

Homothetic tube-based robust Economic MPC with integrated Moving Horizon Estimation

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Abstract—This paper considers homothetic tube-based Economic Model Predictive Control synthesis for constrained linear discrete time systems. Since, in practical systems, full state measurement is seldom available, the proposed method integrates a moving horizon estimator to achieve closed-loop stability and constraint satisfaction despite system disturbances and output measurement noise. In contrast to existing approaches, the worst cost within a single homothetic tube around the solution of the associated nominal system is minimized, which, at the same time, tightens the bound on the set of potential states compatible with past output and input data. We show that the designed optimization problem is recursively feasible and adoption of homothetic tubes leads to less conservative economic performance bounds. Thanks to the use of strict dissipativity of the nominal system with a suitable supply rate, the closed-loop system is shown to be asymptotically stable, in the sense that the state trajectory is driven to an optimal robust invariant set.

Index Terms—Robust economic Model Predictive Control, Predictive control for linear systems, Output feedback and Observers, Optimal control, Robust Control.

I. INTRODUCTION

ECONOMIC Model Predictive Control (EMPC) is an optimization based control method, that coherently takes the economic criteria into consideration by adopting cost functionals which are not necessarily positive definite with respect to any equilibrium point, see e.g., [1], [2], [3], [4]. In EMPC, an optimization problem is repeatedly solved to compute the feedback control policy at every time instant, based on knowledge of the current plant's state. In the paper [2], an approach on EMPC controller without terminal constraints is presented and analyzed based on certain turnpike and suitable controllability properties. Alternatively, strong duality or dissipativity assumptions allow to define rotated positive definite versions of the economic cost, and stability analysis to be carried out by adopting terminal equality or inequality constraints as in standard tracking MPC [3], [4], [5].

If disturbances are affecting the real system, predicted states will be different from actual states and the determined optimal control may even violate constraints and cause the optimization problem at the next time step to become infeasible. To deal with uncertainties, various approaches have been proposed in the literature. In [6], a min-max EMPC approach for linear systems is discussed which recasts the underlying optimization

problem as a second-order cone program to improve computational efficiency of its solution. In [7], [8], the authors use the concept of robust invariant set and adopt the standard tube-based MPC scheme to an EMPC setting by averaging the economic cost function within the invariant tube. Another stream of research, which explicitly considers the stochastic information of the disturbance, includes chance-constrained stochastic EMPC [9] and multi-stage EMPC [10].

Tube-based EMPC formulates an state-feedback control policy of the form $u = v + K(x - z)$, where v is the nominal control input, u is the applied input to the true plant, x and z are the true and nominal states of the system. The feedback gain K is a pre-defined parameter and used to compute the rigid tube which is a robust positively invariant (RPI) set under the feedback affine policy. In contrast to the tube-based MPC, the economic framework employs a modified economic stage cost on which the optimal equilibrium is defined [7], [8].

Unless the full state information is directly available at every time instant, an observer should be used to estimate the states. In addition, for constrained systems, a direct application of the separation principle, which obtains the control inputs based on current estimates from observers, such as high gain observer, Kalman filtering, or moving horizon estimation (MHE), is not possible as the estimation error may cause constraint violation. Moreover, this estimate is optimal in the sense that it maximizes a criterion that captures the likelihood of the estimate for the given measurements. However, this optimal state estimate may not be beneficial from the economic point of view. Therefore, robust output feedback control policies can be adopted to cope with model uncertainties and estimation errors.

To the best of our knowledge, there are no works on robust EMPC design combined with state observer in a single pure economic optimization problem which achieves closed-loop asymptotic stability and a robust guaranteed upper-bound on the long-run average cost. Even for tracking MPC, state estimation and optimal control problems are usually formulated separately, as in [11] and [12], where a robust output feedback control policy is adopted. In [13], [14] and [15], the approach proposed is to solve both the MPC and MHE problems simultaneously in a single min-max optimization problem under the assumption of existence of a saddle-point. Authors additionally derive that existence of saddle-point equilibrium for a linear system with quadratic cost is satisfied if the system is observable and weights in the cost function are appropriately chosen. For comparison, [7] and [8] present an expected closed-loop performance bound and asymptotic stability results for systems with optimal steady-state operation

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under full state knowledge. Motivated by this approach, the present paper shows: (i) it is in fact possible to design a tube-based robust EMPC algorithm combined with a MHE state observer by using a nominal dynamic model and a homothetic tube around the nominal states with guaranteed recursive feasibility; (ii) closed-loop asymptotic stability holds both for optimal steady-state operation, provided that dissipativity with respect to nominal dynamics is fulfilled with a suitable supply rate; (iii) the use of the homothetic tube provides a less conservative robust economic performance bound.

The reminder of this note is organized as follows. Section II introduces the basic notions and setup. Section III addresses the homothetic tube-based robust EMPC algorithm with its formulation in Section III-A and recursive feasibility analysis in Section III-B. The asymptotic stability and average performance of the closed-loop system adopting this EMPC controller are analyzed in Section IV. Finally, illustrative examples using the proposed EMPC algorithm are presented in Section V and Section VI concludes this paper.

A preliminary work on tube EMPC was presented in [16]. This, however, was based on exact availability of state measurements and a state feedback control policy was adopted to achieve closed-loop asymptotic stability and robust bounds on the long run average cost. Here, we extend the results to the case that the current system state is no longer available and a moving horizon estimation technique is used. By solving an integrated estimation and optimal control problem based on the homothetic tube, a less conservative asymptotic performance bound is achieved. In addition, the requirements to obtain the performance bound are relaxed.

Notations: Let symbols $\mathbb{R}$ and $\mathbb{I}$ denote the sets of real numbers and integers, respectively. $\mathbb{R}_{\geq a}$ ($\mathbb{I}_{\geq a}$) denotes the real numbers (integers) greater or equal to a and $\mathbb{I}_{[a,b]}$ denotes the integers $\{a, a+1, \dots, b\}$ for some $a \in \mathbb{I}$ and $b \in \mathbb{I}_{\geq a}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$, if it is zero at zero and strictly increasing. It is of *class* $\mathcal{K}_{\infty}$ if in addition $\alpha(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a *class* $\mathcal{KL}$ function if $\beta(\cdot, t)$ is a *class* $\mathcal{K}$ function for every fixed $t \in \mathbb{I}_{\geq 0}$, and $\beta(s, \cdot)$ is strictly decreasing with $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ for every fixed $s > 0$. For sets $X, Y \in \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \in \mathbb{R}^n : x \in X, y \in Y\}$, in particular, we write $x \oplus X$ instead of $\{x\} \oplus X$. The Pontryagin set difference is defined as $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$. A set $X \subset \mathbb{R}^n$ is a *C* set if it is a compact, convex set that contains the origin in its (non-empty) interior. The Euclidean norm of x is $|x|$. The distance of a point $x \in \mathbb{R}^n$ to a set Π is denoted as $|x|_{\Pi} := \min_{y \in \Pi} |x - y|$. Let $\rho(A)$ denote spectral radius of a given matrix $A \in \mathbb{R}^{n \times n}$.

II. SYSTEM PRELIMINARIES

We consider finite dimensional discrete-time linear systems described by difference equations

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + \omega_t \\ y_t &= Cx_t + \eta_t \end{aligned} \quad (1)$$

where $x_t \in \mathbb{R}^{n_x}$ and $u_t \in \mathbb{R}^{n_u}$ are the system state and the control input, respectively, at time $t \in \mathbb{I}_{\geq 0}$. The system is subject to pointwise-in-time constraints

$$(x_t, u_t) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (2)$$

where the set $\mathbb{Z}$ is compact. Moreover, the variables $\omega_t \in \mathbb{W} \subseteq \mathbb{R}^{n_x}$ and $\eta_t \in \mathbb{H} \subseteq \mathbb{R}^{n_y}$, where $\mathbb{W}$ and $\mathbb{H}$ are compact sets containing zero, denote the state disturbance and the measurement noise, respectively. The signal $y_t \in \mathbb{R}^{n_y}$ is known as the measured output that is available for feedback. Throughout this paper, we adopt the following assumptions about the system (1):

Assumption 1. *The pairs (A, B) and (C, A) are stabilizable and detectable, respectively.*

Our objective is to enhance profitability by minimizing the economic costs accumulated in the long term system operation

$$\sum_t \ell(x_t, u_t), \quad (3)$$

where $t \in \mathbb{I}_{\geq 0}$ and $\ell(x, u) : \mathbb{Z} \rightarrow \mathbb{R}$ is the economic stage cost, which may take arbitrary form coherently with the economic MPC setup (see [3]), and need not be positive definite with respect to any equilibrium state.

