

GENERALIZED POLYNOMIAL EQUATIONS

Thesis submitted by

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Abstract

A generalized polynomial equation has complex exponents as well as coefficients. It is shown that such an equation, with $M + 2$ terms, has an M -ply^c infinity of solutions and an explicit expression is given for any selected one of them. This follows discussion of ordinary polynomial equations. A general chain of convergence conditions is established by examining trinomial, tetranomial and pentanomial cases.

Some tables for solving cubic equations are given.

An algorithm^o is developed for computation, but it is of restricted application. The general solution is stated in matrix form.

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Preface:

Motivation and Acknowledgments

After meeting the Fundamental Theorem of Algebra, it occurred to me that since $x^3 + x + 1$ has 3 zeros,
and $x^{3.14159} + x + 1$ has ^{3/4/59} ~~100000~~,
then $x^\pi + x + 1$ should have a denumerable infinitude of zeros,
6/ as also should $x^{(-3)} + x + 1$. They do; and the way to find them is here given.

After learning that polynomial equations of degree greater than 4 are not generally solvable by radicals, I felt that some other kind of explicit expression for ALL the zeros of ANY polynomial must exist. Such an expression is given for SOME (at least one, perhaps all) of the zeros, and the way shown to find the others.

In 1937, my friend E. H. PALFREYMAN of PMG Research (Australia) told me that linesmen would, over the telephone, state a cubic equation and expect to hear the roots almost instantly. For real coefficients, this is easily done by nomogram and some simple arithmetic. I devised two sets of series for other cases, and the way seemed open to extend the method to the general problems indicated above.

However, my pupils' claims on me prevented my further exploration of these matters, until I was vouchsafed a period of incapacity and convalescence. Then, as was to be expected, further exciting (but unsolved) problems arose, e.g.

(1) (Algebra) Do all critical cases of convergence (to the zeros ~~100/100~~) imply solvability by radicals?

- (2) (Topology) How many, and of what kind, are the domains of incomplete solution ~~of the form~~?
- (3) (Functional Analysis) How much light will be shed on the zeros of integral functions by the pursuit, into Hilbert space, of the general solution here given? It is clear that the necessity of $\Pi(i j k)$ will persist.

It has been my very great good fortune to have had as supervisor of my studies Professor W. G. BICKLEY. His patience, encouragement, wise counsel and sympathy, but above all, the inspiration of his life and achievements have meant to me more than I can say.

I am grateful for the privilege of having been a guest-worker at the National Physical Laboratory, Teddington, and for the understanding of my associates there when ill-health compelled me to suspend my activity. (I had hoped to programme some "computations by swathes" on DEUCE.) It was there that I was able to read, before publication, J. H. WILKINSON's paper on the ill-conditioning of $\prod_{n=1}^{\infty} (x + n) = 0$, and I hope that the treatment of Tetranomials herein may go some way towards explaining why a small change in a coefficient will thoroughly upset the pattern of the zeros.

I owe a great debt of gratitude to Miss B. E. LABY who computed the F and S tables for the cubic.

This has been typewritten on an IMPERIAL DUAL, by Miss A. V. SOUTHWELL. It would be impossible to pay too high a tribute to her skill or to her devotion in the performance - one of extraordinary difficulty.

Chapter II: Notation

Unmodified Roman capitals, also i and j , are natural numbers.

{Greek min^uiscules except ι , π , σ , ϕ , ω } $\subset \mathbb{C}$, the complex field,

{ k , k' , k_1 } $\subset \mathbb{Z}$, the domain of integers.

$\ln z$ is $\log_e |z|$; $-\pi < \text{ph } z \leq \pi$;

$$z_k^\nu \text{ is defined as } \exp \nu \{ \ln z + i(\text{ph } z + 2k\pi) \}, \quad (1)$$

$$\text{and } z^\nu \text{ as } z_0^\nu \quad (2)$$

$$\text{Then } z_k^\nu = z_1^\nu z_k^\nu = z^\nu \text{ cis } 2k\pi \quad (3)$$

$$\text{and } z_k^\mu z_k^\nu = z^{\mu+\nu} \quad (4)$$

$$\text{but } (\zeta\eta)_k^\nu = \zeta^\nu \eta^\nu z_k^\nu, \text{ where } k' \in \{-1, 0, 1\}. \quad (5)$$

A Generalized Polynomial Equation has the form $\sum_{i=1}^M \alpha_i z_{k_i}^{\nu_i} = 0.$

(6)

$$z^{(N)} = z(z-1)(z-2) \dots (z-N+1) \quad (\text{Def.}) \quad (7)$$

This is read "z faculty N".

$$\text{Then } M < N \Rightarrow z^{(M)} = 0. \quad (8)$$

$$z^{(M+N)} = z^{(M)} (z-M)^{(N)}, \text{ by simple counting,} \quad (9)$$

$$\text{and } z^{(N)} = z^{(N-M)} (z-N+M)^{(M)} \quad (10)$$

To interpret $z^{(0)}$ put $M = 0$ in (9): $z^{(N)} = z^{(0)} z^{(N)}$

$$\text{so } z^{(0)} = 1. \quad (\text{Def.}) \quad (11)$$

To interpret $z^{(-N)}$, put $M = -N$ in (9): $z^{(0)} = z^{(-N)} (z+N)^{(N)}$

$$\text{so } z^{(-N)} = \frac{1}{(z+1)(z+2)\dots(z+N)}, \quad z \notin \mathbb{Z}^-, \quad (\text{Def.}) \quad (12)$$

$$\text{Then } 0^{(-N)} = 1/N!; \quad z^{(-1)} = 1/(z+1). \quad (13)$$

$$\text{Now } M^{(N)} = \frac{M!}{(M-N)!} = \frac{\Gamma(M+1)}{\Gamma(M-N+1)}, \quad (14)$$

which leads to the further interpretation for all z :

$$z^{(\nu)} = \frac{\Gamma(z+1)}{\Gamma(z-\nu+1)} = \frac{z!}{(z-\nu)!} \quad (15)$$

$$\text{whence } 0^{(0)} = 1. \quad (16)$$

$$\text{Where } D = d/dz, \quad D^N z_k^\nu = \nu^{(N)} z^{\nu-N} \cdot 1_k^\nu, \quad (17)$$

whence the indefinite general derivative:

$$D^\mu z_k^\nu = \nu^{(\mu)} z^{\nu-\mu} \cdot 1_k^\nu, \quad (\text{Def.}) \quad (18)$$

$$\text{then } f(z) = \sum_{i=0}^N a_i z_0^i \Rightarrow D^\mu f(z) = \sum_{i=0}^N a_i i^{(\mu)} z^{i-\mu} \quad (19)$$

$$\text{For arbitrary fixed } \zeta, \text{ as } z \rightarrow \infty, \quad \Gamma(z + \zeta) \sim z^\zeta \Gamma(z) \quad (20)$$

[LOESCH-SCHOBLIK, Die Fakultät, p. 51.]

Where $(\alpha, k) = \alpha(\alpha+1) \dots (\alpha+k)$, and $\theta = 1 + n - m$,

$$\sum_{k=0}^{\infty} \frac{(\alpha_1, k) \dots (\alpha_m, k)}{(\beta_1, k) \dots (\beta_n, k)} \cdot \frac{x^k}{k!} \quad (21)$$

denoted by ${}_mF_n(\{\alpha_i\}; \{\beta_i\}; x)$,

is a hypergeometric function of order n , and class θ , called "complete" if $\theta = 0$.

LAGRANGE's inversion theorem: Under appropriate convergence conditions

If $z = y + x \phi(z)$

$$\text{then } f(z) = f(y) + \sum_{1}^{\infty} \frac{x^N}{N!} \frac{d^{N-1}}{dy^{N-1}} \left(f'(y) \{ \phi(y) \}^N \right) \quad (22)$$

$$\text{i.e. } \zeta = \phi(\zeta) \Rightarrow f(\zeta) = f(0) + \left(\sum_{1}^{\infty} D^{N-1} f'(\zeta) \{ \phi(\zeta) \}^{N/N!} \right)_{\zeta=0} \quad (23)$$

$$\text{or } w = 1 + \phi(w) \Rightarrow f(w) = f(0) + \left(\sum_{1}^{\infty} D^{N-1} f'(w) \{ \phi(w) \}^{N/N!} \right)_{w=1} \quad (24)$$

Chapter II: Trinomials

Let $ax^P + bx^Q + 1 = 0$, $\{a, b\} \subset \mathbb{C}$, P and Q are prime, $P > Q$.
Put $ax^P = -w$, $\sqrt[P]{(|a|)} = 1/a$, $\epsilon^P = \hat{a} = |a|/(-a)$, and $pP = 1$.

a is one-valued, ϵ is P -valued.

Then $x = a\epsilon w^{1/P}$, ~~$x = a\epsilon w^{1/P}$~~

and $w = 1 + bx^Q = \beta w_0^{QP}$, say $\phi(w)$, where $\beta = b(a\epsilon)^Q$;

also let $c = \beta^P = b^P(a\epsilon)^{PQ} = b^P/(-a)^Q$.

LAGRANGE's theorem gives, if convergence obtains:

$$\begin{aligned} x/a\epsilon &= 1 + p \sum_{N=1}^{\infty} (D^{N-1} w^{p-1} \{\beta w^{QP}\}^N / N!) \Big|_{w=1}, \quad D = d/dw, \\ &= 1 + p \sum_{N=1}^{\infty} (\epsilon^N / N!) D^{N-1} w^{NQp+p-1} \Big|_{w=1} \\ &= 1 + p \sum_{N=1}^{\infty} (\beta^N / N!) (NQp + p - 1)^{(N-1)} \\ x &= a\epsilon^p \sum_{N=0}^{\infty} \frac{\beta^N}{N!} \left(\frac{NQ + 1 - P}{P} \right)^{(N-1)}, \end{aligned}$$

therefore the value of $Px/a\epsilon$ is

$$\begin{aligned} P + \beta + \frac{\beta^2}{2!} \left(\frac{2Q + 1 - P}{P} \right) + \frac{\beta^3}{3!} \left(\frac{3Q + 1 - P}{P} \right) \left(\frac{3Q + 1 - 2P}{P} \right) \\ + \frac{\beta^4}{4!} \left(\frac{4Q + 1 - P}{P} \right) \left(\frac{4Q + 1 - 2P}{P} \right) \left(\frac{4Q + 1 - 3P}{P} \right) + \dots \end{aligned}$$

For $Q > 0$, whenever $P \nmid (NQ + 1)$, $\left(\frac{NQ + 1 - P}{P} \right)^{(N-1)} = 0$, by equation (8),
so every P^{th} term vanishes (except that, when $P = 2$, the first term to
vanish is the fourth; the second term, $\epsilon\beta$, contributes $b'(a\epsilon)^2/P$ or
 $-b/2a$, to the value of x). Also, $\beta^{N+P} = c\beta^N$; $\beta^{N+2P} = c^2\beta^N$, and so on,
so that the series falls into $(P-1)$ sub-series, each having a power of ϵ ,
OTHER THAN UNITY, as a nominal factor. (However, in the end, ϵ may
take unity itself as a value, if a is real.)

Thus Px/a is given by

$$\begin{aligned} & \epsilon \left\{ P + \frac{a}{P!} \left(\frac{PQ+1-P}{P} \right)^{(P-1)} + \frac{a^2}{(2P)!} \left(\frac{2PQ+1-P}{P} \right)^{(2P-1)} + \dots \right\} \\ & + \epsilon^{Q+1} b \left\{ 1 + \frac{a}{(P+1)!} \left(\frac{P+1Q+1-P}{P} \right)^{(P)} + \dots \right\} \\ & + \epsilon^{2Q+1} b^2 a^{2Q} \left\{ \frac{1}{2!} \left(\frac{2Q+1-P}{P} \right) + \dots \right\} \\ & + \epsilon^{3Q+1} b^3 a^{3Q} \left\{ \frac{1}{3!} \left(\frac{3Q+1-P}{P} \right)^{(2)} + \dots \right\} \\ & + \epsilon^{P-1Q+1} b^{P-1} a^{P-1Q} \left\{ \frac{1}{(P-1)!} \left(\frac{P-1Q+1-P}{P} \right)^{(P-2)} + \dots \right\} \end{aligned}$$

the row with vanishing coefficients also containing the only factor

ϵ^{NQ+1} which might equal unity, though if P were even, there would then

be a non-vanishing row containing a power of a which would equal -1 .

If convergent, this gives x as $\frac{a}{P} \{ F_1 \epsilon + F_2 \epsilon^2 + \dots + F_{P-1} \epsilon^{P-1} \}$

and by choosing as ϵ the various P values of $a^{1/P}$ in turn, the P roots

of the equation may be found SIMULTANEOUSLY, and to as great a degree

of accuracy as required.

If Q is replaced by $-Q$, none of the coefficients vanishes,

and x is of the form $\frac{a}{P} (F_1 \epsilon + F_2 \epsilon^2 + \dots + F_P \epsilon^P)$.

$\{F_i\}$ are complete hypergeometric functions of order $P-1$. No general tables for them exist.

