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**Applications of modal decomposition algorithms
to dynamic estimation and reconstruction of fluid
flows**

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Declaration

I hereby declare that work presented in this thesis is my own except where appropriated referenced and acknowledged.

Jakub Krol

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Abstract

In this thesis, modal decomposition algorithms are utilised to construct reduced-order models of the fluid flows with applications to estimator design and extraction of the dynamical information from the irregularly sampled data.

A particular implementation of such algorithms, the Optimal Mode Decomposition (OMD), is used to construct the estimator frameworks for bluff body flows, more specifically past a circular cylinder and bullet-shaped object. Estimator design is motivated by feedback flow control applications in which online knowledge of the system is paramount. Reduced order models are constructed and combined with stochastic estimation techniques using Moving Horizon Estimation (MHE) algorithm to achieve accurate representations of primary oscillatory and transient features of the cylinder flow across all of its dynamically distinct regimes. The designed estimator is subsequently extended to turbulent flow past a three-dimensional axisymmetric bluff body by introducing rotation-dependent framework. This is able to approximate the main oscillatory features of the flow despite the dynamical complexity of the system.

Furthermore, an extension to the OMD algorithm is introduced, referred to as the is-OMD algorithm, which allows modal decomposition techniques to be applied to vector-valued signal, such as an ensemble of PIV snapshots, sampled at random time instances. A mapping between underlying time-resolved snapshots and known measurements is defined and data is projected onto a low-order subspace. Subsequently, dynamical characteristics, such as frequencies and growth rates, are approximated by iteratively solving two optimisation sub-problems: consecutively updating a reduced-order dynamical model, then subsequently approximating the ‘missing’ data at intermediate snapshots between the known values. The methodology is demonstrated on three dynamical systems: a synthetic sinusoid, the cylinder wake at $Re = 60$ and turbulent flow past an axisymmetric bullet-shaped body. In all cases the algorithm is

shown to correctly identify the characteristic frequencies and oscillatory structures present in the flow from downsampled data ensemble.

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List of Acronyms

ADMM	Alternative Direction Method of Multipliers
CoSaMP	Compressive Sampling Matching Pursuit
CS-DMD	Compressive Sampling with Dynamic Mode Decomposition
fb-DMD	Forward-Backward Dynamic Mode Decomposition
DMD	Dynamic Mode Decomposition
DMDc	Dynamic Mode Decomposition with Control
EDMD	Extended Dynamic Mode Decomposition
is-OMD	Irregularly Sampled Optimal Mode Decomposition
LSE	Linear Stochastic Estimation
MHE	Moving Horizon Estimation
MPC	Moving Predictive Control
nc-DMD	Noise-Corrected Dynamic Mode Decomposition
ODE	Ordinary Differential Equation
OMD	Optimal Mode Decomposition
OMP	Orthogonal Matching Pursuit
opt-DMD	Optimised Dynamic Mode Decomposition
PIV	Particle Image Velocimetry

POD	Proper Orthogonal Decomposition
QSE	Quadratic Stochastic Estimation
ROM	Reduced Order Model
SE	Stochastic Estimation
SNR	Signal-to-Noise Ratio
TDMD	Total Dynamic Mode Decomposition
tlsDMD	Total Least Squares Dynamic Mode Decomposition
tomo-PIV	Tomographic Particle Image Velocimetry
TRPIV	Time-Resolved Particle Image Velocimetry

Chapter 1

Introduction

1.1 Introduction

Very often in the analysis of nonlinear dynamical systems, such as fluid flows, it is the case that one is presented with an overabundance of data. In such cases it is common that the underlying system has such complicated dynamics that it is difficult to identify key, or practically useful, features even with a large numerical or experimental data set. Because of these difficulties there is an increased interest in finding the best way to construct models which are low order but at the same time retain information about the most important dynamical features. Reduced order models are already used in a variety of applications, primarily to aid controller design or to facilitate analysis of flow dynamics. In control applications, for example, emphasis is placed upon balancing the desired accuracy of the model but at the same time retaining its low-dimensionality to ensure computational efficiency of the control methodology.

Although most practical fluid flows have high-dimensional dynamics described by nonlinear equations, in many examples it is possible to isolate certain *coherent structures* (alternatively referred to as *modes*) which are associated with some important features of the underlying dynamical system. Such modes will have the form of a time-independent scalar or vector field consisting of, for example, pressure values or velocity vectors at different spatial locations. In many canonical examples the existence of coherent structures is well established. For example, vortex shedding modes in bluff body flows [1] or spatial streaks in wall-bounded flows [2]. In order to provide an intuitive understanding of what a mode is and what criteria can be used to identify them, let us first look more closely at these examples to see what type of analysis

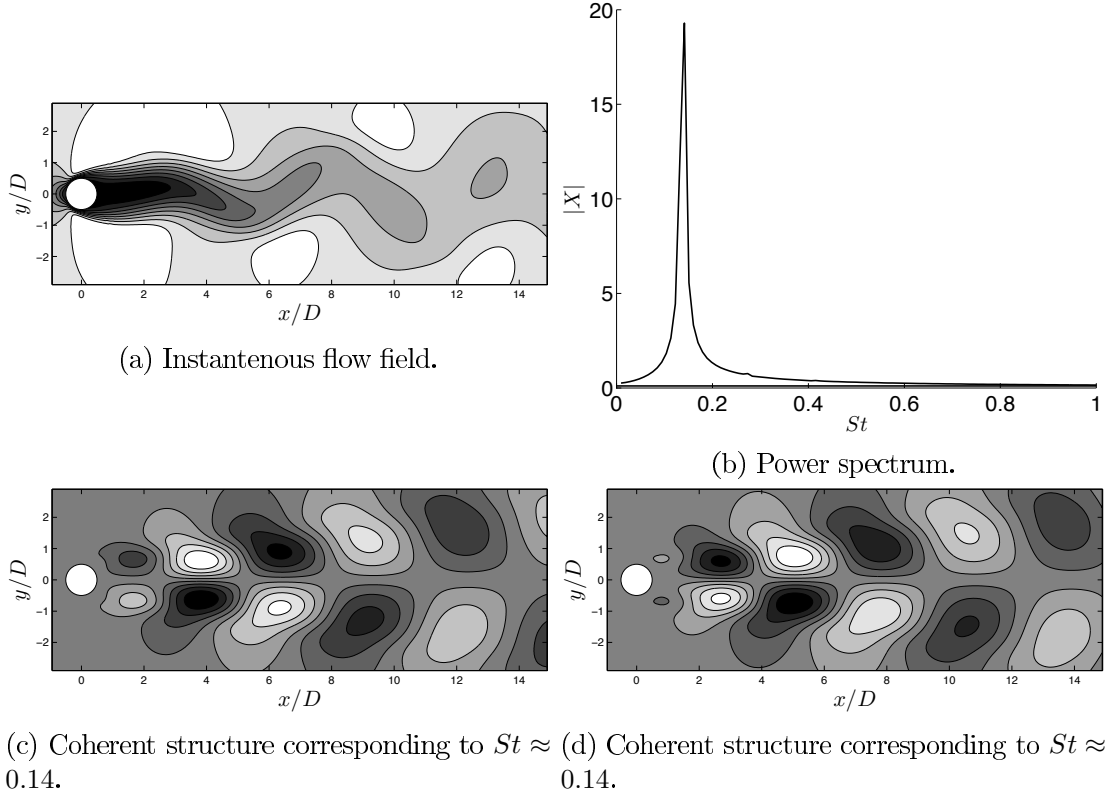


Figure 1.1: The instantaneous streamwise velocity distribution of the cylinder flow (a); the power spectrum of the vertical velocity component at centreline of the flow $1D$ in the streamwise direction (b); streamwise velocity components of two structures corresponding to a frequency with maximum magnitude of the Fourier coefficient at $St \approx 0.14$ (c)-(d).

can be facilitated through an identification of coherent structures.

A canonical example of a bluff body flow is flow past a circular cylinder. At low Reynolds number $Re = (UD)/\nu$ for a flow with free-stream velocity U , cylinder diameter D and kinematic viscosity ν , this system is characterised by a periodic oscillation of the wake, clearly visible in figure 1.1(a), which shows the streamwise velocity distribution at an arbitrary time instance. A probe in the wake allows us to obtain the Fourier spectrum of a point velocity signal (figure 1.1(b)) which, for $Re = 60$, indicates the presence of a single primary Strouhal number $St \approx 0.14$, where $St = (fD)/U$ and f is the frequency. The fact that the frequency spectrum is relatively simple suggests that the flow dynamics can be well approximated by a model of relatively low-order. It is in fact possible to identify Fourier-like structures associated with this frequency. Examples of methodologies which can be used for this purpose will be given in detail in §2.1 and, for now, only the resulting structures are presented in figure 1.1(c)-(d). This pair of *modes* appear to be offset by approximately one quarter of their lengthscale which allows us to approximate the propagation of the main spatial flow features in the streamwise direction.

In fact, using the mean flow and the presented structures, it is possible to reconstruct the dominant shedding behaviour using a simple linear combination of modes with time-varying coefficients. Let us define a time-discretised vector field $\mathbf{u}_i = \mathbf{u}(\mathbf{x}, t_i) \in \mathbb{R}^{N_u}$ where $\mathbf{x} = (x_j)_{j=1}^{N_u} \subseteq \mathbb{R}^2$ is a set of N_u co-ordinates in two-dimensional space. For now, let us assume that $t_i = i\Delta t$ where Δt is a constant time step and $i = 0, 1, \dots, N$. A coherent structure will be denoted by $\Phi_j(\mathbf{x}) \in \mathbb{R}^{N_u}$ and is evaluated at exactly the same spatial locations as $\mathbf{u}_i$. Note that Φ_j is time-invariant. However, information about its dynamics can be incorporated by associating Φ_j with a time-varying coefficient $a_j(t_i)$, which can be interpreted as the “importance” of each mode at each time t_i . Consequently, assuming the chosen coherent structures span the possible space of observed flow velocity fields, it is possible to decompose the original vector field such that

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{\infty} a_j(t) \Phi_j(\mathbf{x}). \quad (1.1)$$

Due to the potentially complex dynamics of the underlying system, the decomposition (1.1) is in general infinite. Nonetheless, in many practical flows it is reasonable to select finite number of Φ_j and a_j to obtain r -dimensional approximation $\hat{\mathbf{u}}$ of the form

$$\mathbf{u}(\mathbf{x}, t) \approx \hat{\mathbf{u}}(\mathbf{x}, t) = \sum_{j=1}^r a_j(t) \Phi_j(\mathbf{x}). \quad (1.2)$$

For example in the cylinder flow, if Φ_j are set to structures seen in figure 1.1(c)-(d), the coefficients $a_j(t)$ can be chosen in such a way so that the main shedding behaviour is approximated with a relatively small least-squares error

$$\frac{\sum_{i=0}^N \|\hat{\mathbf{u}}_i - \mathbf{u}_i\|_2^2}{\sum_{i=0}^N \|\mathbf{u}_i\|_2^2} \cong 0.04, \quad (1.3)$$

where $\|\cdot\|$ is l_2 -norm. It is reiterated that this accuracy is achieved only for $r = 3$ where spatial structures consist of those shown in figure 1.1(c)-(d) and the mean flow field. For a given set of $a_j(t)$, if it is possible to find $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$ such that trajectories of the differential equation

$$\frac{d\mathbf{a}}{dt} = f(\mathbf{a}), \quad \mathbf{a} = (a_j)_{j=1}^r, \quad (1.4)$$

fit those of the coefficients a_j in (1.2) then a model will be obtained which approximates the

dynamics using a significantly smaller number of degrees of freedom.

The simple analysis above already provides us with an indication of why coherent structures can be useful in practical problems. The first immediately obvious application is data compression. For a temporal series of N velocity fields $\mathbf{u}_i$ with two components and spatial resolution of N_u points, $2N_u N$ pieces of information have to be stored. However, the high accuracy (1.3) was achieved through a linear composition of just three $2N_u$ -dimensional modes. In such a situation only the time-varying coefficients need to be stored at each instance of time, implying that only data of dimension $6N_u + 3N$ is required to be stored. For large N the corresponding savings can grow drastically. Additionally, let us consider a criterion used to choose the structures in figure 1.1(c)-(d). The presented modes correspond to a single frequency of $St \approx 0.14$ and the corresponding time-varying coefficients a_i in the decomposition (1.2) exhibit sinusoidal-like behaviour. Consequently, although the fluid flow is modelled by a nonlinear PDE, it is possible to isolate modes which could be modelled using linear, low-dimensional, linear oscillator-like dynamical model. Such a radical simplification of dynamics and reduction of degrees of freedom provides us with a dynamical model which is simple enough to be used in *online* control and estimation applications.

Areas where the use of coherent structures are valuable extend further than data compression and control applications. Often the main purposes is to simply facilitate our dynamical understanding of complicated fluid flows. One example is a turbulent boundary layer where, for example, in [3] coherent structures were extracted to obtain an insight into how different lengthscales change with varying wall roughness. The dynamics of the turbulent boundary layer is known to be complicated [4] but through calculation of modes it is possible to separate structures exhibiting different spatial lengthscales. Based on these modes it is significantly easier to, for example, examine the intensity of different lengthscales in the boundary layer or separate different frequencies in the flow for different types of surfaces. A different objective of using coherent structures in the analysis of boundary layers was presented in [5] where structures with the dynamics characterised by a positive exponential growth rates were identified. Such dynamics often govern the development of instabilities from certain desirable configurations. For example, in [5] those modes were used to examine the transition from laminar to turbulent regimes.

The structures discussed so far are only two examples among a wide range of dynamical systems where the calculation of modes is advantageous. Examples of fluid flows where coherent structures facilitated an analysis of dynamical system include jets [6, 7], turbulent cavity flow [8], combustion instabilities [9], pipe flow [10] and wind turbine wakes [11]. We should also note that the appropriate choice of structures will vary depending on the objective. For example, looking for values of Φ_i which provide the most accurate linear decomposition in (1.2) may not be the most optimal approach if the objective is to isolate structures of certain frequencies or unstable modes (which can be of relatively low energy [12]). Often, especially for estimation and control purposes, more sophisticated, albeit still low-dimensional, dynamical models are required rather than elementary knowledge of frequencies and growth rates. In such cases, selection of structures which would facilitate the construction of a dynamical model is essential, but this task is often non trivial. For instance, in the example of the cylinder flow above an oscillator-like approximation of the dynamical system was proposed, however, such a model would be valid only when the shedding in the wake is fully developed and is not necessarily appropriate to simulate the development of perturbations from a steady-state flow.

A typical systematic approach to construct a more general low-dimensional model would be to attempt to obtain dynamics of coefficients a_i in (1.2) using Galerkin projection [13]. In this procedure, first, the decomposition (1.2) is substituted into Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{Re} \Delta \mathbf{u} - \nabla p, \quad (1.5a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (1.5b)$$

and inner products are taken between such modified equations and chosen structures Φ_i to obtain a set of ordinary differential equations describing the form of temporal derivatives $\dot{a}_i$. Since (1.5a) contains the quadratic term $(\mathbf{u} \cdot \nabla) \mathbf{u}$ the resulting set of equations will have the form

$$\dot{a}_j(t) = A_{jk} a_k(t) + B_{jkl} a_k(t) a_l(t), \quad (1.6)$$

where summation over repeated indices is implied. The coefficients A_{jk} , B_{jkl} are functions of Φ_j with their exact form given in numerous references (for example in [14]). However, their calculation, which requires first and second-order spatial derivatives of Φ_j to be computed, may

often be sufficiently sensitive that significant dynamical inaccuracies could arise [14]. An alternative is to use historical values of a_j and obtain the values of coefficients A_{jk} , B_{jkl} numerically, for example, by solving an optimisation problem related to (1.6). Even then, different choices of structures Φ_j (and consequently a_j) or of historical data may lead to dynamical models providing a varied range of behaviours and accuracies. In most cases deciding a priori about the most optimal Φ_j for accurate model construction is not straightforward.

1.2 Thesis Objectives

The overall objective of this thesis is to investigate different applications of the approximation (1.2) and explore the suitability of different approaches for construction of coherent structures to obtain appropriate dynamical information, whether with an objective of system modelling or obtaining an insight into dynamics of the underlying systems. The analysis will be performed on two different bluff body flows: cylinder flow at low Re and a turbulent flow past a bullet-shaped object. Aside from applications in drag reduction, both systems are characterised by features which make those test cases non-trivial. In particular, this thesis attempts to answer the following questions:

Q1 : What is a systematic way to rank dynamical modes based on combined energetic and dynamic considerations?

In §3.1 a criterion which can be used to identify coherent structures of importance is outlined. Nonetheless, the modes in the presented examples were selected based on the information available *a-priori*, for instance, in the example of the cylinder wake, structures in figure 1.1(c)-(d) were chosen based on association with frequency known to exist in the flow. However, in many situations a correct approach of identifying r dynamically relevant structures out of infinite series (1.1) is not so obvious. In §3.1 a methodology will be proposed which ranks coherent structures by combining energetic and dynamical considerations which can then be used to truncate the dimensionality of the problem accordingly.

Q2 : What is an appropriate form of the low-dimensional model of a flow transitioning from unstable equilibrium to limit cycle and what approaches should be used to design the robust and accurate estimator of the system's most dominant features?

One of the canonical examples of the dynamical system which undergoes such an intrin-

sically nonlinear behaviour is the cylinder flow. Analysis of the associated Hopf bifurcation implies that accurate description of such dynamics can be achieved using third-order model (with details given in chapter 4). Nonetheless, it was observed previously that Galerkin projection of the Navier-Stokes equations shows that dynamical models of the form $\dot{\mathbf{a}} = f(\mathbf{a})$ approximating fluid flows should be quadratic (1.6). The visible inconsistency poses a question about the appropriate modelling approach for such systems. In chapter 4 analysis will be performed on the cylinder flow at low Re to establish which model complexity is the most appropriate to describe dynamics of this system. Furthermore a framework is developed to provide a mapping between different dynamical features and pressure measurements on the base of the object. Since the cylinder flow is characterised by distinct regimes with different dynamics, an analysis is conducted to assess what impact a choice of data used for modelling has on the accuracy of the estimation.

Q3: Is it possible to extract meaningful coherent structures of a three-dimensional, chaotic, noisy system using two-dimensional velocity fields and use such structures to design an accurate velocity field estimator?

Despite its nonlinear behaviour, the cylinder flow at low Re is characterised by a relatively simple dynamics compared to other fluid flows. Consequently, the estimation analysis is extended to the bluff-body wake with more complicated, three-dimensional behaviour. A suitable example for such a study is the bullet flow at high Re , whose dynamics are chaotic due to the nature of turbulence. The challenge of the accurate dynamical modelling is further increased by the fact that the analysis of this three-dimensional system is performed using two-dimensional experimental measurements. Those difficulties invoke the question of whether accurate dynamical estimation of the flow field in such situations is even possible. In chapter 5, it will be shown what known features of the flow can be extracted from 2D data and a framework will be developed to estimate three dimensional dynamics of the flow in a non-intrusive manner.

Q4 : Can modes be extracted accurately from temporally downsampled data ensembles?

In many applications, it is advantageous to identify coherent structures based on dynamical considerations such as frequency or growth rate criteria. So far it was implicitly assumed that underlying data $\mathbf{u}_i$ is sampled at a constant timestep Δt , which is small enough to contain

all dynamical information of interest. If the data used for the analysis does not meet those requirements, e.g. if it is under-sampled or sampled at non-constant timestep, isolation of modes associated with dynamical information becomes significantly more challenging. In order to facilitate identification of coherent structures and dynamics under those circumstances a framework is developed in chapter 6 which facilitates construction of modes corresponding to single frequencies and growth rates from irregularly sampled datasets. The novel methodology will be analysed on the bluff body systems outlined above. The work presented in chapter 6 was published in the Journal of Fluid Mechanics.

1.3 Dynamical Estimation of Bluff Bodies

Bluff body flows are prominent in a variety of applications, with examples including flow past automobiles, submarines or even high-rise buildings. At low Reynolds numbers the primary drag force is a result of skin friction. However, for higher values of Re commonly encountered in practice, unsteady vortex shedding creates a significant pressure drop downstream of the body. This phenomenon typically leads to an increase in drag, structural vibrations, acoustic resonance or increases in lift fluctuations [1].

Due to these implications of the vortex shedding dynamics, significant effort has been made to alter or even suppress them. Most notably, passive and active open loop control has been used for this purpose, with examples including end plates, splitter plates, base bleed or rotary oscillations. An extensive review of the open loop control strategies applied to bluff bodies can be found for example in [1, 15, 16]. Despite open loop approaches being somewhat effective and easily implementable, in recent years active feedback control methods have attracted increasing attention due to the advancement of microelectromechanical systems, developments in control theory and improvements in computer power. In addition to actuation, feedback methods require sensing techniques in order to obtain instantaneous information about the state of the wake. A schematic representation of how a control input $\mathbf{v}$ in such a case can be chosen is shown in figure 1.2. Measurements $\mathbf{y}$ of the high-dimensional system, e.g. pressure values, are fed into an estimator, which combines the observations with a dynamical model to give an estimated state $\hat{\mathbf{x}}$ of the flow. Subsequently, based on the obtained estimate, and in some cases values of $\mathbf{y}$, the controller determines an input to be applied to the dynamical system based on a certain desirable objective such as suppression of wake unsteadiness. In chapters 4-5 emphasis

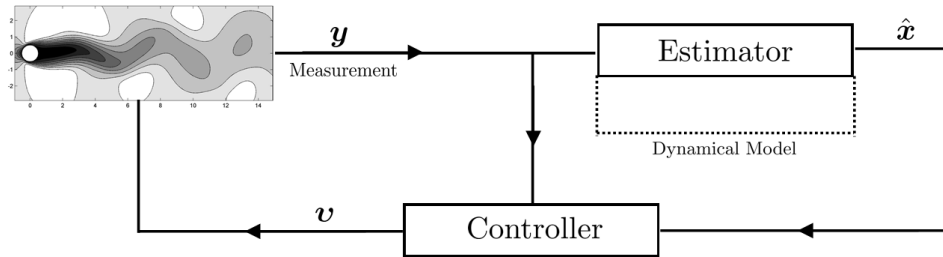


Figure 1.2: A schematic representation of the feedback control methods.

will be put on designing accurate low-dimensional estimators for wake dynamics. An important consideration concerns the number of sensors and the dimension of the state space system such that it is low enough to obtain estimated state $\hat{x}$ and apply control input v in a timely manner but high enough to capture sufficient dynamical information. One way to achieve this goal is to construct reduced order models (ROMs) based on the decomposition (1.2), which can be used to develop simpler and computationally more efficient control strategies than those based on full flow field information.

A lot of research has focused on control strategies for bluff body flows in which ROMs were constructed by setting coherent structures Φ_j to Proper Orthogonal Decomposition (POD) modes, obtained by decomposing an ensemble of snapshots into orthogonal flow fields ranked by their energetic content (a more detailed description of this methodology is given in §2.1.1). For example, a POD-based ROM combined with a neural network controller was used in [17] to suppress vortex shedding oscillations in a cylinder flow. In [18] a controller was designed to modify the cylinder wake at $Re \approx 100$, although it was shown that instabilities can be suppressed the prediction errors of the ROM increased during control. Such dynamical inaccuracies may cause controller actions which lead to undesirable system configurations. The same problem as in [18] has been analysed in [19] who used only the first POD mode and a linear proportional and differential feedback controller to stabilise the wake using traverse cylinder movements. Despite these achievements, certain issues concerning construction of ROMs remain. Since the described models are not mathematically equivalent to the Navier-Stokes equations, a question remains as to whether the chosen structures and resulting dynamical models are able to accurately and robustly predict the dynamics of the system. In particular, in all approaches above Φ_j were chosen to be POD modes. This methodology is a statistical tool (as outlined in §2.1.1) and dynamical considerations are not used in the selection of structures, meaning that

POD modes are not necessarily guaranteed to provide the most optimal basis for dynamical modelling.

If the data is sampled at a constant timestep, Dynamic Mode Decomposition (DMD) [20] can be applied to incorporate temporal information when extracting modes. The DMD algorithm constructs a linear model approximating the dynamics of the projection coefficients $a_i(t)$ when Φ_j are chosen to represent the POD structures. Subsequently the new DMD modes are obtained as a linear combination of POD structures where the scalars will be dependent on eigenvectors of a particular linear operator in order to produce complex modes associated with a single eigenvalue. A number of generalisations and extensions of the DMD algorithm have been proposed recently and an outline of these different methodologies is presented in §2.1. In this thesis, particular emphasis is placed on one generalisation of the DMD algorithm, the Optimal Mode Decomposition (OMD) algorithm [21], which instead of fitting a linear operator to the POD projection coefficients, looks for the low-dimensional projection space which facilitates the construction of the linear operator.

In chapters 4-5 we construct low-order dynamical models using OMD modes which, when combined with pressure measurements, can be used to obtain accurate estimates of the full flow field. We demonstrate that the OMD modes due to their dynamical interpretation perform better than POD modes in a majority of cases. The analysis is performed on two types of bluff bodies: a flow past a canonical cylinder flow at $Re = 60$ and the turbulent flow past a bullet-shaped object at $Re = 1.88 \times 10^5$. The first case is a widely analysed flow with well understood dynamics and the analysis is performed using numerical data allowing us to perform a reliable assessment of different methodologies. The choice of the latter system is motivated by the question of drag reduction for bluff-bodies at high Re such as road vehicles, where the aerodynamic drag has a significant impact on fuel consumption and CO₂ emissions [22]. Consequently, implementation of active control strategies in a non-intrusive manner to alleviate those issues is particularly important. This dynamical system is turbulent and the dynamical modelling is performed based on the experimental data making a dynamical estimation significantly more challenging. For both systems an investigation is performed into the appropriate choice of dynamical modes linking the low and high dimensional systems and an appropriate modelling technique in the low dimensional space.

We begin by discussing the fundamental features of the two flows before attempting to answer these questions in subsequent chapters.

1.3.1 Cylinder flow at $Re = 60$

The flow past a circular cylinder at $Re = 60$ is characterised by three distinct dynamical regimes as presented in figure 1.3; an unstable equilibrium, which, subject to a perturbation, transitions into an oscillatory limit cycle corresponding to vortex shedding. In order to find a corresponding ROM an obvious approach would be to construct a quadratic model based on the form of the Galerkin projection of Navier-Stokes equations (1.6). However, the behaviour visible in figure 1.3 is a result of a Hopf bifurcation which cylinder flow undergoes at the critical Reynolds number $Re_C \approx 46$ when the equilibrium point becomes unsteady and vortex shedding develops [23]. This dynamical feature can be observed through an examination of the eigenvalue spectrum of the linearised Navier-Stokes equations, where for $Re < Re_C$ all the eigenvalues are found in the stable left-half-plane, whilst for $Re > Re_C$ an unstable pair of complex conjugate eigenvalues is observed.

This Hopf bifurcation can be described by a cubic Stuart-Landau equation

$$\frac{dA}{d\tau} = \tilde{\lambda}A - \mu A|A|^2, \quad (1.7)$$

which is the normal form of a Hopf bifurcation. Here, $\tilde{\lambda}, \mu \in \mathbb{C}$, $\tau = \epsilon t$ is a long timescale such that $\epsilon = Re_C^{-1} - Re^{-1} \ll 1$, whilst $A(\tau)$ is the amplitude of the state corresponding to the global mode. The global modes of the cylinder flow are coherent structures corresponding to the most unstable eigenvalue pair of the Navier-Stokes operator linearised around the steady-state solution and are known to be primarily responsible for the vortex shedding [24]. It is noted that theoretically (1.7) is valid only when Re is close to Re_C . Nonetheless, it will be shown in chapter 4 that (1.7) can be used for modelling of the cylinder flow at $Re = 60$.

The structures approximating global modes were already presented in figure 1.1(c)-(d). $A(\tau)$ in essence governs the long term dynamics of the global modes. If those structures are associated with the coefficient a_i in (1.2) their dynamics will have a form $a_j(t) = A(\tau)e^{i\theta t}$, where $A(\tau)$ satisfies (1.7) and $i\theta$ corresponds to the asymptotic eigenvalues of the linearised Navier-Stokes operator. All the subsequent harmonics of the flow can be modelled using coupling

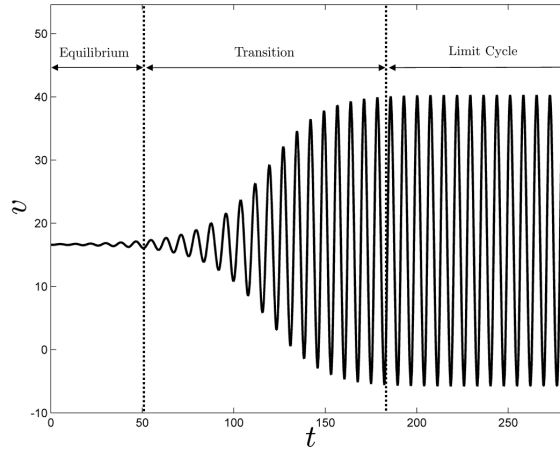


Figure 1.3: The temporal evolution of vertical component of the velocity, v , on the cylinder's centerline at a location just downstream of the base.

between previous sub-harmonics and appropriate powers of A . Further analysis of (1.7) will be given in chapter 4.

It is notable that (1.6) and (1.7) have different orders, i.e. (1.6) is quadratic while (1.7) is cubic creating ambiguity in the construction of the appropriate ROM. It can also be the case that a linear model combined with an appropriate estimation technique is sufficient to obtain state estimates of acceptable accuracy. Consequently, in chapter 4 dynamic estimation based on the ROMs of different orders of complexity is analysed. It will be shown that a cubic model gives estimates of greater accuracy than equivalent results for linear and quadratic models, where in the latter cases majority of the models of the form (1.6) are found to be unstable. In addition, a qualitative comparison is performed between POD and OMD modes in terms of suitability for dynamical modelling. If the dynamical models are obtained based on the limit cycle the POD basis gives a smaller estimation error, whilst, if data sets contains other regimes, OMD modes are more preferable. Additionally, the OMD basis proved to be more accurate for modelling of low-energy dynamics.

1.3.2 Flow past a bullet-shaped object at high Re

The second flow considered in this thesis is the flow past a bullet shaped object at $Re \approx 1.88 \times 10^5$. The analysis is performed based on two-dimensional PIV measurements in the wake and pressure measurements at the base of the object, as shown in figure 1.4. For this system extraction of the appropriate coherent structures is significantly more challenging than for the low Re cylinder flow, since at $Re > 1900$ the bullet flow is turbulent [25] with three

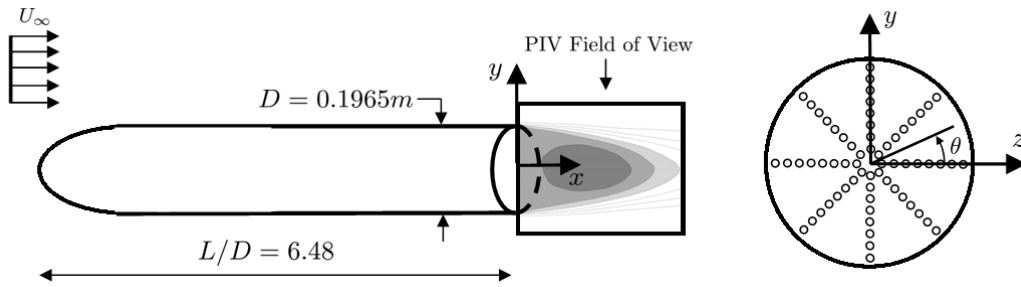


Figure 1.4: Schematic representation of the experimental model and PIV field of view (left) and the location of the 64 pressure tappings on the base of the model (right). The velocity field is defined as $\mathbf{u} = (u, v)$ where u and v are velocity components along x and y axis respectively.

dimensional dynamics. Despite the chaotic nature of the dynamics, it was shown in [26] that the system retains key features which can be observed at a lower Re : more specifically, steady symmetry breaking at $Re \approx 450$; vortex shedding developing at $Re \approx 590$; and a pulsation of the vortex cores producing a second fundamental frequency peak at $Re \approx 690$. In particular, at the Re analysed here, two fundamental frequencies can be observed: the shedding dynamics at $St \approx 0.2$; and pulsation dynamics at $St \approx 0.06$ [26]. It is furthermore noted that the structures corresponding to these dynamics are symmetric with respect to a certain reflectional plane with a random azimuthal orientation. It was also observed that the reflectional symmetry plane changes its reorientation at approximately $St \approx 0.002$, and all the other dynamics are invariant with respect to this rotation.

In chapter 5 dynamical estimation is performed to estimate dynamics of the structures corresponding to both $St \approx 0.06$ and $St \approx 0.2$ from pressure measurements. We show that it is possible to identify three-dimensional features corresponding to those frequencies from two dimensional data. A main challenge in dynamical estimation relates to the fact that shedding frequency has a low imprint on the pressure dynamics [26], which is confirmed in chapter 5 where the estimation of the dynamics using linear stochastic estimation (LSE) shows that only the low frequency dynamics are estimated accurately. Furthermore, dynamical estimation is even more challenging due to a clear directional offset present in the base pressure structures depending on the reorientations of the symmetry plane. Consequently, a single, time-constant set of LSE coefficients is not necessarily suitable for all system configurations. In chapter 5 a framework is developed which allows us to accurately estimate key dynamics present in the system by constructing an orientation-dependent method of selecting the measurements and combining it with appropriate dynamical estimation technique.

Due to the fact that estimation of cylinder and bullet flows will make use of similar types of techniques, an outline of the methodologies applied to both systems will be given in §2.2.

1.4 Dynamic reconstruction from undersampled data

Modal decomposition algorithms can be used in a numerous non-obvious applications extending beyond low-order modelling for control, with one of them presented here. A key feature of the approximation (1.2) is the assumption that it is close to equality for a truncated number of modes. This assumption effectively implies that despite the high-dimensionality of the system the underlying dynamics is in fact *sparse* with respect to the number of coherent structure or states required to well describe its behaviour. Such sparsity of the system has interesting applications and is the focus of the research field of *compressive sampling* [27, 28, 29] where it has been demonstrated that if a signal is sparse then dynamical information can be extracted even if the signal is significantly undersampled. This is especially important in situations where the acquisition of measurements at a desired frequency is not possible, data is lost, or the cost requirements of acquisition are prohibitive.

Extraction of dynamical information from undersampled data is also relevant to fluid flows. The acquisition rate of a classical PIV system does not typically exceed 15 Hz implying an upper limit upon the sampling rate which is not sufficient to investigate temporal characteristics of turbulent flows. Time-resolved PIV (TRPIV) systems exist that are capable of producing a sampling rate in the range of 1-10 kHz [30], nonetheless, they still may not be sufficient to investigate frequencies of interest. Additionally, tomographic PIV (tomo-PIV) was introduced recently [31], allowing the measurement of three velocity components. In such a case, the number of measurements drastically increases, especially in a combination with TRPIV systems, leading to a high computational cost of data processing and consequently establishing a need to obtain dynamical information from potentially undersampled data.

The methodologies which decompose flow fields using a finite number of coherent structures in essence assume that the dynamical systems are sparse in a certain basis. Consequently in chapter 6 it is investigated whether those types of algorithms can be extended to applications of extracting dynamical information from undersampled data. One of the attempts to extend the DMD algorithm to regularly undersampled data sets is presented in [32] where a low-dimensional state space is constructed using POD modes, *missing* snapshots are reconstructed

using an Expectation-Maximisation algorithm and DMD modes are estimated as a solution to a multiple regression problem. The most widely employed compressive sampling methodologies are still relatively new in the fluid dynamics community primarily due to the fact they are most effective in the situations where data snapshots are sampled randomly in time. An attempt to combine modal decomposition algorithms with compressive sampling theory was presented in [33] where compressive sampling is directly performed on a downsampled velocity field data ensemble, then subsequently DMD is used to extract coherent structures.

In chapter 6 an alternative is proposed, the irregularly sampled Optimal Mode Decomposition (is-OMD) algorithm. The motivation for our approach is to extend traditional modal extraction methods so that they can be applied to data sampled at non-constant timesteps. In chapter 6 we show that is-OMD provides a flexible method for the analysis of high-dimensional temporally subsampled data ensembles. Further on, the proposed method makes use of already existing DMD-like algorithms, making it readily and easily implementable. An outline of the existing methodologies with an aim to perform dynamical reconstruction from undersampled data is presented in §2.3.1, whilst the description and the performance assessment of the is-OMD is presented in chapter 6. The analysis is performed on a synthetic sinusoidal flow and on flow past a circular cylinder at a low Re , showing improvement over compressive sampling based mode extraction methodologies. Furthermore, an analysis on turbulent bullet flow demonstrates the potential of the is-OMD algorithm to be applicable to more complicated dynamical systems.

Chapter 2

Literature Review

2.1 Modal Decomposition Algorithms

Since this thesis primarily aims to apply low-order modelling techniques to high dimensional fluid flows, a brief description of various coherent structure extraction methodologies is given here. It was mentioned before that for canonical flows, coherent structures Φ such as vortex shedding modes of bluff bodies flows [1] or spatial streaks in wall-bounded flows [2] are well known. However, for less-studied flows, coherent structures may not be known *a-priori* and must therefore be sought directly from the snapshot data, necessitating the development of automatic, data-driven, structure extraction algorithms.

All of the algorithms considered below are data driven aimed to be applied to spatially discrete vector fields without a knowledge of the underlying equations. Let us denote time-dependent, discretised velocity vector field such that $\mathbf{u}_i = \mathbf{u}(\mathbf{x}, t_i) \in \mathbb{R}^{N_u}$ where $\mathbf{x} = (x_j)_{j=1}^{N_u} \subseteq \mathbb{R}^3$ and $t_i = i\Delta t$ where Δt is a constant time step and $i = 0, 1, \dots, N$. For the ease of presentation let us express the full data ensemble in matrix form as

$$\mathbf{U} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{u}_0 & \mathbf{u}_1 & \cdots & \mathbf{u}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \in \mathbb{R}^{N_u \times (N+1)}. \quad (2.1)$$

Additionally, let us define a matrix where columns consist of first N snapshots $\mathbf{B} = [\mathbf{u}_0, \dots, \mathbf{u}_{N-1}] \in \mathbb{R}^{N_u \times N}$ and a matrix $\mathbf{A}$ containing snapshots a constant time step Δt later in time than the data in $\mathbf{B}$ such that $\mathbf{A} = [\mathbf{u}_1, \dots, \mathbf{u}_N] \in \mathbb{R}^{N_u \times N}$.

2.1.1 Proper Orthogonal Decomposition

First introduced in the context of fluid flows in [34], Proper Orthogonal Decomposition (POD) has become perhaps the most widely used coherent structures extraction technique. An extensive review of numerous examples and applications is found in [13]. The appeal of the methodology is that it provides r structures Φ_j which give the optimal approximation of $\mathbf{u}_i$ as in (1.2) with respect to certain norm. For example, if the l_2 -norm is used on the velocity vectors $\mathbf{u}_i$ the approximation can be interpreted as being optimal with respect to kinetic energy. In this thesis POD modes are always obtained based on an l_2 -norm criterion, however, it is noted that in some cases, such as compressible flows, other norm choices may be more suitable [35].

For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^p$ the inner product is defined as

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^\top \mathbf{w}, \quad (2.2)$$

and a corresponding norm is $\|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. Subsequently, a correlation matrix, $\mathbf{C} \in \mathbb{R}^{(N+1) \times (N+1)}$ of velocity snapshots is calculated

$$\mathbf{C} = (C_{mn}) := \langle \mathbf{u}_m, \mathbf{u}_n \rangle. \quad (2.3)$$

The matrix $\mathbf{C}$ is a symmetric, positive semi-definite Gramian matrix whose eigenvalues are real and nonnegative. The eigenvalues can be sorted in decreasing order, such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N+1} \geq 0$. Since $\mathbf{C}$ is symmetric, there exists a set of orthonormal eigenvectors $\mathbf{z}_1, \dots, \mathbf{z}_{N+1}$ which satisfy $\mathbf{C}\mathbf{z}_j = \lambda_j \mathbf{z}_j$ and $\mathbf{z}_j^\top \mathbf{z}_k = \delta_{jk}$, where δ_{jk} is the Dirac delta function. The POD modes are defined as a linear combination of the snapshots $\mathbf{u}_i$ according to the eigenvectors $\mathbf{z}_j = (z_k)_{k=1}^{N+1} \in \mathbb{R}^{N+1}$ such that

$$\Phi_j^{POD} = \frac{1}{\sqrt{\lambda_j(N+1)}} \mathbf{U} \mathbf{z}_j. \quad (2.4)$$

where $\mathbf{U}$ is given by (2.1). Alternatively, Φ_j^{POD} can be obtained through the singular value decomposition of the underlying data $\mathbf{U}$, such that $\mathbf{U} = \mathbf{U}_p \mathbf{\Sigma} \mathbf{V}^\top$, where columns of $\mathbf{U}_p$ are the POD modes Φ_j^{POD} . Since the eigenvectors $\mathbf{z}_j$ are themselves orthogonal, the POD modes Φ_j^{POD} possess the same property. Utilising this property, the projection coefficients at every

time instance corresponding to a specific Φ_j^{POD} can be obtained as $a_j(t_i) = \langle \Phi_j^{POD}, \mathbf{u}_i \rangle$.

The number of POD modes Φ_j^{POD} can be truncated accordingly, retaining r modes corresponding to the largest singular values in Σ . Defining a residual as a difference between the velocity snapshots and the truncated expansion

$$\mathbf{r}(t_i) = \mathbf{u}_i - \sum_{j=1}^r a_j(t_i) \Phi_j^{POD} \quad (2.5)$$

it can be shown that the expansion using POD structures minimises, over the set of all possible choices of r mode shapes, the time-averaged truncation error $1/(N+1) \sum_{i=0}^N \|\mathbf{r}(t_i)\|^2$ [13]. This optimality of the POD modes provides an attractive methodology for low-order modelling of high-dimensional systems. Nonetheless, dynamical information is not directly used in the calculation of Φ_j^{POD} and often one Φ_j^{POD} will be associated with juxtaposition of different frequencies and growth rates [6]. In order to provide a basis of structures where the fact that snapshots come from time resolved data is taken into account, different dynamic modal decomposition algorithms were proposed in [20, 21, 36, 37].

2.1.2 Dynamic Mode Decomposition

In order to obtain a set of coherent structures based on dynamical considerations, Schmid [20] proposed a methodology referred to as Dynamic Mode Decomposition (DMD) where the modes are selected as eigenmodes of the finite linear operator approximating the nonlinear behaviour of some observable quantities such as velocity or pressure field. The algorithm has received much recent interest in terms of its application to analysis of different classes of flows (such as bluff body flows [24, 38, 39], boundary layers [5, 40], cavity flows [8, 20, 41] or jet flows [6, 20, 42, 43]). A brief description of the ideas behind the algorithm is presented here.

Recalling that $\mathbf{B} \in \mathbb{R}^{N_u \times N}$ consists of N velocity measurements at discrete temporal instances $\mathbf{u}_i = \mathbf{u}(\mathbf{x}, i\Delta t)$ in $\mathbb{R}^{N_u}$, the adjacent columns of $\mathbf{B}$ show the evolution of data a constant time step Δt later in time. Assuming that Δt is sufficiently small, the transformation of data between $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ is approximately linear and it is justifiable to express the dynamics with a linear operator $\mathbf{X}$, such that

$$\mathbf{u}_{i+1} = \mathbf{X}\mathbf{u}_i, \quad (2.6)$$

where $\mathbf{X} \in \mathbb{R}^{N_u \times N_u}$. A natural way of obtaining $\mathbf{X}$ would be to solve an associated optimisation

problem

$$\min_{\mathbf{X}} \sum_i \|\mathbf{u}_{i+1} - \mathbf{X}\mathbf{u}_i\|^2, \quad (2.7)$$

however, since in typical applications $N_u \gg N$ such a problem results in an overfit of data, since the number of decision variables, N_u^2 , is orders of magnitude larger than the number of data points, NN_u . Consequently, additional constraints have to be imposed on $\mathbf{X}$ in order to obtain a meaningful solution.

A fundamental assumption of the DMD methodology is that after a sufficient amount of time $\mathbf{B}$ will capture all of the fundamental dynamical features of the system implying that any subsequent measurement will be approximately contained in the span of the columns of $\mathbf{B}$, i.e.

$$\mathbf{u}_N = \sum_{i=0}^{N-1} c_i \mathbf{u}_i + \mathbf{r} = \mathbf{B}\mathbf{c} + \mathbf{r}, \quad (2.8)$$

where $\mathbf{c} = [c_0, \dots, c_{N-1}]^\top$. Recalling that the matrix $\mathbf{A}$ contains snapshots a constant time step Δt later in time than the data in $\mathbf{B}$, the following expression derives from (2.8)

$$\mathbf{A} = \mathbf{X}\mathbf{B} = \mathbf{B}\mathbf{S} + \mathbf{r}\mathbf{e}_N^\top, \quad (2.9)$$

where $\mathbf{S} \in \mathbb{R}^{N \times N}$ takes the form

$$\mathbf{S} = \begin{bmatrix} 0 & & & c_0 \\ 1 & 0 & & c_1 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 & c_{N-2} \\ & & & 1 & c_{N-1} \end{bmatrix}, \quad (2.10)$$

and $\mathbf{e}_N$ is the N th unit vector in the standard basis for $\mathbb{R}^N$. Notice that now the only unknowns in (2.9) are the coefficients $\mathbf{c}$, however, those can be easily obtained by solving a least-squares problem such that $\mathbf{r}$ in (2.8) is minimised. The eigenvalues of the resulting matrix $\mathbf{S} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ allows us to approximate some of the eigenvalues of the high-dimensional operator $\mathbf{X}$ [20] and a corresponding dynamical mode can be obtained as the lifting of the eigenvalues of $\mathbf{S}$ back to

the high-dimensional space such that

$$\Phi_S^{DMD} := \mathbf{B}\mathbf{P}. \quad (2.11)$$

Although mathematically sound, the formulation outlined above is numerically unstable and typically only the first or first two most dominant modes can be recovered [20]. Instead, it was suggested in [20] that a linear approximation of temporal evolution between $\mathbf{A}$ and $\mathbf{B}$ should have a form

$$\mathbf{A} = \mathbf{B}\tilde{\mathbf{S}} + \mathbf{r}, \quad (2.12)$$

where

$$\tilde{\mathbf{S}} := \mathbf{U}_p^T \mathbf{A} \mathbf{V} \mathbf{\Sigma}^{-1}, \quad (2.13)$$

and $\mathbf{B} = \mathbf{U}_p \mathbf{\Sigma} \mathbf{V}^T$ is the singular value decomposition of $\mathbf{B}$.

In order to establish a link between (2.6) and (2.12), let us assume that $\mathbf{A} \approx \mathbf{X}\mathbf{B}$. Substituting the singular value decomposition $\mathbf{B} = \mathbf{U}_p \mathbf{\Sigma} \mathbf{V}^T$ gives $\mathbf{A} \approx \mathbf{X}\mathbf{U}_p \mathbf{\Sigma} \mathbf{V}^T$. Noting that $\mathbf{V}^T \mathbf{V} = \mathbf{I}$ and $\mathbf{\Sigma}$ is a diagonal matrix, this expression can be rearranged such that $\mathbf{U}_p^T \mathbf{X} \mathbf{U}_p = \mathbf{U}_p^T \mathbf{A} \mathbf{V} \mathbf{\Sigma}^{-1} = \tilde{\mathbf{S}}$, demonstrating that $\tilde{\mathbf{S}}$ is a projection of the high-dimensional linear operator $\mathbf{X}$ onto the POD modes of the data set $\mathbf{B}$. Consequently, (2.6) can be rewritten as

$$\mathbf{A} = \mathbf{U}_p \tilde{\mathbf{S}} \mathbf{U}_p^T \mathbf{B} + \mathbf{r}. \quad (2.14)$$

In this case, instead of providing a transformation forward in time of $\mathbf{u}_i$, $\tilde{\mathbf{S}}$ approximates an evolution of the POD projection coefficients $a(t_i) = \mathbf{U}_p^T \mathbf{u}_i$. In order to get the dynamic modes, $\tilde{\mathbf{S}}$ is eigendecomposed as $\tilde{\mathbf{S}} = \tilde{\mathbf{P}} \tilde{\mathbf{\Lambda}} \tilde{\mathbf{P}}^{-1}$ and the DMD modes are defined as a lifting of the eigenvectors back to the high-dimensional space,

$$\Phi^{DMD} := \mathbf{U}_p \tilde{\mathbf{P}}, \quad (2.15)$$

where $\tilde{\mathbf{P}} = [\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_N] \in \mathbb{C}^{N \times N}$. Since each DMD mode $\Phi_j^{DMD} = \mathbf{U}_p \tilde{\mathbf{p}}_j$, is associated with the eigenvalue $\lambda(\tilde{\mathbf{S}})_j$, where $\tilde{\mathbf{\Lambda}} = (\lambda(\tilde{\mathbf{S}})_{jj})_{j=1}^N$, the DMD modes Φ_j^{DMD} have a certain dynamical

significance. Defining the *DMD eigenvalues* as

$$\lambda_j^{DMD} := \frac{\log \lambda(\tilde{\mathbf{S}})_j}{\Delta t}, \quad (2.16)$$

$\text{Re}(\lambda_j^{DMD})$ and $\text{Im}(\lambda_j^{DMD})$ describe respectively the growth rate and frequency of the DMD mode Φ_j^{DMD} .

Although the construction of the linear operator $\tilde{\mathbf{S}}$ through (2.13) is numerically more stable than through the companion matrix formulation (2.10) [20] there are interesting mathematical developments stemming from the latter approach. It is known that the companion matrix can be decomposed such that $\mathbf{S} = \mathbf{V}^{-1}\mathbf{\Lambda}\mathbf{V}$, where $\mathbf{\Lambda} = (\lambda(\mathbf{S})_{jj})_{j=1}^N$ contains the eigenvalues of $\mathbf{S}$ whilst $\mathbf{V}$ is a Vandermonde matrix of the form

$$\mathbf{V} := \begin{bmatrix} 1 & \lambda(\mathbf{S})_{11} & \lambda(\mathbf{S})_{11}^2 & \dots & \lambda(\mathbf{S})_{11}^{N-1} \\ 1 & \lambda(\mathbf{S})_{22} & \lambda(\mathbf{S})_{22}^2 & \dots & \lambda(\mathbf{S})_{22}^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda(\mathbf{S})_{NN} & \lambda(\mathbf{S})_{NN}^2 & \ddots & \lambda(\mathbf{S})_{NN}^{N-1} \end{bmatrix}. \quad (2.17)$$

Since $\mathbf{\Lambda}$ consists of eigenvalues then $\mathbf{S} = \mathbf{V}^{-1}\mathbf{\Lambda}\mathbf{V}$ is in fact an eigendecomposition of $\mathbf{S}$ where the eigenvectors are the columns of $\mathbf{V}^{-1}$ and $\mathbf{P} = \mathbf{V}^{-1}$. Consequently, following the definition (2.11) the DMD modes are given by $\Phi_S^{DMD} = \mathbf{B}\mathbf{V}^{-1}$. Substituting $\mathbf{S} = \mathbf{V}^{-1}\mathbf{\Lambda}\mathbf{V}$ into (2.9) the following expression is obtained

$$\mathbf{A} = \mathbf{B}\mathbf{S} + \mathbf{r}\mathbf{e}_N^\top = \mathbf{B}\mathbf{V}^{-1}\mathbf{\Lambda}\mathbf{V} + \mathbf{r}\mathbf{e}_N^\top = \Phi_S^{DMD}\mathbf{\Lambda}\mathbf{V} + \mathbf{r}\mathbf{e}_N^\top, \quad (2.18)$$

which in turn implies that

$$\mathbf{u}_i = \sum_{j=1}^N \lambda_j^{i-1} (\Phi_S^{DMD})_j, \quad i = 1, \dots, N-1; \quad (2.19a)$$

$$\mathbf{u}_N = \sum_{j=1}^N \lambda_j^{N-1} (\Phi_S^{DMD})_j + \mathbf{r}, \quad (2.19b)$$

where $\mathbf{r} \perp \text{span}(\mathbf{x}_0, \dots, \mathbf{x}_{N-1})$ [6].

In recent years an active effort developed to establish a link between the decomposition

(2.19) and a particular mathematical object, the *Koopman operator* [6, 24, 44]. A Koopman operator is an infinite-dimensional linear operator which can be used to exactly describe nonlinear dynamics. If such a link is established it can provide additional theoretical motivation for using modal decompositions in practical applications as they would not only provide a linear approximation to the dynamical system but instead can be viewed as a finite subset of infinite Koopman spectrum.

In order to provide formal definition consider a discrete dynamic system evolving between different dynamical states, $\mathbf{x}_i$ as

$$\mathbf{x}_{i+1} = f(\mathbf{x}_i),$$

where f describes the mapping between initial state $\mathbf{x}_i$ and a state at later point in time $\mathbf{x}_{i+1}$ where $\mathbf{x}_i \in Q$ for all i and Q is the underlying space containing all states of the system. The Koopman operator $\mathcal{K}$ is a linear operator which acts on vector-valued observable functions $g : Q \rightarrow \mathbb{R}^{N_u}$ such that

$$[\mathcal{K}g](\mathbf{x}) := g(f(\mathbf{x})), \quad \mathbf{x} \in Q.$$

For example, in fluid flows applications an observable function $\mathbf{g}$ can contain discretised velocity field information at different spatial locations i.e. $[u(x_1), \dots, u(x_{N_u})]^T \in \mathbb{R}^{N_u}$ or simply be a function of the velocity $u(x_i)$ at a given location. In simple terms a Koopman operator describes the time evolution of such observable functions. It is a linear operator as it satisfies the property $\mathcal{K}(\alpha g_1 + \beta g_2)(\mathbf{x}) = \alpha \mathcal{K}g_1(\mathbf{x}) + \beta \mathcal{K}g_2(\mathbf{x})$ for any g_1, g_2 and scalars α, β .

Since $\mathcal{K}$ is linear, it is reasonable to obtain its eigenfunctions, φ , and eigenvalues, λ ,

$$\mathcal{K}\varphi_j(\mathbf{x}) = \lambda_j\varphi_j(\mathbf{x}).$$

Assuming that the spectrum of $\mathcal{K}$ is purely discrete and that the components of g lie in the span of the eigenfunctions of $\mathcal{K}$, the observable function can be expanded as the linear combination of the eigenfunctions φ_i with $\mathbf{v}_j$ as the vector coefficients of the expansion

$$g(\mathbf{x}) = \sum_{j=1}^{\infty} \varphi_j(\mathbf{x}) \mathbf{v}_j. \quad (2.20)$$

The vectors $\mathbf{v}_j$ are referred to as *Koopman modes of the dynamics system f corresponding to*

the observable g .

As the eigenvalues of the Koopman operator characterize the temporal behaviour of each of the Koopman modes it is possible to use the expansion (2.20) to define the value of an observable at each point in time

$$g(\mathbf{x}_i) = \sum_{j=1}^{\infty} \mathcal{K}^i \varphi_j(\mathbf{x}_0) \mathbf{v}_j = \sum_{j=1}^{\infty} \lambda_j^i \varphi_j(\mathbf{x}_0) \mathbf{v}_j. \quad (2.21)$$

Note that this expansion is similar to the approximation (2.19) which was created using DMD modes and eigenvalues. It is therefore argued in [6, 44] that the DMD method serves to approximate the Koopman modes. It is thought that if it is possible to design an algorithm which reproduces the Koopman decomposition it would provide a well-motivated linear description of the nonlinear dynamics. Nonetheless it must be noted that the Koopman expansion consists of infinite number of terms and for practical problems using as few modes as possible is preferable. Additionally, the Koopman expansion does not possess any clear ordering of the modes therefore there is no clear way of choosing the most important ones.

Following the description of DMD it is noted that a number of generalizations have recently been proposed with each seeking to address various deficiencies of the algorithm. Examples of the recent developments include: establishing an appropriate way of selecting the most important DMD modes [36, 45], improving the robustness of the method to observation noise [21, 46], establishing a stronger link to Koopman operator [37] or accounting for input/output systems [47]. A more detailed description of the methodologies outlined above will be presented in §2.1.4. One modification to the DMD algorithm, Optimal Mode Decomposition (OMD), is extensively used throughout this thesis and consequently the more detailed description of this particular methodology is justifiable and is given in §2.1.3.

2.1.3 Optimal Mode Decomposition

Although the DMD decomposition is an adequate foundation for low-order modelling it is not necessarily obvious whether a choice of the POD modes is most appropriate for construction of the low-dimensional basis in (2.14). For instance, if there exists a certain structure which is dynamically relevant but at the same time of low-energy, the truncated POD basis used to calculate $\tilde{\mathbf{S}}$ will discard these dynamics. In order to introduce additional flexibility, instead of

considering a linear operator of the form (2.13), it was proposed in [21] to solve an optimisation problem of the form

$$\begin{aligned} \min_{\mathbf{L}, \mathbf{M}} & \|\mathbf{A} - \mathbf{L}\mathbf{M}\mathbf{L}^T\mathbf{B}\|_F, \\ \text{s.t. } & \mathbf{L}^T\mathbf{L} = \mathbf{I}, \\ & \mathbf{M} \in \mathbb{R}^{r \times r}, \mathbf{L} \in \mathbb{R}^{N_u \times r}, \end{aligned} \quad (2.22)$$

where columns of $\mathbf{A}$, $\mathbf{B}$ are defined as in §2.1, above, and $\|\cdot\|_F$ is the Frobenius norm. A comparison with (2.14) shows that $\mathbf{L}$, $\mathbf{M}$ have an equivalent interpretation to $\mathbf{U}_p$, $\tilde{\mathbf{S}}$ in (2.14). Following the DMD formulation, the OMD modes and eigenvalues will be defined as $\Phi^{OMD} = \mathbf{L}\mathbf{P}_M$ and $\lambda_j^{OMD} = \frac{\log \lambda(\mathbf{M})_j}{\Delta t}$, respectively, where $\mathbf{M} = \mathbf{P}_M\mathbf{\Lambda}_M\mathbf{P}_M^{-1}$ and $\mathbf{\Lambda}_M = (\lambda(\mathbf{M})_{jj})_{j=1}^N$. Notice that the DMD formulation based on the companion matrix in §2.1.2 implies a consecutive sequence of measurements in adjacent columns of $\mathbf{A}$ or $\mathbf{B}$. However, the objective function (2.22) implies that this characteristic is not necessary. The only essential requirement is that the timestep Δt between corresponding entries in $\mathbf{A}$ and $\mathbf{B}$ is constant.

The optimisation (2.22) is performed over two variables, $\mathbf{L}$ and $\mathbf{M}$. It is noted that the objective function in (2.22) is in fact quadratic with respect to $\mathbf{M}$. Consequently, a derivative with respect to $\mathbf{M}$ is calculated

$$\frac{\partial \|\mathbf{A} - \mathbf{L}\mathbf{M}\mathbf{L}^T\mathbf{B}\|_F^2}{\partial \mathbf{M}} = 2[-\mathbf{L}^T\mathbf{A}\mathbf{B}^T\mathbf{L} + \mathbf{M}\mathbf{L}^T\mathbf{B}\mathbf{B}^T\mathbf{L}], \quad (2.23)$$

which can be subsequently set to zero, in order to obtain the expression for the optimal $\mathbf{M}$ as a function of $\mathbf{L}$

$$\mathbf{M}^*(\mathbf{L}) = \mathbf{L}^T\mathbf{A}\mathbf{B}^T\mathbf{L}(\mathbf{L}^T\mathbf{B}\mathbf{B}^T\mathbf{L})^{-1}. \quad (2.24)$$

It is noted that by substituting $\mathbf{B} = \mathbf{U}_p\mathbf{\Sigma}\mathbf{V}^T$ into an expression for $\mathbf{M}^*(\mathbf{U}_p)$, that $\mathbf{M}^*(\mathbf{U}_p) = \tilde{\mathbf{S}}$ is recovered, meaning that OMD reduces to DMD in the case $\mathbf{L} = \mathbf{U}_p$.

Hence, the only variable which has to be calculated to minimise (2.22) is $\mathbf{L}$. Unfortunately, due to nonlinearity of (2.22) with respect to $\mathbf{L}$, an analytic solution is not available and iterative algorithms have to be employed. It must be noted that we are not looking for the unique value of $\mathbf{L}$ but instead for the space spanned by its columns. This is because (2.22) is invariant under rotations of $\mathbf{L}$. In particular, for any orthogonal matrix $\tilde{\mathbf{R}}$ such that $\tilde{\mathbf{R}}^T\tilde{\mathbf{R}} = \mathbf{I}$, the pairs $[\mathbf{L}, \mathbf{M}]$

and $[L\tilde{\mathbf{R}}, \tilde{\mathbf{R}}^\top M \tilde{\mathbf{R}}]$ have equivalent costs. Consequently, the optimisation has to be performed over the Grassmann manifold G_{r,N_u} which is a collection of all r -dimensional linear subspaces of a N_u -dimensional vector space. Therefore, the optimisation process is equivalent to finding an r -dimensional subspace of $\mathbb{R}^p$. Algorithms which implement different iterative methods for a problem on Grassmanian manifolds are presented in [48]. The conjugate gradient method which we use to solve OMD problem can be found in Appendix A.

2.1.4 Alternative DMD generalisations

Many recent generalisations of the DMD algorithm have been developed aiming to improve the robustness and accuracy of the methodology. Some of the key examples are presented here.

Ranking of dynamic modes

One of the practical issues mentioned previously in the application of the DMD algorithm is an ambiguity in selecting the most appropriate modes among the algorithm's output. The most straightforward application using a companion matrix (2.10) or calculation of $\tilde{\mathbf{S}}$ via (2.13) using all available POD modes of $\mathbf{B}$ induces the same number of DMD modes as the number of snapshots. In [21] it was proposed that a smaller number of DMD modes can be obtained by truncating the POD basis. However, this approach may discard low-energy structures. A few alternative methodologies of calculating small number of DMD-like modes were proposed and are outlined below.

In [36] a new methodology referred to as optimised dynamical mode decomposition (opt-DMD) was proposed, which utilises the observation that snapshots can be decomposed as in (2.19). Due to the fact that the number of DMD modes Φ_j^{DMD} is equal to N , the decomposition of first N snapshots is satisfied exactly and the residual is minimised only over the $(N + 1)$ th snapshot. Instead, it was proposed in [36] to constrain the number of modes to r and minimise the following problem motivated by (2.19)

$$\min_{\lambda_j, \Phi_j} \sum_{i=1}^N \|\mathbf{u}_i - \sum_{j=1}^r \lambda_j^{i-1} \Phi_j\|_2 \quad (2.25)$$

where λ_j and Φ_j are the opt-DMD eigenvalues and eigenvectors. The opt-DMD algorithm was analysed on the cylinder flow at $Re = 60$ and it was shown that the new method provides an

improvement over DMD in calculating the dominant system frequencies. Unfortunately, the optimisation problem (2.25) is highly-nonlinear and a numerical algorithm allowing to obtain a solution in an efficient and elegant manner is not known. The solution to (2.25) in [36] was obtained using an algorithm combining simulated annealing and Nelder-Mead simplex method; a methodology which is significantly more complex than the one used to compute either the DMD or OMD algorithms.