A. Feedback control and observer

One natural approach to control system (1), while ensuring the satisfaction of the constraints (2), is to apply an optimization-based feedback control policy. However, in most literature, Economic MPC is formulated assuming full-state feedback, whereas, in practical cases, the full state measurement is often not available. This motivates the use of a pre-defined state observer

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + L(y_t - C\hat{x}_t) \quad (4)$$

where $\hat{x}_t \in \mathbb{R}^{n_x}$ is the observer state and L satisfies $\rho(A - LC) < 1$, which will be used in the computation of the input u_t . Therefore, as in [12], [17], [18], the state x_t can be decomposed into different components:

$$x_t = \hat{x}_t + e_t = z_t + \xi_t + e_t, \quad (5)$$

where these three terms are later referred to as nominal state, observer state error, and estimation error, respectively. The nominal state z_t depends on the nominal input v_t and fulfills the dynamic

$$z_{t+1} = Az_t + Bv_t. \quad (6)$$

In order to determine the optimal control input u_t at each sampling time instant, an optimization problem is solved with the nominal state and input sequences as decision variables. As customary in tube-based EMPC, the optimal nominal state and input pair (z_t, v_t) are used to calculate the control policy

$$u_t = v_t + K(\hat{x}_t - z_t), \quad (7)$$

which is implemented to the systems (1) and (4), where K is a pre-defined feedback gain such that $\rho(A + BK) < 1$.

The state estimation error $e \in \mathbb{R}^{n_x}$ is defined as $e_t := x_t - \hat{x}_t$ and governed by

$$e_{t+1} = (A - LC)e_t + \omega_t - L\eta_t, \quad (8)$$

Moreover, ξ_t the difference between the observer state $\hat{x}_t$ and its nominal counterpart z_t follows the difference equation

$$\xi_{t+1} = (A + BK)\xi_t + L(Ce_t + \eta_t). \quad (9)$$

At this point, we recall the standard definition of robust positively invariant set from [17]:

Definition 1. A set $S \subseteq \mathbb{R}^n$ is robust positively invariant for the system $x^+ = f(x, \omega)$ and the constraint set $(\mathbb{X}, \mathbb{W})$ if $S \subseteq \mathbb{X}$ and $f(x, \omega) \in S, \forall \omega \in \mathbb{W}, \forall x \in S$.

Since $\rho(A - LC) < 1$, there exists a C set that is finite time computable and robustly positively invariant [19] for the estimation error system (8). Then, following the ideas in [17], we can bound e_t and ξ_t using sets E and Ξ , respectively, provided that the system state x_0 , observer state $\hat{x}_0$ and nominal system state z_0 satisfy $e_0 = x_0 - \hat{x}_0 \in E$ and $\xi_0 = \hat{x}_0 - z_0 \in \Xi$. Furthermore, it can be indicated that the system state x_t fulfills $x_t \subseteq z_t \oplus \Omega = z_t \oplus (E \oplus \Xi)$.

Remark 1. Notice that sets E , Ξ and Ω for a linear system (1) and given convex polyhedrons $\mathbb{W}$ and $\mathbb{H}$ are, in turn, convex polyhedrons and can be computed based on MPT3 toolbox. Moreover, paper [12] analyzed the conservatism of using separate sets for e_t and ξ_t , and proposed a single set on an augmented system $\delta_t = [\xi_t \ e_t]^T$ to obtain tighter approximations on the worst case effect of the uncertainties. However, the computation of RPI sets is performed on a space with double dimension, which reduces the applicability of the approach for high dimensional systems.

In order to ensure that the unknown state satisfies the state constraint, we need to introduce the tightened constraint for the nominal system (6) using the sets E and Ξ

$$\mathcal{Z} = \{(z, v) \mid (z + \xi + e, v \oplus K\xi) \subseteq \mathbb{Z}, \forall \xi \in \Xi, \forall e \in E\}, \quad (10)$$

and, accordingly, this tightened constraint guarantees that

$$(z_t, v_t) \in \mathcal{Z} \Rightarrow (x_t, u_t) \in \mathbb{Z}. \quad (11)$$

For later use, the projection of $\mathcal{Z}$ on $\mathbb{X}$ is denoted by $\mathcal{X}$.

Since the set Ω may not provide a tight estimate of the real system state value, it may be possible to find a less conservative bound for the economic cost incurred in operation. Indeed, one might take advantage of the past measured outputs in order to further reduce the region where the real system state belongs. For instance, it is possible to find the best scalar $\gamma \in [0, 1]$ such that the state is guaranteed to fulfill $x \in z \oplus \gamma\Omega$. Notice that this scaling parameter is only used to derive a tighter performance bound and it will not affect the robust optimal regime of operation which will be discussed in the next subsection. Based on the available information at time t , we are able to recursively define the reachable sets

$$\begin{aligned} \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) = \{x_t \mid \exists x_{t-1} \in \mathcal{R}_{t-1}(y_{0:t-1}, \hat{x}_0, u_{0:t-2}), \\ \exists \omega_{t-1} \in \mathbb{W}, y_t - Cx_t \in \mathbb{H} \\ x_t = Ax_{t-1} + Bu_{t-1} + \omega_{t-1}\} \end{aligned} \quad (12)$$

where $\mathcal{R}_0(y_0, \hat{x}_0) = \{x_0 \mid x_0 \in \hat{x}_0 \oplus E, y_0 - Cx_0 \in \mathbb{H}\}$.

B. Robust optimal operation regimes

Clearly, the presence of an additive persistent disturbance ω and measurement noise η acting on system (1) means it is not possible to guarantee asymptotic convergence to the generally defined optimal steady-state, i.e.

$$(x^*, u^*) = \arg \min_{(x, u) \in \mathbb{Z}, x = Ax + Bu} \ell(x, u). \quad (13)$$

Hence, an achievable goal is to steer the system state to a neighborhood of a certain steady-state which may differ from (13). When the deviation of the true state x from the nominal state z is bounded by $\gamma\Omega$, the worst economic cost, over the homothetic set centered at z , is defined as

$$\tilde{\ell}(z, v, \gamma) = \max_{\delta \in \gamma\Omega, \xi \in \Xi} \ell(z + \delta, v + K\xi). \quad (14)$$

This modification takes the worst cost into account and it provides an upper bound of the real system cost regardless of the state disturbance and measurement uncertainty realizations. Readers may refer to [7] for the motivations to consider a modified economic stage cost. We should also notice that $\tilde{\ell}(z, v, \gamma)$ is monotonically non-decreasing with respect to γ , so that the tightest estimates of the state region (corresponding to small γ) provide a lower upper-bound to the economic cost.

At this point, we are able to define the robust optimal steady-state operation by

$$(z_s, v_s) = \arg \min_{(z, v) \in \mathcal{Z}, z = Az + Bv} \tilde{\ell}(z, v, 1). \quad (15)$$

Given this robust optimal state and input pair, the worst cost within $z \oplus \gamma\Omega$ is

$$\tilde{\ell}_{av}^*(\gamma) = \tilde{\ell}(z_s, v_s, \gamma), \quad (16)$$

in particular, the robust optimal average economic cost $\tilde{\ell}_{av}^*(1) = \tilde{\ell}(z_s, v_s, 1)$. Notice that $\tilde{\ell}_{av}^*(1)$ is independent of the parameter γ , so it denotes an a-priori upper bound to the closed-loop average cost.

Although a steady-state often provides the minimal long run average economic cost, sometimes more complex operations analyzed in [20] and [21], such as a periodic solution, can outperform the steady-state, for the nominal system (6) with respect to the modified robust cost function (14).