Considering now the sub-series constituting any one row of this expression, the following treatment of the D'ALEMBERT ratio is didactically simpler than that using asymptotic gamma functions:

$$\begin{aligned} \frac{u_{N+P}}{u_N} &= c \frac{N!}{(N+P)!} \left(\frac{N+P+1}{P} \right)^{(N+P-1)} / \left(\frac{NQ+1}{P} \right)^{(N-1)} \\ &= \frac{c}{(N+P)^{(P)}} \left(N \frac{Q}{P} + Q + \frac{1}{P} - 1 \right)^{(N+P-1)} / \left(N \frac{Q}{P} + \frac{1}{P} - 1 \right)^{(N-1)} \end{aligned}$$

and by several applications of equation (10), this

$$\begin{aligned} &= c \frac{\left(N \frac{Q}{P} + Q + \frac{1}{P} - 1 \right)^{(Q)} \left(N \frac{Q}{P} + \frac{1}{P} - 1 \right)^{(N-1)} \left(N \frac{Q}{P} - N + \frac{1}{P} \right)^{(P-Q)}}{(N+P)^{(Q)} \left(N \frac{Q}{P} + \frac{1}{P} - 1 \right)^{(N-1)} (N+P-Q)^{(P-Q)}} \\ &= c \frac{\left(N \frac{Q}{P} + Q + \frac{1}{P} - 1 \right)^{(Q)} \left(N \frac{Q}{P} - N + \frac{1}{P} \right)^{(P-Q)}}{(N+P)^{(Q)} (N+P-Q)^{(P-Q)}} \end{aligned}$$

then $\lim_{N \rightarrow \infty} \frac{u_{N+P}}{u_N} = c \left(\frac{Q}{P} \right)^Q \left(\frac{Q}{P} - 1 \right)^{P-Q} = C$, say,

those containing the terms in each factor, other than N , being negligible.

$$\begin{aligned} C &= \frac{b^P}{(-a)^Q} \frac{Q^Q}{P^Q} \frac{(Q-P)^{P-Q}}{P^{P-Q}} = \frac{(-b)^P Q^Q (P-Q)^{P-Q}}{a^Q P^P} \\ &= \left\{ \frac{-b(P-Q)}{P} \right\}^P / \left\{ \frac{a(P-Q)}{Q} \right\}^Q. \quad \text{Convergence follows if } |C| < 1. \end{aligned}$$

The critical case of $|C| = 1$ is "at its worst" for $C = 1$, but that is happily a case for a solution, at least in part, by radicals, for $C = +1$ is satisfied by

$$\frac{a(P-Q)}{Q} = \lambda^P, \quad \frac{-b(P-Q)}{P} = \lambda^Q,$$

reducing the original equation to

$$Q \lambda^P x^P - P \lambda^Q x^Q + P - Q = 0, \quad \text{say } g(x) = 0,$$

which is clearly satisfied by $x\lambda = 1$. Then $g'(x) = 0$, is

$$PQ \lambda^P x^{P-1} - PQ \lambda^Q x^{Q-1} = 0,$$

$$\text{or } x^{Q-1} (\lambda^{P-Q} x^{P-Q} - 1) = 0,$$

which is also satisfied by $x\lambda = 1$,

so, under this condition, two equal roots $\left\{ \frac{Q}{a(P-Q)} \right\}^{1/P}$ are at once obtained, and the degree of the equation is reduced by 2.

Conjecture: all P roots of $ax^P + bx^Q + 1 = 0$ are expressible in terms of radicals if $(-b)^{P/Q}(P-Q)^{P-Q} = a^Q P^P$. By WEIERSTRASS's theorem, the equation is irreducible in the "rational" field, but since $C = 1$, at least one root is representible by radicals, so the GALOIS group of the equation is soluble. [van der WAERDEN, B.L. Modern Algebra (trans. F. BLUM). vol. I, p.74, §24, Irreducibility criteria, p. 173, Fundamental Theorem on Equations Soluble by Radicals.]

Regarding the series for x as the sum of power-series each in $c (= \beta^P)$, the radius of convergence of each sub-series, and of the whole series, is $(\frac{P}{P-Q})^P (\frac{P-Q}{Q})^Q$, say R . Reversion of $g(\frac{x}{R}) = 0$ to $x = g^{-1}(\frac{c}{R}) = G(c)$ is impossible at $c = 0$, for $\frac{d^2 G}{dc^2} = 0$ there.

[COPSON, E.T. Theory of Functions of a Complex Variable, p. 121, §6.22, Inverse Functions] However, $c = R$ is the only singularity of $G(c)$, and so the series converges at every other point ON the circle $|c| = R$. [TITCHMARSH, E.C. The Theory of Functions, 2nd Edn. p. 218, §7.31.]

If $|C| > 1$, write the equation as $bx^Q + 1 = -ax^P$,

$$\text{putting } bx^Q + 1 = -\eta, \text{ i.e. } x = \left(\frac{1 + \eta}{-b} \right)^{1/Q} = \beta^* \epsilon^* (1 + \eta)^{1/Q}$$

$$\text{where } \beta^* = Q/|b|, \quad \delta = |b|/(-b)$$

and ϵ^* is some specific value of $\delta^{1/Q}$

$$\text{then } \eta = ax^P = a\{\beta^* \epsilon^* (1 + \eta)^{1/Q}\}^P$$

$$= \alpha^* (1 + \eta)^{P/Q}, \text{ where } \alpha^* = a(\beta^* \epsilon^*)^P$$

$$\text{and let } \alpha^* = (\alpha^*)^Q = a^Q/(-b)^P.$$

Then $\eta = \alpha^* (1 + \eta)^{P/Q}$ and $x = \beta^* \epsilon^* (1 + \eta)^{1/Q}$ give the expansion

$$x = \frac{\beta^* \epsilon^*}{Q} \sum_{N=0}^{\infty} \frac{(\alpha^*)^N}{N!} \left(\frac{NP + 1 - Q}{Q} \right)^{(N-1)}, \text{ similarly to the above,}$$

and this converges if $\left| \frac{a^Q}{(-b)^P} \left(\frac{P}{Q} - 1 \right)^{Q-P} \right| < 1$

$$\text{i.e. } \left| \frac{a^Q}{(-b)^P} \frac{P^P}{Q^P} \frac{(P-Q)^{Q-P}}{Q^{Q-P}} \right| < 1 \quad \text{or} \quad \left| \frac{a^Q}{(-b)^P} \frac{P^P}{Q^Q (P-Q)^{P-Q}} \right| < 1$$

i.e. $1/|C| < 1$, which is satisfied, with the critical case as before,

but now none of the faculty coefficients vanishes.

$$\text{Then } x = \frac{\beta^*}{Q} \left\{ G_0 + G_1 \varepsilon^* + G_2 (\varepsilon^*)^2 + \dots + G_{Q-1} (\varepsilon^*)^{Q-1} \right\}$$

and, giving ε^* its various Q values as Q^{th} roots of $\bar{S}$,

Q roots of the original equation are obtained.

To find the remaining $P-Q$ roots, write the equation as

$$ax^{P-Q} + b + x^{-Q} = 0,$$

$$\text{or } \frac{a}{b} x^{P-Q} + \frac{1}{b} x^{-Q} + 1 = 0,$$

and put $\frac{a}{b} x^{P-Q} + 1 = -\xi$, with $x = \left(\frac{1+\xi}{-a/b} \right)^r$ where $1/r = P-Q$

i.e. $x = \bar{a} \bar{\varepsilon} (1+\xi)^r$, where $\bar{a} = \left| \frac{a}{b} \right|^{1/Q}$ and $\bar{\varepsilon}$ is some value of $\left(\left| \frac{b}{a} \right| (-a/b) \right)^{1/Q}$.

$$\text{Then } \xi = \frac{1}{b} x^{-Q} = \frac{1}{b} \{ \bar{a} \bar{\varepsilon} (1+\xi)^r \}^{-Q} = \bar{\beta} (1+\xi)^{-Qr}$$

$$\text{where } 1/\bar{\beta} = b(\bar{a} \bar{\varepsilon})^Q \quad \text{and} \quad \bar{a} = \bar{\beta}^{P-Q} = \left\{ \frac{1}{b(\bar{a} \bar{\varepsilon})^Q} \right\}^{P-Q}$$

$$\frac{a^Q}{b^P} = b^{Q-P} (-b/a)^{-Q} = (-1)^{Q-Q} a^Q/b^P = (-a)^Q/b^P.$$

So $x = \bar{a} \bar{\varepsilon} (1+\xi)^{Pr}$, $\xi = \bar{\beta} (1+\xi)^{-Qr}$ give

$$\begin{aligned} x &= \bar{a} \bar{\varepsilon} r \sum_{N=0}^{\infty} \frac{\bar{\beta}^N}{N!} (-NQr + r - 1)^{(N-1)} \\ &= \bar{a} \bar{\varepsilon} r \sum_{N=0}^{\infty} \frac{\bar{\beta}^N}{N!} \left(\frac{-NQ + 1 - P - Q}{P - Q} \right)^{(N-1)} \end{aligned}$$

and none of the faculty coefficient vanishes.

$$\frac{U_{N+P-Q}}{U_N} = \bar{a} \left(\frac{-Q}{P-Q} \right)^{-Q} \left(\frac{-P}{P-Q} \right)^P = \frac{(-a)^Q}{b^P} \left(\frac{P-Q}{-Q} \right)^Q \left(\frac{-P}{P-Q} \right)^P = 1/C,$$

whence convergence and simultaneous production of $P-Q$ roots of the form

$$\frac{\bar{a}}{P-Q} \left\{ H_0 + H_1 \bar{\varepsilon} + H_2 \bar{\varepsilon}^2 + \dots + H_{P-Q-1} \bar{\varepsilon}^{P-Q-1} \right\}$$

on the substitution for $\bar{\varepsilon}$ of its various $(P-Q)$ values.

The simplest trinomial is the quadratic, say $ax^2 + bx + 1 = 0$, and the condition $C = 1$, as above, is that $b^2 = 4a$, and the roots are equal. If a and b are not both real, the inequality of b^2 and $4a$ is not directed, but the conditions $|C| \leq 1$ or $|C| \geq 1$ indicate the pattern of the roots: there is either a 2-let (twin-roots, as it were) of the form $-b/2a \pm \epsilon$ where $\epsilon^2 = |a|/(-a)$ or else two separate single roots. It cannot be too hastily added that the calculation of the roots in this way, though valid, is not a "practical proposition".

For a trinomial cubic, say $ax^3 + bx + 1 = 0$, the condition $C = 1$ is that $27a = -4b^3$, and two roots are equal. Again, if a and b are not both real, no directed inequality can hold between $27a$ and $-4b^3$ and the criterion for CARDAN's "irreducible" case fails. Instead, $|C| \leq 1$ gives a 3-let of roots of the form $F_1\epsilon + F_2\epsilon^2$, where $\epsilon^3 = |a|/(-a)$, and $|C| \geq 1$ gives a 1-let and a 2-let, found by expanding from the first terms in $bx + ax^3 + 1 = 0$ and $(a/b)x^2 + (1/b)x^{-1} + 1 = 0$ respectively.

As a practical proposition one is in duty bound to solve WALLIS's equation: $x^3 - 2x - 5 = 0$. [WHITTAKER, E.T. and ROBINSON, G. The Calculus of Observations, p. 86, §44, note this traditional duty, but in the same chapter, in commenting on the 101 decimal places achieved by HORNER's method, make no reference to the complex roots.]

The general cubic readily reduces to the trinomial form $ax^3 + bx + 1 = 0$, giving, when $\left| \frac{-b^3 \cdot 4}{a \cdot 27} \right| \leq 1$, and where $\alpha = \sqrt[3]{|1/a|}$, $\epsilon^3 = |a|/(-a)$, $c = -b^3/a$

$$\frac{3x}{\alpha} = \epsilon \left\{ 3 + \frac{c}{3!} \left(\frac{1}{3}\right)^{(2)} + \frac{c^2}{6!} \left(\frac{4}{3}\right)^{(5)} + \frac{c^3}{9!} \left(\frac{7}{3}\right)^{(8)} + \dots \right\} \\ + \epsilon^2 b \alpha \left\{ 1 + \frac{c}{4!} \left(\frac{2}{3}\right)^{(3)} + \frac{c^2}{7!} \left(\frac{5}{3}\right)^{(6)} + \frac{c^3}{10!} \left(\frac{8}{3}\right)^{(9)} + \dots \right\}$$

or $x = \frac{\epsilon}{3} \alpha \{ 3 - \sum F_N c^N \} + \frac{\epsilon^2}{3} b \alpha^2 \{ 1 + \sum S_N c^N \}$, say.

$1 - \sum F_N c^N / 3 = {}_2F_1(-\frac{1}{6}, \frac{2}{6}; \frac{2}{3}, \frac{4c}{9})$ and $1 + \sum S_N c^N = {}_2F_1(\frac{1}{6}, \frac{4}{6}; \frac{4}{3}, \frac{4c}{9})$, but no general tables exist.

