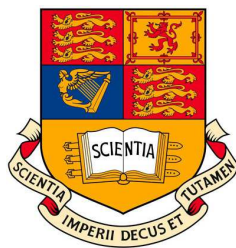


Linearised Navier-Stokes equations and near-wall streaks in turbulent flow subject to drag reduction by wall oscillations

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Abstract

Spanwise wall-oscillation is a promising drag reduction technique in turbulent flows. A major factor in drag reduction is the effect of the oscillations on the boundary layer coherent structures, and particularly the weakening of the near-wall streaks. This thesis studies the streaks in various wall oscillations configurations, based on a linear approach to the turbulent flow. The two main aspects are the study of the structures themselves, and an attempt to predict drag.

The streak structure is studied mainly in turbulent channel flow subject to harmonic wall forcing, using an optimal perturbation technique. It is shown that the streaks have an angle to the main flow direction, which is almost constant during half an oscillation period and experiences a jump in sign and magnitude twice per period. The linear theory shows that this phenomenon is due to the existence of a structure which is dominant during half a period and has a constant angle. Other features of the linear optimal perturbations are studied, such as their comparison with conditionally averaged turbulent structures.

In order to predict drag, the optimal perturbation approach is found to be unsuitable. A more appropriate technique is to use the linearised Navier-Stokes equation subject to random forcing. This was done for a turbulent channel flow subject to travelling wave wall oscillations, thus offering a wide range of comparison with direct numerical simulations, including known regions of drag reduction and drag increase. The main finding is that in the area where drag increase is observed in turbulent flow, the linear operator is unstable. In area where the operator is stable, drag reduction is always predicted.

These two topics are the core of this thesis. Other aspects include the derivation and implementation of an optimal perturbation algorithm and a linear solver. Certain theoretical aspects of the optimal perturbation approach were also investigated.

Preface

This PhD was part of a group project involving the universities of Warwick and Cardiff as well as Imperial College London, and funded by EPSRC and Airbus UK. I guarantee that the work presented in this thesis is the result of my own research. Information obtained from other published or unpublished work are acknowledged and referenced in the text and in the bibliography. The chapter 5 is currently being written for publication, and the chapter 6 is under review for publication in *Journal of Fluid Mechanics*. Some results from chapter 6 have also been presented at the *seventh symposium on turbulent and shear flow phenomena*.

I am grateful to my supervisor Sergei Chernyshenko for his continuous help during this project. He was always ready to spend time with me, even when he was already very busy. Also, the discussions we had were always interesting and constructive, and have been fundamental to make new ideas emerge.

I would like to thank the members of the research group. Among other things, the interactions I had with them have been particularly useful to understand the simulations needed for the last chapter of this thesis. Sylvain Lardeau was really helpful in providing a program that could be adapted for the purpose of this research, and spending time with me talking about many of its numerical aspects. Luca Burini, supervised by Tamer Zaki, also contributed to this thesis by spending time on a cross-validation of our adjoint optimisation programs.

Too many people have been involved more or less directly to this thesis to be all named here. My officemates have been helpful for science related issues, but also mostly for making the few years I spent there an enjoyable and unforgettable experience. Finally, my family and friends far from London have been of great support, helping me to go through difficult moments and always understanding my decisions. Now at the end of this project, I realise how so many people made this experience possible and rewarding.

Olivier Blesbois

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Chapter 1

Introduction and main concepts

Turbulent flows are omnipresent in our modern society, with applications in fields as various as aeronautics, shipping or oil and gas industry. The Aeronautics industry itself represents today 2% of the CO_2 emissions on the planet. This can be considered as a relatively small share of the total emissions, but with the expected increase in air travel, this number could grow significantly. However, the European commission vision for aviation ACARE2050 (Advisory Council for Aeronautical Research in Europe) sets a target of decreasing the total emissions from aviation industry by 50% in 2050 compared to 2005. Achieving such an ambitious goal will require a tremendous effort in every field where improvement is possible. Progress has to be made in various areas such as structures, air traffic management or aerodynamics.

In aerodynamics, major improvements can be made by decreasing the friction drag of aircraft. In standard cruise configuration, about 60% of the total drag is due to viscous friction (Iwamoto et al., 2005). The best possible drag reduction could be achieved by having a laminar flow over the whole surface of aircraft skin. There is a lot of effort made in laminar wing design, as well on the fluid mechanics point of view as for the required material improvements. However, if a fully laminar wing might eventually exist, a significant part of the aircraft skin will still be subject to turbulent flow. This is particularly the case of the fuselage on which, due to its length, transition from laminar to turbulent flow will occur near to the front of the aircraft. This will happen no matter how smooth the material used are, and justifies a research effort to mitigate the drag due to turbulence.

The increase in drag between a laminar and an equivalent turbulent flow field is

due to the energy dissipation by turbulent structures. Some structures of interest are the near-wall streaks. They are universal structures of turbulent boundary layers, elongated in the streamwise direction. Their presence increases the flow shear and turbulent energy generation, and therefore the drag. If a way were found to decrease the streaks energy, then some drag reduction could be expected.

Many possible techniques can be used to decrease turbulent drag. Drag reduction by transverse wall-oscillation is one of them. It has shown to be promising, with drag reduction of up to 40%, and energy gain of up to 20% at modest Reynolds number (Quadrio et al., 2009). This thesis aims at studying the near-wall streaks in configuration with transverse wall-oscillations, in order to better understand the streaks mechanisms and their link to drag reduction. To do so, a linearised treatment of the Navier-Stokes equations will be used. This approach is simpler than studying the fully non-linear Navier-Stokes operator and is expected to give better insight into the streaks physical mechanisms.

In this introduction, a state of the art of the various fields related to this thesis is given. Near-wall streaks and other coherent structures of turbulent flows are described, as well as the effect that transverse wall-oscillations can have on them. It is then explained why the use of linearised equations in turbulent flow can be justified, and how this can be used to understand the effect of wall-oscillations on near-wall streaks.

1.1 Turbulent boundary layer structures

Turbulent boundary layers have been studied for many decades. Their global behaviour is well known (Pope, 2000; Schlichting, 1955; Jiménez and Pinelli, 1999), but a significant amount of the underlying physics is still poorly understood. They are composed of two main regions of interest; the inner layer, close to the wall, and where the direct effects of viscosity are significant, and farther from the wall the outer layer where the dissipation is mostly due to the turbulent Reynolds stress. In the outer layer, the appropriate dimension to describe the problem is based on the the usual domain width, bulk velocity and viscosity, and in the inner layer the appropriate dimensions are wall units. Wall units are denoted in this thesis by the subscript $^+$, and are based on the viscosity ν , the shear stress at the wall and the fluid density. Three areas of interest are present in the inner layer. If y is the wall-normal coordinate, the area situated at

$y^+ < 5$ is the viscous sublayer, where viscous effects are dominant. The flow is turbulent in this area, but it has the mean velocity of a laminar profile, which is equal to the distance to the wall if expressed in wall units. Farther from the wall, for $y^+ \geq 30$, the log layer is dominated by inertial phenomenon, and the mean profile can be calculated assuming the dominance of the Reynolds stress. In this area, the turbulence production is equal to the dissipation. Between these two regions, the buffer layer is situated in the region $5 \leq y^+ \leq 30$.

The description of these main regions and the mean flow is relatively simple. However, if looking at the temporal evolution of the flow, many coherent structures are present. Some of the most relevant ones are hairpin vortices, horseshoe vortices and near-wall streaks. An extensive literature the structures of near-wall turbulence exists, and was reviewed by Robinson (1991), Alfonsi (2006) and Panton (1997, 2001).

Some of the coherent structures which will be relevant for this thesis are the large scale structures. They are present in the outer layer, in the outer part of the log-layer. They are elongated in the streamwise direction, and are characterised by a deficit or surplus in streamwise velocity. They are scaled in outer variables (Tomkins and Adrian, 2005); in a channel flow where the walls are spaced by a distance h , their spacing in the spanwise direction is around $2h$. Literature about these structures can be found in Hutchins et al. (2011), Jiménez (1998) and del Álamo and Jiménez (2003). The main interest in these structures for this thesis is the modulation effect they can have on near-wall streaks, this effect becoming more important as the Reynolds number increases (Hutchins and Marusic, 2007; Hunt and Morrison, 2000).

The existence of near-wall streaks has been known for decades. A first hint of their presence was given by an observation of a spanwise reorganisation of the velocity by Ferrell et al. (1955), who used dye visualisation. Their existence was then clearly showed by Kline et al. (1967) in visualisations using hydrogen bubbles. The streaks are elongated regions in the streamwise direction of smaller and larger than average streamwise velocity. They are inner layer structures, and their characteristics are therefore expressed in wall units. Their length is of about 1000 wall units, and their spacing in the spanwise direction varies with the distance to the wall (Smith and Metzler, 1983). Present from the viscous sublayer to the lower part of the log layer, their average spacing close to the wall is of 80 wall units with a most probable spacing of 100 wall units. Their spacing increases with the distance to the wall. For a distance to the wall

$y^+ < 50$, Kim et al. (1987) showed that the streaks parameters are independent of the Reynolds number (if expressed in wall units). The main effect of the Reynolds number on the streaks characteristics seems to be a modulation by the large scale structures of the outer layer region, as these large scale structures are dependent on the Reynolds number and become more energetic when the Reynolds number is increased (Mathis et al., 2009). The life cycle of the streaks is described as structures appearing close to the wall, then moving farther from the wall as their energy increases, and finally due to nonlinear effects they burst generating turbulent energy (Kim et al., 1971).

The mechanisms leading to streak formation are not yet well understood, but it is already known that they rely on the inner-layer turbulence. Numerical experiments showed for example that no outer layer structure is needed for the streaks to exist (Jiménez and Moin, 1991; Hutchins et al., 2011), and therefore their existence relies on autonomous mechanisms. It is also known that the linear terms in the Navier-Stokes operator are of prime importance to the streak formation, as a numerical experiment by Kim and Lim (2000) showed that the term $v\partial U/\partial y$, with U the mean streamwise velocity and v the wall-normal fluctuation velocity, is responsible for their formation. These observations are however far from sufficient to explain the physical mechanisms leading to the streaks formation. Many ideas have been suggested relying on the cyclic appearance of the streaks, for example streamwise vortices generating streaks by lift-up mechanisms, and then streaks generating vortices by bursting. An extensive literature about these ideas and their link to the near-wall turbulence regeneration cycle is available (Hamilton et al., 1995; Waleffe, 1998; Jiménez and Pinelli, 1999; Schoppa and Hussain, 2002).

Explaining streaks formation mechanisms by the effect of other structures such as vortices is however a difficult problem. The correlation between the pattern of streaks and pattern of vortices was studied experimentally. While streaks are relatively easy to identify in an experiment, identifying vortical structures is much more difficult and requires development of sophisticated techniques involving nonlinear transformations of the data (Adrian et al., 2000; Alfonsi, 2006). Using one such technique, namely, the swirling strength criterion, a clear association of vortices and streaks was obtained by Tomkins and Adrian (2003). To assess the robustness of the method, Chernyshenko et al. (2006) applied exactly the same technique to the velocity field artificially synthesized as a linear superposition of two independent and uncorrelated velocity fields:

a velocity field containing streaks but no vortices and a velocity field containing randomly located vortices but no streaks. Surprisingly, the same association of vortices and streaks as reported by Tomkins and Adrian (2003) was observed. This result was traced to the aberrant consequence of the nonlinearity of the swirling strength criterion, and allowed to conclude that the association between vortices and streaks detected by the swirling strength criterion and, by proxy, other similar criteria, might be (but not necessarily is!) an artefact of the nonlinear data-handling techniques rather than the property of the actual data. The situation is further complicated by the existence of the large-scale structures which might be created by a different mechanism. In any case studies using the idea that the pattern of streaks is dictated by the pattern of vortices have limited predictive ability, since neither the dependence of streak spacing on the distance to the wall (Smith and Metzler, 1983), nor the variation of the streak spacing at the same distance to the wall but for passive scalars with different mean concentration distribution (Baig and Chernyshenko, 2004) can be explained.

A promising idea to solve the streaks formation mechanism problem is to use travelling waves solutions. At its origin, the presence of a near-wall turbulence cycle led to looking for exact solutions of the Navier-Stokes equation showing some temporal periodicity properties. Travelling wave solutions, constructed for a number of canonical flows (see, for example, Waleffe, 2001; Faisst and Eckhardt, 2003) can be considered as a convenient model of the near-wall turbulence cycle. Moreover, it is possible that such solutions are more than just ingredients of a model. It can be conjectured that from the dynamical system viewpoint travelling wave solutions are unstable periodic orbits inside the turbulent attractor, so that the trajectory of the dynamical system wanders between the vicinities of these periodic orbits. Various features of turbulence can then be understood in terms of the properties of travelling wave solutions. For example, Kawahara (2009) recently reviewed the mechanisms of the regeneration of coherent structures, relying, in addition to theoretical understanding, mostly on low-to-moderate Reynolds number calculations, and described an application of these ideas for explaining secondary flows of Prandtl's second kind. However, Kerwell and Tutty (2007) on the basis of numerical calculations came to the conclusion that the visits of travelling wave vicinities are not frequent enough to view turbulence purely as the random switching between the neighbourhoods of travelling waves.

More hope is associated with the extension of travelling waves solutions to include

the so called relative periodic orbits, that is solutions that are time periodic in a frame of reference moving with a suitable speed. There are reasons to believe (Cvitanović and Gibson, 2010) that with sufficient number of travelling waves and periodic orbits, the accuracy of the representation of the turbulent flow they provide can be high. The first relative periodic orbits for a fluid flow were calculated by Viswanath (2007) for the case of a Couette flow, and Duguet et al. (2008) for the case of a pipe flow. The approach of calculating a large number of relative periodic orbits remains, however, prohibitively expensive from the computational viewpoint.

A promising alternative approach to the travelling wave solutions is to use a linearised treatment of the Navier-Stokes equation. Linearising involves approximation of the physics, and therefore the accuracy of predictions can not be expected to be perfect. However this method is a good trade-off between the accuracy and the computational efficiency. In Chernyshenko and Baig (2005), this approach allowed quantitative predictions for velocity and passive scalar streak spacing as a function of the distance to the wall, Reynolds number, and the shape of the mean profile of the velocity and of the passive scalar. More than 35 comparisons exploring this dependence were made, with the streak spacing varying by more than one order of magnitude. The prediction error was rarely larger than 30% when the distance to the wall was below 50 wall units. However, the approach of Chernyshenko and Baig (2005) is limited in scope, as it does not allow to predict, for example, the amplitude of the streaks, as well as many other parameters of the turbulent flow. The method developed in Chernyshenko and Baig (2005) constitute the starting point of this thesis, and will be explained later.

1.2 Drag reduction by wall-oscillations

Many techniques exist to decrease the viscous drag in turbulent boundary layers. Some of these methods are based on a modification of the fluid properties, such as the use of polymer and surfactant addition. Most of them however rely on the modification of some wall properties, such as the use of compliant surfaces and MEMS (Kasagi et al., 2009). One of the most famous techniques is the use of riblets, elongated objects in the streamwise direction, which modify the near-wall streaks (Bechert and Bartenwerfer, 1989; Choi, 1989; Chu and Karniadakis, 1993; Viswanath, 2002). This technique has showed to be efficient, but so far has never been used for industrial application due

to the excessive cost of manufacturing and maintaining such surfaces. So far, no drag reduction technique has proven to be usable for a wide range of applications, and the physical mechanisms leading to drag reduction are themselves not well understood yet. Reviews of various drag reduction techniques can be found in (Bushnell, 2003; Karniadakis and Choi, 2003).

One of the drag-reduction methods widely studied by the fluid mechanics community, and which is considered throughout this thesis, is the drag reduction by spanwise wall-oscillations. The ability of spanwise oscillations of the wall to reduce the skin friction in turbulent flow was first demonstrated numerically by Jung et al. (1992), following earlier experimental (Bradshaw and Pontikos, 1985) and numerical observations (Moin et al., 1990) of transient drag reduction due to imposed spanwise shear. An experimental verification was obtained soon (Laadhari et al., 1994). This generated large interest prompting numerous investigations and resulting in significant progress. In particular, it was demonstrated numerically that not only can drag be reduced, but also the overall power balance can be positive (Baron and Quadrio, 1996). It was shown by Choi and Clayton (2001) that the near-wall burst activity is decreased when the wall oscillates. Within a certain range of situations, the drag reduction level achieved by spanwise oscillations was demonstrated (Quadrio and Ricco, 2004) to collapse as a function of an empirical parameter proposed by Choi (2002), usually denoted S^+ and which combines the oscillation period, its amplitude, penetration depth of the Stokes layer and the spanwise acceleration at a certain distance to the wall, all scaled in wall units. It was also shown that the drag reduction level decreases as the Reynolds number increases (Ricco and Quadrio, 2008; Toubert and Leschziner, 2012). Quadrio et al. (2009) investigated a more complicated form of the in-plane wall motion when the spanwise velocity has the form of a travelling wave $w_w = A \sin(kx - \omega t)$, where x is the coordinate in the mean flow direction, which is also the direction in which the wave is travelling. Calculations over a range of the wavenumbers k and frequencies ω revealed a rich behaviour with regions of both drag reduction and drag increase. The maximum drag reduction and net power saving achieved in channel flow at $Re_\tau = 200$ were 48% and 18% respectively. Such high values are quite encouraging.

The mechanisms of drag reduction by spanwise wall-oscillations remain unclear (Toubert and Leschziner, 2012). However, it is generally agreed that these mechanisms are somehow related to the organised structures observed in near-wall turbulent flows,

and in particular to streaks. The streaks are indeed generating most of the near-wall turbulence energy and therefore a significant amount of the turbulent drag. If, as many authors suggested, spanwise oscillations disrupt the turbulence regeneration cycle, the potential decrease in the streaks amplitude would explain lower level of turbulence and as a consequence smaller drag. It is therefore interesting to study the effect of wall-oscillations on near-wall turbulence. A review of these effects is given by Karniadakis and Choi (2003). Experimental and numerical observations show that streaks become oblique, that is they are elongated in a direction at a nonzero angle to the main flow direction. The streak angle varies with time. The streak strength varies with time strongly, so that during certain time intervals streaks are difficult to identify. The near-wall streaky structure is also affected by the large-scale motions in the outer part of the flow (Touber and Leschziner, 2012). Choi et al. (1998) explain the streak angle variation by the advection of the streaks associated with the spanwise motion (from the viewpoint of streaky pattern being dictated by the pattern of vortices, although their explanation can equally be applied to the streaks). Their explanation corresponds to the situation when the oscillating wall is adjacent to a non-oscillating section upstream. When streaks (or vortices that create the streaks within the streak formation mechanisms they presume) approach the oscillating wall, the downstream ends of the streaks are moved sideways in the direction of the wall motion, turning the streak (or the vortex). This mechanism does work near the leading edge of the oscillating plate, as it can clearly be seen from the figures included in their paper. However, further downstream the angle actually observed have the sign opposite to the sign of the angle predicted by this mechanism (Touber and Leschziner, 2012). Ricco (2004) suggested a different mechanism, pointing out that typical streaks are inclined in the wall-normal direction, so that their downstream end is further away from the wall than the upstream end. Since the spanwise velocity induced by the oscillating wall decreases with the distance from the wall, the amplitude of the oscillations of the upstream end of the streak is greater than that of the downstream end. This explanation gives the streak angle of the same sign as that observed in reality. However, both in numerical and physical experiments, the streak angle variation is usually identified by visualising streaks in a plane parallel to the wall, where the spanwise velocity induced by wall-oscillations is the same near the upstream and downstream end of the streak, so that the proposed mechanism should not work; yet, streak angle variation is observed. To overcome this

difficulty, one could try to complement this mechanism with viscosity effects leading to the velocity diffusion in wall-normal direction. In any case, so far none of the above mechanisms has been used to make quantitative predictions of the streak angle. Such predictions are made in this thesis, and the mechanisms are discussed.

The tool used in this thesis to study the effect of wall-oscillations on streaks is the linearised approach of Chernyshenko and Baig (2005). As seen in section 1.1, linear techniques can be used to predict the streaks in unforced flow, and are therefore a good candidate to analyse the streaks in flow subject to wall-oscillation and attempt to predict drag reduction. The study of Duque-Daza et al. (2012) gives a strong support to this approach, as they used only linearised equations and demonstrated quantitatively that the complicated dependence of drag reduction on k and ω is very similar to the behaviour of the magnitude of the transient growth of a certain initial disturbance. Moarref and Jovanović (2012) also used the linearised equations to predict the effect of spanwise ($k = 0$) wall-oscillations on the drag. These mechanisms provide good qualitative descriptions of the effect of wall-oscillations, and support the idea that drag reduction can be at least partially explained by linear phenomenon. However, they do not provide any relevant physical mechanisms. This is one of the motivations of the present study, and some new ideas will be developed thorough this thesis.

1.3 Linearised treatment of the Navier-Stokes equations

The application of linearised equations to turbulent flows has a very long history. Turbulence is a nonlinear phenomenon, and it is of particular interest here to discuss the motivation for such studies.

The most natural justification for the use of linearised equations rests on the assumption that the nonlinear terms are small. This can be justified - say, if the fluid undergoes a rapid distortion, for example, when passing through a sudden constriction of a pipe. The corresponding Rapid Distortion Theory (often referred to as RDT) is usually attributed to Batchelor and Proudman (1954), although Hunt (1973) also refers to earlier works by Prandtl and Taylor, while Hunt and Kevlahan (1993) also refer to much earlier work by Kelvin. A more recent review is given in Cambon and

Scott (1999). Among other results, the rapid distortion theory was able to predict the existence of streaks (Lee et al., 1990). The literature on rapid distortion theory and its applications is very extensive; the above reviews give more details. In the case when statistically-stationary turbulence is considered, rapid distortion theory represents the velocity as a sum of the mean velocity and the fluctuation. Often, the linear terms in the equations are collected on the left-hand side, while the nonlinear terms are moved to the right-hand side and considered as forcing of a linear system. One can assume the forcing to be small, but other assumptions are possible. For example, Landahl (1977) assumes that the characteristic length scale of forcing is much smaller than the length scale of the mean flow, while Jovanović and Bamieh (2001) and Jovanović and Georgiou (2010) simply prescribe particular statistical characteristics of the forcing and investigate how much new information can thus be obtained (it is of course possible to prescribe the forcing in such a way that the solution of the linearised equations has all the characteristics of the actual turbulent flow, but then the approach would have no predictive ability). The overall outcome of the studies is that certain, but not all, features of real turbulent flows can be predicted reasonably well on the basis of such approaches, but also the predictions are often inaccurate. This is not surprising, of course, since in near-wall turbulence the nonlinear terms are not negligible, and their characteristic scales are comparable to the scales of the mean flow, so that one needs to look for an alternative justification for the use of linearised equations.

Another possible justification to use linearised equation in turbulent flow problems follows from the observation that in many situations, for example in the case of a fluid layer heated from below, the patterns observed in nonlinearly-saturated systems are close, qualitatively and sometimes also quantitatively, to the patterns of the fastest-growing linear instabilities. While the reasons for this behaviour are not fully understood, one can simply use this as an assumption instead of making the assumption that the nonlinear terms are small. In most cases, however, the pattern of turbulent flow is markedly different from that of the linear instability modes of the corresponding laminar flow. It is only one step then to replace the laminar base flow by the mean velocity profile of the turbulent flow and to investigate the linear stability of such a system. Malkus (1956) went even further by proposing that the mean velocity profile in a turbulent flow should adjust itself until all small perturbations are either stable or neutrally stable while the smallest scale perturbations are neutrally stable. This amounts

to a theory of turbulence, since, if it were true, it would allow finding the time-averaged characteristics of a turbulent flow. Reynolds and Tiederman (1967) showed, however, that for the channel flow with mean turbulent profile any small perturbation decays. Reynolds and Hussain (1972) replaced the laminar viscosity with an eddy viscosity in this approach, and showed that, with the particular turbulent viscosity model they used, the flow was also linearly stable. Later, Sen and Veeravalli (1998, 2000) discovered that instabilities could be found when the turbulence model was upgraded to take into account the anisotropy effects. They obtained a certain agreement between the behaviour of the unstable modes and the experimental measurements of the phase-averaged evolution of perturbation introduced in an experiment. Gaster et al. (1985) investigated the inviscid instability of a base flow taken as the mean velocity field in a turbulent mixing layer, and found a good agreement with experiment when large-scale perturbations were introduced in that flow.

The above-mentioned studies stemmed from the classical modal linear stability theory. The theory of optimal perturbations (Butler and Farrell (1992), Henningson et al. (1993), Reddy and Henningson (1993), Schmid and Henningson (1994), Andersson et al. (1999), Luchini (2000), Trefethen et al. (1993), and Schmid (2007)) describing only transient, but very large growth of initial perturbations in laminar flows was also used as a prototype for several attempts to explain the patterns in turbulent flows. An optimal perturbation is a solution of the linearised Navier-Stokes equations with maximal possible transient growth that is the maximum of the ratio of the perturbation energy to its initial value over time and all possible initial conditions. Butler and Farrell (1993) replaced the laminar parabolic profile in a channel flow with the mean turbulent velocity profile and found the optimal perturbation. The streak spacing thus obtained was much larger than that observed in developed turbulent flows. They then suggested performing the optimisation only over a limited time period, not longer than the so-called eddy turnover time. This brought the results in closer agreement with experiments, but at the expense of introducing an adjustable parameter. Chernyshenko and Baig (2005), Lockerby et al. (2005), and Carpenter et al. (2007) calculated perturbations that maximized not the entire energy of the perturbation, but different measures of its magnitude, and obtained agreement with the results of various numerical and physical experiments. Optimal perturbations with the laminar viscosity replaced by an eddy viscosity were calculated by Del Álamo and Jiménez (2006), whose

results were further refined by Pujals et al. (2009) and Cossu et al. (2009). Interestingly, the optimal energy growth was found to have two peaks in the spanwise wavelength, one scaling in outer units, supposedly corresponding to large-scale structures, and a secondary peak scaling in inner units, supposedly corresponding to near-wall streaks spaced of about 100 wall units.

The numerous results summarised above paint a mixed picture. In many cases, the results obtained using linearised equations agree with Direct Numerical Simulations (DNS) and experiments, but in many other cases they disagree significantly. In some cases, no attempt is made to justify the linearised approach, and in other cases the assumptions used to justify it are not applicable to near-wall turbulent flow. The present work is based on the justification introduced and explored by Chernyshenko and Baig (2005) - namely, that the Navier-Stokes equations linearised around the mean turbulent velocity profile have selectivity, or filtering, properties, which vary significantly depending on what particular flow characteristic is considered as the output of the filter. Depending on the particular flow characteristic to be predicted, the filtering properties can be strong or weak, and only predictions for those properties for which the linearised equations describe a strong filter can be made on the basis of these equations together with the assumption of the broadband nature of the forcing. It appears that for many cases in which the linearised approach was applied to predicting the patterns in developed turbulent flows, the resulting agreement or disagreement can be explained on the basis of this justification.

In the light of the above discussion, it is justifiable to use the generalised optimal perturbation approach of Chernyshenko and Baig (2005) in an attempt to explain at least some of the properties of the patterns observed in turbulent flows past spanwise-oscillating walls.

1.4 Thesis overview

This thesis is a step towards improving the understanding of the predictive abilities of the linearised Navier-Stokes equation in turbulent flow. Two main kind of wall-oscillations will be used; harmonic wall-oscillations where the wall has a rigid movement in the transverse direction, and travelling waves where the wall transverse motion depends on the position in the streamwise direction as well as on time. Using wall-

oscillations, more parameters are available for investigation than in simple turbulent flow with fixed walls. Two main features of the flow will be studied; the structures of the streaks themselves as well as their formation mechanisms, and the drag.

To study the streaks, the generalised optimal perturbation approach as defined in Chernyshenko and Baig (2005) is the appropriate tool. It will first be used in chapter 5 in the case of a simple turbulent channel flow to predict passive scalar streaks profile. This is a complement of the study of passive scalar streaks made in Chernyshenko and Baig (2005), showing the limits of some assumptions used in this paper and suggesting a solution to guarantee the quality of the results.

The generalised optimal perturbation approach will then be adapted to the case of time dependent mean flow and used in chapter 6 for the case of turbulent flows subject to harmonic wall forcing. The streaks will be predicted for different forcing frequencies and at different observation times. The prediction will show a reasonable agreement observations from direct numerical simulations, particularly for the estimation of the streaks angle. The conclusions of this chapter have been presented at the *seventh symposium on turbulence and shear flow phenomenon* (Blesbois and Chernyshenko, 2011), and are under review for publication in *Journal of fluid mechanics* (Blesbois et al., 2012).

It will be shown that to predict drag reduction, the generalised optimal perturbation approach is not the most suitable tool. A method similar to the generalised optimal perturbation approach will be defined, where the measure of structures energy is the same, but a random forcing of the linear equation is used instead of calculating optimal perturbations. This approach seems more appropriate to predict the total turbulent energy, and therefore the drag. It will be used in chapter 7 for turbulent flow subject to travelling wave spanwise oscillations. The more important number of parameters defining a travelling wave wall-oscillation compared to an harmonic oscillation allow for a large number of comparisons to be made. It will be shown that to some extent the linear analysis can predict drag reduction, but more importantly some properties of the linear operator in this situation will be revealed. For some oscillation parameters, it will be showed that the linear operator becomes unstable.

Before using the linear analysis in turbulent flow, this thesis starts with a more theoretical part. In chapter 2, the two methods used; the generalised optimal perturbation approach and the one using random forcing are described, and the equations needed

to solve all the problems are derived. The difference between these two approaches and the reason why one is more appropriate to predicting structures and the other one more appropriate to describe drag are given. In chapter 3, the mathematical procedure used to calculate optimal perturbations when the generalised optimal perturbation approach is used is described. Finally in chapter 4, the numerical code used throughout this thesis is described, and its validation is explained.

Chapter 2

Linearised Navier-Stokes equations in turbulent flows

2.1 Linearised Navier-Stokes equations, a filter analogy

In various science and technology fields, the use of linearised equations can be easily justified when nonlinear terms are small and can be neglected. However, in many applications these terms are not small and no rigorous mathematical justification can be made to remove them from the governing equation. Many examples of such use are present for example in finance and economy, where linear regressions are used to estimate the behaviour of some products, when the governing equations are not known and therefore there is no reason to assume they are linear. The fact that a linear equation can describe properly the behaviour of some variables of a nonlinear system can sometimes be qualitatively justified using a “filter” analogy.

Consider, for example, a nonlinear electronic device containing a linear filter and a number of nonlinear elements. Suppose that in this system there are broadband fluctuations. Suppose also that the filter is a narrow-pass filter. Then, if at the input of the filter the signal is broadband, the output of the filter will be a narrow-band signal; it will have a dominant frequency. Now, one can disassemble the device, take the filter into the laboratory, and study it there, separately. As a result of this linear analysis (the filter itself is linear, and no nonlinear components were brought into the

laboratory) one will determine the band-pass properties and, hence, will be able to predict the dominant frequency at the outlet of the filter when it works as a part of the nonlinear system. For this, only a linear analysis is required, and the nonlinear effects in the full system need not be small.

In fluid mechanics, the use of linearised equations has been a common tool to study the transition to turbulence, as in this case the nonlinear terms are small. Using linearised equations in turbulent flow is more recent, as no simple justification of the validity of this approach can be made. The justification made here is based on the idea of filtering. Its origin is approximately thirty years old, but it is difficult to find exactly when this concept was first introduced. Two complementary approaches to the filtering concept are commonly used: The nonlinear part of the equation is either replaced by a stochastic forcing, or removed and the linear part of the operator is then studied by some optimal perturbation theory. Replacing the nonlinear term by a random forcing has been done to predict turbulent flow statistics (Bamieh and Dahleh, 2001; Hwang and Cossu, 2010b). The second approach, removing the nonlinear term and using an optimal perturbation theory, has first been used by Butler and Farrell (1993) to predict streaks in turbulent channel flows. Many similar methods have been used since then as tools to attempt to explain the patterns in various turbulent flow configurations (Del Álamo and Jiménez, 2006; McKeon and Sharma, 2010; Hwang and Cossu, 2010a).

To explain more clearly the idea of filtering in the case of fluid flows, the Navier-Stokes equations have to be considered. Being in an incompressible turbulent flow, the velocity $\mathbf{u}$ can be represented as $\mathbf{u} = \mathbf{U} + \mathbf{u}'$, the sum of a base flow $\mathbf{U}$ and a fluctuation $\mathbf{u}'$. One can then rewrite the full Navier-Stokes equations with the fluctuations as new variables. The linearized terms can be collected on the left hand side, and the remaining nonlinear terms can be placed on the right hand side:

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} + \nabla p - \frac{1}{Re} \Delta \mathbf{u}' = \mathbf{F}, \quad (2.1)$$

$$\mathbf{F} = -\frac{\partial \mathbf{U}}{\partial t} - \mathbf{U} \cdot \nabla \mathbf{U} - \mathbf{u}' \cdot \nabla \mathbf{u}' - \nabla P + \frac{1}{Re} \Delta \mathbf{U}. \quad (2.2)$$

The base flow $\mathbf{U}$ is assumed incompressible and satisfying the same boundary conditions as $\mathbf{u}$, so that the fluctuation velocity $\mathbf{u}'$ satisfies homogeneous boundary conditions as

well as the continuity condition:

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u}' = 0.$$

A note regarding terminology has to be made; in this thesis, $\mathbf{u}'$ is referred to as a fluctuation velocity when it corresponds to a real turbulent flow field, and as a perturbation velocity when it is used in the linearised Navier-Stokes equation framework.

A justification of the filtering approach was suggested by Chernyshenko and Baig (2005). It states that under the hypothesis that the linearized equation has filtering properties, and the right hand side $\mathbf{F}$ is broadband, then studying only the linearized equation is sufficient to understand some of the characteristics of the turbulent flow. This idea is represented in figure 2.1; if the output of the filter is weakly dependent

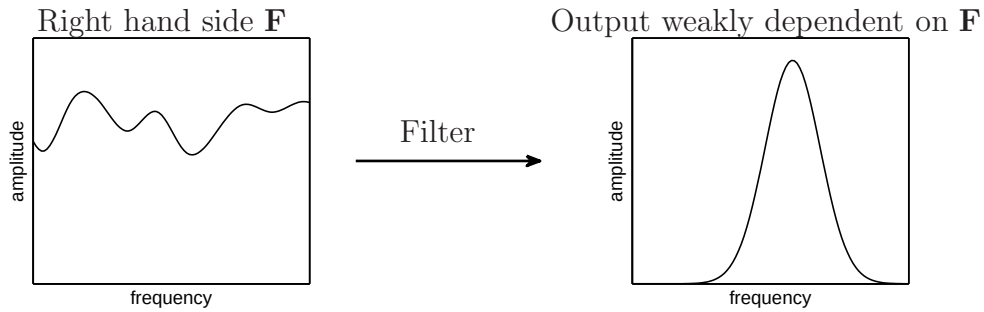


Figure 2.1: Filter analogy: if the linear operator has strong enough selectivity properties, the output of the filter is not strongly dependent on the term $\mathbf{F}$.

on the form of $\mathbf{F}$, then this term can be replaced by any reasonable function without significantly modifying the results. The definition of what constitutes a reasonable function remains open, and the whole approach can be justified only by the quality of the predictions made on a sufficiently large set of examples.

In this thesis, the filter analogy will be applied to the study of near-wall streaks. The amplitude of the filter as described in figure 2.1 will be a measure of the streaks energy, and two different forms of the right hand side $\mathbf{F}$ will be used. The first one is based on the generalised optimal perturbation model developed in Chernyshenko and Baig (2005) and used to predict the most energetic streaks. The structure found in this approach is the one maximising the amplitude of the filter output, this approach therefore determines the centre of the filter passband and the height of its peak (fig-

ure 2.2(a)). To study the width of the passband (figure 2.2(b)), the same measure of the filter output amplitude is kept, but a different forcing term will be used. In that case, $\mathbf{F}$ will simply be a white noise in space and time.

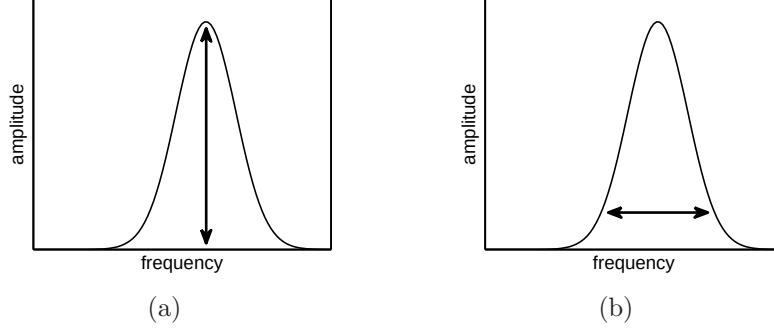


Figure 2.2: filter properties: (a) represents the structures selected by the generalised optimal perturbation approach, which is at the centre of the filter passband. Using a white noise random forcing would rather lead to finding the passband of the filter (b).

The two forcings used in this thesis will be complementary to each other. The generalised optimal perturbation approach will be a useful tool to predict the structure of streaks and explain their formation mechanisms. It will be used in chapter 5 to study passive scalar streaks in turbulent channel flow, and in chapter 6 to clarify the streaks formation mechanisms in turbulent flow subject to harmonic wall forcing. This approach will, however, have no ability to predict the drag, as it focuses only on the peak of the filter and does not take into account any other property such as the passband. For drag reduction purposes, using white noise forcing will be a more appropriate tool. This will be done in chapter 7 for turbulent channel flow subject to travelling wave wall oscillations. The remaining part of this chapter focuses on the generalised optimal perturbation approach and describes it in details.

2.2 The generalised optimal perturbation approach

The generalised optimal perturbation approach of Chernyshenko and Baig (2005) was designed to study the near-wall streaks. To study the linear filter, the right hand side $\mathbf{F}$ was replaced by a delta-correlated function of time $\mathbf{F} = \delta(t - t_0)\mathbf{u}_0$. This temporal forcing problem amounts in this condition to solving an initial value problem for the

linearised equation. The initial value $\mathbf{u}_0$ was determined using an optimal perturbation approach, and the criteria of this optimisation was to maximise a specific measure of the streaks energy.

Using the filter analogy and the representation of figure 2.1, it is interesting to consider which property of the filter is extracted by doing such an optimisation. The “amplitude” in this figure would corresponds to the measure of the streak energy in the present case, and the “frequency” would correspond to a measure of how broadband is the input (in space and time). Choosing a delta-correlated forcing in time could be done without loss of generality, as a continuous forcing in time can be written as a convolution product involving such functions. The more restrictive aspect of this approach is to use an optimal perturbation theory to calculate which forcing maximises the amplitude of the output. After optimisation, the specific right hand side $\mathbf{F}$ can no longer be qualified as broadband, and as a result not all of the filter properties will be extracted. Instead of that, the structure obtained is at the centre of the filter passband, with its energy being the maximum of the filter amplitude curve (figure 2.2(a)). This structure is then the most energetic streaks possible. Because of that, it will be referred as the “most probable streak” later in this thesis.

The validity of using the filter analogy in turbulent flows was justified by the results obtained with the generalised optimal perturbation approach in Chernyshenko and Baig (2005). The ability of the structure predicted to represent the streaks was assessed by calculating streaks spacing of passive scalar profiles. Predictions were within 30% of the streaks spacing obtained in direct numerical simulations, over more than one order of magnitude of variation in streaks spacing. This validates the use of linear approaches in turbulent flows, and also highlights their limitations. They capture fundamental physical mechanisms, but due to the assumptions made can not have a very high accuracy.

In this section, most theoretical properties of the generalised optimal perturbation approach will be described. The norms used to measure the streaks energy are first defined in the case of turbulent channel flow. They are then extended to the case of channel flow subject to spanwise wall oscillations. Finally some additional information about these structures, the mean flow and the linearised equation will be given.

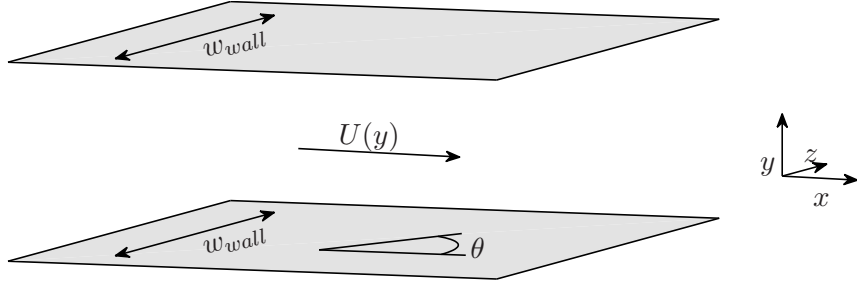


Figure 2.3: Flow configuration. When there is wall oscillations, the two walls are moving in phase, with the wall velocity w_{wall} depending on the time t and the streamwise direction x . In all the simulations performed in this thesis, the base flow in the streamwise direction U does not depend on time. θ is a measure of the angle between structures and the streamwise direction.

2.2.1 Generalised optimal perturbation approach in turbulent channel flows

The generalised optimal perturbation approach is described here for the case of turbulent channel flows, as it was used in Chernyshenko and Baig (2005). As it is based on the optimisation of a linear problem, the measure of the perturbation velocity amplitude has to be a ratio of two norms taken at different instants. Here, it will be the ratio of a norm at an “observation time” with a norm at “initial time”. The initial time is the instant when the perturbation is introduced, and the observation time is a later instant at which the transient energy growth is measured.

The norm at observation time is meant to take into account the fact that the streaks in turbulent flows are defined by their excess or deficit of streamwise velocity. It also takes into account their dependence on the distance from the wall. This norm is therefore a measure the streamwise velocity energy at a given distance to the wall and is defined by:

$$\|\mathbf{u}'\|_f^2 = \frac{1}{S} \int_{y=y_0} u^2 dx dz. \quad (2.3)$$

Here the streamwise direction is denoted x , the wall normal direction is y , the spanwise direction is z (figure 2.3), and the corresponding components of the perturbation velocity vector are $\mathbf{u}' = (u, v, w)$. It is implied that a finite computational domain is considered. In equation (2.3), y_0 is the coordinate of the particular plane where the streaks energy is measured, and S is the area of the cross-section at $y = y_0$ of the com-

putational domain. In other words, the final norm squared equals twice the average of the energy of the longitudinal component of the perturbation at the distance y_0 from the wall.

The norm used at initial time is different. To use as little assumptions as possible on the form of the initial condition, a volume energy norm was chosen. In Chernyshenko and Baig (2005), two such norms were defined. The first one is the simple energy norm defined by:

$$\|\mathbf{u}'\|_b^2 = \frac{1}{V} \int_V (v^2 + w^2) dx dy dz, \quad (2.4)$$

where V is the volume of the domain. To use such a norm, the streamwise velocity at initial time has to be equal to zero. The choice to not take the streamwise velocity into account in this norm is justified, as the transient growth mechanisms rely on the transmission of crossflow energy to the streamwise direction via the coupling term $v\partial U/\partial y$ (with the mean flow used, the streamwise velocity equation is an independent equation except from this term). From this fact, it is possible to know that the optimal solution has no component in the streamwise direction at the initial time, and it is therefore not necessary to include this component in the initial time norm. The use of the norm $\|\cdot\|_b$ has the advantage to be simple and can give good results, but it does not prevent the initial time crossflow components from having high energy very close to the wall. To ensure the initial crossflow components have more physical meaning, an alternative initial norm was also defined as:

$$\|\mathbf{u}'\|_c^2 = \frac{1}{V} \int_V \left(\frac{v^2}{\langle v^2 \rangle} + \frac{w^2}{\langle w^2 \rangle} \right) dx dy dz, \quad (2.5)$$

with $\langle v^2 \rangle(y)$ and $\langle w^2 \rangle(y)$ being components of the Reynolds stress. At the same initial energy value, using $\|\cdot\|_c$ will favour initial perturbation having bigger velocity farther to the wall compared to $\|\cdot\|_b$, where $\langle v^2 \rangle$ and $\langle w^2 \rangle$ are large.

The structure experiencing the maximum transient growth is then obtained by maximising the ratio of the norm at the observation time relative to an initial time norm, over all possible observation time t and initial velocity field $\mathbf{u}'_0 = (0, v_0, w_0)$.

$$A_I(y) = \max_{\mathbf{u}'_0, t > 0} \frac{\|\mathbf{u}'(t, y)\|_f^2}{\|\mathbf{u}'_0\|_I^2}. \quad (2.6)$$

Here, the subscript I denotes the use of either $\|\cdot\|_b$ or $\|\cdot\|_c$. Since the standard optimal perturbation problem uses the same norm for both input and output, the new technique was named *generalised optimal perturbation*. Both initial norms have been successfully used in Chernyshenko and Baig (2005) to predict the streaks spacing and their dependence on the distance to the wall. This was done for turbulent channel flow streaks, and also for passive scalar streaks.

In the chapter 5 of this thesis, the passive scalar streaks will be analysed using these same norms, as in Chernyshenko and Baig (2005). Different optimisation methods than in Chernyshenko and Baig (2005) will be used, and it will be showed that some of the approximations made in that paper in the optimisation process are not valid. It will be shown that even with this issue the results that were obtained are still good, and a method to improve the optimisation method will be suggested.

2.2.2 Generalised optimal perturbation approach in flows subject to wall oscillations

The main purpose of this thesis is to study turbulent channel flow subject to transverse wall oscillations. For that, the norms developed in Chernyshenko and Baig (2005) and described in section 2.2.1 have to be adapted. The norm $\|\cdot\|_f$ at the observation time is kept as described in equation (2.3), the streaks still being defined by their streamwise velocity excess or deficit. The simple initial norm descried in equation (2.4) however had to be modified to take into account every component of the velocity energy at initial time. In the case of flow subject to wall oscillations, it is defined as:

$$\|\mathbf{u}'\|_i^2 = \frac{1}{V} \int_V (u^2 + v^2 + w^2) dx dy dz, \quad (2.7)$$

This is similar to $\|\cdot\|_b$, but all three velocity components are included here instead of only v and w , since we cannot expect that the initial condition of the optimal perturbation will have $u = 0$ in the this case. This is mainly due to the fact that the streaks are no longer supposed to be aligned with the streamwise direction, and the more complex behaviour of the linearised Navier-Stokes equation does not allow for a further simplification of this norm. However, it can be observed that if a channel flow with no wall oscillations is used, performing the optimisation problem of equation (2.6)

using $\|\cdot\|_b$ or $\|\cdot\|_i$ will result in exactly the same optimal solution. This is easy to show; with no wall oscillations and as explained in section 2.2.1, the solution experiencing the largest transient growth has no component in the streamwise direction at initial time, and therefore the only term differing in these two norms will be equal to zero in this case. The other possible initial norm $\|\cdot\|_c$ could similarly be adapted to the case of flows subject to wall oscillations, but this was not done in the present study. The main reason is that the calculations in the presence of wall oscillations have been performed without using data from direct numerical simulation, and therefore the turbulent Reynolds stress was not available.

Another difference due to the time dependence of the mean flow is the definition of the optimisation procedure. As the mean flow depends on time, the streaks observed in the real turbulent case will also depend on the observation time. Statistics can be done in this case by considering a phase average, but not with a time average which would remove a significant amount of information. To take this into account, the generalised optimal perturbation at an observation time t is defined as the solution $\mathbf{u}'$ of the following optimisation problem:

$$A(t_f) = \max_{\mathbf{u}'(t_i), t_i < t_f - \varepsilon} \frac{\|\mathbf{u}'(t_f)\|_f^2}{\|\mathbf{u}'(t_i)\|_i^2}, \quad (2.8)$$

where $\varepsilon > 0$ is a small suitably selected constant. Optimising over t_i is appropriate because the solution for a general form of $\mathbf{F}$ can be represented as a convolution product of solutions with impulse forcing. The generalised optimal perturbation is dependent on the time (or phase) t_f , and is the perturbation achieving the largest possible amplification factor A at $t = t_f$.

For $\varepsilon = 0$ the solution of this optimisation problem is $t_i = t_f$, $A = \infty$, and $\mathbf{u}'$ is zero everywhere except the plane $y = y_0$, where it has a non-zero u component. Such a solution is physically meaningless, since it corresponds to the case when $\mathbf{F}$ is concentrated in the plane $y = y_0$, and regardless how little we might know about $\mathbf{F}$, such a possibility can be excluded. Avoiding this issue could be achieved by modifying $\|\cdot\|_i$. However, since the solution with such initial conditions quickly decays, it is technically easier to eliminate it by introducing $\varepsilon > 0$ into the definition. Note that for values of ε such that the optimisation procedure gives $t_f - t_{i,opt} > \varepsilon$, where $t_{i,opt}$ is the optimal initial time, the result becomes independent of the particular value of ε . We always

select ε so that this is the case.

The optimal perturbation thus defined depends on the final time, or, as we will also call it, the observation time (or phase) t_f , because the mean profile is time-dependent. If the mean flow does not depend on time, the result would not depend on the final time and optimising over t_i would give the same result as the commonly used optimisation over t_f (in this case, the optimisations of equation (2.6) and equation (2.8) give the same result). The dependence of the optimal perturbation on t_f is crucial for comparisons with direct numerical simulation. An optimal perturbation at an observation time t_f will be compared to the direct numerical simulation result at the same observation time t_f .

