SCALE-COVARIANT QUANTUM

FIELD THEORY

by

John M. Ebbutt

Theoretical Physics Group
Department of Physics
Imperial College of Science and Technology

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DEDICATION

I dedicate this work to my Lord, Jesus Christ, whose hand of friendship I accepted whilst at Imperial College.

For, like everyone else, I *"have sinned and fall short of the Glory of God"*[1] and because of my stubborn ignorance of God, could only reasonably expect to be *"punished with everlasting destruction and shut out from the presence of the Lord"*.[2] How fantastic, then, to realise that, *"because of His great love for us, God, who is rich in mercy made us alive with Christ, even when we were dead in transgressions"*.[3]

The simple truth is that *"Christ died for sins, once for all, the righteous for the un-righteous, to bring you to God"*.[4]

Equally amazing, I found, is that to be made spiritually alive, knowing God, is a <u>gift</u> that I only have to respond to, and can in no way earn - *"For it is by grace you have been saved, through faith - and this not from yourselves, it is the gift of God - not by works, so that no-one can boast"*.[5]

I firmly believe that *"God has raised this Jesus to life"*[6] and that *"to all who received Him, to those who believed in His name, He gave the right to become Children of God"*.[7]

So you see, it is true that *"God has given us eternal life, and this life is in His son. He who has the son has life; he who does not have the son of God does not have life"*.[8]

May all who profess to wisdom come to understand that true Wisdom has its origin in God alone, and is personified in Jesus Christ.

I offer this to my Mother, and to my late Father, because of their belief in me, and their hopes for me.

Finally, to all who have encouraged me along the way, please feel that you also share in this work, for *"an anxious heart weighs a man down, but a kind word cheers him up"*.[9]

(1) *Rom. 3:23*
(2) *2 Thess. 1:9*
(3) *Eph. 2:4/5*
(4) *1 Peter 3:18*
(5) *Eph. 2:8/9*
(6) *Acts 2:32*
(7) *John1:12*
(8) *1 John 5:11/12*
(9) *Prov. 12:25*

Author's Note

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CONTENTS

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ABSTRACT

This thesis is an investigation into various aspects of the scale-covariant formulation of quantum field theory pioneered by J.R.Klauder over the past decade (*K1*). Because this subject is not widely known, we spend some time reviewing basic concepts in the Introduction. For simplicity, we restrict ourselves to the scale-covariant formulation of the $\lambda\phi^{p}$ scalar theory.

In Part 1 we consider the possibility of developing unique perturbation series for the Green functions of the theory, using only the scale-covariant coupled Green function equations. It is shown that this is not possible, and extra information is required, not present in these equations. A Renormalisation-Group type of equation and an extra dynamical Green function equation are examined as possible candidates. Both are shown to remove the degeneracy, permitting unique perturbation series in principle.

An alternative formulation of the scale-covariant theory, the *Augmented theory*, is considered in Part 2. The extra dynamical equation is shown to arise naturally. A Diagrammatic expansion is performed for the pseudofree $(\lambda = 0)$ theory and a self-consistent additive mass renormalisation predicted. This is further reinforced by an analysis of the extra dynamical equation, and the functional differential equations for the pseudofree generating functional. The retention of only leading singularity point insertion diagrams is shown to be equivalent to performing a N^{-1} expansion and retaining the large-N limit. We consider the stability of scale-covariant theories in Part 3, where we show that two mechanisms exist giving stable ground states, i.e. the large-N limit of the N^{-1} expansion, and the strong coupling expansion.

In Part 4, renormalisation of the scale-covariant theory is considered. We show that the ultraviolet singularities present in the N^{-1} expansion, due to the change of functional measure, are controllable in just those dimensions where the scale covariant formalism is necessary.

Some general conclusions are presented in Part 5.

INTRODUCTION

In this thesis we shall be examining various aspects of scale-covariant Quantum Field Theory. It is hoped in this introduction to provide the necessary motivation for so doing, besides giving an overview of our recent work. We shall therefore present here the philosophy behind the study of such theories. We make no apologies for drawing heavily from references *(K1, K8)* since these are the pioneering works in the study of the scale-covariant formalism.

A - Introducing and Motivating the scale-covariant formalism

(A1) Calculation-Independent Criteria for Non-Renormalisability

The study of scale-covariant Quantum Field Theory arose from the desire to obtain a generic understanding of ultraviolet non-renormalisability of canonical (translation-covariant) field theories

For these latter, it is argued that, for a given space-time dimension, their renormalisability (i.e. the controlling of ultra-violet divergencies in Green functions) depends upon the scheme of calculation we choose. For example, the canonical self-interacting scalar field theory $\lambda\phi^4$ is renormalisable in the $\hbar$ expansion (Loop expansion) for $d \leq 4$ dimensions, whilst in the N^{-1} expansion, the large-N limit of $\lambda(\underline{\phi}^2)^2$ is renormalisable for $d < 6$. We require a statement of non-renormalisability that depends only on the interaction chosen, not on the calculational scheme adapted for solving the model.

Such a criterion is at hand in the form of certain inequalities relating the kinetic and potential terms for a given model. We illustrate this as follows.

Let us consider a self-interacting $\lambda\phi^p$ scalar field theory. The Euclidean Vacuum generating functional for this is given by

$$S[h] = N\int \mathcal{D}[\phi]\, \exp -\int\left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^p - ih\phi\right)dx \qquad \text{—(A1.1)}$$

Here, the external source current $h(x)$ is a smooth function (C_0^∞), $\phi(x)$ is a random field, and the functional measure is given by

$$\mathcal{D}[\phi] = \prod_x d\phi(x) \qquad \text{—(A1.2)}$$

N is a normalisation constant, chosen such that

$$S[h=0] = 1 \qquad \text{—(A1.3)}$$

If we now take the limit $\lambda_0 \to 0^+$, it seems reasonable that we should obtain the *free* generating functional given by

$$S_0[h] = N \int \mathcal{D}[\phi] \exp - \int \left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} m_0^2 \phi^2 - i h \phi \right) dx$$

$$= N \exp - \tfrac{1}{2} \int dx\, dy\, h(x) G_0(x,y) h(y) \qquad \text{—(A1.4)}$$

where

$$\left(-\nabla_x^2 + m_0^2 \right) G_0(x,y) = \delta(x-y) \qquad \text{—(A1.5)}$$

The assumption that $S[h]$ tends to $S_0[h]$ as $\lambda_0 \to 0^+$ is the basis of all canonical perturbation theory.

However, there are conditions under which $S[h]$ does not pass to $S_0[h]$ as $\lambda_0 \to 0^+$, and hence perturbation theory based on $S_0[h]$ becomes invalid.

Such behaviour occurs whenever random fields $\phi(x)$ are included for which the *free* term is *finite*, but the *interaction* term is infinite, i.e., the interaction is not bounded by the free part (*K9*). This is shown by the Sobolev-type inequality

$$\left(\int |\phi(x)|^p d^n x \right)^{2/p} \leq K \int \left((\nabla\phi)^2 + m_0^2 \phi^2 \right) d^n x \qquad \text{—(A1.6)}$$

provided

$$p \leq \frac{2n}{n-2} \qquad \text{—(A1.7)}$$

where K is a finite constant.

The condition (1.7) is just that of renormalisability of a canonical $\lambda\phi^p$ theory in n-dimensions in the $\hbar$ expansion. However, the criterion (1.7) has been derived without mention of calculational schemes.

Analogously, it can be shown that, provided $p \leq \frac{2n}{n-2}$, the topology of the interacting fields is the same as in the free case, but is vastly different if $p > \frac{2n}{n-2}$, certain fields being excluded.

These observations lead to the following notion of non-remormalisability as being responsible for discontinuous perturbation.

We can perhaps see this more clearly by the following argument due to Klauder (*K12*).

Let $\chi[\phi]$ denote a characteristic (indicator) function such that:

$$\chi[\phi] = 1 \qquad \mathcal{A}_0[\phi] < \infty \qquad V[\phi] < \infty$$

$$\chi[\phi] = 0 \qquad \mathcal{A}_0[\phi] < \infty \qquad V[\phi] = \infty$$

where $\mathcal{A}_0$, V are the free and interacting parts of the action, respectively. Then we can write for our Euclidean generating functional

$$S'[h] = \int \mathcal{D}[\phi]\,\chi[\phi]\exp -\left(\mathcal{A}_0[\phi] + \lambda V[\phi] - i\int h\phi\right)$$

As $\lambda \to 0^+$, this becomes

$$S_0'[h] = \int \mathcal{D}[\phi]\,\chi[\phi]\exp - \mathcal{A}_0[\phi] + i\int h\phi$$

Now if $\chi[\phi] = 1$ this is simply the *free* theory; but if there are configurations for which $\chi[\phi] = 0$ (i.e. certain paths are projected out of the integration) then we have the *pseudo-free* theory, given by

$$S_0'[h] = \int \mathcal{D}[\phi]\, \chi[\phi] \exp - \sigma_0[\phi] + i\int h\phi$$

(A2) Non-Renormalisability as a Discontinuous Perturbation

For a given space-time dimension, n , as we increase the interaction power we approach the point where

$$p > \frac{2n}{n-2} \qquad \text{—(A2.1)}$$

When this condition is achieved, we have a topology of the fields differing from that of the free theory in that some fields have been excluded. These are not recovered as we take the limit $\lambda_0 \to 0^+$ So the interacting theory for which (2.1) holds is not continuously connected to the *free* theory $(\lambda_0 = 0)$ but rather to a *Pseudofree theory* obtained in the limit $\lambda_0 \to 0^+$

From the path integral point of view, it means that when (2.1) is fulfilled, certain paths are projected out of the integration and are not recovered as we take the $\lambda_0 \to 0^+$ limit.

This shows us that when (2.1) is fulfilled, canonical $\hbar$ perturbation theory is in fact perturbation theory about a free theory that is not continuously connected to the interacting theory. This results in the inability to control the ultraviolet divergences. In this case, perturbation theory must utilise the Pseudofree $(\lambda_0 \to 0^+)$ limit) theory as its basis.

We see that the interaction satisfying (2.1) acts as a 'hard-core', projecting out certain paths that would otherwise be allowed by the free theory.

Fuller details giving examples of discontinuous perturbations in simple quantum mechanical examples may be found in reference (*K9*).

We emphasise that, although condition (2.1) coincides with that of Non-renormalisability for canonical $\hbar$ -perturbation theory, it is independent of calculational schemes. When (2.1) is fulfilled, the theory is generically non-renormalisable and an alternative formulation must be sought.

(A.3) The Scale-Covariant Quantum Field Theory

In this Section we shall outline the reasons why the scale-covariant theory, differing from the canonical theory in its use of a *scale covariant* functional measure for path integrals (as opposed to the more usual *translation-invariant* one) is the chosen candidate for supplementing the canonical theory where renormalisation of the latter breaks down.

Essentially, this is the only alternative to the canonical theory in that it gives sensible equations of motion when field products are defined by the operator product expansion in a particular way that we shall now discuss.

Its use is exemplified by the solution of a severely non-renormalisable model. We shall consider each of these in turn.

(a) The Operator Product Expansion

In a Lagrangian density there are local operator field products, e.g. $\phi^2(x)$, $\phi^4(x)$. Now field operators are distributional in character, and so some prescription is required for defining such products. There are two common ways of making sense of such products namely, normal ordering where all creation operators are set to the left of the annihilation operators, and the Operator Product Expansion, O.P.E. which takes the form

$$A(x+y)\,B(x-y) = f(y)\,C(x) + \cdots\cdots \qquad \text{—(A3.1)}$$

where A, B and C are local operators and f is a c-number coefficient having the leading divergent behaviour as $y \to 0$. When an expansion such as (3.1) holds and C is a non-trivial local operator, we suppose that it is consistent to say that the local product of A and B is C, written

$$C(x) = \frac{1}{f(0)}\,A(x)\,B(x) = O_{AB}\,A(x)\,B(x) \qquad \text{—(A3.2)}$$

where O_{AB} has zero numerical value. Let us consider the equations of motion derived for the canonical $\lambda_0 \phi^4(x)$ theory where we use the above to define local operator field products.

The action is given by

$$\sigma = \int dx \left[\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2(x) - \lambda_0 \phi^4(x) \right] \qquad \text{—(A3.3)}$$

and we derive the equations of motion from stationarity of the action under field translation.

Now according to the above $\phi^2(x)$ is defined by

$$\phi^2(x) = O_{\phi\phi}\,\phi(x)\,\phi(x) \qquad \text{—(A3.4)}$$

Under translation of the field, i.e. $\phi(x) \longrightarrow \phi(x) + \Lambda(x)$

$$\phi^2(x) = O_{\phi\phi}\phi(x)\phi(x) \longrightarrow \phi^2(x)$$

Similarly $\phi^4(x) \to \phi^4(x)$ and we obtain no equations of motion.

If then we wish to retain the above definition of field products we must seek an alternative form of quantisation.

Consider now a formalism in which the equations of motion are derived due to invariance of the measure under field-*scaling*,

$$\text{i.e.} \qquad \phi(x) \rightarrow s(x)\,\phi(x) \qquad s(x) > 0 \qquad \text{—(A3.5)}$$

where $s(x)$ is a c-number field.

Then

$$\phi^2(x) = O_{\phi\phi}\,\phi(x)\,\phi(x) \rightarrow O_{\phi\phi}\,(s(x)\phi(x))(s(x)\phi(x))$$

$$= O_{\phi\phi}\,\phi(x)\phi(x)\,s^2(x)$$

$$= s^2(x)\,\phi^2(x) \qquad \text{—(A3.6)}$$

Similarly, for higher-order products

$$\phi^n(x) \rightarrow s^n(x)\,\phi^n(x) \qquad \text{—(A3.7)}$$

Stationarity of the action (A3.3) under the scale-transformation (A3.5) yields the operator equation of motion (including h)

$$\phi\,(\Box + m_0^2)\,\phi + 4\lambda_0\,\phi^4 - h\,\phi = 0 \qquad \text{—(A3.8)}$$

Now for a strictly c-number (classical) theory, we could remove a factor of ϕ to obtain the equation of motion of the canonical theory, i.e.

$$(\Box + m_0^2)\,\phi + 4\lambda\,\phi^3 - h = 0 \qquad \text{—(A3.9)}$$

Then this so-called scale-covariant formulation would be equivalent to canonical quantum field theory. However, for operator fields as in the above, (A3.8) and (A3.9) are *not* equivalent.

We have shown then how the scale-covariant formulation of quantum field theory yields sensible equations of motion, unlike its canonical counterpart, if we insist on the above as a prescription for local operator products.

We now ask the question: is there any other type of field transformation that gives sensible field equations, or is the scale-covariant formulation unique? As an example, consider the transformation

$$\phi(x) \longrightarrow S(x)\,\phi^3(x) \qquad S(x) > 0 \qquad \text{—(A3.10)}$$

Then,

$$\phi^2(x) = O_{\phi\phi}\,\phi(x)\,\phi(x) \longrightarrow O_{\phi\phi}\left(S(x)\,\phi^3(x)\right)\left(S(x)\,\phi^3(x)\right)$$

$$= S^2(x)\,O_{\phi\phi}\,\phi^6(x) \qquad \text{—(A3.11)}$$

Now $O_{\phi\phi}$ is only appropriate for $\phi^2(x)$. It kills lower order powers, but for $\phi^6(x)$ is unlikely to yield a finite multiple.

Hence scale-covariance assumes special importance, being the only transformation to yield sensible results in the presence of the above.

We note here that, in contrast to the above conclusion, when normal ordering is implemented then only the canonical formalism is appropriate.

Consequences of scale-covariance arise for the form of commutation relations, and also for the path-integral formulation of quantum field theory. Equal-time commutation relations now are not the canonical ones

$$[A(x), B(y)] = -i\delta(x-y) \qquad \text{—(A3.12)}$$

but are the so-called *Affine* commutation relations

$$[A(x), B(y)] = -i\delta(x-y)B(y) \qquad \text{—(A3.13)}$$

Further details of this can be found in references (*K1, K8*).

For the Path-Integral description, scale-covariance implies that the functional measure be covariant under field scaling. The canonical measure, invariant under field translation, is formally given as

$$\mathcal{D}[\phi] = \prod_x d\phi(x) \qquad \text{—(A3.14)}$$

The scale-covariant measure, by contrast, is given by

$$\mathcal{D}'[\phi] = \prod_x \frac{d\phi(x)}{|\phi(x)|^\beta} = \frac{\mathcal{D}[\phi]}{\prod_x |\phi(x)|^\beta} \qquad \text{—(A3.15)}$$

The measure (A3.15) is very singular, and one aspect of the scale-covariant theory examined in this thesis is the stability of the ground state(s) of such theories.

We shall now briefly mention how scale-covariance is exemplified by an explicitly solvable model.

(b) Solution of the Independent Value Model (IVM).

The Independent Value Model differs from a conventional model in that it has no gradient terms, e.g. for a self-interacting $\lambda_o\phi^4$ theory, the IVM has action,

$$\sigma\tau = -\int dx\left[\frac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4\right] \qquad \text{—(A3.16)}$$

It is obviously an unphysical model because without gradients the field at one point cannot influence any other point. However, it is a severely non-renormalisable model following the calculation-independent criterion of non-renormalisability given in Section 1. The more general expression of this criterion is derived from the Sobolev inequality

$$\left(\int|\phi(x)|^p d^n x\right)^{2/p} \leq K \int\left(|k|^{2\epsilon} + m_0^2\right)|\tilde{\phi}(k)|^2 d^n k \qquad \text{—(A3.17)}$$

provided that,

$$p \leq \frac{2n}{n-2\epsilon} \qquad \text{—(A3.18)}$$

This allows us to treat spinor fields for which $\epsilon = \frac{1}{2}$, as well as boson fields $(\epsilon = 1)$. Further details of this generalisation can be found in reference *(K1)*.

Now the parameter 2ϵ is the power of the derivative in the relevant equation of motion. For the IVM, we therefore have $\epsilon = 0$ Non-renormalisability sets in for this model for $p > 2$, independent of the space-time dimension, n.

It is known *(K1)* that the IVM Euclidean generating functional,

$$S[h] = N\int\mathcal{D}[\phi]\exp-\int\left[\frac{1}{2}m_0^2\phi^2 + \lambda_0\phi^p - ih\phi\right]dx \qquad \text{(A3.19)}$$

has the *exact* solution

$$S[h] = \exp - b\int dx \int \frac{du}{|u|}\left(1-\cos(uh(x))\right) e^{-\frac{1}{2}bm^2u^2 - \lambda b^{p-1} u^p}$$

—(A3.20)

where
$$m^2 = \frac{m_0^2}{b}\,\delta(0) \qquad \text{(a)}$$

and
$$\lambda = \frac{\lambda}{b^3}\,\delta^3(0) \qquad \text{(b)} \qquad \text{—(A3.21)}$$

A conventional lattice-space approach, using the canonical measure, $\prod_k d\phi_k$, yields a trivial (free) result. However, a lattice-space regularisation utilising the measure

$\prod_k \frac{d\phi_k}{|\phi_k|^{1-2b\Delta}}$, converges to the result (A3.24)

in the continuum limit $\Delta \to 0$. We see that this latter measure is scale-invariant in the continuum limit, and is therefore a particular example of a scale covariant measure.

So the use of the scale-invariant measure

$$\mathcal{D}'[\phi] = \prod_x \frac{d\phi(x)}{|\phi(x)|} \qquad \text{—(A3.22)}$$

is crucial for the solution of the IVM, a severely non-renormalisable model. This suggests its wider use, and that for non-renormalisable theories in general, we should use the scale-covariant measure in their path integral quantisations. We also note here that in the Fock space derivation (*K1*) of equation (A3.20), the realisation of the field operator $\phi(x)$ readily leads to the definition of local operator products via the operator product expansion discussed earlier.

Having outlined the reasons for using scale-covariant quantum field theory, we now proceed to introduce the contents of this thesis.

B - Further Investigation of Scale-covariant Quantum Field Theory - Introduction.

In this thesis we shall examine some of the problems of the pseudofree and interacting scale-covariant theories. We restrict ourselves to scalar fields only.

In Part 1 we take up the notion of non-renormalisability being viewed as a discontinuous perturbation. The possibility of developing λ -perturbation series about the *pseudofree* theory, rather than the disconnected free theory, is considered. It has been argued (*Y1*) that the scale-covariant coupled Green function equations are so degenerate that a perturbation series cannot be developed from them alone, as it can be for the canonical theory. We examine this claim, finding that indeed the branching equations of the scale-covariant field theory are degenerate.

The question naturally arises as to whether there is an additional equation that is able to remove the degeneracy of the branching equations. We consider a renormalisation-group type of equation and show that this removes the degeneracy. Also an extra dynamical equation is studied, motivated by the augmented formulation of the scale-covariant theory. This equation also removes the degeneracy, allowing unique perturbation series in principle. The augmented formalism is taken up in Part 2. This consists of introducing a functional integration over an auxiliary field to replace the denominator of the scale-covariant functional measure, i.e.

$$\mathcal{D}'[\phi] = \prod_x \frac{d\phi(x)}{|\phi(x)|^\beta} = \frac{\mathcal{D}[\phi]}{\prod_x |\phi(x)|^\beta} \qquad \text{—(B1.1)}$$

In particular, for $\beta = 1$,

using the formal identity

$$\left[\prod_x \phi(x) \right]^{-1} = \int \mathcal{D}[\chi] \exp - \tfrac{1}{2} \int \chi^2 \phi^2 dx \qquad \text{—(B1.2)}$$

where $\mathcal{D}[\chi]$ is a canonical (translation-invariant) measure, we have the result

$$\int \mathcal{D}'[\phi] = \int \mathcal{D}[\phi] \mathcal{D}[\chi] \exp - \tfrac{1}{2} \int \chi^2 \phi^2 dx \qquad \text{—(B1.3)}$$

We examine the branching equations of this augmented formalism and, because of the now canonical functional measures, are able to perform diagrammatic expansions.

The stability of scale-covariant theories is discussed in Part 3. This is prompted by the observation that the scale-covariant measure can also be re-expressed as

$$\mathcal{D}'[\phi] = \mathcal{D}[\phi] \exp - \beta/_2 \, \delta(0) \int dx \ln \phi^2 \qquad \text{—(B1.4)}$$

The classical potential of the canonical theory then is changed, due to the use of this measure, from $V(\phi)$ to

$$V'(\phi) = V(\phi) + \beta/_2 \, \delta(0) \ln \phi^2 \qquad \text{—(B1.5)}$$

As $\phi \to 0$, $V'(\phi)$ becomes unbounded below, indicating instability.

In Part 4 we look at the self-interacting $\lambda \phi^4$ scale-covariant theory, having been preoccupied up till then with the pseudofree theory. Using the large-N limit of the N^{-1} expansion for its O(N) generalisation $\frac{\lambda_0}{N}(\underline{\phi}^2 \ )^2$ we examine the renormalisability of the theory for those dimensions where it becomes obligatory to use the scale-covariant formalism, according to the criterion laid down earlier in this introduction.

In conclusion, we mention that the scale-covariant formalism may prove of great use in quantum gravity, where renormalisation is not well understood (*K11, I1*).

Part 1 - The Pseudoperturbation Expansion

(1.1) Introduction

In this part we will consider the possibility of developing the Interacting Theory as a perturbation about the Pseudofree Theory. This will involve us in the use of coupled Green function equations, derived from the formal Path Integral. We therefore need to explain how these equations are derived.

Consider first of all the conventional theory of a self-interacting scalar field theory, $\lambda\phi^4$, described by the usual Euclidean Vacuum generating function

$$S[h] = N\int \mathcal{D}[\phi]\exp -\int d^4x\left[\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4 - ih\phi\right] \qquad \text{—(1.1.1)}$$

This is the generator of unconnected Green functions via multiple functional differentiation, i.e.

$$i^n G_n(x_1 x_2 \cdots x_n) = \left.\frac{\delta^n S[h]}{\delta h(x_1)\delta h(x_2)\ldots\delta h(x_n)}\right|_{h=0} \qquad \text{—(1.1.2)}$$

Now, because the functional measure of (1.1) is invariant under field translation, i.e.

$$\mathcal{D}[\phi+\Lambda] = \mathcal{D}[\phi] \qquad \forall\ \Lambda(x)$$

we have also

$$S[h] = S[h]_\Lambda = N\int\mathcal{D}[\phi]\exp -\int d^4x\left[\tfrac{1}{2}(\nabla(\phi+\Lambda))^2 + \tfrac{1}{2}m_0^2(\phi+\Lambda)^2 + \lambda_0(\phi+\Lambda)^4 - ih(\phi+\Lambda)\right] \qquad \text{—(1.1.3)}$$

$S[h]_\Lambda$ does not really depend on Λ at all and so

$$\left.\frac{\delta}{\delta\Lambda(x)} \cdot S[h]_\Lambda \right)_{\Lambda=0} = 0$$

Hence we obtain the functional differential equation for $S[h]$,

$$\left\{ h(x) + K_x \frac{\delta}{\delta h(x)} - 4\lambda_0 \frac{\delta^3}{\delta h^3(x)} \right\} S[h] = 0 \qquad \text{—(1.1.4)}$$

(where $K_x \equiv -\nabla_x + m_0^2$).

Further differentiation, on setting the source current to zero, yields the Schwinger-Dyson coupled Green function equations, i.e.

$$\delta(x-x_1) - K_x G_2(xx_1) - 4\lambda_0 G_4(xxxx_1) = 0 \qquad \text{(a)}$$

$$\left[\delta(x-x_1) G_2(x_2x_3) + \delta(x-x_2) G_2(x_1x_3) + \delta(x-x_3) G_2(x_1x_2) \right]$$
$$- K_x G_4(xx_1x_2x_3) - 4\lambda_0 G_6(xxxx_1x_2x_3) = 0 \qquad \text{(b)}$$

$$\left[\delta(x-x_1) G_4(x_2x_3x_4x_5) + \delta(x-x_2) G_4(x_1x_3x_4x_5) + \delta(x-x_3) G_4(x_1x_2x_4x_5) \right.$$
$$\left. + \delta(x-x_4) G_4(x_1x_2x_3x_5) + \delta(x-x_5) G_4(x_1x_2x_3x_4) \right]$$
$$- K_x G_6(xx_1x_2x_3x_4x_5) - 4\lambda_0 G_8(xxxx_1x_2x_3x_4x_5) = 0 \qquad \text{(c)}$$

$$\sum_{r=1}^{2m+1} \delta(x-x_r) G_{2m}(x_1 \ldots \hat{x}_r \ldots x_{2m+1}) - K_x G_{2m+2}(xx_1 \ldots x_{2m+1})$$
$$- 4\lambda_0 G_{2m+4}(xxxx_1 \ldots x_{2m+1}) = 0 \qquad \text{(d)}$$

—(1.1.5)

For the scale-covariant theory, by contrast, the functional measure is invariant under field scaling transformations, i.e.

$$\mathcal{D}'[s\phi] = \mathcal{D}'[\phi] \qquad \forall\, s(x) > 0$$

Following through in the same way, we obtain the functional differential equation for $\mathcal{S}'[h]$, the Euclidean vacuum generating functional for the scale covariant theory

$$\left\{ h(x)\frac{\delta}{\delta h(x)} + :\frac{\delta}{\delta h(x)} K_x \frac{\delta}{\delta h(x)}: - 4\lambda_0 :\frac{\delta^4}{\delta h^4(x)}: \right\} \mathcal{S}'[h] = 0 \qquad \text{—(1.1.6)}$$

where : : denotes a subtraction defined by

$$:\frac{\delta^p}{\delta h^p(x)}: \mathcal{S}'[h] \equiv \frac{\delta^p}{\delta h^p(x)} \mathcal{S}'[h] - \frac{\delta^p}{\delta h^p(x)} \mathcal{S}'[h]\Big|_{h=0} \cdot \mathcal{S}'[h]$$

—(1.1.6a)

This subtraction is motivated by the lattice formulation of the Independent Value Model (IVM) in which K_x is replaced by m_0^2 (as in Eq.(A3.23)), but is necessary, in general, for scale covariant measures for which $\beta \neq 1$.

So we find the subtracted scale covariant branching equations

$$[\delta(x-x_1) + \delta(x-x_2)]\, G_2'(x_1x_2) - K_x G_4'(xx^*x_1x_2) - 4\lambda_0 G_6'(xxxxx_1x_2) = 0$$

(a)

$$[\delta(x-x_1)+\delta(x-x_2)+\delta(x-x_3)+\delta(x-x_4)]\, G_4'(x_1x_2x_3x_4) - K_x G_6'(xx^*x_1x_2x_3x_4)$$
$$- 4\lambda_0 G_8'(xxxxx_1x_2x_3x_4) = 0$$

(b)

$$\left[\sum_{r=1}^{2m} \delta(x-x_r)\right] G_{2m}'(x_1\ldots x_{2m}) - K_x G_{2m+2}'(xx^*x_1\ldots x_{2m}) \qquad \text{(c)}$$
$$- 4\lambda_0 G_{2m+4}'(xxxxx_1\ldots x_{2m}) = 0 \qquad \text{—(1.1.7)}$$

$$m \geq 1$$

where $x \overset{*}{\longrightarrow} x$ *after* the operation of K_x. These are represented in Fig.(1.1).

As can be seen, the significant difference between sets (1.5), (1.7) is the absence of an inhomogeneous equation analogous to (1.5a) for the scale-covariant theory. This is entirely due to the subtraction procedure. Also, because of the definition of operator products via the operator product expansion (as distinct from normal ordering) in scale covariant models, the branching equations (1.1.7), written for unconnected Green functions, are the same as for the connected functions. This is because the O.P.E. of Green functions with coincident points, will retain only the leading singularities, which are just the connected parts.

Having derived both sets of branching equations for purposes of contrast, we now move on to examine the perturbation series.

(1.2) Perturbation Theory

First of all, we will examine the Schwinger-Dyson equations and develope a perturbation series, about the free theory, for the interacting $2n$ -point Green functions, G_{2n} .

Generally, we write

$$G_{2n} = \sum_{r=0}^{\infty} \lambda_0^r G_{2n}^{(r)} \qquad \text{—(1.2.1)}$$

where $G_{2n}^{(r)}$ is the $2n$ -point Green function in r^{th}-order. Substituting this into set (1.5) we find

$$\sum_{r=1}^{2m+1} \delta(x-x_r)\left[\sum_{p=0}^{\infty} \lambda_0^p G_{2m}^{(p)}(x_1 \ldots \hat{x}_r \ldots x_{2m+1})\right] - K_x \sum_{p=0}^{\infty} \lambda_0^p G_{2m+2}^{(p)}(xx_1 \ldots x_{2m+1})$$

$$- 4\lambda_0 \sum_{p=0}^{\infty} \lambda_0^p G_{2m+4}^{(p)}(xxxx_1 \ldots x_{2m+1}) = 0$$

—(1.2.2)

Equating powers of λ_0 leads to the perturbation relations

$$\sum_{r=1}^{2m+1} \delta(x-x_r) G_{2m}^{(p)}(x_1 \ldots \hat{x}_r \ldots x_{2m+1}) - K_x G_{2m+2}^{(p)}(xx_1 \ldots x_{2m+1})$$

$$- 4 G_{2m+4}^{(p-1)}(xxxx_1 \ldots x_{2m+1}) = 0$$

$$(m = 0, 1, 2, \ldots)$$

—(1.2.3)

In particular

$$\delta(x-x_1) - K_x G_2^{(0)}(xx_1) = 0$$

$$\delta(x-x_1) G_2^{(0)}(x_2x_3) + \delta(x-x_2) G_2^{(0)}(x_1x_3) + \delta(x-x_3) G_2^{(0)}(x_1x_2)$$

$$- K_x G_4^{(0)}(xx_1x_2x_3) = 0$$

$$K_x G_2^{(1)}(xx_1) + 4 G_4^{(0)}(xxxx_1) = 0$$

So $G_2^{(1)}$ can be found in terms of $G_2^{(0)}$. It is easy to see that, in general, $G_{2n}^{(p)}$ can be expressed in terms of $G_2^{(0)}$. This is just the statement that the Feynman perturbation expansion (in terms of the free 2-point function) exists.

