LINEAR AND NONLINEAR STABILITY PROBLEMS IN THE FLOW BETWEEN TWO ~~ISOTHERMAL~~ ROTATING ECCENTRIC CYLINDERS

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ABSTRACT

The work in this thesis is concerned with the stability of the flow between two long eccentric rotating cylinders.

Part I of the thesis deals with the wavy vortices of the flow between two eccentric cylinders where no Taylor vortex flow is present, and where the eccentricity is small and the clearance ratio is very small. An extension to Di Prima and Stuart's work (1972b) is made by the inclusion of the time variable τ in the amplitude equation for $B(\phi,\tau)$. The results are compared with the observed critical speeds of Vohr (1967) for development of wave instability on Taylor vortices. The results are qualitatively similar, but differ in magnitude from experiment due to the absence of nonlinear effects.

In Part II we make an analysis of the influence exerted by a nonlinear Taylor vortex mode with amplitude $B(\phi)$ on a linear wavy vortex mode with amplitude $A(\phi)$. This problem can be seen as an extension of the paper of Davey, Di Prima and Stuart (1968), where a similar problem has been treated for the concentric case. We take $v = v' + v_{TV}$, where v_{TV} is the Taylor vortex mode and v' is the wavy vortex mode. We define a new parameter $\gamma = T - T_0 - \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1}$, which is the amount above the critical Taylor number. We make the approximation $\gamma \cong \varepsilon T_1$, and expand v_{TV} and the growth rate $\bar{\sigma}$ in powers of $\gamma^{1/2}$.

The problem is solved by using a double parameter expansion, first the expansion in $\gamma^{1/2}$ from γ^0 to the γ order in the equations and afterwards the expansion in ε from ε^0 to ε^2 in certain integrals. A formula for the critical Taylor number is obtained. For $T_1 \to \infty$ (or $k \to \infty$), $\varepsilon \to 0$, $\delta \to 0$ and m large we retrieve the Taylor critical for the concentric case.

Qualitative and consistent agreement is obtained with the experimental data of Vohr for the occurrence of wavy instability though the theoretical results are a little bit lower due to the absence of the quintic terms in the expansions of the nonlinear Taylor vortex mode.

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L I N E A R A N D N O N L I N E A R

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INTRODUCTION

In Couette flow between concentric rotating cylinders we are confronted with two different kinds of transition. The first, the so-called transition by spectral evolution, occurs when the inner cylinder has a larger angular velocity than the outer one, while the second kind of transition, the so-called catastrophic transition, is characterised when the outer cylinder has a larger speed than the inner one. We define a Taylor number proportional to the square of the angular velocity of the inner cylinder. The resulting laminar circular Couette flow becomes unstable when the Taylor number exceeds a critical value $T_c(0) = T_0$. The laminar flow is then replaced by a toroidal or Taylor vortex system, superimposed on a modified circumferential flow, the vortices being spaced regularly along the axis with a definite periodicity, neighbours having opposite senses of rotation. At still higher values of T, the Taylor vortices are modified by a waviness in the azimuth, and in fact become travelling waves in the circumferential direction.

As far as the historical background is concerned, before the beginnings of modern experiments in fluid mechanics, the steady motion of a viscous fluid between concentric rotating cylinders was thought to be a simple linear combination of rotation like a solid body and circulation like a potential vortex, the proportion of each component being dependent on the angular velocities of the two cylinders. The purpose of these early experiments was to check the validity of the Newtonian stress approximation in the Navier-Stokes equations, and this method was the first to verify the viscosity property of the fluid.

Some early experiments, including work by Hagen, Reynolds and Mallock, also brought evidence that elementary laminar flow was

always replaced at sufficiently high speeds by some more complicated flow. Particularly, the experiments by Couette (1890) and Mallock (1888) revealed that the critical speed was very much lower when the inner cylinder was rotating and the outer cylinder fixed than vice versa. The mechanism of instability for the Couette flow, that means the transition from Couette flow to the vortex flow, is characterised by a lack of equilibrium among the forces which act on any given fluid element. In general for Couette flow, the radial pressure gradient and the viscous shear forces are balanced by the inertial forces of the rotating fluid.

Taylor's (1923) considerable contribution to the study of fluid mechanics in including the viscosity in the stability analysis improved the stability criterion for Couette flow and gave an exact description of the secondary flow which appears after the first onset of instability. Although Taylor's work had applications in the general field of inertial instability, his analytical solution was based on the so-called "narrow-gap" assumption, that means $R_1 \gg R_2 - R_1$. In 1961 Chandrasekhar was able to find a mathematical solution for the eigenvalues in the Taylor vortex problem for the "wide-gap" case. In his work Chandrasekhar solved the wide-gap problem theoretically for $R_1/R_2 = 0.50$, and his results were confirmed experimentally by Donnelly and Fultz (1960) for angular velocity ratios ranging from $-7.171 \leq \frac{\Omega_2}{\Omega_1} \leq 0.2499$. Roberts (1965) brought a contribution to Chandrasekhar's research by solving the problem theoretically for the eigenvalues for several values of R_1/R_2 ranging from 0.20 to 0.975. After the completion of Robert's research, the theoretical and experimental confirmation of the onset of Taylor vortex flow was well established under a wide range of conditions. A brief mention of the end effects or end boundary conditions is desirable. Most of

the theoretical research is based on the assumption of infinite length cylinders. Obviously, as an infinite cylinder cannot be constructed and thus one has to rely on trends indicated by the data as the infinite cylinder is approached. Let us define the "effective" wavelength as follows:

$$\lambda_{eff} = \frac{\text{fluid column length} - \text{length of end cells}}{(\text{number of rings}) \cdot (\text{gaps})}$$

where the length of the end cells is the sum of both top and bottom cell lengths. In conclusion λ_{eff} is dependent on the end cell length. Under certain end boundary conditions the end cell length can become much larger than any other cell in the fluid column, and thus gives the impression that the other cells are shrinking, or λ_{eff} seems to decrease.

Burkhalter and Koschmieder (1972) found that the wavelength of singly periodic Taylor vortices in an infinitely long cylinder, remains constant between $T/T_c = 1$ and $T/T_c \approx 80$ in experiments with radius ratios $\eta = 0.505$ and $\eta = 0.727$, provided that proper account was taken of the end effects.

Coles (1965) performed some interesting experiments, beside showing that the wavelength of cells was nonunique at some given supercritical Taylor number, he was able to find that the Taylor vortices became wavy in the azimuthal direction provided that the Taylor number was large enough. In fact these vortices became travelling waves in the azimuthal direction. From now on we call these vortices as wavy vortices. When these azimuthal waves appear the flow is said to be "doubly periodic". Coles found that the tangential waves have a nearly constant velocity of 0.34 times the angular velocity of the inner cylinder. Let a given flow between concentric rotating cylinders be denoted by m/n, where m is the number of

Taylor vortices and n is the number of azimuthal waves. In the case of length/gap = 27.9 and gap/inner radius = 0.144, Coles obtained at a sequence of rising speeds of the inner cylinder the following discrete spectrum of states 28/0 (Taylor vortices), 28/4 (wavy vortices about $1.5T_c$), 24/5, 22/5, 22/6 . . . where $n \neq 0$ the boundary between neighbouring cells were always wavy. Coles was also able to illustrate the experimental fact, that in the range of speeds for which doubly-periodic flows were possible, the steady Couette flow of a given fluid in an apparatus was not uniquely determined by the final speed of rotation, but was dependent upon the manner in which the final speed was attained.

A very brief mention could be made on the paper of Donnelly and Simon (1959), where they determined an empirical torque relation $G = a\Omega_1^{-1} + b\Omega_1^{1.36}$ for supercritical flow between rotating cylinders, where Ω_1 is the angular velocity of the inner cylinder and a and b are constants. This formula is valid to a range of values of Ω_1 above the onset of instability extending to about ten times the critical angular velocity. The experiments also illustrate that the finite amplitude analysis of Stuart (1958) gives the correct torque variation over a short range above the critical speed. Schwarz, Springett and Donnelly (1964) distinguished three-modes of instability in spiral flow between rotating concentric cylinders (the inner) with a narrow annulus: the familiar axisymmetric mode, an azimuthal mode with the predicted $\exp i(\theta - \omega t)$ angular dependence, and a completely non-symmetric instability apparently dependent on the interaction of the $m = 1$ and axisymmetric modes.

Cole was the first to start experiments on the flow between two eccentric cylinders. One interesting conclusion supported by Cole (1957) was that at high eccentricities the flow between eccentric

rotating cylinders should be more stable to disturbances of the Taylor vortex type than is the flow between concentric cylinders. Cole (1974 - Nature, December 20/27) discovered the crucial effect that the cylinder (length/clearance) ratio L/C, has on the wavy vortex stability for concentric cylinders. His visualisation observations as well as his torque measurements were in complete agreement in showing little effect of annulus length on the first critical speed but a crucial effect on the second critical speed or speed for the onset of wavy vortices. In conclusion the research based on the assumption of infinite length has not the same accuracy for the wave instability as for the Taylor vortex instability.

Vohr (1968) made a remarkable contribution to the experimental study of Taylor vortices in flow between eccentric rotating cylinders. It is a known fact that the occurrence of Taylor vortices in the flow between rotating cylinders results in a discontinuity in the slope of the torque versus rotational speed, and this discontinuity is a very proper means of determining the critical speed for onset of vortices. Up to a rotational speed of 2.875 rps, the curve of Vohr is perfectly linear with torque, indicative of laminar Couette flow, and at 2.875 rps the slope of the torque versus rotational speed curve changes sharply to a steeper value, indicating the development of Taylor vortices in the flow. Other experimental results of Vohr were that the change in slope of the torque curve at the Taylor transition point becomes progressively less sharp as eccentricity increases. It would be sensible to assume that the precision with which the critical Taylor speed could be determined from the torque curves will decrease as eccentricity is increasing. Vohr found also that the transition speed for onset of instability between nonconcentric rotating cylinders increases with eccentricity ratios, and that this speed depends on the clearance

ratio at higher eccentricity ratios. One difference between the cases of rotating concentric and the rotating eccentric cylinders was that for the latter case the end boundary vortices were non-existent, a situation which may be explained by the axial flow induced by pressure which occurs at the end of the nonconcentric rotating cylinders and stabilizes the circumferential flow there against development of vortices.

An interesting observation of the visual studies of Taylor vortex instability was that vortices would become weakly visible in the flow at speeds substantially below the speed at which torque measurements would indicate their presence. This occurred in cases both of concentric and of eccentric cylinders; the discrepancy between the first visual sighting of vortices and the point at which torque measurements first indicated their presence increases with eccentricity ratios. Vohr suggests that this discrepancy is due to extraneous disturbances of marginally stable flow situations, the discontinuity on the torque versus rotational speed indicating the true point of instability.

Davey, Di Prima and Stuart (1968) investigated the problem of instability of the Taylor vortex flow against perturbations which are periodic both in the axial and azimuthal co-ordinates, where the perturbations with the azimuthal co-ordinate periodicity travel with a certain phase velocity.

Based on some assumptions and approximations it was found that the Taylor vortex flow is stable against perturbations with the same axial wavelength and phase, but unstable against perturbations differing in phase by $\frac{\pi}{2}$. After instability the flow no longer is characterised by planes separating adjoining vortices, but by travelling waves in the azimuthal direction, a situation confirmed also by much experimental evidence.

In this work it is assumed that the wavelength of the vortices in the axial direction is the same as in the case of the linear theory. Although it was found experimentally that the variation of wavelength with increasing T is not large (Donnelly and Schwarz 1965), it was suggested by Meyer (1966) that a suitable chosen variation of wavelength would bring to a better agreement with experiment. Davey, Di Prima and Stuart consider the interaction of two axisymmetric disturbances of different axial phases, with two non-axisymmetric disturbances of different axial phases. Their method involved expansion of the velocity, in powers of an amplitude function of time, $A(\tau)$ up to $A^3(\tau)$, and in this way they derived a system of four nonlinear differential equations for the amplitudes of the fundamental disturbances. The coefficients in the amplitude equations are functions of the parameters appearing in the problem, namely:

(i) $\mu = \frac{\Omega_2}{\Omega_1}$ and $\eta = \frac{R_1}{R_2}$ which represent the laminar velocity and geometry;

(ii) the axial and circumferential wave-numbers of the disturbance and finally,

(iii) the Taylor number, where these coefficients can be calculated by solving a system of linear-ordinary differential equations.

Since this model includes (1) a stability analysis of the Taylor vortex flow with respect to non-axisymmetric disturbances and (2) the possible states of motion that can result from an instability of the Taylor vortex flow, this research work illustrates the concept of successive instabilities of Landau (1944).

Beside the assumptions of truncation at cubic amplitude terms and restriction of the number of basic modes to four, there are also two other basic assumptions, namely: (3) the small-gap approximation

and (4) infinitely long cylinders. Some additional conclusions of this paper are that the azimuthal wave-number m is not properly determined by the theory, but there is a slight preference for $m = 1$, in accordance with experiments of Schwarz et al. (1964). However, this result is not confirmed by the experiments of Coles (1965), which found a preference for $m = 4$. Other conclusion reached is that the critical Taylor number for the instability of the Taylor vortex flow, is about 8% above the critical value for the occurrence of these Taylor vortices, $T_c(0)$, the experiments most closely related indicating a value in the range of 5-20%.

Eagles' (1971) analysis concentrates on the instability of Taylor vortices for $\eta = 0.951$, and was undertaken to find the effect of: (i) using fifth order terms in the amplitude; (ii) avoiding the small gap approximation and using the full equations instead. The expansion is rewritten in a matrix form, allowing more uniformity of treatment of the appearing systems of ordinary differential equations. A very careful treatment of the method by which certain "coefficients of nonlinear terms" and perturbations functions have to be evaluated, is needed, in order to include the fifth-order terms in the amplitude, because the Stuart-Watson method does not uniquely determine the amplitude function $A(t)$. Since different choices of the "coefficients of nonlinear terms" are possibly leading to different determinations of $A(t)$ and to different perturbation functions, which are asumptotically equivalent as $T \to T_c$, Eagles chose the constants in the same way as Reynolds and Potter (1967) for the channel flow problem. By using an expansion of a disturbance to a steady state solution in terms of a small parameter ε, given by $\varepsilon^2 = \frac{T - T_c}{T_c}$, the following equation in the amplitude function $A_1(t')$ is obtained:

$$\frac{dA_1}{dt'} = 2A_1 - cA_1^3$$

where $t' = \varepsilon^2 t$, and the fifth-order terms are avoided in this manner. The conclusion reached is that the fifth order terms would be important as far as the matter of stability is concerned, but qualitatively unimportant as far as the development of Taylor vortices is concerned.

Eagles found a qualitative confirmation for the Davey, Di Prima and Stuart results for the instability of Taylor vortices to non-axisymmetric disturbances, out of phase by $\frac{\pi}{2}$ in the axial direction with the Taylor vortices, at about 10% above the first critical Taylor number.

Consequently, the contribution of the fifth-order terms is quite small, and it can be assumed that these calculations are quite accurate for the theoretical case of infinitely long cylinders considered in that paper.

Di Prima and Stuart (1972a) deal with the viscous, incompressible flow between eccentric rotating cylinders for very small clearance ratios, a journal bearing. In lubrication problems the mean gap between the cylinders $(b-a)$ is very small $(b-a \ll a)$, and the distance between the axes (ae) can be a big part of that mean gap. The eccentricity parameter ε is defined by $ae = \varepsilon(b-a)$, so that $0 \leq \varepsilon < 1$. They used a modified bipolar co-ordinate system. Their analysis shows that the Sommerfeld pressure distribution in a journal bearing can be derived from the Navier-Stokes equations by a straightforward expansion in the small parameter δ , the clearance ratio. First order corrections for the effects of inertia (R_M) and curvature $(\alpha \text{ or } \delta)$ are calculated. There is no restriction on the eccentricity.

Another problem which has been treated by several researchers is the problem of instability of the flow between eccentric cylinders. A number of people have made experiments on the instability of the flow between eccentric cylinders, including Cole (1957, 1965), Kamal (1966),

Vohr (1967, 1968), Castle and Mobbs (1968), Frêné and Godet (1971) etc. During the observations it was found that the critical Taylor number for the occurrence of the instability varies with the eccentricity parameter. In one of his experiments Vohr found (with $b-a = 0.099a$ and $\varepsilon = 0.475$) that the Taylor vortex secondary motion is greatest at an azimuthal position of 50° downstream from the position of maximum gap, when the inner cylinder was rotating and the outer is at rest, and not maximum at the position of widest gap, where the flow is locally most unstable (Di Prima 1963). Di Prima and Stuart (1972b) found a possible mathematical explanation for this phenomenon, as follows. As the basic flow between the cylinders depends strongly upon the azimuthal angle, the linearized equations for the instability of the flow are partial differential equations in the radial and azimuthal co-ordinates, rather than ordinary differential equations. Single valuedness and boundary conditions of no-slip are required.

The method of Di Prima and Stuart (1972b) took into account that the stability depends not only on the local flow at a given azimuthal position, but on the flow field as a whole: in conclusion the problem was global.

A similar problem has long been treated in the theory of boundary-layer instability. In this case the basic flow depends also on two co-ordinates, so that the accurate stability equations are partial differential equations, and here it was even usual to approximate the partial differential equations for stability by the Orr-Sommerfeld ordinary differential equation, a thing known as the parallel flow approximation.

Di Prima and Stuart (1972b) were the first to develop a self-consistent theory which took proper account of the global flow field and which avoided the parallel-flow approximation.

The research of Di Prima and Stuart was based on the linear stability equations. The basic flow was calculated from the equations of motion, written in the modified bipolar co-ordinate system, as used by Wood (1957), in the paper of Di Prima and Stuart (1972a). There the expansion was made in two small parameters, the modified Reynolds number R_M and the surface curvature which is small with respect to the gap between the cylinders.

The zero order approximation is given exactly by the solution of the approximate "Reynolds" equation of classical lubrication theory. The terms of order R_M give first-order inertial correction and the terms of order α give first-order curvature correction. Di Prima and Stuart (1972b) are showing that terms of higher order are not needed for this problem.

In order to discuss the linearized theory of instability of the basic flow, they wrote:

$$u_\rho = \frac{1}{2}\alpha\varepsilon(q_1+q_2)\,\mathcal{U}(x,\phi) + \left(\frac{\nu}{a\alpha}\right)u(x,\phi,\zeta,t)$$

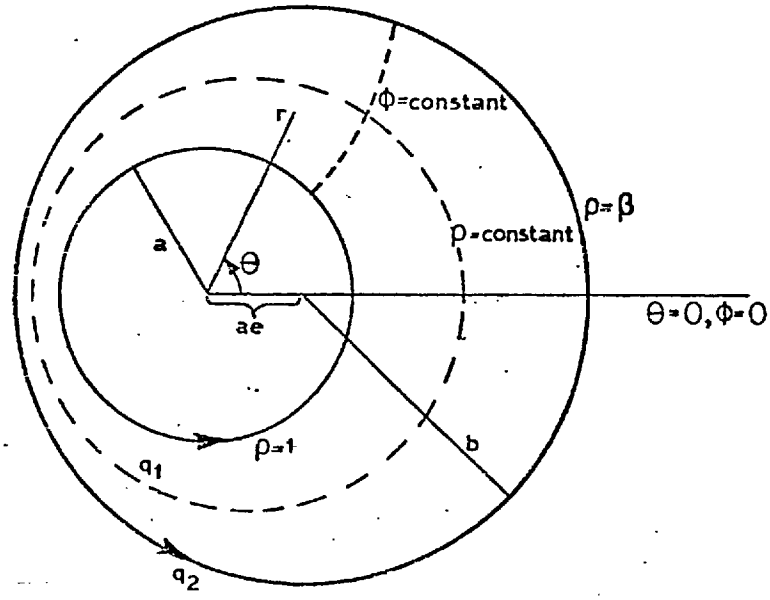
$$u_\phi = \frac{1}{2}(q_1+q_2)\,V(x,\phi) + (q_1-q_2)\,v(x,\phi,\zeta,t)$$

$$u_\zeta = \left(\frac{\nu}{a\alpha}\right)w(x,\phi,\zeta,t)$$

$$p = \left(\frac{\nu q_1}{a\alpha^2}\right)P(x,\phi) + \left(\frac{\nu^2}{a^2\alpha^2}\right)p'(x,\phi,\zeta,t)$$

where $\mathcal{U} = \left[\frac{2J^{1/2}}{\varepsilon\rho(1+\mu)}\right]\left(\frac{\partial\psi}{\partial\phi}\right)$, $V = \left[-\frac{2J^{1/2}}{(1+\mu)}\right]\left(\frac{\partial\psi}{\partial x}\right)$

P denotes the dimensionless basic flow, u, v, w and p' denote the dimensionless perturbation, x, ϕ, t and ζ are the radial, azimuthal co-ordinates, the time and the axial co-ordinate respectively, ν is the kinematic viscosity.



Geometry and co-ordinate systems.

Three dimensional Taylor vortex perturbations depend, in general, upon X, ϕ, ζ and t. For convenience the velocity scale $\frac{1}{2}(q_1+q_2)$ is chosen. The basic flow has components in the axial and azimuthal directions, and it can be seen that the radial component $\frac{\alpha\varepsilon}{2}(q_1+q_2)$ is much smaller than the azimuthal $\frac{1}{2}(q_1+q_2)$, because the ratio $\alpha\varepsilon = 2ck^2\varepsilon^3$.

The above scalings reflect the known fact that, in the concentric problem, the radial and axial perturbation velocity components are smaller than the azimuthal component by a Reynolds number factor (Stuart 1958). The appropriate velocity scales which are $\frac{\nu}{ad}$ radially and axially, and q_1-q_2 azimuthally, are known from studies of the nonlinear problem in the concentric case, (Stuart 1958, Davey 1962).

The scales of the kinematic pressure follow from the lubrication theory (I), and from the knowledge of the stability problem in the concentric case.

The dimensionless stream function Ψ of the basic flow is given by $\Psi = \Psi_{00}(x,\phi;\varepsilon,\mu) + R_M \Psi_{10}(x,\phi;\varepsilon,\mu) + \alpha\Psi_{01}(x,\phi;\varepsilon,\mu) + R_M^2 \Psi_{20}(x,\phi;\varepsilon,\mu) + O(\alpha^2, \alpha R_M, R_M^2)$

As Ψ_{20} is in fact proportional to ε, so that $R_M^2 \Psi_{20}$ is of order $\alpha\varepsilon = 2ck^2\varepsilon^3$, it has not been calculated in detail.

Therefore, the assumption has been made that

$u' = e^{\sigma\tau} u_1(x,\phi)\cos\lambda\xi$ (The linearized theory predicts a
$v' = e^{\sigma\tau} v_1(x,\phi)\cos\lambda\xi$ disturbance which increases
$w' = e^{\sigma\tau} w_1(x,\phi)\sin\lambda\xi$ exponentially with time)
$p' = e^{\sigma\tau} p_1(x,\phi)\cos\lambda\xi$

where ζ and t are defined as follows

$$\zeta = \alpha\xi \quad \text{and} \quad t = \left(\frac{a^2\alpha^2}{\nu}\right)\tau$$

σ and λ representing a non-dimensional growth rate and wavenumber respectively.

By using the basic flow known to order α and R_M, the partial differential equations of stability against perturbations periodic along the axis are formulated.

A model equation is considered as an asymptotic approximation relating α, R_M, ε and a Taylor number T as follows:

$$(D^2-\lambda^2)\left[D^2-\lambda^2-\sigma-R_a V_0(x)\left(\frac{\partial}{\partial\phi}\right)\right]^2 v + \lambda^2 T\left[V_0(x)+\varepsilon V_1(x)\cos\phi\right] v = 0$$

In this model equation we are confronted with a similar situation as in the real problem in that it models how the basic flow depends upon ϕ and how differentiation with respect to ϕ enters into the problem. It is obvious that for $\varepsilon = 0$ and $\frac{\partial}{\partial\phi} \equiv 0$, this model equation has the form of the classical stability equation for the case of a small gap:

$$(D^2-\alpha^2)(D^2-\alpha^2-\sigma)^2 v + \alpha^2 T V_0(x) v = 0$$

If the "parallel-flow" assumption is used, then the expression $[V_0(x)+\varepsilon V_1(x)\cos\phi]$ would be regarded as a function of X only and with ϕ as a parameter, while $\frac{\partial}{\partial\phi}$ had to be neglected. In this case the eigenvalue T of the homogeneous differential system would depend upon ϕ implicitly.

Di Prima (1963) and Ritchie (1968) made calculations of this kind, and they obtained a value of T for each value of ϕ, that means a local criterion of instability. As R_a is small in lubrication theory ($R_a = \frac{1+\mu}{2} R_M$) it would appear that $R_a V_0(x) \frac{\partial}{\partial\phi}$ could be neglected, but exactly that approximation would bring in again the "parallel-flow approximation".

Di Prima and Stuart found a way out of this difficulty by allowing $\frac{\partial}{\partial\phi}$ to take an adequate magnitude, relative to the ϕ dependence dictated by the term $\varepsilon\cos\phi$.

From the equations

$$R_a = \frac{1}{2}(1+\mu) R_M = \frac{1}{2}(1+\mu)\left(\frac{g_M a}{\nu}\right)\alpha^2$$

and $$T = \left(\frac{g_M a}{\nu}\right)^2 \alpha^3 (1-\mu^2)$$

it can be seen that $R_a \equiv (T\alpha/2c)^{1/2}$

The model equation indicates that the response of the term proportional to $\frac{\partial}{\partial\phi}$ is proportional to the ϕ dependence dictated by $\varepsilon\cos\phi$ if $R_a \approx \varepsilon \approx \alpha^{1/2}$, and as a result they set $\alpha^{1/2} = (2c)^{1/2} k\varepsilon$ where $(2c)^{1/2}$ is put for convenience and k is a fixed parameter. By expanding u_1, v_1, w_1, p_1 and T in powers of ε as follows:

$$u_1 = u_{10}(x,\phi) + \varepsilon u_{11}(x,\phi) + \varepsilon^2 u_{12}(x,\phi) + \cdots$$

$$v_1 = v_1(x,\phi) + \varepsilon v_{11}(x,\phi) + \varepsilon^2 v_{12}(x,\phi) + \cdots$$

$$w_1 = w_{10}(x,\phi) + \varepsilon w_{11}(x,\phi) + \varepsilon^2 w_{12}(x,\phi) + \cdots$$

$$p_1 = p_{10}(x,\phi) + \varepsilon p_{11}(x,\phi) + \varepsilon^2 p_{12}(x,\phi) + \cdots$$

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots$$

and by taking the solution of the simplified perturbation equations of order $\varepsilon^0, \varepsilon$ and ε^2

they obtained the following result:

$$u_1 = -B(\phi)\left\{f_0(x) + \varepsilon\left[f_1\cos\phi + f_0\left(\Gamma_1\cos\phi + \frac{\Gamma_2}{2kT_0^{1/2}}\sin 2\phi\right)\right]\right\}$$

$$v_1 = B(\phi)\left\{g_0(x) + \varepsilon\left[g_1\cos\phi + g_0\left(\Gamma_1\cos\phi + \frac{\Gamma_2}{2kT_0^{1/2}}\sin 2\phi\right)\right]\right\}$$

$$w_1 = \lambda^{-1}B(\phi)\left\{Df_0(x) + \varepsilon\left[Df_1\cos\phi + (\Gamma_1 - 1)Df_0\cos\phi + Df_0\left(\frac{\Gamma_2}{2kT_0^{1/2}}\right)\sin 2\phi - c\Gamma g_0\cos\phi\right]\right\}$$

and the Taylor number is given by

$$T = T_0 + \varepsilon^2(T_{21} + k^2 T_{22})$$

where T_{21} is the eccentricity effect and T_{22} the effect of surface curvature, and $B(\phi) = B_0(k)\, e^{\frac{\Gamma}{kT_0^{1/2}}(\sin\phi - 1)}$

In all this analysis there are two small parameters δ (or α) the clearance ratio and ε, the eccentricity.

The problem was solved by considering an appropriate limit as $\delta \to 0$ and $\varepsilon \to 0$ for fixed T, in such a manner as to take proper account of the ϕ variation.

For the limit case $k \to \infty$, which implies that ε is very much smaller than $\delta^{1/2}$, one retrieves the concentric case $B = B_0(\infty)$

Di Prima and Stuart (1975) dealt with the nonlinear calculation of Taylor-vortex flow between two long eccentric rotating cylinders. Researchers are confronted with this problem in lubrication technology,

and it represents an extension of Di Prima and Stuart's earlier work (II) on the linear eccentric and of even earlier work done by Davey and others on the nonlinear concentric research of Taylor-vortex instability. In order to develop a consistent perturbation scheme, relationships are specified between the four small parameters in the analysis: the mean clearance ratio δ, the eccentricity ε, the Taylor vortex amplitude and the amount by which the Taylor number exceeds its critical value, and in this way a global solution to the nonlinear stability problem is obtained.

In the nonlinear analysis the authors were confronted with an additional difficulty, namely to take proper account for the large pressure development due to Reynold's lubrication effect. It is known from the lubrication theory that the requirement of no-slip boundary condition provides for the development of an additional and usually strong pressure field, and Reynolds (1886) lubrication equation for the pressure can be derived from that requirement. They found a way out of this difficulty by solving the continuity equation to a higher order than is necessary for the momentum equations. This difficulty was also anticipated from previous papers on the concentric problem (Stuart 1958; Davey 1962; Reynolds and Potter 1966; Kirchgässner and Sorger 1969), that an additional component of the circumferential mean flow is developed by the Reynold stresses of the Taylor vortices.

The basic nonlinear equations in the paper of Di Prima and Stuart (1975) follow the paper (1972b) but with some differences in notation. As a model equation they considered the steady equation

$$\nabla^2\left(\nabla^2 - R_a\frac{\partial}{\partial\phi}\right)^2 v - T(1+\varepsilon\cos\phi)\frac{\partial^2 v}{\partial\xi^2} = \frac{\partial v^2}{\partial x}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\xi^2}$

This model equation is different from the model equation in (1972b), since it includes a quadratic nonlinear term on the right-hand side and the replacement of $-\lambda^2 v$ by $\frac{\partial^2 v}{\partial \xi^2}$, while in (1972b) the linear form of the model equation was solved by the following expansion in ε :

$$v = A\left[v^{(0)}(x,\phi) + \varepsilon v^{(1)}(x,\phi) + O(\varepsilon^2)\right]\cos\lambda\xi$$

and $T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots\cdots$

where $v^{(0)}, v^{(1)}$ are functions and T_0, T_1, T_2 are numbers to be determined, in the nonlinear analysis it is necessary to determine A and to allow for the generation of harmonics and mean effects caused by the nonlinear term.

While the fundamental $\cos\lambda\xi$ has small amplitude A, the mean and first harmonic $\cos 2\lambda\xi$ appear with magnitude A^2, the correction to the fundamental and correction will have magnitude A^3. Since for the linear problem the amplitude function $B(\phi)$ is determined with $v^{(1)}$ at order εA, and for the nonlinear effect on the fundamental, which is of order A^3, there is need to make the identification $\varepsilon A = A^3$ in order to allow for a simultaneous correction due to eccentricity with the nonlinear effect on the fundamental. Consequently, they wrote the velocity field as follows:

$$u = \varepsilon^{1/2}\left[u_0(x,\phi,\xi) + \varepsilon^{1/2}u_1(x,\phi,\xi) + \varepsilon u_2(x,\phi,\xi) + \varepsilon^{3/2}u_3(x,\phi,\xi) + \varepsilon^2 u_4(x,\phi,\xi)\right] + O(\varepsilon^3)$$

$$v = \varepsilon^{1/2}\left[v_0(x,\phi,\xi) + \varepsilon^{1/2}v_1(x,\phi,\xi) + \varepsilon v_2(x,\phi,\xi) + \varepsilon^{3/2}v_3(x,\phi,\xi) + \varepsilon^2 v_4(x,\phi,\xi)\right] + O(\varepsilon^3)$$

$$w = \varepsilon^{1/2}\left[w_0(x,\phi,\xi) + \varepsilon^{1/2}w_1(x,\phi,\xi) + \varepsilon w_2(x,\phi,\xi) + \varepsilon^{3/2}w_3(x,\phi,\xi) + \varepsilon^2 w_4(x,\phi,\xi)\right] + O(\varepsilon^3)$$

Lubrication theory requires that p should be expanded as

$$p = \frac{T^{1/2}\varepsilon^{-2}}{4ch^3}\left[\int_0^\phi q_0(\phi)\,d\phi + \varepsilon\int_0^\phi q_1(\phi)\,d\phi + O(\varepsilon^2)\right] +$$

$$+\varepsilon^{1/2}\left[p_0(x,\phi,\xi)+\varepsilon^{1/2}p_{11}(x,\phi,\xi)+\varepsilon p_2(x,\phi,\xi)+\varepsilon^{3/2}p_3(x,\phi,\xi)+\varepsilon^2 p_4(x,\phi,\xi)+O(\varepsilon^{5/2})\right]+ \text{constant}$$

By solving the equations of order $\varepsilon^{1/2}$, order ε and order $\varepsilon^{3/2}$ the following perturbation velocity field is obtained

$$u=-\varepsilon^{-1/2}B(\phi)f_0(x)\cos\lambda\xi_f-\varepsilon B^2(\phi)f_{12}(x)\cos 2\lambda\xi_f+O(\varepsilon^{3/2})$$

$$v=\varepsilon^{1/2}B(\phi)g_0(x)\cos\lambda\xi_f+\varepsilon\left[B^2(\phi)g_{10}(x)+(3Q_0\Gamma_3T_1/\Gamma_4)(x^2-\tfrac{1}{4})+\right.$$
$$\left.+B^2(\phi)g_{12}(x)\cos 2\lambda\xi_f\right]+O(\varepsilon^{3/2})$$

$$w=\varepsilon^{1/2}\lambda^{-1}B(\phi)Df_0(x)\sin\lambda\xi_f+\frac{\varepsilon}{2\lambda}B^2(\phi)Df_{12}(x)\sin 2\lambda\xi_f+O(\varepsilon^{3/2})$$

The above mentioned functions f and g are connected to those of Davey's (1962) paper ($\mathcal{M}\to 1, m=0$) as follows:

$$f_0(x)\equiv\bar{u}_1(D),\ g_0(x)\equiv v_1(D),\ f_{12}(x)\equiv-\tfrac{1}{2}\bar{u}_2(D),\ g_{12}(x)=-\tfrac{1}{2}\bar{v}_2(D),$$

$$g_{10}(x)=-\tfrac{1}{2}F_0(x)-3Q_0(x^2-\tfrac{1}{4})=-\tfrac{1}{2}F_1(D)-3Q_0(x^2-\tfrac{1}{4})$$

where D is a symbol for Davey's functions.

As far as the pressure perturbation is concerned:

$$p=-\frac{3T^{1/2}\varepsilon^{-2}}{2ck^3}Q_0\left[\frac{k}{\omega_4'}\ln\left\{1+\frac{(E(2\pi)-1)\int_0^{\phi}E(\phi)d\phi}{\int_0^{2\pi}E(\phi)d\phi}\right\}-\omega_3\frac{T_1\phi}{\omega_4}+O(\varepsilon)\right]-$$

$$-\varepsilon^{1/2}\lambda^{-2}B(\phi)(D^2-\lambda^2)Df_0(x)\cos\lambda\xi_f+\varepsilon\left[B^2(\phi)\ell_{10}(x)+\frac{6Q_0\omega_3T_1}{\omega_4}\right]\ell_{100}(x)+$$

$$+B^2(\phi)\ell_{12}(x)\cos 2\lambda\xi_f\Big]+O(\varepsilon^{3/2})$$

The linear theory is retrieved by letting $T_1\to 0$ with k fixed. The constant $B_0(k)$ of the linear theory is determined

$$B_0^2(k,T_1)=\frac{T_1\omega_3 e^{\frac{\omega}{k}}}{\omega_4 I_0(\frac{\omega}{k})}$$

If the alternative or second limit $T_1\to\infty$ is considered, the nonlinear solutions of Stuart (1958) and Davey (1962) are retrieved. The parameter T_1 lies in the range $0\le T_1<\infty$. The lower limit yields a non-

linear alteration to the results of linear theory for the eccentric case, while the upper end represents an approach to the concentric solution.

Like the works of Di Prima and Stuart (1972a), (1972b), (1974) and (1975), Hall's Ph. Thesis (1973) is based on the assumption of (the rate of change) small and (the magnitude deviation) small. Hall in Part II of his Ph.D. Thesis (1973) investigates the stability of the flow between concentric cylinders, when the outer cylinder is at rest and the inner one has angular speed $\Omega_1(1+\varepsilon\cos\omega t)$. He obtains solutions of the linear and non-linear stability problems in the limits of ω tending to zero and infinity. Using the same methods as for the cylinders problem, he deals with the related Benard convection problem.

Later we shall summarize some works based on the assumption of (the rate of change) small and (the magnitude deviation), defined by $\varepsilon = O(1)$. In all these works, known better as the "slowly-varying" problems, the WKB method was used. In the following thesis there is an analysis for linear and non-linear stability problems in the flow between two rotating eccentric cylinders. In the linear problem there is an analysis of the wavy vortices between two eccentric cylinders. This mathematical analysis can be seen as a contribution to the previous papers of Roberts (1965), Di Prima (1963), Davey, Di Prima and Stuart (1968) and the paper of Di Prima and Stuart (1972b). The results are compared with the experimental results of Vohr (1968) for eccentric cylinders and a qualitative similarity has been found. Nevertheless, a difference in magnitude from experiment is obtained due mainly to the absence of nonlinear effects. The analysis of the wavy vortices of the flow between two eccentric cylinders where the Taylor vortex is absent, is based on the assumption of ε, the eccentricity, small and δ, the clearance ratio, very small. The method of

Di Prima and Stuart's paper (1972b) has been extended by the inclusion of the time variable τ in the amplitude equation for $B(\phi,\tau)$. By defining a new time variable τ_1 as follows:

$$\tau_1 = R_a \tau$$

the mathematical relation $t = \frac{\tau_1}{\frac{1}{2}(\Omega_1+\Omega_2)}$ is obtained,

where $\frac{1}{2}(\Omega_1+\Omega_2)$ is the mean angular velocity.

By defining also the growth rate $\sigma \to R_a \frac{\partial}{\partial \tau_1}$, the new time variable τ_1 is included in the amplitude equation, giving the following linear partial differential equation in the azimuthal and new time variables:

$$k T_0^{1/2}\left[\frac{\partial B(\phi,\tau_1)}{\partial \phi} + \bar{c}\,\frac{\partial B(\phi,\tau_1)}{\partial \tau_1}\right] = (\Gamma\cos\phi + T_1\bar{\Gamma}_1)B(\phi,\tau_1)$$

These new definitions are connected with travelling wave considerations, while the definitions of Di Prima and Stuart (1972b) follow from perturbation growth considerations. The neutral case of this problem is considered and the final result obtained is as follows:

$$T = T_0 + \varepsilon^2(T_{21} + k^2 T_{22}) + \varepsilon^2 m^2 k^2 T_0 \quad \frac{\Gamma_{10}}{\bar{\Gamma}_1} = T_0 + \varepsilon^2(T_{21} + k^2 T_{22}) + k_D^2 T_{C1}$$

where k_0 is the parameter used by Davey, Di Prima and Stuart (1968) when no eccentricity was present. In conclusion T is dependent on m. and the result can be seen as an addition of time dependence and waviness to Di Prima and Stuart's problem (1972), or as an addition of eccentricity and surface curvature effects to Davey, Di Prima and Stuart's problem (1968), that means as an unification of Di Prima and Stuart and Davey, Di Prima and Stuart's papers, where the various effects are linearly additive in T_2.

The nonlinear part of the thesis deals with an interaction between a nonlinear Taylor vortex mode, with amplitude $B(\phi)$ and a linear wavy vortex mode, with amplitude $A(\phi)$, in the flow between two eccentric rotating cylinders. It can be looked upon this problem as an extension of the paper of Davey, Di Prima and Stuart (1968) where a similar problem has been treated in the concentric case.

$v = v' + v_{TV}$ is defined, where v_{TV} is the Taylor vortex mode and

v' is the wavy vortex mode.

A new parameter $\gamma = T - T_0 - \varepsilon^2 T_2(0) = T - T_0 - \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1}$, which is the amount above the critical Taylor number, is also introduced.

The mathematical analysis is based on the approximation $\gamma \cong \varepsilon T_1$, where the Taylor vortex mode is expanded in powers of γ as follows:

$$v_{TV} = \frac{\gamma^{1/2}}{T_1^{1/2}} B(\Phi) g_0(X) \cos\lambda\xi + \frac{\gamma}{T_1}\left[B^2(\Phi) g_{10} + \frac{\sigma_0}{2}(X^2 - \frac{1}{4}) + \cos 2\lambda\xi \cdot B^2(\Phi) g_{112}\right] + \cdots$$

$$= \gamma^{1/2} v_{TV1} + \gamma v_{TV2} + \cdots\cdots$$

The wavy vortex mode and growth rate $\bar{\sigma}$ are also expanded in powers of γ as follows:

$$v' = v^{(1)} + \gamma^{1/2} v^{(2)} + \gamma v^{(3)} + \cdots\cdot$$

$$\bar{\sigma} = \bar{\sigma}_0 + \gamma^{1/2} \bar{\sigma}_1 + \gamma \bar{\sigma}_2 + \cdots$$

where v' has a $e^{i\bar{\sigma}\tau}$ dependence on the time τ.

Since some inconsistencies appeared by expanding directly in the eccentricity ε, we found that a possible way to solve this problem was to use a double parameter expansion; first by expanding in $\gamma^{1/2}$ from γ^0 to γ order in the equations, and later from ε^0 to ε^2 in certain integrals, the parameter γ having a dual role, in the sense of being a parameter, and secondly, being connected with ε as well.

The momentum and continuity equations will give us a nonhomogeneous sixth order system of differential equations, in the radial and azimuthal co-ordinates. In order to assure a solution of this problem, an orthogonality condition involving the suitably chosen partial differential equations adjoint has to be satisfied as follows:

$$-\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} u_j L^+_{ij} v_i^+ \, dx\, d\phi = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} v_i^+ S_i \, dx\, d\phi = 0$$

where $L_{ij} u_j = S_i$ and L^+_{ij} is the adjoint differential operator.

The final result obtained by using the condition of stability $Re[i\bar{\sigma}] = 0$ is as follows: $T'(m) = T_0 + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1} + \frac{k_0^2 T_{c1}}{\ell_0 k_0^2 + \varepsilon^2 F_{cs}(k, T_1)}$

where the second term $\varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1}$ represents Di Prima and Stuart's (1972b) linear Taylor vortex effect and the last term represents a nonlinear Taylor vortex effect, $F_{cs}(k, T_1)$ being a complicated numerical function of k and T_1.

For the limit case $T_1 \to 0$ one retrieves the formula of Di Prima and Stuart (1972b), while for the simultaneous limits $T_1 \to \infty$, $\varepsilon \to 0$ and m *large* (connected by $\varepsilon T_1 = \gamma$), one retrieves the formula

$$T'(m) = T_0 + \frac{T_{c1}}{\ell_0} = T_0 + 125$$

of Davey, Di Prima and Stuart's (1968) paper.

A different kind of approach to the concentric case could be found by letting $k \to \infty$.

It can easily be seen, that the m^2 term appearing in the formula for the critical Taylor number of Davey, Di Prima and Stuart (5.22) does not appear in the present one, and to get the correction term in m^2 there is need to go to a higher order expansion, namely in $\gamma^{1/2}$ from γ^0 to γ^2 in the equations, and later in ε from ε^0 to ε^4 in certain integrals.

Before bringing a short review on the slowly varying problems, there is a need to mention the wavy vortex geometry. Cole (1974 - Nature Dec. 20/27) illustrated the crucial effect that the cylinder (length/clearance) ratio (L/C), has on the wavy vortex stability, for concentric cylinders.

He brought some important revelations in this field, especially that no stability analysis seem to exist for an annulus of finite length and most experimental researchers preferred to work with long cylinders although short cylinders are more likely to be met in engineering applications. Cole concentrated on two kinds of experiments to find out the effect of annulus length L on critical speed N, namely (i)

visual observations and (ii) torque measurements. His visualisation results were for a rotor of diameter 91.42 mm concentrically positioned inside a stator of bore 103.2 mm, effective length 250 mm and thickness 19 mm, so that $R_1/R_2 = 0.894$. The test fluid was a silicone oil of kinematic viscosity $\nu = 19.3$ centistoke, and the vortices were made visible by adding some small aluminium particles. In the case of the visualisation technique the critical speeds were determined with an accurate precision of 0.5%, the annulus length being varied by changing the depth of the test fluid, where the lower end is a stationary solid surface and the upper end a free liquid surface.

Figure I illustrates that the cylinder length has a small influence on the speed at which Taylor vortices appear but a crucial effect on the speed at which wavy vortices appear: at L/C = 25 the second critical speed is about 10% higher than the calculated value and at L/C = 10 it is about 40% higher.

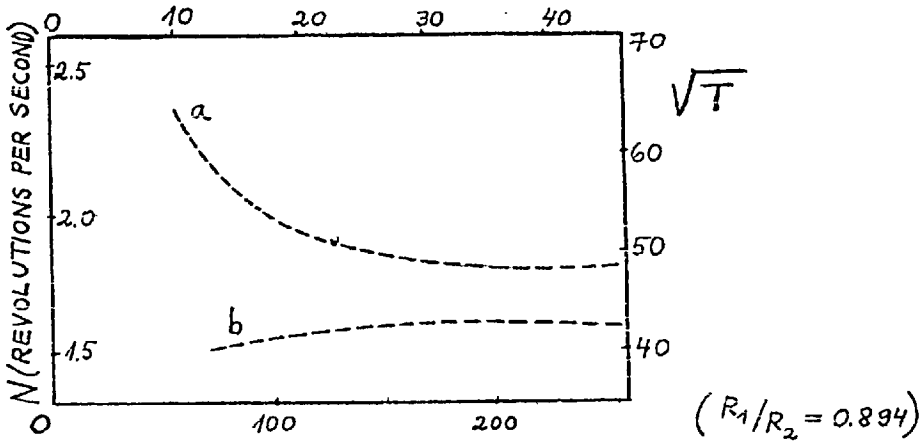


Fig. I Effect of the cylinder length L on critical speeds determined from visual observations (a) wavy vortices
(b) Taylor vortices

The results in the case of torque measurements ($R_1/R_2 = 0.954$) are generally similar to those in Fig. I, with the only marked difference that the very small variation in the Taylor critical speed with length is of opposite sign to that in Fig. I.

The visualisation observations as well as the torque measurements are in complete agreement in illustrating little effect of annulus length on the first critical speed but a crucial effect on the second critical speed or speed for the onset of wavy vortices. Consequently the theoretical research based on the assumption of infinite length is far less accurate for the wave instability than for the Taylor instability.

Some mention is needed about the slowly varying problems, where generally the rate of change is small and the magnitude deviation = $0(1)$, the usual method of analysis is the WKBJ method. In general, the determination of the stability of a given flow to a perturbation requires the evaluation of a system of partial differential equations. It is understood that any general perturbation can be split into normal modes, the stability or instability of the flow depending on the behaviour of the least stable mode.

In the case that the mean flow field is defined as a function of only one spatial variable, the system of partial differential equations defining the perturbation can be separated, and the problem is reduced to that of solving a system of ordinary differential equations. Plane Poiseuille flow can be taken as an example of such a flow.

Rosenblat and Herbert (1970) consider thermal convection in which the temperature of one boundary is slowly oscillating in time, or more explicitly they deal with a Boussinesq fluid heated from below, where the applied temperature gradient consists of a steady and a low-frequency sinusoidal component.

Their analysis was undertaken to achieve two objectives: (i) to find an asymptotic solution of the problem which describes the behaviour of infinitessimal disturbances to this configuration in the case of very small frequency, with arbitrary amplitude ratio; (ii) to find an interpretation of this solution from the point of view of the stability.

They compared their results with some experimental results.

In practice we are often confronted with cases where the dependence on a second variable is weak and the approximation is made that the stability characteristics are in some sense related to those of the equivalent one-dimensional system. As an example of this type can be cited boundary layers, where the mean flows are almost one-dimensional and their stability characteristics are based on the approximation that the flow is parallel over some region. The ordinary differential equation obtained after the separation is known as the Orr-Sommerfeld equation.

The difficulty which has to be overcome in correctly formulating the stability problem for almost parallel flows has led to a serious research effort being directed towards genuine parallel flows. For a parallel mean flow defined by $U(y)$ the adequate perturbation stream function is given by

$$\psi(x, y, t) = \phi(y) \exp i(\alpha x - \beta t)$$

where the wavenumber α is in general real and $\beta = \beta_r + i\beta_i$, the frequency parameter, is complex. A positive value of β_i will denote instability to the perturbation. Schlichting obtained the eigenvalues α and β and the eigenfunction $\phi(y)$ for a range of Reynolds number R, by using the parallel mean flow approximation for the flat-plate boundary layer. In the Orr-Sommerfeld equation the Reynolds number is considered as a parameter. Schlichting's suggestion based on the local growth-rate information was that the overall amplification of a disturbance of a fixed frequency was given by the following integral

$$\int_{x_0}^{x_1} \frac{\beta_i(x)}{\partial\beta_r(x)/\partial\alpha}\, dx$$

where the factor $\frac{\partial\beta_r}{\partial\alpha}$ relates the spatial growth in the physical situation to the temporal growth employed in the theory. A better descrip-

tion of the behaviour of regular periodic waves is given by the spatial modes, for which α is treated as complex and β as real (see Gaster 1965). The growth of these waves is given by $-\int_{x_0}^{x_1}\alpha_i(x)dx$. Gaster (1962) has shown that the two formulae for the growth rate are equivalent in the case of weakly amplified or damped waves. Schubauer and Skramstad related the measured growth with theory by obtaining the amplification form $\int \frac{\beta_i(\alpha)}{\beta_r/\alpha} dx$, where β_r/α is the phase velocity. The differences between all these relations are less significant than errors introduced by using eigenvalues obtained from the asymptotic solutions of the Orr-Sommerfeld equation.

In the analysis on wave packets (Gaster and Grant 1974) it was necessary to obtain a theoretical estimate of the phase as well as the amplitude for each travelling wave mode. The stream function chosen to define these waves was given by

$$\Psi = A(x)\Psi(y,x)\exp i\{\int\alpha(x)dx - \beta t\}$$

where α and Φ were functions of x evaluated locally from the Orr-Sommerfeld equation. The factor A has been included to take some account of the variations of the wavenumber and eigenfunction with A conservation relation such as wave action could enable the determination of $A(x)$. This is the kind of technique commonly used to reduce a partial differential equation with a weak dependence on one variable to an ordinary differential equation. Bouthier (1972) deals with the stability of almost parallel flows by using the WKB method. He obtains perturbation waves which can be locally identified as Tollmien-Schlichting waves, but the characteristics of the waves such as phase, amplitude and propagation speed are slowly varying in the flow. A special analysis is done for the case of the Blasius boundary layer. Finally the asymptotic behaviour of a wave packet in an almost parallel

flow is given. In the second paper (1973) Bouthier applies the slowly varying approximation to the stability of the boundary layer on a flat plate. The numerical results show that the rate of amplification is maximum at the wall; consequently the instability regions are two-dimensional and do not longer spread to infinity in the potential flow, moreover these results indicate increased destabilisation of the inner boundary layer and fit better with the experimental data than those of previous theories. Bouthier chose to define the amplification rate using the growth of the local kinetic energy $\bar{u}^2+\bar{v}^2$. He obtained the lowest Reynolds number where the amplification first reached zero and an upper limit where the growth was positive at every point through the boundary layer.

Gaster (1974) describes the stability of small travelling wave disturbances in the flow over a flat plane. A direct perturbation expansion in the small parameter $R^{-1/2}$ characterising the mean flow divergence is used. The Orr-Sommerfeld parallel flow solution has been used as a zero-order approximation to the full equations of motion and a correction in the form of an amplitude scaling function has been obtained together with an inhomogeneous equation for the first correction to the eigenfunction. This equation has not been solved and consequently the overall motion has only been obtained to zero order, nevertheless, the determination of the amplification rate and wavelength, which is based on the knowledge of the disturbance to lowest order, is made to order $R^{-1/2}$. The amplitude equation to this order is identical to that obtained by Bouthier.

