Green's function in two dimensions [cf. (2.11)], which couples the aperture and outer fields to leading algebraic order. We conclude that the notion of effective length should be applied with care, especially in two dimensions. (We have not found in the literature expressions for the end correction of a two-dimensional slit opening. Given the dependence upon m in (2.37), this is hardly surprising. We note that the end correction given in Ward *et al.* [119] for a slit is actually the end correction for the opening of a three-dimensional circular pipe [64, 51].)

2.6 Comparison with experiments

As for experiments, in figures 2.6 and 2.7 we replicate the comparison carried out by Suckling *et al.* [112] between their electromagnetic measurements and Takakura's approximation (2.36), with our asymptotic prediction (2.35) overlaid. Our approximation is seen to be in excellent agreement with the experimental

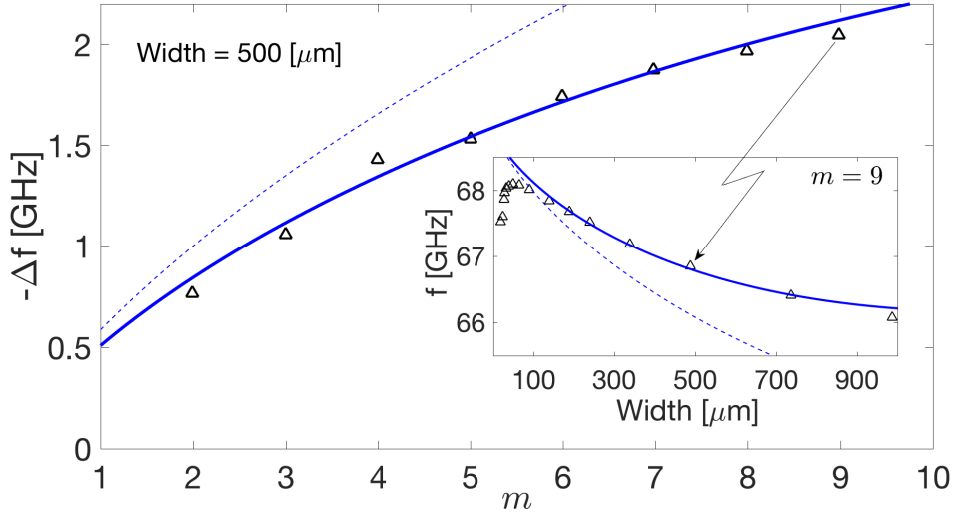


Figure 2.6: Comparison with experiments of Ref. [112] of electromagnetic transmission through a slit in a 19.58mm thick Aluminium plate (triangles — using plot digitizer). Main plot shows shifts of resonance frequencies from the second to ninth Fabry-Pérot mode for a 500 μm wide slit. Inset shows the resonance frequencies for mode $m = 9$ for varying slit width. In both we depict our asymptotic prediction by a solid line and Takakura's approximation (2.36) by a dashed line.

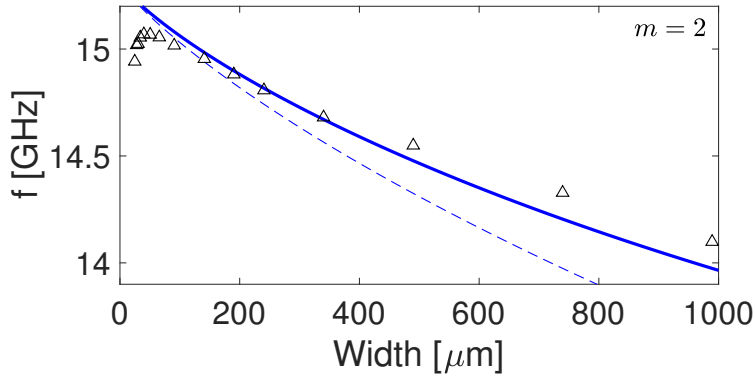


Figure 2.7: Same as the inset of figure 2.6 but for $m = 2$.

data in an intermediate range of slit widths: narrow enough so that h is small yet wide enough such that the metal can be modelled as a perfect conductor. Similar findings are shown in figure 2.8, which compares single-slit acoustic experiments described in [119].

2.7 Extension to a periodic slit array

2.7.1 Formulation

Consider now a more complex configuration where the plate is decorated with an infinite array of slits, say of dimensional period $2dl$. Fig. 2.9 depicts the

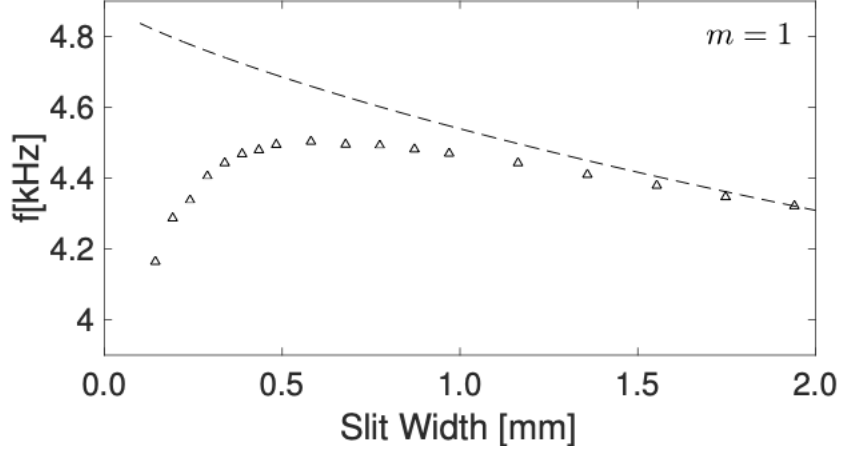


Figure 2.8: Resonance frequencies as a function of slit width for the fundamental resonance of acoustic-wave transmission through a slit in an aluminium plate of thickness 35mm. The dashed black line is calculated from (2.35). Symbols: acoustic experiments [119].

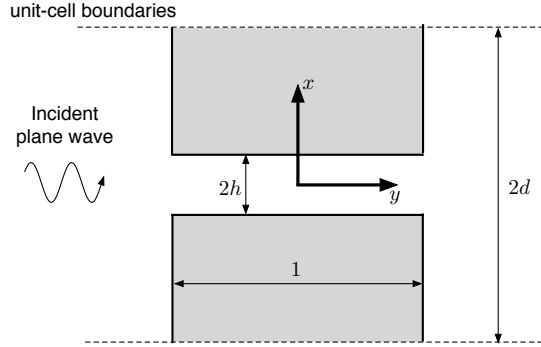


Figure 2.9: Dimensionless schematic of the problem of transmission through a periodic array of slits.

dimensionless geometry, with the origin of the Cartesian coordinates (x, y) placed at the centre of an arbitrary slit. As before, we assume a normally incident plane wave. In that case, the field possesses the same periodicity as the array. Accordingly, the problem can be restricted to the unit-cell $|x| < d$, say, with periodicity conditions applied on the cell boundaries $x = \pm d$.

2.7.2 Analysis and frequency response

We shall extensively build on the near-resonance theory developed in this chapter for a single slit. Thus, in the exterior regions, the periodicity of the problem suggests that the leading $O(1)$ fields $\varphi^\pm$ generalise as [cf. (2.14) and (2.15)]

$$\varphi^- = e^{i\bar{\Omega}y} + e^{-i\bar{\Omega}(1+y)} + Q^- \sum_{n=-\infty}^{\infty} \mathcal{H}_0(\bar{\Omega}r_n^-), \quad \varphi^+ = Q^+ \sum_{n=-\infty}^{\infty} \mathcal{H}_0(\bar{\Omega}r_n^+), \quad (2.39a, b)$$

where we define the shifted radial coordinates $r_n^\pm = \sqrt{(x - 2nd)^2 + (y \mp 1/2)^2}$. We next couple these exterior fields to the bulk slit field, using the matching formulae (2.22) and (2.25)-(2.27). For this purpose, we consider just the slit in the unit-cell $|x| < d$. The expansion of the wave field in the slit region has exactly the same form as in the single-slit case, with $\mathcal{A}$ denoting the complex amplitude of the resonantly excited standing wave at $O(1/h)$. Thus, we find the end values of the slit field at $O(1)$ in the form

$$\mathcal{V}(\pm 1/2) = iQ^\pm \beta + Q^\pm \left[1 + \frac{2i}{\pi} \left(\ln \frac{\bar{\Omega}h}{2} + \gamma_E \right) + \sigma(\bar{\Omega}d) \right] + (1 \mp 1)e^{-i\bar{\Omega}/2}, \quad (2.40)$$

in which the function $\sigma(\bar{\Omega}d)$ is formally given by the conditionally convergent lattice sum

$$\sigma(\bar{\Omega}d) = \sum_{n \neq 0} \mathcal{H}_0(2\bar{\Omega}|n|d). \quad (2.41)$$

For computation, this sum is transformed into the absolutely converging sum (see [77])

$$\sigma(\bar{\Omega}d) = -1 - \frac{2i}{\pi} \left(\gamma_E + \ln \frac{\bar{\Omega}d}{2\pi} \right) - \frac{i}{\chi_0} - i \sum_{n \neq 0} \left(\frac{1}{\chi_n} - \frac{1}{\pi|n|} \right), \quad (2.42)$$

wherein

$$\chi_n = \begin{cases} \sqrt{\pi^2 n^2 - \bar{\Omega}^2 d^2}, & |n| \geq \bar{\Omega}d/\pi, \\ -i\sqrt{\bar{\Omega}^2 d^2 - \pi^2 n^2}, & |n| < \bar{\Omega}d/\pi. \end{cases} \quad (2.43)$$

Along the lines of the single-slit analysis, we solve (2.21), (2.22) and (2.40) for the complex amplitude $\mathcal{A}$. We thereby find the generalised near-resonance approximation

$$\mathcal{A} = \frac{1/\bar{\Omega}}{\Omega' - \frac{2}{\pi} \left(\ln \frac{2\bar{\Omega}h}{\pi} + \gamma_E - 1 \right) + i(1 + \sigma(\bar{\Omega}d))} \times \begin{Bmatrix} i \\ -1 \end{Bmatrix}. \quad (2.44)$$

The corresponding formulae for $Q^\pm$ follow from (2.22).

Similar results were obtained by Lin and Zhang [74]. As in their earlier analysis of the single-slit problem [73], their results involve a constant, denoted α , which is not calculated. In our view, the analysis is highly technical compared to the simple and physically insightful treatment in this chapter using matched asymptotics.

Comparing (2.44) with (2.31), we see that the generalisation to a periodic array enters through the complex-valued function $\sigma(\bar{\Omega}d)$. Its real part contributes to

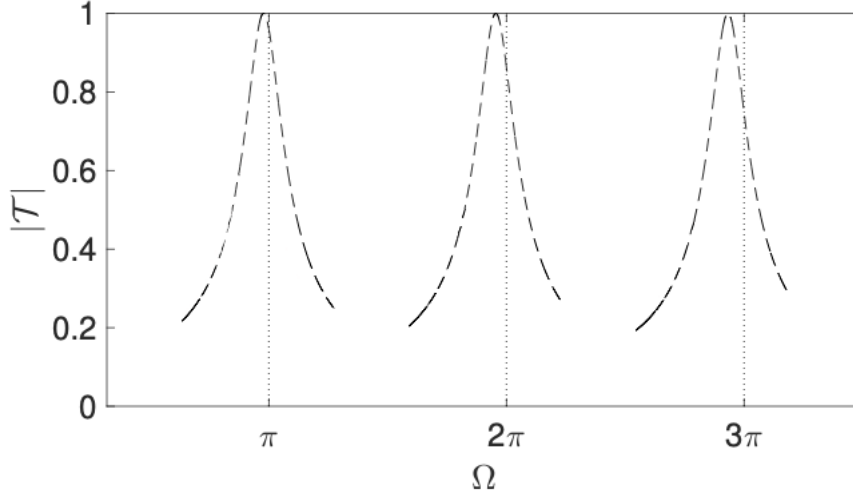


Figure 2.10: Modulus of the transmission coefficient for acoustic transmission through a plate of thickness $l = 19.8\text{mm}$ decorated by a periodic array of slits of width $w = 2hl$ and period $2dl = w + 2.91\text{mm}$ (dimensions from [119]), for $h = 0.01$. Dashed line: approximation (2.47).

radiative damping of the slit mode, while its imaginary part shifts the resonances to lower frequencies. In the above analysis, we tacitly held d fixed; namely, we assumed that the periodicity is comparable with the wall thickness and the wavelength. It can be verified, however, that (2.44) holds as a leading-order approximation for $h \ll d$. We note that the case of an ultra-subwavelength array, $d = O(h)$, was analysed in [75].

In particular, for large d , σ attenuates and (2.44) reduces to (2.31). For small d (subwavelength periodicity), $\sigma(\bar{\Omega}d) \sim (\bar{\Omega}d)^{-1}$; it follows that the slit fields are more strongly damped by radiation than in the single-slit scenario.

2.7.3 Reflection and transmission

Consider next the field at large distances from the plate. To this end, we shall use the asymptotic relation (see [77])

$$\sum_{n=-\infty}^{\infty} \mathcal{H}_0(\bar{\Omega}r_n^{\pm}) \sim \sum_{n=-[\bar{\Omega}d/\pi]}^{[\bar{\Omega}d/\pi]} \frac{e^{i\left\{\frac{\pi n x}{\bar{\Omega}d} \pm \sqrt{\bar{\Omega}^2 d^2 - \pi^2 n^2} \frac{y \mp 1/2}{d}\right\}}}{\sqrt{\bar{\Omega}^2 d^2 - \pi^2 n^2}} \quad \text{as } y \rightarrow \pm\infty. \quad (2.45)$$

In particular, when the period of the array is smaller than the wavelength ($\bar{\Omega}d < \pi$), the finite sum on the right-hand side of (2.45) reduces to a single plane-wave term. In that case, we find from (2.39) the far-field behaviours

$$\varphi^- \sim e^{i\bar{\Omega}y} + \mathcal{R}e^{-i\bar{\Omega}y} \quad \text{as } y \rightarrow -\infty, \quad \varphi^+ \sim \mathcal{T}e^{i\bar{\Omega}y} \quad \text{as } y \rightarrow \infty, \quad (2.46a, b)$$

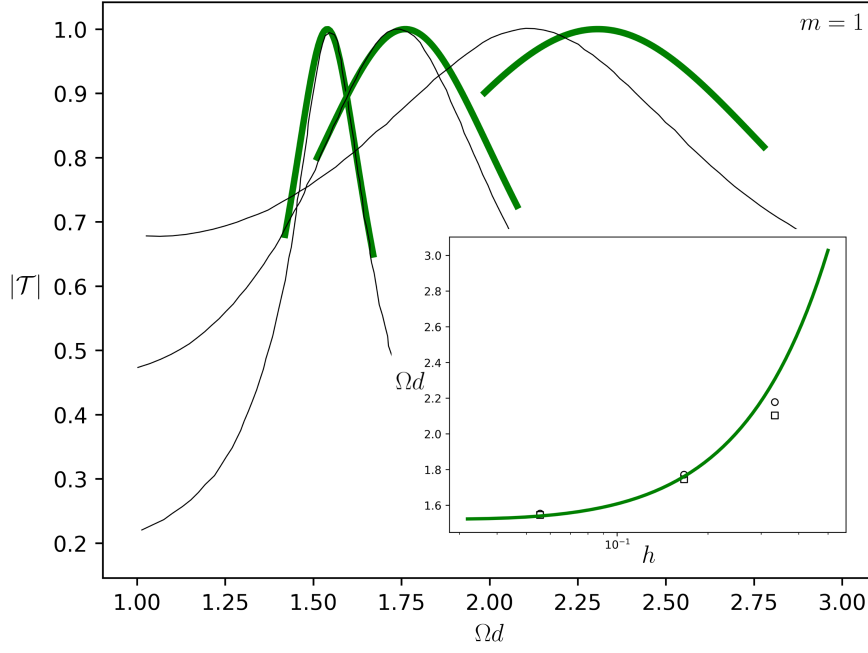


Figure 2.11: Comparison with numerical results (squares and solid black lines) and asymptotic theory (circles) of [39] for the transmission of wave energy through a slit in a wall within a flume. Main plot shows the modulus of the transmission coefficient against Ωd for $h \approx 0.056$ ($d = 0.556$), 0.167 ($d = 0.667$) and 0.333 ($d = 0.833$). Inset shows the enhanced transmission peak frequencies (normalised by d) as a function of h . Solid green lines are the asymptotic approximation (2.47).

where we define the complex-valued reflection and transmission coefficients

$$\mathcal{R} = e^{-i\bar{\Omega}} + \frac{Q^-}{\bar{\Omega}d} e^{-\frac{i\bar{\Omega}}{2}}, \quad \mathcal{T} = \frac{Q^+}{\bar{\Omega}d} e^{-\frac{i\bar{\Omega}}{2}}. \quad (2.47a, b)$$

In Fig. 2.10 we plot $|\mathcal{T}|$ for $h = 0.01$ and with the remaining geometric dimensions from the slit-array acoustic experiments in [119]. We observe perfect transmission at frequencies shifted slightly downwards from the unperturbed standing-wave frequencies. Moreover, Fig. 2.11 compares (2.47) with numerical calculations (based on integral equations) and the asymptotic approximation from [39]. The comparison pertains to the transmission of wave energy through a slit in a wall within a flume. As $h \rightarrow 0$, the agreement with both the numerical and theoretical results improves.

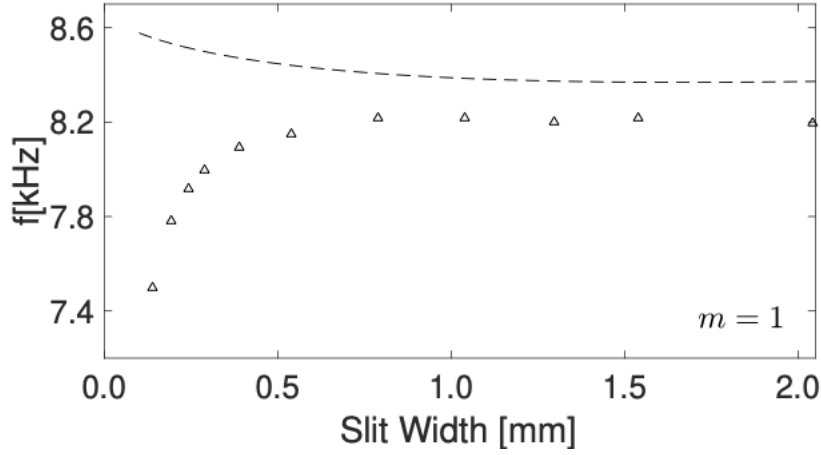


Figure 2.12: Frequency of fundamental resonance as a function of slit width $w = 2hl$ for acoustic-wave transmission through a periodic array of slits, of period $2dl = w + 2.91\text{mm}$, in an aluminium plate of thickness $l = 19.8\text{mm}$. Dashed black line: approximation (2.48). Symbols: acoustic experiments [119].

2.7.4 Comparison with slit-array acoustic experiments

A closed-form approximation for the resonance frequencies can be readily extracted from (2.44). We write this as

$$\frac{\Delta f_m}{f_m} \approx \frac{4h}{\pi} (\ln h + \ln 2m + \gamma_E - 1) + 2h \operatorname{Im} \sigma(md), \quad (2.48)$$

where Δf_m is again the deviation of the m th resonance frequency from the corresponding unperturbed standing-wave frequency f_m . In Fig. 2.12 we test (2.48) against the acoustic slit-array experiments in [119]. The agreement is reasonable only for larger slit widths (still small relative to the plate thickness; in contrast, the asymptotic and experimental results diverge for widths smaller than 0.8mm , despite the small parameter h underlying our analysis becoming smaller). We hypothesise that this divergence is due to the effect of thermoviscous loss).

2.8 Concluding remarks

In conclusion, our asymptotic analysis provides a simple, accurate and physically representative theory of ideal extraordinary transmission through narrow slits, which in particular improves agreement with experiments and highlights the importance of aperture effects close to resonance. As the slit width is reduced, holding the wall thickness and frequency fixed, intrinsic (material) losses ultimately become comparable to the radiative losses captured by our model. This is seen in the electromagnetic experimental data of Ref. [112] [cf. figures 2.6 and 2.7],

as well as in recent acoustic experiments [119, 117], in which most of the data points lie in the latter regime. It is therefore desirable to extend the present theory to incorporate losses, which are of an inherently different nature in the electromagnetic and acoustic cases [112, 119, 85, 23]. We will see in the following chapters that our asymptotic approach offers simplification in capturing non-ideal effects in both of these physical scenarios, which are fundamental to the practical design of acoustic and photonic metamaterials.

CHAPTER 3

Boundary-layer effects on acoustic extraordinary transmission

3.1 Introduction

In this chapter, we continue our focus on two-dimensional slits as resonators in an infinite plate. It is clear that material losses and dissipative effects impair the transmission of wave energy through slits. As structured-wave devices are scaled down, these losses become increasingly significant, impacting transmission considerably. Consequently, the assumptions underlying idealized models may no longer be realistic under these conditions. Acoustic experiments and numerical simulations have shown that, owing to thermal and viscous effects for kHz frequencies in air, the resonant transmission peaks are diminished and shifted to lower frequencies (on top of the diffractive shifts predicted by idealised models) [119]. In this physical scenario, discrepancies with the models are attributed to the effects of dissipative thermoviscous boundary layers, lying in the exterior fluid domain, that are thin compared to the subwavelength width of the slits.

Our goal in this chapter is to analytically investigate the above acoustic boundary-layer effects, for both single-slit and slit-array configurations. In accordance with the relevant experiments, we shall take the incident field to be a plane wave propagating perpendicular to the plate. Furthermore, we shall assume that the fluid is a viscous and thermally conducting ideal gas and that the plate is rigid and isothermal. Our aim is to derive an asymptotic description based on the subwavelength smallness of the slit width in comparison with the wavelength-scale plate thickness, starting from first principles: the linearised Navier–Stokes and energy equations (these fundamental equations were discussed in Chapter 1).

We shall extensively build on the analysis of Chapter 2, which used the method of matched asymptotic expansions [47, 115] to analyse the comparable idealised problem in the case of slits; two important elements in that analysis will also be crucial here. The first is that the analysis focuses on ‘near-resonance’ regimes in frequency and parameter space. In the idealised problem, these regimes are defined by the smallness of the slit width in comparison to the plate thickness, together with the related proximity of the frequency to one of the standing-wave frequencies of the slit. In the present context, we also require that the boundary layers are thin compared to the slit width, so that the resonances are not strongly damped owing to dissipative effects. Without loss of generality, we shall consider a distinguished limit where boundary-layer and diffractive effects are comparable in order of magnitude. The second important element is the systematic treatment of diffractive end effects via matched asymptotics and conformal mapping techniques.

As further discussed in Chapter 2, the analysis and results therein corrected *ad hoc* approximations in the physics literature [113, 43, 20] and simplified and made explicit previous mathematical treatments of the idealised single-slit problem [59, 60, 61, 73] (see also discussion below (3.39)). Furthermore, the asymptotic formulae were shown to be in excellent agreement with numerical solutions of the idealised problem and with a subset of the microwave experiments in [112] and acoustic experiments in [119] for which dissipative effects are minimal. Accordingly, the aim of this chapter is to extend this agreement between asymptotics and experiments to smaller slit widths, including single-slit and slit-array configurations as well as dissipative effects.

The rest of this chapter is structured as follows. In §3.2 we formulate the acoustic problem for transmission of a normally incident plane wave through a single slit. In §3.3 we asymptotically analyse the problem. In §3.5 we extend the theory to a periodic array of slits. For comparisons with experiments, see §3.3.7 and §3.5.3. We give concluding remarks in §3.6.

This chapter is thus based on [12] from joint work with Rodolfo Brandão Macena Lira, which has also served as the basis for Chapter 5 in his thesis [11].

3.2 Formulation for a single slit

3.2.1 Acoustic problem

The acoustic problem is schematically depicted in Fig. 3.1(a). In this problem, the plate is rigid and isothermal (perfectly heat conducting), while the exterior domain

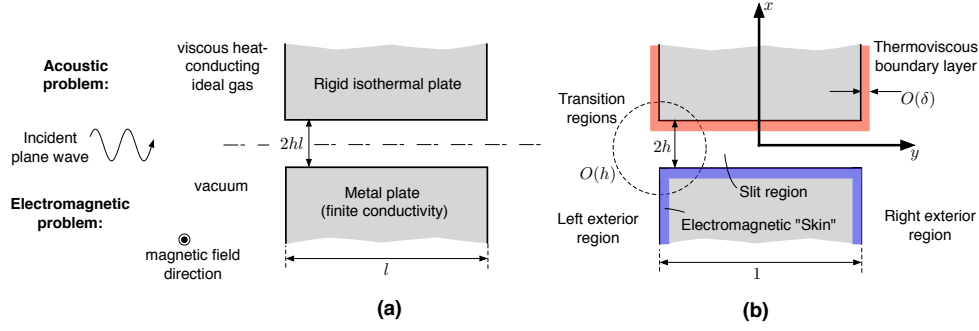


Figure 3.1: Transmission of a normally incident plane wave through a single slit in an infinite plate. (a) Dimensional schematic. (b) Dimensionless schematic showing the regions considered in the asymptotic analysis of §3.3; the boundary-layer scaling $\delta = O(h^2)$ represents a distinguished limit where boundary-layer and diffraction effects are comparable.

is a viscous and heat-conducting ideal gas. We confine the problem to the exterior fluid domain. The governing equations are the continuity, momentum, energy and state equations linearised about an equilibrium state of density ρ_0 , viscosity η , specific heat capacity at constant pressure c_p , sound speed c and heat conductivity κ [90].

In what follows, we adopt a dimensionless convention where lengths are normalised by l . In particular, we will employ the dimensionless Cartesian coordinates (x, y) shown in Fig. 3.1(b), along with the radial coordinates $r^\pm = \sqrt{x^2 + (y \mp 1/2)^2}$ measured from the centres of the slit apertures. We suppress the harmonic time variation $\exp(-i\omega t)$ in the usual way, such that fields are represented by complex-valued counterclockwise phasors. Below, we formulate the acoustic problem for the dimensionless pressure p , temperature perturbation T and velocity field $\mathbf{u}$ (respectively normalised by p_∞ , $p_\infty/(\rho_0 c_p)$ and $p_\infty/(\omega \rho_0 l)$). The reference pressure p_∞ is associated with the incident plane wave, as discussed below.

Following the general theory in Chapter 1, the governing equations are the continuity equation

$$-i\Omega^2 p - i(\gamma - 1)\Omega^2(p - T) + \nabla \cdot \mathbf{u} = 0, \quad (3.1)$$

the momentum equation

$$-i\mathbf{u} = -\nabla p + \delta^2 \left[\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right] \quad (3.2)$$

and the energy equation

$$i(p - T) = \text{Pr}^{-1} \delta^2 \nabla^2 T. \quad (3.3)$$

We also define the dimensionless frequency

$$\Omega = \frac{\omega l}{c}, \quad (3.4)$$

the dimensionless viscous length scale

$$\delta = \frac{l_v}{l}, \quad (3.5)$$

along with the Prandtl number $\text{Pr} = (l_v/l_t)^2$, in which $l_v = \sqrt{\eta/\rho_0\omega}$ and $l_t = \sqrt{\kappa/\rho_0\omega c_p}$ are dimensional viscous and thermal length scales, respectively. The adiabatic index is denoted γ . For air, $\gamma \approx 1.4$ and $\text{Pr} \approx 0.71$.

The above equations are supplemented by boundary and far-field conditions. On the plate boundary, the linearised flow field satisfies the no-slip condition

$$\mathbf{u} = \mathbf{0}, \quad (3.6)$$

while the temperature perturbation satisfies the isothermal boundary condition

$$T = 0. \quad (3.7)$$

The acoustic problem is closed by specifying a normally incident pressure plane wave and requiring the corresponding scattered fields to be outward propagating. Strictly speaking, plane-wave solutions of (3.1)–(3.3) attenuate exponentially; thus, we define the reference value p_∞ as the magnitude of the plane wave at $y = 0$. For small δ , however, sound attenuation in the bulk of the fluid occurs on a length scale that is large compared to the plate thickness [90]. For this reason, it will suffice that at a fixed position, the incident pressure field $p^{(i)}$ satisfies

$$p^{(i)} \rightarrow e^{i\Omega y} \quad \text{as } \delta \rightarrow 0. \quad (3.8)$$

3.2.2 Near-resonance limit

Following the analysis of the idealised problem in Chapter 2, we shall consider the near-resonance regime where $h \ll 1$ and $\Omega - \bar{\Omega} = O(h)$, with

$$\bar{\Omega} = m\pi, \quad m = 1, 2, 3, \dots \quad (3.9)$$

the unperturbed standing-wave, or Fabry–Pérot, frequencies of the slit (i.e., ignoring end-correction and dissipative effects). For this purpose, it is convenient to write

$$\frac{\Omega - \bar{\Omega}}{\bar{\Omega}} \equiv 2h\Omega' \quad (3.10)$$

with Ω' fixed as $h \rightarrow 0$. For this regime, it was shown that the wave field within the slit is approximated by the corresponding one-dimensional standing wave, except close to the slit ends, of magnitude $O(1/h)$ — singularly large compared to the $O(1)$ magnitude of the slit field off-resonance and the $O(1)$ magnitude of the incident wave; concomitantly, the field diffracted from the slit ends was shown to be enhanced from $O(h)$ to $O(1)$.