Now, from the scale-covariant branching equations, we develope the perturbation equations

$$\sum_{r=1}^{2m} \delta(x-x_r) \sum_{p=0}^{\infty} \lambda_0^{p} G_{2m}^{\prime(p)}(x_1 \cdots x_{2m}) - K_x \sum_{p=0}^{\infty} \lambda_0^{p} G_{2m+2}^{\prime(p)}(x x^* x_1 \cdots x_{2m})$$

$$- 4\lambda_0 \sum_{p=0}^{\infty} \lambda_0^{p} G_{2m+4}^{\prime(p)}(xxxx x_1 \cdots x_{2m}) = 0$$

$$(m \geq 1) \qquad \text{—(1.2.4)}$$

i.e., on equating powers of λ_0 ,

$$\sum_{r=1}^{2m} \delta(x-x_r) G_{2m}^{\prime(p)}(x_1 \cdots x_{2m}) - K_x G_{2m+2}^{\prime(p)}(x x^* x_1 \cdots x_{2m})$$

$$- 4 G_{2m+4}^{\prime(p-1)}(xxxx x_1 \cdots x_{2m}) = 0 \qquad \text{—(1.2.5)}$$

$$(m \geq 1)$$

From these equations, it is clear that $G_{2m}^{(p)}$ *cannot* be reduced to a knowledge of $G_2^{(0)}$ alone, because of the absence of an inhomogeneous equation.

We have found, therefore, that for the canonical theory, the perturbation series for n-point functions is unique, order by order. This is not the case for the scale-covariant theory. Now on examining the exact (not perturbation) Schwinger-Dyson equations (1.1.5), we see that G_2 appears to be arbitrary, requiring specification before G_4, G_6 etc. can be determined. How, then, can the perturbation series be unique? We resolve this by the following argument: the requirement that the perturbation series exists, i.e. $\lim_{\lambda_0 \to 0^+} |G_{2n}| < \infty$, can be interpreted as a boundary condition. Also, since the perturbation series in λ_0 is asymptotic, it does not uniquely determine G_2 unless extra information, outside the branching equations, is supplied. So we trade the boundary condition implicit in perturbation expansions, plus the ambiguity of resumming asymptotic series, for the boundary condition G_2 .

In a similar way, for the scale covariant equations (1.1.7) to be soluble, both G_2 *and* G_4 must be given. They are therefore more degenerate than the canonical (Schwinger-Dyson) equations. By imposing the existence of perturbation series for the 2n-point functions, this degeneracy cannot be entirely removed and so the perturbation series are *not* unique.

In order to understand this combinatoric problem of determining higher order functions in terms of lower ones, it is sufficient for us to use simplified versions of the canonical and scale-covariant theories.

(1.3) The Static Ultra Local (UL) and Independent Value (IV) Models

(a) Static Ultra Local Model:

The Ultra-Local Model is simply the canonical theory with the gradient term removed. The vacuum generating functional therefore satisfies an equation similar to (1.1.4) but with K_x replaced by m_0^2, i.e.

$$\left\{ h(x) + m_0^2 \frac{\delta}{\delta h(x)} - 4\lambda_0 \frac{\delta^3}{\delta h(x)^3} \right\} S[h] = 0 \qquad \text{—(1.3.1)}$$

In order that $S[h]$ be non-trivial, the ULM *(C + S 1)* must be regularised by putting it on a lattice of size M^{-1} or by setting $\delta(0) = M^d$ for d-spacetime dimensions. The parameter M cannot be eliminated from the theory. The general solution to (1.3.1) is then

$$S[h] = \exp \int d^d x \, M^d \ln Z_0(h(x)) \qquad \text{—(1.3.2)}$$

where for constant argument h, Z_0 satisfies the differential equation

$$\left(h + m^2 \frac{\partial}{\partial h} - 4\lambda \frac{\partial^3}{\partial h^3} \right) Z_0(h) = 0 \qquad \text{—(1.3.3)}$$

Here $m^2 = M^d m_0^2$, $\lambda = M^{3\delta} \lambda_0$ and $Z_0(0) = 1$.

The *connected* Green functions, defined by

$$i^n W_n (x_1 \cdots x_n) = \frac{\delta^n}{\delta h(x_1) \cdots \delta h(x_n)} \ln S[h] \Big|_{h=0}$$

are then given by

$$W_n (x_1 \cdots x_n) = \omega_n \prod_{r=2}^{n} \delta(x_1 - x_r)$$

where

$$i^n \omega_n = \frac{d^n}{dh^n} \ln Z_0(h) \Big|_{h=0}$$

To establish correspondence with the exact equations (1.1.5) let us define

$$i^n \tau_n = \frac{d^n}{dh^n} Z_0(h) \Big|_{h=0}$$

The branching equations thus derived from (1.3.3) are then, assuming

$\tau_{2p+1} = 0$, $\tau_0 = 1$,

$$\tau_0 - m^2 \tau_2 - 4\lambda \tau_4 = 0 \qquad \text{(a)}$$

$$3\tau_2 - m^2 \tau_4 - 4\lambda \tau_6 = 0 \qquad \text{(b)}$$

$$5\tau_4 - m^2 \tau_6 - 4\lambda \tau_8 = 0 \qquad \text{(c)}$$

$$\vdots \qquad \text{—(1.3.4)}$$

$$(2n+1)\tau_{2n} - m^2 \tau_{2n+2} - 4\lambda \tau_{2n+4} = 0 \qquad n = 0, 1, 2, \ldots$$

We can conveniently express these relations as a matrix equation to derive the results of the previous Section.

$$\begin{pmatrix} -m^2 & -4\lambda & 0 & 0 & \cdots \\ 3 & -m^2 & -4\lambda & 0 & \cdots \\ 0 & 5 & -m^2 & -4\lambda & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \tau_2 \\ \tau_4 \\ \tau_6 \\ \vdots \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \text{—(1.3.5)}$$

i.e. $$(-m^2 \mathbb{1} + L - 4\lambda D)(\tau) = -e_1 \qquad \text{—(1.3.6)}$$

where L is the *lower*-semi matrix

$$L_{ij} = (2j+1)\,\delta_{i,j+1}$$

D is the upper-semi matrix $D_{ij} = \delta_{i+1,j}$ and e_1 is the column matrix $(e_1)_p = \delta_{1,p}$. Now it is a general property of infinite matrices that upper-semi matrices do *not* have unique inverses. Because D is an upper semi-matrix, this means that the matrix equation (1.3.5) is *not* invertible. This was the conclusion of the last Section, i.e. that the exact canonical branching equations are not uniquely solvable (we have freedom in specifying C_{12}).

We cannot find a unique inverse for $(-m^2 \mathbb{1} + L - 4\lambda D)$.

However, since $(-m^2 \mathbb{1} + L)$ is a *lower*-semi-matrix, it does have a *unique* inverse, $(-m^2 \mathbb{1} + L)^{-1}$. We can re-express (1.3.6) as

$$(\tau) = -(-m^2 \mathbb{1} + L)^{-1}(e_1) + 4\lambda(-m^2 \mathbb{1} + L)^{-1}(D)(\tau) \qquad \text{—(1.3.7)}$$

Expanding τ as a power series in λ ,

$$\tau = \sum_{p=0}^{\infty} \tau^{(p)} \lambda^p$$

we ~~get~~ *unique* perturbative solutions

$$(\tau^{p+1}) = 4(-m^2 1 + L)^{-1}(D)(\tau^{(p)})$$

with $\tau^{(0)} = -(-m^2 1 + L)^{-1}(e_1)$

This use of matrices demonstrates how a set of degenerate equations (1.3.4) that permit arbitrary τ_2, have unique perturbation series expansions.

We could have reached the same conclusion by solving equation (1.3.3) directly.

Imposing the boundary conditions,

(1) $Z_0(0) = 1$ (2) $Z_0(h) = Z_0(-h)$

the solution of (1.3.3) takes the form

$$Z_0(h) = \frac{\alpha(m^2,\lambda) Z(m^2,\lambda,h) + \beta(m^2,\lambda) Z(-m^2,\lambda,ih)}{\alpha(m^2,\lambda) Z(m^2,\lambda,0) + \beta(m^2,\lambda) Z(-m^2,\lambda,0)}$$

$$= \frac{Z(m^2,\lambda,h) + \gamma(m^2,\lambda) Z(-m^2,\lambda,ih)}{Z(m^2,\lambda,0) + \gamma(m^2,\lambda) Z(-m^2,\lambda,0)} \qquad \text{—(1.3.8)}$$

where $Z(m^2,\lambda,h) = \int_{-\infty}^{+\infty} du \, \exp\left(-\frac{1}{2}m^2u^2 - \lambda u^4 + ihu\right)$

and $\gamma(m^2,\lambda)$ is an *arbitrary* function of m^2, λ.

Once τ_2 is specified, all the Green functions are fixed. This is equivalent to specifying $\gamma(m^2,\lambda)$.

Now $i^{2p}\tau_{2p} = \frac{d^{2p}}{dh^{2p}} Z_0(h)\Big|_{h=0}$

$$= \frac{\alpha^+_{2p} + \gamma\alpha^-_{2p}}{\alpha^+_0 + \gamma\,\alpha^-_0} \qquad \text{—(1.3.9)}$$

where $$\alpha_{2p}^{+} = \frac{d^{2p}}{dh^{2p}} Z(m^2,\lambda,h)\Big|_{h=0}$$ —(1.3.10)

$$= \left(\frac{1}{32\lambda}\right)^{p/2} \frac{2p!}{p!} \sqcup\left(p, +\sqrt{\frac{m^4}{8\lambda}}\right)$$

and $$\alpha_{2p}^{-} = \frac{dZ^{2p}}{dh^{2p}}(-m^2,\lambda,ih)\Big|_{h=0}$$ —(1.3.11)

$$= (-1)^p\left(\frac{1}{32\lambda}\right)^{p/2} \frac{2p!}{p!} \sqcup\left(p, -\sqrt{\frac{m^4}{8\lambda}}\right)$$

Here the $\sqcup(a, x)$ are the usual parabolic cylinder functions.

We can write (1.3.8) as

$$\tau_{2p} = \frac{1 + \gamma\, \alpha_{2p}^{-}/\alpha_{2p}^{+}}{\alpha_0^{+}/\alpha_{2p}^{+} + \gamma\, \alpha_0^{-}/\alpha_{2p}^{+}}$$ —(1.3.12)

Now for small λ ,

$$\frac{\alpha_{2p}^{-}}{\alpha_{2p}^{+}} = O(\lambda^{-p}\, e^{m^4/16\lambda})$$ —(1.3.13)

For the existence of perturbation series we require that

$$\lim_{\lambda \to 0^+} |\tau_{2p}| < \infty$$ —(1.3.14)

This can only be satisfied from (1.3.12) if

$$\gamma = 0 \qquad \text{or} \qquad \gamma = O(e^{-am^4/16\lambda}) \quad a > 1$$ —(1.3.15)

This is equivalent to requiring $\gamma \bar{\alpha}_{2p}$ to have zero asymptotic series. So the condition that the perturbation series exists (1.3.14) guarantees that it is unique because $(\gamma)_\lambda$ is determined.

(b) Independent Value Model

This is the analogous model in the scale covariant case where we eliminate the gradient term from the action. The functional differential equation for the generating functional $S'[h]$ is then

$$\left\{ h(x)\frac{\delta}{\delta h(x)} + m_0^2 : \frac{\delta^2}{\delta h^2(x)} : - 4\lambda_0 : \frac{\delta^4}{\delta h^4(x)} : \right\} S'[h] = 0 \qquad \text{—(1.3.16)}$$

The general solution for this model has been given by Klauder (*K1*),

$$S'[h] = \exp \int dx \, U(h(x)) \qquad \text{—(1.3.17)}$$

where $U(h(x))$ satisfies the linear subtracted equation

$$h\frac{dU}{dh} + m^2\left(\frac{d^2U}{dh^2} - \frac{d^2U}{dh^2}\bigg|_{h=0}\right) - 4\lambda\left(\frac{d^4U}{dh^4} - \frac{d^4U}{dh^4}\bigg|_{h=0}\right) = 0 \qquad \text{—(1.3.18)}$$

where

$$m^2 = \delta(0)\, m_0^2 \ ; \qquad \lambda = \delta^3(0)\,\lambda_0$$

We introduce renormalised mass $\bar{m}$, and coupling $\bar{\lambda}$ via

$$b\,\bar{m}^2 = m^2 \ , \qquad b^3\bar{\lambda} = \lambda \qquad \text{—(1.3.19)}$$

Now the parameter b plays the same rôle as M in the ULM of the previous Section. The connected Green functions are obtained from $U(h)$ by

$$W_n'(x_1 \dots x_n) = \prod_{r=2}^{n} \delta(x_1 - x_r)\; \omega_n' \qquad \text{—(1.3.20)}$$

where $$i^n \omega_n' = \frac{d^n}{dh^n} U(h)\Big|_{h=0}$$ —(1.3.21)

This is analogous to our analysis of the ULM previously.

Continuing the analogy, we write out the branching equations implied by (1.3.16) and put them into matrix form.

The branching equations arising from (1.3.16) are

$$2\omega_2' - m^2\omega_4' - 4\lambda\omega_6' = 0 \qquad \text{(a)}$$

$$4\omega_4' - m^2\omega_6' - 4\lambda\omega_8' = 0 \qquad \text{(b)}$$

$$\vdots$$

$$2n\,\omega_{2n}' - m^2\omega_{2n+2}' - 4\lambda\omega_{2n+4}' = 0 \qquad (n=1,2,\dots)$$ —(1.3.22)

i.e.

$$\begin{pmatrix} 2 & -m^2 & -4\lambda & 0 & \cdots \\ 0 & 4 & -m^2 & -4\lambda & \cdots \\ 0 & 0 & 6 & -m^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \omega_2' \\ \omega_4' \\ \omega_6' \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

This may be written as

$$(\Lambda - m^2 D - 4\lambda D^2)(\omega') = (0)$$ —(1.3.24)

where $D_{ij} = \delta_{i+1,j}$ (as before), and Λ is the diagonal matrix

$\Lambda_{ij} = 2j\,\delta_{ij}$

Because the matrix D is an upper semi-matrix, (1.3.23) cannot have a unique inverse. In order to more clearly contrast this case with the previous one, let us separate out ω_2 from (1.3.23) and write instead

$$\begin{pmatrix} -m^2 & -4\lambda & 0 & \cdots \\ 4 & -m^2 & -4\lambda & 0 \cdots \\ 0 & 6 & -m^2 & -4\lambda \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \omega_4' \\ \omega_6' \\ \omega_8' \\ \vdots \end{pmatrix} = \begin{pmatrix} 2\omega_2' \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \text{—(1.3.25)}$$

i.e. $(-m^2\, 1 + L' - 4\lambda D)(\varpi) = -(2\omega_2')(e_1)$ —(1.3.26)

where $\varpi = D\omega' = \begin{pmatrix} \omega_4' \\ \omega_6' \\ \omega_8' \\ \vdots \end{pmatrix}$

and L' is the *lower* semi-matrix

$$L'_{ij} = 2(j+1)\,\delta_{i,j+1}$$

As before, because of the upper semi-matrix D, $(-m^2 1 + L' - 4\lambda D)$ is *not* invertible uniquely. So, in accordance with our previous conclusions, even if we are given ω_2', the branching equations (1.3.20) cannot be uniquely solved.

However, $(-m^2 1 + L')$ does have a unique inverse and we can write

$$\varpi = -2\omega_2'(-m^2 1 + L')^{-1}(e_1) + 4\lambda(-m^2 1 + L')^{-1}(D)(\varpi) \qquad \text{—(1.3.27)}$$

i.e. $$\omega_2'^{-1}(\varpi) = -2(-m^2 1 + L')^{-1}(e_1) + 4\lambda(-m^2 1 + L')^{-1}(D)\omega_2'^{-1}(\varpi)$$

—(1.3.28)

We can now expand the combination $\omega_2'^{-1}(\varpi)$ as a power series in λ, i.e.

$$\omega_2'^{-1}(\varpi) = \sum_{p=0}^{\infty} \lambda^p \tilde{\omega}^{(p)} \qquad \text{—(1.3.29)}$$

then $\tilde{\omega}^{(p)}$ is uniquely determined from

$$(\tilde{\omega}^{(p+1)}) = 4(-m^2 1 + L')^{-1}(D)(\tilde{\omega}^{(p)}) \qquad p \geq 1 \tag{1.3.30}$$

where $$\tilde{\omega}^{(0)} = -2(-m^2 1 + L')(e_1) \tag{1.3.31}$$

So we see that unlike the perturbation series of the canonical theory where the existence of perturbation series guaranteed their uniqueness, for the scale-covariant theory, we need to specify the perturbation series for ω_2 in order that all other perturbation series be determined.

As before, let us see how this result appears from the equation (1.3.17) directly.

$$h\frac{du}{dh} + m^2\left(\frac{d^2u}{dh^2} - \frac{d^2u}{dh^2}\Big|_{h=0}\right) - 4\lambda\left(\frac{d^4u}{dh^4} - \frac{d^4u}{dh^4}\Big|_{h=0}\right) = 0 \tag{1.3.17}$$

Imposing the boundary conditions

$$\begin{aligned} &\text{(i)} && u(0) = 0 \\ &\text{(ii)} && u(h) = u(-h) \\ &\text{(iii)} && \omega_0' = \omega_1' = \omega_3' = 0 \end{aligned} \tag{1.3.32}$$

Then $u(h)$ has the form

$$u(h) = a(m^2,\lambda)\,\omega'(m^2,\lambda,h) + c(m^2,\lambda)\,\omega'(-m^2,\lambda,ih) \tag{1.3.33}$$

where
$$\omega'(m^2,\lambda,h) = \int_{-\infty}^{+\infty}\frac{du}{|u|}\left[1-\cos(uh)\right]\exp\left(-\tfrac{1}{2}m^2u^2 - \lambda u^4\right) \tag{1.3.34}$$

Specifying the arbitrary functions $a(m^2,\lambda)$, $c(m^2,\lambda)$ is equivalent to independently specifying ω_2' and ω_4'.

Now, writing ω'_{2p} as

$$i^{2p}\omega'_{2p} = \frac{d^{2p}}{dh^{2p}} W(h)\Big|_{h=0}$$

$$= a(m^2,\lambda)\,\omega^+_{2p} + c(m^2,\lambda)\,\omega^-_{2p} \qquad \text{—(1.3.35)}$$

where

$$\omega^+_{2p} = \frac{d^{2p}}{dh^{2p}}\omega'(m^2,\lambda,h)\Big|_{h=0}$$

$$\omega^-_{2p} = \frac{d^{2p}}{dh^{2p}}\omega'(-m^2,\lambda,ih)\Big|_{h=0} \qquad \text{—(1.3.36)}$$

we find

$$\frac{\omega^-_{2p}}{\omega^+_{2p}} = O\left(\lambda^{-p} e^{m^4/16\lambda}\right) \qquad \text{—(1.3.37)}$$

The existence of the perturbation series, i.e.

$$\lim_{\lambda\to 0^+} |\omega'_{2p}(m^2,\lambda)| < \infty \qquad \text{—(1.3.38)}$$

forces $c\,\omega^-_{2p}$ to have zero asymptotic series in λ . So the asymptotic series for $\omega'_{2p}\,\omega_2'^{-1}$ are now uniquely determined.

(1.4) Renormalisation Group Equations

So far we have contrasted the scale covariant equations with the canonical (Schwinger-Dyson) ones to show that they are more degenerate. The perturbation series for the ULM are unique but for the scale covariant IVM we need to specify the 2-point function in all orders of perturbation theory for all other perturbation series to be uniquely determined.

However, a Lattice calculation of the ULM (*K3*) shows that it is completely determined, rather than just its perturbation series. Similarly for the IVM (*K2*), a Fock space calculation shows it to be completely determined up to a *single* overall scale parameter.

Obviously then, in going from the formal path integral to the branching equations, we lose information.

In this Section we consider the information supplied by parametric equations in m^2, λ.

These are like renormalisation group equations, and they have been examined for the static ultra local and independent-value models in a different context (*M1*).

Consider the formal Euclidean path integral for the ULM,

$$S[h] = N^{-1}\int \mathcal{D}[\phi]\ \exp - \int dx\left(\tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4 - ih\phi\right) \qquad \text{—(1.4.1)}$$

Then,

$$\frac{\partial}{\partial(m_0^2/2)} S[h] = \left(\frac{\partial}{\partial(m_0^2/2)} N^{-1}\right) \cdot \int \mathcal{D}[\phi] \exp - \int dx\left(\tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4 - ih\phi\right)$$

$$- N^{-1}\int \mathcal{D}[\phi]\ \phi^2 \exp - \int dx\left(\tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4 - ih\phi\right) \qquad \text{—(1.4.2)}$$

where the normalisation factor, N^{-1}, must be differentiated because it contains m_0^2.

$$N = \int \mathcal{D}[\phi]\ \exp - \int dx\left(\tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4\right) \qquad \text{—(1.4.3)}$$

$$\frac{\partial}{\partial(m_0^2/2)} N^{-1} = N^{-2}\int dx \int \mathcal{D}[\phi]\ \phi^2(x) \exp - \int dx\left(\tfrac{1}{2}m_0^2\phi^2 + \lambda_0\phi^4\right) \qquad \text{—(1.4.4)}$$

$$= -N^{-1}\frac{\delta^2}{\delta h^2(x)} S[h]\Big|_{h=0}$$

Hence

$$\frac{\partial}{\partial(m_0^2/2)} S[h] = \int dx \left\{ \frac{\delta^2 S[h]}{\delta h^2(x)} - \frac{\delta^2 S[h]}{\delta h^2(x)}\bigg|_{h=0} . S[h] \right\}$$

$$\equiv \int dx : \frac{\delta^2}{\delta h^2(x)} : S[h] \qquad \text{—(1.4.5)}$$

Similarly

$$\frac{\partial}{\partial \lambda_0} S[h] = \int dx : \frac{\delta^4}{\delta h^4(x)} : S[h] \qquad \text{—(1.4.6)}$$

Equations (1.4.5) and (1.4.6) hold equally well for canonical and scale covariant theories. Let us now see the constraints that they impose on $Z_0(h)$ and $U(h)$. Consider first of all the ULM. For we have

$$S[h] = \exp \int d^d x \, M^d \ln Z_0(h(x)) \qquad \text{—(1.3.2)}$$

So

$$\frac{\partial}{\partial(m_0^2/2)} S[h] = \int d^d x \, M^d \frac{1}{Z_0} \frac{\partial}{\partial(m_0^2/2)} Z_0 \;.\; S[h]$$

and

$$\frac{\delta^2}{\delta h^2(x)} S[h] = \frac{\delta}{\delta h(x)} \left\{ M^d \frac{1}{Z_0} \frac{\partial Z_0(h)}{\partial h} \;.\; S[h] \right\}$$

$$= -M^d \frac{1}{Z_0^2} \left(\frac{\partial Z_0(h)}{\partial h} \right)^2 \delta(0) . S[h]$$

$$+ M^d \frac{1}{Z_0} \frac{\partial^2 Z_0(h)}{\partial h^2} \delta(0) \, S[h]$$

$$+ M^d \frac{1}{Z_0} \frac{\partial Z_0(h)}{\partial h} . M^d \frac{1}{Z_0} \frac{\partial Z_0}{\partial h} . S[h]$$

But $\qquad \delta(0) = M^d$

i.e. $$\frac{\delta^2 S[h]}{\delta h^2(x)} = M^{2d} \frac{1}{Z_0(h)} \frac{\partial^2 Z_0(h)}{\partial h^2} . S[h]$$

So our renormalisation group equation (1.4.5) for the ULM becomes, with the mass-renormalisation $m^2 = m_0^2 M^d$,

$$\frac{\partial}{\partial(m^2/2)} Z_0(h) = \frac{\partial^2 Z_0(h)}{\partial h^2} - \frac{\partial^2 Z_0(h)}{\partial h^2}\Big|_{h=0} Z_0(h) \qquad \text{—(1.4.7)}$$

Similarly (1.4.6) becomes

$$\frac{\partial Z_0(h)}{\partial \lambda} = \frac{\partial^4 Z_0(h)}{\partial h^4} - \frac{\partial^4 Z_0(h)}{\partial h^4}\Big|_{h=0} Z_0(h) \qquad \text{—(1.4.8)}$$

These equations can be simplified if we define

$$Z_0(h) = \frac{E(h)}{E(0)} \qquad \text{—(1.4.9)}$$

Then we have $$\frac{\partial E(h)}{\partial(m^2/2)} = \frac{\partial^2 E(h)}{\partial h^2} \qquad \text{—(1.4.10)}$$

and $$\frac{\partial E(h)}{\partial \lambda} = \frac{\partial^4 E(h)}{\partial h^4} \qquad \text{—(1.4.11)}$$

Defining the derivatives of $E(h)$ by

$$i^n \tilde{\tau}_n \equiv \frac{\partial^n E(h)}{\partial h^n}\Big|_{h=0} \qquad \text{—(1.4.12)}$$

it follows that

$$i^n \tau_n \equiv \frac{\partial^n Z_0(h)}{\partial h^n}\Big|_{h=0} = i^n \frac{\tilde{\tau}_n}{\tilde{\tau}_0}$$

i.e. $$\tau_n = \frac{\tilde{\tau}_n}{\tilde{\tau}_0} \qquad \text{—(1.4.13)}$$

Now from (1.4.10) we can functionally differentiate w.r.t. h ,

i.e. $$\frac{\partial}{\partial(m^2/2)} . \frac{\partial^{2p}}{\partial h^{2p}} E(h) = \frac{\partial^{2p+2}}{\partial h^{2p+2}} E(h) \qquad \text{—(1.4.14)}$$

On setting $h=0$ we find, using (1.4.12)

$$\frac{\partial}{\partial(m^2/2)}\tilde{\tau}_{2p} = -\tilde{\tau}_{2p+2} \qquad (p=0,1,2,\dots) \qquad \text{—(1.4.15)}$$

Similarly,

$$\frac{\partial}{\partial\lambda}\tilde{\tau}_{2p} = \tilde{\tau}_{2p+4} \qquad \text{—(1.4.16)}$$

Now from (1.3.9) and (1.4.13) we have

$$\tilde{\tau}_n = \alpha^+_{2p} + \gamma(m^2,\lambda)\alpha^-_{2p} \qquad \text{—(1.4.17)}$$

From the defining equations for $\alpha^{\pm}_{2p}$, (1.3.10, 1.3.11), we see that $\alpha^{\pm}_{2p}$ individually satisfy (1.4.15) and (1.4.16). So we must have

$$\frac{\partial\gamma(m^2,\lambda)}{\partial(m^2/2)} = \frac{\partial\gamma(m^2,\lambda)}{\partial\lambda} = 0 \qquad \text{—(1.4.18)}$$

This has as solution γ = constant. Whereas the Schwinger-Dyson branching equations (1.3.4) gave solutions in terms of the arbitrary *function* $\tau_2(m^2,\lambda)$, imposing the extra conditions (1.4.5) and (1.4.6) gives a family of solutions that depend on the single *constant*, γ . If we now invoke the existence of the perturbation series, then from (1.3.15) we require γ = 0.

Let us now consider the IVM. It is straightforward to see that, given the generating functional

$$S'[h] = \exp\int dx\, \sqcup(h(x)) \qquad \text{—(1.4.19)}$$

and the renormalisation

$$m^2 = b\,\delta(0)\,m_0^2 \quad , \quad \lambda = b^3\delta^3(0)\lambda_0 \qquad \text{—(1.4.20)}$$

the equations (1.4.5) and (1.4.6) for $S'[h]$ become for $\omega(h(x))$,

$$\frac{\partial}{\partial(m^2/2)}\,\omega(h) = -\frac{\partial^2}{\partial h^2}\,\omega(h) \qquad \text{—(1.4.21)}$$

$$\frac{\partial}{\partial\lambda}\,\omega(h) = \frac{\partial^4}{\partial h^4}\,\omega(h) \qquad \text{—(1.4.22)}$$

As for the ULM, substituting for $\omega(h)$ explicitly as in (1.3.31) we find that both $\omega'(m^2, \lambda, h)$ and $\omega'(-m^2, \lambda, ih)$, satisfy (1.4.21) and (1.4.22) separately. So we find,

$$\frac{\partial a}{\partial(m^2/2)} = \frac{\partial c}{\partial(m^2/2)} = 0 \qquad \text{—(1.4.23)}$$

$$\frac{\partial a}{\partial\lambda} = \frac{\partial c}{\partial\lambda} = 0 \qquad \text{—(1.4.24)}$$

i.e. both a, c are constant.

Now if the perturbation series exists, we must have c = 0, as seen from (1.3.35). This leaves

$$\omega(h) = a\int\frac{du}{|u|}\left(1-\cos(uh)\right)\exp-\left(\tfrac{1}{2}m^2u^2+\lambda u^4\right) \qquad \text{—(1.4.25)}$$

Since $a(m^2, \lambda)$ has been restricted to a constant, independent of m^2, λ , due to the imposition of the renormalisation-type group equations (1.4.5) and (1.4.6), these equations have therefore given us extra information on the perturbation series for the IVM. Before the use of these equations, the existence of perturbation series for the IVM forced c = 0, but gave no information on $a(m^2, \lambda)$. Using these equations now shows us that a is independent of λ .

Finally in this Section we briefly mention the consequence of imposing an equation of the Callan-Symanzik form. Whereas in the canonical theory the inhomogeneous equation (1.1.5a) is successfully predicted by the Callan-Symanzik equation, this is not the case for the scale covariant theory. Details may be found in Appendix A.

(1.5) Conclusion

In this part we have examined the branching equations of both the canonical (Schwinger-Dyson) and scale covariant theories. We have seen that although the canonical equations are generate, the existence of perturbation series implies a boundary condition which, together with the ambiguity of resummation resolves the problem of how the exact equations are degenerate, but the perturbation series is unique.

For the scale covariant theory, the degeneracy of the exact branching equations is reduced, but *not* eliminated by requiring the existence of pseudo perturbation series. In order to further reduce the degeneracy, we need to look for an additional dynamical equation, analogous to the inhomogenous equation (1.1.5a). This is the subject of the next part.

Part 2 - The Augmented Formalism

(2.1) Introduction

We will discuss here a different formulation of the scale covariant scalar field theory. For simplicity we will consider only scale-invariant measure. The Augmented Formulation replaces the scale-invariant measure by a functional integration over an auxiliary field by use of the formal relation

$$\mathcal{D}'[\phi] \equiv \frac{\mathcal{D}[\phi]}{\prod_x |\phi(x)|} \propto \mathcal{D}[\phi] \int \mathcal{D}[\chi] \exp -i \int dx \tfrac{1}{2} \chi^2 \phi^2$$

We shall work throughout using the pseudofree theory, i.e. $\lambda_0 \to 0$ limit of the fully interacting scale covariant theory.

We can then write the pseudofree generating functional

$$Z_0'[h] = \int \mathcal{D}'[\phi] \exp i \int dx \left\{ \tfrac{1}{2} (\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2 + h\phi \right\} \quad \text{—(2.1.1)}$$

in its *augmented* form

$$Z_0'[h] = \int \mathcal{D}[\phi] \mathcal{D}[\chi] \exp i \int dx \left\{ \tfrac{1}{2} (\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2 - \tfrac{1}{2} \eta \chi^2 \phi^2 + h\phi \right\} \quad \text{—(2.1.2)}$$

where $\mathcal{D}[\phi]$, $\mathcal{D}[\chi]$ are now translation invariant measures, normalised so that $Z_0'[0] = 1$. The constant η is simply for generality. Integrating out the χ-field in (2.1.2) reproduces (2.1.1). However, we may alternatively perform the ϕ integration first. We shall discuss this later, as it forms the basis of a diagrammatic expansion for the pseudofree augmented theory.

Firstly we shall examine the branching equations derived from the augmented generating functional (2.1.2).

(2.2) The Pseudofree Augmented Branching Equations

For generality, we introduce an external source current, j , for the auxiliary field χ .

i.e. we use

$$Z_0'[h,j] = \int \mathcal{D}[\phi]\mathcal{D}[\chi] \exp i\int dx \left\{ \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m_0^2\phi^2 + h\phi + j\chi - \tfrac{1}{2}\eta\chi^2\phi^2 \right\}$$

From the translation invariance of $\mathcal{D}[\phi]$ and $\mathcal{D}[\chi]$ we obtain as functional differential equations for $Z_0'[h,j]$,

$$\left\{ h(x) - K_x \delta/i\delta h(x) - \eta \frac{\delta^3}{i^3 \delta h(x) \delta j^2(x)} \right\} Z_0'[h,j] = 0 \qquad \text{—(2.2.1)}$$

$$\left\{ j(x) - \eta \frac{\delta^3}{i^3 \delta j(x) \delta h^2(x)} \right\} Z_0'[h,j] = 0 \qquad \text{—(2.2.2)}$$

We see that (2.2.2) is a constraint equation. It is unchanged for the interacting theory. However, (2.2.1) is dynamical and becomes

$$\left\{ h(x) - K_x \delta/i\delta h(x) - \eta \frac{\delta^3}{i^3 \delta h(x) \delta j^2(x)} - 4\lambda_0 \frac{\delta^3}{i^3 \delta h^3(x)} \right\} Z_0' = 0 \qquad \text{—(2.2.3)}$$

on including a $\lambda_0 \phi^4$ self-interaction. The branching equations for the augmented Green functions, defined by

$$G'_{m,n}(x_1 \cdots x_m; y_1 \cdots y_n) \equiv \frac{1}{i^{m+n}} \frac{\delta^{m+n}}{\delta h(x_1) \cdots \delta h(x_m) \delta j(y_1) \cdots \delta j(y_n)} Z_0'[h,j] \Big|_{h=j=0} \qquad \text{—(2.2.4)}$$

are easily found.

