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On the Frobenius morphism and Mori theory

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July 2018
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Submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy in Pure Mathematics of Imperial College London

Declaration of Originality

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Abstract

This dissertation explores the interplay between the Frobenius morphism and the geometry of algebraic varieties. Firstly, it partakes in the development of the positive characteristic Minimal Model Program, with the outcomes of this part comprising: a counterexample to a question posed by Patakfalvi, Schwede, and Tucker on global F-regularity of log Fano varieties, the establishment of new vanishing results on log del Pezzo surfaces, the proof of rationality of Kawamata log terminal three-dimensional singularities in high characteristic, the development of a partial canonical bundle formula with applications to log abundance and low characteristic birational geometry. Secondly, this thesis contains a study of liftings of the Frobenius morphism and their relation to some classical problems in complex geometry. In particular, a conjecture of Buch, Thomsen, Lauritzen, and Mehta is solved, and Winkelmann's theorem is generalised to positive characteristic.

Acknowledgements

First, I would like to express my gratitude for my supervisor, Paolo Cascini, who greatly supported me during this vital first step of my mathematical career. His guidance, advice, and suggestions played a very important role during my PhD, and I benefited a lot from our numerous mathematical conversations.

Moreover, I would like to thank Christopher Hacon for his hospitality, discussions, and collaboration during my visits to the University of Utah. I am grateful to Hiromu Tanaka for our ample conversations, for very helpful suggestions pertaining to my research, and for our studying together at private reading seminars. I would like to thank Piotr Achinger and Maciek Zdanowicz for being always available to answer my questions, and for teaching me a lot about arithmetic aspects of algebraic geometry. I am also very grateful to Fabio Bernasconi, Yoshinori Gongyo, Diletta Martinelli, Mirko Mauri, Yusuke Nakamura, Johannes Nicaise, and Roberto Svaldi for very interesting mathematical conversations.

Last, I would like to thank Michael R. Jimenez for his help with correcting my English and for his great support throughout my PhD.

The code for the table of contents is based on “<https://tex.stackexchange.com/questions/35825/pretty-table-of-contents>”.

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Introduction

FOR HUNDREDS OF YEARS, mathematicians have craved for an understanding of the geometry lurking behind the most basic equations, polynomials. The geometric objects they describe, called *algebraic varieties*, captivated generations of researchers, who devoted their careers to the meticulous study of low dimensional examples. While in the early 20th century members of the famous Italian school of algebraic geometry finally concluded the classification of curves and surfaces, the task of grasping the structure of higher-dimensional varieties seemed to be insurmountable.

The new era of higher-dimensional algebraic geometry started with the celebrated proof by Shigefumi Mori of the Hartshorne Conjecture, which predicted that the only smooth projective variety with an ample tangent bundle is $\mathbb{P}^n$. The techniques he introduced revolutionised algebraic geometry and led to many applications.

These applications came in two major groups. First and foremost, ideas motivated by Mori's results allowed for a comprehensive study of the invariants of the *canonical sheaf* ω_X , laying the foundations for the Minimal Model Program – a program that provides a precise recipe for building every mildly-singular (*Kawamata log terminal*) projective variety from varieties with simple geometries, that is negatively-curved, flat, and positively-curved. The validity of the program in dimension three over complex numbers had been proved by Kollár, Kawamata, and Mori, among others, but the general theory remained open for many years. It was not until 2006, twenty years after the threefolds case had been solved, that a breakthrough was achieved. But that year, news broke out and electrified the community of algebraic geometers: four mathematicians went down in history for cracking the problem for varieties of general type ([BCHM10]). Their landmark result, built upon the work of V. Shokurov (cf. [Cor07]), was a major step in the completion of the Minimal Model Program.

Secondly, the proof of the Hartshorne conjecture provided a framework for the study of the geometry of higher-dimensional varieties through the analysis of deformations of rational curves and, consequently, the invariants of the *cotangent bundle* Ω_X . One of the many applications of Mori's results was Lazarsfeld's positive answer to the conjecture of Remmert and Van de Ven, which states that the only smooth variety that the projective space can map surjectively onto, is the projective space itself. Moreover, abelian, Hermitian-symmetric, and – conjecturally – homogeneous varieties, can also be characterised by way of studying Ω_X . Therefore, Remmert and Van de Ven type problems could be considered for a variety of cases ([OW02], [PS89], etc.).

The major component of the proof of Hartshorne's conjecture came from an observation that, in contrary to the complex setting, every algebraic variety defined over a field of characteristic $p > 0$ admits an endomorphism, called the *Frobenius morphism*, which on the level of rings acts as taking the p -th power:

$$F: R \rightarrow R, \quad r \mapsto r^p, \text{ where } r \in R.$$

Mori's insight was that the study of the embedded deformations of the Frobenius morphism of a curve on a Fano variety, allows to deform (*bend*) and decompose (*break*) the curve into a union of rational curves, thereby showing that smooth Fano varieties defined over a positive characteristic field (and hence Fano varieties defined over $\mathbb{C}$ as well, by means of reduction modulo p) are covered by rational curves. This *bend-and-break* technique, which I believe to be one of the most beautiful results in the history of mathematics, marked the first major application of the Frobenius morphism in birational geometry. Shortly thereafter, new links between Frobenius and classical geometry were discovered.

The main objective of this PhD thesis is to explore further the interplay between classical geometry and the arithmetic properties of the Frobenius morphism, by both discovering new interrelationships between them, and drawing upon the already-established ones to provide geometric applications. Accordingly, the vital part of this thesis pertains to the study of the *canonical sheaf* ω_X , and the *cotangent bundle* Ω_X , by means of techniques inspired by the splittings $F^*\omega_X \rightarrow \omega_X$ of the *Frobenius trace map* and $F^*\Omega_X \rightarrow \Omega_X$ of the *Cartier operator*, where the former may be seen as a linearisation of the latter.

1.1. The Minimal Model Program and Frobenius splittings

Although the foundations of the Minimal Model Program over $\mathbb{C}$ trace back to the positive characteristic bend-and-break technique, until very recently little was known about the Minimal Model Program in characteristic $p > 0$. One of the most fundamental problems in this theory is to show the existence of the basic geometric operations: *flips* and *contractions*. The main tool used in complex birational geometry to achieve this goal is the

Kawamata–Viehweg vanishing theorem (a generalisation of Kodaira vanishing), but, unfortunately, the vanishing results fail in positive characteristic. Nevertheless, at the turn of the millennium, Keel managed to show the existence of contractions of threefolds over $\overline{\mathbb{F}}_p$ with an intriguing application of Artin’s results ([Kee99]). The case of flips, however, remained unsolved for many years, until a new actor emerged on the scene: *a splitting of the Frobenius trace map*.

Splittings of the Frobenius trace map had played a revolutionary role in commutative algebra since 1980s by having allowed for the proofs of difficult results in a surprisingly easy way (cf. [HH90]). Yet, what came about as the most striking is that singularities arising from this theory possess unbelievable similarities to those from the Minimal Model Program. This observation eventually led to breakthroughs in both areas – one of the most fundamental off-springs of this interplay being the proof of the existence of flips, and thereby the development of the Minimal Model Program for threefolds in positive characteristic $p > 5$ ([HX15]). This inspired further advancements: the existence of *Mori fibre spaces* [CTX15], and the validity of the logarithmic MMP [Bir16, BW17] (see [Tan14] for the case of dimension two). The mantra underlying this recent progress is that Frobenius splittings work as a replacement of the vanishing theorems.

Upon the outset of my PhD, the state-of-the-art naturally motivated the following questions:

- (i) *Is it possible to extend the MMP to higher dimensions in large characteristic?*
- (ii) *Can we generalise other results in birational geometry to positive characteristic?*
- (iii) *Is the MMP for threefolds valid in low characteristic?*

In order to explain my contribution to these questions, some definitions need to be introduced.

Definition 1.1.1. We say that a variety X over a perfect field k is *globally F -split* if the Frobenius morphism $F: \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ splits.

This is equivalent to a splitting of the Frobenius trace map by Grothendieck duality. Similarly, one can define a stronger notion of *global F -regularity*. When X is a singularity, we use names *F -purity* and *strong F -regularity*, respectively. The aforementioned fundamental observation about the interplay between Frobenius splittings and birational geometry can be summarised as follows:

$$\begin{aligned}
 \text{globally } F\text{-split} &\implies \text{log Calabi-Yau type} \\
 F\text{-pure} &\implies \text{log canonical} \\
 \text{globally } F\text{-regular} &\stackrel{(A)}{\implies} \text{log Fano type} \\
 \text{strongly } F\text{-regular} &\stackrel{(B)}{\implies} \text{Kawamata log terminal}
 \end{aligned}$$

We say that X is of *log Fano type*, if there exists a $\mathbb{Q}$ -divisor Δ such that (X, Δ) is Kawamata log terminal (klt) and $-(K_X + \Delta)$ is ample. One defines varieties of *log Calabi-Yau type* similarly.

The implications in the other direction are, in general, not true: trying to understand when they *can* be reversed is a major problem in the theory of F-splittings.

1.1.1. Log Fano and globally F-regular varieties in dimension two

This subsection summarises Chapters 3 and 4 of this thesis, which are based on projects in collaboration with P. Cascini and H. Tanaka ([CTW18, CTW17]).

The proof of the existence of flips for threefolds relies on the following results valid in characteristic $p > 5$ only (see [Wat91, Theorem 4.2], [Har98a]):

- (A) One-dimensional log Fano varieties with standard coefficients are globally F-regular.
- (B) Two-dimensional klt singularities are strongly F-regular.

In other words, Implications (A) and (B) can be reversed for $p > 5$ in dimension one and two, respectively, modulo a natural assumption on the coefficients. Here, coefficients of a divisor are called *standard* if they are of the form $1 - \frac{1}{m}$, where $m \in \mathbb{N}$; such coefficients occur in the context of the adjunction formula. Moreover, Cascini, Gongyo, and Schwede showed that this result also holds in the logarithmic setting for arbitrary klt boundaries Δ and big enough characteristic depending only on the set of coefficients of Δ ([CGS16]).

Therefore, with an aim at showing that the Minimal Model Program holds in dimension $n \geq 4$ and characteristic $p \gg 0$, it is natural to ask whether the above results are valid in higher dimensions (see [PST14, Question 5.23]). Unfortunately, this is not true, as shown by us in [CTW18].

Theorem 1.1.2 (Theorem 3.1.1). *Over any algebraically closed field k of characteristic $p > 0$, there exists a projective Kawamata log terminal surface X such that $-K_X$ is ample and X is not globally F-split.*

Theorem 1.1.3 (Theorem 3.1.2). *Over any algebraically closed field k of characteristic $p > 0$, there exists a three-dimensional canonical variety X which is not F-pure.*

What came from the above work was a better understanding of possible counterexamples. In particular, by means of so called *hunt*, we were able to show that these log del Pezzo surfaces defined in characteristic $p \gg 0$ which are *not* globally F-regular can be constructed from a finite, but not explicit, list of deformation classes by way of running a “reversed Minimal Model Program”. Consequently, we obtained:

Theorem 1.1.4 (Theorem 4.1.1). *Let (X, Δ) be a two dimensional projective klt pair defined over an algebraically closed field of characteristic $p \gg 0$ (depending only on the set of coefficients of Δ) such that $-(K_X + \Delta)$ is ample. Then*

1. (X, Δ) is globally F-regular, or

2. *it lifts to characteristic zero together with its singularities (see Theorem 4.1.1 for the precise statement).*

In other words, Implication (A) is almost reversible for surfaces in characteristic $p \gg 0$; all the varieties for which it is not reversible come anyway from characteristic zero. Consequently, using the above result, the Minimal Model Program, and the logarithmic variant of a result of Deligne–Illusie, we were able to prove the Kawamata–Viehweg vanishing theorem for klt log del Pezzo surfaces in large characteristic:

Theorem 1.1.5 (Theorem 4.1.2). *There exists a number $p_0 > 0$ such that Kawamata–Viehweg vanishing holds for any surface S of log Fano type defined over an algebraically closed field of characteristic $p > p_0$.*

Although Theorem 1.1.4 does not allow for a direct generalisation of the existence of flips to higher dimensions, it paves the way for a study of three-dimensional klt singularities; one such application is described in Subsection 1.1.3. But before proceeding to it, let me describe another project related to the above theorems.

1.1.2. Fujita-type bounds for surfaces

This subsection summarises Chapter 5 of this thesis, which is based on [Wit17a].

An important missing piece in the low dimensional positive characteristic birational geometry is Kollar’s effective base point free theorem, as the lack thereof has substantially impeded the development of the boundedness aspects of the Minimal Model Program. A very special case of this problem, one that motivated [Wit17a], is to find a natural number $k \in \mathbb{N}$ such that kK_X is very ample when X is of dimension two and K_X is ample.

The following theorem shows that if X is a Kawamata log terminal surface with K_X ample, and defined in characteristic $p > 5$, then $58mK_X$ is very ample for m being the Gorenstein index.

Theorem 1.1.6. *Let X be a projective surface with Kawamata log terminal singularities defined over an algebraically closed field of characteristic $p > 5$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Then, for an ample Cartier divisor A :*

- $4mK_X + 14mA$ is base point free, and
- $13mK_X + 45mA$ is very ample.

There are two main applications of the above theorem. Firstly, it implies the validity of Matsusaka type bounds for singular surfaces (see Corollary 5.1.2). Secondly, it allows for derivation of explicit bounds on the size of a bounded family of all ϵ -klt log del Pezzo pairs in positive characteristic (see Corollary 5.1.3). Having such explicit bounds facilitates (though, probably is not strictly necessary) a construction of a bounded family of all ϵ -klt log del Pezzo pairs in mixed characteristic, which in turn is an essential component used in the proof of Theorem 1.1.4.

1.1.3. Kawamata log terminal singularities in dimension three

This subsection summarises Chapter 6 of this thesis, which is based on a project in collaboration with C. Hacon ([HW17]).

As mentioned before, in order to apply the techniques of the Minimal Model Program (MMP) it is necessary to consider varieties with Kawamata log terminal singularities (klt). Therefore, it is essential to have a good understanding of their properties, and perhaps the most important of these is the fact that klt singularities are rational and Cohen-Macaulay. This fundamental result was first proved by Elkik for canonical singularities and later generalised to klt singularities (cf. [KM98, Theorem 5.22] and references therein).

The main technical tool used in the proof of Elkik's result is Kawamata-Viehweg vanishing, thus it is natural to ask if one could recover this result for three-dimensional singularities in large characteristic by means of Theorem 1.1.5. Note that strongly F-regular singularities are always rational, but this is not true, in general, for klt singularities; indeed, Gongyo, Nakamura, and Tanaka constructed an example of a klt threefold in characteristic $p = 2$ which is not rational ([GNT16], cf. [Ber17], [Kov17a], and [CT16]).

We managed to give a positive answer to the above question.

Theorem 1.1.7 (Theorem 6.1.1). *There exists a natural number $p_0 > 0$ such that for any three-dimensional Kawamata log terminal pair (X, Δ) defined over an algebraically closed field of characteristic $p > p_0$, we have that X has rational, Cohen-Macaulay singularities.*

One application of this result, which we provide, stems from the Minimal Model Program in mixed characteristic. Given a positive-characteristic variety X , which lifts to characteristic zero, and a basic operation of the MMP $f: X \dashrightarrow Y$ such as a contraction or a flip, it is natural to wonder if f lifts to characteristic zero as well. In particular, this would imply that minimal models of liftable varieties are liftable. As a corollary of Theorem 1.1.7, we showed that the liftability holds for divisorial and flipping contractions of threefolds in large characteristic (see Corollary 6.5.1). Unfortunately, we did not know how to deal with the case of flips.

1.1.4. Log abundance and the canonical bundle formula

This subsection summarises Chapter 7 of this thesis, which is based on [Wit17b]. The techniques described in this chapter do not utilise Frobenius splittings.

So far, we have only discussed issues around vanishing theorems and Frobenius splittings, but the lack of Kawamata-Viehweg vanishing is not the only obstruction to extending results from complex birational geometry to characteristic $p > 0$. Another stumbling block which impeded the development of positive characteristic birational geometry is the failure of the *canonical bundle formula*.

The canonical bundle formula is an important tool used in characteristic zero birational geometry to describe the behaviour of canonical divisors under log Calabi-Yau fibrations. Not only does this formula allow for a calculation of the canonical ring of a klt variety by

means of the MMP, but also plays a vital role in proofs of many fundamental results, like for instance the log abundance conjecture for threefolds over $\mathbb{C}$. Unfortunately, in general the canonical bundle formula is false in positive characteristic, even in a case of a smooth fibration from a surface to a curve.

The goal of Chapter 7 is to embody the idea that the canonical bundle formula in characteristic $p > 0$ can be partially recovered, if we replace the base of a log Calabi-Yau fibration by a purely inseparable cover, and, even more importantly, to provide various applications thereof. The statement of the main theorem of this chapter is presented in a simplified form below.

Theorem 1.1.8 (Theorem 7.1.1 and Theorem 7.1.2). *Let (X, Δ) be an n -dimensional projective log canonical pair defined over an algebraically closed field k of characteristic $p > 3$, let $\varphi: X \rightarrow Z$ be a contraction with the generic fibre of dimension one, and let L be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on Z satisfying $K_X + \Delta \sim_{\mathbb{Q}} \varphi^*L$.*

Then, with additional assumptions on the validity of the MMP and the nef dimension of L , we have that

$$L \sim_{\mathbb{Q}} K_Z + \Delta_Z,$$

up to replacing Z by a purely inseparable cover, where Δ_Z is a pseudo-effective $\mathbb{Q}$ -divisor.

The main application of this canonical bundle formula pertains to the *abundance conjecture*, which is one of the most fundamental open problems left in the study of the birational geometry of threefolds in characteristic $p > 5$, and it has gathered much attention recently, leading to very interesting results by Das, Hacon, Waldron, and Zhang (see [DW16, Wal17, Zha17]). It predicts that a log minimal model is either of general type or admits a “log Calabi-Yau” fibration over a lower dimensional variety.

A special case of abundance, and arguably the most vital part of its proof as well, is the *non-vanishing conjecture* asserting the $\mathbb{Q}$ -effectivity of the canonical divisor of a log minimal model. After the first version of my work ([Wit17b]) had been announced, Xu and Zhang showed the non-vanishing conjecture for terminal threefolds ([XZ18]). Having combined a generalisation of my initial results with their breakthrough theorem, I managed to obtain the non-vanishing conjecture for all three-dimensional klt pairs in characteristic $p > 5$. To the best of my knowledge, the result is new even when the boundary is empty.

Theorem 1.1.9 (Log non-vanishing, Theorem 7.1.3). *Let (X, Δ) be a projective Kawamata log terminal pair of dimension three defined over an algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is pseudo-effective. Then $\kappa(K_X + \Delta) \geq 0$.*

Moreover, building on the work of Zhang and Hacon, I could show the log abundance conjecture when the nef dimension is not maximal (see Subsection 7.2.3 for the definition of the nef dimension).

Theorem 1.1.10 (Log abundance for non-maximal nef dimension, Theorem 7.1.4). *Let (X, Δ) be a projective Kawamata log terminal pair of dimension three defined over an*

algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is nef and $n(K_X + \Delta) \leq 2$. Then $K_X + \Delta$ is semiample.

Theorem 1.1.10 seems to be a valuable step towards the proof of the log abundance conjecture for klt three-dimensional pairs. Indeed, the proof of the log abundance conjecture for three-dimensional klt pairs in characteristic zero (see [KMM94]) consists of three steps: showing abundance for terminal varieties, showing that any effective nef divisor on a klt Calabi-Yau variety is semiample, and showing log abundance for when $n(K_X + \Delta) \leq 2$. The proof of the second step is quite similar in nature to the proof of the first one. The above theorem concludes the third step of the proof of log abundance for threefolds in positive characteristic.

While the proof of log non-vanishing turned out to be the key application of Theorem 1.1.8, the initial motivation for my study stemmed, in fact, from my interest in low characteristic birational geometry, with a focus on working over $\overline{\mathbb{F}}_p$. This is discussed in the next subsection.

1.1.5. The Minimal Model Program over $\overline{\mathbb{F}}_p$

The fundamental problem of birational geometry is to show that certain divisors (like $K_X + \Delta$ in the abundance conjecture) are semiample as this allows for the construction of morphisms. All semiample line bundles are necessarily nef, but the converse is not true, the simplest example being that of a non-torsion point in $\text{Pic}^0(C)$ for an elliptic curve C over an uncountable field. However, constructing such examples over $\overline{\mathbb{F}}_p$ is more complicated given that all numerically trivial line bundles are semiample over this field.

The observation that nef line bundles are more often semiample over $\overline{\mathbb{F}}_p$ played a vital role in the development of the positive characteristic birational geometry, most notably in the aforementioned work of Keel, who showed the base point free theorem for big line bundles on normal threefolds over $\overline{\mathbb{F}}_p$, that is: if both L and $L - (K_X + \Delta)$ are nef and big for an effective $\mathbb{Q}$ -divisor Δ satisfying $[\Delta] = 0$, then L is semiample. His result entails the existence of divisorial and flipping contractions, but it is not strong enough to imply the existence of log Mori fibre spaces.

Employing the canonical bundle formula discussed above, I managed to obtain the full base point free theorem over $\overline{\mathbb{F}}_p$ under the assumption that $p > 2$.

Theorem 1.1.11 (Base point free theorem over $\overline{\mathbb{F}}_p$, Theorem 7.1.5). *Let (X, Δ) be a three-dimensional projective log canonical pair defined over $\overline{\mathbb{F}}_p$ for $p > 2$ and let L be a nef Cartier divisor such that $L - (K_X + \Delta)$ is nef and big. Then L is semiample.*

Although Theorem 1.1.11 was known to hold before when $p > 5$ for any algebraically closed field (see [BW17]), I believe that it is interesting for two-fold reason: to the best of my knowledge, it is the first substantial positive result in the MMP for threefolds in low characteristic since the seminal work of Keel, and further the proof is completely independent of the existence of flips in positive characteristic.

Perversely, Theorem 1.1.11, which is contained in the last article finished during my PhD, is related to [MNW15], which marked the beginning of my mathematical journey in the field of birational geometry. In this article, we studied how far the techniques of Keel can be pushed, hoping that through utilising the flexibility of $\overline{\mathbb{F}}_p$, we could come up with new proofs of foundational results in birational geometry. We provided a proof of concept for this idea by showing that Keel's theorem holds in the log canonical setting. Our proof employed a meticulous study of the *gluing* of semiampleness over $\overline{\mathbb{F}}_p$, that is of when the semiampleness of a line bundle L on a scheme $X = \bigcup X_i$ can be inferred from the semiampleness of $L|_{X_i}$.

Theorem 1.1.12 (Base point free theorem for big line bundles over $\overline{\mathbb{F}}_p$, Theorem [MNW15, Theorem 1.1]). *Let (X, Δ) be a three-dimensional projective log canonical pair defined over $\overline{\mathbb{F}}_p$ and let L be a nef and big Cartier divisor such that $L - (K_X + \Delta)$ is nef and big. Then L is semiample.*

What is surprising about this result is that it is false over $\mathbb{C}$ and all the uncountable algebraically closed fields of positive characteristic. Note that Theorem 1.1.12 accounts for the validity of Theorem 1.1.11 in the log canonical, and not just klt, setting. One of the key components of the proof of the above theorem, besides the *gluing* technique, was the following result.

Theorem 1.1.13 (Theorem [MNW15, Theorem 1.4]). *Let X be a normal projective surface defined over $\overline{\mathbb{F}}_p$ and let Δ be a pseudo-effective $\mathbb{Q}$ -divisor such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample.*

It is important to point out that although our research fell short of the expectation of substantially advancing positive characteristic birational geometry, it was a vital source of inspiration for the work described in Chapter 7. Indeed, Theorem 1.1.13 was indispensable in showing Theorem 1.1.11, while the ideas coming from its proof were vital in establishing Theorems 1.1.9 and 1.1.10.

Last, let me swiftly mention the work conducted in [NW17], which aimed at gaining a better understanding of new phenomena happening over $\overline{\mathbb{F}}_p$ and seeking results around the Minimal Model Program which are valid only over this field. In this article we showed that Theorem 1.1.12 holds for line bundles which are not necessarily big when $p > 5$ (note however that Theorem 1.1.11 supersedes it, as it is valid for $p > 2$), and verified that cone theorem does *not* hold in the log canonical setting over $\overline{\mathbb{F}}_p$. Due to the limitations on the length of theses, the material from [MNW15] and [NW17] could not be appended in this dissertation.

1.2. Splittings of the Cartier operator

The study of the splittings $F^*\Omega_X \rightarrow \Omega_X$ of the *Cartier morphism* (see Subsection 8.2.4 for the definition) traces back to the influential work of Deligne and Illusie ([DI87]) on

the validity of Kodaira vanishing for smooth varieties defined in characteristic $p > 0$ and which are liftable modulo p^2 . The key ingredient in their proof was the analysis of (local) liftings of Frobenius and their relation to the de Rham complex, namely given a lift $\tilde{X}$ over $\mathbb{Z}/p^2$ of a variety X defined over $\mathbb{F}_p$ and an endomorphism $\tilde{F}: \tilde{X} \rightarrow \tilde{X}$ which reduces to the Frobenius morphism modulo p , they constructed an injective homomorphism

$$\xi = \frac{d\tilde{F}}{p}: F^*\Omega_X^1 \longrightarrow \Omega_X^1 \quad (1.1)$$

whose adjoint $\Omega_X^1 \rightarrow F_*\Omega_X^1$ induces the inverse of the Cartier operator. In fact one can show that these two conditions are equivalent, that is, X lifts modulo p^2 together with Frobenius (X is *F-liftable*) if and only if the Cartier operator splits.

In contrast to F-regularity, Frobenius liftings have not yet gathered much attention among the community of classical algebraic geometers. They are, however, ubiquitous in the realm of arithmetic geometry: apart from playing an essential role in the algebraic proof of Kodaira vanishing by Deligne–Illusie, they allow to construct Scholze’s perfectoid spaces from algebraic varieties, provide the canonical lift of abelian varieties in Serre–Tate’s theory, serve as a substitute of hermitian metrics in works of Mochizuki, or even embody the notion of a field with one element $\mathbb{F}_1$ — the Philosopher’s Stone of arithmetic geometry which should transmute the proofs of Weil conjectures into a proof of the Riemann hypothesis.

The research undertaken in [AWZ17] with P. Achinger and M. Zdanowicz, and contained in Chapter 8, aimed for providing a comprehensive geometric treatment of Frobenius lifts for smooth projective varieties, and bringing the machinery of *F*-liftability to the disposal of classical geometers. It is my personal hope that in the near future Frobenius lifts will play as important a role in the study of invariants of Ω_X as *F*-splittings play now with regards to ω_X .

Our work was motivated by the following question:

*Which smooth projective varieties in characteristic $p > 0$ are *F*-liftable?*

Such varieties have been studied for the first time in [BTLM97], where it was shown that they satisfy a much stronger form of Kodaira vanishing called *Bott vanishing*

$$H^j(X, \Omega_X^i \otimes L) = 0 \quad (L \text{ ample, } i \geq 0, j > 0). \quad (1.2)$$

This type of vanishing is extremely restrictive, and hence so is *F*-liftability. In fact, all known examples are in some sense built from toric varieties (where the lifting of Frobenius extends the multiplication by p map on the torus) and ordinary abelian varieties (by the Serre–Tate theory). It is worth noting that the determinant $\det(\xi): F^*\omega_X \rightarrow \omega_X$ of ξ from (1.1) induces a splitting of the Frobenius trace map; in particular, *F*-liftable varieties are *F*-split.

The main goal of our work was to provide evidence for the following conjectural answer to the above question, and to relate it to some open problems in characteristic zero.

Conjecture 1.2.1 (Conjecture 8.1.1). *Let X be a smooth projective variety defined over an algebraically closed field k of characteristic $p > 0$. If X is F -liftable, then, up to a finite étale cover, it is a toric fibration over an abelian variety. In particular, if X is simply connected, then X is a toric variety.*

In layman's terms, the conjecture asserts that what distinguishes positive characteristic, that is the existence of the Frobenius morphism, becomes extremely rare as we move towards characteristic zero. This idea is not new: for example, in more classical algebraic geometry, it is expected [Ame97, Fak03] that complex projective varieties admitting a polarised endomorphism are very scarce.

A few important special cases had already been known for some time. In the paper [MS87], to which our work owed a great deal, Mehta and Srinivas proved (among other things) that if X is F -liftable and the canonical bundle ω_X is numerically trivial, then X is an ordinary abelian variety up to a finite étale cover. In a different direction, the case of homogeneous spaces was considered in the aforementioned paper [BTLM97], where it was shown that many rational homogeneous spaces are not F -liftable, and conjectured that the only F -liftable ones are products of projective spaces.

Although the full proof of Conjecture 1.2.1 was beyond our reach, we settled it in certain special cases. Firstly, we solved the conjecture of [BTLM97].

Theorem 1.2.2 (Theorem 8.1.2). *Conjecture 1.2.1 is true if X is a homogeneous space. More precisely, if X is a smooth projective F -liftable variety whose automorphism group acts transitively, then X is isomorphic to a product of projective spaces and an ordinary abelian variety.*

The proof is based on the study of deformations of rational curves and is, in my personal opinion, the most striking argument included in this thesis. In the crucial case of Picard rank one, our result is in fact stronger: we showed Conjecture 1.2.1 for when T_X is nef.

We also verified Conjecture 1.2.1 in low dimensions.

Theorem 1.2.3 (Theorem 8.1.3). *Conjecture 1.2.1 is true in the following cases.*

- (a) *If $\dim X \leq 2$.*
- (b) *If X is a Fano threefold from the Mori–Mukai classification.*

1.2.1. Remmert and Van de Ven type problems

The most striking feature of Frobenius liftability, which we discovered, is that it is strongly related to Remmert and Van de Ven type problems. Recall the general setting: given

a projective variety Z , we want to determine all *smooth* projective varieties X for which there exists a surjective morphism

$$\varphi: Z \longrightarrow X.$$

This problem has been considered for a variety of cases, for instance, Lazarsfeld showed that if $Z \simeq \mathbb{P}^n$, then $X \simeq \mathbb{P}^n$ as well, while in [Deb89, HM01, DHP08] it was proved that if Z is a complex abelian variety, then X must admit a finite étale cover by a product of an abelian variety and projective spaces. Subsequently, Occhetta and Wiśniewski ([OW02]) extended Lazarsfeld's result to when Z is a toric variety and X has Picard rank one, which in turn led them to pose the following conjecture.

Conjecture 1.2.4 (Occhetta and Wiśniewski). *A smooth complex projective variety X admitting a surjective map $\varphi: Z \rightarrow X$ from a complete toric variety Z is a toric variety.*

What gave momentum to our project was the discovery that Conjecture 1.2.1 provides a uniformised, coherent way of looking at many of the Remmert and Van de Ven type problems in characteristic zero; for instance, by means of reduction modulo p , we proved that it implies Occhetta-Wiśniewski conjecture.

Theorem 1.2.5 (Theorem 8.1.4). *Conjecture 1.2.1 for simply connected varieties implies Conjecture 1.2.4 in characteristic zero.*

1.2.2. Winkelmann's theorem in positive characteristic

Moreover, we studied the relation between Frobenius liftability and triviality of the sheaf $\Omega_X(\log D)$ of differentials with log poles along D , where (X, D) is a normal crossing pair defined over an algebraically closed field k . If k has characteristic zero, then by a result of Winkelmann [Win04], X admits an action of a semi-abelian variety which is transitive on $X \setminus D$; in particular, if Ω_X is free, then X is an abelian variety. However, in positive characteristic X need not be an abelian variety itself when Ω_X is free, but then the main result of [MS87] states that if X is ordinary, then it admits an abelian variety as a finite étale cover. It is not known if the ordinarity assumption is necessary.

We obtained the following generalisation of the theorem of Mehta and Srinivas; here being F -split works as an analogue of ordinarity, and F -liftability of snc pairs (X, D) is defined analogously to the F -liftability of X .

Theorem 1.2.6 (Theorem 8.1.6). *Let (X, D) be a projective nc pair defined over an algebraically closed field k of positive characteristic. The following conditions are equivalent.*

- (i) (X, D) is F -liftable and $\omega_X(D)$ is numerically trivial,
- (ii) X is F -split and $\Omega_X(\log D)$ becomes trivial on a finite étale cover of X ,
- (iii) X is, up to a finite étale cover, a toric fibration over an ordinary abelian variety with toric boundary D .

2

Preliminaries

2.1. Notation and conventions

Throughout this thesis, k is an arbitrary field of positive characteristic $p > 0$, unless stated otherwise. For a k -variety X , we denote the relative canonical divisor $K_{X/\mathrm{Spec} k}$ by K_X (see [Kol13, Definition 1.6]). In fact, we usually assume that k is algebraically closed, but this is not always the case in Chapters 6, 7, and 8.

We refer to [KM98] for basic definitions in birational geometry (e.g., *terminal*, *plt*, *log canonical*, *flip*), and to [Kol13] whenever k is not algebraically closed. We refer to [Tan16c] and [Tan14] for the main results of the Minimal Model Program for surfaces over an excellent field, and to [HX15], [CTX15], [Bir16], and [BW17] in the case of threefolds defined over an algebraically closed field k of characteristic $p > 5$.

- We say that X is a *variety* if it is integral, separated, and of finite type over a field k . A *curve* is a variety of dimension one, and a *surface* is a variety of dimension two.
- We call (X, Δ) a *log pair* if X is a normal variety and Δ is an effective $\mathbb{Q}$ -divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$ -Cartier. If Δ is not effective, then we call (X, Δ) a *sub-log pair*.
- Given a normal variety X and a Weil divisor D on X , we say that a birational morphism $f: Y \rightarrow X$ is a *log resolution* of (X, D) if Y is regular, $\mathrm{Ex}(f)$ has pure codimension one and $(Y, \mathrm{Supp}(f^{-1}(D) + \mathrm{Exc}(f)))$ has simple normal crossings (see [Kol13, Definition 1.8 and 1.12]). If X is of dimension at most three and k is excellent, then a log resolution of (X, D) always exists (see [Lip78], [Cut09], [CP08], and [CP09]).
- We say that a log pair (X, Δ) is *Kawamata log terminal* (*klt*, for short) if for every

birational morphism $g: Y \rightarrow X$, we have that $a_i < 1$, where

$$K_Y + \sum_{i=1}^r a_i E_i = g^*(K_X + \Delta)$$

and $E_1, \dots, E_r$ are the g -exceptional divisors. If a log resolution of $(X, \text{Supp } \Delta)$ exists, it is enough to verify the above condition for g being any such resolution.

- Given a log pair (X, Δ) and a divisorial valuation E over X , we denote by $a(E, X, \Delta)$ the *log discrepancy* of (X, Δ) with respect to E .
- For $\varepsilon > 0$, we say that (X, Δ) is ε -klt if $a(E, X, \Delta) > \varepsilon$ for any divisorial valuation E over X .
- We say that a pair (X, Δ) is a *log Fano* pair if $-(K_X + \Delta)$ is ample. In the case when $\dim(X) = 2$, we say that (X, Δ) is a *log del Pezzo* pair.
- Let $D \subseteq X$ be a divisor on a scheme X , let $\pi: \bar{D} \rightarrow D$ be the normalisation of D , and let L be any $\mathbb{Q}$ -Cartier divisor on X . Then by abuse of notation we denote $\pi^*L|_{\bar{D}}$ by $L|_{\bar{D}}$.
- For a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor D on a normal variety, the *Cartier index* of D is the smallest positive integer m such that mD is Cartier. The *Cartier index* of (X, Δ) is the minimal number $m \in \mathbb{N}$ such that $m(K_X + \Delta)$ is Cartier. If (X, Δ) is klt, then it must be $1/m$ -klt.
- Given a proper morphism $f: X \rightarrow Y$ between normal varieties, we say that two $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisors D_1, D_2 on X are *numerically equivalent over Y* , denoted $D_1 \equiv_f D_2$, if their difference is numerically trivial on any fibre of f .
- Let X be a normal variety, let S be a prime divisor on X such that $K_X + S$ is $\mathbb{Q}$ -Cartier, and let $S' \rightarrow S$ be its normalisation. Then there exists an effective $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor $\text{Diff}_{S'}$ on S' , called the *different* such that

$$(K_X + S)|_{S'} = K_{S'} + \text{Diff}_{S'}.$$

Moreover, if (X, S) is log canonical, then the coefficients of $\text{Diff}_{S'}$ belong to the *standard set*, that is $\{1\} \cup \{1 - \frac{1}{m} \mid m \in \mathbb{Z}_{>0}\}$ (see [K⁺92, Proposition-Definition 16.5] and [Kol13, Definition 4.2 and Proposition 4.5]).

- For any prime p , we denote by $\mathbb{Z}_{(p)}$ the set of rational numbers whose denominator is not divisible by p .
- We denote the base locus of a line bundle $\mathcal{L}$ by $\text{Bs}(\mathcal{L})$.

Note that by abuse of notation we use the notation for line bundles and the notation for divisors interchangeably. Finally, recall that any Kawamata log terminal surface has rational singularities, and is, in particular, $\mathbb{Q}$ -factorial (see [Tan14]).

2.2. F-splittings and F-regularity

Given a scheme X of characteristic $p > 0$, we denote by $F: X \rightarrow X$ the absolute Frobenius morphism, and we say that X is *F-finite* if $F_*\mathcal{O}_X$ is a finitely generated $\mathcal{O}_X$ -module. This is a technical condition ensuring that the definitions of F-splittings and F-regularity are well-behaved. Note that schemes of finite type over F-finite fields (e.g. perfect fields) are always F-finite.

One of the most amazing discoveries of singularity theory, is that properties of the Frobenius map may reflect how singular a variety is. This observation is based on the fact that for a smooth ring R the Frobenius morphism $F^*: R \rightarrow F_*R$ splits, yet this need not hold when R is singular. This led to a definition of *F-split rings*, rings R admitting such a splitting. For log pairs, we have the following.

Definition 2.2.1. Let X be an F-finite scheme defined over a field of characteristic $p > 0$, and let Δ be an effective $\mathbb{Q}$ -divisor. We say that the pair (X, Δ) is *globally F-split* if for every $e \in \mathbb{Z}_{>0}$, the natural map

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$$

splits in the category of sheaves of $\mathcal{O}_X$ -modules.

The definition of an F-splitting may seem a bit mysterious at the first glance. Let us untangle it. The Grothendieck duality gives

$$\mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{sm}}}}(F_*^e \mathcal{O}_{X^{\mathrm{sm}}}, \mathcal{O}_{X^{\mathrm{sm}}}) \simeq F_*^e \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{sm}}}}(\mathcal{O}_{X^{\mathrm{sm}}}, (F^e)^! \mathcal{O}_{X^{\mathrm{sm}}}) \simeq F_*^e H^0(X^{\mathrm{sm}}, \omega_{X^{\mathrm{sm}}}^{1-p^e}),$$

where X^{sm} is the smooth locus of X . This explains the following crucial proposition.

Proposition 2.2.2 ([Sch09, Theorem 3.11, 3.13]). *For a normal variety X , there is a natural bijection.*

$$\left\{ \begin{array}{l} \text{Non-zero } \mathcal{O}_X\text{-linear maps} \\ \varphi: F_*^e \mathcal{O}_X \rightarrow \mathcal{O}_X \text{ up to} \\ \text{pre-multiplication by units.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \text{ such that} \\ (1 - p^e)\Delta \sim -(1 - p^e)K_X. \end{array} \right\}$$

Note, that $(F^e)^*: R \rightarrow F_*^e R$ splits for a ring R if and only if there exists a morphism of R -modules $\varphi: F_*^e R \rightarrow R$ such that $1 \in \varphi(F_*^e R)$. This suggests that one should consider rings which are *F-split* “under all small perturbations”. More formally we say that R is *F-regular* if for every $c \in R \setminus 0$, there exists $e > 0$ and a map $\varphi: F_*^e R \rightarrow R$ such that $1 \in \varphi(F_*^e cR)$.

We can now recall the definitions of most important classes of *F-singularities* (e.g. see [SS10, Definition 3.1]):

Definition 2.2.3. Let X be an *F-finite* normal scheme of characteristic $p > 0$ and let Δ be an effective $\mathbb{R}$ -divisor on X .

1. The pair (X, Δ) is *globally F -split* if for every $e \in \mathbb{Z}_{>0}$, the natural map

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$$

splits in the category of sheaves of $\mathcal{O}_X$ -modules.

2. The pair (X, Δ) is *globally sharply F -split* if there exists $e \in \mathbb{Z}_{>0}$, such that the natural map

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$$

splits in the category of sheaves of $\mathcal{O}_X$ -modules.

3. The pair (X, Δ) is *globally F -regular* if for every effective divisor E on X , there exists $e \in \mathbb{Z}_{>0}$ such that the natural map

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor + E)$$

splits in the category of sheaves of $\mathcal{O}_X$ -modules.

4. The pair (X, Δ) is *purely globally F -regular* if the definition above holds for those E which intersect $\lfloor \Delta \rfloor$ properly (see [Das15, Definition 2.3(2)], [Tak08]).
5. The pair (X, Δ) is *F -pure* (resp. *sharply F -pure*, *strongly F -regular*, *purely F -regular*) if $(\text{Spec } \mathcal{O}_{X,x}, \Delta|_{\text{Spec } \mathcal{O}_{X,x}})$ is globally F -split (resp. globally sharply F -split, globally F -regular, purely globally F -regular) for every point $x \in X$.
6. Given a morphism $f: X \rightarrow Y$, we say that (X, Δ) is F -split, sharply F -split, F -regular, or purely F -regular over Y , if the corresponding splittings hold locally over Y (see [HX15, Definition 2.6]).

If the coefficients of Δ belong to $\mathbb{Z}_{(p)}$, then being sharply F -split coincides with being F -split. The local and global notions coincide for affine varieties.

Frobenius singularities are alleged to be the correct counterparts of birational singularities in positive characteristic. This supposition is propped up by the following results.

Proposition 2.2.4 ([SS10],[HW02]). *Let X be a variety defined over an F -finite field k .*

- *If (X, B) is globally F -regular (globally F -split, resp.), then (X, B) is of log Fano type (log Calabi-Yau type, resp.).*
- *If (X, B) is strongly F -regular (F -pure, resp.), then (X, B) is Kawamata log terminal (log canonical, resp.).*

We say that (X, B) is of *log Fano type* (log Calabi-Yau type, resp.) if there exists a $\mathbb{Q}$ -divisor $B' \geq B$ such that (X, B') is Kawamata log terminal (log canonical, resp.) and $-(K_X + B')$ is ample (trivial, resp.).

Proposition 2.2.5 ([ST17], [Har98b]). *Let X be a Kawamata log terminal surface defined over an F -finite field of characteristic $p > 5$. Then X is strongly F -regular.*

Furthermore, one can show that reductions of complex Kawamata log terminal singularities and projective log Fano varieties modulo all but a finite number of primes p (depending on the considered variety) are strongly F-regular and globally F-regular, respectively.

Last, let us recall some other fundamental results on F-splittings and F-regularity.

Lemma 2.2.6. *Let $f: X \rightarrow Y$ be a proper birational morphism of normal varieties defined over an F-finite field. Let Δ be an effective $\mathbb{Q}$ -divisor on X . If (X, Δ) is globally F-split, then so is $(Y, f_*\Delta)$.*

Proof. Let $e \in \mathbb{Z}_{>0}$. Since (X, Δ) is globally F-split, the Frobenius homomorphism induced by (X, Δ) splits:

$$\text{id}: \mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor) \xrightarrow{\exists \varphi} \mathcal{O}_X.$$

Consider an open subscheme $j: U \hookrightarrow Y$ of codimension at least two, such that the restriction of f to U induces an isomorphism $f^{-1}(U) \rightarrow U$. By restricting to U , we have that the Frobenius homomorphism splits on U :

$$\text{id}: \mathcal{O}_U \xrightarrow{F^e} F_*^e \mathcal{O}_U(\lfloor (p^e - 1)f_*\Delta \rfloor) \xrightarrow{\varphi|_U} \mathcal{O}_U.$$

Since $\text{codim}_Y Y \setminus U \geq 2$, we can conclude by taking the pushforward by j_* . $\square$

By a result of Hacon and Xu, also a variant of the opposite implication holds true:

Lemma 2.2.7 ([HX15, Prop. 2.11]). *Let (X, Δ) be a globally F-regular pair and $g: Y \rightarrow X$ a proper birational morphism of normal varieties defined over an F-finite field such that $g^*(K_X + \Delta) = K_Y + \Delta_Y$, for a $\mathbb{Q}$ -divisor $\Delta_Y \geq 0$ satisfying $g_*\Delta_Y = \Delta$.*

Then (Y, Δ_Y) is globally F-regular.

2.2.1. Frobenius trace map

A key tool used in the theory of Frobenius splittings is the Frobenius trace map (for instance, see [Tan15a]). It is defined as the Grothendieck dual $\text{Tr}_X: F_*\omega_X \rightarrow \omega_X$ of the Frobenius morphism $F^*: \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$. By tensoring this map by $\omega_X^\vee$, we get a morphism $\text{Tr}_X: F_*\omega_X^{1-p} \rightarrow \mathcal{O}_X$, which by abuse of notation we denote by the same symbol.

The following definition generalises this construction to the logarithmic setting. Recall that for an integral divisor D on a normal variety X , Grothendieck duality gives (see [HX15, Lemma 2.9]):

$$\text{Hom}_{\mathcal{O}_X}(F_*^e \mathcal{O}_X(D), \mathcal{O}_X) \simeq F_*^e \mathcal{O}_X(-(p^e - 1)K_X - D). \quad (2.1)$$

Definition 2.2.8. Let Δ be an effective $\mathbb{Q}$ -divisor such that $(p^e - 1)(K_X + \Delta)$ is Cartier for some $e > 0$. We call

$$\text{Tr}_{X, \Delta}^e: F_*^e \mathcal{O}_X(-(p^e - 1)(K_X + \Delta)) \rightarrow \mathcal{O}_X,$$

the *Frobenius trace map*. It is constructed by applying the above isomorphism (2.1) to the morphism

$$\mathrm{Hom}_{\mathcal{O}_X}(F_*^e \mathcal{O}_X((p^e - 1)\Delta), \mathcal{O}_X) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \quad (2.2)$$

being the dual of the composition $\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X((p^e - 1)\Delta)$ (cf. [Sch14, discussion below Proposition 2.5]).

The rank one sheaves in question are not necessary line bundles, but since X is normal, we can always restrict ourselves to the smooth locus. The assumption about the Cartier index can be dropped, but this will not be needed in this thesis. It is easy to verify that this assumption is equivalent to saying that the Cartier index of $K_X + \Delta$ is not divisible by p .

Proposition 2.2.9 ([HX15, Proposition 2.10]). *Let (X, Δ) be a log pair defined over an F -finite field such that the Cartier index of $K_X + \Delta$ is not divisible by p . Then (X, Δ) is F -pure at a point $x \in X$ if and only if the trace map*

$$\mathrm{Tr}_{X, \Delta}^e: F_*^e \mathcal{O}_X(-(p^e - 1)(K_X + \Delta)) \rightarrow \mathcal{O}_X$$

is surjective at x for all divisible enough $e \gg 0$. Moreover, (X, Δ) is globally F -split if and only if the trace map

$$\mathrm{Tr}_{X, \Delta}^e: H^0(X, F_*^e \mathcal{O}_X(-(p^e - 1)(K_X + \Delta))) \rightarrow H^0(X, \mathcal{O}_X)$$

is surjective at the level of global sections.

2.2.2. Inversion of F-adjunction

A fundamental tool for studying Kawamata log terminal singularities in characteristic zero is the inversion of adjunction, which states that a log pair $(X, S + B)$, where S is a prime divisor and $S \not\subseteq \mathrm{Supp} B$, is purely log terminal along S if and only if $(\tilde{S}, B_{\tilde{S}})$ is Kawamata log terminal, where $\tilde{S}$ is the normalisation of S and $B_{\tilde{S}}$ is defined by adjunction

$$(K_X + S + B)|_{\tilde{S}} = K_{\tilde{S}} + B_{\tilde{S}}.$$

A similar result is known for strongly F -regular singularities.

Theorem 2.2.10 ([Das15, Theorem A]). *Let $(X, S + B)$ be a log pair defined over an algebraically closed field k of characteristic $p > 0$, where $\lfloor B \rfloor = 0$ and S is a prime divisor not contained in $\mathrm{Supp} B$. Write $(K_X + S + B)|_{\tilde{S}} = K_{\tilde{S}} + B_{\tilde{S}}$, where $\tilde{S}$ is the normalisation of S . Then $(X, S + B)$ is purely F -regular along S if and only if $(\tilde{S}, B_{\tilde{S}})$ is strongly F -regular. Moreover, under these equivalent conditions, S is normal.*

A key technique which allows to show global F -regularity of projective schemes is the *global inversion of F -adjunction*. It played an important role in the proof of the existence

of three-dimensional flips ([HX15]), and is employed in this thesis extensively. For the convenience of the reader, we give an example of a basic statement of the global inversion of F -adjunction together with a sketch of a proof; a more thorough discussion may be found in Subsection 4.2.2.

Lemma 2.2.11. *Let $(X, S+B)$ be a projective log pair defined over an algebraically closed field k of characteristic $p > 0$, where $[B] = 0$ and S is a prime divisor not contained in $\text{Supp } B$. Assume that*

1. $-(K_X + S + B)$ is ample,
2. the Cartier index of $K_X + S + B$ is not divisible by p ,
3. S is normal and if $K_S + B_S = (K_X + S + B)|_S$ is defined by adjunction, then (S, B_S) is globally F -split.

Then $(X, S + B)$ is globally F -split.

A sketch of the proof. Condition (2) implies that $(p^e - 1)(K_X + S + B)$ is Cartier for sufficiently divisible positive integers e . Let $\mathcal{L} := \mathcal{O}_X(-(p^e - 1)(K_X + S + B))$. One can show the existence of the following diagram:

$$\begin{array}{ccccc} H^0(X, F_*^e \mathcal{L}) & \longrightarrow & H^0(S, F_*^e(\mathcal{L}|_S)) & \longrightarrow & H^1(X, F_*^e(\mathcal{L}(-S))) \\ \downarrow \text{Tr}_{X, S+B}^e & & \downarrow \text{Tr}_{S, B_S}^e & & \\ H^0(X, \mathcal{O}_X) & \xrightarrow{\simeq} & H^0(S, \mathcal{O}_S) & & \end{array}$$

Condition (3) and Proposition 2.2.9 imply that the right vertical arrow Tr_{S, B_S}^e is surjective. By Serre vanishing, after possibly replacing e by a larger multiple, Condition (1) implies that $H^1(X, F_*^e(\mathcal{L}(-S))) = 0$. By a diagram chase, it follows that $\text{Tr}_{X, S+B}^e$ is surjective. Thus, $(X, S + B)$ is globally F -split by Proposition 2.2.9. $\square$

2.3. Liftings

In this dissertation, we study liftability of positive characteristic varieties to characteristic zero, or just modulo p^2 . To formalise this notion for arbitrary perfect fields, we need to use the rings of Witt vectors.

Let k be a perfect field of positive characteristic $p > 0$.

Definition 2.3.1. The ring of Witt vectors $W_n(k)$ of length n is the unique (up to a unique isomorphism) flat $\mathbb{Z}/p^n\mathbb{Z}$ -algebra with an isomorphism $W_n(k)/pW_n(k) \simeq k$. The limit $W(k) := \varprojlim_n W_n(k)$ is the full ring of Witt vectors.

The rings of Witt vectors are mixed characteristic analogues of the rings of infinitesimal thickenings $k[t]/t^n$. In fact they are bijective as sets and have a similar multiplicative structure; one can identify the set of elements of $W_n(k)$ with $k \oplus pk^{1/p} \oplus \dots \oplus p^{n-1}k^{1/p^{n-1}}$, where p^l is a formal identifier, and k^{1/p^l} is a k -algebra isomorphic to k as a ring, but

with the k -multiplication given by $ab^{1/p^l} := (a^{p^l}b)^{1/p^l}$. However, the addition in $W_n(k)$ is not performed componentwise, it is more complicated, for instance, $\underbrace{1 + \dots + 1}_p = p$. The full ring of Witt vectors is a natural generalisation of $\mathbb{Z}_p$: we have $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$ and $W(\mathbb{F}_p) = \mathbb{Z}_p$.

Therewith, we can define the notion of liftability.

Definition 2.3.2. We say that a scheme X defined over a perfect field k of positive characteristic *lifts over* $W_m(k)$ if there exists a flat morphism $\tilde{X} \rightarrow \operatorname{Spec} W_m(k)$ such that $\tilde{X} \times_{W_m(k)} \operatorname{Spec} k \simeq X$. Similarly, one can define liftings of morphisms.

It is important to study how liftability behaves under natural geometric constructions. In particular, we will use the following result in this thesis.

Proposition 2.3.3 ([LS14, Proposition 2.1], [CvS09, Theorem 3.1]). *Let $f: Y \rightarrow X$ be a morphism of schemes defined over a perfect field k of positive characteristic such that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. Assume that Y lifts to a scheme $\tilde{Y}_m$ over $W_m(k)$ for some natural number $m \in \mathbb{Z}_{>0}$. Then f lifts to $\tilde{f}_m: \tilde{Y}_m \rightarrow \tilde{X}_m$ over $W_m(k)$.*

The same is true for formal lifts to the total Witt ring $W(k)$.

3

Klt del Pezzo surfaces which are not globally F -split

based on joint work with P. Cascini and H. Tanaka

3.1. Introduction

THE GOAL OF THIS CHAPTER, as explained in the main introduction, is to show that the natural generalisations of the results in [Har98a, Wat91, CGS16] do not hold true (cf. [PST14, Question 5.23]). Our main result is:

Theorem 3.1.1. *Over any algebraically closed field k of characteristic $p > 0$, there exists a projective Kawamata log terminal surface X such that $-K_X$ is ample and X is not globally F -split.*

Note that, if X is a smooth del Pezzo surface over an algebraically closed field of characteristic $p > 5$, then X is globally F -regular [Har98a, Example 5.5].

By considering the cone over a klt del Pezzo surface which is not globally F -split, as in Theorem 3.1.1, we obtain a three-dimensional klt singularity in arbitrary characteristic which is not F -pure. On the other hand, we give a direct and simpler construction of such singularities. In particular, the varieties that we construct, admit only canonical hypersurface singularities.

Theorem 3.1.2. *Over any algebraically closed field k of characteristic $p > 0$, there exists a three-dimensional canonical variety X , which is not F -pure.*

Theorem 3.1.1 and Theorem 3.1.2 are proved in Section 3.4 and Section 3.5, respectively.

3.2. Preliminaries

We work over an algebraically closed field k of characteristic $p > 0$, unless otherwise mentioned.

In this section, we gather two results on F -splittings which will be used later in the proof of the main theorem. The following result is a consequence of [Wat91] and an elementary calculation.

Lemma 3.2.1. *Let k be an algebraically closed field of characteristic $p > 0$. Then, there exist four distinct closed points $Q_1, \dots, Q_4 \in \mathbb{P}_k^1$ such that the pair*

$$\left(\mathbb{P}_k^1, \frac{1}{2} \sum_{i=1}^4 Q_i\right)$$

is not globally F -split.

Proof. If $p = 2$, then [Wat91, Thm. 4.2(h)] implies that the pair is not globally F -split, for any choice of the points $Q_1, \dots, Q_4$. Thus, we may write $p = 2n + 1$ for some $n \in \mathbb{Z}_{>0}$. Consider the four distinct points of $\mathbb{P}_k^1$:

$$Q_1 = (1 : 0), \quad Q_2 = (0 : 1), \quad Q_3 = (-1 : 1), \quad \text{and} \quad Q_4 = (-\mu : 1)$$

where $\mu \in k \setminus \{0, 1\}$. By [Wat91, Thm. 4.2(h)], the pair

$$\left(\mathbb{P}_k^1, \frac{1}{2} \sum_{i=1}^4 Q_i\right)$$

is globally F -split if and only if the coefficient of x^n in the expansion of $(x + 1)^n(x + \mu)^n$ is not zero.

The coefficient of x^n can be expressed in terms of the following monic polynomial in the variable μ :

$$\sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} \mu^i = \mu^n + n^2 \mu^{n-1} + \dots + 1 \in k[\mu].$$

Since k is algebraically closed, it is enough to show that such a polynomial admits a solution which is different from zero and one. By contradiction, assume that

$$\mu^n + n^2 \mu^{n-1} + \dots + 1 = \mu^a (\mu - 1)^b,$$

for some non-negative integers a and b satisfying $a + b = n$. Clearly $a = 0$, hence we obtain

$$\mu^n + n^2 \mu^{n-1} + \dots + 1 = (\mu - 1)^n \quad \text{in } k[\mu].$$

It follows that $n^2 \equiv -n \pmod{p}$ and, in particular, p divides either n or $n + 1$. Since $n + 1 < 2n + 1 = p$, we get a contradiction. Thus, the claim follows. $\square$

The following result is a consequence of F-adjunction. Here, Diff_C is the different.

Lemma 3.2.2. *Let (Z, C) be a two-dimensional projective plt pair over an algebraically closed field of characteristic $p > 0$, such that C is a smooth prime divisor. If (C, Diff_C) is not globally F -split and the Cartier index of $K_Z + C$ is not divisible by p , then (Z, C) is not globally F -split.*

Proof. This follows by a simpler version of the proof of Lemma 2.2.11. $\square$

3.3. F-purity and the F-pure threshold

The goal of this section is to prove Theorem 3.3.2, which implies that being globally F -split is preserved after taking a limit.

The result was already known in some special cases (e.g. see [Har06, Proposition 2.6], [Her12, Theorem 4.1] and [Sch08, Remark 5.5]). On the other hand, if we consider a pair $(X, \mathfrak{a})$ consisting of a normal variety X and an ideal $\mathfrak{a}$, instead of a divisor, then the result does not hold [TW04, Proposition 2.2(6) and Remark 2.3(2)]. Moreover, note that being globally sharply F -split is not preserved after taking a limit [SS10, Remark 6.2].

We begin by recalling the definitions of the F -pure and the F -split thresholds. The first notion was introduced by Takagi and Watanabe in [TW04]. We refer to [BMS08, BSTZ10, HNBWZ16] for more results in this direction.

Definition 3.3.1. Let X be an F -finite normal scheme of characteristic $p > 0$ and let Δ and D be effective $\mathbb{R}$ -divisors.

1. We define the *F-split threshold* of (X, Δ) with respect to D as
$$\text{fst}(X, \Delta; D) := \sup\{t \in \mathbb{R}_{\geq 0} \mid (X, \Delta + tD) \text{ is globally } F\text{-split}\}.$$
2. We define the *F-pure threshold* of (X, Δ) with respect to D as
$$\text{fpt}(X, \Delta; D) := \sup\{t \in \mathbb{R}_{\geq 0} \mid (X, \Delta + tD) \text{ is } F\text{-pure}\}.$$

The following theorem is used in Section 3.4 to prove Theorem 3.1.1:

Theorem 3.3.2. *Let X be an F -finite noetherian normal scheme of characteristic $p > 0$ and let Δ be an $\mathbb{R}$ -divisor such that (X, Δ) is globally F -split. Let D be an effective $\mathbb{R}$ -divisor on X and let $c = \text{fst}(X, \Delta; D)$.*

Then $(X, \Delta + cD)$ is globally F -split.

Proof. If $c = 0$, then there is nothing to show. Thus, we may assume $c > 0$. We divide the proof into three steps:

Step 1. We first show the theorem, assuming that the following property holds:

1. D and Δ do not have any common component, D has coefficients in $\mathbb{R} \setminus \mathbb{Z}_{(p)}$, and $\text{fst}(X, \Delta; D) = 1$.

Let $e \in \mathbb{Z}_{>0}$. We need to show that

$$\mathcal{O}_X \longrightarrow F_*^e \mathcal{O}_X(\lrcorner(p^e - 1)(\Delta + D) \lrcorner)$$

splits. Since D and Δ do not have any common component, we have

$$\lrcorner(p^e - 1)(\Delta + D) \lrcorner = \lrcorner(p^e - 1)\Delta \lrcorner + \lrcorner(p^e - 1)D \lrcorner.$$

Since the coefficients of D are contained in $\mathbb{R} \setminus \mathbb{Z}_{(p)}$, there exists $t \in (0, 1)$ such that

$$\lrcorner(p^e - 1)D \lrcorner = \lrcorner(p^e - 1)tD \lrcorner.$$

Since $\text{fst}(X, \Delta; D) = 1$, the claim follows.

Step 2. We now show the theorem assuming that the following property holds:

- (2) D is a prime divisor, $D \not\subset \text{Supp } \Delta$, and Δ has coefficients in $\mathbb{Z}_{(p)}$. Moreover, $\text{fst}(X, \Delta; D) \in \mathbb{Z}_{(p)}$.

Since the coefficients of $\Delta + cD$ are contained in $\mathbb{Z}_{(p)}$, it is enough to show that there exists $e \in \mathbb{Z}_{>0}$ such that $(p^e - 1)(\Delta + cD)$ is a Weil divisor and

$$\mathcal{O}_X \longrightarrow F_*^e \mathcal{O}_X(\lrcorner(p^e - 1)(\Delta + cD) \lrcorner)$$

splits. Consider the subset $N_1 \subset \mathbb{Z}_{>0}$ such that $e \in N_1$ if and only if $(p^e - 1)\Delta$ has coefficients in $\mathbb{Z}$ and $(p^e - 1)c$ is an integer. For any $e \in N_1$, we define

$$\nu(e) := \max \{s \in \mathbb{Z}_{\geq 0} \mid \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X((p^e - 1)\Delta + sD) \text{ splits}\}.$$

By the definition of $\nu(e)$ and c , we see that

$$\nu(e) \leq (p^e - 1)c \leq \nu(e) + 1.$$

Thus,

$$\nu(e) \in \{(p^e - 1)c - 1, (p^e - 1)c\}.$$

If $\nu(e) = (p^e - 1)c$ for some $e \in N_1$, then $(X, \Delta + cD)$ is globally F -split and we are done. We now assume by contradiction that

$$\nu(e) = (p^e - 1)c - 1.$$

for every $e \in N_1$.

Pick $e \in N_1$. Then, $2e \in N_1$. Consider

$$\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X((p^e - 1)\Delta).$$

After tensoring by $\mathcal{O}_X((p^e - 1)\Delta + (\nu(e) + 1)D)$, taking the double dual, and applying F_*^e ,

we obtain:

$$F_*^e \mathcal{O}_X((p^e - 1)\Delta + (\nu(e) + 1)D) \rightarrow F_*^{2e} \mathcal{O}_X((p^{2e} - 1)\Delta + p^e(\nu(e) + 1)D).$$

Since

$$\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X((p^e - 1)\Delta + (\nu(e) + 1)D)$$

does not split, neither does

$$\mathcal{O}_X \rightarrow F_*^{2e} \mathcal{O}_X((p^{2e} - 1)\Delta + p^e(\nu(e) + 1)D).$$

Thus,

$$\nu(2e) < p^e(\nu(e) + 1).$$

Since $\nu(d) = (p^d - 1)c - 1$ for $d \in \{e, 2e\}$, and since $(p^e - 1)c \in \mathbb{Z}_{>0}$, we have

$$(p^{2e} - 1)c - 1 < p^e(p^e - 1)c \leq p^e(p^e - 1)c + (p^e - 1)c - 1 = (p^{2e} - 1)c - 1.$$

This is a contradiction. Thus, the claim follows.

Step 3. We now show the theorem in the general case.

We may write

$$\Delta + cD = \sum_{i=1}^r a_i A_i + \sum_{j=1}^s b_j B_j$$

where $a_i \in \mathbb{Z}_{(p)}$, $b_j \in \mathbb{R} \setminus \mathbb{Z}_{(p)}$ and $A_1, \dots, A_r, B_1, \dots, B_s$ are irreducible components.

We first show that $(X, \sum_{i=1}^r a_i A_i)$ is globally F -split. If not, then by Step 2, we can find $a'_1 \in \mathbb{Z}_{(p)}$ such that $0 \leq a'_1 < a_1$ and $(X, a'_1 A_1 + \sum_{i=2}^r a_i A_i)$ is not globally F -split. Proceeding this way, for every $1 \leq i \leq r$, we can find $a'_i \in \mathbb{Z}_{(p)}$ such that $0 \leq a'_i < a_i$ and $(X, \sum_{i=1}^r a'_i A_i)$ is not globally F -split. In particular, there exists $\delta \in (0, c)$ such that

$$\sum_{i=1}^r a'_i A_i \leq \Delta + (c - \delta)D.$$

This contradicts the fact that $(X, \Delta + (c - \delta)D)$ is globally F -split for any $\delta \in (0, c)$. Thus, $(X, \sum_{i=1}^r a_i A_i)$ is globally F -split.

Let

$$c' := \text{fst}(X, \sum_{i=1}^r a_i A_i; \sum_{j=1}^s b_j B_j).$$

If $c' > 1$, then $(X, \Delta + cD)$ is globally F -split, by the definition of the F -split threshold. If $c' = 1$, then the same claim follows from Step 1.

We assume now by contradiction that $0 \leq c' < 1$. For each $j = 1, \dots, s$, there exists

$b'_j \in \mathbb{Z}_{(p)}$ such that $c'b_j < b'_j < b_j$. In particular, the pair

$$(X, \sum_{i=1}^r a_i A_i + \sum_{j=1}^s b'_j B_j)$$

is not globally F -split. By Step 2, it follows that

$$(X, a'_1 A_1 + \sum_{i=2}^r a_i A_i + \sum_{j=1}^s b'_j B_j)$$

is not globally F -split for some $a'_1 \in \mathbb{Z}_{(p)}$ with $0 \leq a'_1 < a_1$. Proceeding this way, we see that

$$(X, \sum_{i=1}^r a'_i A_i + \sum_{j=1}^s b'_j B_j)$$

is not globally F -split for some $a'_i \in \mathbb{Z}_{(p)}$ with $0 \leq a'_i < a_i$ for $i = 1, \dots, r$. We may find $\delta > 0$ such that:

$$\begin{aligned} \sum_{i=1}^r a'_i A_i + \sum_{j=1}^s b'_j B_j &\leq (1 - \delta) \left(\sum_{i=1}^r a_i A_i + c \sum_{j=1}^s b_j B_j \right) \\ &= (1 - \delta)(\Delta + cD) \leq \Delta + (1 - \delta)cD. \end{aligned}$$

Thus, $c = \text{fst}(X, \Delta; D) \leq (1 - \delta)c$, a contradiction. $\square$

As an immediate consequence, we obtain:

Proposition 3.3.3. *Let X be an F -finite noetherian normal scheme of characteristic $p > 0$ and let Δ be an effective $\mathbb{R}$ -divisor such that (X, Δ) is F -pure. Let D be an effective $\mathbb{R}$ -divisor on X and let $c = \text{fpt}(X, \Delta; D)$.*

Then $(X, \Delta + cD)$ is F -pure.

3.4. Non-globally-F-split klt del Pezzo surfaces

The goal of this section is to prove Theorem 3.1.1. To this end, we first construct an unbounded sequence of klt del Pezzo surfaces, starting from a fixed log Calabi-Yau pair (Z, C_Z) , where C_Z is a prime divisor and such that Z admits exactly four singular points along C_Z . The construction was inspired by the “hunt” in [KM99]. Note, in particular, that it holds over any algebraically closed field.

Proposition 3.4.1. *Let k be an algebraically closed field and let*

$$Q_1, \dots, Q_4 \in \mathbb{P}_k^1$$

be four distinct closed points.

Then, there exists a two-dimensional projective plt pair (Z, C_Z) , where C_Z is a prime divisor such that the following properties hold:

1. $2(K_Z + C_Z) \sim 0$,
2. there exists an isomorphism $\iota: C_Z \xrightarrow{\sim} \mathbb{P}_k^1$ such that

$$\iota_* \text{Diff}_{C_Z} = \frac{1}{2} \sum_{i=1}^4 Q_i,$$

and

3. for every $\epsilon \in (0, 1)$, there exist birational morphisms of projective klt surfaces

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow g & & \\ Z & & \end{array}$$

such that $-K_X$ is ample, the strict transform C_Y of C_Z on Y coincides with the exceptional locus of f , and there exists $b \in (1 - \epsilon, 1)$ such that $K_Y + bC_Y = f^*K_X$.

Proof. We divide the proof into three steps:

Step 1. We first construct (Z, C_Z) and prove (1) and (2).

Let $C \simeq \mathbb{P}_k^1$ be the diagonal of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and let

$$\begin{aligned} F_1 &:= \pi_1^*(Q_1), & F_2 &:= \pi_1^*(Q_2), \\ F_3 &:= \pi_2^*(Q_3), & F_4 &:= \pi_2^*(Q_4), \end{aligned}$$

where $\pi_i: \mathbb{P}_k^1 \times \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ denotes the i -th projection, for $i = 1, 2$ (see Figure 3.1).

In particular, the plt pair

$$\left(\mathbb{P}_k^1 \times \mathbb{P}_k^1, C + \frac{1}{2} \sum_{i=1}^4 F_i \right)$$

satisfies $2(K_{\mathbb{P}_k^1 \times \mathbb{P}_k^1} + C + \frac{1}{2} \sum_{i=1}^4 F_i) \sim 0$.

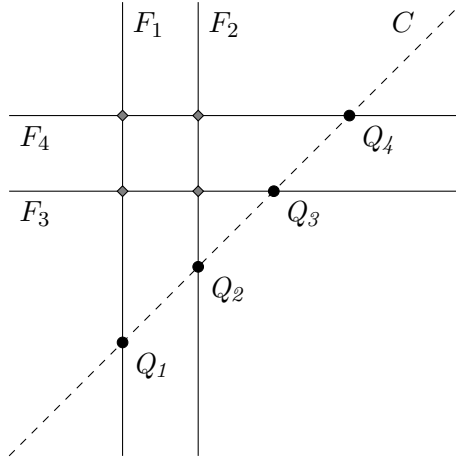


Figure 3.1: Curves on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$

Let

$$\varphi: S \longrightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$$

be the blow-up along the four points in the set

$$(F_1 \cup F_2) \cap (F_3 \cup F_4).$$

By abuse of notation, we denote the proper transforms of $C, F_1, \dots, F_4$ on S by the same symbols. We have

$$2\left(K_S + C + \frac{1}{2} \sum_{i=1}^4 F_i\right) \sim 0.$$

Note that $F_1, \dots, F_4$ are pairwise disjoint (-2) -curves on S . Let

$$\psi: S \rightarrow Z$$

be the contraction of these curves and let $C_Z := \psi_* C$. We obtain

$$2(K_Z + C_Z) = \psi_* \left(2\left(K_S + C + \frac{1}{2} \sum_{i=1}^4 F_i\right) \right) \sim 0.$$

Further, (Z, C_Z) is plt and $\text{Diff}_{C_Z} = \frac{1}{2} \sum_{i=1}^4 Q_i$. Thus, (1) and (2) follow.

Note that, by the Nakai–Moishezon criterion, C_Z is ample. Moreover, Z has exactly four singular points, and all of them lie on the curve C_Z .

Step 2. We now construct X and Y .

Fix an integer $n \in \mathbb{Z}$ such that

$$n \geq 4 \quad \text{and} \quad \frac{2}{n - \frac{7}{2}} < \epsilon.$$

Let $\bar{S} \rightarrow S$ be the birational morphism obtained by blowing-up n -times the point $Q_1 \in C$: by the point Q_1 in the blow-up of S , we mean the unique intersection of the strict transform of C with the exceptional curve. Again, by abuse of notation, we denote the strict transforms of $C, F_1, \dots, F_4$ by the same symbols. Further, we denote the exceptional curves on $\bar{S}$ by $E_1, \dots, E_n$, where E_i is the exceptional curve of the i -th blow-up (see Figure 3.2).

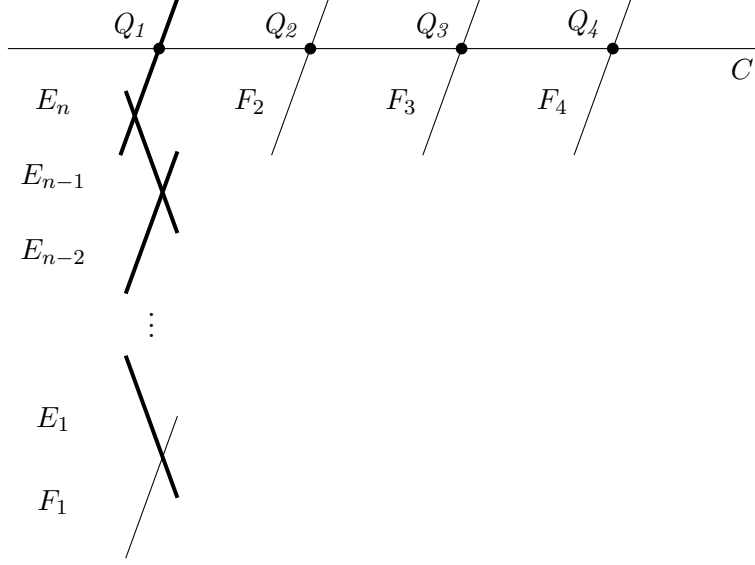
It follows that

$$2\left(K_{\bar{S}} + C + \frac{1}{2} \sum_{i=1}^4 F_i + \frac{1}{2} \sum_{i=1}^n E_i\right) \sim 0.$$

Let

$$h: \bar{S} \longrightarrow Y$$

be the birational contraction of F_2, F_3, F_4 and of the chain of curves $E_{n-1}, \dots, E_1, F_1$.

Figure 3.2: Curves on $\bar{S}$

Thanks to the negativity lemma [KM98, Lemma 3.39] and the $\mathbb{Q}$ -linear equivalence

$$K_{\bar{S}} + C + \frac{1}{2}E_n + \frac{3}{4}\left(\sum_{i=1}^4 F_i + \sum_{i=1}^{n-1} E_i\right) \sim_{\mathbb{Q}} \frac{1}{4}\sum_{i=1}^4 F_i + \frac{1}{4}\sum_{i=1}^{n-1} E_i,$$

we can construct such a birational morphism by running the MMP over Z [Tan14, Theorem 6.5] with respect to the pair

$$\left(\bar{S}, C + \frac{1}{2}E_n + \frac{3}{4}\left(\sum_{i=1}^4 F_i + \sum_{i=1}^{n-1} E_i\right)\right).$$

Let $C_Y := h_*C$ and $E_Y := h_*E_n$. Since all the h -exceptional curves are contracted by the induced birational morphism $\bar{S} \rightarrow S \xrightarrow{\psi} Z$, there exists a birational morphism $g: Y \rightarrow Z$, whose exceptional divisor coincides with E_Y . In particular, C_Y is the strict transform of C_Z on Y . Thus, C_Y is a smooth rational curve and it can be easily checked that

$$C_Y^2 = \frac{7}{2} - n.$$

Because of our choice of n , we have $C_Y^2 < 0$. We denote by $f: Y \rightarrow X$ the contraction of C_Y whose existence is guaranteed by Keel's theorem [Kee99, Proposition 1.6] (cf. [Tan14, Proposition 4.4]). Summarising, we have the following diagram:

$$\begin{array}{ccccc} \bar{S} & \xrightarrow{h} & Y & \xrightarrow{f} & X \\ \downarrow & & \downarrow g & & \\ S & \longrightarrow & Z & & \\ \downarrow & & & & \\ \mathbb{P}_k^1 \times \mathbb{P}_k^1 & & & & \end{array}$$

By taking a pullback to $\overline{S}$, it is easy to check that

$$K_Y + C_Y + \frac{1}{2}E_Y = g^*(K_Z + C_Z) \sim_{\mathbb{Q}} 0.$$

Since (Z, C_Z) is plt, we have that also $(Y, C_Y + \frac{1}{2}E_Y)$ is plt, and Y is klt.

Step 3. We now prove (3).

Let $a \in \mathbb{Q}_{>0}$ be such that

$$g^*C_Z = C_Y + aE_Y.$$

Note that

$$g^*C_Z - \delta E_Y = C_Y + (a - \delta)E_Y$$

is ample for any $0 < \delta \ll 1$, and so f_*E_Y is ample. Since

$$-(K_Y + C_Y) \sim_{\mathbb{Q}} \frac{1}{2}E_Y,$$

we have that

$$-K_X = -f_*(K_Y + C_Y) \sim_{\mathbb{Q}} \frac{1}{2}f_*E_Y$$

is ample. Let $b \in \mathbb{Q}$ be such that

$$K_Y + bC_Y = f^*K_X.$$

By intersecting with C_Y , we have

$$b = \frac{K_Y \cdot C_Y}{-C_Y^2} = 1 - \frac{-(K_Y + C_Y) \cdot C_Y}{-C_Y^2}.$$

Since $K_Y + C_Y + \frac{1}{2}E_Y \equiv 0$ and $C_Y \cdot E_Y > 0$, we have

$$0 < -(K_Y + C_Y + \frac{1}{2}E_Y) \cdot C_Y + \frac{1}{2}E_Y \cdot C_Y = -(K_Y + C_Y) \cdot C_Y \leq 2,$$

where the last inequality follows easily by taking the minimal resolution of Y . Thus, because of our choice of n , we obtain

$$0 < 1 - b = \frac{-(K_Y + C_Y) \cdot C_Y}{-C_Y^2} \leq \frac{2}{n - \frac{7}{2}} < \epsilon.$$

In particular, X is klt and (3) follows. □

We can now prove our main theorem:

Proof of Theorem 3.1.1. Fix an algebraically closed field k of characteristic $p > 0$. If $p = 2$, then the smooth cubic surface of Fermat type satisfies the required properties [Har98a, Example 5.5]. Thus, we may assume $p \geq 3$.

Let $Q_1, \dots, Q_4 \in \mathbb{P}_k^1$ be as in Lemma 3.2.1 and let X, Y, Z and b be as in Proposition 3.4.1 for some $0 < \epsilon \ll 1$. In particular, X is klt and $-K_X$ is ample. By (1) and (2) of Proposition 3.4.1 and Lemma 3.2.2, it follows that the pair (Z, C_Z) is not globally F -split. By Theorem 3.3.2, there exists a rational number β , independent of ϵ , such that $0 < \beta < 1$ and the pair $(Z, \beta C_Z)$ is not globally F -split as well.