It is clear that dissipative effects can only widen the resonances whilst diminishing the above magnitude scalings. In what follows, we shall specifically consider the distinguished boundary-layer scaling $\delta = O(h^2)$ where, on the one hand, dissipative effects are important at leading order, while, on the other hand, the near-resonance scalings are the same as in the idealised problem (see scaling arguments in §3.3.2). This distinguished scaling represents a balance between diffractive energy leakage from the slit and dissipation.

In light of the above, we rescale the dimensionless boundary-layer thickness as

$$\delta = h^2\delta', \quad (3.11)$$

with δ' fixed as $h \rightarrow 0$.

3.3 Asymptotic analysis for a single slit

3.3.1 Matched asymptotic expansions

In this section, we study the single-slit problem formulated in §3.2, considering the near-resonance limit process described in §3.2.2. Following Chapter 2, we shall employ the method of matched asymptotic expansions whereby the physical domain is conceptually decomposed into overlapping regions interconnected by asymptotic matching rules [47, 115]. The regions employed in our analysis are schematically shown in Fig. 3.1(b). These include the ‘bulk’ left and right exterior regions, a slit region and two transition regions close to the slit ends, as well as boundary-layer regions outside the plate. The term bulk is used to indicate characteristic dimensions that are large compared to the boundary-layer thickness. Matching in-between the bulk regions will be carried out based on results from

Chapter 2. Further matching will be needed in order to connect the overlapping boundary-layer and bulk regions.

3.3.2 Boundary-layer effects: scalings

It is constructive to preface the asymptotic analysis by deriving scaling rules describing the effects of the thin boundary layers on wave propagation in the bulk exterior, slit and transition regions. These scaling rules will help to streamline the analysis and justify the distinguished boundary-layer scaling (3.11).

In the acoustic problem, there are viscous and thermal boundary layers on the external side of the plate boundary. The tangential component of the velocity field varies across the boundary layer, from the bulk value, of order $\mathcal{U}$, say, to zero at the boundary; the pressure, of order $\mathcal{P}$, say, is approximately uniform across the boundary layer. Furthermore, the temperature varies from its bulk value, on the order of $\mathcal{P}$, to zero on the boundary. With this construction, the continuity equation (3.1) implies bulk perturbations of the *normal* velocity component of order $\mathcal{U}'_V = (\delta/\Delta_{\parallel})\mathcal{U}$ and $\mathcal{U}'_T = \delta\mathcal{P}$ owing to viscous and thermal effects, respectively, wherein $\Delta_{\parallel}$ is the characteristic scale of the bulk field parallel to the boundary (unity for the exterior and slit regions and h for the transition regions). Given velocity scales like the pressure gradient in the bulk, we have $\mathcal{U} = \mathcal{P}/\Delta_{\parallel}$ and $\mathcal{U}'_{V/T} = \mathcal{P}'_{V/T}/\Delta_{\perp}$, where $\mathcal{P}'_V$ and $\mathcal{P}'_T$ are the orders of the bulk pressure perturbations owing to viscous and thermal effects, respectively. We therefore find the estimates

$$\mathcal{P}'_V = \frac{\delta\Delta_{\perp}}{\Delta_{\parallel}^2}\mathcal{P} \quad \text{and} \quad \mathcal{P}'_T = \delta\Delta_{\perp}\mathcal{P}. \quad (3.12)$$

With $\mathcal{P} = h^{-1}$ in the slit region and unity in the exterior and transition regions, we find $\mathcal{P}'_V = h^2, h^2$ and h in the exterior, slit and transition regions, respectively; for $\mathcal{P}'_T$ we find orders h^2, h^2 and h^3 for the exterior, slit and transition regions, respectively.

The relative smallness of these bulk perturbations may appear to contradict our claim that the assumed scaling $\delta = O(h^2)$ represents a distinguished limit where boundary-layer effects first become important. While these perturbations are indeed negligible in the exterior and transition bulk regions, we know from the analysis of the idealised problem in Chapter 2 that $O(h^2)$ perturbations in the slit region affect the leading $O(h^{-1})$ field in the slit. It is this observation, together with the above scaling rules, by which we have identified the distinguished scaling

(3.11). We note that an alternative scaling analysis could be carried out based on the energy-dissipation balance described in Appendix B.

3.3.3 Exterior regions

Armed with these scaling results, we begin our analysis with the exterior regions, which are defined by the limit process discussed in §3.2.2, together with the additional specification that the coordinates (x, y) are held fixed. We shall distinguish between a left exterior region corresponding to the half-space $y < -1/2$ and a right exterior region corresponding to the half-space $y > 1/2$. We pose the asymptotic expansions

$$p \sim \varphi^\pm(x, y) \quad \text{as } h \rightarrow 0; \quad (3.13)$$

the plus-minus signs indicating the right and left exterior regions, respectively.

The leading-order fields $\varphi^\pm$ satisfy the Helmholtz equations

$$\nabla^2 \varphi^\pm + \bar{\Omega}^2 \varphi^\pm = 0 \quad (3.14)$$

in the respective half-spaces. (3.14) readily follows from combining the leading-order balances of (3.1)–(3.3). Note that the dimensionless frequency is approximated at leading order by $\bar{\Omega}$ in accordance with the frequency rescaling (3.10).

The Helmholtz equations (3.14) are supplemented by effective boundary conditions. At $y = \pm 1/2$, the scaling results of §3.3.2 imply that in the exterior regions, boundary-layer effects are negligible at $O(1)$. This implies the Neumann boundary conditions

$$\left. \frac{\partial \varphi^-}{\partial y} \right|_{y=-1/2} = 0, \quad \left. \frac{\partial \varphi^+}{\partial y} \right|_{y=1/2} = 0, \quad (3.15a, b)$$

for $x \neq 0$. The behaviour of $\varphi^\pm$ as $(x, y) \rightarrow (0, \pm 1/2)$ will be determined by asymptotic matching. As $|y| \rightarrow \infty$, the leading-order scattered fields $\varphi^\pm - \varphi^{(i)}$ satisfy an outward-radiation condition with respect to the incident field [cf. (3.8)]

$$\varphi^{(i)} = e^{i\bar{\Omega}y}. \quad (3.16)$$

Consider the left exterior region. The solution for φ^- is obtained by superposing the incident plane wave (3.16), a reflected plane wave, whose phase is adjusted to satisfy condition (3.15a), and the fundamental singular solutions of the Helmholtz equation (3.14) with origin at $(x, y) = (0, -1/2)$. This singular solution, which represents an outward-radiating cylindrical wave, is proportional to $\mathcal{H}_0(\bar{\Omega}r^-)$, wherein $\mathcal{H}_0$ is the zeroth-order Hankel function [1] defined in Chapter 2 [cf. (2.11)];

it is clearly compatible with (3.15a). Higher-order singular solutions, formed by derivatives of this fundamental solution, can be eliminated via matching, since they imply a slit field larger than $O(1/h)$. We accordingly find

$$\varphi^- = e^{i\bar{\Omega}y} + R e^{-i\bar{\Omega}y} + Q^- \mathcal{H}_0(\bar{\Omega}r^-), \quad (3.17)$$

where $R = e^{-i\bar{\Omega}}$ is a reflection coefficient and Q^- is a diffraction coefficient to be determined.

Similar considerations applied to the right exterior region give

$$\varphi^+ = Q^+ \mathcal{H}_0(\bar{\Omega}r^+), \quad (3.18)$$

where Q^+ is a second diffraction coefficient to be determined.

3.3.4 Slit region

Consider now the slit region, where the stretched transverse coordinate $X = x/h$ is held fixed instead of x . With that rescaling, we find the Helmholtz equation

$$\frac{1}{h^2} \frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial y^2} + \bar{\Omega}^2 \left(1 + 4h\Omega' + 4h^2\Omega'^2 \right) \Phi \approx 0 \quad (3.19)$$

for $|y| < 1/2$ and $|X| < 1$, where $\Phi(X, y)$ stands for p . The governing equations (3.1)–(3.3) together imply (3.19) with $O(\delta^2 \Phi)$ error terms, which are too small to affect the following analysis.

The boundary conditions at $y = \pm 1/2$ are to be determined by asymptotic matching with the transition and exterior regions. As for the boundary conditions at $X = \pm 1$, the scalings derived in §3.3.2 imply that the boundary layers at the top and bottom of the slit affect the bulk slit field only at high orders. Specifically,

$$\frac{\partial \Phi}{\partial X} = O(h\delta \Phi) \quad \text{at} \quad X = \pm 1. \quad (3.20)$$

The precise leading-order form of the right-hand side will be derived in §3.4, by analysing the thermoviscous boundary layers. In the present subsection, we will simply quote this result (at a later point, as it depends on properties of Φ yet to be derived).

As already mentioned, we anticipate an $O(1/h)$ enhancement of the wave field in the slit. We accordingly pose the expansion

$$\Phi = h^{-1} \Phi_{-1}(X, y) + \Phi_0(X, y) + h \Phi_1(X, y) + h^2 \Phi_2(X, y) + \cdots \quad \text{as} \quad h \rightarrow 0. \quad (3.21)$$

The $O(1/h^3)$ and $O(1/h^2)$ balances of (3.19) give the trivial relations

$$\frac{\partial^2 \Phi_{-1}}{\partial X^2} = 0, \quad \frac{\partial^2 \Phi_0}{\partial X^2} = 0. \quad (3.22)$$

Since (3.20) implies the homogeneous Neumann conditions

$$\frac{\partial \Phi_{-1}}{\partial X} = 0, \quad \frac{\partial \Phi_0}{\partial X} = 0 \quad \text{at} \quad X = \pm 1, \quad (3.23a, b)$$

we find that Φ_{-1} and Φ_0 are independent of X , namely $\Phi_{-1} = \Phi_{-1}(y)$ and $\Phi_0 = \Phi_0(y)$. Next, the $O(1/h)$ balance of (3.19) gives

$$\frac{\partial^2 \Phi_1}{\partial X^2} + \frac{d^2 \Phi_{-1}}{dy^2} + \bar{\Omega}^2 \Phi_{-1} = 0. \quad (3.24)$$

Integrating with respect to X and using the homogeneous Neumann condition

$$\frac{\partial \Phi_1}{\partial X} = 0 \quad \text{at} \quad X = \pm 1, \quad (3.25)$$

we find the one-dimensional Helmholtz equation

$$\frac{d^2 \Phi_{-1}}{dy^2} + \bar{\Omega}^2 \Phi_{-1} = 0. \quad (3.26)$$

It follows from (3.24)–(3.26), in turn, that $\Phi_1 = \Phi_1(y)$.

The Helmholtz equation (3.26) is supplemented by the matching conditions

$$\Phi_{-1} = 0 \quad \text{at} \quad y = \pm 1/2, \quad (3.27)$$

which evidently follow from the relative smallness of the wave field in the exterior regions together with the regularity of solutions to (3.26) [48]. The homogeneous problem consisting of (3.26) and (3.27) has nontrivial solutions only for $\bar{\Omega} = m\pi$, in which the integer $m = 1, 2, \dots$ corresponds to the order of the standing wave excited in the slit. We write these standing-wave solutions in the form

$$\Phi_{-1} = \mathcal{A} \times \begin{Bmatrix} \cos(\bar{\Omega}y) \\ \sin(\bar{\Omega}y) \end{Bmatrix}, \quad (3.28)$$

where $\mathcal{A}$ is a complex-valued prefactor; we also carry through the notation used in Chapter 2, where the upper element of the array corresponds to odd m (even standing wave) and the lower to even m (odd standing wave).

At this stage, we need the right-hand side of (3.20). With the knowledge that Φ_{-1} , Φ_0 and Φ_1 are all independent of X , the boundary-layer analysis in §3.4 furnish the impedance-like conditions

$$\frac{\partial \Phi_2}{\partial X} = \pm \mathcal{C} \Phi_{-1} \quad \text{at} \quad X = \pm 1, \quad (3.29)$$

where $\mathcal{C}$ is a complex-valued parameter which is given in the acoustic scenario by

$$\mathcal{C} = \frac{\delta \bar{\Omega}^2}{h^2} \frac{1+i}{\sqrt{2}} \left(1 + \frac{\gamma-1}{\sqrt{\text{Pr}}} \right), \quad (3.30)$$

wherein δ is defined in (3.5). In (3.30), δ can be treated as constant over the near-resonance frequency interval.

Consider next the $O(1)$ balance of (3.19),

$$\frac{\partial^2 \Phi_2}{\partial X^2} + \frac{d^2 \Phi_0}{dy^2} + \bar{\Omega}^2 \Phi_0 + 4\Omega' \bar{\Omega}^2 \Phi_{-1} = 0. \quad (3.31)$$

Integrating across the slit and using the boundary condition (3.29) yields

$$\frac{d^2 \Phi_0}{dy^2} + \bar{\Omega}^2 \Phi_0 = - \left(4\Omega' \bar{\Omega}^2 + \mathcal{C} \right) \Phi_{-1}. \quad (3.32)$$

We shall not need the detailed solution of (3.32). Rather, we derive relations between the leading-order amplitude $\mathcal{A}$ and the end values of Φ_0 , which will be required for asymptotic matching. To this end, we subtract the product of Φ_0 and the complex conjugate of (3.26) from the product of (3.32) and the complex conjugate of Φ_{-1} ; integrating this difference between the two ends of the slit, we find

$$\left[\Phi_0 \frac{d\Phi_{-1}^*}{dy} \right]_{-1/2}^{1/2} = \left(4\bar{\Omega}^2 \Omega' + \mathcal{C} \right) \int_{-1/2}^{1/2} |\Phi_{-1}|^2 dy. \quad (3.33)$$

Substituting (3.28) and using (3.9) gives

$$\mathcal{A} = \frac{2\bar{\Omega} i^m}{4\Omega' \bar{\Omega}^2 + \mathcal{C}} \times \left\{ \begin{array}{c} i\Phi_0(1/2) + i\Phi_0(-1/2) \\ \Phi_0(1/2) - \Phi_0(-1/2) \end{array} \right\}. \quad (3.34)$$

3.3.5 Matching and transition regions

It remains to relate the approximations found in the exterior and slit regions. Given the logarithmic singularity of the leading-order exterior fields, these regions cannot be matched directly. Rather, it is necessary to consider intermediate ‘transition

regions' at $O(h)$ distances from the slit ends. It is clear from the preceding analyses of the slit and exterior regions that p is $O(1)$ in these regions. The scaling arguments in §3.3.2 show that boundary-layer effects are not important at this order. The leading-order aperture problem is therefore the same as the one formulated and solved in Chapter 2, as are the details of the asymptotic matching between the transition regions and the slit and exterior regions. We shall simply quote the results of that chapter that are relevant here.

The first result is a relation between the diffraction amplitudes $Q^\pm$ appearing in the $O(1)$ exterior fields and the rescaled amplitude $\mathcal{A}$ of the enhanced $O(h^{-1})$ standing wave in the slit. This statement of flux conservation through the transition regions can be written as

$$Q^\pm = i^m \bar{\Omega} \mathcal{A} \times \begin{Bmatrix} 1 \\ \mp i \end{Bmatrix}. \quad (3.35)$$

The second result connects the diffraction coefficients $Q^\pm$ with the $O(1)$ discontinuity in the wave field across the aperture, with the singularity of the exterior field subtracted. This relation can be written as

$$\lim_{r^\pm \rightarrow 0} \left(\varphi^\pm - \frac{2i}{\pi} Q^\pm \ln r^\pm \right) - \Phi_0|_{y=\pm 1/2} = iQ^\pm \left(\frac{2}{\pi} \ln \frac{1}{h} - \beta \right), \quad (3.36)$$

where the parameter β depends only on the geometry of the slit opening; for the right-angled aperture considered here, the conformal-mapping analysis in

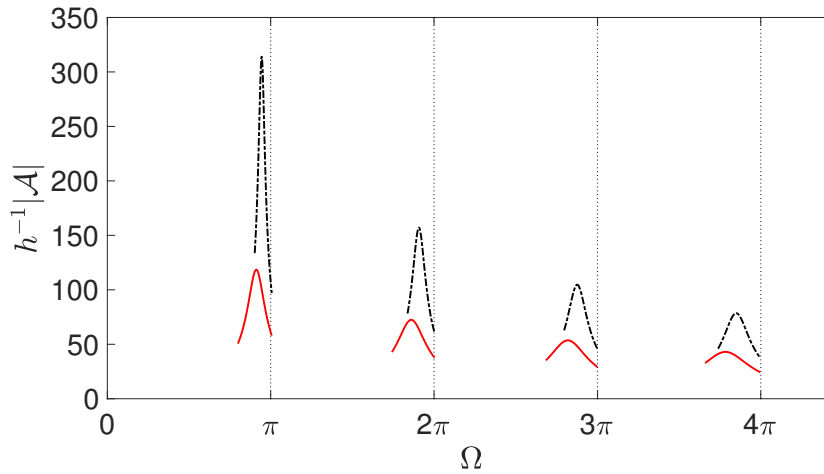


Figure 3.2: Enhancement of the slit field predicted by the near-resonance approximation (3.39) as a function of the dimensionless frequency Ω . The slit aspect ratio is $h = 0.01$. Black dashed lines: idealised problem without dissipation. Red solid lines: thermoviscous-acoustic scenario assuming air at 20 °C and slit length $l = 35\text{mm}$ [119]. The vertical dotted lines mark the unperturbed standing-wave frequencies.

Chapter 2 yields the value

$$\beta = \frac{2}{\pi} \left(\ln \frac{4}{\pi} - 1 \right) \approx -0.4828. \quad (3.37)$$

Substituting the external fields $\varphi^\pm$ from (3.17) and (3.18), and using the asymptotic relation (2.11), we can solve (3.36) for the end values of the $O(1)$ slit field:

$$\Phi_0|_{y=\pm 1/2} = Q^\pm \frac{2i}{\pi} \left(\ln \frac{2\bar{\Omega}h}{\pi} + \gamma_E - 1 - \frac{i\pi}{2} \right) + (1 \mp 1)e^{-i\bar{\Omega}/2}. \quad (3.38)$$

Note the logarithmic dependence upon h in (3.38). This relation assumes an asymptotic convention where logarithmic orders are considered together with the nearest algebraic order [47, 115]. All small- h expansions in this thesis are to be interpreted in this manner.

3.3.6 Frequency response

Solving (3.34), (3.35) and (3.38) for the amplitude $\mathcal{A}$ yields

$$\mathcal{A} = \frac{1/\bar{\Omega}}{\Omega' + \frac{\mathcal{C}}{4\bar{\Omega}^2} - \frac{2}{\pi} \left(\ln \frac{2\bar{\Omega}h}{\pi} + \gamma_E - 1 \right) + i} \times \begin{Bmatrix} i \\ -1 \end{Bmatrix}, \quad (3.39)$$

which provides a closed-form approximation for the near-resonance frequency response. The corresponding results for the diffraction coefficients $Q^\pm$ can be obtained by substituting (3.39) into (3.35).

Recall that (3.39) was derived assuming the distinguished limit $\delta = O(h^2)$. Based on the scaling arguments in §3.3.2, however, it can be shown that (3.39) actually holds as a leading-order approximation over the wider regime $\delta \ll h$; thus, our analysis breaks down only in the alternative distinguished limit $\delta = O(h)$, namely when the boundary-layer thickness is comparable to the slit width, in which case the resonance is no longer weakly damped [23]. In other words, the distinguished limit $\delta = O(h^2)$ was chosen as it describes all regions in $\delta - h$ parameter space where the slit resonances are weakly damped.

To illuminate the different regimes described by (3.39), note that the boundary-layer parameter $\mathcal{C} = O(\delta/h^2)$ and recall definition (3.10) for Ω' . Thus, for $\delta \ll h^2$, dissipation effects are negligible and (3.39) reduces to its lossless counterpart derived in Chapter 2. For $\delta \gg h^2$, however, (3.39) implies a sharp resonance only if $\delta \ll h$; for $h^2 \ll \delta \ll h$, the resonance is weakly damped and limited by dissipation. In that case, the resonance width is $\Omega - \bar{\Omega} = O(\delta/h)$ and the amplitude of the

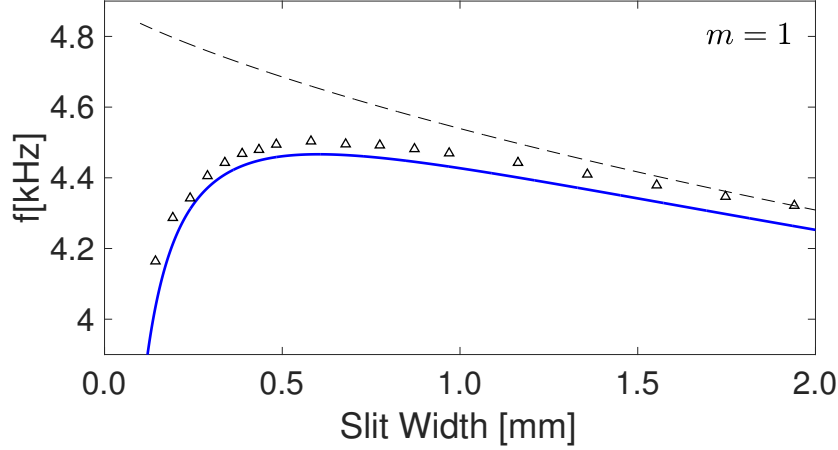


Figure 3.3: Resonance frequencies as a function of slit width for the fundamental resonance of acoustic-wave transmission through a slit in an aluminium plate of thickness 35mm. The thick blue line is calculated from (3.40) with $\mathcal{C}$ calculated from (3.30) for air at 20 °C. Symbols: acoustic experiments [119].

wave in the slit is $O(h/\delta)$. As already noted, for $\delta = O(h^2)$, radiation damping and dissipative losses are comparable and the scalings are therefore the same as in the lossless case.

In Fig. 3.2, we plot the enhancement factor $h^{-1}|A|$ as a function of the dimensionless frequency Ω . As an example, we set $h = 0.01$ and compare the idealised case with the thermoviscous-acoustic scenario assuming the physical parameters of the experiments in [119]. Note that the resonance peaks are shifted from the standing-wave values $\bar{\Omega}$, more so when dissipation is included.

3.3.7 Comparison with single-slit acoustic experiments

A closed-form approximation for the resonance frequencies is readily extracted from (3.39). For the sake of comparison with experimental data, we write this in the form

$$\frac{\Delta f_m}{f_m} \approx \frac{4h}{\pi} (\ln h + \ln(2m) + \gamma_E - 1) - \frac{h}{2\bar{\Omega}^2} \operatorname{Re} \mathcal{C}, \quad (3.40)$$

where Δf_m is the dimensional frequency deviation of the m th resonant peak from the respective standing-wave frequency $f_m = mc/2l$.

In Fig. 3.3, we test (3.40) against kHz-acoustic experiments [119]. The agreement is very good. As already presented in Chapter 2, the lossless approximation with $\mathcal{C} = 0$ is not accurate for very narrow slits ($\lesssim 1.5$ mm).

3.3.8 Dissipation

The formula for the amplitude, $\mathcal{A}$, can also be used to compute the power dissipated by the slit. An approximate energy-dissipation relation, derived in Appendix B, is given as

$$-2 \operatorname{Re} \left(Q^- e^{i\frac{\Omega}{2}} \right) = D + |Q^-|^2 + |Q^+|^2, \quad (3.41)$$

where D represents the rate of dissipation, valid near resonance. Expressing this equation in terms of $\mathcal{A}$ using (3.35), we obtain

$$D = 2\bar{\Omega} \left\{ \begin{array}{c} \operatorname{Re} \\ \operatorname{Im} \end{array} \right\} \mathcal{A} - 2\bar{\Omega}^2 |\mathcal{A}|^2, \quad (3.42)$$

which, after substituting (3.39) and algebraic manipulation, yields

$$D = \frac{1}{2} |\mathcal{A}|^2 \operatorname{Im} \mathcal{C}. \quad (3.43)$$

This result can also be derived through asymptotic evaluation of the dissipation integrals defined in §1.4.4.

3.4 Thermoviscous boundary layer analyses

It remains to derive the effective boundary conditions (3.29) assumed in the slit-region analysis, by matching the thermoviscous boundary layers with the bulk slit region. As a preliminary step, it is convenient to expand the description of the bulk slit region from the single field $p = \Phi$ to include the normalised flow and temperature fields, $\mathbf{u} = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y$ and T , respectively. To this end, we rewrite the slit expansion (3.21) in the form

$$p = h^{-1}\Phi_{-1}(y) + \Phi_0(y) + h\Phi_1(y) + h^2\Phi_2(X, y) + \dots \quad \text{as } h \rightarrow 0. \quad (3.44)$$

Note that the independence of Φ_{-1} , Φ_0 and Φ_1 upon X was determined in §3.3.4 based on the scaling (3.20), which is consistent with the boundary-layer analysis here. Given (3.44), the momentum equation (3.2) implies the expansions

$$u \sim -ih \frac{\partial \Phi_2}{\partial X}, \quad v \sim -ih^{-1} \frac{d\Phi_{-1}}{dy} \quad \text{as } h \rightarrow 0. \quad (3.45a, b)$$

Similarly, the energy equation (3.3) implies the temperature expansion

$$T \sim h^{-1}\Phi_{-1} \quad \text{as } h \rightarrow 0. \quad (3.46)$$

Consider now the $O(\delta)$ boundary layer adjacent to the lower slit boundary $x = -h$. Since $\delta = O(h^2)$, we define the transverse coordinate $X' = (x + h)/h^2$, with the boundary-layer domain being $X' > 0$ ($|y| < 1/2$). The slit-region expansions (3.44)–(3.46) suggest expanding the acoustic fields in the boundary layer as

$$p \sim h^{-1}\bar{P}_{-1}(X', y), \quad T \sim h^{-1}\bar{T}_{-1}(X', y), \quad u \sim h\bar{U}_1(X', y), \quad v \sim h^{-1}\bar{V}_{-1}(X', y). \quad (3.47a-d)$$

The leading $O(h^{-3})$ balance of the x component of the momentum equation (3.2) gives

$$\frac{\partial \bar{P}_{-1}}{\partial X'} = 0, \quad (3.48)$$

showing that $\bar{P}_{-1}$ is independent of X' . With that, straightforward matching of the boundary layer and slit-region pressure fields yields

$$\bar{P}_{-1} = \Phi_{-1}(y). \quad (3.49)$$

With (3.49), the $O(h^{-1})$ balance of the y component of the momentum equation (3.2) can be written

$$\delta'^2 \frac{\partial^2 \bar{V}_{-1}}{\partial X'^2} + i\bar{V}_{-1} = \frac{d\Phi_{-1}}{dy}, \quad (3.50)$$

where we remind that $\delta' = \delta/h^2$. The no-slip condition (3.6) gives the boundary condition

$$\bar{V}_{-1} = 0 \quad \text{at} \quad X' = 0. \quad (3.51)$$

Furthermore, matching the v velocity component yields the far-field condition

$$\bar{V}_{-1} \rightarrow -i \frac{d\Phi_{-1}}{dy} \quad \text{as} \quad X' \rightarrow \infty. \quad (3.52)$$

Solving (3.50) subject to (3.51) and (3.52), we find

$$\bar{V}_{-1} = i \left(e^{-\frac{1-i}{\sqrt{2}} \frac{X'}{\delta'}} - 1 \right) \frac{d\Phi_{-1}}{dy}. \quad (3.53)$$

The leading $O(h^{-1})$ balance of the energy equation (3.3), with (3.49), reads

$$\text{Pr}^{-1} \delta'^2 \frac{\partial^2 \bar{T}_{-1}}{\partial X'^2} + i\bar{T}_{-1} = i\Phi_{-1}. \quad (3.54)$$

From (3.7), we find the boundary condition

$$\bar{T}_{-1} = 0. \quad (3.55)$$

Furthermore, leading-order matching of the temperature field gives the far-field condition

$$\bar{T}_{-1} \rightarrow \Phi_{-1} \quad \text{as} \quad X' \rightarrow \infty. \quad (3.56)$$

Solving (3.54) subject to (3.55) and (3.56), we find

$$\bar{T}_{-1} = \left(1 - e^{-\frac{1-i}{\sqrt{2}} \frac{X' \sqrt{\text{Pr}}}{\delta'}}\right) \Phi_{-1}. \quad (3.57)$$

Consider now the leading $O(h^{-1})$ balance of the continuity equation (3.1),

$$\frac{\partial \bar{U}_1}{\partial X'} = -\frac{\partial \bar{V}_{-1}}{\partial y} + i\bar{\Omega}^2 \bar{P}_{-1} + i(\gamma - 1)\bar{\Omega}^2 (\bar{P}_{-1} - \bar{T}_{-1}), \quad (3.58)$$

which is supplemented by the boundary condition [cf. (3.6)]

$$\bar{U}_1 = 0 \quad \text{at} \quad X' = 0. \quad (3.59)$$

Integrating (3.58) using (3.59), followed by taking the limit $X' \rightarrow \infty$ using the slit-region Helmholtz equation (3.26), yields

$$\bar{U}_1 \rightarrow -\delta' \bar{\Omega}^2 \frac{1-i}{\sqrt{2}} \left(1 + \frac{\gamma-1}{\sqrt{\text{Pr}}}\right) \Phi_{-1} \quad \text{as} \quad X' \rightarrow \infty. \quad (3.60)$$

Matching the velocity component u then furnishes the effective boundary condition

$$\frac{\partial \Phi_2}{\partial X} = -\delta' \bar{\Omega}^2 \frac{1+i}{\sqrt{2}} \left(1 + \frac{\gamma-1}{\sqrt{\text{Pr}}}\right) \Phi_{-1} \quad \text{at} \quad X = -1. \quad (3.61)$$

The corresponding condition at $X = 1$ follows from symmetry. Together these give expression (3.30) for the parameter $\mathcal{C}$.