Gaster obtained the neutral stability boundaries given by the first two terms of the asymptotic series solution and compares them with experimental data. He shows that by using the parallel flow approximation a valid solution is obtained at very large Reynolds numbers.

Eagles and Weissman (1975) deal with the linear stability of a slowly varying flow in a divergent channel by using a modification of the WKB ray or multiple scaling method.

Although the quasi-parallel approach has had great qualitative success (for example Schlichting) recent experiments (Ross et al. 1970; Scotti and Corcos 1972) have shown important differences from the quasi-parallel theory. From the theoretical viewpoint the quasi-parallel approach is deficient in two respects. First, the determination of wave growth or wave decay is made only at a particular point, and the solution cannot be determined as a function of the downstream co-ordinate. Second, it is impossible to estimate the effect of the variation of the basic state on the local growth. The WKB or multiple scaling method enables Eagles and Weissman to remove these deficiencies, and provides a good method of approximation in which the quasi-parallel theory gives the formally correct lowest-order growth rate. However, a measure of caution is needed because the growth rate can be substantially in error if its magnitude has a comparable size to that of the rate of change of the basic state. The method shows that the growth rate, neutral curves, wavenumber depend on the crossstream variable and on the flow quantity under consideration i.e. the stream function, velocity components, kinetic energy etc.

By following Shen's suggestion Eagles and Weissman use a growth rate based on the ratio of disturbance kinetic energy to basic flow kinetic energy. The higher-order corrections in the WKB or ray method which is applied to the divergent channel, show that, in general, the unstable region is substantially widened and the corresponding critical Reynolds number is much lower, than those predicted by quasi-parallel theory.

Eagles and Weissman found that waves of all frequencies eventually

decay as they travel downstream. However, in the unstable region, waves may become large enough for nonlinear effects to be important.

PART I

WAVY VORTICES IN THE FLOW BETWEEN TWO LONG ECCENTRIC ROTATING CYLINDERS : LINEAR THEORY

1.1 The Basic Laminar Flow

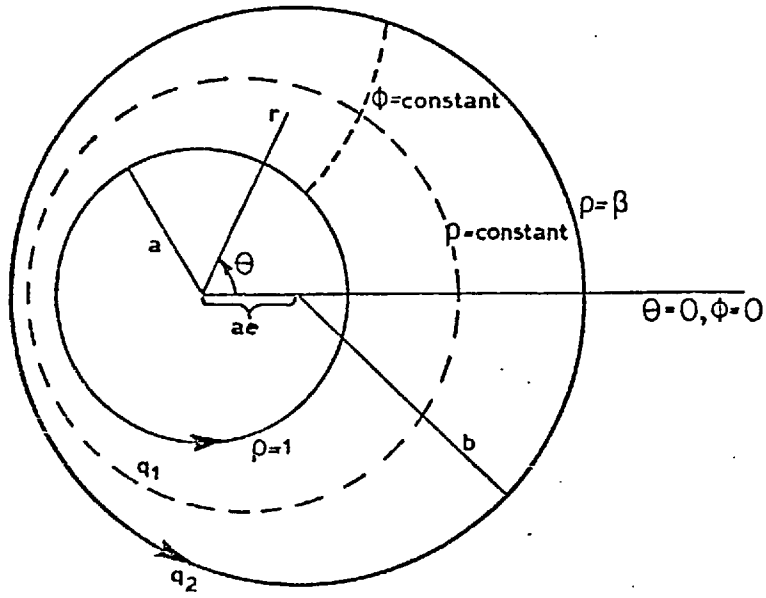


Fig.1 Geometry and co-ordinate systems.

Following Di Prima and Stuart (1972b) one considers two cylinders of radii a and b and linear speeds q_1 and q_2 measured in the anti-clockwise direction. The centres of the cylinders are set a distance ae apart, where $e=\varepsilon\delta$, $\delta=\frac{b-a}{a}$

and $0\leq\varepsilon<1$

Here ε denotes the eccentricity and δ is the mean clearance ratio. Instead of (r,θ) polar co-ordinates, the modified bipolar co-ordinates (ρ,ϕ) of Wood (1957) is used in this analysis. The curves ρ = constant are circles, the inner and outer cylinders being given by ρ = 1 and $\rho=\beta$.

One follows Wood and uses the conformal transformation

$$z=\frac{a(\zeta+\gamma)}{1+\gamma\zeta},\quad z=re^{i\theta},\quad \zeta=\rho e^{i\phi}$$

$$\gamma=\frac{-(1+\beta)+\sqrt{(1+\beta)^2-4\varepsilon^2\beta}}{2\varepsilon\beta}$$

$$\beta=\frac{1+\delta+\varepsilon\delta-\gamma}{1-(1+\delta)\gamma-\varepsilon\delta\gamma}$$

The ray $\phi = 0$ coincides with $\theta = 0$. In the limit case $\varepsilon \to 0$ an identity is obtained between the ρ, ϕ co-ordinates and r, θ co-ordinates, which is a clear advantage of this co-ordinate system compared with the usual bi-polar co-ordinate system. The Jacobian J of the transformation is given by

$$J = \frac{(1+2\gamma\rho\cos\phi+\gamma^2\rho^2)^2}{(1-\gamma^2)^2} \qquad (1.1.5)$$

and the two-dimensional length element is given by

$$ds^2 = dr^2 + r^2 d\theta^2 = \frac{a^2}{J} d\rho^2 + \frac{a^2\rho^2}{J} d\phi^2 \qquad (1.1.6)$$

Following Di Prima and Stuart (1972a) one has the definitions for the basic flow

$$u_\rho = \frac{\alpha q_1 J^{1/2}}{\rho} \frac{\partial\Psi}{\partial\phi}, \quad u_\phi = -q_1 J^{1/2} \frac{\partial\Psi}{\partial\rho} \qquad (1.1.7)$$

These definitions are based on the continuity equation

$$\frac{\partial}{\partial\rho}\left(\frac{\rho}{\sqrt{J}} u_\rho\right) + \frac{\partial}{\partial\phi}\left(\frac{u_\phi}{\sqrt{J}}\right) = 0 \qquad (1.1.8)$$

the boundary conditions are

$$\left.\begin{aligned} &u_\phi = q_1, \; u_\rho = 0 \quad \text{at } \rho = 1 \\ \text{and} \quad &u_\phi = q_2, \; u_\rho = 0 \quad \text{at } \rho = \beta \end{aligned}\right\} \qquad (1.1.9)$$

It is clear that the stream function Ψ can be introduced with

$$u_\rho = \frac{\sqrt{J}}{\rho} \frac{\partial\Psi}{\partial\phi}, \quad u_\phi = -\sqrt{J} \frac{\partial\Psi}{\partial\rho} \qquad (1.1.9a)$$

Assuming that the flow is steady and eliminating the pressure by cross differentiation of the first two momentum equations, we get the following equation for the vorticity ω in the axial direction:

$$\frac{1}{\rho}\left(\frac{\partial\Psi}{\partial\phi}\frac{\partial\omega}{\partial\rho} - \frac{\partial\Psi}{\partial\rho}\frac{\partial\omega}{\partial\phi}\right) = \frac{\nu}{a}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)$$

where (1.1.10)

$$\omega = -\frac{J}{a}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\Psi$$

In order to use the fact that the clearance ratio is extremely small in lubrication problems and also to make easier the algebraic calculations it is convenient to choose $\alpha = \beta - 1$ as the length scale

$$\alpha(\delta,\varepsilon) = \beta - 1 = \delta\sqrt{1-\varepsilon^2}\left[1 - \frac{\delta}{2}(1-\sqrt{1-\varepsilon^2})\right] + O(\delta^3) \qquad (1.1.11)$$

for δ small.

We let $\rho = 1+\alpha(x+\frac{1}{2})$ (1.1.12)

and introduce a dimensionless stream function Ψ and a dimensionless vorticity Ω by

$$\psi = \alpha q_1 \Psi, \quad \omega = \frac{q_1}{a\alpha}\Omega \qquad (1.1.13)$$

The definition $\psi = \alpha q_1 \Psi$ leads to (1.1.7)

The equation for the dimensionless axial vorticity Ω is given by

$$\frac{R_M}{\rho}\left(\frac{\partial\Psi}{\partial\phi}\cdot\frac{\partial\Omega}{\partial x}-\frac{\partial\Psi}{\partial x}\cdot\frac{\partial\Omega}{\partial\phi}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\alpha}{\rho}\frac{\partial}{\partial x}+\left(\frac{\alpha}{\rho}\right)^2\frac{\partial^2}{\partial\phi^2}\right)\Omega \qquad (1.1.8a)$$

where $$\Omega = -J\left(\frac{\partial^2}{\partial x^2}+\frac{\alpha}{\rho}\frac{\partial}{\partial x}+\left(\frac{\alpha}{\rho}\right)^2\frac{\partial^2}{\partial\phi^2}\right)\Psi \qquad (1.1.8b)$$

The parameter R_M the modified Reynolds number, is given by

$$R_M = \left(\frac{q_1 a}{\nu}\right)\alpha^2 \qquad (1.1.9)$$

with the boundary conditions

$$\left.\begin{aligned}&\frac{\partial\Psi}{\partial x} = -J^{1/2},\ \frac{\partial\Psi}{\partial\phi}=0 \text{ at } x=-\tfrac{1}{2}\\ &\frac{\partial\Psi}{\partial x} = -\mu J^{-1/2},\ \frac{\partial\Psi}{\partial\phi}=0 \text{ at } x=\tfrac{1}{2}\end{aligned}\right\} \qquad (1.1.10)$$

In accordance with the scaling $\rho = 1+\alpha(x+\frac{1}{2})$ the pressure p takes the form $$p = \frac{\mu q_1}{a\alpha^2}P \qquad (1.1.11)$$

where $\mu = \sigma\nu$ (1.1.11a) is the viscosity and P is the dimensionless pressure.

One purpose of the paper of Di Prima and Stuart (1972a) was to derive the Sommerfeld pressure distribution and associated flow as a solution of the Navier-Stokes equations in the limit $\alpha\to 0, R_M\to 0$ while the second purpose is to calculate the higher order corrections $O(\alpha)$ and $O(R_M)$ of the limiting solution. While in any lubrication problem it is possible to expect $\alpha \ll 1$, this is not necessarily true for R_M in modern technology despite the factor α^2 which appears in the definition. Beside the practical interest to determine the effect of the

inertia terms $O(R_M)$ on the load and torque, there is also a need to calculate the correction terms $O(\alpha)$ and $O(\alpha R_M)$ for the stability analysis of flow between eccentric rotating cylinders for $\delta \ll 1$. From the stability problem for flow between concentric rotating cylinders ($\alpha = \delta$ for $\varepsilon = 0$) it is known that the Taylor number is

$$T = \left(\frac{q_M a \alpha}{\nu}\right)^2 \alpha = \frac{1}{\alpha} R_M^2 \qquad (1.1.12)$$

For the stability problem for eccentric cylinders it would be appropriate for the case of small gap to study the limiting case $\alpha \to 0$ with T fixed, and this limiting process would require $R_M = O(\alpha^{1/2})$. Consequently there is an argument for calculating the basic velocity through terms $O(\alpha)$ and $O(R_M^2)$. Fortunately, since $R_M^2 \Psi_{20}$ is of order $\alpha\varepsilon$, it turns out that this term does not contribute in the stability analysis.

J is expanded in powers of α as follows:

$$J(X, \Phi; \varepsilon, \alpha) = J_0(\Phi; \varepsilon) + \alpha J_1(X, \Phi; \varepsilon) + O(\alpha^2) \qquad (1.1.13)$$

where $$J_0(\Phi; \varepsilon) = \frac{(1 - \varepsilon\cos\Phi)^2}{1 - \varepsilon^2} \qquad (1.1.14)$$

and $$J_1(X, \Phi; \varepsilon) = 2\frac{1 - \varepsilon\cos\Phi}{\sqrt{1-\varepsilon^2}}\left[\frac{-\varepsilon(\varepsilon - \cos\Phi)}{2(1-\varepsilon^2)} + \frac{1 - \varepsilon\cos\Phi - \sqrt{1-\varepsilon^2}}{\sqrt{1-\varepsilon^2}} X\right] \qquad (1.1.15)$$

The expansions for Ψ and P are of the form

$$\Psi = \Psi_{00}(X, \Phi; \varepsilon, \eta) + R_M \Psi_{10}(X, \Phi; \varepsilon, \eta) + \alpha \Psi_{01}(X, \Phi; \varepsilon, \eta) + O(\alpha^2, R_M^2, \alpha R_M) \qquad (1.1.16)$$

$$P = P_{00}(X, \Phi; \varepsilon, \eta) + R_M P_{10}(X, \Phi; \varepsilon, \eta) + \alpha P_{01}(X, \Phi; \varepsilon, \eta) + O(\alpha^2, R_M^2, \alpha R_M) \qquad (1.1.17)$$

The expansion of Ω obtained from equation (1.1.8b) is

$$\Omega = \Omega_{00}(X, \Phi; \varepsilon, \eta) + R_M \Omega_{10}(X, \Phi; \varepsilon, \eta) + \alpha \Omega_{01}(X, \Phi; \varepsilon, \eta) +$$
$$+ O(\alpha^2, R_M^2, \alpha R_M) = -J_0 \frac{\partial^2 \Psi_{00}}{\partial X^2} - R_M J_0 \frac{\partial^2 \Psi_{10}}{\partial X^2} -$$
$$- \alpha\left[J_0 \frac{\partial^2 \Psi_{01}}{\partial X^2} + J_1 \frac{\partial^2 \Psi_{00}}{\partial X^2} + J_0 \frac{\partial \Psi_{no}}{\partial X}\right] + \cdots \qquad (1.1.18)$$

Substituting these expansions into the differential equation (1.1.8a) and the boundary conditions (1.1.10) and making use of the expansion

of J, we obtain the three problems for Ψ_{oo}, Ψ_{10}, and Ψ_{01} given in the following. Acutally all these problems are derived from the momentum equations. It is also necessary to take into account that the pressure P is single valued in ϕ.

(i) The Reynolds approximation: $\alpha=0, R_M=0$

The general solution which satisfies the differential equation

$$\frac{\partial^4 \Psi_{oo}}{\partial x^4}=0 \qquad (1.1.19)$$

and boundary conditions

$$\left.\begin{aligned} &\frac{\partial \Psi_{oo}}{\partial \phi}=0, \frac{\partial \Psi_{oo}}{\partial x}=-\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\phi} \text{ at } x=0 \\ &\frac{\partial \Psi_{oo}}{\partial \phi}=0, \frac{\partial \Psi_{oo}}{\partial x}=-\frac{\eta\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\phi} \text{ at } x=1 \end{aligned}\right\} \qquad (1.1.20)$$

The general solution which satisfied the boundary conditions

$$\Psi_{oo}(x,\phi)=A_{ooo}-\frac{\sqrt{1-\varepsilon^2}}{2+\varepsilon^2}(\mu+1)(3x^2-2x^3)+\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos\phi}\left[-x+(\mu+2)x^2-(\mu+1)x^3\right] \qquad (1.1.21)$$

P_{oo} is periodic and single valued. This later condition requires

$$\int_0^{2\pi}\left(\partial P_{oo}/\partial\phi\right)d\phi=0 \qquad (1.1.21a)$$

$$P_{oo}(x,\phi)=\frac{6\varepsilon(\mu+1)(2-\varepsilon\cos\phi-\varepsilon^2)\sin\phi}{(2+\varepsilon^2)\sqrt{1-\varepsilon^2}}+G_{oo} \qquad (1.1.22)$$

A_{ooo} and G_{oo} are arbitrary constants, where G_{oo} arises from the integration of $\frac{\partial P_{oo}}{\partial\phi}$ with respect to ϕ.

(ii) The inertial or R_M correction

$$\Psi_{10}(x,\phi)=\sum_{n=0}^{3}A_{10n}(\phi)x^n-\frac{2\varepsilon(1-\varepsilon^2)\sin\phi}{(1-\varepsilon\cos\phi)^2}\sum_{n=0}^{3}\frac{n!B_{10n}(\phi)x^{n+4}}{(n+4)!} \qquad (1.1.23)$$

where the functions $A_{10n}(\phi)$ are determined so that the boundary conditions are satisfied.

$$P_{10}(\phi,\varepsilon)=-\frac{3\varepsilon(1+\mu)^2}{35(2+\varepsilon^2)}\cos\phi+\frac{1}{7}\left(1-\frac{4}{5}\mu+\mu^2\right)\ln(1-\varepsilon\cos\phi)-$$

$$-\frac{27(1+\mu)^2}{35(2+\varepsilon^2)^2}(1-\varepsilon\cos\phi)^2 \qquad (1.1.24)$$

Ψ_{10} is antisymmetric and the pressure P_{10} is symmetric about the line $\phi=0$

(iii) The curvature or α correction:

$$\Psi_{01}(x,\phi)=\sum_{n=0}^{3} A_{01n}(\phi)x^n + B_{010}(\phi)x^4 \qquad (1.1.25)$$

$$P_{01}(\phi,\varepsilon)=\left\{-\frac{8\varepsilon}{2+\varepsilon^2}+(1+\mu)\left[-\frac{\varepsilon}{1-\varepsilon^2}+\frac{6\varepsilon(2-\varepsilon^2)}{(2+\varepsilon^2)\sqrt{1-\varepsilon^2}}\right]\right\}\sin\phi +$$

$$+\left\{\frac{\varepsilon^2}{2+\varepsilon^2}+\frac{1+\mu}{2}\left[\frac{\varepsilon^2}{1-\varepsilon^2}-\frac{6\varepsilon^2}{(2+\varepsilon^2)\sqrt{1-\varepsilon^2}}\right]\right\}\sin 2\phi \qquad (1.1.26)$$

The same as Ψ_{00} and P_{00}, the stream function P_{01} is symmetric and the pressure P_{01} is antisymmetric about the line $\phi=0$.

We consider now the momentum equations and continuity equation of viscous flow in the dimensionless co-ordinate system (ρ,ϕ,ζ), where ζ is different from the complex number used in the conformal transformation (1.2)

$$\frac{du_\rho}{dt}-\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{a\rho}\right)u_\rho u_\phi+\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{a\rho}\right)u_\phi^2=-\frac{J^{1/2}}{a}\frac{\partial p}{\partial\rho}+$$

$$+\frac{\nu J^{1/2}}{a^2}\left\{\frac{\partial}{\partial\rho}\left[\frac{J}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho u_\rho}{J^{1/2}}\right)\right]+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left[J\frac{\partial}{\partial\phi}\left(\frac{u_\rho}{J^{1/2}}\right)\right]+\frac{1}{J^{1/2}}\frac{\partial^2 u_\rho}{\partial\zeta^2}\right\}+$$

$$+\frac{\nu\rho}{a^2}\frac{\partial(\rho^{-2}J,u_\phi)}{\partial(\rho,\phi)} \qquad (1.1.27)$$

$$\frac{du_\phi}{dt}-\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{a\rho}\right)u_\rho u_\phi+\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{a\rho}\right)u_\rho^2=-\frac{J^{1/2}}{a\rho}\frac{\partial p}{\partial\phi}+$$

$$+\frac{\nu J^{1/2}}{a^2}\left\{\frac{\partial}{\partial\rho}\left[\frac{J}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho u_\phi}{J^{1/2}}\right)\right]+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left[J\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{J^{1/2}}\right)\right]+\frac{1}{J^{1/2}}\frac{\partial^2 u_\phi}{\partial\zeta^2}\right\}-$$

$$-\frac{\nu\rho}{a^2}\frac{\partial(\rho^{-2}J,u_\rho)}{\partial(\rho,\phi)} \qquad (1.1.28)$$

$$\frac{du_\rho}{dt}=-\frac{1}{a}\frac{\partial p}{\partial\rho}+\frac{\nu}{a^2}\left\{\frac{J}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_\rho}{\partial\rho}\right)+\frac{J}{\rho^2}\frac{\partial^2 u_\rho}{\partial\phi^2}+\frac{\partial^2 u_\rho}{\partial\zeta^2}\right\} \qquad (1.1.29$$

$$\frac{J}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho u_\rho}{J^{1/2}}\right)+\frac{J}{\rho}\frac{\partial}{\partial\phi}\left(\frac{u_\phi}{J^{1/2}}\right)+\frac{\partial u_\rho}{\partial\zeta}=0 \qquad (1.1.30)$$

where

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\frac{J^{1/2}}{a}u_\rho\frac{\partial}{\partial\rho}+\frac{J^{1/2}}{a\rho}u_\phi\frac{\partial}{\partial\phi}+\frac{u_\rho}{a}\frac{\partial}{\partial\zeta} \qquad (1.1.31)$$

Travelling wave considerations require the following change in the definition of the time variable and of the non-dimensional growth rate σ: $\tau=\frac{\tau_1}{R_a}$ and $\sigma\to R_a\frac{\partial}{\partial\tau_1}$ (1.1.32) (See Introduction)

In order to discuss the linearised theory of instability for the wavy vortices, we shall write:

$$u_\rho = \frac{\alpha\varepsilon}{2}(q_{\rho 1}+q_{\rho 2})U(x,\phi) + \left(\frac{v}{a\alpha}\right)u_1(x,\phi,\tau_1)\cos\lambda\xi$$

$$u_\phi = \frac{q_{\rho 1}+q_{\rho 2}}{2}V(x,\phi) + (q_{\rho 1}-q_{\rho 2})v_1(x,\phi,\tau_1)\cos\lambda\xi$$

$$u_\rho = \left(\frac{v}{a\alpha}\right)w_1(x,\phi,\tau_1)\sin\lambda\xi$$

$$p = \left(\frac{v q_{\rho 1}}{a\alpha^2}\right)P(x,\phi) + \frac{v^2}{a^2\alpha^2}p'(x,\phi,\tau_1)\cos\lambda\xi$$

where $U = \left[\frac{2J^{1/2}}{\varepsilon\rho(1+\mu)}\right]\left(\frac{\partial\Psi}{\partial\phi}\right)$ and $V = \left(-\frac{2J^{1/2}}{1+\mu}\right)\left(\frac{\partial\Psi}{\partial x}\right)$

The following linearised stability equations result:

$$\varepsilon R_a J^{1/2}\left(U\frac{\partial u_1}{\partial x}+\frac{\partial U}{\partial x}u_1\right)+\frac{R_a J^{1/2}}{\rho}\left(V\frac{\partial u_1}{\partial\phi}+\varepsilon c R_a\frac{\partial U}{\partial\phi}v_1\right)-$$

$$-R_a\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)\left(Vu_1+\varepsilon c R_a U v_1\right)+\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)TVv_1 =$$

$$= -J^{1/2}\frac{\partial p_1}{\partial x}+\left(JL^2-\lambda^2-R_a\frac{\partial}{\partial\tau_1}\right)u_1-\alpha^2 H u_1 +$$

$$+cR_a\rho\left\{\alpha\frac{\partial}{\partial\rho}\left(\frac{I}{\rho^2}\right)\frac{\partial v_1}{\partial\phi}-\frac{\partial}{\partial\phi}\left(\frac{I}{\rho^2}\right)\frac{\partial v_1}{\partial x}\right\},$$

$$J^{1/2}\left(\frac{1}{c}\frac{\partial V}{\partial x}u_1+\varepsilon R_a U\frac{\partial v_1}{\partial x}\right)+\frac{1}{\rho}J^{1/2}R_a\left(V\frac{\partial v_1}{\partial\phi}+\frac{\partial V}{\partial\phi}v_1\right)-$$

$$-\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)\left(\frac{1}{c}Vu_1+\varepsilon R_a U v_1\right)+\frac{2\alpha^2\varepsilon}{c}\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)u_1 U =$$

$$= -\frac{J^{1/2}}{\rho}\frac{\alpha^2}{cR_a}\frac{\partial p_1}{\partial\phi}+\left(JL^2-\lambda^2-R_a\frac{\partial}{\partial\tau_1}\right)v_1-\alpha^2 H v_1 +$$

$$+\frac{\rho\alpha^2}{cR_a}\left\{-\alpha\frac{\partial}{\partial\rho}\left(\frac{I}{\rho^2}\right)\frac{\partial u_1}{\partial\phi}+\frac{\partial}{\partial\phi}\left(\frac{I}{\rho^2}\right)\frac{\partial u_1}{\partial x}\right\},$$

$$\varepsilon R_a J^{1/2}U\frac{\partial w_1}{\partial x}+\frac{1}{\rho}J^{1/2}R_a V\frac{\partial w_1}{\partial\phi}=\lambda p_1+\left(JL^2-\lambda^2-R_a\frac{\partial}{\partial\tau_1}\right)w_1,$$

$$J^{1/2}\frac{\partial u_1}{\partial x}-\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)u_1+cR_a\left[\frac{J^{1/2}}{\rho}\frac{\partial v_1}{\partial\phi}-\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)v_1\right]+\lambda w_1=0.$$

Here we have used the definitions:

$$L^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\alpha}{\rho}\frac{\partial}{\partial x} + \left(\frac{\alpha}{\rho}\right)^2 \frac{\partial^2}{\partial \phi^2} \qquad (1.1.42)$$

$$H \equiv \rho J^{1/2}\left\{\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right\}\left(\frac{J^{1/2}}{\rho}\right) \qquad (1.1.43)$$

$$R_a = \left(\frac{1+\mu}{2}\right) R_M = \left(\frac{1+\mu}{2}\right)\left(\frac{q_{M1} a}{\nu}\right)\alpha^2 \qquad (1.1.44)$$

$$T = \left(\frac{q_{M1} a}{\nu}\right)^2 \alpha^3 (1-\mu^2) \qquad (1.1.45)$$

$$c = 2\frac{1-\mu}{1+\mu} \quad \text{and} \qquad (1.1.46)$$

$$\mu = \frac{q_{M2}}{q_{M1}} \qquad (1.1.47)$$

The boundary conditions are that u_1, v_1 and w_1 are zero at $x = \pm\frac{1}{2}$, and that the solution has period 2π in ϕ. In this linearized theory we are dealing with the neutral case.

1.2 Perturbation equations

We expand the perturbation velocity and pressure field as follows:

$$\begin{aligned} u_1 &= u_{10}(x,\phi,\tau_1)+\varepsilon u_{11}(x,\phi,\tau_1)+\varepsilon^2 u_{12}(x,\phi,\tau_1)+\cdots \\ v_1 &= v_{10}(x,\phi,\tau_1)+\varepsilon v_{11}(x,\phi,\tau_1)+\varepsilon^2 v_{12}(x,\phi,\tau_1)+\cdots \\ w_1 &= w_{10}(x,\phi,\tau_1)+\varepsilon w_{11}(x,\phi,\tau_1)+\varepsilon^2 w_{12}(x,\phi,\tau_1)+\cdots \\ p_1 &= p_{10}(x,\phi,\tau_1)+\varepsilon p_{11}(x,\phi,\tau_1)+\varepsilon^2 p_{12}(x,\phi,\tau_1)+\cdots \end{aligned} \tag{1.2.1}$$

For the Taylor number we have also an expansion in the eccentricity ε :

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots \tag{1.2.2}$$

$$R_a = k\varepsilon T^{1/2} = k\varepsilon T_0^{1/2} + k\varepsilon^2 \frac{T_1}{2T_0^{1/2}} + \cdots \tag{1.2.3}$$

Since we shall calculate T only to order ε^2, we keep λ fixed. This assumption is based on a short analysis showing that the variation of λ from its critical value at $\varepsilon = 0(\lambda_0)$ does not affect T through terms of the order of ε^2. (See Chandrasekhar (1961) and Di Prima and Stuart (1972b)). We found that

$$T_{min} = T_0(\lambda_0)+\varepsilon^2 T_2(\lambda_0)+\varepsilon^4\left[T_4(\lambda_0)-\frac{\lambda_2^2}{2}\frac{\partial^2 T_0}{\partial\lambda^2}(\lambda_0)\right]+\cdots \tag{1.2.4}$$

where

$$\begin{cases} \lambda = \lambda_0 + \varepsilon^2\lambda_2 + \varepsilon^4\lambda_4 + \cdots \\ \lambda_1 = \lambda_3 = 0 \end{cases} \tag{1.2.5}$$

and λ_2, λ_4 are specified as follows:

$$\lambda_2 = \frac{-\dfrac{\partial T_2(\lambda_0)}{\partial\lambda}}{\dfrac{\partial^2 T_0(\lambda_0)}{\partial\lambda^2}} \tag{1.2.6}$$

$$\lambda_4 = -\frac{\lambda_2\dfrac{\partial^2 T_0}{\partial\lambda^2}(\lambda_0)+\dfrac{\lambda_2^2}{2}\dfrac{\partial^3 T_0}{\partial\lambda^3}(\lambda_0)+\dfrac{\partial T_4}{\partial\lambda}(\lambda_0)}{\dfrac{\partial^2 T_0}{\partial\lambda^2}(\lambda_0)} \tag{1.2.7}$$

We expand also the basic flow in powers of ε since U and V depend upon α, R_a and ε . We then have

$$V = V_0(X) + \varepsilon V_1(X,\Phi) + \varepsilon^2\left[V_{20}(X) + kT_0^{1/2} V_{21}(X,\Phi) + k^2c^2 V_{22}(X,\Phi)\right] + O(\varepsilon^3) \qquad (1.2.8)$$

$$U = U_0(X,\Phi) + O(\varepsilon) \qquad (1.2.9)$$

In these formulae V_0, V_1, V_{20} and U_0 come from the Ψ_{00} component of the stream function Ψ , while V_{21} comes from Ψ_{10} and V_{22} from Ψ_{01}. $V_0, V_1, V_{20}, V_{21}, V_{22}$ and U_0 have been calculated by Di Prima and Stuart (1972a) as follows.

$$V_0 = 1 - cX \qquad (1.2.10)$$

$$V_1 = 6\left(X^2 - \tfrac{1}{4}\right)\cos\Phi \qquad (1.2.11)$$

$$V_{20} = 3\left(X^2 - \tfrac{1}{4}\right) \qquad (1.2.12)$$

$$V_{21} = \left(X^2 - \tfrac{1}{4}\right)\left[\tfrac{1}{2}\left(1 - \tfrac{c^2}{20}\right)\left(\tfrac{1}{20} - X^2\right) - \tfrac{cX}{5}\left(\tfrac{7}{12} - X^2\right)\right]\sin\Phi \qquad (1.2.13)$$

$$V_{22} = X^2 - \tfrac{1}{4} \qquad (1.2.14)$$

$$U_0 = 2\left(X^2 - \tfrac{1}{4}\right)\left(X - \tfrac{c}{4}\right)\sin\Phi \qquad (1.2.15)$$

By taking into account all the previous expensions in ε , we obtain sets of simplified differential equations, for order ε^0, order ε and order ε^2 . These equations are solved in such a way as to take account of the no-slip conditions at the boundaries and the requirement of single-valuedness in Φ , moreover we study the neutral case, for which no growth or decay is present.

ORDER ε^0

By elimination of p_{10} and w_{10} we have:

$$\lambda^2 T_0 V_0 v_{10} = M^2 u_{10} \qquad (1.2.16)$$

$$-u_{10} = M v_{10} \qquad (1.2.17)$$

where $M \equiv D^2 - \lambda^2$ (1.2.18)

with the boundary conditions

$$u_{10} = v_{10} = Du_{10} = 0 \quad \text{at } x = \pm \frac{1}{2} \qquad (1.2.19)$$

This differential system yields an eigenrelation between λ and T_0. We write the solution in the following form:

$$u_{10} = -B(\phi, \tau_1) \cdot f_0(x) \,, \; v_{10} = B(\phi, \tau_1) \cdot g_0(x) \qquad (1.2.20)$$

where f_0 and g_0 together provide the eigenfunction of the system of ordinary differential equations:

$$\begin{cases} M^2 f_0 + \lambda^2 T_0 V_0 g_0 = 0 \\ f_0 - M g_0 = 0 \\ f_0 = Df_0 = g_0 = 0 \text{ at } x = \pm \frac{1}{2} \end{cases} \qquad (1.2.21)$$

Here $B(\phi, \tau_1)$ is an integration "constant" for the concentric Taylor vortex problem since $\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial \tau_1}$ do not occur, and ϕ and τ_1 are thus parameters here. The determination of $B(\phi, \tau_1)$ requires the consideration of higher order terms in the expansion, as we shall see. Di Prima and Stuart's paper (1972b) has been enlarged by the inclusion of the time variable τ_1 in the amplitude equation for $B(\phi, \tau_1)$.

ADJOINT EQUATION : ORDER ε^0

In order to find a solution to the systems of differential equations, an orthogonality condition involving the adjoint of the system has to be satisfied, namely:

The differential system adjoint to our previous system is given by:

$$\begin{cases} M^2 f_0^+ + g_0^+ = 0 \\ \lambda^2 T_0 V_0 f_0^+ - M g_0^+ = 0 \\ f_0^+ = D f_0^+ = g_0^+ = 0 \text{ at } x = \pm \frac{1}{2} \end{cases} \qquad (1.2.23)$$

The eigenvalues λ, T_0 and c are identical to those of the previous system, but the adjoint eigenfunction (f_0^+, g_0^+) is different from (f_0, g_0) as the form of the equations indicates.

ORDER ε

By elimination of p_{11} and w_{11} use of some previous relations of the $O(\varepsilon^0)$ system, and with the definition $\bar{u}_{11} = -u_{11}$, one obtains:

$$M^2 \bar{u}_{11} + \lambda^2 T_0 V_0 v_{11} = B(\phi, \tau_1)\cos\phi \cdot F_{111}(x) + k T_0^{1/2} \frac{\partial B(\phi,\tau_1)}{\partial \phi} F_{112}(x) + T_1 B(\phi,\tau_1) F_{113}(x) + k T_0^{1/2} \frac{\partial B(\phi,\tau_1)}{\partial \tau_1} F_{114}(x) \qquad (1.2.24)$$

$$\bar{u}_{11} - M v_{11} = B(\phi,\tau_1)\cos\phi \cdot G_{111}(x) + k T_0^{1/2} \left[\frac{\partial B(\phi,\tau_1)}{\partial \phi} G_{112}(x) + \frac{\partial B(\phi,\tau_1)}{\partial \tau_1} G_{114}(x) \right] \qquad (1.2.25)$$

$$\bar{u}_{11} = D\bar{u}_{11} = v_{11} = 0 \text{ at } x = \pm \frac{1}{2} \qquad (1.2.26)$$

where

$F_{111}(x), F_{112}(x), F_{113}(x), G_{111}(x), G_{112}(x)$ are given in the paper of Di Prima and Stuart (1972b) as

$$F_{114}(x) = M f_0(x) \qquad (1.2.27)$$

$$G_{114}(x) = -g_0 \qquad (1.2.28)$$

$$F_{111}(x) = -6\lambda^2 T_0 (x^2 - \tfrac{1}{4}) g_0 + 4 M D^2 f_0 \qquad (1.2.29)$$

$$G_{111}(x) = \left[\left(\frac{12x}{c} + 1 \right) \right] f_0 - 2 D^2 g_0 \qquad (1.2.30)$$

$$F_{112}(x) = V_0 M f_0 \qquad (1.2.31)$$

$$G_{112}(x) = -g_0 \qquad (1.2.32)$$

Once the system to $O(\varepsilon^0)$ has been solved, the right-hand sides of (1.2.24) and (1.2.25) are known, except for the values of τ_1, as yet unknown, and for the amplitude function $B(\phi,\tau_1)$ which is still undetermined. The two operators on the left-hand sides of (1.2.24) and (1.2.25) have no differentials with respect to ϕ and τ_1, consequently ϕ and τ_1 may be considered as parameters as far as (1.2.24) and (1.2.25) are concerned. At this point we have to use the adjoint function pair (f_0^+, g_0^+). Since the homogeneous system (1.2.24) - (1.2.26) has a non-trivial solution, it is known that the non-homogeneous problem will have a solution only if a certain orthogonality condition involving the solution of the adjoint problem is satisfied. That orthogonality condition requires that when we multiply the right-hand sides of (1.2.24) and (1.2.25) by f_0^+ and g_0^+ respectively, add and integrate over $(-\frac{1}{2},\frac{1}{2})$ the result must be zero. This gives a relation between $B(\phi,\tau_1)$, $\frac{\partial B(\phi,\tau_1)}{\partial\phi}$, $\frac{\partial B(\phi,\tau_1)}{\partial\tau_1}$, $\cos\phi$, k and τ_1.
Let us divide it by $I=\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{112}+g_0^+G_{112})dx$
we obtain the following partial differential equation

$$k T_0^{1/2}\left[\frac{\partial B(\phi,\tau_1)}{\partial\phi}+\bar{c}\frac{\partial B(\phi,\tau_1)}{\partial\tau_1}\right]=(\Gamma\cos\phi+\tau_1\bar{\Gamma}_1)B(\phi,\tau_1) \quad (1.2.33)$$

where
$$\bar{c}=\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{114}+g_0^+G_{114})dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{112}+g_0^+G_{112})dx} \quad (1.2.34)$$

To check that $(-\frac{1}{c})$ is the wave velocity, we did the numerical calculations and we obtained that the wave velocity is approximately 1.05, which is a similar result to Davey, Di Prima and Stuart's (1968). (See Table 3). Γ and $\bar{\Gamma}_1$ are defined as follows:

$$\Gamma=-\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{111}+g_0^+G_{111})dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{112}+g_0^+G_{112})dx} \quad (1.2.35)$$

$$\bar{\Gamma}_1=-\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}}f_0^+F_{113}(x)dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+F_{112}+g_0^+G_{112})dx} \quad (1.2.36)$$

Γ is the same as Γ used by Di Prima and Stuart (1975) while $\bar{\Gamma}_1$ corresponds to Γ_3 used by them. Their numerical values are $\Gamma = 23.09$ and

$$\bar{\Gamma}_1 = \Gamma_3 = 0.007356 \qquad (1.2.37)$$

We have a first-order linear partial differential equation for $B(\phi, \tau_1)$. The coefficients can be evaluated in terms of the $O(\varepsilon^0)$ solution and of the adjoint equation. By solving the differential equation with characteristics:

$$B(\phi, \tau_1) = B(\phi, \eta) \qquad (1.2.38)$$

(η: parameter)

where $$\eta = \phi - \frac{\tau_1}{c} \qquad (1.2.39)$$

and by taking into account the periodicity condition in ϕ:

$$B\left(2\pi, 2\pi - \frac{\tau_1}{c}\right) = B\left(0, -\frac{\tau_1}{c}\right) \qquad (1.2.40)$$

we obtain the following solution for the amplitude $B(\phi, \tau_1)$:

$$B(\phi, \eta) = B_{00}\, e^{\frac{\Gamma}{kT_0^{1/2}}(\sin\phi - 1) + \frac{T_1 \bar{\Gamma}_1}{kT_0^{1/2}}(\phi - \eta)} \cdot A(\eta) \qquad (1.2.41)$$

returning to the τ_1 variable we have:

$$B(\phi, \tau_1) = B_{00}\, e^{\frac{\Gamma}{kT_0^{1/2}}(\sin\phi - 1) + \frac{T_1 \bar{\Gamma}_1}{kT_0^{1/2}} \frac{\tau_1}{c}} \cdot A\left(\phi - \frac{\tau_1}{c}\right) \qquad (1.2.42)$$

where $A(\phi - \frac{\tau_1}{c})$ is a unit 2π periodic function

$$A\left(\phi - \frac{\tau_1}{c}\right) = a_0 + a_1 \cos\left(\phi - \frac{\tau_1}{c}\right) + b_1 \sin\left(\phi - \frac{\tau_1}{c}\right) + \cdots + a_m \cos m\left(\phi - \frac{\tau_1}{c}\right) + b_m \sin m\left(\phi - \frac{\tau_1}{c}\right) \qquad (1.2.43)$$

In order to simplify the calculations we shall restrict ourselves to three terms in the unit 2π periodic function which appears in the amplitude formula.

$$B(\phi, \tau_1) = B_{00}\, e^{\frac{\Gamma(\sin\phi - 1)}{kT_0^{1/2}} + \frac{T_1 \bar{\Gamma}_1}{kT_0^{1/2}} \frac{\tau_1}{c}} \left[a_0 + a_m \cos m\left(\phi - \frac{\tau_1}{c}\right) + b_m \sin m\left(\phi - \frac{\tau_1}{c}\right) \right] \qquad (1.2.44)$$

A neutral solution which is neither growing nor decaying in time is obtained by seeking a solution which is periodic in τ_1, which implies $T_1 \equiv 0$.

In this case the amplitude will take the form:

$$B(\phi,\tau_1)=B_{00}\,e^{\frac{\Gamma(\sin\phi-1)}{kT_0^{1/2}}}\left[a_0+a_m\cos m\left(\phi-\frac{\tau_1}{c}\right)+b_m\sin m\left(\phi-\frac{\tau_1}{c}\right)\right] \tag{1.2.45}$$

For $m=0$ we retrieve the formula of the Taylor vortex amplitude (Di Prima and Stuart (1972b)).

Our differential system can be solved as follows:

$$-u_{11}=\bar{u}_{11}=B(\phi,\tau_1)\cdot f_1(x)\cos\phi+mkT_0^{1/2}\bar{B}(\phi)\left[b_m\cos m\left(\phi-\frac{\tau_1}{c}\right)-a_m\sin m\left(\phi-\frac{\tau_1}{c}\right)\right]f_3(x)+B_1(\phi,\tau_1)f_0(x)$$

$$v_{11}=B(\phi,\tau_1)\cdot g_1(x)\cos\phi+mkT_0^{1/2}\bar{B}(\phi)\left[b_m\cos m\left(\phi-\frac{\tau_1}{c}\right)-a_m\sin m\left(\phi-\frac{\tau_1}{c}\right)\right]g_3(x)+B_1(\phi,\tau_1)g_0(x) \tag{1.2.46}$$

where (f_1,g_1) and (f_3,g_3) have to satisfy the following systems of differential equations:

$$\begin{cases} M^2f_1+\lambda^2T_0V_0g_1=F_{111}(x)+\Gamma F_{112}(x) \\ f_1-Mg_1=G_{111}(x)+\Gamma G_{112}(x) \\ f_1=Df_1=g_1=0 \quad at\ x=\pm\frac{1}{2} \end{cases} \tag{1.2.47}$$

$$\begin{cases} M^2f_3+\lambda^2T_0V_0g_3=\left(V_0-\frac{1}{c}\right)Mf_0 \\ f_3-Mg_3=-g_0\left(V_0-\frac{1}{c}\right) \\ f_3=Df_3=g_3=0 \quad at\ x=\pm\frac{1}{2} \end{cases} \tag{1.2.48}$$

The solution is determined up to an additive multiple of the corresponding homogeneous system, and this is illustrated by the inclusion of the terms $B_1(\phi,\tau_1)f_0(x)$ and $B_1(\phi,\tau_1)g_0(x)$ in our expressions respectively, where it is understood that $B_1(\phi,\tau_1)$ is an integration "constant". The

determination of the function $B_1(\phi,\tau_1)$ requires consideration of the ε^2 order.

$\bar{B}(\phi)$ is defined by

$$\bar{B}(\phi) = B_{oo}\, e^{\frac{\Gamma}{kT_o^{1/2}}(\sin\phi - 1)} \qquad (1.2.44a)$$

ORDER ε^2

By elimination of the p_{12} and w_{12} and use of some previous relations obtained from the ε^0 order and ε order systems, together with the definition $u_{12} = -\bar{u}_{12}$, we find:

$$\begin{cases} M^2\bar{u}_{12} + \lambda^2 T_o V_o v_{12} = B_1(\phi,\tau_1)F_{111}(X) + kT_o^{1/2}\left[\dfrac{\partial B_1(\phi,\tau_1)}{\partial\phi}F_{112}(X) + \right. \\ \left. + \dfrac{\partial B_1(\phi,\tau_1)}{\partial\tau_1}F_{114}(X)\right] + B(\phi,\tau_1)F_{123}(X) + k^2 B(\phi,\tau_1) + B(\phi,\tau_1)\cos 2\phi\cdot F_{125}(X) + \\ + kT_o^{1/2}B(\phi,\tau_1)\sin\phi\cdot F_{126}(X) + T_2 B(\phi,\tau_1)F_{127}(X) + \\ + mkT_o^{1/2}\cos\phi\left[b_m\cos m\left(\phi - \tfrac{\tau_1}{\varepsilon}\right) - a_m\sin m\left(\phi - \tfrac{\tau_1}{\varepsilon}\right)\right]F_{129}(X)\cdot\bar{B}(\phi) + \\ + m^2k^2T_o\left[B(\phi,\tau_1) - a_o\bar{B}(\phi)\right]F_{132}(X) \end{cases} \qquad (1.2.49)$$

$$\begin{aligned} \bar{u}_{12} - M v_{12} &= B_1(\phi,\tau_1)\cos\phi\cdot G_{111}(X) + kT_o^{1/2}\left[\frac{\partial B_1(\phi,\tau_1)}{\partial\phi}G_{112}(X) + \right. \\ &\left. + \frac{\partial B_1(\phi,\tau_1)}{\partial\tau_1}G_{114}(X)\right] + B(\phi,\tau_1)G_{123}(X) + k^2B(\phi,\tau_1)G_{124}(X) + \\ &+ B(\phi,\tau_1)\cos 2\phi\cdot G_{125}(X) + kT_o^{1/2}B(\phi,\tau_1)\sin\phi\cdot G_{126}(X) + \\ &+ mkT_o^{1/2}\left[b_m\cos m\left(\phi - \tfrac{\tau_1}{\varepsilon}\right) - a_m\sin m\left(\phi - \tfrac{\tau_1}{\varepsilon}\right)\right]\cdot G_{129}(X)\cdot\bar{B}(\phi) + \\ &+ m^2k^2T_o\left[B(\phi,\tau_1) - a_o\bar{B}(\phi)\right]G_{132}(X) \end{aligned} \qquad (1.2.50)$$

The boundary conditions are:

$$\bar{u}_{12} = D\bar{u}_{12} = v_{12} = 0 \quad \text{at } X = \pm\tfrac{1}{2} \qquad (1.2.51)$$

where $F_{123}(X) = \frac{\Gamma}{2}\left[6(X^2-\frac{1}{4})Mf_o - V_o(3D^2-\lambda^2)f_o - 12f_o + 2cDf_o\right] -$
$-(5D^2-3\lambda^2)D^2f_o - 3\lambda^2T_o(X^2-\frac{1}{4})g_o - \Gamma cMDg_o - 3\lambda^2T_o(X^2-\frac{1}{4})g_1 + \frac{\Gamma V_o}{2}Mf_1$

(1.2.52)

$$F_{124}(x) = -4cMDf_0 + 2\lambda^2 T_0 c\,(x+\tfrac{1}{2})V_0 g_0 - \lambda^2 T_0 c^2 V_{22} g_0 \,, \quad (1.2.53)$$

$$F_{125}(x) = \frac{\Gamma}{2}\left[6(x^2-\tfrac{1}{4})Mf_0 - V_0(3D^2-\lambda^2)f_0 - 12f_0 + 2cDf_0\right] - (3D^2-\lambda^2)D^2 f_0 -$$

$$- \Gamma c MDg_0 + 2MD^2 f_1 + \frac{\Gamma V_0}{2} M f_1 - 3\lambda^2 T_0 (x^2-\tfrac{1}{4}) g_1 \,, \quad (1.2.54)$$

$$F_{126}(x) = 2(x^2-\tfrac{1}{4})(x-\tfrac{c}{4})MDf_0 + (6x^2-cx-\tfrac{1}{2})Mf_0 + V_0(D^2+\lambda^2)f_0 +$$

$$+ 12(f_0 + xDf_0) - \lambda^2 T_0 (x^2-\tfrac{1}{4})\left[\frac{1}{2}(1-\frac{c^2}{12})(\frac{1}{20}-x^2) - \frac{cx}{5}(\frac{7}{12}-x^2)\right]g_0 -$$

$$- 3cMDg_0 - V_0 M f_1 \,, \quad (1.2.55)$$

$$F_{127}(x) = -\lambda^2 V_0 g_0 \,, \quad (1.2.56)$$

$$F_{129}(x) = 3MD^2 f_3 + \Gamma V_0 M f_3 + (V_0 - \tfrac{1}{c})\left[Mf_1 - 2D^2 f_0 - \Gamma c Dg_0\right] +$$

$$+ \left[6(x^2-\tfrac{1}{4}) - V_0\right] Mf_0 + 12xDf_0 - 6\lambda^2 T_0 (x^2-\tfrac{1}{4}) g_3 + cD(Mg_1 - f_1) +$$

$$+ \Gamma c D(Mg_3 - f_3) + c(2Df_0 + 2\Gamma c g_0 - \Gamma V_0 Dg_0 - 2D^3 g_0) \,, \quad (1.2.57)$$

$$F_{132}(x) = (V_0 - \tfrac{1}{c}) M^2 f_3 \,, \quad (1.2.58)$$

$$G_{123}(x) = -\frac{f_0}{2} + \frac{3}{2}D^2 f_0 + \frac{\Gamma}{2}\left[V_0 - 6(x^2-\tfrac{1}{4})\right] g_0 + \frac{f_1}{2} + \frac{6x}{c} f_1 - D^2 g_1 - \frac{\Gamma V_0 g_1}{2} \,, \quad (1.2.59)$$

$$G_{124}(x) = 2(f_0 + cDg_0) \,, \quad (1.2.60)$$

$$G_{125}(x) = -\frac{6x}{c} f_0 + \frac{1}{2}D^2 f_0 + \frac{\Gamma}{2}\left[V_0 - 6(x^2-\tfrac{1}{4})\right] g_0 + \frac{f_1}{2}(1 + \frac{12x}{c}) - D^2 g_1 -$$

$$- \frac{\Gamma V_0 g_1}{2} \,, \quad (1.2.61)$$

$$G_{126}(x) = \frac{f_0}{c}\left[\frac{1}{2}(1-\frac{c^2}{20})(-4x^3 + \frac{3x}{5})\right] + f_0(x^4 - \frac{x^2}{2} + \frac{7}{240}) -$$

$$- 2(x^2-\tfrac{1}{4})(x-\tfrac{c}{4})Dg_0 + 6g_0(x^2-\tfrac{1}{4}) + V_0 g_1 \,, \quad (1.2.62)$$

$$G_{129}(x) = (\frac{12x}{c} + 1) f_3 + (-6x^2 - cx + \frac{5}{2}) g_0 - 2D^2 g_3 - V_0 \Gamma g_3 \,, \quad (1.2.63)$$

$$G_{132}(x) = (V_0 - \frac{1}{c}) g_3 \quad (1.2.64)$$

Again, by multiplying the right-hand side with the adjoint and respectively, adding and integrating over $(-\frac{1}{2}, \frac{1}{2})$, we see that the results must be zero. In this way we obtain the following first-order differential equation for $B_1(\Phi, \tau_1)$:

$$\left\{kT_o^{1/2}\left[\frac{\partial B_1(\phi,\tau_1)}{\partial\phi}+\bar{c}\frac{\partial B_1(\phi,\tau_1)}{\partial\tau_1}\right]-\Gamma\cos\phi\cdot B_1(\phi,\tau_1)\right\}\cdot\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{112}+g_o^+G_{112})dx$$

$$+B(\phi,\tau_1)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{123}+g_o^+G_{123})dx+k^2B(\phi,\tau_1)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{124}+g_o^+G_{124})dx+$$

$$+B(\phi,\tau_1)\cos2\phi\cdot\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{125}+g_o^+G_{125})dx+kT_o^{1/2}B(\phi,\tau_1)\sin\phi\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{126}+g_o^+G_{126})dx$$

$$+T_2B(\phi,\tau_1)\int_{-\frac{1}{2}}^{\frac{1}{2}}f_o^+F_{127}(x)dx+m^2k^2T_o\left[B(\phi,\tau_1)-a_o\bar{B}(\phi)\right]\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{132}+g_o^+G_{132})dx$$

$$+mkT_o^{1/2}\cos\phi\cdot\bar{B}(\phi)\left[b_m\cos m(\phi-\tfrac{\tau_1}{c})-a_m\sin m(\phi-\tfrac{\tau_1}{c})\right]\cdot$$

$$\cdot\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{129}+g_o^+G_{129})dx=0 \qquad (1.2.65)$$

By dividing now this equation with $I=\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{112}+g_o^+G_{112})dx$

and by using the following notations:

$$\Gamma_1=\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{126}+g_o^+G_{126})dx \qquad (1.2.66)$$

$$\Gamma_2=-\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{125}+g_o^+G_{125})dx \qquad (1.2.67)$$

$$\bar{\Gamma}_3=\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{123}+g_o^+G_{123})dx \qquad (1.2.68)$$

$$\bar{\Gamma}_4=\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{124}+g_o^+G_{124})dx \qquad (1.2.69)$$

$$\Gamma_5=\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}f_o^+F_{127}\,dx=-\bar{\Gamma}_1 \qquad (1.2.70)$$

$$\Gamma_7=\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{129}+g_o^+G_{129})dx \qquad (1.2.71)$$

$$\Gamma_{10}=\frac{1}{I}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_o^+F_{132}+g_o^+G_{132})dx \qquad (1.2.72)$$

It is possible to add that the notations conform with Di Prima and Stuart (1972b). One obtains:

$$\left\{kT_o^{1/2}\left[\frac{\partial B(\phi,\tau_1)}{\partial\phi}+\bar{c}\frac{\partial B(\phi,\tau_1)}{\partial\tau_1}\right]-\Gamma\cos\phi\cdot B_1(\phi,\tau_1)\right\}+$$

$$+B(\phi,\tau_1)\left[\bar{\Gamma}_3+k^2\bar{\Gamma}_4-\Gamma_1T_2+kT_o^{1/2}\sin\phi\cdot\Gamma_1-\Gamma_2\cos2\phi+m^2k^2T_o\Gamma_{10}\right]+$$

$$+\bar{B}(\phi)\left\{\left[b_m\cos m(\phi-\tfrac{\tau_1}{c})-a_m\sin m(\phi-\tfrac{\tau_1}{c})\right]mkT_o^{1/2}\cos\phi\cdot\Gamma_7-\right.$$

$$\left.-a_o\,m^2k^2T_o\Gamma_{10}\right\}=0 \qquad (1.2.73)$$

By using again the characteristics method with the parameter $\eta=\phi-\frac{\tau_1}{c}$

and also the periodicity condition in ϕ :

$$B_1\left(2\pi, 2\pi - \frac{\tau_1}{c}\right) = B_1\left(0, -\frac{\tau_1}{c}\right) \qquad (1.2.74)$$

one obtains the following value for $B_1(\phi, \tau_1)$:

$$B_1(\phi,\tau_1) = \frac{\bar{B}(\phi)}{kT_0^{1/2}}\left\{\left[\Gamma_1 kT_0^{1/2}\cos\phi + \frac{\Gamma_2 \sin 2\phi}{2}\right]\left[a_0 + a_m \cos m\left(\phi - \frac{\tau_1}{c}\right) + b_m \sin m\left(\phi - \frac{\tau_1}{c}\right)\right]\right.$$

$$+ \Gamma_7 m k T_0^{1/2} \sin\phi\left[b_m \cos m\left(\phi - \frac{\tau_1}{c}\right) - a_m \sin m\left(\phi - \frac{\tau_1}{c}\right)\right]$$

$$+ \frac{\tau_1}{c}\left\{\left[\bar{\Gamma}_3 + k^2\bar{\Gamma}_4 - \bar{\Gamma}_1 T_2\right]a_0 + \left[a_m \cos m\left(\phi - \frac{\tau_1}{c}\right) + b_m \sin m\left(\phi - \frac{\tau_1}{c}\right)\right]\cdot\right.$$

$$\left.\left.\cdot\left[\bar{\Gamma}_3 + k^2\bar{\Gamma}_4 - \bar{\Gamma}_1 T_2 + m^2 k^2 T_0 \Gamma_{10}\right]\right\} \qquad (1.2.75)$$

For the neutral case, we obtain for wavy vortices

$$a_0 \equiv 0 \qquad (1.2.76)$$

$$\bar{\Gamma}_3 + k^2\bar{\Gamma}_4 - \bar{\Gamma}_1 T_2 + m^2 k^2 T_0 \Gamma_{10} = 0 \Rightarrow T_2 = \frac{\bar{\Gamma}_3 + k^2\bar{\Gamma}_4 + m^2 k^2 T_0 \Gamma_{10}}{\bar{\Gamma}_1} \qquad (1.2.77)$$

Let us denote

$$\begin{cases} T_{21} = \dfrac{\bar{\Gamma}_3}{\bar{\Gamma}_1} \\ T_{22} = \dfrac{\bar{\Gamma}_4}{\bar{\Gamma}_1} \\ T_{23} = \dfrac{\Gamma_{10}}{\bar{\Gamma}_1} \end{cases} \qquad (1.2.77a)$$

Consequently, T_2 is given by

$$T_2 = T_{21} + k^2 T_{22} + m^2 k^2 T_0 T_{23} \qquad (1.2.77b)$$

From Davey, Di Prima and Stuart (1968), page 37, we have a formula for the critical Taylor number for wavy vortices

$$T_c(m) = T_c(0) + k_0^2 T_{c1} + \cdots \qquad (1.2.78)$$

Moreover, we have the known expansion

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots \qquad (1.2.79)$$

and using the relation $\delta = 4k^2\varepsilon^2$ (1.2.80), we have

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots\cdots = T_0 + \varepsilon T_1 + \frac{\delta}{4k^2} T_2 + \cdots \quad (1.2.81)$$

We take into account Davey, Di Prima and Stuart's relation

$$m = 2k_0 (T\delta)^{-1/2} \quad (1.2.82)$$

and we obtain $T_{c1} \equiv T_{23}$ (1.2.83)

We can see our final result as an addition of time-dependence and waviness to Di Prima and Stuart's problem (1972) or as an addition of eccentricity and surface curvature effects to Davey, Di Prima and Stuart's problem. In conclusion we can see our result in T_2 as a unification of Davey, Di Prima and Stuart, Di Prima and Stuart's papers. The various effects are linearly additive in T_2 in this linear problem.