Let us consider the more general case where system (1) is optimally operated at some periodic orbit with period $P \in \mathbb{I}_{\geq 1}$. Solving the following optimization problem,

$$\begin{aligned} \min \quad & \frac{\sum_{i=0}^{P-1} \tilde{\ell}(z_i, v_i, 1)}{P}, \\ \text{s.t.} \quad & z_{i+1} = Az_i + Bv_i \quad \forall i \in \mathbb{I}_{[0, P-2]} \\ & z_0 = Az_{P-1} + Bv_{P-1} \\ & (z_i, v_i) \in \mathcal{Z} \quad \forall i \in \mathbb{I}_{[0, P-1]} \end{aligned}$$

we obtain the best P -periodic solution

$$\Pi = \{(z_{s,0}, v_{s,0}), (z_{s,1}, v_{s,1}), \dots, (z_{s,P-1}, v_{s,P-1})\}, \quad (17)$$

with corresponding optimal periodic orbit $\Pi_{\mathbb{X}} = \{z_{s,0}, z_{s,1}, \dots, z_{s,P-1}\}$, and its robust average performance can be denoted by

$$\tilde{\ell}_{av}^*(\gamma) = \frac{\sum_{i=0}^{P-1} \tilde{\ell}(z_{s,i}, v_{s,i}, \gamma)}{P}. \quad (18)$$

Remark 2. It should be stressed that the proposed algorithm is focusing on the time-invariant scheme with optimal steady-state operation, and the RPI set Ω is selected as a convex polyhedron. Although the formulation can be further applied to systems with optimal periodic operation, solving the corresponding optimization problem could be extremely challenging using standard mathematical programming techniques due to non-convexity of constraints and cost functions.

Next, we define the notion of optimal operation at the steady-state for the nominal dynamic:

Definition 2. The nominal system (6) is optimally operated at a set Π with respect to the modified cost function $\tilde{\ell}$, if for any solution such that $(z_t, v_t) \in \mathcal{Z}, \forall t \in \mathbb{I}_{\geq 0}$, it holds

$$\liminf_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_t, v_t, 1)}{T} \geq \tilde{\ell}_{av}^*(1), \quad (19)$$

where $\tilde{\ell}_{av}^*(1)$ is the average modified cost of the optimal robust invariant set for $\gamma = 1$ as in (16) and (18).

Definition 2 means that each feasible solution will result in an asymptotic average performance which is as good as or worse than the average performance of the optimal steady-state operation.

After clarifying the robust optimal operation of the set Π , we make an assumption based on the definition of strict dissipativity for disturbance-free systems as given in [22], [23]:

Assumption 2. There exists a continuous storage function $\lambda : \mathcal{X} \rightarrow \mathbb{R}$ bounded from below on $\mathcal{X}$, and a class $\mathcal{K}$ function $\alpha_1(\cdot)$ such that

$$\lambda(Az + Bv) - \lambda(z) \leq \tilde{\ell}(z, v, \gamma) - \tilde{\ell}_{av}^*(\gamma) - \alpha_1(|z - z_s|), \quad (20)$$

for all $(z, v) \in \mathcal{Z}$ and $\gamma \in [0, 1]$.

Remark 3. The authors in [24] propose a computational method for automatic verification of dissipativity for discrete time systems with polynomial dynamics and stage cost. Assumption 2 requires that inequality (20) holds for all $\gamma \in [0, 1]$, whereas in the standard definition of dissipativity $\gamma = 0$ and the supply rate is based on the original cost function ℓ , rather than $\tilde{\ell}$. This rather strong condition, ensures an optimal operation regime that is independent of our confidence level γ on the location of the true system's state. In particular, provided that the stage cost ℓ is convex, the modified function $\tilde{\ell}$ is convex as well, and under special symmetry conditions, this may imply the optimal operation is independent of the factor γ (see Example V-A).

III. THE HOMOTHETIC TUBE-BASED ECONOMIC MPC ALGORITHM

A. Problem formulation

In this section, a robust economic MPC controller is designed. To this end, further notation has to be introduced:

$z_{0:t+N|t}$ is the sequence of nominal states from sampling instants 0 to $t + N$, considered at time t ; $v_{t:t+N-1|t}$ is the sequence of future nominal control moves; $u_{0:t-1}$ (or $v_{0:t-1}$ equivalently), $y_{0:t}$ and $\gamma_{0:t-1}$ are the (known) past inputs, measured outputs and scaling factors, respectively, at time t . Because the observer states are auxiliary variables processed within the controller, the sequence $\hat{x}_{0:t}$ is known at time t .

Notice that the Luenberger observer only comes with an a-priori (constant) bound E of the estimation error, which, in some sense, corresponds to the worst case disturbance realization. The moving horizon estimation, instead, is a set membership estimator that, depending on disturbance realization, may allow a much tighter bound on the estimation error. This is beneficial for the economic performance as tighter characterizations of the state whereabouts allow sharper optimization of the economic criterion.

Before presenting the EMPC algorithm, we first make the following assumption to restrict the error between the real system and the estimates:

Assumption 3. The initial observer state $\hat{x}_0$ is provided along with the minimal RPI set E , such that the real system state x_0 is initially known to satisfy $x_0 \in \hat{x}_0 \oplus E$.

Next we propose a feedback control law by solving online the following optimization problem with a prediction horizon $N \in \mathbb{I}_{\geq 1}$,

$$\min_{\mathbf{d}_t} J_N(z_{t:t+N|t}, v_{t:t+N-1|t}, \gamma_{t:t+N|t}) \quad (21)$$

$$= \lambda(z_{t|t}) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_{k|t}, v_{k|t}, \gamma_{k|t}) + V_f(z_{t+N|t})$$

$$s.t. \quad z_{t|t} \in \hat{x}_t \oplus \Xi, \quad (21a)$$

$$z_{j+1|t} = Az_{j|t} + Bv_{j|t}, \quad (21b)$$

$$(z_{j|t}, v_{j|t}) \in \mathcal{Z}, \quad \forall j \in \mathbb{I}_{[0, t-1]} \quad (21c)$$

$$y_j - Cz_{j|t} \in \gamma_j C\Omega \oplus \mathbb{H}, \quad (21d)$$

$$\mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) \subseteq z_{t|t} \oplus \gamma_{t|t}\Omega, \quad (21e)$$

$$A(z_{k|t} \oplus \gamma_{k|t}\Omega) \oplus B(v_{k|t} \oplus K\Xi) \oplus \mathbb{W} \\ \subseteq z_{k+1|t} \oplus \gamma_{k+1|t}\Omega, \quad (21f)$$

$$z_{k+1|t} = Az_{k|t} + Bv_{k|t}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]} \quad (21g)$$

$$(z_{k|t}, v_{k|t}) \in \mathcal{Z}, \quad (21h)$$

$$0 \leq \gamma_{k|t} \leq 1, \quad \forall k \in \mathbb{I}_{[t, t+N]}, \quad (21i)$$

$$z_{t+N|t} \in \mathcal{X}_f, \quad (21j)$$

where $\mathbf{d}_t = [z_{0:t+N|t}, v_{t:t+N-1|t}, \gamma_{t:t+N|t}]$ is the vector of decision variables, J_N is the economic objective function with initial and terminal penalty costs λ and V_f , and set $\mathcal{X}_f$ is an appropriate time-varying terminal constraint set. Note that the initial weighting function is the storage function defined in Assumption 2, and it has been verified that this function is crucial in the proof of stability as shown in [16]. This optimization problem utilizes the full past information, which is not practically realizable as the number of decision variables increases linearly with time. One approach to overcome this issue is adopting moving horizon estimation which only considers the past data within a finite window horizon $N_E \in \mathbb{I}_{\geq 1}$ after the time instant $t = N_E$. In particular, the decision

variables $z_{0:t-N_E-1|t}$ are removed and the index of (21b)-(21d) becomes $j \in \mathbb{I}_{[t-N_E, t-1]}$. In fact, the reachable set $\mathcal{R}_t$ might often be trimmed by the new output measurements, so that discarding past measurements could avoid its number of vertices growing unbounded. Therefore, since the main results are not affected by the length of estimation, in the remaining sections, we will keep the notion of full information estimation.

Remark 4. In particular, rather than tightening the nominal state-input pairs by the homothetic tube, the constraint (21c) is enforced because it is not always satisfied that $\hat{x} - z \in \gamma \Xi$ in which case the input constraint is violated. The constraint (21d) is to select the nominal dynamic compatible with the output measurements, and the restriction of scalar factors along the prediction horizon is described by the constraint (21f) as in [25].

Remark 5. The constraint (21e) employs a homothetic tube with scalar factor $\gamma_{t|t}$ to circumscribe the set of potential states $\mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1})$. This ensures that $\gamma_{t|t} \in [0, 1]$ is selected so that the real system state fulfills $x_t \in z_{t|t} \oplus \gamma_{t|t} \Omega$. Moreover, it is less conservative compared to $x_t \in z_{t|t} \oplus \Omega$, and may result in better economic performance due to the monotonicity of $\bar{\ell}$ with respect to γ . To mitigate the computational difficulty of this set constraint, only its vertices are considered as the computed RPI sets are convex polyhedrons.