Thus, we may assume $\epsilon < 1 - \beta$, and in particular $b > \beta$. Since $(Z, \beta C_Z)$ is not globally F -split, Lemma 2.2.6 implies that $(Y, \beta C_Y)$ is not globally F -split. Thus, (Y, bC_Y) is not globally F -split and, by Lemma 2.2.7, also X is not globally F -split. $\square$

3.5. Non-F-pure canonical threefolds

The goal of this section is to prove Theorem 3.1.2. We begin with the following Lemma.

Lemma 3.5.1. *Let k be an algebraically closed field of characteristic $p > 5$. Let $f(x, y, z) \in k[x, y, z]$ be a homogeneous cubic polynomial such that $\text{Proj } k[x, y, z]/(f(x, y, z))$ is a smooth elliptic curve.*

Then, for every $n \in \mathbb{Z}_{\geq 0}$,

$$X_n = \text{Spec } k[x, y, z, w]/(f(x, y, z) + w^n)$$

has at worst canonical singularities.

Proof. We divide the proof into three steps:

Step 1. We first show that X_n is smooth outside the origin and, in particular, X_n is smooth if $n \leq 1$.

Let $(a, b, c, d) \in X_n \subset \mathbb{A}_k^4$ be a singular point. Since f is homogeneous of degree 3, we have

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 3f.$$

Since $p > 3$, a singular point (a, b, c, d) satisfies the following equation:

$$f(a, b, c) = \frac{\partial f}{\partial x}(a, b, c) = \frac{\partial f}{\partial y}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c) = 0.$$

This implies $a = b = c = 0$, hence $d = 0$, as claimed.

Step 2. We now show that X_2 is terminal.

Since $p > 3$, by writing the cubic f in Weierstrass form, we may assume that there exist $a, b \in k$ such that

$$f(x, y, z) + w^2 = y^2 z + x^3 + axz^2 + bz^3 + w^2.$$

After taking a linear transformation $x \mapsto x + cz$ for some $c \in k$, we may write

$$f(x, y, z) + w^2 = y^2 z + x^3 + \alpha x^2 z + \beta x z^2 + \gamma z^3 + w^2$$

for some $\alpha, \beta, \gamma \in k$ with $\gamma \neq 0$. Consider the hypersurface S in X_2 obtained by cutting with the hyperplane $\{x = 0\}$, i.e.

$$S = \operatorname{Spec} k[y, z, w]/(y^2z + \gamma z^3 + w^2).$$

Then S admits a unique singularity which is du Val of type D_4 , and in particular S is canonical. Since $p > 5$, S is strongly F -regular [Har98b, Theorem 1.1] and inversion of adjunction (Theorem 2.2.10) implies that the pair (X_2, S) is plt. Since S is a Cartier divisor, it follows that X_2 is terminal, as claimed.

Step 3. We now show that X_n is canonical for all $n \in \mathbb{Z}_{\geq 0}$.

We may assume $n \geq 3$. Consider the blow-up $f: Y \rightarrow X_n$ at the origin. Then Y is covered by four affine open subsets:

$$\begin{aligned} Y &= \operatorname{Spec} k[x, y, z, w]/(f(1, y, z) + x^{n-3}w^n) \\ &\cup \operatorname{Spec} k[x, y, z, w]/(f(x, 1, z) + y^{n-3}w^n) \\ &\cup \operatorname{Spec} k[x, y, z, w]/(f(x, y, 1) + z^{n-3}w^n) \\ &\cup \operatorname{Spec} k[x, y, z, w]/(f(x, y, z) + w^{n-3}). \end{aligned}$$

We first show that the first three affine open subsets are smooth. By symmetry, it is enough to consider the first one. By Step 1, we just need to consider points in the exceptional locus $\{x = 0\}$. Since $n \geq 3$, a singular point (x, y, z, w) satisfies the equations:

$$f(1, y, z) = \frac{\partial f(1, y, z)}{\partial y} = \frac{\partial f(1, y, z)}{\partial z} = 0.$$

Since $\operatorname{Spec} k[y, z]/(f(1, y, z))$ is a smooth affine curve, there are no such points. Thus, the singular locus of Y is contained in the last affine open subset

$$\operatorname{Spec} k[x, y, z, w]/(f(x, y, z) + w^{n-3}) \simeq X_{n-3}.$$

Moreover, the multiplicity of X_n at the origin is three. Thus, f is crepant, i.e.

$$K_Y = f^*K_{X_n}.$$

Thus, it is enough to show that X_{n-3} is canonical. Repeating the same procedure finitely many times, Step 1 and Step 2 imply the claim. This completes the proof of this lemma. $\square$

Lemma 3.5.2. *Let k be an algebraically closed field of characteristic $p > 0$. Let $f(x, y, z) \in k[x, y, z]$ be a homogeneous cubic polynomial such that $\operatorname{Proj} k[x, y, z]/(f(x, y, z))$ is a smooth elliptic curve.*

Let $n \in \mathbb{Z}$ be such that $n \geq p$, and let

$$X_n = \operatorname{Spec} k[x, y, z, w]/(f(x, y, z) + w^n).$$

Then, X_n is F -pure if and only if $\operatorname{Proj} k[x, y, z]/(f(x, y, z))$ is an ordinary elliptic curve.

Proof. We consider the following conditions.

1. $\operatorname{Proj} k[x, y, z]/(f(x, y, z))$ is an ordinary elliptic curve.
 2. $\operatorname{Spec} k[x, y, z]/(f(x, y, z))$ is F -pure.
 3. $f(x, y, z)^{p-1} \notin (x^p, y^p, z^p)k[x, y, z]$.
- (a) X_n is F -pure.
- (b) $(f(x, y, z) + w^n)^{p-1} \notin (x^p, y^p, z^p, w^p)k[x, y, z, w]$.

It is well known that (1), (2) and (3) (respectively, (a) and (b)) are equivalent. Since, by assumption: $n \geq p$, it is easy to check that (3) and (b) are equivalent. Thus, the claim follows. $\square$

We can now prove Theorem 3.1.2.

Proof of Theorem 3.1.2. If $p \leq 5$, then

$$X := \operatorname{Spec} k[x, y, z, w]/(x^2 + y^3 + z^5)$$

is canonical but not F -pure [Har98b, (4.4)].

Thus, we may assume $p > 5$. Let $f(x, y, z) \in k[x, y, z]$ be a homogeneous cubic polynomial such that $\operatorname{Proj} k[x, y, z]/(f(x, y, z))$ is a supersingular elliptic curve. Let

$$X := k[x, y, z, w]/(f(x, y, z) + w^p).$$

By Lemma 3.5.1, X is canonical. By Lemma 3.5.2, X is not F -pure. $\square$

4

On log del Pezzo surfaces in large characteristic

based on joint work with P. Cascini and H. Tanaka

4.1. Introduction

THE PROOF OF THE EXISTENCE OF FLIPS in positive characteristic relies on the fact that if $\text{char } k > 5$ and (X, B) is a one-dimensional projective klt pair such that $-(K_X + B)$ is ample and the coefficients of B are contained in the standard set $\{1 - \frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$, then (X, B) is globally F -regular [Wat91, Theorem 4.2]. The main theorem of this chapter provides a partial generalisation of this result to the case of surfaces.

Theorem 4.1.1. *Let $I \subseteq (0, 1) \cap \mathbb{Q}$ be a finite set. Then there exists a positive integer $p(I)$ which satisfies the following property:*

Let (X, B) be a two dimensional projective klt pair over an algebraically closed field of characteristic $p > p(I)$ such that $-(K_X + B)$ is ample and the coefficients of B are contained in I . Then one of the following properties holds:

1. (X, B) is globally F -regular, or
2. *there exists a log resolution $\mu: V \rightarrow X$ of (X, B) such that $(V, \text{Exc}(\mu) \cup \mu_*^{-1}(\text{Supp } B))$ lifts to characteristic zero over a smooth base (cf. Definition 4.2.15).*

Note that we do not know whether there exists a klt del Pezzo surface in large characteristic which violates (2) of Theorem 4.1.1.

Theorem 4.1.2. *There exists a positive integer p_0 which satisfies the following property:*

Let (X, Δ) be a two dimensional projective klt pair over an algebraically closed field of characteristic $p > p_0$. Suppose that there exists an effective $\mathbb{Q}$ -divisor B such that (X, B) is klt and $-(K_X + B)$ is nef and big. If D is a $\mathbb{Z}$ -divisor on X such that $D - (K_X + \Delta)$ is nef and big, then

$$H^i(X, \mathcal{O}_X(D)) = 0 \quad \text{for any } i > 0.$$

Note that Theorem 4.1.1 imposes a condition on the coefficients of the boundary divisor B , but Theorem 4.1.2 does not need such an assumption.

Finally, we show that Theorem 4.1.1 does not hold in characteristic two.

Theorem 4.1.3. *There exists a projective klt surface X over $\overline{\mathbb{F}}_2$ which satisfies the following properties:*

1. $-K_X$ is ample,
2. X is not globally F -split, and
3. for any log resolution $h: W \rightarrow X$ of X , the pair $(W, \text{Exc}(h))$ does not lift to characteristic zero over a smooth base (cf. Definition 4.2.15).

We prove Theorem 4.1.1 in Section 4.5, Theorem 4.1.2 in Section 4.6 and Theorem 4.1.3 in Section 4.7.

4.1.1. Sketch of the proof

We now give an overview of some of the steps of the proof of Theorem 4.1.1. The reader may also consult Diagram 4.1 at the very end of this chapter, where the main ideas of the proof are laid down.

For simplicity, we assume that $I = \emptyset$. Let X be a projective klt surface in large characteristic and such that $-K_X$ is ample. We want to show that at least one of the statements (1) and (2) in Theorem 4.1.1 holds true. The idea is that, choosing a suitable $\epsilon \in \mathbb{Q}_{>0}$, we consider two different cases:

- (a) X is ϵ -klt.
- (b) X is not ϵ -klt.

(a) It is known that, for a fixed algebraically closed field k , the set of all the ϵ -klt del Pezzo surfaces over k forms a bounded family [Ale94]. We generalise this boundedness result to mixed characteristic, i.e. there exists a projective morphism $\mathcal{X} \rightarrow S$ of schemes of finite type over $\text{Spec } \mathbb{Z}$, depending only on ϵ , such that an arbitrary ϵ -klt del Pezzo surface X over an arbitrary algebraically closed field k of characteristic $p > 5$ is isomorphic to some fibre $\mathcal{X}_s$ up to a base change of the base field. By noetherian induction, we can show that for $p \gg 0$, any ϵ -klt del Pezzo surface X over an algebraic closed field of characteristic p admits a log resolution $\mu: V \rightarrow X$ such that $(V, \text{Exc}(\mu))$ lifts to characteristic zero over a smooth base. For more details, see Section 4.3.

(b) Assume that X is not ϵ -klt. In this case, the proof consists of the following three steps:

- (I) Since X is not ϵ -klt, we can extract a prime divisor C with log discrepancy $a := a(C, X, 0) \leq \epsilon$ by:

$$f: Y \rightarrow X, \quad \text{so that} \quad K_Y + (1 - a)C = f^*K_X.$$

- (II) We run a $-(K_Y + C)$ -MMP with scaling of C :

$$(Y, C) =: (Y_0, C_0) \rightarrow (Y_1, C_1) \rightarrow \cdots \rightarrow (Y_n, C_n) =: (Z, C_Z).$$

Note that since (Y, Δ) is log del Pezzo for some effective $\mathbb{Q}$ -divisor Δ , we may run a D -MMP for any $\mathbb{Q}$ -Cartier divisor D . Indeed, let $s > 0$ be a sufficiently small rational number such that $A := sD - (K_Y + \Delta)$ is ample. Then,

$$sD = K_Y + \Delta + A$$

and in particular, a D -MMP coincides with a $(K_Y + \Delta + A)$ -MMP. Furthermore, if ϵ is sufficiently small, then since $1 - a \in [1 - \epsilon, 1)$, it follows that each pair (Y_i, C_i) is log canonical by ACC for the log canonical threshold [Ale93, Theorem 1.1].

- (III) We now consider five different cases as follows:

- (i) $\kappa(Z, -(K_Z + C_Z)) = 2$.
- (ii) $\kappa(Z, -(K_Z + C_Z)) = 1$.
- (iii) $\kappa(Z, -(K_Z + C_Z)) = 0$ and (Z, C_Z) is not plt.
- (iv) $\kappa(Z, -(K_Z + C_Z)) = 0$ and (Z, C_Z) is plt.
- (v) $\kappa(Z, -(K_Z + C_Z)) = -\infty$.

If one of the cases (i), (ii), or (iii) holds, then we can show that X is globally F -regular, i.e. (1) in Theorem 4.1.1 holds. If (iv) holds, then (2) in Theorem 4.1.1 holds. Finally, we show that (v) does not occur if ϵ is sufficiently small.

We now give some details of the methods we use in the steps above. After Step (I), there exists a rational number $b \in (1 - a, 1)$, such that (Y, bC) is klt and $-(K_Y + bC)$ is ample. The easiest case is if $-(K_Y + C)$ is ample and $(p^e - 1)(K_Y + C)$ is Cartier for some positive integer e . In this case, Lemma 4.2.7 implies immediately that (Z, C_Z) is globally F -split, which in turn implies that X is globally F -regular. However, $-(K_Y + C)$ is not ample in general, but by running an MMP in Step (II), we can get closer to this situation. Indeed, in case (i) of Step (III), we have that $-(K_Z + C_Z)$ is nef and big, and after contracting the curves Γ with $(K_Z + C_Z) \cdot \Gamma = 0$, we may assume that $-(K_Z + C_Z)$ is ample. By using Lemma 4.2.7, we can show that X is globally F -regular. We can apply a similar argument to show that X is globally F -regular in cases (ii) and (iii) of Step (III).

Let us assume now that (iv) of Step (III) holds. In this case, we first show that the set of pairs (Z, C_Z) as in (iv) forms a bounded family. By noetherian induction, we show that for $p \gg 0$, any (Z, C_Z) has a log resolution $\pi: W \rightarrow Z$ such that $(W, \pi_*^{-1}C_Z \cup \text{Exc}(\pi))$ lifts to characteristic zero over a smooth base. If W already dominates Y , then the induced birational morphism $W \rightarrow X$ is a log resolution as in (2) of Theorem 4.1.1. In general, we cannot hope that W dominates Y , however we can show that (2) of Theorem 4.1.1 holds after taking some blow-ups of W . For more details, we refer to Section 4, especially Lemma 4.4.3.

Finally, let us assume that (v) of Step (III) holds. In particular, Z admits a $-(K_Z + C_Z)$ -negative Mori fibre space. Let us assume for simplicity that $\rho(Z) = 1$ and $K_Z + C_Z$ is ample. Since $-(K_Y + bC)$ is ample, so is its push-forward $-(K_Z + bC_Z)$. Thus, we may find $b' \in (b, 1)$ such that

$$K_Z + b'C_Z \equiv 0.$$

In particular, $b' \in (1 - \epsilon, 1)$, and after possibly replacing ϵ by a smaller number, we derive a contradiction thanks to ACC for the log Calabi–Yau pairs [Ale93, Theorem 5.3] (cf. Lemma 4.2.3).

4.2. Preliminaries

If not stated otherwise, we work over an algebraically closed field k of characteristic $p > 0$.

Given a subset $I \subseteq [0, 1]$, we say that I is an *ACC* (resp. *DCC*) set if it satisfies the ascending chain condition (resp. the descending chain condition). Given a subset $I \subseteq [0, 1]$, we define:

$$I_+ := \left\{ \sum_{j=1}^m i_j \mid i_j \in I \text{ for } j = 1, \dots, m \right\} \cap [0, 1],$$

and

$$D(I) := \left\{ \frac{m-1+f}{m} \mid m \in \mathbb{Z}_{>0}, f \in I_+ \right\} \cap [0, 1].$$

4.2.1. Log canonical surface singularities

Let X be a normal surface. A singular point $q \in X$ is said to be *cyclic* if the exceptional locus of the minimal resolution of X at q is a chain of rational curves $C_1, \dots, C_m$. We denote by Γ_q the intersection matrix $(-C_i \cdot C_j)$ associated to $C_1, \dots, C_m$. In particular, if d is a positive integer, then there are only finitely many possibilities for Γ_q so that $\det \Gamma_q = d$ [Kol13, pag. 116].

A singular point $q \in X$ is said to be *dihedral* if the exceptional locus of the minimal resolution of X at q is the union of rational curves $C_1, \dots, C_m$, with $m \geq 4$, where $C_1, \dots, C_{m-2}$ is a chain of curves and C_{m-1}, C_m are (-2) -curves which meet C_{m-2} transversally in two distinct points away from $C_{m-3} \cap C_{m-2}$ [Kol13, 3.35(3)].

Lemma 4.2.1. *Let $f: X \rightarrow Y$ be a proper birational morphism between klt surfaces. If L is a Cartier divisor on X such that $L \equiv_f 0$, then there exists a Cartier divisor L_Y on Y*

such that $\mathcal{O}_X(L) \simeq \mathcal{O}_X(f^*L_Y)$. In particular, if D is a Weil divisor on Y such that f^*D is Cartier, then D is Cartier.

Proof. Since a klt surface is automatically quasi-projective (e.g. see [Fuj12, Lemma 2.2]), by taking suitable compactifications, we may assume that X and Y are projective. Let $h: \tilde{X} \rightarrow X$ be the minimal resolution and let $g: \tilde{X} \rightarrow Y$ be the induced morphism. It is enough to show that there exists a Cartier divisor L_Y on Y such that $\mathcal{O}_{\tilde{X}}(h^*L) \simeq \mathcal{O}_{\tilde{X}}(g^*L_Y)$. Let E be the sum of the exceptional prime divisors of g . By running a $(K_{\tilde{X}} + E)$ -MMP over Y , we may assume that f is a birational morphism between projective klt surfaces such that $\rho(X/Y) = 1$.

By using the base point free theorem over surfaces [Tan15b, Theorem 3.2 and Corollary 3.6], we may apply the same argument as [KM98, Theorem 3.7(4)] to conclude. $\square$

Lemma 4.2.2. *Let (X, C) be a two dimensional plt pair, where C is a prime divisor. If $q \in C$ is a singular point of X , then q is cyclic and the coefficient of Diff_C at q is given by $1 - \frac{1}{m_q}$, where $m_q := \det(\Gamma_q)$. Further, for each Weil divisor D on X , we have that $m_q D$ is Cartier around q .*

Proof. The first part of the lemma follows from [Kol13, Theorem 3.36]. We are left to show that if D is a Weil divisor on X , then $m_q D$ is Cartier at q . Let $f: Y \rightarrow X$ be the minimal resolution of X at q . Then the exceptional divisor is a chain of rational curves $C_1, \dots, C_r$. We may write

$$f^*D = D_Y + \sum_{i=1}^r b_i C_i,$$

where D_Y is the strict transform of D in Y and $b_1, \dots, b_r \in \mathbb{Q}$. Thus, for any $j = 1, \dots, r$, we have

$$\sum_{i=1}^r b_i C_i \cdot C_j = -D_Y \cdot C_j.$$

Since $D_Y \cdot C_j \in \mathbb{Z}$, it follows that $m_q b_i \in \mathbb{Z}$ for each $i = 1, \dots, r$. Since X is klt at q , Lemma 4.2.1 implies the claim. $\square$

We need the following version of ACC for the nef threshold in dimension one and two:

Lemma 4.2.3. *Let $I \subseteq (0, 1] \cap \mathbb{Q}$ be a DCC set. Then there exists a finite subset $J \subseteq I$ which satisfies the following property:*

Let (X, B) be a projective log canonical pair and let $\pi: X \rightarrow T$ be a projective morphism onto a normal variety T such that $\pi_\mathcal{O}_X = \mathcal{O}_T$ and*

- $\dim X \leq 2$,
- $\dim T < \dim X$,
- the coefficients of B belong to I , and
- $K_X + B \equiv_\pi 0$.

Then the coefficient of any π -horizontal component of B belongs to J .

Proof. See [Ale93, Theorem 5.3] and [Bir16, Proposition 11.7]. $\square$

To show Proposition 4.2.5, we need the following:

Lemma 4.2.4. *Let (X, Δ) be a two dimensional klt pair such that X is smooth away from a closed point $x \in X$. Let $f: Y \rightarrow X$ be the minimal resolution of X , with exceptional divisors $E_1, \dots, E_n$ and let $a_i = a(E_i, X, \Delta)$. Assume that $n \geq 8$ and let σ be a permutation of $\{1, \dots, n\}$ such that*

$$a_{\sigma(1)} \leq a_{\sigma(2)} \leq \dots \leq a_{\sigma(n)}.$$

Let ℓ be a positive integer such that $\ell\Delta$ is a $\mathbb{Z}$ -divisor and

$$\ell a_{\sigma(i)} \in \mathbb{Z} \quad \text{for } i = 1, \dots, 8.$$

Then $2\ell(K_X + \Delta)$ is Cartier.

Proof. We have

$$K_Y + f_*^{-1}\Delta + \sum_{i=1}^n (1 - a_i)E_i = f^*(K_X + \Delta).$$

Since X is klt and $\ell f_*^{-1}\Delta$ is integral, by Lemma 4.2.1, it is enough to show that $2\ell a_j \in \mathbb{Z}$ for any $j = 1, \dots, n$.

We may assume that $x \in X$ is a cyclic or a dihedral singularity, as otherwise, by the classification of two dimensional klt singularities, we have that $n \leq 8$ (e.g. see [KM98, Theorem 4.16]), and the claim follows immediately.

We first assume that $x \in X$ is a cyclic singularity and $E_1, \dots, E_n$ is a chain of rational curves. For each $i \in \{2, \dots, n-1\}$, by taking the intersection with E_i on both sides of the equality above, we obtain

$$(-E_i^2)a_i + f_*^{-1}\Delta \cdot E_i = a_{i-1} + a_{i+1}.$$

In particular, the numbers $a_1, \dots, a_n$ satisfy the following convexity inequality:

$$a_i \leq \frac{a_{i-1} + a_{i+1}}{-E_i^2} \leq \frac{a_{i-1} + a_{i+1}}{2}.$$

Thus, after possibly replacing $\sigma(1)$ by another index $j \in \{1, \dots, n\}$ such that $a_{\sigma(1)} = a_j$, we may assume that $\sigma(2)$ is equal to $\sigma(1) - 1$ or $\sigma(1) + 1$, say $\sigma(1) + 1$. In particular, $\ell a_{\sigma(1)}, \ell a_{\sigma(1)+1} \in \mathbb{Z}$.

Thus, for each $i = 2, \dots, n-1$, we have

$$(-E_i^2)\ell a_i + (\ell f_*^{-1}\Delta) \cdot E_i = \ell a_{i-1} + \ell a_{i+1},$$

and it follows inductively that $\ell a_j \in \mathbb{Z}$ for any $j = 1, \dots, n$. Thus, the claim follows.

Let us assume now that $x \in X$ is a dihedral singularity. Let E_{n-1} and E_n be two tails of self-intersection (-2) and let $E_1, E_2, \dots, E_{n-2}$ be the remaining chain of rational curves

[Kol13, 3.35(3)], so that E_{n-2} intersects E_{n-1} and E_n . For $m \in \{n-1, n\}$, by taking the intersection with E_m on both sides of the equality

$$K_Y + f_*^{-1}\Delta + \sum_{i=1}^n (1 - a_i)E_i = f^*(K_X + \Delta),$$

we obtain

$$2a_m + f_*^{-1}\Delta \cdot E_m = a_{n-2} + 1.$$

Thus, it is enough to show that $\ell a_i \in \mathbb{Z}$ for any $i = 1, \dots, n-2$. By assumption, it follows that

$$\ell \min_{1 \leq i \leq n-2} a_i, \ell \min_{\substack{1 \leq i \leq n-2 \\ i \neq j_1}} a_i \in \mathbb{Z}$$

for some $j_1 \in \{1, \dots, n-2\}$ such that $a_{j_1} = \min_{1 \leq i \leq n-2} a_i$. By applying the same argument as above to the chain $E_1, E_2, \dots, E_{n-2}$, we obtain that $\ell a_j \in \mathbb{Z}$ for every $j \in \{1, \dots, n-2\}$. Thus, the claim follows. $\square$

Proposition 4.2.5. *Let $I \subseteq (0, 1] \cap \mathbb{Q}$ be a DCC set. Then there exists a positive integer m_0 which satisfies the following property:*

Let (X, Δ) be a two dimensional projective log canonical pair such that

- *X is klt,*
- *the coefficients of Δ are contained in I , and*
- *$K_X + \Delta \equiv 0$.*

Then $m_0(K_X + \Delta)$ is Cartier.

Proof. By Lemma 4.2.3, we may assume that I is a finite set. Further, we may assume that $1 \in I$. Let (X, Δ) be a log pair as in the proposition.

First, we show that there exists $m_1 \in \mathbb{Z}_{>0}$, depending only on I , such that $m_1(K_X + \Delta)$ is Cartier around any closed point $q \in X$ such that (X, Δ) is not klt at q . By Lemma 4.2.1, after taking a dlt modification, we may assume that (X, Δ) is dlt at q . If the support of $\lfloor \Delta \rfloor$ is singular at q , then X is smooth at q [KM98, Theorem 4.15(1)], and there is nothing to show. Thus, we may assume that (X, Δ) is plt around q . Let C be the component of $\lfloor \Delta \rfloor$ containing q . Since (X, Δ) is plt around q , we know that C is smooth, and so we may write

$$K_C + \Delta_C = (K_X + \Delta)|_C \equiv 0,$$

where, by inversion of adjunction, Δ_C is a $\mathbb{Q}$ -divisor on C such that (C, Δ_C) is klt. By [MP04, Lemma 4.3], Δ_C has coefficients in $D(I)$. Since $D(I)$ is a DCC set [MP04, Lemma 4.4] and $\text{Diff}_C \leq \Delta_C$, Lemma 4.2.3 implies that there are only finitely many possibilities for the coefficients of Diff_C at q , and the existence of m_1 follows from Lemma 4.2.2.

Now we show that there exists $m_2 \in \mathbb{Z}_{>0}$, depending only on I , such that $m_2(K_X + \Delta)$ is Cartier around any closed point $q \in X$ such that (X, Δ) is klt at q . Let $f: Y \rightarrow X$ be

the minimal resolution at q . We have

$$K_Y + f_*^{-1}\Delta + \sum_{i=1}^n (1 - a_i)E_i = f^*(K_X + \Delta)$$

where $E_1, \dots, E_n$ are the f -exceptional prime divisors, ordered so that

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

By Lemma 4.2.4, if ℓ is a positive integer such that

$$\ell a_1, \ell a_2, \dots, \ell a_8 \in \mathbb{Z}$$

and such that $\ell\Delta$ is a $\mathbb{Z}$ -divisor, then $2\ell(K_X + \Delta)$ is Cartier around q . Thus, it is enough to find $n_0 \in \mathbb{Z}_{>0}$, depending only on I , such that $n_0 a_1, \dots, n_0 a_8 \in \mathbb{Z}$.

To this end, we first extract a prime divisor E_1 such that $a(E_1, X, \Delta) = a_1$ and we obtain a birational morphism $g: Z \rightarrow X$ such that

$$K_Z + g_*^{-1}\Delta + (1 - a_1)E_1 = g^*(K_X + \Delta) \equiv 0.$$

By ACC for the minimal log discrepancy [Ale93, Theorem 3.2], the set $\{a_1\}_{(X, \Delta), x \in X}$ is an ACC set, hence $\{1 - a_1\}$ is a DCC set. By Lemma 4.2.3, there are finitely many possibilities for a_1 . Then, we extract E_2 with $a(E_2, X, \Delta) = a_2$, and apply the same argument, to show that there are only finitely many possibilities for a_2 . Repeating the same argument eight times, we see that there are finitely many possibilities for $a_1, \dots, a_8$. Thus, we may find a positive integer n_0 as above and the claim follows. $\square$

The following result is well known at least in characteristic zero [Pro01, Lemma 8.3.15]. We include the proof for the sake of completeness.

Lemma 4.2.6. *Let $(Z, C + B)$ be a two dimensional $\mathbb{Q}$ -factorial log canonical pair, where C is a prime divisor and B is an effective $\mathbb{Q}$ -divisor. Let $\pi: Z \rightarrow T$ be a projective morphism onto a smooth curve T such that $\pi_*\mathcal{O}_Z = \mathcal{O}_T$. Assume that*

- C is π -horizontal,
- $K_Z + C + B \equiv_{\pi} 0$, and
- $\rho(Z/T) = 1$.

Then the following hold:

1. *For every closed point $t \in T$, $\pi^{-1}(t)_{\text{red}}$ is isomorphic to $\mathbb{P}^1$.*
2. *$[K(C) : K(T)] \leq 2$.*
3. *Assume that C is normal, and let us define $K_C + B_C = (K_Z + C + B)|_C$ by adjunction. If $[K(C) : K(T)] = 2$ and the field extension $K(C)/K(T)$ is separable, then B_C is invariant under the action of the Galois group $\text{Gal}(K(C)/K(T))$.*

Proof. Let $F := \pi^{-1}(t)_{\text{red}}$. Note that F is irreducible, because Z is $\mathbb{Q}$ -factorial and $\rho(Z/T) = 1$. Since C is π -horizontal, we obtain

$$(K_Z + F) \cdot F = K_Z \cdot F < (K_Z + C + B) \cdot F = 0.$$

Thus, [Tan14, Theorem 3.19(1)] implies (1).

Let G be a general fibre of π . Then G is integral [Băd01, Corollary 7.3]. Thus, $G \simeq \mathbb{P}^1$ by (1). In particular, it follows that

$$(K_Z + B) \cdot G \geq K_Z \cdot G = (K_Z + G) \cdot G = \deg(K_G + \text{Diff}_G) \geq -2,$$

which implies $C \cdot G = -(K_Z + B) \cdot G \leq 2$. Thus, (2) holds.

We now prove (3). Note that B is π -vertical, since

$$0 = (K_Z + C + B) \cdot G \geq -2 + 2 + B \cdot G = B \cdot G,$$

where G is a general fibre of π . Since $\rho(Z/T) = 1$, we obtain $K_Z + C \equiv_{\pi} 0$. Since the divisor $\pi^*(t)|_C$ is $\text{Gal}(K(C)/K(T))$ -invariant for every closed point $t \in T$, so is $B|_C$. In particular, we may assume $B = 0$ and $B_C = \text{Diff}_C$.

Let $Q \in T$ be such that $\pi^{-1}(Q) \cap C$ consists of two distinct points Q_1, Q_2 . It is enough to show that the coefficients of Diff_C at the points Q_1 and Q_2 coincide. We may write $\pi^*Q = mF$ where F is a prime divisor and m is a positive integer. By (1), $F \simeq \mathbb{P}^1$.

We show that the pair $(Z, C + F)$ is log canonical. We have

$$0 \sim_{\mathbb{Q}} (K_Z + C + F)|_F = K_F + \text{Diff}_F + C|_F.$$

Since $(Z, C + F)$ is not plt at the points Q_1 and Q_2 , inversion of adjunction implies that the pair $(F, \text{Diff}_F + C|_F)$ is not klt at the points Q_1 and Q_2 . Thus,

$$\text{Diff}_F + C|_F \geq Q_1 + Q_2,$$

and since $\deg(\text{Diff}_F + C|_F) = \deg(-K_F) = 2$, equality holds. Thus, inversion of adjunction implies that $(Z, C + F)$ is log canonical.

Since $\pi|_C: C \rightarrow T$ is étale over Q , we have $mF|_C = \pi^*Q|_C = Q_1 + Q_2$ and

$$(K_Z + C + F)|_C = K_C + \text{Diff}_C + \frac{1}{m}(Q_1 + Q_2).$$

In particular, by inversion of adjunction again, the coefficients of Diff_C at Q_1 and Q_2 are equal to $1 - \frac{1}{m}$, and the claim follows. $\square$

4.2.2. Global F-adjunction

We now summarise some known results on F -adjunction which we will use in this subsection (cf. Subsection 2.2.2). Let X be a normal variety, S a prime divisor and B an effective

$\mathbb{Q}$ -divisor on X such that $S \not\subseteq \text{Supp } B$. Let $\nu: S' \rightarrow S$ be the normalisation. Assume that $(p^e - 1)(K_X + S + B)$ is Cartier for some $e \in \mathbb{Z}_{>0}$ (this is equivalent to saying that the Cartier index of $K_X + S + B$ is not divisible by p). By adjunction, we may write

$$\nu^*((K_X + S + B)|_S) = K_{S'} + B_{S'}.$$

There are natural homomorphisms

$$\begin{aligned} \text{Tr}_{X,S+B}: F_*^e \mathcal{O}_X(-(p^e - 1)(K_X + S + B)) &\longrightarrow \mathcal{O}_X, \text{ and} \\ \text{Tr}_{S',B_{S'}}: F_*^e \mathcal{O}_{S'}(-(p^e - 1)(K_{S'} + B_{S'})) &\longrightarrow \mathcal{O}_{S'}. \end{aligned}$$

Let

$$\mathcal{L} := \mathcal{O}_X(-(p^e - 1)(K_X + S + B)).$$

Since S is an F -pure centre of $(X, S+B)$ ([Sch14, Definition 5.1]), we obtain a commutative diagram [ibid, Remark 5.2]:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_*^e(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-S)) & \longrightarrow & F_*^e \mathcal{L} & \longrightarrow & F_*^e(\mathcal{L}|_S) \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{Tr}_{X,S+B} & & \downarrow \psi \\ 0 & \longrightarrow & \mathcal{O}_X(-S) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_S \longrightarrow 0. \end{array}$$

We also obtain the following commutative diagram [Sch09, Lemma 8.1]:

$$\begin{array}{ccc} F_*^e(\mathcal{L}|_S) & \longrightarrow & F_*^e(\nu^*(\mathcal{L}|_S)) \\ \downarrow \psi & & \downarrow \text{Tr}_{S',B_{S'}} \\ \mathcal{O}_S & \longrightarrow & \mathcal{O}_{S'}. \end{array}$$

Note that the right vertical arrow coincides with $\text{Tr}_{S',B_{S'}}$ by [Das15, Theorem 5.3].

Under some additional assumptions, we will show that if $-(K_X + S + B)$ is nef and $(S', B_{S'})$ is globally F -split, then $(X, S + B)$ is globally F -split as well. The following three lemmas correspond to the cases

$$\kappa(X, -(K_X + S + B)) = 2, 1, \text{ and } 0$$

respectively.

Lemma 4.2.7. *Let $(X, S + B)$ be a projective log pair where S is a prime divisor and B is an effective $\mathbb{Q}$ -divisor such that $S \not\subseteq \text{Supp } B$. Assume that*

1. $-(K_X + S + B)$ is ample,
2. S is normal and if $K_S + B_S = (K_X + S + B)|_S$ is defined by adjunction, then (S, B_S) is globally F -split, and
3. $(p^e - 1)(K_X + S + B)$ is Cartier for some positive integer e .

Then $(X, S + B)$ is globally F -split.

Proof. This follows from Lemma 2.2.11. $\square$

Lemma 4.2.8. *Assume that $\text{char } k > 2$. Let $(Z, C + B)$ be a two dimensional projective $\mathbb{Q}$ -factorial log canonical pair, where C is a prime divisor and B is an effective $\mathbb{Q}$ -divisor. Let $\pi: Z \rightarrow T$ be a morphism onto a smooth projective curve T such that $\pi_* \mathcal{O}_Z = \mathcal{O}_T$. Assume that*

1. C is π -horizontal,
2. $-(K_Z + C + B) \equiv \pi^* A$ for some ample $\mathbb{Q}$ -divisor A on T ,
3. C is normal and if $K_C + B_C = (K_Z + C + B)|_C$ is defined by adjunction, then (C, B_C) is globally F -split,
4. $(p^e - 1)(K_Z + C + B)$ is Cartier for some positive integer e , and
5. $\rho(Z/T) = 1$.

Then $(Z, C + B)$ is globally F -split.

Proof. Let $\mathcal{L} := \mathcal{O}_Z(-(p^e - 1)(K_Z + C + B))$. We have the following commutative diagram:

$$\begin{array}{ccc} H^0(Z, F_*^e \mathcal{L}) & \longrightarrow & H^0(C, F_*^e \mathcal{L}|_C) \\ \downarrow \text{Tr}_{Z, C+B} & & \downarrow \text{Tr}_{C, B_C} \\ H^0(Z, \mathcal{O}_Z) & \longrightarrow & H^0(C, \mathcal{O}_C). \end{array}$$

By (3), the right vertical arrow Tr_{C, B_C} is surjective.

By (2) and (4), we can find an ample Cartier divisor A' on T such that $\mathcal{L} \simeq \mathcal{O}_Z(\pi^* A')$ [Tan15b, Theorem 0.4]. Since $\pi_* \mathcal{O}_Z = \mathcal{O}_T$, we have that

$$H^0(Z, \mathcal{L}) \simeq H^0(T, \mathcal{O}_T(A')),$$

and so the image of the upper horizontal arrow is exactly the G -invariant part, that is $H^0(C, F_*^e \mathcal{L}|_C)^G$, where G is the Galois group of $K(C)/K(T)$.

Let $f \in H^0(C, F_*^e \mathcal{L}|_C)$ be such that $\text{Tr}_{C, B_C}(f) = 1$. Since B_C is G -invariant by Lemma 4.2.6, we have that Tr_{C, B_C} is G -equivariant. In particular,

$$\text{Tr}_{C, B_C} \left(\frac{1}{|G|} \sum_{g \in G} g(f) \right) = 1.$$

We can divide by $|G|$, because $\text{char } k > 2$ and $|G| \leq 2$ by Lemma 4.2.6. Thus, $\text{Tr}_{Z, C+B}$ is surjective and $(Z, C + B)$ is globally F -split (see Proposition 2.2.9). $\square$

Lemma 4.2.9. *Let $(Z, C + B)$ be an n -dimensional projective log canonical pair with $n \leq 2$, where C is a prime divisor and B is an effective $\mathbb{Q}$ -divisor. Assume that*

1. $(p^e - 1)(K_Z + C + B) \sim 0$ for some positive integer e ,
2. $H^1(Z, \mathcal{O}_Z) = 0$, and
3. if $n = 2$, then $(Z, C + B)$ is not plt.

Then $(Z, C + B)$ is globally F -split.

Proof. We only prove the case $n = 2$, as the case $n = 1$ is easier.

Let $\nu: C' \rightarrow C$ be the normalisation. We may write

$$\nu^*((K_Z + C + B)|_C) = K_{C'} + B_{C'}$$

for some effective $\mathbb{Q}$ -divisor $B_{C'}$ on C' . By (3) and by inversion of adjunction, it follows that $\lfloor B_{C'} \rfloor \neq 0$. Thus, by (1), it follows that C' is isomorphic to $\mathbb{P}^1$. Therefore, by the same lemma in dimension one, $(C', B_{C'})$ is globally F -split.

Let e be a positive integer satisfying (1) and let

$$\mathcal{L} := \mathcal{O}_Z(-(p^e - 1)(K_Z + C + B)).$$

By (1), we have that $\mathcal{L} \simeq \mathcal{O}_Z$. We consider the commutative diagram

$$\begin{array}{ccccc} H^0(Z, F_*^e \mathcal{L}) & \longrightarrow & H^0(C, F_*^e(\mathcal{L}|_C)) & \longrightarrow & H^1(Z, F_*^e(\mathcal{L}(-C))) \\ \downarrow \text{Tr}_{Z, C+B} & & \downarrow \psi & & \\ H^0(Z, \mathcal{O}_Z) & \xrightarrow{\simeq} & H^0(C, \mathcal{O}_C) & & \end{array}$$

We now show that ψ is surjective. We have the following commutative diagram:

$$\begin{array}{ccc} H^0(C, F_*^e(\mathcal{L}|_C)) & \xrightarrow{\alpha} & H^0(C', F_*^e(\nu^*(\mathcal{L}|_C))) \\ \downarrow \psi & & \downarrow \text{Tr}_{C', B_{C'}} \\ H^0(C, \mathcal{O}_C) & \xrightarrow{\simeq} & H^0(C', \mathcal{O}_{C'}). \end{array}$$

Since α is induced from the natural homomorphism $\mathcal{O}_C \rightarrow \nu_* \mathcal{O}_{C'}$ and $\mathcal{L} \simeq \mathcal{O}_Z$, it follows that α is an isomorphism. Since $(C', B_{C'})$ is globally F -split, $\text{Tr}_{C', B_{C'}}$ is surjective. Thus, ψ is surjective as well.

Since $\mathcal{L} \simeq \mathcal{O}_Z$, (2) implies

$$\begin{aligned} H^1(Z, F_*^e(\mathcal{L}(-C))) &\simeq H^1(Z, \mathcal{L}(-C)) \\ &\simeq H^1(Z, \mathcal{O}_Z(-C)) \hookrightarrow H^1(Z, \mathcal{O}_Z) = 0. \end{aligned}$$

Since ψ is surjective, by a diagram chase it follows that $\text{Tr}_{Z, C+B}$ is surjective as well, and, in particular, $(Z, C + B)$ is globally F -split. Thus, the claim follows. $\square$

4.2.3. Perturbations

Let $(X, S + B)$ be a log pair such that S is a prime divisor and B is an effective $\mathbb{Q}$ -divisor such that $S \not\subseteq \text{Supp } B$. If the Cartier index of $K_X + S + B$ is divisible by p , then, in order to apply the results from the previous subsection, we need to perturb the coefficients of B .

We begin with the following well known result:

Lemma 4.2.10. *Let (X, Δ) be a quasi-projective globally sharply F -split pair. Then there exists an effective $\mathbb{Q}$ -divisor H such that $(X, \Delta + H)$ is globally sharply F -split and*

$$(p^e - 1)(K_X + \Delta + H) \sim 0$$

for some positive integer e .

Proof. This follows from the same proof as [SS10, Theorem 4.3(ii)]. $\square$

Lemma 4.2.11. *Let (X, Δ) be a two-dimensional quasi-projective sharply F -pure log pair. Then there exists an effective $\mathbb{Q}$ -divisor H such that $(X, \Delta + H)$ is sharply F -pure, and*

$$(p^e - 1)(K_X + \Delta + H)$$

is Cartier for some positive integer e .

Proof. Let $U \subseteq X$ be an affine open subset such that none of the irreducible components of Δ are contained in $X \setminus U$ and such that, around each point of $X \setminus U$, we have that X is smooth, the support of Δ is simple normal crossing and (X, Δ) is plt. Since (X, Δ) is sharply F -pure and U is affine, it follows that $(U, \Delta|_U)$ is globally sharply F -split.

By Lemma 4.2.10, there exists an effective $\mathbb{Q}$ -divisor H' on X such that $(U, (\Delta + H')|_U)$ is globally sharply F -split and

$$(p^e - 1)(K_U + (\Delta + H')|_U) \sim 0,$$

for some positive integer e . Since X is smooth around each point of $X \setminus U$, it follows that $(p^e - 1)(K_X + \Delta + H')$ is Cartier on X . For every positive integer d , we have

$$(K_X + \Delta + H') - (K_X + \Delta + \frac{1}{p^d + 1}H') = \frac{p^d}{p^d + 1}H'.$$

Thus, for every sufficiently large positive integer d , there exists $e(d) \in \mathbb{Z}_{>0}$ such that

$$(p^{e(d)} - 1)(K_X + \Delta + \frac{1}{p^d + 1}H')$$

is Cartier.

Let $H := \frac{1}{p^d + 1}H'$. It is enough to show that, after possibly replacing d by a larger value, the pair $(X, \Delta + H)$ is sharply F -pure around any point $q \in X \setminus U$. By assumption, around q we have that X is smooth, the support of Δ is simple normal crossing and (X, Δ) is plt. Thus, if q is not contained in the support of $\lrcorner \Delta \lrcorner$ and d is sufficiently large, then $(X, \Delta + H)$ is strongly F -regular. On the other hand, if q is contained in the support of $\lrcorner \Delta \lrcorner$ then the claim follows from inversion of adjunction (e.g. see [Sch09, Main Theorem], Theorem 2.2.10). $\square$

Lemma 4.2.12. *Assume that $\text{char } k > 2$. Let $(Z, C + B)$ be a two dimensional projective sharply F -pure pair, where C is a prime divisor and B is an effective $\mathbb{Q}$ -divisor. Let $\pi: Z \rightarrow T$ be a morphism onto a smooth projective curve T such that $\pi_* \mathcal{O}_Z = \mathcal{O}_T$. Assume that*

1. (Z, B) is klt,
2. $-(K_Z + C + B) \equiv \pi^* A$ for some ample $\mathbb{Q}$ -divisor A on T ,
3. C is π -horizontal,
4. $\rho(Z/T) = 1$, and
5. $(p^d - 1)B$ is a $\mathbb{Z}$ -divisor for some positive integer d .

Then there exists an effective $\mathbb{Q}$ -divisor E , whose support is contained in some fibres of π , such that $(Z, C + B + E)$ is log canonical and

$$(p^e - 1)(K_Z + C + B + E)$$

is Cartier for some positive integer e .

Proof. By Lemma 4.2.6, there exists a non-empty open subset $T^0 \subseteq T$ such that the induced morphism $\pi^{-1}(T^0) \rightarrow T^0$ is a $\mathbb{P}^1$ -bundle. By (2) and (5), it follows that, after possibly shrinking T^0 , there exists a positive integer d such that

$$(p^d - 1)(K_Z + C + B)|_{\pi^{-1}(T^0)} \sim 0.$$

By (1) of Lemma 4.2.6, every fibre of π is irreducible. Thus, we may write

$$(p^d - 1)(K_Z + C + B) \sim \sum_{i=1}^r a_i F_i + \sum_{j=1}^s b_j F'_j,$$

where $a_i, b_j \in \mathbb{Z}$, $F_i = \pi^{-1}(t_i)_{\text{red}}$, and $F'_j = \pi^{-1}(t'_j)_{\text{red}}$, with

$$t_1, \dots, t_r, t'_1, \dots, t'_s \in T,$$

so that $(Z, C + B)$ is plt (resp. not plt) along F_i (resp. at some point of F'_j). Since $\lfloor C + B \rfloor = C$ is π -horizontal and (Z, B) is klt, by inversion of adjunction we can find $\alpha_i \in \mathbb{Q}_{>0}$ for $i = 1, \dots, r$ such that $(Z, C + B + \sum_{i=1}^r \alpha_i F_i)$ is log canonical and $(p^{e_1} - 1)(K_Z + C + B + \sum_{i=1}^r \alpha_i F_i)$ is Cartier around $\bigcup_{i=1}^r F_i$ for some $e_1 \in \mathbb{Z}_{>0}$.

Fix an index $1 \leq j \leq s$. It is enough to find $e_2 \in \mathbb{Z}_{>0}$ such that $(p^{e_2} - 1)b_j F'_j$ is Cartier. By construction, there exists a zero-dimensional log canonical centre $z \in F'_j$ of $(Z, C + B)$. Since (Z, B) is klt, it follows that $z \in C$. Let m_z be the Cartier index of F'_j at z . Since $z \in C$, it follows that $m_z F'_j \cdot C \geq 1$. We can write $\pi^*(\pi(z)) = m F'_j$ for some

positive integer m . Lemma 4.2.6 implies that $mF'_j \cdot C = \pi^*(\pi(z)) \cdot C \leq 2$. Thus,

$$\frac{1}{m_z} \leq F'_j \cdot C \leq \frac{2}{m},$$

and in particular $2m_z \geq m$. Since m_z divides m , it follows that either $m = m_z$ or $m = 2m_z$.

By applying Lemma 4.2.10 to an affine open neighbourhood U of z , we can find $e_3 \in \mathbb{Z}_{>0}$ and an effective $\mathbb{Q}$ -divisor H on U such that $(U, (C + B)|_U + H)$ is sharply F -pure and

$$(p^{e_3} - 1)(K_U + (C + B)|_U + H) \sim 0.$$

Since $z \in Z$ is a zero-dimensional log canonical centre of $(Z, C + B)$, it follows that z is not contained in the support of H . Therefore, $(p^{e_3} - 1)(K_Z + C + B)$ is Cartier around z . Since

$$(p^d - 1)(K_Z + C + B) \sim \sum_{i=1}^r a_i F_i + \sum_{j=1}^s b_j F'_j,$$

it follows that $(p^{e_4} - 1)b_j F'_j$ is Cartier around z for some $e_4 \in \mathbb{Z}_{>0}$. Since $m \in \{m_z, 2m_z\}$, we have that $2(p^{e_4} - 1)b_j F'_j$ is Cartier. Since $\text{char } k \neq 2$, we can find $e_2 \in \mathbb{Z}_{>0}$ such that $(p^{e_2} - 1)b_j F'_j$ is Cartier. Thus, the claim follows. $\square$

4.2.4. Flat families of log pairs and liftability

Definition 4.2.13. Given a reduced noetherian separated scheme T , a *flat family of log pairs* $(\mathcal{X}, \mathcal{B})$ over T consists of the data of

- a normal scheme $\mathcal{X}$ and an effective $\mathbb{Q}$ -divisor $\mathcal{B}$ on $\mathcal{X}$ such that $K_{\mathcal{X}} + \mathcal{B}$ is $\mathbb{Q}$ -Cartier,
- a separated and flat morphism of finite type $\pi: \mathcal{X} \rightarrow T$ such that all the fibres of π are geometrically normal, and
- for each irreducible component $\mathcal{B}_i$ of $\mathcal{B}$, the induced morphism $\pi|_{\mathcal{B}_i}: \mathcal{B}_i \rightarrow T$ is flat and all the fibres of $\pi|_{\mathcal{B}_i}$ are geometrically integral.

Definition 4.2.14. Let $(\mathcal{X}, \mathcal{B})$ be a flat family of log pairs over a reduced noetherian separated scheme T and let (X, B) be a log pair over an algebraically closed field k . We say that (X, B) is a *geometric fibre* of $(\mathcal{X}, \mathcal{B})$, if there exists a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Spec } k & \xrightarrow{\alpha} & T \end{array}$$

such that $\beta^*(\mathcal{B}) = B$, where if we write $\mathcal{B} = \sum_{i=1}^r b_i \mathcal{B}_i$ with $\mathcal{B}_1, \dots, \mathcal{B}_r$ prime components, then we define

$$\beta^*(\mathcal{B}) := \sum_{i=1}^r b_i \beta^{-1}(\mathcal{B}_i).$$

Note that, for each $i = 1, \dots, r$, the scheme-theoretic inverse image $\beta^{-1}(\mathcal{B}_i)$ is a prime divisor on X .

Definition 4.2.15. Let X be a smooth variety over a perfect field k of characteristic $p > 0$, and let D be a simple normal crossing divisor on X . Write $D = \sum_{i=1}^r D_i$, where D_i are the irreducible components of D . We say that the pair (X, D) *lifts to characteristic zero over a smooth base* if there exist

- a scheme T smooth and separated over $\text{Spec } \mathbb{Z}$,
- a smooth and separated morphism $\mathcal{X} \rightarrow T$,
- effective Cartier divisors $\mathcal{D}_1, \dots, \mathcal{D}_r$ on $\mathcal{X}$ such that the scheme-theoretic intersection $\bigcap_{i \in J} \mathcal{D}_i$ for any subset $J \subseteq \{1, \dots, r\}$ is smooth over T , and
- a morphism $\alpha: \text{Spec } k \rightarrow T$,

such that the base changes of the schemes $\mathcal{X}, \mathcal{D}_1, \dots, \mathcal{D}_r$ over T by $\alpha: \text{Spec } k \rightarrow T$ are isomorphic to $X, D_1, \dots, D_r$, respectively.

Remark 4.2.16. Under the same assumption as in Definition 4.2.15, if (X, D) lifts to characteristic zero over a smooth base, then (X, D) also lifts to $W_2(k)$. Indeed, since $W_2(k)$ is a henselian local ring [Fu15, Proposition 2.8.4], and T is smooth over $\text{Spec } \mathbb{Z}$, the morphism $\text{Spec } k \rightarrow T$ lifts to a morphism $\text{Spec } W_2(k) \rightarrow T$ by [Fu15, Proposition 2.8.13]:

$$\begin{array}{ccc} \mathcal{X} \times_T W_2(k) & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Spec } W_2(k) & \longrightarrow & T. \end{array}$$

4.3. Proof of the main theorem for ϵ -klt log del Pezzo

The goal of this section is to prove the main theorem for ϵ -klt log del Pezzo pairs, for any fixed $\epsilon > 0$ (Proposition 4.3.2). The idea of the proof is that the family of ϵ -klt log del Pezzos over $\text{Spec } \mathbb{Z}$ is bounded (Lemma 4.3.1). The same result is known to hold over a fixed algebraically closed field [Ale94, Theorem 6.9]. Our argument follows the same methods, however we include the proof for completeness.

Let S be a noetherian separated scheme, let Y be a scheme which is projective and flat over S and let H be an invertible sheaf on Y , which is ample over S . Let $\varphi \in \mathbb{Q}[t]$ be an arbitrary polynomial. We define $\text{Hilb}_{Y/S}^{\varphi, H}$ to be the Hilbert functor such that, for any scheme T over S , $\text{Hilb}_{Y/S}^{\varphi, H}(T)$ is the set of closed subschemes in $Y \times_S T$ which are flat over T , and with Hilbert polynomial equal to φ with respect to the pull-back of H on $Y \times_S T$ (see [FGI⁺05, Section 5.1] for more details). We also define $\text{Div}_{Y/S}^{\varphi, H}$ to be the functor such that, for any scheme T over S , $\text{Div}_{Y/S}^{\varphi, H}(T)$ is the set of effective Cartier divisors on $Y \times_S T$ which are flat over T , and with Hilbert polynomial equal to φ with respect to the pull-back of H on $Y \times_S T$ (see [ibid, Section 9.3]).

Since $\text{Hilb}_{Y/S}^{\varphi, H}$ is representable by a projective scheme over S [ibid, Theorem 5.14 and Subsection 5.1.3], it follows that $\text{Div}_{Y/S}^{\varphi, H}$ is representable by a quasi-projective scheme over S [ibid, Theorem 9.3.7].

Lemma 4.3.1. *Let $I \subseteq (0, 1) \cap \mathbb{Q}$ be a finite set and let $\epsilon > 0$.*

Then there exists a flat family of log pairs $(\mathcal{X}, \mathcal{B})$ over a reduced quasi-projective scheme T over $\text{Spec } \mathbb{Z}$, such that every ϵ -klt log del Pezzo pair (X, B) over any algebraically closed field k of characteristic $p > 5$, with the coefficients of B contained in I , is a geometric fibre of $(\mathcal{X}, \mathcal{B})$.

Proof. There exist positive integers b and n which satisfy the following property (see Corollary 5.1.3 and Remark 5.6.4): for every (X, B) over k as in the lemma, we can find a very ample divisor H on X such that

- $|H|$ embeds X into $\mathbb{P}_k^b$,
- nB is Cartier,
- $|\varphi_{X,i}| \leq b$ for all $i \geq 0$ where $\varphi_X = \sum_{i \geq 0} \varphi_{X,i} t^i \in \mathbb{Q}[t]$ is the Hilbert polynomial of X with respect to H , and
- $|\psi_{nB,i}| \leq b$ for all $i \geq 0$ where $\psi_{nB} = \psi_{nB,1}t + \psi_{nB,0} \in \mathbb{Z}[t]$ is the Hilbert polynomial of nB with respect to H .

We may assume that $I = (0, 1) \cap \frac{1}{n}\mathbb{Z}$. Further, since $\dim X = 2$, we have that $2\varphi_X \in \mathbb{Z}[t]$ and $\deg \varphi_X = 2$.

In particular, if G is the functor defined by

$$G := \bigcup_{\varphi} \text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^b/\mathbb{Z}}^{\varphi, \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^b/\mathbb{Z}}(1)}$$

where the union is taken over all $\varphi = \sum \varphi_i t^i \in \mathbb{Q}[t]$ such that $2\varphi \in \mathbb{Z}[t]$, $\deg \varphi = 2$, and $|\varphi_i| \leq b$ for all $i \geq 0$, then $X \in G(\text{Spec } k)$.

By [FGI⁺05, Theorem 5.14 and Subsection 5.1.3], the functor G is representable by a projective scheme S over $\text{Spec } \mathbb{Z}$. Let $\mathcal{U} \subseteq \mathbb{P}_{\mathbb{Z}}^b \times_{\mathbb{Z}} S$ be its universal closed subscheme. Let $\mathcal{H}$ be the pull-back of $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^b/\mathbb{Z}}(1)$ to $\mathcal{U}$. Then $\mathcal{H}$ is ample over S . We have

$$(nB \subseteq X) \in \bigcup_{\psi} \text{Div}_{\mathcal{U}/S}^{\psi, \mathcal{H}}(\text{Spec } k),$$

where the union is taken over all $\psi = \psi_1 t + \psi_0 \in \mathbb{Z}[t]$ such that $|\psi_0| \leq b$ and $|\psi_1| \leq b$.

Let T be the reduction of the scheme representing $\bigcup_{\psi} \text{Div}_{\mathcal{U}/S}^{\psi, \mathcal{H}}$. Note that T is quasi-projective over $\text{Spec } \mathbb{Z}$. Let $\mathcal{B}'$ be the universal effective Cartier divisor on $\mathcal{X} := \mathcal{U} \times_S T$. Let $\mathcal{B} := \frac{1}{n}\mathcal{B}'$. Take a generic point ζ of T . By taking a base change of some finite morphism of an open subset of T , we may assume that the fibres of the irreducible components of $\mathcal{B}$ over ζ are geometrically integral. Thus, there exists an open subscheme $U \subseteq T$, such that $(\mathcal{X}|_U, \mathcal{B}|_U)$ is a flat family of log pairs. By replacing T by $U \amalg T \setminus U$ and repeating the same argument to $T \setminus U$, we may conclude the proof. $\square$

We now show the main result of this section.

Proposition 4.3.2. *Let $I \subseteq (0, 1) \cap \mathbb{Q}$ be a finite set and let $\epsilon > 0$. Then, there exists a positive integer $p(I, \epsilon)$ which satisfies the following property:*

Let (X, B) be an ϵ -klt log del Pezzo pair over an algebraically closed field of characteristic $p > p(I, \epsilon)$ such that the coefficients of B are contained in I . Then there exists a log resolution $\mu: V \rightarrow X$ of (X, B) such that $(V, \text{Exc}(\mu) \cup \mu_^{-1}(\text{Supp } B))$ lifts to characteristic zero over a smooth base.*

Proof. We fix a positive integer $p(I, \epsilon) > 5$. We will replace $p(I, \epsilon)$ by a larger number if necessary. By Lemma 4.3.1, there exists a flat family of log pairs $(\mathcal{X}, \mathcal{B})$ over a reduced quasi-projective scheme T over $\text{Spec } \mathbb{Z}$, such that every pair (X, B) as in the proposition is a geometric fibre of the pair $(\mathcal{X}, \mathcal{B})$.

Claim: Let T' be an irreducible component of T such that the field $K(T')$ is of characteristic zero.

Then there exists a dominant morphism $S \rightarrow T'$ from an integral scheme S which satisfies the following properties:

- (a) S is smooth over $\text{Spec } \mathbb{Z}$.
- (b) If we set $\mathcal{X}_S := \mathcal{X} \times_T S$, then there exists a projective birational morphism $\mu_S: \mathcal{V}_S \rightarrow \mathcal{X}_S$ over S such that the induced morphism $\mu_{S,s}: \mathcal{V}_s \rightarrow \mathcal{X}_s$ between any fibres over $s \in S$ is birational, $\mathcal{V}_S$ is smooth over S , and

$$\text{Exc}(\mu_S) \cup (\mu_S)_*^{-1}(\text{Supp } \mathcal{B})$$

is simple normal crossing over S , i.e. all the strata are smooth over S .

We first show the proposition, assuming this claim. By noetherian induction, we can find a surjective morphism

$$\left(\coprod_{1 \leq i \leq \alpha} S_i \right) \amalg \left(\coprod_{1 \leq j \leq \beta} S'_j \right) \rightarrow T$$

such that all S_i and S'_j are separated integral schemes of finite type over $\text{Spec } \mathbb{Z}$, any $K(S_i)$ (resp. $K(S'_j)$) is of characteristic zero (resp. $p_j > 0$), and each S_i satisfies properties (a) and (b). Then the proposition follows after possibly increasing $p(I, \epsilon)$ so that $p(I, \epsilon) \geq p_j$ for all $j = 1, \dots, \beta$.

We now show the claim. We set $S := T'$. It is enough to show that properties (a) and (b) hold after possibly replacing S by a finite cover (i.e. a finite surjective morphism from an integral scheme) of an open subset.

By our assumption, the field $K(S) = \mathcal{O}_{S, \xi}$ is of characteristic zero, where ξ is the generic point of S . Thus, after replacing S by an open subset, we may assume that S is smooth over $\text{Spec } \mathbb{Z}$, hence (a) holds.

Let $(X_{\bar{\xi}}, B_{\bar{\xi}})$ be the base change to the algebraic closure $\bar{\xi}$ of the generic fibre (X_{ξ}, B_{ξ}) and take a log resolution $\mu_{\bar{\xi}}: V_{\bar{\xi}} \rightarrow X_{\bar{\xi}}$ of $(X_{\bar{\xi}}, B_{\bar{\xi}})$. After replacing S by a finite cover of

an open subset, we may assume that there exists a projective birational morphism over S

$$\mu_S: \mathcal{V}_S \rightarrow \mathcal{X}_S$$

whose base change to $\bar{\xi}$ is the same as $V_{\bar{\xi}} \rightarrow X_{\bar{\xi}}$, where $\mathcal{X}_S := \mathcal{X} \times_T S$.

We check that property (b) holds, after replacing S by an open subset. Indeed, since $\mu_{\bar{\xi}}: V_{\bar{\xi}} \rightarrow X_{\bar{\xi}}$ is a projective birational morphism of surfaces, so are the morphisms $\mathcal{V}_{S,s} \rightarrow \mathcal{X}_{S,s}$ between the geometric fibres, for any geometric point s in an open neighbourhood of ξ . Moreover, $V_{\bar{\xi}}$ and any stratum of $\text{Exc}(\mu_{\bar{\xi}}) \cup (\mu_{\bar{\xi}})_*^{-1}(\text{Supp } \mathcal{B}_{\bar{\xi}})$ are smooth, hence $\mathcal{V}_S$ and all the strata of $\text{Exc}(\mu_S) \cup (\mu_S)_*^{-1}(\text{Supp } \mathcal{B})$ are smooth. $\square$

Remark 4.3.3. By Proposition 4.3.2, it follows that ϵ -klt log del Pezzo pairs in large characteristic satisfy (2) of Theorem 4.1.1. By [SS10, Theorem 1.2] and noetherian induction, it follows that (1) of Theorem 4.1.1 holds for ϵ -klt log del Pezzo pairs in large characteristic. However, we do not use this fact in this paper.

4.4. Liftability to characteristic zero

The goal of this section is to study plt pairs $(Z, C_Z + B_Z)$, such that $\lfloor C_Z + B_Z \rfloor = C_Z$ is a prime divisor and $K_Z + C_Z + B_Z \equiv 0$. These pairs appear in the proof of Theorem 4.1.1, after running a suitable MMP, starting from some model over a log del Pezzo pair (X, B) (cf. Section 4.5).

We first show that, in large characteristic, such a pair $(Z, C_Z + B_Z)$ admits a log resolution which lifts to characteristic zero (Proposition 4.4.2). Then, in order to show that also the pair (X, B) admits a log resolution which lifts to characteristic zero, we study the behaviour of such a liftability property under blow-ups (Lemma 4.4.3).

The following result is a consequence of Lemma 4.3.1:

Lemma 4.4.1. *Let $I \subseteq (0, 1) \cap \mathbb{Q}$ be a DCC set.*

Then there exists a flat family of log pairs $(\mathcal{X}, \mathcal{C} + \mathcal{B})$ over a reduced quasi-projective scheme T over $\text{Spec } \mathbb{Z}$, such that any two dimensional projective plt pair $(X, C + B)$, over any algebraically closed field k of characteristic $p > 5$, satisfying

- $\lfloor C + B \rfloor = C$,
- *the coefficients of B are contained in I ,*
- $K_X + C + B \equiv 0$, *and*
- C *is ample,*

is a geometric fibre of $(\mathcal{X}, \mathcal{C} + \mathcal{B})$.

Proof. As in the proof of Proposition 4.2.5, there exists a positive integer m depending only on I such that

$$m(K_X + C + B) \sim 0 \quad \text{and} \quad mC \text{ is Cartier}$$

for any pair $(X, C + B)$ satisfying the assumptions in the Lemma.

In particular, $2m(K_X + \frac{1}{2}C + B)$ is Cartier, and if $\epsilon \in (0, \frac{1}{2m})$ then $(X, \frac{1}{2}C + B)$ is ϵ -klt. Thus, Lemma 4.3.1 implies that there exists a flat family $(\mathcal{X}, \frac{1}{2}C + B)$ of log pairs such that $(X, \frac{1}{2}C + B)$ is a geometric fibre of $(\mathcal{X}, \frac{1}{2}C + B)$. In particular, $(\mathcal{X}, C + B)$ is the required family. $\square$

Proposition 4.4.2. *Let $I \subseteq (0, 1) \cap \mathbb{Q}$ be a finite set.*

Then there exists a positive integer $p(I)$ which satisfies the following property:

For any two dimensional projective plt pair $(Z, C + B)$ over an algebraically closed field of characteristic $p > p(I)$ such that

- $\lfloor C + B \rfloor = C$,
- C is ample,
- $K_Z + C + B \equiv 0$, and
- the coefficients of B are contained in I ,

there exists a birational morphism $\pi: W \rightarrow Z$ such that

- W is a smooth projective surface and $\text{Supp}(\pi_*^{-1}(C + B)) \cup \text{Exc}(\pi)$ is simple normal crossing, and
- $(W, \text{Supp}(\pi_*^{-1}(C + B)) \cup \text{Exc}(\pi))$ lifts to characteristic zero over a smooth base.

Proof. We can apply the same argument as in the proof of Proposition 4.3.2 after replacing (X, B) (resp. Lemma 4.3.1) by $(Z, C + B)$ (resp. Lemma 4.4.1). $\square$

Lemma 4.4.3. *Let Z be a smooth projective surface over an algebraically closed field k of characteristic $p > 0$ and let D be a reduced simple normal crossing divisor on Z . Let $\pi: Z' \rightarrow Z$ be the blow up at a point q contained in the singular locus of D and let E be the π -exceptional (-1) -curve. If (Z, D) lifts to characteristic zero over a smooth base, then so does $(Z', \pi_*^{-1}D + E)$.*

Proof. We may write $D = \sum_{i=1}^n D_i$, where $D_1, \dots, D_n$ are the irreducible components. Since (Z, D) lifts to characteristic zero, there exists a pair $(\mathcal{Z}, \mathcal{D} = \sum_{i=1}^n \mathcal{D}_i)$ over a smooth separated scheme T over $\text{Spec } \mathbb{Z}$ and a morphism $\alpha: \text{Spec } k \rightarrow T$, such that the base changes of the schemes $\mathcal{X}, \mathcal{D}_1, \dots, \mathcal{D}_n$ over T by $\alpha: \text{Spec } k \rightarrow T$ are isomorphic to $X, D_1, \dots, D_n$ respectively and satisfy the same properties as in Definition 4.2.15.

By assumption, there exist unique D_i, D_j such that $q \in D_i \cap D_j$. Let $\mathcal{C}$ be the irreducible component of $\mathcal{D}_i \cap \mathcal{D}_j$ such that $q \in \mathcal{C}$. Let $\bar{\pi}: \mathcal{Z}' \rightarrow \mathcal{Z}$ be the blow-up along $\mathcal{C}$. Then, $\mathcal{Z}'$ is a lift of Z' . Indeed, since $\mathcal{D}_i \cap \mathcal{D}_j$ is smooth over T , so is $\mathcal{C}$. It follows that $\mathcal{Z}'$ is smooth over T and has connected fibres over T (e.g. see the proof of [Liu02, Section 8, Theorem 1.19]). By [ibid], the exceptional divisor $\mathcal{E}$ of $\bar{\pi}$ is a $\mathbb{P}^1$ -bundle over $\mathcal{C}$. Thus, all the assumptions in Definition 4.2.15 hold true for the pairs $(Z', \pi_*^{-1}D + E)$ and $(\mathcal{Z}', \bar{\pi}_*^{-1}\mathcal{D} + \mathcal{E})$ over T . $\square$

Remark 4.4.4. By Lemma 4.4.3, we may assume that the log resolution appearing in (2) of Theorem 4.1.1 factors through the terminalisation $\eta: W \rightarrow X$ of (X, B) , i.e. if we write

$$K_W + B_W = \eta^*(K_X + B),$$

then (W, B_W) is terminal.

4.5. Non- ϵ -klt case and proof of Theorem 4.1.1

In Section 4.3, we showed that ϵ -klt log del Pezzo surfaces form a bounded family. The goal of this section is to study log del Pezzo surfaces (X, B) over an algebraically closed field k , which are not ϵ -klt for $0 < \epsilon \ll 1$. In particular, this yields a proof of our main Theorem (Theorem 4.1.1).

Our method consists of constructing a log canonical pair $(Z, C_Z + B_Z)$ from (X, B) , such that $[C_Z + B_Z] = C_Z$ is prime and $-(K_Z + C_Z + B_Z)$ is nef. If $K_Z + C_Z + B_Z$ is not numerically trivial, then, by using global F -adjunction (see Subsection 4.2.2), we show that $(Z, C_Z + B_Z)$ is globally F -split, provided that the characteristic of k is large enough. This in turn implies that (X, B) is globally F -regular.

Unfortunately, if $K_Z + C_Z + B_Z$ is numerically trivial, then $(Z, C_Z + B_Z)$ does not need to be globally F -split [CTW18, Theorem 1.1]. Thus, we need to consider two different cases, depending on whether $(Z, C_Z + B_Z)$ is plt or not. If the pair is plt (resp. non-plt), then we show that condition (2) (resp. condition (1)) of Theorem 4.1.1 holds.

Proposition 4.5.1. *Let $I \subseteq (0, 1) \cap \mathbb{Q}$ be a finite set.*

Then there exists a rational number $\epsilon(I) > 0$ which satisfies the following property:

Let (X, B) be a two dimensional projective klt pair over an algebraically closed field such that

- *$-(K_X + B)$ is ample,*
- *(X, B) is not $\epsilon(I)$ -klt, and*
- *the coefficients of B are contained in I .*

Then there exist birational morphisms

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow f & & \\ X & & \end{array}$$

*of projective klt surfaces such that if we denote $C = \text{Exc}(f)$, $C_Z = g_*C$, $B_Y = f_*^{-1}B$ and $B_Z = g_*B_Y$, then C is a prime divisor, $C_Z \neq 0$,*

1. *if $a = a(C, X, B)$ so that*

$$K_Y + (1 - a)C + B_Y = f^*(K_X + B),$$

then $a \in (0, \epsilon(I)]$,

2. there exists a $\mathbb{Q}$ -divisor $B_Y^+ \geq B_Y$ such that

$$K_Y + C + B_Y^+ = g^*(K_Z + C_Z + B_Z)$$

and

$$\text{Exc}(g) = \text{Supp}(B_Y^+ - B_Y),$$

3. $-(K_Z + C_Z + B_Z)$ is nef,

4. $(Z, C_Z + B_Z)$ is log canonical, and

5. $(Z, bC_Z + B_Z)$ is klt and $-(K_Z + bC_Z + B_Z)$ is ample for some rational number $b \in (1 - a, 1)$.

Proof. Pick any rational number $\epsilon(I) \in (0, 1)$ such that $\epsilon < 1 - c$ for any $c \in I$. We will replace $\epsilon(I)$ by a smaller number, if necessary.

Let (X, B) be a log pair which satisfies the assumptions in the proposition. Let C be an exceptional divisorial valuation such that $a := a(C, X, B)$ is minimal and let $f: Y \rightarrow X$ be a projective birational morphism such that $\text{Exc}(f) = C$. By assumption, $a \in (0, \epsilon(I)]$. Thus, (1) holds.

Since $-(K_X + B)$ is ample, we can find a rational number $b \in (1 - a, 1)$ such that $(Y, bC + B_Y)$ is klt, and $-(K_Y + bC + B_Y)$ is ample. We run a $-(K_Y + C + B_Y)$ -MMP with scaling of C :

$$\begin{aligned} (Y, C + B_Y) &=: (Y_0, C_0 + B_0) \xrightarrow{g_0} (Y_1, C_1 + B_1) \xrightarrow{g_1} \cdots \\ &\cdots \xrightarrow{g_{n-1}} (Y_n, C_n + B_n) =: (Z, C_Z + B_Z), \end{aligned}$$

where $C_i := (g_{i-1})_* C_{i-1}$ and $B_i := (g_{i-1})_* B_{i-1}$.

By definition of the MMP with scaling, we get a sequence of rational numbers

$$b < b_0 \leq b_1 \leq \cdots \leq b_{n-1}$$

such that

$$b_i := \max\{t > b \mid -(K_{Y_i} + tC_i + B_i) \text{ is nef}\}$$

and $g_i: Y_i \rightarrow Y_{i+1}$ is a birational morphism, such that $\rho(Y_i/Y_{i+1}) = 1$ and

$$K_{Y_i} + b_i C_i + B_i = g_i^*(K_{Y_{i+1}} + b_i C_{i+1} + B_{i+1}).$$

Since $-(K_{Y_i} + bC_i + B_i)$ is ample for any $i = 0, \dots, n$, each step of the MMP is $(K_{Y_i} + bC_i + B_i)$ -negative, and in particular $(Y_i, bC_i + B_i)$ is klt. Thus, (5) holds.

Since the coefficients of B_Z belong to the finite set I and $(Z, bC_Z + B_Z)$ is klt, by ACC for the log canonical threshold in dimension two [Ale93, Theorem 1.1], after possibly replacing $\epsilon(I)$ by a smaller value depending only on I , we may assume that the pair

$(Z, C_Z + B_Z)$ is log canonical, hence (4) holds.

Since C_i is g_i -ample for all $i = 0, \dots, n-1$, it follows that $C_Z \neq 0$. We may write

$$g^*(K_Z + C_Z + B_Z) = K_Y + C + B_Y + R$$

for some g -exceptional $\mathbb{Q}$ -divisor R on Y . Since the MMP is $-(K_Y + C + B_Y)$ -negative, it follows that $R \geq 0$ and the support of R coincides with the exceptional locus of g . In particular, if $B_Y^+ = B_Y + R$, then (2) holds.

Thus, it is enough to show (3). We assume by contradiction that $-(K_Z + C_Z + B_Z)$ is not nef. Thus, there exists a $-(K_Z + C_Z + B_Z)$ -negative Mori fibre space $\pi: Z \rightarrow T$ and, in particular,

$$(K_Z + C_Z + B_Z) \cdot F > 0$$

for any curve F contained in a fibre of π . On the other hand, since $-(K_Z + bC_Z + B_Z)$ is ample, it follows that

$$(K_Z + bC_Z + B_Z) \cdot F < 0.$$

Thus, C_Z is π -horizontal and we can find a rational number $b' \in (b, 1)$ such that

$$K_Z + b'C_Z + B_Z \equiv_\pi 0.$$

Since $1 - \epsilon(I) < b < b' < 1$, after possibly replacing $\epsilon(I)$ by a smaller number depending only on I , we get a contradiction by Lemma 4.2.3. Thus, (3) holds. $\square$

Before we proceed with the proof of Theorem 4.1.1, we recall the following criterion for global F -regularity:

Lemma 4.5.2. *Let $(Z, C + B)$ be a log pair, where $\lfloor C + B \rfloor = C$ is a reduced divisor. Let H be an effective $\mathbb{Q}$ -divisor on Z such that $Z \setminus \text{Supp}(H + C)$ is affine. Assume that*

1. (Z, B) is strongly F -regular, and
2. there exists $\delta > 0$ such that $(Z, C + B + \delta H)$ is globally F -split.

Then $(Z, \lambda C + B)$ is globally F -regular for every $\lambda \in [0, 1]$.

Proof. This follows from [SS10, Theorem 3.9]. $\square$

We now prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Let $\epsilon(I) > 0$ be the positive rational number as in Proposition 4.5.1 and let $p(I) := p(I, \epsilon(I))$ be the positive integer as in Proposition 4.3.2. We will replace $\epsilon(I)$ (resp. $p(I)$) by a smaller (resp. larger) value if necessary.

Let (X, B) be a log del Pezzo pair over an algebraically closed field of characteristic $p > p(I)$. If (X, B) is $\epsilon(I)$ -klt, then Proposition 4.3.2 implies (2) of Theorem 4.1.1. Thus, we may assume that (X, B) is not $\epsilon(I)$ -klt. By Proposition 4.5.1, there exist birational morphisms

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow f & & \\ X & & \end{array}$$

which satisfy the properties of Proposition 4.5.1.

Claim: After possibly replacing $p(I)$ by a larger value depending only on I , the following hold:

- (a) $(p^e - 1)B_Z$ is a $\mathbb{Z}$ -divisor for some $e \in \mathbb{Z}_{>0}$,
- (b) $(Z, C_Z + B_Z)$ is sharply F -pure and $(Z, \lambda C_Z + B_Z)$ is strongly F -regular for any $\lambda \in [0, 1)$, and
- (c) if $(Z, \lambda C_Z + B_Z)$ is globally F -regular for any $\lambda \in [0, 1)$, then (X, B) is globally F -regular.

We now prove the claim. (a) is clear. (4) and (5) of Proposition 4.5.1 imply that $(Z, \lambda C_Z + B_Z)$ is klt for any $\lambda \in [0, 1)$. Thus, after possibly replacing $p(I)$ by a larger number depending only on I , by [CGS16, Theorem 1.1], we may assume that $(Z, \lambda C_Z + B_Z)$ is strongly F -regular. By [CTW18, Proposition 3.3], $(Z, C_Z + B_Z)$ is F -pure, hence it is sharply F -pure by (a). Thus (b) holds.

We now show (c). Assume that $(Z, \lambda C_Z + B_Z)$ is globally F -regular for any $\lambda \in (0, 1)$. By (2) of Proposition 4.5.1, if $\lambda \in (0, 1)$ is a rational number which is sufficiently close to 1, then

$$K_Y + \lambda C + B'_Y = g^*(K_Z + \lambda C_Z + B_Z),$$

for some $\mathbb{Q}$ -divisor $B'_Y \geq B_Y$. By [HX15, Proposition 2.11], it follows that $(Y, \lambda C + B_Y)$ is globally F -regular for some $\lambda \in (0, 1)$. By [CTW18, Lemma 2.2], it follows that (X, B) is globally F -regular as well. Thus, (c) holds and this completes the proof of the claim.