$T_{21} = \frac{\bar{\Gamma}_3}{\bar{\Gamma}_1}$ = eccentricity effect (1.2.84) } Di Prima & Stuart (1972b) eccentric, steady

$k^2 T_{22} = k^2 \frac{\bar{\Gamma}_4}{\bar{\Gamma}_1}$ = surface curvature effect (1.2.85) } Addition

$m^2 k^2 T_0 T_{23} = m^2 k^2 T_0 \frac{\Gamma_{10}}{\bar{\Gamma}_1}$ (1.2.86) Davey, Di Prima and Stuart concentric, unsteady and non-axisymmetric

In the neutral case we get the following result for $B(\phi, \tau_1)$:

$$B_1(\phi, \tau_1) = B_{00}\, e^{\frac{\Gamma(\sin\phi - 1)}{k T_0^{1/2}}} \left\{ \left[\Gamma_1 \cos\phi + \frac{\Gamma_2 \sin 2\phi}{2k T_0^{1/2}} \right] \cdot \left[a_m \cos m\left(\phi - \frac{\tau_1}{c}\right) + b_m \sin m\left(\phi - \frac{\tau_1}{c}\right) \right] + \right.$$

$$\left. + m \Gamma_7 \sin\phi \cdot \left[b_m \cos m\left(\phi - \frac{\tau_1}{c}\right) - a_m \sin m\left(\phi - \frac{\tau_1}{c}\right) \right] \right\} \quad (1.2.87)$$

In our calculations we have seen that $T_{23} = T_{c1} = \frac{\Gamma_{10}}{\bar{\Gamma}_1}$ (1.2.88)

and we shall justify this result.

1.3 Justification of $T_{23}=T_{c1}$

We shall now show that T_{c1} which has a numerical value of is also given as a ratio of these two integrals in the paper of Davey, Di Prima and Stuart (1968).

The linearized disturbance equations of Davey, Di Prima and Stuart (1968) for the concentric, unsteady and non-axisymmetric case are as follows:

$$L\bar{M}Lv = T\Omega_e(x)\frac{\partial^2 v}{\partial \zeta^2} \qquad (1.3.1)$$

$$Lv = u \qquad (1.3.2)$$

$$\frac{\partial u}{\partial x} - \alpha\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial \zeta} = 0 \qquad (1.3.3)$$

where $\Omega_e(x) = 1-\alpha x = V_0(x)$ (1.3.4)

$$\alpha = 2\frac{1-\mu}{1+\mu} = c \qquad (1.3.5)$$

$$\bar{M} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \zeta^2} \qquad (1.3.6)$$

$$L = \bar{M} - \frac{\partial}{\partial \tau_1} - \Omega_e(x)\frac{\partial}{\partial \phi} \qquad (1.3.7)$$

Equations for velocities u, v and w:

$$\begin{cases} u = u_{c11}(x)\cdot\cos\lambda\zeta\cdot e^{ik_D(\phi-\frac{\tau_1}{c})} \\ v = v_{c11}(x)\cos\lambda\zeta\cdot e^{ik_D(\phi-\frac{\tau_1}{c})} \\ w = w_{c11}(x)\sin\lambda\zeta\cdot e^{ik_D(\phi-\frac{\tau_1}{c})} \end{cases} \qquad (1.3.8)$$

where

$$u_{c11} = u_{c10} + k_D u^{(1)} + k_D^2 u^{(2)} + \ldots$$

$$v_{c11} = v_{c10} + k_D v^{(1)} + k_D^2 v^{(2)} + \ldots \qquad (1.3.9)$$

$$w_{c11} = w_{c10} + k_D w^{(1)} + k_D^2 w^{(2)} + \ldots$$

$$\text{and } T = T_0 + k_D T^{(1)} + k_D^2 T^{(2)} + \ldots \qquad (1.3.10)$$

We have $L\bar{M}u = T\Omega_e(x)\frac{\partial^2 v}{\partial y^2}$ (1.3.11)

k_D° ORDER EQUATION

$$(D^2-\lambda^2)^2 u_{c10}(x) + \lambda^2 T_0 \Omega_e(x) v_{c10}(x) = 0 \quad (1.3.12)$$

$$(D^2-\lambda^2) v_{c10}(x) = u_{c10}(x) \quad (1.3.13)$$

$$u_{c10}(x) = Du_{c10}(x) = v_{c10}(x) = 0 \text{ at } x = \pm\frac{1}{2} \quad (1.3.14)$$

where u_{c10} and v_{c10} satisfy the relation

$$\begin{cases} u_{c10}(x) = f_0(x) \\ v_{c10}(x) = g_0(x) \end{cases} \quad (1.3.15)$$

since the equations and boundary conditions of (u_{c10}, v_{c10}) are identical to those of (f_0, g_0)

In conclusion, the k_D° order equation may be written as follows:

$$\begin{cases} M^2 f_0 + \lambda^2 T_0 \Omega_e(x) g_0 = 0 & (1.3.16) \\ f_0 - M g_0 = 0 & (1.3.17) \\ f_0 = Df_0 = g_0 = 0 \text{ at } x = \pm\frac{1}{2} & (1.3.18) \end{cases}$$

where $M \equiv D^2 - \lambda^2$

If we assume a disturbance on $\bar{c}$ ($\frac{1}{\bar{c}}$ the wave velocity) with respect to k_D we have the following perturbation expansion

$$\bar{c} = \bar{c}_0 + k_D \bar{c}_1 + \cdots$$

$$\frac{1}{\bar{c}} = \frac{1}{\bar{c}_0 + k_D \bar{c}_1} = \frac{1}{\bar{c}_0} - k_D \frac{\bar{c}_1}{\bar{c}_0^2} \quad (1.3.19)$$

We get the differential system

$$\begin{cases} M^2 u^{(1)} + \lambda^2 T_0 \Omega_e(x) v^{(1)} = i\left[\Omega_e(x) - \frac{1}{\bar{c}_0}\right] M f_0 - \lambda^2 T^{(1)} \Omega_e(x) g_0 & (1.3.20) \\ u^{(1)} - M v^{(1)} = -i g_0 \left[\Omega_e(x) - \frac{1}{\bar{c}_0}\right] & (1.3.21) \\ u^{(1)} = Du^{(1)} = v^{(1)} = 0 \text{ at } x = \pm\frac{1}{2} & (1.3.22) \end{cases}$$

We multiply the right hand side of (1.3.20) and (1.3.21) by f_0^+ and g_0^+ respectively, add and integrate over $(-\frac{1}{2}, \frac{1}{2})$, and the result must be zero for a solution of the differential system to exist. This orthogonality condition yields:

$$\lambda^2 T^{(1)} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ \Omega_e(x) g_0(x)\, dx =$$

$$= i\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{112} + g_0^+ G_{112})\, dx - \frac{1}{c_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{114} + g_0^+ G_{114})\, dx\right] \quad (1.3.23)$$

From our previous calculations (see equation (1.2.34)) we can see that the right hand side of this equation is equal to zero.

That implies $T^{(1)} \equiv 0$ (1.3.24)

Since we deal with the linearized problem, it follows that (1.3.20) - (1.3.22) can be solved in the form

$$\begin{aligned} u^{(1)} &= i f_3(x) \\ v^{(1)} &= i g_3(x) \end{aligned} \quad (1.3.25)$$

where f_3 and g_3 satisfy the differential system:

$$\begin{cases} M^2 f_3 + \lambda^2 T_0 \Omega_e(x) g_3 = \left[\Omega_e(x) - \frac{1}{c_0}\right] M f_0 \\ f_3 - M g_3 = -f_0\left[\Omega_e(x) - \frac{1}{c_0}\right] \\ f_3 = D f_3 = g_3 = 0 \text{ at } x = \pm\frac{1}{2} \end{cases} \quad (1.3.26)$$

We can easily see that the definition of f_3 and g_3 in (1.3.26) is identical with the definition of f_3 and g_3, as given by the equations (1.2.48)

ORDER k_0^2

$$\begin{cases} M^2 u^{(2)} + T_0 \Omega_e \lambda^2 v^{(2)} = i\left[\Omega_e(x) - \frac{1}{c_0}\right] M u^{(1)} + \frac{i\bar{c}_1}{\bar{c}_0^2} M u_{c10} - \lambda^2 T^{(2)} \Omega_e(x) g_0(x) \\ u^{(2)} - M v^{(2)} = -i\left[\Omega_e(x) - \frac{1}{c_0}\right] v^{(1)} - i v_{c10} \cdot \frac{\bar{c}_1}{\bar{c}_0^2} \\ u^{(2)} = D u^{(2)} = v^{(2)} = 0 \text{ at } x = \pm\frac{1}{2} \end{cases} \quad (1.3.27)$$

We multiply this pair of equations by the adjoint pair (f_0^+, g_0^+) respectively, add and obtain:

$$T^{(2)}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ F_{113}(x)\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{141} + g_0^+ G_{141})\,dx +$$

$$+ \frac{i\bar{c}_1}{c_0^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{142} + g_0^+ G_{142})\,dx = 0 \qquad (1.3.28)$$

where we define

$$\begin{cases} F_{141}(x) = -\left[\Omega_\ell(x) - \frac{1}{c_0}\right] M f_1 = -\left(V_0 - \frac{1}{c}\right) M f_1 & (1.3.29) \\ G_{141}(x) = \left[\Omega_\ell(x) - \frac{1}{c_0}\right] g_1 = \left(V_0 - \frac{1}{c}\right) g_1 & (1.3.30) \end{cases}$$

$$F_{142}(x) = M f_0(x) \qquad (1.3.31)$$

$$G_{142}(x) = -g_0(x) \qquad (1.3.32)$$

If we denote the first integral by I_1, the second by I_2 and the third by I_3, one obtains:

$$T^{(2)} I_1 + I_2 + \frac{i\bar{c}_1}{c_0^2} I_3 = 0 \qquad (1.3.33)$$

I_1, I_2 and I_3 are real quantities, and $T^{(2)}$ is also a real quantity, it follows then that $\bar{c}_1 \equiv 0$ (1.3.34)

Finally, one obtains

$$T^{(2)} = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{141} + g_0^+ G_{141})\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ F_{113}(x)\,dx} = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} (f_0^+ F_{132} + g_0^+ G_{132})\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ F_{113}(x)\,dx} = \frac{\Gamma_{10}}{\Gamma_1} \qquad (1.3.35)$$

$T^{(2)}$ is the k_0^2 term in the perturbation expansion of T as defined by (1.3.10). T_{c1} is the k_0^2 term in the expansion of $T_c(m)$ as illustrated by (1.2.78). It is obvious that this result for $T^{(2)}$ as given by (1.3.35), is identical to the result for $T_{23} = T_{c1}$ as given by (1.2.77a). The results are compared with the observed critical speeds of Vohr (1967) for development of wave instability on Taylor vortices. As seen by the figure there is qualitative similarity, but they differ in magni-

tude from experiment due to the absence of nonlinear effects of the Taylor vortex.

PART II

WAVY VORTICES IN THE FLOW BETWEEN TWO LONG ECCENTRIC ROTATING CYLINDERS : NONLINEAR THEORY

2.1 Introduction

Coles (1965) investigated the occurrence of wavy vortices in the flow between two rotating concentric cylinders. Vohr and Cole did some crucial experiments in connection with the flow between two rotating eccentric cylinders. Cole (1957) investigated the effect of the eccentricity on the stability of the flow between two rotating eccentric cylinders. Vohr (1967, 1968) reported that the critical Taylor number for the occurrence of the instability varies with the eccentricity parameter. Second, he found at least in one experiment (with $\mathcal{E} = 0.475, b-a = 0.099a$) that the Taylor-vortex secondary motion appeared to have its maximum activity at an azimuthal angle 50° downstream from the position of maximum gap when the inner cylinder was rotating and the outer was at rest. It was a serious discrepancy between this result and the result obtained by Di Prima (1963), where the most unstable zone appeared to be that where the gap between the cylinders is greatest. This discrepancy was explained by the paper of Di Prima and Stuart (1972b) where their research took account of the non-local effects in the stability of flow between eccentric rotating cylinders, in such a way that the flow fields at all points must affect the stability characteristics and show, in some way, the observed position of maximum Taylor-vortex activity. All the previous papers were based on the local stability, where it has been customary to approximate the partial differential equations for stability by an ordinary differential equation, namely the Orr-Sommerfeld equation with the local boundary-layer velocity profile, known as the parallel-flow approximation.

Davey, Di Prima and Stuart (1968) dealt with the theory for wavy vortices in the case of the rotating concentric cylinders. They introduced a method of investigating the interaction of certain axisymmetric

and nonaxisymmetric modes of disturbance of the basic Couette flow, which enables an analysis of possible flows consequent upon an instability. For $\eta = 0.951$ and a Taylor number of about $1.07T_c$ they found that the Taylor-vortex flow becomes unstable to a disturbance with one complete wave in the azimuth, although the preference for one wave is not strong. The validity of this method is restricted to Taylor numbers near the first critical value T_c, so that Cole's experiments were not considered, since they were probably outside this range of validity. This method used an expansion of the velocity in powers of an amplitude function of time, $A(t)$ up to $A^3(t)$. Davey and Di Prima and Stuart also made the "small-gap approximation".

Eagles (1971) extended the research of the concentric cylinders problem by using fifth-order terms in the amplitude and using the full equations rather than the "small-gap" equations. He used a matrix form expansion procedure, which enables more uniformity of treatment of the various appearing ordinary differential equations. Di Prima and Stuart (1972b) deal with the linear stability of the flow between two long eccentric rotating circular cylinders. The problem, which is connected with lubrication technology, can be seen as an extension of the classical Taylor problem for concentric cylinders. Since the basic flow has components in the radial and azimuthal directions and depends on both of these co-ordinates, the linearized stability equations are consequently partial differential equations rather than ordinary differential equations. By letting the two small parameters appearing in the problem, namely δ, the clearance ratio, and ε, the eccentricity, tend to zero in such a way that $\delta^{1/2}$ is proportional to ε, a global solution to the stability problem is obtained without the need of the "local instability" or "parallel-flow" approximation, which is used commonly in boundary-layer stability theory.

The purpose of this work is to unite the theory for the stability of the flow between two long concentric rotating circular cylinders, which has been developed by Davey, Di Prima and Stuart (1968) and Eagles (1971) with Di Prima and Stuart (1972b) linear theory for the eccentric rotating cylinders into a single theory. The second object of this work will be to find the critical Taylor number for the wavy vortices $T_c(WV)$ as a function of ε^2 .

2.2 Non-linear analysis of the Taylor vortex mode

The object of the linear theory given in the first part of this thesis was to investigate the wavy vortices when the Taylor vortex mode was absent. In the non-linear part of this work we make an analysis of the influence exerted by a non-linear Taylor vortex mode with amplitude $B(\Phi)$, on a linear wavy vortex mode with amplitude $A(\Phi)$. The non-linear analysis of the Taylor vortex flow between eccentric cylinders has been done by Di Prima and Stuart (1975). The velocity field is given by:

$$u = \varepsilon^{1/2}\left[u_{10}(X,\Phi,\xi)+\varepsilon^{1/2}u_{11}(X,\Phi,\xi)+\varepsilon u_{12}(X,\Phi,\xi)+\varepsilon^{3/2}u_{13}(X,\Phi,\xi)+\right.$$
$$\left.+\varepsilon^{2}u_{14}(X,\Phi,\xi)\right]+O(\varepsilon^{3}) \qquad (2.2.1)$$

$$v = \varepsilon^{1/2}\left[v_{10}(X,\Phi,\xi)+\varepsilon^{1/2}v_{11}(X,\Phi,\xi)+\varepsilon v_{12}(X,\Phi,\xi)+\varepsilon^{3/2}v_{13}(X,\Phi,\xi)+\right.$$
$$\left.+\varepsilon^{2}v_{14}(X,\Phi,\xi)\right]+O(\varepsilon^{3}) \qquad (2.2.2)$$

$$w = \varepsilon^{1/2}\left[w_{10}(X,\Phi,\xi)+\varepsilon^{1/2}w_{11}(X,\Phi,\xi)+\varepsilon w_{12}(X,\Phi,\xi)+\varepsilon^{3/2}w_{13}(X,\Phi,\xi)+\right.$$
$$\left.+\varepsilon^{2}w_{14}(X,\Phi,\xi)\right]+O(\varepsilon^{3}) \qquad (2.2.3)$$

The most interesting situation was when $T-T_c \sim$ eccentricity.

In our analysis we restrict the expansion of the Taylor vortex mode to the $\varepsilon^{3/2}$ order. By using the non-dimensional form of the time t which is given by $\tau = \frac{\nu t}{a^2(1-\chi^2)}$, one obtains the following equations of motion andthe continuity equation in the modified bipolar co-ordinate system:

$$\frac{\partial u}{\partial \tau}+\varepsilon R_a J^{1/2}\left[U\frac{\partial u}{\partial X}+u\frac{\partial U}{\partial X}\right]+\frac{R_a J^{1/2}}{\rho}\left[V\frac{\partial u}{\partial \Phi}+\varepsilon c R_a v\frac{\partial U}{\partial \Phi}\right]-$$
$$-R_a\frac{\partial}{\partial \Phi}\left(\frac{J^{1/2}}{\rho}\right)(Vu+\varepsilon c R_a v U)+\rho\frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right)TVv+$$
$$+\left[J^{1/2}u\frac{\partial u}{\partial X}+\frac{J^{1/2}}{\rho}cR_a v\frac{\partial u}{\partial \Phi}+w\frac{\partial u}{\partial \xi}-\frac{1}{\rho}\frac{\partial}{\partial \Phi}(J^{1/2})cR_a uv+\frac{cT\rho}{2}\frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right)v^2\right]=$$
$$=-J^{1/2}\frac{\partial p}{\partial X}+\left(JL^2+\frac{\partial^2}{\partial \xi^2}\right)u-\alpha^2 Hu+cR_a\rho\left\{\alpha\frac{\partial}{\partial \rho}\left(\frac{J}{\rho^2}\right)\frac{\partial v}{\partial \Phi}-\frac{\partial}{\partial \Phi}\left(\frac{J}{\rho^2}\right)\frac{\partial v}{\partial X}\right\}$$

(2.2.4)

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^{1/2}\left[\frac{1}{c}u\frac{\partial V}{\partial x} + \varepsilon R_a U\frac{\partial v}{\partial x}\right] + \frac{1}{\rho}J^{1/2}R_a\left[V\frac{\partial v}{\partial \phi} + v\frac{\partial V}{\partial \phi}\right] - \\ -\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{I^{1/2}}{\rho}\right)\left[\frac{1}{c}Vu + \varepsilon R_a Uv\right] + \frac{2\alpha^2\varepsilon}{c}\frac{\partial}{\partial\phi}\left(\frac{I^{1/2}}{\rho}\right)uV + \\ +\left[J^{1/2}u\frac{\partial v}{\partial x} + \frac{I^{1/2}}{\rho}cR_a v\frac{\partial v}{\partial\phi} + w\frac{\partial v}{\partial\xi} - \rho\frac{\partial}{\partial\rho}\left(\frac{I^{1/2}}{\rho}\right)\alpha uv + \frac{1}{\rho}\frac{\partial}{\partial\phi}(J^{1/2})\frac{\alpha^2}{cR_a}u^2\right] = \\ = -\frac{I^{1/2}}{\rho}\frac{\alpha^2}{cR_a}\frac{\partial p}{\partial\phi} + \left(JL^2 + \frac{\partial^2}{\partial\xi^2}\right)v - \alpha^2 Hv + \\ +\frac{\rho\alpha^2}{cR_a}\left\{-\alpha\frac{\partial}{\partial\rho}\left(\frac{I}{\rho^2}\right)\frac{\partial u}{\partial\phi} + \frac{\partial}{\partial\phi}\left(\frac{I}{\rho^2}\right)\frac{\partial u}{\partial x}\right\} \end{cases} \qquad (2.2.5)$$

$$\begin{cases} \frac{\partial w}{\partial\tau} + \varepsilon J^{1/2}R_a U\frac{\partial w}{\partial x} + \frac{1}{\rho}J^{1/2}R_a V\frac{\partial w}{\partial\phi} + \left[J^{1/2}u\frac{\partial w}{\partial x} + \frac{I^{1/2}}{\rho}cR_a v\frac{\partial w}{\partial\phi} + \right. \\ \left. + w\frac{\partial w}{\partial\xi}\right] = -\frac{\partial p}{\partial\xi} + \left(JL^2 + \frac{\partial^2}{\partial\xi^2}\right)w \end{cases} \qquad (2.2.6)$$

$$J^{1/2}\frac{\partial u}{\partial x} - \alpha\rho\frac{\partial}{\partial\rho}\left(\frac{I^{1/2}}{\rho}\right)u + cR_a\left[\frac{I^{1/2}}{\rho}\frac{\partial v}{\partial\phi} - v\frac{\partial}{\partial\phi}\left(\frac{I^{1/2}}{\rho}\right)\right] + \frac{\partial w}{\partial\xi} = 0 \qquad (2.2.7)$$

We are now in a position to write down the systems of steady state equations which are solved at successive orders.

ORDER $\varepsilon^{1/2}$

By elimination of p_{10} and w_{10} we obtain a set of equations:

$$\begin{cases} -u_{10} = (D^2 - \lambda^2)v_{10} \\ (D^2-\lambda^2)^2 u_{10} - \lambda^2 T_0 V_0 v_{10} = 0 \end{cases} \qquad (2.2.8)$$

subject to the boundary conditions $u_{10} = v_{10} = Du_{10} = 0$ at $x = \pm\frac{1}{2}$

Davey, Di Prima and Stuart (1968) investigated the interaction between a nonlinear Taylor vortex mode with a $\cos\lambda\xi$ dependence in the axial co-ordinate and a linear wavy vortex mode with a $\sin\lambda\xi$ dependence in the axial co-ordinate. We assume the same here, and we take a $\cos\lambda\xi$ dependence for the nonlinear Taylor vortex mode.

Consequently we choose the solution given by:

$$\begin{cases} u_{10}(x,\phi,\xi) = -B(\phi)f_0(x)\cos\lambda\xi \\ v_{10}(x,\phi,\xi) = B(\phi)g_0(x)\cos\lambda\xi \\ w_{10}(x,\phi,\xi) = \frac{B(\phi)Df_0}{\lambda}\sin\lambda\xi \end{cases} \qquad (2.2.9)$$

where f_0 and g_0 provide the eigenfunction of a differential system:

$$\begin{cases} M^2 f_0 + \lambda^2 T_0 V_0 g_0 = 0 \\ f_0 - M g_0 = 0 \\ f_0 = Df_0 = g_0 = 0 \text{ at } x = \pm \frac{1}{2} \end{cases} \qquad (2.2.10)$$

In the nonlinear theory of the Taylor-vortex mode proper account has to be taken of the lubrication pressure, which is defined as follows:

$$p' = \frac{T^{1/2}\varepsilon^{-2}}{4ck^3}\left[\int_0^\Phi g_0(\Phi)\,d\Phi + \varepsilon\int_0^\Phi g_1(\Phi)\,d\Phi + \cdots\right] +$$

$$+\varepsilon^{1/2}\left[p_{10}(x,\Phi,\xi) + \varepsilon^{1/2} p_{11}(x,\Phi,\xi) + \varepsilon p_{12}(x,\Phi,\xi) + \cdots\right] \qquad (2.2.11)$$

ORDER ε

$$-\frac{cT_0}{2} v_{10}^2 + u_{10}\frac{\partial u_{10}}{\partial x} + \frac{\partial u_{10}}{\partial \xi} w_{10} - T_0 V_0 v_{11} = -\frac{\partial p_{11}}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2}\right) u_{11} \qquad (2.2.12)$$

$$\frac{\partial v_{10}}{\partial x} u_{10} + \frac{\partial v_{10}}{\partial \xi} w_{10} - u_{11} = -g_0(\Phi) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2}\right) v_{11} \qquad (2.2.13)$$

$$-\frac{\partial p_{11}}{\partial \xi} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2}\right) w_{11} = u_{10}\frac{\partial w_{10}}{\partial x} + w_{10}\frac{\partial w_{10}}{\partial \xi} \qquad (2.2.14)$$

$$\frac{\partial u_{11}}{\partial x} + \frac{\partial w_{11}}{\partial \xi} = 0 \qquad (2.2.15)$$

By elimination of p_{11} and w_{11}, one obtains:

$$\begin{aligned} u_{11} &= -B^2(\Phi) f_{112}(x) \cos 2\lambda\xi \\ v_{11} &= v_{111}(x,\Phi) + B^2(\Phi) g_{112}(x) \cos 2\lambda\xi \\ w_{11} &= \frac{1}{2\lambda} B^2(\Phi) Df_{112}(x) \sin 2\lambda\xi \\ p_{11} &= p_{111}(x,\Phi) + B^2(\Phi) \ell_{112}(x) \cos 2\lambda\xi \end{aligned} \qquad (2.2.16)$$

In these formulae f_{112} and g_{112} are defined by the non-homogeneous ordinary differential system:

$$\begin{cases} f_{112} - (D^2 - 4\lambda^2) g_{112} = \frac{1}{2}(f_0 Dg_0 - g_0 Df_0) \\ (D^2 - 4\lambda^2)^2 f_{112} + 4\lambda^2 T_0 V_0 g_{112} = -f_0 D^3 f_0 + Df_0 \cdot D^2 f_0 - c\lambda^2 T_0 g_0^2 \\ f_{112} = Df_{112} = g_{112} = 0 \text{ at } x = \pm\frac{1}{2} \end{cases} \qquad (2.2.17)$$

and the harmonic pressure function $\ell_{112}(x)$ is defined by

$$\ell_{112} = \frac{1}{4\lambda^2}\left[(Df_0)^2 - f_0 D^2 f_0 - (D^2 - 4\lambda^2) D f_{112}\right] \quad (2.2.18)$$

In order to calculate the mean field, that means $v_{111}(x,\phi)$, $q_0(\phi)$ and $p_{111}(x,\phi)$ we have to turn our attention to equation (2.2.13):

One obtains:

$$-\frac{1}{2} B^2(\phi)\, D(f_0 g_0) = -q_0(\phi) + D^2 v_{111} \quad (2.2.19)$$

where $q_0(\phi)$ is yet unknown.

By taking into account the boundary conditions at $x = \pm\frac{1}{2}$, we obtain:

$$v_{111} = \frac{1}{2} q_0(\phi)\cdot(x^2 - \tfrac{1}{4}) - \frac{B^2(\phi)}{2} F_0(x) \quad (2.2.20)$$

$q_0(\phi)$ is a pressure gradient which contributes a "lubrication" or "plane Poiseuille" effect in the second $O(\varepsilon)$ momentum equation (2.2.13).

$$F_0(x) = \int_{-\frac{1}{2}}^{x} f_0 g_0\, dx - (x + \tfrac{1}{2}) \int_{-\frac{1}{2}}^{\frac{1}{2}} f_0 g_0\, dx \quad (2.2.21)$$

From the ε^2 order continuity equation we get

$$u_{130} = -ck T_0^{1/2} \int_{-\frac{1}{2}}^{x} \frac{\partial v_{111}}{\partial \phi}\, dx \quad (2.2.22)$$

subject to the boundary condition $u_{130} = 0$ at $x = \pm\frac{1}{2}$ (2.2.23)

We determine $q_0(\phi)$ by satisfaction of the boundary condition on u_{130}:

$$u_{130} = -ck T_0^{1/2} \int_{-\frac{1}{2}}^{x} \left[\frac{1}{2}(x^2 - \tfrac{1}{4}) \frac{dq_{10}(\phi)}{d\phi} - F_0 B \frac{dB}{d\phi}\right] dx \quad (2.2.24)$$

From the boundary condition (2.2.23) it follows that

$$q_0(\phi) = \sigma_0 - 6 Q_0 B^2(\phi) \quad (2.2.25)$$

while from the periodicity condition involving the pressure, follows

$$\sigma_0 = \frac{3Q_0}{\pi} \int_0^{2\pi} B^2(\phi)\, d\phi \quad (2.2.26)$$

In conclusion, one obtains

$$v_{111} = B^2(\phi) g_{10}(x) + \frac{\sigma_0}{2}(x^2 - \tfrac{1}{4}) \quad (2.2.27)$$

where $g_{10}(X) = -3Q_0(X^2 - \frac{1}{4}) - \frac{F_0(X)}{2}$ (2.2.27a)

From (2.2.12) we get $p_{111}(X,\Phi) = B^2(\Phi)\ell_{111}(X) + \sigma_0 \ell_{100}(X)$ (2.2.27b)

where $\ell_{111} = \int_{-\frac{1}{2}}^{X} \left[T_0 V_0(X) g_{10} - f_0 D f_0 + \frac{1}{4} c T_0 g_0^2 \right] dX + \text{constant}$, (2.2.27c)

and $\ell_{100} = \frac{T_0}{2} \int_{-\frac{1}{2}}^{X} V_0(X) \cdot (X^2 - \frac{1}{4}) dX + \text{constant}$ (2.2.27d)

Consequently,

$$v_{111}(X,\Phi) = -\frac{1}{2}B^2(\Phi)F_0(X) + \left[\frac{1}{2}\sigma_0 - 3Q_0 B^2(\Phi) \right](X^2 - \frac{1}{4}) \quad (2.2.27e)$$

where the first term $F_0(X)$ represents the contribution to v_{111} from the Reynolds stress of the Taylor vortex, and the second term is the plane-Poisseuille contribution and its origin can be detected from the Reynolds or lubrication effect. From (2.2.9), (2.2.16) and (2.2.27) we are able to calculate the nonlinear terms which arise in the order momentum and continuity equations. It is obvious, from the interactions between $O(\varepsilon^{1/2})$ and $O(\varepsilon)$ terms, that the fundamental wave-number λ and the second harmonic 3λ are generated by the nonlinearities.

It is convenient to separate these different harmonics by writing:

$$\bar{u}_{12} = \bar{u}_{121}\cos\lambda\xi_f + \bar{u}_{122}\cos 3\lambda\xi_f \quad (2.2.27f)$$

$$v_{12} = v_{121}\cos\lambda\xi_f + v_{122}\cos 3\lambda\xi_f$$

$$(\bar{u}_{121} = -u_{121})$$

with similar expressions for w_{12} and p_{12} :

The momentum and continuity equations at order $\varepsilon^{3/2}$ are as follows:

$$\begin{cases} -T_0 V_0 v_{12} + kT_0^{1/2} V_0 \frac{\partial u_{10}}{\partial \Phi} - T_0 V_1 v_{10} - T_1 V_0 v_{10} + u_{10}\frac{\partial u_{11}}{\partial X} + u_{11}\frac{\partial u_{12}}{\partial X} + \\ + w_{10}\frac{\partial u_{11}}{\partial \xi_f} + w_{11}\frac{\partial u_{10}}{\partial \xi_f} - cT_0 v_{10} v_{11} = -\frac{\partial p_{12}}{\partial X} + \frac{\partial p_{10}}{\partial X}\cos\Phi + \\ + \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial \xi_f^2}\right)u_{12} - 2\frac{\partial^2 u_{10}}{\partial X^2}\cos\Phi \end{cases} \quad (2.2.28)$$

$$\begin{cases} -u_{12} + u_{10}\cos\Phi + kT_0^{1/2} V_0 \frac{\partial v_{10}}{\partial \Phi} + \frac{1}{c}\frac{\partial V_1}{\partial X} u_{10} + u_{10}\frac{\partial v_{11}}{\partial X} + u_{11}\frac{\partial v_{10}}{\partial X} + \\ + w_{10}\frac{\partial v_{11}}{\partial \xi_f} + w_{11}\frac{\partial v_{10}}{\partial \xi_f} = \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial \xi_f^2}\right)v_{12} - 2\frac{\partial^2 v_{10}}{\partial X^2}\cos\Phi \end{cases} \quad (2.2.29)$$

$$\begin{cases} kT_0^{1/2} V_0 \frac{\partial w_{10}}{\partial \phi} + u_{10}\frac{\partial w_{11}}{\partial x} + u_{11}\frac{\partial w_{10}}{\partial x} + w_{10}\frac{\partial w_{11}}{\partial \xi_f} + w_{11}\frac{\partial w_{10}}{\partial \xi_f} = \\ = -\frac{\partial p_{12}}{\partial \xi_f} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi_f^2}\right) w_{12} - 2\frac{\partial^2 w_{10}}{\partial x^2}\cos\phi \end{cases} \quad (2.2.30)$$

together with

$$\frac{\partial u_{12}}{\partial x} - \frac{\partial u_{10}}{\partial x}\cos\phi + ckT_0^{1/2}\frac{\partial v_{10}}{\partial \phi} + \frac{\partial w_{12}}{\partial \xi_f} = 0 \quad (2.2.31)$$

For the fundamental component, the system (2.2.28) - (2.2.31) will yield:

$$\begin{cases} u_{121} - (D^2-\lambda^2)v_{121} = B(\phi)\cos\phi \cdot G_{111}(x) + kT_0^{1/2}\frac{dB(\phi)}{d\phi}G_{112}(x) + \\ \qquad + \sigma_0 B(\phi)\cdot x f_0(x) + B^3(\phi) G_{N21}(x) \end{cases} \quad (2.2.32)$$

$$\begin{cases} (D^2-\lambda^2)^2 u_{121} + \lambda^2 T_0 V_0 v_{121} = B(\phi)\cos\phi \cdot F_{111}(x) + kT_0^{1/2}\frac{dB(\phi)}{d\phi}F_{112}(x) + \\ \qquad + B^3(\phi)F_{N21}(x) + T_1 B(\phi)F_{113}(x) - \\ \qquad - \frac{\sigma_0}{2} c\lambda^2 T_0 g_0(x)\cdot(x^2-\frac{1}{4})B(\phi) \end{cases}$$

$$u_{121} = Du_{121} = v_{121} = 0 \text{ at } x = \pm\frac{1}{2} \quad (2.2.33)$$

The functions F_{N21} and G_{N21} are given by

$$F_{N21} = -\frac{1}{4}f_0(D^2-4\lambda^2)Df_{112} - \frac{1}{2}Df_0(D^2-4\lambda^2)f_{112} + \frac{1}{2}f_{112}(D^2-\lambda^2)Df_0 + \\ + \frac{1}{4}Df_{112}(D^2-\lambda^2)f_0 - \lambda^2 cT_0 g_0(g_{10} + \frac{1}{2}g_{112}) \quad (2.2.34)$$

$$G_{N21} = f_0 Dg_{10} + \frac{1}{2}f_0 Dg_{112} + \frac{1}{2}f_{112}Dg_0 + g_{112}Df_0 + \frac{1}{4}g_0 Df_{112} \quad (2.2.35)$$

subject to the boundary conditions

$$u_{121} = Du_{121} = v_{121} = 0 \text{ at } x = \pm\frac{1}{2} \quad (2.2.36)$$

The eigenvalue problem at $O(\varepsilon^{1/2})$, whose solution is (2.2.9) is the homogeneous problem corresponding to (2.2.32), (2.2.33) and (2.2.36). Consequently, in order for the differential system (2.2.32-33) and

(2.2.36) to have a solution it is necessary for the orthogonality condition involving the adjoint to be satisfied, namely, that by multiplying the right hand sides of (2.2.32) and (2.2.33), adding and integrating over $(-\frac{1}{2}, \frac{1}{2})$ the result must be zero. The same situation as in the paper of Di Prima and Stuart (1972b), ϕ derivatives do not occur on the left-hand sides of (2.2.32) and (2.2.33) and ϕ is considered merely as a parameter in the integration, so that the orthogonality condition yields a nonlinear differential equation for $B(\phi)$, which represents a generalisation of the linear amplitude equation of Di Prima and Stuart (1972b), and may be written as:

$$kT_0^{1/2}\frac{dB(\phi)}{d\phi} = \Gamma B(\phi)\cos\phi + (T_1\bar{\Gamma}_1 + \sigma_0\Gamma_5)B(\phi) - \Gamma_4' B^3(\phi) \qquad (2.2.37)$$

where $\Gamma, \bar{\Gamma}_1, \Gamma_4'$ and Γ_5 are defined as follows:

$$\begin{aligned}
\Gamma_0 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (F_{112} f_0^+ + G_{112} g_0^+)\,dx \\
\Gamma &= -\Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} (F_{111} f_0^+ + G_{111} g_0^+)\,dx \\
\bar{\Gamma}_1 &= -\Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_0^+ F_{113}(x)\,dx \\
\Gamma_4' &= \Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} (F_{N21} f_0^+ + G_{N21} g_0^+)\,dx \\
\Gamma_5 &= \Gamma_0^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{x^2}{2} c T_0 g_0 f_0^+ (x^2 - \tfrac{1}{4}) - x f_0 g_0^+\right] dx
\end{aligned} \qquad (2.2.38)$$

The Bernoulli type amplitude equation is integrated to yield

$$B^2(\phi) = \frac{k}{\omega_4'}\frac{[E(2\pi)-1]}{\int_0^{2\pi} E(\phi)\,d\phi + [E(2\pi)-1]\int_0^{\phi} E(\phi)\,d\phi} \qquad (2.2.39)$$

where

$$E(\phi) \equiv \exp\left[\frac{\omega}{k}\sin\phi + \frac{1}{k}(T_1\omega_3 + \sigma_0\omega_5)\phi\right] \qquad (2.2.40)$$

and

$$(\omega, \omega_3, \omega_4', \omega_5) \equiv 2T_0^{-1/2}(\Gamma, \bar{\Gamma}_1, \Gamma_4', \Gamma_5) \qquad (2.2.41)$$

In order to ensure that the velocity and velocity fields (2.2.1.), (2.2.2), (2.2.3), (2.2.11) and (2.2.16) are single valued in ϕ, the

integration constant has to be chosen so that $B^2(\phi)$ is periodic with period 2π in ϕ.

From (2.2.26) one obtains

$$\sigma_0 = \frac{6Q_0 T_1 \omega_3}{\omega_4' - 6Q_0\omega_5} \qquad (2.2.42a)$$

or $$\sigma_0 = \frac{6Q_0 T_1 \omega_3}{\omega_4} \qquad (2.2.42b)$$

where $$\omega_4 = \omega_4' - 6Q_0\omega_5 \qquad (2.2.43)$$

and $$\omega_4 = 2T_0^{-1/2}\Gamma_4 \qquad (2.2.44)$$

Therefore, we have

$$T_1\omega_3 + \sigma_0\omega_5 = T_1\frac{\omega_3\omega_4'}{\omega_4} \qquad (2.2.45)$$

In this way the function $E(\phi)$ could be defined by

$$E(\phi) = \exp\left[\left(\frac{\omega_1}{k}\right)\sin\phi + \left(\frac{T_1\omega_3\omega_4'}{k\omega_4}\right)\phi\right] \qquad (2.2.46)$$

With known numerical values of the constants Q_0, Γ, Γ_0, Γ_3, Γ_4', Γ_5, Γ_4 and solution of the associated differential systems, the velocity field can be given to order ε. This velocity field can be seen as a nonlinear extension of the zeroth-order results of the linear theory of Di Prima and Stuart (1972b), which correspond to $O(\varepsilon^{1/2})$ here, As far as the pressure is concerned, the dominant $O(\varepsilon^{-2})$ lubrication pressure has been calculated and also the "Taylor-vortex" pressure to $O(\varepsilon)$ which is less important relative to the previous one.

It is important also to give the numerical value of the above-mentioned constants. Most of the information has been obtained from the paper of Di Prima and Stuart (1975).

$$\left.\begin{aligned} &T_0 = 1695\,,\ \lambda = 3.13\,,\ Q_0 = -0.1365\,,\ \omega_3 = 0.0003573 \\ &\Gamma = 23.09\,,\ \Gamma_0 = 16.90\,,\ \overline{\Gamma}_1 = 0.007356 \\ &\Gamma_4 = 38.27\,,\ \Gamma_5 = -2.38\,,\ \Gamma_4' = 40.22 \\ &\omega = 1.122\,,\ \omega_4 = 1.859\,,\ \omega_4' = 1.954 \end{aligned}\right\} \qquad (2.2.47)$$

Finally, we now have the perturbation velocity field of (2.2.1), (2.2.2) and (2.2.3) to $O(\varepsilon)$:

$$u = -\varepsilon^{1/2} B(\Phi) f_0(X) \cos\lambda\xi - \varepsilon B^2(\Phi) f_{112}(X) \cos 2\lambda\xi + O(\varepsilon^{3/2}) \quad (2.2.48)$$

$$v = \varepsilon^{1/2} B(\Phi) g_0(X) \cos\lambda\xi + \varepsilon\left[B^2(\Phi) g_{10}(X) + (3Q_0\Gamma_3 T_1/\Gamma_4)(X^2 - \tfrac{1}{4}) + B^2(\Phi) g_{112}(X) \cos 2\lambda\xi\right] + O(\varepsilon^{3/2}) \quad (2.2.49)$$

$$w = \varepsilon^{1/2}\lambda^{-1} B(\Phi) D f_0(X) \sin\lambda\xi + \frac{\varepsilon}{2\lambda} B^2(\Phi) D f_{112}(X) \sin 2\lambda\xi + O(\varepsilon^{3/2}) \quad (2.2.50)$$

The functions f and g are connected to certain functions of Davey's (1962) paper ($\mathcal{Y} \to 1/m = 0$) as follows:

$$\left.\begin{aligned} f_0(X) &\equiv \bar{u}_1(D),\ g_0(X) \equiv v_1(D),\ f_{112}(X) \equiv -\tfrac{1}{2}\bar{u}_2(D),\ g_{112} \equiv -\tfrac{1}{2}v_2(D) \\ g_{10}(X) &= -\tfrac{1}{2}F_0(X) - 3Q_0(X^2 - \tfrac{1}{4}) = -\tfrac{1}{2}F_1(D) - 3Q_0(X^2 - \tfrac{1}{4}) \end{aligned}\right\} \quad (2.2.51)$$

where the symbol in parentheses denotes Davey's functions. The pressure perturbation is given as follows:

$$p = \frac{-3T^{1/2}\varepsilon^{-2}}{2ck^3} Q_0 \left[\frac{k}{\omega_4'} \ln\left\{1 + \frac{[E(2\pi) - 1]\int_0^{\Phi} E(\Phi)\,d\Phi}{\int_0^{2\pi} E(\Phi)\,d\Phi}\right\} - \frac{\omega_3 T_1 \Phi}{\omega_4} + O(\varepsilon)\right] -$$

$$- \frac{\varepsilon^{1/2}}{\lambda^2} B(\Phi)(D^2 - \lambda^2) D f_0(X) \cos\lambda\xi + \varepsilon\left[B^2(\Phi) \ell_{111}(X) + \frac{6Q_0\omega_3 T_1}{\omega_4} \ell_{100}(X) + B^2(\Phi) \ell_{112}(X) \cos 2\lambda\xi\right] + O(\varepsilon^{3/2}) \quad (2.2.52)$$

The $O(\varepsilon^{-2})$ pressure perturbation is the dominant lubrication term. The next important term after the $O(\varepsilon^{-2})$ term is the $O(\varepsilon^{-1})$ term, involving $g_1(\Phi)$, and which is not given here. T_1 defines the amount by which the Taylor number exceeds the critical value, which is T_0 when the term $\varepsilon^2 T_2$ of the Taylor number expansion is neglected. In the case that $T_1 \to 0$ it is possible to retrieve the results of the linear theory (Di Prima and Stuart (1972b)).

In order to analyse the interaction between a nonlinear Taylor vortex mode and a linear wavy vortex mode, we define a new parameter

$$\gamma = T - T_c(0,\varepsilon) \qquad (2.2.53)$$

and we make the approximation

$$\gamma = \varepsilon T_1 = T - T_c(0,\varepsilon) \qquad (2.2.54)$$

where $T_c(0,\varepsilon) = T_0 + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2\bar{\Gamma}_4}{\bar{\Gamma}_1}$ (2.2.55)

is the critical Taylor number for the Taylor vortex mode. γ is the amount above the critical and is usually different from zero, because $T_1 \neq 0$ in the non-linear theory.

We also define $v = v_{TV} + v'$ (2.2.56)

where v_{TV} is the Taylor vortex mode, whose perturbation expansion is given above by (2.2.49) and v' is the wavy vortex mode.

A method to solve this problem is a double expansion in parameters, first to expand in γ in the momentum and continuity equations, and afterwards in ε (the eccentricity) in certain integrals. We expand the wavy vortex mode v' and the growth rate $\bar{\sigma}$ in powers of $\gamma^{1/2}$, as follows:

$$v' = v'^{(1)} + \gamma^{1/2} v'^{(2)} + \gamma v'^{(3)} + \cdots \qquad (2.2.57)$$

$$\bar{\sigma} = \bar{\sigma}_0 + \gamma^{1/2}\bar{\sigma}_1 + \gamma\bar{\sigma}_2 + \cdots \qquad (2.2.58)$$

It is important to mention that v_{TV} is of order $\gamma^{1/2}$ and $v' \ll v_{TV}$.