To use this numerically, when a is rational, the cube roots of primes (and their squares) calculated by DRACH, S. M. (Mess. Math. V. 7, 1877, pp. 86-88, as corrected later by R. LIENARD) to 30 decimal places, were used, but the coefficients $\frac{1}{3} F_N$ and $\frac{1}{3} S_N$ were computed to 45 places, and are appended. To solve $ax^3 + bx + 1 = 0$ when $|4b^3/27a| > 1$, take

$$(a/b)x^3 + (1/b)x^{-1} + 1 = 0,$$

write $\bar{a} = \sqrt[3]{(|b/a|)}$, $\bar{e}^3 = |b/a|(-a/b)$, $\bar{c} = -a/b^3$

$$\begin{aligned} \text{then } x &= \frac{\bar{a}\bar{e}}{2} \left\{ 2 + \frac{\bar{c}}{2!} \left(-\frac{3}{2}\right) + \frac{\bar{c}^2}{4!} \left(-\frac{5}{2}\right)(3) + \dots \right\} + \frac{\bar{e}^2 \bar{c}^2 \bar{a}^2}{2} \left\{ 1 + \frac{\bar{c}}{2!} (-2)(2) + \frac{\bar{c}^2}{5!} (-3)(4) + \dots \right\} \\ &= \bar{a}\bar{e} {}_3F_2\left(-\frac{1}{6}, \frac{1}{6}, \frac{3}{6}; \frac{1}{2}, \frac{1}{2}; 3\bar{c}^2/2^2\right) + (\bar{e}^2 \bar{b}^3 \bar{a}^2/2) {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; 2\bar{c}\right) \end{aligned}$$

The third root is $-(p_1 + p_2)$, but if we were to solve $bx + ax^3 + 1 = 0$ (with $a^* = \sqrt[3]{(|1/b|)}$ and $e^* = |b|/(-b)$, $c^* = -a/b^3 = c$, above)

$$x = a^* e^* \left(1 + c^* + \frac{c^{*2}}{2!} 6 + \frac{c^{*3}}{3!} 9(a) + \frac{c^{*4}}{4!} 12(a^2) + \dots \right), \text{ which agrees.}$$

In WALLIS' equation, $a = -1/5$, $b = 2/5$, $c = \sqrt[3]{5}$, $e^3 = 1$,

$c = (2/5)^3 \cdot 5 = 0.32$; the tables of F_N and S_N were used to construct the sums $\sum F_N c^N$, $\sum S_N c^N$, giving

$$\begin{aligned} x &= e \sqrt[3]{5} \left\{ 1 - E(0.32) F_N/3 \right\} + e^2 (0.4) \sqrt[3]{25} \left\{ 1/3 - E(0.32) S_N/3 \right\} \\ &= e \sqrt[3]{5} (1 - E) + e^2 (0.4) \sqrt[3]{25} (1 + S), \text{ say,} \\ &= eU + e^2 V, \text{ say} \end{aligned}$$

so $x = U + V$, $-\frac{1}{2}\{(U+V) \pm \sqrt[3]{5}(U-V)\}$, just as in CARDAN's form

[SMITH, C. Treatise on Algebra, Ch. XLII, Theory of Equations §464]

$$\begin{aligned} \text{where } \{U^3, V^3\} &= -\frac{1}{2a} \pm \sqrt{\left(\frac{1}{2a}\right)^2 + \left(\frac{b}{3a}\right)^3} \\ &= -\frac{1}{2a} \left(1 \pm \sqrt{1 + \frac{4b^3}{27a}} \right). \end{aligned}$$

$$\sqrt{3} = 1.73205 \quad 08075 \quad 68877 \quad 29352 \quad 74463 \quad 41505 \quad 872$$

(Wash. Acad. Sci. Vol. 12, No. 19, p.435) together with DRACH's

$$\sqrt[3]{3} = 1.70997 \quad 59466 \quad 76696 \quad 98935 \quad 31088 \quad 72543 \quad 860$$

$$\sqrt[3]{25} = 2.92401 \quad 77382 \quad 12866 \quad 06550 \quad 67873 \quad 60137 \quad 923$$

and the calculated coefficients as appended, yield

$$U = 1.70311 \quad 09415 \quad 19889 \quad 61889 \quad 66362 \quad 20208 \quad 409$$

$$V = 0.39144 \quad 05400 \quad 22436 \quad 97258 \quad 57503 \quad 20370 \quad 897$$

$$\text{so } x = 2.09455 \quad 14815 \quad 42326 \quad 59148 \quad 23865 \quad 40579 \quad 306$$

$$\text{or } -1 \quad 04727 \quad 57407 \quad 71163 \quad 29574 \quad 11932 \quad 70289 \quad 653$$

$$\pm i \quad 1.13593 \quad 98890 \quad 88928 \quad 18624 \quad 54926 \quad 29029 \quad 439$$

Some 10 hours' work, by an expert, on an electric machine; the F and S tables, taking 7 hours, and the complex roots, at the end, 7 minutes.

The following examination question is of some interest:

$$x^5 + x - 1 = 0 \quad \text{here } a = -1, b = -1, c = 1, e^5 = 1, c = -1,$$

$$\begin{aligned} \frac{5x}{e} = \{ & 5 & - \frac{1}{5!} \left(\frac{1}{5}\right)^{(4)} & \dots\dots & \} \\ & -4e \{ 1 & - \frac{1}{6!} \left(\frac{2}{5}\right)^{(5)} & \dots\dots & \} \\ & + e^2 \{ \frac{1}{2!} \left(\frac{-2}{5}\right) & - \frac{1}{7!} \left(\frac{3}{5}\right)^{(6)} & \dots\dots & \} \\ & - e^3 \{ \frac{1}{3!} \left(\frac{-1}{5}\right)^{(2)} & - \frac{1}{8!} \left(\frac{4}{5}\right)^{(7)} & \dots\dots & \} \end{aligned}$$

$$\begin{aligned} \text{i.e. } x = e \{ & 1 & - 3.7/5^6 & \dots\dots & \} \\ & -e^2 \{ 1/5 & - 6.13/5^6 & \dots\dots & \} \\ & +e^3 \{ -1/5^2 & + 11.17/5^8 & \dots\dots & \} \\ & -e^4 \{ 1/5^3 & - 11.26/5^{10} & \dots\dots & \} \end{aligned}$$

$$e = 0.2e^2 - 0.04e^3 - 0.008e^4,$$

giving all the roots correct to 2 decimal places after the mental

calculation of only four coefficients (namely those to the left of the dotted vertical line), which are capable of being written down on sight.

$$\begin{aligned}
 x &= \varepsilon \quad (1.00130 \quad 72491 \quad 47654) \\
 &\quad -\varepsilon^2 \quad (\cdot 19903 \quad 27905 \quad 80582) \\
 &\quad -\varepsilon^3 \quad (\cdot 03953 \quad 77984 \quad 63965) \\
 &\quad -\varepsilon^4 \quad (\cdot 00785 \quad 90428 \quad 51922)
 \end{aligned}$$

correct to 16 significant figures.

Some 8 hours' work on electric machines.

F ₁	•01234	56790	12345	67901	23456	79012	34567	90123
F ₂	60	96631	61103	49032	15973	17482	09114	46426
F ₃	4	82963	61527	74563	24722	15479	54837	25694
F ₄		46073	97003	20805	69698	52206	24198	84142
F ₅		4859	30054	30659	93065	20567	78354	65725
F ₆		545	80378	89088	89078	25113	39907	18009
F ₇		64	01402	46251	16620	28871	32458	24952
F ₈		7	74834	27552	14062	19796	57330	59469
F ₉			96067	34976	65561	15677	91062	13466
F ₁₀			12138	26709	69407	64211	66618	66393
F ₁₁			1557	30152	35903	94061	66789	10889
F ₁₂			202	33018	03136	25565	91922	62967
F ₁₃			26	56677	08144	09674	69734	63029
F ₁₄			3	51983	71449	38348	95116	90374
F ₁₅				46997	00250	76130	46789	08453
F ₁₆				6317	49830	13582	81243	58130
F ₁₇				854	26157	13238	28589	36444
F ₁₈				116	12138	03502	53584	68158
F ₁₉				15	85849	63186	03043	39207
F ₂₀				2	17486	05330	01337	19171
F ₂₁					29939	33854	66902	86311
F ₂₂					4135	62476	68742	51757
F ₂₃					573	05387	54516	00955
F ₂₄					79	63233	44574	98017
F ₂₅					11	09485	40460	70361
F ₂₆					1	54953	84518	90095
F ₂₇						21689	64145	33541
F ₂₈						3042	29241	26897
F ₂₉						427	54799	96251
F ₃₀						60	19319	13469
F ₃₁						8	48862	30486
F ₃₂						1	19896	97435
F ₃₃							16959	70471
F ₃₄							2402	31083
F ₃₅							340	72661
F ₃₆							48	38569
F ₃₇							6	87911
F ₃₈								97910
F ₃₉								13950
F ₄₀								1990
F ₄₁								284
F ₄₂								41
F ₄₃								6
F ₄₄								1

17
SN/3

S ₁	•01234	56790	12345	67901	23456	79012	34567	90123
S ₂	•00076	20789	51379	36290	19966	46852	61393	08032
S ₃	6	52314	49336	17591	91832	52076	27312	65873
S ₄		64735	86377	61669	85324	30999	02235	68171
S ₅		6993	07170	42155	69402	31743	72155	08907
S ₆		798	21286	29243	24252	76614	45554	63113
S ₇		94	70539	01818	18173	32803	38257	24339
S ₈		11	56341	12209	65206	59523	77103	26095
S ₉		1	44344	36503	53681	48171	00571	18895
S ₁₀			18337	65529	52140	33957	20861	09489
S ₁₁			.2363	18103	49414	26968	01156	72128
S ₁₂			308	17815	18706	28753	19692	60991
S ₁₃			40	59288	66708	98583	16088	66571
S ₁₄			5	39273	86049	39933	17985	77533
S ₁₅				72173	34901	58359	28046	76328
S ₁₆				9721	76284	14188	16586	54593
S ₁₇				1316	98088	88017	78125	37701
S ₁₈				179	30910	76685	09958	50937
S ₁₉				24	52337	59765	31666	03350
S ₂₀				3	36755	93199	91243	75708
S ₂₁					46412	82704	76453	38524
S ₂₂					6418	04256	96150	94050
S ₂₃					890	18904	99133	33496
S ₂₄					123	81300	71254	67730
S ₂₅					17	26463	37421	21751
S ₂₆					2	41306	81058	29632
S ₂₇						33800	76885	50082
S ₂₈						4744	17057	61947
S ₂₉						667	12890	54763
S ₃₀						93	97668	97553
S ₃₁						13	25995	27001
S ₃₂						1	87383	11356
S ₃₃							26518	25073
S ₃₄							3757	92543
S ₃₅							533	21999
S ₃₆							75	75105
S ₃₇							10	77374
S ₃₈							1	53396
S ₃₉								21863
S ₄₀								3119
S ₄₁								445
S ₄₂								64
S ₄₃								9
S ₄₄								1

$$\frac{18}{(0.32)^N}$$

1	0.32								
2	1024								
3	03276	8							
4	01048	576							
5	335	54432							
6	107	37418	24						
7	34	35973	8368						
8	10	99511	62777	6					
9	3	51843	72088	832					
10	1	12589	99068	42624					
11		36028	79701	89639	68				
12		11529	21504	60684	6976				
13		3689	34881	47419	10323	2			
14		1180	59162	07174	11303	424			
15		377	78931	86295	71617	9568			
16		120	89258	19614	62917	47062			
17		38	68562	62276	68133	59060			
18		12	37940	03928	53802	7490			
19		3	96140	81257	13216	880			
20		1	26765	06002	28229	40			
21			40564	81920	73033	4			
22			12980	74214	63371				
23			4153	83748	6828				
24			1329	22799	578				

$$\frac{19}{(0.32)^{N_F} N/3}$$

1	00395	06172	83950	61728	39506	17283	95062
2	6	24295	07696	99740	89315	65310	16613
3		15825	75174	54116	88848	49556	83384
4		483	12059	20035	89154	52197	47012
5		16	30510	69639	87093	58189	15483
6			58605	23558	49141	52869	50513
7			2199	50513	80017	16854	60859
8			85	19392	95535	17902	38611
9			3	38006	93797	74478	34735
10				13666	47379	36764	94488
11				561	07700	49076	16351
12				23	32708	15914	55984
13					98014	08441	56623
14					4155	49023	96041
15					177	54965	55498
16					7	63738	63119
17						33047	64385
18						1437	51306
19						62	82198
20						2	75696
21							12145
22							537
23							24
24							1

*|- F

99598 53206 29714 49013 97446 83381 28333

20

$$(0.32)^{N_{S_N}/3}$$

1	•00395	06172	83950	61728	39506	17283	95062
2	7	80368	84621	24676	11644	56637	70767
3		21375	04131	84781	25197	96804	03532
4		678	80473	09495	80728	03423	67231
5		23	46485	48970	22543	68513	41089
6			85707	45353	76625	89713	25571
7			3254	05242	86866	28386	09892
8			127	14105	09420	67179	21390
9			5	07866	58483	30584	71702
10				20646	36438	85936	31538
11				851	42569	82696	98686
12				35	53052	18541	64269
13				1	49761	31832	62324
14					6366	62200	97114
15					272	66320	34791
16					11	75289	01112
17						50948	23041
18						2219	73924
19						97	14710
20						4	26889
21							18827
22							833
23							37
24							1

1 + S

1•00403 08619 88756 23754 99490 34613 54332

Chapter III: Tetranomials

The Tetranomial Case, $ax^P + bx^Q + cx^R + 1 = 0$,

$0 \notin \{a, b, c\}$, $\{a, b, c\} \subset \mathbb{C}$, $\{P, Q, R\} \subset \mathbb{N}$, $(P, Q, R) = 1$ and $P > Q > R$.

Let $ax^P + 1 = -\zeta$, $P\sqrt[P]{|a|} = 1/\alpha$, and $\varepsilon^P = |a|/(-a)$, also put $Pp = 1$,

then $x = \alpha\varepsilon(1 + \zeta)_0^P$, say $f(\zeta)$

and $\zeta = bx^Q + cx^R$

$= \beta\varepsilon^Q(1 + \zeta)^{Qp} + \gamma\varepsilon^R(1 + \zeta)^{Rp}$, say $\phi(\zeta)$

where $\beta = b\alpha^Q$, $\gamma = c\alpha^R$.

Then $\{\phi(\zeta)\}^N = \sum_{L+M=N} \frac{N!}{L! M!} \beta^L \gamma^M \varepsilon^{LQ+MR} (1 + \zeta)^{LQp+MRp}$

since $f(\zeta) = f(0) + \left[\sum_1^\infty D^{N-1} f'(\zeta) \{\phi(\zeta)\}^N / N! \right]_{\zeta=0}$ $D \equiv d/d\zeta$

$x/\alpha\varepsilon = 1 + \sum_{N=1} \left[D^{N-1} \left\{ p(1 + \zeta)^{P-1} \sum_{L+M=N} \frac{1}{L! M!} \beta^L \gamma^M \varepsilon^{LQ+MR} (1 + \zeta)^{LQp+MRp} \right\} \right]_{\zeta=0}$

$x = \alpha\varepsilon + p \sum_{N=1} \sum_{L+M=N} \frac{\alpha\beta^L \gamma^M}{L! M!} \varepsilon^{LQ+MR+1} (LQp + MRp + p - 1)^{(N-1)}$.

Now, quite generally, with $N = \Sigma L$, $1 + \Sigma LQ < 1 + Q\Sigma L < 1 + NQ \leq 1 + N(P-1) < NP$,

so that if $LQ + MR + 1 = KP$, and $K > 0$,

then $K < N$, so $K - 1 < N - 1$, and $(K - 1)^{(N-1)} \equiv 0$,

i.e. $P | (LQ + MR + 1) \Rightarrow (LQp + MRp + p - 1)^{(L+M-1)} \equiv 0$,

and every P^{th} term or coefficient of ε^{KP} vanishes.