By (3) of Proposition 4.5.1, it follows that $-(K_Z + C_Z + B_Z)$ is nef. By [Tan14, Theorem 1.2], it is semi-ample. In order to prove the theorem, we consider the following four cases separately:

- $\kappa(Z, -(K_Z + C_Z + B_Z)) = 2$,
- $\kappa(Z, -(K_Z + C_Z + B_Z)) = 1$,
- $\kappa(Z, -(K_Z + C_Z + B_Z)) = 0$ and $(Z, C_Z + B_Z)$ is not plt,
- $\kappa(Z, -(K_Z + C_Z + B_Z)) = 0$ and $(Z, C_Z + B_Z)$ is plt.

If one of the first three cases (resp. if the last case) holds, then we will show that (1) (resp. (2)) of Theorem 4.1.1 holds. If C_Z is normal, then we define (C_Z, B_{C_Z}) by adjunction

$$(K_Z + C_Z + B_Z)|_{C_Z} = K_{C_Z} + B_{C_Z}.$$

Case 1: $\kappa(Z, -(K_Z + C_Z + B_Z)) = 2$.

Since $-(K_Z + C_Z + B_Z)$ is semi-ample, we can contract the curves Γ with $(K_Z + C_Z + B_Z) \cdot \Gamma = 0$. In this contraction, C_Z is not contracted, because if $(K_Z + C_Z + B_Z) \cdot C_Z = 0$, then $C_Z^2 < 0$, which contradicts

$$(1 - b)C_Z^2 = (K_Z + C_Z + B_Z) \cdot C_Z - (K_Z + bC_Z + B_Z) \cdot C_Z > 0.$$

Thus, by [HX15, Proposition 2.11], we may assume that $-(K_Z + C_Z + B_Z)$ is ample. By [Tan14, Theorem 3.19] and

$$(K_Z + C_Z) \cdot C_Z \leq (K_Z + C_Z + B_Z) \cdot C_Z < 0,$$

it follows that $C_Z \simeq \mathbb{P}^1$.

By Lemma 4.2.11, there exists an effective $\mathbb{Q}$ -divisor H such that $(Z, C_Z + B_Z + H)$ is log canonical and the Cartier index of $K_Z + C_Z + B_Z + H$ is not divisible by p . After replacing H by $H + H'$ for some ample effective $\mathbb{Q}$ -divisor H' , we may assume that $Z \setminus \text{Supp } H$ is affine. After replacing H by a smaller multiple, we may assume that $-(K_Z + C_Z + B_Z + H)$ is ample.

We now show that $(C_Z, B_{C_Z} + H|_{C_Z})$ is globally F -split. If $(C_Z, B_{C_Z} + H|_{C_Z})$ is not klt, then Lemma 4.2.7 implies that $(C_Z, B_{C_Z} + H|_{C_Z})$ is globally F -split. If $(C_Z, B_{C_Z} + H|_{C_Z})$ is klt, then by [CGS16, Corollary 4.1], after possibly replacing $p(I)$ by a larger value depending only on I , we may assume that (C_Z, B_{C_Z}) is globally F -regular. After possibly replacing H by a smaller multiple again, we may assume that $(C_Z, B_{C_Z} + H|_{C_Z})$ is globally F -split.

By Lemma 4.2.7, it follows that $(Z, C_Z + B_Z + H)$ is globally F -split. Thus, (b) of the claim and Lemma 4.5.2 imply that $(Z, \lambda C_Z + B_Z)$ is globally F -regular for any $\lambda \in [0, 1]$. By (c) of the claim, it follows that (X, B) is globally F -regular.

Case 2: $\kappa(Z, -(K_Z + C_Z + B_Z)) = 1$.

Since $-(K_Z + C_Z + B_Z)$ is semi-ample, there exist a morphism $\pi: Z \rightarrow T$ onto a smooth projective curve T and an ample $\mathbb{Q}$ -divisor A on T such that $\pi_* \mathcal{O}_Z = \mathcal{O}_T$ and $-(K_Z + C_Z + B_Z) \equiv \pi^* A$. By (5) of Proposition 4.5.1 and by [HX15, Proposition 2.11], after running a $(K_Z + bC_Z + B_Z)$ -MMP over T , we may assume that $\pi: Z \rightarrow T$ is a $(K_Z + bC_Z + B_Z)$ -negative Mori fibre space. In particular, C_Z is π -horizontal. By [Tan14, Theorem 3.19] and

$$(K_Z + C_Z) \cdot C_Z \leq (K_Z + C_Z + B_Z) \cdot C_Z < 0,$$

it follows that $C_Z \simeq \mathbb{P}^1$.

By (a) of the claim, we can apply Lemma 4.2.12, and there exists an effective $\mathbb{Q}$ -divisor E , whose support is contained in some fibres of π , and such that $(Z, C_Z + B_Z + E)$ is log canonical and the Cartier index of $K_Z + C_Z + B_Z + E$ is not divisible by p . After possibly replacing E by a smaller multiple, we may assume that $-(K_Z + C_Z + B_Z + E) \equiv \pi^* A'$ for some ample $\mathbb{Q}$ -divisor A' on T . In particular, $-(K_{C_Z} + B_{C_Z} + E|_{C_Z})$ is ample.

By the same argument as in Case 1, $(C_Z, B_{C_Z} + E|_{C_Z})$ is globally F -split. Thus, by Lemma 4.2.8, $(Z, C_Z + B_Z + E)$ is globally F -split. Again by the same argument as in Case 1, it follows that (X, B) is globally F -regular.

Case 3: $\kappa(Z, -(K_Z + C_Z + B_Z)) = 0$ and $(Z, C_Z + B_Z)$ is not plt.

By Proposition 4.2.5, after possibly replacing $p(I)$ by a larger value depending only on I , we may assume that $(p^e - 1)(K_Z + C_Z + B_Z)$ is Cartier for some positive integer e . Thus, Lemma 4.2.9 implies that $(Z, C_Z + B_Z)$ is globally F -split. By (5) of Proposition 4.5.1, it follows that $-(K_Z + bC_Z + B_Z)$ is ample, and in particular also C_Z is ample. Thus, (b) of the claim and Lemma 4.5.2 imply that $(Z, \lambda C_Z + B_Z)$ is globally F -regular for any $\lambda \in [0, 1)$. By (c) of the claim, it follows that (X, B) is globally F -regular.

Case 4: $\kappa(Z, -(K_Z + C_Z + B_Z)) = 0$ and $(Z, C_Z + B_Z)$ is plt.

By Proposition 4.4.2, after possibly replacing $p(I)$ by a larger number depending only on I , there exists a log resolution $\pi: W \rightarrow Z$ of $(Z, C + B)$ such that

$$(W, D_W := \text{Supp}(\pi_*^{-1}(C + B)) \cup \text{Exc}(\pi))$$

lifts to characteristic zero over a smooth base. We may write

$$K_W + D'_W = \pi^*(K_Z + C_Z + B_Z).$$

for some $\mathbb{Q}$ -divisor D'_W on W . Note that $\text{Supp } D'_W \subseteq \text{Supp } D_W$. By (2) of Proposition 4.5.1, the birational morphism $g: Y \rightarrow Z$ only extracts prime divisors E such that $a(E, W, D'_W) = a(E, Z, C_Z + B_Z) < 1$. Thus, there exists a sequence of blow-ups

$$\varphi: V := W_m \xrightarrow{\varphi_{m-1}} \dots \xrightarrow{\varphi_1} W_1 \xrightarrow{\varphi_0} W_0 := W$$

which satisfies the following properties:

- We define a divisor D_i on W_i inductively by

$$\begin{aligned} D_0 &:= D_W \\ K_{W_{i+1}} + D_{i+1} &= \varphi_i^*(K_{W_i} + D_i). \end{aligned}$$

- Each φ_i is the blow-up of a point contained in $\text{Sing}(\text{Supp } D_i)$.
- The composite arrow $\psi: V \xrightarrow{\varphi} W \xrightarrow{\pi} Z$ factors through Y .

Note that each D_i is a reduced simple normal crossing divisor. Therefore, by Lemma 4.4.3, $(V, \psi_*^{-1}(C_Z + B_Z) \cup \text{Exc}(\psi))$ lifts to characteristic zero over a smooth base. Thus, (2) of the theorem holds. $\square$

4.6. Kawamata–Viehweg vanishing for log del Pezzo surfaces

The goal of this section is to prove Theorem 4.1.2. We begin with the following:

Lemma 4.6.1. *Assume that $\text{char } k > 2$. Let (X, Δ) be a two dimensional projective klt pair. Suppose that there exists a log resolution $\mu: V \rightarrow X$ of (X, Δ) such that*

$$(V, \text{Exc}(\mu) \cup \mu_*^{-1}(\text{Supp } \Delta))$$

lifts to $W_2(k)$. Let D be a $\mathbb{Z}$ -divisor on X such that $D - (K_X + \Delta)$ is ample.

Then,

$$H^i(X, \mathcal{O}_X(D)) = 0 \quad \text{for any } i > 0.$$

Proof. Let

$$M = K_V + {}^\top \mu^*(D - (K_X + \Delta))^\top.$$

We may find a μ -exceptional $\mathbb{Q}$ -divisor $E \geq 0$ such that

- $M = K_V + {}^\top \mu^*(D - (K_X + \Delta)) - E^\top$, and
- $\mu^*(D - (K_X + \Delta)) - E$ is ample.

Note that $\{\mu^*(D - (K_X + \Delta)) - E\}$ is simple normal crossing and lifts to $W_2(k)$. Since $\text{char } k > 2$, [Har98a, Corollary 3.8] implies that

$$H^i(V, \mathcal{O}_V(M)) = 0$$

for every $i > 0$. Consider the Leray spectral sequence:

$$E_2^{i,j} := H^i(X, R^j \mu_* \mathcal{O}_V(M)) \Rightarrow H^{i+j}(V, \mathcal{O}_V(M)) =: E^{i+j}.$$

We have

$$\begin{aligned} & \mu_* \mathcal{O}_V(M) \\ &= \mu_* \mathcal{O}_V({}^\top \mu^*(D) + K_V - \mu^*(K_X + \Delta)^\top) \\ &= \mu_* \mathcal{O}_V({}_\perp \mu^*(D) \lrcorner + (\text{effective exceptional } \mathbb{Z}\text{-divisor})) \\ &\simeq \mathcal{O}_X(D), \end{aligned}$$

where the second equality holds because (X, Δ) is klt and D is a $\mathbb{Z}$ -divisor. Since $E_2^{i,j} = 0$ for $j > 0$ by the relative Kawamata–Viehweg vanishing theorem for a proper birational morphism between surfaces, we obtain

$$H^i(X, \mathcal{O}_X(D)) = E_2^{i,0} \simeq E^i = H^i(V, \mathcal{O}_V(M)) = 0$$

for $i > 0$. Thus, the claim follows. $\square$

Proof of Theorem 4.1.2. After possibly perturbing B and Δ , we may assume that $-(K_X + B)$ and $D - (K_X + \Delta)$ are ample. Given $I := \emptyset$, let $p_0 := p(I)$ be the positive integer whose existence is guaranteed by Theorem 4.1.1. We divide the proof into five steps:

Step 1. We first prove the theorem assuming that $\Delta = B = 0$.

Theorem 4.1.1 implies that X is globally F -regular or X admits a log resolution $\mu: V \rightarrow X$ such that $(V, \text{Exc}(f))$ lifts to $W_2(k)$. In the first case, by Serre vanishing, if e is a sufficiently large positive integer and $i > 0$, we have

$$\begin{aligned} H^i(X, \mathcal{O}_X(D)) &\hookrightarrow H^i(X, F_*^e \mathcal{O}_X(p^e D - (p^e - 1)K_X)) \\ &\simeq H^i(X, \mathcal{O}_X(D + (p^e - 1)(D - K_X))) = 0. \end{aligned}$$

On the other hand, if X admits a log resolution $\mu: V \rightarrow X$ such that $(V, \text{Exc}(f))$ lifts to $W_2(k)$, then Lemma 4.6.1 implies the claim.

Step 2. We now prove the theorem under the assumption that D is nef.

Let $f: X \rightarrow Y$ be a proper birational contraction obtained by running a $(-K_X)$ -MMP. Since (X, B) is log del Pezzo, it follows that Y is klt and $-K_Y$ is big and semi-ample. Moreover, there exists an f -exceptional $\mathbb{Q}$ -divisor $\Delta_1 \geq 0$ such that

$$K_X + \Delta_1 = f^* K_Y.$$

Let $g: Y \rightarrow Z$ be the morphism induced by $-K_Y$ and let

$$h: X \xrightarrow{f} Y \xrightarrow{g} Z$$

be the composite morphism. Then $-K_Z$ is ample and

$$K_X + \Delta_1 = h^* K_Z.$$

By Theorem 4.1.1, Z is globally F -regular or it admits a log resolution $\mu: V \rightarrow Z$ such that $(V, \text{Exc}(\mu))$ lifts to characteristic zero over a smooth base.

If Z is globally F -regular, then Lemma 2.2.7 implies that (X, Δ_1) is also globally F -regular. Thus, we may find a $\mathbb{Q}$ -divisor $\Delta_2 \geq \Delta_1$ such that (X, Δ_2) is globally F -regular, $-(K_X + \Delta_2)$ is ample, and $(p^{e_1} - 1)(K_X + \Delta_2)$ is Cartier for some $e_1 \in \mathbb{Z}_{>0}$. Since D is nef, by Fujita vanishing, if e is a sufficiently divisible positive integer, for any $i > 0$ we have

$$H^i(X, \mathcal{O}_X(D)) \hookrightarrow H^i(X, F_*^e \mathcal{O}_X(p^e D - (p^e - 1)(K_X + \Delta_2))) = 0,$$

and the theorem follows.

Thus, we may assume the existence of the log resolution μ , as above. Since $\Delta_1 \geq 0$, the birational morphism $h: X \rightarrow Z$ only extracts prime divisors E with $a(E, Z, 0) \in (0, 1]$. By Remark 4.4.4, we may assume that $\mu: V \rightarrow Z$ factors through X , after possibly replacing V by some blow-up of V :

$$\mu: V \xrightarrow{\mu_X} X \xrightarrow{h} Z.$$

There exists an h -exceptional $\mathbb{Q}$ -divisor $\Delta_2 \geq \Delta_1$ such that (X, Δ_2) is klt and $-(K_X + \Delta_2)$ is ample. Since $\mu_X^{-1}(\Delta_2) \subseteq \text{Exc}(\mu)$, Lemma 4.6.1 implies the theorem.

Step 3. We now show that we may assume that there exists a D -negative Mori fibre space $g: X \rightarrow Z$ onto a smooth projective curve Z .

Let $f: X \rightarrow Y$ be the birational contraction of a curve E such that $D \cdot E \leq 0$. We want to show

$$H^i(X, \mathcal{O}_X(D)) = H^i(Y, \mathcal{O}_Y(f_*D)).$$

Let $D_Y = f_*D$. Then $D = f^*D_Y + cE$ for some $c \in \mathbb{Q}_{\geq 0}$, which implies

$$f_*\mathcal{O}_X(D) = \mathcal{O}_Y(D_Y).$$

Since $D - (K_X + \Delta)$ is ample, the relative Kawamata–Viehweg vanishing in the birational case implies

$$H^i(X, \mathcal{O}_X(D)) \simeq H^i(Y, f_*\mathcal{O}_X(D)) = H^i(Y, \mathcal{O}_Y(D_Y)).$$

Since (X, B) is log del Pezzo, we may run a D -MMP and may assume that one of the following holds:

- D is nef,
- $\rho(X) = 2$, and there exists a D -negative Mori fibre space $X \rightarrow Z$ onto a smooth projective curve Z , or
- $\rho(X) = 1$, and $-D$ is ample.

If D is nef, then we may apply Step 2. In the last case, we may assume that $\Delta = B = 0$, and we may apply Step 1. Thus, we may assume that there exists a D -negative Mori fibre space $g: X \rightarrow Z$ onto a smooth projective curve Z .

Step 4. We now show that we may assume that the following hold:

- (a) There exists a curve E on X such that $E^2 < 0$,
- (b) $B = bE$ for some $b \in [0, 1)$, and
- (c) $\Delta = 0$.

If every curve C on X is nef, then we may assume that $\Delta = B = 0$, and apply Step 1. Thus, we may assume that there exists a curve E on X such that $E^2 < 0$ and (a) holds.

By Step 3, we may assume that there exists a D -negative Mori fibre space $g: X \rightarrow Z$ onto a smooth projective curve Z . In particular, $\rho(X) = 2$, and any curve C on X , which is different from E , is nef. Thus, we may assume that (b) holds. Similarly, we may assume that $\Delta = \delta E$ for some $\delta \in [0, 1)$. Let F be a general fibre of g . Then $D \cdot F < 0$. Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D + F) \rightarrow \mathcal{O}_F(D|_F) \rightarrow 0.$$

Since $H^0(F, \mathcal{O}_F(D|_F)) = H^1(F, \mathcal{O}_F(D|_F)) = 0$, it is enough to show that $H^i(X, \mathcal{O}_X(D + F)) = 0$ for any $i > 0$. Repeating the same method finitely many times, it is enough to

show that $H^i(X, \mathcal{O}_X(D + nF)) = 0$ for some positive integer n and for any $i > 0$. Since $D - (K_X + \delta E)$ is ample, the divisor

$$D + nF - K_X = (D - (K_X + \delta E)) + (nF + \delta E)$$

is also ample for $n \gg 0$. Therefore, after possibly replacing D by $D + nF$, we may assume that $\Delta = 0$ and (c) holds.

Step 5. We now prove the theorem in the general case.

We may assume that (a), (b) and (c) of Step 4 hold. If $K_X \cdot E < 0$, then $-K_X$ is ample by Kleiman's criterion and we may assume that $B = 0$. Thus, the theorem follows by Step 1.

Therefore, we may assume that $K_X \cdot E \geq 0$. Let $f: X \rightarrow Y$ be the birational morphism which contracts E . We may write

$$K_X + b'E = f^*K_Y$$

for some $b' \in [0, b)$. Thus, by Theorem 4.1.1 and Remark 4.4.4, Y is globally F -regular or it admits a log resolution $V \rightarrow Y$ which factors through X such that $(V, \text{Exc}(\mu))$ lifts to $W_2(k)$:

$$\mu: V \xrightarrow{\mu_X} X \rightarrow Y.$$

In the latter case, since $\text{Exc}(\mu_X) \subseteq \text{Exc}(\mu)$, we may apply Lemma 4.6.1. Thus, we may assume that Y is globally F -regular. By Lemma 2.2.7, $(X, b'E)$ is globally F -regular, and so is X . Step 4(c) implies that $D - K_X$ is ample, Therefore, by Serre vanishing, if e is a sufficiently large positive integer, we have

$$H^i(X, D) \hookrightarrow H^i(X, p^e D - (p^e - 1)K_X) = H^i(X, K_X + p^e(D - K_X)) = 0.$$

Thus, the theorem follows. □

4.7. Example in characteristic two

The goal of this section is to show Theorem 4.1.3. We begin with the following two preliminary results:

Lemma 4.7.1. *Let $T \rightarrow \text{Spec } \mathbb{Z}$ be a smooth morphism from an integral scheme T and let $\mathcal{W} \rightarrow T$ be a flat projective morphism. Fix a morphism $\alpha: \text{Spec } \overline{\mathbb{F}}_p \rightarrow T$ and an embedding $K(T) \hookrightarrow \mathbb{C}$. Let $W_{\overline{\mathbb{F}}_p} := \mathcal{W} \times_T \overline{\mathbb{F}}_p$, and $W_{\mathbb{C}} := \mathcal{W} \times_T \mathbb{C}$. If $W_{\overline{\mathbb{F}}_p}$ is a smooth projective rational surface, then the following hold:*

1. $W_{\mathbb{C}}$ is a smooth projective rational surface, and
2. $\rho(W_{\mathbb{C}}) = \rho(W_{\overline{\mathbb{F}}_p})$.

Proof. Since being smooth and geometrically integral are open properties, it follows that $W_{\mathbb{C}}$ is a smooth projective surface.

By upper semi-continuity, it follows that

$$\begin{aligned} H^i(W_{\mathbb{C}}, \mathcal{O}_{W_{\mathbb{C}}}) &= 0 \text{ for any } i > 0, \text{ and} \\ H^2(W_{\mathbb{C}}, \mathcal{O}_{W_{\mathbb{C}}}(-K_{W_{\mathbb{C}}})) &= 0. \end{aligned}$$

Thus, by the rationality criterion, it follows that $W_{\mathbb{C}}$ is a rational surface and (1) holds.

Note that, for any sufficiently large positive integer e , the morphism α factors as follows

$$\alpha: \operatorname{Spec} \bar{\mathbb{F}}_p \rightarrow \operatorname{Spec} \mathbb{F}_{p^e} \rightarrow T.$$

Let $W_{\mathbb{F}_{p^e}} := \mathcal{W} \times_T \mathbb{F}_{p^e}$. After possibly replacing e by a larger number, we may assume that $W_{\mathbb{F}_{p^e}}$ is obtained as a sequence of blow-ups of $\mathbb{P}_{\mathbb{F}_{p^e}}^2$ or a $\mathbb{P}^1$ -bundle over $\mathbb{P}_{\mathbb{F}_{p^e}}^1$, whose centres are $\mathbb{F}_{p^e}$ -rational points. By counting the rational points, it is easy to check that the zeta function of $W_{\mathbb{F}_{p^e}}$ can be written as

$$Z_{W_{\mathbb{F}_{p^e}}}(t) = \frac{1}{(1-t)(1-p^et)^{\rho(W_{\bar{\mathbb{F}}_p})}(1-p^{2et})}.$$

On the other hand, we obtain

$$\rho(W_{\bar{\mathbb{F}}_{p^e}}) = \deg(1 - p^et)^{\rho(W_{\bar{\mathbb{F}}_p})} = \dim_{\mathbb{C}} H^2(W_{\mathbb{C}}, \mathbb{C}) = \rho(W_{\mathbb{C}}),$$

where the second equality follows from a consequence of the Weil conjecture [FK88, Chapter IV, Remark (b) after Theorem 1.2]. Thus, (2) holds. $\square$

Lemma 4.7.2. *Let k be an algebraically closed field of characteristic two. Then there exist six distinct points $q_0, \dots, q_5 \in \mathbb{P}_k^2$ which satisfy the following properties:*

1. *if $g: Z \rightarrow \mathbb{P}_k^2$ is the blow-up at $q_0, \dots, q_5$, then Z is a smooth del Pezzo surface,*
2. *Z is not globally F -split, and*
3. *if C is the smooth conic passing through $q_1, \dots, q_5$, then, for every point $q \in C$, the line L passing through q_0 and q is tangent to C .*

Proof. The smooth cubic surface Z of Fermat type is not globally F -split [Har98a, Example 5.5]. We can find six points $q_0, \dots, q_5 \in \mathbb{P}_k^2$ such that the blow-up along these points is isomorphic to Z . Let $g: Z \rightarrow \mathbb{P}_k^2$ be the induced morphism. Then (1) and (2) hold.

By contradiction, we assume that there exists a line L on $\mathbb{P}_k^2$, passing through q_0 , such that $C + L$ is simple normal crossing. Let C_Z and L_Z be the proper transforms on Z of C and L , respectively. By (1), there exists a point $q \in C \cap L$ with $q \neq q_i$ for any $i = 1, \dots, 5$, and, in particular, $C_Z \cap L_Z \neq \emptyset$. There exists a reduced g -exceptional divisor E_Z on Z such that

$$K_Z + C_Z + L_Z + E_Z = g^*(K_{\mathbb{P}_k^2} + C + L) \sim 0,$$

and $(Z, C_Z + L_Z + E_Z)$ is log canonical but not plt. By Lemma 4.2.9, $(Z, C_Z + L_Z + E_Z)$ is globally F -split. Thus, Z is globally F -split, contradicting (2). Thus, (3) holds. $\square$

We now prove the main result of this section.

Proof of Theorem 4.1.3. Let $q_0, \dots, q_5 \in \mathbb{P}_{\mathbb{F}_2}^2$ be as in Lemma 4.7.2. For any $i = 1, \dots, 5$, let L_i be the line in $\mathbb{P}_{\mathbb{F}_2}^2$ passing through q_0 and q_i . Note that each L_i is tangent to C by (3) of Lemma 4.7.2. Let

$$\pi: Y \rightarrow \mathbb{P}_{\mathbb{F}_2}^2$$

be the birational morphism constructed as follows. First, we consider the blow-up $Z \rightarrow \mathbb{P}_{\mathbb{F}_2}^2$ at the points $q_0, \dots, q_5$. Then, we consider the blow-up $Y \rightarrow Z$ at the points $q'_1, \dots, q'_5$, where q'_i is the intersection point of the proper transforms on Z of C and L_i . In particular, $\rho(Y) = 12$. By the same argument as in [KM99, Section 9, An interesting example in non-zero characteristic], we can find curves $E_2, E_3, \dots, E_{12}$ on Y with

$$E_2^2 = -6, \quad E_m^2 = -2, \quad E_i \cdot E_j = 0$$

for any $m = 3, \dots, 12$ and distinct $i, j = 2, \dots, 12$. Let $f: Y \rightarrow X$ be the birational morphism which contracts all these curves. Then X is a projective klt surface with $\rho(X) = 1$. By [ibid], $-K_X$ is ample. Thus, (1) holds.

By (2) of Lemma 4.7.2, Z is not globally F -split. By [CTW18, Lemma 2.2], it follows that Y is not globally F -split and by Lemma 2.2.7, it follows that X is not globally F -split. Thus, (2) holds.

We now show (3). By contradiction, we assume that there exists a log resolution $h: W \rightarrow X$ such that $(W, \text{Exc}(h) = \sum_{i=2}^n C_i)$ lifts to characteristic zero over a smooth base, i.e. there exists a smooth morphism $T \rightarrow \text{Spec } \mathbb{Z}$, a closed immersion of schemes $\mathcal{C}_i \subseteq \mathcal{W}$ projective and flat over T for any $i = 2, \dots, n$, and a morphism $\alpha: \text{Spec } \overline{\mathbb{F}_2} \rightarrow T$ such that the base changes of $\mathcal{C}_i \subseteq \mathcal{W}$ by α is $C_i \subseteq W$ and such that $\mathcal{W}$ and all the strata of $\{\mathcal{C}_i\}$ are smooth over T . We may assume that T is an integral scheme. Note that $K(T)$ is of characteristic zero. Fix an embedding $K(T) \hookrightarrow \mathbb{C}$, and let $W_{\mathbb{C}} := \mathcal{W} \times_T \mathbb{C}$. By Lemma 4.7.1, we have

$$n = \rho(W) = \rho(W_{\mathbb{C}}).$$

The birational morphism $f: Y \rightarrow X$ constructed above is the minimal resolution of X . Thus, h factors as

$$h: W \xrightarrow{g} Y \xrightarrow{f} X$$

and may assume that $g_* C_i = E_i$ for any $i = 2, \dots, 12$. We can find a sequence of blow-ups

$$g: W =: W_n \xrightarrow{g_n} W_{n-1} \xrightarrow{g_{n-1}} \dots \xrightarrow{g_{13}} W_{12} := Y$$

with $\rho(W_r) = r$ for any $r = 12, \dots, n$. We may assume that the proper transform of $\text{Exc}(g_r)$ on W is C_r .

By invariance of the intersection numbers, the two intersection matrices $(C_i \cdot C_j)$ and $(C_{i,\mathbb{C}} \cdot C_{j,\mathbb{C}})$ coincide. Thus, we can construct the corresponding sequence over $\mathbb{C}$, i.e. a sequence of blow-ups:

$$g_{\mathbb{C}}: W_{\mathbb{C}} =: W_{n,\mathbb{C}} \xrightarrow{g_{n,\mathbb{C}}} W_{n-1,\mathbb{C}} \xrightarrow{g_{n-1,\mathbb{C}}} \dots \xrightarrow{g_{13,\mathbb{C}}} W_{12,\mathbb{C}} := Y_{\mathbb{C}}$$

such that $\rho(W_{r,\mathbb{C}}) = r$ for every $r = 12, \dots, n$ and such that the proper transform of $\text{Exc}(g_{r,\mathbb{C}})$ on $W_{\mathbb{C}}$ is $C_{r,\mathbb{C}}$. Indeed, it is easy to check that the push-forward of $C_{i,\mathbb{C}}$ on $W_{i,\mathbb{C}}$ is a (-1) -curve and that the curves $E_{2,\mathbb{C}} := (g_{\mathbb{C}})_* C_{2,\mathbb{C}}, \dots, E_{12,\mathbb{C}} := (g_{\mathbb{C}})_* C_{12,\mathbb{C}}$ satisfy

$$E_{2,\mathbb{C}}^2 = -6, \quad E_{m,\mathbb{C}}^2 = -2, \quad E_{i,\mathbb{C}} \cdot E_{j,\mathbb{C}} = 0$$

for any $m = 3, \dots, 12$ and distinct $i, j = 2, \dots, 12$. Note also that $E_{\ell,\mathbb{C}}$ is isomorphic to $\mathbb{P}_{\mathbb{C}}^1$ for any $\ell = 2, \dots, 12$. Let

$$f_{\mathbb{C}}: Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$$

be the birational morphism which contracts $E_{2,\mathbb{C}}, \dots, E_{12,\mathbb{C}}$. Then $X_{\mathbb{C}}$ is a projective klt surface with exactly 11 singular points. Moreover, since $\rho(W_{\mathbb{C}}) = n$, it follows that $\rho(X_{\mathbb{C}}) = 1$.

Further, for any invertible sheaf $\mathcal{A}$ on $\mathcal{W}$ which is ample over T , we have

$$0 > g^* f^* K_X \cdot \mathcal{A}|_W = g_{\mathbb{C}}^* f_{\mathbb{C}}^* K_{X_{\mathbb{C}}} \cdot \mathcal{A}|_{W_{\mathbb{C}}}.$$

Thus, $-K_{X_{\mathbb{C}}}$ is ample. On the other hand, [KM99, Theorem 9.2] implies that $X_{\mathbb{C}}$ admits at most 6 singularities, a contradiction. Thus, (3) holds. $\square$

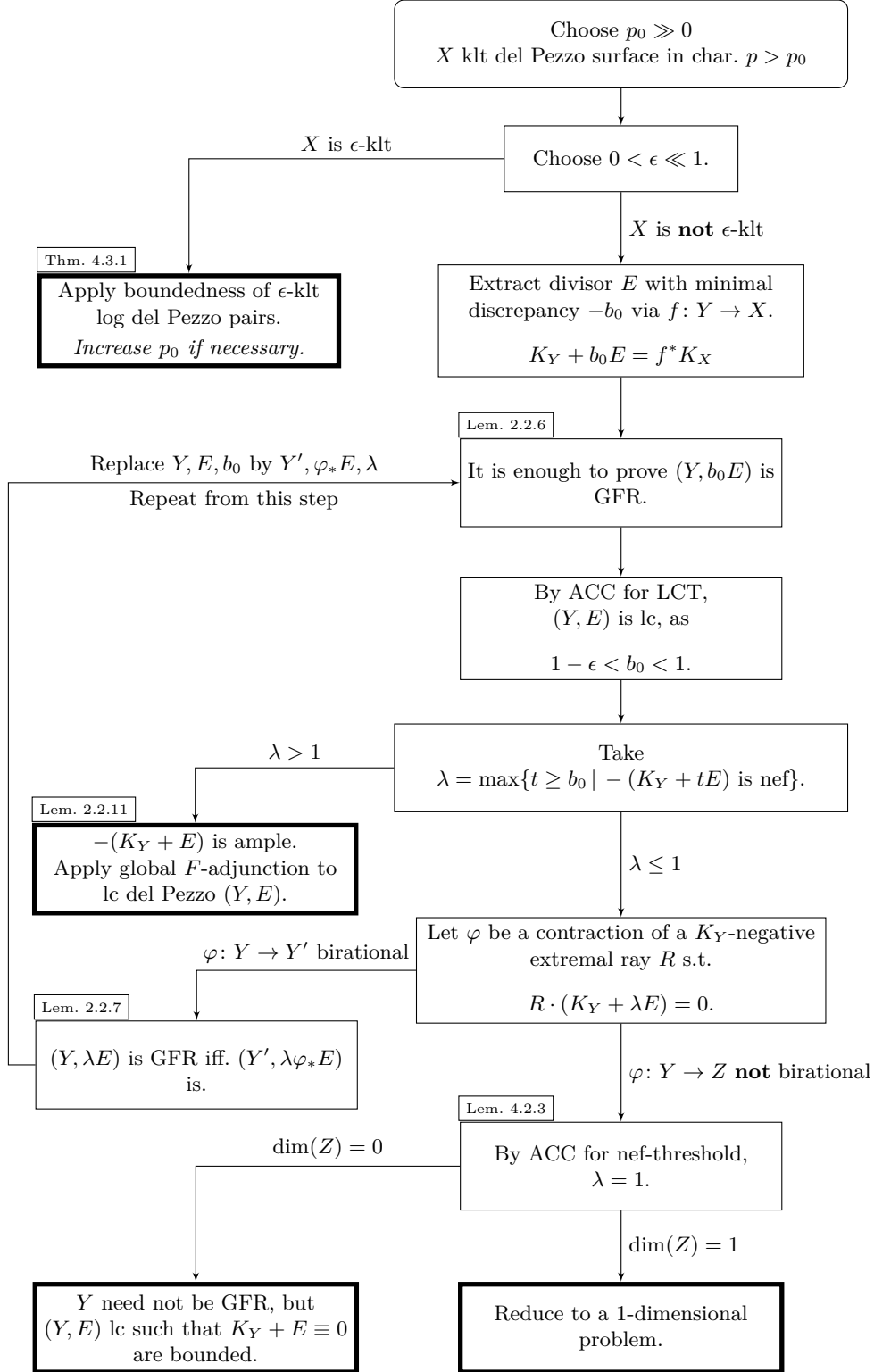


Figure 4.1: Diagram explaining the proof of Theorem 4.1.1

5

Effective bounds on singular surfaces in positive characteristic

5.1. Introduction

IN THIS CHAPTER, we study Fujita-conjecture-type bounds for singular surfaces. The main theorem is the following.

Theorem 5.1.1. *Let X be a projective surface with Kawamata log terminal singularities defined over an algebraically closed field of characteristic $p > 5$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Then, for an ample Cartier divisor A :*

- $4mK_X + 14mA$ is base point free, and
- $13mK_X + 45mA$ is very ample.

This theorem is based on the results of Di Cerbo and Fanelli ([DCF15]), who proved among other things that $2K_X + 4A$ is very ample, where A is ample, and X is a smooth surface of general type in characteristic $p \geq 3$.

The bounds are not sharp. See Theorem 5.4.1 for a slightly more general statement. Instead of assuming that X is Kawamata log terminal and $p > 5$, it is enough to assume that X is F -pure and $\mathbb{Q}$ -factorial. The theorem also holds in characteristic zero, but in such a case the existence of effective bounds follows in an easier way by Kollár's effective base point free theorem ([Kol93]) and the Kodaira vanishing theorem.

The proof consists of three main ingredients. First, we apply the results of Di Cerbo and Fanelli on a desingularisation of X . This shows that the base locus of $2mK_X + 7mA$ is zero dimensional. Then, we apply the technique of Cascini, Tanaka and Xu (see [CTX15, Theorem 3.8]) to show that the base locus of $2(2mK_X + 7mA)$ is empty.

Therewith, the very ampleness of $13mK_X + 45mA$ follows from a generalization of a result of Keeler ([Kee08, Theorem 1.1]) in the case of F -pure varieties.

As a corollary to the main theorem, we obtain the following Matsusaka-type bounds.

Corollary 5.1.2. *Let A and N be, respectively, an ample and a nef Cartier divisor on a Kawamata log terminal projective surface defined over an algebraically closed field of characteristic $p > 5$. Let $m \in \mathbb{N}$ be such that mK_X is Cartier. Then $kA - N$ is very ample for any*

$$k > \frac{2A \cdot (H + N)}{A^2} ((K_X + 3A) \cdot A + 1),$$

where $H := 13mK_X + 45mA$.

One of the fundamental results in birational geometry is Borisov-Alexeev Boundedness conjecture, which says that ϵ -klt log Fano varieties are bounded. In dimension two and any characteristic it was proved by Alexeev ([Ale94, Theorem 6.9]). One of the ingredients of the proof is the beforementioned result about boundedness of polarised surfaces by Kollár ([Kol85, Theorem 2.1.2]). Further, explicit bounds on the volume have been obtained by Lai ([Lai16, Theorem 4.3]) and Jiang ([Jia13, Theorem 1.3]).

For characteristic $p > 5$, in the proof of the boundedness of ϵ -klt log del Pezzo pairs, we can replace one of Kollár's results by Theorem 5.1.1, and hence obtain rough, but explicit bounds on the size of the bounded family.

Corollary 5.1.3. *For any $\epsilon \in \mathbb{R}_{>0}$ and a finite set $I \subseteq [0, 1] \cap \mathbb{Q}$, there exist effectively computable natural numbers $b(\epsilon, I)$ and $n(\epsilon, I)$ satisfying the following property.*

Let (X, Δ) be an ϵ -klt log del Pezzo surface with the coefficients of Δ contained in I , defined over an algebraically closed field of characteristic $p > 5$. Then, there exists a very ample divisor H on X such that $H^i(X, H) = 0$ for $i > 0$ and

$$|H^2|, |H \cdot K_X|, |H \cdot \Delta|, |K_X \cdot \Delta|, |\Delta^2|$$

are smaller than $b(\epsilon, I)$. Further, H embeds X into $\mathbb{P}^k$, where $k \leq b(\epsilon, I)$, and $n(\epsilon, I)\Delta$ is Cartier.

The chapter is organized as follows. The second section consists of basic preliminaries. In the third section we give a proof of the base point free part of the main theorem. In the fourth section, we prove a technical generalization of the main theorem to arbitrary characteristic. In the fifth section we show Matsusaka-type bounds (Corollary 5.1.2). In the sixth section we derive effective bounds on log del Pezzo pairs.

5.2. Preliminaries

We always work over an algebraically closed field k of positive characteristic $p > 0$.

We will repeatedly use, without mentioning directly, that $K_X + 3A$ is nef, where X is a projective normal surface and A is an ample Cartier divisor. This is a direct consequence

of the cone theorem ([Tan14, Proposition 3.15]).

The following facts are used in the proofs in this paper.

Theorem 5.2.1 (Mumford regularity, [Laz04a, Theorem 1.8.5]). *Let X be a projective variety, and let M be a globally generated ample line bundle on X . Let $\mathcal{F}$ be a coherent sheaf on X such that $H^i(X, \mathcal{F} \otimes M^{-i}) = 0$ for $i > 0$. Then $\mathcal{F}$ is globally generated.*

Theorem 5.2.2 (Fujita vanishing, [Laz04a, Theorem 1.4.35] and [Laz04a, Remark 1.4.36]). *Let X be a projective variety, and let H be an ample divisor on X . Given any coherent sheaf $\mathcal{F}$ on X , there exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0,$$

for all $i > 0$, $m \geq m(\mathcal{F}, H)$ and any nef Cartier divisor D on X .

Theorem 5.2.3 (Log-concavity of volume). *For any two big Cartier divisors D_1 and D_2 on a normal variety X of dimension n we have*

$$\text{vol}(D_1 + D_2)^{\frac{1}{n}} \geq \text{vol}(D_1)^{\frac{1}{n}} + \text{vol}(D_2)^{\frac{1}{n}}.$$

Recall that

$$\text{vol}(D) := \limsup_{m \rightarrow \infty} \frac{H^0(X, mD)}{m^n/n!}.$$

Proof. See [DCF15, Theorem 2.2] (cf. [Laz04b, Theorem 11.4.9] and [Tak07]). $\square$

We will also need the following lemma.

Lemma 5.2.4. *Let X be a $\mathbb{Q}$ -factorial variety, which is F -pure at a point $x \in X$. Then, there exists an effective $\mathbb{Q}$ -divisor B such that*

$$(p^e - 1)(K_X + B)$$

is Cartier for enough divisible $e \gg 0$, and (X, B) is F -pure at x . One can take the coefficients of B to be as small as possible.

More precisely, if we can find B as above, then there exists a sequence $\lim_{j \rightarrow \infty} a_j = 0$ such that $a_j B$ satisfies the conditions of this lemma.

Note that if the Gorenstein index of X is not divisible by p , then one can just take $B = 0$.

Proof. By Proposition 2.2.2, we know that there exists an effective $\mathbb{Q}$ -divisor Δ such that (X, Δ) is F -pure at $x \in X$, and the Cartier index of $K_X + \Delta$ at x is not divisible by p .

In particular, for enough divisible $e \gg 0$, there exists a $\mathbb{Q}$ -divisor $D \sim (p^e - 1)(K_X + \Delta)$ such that $x \notin \text{Supp } D$. Further, we can find a Cartier divisor E disjoint with x for which $E + D$ is effective.

Notice that $K_X + \Delta + D + E \sim p^e(K_X + \Delta) + E$ has Cartier index indivisible by p for $e \gg 0$. Take

$$B = \frac{1}{p^n + 1}(\Delta + D + E)$$

for $n \gg 0$. Then

$$K_X + B \sim K_X + \Delta + D + E - \frac{p^n}{p^n + 1}(\Delta + D + E),$$

where both $K_X + \Delta + D + E$ and $p^n(\Delta + D + E)$ have Cartier indices not divisible by p . $\square$

5.2.1. Reider's analysis

Reider's analysis is a method of showing that divisors of the form $K_X + L$ are globally generated or very ample, where L is a big and nef divisor on a smooth surface X .

The idea is that a base point of $K_X + L$ provides us with a rank two vector bundle $\mathcal{E}$ which does not satisfy the Bogomolov inequality $c_1(\mathcal{E})^2 \leq 4c_2(\mathcal{E})$. In characteristic zero such vector bundles are unstable. Using the instability, one can deduce a contradiction to the existence of a base point, when L is “numerically-ample enough”.

In positive characteristic, the aforementioned fact about unstable vector bundles on smooth surfaces is not true in general. However, Shepherd-Barron proved it for surfaces which are neither of general type nor quasi-elliptic of Kodaira dimension one ([SB91]). This leads to the following.

Proposition 5.2.5 ([Ter99, Theorem 2.4]). *Let X be a smooth projective surface neither of general type nor quasi-elliptic with $\kappa(X) = 1$, and let D be a nef divisor such that $D^2 > 4$. Assume that $q \in X$ is a base-point of $K_X + D$. Then, there exists an integral curve C containing q , such that $D \cdot C \leq 1$.*

In particular, for such surfaces, $K_X + 3A$ is base point free for an ample divisor A .

The goal of this subsection is to prove the following proposition, which covers all types of surfaces.

Proposition 5.2.6. *Let X be a smooth projective surface, and let D be a nef and big divisor on it. Assume that $q \in S$ is a base point of $K_X + D$. Further, suppose that*

1. $D^2 > 4$, if X is quasi-elliptic with $\kappa(X) = 1$,
2. $D^2 > \text{vol}(K_X) + 4$, if X is of general type and $p \geq 3$,
3. $D^2 > \text{vol}(K_X) + 6$, if X is of general type and $p = 2$, or
4. $D^2 > 4$, otherwise.

Then, there exists a curve C containing q such that

(1a) $D \cdot C \leq 5$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$,

(1b) $D \cdot C \leq 7$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$,

- (2) $D \cdot C \leq 1$, if X is of general type and $p \geq 3$,
- (3) $D \cdot C \leq 7$, if X is of general type and $p = 2$, or
- (4) $D \cdot C \leq 1$, otherwise.

Note that the case (4) is nothing else but Proposition 5.2.5. The proof follows step-by-step the proof by Di Cerbo and Fanelli ([DCF15]). The only addition is that the curve C must contain q . The idea that this must hold has been established in ([Sak90]) based on ([Ser87]), but, for the convenience of the reader, we present the full proof below.

The following is crucial in the proof of Proposition 5.2.6.

Proposition 5.2.7 ([DCF15]). *Consider a birational morphism $\pi: Y \rightarrow X$ between smooth projective surfaces X and Y which are either of general type, or quasi-elliptic with $\kappa(X) = 1$. Let $\bar{D}$ be a big divisor on Y such that $H^1(Y, \mathcal{O}_Y(-\bar{D})) \neq 0$ and $\bar{D}^2 > 0$. Further, suppose that $D := \pi_*\bar{D}$ is nef, and*

- 1. $\bar{D}^2 > \text{vol}(K_X)$, if X is of general type and $p \geq 3$, or
- 2. $\bar{D}^2 > \text{vol}(K_X) + 2$, if X is of general type and $p = 2$.

Then, there exists a non-zero non-exceptional effective divisor $\bar{E}$ on Y , such that

$$\begin{aligned} k\bar{D} - 2\bar{E} &\text{ is big,} \\ (k\bar{D} - \bar{E}) \cdot \bar{E} &\leq 0, \text{ and} \\ 0 \leq D \cdot E &\leq \frac{k\alpha}{2} - 1, \end{aligned}$$

where $E := \pi_*\bar{E}$, $\alpha := D^2 - \bar{D}^2$, and

- $k = 3$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$,
- $k = 4$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$,
- $k = 1$, if X is of general type and $p \geq 3$, or
- $k = 1$ or $k = 4$, if X is of general type and $p = 2$.

Proof. It follows directly from [DCF15, Proposition 4.3], [DCF15, Theorem 4.4], [DCF15, Proposition 4.6] and [DCF15, Corollary 4.8]. $\square$

Further, we need the following lemma.

Lemma 5.2.8 ([Sak90, Lemma 2]). *Let D be a nef and big divisor on a smooth surface S . If*

$$D \equiv D_1 + D_2$$

for numerically non-trivial pseudo-effective divisors D_1 and D_2 , then $D_1 \cdot D_2 > 0$.

Now, we can proceed with the proof of the main proposition in this subsection.

Proof of Proposition 5.2.6. The first case is covered by Proposition 5.2.5, so we may assume that X is of general type or quasi-elliptic with $\kappa(X) = 1$.

Let $\pi: Y \rightarrow X$ be a blow-up at $q \in X$ with the exceptional curve F . Given that q is a base point of $K_X + D$, from the exact sequence

$$0 \rightarrow \mathcal{O}_Y(\underbrace{\pi^*(K_X + D) - F}_{K_Y + \pi^*D - 2F}) \rightarrow \mathcal{O}_Y(\pi^*(K_X + D)) \rightarrow \mathcal{O}_F(\pi^*(K_X + D)) \rightarrow 0$$

we obtain that

$$H^1(Y, \mathcal{O}_Y(-\pi^*D + 2F)) = H^1(Y, \mathcal{O}_Y(K_Y + \pi^*D - 2F)) \neq 0.$$

Set $\overline{D} := \pi^*D - 2F$. Since

$$\overline{D}^2 = D^2 - 4,$$

we have $\overline{D}^2 > 0$, and the assertions (1) and (2) in Proposition 5.2.7 are satisfied. Further, using that $H^0(Y, \overline{D}) = H^0(X, \mathcal{O}_X(D) \otimes m_q^2)$ and $\text{vol}(D) > 4$, one can easily check that $\overline{D}$ is big. Hence, by Proposition 5.2.7, there exists a non-zero non-exceptional effective divisor $\overline{E}$ on Y , such that

$$\begin{aligned} k\overline{D} - 2\overline{E} &\text{ is big, and} \\ (k\overline{D} - \overline{E}) \cdot \overline{E} &\leq 0, \text{ and} \\ 0 &\leq D \cdot E \leq 2k - 1, \end{aligned}$$

where $E = \pi_*\overline{E}$.

To finish the proof, it is enough to show that $\overline{E}$ contains a component, which intersects F properly. Its pushforward onto X would be the sought-for curve C .

Assume that the claim is not true, that is $\overline{E} = \mu^*E + aF$ for $a \geq 0$. We have that

$$0 \geq (k\overline{D} - \overline{E}) \cdot \overline{E} = (kD - E) \cdot E + (2k + a)a.$$

This implies $kD \cdot E \leq E^2$. Since $D \cdot E \geq 0$, it holds that $E^2 \geq 0$. Given $kD - 2E$ is big, we may apply Lemma 5.2.8 with $kD = (kD - 2E) + 2E$, and obtain $kD \cdot E > 2E^2$. This is a contradiction with the other inequalities in this paragraph. $\square$

5.3. Base point freeness

The goal of this section is to prove the base point free part of Theorem 5.1.1.

Lemma 5.3.1. *Let L be an ample Cartier divisor on a normal projective surface X . Let $\pi: \tilde{X} \rightarrow X$ be the minimal resolution of singularities. Then $K_{\tilde{X}} + 3\pi^*L$ is nef, and $K_{\tilde{X}} + n\pi^*L$ is nef and big for $n \geq 4$.*

Proof. Take a curve C . We need to show that $(K_{\tilde{X}} + 3\pi^*L) \cdot C \geq 0$. If $K_{\tilde{X}} \cdot C \geq 0$, then the inequality clearly holds. Thus, by cone theorem ([Tan14, Theorem 3.13] and [Tan14, Remark 3.14]), we need to prove it, when C is an extremal ray satisfying $K_{\tilde{X}} \cdot C < 0$. In such a case, we have that $K_{\tilde{X}} \cdot C \geq -3$.

If C is not an exceptional curve, then $3\pi^*L \cdot C \geq 3$, and so the inequality holds. But C cannot be exceptional, because then its contraction would give a smooth surface (see [KM98, Theorem 1.28]), and so $\tilde{X}$ would not be a minimal resolution. This concludes the first part of the lemma.

As for the second part, $K_{\tilde{X}} + n\pi^*L$ is big and nef for $n \geq 4$, since adding a nef divisor to a big and nef divisor gives a big and nef divisor. $\square$

The following proposition yields the first step in the proof of Theorem 5.1.1.

Proposition 5.3.2. *Let X be a normal projective surface defined over an algebraically closed field of characteristic $p > 3$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Let A be an ample Cartier divisor on X . Then*

$$\text{Bs}(m(aK_X + bA) + N) \subseteq \text{Sing}(X),$$

for any nef Cartier divisor N , where $a = 2$ and $b = 7$.

The proposition is even true for $a = 2$ and $b = 6$, but in this case, $aK_X + bA$ need not be ample.

Proof. Let $\pi: \bar{X} \rightarrow X$ be the minimal resolution of singularities with the exceptional locus E . First, we claim that

$$\text{Bs}(2K_{\bar{X}} + 7\pi^*A + M) \subseteq E.$$

for any nef Cartier divisor M on $\bar{X}$. Assume this is true. If $m = 1$, then the proposition follows automatically. In general, we note that $2K_{\bar{X}} + 7\pi^*A$ is nef by Lemma 5.3.1, and so setting $M = (m - 1)(2K_{\bar{X}} + 7\pi^*A) + \pi^*N$ yields

$$\text{Bs}(m(2K_{\bar{X}} + 7\pi^*A) + \pi^*N) \subseteq E.$$

In particular, $\text{Bs}(m(2K_X + 7A) + N) \subseteq \pi(E)$, which concludes the proof.

Hence, we are left to show the claim. Assume by contradiction that there exists a base point $q \in \bar{X}$ of $2K_{\bar{X}} + 7\pi^*A + M$ such that $q \notin E$.

We apply Proposition 5.2.6 for $D = K_{\bar{X}} + 7\pi^*A + M$. The assumptions are satisfied, because, by Lemma 5.3.1, D is big and nef, and, by Theorem 5.2.3,

$$\begin{aligned} \text{vol}(D) &\geq \text{vol}(K_{\bar{X}}) + 49, \text{ if } X \text{ is of general type, and} \\ \text{vol}(D) &\geq \text{vol}(K_{\bar{X}} + 4\pi^*A) + \text{vol}(3\pi^*A) > 9 \text{ in general.} \end{aligned}$$

Here, we used that $K_{\bar{X}} + 4\pi^*A$ is nef and big by Lemma 5.3.1.

Therefore, there exists a curve C containing q such that

$$C \cdot D \leq 1.$$

We can write $D = (K_{\overline{X}} + 3\pi^*A + M) + 4\pi^*A$. As C is not exceptional, $C \cdot \pi^*A > 0$. Thus, we obtain a contradiction. $\square$

Applying above Proposition 5.3.2 and Proposition 2.2.5, the base point free part of Theorem 5.1.1 follows from the following proposition by taking $L := m(aK_X + bA) - K_X$.

Proposition 5.3.3. *Let X be an F -pure $\mathbb{Q}$ -factorial projective surface defined over an algebraically closed field of characteristic $p > 0$. Let L be an ample $\mathbb{Q}$ -divisor on X such that $K_X + L$ is an ample Cartier divisor and*

$$\text{Bs}(K_X + L + M) \subseteq \text{Sing}(X),$$

for every nef Cartier divisor M . Then $2(K_X + L) + N$ is base point free for every nef Cartier divisor N .

If we assume that $\dim \text{Bs}(K_X + L + M) = 0$, then the same proof will give us that $3(K_X + L) + N$ is base point free.

Before proceeding with the proof, we would like to give an example explaining an idea of how to show the proposition if we worked in characteristic zero.

Remark 5.3.4. Here, X is a smooth surface defined over an algebraically closed field k of characteristic zero, and L is an ample Cartier divisor on it. The goal of this remark is to prove the following statement by applying a well-known strategy:

if $K_X + L$ is ample and $\dim \text{Bs}(K_X + L) = 0$, then $3(K_X + L)$ is base point free.

Take any point $q \in \text{Bs}(K_X + L)$. It is enough to show that $3(K_X + L)$ is base point free at q . By assumptions, $K_X + L$ defines a finite map outside of its zero dimensional base locus, and so there exist divisors $D_1, D_2, D_3 \in |K_X + L|$ without common components such that the multiplier ideal sheaf $\mathcal{J}(X, \Delta)$ for $\Delta = \frac{2}{3}(D_1 + D_2 + D_3)$ satisfies

$$\begin{aligned} \dim \mathcal{J}(X, \Delta) &= 0, \text{ and} \\ q &\in \mathcal{J}(X, \Delta). \end{aligned}$$

Note that $\Delta \sim_{\mathbb{Q}} 2(K_X + L)$.

Let W be a zero-dimensional subscheme defined by $\mathcal{J}(X, \Delta)$. We have the following exact sequence

$$0 \rightarrow \mathcal{O}_X(3(K_X + L)) \otimes \mathcal{J}(X, \Delta) \rightarrow \mathcal{O}_X(3(K_X + L)) \rightarrow \mathcal{O}_W(3(K_X + L)) \rightarrow 0.$$

Since $3(K_X + L) \sim_{\mathbb{Q}} K_X + \Delta + L$, by Nadel vanishing theorem ([Laz04b, Theorem

9.4.17])

$$H^1(X, \mathcal{O}_X(3(K_X + L)) \otimes \mathcal{J}(X, \Delta)) = 0,$$

and so

$$H^0(X, \mathcal{O}_X(3(K_X + L))) \longrightarrow H^0(W, \mathcal{O}_W(3(K_X + L)))$$

is surjective. Since $\dim W = 0$, we get that $3(K_X + L)$ is base point free along W , and so it is base point free at q .

Proof of Proposition 5.3.3. Take an arbitrary closed point $q \in X$. We need to show that $q \notin \text{Bs}(2(K_X + L) + N)$. By taking $M = K_X + L + N$ in the assumption, we get that $\text{Bs}(2(K_X + L) + N) \subseteq \text{Sing}(X)$. Hence, we can assume $q \in \text{Sing}(X)$.

By assumptions, $K_X + L$ defines a finite map outside of its zero dimensional base locus, so there exist divisors $D_1, D_2 \in |K_X + L|$ such that $\dim(D_1 \cap D_2) = 0$ and $q \in D_1 \cap D_2$. Let W be the scheme defined by the interesection of D_1 and D_2 . By definition, $I_W = I_{D_1} + I_{D_2}$.

By Theorem 5.2.2, we can choose $e > 0$ such that

$$H^1\left(X, \mathcal{O}_X\left(\frac{p^e - 1}{2}L + M\right) \otimes I_W\right) = 0$$

for any nef Cartier divisor M .

By Lemma 5.2.4, we know that there exists an effective $\mathbb{Q}$ -divisor B such that

$$(p^e - 1)(K_X + B)$$

is Cartier, and

$$\text{Tr}_{X,B}: F_*^e \mathcal{O}_X(-(p^e - 1)(K_X + B)) \longrightarrow \mathcal{O}_X$$

is surjective at q , for enough divisible $e \gg 0$. If the Gorenstein index of X is not divisible by p , then we can take $B = 0$. Further, we may assume that $\frac{1}{2}L - B$ is ample.

Now, take maximal $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}$ such that

$$\text{Tr}_{X,\Delta}: F_*^e \mathcal{L} \rightarrow \mathcal{O}_X$$

is surjective at the stalk $\mathcal{O}_{X,q}$, where

$$\begin{aligned} \mathcal{L} &:= \mathcal{O}_X(-(p^e - 1)(K_X + B) - \lambda_1 D_1 - \lambda_2 D_2), \text{ and} \\ \Delta &:= B + \frac{\lambda_1}{p^e - 1} D_1 + \frac{\lambda_2}{p^e - 1} D_2. \end{aligned}$$

The pair (X, Δ) is F -pure by Proposition 2.2.9. We want to show the existence of the following diagram:

$$\begin{array}{ccc} F_*^e \mathcal{L} & \longrightarrow & F_*^e (\mathcal{L}|_W) \\ \downarrow \text{Tr}_{X,\Delta} & & \downarrow \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X,q}/m_q \end{array}$$

To show that such a diagram exists we need to prove that the image of $F_*^e(\mathcal{L} \otimes I_W)$ under $\text{Tr}_{X,\Delta}$ is contained in m_q . This follows from the fact that $I_W = \mathcal{O}(-D_1) + \mathcal{O}(-D_2)$ and from the maximality of λ_1, λ_2 . More precisely the image of

$$F_*^e \mathcal{O}_X(-(p^e - 1)(K_X + B) - (\lambda_1 + 1)D_1 - \lambda_2 D_2)$$

must be contained in m_q , and analogously for λ_2 replaced by $\lambda_2 + 1$.

So, we tensor this diagram by the line bundle $\mathcal{O}_X(K_X + \Delta + H)$, where

$$H := 2(K_X + L) - (K_X + \Delta) + N,$$

and take H^0 to obtain the diagram

$$\begin{array}{ccc} H^0(X, F_*^e \mathcal{O}_X(K_X + \Delta + p^e H)) & \longrightarrow & H^0(W, \mathcal{O}_W) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(K_X + \Delta + H)) & \longrightarrow & H^0(q, \mathcal{O}_{X,q}/m_q), \end{array}$$

Note that $K_X + \Delta + H = 2(K_X + L) + N$. Further, by Proposition 2.2.4, (X, Δ) is log canonical at $q \in X$, and so by Lemma 5.3.5 we get

$$\frac{\lambda_1}{p^e - 1} + \frac{\lambda_2}{p^e - 1} \leq 1.$$

Therefore, H is ample and

$$K_X + \Delta + p^e H \sim \frac{p^e - 1}{2} L + \underbrace{(p^e - 1) \left(\frac{1}{2} L - B \right) + (p^e + 1 - \lambda_1 - \lambda_2)(K_X + L) + p^e N}_{\text{nef}}.$$

The right vertical arrow is surjective, since $\text{Tr}_{X,\Delta}: F_*^e \mathcal{L} \rightarrow \mathcal{O}_X$ is surjective, and $\dim W = 0$. The upper horizontal arrow is surjective as

$$H^1\left(X, \mathcal{O}_X\left(\frac{p^e - 1}{2} L + M\right) \otimes I_W\right) = 0$$

for any nef Cartier divisor M . Thus, the lower horizontal arrow is surjective, and so the proof of the base point freeness is completed. $\square$

The following lemma was used in the proof.

Lemma 5.3.5. *Let $(X, B + a_1 D_1 + a_2 D_2)$ be a log canonical two dimensional pair, such that B is an effective $\mathbb{Q}$ -divisor, $a_1, a_2 \in \mathbb{R}_{\geq 0}$, and D_1 together with D_2 are Cartier divisors intersecting at a singular point $x \in X$. Then $a_1 + a_2 \leq 1$.*

Of course, the lemma is not true, when x is a smooth point.

Proof. Consider a minimal resolution of singularities $\pi: \tilde{X} \rightarrow X$. Write

$$K_{\tilde{X}} + \Delta_{\tilde{X}} + \pi^*(B + a_1 D_1 + a_2 D_2) = \pi^*(K_X + B + a_1 D_1 + a_2 D_2).$$

Since π is a minimal resolution, we have that $\Delta_{\tilde{X}} \geq 0$.

Take an exceptional curve C over x . Since D_1 and D_2 are Cartier, the coefficient of C in $\Delta_{\tilde{X}} + \pi^*(B + a_1 D_1 + a_2 D_2)$ is greater or equal $a_1 + a_2$. Since $(\tilde{X}, \Delta_{\tilde{X}} + \pi^*(B + a_1 D_1 + a_2 D_2))$ is log canonical, this concludes the proof of the lemma. $\square$

5.3.1. Very ampleness

The goal of this subsection is to show the following proposition and finish off the proof of Theorem 5.1.1.

Proposition 5.3.6 (cf. [Sch14, Corollary 4.5], [Kee08, Theorem 1.1]). *Let X be an F -pure projective variety of dimension n . Let D be an ample $\mathbb{Q}$ -Cartier divisor such that $K_X + D$ is Cartier, and let L be an ample globally generated Cartier divisor. Then $K_X + (n+1)L + D$ is very ample.*

The proof follows very closely the strategy described in [Kee08]. Theorems of a similar flavour have been obtained by Schwede ([Sch14]), Smith ([Smi97] and [Smi00]) and Hara ([Har05]).

First, we need a slight generalization of [Laz04a, Example 1.8.22].

Lemma 5.3.7 (c.f. [Laz04a, Examples 1.8.18 and 1.8.22]). *Let X be a normal projective variety of dimension n . Consider a coherent sheaf $\mathcal{F}$ and a point $x \in X$. Let B be a globally generated ample line bundle. If*

$$H^{i+k-1}(X, \mathcal{F} \otimes B^{-(i+k)}) = 0,$$

for $1 \leq i \leq n$ and $1 \leq k \leq n$, then $\mathcal{F} \otimes m_x$ is globally generated.

Proof. Set $\mathcal{F}(-i) := \mathcal{F} \otimes B^{-i}$. Our goal is to prove that

$$H^i(X, \mathcal{F}(-i) \otimes m_x) = 0$$

for all $i > 0$. Then, Theorem 5.2.1 would imply the global generatedness of $\mathcal{F} \otimes m_x$.

Since B is ample and globally generated, it defines a finite map and so there exist sections $s_1, s_2, \dots, s_n \in H^0(X, B)$ intersecting in a zero dimensional scheme W containing x . By the same argument as in [Laz04a, Example 1.8.22], using [Laz04a, Proposition B.1.2(ii)] we get that

$$H^i(X, \mathcal{F}(-i) \otimes I_W) = 0, \text{ for } i > 0.$$

To conclude the proof of the lemma, we consider the following short exact sequence

$$0 \longrightarrow I_W \longrightarrow m_x \longrightarrow m_x/I_W \longrightarrow 0,$$

and tensor it by $\mathcal{F}(-i)$, to get a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}(-i) \otimes I_W \longrightarrow \mathcal{F}(-i) \otimes m_x \longrightarrow \mathcal{F}(-i) \otimes (m_x/I_W) \longrightarrow 0,$$

where the term

$$\mathcal{G} := \ker (\mathcal{F}(-i) \otimes I_W \longrightarrow \mathcal{F}(-i) \otimes m_x)$$

comes from the fact that $\mathcal{F}$ may not be flat. Since m_x/I_W is flat off W , we have that

$$\dim \text{Supp} (\text{Tor}^1(\mathcal{F}(-i), m_x/I_W)) = 0,$$

and so $H^i(X, \mathcal{G}) = 0$ for $i > 0$. A simple diagram chasing shows that $H^i(X, \mathcal{F}(-i) \otimes m_x) = 0$ for $i > 0$, and so we are done. $\square$

Proof of Proposition 5.3.6. Choose a point $q \in X$. To prove the theorem, it is enough to show that $\mathcal{O}_X(K_X + (n+1)L + D) \otimes m_x$ is globally generated at q for all $x \in X$. Set $\mathcal{L} := \mathcal{O}_X(L)$.

Since X is F -pure, Proposition 2.2.9 and Lemma 5.2.4 imply that there exists an effective $\mathbb{Q}$ -divisor B such that

$$(p^e - 1)(K_X + B)$$

is Cartier for divisible enough $e > 0$ and $\text{Tr}_{X,B}^e$ is surjective at q . If the Gorenstein index of X is not divisible by p , then we can take $B = 0$. Further, we may assume that $D - B$ is ample.

Tensoring $\text{Tr}_{X,B}^e((n+1)L + D - B)$ by m_x , we obtain a morphism

$$F_*^e \mathcal{O}_X(K_X + B + p^e((n+1)L + D - B)) \otimes m_x \longrightarrow \mathcal{O}_X(K_X + (n+1)L + D) \otimes m_x,$$

which is surjective at q , and so it is enough to show that $F_*^e \mathcal{O}_X(K_X + B + p^e((n+1)L + D - B)) \otimes m_x$ is globally generated for divisible enough $e > 0$.

However, this follows from Lemma 5.3.7 as

$$\begin{aligned} H^{i+k-1}(X, F_*^e \mathcal{O}_X(K_X + B + p^e((n+1)L + D - B)) \otimes \mathcal{L}^{-(i+k)}) \\ = H^{i+k-1}(X, \mathcal{O}_X(K_X + B + p^e((n+1-i-k)L + D - B))) = 0 \end{aligned}$$

for $e \gg 0$ and $1 \leq i+k-1 \leq n$, by Serre vanishing. $\square$

Now, the proof of the main theorem is straightforward.

Proof of Theorem 5.1.1. It follows directly from Proposition 2.2.5, Proposition 5.3.2, Proposition 5.3.3 and Proposition 5.3.6. $\square$

5.4. Generalisations of the main theorem

In this section we present a technical generalisation of Theorem 5.1.1.

Theorem 5.4.1. *Let X be an F -pure $\mathbb{Q}$ -factorial projective surface defined over an algebraically closed field of characteristic $p > 0$. Assume that mK_X is Cartier for some*

$m \in \mathbb{N}$. Let L be an ample Cartier divisor on X and let N be any nef Cartier divisor. The following holds.

- If X is neither of general type nor quasi-elliptic with $\kappa(X) = 1$, then

$2mK_X + 8mL + N$ is base point free, and

$7mK_X + 27mL + N$ is very ample.

- If $p = 3$ and X is quasi-elliptic with $\kappa(X) = 1$, then

$2mK_X + 12mL + N$ is base point free, and

$7mK_X + 39mL + N$ is very ample.

- If $p = 2$ and X is quasi-elliptic with $\kappa(X) = 1$, then

$2mK_X + 16mL + N$ is base point free, and

$7mK_X + 51mL + N$ is very ample.

- If $p \geq 3$ and X is of general type, then

$4mK_X + 14mL + N$ is base point free, and

$13mK_X + 45mL + N$ is very ample.

- If $p = 2$ and X is of general type, then

$4mK_X + 22mL + N$ is base point free, and

$13mK_X + 69mL + N$ is very ample.

The bounds are rough. The theorem is a direct consequence of the following proposition.

Proposition 5.4.2. *Let X be a normal projective surface defined over an algebraically closed field of characteristic $p > 0$. Assume that mK_X is Cartier for some $m \in \mathbb{N}$. Let A be an ample Cartier divisor on X . Then*

$$\text{Bs}(m(aK_X + bA) + N) \subseteq \text{Sing}(X),$$

for any nef Cartier divisor N , where

- $a = 1, b = 4$, if X is neither of general type nor quasi-elliptic with $\kappa(X) = 1$,
- $a = 1, b = 6$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$,
- $a = 1, b = 8$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$,
- $a = 2, b = 7$, if X is of general type and $p \geq 3$,
- $a = 2, b = 11$, if X is of general type and $p = 2$.

Proof. This follows from Proposition 5.2.6, by exactly the same proof as of Proposition 5.3.2. $\square$

Proof of Theorem 5.4.1. This follows directly from Proposition 2.2.5, Proposition 5.4.2, Proposition 5.3.3 and Proposition 5.3.6. $\square$

5.5. Matsusaka-type bounds

The goal of this section is to prove Corollary 5.1.2. The key part of the proof is the following proposition.

Proposition 5.5.1. *Let A be an ample Cartier divisor and let N be a nef Cartier divisor on a normal projective surface X . Then $kA - N$ is nef for any*

$$k \geq \frac{2A \cdot N}{A^2}((K_X + 3A) \cdot A + 1) + 1.$$

Proof. The proof is exactly the same as [DCF15, Theorem 3.3]. The only difference is that for singular surfaces, the cone theorem is weaker, so we have $K_X + 3D$ in the statement, instead of $K_X + 2D$. $\square$

The following proof is exactly the same as of [DCF15, Theorem 1.2].

Proof of Proposition 5.1.2. By Theorem 5.1.1, we know that H is very ample. By the above proposition, we know that $kA - (H + N)$ is a nef Cartier divisor. Thus, by the proof of Theorem 5.1.1

$$H + \underbrace{(kA - (H + N))}_{\text{nef}} = kA - N$$

is very ample. $\square$

Applying Theorem 5.4.1, we obtain the following.

Corollary 5.5.2. *Let A and N be respectively an ample and a nef Cartier divisor on an F -pure $\mathbb{Q}$ -factorial projective surface defined over an algebraically closed field of characteristic $p > 0$. Let $m \in \mathbb{N}$ be such that mK_X is Cartier. Then $kA - N$ is very ample for any*

$$k > \frac{2A \cdot (H + N)}{A^2}((K_X + 3A) \cdot A + 1),$$

where

- $H := 7mK_X + 27mA$, if X is neither quasi-elliptic with $\kappa(X) = 1$, nor of general type,
- $H := 7mK_X + 39mA$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$,
- $H := 7mK_X + 51mA$, if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$,
- $H := 13mK_X + 45mA$, if X is of general type and $p \geq 3$,
- $H := 13mK_X + 69mA$, if X is of general type and $p = 2$.

5.6. Bounds on log del Pezzo pairs

The goal of this section is to prove Corollary 5.1.3. We need the following facts.

Proposition 5.6.1. *Let (X, Δ) be an ϵ -klt log del Pezzo pair for $0 < \epsilon < 3^{-1/2}$. Let $\pi: \bar{X} \rightarrow X$ be the minimal resolution. Then*

- (a) $0 \leq (K_X + \Delta)^2 \leq \max\left(9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor}\right)$,
- (b) $\text{rk Pic}(\bar{X}) \leq 128(1/\epsilon)^5$,
- (c) $2 \leq -E^2 \leq 2/\epsilon$ for any exceptional curve E of $\pi: \bar{X} \rightarrow X$,
- (d) if m is the $\mathbb{Q}$ -factorial index at some point $x \in X$, then

$$m \leq 2(2/\epsilon)^{128/\epsilon^5}.$$

Proof. Point (a) follows from [Jia13, Theorem 1.3]. Points (b) and (c) follow from [AM04, Corollary 1.10] and [AM04, Lemma 1.2], respectively. Last, (d) follows from the fact that the Cartier index of a divisor divides the determinant of the intersection matrix of the minimal resolution of a singularity (see also the paragraph below [Lai16, Theorem A]). $\square$

Further, we need to prove the following:

Lemma 5.6.2. *Let (X, Δ) be a klt log del Pezzo pair such that $m(K_X + \Delta)$ is Cartier for some natural number $m \geq 2$. Then*

- 1. $0 \leq (K_X + \Delta) \cdot K_X \leq 3m \max\left(9, 2m + 4 + \frac{2}{m}\right)$, and
- 2. $|K_X^2| \leq 128m^5(2m - 1)$.

Recall that if a log del Pezzo pair (X, Δ) , with Cartier index m , is klt, then it must be $1/m$ -klt.

Proof. The non-negativity in (1) is clear, since

$$(K_X + \Delta) \cdot K_X = (K_X + \Delta)^2 - (K_X + \Delta) \cdot \Delta \geq 0.$$

Further, by cone theorem, $K_X - 3m(K_X + \Delta)$ is nef, and thus

$$(K_X + \Delta) \cdot (K_X - 3m(K_X + \Delta)) \leq 0.$$

This, together with (a) in Proposition 5.6.1, implies (1).

To prove (2), we proceed as follows. Let $\pi: \bar{X} \rightarrow X$ be the minimal resolution of singularities of X . By (b) in Proposition 5.6.1, we have $\text{rk Pic}(\bar{X}) \leq 128m^5$, and so $-9 \leq -K_{\bar{X}}^2 \leq 128m^5$. Indeed, the self intersection of the canonical bundle on a minimal model of a rational surface is 8 or 9, and each blow-up decreases it by one.

Write

$$K_{\bar{X}} + \sum a_i E_i = \pi^* K_X,$$

where E_i are the exceptional divisors of π . Notice, that since $\overline{X} \rightarrow X$ is minimal and X is klt, we have $0 \leq a_i < 1$. By applying (b) and (c) from Proposition 5.6.1, we obtain

$$\begin{aligned} |K_X^2| &= |(K_{\overline{X}} + \sum a_i E_i) \cdot K_{\overline{X}}| \\ &\leq |K_{\overline{X}}|^2 + 128m^5(2m-2) \\ &\leq 128m^5(2m-1). \end{aligned} \quad \square$$

Proof of Corollary 5.1.3. By Proposition 5.6.1, the $\mathbb{Q}$ -factorial index of X is bounded with respect to ϵ . Indeed, the $\mathbb{Q}$ -factorial index is bounded at each point by (d), and the number of singular points is bounded by (b). Hence, we can assume that there exists $m \in \mathbb{N}$ bounded with respect to ϵ and I , such that mK_X , $m\Delta$, and $m(K_X + \Delta)$ are Cartier.

Set $a = 13m$ and $b = 45m^2$. By Theorem 5.1.1 and Remark 5.6.7, the divisor $H := aK_X - b(K_X + \Delta)$ is very ample, and $H^i(X, H) = 0$ for $i > 0$. Proposition 5.6.1(a) and Lemma 5.6.2 imply that H^2 , $|H \cdot K_X|$, $H \cdot \Delta$, $|K_X \cdot \Delta|$, and $|\Delta^2|$ are bounded with respect to m .

The ample divisor H embeds X into a projective space of dimension $\chi(\mathcal{O}_X(H)) = H^0(X, \mathcal{O}_X(H))$, which is bounded with respect to m by the Riemann-Roch theorem. $\square$

If mK_X and $m(K_X + \Delta)$ are Cartier for $m > 1$, then one can easily calculate, that is enough to take

$$b(\epsilon, I) = \left(a^2 + b^2\right) \left(128m^5(2m-1) + \max\left(9, 2m+4 + \frac{2}{m}\right)\right),$$

where $a = 13m$ and $b = 45m^2$.

Remark 5.6.3. Corollary 5.1.3 and the Riemann-Roch theorem for surfaces with rational singularities imply that the absolute values of the coefficients of the Hilbert polynomial of X with respect to H are bounded with respect to ϵ and I . Further, let $n \in \mathbb{N}$ be such that $n\Delta$ is Cartier. Then the absolute values of the coefficients of the Hilbert polynomial of $n\Delta$ with respect to $H|_{n\Delta}$ are bounded with respect to ϵ , I , and n . Indeed,

$$\chi(\mathcal{O}_{n\Delta}(mH)) = mn\Delta \cdot H - \frac{1}{2}n\Delta \cdot (n\Delta + K_X)$$

for $m \in \mathbb{Z}$, by the Riemann-Roch theorem and the adjunction formula.

Remark 5.6.4. One of the reasons for our interest in the above corollary is that it provides bounds on ϵ -klt log del Pezzo surfaces which are independent of the characteristic. In particular, it shows the existence of a bounded family of ϵ -klt log del Pezzo surfaces over $\text{Spec } \mathbb{Z}$ (see Lemma 4.3.1). We were not able to verify whether Kollár's bounds depend on the characteristic or not. We believe that stating explicit bounds might ease the life of future researchers, wanting to handle questions related to the behavior of log del Pezzo surfaces in mix characteristic or for big enough characteristic.

5.6.1. Effective vanishing of H^1

The goal of this subsection is to give a proof of the following proposition.

Proposition 5.6.5. *Let X be a normal projective surface. Then*

$$H^i(X, \mathcal{O}_X(D)) = 0 \text{ for } i > 0,$$

where $D = 3K_X + 14A + N$ is Cartier for an ample Cartier divisor A and a nef $\mathbb{Q}$ -Cartier divisor N .

It was pointed to us by the anonymous referee that the proposition follows from a result of Kollár. We present this approach below.

First, let us recall the aforementioned result.

Theorem 5.6.6 ([Kol96, Theorem II.6.2 and Remark II.6.7.2]). *Let X be a normal, projective variety defined over a field of characteristic p . Let L be an ample $\mathbb{Q}$ -Cartier Weil divisor on X satisfying $H^1(X, \mathcal{O}_X(-L)) \neq 0$. Assume that X is covered by a family of curves $\{D_t\}$ such that X is smooth along a general D_t and*

$$((p-1)L - K_X) \cdot D_t > 0.$$

Then, through every point $x \in X$ there is a rational curve $C \subseteq X$ such that

$$L \cdot C \leq 2 \dim X \frac{L \cdot D_t}{((p-1)L - K_X) \cdot D_t}.$$

Proof of Proposition 5.6.5. Set $L = 2K_X + 14A + N$. Since D is Cartier and ω_X is reflexive, we get that $\omega_X \otimes \mathcal{O}_X(-D) = \mathcal{O}_X(-L)$. Thus, by Serre duality, we need to show $H^i(X, \mathcal{O}_X(-L)) = 0$ for $i < 2$. However, by cone theorem, we have that $K_X + 3A$ is nef, and so L is ample, giving in particular that $H^0(X, \mathcal{O}_X(-L)) = 0$. Hence, we are left to show $H^1(X, \mathcal{O}_X(-L)) = 0$.

To this end, we suppose by contradiction that $H^1(X, \mathcal{O}_X(-L)) \neq 0$ and apply Theorem 5.6.6 for a general pencil of curves $\{D_t\}$ in some very ample linear system. Since $(p-1)L - K_X$ is ample, the assumptions of the theorem are satisfied. In particular, we get a curve C such that

$$L \cdot C \leq 4 \frac{L \cdot D_t}{((p-1)L - K_X) \cdot D_t},$$

and, as L is ample, this in turn gives

$$L \cdot C \leq 4 \frac{L \cdot D_t}{(L - K_X) \cdot D_t}.$$

Since $K_X + 3A$ is nef, we have $L \cdot C \geq 8$. Further, $K_X \cdot D_t < (L - K_X) \cdot D_t$, as

$L - K_X = K_X + 14A + N$. Therefore,

$$8 \leq L \cdot C < 4 \left(\frac{2(L - K_X) \cdot D_t}{(L - K_X) \cdot D_t} \right) = 8,$$

which is a contradiction. □

Remark 5.6.7. Proposition 5.6.5 shows, under assumptions and notation of Theorem 5.1.1, that the very ample divisor $H := 13mK_X + 45mA$ satisfies $H^i(X, H) = 0$ for $i > 0$. Similar statements hold for very ample divisors considered in Theorem 5.4.1.

