

Pseudo-Differential Calculus on Generalized Motion Groups

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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ABSTRACT

In recent years, effort has been put into following the ideas of M. Ruzhansky and V. Turunen to construct a global pseudo-differential calculus on Lie groups. By this, we mean a collection of operators containing the left-invariant differential calculus with the additional requirement that it be stable under composition and adjunction. Moreover, we would like these operators to have adequate boundedness properties between Sobolev spaces. Our approach consists in using the group Fourier transform to define a global, operator-valued symbol, yielding pseudo-differential operators via an analogue of the Euclidean Kohn-Nirenberg quantization.

The present document treats the case of the Euclidean motion group, which is the smallest subset of Euclidean affine transformations containing translations and rotations. As our representations are infinite-dimensional, the proofs of the calculus properties are more naturally carried out on the kernel side, which means that particular care is required to treat the singularity at the origin. The key argument is a density result which allows us to approximate singular kernels via smooth ones and is proved herein via purely spectral arguments without using classical estimates on the heat kernel.

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¹ After carefully voiding my warranty and removing all Windows stickers.

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1 | INTRODUCTION

Since their inception in the 1960s, pseudo-differential operators have become standard and natural tools in the study of partial differential equations, particularly when elliptic or hypoelliptic operators are concerned.

On $\mathbb{R}^n$, these operators are defined *globally* as arising from smooth maps called *symbols* via the Euclidean Fourier Transform, the latter generalizing characteristic polynomials. Their usefulness arises from the fact that they enjoy similar properties as their differential counterparts, such as a notion of *order* that behaves well under composition and adjunction, as well as expected boundedness properties between the adequate Sobolev spaces. Crucially, the *adjunction* and *composition* formulae claim that both these operations applied to pseudo-differential operators can be approximately expressed on the symbolic side via a formula that is formally virtually identical to the corresponding rule for characteristic polynomials. In particular, this means that elliptic pseudo-differential operators admit parametrices which are contained in the pseudo-differential calculus. The literature on the subject is vast and overviews can be found, for example, in [Shuo1; Hör07].

While pseudo-differential operators can be defined locally via local charts on any connected manifold, two issues arise. Firstly, operators are usually defined provided that a condition on the *type* such as

$$1 - \rho \leq \delta < \rho$$

holds (see e.g. [Shuo1, Section 4]), while their Euclidean counterparts make sense without restrictions on ρ and δ other than $\rho \geq \delta$. Secondly, a global notion of *symbol* cannot be invariantly defined, making results which rely on a full symbol (such as Gårding's inequality) difficult to establish.

The first successful attempt at defining a global pseudo-differential calculus on a non-Euclidean manifold is due to M. Ruzhansky and V. Turunen, and their treatment of compact Lie groups. They defined the global symbol of an operator via the group Fourier transform of

the right-convolution kernel, which in particular is *matrix*-valued due to the non-commutativity of the group law. Particularly instrumental to their success was their definition of *difference operators*, which generalize derivatives in Fourier variables from the Euclidean case. It is worth remembering that, while good properties of left-invariant pseudo-differential operators are ensured by the Plancherel theory, it is the estimates on the derivatives of the symbol and thus also in the Fourier variables, that extend the desired properties onto more general pseudo-differential operators. More specifically, their concepts of difference operators allowed them to firstly ensure pseudo-differential operators have Calderón-Zygmund kernels, and secondly control both the first terms and the remainders of asymptotic sums which arise from the composition and adjunction of operators.

Since then, many results and applications have been obtained on compact Lie groups by several authors including M. Ruzhansky, J. Wirth, V. Turunen and J. Delgado. To cite a few, these applications include criteria for ellipticity and global hypoellipticity in terms of the global symbols [RT10], a proof of a sharp Gårding inequality [RT11], a global functional calculus [RW14], a study of $L^p(G)$ Fourier multipliers [RW15], and many others.

It is natural to wonder whether a global pseudo-differential calculus can be defined on other, more complicated, settings. To this end, we observe that the Ruzhansky-Turunen theory relies on a Fourier-Plancherel theory, while good properties of the operators rely on an understanding of the theory of singular integrals. Therefore, the natural settings to investigate are Lie groups with polynomial growth of the volume.

In 2013, V. Fischer and M. Ruzhansky treated the case of graded nilpotent Lie groups. In particular, they dealt with the technical issues which arise from working with infinite-dimensional representations, and more crucially, non-central sub-Laplacians or positive Rockland operators. They also present a strategy to obtain the composition and adjunction formulae, which consists of proving symbolic estimates on the functional calculus for the sub-Laplacian or the Rockland operator, and subsequently proving kernel estimates for general symbols.

The aim of this thesis is to study the case of the Euclidean motion group, which is the smallest group of affine transformations on a Euclidean space containing both translations and rotations. Our

analysis will encounter difficulties due to the absence of properties such as compactness and commutativity of the group law, and the lack of a dilation structure. Furthermore, the infinite-dimensional nature of the representations, and the fact that the Laplacian is not central, also complicates matters. Nevertheless, our interest in this group is justified for two reasons: firstly, it is one of the more elementary non-compact groups not yet studied in the context of global pseudo-differential calculus; secondly, by its very definition, it naturally describes rigid motions such as the position and orientation of a robot arm or polymer chains like DNA. As a result, many partial differential equations from engineering, biology and physics are naturally expressed on the motion group. More detail on the different applications can be found in [CW04; CK00; Chi13].

1.1 SURVEY OF THE COMPACT AND GRADED CASES

In this section, we informally discuss an overview of the theory on compact and graded Lie groups. For more detail, the reader is invited to consult [RT10] for the compact case¹ and [FR16] for graded Lie groups. To avoid repetition, we shall discuss both settings simultaneously.

Essentially, [Fis15; FR16] and the present document all follow the following steps to develop a global pseudo-differential calculus.

- STEP 1.** Study the (sub-)Laplacian or positive Rockland operator, and the corresponding Sobolev spaces.
- STEP 2.** Study and generalize the group Fourier Transform.
- STEP 3.** Find a family of smooth functions which have a Leibniz-like rule and can be used for a Taylor development.
- STEP 4.** Define the symbol classes.
- STEP 5.** Study the functional calculus of the (sub-)Laplacian or positive Rockland operator.
- STEP 6.** Obtain kernel estimates.

¹ The adjunction and composition formulae are stated therein, although their corresponding proofs contain a non-trivial gap which has since been filled but not published.

STEP 7. Prove the composition and adjunction formulae.

We now present the above steps in more detail. Unless stated otherwise, G will denote a compact or a graded Lie group until the end of this outline.

Step 1 – Sobolev spaces

We start by fixing a (sub-)Laplacian or, in the graded case, a positive Rockland operator. This allows us to define *Sobolev spaces*. We then need to show their basic properties, the most important of which is the *Sobolev embedding theorem*.

As our kernels and symbols will be linked via the Fourier transform, the *Plancherel theory* will allow us to obtain $L^2(G)$ and more generally $L_s^2(G)$ estimates on the kernel from symbolic estimates. We shall therefore rely on the Sobolev embedding to obtain further regularity results (e.g. continuity). The role of the Sobolev embedding theorem is thus pivotal in the study of the regularity and the singularities of the kernel.

Moreover, as we require our pseudo-differential operators to be bounded maps between suitable Sobolev spaces, we expect these spaces to be somehow part of our symbol classes definition.

In the graded case, Sobolev spaces are studied in [FR16, Chapter 4], following the proofs of the stratified case (see [Fol75]).

In the case of compact groups, the *Laplace-Beltrami* operator associated with a normalized bi-invariant metric appears the natural candidate to define Sobolev spaces, since its eigenspaces arise naturally from the group representations.

From this point onwards, $\mathcal{L}_G$ will denote the operator with respect to which we have defined our Sobolev spaces.

Step 2 – Fourier analysis

The main idea is to define an analogue of the Euclidean Kohn-Nirenberg quantization, whereby the *symbol* of a pseudo-differential operator is the *Fourier transform* of its *convolution kernel*. Our first step is thus to extend the group Fourier transform to a suitable subset of tempered distributions.

On our settings, the Fourier transform is defined on $L^1(G, dg)$ via

$$\mathcal{F}_G f(\xi) \stackrel{\text{def}}{=} \int_G f(g) \xi(g)^* dg,$$

where dg denotes a *Haar measure*, $\xi \in \widehat{G}$, and $\widehat{G}$ is the *unitary dual* of G , i.e. the set of all equivalence classes of irreducible, strongly continuous and unitary representations of G .

If $f \in L^1(G) \cap L^2(G)$, we have the following *Plancherel formula*

$$\int_G |f|^2 dg = \int_{\widehat{G}} \text{tr}(\mathcal{F}_G f(\xi) \mathcal{F}_G f(\xi)^*) d\mu_{\widehat{G}}(\xi),$$

where $d\mu_{\widehat{G}}$ denotes the so-called *Plancherel measure*, while any $f \in \mathcal{S}(G)$ may be recovered from its Fourier Transform via the following *inverse formula*

$$f(g) \stackrel{\text{def}}{=} \int_{\widehat{G}} \text{tr}(\xi(g) \mathcal{F}_G f(\xi)) d\mu_{\widehat{G}}(\xi), \quad g \in G.$$

The problem of defining an extension of the Fourier Transform to make sense of a *Kohn-Nirenberg quantization* procedure was solved in [FR16, Subsection 5.1.1] for a very large class of groups, since their solution relies exclusively on the *Plancherel* theorem and the definition of Sobolev spaces.

Step 3 - Difference operators

In [RT10], M. Ruzhansky and V. Turunen introduce the following notion of difference operator to generalize the derivatives in Fourier variables which appear in the Euclidean definition of symbol classes. If $q \in C^\infty(G)$, they define

$$\Delta_q^G \mathcal{F}_G f \stackrel{\text{def}}{=} \mathcal{F}_G \{qf\}.$$

When $G = \mathbb{R}^n$, we recover the classical derivatives in frequency with the collection $\Delta \stackrel{\text{def}}{=} \{x_1, \dots, x_n \in C^\infty(\mathbb{R}^n)\}$, where x_i is the function that maps x onto its i -th coordinate.

We now need to find the properties that a family of functions

$$\Delta \stackrel{\text{def}}{=} \{q_1, \dots, q_{n_{\Delta, G}} \in C^\infty(G)\}$$

must satisfy in order to develop a global pseudo-differential calculus. We then let

$$q^\alpha \stackrel{\text{def}}{=} q_1^{\alpha_1} \dots q_{n_{\Delta,G}}^{\alpha_{n_{\Delta,G}}}, \quad \Delta_G^\alpha \stackrel{\text{def}}{=} \Delta_{q^\alpha(\cdot, -1)}^G.$$

Upon inspection of the different proofs in the Euclidean [Ste93], compact [RT10; Fis15] and graded [FR16] cases, the main requirements appear to be the following.

- We can use a *Taylor development* at the origin. This is necessary because the adjunction and the composition are difficult to express on the symbolic side without using a Taylor development.
- We have a reasonable *Leibniz rule* for the associated difference operators. This ensures, for example, that the order of the composition of two left-invariant pseudo-differential operators is the sum of the orders.
- We would like to ensure that our pseudo-differentials operators of order at most 0 are *Calderón-Zygmund*. While we can reasonably expect the Plancherel theory to ensure $L^2(G)$ boundedness, we rely on the theory of singular integrals to have the boundedness on $L^p(G)$.

The first couple of conditions naturally require that each q_j vanish at e_G and that $\{dq_j(e) : 1 \leq j \leq n_{\Delta,G}\}$ span the tangent space $T_{e_G}G$. Such a family is called *admissible* in [RTW14] and subsequent literature.

Intuitively, our definition of symbol classes will ensure that the regularity of the kernel will increase every time we multiply it by one of the $q_j \in \Delta$, exactly like in the Euclidean case. In particular, this means that our kernels will be smooth outside of

$$\bigcap_{q \in \Delta} \{g \in G : q(g) = 0\}, \tag{1}$$

while the above set may contain singularities. However, the Calderón-Zygmund theory requires that the singularity be confined to the identity. Therefore, [RTW14] defines Δ to be *strongly admissible* if (1) equals $\{e_G\}$.

The compact case

In [RTW14], the authors show that a well-chosen finite subfamily Δ of

$$\{(\tau - \tau(e_G))_{mn} : \tau \in \widehat{G}, 1 \leq m, n \leq d_\tau\}$$

can be strongly admissible.

Importantly, this family yields the following *Leibniz-like* rule

$$\Delta_G^\alpha(\mathcal{F}_G f_1 \mathcal{F}_G f_2) = \sum_{|\alpha| \leq |\alpha_1| + |\alpha_2| \leq 2|\alpha|} c_{\alpha_1, \alpha_2}^\alpha(\Delta_G^{\alpha_1} \mathcal{F}_G f_1)(\Delta_G^{\alpha_2} \mathcal{F}_G f_2),$$

which is good enough to establish the calculus, but slightly complicates some proofs related to the construction of parametrices.

The compactness of the group allows us to show that the calculus does *not* depend on the choice of strongly admissible difference operators [RTW14; Fis15]. Such a proof relies heavily on the *finite volume* of our group, and it is important to note at this stage that the result does not hold even on $\mathbb{R}^n$.

The graded case

The behaviour of difference operators in the graded case is somehow closer to the Euclidean setting. In [FR16], the authors consider the difference operators associated with *homogeneous polynomials*. The dilation structure shows that we have a Leibniz rule if we consider a sum on homogeneous degrees (see [FR16, Subsection 5.2.1]), while the strong admissibility is easily obtained by choosing a suitable basis.

Step 4 – Symbol and operator classes

Very informally, we say that $\sigma(g, \xi)$ is a *symbol* if there exists $\kappa \in C^\infty(G, \mathcal{S}'(G))$ such that

$$\sigma(g, \xi) \stackrel{\text{def}}{=} \mathcal{F}_G\{\kappa(g, \cdot)\}(\xi)$$

is defined. The distribution κ is called the *kernel associated with* σ .

Moreover, given $m \in \mathbb{R}$, we shall say that $\sigma \in S_{\rho,\delta}^m(G)$ if

$$\left\| \xi(I - \mathcal{L}_G)^{\frac{-m+\rho|\alpha|-\delta|\beta|+\gamma}{2}} X^\beta \Delta_G^\alpha \sigma(g, \xi) \xi(I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(\mathcal{H}_\xi)} < \infty$$

uniformly in $g \in G$ and essentially uniformly in $\xi \in \widehat{G}$ (with respect to the Plancherel measure). Note that if $\mathcal{L}_G$ is central, we can assume $\gamma = 0$. It should be clear that when $G = \mathbb{R}^n$, we obtain the classical definition of symbol classes.

We also let

$$S^{-\infty}(G) \stackrel{\text{def}}{=} \bigcap_{m \in \mathbb{R}} S_{\rho,\delta}^m(G)$$

denote the set of *smoothing symbols*.

We define a *pseudo-differential operator* to be an operator arising from a symbol via an analogue of the *Kohn-Nirenberg quantization*. More precisely, the operator

$$\text{Op}_G(\sigma)\phi(g) \stackrel{\text{def}}{=} \mathcal{F}_G^{-1} \left\{ \xi \in \widehat{G} \mapsto \xi(g) \sigma(g, \xi) \mathcal{F}_G \phi(\xi) \right\} (g)$$

is the *pseudo-differential operator* associated with the symbol σ . We easily check that $\text{Op}_{\mathbb{R}^n}$ is the usual Kohn-Nirenberg quantization.

Naturally, the quantization allows us to define *operator classes* via

$$\begin{aligned} \Psi_{\rho,\delta}^m(G) &\stackrel{\text{def}}{=} \text{Op}_G(S_{\rho,\delta}^m(G)), \\ \Psi^{-\infty}(G) &\stackrel{\text{def}}{=} \text{Op}_G(S^{-\infty}(G)). \end{aligned}$$

For more information, see [FR16, Section 5.1] for the graded case, and [Fis15, Section 3.1] for the compact case.

Step 5 - Functional calculus

On $\mathbb{R}^n$, the composition and adjunction formulae can be proved by carrying out our computations on the frequency space $\widehat{\mathbb{R}^n} = \mathbb{R}^n$, exploiting the fact that the Euclidean space is *abelian* and we have explicit expressions of our difference operators. Since a group Fourier transform is *operator-valued* in more complicated settings, it makes sense to attempt to do our calculations on the kernel side, especially since this is how difference operators were defined. Unfortunately,

kernels may have a singularity at the origin, unlike symbols which are smooth. Therefore, two questions naturally arise.

- What is the *strength* of the singularity of the kernel at the origin?
- Can we *approximate* a symbol σ by a sequence of smoothing symbols $(\sigma_n)_{n \in \mathbb{N}}$ so that the kernel can be approximated by the associated sequence of smooth kernels?

Looking at pseudo-differential operators such as $(I - \mathcal{L}_G)^{\frac{m}{2}}$ suggests that *functional calculus* is the natural framework to answer such questions.

A crucial step in our analysis is to construct a family of smoothing symbols approximating the identity symbol. Using the algebra structure of our symbol classes, this will yield an important density result, as we will be able to approximate any symbol with smoothing symbols, and by extension any kernel with smooth kernels.

Ideally, we would like to create a family $\{\eta_\varepsilon = \mathcal{F}_G p_\varepsilon\}_{\varepsilon \in (0,1]} \subset S^{-\infty}(G)$ such that

$$\|\eta_\varepsilon\|_{S_{\rho,\delta}^m(G),N} \leq C\varepsilon^{\frac{m}{\nu}}, \quad m < 0, \quad (2)$$

$$\|I - \eta_\varepsilon\|_{S_{\rho,\delta}^m(G),N} \leq C\varepsilon^{\frac{m}{\nu}}, \quad m \geq 0, \quad (3)$$

where ν is the order of the operator $\mathcal{L}_G$. Moreover, to prove the Calderón-Zygmund property, we need very good estimates on the kernels p_ε . Therefore, the natural candidate is the Fourier transform of the *heat kernel*.

The family of symbols is used to prove the following results.

KERNEL ESTIMATES Given a symbol $\sigma \in S_{\rho,\delta}^m(G)$, we estimate the singularity of its kernel at the origin via

$$|\kappa_g(h)| \leq |\kappa_g \star p_\varepsilon(h)| + |\kappa_g \star (\delta_{e_G} - p_\varepsilon)(h)|,$$

where we choose ε so that ε^ν equals the distance between h and e_G .

The second term of the right-hand side is the one containing the singularity, and we need (3) to control it.

CONSTRUCTION OF ASYMPTOTIC LIMITS One of the inherent characteristics of symbolic calculus is that compositions and adjunctions can only be expressed via infinite sums from which we

cannot reasonably expect convergence. As certain applications, such as the construction of parametrices, rely on constructing a limit of these sums in a suitable sense, it is important that we define a notion of *asymptotic convergence*.

The idea is the same as in the Euclidean case. We know that cutting off “low frequencies” will not affect the result modulo $S^{-\infty}(G)$. Therefore, given a sequence of symbols σ_j with strictly decreasing order, we define

$$\sum_{j=0}^{+\infty} \sigma_j \stackrel{\text{def}}{\sim} \sum_{j=0}^{+\infty} \sigma_j (I - \eta_{\varepsilon_j}) \quad \text{modulo } S^{-\infty}(G),$$

for a well chosen sequence $(\varepsilon_j) \subset (0, 1]$ decreasing to 0.

APPROXIMATION BY SMOOTHING SYMBOLS Letting

$$\sigma_\varepsilon \stackrel{\text{def}}{=} \sigma \eta_\varepsilon$$

allows us to approximate σ by smoothing symbols, while the error term

$$\sigma - \sigma_\varepsilon = (I - \eta_\varepsilon)\sigma$$

can be controlled via our symbolic estimates on $I - \eta_\varepsilon$.

In particular, this allows us to show the composition and adjunction formulae only for smoothing symbols before taking the limit $\varepsilon \rightarrow 0$.

The compact case

Following the classical heat kernel methods (see e.g. [FMVo6; VSC92]), V. Fischer shows symbolic estimates for the functional calculus of the Laplace-Beltrami operator in [Fis15]. However, she proceeds slightly differently by letting

$$\eta_\varepsilon \stackrel{\text{def}}{=} \chi(\varepsilon \tilde{\zeta}(\mathcal{L}_G)),$$

where $\chi : \mathbb{R}^+ \rightarrow \mathbb{R}$ equals 1 on $[0, 1]$ and has compact support. To estimate the remainder, she uses a *Littlewood-Paley* decomposition

$$I - \eta_\varepsilon = \sum_{j=1}^{+\infty} \eta_{\varepsilon_j},$$

where $\eta_{\varepsilon,j} \stackrel{\text{def}}{=} \eta_{2^{-j}\varepsilon} - \eta_{2^{-j+1}\varepsilon}$, and the convergence is understood in the strong operator topology.

The graded case

The existence of such a family η_ε is ensured directly by the *Hulanicki Theorem* (see [FR16, Theorem 4.5.1]). We can define

$$\eta_\varepsilon(\xi) \stackrel{\text{def}}{=} \mathcal{F}_G\{e^{-\varepsilon\mathcal{L}_G}\delta_{e_G}\}(\xi), \quad \xi \in \widehat{G},$$

and the aforementioned theorem ensures that this defines a smoothing symbol satisfying the appropriate symbolic estimates.

Steps 6 and 7 – Kernel estimates and calculus proofs

As soon as we have obtained a good approximation of the identity symbol, whether it is via a family $\eta_\varepsilon, \varepsilon \in (0, 1]$ or a Littlewood-Paley decomposition, the proofs of the kernel estimates and subsequently the composition and adjunction formulae do not depend so much on the setting. Note that only the case $\rho > \delta$ will be treated herein.

In order to better understand the importance of the kernel estimates, let us sketch the proof of the *adjunction formula* while keeping in mind that the arguments for the composition formula are very similar.

Suppose that $T \stackrel{\text{def}}{=} \text{Op}_G(\sigma) \in \Psi_{1,0}^m(G)$.

1. As we intend to use a *density* argument, we assume first that σ is smoothing.
2. We perform an exact computation of the kernel $\tilde{\kappa}$ of T^* in terms of the kernel κ of T .

More precisely, we show that T^* has right-convolution kernel

$$\tilde{\kappa}_g(h) \stackrel{\text{def}}{=} \bar{\kappa}_{gh^{-1}}(h^{-1}).$$

3. In particular, its *symbol* is given by taking the Fourier transform of the above

$$\tilde{\sigma}(g, \xi) \stackrel{\text{def}}{=} \mathcal{F}_G \tilde{\kappa}_g(\xi) = \int_G \bar{\kappa}_{gh^{-1}}(h^{-1}) \xi(h)^* dh.$$

4. In the integral above, we take the *Taylor development* with respect to the variable h appearing in the subscript of the kernel

$$\begin{aligned}\tilde{\sigma}(g, \xi) &= \int_G \left(\sum_{|\alpha| \leq N} q^\alpha(h^{-1}) X_g^\alpha \bar{\kappa}_g(h^{-1}) + R_N(h) \right) \xi(h)^* dh \\ &= \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha X^\alpha \sigma^*(g, \xi) + \int_G R_N(h) \xi(h)^* dh.\end{aligned}$$

5. Using the Taylor remainder formula, the integrand on the right can be estimated by an expression such as

$$|R_N(h)| \leq Cd(h, e_G)^{N+1} \sup_{\beta \in B} \sup_{\tilde{h} \in \Gamma_h} \left| X^\beta \kappa_g(\tilde{h}) \right|$$

so that the integral can be controlled via good estimates on κ_g .

6. We now conclude the proof by density.

For the compact case, see [Fis15, Sections 6.3 and 7.3]; for the graded case, see [FR16, Sections 5.4 and 5.5].

1.2 OVERVIEW

We essentially follow the strategy mentioned in the previous section, and apply it for the motion group.

In Chapter 2, we introduce the necessary material to follow the arguments developed in subsequent chapters. In particular, we recall the definition of Fourier Transform on Lie groups of type I.

In Chapter 3, we introduce the Euclidean motion groups, study their representation theory and associated Fourier transform. We also comment on the formal but deceptive resemblance between the Fourier theory of the motion group $\mathbb{R}^n \ltimes \mathrm{SO}(n)$ and the one of the direct product $\mathbb{R}^n \times \mathrm{SO}(n)$.

Chapter 4 will contain the main results of our analysis. We first define a good family of Ruzhansky-Turunen difference operators, and use them to define our symbol and operator classes.

In Section 4.4, we define our symbol family which will allow the crucial density argument mentioned in the previous section. Following the ideas of [Shu01; RW14], we define

$$\eta_\varepsilon(\tilde{\zeta}) \stackrel{\text{def}}{=} \frac{1}{i2\pi} \int_{\Gamma} e^{-\varepsilon z} (\tilde{\zeta}(I - \mathcal{L}_G) - zI)^{-1} dz, \quad \tilde{\zeta} \in \hat{G}$$

with an appropriate contour $\Gamma \subset \mathbb{C}$. We also show the symbolic estimates on η_ε and $I - \eta_\varepsilon$ via the exponential decay on the contour and elementary estimates on the resolvent, in particular *without using the classical estimates on the heat kernel*.

This strategy generalizes verbatim to more general settings (provided that we have good properties for difference operators), but also works on the compact group.

This approach has, however, at least two drawbacks.

- We only show symbolic estimates for the functional calculus associated with the functions e^{tz} , $1 - e^{tz}$, and z^γ , while [FR16] and [Fis15] prove a much more general result in their respective cases.
- The kernel estimates in the case $\rho = 1$, essential to show the Calderón-Zygmund property, are trickier to obtain and will require better estimates on the kernel. As a result, we will have to follow [Fis15] to prove the $L^p(G)$ boundedness.

It is nevertheless interesting to observe that this approach provides a shorter and more elementary way to establish a global pseudo-differential calculus if compared with [Fis15; FR16].

The rest of Chapter 4 is dedicated to showing the main result of the thesis, i.e. the composition and adjunction formulae, but also the $L^2(G)$ boundedness. We then conclude with the classical application of pseudo-differential calculus to the construction of parametrices.

NOTATION

SETS AND SYMBOLS

- $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of: all, positive (including 0), and negative (including 0) real numbers respectively.
- $\mathbb{Z}$ denotes the set of all integers, while $\mathbb{N}$ contains all positive integers including 0.
- $\delta_{i,j}$ is the *Kronecker delta*; it is equal to 1 if $i = j$, and 0 otherwise.
- ∇_V denotes the *gradient* on the vector space V .
- $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is the *Lie algebra* of G .
- $\mathcal{L}_G$ denotes the *Laplace operator* on G .
- T^* denotes the *adjoint* of T .
- e_G is the identity element of a group G .
- $e_1, \dots, e_{\dim V}$ denotes an orthonormal basis of V .
- $f \star g$ represents the convolution of f and g .
- δ_g is the delta-distribution at $g \in G$.
- $\mathcal{S}(G)$ is the set of *Schwartz functions* on G .
- $\mathcal{S}'(G)$ is the set of *tempered distributions* on G .
- $C^\infty(G)$ denotes the set of *smooth functions* on G .
- $L^p(G)$ will denote the set of all μ -measurable complex functions f on G such that $|f|^p$ is μ -integrable; here, μ is a Haar measure on G .
- $L_s^2(G)$ is the *2-Sobolev space* of order s on G .
- $L_s^p(G)$ is the *p-Sobolev space* of order s on G .
- $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denotes a bounded linear map between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$; also, we write $\mathcal{L}(\mathcal{H}) \stackrel{\text{def}}{=} \mathcal{L}(\mathcal{H}, \mathcal{H})$.

- $S_p(\mathcal{H})$ denotes the p -Schatten classes on the Hilbert space $\mathcal{H}$.
- $\widehat{G}$ denotes the *unitary dual* of G .
- $\mathcal{F}_G$ denotes the (unitary) *Fourier transform* on G .
- $\mathcal{F}_G^{-1}$ denotes the *inverse Fourier transform* on G .
- $\langle \cdot \rangle_{\widehat{G}}$ denotes the *Japanese bracket*.
- $S_{\rho, \delta}^m(G)$ denotes the class of symbols with order m and type (ρ, δ) on G .
- $\Psi_{\rho, \delta}^m(G)$ denotes the class of operators with order m and type (ρ, δ) on G .

2 | PRELIMINARIES

This chapter aims to very briefly introduce key concepts which will be used in subsequent chapters, and as a result we shall often omit the proofs. For a more comprehensive treatment, the reader is invited to consult the references given at the beginning of each section.

2.1 LIE GROUPS

For more details see e.g. [Kna02; RT10].

To develop a *pseudo-differential calculus* on groups, a reasonable pre-requisite is that the group be equipped with a *differential* structure. This naturally leads us to the notion of *Lie group*.

Definition 2.1.1 (Lie group). *Let G be a group. We say that G is a Lie group if G is a smooth manifold and the map*

$$G \times G \rightarrow G : (g_1, g_2) \mapsto g_1^{-1}g_2$$

is smooth.

Moreover, if G is (locally) compact as a manifold, then we shall say that G is a (locally) compact Lie group.

In the sequel, G and K will be used to denote Lie groups, with the additional assumption that K is compact. Moreover, we shall denote the *identity element* of G by e , or e_G should an ambiguity arise.

Closed Lie subgroups consisting of invertible matrices form an important class of Lie groups called *linear Lie groups*, a non-exhaustive list of which is now given.

Example 2.1.2 (Linear Lie groups). *Let V be a Euclidean space.*

1. *The set $\text{GL}(V)$ of invertible linear maps, equipped with the composition of functions, naturally forms a Lie group called the general linear group of V .*

2. The set

$$\mathrm{SO}(V) \stackrel{\text{def}}{=} \{A \in \mathcal{L}(V) : AA^T = A^T A = I_V \text{ and } \det A = 1\}$$

is a compact Lie subgroup of $\mathrm{GL}(V)$ called the special orthogonal group of V .

2.1.1 Lie algebra

Definition 2.1.3 (Lie algebra). A (real) Lie algebra is a (real) vector space $\mathfrak{g}$ equipped with a bilinear map

$$[\cdot, \cdot]_{\mathfrak{g}} : V \times V \rightarrow V,$$

called the Lie bracket or commutator, such that

1. $[X, X]_{\mathfrak{g}} = 0$ for every $X \in \mathfrak{g}$;
2. For every $X, Y, Z \in \mathfrak{g}$, we have the following Jacobi identity

$$[X, [Y, Z]_{\mathfrak{g}}]_{\mathfrak{g}} + [Y, [Z, X]_{\mathfrak{g}}]_{\mathfrak{g}} + [Z, [X, Y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0.$$

Given a Lie group G , let us denote by $\mathfrak{X}(G)$ the set of all smooth vector fields on G .

Definition 2.1.4 (Left-invariant vector fields). Let G be a Lie group. We shall say that $X \in \mathfrak{X}(G)$ is left-invariant if for every $g \in G$,

$$X \circ L_g = dL_g \circ X,$$

where

$$L_g : G \rightarrow G : h \mapsto gh \tag{4}$$

is the left translation on G by g . We shall denote the set of left-invariant vector fields by $\mathfrak{X}_L(G)$.

Example 2.1.5 (Left-invariant vector fields). Let G be a Lie group. Given two left-invariant vector fields $X, Y \in \mathfrak{X}_L(G)$, we define

$$[X, Y]_{\mathfrak{X}_L(G)} f(x) \stackrel{\text{def}}{=} XYf(x) - YXf(x), \quad f \in C^\infty(G)$$

and we can show that $[X, Y]_{\mathfrak{X}_L(G)}$ is also a left-invariant vector field. Therefore, $\mathfrak{X}_L(G)$ is a Lie algebra.

Let $X, Y \in T_e G$. Defining for each $g \in G$

$$\tilde{X}(g) \stackrel{\text{def}}{=} dL_g(X), \quad \tilde{Y}(g) = dL_g(Y),$$

we check that $\tilde{X}, \tilde{Y} \in \mathfrak{X}_L(G)$.

The Lie algebra structure of $\mathfrak{X}_L(G)$ allows us then to define

$$[X, Y]_{T_e G} \stackrel{\text{def}}{=} [\tilde{X}, \tilde{Y}]_{\mathfrak{X}_L(G)}(e). \quad (5)$$

Definition 2.1.6 (Lie algebra of Lie group). *Let G be a Lie group. The Lie algebra of G , denoted by $\mathfrak{Lie}(G)$, is the tangent space at the identity $T_e G$ with the Lie bracket $[\cdot, \cdot]_{T_e G}$ defined in (5).*

Example 2.1.7 (Lie algebra of $GL(n)$). *Suppose $G \stackrel{\text{def}}{=} GL(n)$. Its Lie algebra is*

$$\mathfrak{g} \stackrel{\text{def}}{=} \mathbb{R}^{n \times n}$$

equipped with the Lie bracket

$$[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} : (A, B) \mapsto AB - BA.$$

Proposition 2.1.8 (Lie algebra of Lie subgroups). *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $F \subset G$ is a closed Lie subgroup of G , then its Lie algebra $\mathfrak{Lie}(F)$ is a subset of $\mathfrak{g}$, and the Lie bracket on $\mathfrak{Lie}(F)$ is the restriction of that of $\mathfrak{g}$.*

Example 2.1.9 (Lie algebra of $SO(n)$). *Suppose $K \stackrel{\text{def}}{=} SO(n)$. Its Lie algebra is*

$$\mathfrak{k} \stackrel{\text{def}}{=} \text{Skew}(n)$$

equipped with the Lie bracket

$$[\cdot, \cdot]_{\mathfrak{k}} : \mathfrak{k} \times \mathfrak{k} \rightarrow \mathfrak{k} : (A, B) \mapsto AB - BA.$$

Lemma 2.1.10 (Orthonormal basis on $\mathfrak{k}$). *Let K be a compact Lie group. There exists an inner product on $\mathfrak{k} = \mathfrak{Lie}(K)$ such that any orthonormal basis $X_1, \dots, X_{\dim K} \in \mathfrak{k}$ satisfies*

$$\mathcal{L}_K = \sum_{j=1}^{\dim K} X_j^2,$$

where $\mathcal{L}_K$ is the Laplace-Beltrami operator associated with the normalized bi-invariant metric.

Suppose that G is a Lie group and that $\mathfrak{g}$ is its Lie algebra. If $X_1, \dots, X_{\dim G} \in \mathfrak{g}$ is a given basis, and $\alpha \in \mathbb{N}^{\dim G}$ we let

$$X^\alpha \stackrel{\text{def}}{=} X_1^{\alpha_1} \dots X_{\dim G}^{\alpha_{\dim G}}.$$

Exponential map

Definition 2.1.11 (Exponential map). *Let G be a Lie group, and fix $X \in \mathfrak{g}$. We define the exponential of X via*

$$\exp_G(X) \stackrel{\text{def}}{=} \gamma_X(1),$$

where $\gamma_X : I \subset \mathbb{R} \rightarrow G$ is the unique solution to the following ordinary differential equation

$$\begin{cases} \partial_t \gamma_X(t) &= \text{d}L_{\gamma_X(t)}|_e(X) \\ \gamma_X(0) &= e. \end{cases}$$

In the above, L_g designates the left-translation by g on G as defined in (4).

Example 2.1.12 (Exponential map on $\text{GL}(n)$). *Suppose that $G \stackrel{\text{def}}{=} \text{GL}(n)$, so that $\mathfrak{g} = \mathbb{R}^{n \times n}$ by Example (2.1.7). The exponential map on G is*

$$\exp_{\text{GL}(n)} : \mathbb{R}^{n \times n} \rightarrow \text{GL}(n) : A \rightarrow \sum_{k=0}^{+\infty} \frac{1}{k!} A^k. \quad (6)$$

The convergence of the series (6) can be shown using a *submultiplicative norm* on $\mathbb{R}^{n \times n}$ and the convergence of the exponential power series for real numbers.

Example 2.1.13 (Exponential map on $\text{SO}(n)$). *Suppose that $K \stackrel{\text{def}}{=} \text{SO}(n)$. By Example 2.1.9, its Lie algebra is $\mathfrak{k} \stackrel{\text{def}}{=} \text{Skew}(n)$. Given $A \in K$, its exponential is given by*

$$\exp_K(A) = \sum_{k=0}^{+\infty} \frac{1}{k!} A^k.$$

Proposition 2.1.14. *Let G be a connected and simply connected Lie group. The exponential map is a diffeomorphism from $\mathfrak{g} \stackrel{\text{def}}{=} \mathfrak{Lie}(G)$ onto G .*

2.1.2 Haar measure

We need to choose a measure with respect to which we shall integrate. On locally compact topological groups, there exist measures which are invariant under the left or right group action. This property is crucial for the Fourier analysis.

Remark 2.1.15. Every Lie group G is also a topological space. As a result, all the topological definitions apply to Lie groups.

Definition 2.1.16 (Haar measure). Let G be a Lie group. A positive Radon measure on G is called a Haar measure if it is in addition left-invariant, i.e. for each $g \in G$ and each Borel set $A \subset G$, we have

$$\mu(gA) = \mu(A).$$

Example 2.1.17 (Lebesgue measure). The Lebesgue measure on $(\mathbb{R}^n, +)$ is a Haar measure.

Proposition 2.1.18 (Haar measure). If G is a locally compact Lie group, then there exists a non-zero Haar measure μ on G .

Moreover, if ν is another left-invariant Radon measure on G , then we can find $c \geq 0$ such that $\nu = c\mu$.

Definition 2.1.19 (Unimodular group). Let G be a locally compact Lie group. If a non-zero Haar measure on G is also right-invariant, or equivalently if all Haar measures are right-invariant, we shall say that G is unimodular.

Proposition 2.1.20. Let G be a Lie group.

1. If G is compact, then G is unimodular.
2. If G is abelian and locally compact, then G is unimodular.

Definition 2.1.21 (Lebesgue measure). Let $(V, (\cdot, \cdot)_V)$ be a Euclidean space. If $e_1, \dots, e_{\dim V} \in V$, we call the Lebesgue measure the unique Haar measure such that the set

$$\{x \in V : 0 \leq (x, e_i)_V \leq 1 \text{ for each } i = 1, \dots, \dim V\}$$

has measure 1.

From now on, everytime we integrate on a Euclidean space, we do so with respect to the Lebesgue measure.

Given a locally compact topological group G , we denote by $\mathcal{M}(G)$ the set of all complex-valued functions that are measurable with respect to a chosen Haar measure.