We emphasise that the effective condition (3.61) relies on the one-dimensional nature of the wave propagation in the bulk slit region and the related independence of Φ_{-1} , Φ_0 and Φ_1 on the transverse coordinate X . When the bulk field is multi-dimensional, the viscous effect gives a contribution to the effective boundary condition proportional to the surface Laplacian of the bulk pressure, rather than the bulk pressure itself [13]. Given (3.26), in the present case, Φ_{-1} and its surface Laplacian are in fact proportional.

3.5 Extension to a periodic slit array

3.5.1 Frequency response

We enhance the near-resonance theory developed in Chapter 2 §2.7 for a slit array by incorporating thermoviscous effects. The exterior regions remain unchanged; it's only the potential in the slits that undergoes modification. Thus, solving (2.40), (3.34) and (3.35), we find the generalised near-resonance approximation for the loss-adjusted complex amplitude

$$\mathcal{A} = \frac{1/\bar{\Omega}}{\Omega' + \frac{\mathcal{C}}{4\bar{\Omega}^2} - \frac{2}{\pi} \left(\ln \frac{2\bar{\Omega}h}{\pi} + \gamma_E - 1 \right) + i(1 + \sigma(\bar{\Omega}d))} \times \begin{Bmatrix} i \\ -1 \end{Bmatrix}. \quad (3.62)$$

The corresponding formulae for the diffraction amplitudes $Q^\pm$ follow from (3.35).

As in §2.7.2, we tacitly held d fixed, assuming that the periodicity is comparable with the wall thickness and the wavelength. It can be verified, however, that (3.62) holds as a leading-order approximation for $\delta \ll h \ll d$. In particular, for large d , σ attenuates and (3.62) reduces to its single-slit counterpart (3.39). For small d (subwavelength periodicity), $\sigma(\bar{\Omega}d) \sim (\bar{\Omega}d)^{-1}$; it follows again that the slit fields are more strongly damped by radiation than in the single-slit scenario. In that case, dissipative and radiative losses are comparable for $\delta = O(h^2/d)$, which is thicker than implied by the single-slit, or order-unity d , distinguished scaling. Then, if $\delta \gg h^2/d$, dissipation is the dominant loss mechanism and the resonance width is $|\Omega - \bar{\Omega}| = O(\delta/h)$. If $\delta \ll h^2/d$, then the dominant loss mechanism is the enhanced radiation damping; the resonance width is then $|\Omega - \bar{\Omega}| = O(h/d)$.

3.5.2 Reflection and transmission

As in Chapter 2, the complex-valued reflection and transmission coefficients can be written

$$\mathcal{R} = e^{-i\bar{\Omega}} + \frac{Q^-}{\bar{\Omega}d} e^{-\frac{i\bar{\Omega}}{2}}, \quad \mathcal{T} = \frac{Q^+}{\bar{\Omega}d} e^{-\frac{i\bar{\Omega}}{2}}, \quad (3.63a, b)$$

with the diffraction coefficients $Q^\pm$ now provided by substituting (3.62) into (3.35).

In Figs. 3.4 and 3.5 we plot $|\mathcal{R}|$ and $|\mathcal{T}|$, respectively, for $h = 0.01$ and with the remaining geometric dimensions from the slit-array acoustic experiments in [119]. In particular, we compare the lossless case $\mathcal{C} = 0$ and the thermoviscous-acoustic scenario where $\mathcal{C}$ is provided by (3.30) for air at 20 °C. The thermal and viscous effects are seen to diminish the reflection valleys and transmission peaks and

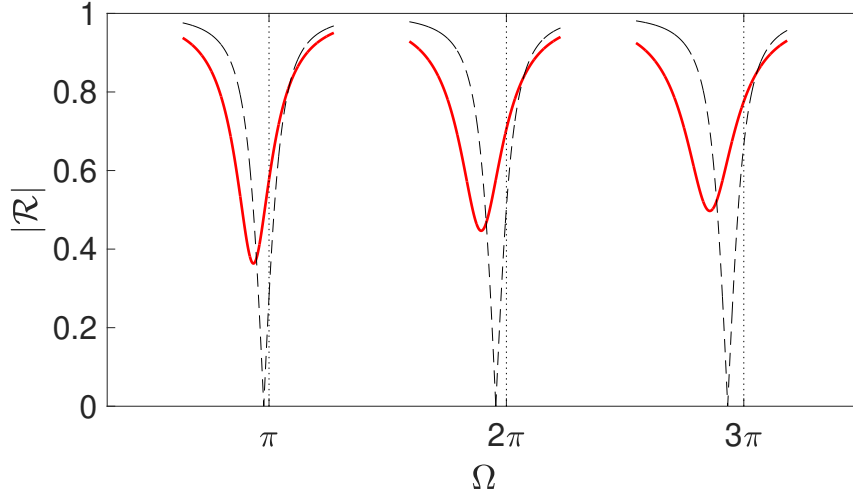


Figure 3.4: Modulus of the reflection coefficient for acoustic transmission through a plate of thickness $l = 19.8\text{mm}$ decorated by a periodic array of slits of width $w = 2hl$ and period $2dl = w + 2.91\text{mm}$ (dimensions from [119]), for $h = 0.01$. Solid line: approximation (2.47) with $\mathcal{C}$ given by (3.30) for air at 20°C . Dashed line: approximation (2.47) for the lossless case $\mathcal{C} = 0$.

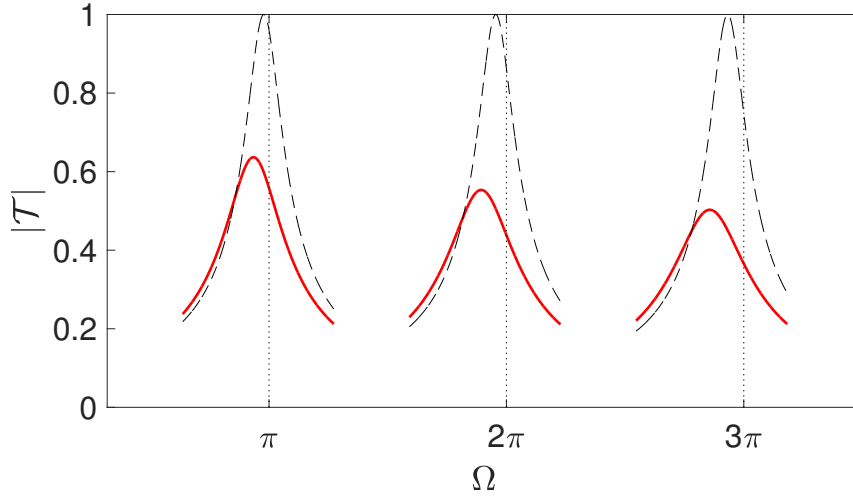


Figure 3.5: Modulus of the transmission coefficient for acoustic transmission through a plate of thickness $l = 19.8\text{mm}$ decorated by a periodic array of slits of width $w = 2hl$ and period $2dl = w + 2.91\text{mm}$ (dimensions from [119]), for $h = 0.01$. Solid line: approximation (2.47) with $\mathcal{C}$ given by (3.30) for air at 20°C . Dashed line: approximation (2.47) for the lossless case $\mathcal{C} = 0$.

shift these to yet lower frequencies. Fig. 3.6 additionally shows that as h becomes smaller, dissipation increases and these transmission peaks decrease and become absorption peaks.

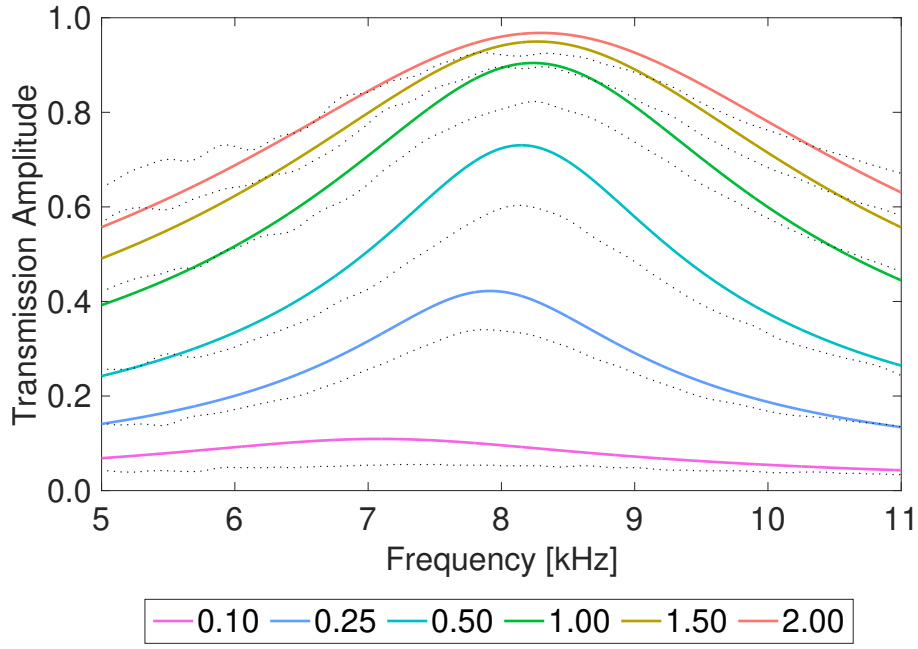


Figure 3.6: The experimental acoustic response of a plate of thickness $l = 19.8\text{mm}$ decorated by a periodic array of slits of width $w = 2hl$ and period $2dl = w + 2.91\text{mm}$ (dimensions from [119]), for varying h (here only the first Fabry–Pérot peak is shown). Dotted black lines: acoustic experiments [119].

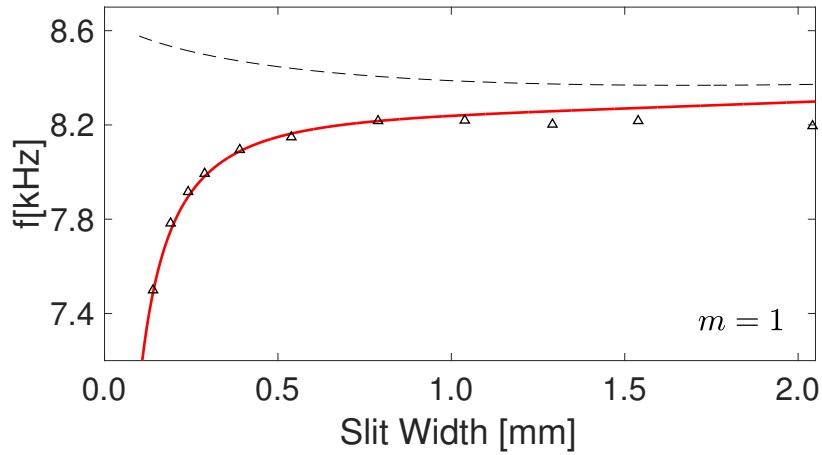


Figure 3.7: Frequency of fundamental resonance as a function of slit width $w = 2hl$ for acoustic-wave transmission through a periodic array of slits, of period $2dl = w + 2.91\text{mm}$, in an aluminium plate of thickness $l = 19.8\text{mm}$. Thick red line: approximation (3.64) with $\mathcal{C}$ given by (3.30) for air at 20°C . Dashed black line: approximation (3.64) for the lossless case $\mathcal{C} = 0$. Symbols: acoustic experiments [119].

3.5.3 Comparison with slit-array acoustic experiments

An adjusted and closed-form approximation, inclusive of material losses, for the resonance frequencies can be readily extracted from (3.62). We write this as

$$\frac{\Delta f_m}{f_m} \approx \frac{4h}{\pi} (\ln h + \ln 2m + \gamma_E - 1) - \frac{h}{2\pi^2 m^2} \operatorname{Re} \mathcal{C} + 2h \operatorname{Im} \sigma(md), \quad (3.64)$$

where Δf_m is again the deviation of the m th resonance frequency from the corresponding unperturbed standing-wave frequency f_m . In Fig. 3.7 we test (3.64) against the acoustic slit-array experiments in [119]. As in the single-slit case, we see excellent agreement.

3.6 Concluding remarks

We have developed an asymptotic theory describing resonant acoustic-wave transmission through slitted plates. The theory provides simple scaling rules and analytical approximations that accurately capture both diffractive and dissipative effects, as demonstrated through a comparison with experimental results in the literature. We hope that this work showcases the power of scaling arguments, matched asymptotics and near-resonance expansions for modelling structured-wave devices including thermoviscous effects. Thus, the present theory can be readily adapted to other configurations where resonant slits are used to manipulate or absorb waves [85, 117, 101]. A similar approach could also be developed for three-dimensional hole resonators [118], analysis for which we will see in later chapters. An analogous theory for acoustic Helmholtz resonators embedded in a wall and arrays thereof has been developed in [14].

CHAPTER 4

Electromagnetic analogy

4.1 Introduction

As demonstrated in Chapter 2, in the simplest version of resonant transmission through narrow slits in an infinite plate, there exists a precise electromagnetic-acoustic analogy in which the wave is represented by a two-dimensional scalar field governed by the Helmholtz equation along with a homogeneous Neumann condition on the plate boundary. There are no dissipative mechanisms in this idealised analogue model. In the electromagnetic scenario, it is valid for transverse-magnetic (TM) polarised waves, assuming that the plate is perfectly conducting. In the acoustic scenario, it is required that the plate is rigid and that viscous and thermal effects are negligible. This analogy naturally extends with appropriate assumptions to other physical scenarios such as water waves [39].

For a single narrow slit, the idealised analogue model predicts enhanced transmission in narrow intervals about a discrete set of resonance frequencies, which owing to diffractive effects are slightly lower than the standing-wave frequencies of the slit calculated with end effects ignored. For an array of narrow slits, similar resonances enable perfect transmission for a discrete set of frequencies.

Qualitatively similar to the acoustic problem in Chapter 3, electromagnetic experiments with metallic plates in the GHz-microwave regime, together with numerical simulations, have shown that owing to the finite conductivity of the plate, the resonant transmission peaks are diminished and shifted to lower frequencies (on top of the diffractive shifts predicted by the idealised model) [112]. In both physical scenarios, these discrepancies with the analogue model are attributed to the effects of dissipative boundary layers that are thin compared to the subwavelength width

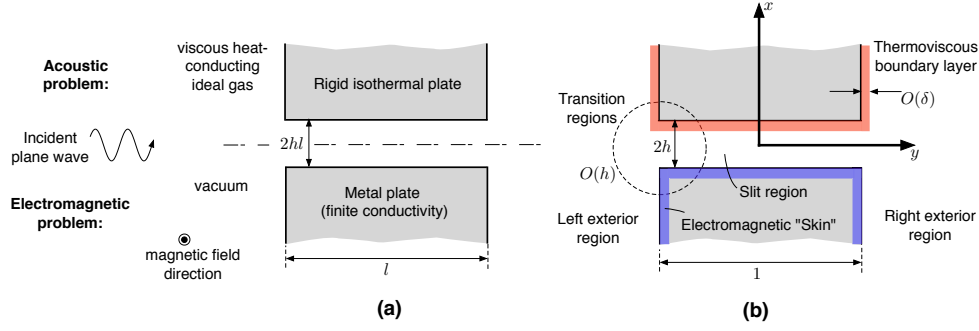


Figure 4.1: Transmission of a normally incident plane wave through a single slit in an infinite plate (upper half: acoustic problem, lower half: electromagnetic problem). (a) Dimensional schematic. (b) Dimensionless schematic showing the regions considered in the asymptotic analysis of §4.3.2; the boundary-layer scaling $\delta = O(h^2)$ represents a distinguished limit where boundary-layer and diffraction effects are comparable.

of the slits. In contrast to acoustics, the boundary layer, aka ‘skin’, lies within the plate in the electromagnetic scenario.

Our goal in this chapter is to analytically investigate the above boundary-layer effects in the electromagnetic setting and show that, while in appearance the two formulations appear very different, the physical analogy between the idealised electromagnetic and acoustic problems can be essentially carried over to the corresponding dissipative problems. In accordance with the relevant experiments, we focus on a single slit in a plate, take the incident field to be a plane wave propagating perpendicular to that plate, assume the TM polarisation and that the plate is metallic.

This chapter is also based on [12] from joint work with Rodolfo Brandão Macena Lira and has also served as the basis for Chapter 5 in his thesis [11].

4.2 Electromagnetic problem formulation

The electromagnetic problem is schematically depicted in Fig. 4.1(a). We consider an electromagnetic plane wave of angular frequency ω normally incident on an infinite metallic plate of thickness l and relative (complex-valued) permittivity ϵ . The plate is bisected by a single perpendicular slit of width $2h$. We consider a two-dimensional problem where the electric and magnetic fields are respectively parallel and perpendicular to the plane in Fig. 4.1; both fields being invariant in the perpendicular direction (TM polarisation). The background medium is assumed to be vacuum.

We adopt the same dimensionless and phasor-field conventions as in the acoustic problem in §3.2.1, with c in (3.4) now denoting the speed of light in vacuum.

With this convention, ϵ is restricted to the second quadrant of the complex plane. We shall formulate the problem for the two-dimensional scalar fields H and $\bar{H}$, the induction fields (normalised by the amplitude of the incident wave) in the background and plate domains, respectively. In Appendix C, we present a comprehensive derivation, starting from the macroscopic form of Maxwell's equations. Note that Ω in (3.4) is of order unity in the case of interest where the wavelength is commensurate with the plate thickness.

The fields H and $\bar{H}$ satisfy the reduced-wave equations

$$\nabla^2 H + \Omega^2 H = 0, \quad \nabla^2 \bar{H} + \epsilon \Omega^2 \bar{H} = 0, \quad (4.1a, b)$$

respectively in the background and plate domains. On the plate boundary, we have the transmission conditions

$$H = \bar{H}, \quad \frac{\partial H}{\partial n} = \frac{1}{\epsilon} \frac{\partial \bar{H}}{\partial n}, \quad (4.2a, b)$$

where $\partial/\partial n$ represents the normal derivative. The problem is closed by specifying the incident field

$$H^{(i)} = e^{i\Omega y} \quad (4.3)$$

along with the condition that the scattered field $H - H^{(i)}$ is outward radiating. (In contrast to the acoustic scenario, here the incident field can be prescribed exactly, as the background medium is lossless).

An important dimensional length scale is the skin depth $l_s = \sqrt{2\rho/\omega\mu_0}$, where ρ is the metal resistivity and μ_0 the metal permeability, which is equal to the vacuum permeability. For a good conductor, l_s is the characteristic scale on which electromagnetic fields attenuate inside the metal. For our purposes, it is useful to define the dimensionless skin depth

$$\delta = \frac{l_s}{l} = \frac{1}{\Omega} \sqrt{\frac{2}{\epsilon''}}, \quad (4.4)$$

in which ϵ'' denotes the imaginary component of ϵ . The relative smallness of the skin depth is determined by the largeness of ϵ'' . In particular, for the GHz-microwave frequencies in the experiments in [112], $\epsilon \approx i\epsilon''$ with $\epsilon'' \simeq 10^6 - 10^7$.

4.3 Near-resonance limit

It again follows from scalings to consider the same near-resonance limit as in §3.2.2; the distinguished limit where dissipation and radiation damping are comparable.

In the electromagnetic problem, δ is implicit in the problem formulation. Thus, in that problem it is more convenient to hold fixed the complex-valued parameter

$$\alpha = h^2 \epsilon^{1/2}, \quad (4.5)$$

where the principal branch of the square-root function is assumed.

4.3.1 Boundary-layer effects: scalings

The scaling of h with δ , corresponding to a distinguished balance between dissipation and radiation damping, is more readily argued in the electromagnetic scenario than in the acoustics scenario (cf. §3.3.2 in Chapter 3); in which the electromagnetic boundary layer lies inside the metal plate. Let the scaling of H in a given bulk region be $\mathcal{H}$; then the field $\bar{H}$ in the electromagnetic boundary layer is comparable in magnitude, attenuating into the metal domain on the boundary-layer scale δ . With this construction, the transmission boundary condition (4.2b) together with (4.4) implies a bulk perturbation of order

$$\mathcal{H}' = \delta \Delta_{\perp} \mathcal{H}, \quad (4.6)$$

where $\Delta_{\perp}$ denotes the characteristic length scale of the bulk field normal to the boundary (unity for the exterior regions and h for the slit and transition regions). We assume that, as in the near-resonance lossless analysis in Chapter 2, $\mathcal{H} = h^{-1}$ in the slit region and unity in the exterior and transition regions. Then, with the distinguished scaling $\delta = O(h^2)$, we find $\mathcal{H}' = h^2, h^2$ and h^3 for the exterior, slit and transition regions, respectively.

4.3.2 Asymptotic analysis

The asymptotic theory established in §3.3 and §3.5 generalises to the electromagnetic problem as follows. The asymptotic expansions for the exterior and bulk-slit fields retain the same form as in the acoustic scenario [cf. (3.17), (3.18) and (3.21)], with the magnetic field in place of pressure; in particular, the boundary conditions (3.29) are still of the form

$$\frac{\partial \Phi_2}{\partial X} = \pm \mathcal{C} \Phi_{-1} \quad \text{at} \quad X = \pm 1, \quad (4.7)$$

but with the complex-valued parameter $\mathcal{C}$ now given by

$$\mathcal{C} = \frac{i\bar{\Omega}}{h^2 \epsilon^{1/2}}. \quad (4.8)$$

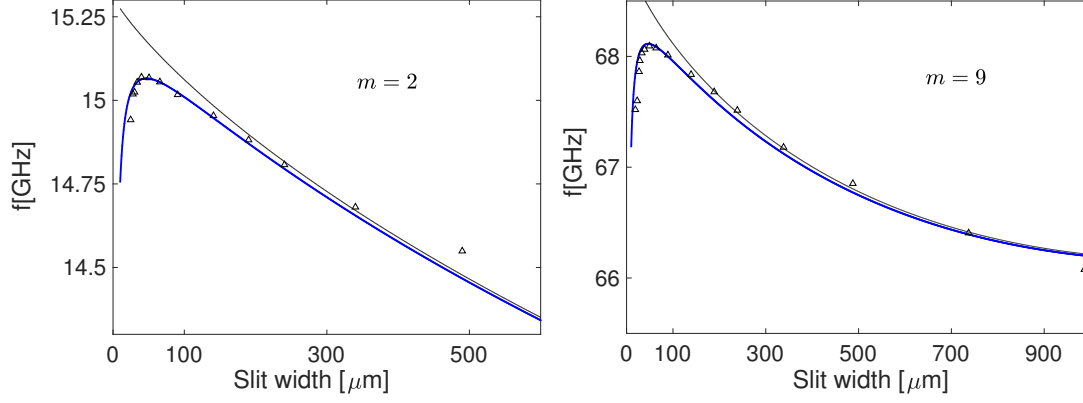


Figure 4.2: Resonance frequencies as a function of slit width for electromagnetic-wave transmission through a single slit in an aluminium plate of thickness $l = 19.58\text{mm}$. Thick blue lines: approximation (4.9) with $\mathcal{C}$ given by (4.8). Dashed black lines: approximation (4.9) for the lossless case $\mathcal{C} = 0$ [48]. Symbols: microwave experiments [112]. The relative permittivity of the metal is $\epsilon \approx i4.2 \times 10^7$ and $\epsilon \approx i4.5 \times 10^6$ for modes $m = 2$ and $m = 9$, respectively.

The leading-order, impedance-like conditions will be derived in §4.4 by analysing the skin layers. In (4.8), ϵ can be treated as constant over the near-resonance frequency interval. In contrast to the acoustic problem, the boundary layers are more simply coupled to the bulk regions, through boundary conditions (4.2) rather than asymptotic matching, in the electromagnetic problem. The results are accordingly of the same form as in Chapter 3, only with $\mathcal{C}$ different.

4.3.3 Comparison with single-slit electromagnetic experiments

We restate the closed-form approximation of resonance frequencies for comparison with experimental data

$$\frac{\Delta f_m}{f_m} \approx \frac{4h}{\pi} (\ln h + \ln(2m) + \gamma_E - 1) - \frac{h}{2\bar{\Omega}^2} \text{Re } \mathcal{C}, \quad (4.9)$$

where Δf_m is the dimensional frequency deviation of the m th resonant peak from the respective standing-wave frequency $f_m = mc/2l$ and $\mathcal{C}$ is now given by (4.8).

In Fig. 4.2, we test (4.9) against GHz-microwave experiments [112], using published data without any fitting. The agreement is excellent, in contrast to the lossless approximation with $\mathcal{C} = 0$ which is not accurate for very narrow slits (below approximately $200\mu\text{m}$ in the microwave experiments).

4.4 Electromagnetic skin layer analyses

Deriving the effective boundary conditions (4.7) is simpler in the electromagnetic scenario than in its acoustic counterpart. We begin by considering the internal field $\bar{H}$ in the electromagnetic skin layers, namely at distances $O(\delta)$ from the slit boundaries. Because of the symmetry of the problem, it suffices to consider the skin layer adjacent to the lower boundary of the slit $x = -h$. Since $\delta = O(h^2)$, we define the strained transverse coordinate $X' = (x + h)/h^2$ and consider the domain $X' < 0$ ($|y| < 1/2$).

Recall the slit-field expansion (3.21) for $H = \Phi$. In particular, the $O(h^{-1})$ enhancement of the slit field suggests the skin-layer expansion

$$\bar{H} \sim h^{-1} \bar{H}_{-1}(X', y) \quad \text{as } h \rightarrow 0. \quad (4.10)$$

The equation governing $\bar{H}_{-1}$ is obtained from the leading-order balance of (4.1b). This gives

$$\frac{\partial^2 \bar{H}_{-1}}{\partial X'^2} + \alpha^2 \bar{\Omega}^2 \bar{H}_{-1} = 0 \quad \text{for } X' < 0, \quad (4.11)$$

where the complex-valued parameter α was defined in (4.5). From the $O(1/h)$ balance of the continuity condition (4.2a), we find the boundary condition

$$\bar{H}_{-1} = \Phi_{-1} \quad \text{at } X' = 0. \quad (4.12)$$

Solving (4.11) subject to (4.12) and the condition that $\bar{H}_{-1}$ decays as $X' \rightarrow -\infty$ gives

$$\bar{H}_{-1} = e^{-i\alpha \bar{\Omega} X'} \Phi_{-1}. \quad (4.13)$$

Consider now the $O(h)$ balance of the transmission condition (4.2b),

$$\left. \frac{\partial \Phi_2}{\partial X} \right|_{X=-1} = \frac{1}{\alpha^2} \left. \frac{\partial \bar{H}_{-1}}{\partial X'} \right|_{X'=0}. \quad (4.14)$$

Substituting (4.13), we obtain the effective boundary condition

$$\frac{\partial \Phi_2}{\partial X} = -\frac{i\bar{\Omega}}{\alpha} \Phi_{-1} \quad \text{at } X = -1. \quad (4.15)$$

The corresponding condition at $X = 1$ follows from symmetry. Using definition (4.5) for α , (4.15) implies the effective boundary condition (4.7) with $\mathcal{C}$ given by (4.8).

4.5 Concluding remarks

We conclude with comments regarding the generalised electromagnetic-acoustic analogy identified in this chapter. First, this analogy is asymptotic, in contrast to the exact analogy between the respective lossless problems. Second, the generalised analogy relies on the findings that (i) boundary-layer effects are only appreciable in the slit region and (ii) the propagation within the bulk of the slit is effectively one-dimensional. This suggests that the analogy could be extended to other slit configurations where boundary-layer effects are important. In general, however, thermoviscous boundary layers and electromagnetic skin layers are described by different types of effective boundary conditions (see here [7, 58] for more general thermoviscous boundary conditions without the assumption of one-dimensional propagation).

CHAPTER 5

Extraordinary transmission through a slender cylindrical hole including boundary-layer effects

5.1 Introduction

In this thesis, we have shown that material losses and dissipative effects significantly hinder the efficient transmission of wave energy through narrow slits. Naturally, this prompts a progression into three-dimensional generalisations. While three-dimensional subwavelength waveguide structures offer greater flexibility in practical fabrication and can realise resonators with higher Q-factors [45, 78, 42, 16, 50], mathematical theories have predominantly focused on two-dimensional set-ups, resulting in comparatively fewer systematic studies of resonances in three dimensions.