An alternative way of reducing the number of DMD modes was proposed in [45] through sparsity-promoting DMD. Note that the decomposition in (2.19a) can be expressed with respect to the matrix $\tilde{\mathbf{S}}$ such that for $i = 1, \dots, N$

$$\mathbf{u}_i = \mathbf{U}_p \tilde{\mathbf{S}}^{i-1} \mathbf{U}_p^\top \mathbf{u}_1 = \mathbf{U}_p \tilde{\mathbf{P}} \tilde{\mathbf{\Lambda}}^{i-1} \tilde{\mathbf{P}}^{-1} \mathbf{U}_p^\top \mathbf{u}_1 = \mathbf{\Phi}^{DMD} \tilde{\mathbf{\Lambda}}^{i-1} \boldsymbol{\alpha} \quad (2.26)$$

where $\mathbf{U}_p$ contains *all* POD modes of $\mathbf{B}$ and $\boldsymbol{\alpha} = \tilde{\mathbf{P}}^{-1} \mathbf{U}_p^\top \mathbf{u}_1 \in \mathbb{C}^N$ is a projection of the first snapshot onto available modes with entries signifying the amplitudes of the DMD modes (noting that columns of $\mathbf{\Phi}^{DMD}$ are normalised when $\mathbf{\Phi}^{DMD} = \mathbf{U}_p \tilde{\mathbf{P}}$). The equation (2.26) is true only if N DMD modes are used. In order to choose a subset of the most important modes it was proposed in [45] to solve the following optimisation problem

$$\min_{\boldsymbol{\alpha}} \sum_{i=1}^N \|\mathbf{u}_i - \mathbf{\Phi}^{DMD} \tilde{\mathbf{\Lambda}}^{i-1} \boldsymbol{\alpha}\|_2^2 + \gamma \|\boldsymbol{\alpha}\|_1. \quad (2.27)$$

The purpose of the additional term $\|\boldsymbol{\alpha}\|_1$ is to induce sparsity of $\boldsymbol{\alpha}$ and the DMD eigenvalues/modes corresponding to $\alpha_j \approx 0$ can be neglected. The benefit of solving (2.27) over (2.25) is the knowledge of the eigenvalues (which are obtained using the standard DMD algorithm) allowing (2.27) to be solved using standard convex optimisation methodologies. Nonetheless, this approach in essence requires solving an additional optimisation problem in addition to DMD. For example, in [45] the optimisation problem (2.27) was solved using alternative direction method of multipliers (ADMM) [49]; a first order method which can converge slowly in certain situations. In §3.1 a different methodology is introduced which provides a compromise, i.e. dynamically relevant modes are selected based on a certain energetic criterion involving l_2 -norm but without the need for solving an additional optimisation problem, hence improving computational efficiency.

Robustness to noise

One of the disadvantages of the DMD algorithm is its lack of noise robustness which was recognised for example in [46, 50, 51] and several methodologies were developed to suppress the effect of noise. A contribution of [50] was the demonstration that in the presence of process noise the DMD eigenvalues λ_n , where $\text{Re}(\lambda_n) = 0$ can be approximated as

$$\lambda_n \approx jm\omega - \epsilon \frac{|S|}{2T} \omega^2 m^2 \quad (2.28)$$

where λ_n corresponds to m th subharmonic of the fundamental frequency ω , ϵ is a noise magnitude, T is a timescale required for a system to approach the oscillatory state in the absence of noise and S is the sensitivity to white noise. Interestingly, the same pattern can be observed in the presence of measurement noise [50, 51].

In order to improve the robustness of the DMD algorithm to noise, a methodology referred to as total DMD (TDMD) was developed in [46]. Noting that the DMD output is obtained based on the operator $\mathbf{U}_p^\top \mathbf{X} \mathbf{U}_p$ where $\mathbf{B} = \mathbf{U}_p \mathbf{\Sigma} \mathbf{V}^\top$, it is implicitly assumed that $\mathbf{B}$ is noise free. In order to eliminate this assumption it is proposed to perform a decomposition of the augmented matrix $[\mathbf{B}^\top, \mathbf{A}^\top]^\top = \tilde{\mathbf{U}}_p \tilde{\mathbf{\Sigma}} \tilde{\mathbf{V}}^\top$ and retaining only n columns of $\tilde{\mathbf{V}}$. Subsequently, $\mathbf{B}$ and $\mathbf{A}$ are preconditioned such that $\tilde{\mathbf{B}} = \mathbf{B} \tilde{\mathbf{V}}_n \tilde{\mathbf{V}}_n^\top$ and $\tilde{\mathbf{A}} = \mathbf{A} \tilde{\mathbf{V}}_n \tilde{\mathbf{V}}_n^\top$, where $\tilde{\mathbf{V}}_n$ is truncated $\tilde{\mathbf{V}}$, and the DMD algorithm is applied to $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ instead. This approach is most effective if the snapshots are assembled in a non-sequential manner, i.e. elements of $\mathbf{B}$ are not necessarily included in $\mathbf{A}$, however, the choice of n is not necessarily obvious.

Multiple modifications to the DMD methodologies aimed at removing observation noise from data were proposed in [51]. A methodology called noise-corrected DMD (ncDMD) provides a modification to matrix $\tilde{\mathbf{S}}$ constructed based on the noise covariance matrix. Unfortunately, knowledge of noise-covariance is usually not available, especially in the analysis of experimental data. A second methodology is forward-backward DMD (fbDMD) which relies on the assumption that the dynamics of the original system are invertible. In addition to a standard DMD implementation, the DMD problem is solved backwards in time, i.e. snapshots in $\mathbf{B}$ and $\mathbf{A}$ are swapped, to obtain a matrix $\tilde{\mathbf{S}}_{back}$. For the ideal, noise-free, system $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}_{back}^{-1}$. Based on the observation in [50], noise decreases the value of $\text{Re}(\lambda)$ obtained by DMD relative to

the true eigenvalues for both $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{S}}_{back}$, which implies that $\tilde{\mathbf{S}}_{back}^{-1}$ will have corresponding $\text{Re}(\lambda)$ artificially increased. Consequently, the detrimental effect of noise can be averaged out through calculation of the $\tilde{\mathbf{S}}^*$ such that $(\tilde{\mathbf{S}}^*)^2 = \tilde{\mathbf{S}}_{back}^{-1} \tilde{\mathbf{S}}$. This methodology showed a great promise in [51], and can be easily implemented in the OMD algorithms to obtain further noise reduction. The last methodology is total least-squares DMD (tlsDMD) which is a modification to TDMD described above, where for $[\mathbf{B}^\top, \mathbf{A}^\top]^\top = \tilde{\mathbf{U}} \tilde{\Sigma} \tilde{\mathbf{V}}^\top$ instead of truncating $\tilde{\mathbf{V}}$ data is preconditioned using truncated POD modes $\tilde{\mathbf{U}}$. The comparison of noise sensitivity between some of the methodologies introduced in [51] and OMD is performed in §3.2.

Link to the Koopman operator

An alternative approach to further developing the DMD methodology is to establish a significantly stronger link to the Koopman operator. One of the most prominent examples was proposed in [37] by introducing extended DMD (EDMD). The new algorithm relies on the observation that eigenfunctions of the Koopman operator φ_j do not have to be linear function of the state and can have a nonlinear form. For this purpose, a dictionary of functions $\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \dots, \psi_K(\mathbf{x})]^\top \in \mathbb{C}^K$ is defined and it is assumed that $\varphi_j \in \text{span}(\Psi)$. In order to find an appropriate form of φ first a finite-dimensional approximation $\mathbf{K}$ of Koopman operator $\mathcal{K}$ is required which can be obtained as a solution to

$$\min_{\mathbf{K}} \|\Psi(\mathbf{A}) - \mathbf{K} \Psi(\mathbf{B})\|_2. \quad (2.29)$$

where $\mathbf{A}$ and $\mathbf{B}$ contain state measurements, $\Psi(\mathbf{A}), \Psi(\mathbf{B}) \in \mathbb{C}^{K \times N}$ are dictionaries of functions evaluated at each time instance, and $\mathbf{K} \in \mathbb{C}^{K \times K}$. The eigenpairs of the Koopman operator are then approximated through the eigendecomposition of $\mathbf{K}$, i.e. for a pair of eigenvalues λ_j, λ_j^* and eigenvectors $\mathbf{z}_j, \mathbf{z}_j^*$ the first are the approximations of the Koopman eigenvalues whilst

$$\varphi_j = \Psi(\mathbf{x}) \mathbf{z}_j, \quad (2.30)$$

is the Koopman eigenfunction. The remaining terms in the Koopman decomposition (2.21) which have to be computed are Koopman modes $\mathbf{v}_j$, which provide a link back to the high-dimensional state $\mathbf{x}$. In [37] it was assumed that the matrix $\mathbf{W} \in \mathbb{R}^{N_u \times K}$ containing weights

necessary to reconstruct state $\mathbf{x}$ from elements of function dictionary $\Psi(\mathbf{x})$ is known. In such case the Koopman modes are expressed as

$$\mathbf{v}_j = \mathcal{W}\boldsymbol{\xi}_j \quad (2.31)$$

where $\boldsymbol{\xi}_j$ are left eigenvectors of $\mathbf{K}$. If the matrix $\mathcal{W}$ is not known, the Koopman modes can be calculated using past data by solving an appropriate least-squares problem.

It was demonstrated in [37] that this methodology successfully approximates Koopman eigenfunctions for some examples of nonlinear systems. Perhaps the biggest difficulty of this methodology is the fact that the dictionary of functions $\Psi(\mathbf{x})$ has to be specified by the user, which either necessitates some knowledge of the analysed system's dynamics or constructing $\Psi(\mathbf{x})$ to be sufficiently "rich" so that assumption $\varphi_j \in \text{span}(\Psi)$ is close to being true. In the latter case this can lead to a numerically cumbersome algorithm or dimensional reduction which is not sufficiently large. Some approaches addressing those problems are outlined in [52, 53].

DMD with control

One of the applications where the linearity and low-dimensionality of the DMD-like operator is particularly enticing is system control. Nonetheless, those algorithms only construct the matrices approximating the dynamics of the states without assuming any information about the link between control input and the state. In a situation where this relationship is not known, a methodology called DMD with control (DMDc) was proposed in [47] where the following optimisation is solved

$$\min_{\tilde{\mathbf{S}}, \tilde{\mathbf{B}}} \|\mathbf{A}_{ld} - [\tilde{\mathbf{S}}, \tilde{\mathbf{B}}][\mathbf{B}_{ld}^\top, \boldsymbol{\Upsilon}^\top]^\top\|_2. \quad (2.32)$$

The matrix $\boldsymbol{\Upsilon}$ contains the time-series of inputs applied to the system and $\tilde{\mathbf{B}}$ is the matrix providing the relationship between inputs and the state. The matrices $\mathbf{A}_{ld}$, $\mathbf{B}_{ld}$ are low dimensional projections of the full flow fields $\mathbf{A}$, $\mathbf{B}$, i.e. $\mathbf{A}_{ld} = \mathbf{L}^\top \mathbf{A}$ and $\mathbf{B}_{ld} = \mathbf{L}^\top \mathbf{B}$, where $\mathbf{L}$ can contain POD modes or can be incorporated as an unknown into the optimisation problem (2.32) to obtain OMD-like procedure. The main benefit of the DMDc methodology is that it allows us to separately construct $\tilde{\mathbf{B}}$, whose form is usually not known. Unfortunately, the methodology does not give any indication about an input that should be chosen for a specific

objective and values of Υ need to be specified by the user based on knowledge of the system. It is noted that in chapters 4-5 dynamical estimation problems are considered where $\Upsilon = 0$.

2.2 Dynamical Estimation Techniques

It was previously mentioned in §1.3 that active fluid flow control requires knowledge of the system's dynamics, which when combined with online measurements may lead to the successful implementation of a given control strategy. Most often the dynamical system is approximated using the state-space form

$$\dot{\mathbf{x}} = A(\mathbf{x}) + \mathbf{w}, \quad (2.33a)$$

$$\mathbf{y} = C(\mathbf{x}) + \mathbf{v}, \quad (2.33b)$$

where $\mathbf{x} \in \mathbb{R}^r$ denotes a low-dimensional state of the system and $\mathbf{y} \in \mathbb{R}^{N_y}$ are the measurable quantities of the system. By $\mathbf{w}$ and $\mathbf{v}$, we denote state disturbances and measurement noise. Subsequently, we define functions $A(\cdot)$ and $C(\cdot)$ which, respectively, describe the dynamics of the system and a relationship between the state $\mathbf{x}$ and measurable quantity $\mathbf{y}$. The overall objective is, given measurements $\mathbf{y}$ and dynamical models governed by $A(\mathbf{x})$ and $C(\mathbf{x})$, to solve a certain optimisation problem associated with (2.33) and obtain state estimate $\hat{\mathbf{x}}$ which approximates a true state $\mathbf{x}$ as closely as possible.

In order to employ control strategies in an efficient manner it is required that the dimensions of $\mathbf{x}$ and $\mathbf{y}$ are not large, necessitating dimensional reduction of systems with large number of states, which is typically necessary for accurate modelling of fluid flows. This procedure is not obvious. One possible choice is the projection coefficients $a_i(t)$, as defined in (1.2), corresponding to spatial structures which represent key dynamical features facilitating a construction of $A(\cdot)$ and $C(\cdot)$. Methodologies which can be used to obtain spatial structures necessary for the computation of $\mathbf{x}$ were outlined in §2.1. The construction of associated dynamical models $A(\cdot)$ and $C(\cdot)$ can be performed either by solving an optimisation problem or making use of the underlying structure of the Navier-Stokes equations as described in §1.1. The methodologies which are employed to construct dynamical models for the cylinder and bullet flows will be described in the relevant sections of chapters 4-5. Subsequently, a methodology needs to be established which uses the knowledge of the dynamical system and combines it with online

measurements $\mathbf{y}$ to solve for the state estimates $\hat{\mathbf{x}}$. The methodologies which will be used for this purpose in the analysis of both fluid flows are outlined in §2.2.1 - §2.2.2.

2.2.1 Stochastic estimation

A straightforward approach in the calculation of $\hat{\mathbf{x}}$ is to ignore the dynamical model and obtain a state estimate directly from the measurement quantities, i.e. $\hat{\mathbf{x}}_i = C^{-1}(\mathbf{y}_i)$. Note that by $C^{-1}(\mathbf{y}_i)$ we denote the reverse mapping from $\mathbf{y}$ to the state estimate and not necessarily an inverse of $\mathbf{y} = C(\mathbf{x})$ which may not exist in the strictest sense. The corresponding question concerns the construction of the appropriate form of $C^{-1}(\mathbf{y}_i)$. One of the widely used approaches is stochastic estimation, first proposed in the fluid dynamics community by [54] and since then successfully used to describe a range of turbulent flows such as shear layers [55], cavity flows [56], wake flows [57] or jet flows [58, 59].

Stochastic estimation is an approach to estimate a value of one quantity given a set of measurements via the means of conditional average. Let us assume that past knowledge of certain dependent variables x_j and some independent variables c_j is known. It was proposed in [60] that the conditional average can be approximated with a Taylor expansion

$$\langle x_j | \mathbf{c} \rangle \approx A_{jk} c_k + B_{jkl} c_k c_l + \dots \quad (2.34)$$

where $\mathbf{c}$ consists of the independent variables c_j and coefficients A_{jk} and B_{jkl} are calculated using past data. In (2.34), summation over repeated indices is implied. Subsequently, based on future measurements of $\mathbf{c}$ it is possible to find an estimate $\hat{x}_j$ of x_j . Stochastic estimation can be used to construct different orders of approximation. In chapter 4 an analysis is performed using approximations of two different orders, Linear Stochastic Estimation (LSE) where only coefficients A_{jk} corresponding to the linear part are retained and Quadratic Stochastic Estimation (QSE) where both A_{jk} and B_{jkl} are used. LSE has proved successful in numerous fluid flows applications (an extensive review can be found in [61]). However, it was demonstrated in [62] that QSE provides a significant improvement when the velocity field in a turbulent boundary layer is estimated from pressure measurements and this improvement is linked to the generation of wall-pressure by a nonlinear mechanism of the velocity field. This situation is in a sense analogous to the test case in chapter 5 where velocity is estimated from pressure

measurements making QSE potentially a viable choice for the estimation.

The values of A_{jk} and B_{jkl} can be directly obtained using a least-squares minimisation on the training set $\mathcal{T}$. For example in the LSE estimation a following problem is solved

$$\min_{\mathbf{A}} \sum_{i \in \mathcal{T}} \|\mathbf{x}_i - \mathbf{A}^\top \mathbf{c}_i\|_2. \quad (2.35)$$

where $\mathbf{x} \in \mathbb{R}^{N_x}$ contains N_x conditional variables of interest, $\mathbf{c} \in \mathbb{R}^{N_c}$ is a vector of N_c unconditional variables and $\mathbf{A} \in \mathbb{R}^{N_c \times N_x}$ consists of LSE coefficients, A_{jk} . For all entries in (2.35), $(\cdot)_i$ denotes a temporal measurement at certain instance in time. In an analogous manner, QSE estimates can be obtained via

$$\min_{\mathbf{A}, \mathbf{B}} \sum_{i \in \mathcal{T}} \|\mathbf{x}_i - \mathbf{A}^\top \mathbf{c}_i - \mathbf{B}^\top \mathbf{c}_i^*\|_2^2, \quad (2.36)$$

where $\mathbf{c}^*$ represents a matrix of appended unconditional variables, i.e.

$$\mathbf{c}^* = [c_1 c_1, \dots, c_1 c_{N_c}, c_2 c_1, \dots, c_{N_c} c_{N_c}]^\top$$

and $\mathbf{B} \in \mathbb{R}^{N_c^2 \times N_x}$ are B_{jkl} coefficients corresponding to quadratic terms in (2.34). It is noted that if N_c is not small then an overfit can occur, i.e. the minimised objective function (2.36) is small for $i \in \mathcal{T}$ but drastically increases for $i \notin \mathcal{T}$. In order to prevent this behaviour it is possible to add a sparsity-inducing l^1 -norm penalty in (2.36), such that

$$\min_{\mathbf{A}, \mathbf{B}} \sum_{i \in \mathcal{T}} \|\mathbf{x}_i - \mathbf{A}^\top \mathbf{c}_i - \mathbf{B}^\top \mathbf{c}_i^*\|_2^2 + \gamma (|\mathbf{A}|_1 + |\mathbf{B}|_1) \quad (2.37)$$

and setting an appropriate value of a weighting coefficient $\gamma > 0$. Consequently, given a solution to (2.35)-(2.37), the LSE and QSE estimates are obtained as $\hat{\mathbf{x}}_i = \mathbf{A}^\top \mathbf{c}_i$ and $\hat{\mathbf{x}}_i = \mathbf{A}^\top \mathbf{c}_i + \mathbf{B}^\top \mathbf{c}_i^*$.

2.2.2 Moving Horizon Estimation

It is noted that the stochastic estimation technique outlined in §2.2.1 is purely a pointwise-in-time statistical estimate which relies heavily on the accuracy of the measurements, i.e. given a measurement with significant statistical noise at a certain time instance, the corresponding estimate will contain significant error [63]. However, if the dynamics of the conditional variable

$\mathbf{x}$ is known, it can be used to improve the accuracy of the estimate $\hat{\mathbf{x}}_i$. One of the approaches which combines the dynamics of $\mathbf{x}$ together with real time measurement is Moving Horizon Estimation (MHE). First introduced in numerical weather prediction as 4D-Var [65, 66], this methodology is still relatively new in the fluid dynamics community. However, its application has had much success in the process control community when used for state estimation [67], data reconciliation [68, 69] or fault detection [70] and promises potential benefits when applied to fluid flows. Additionally, methodologies based on similar ideas, such as Kalman filtering, were successfully applied to fluid flows for estimation in atmospheric forecasting [71], flat plate flows [63, 72] or channel flows [73]; and model predictive control (MPC) has been employed to control turbulent channel flow [74, 75].

MHE is based on solving an optimisation problem of the form

$$\min_{\substack{\{\mathbf{w}_{i-N_h}, \dots, \mathbf{w}_{i-1}\} \\ \{\mathbf{v}_{i-N_h}, \dots, \mathbf{v}_i\} \\ \mathbf{x}_{i-N_h}}} \sum_{j=i-N_h}^{i-1} \mathbf{w}_j^\top \mathbf{R}^{-1} \mathbf{w}_j + \sum_{j=i-N_h}^i \mathbf{v}_j^\top \mathbf{Q}^{-1} \mathbf{v}_j, \quad (2.38a)$$

$$\text{s.t. } \mathbf{x}_{i+1} = A(\mathbf{x}_i) + \mathbf{w}_i, \quad i = k - N_h, \dots, k - 1, \quad (2.38b)$$

$$\mathbf{y}_i = C(\mathbf{x}_i) + \mathbf{v}_i, \quad i = k - N_h, \dots, k, \quad (2.38c)$$

at each timestep i using a finite number of past measurements collected over N_h previous time instances, where $N_h \in \mathbb{N}$ is known as the horizon length. It is noted that constraints (2.38b)-(2.38c) follow a state space formulation (2.33), where (2.38b) models the dynamics of $\mathbf{x}$ whilst (2.38c) provides an output model between $\mathbf{x}$ and some measurements $\mathbf{y}$. Noting that models $A(\mathbf{x})$, $C(\mathbf{x})$ are often inaccurate and $\mathbf{y}$ contains noise, the objective function (2.38a) attempts to balance two types of information to get an improved estimate. The weighting matrices $\mathbf{R}, \mathbf{Q} \succeq 0$ are introduced providing the measure of confidence in the dynamical and output models. The flexibility of (2.38) allows for introduction of additional constraints on the state estimates, for example bounds on $\mathbf{x}$ through inequality constraints. It can be demonstrated that MHE reduces to Kalman filter for linear forms of $A(\mathbf{x})$ and $C(\mathbf{x})$, and unconstrained state variables [76].

In this thesis both linear and nonlinear forms of $A(\mathbf{x})$, $C(\mathbf{x})$ will be used. In cases where $A(\mathbf{x})$, $C(\mathbf{x})$ will be replaced with linear matrices $\mathbf{A} \in \mathbb{R}^{r \times r}$, $\mathbf{C} \in \mathbb{R}^{N_y \times r}$, (2.38) can be rewritten

as the least-squares problem

$$\min_{\boldsymbol{\xi}} \left\| \begin{bmatrix} \mathbf{A}_B \\ \mathbf{C}_B \end{bmatrix} \boldsymbol{\xi} - \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} \right\|_2^2 \quad (2.39)$$

where

$$\mathbf{A}_B = \begin{bmatrix} \mathbf{A} & -\mathbf{I} & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \mathbf{A} & -\mathbf{I} \end{bmatrix} \in \mathbb{R}^{(N_h-1)r \times N_h r}, \quad \mathbf{C}_B = \begin{bmatrix} \mathbf{C} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{C} \end{bmatrix} \in \mathbb{R}^{N_h r \times N_h N_y},$$

and $\mathbf{b} = \left(\mathbf{y}_{i-N_h}^\top \dots \mathbf{y}_i^\top \right)^\top \in \mathbb{R}^{N_h N_y}$ and $\boldsymbol{\xi} = \left(\hat{\mathbf{x}}_{i-N_h}^\top \dots \hat{\mathbf{x}}_i^\top \right)^\top \in \mathbb{R}^{N_h r}$. Consequently, (2.39) admits the solution provided the block matrix $[\mathbf{A}_B^\top, \mathbf{C}_B^\top]^\top$ is full rank. In order to vary the weighting between dynamical and observation optimisation functions, analogous to (2.38) each row of $\mathbf{A}_B$ and $\mathbf{C}_B$ is pre-multiplied by $\mathbf{R}^{-1/2}$ and $\mathbf{Q}^{-1/2}$ respectively. Problem (2.39) is solved at each discrete time i and only the last element of $\boldsymbol{\xi}$, $\hat{\mathbf{x}}_i$ is retained as a state estimate at that instance.

When $A(\mathbf{x})$, $C(\mathbf{x})$ have nonlinear form, the MHE optimisation problem (2.38) will be solved using a methodology similar to the one presented in [77] where a one-step Gauss-Newton procedure is used. At each time-step i an iterative mapping $\mathcal{I}_i$ is defined based on Gauss-Newton algorithm which acts on dummy state variables $\boldsymbol{\eta}^k = [\eta_{i-N_h}^k, \dots, \eta_i^k]$ such that $\mathcal{I}_i(\boldsymbol{\eta}_i^k) = \boldsymbol{\eta}_i^{k+1}$. A sequence of vectors is generated which converges to an optimal solution at time i , $\boldsymbol{\eta}_i^k \rightarrow \boldsymbol{\eta}_i^*$. However, in the algorithm, $\mathcal{I}_i$ is applied only once. Instead, an initial condition at a next timestep is defined based on a solution at time i through a shift map S_k , $\boldsymbol{\eta}_{i+1}^0 = S_k(\boldsymbol{\eta}_i^1)$. In this way the dynamical information is passed between successive time-steps through S_k , compensating for one iteration used in the Gauss-Newton implementation, and allowing for a fast converging algorithm which can be implemented in online applications. For more details about the implementation of the MHE algorithm, the reader is referred to [78].

2.3 Reconstruction Algorithms from undersampled data

In chapter 6 the is-OMD algorithm is introduced which provides a viable way of extracting dynamical information from undersampled data. Currently, perhaps the most widely used

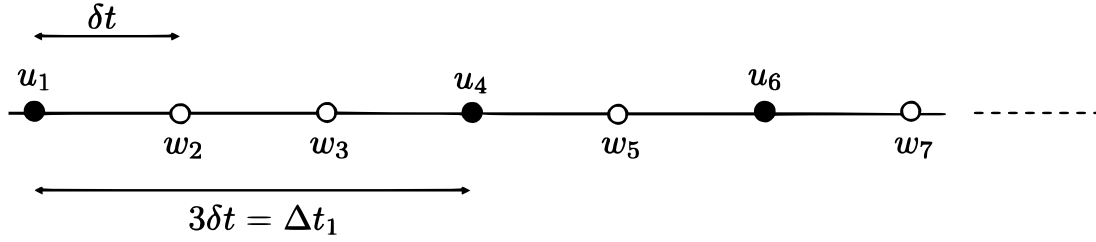


Figure 2.1: Downsampled and regularly sampled data.

methodology in such applications is Compressive Sampling. A brief summary of this algorithm is given here together with existing attempts to extend the DMD algorithm to undersampled data.

Both methodologies rely on the information being randomly sampled in time, consequently requiring us to introduce a new particular problem setup. Consider the snapshots $\mathcal{W} = (\mathbf{w}_i)_{i=0}^N \subset \mathbb{R}^p$ sampled with constant timestep δt , which can be viewed as a *minimal timestep* representing the highest possible frequency of data collection. It is assumed that only a subset $\mathcal{U} \subset \mathcal{W}$ of snapshots may be used for analysis. We refer to $\mathcal{U}$ as the *analysis data ensemble* and define it by choosing a subset $\mathcal{J} \subset \{0, 1, \dots, N\}$, re-labelling $\mathbf{u}_i := \mathbf{w}_i$ for each index $i \in \mathcal{J}$, and setting $\mathcal{U} = \{\mathbf{u}_i : i \in \mathcal{J}\}$. Finally, let Δt_i be the sample times between successive snapshots in $\mathcal{U}$ which are, possibly non-constant, integer multiples of δt . The schematic representation of such a sampling strategy is presented in figure 2.1.

It will be assumed that $\mathcal{W}$ is sampled at a sufficiently high rate that it contains all dynamical information of interest concerning the state trajectory $f(\mathbf{x}, t)$. The question which we aim to address is to what extent $\mathcal{W}$ can be reconstructed from knowledge of the analysis data set $\mathcal{U}$ only. Compressive sampling presents a standard method of answering such a problem.

2.3.1 Compressive Sampling

Extracting coherent dynamical information from a downsampled data ensemble $\mathcal{U}$ may be possible if the underlying system is sparse, or *compressible*, when expressed in particular temporal basis. To describe this, first collect the high-frequency sampled snapshots $\mathcal{W}$ into a matrix

$$\mathbf{W} := \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{w}_0 & \cdots & \mathbf{w}_N \\ \downarrow & & \downarrow \end{bmatrix} \in \mathbb{R}^{N_u \times (N+1)}.$$

The state trajectory $\mathbf{W}$ is said to be compressible if there exists a matrix $\mathbf{\Xi} = (\xi_{jk}) \in \mathbb{R}^{(N+1) \times (N+1)}$ such that it can be approximately represented in the form

$$\mathbf{W} \approx \hat{\mathbf{W}}\mathbf{\Xi}, \quad (2.40)$$

where $\hat{\mathbf{W}} \in \mathbb{R}^{N_u \times N}$ is sparse in the sense that it has only k nonzero columns, with $k \ll N$. Letting $\hat{\mathbf{w}}_i$ denote the columns of $\hat{\mathbf{w}}$, and rewriting the decomposition (2.40) as

$$\mathbf{w}_i \approx \sum_{j=0}^N \hat{\mathbf{w}}_j \xi_{ji}, \quad i = 0, \dots, N,$$

it is clear that the each column $\hat{\mathbf{w}}_j \in \mathbb{R}^{N_u}$ represents a spatial structure, while the respective row $(\xi_{ji})_{i=0}^N$ of $\mathbf{\Xi}$ describes its temporal evolution. For this discussion we make the assumption that a temporal basis $\mathbf{\Xi}$ is known for which a column-wise sparse solution $\hat{\mathbf{W}}$ to (2.40) exists.

Next, write the analysis data ensemble $\mathcal{U}$ in matrix form

$$\mathbf{U} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{u}_1 & \dots & \mathbf{u}_m \\ \downarrow & & \downarrow \end{bmatrix} = \mathbf{W}\mathbf{\Theta}, \quad (2.41)$$

by defining an appropriate binary matrix $\mathbf{\Theta} \in \mathbb{R}^{(N+1) \times m}$, $m = |\mathcal{J}|$, which encodes the particular timesteps at which data are available. The aim of compressive sampling is to determine $\mathbf{W}$ from knowledge of $\mathbf{U}$ only. In view of (2.40) and (2.41), one way to achieve this is to seek a column-wise sparse solution $\hat{\mathbf{W}}$ to

$$\mathbf{U} = \hat{\mathbf{W}}\mathbf{\Xi}\mathbf{\Theta}, \quad (2.42)$$

which would then provide an approximation to the full signal via $\mathbf{W} \approx \hat{\mathbf{W}}\mathbf{\Xi}$.

Note that, since $m \leq N$, (2.42) is overdetermined meaning that the imposition of sparsity is necessary to obtain a well-posed problem. The most widely used approach to sparsity-promotion is to penalize the ℓ^1 -norm of $\hat{\mathbf{W}}$ subject to the constraint (2.42). However, apart from in the scalar valued case $N_u = 1$, this will impose entrywise sparsity rather than the desired columnwise sparsity of $\hat{\mathbf{W}}$. Methods to avoid this problem typically use *greedy* algorithms such as Orthogonal Matching Pursuit (OMP) [79] or Compressive Sampling Matching Pursuit

(CoSaMP) [80]. Rather than minimising the ℓ^1 -norm of $\hat{\mathbf{W}}$ such methodologies analyse the correlation of the matrix $\mathbf{\Xi}\mathbf{\Theta}$ with the downsampled data ensemble $\mathbf{U}$. They are iterative algorithms in which, upon each iteration, one sparse vector (a column of $\mathbf{\Xi}$) is added to the solution. Hence, the outputs of the algorithms are sparse, with the sparsity of the solution equal to the number of iterations.

An important fact is that the sampling strategy, encoded by $\mathbf{\Theta}$, plays a significant role in determining the performance of compressive sampling algorithms. In particular, a known requirement is that the chosen temporal basis $\mathbf{\Xi}$ has a low correlation (in compressive sampling terminology referred to as coherence) with the sampling strategy $\mathbf{\Theta}$ (where coherence is defined $\max_{j,k} |\langle (\theta_{jl})_{l=0}^N, (\xi_{lk})_{l=0}^N \rangle|$) [81]. A common way to ensure low correlation is to use Bernoulli and Gaussian random measurements, since it is highly probable that random matrices will have a low correlation with any basis, including the canonical Fourier basis [81]. That is, if periodic structures are sought in time, then an aperiodic sampling strategy is beneficial for extracting them from downsampled data. We reiterate that DMD-type modal extraction methods do not currently support sampling where the time intervals between corresponding entries in $\mathbf{A}$ and $\mathbf{B}$ are non-constant, meaning that any potential benefits of such a strategy may not be realised using current technique.

A viable technique allowing the use of compressive sampling and at the same time obtain DMD-like output was proposed in [33]. The compressive sampling to $\mathcal{U}$ to form an approximation $\hat{\mathcal{W}}$ to the fully-resolved data ensemble $\mathcal{W}$. This is achieved by choosing and fixing a temporal basis $\mathbf{\Xi}$ and then successively adding a k number of elements of this basis which best correlate with the available data. Motivated by the fact that many fluid flows are sparse in the Fourier basis, it was proposed in [33] to set $\mathbf{\Xi}$ to the inverse Fourier Transform and obtain approximations $\hat{\mathcal{W}}_k \approx \mathcal{W}$. DMD is then applied to each $\hat{\mathcal{W}}_k$ to determine the coherent structures corresponding to the identified eigenvalue information. In chapter 6 the methodology proposed by [33] will be compared against the newly introduced is-OMD technique.

Chapter 3

On some benefits of OMD over established techniques

The methodologies used in this thesis will incorporate the OMD technique to obtain structures of dynamical significance. The algorithm is chosen over the DMD approach due to benefits resulting from the additional flexibilities in choosing projection space $\mathbf{L}$. Some of the advantages are presented here on a synthetic system consisting of three structures (figure 3.1) similar to the most energetic structures present in the cylinder flow. The system is constructed such that

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^3 a_j(t) \Phi_j(\mathbf{x})$$

where the structure in figure 3.1(a) is used to represent a mean mode with a time-independent dynamics of $a_1(t) = 10$, $t \geq 0$ whilst two other structures 3.1(b)-(c) will be associated with dynamics $a_2(t)$, $a_3(t)$ which follow a third-order Stuart-Landau equation

$$\frac{dA}{dt} = \lambda^* A - \mu^* A |A|^2 \quad (3.1)$$

where $A = a_2 + ja_3$ and λ^* , μ^* are set to $\lambda^* = 0.006 \pm j0.25$ and $\mu^* = 0.0004 \pm j0.001$. The equation (3.1) admits solutions, $A = 0$ and $|A|^2 = \lambda^*/\mu^*$, where the latter solution corresponds to the limit cycle. The values λ^* , μ^* were chosen here in such a way to achieve the slow transition between two steady states. Equation (3.1) is solved using a 4th-order Runge Kutta method with randomly generated complex initial condition $A(t_0) \sim 10^{-3}$ to obtain the temporal evolution

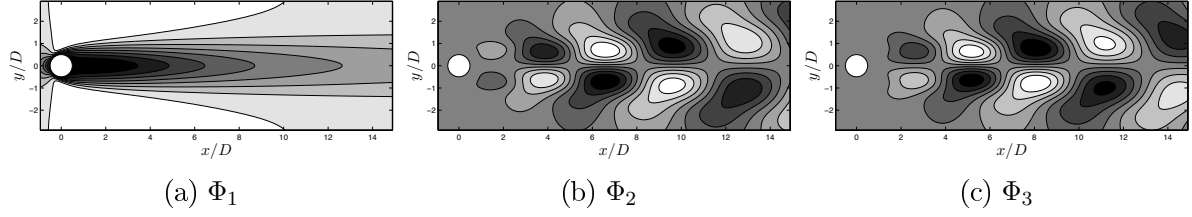


Figure 3.1: Velocity field structures used to construct a synthetic dynamical system.

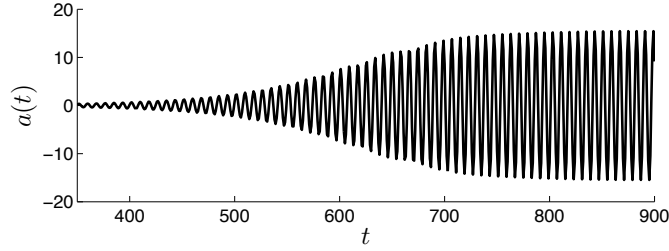


Figure 3.2: The temporal evolution of vertical component of the velocity, v , at the base of a cylinder in the three dimensional synthetic model.

shown in figure 3.2. The synthetic flow fields are reconstructed as in (1.2) to obtain $N = 1050$ snapshots. Applying the OMD or DMD algorithms and constraining the rank to $r = 30$, the structures presented in figure 3.1 are recovered with eigenvalues of $\lambda = 0$ corresponding to Φ_1 and $\lambda = \pm j0.27$ with real and imaginary parts of the associated modes consisting of Φ_2 and Φ_3 .

Two benefits of the OMD algorithm will be presented based on this system: the ability of the OMD algorithm to approximate the system with fewer structures; and a decreased sensitivity of the eigenvalue information to measurement noise.

3.1 A new method of mode ranking

Noting that the information about the number of structures present in the system is not available in practical situations, both the OMD and the DMD algorithms are applied to the synthetic system to obtain an approximation of rank $r = 1049$, equivalent to $r = N - 1$. Because the synthetic system consists only of three structures, for $r > 3$ both OMD and DMD are likely to become numerically unstable. For example, in the OMD algorithm the term $L^T B B^T L$ in (2.24) will be of rank three while for DMD the majority of diagonal entries of Σ in (2.13) will be close to zero. Since the inverse of both terms is required to obtain the output, the algorithms will be ill-conditioned. In order to prevent this behaviour a white noise with standard deviation $\sigma = 10^{-4}$ is added to the snapshots. Such noise magnitude is equivalent to

0.026% of the average amplitude of the velocity oscillations when system is in the limit cycle regime and does not affect significantly the OMD or DMD output.

Despite the full rank of $\mathbf{A}$ and $\mathbf{B}$ after noise addition, the system is still effectively low-dimensional and it is expected that the output of the OMD and DMD for $r = 1049$ will contain numerous redundant modes. In order to identify coherent structures, a criterion indicating the importance of structures and eigenvalues among the output of the modal decomposition algorithms is required. In [36] the DMD modes are ranked based on their energetic content, $E = \sqrt{N} \|\Phi_j\|_2 / \|\mathbf{A}\|_F$ where $\|\cdot\|_F$ denotes the Frobenius norm, estimating the relative contribution of each Φ_j into the overall energy of all snapshots. This approach assumes that Φ_j are not normalised which is the case for the DMD methodology if an interpretation as in (2.19) is used. For $r = N - 1$, the DMD modes Φ_i^{DMD} satisfy $\mathbf{u}_k = \sum_{j=1}^r \lambda_j^{k-1} \Phi_j^{DMD}$ exactly. In this case

$$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \vdots \\ \mathbf{u}_N \end{bmatrix} = \begin{bmatrix} I & I & \cdots & I \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \vdots \\ \Phi_N \end{bmatrix}. \quad (3.2)$$

is solved for $[\Phi_1^\top, \Phi_2^\top, \Phi_3^\top, \dots, \Phi_N^\top]^\top$, which will be in the form of $\Phi_j = k_j \Phi_j^{DMD}$ where $k_j > 0$ and $\Phi_j^{DMD} = \mathbf{U}_p \tilde{\mathbf{P}}$ (2.15) are normalised DMD modes. The value of k_j corresponding to Φ_j is a scaling which matches energetic content of Φ^{DMD} to data $\mathbf{B}$. In the case of the OMD methodology, an equivalent reasoning cannot be reproduced since the relationship between subsequent snapshots is assumed to have a form $\mathbf{u}_{i+1} = \mathbf{L} \mathbf{M} \mathbf{L}^T \mathbf{u}_i + \epsilon_i$ allowing for discontinuous data sets. Furthermore, columns of $\Phi^{OMD} = \mathbf{L} \mathbf{P}_M$, where $\mathbf{M} = \mathbf{P}_M \mathbf{\Lambda}_M \mathbf{P}_M^{-1}$, are normalised. Instead, consider the projection onto a data set $\mathbf{L} \mathbf{L}^T \mathbf{B}$, which can be expressed as

$$\mathbf{L} \mathbf{L}^T \mathbf{B} = \mathbf{L} \mathbf{P}_M \mathbf{P}_M^{-1} \mathbf{L}^T \mathbf{B} = \Phi [\mathbf{P}_M^{-1} \mathbf{L}^T \mathbf{B}]. \quad (3.3)$$

The expression for $\mathbf{P}_M^{-1} \mathbf{L}^T \mathbf{B}$ in terms of the original matrix entries is complicated. Therefore,

let us rewrite (3.3) as

$$\mathbf{L}\mathbf{L}^T\mathbf{B} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \Phi_1 & \Phi_2 & \cdots & \Phi_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1N} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rN} \end{bmatrix}}_{:=\mathbf{P}_M^{-1}\mathbf{L}^T\mathbf{B}}.$$

Consequently, $\mathbf{L}\mathbf{L}^T\mathbf{B}$ can be expressed in terms of OMD modes as

$$\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{L}\mathbf{L}^T\mathbf{u}_1 & \mathbf{L}\mathbf{L}^T\mathbf{u}_2 & \cdots & \mathbf{L}\mathbf{L}^T\mathbf{u}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \sum_{j=1}^r \alpha_{j1}\Phi_j & \sum_{j=1}^r \alpha_{j2}\Phi_j & \cdots & \sum_{j=1}^r \alpha_{jN}\Phi_j \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

Note that coefficients multiplying the j^{th} mode always come from the j^{th} row of $\mathbf{P}_M^{-1}\mathbf{L}^T\mathbf{B}$. Hence, in order to estimate how much of the data is spanned by each mode, it is reasonable to take the norm of the corresponding row of $\mathbf{P}_M^{-1}\mathbf{L}^T\mathbf{B}$, i.e $k_j = \sum_{l=1}^N \alpha_{lj}^2$.

It should be noted that this approach only takes into account the energetic contribution of the modes. In order to add the dynamic impact of each Φ_j note that $\mathbf{L}\mathbf{L}^T\mathbf{u}_{i+1} \approx \mathbf{L}\mathbf{P}_M\mathbf{\Lambda}_M\mathbf{P}_M^{-1}\mathbf{L}^T\mathbf{u}_i$ which implies

$$\begin{bmatrix} \uparrow & \uparrow \\ \mathbf{L}\mathbf{L}^T\mathbf{u}_2 & \cdots & \mathbf{L}\mathbf{L}^T\mathbf{u}_N \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ \sum_{j=1}^r \lambda_j \alpha_{j1}\Phi_j & \cdots & \sum_{j=1}^r \lambda_j \alpha_{jN}\Phi_j \\ \downarrow & \downarrow \end{bmatrix}.$$

The importance of each mode which combines both the energetic and dynamic impact can be estimated by computing the norms of corresponding rows in $\mathbf{\Lambda}_M\mathbf{P}_M^{-1}\mathbf{L}^T\mathbf{B}$ such that

$$E_j = \frac{\sum_{k=1}^N |\lambda_j \alpha_{jk}|^2}{\sum_{j=1}^r \sum_{k=1}^N |\lambda_j \alpha_{jk}|^2}. \quad (3.4)$$

Such a scaling incorporates the magnitude of each eigenvalue, implying, for example, that fast-decaying modes will be penalised through a smaller scaling.

Using the output of the synthetic system for $r = 1049$, the values of E_j signifying the

importance of each mode are calculated using the methodologies associated with each algorithm, i.e. for OMD using (3.4) and for DMD setting $E_j = k_j$ where k_j is obtained from the solution to (3.2). The plots of E_j against the $|\text{Im}(\lambda)|$ for both OMD and DMD are shown in figure 3.3. In both results there is a clear region at $|\text{Im}(\lambda)| \approx 0.27$ where E_j is significantly larger than values corresponding to spurious frequencies. However, for the OMD output this region contains only two eigenvalues while in the DMD case it appears to be distributed over a greater number of modes. An increased contribution of the redundant modes is also visible in figure 3.3. In the case of the OMD, the dynamically significant modes are approximately 10^{12} times larger than the spurious structures, while in the case of DMD this difference is only of 10^3 .

It is noted that as the value of r approaches $r = 3$ the energetic contents of the OMD and DMD outputs progressively become similar. For example, figure 3.3(c) presents the ranking of the dynamical modes for the DMD output with $r = 30$, where the eigenvalues close to $|\text{Im}(\lambda)| \approx 0.27$ are retained and the number of spurious frequencies is reduced. Nonetheless, the ranking methodology implies that all eigenvalues in the neighbourhood of $|\text{Im}(\lambda)| \approx 0.27$ have a similar energetic content. The analogous OMD result for $r = 30$ contain spurious frequencies, however, their relative content is $E_j \sim 10^{-12}$ and it is immediately obvious that they can be discarded. When $r = 3$ both algorithms identify eigenvalues of importance, however, when the underlying differential equations governing the flow are not explicitly known, it is preferable to have an automatic indication of which values are dynamically relevant.

3.2 Sensitivity to noise

The sensitivity of eigenvalue information to measurement noise is analysed on a limit cycle of the Stuart-Landau system (3.1) using a data set consisting of 350 snapshots covering approximately 15 periods of oscillation. The comparison is performed between OMD, DMD, tIsDMD and fbDMD, with the last two methods following the implementation in [51] outlined in §2.1.4. When all algorithms are applied to noise-free data with an approximation rank of $r = 3$, consistent reference eigenvalues, $\lambda_{ref} = \pm j0.27$, are obtained. Subsequently, white noise is added to the data set for a range of standard deviations σ varying between $\sigma = 0.001$ and $\sigma = 0.9$, which is equivalent to 0.26% and 230% of the average velocity oscillation amplitude, and all four algorithms are applied to noisy data. The relative error is calculated such that $\epsilon = |\lambda - \lambda_{ref}|/|\lambda_{ref}|$. In order to obtain a clearer statistical dependence on noise magnitude,

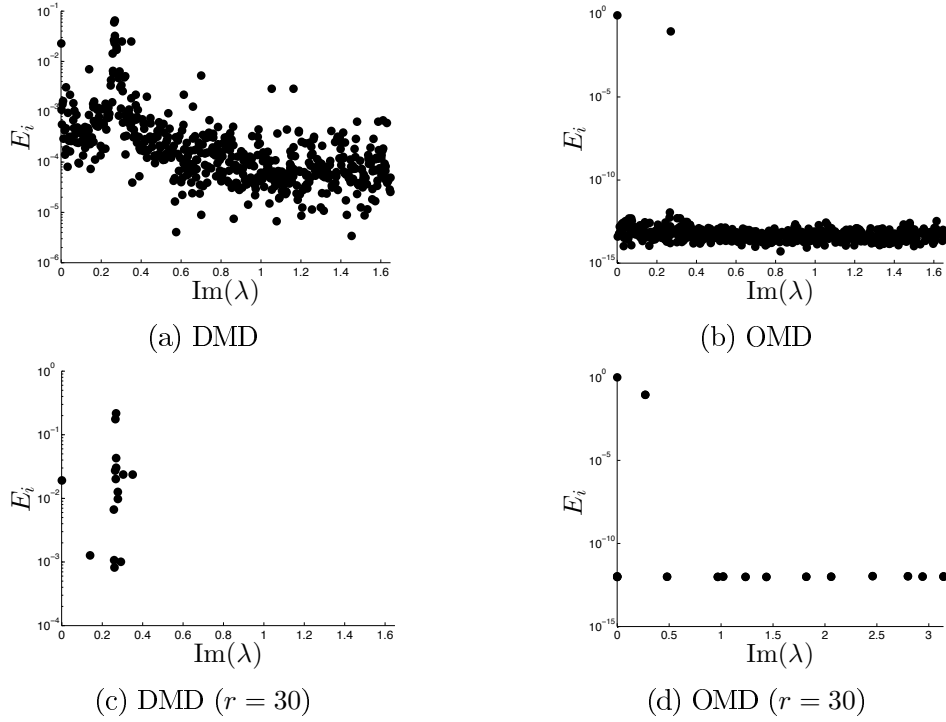


Figure 3.3: Energetic contribution of the dynamic modes to the synthetic system for $r = N - 1$ (a)-(b) and for $r = 30$ (c)-(d).

the procedure was repeated 1000 times, at every standard deviation value σ , and values of ϵ averaged accordingly.

The plot of the averaged error $\bar{\epsilon}$ corresponding to the different algorithms for varying noise standard deviations σ is presented in figure 3.4. It is interesting to note that different methodologies provide most accurate eigenvalues at different levels of noise. For small values of $\sigma \sim 10^{-2}$ the accuracy of all algorithms is comparable. However, when σ increases the discrepancies become more prominent. For “intermediate” level of noise, $\sigma \approx 0.1$, the fbDMD algorithm outputs the most accurate eigenvalues whilst tlsDMD and OMD induce similar values of ϵ . When σ is increased even further, more specifically when $\sigma \approx 0.4$ which is equivalent to signal-to-noise ratio $SNR \approx 1$, the error corresponding to fbDMD increases rapidly. It is further noted that the standard deviation of ϵ corresponding to fbDMD increases for $SNR > 1$ requiring a larger number of measurement to converge to expected value of ϵ within 0.01% accuracy. At the same time, for $SNR > 1$, the accuracy of the OMD eigenvalues improves relative to tlsDMD and figure 3.4 demonstrates that OMD gives most accurate results for extreme levels of noise. In the analysed test case all methodologies “collapse” at the same level of noise, $\sigma = 1.5$, where no algorithm is able to extract the frequencies in the neighbourhood

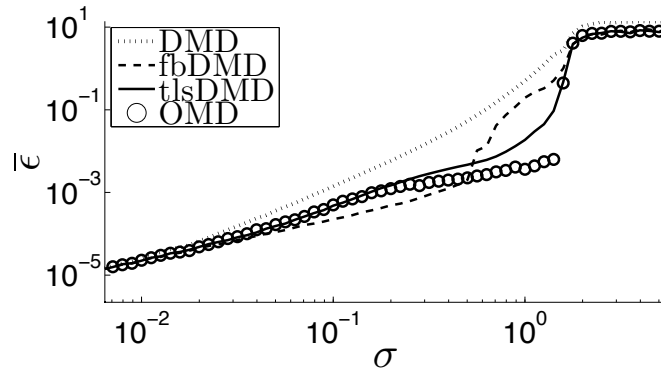


Figure 3.4: The eigenvalue error ϵ for noise with different standard deviation σ .

of the reference values.

It is noted that the analysis was performed here on a very simple system which itself is of low rank and the analysis domain encompassed only the limit cycle (a regime where the system is almost linear). For more complicated systems typically encountered in practical applications the discrepancies are expected to increase. Furthermore, even though for $SNR < 1$ fbDMD is the most accurate it is straightforward to extend this algorithm to implement the features of OMD. In such case it is likely that the ϵ will decrease even further albeit at a greater computational cost.

Chapter 4

Dynamical Estimation of the Cylinder Flow

4.1 Introduction

One of the crucial requirements of active fluid flow control is a need to have online information about the system's state in order to decide upon an appropriate control input. Such knowledge can be obtained by combining a dynamical model with online measurements. Some examples where this approach has proved to be successful are in the dynamical estimation of flat plate flow [63, 72] or channel flow [73], as described in chapter 1. In this section, an analysis of this type is performed on the canonical cylinder flow at the Reynold number $Re = 60$ with the overall objective to construct an accurate low-dimensional model, which when combined with certain estimation algorithms, allows us to accurately reconstruct the full flow field.

The cylinder flow is an example of a well-understood and relatively simple bluff-body flow with an excellent summary of its major physical features presented in the review article [82]. Nonetheless, it is characterised by the evolution from unstable equilibrium to the limit cycle; a behaviour which is intrinsically nonlinear making the construction of a dynamical model which fully approximates its dynamics not trivial. Additionally, despite its relative simplicity, any insights gained in the dynamical estimation of the cylinder flow may be valuable for the study of more complicated bluff body flows.

A detailed description of cylinder flow is given in §4.2 where the main dynamics and corresponding structures are outlined. Based on the observations in §4.2 the approaches used

to obtain the dynamical models of the cylinder flow are given in §4.3. In order to assess the long timescale accuracy of the dynamical models with different orders of complexity, initial value problems are solved and the solutions are presented in §4.4. Subsequently, an estimator is designed in §4.5 and robustness of the proposed estimation methodology is analysed in §4.6. The impact on the accuracy of the estimates due to certain estimation parameters (such as number of pressure measurements or rank of the state space) is analysed in §4.7.

4.2 Description of the dynamical system

For $Re < Re_c = 46.6$ the cylinder flow is asymptotically stable, however, once Re increases beyond this value the system undergoes a Hopf bifurcation. In dynamical system terminology a Hopf bifurcation is a critical point at which a fixed point loses its stability and a periodic limit cycle develops towards which the system approaches asymptotically. For cylinder flow, the relevance of the Hopf bifurcation can be demonstrated analytically via spectral analysis of the linearised Navier-Stokes operator around an equilibrium. For $Re < Re_c$ it can be observed that all eigenvalues of the linearised operator are contained in the stable left-half plane while for $Re > Re_c$ a pair of eigenvalues with positive real part is observed, signifying instability of the steady equilibrium point of the Navier-Stokes equation. The eigenmodes corresponding to these unstable eigenvalues are often referred to as global modes [23]. For $Re > Re_c$ the consequence of the Hopf bifurcation is represented by three distinct regimes demonstrated in figure 4.1(a) through the temporal evolution of the v velocity component close to the cylinder base: the unstable equilibrium 4.1(b) which, subject to perturbation, converges onto the limit cycle represented by von Karman vortex shedding 4.1(c).

It was shown in [23] using weakly nonlinear analysis that the dominant dynamics of the cylinder flow at Re close to Re_c can be approximated by a third-order Stuart-Landau equation of the form

$$\frac{dA}{d\tau} = \lambda A - \mu A|A|^2, \quad (4.1)$$

where $\lambda, \mu \in \mathbb{C}$, A represents the amplitude of the global mode and $\tau = \epsilon t$ is a long time scale, where ϵ quantifies the divergence away from Re_c such that

$$\epsilon = \frac{1}{Re_c} - \frac{1}{Re}.$$

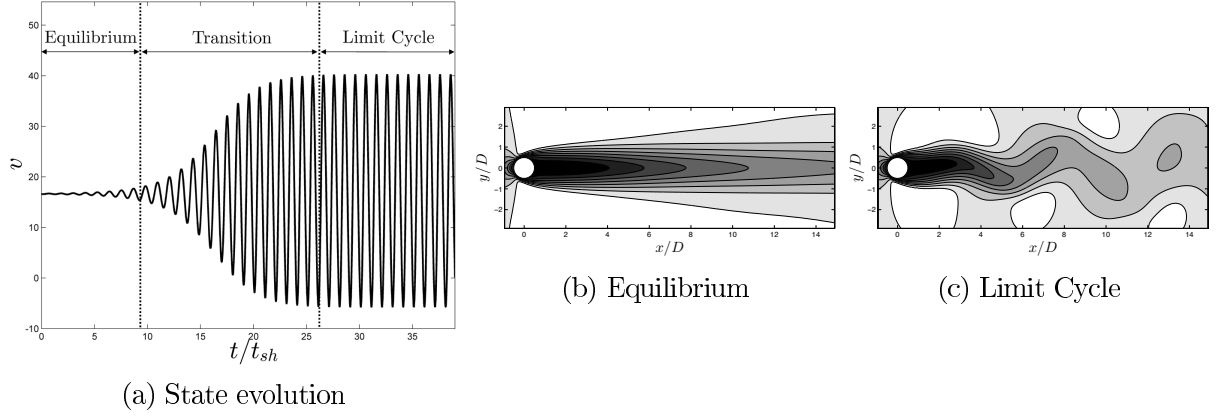


Figure 4.1: The temporal evolution of v -component of the velocity, v , at the base of a cylinder in its centerline plane (a); the u -component of the velocity field at an instance when the system is close to the equilibrium (b) and when it converges onto a limit cycle (c).

The essence of the weakly nonlinear analysis is the assumption that the velocity field can be expressed as summation of multiscale flow fields

$$\mathbf{u}(t) = \mathbf{u}_s + \epsilon^{1/2}\mathbf{u}_1(\tau, t) + \epsilon\mathbf{u}_2(\tau, t) + \epsilon^{3/2}\mathbf{u}_3(\tau, t) + \dots \quad (4.2)$$

where $\mathbf{u}_s$ is a solution to steady Navier-Stokes equations. The decomposition (4.2) is substituted into Navier-Stokes formulation and a system of equations is successively solved at each order $\epsilon^{j/2}$ to obtain structures and dynamics associated with each $\mathbf{u}_j(\tau, t)$. The Stuart-Landau equation emerges as a compatibility condition necessary to obtain a solution corresponding to $\epsilon^{3/2}$. Following [23] the leading order term of (4.2) is expressed as

$$\mathbf{u}_1(\tau, t) = A(\tau) \exp(i\omega_0 t) \bar{\mathbf{u}} + \text{c.c} \quad (4.3)$$

where ω_0 and $\bar{\mathbf{u}}$ are the global mode eigenvalue and eigenvector of the linearised Navier-Stokes equations at Re_C and ‘c.c’ signifies complex conjugate.

The dynamics of $A(\tau)$ are governed by (4.1) which provides an approximation of the Hopf bifurcation. In order to provide an intuitive understanding of this dynamical feature let us

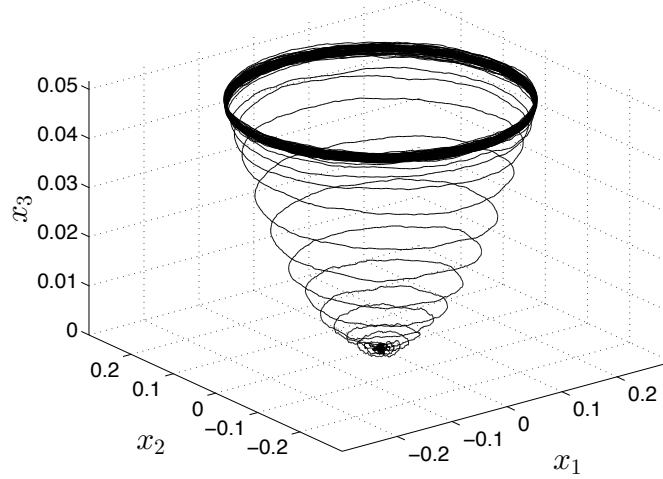


Figure 4.2: Schematic representation of the Hopf bifurcation.

consider a simple system consisting of three states $\mathbf{x} = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ governed by

$$\frac{d\tilde{A}}{dt} = (\mu - x_3)\tilde{A}, \quad (4.4a)$$

$$\frac{d\Phi}{dt} = 1, \quad (4.4b)$$

$$\frac{dx_3}{dt} = -x_3 + \tilde{A}^2 \quad (4.4c)$$

where $\tilde{A} := \sqrt{x_1^2 + x_2^2}$, $\Phi = \arctan(x_2/x_1)$ and $\mu = 0.05$. The system (4.4) also represents a Hopf bifurcation and, in fact, by eliminating x_3 it is possible to recover (4.1). The set of ODEs (4.4) is solved using fourth order Runge-Kutta method with additive process noise with standard deviation $\sigma = 1 \times 10^{-4}$. The addition of process noise is to reproduce a typical situation in practical applications in which a system is subject to random perturbations. The solution to (4.4) with initial condition $\mathbf{x}_0 = 0$ is showed in figure 4.2. It can be seen that $\mathbf{x} = 0$ corresponds to an equilibrium analogous to the one in the cylinder flow visible in figure 4.1(b). Once subject to a perturbation the system spirals out and converges to an oscillatory attractor, analogous to the vortex shedding in figure 4.1(c). It is possible to notice in figure 4.2 that there is a certain variation in the value of x_3 when system settles on the attractor, which is a result of the additive process noise. Nonetheless, despite continuous perturbations, the system oscillates around $x_3 = 0.05$ demonstrating that the attractor is stable. The same behaviour is observed in the vortex shedding of the bluff-body flows.

The Stuart-Landau equation has three equilibrium points: the trivial one $A = 0$ and

$|A|^2 = \frac{\lambda}{\mu}$. For $|A| \ll 1$ the linear part is dominant and if $\text{Re}(\lambda) > 0$ (as is the case in the cylinder flow for $Re > Re_c$) the equilibrium is unstable. In order to obtain further understanding of (4.1) let us express A in terms of polar coordinates $A(\tau) = r(\tau)e^{j\theta(\tau)}$. In this way (4.1) can be reformulated as two separate ODEs describing the evolution of r and θ respectively

$$\frac{dr}{d\tau} = \lambda_R r - \mu_R r^3, \quad (4.5a)$$

$$\frac{d\theta}{d\tau} = \lambda_I - \mu_I r^2. \quad (4.5b)$$

where $\lambda = \lambda_R + j\lambda_I$ and $\mu = \mu_R + j\mu_I$. Assuming initial conditions of $A_0 = r_0 e^{j\theta_0}$, (4.5a) and (4.5b) can be solved to obtain expressions for $r(\tau)$ and $\theta(\tau)$

$$r(\tau) = \sqrt{\frac{\lambda_R r_0^2}{\mu_R r_0^2 + (\lambda_R - \mu_R r_0^2) \exp(-2\lambda_R \tau)}}, \quad (4.6a)$$

$$\theta(\tau) = \theta_0 + \omega \tau + \beta \ln \frac{r(\tau)}{r_0}. \quad (4.6b)$$

Here, $\omega = (\lambda_I - \lambda_R \frac{\mu_I}{\mu_R})$ and $\beta = \frac{\mu_I}{\mu_R}$. Letting $\tau \rightarrow \infty$ gives

$$r(\tau) \rightarrow \sqrt{\frac{\lambda_R}{\mu_R}}, \quad \tau \rightarrow \infty, \quad (4.7a)$$

$$\frac{\theta(\tau)}{\tau} \rightarrow (\lambda_I - \lambda_R \frac{\mu_R}{\mu_I}), \quad \tau \rightarrow \infty, \quad (4.7b)$$

which leads to the full state equation

$$A(\tau) \sim \sqrt{\frac{\lambda_R}{\mu_R}} \exp(i(\lambda_I - \lambda_R \frac{\mu_R}{\mu_I})\tau), \quad \tau \rightarrow \infty \quad (4.8)$$

Equation (4.8) clearly demonstrates that as $\tau \rightarrow \infty$ the dominant dynamics of the system approaches a limit cycle with a constant amplitude and phase. The corresponding values of λ , μ can be obtained analytically via integrals involving eigenvectors and adjoint eigenvectors of the linearised Navier-Stokes equations with forcing terms arising at order $\epsilon^{3/2}$ (the detailed procedure can be found in [23]) or through a numerical fit to the data. It must be noted that the Stuart-Landau equation is valid only for the Reynolds number in the vicinity of the first Hopf bifurcation (i.e. near critical value $Re_c = 46.6$). Nonetheless, $Re = 60$ is sufficiently close

for (4.1) to provide a viable approximation.

The objective here is to capture the key features of the dynamics using a low-dimensional dynamic model in the state-space form

$$\dot{\mathbf{x}} = A(\mathbf{x}) + \mathbf{w}, \quad (4.9a)$$

$$\mathbf{y} = C(\mathbf{x}) + \mathbf{v}, \quad (4.9b)$$

where $\mathbf{x}$ denotes a low-dimensional state of the system and $\mathbf{y}$ are some measurable quantities. By $\mathbf{w}$ and $\mathbf{v}$ we denote state disturbances and measurement noise. Functions $A(\mathbf{x})$ and $C(\mathbf{x})$ describe the dynamics of the system and a relationship between the state $\mathbf{x}$ and measurable quantity $\mathbf{y}$.

Two fundamental questions are: 1) the appropriate construction of the low-dimensional state $\mathbf{x}$; and 2) the choice of the dynamical model order. In order to address the first question, the state will be constructed as a projection of temporal snapshots onto different coherent structures. The analysis will be performed to assess suitability for this purpose of the output from the established modal decomposition methodologies outlined in §2.1 such as the POD, DMD and OMD algorithms. In order to establish an appropriate form of $A(\mathbf{x})$, $C(\mathbf{x})$ the structure of the equations governing the cylinder flow will be taken into account. Even though the Navier-Stokes equations are inherently quadratic the Stuart-Landau equation outlined above is of the third order. Consequently, $A(\mathbf{x})$, $C(\mathbf{x})$ will be approximated using low-order models of the linear, quadratic and cubic form.