Remark 6. Notice that we are not applying certainty equivalence or any kind of separation principle in order to compute state-estimates, and control policies. On the contrary, a unique “coupled” optimization problem of “pure” economic nature is solved in order to simultaneously compute them. By integrating the MHE with EMPC optimization problems and adopting a pure economic cost, there is no need to maximize the likelihood but only to select the nominal state sequence compatible with the known measurements, so that the economic criterion is optimized within the considered prediction horizon.

Remark 7. The proposed approach seeks to optimize performance only assuming knowledge of the (compact) disturbance support. If disturbances display features that one typical of stochastic processes, such as zero mean, or stationary invariant disturbances (i.e. uniform or Gaussian) adopting a deterministic disturbance model may lead to performance deterioration and conservative state estimates. That would be significantly improved by using likelihood criteria in the estimation process and could benefit expected performance measures.

For the terminal set and penalty function, we make the following assumptions:

Assumption 4. There exist terminal sets $\mathcal{X}_f \subseteq \mathbb{X} \ominus \Omega$, containing the nominal optimal steady-state z_s in their interior, a terminal control policy $\kappa_f(\cdot)$, and a continuous terminal penalty function $V_f : \mathcal{X}_f \rightarrow \mathbb{R}$, such that for all $z \in \mathcal{X}_f$ and $\gamma \in [0, 1]$, it holds

- (i) $(z, \kappa_f(z)) \in \mathcal{Z}$
- (ii) $Az + B\kappa_f(z) \in \mathcal{X}_f$
- (iii) $V_f(Az + B\kappa_f(z)) - V_f(z) \leq \tilde{\ell}_{av}(\gamma) - \tilde{\ell}(z, \kappa_f(z), \gamma)$.

When the system enters the optimization routine at time t , the following sub-optimal and feasible solution which is generated from the previous optimal solution can be adopted as an initial guess

$$\begin{aligned} z_{0:t+N|t}^{sub} &= [z_{0|t-1}^*, z_{1|t-1}^*, \dots, z_{t+N-1|t-1}^*, \\ &\quad Az_{t+N-1|t-1}^* + B\kappa_f(z_{t+N-1|t-1}^*)], \\ v_{t:t+N-1|t}^{sub} &= [v_{t|t-1}^*, v_{t+1|t-1}^*, \dots, v_{t+N-2|t-1}^*, \\ &\quad \kappa_f(z_{t+N-1|t-1}^*)], \\ \gamma_{t:t+N|t}^{sub} &= [\gamma_{t|t-1}^*, \gamma_{t+1|t-1}^*, \dots, \gamma_{t+N-1|t-1}^*, 1]. \end{aligned} \quad (22)$$

Finally, the optimization solver converges to the optimal solution $\mathbf{d}_t^*$ which is

$$\begin{aligned} z_{0:t+N|t}^* &= [z_{0|t}^*, z_{1|t}^*, \dots, z_{t|t}^*, \dots, z_{t+N|t}^*], \\ v_{t:t+N-1|t}^* &= [v_{t|t}^*, v_{t+1|t}^*, \dots, v_{t+N-1|t}^*], \\ \gamma_{t:t+N|t}^* &= [\gamma_{t|t}^*, \gamma_{t+1|t}^*, \dots, \gamma_{t+N|t}^*], \end{aligned} \quad (23)$$

and the corresponding optimal value function is

$$\begin{aligned} V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ = J_N(z_{t:t+N|t}^*, v_{t:t+N-1|t}^*, \gamma_{t:t+N|t}^*). \end{aligned} \quad (24)$$

For an initial real system state x , let us denote the admissible set fulfilling constraints (21a)-(21j) by $\mathcal{Z}_N(x)$ and the collection of admissible states x ,

$$\mathcal{X}_N = \{x \in \mathbb{R}^{n_x} \mid \exists (z, v, \gamma) \in \mathcal{Z}_N(x)\}. \quad (25)$$

Next, following the manner of feedback algorithm in EMPC, the optimal control implemented to the system (1) is

$$u_t = v_{t|t}^* + K(\hat{x}_t - z_{t|t}^*), \quad (26)$$

and the resulting closed-loop dynamic is

$$x_{t+1} = Ax_t + Bu_t + \omega_t. \quad (27)$$

In addition, the optimal value $\gamma_{t|t}^*$ is recorded as γ_t and will be used in future optimization problems. If the optimal solution $\mathbf{d}_{t|t}^*$ is non-unique, we may take any sequence that achieves the optimum.

B. Recursive feasibility

Note that, provided the optimization problem is feasible, the optimal nominal state-input pair $(z_{t|t}^*, v_{t|t}^*)$ is restricted in the set $\mathcal{Z}$ defined in (10). The rationale is that true state and input solutions, in the usual spirit of tube MPC, will then fulfill

$$(x_t, v_{t|t}^* + K(\hat{x}_t - z_{t|t}^*)) \in \mathbb{Z}, \quad \forall t \in \mathbb{I}_{\geq 0}, \quad (28)$$

whatever the disturbance and noise signals are until time t . Therefore, the real system's solutions fulfill pointwise-in-time constraints.

Proposition 1. Let Assumption 1, 3, 4.(i), and 4.(ii) hold, then the economic MPC optimization problem (21) is recursively feasible.

Proof. The main idea to prove recursive feasibility is to explicitly construct feasible solutions of problems (21), at the current sampling time, given the previous feasible and optimal

solutions. Specifically, the sequences (22) will be shown as a feasible solution.

Notice that the sequences (22) are generated from the optimization problem at time $t - 1$, they fulfill the initial nominal state constraint (21a), the nominal state dynamic equality constraints (21b) and (21g), the pointwise-in-time constraints (21c) and (21h), and the scalar factor constraints (21f) and (21i).

Because the set Ω is invariant with respect to system disturbance and measurement noise, and the homothetic sets circumscribe the reachable states, it holds

$$x_{t-1} \in z_{t-1|t-1}^* \oplus \gamma_{t-1|t-1}^* \Omega, \text{ and } x_t \in z_{t|t-1}^* \oplus \gamma_{t|t-1}^* \Omega$$

which further imply

$$y_{t-1} - Cz_{t-1|t-1}^* \in \gamma_{t-1|t-1}^* C\Omega \oplus \mathbb{H}, \\ \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) \subseteq z_{t|t-1}^* \oplus \gamma_{t|t-1}^* \Omega.$$

In addition, satisfaction of the terminal constraint follows by Assumption 4, viz.

$$z_{t+N-1|t-1}^* \in \mathcal{X}_f \\ \Rightarrow Az_{t+N-1|t-1}^* + B\kappa_f(z_{t+N-1|t-1}^*) \in \mathcal{X}_f.$$

Therefore, the optimization problem at current time t has at least one solution, and recursive feasibility of this optimization problem is ensured. $\square$

IV. STABILITY AND ASYMPTOTIC PERFORMANCE ANALYSIS

This section is dedicated to the analysis of closed-loop behaviors of the system using the proposed EMPC controller. The first part is to prove the stability of the optimal robust invariant set, which is inferred from the asymptotic convergence of the nominal state sequence. Then, an upper bound of the asymptotic average performance, regardless of the disturbance realizations is derived. Before discussing these two closed-loop properties, we need to make the following assumption:

Assumption 5. *The solution in (15) and (17) are the unique optimal steady-state and the optimal period- P operation, respectively, for any $\gamma \in [0, 1]$.*

Remark 8. *This assumption ensures that the optimal regime of operation is independent of the choice of homothetic tube. It is worth to notice that the stability result discussed later is not valid if this condition does not hold. Nevertheless, the use of homothetic tube may provide economic profits, which will be indicated by examples in Section V.*

A. Stability analysis

Asymptotic stability of economic MPC employing Lyapunov function was first proved in [5] under the assumption of strong duality. Such condition is subsequently relaxed by dissipativity in [3] and [26], where it is shown how to rotate the economic cost to standard positive definite function and choosing the rotated cost-to-go as a candidate Lyapunov function. In analogy, this section explores the asymptotic stability of the closed-loop nominal sequence under EMPC control actions.