2.2.3 Generalised optimal perturbation and turbulent flow streaks

In this thesis, the streaks predicted with the generalised optimal perturbation approach will be often compared to turbulent flow streaks. Before doing such comparisons, it is important to know to which extent the generalised optimal perturbation can represent the turbulent flow streak. Using the filter analogy, the output was reduced to a function of a single variable by defining a norm to measure the structures energy. The important point of the generalised optimal perturbation approach is the observation that the filtering properties vary strongly depending on the nature of this reduction. It is crucial to take this into account when predictions are made.

When the filtering properties of the linearized operator are studied using the optimal perturbations, the solution of the optimisation problem is the input that maximises the transient growth of the output. The corresponding solution of the linearized Navier-Stokes equations can readily be interpreted as a fluid flow. The crucial point to appreciate is the nature of the relationship between this flow and the turbulent flow described by equations (2.1,2.2). The idea of a filter explains why one can expect that the output of the filter that was maximised in calculating the optimal perturbation should be similar to the output of the same filter when it is a part of the full non-linear Navier-Stokes system. However, the other parts of the optimal perturbation, whether they can be interpreted as inputs or as outputs of the other filters constituting the linearized operator, might differ arbitrarily from the corresponding parts of the

turbulent flow. The obvious example is the classical optimal perturbation consisting of longitudinal vortices and streaks. The streaks of the optimal perturbation are the features whose magnitude is maximised over all possible inputs, and therefore, they can be expected to be observed in the turbulent flow as well. The longitudinal vortices of the optimal perturbation are not strong (in fact they usually decay monotonously). Hence there is no reason to expect that such vortices would be observed in a turbulent flow. This is, indeed, the case, as visualisations of instantaneous near-wall turbulent flow do reveal long streaks but do not reveal equally long vortices (Adrian, 2007).

2.2.4 Calculation of the mean flow

To conclude the description of the linearised methods used in this thesis, a note has to be made on the choice of the mean flow. When linearising a laminar flow to study the evolution of small perturbation, the choice of the mean flow is imposed. However, in the case of a linearisation in a turbulent flow there is no obvious choice. As the fluctuation velocity itself is of the same order of magnitude as the mean flow, no strong mathematical argument can be made to justify the choice of a mean flow. However, the choice of the base flow affects both the properties of the linearized system in the left-hand side of equation (2.1) and the properties of the right hand side $\mathbf{F}$. Different choice of base flow would lead to completely different results, therefore some justification of the profile has to be done. Since the left-hand side of equation (2.1) does not contain quadratic terms, it is reasonable to select $\mathbf{U}$ in such a way as to minimise $\langle |\mathbf{u}'|^2 \rangle$, where $\langle . \rangle$ denotes ensemble averaging. It is easy to show that this is achieved if $\mathbf{U} = \langle \mathbf{u} \rangle$. In the case of a periodic oscillation of the wall, it makes sense to select the statistical ensemble in such a way that the phase information is preserved, so that $\langle . \rangle$ denotes phase-averaging. In general, one could simply assume that taking the base flow to coincide with the mean (or phase-averaged) flow is part of the empirical observation of the filtering property on which the generalised optimal perturbation approach is based.

Now the definition of how to obtain the base flow has been given, the way to calculate it is described. Two possible methods to do so are possible. It can be either extracted from direct numerical simulation by calculating phase or temporal average of the turbulent flow, or calculated analytically. Here, the analytical method will be preferred for two reasons. First, obtaining direct numerical simulation data for all the

cases considered would be numerically very demanding, and even in the comparisons performed in this thesis where direct numerical simulation data was available, the convergence was not sufficiently adequate to provide a meaningful mean flow. This is especially the case for the spanwise component of the profile. Second, one of the main advantages of linearised approaches is to be computationally much cheaper than equivalent direct numerical simulations. If linearised approaches prove to have sufficient predictive abilities, then flow control systems could be designed without the need to perform turbulent calculations. With this in mind, it is clear that to fully test the predictive ability of the linearised approach, the base flow used should not be the mean profile from turbulent calculations, but calculated analytically.

The expression of the mean flows used in this thesis will be given in the respective chapters where they are being used. The global configuration can however already be described and is showed in figure 2.3. The system of coordinates used in this figure has already been introduced in the definition of the various norms used, and will be the same for all the results showed. Three kind of mean flow will be considered; the simplest is the usual turbulent channel flow with no wall oscillations. The two other are harmonic wall forcing, where the wall movement depends only on time, and travelling wave, where the wall movement depends on the time t and on the streamwise position x . For all these cases, the spanwise component $W(x, y, t)$ of the mean flow will be calculated analytically, and the derivation will be explained. The streamwise component of the base flow $U(y)$ will be the one defined in Reynolds and Tiederman (1967). the angle θ between the streamwise direction and structures shown on figure 2.3 will be used to study the generalised optimal perturbation structures in the case of harmonic wall forcing.

2.3 Form of the linearised equation used in this thesis

Whether the wall oscillations follow a harmonic movement or travelling waves, the linear problem is homogeneous in the spanwise direction. Solving the linearised operator is then equivalent to solving a sequence of two dimensional problems, where the spanwise direction is decomposed into wavenumbers. This significantly decreases the complex-

ity of the problem, as solving two-dimensional linear equations can be easily done on a single CPU. Several ways to perform the spanwise wavenumber decomposition exist, with the simplest one being to use complex exponentials. However, the program adapted for use in this thesis was designed to solve turbulent flow problems. As this code was not designed to use complex numbers, another approach was preferred: using a trigonometric decomposition of the perturbation velocity. The perturbation velocity corresponding to a spanwise wavenumber β_z can be written as:

$$\mathbf{u}'_{\beta}(x, y, z, t) = \begin{pmatrix} u_c(x, y, t) \\ v_c(x, y, t) \\ w_c(x, y, t) \end{pmatrix} \cos(\beta_z z) + \begin{pmatrix} u_s(x, y, t) \\ v_s(x, y, t) \\ w_s(x, y, t) \end{pmatrix} \sin(\beta_z z). \quad (2.9)$$

From that and equation (2.1), the momentum equation can be calculated. Also taking into account the fact that the sine and cosine modes are orthogonal to each other (when a norm relative to the standard energy norm is used), the modes can be separated, and two momentum equations are obtained. The component of the momentum equation projected on cosines is:

$$\frac{\partial u_c}{\partial t} + U \frac{\partial u_c}{\partial x} + W \beta_z u_s + v_c \frac{\partial U}{\partial y} = -\frac{\partial p_c}{\partial x} + \frac{1}{R} \Delta_{\beta_z} u_c, \quad (2.10a)$$

$$\frac{\partial v_c}{\partial t} + U \frac{\partial v_c}{\partial x} + W \beta_z v_s = -\frac{\partial p_c}{\partial y} + \frac{1}{R} \Delta_{\beta_z} v_c, \quad (2.10b)$$

$$\frac{\partial w_c}{\partial t} + U \frac{\partial w_c}{\partial x} + W \beta_z w_s + u_c \frac{\partial W}{\partial x} + v_c \frac{\partial W}{\partial y} = -\beta_z p_s + \frac{1}{R} \Delta_{\beta_z} w_c, \quad (2.10c)$$

where the diffusive term is $\frac{1}{R} \Delta_{\beta_z} f = \frac{1}{R} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \beta_z^2 f \right)$. The momentum equation projected on sines is:

$$\frac{\partial u_s}{\partial t} + U \frac{\partial u_s}{\partial x} - W \beta_z u_c + v_s \frac{\partial U}{\partial y} = -\frac{\partial p_s}{\partial x} + \frac{1}{R} \Delta_{\beta_z} u_s, \quad (2.11a)$$

$$\frac{\partial v_s}{\partial t} + U \frac{\partial v_s}{\partial x} - W \beta_z v_c = -\frac{\partial p_s}{\partial y} + \frac{1}{R} \Delta_{\beta_z} v_s, \quad (2.11b)$$

$$\frac{\partial w_s}{\partial t} + U \frac{\partial w_s}{\partial x} - W \beta_z w_c + u_s \frac{\partial W}{\partial x} + v_s \frac{\partial W}{\partial y} = \beta_z p_c + \frac{1}{R} \Delta_{\beta_z} w_s. \quad (2.11c)$$

To obtain these equations, it was also taken into account that for the wall oscillation configurations used the mean flow has no wall-normal component V , and the streamwise component of the mean flow U depends only on the wall-normal coordinate y . Similarly, two continuity equations are obtained:

$$\frac{\partial u_c}{\partial x} + \frac{\partial v_c}{\partial y} + \beta_z w_s = 0, \quad (2.12a)$$

$$\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} - \beta_z w_c = 0. \quad (2.12b)$$

In a simple case with no wall oscillations, the mean velocity W would disappear from the equations, and these two sets of equations (projection on sine and projection on cosine) would be identical to each other. However, for the cases with spanwise oscillations, a coupling between the modes is present, and it is necessary to solve the six momentum equations together, as well as the two continuity equations. Numerically, this does not involve additional cost compared to using a complex exponential wavenumber decomposition of the problem.

Chapter 3

Optimisation procedure to extract the dominant near-wall structure

The main purpose of this chapter is to derive a method to solve the optimisation problem necessary to find the perturbation experiencing the largest transient growth when the generalised optimal perturbation approach is used. The optimisation problem of equation (2.6) or equation (2.8) can be decomposed into two substeps. First, the initial time t_i , final time t_f and the spanwise wavenumber β_z are set and the initial solution leading to the maximum growth at observation time t_f is found. Then, an optimisation over these parameters is performed. This chapter focuses on the first step of this optimisation. For the remaining part of this chapter, the parameters t_i , t_f and β_z are set. The method used is adjoint optimisation, as it usually converges relatively quickly and does not require too heavy modifications to a Navier-Stokes solver to be implemented.

Adjoint optimisation is a common tool to calculate optimal disturbances of linear problems in fluid mechanics. It has been used in a wide range of examples, from the study of oceanographic flows (Farrell and Moore, 1991) to the transition to turbulence (Farrell and Moore, 1991; Hill, 1995; Luchini, 2000; Schmid, 2007; Biau and Bottaro, 2009). Practically, it consists in starting a simulation with a random initial guess, solving the linear problem forward in time, and then solving the so called ‘adjoint problem’ backward in time. This is one step of the optimisation problem, after which a new initial perturbation is obtained. The algorithm is iterated several times, and

at each iteration the resulting initial perturbation is going to be closer to the optimal initial perturbation. In this chapter, the fundamentals of the adjoint optimisation technique used are given in detail.

3.1 Adjoint optimisation in finite dimensions

In this section, the problem is described as a finite dimensional problem. This is the case when the flow is discretised, and therefore covers the conditions of use of the adjoint optimisation approach in this thesis. Solving the linearised Navier-Stokes equations between instants t_i and t_f in a computer program is equivalent to multiplying the initial perturbation vector by a matrix. To derive the optimisation algorithm, we will suppose that this matrix is known. If this operator is denoted by R , the perturbation velocity at observation time is linked to the perturbation velocity at initial time by the equation:

$$\mathbf{u}'(t_f) = R(t_f, t_i)\mathbf{u}'_i. \quad (3.1)$$

Here, $(R(t_f, t_i) \in \mathcal{M}_p(\mathbb{R}))$ is a square matrix. To simplify the notation of the problem, the notations $\mathbf{u}'_f = \mathbf{u}'(t_f)$, $\mathbf{u}'_i = \mathbf{u}'(t_i)$, and $R = R(t_i, t_f)$ are used. The equation then becomes:

$$\mathbf{u}'_f = R\mathbf{u}'_i. \quad (3.2)$$

With this notation, the problem to solve is to find the vector $\mathbf{u}'_i$ that is the most amplified by the operator R for a given norm. The theory of linear finite dimensional operators guarantees that this vector exists (R is a continuous function, and the optimisation is made over a compact set). In the remaining part of this section, R is supposed to be known, and an algorithm to find the most amplified initial perturbation is developed.

3.1.1 Principle of the algorithm

Adjoint iteration

The algorithm is first derived supposing that the norm L^2 of $\mathbf{u}'_f$ is optimised, given a specific norm L^2 of $\mathbf{u}'_i$. This algorithm will then be modified for the case where different norms at the initial time and at the observation time are used. The scalar

product associated to the L^2 norm is written $\langle \cdot, \cdot \rangle$. Using the norm L^2 , the maximisation problem can be written as:

$$\max_{\mathbf{u}'_i} \frac{\|\mathbf{u}'_i\|_2^2}{\|\mathbf{u}'_i\|_2^2} = \max_{\mathbf{u}'_i} \frac{\langle R\mathbf{u}'_i, R\mathbf{u}'_i \rangle}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle} = \max_{\mathbf{u}'_i} \frac{\langle \mathbf{u}'_i, R^*R\mathbf{u}'_i \rangle}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle}, \quad (3.3)$$

with R^* the adjoint of R . The quantity R^*R is a real-valued quadratic form, therefore diagonalisable in orthonormal basis, and its eigenvalues are either positive or equal to zero. It can now be proven that solving the optimisation problem is equivalent to finding the biggest eigenvalue of R^*R .

As R^*R is diagonalisable in orthonormal basis, there exist a unitary operator A transforming the canonical basis of $\mathbb{R}^p$ into an orthonormal basis. Due to this property, AR^*RA^{-1} is then a diagonal operator. It is reminded that AR^*RA^{-1} has the same eigenvalues as R^*R , $A^{-1} = A^*$, and $\|A\mathbf{v}\|_2 = \|\mathbf{v}\|_2$ for any vector $\mathbf{v}$. The equation (3.3) can be written:

$$\max_{\mathbf{u}'_i} \frac{\langle \mathbf{u}'_i, R^*R\mathbf{u}'_i \rangle}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle} = \max_{A\mathbf{u}'_i} \frac{\langle A\mathbf{u}'_i, AR^*RA^{-1}A\mathbf{u}'_i \rangle}{\langle A\mathbf{u}'_i, A\mathbf{u}'_i \rangle} = \max_{\mathbf{u}'_i} \frac{\langle \mathbf{u}'_i, AR^*RA^{-1}\mathbf{u}'_i \rangle}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle}. \quad (3.4)$$

As AR^*RA^{-1} is diagonal, it is now easy to see that the optimal solution $\mathbf{u}'_i$ is any vector belonging to the maximum eigenvalue eigenspace, and that the norm amplification is equal to the maximum eigenvalue. From that we can conclude that the maximum energy increase of an initial perturbation is the square root of the maximum eigenvalue of R^*R .

Now, the optimisation algorithm can be derived. It is developed using the operator R^*R as it is what is going to be used for the numerical resolution of the problem. However, it is easier to understand how it works by considering the diagonal operator AR^*RA^{-1} , as the two formulation are identical. The reader can have in mind this second formulation of the problem, or imagine that R^*R is diagonal for a better understanding.

The basic idea of the algorithm is that iterating R^*R on a ‘random’ initial vector $\mathbf{u}'_{i,0}$ that has components in all the eigenspaces of the operator, the direction of the vector obtained after n iteration is going to converge towards a direction within the maximum eigenvalue eigenspace. For example, if E_{max} is the maximum eigenvalue and E_2 is the second biggest eigenvalue, the result at iteration $n+1$ is going to be E_{max}/E_2

more in a maximum eigenspace direction than at iteration n . In the real algorithm, we normalise the solution at each step to avoid a quick divergence of the result (if the maximum eigenvalue is greater than one). The following sequence is built:

$$\mathbf{u}'_{i,n} = \frac{(R^*R)^n \mathbf{u}'_{i,0}}{\|(R^*R)^n \mathbf{u}'_{i,0}\|_2}. \quad (3.5)$$

From this formulation, it is easy to see that $\mathbf{u}'_{i,n}$ converges towards a vector $\mathbf{u}'_{i,\infty}$ which has zero components in all directions orthogonal to the maximum eigenvalue subspace. The vector $\mathbf{u}'_{i,\infty}$ corresponds to one optimum of the figure 3.3. If the maximum eigenspace is of dimension one, it is the only solution of norm one.

To numerically perform the optimisation process, $\mathbf{u}'_{i,n}$ has to be calculated iteratively. It is already known that $R\mathbf{u}'$ represent the linear Navier-Stokes solver. The operator R^* would be easy to calculate if all the elements of R were known. However calculating these terms would be computationally too demanding, except for very simple configurations. Another approach will be used to calculate R^* in the next section. It will be an operator similar to the linear Navier-Stokes operator, but which has to be solved backwards in time. As no explicit formulation of the operators is available, the only things that can be calculated are the quantities $R\mathbf{v}$ and $R^*\mathbf{v}$ for arbitrary vector $\mathbf{v} \in \mathbb{R}^p$. With that in mind, the following algorithm to calculate $\mathbf{u}'_{i,n+1}$ from $\mathbf{u}'_{i,n}$ is described:

- Step 1: Direct problem

The initial solution $\mathbf{u}'_{i,n}$ is known. The linear Navier-Stokes equation is solved between t_i and t_f . The final solution $\mathbf{u}'_f$ is then obtained:

$$\mathbf{u}'_{f,temp} = R\mathbf{u}'_{i,n}$$

- Step 2: Adjoint problem

The adjoint problem is solved from the equation which will be derived in the next section, backward in time from t_f to t_i . The ‘initial’ solution $\mathbf{u}'_i$ is obtained:

$$\mathbf{u}'_{i,temp} = R^*\mathbf{u}'_{f,temp}$$

- Step 3: Renormalisation

To guarantee the next initial solution is still of norm L^2 equal to 1:

$$\mathbf{u}'_{i,n+1} = \frac{\mathbf{u}'_{i,temp}}{\|\mathbf{u}'_{i,n}\|_2}$$

From these substeps, it is easy to prove by recurrence that the initial solution $\mathbf{u}'_n$ after n iterations is the same as the one given in equation (3.5):

$$\mathbf{u}'_{i,n+1} = \frac{\mathbf{u}'_{i,temp}}{\|\mathbf{u}'_{i,n}\|_2} = \frac{R^* R \mathbf{u}'_{i,n}}{\|\mathbf{u}'_{i,n}\|_2} = \frac{(R^* R)^{n+1} \mathbf{u}'_i}{\|(R^* R)^n \mathbf{u}'_i\|_2}.$$

Now the algorithm has been derived, a criteria has to be chosen to estimate when $\mathbf{u}'_{i,n}$ is ‘close enough’ to the limit $\mathbf{u}'_{i,\infty}$. The value $\mathbf{u}'_{i,\infty}$ is obviously not known, but the sequence $(\mathbf{u}'_{i,n})_{n>0}$ can be shown to converge towards this value as an exponential (the convergence is geometric). It will then be sufficient to stop the algorithm when the rate of variation of the energy increase between two iterations becomes small enough.

Use of a different final norm

The L^2 energy norm for which the algorithm has been described corresponds to the energy norm over the whole domain. This is the norm commonly used when adjoint optimisation of the linearised Navier-Stokes operator is performed, but is different from the norms used in this thesis. The optimisation algorithm has to be adapted to the use of different norms at initial and observation time, as well as to the fact that the observation time norm $\|\cdot\|_f$ used in this thesis is not a norm, but a semi norm. In a first step, the initial time norm is still supposed to be the usual energy norm, and the final norm as the is written as the L^2 norm of a projector:

$$\|\mathbf{u}'\|_f = \|P\mathbf{u}'\|_2.$$

Here P is the projector on the observation layer at a distance $y = y_0$ from the wall. As it is orthogonal, the following equality is obtained:

$$\|\mathbf{u}'\|_f = \sqrt{\langle \mathbf{u}', P^* P \mathbf{u}' \rangle} = \sqrt{\langle \mathbf{u}', P P \mathbf{u}' \rangle} = \sqrt{\langle \mathbf{u}', P \mathbf{u}' \rangle}$$

The optimisation problem then becomes:

$$\max_{\mathbf{u}'_i} \frac{\|\mathbf{u}'_f\|_f^2}{\|\mathbf{u}'_i\|_2^2} = \max_{\mathbf{u}'_i} \frac{\langle R\mathbf{u}'_i, PR\mathbf{u}'_i \rangle}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle} = \max_{\mathbf{u}'_i} \frac{\langle \mathbf{u}'_i, R^*PR_0 \rangle}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle}. \quad (3.6)$$

R^*PR is still a quadratic form. Therefore all what has been previously presented remains valid, and only an additional step needs to be added between the step 1 and the step 2 of the optimisation algorithm to take into account the projector:

- Step 1.bis:

$$\mathbf{u}'_{f,temp} \leftarrow P\mathbf{u}'_{f,temp}$$

Use of a different initial norm

To use a different initial norm, a slightly different strategy is needed. Due to the division by the initial norm in the renormalisation step, a seminorm can not be used. The new norm is then written as a multiplication by a nonsingular matrix in the norm L^2 :

$$\|\mathbf{u}'\|_i = \|B\mathbf{u}'\|_2$$

The scalar product associated to this norm can be written as $\langle \mathbf{u}', \mathbf{v} \rangle_B = \langle B\mathbf{u}', B\mathbf{v} \rangle$. The optimisation problem using the new initial norm becomes:

$$\max_{\mathbf{u}'_i} \frac{\|\mathbf{u}'_f\|_2^2}{\|\mathbf{u}'_i\|_i^2} = \max_{\mathbf{u}'_i} \frac{\langle B^{-1}R\mathbf{u}'_i, B^{-1}R\mathbf{u}'_i \rangle_B}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle_B} = \max_{\mathbf{u}'_i} \frac{\langle \mathbf{u}'_i, R^*B^{*-1}B^{-1}R\mathbf{u}'_i \rangle_B}{\langle \mathbf{u}'_i, \mathbf{u}'_i \rangle_B}. \quad (3.7)$$

In this case, two things have to be changed in the algorithm. As for the change of final norm, a step 1bis has to be added. Moreover, the renormalisation step has to be modified:

- Step 1.bis:

$$\mathbf{u}'_{f,temp} \leftarrow B^{*-1}B^{-1}\mathbf{u}'_{f,temp},$$

- Step 3:

$$\mathbf{u}'_{i,n+1} = \frac{\mathbf{u}'_{i,temp}}{\|\mathbf{u}'_{i,n}\|_i},$$

If both the initial and final norm are changed, the operation performed in the step 1.bis is: $\mathbf{u}'_f \longleftarrow B^{*-1}PB^{-1}\mathbf{u}'_f$

3.2 Derivation of the adjoint equation

Now that the optimisation algorithm has been described, the adjoint R^* of the linearised Navier-Stokes operator R has to be calculated. As the direct problem $R\mathbf{u}'$ is calculated using a linear Navier-Stokes solver, a similar approach will be used to calculate the solution of the adjoint problem $R^*\mathbf{u}'$. The equation solved at each time step of the linear solver has the form:

$$\mathcal{L}\tilde{\mathbf{u}} + \mathcal{M}\tilde{p} = 0, \quad (3.8a)$$

$$\mathcal{M}.\tilde{\mathbf{u}} = 0, \quad (3.8b)$$

$$\tilde{\mathbf{u}} = \mathbf{0} \text{ at the boundary.} \quad (3.8c)$$

Here the solutions of the continuous problem (before discretisation) are used. To differentiate them from the discretised case, the operators are curled, and vectors have a superscript $\sim$. The same notation will be used throughout this thesis when infinite dimensional operators are used. Here, equation (3.8a) is the momentum equation and equation (3.8b) is the mass conservation equation. The purpose is to find a similar equation for the adjoint problem, which once solved in time would be equivalent to calculating $R^*\mathbf{u}'$. It is supposed that the governing partial differential equation of this adjoint problem has the form:

$$\mathcal{L}^*\tilde{\mathbf{u}} + \mathcal{M}^*p = 0, \quad (3.9a)$$

$$\mathcal{M}^*.\tilde{\mathbf{u}} = 0, \quad (3.9b)$$

$$\text{boundary conditions for } \tilde{\mathbf{u}}. \quad (3.9c)$$

In this section, standard mathematical tools such as Green functions will be used to obtain a method to derive the adjoint equation. It will be shown that the boundary conditions for this equation are the same as for the linearised Navier-Stokes equations. It will also be shown why the adjoint problem has to be solved backward in time to calculate $R^*\mathbf{u}'$. The main tools necessary for the use of Green unction to solve physical

problems can be found in Morse and Feshbach (1953). Some information about adjoint functions can also be found in Courant and Hilbert (1962) and Hill (1995).

3.2.1 Green functions and Lagrange equation

Green functions are a widely-used tool to find solutions of partial differential equations. For a linear partial differential equation, a Green function is usually a function of the homogeneous equation with a delta function δ as the right hand side. The purpose is usually to derive solutions of the non-homogeneous problem, or of the problem with different boundary conditions using the known Green function. In our case, Green functions will not be used to calculate solutions, but will be useful to find the properties of the adjoint problem. For the linearised Navier-Stokes equation, it is supposed that there exist a Green function which provides the solution at any time t and position $\mathbf{r}$, given an impulsional right hand side at time t_0 , position $\mathbf{r}_0$, and in the direction $\mathbf{e}$:

$$\mathcal{L}\tilde{\mathbf{g}}_u(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) + \mathcal{M}\tilde{g}_p(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) = 4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\mathbf{e}, \quad (3.10)$$

$$\mathcal{M}.\tilde{\mathbf{g}}_u(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) = 0. \quad (3.11)$$

Here, $\tilde{\mathbf{g}}_u(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e})$ is the Green function for the velocity component, and the Green function for the pressure component is $\tilde{g}_p(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e})$. The Green function for the linearised equation is $G(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) = (\tilde{\mathbf{g}}_u, \tilde{g}_p)(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e})$. For this problem, it is supposed that there exist a causality relation: if $t < t_0$, then $G(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) = 0$. This equation can also be written in the more condensed operator form:

$$\begin{pmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{M}' & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{g}}_u(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) \\ \tilde{g}_p(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) \end{pmatrix} = \begin{pmatrix} 4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\mathbf{e} \\ 0 \end{pmatrix} \quad (3.12)$$

Where $\mathcal{M}'$ is the transpose of the operator $\mathcal{M}$. This formula is equivalent to the even more condensed form:

$$\mathcal{A} \mathbf{G}(\mathbf{r}, t|\mathbf{r}_0, t_0; \mathbf{e}) = \begin{pmatrix} 4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\mathbf{e} \\ 0 \end{pmatrix} \quad (3.13)$$

A central tool in the use of Green function is the Green theorem. It is used after the linear equation and its ‘adjoint’ are written in the form of a Lagrange identity:

$$\tilde{\mathbf{v}}^* \cdot \mathcal{A} \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \mathcal{A}^* \tilde{\mathbf{v}}^* = \nabla \cdot \mathbf{P}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}^*), \quad (3.14)$$

where ∇ is a generalised gradient, $\tilde{\mathbf{u}}$ a solution of the linear problem, $\tilde{\mathbf{v}}^*$ a solution of the adjoint problem, and the dot designates the usual scalar product. If the Lagrange equation is integrated over all dimensions, and if after application of the Green theorem the right hand side is equal to zero, we see that there is a close link between the solution of the linear problem, and the solution for its adjoint problem. In the present case, the time has a different effect from the space dimensions. This can be seen for example from the causality relation. Therefore, a distinction is kept in the Lagrange equation. Using the form of the adjoint problem that was supposed in equations (3.9a)-(3.9b), the Lagrange equation can be written:

$$\begin{aligned} & [\tilde{\mathbf{v}}^* \cdot (\mathcal{L} \tilde{\mathbf{u}} + \mathcal{M} \tilde{p}_u) + \tilde{p}_v^* \mathcal{M} \cdot \tilde{\mathbf{u}}] + [\tilde{\mathbf{u}} \cdot (\mathcal{L}^* \tilde{\mathbf{v}}^* + \mathcal{M}^* \tilde{p}_v^*) + \tilde{p}_u \mathcal{M}^* \cdot \tilde{\mathbf{v}}^*] \\ &= \frac{\partial \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^*}{\partial t} + \nabla \cdot J(\mathbf{u}', \tilde{\mathbf{v}}^*, \tilde{p}_u, \tilde{p}_v^*). \end{aligned} \quad (3.15)$$

This equation is used to derive the adjoint operator. This is done in appendix A. It will also be proved there that to have a zero value for the spatial part of the right hand side after integration in volume and application of the Green theorem, the boundary condition must be zero for the adjoint velocity, and does not need to be specified for the adjoint pressure.

3.2.2 Adjoint equations for the linearised Navier-Stokes problem

The equations for the adjoint problem are derived in appendix A, but the result is shown here. The structure is globally the same as for the forward linear problem. There are two distinct groups of equations; the first two equations are equivalent to the mass conservation equations, and the remaining ones are equivalent to the momentum

equations. The mass conservation equations are given by:

$$\begin{aligned}\frac{\partial u_c^*}{\partial x} + \frac{\partial v_c^*}{\partial y} + \beta_z w_s^* &= 0 \\ \frac{\partial u_s^*}{\partial x} + \frac{\partial v_s^*}{\partial y} - \beta_z w_c^* &= 0\end{aligned}\tag{3.16}$$

The first part of the momentum equation can be linked to the projection on cosine of the linear momentum equation:

$$\frac{\partial u_c^*}{\partial t} + \frac{\partial U u_c^*}{\partial x} + W \beta_z u_s^* - w_c^* \frac{\partial W}{\partial x} = -\frac{\partial p_c^*}{\partial x} - \frac{1}{R} \Delta_{\beta_z} u_c^* \tag{3.17a}$$

$$\frac{\partial v_c^*}{\partial t} + \frac{\partial U v_c^*}{\partial x} + W \beta_z v_s^* - u_c^* \frac{\partial U}{\partial y} - w_c^* \frac{\partial W}{\partial y} = -\frac{\partial p_c^*}{\partial y} - \frac{1}{R} \Delta_{\beta_z} v_c^* \tag{3.17b}$$

$$\frac{\partial w_c^*}{\partial t} + \frac{\partial U w_c^*}{\partial x} + W \beta_z w_s^* = -\beta_z p_s^* - \frac{1}{R} \Delta_{\beta_z} w_c^* \tag{3.17c}$$

$$\tag{3.17d}$$

The second part of the momentum equation can be similarly linked to the projection on sine:

$$\frac{\partial u_s^*}{\partial t} + \frac{\partial U u_s^*}{\partial x} - W \beta_z u_c^* - w_s^* \frac{\partial W}{\partial x} = -\frac{\partial p_s^*}{\partial x} - \frac{1}{R} \Delta_{\beta_z} u_s^* \tag{3.18a}$$

$$\frac{\partial v_s^*}{\partial t} + \frac{\partial U v_s^*}{\partial x} - W \beta_z v_c^* - u_s^* \frac{\partial U}{\partial y} - w_s^* \frac{\partial W}{\partial y} = -\frac{\partial p_s^*}{\partial y} - \frac{1}{R} \Delta_{\beta_z} v_s^* \tag{3.18b}$$

$$\frac{\partial w_s^*}{\partial t} + \frac{\partial U w_s^*}{\partial x} - W \beta_z w_c^* = \beta_z p_c^* - \frac{1}{R} \Delta_{\beta_z} w_s^* \tag{3.18c}$$

$$\tag{3.18d}$$

The form of these equations is very similar to the one of the linear problem. It can be seen that the same coupling between the projection on sine equations and the projection on cosine equations is present. The first coupling is in the convective term between the sine and cosine velocities, and the second coupling is at the pressure level and in the mass conservations equation. In the case where there is no wall normal component for the mean velocity ($W = 0$), the coupling disappears in the same way as for the linear problem. In this particular case, one set of independent linear equations has for adjoint one set of the independent adjoint equations, and the other set of linear

equations is linked to the other set of adjoint equations. Therefore in that case there are two decoupled problems.

The main mathematical difference between the adjoint equation and the linear equation is the role of the diffusive term. The sign is inverted. It means that as the linear equation, the adjoint problem is not revertible in time. Instead of being solved forward in time, this system has a well defined solution only if it is solved backward in time. Green functions can be derived for these equations the same way as for the linear equations. But due to the fact that the equations are solved backward in time, the Green functions will give the response of a non-homogeneous impulse at time t_1 to the solution at time $t_0 < t_1$. The differential equation governing the Green function is:

$$\mathcal{L}^* \tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}) + \mathcal{M}^* \tilde{\mathbf{g}}_p^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}) = 4\pi \delta(\mathbf{r}_1 - \mathbf{r}) \delta(t_1 - t) \mathbf{e}, \quad (3.19)$$

$$\mathcal{M}^* \tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}) = 0. \quad (3.20)$$

As for the linear problem, the causality relation is not derived, but its existence is assumed due to the similarity between the two problems. It is written as:

$$\tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}) = 0 \quad \text{if } t > t_1 \quad (3.21)$$

The backward dependence in time is also present in this equation. The Green function has effect at time smaller than t_1 but has no effect forward in time.

Example of use of the Green theorem

In the remaining part of this chapter, the purpose is to prove that solving numerically the adjoint equation is a way to calculate $R^* \mathbf{u}'_f$. For that the Green functions and their properties will be used. A fundamental equation is first derived, which results from the use of the Lagrange equation and the causality relations. Then, this equation is used to prove the reciprocity relation; a link between the direct and adjoint problem Green functions. It will also be used to express the solution of the homogeneous equation with given initial solution in term of Green function.

As described earlier, it is usual to integrate the Lagrange equation over the volume to derive useful properties of the Green functions. The basic calculations are performed here, as they will be needed later. Equation (3.15) is integrated over the spatial domain

Ω , and between instant t_i^- and t_f^+ . This leads to:

$$\begin{aligned} \int_{\Omega} \int_{t=t_i^-}^{t_f^+} [\tilde{\mathbf{v}}^* \cdot (\mathcal{L}\tilde{\mathbf{u}} + \mathcal{M}\tilde{p}_u) + \tilde{p}_v^* \mathcal{M} \cdot \tilde{\mathbf{u}}] + [\tilde{\mathbf{u}} \cdot (\mathcal{L}^* \tilde{\mathbf{v}}^* + \mathcal{M}^* \tilde{p}_v^*) + \tilde{p}_u \mathcal{M}^* \cdot \tilde{\mathbf{v}}^*] d\Omega dt \\ = \int_{\Omega} \int_{t=t_i^-}^{t_f^+} \frac{\partial \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^*}{\partial t} + \nabla \cdot J(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}^*, \tilde{p}_u, \tilde{p}_v^*) d\Omega dt \end{aligned}$$

If the functions $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}^*$ satisfy the direct and adjoint equation with a right hand side $(\tilde{\mathbf{a}}_u, 0)$ and $(\tilde{\mathbf{a}}_v, 0)$, this can be simplified:

$$\int_{\Omega} \int_{t=t_i^-}^{t_f^+} [\tilde{\mathbf{v}}^* \cdot \tilde{\mathbf{a}}_u] + [\tilde{\mathbf{u}} \cdot \tilde{\mathbf{a}}_v] d\Omega dt = \int_{\Omega} \int_{t=t_i^-}^{t_f^+} \frac{\partial \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^*}{\partial t} + \nabla \cdot J(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}^*, \tilde{p}_u, \tilde{p}_v^*) d\Omega dt$$

As proved in appendix A, $J(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}^*, \tilde{p}_u, \tilde{p}_v^*) = 0$ at the spatial boundaries of the domain, the application of the Green theorem implies:

$$\int_{\Omega} \nabla \cdot J(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}^*, \tilde{p}_u, \tilde{p}_v^*) d\Omega = \int_{\partial\Omega} \mathbf{n} \cdot J(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}^*, \tilde{p}_u, \tilde{p}_v^*) dS = 0 \quad (3.22)$$

with $\mathbf{n}$ the normal to the surface $\partial\Omega$. Therefore, using the Green's theorem, the following equality is obtained:

$$\int_{\Omega} \int_{t=t_i^-}^{t_f^+} [\tilde{\mathbf{v}}^*(r, t) \cdot \tilde{\mathbf{a}}_u(r, t)] + [\tilde{\mathbf{u}}(r, t) \cdot \tilde{\mathbf{a}}_v(r, t)] d\Omega dt = \int_{\Omega} [\tilde{\mathbf{u}}(r, t) \cdot \tilde{\mathbf{v}}^*(r, t)]_{t=t_i^-}^{t_f^+} d\Omega \quad (3.23)$$

Reciprocity relation

One of the use of the Green theorem is to obtain a link between the Green functions for the direct and for the adjoint problem. To do that, equation (3.23) is used with $\tilde{\mathbf{u}}(r, t) = \tilde{\mathbf{g}}_u(\mathbf{r}, t | \mathbf{r}_0, t_0; \mathbf{e}_i)$ and $\tilde{\mathbf{v}}^*(r, t) = \tilde{\mathbf{g}}_v^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}_j)$, where $\mathbf{e}_i$ and $\mathbf{e}_j$ are two vectors of unit norm in $\mathbb{R}^3$. The right hand sides of the direct and adjoint equations are then $\tilde{\mathbf{a}}_u(r, t) = 4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\mathbf{e}_i$ and $\tilde{\mathbf{a}}_v(r, t) = 4\pi\delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)\mathbf{e}_j$. The

equation becomes:

$$\begin{aligned}
& \int_{\Omega} \int_{t=t_0^-}^{t_1^+} [\tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}_j) \cdot 4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\mathbf{e}_i] \\
& \quad + [\tilde{\mathbf{g}}_u(\mathbf{r}, t | \mathbf{r}_0, t_0; \mathbf{e}_i) \cdot 4\pi\delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)\mathbf{e}_j] d\Omega dt \\
& = \int_{\Omega} [\tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}_j) \cdot \tilde{\mathbf{g}}_u(\mathbf{r}, t | \mathbf{r}_0, t_0; \mathbf{e}_i)]_{t=t_0^-}^{t_1^+} d\Omega
\end{aligned} \tag{3.24}$$

Integrating the right hand side, and using the causality relation for $\tilde{\mathbf{g}}_u$ and $\tilde{\mathbf{g}}_u^*$, the reciprocity relation can be obtained as a scalar identity:

$$\tilde{\mathbf{g}}_u^*(\mathbf{r}_0, t_0 | \mathbf{r}_1, t_1; \mathbf{e}_j) \cdot \mathbf{e}_i + \tilde{\mathbf{g}}_u(\mathbf{r}_1, t_1 | \mathbf{r}_0, t_0; \mathbf{e}_i) \cdot \mathbf{e}_j = 0 \tag{3.25}$$

Final solution from the initial value

Another classical use of the Lagrange identity is to find the solution of the homogeneous direct problem with arbitrary initial condition from the Green functions of the adjoint problem. It is done here using equation (3.23), with $\tilde{\mathbf{u}}(r, t)$ the solution of the homogeneous linear problem which initial condition $\tilde{\mathbf{u}}(r_0, t_0)$ is given, and $\tilde{\mathbf{v}}^*(r, t) = \tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e})$. The right hand sides of the direct and adjoint equations are then $\tilde{\mathbf{a}}(r, t) = 0$ and $\tilde{\mathbf{a}}(r, t) = 4\pi\delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)\mathbf{e}$. The equation becomes:

$$\begin{aligned}
& \int_{\Omega} \int_{t=t_0^-}^{t_1^+} \tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e}) \cdot \mathbf{0} + \mathbf{u}'(r, t) \cdot 4\pi\delta(\mathbf{r} - \mathbf{r}_1)\delta(t - t_1)\mathbf{e} d\Omega dt \\
& = \int_{\Omega} [\tilde{\mathbf{u}}(r, t) \cdot \tilde{\mathbf{g}}_u^*(\mathbf{r}, t | \mathbf{r}_1, t_1; \mathbf{e})]_{t=t_0^-}^{t_1^+} d\Omega
\end{aligned} \tag{3.26}$$

After integrating the Dirac function in volume and time, and with the right hand side simplified using the direct and adjoint causality relations:

$$\tilde{\mathbf{u}}(r_1, t_1) \cdot \mathbf{e} = - \int_{\Omega} \tilde{\mathbf{u}}(r, t_0) \cdot \tilde{\mathbf{g}}_u^*(\mathbf{r}, t_0 | \mathbf{r}_1, t_1; \mathbf{e}) d\Omega. \tag{3.27}$$

Here, the projection of the solution $\tilde{\mathbf{u}}(r_1, t_1)$ on a vector $\mathbf{e}$ has been obtained. To have the whole solution, the same projection has to be done on an orthogonal basis of the space to have all the components of the vector $\tilde{\mathbf{u}}(r_1, t_1)$.

Doing a similar demonstration, a solution of the homogeneous adjoint problem with given ‘final’ condition can be obtained, as an integral of the direct problem Green functions:

$$\tilde{\mathbf{v}}^*(r_0, t_0) \cdot \mathbf{e} = \int_{\Omega} \tilde{\mathbf{v}}^*(r, t_1) \cdot \tilde{\mathbf{g}}_u(\mathbf{r}, t_1 | \mathbf{r}_0, t_0; \mathbf{e}) d\Omega \quad (3.28)$$

3.2.3 Application to the optimisation problem

For the numerical solution of the linear Navier-Stokes operator, the operation $\mathbf{u}'_f = R\mathbf{u}'_i$ is the projection on a finite dimension space of the real linear Navier-Stokes operator. This section is an attempt to prove that the operation $\mathbf{u}'_{i,temp} = R^*\mathbf{u}'_{f,temp}$ can be performed in a similar way by the projection on a finite dimension space of the adjoint operator.

If the optimisation is done for a channel flow configuration, the spatial domain is $\Omega = [0, L_x] \times [-L_y/2, L_y/2]$, and the velocity vector is in the function space of continuous function $\mathcal{C}(\Omega, \mathbb{R}^3)$. The vector space $\mathcal{C}(\Omega, \mathbb{R}^3)$ with its scalar product defined as $\langle \tilde{\mathbf{u}}, \tilde{\mathbf{v}}^* \rangle = \int_{\Omega} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}^*$ is a complete space (Ω is bounded). Therefore the Riesz theorem applies which prove the existence of an adjoint to the operator $\mathcal{R}$ such that:

$$\langle \mathcal{R}\tilde{\mathbf{v}}^*, \mathcal{R}\tilde{\mathbf{u}} \rangle = \langle \tilde{\mathbf{v}}^*, \mathcal{R}^*\mathcal{R}\tilde{\mathbf{u}} \rangle \quad (3.29)$$

Here, the purpose is to prove that applying the adjoint of $\mathcal{R}$ is equivalent to solving

the adjoint problem. This can be done, using the relations (3.23),(3.27) and (3.28):

$$\begin{aligned}
\langle \mathcal{R}\tilde{\mathbf{v}}^*, \mathcal{R}\tilde{\mathbf{u}} \rangle &= \sum_{1 \leq i \leq 3} \int_{\Omega_1} (\tilde{\mathbf{v}}^*(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i) (\tilde{\mathbf{u}}(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i) d\Omega_1 \\
&= - \sum_{1 \leq i \leq 3} \int_{\Omega_1} \tilde{\mathbf{v}}^*(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i \int_{\Omega} \tilde{\mathbf{u}}(\mathbf{r}, t_0) \cdot \tilde{\mathbf{g}}_u^*(\mathbf{r}, t_0 | \mathbf{r}_1, t_1; \mathbf{e}_i) d\Omega d\Omega_1 \\
&= - \sum_{1 \leq i \leq 3} \int_{\Omega} \int_{\Omega_1} (\tilde{\mathbf{v}}^*(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i) \left(\sum_{1 \leq j \leq 3} (\tilde{\mathbf{u}}(\mathbf{r}, t_0) \cdot \mathbf{e}_j) (\tilde{\mathbf{g}}_u^*(\mathbf{r}, t_0 | \mathbf{r}_1, t_1; \mathbf{e}_i) \cdot \mathbf{e}_j) \right) d\Omega_1 d\Omega \\
&= - \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} \int_{\Omega} \int_{\Omega_1} (\tilde{\mathbf{v}}^*(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i) (\tilde{\mathbf{u}}(\mathbf{r}, t_0) \cdot \mathbf{e}_j) (\tilde{\mathbf{g}}_u^*(\mathbf{r}, t_0 | \mathbf{r}_1, t_1; \mathbf{e}_i) \cdot \mathbf{e}_j) d\Omega_1 d\Omega \\
&= \sum_{1 \leq i \leq 3} \sum_{1 \leq j \leq 3} \int_{\Omega} \int_{\Omega_1} (\tilde{\mathbf{v}}^*(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i) (\tilde{\mathbf{u}}(\mathbf{r}, t_0) \cdot \mathbf{e}_j) (\tilde{\mathbf{g}}_u(\mathbf{r}_1, t_1 | \mathbf{r}, t_0; \mathbf{e}_j) \cdot \mathbf{e}_i) d\Omega_1 d\Omega \\
&= \sum_{1 \leq j \leq 3} \int_{\Omega} \int_{\Omega_1} \left(\sum_{1 \leq i \leq 3} (\tilde{\mathbf{v}}^*(\mathbf{r}_1, t_1) \cdot \mathbf{e}_i) (\tilde{\mathbf{g}}_u(\mathbf{r}_1, t_1 | \mathbf{r}, t_0; \mathbf{e}_j) \cdot \mathbf{e}_i) \right) (\tilde{\mathbf{u}}(\mathbf{r}, t_0) \cdot \mathbf{e}_j) d\Omega_1 d\Omega \\
&= \int_{\Omega} \sum_{1 \leq j \leq 3} (\tilde{\mathbf{v}}^*(\mathbf{r}, t_0) \cdot \mathbf{e}_j) (\tilde{\mathbf{u}}(\mathbf{r}, t_0) \cdot \mathbf{e}_j) d\Omega \\
&= \int_{\Omega} \tilde{\mathbf{v}}^*(\mathbf{r}, t_0) \cdot \tilde{\mathbf{u}}(\mathbf{r}, t_0) d\Omega \\
&= \langle \tilde{\mathbf{v}}^*, R^* R \tilde{\mathbf{u}} \rangle
\end{aligned} \tag{3.30}$$

It has therefore been proved that in infinite dimension, the adjoint of the direct problem operator is the operator of the adjoint problem.

To complete the proof, it should be showed that calculating $R^* R \mathbf{u}'_i$ as done in the optimisation algorithm is equivalent to solving numerically $\tilde{\mathbf{u}}_{\mathbf{f}} = \mathcal{R} \tilde{\mathbf{u}}_{\mathbf{f}}$ and then $\mathcal{R}^* \tilde{\mathbf{u}}_{\mathbf{f}}$. R is defined as the discrete solution of the infinite dimension problem $\mathcal{R}$. For the adjoint problem, R^* is defined as the transpose of R , and what we solve numerically is R^*_c , a discretisation of the infinite dimension operator $\mathcal{R}^*$. In the ideal case, the equality $R^*_c = R^*$ should hold to prove that the algorithm using R^*_c instead of R^* is working. Having the equality between these matrices would mean that the discretisation and adjoint calculations are ‘commutative’. However, this is not the case, as will be showed in later chapters. This non-comutativity implies that the problem solved numerically is not exactly the maximisation of a quadratic form and that some error will be present.

These errors will be estimated in the program validation section, and will be judged to be small enough for the method to be viable.

Chapter 4

Numerical methods

A program was implemented to solve the linearised Navier-Stokes equation derived in chapter 2, the adjoint equation obtained in chapter 3, as well as the adjoint optimisation procedure. This chapter describes all the distinct parts of that program. First, the program used as a basis to develop the linear solvers is described. Then the modification needed to solve the linearised direct and adjoint problem are shown. Finally various validations are performed. The linear and adjoint problems are showed to work properly, with different validation tests for different kind of wall oscillations. The adjoint optimisation procedure is also validated, and its accuracy is discussed.

4.1 Original non-linear algorithm

The original program was a finite difference solver for the incompressible Navier-Stokes equation, in two or three dimensions. It is among other things designed to calculate turbulent channel flows and boundary layers flows. The spatial discretisation is using compact schemes. The pressure is calculated on a staggered grid, and the Poisson solver is performed in spectral space. Full description of this algorithm is available in Laizet (2005); Laizet and Lamballais (2009).

In this section, the most important features of the program are described. As a more accurate description can be found in the literature cited above, the most important points are highlighted without going too deeply into the details. For example, the 3D possibilities of the program are ignored, and all the attention is focussed on the two

dimensional solver. The configuration is a channel flow, with the periodic boundary condition in the streamwise direction x , and Dirichlet boundary condition in $y = \pm l_y/2$ (no slip condition at the wall, l_y being the distance between the walls).

4.1.1 Description of the numerical method

The time marching scheme uses a classical fractional step method (Kim and Moin, 1985), with a convection/diffusion step and a pressure correction step. In the following equations the time is discretised as $t = kdt$, with dt the time step and k the index of the current iteration. If the velocity is known at iteration k , the iteration $k+1$ is found by solving the equations:

$$\frac{\mathbf{u}'^* - \mathbf{u}'^k}{dt} = a_k \mathbf{F}^k + b_k \mathbf{F}^{k-1} \quad (4.1)$$

$$\frac{\mathbf{u}'^{k+1} - \mathbf{u}'^*}{dt} = -c_k \nabla p^{k+1} \quad (4.2)$$

where $\mathbf{F}$ contains the convective and diffusive terms. There are n_k substeps, and the coefficients a_k , b_k and c_k can be calculated using either low storage Runge-Kutta methods (Williamson, 1980), or a second order Adams-Bashforth scheme. In the simulation performed in this thesis, the Runge-Kutta method is used.