Thus the non-vanishing terms of the double series for x fall into $P-1$ lots, having the respective common factors $\{\varepsilon^K\}_1^{P-1}$, and, if there is convergence, $x = \varepsilon F_1 + \varepsilon^2 F_2 + \dots + \varepsilon^{P-1} F_{P-1}$; and, by giving ε , in turn, each of its P values, P different values are found.

pas	1	$\beta \varepsilon^Q$	$(\beta \varepsilon^Q)^2/2!$	$(\beta \varepsilon^Q)^3/3!$	...	$(\beta \varepsilon^Q)^L/L!$
1	P	<u>$(Qp + p - 1)^{(0)}$</u>	$(2Qp + p - 1)^{(1)}$	$(3Qp + p - 1)^{(2)}$	...	$(LQp + p - 1)^{(L-1)}$
$y \varepsilon^R$	<u>$(Rp + p - 1)^{(0)}$</u>	$(Qp + Rp + p - 1)^{(1)}$	$(2Qp + Rp + p - 1)^{(2)}$	$(3Qp + Rp + p - 1)^{(3)}$	...	$(LQp + Rp + p - 1)^{(L)}$
$(y \varepsilon^R)^2/2$	$(2Rp + p - 1)^{(1)}$	$(Qp + 2Rp + p - 1)^{(2)}$	...	...	...	$(LQp + 2Rp + p - 1)^{(L+1)}$
...	...	...	...	...	...	...
$(y \varepsilon^R)^M/M!$	<u>$(MRp + p - 1)^{(M-1)}$</u>	...	...	...	...	$(LQp + MRp + p - 1)^{(LM-1)}$

All the terms of the double-series may be arranged at the lattice-points of the complete quarter-plane, as set out above: At the point (L, M) is named the coefficient of the factors named at $(L, -1)$, $(-1, -1)$, and $(-1, M)$. (Each of the underlined terms is unity.)

This double series is the sum of P^2 double-series, each consisting of every P -th term along each P -th row and each P -th column beginning from some element in the first $P \times P$ square of the array. Each converges as will be shown, by HORN's lemma [Mathematische Annalen, t.34, 1889, p. 544, "On the convergence of hypergeometric series of two and three variables"] namely, that convergence follows from the D'ALEMBERT test being satisfied by each row and each column. Therefore the double-series, and every similarly-constructed and tested multiple-series, converges.

$$\begin{aligned} \frac{u_{N+P}}{u_N} &= (\beta \epsilon^Q)^P \frac{(\overline{L+P} \ Qp + MRp + p - 1)^{(L+P+M+1)}}{(L+P)!} \bigg/ \frac{(LQp + MRp + p - 1)^{(L+M-1)}}{L!} \\ &= \frac{b^P}{(-a)^Q} \cdot \frac{(LQp + MRp + p - 1 + Q)^{(L+M-1)}}{(L+P)^{(P)} (LQp + MRp + p - 1)^{(L+M-1)}} \quad \text{which by applying equation (10)} \\ &= \frac{b^P}{(-a)^Q} \frac{(LQp + MRp + p - 1 + Q)^{(Q)} (LQp + MRp + p - 1)^{(L+M-1)} (LQp + MRp + p - L - M)^{(P-Q)}}{(L+P)^{(Q)} (LQp + MRp + p - 1)^{(L+M-1)} (L+P-Q)^{(P-Q)}} \end{aligned}$$

This, for ANY fixed M , as $L \rightarrow \infty$, tends to $\frac{b^P}{(-a)^Q} (Qp)^Q (Qp - 1)^{P-Q}$

i.e. $\frac{(-b)^P Q^Q (P-Q)^{P-Q}}{a^Q P^P}$

or $(-1)^P \left\{ \frac{(bQ)^{1/Q}}{(aP)^{1/P}} \left(\frac{1}{Q} - \frac{1}{P} \right)^{1/Q - 1/P} \right\}^{PQ}$, say C_{PQ} .

So, convergence of EVERY sub-series in EVERY row is ensured by

$$\left| \frac{(aP)^{1/P}}{(bQ)^{1/Q}} \right| > \left(\frac{1}{Q} - \frac{1}{P} \right)^{1/Q - 1/P}.$$

Similarly the convergence of any one sub-series of P -th terms in one column, and hence of every such sub-series in every column is

$$\left| \frac{(aP)^{1/P}}{(cR)^{1/R}} \right| > \left(\frac{1}{R} - \frac{1}{P} \right)^{1/R - 1/P}, \text{ or } C_{PR} < 1;$$

where $C_{PR} = \frac{(-c)^P R^R (P-R)^{P-R}}{a^R P^P}$.

Similarly $C_{QR} = \frac{(-c)^Q R^R (Q-R)^{Q-R}}{b^R Q^Q}$.

Regarding any row of terms in the double series as a power-series in β , there will be convergence for $|C_{PQ}| \leq 1$ except at $C_{PQ} = 1$, as shown in the trinomial case. Call the condition (PQ), and call by (PR) the totality of conditions for the convergence of terms in any column, regarded as a power-series in γ ; i.e. (PR) means that $|C_{PR}| \leq 1$ except for $C_{PR} = 1$, or it may be said that $|C_{PR}| \leq 1$ a.e. Let (QP) mean $|C_{PQ}| \geq 1$ a.e. (i.e. except for $C_{PQ} = 1$), and (RP) mean that $|C_{PR}| \geq 1$ a.e. (except for $C_{PR} = 1$), and so on. Then (PQ).(PR) ensure a P-set of roots. (Hereafter, the . means the logical "and".)

The putting of $ax^P + 1 = -\zeta$, and proceeding as above, is to "solve" with ax^P as PIVOT".

NO CONVERGENCE CONDITION IS MODIFIED BY SUBSTITUTION OF A MULTIPLE OF THE VARIABLE IN THE ORIGINAL EQUATION: e.g. C_{PQ} for a $(\lambda x)^P + b(\lambda x)^Q + 1 = 0$ contains $(-b\lambda^Q)^P / (a\lambda^P)^Q$, which is still $(-b)^P / a^Q$. Moreover, if $wx = 1$, the original equation becomes $aw^{-P} + bw^{-Q} + cw^{-R} + 1 = 0$, but the condition for convergence of P roots is not, as might be hoped,

$$\left| \frac{b^{-P}(-Q)^{-Q}(-P+Q)^{-P+Q}}{a^{-Q}(-P)^{-P}} \right| < 1, \text{ i.e. (QP) in effect, but (PQ) as before!}$$

That is, divergence for an x-solution implies divergence in solving for $1/x$.

Write $\sqrt[P]{(|aP|)} = A, \quad \sqrt[Q]{(|bQ|)} = B, \quad \sqrt[R]{(|cR|)} = C,$

and $Pp = Qq = Rr = 1,$

so that (PQ), or $|C_{PQ}| \leq 1$ is $A/B \geq (\frac{1}{Q} - \frac{1}{P})^{\frac{1}{Q} - \frac{1}{P}} = (q-p)^{q-p} = n$, say,

and (PR), or $|C_{PR}| \leq 1$ is $A/C \geq (r-p)^{r-p} = m$, say,

and (QR), or $|C_{QR}| \leq 1$ is $B/C \geq (r-q)^{r-q} = l$, say.

(In the above, $\geq$ is to be interpreted as " $\geq$ a.e.", excluding the case

$C_{JK} = 1$, and that case only, for all J, K.)

It will be noted that the discussion of the various possibilities arising from $f_4 = ax^P + bx^Q + cx^R + 1 = 0$ appears to involve the space $\mathbb{C}^3$, since $\{a, b, c\} \subset \mathbb{C}$, but the convergence conditions obtained reduce, as will be seen, first to an octant in $\mathbb{R}^3$, then to a plane equilateral triangle.

Lemma:

If $0 < x < y$, then $xy < x + y$, so $\log x + \log y < \log (x + y)$

and $x \log x + y \log y < (x + y)(\log x + \log y) < (x + y) \log (x + y)$
then $x^x y^y < (x + y)^{x+y}$.

So $(r - q)^{r-q} (q - p)^{q-p} < (r - p)^{r-p}$ or $\boxed{1 \leq m}$

Theorem:

$$(QP).(PR) \Rightarrow (QR)$$

For (QP) is $A/B \leq (q - p)^{q-p}$,

(PR) is $A/C \geq (r - p)^{r-p}$,

jointly entailing $B/C \geq (r - p)^{r-p} / (q - p)^{q-p} > (r - q)^{r-q}$ i.e. (QR) .

If (PR) but not (PQ) , then $(PR).(QP)$, so (QR) .

In $f_4 = 0$, take bx^Q as pivot;

$$\text{i.e. } bx^Q + 1 = -\zeta, \quad \beta = Q/|1/b|, \quad \epsilon^Q = |b|/(-b), \quad Qq = 1,$$

$$x = \beta \epsilon (1 + \zeta)^Q, \text{ say } f(\zeta).$$

$$\zeta = ax^P + cx^R = a\beta^P \epsilon^P (1 + \zeta)^{PQ} + c\beta^R \epsilon^R (1 + \zeta)^{RQ}, \text{ say } \phi(\zeta),$$

and so on, as before, leading to a convergent double-series

if (QP) and (QR) , which obtain. However, this gives only a Q -lot of roots.

In $f_4 = 0$ divide throughout by bx^Q , to get $(a/b)x^{P-Q} + 1 + (c/b)x^{R-Q} + (1/b)x^Q = 0$,

and take the first term as pivot, (i.e. equating the first two terms to $-\zeta$

and proceeding as before) to get a $(P-Q)$ -lot of roots from the double-series

which converges under the conditions $(P-Q, R-Q)$ and $(P-Q, -Q)$, naming them according to the previous pattern.

However, $(P-Q, -Q)$ means that, a.e. $1 \geq \left| \frac{(1/b)^{P-Q} (-Q)^{-Q} P^P}{(a/b)^{-Q} (P-Q)^{P-Q}} \right| = \left| \frac{a^Q P^P}{b^P Q^Q (P-Q)^{P-Q}} \right|$
i.e. (QP) ;

But $(P-Q, R-Q)$ means that, a.e., $1 \geq \left| \frac{(c/b)^{P-Q} (R-Q)^{R-Q} (P-R)^{P-R}}{(a/b)^{R-Q} (P-Q)^{P-Q}} \right|$

i.e. $1 \geq \left| \left(\frac{c}{P-Q} \right)^{P-Q} \left(\frac{a}{Q-R} \right)^{Q-R} \left(\frac{P-R}{b} \right)^{P-R} \right|$,

say (PR, Q) , or $C_{PR,Q} \leq 1$;

multiplying "above and below", appropriately, by $P^P Q^Q R^R$, $C_{PR,Q}$ is found to be $C_{QP} \cdot C_{PR} \cdot C_{RQ}$.

If we put $R = 0$ and $c = 1$, $(PR, Q) = (PR)$.

The formal expression for (QP, R) , i.e. $(Q-R, P-R)$ also reduces to the same form as (PR, Q) , as does that for (RQ, P) , so that all that matters is the cyclic order of the indices, all three forms being representible by (QPR) .

Also (PQR) or (QRP) or (RPQ) denotes the contrary case, namely $|C_{PQ} \cdot C_{QR} \cdot C_{RP}| \leq 1$

i.e. $(PQ) \cdot (QR) \cdot (RP) \Rightarrow (PQR)$

So, when (QP) and (QP, R) , there is a $(P-Q)$ -lot of roots.

To recapitulate: $(PQ) \cdot (PR) \Rightarrow [P]$, the existence of a complete P -lot

of roots of the form $\epsilon F_1 + \dots \epsilon^{P-1} F_{P-1}$

$(QP) \cdot (PR) \Rightarrow (QP) \cdot (QR) \Rightarrow [Q]$, a complete Q -lot

of roots, etc.

$(QP) \cdot (PR, Q) \Rightarrow [P-Q]$, a complete $(P-Q)$ -lot.

When these conditions fail, take cx^R as pivot, and the resulting double-
series converges to give an R-lot of roots if (RP).(RQ); then divide
 $f_4 = 0$ throughout by cx^R , to get

$$(a/c)x^{P-R} + (b/c)x^{Q-R} + 1 + (1/c)x^{-R} = 0,$$

whence, with the first term as pivot, a (P-R)-lot of roots is found when
(P-R, Q-R) and (P-R, -R), these two conditions being the same as (PQ,R)
and (RP).

$$\text{But } (QP).(RQ) \Rightarrow (A/B = n).(B/C \leq 1) \Rightarrow (A/C \leq ln < m) \Rightarrow (RP)$$

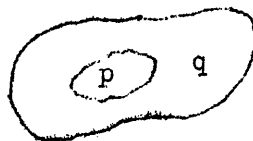
$$\text{So, } (QP).(RQ) \Rightarrow (RP).(PQ) \Rightarrow [R]$$

$$\text{and } (RP).(PQR) \Rightarrow [P-R].$$

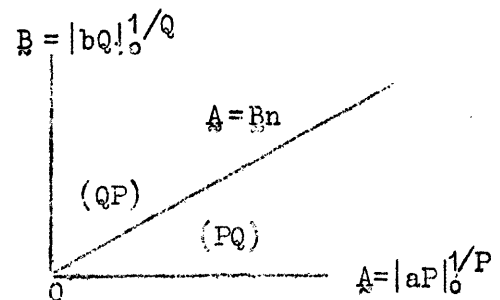
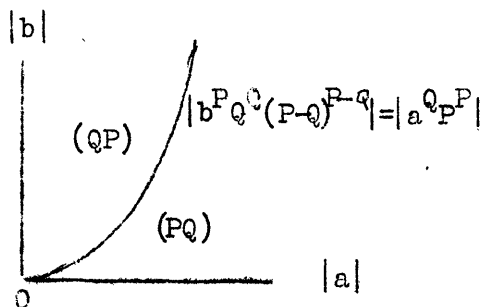
The last possibility of solving by this method is to take the second term
as pivot in $(a/c)x^{P-R} + (b/c)x^{Q-P} + 1 + (1/c)x^{-R} = 0$; a [Q-R] lot is then
obtained if (Q-R, P-R) and (Q-R, -R) which means (QPR) and (RQ).