An analysis is performed using numerical velocity and pressure values. Numerical data was obtained using the in-house solver Pantarhei, which is validated for cylinder flow in [83, 84]. The code uses the finite volume method on an unstructured grid with C-grid topology and collocated variable arrangement. The convective and viscous terms of Navier-Stokes equations are discretised using a second-order central scheme and the system is advanced in time using a second-order, three-point backward scheme. The convective term is linearised using a second order extrapolation of the vertical component of the velocity from the previous time instants whilst the streamwise component is treated implicitly. Additionally, viscous terms are also treated implicitly. In order to compute the velocities at the cell surfaces the momentum interpolation method used in [85] is employed. Continuity is enforced via the Pressure Implicit

with Splitting Operator (PISO) algorithm [86]. In order to obtain the solution of the pressure equation, the GMRES method is used together with the BoomerAMG algebraic multigrid preconditioner of the Hypre package [87].

The domain used for the simulations was $48D$ long and $32D$ wide, where D is the cylinder diameter. The distance between cylinder location and upstream and downstream boundaries is respectively $16D$ and $32D$. The total number of cells is 62,791, with sizes locally refined in the wake region and close to the cylinder boundary. The minimum grid spacing close to the cylinder is set to $h \approx D/60$ and orientation modified such that cell faces are aligned with the cylinder walls. Uniform inflow ($U_\infty = 1$), convective outflow $\frac{\partial \eta}{\partial t} + U_\infty \frac{\partial \eta}{\partial x} = 0$, where η is any variable convected out of the domain, and symmetric boundary conditions on the lateral sides of the domain are imposed. Additionally, the no-slip condition is imposed on the cylinder boundary. Density and viscosity of the flow are modified accordingly to achieve the desired Reynolds number. The velocities and spatial dimensions are non-dimensionalised using respectively U_∞ and D , whilst the advection time that is given by $t = \hat{t}U_\infty/D$. For the subsequent analysis every 25th timestep is retained giving on average 29 snapshots per shedding period.

4.2.1 Structures present in the system

In order to obtain insight into the structures present in the system POD, DMD and OMD algorithms are applied to 1150 snapshots encompassing all three regimes presented in figure 4.1(a). The u -components of the mean flow and the first five POD modes, $(\mathbf{U}_p)_i$, are presented in figure 4.3(a)-(f). The presentation of the analogous structures obtained using the DMD and OMD algorithms is omitted for now due to lack of obviously significant visual differences. Additionally, temporal evolution of the selected projection coefficients is shown in figure 4.3(g)-(i). In this chapter we use a nondimensional time t/t_{sh} where t_{sh} is a timescale corresponding to the shedding frequency obtained from velocity spectra at a downstream wake location.

Based on past literature, for example [12, 24, 36], it is possible to identify the dynamical meaning of the POD modes presented in figure 4.3. The mean flow represents the recirculation zone with a varying length depending on the regime over which the mean was calculated [12]. For $Re = 60$, if the mean is calculated using data in the limit cycle regime the length of the recirculation zone is approximately $2.6D$ whilst if data is averaged over an interval close to the equilibrium, it is found to be approximately $4D$. In [12] it was observed that in order to

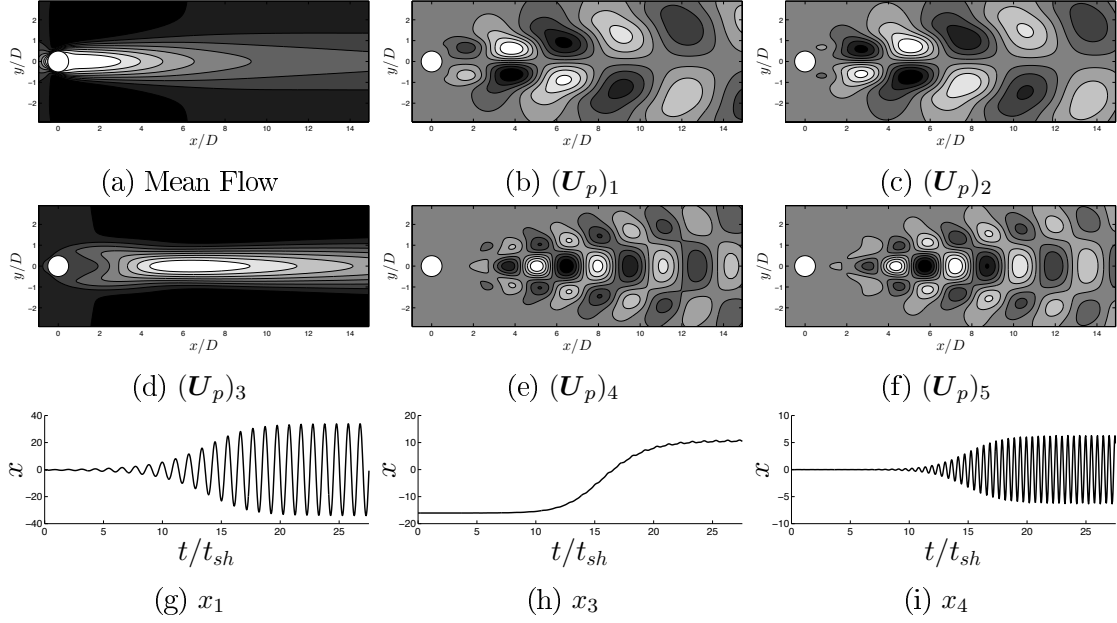


Figure 4.3: The mean flow (a) and first five POD modes (b)-(f) of the cylinder flow; examples of the temporal evolution of the projection coefficients $x_j = (\mathbf{U}_p)_j^\top \mathbf{u}$ (g)-(i)

accurately predict the transition from unsteady equilibrium to the time averaged mean on the limit cycle a structure corresponding to a mean-field correction referred to as *shift mode* should be introduced. In particular, the shift mode is calculated by taking the difference between the mean over the snapshots in the limit cycle and the steady-state solution. The name refers to a fact that this structure represents a *shift* away from the POD vector space in the limit cycle regime. The emergence of this structure is linked to the nonlinear term in the Navier-Stokes equations [88]. More specifically, using a weakly nonlinear analysis it can be shown that the shift mode is a result of the interaction between a global mode and its complex conjugate [23]. If POD modes or OMD modes are calculated using data covering all three regimes the shift mode is in fact present among the extracted structures (figure 4.3(d)).

Among the outputs of the DMD and OMD algorithms, the dynamic modes which most closely resemble the presented POD structures are identified based on a spatial correlation. The OMD mode corresponding to the structure in 4.3(d) is real with eigenvalue of $\lambda = -0.0018$. This is consistent with the temporal evolution in figure 4.3(h) and the interpretation in [12]. At the equilibrium the shift mode is dominant, since it is required to describe the steady state solution to the NSEs which is different to the time-averaged mean flow on the limit cycle and its influence diminishes once the system approaches the limit cycle (with the magnitude of the projection coefficient $|x_3|$ decreasing by approximately 35% as $t \rightarrow \infty$).

The POD modes presented in figure 4.3(b)-(c), (e)-(f) represent oscillatory dynamics in the system, where the modes in figure 4.3(b)-(c) are the global modes. It is noted that the nonlinear term of the Navier-Stokes induces further subharmonics of the global mode. Figure 4.3(e)-(f) represent the first subharmonic which emerges through the interaction of the global mode with itself [23]. Further subharmonics will result from nonlinear interactions between subsequent oscillatory structures and their presentation is omitted here. The oscillatory POD modes come in pairs and represent the same structures offset downstream by approximately a quarter of their lengthscale. It is possible to see further oscillatory POD modes whose lengthscale decreases as they are progressively lower ranked energetically.

The structures seen in figures 4.3(b)-(c) and 4.3(d)-(e) correspond respectively to eigenvalues of $\lambda = 0.003 \pm j0.855$ ($St = 0.136$) and $\lambda = 0.003 \pm j1.71$ ($St = 0.272$). The OMD modes corresponding to $(\mathbf{U}_p)_2$, $(\mathbf{U}_p)_3$ are those with eigenvalues possessing the smallest frequencies out of the spectrum corresponding to the OMD matrix $\mathbf{M}$ and are consistent with the shedding frequency obtained from the velocity spectra downstream of the cylinder base. This confirms that these structures correspond to the global mode. Since the frequency of modes corresponding to $(\mathbf{U}_p)_5$, $(\mathbf{U}_p)_6$ is its integer multiple it is clear that structures in figure 4.3(d)-(e) are the first harmonic of the global mode. It is notable that the growth rates of all harmonics are approximately the same suggesting that they transition to the limit cycle in a similar manner.

The difficulty of accurately extracting structures in figure 4.3 varies depending on the regime contained in the data set chosen to which the modal decomposition algorithms are to be applied. For example, the energetic content of the global mode subharmonics is low close to the equilibrium and progressively increases as the system converges to the limit cycle. Close to the equilibrium it was observed that if a sufficiently large amount of data is not available, such oscillatory structures contain spatial inaccuracies. On the other hand, for the data set on the limit cycle, the temporal range of one shedding period is sufficient to obtain accurate modes.

The opposite is true if the objective is to identify the shift mode. If one of the two algorithms is applied to the data set close to equilibrium, this structure is relatively energetic and is ranked as the third of the most importance, where the POD modes are ranked based on the singular values σ of the singular value decomposition $\mathbf{U} = \mathbf{U}_p \mathbf{\Sigma} \mathbf{V}^T$, where $\mathbf{\Sigma} = (\sigma_{jj})_{j=1}^N$, whilst ranking of the OMD modes follows the methodology outlined in §3.1. Based on the

singular values σ , the shift mode energy as a proportion of full snapshot energy has a value of 0.6%. Nonetheless, the energy of the global mode subharmonics are at least an order of magnitude smaller than that of the shift mode and consequently it is relatively easy to identify this non-oscillatory mode.

If the analysis is performed on the limit cycle, or close to it, the extraction of the shift mode is less trivial. In this case, instead of an energetic criterion, dynamic considerations should be taken into account. The spectrum of the OMD matrix in figure 4.4(a) applied to the full domain for $r = 3$ shows a non-oscillatory eigenvalue with negative growth rate which corresponds to the shift mode. A similar eigenvalue is visible if the OMD algorithm is applied to a data set close to and containing the limit cycle. For example figures 4.4(b)-(c) show the OMD spectra for $t/t_{sh} > 18$ and $t/t_{sh} > 23$ obtained for $r = 7$ and $r = 11$ respectively. In the latter case, as visible in figure 4.1(a), the system is characterised by a fully developed limit cycle, whilst the first case resembles a situation encountered in an analysis of experimental data. In practical applications, the systems would not solely evolve on the limit cycle but instead would be continuously perturbed away from the attractor and converge back onto it. Successively larger values of r were set in the OMD approximation because the shift mode is of relatively lower energy relative to subharmonics of the global mode, such that for $t/t_{sh} > 18$ it contains 0.22% energy whilst for $t/t_{sh} > 23$ it corresponds to 0.0028%, and for smaller values of r the real eigenvalue is not present. If r is increased even further, additional oscillatory structures are obtained, including spurious complex values. Nonetheless, taking into account the knowledge that the shift mode is non-oscillatory with a negative growth rate, figures 4.4(b)-(c) show that, even on the limit cycle, it is possible to identify the approximation of the shift mode based on these dynamic considerations. The shift mode structures corresponding to the OMD modes obtained for $t/t_{sh} > 18$ and $t/t_{sh} > 23$ are presented in figures 4.4(b)-(c), which show that due to its low-energy the extracted modes contain some spatial inaccuracies downstream in the wake. Nonetheless, the main features of the reference shift mode in figure 4.3(d) are retained.

It is noted that although the DMD and OMD modes were not presented here due to apparent lack of visual differences between the structures, the modes obtained from different algorithms are not exactly equivalent. This can be demonstrated by investigating the spatial

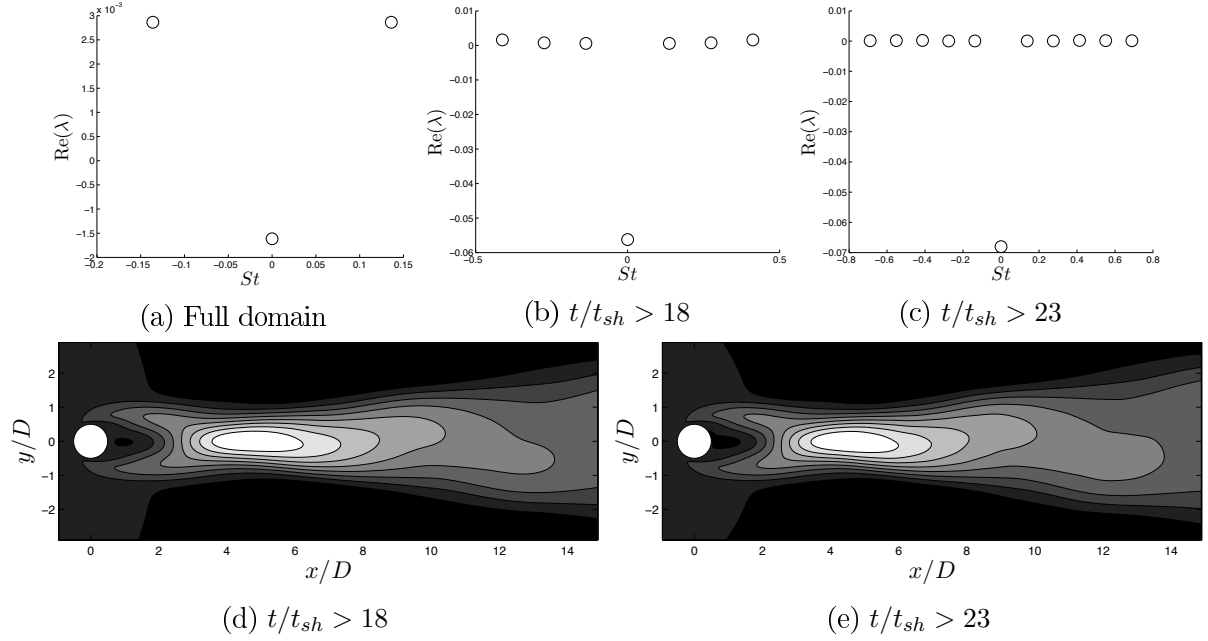


Figure 4.4: The OMD spectra when the algorithm is applied to data sets respectively consisting of all time instances (a), for $t/t_{sh} > 18$ (b) and for $t/t_{sh} > 23$ (c). The dynamic mode corresponding to non-oscillatory eigenvalue for $t/t_{sh} > 18$ (d) and for $t/t_{sh} > 23$ (e).

correlation statistic

$$\rho_{jk} = (\Phi_j^\top \Phi_k) / (\|\Phi_j\|_2 \|\Phi_k\|_2) \quad (4.10)$$

between different outputs. The correlation between DMD and POD modes in this case is $\rho = 0.999$ for global modes and $\rho = 0.995$ for the subsequent harmonic which shows that even though they are not equivalent the DMD structures are very similar to analogous POD modes. On the other hand the correlation between OMD and POD modes is $\rho = 0.937$ for global modes and $\rho = 0.956$ for the higher harmonic, signifying that the OMD algorithm outputs modes subtly different from the POD methodology. Although, the correlation between OMD and POD structures appears high it will be demonstrated in subsequent sections that those differences are sufficiently significant to induce significant differences in the estimation of the cylinder wake.

4.3 Dynamical modelling approach

The output of the mode extraction techniques will be used to project the high-dimensional velocity field data into a low-dimensional subspace of $\mathbb{R}^r$ via the mapping $\mathbf{u} \rightarrow \Phi^\top \mathbf{u}$. We let $\mathbf{x} = \Phi^\top \mathbf{u}$ denote the vector in the low dimensional subspace corresponding to the velocity field $\mathbf{u}$. In cases when the OMD output is used, due to the complex form of the modes an augmented

state space is used, i.e. $\mathbf{x} = [\text{Re}(\mathbf{\Phi})^\top, \text{Im}(\mathbf{\Phi})^\top]^\top \mathbf{u}$. A five-dimensional formulation of the state, $\mathbf{x} \in \mathbb{R}^{N_x}$ where $N_x = 5$, is used, where velocity fields are projected onto two global modes, the shift mode and two modes corresponding to the first subharmonic of the global mode. For POD modes this is equivalent to a projection onto structures seen in figure 4.3(b)-(f). The OMD modes are obtained by applying the OMD algorithm with the approximation rank of $r = 10$ to velocity fields $\mathbf{u}_i$ for $i \in \mathcal{T}_{fd}$, where $\mathcal{T}_{fd} = \{i \in \mathbb{N} : t_i/t_{sh} < 27.5\}$ for $t_i = t(i\Delta t)$ encompasses all three cylinder regimes. The dynamical modes are ranked using the methodology outlined in §3.1 to select the five most dominant modes. Furthermore, in order to confirm that those structures correspond to the same dynamical features, spatial correlations ρ_{jk} , defined in (4.10), between the selected POD and OMD modes are calculated and give values $\rho_{jk} > 0.95$. Although ρ_{jk} implies that both algorithms output very similar structures, nonetheless it will be shown that those discrepancies are large enough to induce differences in the accuracy of the dynamical models.

Taking into account the mathematical models of the cylinder flow (Navier-Stokes and Stuart-Landau equations), numerical models will be created with progressively higher levels of complexity, i.e. using linear, quadratic and cubic formulations. Since in the linear case, a continuous model

$$\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i \quad (4.11)$$

can be readily and exactly converted into a discrete form

$$\mathbf{x}_{i+1} = \tilde{\mathbf{A}}\mathbf{x}_i \quad (4.12)$$

where $\tilde{\mathbf{A}} = \exp(\mathbf{A}\Delta t)$, and Δt is the discretised timestep, linear modelling will be performed using the formulation (4.12) which does not require the approximation of the temporal derivatives. On the other hand, quadratic and cubic models will be expressed in a continuous time via

$$\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \sum_j (x_j)_i (\mathbf{Q}_q)_j \mathbf{x}_i, \quad (4.13a)$$

$$\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \sum_{j,k} (x_j)_i (x_k)_i (\mathbf{Q}_c)_{jk} \mathbf{x}_i. \quad (4.13b)$$

where $\mathbf{x}_i = [(x_1)_i, (x_2)_i, \dots, (x_r)_i]^\top$ and matrices $\mathbf{Q}_q \in \mathbb{R}^{r \times r}$ and $\mathbf{Q}_c \in \mathbb{R}^{r \times r}$ contain quadratic and cubic coefficients respectively.

In order to take into account any model inaccuracies, the state prediction will combine the considered dynamical model with *online* measurements by solving a moving horizon estimation (MHE) problem

$$\min_{\substack{\{\mathbf{w}_{i-N_h}, \dots, \mathbf{w}_{i-1}\} \\ \{\mathbf{v}_{i-N_h}, \dots, \mathbf{v}_i\} \\ \mathbf{x}_{i-N_h}}} \gamma \sum_{j=i-N_h}^{i-1} \mathbf{w}_j^\top \mathbf{w}_j + (1-\gamma) \sum_{j=i-N_h}^i \mathbf{v}_j^\top \mathbf{v}_j, \quad (4.14a)$$

$$\text{s.t. } \mathbf{x}_{i+1} = A(\mathbf{x}_i) + \mathbf{w}_i, \quad i = k - N_h, \dots, k - 1, \quad (4.14b)$$

$$\mathbf{y}_i = C(\mathbf{x}_i) + \mathbf{v}_i, \quad i = k - N_h, \dots, k, \quad (4.14c)$$

at every time instance as outlined in §2.2.2. The major difference here is the substitution of covariance matrices $\mathbf{R}$ and $\mathbf{Q}$ with the weighting factor $0 < \gamma < 1$, since it is assumed that all entries of $\mathbf{x}$ are of comparable importance and γ only balances the emphasis put between the accuracy of the dynamical model and confidence in the measurements $\mathbf{y}$. At each time instance, $\hat{\mathbf{x}}_i$ is retained, which corresponds to optimal decision variable $\hat{\mathbf{x}}_{i-N_h}$ integrated N_h timesteps forward in time based on the optimal decision variables $\mathbf{w}_{i-N_h}, \dots, \mathbf{w}_{i-1}$ of (4.14). If nonlinear models are used, in order to improve the computational efficiency (4.14) is modified such that $\mathbf{w}_i$ instead of being decision variables are set to $\mathbf{w}_i = 0$ implying that dynamical model $A(\mathbf{x}_i)$ is imposed as an equality constraint.

In order to perform a quantitative comparison between different models let us define two quantities: $\hat{\mathbf{x}}$ which contains values retained from the estimation problem (4.14); and $\tilde{\mathbf{x}}$ which is a solution to initial value problem satisfying $\dot{\mathbf{x}}_i = A(\mathbf{x}_i)$ for $A(\mathbf{x}_i)$ of the form (4.12)-(4.13) and initial conditions $\mathbf{x}_0$. We define state error metrics

$$(\epsilon_x)_i = \frac{\|\mathbf{x}_i - \hat{\mathbf{x}}_i\|_2^2}{\sum_i \|\mathbf{x}_i\|_2^2}, \quad (4.15)$$

$$\epsilon_x = \sum_i (\epsilon_x)_i, \quad (4.16)$$

with equivalent definitions for $\tilde{\mathbf{x}}$. Additionally, one of the objectives in the subsequent analysis will be to establish how data obtained from different dynamical regimes impacts state estimation

outside of that region. For this purpose, different *training sets* $\mathcal{T} \subseteq \mathcal{T}_{fd}$ will be defined such that dynamical model calculation will be performed only based on $\mathbf{x}_i$ for $i \in \mathcal{T}$. Unless specified otherwise dynamical estimation will be performed on the domain encompassing all three dynamical regimes as per figure 4.1(a).

In the linear case, models are obtained by solving a least-squares problem

$$\min_{\tilde{\mathbf{A}}} \sum_{i \in \mathcal{T}} \left\| \mathbf{x}_i - \tilde{\mathbf{A}} \mathbf{x}_{i-1} \right\|_2^2, \quad (4.17)$$

using only data set contained in $\mathcal{T}$. For the quadratic model of the form (4.13a), state space is augmented with quadratic terms and model coefficients are obtained from the solution to

$$\min_{\mathbf{A}, (\mathbf{Q}_q)_j} \sum_{i \in \mathcal{T}} \left\| \dot{\mathbf{x}}_i - \mathbf{A} \mathbf{x}_i - \sum_{j=1}^r (x_j)_i (\mathbf{Q}_q)_j \mathbf{x}_i \right\|_2^2 + \gamma \sum_{j=1}^r \|(\mathbf{Q}_q)_j\|_1, \quad (4.18a)$$

where $\gamma \geq 0$, $\mathbf{x}_i = [(x_1)_i, (x_2)_i, \dots, (x_r)_i]^\top$ and $\dot{\mathbf{x}}_i$ are state derivatives calculated using fourth order central finite difference. In order to improve the stability of the dynamical model, sparsity is induced in the nonlinear terms by adding an l^1 -norm penalty. In order to select an appropriate value of γ the following preliminary analysis will be performed for every choice of $\mathcal{T}$: Quadratic models are calculated for a range of γ and starting from known initial conditions $\mathbf{x}_0$, i.e. the first element of $\mathcal{T}$, a dynamical model is integrated forward in time using a 4th order Runge-Kutta method to obtain $\tilde{\mathbf{x}}$ and corresponding state errors $(\epsilon_x)_i$. The value of γ which minimises ϵ_x will be chosen for the subsequent analysis. It is possible to conduct a similar analysis by performing an MHE analysis on $\mathcal{T}$, however, the initial value problem is chosen over MHE due to an associated lower computational cost.

In order to obtain a cubic model a continuous time formulation (4.13b) is used where, as previously, derivatives are approximated using a fourth order central finite difference. The model is obtained by solving an augmented state space optimisation problem

$$\min_{\mathbf{A}, (\mathbf{Q}_c)_{jk}} \sum_i \left\| \dot{\mathbf{x}}_i - \mathbf{A} \mathbf{x}_i - \sum_{j,k} (x_j)_i (x_k)_i (\mathbf{Q}_c)_{jk} \mathbf{x}_i \right\|_2^2 + \gamma \sum_j \|(\mathbf{Q}_c)_{jk}\|_1. \quad (4.19)$$

with $\gamma \geq 0$ and $\mathbf{x}_i = [(x_1)_i, (x_2)_i, \dots, (x_r)_i]^\top$. Furthermore, using weakly nonlinear analysis and the form of the Stuart-Landau equation a certain structure can be imposed on the form of

the cubic model. The long-term dynamics of the global modes is governed by (4.3). Continuing the analysis to the subsequent orders until the expansion contains terms corresponding to all of the structures presented in figure 4.3 gives

$$\mathbf{u}(t) = \Phi_S + A(\tau) \exp(j\omega_0 t) \Phi_{GM} + \epsilon |A(\tau)|^2 \Phi_{SM} + A(\tau)^2 \exp(j2\omega_0 t) \Phi_{HH} + \dots \quad (4.20)$$

where by Φ_S , Φ_{GM} , Φ_{SM} and Φ_{HH} we respectively denote steady-state flow, global mode, shift mode and first subharmonic of the global mode. In order to associate an appropriate term for the shift mode, its known dynamical properties are considered. The two leading non-oscillatory terms are identified as $\epsilon \mathbf{U}^1$ and $\epsilon |A|^2 \mathbf{U}^{|A|^2}$, where $\mathbf{U}^{f(A)}$ denote structures associated with dynamics $f(A)$ in the weakly nonlinear decomposition [23]. The first one is time-invariant and corresponds to the change of the steady-state flow when Re differs from Re_c . Consequently, $\Phi_S = \Phi_S^* + \epsilon \mathbf{U}^1$, where Φ_S^* is a steady-state solution for Re_c . On the other hand, Φ_{SM} is associated with time varying modification to the mean flow through $|A|^2$. The fact, that Φ_{SM} is associated with $|A|^2$ suggests that a fourth order model should accurately describe the dynamics of the most dominant structures. In order to associate the subharmonic of the global mode with long-term dynamics, the information that this flow evolves at twice the frequency of the global mode was taken into account demonstrating that its dynamics is governed by $A(\tau)^2$.

In order to incorporate this information into dynamical modelling, equality constraints are imposed on (4.19) in order to provide a stronger link to the Stuart-Landau equation. Let us assume that the first two projection coefficients correspond to the global mode, in which case $A = x_1 + jx_2$. Denoting $\lambda = \lambda_R + j\lambda_I$, $\mu = \mu_R + j\mu_I$ the expression from Stuart-Landau equation (4.1) for the derivative $\dot{A}$ in matrix form can be written as

$$\dot{A} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda_R & -\lambda_I \\ \lambda_I & \lambda_R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \mu_R & \mu_R & -\mu_I & -\mu_I \\ \mu_I & \mu_I & \mu_R & \mu_R \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_1 x_2^2 \\ x_2 x_1^2 \\ x_2^3 \end{bmatrix} \quad (4.21)$$

Of course, exact values of λ and μ are not known. However, the Stuart-Landau form will be imposed by applying the equality constraints implied by (4.21) to (4.19) in the entries of $\mathbf{A}$ and $(\mathbf{Q}_c)_{jk}$ corresponding to global modes. It will be demonstrated in latter sections that the

cubic models structured in such a way are stable and consequently it is possible to set $\gamma = 0$ in (4.21).

4.4 Initial value problem

Before conducting a performance analysis of the MHE algorithms, the accuracy of the models of different complexities is assessed. The quadratic and cubic models will be obtained for $i \in \mathcal{T}_{fd}$ and an initial value problem is solved on the same temporal domain, setting $\mathbf{x}_0$ to the first element of $\mathcal{T}_{fd}$. An analogous analysis of the linear model is omitted here since, depending on the eigenvalues, the linear dynamics will be characterised either by exponential growth or decay. It is noted that the open loop accuracy is not necessary to obtain the dynamic estimation with a small error. Nonetheless, for cubic and quadratic models, open loop accuracy is instructive with respect to the significant challenge of nonlinear model fitting.

For clarity of presentation, only the magnitudes corresponding to different dynamical features will be shown, i.e. in the case of global modes and higher harmonics $A_{gm} := \sqrt{(x_1)^2 + (x_2)^2}$ and $A_{hh} := \sqrt{(x_4)^2 + (x_5)^2}$ are presented where x_1, x_2 are projections onto global mode structures and analogously x_4, x_5 correspond to subharmonic states. For the shift mode, whose dynamics are non-oscillatory, $A_{sm} := x_3$. Since solutions to initial value problems are likely to induce a phase mismatch with the reference results, evaluating performance of different models using state error ϵ_x is not appropriate. For example, if the model accurately predicts the long-term dynamics but the solution to the initial value problem is out of phase, the corresponding error ϵ_x will be large. Consequently, here, a comparison will be performed based on the amplitude error defined as

$$\epsilon_A = \frac{\sum_i \|(\epsilon_{gm})_i\|_2^2 + \sum_i \|(\epsilon_{sm})_i\|_2^2 + \sum_i \|(\epsilon_{hh})_i\|_2^2}{\sum_i (A_{gm}^2)_i + \sum_i (A_{sm}^2)_i + \sum_i (A_{hh}^2)_i}, \quad (4.22)$$

where $(\epsilon_{gm})_i = (\tilde{A}_{gm})_i - (A_{gm})_i$, $(\epsilon_{sm})_i = (\tilde{A}_{sm})_i - (A_{sm})_i$ and $(\epsilon_{hh})_i = (\tilde{A}_{hh})_i - (A_{hh})_i$.

The temporal evolution of the key dynamical features corresponding to $\tilde{\mathbf{x}}$ obtained using the quadratic model on the OMD projection space with $\gamma = 0.1$ in (4.18) are presented in figures 4.5(a),(c),(e). The solution to the initial value problem $\tilde{\mathbf{x}}$ closely follows the reference trajectory of the oscillatory structures if the quadratic model is used. Nonetheless, there are certain inaccuracies visible, primarily in the shift mode and in the fact that the system does not

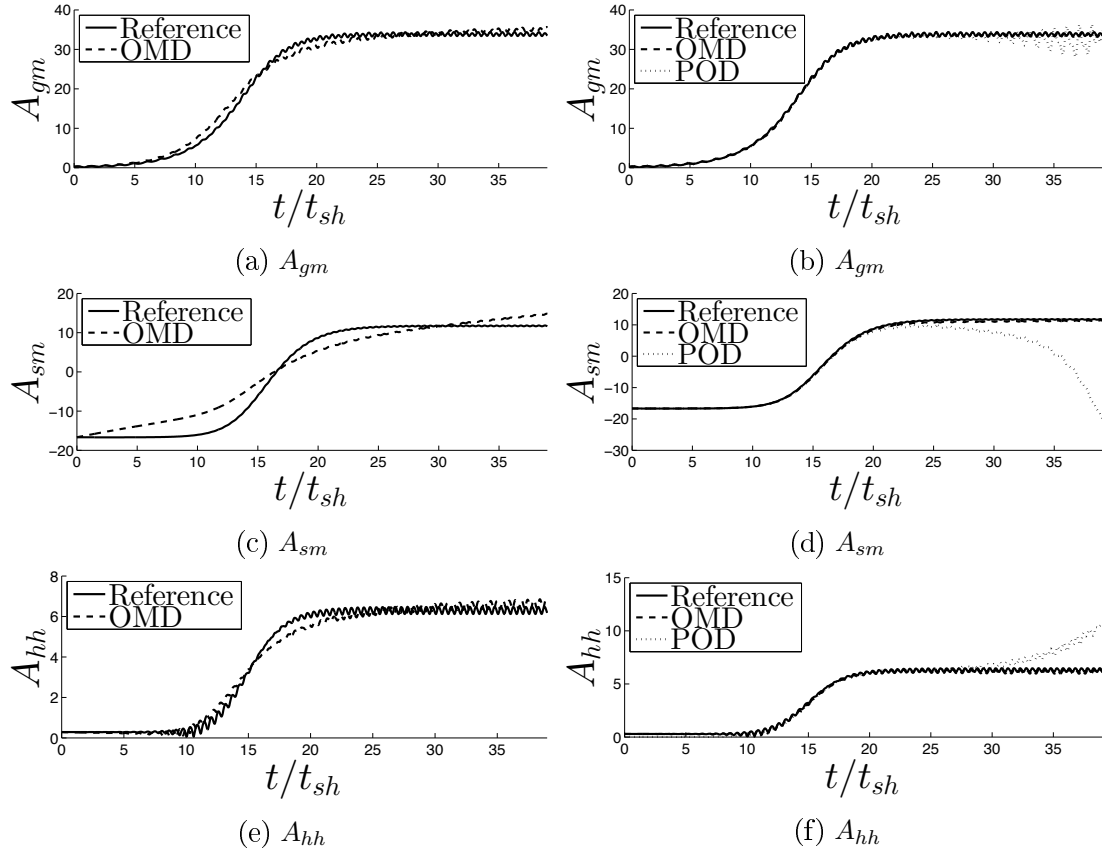


Figure 4.5: Temporal evolution of the amplitude of the main features of the solution to the initial value problem $\hat{\mathbf{x}}$ corresponding to the quadratic model (a),(c),(e) and to the cubic model (b),(d),(f).

appear to converge to a limit cycle with the correct amplitude. Additionally, the linear growth rate close to equilibrium is overestimated. For the quadratic model, results corresponding to OMD and POD modes are visually very similar and presentation of the latter is omitted. The amplitude errors ϵ_A for different projection spaces and dynamical models are presented in table 4.1.

Subsequently, an analogous analysis is performed using models of the third order (4.19) obtained for $i \in \mathcal{T}_{fd}$. The accuracy is assessed by solving the corresponding initial value problems. The value of γ in (4.19) is set to $\gamma = 0$, indicating that any sparsity-promoting terms are neglected. The solution to initial value problem for the cubic models are shown in figure 4.5(b),(d),(f). For the OMD basis, it can be seen that the amplitudes of the dynamical features are reproduced very accurately, where the solution to initial value problem is visually indistinguishable from the reference states. In the case of the POD modes, the system reproduces the initial transitional trajectory well, however, at $t \approx 150$ the solution starts to diverge away from the attractor. For the global modes this behaviour is characterised by increasing oscillation of

	Quadratic	Cubic
POD	0.0715	0.7316
OMD	0.0753	0.0206

Table 4.1: The amplitude errors ϵ_A corresponding to solutions to initial value problems for quadratic and cubic models.

A_{gm} , whilst for the other states exponential divergence is observed. It is noted that if the equality constraints corresponding to Stuart-Landau equation are not imposed in (4.19) both OMD and POD bases output unstable models and a tedious selection of γ is required to obtain an accurate solution to initial value problem. The values of ϵ_A corresponding to results presented in figure 4.5(b)-(f) are presented in table 4.1.

4.4.1 Fourth order modelling

A note is required about the robustness of the dynamical model corresponding to the OMD modes presented in figure 4.5(b),(d),(f), where it can be seen that the shift mode is modelled accurately despite the fact that it was established in (4.20) that its dynamics are of fourth-order. However, if the subharmonic states are not included in the modelling, the accuracy of the initial value problem deteriorates. An example is visible in figure 4.6, where exactly the same methodology is followed but $\mathbf{x} \in \mathbb{R}^3$ consists of two global mode and the shift mode states. In particular, the obtained cubic model does not represent the shift mode dynamics accurately. Not only does transition to limit cycle occur earlier in time but also the final amplitude of the limit cycle is underestimated. Such inaccuracies have an adverse effect on the accuracy of the global mode where, for example, when compared with figure 4.5(b), the linear growth rate is underestimated and a small steady state error is induced on the limit cycle. These results indicate that the accuracy of the shift mode in figure 4.5(d) is partially driven through coupling with the higher harmonic of the global mode.

If the requirement is to obtain an accurate model for only the first three most dynamically important states a fourth-order model can be used instead

$$\begin{aligned}
\min_{\mathbf{A}, (\mathbf{Q}_c)_{jk}, (\mathbf{Q}_{qt})_{jkl}} \sum_i \left\| \dot{\mathbf{x}}_i - \mathbf{A}\mathbf{x}_i - \sum_{j,k} (x_j)_i (x_k)_i (\mathbf{Q}_c)_{jk} \mathbf{x}_i - \sum_{j,k,l} (x_j)_i (x_k)_i (x_l)_i (\mathbf{Q}_{qt})_{jkl} \mathbf{x}_i \right\|_2^2 \\
+ \gamma \left(\sum_{j,k,l} \|(\mathbf{Q}_{qt})_{jkl}\|_1 + \sum_{j,k} \|(\mathbf{Q}_c)_{jk}\|_1 \right) \quad (4.23)
\end{aligned}$$

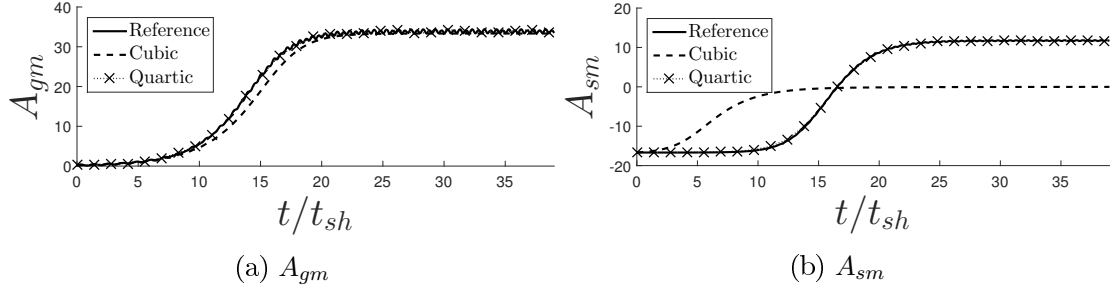


Figure 4.6: Temporal evolution of the dynamical features of the three-dimensional system obtained by solving an initial value problem using cubic and quartic models obtained based on data set $\mathcal{T}_{fd}$.

where $\mathbf{Q}_{qt} \in \mathbb{R}^{3 \times 3}$ are matrices containing coefficients corresponding to quartic terms and the remaining terms retain previous interpretation. Here, sparsity of the model was induced by setting $\gamma = 0.5$.

The solution to the initial value problem obtained using a quartic model and known initial condition $\mathbf{x}_0$ is presented in figure 4.6. It can be seen that the amplitude of the dynamical features are reproduced very accurately, where the solution to initial value problem is visually indistinguishable from the reference values. The results in figure 4.6 give $\epsilon_A = 0.0722$ for the cubic model and $\epsilon_A = 0.0093$ for the quartic. It is noted that in the case of OMD a small error is obtained despite neglecting quadratic terms which appear in the dynamics of $|A|^2$.

Although figure 4.6 demonstrates a high accuracy of the quartic model, the dimension of such a model increases significantly with the dimension of $\mathbf{x}$. The number of repeated quartic variables is r^4 , which means that the dimension of the augmented state space in (4.23) for $x \in \mathbb{R}^3$ is 111. Such a complex model, although accurate, may only be viable for practical estimation when the dimension of $\mathbf{x}$ is low.

4.5 Estimator design

Noting that the accurate value of the initial condition $\mathbf{x}_0$ is in most practical cases not available the dynamical models obtained in §4.4 will be combined with observation $\mathbf{y}$ about the state of the system via MHE algorithm. Initially, analysis is performed on the temporal domain $\mathcal{T}_{fd}$ encompassing all three cylinder regimes, whilst a robustness analysis of the dynamical models with respect to the choice of the training domain will be given in §4.6.

In order to choose appropriate values of $\mathbf{y}$ a qualitative comparison between the dynamics of the states and pressure measurements is required. The need for such an analysis stems not

only from the different nature of the states and measurements, where the first is intrinsically linked to velocity. The measurement vector $\mathbf{y}$ will be constructed based on point measurements whilst $\mathbf{x}$ will represent global dynamics of the system where the projection onto corresponding modes includes values at all of the spatial locations. Consequently, the dynamics close to the cylinder base are not necessarily the same as the global behaviour.

First, let us assume that the dynamics of the global mode is well approximated with a Stuart-Landau equation (4.1) and the values of the complex coefficients λ , μ are found by fitting the Stuart-Landau model to the underlying data. It is assumed that $A = x_1 + jx_2$, i.e. the projection onto the OMD global mode, which, in order to facilitate an appropriate comparison with pressure dynamics, is rescaled as $A/\|A\|_\infty$ such that on the limit cycle $|A| = 1$. The derivatives $dA/d\tau$ are approximated using a second-order finite difference and λ , μ are found by solving a least squares problem

$$\min_{\lambda^*, \mu^*} \sum_{i \in \mathcal{T}_{fd}} \left\| \left(\frac{dA}{dt} \right)_i - \lambda^* A_i - \mu^* A_i |A_i|^2 \right\|_2^2. \quad (4.24)$$

where $\lambda^* = \epsilon\lambda$ and $\mu^* = \epsilon\mu$, to give $\lambda^* = 0.0108 + j0.1858$ and $\mu^* = -0.0109 + j0.0283$ signifying a linear growth rate $\zeta = \text{Re}(\lambda^*)$ of approximately $\zeta = 0.0108$. It is noted that the third order fit corresponding to (4.24) is accurate to 2.5%.

Subsequently, since pressure values follow a similar transition to the limit cycle, pressure measurements at a polar position $\theta = 0$ are analysed to establish corresponding values of the Stuart-Landau coefficients. Since pressure values are real, in order to solve a similar problem to (4.24), a complex analytic signal p_a is constructed using a Hilbert transform [89] and is subsequently rescaled as $p_a/\|p_a\|_\infty$. The solution of (4.24) for such a modified signal gives $\lambda^* = 0.0057 + j0.1973$ and $\mu^* = -0.0062 + j0.0195$ indicating that the growth rate corresponding to pressure is half of the one characterising the global mode dynamics. Similar observations can be made for pressure values at other locations close to the cylinder boundary. Furthermore, it is observed that there is a slight difference in the imaginary parts of λ^* obtained using pressure and state values, indicating a small phase difference between point measurements close to the cylinder and global projection of the snapshots. The amount of phase offset varies depending on the location of the analysed pressure measurement. Consequently, due to such differences, a simple linear mapping between a single pressure measurement and global mode state will

introduce an error in the estimation.

In addition to the growth rate analysis, a spectral study of the pressure values is performed to assess their suitability to approximate frequency characteristics of the oscillatory structures. It is observed that all pressure values close to the cylinder boundary are primarily characterised by a frequency in the neighbourhood of the shedding frequency, making those values suitable for approximation of the spectral characteristics of the global mode. On the other hand, the subharmonic oscillates at double the frequency of the global mode and, in order to facilitate estimation of the corresponding state, $\mathbf{y}$ can be set to contain a certain quadratic function of pressure signals. This choice stems from the fact that assuming pressure values follow dynamics of the form $A_p(\tau) \exp(j\omega_0 t)$, where $A_p(\tau)$ satisfies Stuart-Landau equation (not necessarily with the same coefficients λ^*, μ^* as global modes), squaring of the measurements will lead to $A_p^2(\tau) \exp(j2\omega_0 t)$ which has the same frequency content as the higher harmonic.

Following the analysis above, the measurement vector $\mathbf{y}$ will be constructed based on pressure values $\mathbf{p} \in \mathbb{R}^{N_p}$ equispaced on the cylinder boundary $(X_j, Y_j) = (0.5 \cos(\theta_j), 0.5 \sin(\theta_j))$, where $\theta = -10\pi/20, -9\pi/20, \dots, 10\pi/20$. A discussion of the robustness of the estimation accuracy for smaller number of pressure measurements will be given in §4.7.1. The difference in the values of the growth rates discussed above indicate that more refined methodologies for setting $\mathbf{y}$ are required to accurately estimate long-time dynamics of the global modes. The change in the growth rate values can be achieved, for example, through an appropriate summation of multiple pressure signals. Following the practical considerations, it is possible to acquire multiple measurements at the base of the object, sometimes with a number of measurements larger than the dimension of the estimated state. In such cases, a use of more sophisticated approaches, such as Linear Stochastic Estimation (LSE) outlined in §2.2.1, is justified to obtain an accurate mapping between measurements and state, even at the cost of decreasing the dimension of $\mathbf{y}$, as opposed to naive $\mathbf{y} = \mathbf{p}$. Consequently, LSE is used to obtain an approximation of the global and shift mode states by solving

$$\min_{\Psi_{LSE}} \sum_{i \in \mathcal{T}_{fd}} \|(\mathbf{x}_{1-3})_i - \Psi_{LSE}^\top \mathbf{p}_i\|_2^2, \quad (4.25)$$

where $\mathbf{x}_{1-3} = [x_1, x_2, x_3]^\top$ and $\Psi_{LSE} \in \mathbb{R}^{N_p \times 3}$ are vectors containing LSE coefficients. Additionally, in order to obtain observables corresponding to subharmonics the following QSE problem

is solved

$$\min_{\Psi_{QSE}} \sum_{i \in \mathcal{T}_{fd}} \|(\mathbf{x}_{4-5})_i - \Psi_{QSE}^\top \mathbf{p}_i^*\|_2^2, \quad (4.26)$$

where $\mathbf{x}_{4-5} = [x_4, x_5]^\top$, $\mathbf{p}^* = [p_1^2, p_2^2, \dots, p_{N_p}^2]^\top$ and $\Psi_{QSE} \in \mathbb{R}^{N_p \times 2}$ contains QSE coefficients. The quadratic form of $\mathbf{p}^*$ is used here to facilitate construction of the signal with double the frequency of the pressure measurements. Note that quadratic coupling between pressure measurements at different spatial locations was neglected in (4.26). However, it will be shown that the choice of $\mathbf{p}^*$ is sufficient to obtain accurate estimates. Consequently, in the subsequent MHE analysis $\mathbf{y}$ is set to

$$\mathbf{y}_i = \begin{bmatrix} \Psi_{LSE}^\top \mathbf{p}_i \\ \Psi_{QSE}^\top \mathbf{p}_i^* \end{bmatrix} \in \mathbb{R}^r \quad (4.27)$$

The MHE problem (4.14) is solved for $N_h = 10$ and $\gamma = 0.5$ where $\mathbf{y}_i = \Psi^\top \mathbf{p}_i$, $\mathbf{C} = \mathbf{I}$ and dynamical models presented in §4.4 will be used. The state errors ϵ_x corresponding to dynamical models constructed using both OMD and POD projection bases are given in table 4.2. In order to perform a comparison with stochastic estimation (SE), table 4.2 includes a corresponding error defined as

$$\epsilon_x = \frac{\sum_{i \in \mathcal{T}_{fd}} \|\mathbf{y}_i - \mathbf{x}_i\|_2^2}{\sum_{i \in \mathcal{T}_{fd}} \|\mathbf{x}_i\|_2^2}, \quad (4.28)$$

where $\mathbf{y}$ is given by (4.27).

Noting that values in table 4.2 are all small and of comparable magnitude, visual presentation analogous to figure 4.5 is omitted. Table 4.2 demonstrates that stochastic modelling of the states outlined above provides an accurate approximation of the system. The primary benefit of the dynamical models in the MHE algorithm is suppression of high-frequency noise present in $\mathbf{y}$ which is an artefact of stochastic estimation. It is notable that the largest improvement was achieved using a quadratic model despite the fact that it was demonstrated in §4.4 that the cubic model provides the most accurate model of the dynamics. In order to obtain an insight into the source of this discrepancy an analysis of the impact of the MHE parameters on the state error is performed now.

In the analysis presented so far, values of γ and N_h were chosen arbitrarily. For linear estimation it was observed that the value of γ does not significantly influence the estimation accuracy. For example, in the linear estimation corresponding to results in table 4.2, a variation

	SE	Linear	Quadratic	Cubic
POD	0.0193	0.0182	0.0128	0.0145
OMD	0.0188	0.0177	0.0125	0.0144

Table 4.2: The state errors ϵ_x corresponding to stochastic estimates (SE) and MHE estimates when dynamical modelling is performed using data in $\mathcal{T}_{fd}$.

between $\gamma = 0.01$ and $\gamma = 0.99$ induces a POD error of $\epsilon_x = 0.0182 \pm 2.6\%$. On the other hand, the impact of N_h consists primarily of suppressing high-frequency noise when N_h is increased. Nonetheless, an interesting impact of changing a value of N_h can be observed when the MHE problem is modified such that

$$\min_{\substack{\{\mathbf{v}_{i-N_h}, \dots, \mathbf{v}_i\} \\ \mathbf{x}_{i-N_h}}} \sum_{j=i-N_h}^i \mathbf{v}_j^\top \mathbf{v}_j, \quad (4.29a)$$

$$\text{s.t. } \mathbf{x}_{i+1} = A(\mathbf{x}_i), \quad i = k - N_h, \dots, k - 1, \quad (4.29b)$$

$$\mathbf{y}_i = C(\mathbf{x}_i) + \mathbf{v}_i, \quad i = k - N_h, \dots, k. \quad (4.29c)$$

i.e. the dynamical model $A(\mathbf{x})$ is imposed as an equality constraint, analogous to the situation used to obtain estimates where $A(\mathbf{x})$ is nonlinear. Figure 4.7 presents the state error ϵ_x when (4.29) is solved for $N_h < 30$ using dynamical models of different complexities. It can be observed that: for a linear model, ϵ_x progressively increases with N_h ; for a quadratic model it is minimised around $N_h = 7$; whilst for a cubic model it monotonically decreases for the analysed values. This observation can be linked to the accuracy of the models established in §4.7.1. In (4.29) the state trajectory is forced to follow the dynamics of the model and since the cubic model was most accurate it is beneficial in such cases to set a large value of N_h . On the other hand, the linear model does not provide an accurate approximation of the transition region and consequently it is beneficial to ensure that in this case estimation is primarily driven by the measurement $\mathbf{y}$. The quadratic model, as visible in figure 4.5, provides a compromise between two other models and the value of N_h can be tuned appropriately to minimise ϵ_x . It is expected that for $N_h > 30$ a similar minimum can be observed for the cubic model. Additionally, it is noted that the value of ϵ_x obtained using the cubic model for $N_h = 30$ is slightly lower than the smallest value corresponding to the quadratic model at $N_h = 7$.

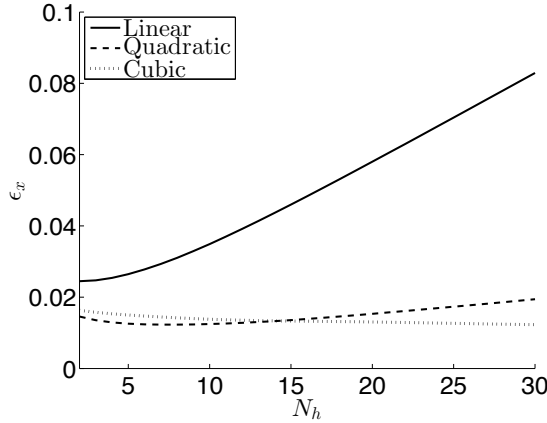


Figure 4.7: The state errors ϵ_x corresponding to the MHE estimates for a range of N_h and different dynamical models.

4.6 Robustness of the estimation strategy

Following the design of the estimator which gives accurate MHE estimates when the training domain encompasses all three dynamical regimes an investigation is performed to assess whether the accuracy of the estimate can be reproduced if the dynamical models are obtained using only a subset of data in $\mathcal{T}_{fd}$. For this reason the analysis from §4.5 will be reproduced by constructing dynamical models and observables $\mathbf{y}$ close to equilibrium, $\mathcal{T}_{eq} = \{i \in \mathcal{T}_{fd} : t_i/t_{sh} < 17\}$ and close to the limit cycle, $\mathcal{T}_{lc} = \{i \in \mathcal{T}_{fd} : t_i/t_{sh} > 17\}$.

4.6.1 Equilibrium

All three models of increasing complexity (4.17)-(4.19) are first solved using the data set $\mathcal{T}_{eq}$. For nonlinear models, the variable γ controlling the sparsity level must be selected. For this purpose, initial value problems are solved on $\mathcal{T}_{eq}$. For the cubic model it is observed that the error ϵ_x increases rapidly once the l_1 -norm terms are added to the optimisation. Nonetheless, the structure imposed on the nonlinear terms through the equality constraints is sufficient to induce stability and accuracy of the model, therefore, in this case $\gamma = 0$. On the other hand, all of the quadratic models obtained for $0 \leq \gamma \leq$ are unstable. In such cases the time indices at which $\tilde{\mathbf{x}} \rightarrow \infty$ are recorded and the model which gives an unstable state trajectory at the latest time is chosen, giving $\gamma = 0.8$. It is noted that the stability of the estimation will be induced by the MHE algorithm through $\|\mathbf{y}_i - \mathbf{C}\hat{\mathbf{x}}_i\|$ in (4.14) which penalises a deviation from the measurement $\mathbf{y}$.

State error statistics ϵ_x for the MHE estimates based on the models trained close to equi-

	SE	Linear	Quadratic	Cubic
POD	0.4191	0.4054	0.3744	0.3684
OMD	0.3530	0.3382	0.3386	0.2977

Table 4.3: The state errors ϵ_x corresponding to stochastic estimates (SE) and MHE estimates when dynamical modelling is performed using data in $\mathcal{T}_{eq}$.

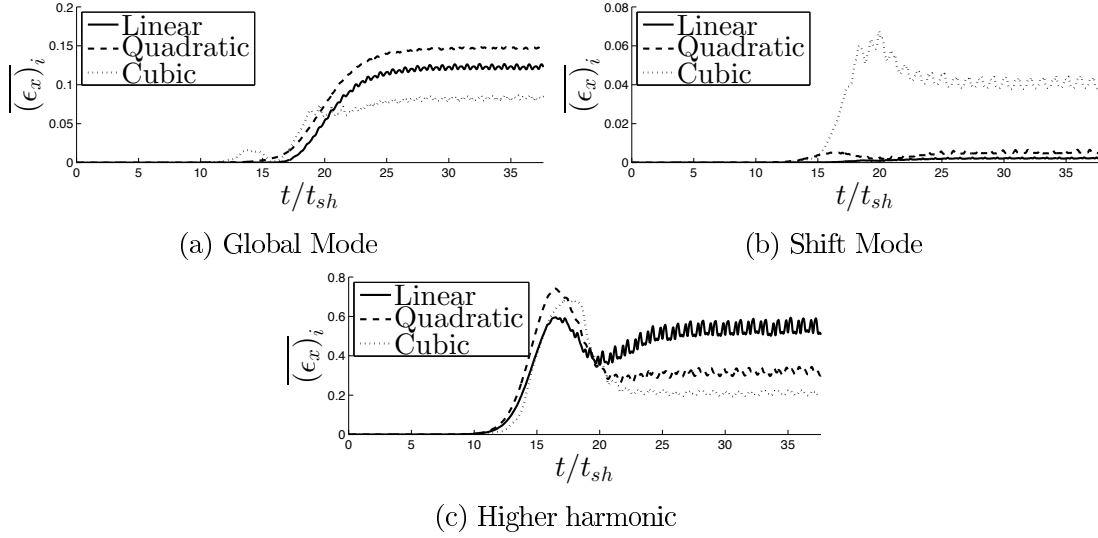


Figure 4.8: Temporal evolution of the amplitude of the main dynamical features corresponding to the MHE estimates when dynamical modelling is performed using data in $\mathcal{T}_{eq}$.

librium are given in table 4.3. Additionally, temporal evolution of the averaged state errors $\overline{(\epsilon_x)_i}$ corresponding to each dynamical feature are shown in figure 4.8. In order to aid clarity of presentation, state errors $(\epsilon_x)_i$ are averaged over the shedding times scale t_{sh} such that

$$\overline{(\epsilon_x)_i} = \frac{1}{t_{sh}} \sum_{i-\frac{t_{sh}}{2}}^{i+\frac{t_{sh}}{2}} (\epsilon_x)_i.$$

Table 4.3 shows that the error progressively decreases when the complexity of the system is increased and; in all cases, an OMD basis provides an improvement over the one constructed using POD modes. Insight into how the error evolves depending on the particular estimated mode can be found based on temporal evolution of errors presented in figure 4.8. For the shift mode, the most accurate estimate is obtained when the quadratic model is used. This is not surprising since as it is demonstrated in (4.20) the long-term dynamics of this structure is governed by $|A|^2$, which consists of quadratic and quartic term. The performance of the quadratic model deteriorates in the estimation of the oscillatory structures. For both the global mode and its subharmonic, the smallest error in the limit cycle regime, i.e. outside of

	SE	Linear	Quadratic	Cubic
POD	0.0754	0.0789	0.0755	0.0686
OMD	0.0718	0.0674	0.0649	0.0789

Table 4.4: The state errors ϵ_x corresponding to stochastic estimates (SE) and MHE estimates when dynamical modelling is performed using data in $\mathcal{T}_{lc}$.

the training domain, are given when the cubic model is used. It is noteworthy that for the global mode this accuracy is achieved despite the fact that cubic model provides the largest error in the transitional section of the domain. Furthermore, we note that none of the dynamical models is able to accurately model transition of the subharmonic dynamics when the estimators are trained close to equilibrium, with errors outside of the training domain $\overline{(\epsilon_x)_i} > 0.2$.

4.6.2 Limit Cycle

Similarly, an analysis is performed to assess the robustness of the estimation technique close to the limit cycle. The optimisation problems (4.17)-(4.19) corresponding to models of different order are solved using data set in $\mathcal{T}_{lc}$. For the quadratic model $\gamma = 0.08$ was established to give the most accurate and stable model based on the solutions to initial value problems. The smaller value of γ relative to the one obtained in the equilibrium analysis is due to the fact that the dynamics close to the limit cycle is characterised by asymptotic convergence on to the attractor. Consequently, dynamic models trained on such a domain are more likely to be stable than around equilibrium, where the exponential growth of perturbations is prominent. The state errors ϵ_x corresponding to MHE estimates obtained using different dynamical models on the POD and OMD basis are given in table 4.4. It can be seen that all errors are significantly smaller than equivalent values corresponding to training on $\mathcal{T}_{eq}$. Errors corresponding to the designed stochastic estimation given by (4.27) show that construction of the measurements close to limit cycle gives a robust measurement $\mathbf{y}$ and the dynamical models given here do not provide significant improvement over such estimates. Nonetheless, state errors corresponding to the analysed dynamical features are presented in figure 4.9. It can be seen that the main sources of errors arise from steady states errors close to the equilibrium, predominantly in the global mode subharmonic.

Figures 4.9(a)-(b) appear to indicate an error in the transitional part of the global mode. However, it is noted that in that regime $\epsilon_x < 0.01$ which means that the transition is estimated accurately despite the corresponding data not being included in $\mathcal{T}_{lc}$. This estimation characte-

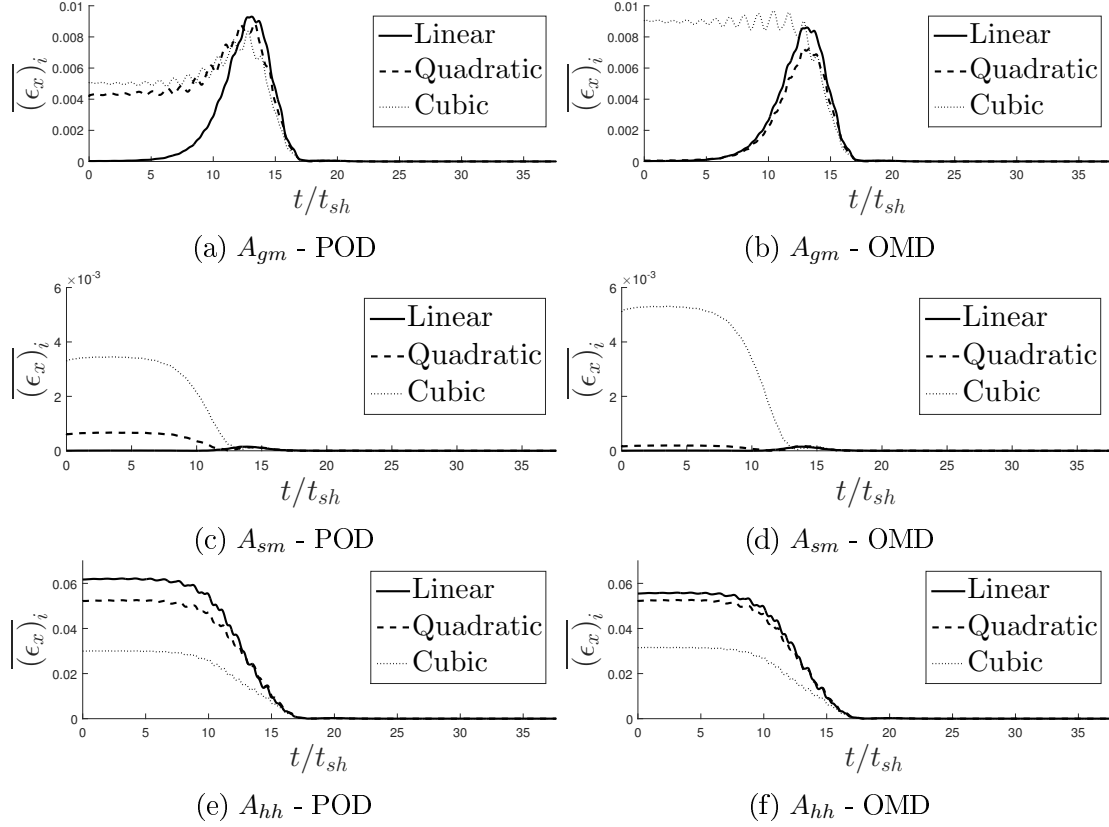


Figure 4.9: Temporal evolution of the amplitude of the main dynamical features corresponding to the MHE estimates when dynamical modelling is performed using data in $\mathcal{T}_{lc}$.

istic is not necessarily intuitive and warrants a further explanation. A potential explanation can be provided through a link to the Koopman analysis on the limit cycle performed in [50].

The Koopman operator is a linear operator acting on a space of observables $g(\mathbf{q})$, where $\mathbf{q}$ is a dynamical state of the system. Note that $\mathbf{q}$ does not necessarily have the same interpretation as $\mathbf{x}$ used here for dynamic estimation. In the subsequent analysis let us set $\mathbf{q} = A$ and functions of A contained in the weakly-nonlinear expansion (4.20) or Stuart Landau equation (4.1), e.g. A , $|A|^2$, $A|A|^2$, are chosen to be observables $g(A) : \mathbb{C} \rightarrow \mathbb{C}$. Following the Koopman formulation (2.21), it is assumed that $g(A) \in \text{span}(\varphi_j)$ where φ_j are the eigenfunctions of the Koopman operator and $g(A)$ can be expanded as

$$g(A) = \sum_{j=1}^{\infty} \alpha_j^g \varphi_j \exp(\lambda_j \tau) \quad (4.30)$$

where α_j^g are Koopman modes corresponding to observables $g(A)$ and λ_j are the eigenvalues of the Koopman operator. The objective here is to find a similar analytic expression for $g(A)$ but deduced from Stuart-Landau formulation.

