NUMERICAL ANALYSIS FOR A SYSTEM COUPLING CURVE EVOLUTION TO REACTION-DIFFUSION ON THE CURVE

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Abstract. We consider a finite element approximation for a system consisting of the evolution of a closed planar curve by forced curve shortening flow coupled to a reaction-diffusion equation on the evolving curve. The scheme for the curve evolution is based on a parametric description allowing for tangential motion, whereas the discretisation for the PDE on the curve uses an idea from [6]. We prove optimal error bounds for the resulting fully discrete approximation and present numerical experiments. These confirm our estimates and also illustrate the advantage of the tangential motion of the mesh points in practice.

Key words. surface PDE, forced curve shortening flow, diffusion induced grain boundary motion, parametric finite elements, tangential motion, error analysis

AMS subject classifications. 65M60, 65M15, 35K55, 53C44, 74N20

1. Introduction. The aim of this paper is to analyse a fully discrete numerical scheme for approximating a solution of the following system: find a family of planar, closed curves $(\Gamma(t))_{t\in[0,T]}$ and a function $w : \bigcup_{t\in[0,T]}\big(\Gamma(t)\times\{t\}\big) \to \mathbb{R}$ such that

$$v = \kappa + f(w) \qquad \text{on } \Gamma(t), \quad t \in (0,T], \tag{1.1a}$$

$$\partial_t^\bullet w = d\, w_{ss} + \kappa\, v\, w + g(v,w) \qquad \text{on } \Gamma(t), \quad t \in (0,T], \tag{1.1b}$$

subject to the initial conditions

$$\Gamma(0) = \Gamma^0, \qquad w(\cdot,0) = w^0 \text{ on } \Gamma^0. \tag{1.2}$$

Here, v and κ are the normal velocity and the curvature of $\Gamma(t)$ corresponding to the choice $\vec\nu$ of a unit normal, while s is the arclength parameter on $\Gamma(t)$. Furthermore, $\partial_t^\bullet w$ denotes the material derivative of w, i.e. $\partial_t^\bullet w = w_t + v\,\frac{\partial w}{\partial \nu}$. Finally, $d \in \mathbb{R}_{>0}$, $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the closed curve Γ^0 and $w^0 : \Gamma^0 \to \mathbb{R}$ are all given. The system (1.1a,b) couples the evolution of the curves $(\Gamma(t))_{t\in[0,T]}$ by forced curve shortening flow to a parabolic PDE on the moving curves. It occurs for example as a sharp–interface model for diffusion induced grain boundary motion: in this setting $\Gamma(t)$ represents a grain boundary separating the crystals of a thin polycrystalline film of metal that is placed in a vapour containing another metal; atoms from the vapour diffuse into the film along the grain boundaries causing them to move. A thorough description of the physical set-up can be found in [11], while an existence and uniqueness result has been obtained in [12].

In what follows we shall describe the evolving curves $\Gamma(t)$ with the help of a parametrisation $\vec x(\cdot,t) : \mathbb{I} \to \mathbb{R}^2$, where $\mathbb{I} := \mathbb{R}\setminus\mathbb{Z}$ is the periodic unit interval. Then

$$\vec\tau = \vec x_s = \frac{\vec x_\rho}{|\vec x_\rho|}, \qquad \vec\nu = \vec\tau^{\,\perp} \tag{1.3}$$

are a unit tangent and unit normal to $\Gamma(t)$ respectively, where $(\cdot)^\perp$ denotes counter-clockwise rotation by $\frac{\pi}{2}$. The normal and tangential velocities of $\Gamma(t)$ are then

$$v = \vec x_t \cdot \vec\nu, \qquad \psi = \vec x_t \cdot \vec\tau. \tag{1.4}$$

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Note that (1.1a) only prescribes v and with it the shape of $\Gamma(t)$, so that there is a certain freedom in choosing the tangential velocity. Since $\vec{x}_{ss} = \kappa\,\vec{\nu}$ one may consider

$$\vec{x}_t = \vec{x}_{ss} + f(\widetilde{w})\,\vec{\nu} = \frac{1}{|\vec{x}_\rho|}\left(\frac{\vec{x}_\rho}{|\vec{x}_\rho|}\right)_\rho + f(\widetilde{w})\,\vec{\nu}, \tag{1.5}$$

where $\widetilde{w}(\rho,t) := w(\vec{x}(\rho,t),t), (\rho,t) \in \mathbb{I}\times[0,T]$. Clearly, (1.1a) holds with the additional property that the velocity vector $\vec{x}_t$ points in the normal direction. Coupling (1.5) to the PDE satisfied by $\tilde{w}$, Pozzi and Stinner have derived and analysed in [13] a finite element scheme for (1.1a,b) (with $g \equiv 0$) based on continuous piecewise linears. They are able to prove the following error bounds in the spatially discrete case:

$$\sup_{[0,T]} \int_{\mathbb{I}} \left(|\vec{x}_\rho - \vec{x}^h_\rho|^2 + |\widetilde{w} - \widetilde{w}^h|^2\right)\,\mathrm{d}\rho + \int_0^T \int_{\mathbb{I}} \left(|\vec{x}_t - \vec{x}^h_t|^2 + |\widetilde{w}_\rho - \widetilde{w}^h_\rho|^2\right)\,\mathrm{d}\rho\,\mathrm{d}t \le C\,h^2.$$

A major difficulty in the analysis arises from the fact that (1.5) is only weakly parabolic. Following [5], this problem is solved in [13] by deriving additional equations for the continuous and discrete length elements and by splitting the error $\vec{x}_\rho - \vec{x}^h_\rho$ into a tangent part and a length element part. Apart from these analytical difficulties, the motion in purely the normal direction also may lead to the accumulation of mesh points in numerical simulations. A natural way to handle the above mentioned difficulties is to introduce a tangential part in the velocity, which can be seen as a reparametrisation, an approach that has recently been explored in a systematic way by Elliott and Fritz in [7]. The underlying idea uses the DeTurck trick in coupling the motion of the curve to the harmonic map heat flow, which results in the following equation replacing (1.5) (cf. (3.1) in [7]):

$$\alpha\,\vec{x}_t + (1-\alpha)\,(\vec{x}_t\cdot\vec{\nu})\,\vec{\nu} = \frac{\vec{x}_{\rho\rho}}{|\vec{x}_\rho|^2} + f(\widetilde{w})\,\vec{\nu}. \tag{1.6}$$

In the above, $\alpha \in (0,1]$ is a parameter so that $\frac{1}{\alpha}$ corresponds to the diffusion coefficient in the harmonic map heat flow. Note that we obtain (1.1a) by taking the scalar product of (1.6) with $\vec{\nu}$. It turns out that $\vec{x}(\cdot,t)$ gets closer to a parametrisation proportional to arclength as α gets small. At the numerical level this means that mesh points along the curve become more and more equidistributed. Setting formally $\alpha = 0$ one recovers an approach introduced by Barrett, Garcke and Nürnberg, see [1], [2]. A nice feature of (1.6) is that the problem now is strictly parabolic allowing for a more straightforward error analysis; for the spatially discrete case with $f = 0$, see [3], [4] for $\alpha = 1$, and [7] for $\alpha \in (0,1]$.

It remains to derive the equation for $\widetilde{w}$, which in contrast to [13] will also involve the tangential velocity $(\vec{x}_t\cdot\vec{\tau})$. It is easily seen that

$$\begin{aligned}\widetilde{w}_t(\rho,t) &= \frac{d}{dt}[w(\vec{x}(\rho,t),t)]\\ &= \partial_t^\bullet w(\vec{x}(\rho,t),t) + (\vec{x}_t(\rho,t)\cdot\vec{\tau}(\vec{x}(\rho,t)))\,(\nabla_\Gamma w(\vec{x}(\rho,t),t)\cdot\vec{\tau}(\vec{x}(\rho,t)))\,,\end{aligned}$$

where $\nabla_\Gamma w$ is the tangential gradient of w. Inserting this relation into (1.1b) and recalling (1.4), we obtain

$$\widetilde{w}_t - (\vec{x}_t\cdot\vec{\tau})\,\frac{1}{|\vec{x}_\rho|}\,\widetilde{w}_\rho - \frac{d}{|\vec{x}_\rho|}\left(\frac{\widetilde{w}_\rho}{|\vec{x}_\rho|}\right)_\rho - \kappa\,v\,\widetilde{w} = g(v,\widetilde{w}). \tag{1.7}$$

The initial conditions for (1.6), (1.7) now are: $\vec{x}(\rho,0) = \vec{x}^0(\rho)$, $\widetilde{w}(\rho,0) = \widetilde{w}^0(\rho) := w^0(\vec{x}^0(\rho))$, $\rho \in \mathbb{I}$, where $\vec{x}^0$ is a parametrization of Γ^0. We shall derive in Section 2 a weak formulation for (1.6), (1.7) which forms the basis for a discretisation by continuous piecewise linear finite elements in space and a backward Euler scheme in time. As the main result of our paper we shall present an error analysis for the resulting fully discrete scheme. The precise result will be formulated at the end of Section 2, while the proof is carried out in detail in Section 3. In Section 4 we present a series of test calculations that confirm our theoretical results and demonstrate the above mentioned improvement of the mesh quality for small values of α.

Finally, we end this section with a few comments about notation. We adopt the standard notation for Sobolev spaces, denoting the norm of $W^{\ell,p}(G)$ ($\ell \in \mathbb{N}$, $p \in [1,\infty]$ and G a bounded interval in $\mathbb{R}$) by $\|\cdot\|_{\ell,p,G}$ and the semi-norm by $|\cdot|_{\ell,p,G}$. For $p=2$, $W^{\ell,2}(G)$ will be denoted by $H^\ell(G)$ with the associated norm and semi-norm written, as respectively, $\|\cdot\|_{\ell,G}$ and $|\cdot|_{\ell,G}$. For ease of notation, in the common case when $G \equiv \mathbb{I}$ the subscript "$\mathbb{I}$" will be dropped on the above norms and semi-norms. The above are naturally extended to vector functions, and we will write $[W^{\ell,p}(G)]^2$ for a vector function with two components. In addition, we adopt the standard notation $W^{\ell,p}(a,b;X)$ ($\ell \in \mathbb{N}$, $p \in [1,\infty]$, (a,b) an interval in $\mathbb{R}$, X a Banach space) for time dependent spaces with norm $\|\cdot\|_{W^{\ell,p}(a,b;X)}$. Once again, we write $H^\ell(a,b;X)$ if $p=2$. Furthermore, C denotes a generic constant independent of the mesh parameter h and the time step Δt, see below.

2. Weak formulation and finite element approximation. We shall assume that the data and the solution of (1.6), (1.7) (writing again w instead of $\widetilde{w}$ for ease of notation) satisfy

(2.1a) $f,\, g \in C(\mathbb{R},\mathbb{R})$ such that $\forall\ L>0,\ \exists\ C_L \geq 0$:
$$|f(w_1)-f(w_2)| \leq C_L\,|w_1-w_2|, \quad \forall\ |w_1|,\,|w_2| \leq L,$$
$$|g(v_1,w)-g(v_2,w)| \leq C_L\,|v_1-v_2|, \quad \forall\ |w| \leq L,\ \forall\ v_1,\,v_2,$$
$$|g(v,w_1)-g(v,w_2)| \leq C_L\,|w_1-w_2|, \quad \forall\ |w_1|,\,|w_2|,\,|v| \leq L;$$

(2.1b) $\vec{x} \in W^{1,\infty}(0,T;[H^2(\mathbb{I})]^2) \cap H^2(0,T;[H^1(\mathbb{I})]^2) \cap W^{2,\infty}(0,T;[L^2(\mathbb{I})]^2)$;

(2.1c) $w \in C([0,T];H^2(\mathbb{I})) \cap W^{1,\infty}(0,T;H^1(\mathbb{I})) \cap H^2(0,T;L^2(\mathbb{I}))$;

(2.1d) $0 < m \leq |\vec{x}_\rho| \leq M$ on $\mathbb{I}\times[0,T]$, for some $m,\,M \in \mathbb{R}_{>0}$.