Definition 2.1.22 (Lebesgue spaces). *Let G be a locally compact group and assume that $1 \leq p \leq \infty$.*

- *If $p < +\infty$, we let*

$$L^p(G) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{M}(G) : \int_G |f|^p \, dg < \infty \right\},$$

and the map

$$\|f\|_{L^p(G)} \stackrel{\text{def}}{=} \left(\int_G |f|^p \, dg \right)^{\frac{1}{p}}, \quad f \in L^p(G)$$

defines a norm, provided we identify the elements which are equal almost everywhere with respect to dg .

- *If $p = \infty$,*

$$L^\infty(G) \stackrel{\text{def}}{=} \{f \in \mathcal{M}(G) : \text{ess sup } |f| < \infty\}.$$

and the map

$$\|f\|_{L^\infty(G)} \stackrel{\text{def}}{=} \text{ess sup } |f|, \quad f \in L^\infty(G)$$

defines a norm, provided we identify the elements which are equal almost everywhere with respect to dg .

When $1 < p < +\infty$, the space $L^p(G)$ is well suited to functional analysis because its norm is *smooth* and *uniformly convex* (see e.g. [Wil13, Proposition 5.4.1 and Proposition 5.4.2]). In particular, its dual is described by the James-Ghika representation theorem [Wil13, Theorem 5.2.6] to be

$$(L^p(G))' = L^q(G), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In particular, $L^p(G)$ is equal to its bidual so that the Banach-Alaoglu theorem ([Rud91, Theorem 3.15]) applies.

The most interesting case is of course when $p = 2$, since $L^2(G)$ is a Hilbert space. Moreover, we can often expect a *Plancherel formula* (see [FR16, Subsection 1.8.2]).

2.2 REPRESENTATION THEORY

For a more detailed treatment, see e.g. [RT10].

To generalize the Fourier Transform to other Lie groups, we need an adequate substitute for the complex exponentials. Many of the properties of the Fourier Transform rely on the fact that the maps

$$x \in \mathbb{R}^n \mapsto e^{i2\pi(\xi, x)_V}, \quad \xi \in \mathbb{R}^n$$

are group homomorphisms $\mathbb{R}^n \rightarrow \mathrm{U}(1)$ for each fixed $\xi \in \mathbb{R}^n$.

Definition 2.2.1 (Unitary representations). *Let G be a group and $\mathcal{H}$ be a Hilbert space. A map*

$$\xi : G \mapsto \mathrm{Hom}(\mathcal{H})$$

is called a unitary representation (on $\mathcal{H}$) if

1. *For each $g \in G$, the map $\xi(g)$ is unitary:*

$$\xi(g)^{-1} = \xi(g)^*;$$

2. *If $g, h \in G$, then we have $\xi(gh) = \xi(g)\xi(h)$.*

The dimension of ξ is that of $\mathcal{H}$. If $\mathcal{H}$ is finite-dimensional, we let $d_\xi \stackrel{\text{def}}{=} \dim \mathcal{H}$ denote the dimension of ξ .

Example 2.2.2 (Right-regular representation). *Let G be a unimodular topological group. The right-regular representation is the representation*

$$\pi_K^R : G \rightarrow \mathrm{Hom}(L^2(G))$$

defined via

$$\pi_K^R(h)f(g) = f(gh)$$

for every $g, h \in G$.

Definition 2.2.3 (Invariant subspaces). *Let G be a group and ξ be a unitary representation of G on a Hilbert space $\mathcal{H}$. We shall say that a vector subspace $W \subset \mathcal{H}$ is invariant under ξ if for each $g \in G$, we have $\xi(g)W \subset W$.*

Definition 2.2.4 (Irreducibility). *Let G be a group and ξ be a unitary representation of G on a Hilbert space $\mathcal{H}$.*

1. If the only invariant subspaces of ξ are $\{0\}$ and $\mathcal{H}$, then ξ is said to be irreducible.
2. Otherwise, if there exists a non-trivial invariant subspace, then ξ is reducible.

Definition 2.2.5 (Equivalent representations). Let G be a group and $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Suppose ξ_1, ξ_2 are representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We shall say that ξ_1 and ξ_2 are equivalent if there exists an invertible linear map

$$A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$$

such that for each $g \in G$, we have

$$\xi_2(g) = A \circ \xi_1(g) \circ A^{-1}.$$

In this case, A is called an intertwining operator.

Definition 2.2.6 (Strongly continuous representations). Suppose that G is a group and let $\mathcal{H}$ be a Hilbert space. Suppose further that ξ is a representation of G on $\mathcal{H}$. We shall say that ξ is strongly continuous if for each $x \in \mathcal{H}$, the map

$$G \rightarrow \mathcal{H} : g \mapsto \xi(g)v$$

is continuous.

Definition 2.2.7 (Unitary dual). Let G be a locally compact topological group. The unitary dual of G , denoted by $\widehat{G}$, is the set of all equivalence classes of strongly continuous, irreducible, unitary representations of G .

Remark 2.2.8. Let G be a locally compact topological group. We shall often abuse notation and use $\widehat{G}$ to denote a set consisting of exactly one representation in each equivalence class of the actual unitary dual.

Example 2.2.9 ($\widehat{V}$). For each $\lambda \in V$, define

$$\xi_\lambda : V \rightarrow \mathbf{U}(1) : x \mapsto e^{i2\pi(\lambda, x)_V}. \quad (7)$$

It can be shown that

$$\widehat{V} = \{\xi_\lambda : \lambda \in V\}.$$

Therefore, the map

$$\lambda \mapsto \xi_\lambda \quad (8)$$

is a group isomorphism which allows us to give $\widehat{V}$ a vector space structure.

2.3 DISTRIBUTIONS

For this section, we assume that V is a vector space and K is a compact Lie group. Also, when we integrate on V , K or $V \times K$, we always mean with respect to a *Haar measure*.

We shall denote by $\mathfrak{k}$ the Lie algebra of K . We fix a basis $Y_1, \dots, Y_{\dim K}$ of $\mathfrak{k}$ and write

$$Y^\beta \stackrel{\text{def}}{=} Y_1^{\beta_1} \dots Y_{\dim K}^{\beta_{\dim K}}, \quad \beta \in \mathbb{N}^{\dim K}.$$

Definition 2.3.1 (Schwartz space). *We shall say that $f \in C^\infty(V \times K)$ is rapidly decreasing if for every $N \in \mathbb{N}$,*

$$\|f\|_{\mathcal{S}(V \times K), N} \stackrel{\text{def}}{=} \sup_{|\alpha|, |\beta| \leq N} \left| (1 + |x|)^N \frac{\partial^{|\alpha|}}{\partial x^\alpha} Y_k^\beta f(x, k) \right| < \infty.$$

The set of all rapidly decaying functions will be denoted by $\mathcal{S}(V \times K)$. The family $\{\|\cdot\|_{\mathcal{S}(V \times K), N} : N \in \mathbb{N}\}$ gives $\mathcal{S}(V \times K)$ the structure of a Fréchet space.

Definition 2.3.2 (Tempered distributions). *We shall say that*

$$\kappa : \mathcal{S}(V \times K) \rightarrow \mathbb{C}$$

is a tempered distribution if it is linear and continuous. The set of all tempered distributions will be denoted by $\mathcal{S}'(V \times K)$.

As usual, if $\kappa \in \mathcal{S}'(V \times K)$ and $\phi \in \mathcal{S}(V \times K)$, we let

$$\langle \kappa, \phi \rangle_{V \times K} \stackrel{\text{def}}{=} \kappa(\phi)$$

We shall also write

$$\int_{V \times K} \kappa(x, k) \phi(x, k) d(x, k) \stackrel{\text{def}}{=} \langle \kappa, \phi \rangle_{V \times K}$$

(which is motivated by the inclusion $L^1(V \times K) \subset \mathcal{S}'(V \times K)$) and we shall say that the above integral is interpreted *in the sense of distributions*.

Theorem 2.3.3 (Schwartz Kernel Theorem [Trè67]). *If the map*

$$T : \mathcal{S}(V \times K) \rightarrow \mathcal{S}'(V \times K)$$

is a continuous linear operator, then there exists a unique distribution $\kappa \in \mathcal{S}'(V \times K \times V \times K)$ such that

$$T\phi(x, k) = \int_{V \times K} \kappa(x, k; y, l) \phi(y, l) \, d(y, l)$$

in the sense of distributions.

2.4 CONVOLUTIONS

As the notion of convolution depends on the group structure, we introduce it in the abstract, general case of a locally compact unimodular Lie group G . Naturally, every integration on G is performed with respect to a Haar measure.

Definition 2.4.1 (Convolution). *Let $f_1, f_2 \in L^1(G)$. We define the convolution of f_1 and f_2 , denoted by $f_1 \star f_2$, via*

$$f_1 \star f_2(g) \stackrel{\text{def}}{=} \int_G f_1(h) f_2(h^{-1}g) \, dh \quad (9)$$

$$= \int_G f_1(gh^{-1}) f_2(h) \, dh. \quad (10)$$

Proposition 2.4.2. *Let $f_1, f_2 \in L^1(G)$. The convolution of f_1 and f_2 , $f_1 \star f_2$, belongs to $L^1(G)$. Moreover, we have*

$$\|f_1 \star f_2\|_{L^1(G)} \leq \|f_1\|_{L^1(G)} \|f_2\|_{L^1(G)}.$$

Proof. By the positivity of the integral, we get

$$\int_G \left| \int_G f_1(h) f_2(h^{-1}g) \, dh \right| \, dg \leq \int_G \int_G |f_1(h)| |f_2(h^{-1}g)| \, dh \, dg.$$

By the Fubini-Tonelli Theorem, we can change the order of integration above and therefore substitute g for hg , which yields

$$\begin{aligned} \int_G \left| \int_G f_1(h) f_2(h^{-1}g) \, dh \right| \, dg &\leq \int_G \int_G |f_1(h)| |f_2(g)| \, dg \, dh \\ &\leq \|f_1\|_{L^1(G)} \|f_2\|_{L^1(G)}, \end{aligned}$$

concluding the proof. $\square$

Lemma 2.4.3 (Young's inequality). *Let $f_1, f_2 \in L^2(G)$. The map*

$$f \stackrel{\text{def}}{=} f_1 \star f_2$$

is continuous.

Proof. Let $g \in G$. For each $h \in G$, we have

$$\begin{aligned} |f(h) - f(g)| &\leq \left| \int_G f_1(hl^{-1}) f_2(l) \, dl \right| \\ &\leq \|f_1\|_{L^2(G)} \|f_2(h\cdot) - f_2(g\cdot)\|_{L^2(G)}. \end{aligned}$$

By [Wil13, Lemma 4.3.8],

$$\lim_{h \rightarrow g} \|f_2(h\cdot) - f_2(g\cdot)\|_{L^2(G)} = 0,$$

showing that f is continuous at g . We conclude the proof by observing that g is arbitrary. $\square$

2.4.1 Convolutions of distributions

In this subsection, we assume that $G = V \ltimes K$, where V is a Euclidean space and $K \subset \text{SO}(V)$.

Suppose that $\kappa, \phi \in \mathcal{S}(G)$. It follows that (10) can be rewritten as

$$\begin{aligned} \phi \star \kappa &= \int_G \phi(gh^{-1}) \kappa(h) \, dh \\ &= \int_G \kappa(h) \pi_K^R(g^{-1}) \iota_G \phi(h) \, dg, \end{aligned}$$

where $\iota_G \phi(h) \stackrel{\text{def}}{=} \phi(h^{-1})$ for each $h \in G$. Similarly, we show that (9) becomes

$$\begin{aligned} \kappa \star \phi &= \int_G \kappa(h) \phi((g^{-1}h)^{-1}) \, dh \\ &= \int_G \kappa(h) \pi_K^L(g^{-1}) \iota_G \phi(h) \, dg. \end{aligned}$$

Both of the above can be generalized for $\kappa \in \mathcal{S}'(G)$ provided that we interpret the above in the sense of distributions, leading to the following definition.

Definition 2.4.4 (Convolution). *Let $\kappa \in \mathcal{S}'(G)$ and $\phi \in \mathcal{S}(G)$. For every $g \in G$, we let*

$$\begin{aligned} \phi \star \kappa(g) &\stackrel{\text{def}}{=} \langle \kappa, \pi_K^R(g^{-1}) \iota_G \phi \rangle_G \\ \kappa \star \phi(g) &\stackrel{\text{def}}{=} \langle \kappa, \pi_K^L(g) \iota_G \phi \rangle_G, \end{aligned}$$

where $\iota_G \phi(h) \stackrel{\text{def}}{=} \phi(h^{-1})$ for each $h \in G$.

Following [FR16, Section 5.1], we define the convolution between two distributions associated with left-invariant operators that are bounded between appropriate Sobolev spaces.

Definition 2.4.5 (Convolution of distributions). *Let $\kappa_1, \kappa_2 \in \mathcal{S}'(G)$ and write*

$$T_i : \phi \in \mathcal{S}(G) \mapsto \phi \star \kappa_i, \quad i = 1, 2.$$

If there exists $a, b, c \in \mathbb{R}$ such that T_1 is bounded from $L_a^2(G)$ to $L_b^2(G)$ and T_2 is bounded from $L_b^2(G)$ to $L_c^2(G)$, then we define $\kappa_1 \star \kappa_2$ to be the right convolution kernel of

$$\phi \in \mathcal{S}(G) \mapsto (T_1 \circ T_2)(\phi).$$

2.5 FOURIER TRANSFORM

If G is separable, locally compact, unimodular and of type I [FR16], we have a Fourier-Plancherel theory.

Definition 2.5.1 (Fourier coefficient). Let $\xi \in \widehat{G}$. If $f \in L^1(G)$, we define the Fourier coefficient of f at ξ , $\mathcal{F}_G f(\xi)$, via

$$\mathcal{F}_G f(\xi) \stackrel{\text{def}}{=} \int_G f(g) \xi(g)^* dg. \quad (11)$$

The map

$$\mathcal{F}_G f : \xi \mapsto \mathcal{F}_G f(\xi)$$

is called the Fourier transform of f .

The Fourier transform satisfies the following properties.

Proposition 2.5.2. Let $f, f_1, f_2 \in L^1(G)$ and $\xi \in \widehat{G}$. The Fourier Transform satisfies the following properties.

1. For each $g \in G$, we have

$$\mathcal{F}_G \{f(\cdot g)\}(\xi) = \xi(g) \mathcal{F}_G f(\xi), \quad \mathcal{F}_G \{f(g \cdot)\}(\xi) = \mathcal{F}_G f(\xi) \xi(g).$$

2. We have

$$\mathcal{F}_G \{f_1 \star f_2\}(\xi) = \mathcal{F}_G f_2(\xi) \mathcal{F}_G f_1(\xi).$$

3. If $f \in \mathcal{S}(G) \cap L^1(G)$ and $X \in \mathfrak{g}$,

$$\mathcal{F}_G \{Xf\}(\xi) = \xi(X) \mathcal{F}_G f(\xi), \quad \mathcal{F}_G \{\tilde{X}f\}(\xi) = \mathcal{F}_G f(\xi) \xi(X).$$

In the above,

$$\xi(X) \stackrel{\text{def}}{=} \left. \frac{d}{dt} \right|_{t=0} \xi(\exp_G(tX))$$

Proof. Let us prove each claim separately.

1. Let $g \in G$. To obtain the first identity, let us observe that

$$\begin{aligned} \mathcal{F}_G \{f(\cdot g)\}(\xi) &= \int_G f(hg) \xi(h)^* dh \\ &= \int_G f(h) \xi(hg^{-1})^* dh, \end{aligned}$$

where we use the invariance of the Haar measure under the group action. Continuing the above calculation, we obtain

$$\mathcal{F}_G \{f(\cdot g)\}(\xi) = \int_G f(h) \xi(g) \xi(h)^* dh = \xi(g) \mathcal{F}_G f(\xi),$$

which is what we intended to show.

Let us proceed similarly for the left-translation. Let us observe that this time

$$\begin{aligned}\mathcal{F}_G\{f(g\cdot)\}(\xi) &= \int_G f(gh)\xi(h)^* dh \\ &= \int_G f(h)\xi(g^{-1}h)^* dh,\end{aligned}$$

where we again use the invariance of the Haar measure under the group action. Continuing the above calculation, we now obtain

$$\mathcal{F}_G\{f(g\cdot)\}(\xi) = \int_G f(h)\xi(h)^* \xi(g) dh = \mathcal{F}_G f(\xi)\xi(g),$$

which is the desired result.

2. By Proposition 2.4.2, $f_1 \star f_2 \in L^1(G)$. Therefore, we can compute its Fourier Transform, yielding

$$\mathcal{F}_G\{f_1 \star f_2\}(\xi) = \int_G \int_G f_1(h)f_2(h^{-1}g)\xi(g)^* dg.$$

Using $\xi(g)^* = (\xi(h)\xi(h^{-1}g))^* = \xi(h^{-1}g)^* \xi(h)^*$ in the above, we obtain

$$\begin{aligned}\mathcal{F}_G\{f_1 \star f_2\}(\xi) &= \int_G \int_G f_2(h^{-1}g)\xi(h^{-1}g)^* dg f_1(h)\xi(h)^* dh \\ &= \mathcal{F}_G f_2(\xi)\mathcal{F}_G f_1(\xi).\end{aligned}$$

3. Let $X \in \mathfrak{g}$. Using integration by parts, we have

$$\mathcal{F}_G\{Xf\}(\xi) = \int_G Xf(g)\xi(g)^* dg = - \int_G f(g)X\xi(g)^* dg.$$

Now, we know that

$$\begin{aligned}X\xi(g)^* &= \frac{d}{dt}\bigg|_{t=0} \xi(g \exp_G(tX))^* = \frac{d}{dt}\bigg|_{t=0} \xi(\exp_G(tX))^* \xi(g)^* \\ &= - \frac{d}{dt}\bigg|_{t=0} \xi(\exp_G(tX))\xi(g)^*,\end{aligned}$$

which implies that

$$X\xi(g)^* = -\xi(X)\xi(g)^*.$$

Substituting this into the Fourier transform above, we obtain

$$\mathcal{F}_G\{Xf\}(\xi) = \int_G f(g)\xi(X)\xi(g)^* dg = \xi(X)\mathcal{F}_G f(\xi).$$

Let us now turn our attention to the case of the right-invariant vector fields. Here,

$$\begin{aligned}\tilde{X}\xi(g)^* &= \frac{d}{dt}\bigg|_{t=0} \xi(\exp_G(tX)g)^* = \frac{d}{dt}\bigg|_{t=0} \xi(g)^*\xi(\exp_G(tX))^* \\ &= -\xi(g)^*\xi(X),\end{aligned}$$

which allows us to conclude that

$$\begin{aligned}\mathcal{F}_G\{\tilde{X}f\}(\xi) &= \int_G \tilde{X}f(g)\xi(g)^* dg = - \int_G f(g)\tilde{X}\xi(g)^* dg \\ &= \int_G f(g)\xi(g)^*\xi(X) dg = \mathcal{F}_G f(\xi)\xi(X).\end{aligned}$$

□

Proposition 2.5.3 (Plancherel formula). *There exists a unique positive σ -finite measure $\mu_{\hat{G}}$ on $\hat{G}$ such that the following property holds: for every $f \in C(G)$ with compact support, the map*

$$\xi \in \hat{G} \mapsto \text{tr}(\mathcal{F}_G f(\xi)\mathcal{F}_G f(\xi)^*)$$

is integrable against $\mu_{\hat{G}}$ and we have

$$\int_G |f|^2 dg = \int_{\hat{G}} \text{tr}(\mathcal{F}_G f(\xi)\mathcal{F}_G f(\xi)^*) d\mu_{\hat{G}}(\xi).$$

2.5.1 Fourier transform of distributions

It is not always possible to extend the Fourier Transform onto the set of tempered distributions, nor is it necessary to have such a general extension for the purpose of pseudo-differential calculus. Here, we briefly present V. Fischer and M. Ruzhansky's construction (see [FR16, Subsection 5.1.2]), which works for a much more general class of groups.

The central idea is the following: the *abstract Plancherel Theorem* [FR16, Subsection 1.8.2] implies that any left invariant operator T belongs to $\mathcal{L}(L^2(G))$ if and only if

$$T\phi(g) = \int_{\widehat{G}} \mathrm{tr}(\xi(g)\sigma(\xi)\mathcal{F}_G\phi(\xi)) \, d\mu_{\widehat{G}}(\xi)$$

for some measurable field $\xi \in \widehat{G} \mapsto \sigma(\xi) \in \mathcal{L}(\mathcal{H}_\xi)$ which satisfies

$$\mathrm{ess\,sup}_{\xi \in \widehat{G}} \|\sigma(\xi)\|_{\mathcal{L}(\mathcal{H}_\xi)} < \infty.$$

Motivated by Proposition 2.5.2, we say that σ is then the Fourier transform of the *right-convolution kernel* associated with T .

If $\kappa \in \mathcal{S}'(G)$ is the right-convolution kernel of a bounded operator

$$T : L_a^2(G) \rightarrow L_b^2(G),$$

we define

$$\mathcal{F}_G\kappa(\xi) \stackrel{\mathrm{def}}{=} \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{b}{2}} \mathcal{F}_G\{(I - \mathcal{L}_G)^{\frac{b}{2}} (I - \tilde{\mathcal{L}}_G)^{-\frac{a}{2}} \kappa\} \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{a}{2}},$$

and check we would obtain the same result with different $a, b \in \mathbb{R}$ such that

$$T : L_a^2(G) \rightarrow L_b^2(G)$$

is bounded. For more detail, see [FR16].

2.6 COMPACT LIE GROUPS

In the sequel, K will denote a compact Lie group, and dk the associated normalized Haar measure. This section is devoted to recalling elementary facts on compact Lie groups. For more detail, see e.g. [RT10; Kna02].

We equip K with a normalized bi-invariant Riemannian metric, and denote by $\mathcal{L}_K$ the associated *Laplace-Beltrami* operator.

Given $\tau \in \widehat{K}$, we let d_τ denote the *dimension* of K .

The following result is fundamental on compact Lie groups.

Theorem 2.6.1 (Peter-Weyl). *The set*

$$\mathcal{B} \stackrel{\text{def}}{=} \{d_\tau \tau_{mn} : \tau \in \widehat{K}, m, n = 1, \dots, d_\tau\} \subset L^2(K)$$

forms an orthonormal basis of $L^2(K)$.

As the set

$$V_\tau \stackrel{\text{def}}{=} \{\tau_{mn} : m, n = 1, \dots, d_\tau\}$$

is invariant under the left- and the right-regular representation, they are eigenspaces of the *Casimir operator*, which coincides with $\mathcal{L}_K$.

The Plancherel formula takes the following form.

Proposition 2.6.2 (Plancherel formula on K). *Let $f \in L^2(K)$. The following formula holds*

$$\int_K |f|^2 dk = \sum_{\tau \in \widehat{G}} d_\tau \operatorname{tr}(\mathcal{F}_K f(\tau) \mathcal{F}_K f(\tau)^*).$$

We shall sometimes use the following notation hereafter.

Definition 2.6.3 (Japanese bracket). *Let $\tau \in \widehat{K}$. We let*

$$\langle \tau \rangle_{\widehat{K}} \stackrel{\text{def}}{=} (1 + \lambda_\tau)^{\frac{1}{2}},$$

where λ_τ is such that $-\mathcal{L}_K \tau_{11} = \tau_{11}$.

3 | MOTION GROUPS

From now on, V will denote a finite-dimensional vector space over $\mathbb{R}$, while K is a compact Lie subgroup of $\mathrm{GL}(V)$. We then define the *motion group* as the *semi-direct product* between V and K .

In this chapter, we start by observing that the compactness of K is a very rigid condition, as it implies that $K \subset \mathrm{SO}(V)$. Using the fact that both $\mathcal{L}_V$ and the Lebesgue measure are invariant under $\mathrm{SO}(V)$, we show that the motion group has the *same Haar measure* and *same Laplacian* as $V \times K$. Moreover, the representations of the motion group seem formally very close to those of $V \times K$, provided that they are grouped on both sides.

All these observations lead to the conclusion that although the Fourier theory on our group might seem identical to the one of $V \times K$, the resemblance is actually very deceptive.

3.1 MOTION GROUPS

Given $x, y \in V$, let us denote by $g_{x,k}$ the affine transformation on V consisting in a rotation by k followed by a translation by x . Naturally, such transformations compose in the following way

$$g_{x,k} \circ g_{y,l} = g_{x+ky,kl},$$

where $x, y \in V$ and $k, l \in \mathrm{SO}(V)$.

This motivates the following definition.

Definition 3.1.1 (Motion group). *Let G be a group. We shall say that G is a motion group if there exists a finite dimensional real vector space V and a compact connected Lie group $K \subset \mathrm{GL}(V)$ such that G is the semi-direct product $V \rtimes K$. More precisely, $G = V \times K$ as a set, and*

$$(x, k)(y, l) \stackrel{\text{def}}{=} (x + ky, kl)$$

for every $(x, k), (y, l) \in G$.

Remark 3.1.2. Let $x \in V$ and $k \in K$. We will never identify x with $(x, I_V) \in V \ltimes K$, or k with $(0, k) \in V \ltimes K$. Therefore, when we write kx , it will always mean the vector obtained by rotating x by k , i.e. $k(x)$.

Example 3.1.3 (Vector spaces). Suppose that $K = \{I_V\}$. It follows that $G \stackrel{\text{def}}{=} V \ltimes K$ is isomorphic to V .

Example 3.1.4 (Euclidean Motion Groups). For each $n \in \mathbb{N}$, let

$$\text{SE}(n) \stackrel{\text{def}}{=} \{g \in \text{Affine}(\mathbb{R}^n) : g(x) = Ax + y, A \in \text{SO}(n), x, y \in \mathbb{R}^n\}.$$

The elements of $\text{SE}(n)$ are called rigid motions, while $\text{SE}(n)$ is called the Euclidean motion group.

It is easily shown that associating $(x, k) \in \mathbb{R}^n \ltimes \text{SO}(n)$ to the motion

$$g_{(x,k)} : \mathbb{R}^n \rightarrow \mathbb{R}^n : y \mapsto x + ky$$

defines a group isomorphism between $\mathbb{R}^n \ltimes \text{SO}(n)$ and $\text{SE}(n)$. We shall therefore identify $\text{SE}(n)$ with $\mathbb{R}^n \ltimes \text{SO}(n)$ from now on.

Example 3.1.5. Let $n \in \mathbb{N}$. Consider the group

$$\{g \in \text{Affine}(\mathbb{C}^n) : g(x) = Ux + y, U \in \text{SU}(n), y \in \mathbb{C}^n\}$$

where the law is the composition of functions. Arguing as in Example 3.1.4, the above group can be identified with $\mathbb{C}^n \ltimes \text{SU}(n)$.

Remark 3.1.6. Since in our examples (e.g. Example 3.1.5) our vector space might be $\mathbb{C}^n$, we choose to use V to denote the vector space instead of simply $\mathbb{R}^n$ to avoid any confusion.

Up until now, the vector space V was not given a Euclidean structure. The following Lemma shows that V can be endowed with an inner product that behaves well under the action of K .

Lemma 3.1.7 (K -invariant inner product). Let V be a vector space, and K be a compact Lie group acting on V . There exists an inner product $(\cdot, \cdot)_V : V \times V \rightarrow \mathbb{R}$ which is K -invariant, i.e. for each $k \in K$ and every $x, y \in V$, we have

$$(x, y)_V = (kx, ky)_V.$$

Proof. Let $Q : V \times V \rightarrow \mathbb{R}$ be an arbitrary inner product. Given $x, y \in V$, we let

$$(x, y)_V \stackrel{\text{def}}{=} \int_K Q(kx, ky) dk,$$

where dk is the normalized Haar measure on K .

It follows that if $k \in K$, then using the right-invariance of the Haar measure on compact groups (for this, see for example [RT10, Theorem 7.4.21]), we obtain

$$(kx, ky)_V = \int_K Q(hkx, hky) dh = \int_K Q(hx, hy) dh = (x, y)_V,$$

i.e. $(\cdot, \cdot)_V$ is K -invariant.

The fact that $(\cdot, \cdot)_V$ is bilinear and non-negative definite follows immediately from the fact that Q has those properties. Now, if $x \in V$ are such that $(x, x)_V = 0$, then it follows that $Q(kx, kx) = 0$ for almost every $k \in K$, hence for at least one such k . However, that means that $kx = 0$, hence $x = 0$ as k is invertible. $\square$

From now on, V will be given the structure of *Euclidean space* with a K -invariant inner product whose existence is given by Lemma 3.1.7. To obtain a *unitary Fourier Transform*, we normalise the *Haar measure* on V as follows.

Definition 3.1.8 (Lebesgue measure). *We call the Lebesgue measure on V the unique Haar measure μ on V such that*

$$\mu(\{x \in V : (x, x)_V \leq 1\}) = \frac{\pi^{\dim V/2}}{\Gamma(\frac{\dim V}{2} + 1)}.$$

From now on, integration on V will always be performed with respect to the above Lebesgue measure.

Lemma 3.1.7 easily implies the following.

Lemma 3.1.9. *With the inner product defined in Lemma 3.1.7, K is a Lie subgroup of $\text{SO}(V)$. In particular, any Haar measure on V is also invariant under the action of K .*

From the invariance of the Lebesgue measure on V under the action of K , we deduce that the *Haar measure* of $G = V \ltimes K$ is that of the direct product $V \times K$.

Lemma 3.1.10 (Haar measure). *If dx is the Lebesgue measure on V and dk is the normalized Haar measure on K , then the product measure $dx dk$ is a Haar measure on $G = V \ltimes K$, which is both left and right-invariant.*

Proof. Let $(x, k) \in G$ and $f \in L^1(G)$.

$$\int_G f((x, k)(y, l)) d(y, l) = \int_V \int_K f(x + ky, kl) dl dy$$

Now, let us substitute y for $k^{-1}(y - x)$ and l for $k^{-1}l$ in the above. As the Lebesgue measure is invariant under $SO(V)$ and under translations, and because the Haar measure dl is left-invariant, we obtain

$$\begin{aligned} \int_G f((x, k)(y, l)) d(y, l) &= \int_V \int_K f(y, l) dl dy \\ &= \int_G f(y, l) d(y, l), \end{aligned}$$

showing that $dy dl$ is indeed a Haar measure on G .

Since by Proposition 2.1.20, dl is also right-invariant, arguing similarly shows that $dy dl$ is also right-invariant. $\square$

The following proposition summarizes how K affects key objects on V .

Proposition 3.1.11. *Let V be a finite dimensional vector space and K be a subgroup of $SO(V)$. The following properties hold.*

1. *The Lebesgue measure on V is invariant under K , i.e. for every $k \in K$ and each Borel set $A \subset V$, we have*

$$\int_A 1 dx = \int_{kA} 1 dx.$$

2. *The Laplacian on V is invariant under K , i.e. for every $k \in K$ and every $\phi \in C^\infty(V)$, we have*

$$\mathcal{L}_V(\phi \circ k)(x) = \mathcal{L}_V \phi(kx).$$

3. *The action of K on V commutes with the dilation structure of V .*

Proof. 1. This follows easily from the change of variables formula,

$$\int_{kA} 1 dx = \int_A (1 \circ k) \det k dx = \int_A 1 dx,$$

where we use the fact that $\det k = 1$ since $k \in SO(V)$.

2. First, observe that

$$d(\phi \circ k)(x) = d\phi(kx)k$$

which implies that $\nabla_V(\phi \circ k)(x) = k^{-1}\nabla_V\phi(kx)$.

From there, using the fact that $\text{Hess} = d\nabla_V$,

$$\text{Hess}(\phi \circ k)(x) = (d\nabla_V(\phi \circ k))(x) = k^{-1}(\text{Hess}\phi(kx))k.$$

Therefore, we conclude by observing that

$$\mathcal{L}_V(\phi \circ k)(x) = \text{tr}(\text{Hess}(\phi \circ k)(x)) = \text{tr}(\text{Hess}\phi(kx)) = \mathcal{L}_V\phi(kx).$$

3. This is a direct consequence of the fact that the dilation structure on V is scalar.

□

3.2 LIE ALGEBRA STRUCTURE

The difference between the semi-direct structure and its direct counterpart appears naturally when we study the *Lie algebra*.

We shall henceforth denote by $\mathfrak{k}$ the Lie algebra of K .

Definition 3.2.1 (Lie algebra of G). *We shall call the set*

$$\mathfrak{g} \stackrel{\text{def}}{=} V \oplus \mathfrak{k}$$

the Lie algebra of G . Given $(X_1, Y_1), (X_2, Y_2) \in \mathfrak{g}$, we define its Lie bracket via

$$[(X_1, Y_1), (X_2, Y_2)]_{\mathfrak{g}} \stackrel{\text{def}}{=} (Y_1X_2 - Y_2X_1, [Y_1, Y_2]_{\mathfrak{k}}).$$

The Lie algebra $\mathfrak{g}$ is characterized by the following *commutation relations*.

Lemma 3.2.2 (Commutation relations). *Suppose that $X_1, X_2, X \in V$ and $Y_1, Y_2, Y \in \mathfrak{k}$. We have the following commutation relations*

$$[X_1, X_2]_{\mathfrak{g}} = 0, \quad [Y, X]_{\mathfrak{g}} = (YX, 0), \quad [Y_1, Y_2]_{\mathfrak{g}} = [Y_1, Y_2]_{\mathfrak{k}}.$$

On the motion group, the *exponential map* has the following expression.

Definition 3.2.3 (Exponential map). *The exponential map on G is the map*

$$\exp_G : \mathfrak{g} \rightarrow G : (X, Y) \mapsto \left(\sum_{k=0}^{\infty} \frac{Y^k}{(k+1)!} X, \exp_K Y \right).$$

Definitions 3.2.3 and 3.2.1 are motivated by the following proposition.

Proposition 3.2.4. *Fix an orthonormal basis $e_1, \dots, e_n$ of V , and consider the map*

$$\Phi : G \rightarrow \mathrm{GL}(n+1) : (x, k) \mapsto \begin{pmatrix} \tilde{k} & \tilde{x} \\ 0 & 1 \end{pmatrix}$$

where $\tilde{k} \in \mathrm{SO}(n)$ and $\tilde{x} \in \mathbb{R}^n$ satisfy

$$\tilde{k}_{ij} = (ke_j, e_i)_V \text{ and } \tilde{x}_i = (x, e_i)_V$$

for all $i, j \in \{1, \dots, n\}$.

The following properties hold.

1. Φ is a faithful representation of G .
2. The map

$$\mathrm{d}\Phi|_{(0, I_V)} : \mathfrak{g} \rightarrow \mathfrak{Lie}(\Phi(G))$$

is an isomorphism of Lie algebras.

3. The exponential map $\exp_{\mathfrak{g}}$ satisfies

$$(\Phi \circ \exp_{\mathfrak{g}})(X) = \sum_{j=0}^{+\infty} \frac{(\mathrm{d}\Phi|_{(0, I_V)}(X))^j}{j!}$$

for every $X \in \mathfrak{g}$.

Proof. Assume without loss of generality that $V = \mathbb{R}^n$ and that the chosen basis is the canonical one so that $k = \tilde{k}$ and $x = \tilde{x}$.

1. Suppose $(x, k), (y, l) \in G$. It follows that

$$\begin{aligned}\Phi(x, k)\Phi(y, l) &= \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l & y \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} kl & ky + x \\ 0 & 1 \end{pmatrix} = \Phi((x, k)(y, l)),\end{aligned}$$

so Φ is indeed a group representation. The fact that Φ is faithful is trivial.

2. Suppose $X \oplus Y \in \mathfrak{g}$. We can check that

$$\begin{aligned}\mathbf{d}\Phi|_{(0, I_V)}(X \oplus Y) &= [(X + Y)\Phi](0_V, I_V) \\ &= \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} Y & X \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Then, we can check that

$$\begin{aligned}\mathbf{d}\Phi|_{(0, I_V)}([(X_1, Y_1), (X_2, Y_2)]_{\mathfrak{g}}) &= \mathbf{d}\Phi|_{((0, I_V))}(Y_1 X_2 - Y_2 X_1, Y_1 Y_2 - Y_2 Y_1) \\ &= \begin{pmatrix} Y_1 Y_2 & Y_1 X_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} Y_2 Y_1 & Y_2 X_1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} Y_1 & X_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_2 & X_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} Y_2 & X_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 & X_1 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

which means that we have

$$\begin{aligned}\mathbf{d}\Phi|_{(0, I_V)}([(X_1, Y_1), (X_2, Y_2)]_{\mathfrak{g}}) &= [\mathbf{d}\Phi|_{(0, I_V)}(X_1, Y_1), \mathbf{d}\Phi|_{(0, I_V)}(X_2, Y_2)]_{\mathbf{d}\Phi|_{(0, I_V)}(\mathfrak{g})}.\end{aligned}$$

3. Suppose again that $X \oplus Y \in \mathfrak{g}$. We can easily check by induction that

$$(\mathbf{d}\Phi|_{(0, I_V)}(X, Y))^k = \begin{pmatrix} Y^k & Y^{k-1}X \\ 0 & 0 \end{pmatrix}$$

for $k \geq 1$.

Therefore, it follows that

$$\begin{aligned} \sum_{k=0}^{+\infty} \frac{1}{k!} (\mathrm{d}\Phi|_{(0, I_V)}(X, Y))^k &= \begin{pmatrix} I_V & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=1}^{+\infty} \frac{1}{k!} \begin{pmatrix} Y^k & Y^{k-1}X \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \exp_t Y & \sum_{k=0}^{+\infty} \frac{1}{(k+1)!} Y^k X \\ 0 & 1 \end{pmatrix} \\ &= (\Phi \circ \exp_{\mathfrak{g}})(X, Y), \end{aligned}$$

which is what we wanted to show. □

Naturally, we can define differential operators which are left- or right-invariant via the exponential map.

Definition 3.2.5 (Left and right-invariant vector fields). *Let $X \in \mathfrak{g}$. We define X , the left-invariant differential operator associated with X , via*

$$Xf(g) \stackrel{\text{def}}{=} \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} f(g \exp_G(tX)),$$

for each $f \in C^\infty(G)$.

Similarly, we define $\tilde{X}$, the right-invariant differential operator associated with X , via

$$\tilde{X}f(g) \stackrel{\text{def}}{=} \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} f(\exp_G(tX)g),$$

where $f \in C^\infty(G)$.