Noting that A is a periodic function of θ , $g(A)$ has the same property and can be consequently expressed as a Fourier series

$$g(A) = \sum_{m=-\infty}^{\infty} \hat{g}(r) \exp(jm\theta). \quad (4.31)$$

The expression for θ is known to be (4.6b) and when substituted into (4.31) gives

$$g(A) = \sum_{m=-\infty}^{\infty} \hat{g}(r) \left(\frac{r}{r_0}\right)^{jm\beta} \exp(jm(\theta_0 + \omega\tau)), \quad (4.32)$$

where $\omega = (\lambda_I - \lambda_R \frac{\mu_I}{\mu_R})$. Furthermore, the expression for $r(\tau)$ is given as (4.6a) and can be rearranged to the form

$$r(\tau) = \sqrt{\frac{\lambda_R}{\mu_R + y(\tau)}} \quad (4.33)$$

where $y(\tau) = \left(\frac{\lambda_R}{r_0^2} - \mu_R\right) \exp(-2\lambda_R\tau)$. Note that $y = 0$ maps $r(\tau)$ onto $\sqrt{\frac{\lambda_R}{\mu_R}}$ which corresponds to the limit cycle amplitude. Consequently, substituting (4.33) into (4.32) and performing a Taylor expansion around $y = 0$ gives an expression of the form

$$g(A) = \sum_k f_k(y)_{y=0} y^k = \sum_k f_k^*(y)_{y=0} \left(\frac{\lambda_R}{r_0^2} - \mu_R\right)^k \exp(-2\lambda_R k\tau + jm(\theta_0 + \omega\tau)). \quad (4.34)$$

Reiterating that (4.34) is obtained through a Taylor expansion around the limit cycle, it can be seen that coefficient λ_R governing a linear growth rate close to equilibrium has a strong influence on the manner in which the dynamics relax onto the limit cycle. In fact, the first order terms ($k = 1$) show that the system approaches the oscillatory attractor exponentially at a rate $-2\lambda_R$. This information provides us with a reasonable direction to pursue when trying to establish why modelling the dynamics close to the limit cycle provides better accuracy. Two predominant features of the estimates which were analysed here were the reproduction of the linear growth rate and accuracy of the limit cycle amplitude. When dynamic modelling is performed close to equilibrium, the first characteristic is well estimated. However, the latter one always contained substantial error. Noting that the limit cycle is a function of λ_R and μ_R , close to the equilibrium the term multiplying μ_R , i.e. $A|A|^2$ is of relatively low energy making it possible that the approximation of μ_R might contain significant error. On the other hand, the information about μ_R and λ_R is strongly embedded in dynamics close to the limit cycle,

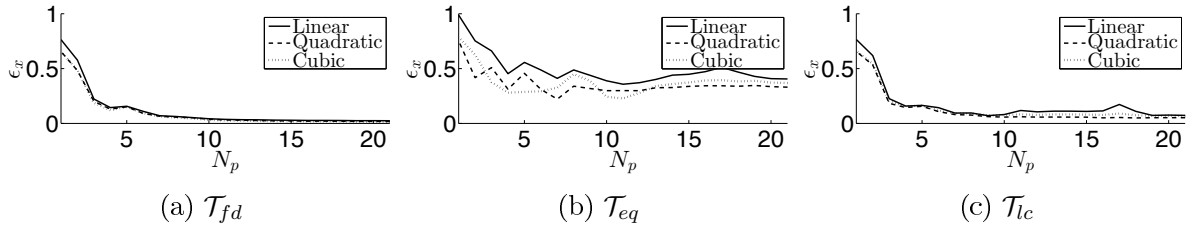


Figure 4.10: The state errors ϵ_x corresponding to MHE using POD modes for different number of pressure measurements N_p for the training domains of $\mathcal{T}_{fd}$, $\mathcal{T}_{eq}$, $\mathcal{T}_{lc}$.

with the first constant governing the limit cycle amplitude (4.34) and the latter convergence onto it. Consequently, the strong embedding is likely to carry into dynamical model obtained based on the data in this regime. We note that this reasoning appears to be true not only for the cubic models but also for the linear and quadratic ones as presented in figure 4.9.

4.7 Parameter Sensitivity

4.7.1 Number of pressure measurements

First, an assessment on how the accuracy of the estimation changes with the number of measurements is performed. For this reason, the analyses on $\mathcal{T}_{fd}$, $\mathcal{T}_{eq}$, $\mathcal{T}_{lc}$ outlined in §4.5-§4.6 are repeated by varying number of pressure measurements $\mathbf{p} \in \mathbb{R}^{N_p}$ between $1 \leq N_p \leq 21$. The locations of pressure measurements are chosen such that their azimuthal positions are equispaced between $-\pi/2 \leq \theta \leq \pi/2$. The observable vector $\mathbf{y}$ is constructed as previously using a combination of LSE and QSE as in (4.25)-(4.27) and the MHE set-up follows the one used in §4.5-§4.6. State errors ϵ_x are calculated for all three different order models and values of ϵ_x corresponding to different numbers of measurements are presented in figure 4.10. Figure 4.10 shows the results for the POD modes and a presentation of the results corresponding to the OMD modes is omitted due to the lack of qualitative differences.

It is seen that estimation based on $\mathcal{T}_{fd}$ and $\mathcal{T}_{lc}$ has the same pattern with respect to N_p . In both cases there is a decrease in error as $N_p \geq 5$. The reasons behind this observation were already discussed in §4.5 and are related to the fact that the growth rates of the pressure values close to the equilibrium are approximately half of the ones corresponding to velocity-based states. Additionally, it can be seen that ϵ_x stays approximately constant for large values of N_p . For $\mathcal{T}_{fd}$ the value of ϵ_x changes by less than 1% when $N_p > 10$, showing that similar accuracy can be achieved using a smaller number of pressure values than N_p used in §4.5-§4.6. For $\mathcal{T}_{lc}$ such a change is observed when $N_p > 15$ suggesting a larger number of measurements

is required to achieve a robust estimation.

Slightly more complex behaviour is observed in figure 4.10(b) which shows the analysis performed on $\mathcal{T}_{eq}$. It is seen that the state errors ϵ_x do not monotonically decrease as N_p increases. For example, the smallest errors are obtained at $N_p = 11$, $N_p = 7$ and $N_p = 11$ respectively, for linear, quadratic and cubic models. This change of behaviour can be linked to the fact that the analysed dynamical features are of low energy close to the equilibrium and due to the fact that the stochastic estimation may be less robust with respect to the training domain than, for example, approximation on the limit cycle. For $\mathcal{T}_{eq}$ and large N_p the stochastic estimation methodologies in a sense ‘overfit’ the data, meaning that ϵ_x is small in the training domain and substantially increases outside of it.

4.7.2 Higher state dimension

So far the state space was chosen to consist of five projection coefficients. Here, an analysis is performed to assess how increasing the number of modes impacts the accuracy of the estimation. For this purpose, state spaces of dimensions $r = \{2j - 1 : j \in \mathbb{N}, j < 50\}$ are considered. Odd values of r are chosen based on the assumption that the dynamics of the cylinder flow primarily consists of the single shift mode structure and pairs of the oscillatory subharmonics. It is noted that in the cases where the system’s eigenvalue spectrum consists of complex conjugate eigenvalue pairs together with a single real eigenvalue, even values of r will give an OMD output containing spurious eigenvalues. The r most dominant POD modes are used to construct a low-dimensional state space while in the case of the OMD, the output of this algorithm applied to the data set with approximation rank of r is used. For this analysis, only linear models are considered. There are two reasons behind the omission of nonlinear models: 1. the computational cost with increasing r grows exponentially making the MHE time-prohibitive; 2. once the dynamically less important modes are added, it is more difficult to induce stability of the cubic model through equality constraints. A similar observation is made concerning the quadratic model, and selecting an appropriate value of the sparsity-inducing constant γ is difficult.

The dynamical models are obtained by solving (4.17) on $\mathcal{T}_{fd}$ and $N_h = 10$ is used in (4.14). A comment about the selection of $\mathbf{y}$ is required. In §4.5 a combination of LSE and QSE was used with the latter approximating the subharmonic structure associated with a frequency twice

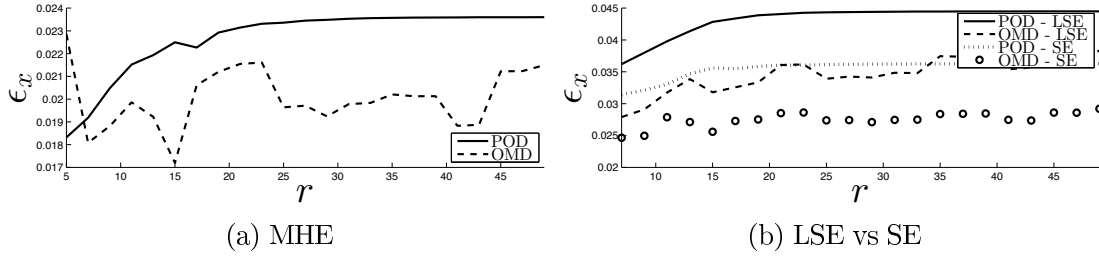


Figure 4.11: The state errors ϵ_x corresponding to linear MHE for $r = \{2j - 1 | j \in \mathbb{N}, j < 50\}$ (a); The state errors ϵ_x corresponding to LSE and stochastic estimation outlined in (4.36) (b).

the value of the one dominant in the pressure signal. Here, in order to take into account both complexities of the stochastic estimation the following problem is solved

$$\min_{\Psi_{SE}} \sum_{i \in \mathcal{T}_{fd}} \|\mathbf{x}_i - \Psi_{SE}^\top \tilde{\mathbf{p}}_i\|_2^2, \quad (4.35)$$

where $\tilde{\mathbf{p}} = [p_1, \dots, p_{N_p}, p_1^2, \dots, p_{N_p}^2]^\top$, $\Psi_{SE} \in \mathbb{R}^{2N_p \times r}$ and $\mathbf{y}_i = \Psi_{SE}^\top \tilde{\mathbf{p}}_i$. The values of ϵ_x corresponding to MHE results are presented for a range of r in figure 4.11(a).

It can be seen that when lower-ranked structures are added to the estimation, the discrepancies between POD and OMD vector spaces become more visible and the latter algorithm outputs structures which are more suitable for estimation using a linear model. Inclusion of additional POD modes increases values of ϵ_x . Since POD modes are obtained based on an energy criterion, the lower-ranked POD modes are obtained in such a way to ‘fit’ a finite linear decomposition of snapshots as closely as possible and low energy modes are progressively less likely to incorporate dynamical information and might contain only statistical information. For the OMD method the error is minimised for $r = 15$ while for $r > 15$ the inaccuracies in modelling dynamics and stochastic estimates of low-ranked states induce inaccuracies in the estimation of more dynamically relevant ones.

When the value of r is increased, it is expected that subsequent subharmonics of the global mode will be added to the state space. Since corresponding frequencies will be integer multiples of the fundamental frequency it is not necessarily obvious that LSE or QSE of pressure signals would provide the most appropriate approximation of such dynamics. In fact, perhaps a more appropriate approach would be to progressively increase the order of the stochastic estimation. Nonetheless, in such cases the number of the conditional variables increases exponentially. In order to decrease the rate at which this happens, an approach similar to the one used in §4.5

can be used, i.e. based on spectral characteristics of the state only a certain subset of the nonlinear variables in stochastic expansion (§2.2.1) will be retained.

Let us define the fundamental frequency of the pressure signal as f_f . Subsequently, spectral characteristics of each state are obtained using a training domain to give the frequency f_{max} associated with the highest magnitude Fourier peak. Subsequently let us define an exponent $k = \lfloor f_{max}/f_{sh} \rfloor \in \mathbb{Z}$ where $\lfloor \cdot \rfloor$ is the nearest integer function. If $k = 0$, the value is set to $k = 1$ instead. Consequently a set of exponents $\mathcal{K} = (k_j)_{j=1}^r$ is obtained and the following optimisation is solved

$$\min_{\Psi_j} \sum_{i \in \mathcal{T}_{fd}} \|(x_j)_i - \Psi_j^\top(\bar{\mathbf{p}}_j)_i\|_2^2, \quad (4.36)$$

where $\bar{\mathbf{p}}_j = [p_1^{k_j}, \dots, p_{N_p}^{k_j}]^\top$. For $k = 1$ and $k = 2$, LSE and QSE are recovered respectively. Figure 4.11(b) presents a comparison of this approach with LSE where it can be observed that, although the same patterns are visible, an improvement in ϵ_x is obtained. It is noted that this approach was applied in an automatic fashion and potentially further reduction can be achieved through a closer examination of the higher harmonic dynamics. Furthermore, it is noted that here Ψ_j will have a twice smaller dimension than Ψ_{SE} in (4.35) and can be treated as a compromise between LSE and the approach given in (4.35).

4.8 Conclusions

In this chapter an estimator of the cylinder flow dynamics was constructed. The assessment included appropriate construction of observables $\mathbf{y}$ where it was demonstrated that when using a combination of LSE and QSE it is possible to obtain an accurate approximation of key dynamical features from pressure measurements. Additionally, observables $\mathbf{y}$ were combined using an MHE algorithm with dynamic models of progressive orders of complexity, i.e. linear, quadratic and cubic models. All models showed an improvement over stochastic estimation, where the structured cubic model proved to be the most accurate in the majority of the situations, e.g. when dynamical models are obtained using $\mathcal{T}_{eq}$ or $\mathcal{T}_{lc}$. The dimension of the analysed state space was sufficiently low that even a relatively complex cubic model can be used efficiently with an MHE algorithm. However, if the dimension of state space is increased the stability of the nonlinear models deteriorates unless the sparsity level of the model is carefully chosen. For state estimates with dimension larger than five, a linear model can be chosen to balance model

accuracy with model stability.

In addition to model complexity analysis concerning the robustness of the estimation approach with respect to training domain was performed. It was observed that the models obtained close to the limit cycle are more robust compared to those based on data close to the equilibrium, where the latter often induced considerable error in the amplitude of the limit cycle. The likely justification of this behaviour was outlined in §4.6.2 which shows that information about the linear growth rate is strongly embedded in the dynamics of the limit cycle.

An assessment was performed between suitability of the state spaces induced by POD and OMD modes on the accuracy of the dynamical model and estimation. Based on the analysed domains spanning different dynamical regimes of the flow, it appears that the OMD low-dimensional space induces a more accurate dynamical model in almost all of the analysed test cases. For example, §4.4 demonstrated that even slight differences in the shapes of the modes are sufficient for the OMD to induce a more accurate cubic model. Additionally, as was presented in figure 4.11, when a large number of modes are used to construct the state space OMD modes provide a more suitable basis for estimation of low-energy structures.

Chapter 5

Dynamical Estimation of turbulent flow past an axisymmetric bluff body

5.1 Introduction

It was demonstrated in chapter 4 that it is possible to accurately estimate the velocity field of flow past a circular cylinder at low Reynolds number Re using only pressure data on the cylinder boundary. Although the dynamics of this system are nonlinear, the analysis was performed on a non-turbulent flow using numerical data. Additionally, the analysed Re was low enough that the system was two-dimensional, i.e. the dynamics are invariant with respect to translation in the direction perpendicular to the cylinder plane, which implies that all of the information necessary for the modeling is contained in the spatial domain of the snapshots. This provided a certain degree of robustness in dynamic estimation, ensuring that the extracted modes and model were not significantly corrupted by noise and unmodelled dynamics. Here, a significantly more complex system, but at the same time more practically relevant, is analysed; flow past a bullet-shaped axisymmetric body at $Re \approx 2 \times 10^5$.

The choice of this system is motivated by the question of drag reduction for bluff-bodies such as road vehicles, where aerodynamic drag has significant impact on fuel consumption and CO₂ emissions [22]. Drag reduction approaches are typically divided into passive and active flow control strategies. Passive flow control techniques usually consist of modifications to the flow geometry and, although proven effective for bluff-bodies [90], increased demand for fuel efficiency has encouraged use of active control methodologies, such as closed-loop control.

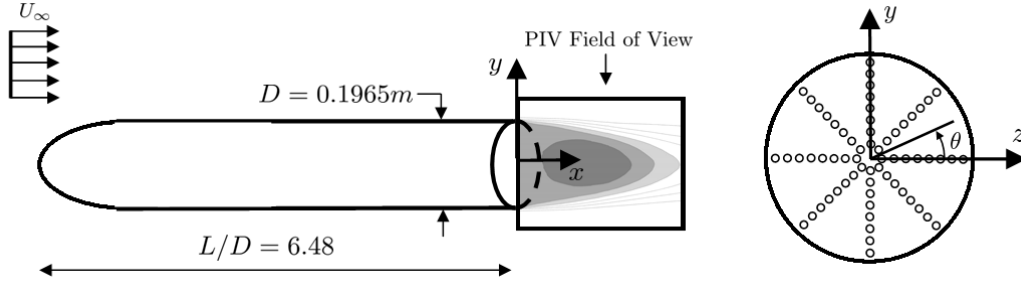


Figure 5.1: Schematic representation of the experimental model and PIV field of view (left) and the location of the 64 pressure tappings on the base of the model (right). The velocity field is defined as $\mathbf{u} = (u, v)$ where u and v are velocity components along x and y axis respectively.

Closed-loop control requires online knowledge of the system states from, typically, velocity or pressure measurements and a dynamical model. In the following section insight into such an estimation problem will be given.

The analysis will be performed using experimental measurements obtained by [91]. A detailed description of the experimental set up is presented in [91, 92] and consists of a bullet-shaped model with base diameter $D = 0.1965m$ with a length-to-diameter ratio of $L/D = 6.48$. The inflow velocity was set to $U_\infty = 15ms^{-1}$, which gives the Reynolds number based on the body diameter of $Re = 1.88 \times 10^5$. Two types of measurements are acquired in the experiment, pressure at the base and velocity fields downstream of the object. The base is mounted with 64 pressure sensors, located at an equal distance in the radial directions and separated by an angle of $\theta = \pi/4$, as presented in figure 5.1. In this way pressure is measured at eight angular and eight radial locations. The wake velocity is measured inside a PIV field of view extending $1.72D$ in the streamwise direction and being $1.5D$ wide. A schematic representation of the experimental setup is shown in figure 5.1. Data was collected in a series of independent experiments lasting $3.79s$ at a frequency of 720 Hz .

In this chapter the main oscillatory dynamics of the wake will be estimated using pressure measurements at the base of the bullet-shaped object. Section 5.2 provides a detailed description of the system based on the existing literature and information obtained from the modal decomposition algorithms applied to experimental data. In §5.3 a summary of the developed estimation approaches is given. First, in order to obtain benchmark metrics for the estimation accuracy, in §5.4 oscillatory dynamics are estimated using wake velocity measurements. Although the estimates obtained in this way show high accuracy, it is difficult to implement such an approach in a non-intrusive manner. Consequently, significantly more challenging estimation

based on the pressure measurements on the base of the object is performed in §5.5. Following certain inaccuracies of the estimates in §5.5, the used technique is altered in §5.6 in order to take into account the azimuthal bias of the pressure measurements. In §5.6.1 an investigation into viability of the methodology where pressure values are rotated into fixed configurations is performed whilst in §5.6.2 different pressure modes are used depending on the azimuthal location of the centre of pressure. It is demonstrated that the latter approach outputs estimates of the greatest accuracy.

5.2 Description of the dynamical system

At low Re the wake of a bullet-shaped object is steady and axisymmetric but once Re is increased the system undergoes a series of bifurcations leading to dynamics with fully three dimensional effects [93]. The bifurcation sequence is extensively analysed numerically in [94] for $L/D = 7$ (similar to the model ratio used in here) and a schematic representation of the observed system states is presented through streamwise vorticity distribution sketches in figure 5.2. At $Re \approx 450$ the flow loses its axisymmetry and undergoes a steady bifurcation which leads to double-threaded wake structure which is symmetric about a fixed reflectional plane (figure 5.2(a)). The reflectional plane is not $z = 0$ but instead offset towards the edge of the object as seen from the corresponding pressure distribution in figure 5.2(b). It is noted that the azimuthal orientation of the reflectional plane is random. Beyond $Re \approx 590$ a Hopf bifurcation induces an oscillatory state (figure 5.2(c)) where alternating hairpin structures are periodically shed at a unique fundamental shedding frequency. Nonetheless, the reflectional symmetry of the steady is still retained. At $Re \approx 690$ pulsating vorticity bursts appear, which induce transient waving of hairpin structures in the reflectional symmetry plane at a frequency an order of magnitude smaller than the shedding dynamics leading to the double frequency state (figure 5.2(d)). Beyond $Re \approx 750$, broadband noise appears in the frequency spectrum of the drag coefficient signifying a transition of the system into chaotic dynamics. Finally, at $Re \approx 900$ the flow develops a mechanism that allows for random changes in the reorientation of the symmetry plane. Similar dynamics were observed for a sphere [95, 96] and a disk [97] indicating that the analogous sequence of bifurcations may be universal for all axisymmetric bluff bodies.

In [26] an analysis of pressure measurements taken at the model base for the same

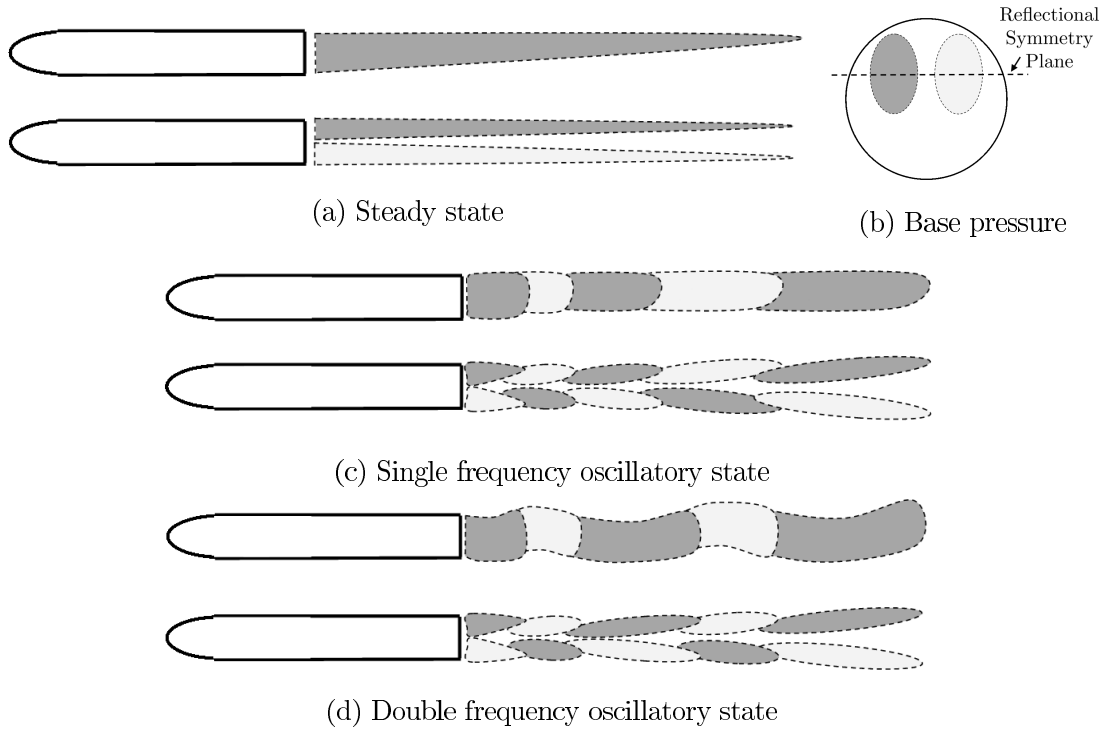


Figure 5.2: Schematic sequence of bifurcations characterising flow past the axisymmetric object, as described in [94]. In all cases the top diagram represents a field of view orthogonal to the reflectional symmetry plane while the bottom view shows the corresponding parallel view. Steady state with reflectional symmetry (a) with a side view of the top diagram showing the approximate pressure distribution on the base of the object (b); oscillatory state with hairpin structures being shed in the wake of the body (c); double frequency state combining shedding dynamics with low-frequency transient oscillations in the reflectional symmetry plane (d).

Reynolds number as analysed here, $Re = 1.88 \times 10^5$ was performed. Although this flow is fully turbulent it was demonstrated that large-scale structures associated with the aforementioned instabilities persist in the fully developed turbulent flow. In particular, there exist large-scale anti-symmetric oscillations with an associated vortex shedding frequency $St \approx 0.2$, where $St = \frac{fL}{D}$, which are linked to the asymmetric mean pressure distribution. This pressure distribution randomly switches its angular reorientation around the axis of the body at frequency $St \approx 0.002$. Due to random nature of this change, when an increasing number of data samples is acquired the axisymmetry of the mean flow is recovered. Additionally, a peak in the spectrum at $St \approx 0.06$ was observed which can be linked to an axisymmetric pressure structure and the pulsation dynamics observed in double frequency oscillatory state [94]. Since the analysis in [26] was performed based on pressure measurements an initial insight on the observability of the pressure structures can be obtained. In the analysis of the power spectrum of the pressure measurements, three spectral peaks were observed at $St \approx 0.002$, $St \approx 0.06$ and $St \approx 0.2$. The magnitude of

the power spectra at the first of the two low-frequencies was orders of magnitude larger than the magnitude corresponding to $St \approx 0.2$, signifying only a low-energy imprint of the shedding dynamics on the base pressure measurements. Noting that a large number (approximately 4.3×10^6) of temporal measurements were needed to recover the shedding spectral peak, this suggests an intrinsic difficulty of recovering shedding dynamics from data covering orders of magnitude shorter time intervals, as we will attempt here. On the other hand, the dominant nature of the low-frequency dynamics in the pressure signals suggest that those dynamical features should be relatively simpler to estimate from the base measurements.

5.2.1 Structures present in system

This section aims to address the question of whether it is possible to identify velocity structures associated with the frequencies described in [26] using modal decomposition techniques applied to wake PIV data. One of the major features of the system, which is likely to make the dynamical estimation significantly more difficult relative to the cylinder flow, is the three-dimensionality of the dynamics. In [26] it was demonstrated that the shedding reflectional plane rotates randomly around the axis of the body with $St \approx 0.002$. Since the PIV field of view presented in figure 5.1 is only two-dimensional, the observed view of the shedding structure will change depending on the orientation of the reflectional symmetry plane. In order to verify this hypothesis, mode extraction algorithms will be applied separately to two different data sets: one for which the point of view is parallel to the symmetry plane, and one for which the point of view is perpendicular to it.

To do this, an indication of the system configuration can be determined from the base pressure measurements; more specifically the azimuthal component, θ , of the centre of pressure as defined in figure 5.1. Two sets of data are constructed which will be used to identify shedding and pulsation modes for system configurations where $\theta \approx \pi$ and $\theta \approx \pi/2$. The first orientation corresponds to a PIV field of view parallel to the reflectional symmetry plane, i.e. the bottom diagrams in figure 5.2, whilst the latter data set shows a PIV field of view perpendicular to the symmetry plane. The azimuthal component of the centre of pressure is calculated for every available snapshot, and velocity snapshots are selected when $\theta \in [\pi - \pi/16, \pi + \pi/16]$. These snapshots are then split into two subsets $\mathbf{A}_\pi \in \mathbb{R}^{N_u \times 2125}$ and $\mathbf{B}_\pi \in \mathbb{R}^{N_u \times 2125}$, where $\mathbf{B}_\pi$ contains data sets at t_i and $\mathbf{A}_\pi$ at $t_i + \delta t$. Such data, although discontinuous, can be easily

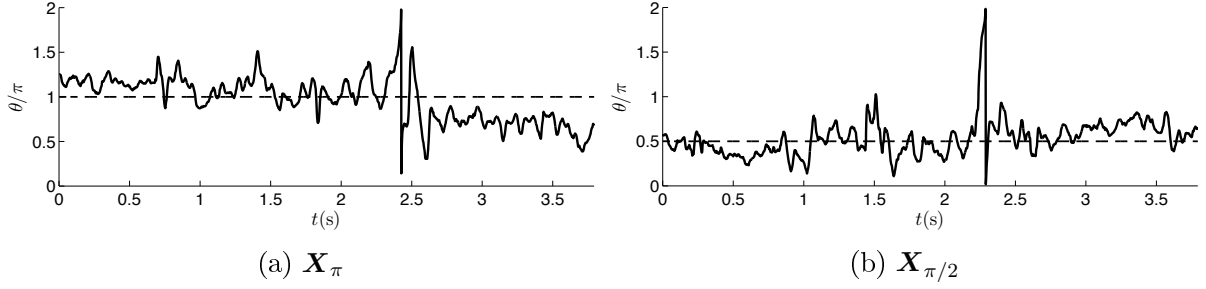


Figure 5.3: Evolution of the azimuthal coordinate of the centre of pressure in $\mathbf{X}_\pi$ (a) and in $\mathbf{X}_{\pi/2}$ (b).

used as the input to the POD and OMD algorithms [21]. An analogous procedure is repeated for $\theta \in [\pi/2 - \pi/16, \pi/2 + \pi/16]$ to obtain datasets $\mathbf{A}_{\pi/2} \in \mathbb{R}^{N_u \times 2882}$ and $\mathbf{B}_{\pi/2} \in \mathbb{R}^{N_u \times 2882}$.

The dynamical information associated with identified structures can be inferred based on the OMD eigenvalues. However, in order to gain an additional insight into the dynamics of the identified structures, it is desirable to analyse the dynamics of their projection coefficients. Since continuous data sets are required for this purpose, two additional sets of data are introduced, $\mathbf{X}_\pi$ and $\mathbf{X}_{\pi/2}$, which were selected based on how close the mean values of θ corresponding to $\mathbf{X}_\pi$ and $\mathbf{X}_{\pi/2}$ are to π and $\pi/2$ respectively. The temporal evolution of θ corresponding to the data sets $\mathbf{X}_\pi$ and $\mathbf{X}_{\pi/2}$ are presented in figure 5.3. Additionally, $\mathcal{X}$ denotes a set of temporal indices corresponding to defined data sets, e.g. if $i \in \mathcal{X}_\pi$ then the corresponding snapshot is included in $\mathbf{X}_\pi$.

The data set $\mathbf{X}_{\pi/2}$, whose azimuthal component θ is shown in figure 5.3(b) contains velocity fields where the system reorientation is close to $\theta = \pi/2$, with a mean value of 0.53π and standard deviation of 0.20π . The dynamics associated with $\mathbf{X}_\pi$ (figure 5.3(a)) is more complex. Until $t \approx 2.5$, θ evolves about a mean value of 1.1π , upon which, it switches its reorientation to approximately 0.7π . The analysis presented in this section is performed on $\mathbf{X}_\pi$ truncated to $t \in [0, 2.5]$, noting that estimation in §5.4-§5.5 will be carried out on a full domain.

5.2.1.1 Velocity Modes

The first three and the fifth POD modes, Φ_j^{POD} , obtained from the dataset $\mathbf{A}_\pi$ are presented in figure 5.4. Modes Φ_1^{POD} and Φ_3^{POD} both contain anti-symmetric features in the u -component of their velocities and both of them appear to be spatially out of phase by approximately a quarter of their streamwise lengthscale. On the other hand the u -component of Φ_2^{POD} is symmetric about the plane $y = 0$. It is noted that the opposite features are visible

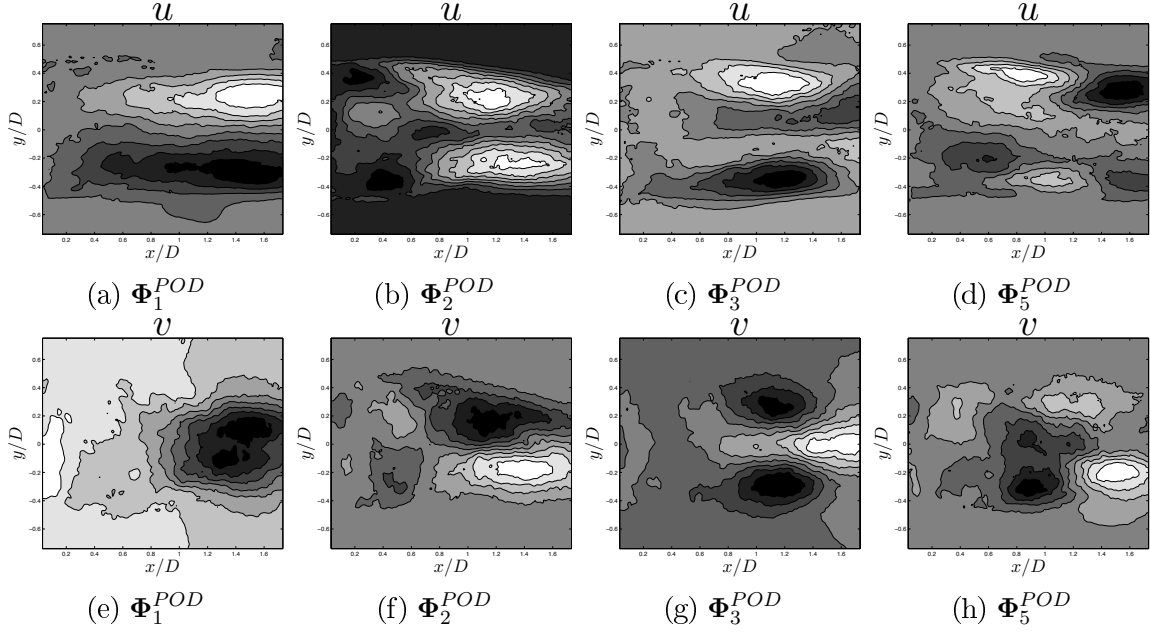


Figure 5.4: The first three and the fifth POD modes obtained from dataset $\mathbf{A}_\pi$. The u -components of the velocities are presented in (a)-(d), whilst their v -components are shown in (e)-(h).

in their v -components, i.e. values corresponding to Φ_1^{POD} and Φ_3^{POD} are symmetric whilst those corresponding to Φ_2^{POD} are anti-symmetric. A combination of the u and v components of Φ_1^{POD} , Φ_3^{POD} would result in the anti-symmetric vorticity plots which are consistent with the shedding structure which appears in the single frequency oscillatory state in figure 5.2(c) and persists in a double frequency oscillatory state (figure 5.2(d)) indicating that these two modes are likely to be associated with the shedding dynamics. It is noted that the pulsation dynamics developing in the second oscillatory bifurcation seen in figure 5.2(d) for the π configuration is characterised by a movement of the wake in and out of PIV field of view, resulting in the oscillatory change in the magnitude of the u -component. Such dynamics could theoretically be captured using a symmetric mode of a similar shape as Φ_2^{POD} .

Since the POD methodology does not directly provide dynamical information associated with each mode, and datasets $\mathbf{A}_\pi$ and $\mathbf{B}_\pi$ are discontinuous making it impossible to analyse the dynamics of the corresponding projection coefficients, the OMD algorithm is applied to $\mathbf{A}_\pi$ and $\mathbf{B}_\pi$ with $r = 50$ to assign dynamical meaning to the structures shown in figure 5.4. In order to identify the dynamical modes which most closely resemble the structures presented in figures 5.4(a)-(b), spatial correlations between Φ_1^{POD} , Φ_2^{POD} and the OMD modes Φ_j^{OMD} are analysed.

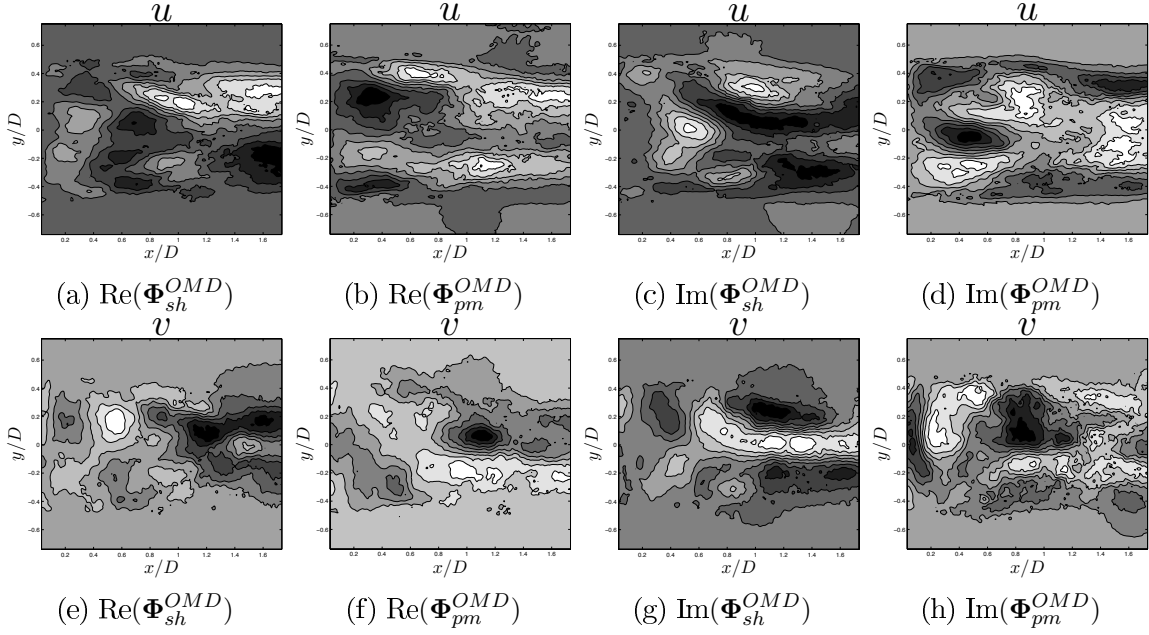


Figure 5.5: The OMD modes which have similar spatial characteristics to Φ_1^{POD} and Φ_3^{POD} in u and v components is shown in (a)-(c) and (e)-(g) respectively, whilst the OMD mode which most closely resembles Φ_2^{POD} and Φ_5^{POD} is shown in (b)-(d) and (f)-(h).

It is in fact possible to identify two modes shown in figure 5.5 which have real parts closely correlated with Φ_1^{POD} and Φ_2^{POD} and will be denoted as Φ_{sh}^{OMD} and Φ_{pm}^{OMD} respectively. The reason behind such notation stems from the associated eigenvalue information which further confirms that the first of the structures is associated with shedding dynamics whilst the second one is the pulsation mode. The OMD mode, Φ_{sh}^{OMD} , most closely correlated with Φ_1^{POD} corresponds to an OMD eigenvalue satisfying $St = \frac{D}{2\pi U_\infty} \text{Im}(\lambda) = 0.167$, where λ is defined as in §2.1.3, whilst Φ_{pm}^{OMD} which has a high spatial correlation with Φ_2^{POD} is associated with frequency $St = 0.069$. Referring to [26], wake structures are known to possess frequency content at approximately these values, specifically a shedding frequency ($St \approx 0.2$) and pulsation of vortex cores ($St \approx 0.06$). Additionally, in [26] it is indicated that shedding and pulsation frequencies are associated with antisymmetric and symmetric features respectively, which are clearly seen in figure 5.4.

It is also noted that $\text{Im}(\Phi_{sh}^{OMD})$ closely resembles the second antisymmetric mode Φ_3^{POD} , confirming that Φ_1^{POD} and Φ_3^{POD} represent the same structure. On the other hand, Φ_5^{POD} was identified to correspond to $\text{Im}(\Phi_{pm}^{OMD})$ based on the spatial correlations. Although the spatial differences between Φ_5^{POD} and $\text{Im}(\Phi_{pm}^{OMD})$ are significantly larger than those of the other analysed structures it is nonetheless possible to identify similar spatial features in their

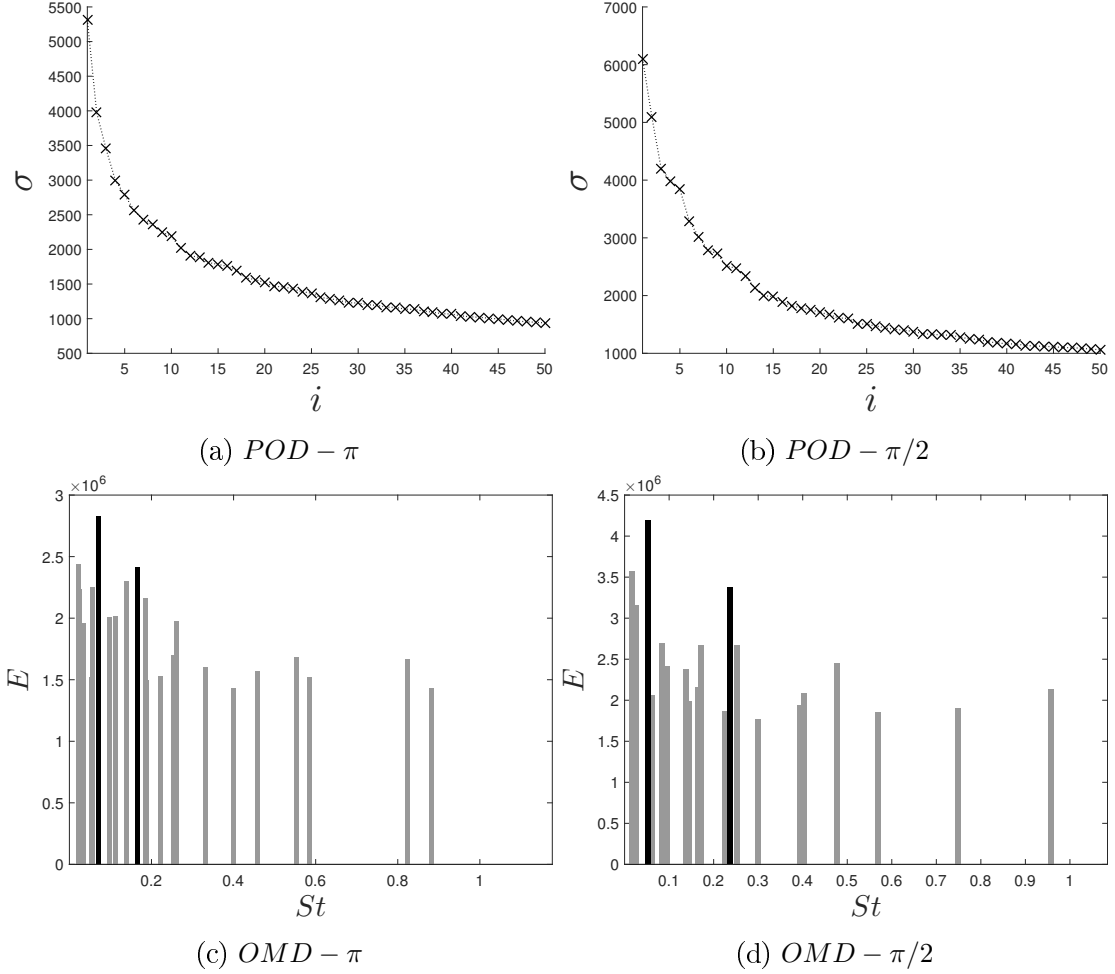


Figure 5.6: Singular values of the POD modes for two sytem orientations ((a)-(b)) and OMD energies obtained using a technique presented in §3.1 for different values of St ((c)-(d)).

u and v components (figures 5.4(d), 5.5(d) and 5.4(h), 5.5(h) respectively).

Additionally, the energies of the structures obtained by applying POD and OMD algorithms to A_π , B_π are respectively showed in figures 5.6(a), 5.6(c). In the case of POD modes the representation of energies is given through a plot of the singular values, whilst in the case of the OMD structures the plot represents energies obtained using the technique outlined in §3.1 and the values of St are retrieved using the imaginary part of the OMD eigenvalues. It is noted that for POD modes the magnitudes of the singular values corresponding to the 10 most dominant modes change on average only by 0.01 % with the inclusion of an additional snapshot. For the OMD algorithm the corresponding change is approximately 0.7 %. It is furthermore noted that in the case of the OMD the energetic values are not only sensitive to the number of snapshots but also to the choice of r which induces variation not only in the mode shapes but also in the value of St . Nonetheless, it will be discussed later that greater difficulties in

the convergence of the OMD mode shapes relative to POD structures are counterbalanced by obtaining a structure which better isolates the desired St and reduces ‘frequency-mixing’ in the projection coefficients.

Subsequently, an investigation is conducted to see how a rotation of the symmetry plane around the axis of the body affects our view of the structures present in the flow by performing the same analysis as above on data sets $\mathbf{A}_{\pi/2}$ and $\mathbf{B}_{\pi/2}$. The first two, fifth and sixth POD modes of $\mathbf{A}_{\pi/2}$ are presented in figure 5.7 whilst the OMD modes associated with frequencies of interest are shown in figure 5.8. The reason behind the presentation of these particular POD modes are their visible spatial symmetries and the similarities with the presented OMD modes. The OMD mode shown in figure 5.8(a)-(b),(e)-(f) was identified to be associated with frequency of $St = 0.24$ indicating that this mode corresponds to shedding dynamics and will be analogously denoted as Φ_{sh}^{OMD} . The POD modes most closely resembling Φ_{sh}^{OMD} are Φ_1^{POD} , Φ_2^{POD} and are shown in figure 5.7(a)-(b),(e)-(f). The OMD mode presented in figure 5.8(c)-(d),(g)-(h) is approximately symmetric about $y = 0$ and associated with $St = 0.052$, indicating that it represents the pulsation mode Φ_{pm}^{OMD} . Based on spatial correlation, the POD modes which most closely resemble $\text{Re}(\Phi_{pm}^{OMD})$ and $\text{Im}(\Phi_{pm}^{OMD})$ are Φ_5^{POD} , Φ_6^{POD} , respectively and are presented in figure 5.7(c)-(d),(g)-(h). It is noted that the spatial correlation in this case is only 0.4 and large differences are visible in their shapes, especially for $\text{Im}(\Phi_{pm}^{OMD})$, indicating that the POD methodology does not necessarily approximate this structure well for $\theta = \pi/2$.

For Φ_1^{POD} and Φ_2^{POD} , although similar antisymmetries are visible as in the POD modes obtained from $\mathbf{A}_\pi$ it is possible to see that the structures for $y \geq 0$ have greater magnitudes than the analogous structures at $y \leq 0$ for both u and v components. This feature is linked to an offset in the symmetry plane away from $y = 0$ towards the edge of the model seen in figure 5.2. Since now the point of view corresponds to the top diagrams in figure 5.2 it is expected that those asymmetries are captured in the shape of the coherent structures. For the pulsation modes, in both OMD and POD modes (figure 5.7(c) and 5.8(c)) there are two coherent structures of the opposite signs visible at $y \approx 0.4$ and $y \approx 0.2$; a feature which was not as prominent in the equivalent modes for the $\theta = \pi$ orientation. It is reiterated that for the $\theta = \pi$ orientation the pulsation dynamics primarily consist of motion in and out of plane of view, whilst here, for the $\theta = \pi/2$ orientation, such dynamics will be viewed as oscillations

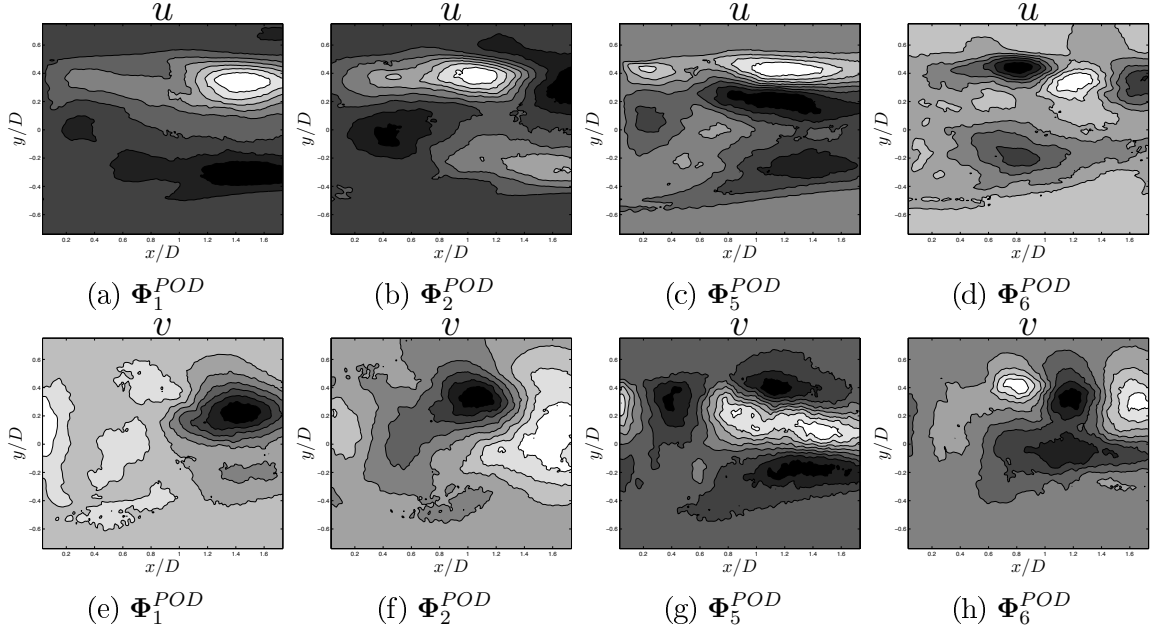


Figure 5.7: The first two, fifth and sixth POD modes obtained from the dataset $\mathbf{A}_{\pi/2}$. The u -components of the velocities are presented in (a)-(d), whilst the v -components are shown in (e)-(h).

along the y -axis. This is consistent with the dynamical mode in figure 5.7(c) where the presence of two structures with the opposite sign adjacent to each other in the vertical direction signifies the ability of the mode to describe transient oscillations.

The energies of the POD and OMD structures corresponding to $\theta = \pi/2$ are presented in figures 5.6(b) and 5.6(d) respectively. The percentage changes of those values with respect to inclusion of an additional snapshot are 0.01% for POD modes and 0.2% for OMD structures.

A comment has to be made about different rates of convergence and the visible differences in the coherency of the analogous POD and OMD structures, where in all cases the OMD modes appear to possess finer-scale structures. Here, the OMD results were obtained for the approximation rank $r = 50$ and it is noted that if the value of r is decreased the OMD modes will retain a coherency similar to the POD structures. Nonetheless, it will be shown in §5.2.1.2 that the OMD modes as presented here perform relatively better in terms of isolating the frequencies of interest. If the value of r is decreased the corresponding dynamics of the projection coefficients will be more similar to the POD values. Consequently, when OMD is applied to experimental data it is necessary to take into account the compromise between the spatial coherency of the modes and accuracy of the associated dynamical features through an appropriate choice of r .

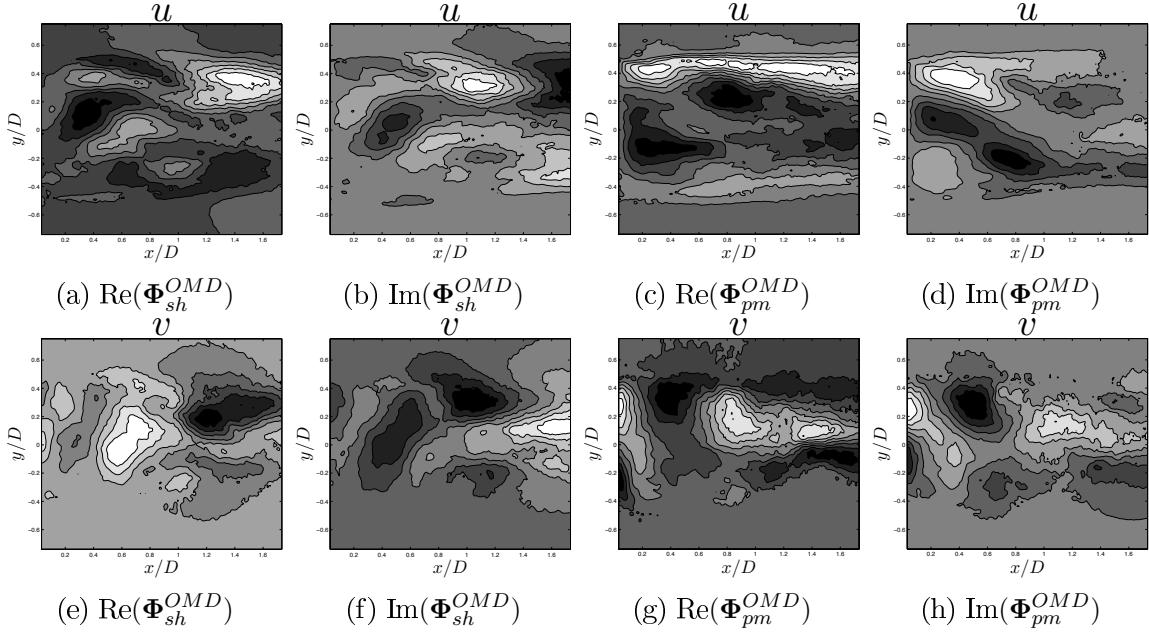


Figure 5.8: The OMD modes which have similar spatial characteristics to Φ_1^{POD} and Φ_2^{POD} in u and v components is shown in (a)-(b) and (e)-(f) respectively, whilst the OMD mode which most closely resembles Φ_5^{POD} and Φ_6^{POD} is shown in (c)-(d) and (g)-(h).

5.2.1.2 Dynamics of the coherent structures

The spatial similarities between the OMD and POD modes in the results presented in §5.2.1.1 indicate that modes with potential dynamic significance can be identified using either of the algorithms. In order to assess the suitability of both types of modes for dynamical estimation, the power spectra of the projection coefficients, $\mathbf{x}_{sh}$ and $\mathbf{x}_{pm}$, corresponding to continuous datasets introduced earlier, $\mathbf{X}_\pi$ and $\mathbf{X}_{\pi/2}$, are analysed. It is noted that two separate structures with different dynamical characteristics were identified for $\theta = \pi$ and $\theta = \pi/2$ using both POD and OMD algorithms, giving in total eight different modes. A dynamical comparison for the same system configuration is performed, i.e. structures obtained from $\mathbf{A}_\pi$ and $\mathbf{B}_\pi$, Φ_{pm} and Φ_{sh} , are projected onto $\mathbf{X}_\pi$ such that $\mathbf{x}_{sh} = \Phi_{sh}^\top \mathbf{X}_\pi$ and $\mathbf{x}_{pm} = \Phi_{pm}^\top \mathbf{X}_\pi$. Likewise similar projection coefficients are obtained using structures corresponding to $\mathbf{A}_{\pi/2}$ and $\mathbf{B}_{\pi/2}$ and projecting them onto $\mathbf{X}_{\pi/2}$.

The power spectra of the real part of the OMD projection coefficients and the equivalent POD mode projections are calculated using the multitaper method [98]. As opposed to the Fourier transform, a multitaper method obtains *multiple* estimates of the power spectra from the same sample using data tapers — mutually orthogonal vectors containing desirable spectral concentrations — resulting in an output less sensitive to noise. Each data taper is used to re-

trieve a windowed trial from which a power spectrum of each frequency component is estimated and the final power spectrum is obtained as an average of windowed spectra. For a noisy signal, like the one analysed here, the multitaper method provides a significantly smoother power spectrum than the Fourier transform, giving a better indication on the range of frequencies dominant in the system.

The power spectra corresponding to the orientation $\theta = \pi$ for structures associated with pulsation and shedding dynamics are presented in figures 5.9(a)-(b) while the equivalent results for $\theta = \pi/2$ are shown in figures 5.9(c)-(d). It is noted that power spectra corresponding to the OMD results are obtained using real parts of the projection coefficients and all power spectra presented in figure 5.9 are normalised in order to assess only the relative magnitudes associated with frequencies of interest.

Figure 5.9(a) shows that the modes Φ_{pm} , identified using both the OMD and POD algorithms from $\mathbf{A}_\pi$, $\mathbf{B}_\pi$, have very similar spectra. This is not surprising, since the overall spatial features visible in figures 5.4(c),(g) and 5.5(b),(f) are broadly similar. For the structures where differences in spatial features are more significant the disparities in spectra are also clearer, as seen in figures 5.9(b)-(d). For example, figure 5.9(b) shows that the shedding mode obtained using the OMD algorithm from data at $\theta = \pi$ clearly isolates the shedding frequency, while the equivalent POD structure produces a low-frequency peak of significant amplitude. Similar features are visible for $\theta = \pi/2$ in figure 5.9(c)-(d). In both cases the modes obtained using the OMD algorithm clearly isolate the frequencies of interest, while POD modes provide the dynamics where both values corresponding to shedding and pulsation dynamics are of comparable magnitude. It is noted that a similar observation was made before for other fluid flows, for example in [6] for the POD modes of a jet in crossflow. Due to the “frequency-mixing” associated with POD modes the OMD structures provide a better approximation of the most dynamically important structures in the flow and are more appropriate for the state construction in the estimation problem to be pursued subsequently in §5.4-§5.6.

For estimation purposes, at each instance in time, a measurement has to be acquired which will provide sufficient information to obtain accurate estimated state values. In order to provide insight into observability of the OMD modes presented in figure 5.8, point correlations were computed between the temporal evolution of the projection coefficients, $\mathbf{x}_j = (\Phi_j^{OMD})^\top \mathbf{X}_{\pi/2}$,

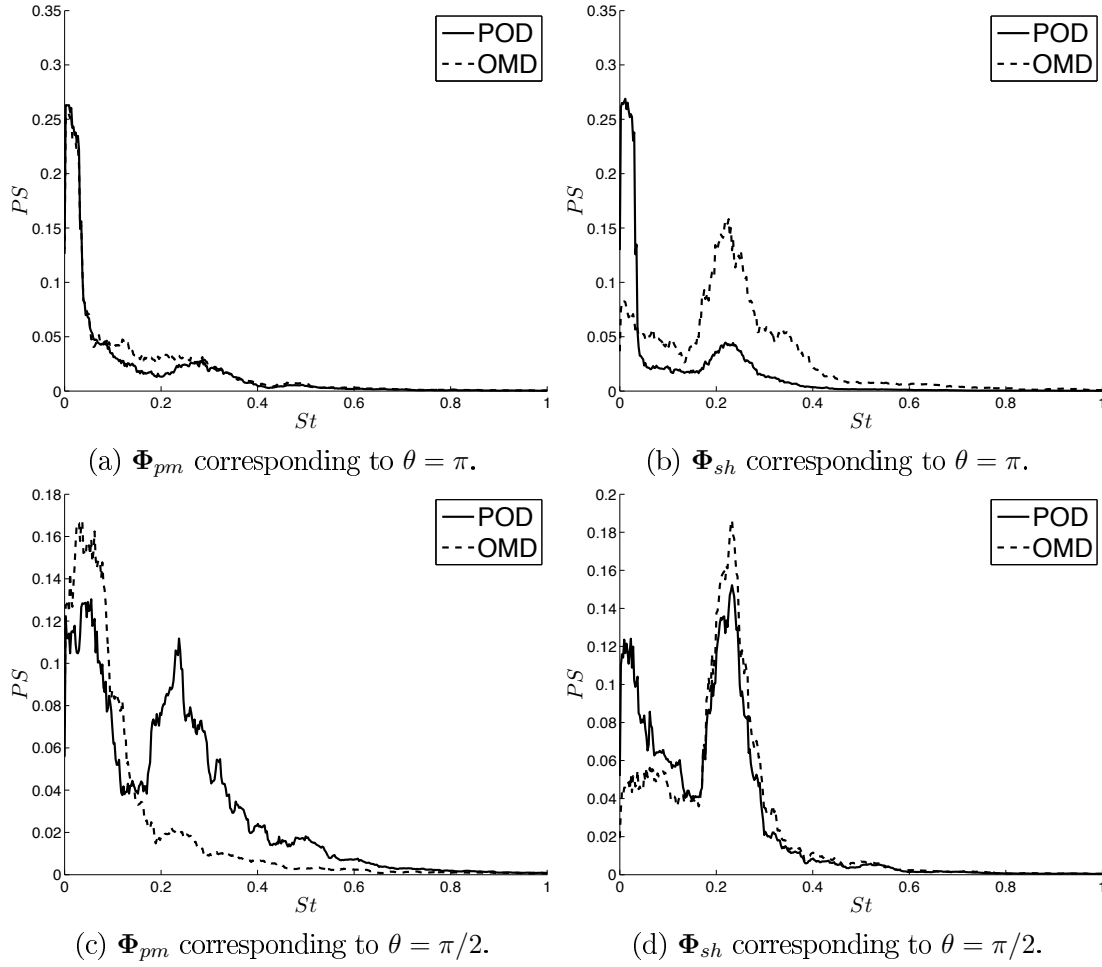


Figure 5.9: Power spectra of the projection coefficients corresponding to the POD modes and real parts of the OMD modes obtained from datasets where $\theta = \pi$ and $\theta = \pi/2$. Projection of structures identified as pulsation and shedding modes at $\theta = \pi$ onto $\mathbf{X}_\pi$ are shown in (a) and (b) respectively. Projection of structures identified as pulsation and shedding modes at $\theta = \pi/2$ onto $\mathbf{X}_{\pi/2}$ are shown in (c) and (d) respectively.

and of the velocity values at every spatial location in $\mathbf{X}_{\pi/2}$. In particular, correlations with the v -components of the velocity are presented in figure 5.10. The correlations corresponding to the shedding structure in figure 5.10(a)-(b) show that shedding dynamics are primarily visible in the regions far downstream from the object and are close to negligible near the object base. On the other hand, for the low-frequency mode (5.10(c)-(d)), there is a region by the base of the model where the correlation is approximately 0.47 suggesting that Φ_{pm}^{OMD} has more imprint on the base-pressure dynamics. Although velocity probes located far-downstream might be necessary to estimate shedding dynamics, it should be possible to estimate the value of low-frequency dynamics purely from pressure measurements.

It is noted that the same conclusions can be reached based on the analysis of the correlations with streamwise velocity components, between corresponding power spectra and analysing the

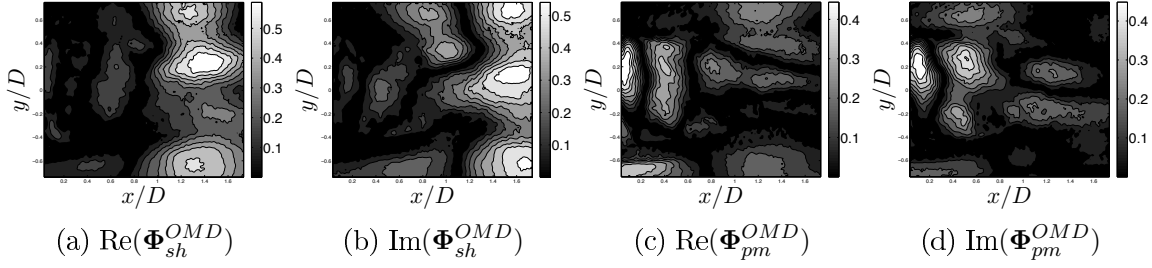


Figure 5.10: Point correlations between projection coefficients of the OMD modes $\mathbf{x}_j = \Phi_j^\top \mathbf{u}$ and v -components of the velocity at every spatial location.

OMD modes obtained from $\mathbf{X}_{\pi}$, therefore, the presentation of these results is omitted.

5.2.1.3 Pressure Structures

An analysis is performed to obtain insight into the shape of the pressure structures at the base of the object. Recalling that base pressure measurements were taken simultaneously with the velocity fields, two discontinuous pressure datasets are constructed for $\theta = \pi$ and $\theta = \pi/2$ using the same approach as in §5.2.1 and POD modes are calculated for both orientations. The mean pressure and the first three POD modes corresponding to $\theta = \pi$ are presented in figure 5.11(a)-(d). Among the POD modes corresponding to the $\theta = \pi/2$ orientation, structures are selected which most closely resemble those in figure 5.11(a)-(d) and are shown in figure 5.11(e)-(h). The main structures show a clear directional preference: for both values of θ they consist of the same spatial features but offset by an azimuthal angle of approximately $\pi/2$. Noticeable is the lack of the axisymmetric POD mode presented in [26]. However, this is not surprising since the system recovers axisymmetry only in the statistical sense for a long time domain and a whole range of θ .