Assumption 6. *The terminal regions $\mathcal{X}_f$ are compact. In addition, the storage function and the terminal cost function satisfy $\lambda(z_s) + V_f(z_s) = 0$, and $\lambda(z) + V_f(z) \geq 0$ for all $z \in \mathcal{X}_f \setminus \{z_s\}$.*

We are now in position to state the first main result:

Theorem 1. *Let Assumptions 1, 2, 3, 4, 5, and 6 hold, under the application of economic MPC feedback control policy (26), the set $z_s \oplus \Omega$ is asymptotically stable for the closed-loop system (27) with region of attraction $\mathcal{X}_N$.*

Proof. Following the usual way to prove asymptotic stability of a closed-loop system under an EMPC controller with the aid of dissipativity assumptions, we first introduce the rotated stage cost function, rotated terminal cost function as

$$L(z, v, \gamma) = \tilde{\ell}(z, v, \gamma) - \tilde{\ell}_{av}^*(\gamma) + \lambda(z) - \lambda(Az + Bv) \geq 0, \\ \bar{V}_f(z) = V_f(z) + \lambda(z) \geq 0.$$

Accordingly, the rotated objective function at time t is

$$\bar{J}_N(z_{t:t+N|t}, v_{t:t+N-1|t}, \gamma_{t:t+N|t}) \\ = \sum_{k=0}^{N-1} L(z_{t+k|t}, v_{t+k|t}, \gamma_{t+k|t}) + \bar{V}_f(z_{t+N|t}).$$

With the notion of rotated cost functions, readers can refer to [26, Theorem 15] and prove

$$\bar{V}_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) - \bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ \leq -L(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) \leq -\alpha_1(|z_{t|t}^* - z_s|),$$

where

$$\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ = \bar{J}_N(z_{t:t+N|t}^*, v_{t:t+N-1|t}^*, \gamma_{t:t+N|t}^*).$$

Thus, the optimal value of the rotated value function $\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})$ is non-increasing along the trajectory of closed-loop system (27) which implies $|z_{t|t}^* - z_s| \rightarrow 0$ as $t \rightarrow +\infty$.

Next, we provide lower and upper bounds for the function $\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})$. Because of the non-negativity of rotated cost functions, by construction, the optimal rotated objective function fulfills

$$\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ = L(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + \sum_{k=1}^{N-1} L(z_{t+k|t}^*, v_{t+k|t}^*, \gamma_{t+k|t}^*) \\ + \bar{V}_f(z_{t+N|t}) \\ \geq L(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) \geq \alpha_1(|z_{t|t}^* - z_s|).$$

On the other hand, it holds

$$\bar{V}_1(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ \leq L(z_{t|t}^*, \kappa_f(z_{t|t}^*)) + \bar{V}_f(Az_{t|t}^* + B\kappa_f(z_{t|t}^*)) \leq \bar{V}_f(z_{t|t}^*).$$

By induction, for any $N \in \mathbb{I}_{\geq 2}$ and $z_{t|t}^* \in \mathcal{X}_f$, we have

$$\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \\ \leq L(z_{t|t}^*, \kappa_f(z_{t|t}^*)) + \bar{V}_{N-1}(Az_{t|t}^* + B\kappa_f(z_{t|t}^*)) \\ \leq L(z_{t|t}^*, \kappa_f(z_{t|t}^*)) + \bar{V}_f(Az_{t|t}^* + B\kappa_f(z_{t|t}^*)) \leq \bar{V}_f(z_{t|t}^*)$$

Suppose that Assumption 6 is satisfied, the continuous function $\bar{V}_f(z_{t|t}^*)$ fulfills $\bar{V}_f(z_{t|t}^*) \leq \alpha_2(|z_{t|t}^* - z_s|)$, which further yields $\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \leq \alpha_2(|z_{t|t}^* - z_s|)$ [29, Proposition 2.16], for all states within $\mathcal{X}_N$, where $\alpha_2(\cdot)$ is a *class* $\mathcal{K}$ function. Hence, the optimal rotated objective function is bounded globally from above and below

$$\alpha_1(|z_{t|t}^* - z_s|) \leq \bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) \leq \alpha_2(|z_{t|t}^* - z_s|).$$

Therefore, the function $\bar{V}_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})$ is a Lyapunov function and the optimal steady-state is Lyapunov stable. Together with the global attractivity, the steady-state z_s is asymptotic stable in the sense that there exists a *class* $\mathcal{KL}$ function β such that $|z_{t|t}^* - z_s| \leq \beta(|z_{0|0}^* - z_s|, t)$ for all $t \in \mathbb{I}_{\geq 0}$, which in turn implies the set $z_s \oplus \Omega$ is asymptotically stable for the real system. $\square$

B. Closed-loop asymptotic performance

Due to convergence to the optimal regime of operation, the average cost of asymptotically stable closed-loop systems does not deserve special attention. Therefore, in the following we analyze asymptotic average cost of more general systems in which the stability conditions, such as Assumption 2 and Assumption 4.(iii) for $\gamma \in [0, 1)$, are not required.

Next, we state our second main result in this note:

Theorem 2. *Suppose Assumptions 1, 3, 4.(i), 4.(ii) and 4.(iii) only for $\gamma = 1$ are satisfied. Let us choose the initial penalty function $\lambda = 0$, then the asymptotic average performance of the closed-loop system (27), regardless of the system disturbance and measurement noise, fulfills,*

$$\begin{aligned} \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \\ \leq \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \tilde{\ell}_{av}^*(1). \end{aligned} \quad (29)$$

Proof. Given the sub-optimal control input sequence and its resulting state trajectory at time instant $t + 1$ in the same structure as (22), for all disturbance $\omega_t \in \mathbb{W}$ and measurement noise $\eta_t \in \mathbb{H}$, the optimal cost-to-go functions fulfill $V_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) \leq J_N(z_{0:t+N|t+1}^{sub}, v_{0:t+N|t+1}^{sub}, \gamma_{0:t+N|t+1}^{sub})$, which is

$$\begin{aligned} & V_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) \\ & \leq V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) - \lambda(z_{t|t}^*) + \lambda(z_{t+1|t}^*) \\ & \quad + V_f(Az_{t+N|t}^* + B\kappa_f(z_{t+N|t}^*)) - V_f(z_{t+N|t}^*) \\ & \quad + \tilde{\ell}(z_{t+N|t}^*, \kappa_f(z_{t+N|t}^*), 1) - \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) \\ & \leq V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1}) - \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*) + \tilde{\ell}_{av}^*(1), \end{aligned}$$

where the last inequality follows considering $\lambda = 0$ and Assumption 4.(iii) for $\gamma = 1$. Then, integrating the previous

inequalities, for any time $T \in \mathbb{I}_{\geq 0}$, we have,

$$\begin{aligned} & V_N(u_0, \hat{x}_0, y_{0:1}, \gamma_0) - V_N(\hat{x}_0, y_0) \\ & \quad + V_N(u_{0:1}, \hat{x}_0, y_{0:2}, \gamma_{0:1}) - V_N(u_0, \hat{x}_0, y_{0:1}, \gamma_0) \\ & \quad + \cdots + V_N(u_{0:T}, \hat{x}_0, y_{0:T+1}, \gamma_{0:T}) \\ & \quad - V_N(u_{0:T-1}, \hat{x}_0, y_{0:T}, \gamma_{0:T-1}) \\ & = \sum_{t=0}^{T-1} [V_N(u_{0:t}, \hat{x}_0, y_{0:t+1}, \gamma_{0:t}) \\ & \quad - V_N(u_{0:t-1}, \hat{x}_0, y_{0:t}, \gamma_{0:t-1})] \\ & \leq \sum_{t=0}^{T-1} [\tilde{\ell}_{av}^*(1) - \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)]. \end{aligned} \quad (30)$$

Finally, by applying $\liminf$ on both sides of (30) and exploiting boundness of solutions, we obtain

$$\limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \tilde{\ell}_{av}^*(1).$$