Since our error analysis will also include the discretisation in time our regularity assumptions are slightly stronger than those made in [13], see Assumption 2.2 there. Let us fix $\alpha \in (0,1]$. For a test function $\vec{\xi} \in [H^1(\mathbb{I})]^2$, we obtain on multiplying (1.6) by $|\vec{x}_\rho|^2\,\vec{\xi}$ that

$$\int_{\mathbb{I}} |\vec{x}_\rho|^2\,[\alpha\,\vec{x}_t + (1-\alpha)\,(\vec{x}_t\cdot\vec{\nu})\,\vec{\nu}]\cdot\vec{\xi}\,\mathrm{d}\rho + \int_{\mathbb{I}} \vec{x}_\rho\cdot\vec{\xi}_\rho\,\mathrm{d}\rho = \int_{\mathbb{I}} |\vec{x}_\rho|^2\,f(w)\,\vec{\nu}\cdot\vec{\xi}\,\mathrm{d}\rho. \tag{2.2}$$

In order to derive a weak formulation for (1.7) we employ an idea from [6], [9], and calculate for $\eta \in H^1(\mathbb{I})$ with the help of integration by parts and (1.3)

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{I}} |\vec{x}_\rho|\,w\,\eta\,\mathrm{d}\rho = \int_{\mathbb{I}} |\vec{x}_\rho|\,w_t\,\eta\,\mathrm{d}\rho + \int_{\mathbb{I}} \frac{\vec{x}_\rho\cdot\vec{x}_{\rho,t}}{|\vec{x}_\rho|}\,w\,\eta\,\mathrm{d}\rho \\
&= \int_{\mathbb{I}} |\vec{x}_\rho|\,w_t\,\eta\,\mathrm{d}\rho - \int_{\mathbb{I}} (\vec{x}_t\cdot\vec{\tau})\,w_\rho\,\eta\,\mathrm{d}\rho - \int_{\mathbb{I}} (\vec{x}_t\cdot\vec{\tau})\,w\,\eta_\rho\,\mathrm{d}\rho - \int_{\mathbb{I}} \kappa\,(\vec{x}_t\cdot\vec{\nu})\,w\,\eta\,|\vec{x}_\rho|\,\mathrm{d}\rho \\
&= -d\int_{\mathbb{I}} \frac{w_\rho\,\eta_\rho}{|\vec{x}_\rho|}\,\mathrm{d}\rho - \int_{\mathbb{I}} \psi\,w\,\eta_\rho\,\mathrm{d}\rho + \int_{\mathbb{I}} |\vec{x}_\rho|\,g\,(v,w)\,\eta\,\mathrm{d}\rho,
\end{aligned} \tag{2.3}$$

where we have also used (1.4), (1.7) and the fact that $\vec{\tau}_\rho = \kappa\,\vec{\nu}\,|\vec{x}_\rho|$.

We now use (2.2), (2.3) in order to discretise our system and begin by introducing the decomposition $\mathbb{I} = \cup_{j=1}^J \overline{\sigma_j}$, where $\sigma_j = (\rho_{j-1}, \rho_j)$. We set $h := \max_{j=1,\ldots,J} h_j$, where $h_j := \rho_j - \rho_{j-1}$ and assume that

$$h \leq C\,h_j, \qquad j = 1, \ldots, J. \tag{2.4}$$

Let

$$V_1^h := \{\chi \in C(\mathbb{I}) : \chi|_{\sigma_j} \text{ is affine, } j = 1, \ldots, J\} \subset H^1(\mathbb{I}) \tag{2.5}$$

and $I^h : C(\mathbb{I}) \to V_1^h$ be the standard Lagrange interpolation operator such that $(I^h\eta)(\rho_j) = \eta(\rho_j)$, $j = 1, \ldots, J$. We require also the local interpolation operator $I_j^h \equiv I_{|\sigma_j}^h$, $j = 1, \ldots, J$ and recall for $p \in (1, \infty]$, $k \in \{0, 1\}$, $\ell \in \{1, 2\}$ and $j = 1, \ldots, J$ that

$$h_j^{\frac{1}{p}}\,|\eta^h|_{0,\infty,\sigma_j} + h_j\,|\eta^h|_{1,p,\sigma_j} \leq C\,|\eta^h|_{0,p,\sigma_j} \qquad \forall\ \eta^h \in V_1^h, \tag{2.6a}$$

$$|(I - I_j^h)\eta|_{k,p,\sigma_j} \leq C\,h_j^{\ell-k}\,|\eta|_{\ell,p,\sigma_j} \qquad \forall\ \eta \in W^{\ell,p}(\sigma_j), \tag{2.6b}$$

$$|(I - I_j^h)\eta|_{\ell-1,\infty,\sigma_j} \leq C\,h_j^{\frac{1}{2}}\,|\eta|_{\ell,\sigma_j} \qquad \forall\ \eta \in H^\ell(\sigma_j). \tag{2.6c}$$

As well as the standard $L^2(\mathbb{I})$ inner product $(\cdot,\cdot)$, we introduce the discrete inner product $(\cdot,\cdot)^h$ defined by

$$(\eta_1, \eta_2)^h := \sum_{j=1}^J \int_{\sigma_j} I_j^h(\eta_1\,\eta_2), \tag{2.7}$$

where η_i are piecewise continuous functions on the partition $\cup_{j=1}^J \overline{\sigma_j}$ of $\mathbb{I}$. We note for $j = 1, \ldots, J$ and for all $\eta^h, \chi^h \in V_1^h$ that

$$\int_{\sigma_j} |\eta^h|^2\ \mathrm{d}\rho \leq \int_{\sigma_j} I_j^h\left[|\eta^h|^2\right]\ \mathrm{d}\rho \leq 3\int_{\sigma_j} |\eta^h|^2\ \mathrm{d}\rho, \tag{2.8a}$$

$$\left|\int_{\sigma_j} (I - I_j^h)(\eta^h\,\chi^h)\ \mathrm{d}\rho\right| \leq C\,h_j^2\,|\eta^h|_{1,\sigma_j}\,|\chi^h|_{1,\sigma_j} \leq C\,h_j\,|\eta^h|_{1,\sigma_j}\,|\chi^h|_{0,\sigma_j}. \tag{2.8b}$$

The result (2.8b) follows immediately from (2.6a,b). The inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)^h$ are naturally extended to vector functions. In addition to the above spatial discretisation, let $0 \equiv t_0 < t_1 < \cdots < t_{N-1} < t_N \equiv T$ be a partitioning of $[0, T]$ with time steps $\Delta t_n := t_n - t_{n-1}$, $n = 1, \ldots, N$, and $\Delta t := \max_{n=1,\ldots,N} \Delta t_n$. Before we define our scheme we assign to an element $\vec{X}^n \in [V_1^h]^2$ (the upper index referring to the time level n) a piecewise constant discrete unit tangent and normal by

$$\vec{\mathcal{T}}^n = \frac{\vec{X}_\rho^n}{|\vec{X}_\rho^n|}, \qquad \vec{\mathcal{V}}^n = (\vec{\mathcal{T}}^n)^\perp \quad \text{on } \sigma_j,\ j = 1, \ldots, J. \tag{2.9}$$

Our discretisation of (2.2) now reads: given $\vec{X}^{n-1} \in [V_1^h]^2$ and $W^{n-1} \in V_1^h$, find $\vec{X}^n \in [V_1^h]^2$ such that

$$\begin{aligned}\left(|\vec{X}_\rho^{n-1}|^2\left[\alpha\,D_t\vec{X}^n + (1-\alpha)\left(D_t\vec{X}^n \cdot \vec{\mathcal{V}}^{n-1}\right)\vec{\mathcal{V}}^{n-1}\right], \vec{\xi}^h\right)^h + \left(\vec{X}_\rho^n, \vec{\xi}_\rho^h\right)\\ = \left(|\vec{X}_\rho^{n-1}|^2\,f(W^{n-1})\,\vec{\mathcal{V}}^{n-1}, \vec{\xi}^h\right)^h \qquad \forall\ \vec{\xi}^h \in [V_1^h]^2.\end{aligned} \tag{2.10}$$

Here, and in what follows, we abbreviate $D_t a^n := \frac{a^n - a^{n-1}}{\Delta t_n}$. We next use the solution $\vec{X}^n$ of (2.10) in order to discretise (2.3). To do so, we define approximations V^n, Ψ^n of the normal and tangential velocities by

$$V^n = D_t \vec{X}^n \cdot \vec{\mathcal{V}}^n \quad \text{and} \quad \Psi^n = D_t \vec{X}^n \cdot \vec{\mathcal{T}}^n \quad \text{on } \sigma_j,\ j = 1, \dots, J. \tag{2.11}$$

Then find $W^n \in V_1^h$ such that

$$\begin{aligned} D_t \left[\left(|\vec{X}_\rho^n|\, W^n, \eta^h \right)^h \right] + d \left(\frac{W_\rho^n}{|\vec{X}_\rho^n|}, \eta_\rho^h \right) + \left(\Psi^n\, W^n, \eta_\rho^h \right)^h \\ = \left(|\vec{X}_\rho^n|\, g(V^n, W^{n-1}), \eta^h \right)^h \qquad \forall\ \eta^h \in V_1^h. \end{aligned} \tag{2.12}$$

Let us formulate the main result of this paper, which will be proved in Section 3.

THEOREM 2.1. *Let $\vec{X}^0 = I^h \vec{x}^0 \in [V_1^h]^2$ and $W^0 = I^h w^0 \in V_1^h$. There exists $h^\star > 0$ such that for all $h \in (0, h^\star]$ and $\Delta t \le C\, h$ the discrete problem (2.10), (2.12) has a unique solution $(\vec{X}^n, W^n) \in [V_1^h]^2 \times V_1^h$, $n = 1, \dots, N$, and the following error bounds hold:*

$$\begin{aligned} \sup_{n=0,\dots,N} \left[|\vec{x}^n - \vec{X}^n|_1^2 + |w^n - W^n|_0^2 \right] \\ + \sum_{n=1}^{N} \Delta t_n \left[\left| \vec{x}_t^n - D_t \vec{X}^n \right|_0^2 + |w^n - W^n|_1^2 \right] \le C\, h^2, \end{aligned} \tag{2.13}$$

where $\vec{x}^n := \vec{x}(\cdot, t_n)$, $w^n := w(\cdot, t_n)$, $\vec{x}_t^n := \vec{x}_t(\cdot, t_n)$, $n = 0, \dots, N$.

3. Error analysis. To begin, it follows from (2.1b–d) and (1.4) that for $n = 0, \dots, N$

$$\|\vec{x}^n\|_2 + \|\vec{x}_t^n\|_1 + \|\vec{\tau}^n\|_1 + \|v^n\|_1 + \|\psi^n\|_1 + \|w^n\|_2 \le C, \tag{3.1}$$

where $\vec{\tau}^n := \vec{\tau}(\cdot, t_n)$, $v^n := v(\cdot, t_n)$, $\psi^n := \psi(\cdot, t_n)$. We abbreviate for $n = 0, \dots, N$

$$\vec{E}^n := I^h \vec{x}^n - \vec{X}^n \in [V_1^h]^2 \qquad \text{and} \qquad Z^n := I^h w^n - W^n \in V_1^h. \tag{3.2}$$

Our proof of Theorem 2.1 will be based on induction. We assume for an $n \in \{1, \dots, N\}$ that

$$|\vec{E}^{n-1}|_1^2 + \beta^2 \left(|\vec{X}_\rho^{n-1}|\, Z^{n-1}, Z^{n-1} \right)^h \le h^2\, e^{\gamma \int_0^{t_{n-1}} \zeta(t)\, \mathrm{d}t}, \qquad \text{for } \ h \in (0, h^\star], \tag{3.3}$$

where the function ζ is defined by

$$\zeta(t) := 1 + \|\vec{x}_{tt}(t)\|_1^2 + |w_{tt}(t)|_0^2 \tag{3.4}$$

and $h^\star > 0$ is chosen so small that

$$(h^\star)^{\frac{1}{2}}\, e^{\gamma K} \le \beta \quad \text{with} \quad K := \int_0^T \zeta(t)\, \mathrm{d}t. \tag{3.5}$$

Here, $\beta \in (0, 1]$ and $\gamma > 0$ are independent of h and Δt, and will be chosen a posteriori. Clearly, (3.3) holds for $n = 1$ in view of our choice of initial data for $\vec{X}^0$ and W^0. It follows from (2.4), (2.6c), (3.3), (3.1) and (3.5) that

$$\begin{aligned} |\vec{x}^{n-1} - \vec{X}^{n-1}|_{1,\infty} \le |\vec{E}^{n-1}|_{1,\infty} + |(I - I^h)\vec{x}^{n-1}|_{1,\infty} \le C\, h^{-\frac{1}{2}}\, |\vec{E}^{n-1}|_1 + C\, h^{\frac{1}{2}}\, |\vec{x}^{n-1}|_2 \\ \le C h^{\frac{1}{2}} \left(e^{\frac{\gamma K}{2}} + 1 \right) \le C\, (h^\star)^{\frac{1}{2}} e^{\gamma K} \le C\, \beta \le \min \left\{ \frac{m}{2}, M \right\}, \end{aligned} \tag{3.6}$$

provided that β is chosen small enough. Combining this inequality with (2.1d) we infer that

$$0 < \frac{m}{2} \le |\vec{X}_\rho^{n-1}| \le 2\,M \qquad \text{on } \mathbb{I} \text{ for } h \in (0, h^\star]. \tag{3.7}$$