Because the Calculus of Propositions is isomorphic with the
Theory of Sets, it is convenient to display pictorially the interrelations
of the inequality propositions, using VENN diagrams.

$p \Rightarrow q$ becomes:



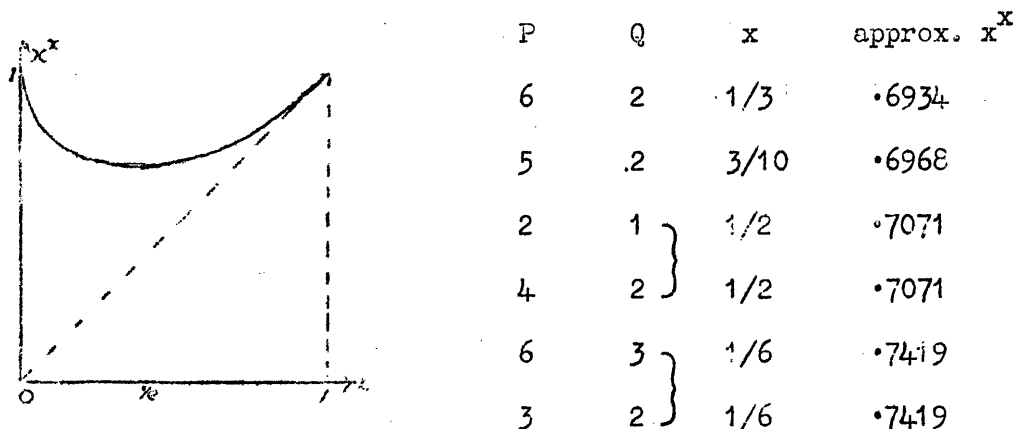
and $p.q$



The values (and bounds) of l, m, n are most rapidly assessed for discussion from the real branch of the graph of $y = x^x$, supplied for reference:

The actual minimum at $x = 1/e$ is about 0.6908.

Some of the possible small values arising from certain P and Q , are



The presentation of (PQ) , (QR) , (RP) , (PQR) in the direct style (as at the left, above, requires three parabolic half-cylinders and a generalized cone, whereas in the transformed style, there are three planes and a generalized cone. The equation to this cone is directly

$$1 = C_{PQR} = \left| \frac{Q-R}{a} \right|^{Q-R} \left| \frac{b}{P-R} \right|^{P-R} \left| \frac{P-Q}{c} \right|^{P-Q} = \left| \frac{P-Q}{P-R} \cdot \frac{b}{c} \right|^{P-Q} \left| \frac{Q-R}{P-R} \cdot \frac{b}{a} \right|^{Q-R},$$

showing the generators to have the form $\left| \frac{P-Q}{P-R} \cdot \frac{b}{c} \right| = k^{Q-R}$, $\left| \frac{Q-R}{P-R} \cdot \frac{b}{a} \right| = (1/k)^{P-Q}$,

or, transformed, $1 = C_{PQR} = C_{PR} \cdot C_{RQ} \cdot C_{QP}$
 $= (Bn/A)^{PQ} (c1/B)^{QR} (A/cn)^{PR},$

since the factors are respectively congruent,

i.e. $1 = (Bn/A)^{PQ-PR} (c1/B)^{PR-QR} (ln/m)^{TR}:$

a generalized cone with axis along the axis of B , ^{which} ~~and~~, for all real values of k , has the generator

$$Bn/A = (k)^{PR-QR} \left(\frac{n}{ln} \right)^{\frac{R}{P-R}}, \quad B/c1 = (1/k)^{PQ-PR} \left(\frac{m}{ln} \right)^{\frac{P}{P-R}}$$

On the faces of the unit cube
which are remote from the
origin are shown the traces of

$A = Bn$, separating (PQ) and (QP)

$B = Cl$, " (RQ) and (QR)

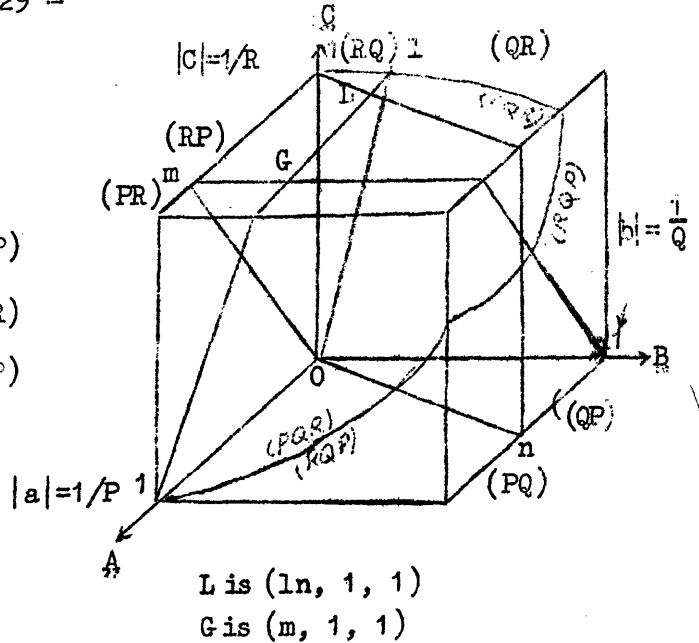
$A = Cn$, " (PR) and (RP)

and the generalized cone,

$1 = C_{PQR}$, separating (PQR),

above and to the left,

from (PRQ), below.



The diagram is distorted to display the nature of the configuration.

Actually, the most "open" one is for

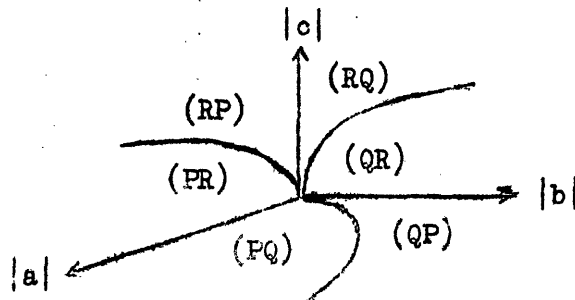
$$P = 3, \quad Q = 2, \quad R = 1,$$

$$l = (1/2)^{1/2} = \sqrt[6]{(1/8)} \approx 0.74$$

$$m = (2/3)^{2/3} \approx \sqrt[6]{(1/5)} \approx 0.96$$

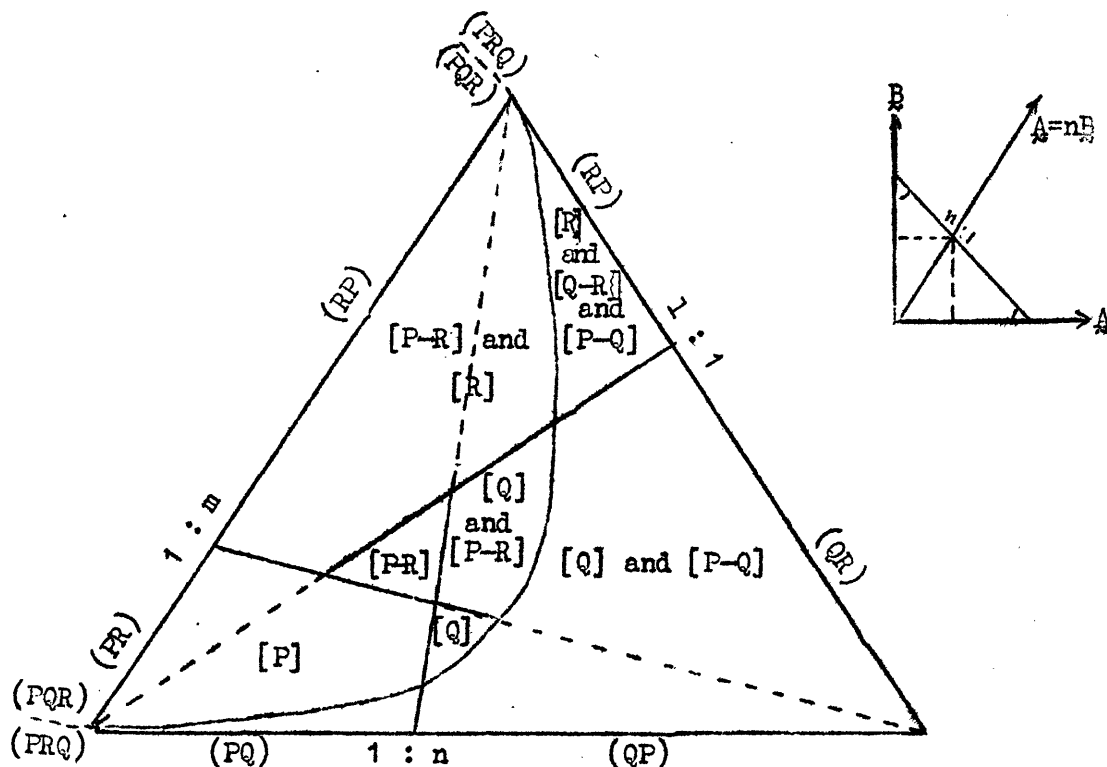
$$n = \sqrt[6]{(1/6)} \approx 0.74.$$

Representing $C_{PQR} = 1$ directly, generalized parabolic cylinders would
have traces as shown:



all these configurations, direct or transformed, are homeomorphic, the
effect of transformation being, near the origin, a clustering and,
further out, a dispersion of the meets of the surfaces.

A simple VENN diagram results from the section of either configuration by a plane. Again, all plane sections are homeomorphic, but it is convenient to take equal axial intercepts and the transformed case, thus:



This "convergence diagram" illustrates these propositions concerning inequalities:

- I $(QP).(PR) \Rightarrow (QR) \quad ; \quad (PQ).(RP) \Rightarrow (PQR)$
- II $(PR).(RQ) \Rightarrow \begin{cases} (i) (PQ) \\ (ii) (PQR) \end{cases}$
- III $(PQ).(RQ) \Rightarrow (PQR)$
- IV $(PRQ).(RQ) \Rightarrow (QP).(RQ) \Rightarrow RP$
- V $(PRQ).(PR) \Rightarrow (PRQ).(PQ) \Rightarrow QR$
- VI $\{1, 1, 1\} \not\subset \{|c_{PQ}|, |c_{QR}|, |c_{PQR}|\}$; no three boundaries concur.

At every point (i.e. for any given a, b, c) there is convergence to at least one set of roots; but for the combination $(PQR).(RP).(QP).(QR)$, both $[P-R]$ and $[Q]$ lots exist. Now $P-R + Q > P$, and by the Fundamental Theorem of Algebra, there are P roots and no more. So, although all the $P-R$ roots are different and all the Q roots are different, some of the roots of these two sets must coincide. The question then arises as to whether a $[Q]$ -lot and a $[P-Q]$ -lot may not have some roots in common, and similar questions about $[R]$ and $[P-R]$ lots, and about $[R]$ and $[Q-R]$ and $[P-Q]$ lots, and also as to where the unmentioned roots are to be found in the "[P-R]" and "[Q]" sectors, ^{the three regions $[P-R], [Q], [P-R] \& [Q]$} ~~these last~~ ^{interior} being regarded as "unstable".

In any "practical" case, where fewer than P roots are found there are several methods by which an equation may be found containing $[B]$ the balance of the roots, and then solved by this method or its extension later described. So in a "[P-R]" case, the equation containing $[B]$, the balance of the R roots, may be solvable NOT by an $[R]$ -lot, but by a $[B_1]$ -lot, a $[B_2]$, and so on, and similarly for a "[Q]" case. THE MAIN POINT OF THIS "CONVERGENCE DIAGRAM" IS TO SHOW HOW A SLIGHT VARIATION IN ONE OF THE COEFFICIENTS MAY SUBSTANTIALLY CHANGE THE PATTERN OF THE ROOTS. The totality of ^{interior} "unstable" sectors ~~and boundaries of regions~~ will be called the CORE. An examination of the values of l, m, n , (together with the realization that the diagram has been twice distorted to magnify this core in order to exhibit its structure), shows that the proportion of pathological cases is very small.

There are six special families of equations determined by the vertices of the core, but they contain only six singular equations corresponding to the simultaneous equality to unity of any two of C_{PQ} , C_{QR} , C_{RP} , C_{PQR} . These are conjectured to be solvable by radicals over the rational field of the coefficients - NOT by having double roots, however, as in the trinomial case for $C_{PQ} = 1$, but by more unexpected phenomena. For instance, if $P = Q + R$, the conditions $C_{PQ} = 1 = C_{PR}$ may be written

$$\frac{(-b)^{PQR}}{a^{QPR}} = 1 = \frac{(-c)^{PRQ}}{a^{RQP}}$$

$$\text{or } \lambda^P = \frac{P^P}{Q^Q R^R} = \frac{(-b)^P}{a^Q} = \frac{(-c)^P}{a^R}$$

$$\text{or } -b = \lambda a^{Q/P}, \quad -c = \lambda a^{R/P}, \quad (\text{taking P.V.s})$$

putting $y^P = ax^P$, the original equation becomes

$$y^P - \lambda y^Q - \lambda y^R + 1 = 0,$$

which, if P is odd, is satisfied by $y = -1$, or $x = -1/\sqrt[P]{a}$.

The general form, in this singular case, of "one" algebraic root

(as described in the trinomial case) necessary for full solvability

by radicals, is not yet known.