The object of study in this chapter is a subwavelength circular hole perforated in a rigid plate of finite thickness, which generates extraordinary enhancement of acoustic transmission at resonant frequencies. This phenomenon has been studied using mode matching [105], an asymptotic analysis of the boundary integral equation (BIE) method [70], and a Fourier-matching method [79]. Although these approaches all predict significant field enhancement at resonance, none have included thermoviscous effects.

In this chapter, our aim is to extend the asymptotic approach developed in earlier chapters to the above-mentioned three-dimensional setting. This method, based on matched asymptotic expansions, has already provided important new physical insights, particularly in facilitating a systematic treatment of losses. The transmission problem for a single slender hole, absent of dissipation, is formulated in §5.2. In §5.3–§5.5 we analyse this lossless problem, and in §5.6 we incorporate thermoviscous effects.

5.2 Transmission problem in the absence of losses

Consider the acoustic transmission problem of a normally incident pressure wave, at fixed angular frequency ω , through a cylindrical hole of radius hl in a rigid plate of thickness l . We first presume that thermoviscous effects are negligible, addressing this later in the chapter. We omit the time dependence $e^{-i\omega t}$, where t is time, in the usual way (see §1.4.3) and adopt a dimensionless convention where lengths are normalised by l , and φ denotes the pressure, normalised by the reference amplitude of the incident wave. Figure 5.1 defines cylindrical coordinates (r, ϕ, z) such that r represents the radial distance, ϕ is the azimuthal angle, and z is the axial distance.

In the fluid domain, the field φ is governed by the Helmholtz equation

$$\nabla^2 \varphi + \Omega^2 \varphi = 0, \quad (5.1)$$

where we define the dimensionless frequency

$$\Omega = \frac{\omega l}{c}, \quad (5.2)$$

with c as the speed of sound; the Neumann boundary condition

$$\frac{\partial \varphi}{\partial n} = 0 \quad (5.3)$$

on the rigid surfaces; and a radiation condition applied to $\varphi - \varphi^{(i)}$ at large distances from the hole, where the incident wave is defined as $\varphi^{(i)} = e^{i\Omega z}$.

Given the axial symmetry of the geometry and transmission problem, it follows that the wave field φ is also axisymmetric, i.e., independent of ϕ . In the following sections, we will study this problem (5.1)-(5.3) in the limit $h \rightarrow 0$.

We shall see that the analysis in the subsequent sections follows a methodology very similar to that in Chapter 2. In particular, the limit $h \rightarrow 0$ is again non-uniform in the frequency, owing to resonances; we shall see, however, that resolving this non-uniformity in three dimensions entails a more intricate set of distinguished frequency intervals, defined by their width around resonance, which we will separately analyse.

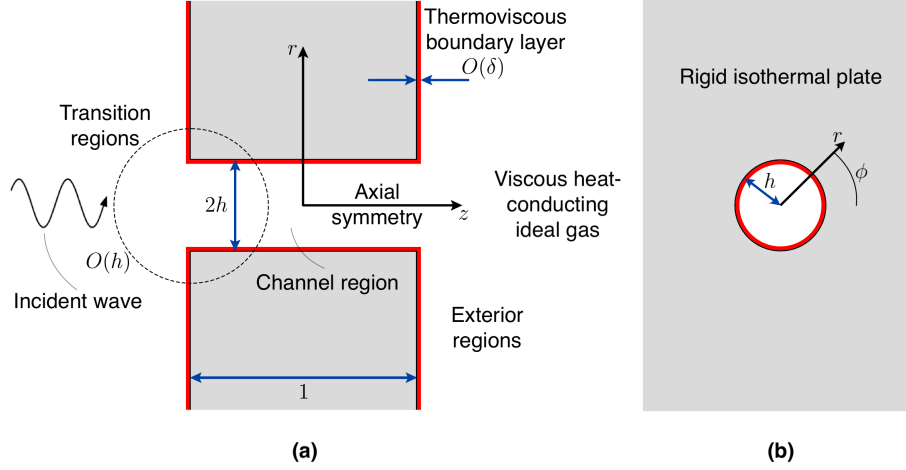


Figure 5.1: Dimensionless schematic of the transmission problem, showing the regions considered in the asymptotic analyses of §5.5 and §5.6.3; the boundary-layer scaling $\delta = O(h^3)$ represents a distinguished limit where boundary-layer and diffraction effects are comparable. (a) Section view. (b) Front view (radially infinite).

5.3 Off-resonance limit

Closely following the approach in Chapter 2, we begin by studying the slender-channel limit as $h \rightarrow 0$ with Ω held fixed (note that Ω being of order unity implies that the wavelength $\simeq$ the plate thickness). We anticipate that this naive limit breaks down as Ω approaches certain critical values. Nonetheless, the analysis is instructive, as the breakdown helps to inform the subsequent systematic studies of near- and on-resonance frequencies.

Analogous to Chapter 2, the analysis involves three asymptotic regions: a channel and two exterior regions (see Figure 5.1(a)), employing the method of matched asymptotic expansions.

5.3.1 Channel region

We start by examining the channel region, corresponding to a limit process where the stretched coordinate

$$R = r/h \quad \text{for} \quad R < 1 \quad (5.4)$$

is held fixed, with $|z| < 1/2$. The channel pressure is accordingly written as $\varphi(r, z) = \Phi(R, z)$, with the problem consisting of

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \Phi}{\partial R} \right) + h^2 \frac{\partial^2 \Phi}{\partial z^2} + h^2 \Omega^2 \Phi = 0, \quad (5.5)$$

the Neumann boundary condition

$$\frac{\partial \Phi}{\partial R} = 0 \quad \text{on} \quad R = 1, \quad (5.6)$$

and asymptotic matching as $z \rightarrow \pm 1/2$.

We assume that the channel pressure is comparable in magnitude to the order-unity incident field, subject to validation through matching. Thus, we propose an expansion in the form

$$\Phi(R, z) = \Phi_0(R, z) + \dots \quad \text{as} \quad h \rightarrow 0, \quad (5.7)$$

where, from the order balances of (5.5) and (5.6), it is easily shown that the leading-order field is independent of R , such that $\Phi_0 = \Phi_0(z)$, and satisfies

$$\frac{d^2 \Phi_0}{dz^2} + \Omega^2 \Phi_0 = 0, \quad (5.8)$$

with solution

$$\Phi_0 = A \cos(\Omega z) + B \sin(\Omega z), \quad (5.9)$$

which is analogous to the leading-order approximation (2.7) in Chapter 2 for the case of a two-dimensional narrow slit.

5.3.2 Exterior regions

Next, we consider the fluid regions exterior to the hole. At sufficiently large distances, the hole openings effectively shrink to points, rendering diffraction from the hole negligible. This suggests we can consider the pressure field left of the substrate, $z < -1/2$, to be

$$\varphi^- \approx e^{i\Omega z} + e^{-i\Omega(1+z)}, \quad (5.10)$$

where the reflection term's coefficient follows from (5.3) and $\approx$, as in previous chapters, denotes a relative asymptotic error of algebraic order. Right of the substrate, $z > 1/2$, there are no incident or reflected waves, only weak diffraction from the hole and the pressure, φ^+ , is accordingly algebraically small.

5.3.3 Leading-order approximation

We match the channel as $z \rightarrow \pm 1/2$ with the limiting exterior expansions to determine the constant solutions

$$A = e^{-\frac{i\Omega}{2}} \sec\left(\frac{\Omega}{2}\right), \quad B = -e^{-\frac{i\Omega}{2}} \csc\left(\frac{\Omega}{2}\right). \quad (5.11)$$

Based on this result, the dimensionless flux density in the z direction, $\partial\varphi/\partial z$, is $O(1)$ in the channel; at the right end of the hole, the corresponding net flux is $\approx \pi h^2 \Phi'_0(1/2)$. This flux is approximately conserved on the small scale of the opening, implying the exterior pressure φ^+ can be written in terms of a Green's function as

$$\varphi^+ \approx h^2 Q \frac{e^{i\Omega s^+}}{s^+}, \quad (5.12)$$

where $s^\pm = \sqrt{r^2 + (z \mp 1/2)^2} > 0$ and

$$Q = \frac{1}{\pi} e^{-\frac{i\Omega}{2}} \csc(\Omega). \quad (5.13)$$

The coefficients A , B and Q [cf. (5.11) and (5.13)], are singular at the Fabry–Pérot frequencies

$$\bar{\Omega} = m\pi, \quad m = 1, 2, \dots, \quad (5.14)$$

as is the leading-order channel pressure Φ_0 . Therefore, this section's analysis is limited to frequencies that are sufficiently far from these critical frequencies. In the subsequent sections, we define the range of validity for the current off-resonance analysis and, for frequencies close to resonance, derive new valid approximations for the diffraction amplitudes and channel pressure.

5.4 Near-resonance limit

The acoustic response of the hole is constrained only by radiation damping in the absence of dissipation. Consequently, the predicted blow-up from the off-resonance analysis in §5.3 arises from the small impact of radiation damping during that limiting process. We can estimate the width of the frequency interval around resonance, where radiation damping becomes significant, by examining the rate at which the off-resonance approximations (5.11) and (5.13) diverge as $\Omega \rightarrow \bar{\Omega}$. Specifically, it can be observed from the off-resonance results that the amplitude of the diffracted exterior waves, proportional to $h^2/(\Omega - \bar{\Omega})$, becomes $O(1)$, meaning that it is comparable to the incident field when $\Omega - \bar{\Omega} = O(h^2)$. Nevertheless, the

off-resonance analysis breaks down in broader frequency intervals, $\Omega - \bar{\Omega} = O(h)$ (near resonance), where the resonance still exhibits singular behaviour. The on-resonance $O(h^2)$ frequency interval will be analysed in §5.5.

To analyse the near-resonance regime, we adopt a modified limiting process; expressing Ω as

$$\Omega = \bar{\Omega} + h\Omega' \quad (5.15)$$

and holding Ω' (and the mode number m , on which $\bar{\Omega}$ depends) fixed as $h \rightarrow 0$. In this limit, we anticipate the diffracted waves in the exterior regions to be $O(h)$, i.e.,

$$\varphi^- \approx e^{i\bar{\Omega}z} + e^{-i\bar{\Omega}(1+z)} + ih\Omega' \left(ze^{i\bar{\Omega}z} - e^{-i\bar{\Omega}(1+z)}(z+1) \right) + hQ^- \frac{e^{i\bar{\Omega}s^-}}{s^-} \quad (5.16)$$

and

$$\varphi^+ \approx hQ^+ \frac{e^{i\bar{\Omega}s^+}}{s^+}, \quad (5.17)$$

where the diffraction coefficients $Q^\pm$ remain to be determined. The pressure in the hole is comparably enhanced,

$$\Phi(R, z) \approx h^{-1}\Phi_{-1}(z) + \Phi_0(z), \quad (5.18)$$

where the uniformity of these leading-order terms is established as discussed in §5.3.1.

The field Φ_{-1} satisfies

$$\frac{d^2\Phi_{-1}}{dz^2} + \bar{\Omega}^2\Phi_{-1} = 0, \quad (5.19)$$

subject to boundary conditions $\Phi_{-1}(\pm 1/2) = 0$, implied from the negligible magnitude of the exterior pressures. The resulting homogeneous problem for $\Phi_{-1}(z)$, by construction, has non-trivial solutions

$$\Phi_{-1}(z) = \mathcal{A}_{-1} \times \begin{Bmatrix} \cos(\bar{\Omega}z) \\ \sin(\bar{\Omega}z) \end{Bmatrix}, \quad (5.20)$$

where $\mathcal{A}_{-1}$ is a complex-valued prefactor and, again, the upper element of the array corresponds to odd m (even standing wave) and the lower to even m (odd standing wave).

Next, it is straightforward to show from the order-unity channel problem that $\Phi_0(z)$ satisfies

$$\frac{d^2\Phi_0}{dz^2} + \bar{\Omega}^2\Phi_0 + 2\bar{\Omega}\Omega'\Phi_{-1} = 0. \quad (5.21)$$

We forgo solving (5.21) explicitly, instead deriving relations between the leading-order amplitude $\mathcal{A}_{-1}$ and end values of Φ_0 , needed for asymptotic matching. We construct a difference equation by subtracting the product of Φ_0 and the complex conjugate of (5.19) from (5.21) times the complex conjugate of Φ_{-1} . Integrating this over the length of the channel yields the solvability condition

$$\left(\Phi_0 \frac{d\Phi_{-1}^*}{dz} \right)_{z=\frac{1}{2}} - \left(\Phi_0 \frac{d\Phi_{-1}^*}{dz} \right)_{z=-\frac{1}{2}} = 2\bar{\Omega}\Omega' \int_{-\frac{1}{2}}^{\frac{1}{2}} |\Phi_{-1}|^2 dz. \quad (5.22)$$

Substituting (5.20) gives

$$\mathcal{A}_{-1} = \frac{i^m}{\Omega'} \times \left\{ \begin{array}{c} i\Phi_0(\frac{1}{2}) + i\Phi_0(-\frac{1}{2}) \\ \Phi_0(\frac{1}{2}) - \Phi_0(-\frac{1}{2}) \end{array} \right\}. \quad (5.23)$$

Relating the approximations derived in the channel and exterior regions remains to be addressed. As in the off-resonance analysis, via a conservation argument, the leading fluxes at the hole ends can be connected with the corresponding diffraction terms in the exterior regions; we thereby find

$$Q^\pm = \frac{i^m \bar{\Omega} \mathcal{A}_{-1}}{2} \times \left\{ \begin{array}{c} -i \\ \mp 1 \end{array} \right\}. \quad (5.24)$$

Unlike in the off-resonance analysis, however, the singularity of the exterior fields as $s^\pm \rightarrow 0$ means that the channel and exterior regions cannot be matched directly.

5.4.1 Transition regions

To connect the channel and exterior regions, we must consider intermediate transition regions at $O(h)$ distances from the hole ends. To analyse these transition regions we introduce stretched coordinates $S^\pm = s^\pm/h$ and $Z^\pm = \pm(z \mp 1/2)/h$ (Z^- grows in the opposite direction to z) and the pressure field $\bar{\Phi}^\pm(S^\pm, Z^\pm)$. The transition regions are defined by the limit $h \rightarrow 0$, while keeping $S^\pm$ and $Z^\pm$ fixed. The limiting geometries correspond to semi-infinite channels ($S^\pm < 1$ and $Z^\pm < 0$) connected to half-spaces $Z^\pm > 0$. It is evident from the form of the channel and exterior expansions that the pressure is $O(1)$ in the transition regions. We therefore write $\varphi \approx \bar{\Phi}_0^\pm(S^\pm, Z^\pm)$, where $\bar{\Phi}_0^\pm$ satisfy Laplace's equation

$$\frac{1}{S^\pm} \frac{\partial}{\partial S^\pm} \left(S^\pm \frac{\partial \bar{\Phi}_0^\pm}{\partial S^\pm} \right) + \frac{\partial^2 \bar{\Phi}_0^\pm}{\partial Z^{\pm 2}} = 0, \quad (5.25)$$

which correspond to the leading-order balances of the Helmholtz equation; the Neumann boundary conditions

$$\frac{\partial \bar{\Phi}_0^\pm}{\partial S^\pm} = 0 \quad \text{at} \quad S^\pm = 1, Z^\pm < 0, \quad \frac{\partial \bar{\Phi}_0^\pm}{\partial Z^\pm} = 0 \quad \text{at} \quad S^\pm > 1, Z^\pm = 0; \quad (5.26a, b)$$

and asymptotic matching with the exterior (5.16)-(5.17) and channel (5.18) regions, which respectively give

$$\bar{\Phi}_0^\pm \sim (1 \mp 1) e^{-\frac{i\bar{\Omega}}{2}} + o(1) \quad \text{as} \quad S^\pm \rightarrow \infty \quad (5.27)$$

and

$$\bar{\Phi}_0^\pm \sim -2Q^\pm Z^\pm + \Phi_0 \left(\pm \frac{1}{2} \right) + o(1) \quad \text{as} \quad Z^\pm \rightarrow -\infty. \quad (5.28)$$

To determine $\mathcal{A}_{-1}$, we require additional relations that connect the constant terms in (5.27) and (5.28). To this end, we take advantage of the fact that $-\bar{\Phi}_0^\pm/2Q^\pm$ solve a similar canonical Laplace problem (uniquely solvable up to an additive constant) where the difference

$$\beta \equiv \lim_{Z^\pm \rightarrow -\infty} \left(-\frac{\bar{\Phi}_0^\pm}{2Q^\pm} - Z^\pm \right) - \lim_{S^\pm \rightarrow \infty} \left(-\frac{\bar{\Phi}_0^\pm}{2Q^\pm} \right) \quad (5.29)$$

is a pure number (≈ -0.8217), which depends only on the geometry of the opening [95, 31, 32, 63, 87, 88, 121, 51, 13]. For completeness, we solve this canonical problem via a mode matching method in Appendix D. Using (5.29) and the boundary conditions (5.27) and (5.28) we derive expressions for the end values of the channel field Φ_0 ,

$$\Phi_0 \left(\pm \frac{1}{2} \right) = (1 \mp 1) e^{-\frac{i\bar{\Omega}}{2}} - 2Q^\pm \beta, \quad (5.30)$$

which facilitates the determination of the amplitude $\mathcal{A}_{-1}$.

5.4.2 Leading-order approximation

Solving (5.23), (5.24) and (5.30) for the amplitude $\mathcal{A}_{-1}$ yields

$$\mathcal{A}_{-1} = \frac{2}{\Omega' - 2\bar{\Omega}\beta} \times \left\{ \begin{array}{c} i \\ -1 \end{array} \right\}, \quad (5.31)$$

which provides a closed-form approximation for the near-resonance frequency response. From this main result, $\mathcal{A}_{-1}$ and, consequently, Φ_{-1} are shown to be

singular at

$$\bar{\Omega}' = 2\bar{\Omega}\beta, \quad (5.32)$$

which corresponds to an $O(h)$ correction to the leading-order resonance frequency (5.14).

5.5 On-resonance limit

The singularity observed in the leading-order approximation (5.31) for the pressure amplitude in the near-resonance limit is unsurprising, given the negligible influence of radiation damping. As a consequence, we now turn our attention to the on-resonance regime, corresponding to a narrower frequency band of $O(h^2)$, centred around the adjusted resonance frequency suggested by our near-resonance analysis.

To examine the on-resonance limit, where we anticipate radiation damping to be appreciable, we express Ω as

$$\Omega = \bar{\Omega} + h\bar{\Omega}' + h^2\Omega'' \quad (5.33)$$

and analyse the limit $h \rightarrow 0$ with Ω'' held fixed. In the on-resonance regime, the expansions for the pressure in the exterior regions become

$$\varphi^- \approx e^{i\bar{\Omega}z} + e^{-i\bar{\Omega}(1+z)} + Q^- \frac{e^{i\bar{\Omega}s^-}}{s^-} \quad (5.34)$$

and

$$\varphi^+ \approx Q^+ \frac{e^{i\bar{\Omega}s^+}}{s^+}, \quad (5.35)$$

where the spherical waves are comparable to the incident and reflected waves. The expansion for the channel pressure is similarly enhanced,

$$\Phi(R, z) \approx h^{-2}\Phi_{-2}(z) + h^{-1}\Phi_{-1}(z) + \Phi_0(z). \quad (5.36)$$

The justification for these leading terms' uniformity constitutes a further generalisation of that presented in §5.3.1. It can be readily demonstrated, as in the near-resonance analysis, that the leading-order term in the channel, $\Phi_{-2}(z)$, exhibits solutions

$$\Phi_{-2}(z) = \mathcal{A}_{-2} \times \left\{ \begin{array}{l} \cos(\bar{\Omega}z) \\ \sin(\bar{\Omega}z) \end{array} \right\}, \quad (5.37)$$

where $\mathcal{A}_{-2}$ is a prefactor. Similarly, the $O(h^{-1})$ channel problem shows that $\Phi_{-1}(z)$ satisfies

$$\frac{d^2\Phi_{-1}}{dz^2} + \bar{\Omega}^2\Phi_{-1} + 2\bar{\Omega}\bar{\Omega}'\Phi_{-2} = 0. \quad (5.38)$$

Employing a reciprocity argument similar to that used in §5.4, we obtain the solvability condition

$$\left(\Phi_{-1}\frac{d\Phi_{-2}^*}{dz}\right)_{z=\frac{1}{2}} - \left(\Phi_{-1}\frac{d\Phi_{-2}^*}{dz}\right)_{z=-\frac{1}{2}} = 2\bar{\Omega}\bar{\Omega}' \int_{-\frac{1}{2}}^{\frac{1}{2}} |\Phi_{-2}|^2 dz. \quad (5.39)$$

Substituting (5.37) yields

$$\mathcal{A}_{-2} = \frac{i^m}{\bar{\Omega}'} \times \left\{ \begin{array}{l} i\Phi_{-1}(\frac{1}{2}) + i\Phi_{-1}(-\frac{1}{2}) \\ \Phi_{-1}(\frac{1}{2}) - \Phi_{-1}(-\frac{1}{2}) \end{array} \right\}. \quad (5.40)$$

In the near-resonance regime, where the frequency detuning is $O(h)$, the leading-order terms capture the system's behaviour, and radiation damping is negligible. There, deriving $\mathcal{A}_{-1}$ from a single solvability condition, based on the end values of Φ_0 , suffices. However, in the on-resonance regime, with frequency deviation scaled as $O(h^2)$, pressure amplitudes are significantly enhanced, and radiation damping becomes pronounced. Thus, another solvability condition is needed here to account for higher-order effects and asymptotic matching between the channel and exterior fields, incorporating the influence of radiation damping on the resonance characteristics.

To proceed, we express the general solution of the forced equation (5.38) as

$$\Phi_{-1}(z) = \mathcal{A}_{-1} \left\{ \begin{array}{l} \cos(\bar{\Omega}z) \\ \sin(\bar{\Omega}z) \end{array} \right\} + \mathcal{B}_{-1} \left\{ \begin{array}{l} \sin(\bar{\Omega}z) \\ \cos(\bar{\Omega}z) \end{array} \right\} + \mathcal{A}_{-2}\bar{\Omega}'z \left\{ \begin{array}{l} -\sin(\bar{\Omega}z) \\ \cos(\bar{\Omega}z) \end{array} \right\}, \quad (5.41)$$

where $\mathcal{A}_{-1}$ and $\mathcal{B}_{-1}$ are constant coefficients. Then, extending the channel region analysis to one further algebraic order, Φ_0 is shown to satisfy

$$\frac{d^2\Phi_0}{dz^2} + \bar{\Omega}^2\Phi_0 + 2\bar{\Omega}\bar{\Omega}'\Phi_{-1} + 2\bar{\Omega}\bar{\Omega}''\Phi_{-2} + \bar{\Omega}'^2\Phi_{-2} = 0, \quad (5.42)$$

and applying a reciprocity argument once more yields a further solvability condition

$$\left(\Phi_0\frac{d\Phi_{-2}^*}{dz}\right)_{z=\frac{1}{2}} - \left(\Phi_0\frac{d\Phi_{-2}^*}{dz}\right)_{z=-\frac{1}{2}} = 2\bar{\Omega}\bar{\Omega}' \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_{-1}\Phi_{-2}^* dz + (2\bar{\Omega}\bar{\Omega}'' + \bar{\Omega}'^2) \int_{-\frac{1}{2}}^{\frac{1}{2}} |\Phi_{-2}|^2 dz. \quad (5.43)$$

Substituting (5.37) and (5.41), we find

$$\mathcal{A}_{-2} = \frac{i^m}{\Omega''} \times \left\{ \begin{array}{c} i\Phi_0(\frac{1}{2}) + i\Phi_0(-\frac{1}{2}) \\ \Phi_0(\frac{1}{2}) - \Phi_0(-\frac{1}{2}) \end{array} \right\} - \frac{\bar{\Omega}'}{\Omega''} \mathcal{A}_{-1}. \quad (5.44)$$

We then relate the approximations obtained for the channel and exterior regions. The leading-order fluxes can be directly connected via flux conservation, to give

$$Q^\pm = \frac{i^m \bar{\Omega} \mathcal{A}_{-2}}{2} \times \left\{ \begin{array}{c} -i \\ \mp 1 \end{array} \right\}, \quad (5.45)$$

but, otherwise, the singularity of the leading-order exterior fields prevents direct matching of the regions.

5.5.1 Transition regions

Similar to the near-resonance analysis, we must consider transition regions where the field enhancement is now $O(h^{-1})$. Given (5.44), we must also consider an additional order:

$$\varphi \approx h^{-1} \bar{\Phi}_{-1}^\pm(S^\pm, Z^\pm) + \bar{\Phi}_0^\pm(S^\pm, Z^\pm), \quad (5.46)$$

where the two leading fields are both governed by Laplace's equation (5.25) along with the Neumann conditions (5.26) (with $\bar{\Phi}_{-1}^\pm$ replacing $\bar{\Phi}_0^\pm$) and matching conditions

$$\bar{\Phi}_{-1}^\pm \sim o(1/S^\pm), \quad \bar{\Phi}_0^\pm \sim (1 \mp 1)e^{-\frac{i\Omega}{2}} + iQ^\pm \bar{\Omega} + o(1) \quad \text{as } S^\pm \rightarrow \infty \quad (5.47a, b)$$

and

$$\bar{\Phi}_{-1}^\pm \sim -2Q^\pm Z^\pm + \Phi_{-1}\left(\pm \frac{1}{2}\right) + o(h^{-1}), \quad (5.48)$$

$$\bar{\Phi}_0^\pm \sim \pm \left(\frac{d\Phi_{-1}}{dz} \right)_{\pm \frac{1}{2}} Z^\pm + \Phi_0\left(\pm \frac{1}{2}\right) + o(1) \quad \text{as } Z^\pm \rightarrow -\infty, \quad (5.49)$$

where

$$\pm \left(\frac{d\Phi_{-1}}{dz} \right)_{\pm \frac{1}{2}} = i^m \left(\mathcal{A}_{-1} \bar{\Omega} + \mathcal{A}_{-2} \bar{\Omega}' \right) \times \left\{ \begin{array}{c} i \\ \pm 1 \end{array} \right\} \quad (5.50)$$

and $Z^{\pm 2}$ is not a solution of Laplace (the two orders constitute the conservation argument used to derive (5.45)). As equivalently shown in the near-resonance analysis, the Laplace problems for $\bar{\Phi}_{-1}^\pm$ and $\bar{\Phi}_0^\pm$ are uniquely solvable, up to an arbitrary additive constant, representing the differences in the constant terms in far-field conditions (5.47), (5.48) and (5.49). By exploiting the canonical problem in

Appendix D again, we derive expressions for the end values of the channel fields Φ_{-1} and Φ_0 ,

$$\Phi_{-1} \left(\pm \frac{1}{2} \right) = -2Q^\pm \beta \quad (5.51)$$

and

$$\Phi_0 \left(\pm \frac{1}{2} \right) = i^m \beta \left(\mathcal{A}_{-1} \bar{\Omega} + \mathcal{A}_{-2} \bar{\Omega}' \right) \times \left\{ \begin{array}{c} i \\ \pm 1 \end{array} \right\} + (1 \mp 1) e^{-\frac{i\bar{\Omega}}{2}} + iQ^\pm \bar{\Omega}. \quad (5.52)$$

5.5.2 Leading-order approximation

Solving the system of equations (5.40), (5.44), (5.45), (5.51) and (5.52), we find the complex amplitude of the channel wave as

$$\mathcal{A}_{-2} = \frac{2}{\Omega'' - 2\beta \bar{\Omega}' + i\bar{\Omega}^2} \times \left\{ \begin{array}{c} i \\ -1 \end{array} \right\}. \quad (5.53)$$

In contrast to the off- and near-resonance limits, the diffraction amplitudes $Q^\pm$ and channel pressure remain finite at resonance, due to the presence of the imaginary term in the denominator of (5.53), representing radiation damping. It follows from (5.53) that the frequency deviations Δf , of the resonant peaks from the Fabry–Pérot frequencies f_m [cf. (5.14)], are given as

$$\frac{\Delta f}{f_m} \approx 2h\beta + 4h^2\beta^2. \quad (5.54)$$

5.6 Thermoviscous dissipation effects

5.6.1 Thermoviscous model

In the preceding sections of this chapter, we focused on analysing the lossless formulation in §5.2, where the only damping mechanism was energy leakage through radiation. As revealed in §5.5, this weak mechanism results in sharp resonances of $O(h^{-2})$ within a frequency interval of $O(h^2)$. However, it is important to note (at this stage) that often dissipation is dominant or at least comparable to radiation damping. In this section, we will incorporate thermoviscous dissipation effects, building on the lossless results from the previous sections. The analysis will closely follow the methodology outlined in Chapter 3.

The physical problem is as described in §5.2, where a normally incident wave interacts with a single hole embedded in a rigid substrate. We now include thermoviscous effects according to the classical model outlined in §1.4 and used in Chapter 3. Assuming the fluid is an ideal gas, losses are attributed to viscous and thermal dissipation. The governing equations are the continuity, momentum, energy and state equations linearised about an equilibrium state of density ρ_0 , viscosity η , specific heat capacity at constant pressure c_p and heat conductivity κ [90]. In the extended model, the pressure p , temperature perturbation T , and velocity field $\mathbf{u}$ are defined as dimensionless time-harmonic fields, normalised respectively by p_∞ , $p_\infty/(\rho_0 c_p)$ and $p_\infty/(\omega \rho_0 l)$.