If we use (3.7) and argue similarly as in (3.6) we further obtain for any $h \in (0, h^\star]$

$$|\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|_0 + |\vec{\nu}^{n-1} - \vec{\mathcal{V}}^{n-1}|_0 \le C\,|\vec{x}^{n-1} - \vec{X}^{n-1}|_1 \le C\left[|\vec{E}^{n-1}|_1 + h\right], \tag{3.8a}$$

$$\begin{aligned} |\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|_{0,\infty} + |\vec{\nu}^{n-1} - \vec{\mathcal{V}}^{n-1}|_{0,\infty} &\le C\,|\vec{x}^{n-1} - \vec{X}^{n-1}|_{1,\infty} \\ &\le C\,h^{-\frac{1}{2}}\left[|\vec{E}^{n-1}|_1 + h\right]. \end{aligned} \tag{3.8b}$$

In the same way as (3.6), we obtain from (2.8a), (3.7), (3.3) and (3.5) that

$$|Z^{n-1}|_0^2 \le C\left(|\vec{X}_\rho^{n-1}|\,Z^{n-1}, Z^{n-1}\right)^h \le \frac{C}{\beta^2}\,h^2\,e^{\gamma K} \le \frac{C}{\beta^2}\,h\,h^\star e^{2\gamma K} \le C\,h, \tag{3.9}$$

which combined with (2.6a,c), (2.4) and (3.1) yields for $h \in (0, h^\star]$

$$|W^{n-1}|_{0,\infty} \le |Z^{n-1}|_{0,\infty} + |I^h w^{n-1}|_{0,\infty} \le C\,h^{-\frac{1}{2}}|Z^{n-1}|_0 + C \le C. \tag{3.10}$$

3.1. The curve equation. We assume throughout that $h \in (0, h^\star]$. Since (2.10) forms a linear problem it is easily seen that (3.7) implies the existence of a unique solution $\vec{X}^n \in [V_1^h]^2$ to (2.10). We deduce from (2.10) and (2.2) with $\vec{\xi} = \vec{\xi}^h = \Delta t_n\,D_t\vec{E}^n \in [V_1^h]^2$ that

$$\begin{aligned} LHS &:= \Delta t_n \left(|\vec{X}_\rho^{n-1}|^2 \left[\alpha\,D_t\vec{E}^n + (1-\alpha)\left(D_t\vec{E}^n \cdot \vec{\mathcal{V}}^{n-1}\right)\vec{\mathcal{V}}^{n-1}\right], D_t\vec{E}^n\right)^h \\ &\quad + \Delta t_n \left(\vec{E}_\rho^n, (D_t\vec{E}^n)_\rho\right) \\ &= \Delta t_n \left(|\vec{X}_\rho^{n-1}|^2 \left[\alpha\,D_t\vec{x}^n + (1-\alpha)\left(D_t\vec{x}^n \cdot \vec{\mathcal{V}}^{n-1}\right)\vec{\mathcal{V}}^{n-1}\right], D_t\vec{E}^n\right)^h \\ &\quad + \Delta t_n \left(\vec{x}_\rho^n, (D_t\vec{E}^n)_\rho\right) - \Delta t_n \left(|\vec{X}_\rho^{n-1}|^2\,f(W^{n-1})\,\vec{\mathcal{V}}^{n-1}, D_t\vec{E}^n\right)^h \\ &= \Delta t_n \Bigg[\left(|\vec{X}_\rho^{n-1}|^2 \left[\alpha\,D_t\vec{x}^n + (1-\alpha)\left(D_t\vec{x}^n \cdot \vec{\mathcal{V}}^{n-1}\right)\vec{\mathcal{V}}^{n-1}\right], D_t\vec{E}^n\right)^h \\ &\quad - \left(|\vec{x}_\rho^n|^2\left[\alpha\,\vec{x}_t(t_n) + (1-\alpha)\left(\vec{x}_t(t_n)\cdot\vec{\nu}^n\right)\vec{\nu}^n\right], D_t\vec{E}^n\right)\Bigg] \\ &\quad + \Delta t_n \left[\left(|\vec{x}_\rho^n|^2\,f(w^n)\,\vec{\nu}^n, D_t\vec{E}^n\right) - \left(|\vec{X}_\rho^{n-1}|^2\,f(W^{n-1})\,\vec{\mathcal{V}}^{n-1}, D_t\vec{E}^n\right)^h\right] \\ &=: A_1 + A_2. \end{aligned} \tag{3.11}$$

Using (3.7) and (2.8a) we find, with the help of an elementary calculation, that

$$LHS \ge \Delta t_n\,\alpha\,\frac{m^2}{4}\left|D_t\vec{E}^n\right|_0^2 + \frac{1}{2}\left[|\vec{E}^n|_1^2 + |\vec{E}^n - \vec{E}^{n-1}|_1^2 - |\vec{E}^{n-1}|_1^2\right]. \tag{3.12}$$

Let us analyse the A_1 term defined in (3.11) and note that

$$\begin{aligned}
A_1 = \Delta t_n \Big[\Big(|\vec{X}_\rho^{n-1}|^2 \left[\alpha\, D_t(I^h \vec{x}^n) + (1-\alpha) \left(D_t(I^h \vec{x}^n) \cdot \vec{\mathcal{V}}^{n-1} \right) \vec{\mathcal{V}}^{n-1} \right], D_t \vec{E}^n \Big)^h \\
- \Big(|\vec{X}_\rho^{n-1}|^2 \left[\alpha\, D_t(I^h \vec{x}^n) + (1-\alpha) \left(D_t(I^h \vec{x}^n) \cdot \vec{\mathcal{V}}^{n-1} \right) \vec{\mathcal{V}}^{n-1} \right], D_t \vec{E}^n \Big) \Big] \\
+ \Delta t_n \Big[\alpha \Big(|\vec{X}_\rho^{n-1}|^2 \left[D_t(I^h \vec{x}^n) - \vec{x}_t^n \right], D_t \vec{E}^n \Big) \\
+ (1-\alpha) \Big(|\vec{X}_\rho^{n-1}|^2 \left(\left[D_t(I^h \vec{x}^n) - \vec{x}_t^n \right] \cdot \vec{\mathcal{V}}^{n-1} \right) \vec{\mathcal{V}}^{n-1}, D_t \vec{E}^n \Big) \Big] \\
+ \Delta t_n \Big[(1-\alpha) \Big(|\vec{X}_\rho^{n-1}|^2 \left[\left(\vec{x}_t^n \cdot \vec{\mathcal{V}}^{n-1} \right) \vec{\mathcal{V}}^{n-1} - (\vec{x}_t^n \cdot \vec{\nu}^n)\, \vec{\nu}^n \right], D_t \vec{E}^n \Big) \\
+ \Big(\left[|\vec{X}_\rho^{n-1}|^2 - |\vec{x}_\rho^n|^2 \right] \left[\alpha\, \vec{x}_t^n + (1-\alpha)\, (\vec{x}_t^n \cdot \vec{\nu}^n)\, \vec{\nu}^n \right], D_t \vec{E}^n \Big) \Big] =: \sum_{i=1}^{3} A_{1,i}.
\end{aligned} \tag{3.13}$$

We now bound the terms $A_{1,i}$ defined in (3.13) on recalling (2.1b,d), (3.7), (1.3) and (2.9). It follows from (2.8b) and (2.6b) that

$$|A_{1,1}| \leq C\, h\, \Delta t_n\, |D_t(I^h \vec{x}^n)|_1\, |D_t \vec{E}^n|_0 \leq C\, h\, \Delta t_n\, |D_t \vec{E}^n|_0, \tag{3.14}$$

since $|D_t(I^h \vec{x}^n)|_1 \leq |D_t \vec{x}^n|_1 \leq \|\vec{x}_t\|_{L^\infty(0,T;[H^1(\mathbb{I})]^2)} \leq C$. Similarly,

$$\begin{aligned}
|A_{1,2}| &\leq C\, \Delta t_n \left[|(I - I^h) D_t \vec{x}^n|_0 + |D_t \vec{x}^n - \vec{x}_t^n|_0 \right] |D_t \vec{E}^n|_0 \\
&\leq C\, \Delta t_n \left[h + \left(\Delta t_n \int_{t_{n-1}}^{t_n} |\vec{x}_{tt}|_0^2 \, \mathrm{d}t \right)^{\frac{1}{2}} \right] |D_t \vec{E}^n|_0.
\end{aligned} \tag{3.15}$$

Next, we have from Sobolev embedding, (2.6b) and (3.1) that

$$\begin{aligned}
|A_{1,3}| &\leq C\, \Delta t_n \left[|\vec{\nu}^n - \vec{\mathcal{V}}^{n-1}|_0 + \left| |\vec{x}_\rho^n|^2 - |\vec{X}_\rho^{n-1}|^2 \right|_0 \right] |D_t \vec{E}^n|_0 \\
&\leq C\, \Delta t_n \left[\Delta t_n\, |D_t \vec{x}^n|_1 + |\vec{x}^{n-1} - \vec{X}^{n-1}|_1 \right] |D_t \vec{E}^n|_0 \\
&\leq C\, \Delta t_n \left[h + |\vec{E}^{n-1}|_1 \right] |D_t \vec{E}^n|_0,
\end{aligned} \tag{3.16}$$

since $\Delta t_n \leq Ch$. We now turn our attention to the A_2 term in (3.11) and note that

$$\begin{aligned}
A_2 = \Delta t_n \left(|\vec{x}_\rho^n|^2\, f(w^n) \left(\vec{\nu}^n - \vec{\mathcal{V}}^{n-1} \right) + \left(|\vec{x}_\rho^n|^2 - |\vec{X}_\rho^{n-1}|^2 \right) f(w^n)\, \vec{\mathcal{V}}^{n-1}, D_t \vec{E}^n \right) \\
+ \Delta t_n \Big[\left(|\vec{X}_\rho^{n-1}|^2\, f(W^{n-1})\, \vec{\mathcal{V}}^{n-1}, D_t \vec{E}^n \right) \\
- \left(|\vec{X}_\rho^{n-1}|^2\, f(W^{n-1})\, \vec{\mathcal{V}}^{n-1}, D_t \vec{E}^n \right)^h \Big] \\
+ \Delta t_n \left(|\vec{X}_\rho^{n-1}|^2 \left(f(w^n) - f(W^{n-1}) \right) \vec{\mathcal{V}}^{n-1}, D_t \vec{E}^n \right) =: \sum_{i=1}^{3} A_{2,i}.
\end{aligned} \tag{3.17}$$

We bound the terms $A_{2,i}$ defined in (3.17) on recalling (2.1a–d), (3.7), (3.10), (1.3) and (2.9). Since $|f(w^n)|_{0,\infty} \leq C$, it follows similarly to (3.16) that

$$|A_{2,1}| \leq C\, \Delta t_n\, |\vec{x}^n - \vec{X}^{n-1}|_1\, |D_t \vec{E}^n|_0 \leq C\, \Delta t_n \left[h + |\vec{E}^{n-1}|_1 \right] |D_t \vec{E}^n|_0. \tag{3.18}$$

Next, we have from (2.7), (2.8b), (2.6a,b), (2.4) and (3.1) that

$$
\begin{aligned}
|A_{2,2}| &\le C\,\Delta t_n\,\big[\,|(I-I^h)f(W^{n-1})|_0 + h\,|I^h[f(W^{n-1})]|_1\big]\,|D_t\vec{E}^n|_0\\
&\le C\,\Delta t_n\,h\,|W^{n-1}|_1\,|D_t\vec{E}^n|_0 \le C\,\Delta t_n\,h\,\big[\,|I^h w^{n-1}|_1 + |Z^{n-1}|_1\big]\,|D_t\vec{E}^n|_0\\
&\le C\,\Delta t_n\,\big[\,h + |Z^{n-1}|_0\big]\,|D_t\vec{E}^n|_0. \qquad (3.19)
\end{aligned}
$$

Finally, it follows from (2.6b), (3.1) and as $\Delta t_n \le Ch$ that

$$
\begin{aligned}
|A_{2,3}| &\le C\,\Delta t_n\,|f(w^n)-f(W^{n-1})|_0\,|D_t\vec{E}^n|_0 \le C\,\Delta t_n\,|w^n - W^{n-1}|_0\,|D_t\vec{E}^n|_0\\
&\le C\,\Delta t_n\,\big[\Delta t_n + h\,|w^{n-1}|_1 + |Z^{n-1}|_0\big]\;|D_t\vec{E}^n|_0\\
&\le C\,\Delta t_n\,\big[h + |Z^{n-1}|_0\big]\,|D_t\vec{E}^n|_0. \qquad (3.20)
\end{aligned}
$$

If we combine (3.13)–(3.20) with Young's inequality and the definition of ζ we infer, as $\Delta t_n \le C\,h$, that