The Poisson equation is used to calculate the pressure term p^{k+1} . It is obtained by calculating the divergence of equation (4.2), and enforcing the divergence free condition for the final velocity $\mathbf{u}'^{k+1}$:

$$\Delta p^{k+1} = \frac{\nabla \cdot \mathbf{u}'^*}{c_k dt} \quad (4.3)$$

4.1.2 Spatial discretisation

The schemes used for the spatial discretisation are sixth order compact schemes. Compact schemes are convenient in turbulent flow simulations because of their good ability to represent a large range of scales (Lele, 1992). In this case, calculating first or second derivative with a sixth order compact scheme implies the inversion of a tridiagonal or pentadiagonal matrix, which can be done at a relatively low computational cost. For

the first and second derivatives, the sixth order compact formulation can be written:

$$\alpha_1 f'_{i-1} + f'_i + \alpha_1 f'_{i+1} = a_1 \frac{f_{i+1} - f_{i-1}}{2dx} + b_1 \frac{f_{i+2} - f_{i-2}}{4dx}, \quad (4.4)$$

$$\alpha_2 f''_{i-1} + f''_i + \alpha_2 f''_{i+1} = a_2 \frac{f_{i+1} - 2f_i + f_{i-1}}{dx^2} + b_2 \frac{f_{i+2} - 2f_i + f_{i-2}}{4dx^2} + c_2 \frac{f_{i+3} - 2f_i + f_{i-3}}{9dx^2}, \quad (4.5)$$

where f is the function to differentiate, and dx is the grid spacing (uniform Cartesian grids are used). Correct values of the coefficients guarantee the sixth order of the schemes.

The boundary conditions in the x -direction are periodic for the flow studied. Therefore the matrix that has to be inverted to calculate the first and second derivative is cyclic, and the global order of the method in the streamwise direction is six. The situation is different for the boundary conditions at $y = \pm l_y/2$, as there is a Dirichlet boundary condition. In this case, the derivative next to the boundary is one sided (Lele, 1992; Laizet and Lamballais, 2009). The order of the scheme at the wall is four which renders the order of the whole algorithm also four, even if the higher accuracy inside the domain should allow for better resolution of turbulent flows.

4.1.3 Pressure correction step

To obtain the Poisson equation, the divergence of the intermediate velocity $\mathbf{u}'^*$ has to be calculated. A staggered grid is used to solve the Poisson equation, and therefore the calculation of the divergence is performed at the same time as interpolating the results on the staggered grid. Here, the method to obtain the component $\partial u / \partial x$ of $\nabla \cdot \mathbf{u}'^*$ is explained.

The velocity component is first interpolated from the velocity grid to a staggered grid in y using a sixth order compact interpolation. Then the derivative is calculated from this semi-staggered grid to the complete staggered grid used for the Poisson equation (the grid is also staggered in the x -direction), using also a sixth order compact scheme. The two steps follow the schemes:

$$\alpha_I f_{i-1/2} + f_{i+1/2} + \alpha_I f_{i+3/2} = a_I \frac{f_{i+1} + f_{i-1}}{2dy} + b_I \frac{f_{i+2} + f_{i-2}}{4dy}, \quad (4.6)$$

$$\alpha_{I1}f'_{j-1/2} + f'_{j+1/2} + \alpha_{I1}f'_{j+3/2} = a_{I1}\frac{f_{j+1} - f_{j-1}}{2dx} + b_{I1}\frac{f_{j+2} - f_{j-2}}{4dx}. \quad (4.7)$$

These schemes are used far from the boundaries. For periodic boundaries conditions in x , the scheme is periodic and remains sixth order, and for the Dirichlet boundary condition in y , a one sided scheme of order four is used next to the walls.

Once the term $\nabla \cdot \mathbf{u}'^*$ is known on the staggered pressure grid, the Fourier transform from the spatial space to the Fourier space is calculated. The numeric formulae for the direct and inverse transform are, for the transform in the x -direction:

$$\hat{f}_l = \frac{1}{n_x} \sum_{i=1}^{n_x} f_i e^{-ik_x x_i} \quad -n_x/2 \leq l \leq n_x/2 - 1 \quad (4.8)$$

$$f_i = \sum_{l=-n_x/2}^{n_x/2-1} \hat{f}_l e^{ik_x x_i} \quad (4.9)$$

with $i = \sqrt{-1}$, and $k_x = 2\pi l/l_x$ the wavenumber. This formula is assuming that the function f is periodic. For non-periodic function (case in the y -direction), another Fourier transform is defined in Laizet and Lamballais (2009), assuming the function is symmetric with a period twice as big as the domain width l_y . This definition is possible because of the Dirichlet condition in $y = \pm l_y/2$.

Once in the Fourier space, the differentiation is equivalent to a multiplication of each Fourier coefficient by the correct wavenumber:

$$\hat{f}'_l = ik'_x \hat{f}_l \quad (4.10)$$

This makes the inversion of the Laplacian very easy and computationally efficient. The rapidity to solve the Poisson equation is what makes this kind of spectral method so attractive.

However, the wave number multiplication will not be used as such to calculate the derivatives. As the Poisson solver has to be coherent with the spatial schemes used everywhere else in the program, and particularly when $\nabla \mathbf{u}'^*$ is calculated, the method needs to be modified. To do so, the idea is to use a modified wavenumber, which once multiplied by the Fourier coefficient will give a derivative equivalent to what would have

been found if it had been calculated in the spatial space (Moin, 2001). This means that if the derivative is calculated in the spatial space or in the Fourier space, the result is going to be exactly the same (the precision is limited to the machine accuracy). It can be shown for example that the modified wavenumber for the first derivative can be linked to the actual wavenumber by the relation:

$$k'_x dx = \frac{a_{I1} \sin(k_x dx) + (b_{I1}/2) \sin(2k_x dx)}{1_2 \alpha_{I1} \cos(k_x dx)}. \quad (4.11)$$

It can be noticed that the same notion of modified wavenumber holds to take into account the interpolation between the velocity grid and the staggered grid. The first derivative in the x -direction will therefore be written:

$$\widehat{f}'_l = i T_y k'_x \widehat{f}_l, \quad (4.12)$$

where T_y is the modified wavenumber for the interpolation described in equation (4.7), and k'_x is the modified wavenumber for the derivative (4.6)

To solve the full Poisson equation, the Fourier transform has to be performed in the two directions x and y . One Fourier coefficient for the pressure is given by the transformation of the spatial expression of the pressure:

$$\widehat{p}_{lm} = \frac{1}{n_x n_y} \sum_i \sum_j p_{ij} W_x(k_x x_i) W_y(k_y y_j), \quad (4.13)$$

with $W_x(x) = \exp(-ix)$. The inverse expression is given by:

$$p_{ij} = \frac{1}{n_x n_y} \sum_l \sum_m \widehat{p}_{lm} W_x(k_x x_i) W_y(k_y y_j). \quad (4.14)$$

To solve the Poisson equation, the same method is used to calculate the Fourier coefficients $\widehat{D}_{lm}$ of the divergence of the intermediate velocity $\nabla \cdot \mathbf{u}'^*$. Then the Poisson equation (4.3) can be solved for each Fourier coefficient by the multiplication:

$$\widehat{p}_{lm}^{k+1} = \frac{\widehat{D}_{lm}}{F_{lm}}, \quad (4.15)$$

where F_{lm} take into account the modified wave number and the interpolation wavenum-

ber:

$$F_{lm} = -[(k'_x T_y)^2 + (k'_y T_x)^2]. \quad (4.16)$$

After doing that and interpolating back from the Fourier space to the spatial non-staggered velocity grid using the same kind of interpolation and derivative as in (4.7) and (4.6), the pressure gradient (∇p) is obtained. It is guaranteed for the velocity $\mathbf{u}'^{k+1}$ to be divergence free up to the machine accuracy.

4.2 Implementation of the linear solver

The modifications of the program described here is made only for the implementation of the forward linear problem. Solving the adjoint problem is very similar to solving the linear problem, and therefore will not be described. As for the adjoint optimisation loop, its implementation is straightforward by following the steps described in chapter 3. The most important thing about the adjoint optimisation is the validation of the accuracy of the method, and this will be presented in the validation section.

The system of linear equations is in a sense more complex than the full Navier-Stokes equation, as there are eight equations to solve instead of three. However, their structure remains similar as the pressure and convective/diffusive part play similar role as for the Navier-Stokes operator. Therefore most of the initial program structure is usable. In this section, the linearised Navier-Stokes equations and mass conservation equations ((2.10), (2.11), (2.12)) are written using a simplified notation for the convective/diffusive term:

$$\frac{\partial u_c}{\partial x} + \frac{\partial v_c}{\partial y} + \beta_z w_s = 0, \quad (4.17a)$$

$$\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} - \beta_z w_c = 0, \quad (4.17b)$$

$$\frac{\partial u_c}{\partial t} - L_{c,x}(\mathbf{u}_c, \mathbf{u}_s) = -\frac{\partial p_c}{\partial x}, \quad (4.18a) \quad \frac{\partial u_s}{\partial t} - L_{s,x}(\mathbf{u}_s, \mathbf{u}_c) = -\frac{\partial p_s}{\partial x}, \quad (4.18d)$$

$$\frac{\partial v_c}{\partial t} - L_{c,y}(\mathbf{u}_c, \mathbf{u}_s) = -\frac{\partial p_c}{\partial y}, \quad (4.18b) \quad \frac{\partial v_s}{\partial t} - L_{s,y}(\mathbf{u}_s, \mathbf{u}_c) = -\frac{\partial p_s}{\partial y}, \quad (4.18e)$$

$$\frac{\partial w_c}{\partial t} - L_{c,z}(\mathbf{u}_c, \mathbf{u}_s) = -\beta_z p_s, \quad (4.18c) \quad \frac{\partial w_s}{\partial t} - L_{s,z}(\mathbf{u}_s, \mathbf{u}_c) = +\beta_z p_c. \quad (4.18f)$$

These equations of projection on cosine and sine space can not be considered separately because of the coupling term in the mass conservation equations and in the $\partial/\partial z$ component of the convective term. However, differentiating the momentum equations and using the mass conservation in a similar way as is applied for the nonlinear Navier-Stokes equations, it is possible to obtain two Poisson equations for the pressures p_c and p_s .

To obtain the Poisson equation for p_c , the sum of the derivative of (4.18a) in the x -direction, the derivative of (4.18b) in the y -direction, and (4.18f) multiplied by β_z , is used. It is then simplified using the mass conservation equation (4.17a). To obtain the Poisson equation for p_s , a similar calculation using the three remaining momentum equations and the remaining continuity equation is made. As the way to solve the problem is very similar to the Navier-Stokes case, the same fractional step method is used. In the first step of each Runge-Kutta iteration, the convective and diffusive terms are calculated to obtain the intermediate velocities. Then the pressures p_c and p_s are obtained using the two Poisson equations. Finally the gradient of the pressure is used to obtain the velocities u_c^{k+1} and u_s^{k+1} .

The first convective/diffusive step can be written in the following way:

$$u_c^* = u_c + dt L_{c,x}(\mathbf{u}_c^k, \mathbf{u}_s^k), \quad (4.19a) \quad u_s^* = u_s + dt L_{s,x}(\mathbf{u}_s^k, \mathbf{u}_c^k), \quad (4.19d)$$

$$v_c^* = v_c + dt L_{c,y}(\mathbf{u}_c^k, \mathbf{u}_s^k), \quad (4.19b) \quad v_s^* = v_s + dt L_{s,y}(\mathbf{u}_s^k, \mathbf{u}_c^k), \quad (4.19e)$$

$$w_c^* = w_c + dt L_{c,z}(\mathbf{u}_c^k, \mathbf{u}_s^k), \quad (4.19c) \quad w_s^* = w_s + dt L_{s,z}(\mathbf{u}_s^k, \mathbf{u}_c^k). \quad (4.19f)$$

From the intermediate velocities, the Poisson equations are obtained:

$$\frac{\partial u_c^*}{\partial x} + \frac{\partial v_c^*}{\partial y} + \beta_z w_s^* = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta_z^2 \right) p_c^{k+1}, \quad (4.20a)$$

$$\frac{\partial u_s^*}{\partial x} + \frac{\partial v_s^*}{\partial y} - \beta_z w_c^* = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta_z^2 \right) p_s^{k+1}. \quad (4.20b)$$

After solving the two Poisson equations, the last substep is:

$$u_c^{k+1} = -dt \frac{\partial p_c}{\partial x} + u_c^*, \quad (4.21a) \quad u_s^{k+1} = -dt \frac{\partial p_s}{\partial x} + u_s^*, \quad (4.21d)$$

$$v_c^{k+1} = -dt \frac{\partial p_c}{\partial y} + v_c^*, \quad (4.21b) \quad v_s^{k+1} = -dt \frac{\partial p_s}{\partial y} + v_s^*, \quad (4.21e)$$

$$w_c^{k+1} = -dt \beta_z p_s + w_c^*, \quad (4.21c) \quad w_s^{k+1} = +dt \beta_z p_c + w_s^*. \quad (4.21f)$$

To adapt the existing program to the linear problem, the global structure of the fractional step method can be kept, as well as the temporal scheme, and the spatial schemes to calculate the derivatives of functions. Mainly, just the equation implementation needs to be changed. The convection/diffusion step needs to be completely rewritten with the six velocity variables. The Poisson step must be doubled and be used once to find the pressure p_c and once for the pressure p_s . The z -derivative also needs to be added to the Poisson solver. The inversion of the Laplacian (equation (4.15)) has to use a modified wavenumber which takes into account this derivative (multiplication by $-\beta_z^2$):

$$F_{lm} = -[(k'_x T_y)^2 + (k'_y T_x)^2 + (T_x T_y \beta_z)^2]. \quad (4.22)$$

Regarding the stability conditions, the CFL number is going to be based on the mean flow maximum value, and the diffusive stability number is going to be based on each projection of both perturbation velocities $\mathbf{u}'_c$ and $\mathbf{u}'_s$.

4.3 Validation of the program

The main changes in the program were the modification of the equation to solve, with the implementation of the direct and adjoint linear solvers instead of the nonlinear Navier-Stokes solver. Further modifications also had to be made, and particularly the implementation of the projection step for the measure of the final norm. In this section, these last implementations as well as a series of tests to validate the program for the different configurations it will be used for are performed.

In this thesis, the adjoint optimisation procedure to find the generalised optimal perturbation will be used only in case of harmonic wall forcing, and therefore has to be fully validated for these cases. For that, two tests will be run. First, the adjoint optimisation for the case of a base flow subject to harmonic wall oscillations will be compared with the results obtained by another research group in the case where both initial and final norm are the standard energy norm. Then, the velocity projection step needed to take into account the specificity of the observation time norm $\|\cdot\|_f$ used (substep 1bis in section 3.1.1) will be validated by reproducing the streaks spacing prediction of Chernyshenko and Baig (2005).

Once these two tests are done, if the code is used with mean flow subject to travelling

waves, it is enough to validate the direct and adjoint solvers for these mean flow, as the optimisation procedure will already be known to be working. However, no adjoint optimisation will be required for these cases in this thesis, so it will be enough to validate the linear solver only. To do so, a manufactured solution technique will be used.

4.3.1 Adjoint optimisation with harmonic wall oscillation

As a first validation, a very simple configuration was used, taken from the results of Butler and Farrell (1992). In that paper, using the usual energy norm for initial and final condition, the maximum transient growth of a perturbation in a laminar channel flow is calculated. It is found that at Reynolds number $Re = 5000$, the optimum is achieved for a perturbation infinitely long in the streamwise direction, growing during 379 advective time units and of spanwise wavenumber $\beta_z = 2.044$. Using a parabolic profile at the same Reynolds number, as well as the same target time and spanwise wavenumber, the results found with the program implemented here is within 1% of the value found by Butler and Farrell (1992).

A more relevant validation of the adjoint optimisation procedure has to focus on a case where the mean flow has the same characteristics as the one for which the code will be used. The main interest is then to validate the code for the case where an energy norm is used for both initial and final condition, but where there is a time dependent flow in the spanwise direction. A code was available from a research group at Imperial College London (Burini and Zaki, 2011), calculating the optimal perturbation in a boundary layer subject to spanwise wall oscillations. In this code, a wavenumber decomposition is used in the spanwise as well as in the streamwise direction; the validation can then be made for perturbation velocities of the form $\mathbf{u}' = \mathbf{u}'_0(t, y)e^{i(\beta_x x + \beta_z z)}$. Working on a boundary layer instead of a channel flow seems very different at first glance. However, the linearised solver uses an equation for the perturbation, and not the full Navier-Stokes equation. For the perturbation, the boundary conditions used at the limits of the domain in the wall-normal direction are standard no slip condition. Therefore, calculating optimal perturbation in a channel flow or in a boundary layer is exactly the same problem, the only change being the mean flow expression.

The mean flow in the streamwise direction is the usual solution for a Blasius bound-

ary layer (Schlichting, 1955). If U_∞ is the free stream velocity, and the new variable $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$ is defined; the differential equation and the boundary conditions are given by:

$$\begin{aligned} ff'' + 2f''' &= 0 \\ f(\eta = 0) &= 0, f'(\eta = 0) = 0, f(\eta = 4.9) = 0.99 \\ U(y) &= U_\infty f'(\eta) \end{aligned}$$

The spanwise wall movement is an harmonic oscillation, and an analytical solution for such boundary conditions is a Stokes layer (Schlichting, 1955). For a period of oscillation T , the wavenumber $\omega = 2\pi/T$ is defined. The maximum velocity of the wall is W_0 . The equation of the Stokes layer is then given by:

$$W(y, t) = W_0 \exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right) \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right).$$

In the present case, the values chosen were $U_\infty = 1$, $W_0 = 0.25$, $T = 200$, $\delta_{99} = 4.9\sqrt{\nu x/U_\infty} = 1$, and the Reynolds number $Re_{\delta_{99}} = U_\infty \delta_{99}/\nu = 800$.

Two sets of simulation were performed. For each of them, a streamwise and spanwise wavenumber are chosen, as well as a sequence of target time t_f (the initial time is $t_i = 0$), and the adjoint optimisation algorithm is run for each of these instants t_f . In the first simulation (figure 4.1(a)), there is no streamwise dependence for the perturbation, and therefore the program is tested only in the wall-normal direction. For the second simulation (figure 4.1(b)), there is a dependence on the streamwise direction, which means that the code from Burini and Zaki (2011) will have a specified streamwise wavenumber whereas the code we are validating will have a grid in the streamwise direction which length was set to be a multiple of the wavelength.

The comparison of the maximum energy growth A are conclusive. For the first case, the energy curves for each optimal perturbation are within a few percent of each other. In the second case, it is also the case for most of the target times t_f , except for a part of the graph at small t_f . The instants where the curves are close to each other are enough to validate the program both with harmonic forcing and streamwise dependence. For the instants where the curves are different in Figure 4.1(b), there is a simple explanation. The code used as a reference is using a Fourier decomposition

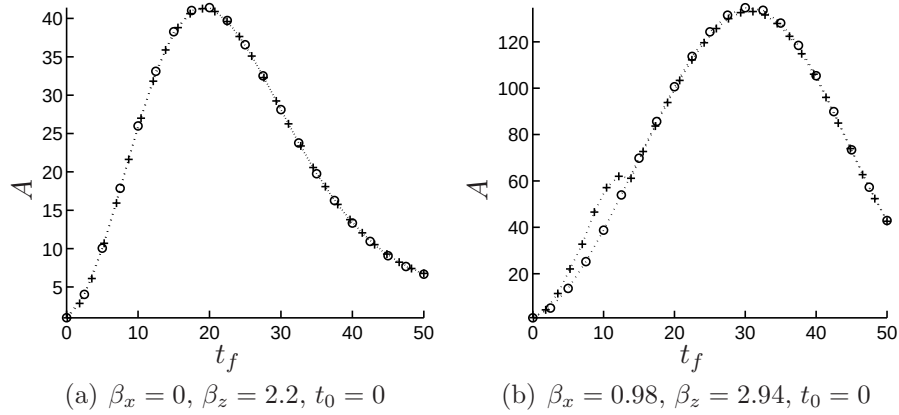


Figure 4.1: Comparison of adjoint optimisation results. (o) simulations from (Burini and Zaki, 2011); (+) adjoint optimisation solver developed for this thesis.

in the streamwise direction, whereas the code to validate uses a grid in this direction. Theoretically, in the presence of harmonic wall forcing, the optimal solution is composed of a single wavenumber in the streamwise direction. However, this optimal wavenumber is not necessarily the one chosen to run the simulation. For target times t_f where it is the case, the program using a grid in the streamwise direction will pick a different more optimal wavenumber which leads to a higher energy growth.

4.3.2 Adjoint optimisation with projection step

Choice of the projection function

The adjoint optimisation algorithm is now validated for the cases where it will be used. It still remains to validate the projection step for cases where the final norm $\|\cdot\|_f$ will be used, as so far only volume energy norm have been used. For that, a form of the projection function will first be determined, and then the results of Chernyshenko and Baig (2005) will be reproduced.

In chapter 3, the equations were described as defined on an infinite dimensional vector space, and the projection function could be considered as the application of a delta function. In numerical situation, the space used is now discrete (the mesh points) and of finite dimension. An intuitive idea would be to define the projection function as the projection on a single mesh point. However, the projection step happens just before the application of the adjoint solver. If such was the case, the initial solution

for the adjoint solver would be numerically non differentiable, and there is no reason to believe that the solution obtained after application of the adjoint solver would be physically relevant. To avoid this issue, the projection function was defined as a unit approximation centred at the projection layer $y = y_0$.

$$p_\delta(y) = f_\delta \exp\left(\frac{\delta^2}{(y - y_0 + \delta/2)(y - y_0 - \delta/2)}\right) \quad \text{for } |y - y_0| < \delta/2, 0 \text{ otherwise,} \quad (4.23)$$

where y_0 is the position of the projection layer, δ is the width of the projection function and f_δ is a renormalisation coefficient to ensure that $\int_{y=-l}^l p_\delta(y) dy = 1$. Two examples of such functions are given in Figure 4.3. Then p_δ converges to the delta function as $\delta \rightarrow 0$. If the mesh resolution is small enough compared to the width δ of the projection function, the velocity field obtained after the projection step will still be differentiable.

To obtain a correct projection step, a compromise had to be found between the width δ of the projection function, and the distance δ_y between two points of the mesh in the wall-normal direction. If δ is too large, the approximation of a delta function is not good enough (for an infinite width $\delta = \infty$, the projection function is the identity, and therefore the final norm is the usual energy norm). If the mesh size is not small enough compared to δ , then the solution obtained after the projection step is not numerically differentiable. If the grid is fine enough and the width of the projection small enough too, then the effect of the projection should be close to the application of a delta function, with the resulting flow field still differentiable. A simple test was to run the adjoint optimisation for a given grid and different width of the projection function. The approximation of a delta function should be good enough if there is a plateau in the energy growth results for a given area of small projection function width. The results are shown in figure 4.2. A few interesting features are visible from this figure. First, the convergence in grid is visible. The fact that a mesh with $n_y = 513$ points or a mesh with $n_y = 1025$ points give the same results whatever the width chosen for the projection function show at which point the convergence is obtained for this problem. As far as the projection function is concerned, a plateau indeed exists for small values of δ , which shows that a good approximation of a delta function can be obtained. However, the more surprising fact is that this plateau goes up to $\delta = 0$, where due to the numerical errors the solution is not expected to be correct. This

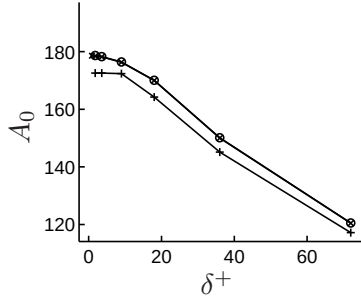


Figure 4.2: Influence of the width δ of the projection function on the maximum energy gain after the adjoint optimisation. Number of mesh point in the wall normal direction: $ny = 129$ (+), $ny = 513$ (o), $ny = 1025$ (x).

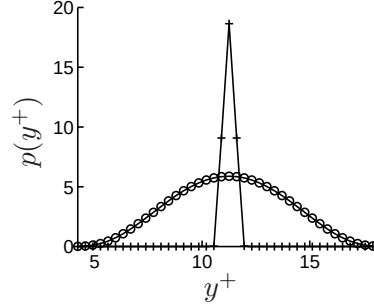


Figure 4.3: Representation of the projection function for two different width $\delta^+ = 1.8$ (+) and $\delta^+ = 18$ (o). Projection centred at $y^+ = 11.25$.

is surprising and not well understood, but shows that it is actually possible to use a projection function of width $\delta = 0$. For the efficiency of the program, $\delta = 0$ will be used for all the simulation performed in this thesis.

Validation of the projection step

Once the choice of the projection function is made, and the function chosen is shown to be close to a delta function projection in the equivalent infinite dimension space, the program using the projection step has to be validated. The only example in the literature where the final norm is the energy over a layer instead of the standard volume energy is in Chernyshenko and Baig (2005). These results therefore have to be reproduced with a good enough accuracy.

The result we are interested in for this validation is the streaks characteristics dependence with the distance to the wall. The mean flow is a turbulent mean profile which analytical expression is obtained from Reynolds and Tiederman (1967). As there is no mean flow in the spanwise direction, the optimal perturbation will be independent of the streamwise direction, and therefore the problem is one dimensional and there is no need to use a computational grid in the streamwise direction.

For each distance to the wall, the calculation of the generalised optimal perturbation is done in two steps. First, the initial time of the calculation is set to $t_i = 0$, and the adjoint optimisation procedure is run for a wide range of parameters ($\lambda_z > 0, t_f >$

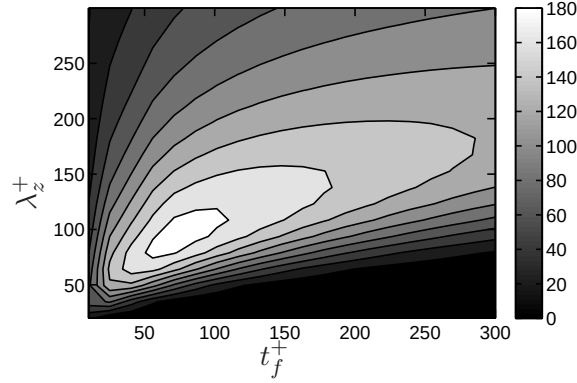


Figure 4.4: Energy growth, after adjoint optimisation for various parameters (λ_z, t_f) . Case of an unforced turbulent mean profile at $Re_\tau = 180$, and for a final norm layer at a distance $y_0^+ = 11$ from the wall.

0). A map of the growth of each parameter is then obtained (Figure 4.3.2). This map is interesting by itself as it gives an idea of the selectivity on the wavenumber and final time of the linearised operator. Similar figures will be used extensively in the case of spanwise harmonic wall forcing, as it provides relevant information about the physical mechanisms involved. Once this map is obtained, the position of the generalised optimal perturbation itself corresponds then to the extremum of the picture, which is in this case $(\lambda_z^+, t_f^+) \approx (100, 80)$, with a growth factor $A \approx 180$. It is found accurately by performing the same mapping on a finer grid around the position of the peak. Usually, two such iterations are needed to obtain an accurate result at a reasonable cost.

Such calculations have been performed for various distances to the wall. The value of the wavenumber, final time and growth of the optimal perturbation can be compared to the results of Chernyshenko and Baig (2005). This is done on figure 4.5. Due to the fact that the projection function has to be at a mesh point, the distances to the wall used are not exactly the same as in Chernyshenko and Baig (2005), and therefore no relative comparison can be made. However, the visual comparison between the two curves shows that the relative error does not exceed a few percent, which is good enough to validate the code.

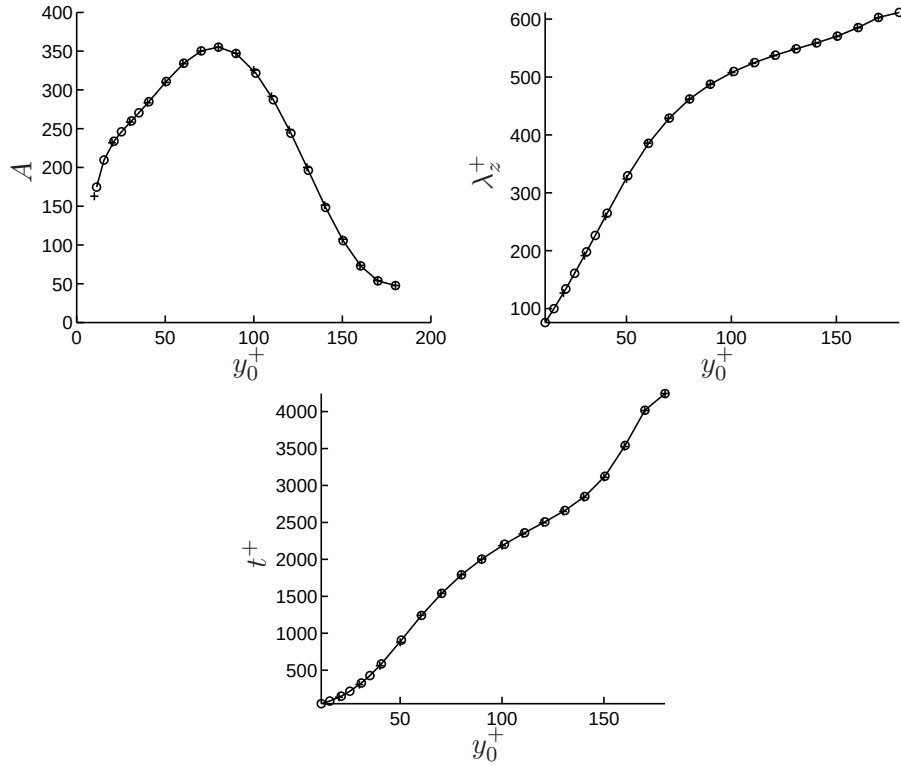


Figure 4.5: Generalised optimal perturbations at different distances to the wall y_0^+ , when the initial norm $\|\cdot\|_b$ is used. (+) results from Chernyshenko and Baig (2005), (o) result from the code implemented for this thesis.

4.3.3 Validation with a manufactured solution, case of travelling waves

The optimisation algorithm has now been extensively validated for the cases where it will be used. The only validation remaining is to verify whether the program is working for mean flow composed of travelling waves in the streamwise direction. The adjoint algorithm and projection step being already known to work properly, it is enough to validate the linear and adjoint solvers in these configurations. However, in this thesis there will be no optimisation performed in the cases where the wall movement is a spanwise travelling wave. For these configurations, only the linear solver for the direct problem will be used, and therefore only this solver needs to be validated here.

Contrary to the two previous validations, there were no available results in the literature for comparison. Because of that, it was decided to use an alternative ap-

proach; the manufactured solution technique. It technically consists in choosing an analytical function of space and time, and derive a correction so that this function satisfies analytically the linear differential equation with the correction term. Then, the initial condition of this function is used as an input of the discretised version of this solver, and the difference between the analytic solution and the solution obtained with the discrete solver is analysed. If both solution are close enough, then the solver is validated.

To formally describe the problem, the linearised Navier-Stokes equation need to be written in a simple enough way. The spatial term is written here as $\mathcal{L}$ if the physical domain is used (continuous), and L if the discretised domain defined for the numeric solver is used. The differential equation to solve the initial value problem then has the form:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \mathcal{L} \tilde{\mathbf{u}}, \quad (4.24a)$$

$$\frac{\partial \mathbf{u}}{\partial t} = L \mathbf{u}, \quad (4.24b)$$

with equation (4.24a) representing the continuous problem and equation (4.24b) representing the discrete problem. In this analysis, operators are curled when they are in continuous space and straight in discrete space, and the vectors are written with a symbol $\tilde{}$ in the continuous space and without in the discrete space. An analytical function of space and time $\tilde{\mathbf{u}}_m$ can be chosen, which will be called the manufactured solution. This function satisfies the linear differential equation:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} = \mathcal{L} \tilde{\mathbf{u}} + \mathcal{F}, \quad (4.25)$$

with $\mathcal{F}$ defined as $\mathcal{F} = \tilde{\mathbf{u}}_m - \mathcal{L} \tilde{\mathbf{u}}_m$, as well as the correct initial condition $\tilde{\mathbf{u}}(t = t_0) = \tilde{\mathbf{u}}_m(t = t_0)$. This equation is now simply the linearised Navier-Stokes equation with an additional forcing term $\mathcal{F}$. Validating the linear solver now consists of showing that the solution $\mathbf{u}_a$ of the equation:

$$\frac{\partial \mathbf{u}_a}{\partial t} = L \mathbf{u}_a + F, \quad (4.26)$$

where F is a discretisation of $\mathcal{F}$, is close enough to the solution of equation (4.25) (the initial condition must obviously be the same). For more simplicity, a projection

operator $\mathcal{Q}$ from the continuous to the discretised domain is defined. The following equalities can then be written: $F = \mathcal{Q}\mathcal{F}$, $\mathbf{u}_a(t = t_0) = \mathcal{Q}\tilde{\mathbf{u}}_m(t = t_0)$ and $\mathbf{u}_m = \mathcal{Q}\tilde{\mathbf{u}}_m$. The purpose now is to determinate the error between the analytical solution and the discrete solution. This error is defined as:

$$\mathbf{u}_{err} = \mathbf{u}_m - \mathbf{u}_a \quad (4.27)$$

To determine the evolution of this error, a projection of the continuous equation into the discretised space has to be performed:

$$\frac{\partial \mathbf{u}_m}{\partial t} = \mathcal{Q}\mathcal{L}\tilde{\mathbf{u}}_m + F$$

The operator L being a discrete approximation of the continuous operator $\mathcal{L}$, it can be written that $\mathcal{Q}\mathcal{L}\tilde{\mathbf{u}}_m = L\mathbf{u}_m + \delta^n \epsilon(t, \delta)$, where n is the order of approximation of the spatial scheme and δ is the spatial characteristic length of the scheme. $\epsilon(\delta, t)$ is bounded function in time as long as $\tilde{\mathbf{u}}_m$ remains regular enough. The evolution equation of the error can then be written:

$$\frac{\partial \mathbf{u}_{err}}{\partial t} = L\mathbf{u}_{err} + \delta^n \epsilon(t, \delta). \quad (4.28)$$

The solution of this equation is easy to calculate and has the form:

$$\mathbf{u}_{err} = \delta^n \int_{s=0}^t e^{L(t-s)} \epsilon(s, \delta) ds. \quad (4.29)$$

If the additional assumption that the operator is stable, the integral can be shown to be bounded in time, independently of δ if δ is small enough. As the operator used here is stable, this assumption will be used. However a rigorous proof would be tedious and is not shown here. The scheme property is clearly visible in this equation, and shows the way of how to complete the validation. The same simulation should be run with a few different grids refinements, which means with different values of δ . The error will then be measured with a usual energy norm, and the dependence of this norm in δ should reflect the order of the spatial discretisation.

One specific example of analytic function $\tilde{\mathbf{u}}_m$ was chosen to perform the validation

of the linear solver. The streamwise and wall-normal components were prescribed as:

$$\begin{aligned} u_c &= \sin(\pi y) \cos\left(\frac{2\pi}{\lambda_x}x\right) \cos\left(\frac{2\pi}{T}t\right), \\ v_c &= \sin(\pi y)^2 \sin\left(\frac{2\pi}{\lambda_x}x\right), \end{aligned}$$

$$\begin{aligned} u_s &= 4y(y^2 - 1) \cos\left(\frac{2\pi}{\lambda_x}x\right) \sin\left(\frac{2\pi}{T}t\right), \\ v_s &= (y^2 - 1)^2 \cos\left(\frac{2\pi}{\lambda_x}x\right). \end{aligned}$$

The spanwise component was then calculated to ensure that the continuity condition is satisfied. This function was chosen to satisfy all the boundary conditions, but also has enough complexity to depend on both spatial parameters x and y as well as on time. The temporal frequency was chosen to be the same as the one of the spanwise oscillation. The mean velocity field used to represent the travelling wave wall oscillation is described in chapter 7. Its exact expression is not necessary to understand this section, but for the record, the simulation is run at $Re_\tau = 180$, with a temporal frequency $T^+ = 200$, a streamwise wavelength $\lambda_x^+ = 500$, and the length of the domain in the streamwise direction is $L_x^+ = 2000$.

A calculation of the error has been performed for two different spatial grid, with the refinement between grids being of a factor two in the streamwise and wall normal direction. Simulations with these two grids have also been performed with various time steps. The results are shown on figure 4.6, where the energy of the error $\|\mathbf{u}_{err}\|_2^2$ is shown. Two groups of simulations are present in this figure; the first one contains four curves (two of which are almost on top of each other) and is situated above $\|\mathbf{u}_{err}\|_2^2 = 10^{-6}$, while the second group contains four curves and is below that value. These two groups allow to evaluate both the spatial and temporal accuracy of the program. In the first group, refining the grid does not change the value of the error; this shows that at this level most of the error comes from the temporal scheme, and the influence of the time step can then be studied. In the second group of curves, where the time step are much smaller, the estimate of the error for $t > 1$ is now independent of the time step. This means that the time steps are small enough for the error due to

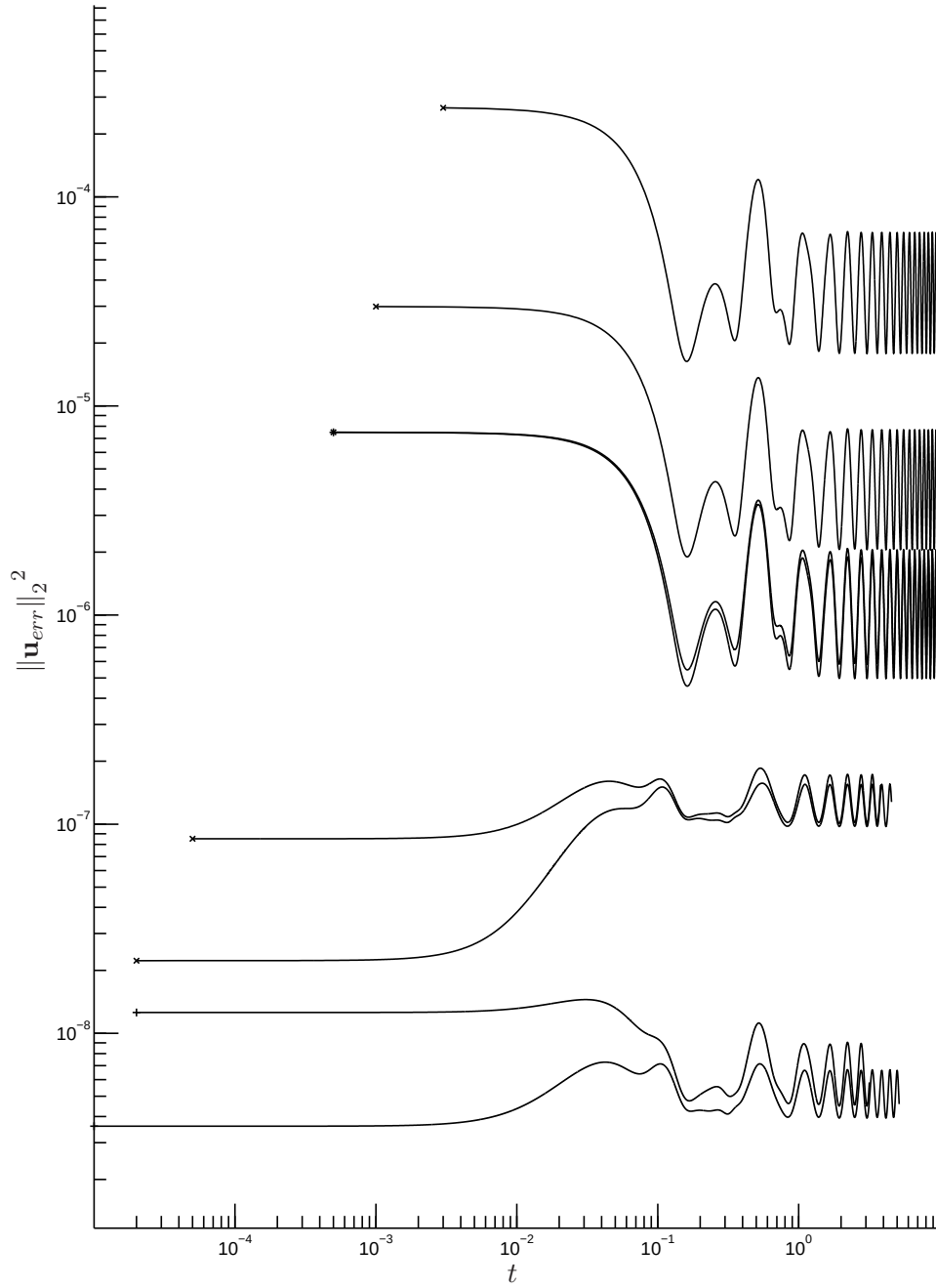


Figure 4.6: Error between the manufactured solution and the solution of the discretised problem. The error $\|\mathbf{u}_{err}\|_2^2$ is calculated at each time step, and therefore the time step of each calculation is the abscissa of the first point of the corresponding curve. Two grids have been used. The coarser one has $(n_x, n_y) = (60, 129)$ points, and corresponding simulation have the symbol ($\times$) at the first instant where the error is calculated. The thinner grid has $(n_x, n_y) = (120, 257)$ points, and corresponding curves have the symbol (+).

the temporal approximation to be negligible compared to the error due to the spatial discretisation. The accuracy of the spatial discretisation can then be studied.

The spatial accuracy of the scheme is first estimated using the second group of curves. From equation (4.29), the error can be approximated by:

$$\|\mathbf{u}_{err}\|_2^2 = \delta^{2n} f(t). \quad (4.30)$$

Between the two grids, the value of δ has been divided by two. This enables to estimate the value of n by $n = \frac{\log f_1(t)/f_2(t)}{2 \log 2}$, if f_1 refers to the curve with the smallest value of δ , and f_2 to the other one. After calculation, the value found in $n \approx 2.33$. This is far from the accuracy that could be expected with a scheme which is of sixth order inside the domain. Even the introduction of a fourth order error at the walls does not explain such a low accuracy. This error is not well understood, but the program is nonetheless validated, even if not as accurate as expected.

For the temporal scheme, a very similar reasoning is made using the first set of curves. The whole proof of how the approximation results in an error of the form $\|\mathbf{u}_{err}\|_2^2 = dt^{2n_t} g(t)$, with n_t the order of the temporal scheme, is not discussed here. However it would be quite similar to the explanation made for the spatial discretisation where the operator was continuous in time and discretised in space, by assuming the spatial operator is independent on the time step and studying only the temporal discretisation. After calculation using the three curves available, it was estimated with a good accuracy that the order of the temporal scheme is $n_t = 1$. This is also very low when considering that the temporal discretisation is made using a third order low storage scheme. However, it can be expected that the global order of the scheme is reduced by the introduction of a correction at the boundaries of the domain during the solver. After the convection/diffusion step and before the Poisson solver, the velocity at the boundary is artificially set to zero. This correction is of first order in time, and probably explains the temporal accuracy of the whole program.

4.4 Complementary properties of the program

4.4.1 Distance to an adjoint problem

The program and adjoint optimisation procedure have already been validated by comparison with various results available. It is therefore known that if a fine enough grid is used, the adjoint optimisation method numeric solver will give the result of the optimisation problem. However, one more problem remains, linked to the definition of the adjoint problem itself, and that could have an influence on the accuracy of the results. If the optimisation is performed between the initial time t_i and the final time t_f , the following equality should hold:

$$\langle P\mathbf{u}_n(t_f), P\mathbf{u}_n(t_f) \rangle = \langle \mathbf{u}_n(t_i), \mathbf{u}_{n+1}(t_i) \rangle, \quad (4.31)$$

where the subscript n denotes the number of adjoint loop performed to obtain $\mathbf{u}_n(t_i)$ ($\mathbf{u}'_0(t_i)$ is the initial guess). This is however not obvious when considering how the adjoint solver equations were derived (see section 3.2.3). If the operator R is a discretised version of the continuous operator, then the solution at t_f can be written $\mathbf{u}'(t_f) = R\mathbf{u}'(t_i)$. The adjoint operator in this case is the transpose of the operator R . Calculating directly the transpose of a linear solver is possible, and called a discrete adjoint technique. The quantity calculated with this approach will be called R_d^* , but this method of calculating the adjoint operator was not the one used in this thesis. In this thesis, the adjoint of the continuous equation was first calculated, and then discretised the same way as the direct problem was. This method is called continuous adjoint. Even if in the limit of fine grid the problem is well represented, for a given grid, the quantity calculated is not exactly the adjoint of R ; it will therefore be called R_c^* . Using this notation, it becomes clear why the equality of equation (4.31) is unlikely to be true, as its left hand side can be rewritten:

$$\begin{aligned} \|\mathbf{u}_n(t_f)\|_f^2 &= \langle P R \mathbf{u}_n(t_i), P R \mathbf{u}_n(t_i) \rangle \\ &= \langle \mathbf{u}_n(t_i), R_d^* P^2 R \mathbf{u}_n(t_i) \rangle, \end{aligned} \quad (4.32)$$

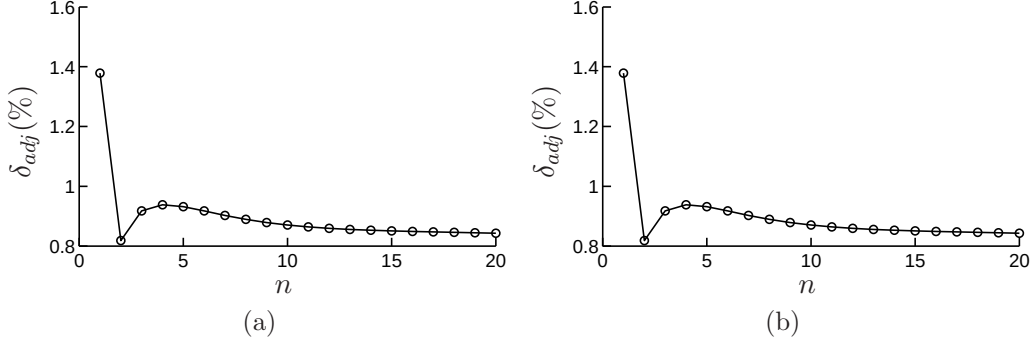


Figure 4.7: Difference between two possible measure of the perturbation velocity energy, dependence on the number n of adjoint loop performed. In (a), E represents $\langle \mathbf{u}_n(t_i), \mathbf{u}_{n+1}(t_i) \rangle$ in the curve with the symbols (o) , and $\|\mathbf{u}_n(t_f)\|_f^2$ in the curve with the symbols $(+)$. In the case used, the mean profile is at $Re_\tau = 180$ with harmonic wall oscillations of period $T^+ = 100$. The optimisation is performed between the initial time $t_i^+ = -23.4$ and the final time $t_f^+ = 0$, at a distance to the wall $y_0^+ = 6.6$.

And its right hand side:

$$\langle \mathbf{u}_n(t_i), \mathbf{u}_{n+1}(t_i) \rangle = \langle \mathbf{u}_n(t_i), R_c^* P^2 R \mathbf{u}_n(t_i) \rangle. \quad (4.33)$$

In the current section, an attempt is made to understand how the discrepancy between the operator R_c^* and the real adjoint operator R_d^* could affects the results.

The main source of comparison is to measure the relative difference between the quantities of equation (4.33) and equation (4.32). A variable is defined as:

$$\delta_{adj} = \frac{\left| \|\mathbf{u}'(t_f)\|_f^2 - \langle \mathbf{u}'(t_i), \mathbf{u}_{n+1}(t_i) \rangle \right|}{\|\mathbf{u}_n(t_f)\|_f} \quad (4.34)$$

The first observation showed on figure 4.7 shows the estimation of this error at each iteration n of the optimisation procedure. As expected, the evolution of the energy growth is exponential (the growth is geometric). A less expected result is the behaviour of the amplitude of the error δ_{adj} , which is of the order of one percent, and which stops decreasing after a few iterations. However, each possible measure of the energy converges to a maximum value, and it is this convergence which will be taken in consideration to stop the optimisation algorithm when the accuracy is supposed to be good enough. This also sets a limit to the expected accuracy on the value of the

energy growth that can be expected. As the two quantities showed on figure 4.7(a) both represent what should be same energy measurement, and as the relative difference between these two quantities is of about one percent, the accuracy on the energy growth obtained can not be expected to be better than one percent.