The fields p , T and $\mathbf{u}$ satisfy the continuity equation

$$-i\Omega^2 p - i(\gamma - 1)\Omega^2(p - T) + \nabla \cdot \mathbf{u} = 0, \quad (5.55)$$

the momentum equation

$$-i\mathbf{u} = -\nabla p + \delta^2 \left[\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right] \quad (5.56)$$

and the energy equation

$$i(p - T) = \text{Pr}^{-1} \delta^2 \nabla^2 T \quad (5.57)$$

(see §1.4.3). We also define the dimensionless viscous length scale

$$\delta = \frac{l_v}{l}, \quad (5.58)$$

along with the Prandtl number $\text{Pr} = (l_v/l_t)^2$, in which $l_v = \sqrt{\eta/\rho_0 \omega}$ and $l_t = \sqrt{\kappa/\rho_0 \omega c_p}$ are dimensional viscous and thermal length scales, respectively. The adiabatic index is denoted by γ . For air, $\gamma \approx 1.4$ and $\text{Pr} \approx 0.71$.

The above thermoviscous equations are supplemented by boundary conditions. On rigid boundaries, the velocity satisfies the no-slip condition

$$\mathbf{u} = 0, \quad (5.59)$$

while the temperature deviation satisfies the isothermal condition

$$T = 0. \quad (5.60)$$

Next, we prescribe the incident wave. While conventional choices, such as plane waves, may exhibit decay, the subsequent asymptotic analysis will show that the

slow exponential attenuation due to thermoviscous effects is negligible in the exterior bulk regions. Therefore, in the limit $\delta \rightarrow 0$, the usual radiation condition in the exterior regions and incident wave can be employed.

5.6.2 Thin boundary layers

The thin-boundary-layer limit, $\delta \ll 1$, is of particular interest, as it results in singular perturbations of the momentum (5.56) and energy (5.57) equations, leading to the formation of oscillatory thermal and viscous boundary layers at the rigid boundary. The tangential velocity and temperature undergo rapid attenuation from their respective bulk values within these boundary layers to satisfy the boundary conditions (5.59) and (5.60). The thickness of these boundary layers, of $O(\delta)$, is considered thin in comparison to the smallest geometric feature, specifically the radius of the hole, depicted in Figure 5.1(b).

We apply the matched-asymptotic-expansions formalism established in the lossless analysis, retaining the definitions of the exterior, channel, and transition regions, to study the thin-boundary-layer limit. Additionally, the exterior and channel regions must now be considered as bulk regions, since a boundary layer forms adjacent to the rigid surface. Within the boundary-layer fields, the exact boundary conditions (5.59) and (5.60) are enforced, whereas the bulk fields satisfy effective boundary conditions derived through matching with the boundary layer.

Let us estimate the magnitude of bulk-pressure perturbations caused by the presence of thermal and viscous boundary layers. We shall perform a scaling analysis of the thermoviscous model, similar to that in Chapter 3, but applied to the current three-dimensional scenario. The tangential component of the velocity field varies across the boundary layer, from the bulk value, of order $\mathcal{U}$, say, to zero at the boundary; the pressure, of order $\mathcal{P}$, say, is approximately uniform across the boundary layer. Furthermore, the temperature varies from its bulk value, on the order of $\mathcal{P}$, to zero on the boundary. With this construction, the continuity equation (5.55) implies bulk perturbations of the *normal* velocity component of order $\mathcal{U}'_V = (\delta/\Delta_{\parallel})\mathcal{U}$ and $\mathcal{U}'_T = \delta\mathcal{P}$ owing to viscous and thermal effects, respectively, wherein $\Delta_{\parallel}$ is the characteristic scale of the bulk field parallel to the boundary (unity for the exterior and channel regions and h for the transition regions). Given that the velocity scales like the pressure gradient in the bulk, we have $\mathcal{U} = \mathcal{P}/\Delta_{\parallel}$ and $\mathcal{U}'_{V/T} = \mathcal{P}'_{V/T}/\Delta_{\perp}$, where $\mathcal{P}'_V$ and $\mathcal{P}'_T$ are the orders of the bulk pressure perturbations owing to viscous and thermal effects, respectively. Here, $\Delta_{\perp}$ is the characteristic scale of the bulk field perpendicular to the boundary, which is unity in the exterior regions and h in the channel and transition regions.

We therefore find the estimates

$$\mathcal{P}'_V = \frac{\delta \Delta_\perp}{\Delta_\parallel^2} \mathcal{P} \quad \text{and} \quad \mathcal{P}'_T = \delta \Delta_\perp \mathcal{P}. \quad (5.61)$$

With this, we can now derive the distinguished limit at which boundary-layer effects first become important. From the analysis of the idealised problem, we observed that $O(h^2)$ perturbations in the channel region affect its leading $O(h^{-2})$ field. Using this scaling analysis and assuming the ideal scalings from §5.5 ($\mathcal{P} = h^{-2}$ in the channel region, h^{-1} in the transition regions and unity in the exterior regions), we find that $O(h^2)$ viscous and thermal bulk channel perturbations emerge when $\delta = O(h^3)$. For this scaling, thermoviscous perturbations in the exterior and transition bulk regions are not appreciable.

5.6.3 Distinguished limit: resonance limited by both radiation damping and loss

In this section, we explore the distinguished limit where $\delta = O(h^3)$. In this regime, both radiation damping and dissipation within the channel region have comparable impact (critically coupled) in limiting resonance. It is sufficient to examine this limit within the on-resonance frequency interval $\Omega - \bar{\Omega} - h\bar{\Omega}' = O(h^2)$, because it can be readily demonstrated that the lossless analysis results are preserved for both the off- and near-resonance intervals. To facilitate our analysis, we introduce the dimensionless boundary-layer thickness defined as

$$\delta = h^3 \delta', \quad (5.62)$$

and consider the limit $h \rightarrow 0$, while fixing both δ' and Ω'' [cf. (5.33)].

As indicated by our scaling analysis in §5.6.2, the distinction from the lossless analysis lies in the boundary condition applied to the bulk channel pressure at the last order of analysis. This condition must be derived by matching the thermoviscous boundary layer region with the bulk channel region (the exterior and transition regions remain unchanged for the orders involved, even when considering higher orders in the transition regions). An analogous procedure was carried out in Chapter 3 in two dimensions. Similar to the results from that chapter, we find that Φ_{-2} , Φ_{-1} , Φ_0 and Φ_1 satisfy the usual homogeneous Neumann boundary condition. In contrast, Φ_2 satisfies an inhomogeneous boundary condition that we will derive.

We expand the description of the bulk channel region from the single field $p = \Phi$ to include the normalised flow and temperature fields, $\mathbf{u} = u_R \hat{\mathbf{e}}_R + u_\phi \hat{\mathbf{e}}_\phi + u_z \hat{\mathbf{e}}_z$ and T , respectively. To this end, we rewrite the channel expansion (5.36) in the form

$$p = h^{-2}\Phi_{-2}(z) + h^{-1}\Phi_{-1}(z) + \Phi_0(z) + h\Phi_1(z) + h^2\Phi_2(R, z) + \dots \quad \text{as } h \rightarrow 0. \quad (5.63)$$

Note the independence of the leading terms is consistent with the boundary-layer analysis here. Given (5.63), the momentum equation (5.56) implies the expansions

$$u_R \sim -ih \frac{\partial \Phi_2}{\partial R}, \quad u_z \sim -ih^{-2} \frac{d\Phi_{-2}}{dz} \quad \text{as } h \rightarrow 0. \quad (5.64a, b)$$

Similarly, the energy equation (5.57) implies the temperature expansion

$$T \sim h^{-2}\Phi_{-2} \quad \text{as } h \rightarrow 0. \quad (5.65)$$

Consider now the $O(\delta)$ boundary layer adjacent to the channel boundary $r = h$. Since $\delta = O(h^3)$, we define the transverse coordinate $R' = (r - h)/h^3$, with the boundary-layer domain being $R' < 0$ ($|z| < 1/2$). The channel-region expansions (5.63)–(5.65) suggest expanding the boundary layer fields as

$$p \sim h^{-2}P'_{-2}(R', z), \quad T \sim h^{-2}T'_{-2}(R', z), \quad u_R \sim hU'_1(R', z), \quad u_z \sim h^{-2}V'_{-2}(R', z). \quad (5.66a-d)$$

The leading $O(h^{-5})$ balance of the R component of the momentum equation (5.56) gives

$$\frac{\partial P'_{-2}}{\partial R'} = 0, \quad (5.67)$$

showing that P'_{-2} is independent of R' . With this, straightforward matching of the boundary layer and channel-region pressure fields yields

$$P'_{-2} = \Phi_{-2}(z). \quad (5.68)$$

Using (5.68), the $O(h^{-2})$ balance of the z component of the momentum equation (5.56) can be written as

$$\delta'^2 \frac{\partial^2 V'_{-2}}{\partial R'^2} + iV'_{-2} = \frac{d\Phi_{-2}}{dz}. \quad (5.69)$$

The no-slip condition (5.59) gives the boundary condition

$$V'_{-2} = 0 \quad \text{at } R' = 0. \quad (5.70)$$

Furthermore, matching the u_z velocity component yields the far-field condition

$$V'_{-2} \rightarrow -i \frac{d\Phi_{-2}}{dz} \quad \text{as } R' \rightarrow -\infty. \quad (5.71)$$

Solving (5.69) subject to (5.70) and (5.71), we find

$$V'_{-2} = i \left(e^{\frac{1-i}{\sqrt{2}} \frac{R'}{\delta'}} - 1 \right) \frac{d\Phi_{-2}}{dz}. \quad (5.72)$$

The leading $O(h^{-2})$ balance of the energy equation (5.57), with (5.68), reads

$$\text{Pr}^{-1} \delta'^2 \frac{\partial^2 T'_{-2}}{\partial R'^2} + iT'_{-2} = i\Phi_{-2}. \quad (5.73)$$

From (5.60), we find the boundary condition

$$T'_{-2} = 0. \quad (5.74)$$

Furthermore, leading-order matching of the temperature field gives the far-field condition

$$T'_{-2} \rightarrow \Phi_{-2} \quad \text{as } R' \rightarrow -\infty. \quad (5.75)$$

Solving (5.73) subject to (5.74) and (5.75), we find

$$T'_{-2} = \left(1 - e^{\frac{1-i}{\sqrt{2}} \frac{R' \sqrt{\text{Pr}}}{\delta'}} \right) \Phi_{-2}. \quad (5.76)$$

Consider now the leading $O(h^{-2})$ balance of the continuity equation (5.55),

$$\frac{\partial U'_1}{\partial R'} = -\frac{\partial V'_{-2}}{\partial z} + i\bar{\Omega}^2 \Phi_{-2} + i(\gamma - 1)\bar{\Omega}^2 (\Phi_{-2} - T'_{-2}), \quad (5.77)$$

which is supplemented by the boundary condition [cf. (5.59)]

$$U'_1 = 0 \quad \text{at } R' = 0. \quad (5.78)$$

Integrating (5.77) using (5.78), followed by taking the limit $R' \rightarrow -\infty$ using the channel-region leading-order Helmholtz equation, yields

$$U'_1 \rightarrow \delta' \bar{\Omega}^2 \frac{1-i}{\sqrt{2}} \left(1 + \frac{\gamma-1}{\sqrt{\text{Pr}}} \right) \Phi_{-2} \quad \text{as } R' \rightarrow -\infty. \quad (5.79)$$

Matching the velocity component u_r then furnishes the effective boundary condition

$$\frac{\partial \Phi_2}{\partial R} = \mathcal{C} \Phi_{-2} \quad \text{at } R = 1, \quad (5.80)$$

where

$$\mathcal{C} = \frac{\delta \bar{\Omega}^2}{h^3} \frac{1+i}{\sqrt{2}} \left(1 + \frac{\gamma-1}{\sqrt{\text{Pr}}} \right), \quad (5.81)$$

which is written in a form analogous to Chapters 3 and 4. We emphasise again that the effective condition (5.80) relies on the one-dimensional nature of the wave propagation in the bulk channel region and the related independence of its four leading terms on the radial coordinate R .

It follows that this boundary condition enters in the solvability condition (5.43) as

$$\left[\Phi_0 \frac{d\Phi_{-2}^*}{dz} \right]_{z=-\frac{1}{2}}^{z=\frac{1}{2}} = 2\bar{\Omega}\bar{\Omega}' \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_{-1} \Phi_{-2}^* dz + \left(2\bar{\Omega}\Omega'' + \bar{\Omega}'^2 + 2\mathcal{C} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} |\Phi_{-2}|^2 dz. \quad (5.82)$$

As in the lossless on-resonance analysis, we can now find the leading-order channel pressure amplitude as

$$\mathcal{A}_{-2} = \frac{2}{\Omega'' - 2\beta\bar{\Omega}' + i\bar{\Omega}^2 + \bar{\Omega}^{-1}\mathcal{C}} \times \begin{Bmatrix} i \\ -1 \end{Bmatrix}. \quad (5.83)$$

The resonance shifts from Fabry–Pérot follow as

$$\frac{\Delta f}{f_m} \approx 2h\beta + 4h^2\beta^2 - \frac{h^2}{\bar{\Omega}^2} \text{Re}(\mathcal{C}). \quad (5.84)$$

Expressions (5.83) and (5.84) generalise the corresponding lossless results (5.53) and (5.54), serving as the three-dimensional counterparts to (3.39) and (3.40).

5.7 Concluding remarks

We have extended our asymptotic theory from narrow slits to slender holes to describe radiation- and dissipation-limited resonant acoustic transmission. To leading order, the approximations for resonance shifts in all weakly damped regimes — where the channel pressure field remains asymptotically large — can be obtained by extrapolating (5.84). In the regime where $\delta \ll h^3$, resonance is radiation limited. Using the scalings discussed in §5.6.2, we find that the leading-order lossless results in §5.5 still apply. In the distinguished limit $\delta = O(h^3)$, as considered in §5.6.3, resonance occurs under the critical-coupling condition, where radiation and dissipation damping are balanced in the on-resonance frequency interval. Conversely, in the regime where $\delta \gg h^3$, resonance becomes dissipation limited. Notably, there is another distinguished limit, $\delta = O(h^2)$, where the near-resonance and on-resonance frequency intervals converge. Revisiting the

near-resonance analysis of §5.4, with thermoviscous effects in the channel entering at $O(h^2)$ as described in §5.6.3, we find in that regime

$$\frac{\Delta f}{f_m} \approx 2h\beta - \frac{h}{\bar{\Omega}^2} \text{Re}(\mathcal{C}). \quad (5.85)$$

While the systematic treatment in two and three dimensions share similarities in their asymptotic approach and inclusion of losses, several distinctions emerge: (i) In three dimensions, inviscid radiation leakage is weaker, manifesting in h^2 scalings, yet still giving an $O(h)$ correction that results in nested frequency intervals. These narrower frequency intervals around resonances correspond to more pronounced enhancement of channel pressure. (ii) With losses, this means less dissipation is required to balance the reduced radiation leakage, modifying the dissipation scaling limit and revealing greater sensitivity to boundary-layer thickness. (iii) The absence of logarithmic behaviour in three dimensions can be attributed to energy dispersal over spherical surfaces, leading to a $1/r$ pressure decay. In contrast, cylindrical wave propagation in two dimensions exhibits slower attenuation, requiring logarithmic terms in the solution. (iv) End corrections are particularly effective in three dimensions, whereas in two dimensions, they become problematic due to their dependence on mode number beyond the leading logarithmic order [cf. §2.5]. (v) In three dimensions, no direct analogy exists between acoustics and electromagnetics, even in a lossless scenario. This results from intrinsic differences in field characteristics and boundary conditions — acoustics is governed by scalar pressure fields, while electromagnetics requires vector field solutions.

As far as we are aware, no acoustic transmission experiments through cylindrical holes, comparable to the slit-based studies by Ward *et al.* [119], have been conducted for comparison, and previous theoretical results [70, 79] remain non-explicit due to the presence of uncalculated constants, as discussed in Chapter 1. Direct generalisations of our model would include non-cylindrical hole geometries, such as elliptical and rectangular, and configurations formed of multiple hole-resonators, which will be explored in a subsequent chapter.

CHAPTER 6

Complex resonances of a slender cylindrical hole and their excitation

6.1 Introduction

In this chapter, we revisit the acoustic analysis of a cylindrical hole embedded in a rigid slab from Chapter 5, which focused on a specific resonant scattering problem with a wave incident from one side. Here, we aim to study the acoustic resonant response and properties of the configuration more generally. To achieve this, we adopt a spectral approach [2, 35], a framework well-established in wave physics due to its versatility in analysing scattering across various systems. Following Schnitzer and Porter [103], who employed this approach to study the asymptotic nature of quasi-bound states in the continuum (quasi-BIC) of a partially partitioned slit, we begin by analysing an eigenvalue problem governing the natural resonant modes of the system. We then examine how these modes are excited through resonant coupling as the physical frequency approaches the system's natural resonances. In contrast to Schnitzer and Porter's inviscid analysis, we incorporate dissipation effects.

The spectral approach overcomes limitations of the 'direct' asymptotic approach employed in Chapter 5, which involves two small parameters — the hole radius and shift of the frequency about resonance. In the direct method, the slender-hole limit is non-uniform in frequency, necessitating separate analyses for different frequency regimes (on, near, and off resonance). In the spectral approach, this non-uniformity is addressed by splitting the problem into two parts. In the eigenvalue problem, the frequency is treated not as an input parameter but as an emergent output of the problem — namely, the eigenvalue. As such, the eigenvalue problem involves only the normalised hole width as a small parameter. While non-uniformity is

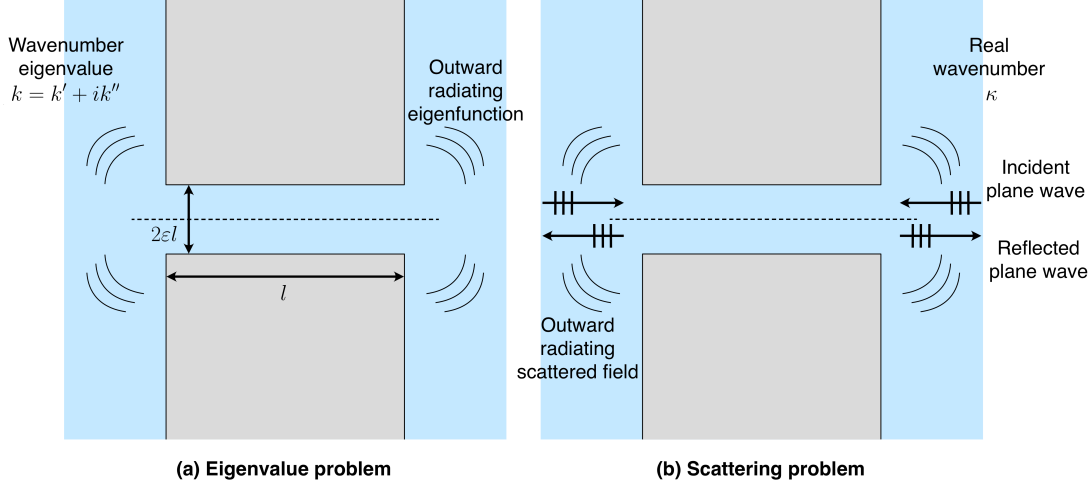


Figure 6.1: (a) Eigenvalue problem governing the modes of a slender cylindrical hole connected to half-spaces. (b) Associated scattering problem with incident plane waves.

absent in the eigenvalue problem itself, it is readily addressed by analysing, at an algebraic level, which eigensolutions are excited, and with what magnitude in a given scattering problem. Our analysis focuses on the low-frequency regime, where the wavelength is comparable to the slab thickness, and accordingly restricts excitation to modes that are uniform in the cross-section. Due to the azimuthal symmetry of the hole and the normal incidence of incoming waves, these modes are effectively axisymmetric.

The asymptotic analysis is performed in two stages. In §6.2, we first address the single-parameter eigenvalue problem, in which no incident field is present, to determine the natural modes of the system. We employ singular perturbation methods [47], similar to previous chapters, but now to derive approximations for wavenumber eigenvalues and eigenfunctions. In §6.3, we investigate the resonant excitation of these modes by analysing the associated scattering problem, with a real wavenumber and incoming normal plane waves present on either side of the slab, as depicted in Figure 6.1. We establish a reciprocal relation, accounting for dissipative effects, which relates the eigensolutions to the scattering problem. This reciprocal relation enables us to find asymptotic approximations that describe the resonant excitation phenomena and coupling dynamics between incident waves and eigenmodes.

6.2 Eigenvalue problem

6.2.1 Problem formulation

We normalise lengths by l and express the wavenumber as k/l , where $k = k' + ik''$ is a complex-valued and dimensionless rescaled wavenumber, serving as the eigenvalue. Figure 6.2 depicts the dimensionless geometry (the hole's radius-to-length ratio $\varepsilon \equiv h$ in Chapter 5) and defines cylindrical coordinates (r, ϕ, z) . The time dependence e^{-ikt/c^l} , where t is time and c is the speed of sound, is suppressed in the usual way (see §1.4.3). In the absence of an energy source, we impose $k'' \leq 0$, preventing the outward-radiating wave field growing over time. Without loss of generality, the eigenfunction u , determined up to a multiplicative constant, is assumed to be dimensionless.

With this convention, the reduced wave field is governed by the Helmholtz equation

$$\nabla^2 u + k^2 u = 0, \quad (6.1)$$

the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad (6.2)$$

and an outward-radiation condition as $r \rightarrow \infty$.

In the following subsections, we study this eigenvalue problem in the limit $\varepsilon \rightarrow 0$, focusing on axisymmetric eigenfunctions.

6.2.2 Channel

In the channel region, we take the eigenfunction u to be of order unity, without loss of generality. To analyse this region, we introduce stretched coordinates $r = \varepsilon R$ and define $u(r, z; \varepsilon) = U(R, z; \varepsilon)$, with $-1/2 < z < 1/2$ and $0 < R < 1$. The eigenfunction U satisfies

$$\mathcal{L}_R^2 U + \varepsilon^2 \left(\frac{\partial^2 U}{\partial z^2} + k^2 U \right) = 0, \quad (6.3)$$

where

$$\mathcal{L}_R^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} \quad (6.4)$$

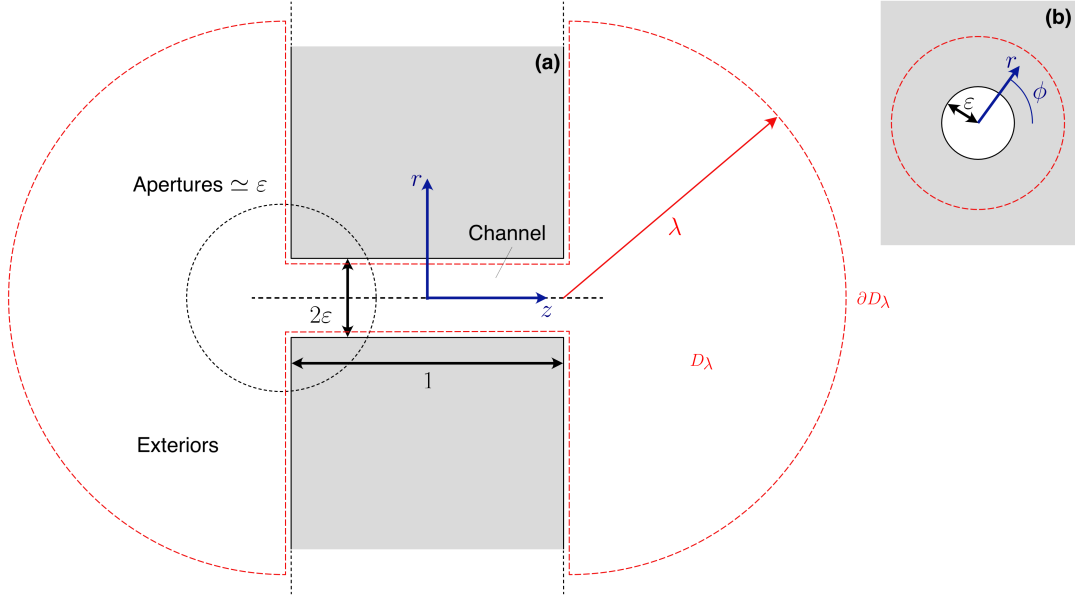


Figure 6.2: Dimensionless sketch of the geometry and asymptotic regions in the limit $\varepsilon \rightarrow 0$. (a) Section view. (b) Front view (radially infinite).

is the radial part of the Laplacian due to the cylindrical symmetry of the problem, and

$$\frac{\partial U}{\partial R} = 0 \quad \text{at} \quad R = 1. \quad (6.5)$$

The conditions at $z \rightarrow \pm 1/2$ are to be derived by asymptotically matching with the aperture regions.

The wavenumber expansion is

$$k = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \dots \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (6.6)$$

Similarly, the eigenfunction is expanded as

$$U(R, z; \varepsilon) = U_0(R, z) + \varepsilon U_1(R, z) + \varepsilon^2 U_2(R, z) + \dots \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (6.7)$$

At orders unity and ε , integrating (6.3) with respect to R and using the respective orders of (6.5), we find $U_0(R, z) = U_0(z)$ and $U_1(R, z) = U_1(z)$. At order ε^2 , integrating (6.3) with respect to R and applying the Neumann condition (6.5) at the same order, we obtain $U_2(R, z) = U_2(z)$ and the one-dimensional Helmholtz equation

$$\frac{d^2 U_0}{dz^2} + k_0^2 U_0 = 0. \quad (6.8)$$

The boundary conditions are determined by anticipating that the eigenfunction becomes asymptotically small within the apertures. This implies the matching

conditions

$$U_0 = 0 \quad \text{at} \quad z = \pm \frac{1}{2}. \quad (6.9)$$

Solving the eigenvalue problem (6.8)-(6.9) yields the leading-order eigenvalues,

$$k_0 = n\pi, \quad n = 1, 2, 3, \dots, \quad (6.10)$$

and corresponding even (n odd) and odd (n even) eigenfunctions, respectively,

$$U_0 = \begin{cases} \cos(k_0 z) \\ \sin(k_0 z) \end{cases}, \quad (6.11)$$

where we set the constant of proportionality to be unity without loss of generality.

The symmetry of the problem setup about the $z = 0$ plane allows consideration of two additional configurations. By restricting the domain to $z > 0$, we can consider either Neumann or Dirichlet conditions at $z = 0$. These configurations correspond to a cylindrical channel opening at one end onto a half-space, where the boundary conditions are rigid except at the closed end, which may be hard (Neumann) or soft (Dirichlet) – thereby selecting only even (symmetric) or odd (antisymmetric) eigenfunctions in z , respectively.

6.2.3 Apertures

We next turn our attention to the aperture regions that connect the channel and exterior regions. To analyse these transition regions, we employ strained coordinates $S^\pm = s^\pm/\varepsilon$, where $s^\pm = \sqrt{r^2 + (z \mp 1/2)^2} > 0$, and $Z^\pm = \pm(z \mp 1/2)/\varepsilon$, where Z^- grows in the opposite direction to z , and define $u(r, z; \varepsilon) = V(S, Z; \varepsilon)$. The eigenfunctions $V^\pm$ satisfy

$$\mathcal{L}_{S^\pm}^2 V^\pm + \frac{\partial^2 V^\pm}{\partial Z^{\pm 2}} + \varepsilon^2 k^2 V^\pm = 0, \quad (6.12)$$

the Neumann boundary condition

$$\frac{\partial V^\pm}{\partial N} = 0, \quad (6.13)$$

where $\partial/\partial N = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_r \partial/\partial S^\pm + \hat{\mathbf{e}}_z \partial/\partial Z^\pm)$, and far-field conditions obtained from matching with both the channel and exterior regions.

The limiting behaviour of U at the apertures suggests $V^\pm \simeq \varepsilon$, where $\simeq$ means 'of asymptotic order'. Accordingly, $V^\pm$ are expanded as

$$V^\pm(S^\pm, Z^\pm; \varepsilon) = \varepsilon V_0^\pm(S^\pm, Z^\pm) + \varepsilon^2 V_1^\pm(S^\pm, Z^\pm) + \dots \quad \text{as } \varepsilon \rightarrow 0. \quad (6.14)$$

From (6.12) and (6.13), $V_0^\pm$ satisfy Laplace's equation

$$\mathcal{L}_{S^\pm}^2 V_0^\pm + \frac{\partial^2 V_0^\pm}{\partial Z^{\pm 2}} = 0 \quad (6.15)$$

and the Neumann condition

$$\frac{\partial V_0^\pm}{\partial N} = 0. \quad (6.16)$$

The expected decay towards the exterior regions implies

$$V_0^\pm \rightarrow 0 \quad \text{as } S^\pm \rightarrow \infty, \quad (6.17)$$

whereas matching the channel expansion (6.7) to order unity with the aperture expansions (6.14) to order ε yields

$$V_0^\pm \sim \pm \frac{dU_0}{dz} \Big|_{z=\pm\frac{1}{2}} Z^\pm + U_1 \Big|_{z=\pm\frac{1}{2}} \quad \text{as } Z^\pm \rightarrow -\infty. \quad (6.18)$$

The Laplace problem for the first term in the normalised expansion of (6.18) determines the potential up to an arbitrary additive constant. We show in Appendix D that $V_0^\pm / \frac{dU_0}{dz} \Big|_{z=\pm\frac{1}{2}}$ solves a joint canonical problem, in which the limit

$$\beta = \lim_{Z^\pm \rightarrow -\infty} \left(V_0^\pm / \pm \frac{dU_0}{dz} \Big|_{z=\pm\frac{1}{2}} - Z^\pm \right) = -0.8217 \quad (6.19)$$

will be needed at the next order of analysis.