(3.21)

$$
|A_1 + A_2| \le \delta\,\Delta t_n\,|D_t\vec{E}^n|_0^2 + C(\delta)\left[\Delta t_n\left(|\vec{E}^{n-1}|_1^2 + |Z^{n-1}|_0^2\right) + h^2\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t\right].
$$

Inserting (3.12) and (3.21) into (3.11) and choosing δ sufficiently small we obtain with the help of (3.7) and (2.8a)

$$
\begin{aligned}
&\Delta t_n\,\alpha\,\frac{m^2}{4}\left|D_t\vec{E}^n\right|_0^2 + |\vec{E}^n|_1^2 + |\vec{E}^n - \vec{E}^{n-1}|_1^2\\
&\le |\vec{E}^{n-1}|_1^2 + C\,\Delta t_n\left[|\vec{E}^{n-1}|_1^2 + (|\vec{X}_\rho^{n-1}|\,Z^{n-1}, Z^{n-1})^h\right] + C\,h^2\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t. \qquad (3.22)
\end{aligned}
$$

The induction hypothesis (3.3) together with the fact that $\zeta \ge 1$ then yields

$$
\begin{aligned}
&\Delta t_n\,\alpha\,\frac{m^2}{4}\left|D_t\vec{E}^n\right|_0^2 + |\vec{E}^n|_1^2 + |\vec{E}^n - \vec{E}^{n-1}|_1^2\\
&\qquad\le h^2\,e^{\gamma\int_0^{t_{n-1}}\zeta(t)\,\mathrm{d}t}\left[1 + C\left(1+\frac{1}{\beta^2}\right)\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t\right] \qquad (3.23)
\end{aligned}
$$

so that

$$
|\vec{E}^n|_1^2 \le h^2\,e^{\gamma\int_0^{t_n}\zeta(t)\,\mathrm{d}t} \quad \text{provided that} \quad \gamma \ge C\left(1+\frac{1}{\beta^2}\right). \qquad (3.24)
$$

In particular, (3.23), this choice of γ and (3.5) imply that

$$
\Delta t_n\,\alpha\,\frac{m^2}{4}\left|D_t\vec{E}^n\right|_0^2 + |\vec{E}^n|_1^2 + |\vec{E}^n - \vec{E}^{n-1}|_1^2 \le h^2\,e^{\gamma K} \le h^{\frac{3}{2}}\,(h^\star)^{\frac{1}{2}}\,e^{\gamma K} \le \beta\,h^{\frac{3}{2}}. \qquad (3.25)
$$

In the same way as in (3.6)–(3.8a,b) we infer from (3.24) for $h \in (0, h^\star]$

$$
0 < \frac{m}{2} \le |\vec{X}_\rho^n| \le 2\,M \quad \text{on } \mathbb{I}, \qquad (3.26a)
$$

(3.26b)

$$
|\vec{\tau}^n - \vec{\mathcal{T}}^n|_0 + |\vec{\nu}^n - \vec{\mathcal{V}}^n|_0 + h^{\frac{1}{2}}\left(|\vec{\tau}^n - \vec{\mathcal{T}}^n|_{0,\infty} + |\vec{\nu}^n - \vec{\mathcal{V}}^n|_{0,\infty}\right) \le C\left[|\vec{E}^n|_1 + h\right].
$$

In addition, we deduce from (1.4), (2.11), (3.1), (3.26b), (2.1b) and as $\Delta t_n \le C\,h$ that

$$
\begin{aligned}
&|v^n - V^n|_0 + |\psi^n - \Psi^n|_0 \\
&\quad \le |\vec{x}_t^n \cdot (\vec{\tau}^n - \vec{\mathcal{T}}^n)|_0 + |\vec{x}_t^n \cdot (\vec{\nu}^n - \vec{\mathcal{V}}^n)|_0 + 2\left|\vec{x}_t^n - D_t(I^h\vec{x}^n)\right|_0 + 2\left|D_t\vec{E}^n\right|_0 \\
&\quad \le C\left[|\vec{E}^n|_1 + h\right] + C\,\Delta t_n\,\|\vec{x}_{tt}\|_{L^\infty(t_{n-1},t_n;[L^2(\mathbb{I})]^2)} + 2\left|D_t\vec{E}^n\right|_0 \\
&\quad \le C\left[|\vec{E}^n|_1 + h + \left|D_t\vec{E}^n\right|_0\right], \qquad (3.27)
\end{aligned}
$$

where the term $\left|\vec{x}_t^n - D_t(I^h\vec{x}^n)\right|_0$ is bounded as in (3.15). Furthermore, we conclude from (2.6a,b), (2.4), (3.27), (3.1), (3.25) and as $\Delta t_n \le C\,h$ that

$$
\begin{aligned}
\Delta t_n\,|\Psi^n|_{0,\infty}^2 &\le 2\,\Delta t_n\left[|I^h\psi^n|_{0,\infty}^2 + |I^h\psi^n - \Psi^n|_{0,\infty}^2\right] \\
&\le 2\,\Delta t_n\left[|\psi^n|_{0,\infty}^2 + C\,h^{-1}\,|I^h\psi^n - \Psi^n|_0^2\right] \\
&\le 2\,\Delta t_n\,|\psi^n|_{0,\infty}^2 + C\,\Delta t_n\,h^{-1}\left[|\psi^n - \Psi^n|_0^2 + |(I - I^h)\psi^n|_0^2\right] \\
&\le C\,h\,\|\psi^n\|_1^2 + C\,\Delta t_n\,h^{-1}\left[|\vec{E}^n|_1^2 + h^2\right] + C\,h^{-1}\Delta t_n\left|D_t\vec{E}^n\right|_0^2 \\
&\le C\,h^{\frac{1}{2}} < 4\,d \qquad (3.28)
\end{aligned}
$$

for $h \in (0, h^\star]$, provided that $h^\star$ is small enough. Finally, on noting (2.6a,b), (2.4), (2.1b) and that $|D_t(I_h\vec{x}^n)|_1 \le C$ as in (3.14), we have

$$
|\vec{X}^n - \vec{X}^{n-1}|_1 \le C\,\Delta t_n\left[|D_t\vec{E}^n|_1 + |D_t(I_h\vec{x}^n)|_1\right] \le C\,\Delta t_n\left[h^{-1}\,|D_t\vec{E}^n|_0 + 1\right]. \qquad (3.29)
$$

3.2. The scalar equation. We assume throughout that $h \in (0, h^\star]$. Let us first establish the existence and uniqueness of $W^n \in V_1^h$. Since the system (2.12) is linear, existence follows from uniqueness which in turn is a consequence of the estimate

$$
\begin{aligned}
\left|\left(\Psi^n\,\eta^h, \eta_\rho^h\right)^h\right| &\le d\left(\frac{\eta_\rho^h}{|\vec{X}_\rho^n|}, \eta_\rho^h\right) + \frac{1}{4\,d}\left(|\vec{X}_\rho^n|\,(\Psi^n)^2\,\eta^h, \eta^h\right)^h \\
&< d\left(\frac{\eta_\rho^h}{|\vec{X}_\rho^n|}, \eta_\rho^h\right) + \frac{1}{\Delta t_n}\left(|\vec{X}_\rho^n|\,\eta^h, \eta^h\right)^h
\end{aligned}
$$

for all non-trivial $\eta^h \in V_1^h$, which follows from (3.28). We deduce from (2.12) with $\eta^h = Z^n$ that

$$
\begin{aligned}
&\left(D_t\left[|\vec{X}_\rho^n|\,Z^n\right], Z^n\right)^h + d\left(\frac{Z_\rho^n}{|\vec{X}_\rho^n|}, Z_\rho^n\right) = \left(D_t\left[|\vec{X}_\rho^n|\,I^h w^n\right], Z^n\right)^h \\
&\qquad + d\left(\frac{(I^h w^n)_\rho}{|\vec{X}_\rho^n|}, Z_\rho^n\right) + \left(\Psi^n\,W^n, Z_\rho^n\right)^h - \left(|\vec{X}_\rho^n|\,g(V^n, W^{n-1}), Z^n\right)^h. \qquad (3.30)
\end{aligned}
$$

Observing that for all a^{n-1}, a^n, b^n, $b^{n-1} \in \mathbb{R}$

$$
D_t[a^n\,b^n]\,b^n = \tfrac{1}{2}D_t[a^n\,(b^n)^2] + \tfrac{1}{2\Delta t_n}\,a^{n-1}\,(b^n - b^{n-1})^2 + \tfrac{1}{2\Delta t_n}\,(a^n - a^{n-1})\,(b^n)^2
$$

and multiplying (3.30) by Δt_n we obtain with the help of (2.3) and (3.26a)

$$
\begin{aligned}
&\tfrac12\,(|\vec X_\rho^n|\,Z^n,Z^n)^h+\frac{d}{2M}\,\Delta t_n\,|Z^n|_1^2+\tfrac12\,(|\vec X_\rho^{n-1}|\,(Z^n-Z^{n-1}),(Z^n-Z^{n-1}))^h\\
&\le \tfrac12\,(|\vec X_\rho^{n-1}|\,Z^{n-1},Z^{n-1})^h-\tfrac12\,\left(\left[|\vec X_\rho^n|-|\vec X_\rho^{n-1}|\right]Z^n,Z^n\right)^h\\
&\quad+\left[\Delta t_n\,\left(D_t[|\vec X_\rho^n|\,I^h w^n],Z^n\right)^h-\Delta t_n\,\left((|\vec x_\rho|\,w)_t\,(t_n),Z^n\right)\right]\\
&\quad+\Delta t_n\,d\left(\frac{(I^h w^n)_\rho}{|\vec X_\rho^n|}-\frac{w_\rho^n}{|\vec x_\rho^n|},Z_\rho^n\right)+\Delta t_n\left[\left(\Psi^n\,(I^h w^n),Z_\rho^n\right)^h-\left(\psi^n\,w^n,Z_\rho^n\right)\right]\\
&\quad-\Delta t_n\,\left(\Psi^n\,Z^n,Z_\rho^n\right)^h+\Delta t_n\left[\left(|\vec x_\rho^n|\,g(v^n,w^n),Z^n\right)-\left(|\vec X_\rho^n|\,g(V^n,W^{n-1}),Z^n\right)^h\right]\\
&=:\tfrac12\,(|\vec X_\rho^{n-1}|\,Z^{n-1},Z^{n-1})^h+\sum_{1=1}^{6}B_i. \qquad (3.31)
\end{aligned}
$$

We now estimate the B_i terms defined in (3.31). In order to treat B_1 we first observe that

$$|\vec X_\rho^n|-|\vec X_\rho^{n-1}|=\vec{\mathcal{T}}^{n-1}\cdot(\vec X_\rho^n-\vec X_\rho^{n-1})+\tfrac12\,|\vec X_\rho^n|\,|\vec{\mathcal{T}}^n-\vec{\mathcal{T}}^{n-1}|^2,$$

so that (2.8b), (2.6a), (2.4) and integration by parts imply

$$
\begin{aligned}
2B_1&\le-\left(\vec{\mathcal{T}}^{n-1}\cdot(\vec X_\rho^n-\vec X_\rho^{n-1})\,Z^n,Z^n\right)^h\\
&\le-\left(\vec{\mathcal{T}}^{n-1}\cdot(\vec X_\rho^n-\vec X_\rho^{n-1})\,Z^n,Z^n\right)+C\,h\,|\vec X^n-\vec X^{n-1}|_1\,|Z^n|_{0,\infty}\,|Z^n|_1\\
&=\left((\vec\tau^{n-1}-\vec{\mathcal{T}}^{n-1})\cdot(\vec X_\rho^n-\vec X_\rho^{n-1})\,Z^n,Z^n\right)+\left(\vec\tau_\rho^{n-1}\cdot(\vec X^n-\vec X^{n-1})\,Z^n,Z^n\right)\\
&\quad+2\left(\vec\tau^{n-1}\cdot(\vec X^n-\vec X^{n-1})\,Z^n,Z_\rho^n\right)+C\,h\,|\vec X^n-\vec X^{n-1}|_1\,|Z^n|_{0,\infty}\,|Z^n|_1.
\end{aligned}
$$