In order to establish the dynamic significance associated with the structures in figure 5.11, an OMD-like procedure is used. The DMD and OMD algorithms are designed to be applied to datasets for which the spatial resolution is significantly greater than the number of temporal measurements, with the dimensional reduction $\mathbf{L}^\top \mathbf{u}$ onto the low order subspace basis, $\mathbf{L}$, being a key feature of both algorithms. Since $\mathbf{p} \in \mathbb{R}^{64}$, the spatial resolution is already low and this step can be omitted. Consequently, a dynamics matrix is calculated through least-squares minimisation

$$\min_{\mathbf{A}_p} \sum_i \|\mathbf{p}_{i+1} - \mathbf{A}_p \mathbf{p}_i\|_2^2. \quad (5.1)$$

In this way the matrix $\mathbf{A}_p$ approximates the temporal evolution of the actual pressure mea-

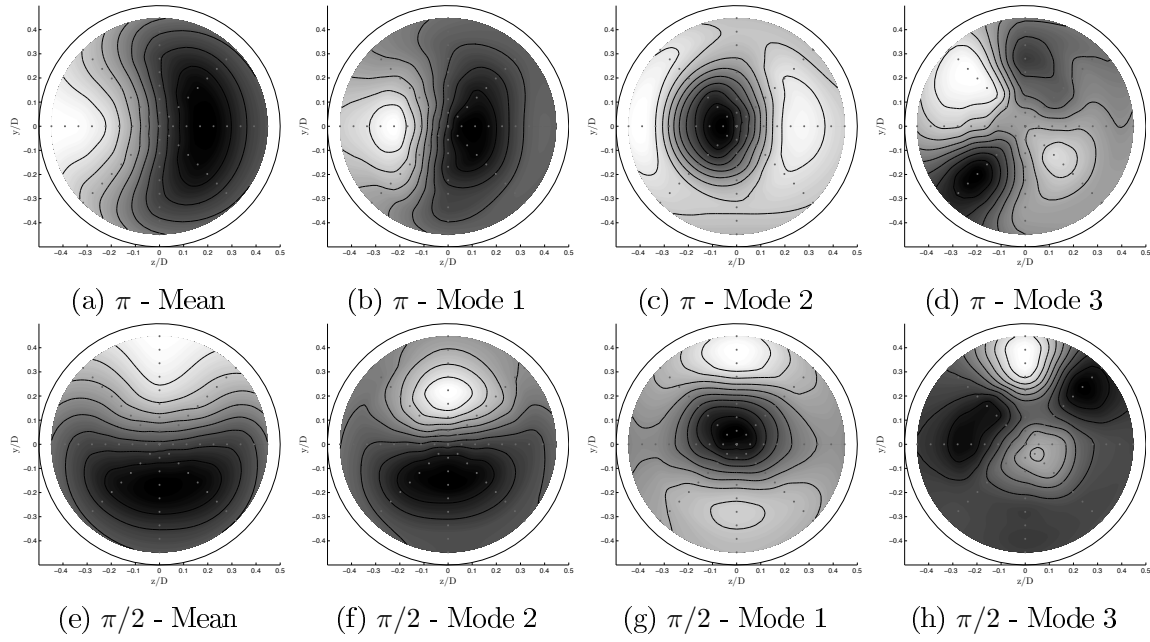


Figure 5.11: The mean pressure (a) and first three pressure POD modes (b)-(d) corresponding to $\theta = \pi$. The mean pressure (e) and the analogous POD modes (f)-(h) corresponding to $\theta = \pi/2$.

measurements instead of certain projection coefficients. Once the dynamic matrix is known, in the same way as in the modal decomposition algorithms, dynamical information is obtained from the eigendecomposition $\mathbf{A}_p = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ where the eigenvalues $\mathbf{\Lambda}$ indicate frequencies present in the system and the associated dynamic structures are represented by the corresponding eigenvectors $\mathbf{V}$.

The eigenvalue spectra of dynamic matrices obtained for $\theta = \pi$ and $\theta = \pi/2$ are presented in figure 5.12. The eigenvalues in figure 5.12 are shaded to represent the energetic contribution of each eigenvalue following a similar methodology to the one in §3.1. First a projection onto the eigenvectors of $\mathbf{A}_p$ is obtained such that $\mathbf{A} = (\alpha_{jk}) := \mathbf{V}^{-1}\mathbf{p}$ and it is observed that the coefficients multiplying j^{th} eigenvector correspond to the j^{th} row of $\mathbf{A}$. Consequently, a relative energetic contribution of each eigenvector is defined as

$$E_j := \frac{\sum_k |\alpha_{jk}|^2}{\sum_{jk} |\alpha_{jk}|^2}. \quad (5.2)$$

Figure 5.12 shows that in both cases the maximum imaginary values are bounded by $St = 0.2$ and the majority of frequencies are smaller than $St = 0.1$. Using spatial correlation, the eigenvectors most closely resembling the structures in figure 5.11 are identified and their corresponding frequencies are obtained. For the modes presented in figure 5.11(b)-(d), the

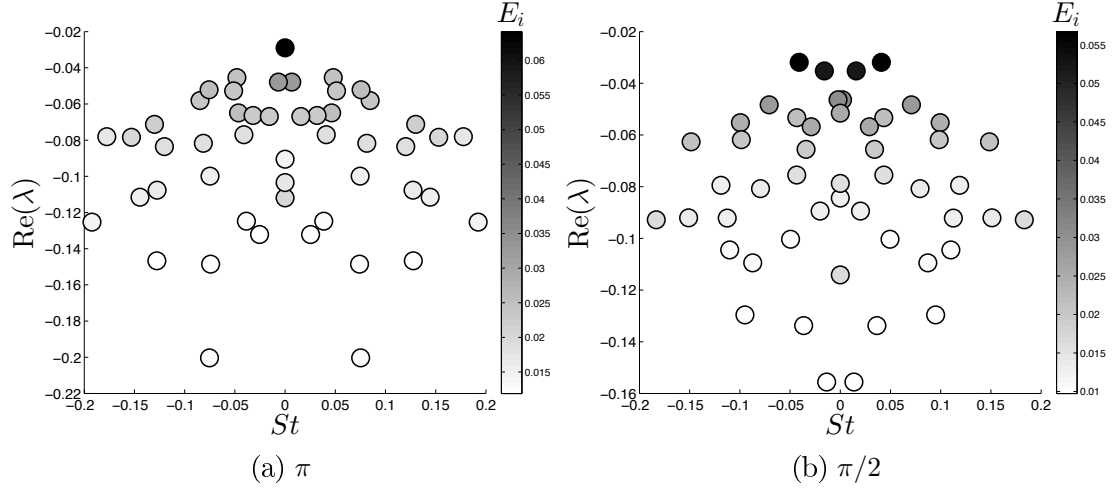


Figure 5.12: Eigenvalues of the least-squares matrix describing the dynamics of the pressure measurements for $\theta = \pi$ (a) and $\theta = \pi/2$ (b).

corresponding frequencies are $St = 0.018, 0.048, 0.045$, whilst for the structures presented in figure 5.11(f)-(h), the corresponding frequencies are $St = 0.048, 0.031, 0.128$. Both of those sets of values indicate that the most energetic POD modes represent structures with low-frequency dynamics.

Both eigenvalue plots in figures 5.12(a) and 5.12(b) show only one pair of eigenvalues with imaginary part in the neighbourhood of the shedding value $St \approx 0.2$. The corresponding eigenvectors are presented in figure 5.13. The two “shedding” structures corresponding to both orientations are considerably different, most likely a result of the susceptibility of the low-energy shedding structure to noise. Nonetheless, in all of the structures it is possible to see that the values of the highest magnitude are concentrated close to the centre of the object. Moreover, the main structures in the dynamic modes shown in figure 5.13(a)-(b) are elongated in the vertical direction whilst those visible in figure 5.13(c)-(d) have a larger horizontal lengthscale. This spatial characteristics might indicate that directional preferences for different θ also exist in high frequency structures.

5.3 Summary of the estimation methodologies

Section 5.2.1.1 demonstrated the existence of two coherent structures with corresponding frequencies $St \approx 0.07$ and $St \approx 0.2$. The objective of dynamic estimation will be to approximate the temporal evolution of both of these features using a small number of measurements.

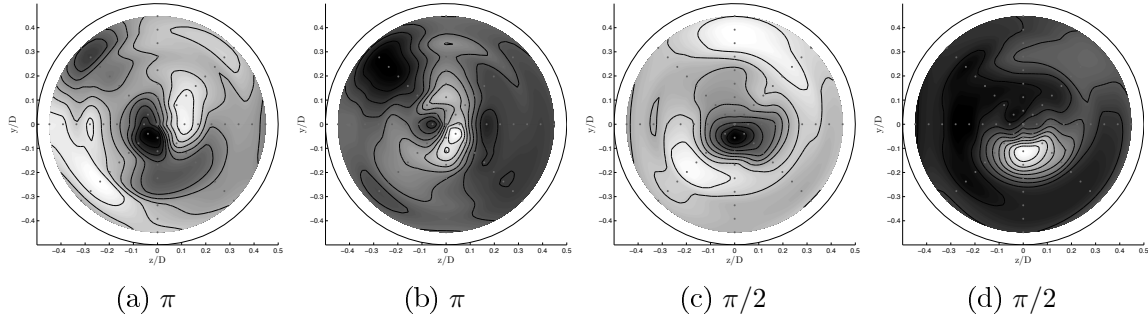


Figure 5.13: The eigenvectors corresponding to eigenvalues in figure 5.12 where $St \approx 0.2$. The real and imaginary parts of the structure corresponding to $\theta = \pi$ are respectively shown in (a) and (b), whilst the real and imaginary parts of the one corresponding to $\theta = \pi/2$ are respectively presented in (c) and (d)

Following a state space formulation of the system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}) + \mathbf{w} \quad (5.3a)$$

$$\mathbf{y} = \mathbf{C}(\mathbf{x}) + \mathbf{v}, \quad (5.3b)$$

the state space vector, $\mathbf{x}$, will be defined as coefficients corresponding to the projection of the velocity field onto the dynamic modes representing pulsation and shedding structures. The overall objective is to construct a measurement vector $\mathbf{y}$ and dynamic matrices, $\mathbf{A}$, $\mathbf{C}$, such that it is possible to solve an optimisation problem associated with (5.3) to obtain $\hat{\mathbf{x}}$ which approximates $\mathbf{x}$ accurately. The key aspects of the approaches used in the following sections are outlined first.

Since, the state space is defined as the projection of the velocity fields onto the selected structures, the choice of an appropriate mode calculation technique is especially important. Two options which were previously analysed in §5.2.1.1 are POD and OMD algorithms. Although the widely used POD methodology identifies structures which are statistically the most energetic, the associated power spectra presented in figure 5.9 show that the OMD modes isolate specific frequencies significantly better, where similar POD structures are associated with a mixture of low and high frequencies. Additionally, the OMD algorithm produces an output where each mode is associated with a growth rate and frequency and since we will choose the dynamics in (5.3) to be governed by linear finite dimensional operators such information will be helpful in the design of the dynamic matrices. Furthermore, it is noted that the reason the OMD algorithm is used over the DMD methodology is the additional flexibility of the first method in

selection of the low-dimensional subspace $\mathbf{L}$ over the latter algorithm where $\mathbf{L} = \text{POD}(\mathbf{u})$ [21].

The estimation of the state vector $\mathbf{x}$ is performed based on $\mathbf{y}$ consisting of a finite number of signals, which can be based on either velocity or pressure measurements. Due to low imprint of the shedding dynamics on base pressure measurements demonstrated in [26] and the lack of a robust shedding pressure structure confirmed in §5.2.1.3, an estimation problem based on velocity measurements is presented in §5.4 where it is shown that accurate estimates can be obtained in a straightforward manner. The results presented in §5.4 will be used as the accuracy benchmark.

Nonetheless, in real applications, acquiring velocity measurements in a non-intrusive manner is challenging, especially at the optimal locations identified in §5.4.2. Therefore, the more practically viable approach is to perform the estimation based on pressure measurements only. The construction of the measurement signal will be performed based on the 64 base pressure measurements and the objective is to construct a signal which will most closely resemble the evolution of the projection coefficients. A standard technique used for this purpose is LSE. However, it will be demonstrated in §5.5.1 that this methodology does not reconstruct well the frequencies in the neighbourhood of $St \approx 0.2$. Nonetheless, a test case from §5.5.1 will be used in §5.5.2 to obtain an insight into an appropriate construction of the dynamical model which puts the stronger emphasis on the features of interest. This step is vital to obtain a model which will provide accurate estimation results in the following sections where alternative forms of $\mathbf{y}$ are used.

The differences between pressure structures for different orientations of the shedding reflectional plane in figure 5.11, namely an azimuthal offset of $\pi/2$, show a clear three dimensionality of pressure dynamics and a hypothesis is put forward that not taking into account this directional bias in the construction of $\mathbf{y}$ significantly hinders the accuracy of the estimation. Two different approaches are subsequently analysed in §5.6, namely a rotation of the pressure measurements and construction of $\mathbf{y}$ as a function of azimuthal centre of pressure θ . In §5.6.1 a certain beneficial impact of pressure measurement rotation is presented. However, it is demonstrated that key aspects of the dynamics are eliminated through this approach. Finally, in §5.6.2 multiple pressure structures are calculated for different values of θ and the measurement signal $\mathbf{y}$ is obtained based on an indicator function. It is demonstrated that through a combi-

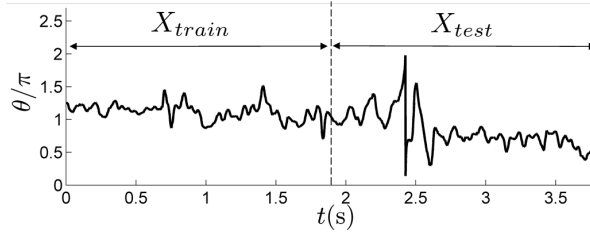


Figure 5.14: Evolution of the azimuthal coordinate of the centre of pressure in X_π .

nation of new $\mathbf{y}$ with the dynamical model constructed in §5.5.2, a reasonably accurate state estimate can be achieved.

5.4 Estimation using velocity probes

In this section dynamical estimation is performed using velocity measurements in order to obtain a benchmark for further comparisons. An analysis will be performed on the continuous data set $\mathbf{X}_\pi$, described in §5.2.1. Noting that an experimental run is 3.79s with data sampled at a frequency of 720 Hz, the data will be split into two equal sets, $\mathbf{X}_{train}$ and $\mathbf{X}_{test}$, as per figure 5.14. The set $\mathbf{X}_{train}$ will be used as a training set to extract dynamic modes and construct dynamical models, while the set $\mathbf{X}_{test}$ will be used to assess the robustness of the estimation technique. It can be seen that in $\mathbf{X}_{train}$ the azimuthal centre of pressure oscillates around 1.1π and switches its configuration to approximately 0.7π in $\mathbf{X}_{test}$ at $t \approx 2.5$. The main objectives of this analysis are to use a training data set, $\mathbf{X}_{train}$, to extract dynamical modes, construct a measurement vector $\mathbf{y}$ and dynamical models which will allow us to robustly estimate the switch in the state dynamics occurring at $t \approx 2.5$.

5.4.1 Dynamical modes

The first objective is to select structures of dynamical importance. For this reason the OMD algorithm is applied to $\mathbf{X}_{train}$ and based on the eigenvalue information and ranking of the OMD modes (§3.1) shedding and pulsation modes, Φ_{sh} and Φ_{pm} , are identified. The u -components of both dynamical modes are presented in figure 5.15(a)-(d). The same spatial features are visible as for the dynamical modes corresponding to $\theta = \pi$ in figure 5.5, albeit with slight symmetry distortions in Φ_{pm} (figure 5.15(c)-(d)), where structures for $y > 0$ have reduced intensity relative to those for $y < 0$. This is most likely related to the fact that on average for the data ensemble $\mathbf{X}_{train}$, $\theta = 1.1\pi$, leading to a slight divergence away from symmetries visible for $\theta = \pi$ in figure 5.4(b).

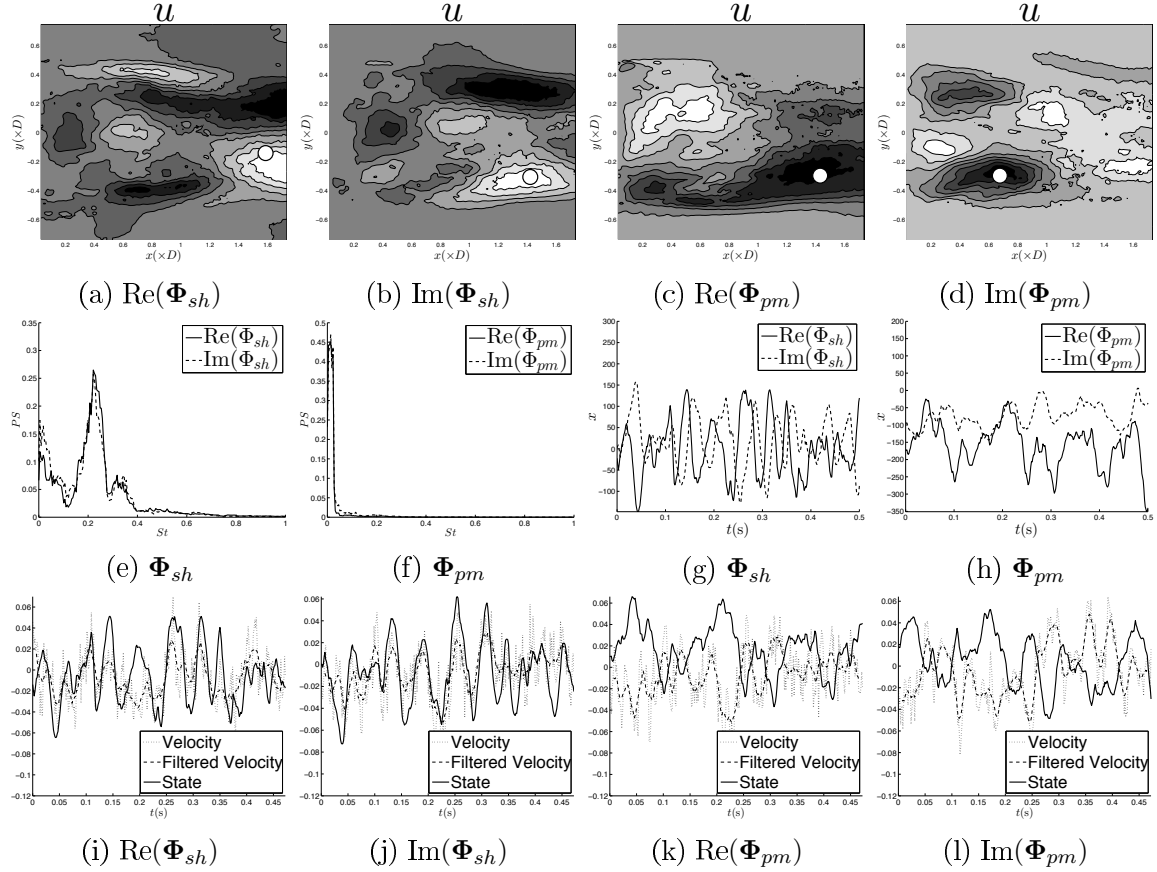


Figure 5.15: The velocity u -components of Φ_{sh} and Φ_{pm} are shown in (a)-(d). The power spectra of the corresponding projection coefficients are presented in (e)-(f), whilst their temporal evolutions are shown in (g)-(h). The locations, where the point correlations between projection coefficients and velocity measurements is maximised, is marked in (a)-(d) with the corresponding temporal evolutions shown in (i)-(l). The absolute values of the temporal correlations between state and velocity evolutions in (i)-(l) are correspondingly $|\rho| = 0.59, 0.64, 0.66, 0.61$.

Additionally, the power spectra and temporal evolution of the corresponding projection coefficients are presented in figure 5.15(e)-(h). The frequencies associated with both modes are confirmed based on the power spectra, where both real and imaginary part of Φ_{sh} show clear peaks around $St \approx 0.2$. On the other hands the analogous plots for Φ_{pm} confirm that this mode is primarily responsible for the dynamics dominated by frequencies an order of magnitude smaller. The temporal dynamics induced by Φ_{sh} in figure 5.15(g) shows a clear period of approximately $0.06s$ (which corresponds to $St \approx 0.22$), whilst the projection coefficients corresponding to Φ_{pm} oscillate at much a lower frequency.

5.4.2 Location of the velocity probes

Since the objective is to perform flow estimation using a small number of velocity measurements, it is crucial to select appropriate spatial locations at which the measurements are

acquired. The leading criterion is a similarity between the dynamics of the velocity at the chosen location and the dynamics of the corresponding states. One of the possible approaches is to analyse the point correlation of the velocities and state values throughout the domain (similar to figure 5.10) and to select the location based on the maximum correlation. The positions which maximise point correlations for each mode are shown in figure 5.15(a)-(d) and the temporal evolutions of the corresponding values of the velocities are presented in figure 5.15(i)-(l).

For all structures identified in §5.4.1, the optimal locations approximately coincide with the the positions where the values of the modes have the highest magnitudes. In each case, temporal evolution of the velocity measurements contains high-frequency oscillations, therefore, for visualisation purposes, the filtered velocity signal is also included in figure 5.15(i)-(l). The filtered velocity was obtained by taking an average of ten measurements at time instances centered around the plotted point. It is noted that despite a noisy signal, high-frequency oscillations can be suppressed by setting a large horizon window N_h in the MHE algorithm.

For the shedding dynamics, figure 5.15(i)-(j) shows that the chosen velocity signal accurately follows the dynamics of the projection coefficients in frequency; albeit with amplitude and phase mismatch at certain time instances. The accuracy of the velocity signal corresponding to low-frequency modes (figure 5.15(k)-(l)) is comparable to the shedding dynamic. However, both signals are π out of phase. Nonetheless, those phase and amplitude differences will be fixed through a construction of the appropriate dynamical model in §5.4.3.

5.4.3 Dynamical model

Analogous to the estimation of the cylinder flow the following form of the MHE optimisation is solved at every time instant

$$\min_{\substack{\{\mathbf{w}_{i-N_h}, \dots, \mathbf{w}_{i-1}\} \\ \{\mathbf{v}_{i-N_h}, \dots, \mathbf{v}_i\} \\ \mathbf{x}_{i-N_h}}} \gamma \sum_{j=i-N_h}^{i-1} \mathbf{w}_j^\top \mathbf{w}_j + (1-\gamma) \sum_{j=i-N_h}^i \mathbf{v}_j^\top \mathbf{v}_j, \quad (5.4a)$$

$$\text{s.t.} \quad \mathbf{x}_{i+1} = A(\mathbf{x}_i) + \mathbf{w}_i, \quad i = k - N_h, \dots, k - 1, \quad (5.4b)$$

$$\mathbf{y}_i = C(\mathbf{x}_i) + \mathbf{v}_i, \quad i = k - N_h, \dots, k. \quad (5.4c)$$

where $0 < \gamma < 1$ is a weighting factor between the two components of the objective function. The accuracy of the estimation techniques will be assessed through the analysis of the following three errors

$$\epsilon_x = \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2}, \quad (5.5)$$

$$\epsilon_X = \frac{\|\mathbf{X} - \hat{\mathbf{X}}\|_2}{\|\mathbf{X}\|_2}, \quad (5.6)$$

$$\epsilon_\theta = \frac{\|\text{Arg}(\hat{\mathbf{x}}_a) - \text{Arg}(\mathbf{x}_a)\|_2}{\|\text{Arg}(\mathbf{x}_a)\|_2}, \quad (5.7)$$

Here, $\mathbf{x}$ contains “true” state values, $\mathbf{X}$ contains power spectra associated with $\mathbf{x}$ and $\mathbf{x}_a$ is the analytical signal defined as $\mathbf{x}_a := \mathbf{x} + jH\mathbf{x}$, where $H\mathbf{x}$ denotes the Hilbert transform [89] of a signal $\mathbf{x}$, and $\text{Arg}(\mathbf{x}_a)$ represents the information about the instantaneous phase of $\mathbf{x}$. It is noted that phase discontinuities close to $\text{Arg}(\mathbf{x}_a) = 2\pi$ may lead to artificially large values of ϵ_θ at certain time instances. In order to prevent this behaviour continuous *unwrapped* phase was obtained, by adding multiples of $\pm 2\pi$ when the absolute difference between consecutive values of $\text{Arg}(\mathbf{x}_a)$ is greater than π . The states $\hat{\mathbf{x}}$, $\hat{\mathbf{X}}$ and $\hat{\mathbf{x}}_a$ have equivalent interpretations but correspond instead to the state estimates. The error statistics (5.5)-(5.7) will be referred to as point, spectral and phase errors respectively. The reason for the inclusion of all three types of statistics is due to the fact that each of them reflects different features of the estimation problem.

The most widely used, ϵ_x , for example, strongly penalises any phase mismatch between reference and estimated values, i.e. if the frequency and amplitude of the oscillation is estimated correctly but the estimate is out of phase, the statistic ϵ_x will be large, and estimates with a suppressed amplitude will, according to this measure, appear to approximate the dynamics more accurately. On the other hand, the spectral error ϵ_X will not cost any phase mismatch between the estimates and reference signals, while the value of ϵ_θ is not affected by any discrepancies in the energetic content of both signals.

In the following analysis the state space is defined as a projection onto the OMD modes. However, since those structures are complex, an extended a four-dimensional state space formulation is used, i.e.

$$\mathbf{x}_i := [\text{Re}(\Phi_{sh}), \text{Im}(\Phi_{sh}), \text{Re}(\Phi_{pm}), \text{Im}(\Phi_{pm})]^\top \mathbf{u}_i \in \mathbb{R}^4. \quad (5.8)$$

The dynamical constraints (5.4b)-(5.4c) take a linear finite dimensional form and are represented using matrices $\mathbf{A} \in \mathbb{R}^{r \times r}$ and $\mathbf{C} \in \mathbb{R}^{N_y \times r}$ such that

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i + \mathbf{w}_i, \quad (5.9a)$$

$$\mathbf{y}_i = \mathbf{C}\mathbf{x}_i + \mathbf{v}_i. \quad (5.9b)$$

where $\mathbf{x}_i \in \mathbb{R}^r$, $\mathbf{y}_i \in \mathbb{R}^{N_y}$ and r is the dimension of state space while N_y represents the number of measurements. Since it was established that both Φ_{sh} and Φ_{pm} correspond to oscillatory dynamics at certain frequencies, an oscillator-like form of the dynamic system (5.9a) can be used. The approximate values of the frequencies can be established from the corresponding OMD eigenvalues or from the power spectra of the projection coefficients on $\mathbf{X}_{train}$ shown in figure 5.15(e)-(f), however since the spectral peaks are relatively broad both the frequency and growth rates of the oscillations are implemented in the least-squares optimisation problem. The dynamic matrix takes the form $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}$, where each $\mathbf{A}_j \in \mathbb{R}^{2 \times 2}$. The matrix $\mathbf{A}$ is obtained as a solution to

$$\min_{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4} \sum_{i \in \mathcal{X}_{train}} \left\| \mathbf{x}_{i+1} - \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} \mathbf{x}_i \right\|_2^2 \quad (5.10a)$$

$$\text{s.t. } \mathbf{A}_1^\top = -\mathbf{A}_1 \quad (5.10b)$$

$$\mathbf{A}_4^\top = -\mathbf{A}_4. \quad (5.10c)$$

In this way both $\mathbf{A}_1$ and $\mathbf{A}_4$ will represent oscillatory matrices with frequencies and growth rates determined from the optimisation procedure, whilst $\mathbf{A}_2$ and $\mathbf{A}_3$ describe the coupling between shedding and low-frequency dynamics.

The measurement vector $\mathbf{y}$ consists of four velocity measurements at the locations visible in figure 5.15(a)-(d). In order to estimate a linear relationship between the state $\mathbf{x}$ and measurements $\mathbf{y}$, a least-squares problem is solved

$$\min_{\mathbf{C}} \sum_{i \in \mathcal{X}_{train}} \|\mathbf{y}_i - \mathbf{C}\mathbf{x}_i\|_2^2, \quad (5.11)$$

where $\mathbf{C} \in \mathbb{R}^{4 \times 4}$, since each entry in the state vector $\mathbf{x}_i$ is associated with independent measurement signal. It is noted that there are no constraints imposed in (5.11) allowing for a coupling between different dynamics, e.g. some information about the low-frequency state can be obtained from signals corresponding to high-frequency dynamics and vice versa.

It is noted that since the measurement vector $\mathbf{y} \in \mathbb{R}^4$ is of exactly the same dimension as $\mathbf{x}$, it is necessary to explain why $\hat{\mathbf{x}}$ cannot be simply approximated through an inversion of matrix $\mathbf{C}$, such that $\hat{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{y}$. Indeed, such a procedure is equivalent to LSE in which the velocity measurements chosen in figure 5.15 are used as variables. Examples of state estimation results where $\tilde{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{y}$ are presented for $\text{Re}(\mathbf{x}_{sh})$ and $\text{Re}(\mathbf{x}_{pm})$ in figure 5.16. The temporal evolution of the state estimates $\tilde{\mathbf{x}}$ in figure 5.16(a)-(b) show a high magnitude of noise. Additionally, the corresponding power spectra in figure 5.16(c)-(d) are broad and the spectral amplitudes associated with frequencies of interest are significantly suppressed. It is observed that the power spectrum for the reference shedding projection coefficient $\text{Re}(\mathbf{x}_{sh})$ contains a low-frequency peak not visible in the previous analysis (figure 5.9). For the particular data set used for estimation, $\mathbf{X}_{test}$, this feature is a result of the change in the value of θ around $t = 2.5$ (figure 5.14) which is reflected in the offset of the mean value of the shedding coefficients necessarily inducing a low-frequency peak in the power spectrum.

Since the results in figure 5.16 demonstrate that the measurements $\mathbf{y}$ contain inaccuracies which prevent direct calculation of $\hat{\mathbf{x}}$ through inversion of matrix $\mathbf{C}$, we consider application of MHE. A suitable configuration of the MHE problem (5.4) is needed through an appropriate selection of the MHE horizon window N_h and the weighting coefficient γ . For this purpose, the MHE estimation is performed on $\mathbf{X}_{train}$ for $\gamma \in [0.01, 0.02, \dots, 0.99]$ and $N_h \in [1, 2, \dots, 30]$ and the errors (5.5)-(5.6) are analysed. The presentation of the phase error ϵ_θ is omitted because for analysed γ and N_h its value stays approximately constant. The contour plots of those values for a range of γ and N_h are shown in figure 5.17.

Both error measures decrease for small values of γ , since if a large γ is set in (5.4) the estimate is constrained to follow the prescribed linear dynamics, not allowing for nonlinear changes in amplitude and phase of $\mathbf{x}$ visible in figure 5.16(a)-(b) or response to observed measurements. It is noted that despite the fact that both ϵ_x and ϵ_X are minimised for small values of γ , if this quantity is set to zero, the LSE estimates shown in figure 5.16 are retained. Additionally, figure

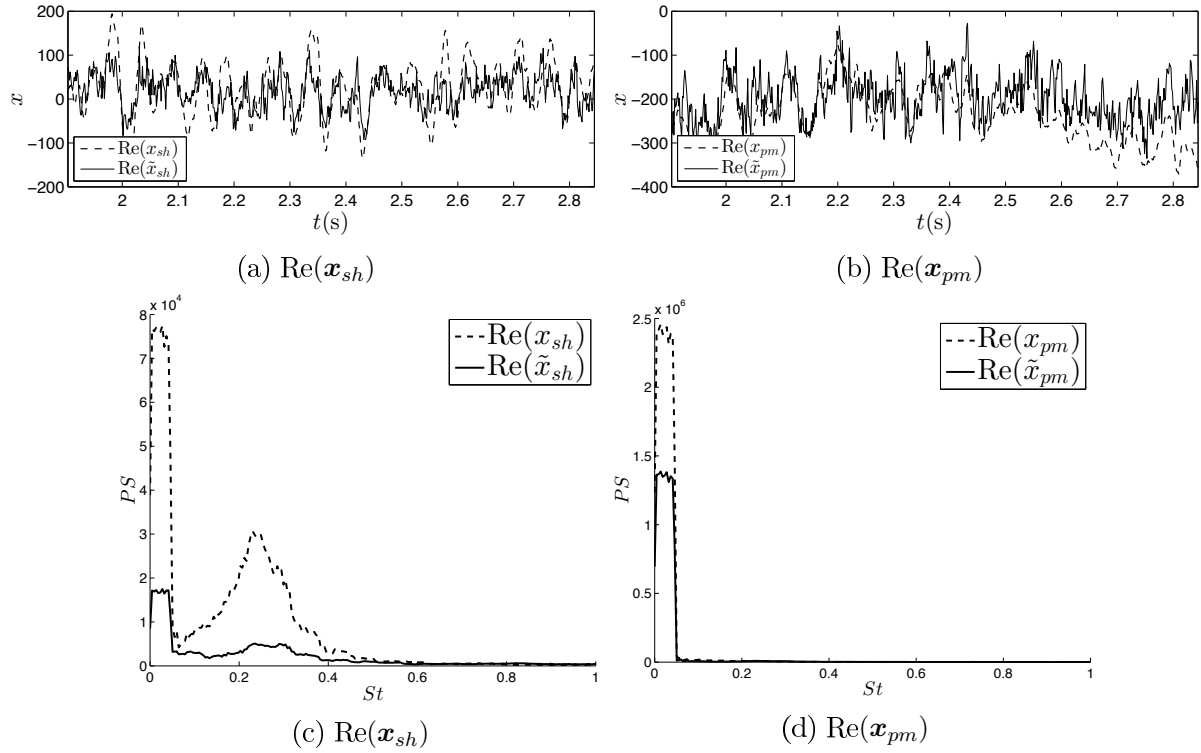


Figure 5.16: Temporal evolution of the reference states, $\mathbf{x}_i$, and the state estimates $\tilde{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{y}$ ((a)-(b)) together with the corresponding power spectra ((c)-(d)).

5.17(a) demonstrates that ϵ_x decreases at large values of N_h for which high-frequency noise is suppressed. It is noted that there is a large area in figure 5.17(a) where ϵ_x is small and does not change significantly for slight variations in γ and N_h and since it coincides with the values at which ϵ_X is minimised ($N_h = 30$, $\gamma = 0.04$) these parameters will be used in the following analysis.

5.4.4 Dynamical estimation results

The model outlined in §5.4.3 is used to perform dynamical estimation on dataset X_{test} , which does not contain any snapshots used for construction of the model. The temporal evolution of both $\mathbf{x}_i$ and $\hat{\mathbf{x}}_i$ are shown in figure 5.18 while the corresponding power spectra are presented in figure 5.19. Additionally, the comparison between the error statistics of the MHE estimation and LSE results in figure 5.16 is shown in table 5.1. It can be seen that the utilisation of the velocity measurements provides accurate estimations of both shedding and low-frequency dynamics. Figure 5.18(a)-(d) shows that the phase of both shedding and pulsation oscillations is, on average, predicted accurately. An accurate estimation of the dominant frequencies is confirmed in the corresponding power spectra in figure 5.19 where the values of the highest

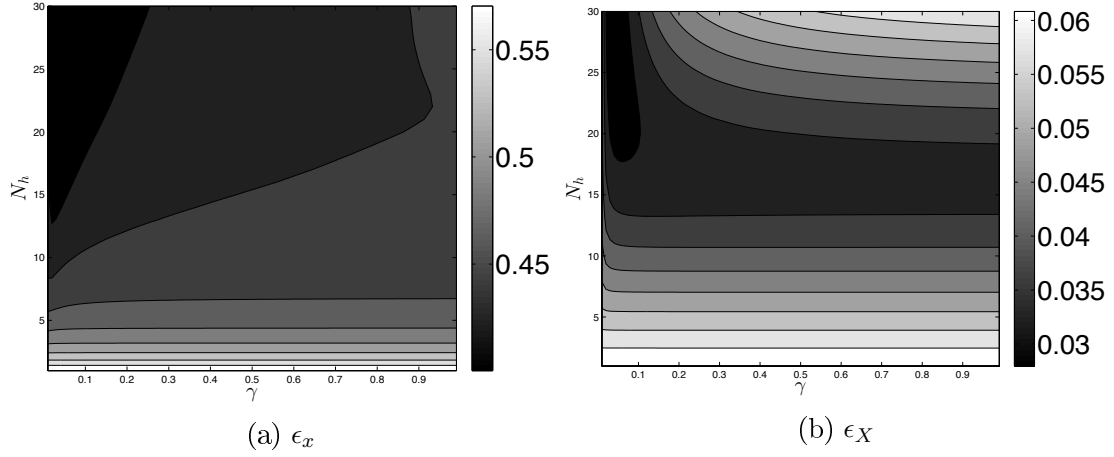


Figure 5.17: Values of ϵ_x (a) and ϵ_X (b) estimated for $\gamma \in [0.01, 0.02, \dots, 0.99]$ and $N_h \in [1, 2, \dots, 30]$.

magnitudes coincide. It is noted that for the estimated shedding spectra the most prominent magnitudes are smaller due to the suppressed amplitude of the shedding oscillations in $\hat{x}_{sh}$.

The observations above are reflected in the error statistics in table 5.1 which shows that the MHE algorithm improves the accuracy of the spectral estimation by suppressing the values corresponding to high-frequency oscillations and making the shedding magnitude more prominent. At the same time the point statistics remain comparable to the LSE estimates and improve for $\text{Im}(\hat{\mathbf{x}}_{sh})$ and $\text{Re}(\hat{\mathbf{x}}_{pm})$.

It is noted that there is a visible offset in the mean values of $\mathbf{x}$ at around $t = 2.6$ apparent in both $\mathbf{x}_{sh}$ and $\mathbf{x}_{pm}$ in figure 5.14 related to the change in the orientation of the shedding reflectional plane. In all cases apart from $\text{Im}(\mathbf{x}_{pm})$ this offset is underestimated by the MHE, as visible in the temporal evolution and further confirmed in figure 5.19(a)-(c) where low-frequency peaks corresponding to state estimates have lower magnitude compared to the reference values. This mismatch between velocity measurements and reference states suggests that the measurement locations estimated on $\mathbf{X}_{train}$ do not capture the corresponding dynamics as accurately for different values of the azimuthal centre of pressure θ .

5.5 Estimation using pressure measurements

5.5.1 LSE of pressure

The MHE results presented in §5.4.4 demonstrate that state values corresponding to key dynamics can be estimated accurately using velocity measurements. It is, however, reiterated that the acquisition of such measurements is not practically viable, instead motivating the use

Point Error, ϵ_x				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig. 5.16)	0.73	0.89	0.30	0.62
MHE (fig. 5.18)	0.76	0.83	0.23	0.66
Spectral Error, ϵ_X				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig. 5.16)	0.80	0.97	0.44	0.72
MHE (fig. 5.19)	0.71	0.80	0.31	0.47
Phase Error, ϵ_θ				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig. 5.16)	0.69	0.69	0.08	0.76
MHE (fig. 5.18)	0.67	0.70	0.08	0.69

Table 5.1: Error statistics for all entries in the state estimate for the LSE estimate when $\mathbf{y}$ consists of the velocity measurement, $\hat{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{y}$, and when $\hat{\mathbf{x}}$ was obtained via the MHE algorithm using four velocity measurements.

of base pressure measurements. Here, LSE (outlined in §2.2.1) of all entries in the state vector $\mathbf{x}_i$ from pressure measurements is attempted first.

Recall that the base of the bullet model is mounted with 64 pressure sensors, equally spaced radially, and separated by an angle of $\theta = \pi/4$. The pressure dynamics at each location exhibit primary oscillations at frequencies an order of magnitude lower than that of the shedding. In order to isolate differing dynamical features, LSE estimation is performed on the projection coefficients $\mathbf{c}_j^p = (\Phi_j^p)^\top \mathbf{p}$ of pressure POD modes $\Phi_j^p = \text{POD}(\mathbf{p})$. This approach separates pressure into structures which have different frequency content. It is observed that the first 10 pressure coefficients have primarily low-frequency content similar to the dynamics associated with Φ_{pm} , whilst only the relatively low-ranked pressure modes, i.e. Φ_j^p where $j > 20$, contain power spectrum in which the most dominant frequencies are close to $St_{sh} \approx 0.2$. The power spectra corresponding to a subset of pressure POD modes demonstrating this behaviour are presented in figure 5.20. Note the order of magnitude change in the value at the most dominant frequencies in the spectrum corresponding to Φ_{25}^p , further confirming a low-energy imprint of the shedding dynamics on the base pressure.

An important consideration concerns the robustness of the LSE estimates outside of the training domain. In this case, we can either underfit or overfit our data, i.e. if not enough POD coefficients $\mathbf{c}_j^p$ are used, the constructed model may not fit the data well enough, but if too many $\mathbf{c}_j^p$ are used, we can obtain a perfect fit in the training domain but may get large

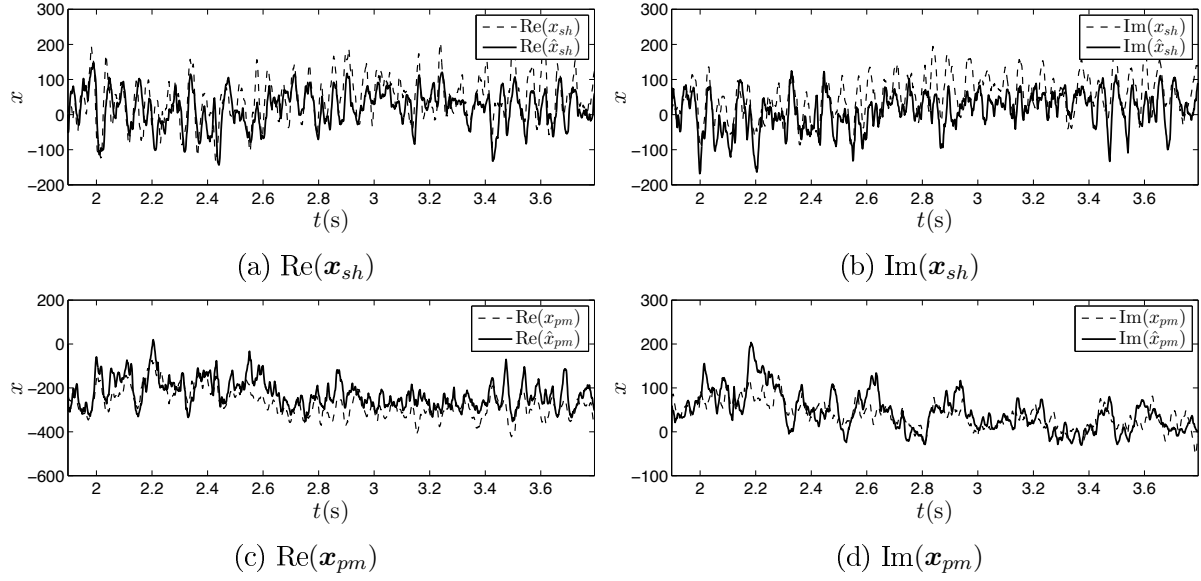


Figure 5.18: Temporal evolutions of the reference states, $\mathbf{x}_i$, and the state estimates $\hat{\mathbf{x}}_i$ obtained through MHE using four velocity measurements.

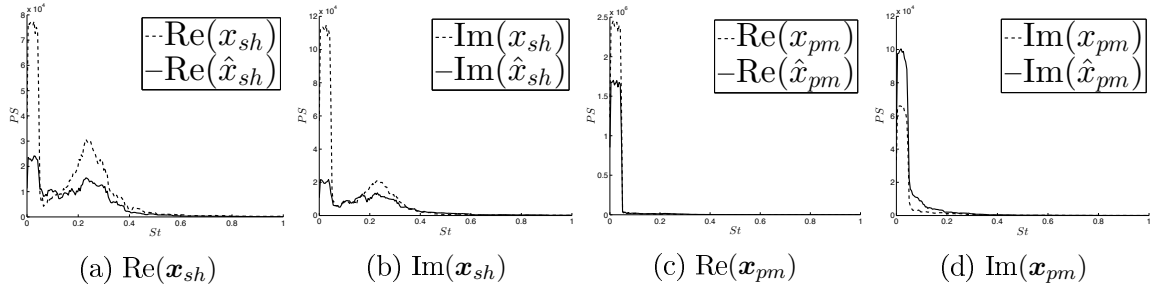


Figure 5.19: The power spectra corresponding to results presented in figure 5.18.

estimation error outside of it. In order to prevent the latter behaviour the number of variables will be reduced by retaining only a certain subset of $\mathbf{c}_j^p$ based on the similarity of power spectra of each $\mathbf{c}_j^p$ to the power spectra of the state $\mathbf{x}$. For this purpose, a power spectrum of each $\mathbf{x}_j$, $(\mathbf{X}_x)_j$, and of each $\mathbf{c}_j^p$, $(\mathbf{X}_c)_j$, is computed using the multitaper method [98] and the following correlations are analysed

$$\rho_{jk} = \frac{(\mathbf{X}_x)_j^\top (\mathbf{X}_c)_k}{\|(\mathbf{X}_x)_j\|_2 \|(\mathbf{X}_c)_k\|_2}. \quad (5.12)$$

A threshold, $\beta < 1$, is set and LSE minimisation will be solved only using coefficients corresponding to pressure POD modes, $\mathbf{c}_l^p$, where $l \in \{k \in \mathbb{N} : \rho_{jk} > \beta\}$. It is reiterated that $\mathbf{c}_l^p$ retained in this way will have similar frequency content to at least one of the states, i.e. contain either low frequencies or frequencies in the neighbourhood of the shedding value and all other pressure modes are consequently discarded. In the following analysis $\beta = 0.75$ retains 27 out of 64 pressure POD coefficients.

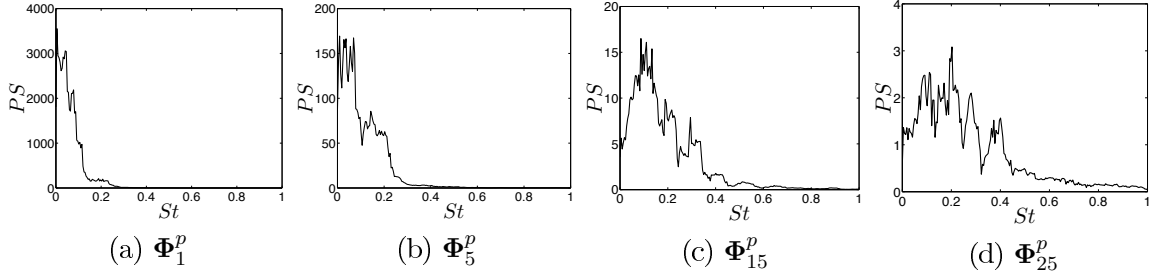


Figure 5.20: Power spectra corresponding to a subset of pressure POD modes Φ^p .

Subsequently all of the retained pressure projection coefficients are assembled into a single matrix $\mathbf{c}_p$ where each row corresponds to the temporal evolution of the retained c_l^p . In order to obtain LSE coefficients which estimate entries in the state $\mathbf{x}$ the following optimisation problem is solved

$$\min_{\Psi_j} \sum_{i \in \mathcal{X}_{train}} \|(x_j)_i - \Psi_j^\top (\mathbf{c}_p)_i\|_2^2, \quad (5.13)$$

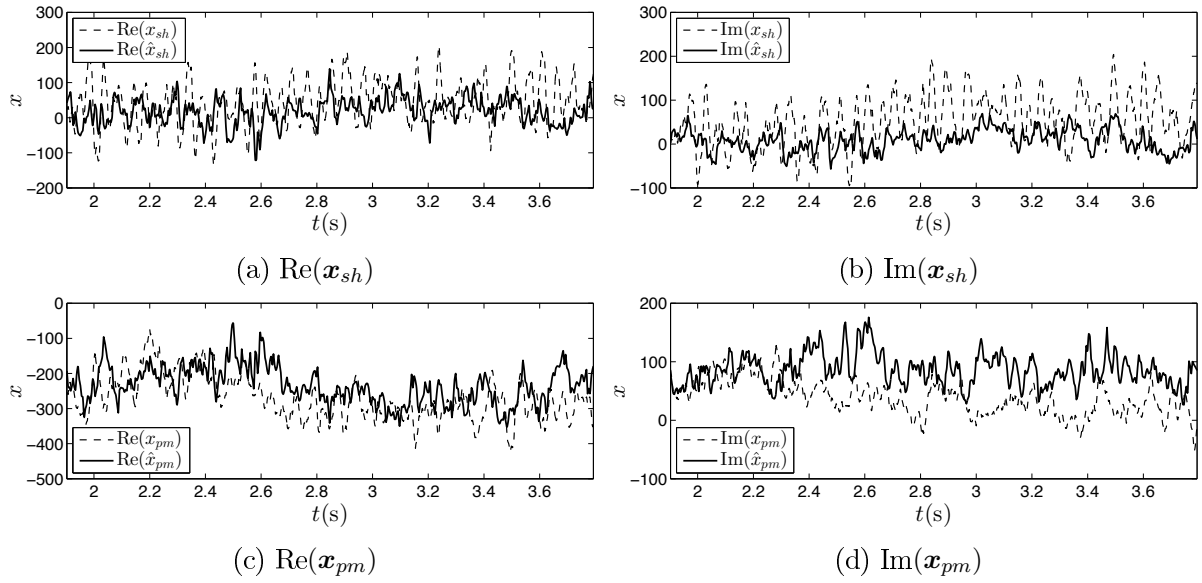
for each $x_j \in \mathbf{x}$, where $\Psi_j \in \mathbb{R}^{N_p^*}$ and N_p^* is the number of retained POD coefficients. The solution contains the LSE coefficients approximating different state entries, giving in total four different vectors Ψ_j . Subsequently, Ψ_j is mapped to the space $\mathbb{R}^{64}$ such that $\phi_j = \tilde{\Phi}_p \Psi_j$ where $\tilde{\Phi}_p \in \mathbb{R}^{64 \times N_p}$ consists of the retained POD modes, i.e. $\Phi_l^p \in \tilde{\Phi}_p$ if $l \in \{k \in \mathbb{N} : \rho_{jk} > \beta\}$ where ρ_{jk} defined as in (5.12). Due to the four-dimensional state space formulation, four different structures ϕ_j are obtained, which can be assembled to form $\Phi_{LSE} = [\phi_1, \dots, \phi_4] \in \mathbb{R}^{64 \times 4}$. The final LSE estimate $\hat{\mathbf{x}}_{LSE}$ is then defined as

$$\hat{\mathbf{x}}_{LSE} = \Phi_{LSE}^\top \mathbf{p}. \quad (5.14)$$

The LSE coefficients between pressure measurements and state vectors were calculated using data corresponding $\mathbf{X}_{train}$ and the result was subsequently applied to $\mathbf{X}_{test}$ to assess the robustness of the methodology. Figure 5.21 presents the estimates $\hat{\mathbf{x}}$ of $\mathbf{x}$ on $\mathbf{X}_{test}$, when $\hat{\mathbf{x}}$ is obtained using LSE. Additionally, the corresponding power spectra are shown in figure 5.22.

Both of the figures show that only $\text{Re}(\mathbf{x}_{pm})$ is estimated with any degree of accuracy. Both values corresponding to the shedding state show a clear mismatch in frequency and amplitude of the oscillations in $\hat{\mathbf{x}}_{sh}$ when compared to $\mathbf{x}_{sh}$. This is especially visible in figure 5.22(a) – (b) where the power spectrum of the estimate around the shedding value is significantly suppressed. Another important dynamical feature visible in the evolution of $\mathbf{x}$ on $\mathbf{X}_{test}$ is the change of

Point Error, ϵ_x				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig.5.21)	0.96	0.93	0.25	3.08
Spectral Error, ϵ_X				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig.5.22)	0.78	0.93	0.25	3.08
Phase Error, ϵ_θ				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig.5.21)	0.80	0.78	0.09	0.98

Table 5.2: Error statistics for all entries in the state estimate $\hat{\mathbf{x}}$ when $\hat{\mathbf{x}} = \Phi_{LSE}^\top \mathbf{p}$.Figure 5.21: Temporal evolutions of the reference states, $\mathbf{x}_i$, and the state estimates $\hat{\mathbf{x}}_i$ obtained through LSE between state and pressure values.

the mean value for $t > 2.5$ due to different reorientation of the shedding plane, which was only predicted for $\text{Re}(\hat{\mathbf{x}}_{pm})$ and underestimated for all the other state entries. This behaviour is reflected in the power spectra in figure 5.22, where low-frequency values associated with this change are substantially smaller for $\hat{\mathbf{x}}$. The error statistics corresponding to the results presented in figures 5.21-5.22 are shown in table 5.2. A comparison against the error metrics corresponding to the MHE estimation using four velocity measurements in table 5.1 shows a decrease in the estimation accuracy for almost all error values.

Furthermore, analogous estimates were also obtained using quadratic stochastic estimation

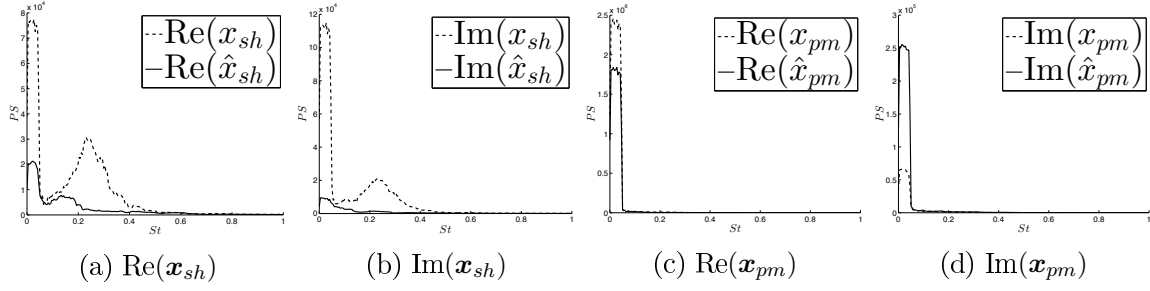


Figure 5.22: The power spectra corresponding to results presented in figure 5.21.

(QSE) with an l^1 -norm penalty in the corresponding optimisation problems

$$\min_{A_{jk}, B_{jkl}} \sum_{i \in \mathcal{X}_{train}} \|(x_j)_i - A_{jk}(c_k^p)_i - B_{jkl}(c_k^p)_i(c_l^p)_i\|_2 + \gamma \left(\sum_k |A_{jk}| + \sum_{k,l} |B_{jkl}| \right), \quad (5.15)$$

where γ is a weighting coefficient. Nonetheless, the analysis demonstrated a difficulty in selecting an appropriate value of γ . If γ is set too low, the estimated state has a large error due to overfit on the training domain, while if γ is set too large the algorithm outputs a state estimate $\hat{\mathbf{x}}$ which is primarily characterised by low frequencies. For the analysed range, $\gamma = 60$ provided the results minimising the error metrics, but even then those values were larger than for the LSE results and we do not report them here.

5.5.2 Moving Horizon Estimation

The MHE estimation methodology is used to investigate if the LSE approximation presented in §5.5.1 can be improved through the inclusion of the dynamical model similar to the one used in §5.4.3. The difference from the approach in §5.4.4 is that the measurement signal $\mathbf{y}$ will take now the form of the LSE estimates demonstrated in §5.5.1, i.e. $\mathbf{y} = \Phi_{LSE}^\top \mathbf{p}$. Noting that here pressure measurements are used instead of velocity values, the appropriate choice of the MHE parameters, γ and N_h , must be re-evaluated. Analysis of the spectral characteristics of the LSE approximations in figure 5.22(a)-(b) shows that the power spectra corresponding to $\hat{\mathbf{x}}_{sh}$ do not exhibit a characteristic shedding frequency peak at $St \approx 0.23$ and that the highest magnitudes are contained in low-frequencies. In order to prevent this behaviour in the MHE estimation, $\gamma = 0.9$ is set to place greater emphasis on the oscillatory dynamical model. Furthermore, although of relatively low magnitude, the temporal evolution of $\hat{\mathbf{x}}_{pm}$ in figure 5.21(c)-(d) contains visible high frequency oscillations not present in the reference state $\mathbf{x}_{pm}$. This dynamical feature can be suppressed by setting a relatively large value of the moving

horizon window, e.g. $N_h = 50$.

Initially, the MHE analysis was performed using the same dynamical model as in §5.4.4, i.e. matrices $\mathbf{A}$ and $\mathbf{C}$ were set to the solutions of (5.10) and (5.11) on $\mathbf{X}_{train}$. However, it was observed that the dominant frequency in the power spectra corresponding to shedding estimate $\hat{\mathbf{x}}_{sh}$ was lower than that of the reference value and lower than the value corresponding to the eigenvalues of the shedding oscillator $\mathbf{A}_1$ in (5.10). The most likely reason for this behaviour is the coupling between shedding and low-frequency dynamics in the model and the smoothing effects of the horizon window N_h . Consequently, the optimisation problem (5.10) is modified such that matrix $\mathbf{A}$ is instead obtained as a solution to

$$\min_{\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4} \sum_{i \in \mathcal{X}_{train}} \left\| \mathbf{x}_{i+1} - \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} \mathbf{x}_i \right\|_2^2, \quad (5.16a)$$

$$\text{s.t. } \mathbf{A}_4^\top = -\mathbf{A}_4, \quad (5.16b)$$

$$\mathbf{A}_1 = \exp \begin{bmatrix} 0 & -\lambda_{sh} \\ \lambda_{sh} & 0 \end{bmatrix}, \quad (5.16c)$$

where $\lambda_{sh} = \frac{2\pi\delta t U_\infty}{D}(St_{ref} + \Delta St)$ such that $St_{ref} = 0.23$ is a target shedding frequency and ΔSt is a parameter introduced to compensate for the low-frequency bias in the shedding estimates. The diagonal terms of $\mathbf{A}_1$ in (5.16c) are set to zero which corresponds to non-decaying dynamics.

In order to investigate the appropriate choice of ΔSt , the MHE estimation results are obtained using $N_h \in [5, 75]$ for a range of $\Delta St \in [0, 0.002, \dots, 0.4]$. Subsequently the spectral error, ϵ_X , of $\text{Re}(\hat{\mathbf{x}}_{sh})$ and $\text{Im}(\hat{\mathbf{x}}_{sh})$ is investigated in the neighbourhood of the shedding value, $St \in [0.045, 0.45]$, selected based on the width of the spectral peaks in figure 5.22 to check what value of parameter ΔSt minimises this error for a given N_h . The corresponding plot for a range of values of N_h is presented in figure 5.23. It can be seen that the value of ΔSt decreases steadily with the increasing N_h . Furthermore, it is noted that ϵ_X corresponding to $\hat{\mathbf{x}}_{sh}$ for the optimal ΔSt also decreases with increasing N_h .

MHE estimation is performed for $N_h = 50$, $\gamma = 0.9$, where $\mathbf{A}$ is a solution to (5.16) for $\Delta St = 0.038$ based on the results presented in figure 5.23. Additionally, $\mathbf{y} = \Phi_{LSE}^\top \mathbf{p}$ and $\mathbf{C}$ is a solution to (5.11). The temporal evolution of the MHE estimates are presented in figure 5.24 and the associated power spectra are shown in figure 5.25. The error statistics associated with

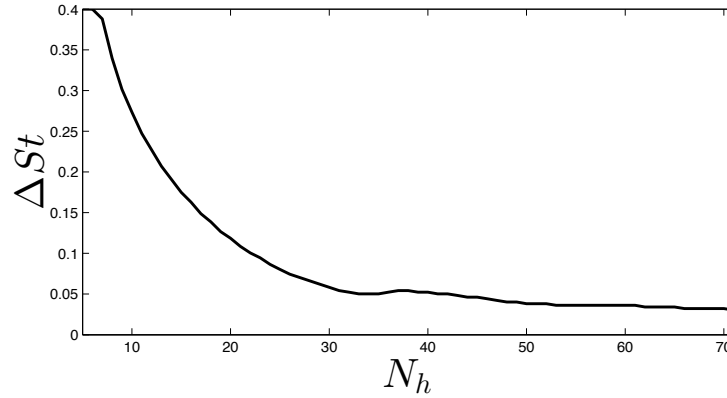


Figure 5.23: The value of ΔSt used to modify $\mathbf{A}$ at which ϵ_X corresponding to $\text{Re}(\hat{\mathbf{x}}_{sh})$ is minimised for a given N_h .

those results are presented in table 5.3.

The presented results show an improved accuracy in the spectral characteristics of the estimates. The power spectra corresponding to shedding estimates in figure 5.25(a)-(b) show visible peaks in the region of the shedding frequency; a feature which was not visible in the LSE estimates in figure 5.22(a)-(b). Even though table 5.3 indicates an increase in the value of ϵ_X corresponding to $\text{Im}(\mathbf{x}_{sh})$, this is due to differences seen in the estimation of the low-frequency spectral peak, while shedding frequencies are estimated more accurately. It is also noted that the MHE results induce a larger point error, ϵ_x , than corresponding LSE results, because of the suppressed amplitude of the shedding estimates in the latter case (as can be seen in figures 5.21 and 5.24). Since table 5.3 shows that in both LSE and MHE estimation the phase mismatch is comparably high, a lower amplitude of the estimate gives a preferential value of ϵ_x .

Furthermore, the inclusion of the dynamical model improves the accuracy of the estimate $\hat{\mathbf{x}}_{pm}$ in the region where the the system orientation is converged onto the values of θ , i.e. where $\theta \approx 1.1\pi$ for $t < 2.5$ and $\theta \approx 0.7\pi$ for $t > 2.8$, by suppressing high frequency oscillations and following the mean value of the reference results more closely. Unfortunately, there is a considerably larger error in the MHE estimation at $t \approx 2.6$ when θ changes its value, which is reflected in the larger values of ϵ_x in table 5.3.

Although results presented in figure 5.24 show that it is possible to improve the accuracy of the dynamic estimates based on the pressure measurements through an appropriate choice of the model, there are certain problems which must be addressed. First is the fact that the change in θ in $\mathbf{X}_{train}$ visible in figure 5.14 is not large and the estimation accuracy is likely to deteriorate for larger disparities from $\theta = \pi$. Additionally, despite high accuracy in the

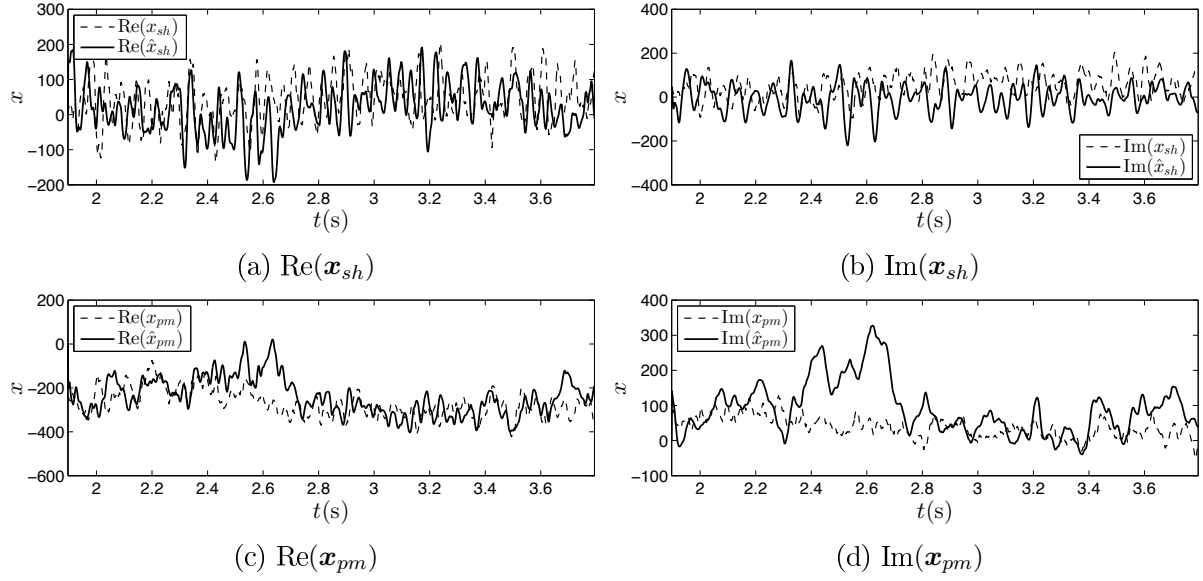


Figure 5.24: Temporal evolutions of the reference states, $\mathbf{x}_i$, and the state estimates $\hat{\mathbf{x}}_i$ obtained through MHE estimation where $\mathbf{y}$ is constructed based on the pressure measurements and dynamic matrix $\mathbf{A}$ is a solution to (5.16).