It would be possible to decrease the error in the adjoint operator definition by refining the spatial grid. This was done on a simple example with no x -dependence of the perturbation velocity on figure 4.8. As expected, the relative error δ_{adj} converged

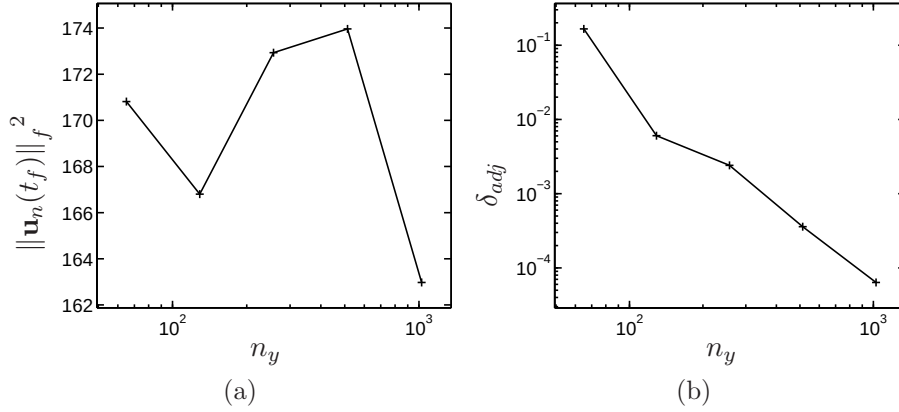


Figure 4.8: The mean flow used is a simple turbulent channel flow mean profile at $Re_\tau = 180$. Test of the evolution of accuracy of the problem with mesh refinement. Results given after 5 adjoint iterations, optimisation at a distance to the wall $y^+ = 10$.

towards zero as the grid is refined. This could be used to define a grid to have the accuracy as good as necessary. However, for the use of the adjoint optimisation procedure in this thesis, the qualitative aspect of the results was more important than the quantitative one, and an error of a few percent on the energy growth was acceptable. As the calculations were relatively expensive, a grid as coarse as possible that enabled a good representation of the physics was kept. The error in the estimation of the energy growth (and therefore in the structures predicted) will never be an issue in this thesis, but gives an idea of the limited accuracy of some quantities.

To test the error in the estimation of the energy growth due to the use of a continuous adjoint problem, the results were again compared to the results of (Burini and Zaki, 2011), as the code used by this group also relies on a continuous adjoint method. The relative error measured is showed in figure 4.9 for two of the cases used, and is representative of all the set of comparisons. The variation of the value of δ_{adj} obtained

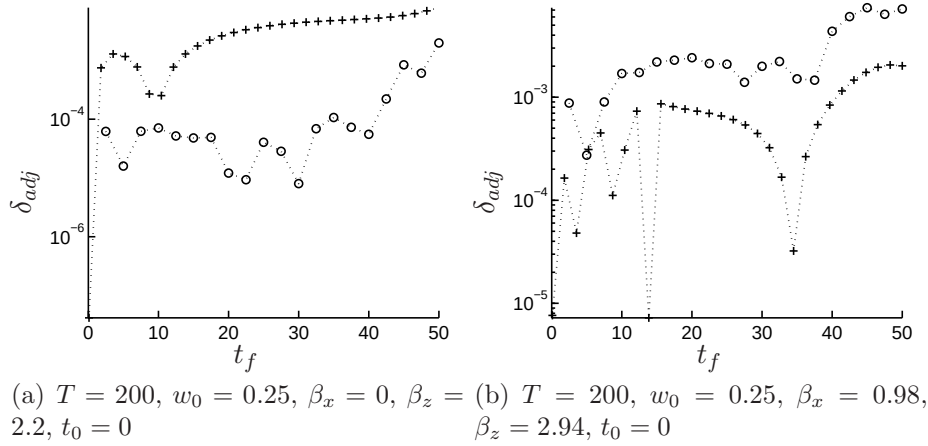


Figure 4.9: Distance to an adjoint problem, in a turbulent boundary layer subject to a spanwise Stokes layer. ($\circ$) simulation from Burini and Zaki (2011), ($+$) program used in this thesis.

with the two programs can be as large as one order of magnitude. This is not due to the choice of a finer grid in one of the codes, as depending on the cases used the more accurate code changes. This shows that both programs have a similar behaviour even if it does not explain the dependence of δ_{adj} on the mean flow used. Overall, the evolution of δ_{adj} remains badly understood. For that reason, this quantity was often estimated for the results obtained in this thesis to guarantee the quality of the optimisation procedure.

4.4.2 A time step dependent instability

To complete the description and validation of the program, this last section describes a not well understood phenomenon. Initially, the time step used to perform simulations (adjoint optimisations and other) at a Reynolds number $Re_\tau = 180$ was set to $dt = 0.003$. This was enough to guarantee the CFL stability and to perform numerous simulations. However, with some specific mean flows, a time step dependent numerical instability appeared. One of these cases is showed on figure 4.10, for an optimal perturbation calculated for harmonic wall oscillations of period $T^+ = 200$, and at a distance $y^+ = 11.25$ to the wall. For time $t^+ < 100$, the energy observed is the one of the optimal perturbation, until suddenly an exponential instability appears. The time step has been changed to observe the dependence of this instability on dt . As it

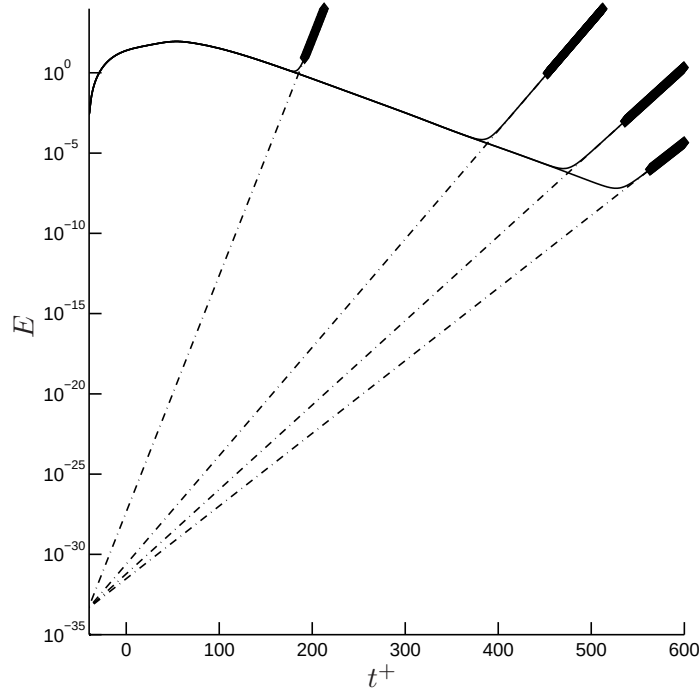


Figure 4.10: Evolution in time of the energy $E = \|\mathbf{u}'(t^+)\|_f^2$ of the generalised optimal perturbation corresponding to a mean flow at $Re_\tau = 180$, with harmonic spanwise oscillations of period $T^+ = 200$. Optimal perturbation at the final time $t_f^+ = 0$. The simulation is run using the same spatial grid, and four different time steps. The divergence becomes visible later as the time step gets smaller. The thick part of each curve is the part that was used to calculate the coefficient of the exponential. The dashed lines are a prolongation of the exponentially growing part up to the initial time t_i .

decreases, the exponential instability takes longer to become visible. To understand the origin of this instability, lines were drawn to continue each exponential part, up to the initial time t_i of the perturbation. All the lines converge to the same area at t_i , which is of the order of magnitude $E = 10^{-33}$. The code used being in double precision, the machine accuracy is of 10^{-16} . When an energy is measured, as is the case in figure 4.10, this error becomes of the order 10^{-32} . Finally taking into account that the energy measured is a ratio of two different norms shows that the exponential divergence is due to the amplification of a perturbation which amplitude was of the order of the machine accuracy at initial time.

As it was easy to calculate the coefficients of the exponential for each time step

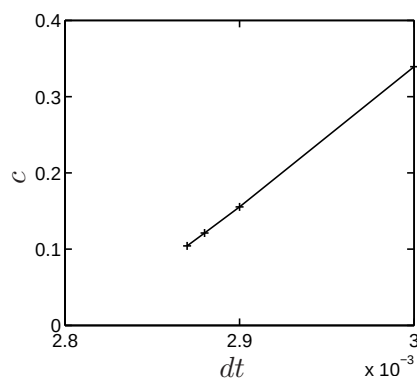


Figure 4.11: Coefficient of the exponential obtained for different time steps. Assuming linearity of the dependence of this coefficient on dt , the intersection of the curve with the axis happens for $dt = 2.81$. For smaller values of dt , the instability disappears.

($E(t) = E_0 e^{ct}$), an attempt could be made to understand at which time step this instability could potentially disappear. This is done in figure 4.11 where the dependence of this coefficient on the time step is showed. Four points are enough there to see that this dependence is linear. This shows that for dt small enough ($dt < 2.81$ for this example), no exponential divergence will appear.

Chapter 5

Passive scalar streaks in turbulent channel flow

One of the central calculations in the work of Chernyshenko and Baig (2005) was to predict passive scalar streaks spacing. In a turbulent channel flow, the linearised equation for a passive scalar is almost the same as the linearised equation for the velocity fluctuation. Due to the mean flow having only one component in the streamwise direction, it can be assumed that when calculating optimal perturbation, the optimal perturbation velocity will not depend on the streamwise direction. With this assumption, the velocity fluctuation equation can be written as:

$$\frac{\partial u}{\partial t} + v \frac{\partial U}{\partial y} = \frac{1}{Re} \nabla^2 u, \quad (5.1a)$$

$$\frac{\partial v}{\partial t} = - \frac{\partial p}{\partial y} + \frac{1}{Re} \nabla^2 v, \quad (5.1b)$$

$$\frac{\partial w}{\partial t} = - \frac{\partial p}{\partial z} + \frac{1}{Re} \nabla^2 w. \quad (5.1c)$$

The streamwise component of this linearised Navier-Stokes equation is very similar to the passive scalar equation, which under the same assumption has the form:

$$\frac{\partial \theta}{\partial t} + v \frac{\partial \Theta}{\partial y} = \frac{1}{Re} \nabla^2 \theta, \quad (5.2)$$

where $\Theta(y)$ is the mean passive scalar profile, and θ the perturbation. The only difference with the velocity momentum equation is the term $v\partial\Theta/\partial y$ which replaces the term $v\partial U/\partial y$. As the velocity momentum equation is not coupled to the crossflow equations in this configuration, the behaviour of the streamwise velocity and a passive scalar are equivalent. The generalised optimal perturbation approach therefore applies as well to passive scalar streaks as to velocity streaks. The advantage of using passive scalar is the greater possibility of mean profile. Using this additional freedom, Chernyshenko and Baig (2005) could test the generalised optimal perturbation approach for streaks spacing varying of more than one order of magnitude. Using such a wide range of parameters is the best way to validate the predictive capabilities of the generalised optimal perturbation approach. Indeed, this approach being approximate by nature, the accuracy of the streaks spacing predicted is expected to be of the order of 30%. Having such a wide range of streaks spacing available then enables to assess that the prediction error are within the acceptable bounds.

In Chernyshenko and Baig (2005), the generalised optimal perturbation was calculated using a semi-analytical Green function approach. Such methods are possible in simple configurations where the perturbation profiles are one dimensional, and have the advantage of being numerically much more efficient than using adjoint optimisation. Velocity streaks spacing were calculated using the two available initial norms, and passive scalar streaks spacing were calculated by using only the initial norm $\|\cdot\|_c$. However it will be shown that some of the assumptions made in that paper to calculate the most amplified perturbation when using this specific norm are not valid. The error made will be estimated using the adjoint optimisation program implemented for this thesis, and finally a new version of the semi-analytical approach will be presented. The passive scalar streaks spacing will also be studied in the case when the initial norm $\|\cdot\|_b$ is being used. In this chapter, as the problem solved is in a channel flow with no wall oscillations, the appropriate optimisation problem to find the generalised optimal perturbation is the one described in equation (2.6). Solutions of this problem will therefore be calculated using the two possible initial norm.

5.1 Background information and problem description

Before explaining the main content of this chapter, a few technical remarks need to be made. As the flow is homogeneous in the spanwise direction, a single spanwise wavelength is considered. For simplicity of notations, this is however not mentioned on the velocity components. In the following equations, the quantity $u(t, y)$ is called perturbation velocity, but is used to describe either the streamwise component of the perturbation velocity, or the passive scalar perturbation. This is possible as they have the same role in their respective linearised equations. The optimisation problem of equation (2.6) can also be decomposed into two stages:

$$A_{I,0}(\beta_z, t) = \max_{\mathbf{u}'_0} \frac{u(t, y)^2}{\|\mathbf{u}'_0\|_i^2}, \quad (5.3a)$$

$$A_I(y) = \max_{\beta_z, t > 0} A_{I,0}(\beta_z, t), \quad (5.3b)$$

where I denotes the kind of initial norm used, $\mathbf{u}'_0 = (0, v_0, w_0)$ is the initial perturbation for the spanwise wavenumber β_z , and $u(t, y)$ the streamwise component of the solution to the linearised problem. The first stage of the optimisation (5.3a) is complex, as it is an optimisation over the shape of the initial condition. The second step (5.3b) is much simpler as it is a parametric optimisation over the spanwise wavenumber β_z and the observation time t . The main focus of this paper is the first step of the optimisation. The remaining part of this section describes the method used in Chernyshenko and Baig (2005) to perform this step of the optimisation. The results will then be compared with the results obtained using adjoint optimisation of the linear operator.

5.1.1 Optimisation with a Green function approach

Among the methods available to solve the optimisation problem of equation (5.3a), one of the most efficient is probably the semi analytical approach used in Chernyshenko and Baig (2005). A summary of this approach is made here, as the accuracy of the results calculated by this method is central to this chapter. Some of the equations displayed here can also be found in Chernyshenko and Baig (2005).

Due to the linearity of the problem, for an initial perturbation $(0, v_0, w_0)$ and the resulting streamwise component $u(t, y)$, there exists a Green function $\mathbf{G}(t, \beta_z, y, \eta) = (G_v(t, \beta_z, y, \eta), G_w(t, \beta_z, y, \eta))$ such that, with the domain boundaries being at $y = -1$ and $y = 1$ in the wall normal direction:

$$u(t, y) = \int_{\eta=-1}^1 [G_v(t, \beta_z, y, \eta)v(\eta) + G_w(t, \beta_z, y, \eta)w(\eta)] d\eta. \quad (5.4)$$

As the flow is incompressible, the initial condition has to satisfy the divergence free condition $\partial v(\eta)/\partial \eta + \beta_z w(\eta) = 0$. Because of that, the Green function used in equation (5.4) is not defined uniquely. This can for example be seen if the spanwise component $G_w(t, \beta_z, y, \eta)$ of the Green function is modified into $\tilde{G}_w(t, \beta_z, y, \eta) = G_w(t, \beta_z, y, \eta) + f(\eta)$, with $f(\eta)$ a differentiable function. It can then be shown using equation (5.4), integrating by part and using the continuity condition that a new Green function $\tilde{\mathbf{G}}(t, \beta_z, y, \eta)$ gives the same result as equation (5.4), with $\tilde{\mathbf{G}}(t, \beta_z, y, \eta)$ defined as:

$$\begin{aligned} \tilde{G}_v(t, \beta_z, y, \eta) &= G_v(t, \beta_z, y, \eta) - \frac{1}{\beta_z} \frac{\partial f}{\partial \eta}(\eta), \\ \tilde{G}_w(t, \beta_z, y, \eta) &= G_w(t, \beta_z, y, \eta) + f(\eta). \end{aligned} \quad (5.5)$$

Apart from its differentiability, there is no condition imposed on the function f , which leads to a large number of possible choices for the Green function. Throughout this chapter, mainly one specific Green function will be used, which was calculated in Chernyshenko and Baig (2005). Its most important characteristic is that it satisfies the divergence equation; $\partial G_v(t, \beta_z, y, \eta)/\partial \eta + \beta_z G_w(t, \beta_z, y, \eta) = 0$.

Initial norm as a standard energy norm

To solve the optimisation problem of equation (5.3a) and calculate $A_{b,0}(\beta_z, t)$ in the case when the norm $\|\cdot\|_b$ for the initial condition is used, the streamwise component has to be maximised under the constraint that $\int_{-1}^1 v_0^2 + w_0^2 = 1$. From equation (5.4), this holds if the velocity field (v_0, w_0) at initial time is parallel to the Green function; $(v(\eta), w(\eta)) = \text{const} \times (G_v(t, \beta_z, y, \eta), G_w(t, \beta_z, y, \eta))$. The solution of the problem is

therefore:

$$(v(\eta), w(\eta)) = \frac{(G_v(t, \beta_z, y, \eta), G_w(t, \beta_z, y, \eta))}{\sqrt{\int_{-1}^1 [G_v(t, \beta_z, y, \eta)^2 + G_w(t, \beta_z, y, \eta)^2] d\eta}}, \quad (5.6)$$

and the energy growth is given by the expression:

$$A_{b,0}(\beta_z, t) = \int_{-1}^1 [G_v(t, \beta_z, y, \eta)^2 + G_w(t, \beta_z, y, \eta)^2] d\eta. \quad (5.7)$$

The issue with this approach is that it solves the optimisation problem over all possible initial conditions, and not only the ones satisfying the continuity condition. However, as seen earlier in this section, the specific Green function calculated in Chernyshenko and Baig (2005) and used to for the calculations in this chapter satisfy the continuity equation. This implies that the solution found in equation (5.6) satisfies the maximisation problem under the initial norm constraint, but also under the continuity constraint. It is therefore the optimal solution. The generalised optimal perturbation at a distance y to the wall is then obtained by maximising the quantity $A_{b,0}(\beta_z, t)$ of equation (5.7) over the parameters (t, β_z) .

Initial norm as a Reynolds stress weighted energy norm

Finding the quantity $A_{c,0}(\beta_z, t)$ in the case where the initial norm $\|\cdot\|_c$ is used is done in a similar way. The constraint on the initial condition is now $\frac{1}{\bar{v}} \int_{-1}^1 v_0^2 / \langle v^2 \rangle + w_0^2 / \langle v^2 \rangle = 1$. To optimise the problem in an orthonormal space, the new variables $\check{v}_0 = v_0 \sqrt{\langle v^2 \rangle}$, and $\check{w}_0 = w_0 \sqrt{\langle v^2 \rangle}$ are defined. The equation (5.4) leading to $u(t, y)$ is then modified to:

$$u(t, y) = \int_{\eta=-1}^1 \left[G_v(t, \beta_z, y, \eta) \sqrt{\langle v(\eta)^2 \rangle} \check{v}_0(\eta) + G_w(t, \beta_z, y, \eta) \sqrt{\langle w(\eta)^2 \rangle} \check{w}_0(\eta) \right] d\eta.$$

To be a solution of the optimisation problem, the initial perturbation $(\check{v}_0, \check{w}_0)$ has to be parallel to the modified Green function, which is:

$$(\check{v}_0, \check{w}_0) = \text{const} \times (G_v(t, \beta_z, y, \eta) \sqrt{\langle v(\eta)^2 \rangle}, G_w(t, \beta_z, y, \eta) \sqrt{\langle w(\eta)^2 \rangle}).$$

This, coupled with the requirement for the initial solution to lie on the unit sphere, leads to an optimal solution at initial time:

$$(v(\eta), w(\eta)) = \frac{(G_v(t, \beta_z, y, \eta) \langle v^2(\eta) \rangle, G_v(t, \beta_z, y, \eta) \langle w^2(\eta) \rangle)}{\sqrt{\int_{-1}^1 [G_v(t, \beta_z, y, \eta)^2 \langle v^2(\eta) \rangle + G_w(t, \beta_z, y, \eta)^2 \langle w^2(\eta) \rangle] d\eta}}, \quad (5.8)$$

and the maximum energy growth is:

$$A_{c,0,Green}(\beta_z, t) = \int_{-1}^1 [G_v(t, \beta_z, y, \eta)^2 \langle v^2(\eta) \rangle + G_w(t, \beta_z, y, \eta)^2 \langle w^2(\eta) \rangle] d\eta. \quad (5.9)$$

With the particular Green function used in Chernyshenko and Baig (2005), the initial condition (equation (5.8)) does not satisfy the continuity condition, and is therefore not a solution to the whole optimisation problem. Because of that, the optimisation of the quantity found in equation (5.9) is called $A_{c,0,Green}(\beta_z, t)$, and not $A_{c,0}(\beta_z, t)$ which is the quantity of interest. An optimisation of the value of $A_{c,0,Green}(\beta_z, t)$ over (t, β_z) can therefore not a priori be expected to give a result close to the optimisation of the correct parameter $A_{c,0}(\beta_z, t)$. It has however been used in Chernyshenko and Baig (2005) in a successful comparison with direct numerical simulation results for the case of passive scalar streaks.

In Chernyshenko and Baig (2005), the use of the energy growth $A_{c,0,Green}(\beta_z, t)$ obtained in equation (5.9) was justified by linking the value obtained to a simplified expression of the streamwise velocity variance $\langle \hat{u}(t) \rangle^2$. Doing so, the problem solved is not exactly the generalised optimal perturbation problem, as maximising the streamwise velocity variance is not the same as maximising the streamwise velocity. However, maximising the variance is supposed to give a good description of the streaks spacing; instead of calculating the most energetic structure (generalised optimal perturbation), the most probable is obtained. From experimental results they are known to be close to each other (Smith and Metzler, 1983).

To find a link between the value of $A_{c,0,Green}(\beta_z, t)$ found in equation (5.9) and the streamwise velocity variance, several hypotheses were made in Chernyshenko and Baig (2005). First, from equation (5.4) and with the hypothesis that a statistical ensemble

of initial condition $(\widehat{v}_0, \widehat{w}_0)$ is used, the expression for the variance of $\widehat{u}$ is:

$$\begin{aligned} \langle \widehat{u}(t) \rangle^2 = & \int_{-1}^1 \int_{-1}^1 G_v(t, \beta_z, y, \eta_1) G_v(t, \beta_z, y, \eta_2) \langle \widehat{v}_0(\eta_1) \widehat{v}_0(\eta_2) \rangle \\ & + G_v(t, \beta_z, y, \eta_1) G_w(t, \beta_z, y, \eta_2) \langle \widehat{v}_0(\eta_1) \widehat{w}_0(\eta_2) \rangle \\ & + G_w(t, \beta_z, y, \eta_1) G_w(t, \beta_z, y, \eta_2) \langle \widehat{w}_0(\eta_1) \widehat{w}_0(\eta_2) \rangle d\eta_1 d\eta_2. \end{aligned} \quad (5.10)$$

With the additional assumption that the characteristics correlations lengths are small and independent of y , and that the cross-correlation $\langle \widehat{v}_0(\eta_1) \widehat{w}_0(\eta_2) \rangle$ is negligible, the expression of the variance can be simplified to:

$$\langle \widehat{u}(t) \rangle^2 = \int_{-1}^1 [G_v(t, \beta_z, y, \eta)^2 \langle \check{v}_0(\eta)^2 \rangle + G_w(t, \beta_z, y, \eta)^2 \langle \check{w}_0(\eta)^2 \rangle] d\eta. \quad (5.11)$$

Assuming then that $(\langle \check{v}_0(\eta)^2 \rangle, \langle \check{w}_0(\eta)^2 \rangle) = \text{const} \times (\langle v^2(\eta) \rangle, \langle w^2(\eta) \rangle)$, the expression of equation (5.11) for the variance and equation (5.9) for the solution of the maximisation problem with the initial norm $\|\cdot\|_c$ are the same. This means that if all the hypothesis made are satisfied, even if the result of the optimisation of $A_{c,0}(\beta_z, t)$ is still not the exact solution to the optimisation problem (the initial solution is not physical as it does not satisfy the continuity equation), it can now be expected to reflect the streaks characteristics via the velocity variance identification.

The hypotheses made here that enable the result of the optimisation to be interpreted as the variance of the streamwise velocity are exactly the same as in Chernyshenko and Baig (2005). The variance comparison is a good justification for using the optimisation to compare the streaks spacing to DNS streaks spacing only if all the assumptions made are correct. In the next section, the results obtained with this method will be compared to proven accurate generalised optimal perturbation results obtained by another method, and then the validity of the assumption will be studied. It will be shown that these assumptions are not valid, and further explanation will be provided.

5.2 Comparison of the approximate streak spacing results with exact solutions

In this section, the streaks spacing in the case where the initial norm $\|\cdot\|_c$ is used will be compared for the two optimisation methods available: the Green function approach and the adjoint optimisation. Comparisons will also be performed using results obtained with the initial norm $\|\cdot\|_b$ and results from DNS. Six passive scalar mean profiles were used in Chernyshenko and Baig (2005), in a channel flow at a Reynolds number $Re_\tau = 360$, with streaks spacing calculations performed at distances to the wall $y^+ = 5$, $y^+ = 20$ and $y^+ = 40$. This led to streaks spacing varying over one order of magnitude, which was sufficient to draw positive conclusions regarding the validity of the generalised optimal perturbation approach. Here, the same passive scalar profiles will be used.

The adjoint optimisation simulations to obtain the exact results have been performed on a very fine grid containing $n_y = 1025$ points in the wall-normal direction. The algorithm was stopped when the difference in the value obtained for $A_{I,0}(\beta_z, t)$ between two consecutive iterations was smaller than 0.1%. This led to relatively expensive calculations, but very accurate results were obtained for comparison (the use of a very fine grid is necessary to have surface norms at a position very close to the position of the surface norm used with the Green function approach). In the comparisons made in this chapter, the streaks spacing will be denoted by λ . The streaks spacing observed in DNS is λ_{DNS} , the streaks spacing calculated with the generalised optimal perturbation approach when the norm $\|\cdot\|_b$ is used is λ_b . In the cases when the norm $\|\cdot\|_c$ is used, the streaks spacing is $\lambda_{c,Green}$ if the Green function method is used, and λ_c if the adjoint optimisation is used. The specific notation $\lambda_{c,Green}$ stress that in the case where the Green function is used, the exact optimisation problem of equation (2.6) is not solved, but rather what was considered as a good approximation by Chernyshenko and Baig (2005).

A first comparison of the streaks spacing results is shown in figure 5.1, where both the passive scalar mean profiles and the streaks spacing obtained from DNS and the two different generalised optimal perturbation techniques are shown. The most interesting comparison in this figure is the difference between $\lambda_{c,Green}$ and λ_c . These results show that at first sight, the streaks spacing calculated are close to each other in absolute

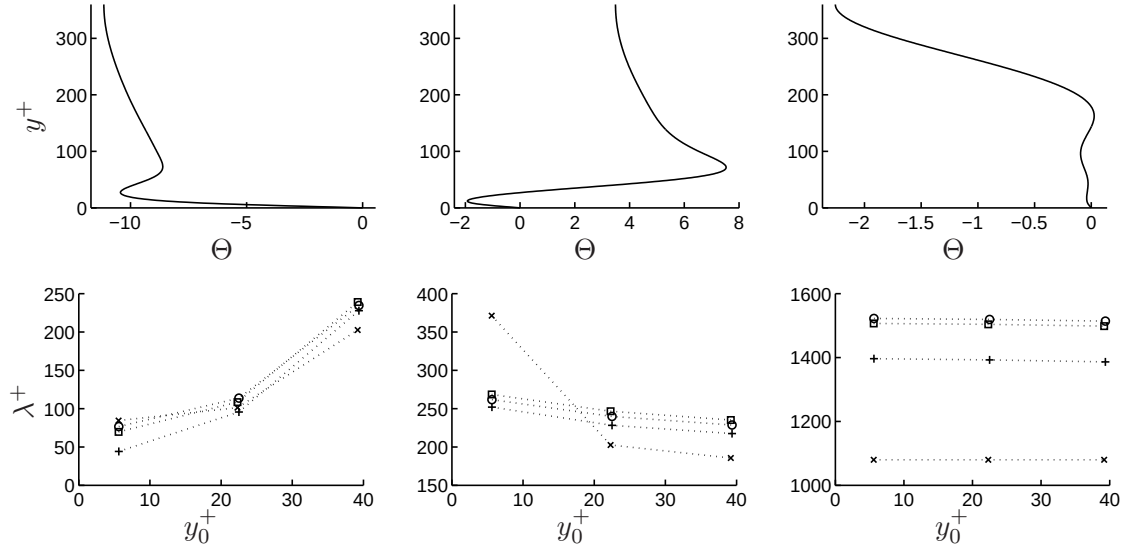


Figure 5.1: Top: passive scalar mean profile Θ , bottom: streaks spacing with respect to the distance to the wall y_0^+ . (x) λ_{DNS}^+ , (+) λ_b^+ , (o) λ_c^+ , (□) $\lambda_{c,Green}^+$. Only three representative cases of the seven passive scalar mean profiles used are shown here.

value. When compared to DNS, both approaches predict the streaks spacing with the same level of accuracy and it is not possible to say that one is better than the other. This is in favour of considering the Green function approximation approach as relevant. It also suggests that the assumption made in section (5.1.1) to assimilate the growth predicted in this case to the streamwise velocity variant as justified, even if it will be shown later that this is not true.

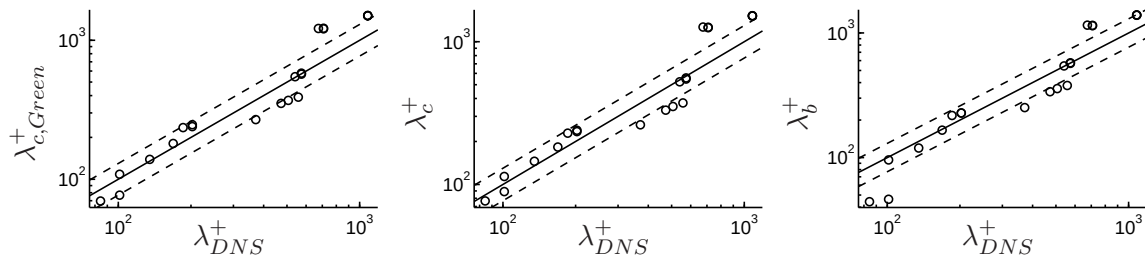


Figure 5.2: Streaks spacing calculated from the three different generalised optimal perturbation methods, compared with the DNS observed streaks spacing. The results for all seven passive scalar profiles used are shown here. (—) represent the 30% error margin.

It is surprising to observe that even if the adjoint optimisation is giving an exact

solution λ_c , and therefore could be expected to provide streaks spacing values closer to the DNS than $\lambda_{c,Green}$ does, there is no visible improvement. In most cases the difference between the two optimisation methods is marginal, and in one case the supposedly more accurate λ_c is farther from the DNS value than $\lambda_{c,Green}$. In figure 5.2, the distance between the generalised optimal perturbation predicted streaks with the two methods and the DNS streaks spacing is shown for the two approaches. In this figure it is also visible that both optimisation approaches predict the DNS streaks spacing with the same order of accuracy; most of the measures being within 30% of the DNS values λ_{DNS} .

5.3 Validity of the assumptions made in the Green function approach

When the generalised optimal perturbation approach is used with the initial norm $\|\cdot\|_c$, the two methods used (exact adjoint optimisation, and approximate Green function approach) have been shown to predict the streaks spacing with the same degree of accuracy when compared with DNS. The relative difference between $\lambda_{c,Green}$ and λ_c , and between either method and λ_{DNS} , is small. This suggests that even if it is approximate, the Green function approach retains enough of the physics to provide very satisfying results. It also supports the assumptions leading to the idea that the value of $A_{c,0,Green}(\beta_z, t)$ calculated by this approach could describe the streamwise velocity variance. In this section it will however be shown that some of the assumptions made in Chernyshenko and Baig (2005) and described in section (5.1.1), leading to this streamwise variance comparison, are not correct. It will also be shown why the results obtained are still good, and a method to improve the Green function approach to calculate the exact result of the optimisation problem will be suggested.

It can be easily proven that at least some of the assumptions leading to the approximate formulae for the streamwise velocity variance obtained in equation (5.11) are not correct. This can be shown by using the non-uniqueness of the Green function linking the initial solution to the solution at an observation time t . Using the degree of freedom given in equation (5.5) by the function f , it can be shown that f can be chosen to obtain any result possible for the energy growth $A_{c,0,Green}(\beta_z, t)$. For example, if f

is chosen to be a constant k (no y -dependence), then the simplified equation (5.11) for the flow variance will become:

$$\begin{aligned} A_{c,0,Green}(\beta_z, t) = & \int_{-1}^1 [G_v(t, \beta_z, y, \eta)^2 \langle \check{v}_0(\eta)^2 \rangle + G_w(t, \beta_z, y, \eta)^2 \langle \check{w}_0(\eta)^2 \rangle] d\eta \\ & + k \int_{-1}^1 [2G_w(t, \beta_z, y, \eta)^2 \langle \check{w}_0(\eta)^2 \rangle] d\eta \\ & + k^2 \int_{-1}^1 [\langle \check{w}_0(\eta)^2 \rangle] d\eta. \end{aligned} \quad (5.12)$$

This is a second order polynomial in k . It is not independent of k , as $\int_{-1}^1 [\langle \check{w}_0(\eta)^2 \rangle] d\eta \neq 0$. Choosing the constant k appropriately could lead to any possible result. This shows that the simplified formula of equation (5.11) is not correct, as it should not depend on the choice of the Green function. As a result, it can be concluded that this formula is not usable and at least one of the assumptions in its derivation is false.

With the assumption leading to the simplified velocity variance of equation (5.11) proven to be incorrect, the streaks spacing $\lambda_{c,Green}$ results obtained with the Green function approach can not be linked to any physical quantity anymore. There is no reason to think they should give a good approximation of the optimisation problem under the constraint that the initial condition is divergence free. In fact, the optimal energy formula of equation (5.9) suffers from the same problem as equation (5.11) with regards to the non-uniqueness of the Green function. Different choices of Green function would lead to different values of $A_{c,0,Green}(\beta_z, t)$, and choosing different Green functions for different parameters (β, t) could lead to a completely different streaks spacing and optimal energy growth $A_c(y)$.

These results suggest that the streaks spacing $\lambda_{c,Green}$ obtained with the formulae of equation (5.9) and the Green function derived in Chernyshenko and Baig (2005) should be discarded. However, the results have already shown to predict streaks spacing in good agreement with DNS streaks spacing. More importantly, these results are very close to the exact generalised optimal perturbation streaks spacing λ_c . Even if not of prime importance, the reasons why these a-priori incorrect results are close to the real optimal values are interesting. This is likely due to the fact that the Green function used in the computations was the Green function that provide the exact result λ_b when the initial time norm used is the usual energy norm $\|\cdot\|_b$.

The energy growth A_b of the optimal perturbation when $\|\cdot\|_b$ is used is given by equation (5.7). For the case with the Reynolds-stress weighted initial energy norm $\|\cdot\|_c$, the formula used to find the optimal growth is given by equation (5.9). The only difference between these two equations is the presence of the Reynolds stress in equation (5.9); the Green function is the same. From this observation, it makes sense to compare the streaks spacing $\lambda_{c,Green}$ and λ_c , but also $\lambda_{c,Green}$ and λ_b .

In figure 5.3, the streaks spacing obtained with the three generalised optimal perturbation methods are shown, as well as the DNS results. λ_c is plotted on the abscissa, and all other data are compared to this as it is the accurate result to which the Green function optimisation approach is compared. Before trying to understand why the value $\lambda_{c,Green}$ is close to the correct value λ_c , it is interesting to compare λ_b with the DNS results, as this has never been done. For most points, the streaks spacing λ_b predicts the DNS streaks spacing with the same order of accuracy as when $\|\cdot\|_c$ is used. The prediction is mostly within 30% of the observed value, for streaks spacing varying over an order of magnitude, and shows that using this initial norm also leads to a good representation of the physical mechanisms involved. A closer analysis shows that the two points significantly off the DNS values, where the streaks spacing predicted λ_b is the smallest, correspond to cases where the observation is performed at a distance to the wall of $y^+ = 5$. It is not surprising as it is in this area that the optimisation with $\|\cdot\|_b$ would favour structures very close to the wall, whereas with the use of $\|\cdot\|_c$ the Reynolds stress weight favours structures located farther from the wall where the Reynolds stress is maximised.

When the observation layer is far from the wall, figure 5.3 shows that the predictions using different initial norms are very close to each other. This shows that the Reynolds stress has little importance in the selectivity mechanisms far from the wall, and it is then not surprising that the optimisation of equation (5.7) and equation (5.9) with regard to (β, t) will give similar results. When the observation layer is close to the wall where the Reynolds stress vanishes, the optimisation of equation (5.9) will favour Green functions which are more concentrated farther from the wall. The streaks spacing predicted will then be bigger than in the case where $\|\cdot\|_b$ is used. The fact that it is so close to the adjoint optimisation results might be a sign that at these distances to the wall the optimisation result is more dependent on the Reynolds stress than on the Green function.

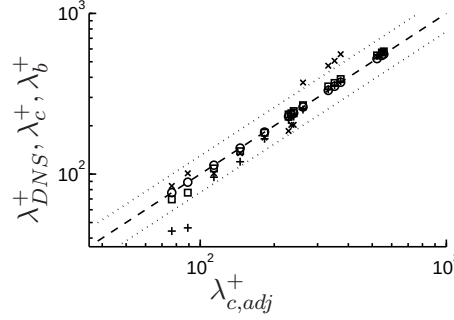


Figure 5.3: Comparison of streaks spacing obtained by various methods to the streaks spacing λ_c^+ obtained with the initial norm $\|\cdot\|_c$ and adjoint optimisation. ($\times$) λ_{DNS}^+ , ($+$) λ_b^+ , ($\square$) $\lambda_{c,Green}^+$. (..) represent the 30% error margin from λ_c^+ .

5.3.1 Suggestion for an efficient and accurate way to find the generalised optimal perturbation

The Green function approach to calculating the solutions of the generalised optimal perturbation is very attractive as it is numerically very efficient. However, the optimisation procedure does not impose the solution to be divergence free, and as a result the optimal found is non-physical unless it also happens to be divergence free. In the case where the initial norm $\|\cdot\|_b$ was used, the optimal solution found was divergence free due to the nature of the Green function used. Using the same Green function with the initial norm $\|\cdot\|_c$ led to a non-physical solution. A suggestion is made here to improve the optimisation procedure in this latter case, still using the numerically efficient Green function approach, but with an extra step ensuring the optimisation is made over the space of divergence free solutions instead of a larger space.

For a given Green function $\mathbf{G}(t, \beta_z, y, \eta)$ leading to a solution $u(t, y)$ at an observation time t from the initial transverse velocity field (v_0, w_0) , the optimal growth in the case when the initial norm $\|\cdot\|_c$ is used is given by equation (5.9), while the initial solution is given by equation (5.8). While the optimal growth calculated is exact, the optimisation space depends on the choice of the Green function. Choosing a Green function for which the optimal initial solution is divergence free would give an exact solution to the constrained optimisation problem. This can be done by using the degree of freedom on the choice of the Green function described in equation (5.5). Replacing a Green function $\mathbf{G}(t, \beta_z, y, \eta)$ in equation (5.8) for the optimal initial condition by the

Green function $\tilde{\mathbf{G}}(t, \beta_z, y, \eta)$ described in equation (5.5), the equation for the optimal initial solution becomes:

$$(v(\eta), w(\eta)) = \text{const} \times \left(\left[G_v(t, \beta_z, y, \eta) - \frac{1}{\beta} \frac{\partial f}{\partial y} \right] \langle v^2(\eta) \rangle, [G_w(t, \beta_z, y, \eta) + f] \langle w^2(\eta) \rangle \right) \quad (5.13)$$

The degree of freedom given by the choice of the function f enables the search for an initial solution satisfying the divergence free solution. Calculating the divergence of equation (5.13), and enforcing it to be equal to zero everywhere in the domain, the following equation is obtained:

$$cf'' = c_1 f' + c_2 f + RHS, \quad (5.14)$$

where:

$$\begin{aligned} c &= \frac{\langle v^2(\eta) \rangle}{\beta}, \\ c_1 &= -\frac{1}{\beta} \frac{\partial \langle v^2(\eta) \rangle}{\partial y}, \\ c_2 &= \beta \langle w^2(\eta) \rangle, \\ RHS &= \frac{\partial G_v(t, \beta_z, y, \eta) \langle v^2(\eta) \rangle}{\partial y} + \beta G_w(t, \beta_z, y, \eta) \langle w^2(\eta) \rangle. \end{aligned} \quad (5.15)$$

This is a linear second order differential equation of the variable η . If the functions $G_v(t, \beta_z, y, \eta)$, $G_w(t, \beta_z, y, \eta)$, $\langle v^2(\eta) \rangle$ and $\langle w^2(\eta) \rangle$ are known with good enough accuracy, a solution of this equation can be numerically found and the exact solution to the constrained optimisation problem obtained. The extra cost brought by this additional step is very small, as the differential equation to solve has only one dimension in space. This step is also relatively straightforward and can be performed using standard software such as Matlab. The whole optimisation using this new accurate Green function approach can be performed in a reasonable amount of time on a desktop computer, whereas the same results obtained from an adjoint optimisation are very time consuming and require the use of a computing cluster. The longer the growth time of the optimal solution, the less attractive the adjoint optimisation approach becomes.

This is because the calculation cost is proportional to the optimal growth time for the adjoint optimisation approach, whereas in the Green function approach the result is independent of the growth time.

5.4 Conclusions

The generalised optimal perturbation applied to predicting passive scalar streaks spacing was used in this chapter. The two possible initial norm have been used; the standard energy norm $\|\cdot\|_b$ and the Reynolds stress weighted energy norm $\|\cdot\|_c$. It was already known from previous research that the method using the Reynolds stress weighted initial norm predicts streaks spacing with a good accuracy. It was shown here that the method based on the standard energy norm also gives good results, except for some cases where the streaks spacing is measured very close to the wall.

The main focus of this chapter was to analyse the case where the Reynolds stress weighted initial norm is used. Previous results obtained using this norm were based on a semi analytical approach using Green functions. Certain assumptions were made, and even if numerically very efficient, this method could not guarantee that the optimal solution found would meet the divergence free criterion requested for continuity. The second method is based on an adjoint optimisation of the linearised Navier-Stokes operator and is numerically more demanding, but it guarantees that the initial optimal solution found is the exact solution to the optimisation problem.

The streaks spacing results for the two methods showed to be very close to each other, and predicted with a good accuracy the streaks spacing observed in DNS. However, it was also shown that some of the assumptions made in the Green function approach were false, and as a result the optimisation outcome is sensitive to the fact that the Green function is not uniquely defined. Because of this issue, results of the Green function based optimisation have no reason to be physically meaningful. However, as the results are close to the exact optimisation results, the conclusions made in previous paper using this approach are still valid. Finally, a complementary step to the Green function approach was derived. This enables the optimisation to be done without the need of the extra hypothesis previously used, allowing for a numerically efficient, but also accurate optimisation process.

Chapter 6

Streaks and harmonic wall forcing

The first study of the linearised Navier-Stokes operator in configuration for drag reduction by wall oscillations is done in this chapter in the case of a channel flow subject to harmonic wall forcing. Different oscillation frequencies are used, where drag reduction occurs in turbulent flow. This kind of configuration has the advantage to have been widely studied in the literature, and direct numerical simulation data are available for comparison. The main focus will be to predict the streaks, and use the simplified linear framework to gain better understanding of the turbulent flow streaks. Also, as data on several wall oscillation regimes are available, and therefore several levels of drag reduction are known, some investigation into the abilities of the linear operator to predict drag will be performed.

In this chapter, only the generalised optimal perturbation approach will be used. When this study was first planned, it was hoped that the generalised optimal perturbation approach could explain the streak structures as well as the drag reduction. This was taking into account the supposed link between structures and drag, but not the fact that the generalised optimal perturbation approach describes only the peak of the filter and not its width. Some observations in this chapter will explain more clearly why this approach is good at predicting structures, but not good at predicting drag. Some other measures based on derivations of the generalised optimal perturbation techniques will also be designed to better take into account the width of the filter. However, the main goal of this chapter is to see to what extent the generalised optimal perturbation approach can predict streaks in these flow configurations and how this helps to understand the streak formation mechanisms.

In the first section, the main characteristics of the simulations and the underlying hypothesis and limitations are given. Then, the structures obtained are compared to direct numerical simulation streaks. In the last part, the streaks formation mechanisms are explained, which leads to a better understanding of the observations made in turbulent flow subject to harmonic wall oscillations.

6.1 Simulations characteristics

6.1.1 Base flow

A flow in a plane channel is considered. The flow is controlled by wall oscillations. The velocity of both walls is prescribed as $w_{wall} = W_m \cos(2\pi t/T)$, with W_m the wall maximum velocity, t the time and T the period of oscillations.

As wall units are the relevant scaling for near wall phenomena, they will be used throughout this chapter. This will however be a source of error when comparing the generalised optimal perturbation results with direct numerical simulation. The direct numerical calculations (Touber and Leschziner, 2012), which will form the basis for our comparisons, were conducted at a fixed bulk Reynolds number. Thus, the skin friction in the controlled flow is not known until the numerical simulation has been performed, and the scaled results are not known either. In order to fully test the predictive capabilities of the generalised optimal perturbation approach, its input in the present study was limited to what is known before the numerical simulation of the controlled flow has been performed, even though this might reduce the accuracy of the predictions. Using the channel half-width as the length-scale, the longitudinal velocity of the uncontrolled mean flow can be approximated by the Reynolds and Tiederman (1967) formula. With the channel wall placed at $y = \pm 1$, the streamwise velocity U^+ is obtained by integration from the following system

$$G(y) = \frac{1}{2} \sqrt{1 + \left(0.525 \frac{Re_\tau}{3} (1 - y^2)(1 + 2y^2) \left[1 - \exp \left(-(1 - |y|) \frac{Re_\tau}{37} \right) \right] \right)^2} - \frac{1}{2},$$

$$\frac{\partial U^+}{\partial y} = \frac{-y Re_\tau}{1 + G(y)}.$$

Here, Re_τ is the Reynolds number based on the friction in the *uncontrolled* flow -

that is, the flow with non-oscillating walls. For non-zero wall oscillations, when the drag is reduced, Re_τ is reduced as well and the mean longitudinal velocity profile is also somewhat different, but this effect was ignored when generalised optimal perturbations were calculated in the present work.

The choice of the spanwise profile was based on the observation (Quadrio et al., 2009; Toubert and Leschziner, 2012) made in direct numerical simulations that if the period of oscillation is small enough ($T^+ \leq 200$), the phase-averaged mean profile is very close to the laminar Stokes layer. For the generalised optimal perturbation simulations, a laminar solution of the Navier-Stokes equation with two walls oscillating in phase was therefore used. The spanwise velocity profile is given by:

$$W(t, y) = C_0 \left\{ e^{-k(y+1)} [\cos(\omega t - k(y-1)) + e^{-2k} \cos(\omega t - k(y-1) + 2k)] + e^{k(y-1)} [\cos(\omega t + k(y-1)) + e^{-2k} \cos(\omega t + k(y-1) + 2k)] \right\},$$

where $\omega = 2\pi/T$, the oscillation wavenumber $k = \sqrt{\omega/(2\nu)}$, and C_0 is a renormalisation constant selected such that at the walls $W(t, \pm 1) = W_m \cos(2\pi t/T)$.

In this chapter, the maximum wall velocity is set to $W_m^+ = 12$, as in Toubert and Leschziner (2012), whose direct numerical simulation results are used for comparison. There is little difference in the spanwise velocity, as the Stokes layer is a good approximation of the spanwise velocity for the oscillation periods used here.

6.1.2 The basis for the comparisons

Since the base flow is homogeneous in the spanwise direction, the optimisation described by (2.8) can be decomposed in two substeps. For the first step, the optimisation is performed for given spanwise wavenumber and initial time, and the second step is a standard optimisation over these two parameters:

$$A_0(t_f, t_i, \lambda_z) = \max_{\mathbf{u}'(t_i)} \frac{\|\mathbf{u}'(t = t_f)\|_f}{\|\mathbf{u}'(t = t_i)\|_i}, \quad (6.1a)$$

$$A(t_f) = \max_{t_i < t_f - \epsilon, \lambda_z} A_0(t_f, t_i, \lambda_z). \quad (6.1b)$$

Here, λ_z is the perturbation wavelength in the spanwise direction. The gain $A_0(t_f, t_i, \lambda_z)$ is calculated by adjoint optimisation, as explained in chapter 3 and chapter 4. The base profile considered in the present work is also homogeneous in the main flow direction, so that the calculations could also be performed separately for each longitudinal wavenumber. Due to this homogeneity in two directions, the optimal solution could be written in the form $\mathbf{u}' = \mathcal{R} \{ \mathbf{u}'_{\beta_x, \beta_z}(y, t) e^{i(\beta_x x + \beta_z z)} \}$. Since this is invariant with respect to translation in the direction $(1/\beta_x, 1/\beta_y, 0)$, the optimal structure consists of infinitely long oblique streaks.