Using (2.6a), (2.4), (3.8a), (3.29), (3.1), the Gagliardo–Nirenberg interpolation inequality $|\eta|_{0,\infty}^2\le C\,|\eta|_0\,\|\eta\|_1$, (3.14) and the fact that $|\vec E^{n-1}|_1\le h^{\frac34}$ (cf. (3.3) and (3.5)) we infer that

$$
\begin{aligned}
B_1&\le C\left[|\vec\tau^{n-1}-\vec{\mathcal{T}}^{n-1}|_0\,|\vec X^n-\vec X^{n-1}|_1\,|Z^n|_{0,\infty}+|\vec X^n-\vec X^{n-1}|_0\,\|Z^n\|_1\right]|Z^n|_{0,\infty}\\
&\le C\,\Delta t_n\left[|\vec E^{n-1}|_1+h\right]\left[h^{-1}|D_t\vec E^n|_0+1\right]|Z^n|_0\|Z^n\|_1\\
&\quad+C\,\Delta t_n\,|D_t(I^h\vec x^n)|_0\,|Z^n|_0^{\frac12}\,\|Z^n\|_1^{\frac32}+C\,\Delta t_n\,|D_t\vec E^n|_0\,h^{-\frac12}\,|Z^n|_0\|Z^n\|_1\\
&\le C\,\Delta t_n\left[h^{-\frac12}\,|D_t\vec E^n|_0+h^{\frac34}\right]|Z^n|_0\,\|Z^n\|_1+C\,\Delta t_n\,|Z^n|_0^{\frac12}\,\|Z^n\|_1^{\frac32}.
\end{aligned}
\qquad (3.32)
$$

Let us postpone the rather complicated analysis of B_2 and first deal with $B_3,\dots,B_6$. To bound B_3, we first note that

$$\left(\frac{(w^n-I^h w^n)_\rho}{|\vec X_\rho^n|},Z_\rho^n\right)=0.$$

It then follows from (2.1b–d), (3.26a), (2.6b) and Sobolev embedding that

$$B_3 = \Delta t_n\, d\left(\frac{\left[|\vec{x}_\rho^n| - |\vec{X}_\rho^n|\right]}{|\vec{x}_\rho^n|\,|\vec{X}_\rho^n|} w_\rho^n, Z_\rho^n\right) \le C\,\Delta t_n\,|\vec{x}^n - \vec{X}^n|_1\,|w^n|_{1,\infty}\,|Z^n|_1$$

$$\le C\,\Delta t_n\left[|\vec{E}^n|_1 + |(I-I^h)\vec{x}^n|_1\right]|Z^n|_1 \le C\,\Delta t_n\left[|\vec{E}^n|_1 + h\right]|Z^n|_1. \tag{3.33}$$

Next we note from (2.8a,b), (2.6a,b), (2.4), Sobolev embedding, (3.1) and (3.27) that

$$\begin{aligned} B_4 &= \Delta t_n\left((\Psi^n - I^h\psi^n)\,I^h w^n, Z_\rho^n\right)^h \\ &\quad + \Delta t_n\left[\left((I^h\psi^n)\,(I^h w^n), Z_\rho^n\right)^h - \left((I^h\psi^n)\,(I^h w^n), Z_\rho^n\right)\right] \\ &\quad + \Delta t_n\left((I^h w^n)\,(I^h - I)\psi^n + \psi^n\,(I^h - I)w^n, Z_\rho^n\right) \\ &\le C\,\Delta t_n\left[|I^h\psi^n - \Psi^n|_0\,|I^h w^n|_{0,\infty}\,|Z^n|_1 + h^2\,|I^h\psi^n|_1\,|I^h w^n|_1\,|Z^n|_{1,\infty}\right] \\ &\quad + C\,\Delta t_n\,h\left[\,|w^n|_{0,\infty}\,|\psi^n|_1 + |\psi^n|_{0,\infty}\,|w^n|_1\right]|Z^n|_1 \\ &\le C\,\Delta t_n\left[\left|D_t\vec{E}^n\right|_0 + |\vec{E}^n|_1 + h\right]|Z^n|_1. \end{aligned} \tag{3.34}$$

On noting (2.7), (2.8a), (2.6a,b), (2.4), Sobolev embedding, (3.1) and (3.27), we have that

$$\begin{aligned} B_5 &= -\Delta t_n\left(I^h\psi^n\,Z^n, Z_\rho^n\right)^h - \Delta t_n\left((\Psi^n - I^h\psi^n)\,Z^n, Z_\rho^n\right)^h \\ &\le C\,\Delta t_n\,|\psi^n|_{0,\infty}\,|Z^n|_0\,|Z^n|_1 + C\,\Delta t_n\,|\Psi^n - I^h\psi^n|_0\,|Z^n|_{0,\infty}\,|Z^n|_1 \\ &\le C\,\Delta t_n\,|Z^n|_0\,|Z^n|_1 + C\,\Delta t_n\left[|\vec{E}^n|_1 + \left|D_t\vec{E}^n\right|_0 + h\right]h^{-\frac{1}{2}}\,|Z^n|_0\,|Z^n|_1 \\ &\le C\,\Delta t_n\,|Z^n|_0\,|Z^n|_1\left[1 + h^{-\frac{1}{2}}\left(|\vec{E}^n|_1 + \left|D_t\vec{E}^n\right|_0\right)\right]. \end{aligned} \tag{3.35}$$

Finally, we deduce from (2.8a,b), the Cauchy–Schwarz inequality for $(\cdot,\cdot)^h$, (2.1a), (3.1) and (2.6b) that

$$\begin{aligned} B_6 &= \Delta t_n\left[\left(\left[|\vec{x}_\rho^n| - |\vec{X}_\rho^n|\right]\,g(v^n, w^n), Z^n\right) + \left(|\vec{X}_\rho^n|\,(I - I^h)g(v^n, w^n), Z^n\right)\right] \\ &\quad + \Delta t_n\left[\left(|\vec{X}_\rho^n|\,I^h\left[g(v^n, w^n)\right], Z^n\right) - \left(|\vec{X}_\rho^n|\,I^h\left[g(v^n, w^n)\right], Z^n\right)^h\right] \\ &\quad + \Delta t_n\left[\left(|\vec{X}_\rho^n|\left[g(I^h v^n, I^h w^n) - g(V^n, W^{n-1})\right], Z^n\right)^h\right] \\ &\le C\,\Delta t_n\left[|\vec{x}^n - \vec{X}^n|_1\,|g(v^n, w^n)|_{0,\infty} + |(I - I^h)g(v^n, w^n)|_0\right]|Z^n|_0 \\ &\quad + C\,\Delta t_n\left[h\,|I^h[g(v^n, w^n)]|_1 + \sqrt{(r^n, r^n)^h}\right]|Z^n|_0 \\ &\le C\,\Delta t_n\left[|\vec{x}^n - \vec{X}^n|_1 + h\left(|v^n|_1 + |w^n|_1\right)\right]|Z^n|_0 \\ &\quad + C\,\Delta t_n\left[h\left(|I^h v^n|_1 + |I^h w^n|_1\right) + \sqrt{(r^n, r^n)^h}\right]|Z^n|_0 \\ &\le C\,\Delta t_n\left[|\vec{E}^n|_1 + h + \sqrt{(r^n, r^n)^h}\right]|Z^n|_0, \end{aligned} \tag{3.36}$$

where we have set $r^n = g(I^h v^n, I^h w^n) - g(V^n, W^{n-1})$. Using (2.1a) together with the fact that $|W^{n-1}|_{0,\infty} \leq C$, recall (3.10), we infer that

$$\begin{aligned}
|r^n| &\leq |g(I^h v^n, I^h w^n) - g(I^h v^n, W^{n-1})| + |g(I^h v^n, W^{n-1}) - g(V^n, W^{n-1})| \\
&\leq C \left[|I^h w^n - W^{n-1}| + |I^h v^n - V^n| \right] \\
&\leq C \left[|I^h (w^n - w^{n-1})| + |Z^{n-1}| + |I^h v^n - V^n| \right],
\end{aligned}$$

from which we deduce that

$$\sqrt{(r^n, r^n)^h} \leq C \big[|I^h (w^n - w^{n-1})|_0 + |Z^{n-1}|_0 + |I^h v^n - V^n|_0 \big]$$

on noting (2.7) and (2.8a). Inserting this bound into (3.36) and recalling (3.1), (2.1c), (2.6b), (3.27) as well as $\Delta t_n \leq Ch$ we have

$$B_6 \leq C\, \Delta t_n \left[|\vec{E}^n|_1 + |Z^{n-1}|_0 + \left| D_t \vec{E}^n \right|_0 + h \right] |Z^n|_0. \tag{3.37}$$

It remains to analyse B_2. We claim that:

$$\begin{aligned}
|B_2| \leq C\, \Delta t_n & \left[\left| D_t \vec{E}^n \right|_0 + |\vec{E}^n|_1 + |\vec{E}^{n-1}|_1 + h \right] \|Z^n\|_1 \\
& + C\, \Delta t_n \left| D_t \vec{E}^n \right|_0^2 h^{-\frac{1}{2}} |Z^n|_0 + C\, |Z^n|_{0,\infty} |\vec{E}^n - \vec{E}^{n-1}|_1^2 \\
& + C\, h \left(\Delta t_n \int_{t_{n-1}}^{t_n} \zeta(t) dt \right)^{\frac{1}{2}} \|Z^n\|_1.
\end{aligned} \tag{3.38}$$

In order to prove (3.38), we first write

$$\begin{aligned}
B_2 = & \left[\left(|\vec{X}_\rho^n|\, I^h w^n - |\vec{X}_\rho^{n-1}|\, I^h w^{n-1}, Z^n \right)^h - \left(|\vec{X}_\rho^n|\, I^h w^n - |\vec{X}_\rho^{n-1}|\, I^h w^{n-1}, Z^n \right) \right] \\
& + \left(|\vec{X}_\rho^n|\, (I^h - I)\, w^n - |\vec{X}_\rho^{n-1}|\, (I^h - I)\, w^{n-1}, Z^n \right) \\
& + \left[\left(|\vec{x}_\rho^n|\, w^n - |\vec{x}_\rho^{n-1}|\, w^{n-1}, Z^n \right) - \Delta t_n \left((|\vec{x}_\rho|\, w)_t\, (t_n), Z^n \right) \right] \\
& + \left(\left[|\vec{X}_\rho^n| - |\vec{x}_\rho^n| \right] w^n - \left[|\vec{X}_\rho^{n-1}| - |\vec{x}_\rho^{n-1}| \right] w^{n-1}, Z^n \right) =: \sum_{i=1}^4 B_{2,i}.
\end{aligned} \tag{3.39}$$

We now bound the $B_{2,i}$ terms defined in (3.39). It follows from (2.8b), (3.7), (2.6b), (3.1), (3.29) and as $w_t \in L^\infty(0,T; H^1(\mathbb{I}))$ that

$$\begin{aligned}
|B_{2,1}| &\leq C\, h\, | |\vec{X}_\rho^n|\, I^h w^n - |\vec{X}_\rho^{n-1}|\, I^h w^{n-1}|_0\, |Z^n|_1 \\
&\leq C\, h \left[|\vec{X}^n - \vec{X}^{n-1}|_1\, |w^n|_{0,\infty} + |I^h (w^n - w^{n-1})|_0 \right] |Z^n|_1 \\
&\leq C\, \Delta t_n \left[\left| D_t \vec{E}^n \right|_0 + h \right] |Z^n|_1.
\end{aligned} \tag{3.40}$$

Similarly, we deduce from (3.7), Sobolev embedding, (2.6b), (2.1c) and (3.29) that

$$\begin{aligned}
|B_{2,2}| &\leq C \left[|\vec{X}^n - \vec{X}^{n-1}|_1\, |(I - I^h) w^n|_{0,\infty} + |(I - I^h)(w^n - w^{n-1})|_0 \right] |Z^n|_0 \\
&\leq C\, h \left[|\vec{X}^n - \vec{X}^{n-1}|_1\, |w^n|_2 + |w^n - w^{n-1}|_1 \right] |Z^n|_0 \\
&\leq C\, \Delta t_n \left[\left| D_t \vec{E}^n \right|_0 + h \right] |Z^n|_0.
\end{aligned} \tag{3.41}$$

In view of (2.1b,c), we have that $\vec{x}_t \in L^\infty(0,T;[W^{1,\infty}(\mathbb{I})]^2)$, $w_t \in L^\infty(0,T;L^\infty(\mathbb{I}))$ and hence, as $\Delta t_n \leq C\,h$,

$$|B_{2,3}| \leq C\,\Delta t_n \left(\Delta t_n \int_{t_{n-1}}^{t_n} |(|\vec{x}_\rho|\,w)_{tt}|_0^2 \,\mathrm{d}t \right)^{\frac{1}{2}} |Z^n|_0$$

$$\leq C\,h \left(\Delta t_n \int_{t_{n-1}}^{t_n} \left[1 + |w_{tt}|_0^2 + |\vec{x}_{tt}|_1^2\right] \,\mathrm{d}t \right)^{\frac{1}{2}} |Z^n|_0. \tag{3.42}$$