Proposition 3.2.6. *Let $X \in \mathfrak{g}$. The differential operator X is the only differential operator satisfying the following properties.*

1. X is left-invariant, i.e. for every $h \in G$, we have

$$(Xf(h \cdot))(g) = (Xf)(hg).$$

2. The vector in $T_e G$ corresponding to the differentiation by X at e is precisely X .
3. Given $X, Y \in \mathfrak{g}$, we have

$$(XY - YX)|_e = [X, Y]_{\mathfrak{g}}.$$

We now describe explicitly the Lie algebra of Euclidean motion groups.

Example 3.2.7 (2-dimensional Euclidean motion group). Assume $G = \mathbb{R}^2 \ltimes \mathbb{T}$. The Lie algebra is the vector space

$$\mathfrak{g} \stackrel{\text{def}}{=} \mathbb{R}^2 \oplus \text{Skew}(\mathbb{R}^2).$$

The vectors

$$\begin{aligned} X_1 &= (1, 0) \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & X_2 &= (0, 1) \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ X_3 &= (0, 0) \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

form a basis of $\mathfrak{g}$ which satisfies the commutation relations

$$[X_1, X_2]_{\mathfrak{g}} = 0, \quad [X_2, X_3]_{\mathfrak{g}} = X_1, \quad [X_3, X_1]_{\mathfrak{g}} = X_2.$$

Moreover, if $f \in C^\infty(G)$, then the associated left-invariant operators act via

$$\begin{aligned} X_1 f(x, t) &= \cos(2\pi t) \frac{\partial f}{\partial x_1}(x, t) + \sin(2\pi t) \frac{\partial f}{\partial x_2}(x, t), \\ X_2 f(x, t) &= -\sin(2\pi t) \frac{\partial f}{\partial x_1}(x, t) + \cos(2\pi t) \frac{\partial f}{\partial x_2}(x, t), \\ X_3 f(x, t) &= \frac{\partial f}{\partial t}(x, t), \end{aligned}$$

where $(x, t) \in \mathbb{R}^2 \ltimes \mathbb{T}$.

Example 3.2.8 (Euclidean motion groups). Assume $G = \mathbb{R}^n \ltimes \text{SO}(n)$. Its Lie algebra is the vector space

$$\mathfrak{g} \stackrel{\text{def}}{=} \mathbb{R}^n \oplus \text{Skew}(\mathbb{R}^n).$$

For each pair of distinct $i, j \in \{1, \dots, n\}$, we let E_{ij} be the only matrix satisfying

$$E_{ij} e_k = \delta_{i,k} e_j - \delta_{j,k} e_i$$

for each $k \in \{1, \dots, n\}$.

A basis of $\mathfrak{g}$ can now be given by

$$\begin{aligned} X_i &\stackrel{\text{def}}{=} e_i \oplus 0_{\mathbb{R}^{n \times n}}, & i = 1, \dots, n \\ X_{ij} &\stackrel{\text{def}}{=} 0_{\mathbb{R}^n} \oplus E_{ij}, & i, j \in \{1, \dots, n\} \text{ with } i < j, \end{aligned}$$

and these vectors satisfy the following commutation relations

$$\begin{aligned} [X_i, X_j]_{\mathfrak{g}} &= 0 \\ [X_{ij}, X_k]_{\mathfrak{g}} &= \delta_{i,k} X_j - \delta_{j,k} X_i \\ [X_{ij}, X_{kl}]_{\mathfrak{g}} &= \delta_{j,l} X_{ik} - \delta_{j,k} X_{il} \end{aligned}$$

where in the above $i < j$, $k < l$ and for the last commutation relation we additionally assume $i < k$.

Let us compute the expression of a left-invariant derivative arising from $V \subset \mathfrak{g}$.

Lemma 3.2.9. *Let $X \in V \subset \mathfrak{g}$. If $\phi \in C^\infty(G)$,*

$$X\phi(x, k) = \left(k^{-1} \nabla_V \phi(x, k), X \right)_V.$$

Proof. By a simple calculation,

$$\begin{aligned} X\phi(x, k) &= \frac{d}{dt} \Big|_{t=0} \phi((x, k)(tX, I_V)) \\ &= \frac{d}{dt} \Big|_{t=0} \phi(x + tkX, k) \\ &= (\nabla_V \phi(x, k), kX)_V. \end{aligned}$$

From there, it follows that:

$$X\phi(x, k) = \left(k^{-1} \nabla_V \phi(x, k), X \right)_V.$$

□

Let us choose a basis of $\mathfrak{g}$. Since K is compact, $\mathfrak{k}$ is *reductive* [Kna02, Corollary 4.25]. Therefore, it follows that

$$\mathfrak{g} = V \oplus \mathfrak{a} \oplus \mathfrak{s},$$

where $\mathfrak{a}$ is *abelian* and $\mathfrak{s}$ is *semisimple*. In particular, we can naturally define an inner product $(\cdot, \cdot)_{\mathfrak{g}}$ on $\mathfrak{g}$ via the Killing form of $\mathfrak{k}$ [Kna02, Corollary 4.26].

From now on, $X_1, \dots, X_{\dim G}$ will denote a basis such that

1. The collection is orthonormal with respect to $(\cdot, \cdot)_g$.
2. If $1 \leq j \leq \dim V$, then $X_j \in V$, otherwise $X_j \in \mathfrak{k}$.

Definition 3.2.10. Let $\alpha \in \mathbb{N}^{\dim G}$. We define the left-invariant differential operator X^α via

$$X^\alpha = X_1^{\alpha_1} \cdots X_{\dim G}^{\alpha_{\dim G}}.$$

The following definition is a very important example of left-invariant differential operator.

Definition 3.2.11 (Left-invariant Laplacian). The left-invariant Laplacian $\mathcal{L}_G$ is the left-invariant differential operator

$$\mathcal{L}_G \stackrel{\text{def}}{=} \sum_{j=1}^{\dim G} X_j^2.$$

Deceptively, the left-invariant Laplacian of G is identical to the one of $V \times K$. It is however important to keep in mind that this operator is *not right-invariant* with the semidirect structure.

Proposition 3.2.12 (Left-invariant Laplacian). Let $f \in C^\infty(G)$. If $(x, k) \in G$, we check that

$$\mathcal{L}_G f(x, k) = \mathcal{L}_V f(x, k) + \mathcal{L}_K f(x, k).$$

Proof. Let $(x, k) \in G$. By Lemma 3.2.9, we check that for each $j \in \{1, \dots, \dim V\}$,

$$\begin{aligned} X_j X_j f(x, k) &= X_j \left(k^{-1} \nabla_V f(x, k), e_j \right)_V \\ &= \left(k^{-1} d \nabla_V f(x, k) [k e_j], e_j \right)_V \\ &= \left(k^{-1} \text{Hess } f(x, k) k e_j, e_j \right)_V. \end{aligned}$$

Summing with respect to j , we get that

$$\sum_{j=1}^{\dim V} X_j X_j f(x, k) = \text{tr Hess } f(x, k) = \mathcal{L}_V f(x, k).$$

The sum over $j = \dim V + 1, \dots, \dim G$ is the *Casimir element* of $\mathfrak{k}$. By [RT10, Subsection 8.3.2], it corresponds to the *Laplace-Beltrami operator* on K . $\square$

It will be important later to be able to establish a link between right- and left-invariant differential operators.

Proposition 3.2.13. *Let $X \in \mathfrak{g}$.*

- *If $X \in V$, then its associated right-invariant vector field acts on $C^\infty(G)$ via*

$$\tilde{X}\phi(x, k) = (\nabla_V \phi(x, k), X)_V,$$

where $(x, k) \in G$ and $\phi \in C^\infty(G)$.

- *If $Y \in \mathfrak{k}$, then its associated right-invariant vector field acts on $C^\infty(G)$ via*

$$\tilde{Y}\phi(x, k) = \sum_{j=1}^{\dim V} (Yx)_j \frac{\partial \phi}{\partial x_j}(x, k) + \tilde{Y}_k \phi(x, k)$$

where $(x, k) \in G$, $\phi \in C^\infty(G)$, and $\tilde{Y}_k$ is the right-invariant vector field associated with Y on the manifold K acting exclusively on the variable $k \in K$.

Proof. Fix $(x, k) \in G$, $\phi \in C^\infty(G)$ and let us check both of the claims separately.

- Using the fact that $\exp_G(tX) = (tX, I_V)$, we obtain

$$\tilde{X}\phi(x, k) = \left. \frac{d}{dt} \right|_{t=0} \phi((tX, I_V)(x, k)) = \left. \frac{d}{dt} \right|_{t=0} \phi(x + tX, k),$$

from which the conclusion follows easily.

- Using the fact that $\exp_G(tY) = (0_V, \exp_K(tY))$, we obtain

$$\begin{aligned} \tilde{Y}\phi(x, k) &= \left. \frac{d}{dt} \right|_{t=0} \phi((0_V, \exp_K(tY))(x, k)) \\ &= \left. \frac{d}{dt} \right|_{t=0} \phi(\exp_K(tY)x, \exp_K(tY)k) \\ &= \sum_{j=1}^{\dim V} (Yx)_j \frac{\partial \phi}{\partial x_j}(x, k) + \tilde{Y}_k \phi(x, k) \end{aligned}$$

as required. □

Corollary 3.2.14. *Let $j \in \{1, \dots, \dim G\}$.*

- If $j \leq \dim V$, then we have

$$\tilde{X}_j = \sum_{i=1}^{\dim V} (ke_i, e_j)_V X_i.$$

- If $j > \dim V$,

$$\tilde{Y} = \sum_{i,j=1}^{\dim V} (Yx)_j (ke_i, e_j)_V X_i + \tilde{Y}_k,$$

where $\tilde{Y}_k$ is the right-invariant vector field associated with Y on the manifold K acting exclusively on the variable $k \in K$.

Corollary 3.2.15. For each $\beta \in \mathbb{N}^{\dim G}$, we can find smooth functions $p_\alpha^\beta \in C^\infty(G)$ such that

$$\tilde{X}_g^\beta = \sum_{|\alpha| \leq N} p_\alpha^\beta(g) X_g^\alpha$$

holds for each $g \in G$.

Moreover, there exists $C_\beta \geq 0$ such that

$$\sup_{|\alpha| \leq N} |p_\alpha^\beta(x, k)| \leq C_\beta (1 + \|x\|_V)^{|\beta|}$$

for each $(x, k) \in G$.

3.3 UNITARY REPRESENTATIONS

The unitary dual of G was first described in [Mac51; Mac52], but a more explicit version, specifically for the generalized motion group, was given in [Itô52].

The Fourier analysis on the motion group was studied in [KO73], where the author observes that the Plancherel formula takes a more elegant form if we group the unitary representations appropriately. His subsequent papers (e.g. [Kum76]) work exclusively with those grouped representations as they are technically more convenient.

We shall therefore make the same choice as [Kum76], and introduce the following unitary, albeit reducible, representations.

Definition 3.3.1. Let $\lambda \in V$. We define a unitary representation $\xi_G^\lambda \in \text{Hom}(G, \text{End}(L^2(K)))$ of G , via

$$\xi_G^\lambda(x, k)F(u) \stackrel{\text{def}}{=} e^{i2\pi(\lambda, ux)_V} F(uk), \quad (12)$$

where $(x, k) \in K$, $F \in L^2(K)$ and $u \in K$.

At this stage, a few comments are necessary.

- If $\lambda = 0$ in (12), then our unitary representation takes the form

$$\xi_G^\lambda(x, k) = \pi_K^R(k) \sim \bigoplus_{\tau \in \hat{K}} \tau^{\otimes d_\tau},$$

and is thus always reducible.

- If $\lambda \neq 0$, then our representation defined in (12) is irreducible if and only if K is *abelian*. This can be deduced easily from its decomposition into irreducible representations (see [KO73]).
- For each $k \in K$ and each $\lambda \in V$, the representations ξ_G^λ and $\xi_G^{k\lambda}$ are equivalent (see Lemma 3.3.3). Again, we choose to work with them nevertheless because they will yield a more elegant Plancherel formula, but will allow us to always be able to differentiate with respect to λ .
- In the literature (see [Itô52; KO73; Kum76; VK95]), the representations are usually defined via

$$\xi_G^\lambda(x, k)F(u) \stackrel{\text{def}}{=} e^{i(\lambda, u^{-1}x)_V} F(k^{-1}u).$$

The choice was made consciously for technical reasons. Like for the Euclidean case, the 2π factor will ensure that the Fourier transform is *unitary* (see Proposition 3.4.5) while the choice to put k on the right of u was made because right-regular representations are more convenient when working with the Peter-Weyl basis of $L^2(K)$.

Example 3.3.2 (2-dimensional Euclidean motion group). Let $\lambda \in \mathbb{R}^2$. Let $(x, t) \in G = \mathbb{R}^2 \ltimes \text{SO}(2)$. If $\lambda \neq 0$, then ξ_G^λ is irreducible.

Using the isomorphism $\text{SO}(2) \sim \mathbb{T}$, (12) takes the form

$$\xi_G^\lambda(x, t)F(u) = e^{i2\pi(\lambda, e^{i2\pi t}x)_V} F(u + t),$$

where $F \in L^2(\mathbb{T})$, $(x, t) \in G$, and $u \in \mathbb{T}$.

Defining $\xi_G^\lambda(x, t)_{mn} \stackrel{\text{def}}{=} (\xi_G^\lambda(x, t)e^{i2\pi n \cdot}, e^{i2\pi m \cdot})_{L^2(\mathbb{T})}$ for every $m, n \in \mathbb{Z}$, it follows that

$$\begin{aligned}\xi_G^\lambda(x, t)_{mn} &= \int_{\mathbb{T}} e^{i2\pi(\lambda, e^{i2\pi u} x)_V} e^{i2\pi[n(u+t)-mu]} \, du \\ &= e^{i2\pi nt} \int_{\mathbb{T}} e^{i2\pi(\lambda, e^{i2\pi u} x)_V} e^{i2\pi(n-m)u} \, du.\end{aligned}$$

Our representations satisfy the following invariance property.

Lemma 3.3.3 (Invariance property). *Let $\lambda \in V$. For each $k \in K$, we have*

$$\xi_G^{k\lambda}(y, l) = \pi_K^L(k) \xi_G^\lambda(y, l) \pi_K^L(k^{-1}).$$

Proof. Let $F \in L^2(K)$ and $u \in K$. It follows that

$$\begin{aligned}\pi_K^L(k) \xi_G^\lambda(y, l) \pi_K^L(k^{-1}) F(u) &= \pi_K^L(k) e^{i2\pi(\lambda, ux)_V} F(kul) \\ &= e^{i2\pi(\lambda, k^{-1}ux)_V} F(ul) = \xi_G^{k\lambda}(y, l) F(u).\end{aligned}$$

□

Representations on $V \times K$

Provided that we also group the representations on $V \times K$, the representations of $V \times K$ and G can seem formally very similar.

Indeed, first observe that the unitary dual is given by

$$\widehat{V \times K} = \left\{ e^{i2\pi(\lambda, \cdot)_V} \otimes \tau : \lambda \in V, \tau \in \widehat{K} \right\}$$

which, if we group the representations of K , becomes

$$e^{i2\pi(\lambda, \cdot)_V} \otimes \left(\bigoplus_{\tau \in \widehat{K}} \tau^{\otimes d_\tau} \right).$$

Using the Peter-Weyl Theorem (Theorem 2.6.1), the representation above is equivalent to the following.

Definition 3.3.4. *Let $\lambda \in V$. We define a unitary representation $\xi_{V \times K}^\lambda \in \text{Hom}(G, \text{End}(L^2(K)))$ of G , via*

$$\xi_{V \times K}^\lambda(x, k) F(u) \stackrel{\text{def}}{=} e^{i2\pi(\lambda, x)_V} F(uk),$$

where $(x, k) \in K$, $F \in L^2(K)$ and $u \in K$.

3.3.1 Infinitesimal representations

Infinitesimal representations are important because they characterize the action of left- (respectively right-) invariant differential operators on the Fourier Transform side. As such, they provide the most elementary example of Fourier transform of distributions that are not integrable functions.

Definition 3.3.5 (Infinitesimal Representation). *Let $X \in \mathfrak{g}$. We define the infinitesimal representation of X as the operator*

$$\tilde{\zeta}_G^\lambda(X) : C^\infty(K) \rightarrow C^\infty(K)$$

defined via

$$\tilde{\zeta}_G^\lambda(X)F(u) \stackrel{\text{def}}{=} \left. \frac{d}{dt} \right|_{t=0} \tilde{\zeta}_G^\lambda(\exp(tX))F(u),$$

where $F \in C^\infty(K)$.

Proposition 3.3.6 (Infinitesimal representations). *Let $\lambda \in V$ and let $j \in \{1, \dots, \dim G\}$. Fix also $F \in C^\infty(K)$ and $u \in K$. The infinitesimal representation of X_j has the following expression.*

1. *If $j \leq \dim V$, then*

$$\tilde{\zeta}_G^\lambda(X_j)F(u) = i2\pi(\lambda, ue_j)_V F(u).$$

2. *If $j > \dim V$, then*

$$\tilde{\zeta}_G^\lambda(X_j)F(u) = X_j F(u),$$

where on the right-hand side X_j is the left-invariant differential operator on K associated with $X_j \in \mathfrak{k}$.

Proof. Let $F \in C^\infty(K)$ and $u \in K$.

1. Fix $j \in \{1, \dots, \dim V\}$. Since $\exp_G(tX_j) = (te_j, I_V)$, it follows that

$$\tilde{\zeta}_G^\lambda(X_j)F(u) = \left. \frac{d}{dt} \right|_{t=0} e^{i2\pi(\lambda, tue_j)_V} F(u) = i2\pi(\lambda, ue_j)_V F(u)$$

which is what we wanted to show.

2. If $j > \dim V$, $X_j \in \mathfrak{k}$ so that

$$\exp_G(tX_j) = (0, \exp_K(tX_j)).$$

From there, it immediately follows that

$$\xi_G^\lambda(X_j)F(u) = \left. \frac{d}{dt} \right|_{t=0} F(u \exp_K(tX_j)),$$

which by definition is $X_j F(u)$.

□

Corollary 3.3.7 (Infinitesimal representation of $\mathcal{L}_G$). *Let $\lambda \in V$. The infinitesimal representation of $\mathcal{L}_G$ is given by*

$$\xi_G^\lambda(\mathcal{L}_G) = -(2\pi)^2 \|\lambda\|_V^2 I_{L^2(K)} + \mathcal{L}_K.$$

3.4 FOURIER TRANSFORM

We have now introduced all the necessary tools to study the Fourier analysis on G .

3.4.1 Definition and elementary properties

Before defining the Fourier Transform, let us adapt the classical definition of Schwartz space.

Definition 3.4.1 (Schwartz space). *We let $\mathcal{S}(G)$ be the space of all $f \in C^\infty(G)$ which are rapidly decaying, i.e. such that for each $N \in \mathbb{N}$,*

$$\|f\|_{\mathcal{S}(G),N} \stackrel{\text{def}}{=} \sup_{|\alpha| \leq N} \left| (1 + |x|)^N X^\alpha f(x, k) \right| < \infty.$$

The maps $(\|\cdot\|_{\mathcal{S}(G),N})_{N \in \mathbb{N}}$ define a collection of semi-norms on $\mathcal{S}(G)$ and give it a structure of a Fréchet space.

We now define a central tool in our analysis: the *Fourier transform*.

Definition 3.4.2 (Fourier transform). *Let $f \in L^1(G)$ and $\lambda \in V$. We define its Fourier coefficient at λ via*

$$\mathcal{F}_G f(\lambda) \stackrel{\text{def}}{=} \int_G f(g) \xi_G^\lambda(g)^* dg.$$

Moreover, the map

$$\mathcal{F}_G f : V \rightarrow \text{End}(L^2(K)) : \lambda \mapsto \mathcal{F}_G f(\lambda)$$

is called the Fourier Transform of f .

To establish the Plancherel and inverse formulae, we need to compute some Schatten norms of our Fourier coefficients.

Lemma 3.4.3. *Let $\lambda \in V$, and $f \in L^1(G)$. The integral kernel of the operator $\mathcal{F}_G f(\lambda)$ is given by*

$$K_f(\lambda; u, k) \stackrel{\text{def}}{=} \mathcal{F}_V f(k^{-1}\lambda, k^{-1}u), \quad (13)$$

i.e. for every $F \in L^2(K)$ and every $u \in K$, we have

$$\mathcal{F}_G f(\lambda)F(u) = \int_K K_f(\lambda; u, k)F(k) \, dk.$$

In particular, the following properties hold.

1. If $f \in \mathcal{S}(G)$, then K_f is smooth, $\mathcal{F}_G f(\lambda)$ is trace class and

$$\text{tr}(\mathcal{F}_G f(\lambda)) = \int_K \mathcal{F}_V f(k\lambda, e) \, dk.$$

2. If $f \in L^1(G) \cap L^2(G)$, then for almost every $\lambda \in V$, $\mathcal{F}_G f(\lambda)$ is Hilbert-Schmidt and

$$\|\mathcal{F}_G f(\lambda)\|_{\mathcal{S}_2(L^2(K))}^2 = \int_K \int_K |\mathcal{F}_V f(k\lambda, u)|^2 \, du \, dk.$$

Proof. Let $F \in L^2(K)$ and $u \in K$. By definition of the Fourier Transform,

$$\mathcal{F}_G f(\lambda)F(u) = \int_V \int_K f(x, k) e^{-i2\pi(ku^{-1}\lambda, x)_V} F(uk^{-1}) \, dk \, dx.$$

Recognizing the Fourier Transform on V in the above, we obtain

$$\begin{aligned} \mathcal{F}_G f(\lambda)F(u) &= \int_K \mathcal{F}_V f(ku^{-1}\lambda, k)F(uk^{-1}) \, dk \\ &= \int_K \mathcal{F}_V f(k^{-1}\lambda, k^{-1}u)F(k) \, dk, \end{aligned}$$

where we substituted k for $k^{-1}u$ to obtain the last line.

From there, it follows that the kernel is indeed given by (13). Let us now prove the two remaining claims.

1. If $f \in \mathcal{S}(G)$, it follows that the integral kernel is smooth. Using [DR14, Corollary 4.1], it follows that $\mathcal{F}_G f(\lambda)$ is trace-class, and

$$\text{tr}(\mathcal{F}_G f(\lambda)) = \int_K K_f(\lambda; k, k) \, dk = \int_K \mathcal{F}_V f(k\lambda, e) \, dk.$$

2. Now, if $f \in L^1(G) \cap L^2(G)$, then $K_f \in L^2(K \times K)$ for almost every $\lambda \in V$. For such λ , it follows by [RS80, Theorem VI.23] that $\mathcal{F}_G f(\lambda)$ is Hilbert-Schmidt and

$$\begin{aligned} \|\mathcal{F}_G f(\lambda)\|_{\mathcal{S}_2(L^2(K))}^2 &= \int_K \int_K |K_f(\lambda; u, k)|^2 \, dk \, du \\ &= \int_K \int_K |\mathcal{F}_V f(k^{-1}\lambda, k^{-1}u)|^2 \, dk \, du. \end{aligned}$$

Substituting k for k^{-1} and then u for $k^{-1}u$ in the above, we obtain

$$\|\mathcal{F}_G f(\lambda)\|_{\mathcal{S}_2(L^2(K))}^2 = \int_K \int_K |\mathcal{F}_V f(k\lambda, u)|^2 \, dk \, du,$$

as required. □

If a function is rotation-invariant in the Euclidean variable, applying the Fourier transform reduces to computing the Fourier transform on $V \times K$.

Lemma 3.4.4. *Let $\lambda \in V$ and $f \in L^1(G)$. If for every $l \in K$,*

$$f(lx, k) = f(x, k), \quad (x, k) \in G, \quad (14)$$

then we have

$$\mathcal{F}_G f(\lambda) = \mathcal{F}_{V \times K} f(\lambda).$$

Proof. By Lemma 3.4.3 and (14), we know that

$$\begin{aligned} \mathcal{F}_G f(\lambda) F(u) &= \int_K \mathcal{F}_V f(\lambda, k^{-1}u) F(k) \, dk \\ &= \int_K \mathcal{F}_V f(\lambda, k) F(uk^{-1}) \, dk, \end{aligned}$$

where the last line is obtained by substituting k for uk^{-1} .

Recognizing a Fourier transform on K with “grouped representations” (see Subsubsection 3.3) in the right-hand side of the above, we obtain

$$\mathcal{F}_G f(\lambda) F(u) = \mathcal{F}_{V \times K} f(\lambda) F(u),$$

which shows that both Fourier transforms are indeed equal. $\square$

3.4.2 Plancherel formula

The following result is of crucial importance, for it establishes a relation between a function and its Fourier transform. Informally speaking, it will later allow us to “translate” between *kernels* and *symbols*, as those notions are linked via the Fourier transform.

Proposition 3.4.5 (Plancherel formula). *Let $f \in L^1(G) \cap L^2(G)$. The following formula holds*

$$\int_G |f|^2 \, dg = \int_V \|\mathcal{F}_G f(\lambda)\|_{\mathcal{HS}(L^2(K))}^2 \, d\lambda. \quad (15)$$

Proof. It follows from Lemma 3.4.3 that for almost every $\lambda \in V$, $\mathcal{F}_G f(\lambda)$ is trace class and

$$\|\mathcal{F}_G f(\lambda)\|_{\mathcal{HS}(L^2(K))}^2 = \int_K \int_K |\mathcal{F}_V f(k\lambda, u)|^2 \, du \, dk.$$

Now, integrating with respect to λ , we obtain

$$\begin{aligned} \int_V \|\mathcal{F}_G f(\lambda)\|_{\mathcal{HS}(L^2(K))}^2 \, d\lambda &= \int_V \int_K |\mathcal{F}_V f(\lambda, k)|^2 \, dk \, d\mu_{\hat{V}}(\lambda) \\ &= \int_V \int_K |f(x, k)|^2 \, du \, dk, \end{aligned}$$

where the last line was obtained by applying the Plancherel formula on V . $\square$

The following result is equivalent to the Plancherel formula, in the sense that proving one allows us to prove the other.

It says that for a very regular function, we can recover a function from its Fourier transform.

Proposition 3.4.6 (Inverse Fourier Transform). *Let $\phi \in \mathcal{S}(G)$. For each $g \in G$, we have*

$$\phi(g) = \int_V \operatorname{tr} \left(\xi_G^\lambda(g) \mathcal{F}_G \phi(\lambda) \right) d\lambda.$$

Proof. Let us assume that $g = e$. By Lemma 3.4.3, we know that $\mathcal{F}_G \phi(\lambda)$ is trace class and

$$\operatorname{tr}(\mathcal{F}_G \phi(\lambda)) = \int_K \mathcal{F}_V \phi(k\lambda, e) dk.$$

Integrating with respect to λ , we obtain

$$\begin{aligned} \int_V \operatorname{tr}(\mathcal{F}_G \phi(\lambda)) d\lambda &= \int_V \int_K \mathcal{F}_V \phi(k\lambda, e) dk d\lambda \\ &= \int_V \mathcal{F}_V \phi(\lambda, e) d\lambda, \end{aligned}$$

where the last line is obtained by a change of variables after permuting the integrals.

Recognizing an inverse Fourier Transform on the right-hand side of the above, we obtain

$$\int_V \operatorname{tr}(\mathcal{F}_G \phi(\lambda)) d\lambda = \phi(0, e),$$

concluding the case $g = e$.

The general case follows immediately, since

$$\phi(g) = \phi(eg) = \int_V \operatorname{tr}(\mathcal{F}_G \{\phi(\cdot g)\}(\lambda)) d\lambda = \int_V \operatorname{tr}(\xi_G^\lambda(g) \mathcal{F}_G \phi(\lambda)) d\lambda,$$

where the last equality is obtained by Proposition 2.5.2. $\square$

The direct product case

In view of Subsubsection 3.3, let us compare the Plancherel formulae.

Given $f \in L^1(V \times K)$ and $\lambda \in V$, we let

$$\begin{aligned} \mathcal{F}_{V \times K} f(\lambda) &\stackrel{\text{def}}{=} \int_{V \times K} f(x, k) \xi_{V \times K}^\lambda(x, k)^* d(x, k), \\ \mathcal{F}_{V \times K} f(\lambda, \tau) &\stackrel{\text{def}}{=} \int_{V \times K} f(x, k) e^{i2\pi(\lambda, x)_V} \tau(k)^* d(x, k). \end{aligned}$$

Note that $d(x, k)$ is also the Haar measure of G . Moreover, it is clear that

$$\mathcal{F}_{V \times K} f(\lambda) \sim \bigoplus_{\tau \in \widehat{K}} (\mathcal{F}_{V \times K} f(\lambda, \tau))^{\otimes d_\tau}. \quad (16)$$

The Plancherel formula therefore takes the following form.

Proposition 3.4.7 (Plancherel formula on $V \times K$). *Let $f \in L^1(V \times K) \cap L^2(V \times K)$. The following formula holds*

$$\int_{V \times K} |f|^2 dg = \int_V \|\mathcal{F}_{V \times K} f(\lambda)\|_{\mathcal{HS}(L^2(K))}^2 d\lambda. \quad (17)$$

Proof. Using the “classical” Plancherel Theorem on $V \times K$, we obtain

$$\int_{V \times K} |f|^2 dg = \int_V \sum_{\tau \in \widehat{K}} d_\tau \|\mathcal{F}_{V \times K} f(\lambda, \tau)\|_{\mathcal{HS}(\mathcal{H}_\tau)}^2 d\lambda.$$

Using (16), it is clear that

$$\|\mathcal{F}_{V \times K} f(\lambda)\|_{\mathcal{HS}(L^2(K))}^2 = \sum_{\tau \in \widehat{K}} d_\tau \|\mathcal{F}_{V \times K} f(\lambda, \tau)\|_{\mathcal{HS}(\mathcal{H}_\tau)}^2,$$

and we conclude easily. $\square$

Naturally, with an identical proof, this means that the *inverse formula* takes the form

$$f(g) = \int_{V \times K} \mathrm{tr}(\xi_{V \times K}^\lambda(g) \mathcal{F}_{V \times K} f(\lambda)) d\lambda,$$

where $f \in \mathcal{S}(V \times K)$ and $g \in V \times K$.

We have thus shown that the Fourier theory and the Plancherel theory are formally very similar on G and $V \times K$ provided we group the representations on either side.

Note that the resemblance is extremely deceptive. To illustrate that last assertion, we observe that

$$\xi_G^\lambda(\mathcal{L}_G) = \xi_{V \times K}^\lambda(\mathcal{L}_{V \times K})$$

when we group the representations on both sides. However, the term appearing on the right-hand side decomposes as a direct sum of *scalar operators* on the irreducible representations, unlike the term appearing on the left. This is a direct consequence of the fact that $\mathcal{L}_G = \mathcal{L}_{V \times K}$ is right-invariant on $V \times K$ but not on G .

3.4.3 Fourier Transform of distributions

Although Subsection 2.5.1 already treated the problem of generalizing the Fourier Transform for the purposes of pseudo-differential calculus, let us mention some results that have been obtained explicitly for the motion group.

For a complete proof of the results stated in this subsection, the reader is invited to consult [EKM80].

Definition 3.4.8 (Image of the Schwartz space). *We shall denote by $\mathcal{S}(\widehat{G})$ the set of all functions $F \in C^\infty(V, \mathcal{L}(L^2(K)))$ satisfying the following properties.*

1. For each $\alpha \in \mathbb{N}^{\dim V}$, $\partial_\lambda^\alpha F$ leaves $C^\infty(K)$ stable.
2. For each $N \in \mathbb{N}$, the quantity

$$\|F\|_{\mathcal{S}(\widehat{G}), N} \stackrel{\text{def}}{=} \sup_{|\alpha|, |\beta|, |\beta'| \leq N} \sup_{\lambda \in V} (1 + |\lambda|)^N \left\| \zeta_G^\lambda(X^\beta) \frac{\partial^{|\alpha|} F}{\partial \lambda^\alpha}(\lambda) \zeta_G^\lambda(X^{\beta'}) \right\|_{\mathcal{L}(L^2(K))}$$

is finite.

3. For every $k \in K$ and each $\lambda \in V$,

$$F(k\lambda) = \pi_K^L(k) F(k) \pi_K^L(k^{-1}).$$

One easily checks the following result.

Lemma 3.4.9. *The space $\mathcal{S}(\widehat{G})$ is a Fréchet space whose topology is given by the seminorms $\|\cdot\|_{\mathcal{S}(\widehat{G}), N}$, $N \in \mathbb{N}$.*

Theorem 3.4.10 ([EKM80]). *The Fourier Transform is a topological isomorphism*

$$\mathcal{F}_G : \mathcal{S}(G) \rightarrow \mathcal{S}(\widehat{G}).$$

It then follows by classical arguments that we may extend the Fourier Transform onto $\mathcal{S}'(G)$.

Proposition 3.4.11 (Fourier transform and duality). *Let $f \in \mathcal{S}(G)$, and $H \in \mathcal{S}(\widehat{G})$. We have the identities*

$$(\mathcal{F}_G f, H)_{\widehat{G}} = (f, \iota \circ \mathcal{F}_G^{-1} H)_G, \quad (\mathcal{F}_G^{-1} H, f)_G = (H, \mathcal{F}_G(\iota \circ f))_{\widehat{G}},$$

where $(\iota \circ f)(g) = f(g^{-1})$.

Definition 3.4.12 (Fourier Transform on $\mathcal{S}'(G)$). Let $f \in \mathcal{S}'(G)$. We define the Fourier Transform of f , $\mathcal{F}_G f$, an element of $\mathcal{S}'(\widehat{G})$, via

$$(\mathcal{F}_G f, H)_{\widehat{G}} \stackrel{\text{def}}{=} \left(f, \iota \circ \mathcal{F}_G^{-1} H \right)_G,$$

for any $H \in \mathcal{S}(\widehat{G})$.

Similarly, let $H \in \mathcal{S}'(\widehat{G})$. We define the tempered distribution $\mathcal{F}_G^{-1} H \in \mathcal{S}'(G)$, called the inverse Fourier Transform of H , via

$$\left(\mathcal{F}_G^{-1} H, f \right)_G = (H, \mathcal{F}_G(\iota \circ f))_{\widehat{G}},$$

for any $f \in \mathcal{S}'(G)$.

Proposition 3.4.13. The Fourier Transform is a topological linear isomorphism of $\mathcal{S}'(G)$ onto $\mathcal{S}'(\widehat{G})$.

3.5 SOBOLEV SPACES

The notion of *order* of a pseudo-differential operator is mainly motivated by boundedness properties between Sobolev spaces. As we wish to define pseudo-differential operators with non-integer orders, we need a notion of Sobolev spaces with general orders.

Definition 3.5.1 (Sobolev norm). Let $s \in \mathbb{R}$. If $\phi \in \mathcal{S}(G)$, we let

$$\|\phi\|_{L_s^2(G)} \stackrel{\text{def}}{=} \left(\int_V \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{s}{2}} \mathcal{F}_G \phi(\lambda) \right\|_{\mathcal{HS}(L^2(K))}^2 d\lambda \right)^{1/2}.$$

Definition 3.5.2 (Sobolev spaces). Let $s \in \mathbb{R}$. We define the Sobolev space of order s to be the completion of $\mathcal{S}(G)$ with the norm $\|\cdot\|_{L_s^2(G)}$.

The Sobolev spaces satisfy the following properties.

Proposition 3.5.3 (Properties of the Sobolev spaces). Let $s \in \mathbb{R}$. The Sobolev space of order s , $L_s^2(G)$, satisfies the following properties.

- If $s = 0$, then $L_s^2(G) = L^2(G)$.
- If $s' \leq s$, then $L_s^2(G) \subset L_{s'}^2(G)$, and the inclusion is continuous.

- If $\alpha \in \mathbb{N}^{\dim G}$, then $X^\alpha f \in L^2_{s-|\alpha|}(G)$ for each $f \in L^2_s(G)$, and the map

$$X^\alpha : L^2_s(G) \rightarrow L^2_{s-|\alpha|}(G)$$

is continuous.

Proof. Let us show all three properties separately.

- Suppose $s = 0$. It is straightforward to check that

$$\|\cdot\|_{L^2_0(G)} = \|\cdot\|_{L^2(G)}$$

by applying the Plancherel formula (Proposition 3.4.5). Since $\mathcal{S}(G)$ is dense in $L^2(G)$ and $L^2(G)$ is complete, we then conclude that $L^2_0(G) = L^2(G)$.

- Suppose that $s' \leq s$. If $\phi \in \mathcal{S}(G)$, it should be clear that

$$\begin{aligned} \|\phi\|_{L^2_{s'}(G)} &\leq \left\| \tau^\lambda_G (I - \mathcal{L}_G)^{\frac{s'-s}{2}} \right\|_{\mathcal{L}(L^2(K))} \left\| \tau^\lambda_G (I - \mathcal{L}_G)^{\frac{s}{2}} \phi \right\|_{L^2(\widehat{G})} \\ &\leq \left\| \tau^\lambda_G (I - \mathcal{L}_G)^{\frac{s}{2}} \phi \right\|_{L^2(\widehat{G})} = \|\phi\|_{L^2_s(G)}. \end{aligned}$$

By a classical density argument (see e.g. [Wil13, Proposition 3.2.3]), it follows that the map

$$I : (\mathcal{S}(G), \|\cdot\|_{L^2(s)}) \rightarrow L^2_{s'}(G) : \phi \mapsto \phi$$

can be continuously extended onto its closure, yielding the required inclusion.

- Let $\alpha \in \mathbb{N}^{\dim G}$. If $\phi \in \mathcal{S}(G)$,

$$\|X^\alpha \phi\|_{L^2_{s-|\alpha|}(G)} \leq \|I_{s,\alpha}\|_{\mathcal{L}(L^2(K))} \left\| \tau^\lambda_G (I - \mathcal{L}_G)^{\frac{s}{2}} \phi \right\|_{L^2(\widehat{G})},$$

where

$$I_{s,\alpha} \stackrel{\text{def}}{=} \tau^\lambda_G (I - \mathcal{L}_G)^{\frac{s-|\alpha|}{2}} \tau^\lambda_G (X)^\alpha \tau^\lambda_G (I - \mathcal{L}_G)^{\frac{-s}{2}}.$$

One easily checks that $\|I_{s,\alpha}\|_{\mathcal{L}(L^2(K))} < \infty$, so that a density argument yields the existence of a unique continuous extension of

$$X^\alpha : L^2_s(G) \rightarrow L^2_{s-|\alpha|}(G).$$

Naturally, because the canonical inclusion

$$L_s^2(G) \hookrightarrow \mathcal{S}'(G)$$

is continuous, the above extended map X^α is the same as the corresponding distributional derivative.