The constraint equation (2.2.2) yields

$$i\delta(x-y) + \eta\, G'_{2,2}(xx;xy) = 0 \qquad \text{(a)}$$

$$i\delta(x-y)\, G'_{m,0}(x_1\cdots x_m) + \eta\, G'_{m+2,2}(xxx_1\cdots x_m;xy) = 0 \qquad \text{(b)}$$

$$i\sum_{r=1}^{q+1} \delta(x-x_r)\, G'_{m,n}(x_1\cdots x_m;y_1\cdots \hat{y}_r\cdots y_{n+1}) + \eta\, G'_{m+2,n+2}(xxx_1\cdots x_m;xyy_1\cdots y_n) = 0 \qquad \text{(c)} \qquad \text{—(2.2.5)}$$

where by symmetry we assume $G'_{2m+1,n} = 0$ and $G'_{m,2n+1} = 0$.

These equations can be displayed in symbolic form (Fig.2.2).

The dynamical equation (2.2.1) gives

$$i\delta(x-y) + K_x G'_{2,0}(xy) + \eta\, G'_{2,2}(xy;xx) = 0 \qquad \text{(a)}$$

$$i\sum_{r=1}^{m} \delta(x-x_r)\, G'_{m-1,0}(x_1\cdots \hat{x}_r\cdots x_m) + K_x G'_{2m+1,0}(xx_1\cdots x_m) + \eta\, G'_{m+1,2}(xx_1\cdots x_m;xx) = 0 \qquad \text{(b)}$$
$$(m \geqslant 2)$$

$$i\sum_{r=1}^{m} \delta(x-x_r)\, G'_{m-1,n}(x_1\cdots \hat{x}_r\cdots x_m;y_1\cdots y_n) + K_x G'_{m+1,n}(xx_1\cdots x_m;y_1\cdots y_n) + \eta\, G'_{m+1,n+2}(xx_1\cdots x_m;xxy_1\cdots y_n) = 0 \qquad \text{(c)} \qquad \text{—(2.2.6)}$$
$$(m,n \geqslant 2)$$

These equations are displayed in (Fig.2.3).

We now compare the augmented formalism with the scale covariant equations of Part 1.

Remember that, motivated by the Lattice formulation of the IVM, the scale covariant equations included the subtractions (*K4*), i.e.

$$\left\{ h(x)\, \delta/_{i\delta h(x)} - :\delta/_{i\delta h(x)}\, K_x\, \delta/_{i\delta h(x)}: \right\} Z'_0[h] = 0 \qquad \text{—(2.2.7)}$$

where

$$:\frac{\delta^p}{\delta h^p(x)}: Z'_0[h] \equiv \frac{\delta^p}{\delta h^p(x)} Z_0[h] - \left.\frac{\delta^p Z_0}{\delta h^p(x)}\right|_{h=0} Z_0[h] \qquad \text{—(2.2.8)}$$

We can derive the analogous equation for the augmented pseudo-free theory by using equations (2.2.1) and (2.2.2). Functionally differentiating (2.2.1) and (2.2.2) w.r.t. $h(y)$ and $j(y)$ respectively and subracting gives

$$\Big\{ h(x)\,\delta/i\delta h(y) - j(x)\,\delta/i\delta j(y) - \delta/i\delta h(y)\,K_x\,\delta/i\delta h(x)$$

$$- \eta\Big(\delta^4/\delta h(x)\delta h(y)\delta j^2(x) - \delta^4/\delta h^2(x)\delta j(x)\delta j(y) \Big)\Big\} Z_0' = 0 \qquad \text{—(2.2.9)}$$

We see that on setting $x = y$ and $j = 0$ we reproduce (2.2.7) except *without* the subtraction procedure. From Part 1 we saw that the subtraction procedure was directly responsible for the absence of an inhomogeneous equation amongst the scale-covariant branching equations, a fact that led to the breakdown of scale covariant perturbation theory.

The augmented formulation gives us cause to doubt the necessity of this subtraction procedure for the particular case of a scale-invariant measure. We resolve this contradiction by proposing that the IVM is a special case of a theory with scale-invariant measure, where the subtraction procedure *must* be implemented, but that for theories where the gradient term is restored such a procedure is absent. This proposal is backed up by the following observation, showing that we cannot accommodate the IVM in the augmented formalism. The general solution for the IVM generating functional

$$S'[h] = N^{-1}\int \mathcal{D}'[\phi]\,\exp - \int \big(\tfrac{1}{2} m_0^2 \phi^2 + \lambda_0 \phi^4 - i h \phi \big)\, dx \qquad \text{—(2.2.10)}$$

is given by *(K1)*

$$S'[h] = \exp -b\int W(h(x))\,dx \qquad \text{—(2.2.11)}$$

where

$$W(h(x)) = \int \frac{du}{|u|}\left(1-e^{ihu}\right)e^{-\frac{1}{2}bm^2u^2-b^3\lambda u^4} \qquad \text{—(2.2.12)}$$

Here b is an arbitrary constant, and we have the infinite multiplicative renormalisations

$$m_0^2 = \frac{bm^2}{\delta(0)} \quad ; \qquad \lambda_0 = \frac{b^3\lambda}{\delta^3(0)} \qquad \text{—(2.2.13)}$$

We can choose our system of units so that $b = 1$ for convenience.

Now the obvious augmentation of (2.2.12) is given by

$$W(h,j) = \int du\,dv\left(1-e^{ihu}\right)e^{-\frac{1}{2}m^2u^2-\lambda u^4-\frac{1}{2}\eta u^2v^2+ivj} \qquad \text{—(2.2.14)}$$

where we have coupled in a source j for the v variable. This can best be written as

$$W(h,j) = \Omega(0,j) - \Omega(h,j) \qquad \text{—(2.2.15)}$$

where $$\Omega(h,j) = \int du\,dv\,\exp\left(-\tfrac{1}{2}m^2u^2-\lambda u^4-\tfrac{1}{2}\eta u^2v^2+ihu+ivj\right) \qquad \text{—(2.2.16)}$$

The augmented functional differential equations for $S'[h,j]$ (in the Euclidean formalism now)

$$\left\{ij(x) - \eta_0\frac{\delta^3}{i^3\,\delta j(x)\,\delta h^2(x)}\right\}S'[h,j] = 0 \qquad \text{—(2.2.17)}$$

$$\left\{ ih(x) - m_0^2\, \delta/i\delta h(x) - \eta_0\, \delta^3/i^3 \delta h(x) \delta j^2(x) - 4\lambda_0 \delta^3/i^3 \delta h^3(x) \right\} S'[h,j] = 0$$

—(2.2.18)

become differential equations for $\omega(h)$ of (2.2.11)

$$ij - \left\{ \eta\, \partial^3 / i^3 \partial j \partial h^2 \right\} \omega(h,j) = 0 \qquad \text{—(2.2.19)}$$

$$ih - \left\{ m^2 \frac{\partial}{i\partial h} + 4\lambda \frac{\partial^3}{i^3 \partial h^3} + \eta \frac{\partial^3}{i^3 \partial h \partial j^2} \right\} \omega(h,j) = 0 \qquad \text{—(2.2.20)}$$

where, in addition to the relations (2.2.13) we also have

$$\eta = \delta^2(o)\, \eta_0 \qquad \text{—(2.2.21)}$$

This latter renormalisation can be seen from the following consideration:

Since $\quad S'[h,j] = \exp - \int dx\, \omega(h(x), j(x))$ $\quad$ it follows that

$$\frac{\delta^3}{\delta j(x) \delta h^2(x)} S'[h,j] = \left\{ -\frac{\partial^3 \omega}{\partial j \partial h^2} \cdot \delta^2(o) + 2 \frac{\partial^2 \omega}{\partial j \partial h} \cdot \frac{\partial \omega}{\partial h} \cdot \delta(o) \right.$$
$$\left. + \frac{\partial \omega}{\partial j} \frac{\partial^2 \omega}{\partial h^2} \delta(o) - \frac{\partial \omega}{\partial j} \left(\frac{\partial \omega}{\partial h} \right)^2 \right\} S'[h,j]$$

So defining $\eta = \delta^2(o) . \eta_0$ $\quad$ we obtain

$$\eta_0 \frac{\delta^3}{\delta j(x) \delta h^2(x)} S'[h,j] \doteq -\eta \frac{\partial^3 \omega}{\partial j \partial h^2} \cdot S'[h,j]$$

Obviously, (2.2.19) and (2.2.20) are not satisfied by (2.2.14).

So we see that the IVM, whose solution depended crucially on the subtraction procedure, cannot be reconciled with the augmented formalism. We interpret this as peculiar to the IVM only, and will now go on to present further support for ignoring the subtraction procedure for gradient-restored theories.

From (2.2.9), on setting $x = y$ and $j = 0$ we find

$$\left\{ h(x)\, \delta/_{i\delta h(x)} - \delta/_{i\delta h(x)} K_x\ \delta/_{i\delta h(x)} \right\} Z_0' = 0 \qquad \text{—(2.2.22)}$$

which is the analogue in the augmented formalism of (2.2.7), the subtracted scale-covariant equation.

Setting $h = 0$, we obtain the new branching equation

$$K_x\, G_2'(x x^*) = 0 \qquad \text{—(2.2.22a)}$$

where we have used the simplified notation $G_p \equiv G_{p,0}$ and $x^* \rightarrow x$ *after* the operation of K_x .

This is the new feature arising from the augmented formalism. As is easily seen, in the interacting case we have for the new equation

$$K_x\, G_2'(x x^*) + 4 \lambda_0\, G_4'(xxxx) = 0 \qquad \text{—(2.2.23)}$$

This is represented diagrammatically in Fig.(2.1). In principle, perturbation theory is now possible since the inclusion of equation (2.2.23) allows all orders $G_{2n}'^{(p)}$ to be expressed in terms of $G_2'^{(o)}$. As we saw in Part 1, the subtracted scale-covariant equations forbade this, requiring *all* orders of G_2' to be specified.

(2.3) Mass Renormalisation

We have seen that the augmented formulation of scale covariant field theory has provided us with greater information than before, at the level of branching equations. This information is (2.2.22), not present for the subtracted case. Let us explore this equation in detail, since Klauder's analysis excluded it as being intrinsically inconsistent.

We assume that $G_2'(xx^*)$ has the spectral representation

$$G_2'(xx^*) = i\int đk \exp ik(x-x^*) \int d\sigma \frac{\rho(\sigma)}{k^2-\sigma}$$

(where $đk \equiv \frac{d^n k}{(2\pi)^n}$)

$$= \int d\sigma\, \Delta(x-x^*;\sigma)\rho(\sigma) \qquad \text{—(2.2.24)}$$

Here $\rho(\sigma)$ is a normalised spectral density.

Then

$$K_x G_2'(xx^*) = (\Box_x + m_0^2) G_2'(xx^*)$$

$$= -i\int đk\,(k^2-m_0^2)\int d\sigma \frac{\rho(\sigma)}{k^2-\sigma} \qquad \text{—(2.2.25)}$$

where we have taken the limit $x^* \to x$ after the operation of K_x.

We can write (2.2.25) as

$$K_x G_2'(xx^*) = -i\int đk\, d\sigma\, \rho(\sigma)\left\{1 + \frac{(\sigma-m_0^2)}{(k^2-\sigma)}\right\} \qquad \text{—(2.2.26)}$$

Now $$i\int đk\, d\sigma\, \rho(\sigma) = i\int đk \equiv i\,\delta(0) \qquad \text{—(2.2.27)}$$

$$i\int đk\, d\sigma \frac{\rho(\sigma)}{k^2-\sigma} = G_2'(0) \qquad \text{—(2.2.28)}$$

Hence

$$K_x G_z'(xx^*) = -i\,\delta(0) + m_0^2 G_z'(0) - m^2 G_z'(0) \qquad \text{—(2.2.29)}$$

where we have *defined*

$$m^2 G_z'(0) = i\int d^4k\, d\sigma\, \frac{\rho(\sigma)\,\sigma}{(k^2-\sigma)} \qquad \text{—(2.2.30)}$$

i.e.

$$m^2 = \frac{\displaystyle\int d^4k\, d\sigma\, \frac{\rho(\sigma)\,\sigma}{(k^2-\sigma)}}{\displaystyle\int d^4k\, d\sigma\, \frac{\rho(\sigma)}{(k^2-\sigma)}} \qquad \text{—(2.2.31)}$$

So (2.2.22a) implies

$$m^2 = m_0^2 + \frac{\delta(0)}{i\,G_z'(0)} \qquad \text{—(2.2.32)}$$

This is a statement of mass renormalisation for the scale covariant theory, where m is the renormalised mass.

Suppose the spectral density was of the form $\rho(\sigma) = N\{\delta(\sigma-m^2) + c(\sigma)\}$ where N is for normalisation, m is the renormalised mass, and $c(\sigma)$ represents the continuum contribution. Then (2.2.32) can be written, $m^2 = m_0^2 + \dfrac{\delta(0)}{i\,G_z'(0)} + \tilde{c}$ —(2.2.33) where $\tilde{c}$ represents the continuum. Similarly (2.2.28) can be written as

$$i\,G_z'(0) = i\,\Delta(0;m^2) + i\,G_z'(\text{CONTINUUM}) \qquad \text{—(2.2.34)}$$

with

$$i\,\Delta(0;m^2) = \int d^4k\, \frac{1}{(k^2+m^2)} \qquad \text{—(2.2.35)}$$

So we see that (2.2.33) is of a self-consistent type since, because of (2.2.34) and (2.2.35), both sides of (2.2.33) involve m^2.

(2.4) The Diagrammatic Expansion of the Augmented Pseudofree Theory

The Augmented Pseudofree Theory is described by the following generating functional

$$Z_0'[h,j] = \int \mathcal{D}[\phi]\,\mathcal{D}[\chi] \exp i\int dx\left\{\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m_0^2\phi^2 - \tfrac{1}{2}\eta\chi^2\phi^2 + h\phi + j\chi\right\} \qquad \text{—(2.4.1)}$$

where $\mathcal{D}[\phi]$ and $\mathcal{D}[\chi]$ are now translation invariant measures, and we have coupled an external source $j(x)$ to the auxiliary field $\chi(x)$. Integrating over the ϕ -field gives

$$Z_0'[h,j] = \int \mathcal{D}[\chi] \exp i\,\mathcal{X}_0[\chi,h] + i\int j(x)\chi(x)dx \qquad \text{—(2.4.2)}$$

where

$$\mathcal{X}_0[\chi,h] = \tfrac{i}{2}\int dx\,dy\,h(x)\,g(x,y;\chi)h(y) + \tfrac{i}{2}\mathrm{Tr}\ln\left(\Box + m_0^2 + \eta\chi^2\right) \qquad \text{—(2.4.3)}$$

and $\left\{\Box_x + m_0^2 + \eta\chi^2(x)\right\} g(x,y;\chi) = -i\delta(x-y)$ —(2.4.4)

Now let $G_0(xy)$ be the free-field Green function satisfying

$$(\Box_x + m_0^2)\,G_0(xy) = -i\delta(x-y) \qquad \text{—(2.4.5)}$$

We can then separate out from (2.4.2) the free-field functional

$$Z_0[h] = \exp -\tfrac{1}{2}\int dx\,dy\,h(x)\,G_0(xy)\,h(y) \qquad \text{—(2.4.6)}$$

to obtain

$$Z_0'[h,j] = Z_0[h] \int \mathcal{D}[\chi] \exp i\,\sigma[\chi,h] + i\int j\chi\, dx \qquad \text{—(2.4.7)}$$

where

$$\sigma[\chi,h] = \sigma_0[\chi,h] - \frac{i}{2}\int dx\, dy\, h(x)\, G_0(xy) h(y) \qquad \text{—(2.4.8)}$$

Now

$$g(x,y;\chi) = \langle x| \frac{1}{\Box + m_0^2 + \eta\chi^2} |y\rangle \qquad \text{—(2.4.9)}$$

$$= \langle x| \frac{1}{\Box + m_0^2} |y\rangle \; -$$

$$- \int dx_1 \langle x| \frac{1}{\Box + m_0^2} |x_1\rangle\langle x_1| \frac{1}{\Box + m_0^2} |y\rangle \eta\chi^2(x_1)$$

$$+ \; \ldots\ldots\ldots\ldots$$

$$= G_0(xy) - \frac{1}{2}\int dx_1\, \eta\, \chi^2(x_1)\, G_0(xx_1)\, G_0(x_1y)$$

$$+ \; \ldots\ldots$$

Also,

$$\ln\langle x|\Box + m_0^2 + \eta\chi^2|y\rangle = \ln\langle x|\Box + m_0^2|y\rangle +$$

$$+ \ln\langle x|1 + \eta\chi^2 \cdot \frac{1}{\Box + m_0^2}|y\rangle \qquad \text{—(2.4.10)}$$

$$= \ln\langle x|\Box + m_0^2|y\rangle - \frac{1}{2}\eta\chi^2(x)\langle x|\frac{1}{\Box + m_0^2}|y\rangle$$

$$+ \frac{1}{3}\eta\chi^2 \frac{1}{(\Box + m_0^2)}\, \eta\chi^2 \frac{1}{(\Box + m_0^2)} + \ldots\ldots$$

Then we see that $\sigma[\chi,h]$ of (2.4.8) can be written as

$$\sigma[\chi,h] = -\tfrac{1}{2}\eta\, G_0(0)\int dx\, \chi^2(x) - \tfrac{1}{2}\eta\int dx\, \chi^2(x)\,\Phi^2(x) + \tfrac{1}{2}\sum_{n=2}^{\infty}\eta^n\int dx_1\cdots dx_n\, V^{(2n)}(x_1\cdots x_n;h)\,\chi^2(x_1)\cdots\chi^2(x_n) \qquad \text{—(2.4.11)}$$

In (2.4.11) we have written

$$\Phi(x) = i\int dy\, \Delta(xy)\, h(y) \qquad \text{—(2.4.12)}$$

and

$$V^{(2n)}(x_1\cdots x_n) = V_0^{(2n)}(x_1\cdots x_n) + V_1^{(2n)}(x_1\cdots x_n;h) \qquad \text{—(2.4.13)}$$

with

$$V_0^{(2n)}(x_1\cdots x_n) = \frac{(-i)^{n+1}}{n}\, G_0(x_1x_2)\,G_0(x_2x_3)\cdots G_0(x_nx_1) \qquad \text{—(2.4.14)}$$

and

$$V_1^{(2n)}(x_1\cdots x_n;h) = (-i)^{n-1}\,\Phi(x_1)\,G_0(x_1x_2)\cdots G_0(x_{n-1}x_n)\,\Phi(x_n) \qquad \text{—(2.4.15)}$$

We can see that $V_1^{(2n)}(x_1\cdots x_n;h)$ is h-dependent through $\Phi(x_i)$. Both $V_0^{(2n)}$ and $V_1^{(2n)}$ are described graphically in Figs.(2.4.).

They are represented by a closed polygon and a chain, respectively.

Now the generating functional for the *connected* Pseudofree Green functions, $W_0'[h]$, is given by

$$i W_o'[h,j] = \ln Z_o'[h,j] \qquad \text{—(2.4.16)}$$

and setting $j = 0$.

From (2.4.7) we see, then, that

$$W_o'[h] = W_o[h] + \Sigma[h] \qquad \text{—(2.4.17)}$$

where $\Sigma[h]$ is the sum of all *connected* Vacuum-to-Vacuum diagrams constructed from a χ -field theory with action $\mathcal{Q}[\chi, h]$ of (2.4.11). This is a theory with propagator

$$\Delta(xy) = \frac{-i}{\eta G_o(o)} \delta(x-y) \qquad \text{—(2.4.18)}$$

,

h -dependent mass insertion (2-point function)

$$M^2(x) = \eta \underline{\Phi}^2(x) \qquad \text{—(2.4.19)}$$

and h -dependent non-local $2n$ -point vertices

$$-\frac{1}{2}\eta^n V^{(2n)}(x_1 \cdots x_n ; h) \qquad \text{—(2.4.19a)}$$

as in Figs. (2.4.14) and (2.4.15).

Having set up the diagrammatic expansion of the pseudofree theory in its augmented formulation, let us consider the pseudofree two-point function $G_2'(xy)$.

This is defined by

$$G_2'(xy) = \frac{\delta^2 W_0'[h]}{i^2 \delta h(x)\delta h(y)}\Bigg|_{h=0} \qquad \text{—(2.4.20)}$$

$$= \frac{\delta^2 W_0[h]}{i^2 \delta h(x)\delta h(y)}\Bigg|_{h=0} + \frac{\delta^2 \sum_1[h]}{i^2 \delta h(x)\delta h(y)}\Bigg|_{h=0} \qquad \text{—(2.4.21)}$$

using (2.4.17). Hence

$$G_2'(xy) = G_0(xy) + \frac{\delta^2 \sum_1[h]}{i^2 \delta h(x)\, \delta h(y)}\Bigg|_{h=0} \qquad \text{—(2.4.22)}$$

Here, $G_0(xy)$ is the *free*-field 2-point function (satisfying (2.4.5)).

Looking at the second term of (2.4.22), there are only two possible types of diagram of $\sum_1[h]$ that can contribute, i.e.

(i) Diagrams with no M^2 insertions. Then all of the full vertices $V^{(2n)}$ except one are replaced by $V_0^{(2n)}$. This one vertex is replaced by $V_1^{(2n)}$ and is then functionally differentiated twice to remove its h-dependence. This procedure is done in all possible ways.

(ii) Diagrams with one M^2 insertion. This insertion is twice differentiated to remove its h-dependence. All vertices $V^{(2n)}$ are replaced by $V_0^{(2n)}$.

Diagrams corresponding to the pseudofree 2-point function are shown in Table 2.1. This table categorises the diagrams according to the number of loops each diagram has. For convenience we have taken

$$\eta\, G_0(0) = 1 \qquad \text{—(2.4.23)}$$

Then, from (2.4.18), the propagator $\Delta(xy)$ is simply a delta-function

$$\Delta(xy) = -i\,\delta(x-y) \qquad \text{—(2.4.24)}$$

and from (2.4.19a), the 2n-point vertices have the coupling $\eta^n = \dfrac{1}{G_o^n(o)}$ associated with them now.

We shall briefly explain Table 2.1. The first column gives the loop order. The second column gives a particular connected vacuum-to-vacuum diagram, constructed from the above propagator and vertices. In the third column, the contribution of this diagram (after it has been differentiated) to the pseudofree 2-point function is given. Dashed lines represent the delta-function propagator $\Delta(xy)$ and solid lines represent the G_o. Contracting the Δ's gives the diagram of column four. The last column gives the formal singularity associated with this contribution to $G_2'(xy)$. This singularity is found as follows:

From the uncontracted diagram of column three, each closed dashed loop gives a factor $\delta(o)$ and each point vertex a factor $1/G_o(o)$. Also, we must include the singularity produced on making the contraction, i.e. the singularity associated with the diagram of column four, e.g. consider 2(b) of Table 1. Because 2(b) has a mass insertion, M^2, this *must* be differentiated, to give the uncontracted diagram shown. This has two dashed loops and three point vertices and so contributes singularity $\delta^2(o)/G_o^3(o)$. Also, on contracting, we gain the additional singularity of the closed solid line, denoted L_2. Generally we denote

$$L_n = \int dx_1\, dx_2 \ldots dx_{n-1}\, G_o(xx_1) \ldots\, G_o(x_{n-1}x) \qquad \text{—(2.4.25)}$$

i.e., an n-link loop fixed at x.

So 2(b) has the overall singularity $L_2 \delta^2(0)/G_0^3(0)$. This can also be seen by calculating directly the uncontracted diagram.

i.e. $$\int d\alpha\, d\beta\, d\gamma\, \delta(0)\frac{1}{G_0^3(0)}\delta^2(\beta-\gamma)\, G_0(\alpha\beta)\, G_0(\beta\alpha)\, G_0(x\gamma)\, G_0(\gamma y)$$

$$= \frac{\delta^2(0)}{G_0^3(0)}\int d\alpha\, d\beta\, G_0(\alpha\beta)\, G_0(\beta\alpha)\, G_0(x\beta)\, G_0(\beta y)$$

$$= \left\{\frac{\delta^2(0)}{G_0^3(0)}\int d\alpha\, G_0(\beta\alpha)\, G_0(\alpha\beta)\right\}\int d\beta\, G_0(x\beta)\, G_0(\beta y)$$

$$= \frac{\delta^2(0)}{G_0^3(0)} L_2 \int d\beta\, G_0(x\beta)\, G_0(\beta y)$$

—(2.4.26)

We see that each singularity occurs at many loop levels. Now if we pick out the most singular diagrams that correspond only to point mass insertions, we can write their sum generally as

$$m^2 = m_0^2 + a_1\,\delta(0)/G_0(0) + \frac{L_2}{G_0(0)}\left(\delta(0)/G_0(0)\right)^2\left\{b_1 + b_2\frac{L_2}{G_0(0)}\frac{\delta(0)}{G_0(0)} + \cdots\right\}$$
$$+ c_1\frac{L_3}{G_0(0)}\left(\delta(0)/G_0(0)\right)^3 + d_1\frac{L_4}{G_0(0)}\left(\delta(0)/G_0(0)\right)^4 + \cdots$$

—(2.4.27)

We will also have cross-terms, $L_2 L_3$ etc., at higher loop levels. The coefficients a_1, b_1, b_2 etc. are uncalculable since each singularity occurs at many loop levels. However, we will show in the next section how this mass, comprising the most singular point mass insertions of the pseudofree 2-point function, can be related to the renormalised mass of Section 3 that was the statement of the extra dynamical equation (2.2.22a).

Concluding this section, we draw attention to Table 2.2. which is the analogous table for the connected pseudofree four-point function, $W_4'(\omega x y z)$. No study of this will be made here.

(2.5) Mass Renormalisation Reconsidered

In this Section we will show how the self-consistent mass renormalisation (2.2.33) is related to the singular diagrammatic series (2.4.27).

Now the series (2.4.27) corresponds to picking out the most singular part of the self mass, with no continuum contributions. This corresponds to taking

$$\rho(\sigma) = \delta(\sigma - m^2) \qquad \text{—(2.5.1)}$$

Then from (2.2.28) we have

$$\begin{aligned} G_2'(0) &= i \int \not{d}k \, \frac{1}{(k^2 - m^2)} \\ &= G_0(0; m^2) \end{aligned} \qquad \text{—(2.5.2)}$$

with $G_0(xy; m^2)$ as in (2.4.5) but with m_0^2 replaced by m^2.

Also (2.2.33) now becomes

$$m^2 = m_0^2 + \delta(0)/i\,G_0(0; m^2) \qquad \text{—(2.5.3)}$$

Now in (2.4.27) $G_0(0)$ is short for $G_0(0; m_0^2)$. In order to show the equivalence of the series (2.4.27) and the statement (2.5.3), we need then to express $G_0(0;m^2)$ in terms of $G_0(0;m_0^2)$

Let $$\delta m^2 = m^2 - m_0^2 \qquad \text{—(2.5.4)}$$

Then, for single point mass insertions, $i\delta m^2$,

$$G_0(0;m^2) = G_0(0;m_0^2) + \int dx_1\, G_0(xx_1;m_0^2)\, i\delta m^2\, G_0(x_1x;m_0^2)$$
$$+ \int dx_1\, dx_2\, G_0(xx_1;m_0^2)\, i\delta m^2\, G_0(x_1x_2;m_0^2)\, i\delta m^2 \cdot G_0(x_2x;m_0^2)$$
$$+ \cdots\cdots$$
$$= G_0(0;m_0^2) + i\delta m^2 L_2 + (i\delta m^2)^2 L_3 + \cdots + (i\delta m^2)^{n-1} L_n + \cdots \qquad \text{—(2.5.5)}$$

where L_n is given in (2.4.25).

Now we know from (2.5.3) that

$$\delta m^2 = \delta(0)/i\, G_0(0;m^2) \qquad \text{—(2.5.6)}$$

Hence $G_0(0;m^2)$ of (2.5.5) can be expanded as continued fractions, i.e.

$$G_0(0;m^2) = G_0(0;m_0^2) + L_2\,\delta(0)/G_0(0;m^2) + L_3\left(\delta(0)/G_0(0;m^2)\right)^2 + \cdots\cdots$$

$$= G_o(o;m_o^2) + \delta(o) L_2 \left\{ G_o(o;m_o^2) + L_2 \delta(o)/G_o(o;m^2) + \dots \right\}^{-1}$$

$$+ \delta^2(o) L_3 \left\{ G_o(o;m_o^2) + L_2 \delta(o)/G_o(o;m_o^2) + \dots \right\}^{-2}$$

$$+ \dots\dots \qquad \text{—(2.5.7)}$$

i.e.

$$G_o(o;m^2) = G_o(o;m_o^2) + \frac{\delta(o)}{G_o(o;m_o^2)} L_2 \left\{ 1 + \frac{\delta(o)}{G_o^2(o;m_o^2)} L_2 \left[1 + \dots\right]^{-1} \right\}^{-1}$$

$$+ \left(\frac{\delta(o)}{G_o(o;m_o^2)}\right)^2 L_3 \left\{ 1 + \frac{\delta(o)}{G_o^2(o;m_o^2)} L_2 \left[1 + \dots\right]^{-1} \right\}^{-2}$$

$$+ \left(\frac{\delta(o)}{G_o(o;m_o^2)}\right)^3 L_4 \left\{ 1 + \frac{\delta(o)}{G_o^2(o;m_o^2)} L_2 \left[1 + \dots\right]^{-1} \right\}^{-3}$$

—(2.5.8)

For (2.5.6) we need the inverse of this expression. We assume this to be

$$G_o^{-1}(o;m^2) = G_o^{-1}(o;m_o^2) \left\{ 1 + \frac{\delta(o)}{G_o^2(o;m_o^2)} L_2 \left[1 + \frac{\delta(o)}{G_o^2(o;m_o^2)} L_2 (1 + \dots)^{-1} \right]^{-1} \right.$$

$$+ \frac{\delta^2(o)}{G_o^3(o;m_o^2)} L_3 \left[1 + \frac{\delta(o)}{G_o^2(o;m_o^2)} L_2 (1 + \dots)^{-1} \right]^{-2}$$

$$\left. + \dots\dots \right\}^{-1}$$

$$= G_0^{-1}(0;m_0^2) - \frac{\delta(0)}{G_0^2(0;m_0^2)} \cdot \frac{L_2}{G_0(0;m_0^2)} + \frac{\delta^2(0)}{G_0^3(0;m_0^2)} \frac{L_2^2}{G_0^2(0;m_0^2)}$$

$$\cdots \quad - \frac{\delta^2(0)}{G_0^3(0;m_0^2)} \frac{L_3}{G_0(0;m_0^2)} + \cdots$$

Hence

$$\delta m^2 = \delta(0)/_i\, G_0(0;m^2)$$

$$= \frac{\delta(0)}{i\,G_0(0;m_0^2)} - \left(\frac{\delta(0)}{G_0(0;m_0^2)}\right)^2 \frac{L_2}{i\,G_0(0;m_0^2)}$$

$$+ \left(\frac{\delta(0)}{G_0(0;m_0^2)}\right)^3 \frac{L_2^2}{i\,G_0^2(0;m_0^2)} + \cdots \qquad \text{—(2.5.9)}$$

$$+ \cdots - \left(\frac{\delta(0)}{G_0(0;m_0^2)}\right)^3 \frac{L_3}{i\,G_0(0;m_0^2)} + \cdots$$

As we see, the singularities arising in (2.5.9) are just those arising from the most singular point mass-insertion diagrams in the straightforward diagrammatic expansion of the previous section. This means that the diagrammatic expansion has reinforced the statement (2.2.32) of mass renormalisation and hence, in turn, has given credence to the new dynamical equation (2.2.22a).

We note that, in order to reproduce the more general statement (2.2.33), we would need, in the diagrammatic expansion, to consider non-leading diagrams. Also, since the diagrams of Table 1 correspond to an effective *non-normal ordered* theory, and equation (2.2.22a) corresponds to a *non-subtracted* theory, it seems that the subtraction procedure (2.2.8) is equivalent to a normal ordering procedure.