It remains to analyse $B_{2,4}$, which requires some more intricate arguments. To begin,

$$\begin{aligned} B_{2,4} &= \left(\left[\vec{X}_\rho^n \cdot \vec{\mathcal{T}}^n - \vec{x}_\rho^n \cdot \vec{\tau}^n \right] w^n - \left[\vec{X}_\rho^{n-1} \cdot \vec{\mathcal{T}}^{n-1} - \vec{x}_\rho^{n-1} \cdot \vec{\tau}^{n-1} \right] w^{n-1}, Z^n \right) \\ &= \left(\vec{X}_\rho^n \cdot (\vec{\mathcal{T}}^n - \vec{\tau}^n)\, w^n - \vec{X}_\rho^{n-1} \cdot (\vec{\mathcal{T}}^{n-1} - \vec{\tau}^{n-1})\, w^{n-1}, Z^n \right) \\ &\quad + \left((\vec{X}^n - \vec{x}^n)_\rho \cdot \vec{\tau}^n\, w^n - (\vec{X}^{n-1} - \vec{x}^{n-1})_\rho \cdot \vec{\tau}^{n-1}\, w^{n-1}, Z^n \right) \\ &=: B_{2,4,1} + B_{2,4,2}. \end{aligned} \tag{3.43}$$

On recalling (2.9), we rewrite

$$\begin{aligned} B_{2,4,1} &= \tfrac{1}{2} \left(|\vec{X}_\rho^n|\,|\vec{\tau}^n - \vec{\mathcal{T}}^n|^2\, w^n - |\vec{X}_\rho^{n-1}|\,|\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|^2\, w^{n-1}, Z^n \right) \\ &= \tfrac{1}{2} \left(\left[|\vec{X}_\rho^n| - |\vec{X}_\rho^{n-1}| \right] |\vec{\tau}^n - \vec{\mathcal{T}}^n|^2\, w^n, Z^n \right) \\ &\quad + \tfrac{1}{2} \left(|\vec{X}_\rho^{n-1}|\,|\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|^2\, (w^n - w^{n-1}), Z^n \right) \\ &\quad + \tfrac{1}{2} \left(|\vec{X}_\rho^{n-1}| \left[|\vec{\tau}^n - \vec{\mathcal{T}}^n|^2 - |\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|^2 \right] w^n, Z^n \right) =: I + II + III. \end{aligned} \tag{3.44}$$

Noting (3.29), (3.26b), Sobolev embedding and (3.25), we obtain that

$$\begin{aligned} |I| &\leq C\,|\vec{X}^n - \vec{X}^{n-1}|_1\,|\vec{\tau}^n - \vec{\mathcal{T}}^n|_0\,|\vec{\tau}^n - \vec{\mathcal{T}}^n|_{0,\infty}\,|Z^n|_{0,\infty} \\ &\leq C\Delta t_n \left[h^{-1}\,|D_t \vec{E}^n|_0 + 1 \right] h^{-\frac{1}{2}} \left[|\vec{E}^n|_1 + h \right]^2 \|Z^n\|_1 \\ &\leq C\,\Delta t_n \left[\left| D_t \vec{E}^n \right|_0 + h \right] \|Z^n\|_1. \end{aligned} \tag{3.45}$$

Since $w_t \in L^\infty(0,T;L^\infty(\mathbb{I}))$ and $|\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|_{0,\infty} \leq 2$, we deduce from (3.7) and (3.8a) that

$$|II| \leq C\,\Delta t_n\,|\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|_0\,|Z^n|_0 \leq C\,\Delta t_n \left[|\vec{E}^{n-1}|_1 + h \right] |Z^n|_0. \tag{3.46}$$

In order to treat term III, we first observe that

$$\begin{aligned} |\vec{\tau}^n - \vec{\mathcal{T}}^n|^2 - |\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|^2 &= -(\vec{\tau}^n - \vec{\tau}^{n-1}) \cdot (\vec{\mathcal{T}}^n + \vec{\mathcal{T}}^{n-1}) \\ &\quad - (\vec{\mathcal{T}}^n - \vec{\mathcal{T}}^{n-1}) \cdot (\vec{\tau}^n + \vec{\tau}^{n-1}). \end{aligned} \tag{3.47}$$

Furthermore, a straightforward calculation shows that

$$\begin{aligned} \vec{\mathcal{T}}^n - \vec{\mathcal{T}}^{n-1} &= \frac{1}{|\vec{X}_\rho^{n-1}|} \left[I - \vec{\mathcal{T}}^n \otimes \frac{\vec{X}_\rho^n + \vec{X}_\rho^{n-1}}{|\vec{X}_\rho^n| + |\vec{X}_\rho^{n-1}|} \right] (\vec{X}_\rho^n - \vec{X}_\rho^{n-1}) \\ &= \frac{1}{|\vec{X}_\rho^{n-1}|} (P^n + R^n)\,(\vec{X}_\rho^n - \vec{X}_\rho^{n-1}), \end{aligned} \tag{3.48a}$$

where

$$P^n = I - \vec{\mathcal{T}}^n \otimes \vec{\mathcal{T}}^n \quad \text{and} \quad |R^n| \le C\,|\vec{X}_\rho^n - \vec{X}_\rho^{n-1}|. \tag{3.48b}$$

Inserting (3.47) and (3.48a,b) into III and observing that $(\vec{\tau}^n - \vec{\tau}^{n-1})\cdot(\vec{\tau}^n + \vec{\tau}^{n-1}) = 0$, we derive

$$\begin{aligned}
2\,III &= -\Big(|\vec{X}_\rho^{n-1}|\,(\vec{\tau}^n - \vec{\tau}^{n-1})\cdot(\vec{\mathcal{T}}^n + \vec{\mathcal{T}}^{n-1})\,w^n, Z^n\Big) \\
&\quad - \Big((P^n + R^n)\,(\vec{X}_\rho^n - \vec{X}_\rho^{n-1})\cdot(\vec{\tau}^n + \vec{\tau}^{n-1})\,w^n, Z^n\Big) \\
&= -\Big(|\vec{X}_\rho^{n-1}|\,(\vec{\tau}^n - \vec{\tau}^{n-1})\cdot(\vec{\mathcal{T}}^n - \vec{\tau}^n + \vec{\mathcal{T}}^{n-1} - \vec{\tau}^{n-1})\,w^n, Z^n\Big) \\
&\quad - \Big(\big[(I^h\vec{x}^n)_\rho - (I^h\vec{x}^{n-1})_\rho\big]\cdot\big(P^n + (R^n)^t\big)\,(\vec{\tau}^n + \vec{\tau}^{n-1})\,w^n, Z^n\Big) \\
&\quad + \Big(P^n(\vec{E}_\rho^n - \vec{E}_\rho^{n-1})\cdot(\vec{\tau}^n + \vec{\tau}^{n-1})\,w^n, Z^n\Big) \\
&\quad + \Big(R^n(\vec{E}_\rho^n - \vec{E}_\rho^{n-1})\cdot(\vec{\tau}^n + \vec{\tau}^{n-1})\,w^n, Z^n\Big) \\
&=: III_1 + III_2 + III_3 + III_4.
\end{aligned} \tag{3.49}$$

It follows from (3.7), (3.8a), (3.26b), Sobolev embedding and (2.1b,c) that

$$\begin{aligned}
|III_1| &\le |\vec{\tau}^n - \vec{\tau}^{n-1}|_{0,\infty}\Big[|\vec{\tau}^n - \vec{\mathcal{T}}^n|_0 + |\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|_0\Big]\,|w^n|_{0,\infty}\,|Z^n|_0 \\
&\le C\,\Delta t_n\Big[\,|\vec{E}^n|_1 + |\vec{E}^{n-1}|_1 + h\Big]\,|Z^n|_0,
\end{aligned} \tag{3.50}$$

as $\vec{x}_t \in L^\infty(t_{n-1}, t_n; [W^{1,\infty}(\mathbb{I})]^2)$. Recalling the definition of P^n we may write

$$P^n(\vec{\tau}^n + \vec{\tau}^{n-1}) = (\vec{\tau}^n - \vec{\mathcal{T}}^n) + (\vec{\tau}^{n-1} - \vec{\mathcal{T}}^n) + \tfrac{1}{2}\Big[|\vec{\tau}^n - \vec{\mathcal{T}}^n|^2 + |\vec{\tau}^{n-1} - \vec{\mathcal{T}}^n|^2\Big]\vec{\mathcal{T}}^n,$$

so, similarly to (3.50), we have from (3.48b), (3.8a), (3.26b), Sobolev embedding and (2.1b,c) that

$$\begin{aligned}
|III_2| &\le C\,\Delta t_n\,\Big[|\vec{\tau}^n - \vec{\mathcal{T}}^n|_0 + |\vec{\tau}^{n-1} - \vec{\mathcal{T}}^{n-1}|_0 + |\vec{X}^n - \vec{X}^{n-1}|_1\Big]\,|w^n|_{0,\infty}\,|Z^n|_0 \\
&\le C\,\Delta t_n\Big[\,|\vec{E}^n|_1 + |\vec{E}^{n-1}|_1 + h\Big]\,|Z^n|_0,
\end{aligned} \tag{3.51}$$

since $\Delta t_n \le C\,h$. Next, performing integration by parts on the subintervals σ_j we derive that

$$\begin{aligned}
III_3 &= -\Big(P^n(\vec{E}^n - \vec{E}^{n-1}), \big[(\vec{\tau}^n + \vec{\tau}^{n-1})\,w^n\big]_\rho\,Z^n + (\vec{\tau}^{n-1} + \vec{\tau}^n)\,w^n\,Z_\rho^n\Big) \\
&\quad - \sum_{j=1}^{J}\big(P^n\,|_{\sigma_{j+1}} - P^n\,|_{\sigma_j}\big)\,(\vec{E}^n - \vec{E}^{n-1})(\rho_j)\cdot(\vec{\tau}^n + \vec{\tau}^{n-1})\,w^n(\rho_j)\,Z^n(\rho_j),
\end{aligned} \tag{3.52}$$

where $\sigma_{J+1} \equiv \sigma_1$. Introducing the nodal basis functions $\chi_j \in V_1^h$ such that $\chi_j(\rho_i) = \delta_{i,j}$, $i, j = 1, \ldots, J$, and then choosing $\vec{\xi}^h = \chi_j\,\vec{\xi}$, for any $\vec{\xi} \in \mathbb{R}^2$, in (2.10) we obtain

$$\begin{aligned}
\Big[\vec{X}_\rho^n\,|_{\sigma_{j+1}} - \vec{X}_\rho^n\,|_{\sigma_j}\Big]\cdot\vec{\xi} &= -\Big(|\vec{X}_\rho^{n-1}|^2\,f(W^{n-1})\,\vec{\mathcal{V}}^{n-1}, \chi_j\,\vec{\xi}\Big)^h \\
&\quad + \Big(|\vec{X}_\rho^{n-1}|^2\Big[\alpha\,D_t\vec{X}^n + (1-\alpha)\Big(D_t\vec{X}^n\cdot\vec{\mathcal{V}}^{n-1}\Big)\vec{\mathcal{V}}^{n-1}\Big], \chi_j\,\vec{\xi}\Big)^h.
\end{aligned}$$

We deduce from this and (3.48b), with the help of (3.7), (3.10), Sobolev embedding and (2.1b), that for $j = 1, \dots, J$

$$\begin{aligned}\left|P^n\,|_{\sigma_{j+1}} - P^n\,|_{\sigma_j}\right| &\le C\left|\vec{X}^n_\rho\,|_{\sigma_{j+1}} - \vec{X}^n_\rho\,|_{\sigma_j}\right| \\ &\le C\,h\left[|f(W^{n-1})|_{0,\infty} + \left|(D_t\vec{X}^n)(\rho_j)\right|\right] \\ &\le C\,h\left[1 + |D_t\vec{x}^n|_{C(\mathbb{I})} + \left|(D_t\vec{E}^n)(\rho_j)\right|\right] \\ &\le C\,h\left[1 + \left|(D_t\vec{E}^n)(\rho_j)\right|\right].\end{aligned}$$

Inserting this estimate into (3.52) and recalling (3.48b), (2.1d), (2.4), (2.8a), Sobolev embedding, (3.1) and (2.6a), we deduce that

$$\begin{aligned}|III_3| &\le C\,\Delta t_n\left|D_t\vec{E}^n\right|_0\,\|Z^n\|_1 + C\,\Delta t_n\left|D_t\vec{E}^n\right|_0^2\,|Z^n|_{0,\infty}. \\ &\le C\,\Delta t_n\left|D_t\vec{E}^n\right|_0\,\|Z^n\|_1 + C\,\Delta t_n\left|D_t\vec{E}^n\right|_0^2\,h^{-\frac{1}{2}}\,|Z^n|_0. \qquad (3.53)\end{aligned}$$