6

On the rationality of Kawamata log terminal singularities

based on joint work with C. Hacon

6.1. Introduction

THIS CHAPTER DISCUSSES rationality of three-dimensional Kawamata log terminal singularities in large characteristic. The following theorem is the main result of this chapter.

Theorem 6.1.1. *There exists a natural number $p_0 > 0$ such that for any three-dimensional Kawamata log terminal pair (X, Δ) defined over an algebraically closed field of characteristic $p > p_0$, we have that X has rational, Cohen-Macaulay singularities. Moreover, if X is $\mathbb{Q}$ -factorial, then, for any divisor D , the sheaf $\mathcal{O}_X(D)$ is Cohen-Macaulay.*

It is not known whether the above theorem holds or is false if we just assume that $p > 2$. The main tool in the proof of this result is the Kawamata-Viehweg vanishing for log del Pezzo surfaces in large characteristic (see Chapter 4). The constant p_0 comes from this result.

Let us also note that it has been recently shown in [Kov17b], that Cohen-Macaulay klt singularities are rational. The results from this paper allowed us to simplify the proof of Theorem 6.1.1.

This chapter is organised as follows. In Section 2, we discuss rational and Cohen-Macaulay singularities, plt blow-ups, and liftings to the rings of Witt vectors. In Section

3, we prove the existence of certain short exact sequences on plt threefolds (see Subsection 6.1.1). In Section 4, we finish the proof of Theorem 6.1.1. In Section 5 we explore applications to the MMP in mixed characteristic.

6.1.1. Sketch of the proof

Assume that X is $\mathbb{Q}$ -factorial. Consider a *plt blow-up* (Proposition 6.2.10) at a closed point $x \in X$, that is a birational morphism $f: Y \rightarrow X$ with an irreducible exceptional divisor S such that

- (Y, S) is plt and $\mathbb{Q}$ -factorial,
- $-(K_Y + S)$ is f -ample,
- f is an isomorphism over $X \setminus \{x\}$.

By adjunction, there exists a $\mathbb{Q}$ -divisor Δ_{diff} such that $(S, \Delta_{\text{diff}})$ is a klt log del Pezzo pair.

When S is Cartier, one can apply the following strategy (cf. [GNT16] and [CT16]). For every $k \geq 0$, consider

$$0 \rightarrow \mathcal{O}_Y(-(k+1)S) \rightarrow \mathcal{O}_Y(-kS) \rightarrow \mathcal{O}_S(G_k) \rightarrow 0,$$

where $G_k \sim -kS|_S$. Since

$$G_k \sim_{\mathbb{Q}} K_S + \Delta_{\text{diff}} - \underbrace{(K_S + \Delta_{\text{diff}})}_{\text{ample}} + G_k,$$

by Theorem 4.1.2, we have that $H^1(S, \mathcal{O}_S(G_k)) = 0$ for every $k \geq 0$, and so $R^1 f_* \mathcal{O}_Y(-(k+1)S) \rightarrow R^1 f_* \mathcal{O}_Y(-kS)$ is surjective. Since $-S$ is f -ample, $R^1 f_* \mathcal{O}_Y(-kS) = 0$ for $k \gg 0$ by Serre vanishing, which implies $R^1 f_* \mathcal{O}_Y = 0$ as well. By [HX15] one sees that Y has F-regular singularities along S and hence Y has rational singularities along S , concluding the proof of the rationality of the singularities of X at x . The fact that X is Cohen-Macaulay at x follows by a similar argument.

When S is not Cartier, the above short exact sequence may fail to be exact, however we show the existence of the following short exact sequence (Proposition 6.3.1)

$$0 \rightarrow \mathcal{O}_Y(-(k+1)S) \rightarrow \mathcal{O}_Y(-kS) \rightarrow \mathcal{O}_S(G_k) \rightarrow 0, \quad (6.1)$$

where G_k is a Weil divisor satisfying $G_k \sim_{\mathbb{Q}} -kS|_S - \Delta_k$ for some effective $\mathbb{Q}$ -divisor Δ_k on S such that $\Delta_k \subseteq \Delta_{\text{diff}}$. Therefore, we can apply the same argument as above with minor changes.

When X is not $\mathbb{Q}$ -factorial, we apply Proposition 6.2.10, Proposition 6.2.11, and the relative F-inversion of adjunction (Lemma 6.2.9).

6.2. Preliminaries

We work over an arbitrary field k of positive characteristic $p > 0$ unless stated otherwise. First, let us recall the statement of the relative Kawamata–Viehweg vanishing.

Theorem 6.2.1 ([Kol13, Theorem 10.4]). *Let X be a regular surface over a field k and let $f: X \rightarrow Y$ be a proper, generically finite morphism with exceptional curves C_i such that $\bigcup_i C_i$ is connected. Let L be a line bundle on X and assume that there exist $\mathbb{Q}$ -divisors N and $\Delta = \sum d_i C_i$ such that:*

- $L \equiv_f K_X + \Delta + N$,
- $N \cdot C_i \geq 0$ for every i ,
- $0 \leq d_i < 1$ for every i .

Then $R^1 f_ L = 0$.*

6.2.1. Rational and Cohen-Macaulay singularities

We say that a variety X has *rational singularities* if there exists a log resolution $f: Y \rightarrow X$ such that $Rf_* \mathcal{O}_Y = \mathcal{O}_X$. Assuming the existence of resolutions of singularities, the phrase *there exists* can be replaced by *for every* (this follows from the main result in [CR15]). Lastly, for the definition of Serre’s condition S_d and Cohen-Macaulay singularities we refer to [Kol13, Definition 5.1 and Definition 5.2].

Proposition 6.2.2. *Let $f: Y \rightarrow X$ be a proper birational morphism of normal varieties and let D be a Weil divisor on Y such that $\mathcal{O}_Y(D)$ is Cohen-Macaulay. Assume that $R^i f_* \mathcal{O}_Y(D) = 0$ and $R^i f_* \mathcal{O}_Y(K_Y - D) = 0$ for all $i > 0$. Then $f_* \mathcal{O}_Y(D)$ is Cohen-Macaulay.*

Proof. Since D is Cohen-Macaulay, so is $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y(D), \omega_Y)$ (see [KM98, Corollary 5.70]). In particular, the latter sheaf is reflexive, and hence it is isomorphic to $\mathcal{O}_Y(K_Y - D)$. Therefore, the proposition follows from [Kol13, Theorem 2.74]. $\square$

The following result is well known.

Proposition 6.2.3. *Let X be a variety with rational Cohen-Macaulay singularities. Then $Rf_* \omega_Y = \omega_X$ for every resolution $f: Y \rightarrow X$.*

Proof. Let $\omega_X^\bullet$ and $\omega_Y^\bullet$ be the dualizing complexes. The assumption that X is Cohen-Macaulay implies that $\omega_X^\bullet \simeq \omega_X[d]$ for $d = \dim X$.

By Grothendieck duality,

$$Rf_* \omega_Y[d] \simeq Rf_* R\mathcal{H}om_Y(\mathcal{O}_Y, \omega_Y^\bullet) \simeq R\mathcal{H}om_X(Rf_* \mathcal{O}_Y, \omega_X^\bullet) \simeq \omega_X^\bullet \simeq \omega_X[d]. \quad \square$$

Let us recall that a rank one sheaf $\mathcal{F}$ on a normal variety X is reflexive if and only if it is divisorial, that is of the form $\mathcal{O}_X(D)$ for some Weil divisor D . Moreover, $i_*(\mathcal{F}|_U) \simeq \mathcal{F}$,

where $i: U \rightarrow X$ is an inclusion of an open subset whose complement is of codimension at least two.

Lemma 6.2.4. *Let X be a normal variety, let $S \subset X$ be a prime divisor, and let D be any Weil divisor. For the following exact sequence*

$$0 \rightarrow \mathcal{O}_X(-S + D) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{E} \rightarrow 0,$$

we have $\text{Supp } \mathcal{E} = S$. Moreover, if $\mathcal{O}_X(-S + D)$ and $\mathcal{O}_X(D)$ satisfy Serre's condition S_3 , then $\mathcal{E}$ is reflexive as a sheaf on S .

Proof. Since the ideal sheaf $\mathcal{I}_S$ of S annihilates $\mathcal{E}$ and the sequence above is the standard restriction exact sequence outside of a closed subset of codimension two, we have that $\text{Supp}(\mathcal{E}) = S$. The second part of the lemma follows from [Kol13, Lemma 2.60]. $\square$

6.2.2. Some results on strongly F-regular singularities

The proof of the main theorem makes use of F-regular singularities. The sole purpose of employing them here is to show, summing up known results, that a plt threefold singularity (X, S) in characteristic $p > 5$ is rational near S and, assuming $\mathbb{Q}$ -factoriality, any reflexive rank one sheaf on X is Cohen-Macaulay near S (Proposition 6.2.6, Proposition 6.2.7, and, for the non- $\mathbb{Q}$ -factorial case, Proposition 6.2.8). This shows that singularities of plt blow-ups possess these desirable properties (see Proposition 6.2.10 and Proposition 6.2.11).

Remark 6.2.5. Assume that (X, Δ) is F-split over Y with respect to $f: X \rightarrow Y$. Then for any affine open subset $U \subseteq Y$, we have that $(f^{-1}(U), \Delta|_{f^{-1}(U)})$ is globally F-split (see [HX15, Proposition 2.10], cf. [CGS16, Remark 2.8]). In particular, if $u: Y \rightarrow Y'$ is affine, then (X, Δ) is also F-split over Y' with respect to $u \circ f$. Analogous statements hold for F-regularity and pure F-regularity.

Proposition 6.2.6. *Let X be a strongly F-regular variety. Then X has rational singularities and is Cohen-Macaulay. Moreover, if X is $\mathbb{Q}$ -factorial and of dimension at least three, then any reflexive sheaf of rank one on it is S_3 .*

In particular, a reflexive sheaf of rank one on a $\mathbb{Q}$ -factorial strongly F-regular threefold is Cohen-Macaulay.

Proof. X is F-rational (see [ST12, Appendix C]), and hence it is rational and Cohen-Macaulay by [Kov17b, Corollary 1.12].

The last statement in the proposition is a consequence of [PS14, Theorem 3.8]. Indeed, let $\mathcal{O}_X(D)$ be a reflexive sheaf, and take a general effective divisor $E \sim -(mp + 1)D$ for $m \gg 0$. Then $(X, \frac{1}{mp+1}E)$ is strongly F-regular, and the result follows. $\square$

Proposition 6.2.7. *Let (X, S) be a plt three-dimensional pair defined over an algebraically closed field of characteristic $p > 5$. Then S is normal and X is strongly F-regular along it.*

Proof. By [Har98b] and Theorem 2.2.10 (cf. Lemma 6.2.9), we have that (X, S) is purely F -regular along S and S is normal. This implies F -regularity of X along S . $\square$

In order to deal with non- $\mathbb{Q}$ -factorial singularities, we also need a relative version of the above result.

Proposition 6.2.8. *Let (X, S) be a plt three-dimensional pair defined over an algebraically closed field of characteristic $p > 5$, and let $f: X \rightarrow Z$ be a proper birational morphism between normal varieties such that $\dim f(S) = 2$. If $-(K_X + S)$ is f -ample, then Z is strongly F -regular along $f(S)$.*

This does not follow from Proposition 6.2.7 directly, because $K_Z + f(S)$ is almost never $\mathbb{Q}$ -Cartier. Further, let us note that we will need to apply Proposition 6.2.8 in the case when X is not $\mathbb{Q}$ -factorial.

Proof. As above, S is normal. The normalization morphism $f(S)^\nu \rightarrow f(S)$ is affine, and so, by [HX15, Theorem 3.1], the pair $(S, \Delta_{\text{diff}})$, defined by adjunction $K_S + \Delta_{\text{diff}} = (K_X + S)|_S$, is globally F -regular over $f(S)$ (see Remark 6.2.5). Lemma 6.2.9 and [HX15, Lemma 2.12] imply that Z is strongly F -regular along $f(S)$. $\square$

In the proposition above, we used a relative version of inversion of F -adjunction. We include the proof here for the convenience of the reader.

Lemma 6.2.9. *Let $(X, S + B)$ be a plt pair where S is a prime divisor, and let $f: X \rightarrow Z$ be a proper birational morphism between normal varieties. Assume that $-(K_X + S + B)$ is f -ample and $(\bar{S}, B_{\bar{S}})$ is globally F -regular over $f(S)$, where $\bar{S}$ is the normalization of S , and $B_{\bar{S}}$ is defined by adjunction $K_{\bar{S}} + B_{\bar{S}} = (K_X + S + B)|_{\bar{S}}$. Then $(X, S + B)$ is purely globally F -regular over a Zariski-open neighbourhood of $f(S) \subseteq Z$.*

Proof. Since the question is local over Z , we can assume that Z is affine, that is $Z = \text{Spec}(R)$ for some ring R . Further, by perturbing the coefficients of B , we can assume that the Cartier index of $K_X + S + B$ is not divisible by p (see for example the proof of Lemma 4.2.10). By Theorem 2.2.10, $(X, S + B)$ is purely F -regular, and S is a normal F -pure centre. By standard arguments (replacing B by $B + \epsilon H$ for $0 < \epsilon \ll 1$ and a very ample divisor H intersecting S properly), it is enough to show that $(X, S + B)$ is F -split over a Zariski-open neighbourhood of $f(S)$ (see the proof of [SS10, Theorem 3.9]).

Set $\mathcal{L} := \mathcal{O}_X((1 - p^e)(K_X + S + B))$ and for a sufficiently divisible integer $e \gg 0$ consider the following diagram

$$\begin{array}{ccccc} H^0(X, F_*^e \mathcal{L}) & \longrightarrow & H^0(S, F_*^e(\mathcal{L}|_S)) & \longrightarrow & H^1(X, F_*^e(\mathcal{L}(-S))) \\ \downarrow \text{Tr}_{X, S+B} & & \downarrow \text{Tr}_{S, B_S} & & \\ H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(S, \mathcal{O}_S), & & \end{array}$$

where $\mathrm{Tr}_{X,S+B}$ and Tr_{S,B_S} are the trace maps, while the horizontal arrows come from the restriction exact sequences. The diagram is well defined and commutes by the fact that S is an F-pure centre and by the equality of the Different and the F-different (see [Das15, Theorem B], cf. Subsection 4.2.2). Note that Tr_{S,B_S} is surjective, since (S, B_S) is F-split over $f(S)$.

Since $e \gg 0$, the relative Serre vanishing implies that $H^1(X, F_*^e(\mathcal{L}(-S))) = 0$. In particular, the upper left horizontal arrow is surjective, and so the composition

$$H^0(X, F_*^e \mathcal{L}) \xrightarrow{\mathrm{Tr}_{X,S+B}} H^0(X, \mathcal{O}_X) \rightarrow H^0(S, \mathcal{O}_S)$$

is surjective as well. But $H^0(X, \mathcal{O}_X) = H^0(Z, \mathcal{O}_Z) = R$, hence this implies that the zero locus of the ideal in R generated by $\mathrm{im}(\mathrm{Tr}_{X,S+B})$ is disjoint from $f(S)$. After removing this locus, the surjectivity of $\mathrm{Tr}_{X,S+B}$ follows. $\square$

6.2.3. Local to global transition

In the course of the proof of the main theorem we will need the following two results that allow for the transition from a local to a global case.

Proposition 6.2.10 (Generalized plt blow-up). *Let (X, Δ) be a Kawamata log terminal three-dimensional pair defined over an algebraically closed field of characteristic $p > 5$, and let $x \in X$ be a closed point. Assume that $X \setminus \{x\}$ is $\mathbb{Q}$ -factorial.*

Then there exists a projective birational morphism $f: Y \rightarrow X$ and an effective $\mathbb{Q}$ -divisor Δ_Y on Y such that

- *f is an isomorphism over $X \setminus \{x\}$,*
- *$S := \mathrm{Exc} f$ is irreducible and anti- f -nef,*
- *$(Y, S + \Delta_Y)$ is a $\mathbb{Q}$ -factorial plt threefold,*
- *$-(K_Y + S + \Delta_Y)$ is f -ample.*

Furthermore, S is normal and Y is strongly F-regular along it.

When $x \in X$ is $\mathbb{Q}$ -factorial, then $-S$ is automatically f -ample.

Proof. Apart from the last statement, this is a direct consequence of [GNT16, Proposition 2.15]. The strong F-regularity of Y and the normality of S follow from Proposition 6.2.7. $\square$

For the non- $\mathbb{Q}$ -factorial case of the main theorem, we will also need the following result.

Proposition 6.2.11. *Under the assumptions of Proposition 6.2.10, the f -relative semi-ample fibration $g: Y \rightarrow Y'$ associated to $-S$ exists. Moreover, g is small, $-S|_S$ is big, and Y' is strongly F-regular along $g(S)$.*

Proof. The first statement follows from the base point free theorem (see [BW17, Theorem 1.2]) after having noticed that $-S = K_Y + \Delta_Y - (K_Y + S + \Delta_Y)$.

For the proof of the bigness of $-S|_S$ it is enough to show that g is small. To this end, run a $(K_Y + S + \Delta_Y)$ -MMP over Y' , and let $Y \dashrightarrow Y_{\min} \xrightarrow{g'} Y'$ be the minimal model. Since $-(K_Y + S + \Delta_Y)$ is g -ample, g' is small. Moreover, as S is g -numerically trivial, none of the steps of the MMP may contract S . Indeed, an MMP preserves $\mathbb{Q}$ -factoriality, and hence a contracted divisor is always relatively anti-ample with respect to a divisorial contraction. Since $\text{Exc } f = S$ is irreducible, this shows that g is small and it does not contract S .

For the last statement, we replace Y by the output of a $-(K_Y + S)$ -MMP over Y' , and so we can assume that $-(K_Y + S)$ is g -nef and g -big. We can run such an MMP, since we can write

$$-\epsilon(K_Y + S) = K_Y + S + (1 + \epsilon)\Delta_Y - (1 + \epsilon)(K_Y + S + \Delta_Y),$$

where $-(K_Y + S + \Delta_Y)$ is g -ample and $(Y, S + (1 + \epsilon)\Delta_Y)$ is plt for $0 < \epsilon \ll 1$. Further, by replacing Y by the image of the g -relative semiample fibration associated to $-(K_Y + S)$, we can assume that $-(K_Y + S)$ is g -ample. The fibration exists by the klt base point free theorem, as S is g -numerically trivial and we can write

$$-(K_Y + S) = K_Y - 2(K_Y + S) + S.$$

Although Y may have ceased to be $\mathbb{Q}$ -factorial, (Y, S) is a well defined plt pair (in particular, $K_Y + S$ is $\mathbb{Q}$ -Cartier), and so Proposition 6.2.8 concludes the proof. $\square$

6.3. The restriction short exact sequence

Proposition 6.3.1. *Let (X, S) be a plt three-dimensional $\mathbb{Q}$ -factorial pair defined over an algebraically closed field k of characteristic $p > 5$, where S is a prime divisor, and let Δ_{diff} be the different. Then for any Weil divisor D on X , there exists an effective $\mathbb{Q}$ -divisor $\Delta_S \leq \Delta_{\text{diff}}$ and a Weil divisor G on S such that $G \sim_{\mathbb{Q}} K_S + \Delta_S + D|_S$ and the short exact sequence*

$$0 \rightarrow \mathcal{O}_X(K_X + D) \rightarrow \mathcal{O}_X(K_X + S + D) \rightarrow \mathcal{O}_S(G) \rightarrow 0$$

is exact.

Let us note that G is only defined up to linear equivalence.

Proof. Note that in proving the proposition, we are free to replace X by an appropriate neighborhood of S . By Proposition 6.2.7, we may assume that X is strongly F-regular and S is normal. Replacing D by a linearly equivalent divisor, we may assume that S is not contained in the support of D and so $D|_S$ is a well defined $\mathbb{Q}$ -divisor. Consider the

following short exact sequence (see Lemma 6.2.4)

$$0 \rightarrow \mathcal{O}_X(K_X + D) \rightarrow \mathcal{O}_X(K_X + S + D) \rightarrow i_*\mathcal{E} \rightarrow 0,$$

where $\mathcal{E}$ is a sheaf supported on S , and $i: S \rightarrow X$ is the inclusion. By Proposition 6.2.6 any reflexive rank one sheaf on X satisfies Serre's condition S_3 and so, by Lemma 6.2.4, $\mathcal{E}$ is reflexive on S .

Let $\pi: \bar{X} \rightarrow X$ be a log resolution of (X, S) and let $\bar{S}$ be the strict transform of S . Define $\Delta_{\bar{X}}$ by

$$K_{\bar{X}} + \bar{S} + \Delta_{\bar{X}} = \pi^*(K_X + S).$$

Consider the following exact sequence

$$0 \rightarrow \mathcal{O}_{\bar{X}}(K_{\bar{X}} + \lceil \pi^*D \rceil) \rightarrow \mathcal{O}_{\bar{X}}(K_{\bar{X}} + \bar{S} + \lceil \pi^*D \rceil) \rightarrow \mathcal{O}_{\bar{S}}(K_{\bar{S}} + \lceil \pi^*D \rceil|_{\bar{S}}) \rightarrow 0.$$

Since (X, S) is plt, we have $\lfloor \Delta_{\bar{X}} \rfloor \leq 0$, and so $\lceil \pi^*D \rceil \geq \lfloor \Delta_{\bar{X}} + \pi^*D \rfloor$. Therefore

$$\begin{aligned} K_{\bar{X}} + \bar{S} + \lceil \pi^*D \rceil &\geq \lfloor K_{\bar{X}} + \bar{S} + \Delta_{\bar{X}} + \pi^*D \rfloor \\ &= \lfloor \pi^*(K_X + S + D) \rfloor, \text{ and} \\ K_{\bar{X}} + \lceil \pi^*D \rceil &\geq \lfloor K_{\bar{X}} + \Delta_{\bar{X}} + \pi^*D \rfloor \\ &\geq \lfloor \pi^*(K_X + D) \rfloor. \end{aligned}$$

In particular,

$$\begin{aligned} \pi_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} + \lceil \pi^*D \rceil) &= \mathcal{O}_X(K_X + D), \text{ and} \\ \pi_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} + \bar{S} + \lceil \pi^*D \rceil) &= \mathcal{O}_X(K_X + S + D). \end{aligned}$$

By applying π_* to the above exact sequence, we have that

$$0 \rightarrow \mathcal{O}_X(K_X + D) \rightarrow \mathcal{O}_X(K_X + S + D) \rightarrow \pi_*\mathcal{O}_{\bar{S}}(\bar{G}) \rightarrow R^1\pi_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} + \lceil \pi^*D \rceil)$$

is exact, where $\bar{G} := K_{\bar{S}} + \lceil \pi^*D \rceil|_{\bar{S}}$. By localizing at one dimensional points on X , Lemma 6.3.2 shows that

$$\dim \text{Supp}(R^1\pi_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} + \lceil \pi^*D \rceil)) \leq 0.$$

Set $\Delta_S := (\pi|_{\bar{S}})_*((\lceil \pi^*D \rceil - \pi^*D)|_{\bar{S}})$ and $G := (\pi|_{\bar{S}})_*\bar{G}$ so that $G \sim_{\mathbb{Q}} K_S + \Delta_S + D|_S$.

We claim that $\mathcal{E} \simeq \mathcal{O}_S(G)$. Indeed, by what we observed above, $\mathcal{E}$ is isomorphic to $\pi_*\mathcal{O}_{\bar{S}}(\bar{G})$ in codimension one on S , which in turn is isomorphic to $\mathcal{O}_S(G)$ in codimension one on S . Since both $\mathcal{E}$ and $\mathcal{O}_S(G)$ are reflexive sheaves of rank one, the isomorphism $\mathcal{E} \simeq \mathcal{O}_S(G)$ follows.

To conclude the proof, it is enough to verify that $\Delta_S \leq \Delta_{\text{diff}}$. This can be checked by localizing at one dimensional points in $\text{Sing}(X)$, and thus we conclude by Lemma 6.3.2 (see also [K⁺92, Corti 16.6.3]). $\square$

The following lemma, used in the above proof, was applied to localisations at non-closed points, and thus we cannot assume that k is algebraically closed. However this has no impact on the proof, because the surfaces under consideration are excellent. We refer to [Kol13] and [Tan16c] for the classification and basic results pertaining to excellent log canonical surfaces. Let us just note that excellent klt surfaces are $\mathbb{Q}$ -factorial by [Tan16c, Corollary 4.10], and if (X, S) is a plt surface pair, then S is regular ([Kol13, 3.35]).

Lemma 6.3.2. *Let (X, S) be a plt surface pair defined over an arbitrary field k , where S is a prime divisor. Let Δ_{diff} be the different, let $\pi: \bar{X} \rightarrow X$ be a log resolution of (X, S) , let $\bar{S}$ be the strict transform of S , let D be any Weil divisor. Then*

- $R^1\pi_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} + \lceil \pi^*D \rceil) = 0$, and
- $\Delta_S \leq \Delta_{\text{diff}}$, where $\Delta_S := (\lceil \pi^*D \rceil - \pi^*D)|_{\bar{S}}$, and S is identified with $\bar{S}$.

Proof. As $\lceil \pi^*D \rceil - \pi^*D = 0$, the vanishing

$$R^1\pi_*\mathcal{O}_{\bar{X}}(K_{\bar{X}} + \lceil \pi^*D \rceil) = 0$$

follows by Theorem 6.2.1.

As for the second statement, we can restrict ourselves to a neighbourhood of S so that all singularities of X lie on S . Therefore, [Kol13, 3.35] implies that each singularity of X is cyclic and S is regular. Moreover, if $x \in \text{Sing}(X)$ and m_x is the $\mathbb{Q}$ -factorial index of X at x , then $m_x D$ is Cartier and Δ_{diff} has coefficient $1 - \frac{1}{m_x}$ at x ([K⁺92, Corti 16.6.3]). Therefore, $\Delta_S \leq \Delta_{\text{diff}}$, because, by definition, Δ_S has coefficients smaller than one, and $m_x \Delta_S$ is Cartier at $x \in \text{Sing}(X)$. □

By applying Proposition 6.3.1 for D replaced by $D - (K_X + S)$, we get that

$$0 \rightarrow \mathcal{O}_X(-S + D) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_S(G) \rightarrow 0$$

is exact for a $\mathbb{Q}$ -divisor $0 \leq \Delta_S \leq \Delta_{\text{diff}}$ and a Weil divisor $G \sim_{\mathbb{Q}} D|_S - \Delta_S$.

Corollary 6.3.3. *Under the assumptions of Proposition 6.3.1, the following sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(K_X - (k+1)S - D) &\rightarrow \mathcal{O}_X(K_X - kS - D) \rightarrow \mathcal{O}_S(G'_k) \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_X(-(k+1)S + D) &\rightarrow \mathcal{O}_X(-kS + D) \rightarrow \mathcal{O}_S(G_k) \rightarrow 0 \end{aligned}$$

are exact for any $k \in \mathbb{Z}_{>0}$, some effective $\mathbb{Q}$ -divisors $\Delta_k, \Delta'_k \leq \Delta_{\text{diff}}$ depending on k , and some integral divisors $G'_k \sim_{\mathbb{Q}} K_S + \Delta'_k - (k+1)S|_S - D|_S$ and $G_k \sim_{\mathbb{Q}} -kS|_S + D|_S - \Delta_k$.

6.4. Proof of the main theorem

Proof of Theorem 6.1.1. Fix p_0 as in Theorem 4.1.2. Since the question is local, we can assume that X is affine, and aim to show that a fixed closed point $x \in X$ is rational

and Cohen-Macaulay. We replace X by a Zariski-open neighbourhood of x whenever convenient. Let $f: Y \rightarrow X$ be a generalized plt blow-up at x as in Proposition 6.2.10. Let $S := \text{Exc}(f)$. Recall that $(Y, S + \Delta_Y)$ is plt, $-S$ is f -nef, and $-(K_Y + S + \Delta_Y)$ is f -ample for some effective $\mathbb{Q}$ -divisor Δ_Y on Y . Moreover, we may assume that Y is strongly F-regular. First, we show that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$.

To this end, we apply Corollary 6.3.3 to get short exact sequences

$$0 \rightarrow \mathcal{O}_Y(-(k+1)S) \rightarrow \mathcal{O}_Y(-kS) \rightarrow \mathcal{O}_S(G_k) \rightarrow 0$$

for all $k \geq 0$ where $G_k \sim_{\mathbb{Q}} -kS|_S - \Delta_k$ for some effective $\mathbb{Q}$ -divisor $\Delta_k \leq \Delta_{\text{diff}}$, where Δ_{diff} is the different. Thus, to show that $R^i f_*\mathcal{O}_Y(-kS)$ vanishes for $k = 0$ and $i > 0$, it is enough to prove that

- $R^i f_*\mathcal{O}_Y(-mS) = 0$ for $i > 0$ and a divisible enough $m \gg 0$, and
- $H^i(S, \mathcal{O}_S(G_k)) = 0$ for $i > 0$ and all $k \geq 0$.

If X is $\mathbb{Q}$ -factorial, then the first vanishing follows from the relative Serre vanishing, as $-S$ is f -ample. In general, we consider the f -relative semiample fibration $Y \xrightarrow{g} Y' \xrightarrow{f'} X$ associated to $-S$. It exists by Proposition 6.2.11. Since Y' is strongly F-regular along $S_{Y'} := g(S)$, we may assume that Y' is rational and Cohen-Macaulay (see Proposition 6.2.6). For $m > 0$ sufficiently divisible, $mS_{Y'}$ is Cartier and $mS = g^*(mS_{Y'})$. By the projection formula, we have $Rg_*\mathcal{O}_Y(-mS) = g_*\mathcal{O}_Y(-mS) = \mathcal{O}_{Y'}(-mS_{Y'})$ and therefore $R^i f_*\mathcal{O}_Y(-mS) = R^i f'_*\mathcal{O}_{Y'}(-mS_{Y'})$, which is zero for $m \gg 0$ by Serre vanishing since $S_{Y'}$ is f' -anti-ample.

Therefore, it suffices to show that $H^i(S, \mathcal{O}_S(G_k)) = 0$ for $i > 0$ and all $k \geq 0$. To this end, we notice that

$$(K_Y + S + \Delta_Y)|_S = K_S + \Delta_{\text{diff}} + \Delta_Y|_S$$

is anti-ample and $(S, \Delta_{\text{diff}} + \Delta_Y|_S)$ is klt. The cohomology in question vanishes because

$$G_k \sim_{\mathbb{Q}} K_S + (\Delta_{\text{diff}} + \Delta_Y|_S - \Delta_k) - \underbrace{(K_S + \Delta_{\text{diff}} + \Delta_Y|_S - kS|_S)}_{\text{ample}},$$

which is zero by Theorem 4.1.2.

By an analogous argument, considering the other short exact sequence in Corollary 6.3.3

$$0 \rightarrow \mathcal{O}_Y(K_Y - (k+1)S) \rightarrow \mathcal{O}_Y(K_Y - kS) \rightarrow \mathcal{O}_S(G'_k) \rightarrow 0,$$

one can show that $R^i f_*\omega_Y = 0$ for $i > 0$. Indeed, $Rg_*\omega_Y = \omega_{Y'}$ by Proposition 6.2.3 applied to both Y and Y' . As S is g -trivial, we have

$$R^i f_*\mathcal{O}_Y(K_Y - mS) = R^i f'_*\mathcal{O}_{Y'}(K_{Y'} - mS_{Y'}) = 0,$$

for $i > 0$ and a divisible enough $m \gg 0$ by the relative Serre vanishing. Moreover, by Theorem 4.1.2

$$H^i(S, \mathcal{O}_S(G'_k)) = 0 \quad \forall i > 0$$

for any $k \geq 0$ and a divisor G'_k such that $G'_k \sim_{\mathbb{Q}} K_S + \Delta'_k - (k+1)S|_S$ where Δ'_k is an effective $\mathbb{Q}$ -divisor satisfying $\Delta'_k \leq \Delta_{\text{diff}}$, because $-S|_S$ is nef and big (see Proposition 6.2.11).

Let $\pi_Y: \bar{Y} \rightarrow Y$ be a log resolution. By Proposition 6.2.6, we have that $R(\pi_Y)_* \mathcal{O}_{\bar{Y}} = \mathcal{O}_Y$, and so, by the composition of derived functors, we get that $R\pi_* \mathcal{O}_{\bar{Y}} = \mathcal{O}_X$, where $\pi := f \circ \pi_Y: \bar{Y} \rightarrow X$. Thus X has rational singularities. Moreover, Proposition 6.2.6 implies that Y is Cohen-Macaulay, and so X is Cohen-Macaulay as well by Proposition 6.2.2.

In order to prove the last statement, we assume that X is $\mathbb{Q}$ -factorial, fix a Weil divisor D on X , and set $D_Y := \lfloor f^*D \rfloor = f^*D - \epsilon S$ for some $0 \leq \epsilon < 1$.

First, we show that $Rf_* \mathcal{O}_Y(D_Y) = \mathcal{O}_X(D)$. By Corollary 6.3.3, we get short exact sequences

$$0 \rightarrow \mathcal{O}_Y(-(k+1)S + D_Y) \rightarrow \mathcal{O}_Y(-kS + D_Y) \rightarrow \mathcal{O}_S(G_k) \rightarrow 0$$

for all $k \geq 0$ where $G_k \sim_{\mathbb{Q}} -(k+\epsilon)S|_S - \Delta_k$ for some effective $\mathbb{Q}$ -divisor $\Delta_k \leq \Delta_{\text{diff}}$. As in the argument above, when $i > 0$ we have

$$\begin{aligned} H^i(S, \mathcal{O}_S(G_k)) &= 0, \text{ and} \\ R^i f_* \mathcal{O}_Y(-mS + D_Y) &= 0 \text{ for divisible enough } m \gg 0, \end{aligned}$$

where the second identity follows from Serre's vanishing as $-S$ is f -ample, and the first one is a consequence of Theorem 4.1.2 given that

$$G_k \sim_{\mathbb{Q}} K_S + (\Delta_{\text{diff}} + \Delta_Y|_S - \Delta_k) - \underbrace{(K_S + \Delta_{\text{diff}} + \Delta_Y|_S - (k+\epsilon)S|_S)}_{\text{ample}}.$$

The claim now follows easily by an argument similar to the one we used in the proof that the singularities are rational.

Analogously, one can show that $R^i f_* \mathcal{O}_Y(K_Y - D_Y) = 0$ for $i > 0$ by considering the other short exact sequence in Corollary 6.3.3

$$0 \rightarrow \mathcal{O}_Y(K_Y - (k+1)S - D_Y) \rightarrow \mathcal{O}_Y(K_Y - kS - D_Y) \rightarrow \mathcal{O}_S(G'_k) \rightarrow 0.$$

Indeed, $R^i f_* \mathcal{O}_Y(K_Y - mS - D_Y) = 0$ for sufficiently divisible $m \gg 0$ by Serre vanishing. Moreover, by Theorem 4.1.2

$$H^i(S, \mathcal{O}_S(G'_k)) = 0 \quad \forall i > 0$$

for any $k \geq 0$ and a divisor G'_k such that $G'_k \sim_{\mathbb{Q}} K_S + \Delta'_k - (k+1-\epsilon)S|_S$ where Δ'_k is an effective $\mathbb{Q}$ -divisor satisfying $\Delta'_k \leq \Delta_{\text{diff}}$.

Since $\mathcal{O}_Y(D_Y)$ is Cohen-Macaulay (see Proposition 6.2.6), Proposition 6.2.2 concludes the proof. $\square$

6.5. Applications and open problems

Corollary 6.5.1. *There exists a constant $p_0 > 0$ with the following property.*

Let $\pi: X \rightarrow Y$ be a proper birational morphism of Kawamata log terminal threefolds defined over an algebraically closed field k of characteristic $p > p_0$. If X lifts to $\tilde{X}_m$ over $W_m(k)$ for some $m \in \mathbb{N}$, then π lifts to $\tilde{\pi}_m: \tilde{X}_m \rightarrow \tilde{Y}_m$.

Proof. This follows from Theorem 6.1.1 and Proposition 2.3.3. $\square$

With Corollary 6.5.1 in mind, it is natural to ask the following.

Question 6.5.2. *Let X be a Kawamata log terminal threefold defined over an algebraically closed field k of characteristic $p \gg 0$ and $X \dashrightarrow X^+$ a flip. If X lifts to $\tilde{X}_m$ over $W_m(k)$ for some $m \in \mathbb{N}$, then does X^+ lift to $\tilde{X}_m^+$ over $W_m(k)$?*

It is also natural to wonder what the optimal value for the constant p_0 is.

Question 6.5.3. *Let X be a three dimensional klt variety defined over an algebraically closed field of characteristic $p > 2$. Does X have rational, Cohen-Macaulay singularities?*

7

On the canonical bundle formula and log abundance

7.1. Introduction

ONE OF VERY IMPORTANT TOOLS used in the characteristic zero birational geometry is the *canonical bundle formula* allowing for the study of canonical divisors under log Calabi-Yau fibrations. As stated in the main introduction, this formula is false in positive characteristic. In this chapter, we describe a way of partially recovering the canonical bundle formula by ways of replacing the base of the fibration by a purely inseparable cover.

Theorem 7.1.1 (see Theorem 7.3.4). *Let (X, Δ) be an n -dimensional projective log canonical pair defined over an algebraically closed field k of characteristic $p > 0$, let $\varphi: X \rightarrow Z$ be a contraction with the generic fibre of dimension one, and let L be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on Z satisfying $K_X + \Delta \sim_{\mathbb{Q}} \varphi^* L$. Assume that $p > 3$, or that Δ is big and $p > 2$.*

Then, there exists a purely-inseparable morphism $f: T \rightarrow Z$ such that

$$f^* L \sim_{\mathbb{Q}} t f^* K_Z + (1 - t)(K_T + \Delta_T),$$

for some rational number $0 \leq t \leq 1$ and an effective $\mathbb{Q}$ -divisor Δ_T on T .

Tanaka (see [Tan16d]) constructed a Mori fibre space in characteristic two and three with a non-klt base; in his example the singularity admits a purely-inseparable cover by a smooth variety. However, note that our theorem does not provide an insight into what the singularities of (T, Δ_T) are.

In view of this result and Tanaka's work, it is compelling to speculate about the existence of a “weighted scheme” living “in-between” T and Z that would appropriately capture the geometry of X and allow for the inductive iteration of the MMP. Alas, we do not know of a suitable category of such objects.

Amongst its various applications, the canonical bundle formula stands out in a synergy with nef reduction maps, that is rational maps $\varphi: X \dashrightarrow Z$ which contract exactly these curves C passing through very general points of X which satisfy $L \cdot C = 0$, where L is a given nef line bundle. The dimension of Z is called the nef dimension of L and denoted by $n(L)$. For the purpose of tackling the base point free theorem and the abundance conjecture in positive characteristic, we derive the following result from Theorem 7.1.1.

Theorem 7.1.2 (see Theorem 7.3.8). *With assumptions as in Theorem 7.1.1, suppose that $K_X + \Delta$ is nef and $n(K_X + \Delta) = n - 1$. Assume that the resolution of singularities and the Minimal Model Program are valid¹ for $(n - 1)$ -dimensional varieties over k . Then, there exists a purely-inseparable morphism $f: T \rightarrow Z$ such that*

$$f^*L \sim_{\mathbb{Q}} K_T + \Gamma,$$

for some pseudo-effective $\mathbb{Q}$ -divisor Γ on T .

One can show that if (X, Δ) is klt and Δ is big, then Γ is big as well. In any case, the $\mathbb{Q}$ -divisor Γ is not canonically defined, and so there is no hope to control the singularities of (T, Γ) .

We provide four applications of the above theorems.

Theorem 7.1.3 (Log non-vanishing). *Let (X, Δ) be a projective Kawamata log terminal pair of dimension three defined over an algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is pseudo-effective. Then $\kappa(K_X + \Delta) \geq 0$.*

Theorem 7.1.4 (Log abundance for non-maximal nef dimension). *Let (X, Δ) be a projective Kawamata log terminal pair of dimension three defined over an algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is nef and $n(K_X + \Delta) \leq 2$. Then $K_X + \Delta$ is semiample.*

This, combined with [Zha17], shows log abundance when $\dim \operatorname{Alb}(X) > 0$ (see Corollary 7.4.13).

Theorem 7.1.5 (Base point free theorem over $\overline{\mathbb{F}}_p$). *Let (X, Δ) be a three-dimensional projective log canonical pair defined over $\overline{\mathbb{F}}_p$ for $p > 2$ and let L be a nef Cartier divisor such that $L - (K_X + \Delta)$ is nef and big. Then L is semiample.*

¹see Remark 7.3.9 for a clarification

Theorem 7.1.6 (Strong non-vanishing in the base point free theorem). *Let (X, Δ) be a three-dimensional projective Kawamata log terminal pair defined over an algebraically closed field of characteristic $p > 2$ and let L be a nef Cartier divisor such that $L - (K_X + \Delta)$ is nef and big. When $n(L) = 0$, assume $\dim \text{alb}(X) \neq 1$. Then $\kappa(L) = n(L)$.*

We finish off this part of the introduction by depicting the historical background. The canonical bundle formula in characteristic zero has been developed by Kodaira, Kawamata, Fujino, Mori, Shokurov, and Ambro, among others (cf. [Amb04]). The case of contractions of relative dimension one with coefficients of Δ greater than $\frac{2}{p}$ in characteristic $p > 0$ has been verified in [CTX15] (see also [DH16]). The canonical bundle formula for when the generic fibre is F-split has been obtained in [DS17], while an analogue of the canonical bundle formula for when the generic fibre is F-pure may be found in [Eji16, Theorem 1.4] (cf. [Pat14]).

To the best of our knowledge, the idea to apply the nef reduction map in the context of the base point free theorem in positive characteristic was employed for the first time in the article of Cascini, Tanaka, and Xu ([CTX15]), to which our work owns a great deal. In fact, a part of our article may be seen as a natural generalisation of [CTX15] to the case of arbitrary coefficients and $p > 2$ (cf. [CTX15, Theorem 1.9]). Certain important special cases of the abundance conjecture have been proven in [Wal17] (Kodaira dimension two), [DW16] (Kodaira dimension one and partially for nef dimension zero), and [Zha17] (when the Albanese dimension is non-zero and the boundary is empty and partially for klt boundaries).

An idea of the proof of Theorem 7.1.1 and Theorem 7.1.2

We start by providing a sketch of the proof of Theorem 7.1.1. By the assumptions on the characteristic, we can assume that the generic fibre is smooth (see Proposition 7.2.2). Let F be a general fibre of φ . If k was of characteristic zero, then $(F, \Delta|_F)$ would be log canonical. Unfortunately, this need not be true in positive characteristic, but in those cases when $(F, \Delta|_F)$ is indeed lc, then the theorem follows by the standard proof of the canonical bundle formula for fibrations of relative dimension one (see Proposition 7.3.2).

Hence, we can assume that $(F, \Delta|_F)$ is not lc. As a consequence, we can show that $\Delta = aT + B$ for an irreducible divisor T such that $\varphi|_T$ is generically purely inseparable of degree p^k for some $k \in \mathbb{N}$, an effective $\mathbb{Q}$ -divisor B such that $T \not\subseteq \text{Supp } B$, and a rational number a such that $0 < \frac{1}{p^k} < a \leq 1$. For simplicity, assume that X , Z , T and φ are smooth, and $\varphi|_T$ is purely inseparable. We are going to show that Theorem 7.1.1 holds for $f = \varphi|_T$.

To this end, we choose $t \in \mathbb{Q}_{\geq 0}$ such that $a = \frac{1}{p^k}t + (1 - t)$ and write

$$K_X + \Delta = t(K_X + \frac{1}{p^k}T) + (1 - t)(K_X + T) + B.$$

By taking a base change X_T of X via f and applying the adjunction, one can show that

$(K_X + \frac{1}{p^k}T)|_T \sim_{\mathbb{Q}} f^*K_Z$ (see Lemma 7.3.3). Therefore,

$$f^*L \sim_{\mathbb{Q}} (K_X + \Delta)|_T \sim_{\mathbb{Q}} tf^*K_Z + (1-t)K_T + B|_T,$$

which concludes the proof.

As for the proof of Theorem 7.1.2, we use the MMP to show that $P := K_Z + \lambda L$ is pseudoeffective for some $\lambda > 0$ (see Lemma 7.3.7). Then

$$\begin{aligned} (1+t\lambda)f^*L &\sim_{\mathbb{Q}} tf^*(K_Z + \lambda L) + (1-t)K_T + B|_T \\ &= (1-t)K_T + tf^*P + B|_T. \end{aligned}$$

Now, the theorem follows by composing f with some power of Frobenius.

An idea of the proof of Theorems 3–6

In what follows, we assume for simplicity that the nef reduction maps $\psi: X \dashrightarrow Z$ of $K_X + \Delta$ are proper.

Let us start with Theorem 7.1.3. We can assume that $K_X + \Delta$ is nef by replacing (X, Δ) by its minimal model. Furthermore, by bounding the lengths of extremal rays (as in [KMM94]), one can see that this theorem follows from the validity of the non-vanishing conjecture for terminal threefolds (shown in [XZ18]) and the log non-vanishing conjecture for when $n(K_X + \Delta) \leq 2$. Thus, it is enough to show Theorem 7.1.4.

We split the proof of Theorem 7.1.4 into three cases depending on the nef dimension. Assume $n(K_X + \Delta) = 0$. If $\text{Alb}(X) \neq 2$, then the log abundance follows from [Zha17]; if $\dim \text{Alb}(X) = 2$, we apply Theorem 7.1.1 to the Albanese map (see Proposition 7.4.4 for details). When $n(K_X + \Delta) = 1$, the proof is a simple consequence of the abundance for surfaces (see Proposition 7.4.5).

We are left to show log abundance when $n(K_X + \Delta) = 2$. A key component used to tackle this case is the following result, the proof of which is based on a careful study of the classification of surfaces.

Lemma 7.1.7 (Lemma 7.5.1, cf. [MNW15, Theorem 1.4]). *Let Z be a normal $\mathbb{Q}$ -factorial projective surface defined over an algebraically closed field k and let Δ_Z be a pseudo-effective $\mathbb{Q}$ -divisor such that $K_Z + \Delta_Z$ is nef of maximal nef dimension. Then $K_Z + \Delta_Z$ is $\mathbb{Q}$ -effective up to numerical equivalence.*

Using Theorem 7.1.2, we can descend $K_X + \Delta$ via the nef reduction map $\psi: X \rightarrow Z$ to an adjoint divisor $K_Z + \Delta_Z$, with Δ_Z pseudo-effective, up to replacing Z by a purely inseparable cover. Then the above lemma gives that $K_Z + \Delta_Z$, and so $K_X + \Delta$, are $\mathbb{Q}$ -effective up to numerical equivalence.

The proof of the semiampleness of $K_X + \Delta$ employs a special dlt modification, abundance for slc surfaces, and the evaporation technique. We refer the reader to Subsubsection 7.4.2 for an explanation of the idea of the proof.

The proofs of Theorem 7.1.5 and Theorem 7.1.6 are of a similar nature. The main difference is that when $n(L) = 3$, we apply [CTX15, Theorem 1.4] and (in the proof of the former theorem) the base-point-free theorem for big line bundles over $\overline{\mathbb{F}}_p$ (see [MNW15, Theorem 1.1] and [Kee99, Theorem 0.5]).

7.1.1. Outline of the chapter

In Section 7.2, we gather preliminary results: we describe the behaviour of the relative canonical divisor under base change (Proposition 7.2.4), recall special properties of varieties over $\overline{\mathbb{F}}_p$ (Subsection 7.2.1), and provide a recap of basic facts about Chow varieties (Subsection 7.2.2) which we then use to define nef reduction maps over countable fields (Subsection 7.2.3). In Section 7.3, we show Theorem 7.1.1 and 7.1.2. In Section 7.4, we prove Theorem 7.1.3, 7.1.4, 7.1.5, and 7.1.6.

7.1.2. Notation and conventions

- We say that a morphism $f: X \rightarrow Y$ is a *contraction* if X and Y are normal varieties, $f_*\mathcal{O}_X = \mathcal{O}_Y$, and f is projective and surjective.
- We call a divisor $D \subseteq X$ *horizontal* with respect to a contraction f if $f|_D$ is dominant; if this is not the case, then we call D *vertical*. We denote the horizontal and the vertical part of a $\mathbb{Q}$ -divisor Δ by Δ^h and Δ^v , respectively.
- We say that a dominant morphism $f: X \rightarrow Y$ is *purely-inseparable* if the extension of fractions fields $K(Y)/K(X)$ is purely-inseparable, in particular the identity is a purely-inseparable morphism.
- We say that a dominant morphism $f: X \rightarrow Y$ is of *relative dimension* k if the generic fibre X_μ is of dimension k , i.e. $\dim X - \dim Y = k$.
- Given a reduced scheme X of finite type over a field and its normalisation $\overline{X} \rightarrow X$, we define the *conductor* $\mathcal{D} \subseteq \overline{X}$ as the scheme defined by the maximal possible ideal sheaf $\mathcal{I}$ satisfying $\mathcal{I}\mathcal{O}_{\overline{X}} \subseteq \mathcal{O}_X$.
- Let D be a $\mathbb{Q}$ -divisor on a scheme X . Then there exists a unique decomposition $D = D_{>0} + D_{<0}$ such that $D_{>0}$ is an effective $\mathbb{Q}$ -divisor (which we call the *positive part* of D) and $D_{<0}$ is an anti-effective $\mathbb{Q}$ -divisor (which we call the *negative part* of D).
- We say that a $\mathbb{Q}$ -divisor D on a normal variety X is *pseudo-effective* if for every ample $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor A the sum $D + A$ is $\mathbb{Q}$ -linearly equivalent to an effective $\mathbb{Q}$ -divisor. Note that we do not assume that D is $\mathbb{Q}$ -Cartier. The image of a pseudo-

effective divisor under any proper birational morphism is pseudo-effective.

7.2. Preliminaries

We refer to [Sta14, Tag 0BRV] for the theory of curves over arbitrary fields.

The observation that semiample can be verified after taking a purely inseparable cover plays a fundamental role in this paper.

Lemma 7.2.1 (cf. [Kee99, Lemma 2.10]). *Let $f: X \rightarrow Y$ be a proper surjective morphism of normal varieties. Then a line bundle L is semiample if and only if f^*L is semiample.*

The following result says that Fano or Calabi-Yau fibrations of relative dimension one have smooth generic fibres in a sufficiently large characteristic.

Proposition 7.2.2. *Let X be a normal variety defined over a field of characteristic $p > 0$, and let $f: X \rightarrow Z$ be a contraction of relative dimension one with the generic fibre X_μ . Assume that*

1. $\deg K_{X_\mu} < 0$ and $p > 2$, or
2. $\deg K_{X_\mu} = 0$ and $p > 3$.

Then X_μ is smooth.

Proof. Since X is normal, X_μ is a regular curve over $\text{Spec } K(Z)$, where $K(Z)$ is the fraction field of Z . Given that $f_*\mathcal{O}_X = \mathcal{O}_Z$, we have $H^0(X_\mu, \mathcal{O}_{X_\mu}) = K(Z)$. By [Sta14, Tag 0BY6, Lemma 49.8.3], the genus of C is zero when $\deg K_{X_\mu} < 0$, and is one when $\deg K_{X_\mu} = 0$. Thus, the proposition follows from [CTX15, Lemma 6.5] and [Zha17, Proposition 2.9]. $\square$

We recall Raynaud's flatification.

Theorem 7.2.3 ([RG71, Thm 5.2.2]). *Let $\pi: X \rightarrow S$ be a morphism of varieties such that there exists an open subset $U \subseteq S$ for which $\pi|_{\pi^{-1}(U)}$ is flat. Then, there exists a birational morphism $h: S' \rightarrow S$ and a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow \pi' & & \downarrow \pi \\ S' & \xrightarrow{h} & S, \end{array}$$

such that π' is flat, where X' is the closure of the generic fibre X_μ of π in $X \times_S S'$.

In particular, if S is normal, then there exists an open subset $U \subseteq S$, whose complement is of codimension at least two, such that π is flat over U . Let us note that every morphism with equidimensional fibres between regular schemes is flat.

Last, we describe the behaviour of the relative canonical divisor under base change.

Proposition 7.2.4 (cf. [CZ15, Proposition 2.1 and Theorem 2.4]). *Let $\varphi: X \rightarrow Z$ be a flat contraction between Gorenstein quasi-projective varieties. Let $f: T \rightarrow Z$ be a generically finite surjective morphism from a Gorenstein quasi-projective variety T such that $X' := X \times_Z T$ is reduced and let $\pi: Y \rightarrow X'$ be the normalisation of X' sitting inside the following commutative diagram*

$$\begin{array}{ccccc}
 & & h & & \\
 & \nearrow & & \searrow & \\
 Y & \xrightarrow{\pi} & X' & \xrightarrow{g} & X \\
 & \searrow & \downarrow \varphi' & & \downarrow \varphi \\
 & & T & \xrightarrow{f} & Z.
 \end{array}$$

*Then $h^*K_{X/Z} = K_{Y/T} + D$, where D is an effective divisor supported on the conductor of π .*

Proof. By [Sta14, Tag 0C02, Lemma 46.26.7], the morphism φ is Gorenstein. Thus φ' is Gorenstein as well ([Sta14, Tag 0C02, Lemma 46.26.8]), and so is X' by [Sta14, Tag 0C02, Lemma 46.26.6]. Further, [CZ15, Proposition 2.3] implies that $g^*\omega_{X/Z} = \omega_{X'/T}$. We can conclude the proof by [Rei94, Proposition 2.3] (cf. [Tan16b, Proposition 2.10]). $\square$

Remark 7.2.5. Note that X' in the statement of Proposition 7.2.4 is reduced if and only if $X \times_{K(Z)} K(T)$ is reduced, where $K(Z)$ and $K(T)$ are the fraction fields of Z and T , respectively. Indeed, since φ is a flat morphism between reduced schemes, it is S_1 , and hence φ' is S_1 as well. Thus X' has no embedded points.

Recall that a finite universal homeomorphism is a morphism which is a homeomorphism under any base change. In particular, purely inseparable morphisms are finite universal homeomorphisms.

Lemma 7.2.6. *Let $f: X \rightarrow Y$ be a finite universal homeomorphism between schemes of finite type over a field k of characteristic $p > 0$ and let L be a nef Cartier divisor on Y . Then L is semiample if and only if f^*L is semiample. In particular, L is semiample if and only if $L|_{Y^{\text{red}}}$ is semiample where Y^{red} is the reduction of Y .*

Proof. This is [Kee99, Lemma 1.4]. Note that the inclusion $Y^{\text{red}} \subseteq Y$ is a finite universal homeomorphism. $\square$

7.2.1. Special properties of varieties over $\overline{\mathbb{F}}_p$

We list some results on line bundles on varieties over $\overline{\mathbb{F}}_p$ which are only valid over this field.

Lemma 7.2.7 ([Kee99, Lemma 2.16]). *Any numerically trivial line bundle on a variety X defined over $\overline{\mathbb{F}}_p$ is torsion.*

On normal projective surfaces over $\overline{\mathbb{F}}_p$, nef line bundles are semiample under very weak conditions. Note that such surfaces are always $\mathbb{Q}$ -factorial (see [Tan14, Theorem 4.5]).

Theorem 7.2.8 ([Art62, Theorem 2.9], cf. [Kee99, Corollary 0.3]). *If D is a nef and big divisor on a normal projective surface S defined over $\overline{\mathbb{F}}_p$, then it is semiample.*

Theorem 7.2.9. *Let S be a normal projective surface defined over $\overline{\mathbb{F}}_p$ and let Δ be a pseudo-effective $\mathbb{Q}$ -divisor on S . Assume that $K_S + \Delta$ is nef. Then it is semiample.*

Proof. This is [MNW15, Theorem 1.4] as we can write $\Delta = N + P$ for a nef $\mathbb{Q}$ -divisor N and an effective $\mathbb{Q}$ -divisor P by Zariski decomposition (see [Laz04a]). $\square$

7.2.2. Chow varieties

In this subsection, we recall basic properties of Chow varieties and algebraic equivalence relations closely following [Kol96, Chapter I]. The theory is used in Subsection 7.2.3 to construct nef reduction maps over countable algebraically closed fields. Let X be a fixed projective variety defined over an algebraically closed field k .

We define an *effective d -dimensional algebraic cycle* on X to be a finite formal sum $\sum a_i[V_i]$, with $a_i \in \mathbb{N}$, and V_i irreducible, reduced, and closed d -dimensional subschemes.

Definition 7.2.10. A *well defined family of effective proper d -dimensional algebraic cycles* of X over a k -variety W , denoted $(g: U \rightarrow W)$, is an effective algebraic cycle $U = \sum m_i[U_i]$ on X (here U_i are its irreducible components) together with a proper morphism $g: \text{Supp } U \rightarrow W$ to a reduced scheme W such that for every i the restriction $g|_{U_i}$ is surjective onto an irreducible component of W with fibres of dimension d . Further, we assume a technical condition necessary for the existence of the cycle theoretic fibre $g^{[-1]}(w)$ at all $w \in W$ (see [Kol96, I, 3.10.4] for details).

We say that a well defined family of algebraic cycles $(g: U \rightarrow W)$ satisfies the *field of definition condition* if $g^{[-1]}(w)$ is defined over $k(w)$ for every $w \in W$. In particular, if $W = \text{Spec } k$ for an algebraically closed field k , then the field of definition condition is automatically satisfied. Accordingly, we can define:

$$\text{Chow}^{\text{small}}(X)(Z) := \left\{ \begin{array}{l} \text{Well defined algebraic families} \\ \text{of proper effective cycles} \\ \text{of } X \times Z \text{ over } Z \text{ which satisfy} \\ \text{the field of definition condition.} \end{array} \right\}.$$

This is a well defined functor on the category of semi-normal schemes ([Kol96, Definition I, 7.2.1]).

Fix an ample Cartier divisor H on X . For $d, d' \in \mathbb{N}$, we define $\text{Chow}_{d,d'}^{\text{small}}(X)$ to be the subfunctor of $\text{Chow}^{\text{small}}(X)$ describing families of effective cycles of dimension d and degree d' with respect to H .

Theorem 7.2.11 ([Kol96, IV, Theorem 4.13]). *Let X be a projective variety over a field k . Then there exists a semi-normal projective scheme $\text{Chow}_{d,d'}(X)$ coarsely representing $\text{Chow}_{d,d'}^{\text{small}}(X)$.*

Unfortunately, the universal family over $\text{Chow}_{d,d'}(X)$ need not satisfy the field of definition condition, and so it is not a cycle in $\text{Chow}_{d,d'}^{\text{small}}(X)$.

Now we turn our attention to equivalence relations defined by subschemes of $\text{Chow}_{1,d'}(X)$ for a normal projective variety X . We refer to [Kol96, IV, Definition 4.2] for the definition of a proalgebraic relation. Given a proper subscheme $V \subseteq \text{Chow}_{1,d'}(X)$ parametrizing connected 1-cycles, one can construct a proalgebraic proper connected relation R (see [Kol96, IV, Theorem 4.8]) which identifies two points $x_1, x_2 \in X$ if and only if there exists a connected curve C such that $x_1, x_2 \in C$ and C is build-up from cycles represented by V (see [Kol96, IV, Definition 4.7] for the formal statement). The proalgebraic relation R is a countable union of proper algebraic relations $R_1, R_2, R_3, \dots$.

Theorem 7.2.12 ([Kol96, IV, Theorem 4.16]). *Let m be any natural number. Then the quotient of a normal projective variety X by $R_1, \dots, R_m$ as above exists. More precisely, there exists an open subvariety $X^0 \subseteq X$ and a proper morphism $\pi: X^0 \rightarrow Z^0$ with connected fibres such that:*

- R is an equivalence relation on X^0 , and
- $\pi^{-1}(z)$ coincides with an R -equivalence class for every $z \in Z^0$.

When R is the union of $R_1, \dots, R_m$, this says that a curve $C \subseteq X$ intersecting X^0 is contracted by π if and only if C is build-up from curves represented by V .

If R is merely a countable union of algebraic relations, then the quotient exists as well (see [Kol96, IV, Theorem 4.17]), but $\pi^{-1}(z)$ coincides with an R -equivalence class only for a very general $z \in Z^0$. This is an easy consequence of the above theorem as the quotients by $R_1, \dots, R_m$ must stabilise for $m \gg 0$.

7.2.3. Nef reduction map

This subsection is based on [BCE⁺02]. The results therein are stated for complex algebraic varieties, but the proofs are valid for all varieties defined over uncountable algebraically closed fields.

Definition 7.2.13. Let X be a normal projective variety defined over an uncountable field k and let L be a nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor. We call a rational map $\varphi: X \dashrightarrow Z$ a *nef reduction map* if Z is a normal projective variety and there exists an open dense subset $V \subseteq Z$ such that

1. φ is proper over V and $\varphi_*\mathcal{O}_{\varphi^{-1}(V)} = \mathcal{O}_V$,
2. $L|_F \equiv 0$ for all fibres F of φ over V , and
3. if $x \in X$ is a very general point and C is a curve passing through it, then $C \cdot L = 0$ if and only if C is contracted by φ .

Theorem 7.2.14 ([BCE⁺02, Theorem 2.1]). *A nef reduction map exists for normal projective varieties defined over an uncountable algebraically closed field k . Further, it is unique up to a birational morphism.*

We call $n(L) = \dim Z$ the *nef dimension* of L , where Z is the target of a nef reduction map $\varphi: X \dashrightarrow Z$.

The following result is fundamental in the proof of Theorem 7.2.14.

Theorem 7.2.15 ([BCE⁺02, Theorem 2.4]). *Let X be a projective variety, non-necessarily normal, defined over an uncountable algebraically closed field k . Let L be a nef line bundle on X . Then L is numerically trivial if and only if any two points in X can be joined by a connected chain C of curves such that $L \cdot C = 0$.*

Now, let X be a normal projective variety defined over a countable algebraically closed field k , for instance $k = \overline{\mathbb{F}}_p$ or $k = \overline{\mathbb{Q}}$, and let L be a nef line bundle on it. The definition of a nef reduction map makes no sense for such X as the set of its very general points is empty.

To circumvent this problem, we choose any uncountable algebraically closed field K containing k , and take X_K, L_K to be the base changes of X and L to $\text{Spec } K$, respectively. Thereafter, we can define the nef dimension as $n(L) := n(L_K)$. In fact, the nef reduction map of L_K is defined over k , as the following result shows.

Proposition 7.2.16. *With notation as above, there exist a rational map $\varphi: X \dashrightarrow Z$ to a normal projective variety Z and an open dense subset $V \subseteq Z$ such that*

1. φ is proper over V and $\varphi_* \mathcal{O}_{\varphi^{-1}(V)} = \mathcal{O}_V$,
2. $L|_F \equiv 0$ for all fibres F of φ over V , and
3. if φ_K is the base change of φ to $\text{Spec } K$, then φ_K is a nef reduction map of L_K .

In particular, $\dim Z = n(L)$.

Morphisms φ as above are independent of the choice of K .

Proof. Let $V' \subseteq \text{Chow}(X_K)$ be the countable union of all irreducible components of the Chow variety parametrising one-cycles C satisfying $L_K \cdot C = 0$. Now, if $\varphi_K: X_K \dashrightarrow Z'$ is any nef reduction map of L_K , then it must be a quotient by the algebraic equivalence relation induced by some irreducible components $V'_1, \dots, V'_r$ of V' . By definition, $\text{Chow}(X_K) \simeq \text{Chow}(X) \times_k \text{Spec } K$, and so there exist irreducible components $V_1, \dots, V_r \subseteq \text{Chow}(X)$ such that $V'_i \simeq V_i \times_k \text{Spec } K$. Then, the assertion of the proposition is satisfied for φ being the quotient of X by the algebraic equivalence relation induced by $V_1, \dots, V_r$ (see Theorem 7.2.12). $\square$

Definition 7.2.17. We call a morphism satisfying the assertions of Proposition 7.2.16 a *nef reduction map* of L on X .

It is natural to ask if a nef line bundle descends under its nef reduction map. This is known to hold when the restriction of L to the generic fibre is torsion, or when $n(L) = 1$.

Lemma 7.2.18. *Let $f: X \rightarrow Z$ be an equidimensional proper morphism between normal projective varieties. Let L be a nef $\mathbb{Q}$ -Cartier divisor on X such that $L|_{X_\mu} \sim_{\mathbb{Q}} 0$, where X_μ is the generic fibre. Then, there exists a $\mathbb{Q}$ -divisor L_Z on Z such that $L \sim_{\mathbb{Q}} f^*L_Z$.*

Proof. By assumptions, there exists a $\mathbb{Q}$ -divisor L_Z on Z and an effective $\mathbb{Q}$ -divisor E on Z such that:

- $L + E \sim_{\mathbb{Q}} f^*L_Z$,
- $\dim f(E) < \dim Z$, and
- for any irreducible component $V \subseteq f(E)$, we have $f^{-1}V \not\subseteq \text{Supp } E$.

By Zariski's lemma (more precisely [Fuj86, Lemma 1.5]), $E = 0$, which concludes the proof. $\square$

Note that f^*L_Z is well defined even when L_Z is not $\mathbb{Q}$ -Cartier, because f has equidimensional fibres.

Lemma 7.2.19. *Let X be a normal projective three-dimensional variety defined over an algebraically closed field k , let C be a smooth curve, let $f: X \rightarrow C$ be a contraction, and let X_μ be the generic fibre. Further, let L be a nef $\mathbb{Q}$ -Cartier divisor on X such that $L|_{X_\mu} \equiv 0$. Then there exists a $\mathbb{Q}$ -divisor D on C such that $L \equiv f^*D$.*

Proof. By [BW17, Lemma 5.3], we have $L \equiv_f 0$. Replacing X by a resolution of singularities, we can assume that X is smooth. Thus we can conclude the proof by [BW17, Lemma 5.2]. $\square$

Last, let us state the following result which allows for applying the nef reduction map in the context of the base point free theorem.

Theorem 7.2.20 ([CTX15, Theorem 1.4]). *Let (X, Δ) be a projective log pair defined over an algebraically closed field k of characteristic $p > 0$, and let L be a nef $\mathbb{Q}$ -Cartier divisor such that $L - (K_X + \Delta)$ is nef and big and $n(L) = \dim X$. Then L is big.*

7.3. A weak canonical bundle formula

The goal of this section is to show Theorem 7.3.4 (cf. Theorem 7.1.1) and Theorem 7.3.8 (cf. Theorem 7.1.2).

7.3.1. The canonical bundle formula for boundaries separable over the base

In this subsection, we derive the weak version of the canonical bundle formula for when $(X_{\bar{\mu}}, \Delta|_{X_{\bar{\mu}}})$ is log canonical where $X_{\bar{\mu}}$ is the geometric generic fibre of φ (Proposition 7.3.2). To this end, we need the following result.

Lemma 7.3.1. *Let (X, Δ) be a quasi-projective log pair defined over an algebraically closed field k of characteristic $p > 0$ and let $\varphi: X \rightarrow Z$ be a contraction. Assume that the generic fibre X_μ of φ is a smooth rational curve, the coefficients of Δ^h are at most one, and $K_X + \Delta \sim_{\mathbb{Q}} \varphi^* L_Z$ for some $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor L_Z on Z . Further, suppose that $\text{Supp } \Delta^h$ is a union of sections of φ over the generic point of Z . Then $L_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$ for some effective $\mathbb{Q}$ -divisor Δ_Z .*

This lemma follows from the standard canonical bundle formula with $\text{Supp } \Delta^h$ separable and tamely ramified over Z , a result which is verified in [CTX15, Subsection 6.2] (see also [DH16, Theorem 4.8] and [PS09]). Thus, we only provide a sketch of the argument.

Proof. We are going to show that $\Delta_Z = \Delta_{\text{div}} + \Delta_{\text{mob}}$, where Δ_{mob} is an effective $\mathbb{Q}$ -divisor and

$$\Delta_{\text{div}} := \sum_{\text{Weil divisor } Q} (1 - c_Q) Q, \text{ for}$$

$$c_Q := \sup \{c \in \mathbb{Q} \mid (X, \Delta + c\varphi^* Q) \text{ is lc over the generic point of } Q\}.$$

As $c_Q \leq 1$, we get that Δ_{div} is effective.

Consider the following diagram:

$$\begin{array}{ccccc} & & & \overline{X} & \\ & & \swarrow \pi_2 & \downarrow \pi_1 & \\ \overline{U}_{0,m} & \xleftarrow{\psi} & U & \dashrightarrow & X \\ & \downarrow \gamma & \searrow \varphi_U & & \downarrow \varphi \\ \overline{M}_{0,m} & \xleftarrow{u} & & & Z, \end{array}$$

where

- $\overline{M}_{0,m}$ is the moduli space of m -pointed stable curves of genus zero,
- $\gamma: \overline{U}_{0,m} \rightarrow \overline{M}_{0,m}$ is a $\mathbb{P}^1$ -bundle constructed by blowing-down divisors on the universal family (see [DH16, Lemma 4.6]),
- $u: Z \dashrightarrow \overline{M}_{0,m}$ is a map given by the fact that $\varphi: X \rightarrow Z$ is generically a $\mathbb{P}^1$ -bundle with m sections $\text{Supp } \Delta^h$,
- $U := \overline{U}_{0,m} \times_{\overline{M}_{0,m}} Z$,
- $\pi_1: \overline{X} \rightarrow X$ is a birational resolution of the rational map $U \dashrightarrow X$.

Since effective divisors extend in codimension one, we can replace Z by a smooth open subset $Z' \subseteq Z$ whose complement is of codimension two and X by $X \times_Z Z'$ so that $u: Z \rightarrow \overline{M}_{0,m}$ is a well defined morphism. Since Z is smooth, $K_Z + \Delta_{\text{div}}$ is a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor.

To prove the lemma, it is enough to show that $K_X + \Delta - \varphi^*(K_Z + \Delta_{\text{div}})$ is semiample. Define $\Delta_{\overline{X}}$ as the log pullback of Δ via π_1 , and set

$$\Delta_U := (\pi_2)_* \Delta_{\overline{X}}.$$

By construction of $\overline{U}_{0,m}$ and by the two-dimensional inversion of adjunction, we have that $(U, \Delta_U^h + \varphi_U^{-1}Q)$ are lc over the generic points of irreducible divisors $Q \subseteq \text{Supp } \Delta_{\text{div}}$ (cf. [DH16, (4.11)]). Since φ_U is a $\mathbb{P}^1$ -fibration, this implies that $\Delta_U^v = \varphi_U^* \Delta_{\text{div}}$. Thus

$$K_U + \Delta_U - (\varphi_U)^*(K_Z + \Delta_{\text{div}}) \sim_{\mathbb{Q}} \psi^*(K_{\overline{U}_{0,m}/\overline{M}_{0,m}} + D)$$

where D is a sum of the canonical sections of γ with rational coefficients in $[0, 1]$ and $K_{\overline{U}_{0,m}/\overline{M}_{0,m}} + D$ is semiample by [DH16, Lemma 4.6]. This finishes the proof of the lemma. $\square$

The following result provides the first step in the proof of Theorem 7.3.4.

Proposition 7.3.2. *Let (X, Δ) be a quasi-projective log pair defined over an algebraically closed field k of characteristic $p > 0$ and let $\varphi: X \rightarrow Z$ be a contraction. Assume that the geometric generic fibre $X_{\bar{\mu}}$ of φ is a smooth curve, $(X_{\bar{\mu}}, \Delta|_{X_{\bar{\mu}}})$ is log canonical, and $K_X + \Delta \sim_{\mathbb{Q}} \varphi^* L_Z$ for some $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor L_Z on Z . Then $L_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$ for an effective $\mathbb{Q}$ -divisor Δ_Z .*

If we only require Δ_Z to be pseudo-effective, then the proposition follows from [Eji16, Theorem 4.5] (cf. [Eji16, Corollary 4.11]). Note that in characteristic zero, if (X, Δ) is log canonical, then so is $(X_{\bar{\mu}}, \Delta|_{X_{\bar{\mu}}})$, but this need not be true in positive characteristic due to the existence of inseparable morphisms.

Proof. By blowing up along components of $\text{Sing}(X)$ of codimension two in X which are not contracted by φ , we can obtain a birational morphism $g: Y \rightarrow X$ such that $g^*(K_X + \Delta) = K_Y + \Delta_Y$ for a $\mathbb{Q}$ -divisor Δ_Y satisfying

$$\text{codim}_Z (\varphi \circ g (\text{Sing } Y \cup \text{Supp } (\Delta_Y)_{<0})) \geq 2,$$

where $(\Delta_Y)_{<0}$ is the negative part of Δ_Y . Indeed, consider the localisations X_{ζ_i} of X at the generic points of the components of $\text{Sing}(X)$ described above. Since X_{ζ_i} is of dimension two, we can construct its minimal resolution by a series of blow-ups. Extending the blown-up loci to X and taking normalisation provides us with Y as above. We replace X by Y and Δ by Δ_Y . Further, we replace Z by a smooth open subscheme $U \subseteq Z$ whose complement is of codimension two, and X by $\varphi^{-1}(U)$. Thus, we can assume that φ is flat, $\Delta \geq 0$, and X, Z are smooth.

Since $(X_{\bar{\mu}}, \Delta|_{X_{\bar{\mu}}})$ is log Calabi-Yau, $X_{\bar{\mu}}$ is either a smooth rational curve or an elliptic curve. In the latter case, $\kappa(K_{X/Z}) \geq 0$ by [CZ15, Subsection 3.1 and Claim 3.1]. Thus, we can write $K_X + \Delta = \varphi^* K_Z + E + \Delta$ for some effective $\mathbb{Q}$ -divisor $E \sim_{\mathbb{Q}} K_{X/Z}$, and by

the Zariski lemma (see [Fuj86, Lemma 1.5]), we get that $E + \Delta = \varphi^* \Delta_Z$ for some effective $\mathbb{Q}$ -divisor Δ_Z on Z . This finishes the proof when $X_{\bar{\mu}}$ is an elliptic curve.

As of now, assume that $X_{\bar{\mu}} \simeq \mathbb{P}^1$, and consider the following Cartesian diagram:

$$\begin{array}{ccc} X_T & \xrightarrow{\psi} & X \\ \downarrow \varphi_T & & \downarrow \varphi \\ T & \xrightarrow{f} & Z, \end{array}$$

where $f: T \rightarrow Z$ is any finite morphism with T normal which generically factors through $\varphi|_{D_i}: D_i \rightarrow Z$ for every irreducible divisor $D_i \subseteq \text{Supp } \Delta^h$ (for instance, set T to be the normal closure of Z in all the fraction fields $K(D_i)$), and X_T is the normalisation of $X \times_Z T$. By replacing Z by a smaller open subscheme with complement of codimension two, we can assume that T is smooth. Then Proposition 7.2.4 implies that

$$\psi^*(K_X + \Delta) \sim_{\mathbb{Q}} K_{X_T/T} + \Delta_T + \varphi_T^* f^* K_Z$$

for some effective $\mathbb{Q}$ -divisor Δ_T . Since the condition that $(X_{\bar{\mu}}, \Delta|_{\bar{\mu}})$ is log canonical is stable under base change, the coefficients of the horizontal part Δ_T^h are at most one. Moreover $\text{Supp } \Delta_T^h$ is a union of sections of φ_T over the generic point of T . Thus by Lemma 7.3.1 we have

$$\psi^*(K_X + \Delta) \sim_{\mathbb{Q}} \varphi_T^*(f^* K_Z + E),$$

where E is some effective $\mathbb{Q}$ -divisor on T . Therefore,

$$L_Z \sim_{\mathbb{Q}} K_Z + \frac{1}{\deg f} f_* E. \quad \square$$

7.3.2. The proof of the weak canonical bundle formula

In the proof of Theorem 7.3.4, we shall need the following lemma.

Lemma 7.3.3 (Purely inseparable relative adjunction). *Let X be a regular quasi-projective variety defined over a field k of characteristic $p > 0$, let $\varphi: X \rightarrow Z$ be a contraction whose generic fibre is a smooth curve, let $S \subseteq X$ be a divisor such that $\varphi|_S: S \rightarrow Z$ is purely inseparable of degree p^k , and let $\tilde{S}$ be the normalisation of S . Then,*

$$(K_{X/Z} + \frac{1}{p^k} S)|_{\tilde{S}} \sim_{\mathbb{Q}} \Delta,$$

for some effective $\mathbb{Q}$ -divisor Δ on $\tilde{S}$.

In fact, the lemma holds if X is only $\mathbb{Q}$ -Gorenstein in codimension two (see the beginning of the proof of Proposition 7.3.2). By means of localisation at codimension two points, one can see that all klt varieties satisfy this condition.

Proof. We can assume that Z is regular and φ is flat by replacing Z by an open regular subset $U \subseteq Z$ whose complement is of codimension two, and X by $\varphi^{-1}(U)$. To ease the

notation, write $T := \tilde{S}$ and $f := \varphi|_{\tilde{S}}$.

Consider the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow \varphi' & & \downarrow \varphi \\ T & \xrightarrow{f} & Z, \end{array}$$

where Y is the normalisation of $X \times_Z T$. The construction yields a natural induced-by- f section $T \rightarrow Y$ whose image by abuse of notation we denote by T .

By Proposition 7.2.4, there exists an effective φ' -vertical divisor D such that

$$g^*(K_{X/Z} + \frac{1}{p^k}S) = K_{Y/T} + T + D.$$

The lemma follows by the standard adjunction on T . $\square$

We are ready to prove the main theorem of this section.

Theorem 7.3.4 (Theorem 7.1.1). *Let (X, Δ) be a quasi-projective log pair defined over an algebraically closed field k of characteristic $p > 0$ and let $\varphi: X \rightarrow Z$ be a contraction with the generic fibre being a smooth curve. Assume that the φ -horizontal divisors in Δ have coefficients at most one, and $K_X + \Delta \sim_{\mathbb{Q}} \varphi^*L_Z$ for some $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor L_Z on Z .*

Then, there exist a finite purely-inseparable morphism $f: T \rightarrow Z$ and an effective $\mathbb{Q}$ -divisor E on T , such that

$$f^*L_Z \sim_{\mathbb{Q}} tK_T + (1-t)f^*K_Z + E,$$

for some rational number $0 \leq t \leq 1$.