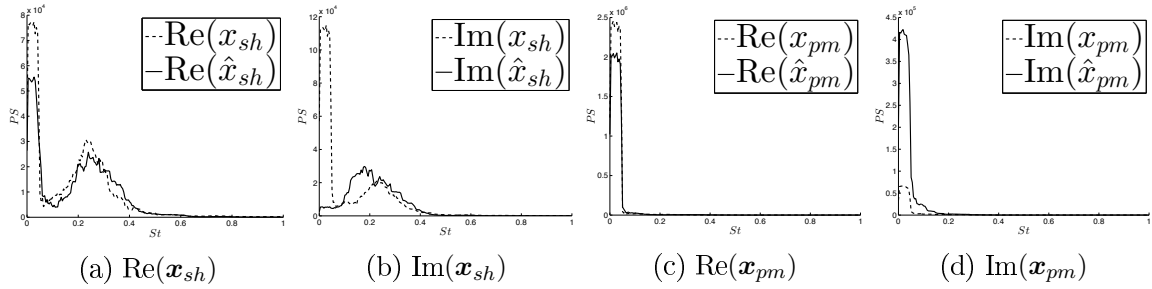


Figure 5.25: The power spectra corresponding to results presented in figure 5.24.

estimation of the power spectrum there exists a persistent non-constant phase mismatch in the shedding dynamics visible in figure 5.24(a)-(b). In the following sections an analysis is performed to assess whether it is possible to construct a measurement vector $\mathbf{y}$ which will allow for more accurate estimation in terms of phase of the oscillations.

5.6 Directional bias of the pressure measurements

5.6.1 Pressure rotation analysis

Figure 5.11 in §5.2.1.3 showed that the major structures in the base pressure measurements exhibit a clear directional bias which depends on a system orientation. Here, it is demonstrated that this feature of the system needs to be taken into account in the dynamical estimation. For this purpose, the dynamical model and LSE coefficients Φ_{LSE} constructed for $\theta \approx \pi$ and used to obtain MHE estimates in figure 5.24 are applied to $\mathbf{X}_{\pi/2}$ shown in figure 5.3(b) where $\theta \approx \pi/2$.

Point Error, ϵ_x				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig.5.21)	0.96	0.93	0.26	1.29
MHE (fig.5.24)	1.12	1.29	0.30	1.87
Spectral Error, ϵ_X				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig.5.22)	0.78	0.93	0.25	3.08
MHE (fig.5.25)	0.39	0.96	0.17	5.33
Phase Error, ϵ_θ				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
LSE (fig.5.21)	0.80	0.76	0.09	0.98
MHE (fig.5.24)	0.73	0.80	0.12	0.86

Table 5.3: Error statistics for all entries in the state estimate $\hat{\mathbf{x}}$ corresponding to different estimation methodologies.

Additionally, the results are compared against the MHE estimation which used $\mathbf{y} = \Phi_{LSE}^\top \tilde{\mathbf{p}}$ where $\tilde{\mathbf{p}}$ corresponds to pressure measurements rotated by $\pi/2$ so that azimuthal centre of pressure of $\tilde{\mathbf{p}}$ is consistent with the data used for the model construction.

Pressure measurements are rotated in the following manner. The matrix denoting the location of each pressure measurement in Cartesian coordinates, $\mathbf{Y}$, and a reference azimuthal direction, $\theta_{ref} = \pi$, are defined. The rotation angle, $\delta\theta$, is set to $\delta\theta = \pi/2$ and a corresponding rotation matrix $\mathbf{R}(\delta\theta)$ is constructed such that

$$\mathbf{R}(\delta\theta) = \begin{bmatrix} \cos(\delta\theta) & -\sin(\delta\theta) \\ \sin(\delta\theta) & \cos(\delta\theta) \end{bmatrix}. \quad (5.17)$$

A new set of coordinates corresponding to the rotated pressure measurements is obtained as $\mathbf{Y}^* = \mathbf{Y}\mathbf{R}$. Since the measurement signal $\mathbf{y}$ is calculated by projection of $\tilde{\mathbf{p}}_i$ onto a single static structure where each entry corresponds to a specific coordinate, it is necessary to map pressure values at $\mathbf{Y}_i^*$ onto $\mathbf{Y}$. This is achieved using a biharmonic spline interpolation [99] providing a set of rotated pressure values $\tilde{\mathbf{p}}_i$. The temporal evolution of state estimates $\text{Re}(\hat{\mathbf{x}}_{sh})$ and $\text{Re}(\hat{\mathbf{x}}_{pm})$ obtained using the MHE algorithm for both $\mathbf{y} = \Phi_{LSE}^\top \mathbf{p}$ and $\mathbf{y} = \Phi_{LSE}^\top \tilde{\mathbf{p}}$ are presented in figure 5.26 whilst the corresponding power spectra are shown in figure 5.27. Additionally, the error statistics corresponding to both methodologies are presented in table 5.4.

Figure 5.26 shows an improvement in accuracy of the estimation for both $\text{Re}(\hat{\mathbf{x}}_{sh})$ and

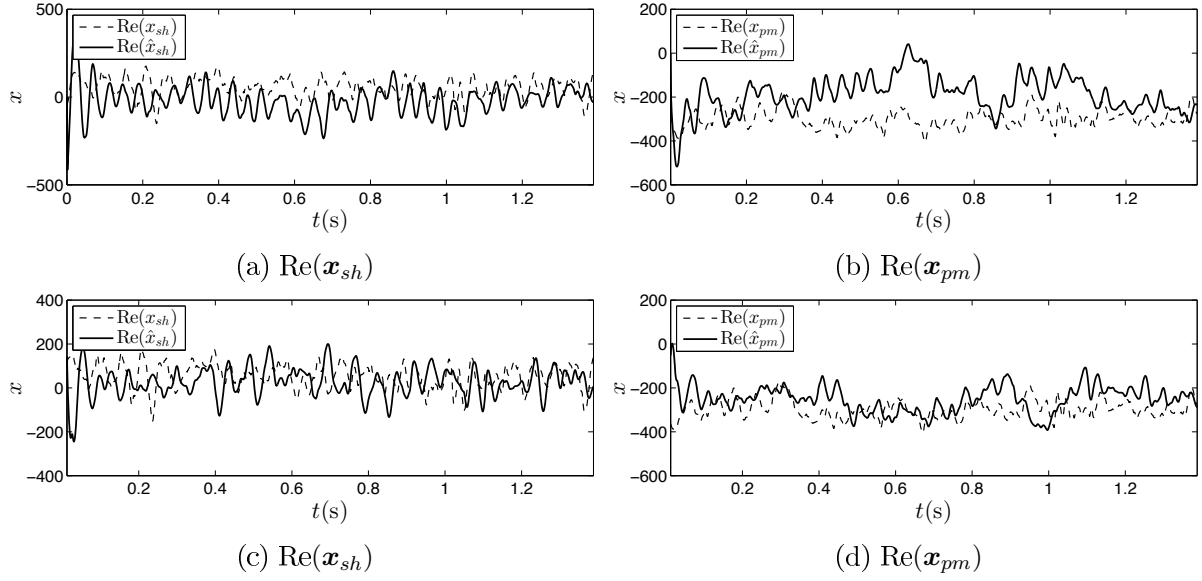


Figure 5.26: Temporal evolutions of the reference states, $\mathbf{x}_i$, and the state estimates $\hat{\mathbf{x}}_i$ obtained through MHE estimation on $\mathbf{X}_{\pi/2}$. The results corresponding to $\mathbf{y} = \Phi_{LSE}^\top \mathbf{p}$ are shown in (a)-(b), whilst those where $\mathbf{y} = \Phi_{LSE}^\top \tilde{\mathbf{p}}$ was used are shown in (c)-(d).

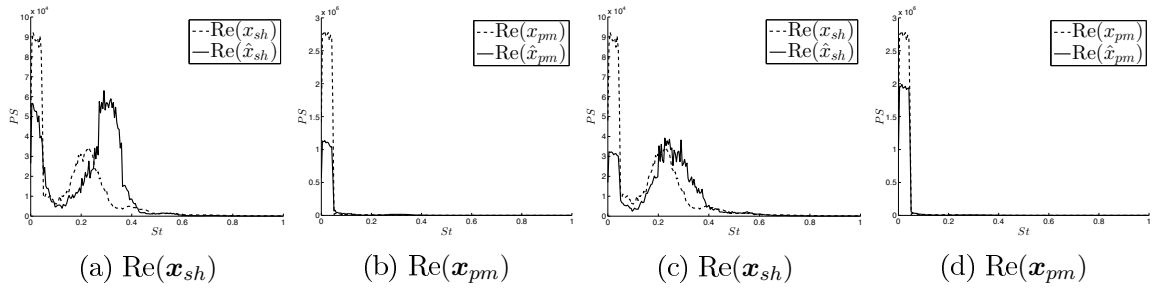


Figure 5.27: The power spectra corresponding to results presented in figure 5.26. The results corresponding to $\mathbf{y} = \Phi_{LSE}^\top \mathbf{p}$ are shown in (a)-(b), whilst those where $\mathbf{y} = \Phi_{LSE}^\top \tilde{\mathbf{p}}$ was used are shown in (c)-(d).

$\text{Re}(\hat{\mathbf{x}}_{pm})$ when rotated pressure values $\tilde{\mathbf{p}}_i$ are used. In both state estimates in figure 5.26(a)-(b) there is a steady state error which is eliminated for results in figure 5.26(c)-(d). Additionally, new estimates now contain a similar energy content to the reference states as demonstrated in the corresponding power spectra where the magnitude of the dominant peaks in figure 5.27(c)-(d) have values which are more comparable. Further investigation into spectra of $\text{Re}(\hat{\mathbf{x}}_{sh})$ shows that the value of the shedding frequency is approximated more accurately in figure 5.27(c), by suppressing the high-frequency oscillations at $St \approx 0.3$ visible in figure 5.27(a). It is reiterated that no modifications to the dynamical model or the MHE optimisation problem were performed here and different shedding frequencies are solely a result of changing the reorientation of the pressure measurements. Those observations are confirmed in error statistics in table 5.4 with an exception of $\text{Im}(\hat{\mathbf{x}}_{sh})$ where the corresponding errors do not decrease.

Point Error, ϵ_x				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
MHE (fig. 5.26(a)-(b))	1.61	1.30	0.51	4.27
MHE (fig. 5.26(c)-(d))	1.38	1.47	0.31	2.47
Spectral Error, ϵ_X				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
MHE (fig. 5.27(a)-(b))	0.80	0.96	0.61	34.28
MHE (fig. 5.27(c)-(d))	0.62	0.96	0.29	9.29
Phase Error, ϵ_θ				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
MHE (fig. 5.26(a)-(b))	0.71	0.78	0.19	0.94
MHE (fig. 5.26(c)-(d))	0.80	0.79	0.10	0.85

Table 5.4: Error comparison for all entries in the state estimate $\hat{\mathbf{x}}$ between the MHE results corresponding to $\mathbf{y} = \Phi_{LSE}^\top \mathbf{p}$ and $\mathbf{y} = \Phi_{LSE}^\top \tilde{\mathbf{p}}$.

The improvement in the estimation of the results presented in figure 5.26 shows that it is necessary to take into account the directional bias of pressure structures in dynamic estimation. The approaches presented above used the pressure rotation when the azimuthal offset between data used for model construction and dynamical estimation was known a priori. In practical applications such knowledge is not available and it is preferable to rotate the pressure values using online measurements, for example by setting a target configuration θ_{ref} and using $\delta\theta = \theta(t_i) - \theta_{ref}$ to construct the appropriate rotation matrix (5.17), where $\theta(t_i)$ is the instantaneous azimuthal component of the centre of pressure. It is noted that in such an approach when θ is constrained to be constant, any dynamical information corresponding to oscillations in azimuthal direction is lost. One way to facilitate this is by setting $\delta\theta = \bar{\theta}(t_i) - \theta_{ref}$, where

$$\bar{\theta} := \frac{1}{n} \sum_{i+1-n}^i \theta(t_i), \quad (5.18)$$

and n is a number of past measurements. For $n = 1$, $\bar{\theta}(t_i) = \theta(t_i)$ is retained. Furthermore, if the assumption is true that system dynamics are identical for different wake orientations it is justifiable to use $\tilde{\mathbf{p}}$ also in the calculation of the Φ_{LSE} and observation matrix $\mathbf{C}$.

In order to assess the applicability of this approach, model construction is performed using both data sets $\mathbf{X}_{\pi/2}$ and $\mathbf{X}_\pi$ and, subsequently, the obtained model is tested on a data set $\mathbf{X}_{3\pi/2}$ with temporal evolution of θ presented in figure 5.28. It is noted that such a test case is especially challenging since two datasets taken independently in time corresponding to

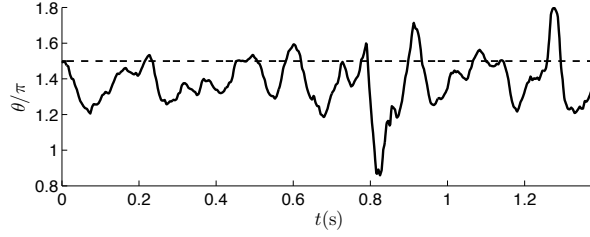


Figure 5.28: Evolution of the azimuthal coordinate of the centre of pressure in $\mathbf{X}_{3\pi/2}$.

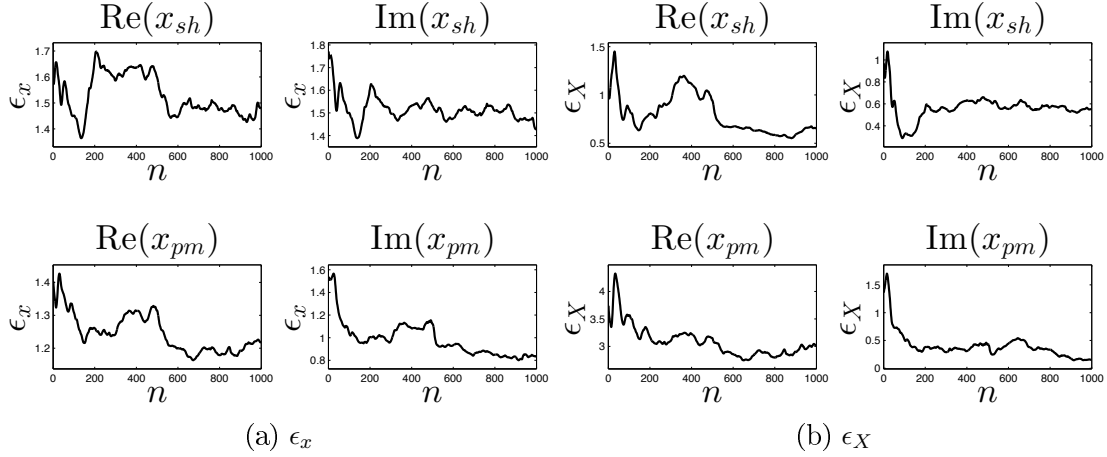


Figure 5.29: The values of ϵ_x and ϵ_X obtained for a range of moving average windows n .

different system reorientations are combined in order to construct the dynamical model which is subsequently tested on data corresponding to θ outside of the range used for the dynamic modeling.

The construction of the dynamical model will follow the methodologies presented in the previous sections. The four-dimensional state formulation (5.8) is used and the matrix $\mathbf{A}$ is obtained as a solution to (5.16) where $\lambda_{sh} = 0.185$. Subsequently, the measurement vector $\mathbf{y}$ is defined as $\mathbf{y} = \Phi_{LSE}^\top \tilde{\mathbf{p}}_i$, where $\tilde{\mathbf{p}}_i$ are the rotated pressure measurements based on a certain number n of past measurements of θ and Φ_{LSE} is calculated using (5.12)-(5.14). Finally, a matrix $\mathbf{C}$ is obtained by solving (5.11). This model is subsequently used to solve the MHE estimation problem (5.4) for $\gamma = 0.9$ and $N_h = 50$ at every time instance in $\mathbf{X}_{3\pi/2}$. The analysis is repeated for different values of n (used to calculate (5.18) to compute an azimuthal offset $\delta\theta$ for a calculation of rotated pressure measurements $\tilde{\mathbf{p}}$). The values of ϵ_x and ϵ_X corresponding to a range of values of n are demonstrated separately for each state entry in figure 5.29.

Figure 5.29 demonstrates that ϵ_x is minimised for $n \approx 150$ for both $\text{Re}(x_{sh})$ and $\text{Im}(x_{sh})$. It is noted that $n\delta t$ where $n \approx 150$ corresponds to $St \approx 0.067$, which is the frequency characterising the pulsation dynamics $\mathbf{x}_{pm}$ [26]. Additionally, further local minima are visible at $n \approx 300$ and

Point Error, ϵ_x				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
MHE (fig. 5.30)	1.32	1.31	1.22	0.85
MHE (fig. 5.32)	1.04	0.84	0.44	0.88
Spectral Error, ϵ_X				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
MHE (fig. 5.31)	0.43	0.38	3.07	0.22
MHE (fig. 5.32)	0.56	0.32	0.41	2.22
Phase Error, ϵ_θ				
	$\text{Re}(\hat{\mathbf{x}}_{sh})$	$\text{Im}(\hat{\mathbf{x}}_{sh})$	$\text{Re}(\hat{\mathbf{x}}_{pm})$	$\text{Im}(\hat{\mathbf{x}}_{pm})$
MHE (fig. 5.30)	0.70	0.67	0.29	0.87
MHE (fig. 5.32)	0.51	0.51	0.27	0.87

Table 5.5: Error statistics for all entries in the state estimate $\hat{\mathbf{x}}$ corresponding to the MHE where both model construction and the dynamical estimation utilised $\tilde{\mathbf{p}}$ and to the MHE where multiple values of Φ_{LSE} are used based on different system orientations and $\mathbf{y}$ takes the form as in (5.20).

$n \approx 450$ corresponding to multiples of $St \approx 0.067$, signifying that the most accurate results for the shedding dynamics are recovered when $\tilde{\mathbf{p}}$ is obtained using a moving average with a timescale corresponding to pulsation dynamics. On the other hand, it is noted that a different trend is visible in the values of ϵ_x corresponding to $\mathbf{x}_{pm}$ where the error decreases for large values of n . Similar trends are visible in the spectral error statistics ϵ_X .

In order to obtain an additional insight the dynamic estimation results corresponding to $n \approx 150$ are presented in figures 5.30-5.31. Additionally, the corresponding error metrics are shown in table 5.5. The power spectra of the shedding estimates in figure 5.31(a)-(b) show that the value of the shedding frequency is estimated accurately. However, there are differences visible in the magnitude of the shedding peaks and the mismatch in the low frequencies. The corresponding temporal evolution in figures 5.30(a)-(b) shows good phase agreement between $\hat{\mathbf{x}}_{sh}$ and $\mathbf{x}_{sh}$ in certain subsets of the temporal domain, for instance when $t \in [0.2, 0.5]$ or when $t \approx 1$. A greater error is visible when the pulsation state estimates $\hat{\mathbf{x}}_{pm}$ are analysed. For both $\text{Re}(\mathbf{x}_{pm})$ and $\text{Im}(\mathbf{x}_{pm})$ there is a considerable steady state error leading to a large point error ϵ_x . The high magnitude of the error is reflected in the corresponding power spectrum in figure 5.30(c) where large differences in the magnitude of low-frequency peaks are visible.

The dynamic estimation of $\mathbf{x}_{pm}$ in figure 5.30(c)-(d) is significantly less accurate when compared with the results presented in the previous sections. The deterioration in the estima-

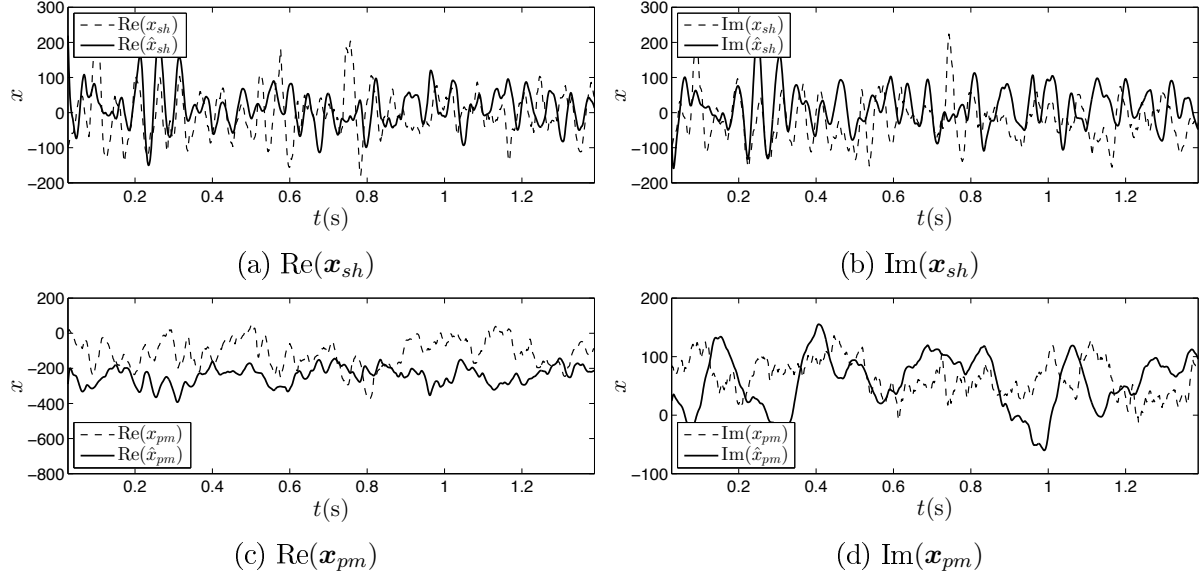


Figure 5.30: Temporal evolution of the reference states, $\mathbf{x}_i$, and the state estimates $\hat{\mathbf{x}}_i$ obtained through MHE where both model construction and the dynamical estimation are using $\tilde{\mathbf{p}}$.

tion of the pulsation dynamics can be linked to the rotation of pressure measurements. Since the timescale used to compute a mean of n past values of θ is comparable to timescale of $\mathbf{x}_{pm}$, the estimation results imply that the pulsation dynamics are effectively “averaged out” through the pressure rotation. Such behaviour would occur if the imprint of this state on the pressure measurements takes the form of variations in the azimuthal direction, which appears to be the case based on the deteriorating accuracy of the estimation. On the other hand the minimisation of error corresponding to $\mathbf{x}_{sh}$ at $n \approx 150$ implies that the imprint of this state is characterised by the changes of magnitude at positions determined through Φ_{LSE} . Consequently, reducing azimuthal variation of pressure measurements is beneficial for the estimation of the shedding dynamics. It is noted that since the dynamic model matrices $\mathbf{A}$ and $\mathbf{C}$ contain a strong coupling between $\mathbf{x}_{sh}$ and $\mathbf{x}_{pm}$, the reduced accuracy of $\hat{\mathbf{x}}_{pm}$ introduces a larger error in $\hat{\mathbf{x}}_{sh}$. Taking into account this observation an analysis will be performed where instead of rotated pressure values $\tilde{\mathbf{p}}$, a separate matrix Φ_{LSE} will be calculated for different values of θ and the subsequent estimation will be performed using measurements $\mathbf{y}$ obtained via a projection onto $\mathbf{p}$ based on an appropriate indicator function.

5.6.2 Multiple pressure modes

The dynamic estimation based on multiple pressure structures corresponding to a range of discrete θ_j values will be performed here. For this purpose, out of the available data, velocity

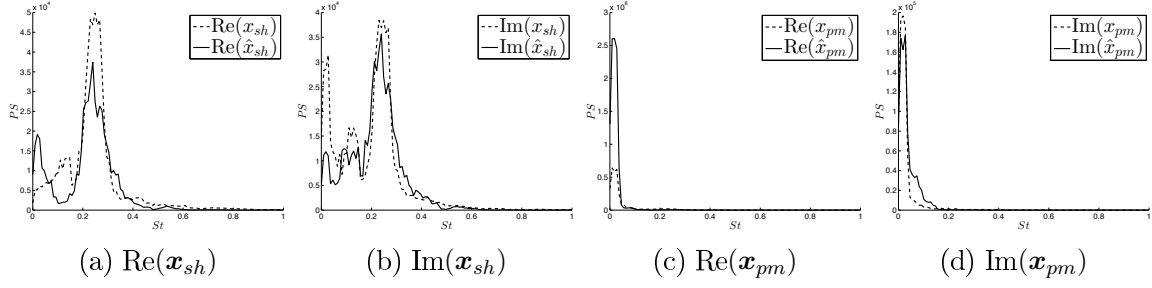


Figure 5.31: The power spectra corresponding to results presented in figure 5.30.

snapshots and corresponding base pressure measurements at different system orientations will be allocated to appropriate data bins which, subsequently, will be used in the calculation of the LSE coefficients. The 16 data bins are constructed based on the azimuthal coordinates and each will contain data close to one of θ_j such that $\theta_j \in 0, \pi/8, \dots, 15\pi/8$. That is, if an instantaneous value $\theta(t_i) \in [\theta_j - \pi/16, \theta_j + \pi/16]$, the corresponding pressure $\mathbf{p}(t_i)$ and velocity measurements $\mathbf{u}(t_i)$ are allocated to the j^{th} data bin.

Previously, the link between pressure measurements and the state entries was established using the LSE methodology on the POD coefficients and in order to prevent overfit, only the POD coefficients which had the same frequency content as at least one of the state entries were retained. Here, the data is discontinuous and it is not possible to perform a similar analysis. Therefore, LSE estimation will be performed using all 64 pressure point measurements. In order to prevent the overfit, it is ensured that a large number of measurements (varied between 1480 and 2679) is included in each data bin for the calculation of the LSE coefficients.

The LSE coefficients are obtained as a solution to

$$\min_{\Phi_{LSE}} \sum_i \|\mathbf{x}_i - \Phi_{LSE}^\top \mathbf{p}_i\|_2, \quad (5.19)$$

where $\mathbf{x}_i$ is defined as in (5.8). It is noted that each $\Phi_{LSE} \in \mathbb{R}^{64 \times 4}$ and 16 different values of Φ_{LSE} are available. The MHE estimation is performed using measurements $\mathbf{y}$ defined as

$$\mathbf{y}(t_i) = \sum_j \mathbb{1}_j(t_i) (\Phi_{LSE})_j^\top \mathbf{p}(t_i) \quad (5.20)$$

where

$$\mathbb{1}_j(t_i) := \begin{cases} 1 & \text{if } \theta(t_i) \in [\theta_j - \pi/16, \theta_j + \pi/16], \\ 0 & \text{otherwise.} \end{cases} \quad (5.21)$$

and $(\Phi_{LSE})_j$ is the solution to (5.19) corresponding to the j^{th} value of $\theta_j \in [0, \pi/8, \dots, 15\pi/8]$.

Matrices $\mathbf{A}$ and $\mathbf{C}$ are estimated using data merged from both datasets $\mathbf{X}_\pi$ and $\mathbf{X}_{\pi/2}$ by solving (5.16) and (5.11) respectively. The model constructed in this way is subsequently used to solve the MHE estimation problem (5.4) for $\gamma = 0.9$ and $N_h = 50$ at every time instance in $\mathbf{X}_{3\pi/2}$ and the temporal evolution of state estimates $\hat{\mathbf{x}}$ and corresponding power spectra are presented in figure 5.32. It is stressed that data contained in $\mathbf{X}_{3\pi/2}$ was not used for the minimisation (5.19). Additionally, a statistical error comparison between results in figure 5.30 and figure 5.32 is presented in table 5.5.

The presented results demonstrate the visible improvement in estimation accuracy if multiple Φ_{LSE} matrices are used compared with the methodology of using rotated pressure measurements $\tilde{\mathbf{p}}$; most visibly in $\text{Re}(\mathbf{x}_{pm})$. Figure 5.32 shows that state estimate $\hat{\mathbf{x}}$ more accurately follows the reference state values, which is also reflected in the point error statistics ϵ_x in table 5.5. Most noticeable is the low value of the phase error ϵ_θ corresponding to the results in figure 5.32. For the shedding state $\mathbf{x}_{sh}$, this value is approximately equal to 0.5 for both real and imaginary parts, signifying that the correct phase is estimated accurately throughout approximately half of the analysed temporal domain. It is noted that $\epsilon_\theta = 0.5$ is significantly lower than for any of the results presented previously and is due to the choice of measurement vector $\mathbf{y}$ (5.20). It is noted that there is an increase in the spectral error ϵ_X for $\text{Re}(x_{sh})$ which reflects the high magnitude peak at a low frequency visible in figure 5.32(e). Nonetheless, spectral characteristics of the frequencies in the neighbourhood of the shedding value are estimated accurately.

5.7 Conclusions

Using experimental two-dimensional velocity fields, the structures corresponding to key three-dimensional features of the dynamics were identified, more specifically those corresponding to shedding and pulsation frequencies, $St \approx 0.2$ and $St \approx 0.06$ respectively. It was additionally observed that the shedding dynamics has a low energy imprint on the base pressure values making the estimation in a non-intrusive manner challenging. The difficulty in constructing an accurate estimator is further increased by the fact that the system does not possess a single fixed symmetry plane. Instead the reflectional plane randomly changes orientation at $St \approx 0.002$. This change of the system reorientation is reflected in the non-axisymmetric pressure structu-

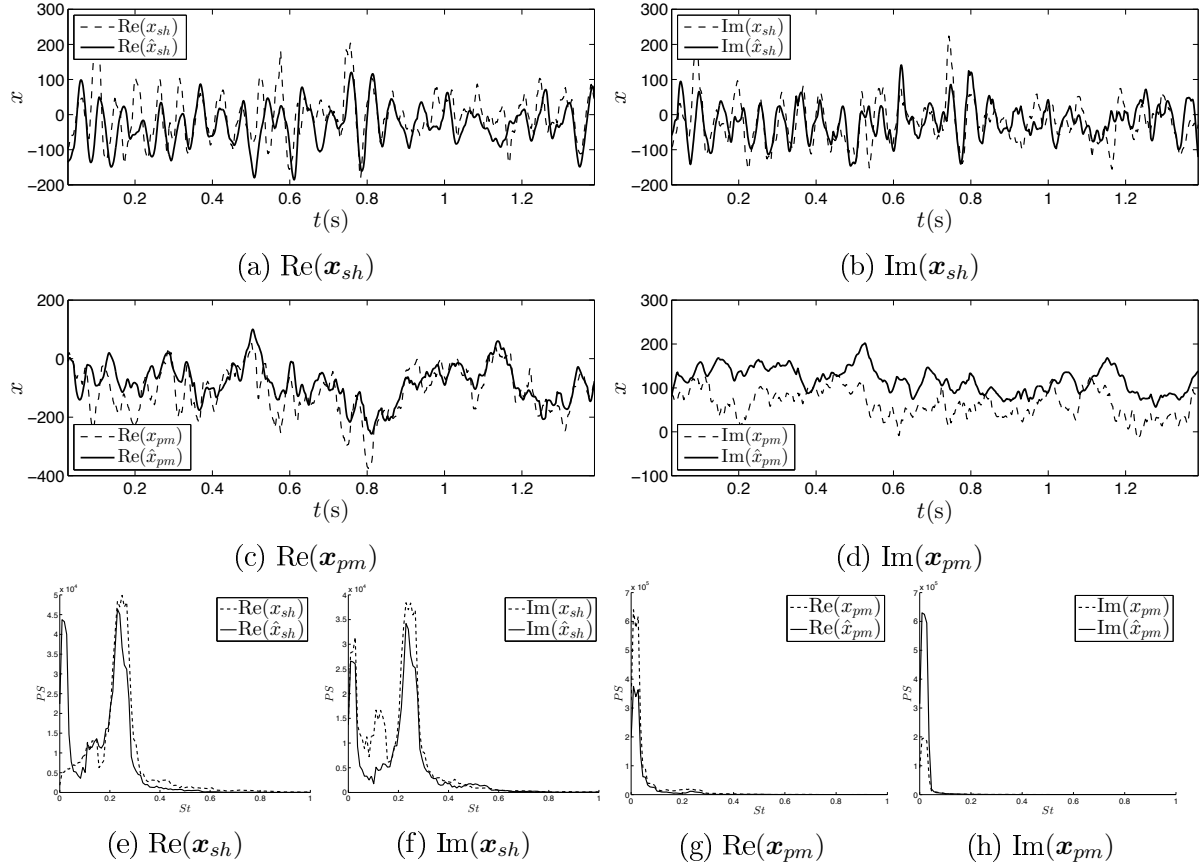


Figure 5.32: Temporal evolution of the reference states, $\mathbf{x}_i$, and the state estimates $\hat{\mathbf{x}}_i$ obtained through MHE using measurements defined based on indicator functions as per (5.20) (a)-(d); the corresponding power spectra (e)-(h)

res which are rotated based on the value of the azimuthal centre of pressure. Consequently utilisation of time-constant pressure structures was deemed not suitable for estimation.

Taking into account those observations an estimation framework is designed which allows us to approximate the pulsation and shedding dynamics. Based on the azimuthal location θ of the centre of pressure, the training domain was split into a number of segments and separate sets of the LSE coefficients were constructed for different values of θ . Subsequently, the measurement vectors $\mathbf{y}$ were constructed based on an indicator function specifying which set of LSE coefficients should be used at each time instance. When the measurement vector $\mathbf{y}$ was combined with the linear dynamic model using the MHE algorithm we were able to obtain state estimates which, on average, reproduced key dynamical features of the system.

Chapter 6

Dynamic reconstruction for sub-sampled or irregularly sampled data

6.1 Introduction

A fundamental requirement of modal decomposition algorithms used so far in this thesis is that they must be applied to a data set of flow snapshots sampled at a constant timestep δt . However, once δt has been chosen, the Nyquist sampling criterion [100] indicates that an inherent limitation is placed upon the maximum frequency that can be extracted from a data set. In order to demonstrate inherent limitations of the DMD algorithm, figure 6.1 shows the error between the true frequencies $(\omega_j)_{j=1}^5 = (j\pi 10^{-2})_{j=1}^5$ and those identified by DMD from snapshots sampled from the one-dimensional sinusoidal system

$$f(\mathbf{x}, t) = \sum_{i=1}^5 \sin(\mathbf{x} + \omega_i t), \quad 0 \leq \mathbf{x} \leq 2\pi, \quad t \geq 0,$$

for varying snapshot spacings $10 \leq \delta t \leq 120$. It can be seen that the error in each of the five system frequencies is zero until, respectively, δt exceeds 100, 50, 33, 25 and 20. This corresponds exactly to the Nyquist condition that the sampling rate must be at least twice the frequency to be identified, that is, $\delta t \leq \frac{\pi}{\omega_j} = \frac{100}{j}$. This presents the natural question of whether it is possible to reduce this error and correctly identify a system's dynamical information in the case of undersampling. Some established methods in signal processing were already outlined in §2.3.1. In §6.2 an extension of the DMD and OMD algorithms is proposed to enable its application to a general irregularly-sampled data ensemble. For clarity, we recall notation specific to describing

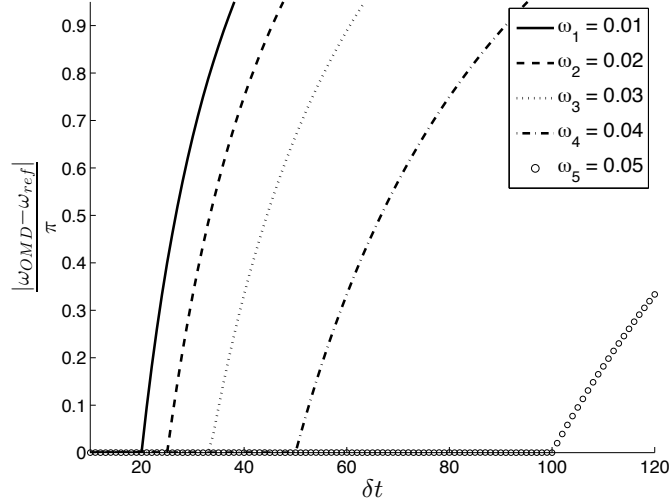


Figure 6.1: The error in frequency estimation for a range of uniform timesteps between consecutive snapshots.

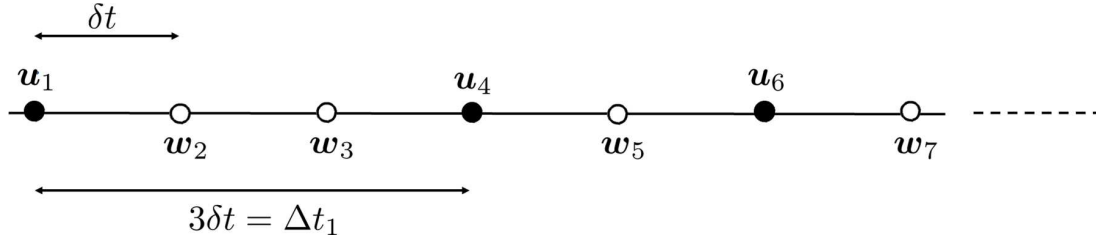


Figure 6.2: Downsampled and regularly sampled data.

irregularly sampled data.

Consider snapshots $\mathcal{W} = (\mathbf{w}_n)_{n=0}^N \subset \mathbb{R}^p$ sampled with constant timestep δt . An alternative to varying δt , as discussed above, is to instead view δt as a *minimal timestep* representing the highest possible frequency of data collection, but assume that only a subset $\mathcal{U} \subset \mathcal{W}$ of snapshots may be used for analysis. We refer to $\mathcal{U}$ as the *analysis data ensemble* and define it by choosing a subset $\mathcal{J} \subset \{0, 1, \dots, N\}$, re-labelling $\mathbf{u}_i := \mathbf{w}_i$ for each index $i \in \mathcal{J}$, and setting $\mathcal{U} = \{\mathbf{u}_i : i \in \mathcal{J}\}$. Finally, let Δt_i be the sample times between successive snapshots in $\mathcal{U}$ which are, possibly non-constant, integer multiples of δt . The schematic representation of such a sampling strategy is presented in figure 6.2.

It will be assumed that $\mathcal{W}$ is sampled at a sufficiently high rate that it contains all dynamical information of interest. The question which we aim to address is to what extent $\mathcal{W}$ can be reconstructed from knowledge of the analysis data set $\mathcal{U}$ only. The algorithm, is-OMD, which achieves such an objective is outlined in §6.2 followed by a rigorous stability analysis in §6.3. A robust extension to the algorithm in terms of estimation of mode shapes and in-

stantaneous unknown snapshots is presented in §6.4. Section 6.5 demonstrates an example of a synthetic system for which is-OMD provides an improvement over compressive sampling in terms of frequency estimation, whilst §6.6 provides a practical discussion of implementation of the algorithm to the canonical example of the cylinder wake. The demonstration of the applicability of the is-OMD algorithm to the dynamically more complicated turbulent axisymmetric wake is presented in §6.7.

6.2 Dynamic Decomposition of Irregularly Sampled Data

The application of OMD to the regularly-sampled data set $\mathcal{W}$, as discussed above, can be obtained by minimizing

$$\sum_{i=1}^N \|\mathbf{w}_{i+1} - \mathbf{L}\mathbf{M}\mathbf{L}^\top \mathbf{w}_i\|_2^2. \quad (6.1)$$

with respect to the dynamics matrix $\mathbf{M}$ and projection space $\mathbf{L}$. Now, if one attempts to apply a similar approach to the irregularly sampled data ensemble $\mathcal{U}$, a natural generalization of (6.1) is to consider the modified cost function

$$\sum_{i=0}^N \|\mathbf{u}_{i+1} - \mathbf{L}\mathbf{M}^{n_i} \mathbf{L}^\top \mathbf{u}_i\|_2^2 = \sum_{i=0}^N \|\mathbf{w}_{i+n_i} - \mathbf{L}\mathbf{M}^{n_i} \mathbf{L}^\top \mathbf{w}_i\|_2^2, \quad (6.2)$$

where $n_i := \frac{\Delta t_i}{\delta t} \in \mathbb{N}$ is the number of minimal timesteps δt between each sample point in $\mathcal{U}$. In other words, if there are n_i minimal timesteps between available data vectors $\mathbf{u}_{i+1}$ and $\mathbf{u}_i$ we seek to model the system evolution over an interval of length $\Delta t_i = n_i \delta t$ by the iterated linear mapping $(\mathbf{L}\mathbf{M}\mathbf{L}^\top)^{n_i} = \mathbf{L}\mathbf{M}^{n_i} \mathbf{L}^\top$.

Such a formulation encounters an immediate problem. Since in general $n_i > 1$, the modified cost (6.2) is at least of quartic order with respect to $\mathbf{M}$ meaning that an analytical solution is no longer available. To attempt to avoid this problem, consider the related optimization

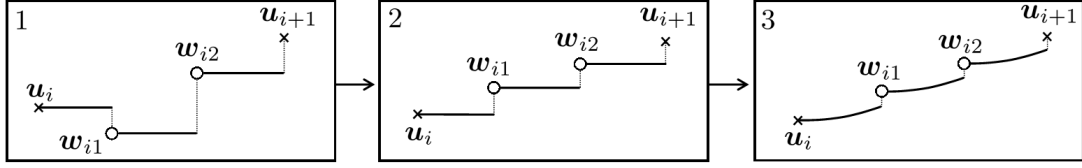


Figure 6.3: Schematic representation of a single iteration of the is-OMD algorithm. Given an initial guess for the dynamics $\mathbf{LML}^\top$ —represented by solid lines—over each timestep and the slack variables (step 1), the slack variables are first updated (step 2) before being fixed to allow an update of $\mathbf{L}$ and $\mathbf{M}$ (step 3).

problem

$$\min_{\hat{\omega}_i, \mathbf{L}, \mathbf{M}} \sum_{i=0}^N \|\hat{\omega}_{i+1} - \mathbf{LML}^\top \hat{\omega}_i\|^2, \quad (6.3a)$$

$$\text{s.t. } \hat{\omega}_i \in \mathbb{R}^{N_u}, \quad i = 0, \dots, N, \quad (6.3b)$$

$$\hat{\omega}_i = \mathbf{u}_i, \quad i \in \mathcal{J}, \quad (6.3c)$$

$$\mathbf{L}^\top \mathbf{L} = \mathbf{I}, \quad \mathbf{L} \in \mathbb{R}^{N_u \times r}, \quad \mathbf{M} \in \mathbb{R}^{r \times r}. \quad (6.3d)$$

Now, the variable space of (6.3) is considerably extended with respect to OMD, since higher powers of the dynamics matrix $\mathbf{M}$ are avoided by introducing “slack” state variables $\hat{\omega}_j \in \mathbb{R}^{N_u}$ corresponding to each underlying minimal timestep $j\delta t$. The analysis data ensemble $\mathcal{U}$ is then employed by enforcing the constraint $\hat{\omega}_i = \mathbf{u}_i, i \in \mathcal{J}$. Finally, again following the OMD formulation, the user-defined parameter $r \leq N - 1$ specifies the rank of the sought approximating dynamics $\mathbf{LML}^\top$. For completeness, we note that (6.3) reduces to the DMD algorithm in the case that $\mathcal{W} = \mathcal{U}, r = N - 1$ and $\mathbf{L} = \text{POD}(\mathbf{w}_0, \dots, \mathbf{w}_{N-1})$. The OMD algorithm is recovered in the case that $\mathcal{W} = \mathcal{U}$.

Despite the removal of higher orders of $\mathbf{M}$ from the cost function, (6.3) cannot be solved directly. Indeed, the inclusion of slack state variables $\hat{\omega}_i \in \mathbb{R}^{N_u}$ makes the problem overdetermined, while the products of decision variables—for example, in the $\mathbf{LML}^\top \hat{\omega}_i$ term—indicate that the problem is non-convex. To begin to address these difficulties, we propose a two-step iterative solution method indicated schematically in figure 6.3: first, the model variables $\mathbf{L}$ and $\mathbf{M}$ are fixed and the slack variables $\hat{\omega}_i$ are updated by solving a least-squares problem; second, the updated slack state variables are fixed, allowing $\mathbf{M}$ and $\mathbf{L}$ to be updated using the standard OMD algorithm applied to the regularly temporally-sampled fields $\hat{\omega}_i$.

The first of these steps—minimizing (6.3a) with respect to $\hat{\omega}_i$ only, subject to constraints

(6.3b), (6.3c)—is simply a least-squares problem. However, since typically $N_u = \mathcal{O}(10^{5-6})$, the problem is memory-intensive and likely to be ill-conditioned. Instead, we only seek to minimize the model error over the subspace $\text{Im}(\mathbf{L}) \subset \mathbb{R}^{N_u}$, where $\mathbf{L}$ is fixed within the slack variable update sub-iteration. This is achieved by considering the projected quadratic cost

$$\|\mathbf{L}\mathbf{L}^\top (\hat{\boldsymbol{\omega}}_{i+1} - \mathbf{L}\mathbf{M}\mathbf{L}^\top \hat{\boldsymbol{\omega}}_i)\|_2^2 = \|\mathbf{L}(\mathbf{L}^\top \hat{\boldsymbol{\omega}}_i - \mathbf{M}\mathbf{L}^\top \hat{\boldsymbol{\omega}}_i)\|_2^2 = \|\mathbf{c}_{i+1} - \mathbf{M}\mathbf{c}_i\|^2, \quad (6.4)$$

where $\mathbf{c}_i := \mathbf{L}^\top \hat{\boldsymbol{\omega}}_i \in \mathbb{R}^r$. Since $r \ll N_u$, minimizing (6.4) with respect to $\mathbf{c}_i$ is advantageous to minimizing (6.3a) with respect to $\hat{\boldsymbol{\omega}}_i$. Consequently, the first step of the pairwise optimization is to solve the constrained least-squares problem

$$\min_{\mathbf{c}_i} \sum_{i=0}^N \|\mathbf{c}_{i+1} - \mathbf{M}\mathbf{c}_i\|^2, \quad (6.5a)$$

$$\text{s.t. } \mathbf{c}_i = \mathbf{L}^\top \mathbf{u}_i, \text{ if } i \in \mathcal{J}, \quad (6.5b)$$

$$\mathbf{c}_i \in \mathbb{R}^r, \quad (6.5c)$$

upon which the slack state variables are updated by lifting up via $\hat{\boldsymbol{\omega}}_i \leftarrow \mathbf{L}\mathbf{c}_i$. After this update, the $\hat{\boldsymbol{\omega}}_i$ are fixed and $\mathbf{L}, \mathbf{M}$ are updated via the OMD algorithm $[\mathbf{L}, \mathbf{M}] \leftarrow \text{OMD}((\hat{\boldsymbol{\omega}}_i)_{i=0}^N)$. It is proved in §6.3.2 that this two-step procedure is guaranteed to decrease the cost (6.3a) which, being bounded below, implies that it will converge asymptotically to a local minimum.

A subtle characteristic of the OMD algorithm (step 6 in algorithm 1) is that the cost function of the minimisation problem (6.7) is invariant under rotations of $\mathbf{L}$ [21]. In particular, for any orthogonal matrix $\tilde{\mathbf{R}}$ such that $\tilde{\mathbf{R}}^\top \tilde{\mathbf{R}} = \mathbf{I}$, the pairs $[\mathbf{L}, \mathbf{M}]$ and $[\mathbf{L}\tilde{\mathbf{R}}, \tilde{\mathbf{R}}^\top \mathbf{M}\tilde{\mathbf{R}}]$ have equivalent costs. The explanation of this behaviour is that OMD identifies a subspace $\text{Im}(\mathbf{L}) \subset \mathbb{R}^{N_u}$ upon which a low-order representation of the system dynamics can be provided, and that each $\mathbf{L}\tilde{\mathbf{R}}$ is just a member of the *same* equivalence class describing this subspace. For visualisation purposes, we therefore propose in steps 10–12 of algorithm 1 to choose a rotation of $\mathbf{L}$ which ranks its columns by their energetic contribution to the *known* data.

The is-OMD procedure is summarised in algorithm 1 which outputs: (i) estimated intermediate snapshots $\mathbf{w}_i, i \notin \mathcal{J}$; (ii) a projection matrix $\mathbf{L}$ and (iii) an approximate dynamics matrix $\mathbf{M}$. The appropriate choice of initial projection and dynamics matrices, $\mathbf{L}_0, \mathbf{M}_0$, will

be discussed in detail in §6.6.1 where it is demonstrated that a sensible choice is to set the initial dynamics matrix to be a small perturbation of the identity matrix with normal distribution of standard deviation σ , $(M_0)_{ij} = N(0, \sigma^2) + \delta_{ij}$, and the projection space to be POD modes of the analysis data ensemble $\mathcal{U}$, $\mathbf{L}_0 = \text{POD}(\mathcal{U})$. Additionally, the beneficial impact of choosing a large value of r will be established in §6.6.2.

Algorithm 1 is-OMD

1: **inputs** $\mathbf{U} = \{\mathbf{u}_i : i \in \mathcal{J}\}$, $\mathbf{L}_0 \in \mathbb{R}^{N_u \times r}$ satisfying $\mathbf{L}_0^T \mathbf{L}_0 = \mathbf{I}$, $\mathbf{M}_0 \in \mathbb{R}^{r \times r}$, $k = 1$, and $\epsilon > 0$.

2: **repeat**

3:

$$(\mathbf{c}_i^k) \leftarrow \arg \min_{(\mathbf{c}_i)} \sum_{i=1}^N \|\mathbf{c}_i - \mathbf{M}_{k-1} \mathbf{c}_{i-1}\|_2^2, \quad (6.6)$$

$$\text{s.t. } \mathbf{c}_i = \mathbf{L}_{k-1}^\top \mathbf{u}_i \text{ if } i \in \mathcal{J}.$$

4: $\mathbf{w}_i^k \leftarrow \mathbf{L}_{k-1} \mathbf{c}_i^k$, $i \notin \mathcal{J}$.

5: $\mathbf{w}_i^k \leftarrow \mathbf{u}_i$, $i \in \mathcal{J}$.

6:

$$[\mathbf{L}_k, \mathbf{M}_k] \leftarrow \arg \min_{\mathbf{L}, \mathbf{M}} \sum_{i=1}^N \|\mathbf{w}_i^k - \mathbf{L} \mathbf{M} \mathbf{L}^\top \mathbf{w}_{i-1}^k\|_2^2, \quad (6.7)$$

$$\text{s.t. } \mathbf{L}^\top \mathbf{L} = \mathbf{I},$$

$$\mathbf{L} \in \mathbb{R}^{N_u \times r}, \mathbf{M} \in \mathbb{R}^{r \times r}.$$

7:

$$S_k \leftarrow \sum_{i=1}^N \|\mathbf{w}_i^k - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_{i-1}^k\|_2^2. \quad (6.8)$$

8: $k \leftarrow k + 1$

9: **until** $(S_k - S_{k-1}) < \epsilon$

10: $\tilde{\mathbf{R}} \Sigma \tilde{\mathbf{V}}^\top \leftarrow \text{svd}(\mathbf{L}_k^\top \mathbf{U})$

11: $\mathbf{M} \leftarrow \tilde{\mathbf{R}}^\top \mathbf{M}_k \tilde{\mathbf{R}}$

12: $\mathbf{L} \leftarrow \mathbf{L}_k \tilde{\mathbf{R}}$

13: **return** $(\mathbf{w}_i)_{i=0}^N$, $\mathbf{M} \in \mathbb{R}^{r \times r}$ and $\mathbf{L} \in \mathbb{R}^{N_u \times r}$.

A natural variant of algorithm 1 is to use DMD in place of OMD in step 6, in which case $\mathbf{L}_k := \text{POD}((\mathbf{w}_i^k)_{i=0}^N)$ would be fixed in step 6 within each iteration. Note that in this case, a final rotation of $\mathbf{L}$ in steps 10 - 12 is unnecessary. Finally, it should be noted that the notion of *dynamic modes* has not yet been defined in the context of algorithm 1. This will be discussed subsequently in section 6.4.1.

6.3 Algorithmic considerations

We now discuss feasibility of the slack variable sub-problem (6.6) and computational complexity of the algorithm 1. Since the model update sub-problem (6.7) is an application of OMD or DMD, we refer to [21] for a discussion of this step's performance. Furthermore a detailed proof demonstrating that the cost (S_k) is monotone decreasing (and hence, convergent) is presented in §6.3.2.

6.3.1 Convexity of the slack variable update

The equality constraint $\mathbf{c}_i = \mathbf{L}^\top \mathbf{u}_i, i \in \mathcal{J}$ of (6.6) represents timesteps at which data is available. Without loss of generality, we consider the problem in which $\mathcal{J} = \{0, N\}$, since the general optimization problem can be broken into a number of independent sub-problems of this form. Now, it is not difficult to prove that (6.6) can be written as the unconstrained least-squares problem

$$\min_{\boldsymbol{\xi}} \|\mathbf{A}\boldsymbol{\xi} - \mathbf{b}\|_2^2$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & 0 & \dots & 0 \\ -\mathbf{I} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mathbf{M} \\ 0 & \dots & 0 & -\mathbf{I} \end{bmatrix} \in \mathbb{R}^{Nr \times (N-1)r}, \mathbf{b} = \begin{bmatrix} \mathbf{L}^\top \mathbf{u}_N \\ 0 \\ \vdots \\ 0 \\ -\mathbf{M}\mathbf{L}^\top \mathbf{u}_1 \end{bmatrix} \in \mathbb{R}^{Nr},$$

and $\boldsymbol{\xi} = \left(\mathbf{c}_{N-1}^\top \dots \mathbf{c}_1^\top \right)^\top \in \mathbb{R}^{(N-1)r}$. Consequently, (6.6) is convex and admits the solution $\boldsymbol{\xi} = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}$ provided the matrix $\mathbf{A}$ is full rank. This is true, and follows from the easily shown fact that $\mathbf{A}\boldsymbol{\xi} = 0 \Leftrightarrow \boldsymbol{\xi} = 0$. Finally, we note that solving a series of independent sub-problems has significant advantages in terms of computational memory savings and improved numerical stability due to a reduced condition number of $(\mathbf{A}^\top \mathbf{A})$ compared to the case where $\boldsymbol{\xi}$ and $\mathbf{b}$ contain $\mathbf{L}^\top \mathbf{u}_i$ and $\mathbf{c}_i$ for all $i \in \mathcal{J}$.

6.3.2 Monotonicity of (S_k)

To facilitate the analysis let $\mathcal{I} = \{0, 1, \dots, N\}$ denote all timesteps of interest, recall that $\mathcal{J} \subset \mathcal{I}$ denotes points of known data and introduce two new subsets: $\mathcal{J}^-$ contains timestep

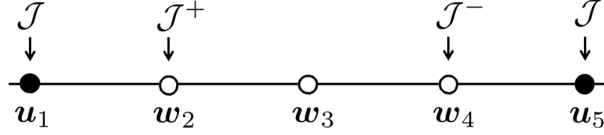


Figure 6.4: A representation of an allocation of indices to different sets in is-OMD problem.

indices immediately preceding those in $\mathcal{J}$; and $\mathcal{J}^+$ which contains indices representing time-steps immediately after those in $\mathcal{J}$. This assignment is indicated in figure 6.4. It is assumed for simplicity that $\mathcal{J}^- \cap \mathcal{J}^+ = \emptyset$ and we note that if $|\mathcal{J}| = m$ then $|\mathcal{J}^- \cup \mathcal{J}^+| = 2(m - 1)$. Recalling that $\mathbf{w}_i^k = \mathbf{u}_i, i \in \mathcal{J}$ and letting $\mathcal{J}^0 := \mathcal{I} \setminus (\mathcal{J} \cup \mathcal{J}^+ \cup \mathcal{J}^-)$, the residual S_k can be written

$$S_k = \sum_{i \in \mathcal{J}^0} \|\mathbf{w}_i^k - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_{i-1}^k\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{w}_i^k - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{u}_{i-1}\|^2 + \sum_{i \in \mathcal{J}^-} \|\mathbf{u}_{i+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_i^k\|^2. \quad (6.9)$$

Theorem 1. *Let S_k be the residual (6.8) from algorithm 1. Then $S_k \geq S_{k+1} \geq 0$ and there exists $S^* \geq 0$ such that $S_k \rightarrow S^*, k \rightarrow \infty$.*

Proof. For each $k \geq 0$, $\mathbf{L}_k \mathbf{L}_k^\top : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is a projection. Estimating the first two terms of (6.9) from below by their projections onto $\text{Im}(\mathbf{L}_k)$ and decomposing the final term into contributions from the image and kernel of $\mathbf{L}_k$ give

$$S_k \geq \sum_{i \in \mathcal{J}^0} \|\mathbf{L}_k \mathbf{L}_k^\top (\mathbf{w}_i^k - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_{i-1}^k)\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{L}_k \mathbf{L}_k^\top (\mathbf{w}_i^k - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{u}_{i-1})\|^2 + \sum_{i \in \mathcal{J}^-} \left[\|\mathbf{L}_k \mathbf{L}_k^\top (\mathbf{u}_{i+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_i^k)\|^2 + \|(I - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{u}_{i+1}\|^2 \right],$$

Now, since $\mathbf{L}_k^\top \mathbf{L}_k = I$, it follows that $\|\mathbf{L}_k \mathbf{v}\| = \|\mathbf{v}\|$, for any $\mathbf{v} \in \mathbb{R}^r$. Hence,

$$S_k \geq \sum_{i \in \mathcal{J}^0} \|\mathbf{L}_k^\top \mathbf{w}_i^k - \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_{i-1}^k\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{L}_k^\top \mathbf{w}_i^k - \mathbf{M}_k \mathbf{L}_k^\top \mathbf{u}_{i-1}\|^2 + \sum_{i \in \mathcal{J}^-} \left[\|\mathbf{L}_k^\top \mathbf{u}_{i+1} - \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_i^k\|^2 + \|(I - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{u}_{i+1}\|^2 \right]$$

Recalling that $\mathbf{c}_i^k = \mathbf{L}_k^\top \mathbf{w}_i^k$ and writing $\mathbf{d}_i = \mathbf{L}_k^\top \mathbf{u}_i$,

$$\begin{aligned}
S_k \geq & \sum_{i \in \mathcal{J}^0} \|\mathbf{c}_i^k - \mathbf{M}_k \mathbf{c}_{i-1}^k\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{c}_i^k - \mathbf{M}_k \mathbf{d}_{i-1}\|^2 \\
& + \sum_{i \in \mathcal{J}^-} \left[\|\mathbf{d}_{i+1} - \mathbf{M}_k \mathbf{c}_i^k\|^2 + \|(\mathbf{I} - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{u}_{i+1}\|^2 \right],
\end{aligned} \tag{6.10}$$

Now, the first three terms of (6.10) are exactly of the form of the optimization problem (6.6) of algorithm 1. Hence, by optimality,

$$\begin{aligned}
S_k \geq & \sum_{i \in \mathcal{J}^0} \|\mathbf{c}_i^{k+1} - \mathbf{M}_k \mathbf{c}_{i-1}^{k+1}\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{c}_i^{k+1} - \mathbf{M}_k \mathbf{d}_{i-1}\|^2 \\
& + \sum_{i \in \mathcal{J}^-} \left[\|\mathbf{d}_{i+1} - \mathbf{M}_k \mathbf{c}_i^{k+1}\|^2 + \|(\mathbf{I} - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{u}_{i+1}\|^2 \right].
\end{aligned}$$

Note that values of $\mathbf{d}_i$ do not change between subsequent iterations due to constraints imposed in the optimisation problem (6.6).

The subsequent step 4 in the algorithm consists of approximating time resolved snapshots $\mathbf{w}_i$ with the “backwards” embedding using the columns of $\mathbf{L}_k$. Again, using the identity $\|\mathbf{L}_k \mathbf{v}\|^2 = \|\mathbf{v}\|^2$, for $\mathbf{v} \in \mathbb{R}^r$,

$$\begin{aligned}
S_k \geq & \sum_{i \in \mathcal{J}^0} \|\mathbf{L}_k \mathbf{c}_i^{k+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{c}_{i-1}^{k+1}\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{L}_k \mathbf{c}_i^{k+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{d}_{i-1}\|^2 \\
& + \sum_{i \in \mathcal{J}^-} \left[\|\mathbf{L}_k \mathbf{d}_{i+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{c}_i^{k+1}\|^2 + \|(\mathbf{I} - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{u}_{i+1}\|^2 \right].
\end{aligned}$$

In step 4 of Algorithm 1, $\mathbf{w}_i^{k+1} = \mathbf{L}_k \mathbf{c}_i^{k+1}$, and, hence, it is true that $\mathbf{L}_k^\top \mathbf{w}_i^{k+1} = \mathbf{c}_i^{k+1}$, which implies that the above expression is equivalent to

$$\begin{aligned}
S_k \geq & \sum_{i \in \mathcal{J}^0} \|\mathbf{w}_i^{k+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_{i-1}^{k+1}\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{w}_i^{k+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{u}_{i-1}\|^2 \\
& + \sum_{i \in \mathcal{J}^-} \left[\|\mathbf{L}_k \mathbf{L}_k^\top \mathbf{u}_{i+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_i^{k+1}\|^2 + \|(\mathbf{I} - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{u}_{i+1}\|^2 \right].
\end{aligned}$$

Again, using the fact that $\|\mathbf{v}\|^2 = \|\mathbf{L}_k \mathbf{L}_k^\top \mathbf{v}\|^2 + \|(\mathbf{I} - \mathbf{L}_k \mathbf{L}_k^\top) \mathbf{v}\|^2$ for any $\mathbf{v} \in \mathbb{R}^p$, the final

two terms in the above inequality can be combined to show that

$$\begin{aligned} S_k \geq & \sum_{i \in \mathcal{J}^0} \|\mathbf{w}_i^{k+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_{i-1}^{k+1}\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{w}_i^{k+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{u}_{i-1}\|^2 \\ & + \sum_{i \in \mathcal{J}^-} \|\mathbf{u}_{i+1} - \mathbf{L}_k \mathbf{M}_k \mathbf{L}_k^\top \mathbf{w}_i^{k+1}\|^2. \end{aligned}$$

Finally, invoking optimality of the $[\mathbf{L}, \mathbf{M}]$ -update optimization sub-problem (6.7) gives

$$\begin{aligned} S_k \geq & \sum_{i \in \mathcal{J}^0} \|\mathbf{w}_i^{k+1} - \mathbf{L}_{k+1} \mathbf{M}_{k+1} \mathbf{L}_{k+1}^\top \mathbf{w}_{i-1}^{k+1}\|^2 + \sum_{i \in \mathcal{J}^+} \|\mathbf{w}_i^{k+1} - \mathbf{L}_{k+1} \mathbf{M}_{k+1} \mathbf{L}_{k+1}^\top \mathbf{u}_{i-1}\|^2 \\ & + \sum_{i \in \mathcal{J}^-} \|\mathbf{u}_{i+1} - \mathbf{L}_{k+1} \mathbf{M}_{k+1} \mathbf{L}_{k+1}^\top \mathbf{w}_i^{k+1}\|^2 \\ = & S_{k+1}. \end{aligned}$$

Hence, $(S_k)_{k \geq 0}$ is a monotone decreasing sequence which is bounded below by 0. Therefore, there exists $S^* \geq 0$ such that $S_k \rightarrow S^*, k \rightarrow \infty$. $\square$

6.3.3 Computational complexity

The computational cost of the algorithm 1 consists of that associated with the two optimisation sub-problems; the least squares problem (6.6) and the modal optimisation (6.7) where the dynamic model is updated via OMD (or DMD). Solving the least squares problem using, for example, the Moore-Penrose pseudoinverse of $\mathbf{A} \in \mathbb{R}^{Nr \times (N-1)r}$ has complexity $\mathcal{O}(N^3 r^3)$. Moreover, as discussed in §6.3.1 the full least squares problem can be broken down into a number of smaller, independent, sub-problems corresponding to N_i intermediate timesteps, each of which can be solved at a low computation burden. For example, $(N_i = 10, r = 40)$ and $(N_i = 20, r = 60)$ can be solved in 0.01s and 0.3s on a standard desktop computer, respectively, and each sub-problem could be parallelised, if required.