To this end, based on the definition of modified cost function, for any $x_t \in z_{t|t}^* \oplus \gamma_{t|t}^* \Omega$, it holds $\ell(x_t, u_t) \leq \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)$, $\forall t \in \mathbb{I}_{\geq 0}$. Therefore, the following inequality is fulfilled

$$\begin{aligned} \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T} \\ \leq \limsup_{T \rightarrow +\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T} \leq \tilde{\ell}_{av}^*(1). \end{aligned}$$

which proves the claim. $\square$

C. Rigid tube-based economic MPC algorithm

As mentioned in Remark 8, in case of Assumption 5 failing, asymptotic stability is not in general guaranteed which might be a dangerous disadvantage of homothetic tube-based economic MPC for some specific applications. However, the asymptotic average performance of the real system can be preserved, and to illustrate this point, a rigid tube-based economic MPC algorithm, in which $\gamma = 1$, defined as following will be employed for comparison,

$$\begin{aligned} \min_{\mathbf{d}_t} \quad & \lambda(z_{t|t}) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_{k|t}, v_{k|t}, 1) + V_f(z_{N|t}) \quad (31) \\ \text{s.t.} \quad & z_{t|t} \in \hat{x}_t \oplus \Xi, \quad (31a) \\ & z_{j+1|t} = Az_{j|t} + Bv_{j|t}, \quad (31b) \\ & (z_{j|t}, v_{j|t}) \in \mathcal{Z}, \quad \left. \vphantom{\sum_{k=t}^{t+N-1}} \right\} \forall j \in \mathbb{I}_{[0, t-1]} \quad (31c) \\ & y_j - Cz_{j|t} \in C\Omega \oplus \mathbb{H}, \quad (31d) \\ & z_{k+1|t} = Az_{k|t} + Bv_{k|t}, \quad (31e) \\ & (z_{k|t}, v_{k|t}) \in \mathcal{Z}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]}, \quad (31f) \\ & z_{t+N|t} \in \mathcal{X}_f, \quad (31g) \end{aligned}$$

where $\mathbf{d}_t = [z_{0:t+N|t}, v_{t:t+N-1|t}]$ is the vector of decision variables. The main advantage of this formulation is reduced computational complexity. Similarly to the previous formulation, recursive feasibility, asymptotic stability and asymptotic performance bounds for this rigid tube-based algorithm can be

proved. However, the modified cost function over a rigid set might reach the sub-optimal solution in some situations which will result into conservatism.

Furthermore, it is also possible to achieve less computational complexity by abandoning moving horizon estimation without losing desired recursive feasibility and stability. However, this may sacrifice the economic performance as the single Luenberger observer for current state estimation is not able to provide a tighter bound on the estimation error. The optimization problem of using Luenberger observer is:

$$\min_{\mathbf{d}_t} \lambda(z_{t|t}) + \sum_{k=t}^{t+N-1} \tilde{\ell}(z_{k|t}, v_{k|t}, 1) + V_f(z_{N|t}) \quad (32)$$

$$s.t. \quad z_{t|t} \in \hat{x}_t \oplus \Xi, \quad (32a)$$

$$z_{k+1|t} = Az_{k|t} + Bv_{k|t}, \quad (32b)$$

$$(z_{k|t}, v_{k|t}) \in \mathcal{Z}, \quad \forall k \in \mathbb{I}_{[t, t+N-1]}, \quad (32c)$$

$$z_{t+N|t} \in \mathcal{X}_f, \quad (32d)$$

which has fewer decision variables $\mathbf{d}_t = [z_{t:t+N|t}, v_{t:t+N-1|t}]$ compared to that in (31).

V. EXAMPLES - CSTR SYSTEM

A. Optimal steady-state operation & Assumption 5 fails

We consider a constant level continuous stirred tank reactor (CSTR) with a single exothermal irreversible first order reaction $A \xrightarrow{k} B$, which has been studied in [7], [27] and [28]. The homothetic tube-based robust EMPC integrated with MHE (MHE-HT) algorithm, the rigid tube-based robust EMPC integrated with MHE (MHE-RT) algorithm and the rigid tube-based robust EMPC integrated with Luenberger observer (LO-RT) are applied to this simplified discrete time linearized CSTR model with sampling time $T_s = 0.1min$,

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + Bu_t + B_\omega\omega_t \\ y_t &= C\mathbf{x}_t + \eta_t \end{aligned}$$

where $\mathbf{x} = [x_1 \ x_2]^T$ and system parameters

$$\begin{aligned} A &= \begin{bmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{bmatrix} \quad B = \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix} \\ B_\omega &= \begin{bmatrix} -0.0002 & 0.0893 \\ 0.1390 & 1.2267 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{aligned}$$

This model consists of two states, the concentration x_1 of the reactant and the temperature x_2 in the reactor. The control input u is the cooling temperature, while the feed flow concentration and the feed flow temperature are regarded as disturbances.

In terms of constraints, the system is subject to

$$\begin{aligned} \mathbb{X} &= \{\mathbf{x} \in \mathbb{R}^2 \mid |x_1| \leq 0.5, |x_2| \leq 5\} \\ \mathbb{U} &= \{u \in \mathbb{R} \mid |u| \leq 15\}. \end{aligned}$$

The disturbance is bounded within the set

$$\mathbb{W} = \{\omega \in \mathbb{R}^2 \mid |\omega_1| \leq 0.2, |\omega_2| \leq 0.05\}$$

and the measurement noise is bounded by

$$\mathbb{H} = \{\eta \in \mathbb{R} \mid |\eta| \leq 0.01\}.$$

By selecting $K = [-84.5145 \ -6.3997]$ and $L = [0.0182 \ 2.6331]^T$, the bounds for errors are shown in Fig. 1, which result into the tightened constraints

$$\mathcal{Z} = \{(z, v) \mid [-0.4811 \ -4.139]^T \leq z \leq [0.4811 \ 4.139]^T, -9.997 \leq v \leq 9.997\}.$$

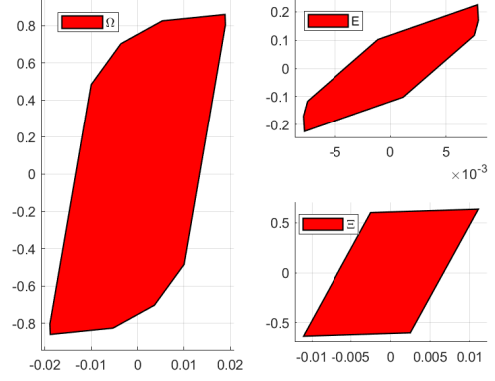


Fig. 1. Forward invariant sets Ω , E and Ξ for state errors.

The stage cost function is written as

$$\ell(\mathbf{x}, u) = x_1 + 0.01x_2^2,$$

where the first term is to boost the reaction from A to B , and the second term can be interpreted as a soft constraint on the temperature, which penalizes the deviation of x_2 from 0. Hence, the modified economic stage cost with a scaling factor γ is

$$\tilde{\ell}(z, v, \gamma) = \max_{\delta \in \Omega} z_1 + \gamma\delta_1 + 0.01(z_2 + \gamma\delta_2)^2.$$

Due to convexity, we know the maximum of the modified cost can be achieved on the vertex of the set Ω , so it is reduced to

$$\tilde{\ell}(z, v, \gamma) = \max_{\delta \in \mathcal{V}} z_1 + \gamma\delta_1 + 0.01(z_2 + \gamma\delta_2)^2.$$

where $\mathcal{V}$ is the collection of vertices of Ω .

According to [7, Theorem 3], the robust optimal regime of operation for the nominal system with $\gamma = 1$ is a steady-state that is

$$(z_s, v_s) = ([-0.0006 \ 0.0337]^T, -0.043954).$$

Note that, for this modified economic cost, different values of γ correspond to different optimal steady-state so that Assumption 5 does not hold. With $\gamma = 1$, a candidate of storage function obtained by using SOSTOOLS is

$$\begin{aligned} \lambda(z) &= 2.67889z_1^2 - 0.04058z_1z_2 - 0.00387z_2^2 \\ &\quad - 5.37833z_1 - 0.00717z_2, \end{aligned}$$

so that the dissipativity inequality holds.

In the simulation, we assume the prediction horizon $N = 12$, initial states $\mathbf{x}_0 = [0.2 \ -3]^T$ and $\hat{\mathbf{x}}_0 = [0.205 \ -2.9]^T$, and use the terminal equality constraint such that Assumption 4 is satisfied and the terminal state is driven to the optimal steady-state in N steps. The optimization is performed using the MATLAB routine *fmincon* with the *sqp* solver. In particular, the computation of sets are based on the Multi-Parametric

Toolbox 3 (MPT3). The following TABLE.I summarizes the computational usage and the asymptotic average economic costs (averaged over 200 simulations with ω and η uniformly distributed over $\mathbb{W}$ and $\mathbb{H}$, respectively) of the three algorithms.