□

A classical strategy to show that a distribution is a continuous function is to show that it belongs to a Sobolev space $L_s^2(G)$ for a sufficiently large $s \in \mathbb{R}$. This strategy can only be successful provided we can prove *Sobolev embeddings*.

Lemma 3.5.4. *Let $s > \dim G$. The map*

$$\lambda \in V \mapsto \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{s}{2}} \right\|_{S_1(L^2(K))}$$

is finite almost everywhere and integrable over V .

Proof. For each $s \in \mathbb{R}$, we let

$$I(s) \stackrel{\text{def}}{=} \int_V \text{tr}(\xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{s}{2}}) d\lambda.$$

Now, fix $s > \dim G$, and let us show that $I(s)$ is finite.

It is clear that

$$I(s) \asymp \int_V \sum_{\tau \in \hat{K}} d_\tau (\|\lambda\|_V + \langle \tau \rangle_{\hat{K}})^{-s} d\lambda,$$

which, after integrating via polar coordinates, yields

$$I(s) \asymp \int_{\mathbb{R}^+} \sum_{\tau \in \hat{K}} d_\tau (r + \langle \tau \rangle_{\hat{K}})^{-s} r^{\dim V - 1} dr.$$

Now, substituting r for $r - \langle \tau \rangle_{\hat{K}}$, we obtain

$$\begin{aligned} I(s) &\asymp \int_{\langle \tau \rangle_{\hat{K}}}^{+\infty} \sum_{\tau \in \hat{K}} d_\tau r^{-s} (r - \langle \tau \rangle_{\hat{K}})^{\dim V - 1} dr \\ &\asymp \int_{\langle \tau \rangle_{\hat{K}}}^{+\infty} \sum_{\tau \in \hat{K}} \sum_{k=0}^{\dim V - 1} d_\tau r^{-s+k} \langle \tau \rangle_{\hat{K}}^{\dim V - k - 1} dr. \end{aligned}$$

Since $s > \dim G$, we can check that

$$\int_{\langle \tau \rangle_{\hat{K}}}^{+\infty} r^{-s+k} dr \approx \langle \tau \rangle_{\hat{K}}^{-s+k+1},$$

which would imply that

$$I(s) \asymp \sum_{\tau \in \widehat{K}} d_\tau \langle \tau \rangle_{\widehat{K}}^{-s + \dim V}.$$

However, [Fis15, Lemma 6.3] implies that the above converges if and only if

$$-s + \dim V < -\dim K,$$

or more simply if and only if $s > \dim G$. $\square$

Proposition 3.5.5 (Sobolev embedding). *If $s > \dim G/2$, then we have the following continuous inclusion*

$$L_s^2(G) \subset C(G) \cap L^\infty(G).$$

More precisely, there exists $C \geq 0$ such that the following property holds: for every $f \in L_s^2(G)$, there exists a continuous function $\tilde{f} \in C(G)$ such that $f = \tilde{f}$ almost everywhere and

$$\|\tilde{f}\|_{C(G)} \leq C \|f\|_{L_s^2(G)}.$$

Proof. Let $s > \dim G/2$. By Lemma 3.5.4 and Proposition 3.4.5, the Bessel potential $\mathfrak{B}_s^G \in L^2(G)$. Therefore,

$$f = (I - \mathcal{L}_G)^{\frac{s}{2}} f \star \mathfrak{B}_s^G,$$

can be written as the convolution of two functions in $L^2(G)$. The conclusion now follows from Proposition 2.4.2. $\square$

The Sobolev embedding theorem allows us to extend the Fourier inversion formula onto Sobolev spaces. To show this, we first show the following technical result.

Lemma 3.5.6. *Let $f \in L_s^2(G)$ for $s > \dim G/2$. For almost every $\lambda \in V$, the operator $\mathcal{F}_G f(\lambda)$ is trace class and we have the estimate*

$$\int_V \|\mathcal{F}_G f(\lambda)\|_{S_1(L^2(K))} d\lambda \leq C \|f\|_{L_s^2(G)}.$$

In particular, if a sequence $(f_j)_{j \in \mathbb{N}} \subset L_s^2(G)$ converges to f in $L_s^2(G)$, then we have

$$\lim_{j \rightarrow \infty} \int_V \operatorname{tr}(\xi_G^\lambda(g) \mathcal{F}_G f_j(\lambda)) \, d\lambda = \int_V \operatorname{tr}(\xi_G^\lambda(g) \mathcal{F}_G f(\lambda)) \, d\lambda$$

for every $g \in G$.

Proof. Suppose that $f \in L_s^2(G)$. By definition, the distribution

$$f_s \stackrel{\text{def}}{=} (I - \mathcal{L}_G)^{\frac{s}{2}} f$$

belongs to $L^2(G)$. By the Cauchy-Schwarz inequality for Schatten classes, we have

$$\|\mathcal{F}_G f(\lambda)\|_{S_1(L^2(K))} \leq \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-s}{2}} \right\|_{S_2(L^2(K))} \|\mathcal{F}_G f_s(\lambda)\|_{S_2(L^2(K))}. \quad (18)$$

The first Hilbert-Schmidt norm on the left-hand side is finite by Proposition 3.5.5, whilst the second one is finite for almost every $\lambda \in V$ by the Plancherel formula (Proposition 3.4.5). Therefore, we have so far shown that

$$\|\mathcal{F}_G f(\lambda)\|_{S_1(L^2(K))} < \infty,$$

i.e. $\mathcal{F}_G f(\lambda)$ is trace class for almost every $\lambda \in V$.

Integrating both sides of (18) with respect to λ , we obtain

$$\begin{aligned} & \int_V \|\mathcal{F}_G f(\lambda)\|_{S_1(L^2(K))} \, d\lambda \\ & \leq \int_V \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-s}{2}} \right\|_{S_2(L^2(K))} \|\mathcal{F}_G f_s(\lambda)\|_{S_2(L^2(K))} \, d\lambda \\ & \leq \left\| \xi_G (I - \mathcal{L}_G)^{\frac{-s}{2}} \right\|_{L^2(\widehat{G})} \|f_s\|_{L^2(G)}, \end{aligned} \quad (19)$$

where the last line was obtained via the Cauchy-Schwarz inequality on $L^2(V)$ and the Plancherel formula. By the Sobolev embeddings (Proposition 3.5.5), the right-hand side of (19) is finite, so that the equation becomes

$$\int_V \|\mathcal{F}_G f(\lambda)\|_{S_1(L^2(K))} \, d\lambda \leq C \|f\|_{L_s^2(G)}. \quad (20)$$

Now, fix a sequence $(f_j)_{j \in \mathbb{N}} \subset L_s^2(G)$ converging to f . By applying (20) to $f_j - f$, we obtain

$$\begin{aligned} & \left| \int_V \operatorname{tr}(\xi_G^\lambda(g) \mathcal{F}_G\{f - f_j\}(\lambda)) \, d\lambda \right| \\ & \leq \int_V \|\mathcal{F}_G\{f - f_j\}(\lambda)\|_{S_1(L^2(K))} \, d\lambda \\ & \leq C \|f - f_j\|_{L_s^2(G)}. \end{aligned}$$

Letting $j \rightarrow \infty$ in the above, we obtain the desired conclusion. $\square$

We are now ready to prove the following generalization of Proposition 3.4.6.

Proposition 3.5.7 (Fourier inverse formula). *Let $f \in L_s^2(G)$ for $s > \dim G/2$. For almost every $\lambda \in V$, the operator $\mathcal{F}_G f(\lambda)$ is trace class and the map*

$$\lambda \in V \mapsto \|\mathcal{F}_G f(\lambda)\|_{S_1(L^2(K))}$$

is integrable on V . Moreover, f is continuous on G and for every $g \in G$, the map

$$G \rightarrow \mathbb{C} : g \mapsto \int_V \operatorname{tr} \left(\xi_G^\lambda(g) \mathcal{F}_G f(\lambda) \right) \, d\lambda$$

is continuous and equal to f almost-everywhere.

Proof. The first claim follows immediately from Lemma 3.5.6.

Now, select a sequence $(f_j)_{j \in \mathbb{N}} \subset \mathcal{S}(G)$ such that

$$\lim_{j \rightarrow \infty} \|f_j - f\|_{L_s^2(G)} = 0.$$

By the inverse formula on $\mathcal{S}(G)$ (Proposition 3.4.6), we have

$$f_j(g) = \int_V \operatorname{tr} \left(\xi_G^\lambda(g) \mathcal{F}_G f_j(\lambda) \right) \, d\lambda,$$

for each $j \in \mathbb{N}$.

By the Sobolev embedding theorem (Proposition 3.5.5), f_j converges uniformly to a continuous function $\tilde{f}$ which is equal to f almost everywhere. In particular, this means that we have

$$\begin{aligned}\tilde{f}(g) &= \lim_{j \rightarrow \infty} f_j(g) \\ &= \lim_{j \rightarrow \infty} \int_V \operatorname{tr} \left(\xi_G^\lambda(g) \mathcal{F}_G f_j(\lambda) \right) d\lambda \\ &= \int_V \operatorname{tr} \left(\xi_G^\lambda(g) \mathcal{F}_G f(\lambda) \right) d\lambda,\end{aligned}$$

concluding the proof. $\square$

Corollary 3.5.8. *Let $s > \dim G/2 + k$, where $k \in \mathbb{N}$. If $f \in L_s^2(G)$, then $f \in C^k(G)$.*

Proof. Let $\alpha \in \mathbb{N}^{\dim G}$ be such that $|\alpha| \leq k$. By Proposition 3.5.3, $X^\alpha f \in L_{s-|\alpha|}^2(G)$.

Applying Proposition 3.5.7, we obtain that $X^\alpha f$ is continuous.

We conclude that $f \in C^k(G)$ by observing that α is arbitrary. $\square$

3.6 TAYLOR FORMULA

As in the Euclidean case, the composition and adjunction formulae rely on a *Taylor expansion*.

Intuitively, we shall say a collection of smooth functions is *admissible* if they can be used to perform a Taylor expansion.

Definition 3.6.1. *A finite collection $q_1, \dots, q_M \in C^\infty(G)$ of smooth functions is said to be admissible if $dq_j(e) \neq 0$ for each $j \in \{1, \dots, M\}$ and if*

$$\operatorname{rank}(dq_1(e), \dots, dq_m(e)) = \dim G.$$

Moreover, if

$$\bigcap_{j=1}^M \{q_j = 0\} = \{e\},$$

we shall say that the collection is strongly admissible.

The notion of *strong admissibility* was introduced in [RTW14] to ensure the kernels of pseudo-differential operators would only have singularities at the origin (see Chapter 4).

From now on, we let $q_1, \dots, q_{\dim G} \in C^\infty(G)$ be a strongly admissible family.

Moreover, we fix a basis $X_1, \dots, X_{\dim G} \in \mathfrak{g}$ such that

$$X_j q_k(\cdot^{-1})(e_G) = \delta_{j,k}$$

for each $j, k \in \{1, \dots, \dim G\}$. To insist upon the fact that the basis is adapted to our family of smooth functions, we shall write

$$X^{(\alpha)} \stackrel{\text{def}}{=} X_1^{\alpha_1} \dots X_{n_{\Delta,G}}^{\alpha_{n_{\Delta,G}}}.$$

Definition 3.6.2 (Distance to the origin). *Let $g = (x, k) \in G$. We define the distance of g to the identity e_G via*

$$\|g\|_G^2 \stackrel{\text{def}}{=} \|x\|_V^2 + d_K(k, e_K)^2,$$

where $d_K(\cdot, \cdot)$ is a left-invariant Riemannian distance.

We now define an analogue of the Euclidean Taylor development and the associated remainder.

Definition 3.6.3 (Taylor development and remainder). *Let $f \in C^\infty(G)$ and $g \in G$. For each $N \in \mathbb{Z}$, we define the Taylor development of f at g of order N to be*

$$P_{g,N}^f(h) \stackrel{\text{def}}{=} \begin{cases} \sum_{|\alpha| \leq N} \frac{1}{\alpha!} q^\alpha(h^{-1}) X^{(\alpha)} f(g) & \text{if } N \geq 0 \\ 0 & \text{if } N < 0, \end{cases}$$

where $h \in G$.

Moreover, we define the Taylor remainder of f of order N at $g \in G$ via

$$R_{g,N}^f(h) \stackrel{\text{def}}{=} f(gh) - P_{g,N}^f(h).$$

The Taylor polynomial was defined so that the following property holds.

Lemma 3.6.4. *Let $N \in \mathbb{N}$, $\beta \in \mathbb{N}^{\dim G}$. For each $f \in C^\infty(G)$, we have*

$$X^{(\beta)} P_{g,N}^f(e_G) = P_{g,N-|\beta|}^{X^{(\beta)} f}(e_G), \quad X^{(\beta)} R_{g,N}^f(e_G) = R_{g,N-|\beta|}^{X^{(\beta)} f}(e_G).$$

By using local charts on K and then subsequently extending onto all of K via a partition of unity, we can show the following estimate on the remainder.

Proposition 3.6.5 (Taylor's theorem). *Let $f \in C^\infty(G)$ and $g \in G$. There exists $\eta \geq 1$ satisfying the following property: for every $N \in \mathbb{N}$, there exists $C \geq 0$ such that*

$$\left| R_{g,N}^f(h) \right| \leq C \|h\|_G^{N+1} \sup_{|\alpha| \leq N+1} \sup_{\|z\|_G \leq \eta \|h\|_G} \left| X^{(\alpha)} f(gz) \right|$$

for every $h \in G$.

4 | SYMBOLIC CALCULUS

4.1 DIFFERENCE OPERATORS

The Euclidean case uses estimates on the derivatives in Fourier variables of the symbol to ensure crucial properties of the convolution kernels, such as the smoothness and Schwartz decay away from the origin, but also the Calderón-Zygmund property of pseudo-differential operators of order 0.

In [RT10], M. Ruzhansky and V. Turunen introduce the concept of *difference operator* to generalize the aforementioned derivatives in Fourier variables and obtain a pseudo-differential calculus on compact Lie groups.

Definition 4.1.1 (Difference operators). *Let $q \in C^\infty(G)$. The difference operator associated with q , Δ_q^G is defined via*

$$\Delta_q^G \mathcal{F}_G f \stackrel{\text{def}}{=} \mathcal{F}_G \{qf\},$$

where $f \in \mathcal{S}(G)$.

Moreover, if q vanishes up to order $k \in \mathbb{N}$, we shall say that Δ_q^G is a difference operator of order k .

In our definition of symbol classes, we shall require that the order of our symbols decrease every time we apply difference operators from a well-chosen family of smooth functions.

It is thus important to show that this condition is not too restrictive, which is the aim of the next statement. As we intend for our symbol classes to contain characteristic polynomials, we show that applying difference operators *reduces* their order.

Lemma 4.1.2 (Difference operators on characteristic polynomials). *Let $q \in C^\infty(G)$. For every $\alpha \in \mathbb{N}^{\dim G}$, we have the formula*

$$\Delta_q^G \xi_G^\lambda(X^\alpha) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} X^\beta q(e_G) \xi_G^\lambda(X^{\alpha-\beta}).$$

In particular, if Δ_q^G is a difference operator of order k , then $\Delta_q^G \xi_G^\lambda(X^\alpha)$ is the characteristic polynomial of a differential operator with order $|\alpha| - k$.

Proof. By definition of difference operator, we have

$$\Delta_q^G \xi_G^\lambda(X^\alpha) = (-1)^\alpha \mathcal{F}_G\{qX^\alpha \delta_{e_G}\}.$$

Let $H \in \mathcal{S}(\widehat{G})$ and write

$$h \stackrel{\text{def}}{=} \iota \circ \mathcal{F}_G^{-1} H.$$

Using the definition of Fourier transform on distributions, we know that

$$\langle \mathcal{F}_G\{qX^\alpha \delta_{e_G}\}, H \rangle_{\widehat{G}} = \langle qX^\alpha \delta_{e_G}, h \rangle_G = (-1)^{|\alpha|} \langle \delta_{e_G}, X^\alpha(qh) \rangle_G. \quad (21)$$

Using the Leibniz rule, we know that

$$\begin{aligned} \langle \delta_{e_G}, X^\alpha(qh) \rangle_G &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \langle \delta_{e_G}, X^\beta q X^{\alpha-\beta} h \rangle_G \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} X^\beta q(e_G) \langle \delta_{e_G}, X^{\alpha-\beta} h \rangle_G, \end{aligned}$$

which, after using the definition of distributional derivative, becomes

$$\langle \delta_{e_G}, X^\alpha(qh) \rangle_G = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} X^\beta q(e_G) \langle X^{\alpha-\beta} \delta_{e_G}, h \rangle_G.$$

Recognizing the definition of distributional Fourier transform again, we obtain

$$\begin{aligned} \langle \delta_{e_G}, X^\alpha(qh) \rangle_G &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} X^\beta q(e_G) \langle \mathcal{F}_G\{X^{\alpha-\beta} \delta_{e_G}\}, H \rangle_{\widehat{G}} \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} X^\beta q(e_G) \langle \xi_G^\lambda(X^{\alpha-\beta}), H \rangle_{\widehat{G}}. \end{aligned}$$

Combining the above with (21), we obtain that

$$\langle \Delta_q^G \xi_G^\lambda(X^\alpha), H \rangle_{\widehat{G}} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\beta|} X^\beta q(e_G) \langle \xi_G^\lambda(X^{\alpha-\beta}), H \rangle_{\widehat{G}}.$$

Since H is arbitrary, we obtain the desired formula. $\square$

With our approach, we will not be able to write the composition or adjunction of operators directly in terms of our symbols, hence it

is important to consider a family with respect to which we can use a *Taylor development*.

Definition 4.1.3. A collection of difference operators is called (strongly) admissible if the associated smooth functions form a (strongly) admissible collection in the sense of Definition 3.6.1.

The existence of a Taylor expansion and the estimates on the remainder are then given by Proposition 3.6.5. We shall see later that the *strong* admissibility will ensure that the singularity of our kernels is confined to the identity e_G .

Another technical requirement is that our collection of smooth functions satisfy a *Leibniz-like* rule. This, amongst other things, ensures that our collection of symbols forms an algebra with respect to point-wise composition.

Definition 4.1.4. We shall say that an admissible collection

$$q_1, \dots, q_{n_{\Delta, G}} \in C^\infty(K)$$

satisfies the Leibniz-like property if for each $j = 1, \dots, n_{\Delta, G}$,

$$q_j(gh) = q_j(g) + q_j(h) + \sum_{1 \leq k, l \leq n_{\Delta, G}} c_{k, l}^j q_k(g) q_l(h)$$

holds for every $g, h \in G$.

It is a routine check to show that the Leibniz-like property for smooth functions becomes the following *Leibniz-like rule* for difference operators.

Lemma 4.1.5 (Leibniz-like rule). Assume that

$$\Delta \stackrel{\text{def}}{=} \{q_1, \dots, q_{n_{\Delta, G}} \in C^\infty(G)\}$$

is a collection of admissible smooth functions satisfying the Leibniz-like property. Writing

$$q^\alpha \stackrel{\text{def}}{=} q_1^{\alpha_1} \dots q_{n_{\Delta, G}}^{\alpha_{n_{\Delta, G}}}, \quad \Delta_G^\alpha \stackrel{\text{def}}{=} \Delta_{q^\alpha}^G(\cdot, -1)$$

for each $\alpha \in \mathbb{N}^{n_{\Delta, G}}$, we have

$$\Delta_G^\alpha (\mathcal{F}_G f_1 \mathcal{F}_G f_2) = \sum_{|\alpha| \leq |\alpha_1| + |\alpha_2| \leq 2|\alpha|} c_{\alpha_1, \alpha_2}^\alpha \Delta_G^{\alpha_1} \mathcal{F}_G f_1 \Delta_G^{\alpha_2} \mathcal{F}_G f_2$$

for each $f_1, f_2 \in \mathcal{S}'(G)$.

We can easily adapt [RTW14, Lemma 4.4] or [Fis15, Corollary 5.13], to show the following result.

Lemma 4.1.6. *There exists a strongly admissible family $q_1, \dots, q_M \in C^\infty(G)$ on G which satisfies the Leibniz-like property and such that*

$$q_j(x, k) = x_j$$

for each $j \in \{1, \dots, \dim V\}$.

Proof. Denote by F the set of all fundamental representations of K [Fis15, p. 3416], i.e. a finite collection such that for any $\tau \in \widehat{K}$, there exists $\tau_1, \dots, \tau_n \in F$ such that

$$\tau = \tau_1 \otimes \dots \otimes \tau_n.$$

We also assume F contains the identity representation.

Now, let us define the following smooth functions

$$\begin{aligned} q_j(x, k) &\stackrel{\text{def}}{=} (x, e_j)_V, & j = 1, \dots, \dim V \\ q_{mn}^\tau(x, k) &\stackrel{\text{def}}{=} (\tau - I_{H_\tau})_{mn}, & \tau \in F, 1 \leq m, n \leq d_\tau. \end{aligned}$$

Now, we let

$$\Delta = \{q_j : 1 \leq j \leq \dim V\} \cup \{q_{mn}^\tau : \tau \in F, 1 \leq m, n \leq d_\tau\}$$

STRONG ADMISSIBILITY Clearly, every $q \in \Delta$ satisfies $q_j(0) = 0$.

By [Fis15, Lemma 5.11], we know that

$$\text{rank}\{dq_{mn}^\tau(e) : \tau \in F, 1 \leq m, n \leq d_\tau\} = \dim K,$$

so we easily check that

$$\text{rank } \Delta = \dim G.$$

By [Fis15, Lemma 5.11], if all $q_{mn}^\tau(x, k)$ vanish, then $k = I_V$, while if all $q_j(x, k)$ vanish, this means $x = 0_V$. Therefore,

$$\bigcap_{q \in \Delta} \{q = 0\} = (0_V, I_V).$$

LEIBNIZ-LIKE FORMULA Using [Fis15, Corollary 5.13], we only need to show that the q_j , $j = 1, \dots, \dim V$ satisfy the Leibniz-like property. For each $j \in \mathbb{N}$,

$$\begin{aligned} q_j((x, k)(y, l)) &= (x + ky, e_j)_V \\ &= q_j(x, k) + q_j(y, l) + ((k - I_V)y, e_j)_V \\ &= q_j(x, k) + q_j(y, l) + \left(y, (k - I_V)^{-1}e_j\right)_V. \end{aligned}$$

Observing that

$$(k - I_V)^T e_j = (k - I_V)_{ij}^T e_i = (k - I_V)_{ji} e_i,$$

we obtain that

$$\begin{aligned} q_j((x, k)(y, l)) &= q_j(x, k) + q_j(y, l) + \sum_{i=1}^{\dim V} (k - I_V)_{ji} (y, e_i)_V \\ &= q_j(x, k) + q_j(y, l) + \sum_{i=1}^{\dim V} (k - I_V)_{ji} q_i(y, l). \end{aligned}$$

Defining

$$\tau(k) \stackrel{\text{def}}{=} k,$$

we observe that $\tau \in F$ and so the above becomes

$$q_j((x, k)(y, l)) = q_j(x, k) + q_j(y, l) + \sum_{i=1}^{\dim V} q_{ji}^\tau(x, k) q_i(y, l).$$

We have thus shown that each $q \in \Delta$ satisfies the Leibniz-like formula.

□

Remark 4.1.7. From now on, we fix an admissible family $q_1, \dots, q_{n_{\Delta, G}}$ on G which satisfies the assumptions of Lemma 4.1.6.

Given $\alpha \in \mathbb{N}^{n_{\Delta, G}}$, we let

$$q^\alpha \stackrel{\text{def}}{=} \prod_{j=1}^{n_{\Delta, G}} q_j^{\alpha_j}.$$

Such maps q^α will be called **polynomials**.

Moreover, we let Δ_G^α be the difference operator associated with the smooth function

$$G \rightarrow \mathbb{C} : g \mapsto q^\alpha(g^{-1}).$$

4.2 SYMBOL CLASSES

Following [RT10] or [FR16], we can now introduce our main definition.

Definition 4.2.1 (Symbol classes). *Let $m \in \mathbb{R}$ and fix $\rho, \delta \in \mathbb{R}$ such that $0 \leq \rho, \delta \leq 1$. We shall say that a map*

$$\sigma : G \times V \mapsto \text{End}(C^\infty(K))$$

is a symbol of order m and of type (ρ, δ) if the following conditions are satisfied.

1. *For each $F \in C^\infty(K)$ and each $k \in K$, the map*

$$(g, \lambda) \mapsto \sigma(g, \lambda)F(k)$$

is smooth.

2. *For each $\beta \in \mathbb{N}^{\dim G}$, $X^\beta \sigma(g, \cdot) \in \mathcal{S}'(\widehat{G})$ for each $g \in G$ and the map*

$$g \in G \mapsto X^\beta \sigma(g, \cdot) \in \mathcal{S}'(\widehat{G})$$

is continuous.

3. *For each $\alpha \in \mathbb{N}^{\dim G}$, $\beta \in \mathbb{N}^m$, and each $\gamma \in \mathbb{R}$, the operator*

$$\zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\rho|\alpha| - m - \delta|\beta| + \gamma}{2}} X^\beta \Delta_G^\alpha \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}}$$

is bounded in $L^2(K)$ by a finite constant that does not depend on $g \in G$ or $\lambda \in V$.

The set $S_{\rho, \delta}^m(G)$ will be used to denote the set of all symbols of order m and type (ρ, δ) .

Let us make a few comments on Definition 4.2.1.

- If $\alpha = \beta = 0$, the *Plancherel theory* (e.g. [FR16, Theorem 1.8.11]) implies that if $\sigma \in S_{\rho,\delta}^m(G)$ does not depend on $g \in G$, then the map

$$\phi \in \mathcal{S}(G) \mapsto \phi \star \mathcal{F}_G^{-1} \sigma$$

extends to a bounded map

$$L_s^2(G) \rightarrow L_{s-m}^2(G)$$

for each $s \in \mathbb{R}$.

It is hoped that the estimates for non-zero $\alpha \in \mathbb{N}^{n_{\Delta,G}}$ and $\beta \in \mathbb{N}^{\dim G}$ will imply a similar result for a general $\sigma \in S_{\rho,\delta}^m(G)$. When $\rho > \delta$, this is proved later in Theorem 4.8.4.

- In the compact or Euclidean case (see [RT10]), we need only check the case $\gamma = 0$. This is due to the fact that in those cases the Laplacian is central. The above definition for non-zero γ was introduced for the first time in [FR14, Section 3.4] and subsequently in [FR16, Section 5.2].

We shall see later, again using the ideas of [FR16], that we also only need to check for $\gamma = 0$ in our case. It is worth keeping in mind that the fact that our Laplacian is not central is due to the *semi-direct* structure.

We can naturally define a collection of seminorms on our symbol classes, and check they endow each $S_{\rho,\delta}^m(G)$ with a structure of *Fréchet space*.

Definition 4.2.2 (Symbol semi-norms). *Let $m \in \mathbb{R}$ and fix $\rho, \delta \in \mathbb{R}$ such that $0 \leq \rho, \delta \leq 1$. If $\sigma \in S_{\rho,\delta}^m(G)$, then for each $N \in \mathbb{N}$, we let*

$$\begin{aligned} \|\sigma(g; \lambda)\|_{S_{\rho,\delta}^m(G), N} &\stackrel{\text{def}}{=} \sup_{|\alpha|, |\beta|, |\gamma| \leq N} \\ &\left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\rho|\alpha| - m - \delta|\beta| + \gamma}{2}} X^\beta \Delta_G^\alpha \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))}. \end{aligned}$$

Moreover, we also let

$$\|\sigma\|_{S_{\rho,\delta}^m(G), N} \stackrel{\text{def}}{=} \sup_{g \in G} \sup_{\lambda \in V} \|\sigma(g; \lambda)\|_{S_{\rho,\delta}^m(G), N}$$

for each $N \in \mathbb{N}$.

Similarly, we let

$$\|\sigma(g; \lambda)\|_{S_{\rho, \delta}^{m, R}(G), N} \stackrel{\text{def}}{=} \sup_{|\alpha|, |\beta| \leq N} \left\| X^\beta \Delta_G^\alpha \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\rho|\alpha| - m - \delta|\beta|}{2}} \right\|_{\mathcal{L}(L^2(K))}$$

and

$$\|\sigma\|_{S_{\rho, \delta}^{m, R}(G), N} \stackrel{\text{def}}{=} \sup_{g \in G} \sup_{\lambda \in V} \|\sigma(g; \lambda)\|_{S_{\rho, \delta}^{m, R}(G), N}$$

for each $N \in \mathbb{N}$.

Using Lemma 4.1.2, we show that our symbol classes contain the left-invariant differential calculus.

Example 4.2.3 (Characteristic polynomials). *Given $m \in \mathbb{N}$, the map*

$$\sigma(g, \lambda) \stackrel{\text{def}}{=} \sum_{|\alpha| \leq m} c_\alpha(g) \zeta_G^\lambda(X)^\alpha, \quad c_\alpha \in C_c^\infty(G)$$

defines a symbol in $S_{1,0}^m(G)$.

Naturally, the requirement that the support be compact in the example above can be relaxed provided that we assume the coefficients have bounded derivatives for all orders.

Equipped with the topology of seminorms defined in Definition 4.2.2, we can easily show the following continuous inclusions.

Lemma 4.2.4 (Inclusion of symbol classes). *Suppose that $m_1, m_2 \in \mathbb{R}$ and $\rho_1, \rho_2, \delta_1, \delta_2 \in [0, 1]$. If the following inequalities hold*

$$m_1 \leq m_2, \quad \delta_1 \leq \delta_2, \quad \rho_1 \geq \rho_2,$$

then $S_{\rho_1, \delta_1}^{m_1}(G) \subset S_{\rho_2, \delta_2}^{m_2}(G)$, and the inclusion is continuous.

Proof. Let $\sigma \in S_{\rho_1, \delta_1}^{m_1}(G)$, and fix $\alpha \in \mathbb{N}^{n_{\Delta, G}}$, $\beta \in \mathbb{N}^{\dim G}$ and $\gamma \in \mathbb{R}$.

It is clear that

$$\zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\rho_2|\alpha| - m_2 - \delta_2|\beta| + \gamma}{2}} X^\beta \Delta_G^\alpha \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} = I_1 I_2,$$

where

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{m_1 - m_2 + (\rho_2 - \rho_1)|\alpha| + (\delta_1 - \delta_2)|\beta|}{2}} \\ I_2 &\stackrel{\text{def}}{=} \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\rho_1|\alpha| - m_1 - \delta_1|\beta| + \gamma}{2}} X^\beta \Delta_G^\alpha \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}}. \end{aligned}$$

Because of our assumption, it is clear that

$$m_1 - m_2 + (\rho_2 - \rho_1) |\alpha| + (\delta_1 - \delta_2) |\beta| \leq 0,$$

so that I_1 satisfies

$$\|I_1\|_{\mathcal{L}(L^2(K))} \leq 1.$$

By our assumption on σ , it is clear that I_2 is bounded on $L^2(G)$ uniformly in $g \in G$ and $\lambda \in G$, so that the same holds for $I_1 I_2$. $\square$

One cannot expect to be able to perform *exact* symbolic computations. Therefore, we shall often attempt to obtain results up to symbols which belong to symbolic classes of “arbitrarily low orders”. Such symbols will be called *smoothing symbols*.

Definition 4.2.5 (Smoothing symbols). *We let*

$$S^{-\infty}(G) \stackrel{\text{def}}{=} \bigcap_{m \in \mathbb{R}} S_{1,0}^m(G).$$

The elements of $S^{-\infty}(G)$ will be called smoothing symbols.

It is a routine check that we can replace $(1, 0)$ in the above definition by any $(\rho, \delta) \in [0, 1]^2$ without affecting the resulting set.

4.3 KERNELS, QUANTIZATION AND OPERATOR CLASSES

We have so far defined a notion of symbol which is similar to the Euclidean case in spirit. Parts of Definition 4.2.1 are directly motivated by the desire to define an analogue of the *Kohn-Nirenberg quantization*.

The process of *quantization* is the association of a symbol with an operator.

Definition 4.3.1 (Quantization). *Let $\sigma \in S_{\rho,\delta}^m(G)$. We define the operator $\text{Op}_G(\sigma)$ via*

$$\text{Op}_G(\sigma)\phi(g) \stackrel{\text{def}}{=} \int_V \text{tr} \left(\xi_G^\lambda(g) \sigma(g, \lambda) \mathcal{F}_G \phi(\lambda) \right) d\lambda,$$

where $\phi \in \mathcal{S}(G)$, and $g \in G$.

Now that we can associate a symbol to an operator, the latter can inherit properties such as the *order* or the *type* from the former.

Definition 4.3.2 (Operator classes). If $T = \text{Op}_G(\sigma)$ for a certain $\sigma \in S_{\rho,\delta}^m(G)$, we shall say that T is an (pseudo-differential) operator of order m and of type (ρ, δ) .

The set of all such operators will be denoted by

$$\Psi_{\rho,\delta}^m(G) \stackrel{\text{def}}{=} \text{Op}_G(S_{\rho,\delta}^m(G)).$$

Naturally, an operator in

$$\Psi^{-\infty}(G) \stackrel{\text{def}}{=} \text{Op}_G(S^{-\infty}(G))$$

is called smoothing.

To compute adjoints or compose pseudo-differential operators, it is useful to know their right *convolution kernel*. Such kernels can be defined directly in terms of the symbols, and we shall subsequently check that they are indeed the right-convolution kernel associated with the pseudo-differential operator.

Definition 4.3.3 (Kernel of a symbol). Let $\sigma \in S_{\rho,\delta}^m(G)$. For each $g \in G$, we let

$$\kappa_g \stackrel{\text{def}}{=} \mathcal{F}_G^{-1}\{\sigma(g, \cdot)\} \in \mathcal{S}'(G).$$

The map

$$\kappa : G \rightarrow \mathcal{S}'(G) : g \mapsto \kappa_g$$

is called the kernel of σ .

The symbol classes were defined partly so that the following result holds.

Lemma 4.3.4. Let $\sigma \in S_{\rho,\delta}^m(G)$, and denote by κ its associated kernel. For each $g \in G$, κ_g is a tempered distribution and

$$g \in G \mapsto \kappa_g \in \mathcal{S}'(G)$$

is smooth, i.e. $\kappa \in C^\infty(G, \mathcal{S}'(G))$.

Proof. Let $g \in G$ and $\beta \in \mathbb{N}^{\dim G}$. By Proposition 3.4.13, $\kappa_g = \mathcal{F}_G^{-1}\{X^\beta \sigma(g, \cdot)\}$ is a tempered distribution.

Since the inverse Fourier transform is also continuous, the continuity of σ in g implies that

$$g \mapsto \kappa_g = \mathcal{F}_G^{-1}\{X^\beta \sigma(g, \cdot)\}$$

is continuous. This concludes the proof. $\square$

We are now ready to show that the *kernel* associated with a symbol is in fact the *right-convolution kernel* of its associated pseudo-differential operator.

Proposition 4.3.5 (Quantization). *Let $\sigma \in S_{\rho, \delta}^m(G)$, and denote by κ its associated kernel. If $\phi \in \mathcal{S}(G)$, then for each $g \in G$, we have*

$$\text{Op}_G(\sigma)\phi(g) = (\phi \star \kappa_g)(g).$$

In other words, κ is the right convolution kernel associated with $\text{Op}_G(\sigma)$.

Proof. Let $g \in G$ and fix $s > \max \dim G/2$. By definition, we know that $\phi \star \kappa_g \in L_s^2(G)$ if and only if

$$\lambda \mapsto \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{s}{2}} \sigma(g, \cdot) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m-s+m}{2}} \mathcal{F}_G \phi$$

belongs to $L^2(\widehat{G})$. To show this, observe that the above implies

$$\left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{s}{2}} \mathcal{F}_G \{\phi \star \kappa_g\} \right\|_{S_2(L^2(K))}^2 \leq C_s \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{s-m}{2}} \mathcal{F}_G \phi \right\|_{S_2(L^2(K))}^2,$$

which, since the right-hand side is integrable on V , means that $\phi \star \kappa_g$ belongs to $L_s^2(G)$. By Proposition 3.5.7, we get that

$$\begin{aligned} \phi \star \kappa_g(g) &= \int_V \text{tr} \left(\zeta_G^\lambda(g) \mathcal{F}_G \{\phi \star \kappa_g\} \right) d\lambda \\ &= \int_V \text{tr} \left(\zeta_G^\lambda(g) \sigma(g; \lambda) \mathcal{F}_G \phi(\lambda) \right) d\lambda, \end{aligned}$$

which concludes the proof, as the right-hand side is exactly $\text{Op}_G(\sigma)\phi(g)$. $\square$

4.3.1 First properties of symbol classes

We now state elementary properties of our symbol classes and their associated pseudo-differential operators. In particular, we claim that applying difference operators or left-invariant differential operators is

a continuous operation between symbol classes, as well as operations such as *pointwise* composition and adjunction.

The proof is a routine check and is left to the reader. For the pointwise composition, we use the *Leibniz-like rule* of Lemma 4.1.5.

Proposition 4.3.6. *Suppose that $\rho, \delta \in \mathbb{R}$ satisfy $0 \leq \rho, \delta \leq 1$. Given $m, m_1, m_2 \in \mathbb{R}$, choose three symbols $\sigma \in S_{\rho, \delta}^m(G)$, $\sigma_1 \in S_{\rho, \delta}^{m_1}(G)$, $\sigma_2 \in S_{\rho, \delta}^{m_2}(G)$. The kernels of the aforementioned symbols will be denoted by κ , κ_1 and κ_2 respectively.*

The following properties hold.

1. *For each $\alpha \in \mathbb{N}^{n_{\Delta, G}}$ and each $\beta \in \mathbb{N}^{\dim G}$,*

$$(g, \lambda) \in G \times V \mapsto X^\beta \Delta_\alpha^G \sigma(g, \lambda)$$

belongs to $S_{\rho, \delta}^{m - \rho|\alpha| + \delta|\beta|}(G)$, and its kernel is given by

$$g \in \lambda \mapsto q^\alpha X^\beta \kappa_g.$$

2. *The pointwise adjunction of σ ,*

$$\sigma^*(g, \lambda) \stackrel{\text{def}}{=} \sigma(g, \lambda)^*,$$

defines a symbol in $S_{\rho, \delta}^m(G)$ whose kernel is given by

$$\kappa_g^*(h) \stackrel{\text{def}}{=} \overline{\kappa_g(h^{-1})},$$

where the above is interpreted in the sense of distributions.