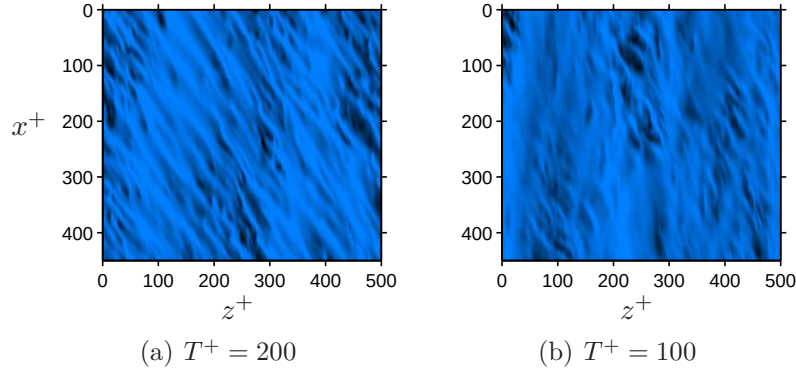


Figure 6.1: Visualisation of the streamwise velocity component at a distance $y_0^+ = 11$ from the wall for the friction Reynolds number $Re_\tau = 500$. Snapshot from the direct numerical simulation of Toubert and Leschziner (2012).

Examples of the flow patterns observed in direct numerical simulations are shown in Figure 6.1. The streaks are clearly visible in Figure 6.1(a), but in Figure 6.1(b) streaks are less clear and virtually absent from parts of the picture. In general, streaks are observed to be less pronounced when the drag reduction is stronger, like in the case of $T^+ = 100$ where the drag reduction is close to the maximum achievable. This can be attributed to streaks having smaller amplitude as compared to the footprint of the outer large scale structures when the drag is reduced. Streaks are observed more clearly during certain parts of the oscillation period and they seem to disappear over the other parts of the period. When streaks are visible their angle varies with time, although the variation is not large. The angle changes sign during the part of the oscillation period when the streaks are not visible. More details of the direct numerical simulation are given in Toubert and Leschziner (2012). The complicated behaviour of the streaky pattern makes comparisons difficult but at the same time allows comparing

more features than only the streak spacing, as it would be in the case of a non-oscillating wall.

The majority of the calculations of generalised optimal perturbation were performed for the Reynolds number $Re_\tau = 180$, for non-oscillating wall case and for oscillating wall cases with oscillation periods $T^+ = 100$ and $T^+ = 200$, at different distances to the wall and for different observation times in the period. Some calculations were also done for $Re_\tau = 500$ to test the dependence of the results on the Reynolds number. In the case $Re_\tau = 180$ the grid was a uniform mesh in the streamwise and wall normal directions composed of $(n_x, n_y) = (193, 180)$ points, for a channel of width 2 and length 10. In the case $Re_\tau = 500$, the length of the channel was equal to 5, and the grid contained $(n_x, n_y) = (513, 200)$ points. Where possible, the results were compared to streaks calculated in the direct numerical simulations of Toubert and Leschziner (2012). The direct numerical simulations were performed at a bulk Reynolds number $Re_b = 9000$, which corresponds to $Re_\tau = 500$ for the baseline flow. The oscillating wall cases with $T^+ = 100$ and $T^+ = 200$, based on the mean profile of the flow past a non-oscillating wall, were calculated for the same bulk Reynolds number. Throughout this chapter, Re_τ and the wall units of direct numerical simulation results are defined using the skin friction in the flow with non-oscillating wall. If the actual skin friction extracted from the oscillating wall direct numerical simulation results were used, the Reynolds number and actuation period would be $Re_{\tau_{ac}} = 420$ and $T_{ac}^+ = 68$ for the case $T^+ = 100$, and $Re_{\tau_{ac}} = 440$ and $T_{ac}^+ = 150$ for the case $T^+ = 200$.

Here, the generalised optimal perturbation results will be shown only in the case $Re_\tau = 180$. These results are compared to the direct numerical simulation performed at a different Reynolds number. However, this is not expected to be a big issue, because the streak characteristics are weakly dependent on the Reynolds number, as long as they are expressed in wall units. Examples of simulations at $Re_\tau = 500$ can be seen in appendix B.

The comparisons of the results in a given plane parallel to the wall amount to the comparison of streak spacing and streak angle of the generalised optimal perturbation as a function of the oscillation phase and the distance to the wall. Comparisons will also be made between the generalised optimal perturbation and the conditionally-averaged results of the direct numerical simulations. It is worth noting that due to the symmetry of the forcing over half a period, the behaviour of the linearised equation

is also symmetric over half a period. As a consequence, all results presented in this chapter will have the same property, which will be called ‘left-right’ symmetry. Streaks spacing will be a periodic function of half the forcing period, and if the generalised optimal perturbation angle $\theta(t_f, y_0)$ is calculated, then $\theta(t_f + T/2, y_0) = -\theta(t_f, y_0)$, where T is the oscillation period.

6.2 Generalised optimal perturbation approach and streaks

6.2.1 Selectivity of the linearized Navier-Stokes equations for the flow past a spanwise-oscillating wall

Prior to considering the optimal perturbation for determining the middle point of the passband, one needs to verify that the idea of a single passband is applicable to the case in question. Indeed, while this idea is valid for the flow past a non-oscillating wall, it might not be true in the case of the oscillating wall. This turns out to be so to a certain degree. Figure 6.2 shows the energy amplification factor $A_0(t_i, t_f, \lambda_z)$ for flows past oscillating and non-oscillating walls. The brightest point in each of these pictures represents the generalised optimal perturbation amplification factor $A(t_f)$ for the corresponding flow, and a comparison between the brightness of the generalised optimal perturbation point and other areas gives an indication of the selectivity of the linearized operator. The temporal window shown in figure 6.2 corresponds to one period for the oscillating cases, and with spanwise wavelengths up to $\lambda_z^+ = 400$. This is sufficient, as preliminary calculations for a few cases showed that no other maximum is observed if a bigger range of parameters (t_i, λ_z) is used.

In the non-oscillating wall case (Figure 6.2(a)) the figure would be the same for all t_f . In this case, the energy amplification factor has one well-defined maximum. This shows that, in this case, the generalised optimal perturbation has a growth time of about $t^+ = 50$ and a wavelength $\lambda_z^+ = 80$. The corresponding streaky structure is indeed known to dominate the real flow. The situation is quite different for the case of oscillating wall showed in Figure 6.2(b). In this case, there are two local maxima, with a relatively small difference in amplitude. This means that a superposition of the two

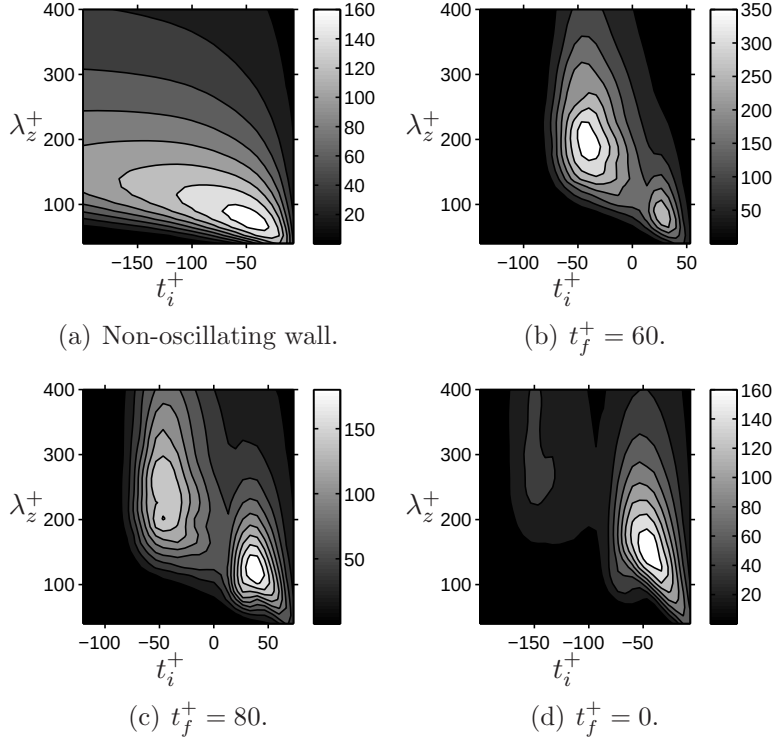


Figure 6.2: Maximum amplification factor $A_0(t_i, \lambda_z, t_f)$; $Re_\tau = 180$, $y_0^+ = 11.25$. The global maximum of each map is $A_0(t_f)$. In the oscillating wall case the period is $T^+ = 200$, and amplitude of the oscillation is $W_m^+ = 12$.

structures corresponding to each of these maxima will be present in the flow. The one for the global maximum $(\lambda_z^+, t_i^+) = (200, -40)$ corresponds to the generalised optimal perturbation (also considered as the most probable structure in the turbulent flow), and the one corresponding to the other local maximum is less pronounced, but given its magnitude it should also contribute to the observed flow structure. This configuration with two local maxima is observed only over a part of the oscillation period, while in the other part of the period one maximum is much larger than the other. The switch of the global maximum from one local maximum to another is illustrated by figures 6.2(b) and 6.2(c). Therefore, in the real turbulent flow, a clear dominant structure can be expected to be present during certain part of the oscillation period, while in other parts of the period two different structures are competing with each other and probably make the situation more difficult to analyse. A more quantitative description of these two dominant structures will be given later.

Another interesting observation can also be made; it is natural to try to relate the amplification factor of the generalised optimal perturbation to the drag. One would expect that the amplification factor is smaller when the wall oscillates and the drag is reduced. In fact, while for the case of non-oscillating wall the energy amplification factor is 160, for the case of oscillating wall at $T^+ = 200$, shown in figure 6.2(c) when the drag reduction is substantial, the energy amplification factor is 400. An explanation of this unexpected result rests on the observation that the width of the A_0 distribution in the t_i direction for the actuated case is much narrower than for the baseline flow. In the non-oscillating wall case, for a wide range of t_i , the most amplified perturbation has the amplification factor close to that of the generalised optimal perturbations, whereas in the case with wall oscillations this is restricted to a very limited range of t_i . This shows that taking into account only the peak of the filter made of the linearised Navier-Stokes equation is not sufficient, as its width is also an important factor.

An attempt can be made to take into account not only the height but also the width of the peaks, thus considering not only the ‘most probable’ streaks, but a collection of possible structures contributing to turbulent energy generation. This can be achieved by integrating the energy A_0 over all possible (λ_z, t_i) . This was done within the available range of (λ_z, t_i) , and was also combined with an averaging over the observation time t_f , as only the average drag reduction over a period is relevant. The integral was found to be equal to $6.4 \cdot 10^4$ in the non-oscillating wall case, $5.1 \cdot 10^4$ in the oscillating wall case at $T^+ = 200$, in which the DNS determined drag reduction is about 25%, and $2.5 \cdot 10^4$ for $T^+ = 100$ in which case the drag reduction is about 32%. This does suggest that the energy amplification factor might be related to drag. Indeed, Duque-Daza et al. (2012) found that there is a correlation between the energy amplification factor and drag reduction over the entire range of oscillation frequencies and longitudinal wavenumbers explored by Quadrio et al. (2009) using direct numerical simulation. However, Duque-Daza et al. (2012) considered the amplification factor for fixed initial conditions, selected to be close to the generalised optimal perturbation for non-oscillating wall case. Moarref and Jovanović (2012) obtained quantitative estimates of the drag reduction using linearised equations, but without the use of optimal perturbations. Taken together with our observations, this does suggest that the energy amplification described by the linearized Navier-Stokes equations is somehow related to drag, but the details of this relationship can not be clarified further using so few

forcing configurations. In the remainder of the chapter we will concentrate on the flow patterns, and the last chapter will be more focussed on drag reduction.

6.2.2 Streak spacing

Streak spacing is one natural way of comparing the generalised optimal perturbation predictions with the results of direct numerical simulations. Streak spacing comparisons were used by Chernyshenko and Baig (2005) in order to demonstrate the predictive ability of the generalised optimal perturbation approach. This was possible, however, only because in that work the comparisons were made in a large number (more than 35) of cases resulting in covering a wide (about an order of magnitude) range of streak spacing. Comparisons over such a range can be conclusive even if the nature of the theoretical approach is such that, as it is the case for generalised optimal perturbations, the quantitative error is relatively large. In the case considered here, the variation of streak spacing with various parameters is limited. At the same time, the uncertainties due to the problem of selecting the mean profile without actually performing a direct numerical simulation (see Section 6.1.1), the absence of a priori information on the distribution of wall normal stresses, which was used in (Chernyshenko and Baig, 2005), and the difficulty of determining the actual streak spacing from the results of direct numerical simulation make the streak spacing comparisons inconclusive. It is, nevertheless, instructive to consider these comparisons in more detail.

To determine the streak spacing from the results of direct numerical simulations a premultiplied spanwise energy spectrum was used. It is defined as $\Phi(\lambda_z) = 1/\lambda_z E(\lambda_z)$, at a given distance to the wall, where E is the standard energy spectrum depending on the spanwise wavelength λ_z . The value of λ_z at which Φ has a maximum is taken as the streak spacing observed in direct numerical simulations. In a case without wall oscillation and in the vicinity of the wall, Φ has a maximum at about $\lambda_z^+ = 100$, which corresponds to the generally accepted average streaks spacing. Other definitions are, of course, possible. In the case of oscillating wall the phase-averaged spectrum was used, so that the streak spacing is time-dependent. The streak spacing obtained from generalised optimal perturbation is also time-dependent, since the optimal perturbations corresponding to different t_f differ. This allows a comparison over a range of t_f^+ .

Two representative examples of pre-multiplied spectra obtained by direct numer-

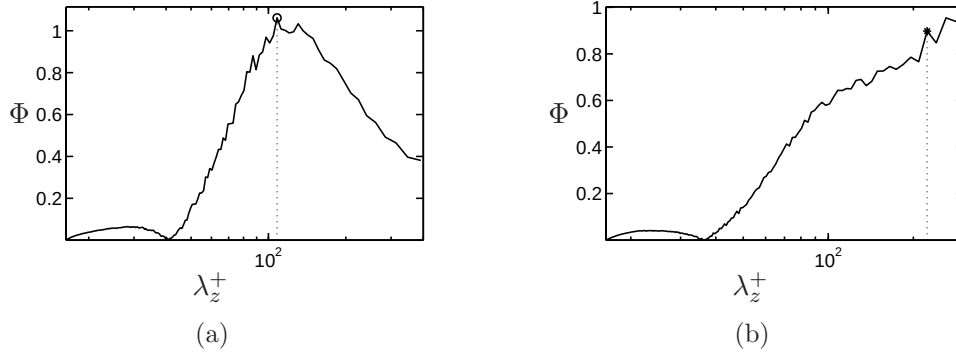


Figure 6.3: Scaled premultiplied direct numerical simulation spectrum $\Phi = 1/\lambda_z E(\lambda_z)$ of the streamwise velocity component in two characteristic cases: (a) $T^+ = 200$, $t_f^+ = 40$, $y_0^+ = 11.17$; (b) $T^+ = 100$, $t_f^+ = 10$, $y_0^+ = 6.52$.

ical simulations are given in Figure 6.3. Figure 6.3(a) represents the most common case, when there is a maximum at a wavelength corresponding to the near-wall streaks spacing. However, in a few cases, all for an oscillation period $T^+ = 100$ when the drag reduction is large, and during the part of the oscillation period when the near-wall streaks are barely detectable, the pre-multiplied spectrum is similar to Figure 6.3(b). We interpret these cases as the cases when near-wall streaks are too weak so that the spectrum is dominated by the large-scale structures. These points will be omitted from the comparisons, and their occurrence can be recognised by the gaps in the numerical data. The spectra in the main flow direction have similar behaviour, but the situation similar to Figure 6.3(b), that is without a clear maximum, is more common. Note that when both characteristic spanwise and streamwise lengths are found one can then calculate also the streak angle magnitude, but not the sign. The sign has to be determined visually. In the present comparison, we include the points corresponding to cases when the pre-multiplied streamwise spectrum has no maximum at small wavelength because, unlike the case of spanwise spectrum, it would be difficult to define an exact criteria for omitting these points. This also enables to perform more comparisons. The points when the angle sign could be determined are shown with open circles, while the points when it could not be determined are shown with filled symbols. In short, open symbols are for cases with clear streaks, while filled symbols are for cases when spanwise characteristic scale can be obtained from direct numerical simulation but the streaks are not very apparent. Due to the left-right symmetry of the wall motion, the streak

spacing should be a periodic function of time, with the period equal to one half that of the wall oscillations. However, since the maximum in the direct numerical simulation spectra is often rather indistinct, the numerical data do not show this property, and the difference between the streak spacing obtained at values of time separated by half an oscillation period may indicate the magnitude of the numerical error. For this reason we plot each data point twice, the second time with a half-period shift. The generalised optimal perturbation results satisfy the symmetry property exactly, of course.

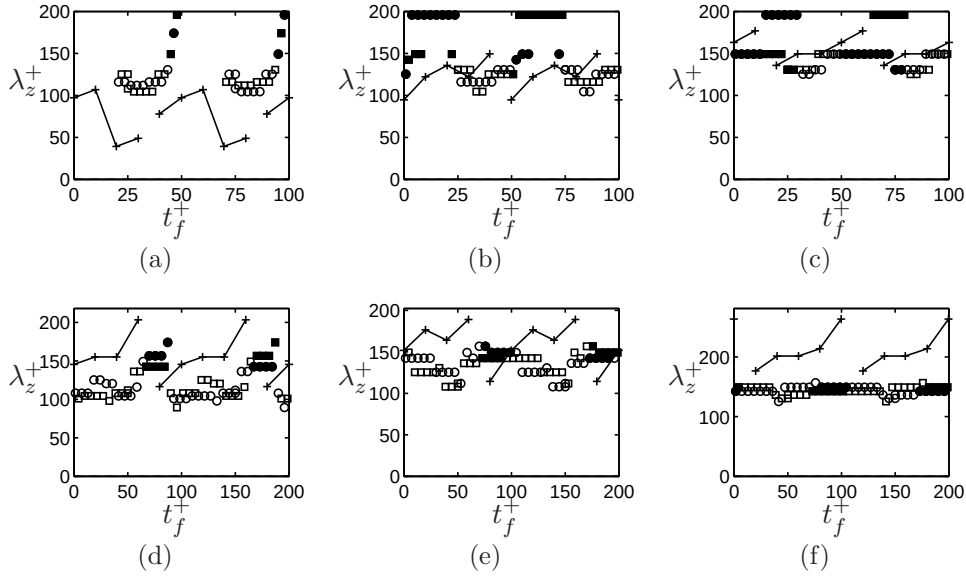


Figure 6.4: Streaks spacing obtained from direct numerical simulations (symbols) and generalised optimal perturbation (lines): (a) $T^+ = 100, y_0^+ \approx 7$; (b) $T^+ = 100, y_0^+ \approx 11$; (c) $T^+ = 100, y_0^+ \approx 16$; (d) $T^+ = 200, y_0^+ \approx 7$; (e) $T^+ = 200, y_0^+ \approx 11$; (f) $T^+ = 200, y_0^+ \approx 16$. Solid symbols are for cases when streaks angle can not be determined from the direct numerical simulation, open symbols are for cases when the angle can be determined. Square symbols are the streaks spacing extracted from direct numerical simulation phase averaged premultiplied spectrum over one period. Circle symbols are the same as the square symbols, shifted by half a period to take into account the mean flow symmetry.

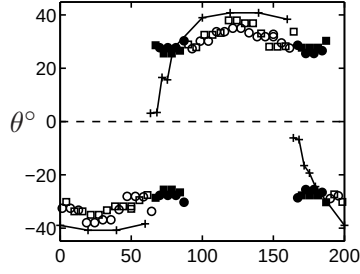
Figure 6.4 shows the comparisons of streak spacing obtained from direct numerical simulation for $T^+ = 100$, that is in the regime with largest drag reduction, when streaks are weak and difficult to detect, and for $T^+ = 200$. Note that the spectrum features noticeable oscillations. Removing these would require phase-averaging over many more periods than it was possible. These oscillations led to streak spacing often having the

same value for several consecutive phases. This is a purely numerical artefact. Several observations can be made. First, the discrepancy between the streak spacing of the generalised optimal perturbations and the direct numerical simulation results is in some instances as large as 50%, which is about the same order of magnitude as the variation of the streak spacing with time and distance to the wall. Given the approximate nature of the filtering effect, which neglects the specific parameters of the forcing, such a discrepancy is not unexpected. This uncertainty, combined with the minor variation in the numerically computed streaks spacing does not allow conclusions to be drawn. This is different from the situation in Chernyshenko and Baig (2005), where the discrepancy was mostly within 30%, while the variation of streak spacing over all cases considered was more than 1000%. Several trends are shared by the generalised optimal perturbation and the numerical results. First, as the distance to the wall increases, the streak spacing increases. Second, the streak spacing experiences a jump in generalised optimal perturbation results within that part of the period when streaks are more difficult to identify in the direct numerical simulation results. Third, there are similarities in the predicted and observed behaviour of streak spacing as a function of time. These observations neither contradict nor strongly support the theoretical approach: the basis for comparisons is simply too narrow, and the existence of two peaks of the amplification factor further complicates the considerations. Fortunately, the comparisons for the streak angle are more informative.

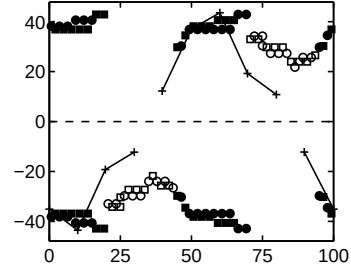
6.2.3 Streak angles

Calculations of the angle were performed for several distances from the wall and for several values of the oscillation phase. For each case, the pre-multiplied energy spectrum in the main flow direction and the pre-multiplied energy spectrum in the spanwise direction were calculated. Then, the maxima of each were found, thus giving the longitudinal λ_x and spanwise λ_z wavelengths. The values obtained were treated as reliable only in cases where a maximum in the pre-multiplied spectrum was present for both $\lambda_z^+ < 300$ and $\lambda_x^+ < 600$. This eliminates the contamination by the footprints of the large-scale structures when the streaks are not energetic enough.

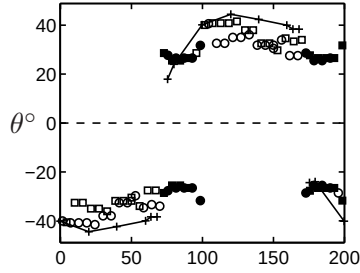
In all cases, where both the values λ_x^+ and λ_z^+ were judged reliable, the streaks could be easily recognised in related visualisations. For these points, the absolute



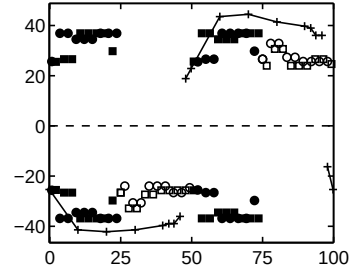
(a) $T^+ = 200$, $y_0^+ = 7.5$, $y_{DNS}^+ = 6.5$



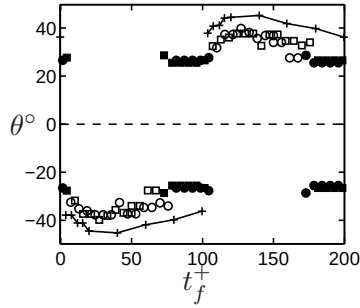
(b) $T^+ = 100$, $y_0^+ = 7.5$, $y_{DNS}^+ = 6.5$



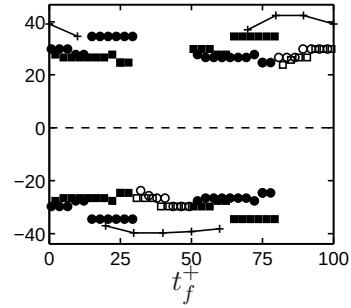
(c) $T^+ = 200$, $y_0^+ = 11.2$, $y_{DNS}^+ = 11.2$



(d) $T^+ = 100$, $y_0^+ = 11.2$, $y_{DNS}^+ = 11.2$



(e) $T^+ = 200$, $y_0^+ = 17$, $y_{DNS}^+ = 16$



(f) $T^+ = 100$, $y_0^+ = 17$, $y_{DNS}^+ = 16$

Figure 6.5: Streak angle θ as a function of the oscillation phase: direct numerical simulation (symbols) at $Re_\tau = 500$, generalised optimal perturbation (lines) at $Re_\tau = 180$. Square symbols are the angle extracted from the direct numerical simulation phase averaged premultiplied spectrum over one period. Circle symbols are the same as the square symbols, shifted by half a period and with the opposite sign. The symbols are filled when the angle measured is not reliable.

value of the angle could be obtained directly from the position (λ_x, λ_z) of the extrema, and its sign was determined from the flow visualisation. Cases where streaks are not easy to see or absent from visualisations all correspond to cases where at least one of the pre-multiplied spectrum plots has no clear local maxima. Only cases where $\lambda_z^+ > 300$ were completely ignored. When $\lambda_x^+ > 600$, the streaks are usually difficult to observe. It is then also difficult to visually identify the sign of the angle. However, the absolute value of an angle can still be extracted from the spectrum. Such points are included in the comparisons twice, with plus and minus sign, and are shown with filled symbols. The generalised optimal perturbation angles are much easier to extract, as the structures are infinitely long in one direction and well defined if the convergence of the optimisation algorithm is good enough. It is however worth noting that they become less reliable when their absolute value is small. This is due to the fact that a grid is used in the streamwise direction, which does not allow for very small angles to be obtained (the streamwise wavelength is then longer than the numerical domain). Due to the left-right symmetry of the mean flow, the absolute values of the angle should be a periodic function of time with the period equal to one half of that of the wall oscillations. This is exactly true for the generalised optimal perturbation, but not for the direct numerical simulation, as in this case the phase-averaging could be done only over a finite time interval. Hence, similarly to the streaks spacing plot, each direct numerical simulation point is plotted twice with a half-period and a sign shift. The discrepancy between the points gives an estimate of the magnitude of the averaging error.

Angle comparisons were undertaken for a large number of cases (different oscillation period and distance to the wall); some of the results are shown in Figure 6.5. Because the streaks are less discernible in the case $T^+ = 100$, there are fewer computational points for this case than for $T^+ = 200$. Note the discontinuity of the predicted angle as a function of the phase t_f^+ . The jump occurs when one local maximum of $A_0(t_i, \lambda_z, t_f)$ becomes higher than the other, so that the global maximum switches (compare Figure 6.2(b)–(c) with Figure 6.5(c)). It is also worth noting that between the jumps the angle variation remains limited. This feature will be discussed in more detail in Section 6.3. To better determine the jump position, more generalised optimal perturbations have been calculated next to the jump position, in cases where it could bring further information. Overall, the comparisons of the angle are favourable and give

support to the generalised optimal perturbation approach.

6.2.4 Conditional average comparison

The filtering idea justifies the comparison of the filter outputs for the optimal perturbations and the real flow, while the inputs can differ arbitrarily. For the particular form of the optimal perturbation considered here, this means that streaks can be compared while vortices cannot be compared¹. However, if the turbulent flow field is conditionally sampled on the presence of the streaks, it might be that the entire optimal perturbation structure will be similar to the conditionally-averaged structure. This can be the case if the conditional averaging is triggered by the same quantity that was maximised by the optimal perturbation, and if certain assumptions about the right hand side $\mathbf{F}$ of equation (2.1) are made. To illustrate this idea, consider the following simple model.

Let $\hat{\mathbf{u}} = L\hat{\mathbf{f}}$, where $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ and $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2)$ are two-dimensional vectors and $L = \{L_{ij}\}$ is a 2×2 matrix, so that $\hat{u}_i = L_{ij}\hat{f}_j$. Let the generalised optimal perturbation be defined as such $\hat{\mathbf{f}}$ of a unit length that $\hat{u}_1$ is maximised. Then, $\hat{\mathbf{f}}_{opt} = (L_{11}, L_{12})/\sqrt{L_{11}^2 + L_{12}^2}$. Now, let $\hat{\mathbf{f}}$ be a random vector represented by a statistical ensemble of its realisations. Let us assume that the probability density function of $\hat{\mathbf{f}}$ is constant inside the circle $|\hat{\mathbf{f}}| = f_{max}$ and is zero outside. This means that all directions of $\hat{\mathbf{f}}$ are equally probable, while the magnitude of $\hat{\mathbf{f}}$ is bounded by f_{max} . Accordingly, in all realisations $\hat{u}_1 < \hat{u}_{1,max} = \sqrt{L_{11}^2 + L_{12}^2}f_{max}$. Next, let this statistical ensemble be conditionally averaged on $\hat{u}_1 > (1 - \epsilon)\hat{u}_{1,max}$, where $\epsilon \ll 1$. This means that all the realisations for which $\hat{u}_1 < (1 - \epsilon)\hat{u}_{1,max}$ are discarded, and the remaining realisations are averaged to give the conditionally-averaged $\hat{\mathbf{f}}$. Since the condition can be satisfied only for those $\hat{\mathbf{f}}$ that are almost parallel to $\hat{\mathbf{f}}_{opt}$, the conditionally-averaged $\hat{\mathbf{f}}$ will also be almost parallel to $\hat{\mathbf{f}}_{opt}$. That is in this case the conditional average and the optimal perturbation, multiplied by a suitable constant, almost coincide.

This model shows that if a randomly forced linear operator is considered, the optimal perturbation can be similar to the conditionally averaged picture. In a real turbulent flow however, the non-linear terms represented by $\mathbf{F}$ in equation (2.1) have a much more complex behaviour. Generalising the simple toy model described here

¹If the optimal perturbation would be obtained by maximising a measure of longitudinal vorticity instead of velocity then, as it was suggested in Chernyshenko and Baig (2005), comparing vortices would be justified

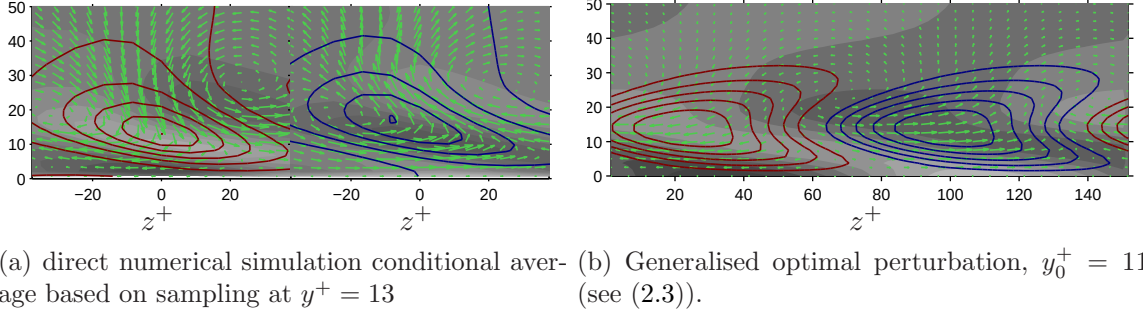


Figure 6.6: Comparison of conditionally averaged flow field and generalised optimal perturbation for $T^+ = 200$ at phase $t_f^+ = 0$. Contour lines are streamwise velocity iso-lines, arrows represent the crossflow velocity, and the background shade (color online) the streamwise vorticity. For comparison, the flow fields have been normalised so that the maximum streamwise velocity is equal to 1 in each case.

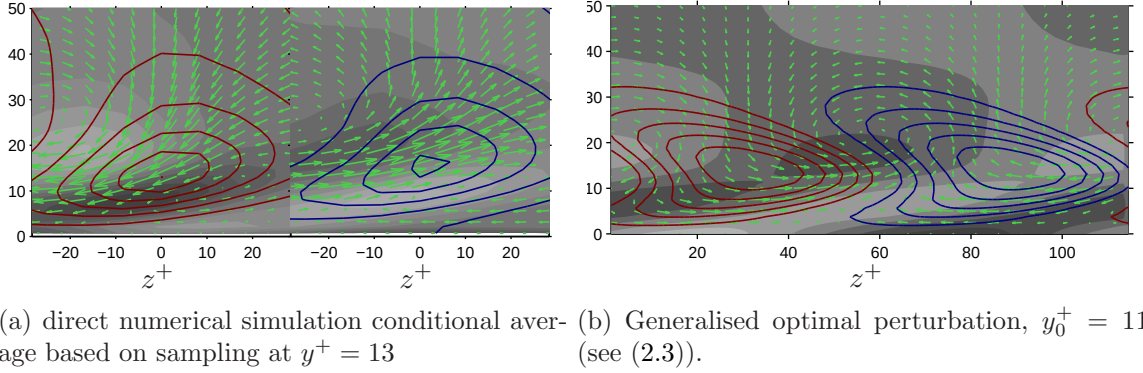


Figure 6.7: Comparison of conditionally averaged flow field and generalised optimal perturbation for $T^+ = 200$ at phase $t_f^+ = 80$.

to turbulent flow would be too difficult and no attempt to do so is made here. The above argument gives motivation for a comparison of the generalised optimal perturbation with conditional average of the turbulent flow field, but is not a proof that such comparison is indeed valid.

Conditional averaging was triggered by extremums of the longitudinal velocity. The sampling threshold was high enough for the conditional averaging structures to be meaningful, and low enough to allow for large samples and good convergence (more details can be found in Touber and Leschziner, 2012). Figure 6.6 shows the comparison representative of the part of the period when streaks are more pronounced (open symbols in Figure 6.5(c)). For $y^+ < 20$, there is a good qualitative agreement between the

shape of the streaky structures, the crossflow velocity field and vorticity. For $y^+ > 20$, the figures differ significantly. This could be the result of a partial visualisation of the outer layer structures, due to the correlation between the streaks and these structures in the direct numerical simulation, which is not accounted for in the generalised optimal perturbation analysis.

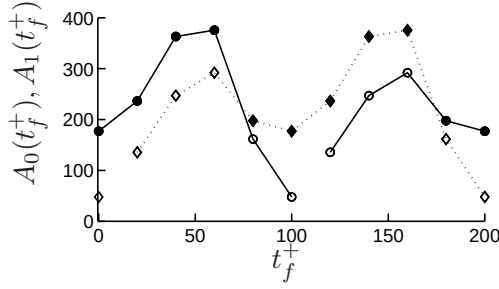
For phases close to the phase at which the angle jump occurs, the agreement is poor (see Figure 6.7). This is expected and has a similar origin as the angle discontinuity. Close to the angle jump position, there are two possible dominant structures, corresponding to the two peaks in Figures 6.2(b)–(c). Both structures should be present in the direct numerical simulation, whereas only the most energetic will be obtained with the generalised optimal perturbation approach. As a result, the conditional average picture will be a combination of the two dominant structures, and the generalised optimal perturbation figure will contain only one of them.

6.3 Generalised optimal perturbation formation mechanisms for the flow past an oscillating wall

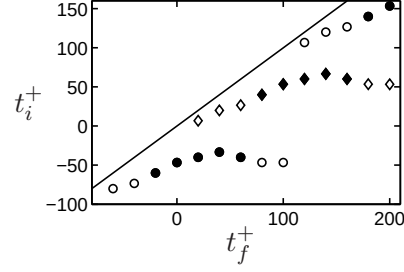
We will now consider the mechanism leading to the existence of the optimal perturbations. An optimal perturbation is a three-dimensional time-dependent solenoidal vector field satisfying the linearized Navier-Stokes equations. It is natural to describe this field as a fluid flow. However, to avoid misunderstanding, we reiterate (see chapter 2) that this velocity field should not be considered as a model of a turbulent flow or an approximation to it, even though certain features (only those corresponding to the output maximised by the generalised optimal perturbation) can indeed be expected to be similar. The flows considered in this section are only a tool used to understand the properties of the linearized Navier-Stokes operator via its generalised optimal perturbation solutions for the case of a flow past a spanwise-oscillating wall.

6.3.1 Universal optimal solution

Generalised optimal perturbations are determined by the filtering properties of the linearized Navier-Stokes operator. These properties depend on the distance between the wall and the observation plane y_0^+ , and on the observation phase t_f^+ where and when



(a) Amplification factor at the local peaks.



(b) Dependence of the initial time on the finite time for the local peaks.

Figure 6.8: Evolution of the local peaks of $A_0(t_f, t_i, \lambda_z)$ (see Figure 6.2) as a function of the observation phase t_f^+ . $Re_\tau = 180$, $T^+ = 200$ and $y_0^+ = 11.25$. The global optimum $A_0(t_f^+)$ is shown with a filled symbol, and the secondary maximum, denoted $A_1(t_f^+)$, is shown with an open symbol. Points corresponding to the same local peak (peaks grouped by similar values of the initial time t_i) are connected with lines.

the energy amplification coefficient is maximised. Here we will set the distance to the wall and analyse the dependence of the filter properties on the observation phase. We first study the dependence of the generalised optimal perturbation on the phase t_f^+ , before analysing the temporal evolution of each generalised optimal perturbation to understand their formation mechanisms.

Figure 6.8 shows the dependence of selected filter properties on the phase. It illustrates the characteristics of the perturbations corresponding to the two local maxima of the energy amplification coefficient map (see again Figure 6.2(b)–(d), the global maximum corresponding to the generalised optimal perturbation). One can clearly see how the global maximum switches between the two local maxima. The curves corresponding to the different local maxima are similar, and indeed differ only by a phase shift equal to half a period. This suggests that the curves represent the same object, appearing repeatedly with a mirror (left/right) symmetry and with a half period shift. As seen before, this is natural since the base flow itself has such symmetry.

Figure 6.8(b) shows that in the time intervals when a local peak is also a global maximum (filled symbol), the corresponding t_i , that is the time when the perturbation should be introduced, remains relatively constant. This suggests that the filter properties are similar for a range of observation phases t_f^+ . To further investigate this idea, an analysis of the time evolution of the generalised optimal perturbations observed during half a period is done. Figure 6.9 shows the energy amplification factor

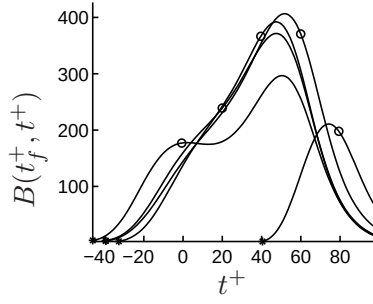


Figure 6.9: Temporal evolution of the energy $B(t_f, t) = \|\mathbf{u}'(t)\|_f^2 / \|\mathbf{u}'(t_i)\|_i^2$ of five generalised optimal perturbations corresponding to observation phases t_f^+ regularly spaced over half a period. For each generalised optimal perturbation, the point $(t_f, B(t_f, t_f)) = (t_f, A(t_f))$ is marked with $(\circ)$ and the point $(t_i, B(t_f, t_i))$ is marked with $(*)$. $Re_\tau = 180$, $T^+ = 200$, and $y_0^+ = 11.25$.

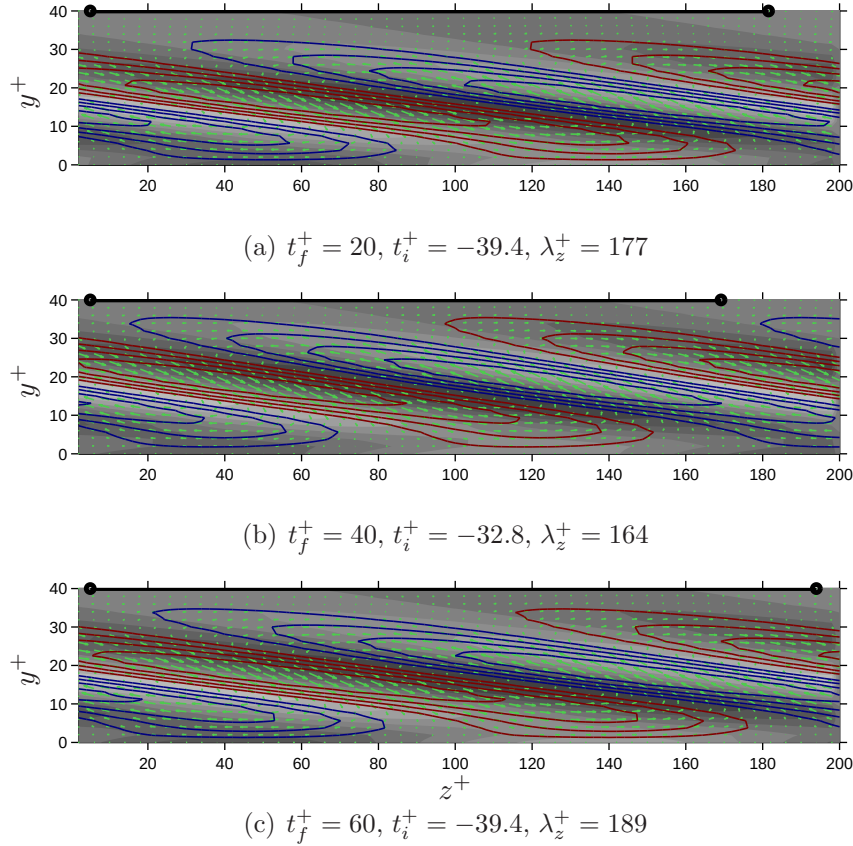


Figure 6.10: Initial conditions corresponding to the three generalised optimal perturbations that are the closest to the “universal optimal” in figure 6.9. Contour lines are streamwise velocity isolines, arrows represent the crossflow velocity, and the background shade the streamwise vorticity. Note that the spanwise wavelength is not exactly the same for these three cases, and is represented by the thick black line on top of each graph.

$B(t_f, t) = \|\mathbf{u}'(t)\|_f^2 / \|\mathbf{u}'(t_i)\|_i^2$ of these generalised optimal perturbations as a function of time, each starting at its initial time t_i^+ , continuing to its observation time t_f^+ for which this perturbation is the optimal (so that $A(t_f) = B(t_f, t_f)$), and then continuing to higher t and eventually decaying. It turns out that certain perturbations originating in a relatively narrow area (the initial time t_i^+ corresponding to these perturbations varies by only $\Delta t_i^+ \approx T^+/20$) of the oscillation half period experience a very fast growth. As a result, when an optimisation of the energy amplification at different values of t_f^+ is done, the optimal initial time to introduce this perturbation happens to be somewhere within this interval, and the optimal perturbation itself (and this includes its angle) remains approximately the same, as seen in figure 6.10 for the initial conditions. This explains why the variation of the streak angle remains limited between the jumps.

Hence, the overall picture can be described now in a simple way. All the generalised optimal perturbations corresponding to half a period between two angle jumps are roughly identical. They therefore all follow the same physical mechanism and can be represented by one universal optimal perturbation. The universal optimal perturbation dominates half of the oscillation period and then gives way to another universal optimal perturbation, shifted in phase by half-period and with the angle of the opposite sign.

6.3.2 The main energy content of the universal generalised optimal perturbation

The universal optimal perturbation structure is infinitely long in one direction, and this direction does not vary in time. If this pattern is studied in a frame of reference where $\tilde{x}$ is aligned with the streak direction, $\tilde{z}$ is orthogonal to it and parallel to the wall, with the wall normal direction y being the third coordinate, the linearized Navier-Stokes equations become simpler. In this new frame of reference all variables are independent of $\tilde{x}$, and therefore their derivatives with respect to $\tilde{x}$ vanish. The momentum equations

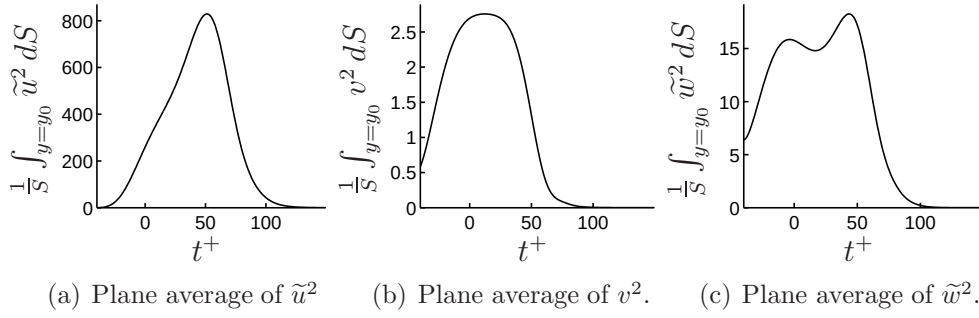


Figure 6.11: Time evolution of $\tilde{u}^2$, v^2 , and $\tilde{w}^2$ averaged over the plane $y = y_0$ for one optimal perturbation. The norm of the initial condition $\|\mathbf{u}'\|_i = 1$. $T^+ = 200$, $Re_\tau = 180$, $y_0^+ = 11.25$, and $t_f^+ = 60$.

then take the form:

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{W} \frac{\partial \tilde{u}}{\partial \tilde{z}} + v \frac{\partial \tilde{U}}{\partial y} = \frac{1}{Re} \Delta \tilde{u}, \quad (6.2a)$$

$$\frac{\partial v}{\partial t} + \tilde{W} \frac{\partial v}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{Re} \Delta v, \quad (6.2b)$$

$$\frac{\partial \tilde{w}}{\partial t} + \tilde{W} \frac{\partial \tilde{w}}{\partial \tilde{z}} + v \frac{\partial \tilde{W}}{\partial y} = -\frac{\partial \tilde{p}}{\partial \tilde{z}} + \frac{1}{Re} \Delta \tilde{w}, \quad (6.2c)$$

where $\tilde{u}$ and $\tilde{w}$ are the components of the perturbation velocity in the new frame of reference, and both mean velocities ($\tilde{U}, \tilde{W}$) are now time-dependent, representing a combination of the turbulent mean profile and the Stokes layer. The continuity equation has the form $\partial v / \partial y + \partial \tilde{w} / \partial \tilde{z} = 0$. Note that the equations (6.2b-c) representing the cross-flow evolution are independent of equation (6.2a).

The universal optimal perturbation will be exemplified here by the generalised optimal perturbation optimal for $t_f^+ = 60$, $y_0^+ = 11.25$, and for a wall oscillation period $T^+ = 200$. Figure 6.11 shows the time evolution of the contribution of each velocity component to the perturbation energy of this perturbation. The surface energy norm of the $\tilde{x}$ -component is greater than the same measure if the usual streamwise direction x is used (compare with Figure 6.9), and much greater than the magnitude of the other two components, that is the wall-normal and cross-pattern components. This result suggests that including or omitting other velocity components in the final norm is relatively unimportant. For example, if the final norm was defined as $u^2 + v^2 + w^2$

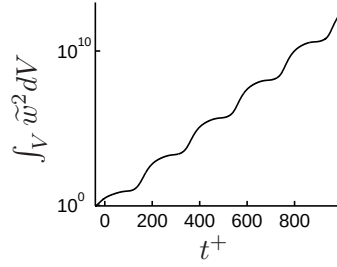


Figure 6.12: Floquet instability with $\tilde{W}\partial/\partial z$ removed from (6.2).

instead of just u^2 , the generalised optimal perturbation would be almost the same, but with the energy curve similar to Figure 6.11(a) instead of Figure 6.9.

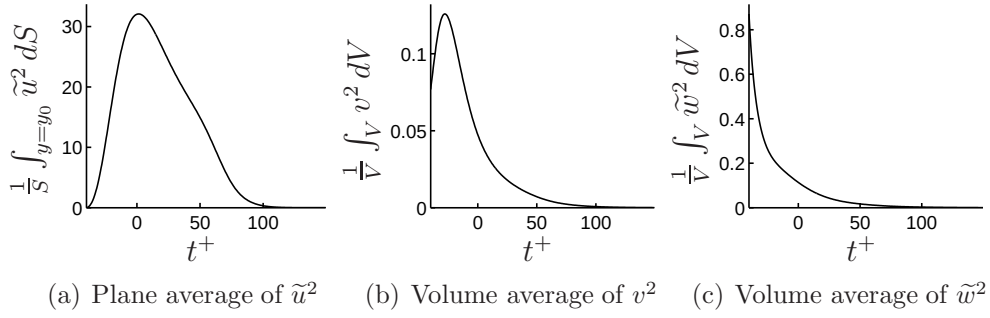
The dominance of $\tilde{u}$ also suggests that the equation (6.2a) should be analysed first to understand the mechanism of energy amplification. Multiplying equation (6.2a) by $\tilde{u}$ and integrating over the flow domain gives the energy equation in the form

$$\frac{d}{dt} \int_V \frac{\tilde{u}^2}{2} dV + \int_V \tilde{u} v \partial \tilde{U} / \partial y dV = -1/Re \int_V (\nabla \tilde{u})^2 dV.$$

The only term contributing to a potential increase in energy stems from the term $v \partial \tilde{U} / \partial y$ in equation (6.2a). This term describes the well-known lift-up effect, which amounts to advection of the mean velocity by the wall-normal component of the velocity perturbation. In the case of an oblique pattern above an oscillating wall, the shear is time-dependent, and one can expect that the growth will be favoured during that part of the oscillation period when $\partial \tilde{U} / \partial y$ is large. Another difference from the non-oscillating wall case is the presence of the similar term $v \partial \tilde{W} / \partial y$ in (6.2c). Because of this term there can be also a transient growth of the cross-flow velocity components, and this is indeed observed in Figures 6.11b,c.

6.3.3 Analysis of the governing equation after removing some terms

In order to estimate the relative importance of various physical effects, calculations were performed with some of the terms removed from the governing equations. First, we removed the terms involving $\partial \tilde{W} / \partial \tilde{z}$ in (6.2). It turned out that the linearized operator becomes unstable without these terms, and the solution diverges along a


 Figure 6.13: Effect of removing $v\partial\tilde{W}/\partial y$ from equation (6.2c).