We assume the same axial phase shift as in the concentric case investigated by Davey, Di Prima and Stuart (1968). There, a $\sin\lambda\xi$ term in the wavy vortex mode corresponds to a $\cos\lambda\xi$ term in the Taylor vortex mode. Thus we assume the following form for $(u'^{(1)}, v'^{(1)}, w'^{(1)}, p'^{(1)})$

$$\left.\begin{aligned} u'^{(1)} &= e^{i\bar{\sigma}_0\tau} u_1'^{(1)} \sin\lambda\xi \\ v'^{(1)} &= e^{i\bar{\sigma}_0\tau} v_1'^{(1)} \sin\lambda\xi \\ w'^{(1)} &= e^{i\bar{\sigma}_0\tau} w_1'^{(1)} \cos\lambda\xi \\ p'^{(1)} &= e^{i\bar{\sigma}_0\tau} p_1'^{(1)} \sin\lambda\xi \end{aligned}\right\} \qquad (2.2.59)$$

We shall consider R_a as a constant in the γ expansion for the equations, and expand it later in powers of ε in certain integrals. By using the approximation $\gamma = \varepsilon T_1$, it is possible to expand $(\bar{u}_{TV}, v_{TV}, w_{TV}, p_{TV})$ in powers of γ :

$$\bar{u}_{TV} = \frac{\gamma^{1/2}}{T_1^{1/2}} B(\Phi) f_0(x) \cos\lambda\xi_f + \frac{\gamma}{T_1} B^2(\Phi) \cdot g_{112} \cos 2\lambda\xi_f + \cdots \quad (2.2.60)$$

$$v_{TV} = \frac{\gamma^{1/2}}{T_1^{1/2}} B(\Phi) g_0(x) \cos\lambda\xi_f + \frac{\gamma}{T_1}\left[B^2(\Phi) g_{10} + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) + \right.$$

$$\left. + \cos 2\lambda\xi_f \cdot B^2(\Phi) g_{112}\right] + \cdots \quad (2.2.61)$$

$$w_{TV} = \frac{\gamma^{1/2}}{T_1^{1/2}} \cdot \frac{B(\Phi) D f_0(x)}{\lambda} \sin\lambda\xi_f + \frac{\gamma}{T_1} \frac{B^2(\Phi)}{2\lambda} D f_{112} \sin 2\lambda\xi_f + \cdots \quad (2.2.62)$$

An alternative way to forumlate $(\bar{u}_{TV}, v_{TV}, w_{TV})$ would be:

$$\bar{u}_{TV} = \gamma^{1/2} \bar{u}_{TV1} + \gamma \bar{u}_{TV2} + \gamma^{3/2} \bar{u}_{TV3} + \cdots \quad (2.2.63)$$

$$v_{TV} = \gamma^{1/2} v_{TV1} + \gamma v_{TV2} + \gamma^{3/2} v_{TV3} + \cdots \quad (2.2.64)$$

$$w_{TV} = \gamma^{1/2} w_{TV1} + \gamma w_{TV2} + \gamma^{3/2} w_{TV3} + \cdots \quad (2.2.65)$$

If u', v', w' are of order η, say, we assume η to be so small that only terms which are linear in η need to be taken into account.

We have to turn our attention once again to the double parameter expansion, first in γ in the equations and afterwards in ε in certain integrals. Although the two parameters γ and ε are connected by the relation $\gamma = \varepsilon T_1$, they are on a different physical footing, the first representing the elevation above the critical Taylor number and the second the eccentricity.

Since we had some inconsistencies by expanding directly in ε we found it necessary to introduce an additional parameter γ, so that we obtain a two stages expansion which takes proper account of the growth rate. By this double parameter expansion we split up the expansion for the growth rate σ so that we get a similar approach to that used by Davey, Di Prima and Stuart (1968).

2.3 Equations of stability

In this chapter we deal with the successive equations of stability for the flow, obtained by the expansion in the parameter γ .

γ° ORDER

The momentum and continuity equations for the γ° order will be identical to the linearised equations of motion and continuity, for the wavy vortices, as given by the first part of this thesis.

$$\begin{cases} \varepsilon R_a J^{1/2}\left(U\frac{\partial u_1^{(1)}}{\partial x}+\frac{\partial U}{\partial x}u_1^{(1)}\right)+\frac{R_a J^{1/2}}{\rho}\left(V\frac{\partial u_1^{(1)}}{\partial \phi}+\varepsilon c R_a\frac{\partial U}{\partial \phi}v_1^{(1)}\right)- \\ -R_a\frac{\partial}{\partial \phi}\left(\frac{J^{1/2}}{\rho}\right)\left(Vu_1^{(1)}+\varepsilon c R_a U v_1^{(1)}\right)+\rho\frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right)(T_0+\varepsilon^2 T_2(0))Vv_1^{(1)}= \\ =-J^{1/2}\frac{\partial p_1^{(1)}}{\partial x}+(JL^2-\lambda^2-ic\bar{\sigma}_0)u_1^{(1)}-\alpha^2 H u_1^{(1)}+ \\ +cR_a\rho\left\{\alpha\frac{\partial}{\partial \rho}\left(\frac{J}{\rho^2}\right)\frac{\partial v_1^{(1)}}{\partial \phi}-\frac{\partial}{\partial \phi}\left(\frac{J}{\rho^2}\right)\frac{\partial v_1^{(1)}}{\partial x}\right\} \end{cases} \tag{2.3.1}$$

$$\begin{cases} J^{1/2}\left(\frac{1}{c}\frac{\partial V}{\partial x}u_1^{(1)}+\varepsilon R_a U\frac{\partial v_1^{(1)}}{\partial x}\right)+\frac{1}{\rho}J^{1/2}R_a\left(V\frac{\partial v_1^{(1)}}{\partial \phi}+\frac{\partial V}{\partial \phi}v_1^{(1)}\right)- \\ -\alpha\rho\frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right)\left(\frac{1}{c}Vu_1^{(1)}+\varepsilon R_a U v_1^{(1)}\right)+\frac{2\alpha^2\varepsilon}{c}\frac{\partial}{\partial \phi}\left(\frac{J^{1/2}}{\rho}\right)u_1^{(1)}U= \\ =-\frac{J^{1/2}}{\rho}\frac{\alpha^2}{cR_a}\frac{\partial p_1^{(1)}}{\partial \phi}+(JL^2-\lambda^2-ic\bar{\sigma}_0)v_1^{(1)}-\alpha^2 H v_1^{(1)}+ \\ +\rho\frac{\alpha^2}{cR_a}\left\{-\alpha\frac{\partial}{\partial \rho}\left(\frac{J}{\rho^2}\right)\frac{\partial u_1^{(1)}}{\partial \phi}+\frac{\partial}{\partial \phi}\left(\frac{J}{\rho^2}\right)\frac{\partial u_1^{(1)}}{\partial x}\right\} \end{cases} \tag{2.3.2}$$

$$\varepsilon R_a J^{1/2}U\frac{\partial w_1^{(1)}}{\partial x}+\frac{1}{\rho}J^{1/2}R_a V\frac{\partial w_1^{(1)}}{\partial \phi}=-\lambda p_1^{(1)}+(JL^2-\lambda^2-ic\bar{\sigma}_0)w_1^{(1)} \tag{2.3.3}$$

$$J^{1/2}\frac{\partial u_1^{(1)}}{\partial x}-\alpha\rho\frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right)u_1^{(1)}+cR_a\left[\frac{J^{1/2}}{\rho}\frac{\partial v_1^{(1)}}{\partial \phi}-\right.$$

$$\left.-\frac{\partial}{\partial \phi}\left(\frac{J^{1/2}}{\rho}\right)v_1^{(1)}\right]-\lambda w_1^{(1)}=0 \tag{2.3.4}$$

This system of partial differential equations can be written as a system involving differential operators:

$$\begin{cases} L_{11}u_1^{1(1)} + L_{12}v_1^{1(1)} + L_{14}p_1^{1(1)} = 0 & (2.3.5) \\ L_{21}u_1^{1(1)} + L_{22}v_1^{1(1)} + L_{24}p_1^{1(1)} = 0 & (2.3.6) \\ L_{33}w_1^{1(1)} + L_{34}p_1^{1(1)} = 0 & (2.3.7) \\ L_{41}u_1^{1(1)} + L_{42}v_1^{1(1)} + L_{43}w_1^{1(1)} = 0 & (2.3.8) \end{cases}$$

where $L_{11}, L_{12}, L_{13}, L_{21}, L_{22}, L_{23}, L_{33}, L_{34}, L_{41}, L_{42}$ and L_{44} are differential operators:

$$L_{11} = \varepsilon R_a J^{1/2}\left(U\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right) + \frac{R_a J^{1/2}}{\rho} V\frac{\partial}{\partial\phi} - R_a\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)V -$$
$$-\left(JL^2 - \lambda^2 - i\bar{\sigma}_0\right) + \alpha^2 H \qquad (2.3.9)$$

$$L_{12} = \frac{\varepsilon c R_a^2 J^{1/2}}{\rho}\frac{\partial U}{\partial\phi} - R_a^2\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)\varepsilon c U + \rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)TV -$$
$$-cR_a\rho\left[\alpha\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial x}\right] \qquad (2.3.10)$$

$$L_{14} = J^{1/2}\frac{\partial}{\partial x} \qquad (2.3.11)$$

$$L_{21} = \frac{J^{1/2}}{c}\frac{\partial V}{\partial x} - \frac{\alpha\rho}{c}V\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) + \frac{2\alpha^2\varepsilon U}{c}\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right) +$$
$$+\frac{\rho\alpha^2}{cR_a}\left[\alpha\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial x}\right] \qquad (2.3.12)$$

$$L_{22} = \varepsilon R_a U J^{1/2}\frac{\partial}{\partial x} + \frac{J^{1/2}R_a}{\rho}\left(V\frac{\partial}{\partial\phi} + \frac{\partial V}{\partial\phi}\right) - \alpha\rho\varepsilon R_a U\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) +$$
$$+\alpha^2 H - \left(JL^2 - \lambda^2 - i\bar{\sigma}_0\right) \qquad (2.3.12a)$$

$$L_{24} = \frac{\alpha^2}{cR_a}\frac{J^{1/2}}{\rho}\frac{\partial}{\partial\phi} \qquad (2.3.12b)$$

$$L_{33} = \varepsilon R_a J^{1/2} U\frac{\partial}{\partial x} + \frac{1}{\rho}J^{1/2}R_a V\frac{\partial}{\partial\phi} - \left(JL^2 - \lambda^2 - i\bar{\sigma}_0\right) \qquad (2.3.13)$$

$$L_{34} = \lambda \qquad (2.3.14)$$

$$L_{41} = J^{1/2}\frac{\partial}{\partial x} - \alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) \qquad (2.3.15)$$

$$L_{42} = \frac{cR_a}{\rho}J^{1/2}\frac{\partial}{\partial\phi} - cR_a\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right) \qquad (2.3.15a)$$

$$L_{43} = -\lambda \qquad (2.3.15b)$$

In order to find a solution of all the partial differential equations systems appearing in our nonlinear work, an orthogonality condition involving the partial differential equations has to be satisfied. Let us assume a partial differential equations system defined by the equation

$$L_{ij}\, u_j = S_i \qquad (2.3.16)$$

where L_{ij} is a differential operator.

We multiply by the adjoint v_i^+ and perform a double integration over $(0, 2\pi)$ in ϕ and $(-\frac{1}{2}, \frac{1}{2})$ in X, as follows:

$$\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} (v_i^+ L_{ij}\, u_j)\, dX\, d\phi = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} v_i^+ S_i\, dX\, d\phi \qquad (2.3.17)$$

If the adjoint and the boundary conditions are properly defined then the orthogonality condition is given by

$$-\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} u_j L_{ij}^+ v_i^+\, dX\, d\phi = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} v_i^+ S_i\, dX\, d\phi = 0 \qquad (2.3.18)$$

where L_{ij}^+ is the adjoint differential operator.

$$L_{ij}^+ v_i^+ = 0 \qquad (2.3.19)$$

together with boundary conditions forms adjoint system. It is at this point, that we introduce the partial differential equations adjoint to our partial differential system, for our momentum and continuity equations. Let us denote the adjoint of $\{u_1^{1(1)}, v_1^{1(1)}, w_1^{1(1)}, p_1^{1(1)}\}$ by $\{u^+, v^+, w^+, p^+\}$

By multiplying the differential equations (2.3.5) to (2.3.8) with (u^+, v^+, w^+, p^+) respectively, and integrating over $(0, 2\pi)$ in ϕ and $(-\frac{1}{2}, \frac{1}{2})$ in X, one obtains:

$$\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[u^+(X,\phi)(L_{11}\, u_1^{1(1)} + L_{12}\, v_1^{1(1)} + L_{14}\, p_1^{1(1)}) +$$

$$+ v^+(X,\phi)(L_{21}\, u_1^{1(1)} + L_{22}\, v_1^{1(1)} + L_{24}\, p_1^{1(1)}) + w^+(X,\phi)(L_{33}\, w_1^{1(1)} + L_{34}\, p_1^{1(1)}) +$$

$$+p^{+}(x,\phi)\left(L_{41}u_1^{1(1)}+L_{42}v_1^{1(1)}+L_{43}w_1^{1(1)}\right)\Big]dx\,d\phi=\int_0^{2\pi}\Big[I_1(x,\phi)\Big]_{x=-\frac{1}{2}}^{x=\frac{1}{2}}d\phi+$$

$$+\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big[I_2(x,\phi)\Big]_{\phi=0}^{\phi=2\pi}dx+\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big[u_1^{1(1)}(x,\phi)\left(\bar{L}_{11}u^{+}+\bar{L}_{12}v^{+}+\bar{L}_{14}p^{+}\right)+$$

$$+v_1^{1(1)}(x,\phi)\left(\bar{L}_{21}u^{+}+\bar{L}_{22}v^{+}+\bar{L}_{24}p^{+}\right)+$$

$$+w_1^{1(1)}(x,\phi)\left(\bar{L}_{33}w^{+}+\bar{L}_{34}p^{+}\right)+$$

$$+p_1^{1(1)}(x,\phi)\left(\bar{L}_{41}u^{+}+\bar{L}_{42}v^{+}+\bar{L}_{43}w^{+}\right)\Big]dx\,d\phi \qquad (2.3.20)$$

where $\bar{L}_{11}$, $\bar{L}_{12}$, $\bar{L}_{14}$, $\bar{L}_{21}$, $\bar{L}_{22}$, $\bar{L}_{24}$, $\bar{L}_{33}$, $\bar{L}_{34}$, $\bar{L}_{41}$, $\bar{L}_{42}$ and $\bar{L}_{43}$ are differential operators as follows:

$$\bar{L}_{11}=\varepsilon R_a\Big[J^{1/2}\frac{\partial U}{\partial x}-J^{1/2}U\frac{\partial}{\partial x}-\frac{\partial}{\partial x}\left(J^{1/2}U\right)\Big]-\frac{R_a}{\rho}\Big[\frac{\partial\left(J^{1/2}V\right)}{\partial\phi}+VJ^{1/2}\frac{\partial}{\partial\phi}\Big]-$$

$$-R_a\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)V+\alpha^2H-\Big[J\frac{\partial^2}{\partial x^2}+2\frac{\partial J}{\partial x}\frac{\partial}{\partial x}+\frac{\partial^2 J}{\partial x^2}-\frac{\alpha J}{\rho}\frac{\partial}{\partial x}-$$

$$-\frac{\partial}{\partial x}\left(\frac{\alpha J}{\rho}\right)+\frac{\alpha^2}{\rho^2}\left(J\frac{\partial^2}{\partial\phi^2}+2\frac{\partial J}{\partial\phi}\frac{\partial}{\partial\phi}+\frac{\partial^2 J}{\partial\phi^2}\right)-\lambda^2-i\bar{\omega}_0\Big] \qquad (2.3.21)$$

$$\bar{L}_{12}=\frac{J^{1/2}}{c}\frac{\partial V}{\partial x}-\frac{\alpha\rho}{c}V\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)+\frac{2\alpha^2\varepsilon U}{c}\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)+\frac{\alpha^2\rho}{cR_a}\frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial x}+$$

$$+\frac{\alpha^2}{cR_a}\frac{\partial}{\partial x}\Big[\rho\frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\Big]-\frac{\rho\alpha^3}{cR_a}\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial\phi}-\frac{\rho\alpha^3}{cR_a}\frac{\partial^2}{\partial\rho\partial\phi}\left(\frac{J}{\rho^2}\right) \qquad (2.3.22)$$

$$\bar{L}_{14}=-J^{1/2}\frac{\partial}{\partial x}-\frac{\partial J^{1/2}}{\partial x}-\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) \qquad (2.3.23)$$

$$\bar{L}_{21}=\frac{\varepsilon cR_a^2J^{1/2}}{\rho}\frac{\partial U}{\partial\phi}-R_a^2\varepsilon cU\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)+\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)T_c(0,\varepsilon)V-$$

$$-cR_a\rho\Big[\frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}\Big(\rho\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\Big)-\alpha\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial}{\partial\phi}-\alpha\frac{\partial^2}{\partial\rho\partial\phi}\left(\frac{J}{\rho^2}\right)\Big] \qquad (2.3.24)$$

where $T_c(0,\varepsilon)$ is defined by (2.1.55)

$$\bar{L}_{22}=-\varepsilon R_a\Big[UJ^{1/2}\frac{\partial}{\partial x}+\frac{\partial\left(UJ^{1/2}\right)}{\partial x}\Big]+\frac{J^{1/2}R_a}{\rho}\frac{\partial V}{\partial\phi}-$$

$$-\alpha\rho\varepsilon R_aU\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)+\alpha^2H-\frac{R_a}{\rho}\Big[J^{1/2}V\frac{\partial}{\partial\phi}+\frac{\partial\left(J^{1/2}V\right)}{\partial\phi}\Big]-$$

$$-\Big[J\frac{\partial^2}{\partial x^2}+2\frac{\partial J}{\partial x}\frac{\partial}{\partial x}+\frac{\partial^2 J}{\partial x^2}-\frac{\alpha J}{\rho}\frac{\partial}{\partial x}-\frac{\partial}{\partial x}\left(\frac{\alpha J}{\rho}\right)+$$

$$+\frac{\alpha^2}{\rho^2}\left(J\frac{\partial^2}{\partial\phi^2}+2\frac{\partial J}{\partial\phi}\frac{\partial}{\partial\phi}+\frac{\partial^2 J}{\partial\phi^2}\right)-\lambda^2-i\bar{\omega}_0\Big] \qquad (2.3.25)$$

$$\bar{L}_{24}=-cR_a\Big[J^{1/2}\frac{1}{\rho}\frac{\partial}{\partial\phi}+2\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)\Big] \qquad (2.3.26)$$

$$\bar{L}_{33} = -\varepsilon Ra\,(J^{1/2} U \frac{\partial}{\partial x} + \frac{\partial (J^{1/2} U)}{\partial x}) - \frac{Ra}{\rho}\Big[V(J^{1/2}\frac{\partial}{\partial \phi} + \frac{\partial J^{1/2}}{\partial \phi}) +$$
$$+ J^{1/2}\frac{\partial V}{\partial \phi}\Big] - \Big[J\frac{\partial^2}{\partial x^2} + 2\frac{\partial J}{\partial x}\frac{\partial}{\partial x} + \frac{\partial^2 J}{\partial x^2} - \frac{\alpha J}{\rho}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}(\frac{\alpha}{\rho} J) +$$
$$+ \frac{\alpha^2}{\rho^2}(J\frac{\partial^2}{\partial \phi^2} + 2\frac{\partial J}{\partial \phi}\frac{\partial}{\partial \phi} + \frac{\partial^2 J}{\partial \phi^2}) - \lambda^2 - i\bar{\sigma}_0\Big] \qquad (2.3.27)$$

$$\bar{L}_{34} = -\lambda \qquad (2.3.28)$$

$$\bar{L}_{41} = -\frac{\partial J^{1/2}}{\partial x} - J^{1/2}\frac{\partial}{\partial x} \qquad (2.3.29)$$

$$\bar{L}_{42} = -\frac{\alpha^2}{cRa\rho}(J^{1/2}\frac{\partial}{\partial \phi} + \frac{\partial J^{1/2}}{\partial \phi}) \qquad (2.3.30)$$

$$\bar{L}_{43} = \lambda \qquad (2.3.31)$$

and where

$$\begin{cases} I_1(x,\phi) = \Big[\varepsilon Ra J^{1/2} V u_1^{1(1)} - J\frac{\partial u_1^{1(1)}}{\partial x} - \frac{\alpha}{\rho} J u_1^{1(1)} + J^{1/2} p_1^{1(1)} + \\ + \frac{cRa}{\rho}\frac{\partial J}{\partial \phi} v_1^{1(1)}\Big] u^+ + \Big[-\frac{\alpha^2}{\rho cRa}\frac{\partial J}{\partial \phi} u_1^{1(1)} + \varepsilon Ra V J^{1/2} v_1^{1(1)} - J\frac{\partial v_1^{1(1)}}{\partial x} - \\ - \frac{\alpha}{\rho} J v_1^{1(1)}\Big] v^+ + \Big[-J\frac{\partial w_1^{1(1)}}{\partial x} - \frac{\alpha}{\rho} J w_1^{1(1)}\Big] w^+ + \Big[J^{1/2} u_1^{1(1)}\Big] p^+ + \\ + u_1^{1(1)}\frac{\partial}{\partial x}(u^+ J) + v_1^{1(1)}\frac{\partial}{\partial x}(v^+ J) + w_1^{1(1)}\frac{\partial}{\partial x}(w^+ J) \end{cases} \qquad (2.3.32)$$

$$I_2(x,\phi) = \Big[Ra\frac{J^{1/2}}{\rho} V u_1^{1(1)} - J\frac{\alpha^2}{\rho^2}\frac{\partial u_1^{1(1)}}{\partial \phi} - cRa\rho\alpha\frac{\partial}{\partial \rho}\Big(\frac{J}{\rho^2}\Big) v_1^{1(1)}\Big] u^+ +$$
$$+ \Big[\frac{\rho\alpha^3}{cRa}\frac{\partial}{\partial \rho}\Big(\frac{J}{\rho^2}\Big) u_1^{1(1)} + \frac{\alpha^2}{cRa}\frac{J^{1/2}}{\rho} p_1^{1(1)} + \frac{Ra J^{1/2}}{\rho} V v_1^{1(1)} - \frac{J\alpha^2}{\rho^2}\frac{\partial v_1^{1(1)}}{\partial \phi}\Big] v^+ +$$
$$+ \Big[\frac{J^{1/2} Ra}{\rho} V w_1^{1(1)} - \frac{J\alpha^2}{\rho^2}\frac{\partial w_1^{1(1)}}{\partial \phi}\Big] w^+ + \Big[\frac{cRa}{\rho} v_1^{1(1)}\Big] p^+ +$$
$$+ \frac{\alpha^2}{\rho^2}\Big[u_1^{1(1)}\frac{\partial}{\partial \phi}(u^+ J) + v_1^{1(1)}\frac{\partial}{\partial \phi}(v^+ J) + w_1^{1(1)}\frac{\partial}{\partial \phi}(w^+ J)\Big] \qquad (2.3.33)$$

The orthogonality condition involving the partial differential equation adjoint will be fulfilled if

$$\int_0^{2\pi}\Big[I_1(x,\phi)\Big]_{x=-\frac{1}{2}}^{x=\frac{1}{2}} d\phi = \int_{-\frac{1}{2}}^{\frac{1}{2}}\Big[I_2(x,\phi)\Big]_{\phi=0}^{\phi=2\pi} dx \equiv 0 \qquad (2.3.34)$$

These two conditions will generate the boundary conditions for the adjoint system as follows:

$$u^+ = v^+ = w^+ = 0 \quad at \quad x = \pm\frac{1}{2} \qquad (2.3.34a)$$

u^+, v^+, w^+ and their derivatives with respect to ϕ, are periodic in with period 2π.

The boundary conditions for p^+ are the same, and they follow automatically from the previous ones.

The adjoint partial differential equations system will be given by

$$\begin{cases} \bar{L}_{11} u^+ + \bar{L}_{12} v^+ + \bar{L}_{14} p^+ = 0 & (2.3.35) \\ \bar{L}_{21} u^+ + \bar{L}_{22} v^+ + \bar{L}_{24} p^+ = 0 & (2.3.36) \\ \bar{L}_{33} w^+ + \bar{L}_{34} p^+ = 0 & (2.3.37) \\ \bar{L}_{41} u^+ + \bar{L}_{42} v^+ + \bar{L}_{43} w^+ = 0 & (2.3.38) \end{cases}$$

As we shall later see, by expanding (u^+, v^+, w^+, p^+) in powers of it is possible to retrieve the adjoint system of Di Prima and Stuart (1972b), page 403, equations 5.7-5.9, and their adjoint system corresponds to our ε^0 order adjoint system.

$\varepsilon^{1/2}$ ORDER

$$\begin{cases} \varepsilon R_a J^{1/2}\left(U\frac{\partial u'^{(2)}}{\partial x} + \frac{\partial U}{\partial x}u'^{(2)}\right) + \frac{R_a J^{1/2}}{\rho}\left(V\frac{\partial u'^{(2)}}{\partial \phi} + \varepsilon c R_a \frac{\partial U}{\partial \phi} v'^{(2)}\right) - \\ - R_a \frac{\partial}{\partial \phi}\left(\frac{J^{1/2}}{\rho}\right)\left(V u'^{(2)} + \varepsilon c R_a U v'^{(2)}\right) + \rho \frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right) T_c(0,\varepsilon) V v'^{(2)} = \\ = -J^{1/2}\frac{\partial p'^{(2)}}{\partial x} + \left(J L^2 - \frac{\partial^2}{\partial \xi^2} - i\bar{\sigma}_0\right)u'^{(2)} - \alpha^2 H u'^{(2)} + \\ + c R_a \rho\left\{\alpha \frac{\partial}{\partial \rho}\left(\frac{J}{\rho^2}\right)\frac{\partial v'^{(2)}}{\partial \phi} - \frac{\partial}{\partial \phi}\left(\frac{J}{\rho^2}\right)\frac{\partial v'^{(2)}}{\partial x}\right\} - i\bar{\sigma}_1 u'^{(1)} + c T_c(0,\varepsilon) v'^{(1)} v_{TV1} + \\ + c R_a \frac{\partial}{\partial \phi}\left(\frac{J^{1/2}}{\rho}\right)\left(u'^{(1)} v_{TV1} + v'^{(1)} u_{TV1}\right) - J^{1/2}\left(u'^{(1)}\frac{\partial u_{TV1}}{\partial x} + \right. \\ \left. + u_{TV1}\frac{\partial u'^{(1)}}{\partial x}\right) - \frac{J^{1/2}}{\rho} c R_a \left(v_{TV1}\frac{\partial u'^{(1)}}{\partial \phi} + v'^{(1)}\frac{\partial u_{TV1}}{\partial \phi}\right) - \\ - w'^{(1)}\frac{\partial u_{TV1}}{\partial \xi} - w_{TV1}\frac{\partial u'^{(1)}}{\partial \xi} \end{cases} \qquad (2.3.39)$$

$$\begin{cases} J^{1/2}\left(\frac{1}{c}\frac{\partial V}{\partial x}u'^{(2)}+\varepsilon R_a U\frac{\partial v'^{(2)}}{\partial x}\right)+\frac{J^{1/2}R_a}{\rho}\left(V\frac{\partial v'^{(2)}}{\partial\phi}+\frac{\partial V}{\partial\phi}v'^{(2)}\right)- \\ -\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)\left(\frac{1}{c}Vu'^{(2)}+\varepsilon R_a Uv'^{(2)}\right)+\frac{2\alpha^2\varepsilon}{c}\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)u'^{(2)}U= \\ =-\frac{J^{1/2}}{\rho}\frac{\alpha^2}{cR_a}\frac{\partial p'^{(2)}}{\partial\phi}+\left(JL^2-\frac{\partial^2}{\partial\xi^2}-i\bar{\sigma}_0\right)v'^{(2)}-\alpha^2Hv'^{(2)}+ \\ +\frac{\rho\alpha^2}{cR_a}\left\{-\alpha\frac{\partial}{\partial\rho}\left(\frac{J}{\rho^2}\right)\frac{\partial u'^{(2)}}{\partial\phi}+\frac{\partial}{\partial\phi}\left(\frac{J}{\rho^2}\right)\frac{\partial u'^{(2)}}{\partial x}\right\}-J^{1/2}\left(u_{TV1}\frac{\partial v'^{(1)}}{\partial x}+\right. \\ \left.+u'^{(1)}\frac{\partial v_{TV1}}{\partial x}\right)-cR_a\frac{J^{1/2}}{\rho}\frac{\partial}{\partial\phi}\left(v_{TV1}\cdot v'^{(1)}\right)-w_{TV1}\frac{\partial v'^{(1)}}{\partial\xi}- \\ -w'^{(1)}\frac{\partial v_{TV1}}{\partial\xi}+\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)\left(u_{TV1}\cdot v'^{(1)}+u'^{(1)}\cdot v_{TV1}\right)- \\ -\frac{2\alpha^2}{cR_a}\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)u_{TV1}\cdot u'^{(1)}-i\bar{\sigma}_1v'^{(1)} \end{cases} \tag{2.3.40}$$

$$\begin{cases} \varepsilon R_a J^{1/2}U\frac{\partial w'^{(2)}}{\partial x}+\frac{1}{\rho}J^{1/2}R_aV\frac{\partial w'^{(2)}}{\partial\phi}=-\frac{\partial p'^{(2)}}{\partial\xi}+\left(JL^2-\frac{\partial^2}{\partial\xi^2}-i\bar{\sigma}_0\right)w'^{(2)}- \\ -i\bar{\sigma}_1w'^{(1)}-J^{1/2}\left(u_{TV1}\frac{\partial w'^{(1)}}{\partial x}+u'^{(1)}\frac{\partial w_{TV1}}{\partial x}\right)- \\ -\frac{cR_a}{\rho}J^{1/2}\left(v_{TV1}\frac{\partial w'^{(1)}}{\partial\phi}+v'^{(1)}\frac{\partial w_{TV1}}{\partial\phi}\right)-\frac{\partial}{\partial\xi}\left(w_{TV1}\cdot w'^{(1)}\right) \end{cases} \tag{2.3.41}$$

$$\begin{cases} J^{1/2}\frac{\partial u'^{(2)}}{\partial x}-\alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)u'^{(2)}+cR_a\left[\frac{J^{1/2}}{\rho}\frac{\partial v'^{(2)}}{\partial\phi}-\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)v'^{(2)}\right]+ \\ +\frac{\partial}{\partial\xi}w'^{(2)}=0 \end{cases} \tag{2.3.42}$$

It can be mentioned that proper account has to be taken of the linearity of v' terms.

It is convenient to separate these different harmonics by writing:

$$u'^{(2)}=u_1'^{(2)}(x,\phi,\tau)\sin\lambda\xi+u_0'^{(2)}(x,\phi,\tau)+u_2'^{(2)}(x,\phi,\tau)\sin 2\lambda\xi \tag{2.3.43}$$

$$v'^{(2)}=v_1'^{(2)}(x,\phi,\tau)\sin\lambda\xi+v_0'^{(2)}(x,\phi,\tau)+v_2'^{(2)}(x,\phi,\tau)\sin 2\lambda\xi \tag{2.3.44}$$

$$w'^{(2)}=w_1'^{(2)}(x,\phi,\tau)\cos\lambda\xi+w_0'^{(2)}(x,\phi,\tau)+w_2'^{(2)}(x,\phi,\tau)\cos 2\lambda\xi \tag{2.3.45}$$

$$p'^{(2)}=p_1'^{(2)}(x,\phi,\tau)\sin\lambda\xi+p_0'^{(2)}(x,\phi,\tau)+p_2'^{(2)}(x,\phi,\tau)\sin 2\lambda\xi \tag{2.3.46}$$

One obtains a differential system for $(u_1'^{(2)}, v_1'^{(2)}, w_1'^{(2)}, p_1'^{(2)})$

$$L_{11}\,u_1^{1(2)} + L_{12}\,v_1^{1(2)} + L_{14}\,p_1^{1(2)} = -i\bar{\sigma}_1\,u_1^{1(1)} \qquad (2.3.47)$$

$$L_{21}\,u_1^{1(2)} + L_{22}\,v_1^{1(2)} + L_{24}\,p_1^{1(2)} = -i\bar{\sigma}_1\,v_1^{1(1)} \qquad (2.3.48)$$

$$L_{33}\,w_1^{1(2)} + L_{34}\,p_1^{1(2)} = -i\bar{\sigma}_1\,w_1^{1(1)} \qquad (2.3.49)$$

$$L_{41}\,u_1^{1(2)} + L_{42}\,v_1^{1(2)} + L_{43}\,w_1^{1(2)} = 0 \qquad (2.3.50)$$

By multiplying the differential equations (2.3.47) - (2.3.50) with the adjoint (u^+, v^+, w^+, p^+) respectively, adding and integrating in X over the interval $(-\frac{1}{2}, \frac{1}{2})$, and Φ over the interval $(0, 2\pi)$, one obtains:

$$0 = -i\bar{\sigma}_1 \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[u_1^{1(1)}u^+ + v_1^{1(1)}v^+ + w_1^{1(1)}w^+\right] dx\,d\Phi \qquad (2.3.51)$$

from that relation we reach the conclusion that $\bar{\sigma}_1 \equiv 0$ (2.3.52)

Differential system for $(u_0^{1(2)}, v_0^{1(2)}, w_0^{1(2)}, p_0^{1(2)})$:

$$L_{110}\,u_0^{1(2)} + L_{12}\,v_0^{1(2)} + L_{14}\,p_0^{1(2)} = 0 \qquad (2.3.53)$$

$$L_{21}\,u_0^{1(2)} + L_{220}\,v_0^{1(2)} + L_{24}\,p_0^{1(2)} = 0 \qquad (2.3.54)$$

$$L_{330}\,w_0^{1(2)} = \frac{J^{1/2}}{2T_1^{1/2}} B(\Phi)\left[f_0\frac{\partial w_1^{1(1)}}{\partial X} - \frac{D^2 f_0}{\lambda}u_1^{1(1)} - \frac{cR_a g_0}{\rho}\frac{\partial w_1^{1(1)}}{\partial \Phi}\right] -$$

$$- \frac{cR_a J^{1/2}}{2\rho\lambda T_1^{1/2}} v_1^{1(1)} \frac{dB(\Phi)}{d\Phi} Df_0 \qquad (2.3.55)$$

$$L_{41}\,u_0^{1(2)} + L_{42}\,v_0^{1(2)} = 0 \qquad (2.3.56)$$

where

$$L_{110} = \varepsilon R_a J^{1/2}\left(U\frac{\partial}{\partial X} + \frac{\partial U}{\partial X}\right) + \frac{R_a J^{1/2}}{\rho} V\frac{\partial}{\partial \Phi} - R_a\frac{\partial}{\partial \Phi}\left(\frac{J^{1/2}}{\rho}\right)V -$$

$$- (JL^2 - i\bar{\sigma}_0) \qquad (2.3.57)$$

$$L_{220} = \varepsilon R_a U J^{1/2}\frac{\partial}{\partial X} + \frac{J^{1/2}R_a}{\rho}\left(V\frac{\partial}{\partial \Phi} + \frac{\partial V}{\partial \Phi}\right) - \alpha\rho\varepsilon R_a U\frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right) +$$

$$+ \alpha^2 H - (JL^2 - i\bar{\sigma}_0) \qquad (2.3.58)$$

$$L_{330} = \varepsilon R_a J^{1/2} U\frac{\partial}{\partial X} + \frac{1}{\rho}J^{1/2}R_a V\frac{\partial}{\partial \Phi} - (JL^2 - i\bar{\sigma}_0) \qquad (2.3.59)$$

It is sensible to assume:

$$\left(u_0^{1(2)}, v_0^{1(2)}, p_0^{1(2)}\right) \equiv 0 \qquad (2.3.60)$$

and $w_0^{1(2)}$ will be defined by the differential equation (2.3.55). It is clear that the interactions between the Taylor vortex terms and the wavy vortex terms, will generate the mean term $w_0^{1(2)}$ and the harmonic 2λ.

The differential system for $(u_2^{1(2)}, v_2^{1(2)}, w_2^{1(2)}, p_2^{1(2)})$:

$$L_{112}\, u_2^{1(2)} + L_{12}\, v_2^{1(2)} + L_{14}\, p_2^{1(2)} = \frac{B(\Phi)}{2T_1^{1/2}}\left\{ cT_c(0,\varepsilon)\, g_0\, v_1^{1(1)} + \right.$$
$$+ cR_a \frac{\partial}{\partial\Phi}\left(\frac{J^{1/2}}{\rho}\right)\left[g_0 u_1^{1(1)} - f_0 v_1^{1(1)}\right] + J^{1/2}\left(u_1^{1(1)} Df_0 + f_0 \frac{\partial u_1^{1(1)}}{\partial x}\right) - \lambda f_0 w_1^{1(1)} -$$
$$\left. - Df_0 \cdot u_1^{1(1)} - \frac{cR_a J^{1/2}}{\rho}\left[g_0 \frac{\partial u_1^{1(1)}}{\partial\Phi} - f_0 \frac{v_1^{1(1)}}{B(\Phi)} \frac{dB(\Phi)}{d\Phi}\right]\right\} \qquad (2.3.61)$$

$$L_{21}\, u_2^{1(2)} + L_{222}\, v_2^{1(2)} + L_{24}\, p_2^{1(2)} = \frac{B(\Phi)}{2T_1^{1/2}}\left[J^{1/2}\left(f_0 \frac{\partial v_1^{1(1)}}{\partial x} - Dg_0 \cdot u_1^{1(1)}\right) - \right.$$
$$- \frac{cR_a J^{1/2} g_0}{\rho B(\Phi)} \frac{\partial}{\partial\Phi}\left(B(\Phi)\cdot v_1^{1(1)}\right) - Df_0 \cdot v_1^{1(1)} + \lambda g_0\, w_1^{1(1)} + \frac{\alpha^2}{cR_a} f_0 u_1^{1(1)} \frac{\partial}{\partial\Phi}\left(\frac{J^{1/2}}{\rho}\right) +$$
$$\left. + \alpha\rho \frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)\left(g_0 u_1^{1(1)} - f_0 v_1^{1(1)}\right)\right] \qquad (2.3.62)$$

$$L_{332}\, w_2^{1(2)} + 2\lambda\, p_2^{1(2)} = \frac{B(\Phi)}{2T_1^{1/2}}\left\{ J^{1/2}\left(f_0 \frac{\partial w_1^{1(1)}}{\partial x} + \frac{D^2 f_0}{\lambda} u_1^{1(1)}\right) - 2Df_0\, w_1^{1(1)} - \right.$$
$$\left. - \frac{cR_a J^{1/2}}{\rho}\left[g_0 \frac{\partial w_1^{1(1)}}{\partial\Phi} - \frac{1}{B(\Phi)} \frac{dB(\Phi)}{d\Phi} \frac{Df_0}{\lambda} v_1^{1(1)}\right]\right\} \qquad (2.3.63)$$

$$L_{41}\, u_2^{1(2)} + L_{42}\, v_2^{1(2)} - 2\lambda\, w_2^{1(2)} = 0 \qquad (2.3.64)$$

where

$$L_{112} = \varepsilon R_a J^{1/2}\left(U\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right) + \frac{R_a J^{1/2}}{\rho} V \frac{\partial}{\partial\Phi} - R_a \frac{\partial}{\partial\Phi}\left(\frac{J^{1/2}}{\rho}\right) V -$$
$$- \left(JL^2 - 4\lambda^2 - i\bar{\sigma}_0\right) \qquad (2.3.65)$$

$$L_{222} = \varepsilon R_a U J^{1/2} \frac{\partial}{\partial x} + \frac{J^{1/2} R_a}{\rho}\left(V\frac{\partial}{\partial\Phi} + \frac{\partial V}{\partial\Phi}\right) - \alpha\rho\varepsilon R_a U \frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) +$$
$$+ \alpha^2 H - \left(JL^2 - 4\lambda^2 - i\bar{\sigma}_0\right) \qquad (2.3.66)$$

$$L_{332} = \varepsilon R_a J^{1/2} U \frac{\partial}{\partial x} + \frac{1}{\rho} J^{1/2} R_a V \frac{\partial}{\partial\Phi} - \left(JL^2 - 4\lambda^2 - i\bar{\sigma}_0\right) \qquad (2.3.67)$$

γ ORDER

$$\begin{cases} L'_{11}u^{1(3)} + L_{12}v^{1(3)} + L_{14}p^{1(3)} = -i\bar{\sigma}_2 u^{1(1)} - \rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)Vv^{1(1)} + \\ + cT_c(0,\varepsilon)\left(v_{TV1}\cdot v^{1(2)} + v_{TV2}\cdot v^{1(1)}\right) + cR_a\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)\left(u_{TV1}\cdot v^{1(2)} + u_{TV2}\cdot v^{1(1)} + \right. \\ \left. + u^{1(1)}\cdot v_{TV2} + u^{1(2)}v_{TV1}\right) - J^{1/2}\frac{\partial}{\partial x}\left(u_{TV1}\cdot u^{1(2)} + u_{TV2}\cdot u^{1(1)}\right) - \\ - \frac{J^{1/2}}{\rho}cR_a\left(v_{TV1}\frac{\partial u^{1(2)}}{\partial\phi} + v_{TV2}\frac{\partial u^{1(1)}}{\partial\phi} + v^{1(1)}\frac{\partial u_{TV2}}{\partial\phi} + v^{1(2)}\frac{\partial u_{TV1}}{\partial\phi}\right) - \\ - w^{1(1)}\frac{\partial u_{TV2}}{\partial\xi_F} - w^{1(2)}\frac{\partial u_{TV1}}{\partial\xi_F} - w_{TV1}\frac{\partial u^{1(2)}}{\partial\xi_F} - w_{TV2}\frac{\partial u^{1(1)}}{\partial\xi_F} \end{cases} \quad (2.3.68)$$

$$\begin{cases} L_{21}u^{1(3)} + L'_{22}v^{1(3)} + L_{24}p^{1(3)} = -i\bar{\sigma}_2 v^{1(1)} - J^{1/2}\left(u_{TV1}\frac{\partial v^{1(2)}}{\partial x} + u_{TV2}\frac{\partial v^{1(1)}}{\partial x} + \right. \\ \left. + u^{1(1)}\frac{\partial v_{TV2}}{\partial x} + u^{1(2)}\frac{\partial v_{TV1}}{\partial x}\right) - \frac{cR_aJ^{1/2}}{\rho}\frac{\partial}{\partial\phi}\left(v_{TV1}\cdot v^{1(2)} + v_{TV2}\cdot v^{1(1)}\right) - \\ - w_{TV1}\cdot\frac{\partial v^{1(2)}}{\partial\xi_F} - w_{TV2}\frac{\partial v^{1(1)}}{\partial\xi_F} - w^{1(1)}\frac{\partial v_{TV2}}{\partial\xi_F} - w^{1(2)}\frac{\partial v_{TV1}}{\partial\xi_F} + \\ + \alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)\left(u_{TV1}\cdot v^{1(2)} + u_{TV2}\cdot v^{1(1)} + u^{1(1)}\cdot v_{TV2} + u^{1(2)}v_{TV1}\right) - \\ - \frac{2\alpha^2}{cR_a}\left(u_{TV1}\cdot u^{1(2)} + u_{TV2}\cdot u^{1(1)}\right)\cdot\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right) \end{cases} \quad (2.3.69)$$

$$\begin{cases} L'_{33}w^{1(3)} + \frac{\partial p^{1(3)}}{\partial\xi_F} = -i\bar{\sigma}_2 w^{1(1)} - 2\frac{\partial}{\partial\xi_F}\left(w_{TV1}\cdot w^{1(2)} + w_{TV2}\cdot w^{1(1)}\right) - \\ - J^{1/2}\left(u_{TV1}\cdot\frac{\partial w^{1(2)}}{\partial x} + u_{TV2}\cdot\frac{\partial w^{1(1)}}{\partial x} + u^{1(1)}\frac{\partial w_{TV2}}{\partial x} + u^{1(2)}\frac{\partial w_{TV1}}{\partial x}\right) - \\ - \frac{cR_aJ^{1/2}}{\rho}\left(v_{TV1}\frac{\partial w^{1(2)}}{\partial\phi} + v_{TV2}\frac{\partial w^{1(1)}}{\partial\phi} + v^{1(1)}\frac{\partial w_{TV2}}{\partial\phi} + v^{1(2)}\frac{\partial w_{TV1}}{\partial\phi}\right) \end{cases} \quad (2.3.70)$$

$$L_{41}u^{1(3)} + L_{42}v^{1(3)} + \frac{\partial}{\partial\xi_F}w^{1(3)} = 0 \quad (2.3.71)$$

where

$$L'_{11} = \varepsilon R_aJ^{1/2}\left(U\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\right) + \frac{R_aJ^{1/2}}{\rho}V\frac{\partial}{\partial\phi} - R_a\frac{\partial}{\partial\phi}\left(\frac{J^{1/2}}{\rho}\right)V - \\ - \left(JL^2 - \frac{\partial^2}{\partial\xi_F^2} - i\bar{\sigma}_0\right) \quad (2.3.72)$$

$$L'_{22} = \varepsilon R_aUJ^{1/2}\frac{\partial}{\partial x} + \frac{J^{1/2}R_a}{\rho}\left(V\frac{\partial}{\partial\phi} + \frac{\partial V}{\partial\phi}\right) - \alpha\rho\varepsilon R_aU\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) - \\ - \left(JL^2 - \frac{\partial^2}{\partial\xi_F^2} - i\bar{\sigma}_0\right) \quad (2.3.73)$$

$$L'_{33} = \varepsilon R_aJ^{1/2}U\frac{\partial}{\partial x} + \frac{1}{\rho}J^{1/2}R_aV\frac{\partial}{\partial\phi} - \left(JL^2 - \frac{\partial^2}{\partial\xi_F^2} - i\bar{\sigma}_0\right) \quad (2.3.74)$$

and where $i\bar{\sigma}_2$ is the coefficient of γ in the growth rate of the wavy vortex mode.

It is again convenient to separate the harmonics by writing:

$$u^{1(3)} = u_1^{1(3)} \sin \lambda\xi + u_2^{1(3)} \sin 2\lambda\xi + u_3^{1(3)} \sin 3\lambda\xi \qquad (2.3.75)$$

$$v^{1(3)} = v_1^{1(3)} \sin \lambda\xi + v_2^{1(3)} \sin 2\lambda\xi + v_3^{1(3)} \sin 3\lambda\xi \qquad (2.3.76)$$

$$w^{1(3)} = w_1^{1(3)} \cos \lambda\xi + w_2^{1(3)} \cos 2\lambda\xi + w_3^{1(3)} \cos 3\lambda\xi \qquad (2.3.77)$$

$$p^{1(3)} = p_1^{1(3)} \sin \lambda\xi + p_2^{1(3)} \sin 2\lambda\xi + p_3^{1(3)} \sin 3\lambda\xi \qquad (2.3.78)$$

Differential system for $(u_1^{1(3)}, v_1^{1(3)}, w_1^{1(3)}, p_1^{1(3)})$:

$$\begin{cases} L_{11} u_1^{1(3)} + L_{12} v_1^{1(3)} + L_{14} p_1^{1(3)} = -i\overline{\sigma_2}\, u_1^{1(1)} - \rho \frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right) V v_1^{1(1)} + \\ + \frac{cT_c(0,\varepsilon)B(\Phi)}{2T_1^{1/2}} g_0 v_2^{1(2)} + \frac{cT_c(0,\varepsilon)}{T_1} v_1^{1(1)} \left[B^2(\Phi) g_{10} + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) - \right. \\ \left. - \frac{B^2(\Phi) g_{112}}{2}\right] + cR_a \frac{\partial}{\partial\Phi}\left(\frac{J^{1/2}}{\rho}\right)\left\{ - \frac{B(\Phi)}{2T_1^{1/2}} g_0 v_2^{1(2)} + \frac{v_1^{1(1)}}{2T_1} g_{112} B^2(\Phi) + \right. \\ \left. + \frac{u_1^{1(1)}}{T_1}\left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4})\right] + \frac{B(\Phi)}{2T_1^{1/2}} g_0 u_2^{1(2)}\right\} - \\ - \frac{J^{1/2} B(\Phi)}{2T_1^{1/2}}\left[\frac{B(\Phi)}{T_1^{1/2}} D(g_{112}\cdot u_1^{1(1)}) - D(g_0 u_2^{1(2)})\right] - \frac{J^{1/2} cR_a}{\rho}\left\{\frac{B(\Phi)}{2T_1^{1/2}} g_0 \frac{\partial u_2^{1(2)}}{\partial\Phi} + \right. \\ + \frac{1}{T_1}\frac{\partial u_1^{1(1)}}{\partial\Phi}\left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4})\right] + \frac{B(\Phi)}{T_1}\frac{dB(\Phi)}{d\Phi} v_1^{1(1)} g_{112} - \\ \left. - \frac{1}{2T_1^{1/2}}\frac{dB(\Phi)}{d\Phi} g_0 v_2^{1(2)}\right\} - \frac{\lambda B^2(\Phi)}{T_1} g_{112}\cdot w_1^{1(1)} - \frac{\lambda B(\Phi) g_0}{T_1^{1/2}}\left(w_0^{1(2)} - \frac{w_2^{1(2)}}{2}\right) + \\ + \frac{B(\Phi) D g_0}{T_1^{1/2}} u_2^{1(2)} - \frac{B^2(\Phi)}{4T_1} u_1^{1(1)} D g_{112} \end{cases} \qquad (2.3.79)$$

$$\begin{cases} L_{21} u_1^{1(3)} + L_{22} v_1^{1(3)} + L_{24} p_1^{1(3)} = -i\overline{\sigma_2}\, v_1^{1(1)} - J^{1/2}\left\{\frac{B(\Phi)}{2T_1^{1/2}}\left[u_2^{1(2)} D g_0 - \right.\right. \\ \left. - g_0 D v_2^{1(2)}\right] + \frac{B^2(\Phi)}{2T_1} g_{112}\frac{\partial v_1^{1(1)}}{\partial x} + \frac{u_1^{1(1)}}{T_1}\left[B^2(\Phi)\cdot D(g_{10} - \frac{g_{112}}{2}) + \sigma_0 x\right]\Big\} - \\ - \frac{cR_a J^{1/2}}{\rho}\frac{\partial}{\partial\Phi}\left\{\frac{B(\Phi)}{2T_1^{1/2}} g_0 v_2^{1(2)} + \frac{v_1^{1(1)}}{T_1}\left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4})\right]\right\} - \\ - \frac{\alpha^2}{cR_a}\frac{\partial}{\partial\Phi}\left(\frac{J^{1/2}}{\rho}\right)\cdot\frac{B(\Phi)}{T_1^{1/2}}\left[\frac{B(\Phi)}{T_1^{1/2}} u_1^{1(1)} g_{112} - g_0 u_2^{1(2)}\right] + \frac{B(\Phi) D g_0 v_2^{1(2)}}{T_1^{1/2}} + \\ + \frac{B^2(\Phi)}{T_1}\left[\lambda w_1^{1(1)} g_{112} - \frac{D g_{112}}{4} v_1^{1(1)}\right] + \frac{\lambda B(\Phi) g_0}{T_1^{1/2}}\left(w_0^{1(2)} - \frac{w_2^{1(2)}}{2}\right) + \\ + \alpha\rho\frac{\partial}{\partial\rho}\left(\frac{J^{1/2}}{\rho}\right)\cdot\left\{ - \frac{B(\Phi)}{2T_1^{1/2}} g_0 v_2^{1(2)} + \frac{v_1^{1(1)}}{2T_1} g_{112}\cdot B^2(\Phi) + \right. \\ \left. + \frac{u_1^{1(1)}}{T_1}\left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4})\right] + \frac{B(\Phi)}{2T_1^{1/2}} g_0 u_2^{1(2)}\right\} \end{cases} \qquad (2.3.80)$$

$$
\begin{cases}
L_{33} w_1^{1(3)} + L_{34} p_1^{1(3)} = -i\bar{\sigma}_2 w_1^{1(1)} + \dfrac{B(\phi)}{2T_1^{1/2}}\left[D f_0 \cdot w_2^{1(2)} - \dfrac{B(\phi)}{2T_1^{1/2}} w_1^{1(1)} D f_{112} - \right. \\
\left. - 2D f_0 \cdot w_0^{1(2)}\right] + J^{1/2} \dfrac{B^2(\phi)}{4\lambda T_1}\left[-u_1^{1(1)} D^2 f_{112} + 2\lambda f_{112} D w_1^{1(1)}\right] + \\
+ J^{1/2} \cdot \dfrac{B(\phi)}{T_1^{1/2}}\left[-u_2^{1(2)} \dfrac{D^2 f_0}{2\lambda} + \dfrac{f_0}{2} D w_2^{1(2)} + f_0 \dfrac{\partial w_0^{1(2)}}{\partial x}\right] - \\
- \dfrac{cR_a J^{1/2}}{\rho}\left\{\dfrac{B(\phi) g_0}{2T_1^{1/2}} \dfrac{\partial w_2^{1(2)}}{\partial \phi} + \dfrac{v_1^{1(1)}}{4\lambda T_1} D f_{112} \dfrac{dB^2(\phi)}{d\phi} + \dfrac{v_2^{1(2)} D f_0}{2\lambda T_1^{1/2}} \dfrac{dB(\phi)}{d\phi} + \right. \\
\left. + \dfrac{B(\phi) g_0}{T_1^{1/2}} \dfrac{\partial w_0^{1(2)}}{\partial \phi} + \dfrac{1}{T_1} \dfrac{\partial w_1^{1(1)}}{\partial \phi}\left[B^2(\phi)\cdot\left(g_{110} + \dfrac{g_{112}}{2}\right) + \dfrac{\sigma_0}{2}\left(x^2 - \dfrac{1}{4}\right)\right]\right\} \quad (2.3.81)
\end{cases}
$$

$$
L_{41} u_1^{1(3)} + L_{42} v_1^{1(3)} + L_{43} w_1^{1(3)} = 0 \qquad (2.3.82)
$$

A shorter way of writing the differential system for $(u_1^{1(3)}, v_1^{1(3)}, w_1^{1(3)}, p_1^{1(3)})$

$$
L_{11} u_1^{1(3)} + L_{12} v_1^{1(3)} + L_{14} p_1^{1(3)} = -i\bar{\sigma}_2 u_1^{1(1)} + \Delta_1 \qquad (2.3.83)
$$

$$
L_{21} u_1^{1(3)} + L_{22} v_1^{1(3)} + L_{24} p_1^{1(3)} = -i\bar{\sigma}_2 v_1^{1(1)} + \Delta_2 \qquad (2.3.84)
$$

$$
L_{33} w_1^{1(3)} + L_{34} p_1^{1(3)} = -i\bar{\sigma}_2 w_1^{1(1)} + \Delta_3 \qquad (2.3.85)
$$

$$
L_{41} u_1^{1(3)} + L_{42} v_1^{1(3)} + L_{43} w_1^{1(3)} = 0 \qquad (2.3.86)
$$

Once again, the compatibility of this differential system (2.3.79) - (2.3.82) requires the fulfilment of an orthogonality condition involving the partial differential equations adjoint (u^+, v^+, w^+, p^+).