Finally in this section we can see how the IVM, which motivated the subtraction procedure in the first instance, is indeed a special case, and that we should not extrapolate from it in using the procedure generally.

For the IVM, $$i G'_{2}(0; m^2) = \frac{\delta(0)}{m^2} \qquad \text{—(2.5.10)}$$

There is no analogue of continuum contributions and hence from (2.5.3),

$$m^2 = m_0^2 + m^2 \qquad \text{—(2.5.11)}$$

This implies that $m_0^2 = 0$. Now a massless pseudofree IVM does not exist, and so once again we see that in the augmented theory we cannot accommodate the IVM. This we expect since the IVM depends upon the subtraction procedure, absent in the augmented formulation.

(2.6) Solving the Augmented Branching Equations

We shall provide even further evidence for taking seriously the constraint equation (2.2.22a) by examining the augmented pseudofree branching equations (2.2.5) and (2.2.6). We shall make assumptions concerning the leading singularities of the Green functions.

First of all, let us consider the constraint equation (2.2.2) and its corresponding branching equations (2.2.5)

$$\left\{ j(x) - \eta \frac{\delta^3}{i^3 \delta j(x) \delta h^2(x)} \right\} Z'_0[h, j] = 0 \qquad \text{—(2.2.2)}$$

$$i\delta(x-y) + \eta\, G'_{2,2}(xx; xy) = 0 \qquad \text{(a)}$$

$$i\delta(x-y)\, G'_{p,0}(x_1 \ldots x_p) + \eta\, G'_{p+2,2}(xxx_1 \ldots x_p; xy) \qquad \text{(b)}$$
$$= 0$$

$$i\sum_{r=1}^{q+1} \delta(x-x_r)\, G'_{p,q}(x_1 \cdots x_p\,;\, y_1 \cdots \hat{y}_r \cdots y_{q+1})$$

$$+ \;\eta\, G'_{p+2,q+2}(x x x_1 \cdots x_p\,;\, x y_1 \cdots y_q) = 0 \qquad \text{(c)} \qquad \text{—(2.2.5)}$$

From (2.2.5a) we see that the singularity arising from making two χ -fields coincident is a delta-function, i.e.

$$G'_{0,2}(xy) \;\propto\; \delta(x-y) \qquad \text{—(2.6.1)}$$

This is compatible with χ being a purely auxiliary field.

Substituting (2.2.5a) into (2.2.5b) we find

$$G'_{p+2,2}(x x x_1 \cdots x_p\,;\, x y) \;=\; G'_{p,0}(x_1 \cdots x_p)\, G_{2,2}(x x\,;\, x y) \qquad \text{—(2.6.2)}$$

showing a high degree of factorisation.

In seeking the simplest solution, we make the assumption that, as far as the leading singularities of the augmented Green functions are concerned, and guided by the observation (2.6.1), the χ -field part of the Green function factorises off, i.e.

$$G'_{2,2}(xx\,;\,xy) \;\approx\; G'_{2,0}(xx)\, G'_{0,2}(xy) \qquad \text{—(2.6.3)}$$

The relevance of treating only leading singularities is that the appropriate definition of local operator products is via the operator product expansion in scale-covariant theories (see Section A3 (a)). The symbol $\approx$ indicates that this is an equation between the most singular parts of the respective functions.

Our assumption indicates that

$$G'_{p+2,2}(x x x_1 \cdots x_p\,;\, x y) \;\approx\; G'_{p+2,0}(x x x_1 \cdots x_p)\, G'_{0,2}(xy) \qquad \text{—(2.6.4)}$$

Hence, from (2.6.2) we find

$$G'_{p+2,0}(xxx_1\cdots x_p)G'_{0,2}(xy) \approx G'_{p,0}(x_1\cdots x_p)G'_{2,0}(xx)G'_{0,2}(xy)$$

or $$G'_{p+2,0}(xxx_1\cdots x_p) \approx G'_{p,0}(x_1\cdots x_p)\,G'_{2,0}(xx) \qquad \text{—(2.6.5)}$$

The leading singularity equations (2.6.1), (2.6.3), (2.6.4) and (2.6.5) are displayed diagrammatically in Fig.2.4.

Now our assumption is, in fact, an assumption of separability for the augmented pseudofree generating function, i.e.

$$Z'_0[h,j] \approx H[h]\,J[j] \qquad \text{—(2.6.6)}$$

where $\approx$ indicates an equality only between the most singular contributions. Let us now turn to the dynamical equations (2.2.6) and see how our assumptions (2.6.1), (2.6.3), (2.6.4) and (2.6.5) work for these – if, indeed, they work at all.

$$i\delta(x-y) + K_x G'_{2,0}(xy) + \eta\, G'_{2,2}(xy\,;xx) = 0 \qquad \text{(a)}$$

$$i\sum_{r=1}^{m}\delta(x-x_r)\,G'_{m-1,0}(x_1\cdots\hat{x}_r\cdots x_m) + K_x G'_{m+1,0}(xx_1\cdots x_m)$$
$$+\,\eta\, G'_{m+1,2}(xx_1\cdots x_m\,;xx) = 0 \qquad \text{(b)}$$

$$i\sum_{r=1}^{m}\delta(x-x_r)\,G'_{m-1,q}(x_1\cdots\hat{x}_r\cdots x_m\,;y_1\cdots y_q) + K_x G'_{m+1,q}(xx_1\cdots x_m\,;y_1\cdots y_q)$$
$$+\,\eta\, G'_{m+1,q+2}(xx_1\cdots x_m\,;xxy_1\cdots y_q) = 0 \qquad \text{(c)} \quad \text{—(2.2.6)} \quad (q\geq 2)$$

Now substituting (2.6.3) into (2.2.6a) we have

$$i\delta(x-y) + K_x G'_{2,0}(xy) + \eta\, G'_{2,0}(xy)\,G_{0,2}(xx) \approx 0 \qquad \text{—(2.6.7)}$$

or $$\left\{ K_x + \eta\, G'_{0,2}(xx) \right\} G'_{2,0}(xy) \approx -i\,\delta(x-y) \qquad \text{—(2.6.8)}$$

Using our assumptions in (2.2.5a) we have

$$i\,\delta(0) + \eta\, G'_{2,0}(xx)\, G'_{0,2}(xx) \approx 0 \qquad \text{—(2.6.9)}$$

and substituting this into (2.6.8) gives

$$\left\{ K_x + \delta(0)/i\,G'_{2,0}(0) \right\} G'_{2,0}(xy) \approx -i\,\delta(x-y) \qquad \text{—(2.6.10)}$$

We can now see how the mass renormalisation (2.5.3) of Section 5 (also (2.2.32) of Section 2) appears. Equation (2.6.10) is a statement about the ϕ -field generating function of (2.6.6), $H[h]$. It says that $H[h]$ is the generating functional of a free field theory in our leading singularity approximation, and that the mass of the ϕ -field is given by the relation

$$K'_x = K_x + \delta(0)/i\,G'_{2,0}(0) \qquad \text{—(2.6.11)}$$

or $$\Box_x + m^2 = \Box_x + m_0^2 + \delta(0)/i\,G'_{2,0}(0) \qquad \text{—(2.6.12)}$$

i.e. $$m^2 = m_0^2 + \delta(0)/i\,G'_{2,0}(0) \qquad \text{—(2.6.13)}$$

For completeness, let us make sure that the other dynamical equations of (2.2.6) can be consistently satisfied by our assumptions.

Consider (2.2.6b) for $m = 3$. This is

$$i\delta(x-x_1)\,G'_{2,0}(x_2x_3) + i\delta(x-x_2)\,G'_{2,0}(x_1x_3) + i\delta(x-x_3)\,G'_{2,0}(x_1x_2)$$

$$+ K_x\,G'_{4,0}(xx_1x_2x_3) + \eta\,G'_{4,2}(xx_1x_2x_3\,;\,xx) = 0$$

—(2.6.14)

Using (2.2.5a) and the assumption (2.6.3) we have

$$i\delta(x-y) \approx -\eta\,G'_{2,0}(xx)\,G'_{0,2}(xy) \qquad \text{—(2.6.15)}$$

Also

$$\eta\,G'_{4,2}(xx_1x_2x_3\,;\,xx) \approx \eta\,G'_{4,0}(xx_1x_2x_3)\,G'_{0,2}(xx)$$

$$\approx G'_{4,0}(xx_1x_2x_3)\,\frac{\delta(0)}{i\,G'_{2,0}(xx)} \qquad \text{—(2.6.16)}$$

Hence (2.6.14) becomes

$$\left\{K_x + \delta(0)/i\,G'_{2,0}(xx)\right\}G'_{4,0}(xx_1x_2x_3) \approx -i\delta(x-x_1)\,G'_{2,0}(x_2x_3)$$

$$-i\delta(x-x_2)\,G'_{2,0}(x_1x_3)$$

$$-i\delta(x-x_3)\,G'_{2,0}(x_1x_2)$$

—(2.6.17)

From (2.6.10) we saw that $H[h]$ described a *free*-field theory. Hence in (2.6.17) we must have

$$G'_{4,0}(xx_1x_2x_3) \approx G'_{2,0}(xx_1)\,G'_{2,0}(x_2x_3) + G'_{2,0}(xx_2)\,G'_{2,0}(x_1x_3)$$

$$+ G'_{2,0}(xx_3)\,G'_{2,0}(x_1x_2) \qquad \text{—(2.6.18)}$$

and so (2.6.17) becomes

$$\left\{K_x + \delta(0)/i\, G'_{2,0}(xx)\right\} G'_{2,0}(xx_1)\, G'_{2,0}(x_2x_3) + \left\{K_x + \delta(0)/i\, G'_{2,0}(xx)\right\} G'_{2,0}(xx_2) \cdot G'_{2,0}(x_1x_3)$$

$$+ \left\{K_x + \delta(0)/i\, G'_{2,0}(xx)\right\} G'_{2,0}(xx_3)\, G'_{2,0}(x_1x_2)$$

$$\approx \; -i\,\delta(x-x_1)\, G'_{2,0}(x_2x_3) - i\,\delta(x-x_2)\, G'_{2,0}(x_1x_3) - i\,\delta(x-x_3)\, G'_{2,0}(x_1x_2) \qquad \text{—(2.6.19)}$$

Equation (2.6.19) is satisfied as a consequence of (2.6.10).

So, in a similar manner, we can show that the assumptions (2.6.1), (2.6.3), (2.6.4), (2.6.5) giving a consistent leading singularity solution of the constraint equations (2.2.5) also satisfy the dynamical branching equations (2.2.6) in a consistent way.

We can clarify these conclusions by substituting our separability assumption (2.6.6) into the augmented functional differential equations (2.2.1) and (2.2.2)

$$\left\{h(x) - K_x \frac{\delta}{i\,\delta h(x)} - \eta \frac{\delta^3}{i^3\,\delta h(x)\,\delta j^2(x)}\right\} Z'_0[h,j] = 0 \qquad \text{—(2.2.1)}$$

$$\left\{ j(x) - \eta \frac{\delta^3}{i^3\,\delta j(x)\,\delta^2 h(x)} \right\} Z'_0[h,j] = 0 \qquad \text{—(2.2.2)}$$

Consider first of all (2.2.2). Substituting for (2.6.6) gives

$$j(x)\, H[h]\, J[j] \approx \eta \frac{\delta^2 H[h]}{i^2\,\delta h^2(x)} \cdot \frac{\delta J[j]}{i\,\delta j(x)} \qquad \text{—(2.6.20)}$$

Setting $h = 0$ with $H[h=0] = 1$, we have

$$j(x)\,\mathcal{J}[j] \approx \eta\, G'_{2,0}(xx)\,\frac{\delta \mathcal{J}[j]}{i\,\delta j(x)} \qquad \text{—(2.6.21)}$$

where $$G'_{2,0}(xx) = \left.\frac{\delta^2 H[h]}{i^2\,\delta h^2(x)}\right|_{h=0} \qquad \text{—(2.6.22)}$$

On putting (2.6.21) back into (2.6.20) we obtain

$$\frac{\delta^2 H[h]}{i^2\,\delta h^2(x)} \approx G'_{2,0}(xx)\,H[h] \qquad \text{—(2.6.23)}$$

Since η is an arbitrary factor, we choose it for convenience to satisfy

$$\eta\, G'_{2,0}(xx) = 1 \qquad \text{—(2.6.24)}$$

Then $\mathcal{J}[j]$ describes a free-field theory with two-point function

$$G'_{0,2}(xy) \approx -i\,\delta(x-y) \qquad \text{—(2.6.25)}$$

since from (2.6.21) we have

$$j(x)\,\mathcal{J}[j] \approx \frac{\delta \mathcal{J}[j]}{i\,\delta j(x)} \qquad \text{—(2.6.26)}$$

Now substituting (2.6.6) into (2.2.1) we find

$$\left\{ h(x) - K_x\,\delta/i\,\delta h(x) \right\} H[h]\,\mathcal{J}[j] \qquad \text{—(2.6.27)}$$

$$\simeq \eta\,\frac{\delta H[h]}{i\,\delta h(x)}\,\frac{\delta^2 \mathcal{J}[j]}{i^2\,\delta j^2(x)}$$

From (2.6.21) we find, on functionally differentiating w.r.t. $ij(x)$

$$-i\delta(0)J[j] + j(x)\frac{\delta J[j]}{i\delta j(x)} \approx \eta G'_{2,0}(xx)\frac{\delta^2 J[j]}{i^2\delta j^2(x)} \qquad \text{—(2.6.28)}$$

i.e. $$-i\delta(0)J[j] + j(x)\left(\frac{j(x)J[j]}{\eta G'_{2,0}(xx)}\right) \approx \eta G'_{2,0}(xx)\frac{\delta^2 J[j]}{i^2\delta j^2(x)}$$

So with $\eta G'_{2,0}(xx) = 1$, equation (2.6.27) becomes

$$\left\{h(x) - K_x\, \delta/_{i\delta h(x)}\right\} H[h] \approx \frac{\delta H[h]}{i\delta h(x)}\left\{\frac{j^2(x)}{G'_{2,0}(xx)} + \frac{\delta(0)}{iG'_{2,0}(xx)}\right\}$$

or $$\left\{h(x) - \left(K_x + \frac{\delta(0)}{iG'_{2,0}(xx)}\right)\delta/_{i\delta h(x)}\right\} H[h] \approx \frac{j^2(x)}{G'_{2,0}(xx)}\,\frac{\delta H[h]}{i\delta h(x)} \qquad \text{—(2.6.29)}$$

Now we assume that the R.H.S. of (2.6.29) is zero because of the infinite $G'_{2,0}(xx)$. This gives us the result

$$\left\{h(x) - \left(K_x + \frac{\delta(0)}{iG'_{2,0}(xx)}\right)\delta/_{i\delta h(x)}\right\} H[h] \approx 0 \qquad \text{—(2.6.30)}$$

This is just the functional differential equation for the generating functional $H[h]$ of a free field with the self-consistent, additive mass renormalisation

$$m^2 = m_0^2 + \frac{\delta(0)}{iG'_{2,0}(xx)} \qquad \text{—(2.6.31)}$$

This was just the conclusion reached by examining the branching equations earlier.

(2.7) The Large N Limit of the O(N) Pseudofree Theory

So far the equation (2.2.32) has played a very important rôle. It has served several purposes by

(i) describing the nature of mass renormalisation

(ii) Implying the existence of the extra dynamical equation (2.2.22a)

(iii) Reflecting the separability assumption for the most singular contributions of the branching equations.

But it is only an approximate equation. Derived from (2.2.22a), it is approximate because we took the spectral density to be of the form

$$\rho(\sigma) = \delta(\sigma - m^2) \qquad \text{—(2.7.1)}$$

Derived from the diagrammatic expansion, it is approximate because we only took the point mass insertions, and derived from the branching equations it is approximate because of our leading singularity approximation, characterised by (2.6.6).

We shall now show that these approximations correspond to taking the large N limit of the O(N) invariant pseudofree theory. This is because the equation (2.2.32) becomes exact in this limit. Since we are going to consider the O(N) invariant scale covariant field theory in detail later, we shall be brief here.

The model has N scalar fields ϕ_i ($i = 1, 2, \cdots N$) and generating functional

$$Z'_0[\underline{h}] = \int \mathcal{D}'[\underline{\phi}] \exp i \int dx \left\{ \tfrac{1}{2}(\partial_\mu \underline{\phi})^2 - \tfrac{1}{2} m_0^2 \underline{\phi}^2 + \underline{h} \cdot \underline{\phi} \right\} \qquad \text{—(2.7.2)}$$

where the measure is $O(N)$ invariant and also invariant under scale transformations, i.e.

$$\mathcal{D}'[\Lambda \underline{\phi}] = \mathcal{D}'[\phi] \qquad \forall \; \Lambda(x) > 0 \tag{2.7.3}$$

In the augmented formalism, (2.7.2) becomes

$$Z_0'[\underline{h}, \underline{j}] = \int \mathcal{D}[\underline{\phi}]\mathcal{D}[\underline{\chi}] \exp i \int dx \Big\{ \tfrac{1}{2}(\partial_\mu \underline{\phi})^2 - \tfrac{1}{2} m_0^2 \underline{\phi}^2 - \tfrac{1}{2}\eta N^{-1} \underline{\chi}^2 \underline{\phi}^2 + \underline{h}.\underline{\phi} + \underline{j}.\underline{\chi} \Big\} \tag{2.7.4}$$

where we are really only interested in $\underline{j} = 0$. In (2.7.4) the measures $\mathcal{D}[\underline{\phi}]$ and $\mathcal{D}[\underline{\chi}]$ are now translation invariant.

We assume that the functional Integral is dominated by those configurations for which $\underline{\phi}^2$ and $\underline{\chi}^2$ are of order N. Hence the N-dependence of the cross-term $\frac{1}{2}\eta N^{-1} \underline{\chi}^2 \underline{\phi}^2$. Making the N-dependence more explicit we write

$$Z_0'[\underline{h}] = \int \mathcal{D}[\underline{\phi}]\mathcal{D}[\underline{\chi}]\mathcal{D}[\sigma]\, \delta(\underline{\chi}^2 - N\sigma) \exp i \int dx \Big\{ \tfrac{1}{2}(\partial_\mu \underline{\phi})^2 - \tfrac{1}{2} m_0^2 \underline{\phi}^2 - \tfrac{1}{2}\eta \sigma \underline{\phi}^2 + \underline{h}.\underline{\phi} \Big\} \tag{2.7.5}$$

where $\delta(\underline{\chi}^2 - N\sigma)$ is a functional delta-function expressing $\underline{\chi}^2$ of order N in terms of a new field σ of order 1. In turn, this functional delta-function can be re-expressed as

$$\delta(\underline{\chi}^2 - N\sigma) = \int \mathcal{D}[\alpha] \exp \frac{i\alpha}{2} \int dx \left(\chi^2(x) - N\sigma \right) \tag{2.7.6}$$

and so (2.7.4) can be written

$$Z'_o[\underline{h}] = \int \mathcal{D}[\underline{\phi}]\,\mathcal{D}[\underline{\chi}]\,\mathcal{D}[\sigma]\,\mathcal{D}[\alpha]\,\exp i\int dx \left\{ \tfrac{1}{2}(\partial_\mu \underline{\phi})^2 - \right.$$

$$\left. - \tfrac{1}{2}(m_o^2 + \eta\sigma)\,\underline{\phi}^2 + \tfrac{1}{2}\alpha(\underline{\chi}^2 - N\sigma) + \underline{h}\cdot\underline{\phi} \right\} \qquad \text{—(2.7.7)}$$

Performing the $\underline{\phi}$, $\underline{\chi}$ integrations gives

$$Z'_o[\underline{h}] = \int \mathcal{D}[\sigma]\,\mathcal{D}[\alpha]\,\exp iNA[\sigma,\alpha;\underline{h}] \qquad \text{—(2.7.8)}$$

where

$$A[\sigma,\alpha;\underline{h}] = \frac{1}{2N}\int dx\,dy\,\underline{h}(x)\,G_2(xy).\underline{h}(y) - \frac{1}{2}\int dx\,\sigma(x)\,\alpha(x)$$

$$+ \frac{i}{2}\,\mathrm{Tr}\,\ln\,\alpha\,G_2^{-1}(xy) \qquad \text{—(2.7.9)}$$

and $$(\Box_x + m_o^2 + \eta\sigma(x))\,G_2(xy) = -i\delta(x-y) \qquad \text{—(2.7.10)}$$

We can now find $Z'_o[\underline{h}]$ in the large N limit by looking for the extremum of $A[\sigma,\alpha;\underline{h}]$. If this extremum occurs at $\sigma_o[\underline{h}]$, $\alpha_o[\underline{h}]$, then, in the large N limit

$$Z'_o[\underline{h}] = \exp iNA[\sigma_o,\alpha_o;\underline{h}] \qquad \text{—(2.7.11)}$$

Generally,

$$Z'_o[\underline{h}] = \exp iW_o[\underline{h}] \qquad \text{—(2.7.12)}$$

and so in this limit

$$W_o[\underline{h}] = NA[\sigma_o,\alpha_o;\underline{h}] \qquad \text{—(2.7.13)}$$

The Effective action, Γ, is the generator of one ϕ -particle irreducible Green functions. If we define the semi-classical fields $\underline{\phi}_c$ by

$$(\phi_c)_i \equiv \frac{\delta W_o[h]}{\delta h_i} \qquad \text{—(2.7.14)}$$

then $\Gamma[\underline{\phi}_c]$ is defined as the Legendre transform of $W_o[\underline{h}]$ via

$$\Gamma[\underline{\phi}_c] = W_o[\underline{h}] - \int \underline{h}(x) \cdot \underline{\phi}_c(x)\, dx \qquad \text{—(2.7.15)}$$

According to (2.7.13) and (2.7.9), (2.7.14) gives that

$$(\Box_x + m_o^2 + \eta\sigma_o(x))\, \underline{\phi}_c(x) = \underline{h}(x) \qquad \text{—(2.7.16)}$$

Defining a generalised effective action from $A[\sigma,\alpha;\underline{h}]$ of (2.7.9) as

$$\Gamma[\underline{\phi}_c,\alpha,\sigma] = -\frac{1}{2}\int dx\ \underline{\phi}_c \cdot (\Box + m_o^2 + \eta\sigma)\, \underline{\phi}_c - \frac{N}{2}\int dx\, \sigma(x)\alpha(x) + \frac{i}{2} N \,\mathrm{Tr} \ln \alpha(x)(\Box + m_o^2 + \eta\sigma) \qquad \text{—(2.7.17)}$$

we then see that $\Gamma[\underline{\phi}_c]$ of (2.7.15) is obtained from $\Gamma[\alpha,\sigma,\underline{\phi}_c]$ by extremising it w.r.t. α, σ. This is exactly analogous to obtaining $A[\sigma_o,\alpha_o;\underline{h}]$ from $A[\sigma,\alpha;\underline{h}]$.

Hence $$\Gamma[\underline{\phi}_c] = \Gamma[\underline{\phi}_c;\alpha_o,\sigma_o] \qquad \text{—(2.7.18)}$$

where $\alpha_o[\underline{\phi}_c]$ and $\sigma_o[\underline{\phi}_c]$ satisfy

$$\left.\frac{\delta\Gamma[\underline{\phi}_c,\sigma,\alpha)}{\delta\alpha}\right|_{\substack{\alpha=\alpha_o\\ \sigma=\sigma_o}} = 0 \qquad \text{—(2.7.19)}$$

and $$\left.\frac{\delta \Gamma[\underline{\phi}_c, \sigma, \alpha]}{\delta \sigma}\right|_{\substack{\alpha=\alpha_0 \\ \sigma=\sigma_0}} = 0 \qquad \text{—(2.7.20)}$$

Equation (2.7.19) gives

$$i\sigma_0 + \frac{\delta(0)}{\alpha_0} = 0 \qquad \text{—(2.7.21)}$$

and (2.7.20) gives

$$\frac{i\eta}{2} \langle x | \frac{1}{\Box + m_0^2 + \eta\sigma_0} | x \rangle - \frac{1}{2}\alpha_0 - \frac{\eta}{2N} \underline{\phi}_0^2 = 0 \qquad \text{—(2.7.22)}$$

To find the one $\underline{\phi}$ -particle irreducible Green Functions, we functionally differentiate (2.7.17) w.r.t. $\underline{\phi}_c$ and then evaluate at α_0 , σ_0 , and $\underline{\phi}_c = 0$.

The two-point function $G_2^{ij}(xy)$ is thus found as the inverse of the one particle irreducible 2-point function, i.e.

$$G_2^{ij}(xy) = \left\{ \frac{\delta^2 \Gamma[\alpha, \sigma, \underline{\phi}_c]}{\delta \phi_c^i(x) \delta \phi_c^j(y)} \right\}^{-1}_{\substack{\alpha=\alpha_0 \\ \sigma=\sigma_0 \\ \underline{\phi}_c=0}} \qquad \text{—(2.7.23)}$$

From (2.7.17) we see that (2.7.23) implies

$$G_2^{ij}(xy) = \delta^{ij} G_2(xy) \qquad \text{—(2.7.24)}$$

where $$(\Box_x + m_0^2 + \eta\sigma_0(x)) G_2(xy) = -i\delta(x-y) \qquad \text{—(2.7.25)}$$

We see that the two-point function is that of a free theory with mass satisfying

$$m^2 = m_0^2 + \eta\,\sigma_0(x) \qquad \text{—(2.7.26)}$$

Now (2.7.21) and (2.7.22) give for $\underline{\phi}_c = 0$ as in the case here when evaluating $G_2^{ij}(xy)$

$$\sigma_0 = \frac{i\,\delta(0)}{i\eta\langle x|(\Box + m_0^2 + \eta\sigma_0)^{-1}|x\rangle} \qquad \text{—(2.7.27)}$$

or
$$\eta\sigma_0 = \frac{\delta(0)}{i\,G_0(0;m^2)} \qquad \text{—(2.7.28)}$$

where
$$(\Box_x + m^2)\,G_0(xy;m^2) = -i\,\delta(x-y) \qquad \text{—(2.7.29)}$$

with m^2 defined as in (2.7.26).

Hence (2.7.26) becomes

$$m^2 = m_0^2 + \frac{\delta(0)}{i\,G_0(0;m^2)} \qquad \text{—(2.7.30)}$$

This is just the familiar mass renormalisation equation and is now exactly satisfied to leading order in N^{-1}.

(2.8) Conclusions

We have seen that from three different lines of attack, each with their own assumptions, we have arrived at the equation of mass renormalisation (2.7.30).

These methods were

(i) the extra dynamical equation,

$$\lim_{x^* \to x} K_x\,G_2(x\,x^*) = 0 \qquad \text{—(2.2.22a)}$$

where we assumed a spectral density without a continuum, i.e.

$$\rho(\sigma) = \delta(\sigma - m^2)$$

(ii) the diagrammatic expansion of the augmented generating functional (2.4.1) where we considered only diagrams with single point mass insertions,

(iii) examining the augmented branching equations for a consistent solution where we used an assumption of separability of Green functions to leading singularity.

All of these assumptions are related and, through the analysis of Section 7, have been shown to correspond to taking the large N-limit of the O(N) pseudofree theory. So the N^{-1} expansion provides a natural way to separate out the most singular part of the hardcore (non-polynomial) interactions due to the scale-invariant measure.

Part 3 - Stability

(3.1) Introduction

In Parts 1 and 2 we have worked either with branching equations for the Green functions of the scale-covariant theory, or we have used the augmented formalism in which the scale-invariant measure, $\mathcal{D}'[\phi]$, is replaced by translation-invariant measures via the formal relation

$$\mathcal{D}'[\phi] = \int \mathcal{D}[\phi]\,\mathcal{D}[\chi]\,\exp -\tfrac{1}{2}\int dx\,\eta\,\phi^2(x)\chi^2(x) \qquad \text{—(3.1.1)}$$

Note that for simplicity we specialised from a covariant to invariant measure when discussing the Augmented theory.

In both approaches we have not considered the fact that the scale-covariant measure is very singular. We can see this by writing

$$\mathcal{D}'[\phi] = \frac{\mathcal{D}[\phi]}{\prod_x |\phi(x)|^\beta} \qquad \text{—(3.1.2)}$$

so when $\phi(x) = 0$ the scale-covariant measure diverges.

Now we can formally write

$$\mathcal{D}'[\phi] = \mathcal{D}[\phi]\,\exp -\frac{\beta}{2}\delta(0)\int dx\,\ln\phi^2(x) \qquad \text{—(3.1.3)}$$

To deduce this, we note that on discretising space time

$$\frac{1}{\prod_x |\phi(x)|^\beta} = \frac{1}{|\phi(x_1)|^\beta}\cdot\frac{1}{|\phi(x_2)|^\beta}\cdots\cdots\frac{1}{|\phi(x_n)|^\beta}\cdots \qquad \text{—(3.1.4)}$$

$$= \frac{1}{(\phi^2(x_1))^{\beta/2}}\cdot\frac{1}{(\phi^2(x_2))^{\beta/2}}\cdots\cdots\frac{1}{(\phi^2(x_n))^{\beta/2}}\cdots \qquad \text{—(3.1.5)}$$

Using the relation

$$\frac{1}{(\phi^2(x_i))^{\beta/2}} = \exp - \frac{\beta}{2} \ln \phi^2(x_i) \qquad \text{—(3.1.6)}$$

we see that

$$\frac{1}{(\phi^2(x_1))^{\beta/2}} \cdots \frac{1}{(\phi^2(x_n))^{\beta/2}} \cdots = \exp - \frac{\beta}{2} \sum_i \ln \phi^2(x_i) \qquad \text{—(3.1.7)}$$

In preparation for taking the continuum limit of (3.1.7) we need to introduce the increment Δ which becomes the integral measure, i.e.

$$\exp - \frac{\beta}{2} \sum_i \ln \phi^2(x_i) = \exp - \frac{\beta}{2\Delta} \sum_i \ln \phi^2(x_i).\Delta \qquad \text{—(3.1.8)}$$

Now taking the limit $\Delta \to 0$, in which case $\frac{1}{\Delta} \to \delta(0)$ we obtain

$$\exp - \frac{\beta}{2} \sum_i \ln \phi^2(x_i) \longrightarrow \exp - \frac{\beta\,\delta(0)}{2} \int dx \ln \phi^2(x) \qquad \text{—(3.1.9)}$$

Hence the result (3.1.3) follows.

Consider the Euclidean generating functional for the scale-covariant theory

$$S'[h] = \int \mathcal{D}'[\phi] \exp - \frac{1}{\hbar} \left\{ \mathcal{A}[\phi] - \int dx\, h(x) \phi(x) \right\} \qquad \text{—(3.1.10)}$$

where $\mathcal{A}[\phi]$ is the classical action

$$\mathcal{A}[\phi] = \int dx \left\{ \frac{1}{2} (\nabla\phi)^2 + V(\phi) \right\} \qquad \text{—(3.1.11)}$$

We can see that writing $\mathcal{D}'[\phi]$ explicitly, as in (3.1.3) has the effect of changing the potential $V(\phi)$ to $V'(\phi)$ given by

$$V'(\phi) = V(\phi) + \frac{\beta\hbar}{2}\,\delta(0)\ln\phi^2(x) \qquad \text{—(3.1.12)}$$

As $\phi(x) \to 0$, this addition causes the potential to become unbounded below and signals an intrinsic instability.

This observation casts doubt on the usefulness of scale-covariant theories. However, we must realise that in quantum field theory it is the *effective potential*, not the classical potential, that is of interest. The latter observation for the classical potential need not be true for the effective potential. Indeed, in this part we shall examine two methods of expansion of the pseudofree generating functional ((3.1.10) for $V(\phi) = 0$) that avoid instability. The first of these is the large-N limit of the O(N) invariant pseudofree theory. This was mentioned briefly in Section 7 of Part 2. Secondly, we evaluate the strong-coupling limit of the pseudofree theory, as a basis for a strong coupling expansion. We are encouraged by the lower boundedness of the effective potential in each case.

Now the interpretation of the effective potential as an energy density is only strictly true for a Minkowski theory. We shall discuss this later.