Furthermore, if $\Delta = B + A$ for an effective $\mathbb{Q}$ -divisor B and an ample $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor A , then E is big.

If $p > 3$, or Δ is big and $p > 2$, then the generic fibre is automatically smooth by Proposition 7.2.2.

Proof. By the same argument as in Proposition 7.3.2 we can blow up along components of $\text{Sing}(X)$ of codimension two in X which are not contracted by φ , and obtain a birational morphism $g: Y \rightarrow X$ such that $g^*(K_X + \Delta) = K_Y + \Delta_Y$ for a $\mathbb{Q}$ -divisor Δ_Y satisfying

$$\text{codim}_Z(\varphi \circ g(\text{Sing } Y \cup \text{Supp}(\Delta_Y)_{<0})) \geq 2,$$

where $(\Delta_Y)_{<0}$ is the negative part of Δ_Y . We replace X by Y and Δ by Δ_Y so that we have $\text{codim}_Z(\varphi(\text{Sing } X \cup \text{Supp } \Delta_{<0})) \geq 2$.

Let $L := K_X + \Delta$ and let F be a general fibre of φ . Note that since $(K_X + \Delta) \cdot F = 0$, we have $\Delta \cdot F \leq 2$. If $(F, \Delta|_F)$ is log canonical, then the theorem follows from Proposition 7.3.2 with $f = \text{id}$. Thus, we can assume that it is not log canonical, and so we can write

$\Delta = aS + B$ for $0 < a \leq 1$, an irreducible divisor S , and an effective $\mathbb{Q}$ -divisor B such that $S \not\subseteq \text{Supp } B$ and

$$(F, aS|_F)$$

is not log canonical. We claim that $\varphi|_S$ is generically purely inseparable. To this end, write $S|_F = \sum_{i=1}^r p^k P_i$ for some distinct points $P_i \in F$, where p^k is the degree of the purely inseparable part of the extension $K(S)/K(Z)$. Since $aS \cdot F \leq 2$, we have $arp^k \leq 2$, and since $(F, aS|_F)$ is not log canonical, $ap^k > 1$ holds. Therefore, $r = 1$ and $\varphi|_S$ is generically purely inseparable of degree $p^k > 1$. Further, $\frac{1}{p^k} < a \leq \frac{2}{p^k}$.

Let $\tilde{S}$ be the normalisation of S and let

$$\begin{array}{ccccc} & & h & & \\ & \nearrow & & \searrow & \\ \tilde{S} & \xrightarrow{g} & T & \xrightarrow{f} & Z. \end{array}$$

be the Stein factorisation of $h := \varphi|_{\tilde{S}}$, wherein g is birational. Since divisors uniquely extend in codimension one and $\text{codim}_Z(\varphi(\text{Sing } X \cup \Delta_{<0})) \geq 2$, we can assume that g is an isomorphism, $\Delta \geq 0$ and X, Z are smooth by replacing Z by an open smooth subset $U \subseteq Z$ whose complement is of codimension two and $X, S, \tilde{S}$ by $\varphi^{-1}(U)$, $\varphi^{-1}(U) \cap S$, and $h^{-1}(U)$, respectively.

Recall that $\Delta = aS + B$, take $0 \leq t < 1$ to be a rational number such that $a = \frac{1}{p^k}t + (1-t)$, and write

$$L = t(K_X + \frac{1}{p^k}S) + (1-t)(K_X + S) + B.$$

By Lemma 7.3.3 and the standard adjunction we get

$$h^*L_Z \sim_{\mathbb{Q}} L|_{\tilde{S}} \sim_{\mathbb{Q}} th^*K_Z + (1-t)K_{\tilde{S}} + E_{\tilde{S}}$$

for some effective $\mathbb{Q}$ -divisor $E_{\tilde{S}}$. Since g is an isomorphism, this finishes the proof of the main part of the theorem.

Last, if $\Delta = B + A$ as in the statement of the theorem, then $A \sim_{\mathbb{Q}} A' + \varphi^*A_Z$ for some sufficiently general ample $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisors A' and A_Z on X and Z , respectively. By applying the above proof to Δ replaced by $B + A'$, we can conclude. $\square$

Example 7.3.5. It is easy to see that the canonical bundle formula is false in positive characteristic even for smooth $\mathbb{P}^1$ -fibrations from surfaces to curves (cf. [Har77, V, Exercise 2.15]). Here, we give an example that in Theorem 7.3.4 it is moreover not enough to take f to be a power of Frobenius.

Let S be a smooth projective surface defined over an algebraically closed field of characteristic two such that K_S is ample, $H^1(S, \omega_S^{-1}) \neq 0$, but $H^1(S, \omega_S^{-2}) = 0$. Such a surface exists by [Eke88, Proposition I.2.14 and Theorem II.1.7]. Let $\mathcal{E}$ be a vector bundle constructed as a non-trivial extension of ω_S by $\mathcal{O}_S$ and let $X := \mathbb{P}_S(\mathcal{E})$. By assumptions, $F^*\mathcal{E}$

splits providing a divisor D on X such that $\pi|_D$ is purely inseparable of degree two, where $\pi: X \rightarrow S$ is the natural projection.

We claim that $L := K_X + D$ is numerically trivial. Since S is of general type, the claim shows that L cannot descend to a $\mathbb{Q}$ -divisor of the form $\lambda K_S + P$, where $\lambda > 0$ and P is pseudo-effective. For the assertion of Theorem 7.3.4 to hold one takes f to be a purely-inseparable cover of S by D .

In order to show that L is numerically trivial, we consider the following Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow \rho & & \downarrow \pi \\ S & \xrightarrow{F} & S, \end{array}$$

where $X' := \mathbb{P}_S(F^*\mathcal{E})$. Since π is smooth, so is ρ . In particular,

$$\varphi^*L = \varphi^*K_{X/S} + \rho^*F^*K_S + 2S = K_{X'} + \rho^*K_S + 2S,$$

wherein $S \subseteq X'$ is a section of ρ such that $S|_S \sim -2K_S$. Thus,

$$\varphi^*L|_S \sim 0,$$

and since π is a $\mathbb{P}^1$ -fibration and L is π -numerically trivial, it must be numerically trivial.

Remark 7.3.6. Theorem 7.3.4 and Proposition 7.3.2 should work over any field k of positive characteristic $p > 0$ (non necessarily algebraically closed) contingent upon the verification that the results in [CZ15, Subsection 3.1 and Claim 3.1] are valid over k .

7.3.3. Low dimensions

In this subsection, we show Theorem 7.3.8 (cf. Theorem 7.1.2). The following lemma is a key component in its proof.

Lemma 7.3.7. *Let X be a normal projective n -dimensional variety defined over an algebraically closed field k of characteristic $p > 0$ and let L be a nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on X such that $n(L) = \dim X$. Assume that $n = 2$, or $p > 5$ and $n = 3$. Then, there exists a rational number $\lambda > 0$ for which $K_X + \lambda L$ is pseudo-effective.*

The lemma is not true when $n(L) < \dim X$.

Proof. Let $\pi: \overline{X} \rightarrow X$ be a resolution of singularities. Assume that there exists $\lambda > 0$ such that $K_{\overline{X}} + \lambda\pi^*L$ is pseudo-effective. Then, $K_X + \lambda L$ is pseudo-effective as well. Thus, by replacing X by $\overline{X}$, we can assume that X is terminal and $\mathbb{Q}$ -factorial. We do not assume that X is smooth for inductive reasons.

First, we show the lemma under an additional assumption that all K_X -negative extremal rays R satisfy $L \cdot R > 0$. To this end, set $\lambda = 2nm$ for the Cartier index $m \in \mathbb{N}$ of L .

It is enough to show that $K_X + \lambda L$ is nef. By contradiction, assume it is not. Then, there exists a curve C such that $(K_X + \lambda L) \cdot C < 0$. Since $L \cdot C \geq 0$, we must have $K_X \cdot C < 0$. By the cone theorem, C is an affine combination of K_X -negative extremal rays. In particular, there must exist a K_X -negative extremal curve R satisfying $(K_X + \lambda L) \cdot R < 0$. However, by the additional assumption, $mL \cdot R \geq 1$, hence

$$(K_X + \lambda L) \cdot R \geq 0,$$

since R belongs to an extremal ray and so it satisfies $R \cdot K_X \geq -2n$. This is a contradiction.

In general we run a K_X -MMP which contracts only those rays R which satisfy $L \cdot R = 0$. Since $n(L) = \dim X$ and X has terminal singularities this MMP terminates with a birational model (cf. [HX15]). Let $f: X \dashrightarrow Y$ be the first step of the MMP and let $L_Y := f_*L$. By induction (the base of the induction is satisfied by the above paragraph), we can assume that $P_Y := K_Y + \lambda L_Y$ is pseudo-effective for some $\lambda > 0$. If f is a flip, then $K_X + \lambda L = f^*P_Y$ is pseudo-effective as well. If f is a contraction of a divisor D , then

$$K_X + \lambda L \sim_{\mathbb{Q}} f^*P_Y + aD$$

for some $a \in \mathbb{Q}$. Since $K_X \cdot R < 0$, we have $a > 0$. This concludes the proof of the lemma. $\square$

Theorem 7.3.8 (Theorem 7.1.2). *Let (X, Δ) be a projective n -dimensional log pair defined over an algebraically closed field k of characteristic $p > 0$ and let $\varphi: X \rightarrow Z$ be a contraction with the generic fibre being smooth and of dimension one. Suppose that the φ -horizontal divisors in Δ have coefficients at most one, $K_X + \Delta$ is nef, and $K_X + \Delta \sim_{\mathbb{Q}} \varphi^*L_Z$ for some $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor L_Z on Z . Further, suppose that $n(K_X + \Delta) = n - 1$ or more generally that $K_Z + \lambda L_Z$ is pseudo-effective for some $\lambda > 0$.*

Assume that $n \leq 3$, or $p > 5$ and $n = 4$. Then, there exists a finite purely-inseparable morphism $f: T \rightarrow Z$, with T normal, and a pseudo-effective $\mathbb{Q}$ -divisor Δ_T on T such that

$$f^*L_Z \sim_{\mathbb{Q}} K_T + \Delta_T.$$

Furthermore, if $\Delta = B + A$ for an effective $\mathbb{Q}$ -divisor B and an ample $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor A , then Δ_T is big.

We do not know if the assumption that $n(K_X + \Delta) = \dim Z$ (or more generally that $K_Z + \lambda L_Z$ is pseudo-effective) is necessary.

Proof. By Theorem 7.3.4, there exists a purely inseparable morphism $f: T \rightarrow Z$ such that

$$L_T := f^*L_Z \sim_{\mathbb{Q}} t f^*K_Z + (1 - t)K_T + E$$

for some effective $\mathbb{Q}$ -divisor E on T and $0 \leq t \leq 1$. If $t = 1$, then $L_Z \sim_{\mathbb{Q}} K_Z + \frac{1}{\deg f} f_*E$. Thus, we can assume that $t < 1$. Lemma 7.3.7 implies that there exists $\lambda > 0$ such that

$P := K_Z + \lambda L_Z$ is pseudo-effective. Therefore,

$$\begin{aligned} (1 + \lambda)L_T &\sim_{\mathbb{Q}} tf^*(K_Z + \lambda L_Z) + (1 - t)K_T + E \\ &= (1 - t)K_T + tf^*P + E. \end{aligned}$$

Thus, there exists a rational number $b > 0$ and a pseudo-effective $\mathbb{Q}$ -divisor Q such that

$$bL_T \sim_{\mathbb{Q}} K_T + Q.$$

By replacing f by its composition with some power of Frobenius, we may assume that

$$\frac{b}{p^k}L_T \sim_{\mathbb{Q}} K_T + Q$$

for $k \geq 0$ such that $\frac{b}{p^k} < 1$. Then

$$L_T \sim_{\mathbb{Q}} K_T + Q + (1 - \frac{b}{p^k})L_T,$$

which concludes the proof. □

Remark 7.3.9. Theorem 7.3.8 is valid for contractions of relative dimension one from a normal projective variety of dimension n contingent upon the validity of the terminal Minimal Model Program in dimension $n - 1$ (including the termination of arbitrary sequences of flips) as well as the existence of resolutions of singularities.

7.4. Applications

The goal of this section is to show Theorem 7.1.3, 7.1.4, 7.1.5, and 7.1.6.

7.4.1. Nef reduction map and the canonical bundle formula

In this subsection, we prepare the toolkit for tackling the case of nef dimension two in the applications of the canonical bundle formula to threefolds; more precisely we deal with the issue that nef reduction maps are well defined only up to a birational modification. On the first reading of this chapter, the reader might consider skipping this subsection by assuming that the encountered nef reduction maps are proper morphisms. With this assumption, Proposition 7.4.2 is a special case of Theorem 7.3.8.

We start by stating a simple lemma used to resolve rational maps.

Lemma 7.4.1. *Let X be a normal projective threefold defined over an algebraically closed field k , and let L be a nef Cartier divisor on it. Let $\varphi: X \dashrightarrow Z$ be a map of relative dimension one which is proper over an open subset $V \subseteq Z$ and such that $\varphi_*\mathcal{O}_{\varphi^{-1}(V)} = \mathcal{O}_V$.*

Assume that $L|_{X_\mu} \sim_{\mathbb{Q}} 0$ where X_μ is the generic fibre. Then the map φ , after possibly replacing Z by a birational modification, admits a birational resolution

$$\begin{array}{ccccc} & & \psi & & \\ & \nearrow & & \searrow & \\ Y & \xrightarrow{g} & X & \dashrightarrow_{\varphi} & Z \end{array}$$

with ψ equidimensional, Y normal, Z smooth, and such that

1. $g^*L \sim_{\mathbb{Q}} \psi^*L_Z$ for a $\mathbb{Q}$ -divisor L_Z on Z , and
2. no curve on Y is contracted by both g and ψ .

In particular, (2) implies that no g -exceptional divisor on Y is contracted by ψ to a point. By a birational modification of Z , we mean an arbitrary surface Z' which is birational to Z .

Proof. First, we pick an arbitrary resolution of φ , that is a birational morphism $g: Y \rightarrow X$ for which there exists a map $\psi: Y \rightarrow Z$ such that $\psi = \varphi \circ g$. Replacing ψ by Raynaud's flatification (see Theorem 7.2.3) we can assume that ψ is equidimensional. Replacing Z by its resolution of singularities, and Y by the normalisation of an appropriate base-change, we can assume that Z is smooth. By Lemma 7.2.18, there exists a $\mathbb{Q}$ -divisor L_Z on Z such that $g^*L \sim_{\mathbb{Q}} \psi^*L_Z$. Hence, (1) holds.

We show that (2) holds if we replace Y by the common factorisation of g and ψ . More precisely, we pick very ample Cartier divisors A_X, A_Z on X and Z , respectively, and replace Y by the image of the semiample fibration associated to the basepoint-free divisor $g^*A_X + \psi^*A_Z$. Therefore, we can assume that $g^*A_X + \psi^*A_Z$ is ample, and so (2) holds. $\square$

Proposition 7.4.2. *Let (X, Δ) be a projective three-dimensional log pair defined over an algebraically closed field k of characteristic $p > 3$ (or $p > 2$ if Δ is big) such that $L := K_X + \Delta$ is nef, $n(L) = 2$, and the coefficients of Δ are at most one. Then there exists a diagram*

$$\begin{array}{ccc} & X & \xleftarrow{g} Y \\ & \downarrow \varphi & \swarrow \psi \\ T & \xrightarrow{f} & Z \end{array}$$

with g being birational, ψ being an equidimensional contraction, f being generically purely inseparable, φ being a nef reduction map for L , the variety Y being normal, and Z, T being smooth such that

1. $g^*L \sim_{\mathbb{Q}} \psi^*L_Z$ for a $\mathbb{Q}$ -divisor L_Z on Z , and
2. $f^*L_Z \sim_{\mathbb{Q}} K_T + \Delta_T - M$ for a pseudo-effective $\mathbb{Q}$ -divisor Δ_T and an effective $\mathbb{Q}$ -divisor M satisfying $f^*L_Z|_{\text{Supp } M} \equiv 0$.

Moreover, if Δ is big and $\lfloor \Delta \rfloor = 0$, then Δ_T is big.

Proof. We shall show the proposition for T being normal. Thereafter, we replace it by its minimal resolution of singularities.

Take φ to be a nef reduction map for L and note that by abundance for curves $L|_{X_\mu} \sim_{\mathbb{Q}} 0$ where X_μ is the generic fibre. Furthermore, X_μ is a smooth curve by Proposition 7.2.2. After possibly replacing Z by a birational modification, Lemma 7.4.1 implies the existence of the diagram as above with Y normal, Z smooth, and ψ a contraction. Moreover, there exists a $\mathbb{Q}$ -divisor L_Z on Z such that $\psi^*L_Z \sim_{\mathbb{Q}} g^*L$. For the divisor $M := \psi(\text{Exc } g)$, we have $L_Z|_M \equiv 0$ and $\text{Exc } g \subseteq \text{Supp } \psi^*M$ (see Lemma 7.4.1(2)). Write $K_Y + \Delta_Y = g^*(K_X + \Delta)$.

Pick a rational number $b > 0$ such that $\Delta_Y + b\psi^*M$ is effective. Then by Theorem 7.3.4 applied to $g^*L + b\psi^*M = K_Y + \Delta_Y + b\psi^*M$, there exists a purely inseparable cover $f: T \rightarrow Z$, a pseudo-effective $\mathbb{Q}$ -divisor Δ_T , and a rational number $0 \leq t \leq 1$ such that

$$f^*(L_Z + bM) \sim_{\mathbb{Q}} tf^*K_Z + (1-t)K_T + \Delta_T. \quad (7.1)$$

If $t = 1$, then in fact we can take $f = \text{id}$. If $t \neq 1$, then

$$(1+\lambda)f^*L_Z \sim_{\mathbb{Q}} (1-t)K_T + tf^*P_Z + \Delta_T - bf^*M,$$

where $\lambda > 0$ is such that $P_Z := K_Z + \lambda L_Z$ is pseudo-effective (see Lemma 7.3.7). We can conclude by composing f with a power of Frobenius as in the proof of Theorem 7.3.8.

Now assume that Δ is big and $[\Delta] = 0$. We can write $\Delta \sim_{\mathbb{Q}} (1-\epsilon)\Delta + \epsilon E + \epsilon A$ where $0 < \epsilon \ll 1$ and $\Delta \sim_{\mathbb{Q}} A + E$ for an ample $\mathbb{Q}$ -divisor A and an effective E . Then Theorem 7.3.4 implies that Δ_T is big. $\square$

Remark 7.4.3. The above proposition holds for $L = K_X + \Delta + N$, where N is any nef and big $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor, by means of perturbation. Indeed, it is enough to show that Equation (7.1) holds in the above proof. To this end, we invoke Theorem 7.3.4 as above but with $g^*L + b\psi^*M = K_Y + \Delta_Y + g^*N + b\psi^*M$ replaced by

$$K_Y + \Delta_Y + (1-\epsilon)g^*N + \epsilon A + b\psi^*M$$

where A is an ample $\mathbb{Q}$ -divisor on Y of relative degree over Z equal to that of g^*N and $0 < \epsilon \ll 1$. By taking $\epsilon \rightarrow 0$, we get (7.1) with Δ_T pseudoeffective.

7.4.2. Non-vanishing and abundance conjectures

In this subsection, we show the log non-vanishing conjecture for klt pairs in characteristic $p > 5$. Furthermore, we prove the log abundance conjecture for klt pairs of non-maximal nef dimension.

We divide the proof into three cases depending on the nef dimension.

Nef dimension zero

Proposition 7.4.4. *Let (X, Δ) be a projective three-dimensional Kawamata log terminal pair defined over an algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta \equiv 0$. Then $K_X + \Delta \sim_{\mathbb{Q}} 0$.*

Proof. Set $L := K_X + \Delta$. By replacing X by its $\mathbb{Q}$ -factorialisation, we may assume that X is $\mathbb{Q}$ -factorial. The theorem is clear when $\dim \text{alb}(X) = 0$ and it follows from the abundance for surfaces and [Zha17, Theorem 4.3 or Theorem 1.2] when $\dim \text{alb}(X) = 1$. If $\dim \text{alb}(X) = 2$, then we apply Theorem 7.3.4 to the connected-fibres part $g: X \rightarrow S$ of the Stein factorisation of the albanese morphism. Note that the generic fibre is smooth by Proposition 7.2.2. Therefore, we get a purely inseparable morphism $f: T \rightarrow S$ and an effective $\mathbb{Q}$ -divisor Δ_T on T satisfying

$$f^*L_S \sim_{\mathbb{Q}} tf^*K_S + (1-t)K_T + \Delta_T,$$

where L_S is such that $g^*L_S \sim_{\mathbb{Q}} L$. Since T and S admit generically finite morphisms to an abelian variety, they cannot be uniruled, and hence $\kappa(S) \geq 0$ and $\kappa(T) \geq 0$. Thus f^*L_S is $\mathbb{Q}$ -effective, and so are L_S and L . Last, when $\dim \text{alb}(X) = 3$, the theorem follows from [DW16, Theorem C] (equivalently, run a K_X -MMP and apply [DW16, Theorem A.1] or [Zha17, Theorem 1.1]). $\square$

Nef dimension one

Proposition 7.4.5. *Let L be a nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on a projective three-dimensional variety defined over an algebraically closed field k , satisfying $n(L) = 1$. Assume that $L \sim_{\mathbb{Q}} K_X + \Delta$ for some log canonical pair (X, Δ) , or $k = \overline{\mathbb{F}}_p$. Then L is semiample.*

Proof. Let $\varphi: X \rightarrow Z$ be a nef reduction map of L . It is well defined as $\dim Z = 1$ and every rational map to a curve extends to a proper morphism. Further, φ is equidimensional.

We claim that there exists a $\mathbb{Q}$ -divisor L_Z on Z such that $\varphi^*L_Z \sim_{\mathbb{Q}} L$. Indeed, when $L \sim_{\mathbb{Q}} K_X + \Delta$, the abundance for surfaces (see [Tan16a, Theorem 1.1]) implies $L|_{X_\mu} \sim_{\mathbb{Q}} 0$ where X_μ is the generic fibre of X , and hence the claim follows from Lemma 7.2.18. Over $\overline{\mathbb{F}}_p$ the claim is a consequence of Lemma 7.2.19 and Lemma 7.2.7.

We can conclude, as $n(L_Z) = 1$ and so the $\mathbb{Q}$ -divisor L_Z is ample. $\square$

Nef dimension two

We are left to show the case of nef dimension two. The idea of the proof is the following. Assume for simplicity that the nef reduction map $\psi: X \dashrightarrow Z$ of $K_X + \Delta$ is proper. Then, using the canonical bundle formula derived in this article, we can descend $K_X + \Delta$ to an adjoint divisor $K_Z + \Delta_Z$, with Δ_Z pseudoeffective, up to replacing Z by a purely inseparable cover. By a careful study of adjoint divisors on surfaces (Lemma 7.5.1), we get that $K_Z + \Delta_Z \equiv D_Z$ for an effective $\mathbb{Q}$ -divisor D_Z , and so $K_X + \Delta \equiv D$ for $D := \psi^*D_Z$.

Now, we modify (X, Δ) so that (X, Δ) is dlt and $\text{Supp } D \subseteq \lfloor \Delta \rfloor$; this is done by replacing Δ with $\max(\Delta, \text{Supp } D)$ and taking a minimal model of an appropriate dlt modification (Lemma 7.4.6). Then $(K_X + \Delta)|_{\text{Supp } D}$ is semiample by abundance for slc surfaces (see [Wal18, Theorem 1.3]), from which we infer that $(K_Z + \Delta_Z)|_{\text{Supp } D_Z}$ is semiample as well (Lemma 7.4.7). Consequently, $\kappa(K_Z + \Delta_Z) \geq 0$ by a careful study of adjoint divisors on surfaces (Lemma 7.5.1). Hence, up to another modification of (X, Δ) we can assume that $D_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$. In this case, we apply the evaporation technique to show that $K_Z + \Delta_Z$ is big (Lemma 7.5.3), procuring that $\kappa(K_X + \Delta) \geq 2$. This concludes the proof of the semiampleness of $K_X + \Delta$ thanks to [Wal17, Theorem 1.3].

The statements and the proofs of the required results for adjoint divisors on surfaces may be found in Section 7.5 (see Lemma 7.5.1).

The following lemma is a variation on the standard result used in the proof of the abundance conjecture for threefolds in characteristic zero.

Lemma 7.4.6 (cf. [K⁺92, Lemma 13.2]). *Let (X, Δ) be a three-dimensional dlt pair defined over an algebraically closed field of characteristic $p > 5$ such that X is $\mathbb{Q}$ -factorial. Suppose that $K_X + \Delta$ is nef and there exists an effective $\mathbb{Q}$ -divisor D satisfying $D \equiv K_X + \Delta$. Then, there exists a three-dimensional dlt pair (Y, Δ_Y) such that*

1. $K_Y + \Delta_Y$ is nef,
2. $n(K_Y + \Delta_Y) = n(K_X + \Delta)$,
3. $\kappa(K_X + \Delta) \leq \kappa(K_Y + \Delta_Y) \leq \kappa(K_X + \Delta + rD)$ for some $r > 0$,
4. $K_Y + \Delta_Y \equiv E$ for an effective $\mathbb{Q}$ -divisor E with $\text{Supp } E \subseteq \lfloor \Delta_Y \rfloor$,
5. $(Y \setminus \text{Supp } E, \Delta_Y) \simeq (X \setminus \text{Supp } D, \Delta)$.

Moreover, if $D \sim_{\mathbb{Q}} K_X + \Delta$, then $K_Y + \Delta_Y \sim_{\mathbb{Q}} E$ in (4).

The pair (Y, Δ_Y) is nothing but a log minimal model of a dlt blow-up of $(X, \tilde{\Delta})$, where $\tilde{\Delta} := \max(\Delta, \text{Supp } D)$. For the convenience of the reader, we append the detailed construction below.

Proof. We will construct the following commutative diagram

$$\begin{array}{ccc} & W & \\ h \swarrow \text{---} & & \searrow g \\ Y & & X. \end{array}$$

Let $g: W \rightarrow X$ be a resolution of singularities of $(X, \text{Supp}(\Delta + D))$ with exceptional locus Exc. Set

$$\Delta_W := \max(g^*\Delta - K_{W/X}, \text{Supp}(g_*^{-1}D) + \text{Exc}),$$

that is Δ_W is constructed from the log pullback of Δ by replacing all the coefficients of irreducible divisors in $\text{Supp}(g_*^{-1}D) + \text{Exc}$ by one. Since (X, Δ) is dlt, we can write

$K_W + \Delta_W \equiv E_W$, where E_W is an effective $\mathbb{Q}$ -divisor with $\text{Supp } E_W \subseteq \text{Supp}(g_*^{-1}D) + \text{Exc} \subseteq \lfloor \Delta_W \rfloor$. By replacing W by the output of a $(K_W + \Delta_W)$ -MMP over X and using again that (X, Δ) is dlt, we can assume that $W \setminus \text{Supp } E_W \simeq X \setminus \text{Supp } D$ and $\text{Supp } E_W = \text{Supp } g^*D$.

Now, run a full $(K_W + \Delta_W)$ -MMP. Since $K_W + \Delta_W$ is pseudoeffective, it terminates with a minimal model $h: W \dashrightarrow Y$; in particular $K_Y + \Delta_Y$ is nef for $\Delta_Y := h_*\Delta$. Furthermore, $K_Y + \Delta_Y \equiv E$ and $\text{Supp } E \subseteq \lfloor \Delta_Y \rfloor$, where $E := h_*E_W$. Hence, (1) and (4) hold. Since h and g are isomorphisms outside of E_W , we get (5).

To show (2) and (3), we find $h': W' \rightarrow W$ such that $h \circ h'$ is a well defined morphism. Replace h by $h \circ h'$ and g by $g \circ h'$. Then

$$g^*(K_X + \Delta) \leq h^*(K_Y + \Delta_Y) \leq g^*(K_X + \Delta + rD)$$

for some $r > 0$. The first inequality follows from the negativity lemma: $g^*(K_X + \Delta) \leq h^*h_*g^*(K_X + \Delta) \leq h^*(K_Y + \Delta_Y)$, while the second inequality is a consequence of (5) as $\text{Supp } g^*D = \text{Supp } h^*E$. Therefore,

$$\begin{aligned} n(K_Y + \Delta_Y) &= n(K_X + \Delta), \text{ and} \\ \kappa(K_X + \Delta) &\leq \kappa(K_Y + \Delta_Y) \leq \kappa(K_X + \Delta + rD), \end{aligned}$$

concluding the proof of the lemma. $\square$

Lemma 7.4.7. *Let X be a three-dimensional projective variety defined over an algebraically closed field k of characteristic $p > 0$, let $\psi: X \rightarrow Y$ be an equidimensional contraction onto a surface Y , let M be a Cartier divisor on Y , and let C be any reduced divisor on Y . Assume that $\psi^*M|_S$ is semiample for the reduction S of $\psi^{-1}(C)$. Then $M|_C$ is semiample.*

Note that C is not necessarily irreducible.

Proof. Let

$$\begin{array}{ccccc} & & \psi|_S & & \\ & \nearrow & & \searrow & \\ S & \xrightarrow{v} & \tilde{C} & \xrightarrow{u} & C \end{array}$$

be the Stein factorisation of $\psi|_S$. Since $(\psi|_S)^*(M|_C) = \psi^*M|_S$ is semiample and $v_*\mathcal{O}_S = \mathcal{O}_{\tilde{C}}$, the projection formula shows that $u^*(M|_C)$ is semiample. Moreover, by the Zariski connectedness theorem, ψ has connected fibres, and so $\psi|_S$ and u have connected fibres as well. This in turn implies that u is a universal homeomorphism; indeed it is a bijective morphism between curves defined over an algebraically closed field (see [Kol97, Lemma 5.2]). Thus $M|_C$ is semiample by Lemma 7.2.6. $\square$

We are ready to tackle the case of nef dimension two. For the clarity of the proof we first show an MMP-free version of Proposition 7.4.10.

Lemma 7.4.8. *Let (X, Δ) be a projective three-dimensional log pair defined over an algebraically closed field k of characteristic $p > 3$ such that the coefficients of Δ are at most one. Assume that $L := K_X + \Delta$ is nef and $n(L) = 2$. Then the following hold:*

1. *there exists an effective $\mathbb{Q}$ -divisor D such that $L \equiv D$,*
2. *if $L|_{\text{Supp } D} \sim_{\mathbb{Q}} 0$ for some D as above, then $\kappa(L) \geq 0$, and*
3. *if $L|_{\text{Supp } D} \not\equiv 0$, or $L|_{\text{Supp } D} \sim_{\mathbb{Q}} 0$ and $L \sim_{\mathbb{Q}} D$, then $\kappa(L) = 2$.*

Proof. Let $\psi: X \dashrightarrow Z$ be a nef reduction map for L . By the assumptions on the characteristic and by Proposition 7.2.2, the generic fibre X_μ is smooth.

Let

$$\begin{array}{ccc} & X & \xleftarrow{g} W \\ & \vdots \varphi & \swarrow \psi \\ T & \xrightarrow{f} & Z \end{array}$$

be a diagram as in Proposition 7.4.2 with L_Z being a $\mathbb{Q}$ -divisor on Z such that $g^*L \sim_{\mathbb{Q}} \psi^*L_Z$ and $f^*L_Z \sim_{\mathbb{Q}} K_T + \Delta_T - M$ for a pseudo-effective $\mathbb{Q}$ -divisor Δ_T and an effective $\mathbb{Q}$ -divisor M satisfying $f^*L_Z|_{\text{Supp } M} \equiv 0$. In particular, Lemma 7.5.1 implies that there exists an effective $\mathbb{Q}$ -divisor D_T on T such that $f^*L_Z \equiv D_T$. Take

$$\begin{aligned} D_Z &:= \frac{1}{\deg f} f_* D_T, \\ D_W &:= \psi^* D_Z, \text{ and} \\ D &:= g(D_Y). \end{aligned}$$

Therewith, $L_Z \equiv D_Z$ and $L \equiv D$, thus (1) holds (here we use Remark 7.4.9).

Now, assume that $L \equiv D'$ for a possibly different effective $\mathbb{Q}$ -divisor D' on X , and let $D'_W := g^*D'$. Since $D' \equiv L$ and D' is effective, we get that $D'|_{X_\mu} \sim_{\mathbb{Q}} 0$, and so Lemma 7.2.18 implies the existence of an effective $\mathbb{Q}$ -divisor D'_Z on Z such that $D'_W \sim_{\mathbb{Q}} \psi^*D'_Z$. Since ψ is a contraction, we can pick D'_Z so that in fact $D'_W = \psi^*D'_Z$. Set $D'_T := f^*D'_Z$.

If $L|_{\text{Supp } D'} \not\equiv 0$, then $L_Z \cdot D'_Z > 0$ and L_Z is big. In particular, $\kappa(L) \geq 2$. Thus, we can assume that $L|_{\text{Supp } D'} \equiv 0$. To show (2) and (3), suppose $L|_{\text{Supp } D'} \sim_{\mathbb{Q}} 0$. By Lemma 7.4.7, we get $L_Z|_{\text{Supp } D'_Z} \sim_{\mathbb{Q}} 0$, and so $f^*L_Z|_{\text{Supp } D'_T} \sim_{\mathbb{Q}} 0$. Hence, we can conclude the proof by Lemma 7.5.1. $\square$

Remark 7.4.9. Let $f: X \rightarrow Y$ be a proper and generically purely inseparable morphism between $\mathbb{Q}$ -factorial varieties and let L be a divisor on X such that $L \equiv_f 0$. Then $L = \frac{1}{\deg f} f^* f_* L$ by the negativity lemma. In particular, if $D_1 \equiv D_2$ for some divisors D_1, D_2 on X , then $f_* D_1 \equiv f_* D_2$.

Proposition 7.4.10. *Let (X, Δ) be a projective three-dimensional Kawamata log terminal pair defined over an algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is nef and $n(K_X + \Delta) = 2$. Then $K_X + \Delta$ is semiample.*

Proof. We only need to show that $\kappa(K_X + \Delta) = 2$; indeed in this case the semiampleness of $K_X + \Delta$ follows from [Wal17, Theorem 1.3]. Note that $\kappa(K_X + \Delta) \leq n(K_X + \Delta) = 2$. By replacing X by its $\mathbb{Q}$ -factorialisation, we can assume that X is $\mathbb{Q}$ -factorial.

By Lemma 7.4.8, there exists an effective $\mathbb{Q}$ -divisor D satisfying $K_X + \Delta \equiv D$. Let (Y, Δ_Y) and E_Y be as in Lemma 7.4.6, that is for some $r > 0$:

- $K_Y + \Delta_Y$ is nef,
- $n(K_Y + \Delta_Y) = n(K_X + \Delta)$ and $\kappa(K_Y + \Delta_Y) \leq \kappa(K_X + \Delta + rD)$,
- $K_Y + \Delta_Y \equiv E_Y$ where E_Y is an effective $\mathbb{Q}$ -divisor such that $\text{Supp } E_Y \subseteq \lfloor \Delta_Y \rfloor$.

By [Wal18, Theorem 1.3], $(K_Y + \Delta_Y)|_{\lfloor \Delta_Y \rfloor}$ is semiample, and therefore $(K_Y + \Delta_Y)|_{\text{Supp } E_Y}$ is semiample as well. By Lemma 7.4.8 applied to (Y, Δ_Y) and E_Y , we see that $\kappa(K_Y + \Delta_Y) \geq 0$.

We claim that in fact $\kappa(K_Y + \Delta_Y) \geq 2$. To this end, we apply Lemma 7.4.6 to (Y, Δ_Y) and a $\mathbb{Q}$ -section of $K_Y + \Delta_Y$, obtaining a dlt pair (Z, Δ_Z) satisfying

- $K_Z + \Delta_Z$ is nef,
- $n(K_Z + \Delta_Z) = n(K_Y + \Delta_Y)$ and $\kappa(K_Z + \Delta_Z) = \kappa(K_Y + \Delta_Y)$,
- $K_Z + \Delta_Z \sim_{\mathbb{Q}} E_Z$ where E_Z is an effective $\mathbb{Q}$ -divisor such that $\text{Supp } E_Z \subseteq \lfloor \Delta_Z \rfloor$.

Arguing as above, we get that $(K_Z + \Delta_Z)|_{\text{Supp } E_Z}$ is semiample. Lemma 7.4.8 (3) applied to (Z, Δ_Z) and E_Z implies $\kappa(K_Y + \Delta_Y) = \kappa(K_Z + \Delta_Z) = 2$.

We are left to show $\kappa(K_X + \Delta) = 2$. Since $\kappa(K_Y + \Delta_Y) = 2$, we have $\kappa(K_X + \Delta + rD) \geq 2$. However, given that $K_X + \Delta \equiv D$, this is only possible when $(K_X + \Delta)|_D \neq 0$. By (3) in Lemma 7.4.8, we can conclude the proof of the proposition. $\square$

Proof of Theorem 7.1.3 and Theorem 7.1.4

Theorem 7.4.11 (Theorem 7.1.3). *Let (X, Δ) be a projective Kawamata log terminal pair of dimension three defined over an algebraically closed field k of characteristic $p > 5$. Assume that $K_X + \Delta$ is pseudo-effective. Then $\kappa(K_X + \Delta) \geq 0$.*

The proof follows from Proposition 7.4.4, Proposition 7.4.5, Proposition 7.4.10, and [XZ18, Theorem 1.1] by ways of [KMM94, Lemma 2.4]. For the convenience of the reader we append the argument of [KMM94] below.

Proof. Since the canonical ring is preserved under any MMP, it is enough to show the theorem for a minimal model of (X, Δ) . In particular, we can replace (X, Δ) by a minimal model (see [Bir16, Theorem 1.2]) and assume that $K_X + \Delta$ is nef. Moreover, by replacing (X, Δ) by its sub-terminalisation (see [Bir16, Theorem 1.7]), we can assume that X is terminal and $\mathbb{Q}$ -factorial.

Let $0 \leq \lambda \leq 1$ be the smallest number such that $K_X + \lambda\Delta$ is nef. Since $K_X + \Delta - (K_X + \lambda\Delta)$ is $\mathbb{Q}$ -effective, it is enough to show that $K_X + \lambda\Delta$ is $\mathbb{Q}$ -effective. If $\lambda = 0$, then the result follows from [XZ18, Theorem 1.1], thus we can assume $\lambda > 0$.

We have $\frac{1}{\lambda}(K_X + \lambda\Delta) = (K_X + \Delta) + \frac{1-\lambda}{\lambda}K_X$, and so there exists a K_X -extremal ray R such that $(K_X + \lambda\Delta) \cdot R = 0$ by [KMM94, Lemma 2.1] (though stated only for $\mathbb{C}$, the result is a direct consequence of the existence of bounds on lengths of extremal rays, true for klt pairs in positive characteristic by [BW17, Theorem 1.1]). In particular, $\lambda \in \mathbb{Q}$.

If $n(K_X + \lambda\Delta) \leq 2$, then $K_X + \lambda\Delta$ is semiample by Proposition 7.4.4, Proposition 7.4.5, and Proposition 7.4.10, thus we can assume $n(K_X + \lambda\Delta) = 3$.

Let $f: X \dashrightarrow Y$ be the contraction of the ray R (or the corresponding flip if the contraction is small). Since $n(K_X + \lambda\Delta) = 3$, the map f is birational. Set $\Delta_Y = f_*\Delta$. It is enough to show that the nef $\mathbb{Q}$ -divisor $K_Y + \lambda\Delta_Y$ is $\mathbb{Q}$ -effective, hence we can replace (X, Δ) by $(Y, \lambda\Delta_Y)$ and proceed as above by choosing a new λ . We can conclude the proof that $\kappa(K_X + \Delta) \geq 0$ as any sequence of flips and divisorial contractions for terminal varieties terminates. $\square$

Remark 7.4.12. As it has been communicated to us by Tanaka, Theorem 7.4.11 holds in the relative setting for projective contractions $\varphi: X \rightarrow Z$ such that $K_X + \Delta$ is relatively pseudo-effective. Indeed, by arguing as above we can assume that $K_X + \Delta$ is relatively nef. By [HNT17], we can compactify φ to a morphism of projective varieties, and, by running the MMP over the base, we can assume that $K_X + \Delta$ is still relatively nef. Using the cone theorem, we can find an ample divisor A on Z such that $K_X + \Delta + \varphi^*A$ is globally nef. By [Tan17, Theorem 1], we can assume that $(X, \Delta + \varphi^*A)$ is klt. Hence, $\kappa(K_X + \Delta + \varphi^*A) \geq 0$ by Theorem 7.4.11, and so $K_X + \Delta$ is relatively $\mathbb{Q}$ -effective.

Proof of Theorem 7.1.4. This follows directly from Proposition 7.4.4, Proposition 7.4.5, and Proposition 7.4.10. $\square$

Further, we obtain the following result.

Corollary 7.4.13. *Let (X, Δ) be a projective Kawamata log terminal pair of dimension three defined over an algebraically closed field k of characteristic $p > 5$. Assume that $\dim \text{Alb}(X) > 0$ and $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample.*

Proof. Let $g: X \rightarrow S$ be the connected-fibres part of the Stein factorisation of the albanese morphism, and let G be the generic fibre of g . Then $(K_X + \Delta)|_G$ is semiample by abundance for curves and surfaces (see [Tan16a, Theorem 1.1]), and so $\kappa((K_X + \Delta)|_G) = n((K_X + \Delta)|_G)$.

In [Zha17, Theorem 1.2], it is shown that $K_X + \Delta$ is semiample, except when:

- $\dim S = 2$ and $n((K_X + \Delta)|_G) = 0$, or
- $\dim S = 1$ and $n((K_X + \Delta)|_G) = 1$.

In both of these cases $n(K_X + \Delta) < 3$, so we can conclude the proof by Theorem 7.1.4. $\square$

Remark 7.4.14. As stated in the main introduction (Chapter 1), the proof of the log abundance conjecture for three-dimensional klt pairs in characteristic zero (see [KMM94]) consists of three steps: showing abundance for terminal varieties, showing that any effective

nef divisor on a klt Calabi-Yau variety is semiample, and showing log abundance for when $n(K_X + \Delta) \leq 2$. Note that Theorem 7.1.4 concludes the third step of the proof of log abundance for threefolds in positive characteristic.

7.4.3. Low characteristic MMP

In this subsection, we show Theorem 7.1.5 and Theorem 7.1.6.

Lemma 7.4.15. *Let (X, Δ) be a projective three-dimensional log pair defined over an algebraically closed field k of characteristic $p > 2$ such that Δ has coefficients at most one. Let L be a $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor on X such that $L - (K_X + \Delta)$ is nef and big, and $n(L) = 2$. Then the following hold:*

- if $\lfloor \Delta \rfloor = 0$, then $\kappa(L) = 2$,
- if $k = \overline{\mathbb{F}}_p$, then L is semiample.

Proof. The proof follows from Proposition 7.4.2, Remark 7.4.3, and Lemma 7.5.1 analogously to the proof of Lemma 7.4.8. $\square$

Note that in general the log canonical base point free theorem is false over uncountable algebraically closed fields (see [MNW15, NW17]).

Theorem 7.4.16. *(Theorem 7.1.5) Let (X, Δ) be a projective three-dimensional log pair defined over $\overline{\mathbb{F}}_p$ for $p > 2$ and let L be a nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor such that $L - (K_X + \Delta)$ is nef and big. Assume that*

- (X, Δ) is log canonical, or
- the coefficients of Δ are at most one and each irreducible component of $\lfloor \Delta \rfloor$ is normal.

Then L is semiample.

Proof. If L is big, then this follows from [MNW15, Theorem 1.1]. Thus, we can assume that L is not big, and hence Theorem 7.2.20 implies that $n(L) \leq 2$. When $n(L) = 2$, we apply Lemma 7.4.15; when $n(L) = 1$, we apply Proposition 7.4.5; and when $n(L) = 0$, we apply Lemma 7.2.7. $\square$

Theorem 7.4.17 (Theorem 7.1.6). *Let (X, Δ) be a three-dimensional projective klt pair defined over an algebraically closed field of characteristic $p > 2$ and let L be a nef $\mathbb{Q}$ -Cartier $\mathbb{Q}$ -divisor such that $L - (K_X + \Delta)$ is nef and big. When $n(L) = 0$, assume $\dim \text{alb}(X) \neq 1$. Then $\kappa(L) = n(L)$.*

Proof. When $n(L) = 3$, this is Theorem 7.2.20; when $n(L) = 2$, this is Lemma 7.4.15; and when $n(L) = 1$, this is Proposition 7.4.5. In the case of $n(L) = 0$, the proof is analogous to that of Proposition 7.4.4 after one notices that when $n(L) = 0$, we have $\dim \text{alb}(X) \leq 2$, since by bend-and-break X is covered in this case by rational curves (see [Kol96, II, Theorem 5.14]). Taking a $\mathbb{Q}$ -factorialisation is not necessary, because we assume that $\dim \text{alb}(X) \neq 1$ when $n(L) = 0$. $\square$

7.5. Positivity of adjoint divisors on surfaces

The following lemma was a fundamental component in the above applications of the canonical bundle formula. On the first reading, the reader might start by assuming that $M = 0$; the reason we consider a non-zero M is due to the fact that nef reduction maps need not be proper.

Lemma 7.5.1. *Let X be a normal $\mathbb{Q}$ -factorial projective surface defined over an algebraically closed field k of characteristic $p > 2$, let Δ be a pseudo-effective $\mathbb{Q}$ -divisor, let L be a nef $\mathbb{Q}$ -divisor such that $n(L) = 2$, and let M be an effective $\mathbb{Q}$ -divisor such that $L|_{\text{Supp } M} \equiv 0$. Suppose that*

$$L \sim_{\mathbb{Q}} K_X + \Delta - M.$$

Then the following hold:

1. *there exists an effective $\mathbb{Q}$ -divisor D such that $L \equiv D$,*
2. *if Δ is big, then L is big,*
3. *if $k = \overline{\mathbb{F}}_p$, then L is semiample.*
4. *if $L|_{\text{Supp } D}$ is semiample for an effective $\mathbb{Q}$ -divisor D such that $D \equiv L$, then $\kappa(L) \geq 0$.
Similarly, if $L|_{\text{Supp } D}$ is semiample for an effective $D \sim_{\mathbb{Q}} L$, then L is big.*

In the proof we implicitly and freely use Remark 7.4.9 for birational morphisms.

Proof. By replacing X (and Δ, M, D) by the minimal resolution of singularities (by the log-pullback or pullback, accordingly), we can assume that X is smooth. By adding a certain multiple of L to Δ , we can suppose henceforth that L is Cartier. Further, by means of the MMP we can contract any K_X -negative extremal ray R satisfying $L \cdot R = 0$. With X replaced by the output of such a partial MMP, we can assume that $K_X + 3L$ is nef thanks to the cone theorem ([Tan14, Proposition 3.15], cf. the proof of Lemma 7.3.7). We use that $n(L) = 2$ to ensure that the output of the MMP is of dimension two. Note, that in Case (4) the assumption that $L|_{\text{Supp } D}$ is semiample might no longer hold, but we can assume that there exists a birational morphism of smooth projective surfaces $f: Y \rightarrow X$ such that $f^*L|_{\text{Supp } f^*D}$ is semiample. Last, we can assume that Δ' and M have no irreducible components in common, where $\Delta = \Delta' + N$ is the Zariski decomposition with Δ' effective and N nef. Note that N is big and nef when Δ is big.

We start by showing that M is nef. To this end, we pick a curve $C \subseteq \text{Supp } M$. By assumptions, $L \cdot C = 0$ and so

$$0 = (K_X + 3L + \Delta - M) \cdot C \geq -M \cdot C.$$

Thus $M \cdot C \geq 0$ and M is nef. Moreover, $M^2 = 0$, as otherwise M is big thereby contradicting $n(L) = 2$, and so $M|_{\text{Supp } M} \equiv 0$.

First, we deal with (3), that is the case of $k = \overline{\mathbb{F}}_p$. By Theorem 7.2.9, the $\mathbb{Q}$ -divisor $L + M \sim_{\mathbb{Q}} K_X + \Delta$ is semiample. Since $n(L) = 2$, the associated semiample fibration is

birational, and as $(L + M)|_M \equiv 0$ it must contract M . This is only possible when $M = 0$, hence L is semiample.

We turn our attention to (1) and (2). First, we will show that L is big if Δ is big or if X is not isomorphic to a rational surface or a ruled surface over an elliptic curve. By contradiction, we can assume that L is not big, i.e. $L^2 = 0$. Since $n(L) = 2$, we must have $\kappa(L) \leq 0$. Furthermore,

$$0 = L^2 = (K_X + \Delta - M) \cdot L = (K_X + \Delta) \cdot L$$

and so $K_X \cdot L \leq 0$. If Δ is big, then $K_X \cdot L < 0$.

By Serre duality $H^2(X, \mathcal{O}_X(mL)) = 0$, and so Riemann-Roch yields

$$H^0(X, \mathcal{O}_X(mL)) = H^1(X, \mathcal{O}_X(mL)) - \frac{1}{2}mL \cdot K_X + \chi(\mathcal{O}_X).$$

Since $\kappa(L) \leq 0$, we have $L \cdot K_X = 0$. This leads to a contradiction when Δ is big, and so (2) holds.

Let us note that since $K_X + 3L$ is nef, we have $(K_X + 3L)^2 \geq 0$, and so $K_X^2 \geq 0$ (recall that $L^2 = K_X \cdot L = 0$). We take $\pi: X \rightarrow X^{\min}$ to be a morphism to a minimal model $X^{\min}$ of X and proceed by classification of surfaces. We cannot have $\kappa(X) \geq 1$ as $K_X \cdot L = 0$ and $n(L) = 2$. If $\kappa(X) = 0$, then $L \cdot \text{Exc}(\pi) = 0$ since $\text{Supp } K_X = \text{Exc}(\pi)$. In particular, X is minimal given that $K_X + 3L$ is nef. We shall now proceed by a case-by-case analysis (cf. [MNW15, Proof of Theorem 1.4 (Case 1.3)]).

If X is a K3 surface, then $\chi(\mathcal{O}_X) \geq 2$ and $h^0(X, \mathcal{O}_X(mL)) \geq 2$ by the Riemann-Roch theorem as above. In particular $h^0(X, \mathcal{O}_X(kmL)) \geq k + 1$ for $k \in \mathbb{N}$ which is impossible as $\kappa(L) \leq 0$. If X is an Enriques surface, then it admits an étale double cover $h: S \rightarrow X$ by a K3 surface S and by Riemann-Roch $h^0(S, \mathcal{O}_S(mh^*L)) \geq 2$. This is again impossible.

If X is an abelian variety, then L is numerically equivalent to some semiample divisor D (see [MNW15, Proposition 3.10]). Since $n(D) = 2$, we have that D is big, and so L is big as well. If X is hyperelliptic, then it admits a finite cover $h: S \rightarrow X$ by an abelian variety S , and the same argument gives that h^*L is big.

Last if X is a quasi-hyperelliptic surface, then it admits a surjective morphism $h: S := C \times \mathbb{P}^1 \rightarrow X$, where C is an elliptic curve. Since $\text{Pic}(S) \simeq \text{Pic}(C) \times \mathbb{Z}$ and $n(h^*L) = 2$, we get that h^*L is ample.

Therefore, we can assume that $\kappa(X) = -\infty$. Since X is irrational, $X^{\min}$ is a ruled surface over a curve C of genus $g(C) \geq 1$. By theory of ruled surfaces, $K_{X^{\min}}^2 = 8(1 - g(C)) \leq 0$. Since $K_X^2 \geq 0$, we get that in fact $X = X^{\min}$ and $g(C) = 1$.

To finish off the proof of (1) we need to show that L is numerically equivalent to an effective divisor if X is rational or is a ruled surface over an elliptic curve. In the latter case, this follows from [MNW15, Proposition 3.13]. In the former, L is in fact $\mathbb{Q}$ -effective; indeed $\chi(\mathcal{O}_X) > 0$, and so by Riemann-Roch $\kappa(L) \geq 0$.

We are left to show (4). By the above proof, we can assume that X is rational or is a ruled surface over an elliptic curve. Let $f: Y \rightarrow X$ be the birational morphism such that $f^*L|_{\text{Supp } f^*D} \sim_{\mathbb{Q}} 0$. Set $L_Y := f^*L$ and $D_Y := f^*D$. Replacing L_Y and D_Y by mL_Y and mD_Y , respectively, for divisible enough $m \in \mathbb{N}$, we can assume that D_Y is an effective divisor. Then Lemma 7.2.6 implies that $(L_Y)|_{D_Y} \sim_{\mathbb{Q}} 0$. Further, since $L^2 = L \cdot K_X = 0$, we have $L_Y^2 = L_Y \cdot K_Y = 0$.

First, we consider the case of $D_Y \equiv L_Y$. Assume by contradiction that $\kappa(L_Y) = -\infty$. Since X is rational or is a ruled surface over an elliptic curve, we have $\chi(\mathcal{O}_Y) \in \{0, 1\}$ and the Riemann-Roch theorem

$$H^0(Y, kL_Y - D_Y) = H^1(Y, kL_Y - D_Y) + \chi(\mathcal{O}_Y)$$

implies $H^1(Y, kL_Y - D_Y) = 0$ for all $k \in \mathbb{Z}_{>1}$. Thus, for the exact sequence

$$H^0(Y, kL_Y) \xrightarrow{\text{res}} H^0(D_Y, (kL_Y)|_{D_Y}) \rightarrow H^1(Y, kL_Y - D_Y),$$

we have that res is surjective. By taking $k \in \mathbb{N}$ such that $kL_Y|_{D_Y} \sim 0$, we get $H^0(Y, kL_Y) \neq 0$, concluding the proof in this case.

When $D_Y \sim_{\mathbb{Q}} L_Y$, we apply Lemma 7.5.3 and obtain that $\kappa(L_Y) \geq 1$. Since $n(L_Y) = 2$, this implies that L_Y is big. $\square$

Remark 7.5.2. With notation as in the above lemma, L need not be $\mathbb{Q}$ -effective when Δ is not big and $k \not\equiv \mathbb{F}_p$. Indeed, consider $X := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$, where C is an elliptic curve and $\mathcal{L} \in \text{Pic}^0(C)$ is a non-torsion line bundle. Let $C_1, C_2 \subseteq X$ be two sections corresponding to $\mathcal{O}_C$ and $\mathcal{L}$. Then $\mathcal{O}_X(C_1 - C_2)|_{C_1} \sim \mathcal{L}$, and so $C_1 - C_2 \equiv 0$. Since $K_X \sim -C_1 - C_2$, we have $C_1 - C_2 \sim K_X + 2C_1$.

Note that a big part of the proof is valid even if $p = 2$; in fact the assumption that $p > 2$ is only used when X is birational to an Enriques surface.

The following lemma, which was used in the above result, is a staple among applications of the so called “evaporation technique”. Its proof is exactly the same as that of Keel (see [Tot09]), but since we work with non-integral curves we decided to append the whole argument for the convenience of the reader.

Lemma 7.5.3 (cf. [Tot09, Theorem 2.1]). *Let X be a smooth projective surface and let D be a nef effective divisor such that $D \cdot K_X = 0$ and $D|_{\text{Supp } D}$ is semiample. Then $\kappa(D) \geq 1$.*

Proof. By Lemma 7.2.6, $D|_D$ is semiample. If $D^2 > 0$, then D is big, so we can assume that $D^2 = 0$ and $D|_D \sim_{\mathbb{Q}} 0$.

By the Riemann-Roch theorem

$$\chi(\mathcal{O}_X(kD)) = \frac{1}{2}kD \cdot (kD - K_X) + \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X)$$

$$\begin{array}{ccccc} H^0(X, \mathcal{O}_X((k-1)D)) & \longrightarrow & H^0(X, \mathcal{O}_X(kD)) & \longrightarrow & H^0(D, \mathcal{O}_D(kD)) \\ \hline H^1(X, \mathcal{O}_X((k-1)D)) & \longrightarrow & H^1(X, \mathcal{O}_X(kD)) & \longrightarrow & H^1(D, \mathcal{O}_D(kD)). \end{array}$$
$$H^1(D, \mathcal{O}_D(kD)) = H^0(D, \mathcal{O}_D(kD)) = 0.$$
$$H^1(X, \mathcal{O}_X((k-1)D)) \simeq H^1(X, \mathcal{O}_X(kD)).$$
$$H^1(X, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X(D)) \simeq \dots \simeq H^1(X, \mathcal{O}_X((m-1)D)).$$
$$\begin{array}{ccccc} H^0(Y, \mathcal{O}_Y(mD_Y)) & \xrightarrow{\text{res}_Y} & H^0(D_Y, \mathcal{O}_{D_Y}(mD_Y)) & \xrightarrow{\delta_Y} & H^1(Y, \mathcal{O}_Y((m-1)D_Y)) \\ f^* \uparrow & & f^* \uparrow & & f^* \uparrow \\ H^0(X, \mathcal{O}_X(mD)) & \xrightarrow{\text{res}} & H^0(D, \mathcal{O}_D(mD)) & \xrightarrow{\delta} & H^1(X, \mathcal{O}_X((m-1)D)). \end{array}$$

Since $f^*H^1(X, \mathcal{O}_X((m-1)D)) = 0$, we have that $\delta_Y(f^*s) = 0$, and so f^*s lies in the image of res_Y . Thus $h^0(Y, \mathcal{O}_Y(mD_Y)) \geq 2$ and $\kappa(D_Y) \geq 1$. By a standard argument (cf. [Kee99, Lemma 2.10]), we have $\kappa(D) \geq 1$. $\square$

8

Liftability of the Frobenius morphism and images of toric varieties

based on joint work with P. Achinger and M. Zdanowicz

8.1. Introduction

ONE OF THE SALIENT FEATURES of algebraic geometry in positive characteristic is the existence of the Frobenius morphism $F_X: X \rightarrow X$ for every $\mathbb{F}_p$ -scheme X . At a philosophical level, this chapter argues that this type of structure becomes extremely rare as we move towards characteristic zero.

Let us recall our main conjecture.

Conjecture 8.1.1. *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. If X is F -liftable, then there exists a finite étale Galois cover $f: Y \rightarrow X$ such that the Albanese morphism of Y*

$$a_Y: Y \longrightarrow \mathrm{Alb}(Y)$$

is a toric fibration. In particular, if X is simply connected (for example, if X is separably rationally connected), then X is a toric variety.

See §8.2.1 for the definition of a toric fibration. Conjecture 8.1.1 *almost* characterises F -liftable smooth projective varieties, see Remark 8.3.7 for a discussion of the converse.

The main theorems of this chapter are the following.

Theorem 8.1.2 (see Theorem 8.6.14). *Conjecture 8.1.1 is true if X is a homogeneous space. More precisely, if X is a smooth projective F -liftable variety whose automorphism group acts transitively, then X is isomorphic to a product of projective spaces and an ordinary abelian variety.*

Our method of proof is based on the study of rational curves on X . In the crucial special case of rational homogeneous spaces of Picard rank one, we analyse the geometry of an étale covering of X induced by the map (1.1) (see Section 1.2), and its restrictions to rational curves belonging to a carefully chosen covering family. In the final step, we apply Mori's characterisation of the projective space. In fact, this method shows that Conjecture 8.1.1 holds if T_X is nef and X is a Fano variety of Picard rank one (see Proposition 8.6.9).

Theorem 8.1.3. *Conjecture 8.1.1 is true in the following cases.*

- (a) *If $\dim X \leq 2$ (see Theorem 8.7.9).*
- (b) *If X is a Fano threefold from the Mori–Mukai classification (see Theorem 8.7.19).*

Note that for ruled surfaces our results do not entirely agree with [Xin16], see Remark 8.7.10. The case of Fano threefolds relies on the Mori–Mukai classification with about 100 families; see the first paragraph of §8.7.2 for a discussion. The case which caused us the most trouble initially was the blow-up of $\mathbb{P}_k^3$ along a twisted cubic. Since F -liftable smooth Fano varieties lift to characteristic zero by ways of Bott vanishing and deformation theory, it is highly likely that one can drop the assumption that X comes from the Mori–Mukai list.

Theorem 8.1.4 (see Theorem 8.4.15). *Conjecture 8.1.1 for simply connected varieties implies Conjecture 1.2.4 in characteristic zero, that is, smooth projective images of complete toric varieties are toric.*

The key step in the proof uses the functoriality of obstruction classes to lifting Frobenius, an idea due to the third author [Zda17]: after reducing a given surjective map $\varphi: Z \rightarrow X$ modulo p^2 , one can relate the obstruction classes to lifting Frobenius on X and Z . Since Z is a toric variety, its reduction modulo p is F -liftable, and so is the reduction of X . By the assumed case of Conjecture 8.1.1, the reduction of X is a toric variety. Finally, one uses a strengthening of Jaczewski's characterisation of toric varieties [Jac94] from [KW15] to conclude that X must be a toric variety. We obtain the following result as a by-product.

Theorem 8.1.5. *Let $\varphi: Z \rightarrow X$ be a surjective morphism from a complete toric variety Z to a smooth projective variety X defined over a field k of characteristic zero. Then X satisfies Bott vanishing (see Section 1.2).*

We do not know of an example of a non-toric smooth projective rationally connected variety in characteristic zero which satisfies Bott vanishing.

Theorem 8.1.6 (see Theorem 8.5.1). *Let (X, D) be a projective nc pair over an algebraically closed field k of positive characteristic. The following conditions are equivalent.*

- (i) (X, D) is F -liftable and $\omega_X(D)$ is numerically trivial,
- (ii) X is F -split and $\Omega_X^1(\log D)$ becomes trivial on a finite étale cover of X ,
- (iii) X admits a finite étale cover $f: Y \rightarrow X$ whose Albanese map $a: Y \rightarrow \text{Alb}(Y)$ is a toric fibration over an ordinary abelian variety with toric boundary $f^{-1}(D)$.

In this generalisation of the main result of [MS87], we say that the pair (X, D) is F -liftable if there exists a lifting $(\tilde{X}, \tilde{D})$ of (X, D) modulo p^2 such that $\tilde{D}$ has relative normal crossings, and a lifting $\tilde{F}$ of Frobenius to $\tilde{X}$ which is compatible with $\tilde{D}$ in the sense that

$$\tilde{F}^* \tilde{D} = p \tilde{D}.$$

The proof is not less complicated than the one in [MS87]. We closely follow their strategy with a few differences. The case $H^1(X, \mathcal{O}_X) = 0$ is handled by lifting to characteristic zero, applying the aforementioned theorem of Winkelmann [Win04], and reducing the setting back to positive characteristic. Our result may be seen as a positive characteristic analogue of Winkelmann's theorem. Interestingly, it also shows that Winkelmann's theorem, in a strict sense, is false in characteristic p , while Winkelmann's theorem indicates that a natural analogue of Theorem 8.5(iii) $\Rightarrow$ (ii) is false in characteristic zero (see Remark 8.5.4). But the most important difference comes from the fact that [MS87] used Yau's work on the Calabi conjecture for a lifting of X to characteristic zero in one of the key steps. In the log setting, such results are unfortunately unavailable, and instead we reduce to the case of a finite ground field, then lift to characteristic zero together with Frobenius, and use results of [NZ10] and [GKP16] on varieties with a polarised endomorphism.

Polarised endomorphisms

A polarised endomorphism of a projective variety X is by definition one which extends to an endomorphism of an ambient projective space. It seems that there are many similarities between Frobenius liftings and polarised endomorphisms. Over a finite field, a power of the Frobenius morphism is a polarised endomorphism. In characteristic zero, the toric Frobenius is an example of such, as is the Serre–Tate canonical lifting of Frobenius on an ordinary abelian variety. As in the case of Frobenius liftability, varieties admitting a polarised endomorphism satisfy $\kappa(X) \leq 0$ (see [NZ10]).

If $\rho(X) = 1$, then every endomorphism is polarised. In this case, a conjecture by Amerik [Ame97] states that if a smooth rationally connected X with $\rho(X) = 1$ admits a polarised endomorphism, then $X \simeq \mathbb{P}^n$. This has been proved in certain cases, for example, when $\dim X \leq 3$ [ARVdV99], X is a hypersurface [Bea01], or X is a rational

homogeneous space [PS89]. Moreover, Zhang showed that a smooth Fano threefold with a polarised endomorphism is rational [Zha10]. In general, it has been asked if every smooth rationally connected variety with a polarised endomorphism is toric (see, for instance, [Fak03]).

8.1.1. Methods and further results

The proofs of the results mentioned above rely on a variety of tools: deformation theory, toric fibrations, and most importantly the study of Frobenius liftings. Let us mention here some of our observations which might be of independent interest. Some of these results are valid for arbitrary (non-necessarily smooth) k -schemes.

The Swiss Army knife in our toolbox is the technique of descending Frobenius liftability (see Theorem 8.3.20): if Y is F -liftable and $\pi: Y \rightarrow X$ is a morphism, then under certain assumptions on π , one can deduce the F -liftability of X . In the special case of a blow-up along a smooth center $Z \subseteq X$, there is an induced Frobenius lifting on X which is suitably compatible with Z (Lemma 8.3.27).

In [Zda17], the third author provided an explicit functorial construction of a lifting modulo p^2 associated to a Frobenius splitting (see Theorem 8.2.12). Using this construction and some properties of the Witt vector ring $W_2(X)$ we prove that there is at most one lifting of X modulo p^2 to which the Frobenius morphism lifts (Theorem 8.3.38). Surprisingly, this lifting is isomorphic to any lifting induced by a Frobenius splitting. This observation allows us to descend Frobenius liftability along good quotients by actions of linearly reductive groups (Theorem 8.3.20(c)). We also employ the Witt vector ring $W_2(X)$ to show a curious base change property of Frobenius liftings (Proposition 8.3.30 and Corollary 8.3.32).

We provide a few results on toric varieties in families. First, if a general fibre of a smooth projective family is toric, then the generic fibre must be toric as well (Corollary 8.4.5). To this end, we need an auxiliary result (Proposition 8.4.2) on the constructibility of the locus where a given vector bundle splits. Second, given a smooth projective family together with a relative normal crossings divisor over a connected base, if a *single* geometric fibre is a toric pair, then the entire family has to be a toric fibration (Proposition 8.4.10). In other words, using the terminology of Hwang and Mok, toric pairs are globally rigid.

8.1.2. Outline of the chapter

In Section 8.2, we gather some preliminary results and notions: toric varieties over an arbitrary base (i.e., toric fibrations) in §8.2.1, deformation theory including the crucial technique of descending deformations in §8.2.2, some basic facts about normal crossing pairs in §8.2.3, a review of the Cartier operator in §8.2.4, and Frobenius splittings (especially in the relative setting) in §8.2.5. Section 8.3 is a standalone mini-treatise on Frobenius liftability. We hoped to provide the reader with a good reference, and so some

statements are much more general than what we need in the sequel. Many of the results are new (to the best of our knowledge). In Section 8.4, we deal with the abstract problem of showing that the generic fibre of a family is toric if a general one is (§8.4.1) and with the opposite problem of proving that if the generic fibre of a degeneration is a toric pair, then so is the special one (§8.4.2). These results are then applied to show a general ‘global rigidity’ result for toric pairs (§8.4.3) and Theorem 8.1.4 (§8.4.4). Sections 8.5, 8.6, and 8.7 deal with Theorems 8.1.6, 8.1.2, and 8.1.3, respectively.

8.1.3. Notation and conventions

In this chapter it is very important to distinguish between the absolute and the relative Frobenius. Therefore we shall use a slightly different notation than that in the other parts of this thesis.

Let p be a prime number. If X is an $\mathbb{F}_p$ -scheme, we denote by $F_X: X \rightarrow X$ its absolute Frobenius morphism (i.e., the identity on the underlying topological space and the p -th power map on the structure sheaf). If $f: X \rightarrow S$ is a morphism of $\mathbb{F}_p$ -schemes, the relative Frobenius $F_{X/S}: X \rightarrow X'$ is the unique morphism making the following diagram commute

$$\begin{array}{ccccc}
 X & & & & \\
 \searrow^{F_{X/S}} & & \xrightarrow{F_X} & & \\
 & X' & \xrightarrow{W_{X/S}} & X & \\
 \downarrow f & \square & \downarrow f & & \\
 S & \xrightarrow{F_S} & S & &
 \end{array} \tag{8.1}$$

The S -scheme X' is called the *Frobenius twist* of X relative to S . Note that if $S = \operatorname{Spec} k$ for a perfect field k , then $W_{X/S}$ is an isomorphism of schemes.

Most of the time we shall be working with a fixed perfect field k of characteristic $p > 0$. We denote by $W_n(k)$ its ring of Witt vectors of length n (see Definition 2.3.1). The unique endomorphism restricting to the Frobenius F_k modulo p is denoted by $\sigma: W_n(k) \rightarrow W_n(k)$ or $\sigma: W(k) \rightarrow W(k)$.

We often set $S = \operatorname{Spec} k$ and $\tilde{S} = \operatorname{Spec} W_2(k)$. If X is an S -scheme, a *lifting of X (modulo p^2)* is a flat $\tilde{S}$ -scheme $\tilde{X}$ with an isomorphism $\tilde{X} \times_{\tilde{S}} S \simeq X$ (see Definition 2.3.2).

A *lifting of Frobenius* on X to $\tilde{X}$ is a morphism $\tilde{F}_X: \tilde{X} \rightarrow \tilde{X}$ restricting to F_X modulo p . We shall give a thorough discussion of these in §8.3.

8.2. Preliminaries

8.2.1. Toric varieties and toric fibrations

A toric variety (see e.g. [Ful93]) is by definition a normal algebraic variety X over an algebraically closed field k together with an effective action of a torus $T \simeq \mathbb{G}_m^n$ with

a dense orbit. Such varieties admit a completely combinatorial description in terms of *rational polyhedral fans* Σ in $N_{\mathbb{R}} = \text{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$. This description is independent of the field k , and in particular every toric variety has a natural model $X(\Sigma)$ over $\mathbb{Z}$. We denote by $D(\Sigma) \subseteq X(\Sigma)$ the *toric boundary* of $X(\Sigma)$, that is, the complement of the open orbit. Sometimes we shall abuse the terminology and say that a variety X is a toric variety meaning that it admits the structure of a toric variety.

In what follows, we will have to deal with families of toric varieties over more general bases (such as the rings $W_2(k)$ and $W(k)$ or the Albanese variety in the statement of Conjecture 8.1.1). There is more than one sensible definition of a ‘toric variety over a base scheme S ,’ and we decided to settle on the following.

Definition 8.2.1. Let S be a scheme.

- (a) A *torus* over S is an S -group scheme T which is étale-locally isomorphic to $\mathbb{G}_{m,S}^n$ for some $n \geq 0$.
- (b) A *toric fibration* over S is a flat S -scheme X together with an action of a torus T such that étale-locally on S there exist isomorphisms $T \simeq \mathbb{G}_{m,S}^n$ and $X \simeq X(\Sigma)_S$ for some rational polyhedral fan $\Sigma \subseteq \mathbb{R}^n$.

In particular, we do not require the torus T to be *split*, i.e. that $T \simeq \mathbb{G}_{m,S}^n$. In general, $\text{Aut}_S(\mathbb{G}_m^n) \simeq GL(n, \mathbb{Z})_S$, and hence tori of dimension n over S are parametrised by the étale non-abelian cohomology $H^1(S, GL(n, \mathbb{Z}))$; if S is connected and normal, this is $\text{Hom}(\pi_1(S, \bar{s}), GL(n, \mathbb{Z}))$. Since $\pi_1(S, \bar{s})$ is pro-finite and $GL(n, \mathbb{Z})$ is discrete, every such homomorphism has finite image, and we see that every torus over a normal S becomes split on a finite étale cover of S . If T is split, then we will say that X is a *split* toric fibration. If $X \rightarrow S$ is a toric fibration for a torus T , the dense open orbit $U \subseteq X$ is a T -torsor over S . If this torsor is trivial (i.e., $U \rightarrow S$ admits a section), we shall say that X is a *trivial* toric fibration. If $X \rightarrow S$ is a split toric fibration under a torus T , then the usual description of toric varieties over algebraically closed fields generalises as follows: there exists a fan Σ in $N_{\mathbb{R}}$ and a T -torsor $U \rightarrow S$ such that

$$X \simeq U \times^T X(\Sigma)_S$$

as S -schemes with a T -action. In particular, if X is split and trivial, then $X \simeq X(\Sigma)_S$. In any case, such an isomorphism exists étale locally on S , and the toric boundaries $D(\Sigma)_S$ glue together to give a global *toric boundary* D , which is a closed subscheme supported on $X \setminus U$.

If X is smooth over S , then D has normal crossings relative to S , and $\Omega_{X/S}^1(\log D)$ is trivial if moreover T is split. We call a pair (X, D) of a smooth S -scheme X and a divisor with relative normal crossings $D \subseteq X$ a *toric pair* if it arises via the above construction. In §8.4.3, we will show that the torus T and its action on X are (essentially) uniquely defined by the pair (X, D) , at least if X is projective over S .

- Example 8.2.2.** (1) The projectivisation $\mathbb{P}_S(E) \rightarrow S$ of a vector bundle E on S has toric fibres, but it admits the structure of a split toric fibration only when E is a direct sum of line bundles.
- (2) Let $S = \mathbb{A}_k^1$ and $X = \mathbb{A}_k^2 \setminus \{(0,0)\}$, treated as an S -scheme via the first projection. Then $X \rightarrow S$ has a natural action of $\mathbb{G}_{m,S}$, its fibres are toric varieties, but it is not a toric fibration.
- (3) Let $S = \operatorname{Spec} \mathbb{Q}$ and let $T = \{x^2 + y^2 = 1\} \subseteq \mathbb{A}_{\mathbb{Q}}^2$ be the circle group. Let $X = \{x^2 + y^2 = z^2\} \subseteq \mathbb{P}_{\mathbb{Q}}^2$ be the projective closure of T . The action of T on itself extends to X , and $D = X \setminus T$ is a point of degree two. The map $X \rightarrow S$ is a toric fibration under T which becomes split and trivial after base change to $\mathbb{Q}(\sqrt{-1})$.

8.2.2. Deformation theory

We use freely the standard results of deformation theory, for which we refer to [Har10, Ill71]. If k is a perfect field of characteristic $p > 0$, we denote by $\mathbf{Art}_{W(k)}(k)$ the category of Artinian local $W(k)$ -algebras with residue field k . If X is a k -scheme, we denote by

$$\operatorname{Def}_X: \mathbf{Art}_{W(k)}(k) \longrightarrow \mathbf{Set}, \quad A \mapsto \left\{ \begin{array}{l} \text{isom. classes of flat } \tilde{X}/\operatorname{Spec} A \\ \text{with an identification } \tilde{X} \otimes_A k \simeq X \end{array} \right\}$$

its deformation functor. Similarly, if $Z \subseteq X$ is a closed subscheme, we denote by $\operatorname{Def}_{X,Z}$ the functor of deformations $\tilde{X}$ of X together with an embedded deformation $\tilde{Z} \subseteq \tilde{X}$ of Z .

Lemma 8.2.3 (Proposition 2.3.3, [LS14, Proposition 2.1]). *Let $\pi: Y \rightarrow X$ be a morphism of k -schemes such that $\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_Y$ and $R^1 \pi_* \mathcal{O}_Y = 0$. Then there exists a natural transformation of deformation functors*

$$\pi_*: \operatorname{Def}_Y \longrightarrow \operatorname{Def}_X,$$

associating to every lifting $\tilde{Y} \in \operatorname{Def}_Y(A)$ a lifting $\tilde{X} = \pi_(\tilde{Y}) \in \operatorname{Def}_X(A)$ together with an A -morphism $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ lifting π . More precisely, the structure sheaf of $\tilde{X}$ is defined by the formula $\mathcal{O}_{\tilde{X}} = \pi_* \mathcal{O}_{\tilde{Y}}$, and consequently the pair $(\tilde{X}, \tilde{\pi})$ is unique up to a unique isomorphism inducing the identity on X and $\tilde{Y}$.*

In the case of blow-ups of smooth varieties in smooth centers, one can say a bit more. Let X be a smooth scheme over k , let $Z \subseteq X$ be a smooth closed subscheme of codimension > 1 , let $\pi: Y \rightarrow X$ be the blowing-up of X along Z , and let $E = \pi^{-1}(Z)$ be the exceptional divisor.

Lemma 8.2.4 ([LS14, Proposition 2.2]). *In the above situation, the natural transformations of deformation functors*

$$\operatorname{Def}_{X,Z} \longrightarrow \operatorname{Def}_{Y,E} \longrightarrow \operatorname{Def}_Y$$

are isomorphisms. The composition $\operatorname{Def}_Y \simeq \operatorname{Def}_{X,Z} \rightarrow \operatorname{Def}_X$ equals π_ .*

8.2.3. Normal crossing pairs

We recall some basics on normal crossings (nc) pairs over general base schemes.

Definition 8.2.5. Let S be a scheme. An *nc pair* over S is a pair (X, D) of a smooth scheme X over S and a divisor $D \subseteq X$ with normal crossings relative to S (see [SGA 1, Exp. XIII §2.1], [Kat70, §4]), that is, such that étale-locally on X there exists an étale map $h: X \rightarrow \mathbb{A}_S^n$ with $D = h^*(\{x_1 \cdots x_n = 0\})$.

Remark 8.2.6. It is often more natural and convenient to work in the framework of log geometry than with nc pairs. We decided not to do so, as we will need only a small subset of the theory which is easily handled by the classical results on nc pairs. A reader familiar with the language of log geometry will notice that everything we do with nc pairs can easily be phrased using log schemes instead, and that many of our results have natural logarithmic analogues. More precisely, the dictionary is as follows.