Floquet orbit, see Figure 6.12. This shows that these terms have a crucial stabilising effect, and that analysing the Navier-Stokes equations by removing some of its terms is not always straightforward.

Removing $v\partial\tilde{W}/\partial y$ from equation (6.2c) reduces significantly the growth of $\tilde{u}$, see Figure 6.13. This term is responsible for energy transfer from the mean flow to the cross-flow component of the perturbation. Without it, the cross-flow volume-averaged energy decays monotonously with time. As has been shown in Subsection 6.3.2, the cross flow contributes to the final norm only indirectly, via the lift-up mechanism. This indirect effect is more properly measured by the volume-averaged energy of the cross-flow than by its plane-average, and for this reason the figure shows the plane average for $\tilde{u}^2$, but volume averages for v^2 and $\tilde{w}^2$. The system is now very similar to the non-oscillating case, and the scenario for the transient energy growth is well understood. Note again that at the start more than 95% of the flow energy is concentrated in the cross-flow components (since the same initial condition is considered). With time, the cross-flow energy decreases monotonously due to the action of viscosity. At the same time, the wall normal component triggers the transient growth of the streamwise component via the term $v\partial\tilde{U}/\partial y$, and this process stops once the velocity v becomes too small. Notably, the growth is much smaller than in the case of the full equation, which suggests that the transient growth of the cross flow is important.

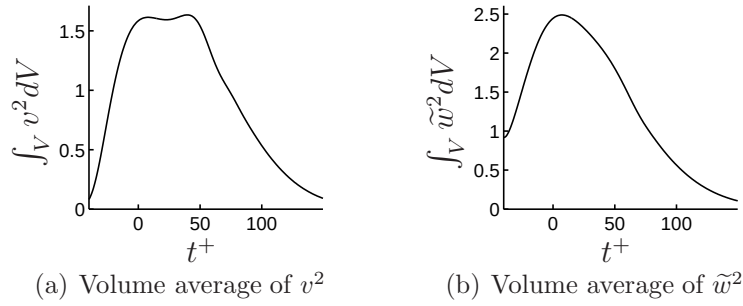


Figure 6.14: Cross-flow volume-average of the squares of both components in a frame of reference linked to the streaks for the universal generalised optimal perturbation.

6.3.4 The cross-flow transient growth is caused by the Orr mechanism

The time evolution of the volume-average of v^2 and $\tilde{w}^2$ for the universal optimal perturbation (no terms removed) is shown in Figures 6.14(a) and 6.14(b). In the frame of reference aligned with the streaks, the equations for v and $\tilde{w}$ are independent of the equation for $\tilde{u}$, so that the flow is essentially two-dimensional. This suggests that the transient growth might be caused by the well-known Orr mechanism as described below. This is further supported by the observation that, at the start, the volume average of the wall-normal velocity squared is small compared to the volume average of $\tilde{w}^2$. Figure 6.15 leaves little doubt: one can clearly see that at the start of the motion the vorticity is organized in sheets inclined against the mean shear (Figure 6.15(a)). The shear then rotates the lines of the constant phase of the vorticity, and as the distance between the vorticity sheets increases they induce larger velocity. Further rotation increases the inclination again, and the velocity induced by the vorticity distribution decreases again. Viscous diffusion, the vicinity of the wall, and the time-dependence and non-uniformity of shear all contribute, making the picture more complicated, but the main features of the Orr mechanism are clearly there.

Overall, one can conclude that the transient growth of the universal generalised optimal perturbation is a combination of the transient growth due to the Orr mechanism in the plane perpendicular to the streaks and the mechanism of lift-up of the longitudinal mean velocity by this transient cross-flow. An excellent description of this mechanism was given by Farrell and Ioannou (1993), who considered a uniform time-

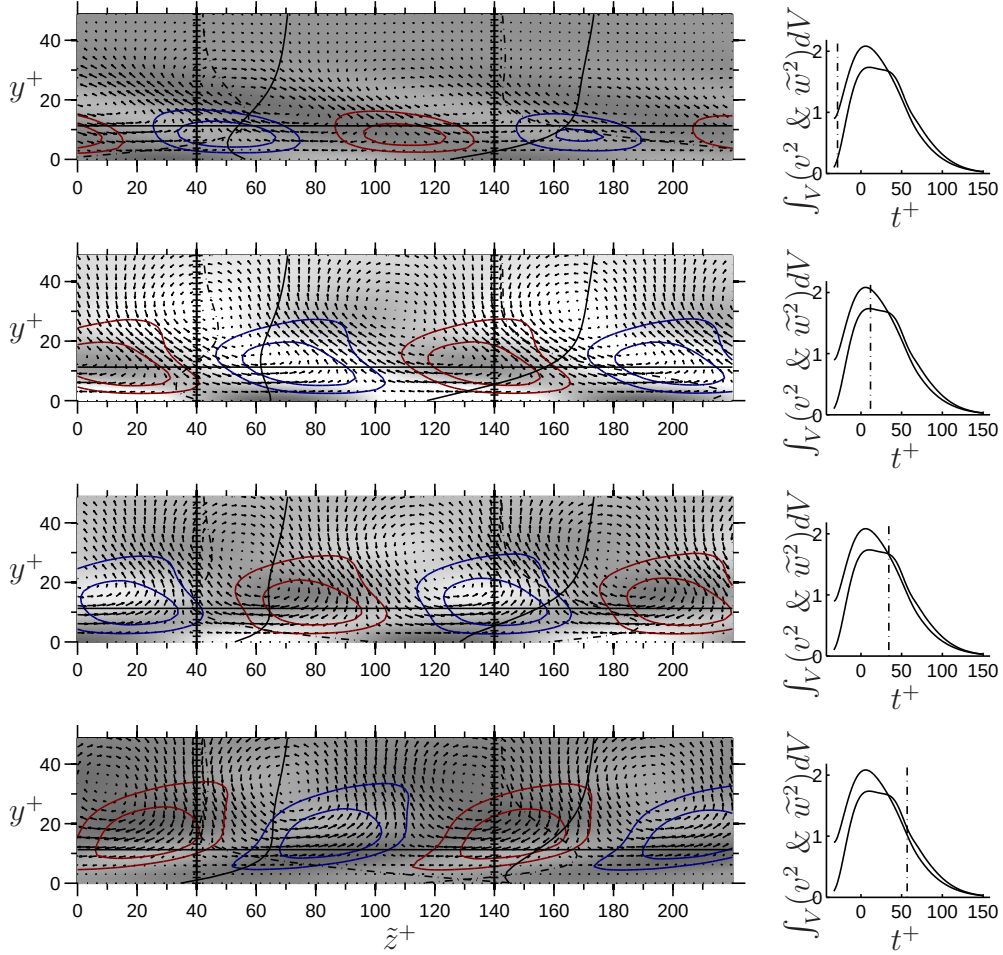


Figure 6.15: Evolution of the universal generalised optimal perturbation in the frame of reference linked to the streak. The horizontal straight line marks the layer $y = y_0$. Vectors show the cross-flow velocity. Grey shades show $\tilde{x}$ -component of vorticity. Closed solid contours are the contours of constant $\tilde{u}$. The base flow components $\tilde{U}$ and $\tilde{W}$ are shown with solid curves, and their derivatives by the dashed curves. In the plots on the right the phase is marked with a vertical dashed line, and the curves are the cross-flow energy components.

independent shear, which allowed them to obtain analytical solutions. In that case the optimal perturbation was aligned with shear and had a pure lift-up mechanism. In order to reveal the interplay between the Orr and lift-up mechanisms, Farrell and Ioannou fixed the inclination angle, thus considering a sub-optimal perturbation. In the case considered in the present work the direction of shear varies with time, so that the optimal perturbation is necessarily oblique and the interplay is always present. Another difference is that we are considering the generalised optimal perturbation.

6.4 Conclusion on the use of the generalised optimal perturbation approach

The generalised optimal perturbation approach was used to study the streaky patterns in turbulent flow subject to a harmonic spanwise wall oscillations in the regime when drag reduction occurs. The generalised optimal perturbation results were obtained without using any quantitative data from direct numerical simulations about the actuated (that is with the wall oscillating) flow, thus emphasising the qualitative predictive ability of the approach at the expense of its quantitative accuracy. The characteristics of the streaks predicted by the generalised optimal perturbation approach were compared to the characteristics of the streaks observed in direct numerical simulations of the equivalent turbulent flow. Some attempts to compare the energy of the generalised optimal perturbation to the level of drag reduction were also performed.

To predict streaks and understand their physical mechanisms, the generalised optimal perturbation approach has been a useful tool. Due to the limited accuracy of using linearised equations in turbulent flow, some measures such as the streaks spacing were performed on a too restricted sample to provide convincing conclusions. However, some other aspects such as the streaks angle were well predicted. The fact that the angle is discontinuous was well explained by the linearised framework, and the physical explanations of this phenomena have been given. Mainly, one structure is dominant during half a period, leaving its symmetric (with opposite angle) being dominant during the next half period.

It was showed that if the generalised optimal perturbation is a good tool to predict turbulent flow streaks, it is however not suitable to predict drag. In fact, the energy of

the generalised optimal perturbation when drag reduction occurs is often larger than the energy of the generalised optimal perturbation for unforced case. This was explained by the fact that the generalised optimal perturbation describes only the most probable structure, and is not a good representation of all the streaks-like structures present in turbulent flow. A measure made available by the generalised optimal perturbation calculation and taking into account more structures was defined, and provided better drag estimations. However, too few comparison points with direct numerical simulation were available to obtain significant conclusions. In the next chapter, another approach to estimating the drag will be used, at the same time simpler and less computationally expensive.

Chapter 7

Linearised Navier-Stokes equations subject to travelling waves wall oscillations

The generalised optimal perturbation approach predicts the most probable streaks observed in DNS and was used successfully to predict and understand some of the turbulent flow streaks characteristics. It was especially useful to discover the presence of an angle jump twice in the oscillation period, and understand its origin. However, when trying to use the generalised optimal perturbation approach to predict drag reduction, the results were deceiving. The generalised optimal perturbation itself is more energetic in the forced case with drag reduction than in the unforced case. Based on this finding, it was concluded that the generalised optimal perturbation as a description of the most probable streaks cannot be used as a representation of all the structures contributing to the skin friction. Seeing the linearised Navier-Stokes as a filter acting on the nonlinear right-hand side, the generalised optimal perturbation describes the peak of the filter, but not the width of its passband.

The generalised optimal perturbation results have also been used in a different way to try to explain the drag reduction curve behaviour. To take into account not only the most probable streaks, but as many structures as possible, the map of most amplified perturbations as a function of the parameters (t_i, β_z) was used. The contribution of all the optimal perturbations on a map was integrated and the number obtained treated

as a “total turbulent energy” and linked to drag. The number obtained compared favourably to the measured drag in DNS. But if the variation in the drag reduction was well predicted, the rate of variation was not well explained and there were far too few points of comparison to draw positive conclusions.

Integrating the (t_i, β_z) map used to find the generalised optimal perturbation can also be limited. It is better to assess the properties of the linearised Navier-Stokes operator, as it takes into account the dependence of this linear filter in (t_i, β_z) . However, for each parameter (t_i, β_z) , only one structure is taken into account and not all the possible structures of same parameter. Similarly to the generalised optimal perturbation case, only the most amplified perturbation at each (t_i, β_z) is taken into account, which will describe the peak of the filter for the given parameters. The filter peak might be a relevant number if its selectivity properties are not dependent of (t_i, β_z) , but it was not possible to prove this assumption to be correct or not. In this chapter a different approach will be used, aimed at studying directly the passband of the filter without using an optimal perturbation approach.

The main idea is similar to the idea which led to an integration in (t_i, β_z) of the optimum perturbations for set values of (t_i, β_z) . To study drag reduction, an assumption can be made that the most amplified perturbation is not a relevant measure of the turbulent energy generated by the linear operator, and that the combination of all the possible structures going through the operator has to be taken into account. Then assuming like in the previous chapters that the turbulent energy is linked to the drag, a comparison with literature on drag reduction can be performed.

The generalised optimal perturbation approach was based on the use of a delta-correlated forcing in time. The linear equation to solve was of the form:

$$\frac{\partial \mathbf{u}'}{\partial t} + L\mathbf{u}' = \delta(t - t_i)F(x, y, z), \quad (7.1)$$

where both t_i and $F(x, y, z)$ were found by the optimisation procedure. To lift all the restrictions linked to the optimisation and predict all the structure possible, an intuitive approach is to replace equation (7.1) with:

$$\frac{\partial \mathbf{u}'}{\partial t} + L\mathbf{u}' = F(x, y, z, t), \quad (7.2)$$

where $F(x, y, z, t)$ is a random generated forcing function, Gaussian and delta-correlated in time and space. The problem being linear, it is still similar to the problem of equation (7.1), except that the shape of the structure is not imposed. Choosing a white noise forcing is the simplest way to have a representation of all the structures possible without doing any physical assumption. A similar approach was very successfully developed in Moarref and Jovanović (2012), where the turbulent viscosity was also used. Here, the turbulent viscosity is not used in any form.

The results obtained in this chapter will be compared to DNS results of Quadrio et al. (2009), where wall oscillations were travelling waves of the form:

$$w_{wall} = W_m \cos(k_x x - \omega t), \quad (7.3)$$

were considered. With travelling waves, not only the oscillation frequency ω is can be varied, but also the streamwise wavelength k_x , which give significantly more comparison points. A summary of the results of Quadrio et al. (2009) is given in figure 7.1. Note

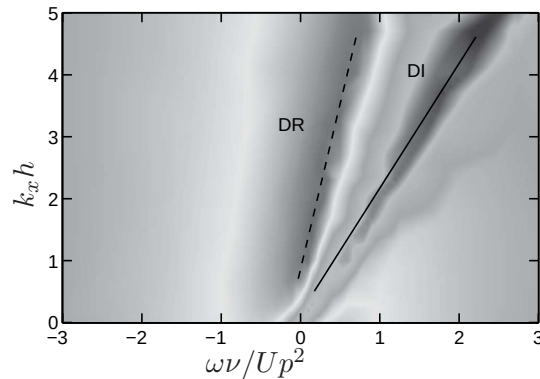


Figure 7.1: Drag reduction map from Quadrio (2011). The dashed line (DR) is the maximum drag reduction line, and the continuous line (DI) is the maximum drag increase line.

the very specific characteristics of this figure which can be used for comparison. As the linearised approach to the turbulent problem is approximate by nature, we will be more interested in reproducing these features, such as the maximum drag reduction line or the maximum drag increase line, than the correct values of drag reduction themselves.

In this chapter, the methods to implement the white noise forcing will first be discussed. The type of forcing used will be tested with a specially implemented one-

dimensional Matlab program, with both temporal forcing and semi analytic calculation of converged statistics. This will be a tool to define the forcing used with the Fortran program described in chapter 4, on more complex two dimensional case and with travelling wave wall oscillations. The results will be first compared with the generalised optimal perturbation results for harmonic wall oscillations to underline the common points and differences between the two techniques. Finally, the results will be compared for the whole range of parameters (k_x, ω) available from Quadrio et al. (2009). The predictive ability, as well as the limits of the approach will be discussed.

7.1 Random forcing on a turbulent channel flow

In the forcing described in equation (7.2), the main issue arises when the calculations are preformed not analytically, but on a numeric grid. If a simple delta-correlated function in space is used on a grid, it is not differentiable using a finite difference scheme, and therefore the results obtained after applying the forcing to the linearised Navier-Stokes solver might be non-physical, only reflecting the effects of a numerical differentiation on a non-smooth function. As the equation used is linear, no problem arises regarding the temporal differentiation; a delta-correlated function in time is equivalent to a sum of uncorrelated initial value problems, and therefore does not suffer from differentiability issue. The attention will therefore be focused on the spatial forcing, which will have to be chosen to be close enough to a delta-correlated function to assure the quality of the results, but also smooth enough to ensure the numerical accuracy (differentiability of the forcing) of the calculations.

This section focuses on the appropriate choice of spatial forcing. As the Fortran program used to solve the problem with travelling waves forcing has a relatively high computational cost, the calculation used to validate the forcing will first be performed on a simple one dimensional program solving the problem of forcing on a simulation dependent only on the wall-normal direction. The advantage of using a code with only one spatial dimension is that it is very fast, and the use of highly efficient languages such as Fortran is no longer mandatory. A Matlab code was written to solve this problem, and has been implemented with different spatial schemes. Using several schemes allowed to validate when the forcing used is smooth enough to be considered differentiable (results become independent of the differentiation scheme used). Another

advantage of the use of Matlab is that the solver could be written on a simplified way as a matrix multiplication. This enabled to calculate converged infinite time statistics at a very small cost. These results are also precious to test the convergence of simulation when the discretisation is done also in time.

Here, the idea of the Matlab code will be described, as well as the method to obtain infinite time statistics. They will then be used with different kind of “smooth” delta-correlated forcing, and the best forcing method will be selected. It will then be used as a base to validate the Fortran program in these configuration for calculations on a two dimensional grid.

7.1.1 Infinite time statistics on a 1D channel

Infinite time statistics have already been used in many cases with the linearised Navier-Stokes equation (Farrell and Ioannou, 1993; Bamieh and Dahleh, 2001; Jovanović and Bamieh, 2001; Moarref and Jovanović, 2012), the main ideas are summarised here. Assume the linearised operator subject to random forcing follows the equation:

$$\frac{\partial \mathbf{u}'_i}{\partial t} = \mathcal{L}_{ij} u_j + \mathcal{F}_{ij} \xi_j, \quad (7.4)$$

where $\mathcal{L}_{ij}$ the linearised Navier-Stokes operator in space (not necessarily discretised at this point), $\mathcal{F}_{ij}$ a forcing correlation function, $\mathbf{u}' = (\mathbf{u}'_i)_{1 \leq i \leq 3}$ the velocity field, and $\xi = (\xi_i)_{1 \leq i \leq 3}$ the Gaussian forcing. In this chapter, the forcing ξ is delta-correlated in space and time, and eventual spatial correlation will be taken into account for by the correlation operator $\mathcal{F}$. This equation can be easily integrated, and the temporal solution is then given by:

$$\mathbf{u}'(t) = e^{\mathcal{L}t} \mathbf{u}'_0 + \int_0^t e^{\mathcal{L}(t-s)} \mathcal{F} \xi(s) ds. \quad (7.5)$$

If the operator $\mathcal{L}$ is stable, the first term $e^{\mathcal{L}t} \mathbf{u}'_0$ vanishes for large time. The equation (7.5) can be simplified by either making this assumption, or assuming the initial condition is zero. Doing so, the second order statistics matrix $\mathcal{V} = \lim_{t \rightarrow +\infty} \langle \mathbf{u}'(t) \mathbf{u}'^*(t) \rangle$

can be calculated:

$$\begin{aligned}
\mathcal{V}^t &= \langle \mathbf{u}'(t) \mathbf{u}'^*(t) \rangle \\
&= \int_0^t \int_0^t e^{\mathcal{L}(t-s)} \mathcal{F} \langle \xi(s) \xi^*(s') \rangle \mathcal{F}^* e^{\mathcal{L}^*(t-s')} ds ds' \\
&= \int_0^t e^{\mathcal{L}(t-s)} \mathcal{F} \mathcal{F}^* e^{\mathcal{L}^*(t-s)} ds.
\end{aligned} \tag{7.6}$$

Here, $\langle . \rangle$ denotes an ensemble average over all the possible functions ξ . Due to ξ being delta-correlated in space and time, the term $\langle \xi(s) \xi^*(s') \rangle$ vanishes unless $s = s'$, where it is the identity tensor. Performing the change of variable $\tau = t - s$, the evolution equation for $\mathcal{V}^t$ becomes:

$$\frac{\partial \mathcal{V}^t}{\partial t} = e^{\mathcal{L}t} \mathcal{F} \mathcal{F}^* e^{\mathcal{L}^*t} \tag{7.7}$$

Assuming that the operator $\mathcal{L}$ is asymptotically stable, this shows that $\lim_{t \rightarrow +\infty} \frac{\partial \mathcal{V}^t}{\partial t} = 0$. This, coupled with the fact that equation (7.7) is integrable ensure that the second order statistic matrix has a limit, $\lim_{t \rightarrow +\infty} \mathcal{V}^t = \mathcal{V}$. The temporal evolution of the second order statistic matrix can also be obtained from equation (7.6):

$$\frac{\partial \mathcal{V}^t}{\partial t} = \mathcal{L} \mathcal{V}^t + \mathcal{V}^t \mathcal{L}^* + \mathcal{F} \mathcal{F}^*. \tag{7.8}$$

Taking the limit of this equation at large time, a Lyapunov equation is obtained:

$$\mathcal{L} \mathcal{V} + \mathcal{V} \mathcal{L}^* = -\mathcal{F} \mathcal{F}^*. \tag{7.9}$$

If the problem is one dimensional, calculating a solution to this equation is much more efficient than aggregating statistics of a linearised solver in time until convergence is achieved. In the remaining part of this section, the forcing used has no spatial correlations. The right hand side in equation (7.9) therefore has the simpler expression $\mathcal{F} \mathcal{F}^* = I$.

The first tests of forced linear solver were performed on a simple one dimensional case with a program implemented in Matlab. The linear solver was designed to be used with a single component of the mean flow, in the streamwise direction x , and depending only on the wall normal coordinate y . The perturbation velocity was restricted to x -independent flow, and decomposed in wavenumber in the spanwise direction z . When

time marching is used, the temporal derivatives are calculated using a simple first order scheme. Four different approaches to the spatial discretisation of the operator were implemented. For calculating derivatives, one method was based on a Cartesian grid and second order finite difference schemes, and the other one based on a pseudospectral method on a Chebyshev grid. The discretisation schemes implemented were either a velocity-pressure formulation of the linearised Navier-Stokes equation, or a velocity-vorticity formulation. The different versions of the code were validated by reproducing the transient growth results of Butler and Farrell (1992) and Chernyshenko and Baig (2005).

The interesting feature of implementing the solver in Matlab is that all the spatial operator discretisations could be written in a Matrix form. This allows for a very efficient time marching if the time is also discretised, but mainly allows to calculate the solution of the Lyapunov equation (7.9). Solving the Lyapunov equation provides the converged statistics at a relatively low cost, and this was therefore the first use made of the program. A first test was to calculate some of the second order statistics for the two spatial discretisation methods used. For this test, the right hand side of equation (7.9) is the identity matrix, which corresponds to a delta-correlated noise in space. The results obtained are shown in figure (7.2). This figure shows that when a real delta-correlated noise is used, using different numerical schemes to calculate derivatives can result in completely different second order statistics. Both the scheme based on finite difference and the one based on Chebyshev polynomials are used on grid fine enough to perform usual simulations such as calculations of optimal perturbation. However, due to the fact that the forcing used is an uncorrelated white noise on the numeric grid, the differentiation error behave differently with the different schemes and the results end up being completely different from each other.

The issue of using an uncorrelated white noise forcing on a discretised physical space could seem obvious, but the example showed in figure (7.2) shows to what extent the predictions can be dependent on the numerical method used. Because of that, it is not possible to use a real white noise forcing in simulations, and the forcing has to be chosen to be a smooth and differentiable function, even though as close as possible to a white noise. The general strategy to achieve such a goal is to use a white noise forcing on the coarser grid than the numerical grid used, and to use some interpolation method to obtain a smooth value on the grid used to compute the linearised Navier-Stokes solver.

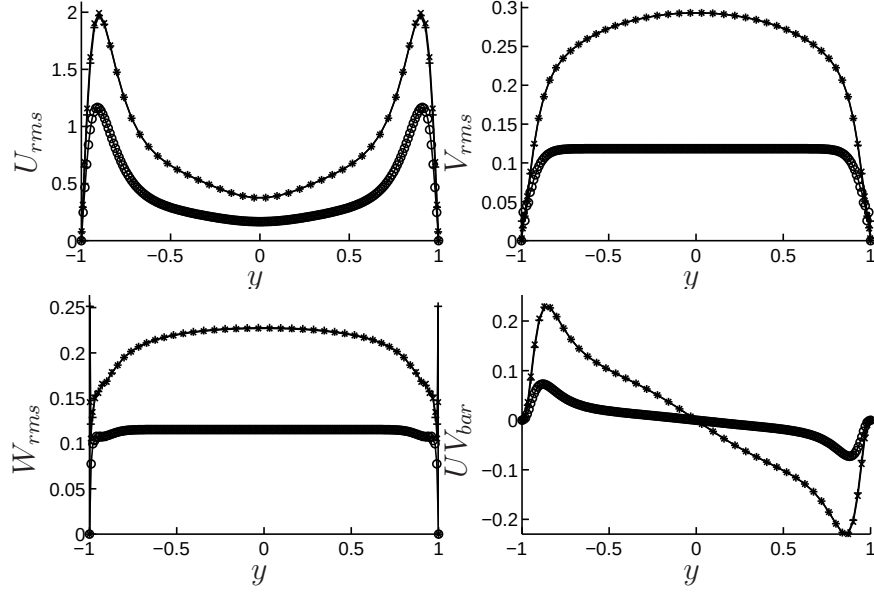


Figure 7.2: Infinite time statistics around a turbulent channel flow mean profile at $Re_\tau = 180$, with three different methods. (-o): finite difference velocity pressure formulation, $n_y=200$, (-x): Chebyshev velocity vorticity formulation, $n_y = 50$, (-+): Chebyshev velocity pressure formulation, $n_y = 50$.

From a practical point of view, using a different grid for the forcing and for the linearised Navier-Stokes solver can be done by creating a spatial correlation matrix $\mathcal{F}$ in equation (7.9). This is possible as long as the relation between the forcing point and the interpolated values is linear. Two such forcing have been tested; one based on a spline interpolation, and one based on a Fourier decomposition. For the spline interpolation, a second order spline interpolation was implemented in a matrix form. The grid used for the forcing in this case is also a Cartesian grid, but with a coarser mesh than the linearised solver. For the second method with a Fourier decomposition, the forcing is given by the formula:

$$F(y) = \sum_{n=1}^N r_c \cos(nk_y y) + r_s \sin(nk_y y),$$

with $k_y = \frac{2\pi}{l_y}$, $l_y = 2$ is the distance between the channel walls, and N is the number of forcing modes. This method is linear and can also be implemented as a matrix multiplication. The only change in calculating the solution of the Lyapunov equation

is that now the right hand side $\mathcal{FF}^*$ in equation (7.9) is no longer identity.

The first comparison using spline forcing is showed on figure 7.3. This simulation as well as all the simulations performed in this forcing validation section are performed around the same turbulent channel flow mean profile at $Re_\tau = 180$. In figure 7.3,

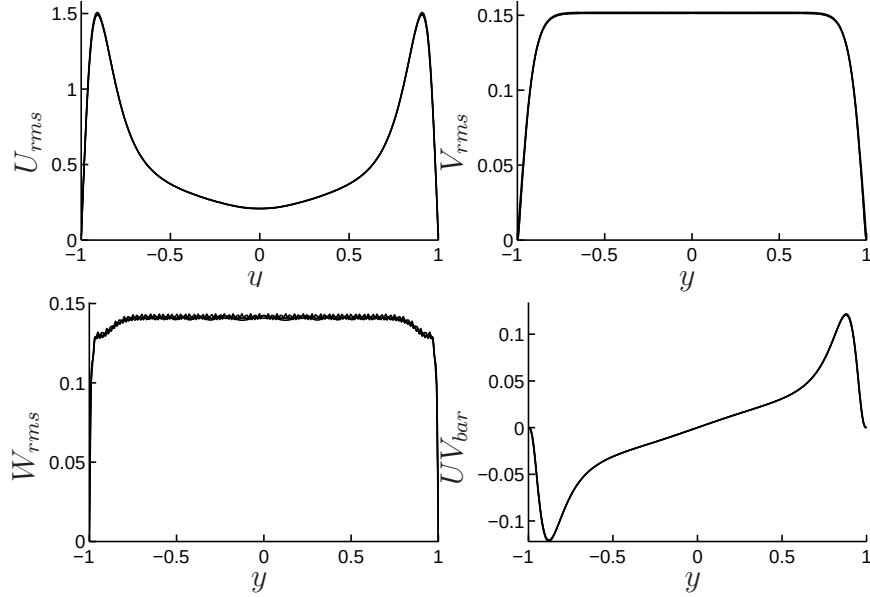


Figure 7.3: Infinite time statistics, spatial forcing with spline interpolation. $dy_{spline}^+ = 3$. The three available numeric methods are showed here, with $n_y = 200$.

contrary to the case of figure (7.2), the two discretisations methods lead to very similar second order statistics. This shows that although a random forcing is used, it has been smoothed enough to be differentiable with both the finite difference and the pseudospectral method. The grid used for the linearised solver being the same as in figure (7.2), it also confirm that the grid used there was fine enough and that the difference in the results on that figure was due to the discretisations error of the random forcing, and not to under-resolved domain. Having this first confirmation that using interpolated forcing improves the results, it remains to choose the best forcing, as well as the best forcing grid and physical grid. Doing so should enable to lower the cost of calculations.

The first choice to make is between the two forcing schemes proposed. To decide which of the spline interpolation and the Fourier forcing is better, both methods have been used with the same linearised Navier-Stokes solver, with the number of forcing

points being relatively small. The results are shown on figure 7.4 and seem to favour the Fourier forcing as a better method. The main difference is on the W_{rms} graph,

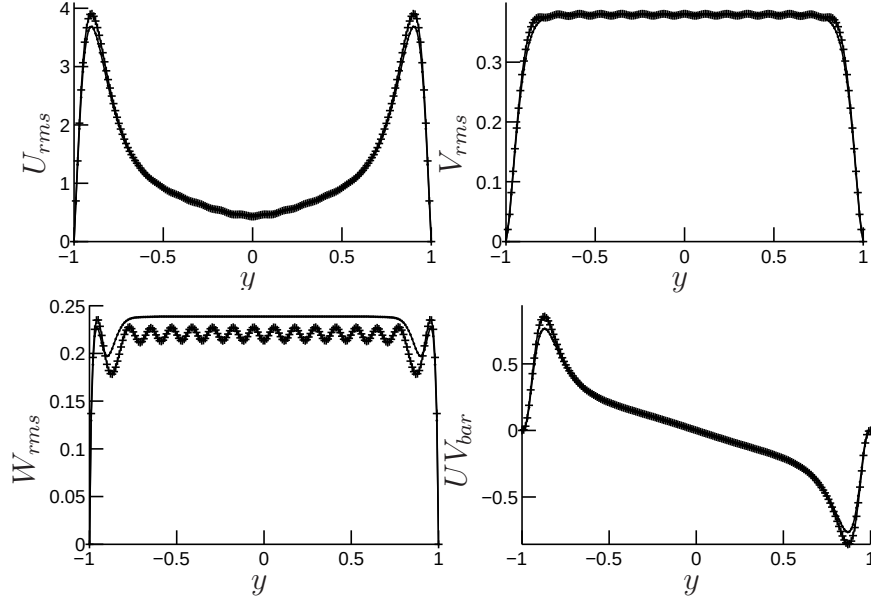


Figure 7.4: Infinite time statistics with the Matlab code. (+) spline interpolation, (.) Fourier forcing. forcing points spacing $dy_{forcing}^+ = 20$. Code with $n_y = 600$ points in the wall normal direction. The more regular results when Fourier forcing is used is an argument in favour of using Fourier forcing rather than spline interpolation.

but on all four graphs showed here, the forcing using the spline interpolation shows some wavy pattern clearly linked to the initial forcing grid. Even though using only 18 forcing points can seem like a small number, the behaviour would be similar if more points were used. This led to select the Fourier forcing as the only forcing function for the remaining part of the thesis.

Once the forcing function has been chosen, the number of forcing modes has to be set. This number should be as small as possible, as it directly influences the number of mesh points needed to solve the linearised Navier-Stokes equation. The more forcing modes are needed, the more grid points will be necessary for the spatial discretisation to be smooth enough and its derivatives well approximated. On figure 7.5, a very fine grid is used for the spatial discretisation, and various numbers of forcing modes are used. Having a very fine grid ensures that the error observed is caused by a lack of forcing points, but not by discretisation approximations due to forcing function varying too

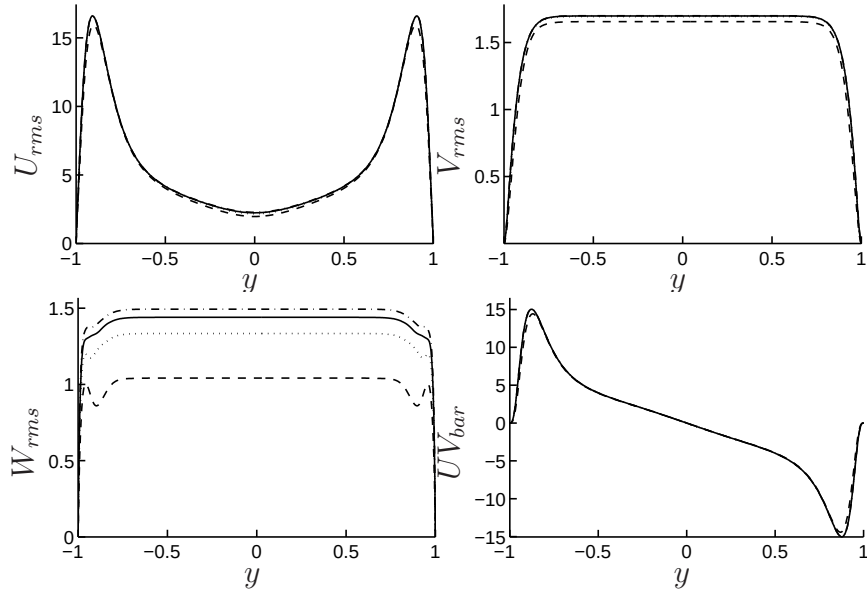


Figure 7.5: Influence of the number of forcing modes for a one dimensional forcing. Results with infinite time statistics (Matlab) and temporal forcing (Fortran). (- -) $nF_y = 18$, (:) $nF_y = 36$, (-) $nF_y = 54$, (- .) $nF_y = 72$

quickly for the mesh used. The purpose is to choose the number of forcing modes large enough to have a good accuracy, but small enough to be computationally efficient. In most cases, the results obtained for the second order statistics are very similar, independently of the number of forcing modes used, with the exception of the spanwise velocity energy W_{rms} . On the graph corresponding to this component, the results seem to converge properly only when a very large number of forcing modes is used ($N > 54$). This suggests using such a big number of modes for future simulations, but eventually only a smaller number will be kept. As in the simulations performed later the main component measured is the streamwise energy U_{rms} , the forcing will be selected such that the graph of U_{rms} only is converged. Doing so, the smallest number of modes showed in this graph $N = 18$ is already sufficient. This will allow to run simulations on relatively coarse grids for the linearised solver, and therefore make important gains in numerical efficiency.

7.1.2 Temporal forcing and convergence rate

Using the Infinite time statistic approach and solving a Lyapunov equation to find the converged second order statistics of the linearised solver was precious for understanding some key features of the forcing, and selecting the most efficient one for a one dimensional case. However, in the case of forcing by travelling wave wall oscillations, the grid used to discretise the flow will be two dimensional, in the streamwise and wall normal direction. For such grid, calculating the solution of the Lyapunov equation would be much more demanding than in the previous subsection, and it was decided to use time marching with the original Fortran program until the flow statistics are properly converged.

As seen previously, the problem of the forcing in time is much simpler than the one of forcing in space. As the problem is linear, forcing the equation at each time step is equivalent to solving as many initial value problems, the initial value being imposed by the forcing at the given time. Because of that, using differentiation in time will not raise any issue with the nature of the forcing, and a simple delta-correlated white noise in time can be used. The only particularity of the forcing used in this thesis is that it is scaled by the time step. If dt is the time step, then the forcing has the form $F(t) = F_0(t)/\sqrt{dt}$. The purpose of this rescaling factor is to take into account that a random walk in one dimension spreads as $\sqrt{n}$, if n is the number of iterations. With this rescaling factor, the results produced become independent of the time step, and in the one dimensional case can also be compared to the Lyapunov equation solution.

The first interesting observation is to study the speed of convergence of the flow statistics. It can initially be done on a simple one dimensional case like in the previous subsection. Such a result is showed on figure 7.6. Showing the converged statistics with the Lyapunov equation solution as well as the statistics obtained from the aggregation of time marching results allows for an accurate estimation of the convergence. Like in previous observations, the convergence seems to be faster for the streamwise component of the flow than for the others. However, the time needed to get an acceptable level of convergence is very long. Even if it can be expected that in the case of travelling wave wall oscillations the convergence will be faster (the structures predicted by the generalised optimal perturbation approach have a shorter life span when the wall is oscillating), the simulations can be expected to require a significant amount of

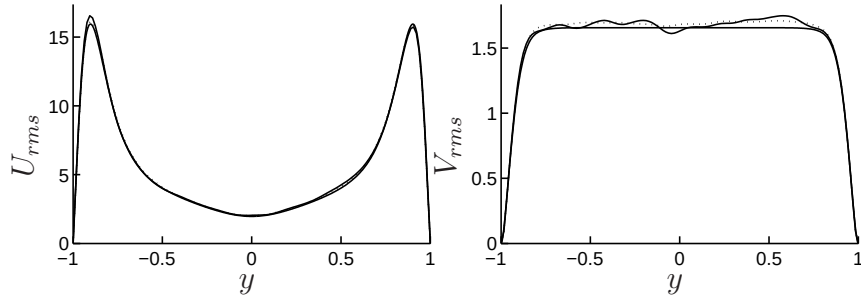


Figure 7.6: Selected statistics to show the convergence of flow statistics. The more regular curve is the semi-analytical converged statistics calculated with Matlab, while the less regular curve is obtained by integrating the energy of a simulation forced by a delta-correlated function of time.

computational time.

Now that the forcing is validated in the case of a one dimensional simulation, the number of forcing modes also has to be chosen in the case of a two dimensional spatial grid. The forcing function is chosen to be also composed of Fourier modes in the two directions, with the number of modes in the wall-normal direction to be $N_y = 18$, like the one chosen in the one dimensional simulations. The results of converged statistics are showed in figure 7.7. For all the second order statistics showed, the convergence with the number of forcing modes in the streamwise direction is very fast, and using $N_x = 20$ forcing modes is already enough to have a result independent of the number of forcing modes. For the simulations with travelling waves wall oscillations, the number of forcing modes used is then set to be $(N_x, N_y) = (24, 18)$.

The number of forcing modes chosen in this thesis is a good compromise between the accuracy of the results and the cost of the simulations. The tests to choose the forcing parameters have all been done in the case of a steady mean flow, and it would not be surprising to have a somehow different situation when the mean flow represent the travelling wave oscillations. This might be a limit to the simulation used in this thesis, but the dependence on the forcing modes could not be performed for all the mean flow used at a reasonable cost. However, the fast convergence with the number of mode increasing in the case of a steady mean flow suggests that the results is weakly dependent on the forcing property, especially in the streamwise direction where the energy will be measured, and it will therefore be assumed later that the forcing used is sufficient.

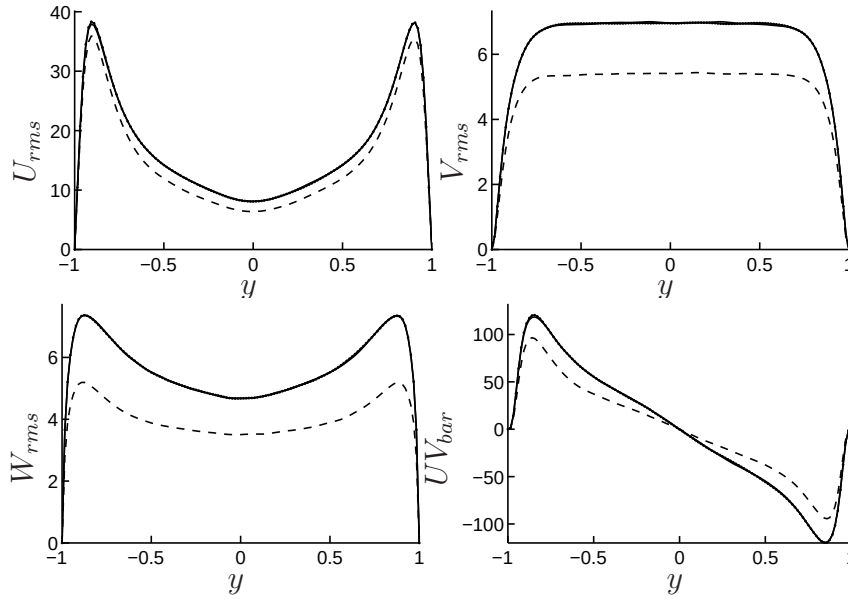


Figure 7.7: Influence of the number of forcing modes in the streamwise direction. Simulations with $(nx, ny) = (160, 129)$ points, final time $t_f = 1000$, forcing with $ny_{forc} = 18$. (—) for $nx_{forc} = 10$, (---) for $nx_{forc} = 20$, (-.-) for $nx_{forc} = 100$

7.2 Travelling wave wall oscillation with random right hand side forcing

In this chapter, the treatment of the linearised Navier-Stokes equation is very different to the one used in the generalised optimal perturbation approach, but when analysing data, some of the assumption made are similar to what was done with the generalised optimal perturbation. The main quantity of interest will be the streamwise velocity energy on a surface at a given distance to the wall. This is an obvious quantity to use as far as streaks characteristics are concerned, as the same measure is used to define the streaks in DNS. However, assumptions are made when comparing this value to the drag reduction measured in DNS. The assumption, similar to the assumption made when drag reduction was compared between the generalised optimal perturbation and DNS, is that the streaks energy is linked to the turbulent energy generation and therefore to the drag reduction. The main question remaining using this approach is to know at which distance to the wall the energy measured is relevant to predict drag reduction. Unfortunately, the approach used will be shown to be not accurate enough to answer

to this question.

Three measures of the streamwise energy will be used in the analysis showed later. The first one is an average in space and in time, and this mean energy is compared to the drag reduction. To understand the temporal evolution of the fluctuation velocity, an average is also performed only in space, and its temporal evolution will provide insight into the structure life span. Finally, to look at the dependence of the response on the forcing, a conditional average is also performed. The forcing having the form $W_{wall} = \cos(k_x x - \omega t)$, the conditional average u_ϕ^2 , with $\phi \in [0, 2\pi]$, is performed by accumulating for each instant t and each position in the streamwise direction x the velocity at points where $k_x x - \omega t \equiv \phi \ [2\pi]$.

In the following comparisons, the white noise forcing results will first be analysed for the simple case of harmonic wall oscillation. This allows for a first understanding of the mechanisms implied, and the results can be compared with the generalised optimal perturbation results. For drag reduction comparison, the white noise forcing results will then be showed on a map in (k_x, ω) , and will be compared to DNS drag reduction measurements from Quadrio et al. (2009).

7.2.1 generalised optimal perturbation and white noise forcing

Comparing the generalised optimal perturbation and right hand side forcing results is interesting as far as a discussion on the filtering properties of the linearised operator is concerned. The generalised optimal perturbation provides only the peak of the filter, and simulations with right hand side forcing use a broadband input and therefore describe the width (or passband) of the filter. The comparison will be performed with the two forced cases studied with the generalised optimal perturbation approach, with harmonic wall oscillation of period $T^+ = 100$ and $T^+ = 200$, at $Re_\tau = 180$.

For the first comparison, the generalised optimal perturbation will be directly compared to random forcing results. Due to the existence of the “universal optimal perturbation”, the analysis is simplified. The generalised optimal perturbation at each observation time corresponds to the observation of the universal optimal perturbation at the same instant. For the case with the oscillation period $T^+ = 200$, this universal optimal is well defined, and the wavelength of the generalised optimal perturbations at

different observation times are very close to each other for most of the period. Because of that, the random forcing simulation can be considered using the same wavelength as the universal optimal wavelength. The comparison is showed in figure 7.8, and it

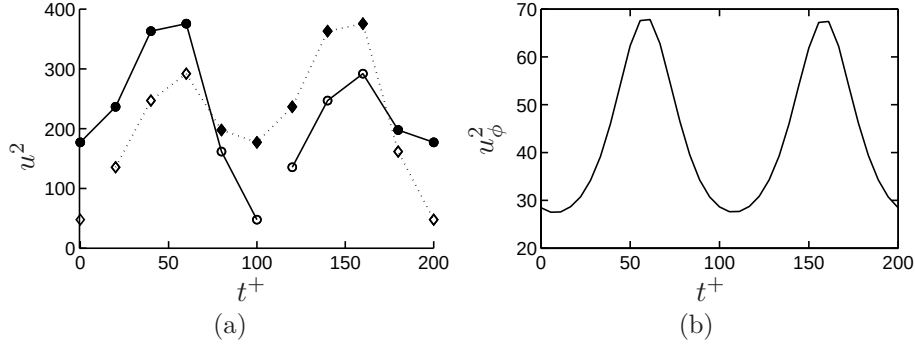


Figure 7.8: For a period of oscillations $T^+ = 200$ and an observation at $y_0^+ = 11.25$. (a) generalised optimal perturbation at each observation time. Each generalised optimal perturbation has a spanwise wavelength relatively close to $\lambda_z^+ = 200$. (b) Phase average energy for the linearised equations subject to right hand side forcing at a spanwise wavelength $\lambda_z^+ = 200$. The similarity between the curves shows that the generalised optimal perturbation is a good representation of the structures present in the forced equation.

can be seen that over a period, the evolution of the energy of the universal optimal perturbation is very similar to the evolution of the phase averaged streamwise energy in the simulation with right hand side forcing. This suggest that in this case the filter described by the linearised operator is relatively narrow; the most likely structure which represents the peak of the filter is also a good representation of all the structures present in the flow, as obtained with right hand side forcing.

As the flow is not composed on only one wavelength, it is interesting to perform comparisons including all possible wavelengths. This is easy for the simulations based on right hand side forcing, as it is sufficient to run simulations for each wavenumber, and then gather the result on a map in (t^+, λ_z^+) , as shown in figure 7.9(a). The situation is more complex in the generalised optimal perturbation case. To take into account the effect of the spanwise wavenumber, at each observation time and spanwise wavenumber, the structure kept is the most amplified over all possible initial time. Instead of optimising on the whole (t_i, λ_z) map as for finding the generalised optimal perturbation, the optimisation is only performed in the initial time t_i direction. Such

results are showed on figure 7.9(b) in the case of a wall oscillation period $T^+ = 200$. The results showed on figure 7.8 would be a slice at $\lambda_z^+ = 150$ of figure 7.9, and therefore the same conclusion can be driven at this specific wavelength. The new interesting information brought by figure 7.8 is the behaviour at large wavelength. The streaks energy predicted by the generalised optimal perturbation approach decrease quickly for $\lambda_z^+ \geq 300$, whereas it keeps increasing on the right hand side forcing figure. If it was concluded that the filter had a narrow passband for wavelength around $\lambda_z^+ = 150$, this is no longer possible for large wavelength. The generalised optimal perturbation approach predict a very small structure energy transient growth, which mean that any initial perturbation will witness only a very small energy increase, if at all. However, the large energy at high wavelength with the right hand side forcing suggests that more structures will be present at these wavelength, and have probably a much longer life span, thus explaining at the same time the high energy value and the smaller energy variation in one period. The results for an oscillation period $T^+ = 100$ are showed in figure 7.10, and very similar conclusions can be done; the filter has a relatively narrow peak at small wavelength and in this case the right hand side forcing and generalised optimal perturbation approach show similar results, but the passband is large at high spanwise wavelength and the two methods then show very different results.