Finally, we have, on noting (3.48b), (3.14), Sobolev embedding and (3.1), that

$$\begin{aligned}|III_4| &\le C\,|\vec{E}^n - \vec{E}^{n-1}|_1\left[|\vec{E}^n - \vec{E}^{n-1}|_1 + |I^h(\vec{x}^n - \vec{x}^{n-1})|_1\right]\,|w^n|_{0,\infty}\,|Z^n|_{0,\infty} \\ &\le C\,|Z^n|_{0,\infty}\,|\vec{E}^n - \vec{E}^{n-1}|_1^2 + C\,\Delta t_n\,|\vec{E}^n - \vec{E}^{n-1}|_1\,\|Z^n\|_1. \qquad (3.54)\end{aligned}$$

Combining (3.44)–(3.46), (3.49)–(3.51), (3.53) and (3.54), we conclude that

$$\begin{aligned}|B_{2,4,1}| &\le C\,\Delta t_n\left[\left|D_t\vec{E}^n\right|_0 + |\vec{E}^n|_1 + |\vec{E}^{n-1}|_1 + h\right]\|Z^n\|_1 \\ &\quad + C\,\Delta t_n\left|D_t\vec{E}^n\right|_0^2\,h^{-\frac{1}{2}}\,|Z^n|_0 + C\,|Z^n|_{0,\infty}\,|\vec{E}^n - \vec{E}^{n-1}|_1^2. \qquad (3.55)\end{aligned}$$

For the second term in (3.43) we obtain with the help of integration by parts that

$$\begin{aligned}B_{2,4,2} &= \Delta t_n\left(D_t[(\vec{X}^n - \vec{x}^n)_\rho \cdot \vec{\tau}^n\,w^n], Z^n\right) \\ &= \Delta t_n\left(D_t[(\vec{X}^n - \vec{x}^n)_\rho] \cdot \vec{\tau}^n w^n, Z^n\right) + \Delta t_n\left((\vec{X}^{n-1} - \vec{x}^{n-1})_\rho \cdot D_t[\vec{\tau}^n w^n], Z^n\right) \\ &= -\Delta t_n\left(D_t(\vec{X}^n - \vec{x}^n), (w^n\,Z^n\,\vec{\tau}^n)_\rho\right) + \Delta t_n\left((\vec{X}^{n-1} - \vec{x}^{n-1})_\rho \cdot D_t[\vec{\tau}^n w^n], Z^n\right).\end{aligned}$$

Similarly to (3.50), it follows from (2.6b), (2.1b–d), (3.1) and Sobolev embedding that

$$\begin{aligned}|B_{2,4,2}| &\le C\,\Delta t_n\,\left[|(I - I^h)D_t\vec{x}^n|_0 + \Delta t_n\left|D_t\vec{E}^n\right|_0\right]\,\|Z^n\|_1 \\ &\quad + C\,\Delta t_n\,\left[\,|(I - I^h)\vec{x}^{n-1}|_1 + |E^{n-1}|_1\right]\,|Z^n|_0 \\ &\le C\,\Delta t_n\left[\left|D_t\vec{E}^n\right|_0 + |\vec{E}^{n-1}|_1 + h\right]\|Z^n\|_1. \qquad (3.56)\end{aligned}$$

Thus, (3.43), (3.55) and (3.56) yield the bound on $B_{2,4}$, which together with (3.39)–(3.42) imply (3.38) on recalling (3.4).

Hence we have bounded all the terms on the right hand side of (3.31), so we obtain from (3.32)–(3.35), (3.37) and (3.38) that

$$
\begin{aligned}
&\tfrac{1}{2}\,(|\vec{X}^n_\rho|\,Z^n,Z^n)^h+\frac{d}{2M}\,\Delta t_n\,|Z^n|_1^2+\tfrac{1}{2}\,(|\vec{X}^{n-1}_\rho|\,(Z^n-Z^{n-1}),(Z^n-Z^{n-1})\,)^h\\
&\quad\le \tfrac{1}{2}\,(|\vec{X}^{n-1}_\rho|\,Z^{n-1},Z^{n-1})^h+C\,\Delta t_n\,h^{-\frac{1}{2}}\left[|\vec{E}^n|_1+\left|D_t\vec{E}^n\right|_0\right]|Z^n|_0\,\|Z^n\|_1\\
&\qquad+C\,\Delta t_n\left[|\vec{E}^n|_1+|\vec{E}^{n-1}|_1+|Z^{n-1}|_0+\left|D_t\vec{E}^n\right|_0+h\right]\|Z^n\|_1\\
&\qquad+\;C\,\Delta t_n\left|D_t\vec{E}^n\right|_0^2\,h^{-\frac{1}{2}}\,|Z^n|_0+C\,|Z^n|_{0,\infty}\,|\vec{E}^n-\vec{E}^{n-1}|_1^2\\
&\qquad+C\,h\left(\Delta t_n\int_{t_{n-1}}^{t_n}\zeta(t)dt\right)^{\frac{1}{2}}\|Z^n\|_1+C\,\Delta t_n\,|Z^n|_0^{\frac{1}{2}}\,\|Z^n\|_1^{\frac{3}{2}}\\
&\quad\le \tfrac{1}{2}\,(|\vec{X}^{n-1}_\rho|\,Z^{n-1},Z^{n-1})^h+\delta\,\Delta t_n\,|Z^n|_1^2+C(\delta)\,\Delta t_n\left[\,|Z^n|_0^2+|Z^{n-1}|_0^2\,\right]\\
&\qquad+C(\delta)\,h^2\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t+C(\delta)\,\Delta t_n\left[|\vec{E}^n|_1^2+|\vec{E}^{n-1}|_1^2+\left|D_t\vec{E}^n\right|_0^2\right]\\
&\qquad+C(\delta)\,\Delta t_n\,h^{-1}\,|Z^n|_0^2\left[|\vec{E}^n|_1^2+\left|D_t\vec{E}^n\right|_0^2\right]+C\,|Z^n|_{0,\infty}\,|\vec{E}^n-\vec{E}^{n-1}|_1^2.
\end{aligned}
\tag{3.57}
$$

Let us examine the last two terms appearing on the right hand side of (3.57). We deduce from (3.9), (3.25), (2.6a) and (2.4) that

$$
\begin{aligned}
&\Delta t_n\,h^{-1}\,|Z^n|_0^2\left[|\vec{E}^n|_1^2+\left|D_t\vec{E}^n\right|_0^2\right]\\
&\qquad\le C\,h^{-1}\left[|Z^{n-1}|_0^2+|Z^n-Z^{n-1}|_0^2\right]\Delta t_n\left[|\vec{E}^n|_1^2+\left|D_t\vec{E}^n\right|_0^2\right]\\
&\qquad\le C\,\Delta t_n\left[|\vec{E}^n|_1^2+\left|D_t\vec{E}^n\right|_0^2\right]+C\,h^{\frac{1}{2}}\,|Z^n-Z^{n-1}|_0^2,
\end{aligned}
\tag{3.58a}
$$

and

$$
\begin{aligned}
|Z^n|_{0,\infty}\,|\vec{E}^n-\vec{E}^{n-1}|_1^2&\le C\,h^{-\frac{1}{2}}\left[|Z^{n-1}|_0+|Z^n-Z^{n-1}|_0\right]|\vec{E}^n-\vec{E}^{n-1}|_1^2\\
&\le C\,|\vec{E}^n-\vec{E}^{n-1}|_1^2+C\,h^{\frac{1}{4}}\,|Z^n-Z^{n-1}|_0|\vec{E}^n-\vec{E}^{n-1}|_1\\
&\le C\,|\vec{E}^n-\vec{E}^{n-1}|_1^2+C\,h^{\frac{1}{2}}\,|Z^n-Z^{n-1}|_0^2.
\end{aligned}
\tag{3.58b}
$$

Combining (3.57) and (3.58a,b) and choosing δ sufficiently small yields, on noting (3.7), (2.8a) and $\Delta t_n\le C\,h$,

$$
\begin{aligned}
&(|\vec{X}^n_\rho|\,Z^n,Z^n)^h+\frac{d}{2M}\,\Delta t_n\,|Z^n|_1^2+(1-C\,h^{\frac{1}{2}})\,(|\vec{X}^{n-1}_\rho|\,(Z^n-Z^{n-1}),(Z^n-Z^{n-1})\,)^h\\
&\quad\le (|\vec{X}^{n-1}_\rho|\,Z^{n-1},Z^{n-1})^h+C\,\Delta t_n\left[(|\vec{X}^{n-1}_\rho|\,Z^{n-1},Z^{n-1})^h+|\vec{E}^{n-1}|_1^2\right]\\
&\qquad+C\,\Delta t_n\left|D_t\vec{E}^n\right|_0^2+C\,|\vec{E}^n-\vec{E}^{n-1}|_1^2+C\,h^2\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t.
\end{aligned}
\tag{3.59}
$$

We now proceed by choosing $h^\star>0$ so small that $C\,(h^\star)^{\frac{1}{2}}\le 1$ in (3.59). Multiplying

(3.59) by β^2 and adding to (3.22) yields that

$$\begin{aligned}
&\Delta t_n\,\alpha\,\frac{m^2}{4}|D_t\vec{E}^n|_0^2+|\vec{E}^n-\vec{E}^{n-1}|_1^2+|\vec{E}^n|_1^2+\beta^2\,(|\vec{X}_\rho^n|\,Z^n,Z^n)^h\\
&\leq|\vec{E}^{n-1}|_1^2+\beta^2\,(|\vec{X}_\rho^{n-1}|\,Z^{n-1},Z^{n-1})^h+C\,\Delta t_n\left[|\vec{E}^{n-1}|_1^2+(|\vec{X}_\rho^{n-1}|\,Z^{n-1},Z^{n-1})^h\right]\\
&\quad+C\,h^2\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t+C\,\beta^2\,\Delta t_n\left|D_t\vec{E}^n\right|_0^2+C\,\beta^2\,|\vec{E}^n-\vec{E}^{n-1}|_1^2. \qquad (3.60)
\end{aligned}$$

Choosing $\beta\in(0,1]$ in such a way that $C\,\beta^2\leq\min\{\alpha\frac{m^2}{4},1\}$ we find, with the help of the induction hypothesis (3.3), that for any $h\in(0,h^\star]$

$$\begin{aligned}
&|\vec{E}^n|_1^2+\beta^2\,(|\vec{X}_\rho^n|\,Z^n,Z^n)^h\\
&\quad\leq\left(1+\frac{C\,\Delta t_n}{\beta^2}\right)\left[|\vec{E}^{n-1}|_1^2+\beta^2\,(|\vec{X}_\rho^{n-1}|\,Z^{n-1},Z^{n-1})^h\right]+C\,h^2\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t\\
&\quad\leq h^2\,e^{\gamma\int_0^{t_{n-1}}\zeta(t)\,\mathrm{d}t}\left[1+C\left(1+\frac{1}{\beta^2}\right)\int_{t_{n-1}}^{t_n}\zeta(t)\,\mathrm{d}t\right]\leq h^2\,e^{\gamma\int_0^{t_n}\zeta(t)\,\mathrm{d}t}, \qquad (3.61)
\end{aligned}$$

provided that $\gamma\geq C\left(1+\frac{1}{\beta^2}\right)$. Hence, on assuming (3.3) for some $n\in\{1,\ldots,N\}$ we have shown that it holds for $n+1$, (3.61), provided that $\beta\in(0,1]$ is chosen as required in deriving (3.6) and (3.61), γ as required by (3.24) and in deriving (3.61), and finally $h^\star$ so that (3.5), (3.28) and the condition for deriving (3.60) are satisfied.

Therefore, under the above constraints on β, γ and $h^\star$, we have that (3.61) holds for $n=0,\ldots,N$, and (3.7) holds for $n=1,\ldots,N+1$. We deduce from these, on noting (2.8a), that

$$\sup_{n=0,\ldots,N}\left[|\vec{E}^n|_1^2+|Z^n|_0^2\right]\leq C\,h^2. \qquad (3.62)$$

Moreover, we have that (3.22) and (3.59) hold for $n=1,\ldots,N$. Summing these from $n=1$ to N, yields, on noting (3.62), (3.7) and (2.8a), that

$$\sum_{n=1}^{N}\Delta t_n\,\left[\left|D_t\vec{E}^n\right|_0^2+|Z^n|_1^2\right]\leq C\,h^2. \qquad (3.63)$$

Finally, (3.62), (3.63), (3.2), (2.6b), (2.1b,c), (3.15) and $\Delta t_n\leq C\,h$ yield the desired error bounds (2.13) of Theorem 2.1.