For the modal optimisation sub-problem (6.7), a discussion of the computational performance of OMD and DMD can be found in [21]. Both the total number of slack variables N and the approximation rank r affect computation time in addition to the underlying dynamic characteristics of the specific system to which the each algorithm is applied. With respect to the examples considered subsequently: for the cylinder flow in §6.6, the is-OMD algorithm converges with a tolerance of 10^{-2} after 80 iterations of algorithm 1, with each call of OMD in

sub-problem (6.7) requiring 8.0s ($N = 500, r = 50$), giving a total computation time for is-OMD of 640s; for the application to PIV snapshots of turbulent flow past an axisymmetric bluff body in §6.7, is-OMD converges to the same tolerance in only 10 iterations but with a higher cost of 13.8s ($N = 120, r = 30$) for each OMD sub-problem, resulting in a total computation time of 138s.

Finally, we note that since both OMD and algorithm 1 are iterative, a possible method to improve algorithmic efficiency would be to apply a (small) fixed number of OMD iterates at each sub-problem (6.7). Alternatively, one may apply DMD in place of OMD in sub-problem (6.7) to reduce the computational cost. This will be explored in future work. All computations were performed using MATLAB on a standard desktop PC with a 3.4 GHz quad-core Intel i7 processor and 16 GB RAM.

6.4 Mode extraction and approximation of intermediate snapshots

In order to describe the performance of algorithm 1, consider first a one-dimensional synthetic sinusoidal system

$$f(\mathbf{x}, t) = \sum_j A_j \sin(k_j \mathbf{x} + \omega_j t) \exp(\gamma_j t), \quad 0 \leq \mathbf{x} \leq 1, t > 0, \quad (6.11)$$

with exponential growth rates γ_j , amplitudes $A_j > 0$, frequencies $\omega_j > 0$ and wavenumbers $k_j > 0$. A reference data set $\mathcal{W} = \{\mathbf{w}_i\}_{i=1}^{900}$ is created where each snapshot

$$\mathbf{w}_i = \left(f\left(\frac{j}{N_u}, i\delta t\right) \right)_{j=0}^{N_u}, \quad i = 1, \dots, 900,$$

is sampled at $N_u = 10^4$ equally spaced points.

Downsampled, analysis data sets $\mathcal{U} = \{\mathbf{w}_i\}_{i \in \mathcal{J}} \subset \mathcal{W}$ are created by defining a sub-sampling strategy $\mathcal{J} \subset \{1, \dots, N\}$, and associated with each downsampled set is an average sampling timestep

$$\Delta t_{av} := \frac{1}{|\mathcal{J}|} \sum_{i \in \mathcal{J}} \Delta t_i,$$

where Δt_i is the timestep between subsequent elements of $\mathcal{U}$. For each frequency ω_j , let

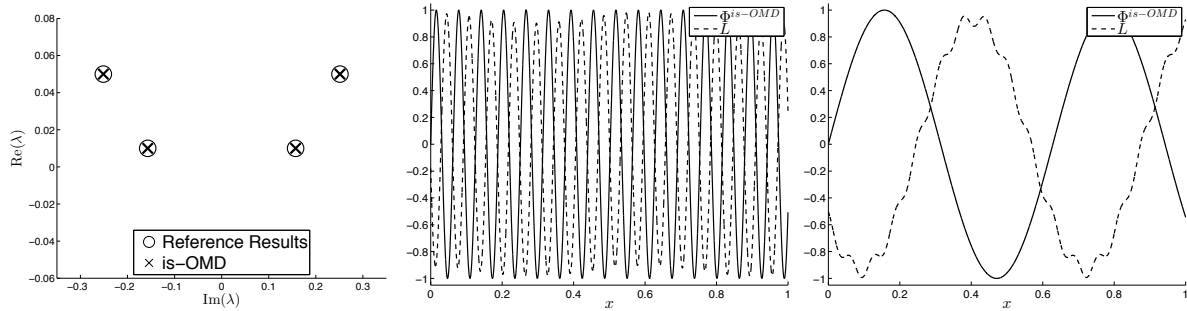


Figure 6.5: Dynamic matrix $\mathbf{M}$ obtained using is-OMD algorithm for the synthetic sinusoidal system with exponential dynamics (left), structures present in the system represented by most energetic columns of $\mathbf{L}$ and is-OMD modes $\Phi^{\text{is-OMD}}$ obtained using algorithm 2 (right).

$\delta t_j^* := \pi/\omega_j$ denote the largest sampling timestep which satisfies the Nyquist criterion for that particular frequency. Of interest will be the situations in which $\Delta t_{av}/\delta t_j^* > 1$, indicating cases where, on average, downsampling may prevent accurate frequency identification.

We consider data sampled from (6.11) in the case of two frequencies $\omega_1 = \frac{\pi}{20}, \omega_2 = \frac{2\pi}{25}$, respective amplitudes $A_1 = 1, A_2 = 0.5$, wavenumbers $k_1 = 10, k_2 = 100$, growth rates $\gamma_1 = 0.01, \gamma_2 = 0.05$ and minimal timestep $\delta t = 1$. The analysis data ensemble $\mathcal{U}$ is selected with an average sampling timestep satisfying

$$\frac{\Delta t_{av}}{\delta t_1^*} = \frac{7}{8}, \quad \frac{\Delta t_{av}}{\delta t_2^*} = \frac{6}{5},$$

meaning that we would expect, according to the Nyquist criterion, to accurately extract ω_1 but not ω_2 . Such a selection of an average sampling timestep provides a downsampled data ensemble $\mathcal{U}$ of cardinality $|\mathcal{U}| = 45$, i.e. only 5% of the time-resolved ensemble $\mathcal{W}$ is retained. Algorithm 1 is initialised with $(M_0)_{ij} = N(0, \sigma^2) + \delta_{ij}$, that is, a small perturbation of the identity matrix with $\sigma = \sqrt{0.3}$, and $\mathbf{L}_0 = \text{POD}(\mathbf{U})$, where $\mathbf{U}$ is the analysis data ensemble in a matrix form.

Figure 6.5 presents the eigenvalues of the matrix $\mathbf{M}$ obtained from the is-OMD algorithm demonstrating that the is-OMD algorithm accurately identifies the oscillatory and exponential parts of the dynamics corresponding to both resolved and undersampled frequencies. For results presented in figure 6.5 the proportional eigenvalue error is $\mathcal{O}(10^{-6})$ for both λ_1 and λ_2 . Additionally, figure 6.5 presents the most energetic columns of $\mathbf{L}$. These structures have the shape of the sinusoids with an amplitude and wavenumber specified by A_j and k_j . Columns of $\mathbf{L}$ retain the prescribed values of k_j , albeit with a visible interaction between k_1 and k_2 .

As discussed in section 6.2, the columns of $\mathbf{L}$ should be viewed as spanning a subspace of $\mathbb{R}^{N_u}$ appropriate to the system dynamics and, consequently, there is no a direct correspondence between eigenvalues of $\mathbf{M}$ and columns of $\mathbf{L}$. In the OMD algorithm, dynamic modes Φ are defined as the embedding of the eigenvectors of $\mathbf{M}$ into a high-dimensional vector space using projection $\mathbf{L}$, i.e. $\Phi = \mathbf{L}\mathbf{P}_M$ where $\mathbf{M} = \mathbf{P}_M\mathbf{\Lambda}_M\mathbf{P}_M^{-1}$, indicating that each mode corresponds directly to a particular eigenvalue. Now, despite the identification of accurate spectral information and projection basis vectors indicated in Figure 6.5, it is the case that the columns of $\mathbf{L}\mathbf{P}_M$ lack coherence. In particular, the lack of time resolved data manifests itself in corrupting the *eigenvectors* of $\mathbf{M}$. Although $\mathbf{L}$ and $\mathbf{M}$ may still be of sufficient quality to construct an approximate low-order model of the system, if one would like to define dynamic modes the following alternative mode extraction post-processing step is proposed.

6.4.1 Dynamic Mode Extraction

Given a data ensemble $(\mathbf{w}_i)_{i=0}^N$, a typical description [6, 20, 36] of dynamic modes Φ_i and eigenvalues λ_i is that they describe the evolution of the first $N - 1$ snapshots according to

$$\mathbf{w}_i = \sum_{j=1}^N \lambda_j^i \Phi_j \Rightarrow \mathbf{W} = \Phi \mathbf{V}, \quad (6.12)$$

where $\mathbf{W} = [\mathbf{w}_0, \dots, \mathbf{w}_{N-1}]$, $\Phi = [\Phi_1, \dots, \Phi_N]$ and $\mathbf{V}$ is the Vandermonde matrix

$$\mathbf{V} := \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{bmatrix}. \quad (6.13)$$

In particular, for $\mathbf{V}$ constructed using DMD eigenvalues, DMD modes can be interpreted as arising from the solution to the minimisation problem

$$\min_{\Phi^{\text{DMD}}} \|\mathbf{W} - \Phi^{\text{DMD}} \mathbf{V}\|. \quad (6.14)$$

In view of the observation that accurate spectral information may be identified by algorithm 1, we seek to adapt this approach to calculate dynamic modes from the output of algorithm

1. In particular, consider the output $\mathbf{M}$ with eigenvalues λ_i obtained from a data ensemble with known data only at points $\mathcal{J} \subset \{0, \dots, N\}$. From this, construct a reduced Vandermonde matrix $\mathbf{V}_{\mathcal{J}}$ containing r rows and only the columns whose indices are in $\mathcal{J}$. The dynamic modes associated with algorithm 1 are given in terms of the solution to the reduced minimization problem

$$\min_{\Phi} \|\mathbf{U} - \Phi \mathbf{V}_{\mathcal{J}}\|^2. \quad (6.15)$$

This is a least-squares problem with solution, Φ , from which the dynamic is-OMD modes are defined as

$$\Phi_j^{\text{is-OMD}} = \frac{\Phi_j}{\|\Phi_j\|}.$$

It is noted that computation of the dynamic modes through an inversion of $\mathbf{V}_{\mathcal{J}}$ can be restrictive in the sense that it is assumed that temporal evolution corresponding to coherent structures is highly periodic. In applications where this is not the case an alternative approach may be required: an example is presented in §6.7 where dynamic modes are obtained by introducing rate constraints to the low-dimensional dynamics.

A technical consideration is that an accurate solution to the problem (6.15) requires $\mathbf{V}$ to be full rank. However, it is known that Vandermonde matrices are typically ill-conditioned [101]. For this reason, it is also desirable to remove *rows* of the Vandermonde matrix corresponding to eigenvalues which are of comparative unimportance to the system dynamics. This can be achieved by taking a QR decomposition with column pivoting $\mathbf{V}_{\mathcal{J}}^{\top} = \mathbf{Q}\mathbf{R}\bar{\mathbf{P}}^{\top}$, where $\mathbf{Q}^{\top}\mathbf{Q} = \mathbf{I}$, $\mathbf{R}$ is an upper triangular matrix and $\bar{\mathbf{P}}$ is a column permutation matrix such that absolute diagonal values of $\mathbf{R}$, $|r_{ii}|$, are decreasing. A tolerance, $0 < \tilde{t} < 1$, is chosen, and the number of rows retained, n , is set to $n = \max\{i : |r_{ii}| > \tilde{t}|r_{11}|\}$. The intention is that the condition number of the reduced matrix is decreased since rows corresponding to dynamically unimportant eigenvalues (which typically possess a large negative real part) are discarded. The number of removed rows will depend on the tolerance $\tilde{t}$ which has to be set such that majority of dynamically irrelevant eigenvalues are discarded but all of the dynamically relevant features are retained. An appropriate way to select $\tilde{t}$ is outlined in §6.6.2, and the post-processing step is summarised in algorithm 2.

Using the eigenvalues obtained previously, the is-OMD modes of the system (6.11) are obtained from (6.15) for a Vandermonde matrix conditioned with $\tilde{t} = 0.01$. Figure 6.5 shows

Algorithm 2 Dynamic Mode Post-Processing

-
- 1: **inputs** $\mathbf{U} = \{\mathbf{u}_i : i \in \mathcal{J}\}, [\lambda_1, \dots, \lambda_r]^\top = \text{eig}(\mathbf{M}_{\text{is-OMD}}), 0 < \tilde{t} < 1.$
 - 2: $\mathbf{V}_{\mathcal{J}} = (v_{\mathcal{J}})_{ji} \leftarrow \lambda_j^{i-1}, \text{ for } i \in \mathcal{J} \text{ and } j = 1, \dots, r.$
 - 3: $\mathbf{QR}\bar{\mathbf{P}}^\top \leftarrow \text{qr}(\mathbf{V}_{\mathcal{J}}^\top), \text{ such that } |r_{jj}| > |r_{ii}| \text{ for } j < i.$
 - 4: $n \leftarrow \max\{i : |r_{ii}| > \tilde{t}|r_{11}|\}.$
 - 5: $\bar{\mathbf{P}} \leftarrow [\bar{\mathbf{p}}_1, \dots, \bar{\mathbf{p}}_n], \text{ where } \bar{\mathbf{p}}_i \text{ are columns of } \bar{\mathbf{P}}.$
 - 6: $\mathbf{V}_{\mathcal{J}} \leftarrow \bar{\mathbf{P}}^\top \mathbf{V}_{\mathcal{J}}.$
 - 7: $\Phi \leftarrow \arg \min_{\Phi} \|\mathbf{U} - \Phi \mathbf{V}_{\mathcal{J}}\|^2.$
 - 8: $\Phi_j^{\text{is-OMD}} \leftarrow \frac{\Phi_j}{\|\Phi_j\|}$
 - 9: $\Lambda^{\text{is-OMD}} \leftarrow [\lambda_1, \dots, \lambda_r]^\top \bar{\mathbf{P}}$
 - 10: **return** $\Phi^{\text{is-OMD}} \in \mathbb{R}^{p \times n}, \Lambda^{\text{is-OMD}} \subset [\lambda_1, \dots, \lambda_r]^\top.$
-

modes corresponding to identified eigenvalues, which accurately match the prescribed spatial wavenumbers k_1 and k_2 . Note the difference between the columns of $\mathbf{L}$ in figure 6.5 where there is a visible interaction between both wavenumbers, a phenomenon which is not evident in $\Phi^{\text{is-OMD}}$. Although $\mathbf{L}$ is not used in the calculation of $\Phi^{\text{is-OMD}}$, its inclusion in the variable space of algorithm 1 allows for greater freedom in the estimation, which may lead to more accurate λ_i and, hence, more accurate dynamical modes.

6.4.2 Intermediate snapshot reconstruction

Despite the fact that the intermediate snapshot variables w_i in algorithm 1 provide the *freedom* with which accurate spectral information can be obtained, these variables typically contain high-frequency content not present in the reference data. However, using the eigenvalues $\Lambda^{\text{is-OMD}} = [\lambda_1, \dots, \lambda_n]$ and dynamic modes $\Phi^{\text{is-OMD}} = [\Phi_1, \dots, \Phi_n]$ obtained from algorithm 2, it is possible to obtain improved approximation of intermediate snapshots corresponding to timesteps $i \notin \mathcal{J}$.

To do this, consider the interpretation of the modal decomposition output described through (6.12). At each of the known analysis data snapshots $(\mathbf{u}_k)_{k \in \mathcal{J}}$ we consider the projection $(\Phi^{\text{is-OMD}})^\top \mathbf{u}_k =: (a_j)_{j=1}^r \in \mathbb{R}^r$ and subsequently extrapolate via

$$\tilde{\mathbf{w}}_i = \sum_{j=1}^n \lambda_j^i a_j \Phi_j, \quad k = 1, \dots, n_k - 1, \quad (6.16)$$

where n_k is the number of minimal timesteps δt between successive snapshots $\mathbf{u}_k$ and $\mathbf{u}_{k+1}$.

The above technique is applied to the system (6.11) with the same parameters as in §6.4.1, with eigenvalues λ_j and dynamic modes Φ_j obtained with algorithm 2, to produce intermediate

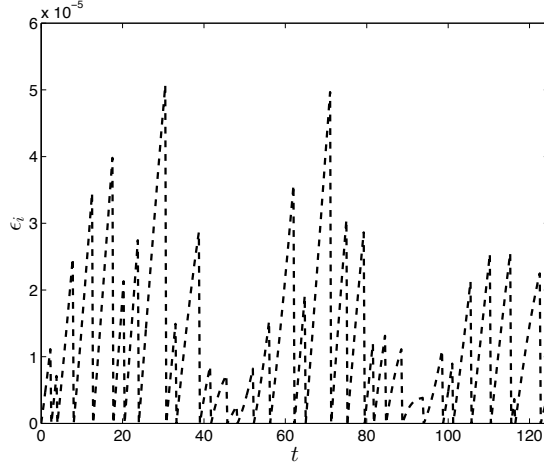


Figure 6.6: Temporal evolution of the snapshots reconstruction error $\epsilon_i = \frac{\|\tilde{\mathbf{w}}_i - \mathbf{w}_i^{ref}\|^2}{\|\mathbf{w}_i^{ref}\|^2}$ where $\tilde{\mathbf{w}}_i$ is defined by (6.16).

snapshots $\tilde{\mathbf{w}}_i$. Figure 6.6 presents the temporal evolution of the snapshot error $\epsilon_i = \|\tilde{\mathbf{w}}_i - \mathbf{w}_i^{ref}\|^2 / \|\mathbf{w}_i^{ref}\|^2$, where it can be seen that due to the linear nature of the system, the error increases linearly when $\tilde{\mathbf{w}}_i$ is estimated further away from known snapshots $\mathbf{u}_i$. However, since the eigenvalues λ are estimated accurately, snapshot extrapolation gives a small error $\epsilon_i < 10^{-4}$.

6.5 A Comparison with Compressive Sampling

The statistical performance of both is-OMD and Compressive Sampling in terms of the ability to extract frequency content from undersampled data will be analysed on the one-dimensional synthetic sinusoidal flow

$$f(\mathbf{x}, t) = \sum_j A_j \sin(k_j \mathbf{x} + \omega_j t), \quad 0 \leq \mathbf{x} \leq 1, \quad t > 0, \quad (6.17)$$

with amplitudes $A_j > 0$, frequencies $\omega_j > 0$ and wavenumbers $k_j > 0$. Similarly to the system in §6.4 reference data set $\mathcal{W} = \{\mathbf{w}_i\}_{i=1}^{900}$ is generated where each snapshot

$$\mathbf{w}_i = \left(f\left(\frac{j}{N_u}, i\delta t\right) \right)_{j=0}^{N_u}, \quad 0 = 1, \dots, 900,$$

is sampled at $N_u = 10^4$ equally spaced points in $0 \leq \mathbf{x} \leq 1$. Algorithm 1 will be compared with the compressive sampling approach developed in [33], which will be referred to here as CS-DMD. CS-DMD is implemented by first obtaining an approximation of the fully-resolved data set, $\mathcal{W}$, from the analysis data set $\mathcal{U}$ using a CoSaMP algorithm [80]. Subsequently, the

DMD eigenvalues corresponding to $\mathcal{W}$ are obtained. Accuracy is described by the proportional error in extracted frequency

$$\epsilon_\lambda := \frac{|\omega_j - \text{Im}(\lambda)|}{\omega_j},$$

where λ are the eigenvalues obtained by each of the two methods.

6.5.1 Robustness of CS-DMD and is-OMD

We consider data sampled from (6.17) in the case of two frequencies $\omega_1 = \frac{\pi}{20}, \omega_2 = \frac{2\pi}{25}$, with respective amplitudes $A_1 = 1, A_2 = 0.5$, and minimal timestep $\delta t = 1$. The analysis data ensemble $\mathcal{U}$ is selected with an average sampling timestep satisfying

$$\frac{\Delta t_{av}}{\delta t_1^*} = \frac{7}{8}, \quad \frac{\Delta t_{av}}{\delta t_2^*} = \frac{6}{5}.$$

as in §6.4.

The first step of CS-DMD is to apply compressive sampling to $\mathcal{U}$ to form an approximation $\hat{\mathcal{W}}$ to the fully-resolved data ensemble $\mathcal{W}$. Recall from §2.3.1 that this achieved by (i) choosing and fixing a temporal basis Ξ , then (ii) successively adding a number, k say, of elements of this basis which best correlate with the available data. Motivated by the fact that the underlying data sampled from (6.17) is sparse in the Fourier basis, we set Ξ to be the inverse Fourier Transform and obtain approximations $\hat{\mathcal{W}}_k \approx \mathcal{W}$ corresponding to extracting $k = 4, 8$ and 20 dominant basis vectors. DMD is then applied to each $\hat{\mathcal{W}}_k$ to determine the identified eigenvalue information.

Since we do not want to assume *a-priori* the number of dynamically important structures present in a given data set, it is desirable that the accuracy of the frequency information provided by CS-DMD behaves predictably with the requested number of basis elements k . Figure 6.7 indicates that this is not the case. Indeed, the frequencies ω_i are accurately identified only in the case $k = 8$, with $k = 4$ failing to identify the higher frequency ω_2 while the choice $k = 20$ results in an under-approximation of ω_1 .

For $k = 4$, inaccuracy arises from the discrete Fourier frequencies implied by the chosen basis Ξ . Since the fully-resolved data ensemble contains 900 samples, the chosen basis contains only discrete frequencies $\omega_F = \left(\frac{2\pi n}{900}\right)_{n=0}^{900}$, k of which are used to express the data. For this particular example, no single ω_F coincides with $\omega_1 = \pi/20$. Instead there are multiple values of

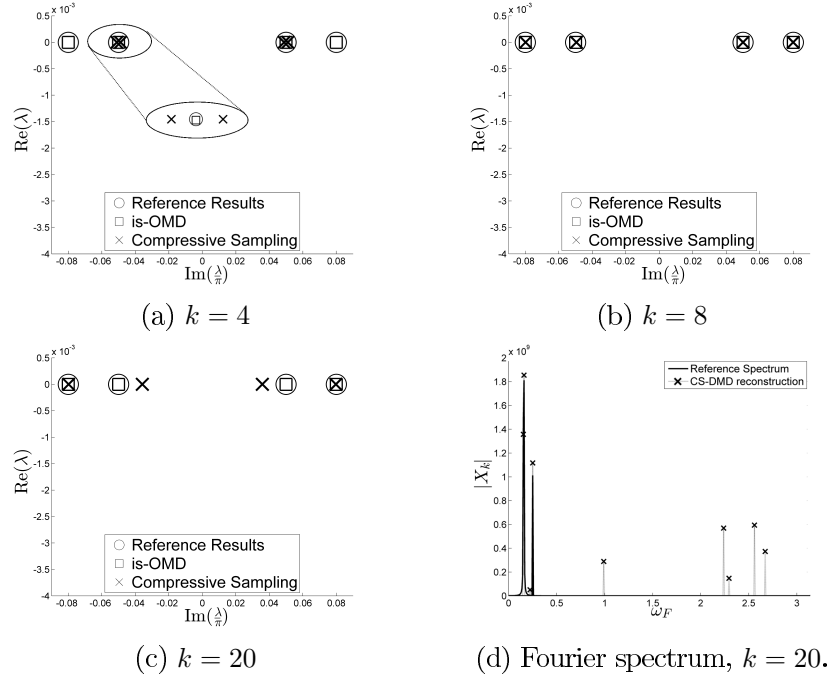


Figure 6.7: The DMD eigenvalue lattice obtained using snapshots reconstructed with CS-DMD for different number of sparse structures (a)-(c), with λ_{is-OMD} is given for $r = 45$. The Fourier spectrum obtained by compressive sampling for $k = 20$ is shown (d)

ω_F in a small neighborhood of ω_1 and, since $A_1 > A_2$, the CoSamp algorithm applied with $k = 4$ (incorrectly) identifies a pair of frequencies ω_F close to ω_1 . The result is improved for $k = 8$, which is sufficiently large to resolve both ω_1 and the higher frequency ω_2 . However, for $k = 20$ it is shown in figure 6.7(d) that compressive sampling identifies Fourier frequencies ω_F not present in the underlying system, in addition to those close to the true values $\omega/\pi = 0.05, 0.08$. Upon application of DMD to $\mathcal{W}_k$, these extra frequencies result in an incorrect identification of ω_1 .

Such a lack of robustness with respect to k is not evident in the application of algorithm 1 to this example, which, as can be seen in figure 6.7 (a)-(c), correctly identifies the two underlying frequencies in the data using a model of order $r = 45$. The is-OMD results in figure 6.7 (a)-(c) present only four eigenvalues of interest. The eigenvalues of relevance were chosen based on the energetic criterion similar to approaches in [6, 36, 45] where the magnitudes of the DMD modes are examined. Here, the magnitude of a solution to (6.15), $\|\Phi_k\|_2$, is assessed and eigenvalues with the largest magnitude of the associated modes are selected. Note that, if $\mathbf{V}_{\mathcal{J}}$ contains dynamically unimportant eigenvalues with large decay rate, the associated modes will have an excessively large magnitude in order to compensate for the low value of λ_j^{i-1} at later timesteps.

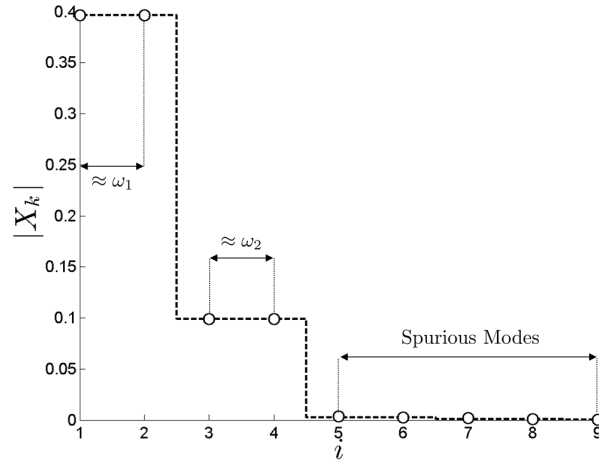


Figure 6.8: The relative contribution of the is-OMD modes.

Therefore, in order to obtain a meaningful ranking of modes based on the energy criterion, it is important to remove dynamically unimportant rows of $\mathbf{V}_{\mathcal{J}}$ using algorithm 2. Here, (6.15) is solved using $\mathbf{V}_{\mathcal{J}}$ conditioned with $\tilde{t} = 10^{-3}$ retaining nine rows of Vandermonde matrix, and consequently nine structures. Figure 6.8 shows a relative energetic contribution of each mode to the system dynamics defined as $|X_k| = \frac{\|\Phi_k\|_2}{\sum_j \|\Phi_k\|_2}$, demonstrating a clearly visible drop in $\|\tilde{\Phi}_k\|_2$ between dynamically important and unimportant eigenvalues and allows the identification of frequencies presented in figure 6.7(a)-(c).

Note that another variable prescribed *a priori* in the CS-DMD methodology is the rank r of the DMD approximation, which is upper bounded by the number of extracted basis vectors k . The CS-DMD results presented in figure 6.7 were obtained with DMD approximation of rank $r = 4$ as opposed to $r = 45$ of the is-OMD results. The reason for selecting this rank for the CS-DMD approximation can be explained if the maximal case $r = k$ is instead considered. With this choice, DMD eigenvalue information will replicate the Fourier frequencies which most accurately correlate with the analysis data ensemble, $\mathcal{U}$. Figure 6.9(a) presents the relative contribution of each mode for $r = k = 20$. Although the frequencies corresponding to the most dominant modes are close to the true values, it is nonetheless not clear from figure 6.9(a) how to choose a threshold which will allow frequencies of dynamical importance to be retained. A similar situation occurs in figure 6.9(c) for $r = k = 8$.

For $r = k = 8$ the eigenvalues coincide with ω_F which lie in a small neighbourhood of ω_1 and ω_2 with a small error ϵ_λ corresponding to those frequencies. However, it is shown in figure 6.7(b) that if a combination $k = 8, r = 4$ is used, ϵ_λ decreases further. Based on these

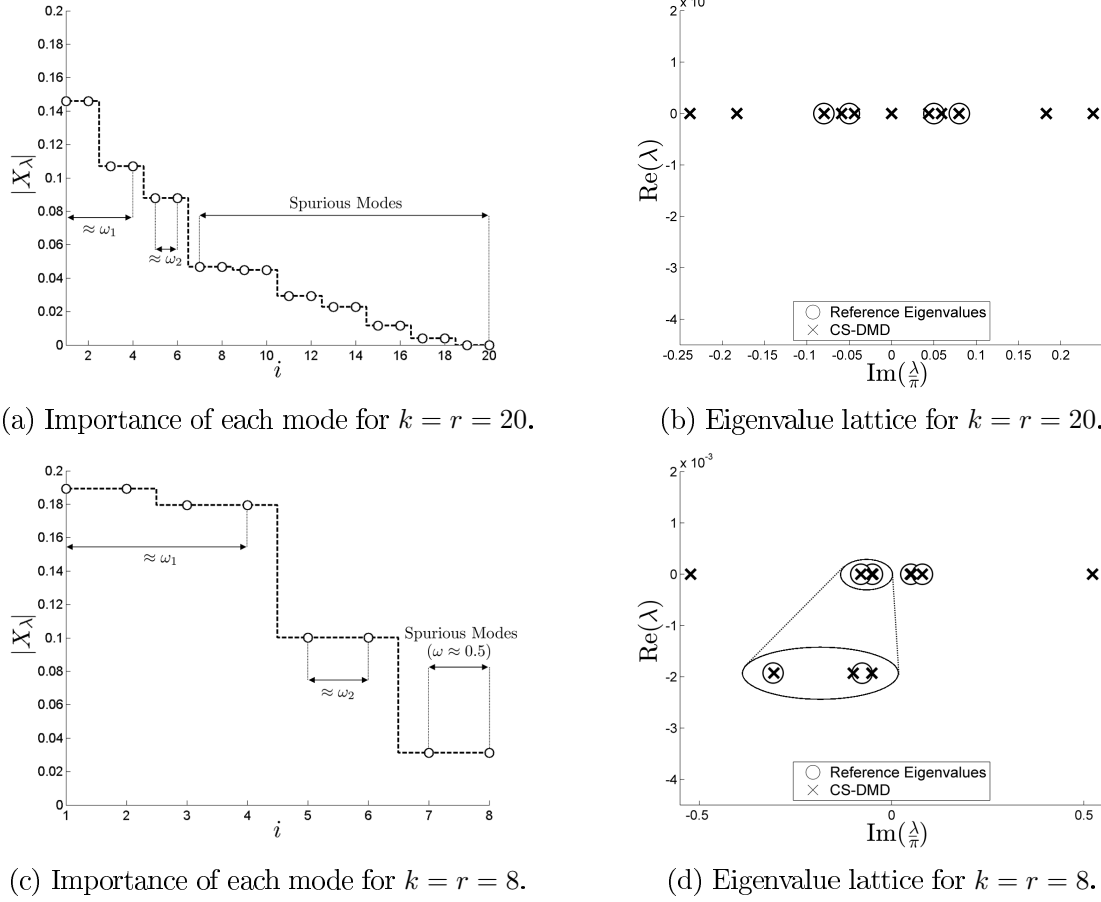


Figure 6.9: The output of the CS-DMD methodology for different values of k and r where $k = r$. The importance of modes and corresponding eigenvalue spectrum for $k = r = 20$ is shown in (a)-(b) while those corresponding to $k = r = 8$ are presented in (c)-(d).

observations, in §6.5.2 where a statistical sample of sinusoidal flows is investigated, for the CS-DMD methodology a combination of $k = 8$ and $r = 4$ is used.

6.5.2 Frequency estimation analysis: a statistical comparison between Compressive Sampling and is-OMD

To study the differences in performance of is-OMD and Compressive Sampling, both methodologies were applied to the system (6.17) for a range of frequencies and average sampling rates. In particular, the lower frequency is fixed, as above, to $\omega_1 = \frac{\pi}{20}$, while the higher system frequency takes one of values

$$\omega_2(n) = \omega_1 + \frac{n\pi}{100}, \quad n = 1, \dots, 20.$$

Subsequently, for each of value of $\omega_2(n)$ a number of downsampled analysis data ensembles $\mathcal{U}_{nm}$ are constructed whose average sampling rates satisfy

$$\frac{\Delta t_{av}(\mathcal{U}_{nm})}{\delta t_2^*} = 1 + \frac{m}{20}, \quad m = 1, \dots, 55,$$

that is, each exceeds the Nyquist criterion to differing degrees. This is achieved by constructing fully-resolved data sets

$$\mathcal{W}_{nm} = \left\{ f \left(\mathbf{x}, i \left(\frac{1 + m/20}{1 + n/5} \right) \right) : 0 \leq \mathbf{x} \leq 1, i = 1, \dots, 900 \right\}$$

and sampling non-uniform analysis data ensembles $\mathcal{U}_{nm} \subset \mathcal{W}_{nm}$, each of cardinality $|\mathcal{U}_{nm}| = 45$. Note that the fully resolved data set is varied for each pair (n, m) in order that analysis data sets with different average downsampling rates (as a proportion of δt_2^*) can be constructed to possess the same proportion, 5%, of elements to the fully-resolved ensemble.

The CS-DMD and is-OMD algorithms are applied to each data set $\mathcal{U}_{nm}$. Motivated by §6.5, we apply the CoSamp algorithm with sparse basis matrix Ξ as the inverse Fourier transform and set the sparsity level to $k = 8$. Algorithm 1 is initialised with $(M_0)_{ij} = N(0, \sigma^2) + \delta_{ij}$, that is, a small perturbation of the identity matrix with $\sigma = \sqrt{0.3}$, and $\mathbf{L}_0 = \text{POD}(\mathcal{U}_{nm})$, with the modelling dimension set as $r = |\mathcal{U}_{nm}| = 45$.

Figures 6.10(a)-(b) show the proportional frequency error ϵ_λ corresponding to ω_1, ω_2 , respectively, averaged over the different values of $\Delta t_{av}(\mathcal{U}_{nm})/\delta t_2^*$. Note that since the data is sampled in such a way to achieve a particular value of $\Delta t_{av}/\delta t_2^*$, the corresponding downsampling ratio for δt_1^* is $\Delta t_{av}/\delta t_1^* = (\Delta t_{av}/\delta t_2^*)(\omega_1/\omega_2)$, meaning that $\Delta t_{av}/\delta t_1^*$ are not similarly grouped. Consequently, a discrete set of values is defined, $\mathcal{S} = \{1.02, 1.04, \dots, 1.92\}$, and for each element $\mathcal{S}_i$ results satisfying $|\Delta t_{av}/\delta t_1^* - \mathcal{S}_i| < 0.01$ are grouped together and averaged accordingly. It is clear from figures 6.10 (a)-(b) that the proportional frequency error is reduced by application of algorithm 1 in comparison to CS-DMD.

Figure 6.10(c) shows the proportional frequency error ϵ_λ corresponding to ω_2 , as ω_2 is varied, each averaged over the 55 different values of the downsampling ratio $\Delta t_{av}/\delta t_2^*$. The CS-DMD error is approximately constant, while to is-OMD error decays as $|\omega_1 - \omega_2|$ increases. It should be noted that such a pattern can be observed if the OMD algorithm is applied to

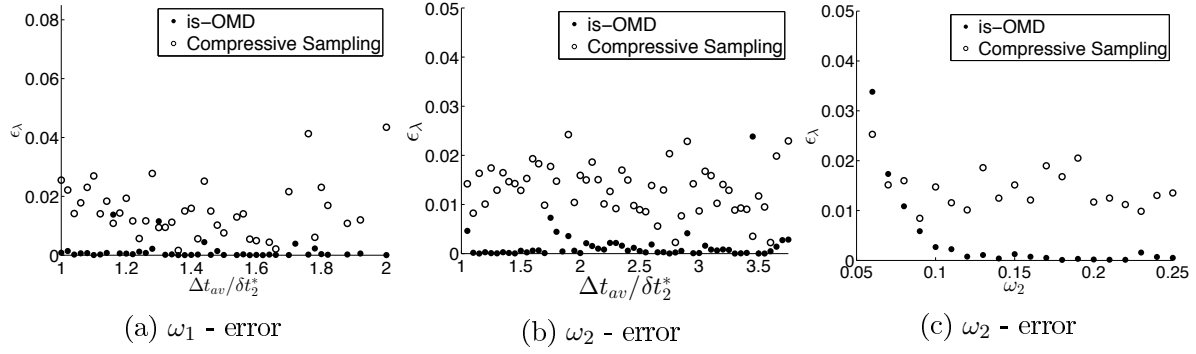


Figure 6.10: Proportional frequency error ϵ_λ averaged over ω_2 in (a)-(b); and downsampling ratio $\Delta t_{av}/\delta t_2^*$ in (c).

fully time-resolved snapshots $\mathcal{W}$ which contain observation noise. This is, pairs of eigenvalues satisfying $|\text{Im}(\lambda_i) - \text{Im}(\lambda_j)| \ll 1$ are sensitive to observation noise.

6.6 Numerical example: Cylinder Wake

Algorithms 1 and 2 are applied to the numerical results of the circular cylinder flow at $\text{Re} = 60$ and their performance is analysed on the limit cycle regime. This dynamical system exhibits a pair of vortices oscillating at a shedding frequency together with a series of higher harmonics; each oscillating at an integer multiple of the shedding frequency. These values can be extracted by applying the OMD algorithm to a time-resolved data ensemble.

The performance of is-OMD will be assessed by comparing the eigenvalue information between eigenvalues retained in the post-processing algorithm 2, $\Lambda^{\text{is-OMD}}$, and λ_{OMD} from the OMD algorithm applied to time-resolved data. Three aspects of algorithm initialisation and their impact on the extraction of dynamic information are discussed here: the initialization of $\mathbf{L}_0$ and $\mathbf{M}_0$; the projection rank r ; and sampling strategy $\mathcal{J}$. A downsampled data set $\mathcal{U}$ is extracted randomly from a reference data set $\mathcal{W}$ uniformly sampled at $\delta t = 0.25s$ containing 30 snapshots per shedding period. Here, it is convenient to introduce the following quantities related to the sampling strategy: $\Delta t_{av} = \frac{\sum_{i \in \mathcal{J}} \Delta t_i}{|\mathcal{J}|}$, $\Delta t_{min} = \min\{\Delta t_i : i \in \mathcal{J}\}$ and $\Delta t_{max} = \max\{\Delta t_i : i \in \mathcal{J}\}$. Unless stated otherwise $\mathcal{U}$ is sampled such that $\Delta t_{av} = 10\delta t$ without constraints on Δt_{min} and Δt_{max} . Applying a Nyquist criterion to data downsampled at a constant $\Delta t = 10\delta t$, such downsampling is sufficient to resolve the vortex shedding frequency but not satisfactory if the objective is to identify higher frequency modes.

6.6.1 Initial condition analysis

To assess the most appropriate properties of $\mathbf{L}_0$ and $\mathbf{M}_0$ to initialise is-OMD, three classes of initial condition are considered:

1. $\mathbf{L}_0$ is set to POD modes of the known data, $\mathbf{U}$, and an identity matrix chosen to approximate $\mathbf{M}_0$, i.e. $\mathbf{L}_0 = \text{POD}(\mathbf{U})$, $\mathbf{M}_0 = \mathbf{I}_r$;
2. $\mathbf{L}_0 = \text{POD}(\mathbf{U})$ and an identity matrix with entries perturbed by noise used for $\mathbf{M}_0$: $(M_0)_{ij} = N(0, \sigma) + \delta_{ij}$;
3. OMD is applied to data undersampled with a constant timestep Δt , $\bar{\mathbf{U}}$, to obtain $\mathbf{L}_0$ and $\mathbf{M}_0$, i.e. $[\mathbf{M}_0, \mathbf{L}_0] = \text{OMD}(\bar{\mathbf{U}})$.

The motivation for $[\mathbf{M}_0, \mathbf{L}_0] = \text{OMD}(\bar{\mathbf{U}})$ is that in systems exhibiting limit cycle behaviour, their dynamics do not qualitatively change with time. In such a case two sets of data can be acquired, one with non-uniform Δt which is used as an input to is-OMD, $\mathbf{U}$, and a second one which is uniformly subsampled, $\bar{\mathbf{U}}$. In such a case $\bar{\mathbf{U}}$ is not sufficient to obtain all the dynamics of interest, however, the output of OMD can still be used for $\mathbf{L}_0$ and $\mathbf{M}_0$.

In order to consider the role of the initial conditions, the same downsampling strategy and approximation rank r are used for all three cases. In particular, $\mathcal{U}$ is sampled non-uniformly, with $\Delta t_{av} = 10\delta t$ and without constraints on Δt_{min} and Δt_{max} , while the projection rank is set to $r = 75$. The dimensions of both data sets $\mathcal{W}$, $\mathcal{U}$ are $\mathbf{W} \in \mathbb{R}^{N_u \times 750}$, $\mathbf{U} \in \mathbb{R}^{N_u \times 75}$, i.e. $|\mathcal{I}| = 750$ and $|\mathcal{J}| = 75$. The $\bar{\mathbf{U}} \in \mathbb{R}^{N_u \times 75}$ was sampled such that $\Delta t_{av} = 10\delta t$ which is sufficient to resolve a timescale of $\frac{2}{3}T$, where T is a shedding frequency. Note, that cylinder flow contains harmonics with timescales of $\frac{T}{n}$ for $n \in \mathbb{Z}^+$, meaning that $\frac{2}{3}T$ is not sufficient to resolve the subharmonic with a timescale of $\frac{T}{2}$.

The extracted dynamical information for $\mathbf{M}_0 = \mathbf{I}_r$, $[\mathbf{M}_0, \mathbf{L}_0] = \text{OMD}(\bar{\mathbf{U}})$ and $(M_0)_{ij} = N(0, \sigma) + \delta_{ij}$ with $\sigma = 10^{-2}, 10^{-3}$ are presented in figure 6.11. In all of the analysed cases, only the eigenvalues retained in algorithm 2 for $\tilde{t} = 0.3$ are presented. The Strouhal number is defined as $St = fD/U_\infty$, where D is the cylinder diameter and U_∞ is the inflow velocity. Assuming that flow field was nondimensionalised using a cylinder diameter and an inflow velocity, the Strouhal number can be extracted from the phase information of each eigenvalue, $St = \text{Im}(\lambda)/(2\pi)$. The magnitude of the eigenvalue provides the information about the growth rate, which is expressed

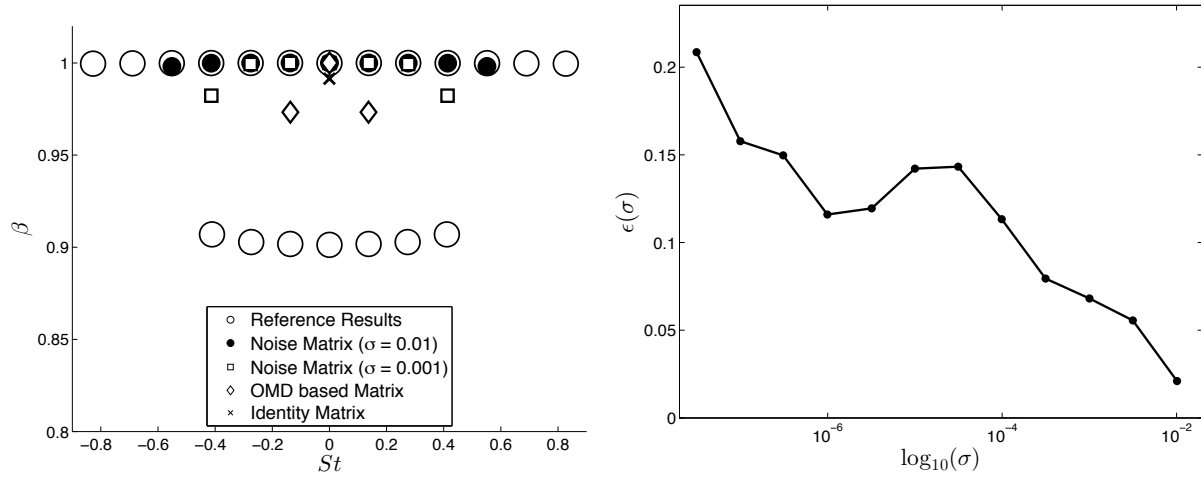


Figure 6.11: Eigenvalues of the dynamic matrix M obtained using the is-OMD algorithm for different initial conditions (left). The error in eigenvalue magnitude for identity matrices with $\sigma \in [10^{-9}, 10^{-2}]$ (right), obtained for the projection rank $r = 75$.

in such a way so that growth rate, β , follows β^t , i.e. $\beta = \exp(\text{Re}(\lambda))$. The reference eigenvalues in figure 6.11 were obtained by applying the OMD algorithm to $\mathbf{W}$ for $r = 30$ and consist of the asymptotic eigenvalues representing the limit cycle behaviour and eigenvalues at $\beta = 0.9$ characterising relaxation of the flow onto its limit cycle [24]. The primary objective is to assess the accuracy at which the is-OMD algorithm approximates the asymptotic eigenvalues since the values at $\beta = 0.9$ correspond to dynamics at extremely low energy for the chosen data set $\mathbf{W}$.

In the case where $\mathbf{M}_0 = \mathbf{I}_r$, a majority of the eigenvalues of $\mathbf{M}$ are characterised by $\text{Re}(\lambda) \ll 0$ and are discarded in algorithm 2 due to the choice of $\tilde{t}$. None of the frequencies present in the system are recovered. For $[\mathbf{M}_0, \mathbf{L}_0] = \text{OMD}(\overline{\mathbf{U}})$ only frequencies within timescale range of $\overline{\mathbf{U}}$ are correctly estimated (i.e. $St \in [-0.2, 0.2]$). Figure 6.11 demonstrates that noise must be added to $\mathbf{M}_0$ in order to extract high frequency harmonics. It is interesting to note that as the standard deviation increases, more frequency information is recovered. In particular, for an order of magnitude increase in σ , an additional conjugate pair of eigenvalues is correctly identified and the growth rate of all eigenvalues, β , is more accurate.

In order to further demonstrate a beneficial influence of noise addition a repeated analysis was performed for $(M_0)_{ij} = N(0, \sigma) + \delta_{ij}$ with $\sigma \in [10^{-9}, 10^{-2}]$. Accuracy in the estimation of seven most dominant eigenvalues is analysed and the proportional error is defined as $\epsilon := \frac{\sum_{i=1}^7 |\lambda_i^{\text{true}} - \lambda_i^{\text{is-OMD}}|}{\sum_{i=1}^7 |\lambda_i^{\text{true}}|}$. Figure 6.11 demonstrates that ϵ decreases as σ increases. It is clear that in order for the is-OMD algorithm to successfully recover dynamical information a certain

degree of flexibility has to be included in $\mathbf{M}_0$. For $\mathbf{M}_0 = \mathbf{I}_r$ and $[\mathbf{M}_0, \mathbf{L}_0] = \text{OMD}(\overline{\mathbf{U}})$ the dynamics corresponding to dominant structures present in the system, i.e. constant mean mode and shedding vortices, are strongly embedded in the initial matrix structure. In such cases the objective function can be significantly reduced in few iterations, indicating that such a $\mathbf{M}_0$ predetermines a local minimum of (6.3). This problem appears to be avoided simply by randomly perturbing the initial matrices. Note that it is possible to adjust a frequency range of the eigenvalue spectrum of $(M_0)_{ij} = N(0, \sigma) + \delta_{ij}$ by adjusting a value of σ , where larger value of σ will cover a larger range of frequencies. If the system exhibits primarily low-frequency dynamics it is desirable to set σ to a lower value such that an eigenvalue spectrum of $\mathbf{M}_0$ is in a neighbourhood of the expected range of the frequencies of interest.

6.6.2 Rank size analysis

It was demonstrated in §6.5.1 that setting a high value of k in the CS-DMD approach induces Fourier modes corresponding to high frequency oscillations not present in the time-resolved data. It is therefore important to consider the impact of the equivalent parameter in the is-OMD algorithm, that is, the dimension r of $\text{Im}(\mathbf{L})$. It will be demonstrated here that larger r improves accuracy in the recovered eigenvalues, which consequently leads to an improved accuracy of the modes $\Phi^{\text{is-OMD}}$ in algorithm 2. An analysis in terms of the eigenvalue and mode estimation is performed for $r = [25, 50, 75]$ using the same sampling strategy as in §6.6.1 and setting $(M_0)_{ij} = N(0, 0.3) + \delta_{ij}$, $\mathbf{L}_0 = \text{POD}(\mathbf{U})$.

Figure 6.12 presents eigenvalue lattices for different values of r and demonstrates a positive impact of increasing r on the accuracy of the dynamic estimation. When r increases, a progressively greater number of asymptotic eigenvalues is recovered and their growth rates β converge towards the correct value of 1. The number of modes which can be calculated is proportional to the number of recovered eigenvalues. For r of 25, 50 and 75 there are respectively one, three, and four conjugate pairs of eigenvalues that are estimated with small error, and with growth rate $\beta > 0.95$. The dynamic modes are estimated as a solution to (6.15) which in turn depends upon $\mathbf{V}_{\mathcal{J}}$ being constructed using accurate eigenvalues. Figure 6.14 shows dynamic modes corresponding to eigenvalues with $\beta > 0.95$ obtained for $r = 75$ and it demonstrates that the mode shapes closely resemble the coherent structures obtained from time-resolved data (figure 6.13). The dynamic modes of similar precision can be obtained for other eigenvalues presented

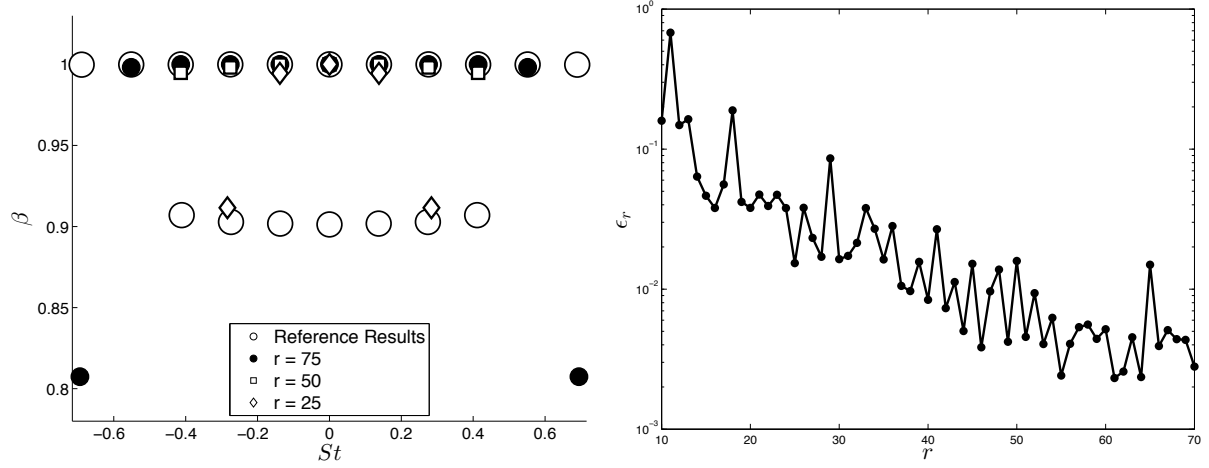


Figure 6.12: Eigenvalues of the dynamic matrix $\mathbf{M}$ for $r = [25, 50, 75]$ (left). The error $\epsilon_r = \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \frac{\|\tilde{\mathbf{W}} - \mathbf{W}^{ref}\|^2}{\|\mathbf{W}^{ref}\|^2}$ for $(M_0)_{ij} = N(0, 0.3) + \delta_{ij}$ with different r (right).

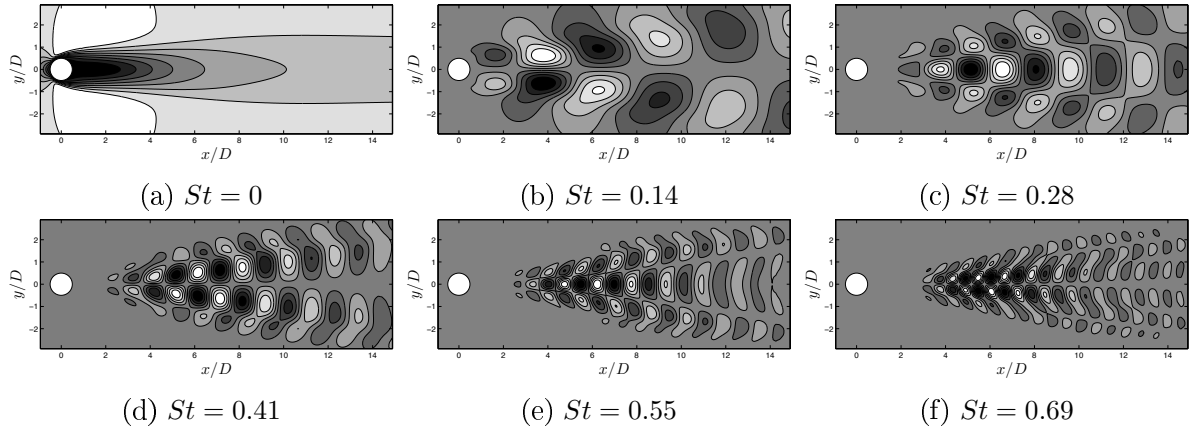


Figure 6.13: Modes $\Phi = \mathbf{L}\mathbf{P}_M$ obtained by applying the OMD algorithm to a time-resolved cylinder flow data.

in 6.12 where $\beta > 0.95$ therefore mode shapes for $r = 25$ and $r = 50$ are omitted here.

Note that for $r = 25$ and $r = 75$ a pair of eigenvalues is recovered which contains an error in β . Figure 6.15 shows the corresponding dynamic modes. Although, these structures are not as coherent as for eigenvalues where $\beta > 0.95$, there is nonetheless a resemblance to the structures present in the flow (figure 6.13). It is expected that when r is increased even further the eigenvalue accuracy improves leading to better precision in the estimation of the dynamic modes. In practice when $\mathbf{L}_0 = \text{POD}(\mathbf{U})$ and $(M_0)_{ij} = N(0, \sigma) + \delta_{ij}$ are used, r is only limited by $|\mathcal{J}|$. This quantity is usually much higher than for the results presented here and the choice of r would be limited only by the processing cost.

When unknown data sets are reconstructed using (6.16), the value of r impacts the accuracy of estimated snapshots, $\tilde{\mathbf{W}} = [\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_N]$ for $i \notin \mathcal{J}$, through the precision and number of

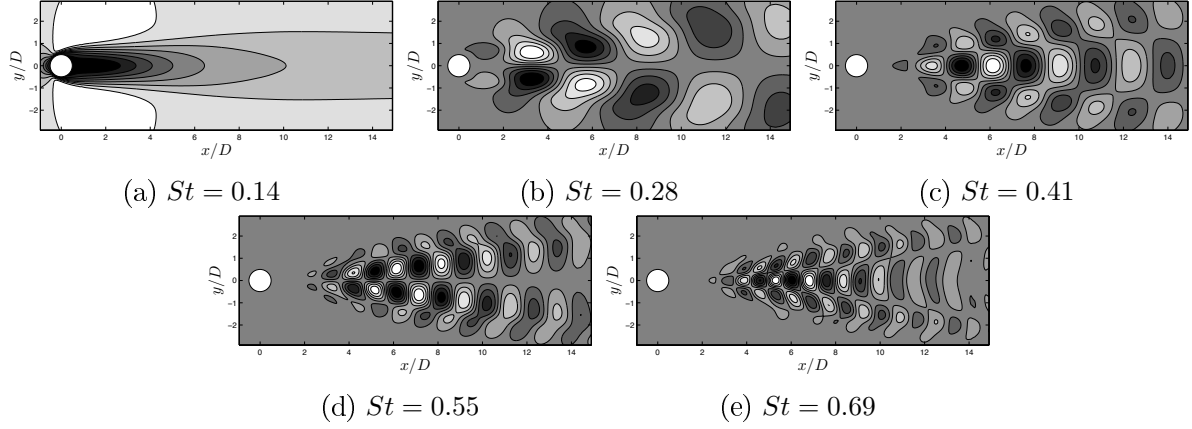


Figure 6.14: Modes Φ^{is-OMD} corresponding to eigenvalues with $\beta > 0.95$ obtained using the is-OMD algorithm for $r = 75$.

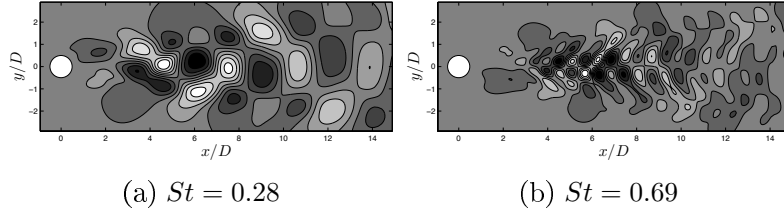


Figure 6.15: Modes Φ^{is-OMD} corresponding to eigenvalues presented in figure 6.12 with $\beta < 0.95$; $St = 0.28$, $\beta = 0.91$ for $r = 25$ (left) and $St = 0.69$, $\beta = 0.81$ for $r = 75$ (right)

reconstructed eigenvalues λ_j and dynamic modes Φ_j and is expected to improve when r is increased. This fact is confirmed in figure 6.12 which presents the evolution of the error norm, $\epsilon_r = \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \frac{\|\tilde{\mathbf{W}} - \mathbf{W}^{ref}\|^2}{\|\mathbf{W}^{ref}\|^2}$ for $r \in [10, 70]$. There is a visible decay in the value of ϵ_r , where ϵ_r is less than 5% for $r \geq 30$. Note that despite being governed by nonlinear Navier-Stokes equations, cylinder flow in the limit cycle is known to be well approximated by a linear model consisting of oscillating structures. For more dynamically complex systems, the value of ϵ_r is expected to have a larger magnitude than values presented in figure 6.12.

Finally, we note that the accuracy of $\tilde{\mathbf{W}}$ depends upon the value of λ_j used in (6.16). The number and choice of λ_j is determined based on the number of retained rows in $\mathbf{V}_{\mathcal{J}}$, which is conditioned with an appropriate value of $\tilde{t}$. When $\tilde{t}$ is small, a majority of λ_j in $\mathbf{V}_{\mathcal{J}}$ are retained including dynamically unimportant values, introducing an error in $\tilde{\mathbf{W}}$. On the other hand, if $\tilde{t}$ is large, dynamically relevant λ_j are removed. In order to find an appropriate value of $\tilde{t}$ and consequently the number of retained λ_j let us compare snapshots estimated using (6.16) with reference measurements. Starting an extrapolation at each $i \in \mathcal{J}$ a snapshot is estimated at the timestep when the next measurements is acquired, i.e. $\tilde{\mathbf{w}}_{i+n_i}$, where $n_i := \frac{\Delta t_i}{\delta t}$ is the number of minimal timesteps δt between the next element of $\mathcal{U}$. Performing a comparison with all of the

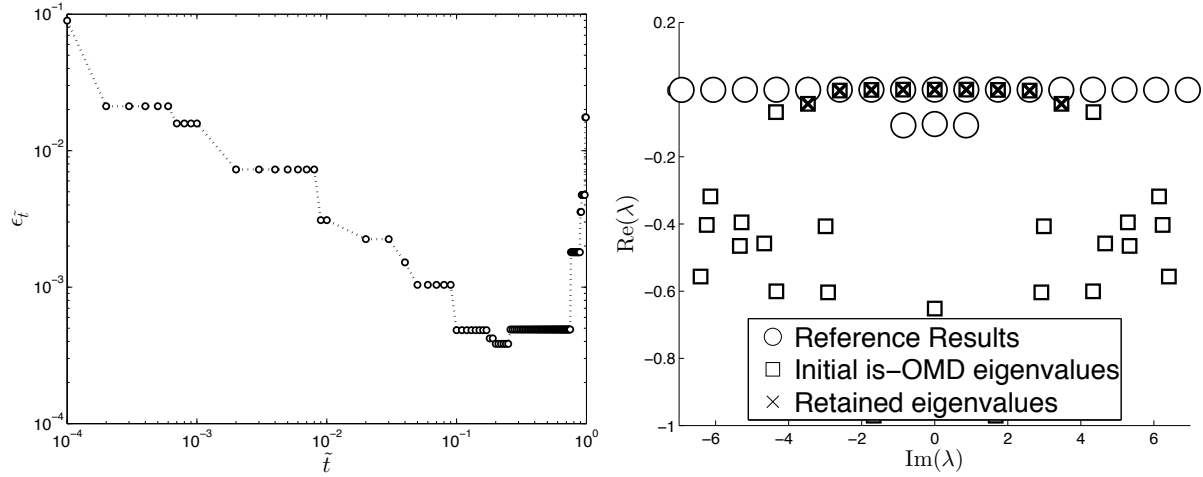


Figure 6.16: The error $\epsilon_{\tilde{t}} = \frac{1}{|\mathcal{J}|} \sum_{i \in \mathcal{J}} \frac{\|\tilde{\mathbf{w}}_{i+n_i} - \mathbf{u}_{i+n_i}\|}{\|\mathbf{u}_{i+n_i}\|}$ for different $\tilde{t}$ (left) and retained is-OMD eigenvalue spectrum for $\tilde{t} = 0.21$ (right).

available measurements, a proportional error metric $\epsilon_{\tilde{t}} = \frac{1}{|\mathcal{J}|} \sum_{i \in \mathcal{J}} \frac{\|\tilde{\mathbf{w}}_{i+n_i} - \mathbf{u}_{i+n_i}\|}{\|\mathbf{u}_{i+n_i}\|}$ is defined, and $\tilde{t}$ is estimated such that the resulting $\tilde{\mathbf{w}}_{i+n_i}$ minimises $\epsilon_{\tilde{t}}$.

An example for the cylinder flow is-OMD results with $r = 70$ is presented in figure 6.16 which shows $\epsilon_{\tilde{t}}$ for $\tilde{\mathbf{w}}_{i+n_i}$ estimated with $\tilde{t} \in [10^{-4}, 1]$. For this particular case $\epsilon_{\tilde{t}}$ is minimised at $\tilde{t} = 0.21$ where $\tilde{\mathbf{W}}$ is calculated using nine most dominant eigenvalues (figure 6.16). Although $\tilde{t} = 0.21$ gives the smallest values of $\epsilon_{\tilde{t}}$ there is a clear trough visible for $\tilde{t} \in [0.01, 0.75]$ and setting $\tilde{t}$ to any value in that range will give a good estimation of snapshots at $i \notin \mathcal{J}$. In figure 6.12, each value of ϵ_r was obtained for $\tilde{\mathbf{W}}$ estimated with the smallest value of $\tilde{t}$.

6.6.3 The importance of random sampling

One of the key properties of the compressive sampling methodology, described in §2.3.1, is its reliance on non-uniform sampling [28]. Here, it is demonstrated that is-OMD can also be improved by random sampling. Three different sampling strategies are analysed, with approximately the same $|\mathcal{J}|$ and Δt_{av} for different Δt_{min} and Δt_{max} . Depending on Δt_{min} , it is possible to use the Nyquist sampling criterion to estimate the range of different timescales that can be recovered as a multiple of the shedding timescale T . Setting $(M_0)_{ij} = N(0, 0.3) + \delta_{ij}$, $\mathbf{L}_0 = \text{POD}(\mathbf{U})$ and $r = 70$, is-OMD is applied to the following sampling strategies:

1. Uniform sampling: $\mathcal{U}$ is sampled uniformly with $\Delta t = 10\delta t$. Such a sampling strategy corresponds to three snapshots per shedding period and is sufficient to resolve timescales greater than $\frac{2}{3}T$. In particular $|\mathcal{J}| = 79$ and $|\mathcal{I}| = 781$.