An *nc log scheme* over a scheme S is an fs log scheme $(X, \mathcal{M}_X)$ over $(S, \mathcal{O}_S^*)$ which is log smooth and such that the underlying map of schemes $X \rightarrow S$ is smooth. Equivalently, an fs log scheme $(X, \mathcal{M}_X)$ over $(S, \mathcal{O}_S^*)$ is nc if and only if étale locally on X , there exists a strict $h: (X, \mathcal{M}_X) \rightarrow \mathbb{A}_S^n = \text{Spec}(\mathbb{N}^n \rightarrow \mathcal{O}_S[x_1, \dots, x_n])$. Thus, there is a natural equivalence of groupoids, compatible with base change

$$(\text{nc log schemes}/(S, \mathcal{O}_S^*)) \xrightarrow{\sim} (\text{nc pairs}/S).$$

Given an nc pair (X, D) over S , one defines the module of differentials $\Omega_{X/S}^1(\log D)$, see [Kat70, §4]. It is locally free and its determinant is isomorphic to $\omega_{X/S}(D)$. Classical deformation theory for smooth schemes has a natural analogue for nc pairs, and in particular is controlled by the groups $\text{Ext}^i(\Omega_{X/S}^1(\log D), \mathcal{O}_X)$ for $i = 0, 1, 2$, see [Kat96]. Moreover, if D is the union of smooth divisors $D_1, \dots, D_r$, there is a short exact sequence

$$0 \longrightarrow \Omega_{X/S}^1 \longrightarrow \Omega_{X/S}^1(\log D) \longrightarrow \bigoplus_{i=1}^r \mathcal{O}_{D_i} \longrightarrow 0.$$

Lemma 8.2.7. *Let (X, D) be an nc pair over a scheme S , let Y be a smooth scheme over S , and let $f: X \rightarrow Y$ be a morphism over S . Then (X, D) is an nc pair over Y if and only if the morphism*

$$f^* \Omega_{Y/S}^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow \Omega_{X/S}^1(\log D)$$

is injective and its cokernel $\Omega_{X/Y}^1(\log D)$ is locally free.

Proof. If (X, D) is an nc pair over Y , then the assertions are clearly satisfied. For the proof in the other direction, we use the language of log geometry. Accordingly, we treat $(X, D) \rightarrow Y$ as a morphism of log schemes. The sheaf $\Omega_{X/Y}^1(\log D)$ is isomorphic to the sheaf of relative log differentials of $(X, D) \rightarrow Y$ and therefore, since it is locally free, we may apply [Kat89, Proposition 3.12] to see that the morphism $(X, D) \rightarrow Y$ is log smooth.

The scheme X is smooth over Y and hence $(X, D) \rightarrow Y$ is an nc log scheme over Y . We conclude using the equivalence between nc log schemes and nc pairs (Remark 8.2.6). $\square$

8.2.4. The Cartier isomorphism

Here, we present basic properties of the Cartier isomorphism and its logarithmic variant. Let $X \rightarrow S$ be a smooth morphism of schemes over k , and let $\Omega_{X/S}^\bullet$ be its de Rham complex. Moreover, let $B_{X/S}^i$ (resp. $Z_{X/S}^i$) be the i^{th} coboundaries (resp. cocycles) in the $\mathcal{O}_{X'}$ -linear complex $F_{X/S*}\Omega_{X/S}^\bullet$, where X' is the Frobenius twist of X relative to S .

More generally, if (X, D) is an nc pair over S , then we denote by $B_{X/S}^i(\log D)$ (resp. $Z_{X/S}^i(\log D)$) the i^{th} coboundaries (resp. cocycles) in the $\mathcal{O}_{X'}$ -linear log de Rham complex $F_{X/S*}\Omega_{X/S}^\bullet(\log D)$. In analogy with the holomorphic Poincaré lemma, the following result describes the cohomology of the de Rham complex in characteristic p .

Theorem 8.2.8 ([Kat70, Theorem 7.2]). *Let $X \rightarrow S$ be a smooth morphism of schemes over k . Then there exists a unique system of isomorphisms of $\mathcal{O}_{X'}$ -modules*

$$C_{X/S}^{-1}: \Omega_{X'/S}^j \xrightarrow{\sim} \mathcal{H}^j(F_{X/S*}\Omega_{X/S}^\bullet)$$

satisfying the conditions

- (i) $C_{X/S}^{-1}(1) = 1$,
- (ii) $C_{X/S}^{-1}(\omega) \wedge C_{X/S}^{-1}(\eta) = C_{X/S}^{-1}(\omega \wedge \eta)$ for local sections $\omega \in \Omega_{X'/S}^i$ and $\eta \in \Omega_{X'/S}^j$,
- (iii) $C_{X/S}^{-1}: \Omega_{X'/S}^1 \rightarrow \mathcal{H}^1(F_{X/S*}\Omega_{X/S}^\bullet)$ is defined by $d(g \otimes 1) \mapsto [g^{p-1}dg]$.

The inverse isomorphisms give rise to short exact sequences

$$0 \longrightarrow B_{X/S}^j \longrightarrow Z_{X/S}^j \longrightarrow \Omega_{X'/S}^j \longrightarrow 0,$$

inducing Cartier morphisms $C_{X/S}: Z_{X/S}^j \rightarrow \Omega_{X'/S}^j$.

Variant 8.2.9. In the logarithmic setting, there exists a system of isomorphisms

$$\Omega_{X'/S}^j(\log D') \xrightarrow{\sim} \mathcal{H}^j(F_{X/S*}\Omega_{X/S}^\bullet(\log D)),$$

where D' is the preimage of D in X' . Moreover, we have short exact sequences

$$0 \longrightarrow B_{X/S}^j(\log D) \longrightarrow Z_{X/S}^j(\log D) \longrightarrow \Omega_{X'/S}^j(\log D') \longrightarrow 0,$$

inducing logarithmic Cartier morphisms $C_{X/S}: Z_{X/S}^j(\log D) \rightarrow \Omega_{X'/S}^j(\log D')$.

Remark 8.2.10. The top Cartier morphism

$$C_{X/S}: F_{X/S*}\omega_{X/S} = Z_{X/S}^r \longrightarrow \omega_{X'/S}$$

for $r = \dim(X/S)$ is nothing but the *Frobenius trace map* $\mathrm{Tr}_{X/S}$. As explained in Subsection 2.2.1, its importance stems from the fact that the natural pairing

$$F_{X/S*} \mathcal{H}om(\mathcal{M}, \omega_{X/S}) \otimes F_{X/S*} \mathcal{M} \longrightarrow \omega_{X'/S}$$

sending $f \otimes m$ to $\mathrm{Tr}_{X/S}(f(m))$ induces the *Grothendieck duality*

$$\mathcal{H}om(F_{X/S*} \mathcal{M}, \omega_{X'/S}) \simeq F_{X/S*} \mathcal{H}om(\mathcal{M}, \omega_{X/S}).$$

8.2.5. Frobenius splittings

Let k be a perfect field of characteristic $p > 0$.

Definition 8.2.11. Let X be a k -scheme.

- (a) In this chapter, we will say that an *F-split scheme* is a pair (X, σ_X) of a k -scheme X and a Frobenius splitting $\sigma_X: F_{X*} \mathcal{O}_X \rightarrow \mathcal{O}_X$ on X ; however, whenever it is convenient, we drop the Frobenius splitting σ_X from the notation;
- (b) A *morphism of F-split schemes* $f: (X, \sigma_X) \rightarrow (Y, \sigma_Y)$ is a morphism $f: X \rightarrow Y$ for which the diagram

$$\begin{array}{ccccc} F_{Y*} f_* \mathcal{O}_X & \xlongequal{\quad} & f_* F_{X*} \mathcal{O}_X & \xrightarrow{f_*(\sigma_X)} & f_* \mathcal{O}_X \\ \uparrow F_{Y*}(f^*) & & & & \uparrow f^* \\ F_{Y*} \mathcal{O}_Y & \xrightarrow{\quad \sigma_Y \quad} & & & \mathcal{O}_Y. \end{array}$$

commutes. This defines the category **F-split** of *F-split schemes*.

Theorem 8.2.12 ([Zda17, Theorem 3.6]). *There exists a functor*

$$(X, \sigma) \mapsto \tilde{X}(\sigma): \mathbf{F}\text{-split} \longrightarrow (\text{flat schemes}/W_2(k))$$

together with a functorial identification $X \simeq \tilde{X}(\sigma) \otimes_{W_2(k)} k$.

For a group G acting on Y , we say that a morphism $\pi: Y \rightarrow X$ is a *good quotient* by G , if it is G -invariant, affine, and $\mathcal{O}_X = (\pi_* \mathcal{O}_Y)^G$. The following lemma will be applied for G being a torus or a finite group of order prime to p .

Lemma 8.2.13. *Let Y be an F-split scheme of finite type over k , let G be a linearly reductive algebraic group acting on Y , and let $\pi: Y \rightarrow X$ be a good quotient. Then there exist splittings $\sigma_Y: F_{Y*} \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ and $\sigma_X: F_{X*} \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $\pi: (Y, \sigma_Y) \rightarrow (X, \sigma_X)$ is a morphism of F-split schemes. In particular, π admits a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$.*

Proof. The relative Frobenius $F_{G/k}: G \rightarrow G'$ is a group homomorphism and the relative Frobenius $F_{Y/k}: Y \rightarrow Y'$ is $F_{G/k}$ -equivariant. It follows that there is a natural linear

G -action on $\mathrm{Hom}((F_{Y/k})_* \mathcal{O}_Y, \mathcal{O}_{Y'})$. Furthermore, the ‘evaluation at one’ map

$$\varepsilon: \mathrm{Hom}((F_{Y/k})_* \mathcal{O}_Y, \mathcal{O}_{Y'}) \longrightarrow H^0(Y', \mathcal{O}_{Y'})$$

is a homomorphism of G -representations.

If Y is integral and proper, then this is a map from a finite-dimensional G -representation to k , and since Y is F -split, this map is surjective. By the linear reductivity of G , this map admits a G -equivariant splitting, and this way we obtain a G -invariant Frobenius splitting of Y .

In general, let $V = \varepsilon^{-1}(k)$, which is the increasing union of G -representations V_i of finite dimension over k , and is endowed with a G -invariant map $\varepsilon: V \rightarrow k$. Moreover, the map ε is surjective since Y is Frobenius split. For some i , the restriction $\varepsilon|_{V_i}: V_i \rightarrow k$ is surjective, and since G is linearly reductive, there exists a G -equivariant splitting $s: k \rightarrow V_i$ of $\varepsilon|_{V_i}$.

In particular, Y admits a G -invariant Frobenius splitting $\sigma_Y = s(1)$. This Frobenius splitting preserves G -invariant sections of $\mathcal{O}_Y$, and hence it descends to a Frobenius splitting on X . Thus $\pi: (Y, \sigma_Y) \rightarrow (X, \sigma_X)$ is a map in the category **F-split**. Applying the canonical lifting functor from Theorem 8.2.12 yields the desired lifting $\tilde{\pi}: \tilde{Y}(\sigma_Y) \rightarrow \tilde{X}(\sigma_X)$ of π . $\square$

The method of proof of Lemma 8.2.13 yields the following interesting result which we will not need in the sequel.

Lemma 8.2.14. *Let Y be an F -split scheme over k and let G be a finite group of order prime to p acting on Y . Then there exists a lifting $\tilde{Y}$ of Y together with a lifting of the G -action.*

We shall need the following lemma describing basic properties of F -split schemes.

Lemma 8.2.15. *Let X be a proper F -split scheme over an algebraically closed field k . Then the following hold.*

- (a) *The map $F_X^*: H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$ is bijective for all $i \geq 0$.*
- (b) *The cohomology groups $H^i(X, B_X^1)$ vanish for all $i \geq 0$, where $B_X^1 = F_{X*} \mathcal{O}_X / \mathcal{O}_X$.*
- (c) *If X is smooth, then the Albanese variety $\mathrm{Alb} X$ is ordinary.*
- (d) *Every étale scheme Y over X is F -split.*

Proof. For (a) and (b), we use the long exact sequence of cohomology associated with

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_{X*} \mathcal{O}_X \longrightarrow B_X^1 \longrightarrow 0,$$

and the fact that an injective p^{-1} -linear endomorphism of a finite dimensional vector space over a perfect field is bijective.

To prove (c), we reason as follows. By [MS87, Lemma 1.3] we see that $H^1(\mathrm{Alb} X, \mathcal{O}_{\mathrm{Alb} X})$ injects into $H^1(X, \mathcal{O}_X)$. By (a) the Frobenius action on the latter group is bijective, and hence it is also bijective on the former. This implies that $\mathrm{Alb} X$ is ordinary.

For (d), let $f: Y \rightarrow X$ be an étale morphism. Then the square

$$\begin{array}{ccc} Y & \xrightarrow{F_Y} & Y \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{F_X} & X \end{array}$$

is cartesian by [SGA 5, XIV=XV §1 n°2, Pr. 2(c)], and hence $f^*F_{X*}\mathcal{O}_X \simeq F_{Y*}\mathcal{O}_Y$ by flat base change. Thus given a splitting $\sigma: F_{X*}\mathcal{O}_X \rightarrow \mathcal{O}_X$, applying f^* we obtain a morphism

$$f^*(\sigma): F_{Y*}\mathcal{O}_Y \simeq f^*F_{X*}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$$

which is a Frobenius splitting on Y . □

We now turn to the study of Frobenius splittings in a relative situation.

Definition 8.2.16. Let $f: X \rightarrow S$ be a morphism of k -schemes. An F -splitting relative to f is an $\mathcal{O}_{X'}$ -linear splitting $\sigma_{X/S}$ of $F_{X/S}^*: \mathcal{O}_{X'} \rightarrow F_{X/S*}\mathcal{O}_X$, where X' is the Frobenius twist of X relative to S .

As in the non-relative situation (see Chapter 2), there is a one-to-one correspondence between Frobenius splittings and splittings of the Frobenius trace map.

Proposition 8.2.17. Let $X \rightarrow S$ be a smooth morphism of k -schemes. Then there is a one-to-one correspondence between splittings $\delta_{X/S}$ of the Frobenius trace map $\mathrm{Tr}_{X/S}$ and F -splittings $\sigma_{X/S}$ relative to f .

Proof. Let $\sigma_{X/S}$ be an F -splitting relative to f . By applying $\mathrm{Hom}(-, \omega_{X'/S})$ to the splitting

$$\mathcal{O}_{X'} \xrightarrow{F_{X/S}^*} F_{X/S*}\mathcal{O}_X \xrightarrow{\sigma_{X/S}} \mathcal{O}_{X'}$$

we get by the Grothendieck duality (see Remark 8.2.10) a splitting

$$\omega_{X'/S} \xrightarrow{\delta_{X/S}} F_{X/S*}\omega_{X/S} \xrightarrow{\mathrm{Tr}_{X/S}} \omega_{X'/S}.$$

The correspondence in the other direction is constructed analogously. □

The following auxiliary base change result shows in particular that the fibres of a smooth relatively F -split morphism are F -split.

Lemma 8.2.18 (cf. [PSZ13, Lemma 2.18]). *Consider a Cartesian diagram of k -schemes*

$$\begin{array}{ccc} W & \xrightarrow{\psi} & X \\ f' \downarrow & & \downarrow f \\ Z & \longrightarrow & S, \end{array}$$

with $f: X \rightarrow S$ smooth, and let $\delta_{X/S}$ be a splitting of $\mathrm{Tr}_{X/S}$. Then there exists a splitting $\delta_{W/Z}$ of $\mathrm{Tr}_{W/Z}$ induced by the following commutative diagram.

$$\begin{array}{ccc} \psi^* \omega_{X'/S} & \xrightarrow{\sim} & \omega_{W'/Z} \\ \psi^* \delta_{X/S} \downarrow & & \downarrow \delta_{W/Z} \\ \psi^* (F_{X/S})_* \omega_{X/S} & \xrightarrow{\sim} & (F_{W/Z})_* \omega_{W/Z}, \end{array}$$

where X' and W' are the Frobenius twists of X and W relative to S and Z , respectively.

Proof. The arrow $\psi^* (F_{X/S})_* \omega_{X/S} \rightarrow (F_{W/Z})_* \omega_{W/Z}$ in the above diagram comes from the cohomological base change for the diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ F_{W/Z} \downarrow & & \downarrow F_{X/S} \\ W' & \longrightarrow & X'. \end{array}$$

To conclude the proof of the lemma it is enough to show that this arrow is an isomorphism, and this is clear because $F_{X/S}^* \omega_{X/S}$ is a vector bundle (f is smooth, so $F_{X/S}$ is finite and flat) and the above diagram is Cartesian. $\square$

As in the non-relative situation (see Proposition 2.2.2), to every F -splitting we can associate a $\mathbb{Q}$ -divisor.

Proposition 8.2.19 ([Sch09] and [PSZ13]). *Let $f: X \rightarrow S$ be a smooth morphism of k -schemes. Then to every splitting $\delta_{X/S}$ of $\mathrm{Tr}_{X/S}$ we can canonically associate an effective $\mathbb{Q}$ -divisor $\Delta_{\delta_{X/S}}$ on X such that*

- (a) $\Delta_{\delta_{X/S}} \sim_{\mathbb{Q}} -K_{X/S}$,
- (b) *if f has connected fibres, then $\Delta_{\delta_{X/S}}$ is horizontal, i.e., it does not contain any fibre.*

Proof. By adjunction, $\delta_{X/S}$ induces a morphism $F_{X/S}^* \omega_{X'/S} \rightarrow \omega_{X/S}$ which is equivalent to $\mathcal{O}_X \rightarrow \omega_{X/S}^{1-p}$, and so we get a divisor

$$D_{\delta_{X/S}} \sim (1-p)K_{X/S}.$$

Set $\Delta_{\delta_{X/S}} = \frac{1}{p-1} D_{\delta_{X/S}}$. The restriction of this $\mathbb{Q}$ -divisor to a fibre of f is non-zero as it corresponds to a splitting of the Frobenius trace map on this fibre (see Lemma 8.2.18). $\square$

For a smooth k -variety X and a splitting δ_X of $\mathrm{Tr}_{X/k}$, we denote the corresponding $\mathbb{Q}$ -divisor by Δ_{δ_X} .

Remark 8.2.20. In the setting of Lemma 8.2.18, we have $\psi^*\Delta_{\delta_{X/S}} = \Delta_{\delta_{W/Z}}$ (see the commutative diagram in the statement of this lemma).

Remark 8.2.21. It is easy to see that the coefficients of $\Delta_{\delta_{X/S}}$ are at most one. When $S = \mathrm{Spec} k$ (which is the only case in which we will apply this observation), this follows from Proposition 2.2.4.

Remark 8.2.22. It is not necessary to assume that $f: X \rightarrow S$ is smooth in order to be able to associate a $\mathbb{Q}$ -divisor to a relative F -splitting. A far more general setting is described in [PSZ13].

8.3. Frobenius liftability

Throughout this section we fix a perfect field k of characteristic $p > 0$ (sometimes assumed algebraically closed). First, set $S = \mathrm{Spec} k$ and $\tilde{S} = \mathrm{Spec} W_2(k)$ (see §8.1.3). If X is a scheme over S and $\tilde{X}$ is a lifting of X to $\tilde{S}$, a *lifting of Frobenius* on X to $\tilde{X}$ is a morphism $\tilde{F}_X: \tilde{X} \rightarrow \tilde{X}$ restricting to F_X on X . It automatically commutes with the map $\sigma: \tilde{S} \rightarrow \tilde{S}$ induced by the Witt vector Frobenius on $W_2(k)$ (see §8.3.5), and hence it fits into the commutative diagram

$$\begin{array}{ccccc}
 & & X & \xrightarrow{\quad} & \tilde{X} \\
 & \nearrow F_X & \downarrow & & \downarrow \tilde{F}_X \\
 X & \xrightarrow{\quad} & \tilde{X} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 S & \xrightarrow{\quad} & \tilde{S} & \xrightarrow{\quad} & \tilde{S} \\
 \downarrow F_S & \nearrow \sigma & & & \\
 S & \xrightarrow{\quad} & \tilde{S} & &
 \end{array}$$

We call a pair $(\tilde{X}, \tilde{F}_X)$, where $\tilde{X}$ and $\tilde{F}_X$ are as above, a *Frobenius lifting* of X . If such a pair exists, we say that X is *F-liftable*. In this situation, we can form a diagram lifting the relative Frobenius diagram (8.1):

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\tilde{F}_X} & \tilde{X} \\
 \tilde{F}_{X/S} \searrow & & \downarrow \\
 \tilde{X}' & \xrightarrow{\quad} & \tilde{X} \\
 \downarrow & \square & \downarrow \\
 \tilde{S} & \xrightarrow{\tilde{F}_S = \sigma} & \tilde{S}
 \end{array} \tag{8.2}$$

Given a lifting $\tilde{X}$, the existence of $\tilde{F}_X$ is thus clearly equivalent to the existence of a lifting $\tilde{F}_{X/S}: \tilde{X} \rightarrow \tilde{X}'$ of the relative Frobenius $F_{X/S}: X \rightarrow X'$.

More generally, the above definitions work in the relative setting, that is when S is an arbitrary k -scheme endowed with a Frobenius lifting $(\tilde{S}, \tilde{F}_S)$. If $\pi: X \rightarrow S$ is a scheme over S and if $\tilde{\pi}: \tilde{X} \rightarrow \tilde{S}$ is a lifting of π , then providing a lifting $\tilde{F}_X$ of F_X to $\tilde{X}$ such that $\tilde{F}_S \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_X$ is equivalent to providing a morphism $\tilde{F}_{X/S}: \tilde{X} \rightarrow \tilde{X}'$ lifting the relative Frobenius $F_{X/S}$, where $\tilde{X}'$ is the base change of $\tilde{X}$ along $\tilde{F}_S$.

If (X, D) is an nc pair over S , a *Frobenius lifting* of (X, D) is a triple $(\tilde{X}, \tilde{D}, \tilde{F}_X)$ where $(\tilde{X}, \tilde{D})$ is an nc pair lifting (X, D) and $\tilde{F}_X$ is a lifting of F_X to X satisfying $\tilde{F}_X^*(\tilde{D}) = p\tilde{D}$. We shall study these in more detail in §8.3.4. In view of Remark 8.2.6, Frobenius liftings of (X, D) correspond to liftings of the associated log scheme $(X, \mathcal{M}_X)$.

8.3.1. Examples of F-liftable schemes

We shall be mostly concerned with smooth schemes in this chapter. Examples of F -liftable singularities were studied in [Zda17].

Example 8.3.1 (Smooth affines). Every smooth affine k -scheme is F -liftable. Indeed, the obstruction class to lifting X together with F_X lies in $\text{Ext}^1(\Omega_X^1, B_X^1) = H^1(X, T_X \otimes B_X^1)$ (see [MS87, Appendix], cf. Proposition 8.3.15(b)), which is zero.

Example 8.3.2 (Toric varieties). Every toric variety $X = X(\Sigma)_k$ over k is F -liftable [BTLM97]. More precisely, let Σ be a fan in $N_{\mathbb{R}}$ for a lattice N and let $X(\Sigma)$ be the associated toric variety over $\text{Spec } \mathbb{Z}$. The multiplication by p map $N \rightarrow N$ preserves Σ and hence it induces a morphism $\tilde{F}: X(\Sigma) \rightarrow X(\Sigma)$. Its restriction to $X(\Sigma)_{\mathbb{F}_p}$ is the absolute Frobenius.

Of course not every Frobenius lifting on a toric variety has to be of this type, e.g. any collection of homogeneous polynomials $f_0, \dots, f_n \in k[x_0, \dots, x_n]$ of degree p defines a lifting of Frobenius on $\mathbb{P}_{W_2(k)}^n$ by

$$\tilde{F}(x_0 : \dots : x_n) = (x_0^p + pf_0(x_0, \dots, x_n) : \dots : x_n^p + pf_n(x_0, \dots, x_n)).$$

The existence of such non-standard liftings is one of the main difficulties in Conjecture 8.1.1, and provides a contrast between its two extreme (toric and abelian) cases.

Example 8.3.3 (Ordinary abelian varieties). An abelian variety A over k is F -liftable if and only if it is ordinary, in which case there exists a unique Frobenius lifting $(\tilde{A}, \tilde{F}_A)$, called the Serre–Tate canonical lifting of A (see [MS87, Appendix]). It has the remarkable property that for every line bundle L on A , there exists a unique up to isomorphism line bundle $\tilde{L}$ on $\tilde{A}$ such that $\tilde{F}_A^* \tilde{L} \simeq \tilde{L}^{\otimes p}$.

Example 8.3.4 (Étale quotients of ordinary abelian varieties). Let A be an ordinary abelian variety and let $f: A \rightarrow Y$ be a finite étale surjective morphism. Replacing f by its Galois closure, we can assume that it is Galois. Then Y is F -liftable by [MS87, Theorem 2], but it need not be an abelian variety.

Example 8.3.5 (Toric fibrations over ordinary abelian varieties). In order to combine Examples 8.3.2 and 8.3.3, let A be an abelian variety, and let $f: X \rightarrow A$ be a toric fibration under a torus T (see §8.2.1). Then X is F -liftable if and only if A is ordinary (see Theorem 8.5.1).

Remark 8.3.6. In fact, in each of the above examples, the Frobenius liftings exist over $W(k)$ and not just over $W_2(k)$. We do not know an example of a smooth k -scheme admitting a Frobenius lifting only over $W_2(k)$. However, it is important to work over $W_2(k)$ as this allows for descending Frobenius liftability under appropriate finite morphisms (see Theorem 8.3.20).

Remark 8.3.7 (Partial converse to Conjecture 8.1.1). Consider the situation of the assertion of Conjecture 8.1.1: let A be an ordinary abelian variety, let $Y \rightarrow A$ be a smooth toric fibration with toric boundary $D \subseteq Y$, and let $Y \rightarrow X$ be a finite étale map. Then (Y, D) is F -liftable by Example 8.3.5, and X is F -liftable if one of the following conditions holds.

- (1) If $X = Y/G$ where G is of order prime to p (see Theorem 8.3.20(c)). Note that since $\pi_1(Y) \xrightarrow{\sim} \pi_1(A)$, passing to the Galois closure we can always assume that $X = Y/G$ for a free action of a finite group G on Y .
- (2) If the toric boundary $D \subseteq Y$ is a pull-back from X (Theorem 8.5.1(iii) $\Rightarrow$ (i)), for example if $Y = A$ (Example 8.3.4).

However, X is not F -liftable in general: if $Y = \mathbb{P}^1 \times C$ where C is an ordinary elliptic curve, and $G = \mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{P}^1$ by $(x : y) \mapsto (x + ay : y)$ and on C by translation, then the diagonal action of G on Y is free, and $X = Y/G$ is not F -liftable (see Lemma 8.7.8 and Proposition 8.7.6). It would be interesting to find an ‘if and only if’ criterion for the F -liftability of quotients $X = Y/G$ as above with G an arbitrary finite group, or even an abelian p -group.

8.3.2. Consequences of F -liftability

As mentioned in the introduction, the existence of a Frobenius lifting has strong consequences for *smooth* schemes. First, let us recall the construction of the map ξ (see (1.1) in Section 1.2, cf. [DI87, Proof of Théorème 2.1, (b)]).

Construction of the map ξ

Let $(\tilde{S}, \tilde{F}_S)$ be a Frobenius lifting of a k -scheme S , let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a k -scheme X , and let $\tilde{\pi}: \tilde{X} \rightarrow \tilde{S}$ be a smooth morphism of $W_2(k)$ -schemes such that $\tilde{F}_S \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_X$. Let $\tilde{F}_{X/S}$ be the induced lifting of the relative Frobenius. By flatness of

$\tilde{X} \rightarrow \tilde{S}$, the differential $d\tilde{F}_{X/S}$ fits into the following diagram with exact rows

$$\begin{array}{ccccccc}
 & & & & F_{X/S}^* \Omega_{X'/S}^1 & & \\
 & & & & \parallel & & \\
 0 & \longrightarrow & F_{X/S}^* \Omega_{X'/S}^1 & \longrightarrow & \tilde{F}_{X/S}^* \Omega_{\tilde{X}'/\tilde{S}}^1 & \longrightarrow & F_{X/S}^* \Omega_{X'/S}^1 \longrightarrow 0 \\
 & & \downarrow dF_{X/S}=0 & & \downarrow d\tilde{F}_{X/S} & & \downarrow dF_{X/S}=0 \\
 0 & \longrightarrow & \Omega_{X/S}^1 & \longrightarrow & \Omega_{\tilde{X}/\tilde{S}}^1 & \longrightarrow & \Omega_{X/S}^1 \longrightarrow 0 \\
 & & \parallel & & & & \\
 & & \Omega_{X/S}^1 & & & &
 \end{array}$$

Using the snake lemma, we obtain a mapping

$$\xi = \frac{d\tilde{F}_{X/S}}{p} : F_{X/S}^* \Omega_{X'/S}^1 \longrightarrow \Omega_{X/S}^1.$$

In simple terms, $\xi(\omega) = \frac{1}{p} d\tilde{F}_X(\tilde{\omega})$, where $\tilde{\omega}$ is any lifting of a local form $\omega \in F_{X/S}^* \Omega_{X'/S}^1$. In particular, if $S = \text{Spec } k$, then $\xi(df) = f^{p-1}df + dg$ where $\tilde{F}_X^*(\tilde{f}) = \tilde{f}^p + pg$.

Since $F_{X/S}^* \Omega_{X'/S}^1$ is isomorphic to $F_X^* \Omega_{X/S}^1$, we may interpret ξ as an $\mathcal{O}_X$ -linear morphism $F_X^* \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^1$.

Proposition 8.3.8 ([DI87, Proof of Théorème 2.1 and §4.1], [BTLM97, Theorem 2]). *The mapping $\xi : F_{X/S}^* \Omega_{X'/S}^1 \rightarrow \Omega_{X/S}^1$ satisfies the following properties:*

- (a) *The adjoint morphism $\xi^{\text{ad}} : \Omega_{X'/S}^1 \rightarrow F_{X/S*} \Omega_{X/S}^1$ has image in the subsheaf $Z_{X/S}^1$ of closed forms and provides a splitting of the short exact sequence*

$$0 \longrightarrow B_{X/S}^1 \longrightarrow Z_{X/S}^1 \xrightarrow{C_{X/S}} \Omega_{X'/S}^1 \longrightarrow 0.$$

- (b) *By taking exterior powers, the morphism ξ^{ad} induces splittings of the short exact sequences*

$$0 \longrightarrow B_{X/S}^i \longrightarrow Z_{X/S}^i \xrightarrow{C_{X/S}} \Omega_{X'/S}^i \longrightarrow 0,$$

as well as a quasi-isomorphism of differential graded algebras

$$\wedge^\bullet \xi^{\text{ad}} : \bigoplus_{i \geq 0} \Omega_{X'/S}^i[-i] \xrightarrow{\sim} (F_{X/S})_* \Omega_{X/S}^\bullet$$

where the maps $\wedge^i \xi^{\text{ad}} : \Omega_{X'/S}^i \rightarrow (F_{X/S})_ \Omega_{X/S}^i$ are split injections. On the level of cohomology, this map induces the Cartier isomorphism.*

- (c) *The determinant*

$$\det(\xi) : F_{X/S}^* \omega_{X'/S}^1 \longrightarrow \omega_{X/S}^1$$

corresponds to a Frobenius splitting σ of X relative to S (see §8.2.5). In particular,

the homomorphism ξ is injective.

Variant 8.3.9 (logarithmic variant of Proposition 8.3.8). If (X, D) is an nc pair over S , and if $(\tilde{X}, \tilde{D}, \tilde{F}_X)$ is a Frobenius lifting of (X, D) , we get a morphism

$$\xi: F_X^* \Omega_{X/S}^1(\log D) \rightarrow \Omega_{X/S}^1(\log D)$$

and the assertions of Proposition 8.3.8 hold in this case (cf. [DI87, §4.2]).

Corollary 8.3.10. *Let X be an F -liftable smooth and proper scheme over k . Then X is ordinary in the sense of Bloch and Kato, i.e., $H^j(X, B_X^i) = 0$ for all $i, j \geq 0$.*

Proof. This follows from the proof of [MS87, Lemma 1.1]. □

Bott vanishing

By Proposition 8.3.8(b), we have $H^j(X, \Omega_X^i \otimes \mathcal{L}) \subseteq H^j(X, \Omega_X^i \otimes \mathcal{L}^p)$ for every line bundle $\mathcal{L}$. This explains the following result.

Theorem 8.3.11 (Bott vanishing, [BTLM97, Theorem 3]). *Let X be a smooth projective F -liftable scheme over k , and let $\mathcal{L}$ be an ample line bundle on X . Then*

$$H^j(X, \Omega_X^i \otimes \mathcal{L}) = 0 \quad \text{for } j > 0, i \geq 0.$$

Moreover, if (X, D) is an F -liftable nc pair, then

$$H^j(X, \Omega_X^i(\log D) \otimes \mathcal{L}) = 0 \quad \text{for } j > 0, i \geq 0.$$

Corollary 8.3.12. *Let X be a smooth projective F -liftable scheme over k . Suppose that X is Fano (i.e., ω_X^{-1} is ample). Then $H^i(X, T_X) = 0$ for $i > 0$. In particular, X is rigid and has unobstructed deformations.*

Proof. Since $T_X \simeq \Omega_X^{n-1} \otimes \omega_X^\vee$, Bott vanishing implies that

$$H^i(X, T_X) = H^i(X, \Omega_X^{n-1} \otimes \omega_X^\vee) = 0. \quad \square$$

The above results suggest that F -liftability is a rare property. For instance, we obtain the following examples of non-liftable varieties.

Example 8.3.13. A smooth hypersurface $X \subseteq \mathbb{P}^n$ of degree $d > 1$, where $n > 1$, is not F -liftable as long as $n + d > 5$. Indeed, we may reason as follows.

- (1) If $d > n + 1$, then X has positive Kodaira dimension, which contradicts Proposition 8.3.8(c).
- (2) If $d = n + 1$, then ω_X is trivial and X is simply connected, contradicting [MS87, Theorem 2].

- (3) If $2 < d < n + 1$, then X is Fano but not rigid, contradicting Corollary 8.3.12.
- (4) If $d = 2$ and $n > 3$, then X is not F -liftable because $H^1(X, \Omega_X^{n-2}(n-3)) \neq 0$, contradicting Bott vanishing (see [BTLM97, §4.1]). See also [Zda17, Theorem 4.15].

The sheaf of ξ -invariant forms

Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a smooth k -scheme X . We can regard the induced map

$$\xi = \frac{1}{p} d\tilde{F}_X: F_X^* \Omega_X^1 \rightarrow \Omega_X^1$$

as a Frobenius-linear endomorphism of Ω_X^1 , that is

$$\xi(f \cdot \omega) = f^p \cdot \xi(\omega) \quad \text{for } f \in \mathcal{O}_X, \omega \in \Omega_X^1.$$

As observed in [MS87], if ω_X is numerically trivial, then ξ is an isomorphism, and therefore Ω_X^1 becomes trivial on a finite étale cover Y of X (see [LS77, Satz 1.4]).

A related geometric idea, which we put to good use in §8.6, is to look at the (étale) subsheaf $(\Omega_X^1)^\xi$ of ξ -invariant forms in Ω_X^1 . By the very definition, $(\Omega_X^1)^\xi$ is the étale sheaf of sections of the fixed point locus $(T^*X)^\xi$ inside the cotangent bundle.

Lemma 8.3.14. *The subscheme $(T^*X)^\xi \subseteq T^*X$ is étale over X and the étale sheaf $(\Omega_X^1)^\xi$ is a constructible sheaf of $\mathbb{F}_p$ -vector spaces on X . Over the dense open subset $U \subseteq X$ where ξ is an isomorphism, $(T^*X)^\xi$ is finite over X of degree $p^{\dim X}$ and $(\Omega_X^1)^\xi$ is an $\mathbb{F}_p$ -locally constant sheaf of rank $p^{\dim X}$.*

Proof. The assertions are local on X , so we may assume that there exists an isomorphism $\mathcal{O}_X^{\dim X} \simeq \Omega_X^1$. Since ξ as an endomorphism of Ω_X^1 is Frobenius-linear, its fixed point locus is described by the following system of equations in variables $[f_1, \dots, f_n]$:

$$[f_1, \dots, f_n] - [f_1^p, \dots, f_n^p] \cdot A = 0,$$

for some $A \in M_{n \times n}(\Gamma(X, \mathcal{O}_X))$. We immediately see that the Jacobian of the above system of equations is the identity matrix, and thus $(T^*X)^\xi$ is étale over X . The other assertions are clear in view of the fact that for a vector space V of dimension r endowed with a Frobenius-linear isomorphism $\xi: F^*V \rightarrow V$, the locus of fixed points V^ξ is isomorphic to $\mathbb{F}_p^r$. $\square$

8.3.3. Deformation theory of the Frobenius morphism

In this subsection, we first discuss obstruction classes to lifting Frobenius, as originally defined by Nori and Srinivas. Then we apply them, together with other tools, to show in Theorem 8.3.20 that under certain conditions on a morphism $\pi: Y \rightarrow X$, if Y is F -liftable, then so is X . We also discuss F -liftability of products, toric fibrations, and divisorial rings.

Obstruction classes and their functoriality

Since we are dealing with deformations of schemes and their Frobenii along the first order thickening $S \hookrightarrow \tilde{S}$, it is natural to seek an obstruction theory for lifting the Frobenius. For smooth schemes over a perfect field k , such a theory was developed by Nori and Srinivas in the appendix to [MS87]. The usual deformation theory of morphisms already tells us that

- given a morphism $f: X \rightarrow Y$ and a lifting $\tilde{Y}$, the obstruction to the existence of $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ is a class in $\text{Ext}^2(L_{X/Y}, \mathcal{O}_X) (\simeq H^2(X, T_{X/Y})$ if f is smooth),
- if we also fix $\tilde{X}$ a priori, then we have an obstruction lying in $\text{Ext}^1(Lf^*L_Y, \mathcal{O}_X) (\simeq H^1(X, f^*T_Y)$ if Y is smooth) to lifting f .

The first obstruction theory does not apply in our context of lifting $f = F_{X/k}: X \rightarrow X'$ as the two liftings $\tilde{X}$ and $\tilde{X}'$ of the source and the target are not independent, they are related by the pullback square in (8.2). The second one does, and produces an obstruction to lifting the Frobenius to a *given* lifting $\tilde{X}$.

If we do not fix $\tilde{X}$ a priori, then the classical theory is of no use, and in fact we cannot even write a suitable functor of Artin rings (one needs a lifting of Frobenius on the base). Nori and Srinivas bypass the difficulty by considering the problem of lifting (X, F_X) only to $W_n(k)$, in which case the lifting σ of Frobenius on $W_n(k)$ is unique. As in their situation, in §8.5 we shall actually need to lift not only to $W_2(k)$ but all the way to $W(k)$ to apply characteristic zero methods.

Their results are as follows.

Proposition 8.3.15 ([MS87, Appendix]). *Let X be a smooth scheme over k , and let (X_n, F_{X_n}) be a Frobenius lifting of X over $W_n(k)$. Then the following holds.*

- (a) *For every lifting X_{n+1} of X_n over $W_{n+1}(k)$ there exists an obstruction class*

$$o_{X_{n+1}}^F \in \text{Ext}^1(\Omega_X^1, F_{X*}\mathcal{O}_X)$$

whose vanishing is sufficient and necessary for the existence of a lifting $F_{X_{n+1}}$ of F_{X_n} to X_{n+1} . If the obstruction vanishes, then the space of such liftings is a torsor under $\text{Hom}(\Omega_X^1, F_{X}\mathcal{O}_X)$.*

- (b) *There exists an obstruction class*

$$o_{X_n} \in \text{Ext}^1(\Omega_X^1, B_X^1)$$

whose vanishing is sufficient and necessary for the existence of a lifting $(X_{n+1}, F_{X_{n+1}})$ of (X_n, F_{X_n}) over $W_{n+1}(k)$. If the obstruction vanishes, then the space of such liftings is a torsor under $\text{Hom}(\Omega_X^1, B_X^1)$.

- (c) *The obstruction class $o_X \in \text{Ext}^1(\Omega_X^1, B_X^1)$ to lifting X over $W_2(k)$ compatibly with the*

Frobenius morphism equals the class of the extension

$$0 \longrightarrow B_X^1 \longrightarrow Z_X^1 \xrightarrow{C_X} \Omega_X^1 \longrightarrow 0.$$

- (d) Let $(X_{n+1}, F_{X_{n+1}})$ be a lifting of (X_n, F_{X_n}) over $W_{n+1}(k)$ and suppose that the Frobenius action on $H^i(X, \mathcal{O}_X)$ is bijective for $i = 1, 2$. Then for every $\mathcal{L}_n \in \text{Pic } X_n$ such that $F_{X_n}^* \mathcal{L}_n = \mathcal{L}_n^{\otimes p}$, there exists a unique $\mathcal{L}_{n+1} \in \text{Pic } X_{n+1}$ such that $\mathcal{L}_{n+1}|_{X_n} \simeq \mathcal{L}_n$ and $F_{X_{n+1}}^* \mathcal{L}_{n+1} = \mathcal{L}_{n+1}^{\otimes p}$.

Variant 8.3.16 (logarithmic variant of Proposition 8.3.15). The above proposition can be repeated word for word for nc pairs (X, D) . In this case, the sheaves of Kähler differentials are replaced by logarithmic differentials $\Omega_X^1(\log D)$. The sheaf B_X^1 is left without change, since the 1-boundaries $B_X^1(\log D)$ of the logarithmic de Rham complex coincide with the 1-boundaries B_X^1 of the standard de Rham complex. We do not give the full proof, but remark about the main ingredients of the logarithmic version of (b) and (c).

1. As in standard deformation theory, for an nc pair (X_n, D_n) over $W_n(k)$ we notice that any two liftings over $W_{n+1}(k)$ (as nc pairs) are Zariski-locally isomorphic and the infinitesimal automorphisms are parametrised by sections of $\mathcal{H}om(\Omega_X^1(\log D), \mathcal{O}_X)$ (see [EV92, Proposition 8.22]).
2. The logarithmic analogue of Proposition 8.3.15(a) holds, and in particular for every Frobenius lifting $F_{X_n}: (X_n, D_n) \rightarrow (X_n, D_n)$ local liftings of the Frobenius morphism over $W_{n+1}(k)$ exist and are torsors under sections of $\mathcal{H}om(\Omega_X^1(\log D), F_{X*} \mathcal{O}_X)$ (see [EV92, Proposition 9.3]).

Now, to construct the obstruction class for lifting (X_n, D_n, F_{X_n}) over $W_{n+1}(k)$ we reason as follows. We take an open affine covering $\{(U_n^i, D_n^i)\}$ of (X_n, D_n) , and then lift (U_n^i, D_n^i) to some nc pairs (U_{n+1}^i, D_{n+1}^i) over $W_{n+1}(k)$. By (2) the nc pairs (U_{n+1}^i, D_{n+1}^i) admit Frobenius liftings $F_{U_{n+1}^i}: (U_{n+1}^i, D_{n+1}^i) \rightarrow (U_{n+1}^i, D_{n+1}^i)$ extending $F_{X_n}|_{U_n^i}$. We set $U_{n+1}^{ij} = U_{n+1}^i \cap U_{n+1}^j$. By (1) we may fix isomorphisms

$$\varphi_{ij}: (U_{n+1}^j, D_{n+1}^j)|_{U_{ij}} \xrightarrow{\sim} (U_{n+1}^i, D_{n+1}^i)|_{U_{ij}} \quad \text{satisfying} \quad \varphi_{ji} = \varphi_{ij}^{-1}.$$

The morphisms $F_{U_{n+1}^i}$ and $\varphi_{ij} \circ F_{U_{n+1}^j} \circ \varphi_{ji}$ are two liftings of $F_{X_n}|_{U_n^{ij}}$ and hence they give rise to local section τ_{ij} of the sheaf $\mathcal{H}om(\Omega_X^1(\log D), F_{X*} \mathcal{O}_X)$. The images of τ_{ij} under the homomorphism $\mathcal{H}om(\Omega_X^1(\log D), F_{X*} \mathcal{O}_X) \rightarrow \mathcal{H}om(\Omega_X^1(\log D), B_X^1)$ does not depend on the choices made, and give a cocycle whose cohomology class we claim is the required obstruction. For the rest of the proof, we may repeat the reasoning of [MS87, Appendix, Proposition 1 (vii)] word for word. For the proof of (c), we observe that the difference of the morphisms ξ induced by the liftings $F_{U_{ij}^i}$ and $\varphi_{ij} \circ F_{U_{ij}^j} \circ \varphi_{ji}$ divided by p is a map $h_{ij}: \Omega_{U_{ij}}^1(\log D) \rightarrow B_{U_{ij}}^1$, and that $\{h_{ij}\}$ is a cocycle representing the log Cartier sequence.

The obstruction classes satisfy the following functoriality properties.

Lemma 8.3.17. *Let $\pi: Y \rightarrow X$ be a morphism of smooth schemes over k .*

- (a) *Let $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ be a lifting of π over $W_2(k)$. Then the obstruction classes $o_{\tilde{X}}^F$ and o_Y^F , treated as morphisms in the appropriate derived categories, fit into the following commutative diagrams*

$$\begin{array}{ccc} \pi^* F_X^* \Omega_X^1 & \xrightarrow{\pi^* o_{\tilde{X}}^F} & \pi^* \mathcal{O}_X[1] \\ F_Y^* d\pi \downarrow & & \parallel \\ F_Y^* \Omega_Y^1 & \xrightarrow{o_Y^F} & \mathcal{O}_Y[1] \end{array} \quad \text{and} \quad \begin{array}{ccc} F_X^* \Omega_X^1 & \xrightarrow{o_{\tilde{X}}^F} & \mathcal{O}_X[1] \\ F_Y^* d\pi \downarrow & & \downarrow \pi^*[1] \\ R\pi_* F_Y^* \Omega_Y^1 & \xrightarrow{R\pi_* o_Y^F} & R\pi_* \mathcal{O}_Y[1]. \end{array} \quad (8.3)$$

- (b) *Suppose that π is smooth. Then the obstruction classes o_X and o_Y fit into the following commutative diagram*

$$\begin{array}{ccc} \pi^* \Omega_X^1 & \xrightarrow{\pi^* o_X} & \pi^* B_X^1[1] \\ d\pi \downarrow & & \downarrow d\pi \\ \Omega_Y^1 & \xrightarrow{o_Y} & B_Y^1[1]. \end{array}$$

Proof. For part (a), see [Zda17, Lemma 4.1]. For part (b), use Proposition 8.3.15(c) and the commutativity of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^* B_X^1 & \longrightarrow & \pi^* Z_X^1 & \longrightarrow & \pi^* \Omega_X^1 \longrightarrow 0 \\ & & d\pi \downarrow & & d\pi \downarrow & & d\pi \downarrow \\ 0 & \longrightarrow & B_Y^1 & \longrightarrow & Z_Y^1 & \longrightarrow & \Omega_Y^1 \longrightarrow 0. \end{array} \quad \square$$

Remark 8.3.18. In the case of singular schemes, the cotangent bundle Ω_X^1 can be substituted with its derived variant $L_{X/k} \in D_{\text{Coh}}^-(\mathcal{O}_X)$ (see [Ill71]). The assertions of Proposition 8.3.15(a) and Lemma 8.3.17(a) remain valid with the standard differential replaced by the derived one.

Descending and lifting Frobenius liftings

With the abstract deformation theory at hand, we can now easily relate F -liftability of two schemes X and Y in the presence of a suitable morphism $\pi: Y \rightarrow X$. In the subsequent sections, we shall frequently use the following results.

Lemma 8.3.19. *Let $\pi: Y \rightarrow X$ be an étale morphism of k -schemes. For every Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ of X , there exists a unique Frobenius lifting $(\tilde{Y}, \tilde{F}_Y)$ of Y and a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ of π such that $\tilde{F}_X \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_Y$.*

Proof. This follows from the equivalence of categories between étale schemes over X and over $\tilde{X}$. $\square$

Theorem 8.3.20 (Descending Frobenius liftability). *Let $\pi: Y \rightarrow X$ be a morphism of schemes (essentially) of finite type over k and let $(\tilde{Y}, \tilde{F}_Y)$ be a Frobenius lifting of Y .*

(a) *Suppose that π admits a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$, and that one of the following conditions is satisfied:*

- i. $\pi^*: \mathcal{O}_X \rightarrow R\pi_*\mathcal{O}_Y$ is a split monomorphism in the derived category,
- ii. π is finite flat of degree prime to p ,
- iii. Y satisfies condition S_2 and π is an open immersion such that $X \setminus Y$ has codimension > 1 in X .

Then F_X lifts to $\tilde{X}$.

(b) *Suppose that one of the following conditions is satisfied:*

- i. $\mathcal{O}_X \xrightarrow{\sim} \pi_*\mathcal{O}_Y$ and $R^1\pi_*\mathcal{O}_Y = 0$,
- ii. X and Y are smooth and π is proper and birational,
- iii. Y satisfies condition S_3 and π is an open immersion such that $X \setminus Y$ has codimension > 2 in X .

Then there exists a unique pair of a Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ of X and a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ of π such that $\tilde{F}_X \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_Y$.

(c) *Suppose that Y is normal and that $\pi: Y \rightarrow X = Y/G$ is a good quotient by an action of a linearly reductive group G on Y . Then there exists a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ of π and a lifting $\tilde{F}_X$ of F_X to $\tilde{X}$.*

In fact, conditions (a.ii) and (b.ii) imply (a.i) and (b.i), respectively. We do not expect $\tilde{F}_X \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_Y$ to hold in general in situations (a) and (c).

Proof. (a) Under condition (i), the right arrow of the right diagram (8.3) is a split injection by assumption. Thus $o_X^F = 0$ if $o_Y^F = 0$. Condition (ii) implies (i), as $R\pi_*\mathcal{O}_Y = \pi_*\mathcal{O}_Y$ and $1/\deg(\pi)$ times the trace map yields a splitting. For (iii), we argue as in [Zda17, Corollary 4.3]. Let K be the fibre of the right arrow in the right diagram (8.3), fitting into an exact triangle

$$K \longrightarrow \mathcal{O}_X[1] \longrightarrow R\pi_*\mathcal{O}_Y[1] \longrightarrow K[1]. \quad (8.4)$$

Since the bottom map in the right diagram (8.3) is zero by assumption, the top map has to factor through K . It is therefore enough to show that $\text{Hom}(F_X^*\Omega_X^1, K) = 0$. Note that $K = R\Gamma_Z(\mathcal{O}_X)[1]$ is the shift by one of the local cohomology complex with supports on $Z = X \setminus Y$ (see [Har67] or [Sta14, Tag 0A39]). Considering the spectral sequence

$$E_2^{p,q} = \text{Hom}\left(F_X^*\Omega_X^1, \mathcal{H}^q(R\Gamma_Z(\mathcal{O}_X))[p]\right) \Rightarrow \text{Hom}\left(F_X^*\Omega_X^1, R\Gamma_Z(\mathcal{O}_X)[p+q]\right),$$

for $p+q = 1$, we see that it suffices to show that the local cohomology vanishes up to degree one, which is implied by the Serre's condition S_2 and [HK04, Proposition 3.3]. Analogous reasoning works if X and Y are not smooth. In this case, as mentioned in Remark 8.3.18, we substitute the sheaf of Kähler differentials with the cotangent complex.

(b) Under condition (i), Lemma 8.2.3 provides a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ defined by the assignment $\mathcal{O}_{\tilde{X}} = \pi_* \mathcal{O}_{\tilde{Y}}$. To obtain a Frobenius lifting on $\tilde{X}$ we just take $F_{\tilde{X}} = \pi_* F_{\tilde{Y}}$. For (ii), we observe that, since X and Y are smooth, [CR15, Theorem 1.1] implies that $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$, and hence we may use (i) to conclude. For (iii), we reason similarly as in (a.iii). More precisely, we consider the long exact sequence of cohomology for (8.4) to see that $R^1 \pi_* \mathcal{O}_Y$ is isomorphic to the second local cohomology supported in $Z = X \setminus U$, which vanishes by condition S_3 and [HK04, Proposition 3.3].

(c) Since Y is normal and F -liftable, it is F -split. Lemma 8.2.13 provides compatible Frobenius splittings σ_Y and σ_X and a lifting $\tilde{\pi}: \tilde{Y}(\sigma_Y) \rightarrow \tilde{X}(\sigma_X)$ of π . By Theorem 8.3.38, $\tilde{Y}(\sigma_Y) \simeq \tilde{Y}$, and we set $\tilde{X} = \tilde{X}(\sigma_X)$, obtaining the required lifting of π . By the definition of a good quotient, $\mathcal{O}_X = (\pi_* \mathcal{O}_Y)^G$ and π is affine, in particular $R^i \pi_* \mathcal{O}_Y = 0$ for $i > 0$. Since G is linearly reductive, $\mathcal{O}_X = (\pi_* \mathcal{O}_Y)^G \rightarrow \pi_* \mathcal{O}_Y$ splits, and hence assumption (a.i) is satisfied. We apply part (a) to conclude. $\square$

Corollary 8.3.21. *Let X and Y be smooth and proper schemes over k . Then $X \times Y$ is F -liftable if and only if X and Y are. Moreover, every Frobenius lifting of $X \times Y$ arises as a product of Frobenius liftings of the factors.*

Proof. If $(\tilde{X}, \tilde{F}_X)$ and $(\tilde{Y}, \tilde{F}_Y)$ are Frobenius liftings of X and Y , respectively, then $(\tilde{X} \times \tilde{Y}, \tilde{F}_X \times \tilde{F}_Y)$ is a Frobenius lifting of $X \times Y$. For the converse, we first use the arguments of [BTLM97, Lemma 1]. The sheaf $\Omega_{X \times Y}^1$ decomposes as a direct sum $\pi_X^* \Omega_X^1 \oplus \pi_Y^* \Omega_Y^1$ and therefore the morphism of de Rham complexes $\Omega_X^\bullet \rightarrow \pi_{X*} \Omega_{X \times Y}^\bullet$ induced by the differential $d\pi_X: \Omega_X^1 \rightarrow \Omega_{X \times Y}^1$ has a natural splitting. This leads to a splitting s of the morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_X^1 & \longrightarrow & Z_X^1 & \longrightarrow & \Omega_X^1 \longrightarrow 0 \\ & & \downarrow d\pi_X \uparrow s & & \downarrow d\pi_X \uparrow s & & \downarrow d\pi_X \uparrow s \\ 0 & \longrightarrow & \pi_{X*} B_{X \times Y}^1 & \longrightarrow & \pi_{X*} Z_{X \times Y}^1 & \longrightarrow & \pi_{X*} \Omega_{X \times Y}^1 \longrightarrow 0. \end{array}$$

Consequently, we see that the upper row is split if the lower row is. This completes the proof of the first assertion.

For the second part, we take a Frobenius lifting $(\widetilde{X \times Y}, \tilde{F}_{X \times Y})$. We already know that X and Y admit Frobenius liftings $(\tilde{X}, \tilde{F}_X)$ and $(\tilde{Y}, \tilde{F}_Y)$, and their product $(\tilde{X} \times \tilde{Y}, \tilde{F}_X \times \tilde{F}_Y)$ is another Frobenius lifting of $X \times Y$. But by Corollary 8.3.39, we must have

$$\widetilde{X \times Y} \simeq \tilde{X} \times \tilde{Y}.$$

Using Proposition 8.3.15 we see that the space of Frobenius liftings on $\widetilde{X \times Y}$ is a torsor

under $H^0(X \times Y, F^*T_{X \times Y})$. The last group equals $H^0(X, F^*T_X) \oplus H^0(Y, F^*T_Y)$ which can be identified with the space of Frobenius liftings of the factors X and Y . This finishes the proof. $\square$

Toric fibrations

The goal of this part of the section is to show that split toric fibrations over an F -liftable base are F -liftable (see Example 8.3.5). We remove the assumption that the fibration is split in Theorem 8.5.1.

Let X be a normal k -scheme, and let $L_1, \dots, L_n$ be line bundles on X . Consider the graded $\mathcal{O}_X$ -algebra

$$\mathcal{R} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda, \quad \mathcal{R}_\lambda = L_1^{\lambda_1} \otimes \dots \otimes L_n^{\lambda_n}, \quad (8.5)$$

with multiplication given by the tensor product, and set $U = \operatorname{Spec}_X \mathcal{R}$. The natural map $U \rightarrow X$ is a torsor under the split torus $T = \mathbb{G}_m^n$. Let $Z = X(\Sigma)_k$ be a toric variety under the torus T and let $Y = U \times^T Z$. The projection $\pi: Y \rightarrow X$ is a split toric fibration with fibre Z , and conversely every split toric fibration over X arises via this construction.

Lemma 8.3.22. *In the above situation, let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of X . Then there exists a Frobenius lifting $(\tilde{Y}, \tilde{F}_Y)$ and a lifting $\tilde{\pi}: \tilde{Y} \rightarrow \tilde{X}$ of π such that $\tilde{F}_X \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_Y$.*

Proof. Let $\tilde{L}_i$ be the unique liftings of L_i to $\tilde{X}$ satisfying $\tilde{F}_X^* \tilde{L}_i \simeq \tilde{L}_i^p$, which exist by Proposition 8.3.15(d) and Lemma 8.2.15(a). We start with the case $Z = \mathbb{G}_m^n$, so $Y = U$. Consider the graded $\mathcal{O}_{\tilde{X}}$ -algebra

$$\tilde{\mathcal{R}} = \bigoplus_{\lambda \in \mathbb{Z}^n} \tilde{\mathcal{R}}_\lambda, \quad \tilde{\mathcal{R}}_\lambda = \tilde{L}_1^{\lambda_1} \otimes \dots \otimes \tilde{L}_n^{\lambda_n},$$

and set $\tilde{U} = \operatorname{Spec}_{\tilde{X}} \tilde{\mathcal{R}}$. The natural map $\tilde{\pi}: \tilde{U} \rightarrow \tilde{X}$ is a $\mathbb{G}_m^n$ -torsor lifting $\pi: U \rightarrow X$. Moreover, the map

$$\tilde{F}_X^* \tilde{\mathcal{R}} = \bigoplus_{\lambda \in \mathbb{Z}^n} \tilde{F}_X^* \tilde{\mathcal{R}}_\lambda \simeq \bigoplus_{\lambda \in \mathbb{Z}^n} (\tilde{\mathcal{R}}_\lambda)^{\otimes p} = \bigoplus_{\lambda \in p\mathbb{Z}^n} \tilde{\mathcal{R}}_\lambda \hookrightarrow \bigoplus_{\lambda \in \mathbb{Z}^n} \tilde{\mathcal{R}}_\lambda = \tilde{\mathcal{R}}$$

induces a map $\tilde{F}_{U/X}: \tilde{U} \rightarrow \tilde{U}'$, which lifts the relative Frobenius $F_{U/X}: U \rightarrow U'$, where $\tilde{U}'$ is the base change of $\tilde{U}$ along $\tilde{F}_X$ and U' is its reduction modulo p .

For the general case, we set

$$\tilde{Y} = \tilde{U} \times^{\tilde{T}} X(\Sigma)_{W_2(k)} \xrightarrow{\tilde{\pi}} \tilde{X}, \quad \text{where } \tilde{T} = \mathbb{G}_{m, W_2(k)}^n,$$

the toric fibration with fibre $X(\Sigma)$ associated to $\tilde{U}$. This is a lifting of Y , and the lifting of Frobenius on $\tilde{U}$ extends to $\tilde{Y}$. $\square$

Divisorial rings

Our final application of the functoriality of the obstruction classes concerns F -liftability of divisorial rings with a view toward the study of Cox rings. A standard reference for the following construction is the book [ADHL15]. Consider a diagram of the form

$$\begin{array}{ccc} Y = \operatorname{Spec} R & \xleftarrow{j} & U \quad \hookrightarrow \quad T = \mathbb{G}_m^n \\ & & \downarrow \pi \\ & & X \end{array}$$

where X is a normal projective k -scheme, $\pi: U \rightarrow X$ is a torsor under a torus $T = \mathbb{G}_m^n$, and $j: U \rightarrow Y$ is an open immersion into a normal affine scheme $Y = \operatorname{Spec} R$ of finite type over k such that the complement $Y \setminus U$ has codimension > 1 . In the previous paragraph we related the F -liftability of X and of U , and now we wish to relate the F -liftability of X and of Y .

The action of T yields a grading $\mathcal{R} = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathcal{R}_\lambda$ on the quasi-coherent $\mathcal{O}_X$ -algebra $\mathcal{R} = \pi_* \mathcal{O}_U$ whose graded pieces $\mathcal{R}_\lambda$ are line bundles on X . By the assumptions on j , we have

$$R = H^0(U, \mathcal{O}_U) = H^0(X, \mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Z}^n} H^0(X, \mathcal{R}_\lambda).$$

Conversely, any choice of line bundles $L_1, \dots, L_n$ on a normal projective variety X yields a diagram as above by setting

$$\mathcal{R}_\lambda = L_1^{\lambda_1} \otimes \dots \otimes L_n^{\lambda_n},$$

$\mathcal{R} = \bigoplus_{\lambda \in M} \mathcal{R}_\lambda$ with multiplication given by the tensor product, $U = \operatorname{Spec}_X \mathcal{R}$, and $Y = \operatorname{Spec} \Gamma(X, \mathcal{R})$, provided that the following assumptions are satisfied:

- $R = \Gamma(X, \mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Z}^n} H^0(X, L_1^{\lambda_1} \otimes \dots \otimes L_n^{\lambda_n})$ is finitely generated, and
- at least one of the line bundles $L_1^{\lambda_1} \otimes \dots \otimes L_n^{\lambda_n}$ is ample (see [BS03, Remark 1.8])

Proposition 8.3.23. *In the above situation, let $\tilde{X}$ be a lifting of X to $\tilde{S} = \operatorname{Spec} W_2(k)$, and let $\tilde{L}_1, \dots, \tilde{L}_n$ be liftings of $L_1, \dots, L_n$ to $\tilde{X}$. Set*

$$\tilde{\mathcal{R}} = \bigoplus_{\lambda \in M} \tilde{L}_1^{\lambda_1} \otimes \dots \otimes \tilde{L}_n^{\lambda_n}$$

and $\tilde{R} = \Gamma(\tilde{X}, \tilde{\mathcal{R}})$. Suppose that $\tilde{R}$ is a flat $W_2(k)$ -algebra. Then $\tilde{Y} = \operatorname{Spec} \tilde{R}$ is a lifting of Y . If $\tilde{F}_X$ is a lift of F_X to $\tilde{X}$ such that $\tilde{F}_X^ \tilde{L}_i \simeq \tilde{L}_i^p$ (see Lemma 8.3.22), then F_Y lifts to $\tilde{Y}$. The liftings of Frobenius on $\tilde{X}$, $\tilde{U}$, and $\tilde{Y}$ commute with the natural maps.*

Proof. By the flatness assumption, $\tilde{Y}$ is a lifting of Y . Then the lifting of Frobenius on $\tilde{U}$ constructed in Lemma 8.3.22 induces a lifting of Frobenius on $\operatorname{Spec} \Gamma(\tilde{U}, \mathcal{O}_{\tilde{U}}) = \tilde{Y}$. $\square$

The flatness assumption on $\tilde{R}$ is satisfied if, for instance, $H^1(X, \mathcal{R}) = 0$, that is,

$$H^1(X, L^{\lambda_1} \otimes \dots \otimes L^{\lambda_n}) = 0 \quad \text{for every } \lambda_1, \dots, \lambda_n \in \mathbb{Z}.$$

Indeed, the cohomology exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \tilde{\mathcal{R}} \longrightarrow \mathcal{R} \longrightarrow 0$$

shows that $0 \rightarrow R \rightarrow \tilde{R} \rightarrow R \rightarrow 0$ is exact as well.

8.3.4. Frobenius liftings compatible with a divisor or a closed subscheme

From the point of view of applications, it is necessary to consider Frobenius liftings of nc pairs (X, D) (see §8.2.3). A *lifting* of (X, D) over $W_2(k)$ is an nc pair $(\tilde{X}, \tilde{D})$ where $\tilde{X}$ is a lifting of X and $\tilde{D} \subseteq \tilde{X}$ is an embedded deformation of D . Note that the requirement that $\tilde{D}$ has normal crossings relative to $W_2(k)$ is not vacuous. Recall that a Frobenius lifting $(\tilde{X}, \tilde{D}, \tilde{F}_X)$ of (X, D) consists of a lifting $(\tilde{X}, \tilde{D})$ of (X, D) and a lifting $\tilde{F}_X$ of F_X to $\tilde{X}$ satisfying $\tilde{F}_X^* \tilde{D} = p\tilde{D}$. In this case, we shall say that $\tilde{F}_X$ is *compatible* with $\tilde{D}$.

Lemma 8.3.24. *Let (X, D) be an nc pair over k , and let $(\tilde{X}, \tilde{D}, \tilde{F}_X)$ be a Frobenius lifting of (X, D) . Let $D_1, \dots, D_r$ be the irreducible components of D . Then for every $i = 1, \dots, r$, the Frobenius lifting $\tilde{F}_X$ induces a Frobenius lifting $\tilde{F}_{D_i}$ on $\tilde{D}_i$ which is compatible with the divisor $(\bigcup_{j \neq i} \tilde{D}_j) \cap \tilde{D}_i \subseteq \tilde{D}_i$.*

Proof. The question is local so we can assume that $X = \text{Spec } R$ and $\tilde{X} = \text{Spec } \tilde{R}$. Moreover, D_j is the zero locus of $f_j \in R$, and $\tilde{D}_j$ is the zero locus of $\tilde{f}_j \in \tilde{R}$ where $1 \leq j \leq r$.

Since $\tilde{F}_X^*(\tilde{f}_j) = \tilde{u}_j \tilde{f}_j^p$ for every $1 \leq j \leq r$ and some $\tilde{u}_j \in \tilde{R}^*$, we get an induced morphism

$$\tilde{F}_{D_i}: \tilde{D}_i \longrightarrow \tilde{D}_i,$$

where $\tilde{D}_i = \text{Spec } \tilde{R}/\tilde{f}_i$, such that $\tilde{F}_{D_i}^*(\tilde{f}_j) = \tilde{u}_j \tilde{f}_j^p$ for $j \neq i$. □

Let $o_{X,D} \in \text{Ext}^1(\Omega_X^1(\log D), B_X^1)$ be the obstruction to the existence of a Frobenius lifting $(\tilde{X}, \tilde{D}, \tilde{F}_X)$ of a simple normal crossing pair (X, D) (see Variant 8.3.16). Consider the short exact sequence

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log D) \longrightarrow \bigoplus_i \mathcal{O}_{D_i} \longrightarrow 0,$$

where $D = \sum D_i$ is a decomposition into irreducible components. Analyzing the construction in Variant 8.3.16 we see that the induced natural morphism

$$\text{Ext}^1(\Omega_X^1(\log D), B_X^1) \longrightarrow \text{Ext}^1(\Omega_X^1, B_X^1) \quad (8.6)$$

maps $o_{X,D}$ into the obstruction o_X to the existence of a Frobenius lifting of X .

Moreover, if $o_{X,D} = 0$, then, after fixing a Frobenius lifting of (X, D) , we can identify $\text{Hom}(\Omega_X^1, B_X^1)$ with the space of Frobenius liftings $(\tilde{X}, \tilde{F}_X)$ (see Proposition 8.3.15). With that, the natural morphism

$$\text{Hom}(\Omega_X^1(\log D), B_X^1) \longrightarrow \text{Hom}(\Omega_X^1, B_X^1) \quad (8.7)$$

maps Frobenius liftings $(\tilde{X}, \tilde{D}, \tilde{F}_X)$ to Frobenius liftings $(\tilde{X}, \tilde{F}_X)$.

As a corollary of the above we obtain the following lemma which describes the behaviour of divisors with a negative normal bundle with respect to Frobenius liftability.

Lemma 8.3.25. *Let (X, D) be a simple normal crossing pair over k such that*

$$H^0(D_i, \mathcal{O}_{D_i}(mD_i)) = 0 \quad \text{for } 1 \leq m \leq p$$

and all irreducible components D_i of D . Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of X . Then there exists a lifting $\tilde{D}$ of D with which $\tilde{F}_X$ is compatible.

In particular, if $Y = \text{Bl}_Z X$ is a blow-up of a smooth k -scheme X in a smooth center $Z \subset X$, then the unique embedded deformation $\tilde{E} \subset \tilde{Y}$ of the exceptional divisor E is preserved by every Frobenius lifting of $\tilde{Y}$.

Proof. Let o_X and $o_{X,D}$ be the obstruction classes as above. Since $o_X = 0$ and $o_{X,D}$ is mapped to it by (8.6), we get that $o_{X,D}$ is the image of an element of

$$\text{Ext}^1 \left(\bigoplus_{i=1}^r \mathcal{O}_{D_i}, B_X^1 \right).$$

Moreover, if this cohomology group vanishes, then (8.7) is surjective. Hence, in order to prove the lemma, it is enough to show that this cohomology group is zero.

To this end, we apply the local to global spectral sequence to see that

$$\text{Ext}^1 \left(\bigoplus_{i=1}^r \mathcal{O}_{D_i}, B_X^1 \right) \simeq H^0 \left(X, \mathcal{E}xt^1 \left(\bigoplus_{i=1}^r \mathcal{O}_{D_i}, B_X^1 \right) \right).$$

Using the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D_i) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0$$

we compute $\mathcal{E}xt^1(\bigoplus_{i=1}^r \mathcal{O}_{D_i}, B_X^1)$ as the cokernel of the mapping:

$$\bigoplus_{i=1}^r \mathcal{H}om(\mathcal{O}_X, B_X^1) \longrightarrow \bigoplus_{i=1}^r \mathcal{H}om(\mathcal{O}_X(-D_i), B_X^1),$$

which is equal to $\bigoplus_{i=1}^r B_X^1(D_i)|_{D_i}$, because B_X^1 is locally free. Since X is F -split (see Proposition 8.3.8), we have

$$B_X^1(D_i)|_{D_i} \subseteq (\mathcal{O}_X(D_i) \otimes F_{X*} \mathcal{O}_X)|_{D_i} = F_{X*}(\mathcal{O}_{pD_i}(pD_i)).$$

Here we used that $(F_{X*}\mathcal{O}_X)|_{D_i}$ is the cokernel of

$$F_{X*}(\mathcal{O}_X(-pD_i)) = \mathcal{O}_X(-D_i) \otimes F_{X*}\mathcal{O}_X \hookrightarrow F_{X*}\mathcal{O}_X,$$

and so it is equal to $F_{X*}\mathcal{O}_{pD_i}$.

Hence, it is enough to show that $H^0(pD_i, \mathcal{O}_{pD_i}(pD_i)) = 0$. This follows by inductively looking at the global sections in the short exact sequences

$$0 \longrightarrow \mathcal{O}_{D_i}((p-m+1)D_i) \longrightarrow \mathcal{O}_{mD_i}(pD_i) \longrightarrow \mathcal{O}_{(m-1)D_i}(pD_i) \longrightarrow 0$$

where $2 \leq m \leq p$. □

Definition 8.3.26. Let X be a k -scheme, let $Z \subseteq X$ be a closed subscheme, let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of X , and let $\tilde{Z} \subseteq \tilde{X}$ be an embedded deformation of Z . We say that $\tilde{F}_X$ is *compatible* with $\tilde{Z}$ if

$$\tilde{F}_X^\#(\mathcal{I}_{\tilde{Z}}) \subseteq \mathcal{I}_{\tilde{Z}}^p,$$

or, in other words, if the image of $\tilde{F}_X^*(\mathcal{I}_{\tilde{Z}}) \rightarrow \mathcal{O}_{\tilde{X}}$ is contained in $\mathcal{I}_{\tilde{Z}}^p$.

In particular, if X is smooth and $D \subseteq X$ is a divisor with normal crossings, then $\tilde{F}_X$ is compatible with a lifting $\tilde{D}$ with relative normal crossings if and only if it is compatible with $\tilde{D}$ as a closed subscheme in the sense of the above definition. In fact, if $\tilde{D} \subseteq \tilde{X}$ is an embedded deformation of D with which $\tilde{F}_X$ is compatible, then $\tilde{D}$ automatically has relative normal crossings, see Corollary 8.3.39.

***F*-liftability and blowing up**

Let X be a smooth scheme over k , let $Z \subseteq X$ be a smooth closed subscheme of codimension > 1 , let $\pi: Y \rightarrow X$ be the blowing-up of X along Z , and let $E = \text{Exc}(\pi)$ be the exceptional divisor. Further, let $\tilde{Y}$ be a lifting of Y over $\tilde{S}$. By the results of §8.2.2, there exist unique liftings of $\tilde{E}$, $\tilde{X}$, and $\tilde{Z}$ fitting inside a commutative square

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow \tilde{\pi} \\ \tilde{Z} & \longrightarrow & \tilde{X}. \end{array}$$

Moreover, by Theorem 8.3.20(b), the map $\tilde{\pi}$ induces an injection

$$\left\{ \text{liftings of } F_Y \text{ to } \tilde{Y} \right\} \longrightarrow \left\{ \text{liftings of } F_X \text{ to } \tilde{X} \right\}.$$

Lemma 8.3.27. *A lifting $\tilde{F}_X$ of F_X to $\tilde{X}$ is in the image of the above map (that is, $\tilde{F}_X$ extends to $\tilde{Y}$) if and only if it is compatible with $\tilde{Z}$ in the sense of Definition 8.3.26.*

Proof. Let $\tilde{F}_X$ be a lifting of F_X to $\tilde{X}$. Suppose that $\tilde{F}_X$ is compatible with $\tilde{Z}$. If $\mathcal{I}$ (resp. $\tilde{\mathcal{I}}$) is the ideal sheaf of Z (resp. $\tilde{Z}$), then $Y = \text{Proj} \bigoplus_{n \geq 0} \mathcal{I}^n = \text{Proj} \bigoplus_{n \geq 0} \mathcal{I}^{np}$ and

$\tilde{Y} = \text{Proj} \bigoplus_{n \geq 0} \tilde{\mathcal{J}}^n = \text{Proj} \bigoplus_{n \geq 0} \tilde{\mathcal{J}}^{np}$. The relative Frobenius $F_{Y/X}$ corresponds to the map of graded $\mathcal{O}_X$ -algebras

$$F_{Y/X}^*: F_X^* \left(\bigoplus_{n \geq 0} \mathcal{J}^n \right) \longrightarrow \bigoplus_{n \geq 0} (\mathcal{J}^n)^{[p]} \longrightarrow \bigoplus_{n \geq 0} \mathcal{J}^{np} \quad (8.8)$$

induced by the inclusion. Since by assumption $\tilde{F}_X^* \tilde{\mathcal{J}}$ maps into $\tilde{\mathcal{J}}^p$, we see that $\tilde{F}_X^* \tilde{\mathcal{J}}^n$ maps into $\tilde{\mathcal{J}}^{np}$ for all $n \geq 0$, and we can define a map of graded $\mathcal{O}_{\tilde{X}}$ -algebras

$$\tilde{F}_{Y/X}^*: \tilde{F}_X^* \left(\bigoplus_{n \geq 0} \tilde{\mathcal{J}}^n \right) = \bigoplus_{n \geq 0} \tilde{F}_X^* (\tilde{\mathcal{J}}^n) \longrightarrow \bigoplus_{n \geq 0} \tilde{\mathcal{J}}^{np} \quad (8.9)$$

again induced by the inclusion. We claim that the rational map induced by $\tilde{F}_{Y/X}^*$ is defined everywhere, and is the identity on the complement of $\tilde{Z}$. It suffices to show that there are no relevant homogeneous prime ideals whose preimage under (8.9) becomes irrelevant. To see this, we first observe that this property holds for the inclusion (8.8), since it induces a well-defined morphism $F_{Y/X}$. The inclusion (8.9) is only a nilpotent extension of (8.8), and hence its action on homogeneous ideals is the same. This finishes the proof of the claim. The composition of $\tilde{F}_{Y/X}$ with the projection $\tilde{Y}' \rightarrow \tilde{Y}$ gives the desired extension of $\tilde{F}_X$ to $\tilde{Y}$. We note that in this part of the proof the smoothness assumptions were not needed.

Now suppose that $\tilde{F}_X$ extends to $\tilde{Y}$. The question whether $\tilde{F}_X$ is compatible with $\tilde{Z}$ is local on $\tilde{X}$, so we can assume that $\tilde{X} = \text{Spec } \tilde{A}$ and $\tilde{\mathcal{J}} = (\tilde{x}_1, \dots, \tilde{x}_c)$ for some $\tilde{x}_i \in \tilde{A}$ such that $(p, \tilde{x}_1, \dots, \tilde{x}_c)$ is a regular sequence and $c > 1$. Write $\tilde{F}_X^* \tilde{x}_i = \tilde{x}_i^p + pf_i$ for $f_i \in \tilde{A} = \tilde{A} \otimes k$. The condition that $\tilde{F}_X^*$ extends to the open subset $\tilde{Y}_i = \text{Spec } \tilde{A}[\tilde{\mathcal{J}}/\tilde{x}_i] \subseteq \tilde{Y}$ is equivalent to the condition

$$\tilde{F}_X^* (\tilde{x}_j / \tilde{x}_i) = \frac{\tilde{x}_j^p + pf_j}{\tilde{x}_i^p + pf_i} = \frac{(\tilde{x}_j^p + pf_j)(\tilde{x}_i^p - pf_i)}{\tilde{x}_i^{2p}} \in \tilde{A}[\tilde{\mathcal{J}}/\tilde{x}_i]$$

which amounts to saying that

$$x_i^p f_j - x_j^p f_i \in \mathcal{J}^{2p} \quad \text{for all } j \neq i,$$

where $x_i \in A$ are the images of $\tilde{x}_i$. These equations imply that

$$f_i \in (\mathcal{J}^{2p} + (x_i^p) : (x_j^p)) = \mathcal{J}^p \quad \text{for } i \neq j. \quad \square$$

In the following proposition, we call a morphism of k -schemes $f: X \rightarrow Y$ *separable* if it is a composition of a generically smooth morphism and a closed immersion.

Proposition 8.3.28. *Let $(\tilde{X}, \tilde{F}_X)$ and $(\tilde{Y}, \tilde{F}_Y)$ be Frobenius liftings of smooth and proper k -schemes X and Y . Let $V \subseteq X \times Y$ be an integral subscheme such that one of the projections π_X or π_Y is separable when restricted to V . Suppose that the lifting of Frobenius*

$\tilde{F}_X \times \tilde{F}_Y$ on $\tilde{X} \times \tilde{Y}$ is compatible with a lifting of V . Then $V = V_X \times V_Y$ for some integral subschemes $V_X \subseteq X$ and $V_Y \subseteq Y$.