(3.2) The Large-N limit of the Pseudofree Effective Potential

Here we consider the O(N)-invariant, scale-covariant pseudofree Euclidean theory of scalar fields ϕ_i ($i = 1, 2, \ldots N$) . This has the generating functional

$$S'[h] = \int \mathcal{D}'[\underline{\phi}] \exp -\frac{1}{\hbar}\int dx \left\{ \frac{1}{2}(\nabla\underline{\phi})^2 + \frac{1}{2} m_0^2 \underline{\phi}^2 - \underline{h}\cdot\underline{\phi} \right\} \qquad \text{—(3.2.1)}$$

where $\mathcal{D}'[\underline{\phi}]$ is invariant under O(N) rotations, and covariant w.r.t.
Scale transformations, i.e.

$$\mathcal{D}'[\Lambda\underline{\phi}] = \mathcal{D}'[\underline{\phi}]\, F[\Lambda] \qquad \forall\ \Lambda(x) > 0 \qquad \text{—(3.2.2)}$$

Such a measure has the form

$$\mathcal{D}'[\underline{\phi}] = \prod_{i=1}^{N} \prod_{x} \frac{d\phi_i(x)}{|\underline{\phi}(x)/N^{1/2}|^{\beta}}$$

$$= \prod_{i=1}^{N} \frac{\mathcal{D}[\phi_i]}{\prod_x |\underline{\phi}(x)/N^{1/2}|^{\beta}} \qquad \text{—(3.2.3)}$$

where $\mathcal{D}[\phi_i]$ is a canonical translation-invariant measure.
Using an analogous equation to (3.1.3) we can write

$$\mathcal{D}'[\underline{\phi}] = \prod_{i=1}^{N} \mathcal{D}[\phi_i] \exp -\frac{\beta N}{2}\delta(0)\int dx \ln(N^{-1}\underline{\phi}^2(x)) \qquad \text{—(3.2.4)}$$

So (3.2.1) can be written

$$S'[h] = \int \prod_{i=1}^{N} \mathcal{D}[\phi_i] \exp -\frac{1}{\hbar}\int dx \left\{ \frac{1}{2}(\nabla\underline{\phi})^2 + \frac{1}{2} m_0^2 \underline{\phi}^2 + \right. \qquad \text{—(3.2.5)}$$
$$\left. + \frac{\beta\hbar}{2} N\delta(0)\ln(\underline{\phi}^2/N) - \underline{h}\cdot\underline{\phi} \right\}$$

We assume that the path integral is dominated by configurations for which $\underline{\phi}^2(x)$ is of order N. This N-dependence can be made explicit by introducing the functional delta function

$$\int \mathcal{D}[\sigma]\, \delta(\underline{\phi}^2 - N\sigma) = \int \mathcal{D}[\sigma]\,\mathcal{D}[\alpha]\, \exp -\frac{i}{2\hbar}\int dx\, \alpha(x)(\underline{\phi}^2 - N\sigma) \qquad \text{—(3.2.6)}$$

where fields σ , α are of order 1.

This gives

$$S'[h] = \int \prod_i \mathcal{D}[\phi_i]\,\mathcal{D}[\sigma]\,\mathcal{D}[\alpha]\, \exp -\frac{1}{\hbar}\int dx \Big\{ \tfrac{1}{2}(\nabla\underline{\phi})^2 + \tfrac{1}{2}m_0^2\underline{\phi}^2 + \frac{i\alpha}{2}(\underline{\phi}^2 - N\sigma) + \frac{\beta\hbar N}{2}\delta(0)\ln\Big(\frac{\underline{\phi}^2}{N}\Big) - \underline{h}\cdot\underline{\phi} \Big\} \qquad \text{—(3.2.7)}$$

Performing the Gaussian $\underline{\phi}$ -integration gives

$$S'[h] = \int \mathcal{D}[\sigma]\,\mathcal{D}[\alpha]\, \exp -\frac{1}{\hbar}\,\mathcal{O}[\sigma,\alpha;\underline{h}] \qquad \text{—(3.2.8)}$$

where

$$\begin{aligned} \mathcal{O}[\sigma,\alpha;\underline{h}] = {} & -\tfrac{1}{2}\int dx\, dy\, \underline{h}(x) \langle x| \frac{1}{-\nabla^2 + m_0^2 + i\alpha} |y\rangle \cdot \underline{h}(y) \\ & + \frac{N\hbar}{2}\,\mathrm{Tr}\ln(-\nabla^2 + m_0^2 + i\alpha) \\ & + \frac{N}{2}\int dx \Big\{ \beta\hbar\,\delta(0)\ln\sigma(x) - i\alpha(x)\sigma(x) \Big\} \end{aligned} \qquad \text{—(3.2.9)}$$

We note that each term in (3.2.9) is of order N (since $\underline{h}^2 = O(N)$). In the limit $N \to \infty$ (keeping $\hbar$ fixed) we assume that $S'[\underline{h}]$ is dominated by a single saddle-point, given by the minimum of $\mathcal{O}[\sigma,\alpha;\underline{h}]$ w.r.t. σ, α .

Let $\sigma_0[\underline{h}]$ and $\alpha_0[\underline{h}]$ be the configurations of σ and α that minimise $\mathcal{O}[\sigma,\alpha;\underline{h}]$. Then we may write for the large-N limit

$$S'[\underline{h}] = \exp -\frac{1}{\hbar}\,\mathcal{O}[\sigma_0,\alpha_0;\underline{h}] \qquad \text{—(3.2.10)}$$

Generally $$S'[\underline{h}] = \exp \frac{1}{\hbar} W'[\underline{h}] \qquad \text{—(3.2.11)}$$

where $W'[\underline{h}]$ is the connected generating functional. So to this order of approximation we have

$$W'[\underline{h}] = -\,\sigma\tau[\sigma_0, \alpha_0 ; \underline{h}] \qquad \text{—(3.2.12)}$$

We now define the *semi-classical* fields ϕ_c^i by

$$\phi_c^i(x) = \frac{\delta W'[h]}{\delta h_i(x)} \qquad \text{—(3.2.13)}$$

From (3.2.12) and (3.2.9) this implies that $\phi_c^i(x)$ satisfies

$$\left(-\nabla_x^2 + m_0^2 + i\alpha_0(x) \right) \phi_c^i(x) = h^i(x) \qquad \text{—(3.2.14)}$$

The *Effective Action* $\Gamma[\underline{\phi}_c]$ is obtained from $W'[\underline{h}]$ by the Legendre transformation

$$\begin{aligned} \Gamma[\phi_c] &= W'[\underline{h}] - \int dx\, \underline{h} \cdot \underline{\phi}_c \qquad \text{—(3.2.15)} \\ &= -\,\sigma\tau[\sigma_0, \alpha_0 ; \underline{h}] - \int dx\, \underline{h} \cdot \underline{\phi}_c \end{aligned}$$

Now from (3.2.14)

$$\left\{ -\nabla_x^2 + m_0^2 + i\alpha_0(x) \right\} \underline{\phi}_c(x) = \underline{h}(x)$$

and hence

$$\int dx\,dy\,\underline{h}(x)\langle x|\frac{1}{-\nabla^2+m_o^2+i\alpha_o}|y\rangle\cdot\underline{h}(y)$$

$$= \int dx\,\underline{\phi}_c(x)\left(-\nabla_x^2+m_o^2+i\alpha_o(x)\right)\cdot\underline{\phi}_c(x) \qquad \text{—(3.2.16)}$$

and

$$\int dx\,\underline{h}\cdot\underline{\phi}_c = \int dx\,\underline{\phi}_c(x)\left(-\nabla_x^2+m_o^2+i\alpha_o(x)\right)\cdot\underline{\phi}_c(x) \qquad \text{—(3.2.17)}$$

With these, we see that $\Gamma[\underline{\phi}_c]$ can be written

$$\Gamma[\underline{\phi}_c] = -\frac{1}{2}\int dx\,\underline{\phi}_c(x)\left(-\nabla_x^2+m_o^2+i\alpha_o(x)\right)\cdot\underline{\phi}_c(x) -$$
$$-\frac{N}{2}\int dx\,\beta\hbar\,\delta(0)\ln\sigma_o(x) + \frac{iN}{2}\int dx\,\alpha_o(x)\sigma_o(x) -$$
$$-\frac{N\hbar}{2}\,\mathrm{Tr}\ln\left(-\nabla^2+m_o^2+i\alpha_o\right) \qquad \text{—(3.2.18)}$$

We can define the generalised effective action $\Gamma[\sigma,\alpha;\underline{\phi}_c]$ by

$$\Gamma[\sigma,\alpha;\underline{\phi}_c] = -\frac{1}{2}\int dx\,\underline{\phi}_c(x)\left(-\nabla_x^2+m_o^2+i\alpha(x)\right)\cdot\underline{\phi}_c(x) -$$
$$-\frac{N}{2}\int dx\,\beta\hbar\,\delta(0)\ln\sigma(x) + \frac{iN}{2}\int dx\,\alpha(x)\sigma(x) -$$
$$-\frac{N\hbar}{2}\,\mathrm{Tr}\ln\left(-\nabla^2+m_o^2+i\alpha\right) \qquad \text{—(3.2.19)}$$

Then

$$\Gamma[\underline{\phi}_c] = \Gamma[\sigma_o,\alpha_o;\underline{\phi}_c] \qquad \text{—(3.2.20)}$$

where σ_o and α_o are given by the extremal conditions

$$\left.\frac{\delta\Gamma[\alpha,\sigma;\underline{\phi}_c]}{\delta\alpha}\right|_{\substack{\alpha=\alpha_o\\ \sigma=\sigma_o}} = 0 \qquad \text{—(3.2.21)}$$

and

$$\frac{\delta \Gamma[\sigma, \alpha ; \underline{\phi}_c]}{\delta \sigma}\bigg|_{\substack{\alpha=\alpha_0 \\ \sigma=\sigma_0}} = 0 \qquad \text{—(3.2.22)}$$

The condition (3.2.21) gives

$$\sigma_0(\underline{\phi}_c^2) - \underline{\phi}_c^2/N - \hbar \int đk \frac{1}{k^2 + m_0^2 + i\alpha_0(\underline{\phi}_c^2)} = 0 \qquad \text{—(3.2.23)}$$

and (3.2.22) gives

$$i\alpha_0(\underline{\phi}_c^2) - \frac{\beta\hbar\,\delta(0)}{\sigma_0(\underline{\phi}_c^2)} = 0 \qquad \text{—(3.2.24)}$$

Note that in the last term of (3.2.23) we used the fact that

$$\mathrm{Tr} \ln(-\nabla^2 + m_0^2 + i\alpha) = \int dx \ln\left(-\nabla^2 + m_0^2 + i\alpha(x)\right) \qquad \text{—(3.2.25)}$$

and hence

$$\frac{\delta}{\delta\alpha(x)} \mathrm{Tr}\ln\left(-\nabla^2 + m_0^2 + i\alpha\right) = i\langle x| \frac{1}{-\nabla^2 + m_0^2 + i\alpha} |x\rangle$$

$$= i \int đk\, đk' \langle x|k\rangle\langle k| \frac{1}{k^2 + m_0^2 + i\alpha} |k'\rangle\langle k'|x\rangle$$

$$= i \int đk\, đk' \exp ix(k-k') \cdot \frac{\delta(k-k')}{k^2 + m_0^2 + i\alpha}$$

$$= i \int đk \frac{1}{k^2 + m_0^2 + i\alpha}$$

The effective potential is obtained from the effective action by considering only constant field configurations. This serves to remove the gradient contribution from the effective action, leaving only the potential, i. e. the effective action has the form

$$\Gamma[\underline{\phi}_c] = \int dx \left\{ \tfrac{1}{2}(\nabla\underline{\phi}_c)^2 - V(\underline{\phi}_c) \right\} \qquad \text{—(3.2.26)}$$

For constant (spacially independent) semi-classical field configurations $\underline{\phi}_c$,

$$\Gamma[\underline{\phi}_c] = \Gamma(\underline{\phi}_c) = -V(\phi_c)\int dx \qquad \text{—(3.2.27)}$$

From (3.2.18) we deduce

$$V(\underline{\phi}_c) = \tfrac{1}{2} m^2(\underline{\phi}_c^2)\,\underline{\phi}_c^2 + \frac{\beta N\hbar}{2}\,\delta(0)\ln\sigma_0(\underline{\phi}_c^2) - \\ - \frac{N}{2}\left(m^2(\underline{\phi}_c^2) - m_0^2\right)\sigma_0(\underline{\phi}_c^2) + \frac{N\hbar}{2}\int dk \ln\left(k^2 + m^2(\underline{\phi}_c^2)\right) \qquad \text{—(3.2.28)}$$

where we have defined

$$m^2(\underline{\phi}_c^2) = m_0^2 + i\,\alpha_0(\underline{\phi}_c^2) \qquad \text{—(3.2.29)}$$

Note that in making $\underline{\phi}_c$ independent of space x, $\sigma_0(\underline{\phi}_c^2)$ and $m^2(\underline{\phi}_c^2)$ are also independent of x. Also the constraints (3.2.23) and (3.2.24) are rederived from the following extremal conditions on the generalised effective potential $V(\sigma, m^2; \underline{\phi}_c)$

$$\left.\frac{\partial V(m^2, \sigma; \underline{\phi}_c)}{\partial \sigma}\right|_{\substack{\sigma = \sigma_0(\underline{\phi}_c^2) \\ m^2 = m^2(\underline{\phi}_c^2)}} = 0 \qquad \text{—(3.2.30)}$$

$$\left.\frac{\partial V(m^2, \sigma; \underline{\phi}_c)}{\partial m^2}\right|_{\substack{\sigma = \sigma_0(\underline{\phi}_c^2) \\ m^2 = m^2(\underline{\phi}_c^2)}} = 0 \qquad \text{—(3.2.31)}$$

Having derived the effective potential for the

O(N)-invariant pseudofree theory in the large-N limit, let us now examine its stability. To do this we must expand $V(\underline{\phi}_c)$ about the global extremum. This extremum $\underline{\phi}_c = \underline{\eta}$ is found from

$$\left.\frac{dV(\underline{\phi}_c)}{d\phi_c^i}\right|_{\underline{\phi}_c=\underline{\eta}} = 0 \qquad \text{—(3.2.32a)}$$

Now

$$\frac{dV(\underline{\phi}_c)}{d\phi_c^i} = \frac{\partial V}{\partial \phi_c^i} + \frac{\partial V}{\partial m^2(\underline{\phi}_c^2)} \cdot \frac{\partial m^2(\underline{\phi}_c^2)}{\partial \phi_c^i} + \frac{\partial V}{\partial \sigma_0} \cdot \frac{\partial \sigma_0}{\partial \phi_c^i}$$

—(3.2.32b)

But from (3.2.30) and (3.2.31)

$$\frac{\partial V}{\partial m^2(\underline{\phi}_c^2)} = \frac{\partial V}{\partial \sigma_0} = 0 \qquad \text{—(3.2.32c)}$$

Hence $$\frac{dV(\underline{\phi}_c)}{d\phi_c^i} = \frac{\partial V(\underline{\phi}_c)}{\partial \phi_c^i} = m^2(\underline{\phi}_c^2).\,\phi_c^i \qquad \text{—(3.2.33)}$$

so the global extremum occurs at $\underline{\phi}_c = 0$.

Then we have, from (3.2.23) and (3.2.24) with $m^2 = m_0^2 + i\alpha_0$

$$\sigma_0(0) = \hbar \int \frac{đk}{k^2 + m^2(0)} \qquad \text{—(3.2.34)}$$

$$m^2(0) = m_0^2 + \frac{\beta\hbar\,\delta(0)}{\sigma_0(0)} \qquad \text{—(3.2.35)}$$

Notice that since

$$\int \frac{đk}{k^2+m^2(0)} = i\,G_0(0;\, m^2(0)) \qquad \text{—(3.2.36)}$$

substituting (3.2.34) into (3.2.35) gives

$$m^2(0) = m_0^2 + \frac{\beta\,\delta(0)}{i\,G_0(0;m^2(0))} \qquad \text{—(3.2.37)}$$

This is just the mass-renormalisation equation discussed extensively in Part 2.

Defining the quantities

$$\tilde{m}^2 = m^2(\underline{\phi}_c^2) - m^2(0)$$

and $$\tilde{\sigma}_0 = \sigma_0(\underline{\phi}_c^2) - \sigma_0(0) \qquad \text{—(3.2.38)}$$

we now develop $V(\underline{\phi}_c)$ as an expansion in $\underline{\phi}_c^2$ about the global extremum $\underline{\phi}_c = 0$. Substituting (3.2.38) into (3.2.28) gives

$$V(\underline{\phi}_c) = \tfrac{1}{2}m^2(0)\underline{\phi}_c^2 + \tfrac{1}{2}\tilde{m}^2(\underline{\phi}_c^2)\underline{\phi}_c^2 + \frac{\beta N\hbar}{2}\delta(0)\ln\sigma_0(0) +$$

$$+ \frac{\beta N\hbar}{2}\delta(0)\ln\left(1+\frac{\tilde{\sigma}_0(\underline{\phi}_c^2)}{\sigma_0(0)}\right) - \frac{N}{2}\left(m^2(0)+\tilde{m}^2(\underline{\phi}_c^2)-m_0^2\right)\cdot\left(\sigma_0(0)+\tilde{\sigma}_0(\underline{\phi}_c^2)\right)$$

$$+ \frac{N\hbar}{2}\int dk\,\ln(k^2+m^2(0)) + \frac{N\hbar}{2}\int dk\,\ln\left(1+\frac{\tilde{m}^2(\underline{\phi}_c^2)}{k^2+m^2(0)}\right)$$

—(3.2.39)

From (3.2.35), $$m^2(0) - m_0^2 = \frac{\beta\hbar\,\delta(0)}{\sigma_0(0)}$$

and so

$$\left(m^2(0) + \tilde{m}^2(\underline{\phi}_c^2) - m_0^2\right)\left(\sigma_0(0) + \tilde{\sigma}_0(\underline{\phi}_c^2)\right)$$

$$= \beta\hbar\,\delta(0) + \tilde{m}^2\sigma_0(0) + \beta\hbar\,\delta(0)\frac{\tilde{\sigma}_0}{\sigma_0(0)} + \tilde{m}^2\tilde{\sigma}_0 \qquad \text{—(3.2.40)}$$

For small fluctuations $\tilde{m}^2$, $\tilde{\sigma}_0$ we can expand the logarithms of (3.2.39). Also, since $V(\underline{\phi}_c)$ is a potential, we can subtract off constant ($\underline{\phi}_c$ - independent) terms. After doing this we find

$$V(\underline{\phi}_c) = \tfrac{1}{2}m^2(0)\underline{\phi}_c^2 + \tfrac{1}{2}\tilde{m}^2(\underline{\phi}_c^2)\underline{\phi}_c^2 - \frac{\beta N\hbar}{4}\delta(0)\frac{\tilde{\sigma}_0^2(\underline{\phi}_c^2)}{\sigma_0^2(0)} -$$

$$- \frac{N}{2}\tilde{m}^2\tilde{\sigma}_0 - \frac{N\hbar}{4}(\tilde{m}^2)^2\int dk \frac{1}{(k^2+m^2(0))^2}$$

+ higher order terms

—(3.2.41)

Now

$$\tilde{\sigma}_0(\underline{\phi}_c^2) = \sigma_0(\underline{\phi}_c^2) - \sigma_0(0)$$

$$= \frac{\underline{\phi}_c^2}{N} + \hbar\int dk \left\{\frac{1}{k^2+m^2(\underline{\phi}_c^2)} - \frac{1}{k^2+m^2(0)}\right\}$$

$$= \frac{\underline{\phi}_c^2}{N} - \hbar\int dk \frac{\tilde{m}^2(\underline{\phi}_c^2)}{(k^2+m^2(\underline{\phi}_c^2))(k^2+m^2(0))}$$

$$= \frac{\underline{\phi}_c^2}{N} - \hbar\tilde{m}^2\int dk \frac{1}{(k^2+m^2(0))^2\left(1+\frac{\tilde{m}^2}{k^2+m^2(0)}\right)}$$

$$= \frac{\underline{\phi}_c^2}{N} - \hbar\tilde{m}^2\left\{\int\frac{dk}{(k^2+m^2(0))^2} - \tilde{m}^2\int\frac{dk}{(k^2+m^2(0))^3} + \right. \qquad \text{—(3.2.42)}$$

$$\left. + (\tilde{m}^2)^2\int\frac{dk}{(k^2+m^2(0))^4} - \cdots + \cdots\right\}$$

Similarly

$$\tilde{m}^2(\underline{\phi}_c^2) = m_0^2 + \frac{\beta\hbar\delta(0)}{\sigma_0(\underline{\phi}_c^2)} - m_0^2 - \frac{\beta\hbar\delta(0)}{\sigma_0(0)}$$

$$= \beta\hbar\delta(0)\left(\frac{1}{\sigma_0(\underline{\phi}_c^2)} - \frac{1}{\sigma_0(0)}\right)$$

$$\simeq -\beta\hbar\delta(0)\frac{\tilde{\sigma}_0(\underline{\phi}_c^2)}{\sigma_0^2(0)} \qquad \text{—(3.2.43)}$$

Now from (3.2.42) let us approximate $\tilde{\sigma}_0$ to

$$\tilde{\sigma}_0(\underline{\phi}_c^2) \simeq \frac{\underline{\phi}_c^2}{N} - \hbar\,\tilde{m}^2 B(m^2(0)) \qquad \text{—(3.2.44)}$$

where we have defined

$$B(m^2) = \int \frac{dk}{(k^2+m^2)^2} \qquad \text{—(3.2.43)}$$

On substituting (3.2.44) into (3.2.43) we have

$$\tilde{m}^2 \simeq -\frac{\beta\hbar\delta(0)}{\sigma_0^2(0)}\left\{\underline{\phi}_c^2/N - \hbar\,\tilde{m}^2 B(m^2(0))\right\} \qquad \text{—(3.2.45)}$$

or

$$\tilde{m}^2\left\{1 - \frac{\beta\hbar^2\delta(0)}{\sigma_0^2(0)}B(m^2(0))\right\} \simeq -\frac{\underline{\phi}_c^2}{N}\cdot\frac{\beta\hbar\delta(0)}{\sigma_0^2(0)}$$

and hence

$$\tilde{m}^2(\underline{\phi}_c^2) = -\frac{\underline{\phi}_c^2}{N}\cdot\frac{\beta\hbar\delta(0)}{\sigma_0^2(0)}\left\{1 - \frac{\beta\hbar\delta(0)}{\sigma_0^2(0)}B(m^2(0))\right\}^{-1}$$

$$+ O((\underline{\phi}_c^2/N)^2) + \cdots\cdots \qquad \text{—(3.2.46)}$$

Using (3.2.46) and (3.2.43) gives

$$\tilde{\sigma}_o(\underline{\phi}_c^2) = \frac{\underline{\phi}_c^2}{N}\left\{1 - \frac{\beta\hbar^2\delta(0)\,B(m^2(0))}{\sigma_o^2(0)}\right\}^{-1} + O\left((\underline{\phi}_c^2/N)^2\right) + \cdots\cdots$$

—(3.2.47)

In order to examine the singularity of $\tilde{m}^2$ and $\tilde{\sigma}_o$, and ultimately $V(\phi_c)$, we introduce an ultra-violet cut-off Λ, in the respective integrals. For a d-dimensional Euclidean theory,

$$\delta(0) = \frac{1}{(2\pi)^d}\int d^dk = O(\Lambda^d) \qquad \text{—(3.2.48)}$$

$$\sigma(0) = \frac{\hbar}{(2\pi)^d}\int\frac{d^dk}{k^2+m^2(0)} = O(\Lambda^{d-2}) \qquad \text{—(3.2.49)}$$

$$B(m^2(0)) = \frac{1}{(2\pi)^d}\int\frac{d^dk}{(k^2+m^2(0))^2} = O(\Lambda^{d-4}) \qquad \text{—(3.2.50)}$$

We see, then, that

$$\tilde{\sigma}_o = \frac{\underline{\phi}_c^2}{N}\,O(1) \qquad \text{—(3.2.51)}$$

and

$$\tilde{m}^2 = \frac{\underline{\phi}_c^2}{N}\cdot O(\Lambda^{4-d}) \qquad \text{—(3.2.52)}$$

Turning to the effective potential (3.2.41) now, we can examine its singularity structure. Taking the terms in order

$$\frac{1}{2}\tilde{m}^2 \frac{\underline{\phi}_c^2}{N} = (\underline{\phi}_c^2/N)^2 \, O(\Lambda^{4-d}) \qquad \text{(a)}$$

$$\frac{\delta(0)\,\tilde{\sigma}_0^2}{\sigma_0^2(0)} = (\underline{\phi}_c^2/N)^2 \, O(\Lambda^{4-d}) \qquad \text{(b)}$$

$$\tilde{m}^2 \tilde{\sigma}_0 = (\underline{\phi}_c/N)^2 \, O(\Lambda^{4-d}) \qquad \text{(c)}$$

$$(\tilde{m}^2)^2 \, B(m^2(0)) = (\underline{\phi}_c/N)^2 \, O(\Lambda^{4-d}) \qquad \text{(d)}$$

—(3.2.53)

It can be shown that when higher order terms are considered, these are down by additional powers of Λ .

We see that for $d > 4$ dimensions, when we take the cut-off limit $\Lambda \to \infty$, only the first term in $V(\underline{\phi}_c)$ of (3.2.41) survives, i.e.

$$V(\underline{\phi}_c) = \frac{1}{2} m^2(0) \, \underline{\phi}_c^2 \qquad \text{—(3.2.54)}$$

where $m^2(0)$ satisfies (3.2.37). So the large-N limit of the pseudo-free theory is a stable free theory for $d > 4$ dimensions. The dimensional dependence is encouraging since it shows that the scale-covariant formalism does not work where it is not obligatory, i.e. for $d < 4$ dimensions.

The non-leading terms in the N^{-1} expansion may provide non-free connections. However, experience with N^{-1} expansions shows that any pathologies occur in leading order. So we do not expect non-leading orders to upset the stability of the theory. However, the N^{-1} can be pathological for non-asymptotically free theories. In Part 4 we shall examine the effect on the stability of introducing a self-interaction, but first let us consider a second method of maintaining a stable pseudo-free theory by considering the strong coupling expansion.

(3.3) The Strong-Coupling Limit of the Pseudofree Theory

We consider here the pseudofree theory of a single scalar field. The strong coupling expansion is an expansion where the kinetic terms serve as the perturbation, instead of the potential terms (more usual is the weak coupling expansion in which the potential terms serve as the perturbation). The basis of the strong coupling expansion of the pseudofree theory, i.e. the strong coupling limit, is the Euclidean generating functional

$$S_0'[j] = \int \mathcal{D}'[\phi] \exp -\frac{1}{\hbar}\int dx \left(\tfrac{1}{2}m_0^2\phi^2 - j\phi\right) \qquad \text{—(3.3.1)}$$

This is simply the generating function for the Independent Value Model (IVM) discussed in Part 1.

It is known that the IVM generating functional (3.3.1) is exactly and non-trivially solvable (*K1, K2*). The scheme of calculation forces the measure to be specifically scale invariant.

An expansion in kinetic terms about (3.3.1) has been attempted (*K5*) and it is argued that it gives rise to a semi-classical theory of tree diagrams with the mass and vertices of the IVM. If this is true, it means that the Euclidean effective potential $V(\phi_c)$ for the theory is just that obtained from the IVM. This is because the restoration of kinetic terms has not given rise to loop contributions, as it usually does. We are therefore motivated to calculate the IVM effective potential.

The IVM generating functional (3.3.1) is exactly solvable (*K1, K2*) as

$$S_0'[j] = \exp \frac{1}{\hbar} W_0'[j(x)] \qquad \text{—(3.3.2)}$$

where

$$W_o'[j(x)] = -\hbar b \int dx \int \frac{du}{|u|} \left(1 - \cosh\left(\frac{u j(x)}{\hbar}\right)\right) \exp -\frac{b}{2\hbar} m^2 u^2 \qquad \text{—(3.3.3)}$$

and

$$m^2 = \frac{\delta(0)}{b} m_o^2 \qquad \text{—(3.3.4}$$

The semi-classical field ϕ_c is now defined

$$\phi_c(x) = \frac{\delta W_o'[j]}{\delta j(x)} \qquad \text{—(3.3.5)}$$

Let

$$W_o'[j] = -\int dx\, U(j(x)) \qquad \text{—(3.3.6)}$$

Then

$$U(j(x)) = \hbar b \int \frac{du}{|u|} \left(1 - \cosh\left(\frac{u j(x)}{\hbar}\right)\right) \exp -\tfrac{1}{2} b m^2 u^2/\hbar \qquad \text{—(3.3.7)}$$

so

$$\phi_c(j) = -\frac{\partial U(j)}{\partial j}$$

$$= b \int du \sinh\left(\frac{uj}{\hbar}\right) \exp -\tfrac{1}{2} b m^2 u^2/\hbar$$

$$= \left(\frac{\pi b \hbar}{2m^2}\right)^{1/2} \exp j^2/2b\hbar m^2 \ \Phi\left(\frac{j}{m\sqrt{2b\hbar}}\right) \qquad \text{—(3.3.8)}$$

where Φ is the error function.

The effective action is defined as the Legendre transform of the connected functional W_o' via the semi-classical field, ϕ_c , i.e.

$$\Gamma[\phi_c] = W_o'[j] - \int dx\, j(x)\, \phi_c(x) \qquad \text{—(3.3.9)}$$

For constant (x-independent) semi-classical field configurations, we obtain the effective potential, $V(\phi_c)$, since then

$$\Gamma[\phi_c] = \Gamma(\phi_c) = -V(\phi_c)\int dx \qquad \text{—(3.3.10)}$$

From (3.3.6), (3.3.9) and (3.3.10) we find

$$V(\phi_c) = \sqcup(j) + j\,\phi_c(j) \qquad \text{—(3.3.11)}$$

so

$$\frac{\partial V}{\partial \phi_c} = \frac{\partial \sqcup}{\partial j}\cdot\frac{\partial j}{\partial \phi_c} + j + \frac{\partial j}{\partial \phi_c}\cdot\phi_c \qquad \text{—(3.3.12)}$$

Now from (3.3.8) $\quad \dfrac{\partial \sqcup}{\partial j} = -\phi_c(j)$

and so

$$\frac{\partial V}{\partial \phi_c} = j(\phi_c) \qquad \text{—(3.3.13)}$$

whence

$$V(\phi_c) = \int^{\phi_c} j(\phi_c')\,d\phi_c' \qquad \text{—(3.3.14)}$$

Because $\phi(j)$ is monotonic as a function of j (as is seen from (3.3.8)), it follows that $V(\phi_c)$ is bounded below by zero. So the Euclidean effective potential shows no sign of instability.

From the asymptotic expressions for (3.3.8) for $j \to \infty$ and $j \sim 0$ (and hence $\phi_c \to \infty$ and $\phi_c \sim 0$) we discover that

$$\phi_c \sim 0 \;, \qquad V(\phi_c) \sim \tfrac{1}{2}m^2\phi_c^2 + \frac{m^4\phi_c^4}{12b\hbar} + O\Big(\frac{\phi^6}{(b\hbar)^2}\Big) \qquad \text{(a)}$$

$$\phi_c \sim \infty \;, \qquad V(\phi_c) \sim (2b\hbar m^2)^{1/2}\phi_c\left\{\ln\sqrt{\frac{2m^2}{\pi b\hbar}}\cdot\phi\right\} \qquad \text{(b)}$$

—(3.3.15)

Consider $\hbar b/m^2 \to \infty$. Then from (3.3.15a) we recover the *free* theory, $V(\phi_c) = \frac{1}{2}m^2\phi_c^2$. This means that on mass scales that are small compared to b (b has the dimensions [mass]^d in d -dimensions), the pseudofree theory is indistinguishable from the free theory. Scale covariant theories seem, then, to possess mass scales below which they behave like canonical theories.

Now, of course, it has been our assumption all along that the IVM effective potential is the effective potential of the complete theory. This is supported in part by (*K5*). However, the analytic regularisation used in (*K5*) may have been too severe. But we believe that the IVM effective potential certainly describes part of the complete picture, and we have performed this calculation of the strong coupling limit to show that there exists more than one mechanism for preserving the stability of scale-covariant theories.

(3.4) The Minkowski Theory

So far we have only considered the Euclidean Pseudofree theory. Strictly, it is only for the Minkowski theory that the effective potential has the interpretation of an energy density (*C2*).

Now for the large-N limit of O(N)-invariant pseudofree theory, the analytic continuation from the Minkowski to the Euclidean theory is trivial and, for $d > 4$, the Minkowski effective potential is still

$$V(\phi_c) = \frac{1}{2}m^2(0)\underline{\phi}_c^2 \qquad \text{—(3.4.1)}$$

with $m^2(0)$ as in (3.2.37).