By multiplying the four differential equations with the adjoint (u^+, v^+, w^+, p^+) respectively, adding and integrating in X over $(-\frac{1}{2}, \frac{1}{2})$, and ϕ over $(0, 2\pi)$ we must obtain zero.

Finally, one obtains:

$$
i\bar{\sigma}_2 = i\bar{\sigma}_2(TV=0) + i\bar{\sigma}_2(TV) = \frac{I_2}{I_1} \qquad (2.3.87)
$$

where $$I_1 = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[u_1^{1(1)} u^+ + v_1^{1(1)} v^+ + w_1^{1(1)} w^+\right] dx\, d\phi \qquad (2.3.88)$$

and $$I_2 = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\Delta_1 u^+ + \Delta_2 v^+ + \Delta_3 w^+\right] dx\, d\phi \qquad (2.3.89)$$

$i\bar{\sigma}_2(TV=0)$ is the growth rate when there is no Taylor vortex mode present, while $i\bar{\sigma}_2(TV)$ denotes the contribution to the growth rate which arises from the influence of the Taylor vortex mode.

2.4 Expansion in powers of ε in the integrals

We expand $(u_1^{\prime(1)}, v_1^{\prime(1)}, w_1^{\prime(1)}, p_1^{\prime(1)})$, (u^+, v^+, w^+, p^+) and $(u_2^{\prime(2)}, v_2^{\prime(2)}, w_2^{\prime(2)}, p_2^{\prime(2)})$ in powers of ε.

$$u_1^{\prime(1)} = \hat{u}_{10} + \varepsilon \hat{u}_{11} + \varepsilon^2 \hat{u}_{12} + \cdots \quad (2.4.1)$$

$$v_1^{\prime(1)} = \hat{v}_{10} + \varepsilon \hat{v}_{11} + \varepsilon^2 \hat{v}_{12} + \cdots \quad (2.4.2)$$

$$w_1^{\prime(1)} = \hat{w}_{10} + \varepsilon \hat{w}_{11} + \varepsilon^2 \hat{w}_{12} + \cdots \quad (2.4.3)$$

$$p_1^{\prime(1)} = \hat{p}_{10} + \varepsilon \hat{p}_{11} + \varepsilon^2 \hat{p}_{12} + \cdots \quad (2.4.3a)$$

$$\overline{\sigma_0} = \sigma_{00} + \varepsilon \sigma_{01} + \varepsilon^2 \sigma_{02} + \cdots \quad (2.4.3b)$$

$$T_c(0,\varepsilon) = T_0 + \varepsilon^2 T_2(0) = T_0 + \varepsilon^2 \frac{\overline{\Gamma}_3 + \ell_2^2 \overline{\Gamma}_4}{\overline{\Gamma}_1} \quad (2.4.4)$$

$$R_a = \left(\frac{T\alpha}{2c}\right)^{1/2} = \left(\frac{\alpha}{2c}\right)^{1/2} (T_c + \gamma)^{1/2} = \left(\frac{\alpha}{2c}\right)^{1/2} \left(T_c^{1/2} + \frac{\gamma}{2T_c^{1/2}} + \cdots\right) \quad (2.4.4a)$$

Thus we proceed to the solution of the ε^0, ε and ε^2 order simplified equations for the differential system (2.3.1) - (2.3.4)

ε^0 ORDER

$$-M\hat{u}_{10} + i\sigma_{00}\hat{u}_{10} - T_0 V_0 \hat{v}_{10} + \frac{\partial \hat{p}_{10}}{\partial x} = 0 \quad (2.4.5)$$

$$-\hat{u}_{10} - M\hat{v}_{10} + i\sigma_{00}\hat{v}_{10} = 0 \quad (2.4.6)$$

$$-M\hat{w}_{10} + i\sigma_{00}\hat{w}_{10} + \lambda \hat{p}_{10} = 0 \quad (2.4.7)$$

$$\frac{\partial \hat{u}_{10}}{\partial x} - \lambda \hat{w}_{10} = 0 \quad (2.4.8)$$

where $i\sigma_{00}$ is the damping rate of the wavy vortex mode when $\varepsilon = 0$ and $T = 1695$.

After the elimination of $\hat{w}_{10}$ and $\hat{p}_{10}$, one obtains:

$$\hat{u}_{10} = -A(\phi) \cdot f_0(x) \quad (2.4.9)$$

$$\hat{v}_{10} = A(\phi) \cdot g_0(x) \quad (2.4.10)$$

where (f_0, g_0) are providing the eigenvalue of the differential system:

$$\begin{cases} M^2 f_0 + \lambda^2 T_0 V_0 g_0 - \sigma_{00} M f_0 = 0 & (2.4.11) \\ f_0 - M g_0 + \sigma_{00} g_0 = 0 & (2.4.12) \\ f_0 = Df_0 = g_0 = 0 \text{ at } x = \pm \frac{1}{2} & (2.4.13) \end{cases}$$

ε ORDER

By eliminating $\hat{w}_{11}$ and $\hat{p}_{11}$, we get the following differential system for $\hat{u}_{11}$ and $\hat{v}_{11}$:

$$\begin{cases} M^2\hat{\bar{u}}_{11} + \lambda^2 T_0 V_0 \hat{v}_{11} + i\sigma_{00} M\hat{u}_{11} = A(\phi)\cos\phi\, F_{111}(x) + \\ \qquad + kT_0^{1/2}\dfrac{dA(\phi)}{d\phi} F_{112}(x) + i\sigma_{01} A(\phi) M f_0(x) \\ \hat{\bar{u}}_{11} - M\hat{v}_{11} + i\sigma_{00}\hat{v}_{11} = A(\phi)\cos\phi\cdot G_{111}(x) + kT_0^{1/2}\dfrac{dA(\phi)}{d\phi} G_{112}(x) - \\ \qquad - i\sigma_{01} A(\phi) g_0(x) \end{cases} \quad (2.4.14) \quad (2.4.15)$$

where $-\hat{u}_{11} = \hat{\bar{u}}_{11}$ (2.4.16)

and where $F_{111}(x)$, $F_{112}(x)$, $G_{111}(x)$ and $G_{112}(x)$ are given in the linear theory. Once again, by multiplying with the adjoint (f_0^+, g_0^+), adding and integrating in X over $(-\frac{1}{2}, \frac{1}{2})$, we get:

$$kT_0^{1/2}\frac{dA(\phi)}{d\phi}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+ F_{112} + g_0^+ G_{112})dx + \cos\phi\cdot A(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+ F_{111} + g_0^+ G_{111})dx$$

$$+ i\sigma_{01} A(\phi)\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+ F_{114} + g_0^+ G_{114})dx = 0 \quad (2.4.17)$$

where $F_{114} = M f_0(x)$ (2.4.18)

$$G_{114} = -g_0(x) \quad (2.4.19)$$

as defined in the linear theory of this thesis.

Dividing the equation (2.3.17) by $\int_{-\frac{1}{2}}^{\frac{1}{2}}(f_0^+ F_{112} + g_0^+ G_{112})$ we have the following ordinary differential equation.

$$kT_0^{1/2}\frac{dA(\phi)}{d\phi} = (\Gamma\cos\phi - i\bar{c}\sigma_{01})A(\phi) \quad (2.4.20)$$

We assume $A(\phi) = e^{im\phi}F(\phi)$ (2.4.21)

Differential equation for $F(\phi)$

$$kT_0^{1/2}\left[\frac{dF(\phi)}{d\phi} + im F(\phi)\right] = (\Gamma\cos\phi - i\bar{c}\sigma_{01r} + \bar{c}\,\sigma_{01i})F(\phi) \quad (2.4.22)$$

where $\sigma_{01} = \sigma_{01r} + i\sigma_{01i}$ (2.4.23)

By making the imaginary parts of the right and left hand side of (2.4.22) equal, one obtains:

$$\sigma_{01r} = -\frac{m k T_0^{1/2}}{c} \qquad (2.4.24)$$

It is necessary also to take into account the periodicity condition in Φ with period 2π, it yields

$$\sigma_{01i} \equiv 0 \qquad (2.4.25)$$

and

$$A(\Phi) = e^{im\Phi} \cdot A_0(k) e^{\frac{\Gamma(\sin\Phi - 1)}{k T_0^{1/2}}} \qquad (2.4.26)$$

Consequently,

$$-\hat{u}_{11} = A(\Phi)\left[\cos\Phi \cdot f_1(x) + im k T_0^{1/2} f_3\right] + A_1(\Phi) f_0 \qquad (2.4.27)$$

$$\hat{v}_{11} = A(\Phi)\left[\cos\Phi \cdot g_1(x) + im k T_0^{1/2} g_3\right] + A_1(\Phi) g_0 \qquad (2.4.28)$$

where (f_1, g_1) and (f_3, g_3) provide the eigen value of the following differential systems, respectively:

$$\begin{cases} M^2 f_1 + \lambda^2 T_0 V_0 g_1 - \sigma_{00} M f_1 = -6\lambda^2 T_0 (x^2 - \frac{1}{4}) g_0 + 4(M - \frac{\sigma_{00}}{2}) D^2 f_0 + \Gamma V_0 M f_0 & (2.4.29) \\ f_1 - M g_1 + \sigma_{00} g_1 = \left(\frac{12x}{c} + 1\right) f_0 - 2 D^2 g_0 - \Gamma V_0 g_0 & (2.4.30) \\ f_1 = D f_1 = g_1 = 0 \quad at \quad x = \pm \frac{1}{2} & (2.4.31) \end{cases}$$

$$\begin{cases} M^2 f_3 + \lambda^2 T_0 V_0 g_3 - \sigma_{00} M f_3 = (V_0 - \frac{1}{c}) M f_0 & (2.4.32) \\ f_3 - M g_3 + \sigma_{00} g_3 = -g_0 (V_0 - \frac{1}{c}) & (2.4.33) \\ f_3 = D f_3 = g_3 = 0 \quad at \quad x = \pm \frac{1}{2} & (2.4.34) \end{cases}$$

We note that (f_3, g_3) are defined in the same way as in the linear part of this work.

ORDER ε^2

By elimination of $\hat{p}_{12}$ and $\hat{w}_{12}$ and use of some previous relations obtained from the ε^0 and ε order systems, one obtains

$$\begin{cases} -M^2\hat{u}_{12}+\lambda^2 T_0 V_0\hat{v}_{12}+i\sigma_{00}M\hat{u}_{12}=A_1(\phi)\cos\phi\cdot F_{111}(x)+ \\ +kT_0^{1/2}\dfrac{dA_1(\phi)}{d\phi}F_{112}(x)-\dfrac{imkT_0^{1/2}}{\bar{c}}Mg_0\cdot A_1(\phi)+A(\phi)F_{123}(x)+ \\ +k^2A(\phi)F_{124}(x)+A(\phi)\cos 2\phi\cdot F_{125}(x)+kT_0^{1/2}A(\phi)\sin\phi\cdot F_{126}(x)+ \\ +T_1\cos\phi\cdot A(\phi)F_{128}(x)+imkT_0^{1/2}A(\phi)\cos\phi\cdot F_{129}(x)+\dfrac{imkT_1}{T_0^{1/2}}A(\phi)F_{130}(x)+ \\ +m^2k^2T_0\left[F_{132}(x)-\dfrac{\Gamma_{10}}{\bar{c}}Mg_0\right]A(\phi)-\lambda^2\dfrac{\bar{\Gamma}_3+k^2\bar{\Gamma}_4}{\bar{\Gamma}_1}A(\phi)V_0g_0 \end{cases} \tag{2.4.35}$$

$$\begin{cases} -\hat{u}_{12}-M\hat{v}_{12}+i\sigma_{00}\hat{v}_{12}=A_1(\phi)\cos\phi\cdot G_{111}(x)+kT_0^{1/2}\dfrac{dA_1(\phi)}{d\phi}G_{112}(x)+ \\ +\dfrac{imkT_0^{1/2}}{\bar{c}}g_0A_1(\phi)+A(\phi)G_{123}(x)+k^2A(\phi)G_{124}(x)+A(\phi)\cos 2\phi\cdot G_{125}(x)+ \\ +kT_0^{1/2}A(\phi)\sin\phi\cdot G_{126}(x)+T_1\cos\phi\cdot A(\phi)G_{128}(x)+ \\ +imkT_0^{1/2}A(\phi)\cos\phi\cdot G_{129}(x)+\dfrac{imkT_1}{T_0^{1/2}}A(\phi)G_{130}(x)+ \\ +m^2k^2T_0\left[G_{132}(x)+\dfrac{\Gamma_{10}}{\bar{c}}g_0\right]A(\phi) \end{cases} \tag{2.4.36}$$

$$\hat{u}_{12}=D\hat{u}_{12}=\hat{v}_{12}=0 \quad \text{at} \quad x=\pm\frac{1}{2} \tag{2.4.36a}$$

The procedure used is similar to that for the order system; it yields:

$$kT_0^{1/2}\frac{dA_1(\phi)}{d\phi}-(\Gamma\cos\phi+imkT_0^{1/2})A_1(\phi)+A(\phi)\left[kT_0^{1/2}\cdot\Gamma_1\sin\phi-\Gamma_2\cos 2\phi+\right.$$
$$+imkT_0^{1/2}\Gamma_7\cos\phi+m^2k^2T_0\Gamma_{10}+T_1\Gamma_6\cos\phi+\frac{imkT_1}{T_0^{1/2}}\Gamma_8+$$
$$\left.+i\bar{\sigma}_{02}\cdot\bar{c}\right]=0 \tag{2.4.37}$$

Using the periodicity condition $A_1(2\pi)=A_1(0)$ (2.4.38)

one obtains $\sigma_{02i}=\dfrac{m^2k^2T_0\Gamma_{10}}{\bar{c}}$ (2.4.39).

$$\sigma_{02r} = -\frac{mkT_1\Gamma_8}{\bar{c}\,T_0^{1/2}} \qquad (2.4.40)$$

where

$$i\sigma_{02} = i\sigma_{02r} - \sigma_{02i} \qquad (2.4.41)$$

$$A_1(\phi) = A(\phi)\left[\Gamma_1\cos\phi + \frac{\Gamma_2\sin 2\phi}{2kT_0^{1/2}} - im\Gamma_7\sin\phi - \frac{T_1\Gamma_6\sin\phi}{kT_0^{1/2}}\right] \qquad (2.4.42)$$

By substituting (2.4.42) in (2.4.27) and (2.4.28), we obtain more explicit formulae for $\hat{u}_{11}$ and $\hat{v}_{11}$:

$$-\hat{u}_{11} = A(\phi)\left[(f_1+\Gamma_1 f_0)\cos\phi + \frac{\Gamma_2 f_0\sin 2\phi}{2kT_0^{1/2}} - \frac{T_1\Gamma_6 f_0\sin\phi}{kT_0^{1/2}} + \right.$$
$$\left. + im(kT_0^{1/2} f_3 - \Gamma_7 f_0\sin\phi)\right] \qquad (2.4.43)$$

$$\hat{v}_{11} = A(\phi)\left[(g_1+\Gamma_1 g_0)\cos\phi + \frac{\Gamma_2 g_0\sin 2\phi}{2kT_0^{1/2}} + im(kT_0^{1/2} g_3 - \Gamma_7 g_0\sin\phi) - \right.$$
$$\left. - \frac{T_1\Gamma_6 g_0\sin\phi}{kT_0^{1/2}}\right] \qquad (2.4.44)$$

As far as $\hat{u}_{12}$ and $\hat{v}_{12}$ are concerned, it is possible to see that

$$-\hat{u}_{12} = A(\phi)\left[f_{10} + k^2 f_{11} + f_{12}\cos 2\phi + kT_0^{1/2}\sin\phi\cdot f_{13} + \right.$$
$$+ imkT_0^{1/2}\cos\phi\cdot f_{14} + T_1\cos\phi\cdot f_{15} + \frac{\Gamma_2(\sin 3\phi+\sin\phi)}{4kT_0^{1/2}} f_1 +$$
$$+ \frac{im(\Gamma_2 f_3 - \Gamma_7 f_1)}{2}\sin 2\phi + \Gamma_7 m^2 kT_0^{1/2} f_3\sin\phi + m^2k^2T_0 f_{18} -$$
$$\left. - \frac{T_1\Gamma_6\sin 2\phi}{2kT_0^{1/2}} f_1 - im\,T_1\Gamma_6 f_3\sin\phi + \frac{imkT_1}{2T_0^{1/2}} f_{16}\right] \qquad (2.4.45)$$

$$\hat{v}_{12} = A(\phi)\left[g_{10} + k^2 g_{11} + g_{12}\cos 2\phi + kT_0^{1/2}\sin\phi\cdot g_{13} + \right.$$
$$+ imkT_0^{1/2}\cos\phi\cdot g_{14} + T_1\cos\phi\cdot g_{15} + \frac{\Gamma_2(\sin 3\phi+\sin\phi)}{4kT_0^{1/2}} g_1 +$$
$$+ \frac{im(\Gamma_2 g_3 - \Gamma_7 g_1)}{2}\sin 2\phi + \Gamma_7 m^2 kT_0^{1/2} g_3\sin\phi + m^2k^2T_0 g_{18} -$$
$$\left. - \frac{T_1\Gamma_6\sin 2\phi}{2kT_0^{1/2}} g_1 - im\,T_1\Gamma_6 g_3\sin\phi + \frac{imkT_1}{2T_0^{1/2}} g_{16}\right] \qquad (2.4.46)$$

and where (f_{10}, g_{10}), (f_{11}, g_{11}), (f_{12}, g_{12}), (f_{13}, g_{13}), (f_{14}, g_{14}),

(f_{15}, g_{15}), (f_{16}, g_{16}) and (f_{18}, g_{18}) provide the eigenfunctions of several differential systems, as follows:

$$\begin{cases} M^2 f_{10} + \lambda^2 T_0 V_0 g_{10} - i\sigma_{00} M f_{10} = F_{123} + \frac{\Gamma_4}{2}(F_{111} + \Gamma F_{112}) + \frac{\Gamma_3}{\Gamma_1} F_{113} \\ f_{10} - M g_{10} + i\sigma_{00} g_{10} = G_{123} + \frac{\Gamma_4}{2}(G_{111} + \Gamma G_{112}) \\ f_{10} = D f_{10} = g_{10} = 0 \quad at \quad x = \pm \frac{1}{2} \end{cases} \tag{2.4.47}$$

$$\begin{cases} M^2 f_{11} + \lambda^2 T_0 V_0 g_{11} - i\sigma_{00} M f_{11} = F_{124} + \frac{\Gamma_4}{\Gamma_1} F_{113} \\ f_{11} - M g_{11} + i\sigma_{00} g_{11} = G_{124} \\ f_{11} = D f_{11} = g_{11} = 0 \quad at \quad x = \pm \frac{1}{2} \end{cases} \tag{2.4.48}$$

$$\begin{cases} M^2 f_{12} + \lambda^2 T_0 V_0 g_{12} - i\sigma_{00} M f_{12} = F_{125} + \frac{\Gamma_4}{2}(F_{111} + \Gamma F_{112}) + \Gamma_2 F_{112} \\ f_{12} - M g_{12} + i\sigma_{00} g_{12} = G_{125} + \frac{\Gamma_4}{2}(G_{111} + \Gamma G_{112}) + \Gamma_2 G_{112} \\ f_{12} = D f_{12} = g_{12} = 0 \quad at \quad x = \pm \frac{1}{2} \end{cases} \tag{2.4.49}$$

$$\begin{cases} M^2 f_{13} + \lambda^2 T_0 V_0 g_{13} - i\sigma_{00} M f_{13} = F_{126} - \Gamma_1 F_{112} \\ f_{13} - M g_{13} + i\sigma_{00} g_{13} = G_{126} - \Gamma_1 G_{112} \\ f_{13} = D f_{13} = g_{13} = 0 \quad at \quad x = \pm \frac{1}{2} \end{cases} \tag{2.4.50}$$

$$\begin{cases} M^2 f_{14} + \lambda^2 T_0 V_0 g_{14} - i\sigma_{00} M f_{14} = F_{129} + \Gamma_1 (V_0 - \frac{1}{c}) M f_0 - \Gamma_7 F_{112} \\ f_{14} - M g_{14} + i\sigma_{00} g_{14} = G_{129} - \Gamma_1 (V_0 - \frac{1}{c}) g_0 - \Gamma_7 G_{112} \\ f_{14} = D f_{14} = g_{14} = 0 \quad at \quad x = \pm \frac{1}{2} \end{cases} \tag{2.4.51}$$

$$\begin{cases} M^2 f_{15} + \lambda^2 T_0 V_0 g_{15} - i\sigma_{00} M f_{15} = F_{128} - \Gamma_6 F_{112} \\ f_{15} - M g_{15} + i\sigma_{00} g_{15} = G_{128} - \Gamma_6 G_{112} \\ f_{15} = D f_{15} = g_{15} = 0 \quad at \quad x = \pm \frac{1}{2} \end{cases} \tag{2.4.52}$$

$$\begin{cases} M^2 f_{16} + \lambda^2 T_0 V_0 g_{16} - i\sigma_{00} M f_{16} = F_{130} - \frac{\Gamma_8}{c} M f_0 \\ f_{16} - M g_{16} + i\sigma_{00} g_{16} = G_{130} + \frac{\Gamma_8}{c} g_0 \\ f_{16} = D f_{16} = g_{16} = 0 \text{ at } x = \pm \frac{1}{2} \end{cases} \qquad (2.4.53)$$

$$\begin{cases} M^2 f_{18} + \lambda^2 T_0 V_0 g_{18} - i\sigma_{00} M f_{18} = F_{132} - \frac{\Gamma_{10}}{c} M f_0 \\ f_{18} - M g_{18} + i\sigma_{00} g_{18} = G_{132} + \frac{\Gamma_{10}}{c} g_0 \\ f_{18} = D f_{18} = g_{18} = 0 \text{ at } x = \pm \frac{1}{2} \end{cases} \qquad (2.4.54)$$

Expansion of the adjoint (u^+, v^+, w^+, p^+) in powers of ε:

$$u^+ = u_0^+ + \varepsilon u_1^+ + \varepsilon^2 u_2^+ + \cdots$$

$$v^+ = v_0^+ + \varepsilon v_1^+ + \varepsilon^2 v_2^+ + \cdots$$

$$w^+ = w_0^+ + \varepsilon w_1^+ + \varepsilon^2 w_2^+ + \cdots$$

$$p^+ = p_0^+ + \varepsilon p_1^+ + \varepsilon^2 p_2^+ + \cdots$$

<u>ε^0 ORDER</u>

$$-M u_0^+ + i\sigma_{00} u_0^+ - v_0^+ - \frac{\partial p_0^+}{\partial x} = 0$$

$$-T_0 V_0 u_0^+ - M v_0^+ + i\sigma_{00} v_0^+ = 0$$

$$-M w_0^+ + i\sigma_{00} w_0^+ - \lambda p_0^+ = 0$$

$$-D u_0^+ + \lambda w_0^+ = 0$$

After elimination of w_0^+ and p_0^+, we obtain:

$$\begin{aligned} u_0^+ &= -\lambda^2 f_0^+(x) \cdot A^+(\phi) \\ v_0^+ &= g_0^+(x) \cdot A^+(\phi) \end{aligned} \qquad (2.4.55)$$

where (f_0^+, g_0^+) provide the eigenvalue for the adjoint system:

$$\begin{cases} M^2 f_0^+ + g_0^+ - i\sigma_{00} M f_0^+ = 0 \\ \lambda^2 T_0 V_0 f_0^+ - M g_0^+ + i\sigma_{00} g_0^+ = 0 \\ f_0^+ = D f_0^+ = g_0^+ = 0 \ \text{at} \ x = \pm \frac{1}{2} \end{cases} \qquad (2.4.56)$$

It is just here that we retrieve the adjoint system of the linear theory of Di Prima and Stuart (1972b), p. 403, eq. 5.7 - 5.9.

ε ORDER

$$-M u_1^+ - v_1^+ + i\sigma_{00} u_1^+ - \frac{\partial p_1^+}{\partial x} = k T_0^{1/2} V_0 \frac{\partial u_0^+}{\partial \phi} - 2\cos\phi \, D^2 u_0^+ - i\sigma_{01} u_0^+ - \left(\frac{12x}{c} + 1\right)\cos\phi \cdot v_0^+ - \cos\phi \cdot \frac{\partial p_0^+}{\partial x}$$

$$-T_0 V_0 u_1^+ - M v_1^+ + i\sigma_{00} v_1^+ = -i\sigma_{01} v_0^+ + 6 T_0 \left(x^2 - \frac{1}{4}\right)\cos\phi \cdot u_0^+ + k T_0^{1/2} V_0 \frac{\partial v_0^+}{\partial \phi} - 2\cos\phi \cdot D^2 v_0^+ + c k T_0^{1/2} \frac{\partial p_0^+}{\partial \phi}$$

$$-M w_1^+ + i\sigma_{00} w_1^+ - \lambda p_1^+ = -i\sigma_{01} w_0^+ + k T_0^{1/2} V_0 \frac{\partial w_0^+}{\partial \phi} - 2\cos\phi \cdot D^2 w_0^+ \qquad (2.4.57)$$

$$-D u_1^+ + \lambda w_1^+ = -\cos\phi \cdot \frac{\partial u_0^+}{\partial x}$$

We eliminate w_1^+ and p_1^+, and we denote $u_1'^+ = -\frac{1}{\lambda^2} u_1^+$ (2.4.58)

An alternative possibility to formulate our differential system:

$$\begin{cases} M^2 u_1'^+ + v_1^+ - i\sigma_{00} M u_1'^+ = A^+(\phi)\cos\phi \cdot F_{111}^+(x) + k T_0^{1/2} \frac{dA^+(\phi)}{d\phi} F_{112}^+(x) + i\sigma_{01} M f_0^+ \cdot A^+(\phi) \\ \lambda^2 T_0 V_0 u_1'^+ - M v_1^+ + i\sigma_{00} v_1^+ = -i\sigma_{01} g_0^+ \cdot A^+(\phi) + A^+(\phi)\cos\phi \cdot G_{111}^+(x) + k T_0^{1/2} \frac{dA^+(\phi)}{d\phi} G_{112}^+(x) \\ u_1'^+ = D u_1'^+ = v_1^+ = 0 \ \text{at} \ x = \pm \frac{1}{2} \end{cases} \qquad (2.4.59)$$

where

$$F_{111}^+(x) = \left(\frac{12x}{c} + 1\right) g_0^+ + 4 M D^2 f_0^+ - 2 i\sigma_{00} D^2 f_0^+ \qquad (2.4.60)$$

$$F_{112}^+(x) = -V_0 M f_0^+ + c D f_0^+ \qquad (2.4.61)$$

$$G^{+}_{111}(x) = -6\lambda^2 T_0 (x^2 - \tfrac{1}{4}) f_0^{+} - 2D^2 g_0^{+} \qquad (2.4.62)$$

$$G^{+}_{112}(x) = V_0 g_0^{+} + cMDf_0^{+} \qquad (2.4.63)$$

The usual straightforward procedure yields:

$$kT_0^{1/2} \frac{dA^{+}(\phi)}{d\phi} = (-\Gamma \cos\phi + i\bar{c}\sigma_{01}) A^{+}(\phi) \qquad (2.4.64)$$

with proper account taken of the periodicity condition

$$A^{+}(2\pi) = A^{+}(0)$$

we obtain the solution

$$A^{+}(\phi) = e^{-im\phi} e^{-\frac{\Gamma(\sin\phi - 1)}{kT_0^{1/2}}} \qquad (2.4.65)$$

Consequently,

$$u_1^{+} = -\lambda^2 \left[A^{+}(\phi) \cdot (f_1^{+} \cos\phi + imkT_0^{1/2} f_3^{+}) + A_1^{+}(\phi) \cdot f_0^{+} \right] \qquad (2.4.66)$$

$$v_1^{+} = A^{+}(\phi)(g_1^{+} \cos\phi + imkT_0^{1/2} g_3^{+}) + A_1^{+}(\phi) g_0^{+} \qquad (2.4.67)$$

where $(f_1^{+}, g_1^{+}), (f_2^{+}, g_2^{+})$ are given by the following differential systems respectively:

$$\begin{cases} M^2 f_1^{+} + g_1^{+} - i\sigma_{00} M f_1^{+} = F^{+}_{111}(x) - \Gamma F^{+}_{112}(x) \\ \lambda^2 T_0 V_0 f_1^{+} - M g_1^{+} + i\sigma_{00} g_1^{+} = G^{+}_{111}(x) - \Gamma G^{+}_{112}(x) \\ f_1^{+} = Df_1^{+} = g_1^{+} = 0 \text{ at } x = \pm\frac{1}{2} \end{cases} \qquad (2.4.68)$$

$$\begin{cases} M^2 f_3^{+} + g_3^{+} - i\sigma_{00} M f_3^{+} = (V_0 - \frac{1}{c}) M f_0^{+} - cDf_0^{+} \\ \lambda^2 T_0 V_0 f_3^{+} - M g_3^{+} + i\sigma_{00} g_3^{+} = -g_0^{+}(V_0 - \frac{1}{c}) - cMDf_0^{+} \\ f_3^{+} = Df_3^{+} = g_3^{+} = 0 \text{ at } x = \pm\frac{1}{2} \end{cases} \qquad (2.4.69)$$

ORDER ε^2

We write down the momentum and continuity equations at order ε^2

$$\begin{cases} -M u_2^+ - v_2^+ + i\sigma_{00} u_2^+ - \frac{\partial p_2^+}{\partial x} = V_0 k T_0^{1/2} \frac{\partial u_1^+}{\partial \phi} + k T_0^{1/2}\left[6(x^2-\frac{1}{4}) - V_0\right]\cos\phi \frac{\partial u_0^+}{\partial \phi} + \\ + V_0 \frac{k T_1}{2 T_0^{1/2}} \frac{\partial u_0^+}{\partial \phi} - i\sigma_{01} u_1^+ - i\sigma_{02} u_0^+ - 2ck^2 \frac{\partial u_0^+}{\partial x} - k T_0^{1/2}\left[6(x^2-\frac{1}{4}) - V_0\right]\sin\phi \cdot u_0^+ + \\ + k T_0^{1/2} \bar{U}_0 \sin\phi \frac{\partial u_0^+}{\partial x} - 2\cos\phi \cdot \frac{\partial^2 u_1^+}{\partial x^2} + \frac{3+\cos 2\phi}{2} \frac{\partial^2 u_0^+}{\partial x^2} - 2k^2 V_0 v_0^+ - \\ - \left(\frac{12x}{c} + 1\right)\cos\phi \cdot v_1^+ + \left[\frac{1}{2}\left(1 + \frac{12x}{c}\cos 2\phi\right) - 2cxk^2 - \frac{1}{c}\frac{d\bar{V}_{21}}{dx}\sin\phi\right] v_0^+ + \\ + \left[2ck^2 p_0^+ - \cos\phi \cdot \frac{\partial p_1^+}{\partial x} + \frac{1}{2}\frac{\partial p_0^+}{\partial x}\right] \end{cases} \qquad (2.4.70)$$

$$\begin{cases} -T_0 V_0 u_2^+ - M v_2^+ + i\sigma_{00} v_2^+ = 2ck T_0^{1/2} \sin\phi \cdot D u_0^+ + 6 T_0 (x^2-\frac{1}{4})\cos\phi \cdot u_1^+ + \\ + V_0 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1} u_0^+ + T_0 u_0^+\left[3(x^2-\frac{1}{4}) + k T_0^{1/2} \bar{V}_{21} \sin\phi + k^2 c^2 (x^2-\frac{1}{4})\right] + \\ + k T_0^{1/2} \bar{U}_0 \sin\phi \cdot \frac{\partial v_0^+}{\partial x} + k T_0^{1/2} \sin\phi \frac{\partial \bar{U}_0}{\partial x} v_0^+ + 6 k T_0^{1/2} (x^2-\frac{1}{4}) \sin\phi + \\ + \frac{k T_1}{2 T_0^{1/2}} V_0 \frac{\partial v_0^+}{\partial \phi} + k T_0^{1/2} V_0 \frac{\partial v_1^+}{\partial \phi} - k T_0^{1/2} \cos\phi \cdot V_0 \frac{\partial v_0^+}{\partial \phi} + \\ + 6 k T_0^{1/2} (x^2-\frac{1}{4}) V_0 \frac{\partial v_0^+}{\partial \phi} \cos\phi - 6 k T_0^{1/2} (x^2-\frac{1}{4}) \sin\phi \cdot v_0^+ + k T_0^{1/2} V_0 \sin\phi \cdot v_0^+ - \\ - 2\cos\phi \frac{\partial^2 v_1^+}{\partial x^2} + \frac{3+\cos 2\phi}{2} \frac{\partial^2 v_0^+}{\partial x^2} - 2ck^2 \frac{\partial v_0^+}{\partial x} - i\sigma_{01} v_1^+ - i\sigma_{02} v_0^+ + \\ + ck T_0^{1/2} \frac{\partial p_1^+}{\partial \phi} + \frac{ck T_1}{2 T_0^{1/2}} \frac{\partial p_0^+}{\partial \phi} - ck T_0^{1/2} \cos\phi \frac{\partial p_0^+}{\partial \phi} + 2ck T_0^{1/2} \sin\phi \cdot p_0^+ \end{cases} \qquad (2.4.71)$$

$$\begin{cases} -M w_2^+ + i\sigma_{00} w_2^+ - \lambda p_2^+ = k T_0^{1/2}\left(\bar{U}_0 \frac{\partial w_0^+}{\partial x} + \frac{\partial \bar{U}_0}{\partial x} w_0^+\right)\sin\phi - \\ - i\sigma_{01} w_1^+ - i\sigma_{02} w_0^+ - 2\cos\phi \cdot \frac{\partial^2 w_1^+}{\partial x^2} + \frac{3+\cos 2\phi}{2} \frac{\partial^2 w_0^+}{\partial x^2} - 2ck^2 \frac{\partial w_0^+}{\partial x} + \\ + k T_0^{1/2}\left[V_0 \frac{\partial w_1^+}{\partial \phi} - V_0 \cos\phi \frac{\partial w_0^+}{\partial \phi} + 6(x^2-\frac{1}{4})\cos\phi \cdot \frac{\partial w_0^+}{\partial \phi} + \sin\phi \cdot V_0 w_0^+ - \right. \\ \left. - 6(x^2-\frac{1}{4})\sin\phi \cdot w_0\right] + \frac{k T_1}{2 T_0^{1/2}} V_0 \frac{\partial w_0^+}{\partial \phi} \\ - \frac{\partial u_2^+}{\partial x} + \lambda w_2^+ = \frac{\partial u_0^+}{\partial x} - \cos\phi \frac{\partial u_1^+}{\partial x} \end{cases} \qquad (2.4.72)$$

where $\bar{U}_0 = 2(x^2-\frac{1}{4})(x^2-\frac{c}{4})$ (2.4.73)

$$\overline{V}_{21} = (x^2 - \tfrac{1}{4})\left[\tfrac{1}{2}(1-\tfrac{c^2}{20})(\tfrac{1}{20} - x^2) - \tfrac{cx}{5}(\tfrac{7}{12} - x^2)\right] \quad (2.4.74)$$

If we denote $u_2^+ = -\lambda^2 u_2'^+$, we obtain the following ε^2 order adjoint system (By elimination of p_2^+ and v_2^+ and knowledge of some previous relations obtained from the ε^0 and ε order adjoint systems):

$$M^2 u_2'^+ - i\sigma_{00} M u_2'^+ + v_2^+ = A_1^+(\phi)\cos\phi \cdot F_{111}^+(x) + kT_0^{1/2}\frac{dA_1^+(\phi)}{d\phi}F_{112}^+(x)$$
$$+ \frac{imkT_0^{1/2}}{c}A_1^+(\phi)\cdot F_{114}^+(x) + A^+(\phi)\cdot F_{123}^+(x) + k^2 A^+(\phi)F_{124}^+(x) +$$
$$+ A^+(\phi)F_{125}^+(x)\cos 2\phi + kT_0^{1/2}A^+(\phi)F_{126}^+(x)\sin\phi + T_1 A^+(\phi)F_{128}^+(x)\cos\phi +$$
$$+ imkT_0^{1/2}F_{129}^+(x)A^+(\phi)\cos\phi + \frac{imkT_1}{T_0^{1/2}}A^+(\phi)F_{130}^+(x) +$$
$$+ m^2k^2T_0 A^+(\phi)\left[F_{132}^+(x) - \frac{\Gamma_{10}}{c}Mf_0^+\right] \quad (2.4.75)$$

$$\lambda^2 T_0 V_0 u_2'^+ - Mv_2^+ + i\sigma_{00}u_2'^+ = A_1^+(\phi)\cos\phi\cdot G_{111}^+(x) +$$
$$+ kT_0^{1/2}\frac{dA_1^+(\phi)}{d\phi}G_{112}^+(x) + \frac{imkT_0^{1/2}}{c}A_1^+(\phi)\cdot G_{114}^+(x) + A^+(\phi)G_{123}^+(x) +$$
$$+ k^2A^+(\phi)G_{124}^+(x) + A^+(\phi)\cos 2\phi\cdot G_{125}^+(x) + kT_0^{1/2}G_{126}^+(x)\cdot A^+(\phi)\sin\phi +$$
$$+ imkT_0^{1/2}G_{129}^+(x)\cdot A^+(\phi)\cos\phi + T_1 G_{128}^+(x)A^+(\phi)\cos\phi +$$
$$+ \frac{imkT_1}{T_0^{1/2}}G_{130}^+(x)\cdot A^+(\phi) + m^2k^2T_0 A^+(\phi)\left[G_{132}^+(x) + \frac{\Gamma_{10}}{c}g_0^+\right] -$$
$$- \lambda^2\frac{\Gamma_3 + k^2\Gamma_4}{\Gamma_1}V_0 f_0^+ \quad (2.4.76)$$

$$u_2'^+ = Du_2^+ = v_2^+ = 0 \quad \text{at } x = \pm\tfrac{1}{2} \quad (2.4.77)$$

where

$$F_{111}^+(x) = 4MD^2f_0^+ - 2i\sigma_{00}D^2f_0^+ + (\tfrac{12x}{c} + 1)g_0^+ \quad (2.4.78)$$

$$F_{112}^+(x) = cDf_0^+ - V_0Mf_0^+ \quad (2.4.79)$$

$$F_{114}^+(x) = -Mf_0^+ \quad (2.4.80)$$

$$F^+_{123}(x) = \frac{1}{2}\left(1+\frac{12x}{c}\right)g^+_1 - (MD^2 - ic\sigma_{00}D^2)(2f^+_0 - f^+_1) + \frac{\Gamma\lambda^2 V_0}{2}(f^+_0 - f^+_1) +$$
$$+ 6\Gamma x D f^+_0 + 3\Gamma(x^2 - \frac{1}{4})Mf^+_0 - g^+_2 + MD^2 f^+_1 - 2D^4 f^+_0 - \frac{3}{2}MD^2 f^+_0 -$$
$$- \frac{\Gamma}{2}D[V_0 \cdot D(3f^+_0 - f^+_1)] \qquad (2.4.81)$$

$$F^+_{124}(x) = 2(g^+_0 + ic\sigma_{00}cDf^+_0) \qquad (2.4.82)$$

$$F^+_{125}(x) = \frac{g^+_1}{2}\left(\frac{12x}{c}+1\right) + 6x\left(\Gamma D f^+_0 - \frac{1}{c}g^+_0\right) + \Gamma\left[3(x^2-\frac{1}{4}) - \frac{V_0}{2}\right]Mf^+_0 +$$
$$+ MD^2 f^+_1 - \frac{\lambda^2\Gamma V_0 f^+_1}{2} - \frac{5}{2}D^4 f^+_0 - \frac{1}{2}(MD^2 - ic\sigma_{00}D^2)(f^+_0 - 2f^+_1) +$$
$$+ \frac{c\Gamma}{2}D(3f^+_0 - f^+_1) \qquad (2.4.83)$$

$$F^+_{126}(x) = 6(x^2-\frac{1}{4})Mf^+_0 + 12xDf^+_0 + \frac{1}{c}\frac{d\bar{V}_{21}}{dx}g^+_0 - D^2\bar{U}_0 Df^+_0 - 2D\bar{U}_0 D^2 f^+_0 +$$
$$+ cD(2f^+_0 - f^+_1) - 2V_0 Mf^+_0 - \bar{U}_0 MDf^+_0 + V_0 Mf^+_1 \qquad (2.4.84)$$

$$F^+_{128}(x) = \frac{\Gamma}{2T_0}(V_0 Mf^+_0 - cDf^+_0) \qquad (2.4.85)$$

$$F^+_{129}(x) = \left[6(x^2-\frac{1}{4}) - V_0\right]Mf^+_0 + \left(V_0 - \frac{1}{c}\right)(Mf^+_1 - 2D^2 f^+_0) + 4MD^2 f^+_3 -$$
$$- 2ic\sigma_{00}D^2 f^+_3 + \left(\frac{12x}{c}+1\right)g^+_3 + cD(3f^+_0 - f^+_1 - \Gamma f^+_3) + \Gamma V_0 Mf^+_3 + 12xDf^+_0 \qquad (2.4.86)$$

$$F^+_{130}(x) = \frac{1}{2}\left(V_0 - \frac{1}{c}\right)Mf^+_0 - \frac{cDf^+_0}{2} \qquad (2.4.87)$$

$$F^+_{132}(x) = cDf^+_3 - \left(V_0 - \frac{1}{c}\right)Mf^+_3 \qquad (2.4.88)$$

$$G^+_{111}(x) = -6\lambda^2 T_0(x^2 - \frac{1}{4})f^+_0 - 2D^2 g^+_0 \qquad (2.4.89)$$

$$G^+_{112}(x) = V_0 g^+_0 + c(MD - ic\sigma_{00}D)f^+_0 \qquad (2.4.90)$$

$$G^+_{113}(x) = -\lambda^2 V_0 f^+_0 \qquad (2.4.91)$$

$$G^+_{114}(x) = g^+_0 \qquad (2.4.92)$$

$$G^+_{123}(x) = -3(x^2-\frac{1}{4})\left[\lambda^2 T_0(f^+_0 + f^+_1) + \Gamma V_0 g^+_0\right] + \frac{\Gamma}{2}(g^+_0 - g^+_1 + c\Gamma D f^+_0)$$
$$+ \frac{c\Gamma}{2}(MD - ic\sigma_{00}D)(2f^+_0 - f^+_1) + \frac{3}{2}D^2 g^+_0 - D^2 g^+_1 + c\Gamma D^3 f^+_0 \qquad (2.4.93)$$

$$G^+_{124}(x) = -c^2\lambda^2 T_0(x^2-\frac{1}{4})f^+_0 - 2cDg^+_0 + 2c\lambda^2(x+\frac{1}{2})T_0 V_0 f^+_0 \qquad (2.4.94)$$

$$G^+_{125}(x) = \frac{\Gamma V_0}{2}(g^+_0 - g^+_1) - 3(x^2-\frac{1}{4})\left[\Gamma g^+_0 + \lambda^2 T_0 f^+_1\right] + \frac{1}{2}D^2(g^+_0 - 2g^+_1)$$
$$+ \frac{c\Gamma}{2}(MD - ic\sigma_{00}D)(2f^+_0 - f^+_1) + \frac{cV_0 Df^+_0\Gamma^2}{2} + c\Gamma D^3 f^+_0 \qquad (2.4.95)$$

$$G^+_{126}(x) = 2cMD f^+_0 - \lambda^2 T_0 \nabla_{21} f^+_0 + D(\bar{U}_0 g^+_0) + V_0 (g^+_0 - g^+_1) +$$
$$+ c(MD - i\sigma_{00} D)(3f^+_0 - f^+_1) + c\Gamma V_0 D f^+_0 \qquad (2.4.96)$$

$$G^+_{128}(x) = -\frac{1}{2T_0}\left[\Gamma V_0 g^+_0 + c\Gamma(MD f^+_0 - i\sigma_{00} D f^+_0)\right] \qquad (2.4.97)$$

$$G^+_{129}(x) = V_0(g^+_0 - \Gamma g^+_3) - 6(x^2 - \tfrac{1}{4})\left[g^+_0 + \lambda^2 T_0 f^+_3\right] - (V_0 - \tfrac{1}{c}) g^+_1 -$$
$$- 2D^2 g^+_3 + c(MD - i\sigma_{00} D)(2f^+_0 - f^+_1 - \Gamma f^+_3) + 2c D^3 f_0 + c\Gamma(2V_0 - \tfrac{1}{c}) D f^+_0$$
$$(2.4.98)$$

$$G^+_{130}(x) = -g^+_2 (V_0 - \tfrac{1}{c}) - \tfrac{c}{2}(MD f^+_0 - i\sigma_{00} D f^+_0) \qquad (2.4.99)$$

$$G^+_{132}(x) = (V_0 - \tfrac{1}{c}) g^+_3 - c(V_0 - \tfrac{1}{c}) D f^+_0 + \qquad (2.4.100)$$
$$+ c(MD - i\sigma_{00} D) f^+_3$$

By the routine multiplication with the adjoint respectively, adding and integrating over $(-\frac{1}{2}, \frac{1}{2})$, we obtain a differential equation for

$$kT_0^{1/2} \frac{dA^+_1(\phi)}{d\phi} + A^+_1(\phi)(\Gamma\cos\phi + imkT_0^{1/2}) +$$
$$+ A^+(\phi)\left[\hat{\Gamma}_2 \cos 2\phi - kT_0^{1/2} \sin\phi \cdot \hat{\Gamma}_1 - imkT_0^{1/2} \cos\phi \cdot \hat{\Gamma}_7 - m^2 k^2 T_0 \Gamma_{10} -\right.$$
$$\left. - i\sigma_{02} \cdot \bar{c} - \frac{T_1}{T_0}\left(\hat{\Gamma}_6 \cos\phi + imkT_0^{1/2} \hat{\Gamma}_8\right)\right] = 0 \qquad (2.4.101)$$

it yields

$$A^+_1(\phi) = A^+(\phi)\left[\hat{\Gamma}_1 \cos\phi + \frac{\hat{\Gamma}_2 \sin 2\phi}{2kT_0^{1/2}} - im\Gamma_7 \sin\phi - \frac{T_1 \hat{\Gamma}_6 \sin\phi}{kT_0^{1/2}}\right]$$
$$(2.4.102)$$