Conspectus of Root-Structure of Tetranomial:

Singular	Solvable by radicals	$\{1,1\}C_{PQ}, C_{QR}, C_{RP}, C_{PQR}$
Stable	6 { P roots	$(PQ).(PR)$
	Q and P-Q	$(QP).(PRQ).(QR)$
	R and Q-P and P-Q	$(PRQ).(RQ)$
Unstable	R and P-R	$(RP).(PQR).(RQ)$
	B P-R	$(PQ).(QR).(RP)$
Metastable	Q B	$(PR).(RPQ).(QP)$
	Q and P-R	$(PQR).(RP).(QP).(QR)$

Programme for decision on pivot term:

PQ		QP					
PR	RP		RQ		QR		
	QR	RQ	(PQR)	(PRQ)	(PRQ)	(PQR)	
						(PR	R RP
[P]	[P-R][B]	[P-R][R]	[R][P-R]	[R][Q-R][P-Q]	[Q][P-Q]	[Q][B]	[Q][P-R]

4 criteria; 6 singularities; 7 other eventualities, 4 of them stable or, more baldly, where $-(PQ)$ means (QP) , $-(PQR)$ means (RQP) , etc.

Test	Reaction					
(PQ)	+	-	-			
(PR)	+			-	-	
(QR)		+		-		-
(PQR)			-		+	-
Result	[P]	[Q]	[P-Q]	[R]	[P-R]	[Q-R]

Chapter IV: Pentanomial and General Cases

The Pentanomial Case, $ax^P + bx^Q + cx^R + dx^S + 1 = 0$, or $f_5 = 0$.

The procedure is strictly analogous to, and an extension of, that of the Tetranomial. With the same notation for convergence conditions, there are corresponding chains of inequalities, e.g.

I: (QP).(PS) $\Rightarrow$ (QS); II: (PS).(SQ) $\Rightarrow$ (PQ); IV: (PSR).(SR) $\Rightarrow$ (SP).

Taking ax^P as pivot, when (PQ).(PR).(PS), and putting, for greater brevity than before, $\alpha \in \{a : -aa^P = 1\}$, $\beta = ba^Q$, $\gamma = ca^R$, $\delta = da^S$, then

$$\frac{x}{a} = 1 + \frac{1}{P} \sum_{k=1}^{\infty} \sum_{L+M+N=K} \frac{\beta^L}{L!} \frac{\gamma^M}{M!} \frac{\delta^N}{N!} \left(\frac{LQ + MR + NS + 1}{P} - 1 \right)^{(K-1)}.$$

Each term of the triple-series may be decomposed into five factors, arranged at the lattice-points $(-1, -1, -1)$, $(L-1, -1)$, $(-1, M-1)$, $(-1, -1, N)$, and (L, M, N) of an octant of 3-dimensional space. Convergence for a sub-series (of every P^{th} term) along any row parallel to an axis entails convergence for every parallel row, but three convergence conditions are required for the whole triple-series thus arranged. If any convergence condition fails, a different pivot term may be used as the table shows:

Test	Reaction									
(PQ)	+	-	-	-	-	-	-	-	-	-
(PR)	+			-	-					
(PS)	+						-	-		
(QR)		+		-		-				
(QS)		+					-		-	
(RS)				+			-			-
(PQR)			-		+	-				
(PQS)			-					+	-	
(PRS)					-			+		-
(QRS)						-			+	-
Result	[P]	[Q]	[P-Q]	[R]	[P-R]	[Q-R]	[S]	[P-S]	[Q-S]	[R-S]
Pivot	ax^P	bx^Q	ax^{P-Q}/b	cx^R	ax^{P-R}/c	bx^{Q-R}/c	dx^S	ax^{P-S}/d	bx^{Q-S}/d	cx^{R-S}/d

Let $Pp = Qq = Rr = Ss = 1$ and let $|aP|^P = A$, $|bQ|^Q = B$, $|cR|^R = C$, $|dS|^S = D$ be co-ordinates in 4-space. Then (PQ) and (QP) are on opposite sides of the hyperplane $A = nB$; (PQR) and (RQP) are on opposite sides of a manifold meeting the hyperplane $D = 0$ in the generalized cone encountered in the Tetranomial Case, and so on. The whole configuration leaves, on any hyperplane making equal intercepts on the axes, traces which dissect a regular tetrahedron, say PQRS, its vertices in lexicographical correspondence with the axes. This dissection is a mapping of the assemblage of convergence domains: (PQ) and (QP) are the portions of the tetrahedron, respectively including P and Q, which are separated by the plane through RS dividing PQ in the ratio 1 : n, as before. There are six of these planes, determining 16 vertices (singularities of f_5) of a complete hexahedron INSIDE the tetrahedron. (PQR) and (RQP) are separated by a ruled surface passing through P and R, having Q on the (RQP) side and the hexahedron just mentioned on the other. There are 3 similar ruled surfaces, determining among themselves 4 more singularities of f_5 . The planes through P determine 3 lines, each of which meets the interface of (QRS) and (SRQ) in a singular point; those through Q determine singularities on each of the 3 ruled surfaces not containing Q, similarly there are 9 singularities on lines through R and another 3 on the lines through S. So that the ruled surfaces contain 28 singularities of f_5 , making 44 altogether.

The number and nature of the regions of the core ~~of the tetrahedron~~ is not yet known, but the 8 regions of complete solution are given as Results on the table above.

[It is uncertain whether the ruled surface through P and S , not containing R , i.e. $(PQS).(SQP)$, and the ruled surface through P and S , not containing Q , i.e. $(PRS).(SRP)$, are identical, but the above count is not invalidated by the uncertainty.]

Now consider $a_k x^k + \dots a_j x^j + \dots a_2 x^2 + a_1 x + 1 = 0$, the indices being consecutive natural numbers. Then if $(j1).(j2) \dots (j, j-1).(j, j+1) \dots (jk)$, with the same meanings as before, a $[j]$ -lot of roots can be found by taking $a_j x^j$ as pivot. If the first failure of these $k-1$ tests is at $a_1 x^1$, transfer the tests, in turn, to the pivots $a_1 x^1$ and $a_k x^{k-1}/a_1$. If failure again occurs, transfer the pivot to the term occasioning failure, and also to its quotient with $a_k x^k$, and so on. Because the mapping of the assemblage of convergence domains exhausts the regular $(k-2)$ -simplex in which they are topologically embedded, there will be at least one lot of roots yielded by this process. In the very rare case in which this is the only lot, remove it by division and proceed similarly with the reduced equation.*

In hope, begin at $a_k x^k$.

In following the drill, it is useful to think of (PQ) as $(PQ0)$, that is, as if the term at the end had been $a_0 x^0$, and reduced to unity by division by a_0 , as indeed it would be on nearly all actual occasions.

So, EVERY convergence test becomes a 3-index one: a $[j-i]$ -lot exists if

$$(ij0).(ij1).(ij2) \dots (ijk).$$

Lastly: if $\{n_i\}_1^k \subset \emptyset$, and $(n_1, n_2, \dots, n_k) = J/K$, $(J, K) = 1$, then $\sum_1^k a_i x^{n_i} + 1 = 0$ reduces to $f_k = 0$, solvable as above.

*For a $(k+1)$ -nomial of k -th degree, a bilinear transformation $xy + ux + vy = 1$ will not complicate, but may complete the transformation pattern.

Chapter V: Irrational Exponents

Let s be real but irrational.

Let k, N and P be integers, and $pP = 1$.

For all ν and μ let z_μ^ν be $\exp \nu \{ \ln |z| + i(\text{ph } z + 2\mu\pi) \}$; $1_\mu^\nu = \text{cis } 2\nu\mu\pi$.

Then (i) $z_0^{p-1} = z_{kP}^{p-1}$

(ii) $(z_\mu^\nu)^N = z_\mu^{N\nu}$.

(iii) $(z_j^p)_k^s = \exp s \{ \ln |z_j^p| + i(\text{ph } z_j^p + 2k\pi) \}, \quad -\pi < \text{ph } z \leq \pi$
 $= \exp s \{ p \ln |z| + i(p \text{ph } z + 2p j\pi + 2k\pi) \}, \quad \text{since } |p| < 1,$
 $= \exp ps \{ \ln |z| + i(\text{ph } z + 2j\pi + 2kP\pi) \}.$
 $= z_{j+kP}^{ps}.$

Consider $ax^P + bx_k^s + 1 = 0$.

As before, take $ax^P + 1 = -\zeta, \quad aa^P = -1, \quad \beta = ba_k^s,$
 $\alpha_k^s = \left\{ \left(\frac{-1}{a} \right)_j^p \right\}_k^s = \left(\frac{-1}{a} \right)_{j+kP}^{ps}; \quad \alpha \beta^N = b^N \left(\frac{-1}{a} \right)_{j+kP}^{Nps}; \quad \beta^P = b^P / (-a)_{j+kP}^s$

then $x = a(1 + \zeta)_0^p = f(\zeta)$, say,

$\zeta = b [a(1 + \zeta)_0^p]_k^s$
 $= \beta(1 + \zeta)_{kP}^{ps}, \quad \text{say } \phi(\zeta)$

and $x = f(0) + \left[\sum_{N=1}^{\infty} D^{N-1} \{ \alpha^P (1 + \zeta)_0^{p-1} \beta^N (1 + \zeta)_{kP}^{Nps} \} / N! \right]_{\zeta=0}$

i.e. $x/a = 1 + p \sum_{N=1}^{\infty} \beta^N (Nsp + p - 1)^{(N-1)} \cdot 1_{kP}^{Nsp - N + 1} / N!$

but $1_{kP}^{Nsp - N + 1} = \text{cis } 2kP(Nsp - N + 1)\pi = \text{cis } 2Nks\pi$

$= 1_k^{Ns}$, which may be incorporated in β^N :

so $\frac{x}{a} \in \left\{ 1 + \frac{1}{P} \sum_{N=1}^{\infty} \frac{b^N}{N!} \left(\frac{-1}{a} \right)_{j+kP}^{Nps} \left(\frac{Ns + 1 - P}{P} \right)^{(N-1)} \right\}_{j=1}^P$ if convergent.

The faculty coefficient = $\Gamma(Nsp + p) / \Gamma(\overline{Nsp-1} + p + 1)$

So, in the series, $u_{N+P}/u_N = \frac{\beta^P}{(N+P)^{(P)}} \frac{\Gamma(Nsp + s + p)}{\Gamma(\overline{Nsp-1} + s - P + p + 1)} \frac{\Gamma(\overline{Nsp-1} + p + 1)}{\Gamma(Nsp + p)}$

Now, for real and constant a ,

$$\lim_{x \rightarrow \infty} \Gamma(x) / \Gamma(x + a) = x^{-a}.$$

$$\begin{aligned} |u_{N+P}/u_N| &\rightarrow \frac{|\beta^P|}{(N+P)^{(P)}} \frac{(Nsp + p)^s}{(\overline{Nsp-1} + p - 1)} \\ &\rightarrow \frac{|\beta^P|}{N^P} \cdot \frac{N^s (sp)^s}{N^{s-P} (sp-1)^{s-P}} \\ &\rightarrow \left| \frac{b^P}{(-a)^s} \frac{s^s}{P^P (s-P)^{s-P}} \right|, \quad \text{say } C_{Ps}. \end{aligned}$$

It would be a very interesting event indeed if $C_{Ps} = 1$. We may again represent convergence to a $[P]$ -lot of roots by (Ps) , for any k ; but none of the P roots for $k = k_1$ will equal any of the P roots for $k = k_2 \neq k_1$. writing $\hat{a}$ for $|a|/(-a)$, the values of $(\hat{a})_{j+kP}^{Nps}$ are everywhere dense on the unit circle but never repeat themselves. For each integral value that k takes, $j \in \{1, 2, \dots, P\}$. The absence of k from the faculty coefficient enables an extension to be made to the equation

$$ax^P + \sum_1^M b_1 x_{k_1}^{s_1} + 1 = 0, \quad \{s_1\} \not\subset \emptyset, \quad \text{but } \{s_1\} \subset \mathbb{P}.$$

if $(Ps_1) \cdot (Ps_2) \cdot \dots \cdot (Ps_M)$

$$\text{and thence to } \sum_1 a_j x^{Pj} + \sum_1 b_1 x_{k_1}^{s_1} + 1 = 0$$

for any condition in the sets $\{(P_j s_1) \cdot (P_j s_2) \dots (P_j s_M) : j \in \{1, 2, \dots, N\}\}$
and $\{(P_i P_j s_1) \cdot (P_i P_j s_2) \dots (P_i P_j s_M) : \{i, j\} \subset \{1, 2, \dots, N\}\}$.

Now we dare to consider taking the second term as pivot in

$$ax^P - bx_k^s + 1 = 0,$$

changing the sign of that term, for convenience. Put $so = 1$.

[Notice that if $\alpha + \beta = \gamma + \delta$, $\zeta_\alpha^\nu \eta_\beta^\nu = \zeta_\gamma^\nu \eta_\delta^\nu$,

since each is $\exp \nu \{ \ln |\zeta| + \ln |\eta| + i(\text{ph } \zeta + \text{ph } \eta + 2[\alpha + \beta]\pi) \}$

whereas $(\zeta \eta)_k^\nu = \zeta_0^\nu \eta_0^\nu 1_{k'}^\nu$, where $k - k' \in \{-1, 0, 1\}$.

But If $z_k^s = w$, then $|w| = |z|_0^s$, $\text{ph } w = s(\text{ph } z + 2k\pi)$

so $(|z| = |w|_0^\sigma$ and $\text{ph } z = \sigma \text{ph } w - 2k\pi = \sigma(\text{ph } w - 2ks\pi)$

then $z = w_{-ks}^\sigma$]

Put $-bx_k^s + 1 = -\zeta$, then $x_k^s = b^{-1}(1 + \zeta)$,

and $x = \{b^{-1}(1 + \zeta)\}_{-ks}^\sigma = b_0^{-\sigma}(1 + \zeta)_0^\sigma 1_{k'}^\sigma$,

where $k' + ks \in \{-1, 0, 1\}$, but when $\zeta = 0$, $k' + ks = 0$,

moreover $1_{k'}^\sigma = \text{cis } 2k'\sigma\pi = 1_{-k'}^{-\sigma}$,

and $b^{-\sigma} 1_{-k}^{-\sigma} = b_{-k}^{-\sigma} = \beta$, say,

so $x = \beta(1 + \zeta)_{k'}^\sigma = f(\zeta)$, say,

$\zeta = ax^P = a\beta^P(1 + \zeta)_0^{P\sigma} = \phi(\zeta)$, say,

Therefore $x/\beta = 1 + \sigma \sum_{N=1}^{\infty} \frac{(a\beta^P)^N}{N!} (NP\sigma + \sigma - 1)^{(N-1)}$ if convergent,

$$x = b_{-ks}^{-1/s} \left\{ 1 + \frac{1}{s} \sum_{N=1}^{\infty} \frac{(ab_{-ks}^{-1/s})^N}{N!} \left(\frac{NP + 1 - s}{s} \right)^{(N-1)} \right\},$$

convergent if (sP) , to a denumerable lot of roots.

$$\left((sP) \Rightarrow C_{sP} = \left| \frac{a_{sP}^P (s-P)^{s-P}}{b^P s^s} \right| < 1 \right)$$

Extension to the general case follows as before.