6.2.4 Exteriors

Before we extend the analysis to higher orders, we briefly consider the exterior regions, where we hold the unstrained coordinates fixed and denote the eigenfunction by u , consistent with the exact problem formulation. The exterior problems are defined in the half-spaces $z > 1/2$ and $z < -1/2$, with singular conditions given by

$$u^\pm(s^\pm, z; \varepsilon) \sim \varepsilon^2 Q^\pm \frac{e^{is^\pm k_0}}{s^\pm} \quad \text{as } \varepsilon \rightarrow 0, \quad (6.20)$$

where the prefactors

$$Q^{\pm} = \left\{ \begin{array}{l} \frac{k_0}{2} \sin(\frac{k_0}{2}) \\ \mp \frac{k_0}{2} \cos(\frac{k_0}{2}) \end{array} \right\} \quad (6.21)$$

are found from leading-order flux connection conservation arguments.

6.2.5 Higher-order analysis

Returning to the channel region, the order ε^3 balance of (6.3), after integration with respect to R and the respective order of (6.5), shows U_1 to satisfy the one-dimensional inhomogeneous Helmholtz equation

$$\frac{d^2 U_1}{dz^2} + k_0^2 U_1 = -2k_0 k_1 U_0. \quad (6.22)$$

This equation is subject to the conditions

$$U_1 = \pm \frac{dU_0}{dz} \Big|_{z=\pm\frac{1}{2}} \beta \quad \text{at} \quad z = \pm\frac{1}{2}, \quad (6.23)$$

which are derived from matching the channel expansion (6.7) and apertures expansions (6.14) to order ε , using (6.19). (It is evident that the channel eigenfunction remains uniform in R to arbitrary powers of ε .)

The problem for U_1 mirrors that of the singular homogeneous problem for U_0 , but with the inclusion of forcing terms. Thus, those forcing terms must satisfy special conditions for a solution to the former to exist. We derive a solvability condition by subtracting the complex conjugate of (6.8) multiplied by U_1 from (6.22) multiplied by U_0^* , where $*$ denotes complex conjugation, and using integration by parts along with the boundary conditions (6.9),

$$\left(U_1 \frac{dU_0^*}{dz} \right)_{z=\frac{1}{2}} - \left(U_1 \frac{dU_0^*}{dz} \right)_{z=-\frac{1}{2}} = 2k_0 k_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} |U_0|^2 dz. \quad (6.24)$$

We find the first wavenumber correction by substituting (6.11) and (6.23) into (6.24),

$$k_1 = 2\beta k_0. \quad (6.25)$$

Thus, we can write the approximation for the wavenumber as

$$\frac{k - k_0}{k_0} \sim 2\varepsilon\beta \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (6.26)$$

To derive a leading-order approximation for k'' , we extend our analysis to higher orders. At ε^4 , we find from the balance of channel equation (6.3) that U_2 satisfies

$$\frac{d^2 U_2}{dz^2} + k_0^2 U_2 = -2k_0 k_1 U_1 - k_1^2 U_0 - 2k_0 k_2 U_0. \quad (6.27)$$

This equation is supplemented by the boundary conditions at the channel ends. These conditions at $z \rightarrow \pm 1/2$ follow from the order ε^2 balance of the aperture regions, where $V_1^\pm$, like $V_0^\pm$, satisfy Laplace's equation and the Neumann boundary condition, but with matching conditions

$$V_1^\pm \rightarrow ik_0 Q^\pm \quad \text{as} \quad S^\pm \rightarrow \infty, \quad (6.28)$$

from exterior expansions (6.20), and

$$V_1^\pm \sim \pm \frac{dU_1}{dz} \Big|_{z=\pm \frac{1}{2}} Z^\pm + U_2 \Big|_{z=\pm \frac{1}{2}} \quad \text{as} \quad Z^\pm \rightarrow -\infty. \quad (6.29)$$

Again employing the solution to the canonical aperture problem (see Appendix D), we find that the solution for $V_1^\pm$ yields the limit

$$\beta = \lim_{Z^\pm \rightarrow -\infty} \left(V_1^\pm / \pm \frac{dU_1}{dz} \Big|_{z=\pm \frac{1}{2}} - Z^\pm \right) - \lim_{S^\pm \rightarrow \infty} \left(V_1^\pm / \pm \frac{dU_1}{dz} \Big|_{z=\pm \frac{1}{2}} \right) = -0.8217, \quad (6.30)$$

and the conditions

$$U_2 = \pm \frac{dU_1}{dz} \Big|_{z=\pm \frac{1}{2}} \beta + ik_0 Q^\pm \quad \text{at} \quad z = \pm \frac{1}{2}, \quad (6.31)$$

which follow from matching the channel expansion (6.7) and apertures expansions (6.14) to order ε^2 , using (6.30).

The problem for U_2 is the same as the problem for U_1 . We derive a further solvability condition by subtracting the complex conjugate of (6.8) multiplied by U_2 from (6.27) multiplied by U_0^* , and using integration by parts along with the boundary conditions (6.9),

$$\left(U_2 \frac{dU_0^*}{dz} \right)_{z=\frac{1}{2}} - \left(U_2 \frac{dU_0^*}{dz} \right)_{z=-\frac{1}{2}} = 2k_0 k_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} U_1 U_0^* dz + (k_1^2 + 2k_0 k_2) \int_{-\frac{1}{2}}^{\frac{1}{2}} |U_0|^2 dz. \quad (6.32)$$

To find the second wavenumber correction, we solve the forced equation (6.22)

$$u_1 = \begin{Bmatrix} \cos(k_0 z) - 2\beta k_0 z \sin(k_0 z) \\ \sin(k_0 z) + 2\beta k_0 z \cos(k_0 z) \end{Bmatrix} \quad (6.33)$$

and substitute this solution, (6.11) and (6.31), into (6.32), to find

$$k_2 = 4\beta^2 k_0 - ik_0^2. \quad (6.34)$$

We can write the approximation for the wavenumber as

$$\frac{k - k_0}{k_0} \sim 2\varepsilon\beta + \varepsilon^2(4\beta^2 - ik_0) \quad \text{as } \varepsilon \rightarrow 0. \quad (6.35)$$

This approximation is asymptotically consistent with the results in §5.5 [cf. (5.54)], in which the real part of the wavenumber agrees with the lossless resonant frequency (dimensionless) peak approximation to the expanded order of ε^2 . We can also compare the imaginary part of this approximation with an exact reciprocal relation. This identity, representing a balance between the time rate of change of the state's energy and the energy lost through radiation to infinity, is derived by applying Green's second identity to u and u^* , and, using (6.2), can be written as

$$k'' = \frac{i}{4k'} \frac{\oint_{\partial\mathcal{D}_\lambda} (u^* \frac{\partial u}{\partial n} - u \frac{\partial u^*}{\partial n}) dA}{\int_{\mathcal{D}_\lambda} |u|^2 dV} \sim -\varepsilon^2 k_0^2 \quad \text{as } \varepsilon \rightarrow 0, \quad (6.36)$$

where dA and dV are infinitesimal area and volume elements, respectively. The domain $\mathcal{D}_\lambda$ is chosen as the union of the channel and subregions $s^\pm < \lambda$ of the exterior hemispheres, for an arbitrary $\lambda > 0$ (see Figure 6.2).

6.3 Scattering problem

Next, we study the associated scattering problem in which the modes are resonantly excited. The same physical assumptions and geometry apply to this problem as those in the eigenvalue problem presented in §6.2; however, it incorporates a real dimensionless wavenumber κ and, in contrast to Chapter 5, includes incident plane waves from either side of the slab, as shown in Figure 6.1(b). Our analysis will focus on resonant excitation in the limit $\varepsilon \rightarrow 0$ and extend the results from the eigenvalue problem.

The wave field v , normalised by some reference amplitude, satisfies the Helmholtz equation

$$\nabla^2 v + \kappa^2 v = 0, \quad (6.37)$$

the Neumann condition

$$\frac{\partial v}{\partial n} = 0 \quad (6.38)$$

and outward radiation conditions as $s^\pm \rightarrow \infty$ in the exterior hemispheres, on $v - v_i$, where $v_i = \alpha^- e^{i\kappa z}$ for $z < -1/2$ and $v_i = \alpha^+ e^{-i\kappa z}$ for $z > 1/2$ are the incident plane waves.

It is useful to decompose the total field in the exterior regions as

$$v = v_a + v_s, \quad (6.39)$$

in which

$$v_a = \alpha^- (e^{i\kappa z} + e^{-i\kappa(1+z)}) \quad \text{for } z < -\frac{1}{2} \quad (6.40)$$

and

$$v_a = \alpha^+ (e^{-i\kappa z} + e^{i\kappa(z-1)}) \quad \text{for } z > \frac{1}{2} \quad (6.41)$$

are the ambient fields, defined as the superposition of the incident fields v_i and their reflections from the rigid slab in the absence of the hole. The ambient fields satisfy the Helmholtz equation (6.37) with Neumann conditions at $z = \pm 1/2$. The scattered fields v_s satisfy the Helmholtz equation (6.37), radiation conditions and inhomogeneous Neumann conditions.

6.3.1 Reciprocal relation

A reciprocal relation between the eigenvalue and scattering problems is derived by applying Green's second identity to the complex conjugate of the total field v^* , and an eigenfunction u over the domain $\mathcal{D}_\lambda$ defined previously (Figure 6.2). Coupled with (6.1), (6.2), (6.37) and (6.38), we find

$$(\kappa^2 - k^2) \int_{\mathcal{D}_\lambda} v^* u dV - \oint_{\partial \mathcal{D}_\lambda} \left(v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) dA = 0. \quad (6.42)$$

6.3.2 Resonance

In analysing the scattering problem, we focus on resonant excitation in the limit $\varepsilon \rightarrow 0$, meaning the scattered fields in the exterior domains can be expanded as

$$v_s = \mathcal{A}u + \dots \quad \text{as } \varepsilon \rightarrow 0, \quad (6.43)$$

and in the channel

$$v = \mathcal{A}u + \dots \quad \text{as } \varepsilon \rightarrow 0, \quad (6.44)$$

where $\mathcal{A}$ is a complex amplitude that is to be determined.

To proceed, we must estimate the integrals in (6.42) asymptotically in the limit $\varepsilon \rightarrow 0$. For (6.43) to hold in the exterior regions, $\mathcal{A}$ must be asymptotically large. As a result, the leading-order approximation for the volume integral in (6.42) is dominated by the channel region. Using the expansion (6.44), this gives

$$\int_{\mathcal{D}_\lambda} v^* u dV \sim \frac{\varepsilon^2 \mathcal{A}^* \pi}{2} \quad \text{as } \varepsilon \rightarrow 0. \quad (6.45)$$

Moving to the surface integral in (6.42), we substitute the exterior total field decomposition (6.39) to obtain

$$\int_{\partial \mathcal{D}_\lambda - \text{chnl}} \left(v_s^* \frac{\partial u}{\partial n} - u \frac{\partial v_s^*}{\partial n} \right) dA + \int_{\partial \mathcal{D}_\lambda - \text{chnl}} \left(v_a^* \frac{\partial u}{\partial n} - u \frac{\partial v_a^*}{\partial n} \right) dA + \int_{\text{chnl}} \left(v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) dA, \quad (6.46)$$

where the leading-order approximations for the first two integrals are evaluated as

$$\int_{\partial \mathcal{D}_\lambda - \text{chnl}} \left(v_s^* \frac{\partial u}{\partial n} - u \frac{\partial v_s^*}{\partial n} \right) dA \sim -2ik_0 k'' \varepsilon^2 \mathcal{A}^* \pi \quad \text{as } \varepsilon \rightarrow 0, \quad (6.47)$$

using (6.43), and

$$\int_{\partial \mathcal{D}_\lambda - \text{chnl}} \left(v_a^* \frac{\partial u}{\partial n} - u \frac{\partial v_a^*}{\partial n} \right) dA \sim \left\{ \begin{array}{l} -2ik_0 \varepsilon^2 \pi (\alpha^- + \alpha^+) \\ -2k_0 \varepsilon^2 \pi (\alpha^- - \alpha^+) \end{array} \right\} \quad \text{as } \varepsilon \rightarrow 0, \quad (6.48)$$

with the latter contributing to forcing in the problem. We omit the third integral in (6.46) until the following subsection, where dissipation effects will be addressed. This simplification is valid since, in the absence of dissipation, the homogeneous Neumann boundary condition on the cylindrical channel wall ensures this term vanishes to arbitrary order.

Substituting (6.45), (6.47) and (6.48) into the reciprocal relation (6.42), we derive the asymptotic amplitude-wavenumber relation

$$\mathcal{A} = \left\{ \begin{array}{l} 2i(\alpha^- + \alpha^+) \\ -2(\alpha^- - \alpha^+) \end{array} \right\} \frac{1}{\kappa - k' - ik''}. \quad (6.49)$$

In the denominator, we cannot assume a leading-order approximation for k' , nor can we truncate its expansion to a finite number of terms. This is due to κ and k' cancelling out up to the algebraically small ε^2 order of k'' . Given that $v_a \simeq 1$ in the

exterior regions, and considering (6.49) along with the scaling of k'' , the resonance conditions (6.43) and (6.44) require

$$\kappa - k' = O(\varepsilon^2). \quad (6.50)$$

We note that in the resonance regime, $\mathcal{A}$ is algebraically large, scaling as

$$\mathcal{A} \simeq \frac{1}{\varepsilon^2}. \quad (6.51)$$

Specifically, at resonance, $\kappa = k'$, $\mathcal{A}$ attains the value [cf. (6.36)]

$$\mathcal{A}^{(r)} = \left\{ \begin{array}{c} (\alpha^- + \alpha^+) \\ i(\alpha^- - \alpha^+) \end{array} \right\} \frac{2}{\varepsilon^2 k_0^2}. \quad (6.52)$$

The resonance conditions (6.43) and (6.44), in conjunction with the asymptotic amplitude-wavenumber relation (6.49) and the asymptotic expansions for the eigenfunctions, enable the derivation of a leading approximation for the scattered field across all regions. Notably, from (6.7) and (6.11), we observe that $|\mathcal{A}|$ represents the magnitude of the standing waves within the channel, which become algebraically large at resonance. Moreover, employing (6.20), it is evident that the scattered fields in the exterior regions exhibit the asymptotic approximations

$$v_s \sim \left\{ \begin{array}{c} i \sin(\frac{k_0}{2})(\alpha^- + \alpha^+) \\ -\cos(\frac{k_0}{2})(\alpha^- - \alpha^+) \end{array} \right\} \frac{k_0 \varepsilon^2}{\kappa - k' - ik''} \frac{e^{is^- k_0}}{s^-} \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for } z < -\frac{1}{2} \quad (6.53)$$

and

$$v_s \sim \left\{ \begin{array}{c} i \sin(\frac{k_0}{2})(\alpha^- + \alpha^+) \\ \cos(\frac{k_0}{2})(\alpha^- - \alpha^+) \end{array} \right\} \frac{k_0 \varepsilon^2}{\kappa - k' - ik''} \frac{e^{is^+ k_0}}{s^+} \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for } z > \frac{1}{2}. \quad (6.54)$$

At resonance, substituting (6.36) shows that (6.53) and (6.54) reduce to

$$v_s^{(r)} \sim \left\{ \begin{array}{c} \sin(\frac{k_0}{2})(\alpha^- + \alpha^+) \\ i \cos(\frac{k_0}{2})(\alpha^- - \alpha^+) \end{array} \right\} \frac{1}{k_0} \frac{e^{is^- k_0}}{s^-} \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for } z < -\frac{1}{2} \quad (6.55)$$

and

$$v_s^{(r)} \sim \left\{ \begin{array}{c} \sin(\frac{k_0}{2})(\alpha^- + \alpha^+) \\ -i \cos(\frac{k_0}{2})(\alpha^- - \alpha^+) \end{array} \right\} \frac{1}{k_0} \frac{e^{is^+ k_0}}{s^+} \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for } z > \frac{1}{2}. \quad (6.56)$$

6.3.3 Dissipative effects

Our analysis thus far has focused on a lossless model formulation, where energy leakage via radiation serves as the sole damping mechanism. However, it can be the case that both radiation damping and dissipation contribute comparably. This critical-coupling regime, where resonance is both loss and radiation limited, will be the subsequent focus.

It follows from our scaling analysis in §5.6.2 that, at leading order, thermoviscous boundary-layer effects enter in the form of an effective boundary condition in the channel

$$\frac{\partial v}{\partial r} - \varepsilon^2 \mathcal{C}v = 0 \quad \text{at} \quad r = \varepsilon, \quad (6.57)$$

where

$$\mathcal{C} = \mathcal{C}' + i\mathcal{C}'' = \frac{\delta k_0^2}{\varepsilon^3} \frac{1+i}{\sqrt{2}} \left(1 + \frac{\gamma-1}{\sqrt{\text{Pr}}} \right) \quad (6.58)$$

is a complex-valued parameter that we derived from first principles in §5.6.3. Here, δ is the dimensionless viscous length scale, Pr is the Prandtl number and γ is the adiabatic index. (We remind that this result depends on the uniformity of the cross section and does not apply as a general condition for a surface.)

Returning to the third integral in (6.46), we find, using (6.57), that

$$\int_{\text{chnl}} \left(v^* \frac{\partial u}{\partial n} - u \frac{\partial v^*}{\partial n} \right) dA \sim -\varepsilon^4 \mathcal{A}^* \mathcal{C}^* |U_0|^2 \pi \quad \text{as} \quad \varepsilon \rightarrow 0. \quad (6.59)$$

Adding (6.59) to the reciprocal relation (6.42) and integrating this between the two ends of the channel, we find the amplitude-wavenumber relation with radiative and dissipative damping

$$\mathcal{A} = \left\{ \begin{array}{c} 2i(\alpha^- + \alpha^+) \\ -2(\alpha^- - \alpha^+) \end{array} \right\} \frac{1}{\kappa - k' - ik'' + \frac{\varepsilon^2 \mathcal{C}'}{k_0}}, \quad (6.60)$$

where, at resonance, $\mathcal{A}$ now attains the value

$$\mathcal{A}^{(r)} = \left\{ \begin{array}{c} (\alpha^- + \alpha^+) \\ i(\alpha^- - \alpha^+) \end{array} \right\} \frac{2}{\varepsilon^2 k_0^2 + \frac{\varepsilon^2 \mathcal{C}''}{k_0}}. \quad (6.61)$$

6.4 Concluding remarks

In this chapter, we have demonstrated that the eigenvalue-based spectral approach provides a more general and simplified framework for analysing acoustic resonant

scattering compared to the direct asymptotic method presented in Chapter 5. This formulation facilitates the extension to multi-resonator systems, offering a more physically intuitive and transparent understanding of complex resonator interactions, which will be leveraged in the next chapter to analyse configurations involving multiple holes.

While our work has focused on axisymmetric modes under normal incidence excitation, natural extensions include the exploration of higher-order and asymmetric modes. Higher-order modes become increasingly relevant in higher-frequency regimes, where the wavelength approaches or is smaller than the hole radius [34, 50]. Asymmetric modes can emerge from controlled geometric variations or oblique incidence of plane waves [56, 57, 103], potentially inducing coupling between various mode types, characterised by energy transfer and mixing of modal characteristics, leading to novel resonance phenomena.

Near-field excitation presents another promising direction for future research. This includes various configurations such as in-channel or near-aperture sources and scattering of a plane wave from particles in the near field. In these arrangements, evanescent waves dominate and strongly interact with the resonant features of acoustic metamaterials, enriching the modal landscape [21]. This interaction enables the excitation of localised acoustic modes in structured materials [84], thereby offering the potential for subwavelength control of sound fields and novel sensing applications.

CHAPTER 7

N interacting holes: multiple scattering and homogenisation

7.1 Introduction

Holes have been effectively used as building blocks in the engineering of acoustic metasurfaces, enabling novel capabilities in sound absorption and control. Inspired by this, our objective is to systematically extend our theory of a slender cylindrical hole developed in Chapter 6 to configurations involving multiple interacting holes.

To generalise the model of a single hole from §6.3 to multiple holes, we employ Foldy’s method [17, 83], a technique accounting for multiple scattering events in a collection of subwavelength scatterers. This method is typically applied to scatterers in a bulk or embedded in a surface interfacing a single exterior domain. Here, we adapt it to the cylindrical holes considered in Chapters 5 and 6, which connect *two* exterior half-spaces. Our multiple-hole model preserves the key assumptions underlying the single-hole model — namely, subwavelength holes and thin thermoviscous boundary layers relative to the hole radii. Importantly, in the limit of a single hole, our multiple-hole model reduces to the model from §6.3.

We further derive a homogenisation approximation of our multiple-scattering model through a multiple scales analysis within an appropriate asymptotic limit, thereby representing the metasurface with an effective impedance condition. Both the homogenised and multiple-scattering models support arbitrary arrangements of non-identical holes, although to illustrate our models, we focus on a square lattice of identical holes, including thermoviscous effects. (Analogous models were derived in [102] to study single-sided, finite-sized acoustic Helmholtz-type metasurfaces’ absorption characteristics.) With our models, we explore the conditions for critical coupling of an infinite metasurface with incoming waves from both sides, enabling

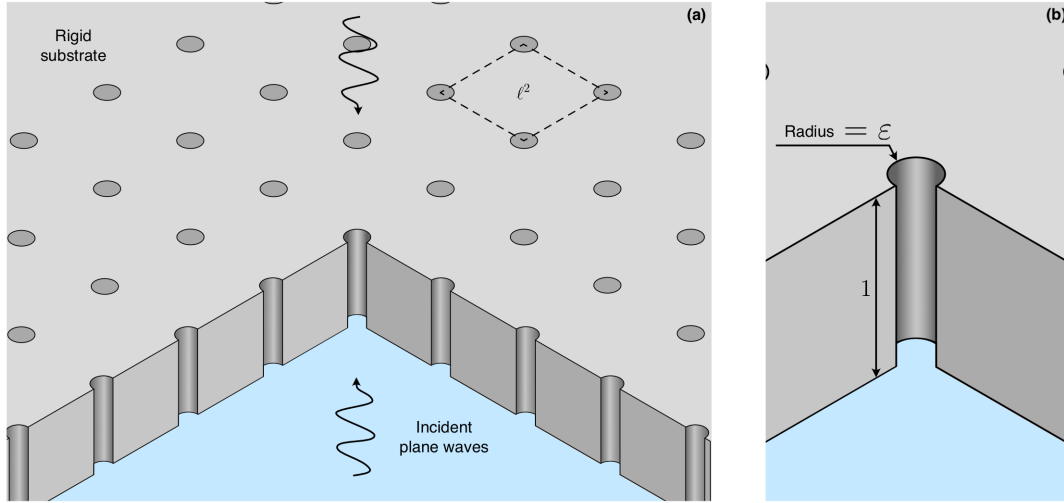


Figure 7.1: (a) A dimensionless sketch of the metasurface, demonstrating a square $\sqrt{N} \times \sqrt{N}$ lattice of identical holes with spacing $\gg \epsilon$. Note that our formulation does not assume periodic. (b) Geometry of the cylindrical holes.

the phenomenon of coherent perfect absorption discussed in Chapter 1. We also study the surface waves supported by infinite metasurfaces, demonstrating their dramatic influence on scattering from finite structures owing to the excitation of ‘lattice resonances’.

7.2 Multiple-scattering model

Consider a flat, rigid slab of thickness l , embedded with N slender, cylindrical holes of radius ϵl , the type considered in Chapters 5 and 6 (see §6.2.1 and Figure 7.1). This configuration forms an acoustic metasurface. We adopt dimensionless quantities, normalising lengths by l , and use the superscripts $+$ and $-$ to differentiate these in the half-space exteriors on either side of the metasurface, $z > 1/2$ and $z < -1/2$, respectively. We define $\mathbf{x}$ as the dimensionless position vector, referenced from the centre of the first hole, and $\mathbf{x}^\pm = \mathbf{x} \mp \hat{\mathbf{e}}_z/2$, where $\hat{\mathbf{e}}_z$ is the unit vector towards the $+$ domain. The locations of the hole openings are $\mathbf{x}^\pm = \mathbf{r}_n^\pm$, with $\mathbf{r}_1^\pm = \mathbf{0}$. As $\epsilon \rightarrow 0$, the hole openings shrink to points, and $n = 1, 2, \dots, N$ indexes the quantities of the n th hole. The wave fields, normalised by a reference amplitude v_∞ , are expressed as $\text{Re}(v^\pm(\mathbf{x}^\pm)e^{-i\omega t})$, where t is time, ω is the angular frequency, and $v^\pm(\mathbf{x}^\pm)$ are the complex-valued, dimensionless reduced fields. External forcing on the metasurface is represented by incident fields $v_i^\pm(\mathbf{x}^\pm)$, and reflected fields $v_r^\pm(\mathbf{x}^\pm)$ are generated by the rigid substrate in the absence of holes. As in §6.3, we define the superposition $v_a^\pm(\mathbf{x}^\pm) = v_i^\pm(\mathbf{x}^\pm) + v_r^\pm(\mathbf{x}^\pm)$ as the ambient fields, which

are defined in the exterior domains. The dimensionless wavenumber k is defined in §6.2, while the dimensionless frequency κ and relative boundary-layer thickness δ are defined as in §6.3.

We assume that the separation ℓ between holes is large relative to their radius, such that

$$|\mathbf{r}_i^\pm - \mathbf{r}_j^\pm| \gg \varepsilon, \quad \text{for all } i, j = 1, 2, \dots, N \ (i \neq j). \quad (7.1)$$

(In our illustrative example, ℓ^2 denotes the area of a cell in the square lattice.) In light of the above, at distances $\gg \varepsilon$ from the slab, we approximate the wave fields as

$$\mathbf{v}^\pm = \mathbf{v}_a^\pm(\mathbf{x}^\pm) + \sum_{n=1}^N \mathbf{v}_{s,n}^\pm(|\mathbf{x}^\pm - \mathbf{r}_n^\pm|) = \mathbf{v}_a^\pm(\mathbf{x}^\pm) + \sum_{n=1}^N \mathcal{A}_n \varepsilon^2 Q^\pm \frac{e^{ik_0|\mathbf{x}^\pm - \mathbf{r}_n^\pm|}}{|\mathbf{x}^\pm - \mathbf{r}_n^\pm|}, \quad (7.2)$$

where we introduce exterior conditions [cf. (6.20)] and the scattering coefficients $\mathcal{A}_n$. These coefficients, associated with spherical waves diffracted from the n th hole, are calculated by modelling the corresponding hole as an isolated entity responding to the effective ambient fields. The n th hole experiences the local ambient fields $\mathbf{v}_{a,n}^\pm$, defined as the sum of the ambient fields $\mathbf{v}_a^\pm$ and the spherical waves diffracted from all other holes, each evaluated at $\mathbf{x}^\pm = \mathbf{r}_n^\pm$,

$$\mathbf{v}_{a,n}^\pm = \mathbf{v}_a^\pm(\mathbf{r}_n^\pm) + \sum_{m \neq n}^N \mathbf{v}_{s,m}^\pm(|\mathbf{r}_n^\pm - \mathbf{r}_m^\pm|) = \mathbf{v}_a^\pm(\mathbf{r}_n^\pm) + \sum_{m \neq n}^N \mathcal{A}_m \varepsilon^2 Q^\pm \frac{e^{ik_0|\mathbf{r}_n^\pm - \mathbf{r}_m^\pm|}}{|\mathbf{r}_n^\pm - \mathbf{r}_m^\pm|}. \quad (7.3)$$

Given (6.60) from the asymptotic single-hole theory, we can use the approximation

$$\mathcal{A}_n = -\frac{\mathbf{v}_{a,n}^- + \mathbf{v}_{a,n}^+}{\kappa - k' - ik'' + \frac{\varepsilon^2 \mathcal{C}}{k_0}} \quad (7.4)$$

to solve a system of linear algebraic equations for $\mathbf{v}_{a,n}^\pm$, and use these solutions to determine the scattering coefficients (where $\mathcal{C}$ is defined in (6.58)). We remark here that equation (7.4) is an approximation appropriate for frequency intervals associated with the resonant excitation of z -even channel modes, which includes the fundamental mode [64, 6]. While both even and odd cases are readily tractable, the subsequent analysis will focus on even modes, particularly the fundamental mode, for the sake of simplification.