To investigate the life span of the structures in the case of simulation with right hand side forcing, a frequential analysis of the signal was also performed, using the streamwise energy averaged on a whole plane. In this case, as no comparison with the generalised optimal perturbation approach is made, a wide range of oscillation periods could be taken into account. The maximum and minimum periods ($T_{structure}$) present in the signal are showed on figure 7.11. They are found using a Welch spectral estimation on a signal of length $\Delta t = 250$. At small spanwise wavelength λ_z , the maximum period observed is half the period of oscillation. This is expected as the dominant structures (mostly the generalised optimal perturbation) have a life span equal to the around oscillation period. Their cumulated energy will then be periodic with a period of half the oscillation period (symmetry of the profile over half a period). At larger frequencies larger period appear. This supports the previous idea according to which structures have a longer life span when the oscillation frequency increases. It also shows that at small frequencies the dynamics is mostly dictated by the transverse oscillations, but at larger frequencies some other mean flow parameter become more important, as the

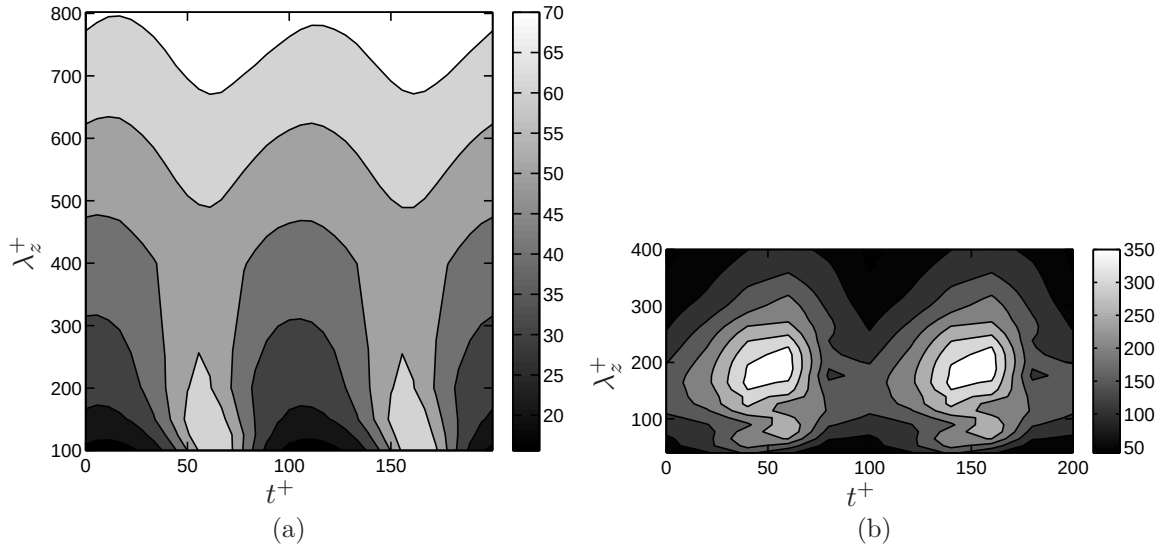


Figure 7.9: (a) Phase average streamwise energy u_ϕ^2 . (b) generalised optimal perturbation most amplified perturbation. Wall oscillation period $T^+ = 200$, observation at a distance $y^+ = 11.25$ to the wall.

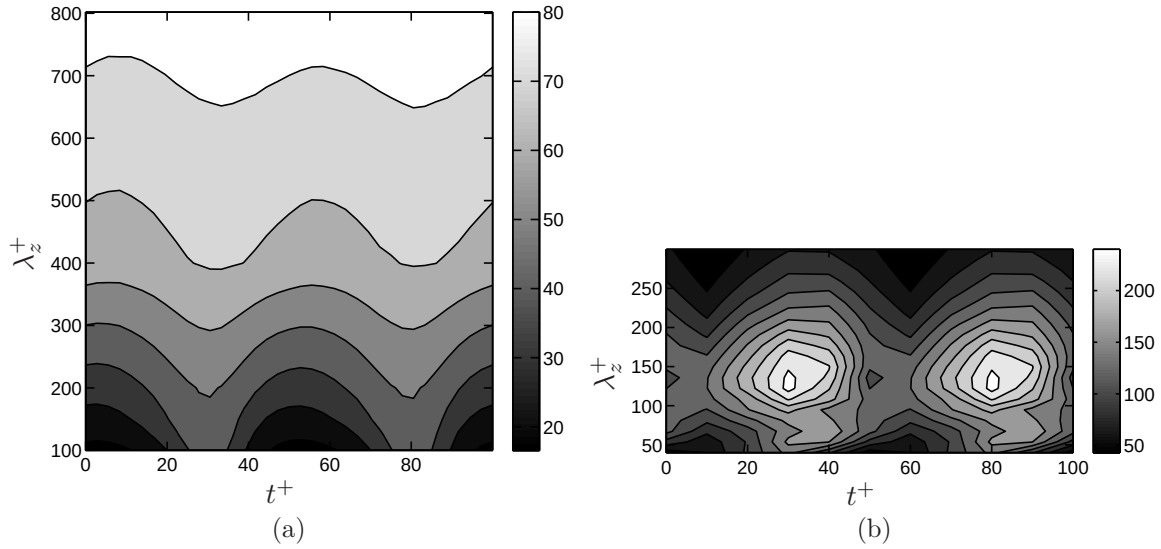


Figure 7.10: (a) Phase average streamwise energy u_ϕ^2 . (b) generalised optimal perturbation most amplified perturbation. Wall oscillation period $T^+ = 100$, observation at a distance $y^+ = 11.25$ to the wall.

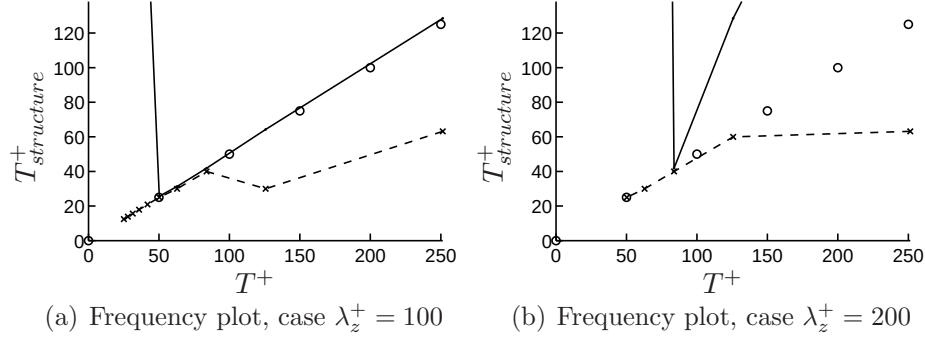


Figure 7.11: Value $T_{structure}^+$ of the smallest and the largest temporal period present in the flow for two spanwise wavenumbers, and for several wall oscillation periods T^+ . The circles represent the imaginary line $T_{structure} = T^+/2$, and corresponds to structures which dynamic is dictated mainly by the wall oscillations (such as for example the generalised optimal perturbations). As the spanwise wavelength λ_z^+ is increasing, the extremums frequencies present in the flow become more dictated by the streamwise component of the mean velocity than by the Stokes layer.

period of the structures is not linked to the oscillation anymore. This is not completely surprising as these large structures in the spanwise direction will also potentially be larger in the wall normal direction and be more influenced by the outer part of the boundary layer where the transverse flow vanishes and the dynamics are governed by the streamwise mean profile.

The analysis of harmonic wall oscillations was interesting not only to compare this method with the generalised optimal perturbation approach. It also gives some idea of the quality of the drag reduction prediction to be expected. The total drag predicted by the linear operator is the total energy of all the structures combined, measured at a certain distance to the wall. Assuming for now this drag is calculated in the case of an oscillation period $T^+ = 200$, using the energy at a distance to the wall $y^+ = 11$. To obtain the “energy” corresponding to the drag, an integral of the energy over time and different spanwise wavenumbers has to be performed:

$$\langle u^2 \rangle(y) = \frac{1}{L_x} \int_{\beta_z=0}^{\infty} \int_{t=0}^{\infty} \int_{x=0}^{L_x} u^2(\beta_z, z, y, t) dx dt d\beta_z. \quad (7.10)$$

This is equivalent to integrating the figure 7.9(a), with more data at larger wavelength and after a change of variable $\lambda_z \leftarrow \beta_z$. The main issue is the importance given to

the large scale structures. It is known in DNS that such large scales are not present so close to the wall, and they must therefore be an artefact of the linearised solver. This might be a limit to the current approach; it is known that this approach is valid for relatively small wavenumbers, as the generalised optimal perturbation theory provided good comparison with DNS, but it seems that using right hand side forcing, too much importance is given by the linearised operator to large scale structures. Better use of this approach might be done by weighting the forcing with some quantities such as the Reynolds stress, but this is not in the scope of this thesis.

7.2.2 Flow subject to travelling wave wall oscillations.

To further analyse the properties of the linear operator, travelling wave wall oscillations were considered. The mean flow under such boundary conditions was determined using an analytical method described in Duque-Daza et al. (2012). The streamwise mean flow is considered to be the standard turbulent mean profile of Reynolds and Tiederman (1967), like in the case of harmonic wall oscillation. From that, only the transverse component of the mean flow has to be determined. Given the form of wall oscillation used, it is assumed that the spanwise phase averaged velocity (also called mean velocity here) has the form $W = \mathcal{R}(\overline{W}(y)e^{i(k_x x - \omega t)})$. With this assumption and using the linearised operator, a differential equation can be obtained for the transverse velocity:

$$\left(\left(U(u) - \frac{\omega}{k_x} \right) i k_x \Re + k_x^2 \right) \overline{W}(y) = \overline{W}''(y)$$

This equation was then solved numerically using Matlab. As in the case of harmonic wall forcing, the mean flow used will not be exactly the same as the real turbulent phase averaged profile, but should be close enough if the oscillation frequency is large enough. Moreover, using an analytical input avoids relying on DNS data, which would lower the potential predictive interest of the method.

In this subsection, the case of single spanwise wavenumbers will first be discussed, and then an integral over all possible wavenumbers will be taken in order to assess the drag reduction capabilities of the method. The quantity of interest from now is the streamwise energy at a given distance to the wall. However, the “total energy” such as described in equation (7.10) is not the only quantity of interest. To study the properties

at a given wavenumber, two other quantities will be used; the simple spatial average $\langle u_{\beta_z}^2 \rangle_x$ in the streamwise direction, and the average in time and space $\langle u_{\beta_z}^2 \rangle_{x,t}$. They are defined by:

$$\langle u_{\beta_z}^2 \rangle_x(\beta_z, t, y) = \frac{1}{L_x} \int_{x=0}^{L_x} u^2(\beta_z, z, y, t) dx, \quad (7.11a)$$

$$\langle u_{\beta_z}^2 \rangle_{x,t}(\beta_z, y) = \frac{1}{L_x} \int_{t=0}^{L_x} \int_{x=0}^{\infty} u^2(\beta_z, z, y, t) dx dt. \quad (7.11b)$$

Streamwise energy for selected wavenumbers

The first interesting case to study is when the spanwise wavenumber of the perturbations is relatively small. Here it was chosen to be $\lambda_z^+ = 38$. The choice of this wavelength is not aimed to be an asymptotic study at small wavenumber, but is more interesting if considered as the lowest end of the range of significant wavelength for streaks study. As such, it can be expected that a significant amount of the turbulent energy generated will be present at this wavelength.

In figure 7.12, the time and space averaged streamwise energy is represented at a few distances to the wall. The simulations were run for time in the interval $t \in [0, 100]$. The time average for the calculation of $\langle u_{\beta_z}^2 \rangle_{x,t}$ was performed for $t \in [2, 100]$; the small initial period is removed to avoid the transitional period from the zero initial condition. The layer supposed to be the most relevant for drag reduction is around $y^+ = 10$, as it is where the streaks closest to the wall are present. However, it is also interesting to look farther from the wall to understand the kind of structures present, and to compare the energy between the different layers at different distances to the wall.

The first important observation is to assess the degree of convergence of the simulations. If the integral in time is much longer than the life span of the structures, enough different structures will be present in the average and the result will be modified only to a small extent by each of them. In this case, the convergence is satisfying, as can be seen by the smoothness of the curves in figures 7.12(a)-7.12(c). On figure 7.12(d), the convergence seems less satisfying, but this is mainly due to the fact that the results are very weakly dependent on the parameters (k_x, ω) and the limits of the figure axis is mainly determined by the convergence approximations errors.

Having in mind the drag reduction map for turbulent flow of figure 7.1, it is interesting to see which of its features can also be seen, for example on figure 7.12(a). The

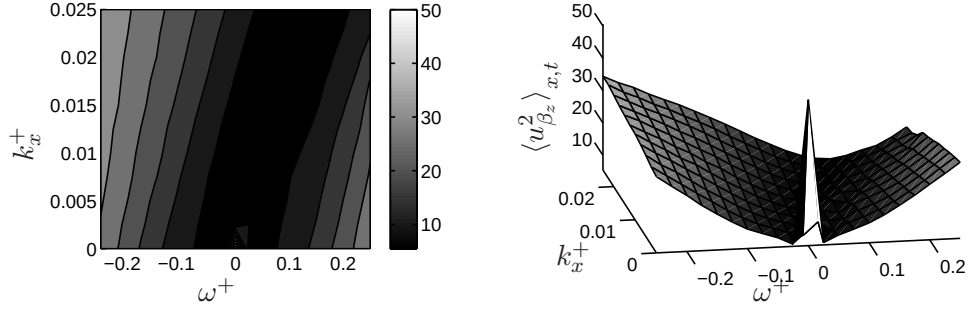
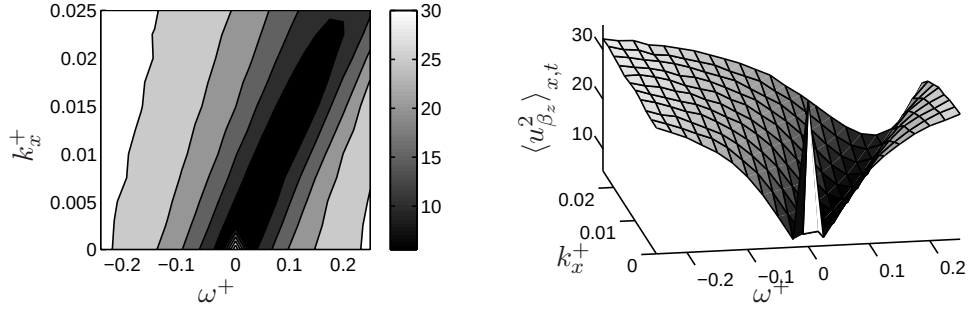
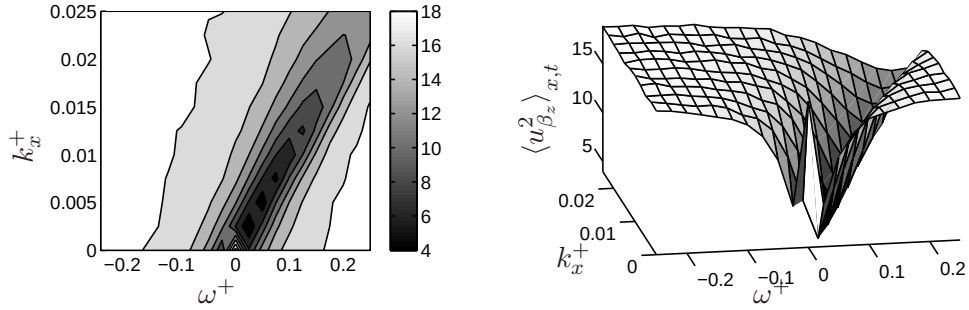
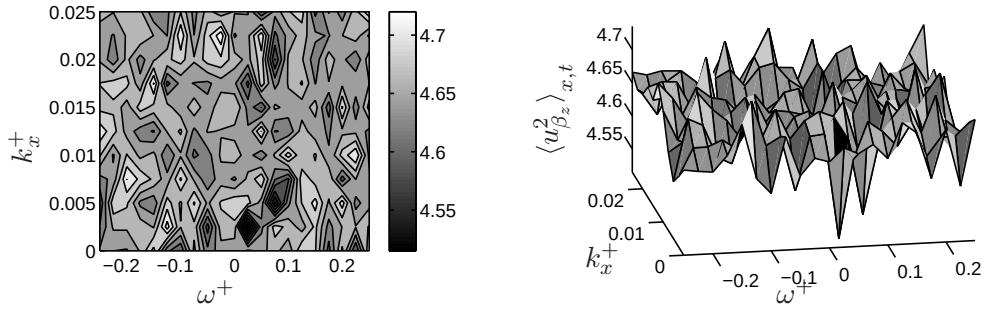
(a) distance to the wall $y^+ = 11.25$ (b) distance to the wall $y^+ = 19.7$ (c) distance to the wall $y^+ = 28.12$ (d) distance to the wall $y^+ = 70$

Figure 7.12: Streamwise energy $\langle u_{\beta_z}^2 \rangle_{x,t}$, averaged in space and time. The spanwise wavenumber used is $\lambda_z^+ = 38$, and the value is measured at different distances from the wall. The graphs on the left and on the right are the same, the left one being a contour plot and right one a surface curve. This allows a better visualisation.

unforced case $(k_x, \omega) = (0, 0)$ of figure 7.12(a) is taken as a base case for comparison. If the mean streamwise energy of one of the forced case $(k_x, \omega) \neq (0, 0)$ is smaller than for the unforced case, the turbulent energy generated is supposed to be smaller, and thus the drag is also smaller. The first striking observation is that whatever forced case is used, the streamwise average energy is smaller than for the unforced case, and there is an apparent discontinuity between the unforced case and the closest forced case. This seems to suggest that whatever forcing property is used, there will be drag reduction. This is obviously not the case, and a closer observation can make more sense. Reminding that a linearised approach to turbulent flow is more suited to be a qualitative approach than quantitative, the main feature to compare would be the line of maximum drag reduction and maximum drag increase. At this wavelength, no drag increase line can be visible, as drag reduction is predicted in all forced case. However, the line of smaller energy can be compared to the maximum drag reduction line of figure 7.1. This line originates at $(k_x, \omega) = (0, 0)$ like in the turbulent drag reduction figure, and its orientation is also towards positive values of k_x and ω . This is a first positive comparison; even if one single wavelength cannot be representative of the whole flow, it already indicates that the line of maximum drag reduction can be predicted.

Beyond the interest of understanding the drag reduction mechanisms, figure 7.12 also gives indications of the property of the linearised operator. It was already mentioned that the average energy is smaller for all the forced cases compared to the unforced case. This seems to be the case independently of the distance to the wall. It will be seen later that this is also the case for other spanwise wavenumbers, and even if this characteristic could not be proved analytically, it seems to be a strong property of the linearised Navier-Stokes equation subject to travelling wave wall oscillations. A last observation can also be made about the dependence of the streamwise mean energy on the distance to the wall. Close to the wall, the whole map is affected by the oscillations, which is not surprising as the transverse flow is affecting the operator properties in this area. Going farther from the wall and going beyond the penetration length of the transverse flow, the structures get less affected and resemble more the unforced case. This starts at larger values of ω , as can be seen by looking at the difference between figure 7.12(b) and figure 7.12(c). Larger values of ω correspond to structures of smaller oscillation periods, and therefore to smaller penetration length of

the transverse mean flow in the wall-normal direction. The terms $\partial W/\partial y$ being the only difference between the unforced and forced operators, it is then logical that when it is small in the region of interest (far from the wall for large spanwise wavenumbers) the result gets closer to the unforced case result.

Some more insight can be gained by looking at the statistics for larger spanwise wavelength. The dependence with the distance to the wall is always similar to what was shown with a wavenumber $\lambda_z^+ = 38$, and therefore only the results for an observation layer at a distance to the wall $y^+ = 11$ will be shown later. The next case analysed is the simulations for a spanwise wavelength $\lambda_z^+ = 68$. The wavelength $\lambda_z^+ = 68$ corresponds roughly to the streaks spacing in the unforced case at this distance to the wall, and therefore the contribution of this wavelength should be significant. The situation is shown on figure 7.13 and is quite similar to the case of figure 7.12. The line

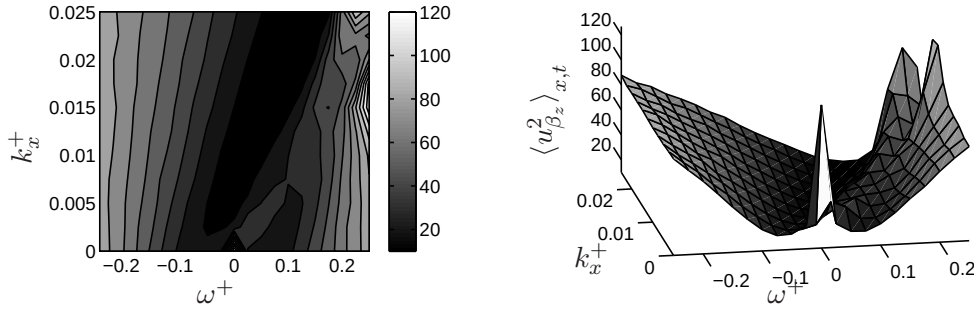


Figure 7.13: Streamwise energy $\langle u_{\beta z}^2 \rangle_{x,t}$ at a distance to the wall $y^+ = 11.25$. The spanwise wavelength is $\lambda_z^+ = 68$.

of maximum drag reduction is still present and with a very similar angle. An important qualitative difference is the presence of a change of variation, with a local extrema along a line with a slightly smaller angle than the line of smaller energy (angles are mentioned in the trigonometric direction in the (ω^+, k_x^+) map). The position of this line, and the fact that the energy is slightly bigger than it would be without the change of variation suggests a link with the maximum drag increase line of figure 7.1 which is at a similar position on the map. Another difference is the few points at high values of both k_x and ω where the surface seems to be “less continuous”, and for which the energy is higher than in the unforced case. There are reasons to believe that the value at these points should not be trusted, and this will be more visible by running simulations at even higher wavenumber.

The case with a spanwise wavelength $\lambda_z^+ = 100$ showed in figure 7.14 is again very similar apart from some points where an important energy growth takes place. Some of the points have a mean energy several orders of magnitude higher than in the unforced case, and have been clipped from the results for better visualisation. It

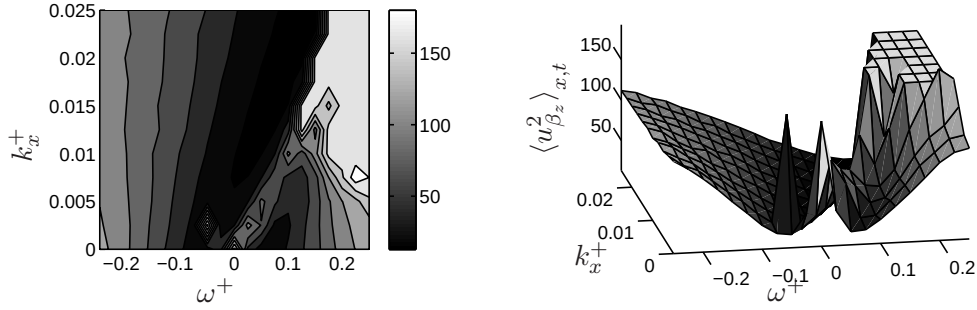


Figure 7.14: Streamwise energy $\langle u_{\beta_z}^2 \rangle_{x,t}$ at a distance to the wall $y^+ = 11.25$. The spanwise wavelength is $\lambda_z^+ = 100$. The value points corresponding to unstable simulations have been truncated to the value for the unforced case.

was showed that the streamwise energy at these points was subject to an exponential instability, and therefore the operator is unstable for the corresponding parameters. This is not surprising given some observation previously made. The work by Togneri (2011) showed that the linearised Navier-Stokes operator around turbulent flow in the streamwise direction, and with a spanwise velocity component being a frozen Stokes layer could be subject to exponential instabilities. Much closer to the current work, the same mean velocity profiles in the (k_x, ω) map as here were used in Duque-Daza et al. (2012). The simulation performed were initial value problems, with the initial perturbation velocity profile being close to the generalised optimal perturbation for the unforced case. For some wall oscillation parameters, a large growth was observed, and the exact value of the growth was clipped from the results. As the mean flow was the same as in this study, the asymptotic behaviour should be the same, even if an initial value equation is solved instead of the equation subject to right hand side forcing. The spanwise wavelength used in the work of Duque-Daza et al. (2012) was $\lambda_z^+ = 74$ which is in between the cases showed on figure 7.13 and figure 7.14. It was not concluded that exponential instabilities are present, as the simulations were run only for a short time. However, later comparisons with the present work (private communication) showed that some points experiencing large growth with the program used in Duque-Daza

et al. (2012) are indeed subject to exponential stabilities with the code described in this thesis. This was at the same time a confirmation of the similarities between the two studies and a cross validation of the results obtained here.

Instabilities of the linearised Navier-Stokes operator

The instabilities discovered with a spanwise wavelength $\lambda_z^+ = 100$ are of prime importance in the study of the linearised operator. This section describes in more details their properties, and the dependence of these instabilities on the spanwise wavelength.

A first clue of the presence of instabilities was the presence of the points with higher mean streamwise than the unforced case in figure 7.13. It could be suspected that these points lie at the threshold of where the simulation becomes unstable. If it was marginally unstable, it would not be possible to determine whether there was an exponential divergence, even if running the simulation for a very long time. To clarify this issue, and as the previous growth figures show that exponential instability appear only for certain wavenumber, a simulation with the parameters $(k_x^+, \omega^+) = (0.025, 0.2)$, one of the potentially marginally unstable case at $\lambda_z^+ = 68$ was run for a large set of spanwise wavenumbers, as shown on figure 7.15. In this simulation, to be able to

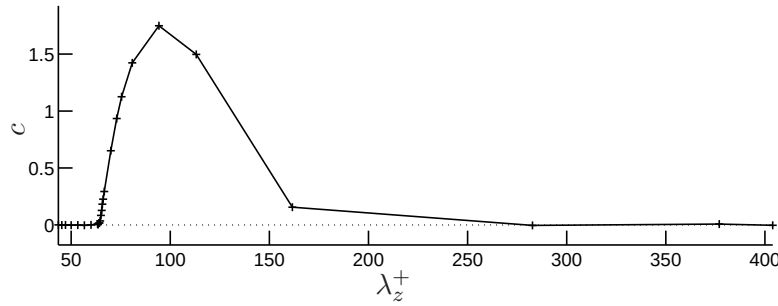


Figure 7.15: Coefficient c of the exponential approximation of the streamwise energy $\langle u_{\beta_z}^2 \rangle_x(\beta_z, t)$, for a range of spanwise wavelengths λ_z . Wall oscillation parameters: $(k_x^+, \omega^+) = (0.025, 0.2)$.

study the evolution in time, the quantity of interest is the average in space $\langle u_{\beta_z}^2 \rangle_x$ of the streamwise energy. It is then possible to approximate the curve in time by an exponential (by a least square regression of the logarithm of the curve), such as:

$$\langle u_{\beta_z}^2 \rangle_x(t, \beta_z) = A(\beta_z)e^{c(\beta_z)t} + \epsilon(\beta_z, t). \quad (7.12)$$

If the coefficient $c(\beta_z)$ of the exponential is close to zero, the operator is stable or marginally unstable. However, if the coefficient is non-zero, the operator is unstable at the given wavelength. The curves in time of the streamwise energy are shown for a similar case on figure 7.16(b), and show that the approximation of the exponential coefficient can be very good with the time scales used for the simulation. The main information in figure 7.15 is that the operator is unstable only over a specific range of wavelength. It was already known that at small wavelength the operator is stable over the whole (k_x, ω) map, and it will be shown later that this is also the case at large wavelengths.

The attention will now focus at the boundary between the stable and the unstable regions, still using the case with the forcing parameters $(k_x^+, \omega^+) = (0.025, 0.2)$. It can be interesting to first understand whether the “less continuous” region in figure 7.13 can be linked to the presence of instability. For that, a simulation was first run at a spanwise wavelength $\lambda_z^+ = 68.5$, with different time steps and different grids, as shown in figure 7.16. The time step used in all the map simulations showed here is

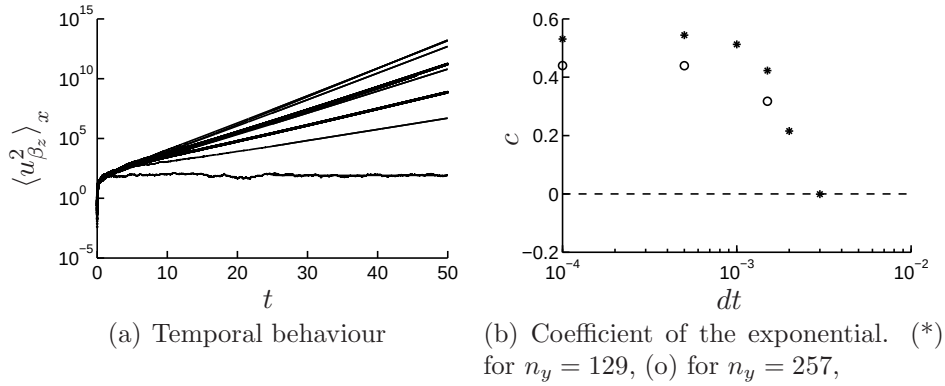


Figure 7.16: Dependence of a potentially unstable case on the time step and on the grid used. This simulation is diverging in time for small time steps. Wall oscillation parameters $\omega^+ = 0.2$ and $k_x^+ = 0.025$, spanwise wavelength $\lambda_z^+ = 68.5$. measure of the energy at a distance $y^+ = 11.25$ from the wall.

$dt = 0.003$, and is the largest time step showed in this figure. At this time step, the simulation seems to be stable, or marginally unstable in a way that can not be detected, even by running simulations for a very long time. This corresponds to the observation of figure 7.13 where the average energy corresponding to this point is of the same order of magnitude as in the unforced case. The surprising observation

in this figure is that the system becomes unstable when the time step is decreased. This suggests some unusual approximations errors in the discretisation scheme. To confirm when the linear operator is well represented by its discrete version, a fine grid $(n_x, n_y) = (400, 257)$ was also used, and time steps were decreased significantly. It was not possible to run simulations with a time step $dt = 0.003$ and the fine grid, as this is beyond the usual CFL stability limit. For smaller time step however, the results predicted by both grids are exponentially divergent, with the coefficient of the exponential becoming independent of the time step. This is in favour of considering the simulation to be representative of the continuous operator for both grids if the time step is small enough. It also shows that the “less continuous” parts of average energy plots on (k_x, ω) are likely to be points for which the time step is too large to be a good approximation of the linear operator and should be discarded.

Common discretisation issues when a time step is too large are usually due to the CFL number being too large, which makes simulations unstable. In this case, when $dt = 0.003$, there is no instability as such, but the physics is still not well described. It is difficult to analyse a case where exponential growth takes place. Due to computational limitations it was not possible to perform large scale simulations with a small time step ensuring a good approximation of the linear operator for every (k_x, ω) parameter. However this was done for one case to see to what extent the results can be considered as reliable. In figure 7.17, the same map calculations have been performed with two different time steps, the usual $dt = 0.003$ and the smaller step $dt = 0.001$, and the spanwise wavelength is $\lambda_z^+ = 68.5$ like in the case previously used. The points where the simulation diverges have been removed from the plot, the purpose is then to analyse the time step dependence only for parameters where the operator remains stable even at small time step. The difference between the energy obtained with the two time steps simulations is shown on figure 7.17(f). The main error occurs for oscillation parameters close to the ones for which the operator is unstable. By slightly increasing the wavenumber, these points would become themselves unstable.

Only observations have been made on the way the linear operator becomes unstable, and no theoretical study was performed. The main conclusions are that the operator is stable over the whole forcing map (k_x, ω) if the spanwise wavelength is small enough or large enough. In between, the operator will become unstable for some forcing parameters and over a specific range of wavelengths. When getting close to

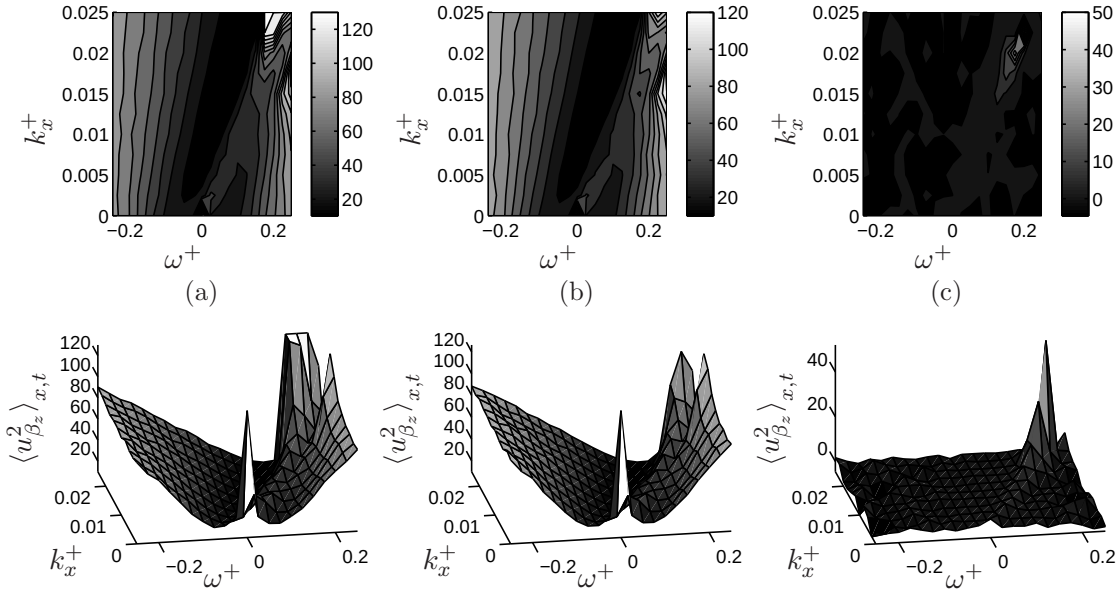


Figure 7.17: Dependence on the time step on the whole range parameters (k_x, ω) , for a spanwise wavelength $\lambda_z^+ = 68.5$. (a) $dt = 0.001$, (b) $dt = 0.003$, (c) is the difference between (a) and (b). On graph (a) three points correspond to unstable simulations while on graph (b) all simulations are stable.

unstable wavelength, the operator becomes more sensitive to the time step.

7.2.3 Drag reduction

A lot has been learned about the response of the linearised operator to right hand side forcing by studying the response of single wavenumbers. However, to pursue the initial purpose of this study and investigate the potential of the method to predict drag reduction, the combined effect of different spanwise wavelengths has to be studied. A single wavenumber can be considered as a component of the Fourier transform of the flow in the spanwise direction. To obtain the full velocity component, it is then sufficient to perform the inverse Fourier transform. The flow being written in a sine/cosine form, the streamwise component of the velocity, for example, can be written as:

$$u(x, y, z, t) = \int_{\beta_z=0}^{\infty} u_c(x, y, \beta_z, t) \cos(\beta_z z) + u_s(x, y, \beta_z, t) \sin(\beta_z z) d\beta_z. \quad (7.13)$$

As the quantity of interest is the streamwise energy at a distance $y = y_0$ from the wall, this expression can be easily obtained using the fact that different spanwise wavelengths are orthogonal to each other, and has the form:

$$\begin{aligned}\langle u^2 \rangle_x(y_0, t) &= \frac{1}{L_x L_z} \int_{\beta_z=0}^{\infty} \int_{x=0}^{L_x} \int_{z=0}^{L_z} u_c^2(x, y_0, \beta_z, t) \cos^2(\beta_z z) + u_s^2(x, y_0, \beta_z, t) \sin^2(\beta_z z) dz dx d\beta_z \\ &= \frac{1}{2L_x} \int_{\beta_z=0}^{\infty} \int_{x=0}^{L_x} u_c^2(x, y_0, \beta_z, t) + u_s^2(x, y_0, \beta_z, t) dx d\beta_z.\end{aligned}\tag{7.14}$$

If only the mean drag is important, then the mean of this energy over one period of oscillations is the main quantity of interest:

$$\begin{aligned}\langle u^2 \rangle(y_0) &= \frac{1}{t_{end}} \int_{t=0}^{t_{end}} \langle u^2 \rangle_x(y_0, t) dt \\ &= \int_{\beta_z=0}^{\infty} \langle u_{\beta_z}^2 \rangle_{x,t}(y_0) d\beta_z,\end{aligned}\tag{7.15}$$

with t_{end} the length of the simulation in time. The aggregated result is then a simple integration of the energy maps previously analysed for a few frequencies. Given the computational power available, the energy for as many spanwise wavelengths as possible was calculated. A sample for a few oscillation parameters is shown in figure 7.18 to see the domain integration used and judge for the accuracy of the integral calculation. Two version of the same figure are used, one in wavenumber as it is the simplest way to calculate the integral, and one in wavelength and pre-multiplied energy, as it is equivalent (shown by a simple change of variables) and more visible. The main limit of the integration procedure is the quality of convergence of individual spanwise wavelength simulations. For small wavelength, the structures have a relatively short life. However, as the wavenumber is increasing, the frequencies present in the energy temporal spectrum are smaller and smaller, up to the point where even with the large time limit used in these simulations ($T = 100$), the convergence is not good (more visible on figure 7.19 at small values of β_z). This phenomenon could be responsible for some lack of accuracy for the integral, but was not an issue as far as a qualitative analysis is concerned.

The integration could be performed only for cases for which the operator is stable

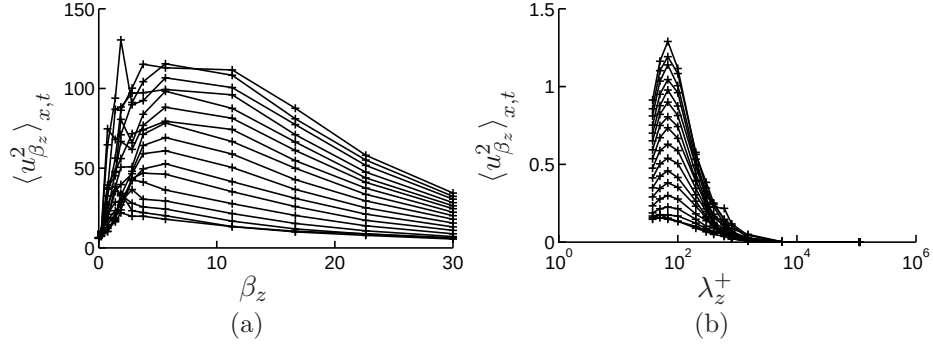


Figure 7.18: Repartition of the energy in wavenumber or wavelength. The example showed corresponds to the range of ω available, for the travelling wave parameter $k_x^+ = 0.025$. Integrating each of these curves provides the total energy $\langle u^2 \rangle$. For large values of λ_z^+ the result obtained is not well converged as the structures life expectancy is getting too large. However this does not change significantly the value of the integral.

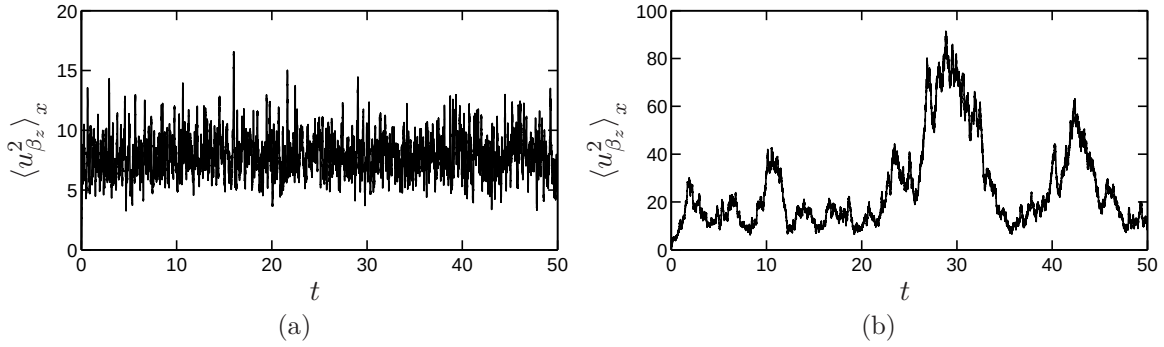


Figure 7.19: Temporal visualisation of the streamwise energy for two spanwise wavenumbers, $\lambda_z^+ = 38$ (a), and $\lambda_z^+ = 802$ (b). The longer time scales present at large spanwise wavelengths mean that it is difficult to obtain convergence. Observation at $y^+ = 11.25$

at all wavelengths. Cases where the operator is unstable for at least one wavelength are shown in figure 7.20. As no integral could be calculated for these cases, the energy

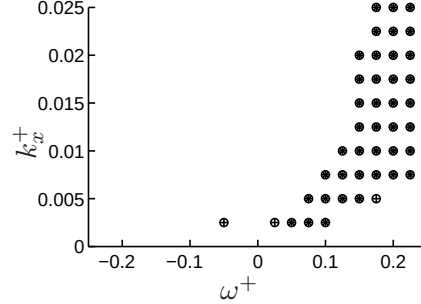


Figure 7.20: Positions for which the simulation was unstable for at least one wavelength λ_z^+ . This includes only the case for which the coefficient c of the exponential approximating the energy $\langle u_{\beta_z}^2 \rangle_x(t)$ was strictly positive, and not the marginally unstable cases.

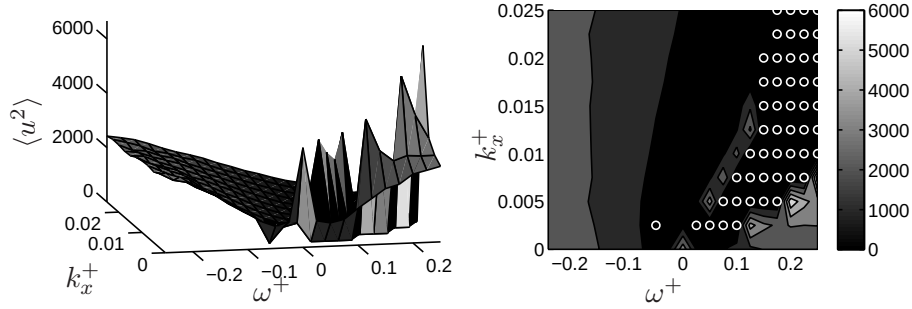


Figure 7.21: Total energy $\langle u^2 \rangle$ after integration over the spanwise wavelength of $\langle u_{\beta_z}^2 \rangle_{x,t}(\beta_z)$. Observation at a distance to the wall $y^+ = 11.25$. Points for which the simulation is unstable for at least one spanwise wavelength have been removed and are marked by white circles on the contour plot. Next to this region, the unsteady and high energy corresponds to marginally unstable points.

at the corresponding points has been arbitrarily set to zero. An example of integral map is shown in figure 7.21, for the distance to the wall $y_0^+ = 11.25$. In this figure, the main observations made for specific wavelengths are still valid. The unforced case presents a point of discontinuity in the curve. For most oscillation parameters where the operator is stable at all spanwise wavelengths, the energy $\langle u^2 \rangle(y_0)$ is smaller than in the unforced case, suggesting drag reduction on the whole map. The line of minimum energy is at a similar position as the line of maximum drag reduction in figure 7.1.

A specific consideration about the cases for which the operator is unstable has to

be made. The interesting observation is that these points lie in a specific region of the map, and that this region covers the region of drag increase in figure 7.1. This could be a clue that the fact that the operator becomes unstable for these parameters has an influence in the drag increase mechanisms, even if no such mechanism is studied here. On both sides next to the unstable region, the energy measured is much more uneven than farther from this region where it is continuous. It is suspected that for these points there exists wavelength for which the linearised Navier-Stokes operator is marginally unstable. These points could have been discarded like the unstable points, but a clear criteria to assess whether a point is valid can not be defined with the limited computational resource available. They have therefore been left.

The previous figure is a measure of the total streamwise energy at the distance to the wall $y_0^+ = 11$. More attention is focussed on this case, as it is believed to lie in the main region of interest for drag reduction purposes. It is nonetheless interesting to observe the situation at different distances to the wall, and a few other cases are showed in figure 7.22. The global behaviour is still the same, with obviously the

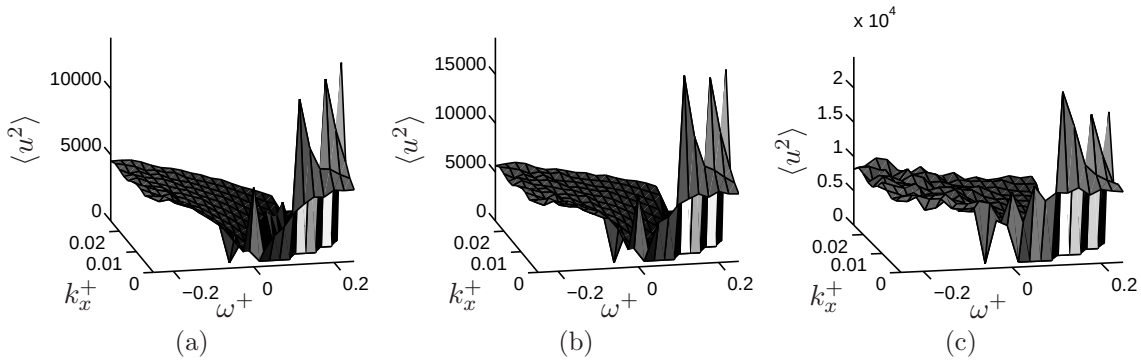


Figure 7.22: measure of the total energy $\langle u^2 \rangle$ at different distances to the wall. (a) $y^+ = 19.7$, (b) $y^+ = 28.12$, (c) $y^+ = 70$. The marginally unstable points are present as in figure 7.21. The convergence deteriorates farther from the wall, as larger structures with a long time of existence become more dominant.

unstable cases being at exactly the same position, as well as the marginally unstable points. The main difference in the zone of stability is the level of energy decrease in the forced cases compared to the unforced cases. Like in figure 7.12 where only one spanwise wavelength was studied, the effect of the oscillations becomes less significant as the distance to the wall is increased. Farther from the wall, the smoothness of

the curve is not as good. This is a direct effect of the structures present for large spanwise wavelengths. Close to the wall, structures with a spanwise wavelength of about $\lambda_z^+ = 100$ contribute the most to the total energy, but farther from the wall, larger structures contribute to the energy. Due to the fact that the simulation are not well converged for large structures, this become visible for the energy graphs far from the wall.

7.3 Improvements and limits of the random forcing approach

The idea of using right hand side white noise forcing came from the fact that the generalised optimal perturbation approach was good at predicting structures, but not drag. Using a white noise forcing instead of an optimal perturbation ensures that all possible structures are taken into account instead of only the most amplified one. This accounts to looking at the passband of the linear filter instead of only its peak and seems to be a better option for drag prediction purposes. However, the quality of the results could not be assessed beforehand. Contrary to the generalised optimal perturbation where it was known from Chernyshenko and Baig (2005) that the method was suitable to predict the streaks structures, using the linearised operator to predict drag has not been done extensively before and the results were more uncertain. It is natural to think that as the linearised operator is suitable to predict streaks, and as streaks are of prime importance in the turbulent drag mechanisms, the linear operator should then be good at predicting drag as it is good at predicting the streaks. However, the choice of forcing to use is not obvious. A white noise forcing is the simplest possible, and is therefore a good candidate for a first investigation. There is however no reason to think that this kind of forcing is better than some forcing weighted in the spatial or temporal spectrum. Indeed, such forcing could possibly take better into account some properties of the real turbulent flow right hand side term.

Knowing the possible limits of the approach, some interesting results have been obtained. A first one is that for every forcing point where the operator is stable, the streamwise energy is smaller than for the unforced case. This tends to predict drag reduction independently of the forcing, but more interestingly the line of minimum

energy seems to be in accordance with the line of maximum drag reduction predicted in Quadrio et al. (2009). The fact that the linearised Navier-Stokes operator becomes unstable for some forcing parameters and some wavelengths is more interesting. In fact, this property is independent of the kind of forcing used, and is even independent from the fact that a forcing is used or an initial value problem is solved. This property is a fundamental characteristic of the operator as it shows its potential and its limits to predict drag reduction. The area where the operator becomes unstable covers the area of drag increase for turbulent flow. This suggests that even if the linear operator can not be used for any quantitative prediction in this area, the fact that drag increase occur could be linked to the fact that this operator becomes unstable. This is only an hypothesis at this point, but it would be interesting to investigate this in further research.

The current forcing approach could not be used for more than a qualitative analysis of the linear operator properties. In the regions where the operator is stable, drag reduction is predicted, but the shape of the curve does not have many common points with the turbulent drag reduction map of Quadrio et al. (2009) apart from the line of maximum drag reduction. This is the case even if considering only the case of harmonic wall oscillations. The operator seems to be always stable when there is no travelling wave, and therefore the whole range of oscillations used is available for analysis. The main issue is the fact that the curve seems discontinuous close to $\omega = 0$. This is not so surprising considering that at constant forcing amplitude, having $\omega \rightarrow 0$ means that the oscillation period T will go to infinity; the mean flow evolves rapidly for small values of ω and therefore the result evolves also quickly and seems discontinuous. The main issue is that this feature is not present in the direct numerical simulation results of Quadrio et al. (2009). This suggests that some mechanisms are not being captured by the forcing used. However this does not discredit the whole linear approach, as some different kind of forcing could be used and maybe not show this apparent discontinuity. This is the case for example by taking into account the turbulent Reynolds stress, such as in Moarref and Jovanović (2012), where a good approximation of drag reduction was obtained for the case of harmonic forcing. Such a weighted forcing could be used as a basis for further studies.

Chapter 8

Final discussion and conclusions

8.1 Discussion

Turbulent flow configurations for drag reduction by wall oscillations are interesting for the potentially significant energy gain they offer and that could be attractive for industrial flows. Such configurations have been studied in the present work, the near-wall streaks being the main focus. Linearised approaches to the equations governing turbulent flow have been used as a tool to predict streaks and understand their physics. They have also been used in a first attempt to predict drag reduction.

One of the main achievements of the present study is the use of the generalised optimal perturbation approach to predict the streaks structure when harmonic wall forcing is used. It was showed that the streaks have an angle that is discontinuous in time, and the linear mechanisms could explain the physics of the angle evolution.

It was showed that the generalised optimal perturbation, even if it describes well the turbulent flow streaks, can not by itself explain the drag. The generalised optimal perturbation represents only the most probable streaks. To consider drag, a combination of all structures possible has to be taken into account, which suggested using randomly forced Navier-Stokes equations.