3. *The pointwise composition*

$$(g, \lambda) \mapsto \sigma_1(g, \lambda) \sigma_2(g, \lambda)$$

defines a symbol in $S_{\rho, \delta}^{m_1 + m_2}(G)$, whose symbol is given by

$$g \in G \mapsto \kappa_{2, g} \star \kappa_{1, g}.$$

All the above operations are continuous with respect to the topology of their respective symbol classes.

To show that the composition of two pseudo-differential operators is well-defined, we shall need the following result. Its proof is a simple adaptation of that of [FR16, Theorem 5.2.15].

Proposition 4.3.7. *Let $m \in \mathbb{R}$ and assume that $1 \geq \rho \geq \delta \geq 0$. If $T \in \Psi^m(G)$ and $\phi \in \mathcal{S}(G)$, then $T\phi \in \mathcal{S}(G)$. Moreover, T acts continuously on $\mathcal{S}(G)$.*

4.4 HOLOMORPHIC FUNCTIONAL CALCULUS

The aim of this section is to show two results concerning functions of symbols which will be essential hereafter.

First, we wish to show that for a well-behaved symbol $\sigma \in S_{\rho,\delta}^m(G)$, its powers σ^s , $s \in \mathbb{R}$, define symbols in $S_{\rho,\delta}^{sm}(G)$.

Secondly, we would like to show that $S^{-\infty}(G)$ is dense in $S_{\rho,\delta}^m(G)$ in a suitable sense (see Proposition 4.5.1), a result whose importance can be compared to that of other density results in analysis. To this end, we want to create a family of smoothing symbols η_ε , $\varepsilon \in (0, 1]$, in the hope that $\sigma\eta_\varepsilon$ converges to σ as ε goes to 0 in a suitable symbol class.

The results presented herein would be significantly more general with the calculus at our disposal. However, the proof of the composition and adjunction formulae rely on the fact that $\zeta_G^\lambda(I - \mathcal{L}_G)^{\frac{\gamma}{2}}$ is a symbol of order γ and on the aforementioned density of $S^{-\infty}(G)$ in the symbol classes $S_{\rho,\delta}^m(G)$.

The main idea of this section is to follow [RW14], which uses the following version of the *Cauchy integral formula*

$$F(\sigma) = \frac{1}{i2\pi} \int_{\Gamma} F(z)(\sigma - zI_{L^2(K)})^{-1} dz. \quad (22)$$

This allows us to carry out our symbolic estimates on the resolvent.

In order to successfully apply (22), we need to identify a class of symbols whose resolvent might give us appropriate bounds. We are thus led to the following definition.

Definition 4.4.1 (Parameter-dependent ellipticity, [RW14]). *Let $\sigma \in S_{\rho,\delta}^m(G)$ with $m > 0$. Given a subset $\Lambda \subset \mathbb{C}$, we shall say that σ is parameter-elliptic with respect to Λ if the following properties holds.*

1. *For each $z \in \Lambda$ and each $(g, \lambda) \in G \times V$, the operator*

$$\sigma(g, \lambda) - zI_{L^2(K)}$$

is invertible.

2. For each $\gamma_1, \gamma_2 \in \mathbb{R}$, the operator

$$I_1(m, m + \gamma_1) I_2(\gamma_2) (\sigma(g, \lambda) - z I_{L^2(K)})^{-1} I_2(-\gamma_2) I_1(m, -\gamma_1)$$

is bounded on $L^2(K)$ uniformly in $g \in G$, $\lambda \in V$ and $z \in \Lambda$, where

$$I_1(m, \gamma) \stackrel{\text{def}}{=} (|z|^{\frac{1}{m}} + \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{1}{2}})^\gamma, \quad (23)$$

$$I_2(\gamma) \stackrel{\text{def}}{=} \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}}. \quad (24)$$

If $m = 0$, we shall say that σ is parameter-elliptic with respect to Λ if

$$\sigma(g, \lambda) - z I_{L^2(K)}$$

is invertible and satisfies

$$\begin{aligned} & \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} (\sigma(g, \lambda) - z I_{L^2(K)})^{-1} \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} \\ & \leq C(1 + |z|)^{-1} \end{aligned}$$

uniformly in $z \in \Lambda$ and $(g, \lambda) \in G \times V$.

The definition above had to be adapted from the compact case to take into account that our left-invariant Laplacian is *not* right-invariant. It makes it somehow difficult to check parameter-ellipticity, but as we shall mainly be concerned with diagonal symbols, this is a price we are willing to pay.

Example 4.4.2. The symbol $\xi_G^\lambda (I - \mathcal{L}_G)$ is parameter-elliptic of order 2 with respect to

$$\Lambda \stackrel{\text{def}}{=} \{re^{i\theta} : r \geq 0 \text{ and } \alpha \leq \theta \leq \beta\} \cup \{z \in \mathbb{C} \mid |z| \leq \frac{1}{2}\}$$

with $0 < \alpha < \beta < 2\pi$.

Lemma 4.4.3 ([RW14], Theorem 3.1. (1)). Let $\sigma \in S_{\rho, \delta}^m(G)$ be a symbol which does not depend on $g \in G$, with $m \geq 0$ and $\rho > 0$.

If $m > 0$ and σ is parameter-elliptic with respect to Λ , then for each $\alpha \in \mathbb{N}^{n_{\Delta, G}}$, $\beta \in \mathbb{N}^{\dim G}$, and $\gamma_1, \gamma_2 \in \mathbb{R}$, the operator

$$\begin{aligned} & I_1(m, m + \gamma_1) I_2(\gamma_2 + \rho |\alpha| - \delta |\beta|) \\ & X^\beta \Delta_G^\alpha (\sigma(\lambda) - z I_{L^2(K)})^{-1} I_2(-\gamma_2) I_1(m, -\gamma_1) \end{aligned}$$

is bounded in $L^2(K)$ uniformly in $g \in G$, $\lambda \in V$ and $z \in \Lambda$. In the above, I_1 and I_2 are defined like in (24).

If $m = 0$, the resolvent $(\sigma - zI_{L^2(K)})^{-1}$ is a symbol in $S_{\rho,\delta}^0(G)$. Moreover, we have the symbolic estimate

$$\left\| (\sigma - zI_{L^2(K)})^{-1} \right\|_{S_{\rho,\delta}^0(G),N} \leq C_N(1 + |z|)^{-1}.$$

Moreover, when we have proved the composition formula, the requirement that the symbol not depend on g can be lifted.

Sketch proof. The proof in the compact case (cf [RW14, Theorem 3.1. (1)]) relies mainly on the *Leibniz-like* rule for difference operators and the Leibniz rule for left-invariant differential operators.

To prove the result, we apply difference operators and differential operators to

$$I = (\sigma(\lambda) - zI_{L^2(K)})(\sigma(\lambda) - zI_{L^2(K)})^{-1}$$

and use the respective Leibniz rules on the right hand side.

The proof in [RW14] holds because a similar Leibniz-like rule holds on the motion group. Moreover, our definition of parameter-ellipticity ensures that for the purposes of evaluating a norm $\|\cdot\|_{\mathcal{L}(L^2(G))}$, we can reorder a product of $I_1(m, \gamma_1)$, $I_2(\gamma_2)$ and the resolvent $(\sigma - I_{L^2(K)})^{-1}$ however we like, exactly like on the compact group. $\square$

We introduce the following notion of *sector*, whose boundary will provide a good contour for (22).

Definition 4.4.4 (Sector). Let $\Lambda \subset \mathbb{C}$. We shall say that Λ is a sector if

$$\Lambda = \Lambda(\theta_1, \theta_2) \stackrel{\text{def}}{=} \{re^{i\theta} \in \mathbb{C} : r \geq 0, \theta_1 \leq \theta \leq \theta_2\}$$

for some $\theta_1 < \theta_2$.

It follows from elementary complex analysis that there exists a determination of $\log$ on the complement of a sector. As a result, we can define

$$z^s \stackrel{\text{def}}{=} \exp(s \log z), \quad z \in \mathbb{C} \setminus \Lambda$$

for each $s \in \mathbb{C}$.

We are now ready to state our main result.

Theorem 4.4.5 ([RW14]). Assume $0 \leq \delta < \rho \leq 1$. Let $\sigma \in S_{\rho,\delta}^m(G)$ be a symbol of strictly positive order, which is self-adjoint and parameter-elliptic with respect to a sector $\Lambda \subset \mathbb{C}$. If F is analytic in $\mathbb{C} \setminus \Lambda_\eta$ with $\Lambda_\eta \stackrel{\text{def}}{=} (\Lambda \cup \{z \in \mathbb{C} : |z| \leq \eta\})$ for some $\eta > 0$ and satisfies

$$|F(z)| \leq C |z|^s, \quad z \in \Gamma \stackrel{\text{def}}{=} \partial\Lambda_\eta$$

for some $s < 0$, then $F(\sigma)(g, \lambda)$ defines a symbol in $S_{\rho,\delta}^{sm}(G)$.

Moreover, for each $N \in \mathbb{N}$, there exists $C \geq 0$ which does not depend on F such that

$$\|F(\sigma)\|_{S_{\rho,\delta}^{sm}(G),N} \leq C \sup_{z \in \Gamma} \frac{|F(z)|}{|z|^s}. \quad (25)$$

If we have not shown the composition formula, we can only show the result for a symbol σ which does not depend on $g \in G$.

Proof. Let us first assume that $-1 < s < 0$ and fix $\alpha \in \mathbb{N}^{n_{\Delta,G}}$, $\beta \in \mathbb{N}^{\dim G}$ and $\gamma \in \mathbb{R}$. We shall show that

$$\zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-sm+\rho|\alpha|-\delta|\beta|+\gamma}{2}} F(\sigma)(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}}$$

is bounded on $\mathcal{L}(L^2(K))$ uniformly in $(g, \lambda) \in G \times V$. For technical convenience, we shall check the boundedness in the Peter-Weyl basis of $L^2(K)$. Therefore, we also fix $\mu, \nu \in \widehat{K}$, and integers $m, n, p, q \in \mathbb{N}$ such that $1 \leq m, n \leq d_\mu$, $1 \leq p, q \leq d_\nu$.

By elementary functional calculus, we have

$$\begin{aligned} & \left| X^\beta \Delta_\alpha^G F(\sigma)(g, \lambda)_{\mu_{mn}, \nu_{pq}} \right| \\ & \leq C \oint_\Gamma |F(z)| \left| X^\beta \Delta_\alpha^G (\sigma(g, \lambda) - I_{L^2(K)})_{\mu_{mn}, \nu_{pq}}^{-1} \right| |dz|. \end{aligned}$$

Using parameter-ellipticity and Lemma 4.4.3, we obtain

$$\left| X^\beta \Delta_\alpha^G (\sigma(g, \lambda) - I_{L^2(K)})_{\mu_{mn}, \nu_{pq}}^{-1} \right| \leq C(|z|^{\frac{1}{m}} + \|\lambda\|_V + \langle \mu \rangle_{\widehat{K}})^{-m} L_{\mu\nu}(\alpha, \beta, \gamma)$$

where

$$L_{\mu\nu}(\alpha, \beta, \gamma) \stackrel{\text{def}}{=} (\|\lambda\|_V + \langle \mu \rangle_{\widehat{K}})^{-\rho|\alpha|+\delta|\beta|+\gamma} (\|\lambda\|_V + \langle \nu \rangle_{\widehat{K}})^{-\gamma}.$$

Therefore,

$$\begin{aligned} & \left| X^\beta \Delta_\alpha^G F(\sigma)(g, \lambda)_{\mu_{mn}, \nu_{pq}} \right| \\ & \leq C \oint_{\Gamma} |z|^s \left(|z|^{\frac{1}{m}} + \|\lambda\|_V + \langle \mu \rangle_{\widehat{K}} \right)^{-m} L_{\mu\nu}(\alpha, \beta, \gamma) |dz|. \end{aligned}$$

It is clear that the above can be estimated by

$$\begin{aligned} \int_{\mathbb{R}^+} r^s \left(r^{\frac{1}{m}} + \|\lambda\|_V + \langle \mu \rangle_{\widehat{K}} \right) dr & \leq \int_{\mathbb{R}^+} ((\|\lambda\|_V + \langle \mu \rangle_{\widehat{K}})^m r)^s (1 + r^{\frac{1}{m}})^{-m} dr \\ & \leq (\|\lambda\|_V + \langle \mu \rangle_{\widehat{K}})^{ms}, \end{aligned}$$

where the inequality on the first line was obtained by substituting r for $(\|\lambda\|_V + \langle \mu \rangle_{\widehat{K}})^m r$, and the integrability of r^s is guaranteed by $-1 < s < 0$.

Going back to our original calculation, we then obtain that

$$\begin{aligned} & \left| X^\beta \Delta_\alpha^G F(\sigma)(g, \lambda)_{\mu_{mn}, \nu_{pq}} \right| \\ & \leq C \sup_{z \in \Gamma} \frac{|F(z)|}{|z|^s} (\|\lambda\|_V + \langle \mu \rangle_{\widehat{K}})^{ms} L_{\mu\nu}(\alpha, \beta, \gamma), \end{aligned}$$

which by definition means that $F(\sigma) \in S_{\rho, \delta}^{ms}(G)$ and that 25 holds.

Suppose now that $s < -1$. Note that we have a determination of the logarithm on our contour, so we can define

$$G(z) \stackrel{\text{def}}{=} F(z)^{\frac{1}{[-s]+1}}$$

and observe that

$$|G(z)| \leq C |z|^{\frac{s}{[-s]+1}}, \quad -1 < \frac{s}{[-s]+1} < 0.$$

By the first part of the proof, $G(\sigma) \in S_{\rho, \delta}^{\frac{ms}{[-s]+1}}(G)$ and

$$\|G(\sigma)\|_{S_{\rho, \delta}^{\frac{ms}{[-s]+1}}(G), N} \leq C \sup_{z \in \Gamma} \frac{|G(z)|}{|z|^{\frac{s}{[-s]+1}}}.$$

We can now conclude by observing that

$$F(\sigma) = G(\sigma)^{[-s]+1} \in S_{\rho, \delta}^{ms}(G),$$

and

$$\|F(\sigma)\|_{S_{\rho,\delta}^{ms}(G),N} \leq C \sup_{z \in \Gamma} \frac{|F(z)|}{|z|^s},$$

where the inclusion in the symbol class above can only be obtained if the symbol does not depend on g . When we have obtained the composition formula, note that this will hold for any symbol σ satisfying the assumptions of the proof. $\square$

Corollary 4.4.6. *Let $\gamma \in \mathbb{R}$. The map*

$$\lambda \in V \mapsto \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}}$$

defines a symbol in $S_{1,0}^\gamma(G)$.

Proof. Since $F(z) = z^s$, with $s < 0$, is holomorphic in $\mathbb{C} \setminus \Lambda$, where

$$\Lambda \stackrel{\text{def}}{=} \Lambda\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$$

and $(I - \mathcal{L}_G)$ is parameter elliptic with respect to Λ , we know that

$$(I - \mathcal{L}_G)^s \in S_{1,0}^{2s}(G),$$

concluding the proof for $\gamma < 0$.

If $\gamma \geq 0$, we let $N \in \mathbb{N}$ be sufficiently large so that

$$s \stackrel{\text{def}}{=} \gamma - 2N < 0,$$

and conclude that

$$(I - \mathcal{L}_G)^{\frac{\gamma}{2}} = (I - \mathcal{L}_G)^{\frac{s}{2}} (I - \mathcal{L}_G)^N \in S_{1,0}^\gamma(G).$$

$\square$

The following result is central in our analysis. We show that the Fourier transform of the heat kernel can be used to define a good approximation of the symbol $\mathcal{F}_G \delta_{e_G}$.

Theorem 4.4.7. *For each $\varepsilon \in (0, 1]$, we let*

$$\eta_\varepsilon(\lambda) \stackrel{\text{def}}{=} e^{-\varepsilon \xi_G^\lambda (I - \mathcal{L}_G)}$$

for each $\lambda \in V$. This defines a smoothing symbol in $S^{-\infty}(G)$. Moreover, we have the following symbol class estimates.

- For each $m < 0$ and each $N \in \mathbb{N}$, there exists $C \geq 0$ such that

$$\|\eta_\varepsilon\|_{S_{\rho,\delta}^m(G),N} \leq C\varepsilon^{\frac{m}{2}}$$

for each $\varepsilon \in (0, 1]$.

- If $0 \leq m \leq 4$, then for each $N \in \mathbb{N}$ there exists $C \geq 0$ such that

$$\|I_{L^2(K)} - \eta_\varepsilon\|_{S_{\rho,\delta}^m(G),N} \leq C\varepsilon^{\frac{m}{2}}$$

for each $\varepsilon \in (0, 1]$.

In particular, for each $m > 0$, η_ε converges to $I_{L^2(K)}$ in $S_{\rho,\delta}^m(G)$ as $\varepsilon \rightarrow 0$.

Proof. In this proof, we consider the following sector

$$\Lambda \stackrel{\text{def}}{=} \Lambda\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$$

and observe that $\xi_G^\lambda(I - \mathcal{L}_G)$ is parameter elliptic with respect to Λ . We also let Γ be the boundary of $\Lambda \cup \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$.

Let $m < 0$ and write $F_\varepsilon(z) = e^{-\varepsilon z}$. We observe that since F_ε decays faster than any polynomial and does not have a pole at 0,

$$|F_\varepsilon(z)| \leq C |\varepsilon z|^{\frac{m}{2}}, \quad z \in \Gamma,$$

where C does not depend on ε . Theorem 4.4.5, with (25), implies that

$$\|\eta_\varepsilon\|_{S_{\rho,\delta}^m(G),N} \leq C_{N,n,s} \varepsilon^{\frac{m}{2}}.$$

Now let $0 < m \leq 4$ and let us check the second symbolic estimate. To this end, we let

$$G_\varepsilon(z) \stackrel{\text{def}}{=} (\varepsilon z)^{-\frac{m}{2}} (1 - e^{-\varepsilon z})$$

and observe that since G_1 vanishes at order 1 at the origin, $|G_\varepsilon(z)| \leq C |\varepsilon z|^{-m/4}$ on Γ where C does not depend on $\varepsilon \in (0, 1]$, because

$$\lim_{z \rightarrow 0} \frac{G_1(z)}{z^{-m/4}} = \lim_{z \rightarrow 0} \frac{1 - e^{-z}}{z^{m/4}}$$

exists and is finite by our condition on m . Using Theorem 4.4.5 with (25), we obtain

$$\left\| G_\varepsilon(\xi_G^\lambda(\mathcal{L}_G)) \right\|_{S_{\rho,\delta}^{-m/2}(G),N} \leq C\varepsilon^{-m/4}.$$

We can now conclude by Corollary 4.4.6 that

$$\begin{aligned}
& \left\| I_{L^2(K)} - \eta_\varepsilon \right\|_{S_{\rho,\delta}^m(G),N} \\
& \leq \left\| G_\varepsilon(\zeta_G^\lambda(I - \mathcal{L}_G)) \right\|_{S_{\rho,\delta}^{-m/2}(G),N'} \left\| \left(\varepsilon \zeta_G^\lambda(I - \mathcal{L}_G) \right)^{m/2} \right\|_{S_{\rho,\delta}^m(G),N'} \\
& \leq C \varepsilon^{-\frac{m}{4} + \frac{m}{2}} = C \varepsilon^{\frac{m}{2}}.
\end{aligned}$$

Now, assume that $m = 0$. Defining

$$G_\varepsilon(z) = (\varepsilon z)^{-1} (1 - e^{-\varepsilon z}),$$

it is clear that

$$|G_\varepsilon(z)| \leq C |\varepsilon z|^{-1},$$

where C does not depend on ε . Using Theorem 4.4.5 with (25), we obtain

$$\left\| G_\varepsilon(\zeta_G^\lambda(\mathcal{L}_G)) \right\|_{S_{\rho,\delta}^{-2}(G),N} \leq C \varepsilon^{-1}.$$

We can now conclude like before that

$$\begin{aligned}
& \left\| I_{L^2(K)} - \eta_\varepsilon \right\|_{S_{\rho,\delta}^0(G),N} \\
& \leq \left\| G_\varepsilon(\zeta_G^\lambda(I - \mathcal{L}_G)) \right\|_{S_{\rho,\delta}^{-2}(G),N'} \left\| \varepsilon \zeta_G^\lambda(I - \mathcal{L}_G) \right\|_{S_{\rho,\delta}^2(G),N'} \\
& \leq C \varepsilon^{-1+1} = C.
\end{aligned}$$

□

4.5 APPROXIMATION OF SYMBOLS

One of the first applications of Theorem 4.4.7 is that we can approximate any symbol $\sigma \in S_{\rho,\delta}^m(G)$ with smoothing symbols, and control the convergence in all symbol classes of order $m_1 > m$.

Proposition 4.5.1. *Assume that $1 \geq \rho \geq \delta \geq 0$. Let $\sigma \in S_{\rho,\delta}^m(G)$. We can construct a family $(\sigma_\varepsilon)_{\varepsilon \in (0,1]} \subset S^{-\infty}(G)$ satisfying the following properties.*

1. For each $\varepsilon \in (0, 1]$, $\sigma_\varepsilon(g, \lambda)$ has compact support in $g \in G$. More precisely, its associated kernel

$$G \times G : (g, h) \mapsto \kappa_{\varepsilon, g}(h)$$

belongs to $\mathcal{S}(G \times G)$ and is also compactly supported in $g \in G$.

2. If $m_1 < m$, then for each $N \in \mathbb{N}$ there exists $C \geq 0$ and $N' \in \mathbb{N}$ such that

$$\|\sigma_\varepsilon\|_{S_{\rho, \delta}^{m_1}(G), N} \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), N'} \varepsilon^{\frac{m_1 - m}{2}}$$

for each $\varepsilon \in (0, 1]$.

3. If $m_1 \geq m$, then for each $N \in \mathbb{N}$ there exists $C \geq 0$ and $N' \in \mathbb{N}$ such that

$$\|\sigma - \sigma_\varepsilon\|_{S_{\rho, \delta}^{m_1}(G), N} \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), N'} \varepsilon^{\frac{\max\{m_1 - m, 4\}}{2}}$$

for each $\varepsilon \in (0, 1]$.

In particular, if $m_1 > m$, σ_ε converges to σ in $S_{\rho, \delta}^{m_1}(G)$ as $\varepsilon \rightarrow 0$.

Proof. Let $\chi \in C^\infty(\mathbb{R}^+, [0, 1])$ be a function with compact support which satisfies $\chi(r) = 1$ for $|r| \leq 1$. Let $\varepsilon \in (0, 1]$. We let

$$\sigma_\varepsilon(g, \lambda) \stackrel{\text{def}}{=} \chi(\varepsilon \|g\|_G^2) \sigma(g, \lambda) \eta_\varepsilon(\lambda),$$

where η_ε was defined in Theorem 4.4.7. It should be clear that the first property holds by our definition of χ_ε and the fact that the symbols η_ε are smoothing (Theorem 4.4.7).

Similarly, the second and third point hold by Theorem 4.4.7 and Part 3 of Proposition 4.3.6. $\square$

These estimates are crucial in the rest of our analysis. We present a short overview of how they will be used.

- If we wish to estimate the singularity of the kernel at the origin, we approximate the kernel κ_g with $\kappa_{\varepsilon, g} \stackrel{\text{def}}{=} \mathcal{F}_G^{-1} \sigma_\varepsilon(g, \cdot)$ and use the bounds provided by Proposition 4.5.1. For more details, see the proof of Theorem 4.6.1.
- We can prove results such as the composition and adjunction formulae first on σ_ε , a smoothing symbol which has a kernel

which is compactly supported in $g \in G$. For more details, see Section 4.7.

- By Proposition 4.5.1,

$$\sum_{j=1}^N \sigma_j = \sum_{j=1}^N (\sigma_j - \sigma_{j,\varepsilon_j}) \quad \text{modulo } S^{-\infty}(G).$$

If, for example, the order of σ_j decreases as $j \rightarrow \infty$ and we choose ε_j carefully, the right-hand side could define a convergent series in $S_{\rho,\delta}^{m_1}(G)$ using the estimates provided by Proposition 4.5.1.

The convergence of the so-called *asymptotic sums* is studied in Proposition 4.7.15 and is crucial to the construction of parametrices (Theorem 4.9.4).

4.6 KERNEL ESTIMATES

Of particular importance is the study of the kernel. Firstly, we want our pseudo-differential operators in $\Psi_{1,0}^0(G)$ to be Calderón-Zygmund, which requires that the singularity only appear at the origin and be of a certain strength. Moreover, we shall compute the composition and adjunction on the kernel side and rely on precise estimates to prove the celebrated *composition and adjunction formulae*.

It is important to note that all the results in this section rely on the reduction of the order through the application of difference operators. Therefore, we shall henceforth assume that $\rho > 0$ and every result concerning or using kernel estimates will have that assumption.

We shall see that the regularity of the kernel increases as the order of the symbol decreases. Since $\rho > 0$, applying difference operators decreases the order of the symbol, which can be reformulated equivalently by saying that multiplying kernels by some suitable polynomial increases its regularity. However, this can only happen if at each $g \in G$, either the kernel is smooth around that point or the polynomial vanishes at g . The reader should therefore expect smoothness and Schwartz decay away from the origin, as the latter is the only point where we allow *all* our polynomials to vanish (see Definition 3.6.1).

Before stating the main result of this section, let us define the quantity

$$\|(x, k)\|_G \stackrel{\text{def}}{=} \|x\|_V + d_K(k, I_V), \quad (x, k) \in G,$$

where d_K is a left-invariant Riemannian distance on K . The above is of course an abuse of notation, as $\|\cdot\|_G$ is not a norm.

The aim of this section is to prove the following result.

Theorem 4.6.1 (Kernel estimates). *Let $\rho, \delta \in \mathbb{R}$ be such that $1 \geq \rho \geq \delta \geq 0$ and assume further that $\rho > 0$. Suppose that $\sigma \in S_{\rho, \delta}^m(G)$, and $\kappa \in C^\infty(G, \mathcal{S}'(G))$ is its associated kernel.*

1. *For every $d > 0$ and every $N \in \mathbb{N}$, there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for every $g, h = (y, l) \in G$ with $\|y\|_V \geq d$, we have*

$$|\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} \|y\|_V^{-N}.$$

2. *If $m > -\dim G$ and $\rho < 1$, there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for every $g, h \in G$ with $h \neq 0$, we have*

$$|\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} \|h\|_G^{-\frac{\dim G + m}{\rho}}.$$

3. *If $m \geq -\dim G$ or $\rho = 1$, then for every $\gamma < -(m + \dim G)/\rho$, there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for every $g, h \in G$ with $h \neq 0$, we have*

$$|\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} \|h\|_G^\gamma.$$

4. *If $m < -\dim G$, κ_g is continuous on G and there exists $C \geq 0$ and $P \in \mathbb{N}$ such that*

$$\sup_{g, h \in G} |\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} < \infty.$$

Remark 4.6.2 (Right semi-norms). *By inspecting the proof of all estimates in Theorem 4.6.1, we check it is possible to replace the symbol semi-norms by $\|\sigma\|_{S_{\rho, \delta}^{m, R}(G), P}$.*

The estimates at the origin in Theorem 4.6.1 are sufficient to obtain the pseudo-differential calculus but not the Calderón-Zygmund property. This is due to the fact that our proof will rely exclusively on the Sobolev inequality.

The following example illustrates very well the relationship between the estimates around the origin of Theorem 4.6.1 and the *Sobolev inequality*. Moreover, it shows that the estimates at the origin in Theorem 4.6.1 are not optimal.

Example 4.6.3 (Bessel potential). *Suppose that*

$$\sigma_m(\lambda) \stackrel{\text{def}}{=} \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m}{2}}, \quad \kappa_m(h) \stackrel{\text{def}}{=} (\mathcal{F}_G^{-1} \sigma_m)(h)$$

for each $m \in \mathbb{R}$. The Bessel potential is smooth away from the origin and satisfies

$$|\kappa_m(h)| \leq \begin{cases} C \|h\|_G^\gamma & \text{if } m > -\dim G \text{ and } \gamma < -(m + \dim G) \\ C & \text{if } m < -\dim G. \end{cases}$$

By carrying direct estimates on the heat kernel (see Subsection 4.8.2), we can show

$$|\kappa_m(h)| \leq C \|h\|_G^{-(m + \dim G)} \quad \text{if } m > -\dim G. \quad (26)$$

Remark 4.6.4. Example 4.6.3 illustrates well why the case $\rho = 1$ needs to be distinguished. As our proof of the kernel estimates at the origin relies exclusively on the Sobolev inequality, we will not be able to prove (26).

4.6.1 L^2 estimates for the kernel

A classical strategy in analysis is to show regularity via the Sobolev embeddings (Proposition 3.5.5). Moreover, the Fourier-Plancherel theory allows to link the $L^2(G)$ -norm of the kernel to the $L^2(\widehat{G})$ -norm of the symbol, and the latter can be easily derived directly from the symbolic estimates.

Proposition 4.6.5. *Let $\sigma \in S_{\rho,\delta}^m(G)$ with $1 \geq \rho \geq \delta \geq 0$, and denote by $\kappa \in C^\infty(G, \mathcal{S}'(G))$ its associated kernel. If $m < -\dim G/2$, then $\kappa_g \in L^2(G)$ for every $g \in G$, and there exists $C \geq 0$ and $P \in \mathbb{N}$ which do not depend on σ such that*

$$\sup_{g \in G} \|\kappa_g\|_{L^2(G)} \leq C \|\sigma\|_{S_{\rho,\delta}^m(G), P}.$$

Proof. Let $g \in G$ and $\lambda \in V$. Clearly,

$$\begin{aligned} & \|\sigma(g, \lambda)\|_{S_2(L^2(K))}^2 \\ &= \left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{m}{2}} \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{2}} \sigma(g, \lambda) \right\|_{S_2(L^2(K))}^2 \\ &\leq \left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{2}} \right\|_{S_2(L^2(K))}^2 \left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{m}{2}} \sigma(g, \lambda) \right\|_{\mathcal{L}(L^2(K))}^2. \end{aligned}$$

Using $\sigma \in S_{\rho, \delta}^m(G)$, the above becomes

$$\|\sigma(g, \lambda)\|_{S_2(L^2(K))}^2 \leq C \left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{2}} \right\|_{S_2(L^2(K))}^2.$$

Integrating both sides of the above with respect to $\lambda \in V$ and using the Plancherel formula (Proposition 3.4.5) on either sides, we get

$$\|\kappa_g\|_{L^2(K)}^2 \leq C \left\| \mathfrak{B}_{-m}^G \right\|_{L^2(K)}^2.$$

We conclude the proof by observing that the right-hand side is finite by assumption on m and Proposition 3.5.5, but also that the right-hand side does not depend on g . $\square$

Corollary 4.6.6. *Let $\sigma \in S_{\rho, \delta}^m(G)$ with $1 \geq \rho \geq \delta \geq 0$, and denote by $\kappa \in C^\infty(G, \mathcal{S}'(G))$ its associated kernel.*

For every $s \in \mathbb{R}$ satisfying

$$s < -\frac{\dim G}{2} - m,$$

the kernel κ_g belongs to $L_s^2(G)$ for every $g \in G$. Moreover, for such $s \in \mathbb{R}$, there exists $C \geq 0$ and $P \in \mathbb{N}$ which do not depend on σ such that

$$\sup_{g \in G} \|\kappa_g\|_{L_s^2(G)} < C \|\sigma\|_{S_{\rho, \delta}^m(G), P}.$$

Proof. The symbol

$$\tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{s}{2}} \sigma(g, \lambda)$$

belongs to $S_{\rho, \delta}^{m+s}(G)$, with $m + s \leq -\dim G/2$ by our assumption.

Therefore, applying Proposition 4.6.5, we know that for every $g \in G$,

$$(I - \mathcal{L}_G)^{\frac{s}{2}} \kappa_g \in L^2(G),$$

or in other words, $\kappa_g \in L_s^2(G)$, and

$$\sup_{g \in G} \|\kappa_g\|_{L_s^2(G)} = \sup_{g \in G} \|(I - \mathcal{L}_G)^{\frac{s}{2}} \kappa_g\|_{L^2(G)} < \infty,$$

concluding the proof. $\square$

We can now prove Part 4 of Theorem 4.6.1. As announced before, we need only apply Corollary 4.6.6 and the Sobolev embeddings.

Corollary 4.6.7. *Let $\sigma \in S_{\rho,\delta}^m(G)$ with $1 \geq \rho \geq \delta \geq 0$, and denote by $\kappa \in C^\infty(G, \mathcal{S}'(G))$ its associated kernel.*

If $m < -\dim G$, then $\kappa_g \in C(G)$ is continuous for every $g \in G$. Moreover, there exists $C \geq 0$ and $P \in \mathbb{N}$ which do not depend on σ such that

$$\sup_{g \in G} \|\kappa_g\|_{C(G)} < \infty.$$

Proof. Choose $s \in \mathbb{R}$ such that

$$\frac{\dim G}{2} < s < -\frac{\dim G}{2} - m,$$

which is possible since $m < -\dim G$.

Since $m < -\dim G/2 - s$, Corollary 4.6.6 implies that $\kappa_g \in L_s^2(G)$ for every $g \in G$,

$$\sup_{g \in G} \|\kappa_g\|_{L_s^2(G)} < \infty.$$

However, since $s > \dim G/2$, the Sobolev inequality (Proposition 3.5.5) implies that

$$\sup_{g \in G} \|\kappa_g\|_{C(G)} \leq C \sup_{g \in G} \|\kappa_g\|_{L_s^2(G)} < \infty.$$

This concludes the proof. $\square$

4.6.2 Estimates at infinity

We start by proving Part 1 of Theorem 4.6.1. In the previous subsection, we have shown the continuity and boundedness of kernels associated with symbols with sufficiently low order.

Applying difference operators to symbols, or equivalently multiplying the associated kernel by a polynomial, reduces the order, thereby improving our chances for the kernel to be bounded.

Sufficiently far from the origin, this is only possible if our kernels have *Schwartz decay*, which we prove below.

Proof of 1 of Theorem 4.6.1. Let $M \in \mathbb{R}$ and choose $p \in 2\mathbb{N}$ sufficiently large so that

$$g \geq M \quad \text{and} \quad m - \rho p + \lceil \dim G/2 \rceil < -\frac{\dim G}{2}.$$

Fix $d > 0$, and let $\chi \in C^\infty(G)$ be a smooth radial function satisfying

$$\chi(y, l) = \begin{cases} 0 & \text{if } \|y\|_V \leq d/2 \\ 1 & \text{if } \|y\|_V \geq d. \end{cases}$$

From now on, we shall write $h = (y, l)$.

By the Sobolev inequality (Proposition 3.5.5), we have

$$\begin{aligned} & \sup_{\|y\|_V \geq d} \left| \|y\|_V^M \kappa_g(h) \right| \\ & \leq \sup_{h \in G} \left| \|y\|_V^M \kappa_g(h) \chi(h) \right| \\ & \leq C \sum_{|\beta| \leq \lceil \dim G/2 \rceil} \left\| X_h^\beta \{ \|y\|_V^M \kappa_g(h) \chi(h) \} \right\|_{L^2(G, dh)}. \end{aligned} \quad (27)$$

Let us do some calculations on the right-hand side of the above. By the Leibniz formula, we have

$$\begin{aligned} & \left| X_h^\beta \{ \kappa_g(h) \chi(h) \} \right| \\ & \leq \sum_{\beta' \leq \beta} \left| X_h^{\beta - \beta'} \{ \|y\|_V^{M-p} \chi(h) \} X_h^{\beta'} \{ \|y\|_V^p \kappa_g(h) \} \right| \\ & \leq \left(\sum_{\beta' \leq \beta} \left| X_h^{\beta'} \{ \|y\|_V^{M-p} \chi(h) \} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{\beta' \leq \beta} \left| X_h^{\beta'} \{ \|y\|_V^p \kappa_g(h) \} \right|^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (28)$$

where the last line was obtained by applying the Cauchy-Schwartz inequality to the sum. Since $p \geq M$ and χ vanishes around $\{0_V\} \times K$, it follows that

$$\sup_{h \in G} \sum_{\beta' \leq \beta} \left| X_h^{\beta'} \{ \|y\|_V^{M-p} \chi(h) \} \right|^2 < \infty,$$

which means that (28) becomes

$$\left| X_h^\beta \{ \kappa_g(h) \chi(h) \} \right| \leq C \left(\sum_{\beta' \leq \beta} \left| X_h^{\beta'} \{ \|y\|_V^p \kappa_g(h) \} \right|^2 \right)^{\frac{1}{2}}.$$

Therefore, if we square both sides of the above and integrate with respect to h , we obtain

$$\begin{aligned} & \int_G \left| X_h^\beta \{ \kappa_g(h) \chi(h) \} \right|^2 dh \\ & \leq C \int_G \sum_{\beta' \leq \beta} \left| X_h^{\beta'} \{ \|y\|_V^p \kappa_g(h) \} \right|^2 dh \\ & \leq C \sum_{\beta' \leq \beta} \sum_{|\alpha|=p} \int_G \left| X_h^{\beta'} \{ q^\alpha \kappa_g \} \right|^2 dh, \end{aligned}$$

where the last line was obtained by using the strong admissibility of our difference operators. Continuing the above calculation, we get

$$\begin{aligned} & \int_G \left| X_h^\beta \{ \kappa_g(h) \chi(h) \} \right|^2 dh \\ & \leq C \sum_{|\alpha|=p} \int_G \left| (I - \mathcal{L}_G)^{\frac{[\dim G/2]}{2}} \{ q^\alpha \kappa_g \} \right|^2 dh. \end{aligned}$$

Combining the above with (27), this yields

$$\sup_{\|y\|_V \geq d} \left| \|y\|_V^M \kappa_g(h) \right| \leq C \sum_{|\alpha|=p} \left\| (I - \mathcal{L}_G)^{\frac{[\dim G/2]}{2}} \{ q^\alpha \kappa_g \} \right\|_{L^2(G)}. \quad (29)$$

In the above, each symbol $\xi_G^\lambda (I - \mathcal{L}_G)^{\frac{[\dim G/2]}{2}} \Delta_G^\alpha \sigma(g, \lambda)$ is of order

$$m - \rho p + [\dim G/2] < -\frac{\dim G}{2}$$

so that its kernel $(I - \mathcal{L}_G)^{\frac{[\dim G/2]}{2}} \{ q^\alpha \kappa_g \}$ belongs to $L^2(G)$ uniformly in $g \in G$ by Proposition 4.6.5. This implies that (29) becomes

$$\sup_{g \in G} \sup_{\|y\|_V \geq d} \left| \|y\|_V^M \kappa_g(h) \right| < \infty,$$

concluding the proof. $\square$

4.6.3 Estimates at the origin

We have shown so far that the regularity of the kernel is linked to the order of its corresponding symbol in the following way: the lower the order, the more regular the kernel is. As applying difference operators strictly *reduces the order* when $\rho > 0$, the distribution

$$q^\alpha \kappa_g \stackrel{\text{def}}{=} \mathcal{F}_G^{-1} \{ \Delta_G^\alpha \sigma(g, \cdot) \}$$

becomes more regular as $|\alpha|$ increases. This means that κ_g can only have singularities at the common zeros

$$\bigcap_{j=1}^{n_{\Delta, G}} \{q_j = 0\} = (0_V, I_V),$$

which happens to simply be the origin as the family $\{q_j : j = 1, \dots, n_{\Delta, G}\}$ was chosen to be *strongly admissible*.