Consequently, one obtains:

$$u^+_1 = -\lambda^2 A^+(\phi)\left[(f^+_1 + \hat{\Gamma}_1 f^+_0)\cos\phi + \frac{\hat{\Gamma}_2 f^+_0 \sin 2\phi}{2kT_0^{1/2}} +\right.$$
$$\left. + im(kT_0^{1/2} f^+_3 - \sin\phi \cdot \hat{\Gamma}_7 f^+_0) - \frac{T_1 \Gamma_6 f^+_0 \sin\phi}{kT_0^{1/2}}\right]$$
$$v^+_1 = A^+(\phi)\left[(g^+_1 + \hat{\Gamma}_1 g^+_0)\cos\phi + \frac{\hat{\Gamma}_2 g^+_0 \sin 2\phi}{2kT_0^{1/2}} + im(kT_0^{1/2} g^+_3 - \hat{\Gamma}_7 g^+_0 \sin\phi) -\right.$$
$$\left. - \frac{T_1 \Gamma_6 g^+_0 \sin\phi}{kT_0^{1/2}}\right] \qquad (2.4.103)$$

and

$$\begin{cases} -u_{12}^{+} = \lambda^2 A^{+}(\phi)\Big[g_{10}^{+} + k^2 g_{11}^{+} + g_{12}^{+}\cos 2\phi + kT_0^{1/2}\sin\phi\, g_{13}^{+} + \\ + imkT_0^{1/2} g_{14}^{+}\cos\phi + \dfrac{\hat{\Gamma}_2(\sin 3\phi + \sin\phi)}{4kT_0^{1/2}} g_{11}^{+} + \dfrac{im(\hat{\Gamma}_2 g_3^{+} - \hat{\Gamma}_7 g_{11}^{+})}{2}\sin 2\phi + \\ + \hat{\Gamma}_7 m^2 kT_0^{1/2} g_3^{+}\sin\phi + T_1 g_{15}^{+}\cos\phi - \dfrac{T_1\hat{\Gamma}_6\sin 2\phi}{2kT_0^{1/2}} g_{11}^{+} - \\ - imT_1\hat{\Gamma}_6 g_3^{+}\sin\phi + \dfrac{imkT_1}{2T_0^{1/2}} g_{16}^{+} + m^2k^2T_0 g_{18}^{+}\Big] \end{cases} \quad (2.4.104)$$

$$\begin{cases} v_{12}^{+} = A^{+}(\phi)\Big[g_{10}^{+} + k^2 g_{11}^{+} + g_{12}^{+}\cos 2\phi + kT_0^{1/2} g_{13}^{+}\sin\phi + \\ + imkT_0^{1/2} g_{14}^{+}\cos\phi + \dfrac{\hat{\Gamma}_2(\sin 3\phi + \sin\phi)}{4kT_0^{1/2}} g_{11}^{+} + \\ + \dfrac{im(\hat{\Gamma}_2 g_3^{+} - \hat{\Gamma}_7 g_{11}^{+})\sin 2\phi}{2} + \hat{\Gamma}_7 m^2 kT_0^{1/2} g_3^{+}\sin\phi + T_1 g_{15}^{+}\cos\phi - \\ - \dfrac{T_1\hat{\Gamma}_6\sin 2\phi}{2kT_0^{1/2}} g_{11}^{+} - imT_1\hat{\Gamma}_6 g_3^{+}\sin\phi + \dfrac{imkT_1}{2T_0^{1/2}} g_{16}^{+} + m^2k^2T_0 g_{18}^{+}\Big] \end{cases} \quad (2.4.105)$$

where

(f_{10}^{+}, g_{10}^{+}), (f_{11}^{+}, g_{11}^{+}), (f_{12}^{+}, g_{12}^{+}), (f_{13}^{+}, g_{13}^{+}), (f_{14}^{+}, g_{14}^{+}), (f_{15}^{+}, g_{15}^{+}), (f_{16}^{+}, g_{16}^{+}) and (f_{18}^{+}, g_{18}^{+}) provide the eigenfunctions of several differential systems, as follows:

$$\begin{cases} M^2 f_{10}^{+} + g_{10}^{+} - i\sigma_{00} M f_{10}^{+} = F_{123}^{+} + \dfrac{\hat{\Gamma}_1}{2}(F_{111}^{+} - \Gamma F_{112}^{+}) \\ \lambda^2 T_0 V_0 f_{10}^{+} - M g_{10}^{+} + i\sigma_{00} g_{10}^{+} = G_{123}^{+} + \dfrac{\hat{\Gamma}_1}{2}(G_{111}^{+} - \Gamma G_{112}^{+}) + \dfrac{\bar{\Gamma}_3}{\bar{\Gamma}_1} G_{113}^{+} \\ f_{10}^{+} = Df_{10}^{+} = g_{10}^{+} = 0 \quad \text{at } x = \pm\frac{1}{2} \end{cases} \quad (2.4.106)$$

$$\begin{cases} M^2 f_{11}^{+} + g_{11}^{+} - i\sigma_{00} M f_{11}^{+} = F_{124}^{+} \\ \lambda^2 T_0 V_0 f_{11}^{+} - M g_{11}^{+} + i\sigma_{00} g_{11}^{+} = G_{124}^{+} + \dfrac{\bar{\Gamma}_4}{\bar{\Gamma}_1} G_{113}^{+} \\ f_{11}^{+} = Df_{11}^{+} = g_{11}^{+} = 0 \quad \text{at } x = \pm\frac{1}{2} \end{cases} \quad (2.4.107)$$

$$\begin{cases} M^2 f_{12}^{+} + g_{12}^{+} - i\sigma_{00} M f_{12}^{+} = F_{125}^{+} + \dfrac{\hat{\Gamma}_1}{2}(F_{111}^{+} - \Gamma F_{112}^{+}) + \hat{\Gamma}_2 F_{112}^{+} \\ \lambda^2 T_0 V_0 f_{12}^{+} - M g_{12}^{+} + i\sigma_{00} g_{12}^{+} = G_{125}^{+} + \dfrac{\hat{\Gamma}_1}{2}(G_{111}^{+} - \Gamma G_{112}^{+}) + \hat{\Gamma}_2 G_{112}^{+} \\ f_{12}^{+} = Df_{12}^{+} = g_{12}^{+} = 0 \quad \text{at } x = \pm\frac{1}{2} \end{cases} \quad (2.4.108)$$

$$\begin{cases} M^2 f_{13}^+ + g_{13}^+ - i\sigma_{00} M f_{13}^+ = F_{126}^+ - \hat{\Gamma}_1 F_{112}^+ \\ \lambda^2 T_0 V_0 f_{13}^+ - M g_{13}^+ + i\sigma_{00} g_{13}^+ = G_{126}^+ - \hat{\Gamma}_1 G_{112}^+ \\ f_{13}^+ = D f_{13}^+ = g_{13}^+ = 0 \quad at\ x = \pm\frac{1}{2} \end{cases} \tag{2.4.109}$$

$$\begin{cases} M^2 f_{14}^+ + g_{14}^+ - i\sigma_{00} M f_{14}^+ = F_{129}^+ + \hat{\Gamma}_1 (F_{112}^+ + \frac{1}{c} F_{114}^+) - \hat{\Gamma}_7 F_{112}^+ \\ \lambda^2 T_0 V_0 f_{14}^+ - M g_{14}^+ + i\sigma_{00} g_{14}^+ = G_{129}^+ + \hat{\Gamma}_1 (G_{112}^+ + \frac{1}{c} G_{114}^+) - \hat{\Gamma}_7 G_{112}^+ \\ f_{14}^+ = D f_{14}^+ = g_{14}^+ = 0 \quad at\ x = \pm\frac{1}{2} \end{cases} \tag{2.4.110}$$

$$\begin{cases} M^2 f_{15}^+ + g_{15}^+ - i\sigma_{00} M f_{15}^+ = F_{128}^+ - \hat{\Gamma}_6 F_{112}^+ \\ \lambda^2 T_0 V_0 f_{15}^+ - M g_{15}^+ + i\sigma_{00} g_{15}^+ = G_{128}^+ - \hat{\Gamma}_6 G_{112}^+ \\ f_{15}^+ = D f_{15}^+ = g_{15}^+ = 0 \quad at\ x = \pm\frac{1}{2} \end{cases} \tag{2.4.111}$$

$$\begin{cases} M^2 f_{16}^+ + g_{16}^+ - i\sigma_{00} M f_{16}^+ = F_{130}^+ - \frac{\hat{\Gamma}_8}{c} M f_0^+ \\ \lambda^2 T_0 V_0 f_{16}^+ - M g_{16}^+ + i\sigma_{00} g_{16}^+ = G_{130}^+ + \frac{\hat{\Gamma}_8}{c} g_0^+ \\ f_{16}^+ = D f_{16}^+ = g_{16}^+ = 0 \quad at\ x = \pm\frac{1}{2} \end{cases} \tag{2.4.112}$$

$$\begin{cases} M^2 f_{18}^+ + g_{18}^+ - i\sigma_{00} M f_{18}^+ = F_{132}^+ + \frac{\Gamma_{10}}{c} F_{114}^+ \\ \lambda^2 T_0 V_0 f_{18}^+ - M g_{18}^+ + i\sigma_{00} g_{18}^+ = G_{132}^+ + \frac{\Gamma_{10}}{c} G_{114}^+ \\ f_{18}^+ = D f_{18}^+ = g_{18}^+ = 0 \quad at\ x = \pm\frac{1}{2} \end{cases} \tag{2.4.113}$$

As far as the ε expansion of $w_0^{1(2)}$ is concernad, we have the results:

$$w_{00}^{1(2)} \equiv 0 \tag{2.4.114}$$

where

$$w_0^{1(2)} = w_{00}^{1(2)} + \varepsilon w_{01}^{1(2)} + \varepsilon^2 w_{02}^{1(2)} \tag{2.4.115}$$

$$w_{01}^{1(2)} = \frac{AB}{2\lambda T_1^{1/2}} \left[h_{10} \cos\phi + imk T_0^{1/2} h_{11} \right] - \frac{ckT_0^{1/2}}{2\lambda T_1^{1/2}} h_{12} A \frac{dB}{d\phi} \tag{2.4.116}$$

where h_{10}, h_{11} and h_{12} are defined by

$$D^2h_{10} = f_0 D^2 f_1 - f_1 D^2 f_0 - f_0 D^2 f_0 - c\Gamma f_0 D g_0 \qquad (2.4.117)$$

$$D^2h_{11} = f_0 D^2 f_3 - f_3 D^2 f_0 - cD(f_0 g_0) \qquad (2.4.118)$$

$$D^2h_{12} = g_0 D f_0 \qquad (2.4.119)$$

$$w_{02}^{1(2)} = \frac{AB}{\lambda T_1^{1/2}}\Big[h_{20} + h_{21}\cos 2\phi + k^2 h_{22} + kT_0^{1/2} h_{23}\sin\phi + \\ + \frac{\sin 3\phi + \sin\phi}{kT_0^{1/2}} h_{24} + m^2 kT_0^{1/2}\sin\phi \cdot h_{25} + m^2 k^2 T_0 h_{26} + \\ + imkT_0^{1/2}\cos\phi \cdot h_{27} + imh_{28}\sin 2\phi\Big] - \frac{c}{4\lambda T_0} imkT_0^{1/2} F_0(x) AB T_1^{1/2} + \\ + \frac{c}{2\lambda T_1^{1/2}} A\frac{dB}{d\phi}\Big[h_{29} kT_0^{1/2}\cos\phi + imk^2 T_0 h_{30} + \\ + h_{12}\Big(\frac{\Gamma_2 \sin 2\phi}{2} - imkT_0^{1/2}\Gamma_7 + \frac{kT_1}{2T_0^{1/2}}\Big)\Big] \qquad (2.4.120)$$

where

h_{20}, h_{21}, h_{22}, h_{23}, h_{24}, h_{25}, h_{26}, h_{27}, h_{28}, h_{29} and h_{30} are given by simple integrations of the following quantities:

$$D^2h_{20} = \frac{c^2 g_0^2}{4}\Gamma^2 + \frac{1}{2}(f_0 D^2 f_{10} - f_{10} D^2 f_0) - \frac{\Gamma_1 f_0 D^2 f_0}{4} - \frac{f_1 D^2 f_0}{4} - \\ - \frac{\Gamma c(\Gamma_1 - 1)}{4} g_0 D f_0 - \frac{c\Gamma}{4}\Big[2g_1 D f_0 - g_0 D f_1 - f_0 D g_1 - \Gamma_1 f_0 D g_0\Big] + \frac{\Gamma V_0 h_{10}}{2} + \\ + \frac{1}{2}(f_0 D^2 f_1 - f_1 D^2 f_0 - f_0 D^2 f_0 - c\Gamma f_0 D g_0) \qquad (2.4.121)$$

$$D^2h_{21} = \frac{c^2 g_0^2}{4}\Gamma^2 + \frac{1}{2}(f_0 D^2 f_{12} - f_{12} D^2 f_0) - \frac{f_1 D^2 f_0}{4} + \frac{c\Gamma}{2} g_0 D f_0 - \\ - \frac{c\Gamma}{4}(f_0 D g_1 + g_0 D f_1) - \frac{c\Gamma\Gamma_1}{4} f_0 D g_0 - \frac{\Gamma_1 f_0 D^2 f_0}{4} - \frac{f_0 D^2 f_0}{4} + \frac{c\Gamma}{4} g_1 D f_0 + \\ + \frac{1}{2} D^2 h_{10} + \frac{\Gamma V_0 h_{10}}{2} - \frac{c\Gamma D f_0}{4}\Big[g_1 + (\Gamma_1 - 1) g_0\Big] \qquad (2.4.122)$$

$$D^2h_{22} = c f_0 D f_0 + \frac{1}{2}(f_0 D^2 f_{11} - f_{11} D^2 f_0) \qquad (2.4.123)$$

$$D^2h_{23} = \frac{1}{2}(f_0 D^2 f_{13} - f_{13} D^2 f_0) - \frac{c^2\Gamma g_0^2}{2} + \frac{c}{2}(f_0 D g_1 - g_0 D f_1) + \\ + \frac{c\Gamma_1}{2}(f_0 D g_0 - g_0 D f_0) + \frac{c}{2} D(f_0 g_0) - \frac{V_0 h_{10}}{2} \qquad (2.4.124)$$

$$D^2h_{24} = \frac{1}{2}(f_0 D^2 f_{15} - f_{15} D^2 f_0) - \frac{\Gamma_2}{8} f_0 D^2 f_0 - \frac{c\Gamma\Gamma_2}{8} D(f_0 g_0) \quad (2.4.125)$$

$$D^2h_{25} = \frac{1}{2}(f_0 D^2 f_{17} - f_{17} D^2 f_0) - \frac{c\Gamma_7}{2} D(f_0 g_0) \quad (2.4.126)$$

$$D^2h_{26} = -\frac{c^2 g_0^2}{2} + \frac{1}{2}(f_0 D^2 f_{18} - f_{18} D^2 f_0) + \frac{c}{2}(f_0 D g_3 + g_0 D f_3) - \\ - \frac{1}{2}(V_0 - \frac{1}{c}) h_{11} \quad (2.4.127)$$

$$D^2h_{27} = -\frac{h_{10}}{2c} - \frac{1}{2} f_{14} D^2 f_0 + c^2\Gamma g_0^2 - \frac{c g_0}{2} D(f_1 + \Gamma_1 f_0) - \\ - \frac{c\Gamma}{2}(f_0 D g_3 + g_0 D f_3) - \frac{c f_0}{2}(D g_1 + \Gamma_1 D g_0) + \frac{f_0 D^2 f_{14}}{2} + D^2 h_{11} + \\ + \frac{c\Gamma_7}{2}(f_0 D g_0 - g_0 D f_0) - \frac{f_3 D^2 f_0}{2} + \frac{V_0}{2}(h_{10} + 2\Gamma h_{11}) \quad (2.4.128)$$

$$D^2h_{28} = \frac{1}{2}(f_0 D^2 f_{16} - f_{16} D^2 f_0) + \frac{\Gamma_7}{4}[f_0 D^2 f_0 + c\Gamma D(f_0 g_0)] \quad (2.4.129)$$

$$D^2h_{29} = D f_0 \cdot [g_1 + (\Gamma_1 - 1) g_0] \quad (2.4.130)$$

$$D^2h_{30} = g_3 D f_0 \quad (2.4.131)$$

If we return to the $(u'^{(2)}, v'^{(2)}, w'^{(2)}, p'^{(2)})$ terms, given by

$$u'^{(2)} = u_1'^{(2)} \sin\lambda\xi_f + u_0'^{(2)} + u_2'^{(2)} \sin 2\lambda\xi_f$$

$$v'^{(2)} = v_1'^{(2)} \sin\lambda\xi_f + v_0'^{(2)} + v_2'^{(2)} \sin 2\lambda\xi_f$$

$$w'^{(2)} = w_1'^{(2)} \cos\lambda\xi_f + w_0'^{(2)} + w_2'^{(2)} \cos 2\lambda\xi_f$$

$$p'^{(2)} = p_1'^{(2)} \sin\lambda\xi_f + p_0'^{(2)} + p_2'^{(2)} \sin 2\lambda\xi_f \quad (2.4.132)$$

and we expand the harmonic $(u_2'^{(2)}, v_2'^{(2)}, w_2'^{(2)}, p_2'^{(2)})$ in powers of ε, as follows

$$u_2'^{(2)} = u_{20}'^{(2)} + \varepsilon u_{21}'^{(2)} + \varepsilon^2 u_{22}'^{(2)} + \cdots\cdots$$

$$v_2'^{(2)} = v_{20}'^{(2)} + \varepsilon v_{21}'^{(2)} + \varepsilon^2 v_{22}'^{(2)} + \cdots\cdots$$

$$w_2'^{(2)} = w_{20}'^{(2)} + \varepsilon w_{21}'^{(2)} + \varepsilon^2 w_{22}'^{(2)} + \cdots\cdots$$

$$p_2'^{(2)} = p_{20}'^{(2)} + \varepsilon p_{21}'^{(2)} + \varepsilon^2 p_{22}'^{(2)} + \cdots\cdots \quad (2.4.133)$$

one obtains

$$u_{20}^{1(2)} = -\frac{2AB}{T_1^{1/2}} f_{112}$$

$$v_{20}^{1(2)} = \frac{2AB}{T_1^{1/2}} g_{112} \qquad (2.4.134)$$

$$w_{20}^{1(2)} = -\frac{AB}{\lambda T_1^{1/2}} Df_{112}$$

We shall later see that it is much more convenient to formulate $(\bar{u}_{21}^{1(2)}, v_{21}^{1(2)})$ with respect to $kT_0^{1/2} A \frac{dB}{d\phi}$, than with respect to AB^3

As for $(\bar{u}_{21}^{1(2)}, v_{21}^{1(2)}, w_{21}^{1(2)}, p_{21}^{1(2)})$ one obtains

$$\bar{u}_{21}^{1(2)} = kT_0^{1/2} A \frac{dB}{d\phi} \cdot \frac{f_{210}}{T_1^{1/2}} + \frac{AB}{T_1^{1/2}} \left[imkT_0^{1/2} f_{211} + f_{112} \left(\frac{\Gamma_2 \sin 2\phi}{kT_0^{1/2}} - 2im\Gamma_7 \sin\phi \right) + \right.$$

$$\left. + f_{212} \cos\phi \right] \qquad (2.4.135)$$

$$v_{21}^{1(2)} = kT_0^{1/2} A \frac{dB}{d\phi} \frac{g_{210}}{T_1^{1/2}} + \frac{AB}{T_1^{1/2}} \left[imkT_0^{1/2} g_{211} + g_{112} \left(\frac{\Gamma_2 \sin 2\phi}{kT_0^{1/2}} - 2im\Gamma_7 \sin\phi \right) + \right.$$

$$\left. + g_{212} \cos\phi \right] \qquad (2.4.136)$$

$$w_{21}^{1(2)} = kT_0^{1/2} A \frac{dB}{d\phi} \cdot \frac{w_{210}}{T_1^{1/2}} + \frac{AB}{T_1^{1/2}} \left[imkT_0^{1/2} w_{211} - \frac{Df_{112}}{2\lambda} \left(\frac{\Gamma_2 \sin 2\phi}{kT_0^{1/2}} - \right. \right.$$

$$\left. \left. - 2im\Gamma_7 \sin\phi \right) + w_{212} \cos\phi \right] \qquad (2.4.137)$$

$$p_{21}^{1(2)} = kT_0^{1/2} A \frac{dB}{d\phi} \frac{p_{210}}{T_1^{1/2}} + \frac{AB}{T_1^{1/2}} \left[imkT_0^{1/2} p_{211} + p_{112} \left(\frac{\Gamma_2 \sin 2\phi}{kT_0^{1/2}} - \right. \right.$$

$$\left. \left. - 2im\Gamma_7 \sin\phi \right) + p_{212} \cos\phi \right] \qquad (2.4.138)$$

where $(f_{210}, g_{210}, w_{210}, p_{210})$, $(f_{211}, g_{211}, w_{211}, p_{211})$ and $(f_{212}, g_{212}, w_{212}, p_{212})$ are given by three systems of ordinary differential equations respectively as follows:

$$(D^2 - 4\lambda^2) f_{210} - T_0 V_0 g_{210} + \frac{\partial p_{210}}{\partial x} = \frac{c f_0 g_0}{2} + 2V_0 f_{112}$$

$$f_{210} - (D^2 - 4\lambda^2) g_{210} = -\frac{c}{4} g_0^2 - 2V_0 g_{112}$$

$$-(D^2 - 4\lambda^2) w_{210} + 2\lambda p_{210} = \frac{1}{\lambda} \left(V_0 Df_{112} + \frac{c g_0 Df_0}{2} \right)$$

$$Df_{210} + 2\lambda w_{210} = 2c g_{112} \qquad (2.4.139)$$

$$\begin{cases} (D^2-4\lambda^2)f_{211} - T_0 V_0 g_{211} + \frac{\partial p_{211}}{\partial x} = \frac{cT_0 g_0 g_3}{2} + 2\left(V_0 - \frac{1}{c}\right) f_{112} \\ f_{211} - (D^2-4\lambda^2) g_{211} = \frac{1}{2}(f_0 Dg_3 + f_3 Dg_0 - g_3 Df_0 - g_0 Df_3) + \frac{c}{4} g_0^2 - 2\left(V_0 - \frac{1}{c}\right) g_{112} \\ -(D^2-4\lambda^2) w_{211} + 2\lambda p_{211} = \frac{1}{\lambda}\Big[\left(V_0 - \frac{1}{c}\right) Df_{112} + \frac{c}{2}(f_0 Dg_0 - g_0 Df_0) - \\ \qquad - \frac{1}{2}(f_3 D^2 f_0 + f_0 D^2 f_3 - 2Df_0 \cdot Df_3)\Big] \\ Df_{211} + 2\lambda w_{211} = 2c g_{112} \end{cases} \qquad (2.4.140)$$

$$(D^2-4\lambda^2) f_{212} - T_0 V_0 g_{212} + \frac{\partial p_{212}}{\partial x} = \frac{1}{2\lambda^2}(Df_0 \cdot D^2 f_0 - f_0 D^3 f_0) -$$

$$- \frac{D^2(D^2-12\lambda^2)}{2\lambda^2} f_{112} + 12\left(x^2 - \frac{1}{4}\right) T_0 g_{112} + \frac{cT_0 g_0 (g_1 + \Gamma_1 g_0)}{2} + \frac{f_0 Df_0}{2} +$$

$$+ 2\Gamma V_0 f_{112}$$

$$f_{212} - (D^2-4\lambda^2) g_{212} = \Gamma_1 (f_0 Dg_0 - g_0 Df_0) - f_0 Dg_0 + \frac{g_0 Df_0}{2} - 4D^2 g_{112} +$$

$$+ 2\left(\frac{12x}{c} + 1\right) f_{112} + \frac{f_0 Dg_1 + f_1 Dg_0 - g_1 Df_0 - g_0 Df_1}{2} - 2\Gamma V_0 g_{112}$$

$$-(D^2-4\lambda^2) w_{212} + 2\lambda p_{212} = \frac{1}{\lambda}\Big[2D^3 f_{112} + \Gamma V_0 Df_{112} + \frac{3}{2} f_0 D^2 f_0 +$$

$$+ \frac{c\Gamma}{2}(f_0 Dg_0 - g_0 Df_0) - \frac{f_0}{c^2} D^2 (f_1 + \Gamma_1 f_0) - \frac{f_1 + \Gamma_1 f_0}{2} D^2 f_0 + Df_0 \cdot D(f_1 - f_0 + \Gamma_1 f_0)\Big]$$

$$Df_{212} + 2\lambda w_{212} = 2(Df_{112} + c\Gamma g_{112}) \qquad (2.4.141)$$

$\bar{u}_{22}^{1(2)}$ and $v_{22}^{1(2)}$ are some lengthy complex functions given by:

$$\bar{u}_{22}^{1(2)} = \frac{AB}{T_1^{1/2}} \Big\{ f_{220} + k^2 f_{221} + k T_0^{1/2} f_{222} \sin\phi + f_{223} \cos\phi +$$

$$+ \frac{\sin 3\phi + \sin\phi}{k T_0^{1/2}} f_{224} + m^2 k T_0^{1/2} f_{225} \sin\phi + m^2 k^2 T_0 f_{226} +$$

$$+ im\left[k T_0^{1/2} \cos\phi \cdot f_{227} + f_{228} \sin 2\phi\right]\Big\} +$$

$$+ \frac{AB T_1^{1/2}}{T_0}\left[f_{229} \cos\phi + im \frac{k T_0^{1/2} \cos\phi}{f_{230}^{-1}}\right] + \frac{1}{T_1^{1/2}} A \frac{dB}{d\phi}\left[\frac{\Gamma_2}{2} \sin 2\phi \cdot f_{210} +\right.$$

$$+ k T_0^{1/2} f_{229} \cos\phi + k T_1 (f_{210} - f_{229}) + im(k^2 T_0 f_{230} -$$

$$\left. - k T_0^{1/2} \Gamma_7 f_{210} \sin\phi)\right] + \frac{k^2 T_0}{T_1^{1/2}} A \frac{d^2 B}{d\phi^2} f_{231} \qquad (2.4.142)$$

$$v_{22}^{1(2)} = \frac{AB}{T_1^{1/2}}\Big\{ g_{220} + k^2 g_{221} + kT_0^{1/2} g_{222}\sin\phi + g_{223}\cos 2\phi + \\ + \frac{\sin 3\phi + \sin\phi}{kT_0^{1/2}} g_{224} + m^2 kT_0^{1/2} g_{225}\sin\phi + m^2 k^2 T_0 g_{226} + \\ + im\Big[kT_0^{1/2}\cos\phi \cdot g_{227} + g_{228}\sin 2\phi\Big]\Big\} + \frac{ABT_1^{1/2}\cos\phi}{T_0}(g_{229} + im kT_0^{1/2} g_{230}) + \\ + \frac{1}{T_1^{1/2}} A \frac{dB}{d\phi}\Big[\frac{\Gamma_2}{2}\sin 2\phi \cdot g_{210} + kT_0^{1/2} g_{229}\cos\phi + kT_1(g_{210} - g_{229}) + \\ + im(k^2 T_0 g_{230} - kT_0^{1/2}\Gamma_7 \cdot g_{210}\sin\phi)\Big] + \frac{k^2 T_0}{T_1^{1/2}} A \frac{d^2 B}{d\phi^2} g_{231} \qquad (2.4.143)$$

where (f_{220}, g_{220}), (f_{221}, g_{221}), (f_{222}, g_{222}), (f_{223}, g_{223}), (f_{224}, g_{224}), (f_{225}, g_{225}), (f_{226}, g_{226}), (f_{227}, g_{227}), (f_{228}, g_{228}), (f_{229}, g_{229}), (f_{230}, g_{230}) and (f_{231}, g_{231}) are known eigenfunctions.

Let us return now to the important formula of the growth rate:

$$i\overline{\sigma_2}(TV=0) + i\overline{\sigma_2}(TV) = \frac{I_2}{I_1} \qquad (2.4.144)$$

where

$$I_1 = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} (u_1^{1(1)} u^+ + v_1^{1(1)} v^+ + w_1^{1(1)} w^+)\, dx\, d\phi = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} (\Sigma_0 + \varepsilon\Sigma_1 + \varepsilon^2\Sigma_2)\, dx\, d\phi =$$

$$= \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[(\hat{u}_{10} u_0^+ + \hat{v}_{10} v_0^+ + \hat{w}_{10} w_0^+) + \varepsilon(u_0^+ \hat{u}_{11} + \hat{u}_{10} u_1^+ + v_0^+ \hat{v}_{11} + \hat{v}_{10} v_1^+ + \\ + w_0^+ \hat{w}_{11} + \hat{w}_{10} w_1^+) + \varepsilon^2(\hat{u}_{12} u_0^+ + \hat{v}_{12} v_0^+ + \hat{w}_{12} w_0^+ + \hat{u}_{11} u_1^+ + \hat{v}_{11} v_1^+ + \\ + \hat{w}_{11} w_1^+ + \hat{u}_{10} u_2^+ + \hat{v}_{10} v_2^+ + \hat{w}_{10} w_2^+)\Big]\, dx\, d\phi \qquad (2.4.145)$$

$$I_2 = \int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[(u_0^+ \Sigma_{10} + v_0^+ \Sigma_{20} + w_0^+ \Sigma_{30}) + \varepsilon(u_1^+ \Sigma_{10} + v_1^+ \Sigma_{20} + w_1^+ \Sigma_{30} + \\ + u_0^+ \Sigma_{11} + v_0^+ \Sigma_{21} + w_0^+ \Sigma_{31}) + \varepsilon^2(u_0^+ \Sigma_{12} + v_0^+ \Sigma_{22} + w_0^+ \Sigma_{32} + \\ + u_1^+ \Sigma_{11} + v_1^+ \Sigma_{21} + w_1^+ \Sigma_{31} + u_2^+ \Sigma_{10} + v_2^+ \Sigma_{20} + w_2^+ \Sigma_{30})\Big]\, dx\, d\phi$$

$$(2.4.146)$$

where $\Sigma_{10}, \Sigma_{20}, \Sigma_{30}, \Sigma_{11}, \Sigma_{21}, \Sigma_{31}, \Sigma_{12}, \Sigma_{22}$ and Σ_{32} are algebraic expressions:

$$\Sigma_{10} = V_0 \hat{v}_{10} + \frac{cT_0 B(\phi)}{2T_1^{1/2}} g_0 v_{20}^{1(2)} + \frac{cT_0}{T_1}\hat{v}_{10}\Big[B^2(\phi) g_{10} + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) - \\ - \frac{B^2(\phi)}{2} g_{112}\Big] - \frac{B(\phi)}{2T_1^{1/2}}\Big[\frac{B(\phi)}{T_1^{1/2}} D(f_{112}\cdot\hat{u}_{10}) - D(f_0 \cdot u_{20}^{1(2)})\Big] - \frac{2B^2(\phi)}{T_1} f_{112}\cdot\hat{w}_{10} \\ + \frac{2B(\phi) f_0}{2T_1^{1/2}} w_{20}^{1(2)} + \frac{B(\phi) D f_0}{T_1^{1/2}} u_{20}^{1(2)} - \frac{B^2(\phi)}{4T_1}\hat{u}_{10}\cdot D f_{112} \qquad (2.4.147)$$

$$\Sigma_{20} = -\frac{B(\Phi)}{2T_1^{1/2}}\left[u_{20}^{1(2)} Dg_0 - g_0 Dv_{20}^{1(2)}\right] - \frac{B^2(\Phi)}{2T_1} g_{112} D\hat{v}_{10} - \frac{\hat{u}_{10}}{T_1}\left[B^2(\Phi)\cdot D\left(g_{10}-\frac{g_{112}}{2}\right)+\right.$$

$$\left.+\sigma_0 x\right] + \frac{B(\Phi) Dg_0}{T_1^{1/2}} v_{20}^{1(2)} - \frac{B^2(\Phi)}{4T_1}\hat{v}_{10} Dg_{112} + \frac{B^2(\Phi)}{T_1}\lambda\hat{w}_{10} g_{112} - \frac{\lambda B(\Phi) g_0 w_{20}^{1(2)}}{2T_1^{1/2}} \quad (2.4.148)$$

$$\Sigma_{30} = \frac{B(\Phi)}{2T_1^{1/2}}\left[Dg_0\cdot w_{20}^{1(2)} - \frac{B(\Phi)}{2T_1^{1/2}}\hat{w}_{10} Dg_{112}\right] + \frac{B^2(\Phi)}{4\lambda T_1}\left[-\hat{u}_{10} D^2 g_{112} + 2\lambda g_{112} D\hat{w}_{10}\right] +$$

$$+ \frac{B(\Phi)}{T_1^{1/2}}\left[-\frac{D^2 g_0}{2\lambda} u_{20}^{1(2)} + \frac{g_0}{\sigma_2} Dw_{20}^{1(2)}\right] \quad (2.4.149)$$

$$\Sigma_{11} = 6\left(x^2-\frac{1}{4}\right)\cos\phi\cdot v_{10} + V_0\hat{v}_{11} + \frac{cT_0 B(\Phi)}{2T_1^{1/2}} g_0 v_{21}^{1(2)} + \frac{cT_0}{T_1}\hat{v}_{11}\left[B^2(\Phi)\cdot\left(g_{10}-\frac{g_{112}}{2}\right)+\right.$$

$$\left.+\frac{\sigma_0}{2}\left(x^2-\frac{1}{4}\right)\right] - \frac{B(\Phi)}{2T_1^{1/2}}\left[\frac{B(\Phi)}{T_1^{1/2}} D\left(g_{112}\cdot\hat{u}_{11}\right) - D\left(g_0\cdot u_{21}^{1(2)}\right)\right] + \cos\phi\cdot\frac{B(\Phi)}{2T_1^{1/2}}\left[\frac{B(\Phi)}{T_1^{1/2}} D\left(g_{112}\hat{u}_{10}\right)-\right.$$

$$\left.- D\left(g_0 u_{20}^{1(2)}\right)\right] - ckT_0^{1/2}\left\{\frac{B(\Phi)}{2T_1^{1/2}} g_0\frac{\partial u_{20}^{1(2)}}{\partial\phi} + \frac{1}{T_1}\frac{\partial\hat{u}_{10}}{\partial\phi}\left[B^2(\Phi)\cdot\left(g_{10}-\frac{g_{112}}{2}\right)+\right.\right.$$

$$\left.\left.+\frac{\sigma_0}{2}\left(x^2-\frac{1}{4}\right)\right] + \frac{B(\Phi)}{T_1}\frac{dB(\Phi)}{d\phi}\hat{v}_{10} g_{112} - \frac{1}{2T_1^{1/2}}\frac{dB(\Phi)}{d\phi} g_0 v_{20}^{1(2)}\right\} - \frac{\lambda B^2(\Phi)}{T_1} g_{112}\hat{w}_{11} -$$

$$- \frac{\lambda B(\Phi) g_0}{T_1^{1/2}}\left(w_{01}^{1(2)} - \frac{w_{21}^{1(2)}}{2}\right) + \frac{B(\Phi) Dg_0}{T_1^{1/2}} u_{21}^{1(2)} - \frac{B^2(\Phi)}{4T_1}\hat{u}_{11} Dg_{112} \quad (2.4.150)$$

$$\Sigma_{21} = \cos\phi\left\{\frac{B(\Phi)}{2T_1^{1/2}}\left[u_{20}^{1(2)} Dg_0 - g_0 Dv_{20}^{1(2)}\right] + \frac{B^2(\Phi)}{2T_1} g_{112} D\hat{v}_{10} +\right.$$

$$\left.+\frac{\hat{u}_{10}}{T_1}\left[B^2(\Phi)\cdot D\left(g_{10}-\frac{g_{112}}{2}\right)+\sigma_0 x\right]\right\} - \left\{\frac{B(\Phi)}{2T_1^{1/2}}\left[u_{21}^{1(2)} Dg_0 - g_0 Dv_{21}^{1(2)}\right] +\right.$$

$$\left.+\frac{B^2(\Phi)}{2T_1} g_{112}\frac{\partial\hat{v}_{11}}{\partial x} + \frac{\hat{u}_{11}}{T_1}\left[B^2(\Phi)\cdot D\left(g_{10}-\frac{g_{112}}{2}\right)+\sigma_0 x\right]\right\} -$$

$$- ckT_0^{1/2}\frac{\partial}{\partial\phi}\left\{\frac{B(\Phi)}{2T_1^{1/2}} g_0 v_{20}^{1(2)} + \frac{\hat{v}_{10}}{T_1}\left[B^2(\Phi)\cdot\left(g_{10}-\frac{g_{112}}{2}\right)+\frac{\sigma_0}{2}\left(x^2-\frac{1}{4}\right)\right]\right\} +$$

$$+\frac{B(\Phi) Dg_0 v_{21}^{1(2)}}{T_1^{1/2}} + \frac{B^2(\Phi)}{T_1}\left[\lambda\hat{w}_{11} g_{112} - \frac{Dg_{112}}{4}\hat{v}_{11}\right] + \frac{\lambda B(\Phi) g_0}{T_1^{1/2}}\left(w_{01}^{1(2)} - \frac{w_{21}^{1(2)}}{2}\right) \quad (2.4.151)$$

$$\Sigma_{31} = \frac{B(\Phi)}{2T_1^{1/2}}\left[Dg_0\cdot w_{21}^{1(2)} - \frac{B(\Phi)}{2T_1^{1/2}}\hat{w}_{11} Dg_{112} - 2Dg_0\cdot w_{01}^{1(2)}\right] +$$

$$+\frac{B^2(\Phi)}{4\lambda T_1}\left[-\hat{u}_{11} D^2 g_{112} + 2\lambda g_{112} D\hat{w}_{11}\right] - \cos\phi\cdot\frac{B^2(\Phi)}{4\lambda T_1}\left[-\hat{u}_{10} D^2 g_{112} +\right.$$

$$\left.+2\lambda g_{112} D\hat{w}_{10}\right] + \frac{B(\Phi)}{T_1^{1/2}}\left[-\frac{D^2 g_0}{2\lambda} u_{21}^{1(2)} + \frac{g_0}{\sigma_2} Dw_{21}^{1(2)} + g_0\frac{\partial w_{01}^{1(2)}}{\partial x}\right] -$$

$$- \cos\phi\cdot\frac{B(\Phi)}{T_1^{1/2}}\left[-\frac{D^2 g_0}{2\lambda} u_{20}^{1(2)} + \frac{g_0}{\sigma_2} Dw_{20}^{1(2)}\right] - ckT_0^{1/2}\left\{\frac{B(\Phi) g_0}{2T_1^{1/2}}\frac{\partial w_{20}^{1(2)}}{\partial\phi} +\right.$$

$$\left.+\frac{\hat{v}_{10}}{4\lambda T_1} Dg_{112}\frac{dB^2(\Phi)}{d\phi} + \frac{v_{20}^{1(2)}\cdot Dg_0}{2\lambda T_1^{1/2}}\frac{dB(\Phi)}{d\phi} + \frac{1}{T_1}\frac{\partial\hat{w}_{10}}{\partial\phi}\left[B^2(\Phi)\cdot\left(g_{10}+\frac{g_{112}}{2}\right)+\right.\right.$$

$$\left.\left.+\frac{\sigma_0}{2}\left(x^2-\frac{1}{4}\right)\right]\right\} \quad (2.4.152)$$

$$\Sigma_{12} = \frac{cT_0 B(\Phi)}{2T_1^{1/2}} g_0 v_{22}^{1(2)} + \frac{c}{2T_1^{1/2}} \frac{\bar{\Gamma}_3 + k^2\bar{\Gamma}_4}{\bar{\Gamma}_1} B(\Phi) g_0 v_{20}^{1(2)} +$$

$$+ \frac{cT_0}{T_1} \hat{v}_{12} \left[B^2(\Phi)(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] + \frac{c}{T_1} \hat{v}_{10} \frac{\bar{\Gamma}_3 + k^2\bar{\Gamma}_4}{\bar{\Gamma}_1} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \right.$$

$$\left. + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] - \frac{\lambda B^2(\Phi)}{T_1} g_{112} \hat{w}_{12} + \frac{\lambda B(\Phi) g_0}{2T_1^{1/2}} w_{22}^{1(2)} + \frac{B(\Phi) D g_0}{T_1^{1/2}} u_{22}^{1(2)} -$$

$$- \frac{B^2(\Phi)}{4T_1} \hat{u}_{12} D g_{112} - \frac{B(\Phi)}{2T_1^{1/2}} \left[\frac{B(\Phi)}{T_1^{1/2}} D(g_{112} \hat{u}_{12}) - D(g_0 u_{22}^{1(2)}) \right] +$$

$$+ \frac{B(\Phi)\cos\Phi}{2T_1^{1/2}} \left[\frac{B(\Phi)}{T_1^{1/2}} D(g_{112} \hat{u}_{11}) - D(g_0 u_{21}^{1(2)}) \right] - \frac{B(\Phi)}{4T_1^{1/2}} \left[\frac{B(\Phi)}{T_1^{1/2}} D(g_{112} \hat{u}_{10}) - D(g_0 u_{20}^{1(2)}) \right] -$$

$$- ckT_0^{1/2} \left\{ \frac{B(\Phi)}{2T_1^{1/2}} g_0 \frac{\partial u_{21}^{1(2)}}{\partial \Phi} + \frac{1}{T_1} \frac{\partial \hat{u}_{11}}{\partial \Phi} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] + \right.$$

$$\left. + \frac{B(\Phi)}{T_1} \frac{dB(\Phi)}{d\Phi} \hat{v}_{11} g_{112} - \frac{1}{2T_1^{1/2}} \frac{dB(\Phi)}{d\Phi} g_0 v_{21}^{1(2)} \right\} - c\left(\frac{kT_1}{2T_0^{1/2}} - kT_0^{1/2}\cos\Phi \right) \left\{ \frac{Bg_0}{2T_1^{1/2}} \frac{\partial u_{20}^{1(2)}}{\partial \Phi} + \right.$$

$$\left. + \frac{1}{T_1} \frac{\partial \hat{u}_{10}}{\partial \Phi} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] + \frac{1}{2T_1} \frac{dB^2(\Phi)}{d\Phi} \hat{v}_{10} g_{112} - \frac{1}{2T_1^{1/2}} \frac{dB(\Phi)}{d\Phi} g_0 v_{20}^{1(2)} \right\}$$

$$+ ckT_0^{1/2} \sin\Phi \left\{ - \frac{B(\Phi)}{2T_1^{1/2}} g_0 v_{20}^{1(2)} + \frac{\hat{v}_{10}}{2T_1} g_{112} B^2(\Phi) + \frac{\hat{u}_{10}}{T_1} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \right. \right.$$

$$\left. \left. + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] + \frac{B(\Phi)}{2T_1^{1/2}} g_0 u_{20}^{1(2)} \right\} \qquad (2.4.153)$$

$$\Sigma_{22} = -\frac{1}{2} \left\{ \frac{B(\Phi)}{2T_1^{1/2}} \left[u_{20}^{1(2)} D g_0 - g_0 D v_{20}^{1(2)} \right] + \frac{B^2(\Phi)}{2T_1} g_{112} D \hat{v}_{10} + \right.$$

$$\left. + \frac{\hat{u}_{10}}{T_1} \left[B^2(\Phi) D(g_{10} - \frac{g_{112}}{2}) + \sigma_0 x \right] \right\} + \cos\Phi \left\{ \frac{B(\Phi)}{2T_1^{1/2}} \left[u_{21}^{1(2)} D g_0 - g_0 D v_{21}^{1(2)} \right] + \right.$$

$$\left. + \frac{B^2(\Phi)}{2T_1} g_{112} D \hat{v}_{11} + \frac{\hat{u}_{11}}{T_1} \left[B^2(\Phi)\cdot D(g_{10} - \frac{g_{112}}{2}) + \sigma_0 x \right] \right\} - \left\{ \frac{B(\Phi)}{2T_1^{1/2}} \left[u_{22}^{1(2)} D g_0 - g_0 D v_{22}^{1(2)} \right] + \right.$$

$$\left. + \frac{B^2(\Phi)}{2T_1} g_{112} D \hat{v}_{12} + \frac{\hat{u}_{12}}{T_1} \left[B^2(\Phi)\cdot D(g_{10} - \frac{g_{112}}{2}) + \sigma_0 x \right] \right\} - ck^2 T_0^{1/2} \frac{\partial}{\partial \Phi} \left\{ \frac{B(\Phi)}{2T_1^{1/2}} g_0 v_{21}^{1(2)} + \right.$$

$$\left. + \frac{\hat{v}_{11}}{T_1} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] \right\} + \frac{B(\Phi) D g_0 v_{22}^{1(2)}}{T_1^{1/2}} + \frac{\lambda B(\Phi) g_0}{T_1^{1/2}} \left(w_{02}^{1(2)} - \frac{w_{22}^{1(2)}}{2} \right) -$$

$$- c\left(\frac{kT_1}{2T_0^{1/2}} - kT_0^{1/2}\cos\Phi \right) \cdot \frac{\partial}{\partial \Phi} \left\{ \frac{B(\Phi)}{2T_1^{1/2}} g_0 v_{20}^{1(2)} + \frac{\hat{v}_{10}}{T_1} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] \right\} +$$

$$+ \frac{B^2(\Phi)}{T_1} \left[\lambda \hat{w}_{12} g_{112} - \frac{D g_{112}}{4} \hat{v}_{12} \right] - 2ck^2 \left\{ - \frac{B(\Phi)}{2T_1^{1/2}} g_0 v_{20}^{1(2)} + \frac{\hat{v}_{10}}{2T_1} g_{112} \cdot B^2(\Phi) + \right.$$

$$\left. + \frac{\hat{u}_{10}}{T_1} \left[B^2(\Phi)\cdot(g_{10} - \frac{g_{112}}{2}) + \frac{\sigma_0}{2}(x^2 - \frac{1}{4}) \right] + \frac{B(\Phi)}{2T_1^{1/2}} g_0 u_{20}^{1(2)} \right\} \qquad (2.4.154)$$

$$\Sigma_{32} = \frac{B(\phi)}{2T_1^{1/2}}\left[Df_0 \cdot w_{22}^{1(2)} - \frac{B(\phi)}{2T_1^{1/2}}\hat{w}_{12} Df_{112} - 2Df_0 \cdot w_{02}^{1(2)}\right] + \frac{B^2(\phi)}{4\lambda T_1}\left[-\hat{u}_{12} D^2 f_{112} + \right.$$

$$\left. + 2\lambda f_{112} D\hat{w}_{12}\right] - \frac{B^2(\phi)\cos\phi}{4\lambda T_1}\left[-\hat{u}_{11} D^2 f_{112} + 2\lambda f_{112} D\hat{w}_{11}\right] + \frac{B^2(\phi)}{8\lambda T_1}\left[-\hat{u}_{10} D^2 f_{112} + \right.$$

$$\left. + 2\lambda f_{112} D\hat{w}_{10}\right] + \frac{B(\phi)}{T_1^{1/2}}\left[-\frac{D^2 f_0}{2\lambda} u_{22}^{1(2)} + \frac{f_0}{\sigma_0} Dw_{22}^{1(2)} + f_0 \frac{\partial w_{02}^{1(2)}}{\partial x}\right] -$$

$$- \frac{B(\phi)\cos\phi}{T_1^{1/2}}\left[-\frac{D^2 f_0}{2\lambda} u_{21}^{1(2)} + \frac{f_0}{\sigma_0} Dw_{21}^{1(2)} + f_0 Dw_{01}^{1(2)}\right] + \frac{1}{2}\frac{B(\phi)}{T_1^{1/2}}\left[-\frac{D^2 f_0}{2\lambda} u_{20}^{1(2)} + \right.$$

$$\left. + \frac{f_0}{\sigma_0} Dw_{20}^{1(2)}\right] - c k T_0^{1/2}\left\{\frac{B(\phi) g_0}{2T_1^{1/2}} \frac{\partial w_{21}^{1(2)}}{\partial \phi} + \frac{\hat{v}_{11} Df_{112}}{4\lambda T_1} \cdot \frac{dB^2(\phi)}{d\phi} + \frac{B(\phi) g_0}{T_1^{1/2}} \frac{\partial w_{01}^{1(2)}}{\partial \phi} + \right.$$

$$\left. + \frac{v_{21}^{1(2)} Df_0}{2\lambda T_1^{1/2}} \frac{dB(\phi)}{d\phi} + \frac{1}{T_1} \frac{\partial \hat{w}_{11}}{\partial \phi}\left[B^2(\phi)\cdot\left(g_{10} + \frac{g_{112}}{2}\right) + \frac{\sigma_0}{2}\left(x^2 - \frac{1}{4}\right)\right]\right\} +$$

$$+ c\left(k T_0^{1/2}\cos\phi - \frac{k T_1}{2T_0^{1/2}}\right)\left\{\frac{B(\phi) g_0}{2T_1^{1/2}} \frac{\partial w_{20}^{1(2)}}{\partial \phi} + \frac{\hat{v}_{10} Df_{112}}{4\lambda T_1} \frac{dB^2(\phi)}{d\phi} + \right.$$

$$\left. + \frac{v_{20}^{1(2)} Df_0}{2\lambda T_1^{1/2}} \frac{dB(\phi)}{d\phi} + \frac{1}{T_1} \frac{\partial \hat{w}_{10}}{\partial \phi}\left[B^2(\phi)\cdot\left(g_{10} + \frac{g_{112}}{2}\right) + \frac{\sigma_0}{2}\left(x^2 - \frac{1}{4}\right)\right]\right\} \quad (2.4.155)$$

We have the following important formula

$$i\bar{\sigma} = i\bar{\sigma}_0 + i\bar{\sigma}_2 \cdot \gamma \qquad (2.4.155a)$$

where γ was previously defined as the amount above the critical.

Stability considerations require that

$$Re\left[i\bar{\sigma}\right] = 0 \qquad (2.4.156)$$

which implies that

$$0 = -\frac{\varepsilon^2 m^2 k^2 T_0 \Gamma_{10}}{\bar{c}} + (T - T_c)\cdot Re\left[i\bar{\sigma}_2(TV) + i\bar{\sigma}_2(TV=0)\right] \quad (2.4.157)$$

If the growth rate is equal to 0, then this will determine $T = T'(m)$

$$T'(m) = T_0 + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2\bar{\Gamma}_4}{\bar{\Gamma}_1} + \frac{\varepsilon^2 m^2 k^2 T_0 \cdot \frac{\Gamma_{10}}{\bar{c}}}{Re\left[i\bar{\sigma}_2(TV=0) + i\bar{\sigma}_2(TV)\right]} \quad (2.4.158)$$

$T'(m)$ is the critical Taylor number for the wavy vortex mode.

As a first check of this formula, it is possible to retrieve the formula for the critical Taylor number of the linear theory of wavy vortices.

$$i\bar{\sigma}_2(TV=0) = \frac{I_2'}{I_1} = \frac{\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} -\rho \frac{\partial}{\partial \rho}\left(\frac{J^{1/2}}{\rho}\right) V v_1^{1(1)} u^+ dx\, d\phi}{I_1} \quad (2.4.159)$$

It can easily be seen that

$$Re\,\frac{I_2'}{I_1} = \frac{\bar{\Gamma}_1}{\bar{c}} + \varepsilon^2(\quad) + \cdots\cdots \qquad (2.4.160)$$

Consequently one obtains

$$T_c(m) = T_0 + \varepsilon^2\left[\frac{\Gamma_3 + k^2\Gamma_4}{\bar{\Gamma}_1} + \frac{m^2k^2T_0\Gamma_{10}}{\bar{\Gamma}_1}\right]$$

which represents the linear formula.

It can be shown that $\mathrm{Re}\left[i\bar{\sigma}_2(TV=0) + i\bar{\sigma}_2(TV)\right] = O(\varepsilon^2)$
Consequently, the last term in (2.4.158) is of magnitude O(1).
If $i\bar{\sigma}_2(TV=0) = \mathcal{F}_0 + \varepsilon\mathcal{F}_1 + \varepsilon^2\mathcal{F}_2 + \cdots\cdots$

and $i\bar{\sigma}_2(TV) = \mathcal{F}_{00} + \varepsilon\mathcal{F}_{10} + \varepsilon^2\mathcal{F}_{20} + \cdots$

$$\mathcal{F}_0 = \frac{\bar{\Gamma}_1}{\bar{c}} = \frac{T_0^{1/2}\omega_3}{2\bar{c}}$$

$$\mathcal{F}_{00} = \frac{\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}(u_0^+\Sigma_{10} + v_0^+\Sigma_{20} + w_0^+\Sigma_{30})\,dx\,d\phi}{-\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}A(\phi)A^+(\phi)\cdot(f_0^+F_{114} + g_0^+G_{114})\,dx\,d\phi}$$

It has been found that

$$v_0^+\Sigma_{20} = \left[\frac{A(\phi)B^2(\phi)}{T_1}G_{N21}(x) + \frac{\sigma_0 A(\phi)}{T_1}G_{N23}(x)\right]g_0^+A^+(\phi)$$

and

$$u_0^+\Sigma_{10} + w_0^+\Sigma_{30} = \frac{A(\phi)A^+(\phi)}{T_1}\left[B^2(\phi)f_0^+F_{N21}(x) + \sigma_0 f_0^+F_{N23}(x)\right]$$

In conclusion

$$\mathcal{F}_0 + \mathcal{F}_{00} = \frac{\bar{\Gamma}_1}{\bar{c}} - \frac{1}{2\pi\bar{c}T_1}\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[B^2(\phi)\cdot(f_0^+F_{N21} + g_0^+G_{N21}) + \right.$$

$$\left. + \sigma_0(f_0^+F_{N23} + g_0^+G_{N23})\right]dx\,d\phi = \frac{\bar{\Gamma}_1}{\bar{c}} - \frac{1}{2\pi\bar{c}T_1}\left(\frac{\sigma_0\pi}{3Q_0}\Gamma_4' - 2\pi\sigma_0\Gamma_5\right) =$$

$$= \frac{\bar{\Gamma}_1}{\bar{c}} + \frac{\sigma_0}{6Q_0T_1\bar{c}}(6Q_0\Gamma_5 - \Gamma_4') \equiv 0$$

where use has been made of formulae (2.2.38) and (2.2.42a).