	MHE-HT	MHE-RT	LO-RT
$\tau_f(\text{seconds})$	4.94332	0.68121	0.186665
n_f	4.05545e+09	3.91911e+08	1.81071e+08
$\ell_{av}(T=20)$	0.06133	0.06138	0.06144
$\bar{\ell}_{av}(T=20)$	0.07167	0.08383	0.07874
$\ell_{av}(T=50)$	0.02383	0.02385	0.02387
$\bar{\ell}_{av}(T=50)$	0.03339	0.04946	0.04725
$\ell_{av}(T=100)$	0.01057	0.01059	0.01059
$\bar{\ell}_{av}(T=100)$	0.0205	0.03786	0.03675

TABLE I

AVERAGE COMPUTATIONAL TIME FOR $fmincon(\tau_f)$, CORRESPONDING NUMBER OF FLOPS (n_f), THE AVERAGED ECONOMIC COSTS

($\ell_{av}(T) = \frac{\sum_{t=0}^{T-1} \ell(x_t, u_t)}{T}$), AND THE UPPER BOUNDS OF ASYMPTOTIC

AVERAGE PERFORMANCE ($\bar{\ell}_{av}(T) = \frac{\sum_{t=0}^{T-1} \bar{\ell}(z_{t|t}^*, v_{t|t}^*, \gamma_{t|t}^*)}{T}$), WHERE $\gamma_{t|t}^* = 1$ IN MHE-RT) BY USING THREE PROPOSED EMPC ALGORITHMS, AVERAGED OVER 200 RANDOMLY GENERATED DISTURBANCE AND NOISE SEQUENCES, WHEN ASSUMPTION 5 FAILS.

As expected, while Assumption 5 does not hold, the approach considering the homothetic tube in the optimization provides a better average performance and bounds at the expense of computational complexity. Moreover, it can be seen that the upper bound of MHE-RT is more conservative than that of LO-RT because more constraints on past information are enforced in the optimization problems.

We also run simulations consisting malicious disturbance and noise sequences, $\omega_t = [0.2 \ 0.05]^T$ and $\eta_t = 0.01$, $\forall t \in \mathbb{I}_{\geq 0}$, which intend to maximize the economic cost in prediction. The simulated results are shown in TABLE.II. For these constant disturbance and noise, the optimal value of the scaling factor $\gamma_{t|t}^*$ converges to 0.8946. Comparing these two tables, it can be observed that the method MHE-HT provides a larger gap in average economic performance relative to other two methods when the disturbances and noises are malicious to the system. The reason is that the modified online optimization is considering the worst cost within the tube, so the results in TABLE.II takes more advantages compared to TABLE.I which corresponds to the random noise realizations.

	MHE-HT	MHE-RT	LO-RT
$\tau_f(\text{seconds})$	5.03089	0.799921	0.165637
n_f	5.21042e+09	7.89297e+08	2.12118e+08
$\ell_{av}(T=20)$	0.0718	0.07539	0.0754
$\bar{\ell}_{av}(T=20)$	0.07727	0.08413	0.08186
$\ell_{av}(T=50)$	0.04031	0.04113	0.04114
$\bar{\ell}_{av}(T=50)$	0.04469	0.04971	0.04875
$\ell_{av}(T=100)$	0.02914	0.02944	0.02944
$\bar{\ell}_{av}(T=100)$	0.03354	0.03798	0.0375

TABLE II

COMPUTATIONAL TIME FOR $fmincon(\tau_f)$, CORRESPONDING NUMBER OF FLOPS (n_f), THE ASYMPTOTIC ECONOMIC COSTS ($\ell_{av}(T)$), AND THE UPPER BOUNDS OF ASYMPTOTIC PERFORMANCE ($\bar{\ell}_{av}(T)$) BY USING THREE PROPOSED EMPC ALGORITHMS, UNDER THE MALICIOUS DISTURBANCE AND NOISE SEQUENCES, WHEN ASSUMPTION 5 FAILS.

Note that, without Assumption 5, the algorithm does not guarantee convergence to the robust optimal equilibrium z_s . For instance, given arbitrary uniformly randomly generated disturbance and noise sequences of ω and η , comparing the sequences of the nominal states shown in Fig.2, Fig.4 and Fig.5, the algorithms MHE-RT and LO-RT always enforce initial nominal states converging to the robust optimal steady state in finite time, whereas they do not have convergence behavior in MHE-HT as the value of γ varies.

Therefore, we can conclude that our proposed algorithm MHE-HT can bring some advantages in economic performance while stability convergence is not a necessary requirement. Moreover, as the total cost is monotone with respect to the initial scaling factor, it is observed that in Fig.3 the optimal current scaling factor $\gamma_{t|t}^*$ coincides with the minimal scaling factor $\underline{\gamma}_t$ defined as

$$\underline{\gamma}_t = \min_{\gamma \in [0,1]} \{ \gamma : \mathcal{R}_t(y_{0:t}, \hat{x}_0, u_{0:t-1}) \subseteq z_{t|t}^* \oplus \gamma \Omega \}.$$

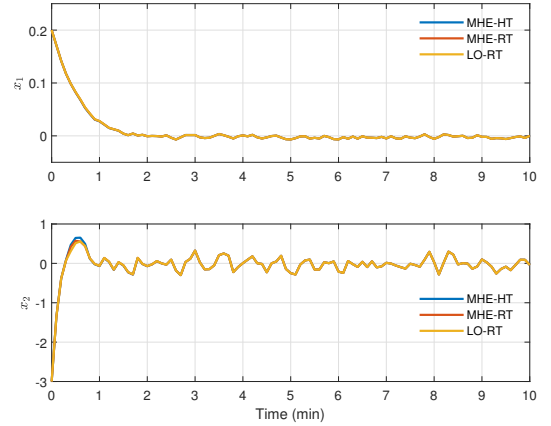


Fig. 2. The real state trajectory in closed-loop when Assumption 5 does not hold.

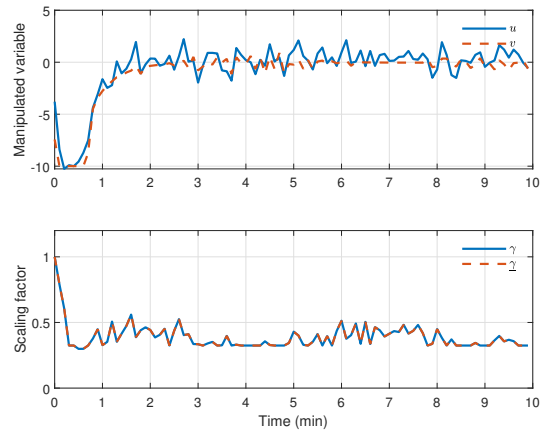


Fig. 3. Sequences optimal control inputs and scaling factors using MHE-HT EMPC algorithm, under randomly generated disturbances and noise sequence, when Assumption 5 does not hold and the optimal operation is a steady-state.

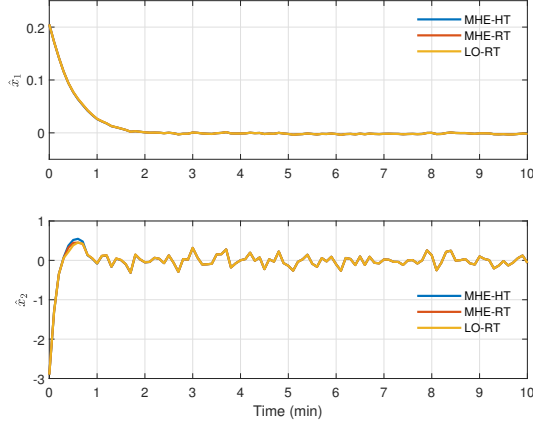


Fig. 4. The estimated state trajectory in closed-loop when Assumption 5 does not hold and the optimal operation is a steady-state.

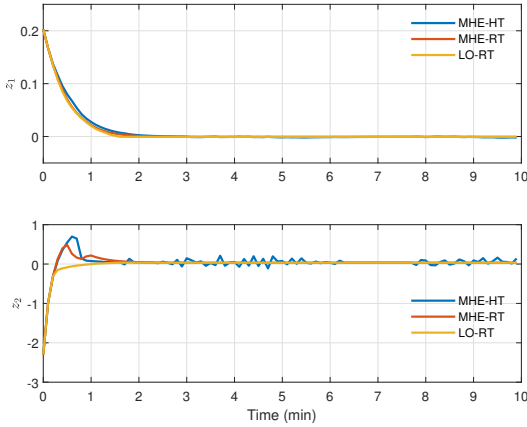


Fig. 5. The nominal state sequence in closed-loop when Assumption 5 does not hold and the optimal operation is a steady-state.