Therefore
$$\sum_{j=1}^N a_j x^{P_j} + \sum_{i=1}^M b_i x_{k_i}^{s_i} + 1 = 0$$

always has infinitely many roots, found by taking "pivot" terms according to the same drill as that given at the ends of chapters III and IV.

If in $\sum_{i=1}^M a_i x_{k_i}^{s_i} + 1 = 0$, no s_i is rational, the substitutions $y^{P_i} = x_{k_i}$ may be made in turn for $i \in \{1, \dots, M\}$, with the appropriate changes in the other exponents, to test for the possible occurrence of a chain of convergence conditions and therefore of solutions. From an arrangement of $\{P_1, \dots, P_N, s_1, \dots, s_M\}$ in descending order of magnitude, the appropriate inequalities e.g. $(P_N P_1) \cdot (P_1 s_1) \Rightarrow (P_N s_1)$ may be written down at sight.

Chapter VI: Complex Exponents

Theorem:

With the generalized definition of z_K^ν as $\exp \nu \{ \ln z + i(\text{ph } z + 2k\pi) \}$,

$$z_K^\nu = w \implies z = w^{\frac{1}{\nu}}_{-K\nu} \text{ for all } \nu, K$$

Let $\nu = n + i n'$, $K = k + i k'$, $w = \exp(s + i\phi)$, $z = \exp(r + i\theta)$:

ϕ and θ need not be the principal phase, only the particular phase involved in the transformation.

$$w = \exp(n + i n') \{ r + i(\theta + 2[k + i k']\pi) \}$$

$$\text{so } \begin{cases} s = n\xi - n'\eta \\ \phi = n'\xi + n\eta \end{cases} \quad \text{where } \begin{cases} \xi = r - 2k'\pi \\ \eta = \theta + 2k\pi \end{cases}$$

$$\text{i.e. } \begin{cases} ns + n'\phi = |\nu|^2 \xi \\ n\phi - n's = |\nu|^2 \eta \end{cases}$$

$$\text{so } r = \frac{ns + n'\phi}{|\nu|^2} + 2k'\pi$$

$$\theta = \frac{n\phi - n's}{|\nu|^2} - 2k\pi$$

$$z = \exp \left(\frac{ns + n'\phi}{|\nu|^2} + 2k' + i \left\{ \frac{n\phi - n's}{|\nu|^2} - 2k\pi \right\} \right)$$

$$= \exp \left(\frac{s}{\nu} + i \left\{ \frac{\phi}{\nu} - 2k\pi \right\} \right) = w^{\frac{1}{\nu}}_{-K\nu};$$

$$\text{and if } z_K^\nu = 1, \quad z = \text{cis}(-2k\pi) = e^{2k'\pi} \text{cis}(2k\pi)$$

which is independent of ν .

Theorem:

$$\begin{aligned} z_\lambda^\mu / z_K^\nu &= \exp [\{ \ln z + i(\text{ph } z + 2\lambda\pi) \} - \nu \{ \ln z + i(\text{ph } z + 2k\pi) \}] \\ &= \exp [(\mu - \nu)(\ln z + i \text{ph } z) + 2(\lambda\mu - \kappa\nu)\pi i] \\ &= z_0^{\mu-\nu} \text{cis } 2(\lambda\mu - \kappa\nu)\pi \end{aligned}$$

Note:

z_λ^μ cannot vanish unless $z = 0$.

Consider now $S(z) = \sum_{i=0}^{M+1} \gamma_i z_{k_i}^{\nu_i} = 0$, $\{\gamma_i, \nu_i\} \subset \Phi$;

$\{k_i\}$ is usually regarded as $\subset \mathbb{J}$, but need not be so restricted.

Since $z_k^\nu = \exp \nu \{ \ln z + i(\text{ph } z + 2k\pi) \} = z_0^\nu \cdot 1_k^\nu$, there is immediately:

$$S(z) = \sum_{i=0}^{M+1} \delta_i z_0^{\nu_i}, \quad \text{with } \delta_i = \gamma_i \text{ cis } 2k_i \nu_i \pi.$$

The set $\{\nu_i\}$ is not "ordered", but the exponents may be named, and given a vocabulary order. Choose ν_0 as a trial "reducing-exponent" and since $\delta_1 z_0^{\nu_1} \neq 0$, divide throughout by this term, making $S(z) = 0$ equivalent to $1 + \sum_{i=1}^{M+1} \beta_i z_0^{\mu_i} = 0$, with $\mu_i = \nu_i - \nu_0$; $\beta_i = \delta_i / \delta_0$.

Choose a "pivot", ~~and~~ call it $-\beta z_0^\mu$ and equate it to $-(1 + \zeta)$, and "order" the remaining terms $\sum_{i=1}^M$.

$$\text{Let } \lambda_\mu = 1 \text{ and } \lambda_1 \mu = \mu_1 \text{ or } \lambda \mu_1 = \lambda_1$$

$$\begin{aligned} \text{then} \quad \text{then } z_0^\mu &= (1 + \zeta) \beta^{-1}, \quad \text{so } z = [(1 + \zeta) \beta^{-1}]^\lambda \\ &= (1 + \zeta)_0^\lambda \beta_0^{-\lambda} 1_k^\lambda = f(\zeta), \quad \text{say.} \end{aligned}$$

Now as $\zeta \rightarrow 0$, $\kappa \rightarrow 0$, and as ζ is destined to vanish, drop all further reference to κ ; and since only principal (or particular) phases are involved, the suffices may be dropped.

$$\text{Also } \beta_1 z^{\mu_1} = \beta_1 \left((1 + \zeta)^\lambda \beta^{-\lambda} \right)^{\mu_1} = \beta_1 (1 + \zeta)^{\lambda_1} \beta^{-\lambda_1} 1_k^{\lambda_1} = f(\zeta), \quad \text{say.}$$

$$S(z) = 0 \text{ is now equivalent to } \zeta = \sum_{i=1}^M \alpha_i (1 + \zeta)^{\lambda_i} = \phi(\zeta), \quad \text{say.}$$

Proceeding from $z = f(\zeta)$ and $\zeta = \phi(\zeta)$ as before

$$\begin{aligned} z \beta^\lambda &= 1 + \lambda \left(\sum_{N=1}^{\infty} D^{N-1} (1 + \zeta)^{\lambda-1} \left\{ \sum_{i=1}^M \alpha_i (1 + \zeta)^{\lambda_i} \right\}^N / N! \right)_{\zeta=0} \\ \Pi &\quad = 1 + \lambda \sum_{N=1}^{\infty} \prod_{\Sigma N_i = N} \frac{\alpha_i^{N_i}}{N_i!} \left(\sum_{i=1}^M \lambda_i N_i + \lambda - 1 \right)^{(N-1)}, \quad \text{if convergent} \end{aligned}$$

The complex faculties will be arranged at the lattice points $(N_1, N_2, \dots, N_M)$ of M -dimensional space, the factors $\alpha_1^{N_1}/N_1!$ at the points $(-1, -1, \dots, N_1, \dots, -1)$, and the factor $\lambda \beta^{-\lambda}$ at the point $(-1, -1, \dots, -1)$ as before, to provide an ordered multiple series, the $\{\alpha_i\}$ having been ordered, temporarily and arbitrarily, to correspond to the $\{\nu_i\}$. If ω is the first transfinite ordinal, then the coefficient named $(\sum \lambda_i N_i + \lambda - 1)^{(N-1)}$ corresponding to the partition $N_1 + N_2 + \dots + N_M = N$ has the ordinal number

$$N_M \omega^{M-1} + N_{M-1} \omega^{M-2} + \dots + N_3 \omega^2 + N_2 \omega + N_1.$$

This ordering is necessary to test for convergence, but irrelevant to the later computation. Consider, as before, a row of terms parallel to the "axis of α_1 ", that is, in which all the factors other than α_1 change. For this line, write Σ for $(\sum_{i=1}^M \lambda_i N_i) - \lambda_1 N_1$, and S for $N - N_1$ (noting that as N_1 , and hence N , $\rightarrow \infty$, S is finite); then for constant $\{\alpha_j\} : j \neq i$

$$\begin{aligned} u_{N_1+1}/u_{N_1} &= \frac{\alpha_1}{N_1+1} \cdot \frac{\Gamma(\lambda_1 N_1 + \lambda_1 + \Sigma + \lambda)}{\Gamma(\lambda_1 N_1 + \lambda_1 + \Sigma + \lambda - N)} \cdot \frac{\Gamma(\lambda_1 N_1 + \Sigma + \lambda - N + 1)}{\Gamma(\lambda_1 N_1 + \Sigma + \lambda)} \\ &= \frac{\alpha_1}{N_1+1} \cdot \frac{\Gamma(\lambda_1 N_1 + \lambda_1 + \Sigma + \lambda)}{\Gamma(\lambda_1 N_1 + \Sigma + \lambda)} \cdot \frac{\Gamma(N_1(\lambda_1 - 1) + F + 1)}{\Gamma(N_1(\lambda_1 - 1) + \lambda_1 + F)} \end{aligned}$$

$$\text{where } \Sigma + \lambda - S = F$$

$$\begin{aligned} &\rightarrow \frac{\alpha_1}{N_1+1} \cdot \frac{(\lambda_1 N_1 + \Sigma + \lambda)^{\lambda_1}}{(N_1(\lambda_1 - 1) + F + 1)^{\lambda_1 - 1}} \quad \text{since } \Gamma(z + z_0) \sim z^{z_0} \Gamma(z), \\ &\rightarrow \frac{\alpha_1 \lambda_1^{\lambda_1}}{(\lambda_1 - 1)^{\lambda_1 - 1}} \equiv C_1, \text{ for all such rows.} \end{aligned}$$

Similarly for some, and hence any, row parallel to each of the other "axes" in turn.

For convergence along α_1 -axis, $|C_1| < 1$, which implies that $|C_1|^\mu < 1$ if $\Re(\mu) > 0$. Now $\Re(\mu) > 0$ and $|C_1| < 1$ jointly imply

$$\begin{aligned} 1 &> \left| \beta_1 \beta^{-\lambda_1} \lambda_1^{\lambda_1} (\lambda_1 - 1)^{\lambda_1 - 1} \right|^\mu, \text{ since } \alpha_1 = \beta_1 \beta^{-\lambda_1} \\ &= \left| \frac{\beta_1^\mu}{\beta^{\mu_1}} \left(\frac{\mu_1}{\mu} \right)^{\mu_1} \left(\frac{\mu_1 - \mu}{\mu} \right)^{\mu_1 - \mu} \right|, \text{ since } \lambda_1 \mu = \mu_1 \\ &= \left| \frac{\beta_1^\mu \mu_1^{\mu_1}}{\beta^{\mu_1} \mu^\mu (\mu_1 - \mu)^{\mu_1 - \mu}} \right| \\ &= \left| \frac{(\delta_1 / \delta_0)^{\nu - \nu_0} (\nu_1 - \nu_0)^{\nu_1 - \nu_0}}{(\delta / \delta_0)^{\nu_1 - \nu_0} (\nu_1 - \nu_0)^{\nu_1 - \nu_0} (\nu_1 - \nu)^{\nu_1 - \nu}} \right|, \text{ supposing } \begin{cases} \beta = \gamma / \gamma_0 \\ \mu = \nu - \nu_0 \end{cases} \\ &= \left| \left(\frac{\delta_1}{\nu - \nu_0} \right)^{\nu - \nu_0} \left(\frac{\nu_1 - \nu_0}{\delta} \right)^{\nu_1 - \nu_0} \left(\frac{\delta_0}{\nu_1 - \nu} \right)^{\nu_1 - \nu} \right| = C_{\nu_1 \nu \nu_0}, \text{ say.} \end{aligned}$$

i.e. $(\nu_1 \nu \nu_0)$, in the three-index notation of Chapter III.

Therefore ν_0 will serve as reducing index, with ν as pivot

$$\begin{aligned} &\text{if } \Re(\nu - \nu_0) > 0 \quad \text{and} \quad \prod_{i=1}^M (\nu_1 \nu \nu_0), \\ &\text{or if } \Re(\nu - \nu_0) < 0 \quad \text{and} \quad \prod_{i=1}^M (\nu_1 \nu_0 \nu). \end{aligned}$$

$C_{\nu_1 \nu \nu_0}$ contains the factor $|\delta^{\nu_1 - \nu_0}|$ which involves $|\text{cis } 2k \nu (\nu_1 - \nu_0) \pi|$,

say $|\text{cis } 2k(\xi + i\eta)\pi|$, or $|\exp 2k(\xi + i\eta)\pi|$, i.e. $e^{-2k\eta\pi}$, which may be

made greater or less than any positive quantity by suitable choice of k .

So $(\nu_1 \nu \nu_0)$ which means $C_{\nu_1 \nu \nu_0} < 1$, may be made so for infinitely

many k_1 and $(\nu_1 \nu_0 \nu)$ will then be true for all other values of k_1 .