After some lengthy calculations one obtains the final result:

$$\mathrm{Re}\left[i\bar{\sigma}_2(TV=0) + i\bar{\sigma}_2(TV)\right] = O(\varepsilon^2) = \varepsilon^2\Sigma_g$$

The O(1) terms are cancelling and the O(ε) terms are purely imaginary.
where Σ_g is defined as follows:

$$\Sigma_g = \left[\alpha_0 + \alpha_{10}\frac{\sigma_0}{T_1}\right] + k^2\left[\alpha_1 + \alpha_{11}\frac{\sigma_0}{T_1}\right] + m^2k^2T_0\left[\alpha_2 + \alpha_{12}\frac{\sigma_0}{T_1}\right] +$$
$$+ \left(\frac{\alpha_{13}}{T_1} + \alpha'_{13}\right)\int_0^{2\pi} B^2(\phi)\cos\phi\,d\phi + \left(\frac{\alpha_{14}}{kT_0^{1/2}T_1} + \frac{kT_0^{1/2}\alpha'_{14}}{T_1}\right)\int_0^{2\pi} B^2(\phi)\sin\phi\,d\phi +$$
$$+ \frac{\alpha_{15}}{T_1}m^2kT_0^{1/2}\int_0^{2\pi} B^2(\phi)\sin\phi\,d\phi + \left(\frac{\alpha_{16}}{T_1} + \alpha'_{16}T_1\right)\int_0^{2\pi} B^2(\phi)\cos 2\phi\,d\phi -$$
$$- \frac{m^2\Gamma_7\hat{\Gamma}_7\Gamma'_4}{4T_1\pi c}\int_0^{2\pi} B^2(\phi)\cos 2\phi\,d\phi + \frac{\alpha_{13}}{kT_0^{1/2}T_1}\int_0^{2\pi} B^2(\phi)\sin 3\phi\,d\phi +$$
$$+ \frac{\Gamma_0(\Gamma_2+\hat{\Gamma}_2)\omega\omega'_4}{32k^2T_0}\int_0^{2\pi} B^2(\phi)\cos 3\phi\,d\phi + \frac{\alpha_{17}}{kT_0^{1/2}}\int_0^{2\pi} B^2(\phi)\sin 2\phi\,d\phi -$$
$$- \frac{\Gamma_2\hat{\Gamma}_2\omega'_4}{16k^2T_0^{1/2}T_1}\int_0^{2\pi} B^2(\phi)\cos 4\phi\,d\phi \qquad (2.4.171)$$

where

$$\alpha'_{13} = \frac{\Gamma}{4T_0}\Big\{\lambda^2 f_0^+\Big[cT_0 g_0\left(g_{229} + \frac{g_{210}}{2}\right) - D(f_0 f_{229}) - \frac{1}{2}D(f_0 f_{210}) +$$
$$+ c(f_0 g_{112} + g_0 f_{112}) + \lambda f_0\left(w_{229} + \frac{w_{210}}{2}\right) + \frac{cf_0 h_{12}}{2} - (f_{210} + 2f_{229})Df_0\Big] -$$
$$- g_0^+\Big[Dg_0\left(f_{229} + \frac{f_{210}}{2}\right) + f_0 D\left(g_{229} + \frac{g_{210}}{2}\right) + 2\left(g_{229} + \frac{g_{210}}{2}\right)Df_0 +$$
$$+ \lambda g_0\left(w_{229} - \frac{w_{210}}{2}\right)\Big] + Df_0^+\Big[\lambda Df_0\left(\frac{w_{210}}{2} - w_{229}\right) + ch_{12}Df_0 +$$
$$+ \left(f_{229} + \frac{f_{210}}{2}\right)D^2 f_0 + \lambda f_0 D\left(\frac{w_{210}}{2} - w_{229}\right) - \frac{cf_0 Dh_{12}}{2}\Big]\Big\} - \frac{\Gamma\hat{\Gamma}_2\Gamma_0\omega'_4}{16k^2T_0^{3/2}} \qquad (2.4.172)$$

$$\alpha'_{16} = \frac{\omega^2\Gamma_0\omega'_4}{64k^2T_0^{3/2}} \qquad (2.4.173)$$

$$\alpha'_0 = f_{10}^+ F_{113} - \lambda^2 f_0^+ g_0 V_{20} - \lambda^2 V_0 f_0^+ g_{10} - \frac{1}{2}\lambda^2\left(f_{11}^+ + \hat{\Gamma}_1 f_0^+\right)V_0(g_1 + \Gamma_1 g_0) \qquad (2.4.174)$$

$$\alpha_{10} = f_{10}^+ F_{N23} - \frac{c\lambda^2}{2}\left(x^2 - \frac{1}{4}\right)f_0^+\left[T_0 g_{10} + \frac{\Gamma_3 g_0}{\Gamma_1} - \frac{\Gamma f_0}{2}\right] +$$
$$+ g_0^+\Big[X\left(f_{10} + \frac{f_0 - f_1 - \Gamma_1 f_2}{2}\right) + \frac{c\Gamma g_0}{4}\left(x^2 - \frac{1}{4}\right) + \frac{c\Gamma D f_0^+}{4}\left(x^2 - \frac{1}{4}\right)\Big[c\Gamma g_0 -$$
$$- D(f_1 - 2f_0 + \Gamma_1 f_0)\Big] + 6_{N23} g_{10}^+ - \frac{c\lambda^2}{4}\left(x^2 - \frac{1}{4}\right)\left(f_{11}^+ + \hat{\Gamma}_1 f_0^+\right)(T_0 g_1 + \Gamma_1 T_0 g_0 + \Gamma f_0)$$
$$- \frac{c\Gamma}{4}\left(x^2 - \frac{1}{4}\right)Df_0 \cdot D\left(f_{11}^+ - f_0^+ + \hat{\Gamma}_1 f_0^+\right) +$$
$$+ \frac{1}{2}\left(g_{11}^+ + \hat{\Gamma}_1 g_0^+\right)\Big[X(f_1 - f_0 + \Gamma_1 f_0) - \frac{c\Gamma}{2}\left(x^2 - \frac{1}{4}\right)g_0\Big] +$$

$$+\frac{2\pi\omega_3}{\omega_4}\Big\{\Big[g_0 g_0^+ + Df_0\cdot Df_0^+ - f_0 Df_0^+ D\Big]h_{20} + \frac{1}{4}\Big(f_0 Df_0^+ Dh_{10} + c\Gamma g_0 h_{10} Df_0^+\Big) +$$

$$+\frac{1}{2}\lambda^2 f_0^+ D(f_0 f_{112}) + \frac{1}{2} f_{112} Dg_0\cdot g_0^+ - \frac{1}{2} f_{112} D^2 f_0\cdot Df_0^+ + \frac{1}{2} c\Gamma\lambda^2 f_{112} g_0 f_0^+ -$$

$$-\frac{c\lambda^2\Gamma_3 g_{112} g_0 f_0^+}{\Gamma_1} + \frac{1}{2} f_0 g_0^+ Dg_{112} + \frac{1}{2} c\Gamma g_0 g_0^+ g_{112} + \frac{1}{4} f_0 D^2 f_{112} Df_0^+ +$$

$$+\frac{1}{4} c\Gamma g_0 Df_{112} Df_0^+ - \frac{1}{4}\lambda^2 f_0^+ D(f_0 f_{112}) + \frac{1}{2} f_0 g_0^+ D\Big(g_{10} - \frac{g_{112}}{2}\Big) - \frac{f_0 Df_0^+}{8} D^2 f_{112} +$$

$$+\frac{1}{2} c\Gamma^2 f_0^+ (f_0 - f_1 - \Gamma_1 f_0)\Big(g_{10} - \frac{g_{112}}{2}\Big) - \frac{c\lambda^2 f_0^+ g_0\Gamma_3}{\Gamma_1}\Big(g_{10} - \frac{g_{112}}{2}\Big) +$$

$$+\frac{c\Gamma g_0 g_0^+}{2}\Big(g_{10} - \frac{g_{112}}{2}\Big) - \frac{1}{4} f_{112} g_0^+ Dg_0 + f_{10}\Big[g_0^+ D\Big(g_{10} - \frac{g_{112}}{2}\Big) -$$

$$-\frac{1}{4} Df_0^+\cdot D^2 f_{112} - \frac{1}{4}\lambda^2 f_0^+ Df_{112}\Big] + \Big[-cT_0\lambda^2 f_0^+\Big(g_{10} - \frac{g_{112}}{2}\Big) - \frac{g_0^+ Df_{112}}{4} -$$

$$-\frac{g_0^+ f_{112}}{2} D\Big] g_{10} + \frac{1}{8} D^2 f_{112}\cdot Df_0^+ (f_1 + \Gamma_1 f_0) + \frac{1}{4}\lambda^2 f_0^+ D\Big[f_{112}(f_1 + \Gamma_1 f_0)\Big] -$$

$$-\Big[\lambda^2 f_0^+ f_{112} + g_0^+ g_{112} + \frac{1}{4} Df_0^+ Df_{112} - \frac{1}{2} f_{112} Df_0^+ D\Big]\Big[\frac{1}{2} D(f_0 - f_1 - \Gamma_1 f_0) + Df_{10} +$$

$$+\frac{1}{2} c\Gamma(g_0 - g_1 - \Gamma_1 g_0)\Big] - \frac{1}{2} c\Gamma g_0^+\Big(g_{10} - \frac{g_{112}}{2}\Big)(f_1 + \Gamma_1 f_0) - \frac{1}{4}\big(f_0 g_0^+ Dg_{222} + c\Gamma g_0 g_0^+ g_{212}\big)$$

$$+\frac{1}{4} f_{112} Df_0^+\Big[c\Gamma Dg_0 - D^2(f_1 - f_0 + \Gamma_1 f_0)\Big] + \frac{c\Gamma Df_0^+}{2}\Big(g_{10} + \frac{g_{112}}{2}\Big)\Big[c\Gamma g_0 - D(f_1 - 2f_0 + \Gamma_1 f_0)\Big] +$$

$$+\frac{1}{4} f_{212}\Big[D^2 f_0\cdot Df_0^+ - g_0^+ Dg_0 - \lambda^2 f_0^+ Df_0 - c\Gamma\lambda^2 f_0^+ g_0 - \lambda^2 f_0 f_0^+\cdot\frac{Df_{212}}{f_{212}}\Big] +$$

$$+\frac{1}{4}\lambda w_{212} Df_0^+ (f_0 + c\Gamma g_0) + \frac{1}{2} g_{221}\Big[2g_0^+ Df_0 - c\lambda^2 T_0 f_0^+ g_0 + \frac{f_0 g_0^+ Dg_{221}}{g_{221}}\Big]$$

$$+\frac{1}{2}\Big[\big(3\lambda^2 f_0^+ Df_0 + g_0^+ Dg_0 - D^2 f_0\cdot Df_0^+\big) + \lambda^2 f_0 f_0^+ D\Big] f_{220} + G_{N21} g_{10}^+ -$$

$$-\frac{\lambda}{2}\Big[\big(\lambda^2 f_0 f_0^+ + f_0 g_0^+ + Df_0\cdot Df_0^+\big) + f_0 Df_0^+ D\Big] w_{220} -$$

$$-\lambda^2 f_{10}^+\Big[cT_0 g_0\Big(g_{10} + \frac{g_{112}}{2}\Big) - \frac{1}{2}\big(f_{112} Df_0 + 3f_0 Df_{112}\big)\Big] -$$

$$-\frac{1}{4}\Big[2f_{112} D^2 f_0 - D(f_0 Df_{112})\Big]\cdot\Big(Df_{10}^+ + \frac{1}{2} Df_0^+ - \frac{1}{2} Df_1^+ - \frac{\Gamma_1}{2} Df_0^+\Big) -$$

$$-\frac{\lambda^2}{2T_1}\big(f_1^+ + \hat{\Gamma}_1 f_0^+\big)\Big\{cT_0\big(g_1 + \Gamma_1 g_0\big)\Big(g_{10} - \frac{g_{112}}{2}\Big) + \frac{1}{2} D\Big[f_{112}(f_1 + \Gamma_1 f_0)\Big] +$$

$$+\frac{1}{2} D(f_0 f_{112}) + f_{112} D(f_1 - f_0 + \Gamma_1 f_0) +$$

$$+\frac{1}{4}(f_1 + \Gamma_1 f_0) Df_{112} + c\Gamma f_0\Big(g_{10} - \frac{g_{112}}{2}\Big) + \frac{1}{2} cT_0 g_0 f_{212} -$$

$$-\frac{1}{2}\big(f_0 Df_{212} + 3f_{212} Df_0\big) + \frac{f_0}{2}\big(\lambda w_{212} - h_{10}\big) - \frac{c}{2} T_0\Gamma g_0 g_{210} +$$

$$+\frac{\Gamma}{2}(f_0 Df_{210}+3f_{210}Df_0)-\frac{\Gamma f_0}{2}(\lambda w_{210}-ch_{12})\}+\frac{1}{2\Gamma_1}(g_1^+ +\hat{\Gamma}_1 g_0^+)\{\frac{1}{2}f_0 Dg_{212}+$$

$$+\frac{1}{2}Dg_0(f_{212}-f_{112})-f_0 D(g_{10}+\frac{g_{112}}{2})-\frac{f_{112}}{2}D(g_1+\Gamma_1 g_0)+$$

$$+(f_1+\Gamma_1 f_0)D(g_{10}-\frac{g_{112}}{2})+\frac{g_0}{2}(h_{10}-\lambda w_{212})+g_{212}Df_0-$$

$$-\frac{1}{4}Df_{112}(g_1+\Gamma_1 g_0)-g_{112}D(f_1-f_0+\Gamma_1 f_0)-\Gamma[cg_0(g_{10}-\frac{g_{112}}{2})+$$

$$+\frac{1}{2}(f_{210}Dg_0+f_0 Dg_{210})+\frac{g_0}{2}(ch_{12}-\lambda w_{210})+g_{210}Df_0]\}-$$

$$-\frac{1}{4\Gamma_1}D(f_1^+-f_0^+\hat{\Gamma}_1 f_0^+)\{\lambda D(f_0 w_{212})+(f_0 Dh_{10}-h_{10}Df_0)+f_{212}D^2 f_0+$$

$$+\frac{1}{2}Df_{112}\cdot D(f_1-f_0+\Gamma_1 f_0)-f_{112}D^2(f_1+\Gamma_1 f_0)+\frac{1}{2}f_0 D^2 f_{112}+$$

$$+\Gamma[c(f_{112}Dg_0+\frac{f_0 Df_{112}}{2})+2cDf_0(g_{10}+\frac{g_{112}}{2})-f_{210}D^2 f_0-$$

$$-c(f_0 Dh_{12}-h_{12}Df_0)-\lambda D(f_0 w_{210})]\}\} \quad (2.4.175)$$

$$\alpha_1 = F_{113}\cdot f_{11}^+ +2c\lambda^2 f_0^+ g_0(X+\frac{1}{2})-\lambda^2 f_0^+ g_{11}-\lambda^2 c^2 f_0^+ g_0 V_{22} \quad (2.4.176)$$

$$\alpha_{11} = F_{N23}\cdot f_{11}^+ -\lambda^2 f_0^+\cdot\frac{c}{2}(X^2-\frac{1}{4})(T_0 g_{11}+g_0\frac{\bar{\Gamma}_4}{\Gamma_1})+Xg_0^+ f_{11}+$$

$$+cf_0 g_0^+(X^2-\frac{1}{4})+G_{N23}\cdot g_{11}^+ +2\pi\frac{\omega_3}{\omega_4}\{2cf_{112}g_0 g_0^+-\frac{c\lambda^2}{\Gamma_1}\bar{\Gamma}_4 g_{112}g_0 f_0^+ +$$

$$+[g_0 g_0^+ +Df_0 Df_0^+ -f_0 Df_0^+ D]h_{22}+2cf_0 g_0^+(g_{10}-\frac{g_{112}}{2})-$$

$$-\frac{c\lambda^2 f_0^+ g_0\bar{\Gamma}_4}{\Gamma_1}(g_{10}-\frac{g_{112}}{2})-cg_0 g_0^+ f_{112}-\frac{1}{2}\lambda^2 g_0^+ D(f_{11}\cdot f_{112})+$$

$$+f_{11}[g_0^+ D(g_{10}-\frac{g_{112}}{2})-\frac{1}{4}Df_0^+ D^2 f_{112}-\frac{\lambda^2}{4}f_0^+ Df_{112}]+$$

$$+[-cT_0\lambda^2 f_0^+(g_{10}-\frac{g_{112}}{2})-\frac{g_0^+ Df_{112}}{4}-\frac{g_0^+ f_{112}}{2}D]g_{11}-$$

$$-[\lambda^2 f_0^+ f_{112}+g_0^+ g_{112}+\frac{1}{4}Df_0^+ Df_{112}-\frac{f_{112}Df_0^+}{2}D](2cf_0+Df_{11})+$$

$$+\frac{1}{2}[(3\lambda^2 f_0^+ Df_0+g_0^+ Dg_0-D^2 g_0\cdot Df_0^+)+\lambda^2 f_0 f_0^+ D]f_{221}+$$

$$+\frac{1}{2}[(2g_0^+ Df_0-c\lambda^2 T_0 f_0^+ g_0)+f_0 g_0^+ D]g_{221}-\frac{\lambda}{2}[(\lambda^2 f_0 f_0^+ +g_0 g_0^+ +Df_0 Df_0^+)+$$

$$+f_0 Df_0^+ D]w_{221}-\lambda^2 f_{11}^+[cT_0 g_0(g_{10}+\frac{g_{112}}{2})-\frac{1}{2}(f_{112}Df_0+3f_0 Df_{112})]+$$

$$+G_{N21}\cdot g_{11}^+ -\frac{Df_{11}^+}{4}[2\lambda f_{112}D^2 f_0-D(f_0 Df_{112})] \quad (2.4.177)$$

Later we shall find the connection between the formula (2.4.128) and the formula

$$T = T_c(0) + \frac{\alpha(0) T_{c1}}{\left[\alpha_1 - \frac{b_{44}\pi_2}{a_1}\alpha(0)\right]} \qquad (2.4.178)$$

Davey, Di Prima and Stuart (1968) for an interaction between the Taylor vortex mode and an out-of-phase non-axisymmetric mode. In order to find this connection, we have to find the asymptotic expansion of (2.4.141) for the limit case $T_1 \to \infty$

For the limit case $T_1 \to \infty$, it can be seen that the terms

$$\varepsilon^2\left(\alpha_0 + \alpha_{10}\frac{T_0}{T_1}\right) \text{ and } \alpha'_{13}\,\varepsilon^2 \int_0^{2\pi} B^2(\phi)\cos\phi\, d\phi$$

can be considered as eccentricity terms and the $\varepsilon^2 k^2\left(\alpha_1 + \alpha_{11}\frac{T_0}{T_1}\right)$ terms can be considered as a surface curvature term.

On the other hand, if we consider the limit case $T_1 \to 0$ we retrieve the result of the linear eccentric problem.

We would like to add that $T'(m)$ is the critical Taylor number for the m-th wavy vortex mode, allowing for the effects of eccentricity and the presence of the Taylor vortex mode.

2.5. The asymptotic expansion of $Re\left[i\bar{\sigma}_2(TV=0)+i\bar{\sigma}_2(TV)\right]$ for the limit case $T_1\to 0$ and the limit case $T_1\to\infty$

From Di Prima and Stuart's paper (1975) we know that

$$\lim_{\substack{T_1\to 0\\ k \text{ fixed}}} B^2(\phi)=\frac{T_1\omega_3}{\omega_4}\cdot\frac{\exp\left[(\omega/k)\sin\phi\right]}{I_0(\frac{\omega}{k})}\left[1+\frac{T_1\omega_3\omega_4'}{k\omega_4}\left(\pi+\phi-\frac{I(\frac{\omega}{k})}{I_0(\frac{\omega}{k})}-\right.\right.$$

$$\left.\left.-\frac{1}{I_0(\frac{\omega}{k})}\int_0^{\phi}\exp\left[\frac{\omega}{k}\sin\phi\right]d\phi+O(T_1^2)\right]\right. \qquad (2.5.1)$$

where $$I_0(x)=\frac{1}{2\pi}\int_0^{2\pi}\exp(x\sin\phi)\,d\phi \qquad (2.5.2)$$

is the modified Bessel function of order zero.

If we take the dominant term of $\int_0^{2\pi}B^2(\phi)\cos 2\phi\,d\phi$, we obtain

$$\lim_{T_1\to 0}\frac{1}{T_1}\int_0^{2\pi}B^2(\phi)\cos 2\phi\,d\phi=\frac{2\pi\omega_3}{\omega_4}\left[1-2\frac{I_0''(\frac{\omega}{k})}{I_0(\frac{\omega}{k})}\right] \qquad (2.5.3)$$

where proper account has been taken of the relations:

$$I_0'(\frac{\omega}{k})=\frac{1}{2\pi}\int_0^{2\pi}\sin\phi\cdot\exp\left(\frac{\omega}{k}\sin\phi\right)d\phi \qquad (2.5.4)$$

and

$$I_0''(\frac{\omega}{k})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\cos 2\phi}{2}\exp\left(\frac{\omega}{k}\sin\phi\right)d\phi \qquad (2.5.5)$$

We have to use also the formula

$$I(x)=\frac{1}{2\pi}\int_0^{2\pi}\phi\exp(x\sin\phi)\,d\phi=\pi I_0(x)-\frac{\pi}{2}L_0(x) \qquad (2.5.6)$$

where $L_0(x)$ is the modified Struve function $L_0(x)$

One obtains:

$$\lim_{T_1\to 0}\frac{1}{T_1}\int_0^{2\pi}B^2(\phi)\cos 2\phi\,d\phi=\frac{2\pi\omega_3}{\omega_4}\left[1-2\frac{I_0''(\frac{\omega}{k})}{I_0(\frac{\omega}{k})}\right]+$$

$$+\frac{2\pi\omega_3^2\omega_4'}{k\omega_4^2}T_1\left(\pi-\frac{I_1}{I_0}\right)\left[1-2\frac{I_0''(\frac{\omega}{k})}{I_0(\frac{\omega}{k})}\right]+\frac{2\pi T_1\omega_3^2\omega_4'}{k\omega_4^2 I_0(\frac{\omega}{k})}\left[I(\frac{\omega}{k})-2I''(\frac{\omega}{k})\right]-$$

$$-\frac{T_1\omega_3^2\omega_4'}{k\omega_4^2 I_0^2(\frac{\omega}{k})}\int_0^{2\pi}e^{\frac{\omega}{k}\sin\phi}\cos 2\phi\left(\int_0^{\phi}e^{\frac{\omega}{k}\sin\phi}\,d\phi\right)d\phi \qquad (2.5.7)$$

By using the definition of the modified Struve function:

$$2\pi S\left(\frac{\omega}{k}\right)=\int_0^{2\pi} e^{\frac{\omega}{k}\sin\phi}\sin\phi\left(\int_0^{\phi} e^{\frac{\omega}{k}\sin\phi}d\phi\right)d\phi \qquad (2.5.8)$$

By using the derivative of the modified Struve function with respect to $\frac{\omega}{k}$, one obtains:

$$\lim_{T_1\to 0}\frac{1}{T_1}\int_0^{2\pi} B^2(\phi)\cos 2\phi\, d\phi=\frac{2\pi\omega_3}{\omega_4}\left[1-2\frac{I_0''\left(\frac{\omega}{k}\right)}{I_0\left(\frac{\omega}{k}\right)}\right]\left[1+\frac{T_1\omega_3\omega_4'}{k\omega_4}\right]+$$

$$+\frac{2\pi T_1\omega_3^2\omega_4'}{k\omega_4^2 I_0\left(\frac{\omega}{k}\right)}\left[I\left(\frac{\omega}{k}\right)-2I''\left(\frac{\omega}{k}\right)\right]-$$

$$-\frac{2\pi T_1\omega_3^2\omega_4'}{k\omega_4^2 I_0^2\left(\frac{\omega}{k}\right)}\left\{\pi\left[I_0^2\left(\frac{\omega}{k}\right)+2I_1^2\left(\frac{\omega}{k}\right)\right]-2S'\left(\frac{\omega}{k}\right)\right\}+O(T_1^2) \qquad (2.5.9)$$

For the calculation of $\lim_{T_1\to 0}\frac{1}{k T_0^{1/2} T_1}\int_0^{2\pi} B^2(\phi)\sin 3\phi\, d\phi$

we take the derivative of the formula

$$\int_0^{2\pi} e^{\frac{\omega}{k}\sin\phi}\cos 2\phi\left(\int_0^{\phi} e^{\frac{\omega}{k}\sin\phi}d\phi\right)d\phi=2\pi^2\left[I_0^2\left(\frac{\omega}{k}\right)+2I_1^2\left(\frac{\omega}{k}\right)\right]-$$

$$-4\pi S'\left(\frac{\omega}{k}\right) \qquad (2.5.10)$$

with respect to $X=\frac{\omega}{k}$

and we take account of the relation

$$\int_0^{\phi} e^{\frac{\omega}{k}\sin\phi}\sin\phi\, d\phi=1-e^{\frac{\omega}{k}\sin\phi}\cos\phi+\frac{\omega}{k}\int_0^{\phi} e^{\frac{\omega}{k}\sin\phi}\cos^2\phi\, d\phi$$

(2.5.11)

One obtains:

$$\lim_{T_1\to 0}\frac{1}{k T_0^{1/2} T_1}\int_0^{2\pi} B^2(\phi)\sin 3\phi\, d\phi=$$

$$=\frac{2\pi\omega_3}{k T_0^{1/2}\omega_4 I_0\left(\frac{\omega}{k}\right)}\left\{(3I_1-4I_0''')+\frac{T_1\omega_3\omega_4'}{k\omega_4}\left\{(3I_1-4I_0''')\left(\pi-\frac{I}{I_0}\right)+\right.\right.$$

$$+3I'-4I'''+\pi\left[I_0 I'-I I_1+17\pi I_0 I_1+16\pi I_1 I_1'-4S\left(\frac{\omega}{k}\right)-\right.$$

$$\left.\left.\left.-8S''\left(\frac{\omega}{k}\right)-8(I_0-I_0'')-\frac{8\pi\omega}{k}(I_0-I_0'')^2\right]\right\}\right\} \qquad (2.5.12)$$

In a similar way it is possible to get

$$\lim_{T_1\to 0} \frac{\omega\omega_4' \Gamma_0(\Gamma_2+\hat{\Gamma}_2)}{32k^2T_0}\int_0^{2\pi} B^2(\phi)\cos 3\phi\, d\phi =$$

$$= \frac{\Gamma_0(\Gamma_2+\hat{\Gamma}_2)T_1^2\pi\omega_3^2\omega_4'^2}{16k^2T_0\omega_4^2}\left\{\left(1-\frac{8k^2}{\omega^2}\right)\left[\frac{I_0\left(\frac{2\omega}{k}\right)}{I_0\left(\frac{\omega}{k}\right)}-I_0\left(\frac{\omega}{k}\right)\right]-\frac{8k^2}{\omega^2}+\right.$$

$$\left.+4\left[I_0''\left(\frac{\omega}{k}\right)-\frac{I_0''\left(\frac{2\omega}{k}\right)}{I_0\left(\frac{\omega}{k}\right)}\right]+\frac{8k}{\omega}\left[\frac{I_0'\left(\frac{2\omega}{k}\right)}{I_0\left(\frac{\omega}{k}\right)}-I_0'\left(\frac{\omega}{k}\right)\right]\right\} \quad (2.5.13)$$

$$\lim_{T_1\to 0}\frac{1}{k^2T_0T_1}\int_0^{2\pi} B^2(\phi)\cos 4\phi\, d\phi = \frac{2\pi\omega_3}{k^2T_0\omega_4 I_0\left(\frac{\omega}{k}\right)}\left[I_0-8I_0''+8I_0^{IV}\right]+$$

$$+\frac{2\pi T_1\omega_3^2\omega_4'}{k^3\omega_4^2T_0 I_0\left(\frac{\omega}{k}\right)}\left\{\left[\left(\pi-\frac{I}{I_0}\right)\left(I_0-8I_0''+8I_0^{IV}\right)+\left(I-8I''+8I^{IV}\right)\right]-\right.$$

$$-\frac{1}{I_0}\left[\pi\left(I_0^2+8I_1^2\right)-2S'\left(\frac{\omega}{k}\right)-\frac{8I_0'''\left(\frac{\omega}{k}\right)}{\omega^2+k^2}\left(\pi\omega^2I_0'''+k^2\right)\right]-$$

$$-\frac{1}{I_0}\left(\frac{\omega k}{\omega^2+k^2}-1\right)\left[I_0I'-II_1+\pi I_1\left(17I_0+16I_1'\right)-4S\left(\frac{\omega}{k}\right)-8S''\left(\frac{\omega}{k}\right)-\right.$$

$$\left.\left.-8\left(I_0-I_0''\right)-\frac{8\pi\omega}{k}\left(I_0-I_0''\right)^2\right]\right\} \quad (2.5.14)$$

It is possible to formulate all the asymptotic expressions in a simpler algebraic form by converting the derivatives $I_0', I_0'', I_0''', I_0^{IV}$, $S'\left(\frac{\omega}{k}\right), S''\left(\frac{\omega}{k}\right), S'''\left(\frac{\omega}{k}\right), S^{IV}\left(\frac{\omega}{k}\right), I'\left(\frac{\omega}{k}\right), I''\left(\frac{\omega}{k}\right), I'''\left(\frac{\omega}{k}\right)$ and $I^{IV}\left(\frac{\omega}{k}\right)$ in terms of the modified Bessel functions $I_0\left(\frac{\omega}{k}\right), I_1\left(\frac{\omega}{k}\right)$ and the modified Struve function $S\left(\frac{\omega}{k}\right)$ as follows:

$$I_0' = I_1 \quad (2.5.15)$$

$$I_0'' = I_0 - \frac{k}{\omega}I_1 \quad (2.5.16)$$

$$I_0''' = I_1\left(1+\frac{2k^2}{\omega^2}\right)-\frac{k}{\omega}I_0 \quad (2.5.17)$$

$$I_0^{IV} = \left(I_0-\frac{2k}{\omega}I_1\right)\left(1+\frac{3k^2}{\omega^2}\right) \quad (2.5.18)$$

$$I'\left(\frac{\omega}{k}\right) = \pi I_1 - \frac{\pi^2}{2}\left(I_0^2+I_1^2\right)+\frac{k}{\omega}\cdot\frac{\pi}{2}\left(S+I_0\right) \quad (2.5.19)$$

$$I''\left(\frac{\omega}{k}\right) = -\frac{\pi}{2}\left[\frac{k^2}{\omega^2}\left(S+I_0\right)-\frac{k\pi}{\omega}\left(I_0^2+3I_1^2\right)+I_1\left(4\pi I_0+\frac{k}{\omega}\right)-2I_0\right] \quad (2.5.20)$$

$$I'''\left(\frac{\omega}{k}\right)=\pi\left[I_1-2\pi\left(I_0^2+I_1^2\right)\right]-\frac{\pi k^2}{2\omega^2}\left[-2I_1-\frac{k}{\omega}(S+I_0)+\pi\left(I_0^2+7I_1^2\right)\right]+$$

$$+\frac{k\pi I_0}{2\omega}(12\pi I_1-1) \qquad (2.5.21)$$

$$I^{IV}\left(\frac{\omega}{k}\right)=\pi\left(I_0-\frac{2k}{\omega}I_1\right)\left(1+\frac{3k^2}{\omega^2}\right)-\frac{\pi}{2}\left\{16\pi I_0 I_1-\frac{kI_1}{\omega}(1+20\pi I_1)-\right.$$

$$-\frac{12\pi}{\omega}kI_0^2+\frac{12\pi}{\omega^2}k^2I_0I_1+\frac{k^2}{\omega^2}\left\{2I_0(1+8\pi I_1)-\frac{k}{\omega}\left[3I_1+\pi\left(I_0^2+15I_1^2\right)\right]+\right.$$

$$\left.\left.+\frac{k^2}{\omega^2}(S+I_0)\right\}\right. \qquad (2.5.22)$$

$$S'\left(\frac{\omega}{k}\right)=-\frac{k}{\omega}S\left(\frac{\omega}{k}\right)+\pi\left[I_0^2\left(\frac{\omega}{k}\right)+I_1^2\left(\frac{\omega}{k}\right)\right]-\frac{k}{\omega}I_0\left(\frac{\omega}{k}\right) \qquad (2.5.23)$$

$$S''\left(\frac{\omega}{k}\right)=\frac{k^2}{\omega^2}(S+I_0)-\frac{k\pi}{\omega}\left(I_0^2+3I_1^2\right)+4\pi I_0I_1-\frac{k}{\omega}I_1 \qquad (2.5.24)$$

$$S'''\left(\frac{\omega}{k}\right)=\frac{k^2}{\omega^2}\left[2I_1-\frac{k}{\omega}(S+I_0)+\pi\left(I_0^2+7I_1^2\right)\right]-\frac{kI_0}{\omega}(1+12\pi I_1)+$$

$$+4\pi\left(I_0^2+I_1^2\right) \qquad (2.5.25)$$

$$S^{IV}\left(\frac{\omega}{k}\right)=16\pi I_0I_1-\frac{kI_1}{\omega}(1+20\pi I_1)-\frac{12\pi}{\omega}kI_0^2+\frac{12k^2}{\omega^2}\pi I_0I_1+$$

$$+\frac{k^2}{\omega^2}\left\{2I_0(1+8\pi I_1)-\frac{k}{\omega}\left[3I_1+\pi\left(I_0^2+15I_1^2\right)\right]+\frac{k^2}{\omega^2}(S+I_0)\right\} \qquad (2.5.26)$$

It is obvious that the dominant term, $O(T_1)$, of (2.5.1) is of the same form as the linear solution

$$B^2(\phi)=B_0^2(k)\exp\left\{\left(\frac{\omega}{k}\right)(\sin\phi-1)\right\} \qquad (2.5.27)$$

but has the arbitrary constant $B_0^2(k)$ replaced by

$$B_0^2(k,T_1)=\frac{T_1\omega_3 e^{\frac{\omega}{k}}}{\omega_4 I_0\left(\frac{\omega}{k}\right)} \qquad (2.5.28)$$

The nonlinear solution for $B^2(\phi)$ is different from the linear solution, because it has a definite value for given values of k and $T_1\geq 0$ The formula (2.5.28) confirms the expectation of Di Prima and Stuart (1972b) about the importance of nonlinearity in the determination of $B_0(k)$. It indicates that the limit $T_1\to 0$ shows a tendency to approach the linearized solution near to the critical Taylor number, but with a well defined amplitude.

If we return for a short while to formula (2.4.127) we can remember the approximation

$$T - T_c = \gamma \cong \varepsilon T_1 \qquad (2.5.29)$$

$T \to T_c$ will correspond to the limit $T_1 \to 0$, and this shows that the critical Taylor number of Di Prima and Stuart (1972b)

$$T = T_c = T_0 + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1}$$

can be obtained as a limit case of the formula (2.4.127) for $T_1 \to 0$ and this constitutes an approach to the critical Taylor number for the steady, eccentric and linearized theory.

It is also very important to consider the second limit $T_1 \to \infty$ As we shall later see, this will place Davey, Di Prima and Stuart's result for the critical Taylor number for an interaction between a Taylor vortex mode and a wavy vortex mode in the concentric case in the context of the present calculations.

The limit $T_1 \to \infty, k$ fixed.

$$\beta^2(\phi) = \frac{k}{\omega_4'} \cdot \frac{\left(e^{\frac{2\pi T_1 \omega_3 \omega_4'}{k\omega_4}} - 1\right) \cdot e^{\frac{\omega}{k}\sin\phi + \frac{T_1 \omega_3 \omega_4'}{\omega_4}\phi}}{\int_0^{2\pi} e^{\frac{\omega}{k}\sin\phi + \frac{T_1\omega_3\omega_4'}{k\omega_4}\phi} d\phi + \left(e^{\frac{2\pi T_1 \omega_3 \omega_4'}{k\omega_4}} - 1\right) \int_0^{\phi} e^{\frac{\omega}{k}\sin\phi + \frac{T_1\omega_3\omega_4'}{k\omega_4}} d\phi} \qquad (2.5.30)$$

where

$$\int_0^{\phi} e^{\frac{\omega}{k}\sin\phi + \frac{T_1\omega_3\omega_4'}{k\omega_4}\phi} d\phi = \int_0^{\phi} E\, d\phi \qquad (2.5.31)$$

let us denote

$$\frac{\omega}{k}\sin\phi + \frac{T_1\omega_3\omega_4'}{k\omega_4}\phi = \frac{T_1\omega_3\omega_4'}{k\omega_4}\psi \qquad (2.5.32)$$

$$\therefore \quad \psi = \phi + \frac{\omega\omega_4}{T_1\omega_3\omega_4'}\sin\phi \qquad (2.5.32a)$$

We shall look for an expansion of the following form:

$$\phi = \psi + \frac{1}{T_1}\phi_1 + \frac{1}{T_1^2}\phi_2 + \frac{1}{T_1^3}\phi_3 + \frac{1}{T_1^4}\phi_4 + \cdots \qquad (2.5.33)$$

By substituting (2.5.33) into (2.5.32a) one obtains:

$$\frac{\omega\omega_4}{T_1\omega_3\omega_4'}\left[\sin\Psi\cos\left(\frac{\phi_1}{T_1}+\frac{\phi_2}{T_1^2}+\frac{\phi_3}{T_1^3}+\frac{\phi_4}{T_1^4}\right)+\cos\Psi\sin\left(\frac{\phi_1}{T_1}+\frac{\phi_2}{T_1^2}+\frac{\phi_3}{T_1^3}+\frac{\phi_4}{T_1^4}\right)\right]+$$

$$+\frac{\phi_1}{T_1}+\frac{\phi_2}{T_1^2}+\frac{\phi_3}{T_1^3}+\frac{\phi_4}{T_1^4}=0 \qquad (2.5.34)$$

The $O(T_1^{-1})$, $O(T_1^{-2})$, $O(T_1^{-3})$ and $O(T_1^{-4})$ equations will define the values of ϕ_1 , ϕ_2 , ϕ_3 and ϕ_4 respectively.

Result for Φ :

$$\Phi=\Psi-\frac{\omega\omega_4}{T_1\omega_3\omega_4'}\sin\Psi+\frac{1}{2}\left(\frac{\omega\omega_4}{T_1\omega_3\omega_4'}\right)^2\sin 2\Psi+\frac{1}{8}\left(\frac{\omega\omega_4}{T_1\omega_3\omega_4'}\right)^3(\sin\Psi-3\sin 3\Psi)+$$

$$+\frac{1}{24}\left(\frac{\omega\omega_4}{T_1\omega_3\omega_4'}\right)^4(7\sin 4\Psi-2\sin 2\Psi) \qquad (2.5.35)$$

This formula is needed in order to derive later formula (2.5.37).

The nonlinear differential equation for $B(\Phi)$

$$k T_0^{1/2}\frac{dB(\Phi)}{d\Phi}=\Gamma B(\Phi)\cos\Phi+(T_1\bar{\Gamma}_1+\sigma_0\Gamma_5)B(\Phi)-\Gamma_4' B^3(\Phi) \qquad (2.5.36)$$

can be written in a different form

$$k\frac{dB^2(\Phi)}{d\Phi}=\omega B^2(\Phi)\cos\Phi+\frac{T_1\omega_3\omega_4'}{\omega_4}B^2(\Phi)-\omega_4' B^4(\Phi) \qquad (2.5.36a)$$

Now, we shall use as a start the formula (5.12) of Di Prima and Stuart (1975)

$$B^2(\Phi)\cong\frac{T_1\omega_3}{\omega_4}+\frac{\omega}{\omega_4}\cos\Phi+\frac{k\omega\omega_4}{T_1\omega_3\omega_4'^2}\sin\Phi-\frac{1}{T_1^2}\left[\frac{k^2\omega\omega_4'^2}{\omega_3^2\omega_4'^3}\cos\Phi+\right.$$

$$\left.+\frac{k\omega^2\omega_4'^2}{2\omega_3^2\omega_4'^3}\sin 2\Phi\right]+Q T_1^{-3} \qquad (2.5.37)$$

It is in our interest to calculate the $O(T_1^{-3})$, $O(T_1^{-4})$ and $O(T_1^{-5})$ terms.

By substituting (2.5.37) into (2.5.36a), and by considering the $O(T_1^{-2})$ equation one obtains

$$Q(\Phi)=\frac{\omega k\omega_4^3}{\omega_3^3\omega_4'^4}\left[\left(\frac{\omega^2}{4}-k^2\right)\sin\Phi+2\omega k\cos 2\Phi+\frac{\omega^2}{4}\sin 3\Phi\right] \qquad (2.5.38)$$

Similarly, one obtains the RT_1^{-4} and ST_1^{-5} terms:

$$R(\phi)=\frac{\omega k\omega_4^4}{\omega_3^4\omega_4'^5}\left[k\left(k^2-\frac{3\omega^2}{4}\right)\cos\phi+\frac{\omega}{2}\left(11k^2-\frac{\omega^2}{2}\right)\sin 2\phi-\right.$$

$$\left.-\frac{9\omega^2 k}{4}\cos 3\phi-\frac{\omega^3}{8}\sin 4\phi\right] \qquad (2.5.39)$$

$$S(\phi)=\frac{\omega k\omega_4^5}{\omega_3^5\omega_4'^6}\left\{\left[\frac{\omega^4}{8}-2\omega^2k^2+k^4\right]\sin\phi+\omega k\left(2\omega^2-13k^2\right)\cos 2\phi+\right.$$

$$\left.+3\omega^2\left(\frac{\omega^2}{16}-4k^2\right)\sin 3\phi+2\omega^3 k\cos 4\phi+\frac{\omega^4}{16}\sin 5\phi\right\} \qquad (2.5.40)$$

The extended formula of $B^2(\phi)$ with known QT_1^{-3}, RT_1^{-4} and ST_1^{-5} terms

$$B^2(\phi)\cong\frac{T_1\omega_3}{\omega_4}+\frac{\omega}{\omega_4}\cos\phi+\frac{k\omega\omega_4}{T_1\omega_3\omega_4'^2}\sin\phi-\frac{1}{T_1^2}\left[\frac{k^2\omega\omega_4^2}{\omega_3^2\omega_4'^3}\cos\phi+\right.$$

$$\left.+\frac{k\omega^2\omega_4^2}{2\omega_3^2\omega_4'^3}\sin 2\phi\right]+\frac{Q}{T_1^3}+\frac{R}{T_1^4}+\frac{S}{T_1^5} \qquad (2.5.41)$$

A characteristic fact of this formula is that $B(\phi)$ becomes uniform in ϕ when $T_1\to\infty$. In this limit, only the leading term $\frac{T_1\omega_3}{\omega_4}$ of (2.5.41) is retained, the velocity field (u,v,w) is independent of ϕ. This leading term can also be obtained from the amplitude equation (2.4.36). The balance is between the third and the fourth terms, $(T_1\Gamma_1+\sigma_0\Gamma_5)B(\phi)=\Gamma_4'B^3(\phi)$ which are of order T_1, while the first and second terms are negligible. It can also be mentioned that the leading term is independent of k, and consequently, the velocity fie ld will also be independent of k. It can easily be seen that in this case $q_0(\phi)=-6Q_0\frac{T_1\omega_3}{\omega_4}+\sigma_0\to 0$ in the limit $T_1\to\infty$, and by using the relation (2.2.42b).

If we consider the role of the parameter T_1, we assume that parameter to lie in the range $0\le T_1<\infty$. $T_1\ge 0$ represents a non-linear modification of the results of the linear theory for the eccentric case (Di Prima and Stuart 1972b). The limit $T_1\to\infty$ will represent an approach to the result of Davey, Di Prima and Stuart

(1968), for an interaction between a Taylor vortex mode and a wavy mode, in the concentric case.

A remark of Di Prima and Stuart (1975) about the range of validity of the formulae (2.5.1) and (2.5.37) could be mentioned, namely, the expectation that (2.5.1) is valid for $T_1 < 100$ approximately, and that (2.5.37) is valid for $T_1 > 10^4$.

Let us return now to the expressions occurring in the present calculations:

$$\lim_{T_1\to\infty}\left(\frac{\alpha_{16}}{T_1}+\alpha'_{16}T_1\right)\int_0^{2\pi}B^2(\phi)\cos 2\phi\,d\phi=\left(\frac{\alpha_{16}}{T_1}+\alpha'_{16}T_1\right)\cdot\frac{2\pi\omega^2k^2\omega_4}{T_1^3\omega_3^3\omega_4'^4}\left[1+\frac{\omega_4^4\left(\omega^2-\frac{13k^2}{2}\right)}{T_1^2\omega_3^2\omega_4'^2}\right]+ \tag{2.5.42}$$

$$+O(T_1^{-3})$$

$$\lim_{T_1\to\infty}\frac{1}{kT_0^{1/2}T_1}\int_0^{2\pi}B^2(\phi)\sin 3\phi\,d\phi=\frac{\pi\omega^3\omega_4^3}{4T_0^{1/2}T_1^4\omega_3^3\omega_4'^4}\left[1+\frac{3\omega_4^2\left(\omega^2-64k^2\right)}{4T_1^2\omega_3^2\omega_4'^2}\right]+O(T_1^{-8}) \tag{2.5.43}$$

$$\lim_{T_1\to\infty}\frac{\Gamma\Gamma_0\Gamma_4'\left(\Gamma_2+\hat{\Gamma}_2\right)}{8k^2T_0^2}\int_0^{2\pi}B^2(\phi)\cos 3\phi\,d\phi=-\frac{9\pi\Gamma_0\left(\Gamma_2+\hat{\Gamma}_2\right)\omega^4\omega_4^4}{128\,T_0T_1^4\omega_3^4\omega_4'^4}+O(T_1^{-5}) \tag{2.5.44}$$

$$\lim_{T_1\to\infty}\frac{1}{k^2T_0T_1}\int_0^{2\pi}B^2(\phi)\cos 4\phi\,d\phi=\frac{2\pi\omega^4\omega_4^5}{T_0T_1^6\omega_3^5\omega_4'^6}+O(T_1^{-7}) \tag{2.5.45}$$

The critical Taylor number is given by the formulae (2.4.128), (2.4.171). The dominant term of (2.4.171) for $T_1\to\infty$ will be defined as follows:

$$\Sigma_{g\,\mathrm{DOMINANT}}=\left(\alpha_0+\alpha_{10}\frac{\sigma_0}{T_1}\right)+k^2\left(\alpha_1+\alpha_{11}\frac{\sigma_0}{T_1}\right)+m^2k^2T_0\left(\alpha_2+\alpha_{12}\frac{\sigma_0}{T_1}\right)+$$

$$+\alpha'_{13}\frac{\pi\omega}{\omega'_4} \tag{2.5.46}$$

An alternative way of describing Σ_g dominant

$$\lim_{T_1\to\infty}\Sigma_g\cong\left(\alpha'_0+\alpha_{10}\frac{\sigma_0}{T_1}\right)+k^2\left(\alpha_1+\alpha_{11}\frac{\sigma_0}{T_1}\right)+m^2k^2T_0\left(\alpha_2+\alpha_{12}\frac{\sigma_0}{T_1}\right)-$$

$$-\frac{\omega^2\hat{\Gamma}_2\omega'_4}{256\,k^2T_0} \tag{2.5.47}$$

One limit has to be added, namely:

$$\lim_{T_1\to\infty}\frac{1}{kT_0^{1/2}}\int_0^{2\pi}B^2(\phi)\sin 2\phi\,d\phi=-\frac{\pi}{2T_0^{1/2}}\cdot\frac{\omega^2\omega_4^2}{T_1^2\omega_3^2\omega_4'^3}\left[1+\frac{\omega_4^2}{T_1^2\omega_3^2\omega_4'^2}\left(\frac{\omega^2}{2}-11k^2\right)\right] \tag{2.5.48}$$

An explanation is needed as far as formula (2.4.171) is concerned:

The $\frac{\alpha_{17}}{k T_0^{1/2}} \int_0^{2\pi} B^2(\phi) \sin 2\phi \, d\phi$ is appearing from the $(u_0^+ \Sigma_{12} + v_0^+ \Sigma_{22} + w_0^+ \Sigma_{32})$ terms. In $u_0^+ \Sigma_{12}$ one term of this kind is obtained as the product of $-\lambda^2 f_0^+ \frac{c T_0}{T_1} A^+ B^2 (g_{10} - g_{112}/2)$ with the term $-\frac{A \Gamma T_1 \sin 2\phi}{4 k T_0^{3/2}} g_1$ of $\hat{v}_{12}$.

The $-\frac{\pi \omega^2 \hat{\Gamma}_2 \Gamma_0 \omega_4'}{128 k^2 T_0}$ term in Σ_g is obtained from the sum $(u_1^+ \Sigma_{11} + v_1^+ \Sigma_{21} + w_1^+ \Sigma_{31})$ for example: from the product of the term $-\lambda^2 A^+ \cdot \frac{\hat{\Gamma}_2 f_0^+ \sin 2\phi}{2 k T_0^{1/2}}$ of u_1^+ with the term

$$-\frac{c T_0}{T_1}\left[B^2(\phi)\cdot(g_{10} - g_{112}/2) + \frac{\sigma_0}{2}(\chi^2 - 1/4)\right] \cdot A \frac{g_2 T_1 \sin\phi}{2 k T_0^{3/2}} \text{ of } \frac{c T_0}{T_1} \hat{v}_{11}\left[B^2(\phi)(g_{10} - g_{112}/2) + \frac{\sigma_2}{2}(\chi^2 - 1/4)\right]$$

Similarly, the term $\alpha_{16}' T_1 \int_0^{2\pi} B^2(\phi) \cos 2\phi \, d\phi$ is obtained for example from the product of the term $\frac{A^+ f_2^+ T_1 \Gamma \sin\phi}{2 k T_0^{3/2}}$ of v_1^+ and the term

$$\frac{A T_1 \Gamma f_0 \sin\phi}{2 k T_0^{3/2}} \cdot \frac{1}{T_1}\left[B^2(\phi)\cdot D(g_{10} - g_{112}/2) + \sigma_0 \chi\right] \text{ of } \frac{\hat{u}_{11}}{T_1}\left[B^2(\phi)\cdot D(g_{10} - g_{112}/2) + \sigma_0 \chi\right].$$

In the limit $T_1 \to \infty$, the combination $\varepsilon T_1 = T - T_c(0)$ determines the magnitude of the supercritical Taylor number T, but subject to the first limit $\varepsilon \to 0$.

$$\lim_{T_1 \to \infty} T = T_0 + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1} + \frac{m^2 k^2 T_0 \frac{\Gamma_{10}}{\bar{\Gamma}_1}}{m^2 k^2 (\alpha_2 + \alpha_{12} \frac{\sigma_0}{T_1}) + (\alpha_0' + \alpha_{10} \frac{\sigma_0}{T_1}) + k^2 (\alpha_1 + \alpha_{11} \frac{\sigma_0}{T_1}) + \Delta_t} \tag{2.5.49}$$

where $\Delta_t = -\frac{\pi \omega^2 \hat{\Gamma}_2 \Gamma_0 \omega_4'}{128 k^2 T_0}$

If we use the notation $k_0^2 = m^2 k^2 \varepsilon^2 T_0$, (2.5.50)

and $T_{c1} = \frac{\Gamma_{10}}{\bar{\Gamma}_1}$ (as known from the linear theory) (2.5.51)

We obtain the alternative formula

$$\lim_{T_1 \to \infty} T = T_c(0) + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2 \bar{\Gamma}_4}{\bar{\Gamma}_1} + \frac{k_0^2 T_{c1}}{k_0^2 l_0 + \varepsilon^2 (l_1 + l_2 k^2 + \frac{l_3}{k^2})}$$

If (2.5.52) is subject to the limit $\varepsilon \to 0$, k fixed (implies $\delta \to 0$) and m large, then the k_0^2 term is the dominant one, and we obtain

$$T = T_c(0) + \frac{T_{c1}}{l_0} \qquad (2.5.53)$$

We shall later show that formula (2.5.53) is identical with the formula

$$T = T_c(0) + \frac{\alpha(0)\, T_{c1}}{\left[\alpha_1 - \frac{b_{11r}}{a_1}\alpha(0)\right]} \qquad (2.5.54)$$

of Davey, Di Prima and Stuart (1958)

There is need to explain that formula (2.5.54) of Davey Di Prima and Stuart (1968) has been obtained only by consideration of the k_0^2 terms. Their m^2 term has been derived from consideration of the k_0^4 terms, as their formula (5.21) clearly indicates.
Our limit case for m large has been taken especially for the calculation of the ratio of the dominant terms, and has nothing to do with the correction term in m^2, which has not been calculated at this stage.

Justification

In order to find the connection between (2.5.53) and (2.5.54) we have to go back to the formula of Davey, Di Prima and Stuart (1968).

$$M(\lambda, b_0+2a_0, k_D)h_4 = 2b_4\{N(\lambda, b_0+a_0, k_D)(D^2-\lambda^2)+ik_D\alpha D\}h_0+\Psi \tag{2.5.55}$$

where

$$\Psi = \left(\frac{\lambda^2}{\alpha}\right)D(f_{20}r_{21}-h_{20}m_{21}-f_{30}r_{31}-2f_{30}z_1+h_{30}m_{31})-$$
$$-\frac{1}{2}\left\{\left(\frac{1}{\alpha}\right)N(\lambda, b_0+2a_0, k_D)(D^2-\lambda^2)D+ik_D(D^2+\lambda^2)\right\}\times[f_{20}r_1+$$
$$+2h_{20}F_1-h_{20}m_1-h_0m_{21}+f_0r_{21}]-$$
$$-\left(\frac{\lambda}{2\alpha}\right)(D^2+\lambda^2)[-f_{20}r_{31}+2f_{20}z_1+h_{20}m_{31}-h_{30}m_{21}+f_{30}r_{21}]-$$
$$-\left\{\frac{\lambda^2}{2}T+k_D^2\alpha D-ik_D N(\lambda, b_0+2a_0, k_D)(D^2-\lambda^2)\right\}\times[2h_0F_1+f_0r_1-$$
$$-h_0m_1]+\frac{\lambda}{2}\left\{\left(\frac{1}{\alpha}\right)N(\lambda, b_0+2a_0, k_D)\times(D^2-\lambda^2)+2ik_D D\right\}\cdot[-2h_{30}F_1-$$
$$-f_0r_{31}+2f_0z_1+h_0m_{31}-h_{30}m_1+f_{30}r_1] \tag{2.5.56}$$

the boundary conditions are

$$h_4, D^2h_4 = 0, \quad N(\lambda, b_0+2a_0, k_D)Dh_4 = b_4Dh_0 \quad \text{at } x=\pm\frac{1}{2} \tag{2.5.57}$$

and where

$$N(\lambda, b_0+a_0, k_D) = D^2-\lambda^2-(b_0+a_0)-ik_D\Omega_e(x) \tag{2.5.58}$$

and

$$M(\lambda, a_{c0}, k_D, T) = N(\lambda, a_{c0}, k_D)(D^2-\lambda^2)N(\lambda, a_{c0}, k_D)+\lambda^2T\Omega_e(x) \tag{2.5.59}$$

The determination of b_4 is as follows. The homogeneous boundary value problem corresponding to the previous equations is $M(\lambda, b_0+2a_0, k_D)h_4=0$ with the boundary conditions $h_4 = D^2h_4 = N(\lambda, b_0+2a_0, k_D)Dh_4 = 0$ at $x=\pm\frac{1}{2}$

For $T \to T_c$, $a_0 \to 0$ one obtains the same equation and boundary conditions as for the linear eigenvalue problem for b_0 and h_0.