B. Optimal steady-state operation & Assumption 5 satisfied

If different values of the scaling factor γ lead to the same optimal equilibrium, which means Assumption 5 holds, we compare the economic performance of using three algorithms and show that the closed-loop nominal sequence of MHE-HT also converges to the robust optimal equilibrium z_s .

Considering the same system in previous section but with a different stage cost function

$$\ell(x, u) = 50x_1$$

which is to accelerate the reaction from A to B without any penalty on the temperature. Thus, the modified stage cost is represented as

$$\tilde{\ell}(z, v, \gamma) = \max_{\delta \in \mathcal{V}} 50(z_1 + \gamma\delta_1).$$

In this case, for any $\gamma \in [0, 1]$, the robust optimal regime of operation is

$$(z_s, v_s) = ([-0.0740 \ 4.1390]^T, -5.4017).$$

Then, a candidate of storage function with respect to this new stage cost is

$$\begin{aligned} \lambda(z) = & 2734.42z_1^2 - 1.9195z_1z_2 - 0.1929z_2^2 \\ & + 144.29z_1 + 1.0853z_2, \end{aligned}$$

For the initial states $x_0 = [-0.2 \ 2.5]^T$ and $\hat{x}_0 = [-0.195 \ 2.6]^T$, and prediction horizon $N = 12$, the simulated results are shown in the following TABLE.III and Fig.6 - Fig.9. Although the algorithm MHE-HT consumes more time in computation, the resulting economic performance is slightly better than other two cases as tighter characterizations of the state whereabouts allow sharper optimization of the economic criterion. In particular, there is an obvious improvement on performance bound.

	MHE-HT	MHE-RT	LO-RT
$\tau_f(\text{seconds})$	2.61487	0.253959	0.0224035
n_f	1.79524e+09	1.85996e+08	1.66222e+07
$\ell_{av}(T = 20)$	-5.269	-5.268	-5.268
$\bar{\ell}_{av}(T = 20)$	-4.341	-3.99	-3.982
$\ell_{av}(T = 50)$	-4.378	-4.377	-4.377
$\bar{\ell}_{av}(T = 50)$	-3.722	-3.25	-3.246
$\ell_{av}(T = 100)$	-4.097	-4.096	-4.096
$\bar{\ell}_{av}(T = 100)$	-3.506	-3.002	-3

TABLE III
AVERAGE COMPUTATIONAL TIME FOR $fmincon$ (τ_f), CORRESPONDING NUMBER OF FLOPS (n_f), THE AVERAGED ECONOMIC COSTS ($\ell_{av}(T)$), AND THE UPPER BOUNDS OF ASYMPTOTIC AVERAGE PERFORMANCE ($\bar{\ell}_{av}(T)$) BY USING THREE PROPOSED EMPC ALGORITHMS, AVERAGED OVER 200 RANDOMLY GENERATED DISTURBANCE AND NOISE SEQUENCES, WHEN ASSUMPTION 5 IS SATISFIED.

An important observation of adopting these three algorithms is that the closed-loop nominal sequences converge to the common robust optimal state and input pair (z_s, v_s) regardless of the disturbances and noises or the optimal scaling factors. This fulfills our expectation because satisfaction of Assumption 5 ensures convergence, and hence the improvement in average economic cost by using MHE-HT might be less evident.

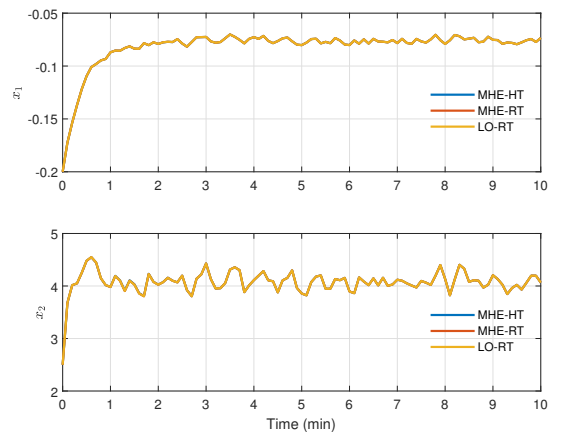


Fig. 6. The real state trajectory in closed-loop when Assumption 5 holds and the optimal operation is a steady-state.

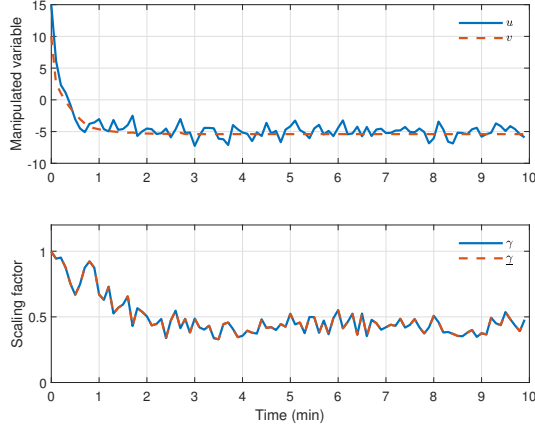


Fig. 7. Sequences optimal control inputs and scaling factors using MHE-HT EMPC algorithm, when Assumption 5 holds and the optimal operation is a steady-state.

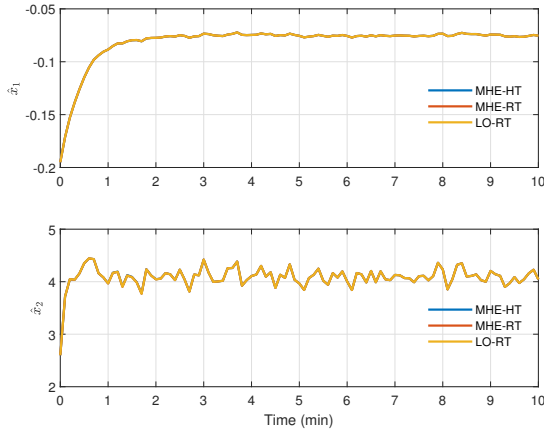


Fig. 8. The estimated state trajectory in closed-loop when Assumption 5 holds and the optimal operation is a steady-state.

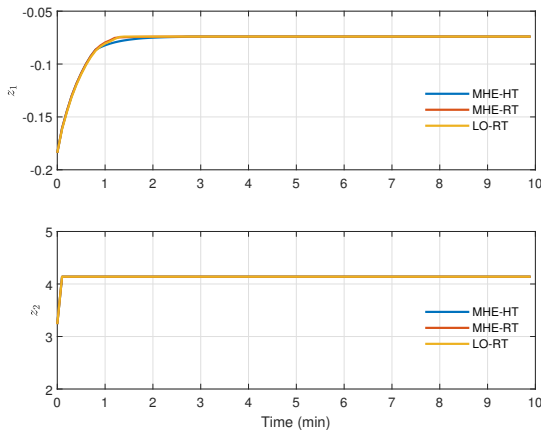


Fig. 9. The nominal state sequence in closed-loop when Assumption 5 holds and the optimal operation is a steady-state.

VI. CONCLUSION

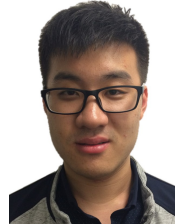
In conclusion, we presented a homothetic tube-based robust economic MPC algorithm integrated with moving horizon estimation for constrained linear discrete time systems. By considering the worst cost within the homothetic tube along the artificial nominal system and a weighting function on nominal initial state, the robustness against the disturbances and measurement noise is guaranteed in our EMPC design. Within this paper, constraint tightening is used to prove recursive feasibility and a less conservative bound of the closed-loop asymptotic average performance is derived. Moreover, the stability of the optimal robust invariant set is inferred from the asymptotic stability of the nominal state sequence.

One of the major drawbacks of the algorithm is its computational burden. We highlight below possible future research directions to mitigate this aspect of the proposed solution. In particular, one may use ellipsoid or polyhedral sets with fixed number of vertices to approximate the robust invariant sets and the reachable sets. Another possibility is discarding past measurements and adoption of finite time-windows. All of these solutions, however, could lead to more conservative results as we are considering the worst cost within an enlarged tube. In this respect, there seems to be, as expected, a trade-off between computational burden and closed-loop system's performance.

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