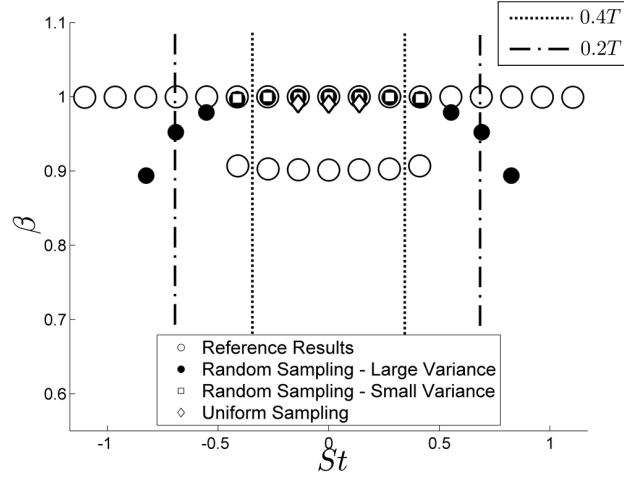


Figure 6.17: Eigenvalue lattice for different sampling strategies.

2. Random sampling (large variance): $\mathcal{U}$ is sampled non-uniformly such that $\Delta t_{min} = 3\delta t$ ($0.2T$) and $\Delta t_{max} = 17\delta t$ ($1.15T$), which results in $|\mathcal{J}| = 80$ and $|\mathcal{I}| = 783$. Together Δt_{min} and Δt_{max} encompass 3% of $|\mathcal{J}|$.
3. Random sampling (small variance): $\mathcal{U}$ is sampled non-uniformly such that $\Delta t_{min} = 6\delta t$ ($0.4T$) and $\Delta t_{max} = 14\delta t$ ($0.95T$), which results in $|\mathcal{J}| = 79$ and $|\mathcal{I}| = 783$. Together Δt_{min} and Δt_{max} encompass 5% of $|\mathcal{J}|$.

Figure 6.17 shows the eigenvalues retained using the algorithm 2 with $\tilde{t} = 0.1$ for all three cases. Recall that the limit cycle dynamics of the cylinder flow are characterised by structures oscillating with frequency $\frac{n}{T}$, where $n \in \mathbb{Z}^+$, and having corresponding timescales of $\frac{T}{n}$. For the cylinder data used at in the analysis, a uniform timestep $\Delta t = 10\delta t$ is high enough to recover T but not sufficient to extract higher frequency dynamics. Figure 6.17 shows that when the is-OMD algorithm is applied to uniformly sampled data, high frequency dynamics are not recovered. When non-uniform sampling strategies are used, although $\Delta t_{av} = 10\delta t$, there is enough phase variation due to changes in Δt to extract further frequencies. Both random strategies recover the dynamics within ranges determined by the Nyquist sampling criterion corresponding to Δt_{min} which are indicated in figure 6.17 with dotted lines. In both cases an additional pair of eigenvalues corresponding to frequency beyond *recoverable* bounds is identified. Additionally, it can be observed that higher deviation from Δt_{av} positively impacts the accuracy of the recovered dynamics, not only in terms of number of recovered frequencies but also in the estimation of β .

6.7 Application to the turbulent wake of an axisymmetric bluff body

The ability of the is-OMD algorithm to recover frequency information from sub-sampled data was demonstrated in §6.5 and §6.6 for systems with dominant periodicity. For a more challenging test-case, we consider the application of is-OMD to PIV snapshots of the turbulent wake of an axisymmetric bluff body. The descriptions of the experimental setup and major dynamical features of the bullet flow were outlined in chapter 4.

It is known [25, 26, 102] that a large-scale anti-symmetric structure corresponding to vortex-shedding is present in the wake and oscillates at the approximate vortex shedding frequency $St_{sh} \approx 0.22$. This is confirmed by examining the power spectrum in figure 6.18(h) of the streamwise-velocity, averaged over the region $D \leq \mathbf{x} \leq 1.72D, -0.375D \leq \mathbf{y} \leq 0.375D$ in which the shedding is expected to be observed. It is possible to identify this frequency and its associated structure by applying the OMD algorithm to the time-resolved data. Indeed, applying OMD to a data ensemble of $N = 1000$ snapshots, sampled at 720 Hz and setting $r = 30$ produces an OMD mode with frequency $\frac{D}{2\pi U_\infty} \text{Im}(\lambda) = 0.21$ whose real and imaginary parts are shown in figure 6.18(a),(d).

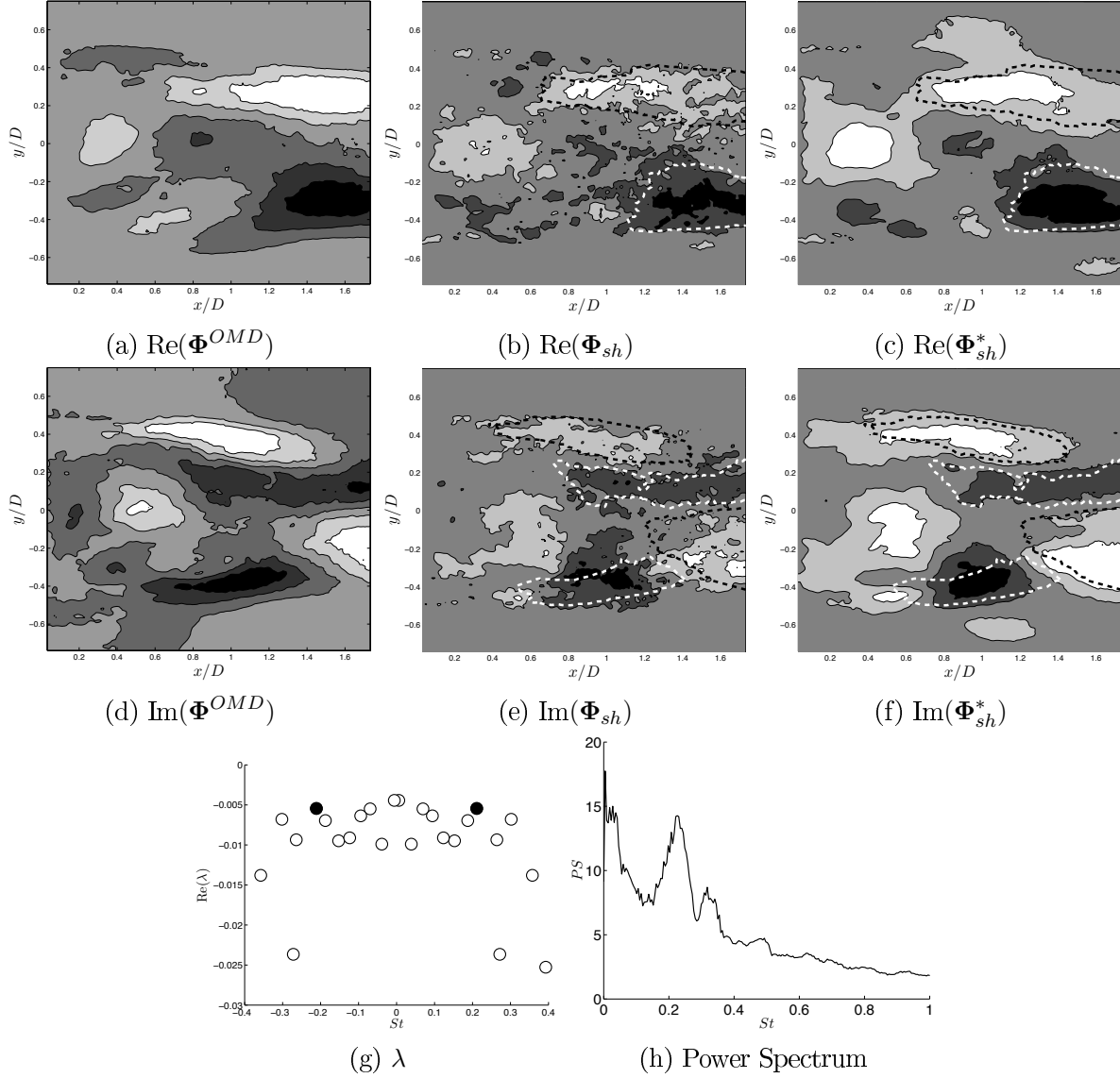


Figure 6.18: The OMD mode Φ^{OMD} corresponding to $St_D \approx 0.22$ obtained from time-resolved data is shown in (a), (d); the corresponding is-OMD mode Φ_{sh} in (b), (e); and its projection Φ_{sh}^* onto first 30 POD modes of U in (c), (f). Eigenvalues of the corresponding dynamic matrix $M^{\text{is-OMD}}$ are shown in (g) with the pair of shedding eigenvalues highlighted; (h) shows the mean power spectrum of streamwise velocities in a selected wake region.

It should be noted that, even when applied to the highly-resolved data ensemble, the frequency content is somewhat sensitive to the particular implementation of the OMD algorithm. For example, varying the length of the data ensemble $100 \leq N \leq 1000$ and the model rank $20 \leq r \leq 200$ provides OMD eigenvalues λ satisfying $0.18 \leq \frac{D}{2\pi U_\infty} \text{Im}(\lambda) \leq 0.26$. Such behaviour is not unexpected due to the relatively broad peak in the spectrum observed in figure 6.18(h) and we note that similar sensitivity was observed when applying the more widely employed DMD algorithm.

We now undersample the data to determine if is-OMD is able to extract the dominant vortex shedding frequency at $St_{sh} \approx 0.22$. In particular, it is assumed that only every 12th snapshot is available, resulting in a data ensemble sampled at 60 Hz. Following section 6.6.3 we apply the is-OMD algorithm to an analysis ensemble $\mathbf{U}$ formed by taking a random sample of the 60 Hz data. Specifically, we let the timestep snapshots in $\mathbf{U}$ be one of the values $\Delta t = j/60$, where $j \in \{1, 2, 3\}$, with a uniform probability of choosing any of the three possible values of Δt . This approach provides an undersampled data ensemble satisfying $\Delta t_{av}/\delta t_{Nyq} \approx 1.12$, that is, the average timestep Δt_{av} is greater than the one specified by the Nyquist criterion corresponding to the St_{sh} .

Frequency Identification: The spectrum $\lambda(\mathbf{M})$ obtained by is-OMD is shown in figure 6.18(g) and contains an eigenvalue pair corresponding to the shedding frequency at $St = 0.23$. This eigenvalue pair was selected based upon the proportion of data spanned by each of the modes (see §3.1) and, in this case, the shedding eigenvalue pair is ranked as the third most important. Two other dynamically significant eigenvalue pairs correspond to low frequency structures with $St = 0.006$ and $St = 0.07$, an order of magnitude smaller than St_{sh} . It was discussed in chapter 5 that wake structures are known to possess frequency content at approximately these values [26], specifically a random reorientation of the shedding reflectional plane ($St \approx 0.002$) and an axisymmetric pulsation of the vortex cores ($St \approx 0.06$).

Mode Extraction: For this example, the contribution of the shedding mode to the total energy of the wake is approximately 75% lower than that of the cylinder wake studied in section 6.6. Furthermore, although a (relatively wide) spectral peak at $St_{sh} \approx 0.22$ is visible in figure 6.18(h) a projection of the shedding structure onto the underlying data will not produce a purely periodic signal. For this reason, obtaining dynamic modes through an inversion of the Vandermonde matrix, which essentially assumes periodic evolution of the temporal coefficients between known snapshots, is overly restrictive. Instead, it is more appropriate to impose a looser rate constraint upon the unknown mode coefficients $\mathbf{c}_i$ appearing in the least squares

problem of the is-OMD algorithm. Specifically, by replacing step 3 of Algorithm 1 by

$$\mathbf{c}_i^k = \arg \min_{\mathbf{c}_i \in \mathbb{R}^r} \sum_{i=1}^N \|\mathbf{c}_i - \mathbf{M}\mathbf{c}_{i-1}\|^2, \quad (6.18a)$$

$$\text{s.t. } \mathbf{c}_i = \mathbf{L}^T \mathbf{u}_i, \text{ if } i \in \mathcal{J}, \quad (6.18b)$$

$$|(c_{i+1})_n - (c_i)_n| \leq \gamma \max_{j \in \mathcal{J}} |(c_j)_n|, \quad n = 1, \dots, r, \quad (6.18c)$$

where $(c_i)_n$ denotes the n^{th} element of $\mathbf{c}_i \in \mathbb{R}^r$ and $\gamma > 1$ is a parameter. In other words, a restriction is placed upon the coefficients $\mathbf{c}_i$ to ensure that between each minimal timestep, each of its elements does not vary by more than a proportion γ of its largest observed value on the known data. Subsequently, we let $\Phi = \mathbf{L}\mathbf{P}_M$ where $\mathbf{M} = \mathbf{P}_M^{-1} \Lambda \mathbf{P}_M$.

The real and imaginary parts of the mode Φ_{Sh} associated with $St = 0.23$ are presented in figure 6.18(b),(e) and were obtained by setting $\gamma = 3.75$. Although less coherent than the OMD modes obtained from time-resolved data, both $\text{Re}(\Phi_{sh})$ and $\text{Im}(\Phi_{sh})$ possess anti-symmetric and spatially out-of-phase structures present in the modes obtained from time-resolved data. For comparison, the spatial locations of the anti-symmetric structures visible in figures 6.18(a),(d) are overlaid and indicate reasonable agreement between the OMD and is-OMD modes. Furthermore, coherency of the mode can be improved through a projection $\Phi_{Sh}^* = \mathbf{U}_p \mathbf{U}_p^\top \Phi_{Sh}$, where $\mathbf{U}_p \subset \text{POD}(\mathbf{U})$, suppressing the spatial noise present in the structure. An example of Φ_{Sh}^* obtained through a projection using the first 30 POD modes of $\mathbf{U}$ is presented in figure 6.18(c),(f). We emphasise that the constraint (6.18c) is significantly less restrictive than the one imposed in the Vandemonde-based post-processing Algorithm 2. An interesting direction of future research will be to determine the optimal form of constraint required for effective spectral and mode information, dependent upon the intermittancy of the underlying data ensemble.

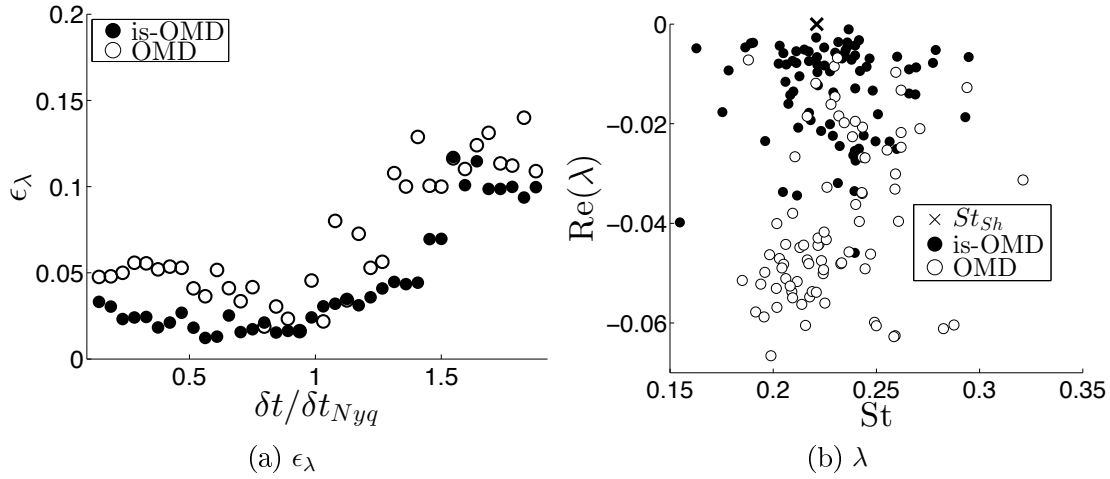


Figure 6.19: (a) Proportional eigenvalue error ϵ_λ for a range of $\delta t / \delta t_{Nyq}$; (b) the eigenvalues obtained for $\delta t / \delta t_{Nyq} < 1$ where the x-axis represents $St = \frac{D}{2\pi U_\infty} \text{Im}(\lambda)$.

Statistical Analysis: To quantify the improvement of is-OMD over OMD applied to downsampled data, a statistical analysis akin to that presented in §6.5.2 is performed. The experimental data is undersampled using a range of uniform timesteps δt to obtain datasets sampled at frequencies in the range 18 Hz – 240 Hz, which in terms of δt_{Nyq} is equivalent to $0.141 \leq \delta t / \delta t_{Nyq} \leq 1.875$. Subsequently, randomly-sampled data ensembles $\mathbf{U}$ are created in an analogous manner to the one used to obtain results in figure 6.18(b),(g), i.e. the timestep between known snapshots is $\Delta t = j\delta t$ where $j \in \{1, 2, 3\}$, which gives the average time step $\Delta t_{av} \approx 2\delta t$. The is-OMD algorithm is applied to $\mathbf{U}$ and the eigenvalue corresponding to the shedding structure is identified using the methodology described in §3.1. Note that due to the chosen sampling strategy $\mathbf{U}$ contains 50% of data used to obtain the OMD approximation. Finally, the error $\epsilon_\lambda = |\lambda_{sh} - \lambda| / |\lambda_{sh}|$ is computed, where λ corresponds to the eigenvalues obtained using either OMD algorithm applied to the uniformly downsampled data or the is-OMD algorithm applied to $\mathbf{U}$. The reference shedding eigenvalue λ_{sh} is defined as $\lambda_{sh} = j \frac{2\pi U_\infty}{D} St_{sh}$.

The eigenvalues obtained for $\delta t / \delta t_{Nyq} < 1$ and the values of ϵ_λ for the considered range of $\delta t / \delta t_{Nyq}$ are presented in figure 6.19. Due to the identified sensitivity of the OMD algorithm to the approximation rank r and of the is-OMD algorithm to the analysis data ensemble, figure 6.19(a) presents an average value of ϵ_λ corresponding to results obtained for $r \in [20, 30, \dots, 60]$ in the case of the OMD algorithm and, in the case of the is-OMD algorithm, for five different values of $\mathbf{U}$ each satisfying $\Delta t_{av} \approx 2\delta t$. Figure 6.19(a) demonstrates that the is-OMD algorithm provides a visible improvement for all values of δt except when $\delta t / \delta t_{Nyq} \approx 1$, where both

methods are similarly accurate. Further insight into the differences between both results can be obtained from figure 6.19(b), where the OMD output gives eigenvalues which are concentrated around a lower frequency than $\text{Im}(\lambda_{sh})$ and have more negative growth rate than is-OMD results. It is reiterated that the is-OMD results were obtained using only half of the data used to obtain the OMD approximation with $\Delta t_{av} > \delta t$, implying a beneficial impact of the randomness in the data acquisition on the accuracy of the eigenvalue estimation.

6.8 Conclusions

Here, a methodology is presented to extract dynamical information from data sampled at irregular timesteps, below the Nyquist-Shannon criterion. Information concerning temporal dynamics is obtained using the proposed is-OMD algorithm, which can be interpreted as solving an OMD optimisation problem where the majority of snapshots are treated as unknowns. It is assumed that no *a priori* information about the dynamics is available and at each iteration two problems are solved in a pairwise manner; modal decomposition (using DMD or OMD) to update an estimate of the dynamics and reconstruction of intermediate snapshot with a least squares technique. Further, mode recovery is improved by assuming that the system can be approximated using a Vandermonde matrix and by solving an appropriate minimisation problem. The is-OMD methodology's performance is analysed on synthetic sinusoidal flow and on flow past a circular cylinder at a low Reynolds number, showing improvement over compressive sampling based mode extraction methodologies. Furthermore, the analysis on the turbulent flow demonstrated a potential of the is-OMD algorithm to be applicable to more complicated dynamical systems.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this thesis the suitability of modal decomposition algorithms to construct reduced-order models of fluid flows, with applications to estimator design and extraction of modal dynamical information from the irregularly sampled data, was investigated. A particular decomposition, OMD, was used throughout the thesis primarily because the modes it outputs are guaranteed to be associated with dynamical information (contrary to POD) and due to its additional flexibility in the selection of the projection vector space relative to DMD. It was demonstrated in §3.2 that the latter feature provides increased robustness in the accuracy of eigenvalue information when applied to noisy data ensembles. Additionally, a metric was developed which facilitates sorting of OMD modes based on energetic and dynamical considerations, providing a systematic criterion with which to truncate the number of extracted modes.

Subsequently, the output of the OMD algorithm was used to develop estimators for bluff body flows. One of the analysed systems, the cylinder wake, where coherent structures were identified which correspond to three dynamically important features were identified: the global mode driving the shedding oscillations; shift mode responsible for change in the recirculation length and the first sub-harmonic of the global mode. Stochastic estimation was used to approximate the main dynamical features of the flow and was combined with a MHE algorithm with three dynamical models of different orders of complexity (linear, quadratic and cubic) to give an accurate estimation of the flow dynamics.

Using these established estimation techniques, analysis of a more practically relevant dy-

namical system, turbulent flow past a bullet-shaped bluff body, was subsequently conducted. This system, despite its complex behaviour, possesses two dominant oscillatory features with known associated frequencies. It was demonstrated that it is possible to extract dynamic modes associated with these frequencies using the OMD algorithm. For this flow, estimator design was significantly more challenging not only due to the turbulence present in the system but also due to rotational dynamics and the significantly lower imprint of the shedding dynamics on the available base pressure measurements. Nonetheless, it was shown that if the rotational features are taken into account then it is possible to obtain estimates which, on average, reproduce main characteristics of the analysed system. One of the important features of the new approach was to establish separate sets of LSE coefficients for different azimuthal orientations of the system and use an indicator function to construct an appropriate output vector $\mathbf{y}$. Such measurable values were implemented in an MHE algorithm to give an estimate of the system reproducing its main dynamical features.

A final significant application considered in this thesis was to provide an extension to modal decomposition algorithms so that they can be applied to the irregularly sampled and potentially sub-sampled data. The main foundation of the new algorithm, is-OMD, is that, in addition to a lack of knowledge about the dynamics, the majority of the underlying snapshots are treated as unknowns. The resulting optimisation problem is split into two sub-problems which are solved in the pairwise manner: first fixing the dynamics and solving for the unknown snapshots; and, subsequently, using the reconstructed data from the previous iteration to update the dynamics. The analysis was performed on the two bluff body flows studied in earlier chapters and it was demonstrated that the is-OMD algorithm is able to extract the main frequencies of the cylinder and bullet flow even from sub-sampled data. Remarkably, it was shown that in certain cases the new algorithm provided an improvement over application of the compressive sampling, as was the case in the analysis of a simple sinusoidal system, and outputs more accurate eigenvalues than OMD applied to regularly undersampled data sets.

7.2 Future Work

In chapter 4, an estimation methodology was constructed which accurately approximates the dynamics of the cylinder flow. The natural direction extending the estimator design is to further analyse the robustness of the established approach. Such an assessment can be per-

med on the same system for larger Re , when the validity of the Stuart-Landau approximation diminishes, or in the presence of the forcing inputs. Major areas of the analysis will concern if spatial characteristics and dynamics of the most important coherent structures change subject to an input, determine whether the created mapping between pressure and states retains its accuracy and establish which dynamical model is the most robust with respect to control input. Additionally, application of DMDC-like methodologies introduced in [47] can be explored in such a scenario.

In chapter 5, where the bullet flow was analysed, it was established that there exist spatial asymmetries in pressure structures. In §5.2.1 it was suggested that the two-dimensional view of the velocity structures also varies depending on the reorientation of the system. Nonetheless, only the estimates obtained using a time-constant mode were presented. An attempted extension of the indicator function approach into velocity structures results in a discontinuous signals with numerous and rapid changes in the state values (most importantly in the signal corresponding to low-frequency mode). In such cases the MHE algorithm has to be repeatedly restarted at each instance when the discontinuity occurs.

This situation can be perhaps prevented if three-dimensional modes can be obtained, which may consequently provide a more robust way of estimating the dynamics of the system. There are two potential approaches which can be used for this purpose. The experimental set-up can be extended to obtain third velocity component, for example using tomographic PIV [31]. This approach will not only allow us to obtain three dimensional modes but also provide more insight into the dynamics of the system. A second potential approach is more ambitious and would rely on establishing a framework which extrapolates three dimensional modes from two-dimensional snapshots. Such a method would rely on a mapping between the available field of view and certain three dimensional domain and corresponding optimisation problem will consist of larger number of unknowns. The investigation would consist of establishing such a mapping given available data and constructing an optimisation problem which is computationally feasible to solve.

The is-OMD methodology presented in chapter 6 performs well when the objective is to extract appropriate eigenvalues of the system and the only major inaccuracies occur when approximating the dynamic modes and their projection coefficients. These imprecisions are

predominant when is-OMD is applied to dynamically complicated systems and a framework to improve the structure of dynamic modes and reconstruction of the projection coefficients is desirable.

One approach was already presented in §6.7 in the shape of inequality constraints imposed on the rate of change of the projection coefficients onto columns of $\mathbf{L}$. Additionally, it is possible to impose inequality constraints on the magnitude of the coefficients. Analogous constraints can be imposed not only using projection onto columns of $\mathbf{L}$ but also on the coefficients of the POD modes preventing the excessive growth of the low-energy structures. Ideally, inequalities corresponding to POD and $\mathbf{L}$ would be applied at the same time providing significantly more information concerning the analysed system into the optimisation problem. There are two challenges associated with this approach: 1) construction of a mapping between projection coefficients of $\mathbf{L}$ and POD; 2) ensuring that an optimisation problem is not overconstrained such that it converges onto the boundary of the feasible set. For the latter problem, the algorithm can be potentially modified such that it starts with lax constraints which are progressively tightened in the subsequent iterations.

7.2.1 Sensitivity to Forcing

One of the potential extension of modal decomposition algorithms is to use them for the assessment of a system's sensitivity to the forcing input. The most widely used approaches often rely on computation of the adjoint modes of the linearised Navier-Stokes operator [103]. Such calculations are performed numerically and cannot be easily extended to the analysis of the experimental data. The OMD approach outputs a certain linear operator which, in a sense, has a similar interpretation to the linearised Navier-Stokes operator, suggesting that the OMD algorithm could perhaps be used to obtain a similar adjoint structure. Investigation into the applicability of such approach and potential ways of modifying modal decomposition algorithms to accommodate discussed objective would be advantageous.

An alternative method is now proposed which is briefly analysed on the cylinder flow. However, more extensive analysis on different examples (including analytic test cases) will be required to confirm its suitability. The main idea behind this method is to project the forcing input onto POD modes and construct modifications $\delta\mathbf{M}$ to the OMD matrix $\mathbf{M}$. Separate matrices $\delta\mathbf{M}$ are computed for forcing at every spatial location. Using left and right eigenvectors

of $\mathbf{M}$, $\boldsymbol{\xi}$ and $\mathbf{w}$, the modification to the eigenvalues can be calculated as

$$\delta\lambda = -\frac{\boldsymbol{\xi}^T \delta \mathbf{M} \mathbf{w}}{\boldsymbol{\xi}^T \mathbf{w}} \quad (7.1)$$

giving an indication on how the linear dynamics of the system changes when forcing is applied at different positions [104]. A more detailed explanation of this methodology with some preliminary results is presented below in §7.2.1.1.

7.2.1.1 Mathematical Formulation

Assume the velocity field decomposition is $\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^r a_j(t) \boldsymbol{\Phi}_j(\mathbf{x})$ where, here, $\boldsymbol{\Phi}_j^\top \boldsymbol{\Phi}_k = \delta_{jk}$ and $\mathbf{x} \subseteq \mathbb{R}^2$. Using such a decomposition to perform a Galerkin projection onto the linearised Naveier Stokes equations with a forcing term $\mathbf{f}$ will give

$$\dot{a}_j = \sum_{k=1}^r a_k \int_{\Omega} \boldsymbol{\Phi}_j \cdot [-(\bar{\mathbf{u}} \cdot \nabla) \boldsymbol{\Phi}_k - (\boldsymbol{\Phi}_k \cdot \nabla) \bar{\mathbf{u}} + \frac{1}{Re} \boldsymbol{\Phi}_k] dx - \int_{\Omega} \boldsymbol{\Phi}_j \cdot \nabla p dx + \int_{\Omega} \boldsymbol{\Phi}_j \cdot \mathbf{f} dx \quad (7.2)$$

where $\mathbf{f}$ is a forcing, $\bar{\mathbf{u}}$ is time-averaged flow, and Ω is the domain of the flow. For the state space representation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\tilde{\mathbf{u}}(t)$, (7.2) shows that the terms corresponding to input matrix $\mathbf{B}$ are given by

$$\mathbf{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix}, \quad \text{where} \quad b_j = \int_{\Omega} \boldsymbol{\Phi}_j \mathbf{f} dx. \quad (7.3)$$

and $\mathbf{x} = [a_1, \dots, a_r]^\top$. Let us assume, for example, that the forcing term has a form of a solely horizontal velocity applied at a point (X, Y) which corresponds to

$$\mathbf{f} = \begin{bmatrix} \delta(x - X, y - Y) \\ 0 \end{bmatrix}. \quad (7.4)$$

Evaluating the entries of matrix $\mathbf{B}$ using (7.3), the following expression is obtained for this type of forcing

$$\begin{aligned}
 b_j &= \int \mathbf{f} \cdot \Phi_j dx \\
 &= \int \mathbf{f} \begin{bmatrix} \Phi_{1j}(x, y) \\ \Phi_{2j}(x, y) \end{bmatrix} dx \\
 &= \int \delta(x - X, y - Y) \Phi_{1j}(x, y) dx \\
 &= \Phi_{1j}(X, Y),
 \end{aligned} \tag{7.5}$$

which shows that the input matrix depends explicitly on the forcing location (X, Y) . Assuming, again arbitrarily, that the forcing is controlled using a feedback loop where input is proportional to the streamwise velocity at the chosen location (X, Y) , i.e. $\tilde{\mathbf{u}}(t) = -k\mathbf{u}(X, Y, t)$, we want to find the gain matrix $\mathbf{K}$ such that $\mathbf{u} = -\mathbf{K}\mathbf{x}$. In order to do so the decomposition of the flow into a sum of modes shapes is used to obtain

$$\mathbf{u}(X, Y, t) = \begin{bmatrix} \Phi_{11}(X, Y) & \dots & \Phi_{1r}(X, Y) \end{bmatrix} \begin{bmatrix} a_1(t) \\ \vdots \\ a_r(t) \end{bmatrix}, \tag{7.6}$$

which suggests that

$$\mathbf{K} = k \begin{bmatrix} \Phi_{11}(X, Y) & \dots & \Phi_{1r}(X, Y) \end{bmatrix} \tag{7.7}$$

Based on those assumptions the expression for the state evolution of the system will have the following form

$$\dot{\mathbf{x}}(t) = (\mathbf{M} - \mathbf{BK})\mathbf{x}(t) \tag{7.8}$$

and the term $-\mathbf{BK}$ can be interpreted as the perturbation to matrix $\mathbf{M}$ in the form

$$\delta\mathbf{M} = k \begin{bmatrix} \Phi_{11}(X, Y) \\ \vdots \\ \Phi_{1r}(X, Y) \end{bmatrix} \begin{bmatrix} \Phi_{11}(X, Y) & \dots & \Phi_{1r}(X, Y) \end{bmatrix} = k(\Phi_{1i}(X, Y)\Phi_{1j}(X, Y))_{i,j=1}^n \tag{7.9}$$

The expression (7.9) provides the term necessary to perform an eigenvalue sensitivity analysis. In order to provide a summary of the necessary terms, it is reminded that the expression for this analysis has the form

$$\delta\lambda_1 = -\frac{\boldsymbol{\xi}^T \delta\mathbf{M}\mathbf{w}}{\boldsymbol{\xi}^T \mathbf{w}} \quad (7.10)$$

where $\delta\mathbf{M}$ has a form (7.9) whilst $\boldsymbol{\xi}$ and $\mathbf{w}$ are the left and right eigenvectors of $\mathbf{M}$ corresponding to λ_1 .

The results presented in this section show an eigenvalue sensitivity analysis for the cylinder flow at $Re = 50$ and $Re = 60$. The comparison is performed against reference results from [105] where the adjoint and global modes were obtained by finding left and right eigenvectors of the linearised Navier-Stokes operators using classical inverse-iteration algorithm [104]. Figure 7.1 presents sensitivity of the eigenvalues corresponding to global modes for cylinder flow at different low Reynolds numbers. Characteristic elliptical features visible for each Re signify the area where the system is the most receptive to the considered control input. It is noted that the elliptical regions are elongated and their centres move further downstream as Re increases. We are looking to reproduce those two features with the outlined methodology.

Figure 7.2 presents the eigenvalue sensitivity of the cylinder flows at $Re = 50$ and $Re = 60$ obtained using (7.9)-(7.10), based on the linear operator constructed with the OMD algorithm applied to a data ensemble drawn from the limit cycle regime of the respective cylinder flow. Each ensemble consists of 250 snapshots encompassing seven shedding periods. The rank of both approximations was set to $r = 40$. The analysis is performed on global modes to obtain a comparison with figure 7.1. In order to obtain the perturbation matrix $\delta\mathbf{M}$ it is assumed that the flow perturbation is decomposed using columns of $\mathbf{L}$ and, subsequently, the point forcing is projected on the vector space spanned by those vectors according to (7.9). Although the methodology used to obtain figure 7.2 relies on significant approximations the two main features visible in figure 7.1 are reproduced. Both contour plots show the characteristic ellipsoidal peaks in the forcing sensitivity and there is visible shift downstream for greater Re . Additionally, when Re increases the peaks are elongated. The major discrepancy concerns the location where the flow is most sensitive to forcing. Based on the results from [105] this location should be at $x \approx 2.5D$ for $Re = 50$ and $x \approx 2.75D$ for $Re = 60$. In the results from figure 7.2 those peaks are located at $x \approx 3.3D$ for $Re = 50$ and $x \approx 3.5D$ for $Re = 60$.

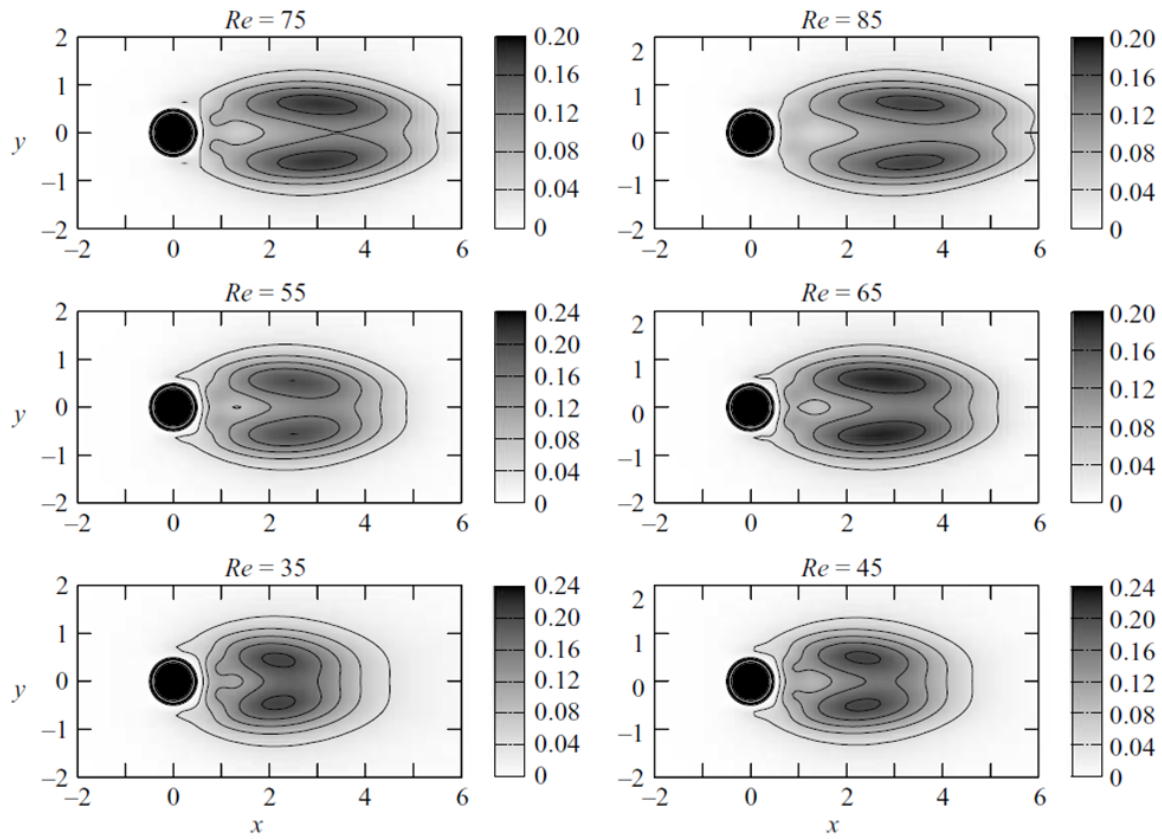


Figure 7.1: The contours of $\delta\lambda_1$ for cylinder flow at various Reynolds numbers obtained from [105]

The presented results demonstrate a potential of the outlined method nonetheless discrepancies between figures 7.1 and 7.2 suggest more analysis is required, including tests on other systems, to assess its applicability.

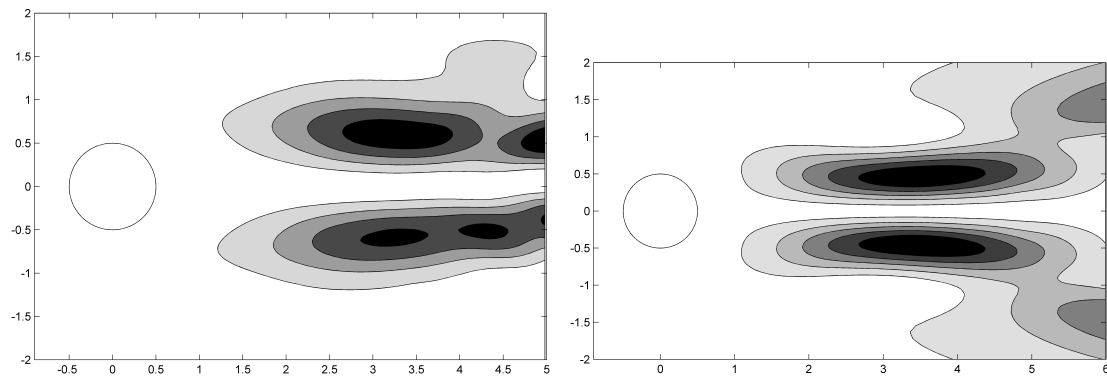


Figure 7.2: The contours of $\delta\lambda_1$ for cylinder flow at $Re = 50$ (left) and $Re = 60$ (right)

Bibliography

- [1] W. P. Choi, H. Jeon and J. Kim. Control of flow over a bluff body. *Annual Review of Fluid Mechanics*, 40(1):113–139, 2008.
- [2] P. Orlandi. On the generation of turbulent wall friction. *Physics of Fluids*, 6:634–641, 1994.
- [3] J. H. Lee, H. J. Sung, and P. A. Krogstad. Direct numerical simulation of the turbulent boundary layer over a cube-roughened wall. *Journal of Fluid Mechanics*, 669:397–431, 2011.
- [4] F. H. Clauser. The turbulent boundary layer. *Advances in Applied Mechanics*, 4:1–51, 1956.
- [5] T. Sayadi, P. J. Schmid, J. W. Nichols, and P. Moin. Reduced-order representation of near-wall structures in the late transitional boundary layer. *Journal of Fluid Mechanics*, 748:278–301, 2014.
- [6] C. W. Rowley, I. Mezic, S. Bagheri, P. Schlatter, and D. S. Henningson. Spectral analysis of nonlinear flows. *Journal of Fluid Mechanics*, 641:115–27, 2009.
- [7] I. Nastase, A. Meslem, and M. El Hassan. Image processing analysis of vortex dynamics of lobed jets from three-dimensional diffusers. *Fluid Dynamics Research*, 43:065503, 2011.
- [8] A. Seena and H. J. Sung. Dynamic mode decomposition of turbulent cavity flows for self-sustained oscillations. *International Journal of Heat and Fluid Flow*, 32:1098–1110, 2011.
- [9] E. Motheau, Y. Mery, F. Nicoud, and T. Poinso. Analysis and modeling of entropy

- modes in a realistic aeronautical gas turbine. *Journal of Engineering for Gas Turbines and Power*, 135:092602, 2013.
- [10] F. Gomez, H. M. Blackburn, M. Rudman, B. J. McKeon, M. Luhar, R. Moarref, and A. S. Sharma. On the origin of frequency sparsity in direct numerical simulations of turbulent pipe flow. *Physics of Fluids*, 26:101703, 2014.
- [11] S. Sarmast, R. Dadfar, R.F. Mikkelsen, P. Schlatter, S. Ivanell, J. N. Sørensen, and D. S. Henningson. Mutual inductance instability of the tip vortices behind a wind turbine. *Journal of Fluid Mechanics*, 755:705–731, 2014.
- [12] B. R. Noack, K. Afanasiev, M. Morzynski, G. Tadmor, and F. Thiele. A hierarchy of low-dimensional models for the transient and post-transient cylinder wake. *Journal of Fluid Mechanics*, 497:335–363, 2003.
- [13] P. Holmes, J. L. Lumley, and G. Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*. Cambridge University Press, 1996.
- [14] L. Cordier, B. A. El Majd, and J. Favier. Calibration of pod reduced-order models using tikhonov regularization. *International Journal for Numerical Methods in Fluids*, 63:269–296, 2010.
- [15] M. M. Zdravkovich. Review and classification of various aerodynamic and hydrodynamic means for suppressing vortex shedding. *Journal of Wind Engineering and Industrial Aerodynamics*, 7:145–189, 1981.
- [16] O. M. Griffin and M. S. Hall. Review-vortex shedding lock-on and flow control in bluff body wakes. *Journal of Fluids Engineering, Transactions of the ASME*, 113:526–537, 1991.
- [17] E. A. Gillies. Low-dimensional control of the circular cylinder wake. *Journal of Fluid Mechanics*, 371:157–178, 1998.
- [18] W. R. Graham, J. Peraire, and K. Y. Tang. Optimal control of vortex shedding using low-order models. ii. model-based control. *International Journal for Numerical Methods in Engineering*, 44:973–990, 1999.

- [19] S. Siegel, K. Cohen, and T. McLaughlin. Numerical simulations of a feedback-controlled circular cylinder wake. *AIAA Journal*, 44:1266–1276, 2006.
- [20] P. J. Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of Fluid Mechanics*, 656:5–28, 2010.
- [21] A. Wynn, D. S. Pearson, B. Ganapathisubramani, and P. J. Goulart. Optimal mode decomposition for unsteady flows. *Journal of Fluid Mechanics*, 733:473–503, 2013.
- [22] W.-H. Hucho. *Aerodynamics of Road Vehicles*. Society of Automotive Engineers, 1998.
- [23] D. Sipp and A. Lebedev. Global stability of base and mean flows: A general approach and its applications to cylinder and open cavity flows. *Journal of Fluid Mechanics*, 593:333–358, 2007.
- [24] S. Bagheri. Koopman-mode decomposition of the cylinder wake. *Journal of Fluid Mechanics*, 726:596–623, 2013.
- [25] A. Sevilla and C. Martínez-Bazán. Vortex shedding in high reynolds number axisymmetric bluff-body wakes: local linear instability and global bleed control. *Physics of Fluids*, 16:3460–3469, 2004.
- [26] G. Rigas, A. R. Oxlade, A. S. Morgans, and J. F. Morrison. Low-dimensional dynamics of a turbulent axisymmetric wake. *Journal of Fluid Mechanics*, 755:R5, 2014.
- [27] E. J. Candes and T. Tao. Near-optimal signal recovery from random projections: Universal encoding strategies? *IEEE Transactions on Information Theory*, 52:5406–5425, 2006.
- [28] E. J. Candes, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *IEEE Transactions on Information Theory*, 52(2):489–509, 2006.
- [29] D. L. Donoho. Compressed sensing. *IEEE Transactions on Information Theory*, 52(4):1289–1306, 2006.
- [30] R. J. Adrian and J. Westerweel. *Particle Image Velocimetry*. Cambridge University Press, 2011.

- [31] G. E. Elsinga, F. Scarano, B. Wieneke, and B. W. van Oudheusden. Tomographic particle image velocimetry. *Experiments in Fluids*, 41:933–947, 2006.
- [32] R. Leroux and L. Cordier. Dynamic mode decomposition for non-uniformly sampled data. *Experiments in Fluids*, 57(5), 2016.
- [33] J. H. Tu, C. W. Rowley, J. N. Kutz, and J. K. Shang. Spectral analysis of fluid flows using sub-Nyquist-rate PIV data. *Experiments in Fluids*, 55:1805, 2014.
- [34] J. Lumley. The structure of inhomogeneous turbulent flows. *Proceedings of the International Colloquium on the Fine Scale Structure of the Atmosphere and its Influence on Radio Wave Propagation*, pages 166–178, 1967.
- [35] C. W. Rowley, T. Colonius, and R. M. Murray. Model reduction for compressible flows using pod and galerkin projection. *Physica D: Nonlinear Phenomena*, 189:115–129, 2004.
- [36] K. K. Chen, J. H. Tu, and C. W. Rowley. Variants of dynamic mode decomposition: Boundary condition, Koopman, and Fourier analyses. *Journal of Nonlinear Science*, 22(6):887–915, 2012.
- [37] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley. A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition. *Journal of Nonlinear Science*, 25:1307–1346, 2015.
- [38] T. W. Muld, G. Efraimsson, and D. S. Henningson. Flow structures around a high-speed train extracted using proper orthogonal decomposition and dynamic mode decomposition. *Computers and Fluids*, 57:87–97, 2012.
- [39] G. Tissot, L. Cordier, N. Benard, and B. R. Noack. Model reduction using dynamic mode decomposition. *Comptes Rendus Mécanique*, 342(6):410–416, 2014.
- [40] M. Grilli, P. J. Schmid, S. Hickel, and N. A. Adams. Analysis of unsteady behaviour in shockwave turbulent boundary layer interaction. *Journal of Fluid Mechanics*, 700:16–28, 2012.

- [41] N. Vinha, F. Meseguer-Garrido, J. De Vicente, and E. Valero. A dynamic mode decomposition of the saturation process in the open cavity flow. *Aerospace Science and Technology*, 52:198–206, 2016.
- [42] P. J. Schmid, D. Violato, and F. Scarano. Decomposition of time-resolved tomographic PIV. *Experiments in Fluids*, 52:1567–1579, 2012.
- [43] O. Semeraro, G. Bellani, and F. Lundell. Analysis of time-resolved PIV measurements of a confined turbulent jet using POD and Koopman modes. *Experiments in Fluids*, 53:1203–1220, 2012.
- [44] I. Mezic. Analysis of fluid flows via spectral properties of the Koopman operator. *Annual Review of Fluid Mechanics*, 45:357–78, 2013.
- [45] M. R. Jovanovic, P. J. Schmid, and J. W. Nichols. Sparsity-promoting dynamic mode decomposition. *Physics of Fluids*, 26:024103, 2014.
- [46] M. S. Hemati, C. W. Rowley, and J. W. Nichols. De-biasing the dynamic mode decomposition for applied Koopman spectral analysis. 2015.
- [47] J. L. Proctor, S. L. Brunton, and J. N. Kutz. Dynamic mode decomposition with control. 2014.
- [48] A. Edelman, T. A. Arias, and S. T. Smith. The geometry of algorithms with orthogonality constraints. *SIAM Journal on Matrix Analysis and Applications*, 20:303–53, 1998.
- [49] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3:1–122, 2010.
- [50] S. Bagheri. Effects of weak noise on oscillating flows: Linking quality factor, Floquet modes, and Koopman spectrum. *Physics of Fluids*, 26:094104, 2014.
- [51] S. T. M. Dawson, M. S. Hemati, M. O. Williams, and C. W. Rowley. Characterizing and correcting for the effect of sensor noise in the dynamic mode decomposition. *Experiments in Fluids*, 57:42, 2016.

- [52] M. O. Williams, C. W. Rowley, and I. G. Kevrekidis. A kernel-based approach to data-driven Koopman spectral analysis. *ArXiv e-prints*, 2014.
- [53] S. L. Brunton, J. L. Proctor, and J. N. Kutz. Sparse identification of nonlinear dynamics with control (sindyc). *IFAC-PapersOnLine*, 49:710–715, 2016.
- [54] R. J. Adrian. On the role of conditional averages in turbulence theory. *Proceedings of the 5th Biennial Symposium on Turbulence in Liquids, AFOSR*, pages 322–332, 1977.
- [55] J. P. Bonnet, D. R. Cole, J. Delville, M. N. Glauser, and L. S. Ukeiley. Stochastic estimation and proper orthogonal decomposition: complementary techniques for identifying structure. *Experiments in Fluids*, 17:307–314, 1994.
- [56] L. Ukeiley and N. Murray. Velocity and surface pressure measurements in an open cavity. *Experiments in Fluids*, 38:656–671, 2005.
- [57] V. Durgesh and J. W. Naughton. Multi-time-delay lse-pod complementary approach applied to unsteady high-reynolds-number near wake flow. *Experiments in Fluids*, 49:571–583, 2010.
- [58] D. Ewing and J. H. Citriniti. Examination of a lse/pod complementary technique using single and multi-time information in the axisymmetric shear layer. *IUTAM symposium on simulation and identification of organized structures in flows*, pages 375–384, 1997.
- [59] C. E. Tinney, F. Coiffet, J. Delville, A. M. Hall, P. Jordan, and M. N. Glauser. On spectral linear stochastic estimation. *Experiments in Fluids*, 41:763–775, 2006.
- [60] R. J. Adrian and P. Moin. Stochastic estimation of organized turbulent structure : homogeneous shear flow. *Journal of Fluid Mechanics*, 190:531–559, 1988.
- [61] R. J. Adrian. Stochastic estimation of conditional structure: a review. *Applied Scientific Research*, 53:291–303, 1994.
- [62] A. M. Naguib, C. E. Wark, and O. Juckenhofel. Stochastic estimation and flow sources associated with surface pressure events in a turbulent boundary layer. *Physics of Fluids*, 13:2611–2626, 2001.

- [63] J. H. Tu, J. Griffin, A. Hart, C. W. Rowley, L. N. Cattafesta, and L. S. Ukeiley. Integration of non-time-resolved PIV and time-resolved velocity point sensors for dynamic estimation of time-resolved velocity fields. *Experiments in Fluids*, 54:1429, 2001.
- [64] K. R. Muske, J. B. Rawlings, and J. H. Lee. Receding horizon recursive state estimation. *Proceedings of the 1993 American Control Conference*, pages 900–904, 1993.
- [65] A. C. Lorenc. Analysis methods for numerical weather prediction. *Quarterly Journal of the Royal Meteorological Society*, 112:1177–1194, 1987.
- [66] P. Courtier, J.-N. Thépaut, and A. Hollingsworth. A strategy for operational implementation of 4D-var, using an incremental approach. *Quarterly Journal of the Royal Meteorological Society*, 120:1367–1387, 1994.
- [67] A. M’Hamdi, A. Helbig, O. Abel, and W. Marquardt. Proceedings of the 13th world congress, international federation of automatic control. vol.m. chemical process control, mineral, mining, metals. pages 121–126, 1997.
- [68] J. S. Albuquerque and L. T. Biegler. Data reconciliation and gross-error detection for dynamic systems. *AIChE Journal*, 42:2841–2856, 1996.
- [69] T. Binder, L. Blank, W. Dahmen, and W. Marquardt. On the regularization of dynamic data reconciliation problems. *Journal of Process Control*, 12:557–567, 2002.
- [70] E. P. Gatzke and F. J. Doyle. Use of multiple models and qualitative knowledge for on-line moving horizon disturbance estimation and fault diagnosis. *Journal of Process Control*, 12:339–352, 2002.
- [71] B. F. Farrell and P. J. Ioannou. State estimation using a reduced-order kalman filter. *Journal of the Atmospheric Sciences*, 58:3666–225, 2001.
- [72] M. Pastoor, L. Henning, B. R. Noack, R. King, and G. Tadmor. Feedback shear layer control for bluff body drag reduction. *Journal of Fluid Mechanics*, 608:161–196, 2008.
- [73] M. Högberg, T. R. Bewley, and D. S. Henningson. Linear feedback control and estimation of transition in plane channel flow. *Journal of Fluid Mechanics*, 481:149–175, 2003.

- [74] S. S. Collis, Y. Chang, S. Kellogg, and R.D. Prablm. Large eddy simulation and turbulence control. *Fluids 2000 Conference and Exhibit*, 2000.
- [75] T. R. Bewley, P. Moin, and R. Temam. DNS-based predictive control of turbulence: an optimal benchmark for feedback algorithms. *Journal of Fluid Mechanics*, 447:179–225, 2001.
- [76] D. G. Robertson, J. H. Lee, and J. B. Rawlings. Moving horizon-based approach for least-squares estimation. *AIChE Journal*, 42:2209–2224, 1996.
- [77] P. Kuhl, M. Diehl, T. Kraus, J. P. Schloder, and H. Bock. A real-time algorithm for moving horizon state and parameter estimation. *Computers and Chemical Engineering*, 35:71–83, 2011.
- [78] A. Wynn, M. Vukov, and M. Diehl. Convergence guarantees for moving horizon estimation based on the real-time iteration scheme. *IEEE Transactions on Automatic Control*, 59:2215–2221, 2014.
- [79] J. A. Tropp and A. C. Gilbert. Signal recovery from random measurements via orthogonal matching pursuit. *IEEE Transactions on Information Theory*, 53:4655–66, 2007.
- [80] D. Needell and J. A. Tropp. Cosamp: iterative signal recovery from incomplete and inaccurate samples. *Applied and Computational Harmonic Analysis*, 26:301–21, 2009.
- [81] E. J. Candes and M. B. Wakin. An introduction to compressive sampling. *IEEE Signal Processing Magazine*, 25(2):21–30, 2008.
- [82] C. H. K. Williamson. Vortex dynamics in the cylinder wake. *Annual review of fluid mechanics*, 28:477–539, 1996.
- [83] L. Lu and G. Papadakis. Investigation of the effect of external periodic flow pulsation on a cylinder wake using linear stability analysis. *Physics of Fluids*, 23:094105, 2011.
- [84] L. Lu and G. Papadakis. An iterative method for the computation of the response of linearised navier-stokes equations to harmonic forcing and application to forced cylinder wakes. *International Journal for Numerical Methods in Fluids*, 74:794–817, 2014.

- [85] C. M. Rhie and W. L. Chow. Numerical study of the turbulent flow past an airfoil with trailing edge separation. *AIAA Journal*, 21:1525–1532, 1983.
- [86] R. I. Issa. Solution of the implicitly discretised fluid flow equations by operator-splitting. *Journal of Computational Physics*, 62:40–65, 1986.
- [87] R. D. Falgout and U. M. Yang. hypre: A library of high performance preconditioners. *Lecture Notes in Computer Science*, 2331:632–641, 2002.
- [88] B. J. A. Zielinska, S. Goujon-Durand, J. Dusek, and J.E. Wesfreid. Strongly nonlinear effect in unstable wakes. *Physical Review Letters*, 79:3893–3896, 1997.
- [89] R. Bracewell. The Hilbert transform. In *The Fourier Transform and its applications*, 3rd ed., chapter 13, pages 267–273. McGraw-Hill, 1999.
- [90] H. Park, D. Lee, W.-D. Jeon, S. Hahn, J. Kim, J. Kim, J. Choi, and H. Choi. Drag reduction in flow over a two-dimensional bluff body with a blunt trailing edge using a new passive device. *Journal of Fluid Mechanics*, 563:389–414, 2006.
- [91] R. Brackston. *Feedback control of three-dimensional bluff body wakes for efficient drag reduction*. PhD dissertation, Imperial College London, 2017.
- [92] A. R. Oxlade, J. F. Morrison, A. Qubain, and G. Rigas. High-frequency forcing of a turbulent axisymmetric wake. *Journal of Fluid Mechanics*, 770:305–318, 2015.
- [93] P. Bohorquez, E. Sanmiguel-Rojas, A. Sevilla, J.I. Jiménez-González, and C. Martínez-Bazán. Stability and dynamics of the laminar wake past a slender blunt-based axisymmetric body. *Journal of Fluid Mechanics*, 2011:1–35, 2011.
- [94] Y. Bury and Jardin T. . Transitions to chaos in the wake of an axisymmetric bluff body. *Physics Letters A*, 376:3219–3222, 2012.
- [95] B. Pier. Local and global instabilities in the wake of a sphere. *Journal of Fluid Mechanics*, 603:39–61, 2008.
- [96] A. G. Tomboulides and S. A. Orszag. Numerical investigation of transitional and weak turbulent flow past a sphere. *Journal of Fluid Mechanics*, 416:45–73, 2000.

- [97] B. Natarajan and A. Acrivos. The instability of the steady flow past spheres and disks. *Journal of Fluid Mechanics*, 254:323–344, 1993.
- [98] D. J. Thomson and P. Moin. Spectrum estimation and harmonic analysis. *Proceedings of the IEEE*, 70:1055–1096, 1982.
- [99] D. T. Sandwell. Biharmonic spline interpolation of GOES-3 and SEASAT altimeter data. *Geophysical Research Letters*, 14:139–142, 1987.
- [100] H. Nyquist. Certain topics in telegraph transmission theory. *Proceedings of the IEEE*, 90(2):280–305, 1928.
- [101] G. Williams. *Linear Algebra with Applications*. Jones and Bartlett Publishers, 2011.
- [102] M. Grandemange, M. Gohlke, V. Parezanović, and O. Cadot. On experimental sensitivity analysis of the turbulent wake from an axisymmetric blunt trailing edge. *Physics of Fluids*, 24:035106, 2012.
- [103] P. Luchini and A. Bottaro. Adjoint equations in stability analysis. *Annual Review of Fluid Mechanics*, 46:493–517, 2014.
- [104] G. H. Golub and C. Van Loan. *Matrix Computations*. Johns Hopkins University Press, 1989.
- [105] F. Giannetti and P. Luchini. Structural sensitivity of the first instability of the cylinder wake. *Journal of Fluid Mechanics*, 581:167–197, 2007.

Appendix A

Conjugate gradient method for solving OMD optimisation problem

This Appendix gives a list of steps to find the low-dimensional subspace spanned by columns of $\mathbf{L}$ which minimises $\|\mathbf{B} - \mathbf{L}\mathbf{M}\mathbf{L}^T\mathbf{A}\|^2$. Using the expression for the optimal $\mathbf{M}^*(\mathbf{L})$ (2.23), the OMD optimisation problem can be rewritten as

$$\|\mathbf{B} - \mathbf{L}\mathbf{M}^*(\mathbf{L})\mathbf{L}^T\mathbf{A}\|^2 = \|\mathbf{A}\|^2 - \|\mathbf{L}^T\mathbf{A}\mathbf{Q}(\mathbf{L})\|^2 \quad (\text{A.1})$$

where $\mathbf{Q}(\mathbf{L}) := \mathbf{B}^T\mathbf{L}(\mathbf{L}^T\mathbf{B}\mathbf{B}^T\mathbf{L})^{-1}\mathbf{L}^T\mathbf{B}$. The expression (A.1) demonstrates that the OMD problem is equivalent to the single-variable maximisation of

$$g(\mathbf{L}) = \|\mathbf{L}^T\mathbf{A}\mathbf{Q}(\mathbf{L})\|^2 \quad (\text{A.2})$$

under conditions $\mathbf{L}^T\mathbf{L} = \mathbf{I}$.

The conjugate-gradient implementation which can be used to maximise (A.2) is given in algorithm 3. For clarity of exposition two terms are defined first: 1) $\mathbf{g}_L := \frac{\partial g}{\partial \mathbf{L}}(\mathbf{L})$ is equal to

$$\begin{aligned} \mathbf{g}_L = & -2\mathbf{B}\mathbf{L}(\mathbf{L}^T\mathbf{B}\mathbf{L})^{-1}(\mathbf{L}^T\mathbf{A}^T\mathbf{L})(\mathbf{L}^T\mathbf{A}\mathbf{L})(\mathbf{L}^T\mathbf{B}\mathbf{L})^{-1} \\ & + 2[\mathbf{A}\mathbf{L}(\mathbf{L}^T\mathbf{B}\mathbf{L})^{-1}(\mathbf{L}^T\mathbf{A}^T\mathbf{L}) + \mathbf{A}^T\mathbf{L}(\mathbf{L}^T\mathbf{A}\mathbf{L})(\mathbf{L}^T\mathbf{B}\mathbf{L})^{-1}] \quad (\text{A.3}) \end{aligned}$$

where $\mathbf{A} := \mathbf{A}\mathbf{A}^\top$, $\mathbf{B} := \mathbf{B}\mathbf{B}^\top$ and 2) conjugate-gradient correction term Δ_{k+1} given by

$$\Delta_{k+1} := \frac{\langle \mathbf{G}_{k+1} - \tau(\mathbf{G}_k), \mathbf{G}_{k+1} \rangle}{\|\mathbf{G}_{k+1}\|^2} \tau(\mathbf{H}_k), \quad (\text{A.4})$$

where

$$\tau(\mathbf{G}_k) := \mathbf{G}_k - (\mathbf{L}_k \mathbf{V} \sin(\Sigma t_k) + \mathbf{U}(\mathbf{I} - \cos(\Sigma t_k))) \mathbf{U}^\top \mathbf{G}_k \quad (\text{A.5a})$$

$$\tau(\mathbf{H}_k) := (-\mathbf{L}_k \mathbf{V} \sin(\Sigma t_k) + \mathbf{U} \cos(\Sigma t_k)) \Sigma \mathbf{V}^\top \quad (\text{A.5b})$$

and $\mathbf{H}_k = \mathbf{U} \Sigma \mathbf{V}^\top$ is a search direction whilst $\mathbf{G}_k$ is a gradient (A.6). The constant $t_k \in [0, 1]$ is a minimiser in the step 4 of algorithm 3.

Algorithm 3 OMD

- 1: Set initial conditions $\mathbf{L}_0$ such that $\mathbf{L}_0^\top \mathbf{L}_0 = \mathbf{I}$.
- 2: Compute initial gradient $\mathbf{G}_0 = (\mathbf{I} - \mathbf{L}_0^\top \mathbf{L}_0) \mathbf{g}_{L_0}$ and search direction $\mathbf{H}_0 = -\mathbf{G}_0$.
- 3: **repeat**
- 4: Find $t_{min} \in [0, 1]$ which minimises $g(\mathbf{L}_k(t))$ along the geodesic curve

$$\mathbf{L}_k(t) = \mathbf{L}_k \mathbf{V} \cos(\Sigma t) \mathbf{V}^\top + \mathbf{U} \sin(\Sigma t) \mathbf{V}^\top, \quad t \in [0, 1]$$

along the search direction $\mathbf{H}_k = \mathbf{U} \Sigma \mathbf{V}^\top$.

- 5: Update subspace $\mathbf{L}_{k+1} = \mathbf{L}_k(t_{min})$.
- 6: Update gradient

$$\mathbf{G}_{k+1} = (\mathbf{I} - \mathbf{L}_{k+1} \mathbf{L}_{k+1}^\top) \mathbf{g}_{L_{k+1}}. \quad (\text{A.6})$$

- 7: Update search direction

$$\mathbf{H}_{k+1} = \mathbf{G}_{k+1} \Delta_{k+1}.$$

- 8: **until** $g(\mathbf{L}_{k+1}) - g(\mathbf{L}_k) < \text{tolerance}$
 - 9: **return** Optimal low order subspace basis $\mathbf{L}_{k+1}$ and dynamics $\mathbf{M}(\mathbf{L}_{k+1})$
-