We can expect a singular behaviour of the solution of the non-homogeneous boundary-value problem, unless b_4 is properly chosen as $T \to T_c$, because the homogeneous boundary value problem has a non-trivial solution.

The solution will behave as a_0^{-1} as $a_0 \to 0$, and in order to remove this singularity, it is necessary and sufficient to set $a_0 \equiv 0$. Here h_0^+ the adjoint of h_0, is the solution of the adjoint eigenvalue problem:

$$M(\lambda, b_0, k_D) h_0^+ = 0$$

$$h_0^+ = Dh_0^+ = (D^2 - 2\lambda^2 - b_0 - ik_D \Omega_\ell) D^2 h_0^+ = 0 \quad \text{at } x = \pm \tfrac{1}{2} \tag{2.5.60}$$

Result for b_4:

$$b_4 = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} h_0^+(x) \Psi \, dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[h_0^+ N(\lambda, b_0, k_D)(D^2 - \lambda^2) h_0 + h_0 N(\lambda, b_0, k_D)(D^2 - \lambda^2) h_0^+ \right] dx} \tag{2.5.61}$$

where the calculation is carried out at $a_0 = 0$.

A brief mention is needed in connection with the symbol k_D, which is a possibility to distinguish between k of Davey, Di Prima and Stuart (1968), written as k_D, and k of Di Prima and Stuart (1972b), written as simple k. $b_{4/2}$ is defined by

$$b_{4/2} = a_1 + k_D^2 b_{4/1} \tag{2.5.62}$$

Let us write

$$\begin{aligned} h_4 &= h_{40} + ik_D h_{41} + (ik_D)^2 h_{42} \\ h_0 &= h_{00} + ik_D h_{01} + (ik_D)^2 h_{02} \\ b_0 &= a_0 + ik_D b_{01} + (ik_D)^2 b_{02} \end{aligned} \tag{2.5.63}$$

From the $M(\lambda, b_0, k_D) h_0 = 0$ equation, one obtains the $(ik_D)^2$ order equation:

$$\begin{cases} (D^2-\lambda^2)h_{oo} = \ell_{oo} \\ (D^2-\lambda^2)^2\ell_{oo} + \lambda^2 T_o \Omega_e h_o = 0 \end{cases} \quad (2.5.64) \qquad \begin{cases} (D^2-\lambda^2)^2 h_{oo}^+ = -\ell_{oo}^+ \\ \lambda^2 T_o \Omega_e h_{oo}^+ - (D^2-\lambda^2)\ell_{oo}^+ = 0 \end{cases} \quad (2.5.64a)$$

It follows that

$$\begin{cases} \ell_{oo} = f_o \\ h_{oo} = g_o \end{cases} \quad (2.5.65)$$

It follows that

$$\begin{cases} h_{oo}^+ = f_o^+ \\ \ell_{oo}^+ = g_o^+ \end{cases} \quad (2.5.65a)$$

If $$\Omega_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[h_o^+ N(\lambda, b_o, k_D)(D^2-\lambda^2)h_o + h_o N(\lambda, b_o, k_D)(D^2-\lambda^2)h_o^+ \right] dx =$$

$$= I_{10} + (ik_D) I_{11} + (ik_D)^2 I_{12} \quad (2.5.66)$$

Consequently, I_{10} can easily be calculated

$$I_{10} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[f_o^+ M^2 g_o + g_o M^2 f_o^+ \right] dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[f_o^+ M f_o - g_o g_o^+ \right] dx =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[f_o^+ F_{114} + g_o^+ G_{114} \right] dx \quad (2.5.67)$$

or, in the double integral form:

$$I_{10} = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[f_o^+ F_{114} + g_o^+ G_{114} \right] A(\phi) A^+(\phi) \, dx \, d\phi \quad (2.5.67a)$$

<u>(ik_D) order for h_o, ℓ_o</u>

By substituting again in the (2.5.59) equation, we have:

$$\begin{cases} (D^2-\lambda^2)^2 \ell_{o1} + \lambda^2 \Omega_e T_o h_{o1} = (b_{o1} + \Omega_e)(D^2-\lambda^2) f_o \\ \ell_{o1} - (D^2-\lambda^2) h_{o1} = -(b_{o1} + \Omega_e) g_o \\ \ell_{o1} = D\ell_{o1} = h_{o1} = 0 \quad \text{at } x = \pm\frac{1}{2} \end{cases} \quad (2.5.68)$$

By comparing with the known systems, it can be seen that

$$\ell_{o1} \equiv f_3$$

$$h_{o1} \equiv g_3 \quad (2.5.69)$$

$(ik_0)^2$ order for h_0, l_0

$$\begin{cases} (D^2-\lambda^2)^2 l_{02} + \lambda^2 \Omega_\ell T_0 h_{02} = (b_{01}+\Omega_\ell)(D^2-\lambda^2) l_{01} + b_{02}(D^2-\lambda^2) g_0 \\ l_{02} - (D^2-\lambda^2) h_{02} = -(b_{01}+\Omega_\ell) h_{01} - b_{02} g_0 \\ l_{02} = D l_{02} = h_{02} = 0 \text{ at } x = \pm \frac{1}{2} \end{cases} \qquad (2.5.70)$$

one obtains:

$$\begin{cases} l_{02} = -f_{18} \\ h_{02} = -g_{18} \end{cases} \qquad (2.5.71)$$

Similarly for the adjoint:

$$\begin{cases} h_{01}^+ = f_3^+ \\ l_{01}^+ = g_3^+ \end{cases} \qquad (2.5.72)$$

$$\begin{cases} h_{02}^+ = -f_{18}^+ \\ l_{02}^+ = -g_{18}^+ \end{cases} \qquad (2.5.73)$$

As a summary, we may write:

$$\begin{aligned} h_0 &= g_0 + (ik_D) g_3 - (ik_D)^2 g_{18} \\ l_0 &= f_0 + (ik_D) f_3 - (ik_D)^2 f_{18} \\ h_0^+ &= f_0^+ + (ik_D) f_3^+ - (ik_D)^2 f_{18}^+ \\ l_0^+ &= g_0^+ + (ik_D) g_3^+ - (ik_D)^2 g_{18}^+ \\ b_0 &= a_0 - \frac{ik_D}{c} + \frac{\Gamma_{10}}{c} (ik_D)^2 \end{aligned} \qquad (2.5.74)$$

Now we are enabled to calculate I_{11} and I_{12} :

$$I_{11} = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\lambda^2 (f_0^+ f_3 + f_0 f_3^+) + g_0^+ g_3 + g_0 g_3^+ + Df_0 \cdot Df_3^+ + Df_0^+ \cdot Df_3 - c g_0 Df_0^+ \right] dx \qquad (2.5.75)$$

$$I_{12} = \int_{-\frac{1}{2}}^{\frac{1}{2}} [g_{18} g_0^+ - f_{18}^+ M f_0 + f_3^+ M f_3 - f_{18} M f_0^+ + g_0 g_{18}^+ - g_3 g_3^+ + c(g_3 D f_0^+ + g_0 D f_3^+)] dx \qquad (2.5.76)$$

If we compare Ω_1 of (2.5.66) with I_1 of (2.5.114), we find that

$$\Omega_1 = -I_1 \qquad (2.5.77)$$

Therefore, it is possible to express all the functions appearing in (3.33) of Davey, Di Prima and Stuart (1968) (or (2.5.56) in this work) with respect to the functions appearing in this work.

If we consider

$w_{01}(x,\tau) = A_c B_s z_1$ from 3.9 (Davey, Di Prima and Stuart (1968))

since only A_c and B_s are appearing in this work, it is possible to find the mathematical correspondence

$$z_1 = -\frac{1}{2c\lambda} h_{11} \qquad (2.5.78)$$

where $D^2 h_{11} = f_0 D^2 f_3 - f_3 D^2 f_0 - c D(f_0 g_0)$ as given in (2.5.88)

It can be seen that

$$m_1 = \frac{g_{112}}{c}$$

$$m_{21} = \frac{f_{112}}{c} \qquad (2.5.79)$$

$$m_{31} = -\frac{1}{2\lambda c} D f_{112}$$

$$F_1 = -\frac{1}{2c} F_0(x) \qquad (2.5.80)$$

$$f_{20} = f_0$$
$$f_{30} = -\frac{D f_0}{\lambda} \qquad (2.5.81)$$

$$h_{20} = f_0 + (ik_0)\left[M f_3 - g_0\left(V_0 - \frac{1}{c}\right)\right] + (ik_0)^2\left[-\frac{\Gamma_{10}}{c} g_0 + M g_{18} - \left(V_0 - \frac{1}{c}\right) g_3\right]$$

$$h_{30} = -\frac{D f_0}{\lambda} + \frac{ik_0}{\lambda}\left[\left(V_0 - \frac{1}{c}\right) D g_0 - M D f_3\right] + \frac{(ik_0)^2}{\lambda}\left[\left(V_0 - \frac{1}{c}\right) D g_3 + \frac{\Gamma_{10}}{c} D g_0 - M D g_{18}\right] \qquad (2.5.82)$$

$$r_1 = \frac{2}{c} g_{112} + \frac{2}{c}(ik_D) g_{211} + \frac{2}{c}(ik_D)^2 g_{226}$$

$$r_{21} = \frac{2}{c} f_{112} + \frac{2}{c}(ik_D) f_{211} + \frac{2}{c}(ik_D)^2 f_{226} \qquad (2.5.83)$$

$$r_{31} = -\frac{1}{c\lambda} D f_{112} - \frac{1}{c\lambda}(ik_D) D f_{211} - \frac{1}{c\lambda}(ik_D)^2 D f_{226}$$

If we take $k_D \to 0$

$b_{4l} \to a_1$

In this case we obtain

$$a_1 = \frac{\left\{ \begin{array}{l} \frac{1}{c}\int_{-\frac{1}{2}}^{\frac{1}{2}} f_{20} D F_1 (D^2-\lambda^2)^2 \theta \, dx + \lambda^2 T_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta f_0 F_1 \, dx + \frac{1}{c}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Psi_1 (D^2-\lambda^2)^2 \theta \, dx \\ + \int_{-\frac{1}{2}}^{\frac{1}{2}} \Psi_3 D\theta \, dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} \Psi_2 (D^2+\lambda^2)\theta \, dx + \frac{1}{2}\lambda^2 T_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta m_1 f_0 \, dx \end{array} \right\}}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \theta (D^2-\lambda^2)^2 f_0 + f_0 (D^2-\lambda^2)^2 \theta \right\} dx} \qquad (2.5.84)$$

where

$$\Psi_1 = \frac{m_{21}}{2} D f_0 + \frac{1}{4} f_0 D m_{21} + m_1 D f_{20} + \frac{1}{2} f_{20} D m_1$$

$$\Psi_2 = \frac{m_{21}}{2c} D f_{20} - \frac{1}{4c} f_{20} D m_{21} \qquad (2.5.85)$$

$$\Psi_3 = \frac{\lambda^2 f_{20}}{c} m_{21} - \frac{1}{2c} D f_{20} \cdot D m_{21}$$

θ is defined as the adjoint of h_0.

One obtains

$$a_1 = -\frac{T_0^{1/2}(\omega_4' - 6Q_0\omega_5)}{2\bar{c}c^2} \qquad (2.5.86)$$

Since $\sigma_0 = \dfrac{6Q_0 T_1 \omega_3}{\omega_4' - 6Q_0\omega_5}$

$$\frac{1}{a_1} = -\frac{\bar{c}c^2}{3T_0^{1/2}\omega_3 Q_0} \cdot \frac{\sigma_0}{T_1} \qquad (2.5.86a)$$

We calculate b_{41r} or the k_D^2 component of $b_{4r}(m)$ from (2.5.61) and we obtain

$$T'(m) = T_c(0) + 125 \qquad (2.5.87)$$

which is similar to formula (5.22) of Davey, Di Prima and Stuart (1968). The difference in the results is due to the missing m^2 term, and to find the m^2 correction we need to go to a higher order, namely to γ^2 in the equations and to ε^4 in the integrals.

A part of the m^2 correction terms are appearing in our calculations, but as they are only a part of them, we cannot sensibly keep them.

2.6 A simultaneous method of approach to the concentric case ($k \to \infty$)

A different kind of approach to the concentric case is obtained by letting $k \to \infty$. This method is a simultaneous approach to the concentric case as obtained by letting $T_1 \to \infty$, and its use has been based on practical experience.

We return again to the formula

$$\beta^2(\phi) = \frac{k}{\omega_4'} \cdot \frac{\left[E(2\pi) - 1\right] E(\phi)}{\int_0^{2\pi} E(\phi)\, d\phi + \left[E(2\pi) - 1\right] \int_0^{\phi} E(\phi)\, d\phi} \qquad (2.6.1)$$

where

$$E(\phi) \equiv \exp\left[\frac{\omega}{k} \sin\phi + \frac{1}{k}(T_1 \omega_3 + \sigma_0 \omega_5)\phi\right] \qquad (2.6.2)$$

and

$$(\omega, \omega_3, \omega_4', \omega_5) \equiv 2T_0^{-1/2} (\Gamma, \bar{\Gamma}_1, \Gamma_4', \Gamma_5) \qquad (2.6.3)$$

If we let $k \to \infty$, the exponent $\dfrac{\omega \sin\phi + (T_1 \omega_3 + \sigma_3 \omega_5)\phi}{k}$

will be small, and in this case we could expand in a series of powers, as follows:

$$E(\phi) \equiv e^{\frac{\omega\omega_4 \sin\phi + T_1 \omega_3 \omega_4' \phi}{k\omega_4}} =$$

$$= 1 + \frac{\omega\omega_4 \sin\phi + T_1 \omega_3 \omega_4' \phi}{k\omega_4} + \frac{(\omega\omega_4 \sin\phi + T_1 \omega_3 \omega_4' \phi)^2}{2k^2\omega_4^2} + \cdots\cdots (2.6.4)$$

By substituting in the formula of $\beta^2(\phi)$, it yields:

$$\lim_{k\to\infty} \beta^2(\phi) = \frac{\frac{T_1\omega_3}{\omega_4}\left\{1 + \frac{1}{k\omega_4}\left[T_1\omega_3\omega_4'(\pi+\phi) + \omega\omega_4 \sin\phi\right] + \frac{1}{k^2\omega_4^2}\left[\begin{array}{l} T_1\omega\omega_3\omega_4\omega_4'(\pi+\phi)\sin\phi + \\ + T_1^2\omega_3^2\omega_4'^2\left(\frac{2\pi^2}{3} + \pi\phi + \frac{\phi^2}{2}\right) \\ + \frac{\omega^2\omega_4^2}{4}(1-\cos 2\phi) \end{array}\right]\right\}}{1 + \frac{T_1\omega_3\omega_4'}{k\omega_4}(\pi+\phi) + \frac{1}{k^2\omega_4^2}\left[\frac{\omega^2\omega_4^2}{4} + T_1^2\omega_3^2\omega_4'^2\left(\frac{2\pi^2}{3} + \pi\phi + \frac{\phi^2}{2}\right) - T_1\omega\omega_3\omega_4\omega_4'\cos\phi\right]} \qquad (2.6.5)$$

By expanding in powers of $\left(\frac{1}{k}\right)$ till $\frac{1}{k^2}$, we get:

$$\lim_{k\to\infty} \beta^2(\phi) = \frac{T_1\omega_3}{\omega_4}\left[1 + \frac{\omega}{k}\sin\phi + \frac{1}{k^2}\frac{\omega}{\omega_4}\left(T_1\omega_3\omega_4'\cos\phi - \frac{\omega\omega_4}{4}\cos 2\phi\right)\right] + \cdots\cdots (2.6.6)$$

By making use of the differential equation

$$k\frac{dB^2(\phi)}{d\phi}=\left(\omega\cos\phi+\frac{T_1\omega_3\omega_4'}{\omega_4}\right)B^2(\phi)-\omega_4'B^4(\phi) \quad (2.6.7)$$

it is possible to calculate the (k^{-3}) and (k^{-4}) terms of $B^2(\phi)$

Let us take

$$\lim_{k\to\infty}B^2(\phi)=\frac{T_1\omega_3}{\omega_4}\left[1+\frac{\omega}{k}\sin\phi+\frac{1}{k^2}\cdot\frac{\omega}{\omega_4}\left(T_1\omega_3\omega_4'\cos\phi-\frac{\omega\omega_4}{4}\cos 2\phi\right)\right]+$$

$$+\frac{\mathcal{R}_3(\phi)}{k^3} \quad (2.6.8)$$

By considering the $0(k^{-2})$ order equation, one obtains

$$\frac{d\mathcal{R}_3(\phi)}{d\phi}=\frac{T_1\omega\omega_3}{\omega_4^2}\left\{\left(\omega\cos\phi+\frac{T_1\omega_3\omega_4'}{\omega_4}\right)\left(T_1\omega_3\omega_4'\cos\phi-\frac{\omega\omega_4}{4}\cos 2\phi\right)-\right.$$

$$\left.-\omega_4'\left[\frac{2T_1\omega_3}{\omega_4}\left(T_1\omega_3\omega_4'\cos\phi-\frac{\omega\omega_4}{4}\cos 2\phi\right)+T_1\omega\omega_3\sin^2\phi\right]\right\} \quad (2.6.9)$$

which yields:

$$\mathcal{R}_3(\phi)=\frac{T_1\omega\omega_3}{\omega_4^2}\left[-\left(\frac{\omega^2\omega_4}{8}+\frac{T_1^2\omega_3^2\omega_4'^2}{\omega_4}\right)\sin\phi+\frac{5}{8}T_1\omega\omega_3\omega_4'\sin 2\phi-\right.$$

$$\left.-\frac{\omega^2\omega_4}{24}\sin 3\phi\right] \quad (2.6.10)$$

In a similar way it is possible to determine the $\frac{\mathcal{R}_4(\phi)}{k^4}$ term, by considering the $0(k^{-3})$ order equation:

$$\mathcal{R}_4(\phi)=\frac{T_1\omega\omega_3}{\omega_4^3}\left[-T_1\omega_3\omega_4'\left(T_1^2\frac{\omega_3^2\omega_4'^2}{\omega_4}+3\frac{\omega^2\omega_4}{16}\right)\cos\phi+\right.$$

$$+\left(\frac{\omega_3^3\omega_4^2}{24}+\frac{17}{16}T_1^2\omega\omega_3^2\omega_4'^2\right)\cos 2\phi-\frac{29}{144}T_1\omega^2\omega_3\omega_4\omega_4'\cos 3\phi+$$

$$\left.+\frac{\omega^3\omega_4^2}{192}\cos 4\phi\right] \quad (2.6.11)$$

Consequently

$$\lim_{k\to\infty}B^2(\phi)=\frac{T_1\omega_3}{\omega_4}\left\{1+\frac{\omega}{k}\sin\phi+\frac{1}{k^2}\frac{\omega}{\omega_4}\left(T_1\omega_3\omega_4'\cos\phi-\frac{\omega\omega_4}{4}\cos 2\phi\right)+\right.$$

$$+\frac{1}{k^3}\frac{\omega}{\omega_4}\left[-\left(\frac{\omega^2\omega_4}{8}+\frac{T_1^2\omega_3^2\omega_4'^2}{\omega_4}\right)\sin\phi+\frac{5}{8}T_1\omega\omega_3\omega_4'\sin 2\phi-\right.$$

$$\left.-\frac{\omega^2\omega_4}{24}\sin 3\phi\right]+\frac{1}{k^4}\cdot\frac{\omega}{\omega_4^2}\left[-T_1\omega_3\omega_4'\left(T_1^2\frac{\omega_3^2\omega_4'^2}{\omega_4}+3\frac{\omega^2\omega_4}{16}\right)\cos\phi+\right.$$

$$\left.+\left(\frac{\omega_3^3\omega_4^2}{24}+\frac{17}{16}T_1^2\omega\omega_3^2\omega_4'^2\right)\cos 2\phi-\frac{29}{144}T_1\omega^2\omega_3\omega_4\omega_4'\cos 3\phi+\frac{\omega^3\omega_4^2}{192}\cos 4\phi\right]\Bigg\} \tag{2.6.12}$$

Let us return to the formula of the critical Taylor number:

$$T'(m)=T_0+\varepsilon^2\frac{\bar{\Gamma}_3+k^2\bar{\Gamma}_4}{\bar{\Gamma}_1}+\frac{\varepsilon^2m^2k^2T_0T_{c1}}{Re\left[i\bar{\sigma}_2(TV=0)+i\bar{\sigma}_2(TV)\right]} \tag{2.6.13}$$

where

$$Re\left[i\bar{\sigma}_2(TV=0)+i\bar{\sigma}_2(TV)\right]=O(\varepsilon^2)=\varepsilon^2\Sigma_g \tag{2.6.14}$$

and Σ_g will be given as a function dependent on k.

$$\Sigma_g=\beta_0+\beta_1k^2+\beta_2m^2k^2T_0+\left(\beta_3+\frac{\beta_3'}{T_1}+\frac{\Gamma\hat{\Gamma}_2\Gamma_4'}{16\pi\bar{c}T_0^2}\cdot\frac{1}{k^2}\right)\int_0^{2\pi}B^2(\phi)\cos\phi d\phi+$$

$$+\left(\frac{\beta_4}{k}+k\beta_4'+\beta_5\frac{m^2k}{T_1}\right)\int_0^{2\pi}B^2(\phi)\sin\phi d\phi+$$

$$+\left(\frac{\beta_6}{T_1}-\frac{m^2\Gamma_7\hat{\Gamma}_7\Gamma_4'}{4\pi\bar{c}T_1}-\frac{\Gamma^2\Gamma_4'T_1}{16\pi\bar{c}T_0^3}\cdot\frac{1}{k^2}\right)\int_0^{2\pi}B^2(\phi)\cos 2\phi d\phi+$$

$$+\frac{\beta_7}{k}\int_0^{2\pi}B^2(\phi)\sin 2\phi\,d\phi+\frac{\beta_8}{kT_1}\int_0^{2\pi}B^2(\phi)\sin 3\phi d\phi-$$

$$-\frac{\Gamma\Gamma_4'(\Gamma_2+\hat{\Gamma}_2)}{16\pi\bar{c}T_0^2}\cdot\frac{1}{k^2}\int_0^{2\pi}B^2(\phi)\cos 3\phi d\phi+\frac{\Gamma_2\hat{\Gamma}_2\Gamma_4'}{16\pi\bar{c}T_1T_0}\frac{1}{k^2}\int_0^{2\pi}B^2(\phi)\cos 4\phi d\phi \tag{2.6.15}$$

By taking into account the expansion obtained for $\lim_{k\to\infty}B^2(\phi)$, we get the following result:

$$\lim_{k\to\infty}\Sigma_g=\beta_0+\beta_1k^2+\beta_2m^2k^2T_0+$$

$$+\left(\beta_3+\frac{\beta_3'}{T_1}+\frac{\Gamma\hat{\Gamma}_2\Gamma_4'}{16\pi\bar{c}T_0^2}\cdot\frac{1}{k^2}\right)\cdot\frac{\pi T_1^2\omega\omega_3^2\omega_4'}{k^2\omega_4^2}\left[1-\frac{1}{k^2}\left(\frac{3\omega^2}{16}+T_1^2\frac{\omega_3^2\omega_4'^2}{\omega_4^2}\right)\right]+$$

$$+\left(\beta_4+k\beta_4'+\beta_5\frac{m^2k}{T_1}\right)\frac{\pi T_1\omega\omega_3}{k\omega_4}\left[1-\frac{1}{k^2}\left(\frac{\omega^2}{8}+\frac{T_1^2\omega_3^2\omega_4'^2}{\omega_4^2}\right)\right]+$$

$$+\frac{\pi\omega}{k^2\omega_4^2}\left(\frac{\beta_6}{T_1}-\frac{m^2\Gamma_7\hat{\Gamma}_7\Gamma_4'}{2\pi\bar{c}}-\frac{\Gamma^2\Gamma_4'T_1}{16\pi\bar{c}T_0^3}\cdot\frac{1}{k^2}\right)\left[-\frac{T_1\omega_4\omega_3}{4}+\right.$$

$$\left.+\frac{T_1\omega_3}{8\omega_4k^2}\left(\frac{\omega_3^3\omega_4^2}{3}+\frac{17}{2}T_1^2\omega\omega_3^2\omega_4'^2\right)\right]+\frac{5\pi\beta_7T_1^2\omega^2\omega_3^2\omega_4'}{\omega_4^2k^3}-$$

$$-\frac{\pi\beta_8\omega^2\omega_3}{24\omega_4k^4}+\frac{1}{k^6}\left[\frac{29\Gamma\Gamma_4'(\Gamma_2+\hat{\Gamma}_2)T_1^2\omega^3\omega_3^2\omega_4'}{2304\bar{c}T_0^2\omega_4^2}+\frac{\Gamma_2\hat{\Gamma}_2\Gamma_4'\omega^4\omega_3}{3072\bar{c}\omega_4}\right] \tag{2.6.15}$$

If we restrict ourselves to the dominant terms in the denominator, we obtain the following critical Taylor number

$$\lim_{k\to\infty} T'(m) = T_o + \varepsilon^2 \frac{\bar{\Gamma}_3 + k^2\bar{\Gamma}_4}{\bar{\Gamma}_1} + \frac{k_0^2 T_{c1}}{\beta_2 k_0^2 + \beta_1 \varepsilon^2 k^2} \qquad (2.6.17)$$

where $\beta_1 = \alpha_1 + \alpha_{11}\frac{T_o}{T_1}$ (2.6.18)

and $\beta_2 = \alpha_2 + \alpha_{12}\frac{T_o}{T_1}$ (2.6.19)

as given by the equation (2.5.171)

To retrieve the concentric case, there is need to subject this expression to the limit $\varepsilon \to 0$, $\delta \to 0$ and m large. (In this case $\delta \gg \varepsilon^2$)

In this case, once again, we retrieve the previous formula:

$$\lim_{\substack{k\to\infty \\ \varepsilon\to 0 \\ \delta\to 0 \\ m \text{ large}}} T'(m) = T_o + \frac{T_{c1}}{\beta_2} = T_o + 125 = 1820 \qquad (2.6.20)$$

the result is the same, only the method of approach was somewhat different.

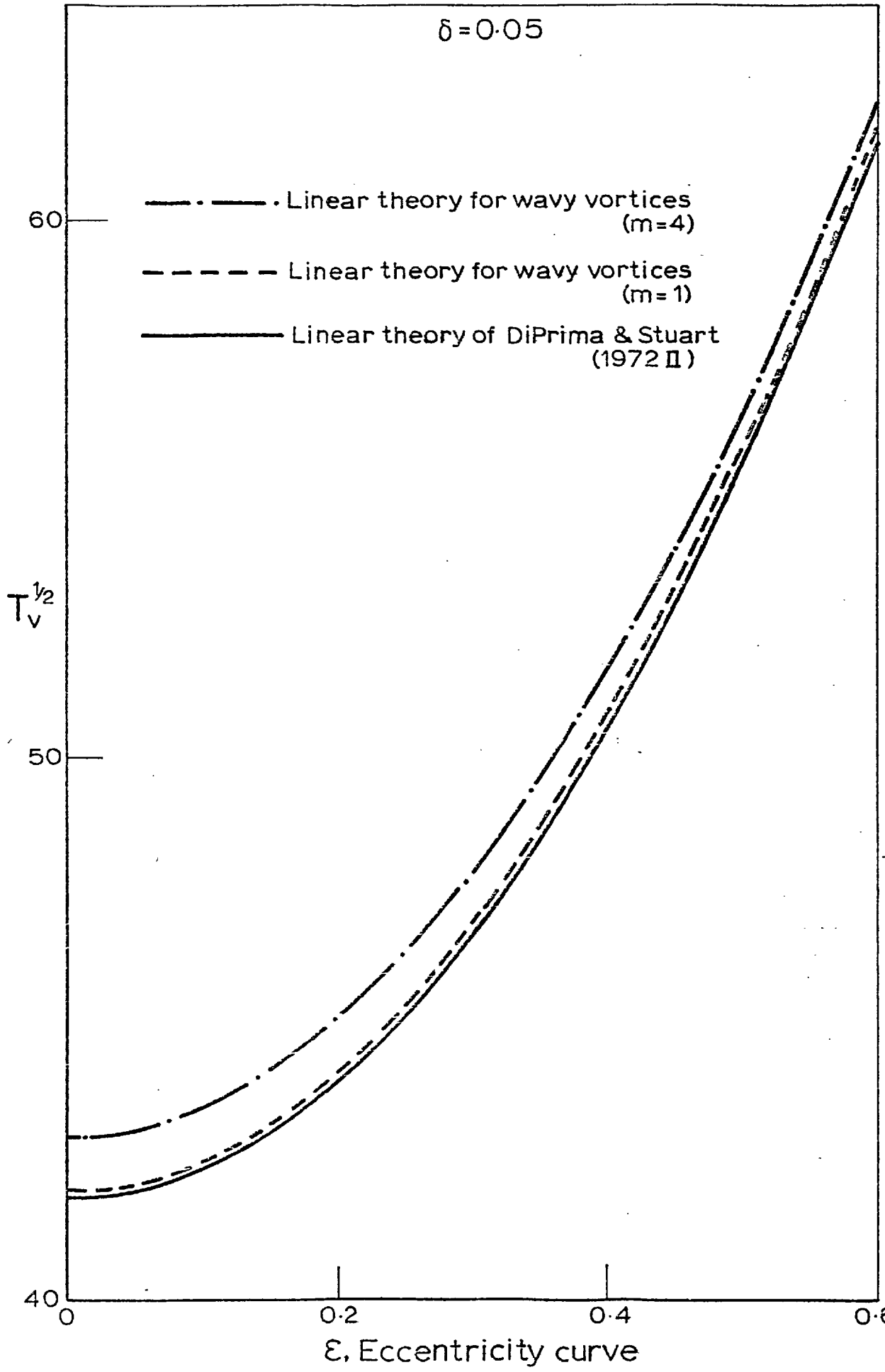


Fig. II Results for the wavy criticals (for m=1 and m=4) obtained from the linear theory of wavy vortices. (δ=0·05)

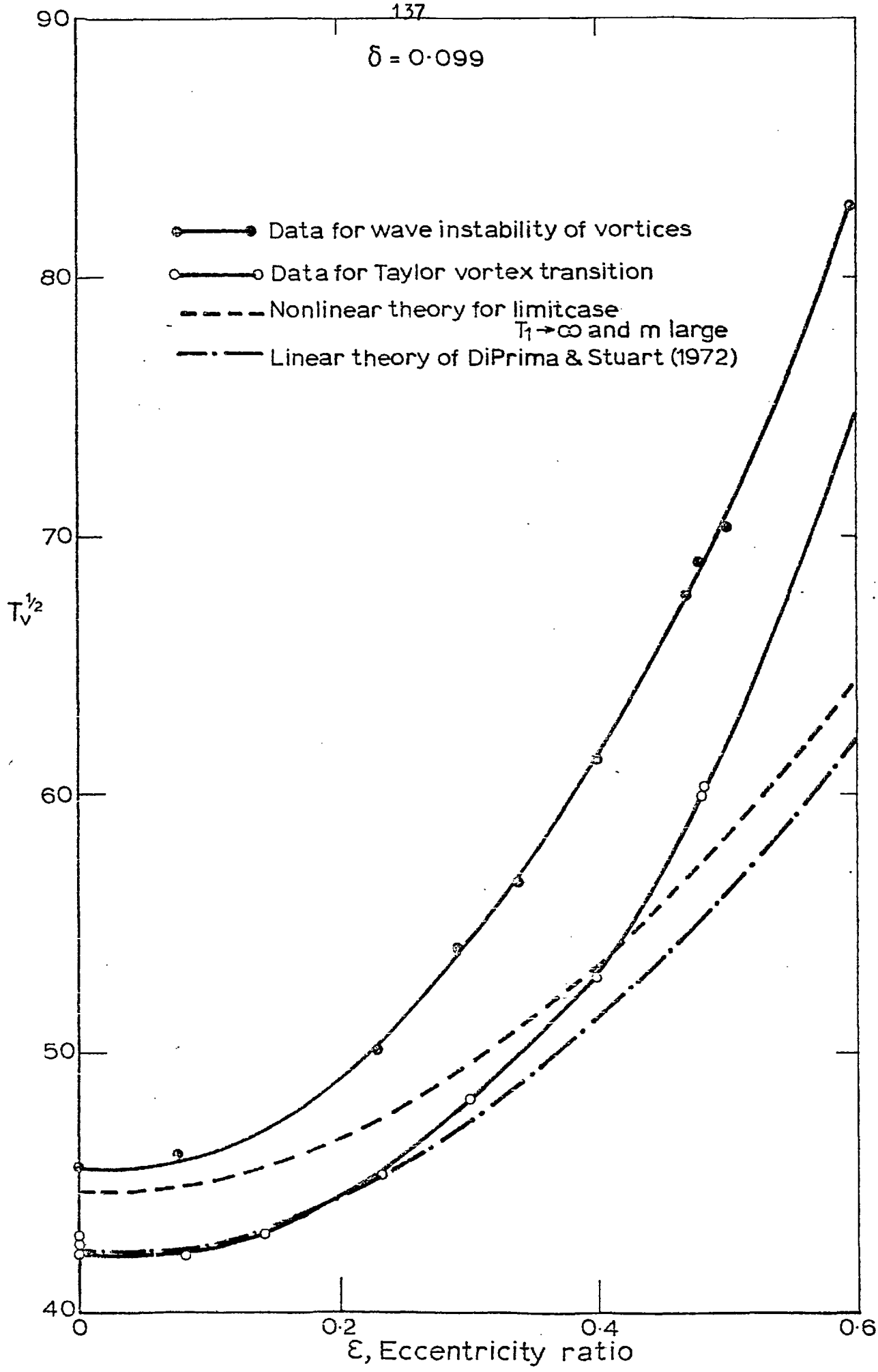


Fig. III Comparison of nonlinear theory with experimental measurements of Vohr (1968) for the occurence of wave instability for δ=0·099

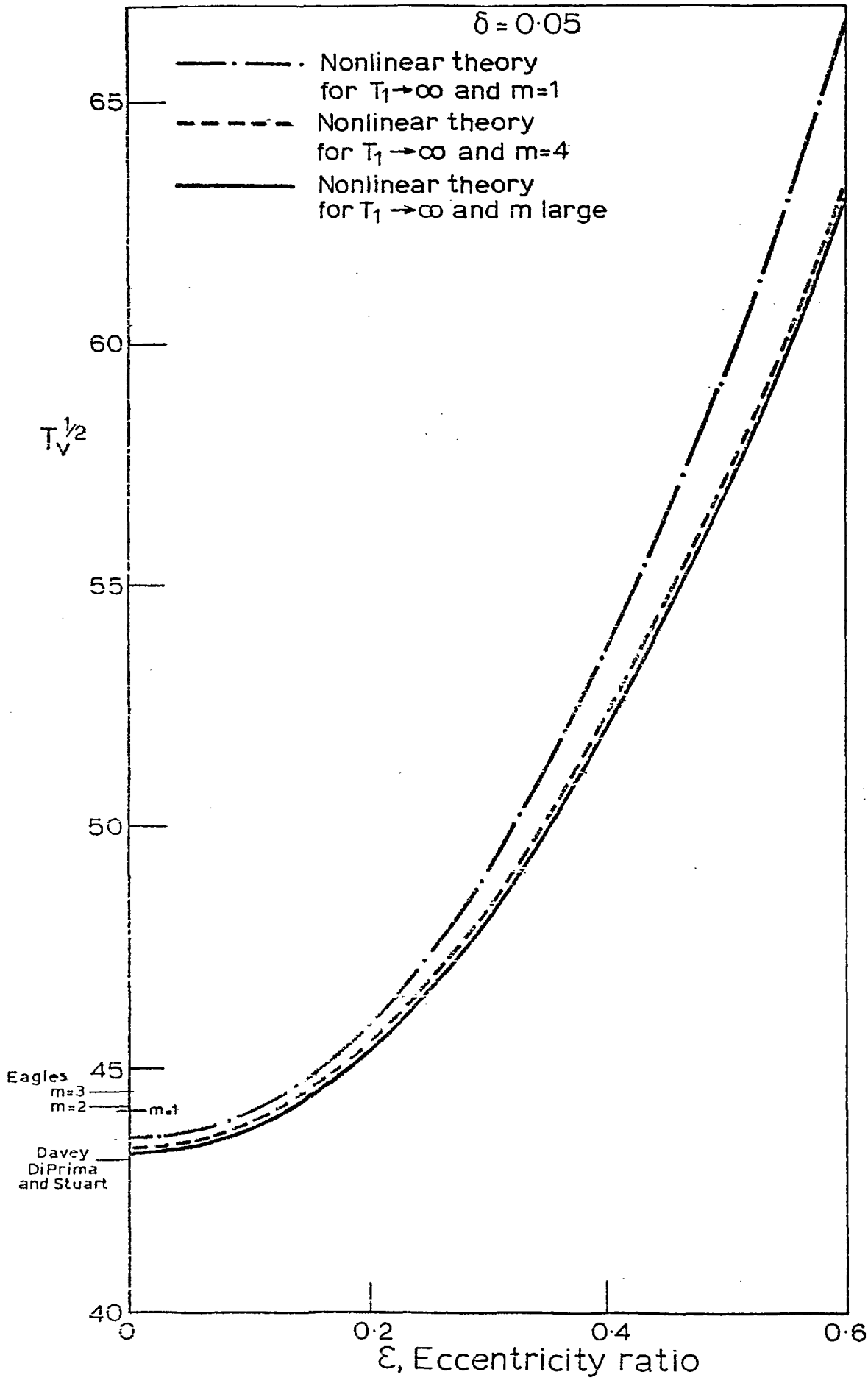


Fig. IV Theoretical results for the occurence of wave instability for $\bar{\delta}$ = 0·05 and different values of m obtained from the nonlinear analysis

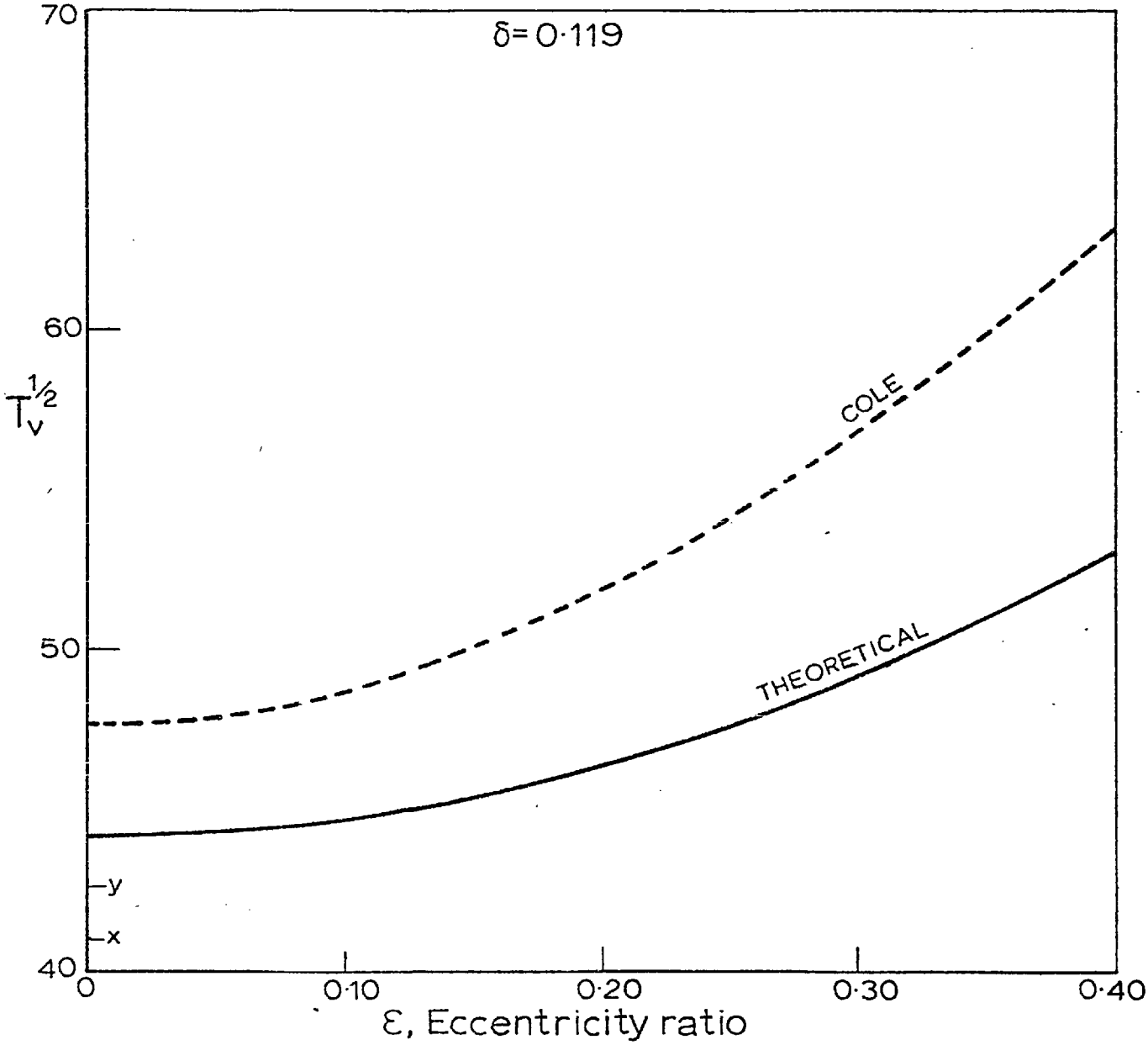


Fig. V Comparison of nonlinear theory with experimental measurements of Cole for wave instability for δ=0·119. x= critical Taylor number for the linear theory of DiPrima and Stuart (1972b) y=critical Taylor number for the nonlinear theory of Davey, DiPrima and Stuart (1968). (δ=0·119)

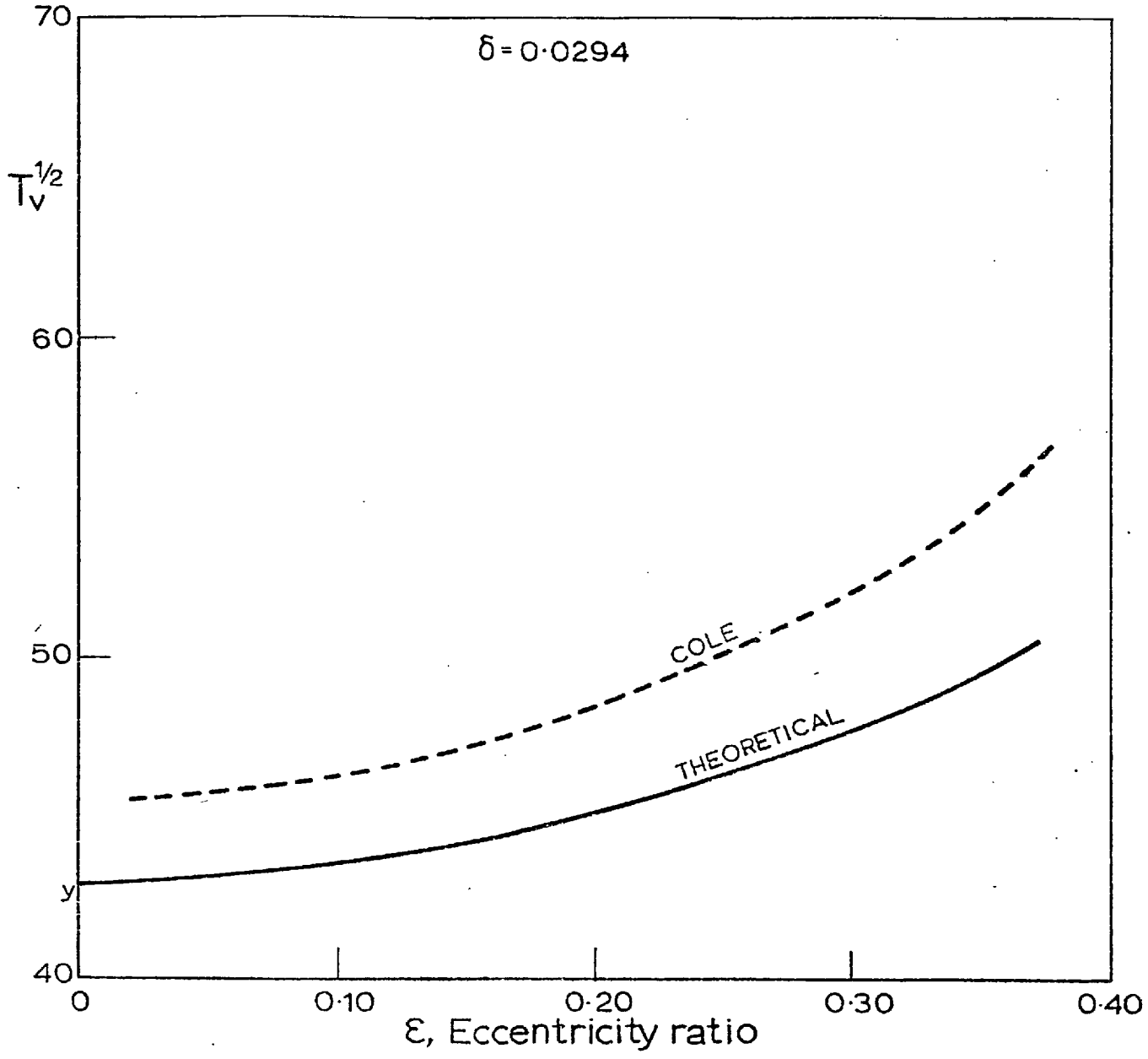


Fig. VI Comparison of nonlinear theory with experimental measurements of Cole for wave instability for δ=0·0294

y = critical Taylor number for the nonlinear theory of Davey, DiPrima and Stuart (1968) (δ= 0·0294)

DISCUSSION

The linear theory of wavy vortices in Eccentric Cylinder Flow, which constitutes the first part of the Thesis, is an extension of the previous papers of Di Prima and Stuart (1972b) on the Taylor vortex flow between two eccentric cylinders, and of Di Prima and Roberts on the linear theory of wavy vortices between two concentric cylinders. The first part of the Thesis deals with the wavy vortices of the flow between two eccentric cylinders where no Taylor vortex flow is present, and where the eccentricity is small and the clearance ratio, is very small. An extension to Di Prima and Stuart's work (1972b) is made by the inclusion of the time variable τ in the amplitude equation for $B(\phi,\tau)$.

The results are compared with the observed critical speeds of Vohr (1967) for development of wave instability on Taylor vortices. The results are qualitatively similar, but differ in magnitude from experiment due to the absence of nonlinear effects.

In this work the stability of the basic flow between rotating eccentric cylinders is analysed. The basic velocity depends upon the independent co-ordinates ρ and ϕ in the radial and azimuthal directions, respectively. The formulation of the linear stability problem leads to partial differential equations rather than ordinary differential equations. The analysis is based on the existence of two small parameters: $\delta(\text{or } \alpha)$, the clearance ratio, and ε, the eccentricity. The problem is solved by considering an appropriate limit as $\delta \to 0$ and $\varepsilon \to 0$ for fixed T, in such a way that proper account is taken of the ϕ variation. The analysis is based on two additional assumptions, namely, the small gap between the eccentric cylinders and cylinders of infinite length. By assuming $\alpha^{1/2} = (2c)^{1/2} k\varepsilon$, an asumptotic result in the limit $\alpha \to 0, \varepsilon \to 0$, with k and T fixed, is obtained.

Three cases of the global result are of interest. The first case is $m \to 0$, which retrieves the formula of the amplitude $B(\phi) = B_0(k) e^{\frac{\Gamma(m\phi - n)}{k T_1^{1/2}}}$ of Di Prima and Stuart (1972b) for the linear theory of Taylor vortices between eccentric cylinders. The second case is $k \to \infty$, which implies that ε is very much smaller than $\delta^{1/2}$, as seen by the formula $\alpha^{1/2} = (2c)^{1/2} k \varepsilon$ and $\alpha(\delta, \varepsilon) = \delta(1-\varepsilon^2)^{1/2}\{1 - \frac{\delta}{2}[1-(1-\varepsilon^2)^{1/2}]\} + O(\delta^3)$ In this case the concentric case $B = B_0(\infty)$ is retrieved. The third case is $k \to 0$, which implies that $\delta^{1/2}$ is very much smaller than ε, in this case

$$B \to 0 \quad \text{for} \quad \phi \neq \frac{\pi}{2}$$

$$B \to B_0(0) \quad \text{for} \quad \phi = \frac{\pi}{2}$$

It is obvious that these results are not derived from local stability considerations. $B_0(k)$ has been determined properly by Di Prima and Stuart (1975) by consideration of the nonlinear problem of the Taylor vortex flow: $B_0^2(k, T_1) = \frac{T_1 \omega_3 e^{\frac{\omega}{k}}}{\omega_4 I_0(\frac{\omega}{k})}$

In the second part of the thesis the perturbing influence exerted by a linear wavy vortex mode upon a nonlinear Taylor vortex mode is investigated. A mathematical perturbation procedure is used in which the eccentricity ε tends to zero subject to two parameters k and T_1 being held fixed; the parameter k relates the clearance ratio and eccentricity, while T_1 determines the value of the supercritical Taylor number. If we consider the limit $T_1 \to 0$ the bifurcation point of linear theory is approached, and we retrieve the critical Taylor number of the linear, steady and eccentric problem of Di Prima and Stuart (1972b).

In order to retrieve the formula of Davey, Di Prima and Stuart (1968) for the Taylor critical number, for an interaction between a nonlinear Taylor vortex mode and a linear wavy vortex mode in the concentric case, we could envisage two simultaneous methods, the first by

letting $T_1 \to \infty$ together with $\varepsilon \to 0$, $\delta \to 0$, m large and where k is kept fixed, and the second by letting $k \to \infty$, together with $\varepsilon \to 0$, $\delta \to 0$, m large and T_1 kept fixed. We note that these two simultaneous methods of approach to the concentric case are based on theoretical research experience, as found recently by Di Prima and Stuart.

The result for m large and T_1 or $k \to \infty$ is compared with the observed critical speeds of Vohr for the occurrence of wavy vortices. There is qualitative and consistent agreement with the experimental results of Vohr, but the theoretical critical speeds are a little bit lower, due maybe to the absence of quintic terms in the nonlinear expansion of the Taylor vortex mode.

The higher order correction in m^2 of the Taylor critical number could constitute a future research subject for the nonlinear problem, but in this case we must expect the involvement of quite heavy algebra. A different idea to include the quintic terms in the nonlinear expansion of the Taylor vortex mode, as used by Eagles (1971) would not be very practical, since the algebra involved would make it very difficult to achieve a real progress. An alternative and interesting method to attack this nonlinear problem would be the use of the W.K.B. method. As we may remember all the investigated problems were based on the assumption of infinite length cylinder. An advance in the investigation of this kind of problem could be made by considering "finite length" rotating cylinders instead of the usual "infinite length" cylinders.

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Addenda.

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