7.2.1 Dissipation

The power dissipated by the metasurface — normalised by $v_\infty^2 \ell^2 \pi / \rho_0 c$ — is a key quantity of interest. To derive a energy-dissipation relation, we apply (1.50) in the domains bounded by $S_\lambda^\pm$, the surfaces of exterior hemispheres of radius λ centred at $\mathbf{r}_n^\pm$, where λ is chosen to exclude other hole opening positions, i.e., $\lambda < \ell$, and external sources. Under these restrictions, the choice of λ is arbitrary. The dimensionless energy-flux vectors (1.51) vanish at the rigid boundary due to impermeability conditions, and, as thermoviscous losses are negligible in the exterior domains to leading order, these vectors reduce to

$$-\frac{1}{2} \text{Im}(v^\pm \nabla v^{\pm*}). \quad (7.5)$$

Following these considerations, the time-averaged rate of energy loss through the hole openings $\mathbf{r}_n^\pm$ is

$$D_n = \frac{1}{2\pi\ell^2} \int_{S_\lambda^+} \text{Im}(v^+ \nabla v^{+*}) \cdot \hat{\mathbf{n}} dS + \frac{1}{2\pi\ell^2} \int_{S_\lambda^-} \text{Im}(v^- \nabla v^{-*}) \cdot \hat{\mathbf{n}} dS. \quad (7.6)$$

Choosing n , we can rewrite the fields as

$$\begin{aligned} v^\pm &= v_a^\pm(\mathbf{r}_n^\pm) + \sum_{m \neq n}^N \mathcal{A}_m \varepsilon^2 Q^\pm \frac{e^{ik_0|\mathbf{r}_n^\pm - \mathbf{r}_m^\pm|}}{|\mathbf{r}_n^\pm - \mathbf{r}_m^\pm|} + \mathcal{A}_n \varepsilon^2 Q^\pm \frac{e^{ik_0|\mathbf{x}^\pm - \mathbf{r}_n^\pm|}}{|\mathbf{x}^\pm - \mathbf{r}_n^\pm|} + \mathcal{O}(|\mathbf{x}^\pm - \mathbf{r}_n^\pm|) \\ &= v_{a,n}^\pm + \mathcal{A}_n \varepsilon^2 Q^\pm \frac{e^{ik_0|\mathbf{x}^\pm - \mathbf{r}_n^\pm|}}{|\mathbf{x}^\pm - \mathbf{r}_n^\pm|} + \mathcal{O}(|\mathbf{x}^\pm - \mathbf{r}_n^\pm|) \quad \text{as } |\mathbf{x}^\pm - \mathbf{r}_n^\pm| \rightarrow 0, \end{aligned} \quad (7.7)$$

where $v_{a,n}^\pm$ are regular at $\mathbf{x}^\pm = \mathbf{r}_n^\pm$. Accordingly, we can evaluate (7.6) by taking the limit $\lambda \rightarrow 0$. Noting that (7.7) gives

$$\text{Im}(v^\pm \nabla v^{\pm*}) \cdot \hat{\mathbf{n}} = -\frac{\varepsilon^2 Q^\pm}{\lambda^2} \left[\varepsilon^2 Q^\pm k_0 |\mathcal{A}_n|^2 + \text{Im}(\mathcal{A}_n^* v_{a,n}^\pm) \right] + o\left(\frac{1}{\lambda^2}\right) \quad \text{as } \lambda \rightarrow 0, \quad (7.8)$$

we find for the fundamental mode

$$D_n = -\frac{\varepsilon^2 \pi}{2\ell^2} \left[\varepsilon^2 \pi^2 |\mathcal{A}_n|^2 + \text{Im}(\mathcal{A}_n^* [v_{a,n}^- + v_{a,n}^+]) \right]. \quad (7.9)$$

Substitution of (7.4) and summing over n yields the formula for the total dimensionless dissipation

$$D = \frac{\varepsilon^4 \mathcal{C}''}{2\ell^2} \sum_{n=1}^N |\mathcal{A}_n|^2, \quad (7.10)$$

as expected.

7.3 Homogenised model

7.3.1 Multiple scales

In our multiple-scattering model, the solution of a linear system of equations can become computationally intensive, particularly for large metasurfaces with many holes. This complexity can be mitigated in certain cases by approximating the metasurface with a homogenised model, where it is characterised by an effective impedance condition. A necessary condition of this homogenised model is subwavelength spacing between holes, i.e., $\ell \ll 1$.

We start by reformulating the multiple-hole model through singularity conditions prescribed at the hole openings $\mathbf{x}^\pm = \mathbf{r}_n^\pm$. Thus, the wave fields satisfy the reduced-wave equation

$$\nabla^2 v^\pm + \kappa^2 v^\pm = 0 \quad \text{in} \quad \hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm \gtrless 0^\pm, \quad (7.11)$$

the Neumann boundary condition

$$\hat{\mathbf{e}}_z \cdot \nabla v^\pm = 0 \quad \text{on} \quad \hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm = 0^\pm \setminus \{\mathbf{x}^\pm = \mathbf{r}_n^\pm\}, \quad (7.12)$$

the singularity conditions

$$v^\pm = \frac{\mathcal{A}_n \varepsilon^2 \mathcal{Q}}{|\mathbf{x}^\pm - \mathbf{r}_n^\pm|} + o\left(\frac{1}{|\mathbf{x}^\pm|}\right) \quad \text{as} \quad \mathbf{x}^\pm \rightarrow \mathbf{r}_n^\pm, \quad (7.13)$$

wherein $\mathcal{A}_n$ is given by equation (7.4) and $\mathcal{Q} = \mathcal{Q}^- = \mathcal{Q}^+$ for even modes, and outward radiation conditions imposed on $v^\pm - v_i^\pm$. In the present formulation, the ambient fields experienced by the n th hole are given in the limit of (7.13):

$$v_{a,n}^\pm = \lim_{\mathbf{x}^\pm \rightarrow \mathbf{r}_n^\pm} \left(v^\pm - \frac{\mathcal{A}_n \varepsilon^2 \mathcal{Q}}{|\mathbf{x}^\pm - \mathbf{r}_n^\pm|} \right). \quad (7.14)$$

For simplicity, we consider identical holes of subwavelength radius, $\varepsilon \ll \ell \ll 1$, positioned at the vertices of a square lattice. This lattice is not necessarily infinite but may instead be a large patch away from boundaries.

We assume that the incident fields $v_i^\pm$ vary on the wavelength scale. Accordingly, we expect that the fields in the ‘wavelength-scale’ regions — those at distances comparable to the wavelength from the surface — will also vary on the same scale. Thus, the associated wavelength-scale (or ‘macroscale’) fields $v^\pm(\mathbf{x}^\pm)$ and the wavelength-scale (or ‘slow’) coordinates $\mathbf{x}^\pm$ are defined as outlined in §7.2. In the

limit $\ell \rightarrow 0$, which necessarily implies $\varepsilon \rightarrow 0$, the problems in the wavelength-scale regions are

$$\nabla_{\mathbf{x}^\pm}^2 v^\pm + \kappa^2 v^\pm = 0, \quad (7.15)$$

subject to matching conditions as $\hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm \rightarrow 0^\pm$, and outward radiation conditions as $z \rightarrow \pm\infty$ on $v^\pm - v_i^\pm$.

We now focus on the ‘near-field’ regions, characterised by distances $\simeq \ell$ from the surface. In these regions, we seek solutions in a multiple-scales form

$$v^\pm(\mathbf{x}^\pm) = V^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel), \quad (7.16)$$

where

$$\mathbf{x}_\parallel = \mathbf{x}^\pm - x_\perp^\pm \hat{\mathbf{e}}_z, \quad x_\perp^\pm = \hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm, \quad \mathbf{X}^\pm = \ell^{-1} \mathbf{x}^\pm, \quad (7.17)$$

with the scattering coefficient

$$\mathcal{A}_n(\mathbf{x}^\pm) = \mathcal{A}(\mathbf{x}_\parallel) \quad (7.18)$$

written as a function of the slow (or ‘long’) scale, due to the discrete positioning of the holes. We express $\mathcal{A}$ as

$$\mathcal{A} = -\frac{V_a^-(\mathbf{x}_\parallel) + V_a^+(\mathbf{x}_\parallel)}{\kappa - k' - ik'' + \frac{\varepsilon^2 \mathcal{C}}{k_0}}, \quad (7.19)$$

where

$$V_a^\pm = \lim_{\mathbf{x}^\pm \rightarrow \mathbf{0}^\pm} \left(V^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel) - \frac{\mathcal{A}(\mathbf{x}_\parallel) \varepsilon^2 \mathcal{Q}}{\ell |\mathbf{X}^\pm|} \right). \quad (7.20)$$

In accordance with the multiple-scales approach, we require that $V^\pm$ are periodic in the ‘fast’ (or ‘short’) coordinates $\mathbf{X}^\pm$,

$$V^\pm(\mathbf{X}^\pm + n_1 \hat{\mathbf{e}}_{X^1} + n_2 \hat{\mathbf{e}}_{X^2}; \mathbf{x}_\parallel) = V^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel), \quad (7.21)$$

where unit vectors $\hat{\mathbf{e}}_{X^1}$ and $\hat{\mathbf{e}}_{X^2}$ lie in the substrate plane, and $\{n_1, n_2\} \in \mathbb{Z}^2$.

Using the chain rule, the transformations for the first and second derivatives are

$$\nabla \rightarrow \ell^{-1} \nabla_{\mathbf{X}^\pm} + \nabla_{\mathbf{x}_\parallel}, \quad \nabla^2 \rightarrow \ell^{-2} \nabla_{\mathbf{X}^\pm}^2 + 2\ell^{-1} \nabla_{\mathbf{X}^\pm} \cdot \nabla_{\mathbf{x}_\parallel} + \nabla_{\mathbf{x}_\parallel}^2. \quad (7.22)$$

We can rewrite (7.11)-(7.14) in multiple-scales form by substituting expressions (7.22),

$$\left(\ell^{-2} \nabla_{\mathbf{X}^\pm}^2 + 2\ell^{-1} \nabla_{\mathbf{X}^\pm} \cdot \nabla_{\mathbf{x}_\parallel} + \nabla_{\mathbf{x}_\parallel}^2 \right) V^\pm + \kappa^2 V^\pm = 0 \quad \text{in } \mathcal{F}^\pm, \quad (7.23)$$

where the near-field analysis is restricted to the regions $\mathcal{F}^\pm$, representing the domains above the areas associated with the vertices $n_1 = n_2 = 0$. This system is subject to the Neumann condition

$$\hat{\mathbf{e}}_z \cdot \ell^{-1} \nabla_{\mathbf{x}^\pm} V^\pm = 0 \quad \text{on} \quad \partial \mathcal{F}^\pm \setminus \{\mathbf{X}^\pm = \mathbf{0}^\pm\}, \quad (7.24)$$

wherein the periodic boundary conditions (7.21) are incorporated, along with the singularity conditions

$$V^\pm = \frac{\mathcal{A}(\mathbf{x}_\parallel) \varepsilon^2 Q}{\ell |\mathbf{X}^\pm|} \quad \text{as} \quad \mathbf{X}^\pm \rightarrow \mathbf{0}^\pm. \quad (7.25)$$

We next need to prescribe the behaviour of the macroscale $\mathcal{A}$ in the limit $\ell \rightarrow 0$. Equation (7.4) indicates that $\mathcal{A}$ is frequency-dependent. In the off-resonance regime, it follows that $\mathcal{A} \simeq 1$, as the weak responses of the holes render the metasurface effectively rigid. A flux balance argument suggests instead considering the regime $\kappa - k' \simeq \varepsilon^2/\ell^2$, where

$$\mathcal{A} \simeq \left(\frac{\ell}{\varepsilon} \right)^2. \quad (7.26)$$

The indeterminacy between ε and ℓ arises because the homogenisation is performed in an intermediate asymptotic limit. This lies between two distinguished limits: one in which the period is $\sim \varepsilon$, and another in which the period is $\sim$ wavelength, corresponding to the Foldy model.

Accordingly, we write

$$\mathcal{A}(\mathbf{x}_\parallel) = \varepsilon^{-2} \ell^2 \mathcal{A}_{-2}(\mathbf{x}_\parallel) + \dots, \quad (7.27)$$

and expand the wavelength-scale fields as

$$\mathbf{v}^\pm(\mathbf{x}^\pm) = \mathbf{v}_0^\pm(\mathbf{x}^\pm) + \dots, \quad (7.28)$$

where

$$\mathbf{v}_0^\pm(\mathbf{x}^\pm) \sim \mathbf{v}_0^\pm(\mathbf{x}_\parallel, 0^\pm) + \mathbf{x}_\perp^\pm \frac{\partial \mathbf{v}_0^\pm}{\partial \mathbf{x}_\perp^\pm}(\mathbf{x}_\parallel, 0^\pm) + \dots \quad \text{as} \quad \mathbf{x}_\perp^\pm \rightarrow 0^\pm, \quad (7.29)$$

since it does not see the singularity; and near fields as

$$V^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel) = V_0^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel) + \ell V_1^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel) + \dots \quad (7.30)$$

and

$$V_a^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel) = V_{a,0}^\pm(\mathbf{X}^\pm; \mathbf{x}_\parallel) + \dots \quad (7.31)$$

Substituting the expansions above into (7.23)-(7.25), we arrive at the leading-order problems

$$\nabla_{\mathbf{X}^\pm}^2 V_0^\pm = 0 \quad \text{in } \mathcal{F}^\pm; \quad (7.32)$$

subject to homogeneous boundary conditions

$$\hat{\mathbf{e}}_z \cdot \nabla_{\mathbf{X}^\pm} V_0^\pm = 0 \quad \text{on } \partial\mathcal{F}^\pm, \quad (7.33)$$

with periodicity and regularity free from singularities in this limit,

$$V_0^\pm = o\left(\frac{1}{|\mathbf{X}^\pm|}\right) \quad \text{as } \mathbf{X}^\pm \rightarrow \mathbf{0}^\pm. \quad (7.34)$$

We can argue that $V_0^\pm$ are independent of $\mathbf{X}^\pm$, and leading-order matching with the wavelength-scale regions yields

$$V_0^\pm(\mathbf{x}_\parallel) = v_0^\pm(\mathbf{x}_\parallel, 0^\pm). \quad (7.35)$$

Using the definitions (7.20) and expansions (7.31), it is evident that $V_a^\pm$ are constant in $\mathbf{X}^\pm$, to leading order, and given by

$$V_{a,0}^\pm = v_0^\pm(\mathbf{x}_\parallel, 0^\pm). \quad (7.36)$$

Consider next, the order ℓ^{-1} balance of (7.23)-(7.25), given by

$$\nabla_{\mathbf{X}^\pm}^2 V_1^\pm = 0 \quad \text{in } \mathcal{F}^\pm, \quad (7.37)$$

subject to heterogeneous boundary conditions

$$\hat{\mathbf{e}}_z \cdot \nabla_{\mathbf{X}^\pm} V_1^\pm = 0 \quad \text{on } \partial\mathcal{F}^\pm \setminus \{\mathbf{X}^\pm = \mathbf{0}^\pm\}, \quad (7.38)$$

with periodicity and the singularity conditions

$$V_1^\pm = \frac{\mathcal{A}_{-2}(\mathbf{x}_\parallel)Q}{|\mathbf{X}^\pm|} \quad \text{as } \mathbf{X}^\pm \rightarrow \mathbf{0}^\pm, \quad (7.39)$$

where

$$\mathcal{A}_{-2} = -\frac{V_{a,0}^-(\mathbf{x}_\parallel) + V_{a,0}^+(\mathbf{x}_\parallel)}{\kappa - k' - ik'' + \frac{\varepsilon^2 c}{k_0}}. \quad (7.40)$$

The remaining boundary conditions are derived by matching with the wavelength-scale regions,

$$V_1^\pm \sim (\hat{\mathbf{e}}_z \cdot \mathbf{X}^\pm) \frac{\partial v_0^\pm}{\partial x_\perp^\pm}(\mathbf{x}_\parallel, 0^\pm) \quad \text{as} \quad \hat{\mathbf{e}}_z \cdot \mathbf{X}^\pm \rightarrow \pm\infty. \quad (7.41)$$

Rather than solving this problem explicitly, we integrate (7.37) over the domains using the divergence theorem

$$\int_{\mathcal{G}^\pm} \nabla_{\mathbf{x}^\pm}^2 V_1^\pm dV_{\mathbf{x}^\pm} = \oint_{\partial\mathcal{G}^\pm} \pm \hat{\mathbf{e}}_z \cdot \nabla_{\mathbf{x}^\pm} V_1^\pm dS_{\mathbf{x}^\pm}, \quad (7.42)$$

which, together with (7.38)-(7.41), gives

$$\frac{\partial v_0^\pm}{\partial x_\perp^\pm}(\mathbf{x}_\parallel, 0^\pm) \pm 2\pi \mathcal{A}_{-2}(\mathbf{x}_\parallel) \mathcal{Q} = 0. \quad (7.43)$$

Finally, combining this result with (7.35), gives

$$\frac{\partial v_0^\pm}{\partial x_\perp^\pm}(\mathbf{x}_\parallel, 0^\pm) \mp \frac{2\varepsilon^2 \pi \mathcal{Q}}{\ell^2 f} \left(v_0^-(\mathbf{x}_\parallel, 0^-) + v_0^+(\mathbf{x}_\parallel, 0^+) \right) = 0, \quad (7.44)$$

where

$$f = \kappa - k' - ik'' + \frac{\varepsilon^2 \mathcal{C}}{k_0}. \quad (7.45)$$

Equations (7.44) constitute effective impedance boundary conditions, representing the metasurface within the homogenised model.

7.3.2 Homogenised model

The homogenised problem for the macroscale fields consists of the Helmholtz equation

$$\nabla^2 v^\pm + \kappa^2 v^\pm = 0 \quad \text{in} \quad \hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm \gtrless 0^\pm, \quad (7.46)$$

the impedance boundary conditions on metasurface patches $\mathcal{M}^\pm$

$$\hat{\mathbf{e}}_z \cdot \nabla v^\pm \mp 2\pi \mathcal{Q} \bar{f}^{-1}(v^- + v^+) = 0 \quad \text{on} \quad \mathcal{M}^\pm, \quad (7.47)$$

where

$$\bar{f} = \frac{\ell^2}{\varepsilon^2} f \simeq \frac{\ell^2}{\varepsilon^2} (\kappa - k') + \bar{\mathcal{C}}, \quad \bar{\mathcal{C}} = \bar{\mathcal{C}}' + i\bar{\mathcal{C}}'' = \frac{\ell^2 \mathcal{C}}{k_0}, \quad (7.48)$$

the Neumann boundary condition on the rigid substrate

$$\hat{\mathbf{e}}_z \cdot \nabla v^\pm = 0 \quad \text{on} \quad \hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm = 0^\pm \setminus \mathcal{M}^\pm, \quad (7.49)$$

and outward radiation conditions as $z \rightarrow \pm\infty$ on the scattered fields $v^\pm - v_i^\pm$, with $v_i^\pm$ being the incident fields. The averaged dissipation per hole can be readily calculated from (7.10) using (7.18).

It is evident from (7.48) that in cases of sufficiently low loss, the homogenised model fails very close to resonance, where ε^2 terms are important. This invalidity can be attributed to the excitation of short-wavelength surface waves, as discussed later.

7.3.3 Infinite metasurface

We subsequently consider the scattering problem for an infinite metasurface subjected to normally incident fields. While Floquet-Bloch theory [52] provides a general approach for infinite metasurfaces, assuming compatibility with the incident fields, we instead apply our homogenisation approach here. The symmetry of the multiple-hole problem implies that the channel fields are equal, and the macroscale fields in the homogenised model depend solely on $x_\perp^\pm$.

Assuming the homogenised model, the macroscale fields can be written as

$$v^\pm = \mathcal{J}^\pm e^{\mp i\kappa \hat{e}_z \cdot x^\pm} + \mathcal{R}^\pm e^{\pm i\kappa \hat{e}_z \cdot x^\pm}, \quad (7.50)$$

where the first terms are the incident plane waves with amplitudes and phases $\mathcal{J}^\pm = \alpha^\pm e^{i\theta^\pm}$, and the second terms represent the reflected plane waves, with reflection coefficients $\mathcal{R}^\pm$. The aggregate effects of the spherical waves radiating from the hole openings are reflected in any deviations of $\mathcal{R}^\pm$ from unity. Applying the impedance boundary conditions (7.47), we find

$$\mathcal{R}^\pm = \frac{(\kappa \bar{f} - 2i\pi Q) \mathcal{J}^\pm - 2i\pi Q (\mathcal{J}^\mp + \mathcal{R}^\mp)}{\kappa \bar{f} + 2i\pi Q}, \quad (7.51)$$

which, upon simplification, becomes

$$\mathcal{R}^\pm = \frac{\kappa \bar{f} \mathcal{J}^\pm - 4i\pi Q \mathcal{J}^\mp}{\kappa \bar{f} + 4i\pi Q}, \quad (7.52)$$

and in matrix form for the fundamental mode, to leading order,

$$\begin{pmatrix} \mathcal{R}^+ \\ \mathcal{R}^- \end{pmatrix} = \frac{1}{\bar{f} + 2i\pi} \begin{pmatrix} \bar{f} & -2i\pi \\ -2i\pi & \bar{f} \end{pmatrix} \begin{pmatrix} \mathcal{J}^+ \\ \mathcal{J}^- \end{pmatrix}. \quad (7.53)$$

We note that an asymptotically equivalent result to (7.53) can be derived from the multiple-scattering model directly, rather than relying on the homogenised

approach. This derivation involves asymptotically approximating a lattice sum, assuming only that the array spacing is subwavelength (as done in [14], for example, in a closely related problem).

Extraordinary transmission

We first use equation (7.53) to investigate extraordinary transmission. Analogous to our periodic slit analysis in §2.7, we set $\mathcal{J}^- = 1$ and $\mathcal{J}^+ = 0$, and define complex-valued reflection and transmission coefficients $\mathcal{R} = \mathcal{R}^-$ and $\mathcal{T} = \mathcal{R}^+$, respectively. Under these conditions, we find

$$\mathcal{R} = \frac{\bar{f}}{\bar{f} + 2i\pi} = 0, \quad (7.54)$$

and

$$\mathcal{T} = \frac{-2i\pi}{\bar{f} + 2i\pi} = -1 \sim \mathcal{O}(1), \quad (7.55)$$

that is, in the absence of loss, extraordinary transmission occurs at resonance, where the wave interacts negligibly with the metasurface, except for a π -phase shift. (This result is consistent with the two-dimensional case.)

Coherent perfect absorption

We can similarly apply this theory to study coherent perfect absorption, which expects the reflection coefficients to vanish, $\mathcal{R}^\pm = 0$. These conditions lead to

$$\begin{pmatrix} \bar{f} & -2i\pi \\ -2i\pi & \bar{f} \end{pmatrix} \begin{pmatrix} \mathcal{J}^+ \\ \mathcal{J}^- \end{pmatrix} = 0, \quad (7.56)$$

where, by examining the determinant of the singular matrix, we derive the necessary conditions

$$\bar{f} = 2i\pi \implies \bar{c}'' = 2\pi, \quad \mathcal{J}^- = \mathcal{J}^+. \quad (7.57)$$

These conditions reveal that coherent perfect absorption occurs at resonance, with losses appropriately tuned with the loss parameter $\delta \simeq \varepsilon^3/\ell^2$, and that the opposing incident waves must have equal magnitude and phase. (These conditions also hold in the symmetric half-space configuration.)

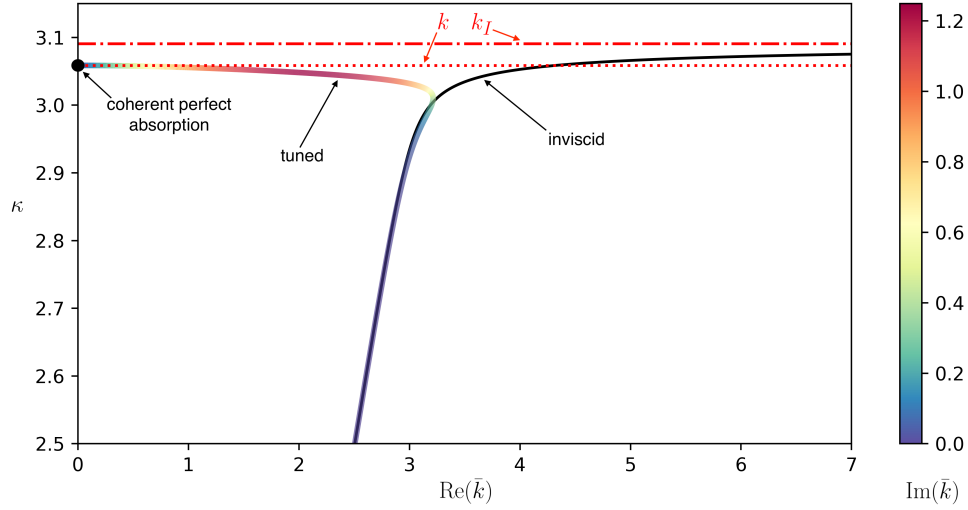


Figure 7.2: Surface wave dispersion relation on infinite metasurfaces: tuned and inviscid. The wavevector is taken in an arbitrary direction in the substrate plane. Colour-gradient curve: tuned (critically coupled) scenario, computed from the dispersion relation (7.63) derived from the homogenised model. Black-solid curve: inviscid scenario, computed from the lossless dispersion relation.

7.4 Surface waves

Let us now examine the behaviour of surface waves on an infinite metasurface, focusing on homogeneous solutions existing for real frequencies without incident fields, exhibiting exponential decay away from the slab. These surface waves are characterised by quasi-periodic solutions, expressed as

$$v^\pm = \psi^\pm(\mathbf{x}^\pm) e^{i\bar{\mathbf{k}} \cdot \mathbf{x}^\pm}, \quad (7.58)$$

where $\psi^\pm(\mathbf{x}^\pm)$ are periodic over the square lattice (with vertices at the limiting hole openings $\mathbf{r}_n^\pm$), and $\bar{\mathbf{k}}$ is the wavevector,

$$\bar{\mathbf{k}} = \bar{k}_1 \hat{\mathbf{e}}_{x^1} + \bar{k}_2 \hat{\mathbf{e}}_{x^2}, \quad (7.59)$$

where $\bar{k}_1$ and $\bar{k}_2$ are its components, parallel to the substrate plane, in the directions $\hat{\mathbf{e}}_{x^1}$ and $\hat{\mathbf{e}}_{x^2}$. For a given real frequency κ , the wavevector serves as an eigenvector, which we expect to be complex-valued, implying exponential attenuation of surface waves parallel to the slab.

Assuming a homogenised model where the wavevector is isotropic, we write $\bar{\mathbf{k}} = \bar{k} \hat{\mathbf{e}}$, where $\bar{k}$ is a complex wavenumber and $\hat{\mathbf{e}}$ represents an arbitrary direction in the substrate plane. By seeking quasi-periodic solutions of (7.46)-(7.49) with $v_i^\pm = 0$ and enforcing decay away from the slab, we determine that the macroscale

fields possess the form

$$v^\pm \propto e^{\mp(\bar{k}^2 - \kappa^2)^{1/2} \hat{\mathbf{e}}_z \cdot \mathbf{x}^\pm + i\bar{\mathbf{k}} \cdot \mathbf{x}^\pm}. \quad (7.60)$$

Based on the solution above, we can identify two characteristic length scales for the decay of surface waves:

$$\ell_n^{-1} = \text{Re}(\bar{k}^2 - \kappa^2)^{1/2} > 0, \quad (7.61)$$

defined as the length scale of decay normal to the substrate, and

$$\ell_p = \frac{1}{\text{Im}(\bar{k})}, \quad (7.62)$$

representing the length scale of attenuation parallel to the substrate.

The homogenised model can be used to study the surface waves supported by the slab in the case where the surface wavelength is long compared to the spacing, with surface wavelength comparable to ℓ only at sufficiently low loss. Thus, substituting (7.60) into the impedance boundary conditions (7.47), we find the approximate dispersion relation

$$(\bar{k}^2 - \kappa^2)^{1/2} = -2\pi Q \bar{f}^{-1}. \quad (7.63)$$

Figure 7.2 shows the real and imaginary components of $\bar{k}$ as a function of κ . As the resonance frequency k is approached, thermoviscous effects enter, causing the dispersion curve to fold and turn around — a feature absent in the inviscid case. In the limit $(\bar{k}, \kappa) \rightarrow (0, k)$, both the decay and propagation lengths increase to infinity, $\ell_n, \ell_p \rightarrow \infty$. From (7.60), we observe that these surface waves transform into incident waves from infinity towards the slab, corresponding to coherent perfect absorption of the incident waves by the metasurface. In this limit, the waves effectively become stationary and are fully absorbed, thus remarkably connecting two seemingly very different phenomena.