The aim of this subsection is therefore to study the strength of the only singularity that a kernel can have, which will lead to the proof of parts 2 and 3 of Theorem 4.6.1.

The ideas are relatively simple. Multiplying the kernels by a polynomial of a sufficiently high order removes the singularity at the origin since it causes the order to decrease. The *strong admissibility* means that polynomials can constitute a good approximation of the distance to the origin.

Theorem 4.6.8 (Kernel estimates at the origin). *Let $\rho, \delta \in \mathbb{R}$ be such that $1 \geq \rho \geq \delta \geq 0$ and assume further that $\rho > 0$. Suppose that $\sigma \in S_{\rho, \delta}^m(G)$, and $\kappa \in C^\infty(G, S'(G))$ denotes its associated kernel.*

1. *If $\rho < 1$ and $m > -\dim G$, there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for every $g, h \in G$ with $h \neq 0$, we have*

$$|\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} \|h\|_G^{-\frac{\dim G + m}{\rho}},$$

where $C \geq 0$ and $P \in \mathbb{N}$ do not depend on σ .

2. *If $\rho = 1$ or if $m = -\dim G$, then for every $\gamma < -(\dim G + m)/\rho$, there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for every $g, h \in G$ with $h \neq 0$, we have*

$$|\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} \|h\|_G^\gamma,$$

where $C \geq 0$ and $P \in \mathbb{N}$ do not depend on σ .

Proof. Fix $h \in G \setminus \{(0, I_V)\}$ such that $\varepsilon \stackrel{\text{def}}{=} \|h\|_G^2 \leq 1$. Define the symbol

$$\sigma_\varepsilon(g, \lambda) \stackrel{\text{def}}{=} \sigma(g, \lambda) \eta_\varepsilon(\lambda), \quad g \in G, \lambda \in V,$$

where η_ε is given by Theorem 4.4.7, and denote by $\kappa_\varepsilon \stackrel{\text{def}}{=} \mathcal{F}_G^{-1}\{\sigma_\varepsilon(g, \cdot)\}$ its associated kernel.

Using

$$|\kappa_g(h)| \leq |\kappa_{\varepsilon, g}(h)| + |\kappa_g(h) - \kappa_{\varepsilon, g}(h)|,$$

the idea of the proof is to estimate both terms of the right-hand side above separately.

As we have shown earlier in Corollary 4.6.6, we can get a bound on the kernel provided the order of its associated symbol is strictly less than $-\dim G$. For κ_ε , this will be easy since σ_ε is smoothing; for $\kappa_g - \kappa_{\varepsilon, g}$, we will have to use difference operators to lower the order.

Note that the case $m = -\dim G$ is a simple consequence of the other estimates, and the inclusion $S_{\rho, \delta}^m(G) \subset S_{\rho, \delta}^{m'}(G)$ when $m \leq m'$. We can therefore assume without loss of generality that $m > -\dim G$.

ESTIMATES ON κ_ε We observe that for each $\gamma \in \mathbb{R}$, we have

$$\begin{aligned} \sigma_\varepsilon(g, \lambda) &= \sigma(g, \lambda) \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{2}} \\ &\quad \eta_\varepsilon(\lambda) \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{\gamma}{2}} \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m+\gamma}{2}} \end{aligned}$$

since η_ε is diagonal.

Using the Hölder inequality on Schatten classes, together with Theorem 4.4.7, we obtain

$$\|\sigma_\varepsilon(g, \lambda)\|_{S_1(L^2(K))} \leq C\varepsilon^{\frac{\gamma}{2}} \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{2}} \right\|_{S_1(L^2(K))} \quad (30)$$

for every $\gamma < 0$.

The idea is now to pick $\gamma < \min\{0, -(\dim G + m)\}$, so that

$$|\kappa_{\varepsilon, g}(h)| \leq \int_V \|\sigma_\varepsilon(g, \lambda)\|_{S_1(L^2(K))} d\lambda \leq C\varepsilon^{\frac{\gamma}{2}}, \quad (31)$$

where the last inequality was obtained by the Sobolev inequality (Proposition 3.5.5).

The idea is now to pick a suitable $\gamma < 0$ and obtain an estimate on the kernel via (31). Keeping in mind that $\varepsilon^2 = \|h\|_G$, let us now treat the different cases separately.

$\rho = 1$. We pick $\gamma < -(\dim G + m)$ to obtain

$$|\kappa_{\varepsilon, g}(h)| \leq C \|h\|_G^\gamma.$$

$\rho < 1$. We pick

$$\gamma \stackrel{\text{def}}{=} -\frac{\dim G + m}{\rho},$$

and check that $\gamma < 0$ by the condition on m and $\gamma < -(\dim G + m)$ by our condition on ρ .

Therefore, (31) yields

$$|\kappa_{\varepsilon, g}(h)| \leq C \|h\|_G^{-\frac{\dim G + m}{\rho}}.$$

Note that we could get a better estimate here, but we choose not to do so, because in this setting our choice of γ will turn out to be appropriate when estimating $\kappa_g - \kappa_{g, \varepsilon}$.

ESTIMATE ON $\kappa_g - \kappa_{g, \varepsilon}$. The strategy of the proof is the same, but slightly more delicate. As γ needs to be positive to apply Theorem 4.4.7, we will need to apply difference operators so that m would be replaced by $m - \rho|\alpha|$, thus making it possible for $\gamma + m - \rho|\alpha|$ to be small enough.

By the Leibniz rule, we have

$$\begin{aligned} \Delta_\alpha^G(\sigma - \sigma_\varepsilon) &= \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \Delta_{\alpha_1}^G \sigma \Delta_{\alpha_2}^G \eta_\lambda \\ &= \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} I_{1, \alpha_1} I_{2, \alpha_2, \gamma} I_{3, \gamma}, \end{aligned}$$

where

$$\begin{aligned} I_{1,\alpha_1} &\stackrel{\text{def}}{=} \Delta_{\alpha_1}^G \sigma(g, \lambda) \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\rho|\alpha_1|}{2}}, \\ I_{2,\alpha_2,\gamma} &\stackrel{\text{def}}{=} \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{m-\rho|\alpha_1|}{2}} \Delta_G^{\alpha_2} (I_{L^2(K)} - \eta_\varepsilon(\lambda)), \\ &\quad \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma+\rho|\alpha_2|+m+\rho|\alpha_1|}{2}}, \\ I_{3,\gamma} &\stackrel{\text{def}}{=} \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma+m-\rho|\alpha|}{2}}. \end{aligned}$$

Since $\sigma \in S_{\rho,\delta}^m(G)$, $\|I_{1,\alpha_1}\|_{\mathcal{L}(L^2(K))}$ is bounded uniformly in α_1 , while

$$\|I_{2,\alpha_2,\gamma}\|_{\mathcal{L}(L^2(K))} \leq C\varepsilon^{\frac{\gamma}{2}}$$

holds for every $\gamma \geq 0$ by Theorem 4.4.7. Therefore, for every $\gamma \geq 0$, we have

$$\begin{aligned} &\left\| \Delta_\alpha^G (\sigma - \sigma_\varepsilon)(g, \lambda) \right\|_{S_1(L^2(K))} \\ &\leq C\varepsilon^{\frac{\gamma}{2}} \left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma+m-\rho|\alpha|}{2}} \right\|_{S_1(L^2(K))}, \end{aligned}$$

where the Schatten norm on the right hand side is not necessarily finite.

Integrating the above with respect to $\lambda \in V$, we obtain

$$|(q^\alpha(\kappa_g - \kappa_{g,\varepsilon}))(h)| \leq C\varepsilon^{\frac{\gamma}{2}},$$

provided that $\gamma \geq 0$ and $\gamma + m - \rho|\alpha| < -\dim G$. Observe that these two conditions can be summarized by the inequality

$$0 \leq \gamma < \rho|\alpha| - (\dim G + m).$$

Since for every $r \in 2\mathbb{N}$, we have

$$\|h\|_G^r |(\kappa_g - \kappa_{g,\varepsilon})(h)| \leq C \sum_{|\alpha|=r} |q^\alpha(h)(\kappa_g - \kappa_{g,\varepsilon})(h)|,$$

we have so far shown, keeping in mind that $\varepsilon = \|h\|_G^2$, that

$$|(\kappa_g - \kappa_{g,\varepsilon})(h)| \leq C \|h\|_G^{\gamma-r}, \quad (32)$$

provided $0 \leq \gamma < \rho r - (\dim G + m)$.

We observe that the best estimate is achieved when

$$r \stackrel{\text{def}}{=} \min\{n \in 2\mathbb{N} : n > \frac{\dim G + m}{\rho}\}$$

because it should be as small as possible, but also the condition on γ must be possible.

Let us now treat the different cases.

$\rho = 1$. Choose $\tilde{\gamma} \stackrel{\text{def}}{=} \gamma - r$ so that $-r \leq \tilde{\gamma} < -(\dim G + m)$.

Clearly, we have

$$|(\kappa_g - \kappa_{g,\varepsilon})(h)| \leq \|h\|_G^{\tilde{\gamma}}.$$

$\rho < 1$. Pick γ as follows

$$\gamma - r = -\frac{\dim G + m}{\rho}.$$

We check that

$$\gamma = r - \frac{\dim G + m}{\rho} > 0,$$

and similarly,

$$\begin{aligned} \gamma - \rho r &= r(1 - \rho) - \frac{\dim G + m}{\rho} \\ &= -\rho r \left(1 - \frac{1}{\rho}\right) - (\dim G + m) \frac{1}{\rho} \\ &< -(\dim G + m) \left(1 - \frac{1}{\rho} + \frac{1}{\rho}\right), \end{aligned}$$

which can be rewritten as $\gamma < \rho r - (\dim G + m)$. Note that we could not have obtained the strict inequality above had we not assumed that $\rho < 1$. We can thus apply (32) to obtain

$$|(\kappa_g - \kappa_{g,\varepsilon})(h)| \leq \|h\|_G^{-\frac{\dim G + m}{\rho}}.$$

The kernel estimates now follow from the triangle inequality. $\square$

Remark 4.6.9. The statement of Theorem 4.6.8 is different than its counterpart in [Fis15; FR16; FR16]. A closer inspection of the cited proofs shows that there are issues for the case $\rho = 1$ and when $m = -\dim G$ (or the

homogenous dimension in the graded case), which we tried to emphasize herein.

Before we are able to describe smoothing operators, we need to recall a property of convolution by tempered distributions. For a proof, see e.g. [FR16, Lemma 3.1.55], which can be easily adapted to our case.

Lemma 4.6.10. *Let $f \in \mathcal{S}'(G)$. There exists $N \in \mathbb{N}$ and $C \geq 0$ such that*

$$|(\phi \star f)(g)| \leq C \|f\|_{\mathcal{S}'(G),N} (1 + \|g\|_G)^N \|\phi\|_{\mathcal{S}(G),N}.$$

We are now able to show that $\Psi^{-\infty}(G)$ are worthy of the name.

Proposition 4.6.11. *Let $T \in \Psi^{-\infty}(G)$. Its convolution kernel*

$$(g, h) \mapsto \kappa_g(h)$$

is smooth on $G \times G$ and given a fixed $g \in G$, $\kappa_g(\cdot) \in \mathcal{S}(G)$.

As a result, the operator T extends to a continuous mapping from $\mathcal{S}'(G)$ to $C^\infty(G)$.

Proof. The smoothness of the kernel and Schwartz decay in the second variable follow directly from the kernel estimates (Theorem 4.6.1). In particular, for each $\beta \in \mathbb{N}^{\dim G}$ and each $N \in \mathbb{N}$, we have

$$\sup_{g \in G} \|X_g^\beta \kappa_g\|_{\mathcal{S}(G),N} \leq C \|\sigma\|_{S_{\rho,\delta}^m(G),N'} \quad (33)$$

for some $N \in \mathbb{N}$.

Fix $\beta \in \mathbb{N}^n$. Using the Leibniz rule for left-invariant differential operators, we check that

$$X^\beta(Tf)(g) = \sum_{\beta=\beta_1+\beta_2}^{\bar{\bar{\beta}}} X_{g_1=g}^{\beta_1} (f \star X_{g_2=g}^{\beta_2} \kappa_{g_2})(g_1).$$

Passing from left-invariant vector fields to their right-invariant counterpart, we obtain

$$\left| X^\beta(Tf)(g) \right| \leq C \sum_{\beta=\beta_1+\beta_2}^{\bar{\bar{\beta}}} (1 + \|g\|_G)^{|\beta|_1} \left| (\tilde{X}_{g_1=g}^{\beta_1} f \star X_{g_2=g}^{\beta_2} \kappa_{g_2})(g_1) \right|.$$

Using Lemma 4.6.10 and the continuity of X^{β_1} on $\mathcal{S}'(G)$, we have

$$\begin{aligned} & \left| X^\beta(Tf)(g) \right| \\ & \leq C \sum_{\beta=\beta_1+\beta_2}^{\infty} (1 + \|g\|_G)^{|\beta|_1+N} \left\| \tilde{X}_{g_1=g}^{\beta_1} f \right\|_{\mathcal{S}'(G),N} \left\| X_{g_2=g}^{\beta_2} \kappa_{g_2} \right\|_{\mathcal{S}'(G),N} \\ & \leq C \sum_{\beta=\beta_1+\beta_2}^{\infty} (1 + \|g\|_G)^{N'} \|f\|_{\mathcal{S}'(G),N'} \left\| X_{g_2=g}^{\beta_2} \kappa_{g_2} \right\|_{\mathcal{S}(G),N'}. \end{aligned}$$

Using the above, together with (33), we conclude easily that T is continuous from $\mathcal{S}'(G)$ to $C^\infty(G)$. $\square$

4.7 ADJOINT AND COMPOSITION FORMULAE

We are now ready to prove our two main results, the *composition* and *adjunction* formulae.

The former ensures it is possible to create parametrices for elliptic operators, while the latter allows to extend our pseudo-differential operators (and in particular our parametrices) to tempered distributions.

We shall show the following results.

- Provided $\rho > \delta$, each operator class $\Psi_{\rho,\delta}^m(G)$ is stable under adjunction.
- Provided $\rho > \delta$, the composition of $T_1 \in \Psi_{\rho,\delta}^{m_1}(G)$ and $T_2 \in \Psi_{\rho,\delta}^{m_2}(G)$ is in $\Psi_{\rho,\delta}^{m_1+m_2}(G)$.

Both of these results are often proved by doing calculations on the Fourier Transform side, exploiting the fact that $\widehat{\mathbb{R}^n} = \mathbb{R}^n$. As our symbols are *operator-valued*, a more suitable strategy would be to carry out our calculations on the kernel side, exploiting our previously obtained kernel estimates (Theorem 4.6.1). Fortunately, the fact that kernels might be singular is not an issue, since we have the density of smoothing symbols (Proposition 4.5.1).

For technical simplicity, we shall perform symbolic estimates only with right symbol seminorms. We then show in Subsection 4.7.3 that the symbol classes can be defined in an equivalent way by only considering the action of the right-invariant Laplacian, which will allow us to conclude in Subsection 4.7.4 that both the composition and the adjunction formulae hold.

Remark 4.7.1. Our family Δ of difference operators shall henceforth be numbered in such a way that

$$q_1, \dots, q_{\dim G}$$

forms a strongly admissible family. Given $\alpha \in \mathbb{N}^{\dim G}$, we let

$$q^\alpha \stackrel{\text{def}}{=} q^{(\alpha, 0)}, \quad \Delta_G^\alpha \stackrel{\text{def}}{=} \Delta_G^{(\alpha, 0)}.$$

Before we start, let us state a result that will be used in the proof of both results.

Lemma 4.7.2. Let $\sigma \in S_{\rho, \delta}^m(G)$ with $1 \geq \rho \geq \delta \geq 0$ with $\rho \neq 0$, and denote by κ its associated kernel. If $\gamma \in \mathbb{R}$ satisfies

$$\gamma > \max \left\{ \frac{m + \dim G}{\rho}, 0 \right\} - \dim G,$$

then we can find $C \geq 0$ and $N \in \mathbb{N}$ such that

$$\int_G \|h\|_G^\gamma |\kappa_g(h)| \, dh \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), N}.$$

In the above, we may replace $\|\sigma\|_{S_{\rho, \delta}^m(G), N}$ by $\|\sigma\|_{S_{\rho, \delta}^m(G), N'}$.

Proof. We split the above integral like this

$$\int_G dh = \int_{\|h\|_G \leq 1} dh + \int_{\|h\|_G \geq 1} dh.$$

INTEGRAL ON $\|h\|_G \geq 1$

By Part (1) of Theorem 4.6.1, for each $M \in \mathbb{N}$ we can find $N \in \mathbb{N}$ such that

$$\sup_{\|h\|_G \geq 1} |\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), N} \|h\|_G^{-M}.$$

Taking M sufficiently large, it implies that

$$\int_{\|h\|_G \geq 1} \|h\|_G^\gamma |\kappa_g(h)| \, dh \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), N},$$

as required.

INTEGRAL ON $\|h\|_G \leq 1$

If $m \geq -\dim K$, then Part (2) of Theorem 4.6.1 implies that if we pick $\tilde{\gamma}$ satisfying

$$\begin{cases} \gamma + \dim G > \tilde{\gamma} > \frac{m+\dim G}{\rho} & \text{if } m \geq \dim G \\ \tilde{\gamma} = 0 & \text{if } m < -\dim G \end{cases}, \quad (34)$$

then there exists $N \in \mathbb{N}$ such that

$$\sup_{0 < \|h\|_G \leq 1} |\kappa_g(h)| \leq C \|\sigma\|_{S_{\rho,\delta}^m(G),N} \|h\|_G^{\tilde{\gamma}}.$$

From there, it follows that

$$\int_{\|h\|_G \leq 1} \|h\|_G^{\gamma} |\kappa_g(h)| \, dh \leq C \|\sigma\|_{S_{\rho,\delta}^m(G),N} \int_{\|h\|_G \leq 1} \|h\|_G^{\gamma+\tilde{\gamma}}.$$

We conclude the proof by observing that (34) implies

$$\gamma + \tilde{\gamma} > -\dim G,$$

which means that the above integral is finite.

The conclusion for right semi-norms is deduced by checking the claim in Remark 4.6.2. $\square$

4.7.1 Composition formula

The first step to obtain the composition formula is to understand the composition of two smoothing operators on the kernel side.

Lemma 4.7.3. *Let $\sigma_1, \sigma_2 \in S^{-\infty}(G)$ and denote by κ_1 and κ_2 their respective associated kernels. We also set*

$$\kappa_g(h) \stackrel{\text{def}}{=} \int_G \kappa_{2,g l^{-1}}(h l^{-1}) \kappa_{1,g}(l) \, dl$$

for each $g, h \in G$, and subsequently define the map

$$\sigma(g, \lambda) \stackrel{\text{def}}{=} \mathcal{F}_G \kappa_g(\lambda), \quad g \in G, \lambda \in V.$$

The map σ defined above is a smoothing symbol for which

$$\sigma(g, \lambda) = \int_G \kappa_{1,g}(l) \tilde{\zeta}_G^\lambda(l)^* \sigma_2(g l^{-1}, \lambda) \, dl \quad (35)$$

holds for all $g \in G$ and $\lambda \in V$.

Proof. It should be clear that κ is smooth on $G \times G$. To show that the Fourier transform of κ_g is well defined, check that $\kappa_g \in L^1(K)$. Indeed,

$$\begin{aligned} \int_G |\kappa_g(h)| \, dh &\leq \int_G \int_G |\kappa_{2,g} l^{-1}(hl^{-1}) \kappa_{1,g}(l)| \, dl \, dh \\ &\leq \max_{g' \in G} \int_G |\kappa_{2,g'}(h')| \, dh' \int_G |\kappa_{1,g}(l)| \, dl. \end{aligned}$$

Let us check that κ_g is the kernel of $\text{Op}_G(\sigma_1)\text{Op}_G(\sigma_2)$. Applying the quantization formula twice (Proposition 4.3.5), we obtain

$$\begin{aligned} \text{Op}_G(\sigma_1)\text{Op}_G(\sigma_2)\phi(g) &= \int_G \text{Op}_G(\sigma_2)\phi(h) \kappa_1(g, h^{-1}g) \, dh \\ &= \int_G \int_G \phi(l) \kappa_2(h, l^{-1}h) \kappa_1(g, h^{-1}g) \, dl \, dh. \end{aligned}$$

In the right hand side of the above, we can let $m = h^{-1}g$ to obtain

$$\begin{aligned} \text{Op}_G(\sigma_1)\text{Op}_G(\sigma_2)\phi(g) &= \int_G \phi(l) \int_G \kappa_2(gm^{-1}, l^{-1}xm^{-1}) \kappa_1(g, m) \, dm \, dl \\ &= \int_G \phi(l) \kappa_g(l^{-1}g) \, dl = \phi \star \kappa_g(g), \end{aligned}$$

which means that $\text{Op}_G(\sigma_1)\text{Op}_G(\sigma_2) = \text{Op}_G(\sigma)$.

From there, let us try to obtain a more explicit formula describing σ . Taking the Fourier Transform, we get

$$\begin{aligned} \sigma(g, \lambda) &= \int_G \kappa_g(h) \tilde{\zeta}_G^\lambda(h)^* \, dh \\ &= \int_G \int_G \kappa_2(gm^{-1}, hm^{-1}) \kappa_1(x, m) \tilde{\zeta}_G^\lambda(m)^* \tilde{\zeta}_G^\lambda(hm^{-1})^* \, dh \, dm \\ &= \int_G \kappa_1(g, m) \tilde{\zeta}_G^\lambda(m)^* \sigma_2(gm^{-1}, \lambda) \, dm, \end{aligned}$$

concluding the proof. $\square$

Definition 4.7.4. Let $\sigma_1, \sigma_2 \in S^{-\infty}(G)$. We let

$$\sigma_1 \circ \sigma_2(g, \lambda) = \int_G \kappa_{1,g}(h) \tilde{\zeta}_G^\lambda(h)^* \sigma_2(gh^{-1}, \lambda) \, dh$$

for each $(g, \lambda) \in G \times V$.

The following lemma shows that provided some technical conditions are met, we obtain an estimate on the remainder in terms of certain seminorms in the *correct* symbol classes. Note that if $\rho > \delta$, the condition (36) below is always possible by choosing N sufficiently large. This is the reason why we shall always assume $\rho > \delta$ hereafter.

Lemma 4.7.5. *Let $m_1, m_2 \in \mathbb{R}$ and suppose $\rho, \delta \in \mathbb{R}$ are chosen such that $1 \geq \rho > \delta \geq 0$. Suppose further that $\beta_0 \in \mathbb{N}^{\dim G}$, and $N \in \mathbb{N}$.*

If there exists $M \in \mathbb{N}$ such that

$$\begin{cases} \frac{m_2 + \delta |\beta_0|}{1 - \delta} \leq 2M < N - \dim G - m_1 - \delta |\beta_0| + \rho(\dim G) \\ m_2 + \delta(N + 1 + |\beta_0|) \leq 2M < -\dim G - m_1 - \delta |\beta_0| + \rho(\dim G + N + 1), \end{cases} \quad (36)$$

then there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for any $\sigma_1, \sigma_2 \in S^{-\infty}(G)$ and any $(g, \lambda) \in G \times V$, we have

$$\begin{aligned} & \left\| X_g^\beta \left(\sigma_1 \circ \sigma_2(g, \lambda) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_1(g, \lambda) X^{(\alpha)} \sigma_2(g, \lambda) \right) \right\|_{\mathcal{L}(L^2(K))} \\ & \leq C \|\sigma_1\|_{S_{\rho, \delta}^{m_1, R}(G), P} \|\sigma_2\|_{S_{\rho, \delta}^{m_2}(G), P}. \end{aligned}$$

Proof. For each $N \in \mathbb{N}$, let us define

$$\tau_N(g, \lambda) \stackrel{\text{def}}{=} \sigma(g, \lambda) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_1(g, \lambda) X^{(\alpha)} \sigma_2(g, \lambda),$$

where $\sigma \stackrel{\text{def}}{=} \sigma_1 \circ \sigma_2$. We want to estimate

$$\begin{aligned} & \tau_N(g, \lambda) \\ &= \int_G \kappa_{1, g}(h) \xi_G^\lambda(h)^* \left(\sigma_2(g h^{-1}, \lambda) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} q_\alpha(h^{-1}) X^{(\alpha)} \sigma_2(g, \lambda) \right) dh \\ &= \int_G \kappa_{1, g}(h) \xi_G^\lambda(h)^* R_{e_G, N}^{\sigma_2(g, \lambda)}(h^{-1}) dh. \end{aligned} \quad (37)$$

Let $\beta_0 \in \mathbb{N}^{\dim G}$. Applying the Leibniz-like rule for difference operators and the Leibniz rule for left-invariant differential operators, we obtain

$$\begin{aligned} X^{\beta_0} \tau_N(g, \lambda) &= \\ & \sum_{\beta_{01} + \beta_{02} = \beta_0} \int_G X_g^{\beta_{01}} \kappa_{1, g}(h) \xi_G^\lambda(h)^* R_{e_G, N}^{X_{g_2}^{\beta_{02}} \sigma_2(g_2, \lambda)}(h^{-1}) dh. \end{aligned}$$

Now, we would like to introduce powers of $\zeta_G^\lambda(I - \mathcal{L}_G)$ near σ_2 , which will allow us to estimate the above in terms of the symbol semi-norms of σ_2 . To this end, observe that if $M \in \mathbb{N}$,

$$\begin{aligned}\zeta_G^\lambda(h)^\star &= \zeta_G^\lambda(h)^\star \zeta_G^\lambda(I - \mathcal{L}_G)^M \zeta_G^\lambda(I - \mathcal{L}_G)^{-M} \\ &= \sum_{|\beta| \leq 2M} \zeta_G^\lambda(h)^\star \zeta_G^\lambda(X)^\beta \zeta_G^\lambda(I - \mathcal{L}_G)^{-M},\end{aligned}$$

which, after using $\zeta_G^\lambda(h)^\star \zeta_G^\lambda(X)^\beta = (-1)^{|\beta|} (\tilde{X}_h^\beta \zeta_G^\lambda(h))^\star$, becomes

$$\zeta_G^\lambda(h)^\star = \sum_{|\beta| \leq 2M} (\tilde{X}_h^\beta \zeta_G^\lambda(h))^\star \zeta_G^\lambda(I - \mathcal{L}_G)^{-M}.$$

Therefore, injecting the above into (37) and using integration by parts yields

$$\begin{aligned}X^{\beta_0} \tau_N(g, \lambda) &= \\ &\sum_{\substack{\beta_{01} + \beta_{02} = \beta_0 \\ |\beta_1| + |\beta_2| \leq 2M}} \int_G \tilde{X}_{h_1=h}^{\beta_1} (X_g^{\beta_{01}} \kappa_{1,g})(h_1) \zeta_G^\lambda(h)^\star \\ &\quad \times \tilde{X}_{h_2=h}^{\beta_2} R_{e_G, N}^{\zeta_G^\lambda(I - \mathcal{L}_G)^{-M} X_{g_2=g}^{\beta_{02}} \sigma_2(g_2, \cdot, \lambda)}(h_2^{-1}) dh.\end{aligned}$$

Now, using Lemma 3.6.4, the above becomes

$$\begin{aligned}X^{\beta_0} \tau_N(g, \lambda) &= \\ &\sum_{\substack{\beta_{01} + \beta_{02} = \beta_0 \\ |\beta_1| + |\beta_2| \leq 2M}} \int_G \tilde{X}_{h_1=h}^{\beta_1} (X_g^{\beta_{01}} \kappa_{1,g})(h_1) \zeta_G^\lambda(h)^\star \\ &\quad \times R_{e_G, N - |\beta_2|}^{\zeta_G^\lambda(I - \mathcal{L}_G)^{-M} X_{g_2=g}^{\beta_{02}} X^{\beta_2} \sigma_2(g_2, \cdot, \lambda)}(h_2^{-1}) dh.\end{aligned}$$

Therefore, if we take the operator norm, we obtain

$$\begin{aligned}\|X^{\beta_0} \tau_n(g, \lambda)\|_{\mathcal{L}(L^2(K))} &\leq \\ &\sum_{\substack{\beta_{01} + \beta_{02} = \beta_0 \\ |\beta_1| + |\beta_2| \leq 2M}} \int_G \left| \tilde{X}_{h_1=h}^{\beta_1} (X_g^{\beta_{01}} \kappa_{1,g})(h_1) \right| \\ &\quad \times \left\| R_{e_G, N - |\beta_2|}^{\zeta_G^\lambda(I - \mathcal{L}_G)^{-M} X_{g_2=g}^{\beta_{02}} X^{\beta_2} \sigma_2(g_2, \cdot, \lambda)}(h_2^{-1}) \right\|_{\mathcal{L}(L^2(K))} dh.\end{aligned}\quad (38)$$

Suppose first that $N - |\beta_2| < 0$. It follows by Lemma 4.7.2 that (38) is bounded by $C \|\sigma_1\|_{S_{\rho,\delta}^{m_1,R}(G),N'} \|\sigma_2\|_{S_{\rho,\delta}^{m_2}(G),N'}$ provided that

$$\begin{cases} m_1 + \delta(|\beta_{01}| + |\beta_1|) + \dim G < \rho(\dim G + |\beta_{02}|) \\ -2M + \delta(|\beta_{02}| + |\beta_2|) \leq -m_2. \end{cases}$$

Using $|\beta_1| \leq 2M - N$, it is sufficient that

$$\begin{cases} m_1 + \delta|\beta_0| + 2M - N + \dim G < \rho(\dim G) \\ -2M + \delta(|\beta_0| + 2M) \leq -m_2, \end{cases}$$

which can be rewritten in one line as

$$\frac{m_2 + \delta|\beta_0|}{1 - \delta} \leq 2M \leq N - \dim G - m_1 - \delta|\beta_0| + \rho(\dim G). \quad (39)$$

Applying the Taylor Remainder Theorem (Proposition 3.6.5), we have

$$\begin{aligned} & \left\| R_{g,N-|\beta_2|}^{\zeta_G^\lambda (I - \mathcal{L}_G)^{-M} X_{g_2=g}^{\beta_{02}} X^{\beta_2} \sigma_2(g_2, \lambda)} (h^{-1}) \right\|_{\mathcal{L}(L^2(K))} \\ & \leq C \sup_{|\gamma| \leq N - |\beta_2| + 1} \|h\|_G^{N - |\beta_2| + 1} S(h, M, \gamma, \beta_{02}, \beta_2), \end{aligned}$$

where

$$\begin{aligned} S(h, M, \gamma, \beta_{02}, \beta_2) & \stackrel{\text{def}}{=} \sup_{\|g_1\|_G \leq \eta \|h\|_G} \\ & \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{-M} X_{g_1}^\gamma X_{g_2=g}^{\beta_{02}} X_{g_1}^{\beta_2} \sigma_2(g_2 g_1, \lambda) \right\|_{\mathcal{L}(L^2(K))}. \end{aligned}$$

Observing that we can replace $X_{g_2}^{\beta_{02}}$ by $\tilde{X}_{g_1}^{\beta_{02}}$ in the above (see Corollary 3.2.15),

$$\begin{aligned} S(h, M, \gamma, \beta_{02}, \beta_2) & \leq C \|h\|_G^{|\beta_{02}|} \\ & \sum_{|\beta'_{02}| \leq |\beta_{02}|} \sup_{g_1 \in G} \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{-M} X_{g_1}^{\gamma + \beta'_{02} + \beta_2} \sigma_2(g_1, \lambda) \right\|_{\mathcal{L}(L^2(K))}. \end{aligned} \quad (40)$$

We have thus shown so far that

$$\begin{aligned}
& \left\| \sigma(g, \lambda) - \sum_{|\alpha| \leq N} \Delta_G^\alpha \sigma_1(g, \lambda) X^{(\alpha)} \sigma_2(g, \lambda) \right\|_{\mathcal{L}(L^2(K))} \\
& \leq \sum_{|\beta_1| + |\beta_2| \leq 2M} \sup_{|\gamma| \leq N - |\beta_2| + 1} S(M, \gamma, \beta_{02}, \beta_2) \\
& \quad \int_G \left| \tilde{X}_{h_1=h}^{\beta_1} (X_g^{\beta_{01}} \kappa_{1,g})(h_1) \right| \|h\|_G^{N - |\beta_2| + 1} dh. \quad (41)
\end{aligned}$$

Combining (40) and (41), we obtain that Lemma (4.7.2) implies that (41) is bounded (up to a constant) by $\|\sigma_1\|_{S_{\rho,\delta}^{m_1,R}(G),N'} \|\sigma_2\|_{S_{\rho,\delta}^{m_2}(G),N'}$ if both

$$\begin{cases} m_1 + \delta |\beta_{01}| + |\beta_1| + \dim G < \rho(\dim G + N - |\beta_2| + 1 + |\beta_{02}|) \\ -2M + \delta(|\gamma| + |\beta'_{02}| + |\beta_2|) \leq -m_2. \end{cases}$$

hold. For that it suffices that (using $\rho \geq \delta$)

$$\begin{cases} m_1 + \delta |\beta_0| + 2M + \dim G < \rho(\dim G + N + 1) \\ -2M + \delta(N + 1 + |\beta_0|) \leq -m_2, \end{cases}$$

which can be rewritten in one line as the following

$$m_2 + \delta(N + 1 + |\beta_0|) \leq 2M < -\dim G - m_1 - \delta |\beta_0| + \rho(\dim G + N + 1). \quad (42)$$

□

We now prove the main result of the subsection, i.e. an estimate on the right symbol semi-norms of the remainder when composing two smoothing symbols.

We will only be able to derive the *composition formula* once we have obtained a similar result for the adjunction.

Proposition 4.7.6. *Assume that $1 \geq \rho > \delta \geq 0$. For each $m_1, m_2 \in \mathbb{R}$, and each $N_0, P \in \mathbb{N}$, there exists $C > 0$ and $P' \in \mathbb{N}$ such that*

$$\begin{aligned}
& \left\| (\sigma_1 \circ \sigma_2)(g, \lambda) - \sum_{|\alpha| \leq N_0} \Delta_G^\alpha \sigma_1(g, \lambda) X^\alpha \sigma_2(g, \lambda) \right\|_{S_{\rho,\delta}^{m_1+m_2-N_0(\rho-\delta),R}(G),P} \\
& \leq C \|\sigma_1\|_{S_{\rho,\delta}^{m_1,R}(G),P'} \|\sigma_2\|_{S_{\rho,\delta}^{m_2}(G),P'}
\end{aligned}$$

holds for every $\sigma_1, \sigma_2 \in S^{-\infty}(G)$.

Proof. Let $N_0, P \in \mathbb{N}$. Consider the symbols

$$\begin{aligned}\tilde{\sigma}_1(g, \lambda) &\stackrel{\text{def}}{=} \sigma_1(g, \lambda) \\ \tilde{\sigma}_2(g, \lambda) &\stackrel{\text{def}}{=} \sigma_2(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m_1 - m_2 + (\rho - \delta)N_0 - \delta|\beta_0|}{2}},\end{aligned}$$

which are symbols of order

$$\begin{aligned}\tilde{m}_1 &\stackrel{\text{def}}{=} m_1 \\ \tilde{m}_2 &\stackrel{\text{def}}{=} -m_1 + (\rho - \delta)N_0 - \delta|\beta_0|.\end{aligned}$$

We will apply Lemma 4.7.5 to $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$.

To this end, fix $N \in \mathbb{N}$ satisfying $N \geq N_0$ so that the conditions (36) make sense. More precisely, select N so that the difference between the right-hand side and the left-hand side is strictly greater than 2 for both lines of (36), and note that this is possible because $\rho > \delta$. In particular, we can find $M \in \mathbb{N}$ such that (36) holds.

By Lemma 4.7.5, it follows that

$$\begin{aligned}&\left\| X_g^{\beta_0} \left(\tilde{\sigma}_1 \circ \tilde{\sigma}_2(g, \lambda) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \tilde{\sigma}_1(g, \lambda) X^{(\alpha)} \tilde{\sigma}_2(g, \lambda) \right) \right\|_{\mathcal{L}(L^2(K))} \\ &\leq C \|\tilde{\sigma}_1\|_{S_{\rho, \delta}^{\tilde{m}_1, R}(G), P} \|\tilde{\sigma}_2\|_{S_{\rho, \delta}^{\tilde{m}_2}(G), P}.\end{aligned}$$

From there, we may deduce that

$$\begin{aligned}&\left\| X_g^{\beta_0} \tau_N(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m_1 - m_2 + (\rho - \delta)N_0 - \delta|\beta_0|}{2}} \right\|_{\mathcal{L}(L^2(K))} \\ &\leq C \|\sigma_1\|_{S_{\rho, \delta}^{m_1, R}(G), P'} \|\sigma_2\|_{S_{\rho, \delta}^{m_2}(G), P'} \\ &\leq C \|\sigma_1\|_{S_{\rho, \delta}^{m_1, R}(G), P''} \|\sigma_2\|_{S_{\rho, \delta}^{m_2}(G), P'''},\end{aligned}$$

where as usual

$$\tau_N(g, \lambda) \stackrel{\text{def}}{=} (\sigma_1 \circ \sigma_2)(g, \lambda) - \sum_{|\alpha| \leq N_0} \Delta_G^\alpha \sigma_1(g, \lambda) X^\alpha \sigma_2(g, \lambda).$$

Using the fact that

$$\Delta_G^{\alpha_0}(\tilde{\sigma}_1 \circ \tilde{\sigma}_2) = \sum_{\alpha_{01}, \alpha_{02}} \Delta_G^{\alpha_{01}} \tilde{\sigma}_1 \circ \Delta_G^{\alpha_{02}} \tilde{\sigma}_2,$$

where the coefficients are exactly the ones from the Leibniz-like rule for differential operators for a pointwise multiplication, we can show that

$$\|\tau_N\|_{S_{\rho,\delta}^{m_1+m_2-N_0(\rho-\delta)}(G),P} \leq C \|\sigma_1\|_{S_{\rho,\delta}^{m_1,R}(G),P''} \|\sigma_2\|_{S_{\rho,\delta}^{m_2}(G),P''}.$$

We conclude by observing that

$$\tau_N - \tau_{N_0} = \sum_{N_0 < |\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_1(g, \lambda) X^{(\alpha)} \sigma_2(g, \lambda) \in S_{\rho,\delta}^{m_1+m_2-N_0(\rho-\delta)}(G),$$

where the right-hand side is dominated appropriately by the symbol seminorms of σ_1 and σ_2 . $\square$

4.7.2 Adjunction formula

We now follow the same steps as in the previous subsection, but this time to study the adjunction of operators. As this involves only one symbol, the proof is somehow slightly easier and less technical.