For the expansion about the IVM, the situation is different. Consider the Minkowski generating functional for the IVM

$$Z_0''[j] = \int \mathcal{D}'[\phi] \exp -\frac{i}{\hbar}\int\left(\tfrac{1}{2}m_0^2\phi^2 - j\,\phi\right)dx \qquad \text{—(3.4.2)}$$

This has the exact evaluation (*K1*, *K2*)

$$Z_0''[j] = \exp \frac{i}{\hbar} W_0''[j] \qquad \text{—(3.4.3)}$$

where

$$W_0''[j] = i b \hbar \int dx \int \frac{du}{|u|}\left(1 - \cos\left(\frac{uj}{\hbar}\right)\right)\exp -\frac{i b m^2 u^2}{2\hbar} \qquad \text{—(3.4.4)}$$

Writing $$W_0''[j] = \int dx \, \sqcup(j(x)) \qquad \text{—(3.4.5)}$$

Then

$$\sqcup(j) = i b \hbar \int \frac{du}{|u|}\left(1 - \cos\left(\frac{uj}{\hbar}\right)\right)\exp -\frac{i b m^2 u^2}{2\hbar} \qquad \text{—(3.4.6)}$$

The semi-classical field ϕ_c is defined by

$$\phi_c(x) = \frac{\delta W_0''[j]}{\delta j(x)} \qquad \text{—(3.4.7)}$$

or
$$\phi_c(j) = \frac{\partial \sqcup(j)}{\partial j}$$
$$= i b \int du \, \sin(uj/\hbar) \exp -\frac{i b m^2 u^2}{2\hbar} \qquad \text{—(3.4.8)}$$

This function is double valued and complex. We see that

$$\begin{aligned} j &\to 0, \quad \phi \to 0 \\ j &\to \infty, \quad \phi \to 0 \end{aligned} \qquad \text{—(3.4.9)}$$

The complexity of ϕ_c implies that the Minkowski effective potential is also complex. This, it might be argued, is an indication of vacuum instability.

An alternative continuation would be to begin with the Euclidean IVM for which

$$S_o'[j] = \int \mathcal{D}'[\phi] \exp - \frac{1}{\hbar} \int dx \left(\tfrac{1}{2} m_o^2 \phi^2 - ij\phi \right) \qquad \text{—(3.4.10)}$$

and continue afterwards. This differs from (3.3.1) because we have replaced $j(x)$ by $ij(x)$. Evaluating (3.4.10) according to (*K1, K2*) we have

$$S_o'[j] = \exp \frac{1}{\hbar} W_o'[j] \qquad \text{—(3.4.11)}$$

where

$$W_o'[j] = -b\hbar \int dx \int \frac{du}{|u|} \left(1 - \cos\left(\frac{uj}{\hbar}\right) \right) \exp - \frac{bm^2u^2}{2\hbar} \qquad \text{—(3.4.12)}$$

The semi-classical field is given by

$$\phi_c = \frac{\hbar b}{m^2} j \, M\left(1 ; \frac{3}{2} ; \frac{-j^2}{2b\hbar m^2} \right) \qquad \text{—(3.4.13)}$$

where M is the confluent hypergeometric function. Here ϕ_c is real, but is double valued. Consequently, the effective potential derived from (3.4.10) is double branched as a function of ϕ_c .

Considering the asymptotic expressions for large and small in (3.4.13), we find that for

$$j \to 0 , \qquad \phi_c \sim j/m^2 \qquad \text{(a)}$$

$$j \to \infty , \qquad \phi_c \sim \frac{b\hbar}{j} \qquad \text{(b)} \qquad \text{—(3.4.14)}$$

Using (3.3.14), for the upper branch (3.4.14a) we have

$$V(\phi_c) \sim \frac{1}{2} m^2 \phi_c^2 \qquad \text{—(3.4.15)}$$

as in the original Euclidean example, i.e. (3.3.15a). But on the lower branch (3.4.14b) we find

$$V(\phi_c) \sim \frac{1}{2} b\hbar \ln(\phi_c^2) \qquad \text{for } \phi_c \text{ small.}$$

So the disturbing logarithmic singularity of (3.1.12) has reappeared on the lower branch, with $\delta(0)$ replaced by b.

The two branches of $V(\phi_c)$ are shown in Fig.(3.1).

As $\hbar b \to \infty$, the two branches decouple, but for $\hbar b$ finite there is the possibility of communication via instantons.

(3.5) Conclusions

In this part we have presented two mechanisms for preserving the stability of scale covariant theories, despite the presence of the $\ln(\phi^2)$ term in the classical potential due to the scale-invariant measure. First of all we considered the large-N limit of the O(N)-invariant pseudofree theory. It was shown that stability is obtained for $d > 4$ dimensions. This dimension specific result is very encouraging because it shows that the scale covariant theory makes sense just when it is obligatory. As mentioned in the introduction, the motivation for examining scale-covariant theories is to provide a criterion for non-renormalisability that is independent of the calculational scheme adopted. Hence in the canonical theory of $\lambda\phi^4$, non-renormalisability sets in for the $\hbar$ expansion for dimensions $d > 4$. Even though renormalisability can be extended to higher dimensions using the N^{-1} expansion, the philosophy of the scale covariant formalism states that the canonical theory is inherently inadequate for $d > 4$ and must be replaced by the analogous scale-covariant theory. The argument for this relies on the classical Sobolev inequalities discussed in the introduction, relating potential and kinetic terms.

The fact that the scale-covariant pseudofree theory makes sense, at least for one calculational scheme, for $d > 4$ dimensions but not for $d < 4$, is therefore encouraging. In the next part we shall be examining the large-N limit of the $\lambda(\underline{\phi}^2)^2$ theory for $d > 4$ to see if the self-interacting scale-covariant theory also is sensible.

In this section we also examined a strong coupling expansion about the IVM in an approximation scheme where only tree diagrams survived. This was seen to give a lower bound to the effective potential, although there is some ambiguity in the continuation from the Euclidean to the Minkowski theory.

As far as scale covariant theories are concerned then, we see no problems for the existence of such stable theories.

Part 4 - The Self-Interacting $\lambda(\underline{\phi}^2)^2$ Scalar Theory

(4.1) Introduction

The problem that we wish to consider is how ultraviolet divergences arise in scale-covariant theories and the circumstances under which they can be controlled. In Part 3 and Section 7 of Part 2 we examined some aspects of the pseudofree theory using the N^{-1} expansion. It was seen that the 'Hard core' ultraviolet effects due to the change of measure are controllable and renormalisation is possible for $d > 4$ dimensions, and not possible for $d < 4$ (we shall consider $d = 4$ in this section). As mentioned in concluding Part 3, scale-covariance becomes obligatory for $\lambda(\underline{\phi}^2)^2$ in $d > 4$ and so the fact that we can at least renormalise the pseudofree theory for $d > 4$ in the large-N scheme is encouraging. We shall use the large-N method to examine the self-interacting scale-covariant theory with action

$$\sigma[\phi] = \int d^d x \left\{ \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda_0}{8} \phi^4 \right\} \qquad \text{—(4.1.1)}$$

and scale-covariant measure $\mathcal{D}'[\phi]$, given by

$$\mathcal{D}'[\phi]_\beta = \frac{\mathcal{D}[\phi]}{\prod_x |\phi(x)|^\beta} \qquad \text{—(4.1.2)}$$

It has been shown *(K6)* that all $\mathcal{D}'[\phi]_\beta$ give rise to the same subtracted scale-covariant branching equations, as were studied in Part 1, i.e. (1.1.7).

Different values of β, a non-classical degree of freedom, correspond to quantising the scale-covariant theory in different ways.

In the following, we shall consider the O(N)-invariant generalisation of (4.1.1) and show how renormalisation can be performed in the large-N limit. We shall be particularly concerned for solutions in $d > 4$ dimensions where scale-covariance is obligatory due to the breakdown of the canonical theory.

(4.2) The Large-N Limit

The $\lambda(\underline{\phi}^2)^2$ scale-covariant theory is characterised by the Euclidean generating functional

$$S'[\underline{h}] = \int \mathcal{D}'[\underline{\phi}]_\beta \exp -\frac{1}{\hbar}\int d^d x \left\{ \frac{1}{2}(\nabla\underline{\phi})^2 + \frac{1}{2}m_0^2\underline{\phi}^2 + \frac{\lambda_0}{8N}(\underline{\phi}^2)^2 - \underline{h}.\underline{\phi} \right\} \qquad \text{—(4.2.1)}$$

Here, $\mathcal{D}'[\underline{\phi}]_\beta$ is both O(N)-invariant and scale-*covariant*. It must have the form

$$\mathcal{D}'[\underline{\phi}]_\beta = \prod_{i=1}^{N} \left(\frac{\mathcal{D}[\phi_i]}{\prod_x |\underline{\phi}(x)/N^{1/2}|^\beta} \right) \qquad \text{—(4.2.2)}$$

or equivalently,

$$\mathcal{D}'[\underline{\phi}]_\beta = \left(\prod_{i=1}^{N}\mathcal{D}[\phi_i]\right) \exp - \frac{N\beta}{2}\delta(0)\int d^d x \ln(\underline{\phi}^2/N) \qquad \text{—(4.2.3)}$$

This is the generalisation $\beta \neq 1$ of (3.2.4).

We now make use of the functional identities

$$\int \mathcal{D}[\rho] \exp \frac{N}{2\lambda_0 \hbar} \int d^d x \left(i\rho - \frac{\lambda_0 \underline{\phi}^2}{2N} \right)^2 = \text{constant (a)}$$

$$\int \mathcal{D}[\sigma] \, \delta(\underline{\phi}^2 - N\sigma)$$

$$= \int \mathcal{D}[\sigma] \mathcal{D}[\alpha] \exp \frac{-i}{2\hbar} \int d^d x \, \alpha (\underline{\phi}^2 - N\sigma) = \text{constant (b)}$$

—(4.2.4)

Inserting these into (4.2.1) gives

$$S'[\underline{h}] = \int \prod_i \mathcal{D}[\phi_i] \mathcal{D}[\sigma] \mathcal{D}[\alpha] \mathcal{D}[\rho] \exp -\frac{1}{\hbar} \mathcal{A}[\sigma, \alpha, \rho ; \underline{h}]$$

—(4.2.5)

where

$$\mathcal{A}[\sigma, \alpha, \rho ; \underline{h}] = \int d^d x \left\{ \frac{1}{2} \underline{\phi} (-\nabla^2 + m_0^2 + i\alpha + i\rho) . \underline{\phi} - \frac{iN\alpha\sigma}{2} + \frac{N\rho^2}{2\lambda_0} + \frac{\hbar N \beta}{2} \delta(0) \ln \sigma \right\}$$

—(4.2.6)

Performing the Gaussian $\underline{\phi}$ -integration, we obtain

$$S'[\underline{h}] = \int \mathcal{D}[\sigma] \mathcal{D}[\alpha] \mathcal{D}[\rho] \exp -\frac{1}{\hbar} \mathcal{A}'[\sigma, \alpha, \rho ; \underline{h}]$$

—(4.2.7)

where

$$\mathcal{A}'[\sigma, \alpha, \rho ; \underline{h}] = -\frac{1}{2} \int dx \, dy \, \underline{h}(x) \langle x | \frac{1}{-\nabla^2 + m_0^2 + i\alpha + i\rho} | y \rangle . \underline{h}(y) +$$

$$+ \frac{N\hbar}{2} \operatorname{Tr} \ln (-\nabla^2 + m_0^2 + i\alpha + i\rho) + \frac{N\hbar\beta\delta(0)}{2} \int dx \ln \sigma(x)$$

$$- \frac{iN}{2} \int dx \, \alpha(x) \sigma(x) + \frac{N}{2\lambda_0} \int dx \, \rho^2(x)$$

—(4.2.8)

This is the generalisation of (3.2.9) for $\beta \neq 1$ and including self-interactions. Each term in (4.2.8) is O(N) since the fields σ, α and ρ are of order 1.

As before, we assume that for $\hbar$ fixed, in the large-N limit, the path integral (4.2.7) is dominated by the single saddle point at $\sigma_0, \alpha_0, \rho_0$, satisfying

$$\frac{\delta \sigma c'}{\delta \sigma}\bigg|_{\sigma_0,\alpha_0,\rho_0} = \frac{\delta \sigma c'}{\delta \alpha}\bigg|_{\sigma_0,\alpha_0,\rho_0} = \frac{\delta \sigma c'}{\delta \rho}\bigg|_{\sigma_0,\alpha_0,\rho_0} = 0 \qquad \text{—(4.2.9)}$$

Then, to leading order in the N^{-1} expansion,

$$S'[\underline{h}] = \exp \frac{1}{\hbar} W[\underline{h}] = \exp -\frac{1}{\hbar} \sigma c'[\sigma_0,\alpha_0,\rho_0,\underline{h}] \qquad \text{—(4.2.10)}$$

and hence

$$W[\underline{h}] = -\sigma c'[\sigma_0[\underline{h}],\alpha_0[\underline{h}],\rho_0[\underline{h}];\underline{h}] \qquad \text{—(4.2.11)}$$

The semi-classical field $\underline{\phi}_c$ is given by

$$\phi_c^i = \delta W[\underline{h}] / \delta h_i$$

$$\phi_c^i = -\frac{\delta \sigma c'[\sigma_0,\alpha_0,\rho_0;\underline{h}]}{\delta h_i} \quad \text{(total derivative)}$$

$$= -\left\{ \frac{\delta \sigma c'}{\delta \sigma_0}\frac{\delta \sigma_0}{\delta h_i} + \frac{\delta \sigma c'}{\delta \alpha_0}\frac{\delta \alpha_0}{\delta h_i} + \frac{\delta \sigma c'}{\delta \rho_0}\frac{\delta \rho_0}{\delta h_i} + \frac{\delta \sigma c'}{\delta h_i} \right\}$$

$$= -\frac{\delta \sigma c'}{\delta h_i} \quad \text{(partial derivative)}$$

$$= \langle x | \frac{1}{-\nabla^2 + m_0^2 + i\alpha_0 + i\rho_0} | x \rangle h_i \qquad \text{—(4.2.12)}$$

or $\left\{-\nabla_x^2+m_0^2+i\alpha_0(x)+i\rho_0(x)\right\}\phi_c^i(x)=h_i(x)$ —(4.2.13)

where, in deriving (4.2.12), we made use of the constraints (4.2.9).

The effective action is the Legendre transform of $W[\underline{h}]$ w.r.t. $\underline{\phi_c}$.

$$\Gamma[\underline{\phi}_c] = W[\underline{h}] - \int \underline{\phi}_c\cdot\underline{h}$$

$$= -\alpha'[\sigma_0,\alpha_0,\rho_0;\underline{h}] - \int\underline{\phi}_c\cdot\underline{h} \qquad \text{—(4.2.14)}$$

Writing α' explicitly from (4.2.8) and using the result

$$\int dx\,\underline{\phi}_c\cdot\underline{h} = \int dx\,\underline{\phi}_c\left(-\nabla^2+m_0^2+i\alpha_0+i\rho_0\right)\cdot\underline{\phi}_c \qquad \text{—(4.2.15)}$$

we obtain

$$\Gamma[\underline{\phi}_c] = -\frac{1}{2}\int dx\,\underline{\phi}_c(x)\left(-\nabla_x^2+m_0^2+i\alpha_0(x)+i\rho_0(x)\right).\,\underline{\phi}_c(x)$$

$$-\frac{N\hbar}{2}\,\mathrm{Tr}\ln\left(-\nabla^2+m_0^2+i\alpha_0+i\rho_0\right)-\frac{N\hbar\beta}{2}\delta(0)\int dx\ln\sigma_0(x)$$

$$+\frac{iN}{2}\int dx\,\sigma_0(x)\alpha_0(x)-\frac{N}{2\lambda_0}\int dx\,\rho_0^2(x) \qquad \text{—(4.2.16)}$$

We can define the generalised effective potential $\Gamma[\sigma,\chi,\rho;\underline{\phi}_c]$ in terms of the *independent* auxiliary fields σ, χ, ρ where we define

$$\chi = m_0^2 + i\alpha + i\rho \qquad \text{—(4.2.17)}$$

Then

$$\Gamma[\sigma, \chi, \rho ; \underline{\phi}_c] = - \int d^d x \left\{ \tfrac{1}{2}(\nabla \underline{\phi}_c)^2 + \tfrac{1}{2}\chi \underline{\phi}_c^2 + \frac{N\hbar\beta\delta(0)}{2} \ln\sigma(x) - \frac{N\sigma(x)}{2}(\chi - m_0^2 - i\rho) + \frac{N\rho^2(x)}{2\lambda_0} \right\} - \frac{N\hbar}{2} \mathrm{Tr} \ln(-\nabla^2 + \chi)$$

—(4.2.18)

We obtain $\Gamma[\underline{\phi}_c]$ from $\Gamma[\sigma, \chi, \rho; \underline{\phi}_c]$ by imposing constraints analogous to (4.2.9), i.e.

$$\left.\frac{\delta\Gamma}{\delta\sigma}\right|_{\sigma_0,\chi_0,\rho_0} = \left.\frac{\delta\Gamma}{\delta\chi}\right|_{\sigma_0,\chi_0,\rho_0} = \left.\frac{\delta\Gamma}{\delta\rho}\right|_{\sigma_0,\chi_0,\rho_0} = 0 \qquad \text{—(4.2.19)}$$

The generalised effective potential is obtained from $\Gamma[\sigma, \chi, \rho ; \underline{\phi}_c]$ by restricting to constant (x-independent) field configurations, $\underline{\phi}_c$. Then

$$\Gamma[\sigma, \chi, \rho ; \underline{\phi}_c] = \Gamma(\sigma, \chi, \rho ; \underline{\phi}_c) = -V(\sigma, \chi, \rho ; \underline{\phi}_c) \cdot \int dx \qquad \text{—(4.2.20)}$$

From (4.2.18) we find

$$V(\sigma, \chi, \rho ; \underline{\phi}_c) = \tfrac{1}{2}\chi \underline{\phi}_c^2 + \frac{N\hbar\beta\delta(0)}{2}\ln\sigma - \frac{N\sigma}{2}(\chi - m_0^2 - i\rho) + \frac{N\rho^2}{2\lambda_0} + \frac{N\hbar}{2}\int dk \ln(k^2 + \chi)$$

—(4.2.21)

The true effective potential $V(\underline{\phi}_c)$ is found from (4.2.21) by imposing constraints analogous to (4.2.9) and (4.2.19), i.e.

$$\left.\frac{\partial V}{\partial\sigma}\right|_{\sigma_0,\chi_0,\rho_0} = \left.\frac{\partial V}{\partial\chi}\right|_{\sigma_0,\chi_0,\rho_0} = \left.\frac{\partial V}{\partial\rho}\right|_{\sigma_0,\chi_0,\rho_0} = 0 \qquad \text{—(4.2.22)}$$

These give the constraint equations

$$\chi_0 - m_0^2 - i\rho_0 - \frac{\beta\hbar\,\delta(0)}{\sigma_0} = 0 \qquad \text{(a)}$$

$$\underline{\phi}_c^2/N - \sigma_0 + \hbar\int \frac{\bar{d}k}{k^2+\chi_0} = 0 \qquad \text{(b)}$$

$$i\sigma_0 + \frac{2\rho_0}{\lambda_0} = 0 \qquad \text{(c)} \qquad \text{—(4.2.23)}$$

(4.3) Renormalisation and the Mass-Gap Equation

By experience, we know that if the effective potential, $V(\underline{\phi}_c)$, can be renormalised, then so can the effective action, $\Gamma[\underline{\phi}_c]$ and hence all Green functions.

As a stepping stone to renormalising the effective potential, we derive the so-called 'mass-gap' equation, the equation determining the ground state of the theory. If this can be renormalised, there is a good chance that $V(\underline{\phi}_c)$ can also; if it cannot be renormalised, then neither can $V(\underline{\phi}_c)$.

Using (4.2.23b) and (4.2.23c) we can write (4.2.23a) as

$$\begin{aligned}\chi_0 &= m_0^2 + \frac{\sigma_0\lambda_0}{2} + \frac{\beta\hbar\,\delta(0)}{\sigma_0} \\ &= m_0^2 + \frac{\lambda_0}{2}\left(\underline{\phi}_c^2/N + \hbar\, G(0,\chi_0)\right) \\ &\qquad + \beta\hbar\,\delta(0)\left(\underline{\phi}_c^2/N + \hbar\, G(0,\chi_0)\right)^{-1}\end{aligned} \qquad \text{—(4.3.1)}$$

where

$$G(xy\,;m^2) = \int \bar{d}k \frac{\exp -ik(x-y)}{k^2+m^2} \qquad \text{—(4.3.2)}$$

is the free Euclidean propagator for a scalar field of mass m .

To interpret this equation, we first observe that the true vacuum of the large-N theory satisfies

$$\frac{dV(\underline{\phi}_c)}{d\phi_c^i} = \frac{\partial V(\sigma_0, \chi_0, \rho_0; \underline{\phi}_c)}{\partial \phi_c^i} = \chi_0 \phi_c^i = 0 \qquad \text{—(4.3.3)}$$

If there is no symmetry breaking (the vacuum has O(N) symmetry) then $\phi_c^i = 0$ and the $\underline{\phi}$ -propagator is given by

$$\mathcal{D}_{ij}(p^2) = \frac{\delta_{ij}}{p^2 + \chi_0(0)} \qquad \text{—(4. 3.4)}$$

where

$$\chi_0(0) = m_0^2 + \frac{\hbar\lambda_0}{2} G(0, \chi_0(0)) + \frac{\beta\,\delta(0)}{G(0, \chi_0(0))} \qquad \text{—(4.3.5)}$$

If there is symmetry breaking, then $\chi_0 = 0$ for the Goldstone mode. For the unbroken degrees of freedom, $\chi_0(0)$ is still the common (mass)^2. Equation (4.3.1) is called the 'mass-gap' equation.

Now for the canonical theory, for which $\beta = 0$, (4.3.5) becomes

$$\chi_0(0) = m_0^2 + \frac{\hbar\lambda_0}{2} G(0, \chi_0(0)) \qquad \text{—(4.3.6)}$$

This is the conventional large-N mass gap equation *(C3)*. On the other hand, for $\beta = 1$, $\lambda_0 = 0$ we have the self-consistent mass-renormalisation equation arising from the 'hard-core' effects of changing the measure

$$\chi_0(0) = m_0^2 + \frac{\delta(0)}{G(0, \chi_0(0))} \qquad \text{—(4.3.7)}$$

This was discussed in detail in Part 2.

From (4.3.5) we see then, that for the self-interacting scale-covariant theory, the singular contributions from (i) the self-interaction, and (ii) the change of measure, are additive. We suppose this to be the case only if we organise the diagrams as in the large-N limit. We shall examine the $\underline{\phi}_c$ -dependent mass-gap equation (4.3.1) for renormalisability, but first let us look at the more simplified equation (4.3.5), for which $\underline{\phi}_c = 0$.

It is convenient to regularise the Euclidean theory by imposing the ultra-violet cut-off $|\underline{k}| < \Lambda$. Defining the regularised quantities,

$$\delta(0)_\Lambda = \int^{|\underline{k}|<\Lambda} đk \qquad \text{(a)}$$

$$G(0,\chi)_\Lambda = \int^{|\underline{k}|<\Lambda} \frac{đk}{k^2+\chi} \qquad \text{(b)} \qquad \text{—(4.3.8)}$$

we form the Table (4.1) of relevant quantities.

The mass-gap equation (4.3.5) then takes the form

$$\chi_0(0) = m_0^2 + \hbar\lambda_0\, O(\Lambda^{d-2}) + \beta\, O(\Lambda^2) \qquad \text{—(4.3.9)}$$

Now for $\lambda(\underline{\phi}^2)^2$ theory, the scale-covariant formalism is obligatory for d > 4 dimensions. The argument for this is independent of the calculational scheme adopted, depending on the classical Sobolev inequalities between potential and kinetic terms of the action (*K1*). This was discussed in the introduction. From (4.3.9) we see that it is just for d > 4 that the ultraviolet singularities due to the "hard-core" change of measure are *less singular* than those of the self-interaction.

If, therefore, we can control these latter singularities, there is a good chance that the 'hard-core' singularities can also be controlled, and hence the scale-covariant theory can be renormalised. For the canonical theory, the N^{-1} expansion organises the ultraviolet singularities for $\lambda(\underline{\phi}^2)^2$ such that the theory is renormalisable in $d < 6$ dimensions. So we expect the scale-covariant theory to be renormalised in the N^{-1} expansion for $4 < d < 6$ dimensions.

For $d < 4$, we see that from (4.3.9) the 'hard-core' singularities are *more* singular than those due to the self-interaction. From our analysis of the pseudofree theory ($\lambda_0 = 0$ for the scale-covariant formalism) Part 3, Section 2, we saw that renormalisation was not possible for $d < 4$ dimensions. For $d < 4$ *and* $\lambda_0 \neq 0$ we are not very interested because an acceptable canonical theory exists.

We shall now study the $\underline{\phi}_c$ -dependent mass-gap equation (4.3.1) and test its renormalisability for different dimensions.

(a) $d > 6$ Dimensions

Equation (4.3.1) can be written as

$$\chi_0/\lambda_0 = m_0^2/\lambda_0 + \tfrac{1}{2}\underline{\phi}_c^2/N + \tfrac{\hbar}{2} a_d \left\{ \Lambda^{d-2} + b_d \Lambda^{d-4}\chi_0 + c_d \Lambda^{d-6}\chi_0^2 + \right.$$
$$\left. + \cdots \right\}$$
$$+ \frac{\beta A_d}{\lambda_0}\left(\Lambda^2 - b_d \chi_0 + \cdots\right) \qquad c_d \neq 0 \qquad \text{—(4.3.10)}$$

The presence of the divergent term $\Lambda^{d-6}\chi_0^2$ makes it impossible to renormalise (4.3.10). This is because, although the $\Lambda^{d-4}\chi_0$ can be combined with the other factors involving χ_0 , and a suitable renormalisation of λ_0 found, this cannot be done also for factors involving χ_0^2. Notice that this term arises from the self-interaction and hence only for the pseudofree theory $(\lambda_0 = 0)$ is $d > 6$ renormalisable.

(b) $d = 6$ Dimensions

We now have

$$\frac{\chi_0}{\lambda_0} = \frac{m_0^2}{\lambda_0} + \frac{1}{2}\frac{\phi_c^2}{N} + \frac{\hbar}{2} a_6 \left\{ \Lambda^4 + b_6 \Lambda^2 \chi_0 + c_6 \chi_0^2 \ln \frac{\chi_0}{\Lambda^2} + \cdots \right\}$$
$$+ \frac{\beta A_6}{\lambda_0} \left(\Lambda^2 - b_6 \chi_0 + \cdots \right) \qquad \text{—(4.3.11)}$$

Again, the presence of the $\chi_0^2 \ln \frac{\chi_0}{\Lambda^2}$ term makes it impossible to renormalise (4.3.11) when $\lambda_0 \neq 0$. So far then, we have fulfilled our expectations of the scale-covariant theory not being renormalisable in the N^{-1} expansion for $d \geq 6$ dimensions.

(c) $d = 5$ Dimensions

$$\frac{\chi_0}{\lambda_0} = \frac{m_0^2}{\lambda_0} + \frac{1}{2}\frac{\phi_c^2}{N} + \frac{\hbar}{2}\frac{\Lambda^3}{36\pi^3}\left(1 - \frac{3\chi_0}{\Lambda^2} + \frac{3\pi \chi_0^{3/2}}{2\Lambda^3} + O(\Lambda^{-4}) \right)$$
$$+ \frac{\beta}{\lambda_0}\left(\frac{3}{5}\Lambda^2 + \frac{9}{5}\chi_0 - \frac{9}{10}\frac{\pi \chi_0^{3/2}}{\Lambda} + O(\Lambda^{-2}) \right)$$
$$- \frac{\beta}{\lambda_0}\frac{\phi_c^2}{\hbar N}\left(\frac{108\pi^3}{5\Lambda} + O(\Lambda^{-2}) \right)$$
$$\text{—(4.3.12)}$$

i.e.

$$\chi_o\left(\frac{1}{\lambda_o}\left(1-\frac{9}{5}\beta\right)+\frac{\hbar\Lambda}{24\pi^3}\right) = \left(\frac{m_o^2}{\lambda_o}+\frac{\hbar\Lambda^3}{72\pi^3}+\frac{3\beta\Lambda^2}{5\lambda_o}\right) +$$

$$+\frac{\phi_c^2}{2N}\left(1-\frac{216\pi^3}{5\hbar\lambda_o\Lambda}\beta\right)$$

$$+\frac{\hbar\chi_o^{3/2}}{48\pi^2}\left(1-\frac{216\pi^3}{5\lambda_o\Lambda\hbar}\beta\right) \qquad \text{—(4.3.13)}$$

Defining the renormalised mass, m, and coupling constant λ by

$$\frac{1}{\lambda} = \left\{\frac{1}{\lambda_o}\left(1-\frac{9\beta}{5}\right)+\frac{\hbar\Lambda}{24\pi^3}\right\}\left(1-\frac{216\beta\pi^3}{5\hbar\Lambda\lambda_o}\right)^{-1} \qquad \text{—(4.3.14)}$$

$$\frac{m^2}{\lambda} = \left(\frac{m_o^2}{\lambda_o}+\frac{\hbar\Lambda^3}{72\pi^3}+\frac{3\beta\Lambda^2}{5\lambda_o}\right)\left(1-\frac{216\beta\pi^3}{5\hbar\lambda_o\Lambda}\right)^{-1} \qquad \text{—(4.3.15)}$$

we then obtain the finite mass-gap equation

$$\frac{\chi_o}{\lambda} = \frac{m^2}{\lambda}+\frac{1}{2}\frac{\phi_c^2}{N}+\frac{\hbar}{48\pi^2}\chi_o^{3/2} \qquad \text{—(4.3.16)}$$

Note that $\hbar\chi_o^{3/2}/48\pi^2 = \frac{\hbar}{2}G_1(0,\chi_o)_F$ where $G_1(0,\chi_o)_F$ is the finite part of $G_1(0,\chi_o)$. Equation (4.3.16) has no explicit β -dependence and hence is just the form obtained had we not changed the measure (*R1*).

(d) d = 4 Dimensions

Now we have

$$\frac{x_0}{\lambda_0} = \frac{m_0^2}{\lambda_0} + \frac{1}{2}\phi_c^2/N + \frac{\hbar}{2}\cdot\frac{1}{16\pi^2}\left(\Lambda^2 + x_0 \ln\left(\frac{x_0}{\Lambda^2}\right) + O(\Lambda^{-2})\right)$$

$$+ \frac{\beta}{\lambda_0}\left(\frac{\Lambda^2}{2} - \frac{x_0}{2}\ln\left(\frac{x_0}{\Lambda^2}\right) + O(\Lambda^{-2})\right)$$

$$- \frac{\beta\phi_c^2}{\lambda_0 \hbar N}\left(8\pi^2 + O(\Lambda^{-2})\right) + \frac{\beta}{\lambda_0}\left(\phi_c^2/\hbar N\right)^2 O(\Lambda^{-2})$$

—(4.3.17)

We now introduce the subtraction point at $p^2 = M^2$ such that $\ln\frac{x_0}{\Lambda^2} = \ln\frac{x_0}{M^2} + \ln\frac{M^2}{\Lambda^2}$. Then

$$x_0\left\{\frac{1}{\lambda_0}\left(1 - \frac{\beta}{2}\ln\left(\frac{\Lambda^2}{M^2}\right)\right) + \frac{\hbar}{32\pi^2}\ln\frac{\Lambda^2}{M^2}\right\} = \left(\frac{m_0^2}{\lambda_0} + \frac{\hbar\Lambda^2}{32\pi^2} + \frac{\beta\Lambda^2}{2\lambda_0}\right) +$$

$$+ \frac{\phi_c^2}{2N}\left(1 - \frac{16\pi^2}{\hbar\lambda_0}\beta\right) +$$

$$+ \frac{\hbar}{32\pi^2}\left(1 - \frac{16\pi^2}{\hbar\lambda_0}\beta\right)x_0 \ln\frac{x_0}{M^2} \qquad \text{—(4.3.18)}$$

Defining renormalised mass and coupling via

$$\frac{1}{\lambda} = \left\{\frac{1}{\lambda_0}\left(1 - \frac{\beta}{2}\ln\frac{\Lambda^2}{M^2}\right) + \frac{\hbar}{32\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)\right\}\left(1 - \frac{16\pi^2}{\hbar\lambda_0}\beta\right)^{-1} \qquad \text{—(4.3.19a)}$$

$$\frac{m^2}{\lambda} = \left\{\frac{m_0^2}{\lambda_0} + \frac{\hbar\Lambda^2}{32\pi^2} + \frac{\beta\Lambda^2}{2\lambda_0}\right\}\left(1 - \frac{16\pi^2}{\hbar\lambda_0}\beta\right)^{-1} \qquad \text{—(4.3.19b)}$$

This gives

$$\frac{x_0}{\lambda} = \frac{m^2}{\lambda} + \frac{\phi_c^2}{2N} + \frac{\hbar}{32\pi^2}x_0 \ln\frac{x_0}{M^2} \qquad \text{—(4.3.20)}$$

Once again, this is β -independent and hence is the same as for the canonical theory. Also,

$\frac{\hbar}{32\pi^2}\chi_0 \ln \frac{\chi_0}{M^2} = \frac{\hbar}{2} G(0,\chi_0)_F$ where $G(0,\chi_0)_F$ is the finite part of $G(0,\chi_0)$.