To study drag, the linearised Navier-Stokes equation subject to white noise random forcing have been used. The choice of a white noise was deliberate, making as little assumptions as possible on the form of the forcing. It was showed that qualitatively, the operator predicts drag reduction where drag reduction is observed in direct numerical

simulation. The quantitative agreement remains limited as could be expected with the forcing chosen.

Overall, this study confirms that linearised approach to the Navier-Stokes equations are useful to predict and understand turbulent flow streaks. A first attempt to use linearised approach to predict drag gives encouraging results and suggests to pursue investigations in this direction.

8.2 Conclusions

In the work described in this thesis:

- i A simple method of satisfying the solenoidality constraint in the Green-function-based approach to generalised optimal perturbation was proposed.
- ii The well-known adjoint optimisation approach was extended to the case of the generalised optimal perturbation.
- iii A code solving the linearised Navier-Stokes equation and calculating adjoint optimisation was written, based on an existing Navier-Stokes solver.
- iv The tools and ideas developed were applied to the problem of turbulent flows, in particular the flows past oscillating walls in the regime with drag reduction.

The following conclusions were obtained:

1. Imposing solenoidality constraint on generalised optimal perturbation is more consistent than the version of the generalised optimal perturbation approach studied by Chernyshenko and Baig (2005).
2. Generalised optimal perturbations with and without solenoidality constraint are quantitatively close to each other at least for the cases considered in Chernyshenko and Baig (2005).
3. Generalised optimal perturbation approach applied without using any data about the turbulent flow past a spanwise oscillating wall was demonstrated to give results in qualitative agreement with direct numerical simulations of the equivalent

turbulent flow past a spanwise oscillating wall, while the quantitative agreement is within the limits of error that can be expected.

4. For the flow past a spanwise oscillating wall, the generalised optimal perturbation was found to be an infinitely long structure having a certain angle to the mean flow direction. The angle varies slowly with phase for half a period, and at a certain instant in the oscillation period changes discontinuously, changing its sign and magnitude.
5. The generalised optimal perturbations angle varies slowly during half of a period because the linear mechanisms select generalised optimal perturbations of similar characteristics. Particularly, their initial time and growth pattern are very similar. From this observation a universal optimal perturbation could be defined, the angle of which does not vary with time and which is a good representation of all the generalised optimal perturbations observed during that half period.
6. The sudden jump in the angle is linked to the symmetry of the mean flow over half a period, as well as to the presence of the universal optimal perturbation. A universal optimal perturbation dominates the flow during half a period, imposing a relatively constant angle. Because of the mean flow symmetries an identical universal optimal, but with the angle of an opposite sign, dominates the flow during the next half period. The jump occurs when the energy of one universal optimal becomes greater than the energy of the previous one.
7. During those parts of the oscillation period when the angle can be reliably extracted from direct numerical simulations it matches approximately the angles predicted by the generalised optimal perturbation theory. Near the jump position, the two dominant structures are superimposed with similar amplitude, which explains why it is difficult to observe streaks in the velocity fields obtained from direct numerical simulations for this part of the period.
8. The spanwise spacing between streaks was compared for all the flow cases available in direct numerical simulations and generalised optimal perturbation results and was shown to be of the same order of magnitude. The range of variation of streak spacing is, however, of the same order as the error that can be reasonably expected from the generalised optimal perturbation approach.

9. A new interpretation of generalised optimal perturbation solutions as a model of conditionally-averaged structures in real flow was proposed and shown to depend on the validity of certain additional assumptions. Comparisons with direct numerical simulation results are inconclusive.
10. The transient growth mechanism of the generalised optimal perturbation is found to be a combination of the Orr mechanism due to the cross-flow shear, acting at the initial stage, and the mechanism of the lift-up of the longitudinal velocity by longitudinal vortices, acting at the later stage. The conditions for the Orr mechanism appear to exist only at a relatively short interval within the period, thus explaining why all optimal perturbations originate at approximately the same instant in the half-period, resulting in the existence of a universal optimal perturbation.
11. When travelling wave wall oscillations are used, for certain parameters of the travelling wave the linearised Navier-Stokes equations is unstable. The region of instability overlaps the region where drag increase occurs.
12. There is a similarity between the amplitude of the response of the linearised Navier-Stokes equation to random forcing and the magnitude of drag reduction effect. Note that this last result is superseded by the recent results of Duque-Daza et al. (2012) and Moarref and Jovanović (2012).

8.3 Future work

Overall it was shown that the linearised approach can predict the most probable structures, and that it has some interesting qualities to predict drag reduction features. The generalised optimal perturbation approach is a well-understood tool. It can be used to study streaks in flows with different wall oscillation characteristics. However, the area where the most improvement can be done is in understanding the behaviour of the forced linearised operator and its ability to predict drag. This includes two main topics. First, the link between the drag increase and the instability of the linear operator should be investigated. Second, when the operator is stable, a better understanding of the adequate form of the forcing to use is needed.

The forcing used in this thesis in the case of flow subject to travelling wave wall oscillations was the very simple white noise Gaussian forcing. It was chosen without doing any physical assumptions, to avoid making any restrictive hypothesis and understand the potential of the approach. It was successful in that way, as the drag reduction predictions compared qualitatively well to turbulent flow drag. It is however obvious that this approach is only a first step and that refinements are needed. This is exemplified by the recent publications of Duque-Daza et al. (2012) and Moarref and Jovanović (2012). In both papers, linearised approaches are used, but assumptions on the form of the forcing are made. For example, using the Reynolds stress such as in Moarref and Jovanović (2012) instead of white noise forcing provides much better results. Overall, the potential of linearised approaches to predict drag is present, but more work is needed to understand to what extent they can be improved and whether they can be usable to design drag reduction devices.

Appendix A

Derivation of the adjoint equations

Operator form of the linearised Navier-Stokes equations

In this appendix, the adjoint of the linearised Navier-Stokes equation will be derived. As seen in chapter 3, the main purpose is to obtain a Lagrange identity, as this is the main equation used in the diverse proofs made in this chapter. Before deriving the adjoint equations, the linearised equations described in equations (2.10)-(2.12) are written in a more convenient operator form. For that, a more compact notation for the vector to describe the velocity and the pressure is used. The perturbation velocity vector is written:

$$\tilde{\mathbf{u}} = (u_c, v_c, w_c, u_s, v_s, w_s),$$

and the pressure vector is written:

$$p = (p_c, p_s).$$

Convenient operators are now defined in order to rewrite the linearised equations. For the mass conservation, the operator $\mathcal{M}$ is defined by:

$$\mathcal{M}(\tilde{\mathbf{u}}) = \begin{pmatrix} \frac{\partial u_c}{\partial x} + \frac{\partial v_c}{\partial y} + \beta_z w_s \\ \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} - \beta_z w_c \end{pmatrix}.$$

For the momentum equations, the operators $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{P}$ are defined as:

$$\mathcal{L}_1(\tilde{\mathbf{u}}) = \begin{pmatrix} U \frac{\partial u_c}{\partial x} + W \beta_z u_s \\ U \frac{\partial v_c}{\partial x} + W \beta_z v_s \\ U \frac{\partial w_c}{\partial x} + W \beta_z w_s \\ U \frac{\partial u_s}{\partial x} - W \beta_z u_c \\ U \frac{\partial v_s}{\partial x} - W \beta_z v_c \\ U \frac{\partial w_s}{\partial x} - W \beta_z w_c \end{pmatrix}, \quad \mathcal{L}_2(\tilde{\mathbf{u}}) = \begin{pmatrix} u_c \frac{\partial U}{\partial x} + v_c \frac{\partial U}{\partial y} \\ 0 \\ u_c \frac{\partial W}{\partial x} + v_c \frac{\partial W}{\partial y} \\ u_s \frac{\partial U}{\partial x} + v_s \frac{\partial U}{\partial y} \\ 0 \\ u_s \frac{\partial W}{\partial x} + v_s \frac{\partial W}{\partial y} \end{pmatrix}, \quad \mathcal{P}(p) = \begin{pmatrix} \frac{\partial p_c}{\partial x} \\ \frac{\partial p_c}{\partial y} \\ \beta_z p_s \\ \frac{\partial p_s}{\partial x} \\ \frac{\partial p_s}{\partial y} \\ -\beta_z p_c \end{pmatrix}.$$

The linearised Navier-Stokes equations in operator form can then be written:

$$\begin{cases} \mathcal{M}(\tilde{\mathbf{u}}) = 0, \\ \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \mathcal{L}_1(\tilde{\mathbf{u}}) + \mathcal{L}_2(\tilde{\mathbf{u}}) + \mathcal{P}(p) + \frac{1}{R} \Delta_{\beta_z} \tilde{\mathbf{u}} = 0. \end{cases}$$

Lagrange identity

As described in chapter 3, the purpose is to find an adjoint equation such that the pair composed of the linear equation and the adjoint equation satisfies the Lagrange identity. For that, it is first supposed that the adjoint equation has the form:

$$\begin{cases} \mathcal{M}^*(\tilde{\mathbf{u}}^*) = 0, \\ \frac{\partial \tilde{\mathbf{u}}^*}{\partial t_1} - \mathcal{L}_1^*(\tilde{\mathbf{u}}^*) - \mathcal{L}_2^*(\tilde{\mathbf{u}}^*) - \mathcal{P}^*(p^*) - \mathcal{D}^*(\tilde{\mathbf{u}}^*) = 0. \end{cases}$$

This form is very similar to the linearised Navier-Stokes equation. The operators are different, but in term of their effect on the system, there is a one to one correspondence. For that reason, it is assumed that a Lagrange identity with a specific right-hand side

can be found. it has the form:

$$\begin{aligned} & \left[\left(\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \mathcal{L}_1(\tilde{\mathbf{u}}) + \mathcal{L}_2(\tilde{\mathbf{u}}) + \mathcal{P}(p) + \mathcal{D}(\tilde{\mathbf{u}}) \right) \cdot \tilde{\mathbf{u}}^* + \mathcal{M}(\tilde{\mathbf{u}}) \cdot p^* \right] + \\ & \left[\left(\frac{\partial \tilde{\mathbf{u}}^*}{\partial t} + \mathcal{L}_1^*(\tilde{\mathbf{u}}^*) + \mathcal{L}_2^*(\tilde{\mathbf{u}}^*) + \mathcal{P}^*(p^*) + \mathcal{D}^*(\tilde{\mathbf{u}}^*) \right) \cdot \tilde{\mathbf{u}} + \mathcal{M}^*(\tilde{\mathbf{u}}^*) \cdot p \right] \\ & = \frac{\partial \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}^*}{\partial t} + \underline{\nabla} \cdot (\mathcal{J}_1(u_c, u_c^*) + \mathcal{J}_2(u_c, u_c^*) + \mathcal{J}_p + \mathcal{J}_d) \end{aligned}$$

In this equation, the terms $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_p$ and $\mathcal{J}_d$ correspond to the operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{P}$, $\mathcal{D}$ and their adjoints. To derive the adjoint operator, each of them will be derived independently from the other. One of the purpose is to have $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_p$ and $\mathcal{J}_d$ equal to zero at the boundary of the domain. This is the condition to be able to apply the Green's theorem the same way as in chapter 3. It will be seen that it is this part which is going to be the origin of the Dirichlet boundary conditions for the adjoint velocity $\tilde{\mathbf{u}}^* = (u_c^*, v_c^*, w_c^*, u_s^*, v_s^*, w_s^*)$, and absence of boundary condition for adjoint pressure $p^* = (p_c^*, p_s^*)$.

Each operator of the adjoint equations is now derived, as well as the corresponding right hand side in the Lagrange identity. The adjoint operator is given so that the right hand side can be written as the form of a divergence.

Inversion of $\mathcal{L}_1$

The part of the Lagrange identity dependent on $\mathcal{L}_1$ is given by the equation:

$$\mathcal{L}_1(\tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}}^* + \mathcal{L}_1^*(\tilde{\mathbf{u}}^*) \cdot \tilde{\mathbf{u}} = \nabla \cdot J_1$$

To find $\mathcal{L}_1^*$ and J_1 , the common equality is used:

$$\frac{\partial f g}{\partial x_i} = f \frac{\partial g}{\partial x_i} + g \frac{\partial f}{\partial x_i}.$$

Doing so, it is easy to write each term of the scalar product $\mathcal{L}_1^*(\tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}}^*$ as the sum of the derivative of a function (right hand side) and an other function which does not

involve derivatives of $\tilde{\mathbf{u}}$. applying that to all the components, one can obtain:

$$\mathcal{L}^*_1(\tilde{\mathbf{u}}) = \begin{pmatrix} \frac{\partial U u_c^*}{\partial x} + W \beta_z u_s^* \\ \frac{\partial U v_c^*}{\partial x} + W \beta_z v_s^* \\ \frac{\partial U w_c^*}{\partial x} + W \beta_z w_s^* \\ \frac{\partial U u_s^*}{\partial x} - W \beta_z u_c^* \\ \frac{\partial U v_s^*}{\partial x} - W \beta_z v_c^* \\ \frac{\partial U w_s^*}{\partial x} - W \beta_z w_c^* \end{pmatrix}, \quad J_1 = \begin{pmatrix} u_c U u_c^* + v_c U v_c^* + w_c U w_c^* + u_s U u_s^* + v_s U v_s^* + w_s U w_s^* \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{aligned} \mathcal{L}_1(\tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}}^* + \mathcal{L}^*_1(\tilde{\mathbf{u}}^*) \cdot \tilde{\mathbf{u}} &= \frac{\partial u_c U u_c^*}{\partial x} + \frac{\partial v_c U v_c^*}{\partial x} + \frac{\partial w_c U w_c^*}{\partial x} \\ &\quad + \frac{\partial u_s U u_s^*}{\partial x} + \frac{\partial v_s U v_s^*}{\partial x} + \frac{\partial w_s U w_s^*}{\partial x}. \end{aligned}$$

Inversion of $\mathcal{L}_2$

A similar method is used as for the case of $\mathcal{L}_1$. The results obtained are:

$$\mathcal{L}_2(\tilde{\mathbf{u}}) \cdot \tilde{\mathbf{u}}^* + \mathcal{L}^*_2(\tilde{\mathbf{u}}^*) \cdot \tilde{\mathbf{u}} = \nabla \cdot J_2(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}^*),$$

$$\mathcal{L}^*_2(\tilde{\mathbf{u}}) = \begin{pmatrix} -u_c^* \frac{\partial U}{\partial x} - w_c^* \frac{\partial W}{\partial x} \\ -u_c^* \frac{\partial U}{\partial y} - w_c^* \frac{\partial W}{\partial y} \\ 0 \\ -u_s^* \frac{\partial U}{\partial x} - w_s^* \frac{\partial W}{\partial x} \\ -u_s^* \frac{\partial U}{\partial y} - w_s^* \frac{\partial W}{\partial y} \\ 0 \end{pmatrix}, \quad \mathcal{J}^*_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Inversion of the diffusive term

$$\mathcal{D}(\tilde{\mathbf{u}})\tilde{\mathbf{u}}^* + \mathcal{D}^*(\tilde{\mathbf{u}}^*).\tilde{\mathbf{u}} = \underline{\nabla}.\mathcal{J}_d(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}^*)$$

On one of the components, for example the part of the scalar product resulting from the projection on $\mathbf{e}_{x,c}$:

$$\mathcal{D}^*(\tilde{\mathbf{u}}^*).\mathbf{e}_{x,c} = - \left(\frac{\partial^2 u_c^*}{\partial x^2} + \frac{\partial^2 u_c^*}{\partial y^2} - \beta_z u_c^* \right)$$

The part of the operator $\mathcal{J}_d$ corresponding to this term is:

$$\frac{\partial}{\partial x} \left(u_c^* \frac{\partial u_c}{\partial x} - u_c \frac{\partial u_c^*}{\partial x} \right) \mathbf{e}_x$$

Inversion of the pressure term

For the pressure, coupling between the pressure gradient in the momentum equations and the mass conservation equations has to be kept. Therefore the operators $\mathcal{P}$ and $\mathcal{M}$ and their respective adjoints are considered together. the following equality has to be found:

$$\mathcal{P}(p).\tilde{\mathbf{u}}^* + \mathcal{M}(\tilde{\mathbf{u}}).p^* + \mathcal{P}^*(p^*).\tilde{\mathbf{u}} + \mathcal{M}^*(\tilde{\mathbf{u}}^*).p = \underline{\nabla}.\mathcal{J}_p,$$

Which is satisfied if the adjoint operators are defined as:

$$\mathcal{M}^*(\tilde{\mathbf{u}}) = \begin{pmatrix} \frac{\partial u_c^*}{\partial x} + \frac{\partial v_c^*}{\partial y} + \beta_z w_s^* \\ \frac{\partial u_s^*}{\partial x} + \frac{\partial v_s^*}{\partial y} - \beta_z w_c^* \end{pmatrix}, \mathcal{P}^*(p) = \begin{pmatrix} \frac{\partial p_c^*}{\partial x} \\ \frac{\partial p_c^*}{\partial y} \\ \beta_z p_s^* \\ \frac{\partial p_s^*}{\partial x} \\ \frac{\partial p_s^*}{\partial y} \\ -\beta_z p_c^* \end{pmatrix}, \mathcal{J}_p = \begin{pmatrix} u_c p_c^* + u_s p_s^* + u_c^* p_c + u_s^* p_s \\ v_c p_c^* + v_s p_s^* + v_c^* p_c + v_s^* p_s \\ 0 \end{pmatrix}.$$

Boundary conditions and final comments

Now that an explicit form for each operator of the adjoint equations has been found, the boundary conditions for the adjoint problem have to be prescribed. Looking at the different components of the right hand side derived above, it can be seen that a simple way to obtain a surface integral equal to zero after application of the Green theorem (see equation (3.22)) is to prescribe the same boundary conditions for the adjoint problem as for the linear problem. The two velocities $\mathbf{u}_c^*$ and $\mathbf{u}_s^*$ have a no slip condition at the walls ($y = \pm l_y/2$) and a periodic boundary condition in the streamwise direction x . For the pressure p^* , no boundary condition is needed.

With the boundary conditions now prescribed for the adjoint equations, it is known that after application of the Green theorem, the equality of equation (3.22) holds. Therefore there is no need to work on the right hand side of the Lagrange identity anymore, and the adjoint equations as well as their boundary conditions can be considered. The complete form of these adjoint equations is displayed in equations (3.16)-(3.18). The diffusive operator change of sign is important, as it is this part of the equations which makes that the adjoint equations has to be solved backward in time, and not forward as the linearised Navier-Stokes equations.

Appendix B

Generalised optimal perturbations in harmonic wall forcing

Some figures that were not shown in chapter 6 are displayed here, so that the reader can have all the necessary data to understand all the cases studied. A first set of figures shows the necessary data to assess the validity of the "universal optimal" perturbation idea. In a second set of figures, the dependence on the Reynolds number is displayed for the few cases where it was studied.

B.1 Universal optimal perturbation

The three main figures that led to the idea of the presence of a universal optimal perturbation are shown. First, with the angle representation, it was seen that the angle remains relatively invariant during half a period. Then, the temporal energy growth curve showed that generalised optimal perturbations corresponding to different observation time t_f have similar growth patterns. Finally, the comparison of initial time and observation time of each generalised optimal perturbation where it was seen that for a wide range of observation time, the initial time lies in a narrow area. These figures are displayed here for all the configurations studied. The idea of the presence of a universal optimal seems convincing for all the cases, except for the case with a period of oscillations $T^+ = 100$ and an observation at a distance $y^+ = 6$ from the wall.

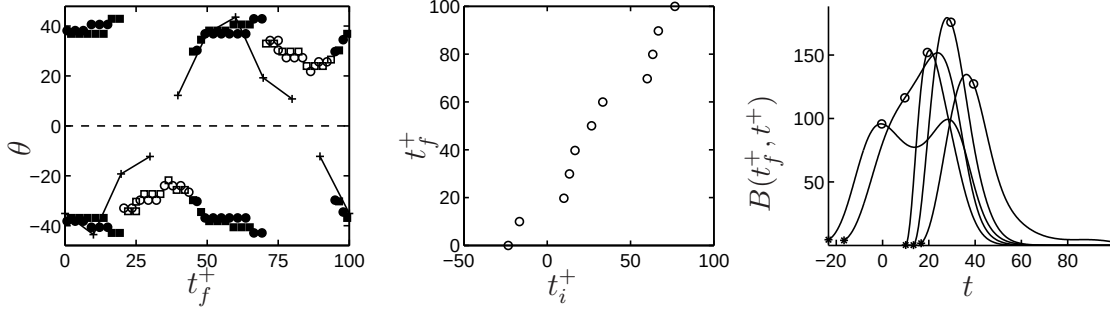


Figure B.1: Properties of the generalised optimal perturbation s optimum at different observation times t_f . Case $y^+ = 6$, $T^+ = 100$.

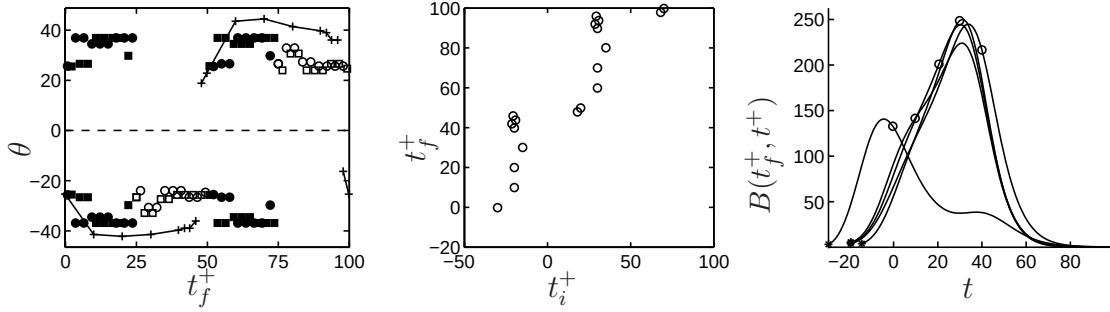


Figure B.2: Properties of the generalised optimal perturbation s optimum at different observation times t_f . Case $y^+ = 11$, $T^+ = 100$.

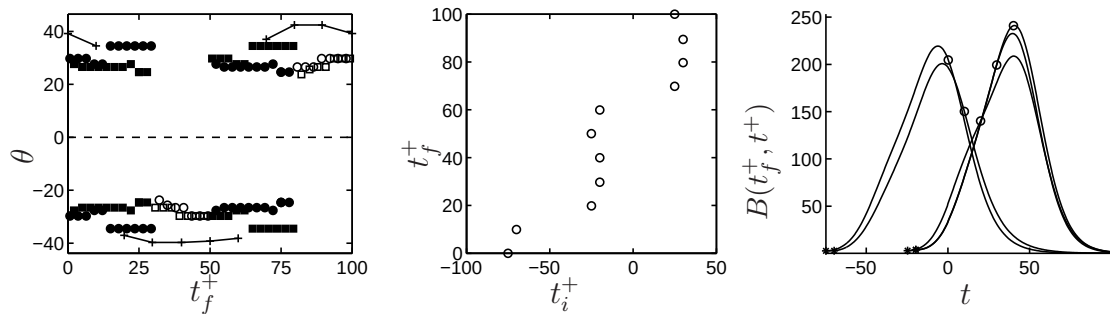


Figure B.3: Properties of the generalised optimal perturbation s optimum at different observation times t_f . Case $y^+ = 16$, $T^+ = 100$.

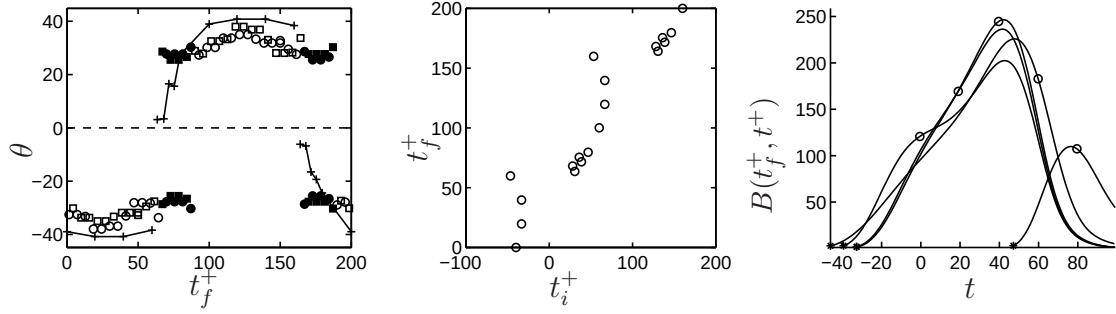


Figure B.4: Properties of the generalised optimal perturbation s optimum at different observation times t_f . Case $y^+ = 6$, $T^+ = 200$.

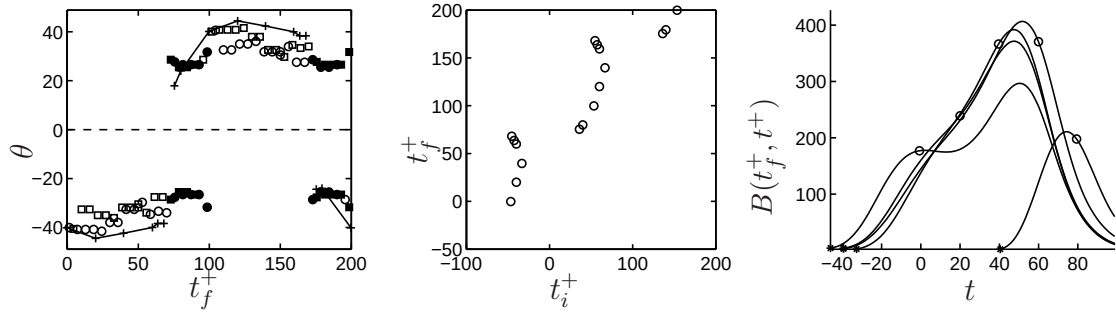


Figure B.5: Properties of the generalised optimal perturbation s optimum at different observation times t_f . Case $y^+ = 11$, $T^+ = 200$.

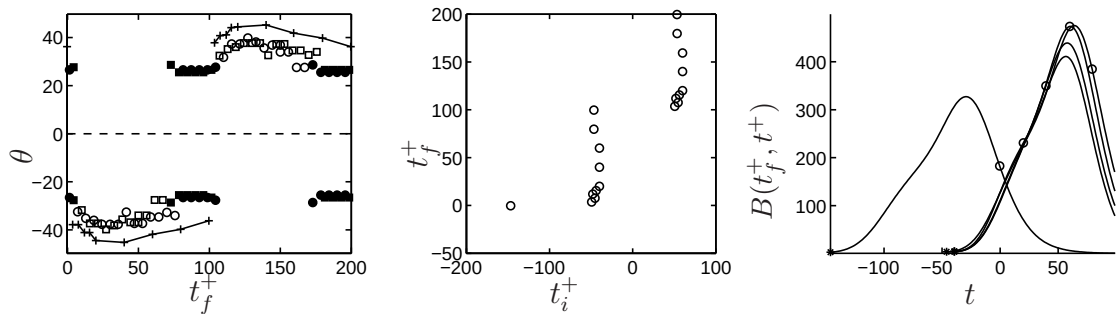


Figure B.6: Properties of the generalised optimal perturbation s optimum at different observation times t_f . Case $y^+ = 16$, $T^+ = 200$.

B.2 Effect of the Reynolds number

The effect of the Reynolds number was not studied in this thesis, and it was assumed that the linear predictions are mostly independent of the Reynolds number if they are performed close enough to the wall. However, some preliminary tests were performed. The adjoint optimisation were performed on maps in (λ_z, t_i) at a Reynolds number $Re_\tau = 500$. The maps would be visually difficult to compare. For that reason, a similar approach to the one used for the streaks spacing calculation is used. The graphs show the maximum of the map in the t_i direction for each λ_z as a function of the spanwise wavelength λ_z . The maximum of the curve corresponding to the case $Re_\tau = 180$ is the quantity that defined the streaks spacing in chapter 6. The direct numerical simulation premultiplied spectrums are also showed in these figures, as they were the quantities used to define the direct numerical simulation streaks spacing.

Each figure shows several curves. In each case, the curves presented are scaled by a given factor. Each figure corresponds to one wall oscillation frequency, and one observation layer. All the graphs are scaled so that the maximum of the generalised optimal perturbation curve at $Re_\tau = 180$ matches the maximum of the direct numerical simulation curve at the same instant. The generalised optimal perturbation results at $Re_\tau = 500$ use the same scaling factor as for the case $Re_\tau = 180$, but multiplied by $180/500$. In some of the curves, the presence of two local maxima is due to the existence of two peaks in the map energy plot.

For some cases, the generalised optimal perturbation results at $Re_\tau = 180$ and $Re_\tau = 500$ are similar, but for some cases they also differ by a large amount. One of the issue could be the fact that the simulations are run on different grids, and with then different approximation errors. Another issue is that the map simulations in the case $Re_\tau = 500$ contained less points than the same map at $Re_\tau = 180$. Overall, these graphs suggest that if there is a dependence on the Reynolds number, it might not be significant. However, these simulations were not sufficient to provide strong conclusion, and a more refined study would be necessary.

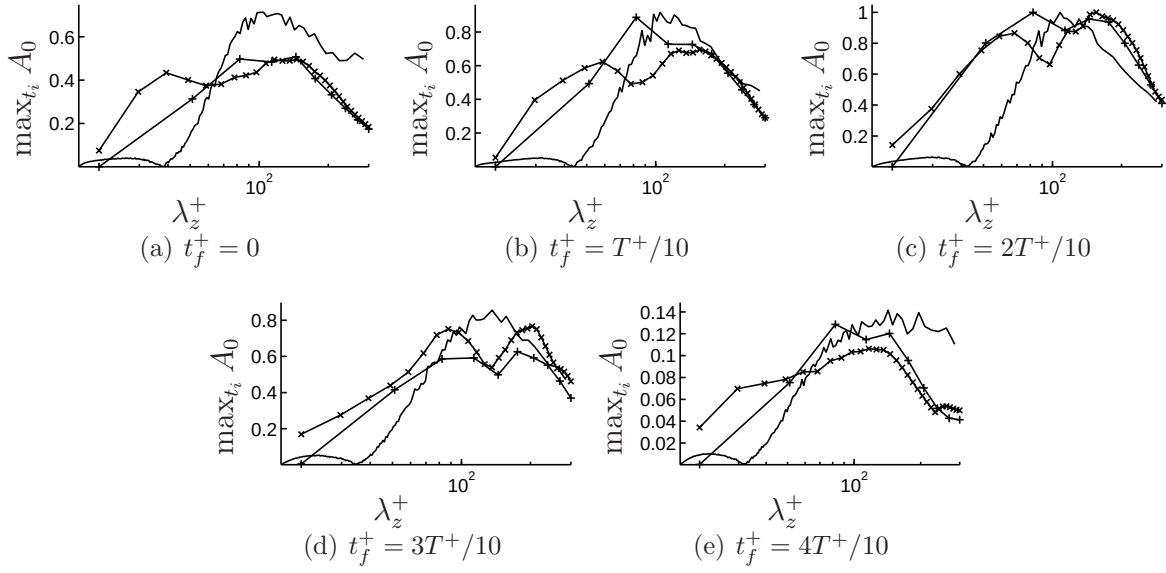


Figure B.7: (—) Generalised optimal perturbation at $R_\tau = 180$, (—+) generalised optimal perturbation at $R_\tau = 500$, (---) premultiplied energy spectrum of the direct numerical simulation at $R_\tau = 500$. Period of oscillation $T^+ = 200$, and distance to the wall $y^+ = 06$.

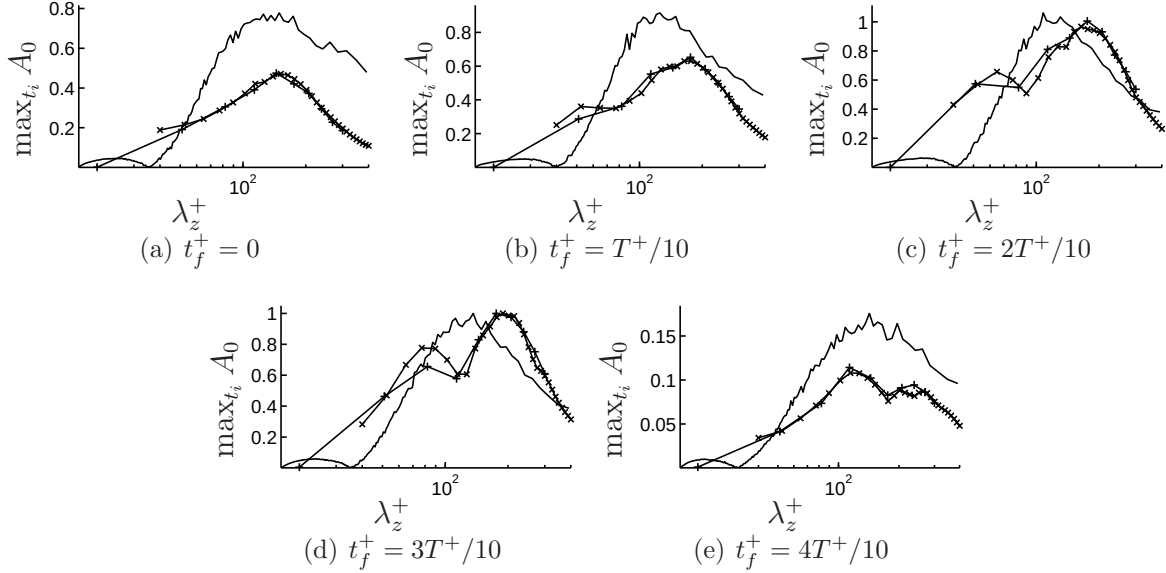


Figure B.8: (—) Generalised optimal perturbation at $R_\tau = 180$, (—+) generalised optimal perturbation at $R_\tau = 500$, (---) premultiplied energy spectrum of the direct numerical simulation at $R_\tau = 500$. Period of oscillation $T^+ = 200$, and distance to the wall $y^+ = 11$.

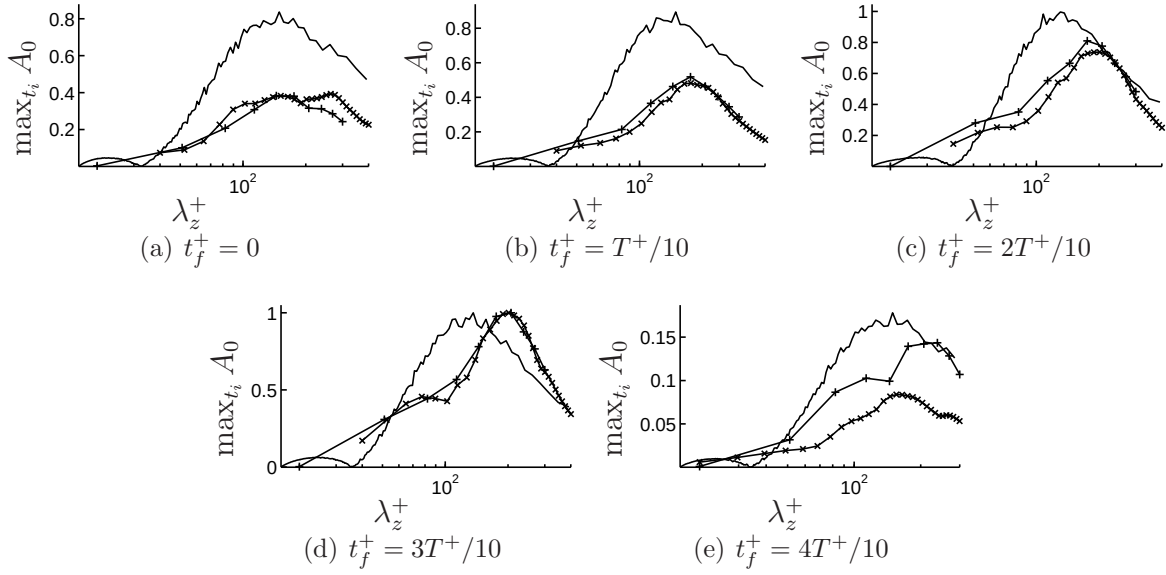


Figure B.9: (—) Generalised optimal perturbation at $R_\tau = 180$, (—+) generalised optimal perturbation at $R_\tau = 500$, (---) premultiplied energy spectrum of the direct numerical simulation at $R_\tau = 500$. Period of oscillation $T^+ = 200$, and distance to the wall $y^+ = 16$.

Appendix C

Conditional average

In chapter 6, comparison between generalised optimal perturbations and conditional averaged direct numerical simulation structures were performed. Only two representative cases were showed there. Here, all the comparisons available are showed. The conditional sampling from the direct numerical simulation results was done for the case with a wall oscillation period $T^+ = 200$, and at a distance to the wall $y^+ = 11$, and on 36 slices in phase. The generalised optimal perturbations are compared to the conditional average pictures corresponding to the closest observation time. In these figure, it can be noticed that the left/right symmetry over half a period is exact for the generalised optimal perturbation, but that due to convergence errors it is not as well satisfied for the direct numerical simulation pictures. Overall, the conclusions made by the observation of these figures are the same as the one made in chapter 6.

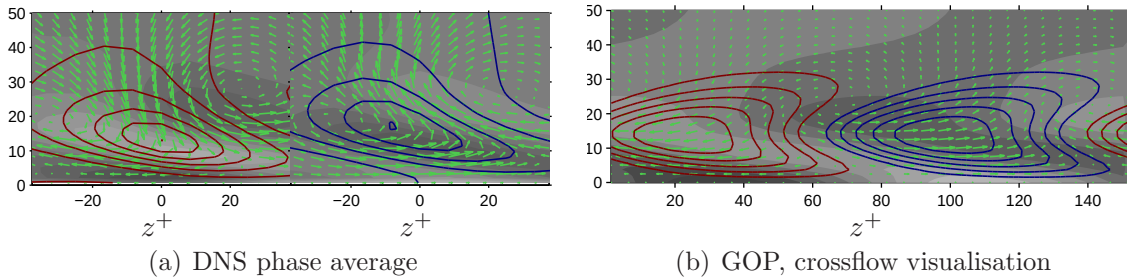


Figure C.1: Time of observation: $t_{GOP}^+ = t_f^+ = -0.72$, $t_{DNS}^+ = 0.00$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

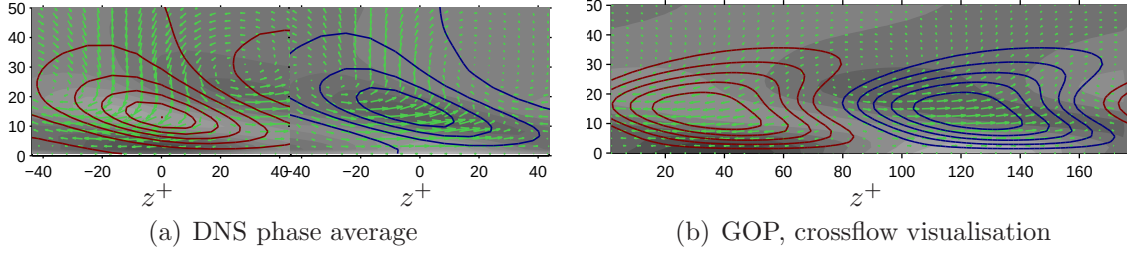


Figure C.2: Time of observation: $t_{GOP}^+ = t_f^+ = 19.44$, $t_{DNS}^+ = 16.67$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

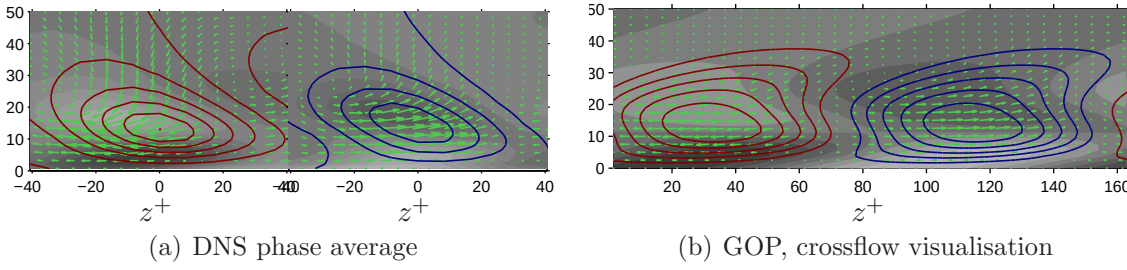


Figure C.3: Time of observation: $t_{GOP}^+ = t_f^+ = 39.06$, $t_{DNS}^+ = 38.89$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

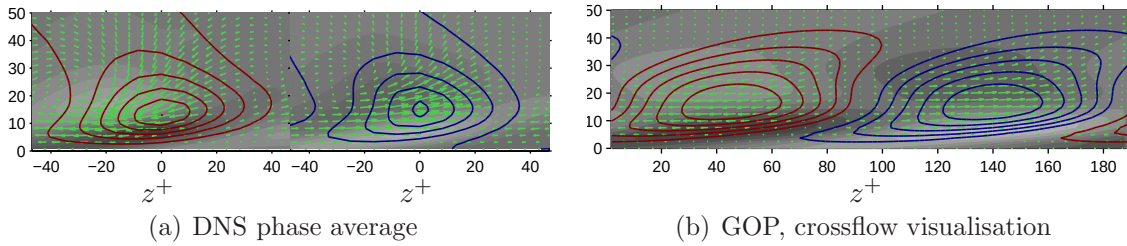


Figure C.4: Time of observation: $t_{GOP}^+ = t_f^+ = 59.40$, $t_{DNS}^+ = 61.11$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

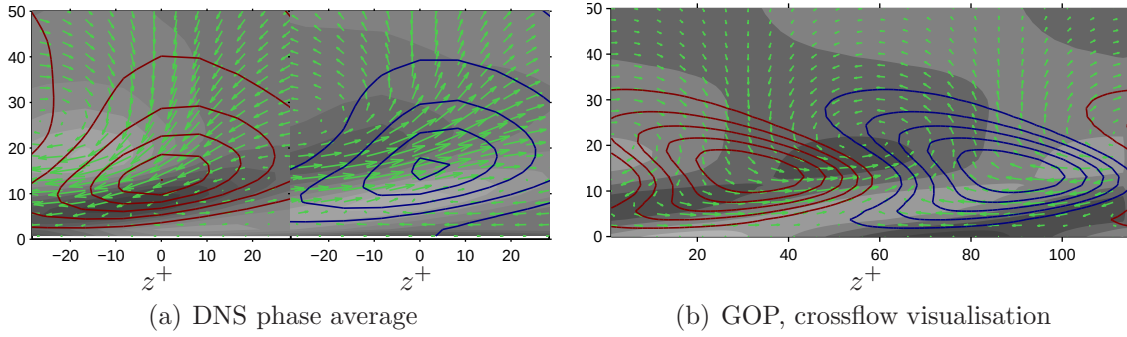


Figure C.5: Time of observation: $t_{GOP}^+ = t_f^+ = 79.38$, $t_{DNS}^+ = 77.78$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

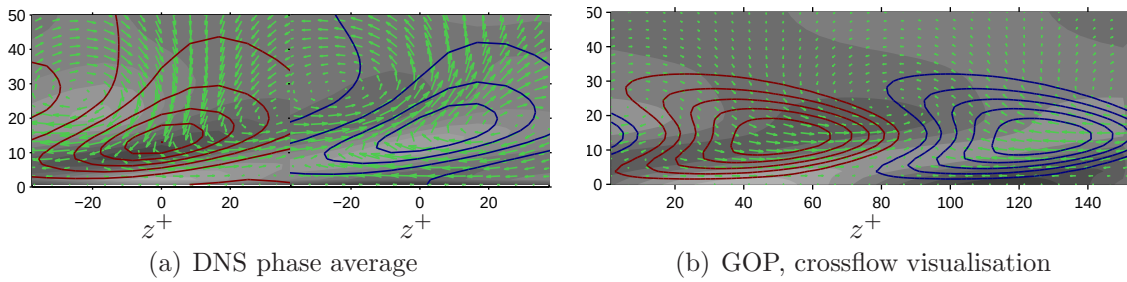


Figure C.6: Time of observation: $t_{GOP}^+ = t_f^+ = 99.28$, $t_{DNS}^+ = 100.00$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

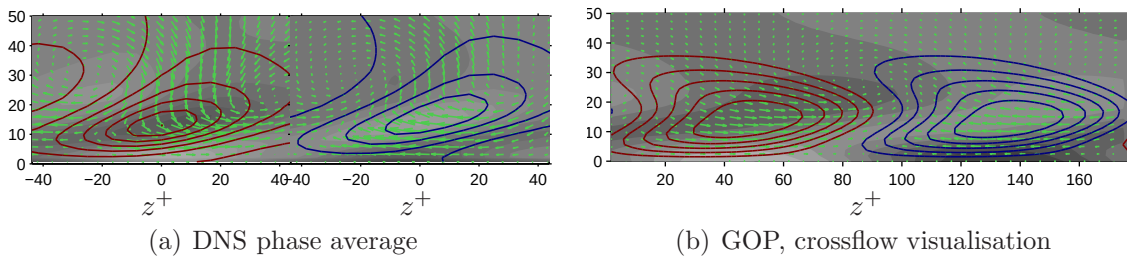


Figure C.7: Time of observation: $t_{GOP}^+ = t_f^+ = 119.44$, $t_{DNS}^+ = 116.67$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

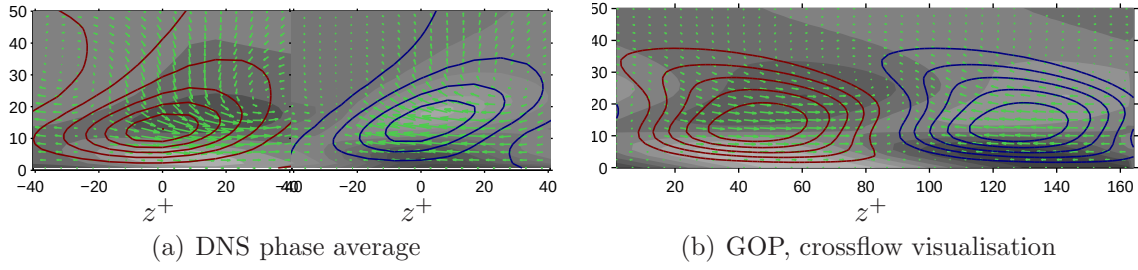


Figure C.8: Time of observation: $t_{GOP}^+ = t_f^+ = 139.06$, $t_{DNS}^+ = 138.89$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

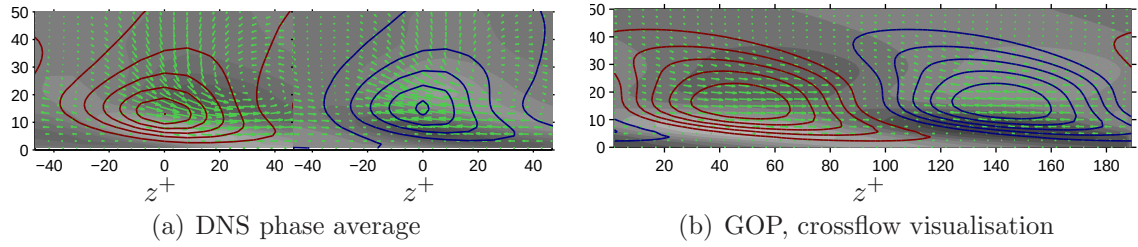


Figure C.9: Time of observation: $t_{GOP}^+ = t_f^+ = 158.86$, $t_{DNS}^+ = 161.11$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

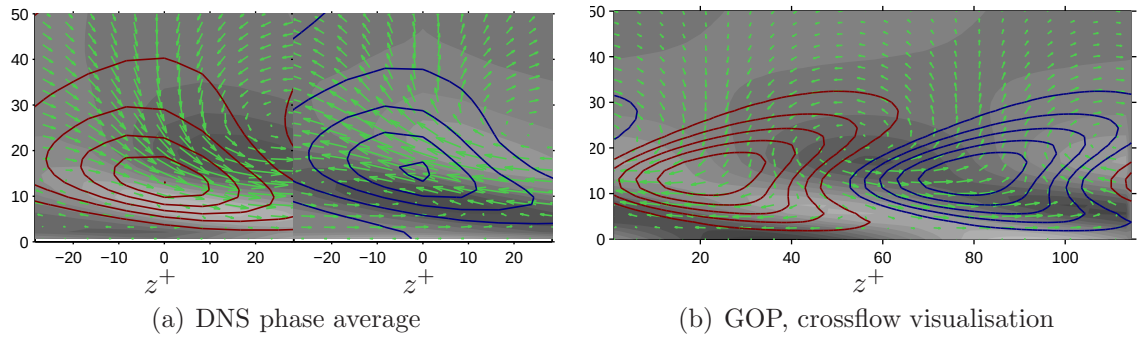


Figure C.10: Time of observation: $t_{GOP}^+ = t_f^+ = 178.84$, $t_{DNS}^+ = 177.78$. Distance of the observation layer from the wall: $y_{DNS}^+ = 13.02$, $y_{GOP}^+ = 11.25$.

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