The first step consists of computing the kernel of the adjoint of a smoothing operator.

Lemma 4.7.7. *Let $\sigma \in S^{-\infty}(G)$ and denote by κ its associated kernel. We let*

$$\kappa_g^{(*)}(h) \stackrel{\text{def}}{=} \bar{\kappa}_{gh^{-1}}(h^{-1}).$$

Then $\kappa^{(*)}$ is smooth on $G \times G$ and the map

$$g \in G \mapsto \kappa_g^{(*)}(\cdot) \in \mathcal{S}(G)$$

is smooth. Moreover, the map

$$\sigma^{(*)}(g, \lambda) \stackrel{\text{def}}{=} \mathcal{F}_G \kappa_g^{(*)}(\lambda), \quad (g, \lambda) \in G \times V,$$

defines a smoothing symbol which satisfies

$$\text{Op}_G(\sigma)^* = \text{Op}_G(\sigma^{(*)}).$$

Proof. Let $\phi, \psi \in \mathcal{S}(G)$ and fix $g \in G$.

$$\begin{aligned} (\text{Op}_G(\sigma)\phi, \psi)_{L^2(G)} &= \int_G \int_G \phi(h) \kappa_g(h^{-1}g) \bar{\psi}(g) \, dg \\ &= \int_G \int_G \phi(h) \bar{\kappa}_{g(h^{-1}g)^{-1}}^{(*)}((h^{-1}g)^{-1}) \bar{\psi}(g) \, dg \end{aligned}$$

by definition of $\kappa^{(*)}$. Continuing our calculation, we get

$$\begin{aligned} (\text{Op}_G(\sigma)\phi, \psi)_{L^2(G)} &= \int_G \int_G \phi(h) \overline{\kappa_h^{(*)}(g^{-1}h)} \psi(g) \, dg \\ &= \int_G \phi(h) (\psi \star \kappa_h^{(*)})(h) \, dh, \end{aligned}$$

which shows that $\kappa^{(*)}$ is the right-convolution kernel of the adjoint. $\square$

Definition 4.7.8. Let $\sigma \in S^{-\infty}(G)$. We let

$$\sigma^{(*)}(g, \lambda) \stackrel{\text{def}}{=} \int_G \bar{\kappa}_{gh^{-1}}(h^{-1}) \zeta_G^\lambda(h)^\star \, dh$$

for each $(g, \lambda) \in G \times V$.

Exactly like Lemma 4.7.5, we give an estimate on the right symbol seminorm of the remainder provided some technical condition is met. Again, this condition is not too restrictive if $\rho > \delta$.

Lemma 4.7.9. Let $m \in \mathbb{R}$ and assume that $1 \geq \rho > \delta \geq 0$. Fix also $\beta_0 \in \mathbb{N}^{\dim G}$ and $N \in \mathbb{N}$. If there exists $M \in \mathbb{N}$ such that $N \geq 2M$ and

$$(\rho - \delta)(N + 1) + \rho(\dim G) > m + 2M + \delta |\beta_0| + \dim G,$$

then there exists $C \geq 0$ and $P \in \mathbb{N}$ such that for any $\sigma \in S^{-\infty}(G)$,

$$\begin{aligned} &\left\| X_g^{\beta_0}(\sigma^{(*)})(g, \lambda) - \sum_{|\alpha| \leq N} \Delta_G^\alpha X^\alpha \sigma^{(*)}(g, \lambda) \zeta_G^\lambda(I - \mathcal{L}_G)^{\frac{M}{2}} \right\|_{\mathcal{L}(L^2(G))} \\ &\leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P} \end{aligned}$$

holds for every $(g, \lambda) \in G \times V$.

Proof. For each $N \in \mathbb{N}$, we define

$$\begin{aligned} \tau_N(g, \lambda) &\stackrel{\text{def}}{=} \sigma^{(*)}(g, \lambda) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha X^\alpha \sigma^{(*)}(g, \lambda) \\ &= \int_G R_{g, N}^{\kappa_g^{(*)}(h)}(h^{-1}) \zeta_G^\lambda(h)^\star \, dh. \end{aligned} \tag{43}$$

Fix $\beta_0 \in \mathbb{N}^{\dim G}$. Applying X^{β_0} on either side of (43), we obtain

$$X^{\beta_0} \tau_N(g, \lambda) = \int_G R_{g,N}^{X^{\beta_0} \kappa_g^{(*)}(h)} (h^{-1}) \zeta_G^\lambda(h)^\star dh.$$

Multiplying both sides by $\zeta_G^\lambda(I - \mathcal{L}_G)^M$, and arguing like in Lemma 4.7.5, we obtain

$$\begin{aligned} & X^{\beta_0} \tau_N(g, \lambda) \zeta_G^\lambda(I - \mathcal{L}_G)^M \\ &= \sum_{|\beta_1| + |\beta_2| \leq 2M} \int_G R_{g_1=e_G, N-|\beta_1|}^{\tilde{X}_{h_2=h}^{\beta_2} X_{g_1}^{\beta_1} X_g^{\beta_0} \kappa_{gg_1}^{(*)}(h_2)} (h^{-1}) \zeta_G^\lambda(h)^\star dh. \end{aligned}$$

Taking the operator norm, we obtain

$$\begin{aligned} & \left\| X^{\beta_0} \tau_N(g, \lambda) \zeta_G^\lambda(I - \mathcal{L}_G)^M \right\|_{\mathcal{L}(L^2(K))} \\ & \leq \sum_{|\beta_1| + |\beta_2| \leq 2M} \int_G \left\| R_{g_1=e_G, N-|\beta_1|}^{\tilde{X}_{h_2=h}^{\beta_2} X_{g_1}^{\beta_1} X_g^{\beta_0} \kappa_{gg_1}^{(*)}(h_2)} (h^{-1}) \right\|_{\mathcal{L}(L^2(K))} dh. \end{aligned}$$

Using Proposition 3.6.5, we can estimate the remainder via

$$\begin{aligned} & \left\| R_{g_1=e_G, N-|\beta_1|}^{\tilde{X}_{h_2=h}^{\beta_2} X_{g_1}^{\beta_1} X_g^{\beta_0} \kappa_{gg_1}^{(*)}(h_2)} (h^{-1}) \right\|_{\mathcal{L}(L^2(K))} \\ & \leq \sum_{|\gamma|=N-|\beta_1|+1} \|h\|_G^{N-|\beta_1|+1} \sup_{\|z\|_G \leq \eta \|h\|_G} \left| X_z^\gamma \tilde{X}_{h_2=h}^{\beta_2} X_z^{\beta_1} X_g^{\beta_0} \kappa_{gg_1}^*(h_2) \right| \end{aligned}$$

Using $X_g^{\beta_0} = \tilde{X}_z^{\beta_0}$ and passing from left to right-invariant derivatives with Corollary 3.2.15,

$$\begin{aligned} & \left\| R_{g_1=e_G, N-|\beta_1|}^{\tilde{X}_{h_2=h}^{\beta_2} X_{g_1}^{\beta_1} X_g^{\beta_0} \kappa_{gg_1}^{(*)}(h_2)} (h^{-1}) \right\|_{\mathcal{L}(L^2(K))} \\ & \leq \sum_{|\gamma|=N-|\beta_1|+1} \|h\|_G^{N-|\beta_1|+1+|\beta_0|} \sup_{\substack{g_1 \in G \\ \beta'_0 \leq |\beta_0|}} \left| X_{g_1}^{\gamma+\beta_1+\beta'_0} \tilde{X}_{h_2=h}^{\beta_2} \kappa_{g_1}^*(h_2) \right|, \end{aligned}$$

which is integrable, with the integral being bounded by $C \|\sigma\|_{S_{\rho,\delta}^m(G),P}$ provided

$$m + |\beta_2| + \delta(\gamma + |\beta_1| + |\beta'_0|) + \dim G < \rho(N - |\beta_1| + 1 + |\beta_0| + \dim G).$$

Naturally, the above condition is satisfied when

$$m + 2M + \delta(N + 1 + |\beta_0|) + \dim G < \rho(N + 1 + \dim G),$$

which can be rewritten

$$(\rho - \delta)(N + 1) + \rho(\dim G) > m + 2M + \delta |\beta_0| + \dim G.$$

□

We are now ready to state and prove the main result of this subsection.

Proposition 4.7.10. *Assume that $1 \geq \rho > \delta \geq 0$. For each $m \in \mathbb{R}$ and each $P, N_0 \in \mathbb{N}$, there exists $C \geq 0$ and $P' \in \mathbb{N}$ such that*

$$\left\| \sigma^{(*)}(g, \lambda) - \sum_{|\alpha| \leq N_0} \frac{1}{\alpha!} \Delta_G^\alpha X^{(\alpha)} \sigma^*(g, \lambda) \right\|_{S_{\rho, \delta}^{m-N_0(\rho-\delta), R}(G), P} \leq C \|\sigma\|_{S_{\rho, \delta}^m(G), P'}$$

holds for each $\sigma \in S^{-\infty}(G)$.

Proof. Let $N_0 \in \mathbb{N}$, $\beta_0 \in \mathbb{N}^{\dim G}$ and $\alpha_0 \in \mathbb{N}^{n_{\Delta, G}}$. For each $N \in \mathbb{N}$, we write

$$\tau_N(g, \lambda) \stackrel{\text{def}}{=} \sigma^{(*)}(g, \lambda) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha X^{(\alpha)} \sigma^*(g, \lambda).$$

We shall apply Lemma 4.7.9 to the symbol

$$\tilde{\sigma}(g, \lambda) = \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m + (\rho - \delta)N_0 - \delta |\beta_0|}{2}}$$

which is of order $\tilde{m} \stackrel{\text{def}}{=} (\rho - \delta)N_0 - \delta |\beta_0|$. Let $M \in \mathbb{N}$ be the smallest natural number such that

$$\frac{-\tilde{m} + (\rho - \delta)N_0 + \delta |\beta_0|}{2} \geq M.$$

We now choose $N \geq \max\{N_0, 2M\}$ such that

$$(\rho - \delta)(N + 1) + \rho(\dim G) > \tilde{m} + 2M + \delta |\beta_0| + \dim G,$$

since it is possible by $\rho > \delta$. By Lemma 4.7.9, we obtain that

$$\begin{aligned} & \left\| X_g^{\beta_0} \tau_N(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m + (\rho - \delta)N_0 - \delta |\beta_0|}{2}} \right\|_{\mathcal{L}(L^2(G))} \\ & \leq C \|\tilde{\sigma}\|_{S_{\rho, \delta}^{\tilde{m}}(G), P} = C \|\sigma\|_{S_{\rho, \delta}^m(G), P}. \end{aligned}$$

Using the fact that difference operators $\Delta_G^\alpha \sigma^{(*)}$ can be written as a linear combination of $\Delta_G^{\alpha'}$ which does not depend on σ , we conclude the proof. $\square$

4.7.3 Simplification of the definition of symbol classes

We shall see in this subsection that it is possible to considerably simplify our definition of symbol classes (Definition 4.2.1) by writing it “without γ ”, i.e. by considering the action of powers of $\xi_G^\lambda (I - \mathcal{L}_G)^{\frac{1}{2}}$ only on the right (resp. only on the left) of σ .

This will allow us to conclude the proof that our collection of operators $\Psi_{\rho,\delta}^m(G)$, $m \in \mathbb{R}$ does behave well under composition and adjunction.

Theorem 4.7.11. *Assume that $1 \geq \rho > \delta \geq 0$ and fix $m \in \mathbb{R}$. A symbol σ is in $S_{\rho,\delta}^m(G)$ if and only if*

$$\sup_{g \in G} \operatorname{ess\,sup}_{\lambda \in V} \left\| X^\beta \Delta_G^\alpha \sigma(g, \lambda) \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\rho|\alpha|-\delta|\beta|}{2}} \right\|_{\mathcal{L}(L^2(K))} < \infty.$$

Moreover, the semi-norms

$$\sigma \in S_{\rho,\delta}^m(G) \mapsto \|\sigma\|_{S_{\rho,\delta}^m(G),N}$$

$$\sigma \in S_{\rho,\delta}^m(G) \mapsto \|\sigma\|_{S_{\rho,\delta}^{m,R}(G),N}$$

yield the same topology on $S_{\rho,\delta}^m(G)$.

Proof. Let $N \in \mathbb{N}$ and $\sigma \in S^{-\infty}(G)$.

It is clear by definition of $S_{\rho,\delta}^m(G)$ -topology that there exists $c_N \geq 0$ such that

$$\|\sigma\|_{S_{\rho,\delta}^m(G),N} \leq \max_{|\gamma| \leq c_N} \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} \sigma \right\|_{S_{\rho,\delta}^{m+\gamma,R}(G),N'}. \quad (44)$$

By Proposition 4.7.6 and Corollary 4.4.6, we know that

$$\sup_{|\gamma| \leq c_N} \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} \sigma \right\|_{S_{\rho,\delta}^{m+\gamma,R}(G),N}$$

is finite.

Since taking the adjoint is continuous in the $S_{\rho,\delta}^{m,R}(G)$ -topology (Proposition 4.7.10), we have

$$\begin{aligned} \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} \sigma \right\|_{S_{\rho,\delta}^{m+\gamma,R}(G),N} &\leq C \left\| \sigma^* \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} \right\|_{S_{\rho,\delta}^{m+\gamma,R}(G),N'} \\ &\leq C \left\| \sigma^* \right\|_{S_{\rho,\delta}^{m,R}(G),N''} = C \left\| \sigma \right\|_{S_{\rho,\delta}^{m,R}(G),N'''} \end{aligned}$$

which, if combined with (44), yields that for every $N \in \mathbb{N}$, there exists $N' \in \mathbb{N}$ such that

$$\left\| \sigma \right\|_{S_{\rho,\delta}^m(G),N} \leq C \left\| \sigma \right\|_{S_{\rho,\delta}^{m,R}(G),N'}.$$

Since it is very clear that

$$\left\| \sigma \right\|_{S_{\rho,\delta}^{m,R}(G),N} \leq \left\| \sigma \right\|_{S_{\rho,\delta}^m(G),N'},$$

It follows that the seminorms $\|\cdot\|_{S_{\rho,\delta}^{m,R}(G),N}, N \in \mathbb{N}$, define the same topology as $\|\cdot\|_{S_{\rho,\delta}^m(G),N}, N \in \mathbb{N}$.

It remains thus to show that $S_{\rho,\delta}^{m,R}(G) \subset S_{\rho,\delta}^m(G)$ as the reverse inclusion is trivial. To this end, fix a symbol $\sigma \in S_{\rho,\delta}^{m,R}(G)$, and for each $\varepsilon \in (0,1]$, let

$$\sigma \stackrel{\text{def}}{=} \sigma \eta_\varepsilon,$$

where η_ε was defined in Theorem 4.4.7.

Using the fact that the right seminorms yield the same topology, we obtain

$$\begin{aligned} \sup_{\varepsilon \in (0,1]} \left\| \sigma_\varepsilon \right\|_{S_{\rho,\delta}^m(G),N} &\leq C \sup_{\varepsilon \in (0,1]} \left\| \sigma_\varepsilon \right\|_{S_{\rho,\delta}^{m,R}(G),N'} \\ &\leq C \left(\left\| \sigma \right\|_{S_{\rho,\delta}^{m,R}(G),N'} + \sup_{\varepsilon \in (0,1]} \left\| \sigma_\varepsilon - \sigma \right\|_{S_{\rho,\delta}^{m,R}(G),N'} \right) < \infty, \end{aligned}$$

where the finiteness is obtained by Proposition 4.5.1.

From the above, since σ_ε converges pointwise to σ on $C^\infty(K)$, it follows that

$$\begin{aligned} &\left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\gamma-\delta|\beta|}{2}} X^\beta \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\gamma-\delta|\beta|}{2}} X^\beta \sigma_\varepsilon(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))}, \end{aligned}$$

which implies that the left-hand side of the above is indeed uniformly bounded in $g \in G$ and $\lambda \in V$.

Assume now that $\alpha \in \mathbb{N}^{n_{\Delta,G}}$ is non-zero. By the Leibniz-like rule, we have

$$X^\beta \Delta_G^\alpha \sigma_\varepsilon(g, \lambda) = X^\beta \Delta_G^\alpha \sigma(g, \lambda) \eta_\varepsilon(\lambda) + L_\varepsilon(g, \lambda),$$

where

$$L_\varepsilon(g, \lambda) \stackrel{\text{def}}{=} \sum_{\substack{|\alpha| \leq |\alpha_1| + |\alpha_2| \leq 2|\alpha| \\ \alpha_2 \neq 0}} X^\beta \Delta_G^{\alpha_1} \sigma(g, \lambda) \Delta_G^{\alpha_2} (I_{L^2(K)} - \eta_\varepsilon(\lambda)).$$

Using the estimates in Theorem 4.4.7, it follows that

$$\liminf_{\varepsilon \rightarrow 0} \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\rho|\alpha|-\delta|\beta|+\gamma}{2}} X^\alpha L_\varepsilon(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(G))} = 0.$$

Therefore, we can conclude that

$$\begin{aligned} & \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\gamma-\delta|\beta|}{2}} X^\beta \Delta_G^\alpha \sigma(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} \\ & \leq \liminf_{\varepsilon \rightarrow 0} \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\gamma-\delta|\beta|}{2}} X^\beta \Delta_G^\alpha \sigma_\varepsilon(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} \end{aligned}$$

is uniformly bounded in g and λ , concluding the proof that $\sigma \in S_{\rho,\delta}^m(G)$. $\square$

Naturally, the following statement holds with an identical proof.

Theorem 4.7.12. *Assume that $1 \geq \rho > \delta \geq 0$ and fix $m \in \mathbb{R}$. A symbol σ is in $S_{\rho,\delta}^m(G)$ if and only if*

$$\sup_{g \in G} \operatorname{ess\,sup}_{\lambda \in V} \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m+\rho|\alpha|-\delta|\beta|}{2}} X^\beta \Delta_G^\alpha \sigma(g, \lambda) \right\|_{\mathcal{L}(L^2(K))} < \infty.$$

Moreover, the semi-norms

$$\sigma \in S_{\rho,\delta}^m(G) \mapsto \|\sigma\|_{S_{\rho,\delta}^m(G),N}$$

$$\sigma \in S_{\rho,\delta}^m(G) \mapsto \|\sigma\|_{S_{\rho,\delta}^{m,L}(G),N}$$

yield the same topology on $S_{\rho,\delta}^m(G)$.

4.7.4 Pseudo-differential calculus

Exactly like in [FR16], we can now use Theorem 4.7.11 and Proposition 4.5.1 to derive the composition and adjunction formulae from Propositions 4.7.6 and 4.7.10.

Theorem 4.7.13 (Composition formula). *Assume that $1 \geq \rho > \delta \geq 0$. Given $m_1, m_2 \in \mathbb{R}$, we let $\sigma_1 \in S_{\rho,\delta}^{m_1}(G)$, $\sigma_2 \in S_{\rho,\delta}^{m_2}(G)$.*

There exists $\sigma \in S_{\rho,\delta}^{m_1+m_2}(G)$ such that

$$\text{Op}_G(\sigma) = \text{Op}_G(\sigma_1) \circ \text{Op}_G(\sigma_2).$$

Moreover, σ is given modulo $S^{-\infty}(G)$ by the asymptotic sum

$$\sigma \sim \sum_{\alpha \in \mathbb{N}^n_{\Delta,G}} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_1 X^{(\alpha)} \sigma_2.$$

More precisely, we have

$$\sigma - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_1 X^{(\alpha)} \sigma_2 \in S_{\rho,\delta}^{m-(\rho-\delta)(N+1)}(G)$$

for each $N \in \mathbb{N}$.

Proof. Let $N \in \mathbb{N}$ and set $m \stackrel{\text{def}}{=} m_1 + m_2$. We write

$$\tau_N \stackrel{\text{def}}{=} \sigma_1 \circ \sigma_2 - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_1 X^{(\alpha)} \sigma_2.$$

For each $\varepsilon \in (0, 1]$, we let

$$\begin{aligned} \sigma_{1,\varepsilon} &\stackrel{\text{def}}{=} \eta_\varepsilon \sigma_1 \\ \tau_{N,\varepsilon} &\stackrel{\text{def}}{=} \sigma_{1,\varepsilon} \circ \sigma_2 - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha \sigma_{1,\varepsilon} X^{(\alpha)} \sigma_2. \end{aligned}$$

We check that

$$\tau_{N,\varepsilon} = \eta_\varepsilon \tau_N - L_\varepsilon,$$

where

$$L_\varepsilon \stackrel{\text{def}}{=} \sum_{|\alpha| \leq N} \sum_{|\alpha| \leq |\alpha_1| \eta_\varepsilon + |\alpha_2| \leq 2|\alpha|, \alpha_1 \neq 0} \frac{1}{\alpha!} \Delta_G^{\alpha_1} \Delta_G^{\alpha_2} \sigma_1 X^{(\alpha)} \sigma_2.$$

Arguing like in the proof of Theorem 4.7.11, we obtain that

$$\liminf_{\varepsilon \rightarrow 0} \left\| X^\beta \Delta_G^\alpha L_\varepsilon(g, \lambda) \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{-m + (\rho - \delta)(N+1) + \rho|\alpha| - \delta|\beta|}{2}} \right\|_{\mathcal{L}(L^2(G))} = 0$$

and thus combining the above with Proposition 4.7.6, we obtain

$$\|\tau_N\|_{S_{\rho, \delta}^{m - (\rho - \delta)(N+1)}(G), P} = \liminf_{\varepsilon \rightarrow 0} \|\tau_{N, \varepsilon}\|_{S_{\rho, \delta}^{m - (\rho - \delta)(N+1)}(G), P} < +\infty,$$

which concludes the proof. $\square$

Theorem 4.7.14 (Adjunction formula). *Assume that $1 \geq \rho > \delta \geq 0$. Let $\sigma \in S_{\rho, \delta}^m(G)$ for some $m \in \mathbb{R}$.*

There exists $\tilde{\sigma} \in S_{\rho, \delta}^m(G)$ such that

$$\text{Op}_G(\tilde{\sigma}) = \text{Op}_G(\sigma)^*.$$

Moreover, $\tilde{\sigma}$ is given modulo $S^{-\infty}(G)$ by the asymptotic sum

$$\tilde{\sigma} \sim \sum_{\alpha \in \mathbb{N}^{n_{\Delta, G}}} \frac{1}{\alpha!} \Delta_G^\alpha X^{(\alpha)} \sigma^*.$$

More precisely, we have

$$\tilde{\sigma} - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha X^{(\alpha)} \sigma^* \in S_{\rho, \delta}^{m - (\rho - \delta)(N+1)}(G)$$

for each $N \in \mathbb{N}$.

Proof. Let $\sigma \in S_{\rho, \delta}^m(G)$ and write

$$\tau_N = \sigma^{(*)} - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \Delta_G^\alpha X^{(\alpha)} \sigma^*$$

Moreover, for each $\varepsilon \in (0, 1]$, we let

$$\begin{aligned} \sigma_\varepsilon &\stackrel{\text{def}}{=} \eta_\varepsilon \sigma, \\ \tau_{\varepsilon, N} &\stackrel{\text{def}}{=} \sigma_\varepsilon^{(*)} - \sum_{\alpha \in \mathbb{N}^{n_{\Delta, G}}} \frac{1}{\alpha!} \Delta_G^\alpha X^{(\alpha)} \sigma_\varepsilon^*. \end{aligned}$$

Using the fact that $\sigma_\varepsilon^{(*)} = \sigma^{(*)} \eta_\varepsilon$ and $\sigma_\varepsilon^* = \sigma^* \eta_\varepsilon$, it follows that

$$\tau_{\varepsilon, N} \stackrel{\text{def}}{=} \tau_N \eta_\varepsilon - L_\varepsilon,$$

where

$$L_\varepsilon = \frac{1}{\alpha!} \sum_{\alpha \in \mathbb{N}^{\Delta, G}} \sum_{\substack{|\alpha| \leq |\alpha_1| + |\alpha_2| \leq 2|\alpha| \\ \alpha_2 \neq 0}} \Delta_G^{\alpha_1} X^{(\alpha)} \sigma^\star \Delta_G^{\alpha_2} \eta_\varepsilon.$$

Arguing like in the proof of Theorem 4.7.11, we obtain that

$$\liminf_{\varepsilon \rightarrow 0} \left\| X^\beta \Delta_G^\alpha L_\varepsilon(g, \lambda) \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-m + (\rho - \delta)(N+1) + \rho|\alpha| - \delta|\beta|}{2}} \right\|_{\mathcal{L}(L^2(G))} = 0$$

and thus combining the above with Proposition 4.7.10, we obtain

$$\|\tau_N\|_{S_{\rho, \delta}^{m - (\rho - \delta)(N+1)}(G), P} = \liminf_{\varepsilon \rightarrow 0} \|\tau_{N, \varepsilon}\|_{S_{\rho, \delta}^{m - (\rho - \delta)(N+1)}(G), P} < +\infty,$$

which concludes the proof. $\square$

4.7.5 Asymptotic sums of symbols

More complex operations on operators like composition and adjunction cannot be written in terms of symbols without using a Taylor development on the kernel side. This shows that infinite sums appear naturally on the symbolic side, and we need to find out whether it converges in a satisfactory sense for our purposes.

The construction of asymptotic limits is essential to construct parametrices.

Proposition 4.7.15 (Asymptotic sum). *Given $1 \geq \rho \geq \delta \geq 0$, let $(\sigma_j)_{j \in \mathbb{N}}$ be a sequence of symbols such that $\sigma_j \in S_{\rho, \delta}^{m_j}(G)$, where $(m_j)_{j \in \mathbb{N}}$ is a strictly decreasing sequence of real numbers.*

There exists a symbol $\sigma \in S_{\rho, \delta}^{m_0}(G)$ such that

$$\sigma - \sum_{j=0}^N \sigma_j \in S_{\rho, \delta}^{m_{N+1}}(G) \quad (45)$$

for each $N \in \mathbb{N}$. Moreover, σ is unique modulo $S^{-\infty}(G)$.

Definition 4.7.16 (Asymptotic sum). *With the notation of Proposition 4.7.15, we shall write*

$$\sigma \sim \sum_{j=0}^{+\infty} \sigma_j$$

when (45) holds for each $N \in \mathbb{N}$.

Proof of Proposition 4.7.15. Using Theorem 4.4.7, for each $l \in \mathbb{N}$ we can find $C_l \geq 0$ and $N_l \in \mathbb{N}$ such that

$$\left\| \sigma_j(I_{L^2(K)} - \eta_\varepsilon) \right\|_{S_{\rho,\delta}^{m_0}(G),l} \leq C_l \left\| \sigma_j \right\|_{S_{\rho,\delta}^{m_0}(G),N_l} \varepsilon^{\frac{\max\{m_0-m_j,4\}}{2}}$$

for each $0 < \varepsilon \leq 1$ and each $j \in \mathbb{N}$. Without loss of generality, we assume that $N_l \geq l$.

Choosing a decreasing sequence of sufficiently small $\varepsilon_j > 0$, $j \in \mathbb{N}$, so that the right-hand side of the above satisfies

$$C_j \left\| \sigma_j \right\|_{S_{\rho,\delta}^{m_0}(G),N_j} \varepsilon_j^{\frac{\max\{m_0-m_j,4\}}{2}} \leq 2^{-j},$$

we now obtain that the symbols

$$\tilde{\sigma}_j \stackrel{\text{def}}{=} \sigma_j(I_{L^2(K)} - \eta_{\varepsilon_j})$$

satisfy the estimate

$$\left\| \tilde{\sigma}_j \right\|_{S_{\rho,\delta}^{m_0}(G),j} \leq 2^{-j}. \quad (46)$$

Fix $l \in \mathbb{N}$. We have

$$\begin{aligned} \sum_{j=0}^{\infty} \left\| \tilde{\sigma}_j \right\|_{S_{\rho,\delta}^{m_0}(G),l} &\leq \sum_{j=0}^l \left\| \tilde{\sigma}_j \right\|_{S_{\rho,\delta}^{m_0}(G),l} + \sum_{j=l+1}^{+\infty} \left\| \tilde{\sigma}_j \right\|_{S_{\rho,\delta}^{m_0}(G),j} \\ &\leq \sum_{j=0}^l \left\| \tilde{\sigma}_j \right\|_{S_{\rho,\delta}^{m_0}(G),l} + \sum_{j=l+1}^{+\infty} 2^{-j} < +\infty, \end{aligned} \quad (47)$$

where the last line was obtained by applying (46). As $S_{\rho,\delta}^{m_0}(G)$ is a Fréchet space, it follows that the sum

$$\sum_{j=0}^{+\infty} \tilde{\sigma}_j$$

converges to some $\sigma \in S_{\rho,\delta}^{m_0}(G)$.

For each $N \in \mathbb{N}$, applying (47) to the sequence $\tilde{\sigma}'_j \stackrel{\text{def}}{=} \tilde{\sigma}_{j+N+1}$, we have thus shown that

$$\sum_{j=N+1}^{+\infty} \tilde{\sigma}_j \in S_{\rho,\delta}^{m_{N+1}}(G). \quad (48)$$

Therefore, it follows that for each $N \in \mathbb{N}$, we have

$$\begin{aligned} \sigma - \sum_{j=0}^N \sigma_j &= \sum_{j=0}^{+\infty} \sigma_j (I_{L^2(K)} - \eta_{\varepsilon_j}) - \sum_{j=0}^N \sigma_j \\ &= - \sum_{j=0}^N \sigma_j \eta_{\varepsilon_j} + \sum_{j=N+1}^{+\infty} \tilde{\sigma}_j, \end{aligned}$$

which, if combined with (48) and Theorem 4.4.7, means that

$$\sigma - \sum_{j=0}^N \sigma_j \in S_{\rho, \delta}^{m_{N+1}}(G),$$

as $\eta_{\varepsilon_j} \in S^{-\infty}(G)$. We have thus shown (45).

Let us now check the uniqueness of σ modulo $S^{-\infty}(G)$. To this end, assume that τ also satisfies (45) and fix $N \in \mathbb{N}$. It is clear that

$$\sigma - \tau = \left(\sigma - \sum_{j=0}^N \sigma_j \right) - \left(\tau - \sum_{j=0}^N \sigma_j \right) \in S_{\rho, \delta}^{m_{N+1}}(G).$$

As N is arbitrary and m_{N+1} decreases to $-\infty$ as $N \rightarrow +\infty$, it follows that $\sigma - \tau \in S^{-\infty}(G)$. $\square$

4.8 L^p BOUNDEDNESS

Pseudo-differential operators partly owe their names to the fact that they behave like their differential counterparts. Since X^β extends to a bounded operator

$$X^\beta : L_s^p(G) \rightarrow L_{s-|\beta|}^p(G),$$

we might wonder whether this is true for pseudo-differential operators.

We shall show in this section that if $T \in \Psi_{\rho, \delta}^m(G)$ with $\rho > \delta$, then T extends to a bounded operator

$$T : L_s^2(G) \rightarrow L_{s-m}^2(G).$$

The condition $\rho > \delta$ is due to the fact that we shall follow [RW14], which provides a proof that depends on a composition formula with asymptotic expansion.

When $(\rho, \delta) = (1, 0)$, we show that the result is true for more general Sobolev spaces by the Calderón-Zygmund theory.

4.8.1 L^2 boundedness

The aim of this subsection is to prove that symbols of order 0 induce bounded operators in $L^2(G)$ when $\rho > \delta$.

These results would naturally generalize to symbols of order m as follows: if $\sigma \in S_{\rho, \delta}^m(G)$, where ρ, δ satisfy either of the conditions above, then $\text{Op}_G(\sigma)$ extends continuously to a continuous operator

$$\text{Op}_G(\sigma) : L_s^2(G) \rightarrow L_{s-m}^2(G)$$

for every $s \in \mathbb{R}$.

Suppose that $\sigma_T \in S_{\rho, \delta}^0(G)$ for $0 \leq \delta < \rho \leq 1$. One way of showing that $T \stackrel{\text{def}}{=} \text{Op}_G(\sigma_T)$ is bounded would be to find $S \in \Psi_{\rho, \delta}^0(G)$ such that

$$T^*T = CI_{S(G)} - S^*S + R, \quad (49)$$

where $R \in \Psi_{\rho, \delta}^m(G)$ for some $m < 0$.

It would then follow that

$$\begin{aligned} \|T\phi\|_{L^2(G)}^2 &= (T^*T\phi, \phi)_{L^2(G)} \\ &= C\|\phi\|_{L^2(G)}^2 - \|S\phi\|_{L^2(G)}^2 + (R\phi, \phi)_{L^2(G)} \\ &\leq C\|\phi\|_{L^2(G)}^2 + \|R\phi\|_{L^2(G)}\|\phi\|_{L^2(G)}, \end{aligned}$$

reducing the boundedness of T to that of R , which is much easier to prove.

A sufficient condition to have (49) would be to have

$$\sigma_T^* \sigma_T = CI_{L^2(K)} - \sigma_S^* \sigma_S,$$

on the symbol side, with $\sigma_S \stackrel{\text{def}}{=} \text{Op}_G^{-1}(S)$. A good candidate for σ_S would therefore be

$$\sigma_S \stackrel{\text{def}}{=} \sqrt{CI_{L^2(K)} - \sigma_T^* \sigma_T}. \quad (50)$$

The condition that $S \in \Psi_{\rho, \delta}^0(G)$ is equivalent to showing that $\sigma_S \in S_{\rho, \delta}^0(G)$. The crucial part of the proof is therefore to show that (50) is a symbol of order 0.

Lemma 4.8.1. *Let $\rho, \delta \in \mathbb{R}$ be such that $1 \geq \rho > \delta \geq 0$. If $\sigma \in S_{\rho, \delta}^0(G)$ is positive definite, invertible, and its inverse satisfies*

$$\sup_{g \in G} \operatorname{ess\,sup}_{\lambda \in V} \left\| \sigma(g, \lambda)^{-1} \right\|_{\mathcal{L}(L^2(G))} < \infty,$$

then its square root

$$(x, k; \lambda) \in G \times V \mapsto \sqrt{\sigma(x, k; \lambda)}$$

also belongs to $S_{\rho, \delta}^0(G)$.

Proof. It is clear that the spectrum of σ is contained in $[c, c^{-1}] \subset \mathbb{R}$ for some $c > 0$. Therefore, for each $z \in \mathbb{R}^-$,

$$\left\| (\sigma(g, \lambda) - zI_{L^2(K)})^{-1} \right\|_{\mathcal{L}(L^2(K))} \leq (c + |z|)^{-1} \leq C(1 + |z|)^{-1}.$$

By Theorem 4.7.11, it follows that for each $\gamma \in \mathbb{R}$,

$$\begin{aligned} & \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} (\sigma(g, \lambda) - zI_{L^2(K)})^{-1} \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} \\ & \leq C(1 + |z|)^{-1}, \end{aligned}$$

which means that σ is parameter-elliptic with order 0.

Since σ is parameter-elliptic with respect to $\mathbb{R}^-$, we can apply Theorem 4.4.5 with $F = z^{-\frac{1}{2}}$, which yields that $F(\sigma) \in S_{\rho, \delta}^0(G)$. By the calculus, we conclude that

$$\sqrt{\sigma} = F(\sigma)\sigma \in S_{\rho, \delta}^0(G).$$

□

Before we can prove our main result, we need to show that for some $m < 0$, operators in $\Psi_{\rho, \delta}^m(G)$ are bounded on $L^2(G)$.

Proposition 4.8.2. *Let $\sigma \in S_{\rho, \delta}^m(G)$, where $m < -\dim G/2$ and $\rho, \delta \in [0, 1]$. The operator associated with σ , $\operatorname{Op}_G(\sigma)$, has a continuous extension*

$$\operatorname{Op}_G(\sigma) : L^2(G) \rightarrow L^2(G).$$

Moreover,

$$\|\operatorname{Op}_G(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C\|\sigma\|_{S_{\rho, \delta}^m(G), N}.$$

Proof. First, let us observe that for each $g_0 \in G$, we have

$$\begin{aligned} \sup_{g \in G} |Tf(g_0g)| &\leq \int_G \left| f(g_0gh^{-1}) \kappa_{g_0g}(h) \right| dh \\ &\leq \left\| (1 + \|\cdot\|_G^2)^{-N} f(g_0\cdot) \right\|_{L^2(G)} \left\| (1 + \|\cdot\|_G^2)^N \kappa_{g_0g} \right\|_{L^2(G)}, \end{aligned}$$

where the last line was obtained by the Cauchy-Schwartz inequality.

Integrating the above with respect to g over $B_G(g_0, 1)$, we obtain

$$\begin{aligned} \int_{B_G(g_0, 1)} |Tf(g)|^2 dg &= \int_{B_G(e, 1)} |Tf(g_0g)|^2 dg \\ &\leq C \|\sigma\|_{S_{\rho, \sigma}^m(G), P}^2 \left\| (1 + \|\cdot\|_G^2)^{-N} f(g_0\cdot) \right\|_{L^2(G)}^2. \quad (51) \end{aligned}$$

To conclude the proof, we shall integrate each side of the above with respect to $g_0 \in G$.