Proof. We set V_X (resp. V_Y) to be the image of V under the projection $\pi_X: X \times Y \rightarrow X$ (resp. $\pi_Y: X \times Y \rightarrow Y$), and assume without loss of generality that π_Y is separable when restricted to V . We claim that the closed immersion $V \hookrightarrow V_X \times V_Y$ is an isomorphism. To see this, we apply Corollary 8.3.32(c) to the projection

$$\pi_{\tilde{Y}}: (\tilde{X} \times \tilde{Y}, \tilde{F}_X \times \tilde{F}_Y) \longrightarrow (\tilde{Y}, \tilde{F}_Y)$$

compatible with the respective Frobenius liftings, and observe that for every $y \in V_Y$ the Frobenius lifting $\tilde{F}_X$ is compatible with a lifting of the subscheme $V_y = V \cap (X \times \{y\})$, when interpreted as a subscheme of X . By the assumptions the projection π_Y is separable and therefore for a general y the subscheme V_y is a union of integral subschemes. Using Corollary 8.3.37 we now notice that there are only finitely many choices for V_y , and therefore they are all isomorphic since V_Y is connected. This implies that the immersion $V \hookrightarrow V_X \times V_Y$ is an isomorphism on the fibres of the projection to V_Y and hence is an isomorphism. $\square$

8.3.5. Base change of a lifting of Frobenius

In this subsection, we show that a morphism from a $W_2(k)$ -liftable scheme to an F -liftable scheme lifts to $W_2(k)$ after composing with the Frobenius. Moreover, if the source is endowed with a lifting of Frobenius, this lifting commutes with the lifting of Frobenius. Although we do not use it much in the sequel, we regard the result as essential for a good understanding of Frobenius liftings.

Witt vector schemes

Since our construction uses the Witt vector schemes $W_2(X)$, let us briefly recall what they are. For an $\mathbb{F}_p$ -scheme X and an integer n , we denote by $W_n(X)$ the scheme of Witt vectors of length n of X ; it has the same underlying topological space as X and $W_n(\mathcal{O}_X)$ as its structure sheaf. In particular, if $X = \operatorname{Spec} A$, then $W_n(X) = \operatorname{Spec} W_n(A)$. We denote by $\sigma_n: W_n(X) \rightarrow W_n(X)$ the map induced by the Witt vector Frobenius on $\mathcal{O}_X$. The construction $X \mapsto (W_n(X), \sigma_n)$ is functorial in X .

We do not know of an easy universal property characterising $W_n(X)$ except for the case of perfect schemes.

Lemma 8.3.29 ([GR03, Lemma 6.5.13 i], [Sch12, Proposition 5.13(i) and Remark 5.14]). *Suppose that S is a perfect $\mathbb{F}_p$ -scheme (i.e., F_S is an isomorphism), and let $n \geq 1$. Then for every flat $\mathbb{Z}/p^n\mathbb{Z}$ -scheme X , the map induced by restriction*

$$\operatorname{Hom}(X, W_n(S)) \longrightarrow \operatorname{Hom}(X \otimes \mathbb{F}_p, S)$$

is bijective.

Proof. We first claim that $L_{S/\mathbb{F}_p} = 0$. Indeed, since S is perfect, its Frobenius morphism is an isomorphism and therefore its differential $dF: F^*L_{S/\mathbb{F}_p} \rightarrow L_{S/\mathbb{F}_p}$ is a quasi-isomorphism. This quasi-isomorphism is also zero and hence the claim is proven. Now the assertion follows by [Ill71, Proposition 2.2.4]. $\square$

Proposition 8.3.30. *Let $(\tilde{Y}, \tilde{F}_Y)$ be a Frobenius lifting of a k -scheme Y . Let $\varphi: Z \rightarrow Y$ be a morphism of k -schemes and let $\tilde{Z}$ be a lifting of Z over $W_2(k)$. Then there exists a morphism $\psi: \tilde{Z} \rightarrow \tilde{Y}$ (canonically defined by (8.12) below) such that $\psi|_Z = F_Y \circ \varphi$:*

$$\begin{array}{ccccc}
 & & \psi & & \\
 & \nearrow & \text{---} & \searrow & \\
 \tilde{Z} & & \tilde{Y} & \xrightarrow{\tilde{F}_Y} & \tilde{Y} \\
 \uparrow & & \uparrow & & \uparrow \\
 Z & \xrightarrow{\varphi} & Y & \xrightarrow{F_Y} & Y
 \end{array}$$

If $\tilde{F}_Z$ is a lifting of F_Z to $\tilde{Z}$, then $\psi \circ \tilde{F}_Z = \tilde{F}_Y \circ \psi$. Further, if $\tilde{\varphi}: \tilde{Z} \rightarrow \tilde{Y}$ is any lifting of φ , then $\psi = \tilde{F}_Y \circ \tilde{\varphi}$.

Recall that if $\tilde{X}$ is any $W_2(k)$ -scheme and $X = \tilde{X} \otimes_{W_2(k)} k$, then there is a canonical affine morphism $\theta_{\tilde{X}}: \tilde{X} \rightarrow W_2(X)$ defined on functions by the formula

$$\theta_{\tilde{X}}^*(f_0, f_1) = (\tilde{f}_0)^p + p\tilde{f}_1$$

where $\tilde{f}_0, \tilde{f}_1 \in \mathcal{O}_{\tilde{X}}$ are any liftings of $f_0, f_1 \in \mathcal{O}_X$.

Moreover, if $\tilde{X}$ is flat over $W_2(k)$, then a lifting $\tilde{F}_X: \tilde{X} \rightarrow \tilde{X}$ of the absolute Frobenius $F_X: X \rightarrow X$ induces an affine morphism $\nu_{\tilde{X}, \tilde{F}_X}: W_2(X) \rightarrow \tilde{X}$ in the opposite direction, defined on functions by

$$\nu_{\tilde{X}, \tilde{F}_X}^*(\tilde{f}) = (f, \delta(f)) \quad \text{for } \tilde{f} \in \mathcal{O}_{\tilde{X}}, \quad (8.10)$$

where f the image of $\tilde{f}$ in $\mathcal{O}_X$ and $\delta(f)$ is the unique element such that $\tilde{F}_X^*(\tilde{f}) = \tilde{f}^p + p\delta(f)$.

We can recover $\tilde{F}_X$ from $\nu_{\tilde{X}, \tilde{F}_X}$ by the formula $\tilde{F}_X = \nu_{\tilde{X}, \tilde{F}_X} \circ \theta_{\tilde{X}}$, while the other composition $\theta_{\tilde{X}} \circ \nu_{\tilde{X}, \tilde{F}_X}$ coincides with the Witt vector Frobenius $\sigma_X: W_2(X) \rightarrow W_2(X)$, given by $\sigma_X^*(f_0, f_1) = (f_0^p, f_1^p)$.

$$\begin{array}{ccccc}
 & & \tilde{F} & & \\
 & \nearrow & \text{---} & \searrow & \\
 W_2(X) & \xrightarrow{\nu_{\tilde{X}, \tilde{F}}} & \tilde{X} & \xrightarrow{\theta_{\tilde{X}}} & W_2(X) \xrightarrow{\nu_{\tilde{X}, \tilde{F}}} \tilde{X} \\
 & \searrow & \text{---} & \nearrow & \\
 & & \sigma_X & &
 \end{array} \quad (8.11)$$

Proof of Proposition 8.3.30. We define the desired lifting $\psi: \tilde{Z} \rightarrow \tilde{Y}$ as the composition

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\theta_{\tilde{Z}}} & W_2(Z) \\ & \downarrow W_2(\varphi) & \\ W_2(Y) & \xrightarrow{\nu_{\tilde{Y}, \tilde{F}_Y}} & \tilde{Y}. \end{array} \quad (8.12)$$

Explicitly, on functions, ψ takes the form

$$\psi^*(\tilde{f}) = \widetilde{\varphi^*(f)}^p + p \cdot \varphi^*(\delta_{\tilde{F}_Y}(\tilde{f})),$$

where $\widetilde{\varphi^*(f)} \in \mathcal{O}_{\tilde{Z}}$ is a local section mapping to $\varphi^*(f) \in \mathcal{O}_Z$. It is thus clear that $\psi|_Z = F_Y \circ \varphi$.

We now check that $\psi \circ \tilde{F}_Z = \tilde{F}_Y \circ \psi$ when $\tilde{Z}$ is flat over $W_2(k)$ and endowed with a Frobenius lifting $\tilde{F}_Z$. This follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} & & \tilde{F}_Z & & \\ & \curvearrowright & & \curvearrowright & \\ \tilde{Z} & \xrightarrow{\theta_{\tilde{Z}}} & W_2(Z) & \xrightarrow{\nu_{\tilde{Z}, \tilde{F}_Z}} & \tilde{Z} & \xrightarrow{\theta_{\tilde{Z}}} & W_2(Z) \\ & \downarrow W_2(\varphi) & & \downarrow W_2(\varphi) & & & \\ & W_2(Y) & \xrightarrow{\nu_{\tilde{Y}, \tilde{F}_Y}} & \tilde{Y} & \xrightarrow{\theta_{\tilde{Y}}} & W_2(Y) & \xrightarrow{\nu_{\tilde{Y}, \tilde{F}_Y}} & \tilde{Y}. \\ & & & & & \curvearrowright & & \\ & & & & & \tilde{F}_Y & & \end{array}$$

Indeed, by (8.11) the composition $\tilde{Z} \rightarrow W_2(Z) \rightarrow \tilde{Z}$ (resp. $\tilde{Y} \rightarrow W_2(Y) \rightarrow \tilde{Y}$) equals $\tilde{F}_Z$ (resp. $\tilde{F}_Y$). To show the commutativity of the diagram, we note that the compositions $W_2(Z) \rightarrow \tilde{Z} \rightarrow W_2(Z)$ and $W_2(Y) \rightarrow \tilde{Y} \rightarrow W_2(Y)$ are the Witt vector Frobenius morphisms σ_Z and σ_Y . These are functorial, which shows that the middle square (and hence the whole diagram) commutes.

The final assertion follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\theta_{\tilde{Z}}} & W_2(Z) & & \\ \tilde{\varphi} \downarrow & & \downarrow W_2(\varphi) & & \\ \tilde{Y} & \xrightarrow{\theta_{\tilde{Y}}} & W_2(Y) & \xrightarrow{\nu_{\tilde{Y}, \tilde{F}_Y}} & \tilde{Y}. \\ & & & \curvearrowright & \\ & & & \tilde{F}_Y & \end{array}$$

Here the square commutes by functoriality of the maps θ . □

Corollary 8.3.31. *Let $f: X \rightarrow Y$ and $\varphi: Z \rightarrow Y$ be morphisms of k -schemes, and let $(\tilde{X}, \tilde{F}_X)$, $(\tilde{Y}, \tilde{F}_Y)$, and $(\tilde{Z}, \tilde{F}_Z)$ be Frobenius liftings of X , Y , and Z , respectively. Let $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ be a lifting of f commuting with the Frobenius liftings, and let $\psi: \tilde{Z} \rightarrow \tilde{Y}$ be*

the lifting of $F_Y \circ \varphi$ given by Proposition 8.3.30. Form the cartesian diagram

$$\begin{array}{ccc} \tilde{W} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \tilde{f} \\ \tilde{Z} & \xrightarrow{\psi} & \tilde{Y}. \end{array} \quad (8.13)$$

Then $\tilde{W}$ admits a Frobenius lifting $\tilde{F}_W$ such that the maps $\tilde{W} \rightarrow \tilde{Z}$ and $\tilde{W} \rightarrow \tilde{X}$ commute with the Frobenius liftings. Moreover, for every subscheme $\tilde{V} \subseteq \tilde{X}$ compatible with $\tilde{F}_X$, its preimage under $\tilde{W} \rightarrow \tilde{X}$ is compatible with $\tilde{F}_W$.

Setting $Z = \operatorname{Spec} k$ (in which case F_Z is an isomorphism), we obtain the following.

Corollary 8.3.32. *Let $(\tilde{Y}, \tilde{F}_Y)$ be a Frobenius lifting of a k -scheme Y .*

(a) *The construction of Proposition 8.3.30 with $Z = \operatorname{Spec} k$ yields a section of the specialisation map*

$$\tilde{Y}(W_2(k)) \longrightarrow Y(k).$$

In particular, every k -point of Y lifts (canonically) to a point of $\tilde{Y}$.

- (b) *Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a k -scheme X , and let $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ be a map commuting with the Frobenius liftings. Then for every $y \in Y(k)$, the fibre $X_y = f^{-1}(y)$ is F -liftable.*
- (c) *In the situation of (b), for every subscheme $Z \subset X$ such that there exists a lifting $\tilde{Z} \subset \tilde{X}$ compatible with $\tilde{F}_X$, the Frobenius lifting of X_y is compatible with the induced lifting of the intersection $f^{-1}(y) \cap Z$.*

Example 8.3.33. The scheme $\tilde{Y} = \{xy = p\} \subseteq \mathbb{A}_{W_2(k)}^2$ does not admit a lifting of Frobenius (not even locally), because the k -point $(0, 0) \in Y = \{xy = 0\}$ does not admit a lifting modulo p^2 . Of course Y admits a Frobenius lifting

$$\tilde{Y}' = \{xy = 0\} \subseteq \mathbb{A}_{W_2(k)}^2, \quad \tilde{F}'_Y(x) = x^p, \quad \tilde{F}'_Y(y) = y^p.$$

In fact, we will see in Corollary 8.3.39, that since Y is F -split, it admits at most one lifting to which F_Y lifts. This phenomenon seems to be related to some aspects of the log crystalline cohomology of Hyodo and Kato [HK94, §4], where it is more natural to consider the ‘hollow’ log structure on the base $\operatorname{Spec} W(k)$ rather than the standard one.

8.3.6. Frobenius liftings and Frobenius splittings

To every Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ of a smooth (or just normal) k -scheme X we can associate a corresponding F -splitting $\sigma_{\tilde{F}_X}$ on X and to every smooth morphism $\tilde{f}: \tilde{Y} \rightarrow \tilde{X}$ commuting with the liftings of Frobenius on $\tilde{Y}$ and $\tilde{X}$ we can associate a relative F -splitting $\sigma_{\tilde{F}_{Y/X}}$ of Y/X (see Proposition 8.3.8). Here $\tilde{F}_{Y/X}$ denotes the induced lifting of the relative Frobenius.

This provides us with $\mathbb{Q}$ -divisors $\Delta_{\tilde{F}_Y}$, $\Delta_{\tilde{F}_X}$, and $\Delta_{\tilde{F}_{Y/X}}$ as in Proposition 8.2.19. Let $f: Y \rightarrow X$ be the reduction of $\tilde{f}$ modulo p .

Lemma 8.3.34. *Let X and Y be smooth k -schemes. Then $\Delta_{\tilde{F}_Y} = \Delta_{\tilde{F}_{Y/X}} + f^* \Delta_{\tilde{F}_X}$, and if f has connected fibres, then $\Delta_{\tilde{F}_{Y/X}}$ (resp. $f^* \Delta_{\tilde{F}_X}$) is horizontal (resp. vertical).*

Proof. Let Y' be the base change of Y along F_X . By the construction of the map ξ , we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^* F_X^* \Omega_X^1 & \longrightarrow & F_Y^* \Omega_Y^1 & \longrightarrow & F_{Y/X}^* \Omega_{Y'/X}^1 \longrightarrow 0 \\ & & \downarrow f^* \xi_X & & \downarrow \xi_Y & & \downarrow \xi_{Y/X} \\ 0 & \longrightarrow & f^* \Omega_X^1 & \longrightarrow & \Omega_Y^1 & \longrightarrow & \Omega_{Y/X}^1 \longrightarrow 0, \end{array}$$

where $F_{Y/X}^* \Omega_{Y'/X}^1 \simeq F_Y^* \Omega_{Y/X}^1$ and $f^* F_X^* \Omega_X^1 \simeq F_Y^* f^* \Omega_X^1$. The first part of the lemma follows since the considered $\mathbb{Q}$ -divisors multiplied by $p - 1$ are equal to $\text{div}(\det \xi_Y)$, $\text{div}(\det \xi_{Y/X})$, and $\text{div}(\det \xi_X)$, respectively. If f has connected fibres, then $\Delta_{\tilde{F}_{Y/X}}$ is horizontal by Proposition 8.2.19. $\square$

The following corollary lists all the properties of $\Delta_{\tilde{F}_X}$ we need in this article. Given a flat morphism $f: Y \rightarrow X$ of normal varieties such that $f_* \mathcal{O}_Y = \mathcal{O}_X$ and a $\mathbb{Q}$ -divisor D on Y , we denote by D^h and D^v the horizontal and the vertical part, respectively.

Corollary 8.3.35. *Let $(\tilde{Y}, \tilde{F}_Y)$ be a Frobenius lifting of a smooth k -scheme Y .*

- (a) *If $D \subseteq Y$ is a smooth irreducible divisor such that $H^0(D, \mathcal{O}_D(mD)) = 0$ for $1 \leq m \leq p$, then $D \leq \Delta_{\tilde{F}_Y}$.*
- (b) *In the situation of Theorem 8.3.20(b.ii), we have $\Delta_{\tilde{F}_X} = \pi_* \Delta_{\tilde{F}_Y}$.*
- (c) *In the situation of Theorem 8.3.20(b.i), assume that $\pi: Y \rightarrow X$ is smooth and let $\tilde{F}_{Y/X}$ be the induced lifting of the relative Frobenius. Then $\Delta_{\tilde{F}_Y}^h$ is the $\mathbb{Q}$ -divisor associated to the relative F -splitting $\sigma_{\tilde{F}_{Y/X}}$. In particular,*

$$\begin{aligned} \Delta_{\tilde{F}_Y}^h &\sim_{\mathbb{Q}} -K_{Y/X}, \text{ and} \\ \Delta_{\tilde{F}_Y}^v &= \pi^* \Delta_{\tilde{F}_X} \sim_{\mathbb{Q}} -\pi^* K_X. \end{aligned}$$

Proof. Statement (a) follows from Lemma 8.3.25. Indeed, we have the morphism

$$\xi_{(X,D)}: F_{X/k}^* \Omega_{X'}^1(\log D') \longrightarrow \Omega_X^1(\log D)$$

(see Variant 8.3.9) such that $\text{div}(\det \xi_X) = \text{div}(\det \xi_{(X,D)}) + (p - 1)D$.

Statement (b) is clear by the construction of $\Delta_{\tilde{F}_Y}$ since $g(\text{Exc } g)$ has codimension at least two. Statement (c) follows from Lemma 8.3.34. $\square$

We now relate certain conditions on the compatibility of subschemes for Frobenius liftings and Frobenius splittings.

Lemma 8.3.36. *Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a quasi-projective smooth k -scheme X . Suppose that $\tilde{F}_X$ is compatible with a lifting of an integral subscheme $Z \subset X$. Then the associated Frobenius splitting is compatible with Z*

Proof. Let $\sigma_X: F_{X*}\mathcal{O}_X \rightarrow \mathcal{O}_X$ be the Frobenius splitting associated with $\tilde{F}_X$. First, we consider the case when Z is smooth. By Lemma 8.3.27 we see that $Y = \text{Bl}_Z X$ admits a Frobenius lifting $(\tilde{Y}, \tilde{F}_Y)$ compatible with the unique lifting of the exceptional divisor E and equipped with a lifting $\tilde{\pi}: (\tilde{Y}, \tilde{F}_Y) \rightarrow (\tilde{X}, \tilde{F}_X)$ of the contraction morphism $\pi: Y \rightarrow X$. By Variant 8.3.9 we see that E is compatible with the Frobenius splitting $\sigma_Y: F_{Y*}\mathcal{O}_Y \rightarrow \mathcal{O}_Y$ of Y induced by $\tilde{F}_Y$. By [BK05, Lemma 1.1.8(ii)] we therefore see that $Z = \pi(E)$ is compatible with the push-forward of σ_Y under π , which is equal to σ_X . This finishes the proof for Z smooth. For an arbitrary integral Z , we just observe that the condition of being compatibly split can be checked at the generic point. $\square$

Corollary 8.3.37. *Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a quasi-projective smooth k -scheme X . Then there are only finitely integral subschemes Z such that $\tilde{F}_X$ is compatible with a lifting of Z .*

Proof. By Lemma 8.3.36 we observe that every subscheme compatible with $\tilde{F}_X$ is compatible with the associated F -splitting. Then we conclude by [Sch09, Theorem 5.8], which states that there are only finitely many subschemes compatible with a given Frobenius splitting. $\square$

Uniqueness of liftings admitting a lifting of Frobenius

So far we have discussed Frobenius splittings arising from Frobenius liftings, but recall from §8.2.5 that a Frobenius splitting gives rise to a canonical lifting to $W_2(k)$. It is natural to ask whether the lifting induced by the Frobenius splitting associated to a Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ of a normal k -scheme X is canonically isomorphic to $\tilde{X}$. In fact much more is true, as indicated by the following simple but surprising result.

Theorem 8.3.38. *Let X be a scheme over k , let σ be a Frobenius splitting on X , and let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of X . Let $\tilde{X}(\sigma)$ be the canonical lifting of X induced by σ (see Theorem 8.2.12). Then one has a canonical isomorphism $\tilde{X}(\sigma) \xrightarrow{\sim} \tilde{X}$ (depending on $\tilde{F}_X$ and σ) of liftings of X .*

Proof. The canonical lifting $\tilde{X}(\sigma)$ admits the following description (see [Zda17, §3.5]). Recall that the ring $W_2(\mathcal{O}_X)$ is a square zero extension of $\mathcal{O}_X$ by $F_*\mathcal{O}_X$. Then $I(\sigma) = \ker(\sigma: F_*\mathcal{O}_X \rightarrow \mathcal{O}_X) \subseteq W_2(\mathcal{O}_X)$ is an ideal, and $\mathcal{O}_{\tilde{X}(\sigma)} = W_2(\mathcal{O}_X)/I(\sigma)$, sitting inside the pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \sigma \downarrow & & i_\sigma^* \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\tilde{X}(\sigma)} & \longrightarrow & \mathcal{O}_X \longrightarrow 0. \end{array}$$

The upshot is that we have a natural closed immersion $i_\sigma: \tilde{X}(\sigma) \rightarrow W_2(X)$ making the following triangle commute

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \tilde{X}(\sigma) & \xrightarrow{i_\sigma} & W_2(X). \end{array}$$

On the other hand, the Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ induces the map (8.10)

$$\nu_{\tilde{X}, \tilde{F}_X}: W_2(X) \longrightarrow \tilde{X}, \quad \nu^*(\tilde{f}) = (\tilde{f} \bmod p, \delta(\tilde{f}))$$

where $\tilde{F}_X^*(\tilde{f}) = \tilde{f}^p + p \cdot \delta(\tilde{f})$. The composition

$$\nu_{\tilde{X}, \tilde{F}_X} \circ i_\sigma: \tilde{X}(\sigma) \longrightarrow \tilde{X}$$

restricts to the identity on X , and hence is an isomorphism of liftings of $\tilde{X}$. This is the required map. $\square$

The proof shows that $\nu_{\tilde{X}, \tilde{F}_X} \circ i_\sigma: \tilde{X}(\sigma) \rightarrow \tilde{X}$ fits into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\tilde{X}} & \longrightarrow & \mathcal{O}_X \longrightarrow 0. \\ & & F_X^* \downarrow & & \nu_{\tilde{X}, \tilde{F}_X}^* \downarrow & & \parallel \\ 0 & \longrightarrow & F_* \mathcal{O}_X & \longrightarrow & W_2(\mathcal{O}_X) & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \sigma \downarrow & & i_\sigma^* \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\tilde{X}(\sigma)} & \longrightarrow & \mathcal{O}_X \longrightarrow 0, \end{array}$$

where the left composition $\sigma \circ F_X^*$ is the identity.

Corollary 8.3.39. *Let X be a normal k -scheme, and let $(\tilde{X}^{(i)}, \tilde{F}_X^{(i)})$ for $i = 1, 2$ be two Frobenius liftings of X . Then $\tilde{X}^{(1)} \simeq \tilde{X}^{(2)}$.*

Remark 8.3.40. If X is smooth, then Corollary 8.3.39 can be seen more directly as follows. The application of $\text{Hom}(\Omega_X^1, -)$ to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_{X*} \mathcal{O}_X \longrightarrow B_X^1 \longrightarrow 0$$

yields a connecting homomorphism $\delta: \text{Hom}(\Omega_X^1, B_X^1) \rightarrow \text{Ext}^1(\Omega_X^1, \mathcal{O}_X)$. The forgetful map

$$(\tilde{X}, \tilde{F}_X) \mapsto \tilde{X}: \quad \{\text{Frobenius liftings of } X\} / \text{isom.} \longrightarrow \{\text{liftings of } X\} / \text{isom.}, \quad (8.14)$$

is a map from a torsor under $\text{Hom}(\Omega_X^1, B_X^1)$ to a torsor under $\text{Ext}^1(\Omega_X^1, \mathcal{O}_X)$ which is equivariant with respect to the map δ . If X is F -liftable, it is F -split, and hence $\delta = 0$. Thus (8.14) is constant.

8.4. Toric varieties in families

In this section, we address the following three questions:

- **Generalisation:** Given a family $f: X \rightarrow S$ such that X_s is a toric variety for a dense set of $s \in S$, can we deduce that the generic fibre is a toric variety? (§8.4.1)
- **Specialisation:** Given a family $f: X \rightarrow S$ whose generic fibre is a toric variety and $S = \operatorname{Spec} R$ for a discrete valuation ring R , when can we deduce that the special fibre is a toric variety as well? (§8.4.2)
- **Global rigidity:** Given a proper nc pair (X, D) over a connected scheme S , if one geometric fibre $(X, D)_{\bar{s}}$ is a toric pair, must (X, D) globally come from a toric fibration? (§8.4.3)

The result of §8.4.1 will be used in the subsequent §8.4.4 to show that Conjecture 1.2.4 follows from (a special case of) Conjecture 8.1.1. The results of §8.4.2–§8.4.3 will be needed in Section 8.5.

8.4.1. Generalisation

We start with a somewhat lengthy proof of the following fact which should be well-known but for which we were unable to find a reference.

Lemma 8.4.1. *Let S be a geometrically unibranch [EGA IV_{II}, 6.15.1] noetherian scheme, and let $\pi: X \rightarrow S$ be a smooth proper morphism whose geometric fibres are connected and satisfy*

$$H^1(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) = H^2(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) = 0. \quad (8.15)$$

Then there exists a finite étale surjective morphism $S' \rightarrow S$ such that $\operatorname{Pic}_{X'/S'}$ is the constant sheaf associated to a finitely generated group on the big étale site of S' , where X' is the base change of X to S' .

Proof. By [Kle05, Theorem 9.4.8], $\operatorname{Pic}_{X/S}$ is representable by the disjoint union of quasi-projective schemes over S . By the deformation theory of line bundles, $H^2(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}})$ is the obstruction space and $H^1(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}})$ is the tangent space of the deformation functor of a line bundle on $X_{\bar{s}}$, and hence (8.15) shows that line bundles deform uniquely over thickenings of $X_{\bar{s}}$. Consequently, $\operatorname{Pic}_{X/S}$ is formally étale, and hence étale, over S . Moreover, $\operatorname{Pic}_{X/S} \rightarrow S$ satisfies the valuative criterion of properness. Thus if P is a connected component of $\operatorname{Pic}_{X/S}$, then $P \rightarrow S$ is a connected étale covering of S , and since S is geometrically unibranch, P is finite over S . We conclude that $\operatorname{Pic}_{X/S}$ is the disjoint union of connected finite étale coverings of S .

We assume without loss of generality that S is connected, and pick a geometric point $\bar{s}$ of S . Let $M = \operatorname{Pic}_{X/S}(\bar{s}) = \operatorname{Pic}(X_{\bar{s}})$, which is a finitely generated abelian group. Pick a finite set of generators $p_1, \dots, p_k \in M$, and for each $i = 1, \dots, k$, let P_i be the connected component of $\operatorname{Pic}_{X/S}$ containing the corresponding geometric point $\bar{p}_i \rightarrow \operatorname{Pic}_{X/S}$. Let

$S' \rightarrow S$ be a finite étale cover after pullback to which each $P_i \rightarrow S$ becomes constant. Replacing S by S' , we can assume that each P_i maps isomorphically onto S , that is, each $p_i \in \text{Pic}_{X/S}(\bar{s})$ is the restriction of a (unique) global section $q_i \in \text{Pic}_{X/S}(S)$. We claim that in this case $\text{Pic}_{X/S}$ is actually constant. Let $P = M \times S$ be the constant group scheme over S associated to M ; we will construct an isomorphism $\text{Pic}_{X/S} \xrightarrow{\sim} P$. The sections p_i and q_i define surjective morphisms of group schemes over S :

$$\alpha: \mathbb{Z}^k \times S \longrightarrow \text{Pic}_{X/S} \quad \text{and} \quad \beta: \mathbb{Z}^k \times S \longrightarrow P.$$

Moreover if the p_i satisfy a relation $\sum a_i p_i = 0$, then so do the sections q_i , and hence there is a surjective morphism $\gamma: \text{Pic}_{X/S} \rightarrow P$. Its kernel K is a closed subscheme of $\text{Pic}_{X/S}$ which is flat, and hence étale, over S . Moreover, $K_{\bar{s}} = \{1\}$ by construction, and hence $K \xrightarrow{\sim} S$. Thus γ is an isomorphism. $\square$

The above assertion need not be true when $H^2(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) \neq 0$, for example for non-isotrivial families of K3 surfaces over complete complex curves [BKPSB98].

Proposition 8.4.2. *Let S be a noetherian excellent scheme and let $\pi: X \rightarrow S$ be a smooth projective morphism whose geometric fibres are connected and satisfy (8.15). Let $\mathcal{E}$ be a locally free sheaf of rank r on X . Then the set*

$$\{s \in S \mid \mathcal{E}_{\bar{s}} \text{ is a direct sum of line bundles on } X_{\bar{s}} \text{ for every geom. pt. } \bar{s} \text{ over } s\} \subseteq S$$

is a constructible subset of S .

Proof. Stratifying S , we may assume that S is connected and regular. By Lemma 8.4.1, after replacing S with some finite étale cover $S' \rightarrow S$, we have $\text{Pic}_{X/S} \simeq M \times S$ for a finitely generated group M , and hence $\text{Pic}(X_s) = \text{Pic}(X_{\bar{s}}) = M$ for all $s \in S$. Given $H, D \in M$, there is a well-defined intersection number $H^{d-1} \cdot D \in \mathbb{Z}$, the same for all fibres, where d is the relative dimension of f .

We can assume that there exist $H_1, \dots, H_s \in M$ which give ample line bundles on every fibre and which span $M \otimes \mathbb{Q}$. If H is one of these, then by Noetherian induction there is a natural number n_H such that $\mathcal{E}(n_H H)$ is globally generated on all fibres. If $L \in M$ is a direct summand of $\mathcal{E}_s$, it follows that $L + n_H H$ is a direct summand of a globally generated sheaf, and hence is effective on X_s , so $(L + n_H H) \cdot H^{d-1} \geq 0$ (independent of s). Thus $L \cdot H^{d-1} \geq -n_H H^d$. Applying the same argument to $\mathcal{E}^\vee$, we get a natural number m_H such that $-L \cdot H^{d-1} \geq -m_H H^d$, i.e. $L \cdot H^{d-1} \leq m_H H^d$. We conclude that there exists a natural number B such that

$$|L \cdot H_i^{d-1}| < B \quad \text{for } i = 1, \dots, s \quad (8.16)$$

whenever L is a direct summand of some $\mathcal{E}_s$.

Because H_i span $M \otimes \mathbb{Q}$, the set K of all $L \in M$ satisfying the inequalities (8.16) is

finite. For $v = (L_1, \dots, L_r) \in K^r$, consider the functor

$$F_v: (\mathbf{Schemes}/S) \longrightarrow \mathbf{Sets}, \quad T \mapsto \mathrm{Isom}_{X \times_S T}((\bigoplus_{i=1}^r L_i)_T, \mathcal{E}_T).$$

Stratifying S further, we may assume that $\pi_* \mathcal{H}om((\bigoplus_{i=1}^r L_i), \mathcal{E})$ is locally free and its formation commutes with base change. This implies that F_v is representable by an open subscheme of the total space of the vector bundle $\pi_* \mathcal{H}om((\bigoplus_{i=1}^r L_i), \mathcal{E})$, and hence is a scheme of finite type over S . Let W be the disjoint union of these schemes, which is still of finite type as K^r is a finite set. The locus where $\mathcal{E}$ splits is the set-theoretic image of $W \rightarrow S$, which is constructible by Chevalley's theorem. $\square$

We shall now apply this to the study of toric varieties in families. As observed by Jaczewski and developed further by Kedzierski and Wiśniewski, smooth toric varieties admit a generalisation of the Euler sequence, and this property in fact characterises smooth projective toric varieties, at least in characteristic zero.

Definition 8.4.3 (cf. [Jac94, Definition 2.1] and [KW15, §1.1]). Let $\pi: X \rightarrow S$ be a smooth projective morphism of schemes. Suppose that S is affine, and that the coherent sheaf $\mathcal{H} = R^1 \pi_* \Omega_{X/S}^1$ is locally free. Then the image of $\mathrm{id}_{\mathcal{H}}$ under the natural identification

$$\begin{aligned} \mathrm{Hom}_S(\mathcal{H}, \mathcal{H}) &\simeq H^0(S, R^1 \pi_*(\pi^* \mathcal{H}^\vee \otimes \Omega_{X/S}^1)) \\ &\simeq H^0(S, \mathcal{E}xt_{X/S}^1(\pi^* \mathcal{H}, \Omega_{X/S}^1)) \simeq \mathrm{Ext}^1(\pi^* \mathcal{H}, \Omega_{X/S}^1) \end{aligned}$$

gives rise to an extension

$$0 \longrightarrow \Omega_{X/S}^1 \longrightarrow R_{X/S} \longrightarrow \pi^* \mathcal{H} \longrightarrow 0. \quad (8.17)$$

The locally free sheaf $R_{X/S}$ is called the *potential sheaf* of X over S .

Theorem 8.4.4 ([KW15, Theorem 1.1 and Corollary 2.9]). *Let K be an algebraically closed field, and let X be a smooth projective integral scheme over K . If X is a toric variety, then the sheaf $R_{X/K}$ splits into the direct sum of line bundles. Conversely, if K has characteristic zero, the potential sheaf $R_{X/K}$ splits into the direct sum of line bundles, and we have the vanishing*

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0,$$

then X is a toric variety.

Corollary 8.4.5. *Let S be an integral noetherian scheme with generic point η of characteristic zero, and let $\pi: X \rightarrow S$ be a smooth and projective morphism. Suppose that the geometric fibre $X_{\bar{s}}$ is a toric variety for a dense set of closed points $s \in S$. Then the geometric generic fibre $X_{\bar{\eta}}$ is a toric variety.*

Proof. Shrinking S , we may assume that S is affine and that $R^1\pi_*\Omega_{X/S}^1$ is locally free, in which case the potential sheaf $R_{X/S}$ is defined. Moreover, since $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for any $i > 0$ and a toric variety Y over a field k , we see that the assumption on $H^i(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}})$ in Proposition 8.4.2 is satisfied. We apply this result to the potential sheaf $R_{X/S}$. Since $R_{Y/k}$ splits for a toric variety Y by the first part of Theorem 8.4.4, we deduce that $R_{X_{\bar{\eta}}/\bar{\eta}} = (R_{X/S})_{\bar{\eta}}$ must be split as well. By the other direction of Theorem 8.4.4, this implies that $X_{\bar{\eta}}$ is a toric variety. $\square$

8.4.2. Specialisation

Let R be a discrete valuation ring with residue field k and fraction field K . We set

$$S = \operatorname{Spec} R, \quad s = \operatorname{Spec} k, \quad \bar{s} = \operatorname{Spec} \bar{k}, \quad \eta = \operatorname{Spec} K, \quad \bar{\eta} = \operatorname{Spec} \bar{K}.$$

Suppose that X is a smooth projective scheme over S whose general fibre X_{η} is a toric variety. It is not true in general that the geometric special fibre $X_{\bar{s}}$ is a toric variety, as the following basic example shows.

Example 8.4.6. Let $s_0, s_1, s_2 \in \mathbb{P}^2(R)$ be three sections over S which give a triple of distinct collinear points in the special fibre $\mathbb{P}^2(k)$, but which are not collinear in $\mathbb{P}^2(K)$. Let X be the blow-up of $\mathbb{P}_S^2$ along the union of these three sections. Then X is a smooth projective surface over S , and X_{η} is a toric variety but $X_{\bar{s}}$ is not.

The above phenomenon cannot happen if we consider deformations of toric varieties together with their toric boundaries. The goal of this subsection is to show Proposition 8.4.9 which says that toric pairs can only degenerate to toric pairs.

Lemma 8.4.7 ([Ful98, Lemma, p.66]). *Let X be a toric variety over a field k and let $D_1, \dots, D_r$ be all of its toric prime divisors. Let $\mathcal{L}$ be a line bundle on X and set*

$$V_{\mathcal{L}} = \left\{ (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}_{\geq 0}^r \mid \mathcal{L} \simeq \mathcal{O}_X(\lambda_1 D_1 + \dots + \lambda_r D_r) \right\}.$$

For $\lambda \in V_{\mathcal{L}}$, let $s_{\lambda} \in H^0(X, \mathcal{L})$ be the (well-defined up to scaling) section whose divisor of zeros is $\lambda_1 D_1 + \dots + \lambda_r D_r$. Then

$$H^0(X, \mathcal{L}) = \bigoplus_{\lambda \in V_{\mathcal{L}}} k \cdot s_{\lambda}.$$

Remark 8.4.8. Choose $1 \leq i \leq r$ and consider the standard exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{L}(-D_i)) \longrightarrow H^0(X, \mathcal{L}) \longrightarrow H^0(D_i, \mathcal{L}|_{D_i})$$

The above lemma shows that

$$H^0(X, \mathcal{L}(-D_i)) = \bigoplus_{\lambda \in V_{\mathcal{L}}, \lambda_i > 0} k \cdot s_{\lambda},$$

and hence $\bigoplus_{\lambda \in V_{\mathcal{L}}, \lambda_i=0} k \cdot s_{\lambda}$ injects into $H^0(D_i, \mathcal{L}|_{D_i})$.

Proposition 8.4.9. *Let X be a smooth projective scheme over S , and let $D \subseteq X$ be a divisor with normal crossings relative to S . If (X_{η}, D_{η}) (resp. $(X_{\bar{\eta}}, D_{\bar{\eta}})$) is a toric pair (see §8.2.1), then so is (X_s, D_s) (resp. $(X_{\bar{s}}, D_{\bar{s}})$).*

Proof. Suppose that (X_{η}, D_{η}) is a toric pair. Let $D = D_1 \cup \dots \cup D_r$ be the decomposition of D into irreducible components. By induction, we can assume that $(D_i, (D - D_i)|_{D_i})$ is a toric pair for every $1 \leq i \leq r$.

Our goal is to show that the assertion of Lemma 8.4.7 holds over R . To this end, for an arbitrary line bundle $\mathcal{L}$ on X we define

$$R_{\mathcal{L}}(X, D) = \bigoplus_{\lambda \in V_{\mathcal{L}}} R \cdot \bar{s}_{\lambda},$$

where

$$V_{\mathcal{L}} = \left\{ (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}_{\geq 0}^r \mid \mathcal{L}_{\eta} \simeq \mathcal{O}_X(\lambda_1(D_1)_{\eta} + \dots + \lambda_r(D_r)_{\eta}) \right\},$$

and $\bar{s}_{\lambda}$ is a formal symbol identifying the λ -th component. There is a natural map

$$R_{\mathcal{L}}(X, D) \longrightarrow H^0(X, \mathcal{L}),$$

sending $\bar{s}_{\lambda}$ to the unique (up to scaling) section corresponding to the divisor $\lambda_1 D_1 + \dots + \lambda_r D_r$.

Note that $\text{Pic } X = \text{Pic } X_{\eta}$. We claim that

$$\begin{aligned} R_{\mathcal{L}}(X, D) \otimes K &\xrightarrow{\sim} H^0(X_{\eta}, \mathcal{L}_{\eta}), \text{ and} \\ R_{\mathcal{L}}(X, D) \otimes k &\longrightarrow H^0(X_s, \mathcal{L}_s) \text{ is injective,} \end{aligned}$$

where $\mathcal{L}_s = \mathcal{L}|_{X_s}$.

The composition of $R_{\mathcal{L}}(X, D) \rightarrow H^0(X, \mathcal{L})$ with the restriction $H^0(X, \mathcal{L}) \rightarrow H^0(X_{\eta}, \mathcal{L}_{\eta})$ gives us, thanks to Lemma 8.4.7, an isomorphism

$$R_{\mathcal{L}}(X, D) \otimes K \simeq H^0(X_{\eta}, \mathcal{L}_{\eta}).$$

In particular, we can write $R_{\mathcal{L}}(X, D) \subseteq H^0(X, \mathcal{L})$, where we identify $\bar{s}_{\lambda}$ with the corresponding section, as defined above.

Now, we want to show that $R_{\mathcal{L}}(X, D) \otimes k \rightarrow H^0(X_s, \mathcal{L}_s)$ is injective. To this end, take any $1 \leq i \leq k$ and let Θ_i be the sum of all D_j 's, except for the i -th one, restricted to D_i . By Lemma 8.4.7 and Remark 8.4.8 we have the following exact sequence:

$$0 \longrightarrow R_{\mathcal{L}(-D_i)}(X, D) \longrightarrow R_{\mathcal{L}}(X, D) \longrightarrow R_{\mathcal{L}|_{D_i}}(D_i, \Theta_i).$$

Indeed, we only need to show that the free module $R_{\mathcal{L}}(X, D)/R_{\mathcal{L}(-D_i)}(X, D)$ injects into $R_{\mathcal{L}|_{D_i}}(D_i, \Theta_i)$, but this can be checked by performing a base change to the generic

fibre. Let us note that tensoring by k preserves the exactness of this exact sequence as $R_{\mathcal{L}(-D_i)}(X, D)$ is a direct summand of $R_{\mathcal{L}}(X, D)$.

Now, assume by contradiction that the image of some $s \in R_{\mathcal{L}}(X, D) \otimes k$ is zero in $H^0(X_s, \mathcal{L}_s)$. By induction, we can assume that $R_{\mathcal{L}|_{D_i}}(D_i, \Theta_i) \otimes k$ injects into

$$H^0((D_i)_s, \mathcal{L}_s|_{(D_i)_s}).$$

In particular, $s \in R_{\mathcal{L}(-D_i)}(X, D) \otimes k$. By subsequently repeating this argument, we obtain that $s \in R_{\mathcal{L}(-mD_i)}(X, D) \otimes k$ for all $m \geq 0$, and so $H^0(X, \mathcal{L}(-mD_i)) \neq 0$. This is a contradiction, and hence the claim is proved.

Finally, we show that if $H^1(X_s, \mathcal{L}_s) = 0$, then $R_{\mathcal{L}}(X, D) \otimes k \xrightarrow{\sim} H^0(X_s, \mathcal{L}_s)$. Since the obstructions for deforming sections of $H^0(X_s, \mathcal{L}_s)$ lie in $H^1(X_s, \mathcal{L}_s)$, we get $h^0(X_s, \mathcal{L}_s) \leq h^0(X_\eta, \mathcal{L}_\eta)$, and so, by semicontinuity, this is an equality. Hence, $R_{\mathcal{L}}(X, D) \otimes k \rightarrow H^0(X_s, \mathcal{L}_s)$ is an isomorphism.

To sum up, X is isomorphic to $\text{Proj}_S(\bigoplus_m R_{\mathcal{L}^m}(X, D))$ for any very ample line bundle $\mathcal{L}$ satisfying $H^1(X_s, \mathcal{L}_s^{\otimes m}) = 0$ for all $m \geq 0$. In particular, (X, D) is a constant toric fibration over S and (X_s, D_s) is toric.

Finally, suppose that $(X_{\bar{\eta}}, D_{\bar{\eta}})$ is a toric pair. Then there exists a finite extension η'/η such that $(X_{\eta'}, D_{\eta'})$ is toric. Let $S' \rightarrow S$ be the normalisation of S in η' , and let s' be the closed point of S' . Applying the above to the base change $X' \rightarrow S'$, we deduce that $(X_{s'}, D_{s'})$ is toric, and hence so is $(X_{\bar{s}}, D_{\bar{s}})$. $\square$

8.4.3. Global rigidity of toric pairs

The following result shows that there are no interesting families of toric pairs.

Proposition 8.4.10. *Let $f: X \rightarrow S$ be a smooth and projective morphism to a non-empty noetherian connected scheme S and let $D \subseteq X$ be a divisor with normal crossings relative to S . The following conditions are equivalent:*

- (i) *The morphism f admits a structure of a toric fibration with toric boundary D .*
- (ii) *Étale-locally on S , there exists a smooth fan Σ and an S -isomorphism*

$$(X, D) \simeq (X(\Sigma), D(\Sigma))_S.$$

- (iii) *For a single geometric point $\bar{s}$ of S , there exists a smooth fan Σ and an isomorphism*

$$(X, D)_{\bar{s}} \simeq (X(\Sigma), D(\Sigma))_{\bar{s}}.$$

Proof. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. For (iii) $\Rightarrow$ (ii), it is enough to consider S irreducible. Consider the scheme

$$I = \text{Isom}_S((X, D), (X(\Sigma), D(\Sigma))_S),$$

which is locally of finite type over S . If $\bar{i}$ is a geometric point of I and $\bar{s}$ is its image in S , then $\Omega_{X_{\bar{s}}/\bar{s}}^1(\log D) \simeq \Omega_{X(\Sigma)_{\bar{s}}}^1(\log D(\Sigma)_{\bar{s}})$ is trivial and $H^i(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) \simeq H^i(X(\Sigma)_{\bar{s}}, \mathcal{O}_{X(\Sigma)_{\bar{s}}}) = 0$ for $i > 0$. Deformation theory shows that $I \rightarrow S$ is formally smooth at $\bar{i}$. We conclude that $I \rightarrow S$ is smooth, and in particular its image is an open subset of S .

We shall now prove that the assertion of (iii) holds for *every* geometric point of S , or in other words, that $I \rightarrow S$ is surjective. By assumption, I is non-empty, and hence the image of $I \rightarrow S$ is a non-empty open subset of S . It is also dense in S (as S is irreducible), so it only remains to show that it is closed under specialisation. By Lemma 8.4.13, it is enough to consider $S = \operatorname{Spec} V$ for a discrete valuation ring V . In this case, Proposition 8.4.9 implies the required assertion. Now since $I \rightarrow S$ is smooth and surjective, it admits sections étale locally on S , which shows (ii).

Finally, assume (ii), and let $T = \operatorname{Aut}_S^0(X, D)$ be the connected component of $\operatorname{Aut}_S(X, D)$. Then by Lemma 8.4.11, étale-locally on S there exists an isomorphism $T \simeq \mathbb{G}_m^n$ and an equivariant isomorphism $(X, D) \simeq (X(\Sigma), D(\Sigma))$. Thus (X, D) is a toric fibration over S . $\square$

Lemma 8.4.11. *Let $\Sigma \subseteq \mathbb{R}^n$ be a smooth complete fan, and let S be a scheme. Then the group scheme $A = \operatorname{Aut}_S((X(\Sigma), D(\Sigma)))_S$ is an extension*

$$1 \longrightarrow \mathbb{G}_{m,S}^n \longrightarrow A \longrightarrow \operatorname{Aut}(\Sigma)_S \longrightarrow 1$$

where $\operatorname{Aut}(\Sigma)$ is the (finite) group of automorphisms of the fan Σ .

Proof. Let $\Sigma(1)$ denote the set of rays (one-dimensional cones) of Σ . Let $D(\Sigma) = \bigcup_{\rho \in \Sigma(1)} D_\rho$ be a decomposition of $D(\Sigma)$ into irreducible components. Then every automorphism of the pair $(X(\Sigma), D(\Sigma))$ over S has to permute the divisors D_ρ , which yields a homomorphism $A \rightarrow \operatorname{Aut}(\Sigma(1))$. Let A^0 denote its kernel and let $\sigma \in \Sigma$ be a top-dimensional cone, corresponding to an open immersion

$$(\mathbb{A}_S^n, \{x_i = 0\}_{i=1,\dots,n}) \hookrightarrow (X(\Sigma), D(\Sigma))_S$$

whose image is preserved by A^0 . A direct calculation shows that $\operatorname{Aut}_S(\mathbb{A}_S^n, \{x_i = 0\}_{i=1,\dots,n})$ can be naturally identified with $\mathbb{G}_{m,S}^n$. $\square$

Remark 8.4.12. In the proof of Lemma 8.4.11, the completeness assumption was only used to find a top-dimensional cone in Σ . On the other hand, $\operatorname{Aut}_S(\mathbb{G}_{m,S})$ (where $\mathbb{G}_{m,S}$ is treated as an S -scheme) might be bigger than the semidirect product of $\mathbb{G}_{m,S}$ and $\mathbb{Z} \times \mathbb{Z}$ (and non-representable) if S is non-reduced. Smoothness is probably not necessary.

Lemma 8.4.13. *Let S be a connected noetherian scheme and let $U \subseteq S$ be an open subset. Suppose that for every discrete valuation ring V , and for every morphism $h: \operatorname{Spec} V \rightarrow S$ mapping the generic point into U , we have $h(\operatorname{Spec} V) \subseteq U$. Then $U = S$.*

Proof. The valuative criterion of properness shows that $U \rightarrow S$ is proper, and hence U is also closed. $\square$

Remark 8.4.14. Smoothness is probably not necessary for Proposition 8.4.10. Properness seems essential: for example, the pair $(\mathbb{A}^2, \mathbb{G}_m \times \{0\} + \{0\} \times \mathbb{G}_m)$ has non-trivial first-order deformations. Projectivity seems to be only an artifact of the proof of Proposition 8.4.9.

8.4.4. Images of toric varieties

We shall now combine the technique of descending F -liftability (Theorem 8.3.20) with the main result of §8.4.1 to show the following.

Theorem 8.4.15. *Suppose that Conjecture 8.1.1 is true for simply connected (e.g. separably rationally connected) varieties, that is, that every smooth projective simply connected F -liftable variety over an algebraically closed field of characteristic $p > 0$ is a toric variety. Then Conjecture 1.2.4 is true, that is, a smooth projective image of a complete toric variety in characteristic zero is a toric variety.*

Proof. Let $\varphi_K: Z_K \rightarrow X_K$ be a surjective morphism from a complete toric variety Z_K to a smooth projective variety X_K defined over an algebraically closed field K of characteristic zero. Reasoning as in [OW02], we can assume that φ_K is finite, in which case it is also flat by “Miracle Flatness” [EGA IV_{III}, 15.4.2], because X_K is smooth and Z_K is Cohen–Macaulay. There exists a finitely generated subring $R \subseteq K$ and a finite flat surjective map $\varphi: Z \rightarrow X$ of schemes over $S = \operatorname{Spec} R$, satisfying the following properties:

- (1) φ_K is the base change of φ to K ,
- (2) S is smooth over $\operatorname{Spec} \mathbb{Z}$,
- (3) Z is a proper constant toric fibration over S (see §8.2.1),
- (4) X is smooth and projective over S ,
- (5) $d = \deg(\varphi_K)$ is invertible on S .

If $\bar{s}: \operatorname{Spec} k \rightarrow S$ is a geometric point of S , then $\varphi_{\bar{s}}: Z_{\bar{s}} \rightarrow X_{\bar{s}}$ is a finite surjective map from a toric variety $Z_{\bar{s}}$ to a smooth projective variety $X_{\bar{s}}$ over $\bar{s}$, of degree invertible in k . Let $\tilde{s}: \operatorname{Spec} W_2(k) \rightarrow S$ be a lifting of $\bar{s} \bmod p^2$ (such a lifting exists thanks to condition (2)). Then $Z_{\tilde{s}}$ is a constant toric fibration over $\tilde{s}$, and hence $F_{Z_{\tilde{s}}}$ lifts to $Z_{\tilde{s}}$. By Theorem 8.3.20(a) applied to $\varphi_{\tilde{s}}: Z_{\tilde{s}} \rightarrow X_{\tilde{s}}$, we get that $F_{X_{\tilde{s}}}$ lifts to $X_{\tilde{s}}$. Moreover, since X_K is simply connected, so is $X_{\bar{s}}$, as the specialisation map $\pi_1(X_K) \rightarrow \pi_1(X_{\bar{s}})$ is surjective [SGA 1, Exp. X, Corollaire 2.3]. The assumed case of Conjecture 8.1.1 implies that $X_{\bar{s}}$ is a toric variety. Thus X_K is toric, by Corollary 8.4.5. $\square$

Remark 8.4.16. If we believe in Conjecture 8.1.1 in its full strength, then the connection between Conjecture 1.2.4 and a special case of Conjecture 8.1.1 suggests that in characteristic zero, images of toric fibrations over abelian varieties should be also of this type, up to a finite étale cover. This would be a common generalisation of Conjecture 1.2.4 and the results of [Deb89, HM01, DHP08] on the images of abelian varieties.

The first obvious obstacle in deducing such a statement from Conjecture 8.1.1 is the ordinary reduction conjecture for abelian varieties. But even assuming that, the method of proof of Theorem 8.4.15 does not apply in this case, as the assumptions of Theorem 8.3.20(a) may no longer be satisfied. More precisely, if $A \rightarrow S$ is an abelian scheme over a base S which is of finite type over $\mathbb{Z}$, if $\bar{s} = \text{Spec } k \rightarrow S$ is a closed geometric point, and $\tilde{s} = \text{Spec } W_2(k) \rightarrow S$ is a lifting of $\bar{s}$, then to apply the argument from the proof of Theorem 8.4.15 we need not only $A_{\tilde{s}}$ to be ordinary, but also $A_{\tilde{s}}$ to be its Serre–Tate canonical lifting. We do not know whether one should expect such $\tilde{s}$ to exist, even if A is an elliptic curve over S .

Using Theorem 8.3.20(c) we can show the following.

Let K be an algebraically closed field of characteristic zero and let Z be a smooth projective variety over K whose Albanese morphism $Z \rightarrow A$ is a toric fibration. Let G be a finite group acting on Z . Suppose that $X = Z/G$ is smooth, and that the abelian variety A satisfies the ordinary reduction conjecture. Assume that Conjecture 8.1.1 is valid. Then X admits a finite étale cover by a variety whose Albanese morphism is a Zariski-locally trivial fibration with toric fibres.

Since the proof is long and technical, we refrained from including it in this article.

Using Galois closures and Albanese mappings it is not difficult to show that if $f: A \rightarrow X$ is a finite morphism from an abelian variety A to a smooth variety X defined over an algebraically closed field k of characteristic zero, then X is a quotient of some (possibly different) abelian variety by a finite group. Therefore, the above result partially recovers the classification of smooth images of abelian varieties under finite morphisms mentioned above contingent upon the validity of Conjecture 8.1.1.

8.5. Structure of F -liftable nc pairs with trivial canonical bundle

8.5.1. Statement of the main result and some preliminaries

In this section, we provide a logarithmic generalisation of [MS87, Theorem 2], settling a special case of Conjecture 8.1.1. More precisely, we prove the following theorem which characterises projective F -liftable nc pairs (X, D) with $\omega_X(D)$ numerically trivial.

Theorem 8.5.1. *Let (X, D) be a projective nc pair over an algebraically closed field k of positive characteristic. The following conditions are equivalent:*

- (i) (X, D) is F -liftable and $\omega_X(D)$ is numerically trivial,
- (ii) X is F -split and $\Omega_X^1(\log D)$ becomes trivial on a finite étale cover of X ,
- (iii) X admits a finite étale cover $f: Y \rightarrow X$ whose Albanese map $a: Y \rightarrow A$ is a toric fibration over an ordinary abelian variety with toric boundary $f^{-1}(D)$.

Proof. We start by showing (i) $\Rightarrow$ (ii). If X is F -liftable, it is F -split, by Proposition 8.3.8(c). Using Variant 8.3.9, we observe that there exists an injective morphism

$$\xi: F^*\Omega_X^1(\log D) \longrightarrow \Omega_X^1(\log D).$$

The determinant of ξ gives rise to a non-zero section of the $(p-1)$ -st power of the numerically trivial bundle $\omega_X(D)$, and hence is an isomorphism. Therefore, we see that $F^*\Omega_X^1(\log D) \simeq \Omega_X^1(\log D)$ and thus by [LS77, Satz 1.4] the bundle $\Omega_X^1(\log D)$ becomes trivial on a finite étale cover $\pi: Y \rightarrow X$.

Now we show (iii) $\Rightarrow$ (i). Replacing A with a finite étale cover and Y with its base change, we may assume that $Y \rightarrow A$ is a split toric fibration. By Lemma 8.3.22, there exists a natural Frobenius lifting of $a: Y \rightarrow A$ over the Serre–Tate canonical lifting $(\tilde{A}, \tilde{F}_A)$ of A , and consequently Y is F -liftable. Moreover, this Frobenius lifting is compatible with the toric boundary $f^{-1}D$ of $Y \rightarrow A$. Further, since $Y \rightarrow A$ is a split toric fibration with toric boundary $f^{-1}(D)$, the bundle $\Omega_{Y/A}^1(\log f^{-1}D)$ is trivial, and hence $\omega_Y(f^{-1}(D))$ is trivial as well. Thus $(Y, f^{-1}D)$ satisfies (i), and hence by what we already proved it also satisfies (ii). Replacing Y by a further finite étale cover, we can thus assume that $\Omega_Y^1(\log f^{-1}D)$ is trivial and that $Y \rightarrow X$ is Galois under a finite group G .

To finish the argument, we argue as in [MS87, Proof of Theorem 2, (i) $\Rightarrow$ (ii)]. By Variant 8.3.16, Frobenius liftings of $(Y, f^{-1}D)$ correspond to splittings of the short exact sequence

$$0 \longrightarrow B_Y^1 \longrightarrow Z_Y^1(\log f^{-1}D) \longrightarrow \Omega_{Y'}^1(\log f^{-1}D') \longrightarrow 0,$$

where Y' is the Frobenius twist of Y . Since $\Omega_Y^1(\log f^{-1}D)$ is trivial and $H^0(Y, B_Y^1) = 0$ (as Y is F -split), the above extension admits a unique splitting. In particular, this splitting is G -invariant, and hence descends to a splitting of the corresponding sequence on X . Thus (X, D) is Frobenius liftable. Since $f^*\omega_X(D) = \omega_Y(f^{-1}D) \simeq \mathcal{O}_Y$, we see that $\omega_X(D)$ is numerically trivial.

Finally, (ii) $\Rightarrow$ (iii) follows from Theorem 8.5.5, whose proof occupies the subsequent subsection. $\square$

The proof of the implication (ii) $\Rightarrow$ (iii) in the case $D = 0$ in [MS87] relied on lifting to characteristic zero and using Yau’s work on the Calabi conjecture. Instead, we use the following two characteristic zero results.

Theorem 8.5.2 ([Win04, Corollary 1]). *Let (X, D) be a projective nc pair defined over $\mathbb{C}$. Then the following conditions are equivalent:*

- (i) *the log cotangent bundle $\Omega_X^1(\log D)$ is trivial,*
- (ii) *there exists a semi-abelian group variety T (an extension of an abelian variety by a torus) acting on X with an open dense orbit $X \setminus D$.*

If $\dim \text{Alb } X = 0$, then (X, D) is a toric pair.

We say that a group G is *virtually abelian* if it contains a finitely generated abelian subgroup of finite index.

Theorem 8.5.3 ([GKP16, Theorem 10.1], cf. [NZ10]). *Let X be a normal projective variety over $\mathbb{C}$ which admits a polarised endomorphism of degree greater than one. Then the topological fundamental group $\pi_1(X(\mathbb{C}))$ is virtually abelian. The same holds for $\pi_1^{\text{ét}}(X)$ since it is the profinite completion of $\pi_1(X(\mathbb{C}))$.*

Remark 8.5.4. Since [MS87] was motivated by the corresponding result in characteristic zero, we must warn the reader that the following statements are false.

“Theorem” A (characteristic zero analogue of Theorem 8.5.1). *Let (X, D) be a projective nc pair over an algebraically closed field k of characteristic zero. The following conditions are equivalent:*

- (i) $\Omega_X^1(\log D)$ becomes trivial on a finite étale cover of X ,
- (ii) X admits a finite étale cover $f: Y \rightarrow X$ whose Albanese map $a: Y \rightarrow A$ is a toric fibration over an ordinary abelian variety with toric boundary $f^{-1}(D)$.

“Theorem” B. *The assertion of Winkelmann’s theorem (Theorem 8.5.2) holds over algebraically closed fields of positive characteristic.*

“Theorem” C. *Let X be a smooth projective scheme over an open subset $S \subseteq \text{Spec } \mathcal{O}_K$ where $\mathcal{O}_K$ is the ring of integers in a number field K . Suppose that for infinitely many closed points $s \in S$, the reduction $X_{\bar{s}}$ has a finite étale cover whose Albanese map is a toric fibration. Then the same holds for $X_{\bar{K}}$.*

In “Theorem” A, we have (i) $\Rightarrow$ (ii) by Theorem 8.5.2, but (ii) does *not* imply (i). A basic counterexample is as follows. Let C be an elliptic curve, let L be a line bundle of non-zero degree on C , and let $E = \mathcal{O}_C \oplus L$. Let $X = \mathbb{P}_C(E)$ and let D be the sum of the two sections $C \rightarrow X$ corresponding to the projections $E \rightarrow \mathcal{O}_C$ and $E \rightarrow L$. Then condition (ii) is satisfied with $Y = X$. Moreover, every finite étale cover of X is of the same type, so if (ii) $\Rightarrow$ (i) were to hold, we could find such an X with $\Omega_X^1(\log D)$ trivial. So suppose that $\Omega_X^1(\log D)$ is trivial. Again by Theorem 8.5.2, the open subset $U = X \setminus D$ (which is the $\mathbb{G}_m$ -torsor corresponding to L) admits a group structure making $U \rightarrow C$ a group homomorphism. But this is only possible if $\deg L = 0$ by the “Barsotti–Weil formula”

$$\text{Pic}^0(C) \simeq \text{Ext}^1(C, \mathbb{G}_m) \quad (\text{see [Ser59, VII.16, Théorème 6]}).$$

Note that in the two extreme cases $A = Y$ and $A = 0$, the implication (ii) $\Rightarrow$ (i) does indeed hold. This construction also provides a counterexample to “Theorem” B (i) $\Rightarrow$ (ii).

For a counterexample to “Theorem” C, we take an elliptic curve C over $\mathbb{Z}[1/N]$ for some N , and the non-split extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

We again set $X = \mathbb{P}_C(E)$. If p is a prime of ordinary reduction of E and $k = \overline{\mathbb{F}}_p$, then X_k satisfies the required property by Remark 8.3.7 and Lemma 8.7.8. On the other hand, the above extension does not become split over any finite étale cover of $C_{\overline{\mathbb{Q}}}$, and consequently no finite étale cover of $X_{\overline{\mathbb{Q}}}$ is a toric fibration over an abelian variety.

8.5.2. F -split nc pairs with trivial cotangent bundle

In this section, we prove the following result generalising [MS87, Theorem 1] and yielding the implication (ii) $\Rightarrow$ (iii) needed above.

Theorem 8.5.5 (Log version of [MS87, Theorem 1]). *Let (X, D) be a projective nc pair over an algebraically closed field k such that $\Omega_X^1(\log D)$ is trivial and X is F -split. Then after an étale covering X admits a structure of a toric fibration over an ordinary abelian variety.*

Even though the proof closely follows the ideas of Mehta and Srinivas, there are many important details that need to be figured out in the logarithmic setting. For this reason, we precede the proof with a sequence of lemmas generalising their results.

Lemma 8.5.6 (Log version of [MS87, Lemma 1.2]). *Let (X, D) be an nc pair satisfying the hypotheses of Theorem 8.5.5, and let $\pi: Y \rightarrow X$ be a finite étale covering. Then $(Y, \pi^{-1}D)$ also satisfies the hypotheses of Theorem 8.5.5.*

Proof. Since π is étale, we see that $\Omega_Y^1(\log \pi^{-1}D)$ is isomorphic to $\pi^*\Omega_X^1(\log D)$ and is therefore trivial. To prove that Y is F -split, we use Lemma 8.2.15(d). $\square$

Lemma 8.5.7 (Log version of [MS87, Lemma 1.4]). *Let (X, D) be an nc pair satisfying the hypotheses of Theorem 8.5.5. Then the Albanese map $f: X \rightarrow \text{Alb } X$ is smooth, surjective, and has connected fibres. Moreover, (X, D) is an nc pair over $\text{Alb } X$ and $\Omega_{X/\text{Alb } X}^1(\log D)$ is trivial.*

Proof. First, we observe that the logarithmic differential $f^*\Omega_{\text{Alb } X}^1 \rightarrow \Omega_X^1(\log D)$ decomposes as

$$f^*\Omega_{\text{Alb } X}^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log D),$$

where the first morphism is the differential and the second is an inclusion (see §8.2.3). Since $H^0(\text{Alb } X, \Omega_{\text{Alb } X}^1) = H^0(X, f^*\Omega_{\text{Alb } X}^1)$, by [MS87, Lemma 1.3] we see that the induced morphism $H^0(X, f^*\Omega_{\text{Alb } X}^1) \rightarrow H^0(X, \Omega_X^1)$ is injective and hence

$$H^0(X, f^*\Omega_{\text{Alb } X}^1) \longrightarrow H^0(X, \Omega_X^1(\log D)) \quad (8.18)$$

is injective as well. As $f^*\Omega_{\text{Alb } X}^1$ and $\Omega_X^1(\log D)$ are trivial, the map $f^*\Omega_{\text{Alb } X}^1 \rightarrow \Omega_X^1(\log D)$ is uniquely determined by (8.18) and is therefore injective. We conclude that f is separable and dominant and that there exists a short exact sequence of trivial bundles

$$0 \longrightarrow f^*\Omega_{\text{Alb } X}^1 \longrightarrow \Omega_X^1(\log D) \longrightarrow \Omega_{X/\text{Alb } X}^1(\log D) \longrightarrow 0.$$

Consequently, the sheaf of relative log differentials $\Omega_{X/\text{Alb } X}^1(\log D)$ is locally free and therefore $f: (X, D) \rightarrow \text{Alb } X$ is an nc pair over $\text{Alb } X$ by Lemma 8.2.7.

To see that the fibres of f are connected, that is, we have $f_*\mathcal{O}_X = \mathcal{O}_{\text{Alb } X}$, we consider the Stein factorisation $X \rightarrow Z \rightarrow \text{Alb } X$. Since $X \rightarrow \text{Alb } X$ is smooth, the morphism $Z \rightarrow \text{Alb } X$ is étale (by [SGA 1, Exp. X, Prop. 1.2] or [EGA III_{II}, 7.8.10 (i)]) and therefore Z is an abelian variety. By the universal property of the Albanese morphism, $Z \rightarrow \text{Alb } X$ is an isomorphism, which finishes the proof. $\square$

Lemma 8.5.8. *Let (X, D) be an nc pair satisfying the hypothesis of Theorem 8.5.5. Then there exists a projective lifting $(\mathcal{X}, \mathcal{D})$ of (X, D) over $W(k)$ together with a lifting of the Frobenius morphism $F_{\mathcal{X}}: (\mathcal{X}, \mathcal{D}) \rightarrow (\mathcal{X}, \mathcal{D})$. Moreover, for every line bundle L on X there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $L \simeq \mathcal{L}|_X$ and $F_{\mathcal{X}}^*\mathcal{L} \simeq \mathcal{L}^{\otimes p}$.*

Proof. We apply Variant 8.3.16 to see that the obstruction classes to lifting the nc pair (X, D) together with the Frobenius morphism over consecutive Witt rings $W_n(k)$ lie in

$$\text{Ext}^1(\Omega_X^1(\log D), B_X^1).$$

Since $\Omega_X^1(\log D)$ is trivial and X is F -split, we see by Lemma 8.2.15 that the latter group satisfies

$$\text{Ext}^1(\Omega_X^1(\log D), B_X^1) = \text{Ext}^1(\mathcal{O}_X^{\oplus n}, B_X^1) = H^1(X, B_X^1)^{\oplus n} = 0$$

and therefore (X, D) deforms to a formal nc pair $\{(X_n, D_n)\}_{n \geq 1}$ over the formal spectrum of $W(k)$ together with a compatible lifting of the Frobenius morphism

$$\{F_{X_n}: (X_n, D_n) \rightarrow (X_n, D_n)\}_{n \geq 1}.$$

Since X is F -split, by Lemma 8.2.15 we see that the Frobenius action on $H^i(X, \mathcal{O}_X)$ is bijective and hence by Proposition 8.3.15(d) we see that every line bundle L on X admits a formal lifting $\{L_n \in \text{Pic}(X_n)\}_{n \geq 1}$ such that $F_{X_n}^*L_n \simeq L_n^{\otimes p}$.

To finish the proof we need to show that the given inductive systems are algebraisable. For this purpose, since every ample line bundle deforms to the formal nc pair $\{(X_n, D_n)\}_{n \geq 1}$, we may just apply Grothendieck's algebraisation theorem (see [EGA III_I, Section 3.4] or [Sta14, Tag 089A]). $\square$

Remark 8.5.9. The lifting of the Frobenius morphism we exhibit above is not a $W(k)$ -linear endomorphism, it is only Frobenius-linear. However, if X is defined over a finite field $k = \mathbb{F}_{p^e}$, then the e^{th} iterate of $F_{\mathcal{X}}$ is in fact a polarised $W(k)$ -endomorphism of $\mathcal{X}$.

The following lemma is the essential part of our argument. It differs substantially from its counterpart in [MS87], and is based on the two theorems stated in §8.5.1.

Lemma 8.5.10. *Let (X, D) be an nc pair satisfying the hypotheses of Theorem 8.5.5. Then either (X, D) is a toric pair (see §8.2.1) or there exists a finite étale covering $Y \rightarrow X$ such that $\text{Alb } Y \neq 0$.*

Proof. First, we assume that there exists an étale covering $Y \rightarrow X$ such that $H^1(Y, \mathcal{O}_Y) = 0$. In this case, we claim that (X, D) is in fact a toric pair. In order to see this, we use Lemma 8.5.8 to obtain a $W(k)$ -lifting $(\mathcal{Y}, \mathcal{E})$ of (Y, E) , where E is the preimage of D . The induced deformation of the log cotangent bundle $\Omega_Y^1(\log E)$ is trivial because the tangent space of its deformation functor is isomorphic to

$$H^1\left(Y, \mathcal{E}nd(\Omega_Y^1(\log E))\right) = H^1(Y, \mathcal{O}_Y)^{\oplus n^2} = 0.$$

This implies that the log cotangent bundle of the generic fibre $\mathcal{Y}_{\bar{\eta}}$ is trivial. By semicontinuity of cohomology we also see that $H^1(\mathcal{Y}_{\bar{\eta}}, \mathcal{O}_{\mathcal{Y}_{\bar{\eta}}}) = 0$ and therefore $(\mathcal{Y}_{\bar{\eta}}, \mathcal{E}_{\bar{\eta}})$ is a toric pair by Theorem 8.5.2. By Proposition 8.4.9 this means that the special fibre (Y, E) is a toric pair as well. To finish the proof of the claim, we show that $Y \rightarrow X$ is an isomorphism. For this purpose, we observe using the Hirzebruch–Riemann–Roch theorem (see [Ful98, Corollary 15.2.1]) that

$$1 = \chi(Y, \mathcal{O}_Y) = d \cdot \chi(X, \mathcal{O}_X),$$

where d is the degree of the finite map $Y \rightarrow X$. This clearly implies that $d = 1$ and hence $Y \rightarrow X$ is an isomorphism.

Now we proceed to the second case, where $H^1(Y, \mathcal{O}_Y) \neq 0$ for every étale covering $Y \rightarrow X$. We follow the strategy of [MS87, Lemma 1.6] substituting the application of the Calabi conjecture with Theorem 8.5.3 describing the algebraic fundamental groups of varieties admitting a polarised endomorphism. We claim that there exists an étale covering $Y \rightarrow X$ such that $\text{Alb } Y \neq 0$.

For the proof, we first use the spreading out technique to reduce to the case of nc pairs defined over finite fields. Let $(\mathcal{X}, \mathcal{D})$ be an nc pair over a spectrum of a finitely generated local $\mathbb{F}_p$ -algebra R such that the geometric generic fibre is isomorphic to (X, D) and the residue field is finite with $q = p^e$ elements. Spreading out the trivialisation of the log cotangent bundle and the Frobenius splitting (which can be interpreted as a morphism of vector bundles on the Frobenius twist of X), we may assume that $(\mathcal{X}, \mathcal{D})_{\mathbb{F}_q}$ satisfies the assumptions of Theorem 8.5.5. Assume now that there exists a finite étale covering

$$\pi_{\mathbb{F}_q} : \mathcal{Y}_{\mathbb{F}_q} \longrightarrow \mathcal{X}_{\mathbb{F}_q}$$

of the geometric special fibre such that $\text{Alb } \mathcal{Y}_{\mathbb{F}_q} \neq 0$. By [SGA 4 $\frac{1}{2}$, Arcata, IV, Proposition 2.2] (or [Sta14, Tag 0BQC]) we see that such a covering extends to a covering of

$\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} R^{\mathrm{sh}}$, where R^{sh} is the strict henselisation of R . Since R^{sh} is the colimit of étale extensions of R we observe that, possibly after taking an étale covering of $\mathrm{Spec} R$, the morphism $\pi_{\mathbb{F}_q}$ arises as a special fibre of a covering $\pi: \mathcal{Y} \rightarrow \mathcal{X}$. The geometric generic fibre of π yields an étale covering of $Y \rightarrow X$ which satisfies $\dim \mathrm{Alb} Y = H_{\mathrm{\acute{e}t}}^1(Y, \mathbb{Q}_\ell) \neq 0$ by smooth base change for étale cohomology (we take $\ell \neq p$). This finishes the reduction step.

For (X, D) defined over a finite field $k = \mathbb{F}_q$, we reason in two steps which we describe briefly. First, we lift the pair (X, D) to characteristic zero and apply Theorem 8.5.3 to prove that $\pi_1^{\acute{e}t}(X)$ is virtually abelian. Consequently, using the approach described in [MS87, Lemma 1.6], we prove that for an étale cover $Y \rightarrow X$ the crystalline cohomology group $H_{\mathrm{crys}}^1(Y/K)$ over the field $K = \mathrm{Frac} W(k)$ is non-zero, and hence $\dim \mathrm{Alb} Y = \dim_K H_{\mathrm{crys}}^1(Y/K)$ is non-zero as well.

Now, we present the details of the first step. Using Lemma 8.5.8 we construct a $W(k)$ -lifting $((\mathcal{X}, \mathcal{D}), F_{\mathcal{X}})$ of $((X, D), F_X)$. We set η to be the generic point of $W(k)$. Then, applying Remark 8.5.9, we observe that $F_X^e: X \rightarrow X$ is in fact a k -linear endomorphism for some $e > 0$, and therefore the geometric generic fibre $F_{\mathcal{X}_{\bar{\eta}}}^e$ of the lifting of F_X^e is a polarised endomorphism of $\mathcal{X}_{\bar{\eta}}$. By Theorem 8.5.3 the geometric fundamental group $\pi_1^{\acute{e}t}(\mathcal{X}_{\bar{\eta}})$ is virtually abelian. Using the surjectivity of the specialisation morphism $\pi_1^{\acute{e}t}(\mathcal{X}_{\bar{\eta}}) \rightarrow \pi_1^{\acute{e}t}(X)$ we see that the same holds for $\pi_1^{\acute{e}t}(X)$ and therefore after taking an étale covering of X we may assume $\pi_1^{\acute{e}t}(X)$ is abelian.

We proceed to the second step. We assume that $\pi_1^{\acute{e}t}(X)$ is abelian and follow [MS87, Lemma 1.6] closely. First, by [Ill79, Chapitre II, Théorème 5.2], to prove that $\dim \mathrm{Alb}(X) = \dim_K H_{\mathrm{crys}}^1(X/K)$ is non-zero it suffices to show that $H_{\mathrm{\acute{e}t}}^1(X, \mathbb{Z}_p)$ is non-torsion. We have

$$H_{\mathrm{\acute{e}t}}^1(X, \mathbb{Z}_p) \simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_1^{\acute{e}t}(X)^{\wedge p}, \mathbb{Z}_p),$$

where $\pi_1^{\acute{e}t}(X)^{\wedge p} = \varprojlim_n \pi_1^{\acute{e}t}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$. The $\mathbb{Z}_p$ -module $\pi_1^{\acute{e}t}(X)^{\wedge p}$ is finitely generated and therefore it is torsion if and only if it is finite. If $\pi_1^{\acute{e}t}(X)^{\wedge p}$ was finite, then there would exist an étale covering $X' \rightarrow X$ such that $H_{\mathrm{\acute{e}t}}^1(X', \mathbb{F}_p) = 0$. This gives a contradiction with $H^1(X', \mathcal{O}_{X'}) \neq 0$. Indeed, using the Artin–Schreier sequence of étale sheaves

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathcal{O}_X \xrightarrow{1-F} \mathcal{O}_X \longrightarrow 0$$

we see that $H^1(X', \mathbb{F}_p)$ is the $\mathbb{F}_p$ -vector space of elements in $H^1(X', \mathcal{O}_{X'})$ fixed by the Frobenius morphism. This is non-zero because Frobenius is bijective on $H^1(X', \mathcal{O}_{X'})$ for an F -split scheme by Lemma 8.2.15(a). $\square$

Lemma 8.5.11 (Log version of [MS87, Lemma 1.7]). *Let (X, D) be an nc pair satisfying the hypothesis of Theorem 8.5.5, and let $Y \rightarrow X$ be a Galois étale cover with $\mathrm{Alb} Y \neq 0$. Then there exists an intermediate Galois étale cover $Z \rightarrow X$ of degree p^m , for some $m \geq 0$, such that $Y \rightarrow Z$ induces an isogeny on Albanese varieties, in particular $\mathrm{Alb} Z \neq 0$.*

Proof. We apply the argument given in the proof of [MS87, Lemma 1.7] to the logarithmic

cotangent bundle instead of cotangent bundle. $\square$

Lemma 8.5.12 (Log version of [MS87, Lemma 1.8]). *Let (X, D) be an nc pair satisfying the hypothesis of Theorem 8.5.5, and let $\pi: X \rightarrow \text{Alb } X$ be the Albanese mapping. Then*

- (a) *all geometric fibres of π are F -split,*
- (b) *for each $i \geq 0$, $R^i \pi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_{\text{Alb } X}$ -module which becomes free on a finite étale cover of $\text{Alb } X$,*
- (c) *$\text{Alb } X$ is an F -split abelian variety.*

Proof. This follows by the same proof with the caveat that the Cartier isomorphism and the Grothendieck duality need to be replaced with their logarithmic versions. For the convenience of the reader, we present a slightly simplified argument below.