From (4.3.19a) rearranging gives

$$\lambda = \frac{\lambda_0 - \frac{16\pi^2}{\hbar}\beta}{1 - \frac{\beta}{2}\ln\left(\frac{\Lambda^2}{M^2}\right) + \frac{\lambda_0\hbar}{32\pi^2}\ln\left(\frac{\Lambda^2}{M^2}\right)} \qquad \text{—(4.3.21)}$$

So, setting $\lambda_0 = 0$ implies $\lambda = 0$. This means that the pseudo-free gap equation can be renormalised for $d = 4$ dimensions.

(e) $d = 3$ Dimensions

For this

$$\begin{aligned}\frac{\chi_0}{\lambda_0} = {} & \frac{m_0^2}{\lambda_0} + \frac{\phi_c^2}{2N} + \frac{\hbar}{4\pi^2}\left(\Lambda - \frac{\pi\chi_0^{1/2}}{2}\right) + \beta\left(\frac{\Lambda^2}{3} + \frac{\pi\Lambda\chi_0^{1/2}}{6} + \frac{\pi^2}{12}\chi_0 + O(\Lambda^{-1})\right) \\ & - \frac{\beta}{\lambda_0}\frac{\phi_c^2}{\hbar N}\left(\frac{2\pi^2}{3}\Lambda + \frac{2\pi^2}{3}\chi_0^{1/2} + O(\Lambda^{-1})\right) + \\ & + \frac{\beta}{\lambda_0}\left(\phi_c^2/\hbar N\right)^2\left(\frac{4\pi^4}{3} + O(\Lambda^{-1})\right)\end{aligned} \qquad \text{—(4.3.22)}$$

i.e.

$$\begin{aligned}\frac{\chi_0}{\lambda_0}\left(1 - \frac{\beta\pi^2}{12}\right) = {} & \left(\frac{m_0^2}{\lambda_0} + \frac{\hbar\Lambda}{4\pi^2} + \frac{\beta\Lambda^2}{3\lambda_0}\right) + \frac{\phi_c^2}{2N}\left(1 - \frac{4\beta\pi^2\Lambda}{3\hbar\lambda_0}\right) - \\ & - \frac{\hbar}{8\pi}\chi_0^{1/2}\left(1 - \frac{4\beta\Lambda\pi^2}{3\hbar\lambda_0}\right) + \frac{4\pi^4}{3\lambda_0\hbar^2}\beta\left(\frac{\phi_c^2}{N}\right)^2 - \\ & - \frac{2\beta\pi^2}{3\lambda_0\hbar}\chi_0^{1/2}\,\phi_c^2/N .\end{aligned}$$

—(4.3.23)

Defining the renormalised parameters by

$$\frac{1}{\lambda} = \frac{1}{\lambda_0}\left(1 - \frac{\beta\pi^2}{12}\right)\left(1 - \frac{4\beta\pi^2}{3\hbar\lambda_0}\Lambda\right)^{-1} = \frac{(1 - \beta\pi^2/12)}{\left(\lambda_0 - \frac{4\beta\pi^2}{3\hbar}\Lambda\right)} \qquad \text{—(4.3.24)}$$

and $$\frac{m^2}{\lambda} = \left(\frac{m_0^2}{\lambda_0} + \frac{\hbar\Lambda}{4\pi^2} + \frac{\beta\Lambda^2}{3\lambda_0}\right)\left(1 - \frac{4\beta\pi^2}{3\hbar\lambda_0}\Lambda\right)^{-1} \qquad \text{—(4.3.25)}$$

From (4.3.24) we see that

$$\lambda_0 = \left(\frac{4\beta\pi^2}{3\hbar}\right)\Lambda + \lambda\left(1 - \frac{\pi\beta}{12}\right) \qquad \text{—(4.3.26)}$$

On taking the limit $\Lambda \to \infty$, we see that the term $\frac{\chi_0^{1/2}}{\lambda_0}\frac{\phi_c^2}{N}$ in (4.3.23) vanishes as does the term $\frac{1}{\lambda_0}(\phi_c^2/N)^2$.
Thus we obtain the finite equation

$$\frac{\chi_0}{\lambda} = \frac{m^2}{\lambda} + \frac{\phi_c^2}{2N} - \frac{\hbar}{8\pi}\chi_0^{1/2} \qquad \text{—(4.3.27)}$$

From (4.3.26) we can see that setting $\lambda_0 = 0$ implies $\lambda = O(\Lambda)$. Thus it is not possible to renormalise the pseudofree theory in d = 3 dimensions, as we mentioned in Part 3, Section 2. It seems that the self-interaction softens the 'hard-core' due to the change of measure, and makes it controllable.

(f) d = 2 Dimensions

This gives

$$\frac{\chi_0}{\lambda_0} = \frac{m_0^2}{\lambda_0} + \frac{1}{2}\frac{\phi_c^2}{N} + \frac{\hbar}{8\pi}\ln\left(\Lambda^2/\chi_0\right) + \frac{\beta\Lambda^2}{\lambda_0 \cdot 4\pi}\left(\frac{\phi_c^2}{\hbar N} + \frac{1}{4}\ln\left(\frac{\Lambda^2}{\chi_0}\right)\right)^{-1} \qquad \text{—(4.3.28)}$$

It is not possible to renormalise this equation; so our analysis of the renormalisability of the ϕ_c^2 -dependent mass-gap equation (4.3.1) has fulfilled our earlier expectations of being renormalisable for $4 \leq d < 6$ and is, in fact, renormalisable for $2 < d < 6$.

Our next step is to examine the renormalisability of the effective potential $V(\phi_c)$ of (4.2.21) (with $\sigma = \sigma_0$, $\chi = \chi_0$, and $\rho = \rho_0$) using the definitions of renormalised parameters m^2 and λ in (c), (d) and (e) above.

As a final note, consider the Independent Value Model (IVM) for which

$$G(0,x) = \frac{\delta(0)}{x} \qquad \text{—(4.3.29)}$$

Equation (4.3.1) becomes for this

$$\frac{m_0^2}{\lambda_0} + \frac{1}{2}\frac{\phi_c^2}{N}\left(1 - \frac{2x^2}{\lambda_0 \hbar\,\delta(0)} + \dots\right) + \frac{1}{2}\frac{\hbar\,\delta(0)}{x} = 0 \qquad \text{—(4.2.30)}$$

No sense can be made of this equation by using the multiplicative renormalisation applicable to the IVM, i.e.

$$m^2 = \frac{\delta(0)}{b} m_0^2 \qquad \text{(a)}$$

$$\lambda = \frac{\delta^3(0)}{b^3} \lambda_0 \qquad \text{(b)} \qquad \text{—(4.2.31)}$$

In order to get non-trivial behaviour for the O(N)-invariant IVM in the large-N limit, we are thus obliged to adopt a different explicit N-dependence for λ_0 than is adopted here (*K7*).

(4.4.) Renormalisation of the Effective Potential

In this section we shall show how the effective potential can be renormalised in d = 3, 4 and 5 dimensions, following on from the renormalisation of the mass-gap equation (4.3.1) in these dimensions.

From (4.2.21) for $\sigma = \sigma_0$, $x = x_0$, $\rho = \rho_0$ and using the constraint equations (4.2.23), we can write $V(\underline{\phi}_c)$ solely in terms of $x_0(\underline{\phi}_c)$ and $\underline{\phi}_c$, i.e.

$$V(\underline{\phi}_c) = \frac{1}{2}\chi_0 \underline{\phi}_c^2 + \frac{N\hbar\beta}{2}\delta(0)\ln\left(\frac{\underline{\phi}_c^2}{N} + \hbar G(0,\chi_0)\right) -$$

$$- \frac{N}{2}\left(\frac{\underline{\phi}_c^2}{N} + \hbar G(0,\chi_0)\right)(\chi - m_0^2) + \frac{\lambda_0 N}{8}\left(\underline{\phi}_c^2/N + \hbar G(0,\chi_0)\right)^2 +$$

$$+ \frac{N\hbar}{2}\int dk \ln(k^2 + \chi_0) \qquad \text{—(4.4.1)}$$

We can subtract off $\underline{\phi}_c$ -independent quantities.

For the relevant dimension, we use the cut-off forms of Table 4.1, the definitions of the renormalised parameters in the previous section, and the renormalised mass-gap equation common to all d,

$$\frac{\chi_0}{\lambda} = \frac{m^2}{\lambda} + \frac{\underline{\phi}_c^2}{2N} + \frac{\hbar}{2} G_F(0,\chi_0) \qquad \text{—(4.4.2)}$$

where G_F is the *finite* part of $G(0,\chi_0)$.

(a) d = 5 Dimensions

From (4.2.35) and (4.2.36) we find

$$\lambda_0 = \left\{\lambda(1 - 9\beta/5) + \frac{216\beta\pi^3}{5\hbar\Lambda}\right\}\left(1 - \frac{\lambda\hbar\Lambda}{24\pi^3}\right)^{-1} \qquad \text{(a)}$$

$$= O(\Lambda^{-1})$$

$$m_0^2 = m^2(1 - 9\beta/5) - \frac{3}{5}\beta\Lambda^2 - \frac{\hbar\Lambda}{72\pi^3}(\Lambda^2 - 3m^2)\lambda_0 \qquad \text{(b)}$$

$$= O(\Lambda^2) \qquad \text{—(4.4.3)}$$

The next step is to insert (4.4.3) into (4.4.1) and use Table 4.1 for the cut-off forms. We also use the renormalised gap equation (4.4.2). We will not give the details of this simple, but lengthy, calculation. The end result is

$$V(\underline{\phi}_c) = \frac{1}{2}\chi_o(\underline{\phi}_c)\underline{\phi}_c^2 - \frac{N\chi_o^2}{2\lambda} + \frac{Nm^2\chi_o}{\lambda} + \frac{N\hbar.5}{24\pi^2}\chi_o^{5/2} \qquad \text{—(4.4.4)}$$

Note that $\frac{N\hbar.5}{24\pi^2}\chi_o^{5/2} = \frac{N\hbar}{2}\left[\int dk \ln(k^2+\chi_o)\right]_F$.

where $[\quad]_F$ denotes the finite part. This result is β -independent and is the conventional translation-covariant result (*R1*).

(b) d = 4 Dimensions

Using (4.2.40) and (4.2.41) we find

$$\lambda_o = \frac{\left\{\lambda\left(1-\frac{\beta}{2}\ln\left(\frac{\Lambda^2}{M^2}\right)\right)+\frac{16\pi^2}{\hbar}\beta\right\}}{\left\{1-\frac{\lambda\hbar}{32\pi^2}\ln\left(\Lambda^2/M^2\right)\right\}} \qquad \text{(a)}$$

$$= \frac{16\pi^2}{\hbar}\beta + O\left(\frac{1}{\ln\frac{\Lambda^2}{M^2}}\right)$$

$$m_o^2 = m^2 + \ln\left(\frac{\Lambda^2}{M^2}\right)\left\{\frac{\lambda_o\hbar\, m^2}{32\pi^2} - \frac{\beta m^2}{2}\right\} - \frac{\hbar\Lambda^2\lambda_o}{32\pi^2} - \frac{\beta\Lambda^2}{2} \qquad \text{(b)}$$

$$= O(\Lambda^2) \qquad \text{—(4.4.5)}$$

Repeating the procedure, we finally obtain once again, the result

$$V(\underline{\phi}_c) = \frac{1}{2}\chi_o\underline{\phi}_c^2 - \frac{N\chi_o^2}{2\lambda} + \frac{Nm^2}{\lambda}\chi_o + \frac{\hbar}{2}\left[\int dk \ln(k^2+\chi_o)\right]_F \qquad \text{—(4.4.6)}$$

where

$$\left[\int dk \, \ln (k^2 + x_o) \right]_F = \frac{x_o^2}{32\pi^2} \left(\ln \frac{x_o}{M^2} - \frac{1}{2} \right) \qquad \text{—(4.4.7)}$$

(c) $d = 3$ Dimensions

Using (4.2.46) and (4.2.47) gives

$$\lambda_o = \lambda \left(1 - \frac{\beta\pi^2}{12}\right) + \frac{4}{3\hbar}\beta\pi^2 \Lambda = O(\Lambda) \qquad (a)$$

$$m_o^2 = m^2\left(1 - \frac{\beta\pi^2}{12}\right) - \frac{\lambda_o \hbar}{4\pi^2}\Lambda - \frac{\beta}{3}\Lambda^2 = O(\Lambda^2) \qquad (b)$$

—(4.4.8)

Substituting for λ_o , m_o^2 and using the gap-equation (4.4.2) yields

$$V(\underline{\phi}_c) = \frac{1}{2} x_o \underline{\phi}_c^2 - \frac{N x_o^2}{2\lambda} + \frac{N m^2}{\lambda} x_o + \frac{N\hbar}{2} \left[\int dk \ln (k^2 + x_o) \right]_F -$$

$$- \frac{N\pi^2}{12} \beta \left\{ \frac{m^2 x_o}{\lambda} + \frac{\hbar}{24\pi} x_o^{3/2} + \frac{2\pi}{\hbar} x_o^{1/2} \left(\underline{\phi}_c^2 / N \right)^2 - \frac{8\pi^2}{3\hbar^2} \left(\underline{\phi}_c^2 / N \right)^3 \right\}$$

—(4.4.9)

where $$\left[\int dk \, \ln (k^2 + x_o) \right]_F = - \frac{x_o^{3/2}}{6\pi} \qquad \text{—(4.4.10)}$$

We now have a β *-dependent result.* The large-N limit of the effective potential is therefore only β -dependent when the 'hard-core' (due to the change of measure) is more singular than the self-interaction.

Finally, we point out that, as we showed the pseudofree theory to be stable for $d > 4$ dimensions (Part 3, Section 2) so we have shown here that the self-interacting theory is stable for $d = 3, 4, 5$ dimensions in the large-N limit.

(4.5) Conclusions

We have seen, in our analysis of the large-N limit of the $O(N)$ invariant scale-covariant $\lambda(\underline{\phi}^2)^2$ theory that in this limit the ultra-violet singularities due to the 'hard-core' are *additive* to those of the self-interaction. This is shown by equation (4.2.28). This allows us to use orthodox renormalisation techniques.

We also saw that, for the $\lambda(\underline{\phi}^2)^2$ theory, the 'hard-core' singularities are *less* singular than those of the self-interaction just for those space-time dimensions, $d > 4$, for which the scale-covariant formalism must be used. This can be shown to be a general result, i.e. suppose we have a $\lambda_0 N^{1-p}(\underline{\phi}^2)^p$ theory; the mass-gap equation (4.2.28) becomes

$$\chi_0(0) = m_0^2 + \lambda_0 G^{p-1}(0,\chi_0) + \frac{\beta\,\delta(0)}{G(0,\chi_0)} \qquad \text{—(4.5.1)}$$

where the factor $G^{p-1}(0,\chi_0)$ arises because we now use a different self-interaction. The factor $\beta\,\delta(0)/G(0,\chi_0)$ corresponds to the 'hard-core' and hence is the same as for the original theory.

In d-dimensions with an ultra-violet cut-off, (4.5.1) becomes

$$\chi_0 = m_0^2 + \lambda_0\, O(\Lambda^{(p-1)(d-2)}) + \beta\, O(\Lambda^2) \qquad \text{—(4.5.2)}$$

The 'hard'-core' is then *less* singular than the self-interaction provided that

$$\Lambda^{(p-1)(d-2)} > \Lambda^{2} \tag{4.5.3}$$

$$\text{i.e.} \quad 2p > \frac{2d}{d-2} \tag{4.5.4}$$

This is just the condition, derived from classical Sobolev inequalities, that the scale-covariant formalism is obligatory for $\lambda_0 N^{1-p}(\underline{\phi}^2)^p$ theory.

Provided that the 'hard-core' is not more singular than the self-interaction, it can be absorbed into the self-interaction. Because for $\frac{\lambda_0}{N}(\underline{\phi}^2)^2$ the large-N limit is renormalisable for $d < 6$ dimensions, this means we have a region $4 \leq d < 6$ in which we have a renormalisable scale-covariant theory. A particular consequence of the fact that the self-interaction absorbs the 'hard-core' for $4 \leq d < 6$ is that, using the measure, $\mathcal{D}'[\underline{\phi}]_\beta$, there is no β-dependence in the large-N limit.

However, for $d < 4$, the 'hard-core' is more singular and results become β-dependent.

Part 5 - General Conclusions

We shall summarise briefly here the results of our investigation of the scale-covariant quantum field theory.

In agreement with reference (*Y1*), we found that the coupled Green function equations were too degenerate to permit perturbation series solutions for the Green functions. This degeneracy was entirely due to a 'subtraction procedure' (1.1.6a), necessary when dealing with scale-covariant measures. Dropping this procedure yielded an extra dynamical equation (2.2.23) that removed the degeneracy of the branching equations allowing, in principle, unique perturbation series.

The justification for ignoring the subtraction procedure for models other than the IVM came when we studied the augmented formalism. The extra dynamical equation (2.2.22a) arose naturally. Furthermore, we showed that the IVM could not be accommodated in this formalism. This shows that the IVM must be regarded as a special case where the subtraction procedure is crucial, but this is not the case in general.

The augmented formalism, because of its use of canonical functional measures, permitted a diagrammatic expansion. We showed that retaining only the most singular single-point mass insertion diagrams was equivalent to performing a N^{-1} expansion of the analogous $O(N)$ invariant theory, and taking the large-N limit.

By studying separately the following three areas:

(a) spectral analysis of the extra dynamical equation (2.2.22a)

(b) diagrammatic expansion of the augmented theory, retaining only the most singular point mass-insertions

(c) leading singularity approximation for the augmented functional differential equations,

we showed that each yielded the same statement of mass renormalisation (2.2.32). This was also shown to be the 'hard-core' (i.e., due to scale-invariant measure) contribution to the mass-gap equation in the large-N limit of the N^{-1} expansion of the theory.

The stability of the theory was established by showing that there are two mechanisms whereby the classical instability, suggested by the augmentation, $\frac{1}{2}\delta(0)\int dx \ln \phi^2(x)$, could be avoided. These were the large-N limit of the N^{-1} expansion, and the strong-coupling limit (which is, in fact, the IVM). Both of these showed stable ground states.

Finally, we showed that scale-covariant theories are sensible, in the sense of being renormalisable, just for those dimensions where they are mandatory according to the calculation-independent criterion given in the introduction. Indeed, we showed that the ultra-violet singularities due to the self-interaction and the 'hard-core' change of measure are additive in the large-N limit. This allowed the use of orthodox renormalisation techniques. We found that the pseudofree theory made sense only for $d \geq 4$ dimensions, where it must be used, and the self-interacting theory for $3 \leq d < 6$, in the large-N limit.

As suggestions for further research, we note that although we have been able to remove the degeneracy of the scale-covariant branching equations by the inclusion of the extra dynamical equation (2.2.23), we have not actually proceeded to establish the perturbation series for the Green functions.

Also, there was some ambiguity over the continuation from

Euclidean to Minkowski space-time when examining the effective potential of the Independent Value Model.

A further area warranting study is the calculation of the non-leading terms in the N^{-1} expansion of the theory, to see if, indeed, the conclusions deduced from the large-N limit about the stability of the model are still valid.

Appendix A

The difficulty in determining the perturbation series from the scale-covariant branching equations arises because of the absence of an inhomogeneous equation, i.e. whereas the set of canonical branching equations commences with

$$\delta(x-x_1) - K_x G_2(xx_1) - 4\lambda_0 G_4(xxxx_1) = 0 \qquad \text{—(1.1.5a)}$$

the scale covariant equations commences with

$$\{\delta(x-x_1) + \delta(x-x_2)\} G_2'(x_1x_2) - K_x G_4'(xx\overset{*}{x}_1x_2) - $$
$$- 4\lambda_0 G_6'(xxxxx_1x_2) = 0 \qquad \text{—(1.1.7a)}$$

We will show that, if we were able to omit (1.1.5a), then imposing a Callan-Symanzik type equation restores it. Unfortunately, a similar situation does not occur for the scale covariant theory. For convenience, we use the ULM.

We can write (1.1.5) as, on making points coincident

$$\delta(0) - m_0^2 G_2(xx) - 4\lambda_0 G_4(xxxx) = 0 \qquad \text{(a)}$$

$$\delta(0) G_2(x_1x_2) + \delta(x-x_1) G_2(xx_2) + \delta(x-x_2) G_2(xx_1)$$
$$- m_0^2 G_4(xxx_1x_2) - 4\lambda_0 G_6(xxxxx_1x_2) = 0 \qquad \text{(b)}$$

$$\left\{\delta(0) + \sum_{r=1}^{2n} \delta(x-x_r)\right\} G_{2n}(x_1 \cdots x_{2n}) - m_0^2 G_{2n+2}(xxx_1 \cdots x_{2n}) -$$
$$- 4\lambda_0 G_{2n+4}(xxxxx_1 \cdots x_{2n}) = 0 \qquad \text{(c)}$$

$$\text{—(1.4.26)}$$

$$(n = 0, 1, 2, \ldots)$$

Now

$$\frac{\partial}{\partial(m_0^2/2)} G_{2n}(x_1 \cdots x_{2n}) = \frac{\partial}{\partial(m_0^2/2)} \left\{ N^{-1} \int \mathcal{D}[\phi]\, \phi(x_1) \ldots \phi(x_{2n}) \exp - \int dx \left(\tfrac{1}{2} m_0^2 \phi^2 + \right.\right.$$
$$\left.\left. + \lambda_0 \phi^4\right)\right\}$$

—(1.4.27)

$$= \left\{ - N^{-2} \frac{\partial}{\partial(m_0^2/2)} N \right\} \int \mathcal{D}[\phi]\, \phi(x_1) \ldots \phi(x_{2n}) \,.$$
$$\cdot \exp - \int dx \left(\tfrac{1}{2} m_0^2 \phi^2 + \lambda_0 \phi^4\right)$$

$$- N^{-1} \int dx \int \mathcal{D}[\phi]\, \phi^2(x)\, \phi(x_1) \ldots \phi(x_{2n}) \,.$$
$$\cdot \exp - \int dx \left(\tfrac{1}{2} m_0^2 \phi^2 + \lambda_0 \phi^4\right)$$

i.e. $$\frac{\partial}{\partial(m_0^2/2)} G_{2n}(x_1 \cdots x_{2n}) = \int dx\, G_2(xx)\, G_{2n}(x_1 \cdots x_{2n}) - \int dx$$

—(1.4.28)

Similarly

$$\frac{\partial}{\partial \lambda_0} G_{2n}(x_1 \cdots x_{2n}) = \int dx\, G_4(xxxx)\, G_{2n}(x_1 \cdots x_{2n}) - \int dx\, G_{2n+4}(xxxxx_1 \cdots x_{2n})$$

—(1.4.29)

Integrating (1.4.26) gives

$$2n G_{2n}(x_1 \cdots x_{2n}) - m_0^2 \int dx\, G_{2n+2}(xxx_1 \cdots x_{2n}) - 4\lambda_0 \int dx\, G_{2n+4}(xxxxx_1 \cdots x_{2n})$$
$$= - \delta(0) \int dx$$

—(1.4.30)

substituting for (1.4.28) and (1.4.29)

$$2n\, G_{2n}(x_1 \cdots x_{2n}) + m_0^2 \frac{\partial}{\partial(m_0^2/2)} G_{2n}(x_1 \cdots x_{2n}) - m_0^2 \int dx\, G_2(xx) G_{2n}(x_1 \cdots x_{2n}) +$$

$$+ 4\lambda_0 \frac{\partial}{\partial \lambda_0} G_{2n}(x_1 \cdots x_{2n}) - 4\lambda_0 \int dx\, G_4(xxxx) G_{2n}(x_1 \cdots x_{2n}) = -\delta(0) \int dx$$

i.e.

$$\left\{ 2n + m_0^2 \frac{\partial}{\partial(m_0^2/2)} + 4\lambda_0 \frac{\partial}{\partial \lambda_0} \right\} G_{2n}(x_1 \cdots x_{2n}) \qquad \text{—(1.4.31)}$$

$$= - \int dx \left\{ \delta(0) - m_0^2 G_2(xx) - 4\lambda_0 G_4(xxxx) \right\} G_{2n}(x_1 \cdots x_{2n})$$

If we then hold to a constraint

$$\left\{ 2n + m_0^2 \frac{\partial}{\partial(m_0^2/2)} + 4\lambda_0 \frac{\partial}{\partial \lambda_0} \right\} G_{2n}(x_1 \cdots x_{2n}) = 0$$

this implies

$$\delta(0) - m_0^2 G_2(xx) - 4\lambda_0 G_4(xxxx) = 0$$

which is just (1.4.26A). So we can essentially swop the inhomogeneous branching equation (1.4.26A) for this Callan-Symanzik equation.

This is not true for the scale covariant theory.

TABLE CAPTIONS

Table (2.1) The zero, one, two and three loop contributions to the two-point pseudofree function, $G_2(xy)$

(2.2) Examples from the one, two and three loop contributions to the *connected* four-point pseudofree function, $W_4(wxyz)$

(4.1) Regularised quantities needed for examining renormalisability of the mass-gap equation and effective potential.

FIGURE CAPTIONS

Fig.(1.1) The diagrammatic representation of the scale-covariant branching equations (1.1.7). Solid lines represent the ϕ-field, dashed lines represent δ-functions and blobs denote unconnected Green functions.

Fig.(2.1) The diagrammatic representation of the extra dynamical equation (2.2.23).

Fig.(2.2) The diagrammatic description of the augmented constraint equations (2.2.5). Solid lines denote the ϕ-field, wavy lines the χ-field, dashed lines are δ-functions and blobs represent unconnected Green functions.

Fig.(2.3) The diagrammatic representation of the augmented dynamical equations (2.2.6).

Fig.(2.4) Polygons describing the non-local interaction vertices $V_0^{(2n)}$ and $V_1^{(2n)}$ of the effective action $\alpha[\chi, h]$ of (2.4.11).

Fig.(2.5) The approximate factorisation equations (2.6.1), (2.6.3), (2.6.4) and (2.6.5). Solid lines represent the ϕ-field, wavy lines represent the χ-field. Blobs denote unconnected Green functions.

Fig.(3.1) The two branches of the Euclidean effective potential derived from the generating functional (3.4.10).

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(8) J.R. Klauder, Scale Covariant Quantum Field Theory, XV Winter School of Theoretical Physics, Karpacz, 146 (University of Wroclaw Press, 1978)

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(11) J.R. Klauder, "Relativity" (Plenum Press, 1970, pp.1-17)

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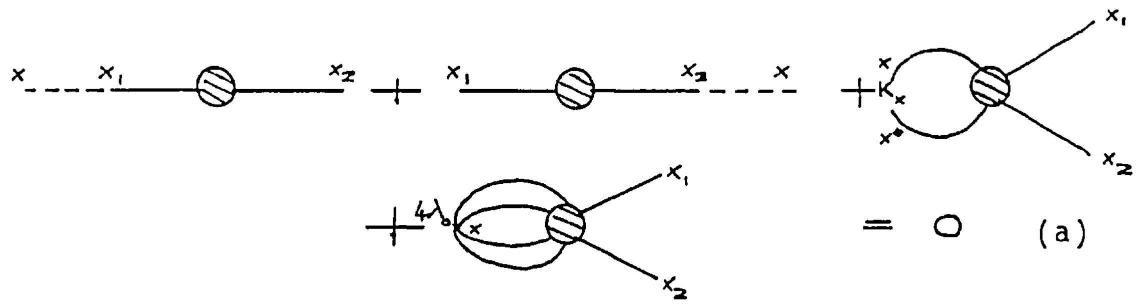
M (1) M. Marinaro, Nuovo Cim. 32A, 355 (1976)

R (1) P. Rembiesa, Phys. Rev. D 17, 2830 (1978)

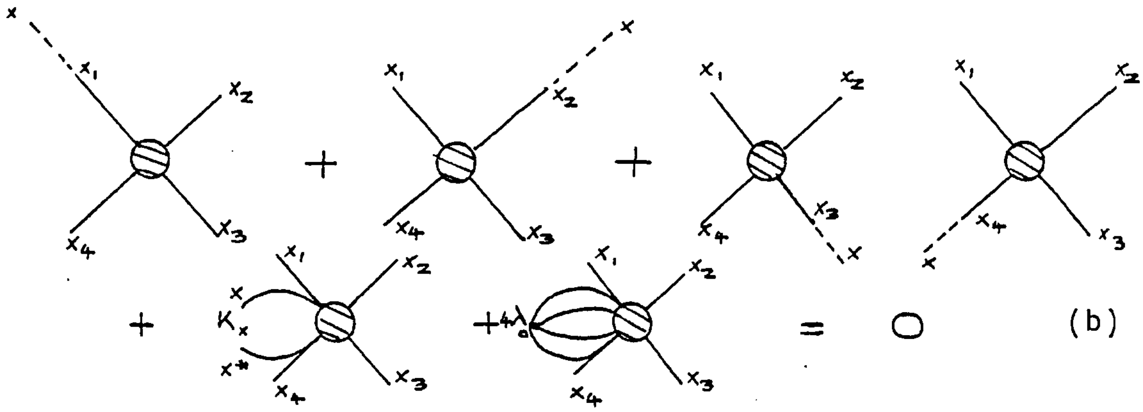
Y (1) T. Yoshimura and M. Nouri-Moghadam, Phys.Rev. D 17, 3284 (1978)

FIGURES

Fig.(1.1)



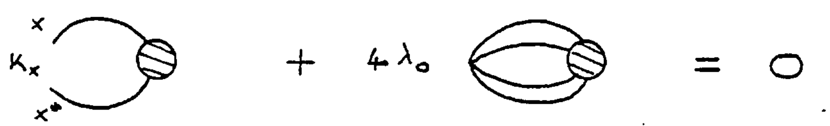
(a)



(b)

(1.1.7)

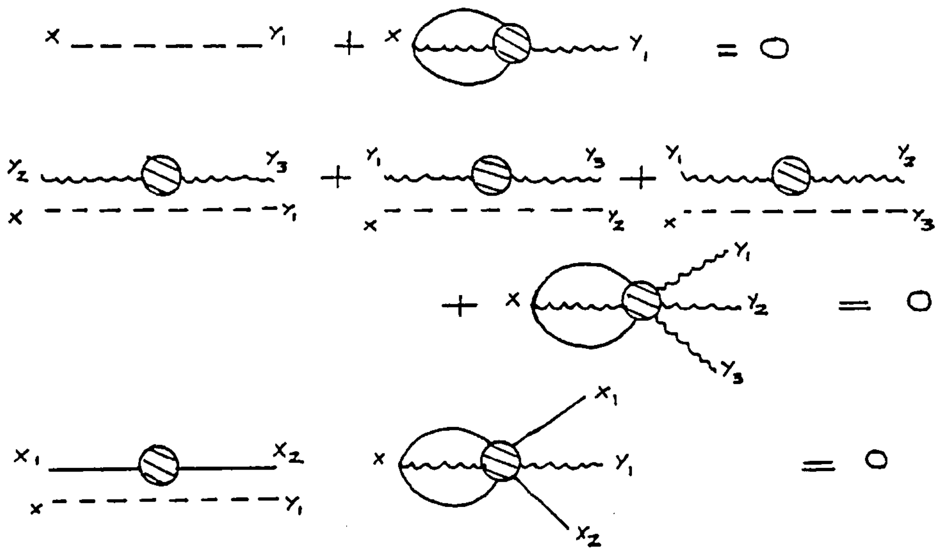
Fig.(2.1)



(2.2.23)

Fig.(2.2)

(a)
(m=0, n=1)