For the left-hand side, a simple application of the Fubini-Tonnelli Theorem yields

$$\begin{aligned} \int_G \int_{B_G(g_0, 1)} |Tf(g)|^2 dg dg_0 &= \int_G \int_G 1_{\|g_0^{-1}g\|_G \leq 1} |Tf(g)|^2 dg_0 dg \\ &= \int_G \int_G 1_{\|g_0^{-1}g\|_G \leq 1} |Tf(g)|^2 dg_0 dg, \end{aligned}$$

so that the change of variable $h = g_0^{-1}g$ yields

$$\int_G \int_{B_G(g_0, 1)} |Tf(g)|^2 dg dg_0 = C \|Tf\|_{L^2(G)}^2. \quad (52)$$

Now, integrating the right-hand side of (51), we obtain

$$\begin{aligned} &\int_G \left\| (1 + \|\cdot\|_G^2)^{-N} f(g_0\cdot) \right\|_{L^2(G)}^2 dg_0 \\ &= \int_G \int_G \left| (1 + \|g\|_G^2)^{-N} f(g_0g) \right|^2 dg dg_0 \\ &= \|f\|_{L^2(G)}^2 \int_G (1 + \|g\|_G^2)^{-2N} dg. \end{aligned}$$

Choosing N sufficiently large, we therefore get that

$$\int_G \left\| (1 + \|\cdot\|_G^2)^{-N} f(g_0\cdot) \right\|_{L^2(G)}^2 dg_0 \leq C \|f\|_{L^2(G)}^2. \quad (53)$$

Therefore, using (53) and (52) in (51), we obtain

$$\|Tf\|_{L^2(G)}^2 \leq C \|\sigma\|_{S_{\rho, \sigma}^m(G), P}^2 \left\| (1 + \|\cdot\|_G^2)^{-N} f(g_0\cdot) \right\|_{L^2(G)}^2,$$

which concludes the proof. $\square$

We can now prove the main result of this section.

Proposition 4.8.3 (L^2 boundedness). *Let $\rho, \delta \in \mathbb{R}$ be such that $1 \geq \rho > \delta \geq 0$. If $\sigma \in S_{\rho, \delta}^0(G)$, then its associated operator $T \stackrel{\text{def}}{=} \text{Op}_G(\sigma)$ is bounded in $L^2(G)$, i.e. we have*

$$\|T\phi\|_{L^2(G)} \leq C\|\phi\|_{L^2(G)}$$

for every $\phi \in \mathcal{S}(G)$.

Proof. **STEP 1.** If $\sigma \in S_{\rho, \delta}^m(G)$ for $m < 0$, then

$$\|\text{Op}_G(\sigma)\|_{\mathcal{L}(L^2(G))} < \infty.$$

Suppose by contradiction that the claim does not hold. Therefore, there exists $m < 0$ and $\sigma \in S_{\rho, \delta}^m(G)$ such that $\text{Op}_G(\sigma)$ is not bounded. In particular, for each $n \in \mathbb{N}$,

$$(\text{Op}_G(\sigma)\text{Op}_G(\sigma)^*)^n$$

is in $\Psi_{\rho, \delta}^{2mn}(G)$ and unbounded. When n is such that $-2mn < -\dim G/2$, this contradicts Proposition 4.8.2.

STEP 2. Conclusion

Let $C \geq 0$ be such that

$$\sup_{(x,k) \in G} \text{ess sup}_{\lambda \in V} \|\sigma(x, k; \lambda)\|_{L^2(K)} \leq C.$$

It follows that

$$4C^2 I_{L^2(K)} - \sigma^* \sigma$$

is an operator of order 0, and so by Lemma 4.8.1, the symbol

$$\sigma_S \stackrel{\text{def}}{=} \sqrt{4C^2 I_{L^2(K)} - \sigma^* \sigma}$$

is also in $S_{\rho, \delta}^0(G)$. By definition, we have

$$\sigma^* \sigma = 4C^2 I_{L^2(K)} - \sigma_S^* \sigma_S. \quad (54)$$

By the adjoint and composition formulae, we have

$$\begin{aligned}\mathrm{Op}_G(\sigma^*\sigma) &= T^*T + R_1 \\ \mathrm{Op}_G(\sigma_S^*\sigma_S) &= S^*S + R_2,\end{aligned}$$

with $R_1, R_2 \in \Psi_{\rho,\delta}^{\rho-\delta}(G)$. Therefore, applying Op_G on both sides of (54) yields

$$T^*T = 4C^2I_{\mathcal{S}(G)} - S^*S + R, \quad R \stackrel{\mathrm{def}}{=} (R_2 - R_1) \in \Psi_{\rho,\delta}^{\rho-\delta}(G).$$

Now, fix $\phi \in L^2(G)$. Since Step 2 implies that R is bounded, we get

$$\begin{aligned}\|T\phi\|_{L^2(G)}^2 &= (T^*T\phi, \phi)_{L^2(G)} \\ &= 4C^2\|\phi\|_{L^2(G)}^2 - \|S\phi\|_{L^2(G)}^2 + (R\phi, \phi)_{L^2(G)} \\ &\leq (4C^2 + \|R\|_{\mathcal{L}(L^2(G))})\|\phi\|_{L^2(G)}^2,\end{aligned}$$

concluding the proof. □

From the previous result, we easily deduce the following corollary.

Theorem 4.8.4 (Boundedness on Sobolev spaces). *Let $\rho, \delta \in \mathbb{R}$ be such that $1 \geq \rho > \delta \geq 0$. If $\sigma \in S_{\rho,\delta}^m(G)$, then its associated operator $T \stackrel{\mathrm{def}}{=} \mathrm{Op}_G(\sigma)$ extends to a bounded map between $L_s^2(G)$ and $L_{s-m}^2(G)$, i.e. there exists $C \geq 0$ such that*

$$\|T\phi\|_{L_{s-m}^2(G)} \leq C\|\phi\|_{L_s^2(G)}$$

for every $\phi \in \mathcal{S}(G)$.

4.8.2 Kernel estimates when $\rho = 1$

In order to show $L^p(G)$ boundedness, we need to improve the estimates in Theorem 4.6.1 when $\rho = 1$. In particular, we need to show that if κ is the kernel associated with a symbol in $S_{1,0}^0(G)$, we have

$$|\kappa_g(h)| \leq C\|h\|_G^{-\dim G}, \quad h \neq e_G,$$

which is problematic because $-\dim G$ is a *critical exponent* due to the Sobolev inequality. We therefore need *sharper estimates*, which can be provided by studying the *heat kernel*.

Remark 4.8.5. *The discussion herein does not appear in [Fis15; FR14; FR16] but the ideas presented are based upon Véronique Fischer's privately communicated proof in the compact and graded setting.*

The following lemma is obtained by a very simple modification of [Fis15, Lemma A.6], which relies mainly on classical heat kernel estimates valid for Lie groups with polynomial growth of the volume, and are thus valid on the motion group.

The arguments in [Fis15, Lemma A.6] are themselves based on [FMVo6; VSC92].

Lemma 4.8.6. *Let $f \in C^\infty(\mathbb{R}^+)$ be such that f vanishes outside of $[C^{-1}, C]$ for some $C > 1$. For each $\alpha \in \mathbb{N}^{n_{\Delta, G}}$ and each $m \in \mathbb{R}$, we have*

$$\left\| X^\beta \{q^\alpha f(t\mathcal{L}_G)\delta_{e_G}\} \right\|_{L^2(G)} \leq C\sqrt{t}^{|\alpha| - |\beta| - \frac{\dim G}{2}}.$$

From there, we can deduce the following corollary.

Corollary 4.8.7. *Let $f \in C^\infty(\mathbb{R}^+)$ be such that f vanishes outside of $[C^{-1}, C]$ for some $C > 1$. For each $\alpha \in \mathbb{N}^{n_{\Delta, G}}$ and each $m \in \mathbb{R}$, we have*

$$\|q^\alpha f(t\mathcal{L}_G)\delta_{e_G}\|_{L_m^2(G)} \leq C\sqrt{t}^{|\alpha| - m - \frac{\dim G}{2}}.$$

Proof. Clearly, the result is true when $m \in 2\mathbb{N}$ by Lemma 4.8.6. The result on $\mathbb{R}^+$ can easily be deduced by interpolation.

Suppose that $n \in \mathbb{N}$. We let

$$f_1(\mu) \stackrel{\text{def}}{=} \begin{cases} \frac{f(\mu)}{\mu^n} & \text{if } \mu \neq 0 \\ 0 & \text{otherwise,} \end{cases} \quad \mu \in \mathbb{R}^+.$$

Suppose now $m < 0$ and pick $n \in \mathbb{N}$ so that $2n - |\alpha| + m \geq 0$. Clearly,

$$\|q^\alpha f(t\mathcal{L}_G)\delta_{e_G}\|_{L_m^2(G)} = t^n \|q^\alpha (\mathcal{L}_G^n f_1(t\mathcal{L}_G)\delta_{e_G})\|_{L_m^2(G)}.$$

Applying the Leibniz rule, we obtain that the right-hand side is bounded by

$$\begin{aligned} & C \sum_{(\alpha_1, \alpha_2) \in A} t^n \|q^{\alpha_1} \mathcal{L}_G^n\|_{\mathcal{L}(L^2_{2n-|\alpha_1|+m}(G), L^2_m(G))} \|q^{\alpha_2} (f_1(t\mathcal{L}_G)\delta_{e_G})\|_{L^2_{2n-|\alpha_1|+m}(G)} \\ & \leq C \sum_{(\alpha_1, \alpha_2) \in A} t^n t^{\frac{|\alpha_2|-2n+|\alpha_1|-m-\dim G/2}{2}} = C\sqrt{t}^{|\alpha|-m-\frac{\dim G}{2}}. \end{aligned}$$

where in the above $A \stackrel{\text{def}}{=} \{(\alpha_1, \alpha_2) : |\alpha| \leq |\alpha_1| + |\alpha_2| \leq 2|\alpha|\}$.

This concludes the proof. $\square$

We are now ready to improve upon our already established kernel estimates.

Theorem 4.8.8 (Kernel estimates when $\rho = 1$). *Let $\sigma \in S^m_{1,\delta}(G)$. If $m > -\dim G$, then its associated kernel κ satisfies*

$$|\kappa_g(h)| \leq C \|h\|_G^{-m-\dim G}$$

for every $h \in G \setminus \{e_G\}$.

Proof. Let $\chi_0, \chi_1, \dots \in C_c^\infty(\mathbb{R}, [0, 1])$ be such that

$$\sum_{j=0}^{+\infty} \chi_j = 1 \quad \text{on } \mathbb{R}^+,$$

$\text{supp } \chi_1 \subset [a, b]$ for some $0 < a < b$, and

$$\chi_j = \chi_1(2^{-j+1}\cdot), \quad j = 1, \dots$$

Treating χ_0 separately and using Corollary 4.8.7, we obtain that

$$\left\| q^\alpha \sqrt{\chi_j}(t\mathcal{L}_G)\delta_{e_G} \right\|_{L^2_m(G)} \leq C 2^{-\frac{j}{2}(|\alpha|-m-\frac{\dim G}{2})}, \quad (55)$$

where the above is understood to hold only for $m = |\alpha| = 0$ when $j = 0$.

Now, let us write

$$\sigma_j(g, \lambda) \stackrel{\text{def}}{=} \sigma(g, \lambda) \chi_j(\xi_G^\lambda \mathcal{L}_G), \quad \kappa_{j,g}(h) = \mathcal{F}_G^{-1}\{\sigma_j(g, \lambda)\}(h)$$

and fix $h \in G \setminus \{e_G\}$ such that $\|h\|_G \leq 1$. Let also $l \in \mathbb{N}$ be such that $\|h\|_G^2 \in (2^{-l-1}, 2^{-l}]$.

We check that

$$\begin{aligned} \sum_{j=0}^l |\kappa_{j,g}(h)| &\leq \sum_{j=0}^l \int_V \left\| \sigma(g, \lambda) \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m-m}{2}} \chi_j(\xi_G^\lambda \mathcal{L}_G) \right\|_{S_1(L^2(K))} d\lambda \\ &\leq C \sum_{j=0}^l \int_V \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{4}} \sqrt{\chi_j}(\xi_G^\lambda \mathcal{L}_G) \right\|_{S_2(L^2(K))}^2 d\lambda, \end{aligned}$$

so that applying the Plancherel formula yields

$$\begin{aligned} \sum_{j=0}^l |\kappa_{j,g}(h)| &\leq C \sum_{j=0}^l \left\| \sqrt{\chi_j}(\mathcal{L}_G) \right\|_{L^2_{-m/2}(G)}^2 \\ &\leq C \sum_{j=0}^l 2^{-\frac{j}{2}(m-\dim G)} \leq C \|h\|_G^{-m-\dim G}. \end{aligned} \quad (56)$$

The sum for $j > l$ is slightly more delicate and requires the use of difference operators. Given $\alpha \in \mathbb{N}^{n_{\Delta,G}}$ and $j \in \mathbb{N}$ satisfying $j > l$, we have

$$\begin{aligned} |q^\alpha(h) \kappa_{j,g}(h)| &\leq \int_V \left\| \Delta_G^{\alpha_1} \sigma(g, \lambda) \Delta_G^{\alpha_2} \chi_j(\xi_G^\lambda \mathcal{L}_G) \right\|_{S_1(L^2(K))} d\lambda \\ &\leq C \int_V \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m-|\alpha_1|}{2}} \Delta_G^{\alpha_2} \chi_j(\xi_G^\lambda \mathcal{L}_G) \right\|_{S_1(L^2(K))} d\lambda, \end{aligned}$$

with the understanding that we sum over α_1 and α_2 according to the *Leibniz-like rule*.

After using the Leibniz-like rule again (and implying the summation again), we have

$$\Delta_G^{\alpha_2} \chi_j = \sum_{\alpha_{21}, \alpha_{22}} \Delta_G^{\alpha_{21}} \sqrt{\chi_j}(\xi_G^\lambda(\mathcal{L}_G)) \Delta_G^{\alpha_{22}} \sqrt{\chi_j}(\xi_G^\lambda(\mathcal{L}_G)),$$

so that the above calculation becomes

$$\begin{aligned} |q^\alpha(h) \kappa_{j,g}(h)| &\leq C \int_V \left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m-|\alpha_1|}{2}} \Delta_G^{\alpha_{21}} \sqrt{\chi_j}(\xi_G^\lambda \mathcal{L}_G) \Delta_G^{\alpha_{22}} \sqrt{\chi_j}(\xi_G^\lambda \mathcal{L}_G) \right\|_{S_1(L^2(K))} d\lambda \\ &\leq C \left\| \Delta_G^{\alpha_{21}} \sqrt{\eta_j}(\xi_G^\lambda(\mathcal{L}_G)) \right\|_{L^2_{m-|\alpha_1|}(G)} \left\| \Delta_G^{\alpha_{22}} \sqrt{\eta_j}(\xi_G^\lambda(\mathcal{L}_G)) \right\|_{L^2(G)}. \end{aligned}$$

Therefore, using the strong admissibility of our difference operators and (55), we have shown that for every $a \in 2\mathbb{N}$,

$$\begin{aligned} \|h\|_G^a |\kappa_{j,g}(h)| &\leq C 2^{-\frac{j}{2}(|\alpha_{21}| - (m - |\alpha_1|) - \dim G/2)} 2^{-\frac{j}{2}(|\alpha_{22}| - \dim G/2)} \\ &\leq C 2^{\frac{j}{2}(m - a + \dim G)}. \end{aligned}$$

Choosing $a \in 2\mathbb{N}$ in the above sufficiently large so that $m - a + \dim G < 0$, the above implies

$$\sum_{j=l+1}^{\infty} |\kappa_{j,g}(h)| \leq C \|h\|_G^{-a} \sum_{j=l+1}^{\infty} 2^{\frac{j}{2}(m - a + \dim G)} \quad (57)$$

$$\leq C \|h\|_G^{-a} 2^{\frac{l}{2}(m - a + \dim G)} = C \|h\|_G^{-(m + \dim G)}. \quad (58)$$

Combining (56) and (58), we obtain that

$$|\kappa_g(h)| \leq C \|h\|_G^{-(m + \dim G)},$$

which concludes the proof. $\square$

4.8.3 Calderón-Zygmund theory

Using the kernel estimates obtained in Theorem 4.8.8, we shall show that the *integral kernel* of a pseudo-differential operator in $\Psi_{1,0}^0(G)$ satisfies the following property.

Definition 4.8.9 (Calderón-Zygmund kernel). *We shall say that a measurable map*

$$\kappa : (G \times G) \setminus \{(g, g) : g \in G\} \rightarrow \mathbb{C}$$

is a Calderón-Zygmund kernel if there exists $\gamma \in (0, 1]$, $C \geq 0$ and $A \geq 0$ such that the estimates

$$\begin{aligned} |\kappa(g, h)| &\leq A \|h^{-1}g\|_G^{-\dim G} \\ |\kappa(g, h) - \kappa(g', h)| &\leq A \frac{\|g^{-1}g'\|_G^\gamma}{\|h^{-1}g\|_G^{\dim G + \gamma}} \quad \text{if } \|g^{-1}g'\|_G \leq \frac{\|h^{-1}g\|_G}{2} \\ |\kappa(g, h) - \kappa(g, h')| &\leq A \frac{\|h^{-1}h'\|_G^\gamma}{\|h^{-1}g\|_G^{\dim G + \gamma}} \quad \text{if } \|h^{-1}h'\|_G \leq \frac{\|h^{-1}g\|_G}{2} \end{aligned}$$

hold for every $g, g', h, h' \in G$ satisfying

$$g \neq h, \quad g' \neq h, \quad g \neq h'.$$

If T is the operator given by

$$T\phi(g) = \int_G \phi(h) \kappa(g, h), \quad g \in G, \phi \in \mathcal{S}(G),$$

in the sense of distributions, we shall say that T is a Calderón-Zygmund operator.

We give here a sufficient condition to be Calderón-Zygmund.

Lemma 4.8.10. *If the map*

$$\kappa : (G \times G) \setminus \{(g, g) : g \in G\} \rightarrow \mathbb{C}$$

is continuously differentiable on its domain and satisfies

$$\begin{aligned} |\kappa(g, h)| &\leq A \|h^{-1}g\|_G^{-\dim G} \\ |X_{j_g} \kappa(g, h)| &\leq A \|h^{-1}g\|_G^{-\dim G - 1} \\ |X_{j_h} \kappa(g, h)| &\leq A \|h^{-1}g\|_G^{-\dim G - 1} \end{aligned}$$

for some $A \geq 0$, then κ is a Calderón-Zygmund.

Proof. Clearly, the first inequality is satisfied.

Using the second inequality in our assumption, we obtain

$$\sup_{\|z\|_G \leq \|g^{-1}g'\|_G} |(X_j)_{g_1=gz} \kappa(g_1, h)| \leq A \sup_{\|z\|_G \leq \|g^{-1}g'\|_G} |h^{-1}gz|^{-\dim G - 1}.$$

Assume now that $2\|g^{-1}g'\|_G \leq \|h^{-1}g\|_G$. It follows that

$$\|z\|_G \leq \frac{1}{2} \|h^{-1}g\|_G.$$

Since $\|\cdot\|_G$ satisfies a triangle inequality, it also satisfies a reverse triangle inequality,

$$\|h^{-1}gz\|_G \geq \|h^{-1}g\|_G - \|z\|_G \geq \frac{1}{2} \|h^{-1}g\|_G.$$

Using the second inequality in our assumption, we have

$$\sup_{\|z\|_G \leq \|g^{-1}g'\|_G} |(X_j)_{g_1=gz} \kappa(g_1, h)| \leq C \|h^{-1}g\|_G^{-\dim G-1}.$$

By the Taylor Remainder theorem, we conclude that

$$|\kappa(g', h) - \kappa(g, h)| \leq C \|g^{-1}g'\|_G \|h^{-1}g\|_G^{-\dim G-1}.$$

The proof of the third inequality is proved similarly. $\square$

We can now state and prove the main result of the section.

Proposition 4.8.11. *Assume that $T \in \Psi_{1,0}^0(G)$. The operator T is Calderón-Zygmund. In particular, for each $1 < p < \infty$ the map T extends to a bounded map*

$$T : L^p(G) \rightarrow L^p(G).$$

Proof. Let $\tilde{\kappa}(g, h) \stackrel{\text{def}}{=} \kappa_g(h^{-1}g)$. It follows that for each $\phi \in \mathcal{S}(G)$, we have

$$T\phi(g) = \int_G \phi(h) \tilde{\kappa}(g, h) \, dh$$

for each $g \in G$, where the right-hand side is interpreted in the sense of distributions.

By the kernel estimates for $\rho = 1$ (Theorem 4.8.8), we have

$$|\tilde{\kappa}(g, h)| \leq |\kappa_g(h^{-1}g)| \leq C \|h^{-1}g\|_G^{-\dim G}.$$

Fix $j \in \{1, \dots, \dim G\}$. Using the kernel estimates (Theorem 4.6.1), it follows that

$$\begin{aligned} |(X_j)_g \tilde{\kappa}(g, h)| &\leq |(X_j)_{g_1=g} \kappa_{g_1}(h^{-1}g)| + |(X_j)_{g_2=g} \kappa_g(h^{-1}g_2)| \\ &\leq C \|h^{-1}g\|_G^{-\dim G-1}, \end{aligned}$$

and similarly

$$|(X_j)_h \tilde{\kappa}(g, h)| \leq |(X_j)_g \kappa_g(h^{-1}g)| \leq C \|h^{-1}g\|_G^{-\dim G-1}.$$

We therefore obtain that T is Calderón-Zygmund by applying Lemma 4.8.10. For the rest, we check easily (see for example [FR16, Proposition 3.2.17]) that being Calderon-Zygmund means $\tilde{\kappa}$ satisfies the assumptions of [FR16, Theorem A.4.4]. $\square$

Naturally, this implies that $T \in \Psi_{1,0}^m(G)$ is bounded between $L_s^p(G)$ and $L_{s-m}^p(G)$ for each $s \in \mathbb{R}$. Since $(I - \mathcal{L}_G)^{\frac{m}{2}} \in \Psi_{1,0}^m(G)$, it follows that these $L^p(G)$ estimates are sharp.

4.9 CONSTRUCTION OF PARAMETRICES

In this section, we show that the composition formula (Theorem 4.7.13) and the construction of asymptotic limits (Proposition 4.7.15) can be used to construct parametrices of so-called *elliptic pseudo-differential operators*.

Before defining the concept of ellipticity, we define the following linear operator

$$E_\lambda(\Lambda) \stackrel{\text{def}}{=} I_{\oplus_{(\tau) \in \mathbb{R}^+} \{|\lambda|^2 \geq \Lambda\}} \text{span}\{\tau_{ij}; i, j=1, \dots, d_\tau\}$$

when $\lambda \in V$ and $\Lambda \geq 0$.

We shall say that a symbol is *elliptic* if it is pointwise invertible for “high frequencies”.

Definition 4.9.1 (Ellipticity). *Let $\sigma \in S_{\rho,\delta}^m(G)$. We shall say that σ is elliptic and of elliptic order m if there exists $\Lambda \geq 0$ such that the following property holds: for each $\gamma \in \mathbb{R}$, there exists $C \geq 0$ such that*

$$\left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} \sigma(g, \lambda) F \right\|_{L^2(K)} \geq C \left\| \tilde{\zeta}_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma+m}{2}} F \right\|_{L^2(K)} \quad (59)$$

holds for every $F \in E_\lambda(\Lambda)C^\infty(K)$ and every $(g, \lambda) \in G \times V$.

From the definition of ellipticity, we can construct an “inverse for high frequencies” which satisfy certain symbolic estimates.

Lemma 4.9.2 (Inverse of an elliptic symbol). *Let $\sigma \in S_{\rho,\delta}^m(G)$ be an elliptic symbol of elliptic order m , with $\Lambda \geq 0$ given by Definition 4.9.1.*

For any $F \in E_\lambda(\Lambda)C^\infty(K)$, let

$$E_\lambda(\Lambda)\sigma(g, \lambda)^{-1}(\sigma(g, \lambda)F) \stackrel{\text{def}}{=} F,$$

and consider the unique extension of $E_\lambda(\Lambda)\sigma(g, \lambda)^{-1}$ onto $C^\infty(K)$ which satisfies

$$E_\lambda(\Lambda)\sigma(g, \lambda)^{-1} \Big|_{(\sigma(g, \lambda)E_\lambda(\Lambda))^\perp} = 0.$$

The map $E_\lambda(\Lambda)\sigma(g, \lambda)^{-1}$ is well-defined. Moreover, for each $\lambda \in \mathbb{R}$, we have

$$\left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{m+\gamma}{2}} E_\lambda(\Lambda)\sigma(g, \lambda)^{-1} \zeta_G^\lambda (I - \mathcal{L}_G)^{-\frac{\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} < \infty. \quad (60)$$

Proof. Suppose that $F_1, F_2 \in E_\lambda(\Lambda)C^\infty(K)$ are such that

$$\sigma(g, \lambda)F_1 = \sigma(g, \lambda)F_2.$$

By definition of ellipticity with $\gamma \stackrel{\text{def}}{=} -m$, it follows that

$$\begin{aligned} \|F_1 - F_2\|_{L^2(K)} &\leq C \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{-\frac{m}{2}} \sigma(g, \lambda)(F_1 - F_2) \right\|_{L^2(K)} \\ &= 0, \end{aligned}$$

so that the map is indeed well-defined.

Now, given $\gamma \in \mathbb{R}$, we easily check by ellipticity that

$$\begin{aligned} &\left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{m+\gamma}{2}} E_\lambda(\Lambda)\sigma(g, \lambda)^{-1} F \right\|_{L^2(K)} \\ &\leq C \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} \sigma(g, \lambda) E_\lambda(\Lambda)\sigma(g, \lambda)^{-1} F \right\|_{L^2(K)} \\ &\leq C \left\| \zeta_G^\lambda (I - \mathcal{L}_G)^{\frac{\gamma}{2}} F \right\|_{L^2(K)}, \end{aligned}$$

which shows (60). $\square$

The previous lemma can be used to construct a symbol which inverts an elliptic symbol modulo $S^{-\infty}(G)$.

Proposition 4.9.3 (Inverse of an elliptic symbol). *Assume that $1 \geq \rho > \delta \geq 0$. Suppose that $\sigma \in S_{\rho, \delta}^m(G)$ is a symbol which is (Λ, m) -elliptic.*

Let $\chi \in C^\infty(\mathbb{R}^+, [0, 1])$ be such that

$$\chi_\Lambda(r) = \begin{cases} 1 & \text{if } r \geq \Lambda_2 \\ 0 & \text{if } r \leq \Lambda_1, \end{cases}$$

for some $\Lambda < \Lambda_1 < \Lambda_2$, and define the symbol

$$\psi_\Lambda(\lambda) \stackrel{\text{def}}{=} \sum_{\tau \in \widehat{K}} \chi_\Lambda(\langle \lambda \rangle_{\widehat{V}} + \langle \lambda \rangle_{\widehat{K}}) I_{\text{span}\{\tau_{ij}; 1 \leq i, j \leq d_\tau\}}.$$

The map

$$\sigma_\Lambda^{-1}(g, \lambda) \stackrel{\text{def}}{=} \psi_\Lambda E_\lambda(\Lambda_1) \sigma(g, \lambda)^{-1}$$

defines a symbol of $S_{\rho, \delta}^{-m}(G)$.

Proof. Let us first observe that

$$\psi_\Lambda = \sigma_\Lambda^{-1}(g, \lambda) \sigma(g, \lambda). \quad (61)$$

Since ψ_Λ commutes with the powers of $\xi_G^\lambda(I - \mathcal{L}_G)$,

$$\begin{aligned} & \left\| \xi_G^\lambda(I - \mathcal{L}_G)^{\frac{m}{2}} \sigma_\Lambda(g, \lambda)^{-1} \right\|_{\mathcal{L}(L^2(K))} \\ & \leq \|\psi_\Lambda\|_{\mathcal{L}(L^2(K))} \left\| \xi_G^\lambda(I - \mathcal{L}_G)^{\frac{m}{2}} E_\lambda(\Lambda) \sigma(g, \lambda)^{-1} \right\|_{\mathcal{L}(L^2(K))} < \infty, \end{aligned}$$

where the finiteness is given by Lemma 4.9.2.

We have thus shown that

$$\sup_{g \in G} \text{ess sup}_{\lambda \in V} \left\| \xi_G^\lambda(I - \mathcal{L}_G)^{\frac{m-\delta|\beta_0|}{2}} X^{\beta_0} \sigma_\Lambda(g, \lambda)^{-1} \right\|_{\mathcal{L}(L^2(K))} < \infty,$$

for $\beta_0 = 0$. Let us show it also holds for β_0 of higher order.

By the Leibniz rule, we have

$$0 = X^{\beta_0} \psi_\Lambda = \sum_{\beta_0 = \beta_1 + \beta_2} \left(X^{\beta_1} \sigma_\Lambda(g, \lambda)^{-1} X^{\beta_2} \sigma(g, \lambda) \right),$$

which implies that

$$X^{\beta_0} \sigma_\Lambda(g, \lambda)^{-1} \sigma(g, \lambda) = - \sum_{\substack{\beta_0 = \beta_1 + \beta_2 \\ |\beta_1| < |\beta_0|}} \left(X^{\beta_1} \sigma_\Lambda(g, \lambda)^{-1} X^{\beta_2} \sigma(g, \lambda) \right).$$

Since $\sigma(g, \lambda)$ is invertible on $E_\lambda(\Lambda_1) C^\infty(K)$, we obtain

$$\begin{aligned} & X^{\beta_0} \sigma_\Lambda(g, \lambda)^{-1} \\ & = - \sum_{\substack{\beta_0 = \beta_1 + \beta_2 \\ |\beta_1| < |\beta_0|}} \left(X^{\beta_1} \sigma_\Lambda(g, \lambda)^{-1} X^{\beta_2} \sigma(g, \lambda) \right) E(\Lambda_1) \sigma^{-1}(g, \lambda), \end{aligned}$$

which allows us to conclude by induction and Lemma 4.9.2.

Let us now work with difference operators. The ideas are the same, but the proof is slightly trickier because we have a *Leibniz-like* rule instead of the classical Leibniz rule.

Applying the Leibniz-like rule to (61), we obtain

$$\begin{aligned}\Delta_j^G \psi_\Lambda(\lambda) &= \Delta_j^G \sigma_\Lambda(g, \lambda)^{-1} \sigma(g, \lambda) + \sigma_\Lambda(g, \lambda)^{-1} \Delta_j^G \sigma(g, \lambda) \\ &\quad + \sum_{1 \leq k, l \leq n_{\Lambda, G}} c_j^{kl} \Delta_k^G \sigma_\Lambda(g, \lambda)^{-1} \Delta_l^G \sigma(g, \lambda).\end{aligned}$$

Reorganising the terms above, and using the fact that σ is invertible on $E_\lambda(\Lambda_1)C^\infty(K)$, we obtain

$$\begin{aligned}\Delta_j^G \sigma_\Lambda(g, \lambda)^{-1} &+ \sum_{1 \leq k, l \leq n_{\Lambda, G}} c_j^{kl} \Delta_k^G \sigma_\Lambda(g, \lambda)^{-1} \Delta_l^G \sigma(g, \lambda) E_\lambda(\Lambda_1) \sigma^{-1}(g, \lambda) \\ &= \Delta_j^G \psi_\Lambda(\lambda) E_\lambda(\Lambda_1) \sigma^{-1}(g, \lambda) - \sigma_\Lambda(g, \lambda)^{-1} \Delta_j^G \sigma(g, \lambda) E_\lambda(\Lambda_1) \sigma^{-1}(g, \lambda).\end{aligned}\tag{62}$$

If $R(g, \lambda)$ is the second line of the above, we check easily that

$$\left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m+\rho+\gamma}{2}} R(g, \lambda) \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} < \infty,$$

we can therefore deduce that

$$\left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m+\rho+\gamma}{2}} \Delta_j^G \sigma_\Lambda(g, \lambda)^{-1} \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} < \infty,$$

by considering the equations (62) for all j simultaneously. Applying Δ_G^α to (61), we can deduce inductively by arguing like above that

$$\left\| \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{m+\rho|\alpha|+\gamma}{2}} \Delta_G^\alpha \sigma_\Lambda(g, \lambda)^{-1} \xi_G^\lambda (I - \mathcal{L}_G)^{\frac{-\gamma}{2}} \right\|_{\mathcal{L}(L^2(K))} < \infty.$$

The general case can be deduced similarly by applying $X^\beta \Delta_G^\alpha$ to (61). $\square$

We are now able to prove that any *elliptic pseudo-differential operator* admits a left parametrix.

Theorem 4.9.4 (Existence of left parametrices). *Let $A \in \Psi_{\rho, \delta}^m(G)$ be an m -elliptic operator of order m with $1 \geq \rho > \delta \geq 0$. There exists an operator $B \in \Psi_{\rho, \delta}^{-m}(G)$ which satisfies*

$$BA = I_{\mathcal{S}(G)} \quad \text{mod} \quad \Psi^{-\infty}(G).$$

Proof. Since $A = \text{Op}_G(\sigma)$ is elliptic, we know that $\sigma_\Lambda(g, \lambda)^{-1} \in S_{\rho, \delta}^{-m}(G)$ and

$$\begin{aligned}\sigma_\Lambda(g, \lambda)^{-1} \sigma(g, \lambda) &= \psi_\Lambda - \sigma_\Lambda(g, \lambda)^{-1} \sigma(g, \lambda) (I_{L^2(K)} - \psi_\Lambda) \\ &= \psi_\Lambda \mod S^{-\infty}(G), \\ &= I_{L^2(K)} \mod S^{-\infty}(G),\end{aligned}$$

since $I_{L^2(K)} - \psi_\Lambda$ is smoothing.

Therefore, letting $B_0 \stackrel{\text{def}}{=} \text{Op}_G(\sigma_\Lambda(g, \lambda)^{-1})$ yields

$$B_0 A = I_{S(G)} - E_0 \quad \text{where } E_0 = \text{Op}_G(e_0) \in \Psi_{\rho, \delta}^{-(\rho-\delta)}(G)$$

by the composition formula. Suppose by induction that we have defined B_j and $E_j = \text{Op}_G(e_j)$ for $j = 0, \dots, k$ such that

$$(B_0 + \dots + B_k)A = I - E_k \quad \text{where } E_k = \text{Op}_G(e_k) \in \Psi_{\rho, \delta}^{-(k+1)(\rho-\delta)}(G).$$

Letting $B_{k+1} \stackrel{\text{def}}{=} \text{Op}_G(E_k \sigma_\Lambda(g, \lambda)^{-1})$, we obtain

$$(B_0 + \dots + B_{k+1})A = I - E_k + B_{k+1}A = I - E_{k+1},$$

where $E_{k+1} \stackrel{\text{def}}{=} E_k - B_{k+1}A$ is an operator in $\Psi_{\rho-\delta}^{-(k+2)(\rho-\delta)}(G)$, again by the composition formula.

Since the order of B_k strictly decreases as k strictly increases, it follows that

$$B \sim \sum_{k=0}^{+\infty} B_k$$

defines an operator in $\Psi_{\rho, \delta}^{-m}(G)$. Moreover, for each $N \in \mathbb{N}$ we have

$$BA - I + E_N = (B - \sum_{k=0}^N B_k)A \in \Psi_{\rho, \delta}^{-(N+1)(\rho-\delta)+m}(G),$$

with $E_N \in \Psi_{\rho, \delta}^{-(N+1)(\rho-\delta)}(G)$. Letting $N \rightarrow \infty$ on both sides of the above, we obtain that

$$BA - I \in \Psi^{-\infty}(G),$$

concluding the proof. $\square$

Combined with the $L^2(G)$ boundedness, the existence of parametrices yields the following result.

Corollary 4.9.5 (A-priori estimates). *Let $m \in \mathbb{R}$ and $1 \geq \rho > \delta \geq 0$. If $L \in \Psi_{\rho,\delta}^m(G)$ is an elliptic pseudo-differential operator and*

$$Lu = f$$

for some $f \in L_s^2(G)$, then $u \in L_{m+s}^2(G)$.

Moreover, we have the following a-priori estimate

$$\|u\|_{L_{m+s}^2(G)} \leq C(\|f\|_{L_s^2(G)} + \|u\|_{L_{-N}^2(G)})$$

for each $N \in \mathbb{R}$.

Proof. By Theorem 4.9.4, there exists a left-parametrix $P \in \Psi_{\rho,\delta}^{-m}(G)$ such that

$$PL = I - R, \quad R \in \Psi^{-\infty}(G).$$

In particular, by the triangular inequality and Theorem 4.8.4 applied to P and R ,

$$\begin{aligned} \|u\|_{L_{s+m}^2(G)} &\leq \|PLu\|_{L_{s+m}^2(G)} + \|Ru\|_{L_{s+m}^2(G)} \\ &\leq C(\|Lu\|_{L_s^2(G)} + \|u\|_{L_{-N}^2(G)}), \end{aligned}$$

concluding the proof. □

Remark 4.9.6. *Let us make a few comments on the last results.*

- In Corollary 4.9.5, when $L \in \Psi_{1,0}^m(G)$ is of type $(1,0)$, then we may replace 2 by any $p \in (1, +\infty)$ thanks to the Calderón-Zygmund theory (Proposition 4.8.11).
- The result Corollary 4.9.5 can be localized by following [FR16, Subsection 5.8.3].

5 | CONCLUSION

Building upon the work of M. Ruzhansky, V. Turunen, V. Fischer and others, we managed to define a global pseudo-differential calculus on the motion group. The main contribution of this document is arguably a shorter, less technical, and purely symbolic proof of the kernel estimates, which is the key result in the strategy devised in [FR16].

Our analysis uses the specific structure of the motion group only when we prove the existence of a finite, strongly admissible family of polynomials satisfying a *Leibniz-like* rule, and with adequate growth at infinity to control the decay of the kernel away from the origin. Going forward, it would be interesting to know which other Lie groups possess such a family of polynomials, or whether the aforementioned conditions can be relaxed in any way.

Unfortunately, the present document is rather academic in the sense that applications of the developed calculus are left untreated. As an example, we expect a sharp Gårding inequality to hold on the motion group. Moreover, the study of the spectral and the propagation of singularities of our pseudo-differential operators is also of interest. This, along with other aspects, could be the object of future work.

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