For the proof of (a) it is sufficient to show that π is relatively F -split, which follows from [Eji17, Theorem 1.2]. To get (b), we reason as in [MS87, Lemma 1.8]. More precisely, let $A = \text{Alb } X$, let X' be the base change of X along F_A , and let $\pi': X' \rightarrow A$ denote the induced projection. By flat base change we obtain an isomorphism $F_A^* R^i \pi_* \mathcal{O}_X \xrightarrow{\sim} R^i \pi'_* \mathcal{O}_{X'}$. Since the fibres of π' are F -split we observe that the natural homomorphisms

$$R^i \pi'_* \mathcal{O}_{X'} \rightarrow R^i \pi'_* F_{X/A*} \mathcal{O}_X$$

induced from the long exact sequence of $R^i \pi'_*$ for

$$0 \longrightarrow \mathcal{O}_{X'} \longrightarrow F_{X/A*} \mathcal{O}_X \longrightarrow B_{X/A}^1 \longrightarrow 0$$

are isomorphisms and therefore $F_A^* R^i \pi_* \mathcal{O}_X \simeq R^i \pi'_* \mathcal{O}_{X'}$. By [MN84, Lemma 1.4] we see that $R^i \pi'_* \mathcal{O}_{X'}$ is locally free, and [LS77, Satz 1.4] implies that it is étale trivialisable. Part (c) follows from Lemma 8.2.15(c). $\square$

Lemma 8.5.13 (Log version of [MS87, Lemma 1.9]). *Let (X, D) be an nc pair satisfying the hypothesis of Theorem 8.5.5, and let $\pi: X \rightarrow \text{Alb } X$ be the Albanese mapping. Let $s \in \text{Alb } X$ be a closed point. Then every finite étale covering of X_s of degree p^m , for any $m \geq 0$, is induced by a covering of X .*

Proof. Given Lemma 8.5.12(b) the proof of [MS87, Lemma 1.9] applies without change. $\square$

Lemma 8.5.14 (Generalisation of [MS87, Remark below Lemma 1.10]). *Let Y be a smooth projective variety and let $\pi: X \rightarrow Y$ be a proper smooth morphism whose geometric fibres are F -split. Then $R^1 \pi_* \mathbb{Q}_\ell$ becomes trivial on a finite étale cover of Y .*

Remark about the proof. Once we know that the Albanese varieties of smooth projective F -split varieties are ordinary (see Lemma 8.2.15), the proof given in [MS87, Remark below Lemma 1.10] can be repeated word for word. $\square$

Proof of Theorem 8.5.5. Let (X, D) be a nc pair satisfying the hypotheses of Theorem 8.5.5. We want to prove that there exists an étale covering admitting a structure of a toric fibration over an ordinary abelian variety with a toric divisor given by the preimage of D . First, following [DHP08, Proof of Theorem 1.1] we consider an étale covering $Y \rightarrow X$ such that $\dim \operatorname{Alb} Y$ is maximal (by Lemma 8.5.7 we know that $\dim \operatorname{Alb} Y \leq \dim Y = \dim X$). Using Lemma 8.5.6 we see that for E defined as the preimage of D the nc pair (Y, E) also satisfies the hypotheses of Theorem 8.5.5.

We claim that the Albanese morphism of Y is a toric fibration with toric boundary E . Indeed, using Lemma 8.5.7 and Lemma 8.5.12 we see that the fibres of $Y \rightarrow \operatorname{Alb} Y$ satisfy the assumptions of Lemma 8.5.10 and therefore they are either toric, and hence the proof is finished by Proposition 8.4.10, or for some fibre Y_s there exists an étale covering $g_s: \bar{Y}_s \rightarrow Y_s$ such that $\operatorname{Alb} \bar{Y}_s \neq 0$. In the latter case, using Lemma 8.5.11, we may assume that $\deg g_s = p^m$, for some $m \geq 0$, and therefore by Lemma 8.5.13 the morphism g_s is induced by a covering $\bar{Y} \rightarrow Y$. We consider the composition $\pi: \bar{Y} \rightarrow \operatorname{Alb} Y$ of the covering and the Albanese morphism of Y . By Lemma 8.5.6 the fibres of π are F -split and hence we may apply Lemma 8.5.14 to see that $R^1 \pi_* \mathbb{Q}_\ell$ is non-zero and étale trivialisable. This means that after an étale covering $\eta: A \rightarrow \operatorname{Alb} Y$ we have an isomorphism

$$\eta^* R^1 \pi_* \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^{2d}$$

for some $d > 0$. Let $\hat{Y} = \bar{Y} \times_{\operatorname{Alb} Y} A$ be the covering of $\bar{Y}$ induced by η , and let $\hat{\pi}: \hat{Y} \rightarrow A$ be the projection in the cartesian diagram

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\quad} & \bar{Y} \\ \hat{\pi} \downarrow & & \downarrow \pi \\ A & \xrightarrow{\quad \eta \quad} & \operatorname{Alb} Y. \end{array}$$

We claim that $\dim \operatorname{Alb} \hat{Y} > \dim \operatorname{Alb} Y$. Indeed, as in [MS87, Proof of Theorem 1] we consider the Leray spectral sequence $H_{\text{ét}}^i(A, R^j \hat{\pi}_* \mathbb{Q}_\ell) \Rightarrow H_{\text{ét}}^{i+j}(\hat{Y}, \mathbb{Q}_\ell)$ to obtain the exact sequence

$$0 \longrightarrow H_{\text{ét}}^1(A, \mathbb{Q}_\ell) \longrightarrow H_{\text{ét}}^1(\hat{Y}, \mathbb{Q}_\ell) \longrightarrow H^0(A, R^1 \hat{\pi}_* \mathbb{Q}_\ell) \longrightarrow H_{\text{ét}}^2(A, \mathbb{Q}_\ell) \longrightarrow H_{\text{ét}}^2(\hat{Y}, \mathbb{Q}_\ell).$$

Since $\hat{\pi}: \hat{Y} \rightarrow A$ admits a multi-section the morphism $H_{\text{ét}}^2(A, \mathbb{Q}_\ell) \longrightarrow H_{\text{ét}}^2(\hat{Y}, \mathbb{Q}_\ell)$ is injective, and therefore we have

$$\begin{aligned} \dim \operatorname{Alb} \hat{Y} &= \frac{1}{2} \dim_{\mathbb{Q}_\ell} H^1(\hat{Y}, \mathbb{Q}_\ell) = \frac{1}{2} \left(\dim_{\mathbb{Q}_\ell} H^1(A, \mathbb{Q}_\ell) + \dim_{\mathbb{Q}_\ell} H^0(A, R^1 \hat{\pi}_* \mathbb{Q}_\ell) \right) \\ &= \dim A + \frac{1}{2} \dim_{\mathbb{Q}_\ell} H^0(A, \eta^* R^1 \pi_* \mathbb{Q}_\ell) \\ &= \dim A + \frac{1}{2} \dim_{\mathbb{Q}_\ell} H^0(A, \mathbb{Q}_\ell^{2d}) \\ &= \dim \operatorname{Alb} Y + d > \dim \operatorname{Alb} Y. \end{aligned}$$

This gives a contradiction with the choice of Y and hence finishes the proof. $\square$

8.6. Homogeneous spaces

In [BTLM97, §4], the authors study Frobenius liftability of rational homogeneous spaces. In many cases, they are able to show that such a variety X is not F -liftable because Bott vanishing

$$H^j(X, \Omega_X^i \otimes L) = 0 \quad (j > 0, \quad L \text{ ample})$$

does not hold. As in this chapter (although using a different argument), they reduce the question to the case of Picard number one. But even under this assumption, finding i, j , and L as above for which $H^i(X, \Omega_X^j \otimes L) \neq 0$ is a difficult task. To this end, the authors use involved results of D. Snow [Sno88, Sno86] on the cohomology of flag varieties of Hermitian symmetric type in characteristic zero. They ask (see [BTLM97, Remark 2]) whether the only F -liftable rational homogeneous spaces are products of projective spaces. As these are precisely the toric ones, this is a special case of our Conjecture 8.1.1, one which we are actually able to settle.

Our proof in the Picard number one case does not rely on the classification of homogeneous spaces or Bott vanishing. In fact, we only need to assume that X is Fano and that the tangent bundle T_X is nef (note that the Campana–Peternell conjecture ([CP91], [Kol96, V, Conjecture 3.10]) predicts that these two conditions should in fact imply that X is a rational homogeneous space). The main ingredients of our proof are Mori’s characterisation of the projective space in terms of rational curves (Theorem 8.6.4) and a careful analysis of the restrictions of the sheaf of ξ -invariant forms (§8.6.2) to some special families of rational curves (Proposition 8.6.9). For the general case, we unfortunately have to look at the classification of homogeneous spaces, but only to check for which vertices of which Dynkin diagrams the corresponding homogeneous space of Picard rank one is a projective space (Lemma 8.6.10). A result of Lauritzen and Mehta [LM97] allows us to exclude the possibility of non-reduced stabilizers. We believe that our ideas could be useful in tackling the Picard rank one case of Conjecture 8.1.1 with the assumption that T_X is nef dropped.

In this section we work over an algebraically closed field k of positive characteristic.

8.6.1. Families of rational curves

We start by recalling basic definitions pertaining to rational curves. The main reference for this subsection is [Kol96, II §2–3]. In what follows we assume that X is a smooth projective k -scheme.

Definition 8.6.1. Let $\varphi: \mathbb{P}^1 \rightarrow X$ be a non-constant morphism. We say that φ is *free* if φ^*T_X is nef, and *very free* if φ^*T_X is ample.

Given a rational curve $C \subseteq X$ we shall say that it is free (resp. very free) if the normalisation $\varphi: \mathbb{P}^1 \rightarrow C \rightarrow X$ is free (resp. very free). Further, for a free $\varphi: \mathbb{P}^1 \rightarrow X$ we

can write

$$\varphi^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus r} \oplus \bigoplus_{i=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad a_i > 0.$$

With that, φ is very free if and only if $r = 0$.

Fix an ample divisor H on X . If X is Fano, we always take $H = -K_X$. For an integer d , we denote by $\text{Hom}_d(\mathbb{P}^1, X) \subseteq \text{Hilb}(\mathbb{P}^1 \times X)$ the scheme parametrising morphisms $\varphi: \mathbb{P}^1 \rightarrow X$ such that $\deg \varphi^*H = d$. We denote by $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X) \subseteq \text{Hom}_d(\mathbb{P}^1, X)$ the subscheme parametrising free $\varphi: \mathbb{P}^1 \rightarrow X$ which are generically injective. We drop the subscript d whenever we do not wish to specify the degree, so $\text{Hom}(\mathbb{P}^1, X) = \bigsqcup_d \text{Hom}_d(\mathbb{P}^1, X)$. By [Kol96, II, Corollary 3.5.4], the natural morphism

$$\mathbb{P}^1 \times \text{Hom}_d^{\text{free}}(\mathbb{P}^1, X) \longrightarrow X$$

is smooth, and so $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)$ is smooth as well.

Let $\text{RatCurves}_d^{\text{free}}(X)$ be the quotient $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)/\text{Aut}(\mathbb{P}^1)$, which exists by [Kol96, II, Comment 2.7] as $\text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)$ is a smooth open subscheme of $\text{Hom}_d(\mathbb{P}^1, X)$, invariant under the $\text{Aut}(\mathbb{P}^1)$ -action (see [Kol96, II, Corollary 3.5.4]). We denote by $\text{Univ}_d(X)$ the universal $\mathbb{P}^1$ -bundle over $\text{RatCurves}_d^{\text{free}}(X)$ so that we have a diagram

$$\begin{array}{ccc} \text{Univ}_d(X) & \xrightarrow{\varphi} & X \\ \pi \downarrow & & \\ \text{RatCurves}_d^{\text{free}}(X) & & \end{array}$$

It is constructed as a quotient of $\mathbb{P}^1 \times \text{Hom}_d^{\text{free}}(\mathbb{P}^1, X)$ by $\text{Aut}(\mathbb{P}^1)$. We have a factorisation

$$\mathbb{P}^1 \times \text{Hom}_d^{\text{free}}(\mathbb{P}^1, X) \longrightarrow \text{Univ}_d(X) \xrightarrow{\varphi} X$$

where the left arrow is smooth and surjective (cf. [Kol96, II, Theorem 2.15]), and so φ is smooth. In particular, both $\text{Univ}_d(X)$ and $\text{RatCurves}_d^{\text{free}}(X)$ are smooth [Sta14, Tag 02K5].

Definition 8.6.2. Let $\varphi: \mathbb{P}^1 \rightarrow X$ be a generically injective morphism which is free and such that $d = \deg \varphi^*H$ is minimal among all $\varphi' \in \text{Hom}^{\text{free}}(\mathbb{P}^1, X)$. We call such φ a *minimal free rational curve*.

The study of deformations of rational curves of minimal degree plays a vital role in the theory of rationally connected varieties by means of the Mori theory (and *varieties of minimal rational tangents*, see e.g. [Keb02, HM04]). The picture becomes slightly simpler if we assume that T_X is nef. As we have already noted, conjecturally this condition should be equivalent to X being a homogeneous space. When T_X is nef, we denote $\text{RatCurves}_d^{\text{free}}(X)$ by $\text{RatCurves}_d(X)$, as all rational curves are free.

Lemma 8.6.3 ([Kol96, II, Proposition 2.14.1]). *With notation as above, suppose that*

T_X is nef and that minimal free rational curves are of degree d . Then $\text{RatCurves}_d(X)$ is proper.

Proof. By [Kol96, II, Theorem 2.15], the above definition of $\text{RatCurves}_d(X)$ coincides with [Kol96, II, Definition – Proposition 2.11] under the assumption that T_X is nef, and hence $\text{RatCurves}_d(X)$ is proper by [Kol96, II, Proposition 2.14.1]. $\square$

Finally, let us recall the celebrated result of Mori, which we present here in a slightly different than that of [Kol96, V, Theorem 3.2] – for the proof of Proposition 8.6.9 we need a variant for minimal free rational curves (cf. [OW02, Families of curves, Theorem]).

Theorem 8.6.4. *Let X be a smooth projective Fano variety of dimension n defined over an algebraically closed field k and let $x \in X$ be a general point. Suppose that every rational curve through x is free, and that every minimal free rational curve through x is very free. Then $X \simeq \mathbb{P}^n$.*

Every rational curve through x is free provided that k is of characteristic zero and x is very general [Kol96, II, Theorem 3.11], or X is F -liftable and $x \in X$ is general (Lemma 8.6.5(a)). Note that there exists a minimal free rational curve through a general point x by [Kol96, II, Corollary 3.5.4.2].

Proof. The degree of a very free rational curve is always at least $n + 1$ (see [Kol96, V, Lemma 3.7.2]). Moreover, there exists a rational curve of degree at most $n + 1$ through x by bend-and-break (see [Kol96, V, Theorem 1.6.1]), hence minimal free rational curves are of degree $n + 1$. Since they are very free by assumption, the rest of the proof follows [Kol96, V, Theorem 3.2] word for word. $\square$

8.6.2. The sheaf of ξ -invariant forms and rational curves

In this subsection we study positivity conditions imposed on the tangent bundle by the existence of a Frobenius lifting. This is the main component of the proof of Theorem 8.1.2 in the Picard rank one case.

Let us fix a Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ of a smooth k -scheme X and consider the induced morphism $\xi: F^*\Omega_X^1 \rightarrow \Omega_X^1$. Recall that it is injective, and hence generically an isomorphism, by Proposition 8.3.8. Let $U \subseteq X$ to be the maximal open subset where ξ is an isomorphism.

The following simple lemma allows for the study of families of rational curves by using the sheaf of ξ -invariant forms denoted by $(\Omega_X^1)^\xi$.

Lemma 8.6.5. *Let $\varphi: \mathbb{P}^1 \rightarrow X$ be a non-constant morphism such that $\varphi(\mathbb{P}^1) \cap U \neq \emptyset$. Then,*

- (a) φ is free (cf. [Xin16, Proposition 5]),
- (b) $\varphi^*(\Omega_X^1)^\xi$ contains a locally constant $\mathbb{F}_p$ -subsheaf of rank $r = h^0(\mathbb{P}^1, \varphi^*\Omega_X^1)$.

Proof. Since $\varphi^*\Omega_X^1$ is locally free and $\varphi^*(\xi): F^*\varphi^*\Omega_X^1 \rightarrow \varphi^*\Omega_X^1$ is generically an isomorphism, $\varphi^*(\xi)$ must be an injection.

Write $\varphi^*\Omega_X^1 = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$, where $a_1 \geq a_2 \geq \cdots \geq a_n$. Since $\varphi^*(\xi)$ is an injection, it induces a non-zero morphism $\mathcal{O}_{\mathbb{P}^1}(pa_1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(a_j)$ for some $1 \leq j \leq n$. This is only possible if $a_1 \leq 0$, and so φ^*T_X is nef, concluding (a).

The morphism $\varphi^*(\xi)$ induces a Frobenius-linear automorphism of the space of global sections $H^0(\mathbb{P}^1, \varphi^*\Omega_X^1)$, and so by taking $\varphi^*(\xi)$ -fixed points

$$\mathcal{G}_\varphi = H^0(\mathbb{P}^1, \varphi^*\Omega_X^1)^{\varphi^*(\xi)} \otimes_{\mathbb{F}_p} (\mathbb{F}_p)_{\mathbb{P}^1}$$

we get a constant $\mathbb{F}_p$ -subsheaf of $\varphi^*(\Omega_X^1)^\xi$ of rank r , where $(\mathbb{F}_p)_{\mathbb{P}^1}$ is the constant sheaf with value $\mathbb{F}_p$ on $\mathbb{P}^1$. $\square$

As mentioned before, for any free rational curve $\varphi: \mathbb{P}^1 \rightarrow X$ we can write $\varphi^*\Omega_X^1 = \bigoplus_{i=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$, where $a_i < 0$. In general, the value of r is upper-semicontinuous under deformations of φ . Here, we show that r is invariant under deformations provided that X is F -liftable and $\varphi(\mathbb{P}^1)$ intersects U .

Proposition 8.6.6. *With notation as above, let $\varphi_i: \mathbb{P}^1 \rightarrow X$ for $i \in \{1, 2\}$ be two rational curves intersecting $U \subseteq X$ and lying in the same irreducible component $\mathcal{M} \subseteq \text{Hom}(\mathbb{P}^1, X)$. Then $h^0(\mathbb{P}^1, \varphi_1^*\Omega_X^1) = h^0(\mathbb{P}^1, \varphi_2^*\Omega_X^1)$.*

Proof. We have the following diagram

$$\begin{array}{ccc} \mathcal{M} \times \mathbb{P}^1 & \xrightarrow{\varphi} & X \\ \pi \downarrow & & \\ \mathcal{M} & & \end{array}$$

Replacing $\mathcal{M}$ by an open subset, we can assume that $\varphi(\{m\} \times \mathbb{P}^1)$ intersects U for every closed point $m \in \mathcal{M}$. Pick any closed point $m \in \mathcal{M}$. Then, as in Lemma 8.6.5 we have

$$h^0(\mathbb{P}^1, \varphi^*\Omega_X^1|_{\pi^{-1}(m)}) = \dim_{\mathbb{F}_p} H^0(\mathbb{P}^1, \varphi^*\Omega_X^1|_{\pi^{-1}(m)})^\xi = \dim_{\mathbb{F}_p} (\pi_*\varphi^*(\Omega_X^1)^\xi)_m,$$

which is lower semi-continuous with respect to m by Lemma 8.6.7. The last equality holds by the proper base change theorem. On the other hand, $h^0(\mathbb{P}^1, \varphi^*\Omega_X^1|_{\pi^{-1}(m)})$ is upper semi-continuous with respect to m by the semi-continuity theorem ([Har77, III, Theorem 12.8]). Hence, it is constant over $\mathcal{M}$. $\square$

We needed the following corollary of the proper base change theorem in the above proof.

Lemma 8.6.7. *Let $f: X \rightarrow S$ be a proper morphism of schemes, let $g: U \rightarrow X$ be a separated étale morphism of finite type, and let $\mathcal{F}$ be the étale sheaf of sections of g . Then*

the function $\varphi: S \rightarrow \mathbb{Z}$ defined as

$$\varphi(s) = |(f_*\mathcal{F})_{\bar{s}}|$$

for any geometric point $\bar{s}$ over s is lower semi-continuous.

Proof. By [SGA 4_{III}, Exposé IX, Corollaire 2.7.1] or [Sta14, Tag 03S8], $\mathcal{F}$ is constructible, and hence $f_*\mathcal{F}$ is constructible as well [SGA 4_{III}, Exposé XIV]. Therefore φ is a constructible function on S , and hence to prove the assertion it is enough to show that if $s, \mu \in S$ are two points such that s lies in the closure of μ , then $\varphi(s) \leq \varphi(\mu)$.

Let $\bar{s}$ be a geometric point lying over s and let $\bar{\mu}$ be a geometric point of the localisation $S_{(\bar{s})}$ lying over μ . Passing to stalks, we obtain the cospecialisation map

$$c_{\bar{\mu} \rightsquigarrow \bar{s}}: (f_*\mathcal{F})_{\bar{s}} \longrightarrow (f_*\mathcal{F})_{\bar{\mu}}.$$

It remains to show that $c_{\bar{\mu} \rightsquigarrow \bar{s}}$ is injective. The proper base change theorem implies that $(f_*\mathcal{F})_{\bar{s}}$ is the set of sections of $U \rightarrow X$ over $X \times_S S_{(\bar{s})}$. Take any two such sections $u_1, u_2: X \times_S S_{(\bar{s})} \rightarrow U$, and suppose they are equal after restricting to $X \times_S S_{(\bar{\mu})} \rightarrow U$. Then $u_1 = u_2$, as $U \rightarrow X$ is separated and $S_{(\bar{\mu})} \rightarrow S_{(\bar{s})}$ has dense image. $\square$

8.6.3. The Picard rank one case

In this subsection we prove Theorem 8.1.2 in the Picard rank one case (Proposition 8.6.9). Before proceeding with the proof we need the following result.

Lemma 8.6.8. *Let X be a smooth projective F -liftable Fano variety over k . Then X is simply connected and $H^0(X, \Omega_X^1) = 0$.*

Proof. In order to prove that X is simply connected, let us consider an étale cover $f: Y \rightarrow X$ of degree d . By the Hirzebruch–Riemann–Roch theorem (see [Ful98, Corollary 15.2.1]), we have $\chi(Y, \mathcal{O}_Y) = d \cdot \chi(X, \mathcal{O}_X)$. Since X is F -liftable, so is Y (Lemma 8.3.19), and hence Kodaira vanishing holds on both X and Y (see Theorem 8.3.11). Thus

$$\chi(Y, \mathcal{O}_Y) = \chi(X, \mathcal{O}_X) = 1,$$

which shows that $d = 1$ and X is simply connected.

Now, we show that $H^0(X, \Omega_X^1) = 0$. Since X is F -liftable, all of its global one-forms are closed by Proposition 8.3.8, and

$$Z_X^1 \simeq B_X^1 \oplus \Omega_X^1,$$

where $Z_X^1 = \ker(d: F_*\Omega_X^1 \rightarrow F_*\Omega_X^2)$ and $B_X^1 = F_*\mathcal{O}_X/\mathcal{O}_X$. In particular, the Cartier operator induces an isomorphism $H^0(X, Z_X^1) \simeq H^0(X, \Omega_X^1)$. Therefore, the assumptions

of [vdGK03, Proposition 4.3] are satisfied and we obtain

$$H^0(X, \Omega_X^1) \simeq \text{Pic}(X)[p] \otimes_{\mathbb{Z}} k.$$

We claim that $\text{Pic}(X)$ is torsion free. Indeed, if L is a numerically trivial line bundle on X , then by Kodaira vanishing we have

$$H^i(X, L) = H^i(X, \omega_X \otimes \omega_X^{-1} \otimes L) = 0 \quad \text{for } i > 0,$$

and therefore $h^0(X, L) = \chi(X, L) = \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$. $\square$

Proposition 8.6.9. *Let X be a smooth projective Fano variety of dimension n defined over an algebraically closed field of positive characteristic. Suppose that T_X is nef, $\rho(X) = 1$, and X is F -liftable. Then $X \simeq \mathbb{P}^n$.*

The idea of the proof is the following. If $X \not\simeq \mathbb{P}^n$, then by Mori theory there exists a rational curve $C \subseteq X$ such that $T_X|_C$ is not very ample. In particular, $h^0(C, \Omega_X^1|_C) > 0$. Using the global lifting of Frobenius we show that the sections of $\Omega_X^1|_{C_t}$, where C_t are deformations of C , glue to global sections of Ω_X^1 . This contradicts the fact that X is Fano (see Lemma 8.6.8).

Proof. Choose a general point $x \in X$. If every minimal free rational curve $\rho: \mathbb{P}^1 \rightarrow X$ passing through x is very free, then $X \simeq \mathbb{P}^n$ by Theorem 8.6.4. Suppose by contradiction that there exists such a non-very-free minimal ρ of degree d . Then $h^0(\mathbb{P}^1, \rho^* \Omega_X^1) = r \geq 1$.

Let $\mathcal{M}$ be the irreducible component of $\text{RatCurves}_d(X)$ containing ρ (see §8.6.1 for the notation). By Lemma 8.6.3, we know that $\mathcal{M}$ is proper and there exist morphisms

$$\begin{array}{ccc} \text{Univ} & \xrightarrow{\varphi} & X \\ \pi \downarrow & & \\ \mathcal{M} & & \end{array}$$

with $\pi: \text{Univ} \rightarrow \mathcal{M}$ being a $\mathbb{P}^1$ -fibration, and φ being smooth. Since X is simply connected (see Lemma 8.6.8), we get that φ has connected fibres. Indeed, let $\text{Univ} \rightarrow Y \rightarrow X$ be the Stein factorisation of φ . Since φ is smooth and proper, the morphism $Y \rightarrow X$ is finite étale (see [SGA 1, Exp. X, Prop. 1.2] or [EGA III_{II}, 7.8.10 (i)]). and so it must be an isomorphism.

We fix a Frobenius lifting $(\tilde{X}, \tilde{F}_X)$ of X and consider the induced morphism

$$\xi: F^* \Omega_X^1 \rightarrow \Omega_X^1.$$

Let $(\Omega_X^1)^\xi$ be the sheaf of ξ -invariant forms and let $U \subseteq X$ be the maximal open subset for which $\xi|_U: F^* \Omega_U^1 \rightarrow \Omega_U^1$ is an isomorphism.

Set $\mathcal{M}^\circ = \pi(\varphi^{-1}(U))$ to be the locus of all $m: \mathbb{P}^1 \rightarrow X$ in $\mathcal{M}$ whose image intersects U . It is open as π is smooth. Take $\text{Univ}^\circ = \pi^{-1}(\mathcal{M}^\circ)$ and let $j: \text{Univ}^\circ \rightarrow \text{Univ}$ denote

the inclusion. We define the following subsheaf of $\varphi^*(\Omega_X^1)^\xi|_{\text{Univ}^\circ}$

$$\mathcal{G} = j^* \pi^* \pi_* \varphi^*(\Omega_X^1)^\xi$$

Given $m: \mathbb{P}^1 \rightarrow X$ in $\mathcal{M}^\circ$, the proper base change theorem implies that $\mathcal{G}|_{\pi^{-1}(m)}$ is isomorphic to the constant $\mathbb{F}_p$ -sheaf with value $H^0(\mathbb{P}^1, m^*(\Omega_X^1)^\xi)$. Lemma 8.6.5(b) and Proposition 8.6.6 show that $\mathcal{G}$ is a locally constant $\mathbb{F}_p$ -sheaf of rank r . Indeed, $(\pi_* \varphi^* \Omega_X^1)|_{\mathcal{M}^\circ}$ is a locally free sheaf of rank r by [Har77, III, Corollary 12.9] and

$$\mathcal{G} = \pi^*(\pi_* \varphi^* \Omega_X^1|_{\mathcal{M}^\circ})^{\pi_* \varphi^* \xi}.$$

We claim that $\text{codim } \text{Univ} \setminus \text{Univ}^\circ \geq 2$. In order to prove this it is enough to show that for an irreducible divisor $D \subseteq X \setminus U$, the codimension of $\varphi^{-1}(D) \setminus \text{Univ}^\circ$ in Univ is at least two. Since φ is smooth and has connected fibres, $\varphi^{-1}(D)$ is irreducible, and hence it is enough to show that $\varphi^{-1}(D) \cap \text{Univ}^\circ \neq \emptyset$. This follows from the fact that $\rho(X) = 1$. Indeed, for a general closed point $m \in \mathcal{M}$, the curve $\varphi(\pi^{-1}(m))$ intersects D , and so $\pi^{-1}(m) \subseteq \text{Univ}^\circ$ intersects $\varphi^{-1}(D)$.

By Zariski–Nagata purity, the sheaf $j_* \mathcal{G}$ is a locally constant $\mathbb{F}_p$ -sheaf of rank r . Since Univ is normal and the complement of Univ° is of codimension at least two, we have $j_* j^* \varphi^* \Omega_X^1 \simeq \varphi^* \Omega_X^1$. In particular,

$$j_* \mathcal{G} \subseteq j_* j^* \varphi^*(\Omega_X^1)^\xi = (j_* j^* \varphi^* \Omega_X^1)^{\varphi^* \xi} \simeq \varphi^*(\Omega_X^1)^\xi.$$

Let T be a fibre of $\varphi: \text{Univ} \rightarrow X$. Since $j_* \mathcal{G}|_T$ is a locally constant subsheaf of the constant sheaf $\varphi^*(\Omega_X^1)^\xi|_T$, we see that $j_* \mathcal{G}|_T$ is constant. In particular, the proper base change theorem implies that $\varphi_* j_* \mathcal{G}$ is a nonzero locally constant subsheaf of $(\Omega_X^1)^\xi$. As X is simply connected, $\varphi_* j_* \mathcal{G}$ is constant, and hence $H^0(X, (\Omega_X^1)^\xi) \neq 0$, contradicting Lemma 8.6.8. $\square$

8.6.4. The general case

We recall some facts about homogeneous spaces over an algebraically closed field k . A projective variety X over k is called *homogeneous* if its automorphism group $\text{Aut}(X)$ acts transitively on X . Note that since X is projective, $\text{Aut}(X)$ is actually (the set of k -points of) a group scheme locally of finite type over k , and in particular its connected component $\text{Aut}^0(X)$ is a group scheme of finite type acting transitively on X .

The decomposition theorem of Borel and Remmert [BR61] (see [SdS03, §5–6] for its extension to positive characteristic) states that X decomposes into a product

$$X \simeq A \times G_1/P_1 \times \dots \times G_r/P_r, \tag{8.19}$$

where A is an abelian variety, G_i are simple linear algebraic groups of adjoint type, and $P_i \subseteq G_i$ are parabolic subgroup schemes (i.e., each P_i contains a Borel subgroup of G_i).

Let G be a simple linear algebraic group over k . A choice of a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$ gives a set D of *simple roots* of G , which is the set of nodes of the Dynkin diagram of G . Following the conventions of [Hum78, p. 58], we number them $1, \dots, n$ where n is the rank of G . Reduced parabolic subgroups of G containing B are in an inclusion-preserving bijection with subsets of D . We can thus represent (possibly ambiguously) rational homogeneous spaces with reduced stabilizers by marked Dynkin diagrams, i.e. Dynkin diagrams with a chosen set of nodes. If $\alpha \in D$ is a simple root, we denote by $P(\alpha)$ the maximal parabolic subgroup corresponding to $D \setminus \{\alpha\}$.

Lemma 8.6.10. *Let G be as above, and let $\alpha \in D = \{1, \dots, n\}$ be a simple root. Suppose that $G/P(\alpha) \simeq \mathbb{P}_k^r$ for some $r > 0$. Then one of the following holds:*

- (i) $r = n$, the group G is of type A_n , and $\alpha = 1$ or n ,
- (ii) $r = 2n - 1 \geq 3$, the group G is of type C_n , and $\alpha = 1$.

In other words, the Dynkin diagram and the node α are as shown below.



Proof. Demazure [Dem77] showed that for most pairs (G, P) of a simple group G of adjoint type and a reduced parabolic subgroup P , the natural morphism

$$G \longrightarrow \operatorname{Aut}^0(G/P)$$

is an isomorphism, and classified the exceptions. The only exception with $\operatorname{Aut}^0(G/P) \simeq \operatorname{PGL}_{r+1}(k)$ is (see *op.cit.*, case (a) on p. 181) $G = \operatorname{PSp}_r(k)$ and P the stabilizer of a line, in which case $G/P \simeq \mathbb{P}_k^r$ (r has to be odd in this case). This is our case (ii). If (ii) does not hold, then $r = n$ and $G \simeq \operatorname{PGL}_{n+1}(k)$, and hence $G/P(\alpha) \simeq \operatorname{Gr}(\alpha, n+1)$. So $\dim G/P(\alpha) = \alpha(n+1-\alpha) = n$, and hence $\alpha = 1$ or $\alpha = n$. $\square$

Below, we denote by $F_{1,n}$ the incidence variety parametrising partial flags $W_1 \subseteq W_n \subseteq k^{n+1}$ where $\dim W_1 = 1$ and $\dim W_n = n$. It is a hypersurface of degree $(1, 1)$ in $\mathbb{P}_k^n \times \mathbb{P}_k^n$, and as a homogeneous space it corresponds to the Dynkin diagram of type A_n with all nodes except for the two endpoints $1, n$ marked.

Lemma 8.6.11. *Let G be a simple algebraic group over k and let $P \subseteq G$ be a reduced parabolic subgroup. Suppose that for every maximal reduced parabolic $Q \subseteq G$ containing P , the homogeneous variety G/Q is isomorphic to $\mathbb{P}_k^r$ for some r . Then one of the following conditions holds:*

- (i) $G/P \simeq \mathbb{P}_k^n$ for some n ,
- (ii) $G/P \simeq F_{1,n}$ for some n .

Proof. This follows from Lemma 8.6.10 and the classification of rational homogeneous spaces with reduced stabilizers. $\square$

Lemma 8.6.12. *The incidence variety $F_{1,n}$ is not F -liftable for $n > 1$.*

Proof. This is shown in [BTLM97, §4.2] by using Bott vanishing. Just for fun, we give an alternative proof, the idea of which is the following. The multi-homogeneous polynomial defining X can be treated as a homogeneous polynomial in the usual sense, thus giving rise to a smooth homogeneous quadric hypersurface $\overline{X} \subseteq \mathbb{A}^{2n+2}$. We show that if X is F -liftable, then so is $\overline{X}$, which contradicts [Zda17, Theorem 4.15].

Formally, we notice that by the short exact sequence for restrictions, we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(a, b)) &= H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(a, b))|_X, \text{ and} \\ H^1(X, \mathcal{O}_X(a, b)) &= 0, \end{aligned}$$

for $a, b \geq 0$. Moreover, $H^0(X, \mathcal{O}_X(a, b)) = 0$ if $a < 0$ or $b < 0$. Thus, by Proposition 8.3.23, if X is F -liftable, then the affine cone $\overline{X} \subseteq \mathbb{A}^{n+1} \times \mathbb{A}^{n+1}$ of X given by

$$\overline{X} = \text{Spec} \bigoplus_{a, b \geq 0} H^0(X, \mathcal{O}_X(a, b)) \cong \text{Spec } k[x_0, \dots, x_n, y_0, \dots, y_n] / (\sum_{i=0}^n x_i y_i)$$

is F -liftable as well, contradicting [Zda17, Theorem 4.15]. $\square$

Theorem 8.6.13 ([LM97]). *Let G be a simple algebraic group over k , let $P \subseteq G$ be a parabolic subgroup scheme, let $X = G/P$, and let $Y = G/P_{\text{red}}$. If X is Frobenius split, then $X \simeq Y$ as varieties.*

We are ready to show the main theorem of this section.

Theorem 8.6.14. *Let X be a projective variety over k whose automorphism group acts transitively on $X(k)$. Then the following are equivalent:*

- (i) X is F -liftable,
- (ii) $X \simeq A \times \mathbb{P}_k^{n_1} \times \dots \times \mathbb{P}_k^{n_r}$ for some $n_1, \dots, n_r$, where A is an ordinary abelian variety,
- (iii) the Albanese variety $A = \text{Alb}(X)$ is ordinary and the fibres of the Albanese map $a_X: X \rightarrow A$ are toric varieties.

Proof. Consider the decomposition (8.19). By Corollary 8.3.21, X is F -liftable if and only if A and $G_1/P_1, \dots, G_r/P_r$ are. Moreover, A is F -liftable if and only if it is ordinary, by Example 8.3.3. The Albanese map of X is simply the projection to the first factor of the decomposition. Since the only homogeneous toric varieties are products of projective spaces, we see that (ii) and (iii) are equivalent and imply (i).

It remains to prove that (i) implies (ii). For this, we can assume that $X = G/P$ where G is a simple linear algebraic group of adjoint type and $P \subseteq G$ is a parabolic subgroup scheme. Since X is Frobenius split (Proposition 8.3.8(c)), Theorem 8.6.13 implies that $X \simeq G/P_{\text{red}}$, so we can assume that P is reduced.

If $Q \subseteq G$ is a reduced maximal parabolic subgroup containing P , then $Z = G/Q$ inherits a Frobenius lifting from X via the map $X = G/P \rightarrow G/Q = Z$ by Theorem 8.3.20(b), as

the fibres are isomorphic to the rational homogeneous space Q/P and $H^i(Q/P, \mathcal{O}_{Q/P}) = 0$ for $i > 0$. Now Z is Fano, $\text{Pic } Z \simeq \mathbb{Z}$, and T_Z is nef. By Proposition 8.6.9, $Z \simeq \mathbb{P}_k^n$ for some n . By Lemma 8.6.11, this implies that $X \simeq \mathbb{P}_k^n$ or X is isomorphic to the incidence variety $F_{1,n}$. But $F_{1,n}$ is not F -liftable (Lemma 8.6.12), so $X \simeq \mathbb{P}_k^n$. $\square$

Remark 8.6.15. By Theorem 8.6.14 and the proof of Theorem 8.4.15, we get that Conjecture 1.2.4 holds if the target variety is a homogeneous space. This observation is not very interesting because this fact almost follows from [OW02]. What is missing is a direct proof that the incidence variety $F_{1,n}$ is not an image of a toric variety.

8.7. F -liftability in low dimensions

This section pertains to the study of F -liftability of low dimensional smooth varieties. By means of the theory developed in Section 8.3, we classify F -liftable surfaces and F -liftable Fano threefolds from the Mori–Mukai list, confirming Conjecture 8.1.1 for these classes of varieties.

We work over an algebraically closed field k of characteristic $p > 0$. Let $(\tilde{X}, \tilde{F}_X)$ be a Frobenius lifting of a smooth k -scheme X . As in §8.3.6, we have the associated effective $\mathbb{Q}$ -divisor

$$\Delta_{\tilde{F}_X} = \frac{1}{p-1} \text{div}(\det(\xi_{\tilde{F}_X})) \sim_{\mathbb{Q}} -K_X.$$

A careful analysis of $\Delta_{\tilde{F}_X}$ plays a vital role in the proofs of the results of this section.

8.7.1. Surfaces

The goal of this subsection is to show Conjecture 8.1.1 for smooth surfaces. With notation as above, we define $D_{\tilde{F}_X} = \lfloor \Delta_{\tilde{F}_X} \rfloor$. Note that $D_{\tilde{F}_X}$ is reduced (see Remark 8.2.21). If X is a toric variety with its standard Frobenius lifting (see Example 8.3.2), then $\Delta_{\tilde{F}_X} = D_{\tilde{F}_X}$ is the toric boundary (the complement of the open torus orbit) of X .

First, we tackle the case of rational surfaces.

Lemma 8.7.1. *Let X be a smooth surface over k , and let*

$$\pi: Y = \text{Bl}_x X \longrightarrow X$$

be the blow-up of X at a closed point $x \in X$. Let $(\tilde{X}, \tilde{F}_X)$ and $(\tilde{Y}, \tilde{F}_Y)$ be Frobenius liftings of X and Y , and let $\tilde{\pi}: \tilde{X} \rightarrow \tilde{Y}$ be a lifting of π satisfying $\tilde{F}_X \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{F}_Y$. Suppose that $\text{Supp } \Delta_{\tilde{F}_X}$ has simple normal crossings at x . Then $x \in \text{Sing } D_{\tilde{F}_X}$.

This is a direct consequence of the fact that x is a log canonical center of $(X, \Delta_{\tilde{F}_X})$ by Corollary 8.3.35(a) (in fact, this shows that the above result is valid in higher dimensions for blow-ups along arbitrary smooth centers). We provide a more elementary explanation below.

Proof. For the exceptional divisor $E = \text{Exc}(\pi)$, we have $E \leq \Delta_{\tilde{F}_Y}$ by Corollary 8.3.35(a). Since $\pi_*\Delta_{\tilde{F}_Y} = \Delta_{\tilde{F}_X}$ (see Corollary 8.3.35(b)), the support of $\pi^*\Delta_{\tilde{F}_X} - \Delta_{\tilde{F}_Y}$ is exceptional. By definition, this $\mathbb{Q}$ -divisor is linearly equivalent to $K_{Y/X}$, which is equal to E by [Har77, Proposition V.3.3], and so

$$\pi^*\Delta_{\tilde{F}_X} - \Delta_{\tilde{F}_Y} = E.$$

In particular, $\pi^*\Delta_{\tilde{F}_X} = 2E + \Delta'$, where $E \not\subseteq \text{Supp } \Delta'$. As $\text{Supp } \Delta_{\tilde{F}_X}$ has simple normal crossings at x and the coefficients of $\Delta_{\tilde{F}_X}$ are at most one (see Remark 8.2.21), this is only possible if $x \in \text{Sing } D_{\tilde{F}_X}$. $\square$

Remark 8.7.2. Let $f: (Y, D_Y) \rightarrow (X, D_X)$ be a toric morphism between two-dimensional toric pairs, which on the level of schemes is a blowing-up of a smooth point $x \in X$. Then $x \in \text{Sing}(D_X)$ and $D_Y = f_*^{-1}D_X + \text{Exc}(f)$. Moreover, the converse is also true, that is the blowing-up of a smooth toric surface at toric points is a toric morphism.

Let us call a pair (X, D) of a normal variety X and a reduced effective divisor D on X *sub-toric* if X admits the structure of a toric variety such that D is invariant under the torus action. If D is the maximal invariant divisor, then we call (X, D) *toric*, see §8.2.1.

Lemma 8.7.3. *Let $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the n^{th} Hirzebruch surface for $n \geq 0$. Then for every Frobenius lifting $(\tilde{\mathbb{F}}_n, \tilde{F})$ of $\mathbb{F}_n$, the pair $(\mathbb{F}_n, D_{\tilde{F}})$ is sub-toric. Moreover, if $n = 0$, then $\text{Supp } \Delta_{\tilde{F}}$ has simple normal crossings, and if $n > 0$, then $\text{Supp } \Delta_{\tilde{F}}$ has simple normal crossings at every point of the negative section $C \subseteq \mathbb{F}_n$.*

Write $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(E)$ for $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. A choice of the splitting $E \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ provides $\mathbb{F}_n$ with a natural toric structure for which the natural morphisms $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) \rightarrow \mathbb{F}_n$ and $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(n)) \rightarrow \mathbb{F}_n$ are toric. Thus $(\mathbb{F}_n, D)$ is a toric pair if and only if

- for $n = 0$, we have $D = G_1 + G'_1 + G_2 + G'_2$, where G_1, G'_1 , and G_2, G'_2 are distinct fibres of the two projections $\pi_1, \pi_2: \mathbb{F}_0 \rightarrow \mathbb{P}^1$.
- for $n > 0$, we have $D = C + C' + G_1 + G_2$, where C is the unique negative section (corresponding to $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) \rightarrow \mathbb{F}_n$), C' is a section disjoint from C , and G_1, G_2 are two distinct fibres of the projection $\pi: \mathbb{F}_n \rightarrow \mathbb{P}^1$.

Proof. Let us fix a Frobenius lifting $(\tilde{\mathbb{F}}_n, \tilde{F})$. If $n = 0$, then, by Corollary 8.3.21, we have $\Delta_{\tilde{F}} = \pi_1^*\Delta_1 + \pi_2^*\Delta_2$, where Δ_1 and Δ_2 are effective $\mathbb{Q}$ -divisors on $\mathbb{P}^1$, and so $\text{Supp } \Delta_{\tilde{F}}$ is simple normal crossing. Since $\omega_{\mathbb{P}^1 \times \mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2)$, both $[\Delta_1]$ and $[\Delta_2]$ consist of at most two irreducible divisors, which concludes the proof.

If $n > 0$, then by Theorem 8.3.20(b.i) we get a compatible Frobenius lifting of $\mathbb{P}^1$ and so Corollary 8.3.35(c) implies that

$$\begin{aligned} \Delta^h &\sim_{\mathbb{Q}} -K_{\mathbb{F}_n/\mathbb{P}^1}, \\ \Delta^v &\sim_{\mathbb{Q}} -\pi^*K_{\mathbb{P}^1}, \end{aligned}$$

where Δ^h and Δ^v are the horizontal and the vertical part of $\Delta_{\tilde{F}}$, respectively. Moreover, Corollary 8.3.35(a) gives $\Delta^h = \Delta' + C$, where Δ' is an effective $\mathbb{Q}$ -divisor such that $C \not\subseteq \text{Supp } \Delta'$.

As $K_{\mathbb{F}_n} + \Delta^v + \Delta' + C \sim_{\mathbb{Q}} 0$, we have

$$\Delta' \cdot C = -(K_{\mathbb{F}_n} + C + \Delta^v) \cdot C = 2 - \Delta^v \cdot C = 0$$

by adjunction, and so Δ' is disjoint from C . Given that $\text{Supp}(\Delta^v + C)$ is simple normal crossing, so is $\text{Supp } \Delta_{\tilde{F}}$ along C . Moreover, for a fibre G of π

$$\Delta' \cdot G = -(K_{\mathbb{F}_n} + \Delta^v + C) \cdot G = -(K_{\mathbb{F}_n} + G) \cdot G - (\Delta^v + C) \cdot G = 1,$$

by adjunction as $G^2 = 0$, and so $[\Delta']$ is zero or is a single section disjoint from C . Since $[\Delta^v]$ consists of at most two distinct fibres, this concludes the proof of the lemma. $\square$

Proposition 8.7.4. *Let Y be a smooth projective rational surface. If Y is F -liftable, then it is toric.*

Proof. Since $\mathbb{P}^2$ is toric, we can assume that $Y \not\cong \mathbb{P}^2$. Every smooth rational surface which is not isomorphic to $\mathbb{P}^2$ admits a birational morphism to a Hirzebruch surface $\pi: Y \rightarrow \mathbb{F}_n$ for some $n \geq 0$, which factors into a sequence of monoidal transformations

$$Y = X_m \xrightarrow{\pi_{m-1}} X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_0} X_0 = \mathbb{F}_n, \quad X_{i+1} = \text{Bl}_{x_i} X_i.$$

We assume that n is *minimal* among such. It follows that if $n > 0$, then $\pi(\text{Exc}(\pi)) \subseteq C$, where $C \subseteq \mathbb{F}_n$ is the negative section. Indeed, the blow-up $\text{Bl}_x \mathbb{F}_n$ at any $x \notin C$ admits a morphism to $\mathbb{F}_{n-1}$ constructed by contracting the strict transform of the fibre through x of the natural projection $\mathbb{F}_n \rightarrow \mathbb{P}^1$.

Let $(\tilde{Y}, \tilde{F}_Y)$ be a Frobenius lifting of $Y = X_m$. By Theorem 8.3.20(b), there exist Frobenius liftings $(\tilde{X}_i, \tilde{F}_i)$ of X_i for $0 \leq i \leq m$, and liftings $\tilde{\pi}_i: \tilde{X}_{i+1} \rightarrow \tilde{X}_i$ such that $\tilde{F}_i \circ \tilde{\pi}_i = \tilde{\pi}_i \circ \tilde{F}_{i+1}$. By Lemma 8.7.3 we know that $(X_0, D_{\tilde{F}_0})$ is sub-toric and $\Delta_{\tilde{F}_0}$ has simple normal crossings at $\pi(\text{Exc}(\pi))$. Therefore, for every $i > 0$ the $\mathbb{Q}$ -divisor $\Delta_{\tilde{F}_i}$ has simple normal crossings at $x_i \in X_i$ (it is contained in the union of the exceptional locus of π and the support of the strict transform of $\Delta_{\tilde{F}_0}$ by Corollary 8.3.35(b)).

By induction we can show that $(X_i, D_{\tilde{F}_i})$ is sub-toric for every $0 \leq i \leq m$. Indeed, if $(X_{i-1}, D_{\tilde{F}_{i-1}})$ is sub-toric, then by Remark 8.7.2 it is enough to show that $x_{i-1} \in \text{Sing } D_{\tilde{F}_{i-1}}$, which follows by Lemma 8.7.1. This concludes the proof. $\square$

Remark 8.7.5. In the course of the proof, we showed that if $(\tilde{X}, \tilde{F}_X)$ is a Frobenius lifting of a smooth projective rational surface $X \not\cong \mathbb{P}^2$, then $(X, [\Delta_{\tilde{F}_X}])$ is sub-toric. This is false for some liftings of Frobenius on $\mathbb{P}^2$.

We now turn our attention to ruled surfaces. We say that a rank two vector bundle E on a curve C is *normalised* if $H^0(C, E) \neq 0$ and $H^0(C, E \otimes \mathcal{L}) = 0$ for every line bundle

$\mathcal{L}$ such that $\deg \mathcal{L} < 0$. Given a ruled surface $X = \mathbb{P}_C(E)$ we can assume that E is normalised by replacing E with $E \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$.

Proposition 8.7.6. *Let $X = \mathbb{P}_C(E)$ be a smooth projective ruled surface for a normalised rank two vector bundle E on an ordinary elliptic curve C . Then X is F -liftable if and only if E is not a non-split extension of $\mathcal{O}_C$ with itself.*

Proof. If E is decomposable (that is, a direct sum of two line bundles), then X is F -liftable by Example 8.3.5. Hence, we can assume that E is indecomposable. By [Har77, Theorem V.2.15], there are only two such ruled surfaces corresponding to E being a non-split extension of $\mathcal{O}_C$ with $\mathcal{O}_C(c)$ where $c \in C$, and E being a non-split extension of $\mathcal{O}_C$ with itself.

In the former case, $X \simeq \text{Sym}^2(C)$ (see for instance [Gar06, Section 6]), and it is easy to see that it is F -liftable. Indeed, let $(\tilde{C}, \tilde{F}_C)$ be the canonical Frobenius lifting of C (see Example 8.3.3), where $\tilde{F}_C: \tilde{C} \rightarrow \tilde{C}$. Then $\tilde{F}_C \times \tilde{F}_C: \tilde{C} \times \tilde{C} \rightarrow \tilde{C} \times \tilde{C}$ is equivariant under the natural $\mathbb{Z}/2\mathbb{Z}$ action by swapping the coordinates, and so it descends to $\text{Sym}^2(\tilde{C}) \rightarrow \text{Sym}^2(\tilde{C})$.

Therefore, we are left to show that X is not F -liftable when E is a non-split extension of $\mathcal{O}_C$ with itself. By contradiction assume that it does admit a Frobenius lifting $(X, \tilde{F}_X)$. By Lemma 8.7.8, we know that X is a quotient of $Y = C' \times \mathbb{P}^1$ by $\mathbb{F}_p$ acting independently on C' and $\mathbb{P}^1$, where C' is the Frobenius twist of C , and the action on $(x : y) \in \mathbb{P}^1(k)$ is defined as $(x : y) \mapsto (x + ly : y)$ for a fixed $l \in \mathbb{F}_p$. This is illustrated by the following diagram

$$\begin{array}{ccccc} \mathbb{P}^1 & \xleftarrow{\rho} & Y & \xrightarrow{V'} & X \\ & & \pi' \downarrow & \square & \downarrow \pi \\ & & C' & \xrightarrow{V} & C. \end{array}$$

Since V' is étale, Lemma 8.3.19 implies that Y admits a Frobenius lifting $(\tilde{Y}, \tilde{F}_Y)$ such that $\Delta_{\tilde{F}_Y} = V'^* \Delta_{\tilde{F}_X}$, where $\Delta_{\tilde{F}_Y}$ and $\Delta_{\tilde{F}_X}$ are the $\mathbb{Q}$ -divisors associated to $\tilde{F}_Y$ and $\tilde{F}_X$, respectively. In particular, $\Delta_{\tilde{F}_Y}$ is $\mathbb{F}_p$ -invariant.

As $\Delta_{\tilde{F}_Y} \sim_{\mathbb{Q}} -K_{C' \times \mathbb{P}^1} = -\rho^* K_{\mathbb{P}^1}$, we have $\Delta_{\tilde{F}_Y} = \rho^* T$ for some $\mathbb{Q}$ -divisor T on $\mathbb{P}^1$. Let G be a fibre of π over a general point $c \in C$ and $\Delta = \Delta_{\tilde{F}_X}|_G$. Since $\Delta_{\tilde{F}_Y}$ is $\mathbb{F}_p$ -invariant and it is a pullback from $\mathbb{P}^1$, we get that Δ is invariant under the action of $\mathbb{F}_p$ on $G \simeq \mathbb{P}^1$.

By Corollary 8.3.35(c) and Remark 8.2.20 applied to π' , we get that Δ is the associated $\mathbb{Q}$ -divisor of an F -splitting of $\mathbb{P}^1$. This contradicts Lemma 8.7.7. $\square$

We needed the following two lemmas in the proof of the above proposition.

Lemma 8.7.7. *There does not exist an $\mathbb{F}_p$ -invariant F -splitting of $\mathbb{P}^1$, where $\mathbb{F}_p$ acts on $\mathbb{P}^1$ via translations, that is $(x : y) \mapsto (x + ly : y)$ for $l \in \mathbb{F}_p$.*

Note that an F -splitting is invariant under an action of a group if and only if the corresponding $\mathbb{Q}$ -divisor is invariant.

Proof. Assume by contradiction that such an F -splitting exists and let Δ be the corresponding $\mathbb{Q}$ -divisor (see Proposition 8.2.19). By the definition of Δ , we get that $(p-1)\Delta$ is an effective integral divisor and $\deg \Delta = 2$. Furthermore, the coefficients of Δ are at most one (see Remark 8.2.21). Since each orbit of the action of $\mathbb{F}_p$ on $\mathbb{P}^1$ is of length p except for the one of the fixed point $\infty \in \mathbb{P}^1$, the only $\mathbb{Q}$ -divisor satisfying the aforementioned properties is

$$\Delta = \left(1 - \frac{1}{p-1}\right)(\infty) + \sum_{i=0}^{p-1} \frac{1}{p-1}(x_i),$$

where $x_i = x_0 + i \in \mathbb{A}^1(k)$. Up to an action of an automorphism we can assume that $x_0 = 0$. This yields a contradiction, because $\mathbb{P}^1$ cannot be F -split with such an associated $\mathbb{Q}$ -divisor Δ by [CGS16, Example 3.4] and [SS10, Proposition 5.3 (2)]. One can check that the trace of an F -splitting of $\mathbb{P}^1$ cannot be equal to $(p-1)\Delta$ directly by noticing that $x(x-y) \cdots (x-(p-1)y)y^{p-2}$ has coefficient zero at $x^{p-1}y^{p-1}$ (see [BK05, Theorem 1.3.8]). $\square$

Lemma 8.7.8. *Let C be an ordinary elliptic curve, and let C' be the Frobenius twist of C . Consider the action of $\mathbb{F}_p$ on $C' \times \mathbb{P}^1$ with $l \in \mathbb{F}_p$ acting as*

$$(c, (x : y)) \mapsto (c + l\alpha, (x + ly : y)),$$

for $c \in C'$ and $(x : y) \in \mathbb{P}^1$, where $\alpha \in C'[p]$ is a fixed p -torsion point. Let X be the quotient $(C' \times \mathbb{P}^1)/\mathbb{F}_p$. Then $X \simeq \mathbb{P}_C(E)$ for E being a non-split extension of $\mathcal{O}_C$ with itself.

Proof. By definition we have the following Cartesian diagram

$$\begin{array}{ccc} C' \times \mathbb{P}^1 & \xrightarrow{V'} & X \\ \pi' \downarrow & & \downarrow \pi \\ C' & \xrightarrow{V} & C, \end{array}$$

where V is the quotient by $C'[p]$. Note that $C' \times \mathbb{P}^1 = \mathbb{P}_{C'}(E')$, where E' is a vector bundle sitting inside a split (but not $\mathbb{F}_p$ -equivariantly so) extension of $\mathbb{F}_p$ -equivariant sheaves

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow E' \longrightarrow \mathcal{O}_{C'} \longrightarrow 0,$$

with $E' \rightarrow \mathcal{O}_{C'}$ corresponding to the $\mathbb{F}_p$ -equivariant section $C' \times (1 : 0)$ of π' . Therefore, $X = \mathbb{P}_C(E)$ for a vector bundle E on C such that V^*E is $\mathbb{F}_p$ -equivariantly isomorphic to E' , and the above short exact sequence descends to C making E a non-split extension of $\mathcal{O}_C$ by itself. $\square$

Having handled rational and ruled surfaces, we are ready to proceed to the general case.

Theorem 8.7.9. *Let X be a smooth projective surface over k . Then X is F -liftable if and only if X is*

- (i) *an ordinary abelian surface,*
- (ii) *a hyperelliptic surface being a quotient of a product of two ordinary elliptic curves,*
- (iii) *a ruled surface $\mathbb{P}_C(E)$ for a normalised rank two vector bundle E over an ordinary elliptic curve C except when E is a non-trivial extension of $\mathcal{O}_C$ with itself, or*
- (iv) *a toric surface.*

In particular, Conjecture 8.1.1 is true for surfaces.

Proof. If X is F -liftable, then it is F -split and ω_X^{1-p} is effective (see Proposition 8.3.8). In particular, we only need to consider the case of $\kappa(X) \leq 0$. If $\kappa(X) = 0$, then K_X is $\mathbb{Q}$ -effective, and hence ω_X^{p-1} is trivial and X is minimal. In this case, the theorem follows from [Xin16, Theorem 1], but for the convenience of the reader we present a simplified argument.

When K_X is torsion, X is F -liftable if and only if it is a quotient of an ordinary abelian surface by a free action of a finite group (see [MS87, Theorem 2] and Proposition 8.3.15(c)). By Hirzebruch–Riemann–Roch, such quotients satisfy $\chi(X, \mathcal{O}_X) = 0$, and so by the classification of surfaces they are abelian, hyperelliptic, or quasi-hyperelliptic. We can exclude the surfaces of the latter type as they contain rational curves (see [Lie13, §7]): indeed, if $\varphi: \mathbb{P}^1 \rightarrow X$ is non-constant, then there exists an injection $T_{\mathbb{P}^1} \rightarrow \varphi^*T_X$, so T_X cannot be étale trivialisable (cf. [Xin16, Proposition 5]). Hyperelliptic surfaces are étale quotients of products of elliptic curves $E_1 \times E_0$, except for case a3) which is an étale quotient of an abelian surface $A = E_1 \times E_0/\mu_2$ (see [BM77, p. 37] for the notation). Therefore they are F -liftable if and only if E_1 and E_0 are ordinary, by [MS87, Theorem 2] and Lemma 8.3.19. Note that A is ordinary if and only if $E_1 \times E_0$ is, as they are isogeneous to each other. This concludes our analysis of Kodaira dimension zero.

As of now, we assume $\kappa(X) = -\infty$. If X is of type (iii) or (iv), then it is F -liftable by Example 8.3.2 and Proposition 8.7.6. To conclude, we assume that X is F -liftable and show that X is of type (iii) or (iv). If X is rational, then it is toric by Proposition 8.7.4. Thus, we can assume that there exists a birational morphism $f: X \rightarrow \mathbb{P}_C(E)$, where $\mathbb{P}_C(E)$ is a ruled surface over $C \not\cong \mathbb{P}^1$. By Theorem 8.3.20(b), the curve C is F -liftable, and so it is an ordinary elliptic curve.

We claim that $X \simeq \mathbb{P}_C(E)$. Otherwise, f factors through a monoidal transformation $\pi: \mathrm{Bl}_x \mathbb{P}_C(E) \rightarrow \mathbb{P}_C(E)$, where $x \in \mathbb{P}_C(E)$. In particular, Theorem 8.3.20(b) implies that $Y = \mathrm{Bl}_x \mathbb{P}_C(E)$ is F -liftable, which is impossible. Indeed, if Y is F -liftable, then so is $(Y, \mathrm{Exc} \pi)$ by Lemma 8.3.25, and hence Y is F -split compatibly with $\mathrm{Exc} \pi$ by Variant 8.3.9. Therefore, we get an F -splitting on C compatible with a point (see [BK05, Lemma 1.1.8(ii)]), which is impossible as C is an elliptic curve.

Since we know that X is a ruled surface over an ordinary elliptic curve, the theorem follows from Proposition 8.7.6. $\square$

Remark 8.7.10. The F -liftability of minimal surfaces has been considered in [Xin16]. Note that our results do not agree when $\kappa(X) = -\infty$, as [Xin16, Theorem 1 (2b)] claims that all ruled surfaces over ordinary elliptic curves are F -liftable. It seems to us that the gluing argument used at the end of the proof of Proposition 9 in *op.cit.* is incomplete, as it is unclear why h extends to a regular function over V .

8.7.2. Fano threefolds

In this subsection, we show that Conjecture 8.1.1 is valid for smooth Fano threefolds from the Mori–Mukai list defined over an algebraically closed field k of characteristic $p > 0$. Note that we do not show that the conjecture holds for all smooth Fano threefolds over k . Indeed, the full Mori–Mukai classification is only known to be true in characteristic zero. However, F -liftable smooth Fano varieties in positive characteristic admit a unique lifting to characteristic zero by Bott vanishing (see Theorem 8.3.11 and Corollary 8.3.12). Given that, we believe that our results could be extended to all smooth Fano threefolds over k with some additional work.

Although a proof based on the classification might seem tedious and not very useful, we embarked on this task for two reasons. First, the study conducted in this section confirms our belief that Frobenius liftability is very rare among smooth projective varieties, and provides substantial empirical evidence for the validity of Conjecture 8.1.1. But what is even more important is that the examples analysed in this section were a source of inspiration in our study of general properties of F -liftable smooth projective varieties.

The main reference for the Mori–Mukai classification used in this section is [Sha99, Tables §12.3–§12.6]. We denote a Fano threefold of Picard rank ρ and whose number in the tables is n as M–M $\rho.n$ (for example, M–M 2.12).

Remark 8.7.11. Let $f: Y \rightarrow X$ be a proper morphism between smooth varieties such that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. By Theorem 8.3.20(b), if Y is F -liftable, then so is X . This observation significantly reduces the number of cases of smooth Fano threefolds which need to be considered. Note that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$ when f is birational, by [CR15, Theorem 1.1].

We start by giving main examples of techniques we apply to deal with F -liftability of Fano threefolds.

Rigidity and Bott vanishing

By Theorem 8.3.11 and Corollary 8.3.12, smooth F -liftable Fano varieties satisfy Bott vanishing, and hence they are rigid. Note that the majority of non-toric smooth Fano threefolds are not rigid (see the proof of Theorem 8.7.19).

The following lemma shows that Bott vanishing does not hold for a certain Fano threefold.

Lemma 8.7.12 (M–M 1.16). *Let $X = \mathrm{Gr}(2, 5) \cap \mathbb{P}^6$, where $\mathrm{Gr}(2, 5) \subseteq \mathbb{P}^9$ is the Grassmannian of planes in $\mathbb{P}^5$ embedded in $\mathbb{P}^9$ by the Plücker embedding and $\mathbb{P}^6 \subseteq \mathbb{P}^9$ is a general linear subspace. Then $H^1(X, \Omega_X^2(1)) \neq 0$, where $\mathcal{O}_X(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}^9}(1)$ to X .*

Proof. The setting admits a natural lifting to characteristic zero, and by the semicontinuity theorem, it is enough to show that the required non-vanishing holds for this lifting. Therefore, we can assume that X is defined over $\mathbb{C}$.

We have $\omega_{\text{Gr}(2,5)} \simeq \mathcal{O}_{\text{Gr}(2,5)}(-5)$, hence $\omega_X \simeq \mathcal{O}_X(-2)$ and therefore $H^1(X, \Omega_X^2(1)) = H^1(X, T_X(-1))$. First, we prove that $H^i(X, T_{\text{Gr}(2,5)}(-1)|_X) = 0$ for all i . Consider the Koszul resolution

$$0 \longrightarrow \mathcal{O}_{\text{Gr}(2,5)}(-3) \longrightarrow \mathcal{O}_{\text{Gr}(2,5)}(-2)^{\oplus 3} \longrightarrow \mathcal{O}_{\text{Gr}(2,5)}(-1)^{\oplus 3} \longrightarrow \mathcal{O}_{\text{Gr}(2,5)} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

and tensor it by $T_{\text{Gr}(2,5)}(-1)$ to get

$$0 \rightarrow T_{\text{Gr}(2,5)}(-4) \rightarrow T_{\text{Gr}(2,5)}(-3)^{\oplus 3} \rightarrow T_{\text{Gr}(2,5)}(-2)^{\oplus 3} \rightarrow T_{\text{Gr}(2,5)}(-1) \rightarrow T_{\text{Gr}(2,5)}(-1)|_X \rightarrow 0.$$

By [Sno86, Theorem, p.171, (3)], all the cohomology groups of $T_{\text{Gr}(2,5)}(-k) \simeq \Omega_{\text{Gr}(2,5)}^5(5-k)$ vanish for $1 \leq k \leq 4$, and so the above exact sequence shows that the same holds for $T_{\text{Gr}(2,5)}(-1)|_X$.

The dual of the conormal exact sequence tensored by $\mathcal{O}_X(-1)$ is

$$0 \longrightarrow T_X(-1) \longrightarrow T_{\text{Gr}(2,5)}(-1)|_X \longrightarrow \mathcal{O}_X^{\oplus 3} \longrightarrow 0.$$

Since the cohomology groups of the middle sheaf vanish, we get

$$H^1(T_X(-1)) = H^0(X, \mathcal{O}_X)^{\oplus 3} \neq 0. \quad \square$$

Blow-ups

We show that some Fano threefolds constructed as blow-ups are not F -liftable. From the viewpoint of our research, smooth Fano threefold 2.27 (see Lemma 8.7.17) was the most intriguing. The study of this rigid Fano variety motivated many results in Section 8.3.

If a blow-up of a smooth variety X along a smooth locus $V \subseteq X$ is F -liftable, so is X by Remark 8.7.11. Moreover, V must be F -liftable as well by results of §8.3.4.

Example 8.7.13 (M–M 2.12). Let X be the blow-up of $\mathbb{P}^3$ along a smooth curve C of degree six and genus three. Then X is not F -liftable, because otherwise $(\mathbb{P}^3, C)$ would be F -liftable by Lemma 8.3.25 and Lemma 8.3.27. In particular, C would be F -liftable, which is impossible as it is of general type. One can also deduce non- F -liftability of X by noticing that it is not rigid.

In many cases, a Frobenius lifting of a product of varieties must be a product of Frobenius liftings (see Corollary 8.3.21).

Example 8.7.14 (M–M 4.6). Let X be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along the tridiagonal curve

$$C = \left\{ (x, x, x) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid x \in \mathbb{P}^1 \right\}.$$

The variety X cannot be F -liftable, because then $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, C)$ would admit a Frobenius lifting by Lemma 8.3.25 and Lemma 8.3.27, which is impossible by Proposition 8.3.28.

In order to be able to deal with the next example, we need the following lemma.

Lemma 8.7.15. *Let $D \subseteq \mathbb{P}^2$ be a smooth conic. Then $(\mathbb{P}^2, D)$ is not F -liftable.*

Proof. Assume by contradiction that $(\mathbb{P}^2, D)$ is F -liftable and fix a Frobenius lifting. Let

$$\xi: F^* \Omega_{\mathbb{P}^2}^1(\log D) \rightarrow \Omega_{\mathbb{P}^2}^1(\log D)$$

be the associated morphism constructed in Section 8.3.2 (see Variant 8.3.9). Choose a general point $x \in D$ and take $C \subseteq \mathbb{P}^2$ to be the line tangent to D at x . Since x is general and ξ is generically an isomorphism (see Proposition 8.3.8), we get that $\xi|_C$ is injective. In particular, if $\Omega_{\mathbb{P}^2}^1(\log D)|_C = \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$ for some $a, b \in \mathbb{Z}$, then $a, b \leq 0$ (see the proof of Lemma 8.6.5(a)).

We will show that $\Omega_{\mathbb{P}^2}^1(\log D)|_C = \mathcal{O}_C(-2) \oplus \mathcal{O}_C(1)$ yielding a contradiction (cf. [Xin16, Lemma 4]). To this end, choose a standard affine chart on $\mathbb{A}^2 \subseteq \mathbb{P}^2$ in which C is described as $y = 0$, and D as $y - x^2 = 0$. In the chart $\mathbb{A}^2 \cap C$, we get that $\Omega_{\mathbb{P}^2}^1(\log D)|_C$ is generated by

$$dx \quad \text{and} \quad \frac{d(y - x^2)}{y - x^2} = -\frac{dy}{x^2} + \frac{2dx}{x}.$$

Now, we take the chart of $\mathbb{P}^2$ having coordinates $\frac{1}{x}$ and $\frac{y}{x}$. In this chart C and D are disjoint, and so in the restriction of this chart to C the vector bundle $\Omega_{\mathbb{P}^2}^1(\log D)|_C$ is generated by

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx \quad \text{and} \quad d\left(\frac{y}{x}\right) = \frac{dy}{x} = 2dx + x\left(\frac{dy}{x^2} - \frac{2dx}{x}\right).$$

Therewith, the coordinate-change matrix is:

$$\begin{bmatrix} -\frac{1}{x^2} & 0 \\ 2 & -x \end{bmatrix},$$

and so $\Omega_{\mathbb{P}^2}^1(\log D)|_C = \mathcal{O}_C(-2) \oplus \mathcal{O}_C(1)$. □

Example 8.7.16 (M–M 2.30). Let $X = \text{Bl}_C \mathbb{P}^3$ be a blow-up of $\mathbb{P}^3$ along a conic C contained in a hyperplane $H \subseteq \mathbb{P}^3$. We show that X is not F -liftable by assuming otherwise and inferring a contradiction.

Let $\overline{H} \simeq \mathbb{P}^2$ be the strict transform of H on X , and let E be the exceptional divisor of the blow-up. By Lemma 8.3.25, we get that $(X, \overline{H} + E)$ admits a Frobenius lifting, and so $(\overline{H}, \overline{H} \cap E)$ is F -liftable by Lemma 8.3.24. Since $C \simeq \overline{H} \cap E$ is a conic, this contradicts Lemma 8.7.15.

A surprising property of F -liftability is that if X is F -liftable and $f: Y \rightarrow X$ is a smooth morphism such that $Rf_* \mathcal{O}_Y = \mathcal{O}_X$, then f is relatively F -split. This is not true in general, if we assume that X is only F -split.

Lemma 8.7.17 (M–M 2.27). *Let $C \subset \mathbb{P}^3$ be the twisted cubic, that is, the image of $\mathbb{P}^1$ under the embedding given by the full linear system $|\mathcal{O}_{\mathbb{P}^1}(3)|$. Then the blow-up $Y = \text{Bl}_C \mathbb{P}^3$ is not F -liftable.*

Proof. Let E be the exceptional divisor of the blow-up $\pi: Y \rightarrow \mathbb{P}^3$. Firstly, we observe that by [SW90, Application 1, page 299] Y is isomorphic to a projective bundle $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^2$ for some non-split rank two vector bundle $\mathcal{E}$. The morphism $f: Y \rightarrow \mathbb{P}^2$ is given by a pencil of quadrics in $\mathbb{P}^3$ containing C .

If Y is F -liftable, then Corollary 8.3.35(a) and (c) imply that $-(K_{Y/\mathbb{P}^2} + E)$ is $\mathbb{Q}$ -linearly equivalent to an effective $\mathbb{Q}$ -divisor. In what follows, we show that this is not true. By the construction of f , we have $Q = f^{-1}(L)$ for some line $L \subseteq \mathbb{P}^2$, where Q is the strict transform of a quadric in $\mathbb{P}^3$ containing C . Since $\omega_Y \simeq \mathcal{O}_Y(E) \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(-4)$, we have $K_Y \sim -2Q - E$, and so:

$$K_{Y/\mathbb{P}^2} + E \sim -2Q + 3f^{-1}(L) \sim Q,$$

which is not $\mathbb{Q}$ -linearly equivalent to an anti-effective $\mathbb{Q}$ -divisor. $\square$

Remark 8.7.18. Using Corollary 8.3.31 and a more intricate version of the above argument one can show that a rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ is decomposable if and only if $\mathbb{P}_{\mathbb{P}^n}(\mathcal{E})$ is F -liftable. This seems intriguing from the viewpoint of Hartshorne’s conjecture predicting that all such vector bundles are decomposable when $n \geq 7$.

F -liftable smooth Fano threefolds are toric

Summing up the above results, we can show the following.

Theorem 8.7.19. *Let X be a smooth projective Fano threefold from the Mori–Mukai list, defined over an algebraically closed field k of characteristic $p > 0$. Then X is F -liftable if and only if X is toric.*

Fano threefolds		Mori–Mukai list
Non-rigid	blow-up of a non-rigid curve	2.28
	negative virtual dimension	1.1–1.14, 2.1–2.25, 3.1–3.12, 4.1, 4.2
Blow-up on a product		3.17, 3.21, 4.3, 4.6, 4.8
Violating Bott vanishing		1.15, 1.16
Inducing a lifting of Frobenius on $\mathbb{P}^2$ compatible with a conic		2.30, 3.22
Mapping to a Frobenius non-liftable variety		2.26, 2.29, 2.31, 3.13–3.16, 3.18–3.20, 3.23, 3.24, 4.4, 4.5, 4.7, 5.1, 5.4–5.8
Toric		2.33–2.36, 3.25–3.31, 4.9–4.12, 5.2, 5.3
Other		2.27, 2.32

Figure 8.1: Fano threefolds.

Proof. We denote a Fano threefold of Picard rank ρ and whose number in [Sha99, Tables §12.2–§12.6] is n as $\rho.n$. We recommend the reader to look at the tables in order to be able to follow the proof.

First, smooth Fano threefold 2.27 is not F -liftable by Lemma 8.7.17, and Fano threefold 2.32 is not F -liftable by Lemma 8.6.12.

Non-rigid. The Hirzebruch–Riemann–Roch formula for threefolds reads

$$\begin{aligned} \chi(X, E) = & \frac{1}{24} \operatorname{rk}(E) c_1(T_X) c_2(T_X) + \frac{1}{12} c_1(E) (c_1(T_X)^2 + c_2(T_X)) \\ & + \frac{1}{4} c_1(T_X) (c_1(E)^2 - 2c_2(E)) + \frac{1}{6} (c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)). \end{aligned}$$

In particular, $\chi(X, \mathcal{O}_X) = c_1(T_X)c_2(T_X)/24$; on the other hand, $\chi(X, \mathcal{O}_X) = 1$ since X satisfies Kodaira vanishing, so $c_1(T_X)c_2(T_X) = 24$. The number $c_3(T_X)$ equals the topological Euler characteristic, $2 + 2\rho(X) - b_3(X)$. We deduce the following formula

$$\chi(X, T_X) = \frac{1}{2}(-K_X)^3 - 18 + \rho(X) - \frac{1}{2}b_3(X).$$

If X is an F -liftable Fano threefold, then by Bott vanishing, we have $H^i(X, T_X) = 0$ for $i > 0$, and hence $\chi(X, T_X) \geq 0$. Since F -liftable Fano threefolds admit a unique lifting to characteristic zero (Theorem 8.3.11), and as Betti numbers are invariant under deformations we can read them off from [Sha99, Tables §12.2–§12.6] (note that $b_3(X) = 2h^2(X, \Omega_X^1)$ in characteristic zero). This allows us to calculate that smooth Fano threefolds 1.1–14, 2.1–25, 3.1–12, and 4.1–2 have $\chi(X, T_X) < 0$. In particular, they are not F -liftable (Corollary 8.3.12).

Smooth Fano threefold 2.28 is the blow-up of $\mathbb{P}_k^3$ along a plane cubic $C \subseteq \mathbb{P}_k^3$. By Lemma 8.2.4, $\operatorname{Def}_{\operatorname{Bl}_C \mathbb{P}_k^3} \simeq \operatorname{Def}_{\mathbb{P}_k^3, C}$. Further, the forgetful transformation $\operatorname{Def}_{\mathbb{P}_k^3, C} \rightarrow \operatorname{Def}_C$ is non-constant. We conclude that $\operatorname{Bl}_C \mathbb{P}_k^3$ is not rigid.

Blow-ups on a product. Fano threefolds 3.17, 3.21, 4.3, 4.6, and 4.8 are not F -liftable by the same argument as in Example 8.7.14. Note that Fano threefold 3.17 is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along the graph of the Segre embedding $\mathbb{P}^1 \rightarrow \mathbb{P}^2$ of degree two.

Violating Bott vanishing. Fano threefolds 1.15 and 1.16 do not satisfy Bott vanishing (Example 8.3.13 and Lemma 8.7.12), and so they are not F -liftable (Theorem 8.3.11).

Inducing a lifting of Frobenius on $\mathbb{P}^2$ compatible with a conic. Fano threefolds 2.30 and 3.22 are not F -liftable by the same argument as in Example 8.7.16.

Mapping to a Frobenius non-liftable variety. Smooth Fano threefolds 2.26, 2.29, 2.31, 3.13–3.16, 3.18–3.20, 3.23–24, 4.4–5, 4.7, and 5.1 are not F -liftable, as they admit a birational morphism to another smooth Fano threefold which is not F -liftable (see Remark 8.7.11).

Fano threefolds 5.4–8 are not F -liftable, as they admit a morphism $f: X \rightarrow S$ such that $Rf_* \mathcal{O}_X = \mathcal{O}_S$, where S is a non-toric rational surface (see Theorem 8.3.20 and Theorem 8.7.9).

Toric. Fano threefolds 2.33–2.36, 3.25–3.31, 4.9–4.12, 5.2, 5.3 are toric (cf. [Sha99, Ch. 12, Remarks (i), p. 216], note that toric varieties 3.25 and 4.12 are missing from this list). $\square$

Combined with Theorem 8.4.15, Theorems 8.7.9 and 8.7.19 show that Conjecture 1.2.4 holds if the target X is a surface or a Fano threefold.

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