7.5 Plane-wave forcing on tuned and detuned large metasurfaces

We present numerical results of the multiple-scattering model for finite metasurfaces, focusing on the response to normally incident plane waves on both sides of the substrate. Two scenarios are considered: a ‘tuned’ metasurface, where the loss parameter δ [cf. (6.58)] is tuned to the critical-coupling condition for coherent

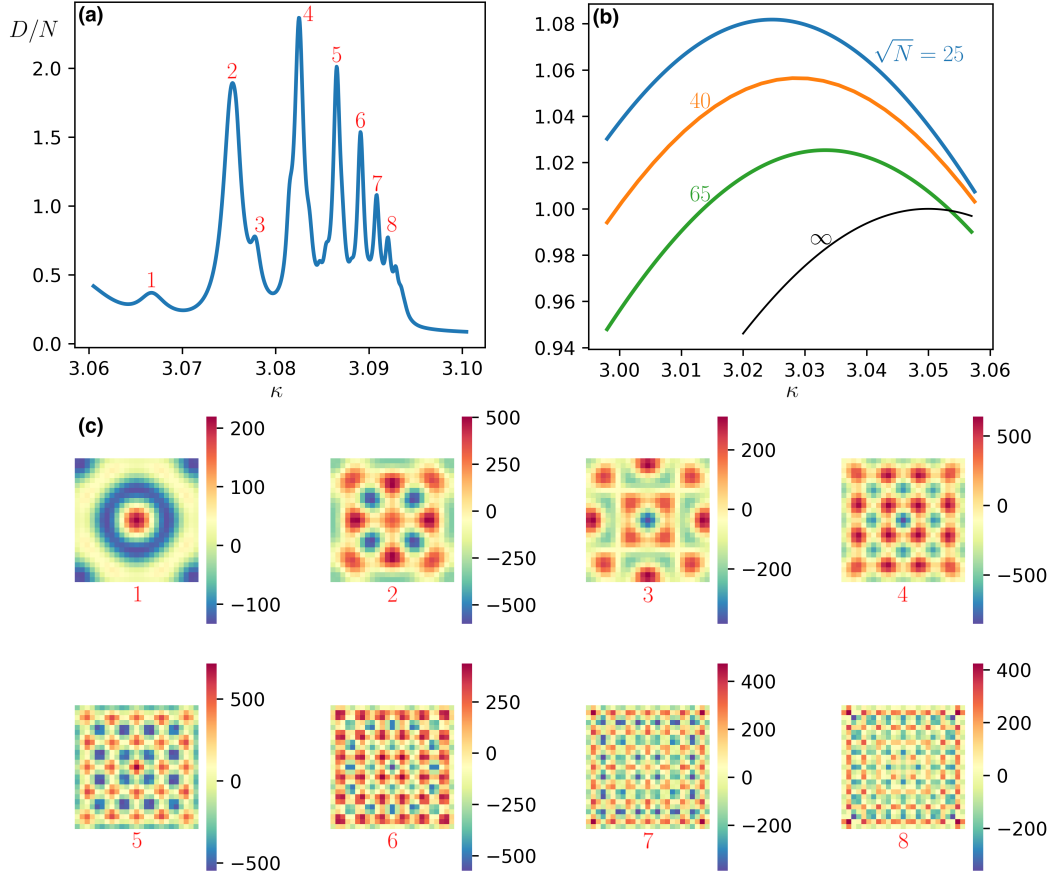


Figure 7.3: Average dissipation per hole for plane waves normally incident on (a) detuned and (b) tuned metasurfaces of finite size $\sqrt{N} = 25$ and $\sqrt{N} = 25, 40, 65$, respectively, computed using the multiple-scattering model, compared to an infinite metasurface approximated by the homogenised model. (c) Imaginary parts of the scattering coefficients $\mathcal{A}_n$ for the detuned scenario, at the meta-resonance peaks indicated in (a).

perfect absorption, assuming an infinite metasurface ($\simeq \varepsilon^3/\ell^2$, as derived in §7.3.3); and a low-loss ‘detuned’ metasurface, where δ is not tuned but rather set to the critical-coupling condition for a single hole ($\simeq \varepsilon^3$, as derived in §6.3.3).

Figures 7.3(a) and 7.3(b) display the average dissipation per hole as a function of frequency for detuned and tuned finite-size metasurfaces ($\sqrt{N} = 25$), respectively. Across most frequencies, the detuned metasurface exhibits significantly lower absorption efficiency than its tuned and theoretically infinite counterparts. (Using equations (7.10), (7.18), (7.50) and (7.53), the tuned infinite metasurface achieves unit dissipation at resonance under our normalisation. The tuned finite metasurfaces are seen to be more efficient, converging to the infinite case with increasing N .) However, just below the resonance frequency κ , a narrow frequency band reveals enhanced absorption in the detuned metasurface, with a series of sharp peaks where the efficiency exceeds that of the tuned metasurface. These peaks can be attributed to the collective meta-resonances of the entire metasurface, which

correspond to the excitation of standing waves by the traversal of surface waves. The above features are demonstrated in Figure 7.3(c), where the standing-wave patterns are shown at eight spectral peaks. As the system approaches the cut-off frequency, we observe a progressive shortening of surface wavelengths.

For plane waves incident from a single side, the absorption profiles closely match those observed for two-sided incidence, albeit with reduced efficiency, as anticipated by approximation (7.4). In the numerical simulations presented in this chapter, we assume subwavelength hole spacing $\ell = 0.1$ and a hole radius $\varepsilon = 0.01$.

7.6 Concluding remarks

Using a *two*-sided variant of Foldy's method and the single-hole model with losses from preceding chapters, we analysed the absorption properties of large, finite metasurfaces with multiple subwavelength holes, relating this to the theory for infinite metasurfaces derived from a homogenised model. For tuned — i.e., critically coupled — metasurfaces, we observe that large, finite arrays exhibit absorption amplitudes comparable to those predicted by their infinite counterparts. Most significantly, we reveal a less expected feature in which, near meta-resonance frequencies, large, detuned — i.e., low-loss — metasurfaces act as efficient absorbers, achieving dissipation levels that surpass those of both tuned and infinite metasurfaces — an effect suggesting different functionalities, absent in infinite systems. These resonances manifest as sharp spectral peaks clustered near the surface wave propagation cut-off frequency, close to the resonance frequency of an isolated hole. This phenomenon is due to the formation of standing-wave patterns by short-wavelength surface waves across the metasurface. Such precise energy absorption at specific frequencies could potentially be exploited in sensing and filtering applications.

CHAPTER 8

Conclusion

In this thesis, we have employed asymptotic methods — including matched asymptotic expansions and homogenisation — together with scaling arguments to systematically develop new models of wave propagation in complex media. These models scale from subwavelength resonators to structured resonant systems, with potential applications to the modelling of real-world acoustic metasurfaces and metamaterials, involving resonance phenomena, complex geometries and realistic loss conditions. A novel aspect of our approach has been the systematic inclusion of thermoviscous dissipation effects from first principles.

Beginning with two-dimensional geometries featuring narrow slits (Chapter 2), we refined existing literature by developing relatively simple yet asymptotically precise models of extraordinary transmission, particularly highlighting the importance of aperture (diffractive) effects near resonance — details that were ignored in previous seminal works [113]. The asymptotic approximations derived in Chapter 2 demonstrated excellent agreement with both experiments [112] and numerical simulations [39] in lossless regimes. In Chapter 3, we extended our approach to include thermoviscous effects — emerging from even smaller scales — leading to improved agreement with experimental results over a larger parameter space [119]. We also established an electromagnetic-acoustic asymptotic analogy (Chapter 4), which, surprisingly, holds in the presence of dissipation owing to the specific nature of wave propagation in slitted domains. Our asymptotic theory admits numerous potential generalisations, including oblique incidence [103] and other geometries such as curled-channel slits, which are central to the design of extreme metamaterials capable of manipulating wave behaviour through coiling space [71, 123].

In Chapter 5, we naturally transitioned to three-dimensional models for slender holes, elucidating key differences between two- and three-dimensional systems. Notably, we identified a more pronounced channel field enhancement in three dimensions and the limitations of end corrections in two-dimensional geometries. In Chapter 6, we introduced a spectral approach that offers a more general, cleaner and physically insightful framework for analysing the response of acoustic resonators embedded in slabs — again, systematically accounting for thermoviscous effects near resonance. While both direct and spectral methods yielded comparable formulae for complex amplitudes and resonance shifts relative to standing wave frequencies (in the absence of radiation and dissipative damping), the spectral approach facilitates theory generalisation with fewer parameters and separable singularities of the problem. As no direct experimental comparisons could be made, future work should focus on numerical computations and extensions to non-cylindrical hole geometries [116].

Finally, in Chapter 7, we scaled up our asymptotic theory to multiple-sided and -holed metasurfaces, enabling an accurate and computationally efficient description of complex resonant structures. This detailed framework allowed us to explore critical-coupling conditions for extraordinary transmission and coherent perfect absorption, as well as the absorption characteristics of large finite and infinite metasurfaces — revealing the significant effect that excited surface waves may have on the power dissipated by the metasurfaces. Many interesting extensions of this work are possible, including the study of varying domain geometries, non-uniform resonator distributions, and interactions between multiple metasurface patches.

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APPENDIX A

Numerical solution

In this appendix, we outline the mode-matching scheme used to validate the asymptotic approximations for a narrow slit developed in Chapter 2. Following previous works [30, 8, 9, 92], we write, on account of the symmetry of the geometry about $y = 0$

$$\varphi = \frac{1}{2}(\varphi^e + \varphi^o), \quad (\text{A.1})$$

where $\varphi^e(x, y) = \varphi^e(x, -y)$ and $\varphi^o(x, y) = -\varphi^o(x, -y)$. The potentials $\varphi^{e,o}$ only need to be found for $y < 0$ in conjunction with the Neumann and Dirichlet boundary conditions

$$\frac{\partial \varphi^e}{\partial y} = \varphi^o = 0 \quad \text{at} \quad y = 0; \quad (\text{A.2})$$

as $y \rightarrow -\infty$, both of the potentials $\varphi^{e,o}$ approach the incident wave $\exp(i\Omega y)$ with any corrections representing waves propagating away from the wall. On account of the symmetry of the problem about $x = 0$, we have that $\varphi^e(x, y) = \varphi^e(-x, y)$ and $\varphi^o(x, y) = \varphi^o(-x, y)$. The transmission efficiency τ [cf. (2.4)] can be calculated as

$$\tau = \frac{1}{8h\Omega} \int_{-h}^h \text{Im} \left(\varphi^{e*} \frac{\partial \varphi^o}{\partial y} \right)_{y=0} dx. \quad (\text{A.3})$$

In the slit, the potentials $\varphi^{e,o}$ can be represented in terms of cosine Fourier series

$$\begin{Bmatrix} \varphi^e \\ \varphi^o \end{Bmatrix} = 2 \sum_{n=0}^{\infty} a_n \begin{Bmatrix} \cos(\lambda_n y) \\ i \sin(\lambda_n y) \end{Bmatrix} \cos\left(\frac{n\pi x}{h}\right), \quad (\text{A.4})$$

where

$$\lambda_n = \begin{cases} \sqrt{(\Omega^2 - \frac{n^2\pi^2}{h^2})}, & \frac{n^2\pi^2}{h^2} < \Omega^2, \\ i\sqrt{(\frac{n^2\pi^2}{h^2} - \Omega^2)}, & \frac{n^2\pi^2}{h^2} > \Omega^2. \end{cases} \quad (\text{A.5})$$

Outside the slit, for $y < -1/2$, the potentials are composed of the incident wave, a reflected wave and a diffractive contribution which can be represented using a Fourier cosine transform:

$$\varphi = e^{i\Omega y} + e^{-i\Omega} e^{-i\Omega y} + \sqrt{\frac{2}{\pi}} \int_0^\infty R(Q) e^{-i\gamma(Q,\Omega)y} \cos(Qx) dQ, \quad (\text{A.6})$$

where

$$\gamma(Q, \Omega) = \begin{cases} \sqrt{\Omega^2 - Q^2}, & \Omega > Q, \\ i\sqrt{Q^2 - \Omega^2}, & 0 \leq \Omega \leq Q. \end{cases} \quad (\text{A.7})$$

The reflected term in (A.6) corresponds to the delta function in the cosine transform and the integral represents diffraction.

Continuity of $\partial\varphi^{e,o}/\partial y$ at $y = -1/2$, for $|x| < h$, in conjunction with the Neumann conditions on the external potentials for $|x| > h$, yields $R(Q)$ as:

$$R(Q) = \frac{2ie^{-i\gamma(Q,\Omega)/2}}{\gamma(Q,\Omega)} \sqrt{\frac{2}{\pi}} \sum_{n=0}^\infty a_n \begin{Bmatrix} \lambda_n \sin(\frac{\lambda_n}{2}) \\ i\lambda_n \cos(\frac{\lambda_n}{2}) \end{Bmatrix} \frac{h^2 Q (-1)^n \sin(hQ)}{h^2 Q^2 - \pi^2 n^2}. \quad (\text{A.8})$$

This expression for $R(Q)$, together with continuity of the potentials at $y = -1/2$, for $|x| < h$, gives an infinite linear set of algebraic equations

$$\begin{aligned} \frac{4ih^4}{\pi} \sum_{n=0}^\infty a_n \begin{Bmatrix} \lambda_n \sin(\frac{\lambda_n}{2}) \\ i\lambda_n \cos(\frac{\lambda_n}{2}) \end{Bmatrix} (-1)^{n+j} \int_0^\infty \frac{Q^2 \sin^2(hQ) dQ}{\gamma(Q,\Omega)(h^2 Q^2 - \pi^2 n^2)(h^2 Q^2 - \pi^2 j^2)} \\ - (1 + \delta_{j,0}) h a_j \begin{Bmatrix} \cos(\frac{\lambda_j}{2}) \\ -i \sin(\frac{\lambda_j}{2}) \end{Bmatrix} = -2e^{-\frac{i\Omega}{2}} h \delta_{j,0}, \quad j = 0, 1, 2, \dots \end{aligned} \quad (\text{A.9})$$

The linear system is solved by controlled-error truncation. In practice, the following re-scalings are useful in order to capture the fast attenuation in magnitude of the coefficients for large n :

$$\begin{Bmatrix} \tilde{a}_n^e \\ \tilde{a}_n^o \end{Bmatrix} = \begin{Bmatrix} a_n^e \cos(\frac{\lambda_n}{2}) \\ a_n^o \sin(\frac{\lambda_n}{2}) \end{Bmatrix}. \quad (\text{A.10})$$

We also note that the integrable singularities appearing in the coefficients in (A.9) can be analytically resolved for improved efficiency. Finally, once (A.9) is solved, the transmission efficiency (A.3) is calculated as

$$\tau = \text{Re} \left(a_0^{e*} a_0^o + \frac{1}{2\Omega} \sum_{n=1}^\infty a_n^{e*} a_n^o \lambda_n \right). \quad (\text{A.11})$$

(The singularities at the corners ($x = \pm h$, $y = \pm 1/2$) are localised and associated with finite energy. Although our numerical solver, which employs regularisation, converges without explicit treatment of these singularities, the edge conditions should enforce the leading-order singularity $\varphi(r, \phi) = r^{2/3}g(\phi)$, where r and ϕ represent the radial and angular coordinates, respectively.)

APPENDIX B

Energy-dissipation balance in narrow slits

B.1 Acoustic energy-dissipation relation

Following the formulation in Chapter 3, let us define the respective pressure fields φ^- and φ^+ in the left ($y < -1/2$) and right ($y > 1/2$) exterior regions, which are separated by a narrow slit. We can express an energy-dissipation relation to leading order [cf. §3.2.2] in terms of these fields

$$D = \frac{1}{2} \int_{\mathcal{D}_\lambda^-} \text{Im} (\varphi^- \nabla \varphi^{-*}) \cdot \hat{\mathbf{n}} d\mathbf{l} + \frac{1}{2} \int_{\mathcal{D}_\lambda^+} \text{Im} (\varphi^+ \nabla \varphi^{+*}) \cdot \hat{\mathbf{n}} d\mathbf{l}, \quad (\text{B.1})$$

where D is the dimensionless dissipated power. Here, the integration regions $\mathcal{D}_\lambda^\pm$ are defined as semicircles with radius $r^\pm = \lambda$, centred at the slit apertures, in $y < -1/2$ ($\mathcal{D}_\lambda^-$) and $y > 1/2$ ($\mathcal{D}_\lambda^+$). The unit vector $\hat{\mathbf{n}}$ denotes the outward-pointing normal on the semicircle boundaries.

The relation (B.1) emerges from employing the exact energy-dissipation relation (1.50) to the domains bounded by $\mathcal{D}_\lambda^\pm$ and the rigid boundaries within $|x| < \lambda$. Along these rigid surfaces, the boundary conditions $\mathbf{u} = \mathbf{0}$ and $T = 0$ ensure that the energy-flux vector vanishes. Furthermore, in the exterior domains, thermoviscous effects are negligible to leading order, simplifying the dimensionless energy-flux vectors (1.51) to

$$-\frac{1}{2} \text{Im} (\varphi^\pm \nabla \varphi^{\pm*}), \quad (\text{B.2})$$

which subsequently reduces (1.50) to (B.1).

B.2 Approximate energy balance

To leading order in the near-resonance regime [cf. §3.3], the exterior field solutions take the form

$$\varphi^- = e^{i\bar{\Omega}y} + Re^{-i\bar{\Omega}y} + Q^-\mathcal{H}_0(\bar{\Omega}r^-), \quad \varphi^+ = Q^+\mathcal{H}_0(\bar{\Omega}r^+), \quad (\text{B.3a}, b)$$

where $\bar{\Omega}$ is a resonance frequency [cf. (3.9)], $R = e^{-i\bar{\Omega}}$ is a reflection coefficient, $Q^\pm$ are diffraction coefficients [cf. (3.35)] and $\mathcal{H}_0$ is the zeroth-order Hankel function [1].

By using these solutions to evaluate the integrals in (B.1), we obtain

$$\frac{1}{2} \int_{\mathcal{D}_\lambda^-} \text{Im}(\varphi^- \nabla \varphi^{-*}) \cdot \hat{\mathbf{n}} d\mathbf{l} = -2 \text{Re} \left(Q^- e^{i\frac{\bar{\Omega}}{2}} \right) - |Q^-|^2 \quad (\text{B.4})$$

and

$$\frac{1}{2} \int_{\mathcal{D}_\lambda^+} \text{Im}(\varphi^+ \nabla \varphi^{+*}) \cdot \hat{\mathbf{n}} d\mathbf{l} = -|Q^+|^2. \quad (\text{B.5})$$

We then find the approximate energy balance

$$-2 \text{Re} \left(Q^- e^{i\frac{\bar{\Omega}}{2}} \right) = D + |Q^-|^2 + |Q^+|^2. \quad (\text{B.6})$$

The left side of (B.6) represents the reduction of the incident field due to the interaction with the system, while the final two terms on the right side correspond to the energy radiated in each exterior region [10].

In the absence of losses, (B.6) implies that radiative effects become significant when $Q^\pm = O(1)$. When dissipative effects are considered, balance among all terms in (B.6) is achieved when $D = O(1)$. Analysis of the boundary layer dissipation integrals in this distinguished limit reveals the scaling $\delta = O(h^2)$, consistent with the analysis in §3.3.2.

APPENDIX C

Maxwell's equations

In this appendix, we outline the equations that govern electromagnetic waves in conducting media. We consider a medium characterised by real-valued permittivity $\epsilon'\epsilon_0$ and permeability $\mu\mu_0$, with $\epsilon' = \mu = 1$ in a vacuum. For time-harmonic fields, we express the fields in phasor form, suppressing the time dependence $\exp(-i\omega t)$, where ω denotes the angular frequency and t represents time.

Let l be a characteristic length scale and $\tilde{H}_\infty$ the reference magnetic field strength. We normalise lengths by l , and define normalised fields $\mathbf{D}$ (electric displacement normalised by $\tilde{H}_\infty/\omega l$), $\mathbf{B}$ (magnetic induction normalised by $\mu_0\tilde{H}_\infty$), $\mathbf{E}$ (the electric field normalised by $\tilde{H}_\infty/\epsilon_0\omega l$) and $\mathbf{H}$ (the magnetic field normalised by $\tilde{H}_\infty$). The normalised macroscopic Maxwell's equations [54] are then

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (\text{C.1})$$

$$\nabla \cdot \mathbf{B} = 0, \quad (\text{C.2})$$

$$\nabla \times \mathbf{E} = i\mathbf{B}, \quad (\text{C.3})$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f - i\mathbf{D}, \quad (\text{C.4})$$

where ρ_f is the free charge density (normalised by $\tilde{H}_\infty/\omega l^2$) and $\mathbf{J}_f$ is the free current density (normalised by $\tilde{H}_\infty/l$).

Assuming the absence of free charges ($\rho_f = 0$), and that the medium both obeys Ohm's law and is linear, we have the following relations

$$\mathbf{D} = \epsilon'\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{J}_f = \frac{1}{\rho}\mathbf{E}, \quad (\text{C.5a} - \text{c})$$

where ρ is the real-valued resistivity (normalised by $1/\epsilon_0\omega$).

Substituting the constitutive relations (C.5) into (C.1)-(C.4) and taking the curl of (C.4), while using (C.2), (C.3) and the vector identity $\nabla \times (\nabla \times \mathbf{H}) = \nabla \cdot (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}$, we obtain the vector reduced-wave equation

$$\nabla^2 \mathbf{H} + \mu \epsilon \mathbf{H} = 0, \quad (\text{C.6})$$

where $\epsilon = \epsilon' + i\epsilon''$ is the complex-valued permittivity, with $\epsilon'' = 1/\rho$.

We now consider the two-dimensional problem from Chapter 4, using Cartesian coordinates (x, y) and restricting to TM-polarized waves. In this context, the magnetic field is perpendicular to the propagation plane, defined as $\mathbf{H} = H(x, y)\hat{\mathbf{e}}_z$. (H is independent of z by (C.2) and $\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y$.) Under these conditions, (C.6) reduces to

$$\nabla^2 H + \mu \epsilon H = 0. \quad (\text{C.7})$$

The normal components $\hat{\mathbf{n}} \cdot \mathbf{D}$ and $\hat{\mathbf{n}} \cdot \mathbf{B}$ must be continuous at the interface between two media with no free charges. For TM polarization, the boundary conditions, in conjunction with (C.5b), suggest that

$$\mu H \text{ is continuous across the interface.} \quad (\text{C.8})$$

Furthermore, using (C.4), (C.5c) and (C.5b), we find that

$$\frac{1}{\epsilon} \nabla H \cdot \hat{\mathbf{n}} \text{ is continuous across the interface.} \quad (\text{C.9})$$

The wave equation (C.7) and these boundary conditions are used in the chapter.

APPENDIX D

Canonical problem for a cylindrical hole opening

D.1 Boundary conditions

In this appendix, we address the canonical problem from Chapters 5 and 6, for a unit-radius cylindrical hole aperture into a half-space, where the potential field φ is governed by Laplace's equation. The problem is defined by the following limiting boundary conditions:

$$\varphi \sim o\left(\frac{1}{R}\right) \quad \text{as } R \rightarrow \infty, \quad (\text{D.1})$$

and

$$\varphi \sim Z + \beta + o(1) \quad \text{as } Z \rightarrow -\infty, \quad (\text{D.2})$$

where β is a constant to be determined as part of the solution. We will apply a mode matching method [63, 121] to solve this boundary value problem.

D.2 Domain solutions

In the exterior domain, the axisymmetric potential $\Phi(R, Z)$ satisfies Laplace's equation,

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \Phi}{\partial R} \right) + \frac{\partial^2 \Phi}{\partial Z^2} = 0. \quad (\text{D.3})$$

We apply the Hankel transform [108] to (D.3) to facilitate the solution of Laplace's equation in cylindrical coordinates

$$-s^2 \hat{\Phi}(s, Z) + \frac{\partial^2 \hat{\Phi}}{\partial Z^2}(s, Z) = 0. \quad (\text{D.4})$$

Solving this transformed equation, we find

$$\hat{\Phi}(s, Z) = A(s)e^{-sZ} + B(s)e^{sZ}, \quad (\text{D.5})$$

where $B(s) = 0$ ensures decay as $Z \rightarrow \infty$. Applying the inverse Hankel transform, we then obtain the external potential solution

$$\Phi(R, Z) = \int_0^\infty A(s)e^{-sZ}J_0(sR)sds, \quad (\text{D.6})$$

where J_0 denotes the Bessel function of the first kind of order zero.

In the interior domain, the potential function $\bar{\Phi}(R, Z)$ also satisfies Laplace's equation,

$$\frac{\partial^2 \bar{\Phi}}{\partial R^2} + \frac{1}{R} \frac{\partial \bar{\Phi}}{\partial R} + \frac{\partial^2 \bar{\Phi}}{\partial Z^2} = 0. \quad (\text{D.7})$$

We apply the method of separation of variables by assuming a solution of the form

$$\bar{\Phi} = \mathcal{R}(R)\mathcal{Z}(Z). \quad (\text{D.8})$$

Substituting this form into (D.7) and separating variables gives

$$\frac{d^2 \mathcal{Z}}{dZ^2} - k^2 \mathcal{Z} = 0 \quad \text{and} \quad \frac{d^2 \mathcal{R}}{dR^2} + \frac{1}{R} \frac{d\mathcal{R}}{dR} + k^2 \mathcal{R} = 0. \quad (\text{D.9})$$

The solution to the $\mathcal{Z}(Z)$ -equation that decays as $Z \rightarrow -\infty$ is

$$\mathcal{Z} \propto e^{kZ}. \quad (\text{D.10})$$

The radial part's solution,

$$\mathcal{R} \propto J_0(kR), \quad (\text{D.11})$$

is a Bessel function of the first kind of order zero, since the solution is axisymmetric and there is a nonsingular requirement at the origin. The constant k is determined by the boundary condition

$$\left. \frac{\partial \bar{\Phi}}{\partial R} \right|_{R=1} = J'_0(k) = 0, \quad (\text{D.12})$$

which implies that $k = \lambda_n$, the zeros of the first derivative of J_0 . For the zeroth positive root $\lambda_0 = 0$,

$$\bar{\Phi} = Z + \beta, \quad (\text{D.13})$$

satisfies (D.2). Thus, the complete solution for the internal potential is given by

$$\bar{\Phi}(R, Z) = Z + \beta + \sum_{n=1}^{\infty} c_n J_0(\lambda_n R) e^{\lambda_n Z}. \quad (\text{D.14})$$

D.3 Continuity and the determination of β

To ensure the continuity of the potential field φ across the exterior and interior domains of the cylindrical aperture, we impose

$$\Phi(R, 0) = \bar{\Phi}(R, 0) \quad \text{for} \quad 0 \leq R < 1. \quad (\text{D.15})$$

Using the completeness property of Bessel functions, we express this continuity condition as integral equations

$$\int_0^1 \Phi(R, 0) J_0(\lambda_m R) R dR = \int_0^1 \bar{\Phi}(R, 0) J_0(\lambda_m R) R dR, \quad (\text{D.16})$$

where for $m = 0$:

$$\int_0^{\infty} A(s) J_1(s) ds = \frac{\beta}{2}; \quad (\text{D.17})$$

and for $m = 1, 2, \dots$:

$$\int_0^{\infty} \frac{-s^2 A(s) J_0(\lambda_m) J_1(s)}{\lambda_m^2 - s^2} ds = \frac{c_m}{2} [J_0(\lambda_m)]^2. \quad (\text{D.18})$$

By also ensuring the continuity of the derivative, we additionally impose

$$\frac{\partial \Phi}{\partial Z}(R, 0) = \frac{\partial \bar{\Phi}}{\partial Z}(R, 0) \quad \text{for} \quad 0 \leq R < 1. \quad (\text{D.19})$$

To express this condition conveniently, we convert it into the integral expression

$$\int_0^{\infty} \left(\frac{\partial \Phi}{\partial Z} \right)_{Z=0} J_0(sR) R dR = \int_0^1 \left(\frac{\partial \bar{\Phi}}{\partial Z} \right)_{Z=0} J_0(sR) R dR, \quad (\text{D.20})$$

with $\partial \Phi / \partial Z(R, 0) = 0$ for $R > 1$. This integral condition guarantees the conservation of total flux across the two domains and yields the expression

$$A(s) = -\frac{J_1(s)}{s^2} - \sum_{n=1}^{\infty} \frac{-c_n \lambda_n J_0(\lambda_n) J_1(s)}{\lambda_n^2 - s^2}. \quad (\text{D.21})$$

Substituting (D.21) into (D.17) and (D.18), we derive equations for $m = 0$:

$$\frac{\beta}{2} - \sum_{n=1}^{\infty} c_n \lambda_n J_0(\lambda_n) \int_0^{\infty} \frac{[J_1(s)]^2}{\lambda_n^2 - s^2} ds = -\frac{4}{3\pi}, \quad (\text{D.22})$$

and $m = 1, 2, \dots$:

$$\frac{c_m}{2} [J_0(\lambda_m)]^2 + \sum_{n=1}^{\infty} c_n \lambda_n J_0(\lambda_n) J_0(\lambda_m) \int_0^{\infty} \frac{s^2 [J_1(s)]^2}{(\lambda_n^2 - s^2)(\lambda_m^2 - s^2)} ds = J_0(\lambda_m) \int_0^{\infty} \frac{[J_1(s)]^2}{\lambda_m^2 - s^2} ds. \quad (\text{D.23})$$

Numerically solving this linear system, we find

$$\beta = -0.8217\dots \quad (\text{D.24})$$

This value of β is consistent with established results in the literature [95, 31, 32, 63, 87, 88, 121, 51, 13].