Thus any term with assigned γ_0, ν_0, k_0 may be used for "reducing"

and " " " " " $\gamma, \nu, k, \text{ " " " as pivots,$

if $\{k_1, k_2, \dots, k_M\}$ may then be chosen. There will be convergence

to a unique root for each of the M -ply infinite proper selections.

For unassigned $\{k_i\}$, the above is a construction for an

$(M+2)$ -ply infinitude of selected solutions.

Chapter VII. Computation by Swathes

If β_z^{μ} is a pivot, the solution of

$$1 - \beta z^{\mu} + \sum_1^M \beta_1 z^{\mu_1} = 0$$

takes the form

$$\text{where } F = \sum_{N=1}^{\infty} \left(\frac{\sum_{N_1=N}^{\infty} \frac{\alpha_1^{N_1}}{N_1!}}{\sum_{N_1=N}^{\infty} \frac{\alpha_1^{N_1}}{N_1!}} \right) \left(\frac{1 - \mu + \sum_1^M N_1 \mu_1}{\mu} \right)$$

$$\epsilon^{-\mu} = \beta, \quad \text{and } \alpha_1 = \beta_1 \epsilon^{\mu_1}.$$

It is only to examine convergence that the terms of F are arranged at the lattice-points of 2^{-M} of M -dimensional space. When μ is a natural number, say m , the terms in the lattice along any row parallel to the α_1 -axis will have as sum a complete hypergeometric function of order $m-1$, and, indeed, the whole of F may be described in terms of LAURICELLA's hypergeometric functions of many variables and parameters, but the computation of F is not thereby advanced.

$$\text{But } F = \left(\sum_n^{\infty} D^{n-1} (1 + \zeta)^{\lambda-1} \left\{ \sum_1^M \alpha_1 (1 + \zeta)^{\lambda_1} \right\}^n / n! \right)_{\zeta=0}, \quad D = d/d\zeta,$$

$$= \sum_n^{\infty} \hat{n}, \quad \text{say, replacing } N \text{ by } n \text{ for present typographical}$$

convenience. $\hat{n}$ is a "swathe", the sum of the terms at the lattice-points contained in the hyperplane which makes on each α_1 -axis the intercept n .

$$\text{Let } \chi = \sum_1^M \alpha_1 (1 + \zeta)^{\lambda_1}, \equiv \chi(\zeta)$$

$$\text{and } \mathfrak{m} = D^m \chi / m! \Big|_{\zeta=0} = \sum_1^M \alpha_1 \lambda_1 (\mathfrak{m}) / m! = \sum_1^M \alpha_1 \binom{\lambda_1}{m}; \quad \mathfrak{Q} = \chi(0),$$

(these are generalized polynomials in ϵ)

$$\text{so that } \chi \equiv \mathfrak{Q} + \mathfrak{P}_1 \zeta + \mathfrak{P}_2 \zeta^2 + \dots = \sum_0^{\infty} \mathfrak{m} \zeta^m.$$

$$\text{Let } n_m = \frac{D^m}{m!} \left(\frac{\chi^m}{n!} \right) \Big|_{\zeta=0}, \quad \text{the coefficient of } \zeta^m \text{ in the TAYLOR}$$

$$\text{series for } \chi^n / n!$$

$$\text{and } \mathfrak{m}_k = \mathfrak{m}^k / k!$$

(N.B. the suffices to n and $\mathfrak{m}$ have different offices.)

The evaluation of n_n involves a consideration of $p_n(m)$, the number of partitions of m into at most n parts, (which is equal to the number of partitions of m into parts which do not exceed n). [$p_n(m)$ is the coefficient of x^m in $\prod_{r=1}^n (1 - x^r)^{-1}$: G. H. HARDY and E. M. WRIGHT The Theory of Numbers, (1938) p. 273.] Since $m \leq n$, the partitions can be written down at sight, or stored for purposes of comparison when needed, and we may write simply $p(m)$.

$$n_0 = x^n/n!|_{\zeta=0} = Q_n.$$

$$n_1 = D(x^n/n!)|_{\zeta=0} = (DX)(x^{n-1}/(n-1)!)|_{\zeta=0} = 1 Q_{n-1}$$

$$n_2 = \{(D^2X)(x^{n-1}/(n-1)!) + (DX)^2(x^{n-2}/(n-2)!)\}/2!|_{\zeta=0} = 2 Q_{n-1} + 1 Q_{n-2}$$

$$n_3 = 3 Q_{n-1} + 2 \cdot 1 Q_{n-2} + 1 Q_{n-3}$$

$$n_4 = 4 Q_{n-1} + 3 \cdot 1 Q_{n-2} + 2 \cdot 2 Q_{n-2} + 2 \cdot 1 Q_{n-3} + 1 Q_{n-4}$$

$$\text{c.f. } p(4) = 4 \quad ; \quad 3+1 \quad , \quad 2+2 \quad ; \quad 2+1+1 \quad ; \quad 1+1+1+1$$

$$n_5 = 5 Q_{n-1} + (4 \cdot 1 + 3 \cdot 2) Q_{n-2} + (3 \cdot 1 + 2 \cdot 2) Q_{n-3} + 2 \cdot 1 Q_{n-4} + 1 Q_{n-5}$$

$$\text{c.f. } p(5) = 5 \quad ; \quad 4+1, \quad 3+2 \quad ; \quad 3+1+1, \quad 2+2+1 \quad ; \quad 2+1+1+1 \quad ; \quad 1+1+1+1+1$$

$$n_6 = 6 Q_{n-1} + (5 \cdot 1 + 4 \cdot 2 + 3 \cdot 2) Q_{n-2} + (4 \cdot 1 + 3 \cdot 2 + 2 \cdot 3) Q_{n-3} + (3 \cdot 1 + 2 \cdot 2 + 1 \cdot 4) Q_{n-4} + 2 \cdot 1 Q_{n-5} + 1 Q_{n-6}$$

and so on.

Now, by LEIBNIZ's theorem for the $(n-1)$ -th derivative of a product

$$\hat{n} = \sum_{r=0}^{n-1} \binom{n-1}{r} D^r (1 + \zeta)^{\lambda-1} \cdot D^{n-1-r} x^n/n!|_{\zeta=0},$$

$$\text{moreover } D^r (1 + \zeta)^{\lambda-1}/r!|_{\zeta=0} = \binom{\lambda-1}{r}$$

$$\text{so } \hat{n} = \sum_{r=0}^{n-1} (n-1)! \binom{\lambda-1}{r} n_{n-1-r}$$

$$= (n-1)! \left\{ n_{n-1} + (\lambda-1) n_{n-2} + \binom{\lambda-1}{2} n_{n-3} + \dots + \binom{\lambda-1}{n-1} n_0 \right\}$$

$$\text{and } F = \sum_{n=1}^{\infty} \hat{n} = \sum_{n=1}^{\infty} \binom{\lambda-1}{n-1} \left[(n-1)! n_0 + n!(n+1)_1 + (n+1)!(n+2)_2 + \dots \right]$$

$$\text{So } [F] = \begin{bmatrix} \binom{\lambda-1}{0} \\ \binom{\lambda-1}{1} \\ \binom{\lambda-1}{2} \\ \binom{\lambda-1}{3} \end{bmatrix}^T \begin{bmatrix} 1_0 & 2_1 & 3_2 & 4_3 & \dots \\ & 2_0 & 3_1 & 4_2 & \dots \\ & & 3_0 & 4_1 & \dots \\ & & & 4_0 & \dots \end{bmatrix} \begin{bmatrix} 0! \\ 1! \\ 2! \\ 3! \end{bmatrix}$$

Moreover, if m_k denote $(m/Q)^k/k!$, with $m = m_1$, $k > 0$, then

$$n_1/n_0 = n^{(1)}$$

$$n_2/n_0 = n^{(2)} + n^{(2)} \cdot 1_2$$

$$n_3/n_0 = n^{(3)} + n^{(2)} \cdot 2_1 + n^{(3)} \cdot 1_3$$

$$n_4/n_0 = n^{(4)} + n^{(2)} \cdot 3_1 + 2_2 + n^{(3)} \cdot 2_1 + n^{(4)} \cdot 1_4,$$

etc.

$$\text{i.e. } \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} = n_0 \begin{bmatrix} 1) \\ 2), 1_2) \\ 3), 2)1) \\ 4), 3)1) + 2_2) \\ 5), 4)1) + 3)2), 3)1_2) + 2_2)1), 2)1_3), 1_5) \end{bmatrix} \begin{bmatrix} n \\ n^{(2)} \\ n^{(3)} \\ n^{(4)} \\ n^{(5)} \end{bmatrix}$$

but this will not be as satisfactory a form for the computer as the previous one, if m_k involves the quotient of two integral functions in s . However, since $s^{\mu} = \beta^{-\mu}$, the quotient of two polynomials which arises when μ is a natural number, is itself a polynomial.

Writing s for $Q = \chi(0) = \Sigma a$

t for $1 = \chi'(0) = \Sigma a\mu$

u for $s/(1-t)$,

when $|t| < 1, |u| < 1$, F takes a simpler form:

The coefficient of $\binom{\lambda-1}{n-1}$ in F is $(n-1)!n_0 + n!(n+1)_1 + (n+1)!(n+2)_2 + \dots$

$$= (n-1)! \frac{s^n}{n!} + n! \frac{ts^n}{n!} + (n+1)! \left(2 \frac{s^{n+1}}{(n+1)!} + \frac{t^2}{2!} \frac{s^n}{n!} \right) \\ + (n+2)! \left(3 \frac{s^{n+2}}{(n+2)!} + 2t \frac{s^{n+1}}{(n+1)!} + \frac{t^3}{3!} \frac{s^n}{n!} \right) \\ + (n+3)! \left(4 \frac{s^{n+3}}{(n+3)!} + [3t + 2_2] \frac{s^{n+2}}{(n+2)!} + 2 \frac{t^2}{2!} \frac{s^{n+1}}{(n+1)!} + \frac{t^4}{4!} \frac{s^n}{n!} \right) + \dots$$

$$= \frac{s^n}{n} \left\{ 1 + nt + \binom{n+1}{2} t^2 + \binom{n+2}{3} t^3 \dots \right\} = s^n(1-t)^{-n}/n \\ + 2s^{n+1} \left\{ 1 + (n+2)t + \binom{n+3}{2} t^2 \dots \right\} + 2s^{n+1}(1-t)^{-n-2} \\ + 3s^{n+2} \{ 1 + (n+3)t + \dots \} + 3s^{n+2}(1-t)^{-n-3} \\ + 2_2 s^{n+3} \{ (n+3) + (n+3)(n+4)t \dots \} + 2_2 s^{n+3}(n+3)(1-t)^{-n-4} \\ + 3_2 s^{n+3} \{ (n+4) + (n+4)(n+5)t \dots \} + 3_2 s^{n+3}(n+4)(1-t)^{-n-5} \\ + 4s^{n+3} \{ 1 + (n+4)t + \dots \} + 4s^{n+3}(1-t)^{-n-4} \\ + \dots + \dots$$

$$= u^n \left\{ \frac{1}{n} + \frac{1}{s} [u^2 2 + u^3 3 + u^4 (4 + \frac{n+3}{3} 2_2) + u^5 (5 + \frac{n+4}{s} 3_2) \right. \\ \left. + u^6 (6 + \frac{n+5}{s} \{ 4_2 2 + 3_2 \} + \frac{(n+5)^{(2)}}{s^2} 2_3) + \dots] \right\}$$

$$= u^n \{ (1/n) + 2u^2 + 3u^3 + [4] + (n+3) \cdot 2_2 \} u^4 \\ + [5] + (n+4) \cdot 3_2 \} u^5 \\ + [6] + (n+5) \{ 4_2 2 + 3_2 \} + (n+5)^{(2)} \cdot 2_3 \} u^6 + \dots \}$$

using m_k) for $(u/s)^k/k!$

$$F = \sum_n \sum_1^{\infty} \binom{\lambda-1}{n-1} u^n \left\{ \frac{1}{n} + \sum_m^{\infty} m u^m + \sum_m^{\infty} (n+m-1) \binom{n-1}{m} u^m \cdot e_{mn} \right\}$$

where e_{mn} is the element remaining in the m -th row and n -th column of the large lower matrix on p. 47 after removing from it ^{the first column, and, elsewhere,} all terms in which the digit 1 is mentioned ~~on p. 47~~, i.e.

$$F = [(1+u)^{\lambda} - 1]/\lambda + u[(1+u)^{\lambda-1} \sum_m^{\infty} m u^m + R]$$

where $R = \sum_n^{\infty} (u d/du)^n (1+u)^{\lambda-1} \sum_m^{\infty} u^{n+m} \cdot e_{mn}$.

If μ is a natural number, then, in the $(M-2)$ -dimensional array of coefficients, except for the hypercuboidal block of 2^{M-2} terms near the origin, either

- (i) each μ -th term will vanish in every row parallel to every axis, or
 - (ii) uniformly spaced rows will vanish, or
 - (iii) complete blocks of terms will vanish throughout the heart of the array,
- and in the remaining terms, clustered along the axes, some clusters will have terms of alternating signs. These facts have emerged from practical use of the method; by HADAMARD's "gap" theorem, they preclude over-convergence. The matrix formulation of F (p. 47) may be abbreviated to $g M f$, where $g = \left[\binom{\lambda-1}{n-1} \right]_{n=1}^{\infty}$, $f = [0! \ 1! \ 2! \ 3! \ \dots]^T$, and

$$M = \begin{bmatrix} 0 & 1Q & 2Q_2 + 1_2Q & 3Q_3 + 21Q_2 + 1_3Q & 4Q_4 + 31Q_3 + 2_2Q_3 + 21_2Q + 1_4Q & \dots \\ & Q_2 & 1Q_2 & 2Q_3 + 1_2Q_2 & 3Q_4 + 21Q_3 + 1_3Q_2 & \dots \\ & & Q_3 & 1Q_3 & 2Q_4 + 1_2Q_3 & \dots \\ & & & Q_4 & 1Q_4 & \dots \\ & & & & Q_5 & \dots \\ & & & & & \dots \end{bmatrix}$$

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