4. Numerical results. Throughout this section we set $d=1$. Let us begin by investigating the experimental order of convergence (eoc). We use the following example from Section 6.4 of [13] and consider for $\rho\in\mathbb{I}$, $t\in[0,1]$

$$\vec{x}(\rho,t)=\begin{pmatrix}(1+\frac{1}{2}\sin(2\pi t))\,\cos(2\pi\rho)\\(1-\frac{1}{2}\sin(2\pi t))\,\sin(2\pi\rho)\end{pmatrix},\quad w(\rho,t)=t\,\cos(8\pi\rho)+(1-t)\,\sin(6\pi\rho),$$

as well as $f(w)=2w$, $g(v,w)=0$. With these choices (1.6), (1.7) are satisfied with additional terms $\vec{S}$, S_w on the right hand side respectively, so that we replace $f(W^{n-1})\,\vec{\mathcal{V}}^{n-1}$ by $f(W^{n-1})\,\vec{\mathcal{V}}^{n-1}+\vec{S}$ in (2.10) and $g(V^n,W^{n-1})$ by $g(V^n,W^{n-1})+S_w$ in (2.12). In what follows we choose a uniform element length $h=1/J$ and, unless

stated otherwise, a uniform time step $\Delta t = h^2$ throughout the computations in this section. We monitor the following errors:

$$\mathcal{E}_1 := \sup_{n=0,\dots,N} |Z^n|_0^2,\ \mathcal{E}_2 := \sum_{n=1}^{N} \Delta t\, |Z^n|_1^2,\ \mathcal{E}_3 := \sup_{n=0,\dots,N} |\vec{E}^n|_1^2,\ \mathcal{E}_4 := \sum_{n=1}^{N} \Delta t \left| D_t \vec{E}^n \right|_0^2.$$

In Tables 4.1 and 4.2 we display the values of $\mathcal{E}_i$, $i = 1, \dots, 4$, for $\alpha = 1$ with $\Delta t = h^2$ and $\Delta t = 0.5h$ respectively. For $\Delta t = 0.5h$ we see eocs close to two, while for $\Delta t = h^2$ we see eocs close to four. When $\Delta t = h^2$ the eocs for $\mathcal{E}_2$ and $\mathcal{E}_3$ are better than the spatial approximation error, thus demonstrating superconvergence. Tables 4.3 and 4.4 show the corresponding results for $\alpha = 0.1$, again we see eocs close to two for $\Delta t = 0.5h$ and eocs close to four for $\Delta t = h^2$. We saw similar results when we computed $\mathcal{E}_i$, $i = 1, \dots, 4$, for the scheme in [13]. The above results confirm the bounds obtained in Theorem 2.1. It will be a subject of future research to rigorously prove the better rates in the case $\Delta t = h^2$.

J	$\mathcal{E}_1 \times 10$	eoc_1	$\mathcal{E}_2$	eoc_2	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4 \times 10$	eoc_4
30	0.1817509	-	2.6729150	-	0.2043562	-	0.1177607	-
60	0.0116485	3.96	0.1712537	3.96	0.0134930	3.92	0.0079150	3.90
120	0.0007333	3.99	0.0107777	3.99	0.0008574	3.98	0.0005056	3.97
240	0.0000459	4.00	0.0006748	4.00	0.0000538	3.99	0.0000318	3.99

TABLE 4.1
$\alpha = 1, \Delta t = h^2$

J	$\mathcal{E}_1 \times 10$	eoc_1	$\mathcal{E}_2$	eoc_2	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4 \times 10$	eoc_4
30	0.5093612	-	5.7947280	-	0.4464560	-	0.2664600	-
60	0.0835780	2.61	0.9498929	2.61	0.0868606	2.36	0.0545729	2.29
120	0.0162823	2.36	0.1859501	2.35	0.0187461	2.21	0.0125972	2.12
240	0.0035578	2.19	0.0408490	2.19	0.0043234	2.12	0.0030531	2.04

TABLE 4.2
$\alpha = 1, \Delta t = 0.5h$

J	$\mathcal{E}_1 \times 10$	eoc_1	$\mathcal{E}_2$	eoc_2	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4 \times 10$	eoc_4
30	0.3227043	-	4.8893170	-	1.0377730	-	0.6239216	-
60	0.0205566	3.97	0.2808019	4.12	0.0562308	4.21	0.0372623	4.07
120	0.0012831	4.00	0.0172030	4.03	0.0033578	4.07	0.0022726	4.04
240	0.0000801	4.00	0.0010698	4.01	0.0002074	4.02	0.0001411	4.01

TABLE 4.3
$\alpha = 0.1, \Delta t = h^2$

J	$\mathcal{E}_1 \times 10$	eoc_1	$\mathcal{E}_2$	eoc_2	$\mathcal{E}_3$	eoc_3	$\mathcal{E}_4 \times 10$	eoc_4
30	1.0152550	-	13.267420	-	3.8386880	-	2.5325450	-
60	0.1996360	2.35	2.4120110	2.46	0.7548792	2.35	0.6895750	1.88
120	0.0433905	2.20	0.5122927	2.24	0.1665889	2.18	0.1772575	1.96
240	0.0100380	2.11	0.1179202	2.12	0.0391495	2.09	0.0449715	1.98

TABLE 4.4
$\alpha = 0.1, \Delta t = 0.5h$

In our second example we present numerical results that highlight the advantage of using $\alpha \ll 1$ for practical choices of J. We set $f(w) = 0.5w^2$, $g(v, w) = 0$,

$$\vec{x}(\rho, 0) = \begin{pmatrix} \cos(2\pi\rho) \\ (0.9\cos^2(2\pi\rho) + 0.1)\sin(2\pi\rho) \end{pmatrix}, \quad w(\rho, 0) = \sin(6\pi\rho), \quad \rho \in \mathbb{I}.$$

For $J = 1200$ the results obtained by plotting $\vec{X}(\rho, t)$, for $t \in [0, 0.15]$, with $\alpha = 1$ and $\alpha = 0.1$ are visually indistinguishable. However when $J = 60$ at $t = 0.15$ there is a marked difference between the two solutions. This can be seen in Figure 4.1 where we display $\vec{X}(\rho, t)$ (red lines) at times $t = 0.05,\ 0.1$ and 0.15 for $\alpha = 1$ (upper plots) and $\alpha = 0.1$ (lower plots). In these results we include $\vec{X}(\rho, t)$ (blue lines), obtained by setting $J = 1200$ and $\alpha = 1$, to act as a comparison to the 'true solution'. We see that the additional tangential motion of the nodes obtained by setting $\alpha = 0.1$ results in a better approximation at $t = 0.15$ to the 'true solution' than the approximation obtained by setting $\alpha = 1$. The importance of the tangential motion of the nodes can be further seen in Figure 4.2 where we display $\vec{X}(\rho, t)$ (red lines) at times $t = 0.05,\ 0.1$ and 0.15 for the scheme in [13] with $J = 60$, in which there is no tangential motion. Again we include the 'true solution' (blue lines) as in Figure 4.1. The absence of tangential motion leads to a severe accumulation of the nodes at $t = 0.15$, however as the value of J increases this accumulation effect becomes less pronounced.

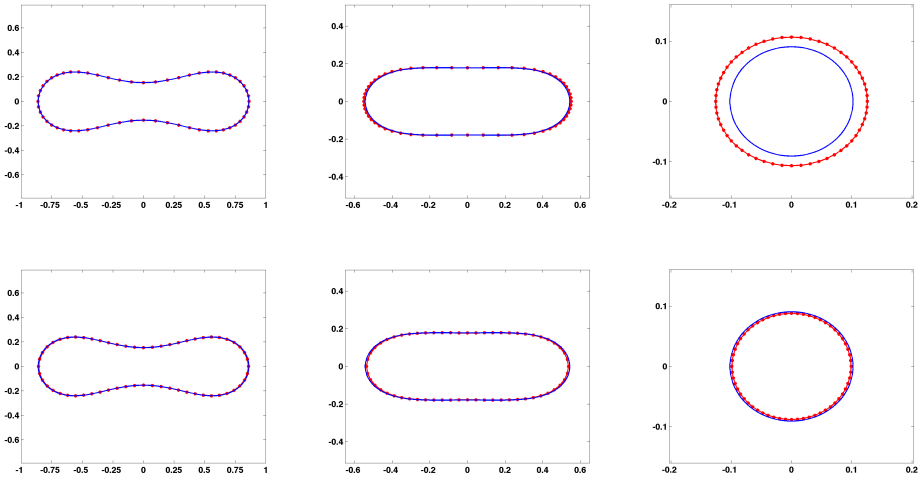


FIG. 4.1. *$\vec{X}(\rho, t)$ at times $t = 0.05,\ 0.1$ and 0.15 with $\alpha = 1$ (upper plots) and $\alpha = 0.1$ (lower plots), $J = 60$ red lines. In addition, $J = 1200$ with $\alpha = 1$ blue lines.*

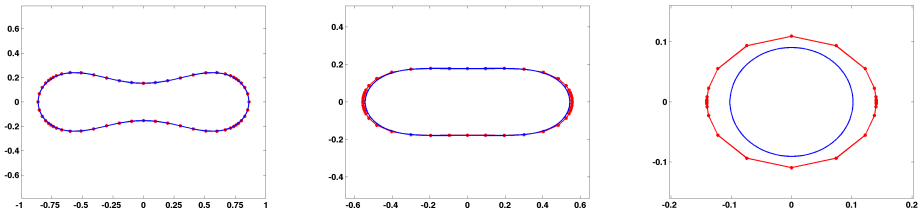


FIG. 4.2. $\vec{X}(\rho,t)$ *at times* $t = 0.05,\ 0.1$ *and* 0.15. *The red line are computed using the scheme in [13] with* $J = 60$ *and the blue lines are computed using (2.10), (2.12) with* $J = 1200$ *and* $\alpha = 1$.

We conclude our numerical results with a simulation of diffusion induced grain boundary motion. With $\vec{\nu}$ being the inward normal, physically meaningful choices for $f(w)$ and $g(v,w)$ in (1.1a,b) are $f(w) = w^2$ and $g(v,w) = (C-w) - |v|\,w$, where C is the concentration of solute in the vapour. Note that these choices satisfy (2.1a). We consider the experimental setup in which the parametrisation of the initial grain boundary and the initial concentration of solute are given by

$$\vec{x}(\rho,0) = \begin{pmatrix} \cos(2\pi\rho) \\ \frac{1}{2}\sin(2\pi\rho) + \sin(\cos(2\pi\rho)) + \sin(2\pi\rho)[\frac{1}{5} + \sin(2\pi\rho)\sin^2(6\pi\rho)] \end{pmatrix},\ \rho \in \mathbb{I}$$

and $w(\rho,0) = 0$, $\rho \in \mathbb{I}$, respectively, while $C = 1$. In the simulations presented we set $\alpha = 0.1$ and $J = 120$. Figure 4.3 displays the grain boundary, $\vec{X}(\rho,t)$, (blue line) at times $t = 0.105$, $t = 0.21$ and $t = 0.315$ together with its initial position (red lines). In this simulation we see bi-direction motion of the grain boundary, see [8], [10]. In Figure 4.4 we plot the concentration $w(\rho,t)$ at $t = 0.105$, $t = 0.21$ and $t = 0.315$.

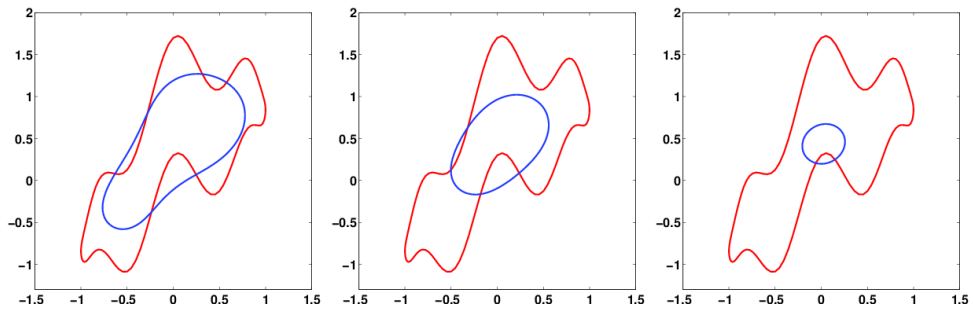


FIG. 4.3. *The grain boundary* $\vec{X}(\rho,t)$ *at times* $t = 0.105,\ 0.21,\ 0.315$ *(blue lines) and at* $t = 0$ *(red lines).*

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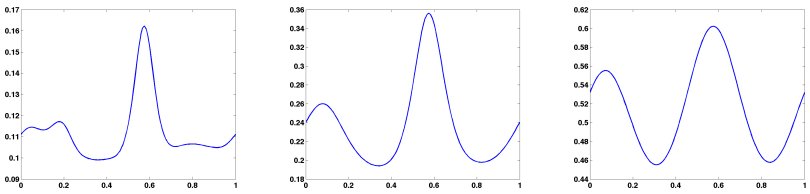


FIG. 4.4. *The concentration $W(\rho, t)$ at times at times $t = 0.105,\ 0.21,\ 0.315$.*

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