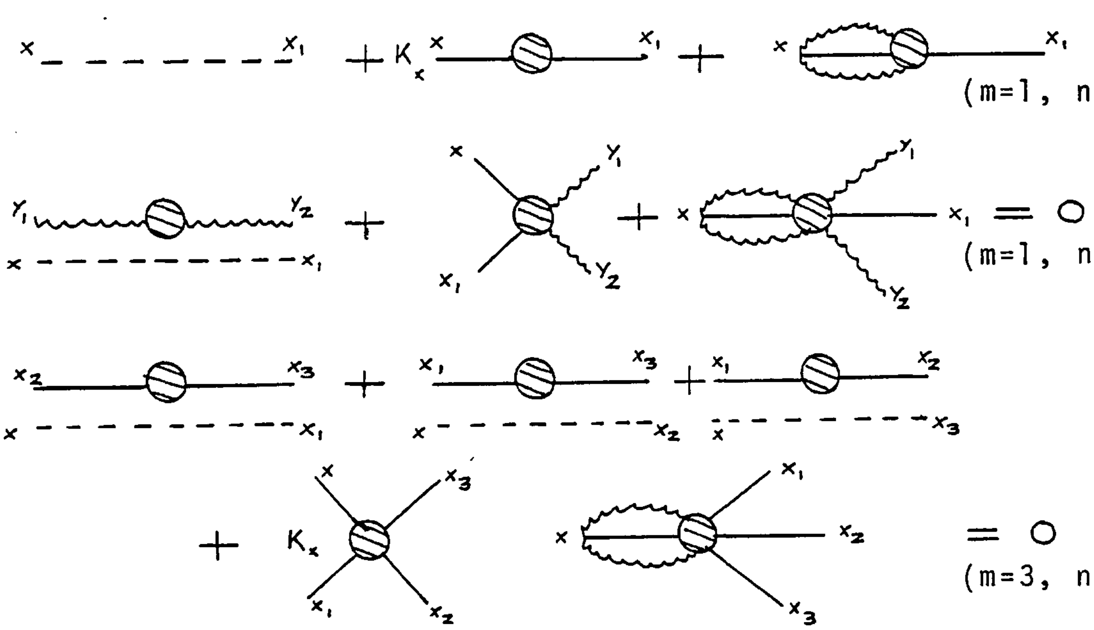


(b)
(m=0, n=3)

(c)
(m=2, n=1)

(2.2.5)

Fig.(2.3)



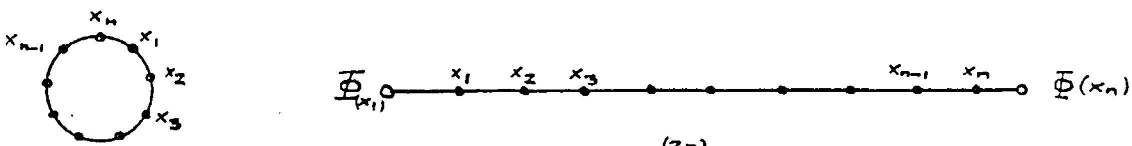
(a)
(m=1, n=0)

(b)
(m=1, n=2)

(c)
(m=3, n=0)

(2.2.6)

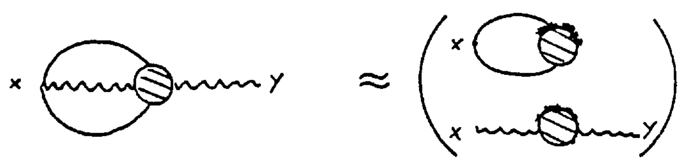
Fig.(2.4)



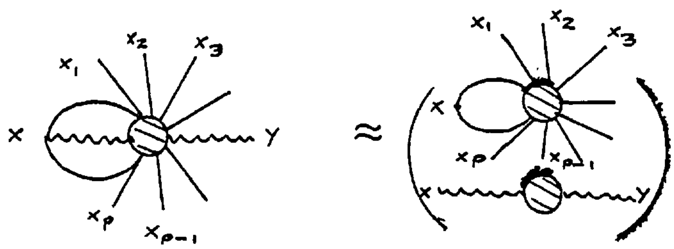
(a) $V_0^{(2n)}(x_1 \cdots x_n)$ (b) $V_1^{(2n)}(x_1 \cdots x_n; h)$

Fig.(2.5)

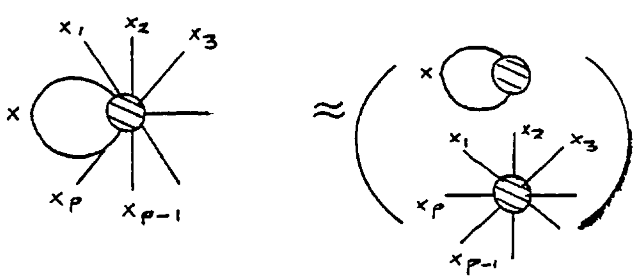
(2.6.1)



(2.6.3)



(2.6.4)



(2.6.5)

Fig.(3.1)

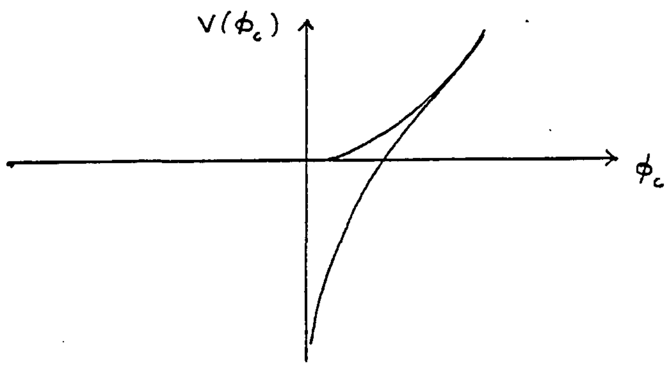
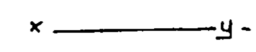
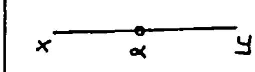
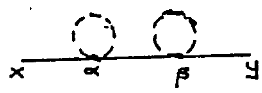
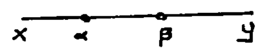
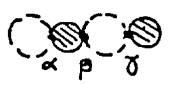
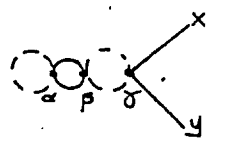
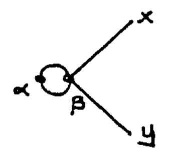
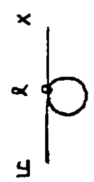
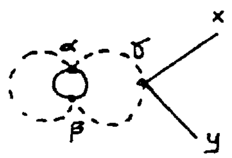
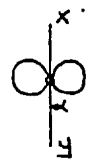
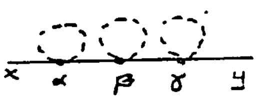
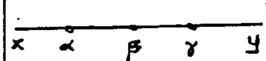
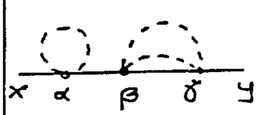
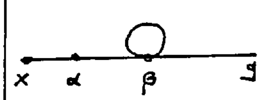
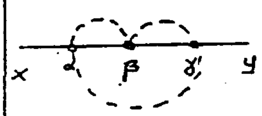
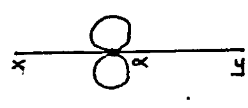
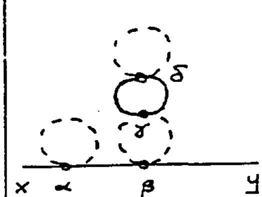
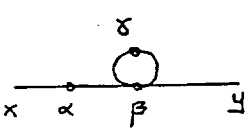
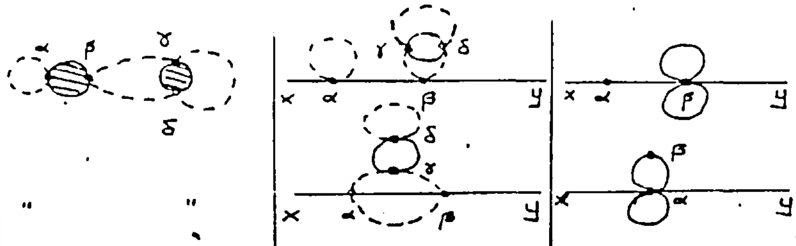


TABLE 2.1

Number of loops	Relevant diagrams of $W[h]$	$G_2(xy) = \left.\frac{-\delta^2 W[h]}{\delta h(x)\,\delta h(y)}\right\|_0$ Uncontracted	Contracted	Singularity
0				1
1				$\delta(0)/G_0(0)$
2(a)				$(\delta(0)/G_0(0))^2$
2(b)				$\frac{L_2}{G_2(0)}(\delta(0)/G_0(0))^2$
2(c)				$\delta(0)/G_0(0)$
2(d)				$\delta(0)/G_0(0)$
3(a)				$(\delta(0)/G_0(0))^3$
3(b)				$(\delta(0)/G_0(0))^2$
3(c)				$\delta(0)/G_0(0)$
3(d)				$\frac{L_2}{G_2(0)}(\delta(0)/G_0(0))^2$

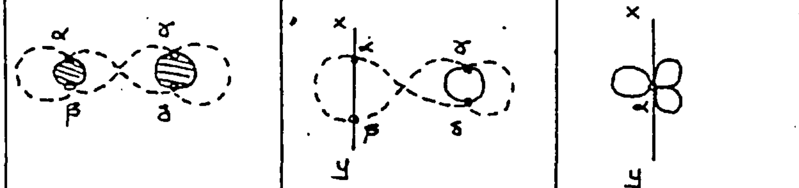
3(e)



$\left(\delta(0)/G_0(0)\right)^2$

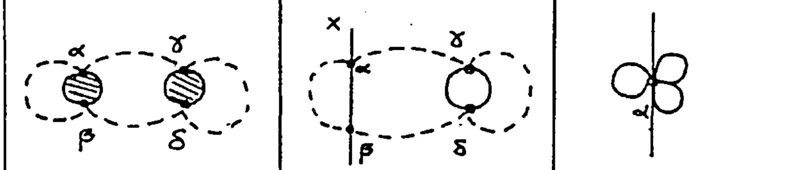
$\frac{L_2}{G_0(0)}\left(\delta(0)/G_0(0)\right.$

3(f)



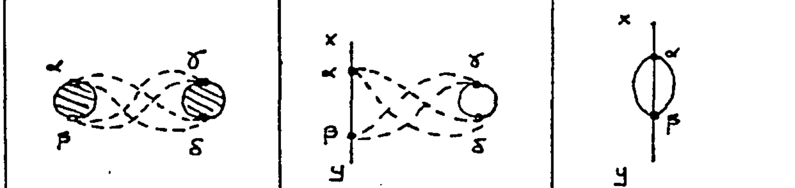
$\delta(0)/G_0(0)$

3(g)



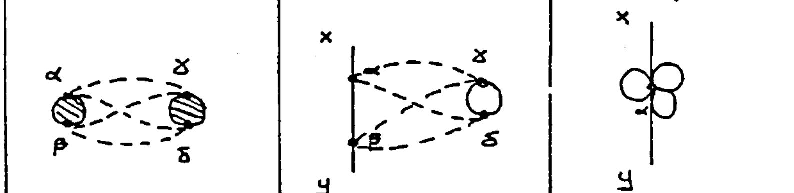
$\delta(0)/G_0(0)$

3(h)



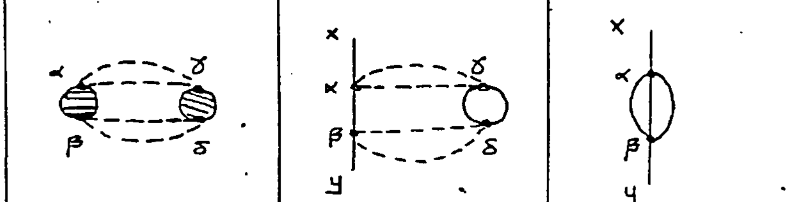
$\frac{D}{(G_0(0))^2}\left(\delta(0)/G_0(0)\right)^2$

3(i)



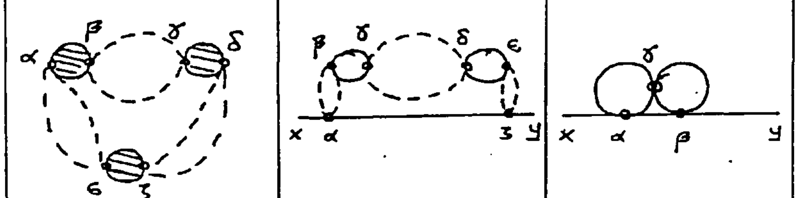
$\delta(0)/G_0(0)$

3(j)



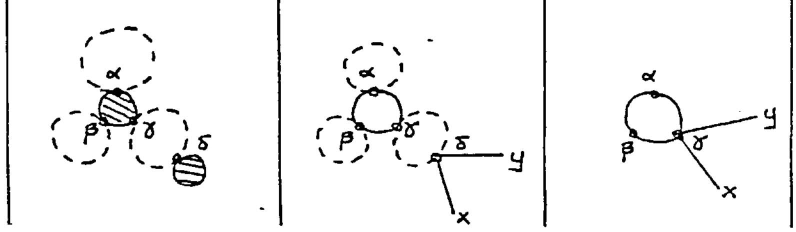
$\frac{D}{(G_0(0))^2}\left(\delta(0)/G_0(0)\right)^2$

3(k)

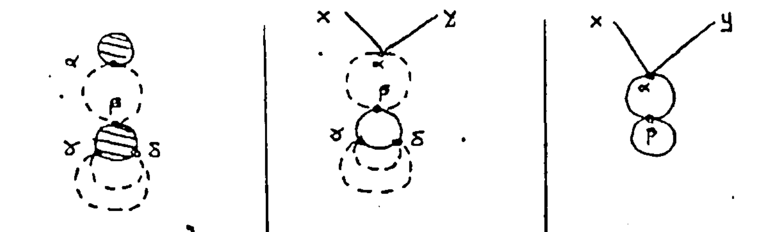
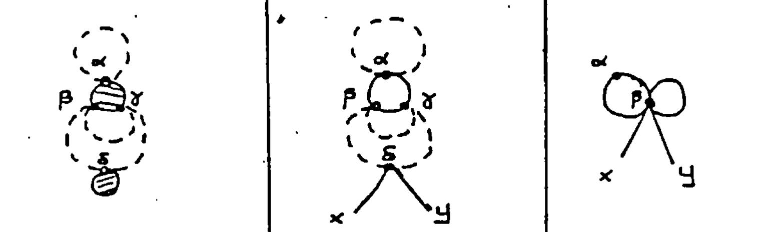
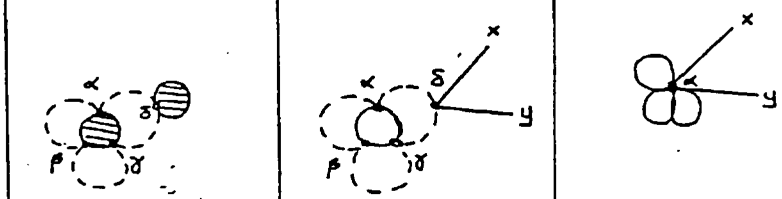
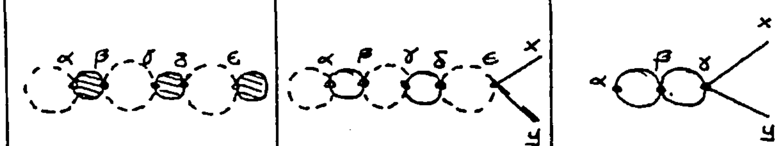
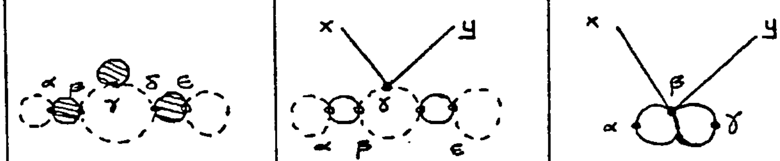
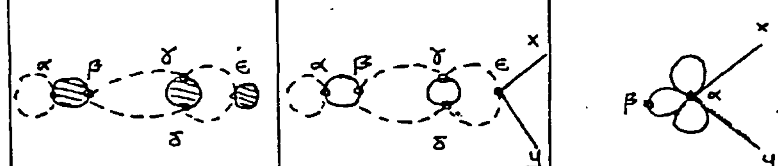
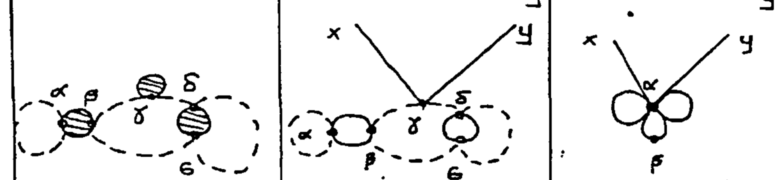
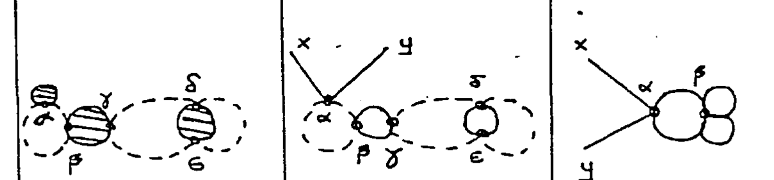


$\frac{E}{(G_0(0))^3}\left(\delta(0)/G_0(0)\right)^2$

3(a_1)



$\frac{L_3}{G_0(0)}\left(\delta(0)/G_0(0)\right)^3$

3(b_1)		$\frac{L_2}{G_2(0)}\left(\delta(0)/G_0(0)\right)^2$
3(b_2)		$\frac{L_2}{G_2(0)}\left(\delta(0)/G_0(0)\right)^2$
3(c_1)		$\delta(0)/G_0(0)$
3(d_1)		$\left(\frac{L_2}{G_0(0)}\right)^2\left(\delta(0)/G_0(0)\right)^3$
3(d_2)		$\left(\frac{L_2}{G_0(0)}\right)^2\left(\delta(0)/G_0(0)\right)^3$
3(e_1)		$\frac{L_2}{G_0(0)}\cdot\left(\frac{\delta(0)}{G_0(0)}\right)^2$
3(e_2)		$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$
3(e_3)		$\frac{L_2}{G_0(0)}\left(\frac{\delta(0)}{G_0(0)}\right)^2$

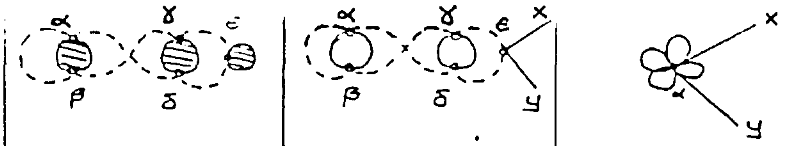
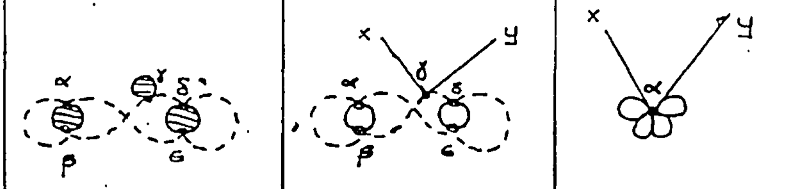
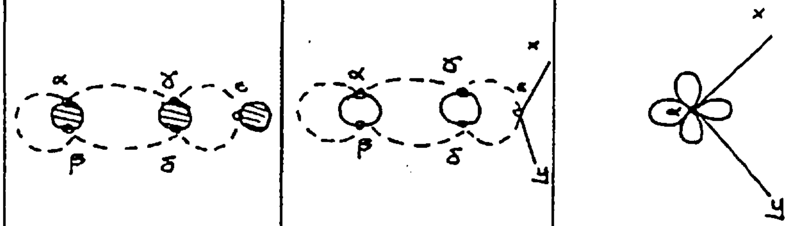
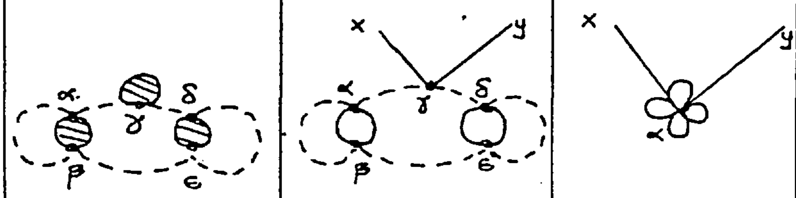
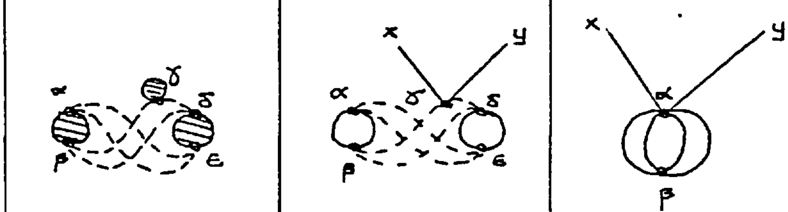
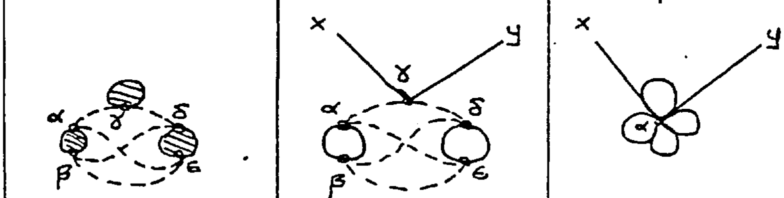
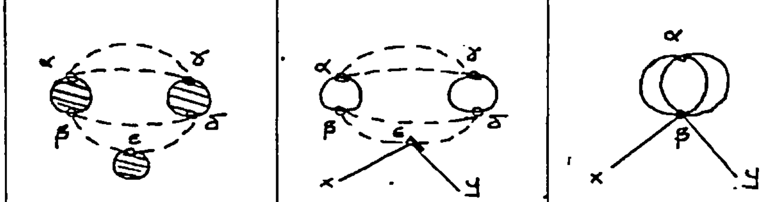
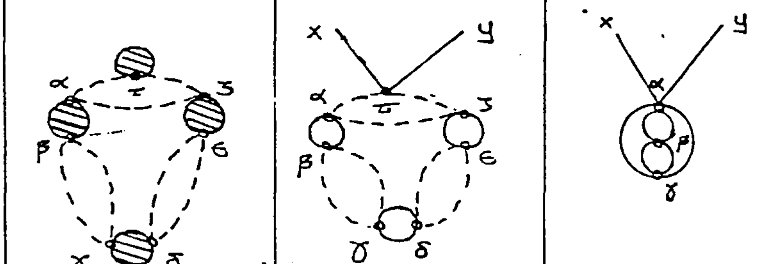
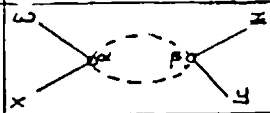
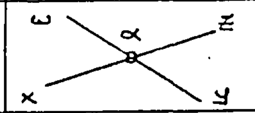
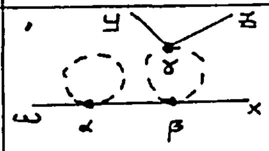
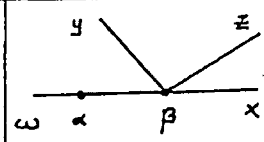
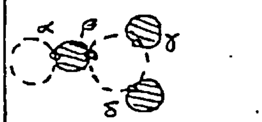
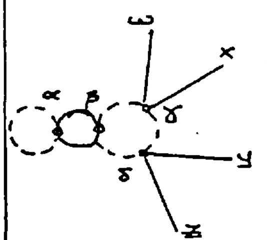
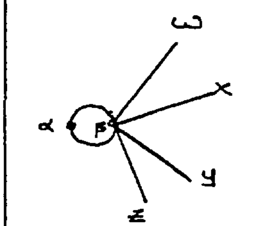
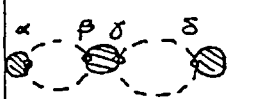
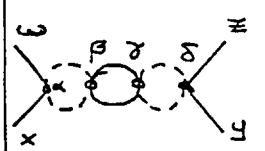
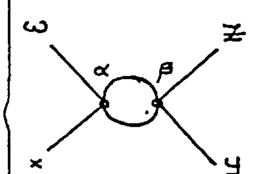
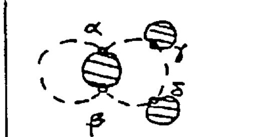
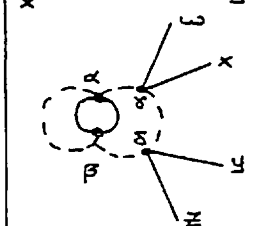
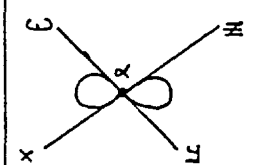
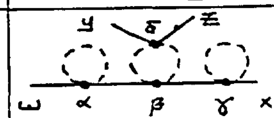
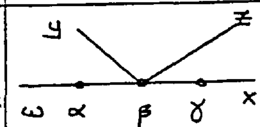
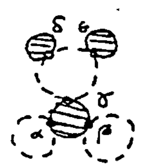
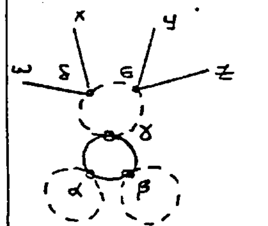
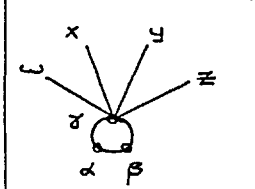
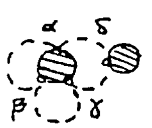
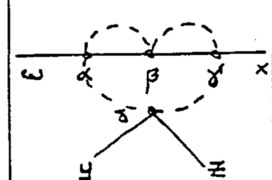
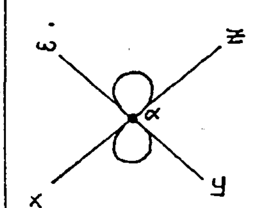
3(f₁)		$\delta(0)/G_0(0)$
3(f₂)		$\delta(0)/G_0(0)$
3(g₁)		$\delta(0)/G_0(0)$
3(g₂)		$\delta(0)/G_0(0)$
3(h₁)		$\frac{B}{(G_0(0))^3}\left(\delta(0)/G_0(0)\right)^2$
3(i₁)		$\delta(0)/G_0(0)$
3(j₁)		$\frac{B}{(G_0(0))^3}\left(\delta(0)/G_0(0)\right)^2$
3(k₁)		$\frac{F}{(G_0(0))^4}\left(\delta(0)/G_0(0)\right)^3$

TABLE 2.2

Number of loops	Relevant diagrams of W[h]	$W_4(\omega xyz) = \left.\frac{\delta^4 W[h]}{\delta h(\omega)\delta h(x)\delta h(y)\delta h(z)}\right\vert_0$ Uncontracted	Contracted	Singularity
1				$\delta(0)/(G_0(0))^2$
$2(a_1)$				$\delta^2(0)/G_0^3(0)$
$2(a_2)$				$\frac{L_2}{(G_0(0))^2}(\delta(0)/G_0(0))^2$
$2(a_3)$				$\frac{E_2}{(G_0(0))^2}(\delta(0)/G_0(0))^2$
$2(b_2)$				$\delta(0)/G_0^2(0)$
$3(a_1)$				$\delta^3(0)/G_0^4(0)$
$3(a_2)$				$\frac{L_3}{G_0^2(0)}(\delta(0)/G_0(0))^3$
$3(c_1)$				$\delta(0)/G_0^2(0)$

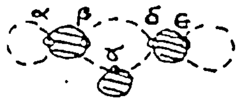
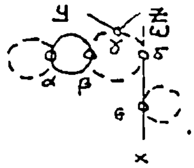
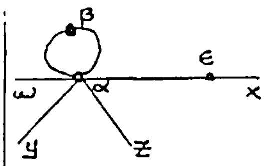
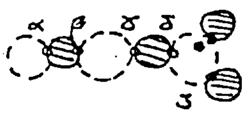
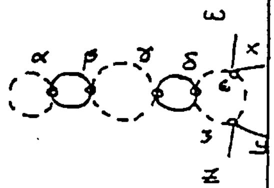
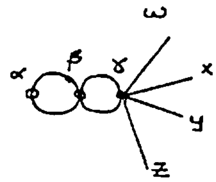
$3(d_2)$				$\frac{L_z}{(G_0(0))^2}(\delta(0)/G_0(0))^3$
$3(d_3)$				$\frac{L_z^2}{(G_0(0))^3}(\delta(0)/G_0(0))^3$

TABLE 4.1

d = 2 Space-time dimensions

$$G(0,\chi)_\Lambda = \frac{1}{4\pi} \ln(\Lambda^2/\chi)$$

$$\delta(0)_\Lambda / G(0,\chi)_\Lambda = \Lambda^2 \left(\ln \frac{\Lambda^2}{\chi}\right)^{-1}$$

d = 3 Space-time dimensions

$$G(0,\chi)_\Lambda = \frac{1}{2\pi^2}\left(\Lambda - \frac{\pi \chi^{1/2}}{2}\right)$$

$$\delta(0)_\Lambda / G(0,\chi)_\Lambda = \frac{\Lambda^2}{3}\left(1 - \frac{\pi \chi^{1/2}}{2\Lambda}\right)^{-1} = \frac{1}{3}\left(\Lambda^2 + \frac{\pi \Lambda \chi^{1/2}}{2} + \frac{\pi^2 \chi}{4} + O(\Lambda^{-1})\right)$$

$$\delta(0)_\Lambda / G^2(0,\chi)_\Lambda = \frac{2\pi^2 \Lambda}{3}\left(1 - \frac{\pi \chi^{1/2}}{2\Lambda}\right)^{-2} = \frac{2\pi^2}{3}\left(\Lambda + \chi^{1/2} + O(\Lambda^{-1})\right)$$

$$\delta(0)_\Lambda / G^3(0,\chi)_\Lambda = \frac{4\pi^4}{3} + O(\Lambda^{-1})$$

$$\delta(0)_\Lambda / G^4(0,\chi)_\Lambda = O(\Lambda^{-1})$$

$$\int_{|\underline{k}|<\Lambda} dk \ln(k^2+\chi) = \frac{1}{2\pi}\left(\Lambda\chi - \frac{\pi}{3}\chi^{3/2}\right) + \text{constant}$$

TABLE 4.1
(continued)

d = 4 Space-time dimensions

$$G(0,\chi)_\Lambda = \frac{1}{16\pi^2}\left(\Lambda^2 + \chi \ln(\chi/\Lambda^2) + O(\Lambda^{-2})\right)$$

$$\delta(0)_\Lambda / G(0,\chi)_\Lambda = \frac{1}{2}\Lambda^2\left(1 + \chi \ln\frac{\chi}{\Lambda^2} + \ldots\right)^{-1}$$

$$= \frac{1}{2}\left(\Lambda^2 + \chi \ln\frac{\chi}{\Lambda^2} + O(\Lambda^{-2})\right)$$

$$\delta(0)_\Lambda / G^2(0,\chi)_\Lambda = 8\pi^2 + O(\Lambda^{-2})$$

$$\delta(0)_\Lambda / G^3(0,\chi)_\Lambda = O(\Lambda^{-2})$$

d = 5 Space-time dimensions

$$G(0,\chi)_\Lambda = \frac{\Lambda^3}{36\pi^3}\left(1 - \frac{3\chi}{\Lambda^2} + \frac{3\pi}{2}\frac{\chi^{3/2}}{\Lambda^3} + O(\Lambda^{-4})\right)$$

$$\delta(0)_\Lambda / G(0,\chi)_\Lambda = \frac{3\Lambda^2}{5}\left(1 - \frac{3\chi}{\Lambda^2} + \frac{3\pi\chi^{3/2}}{2\Lambda^3}\right)$$

$$= \frac{3}{5}\left(\Lambda^2 + 3\chi - \frac{3\pi}{2}\frac{\chi^{3/2}}{\Lambda} + O(\Lambda^{-2})\right)$$

$$\delta(0)_\Lambda / G^2(0,\chi)_\Lambda = \frac{108\pi^2}{5\Lambda} + O(\Lambda^{-2})$$

$$\int_{|k|<\Lambda} dk \ln(k^2+\chi) = \frac{1}{36\pi^3}\left(\Lambda^3\chi - \frac{3}{2}\Lambda\chi^2 + \frac{3}{5}\pi\chi^{5/2}\right) + \text{constant}$$

d $\geq$ 6 Space-time dimensions

d = 6 $\quad G(0,\chi)_\Lambda = a_6\left(\Lambda^4 + b_6\Lambda^2\chi + c_6\chi^2 \ln\frac{\chi}{\Lambda^2} + \ldots\right)$

d > 6 $\quad G(0,\chi)_\Lambda = a_d\left(\Lambda^{d-2} + b_d\Lambda^{d-4}\chi + c_d\Lambda^{d-6}\chi^2 + \ldots\right)$

d $\geq$ 6 $\quad \delta(0)_\Lambda / G(0,\chi)_\Lambda = A_d\left(\Lambda^2 - b_d\chi + \ldots\right)$

$\qquad \delta(0)_\Lambda / G^2(0,\chi)_\Lambda = O(\Lambda^{4-d})$

where a_6, b_6, a_d, A_d, etc. are constant coefficients.