

Nonlinear stability of prestressed stayed columns with multiple crossarm systems

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Declaration

I confirm that this thesis is my own work and that any material from published or unpublished work from others is appropriately referenced.

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Abstract

Prestressed stayed steel columns are increasing in popularity in the construction industry. This type of structure possesses a considerably higher buckling resistance as compared to the conventional column without a commensurate increase in self-weight, owing to the extra stiffnesses provided by the system of crossarms and cable stays.

Most of the previous literature focused on the buckling load of the stayed column while systematic studies on the post-buckling behaviour of the single-bay column have only been conducted more recently. Although the significance of the post-buckling behaviour of such structures has already been demonstrated, certain relevant topics have not been satisfactorily studied for the stayed columns with multiple crossarm systems, which potentially possess even higher buckling resistances as compared to the single crossarm cases. Therefore, the aim of the current thesis is to reveal the post-buckling behaviour of the prestressed stayed column with multiple crossarms along the length.

A finite element model of a prestressed stayed column with multiple crossarms along the length is developed with the initial imperfections and the prestress taken into consideration. Interactive buckling is found to be the most dangerous case particularly when the critical buckling mode is anti-symmetric. It is found that the post-buckling behaviour is strongly linked to the actual geometric arrangement of the structure. Sensitivities to the levels of the prestress and the initial imperfections are also presented. The observations obtained from the FE model are first validated qualitatively by a simplified analytical model. The

simplified model also reveals the process for how the pure buckling modes interact during the post-buckling stage.

To compare with the results of the FE model quantitatively, a more sophisticated analytical model is subsequently developed. The total potential energy of the system is formulated in conjunction with the Rayleigh–Ritz method. The slackening of the stays is also successfully simulated. An identical observation, in that interactive buckling governs the post-buckling response when anti-symmetric buckling is critical, is obtained also. Excellent quantitative agreement is found between the results from the FE and the more sophisticated analytical model. Parametric studies on the geometric configurations where interactive buckling is most significant are also presented.

Finally, a large number of separate cases with different structural geometries and levels of prestress are studied. The actual optimal prestress, which accounts for the load carrying capacity of the column and the required material resistances of different structural components, is determined for the triple-bay stayed column. This demonstrates that the nonlinear response of the stayed column system is practically important from a safety as well as an efficiency perspective.

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Nomenclature

Coordinates, Stress, Strain, Loads and Energy

$\tilde{A}$	Diagonalized normalized total potential energy
F_{ie}, F_{fe}	Initial and final compressive forces in the edge crossarms respectively
F_{im}, F_{fm}	Initial and final compressive forces in the middle crossarm respectively
f_y	Yield stress of the column and the crossarms
$f_{y,req}, f_{s,req}$	Required material resistances of the column and the stays respectively
P_i, P_f	Initial and final axial forces in the column respectively
P_a	External load applied axially to the column
P^C	Critical buckling load of the column
P_1^C, P_2^C	Buckling loads of Modes 1 and 2 respectively
P_y	Squash load of the column
P_E	Euler buckling load of the column
p	Normalized axial load
p_1^C, p_2^C, p_3^C	Normalized buckling load of Modes 1, 2 and 3 respectively
S_i	Stay slackening switch
T	Initial prestressing force in the stays

T_i, T_f	Initial and final prestressing forces in the stays respectively
T_{opt}	Theoretically optimal prestressing force from linear analysis
U	Total strain energy stored in the system
U_{si}	Strain energy stored in stay i
U_{ca}	Axial energy of the column
U_{cb}	Bending energy of the column
$U_{\text{Ri}}, U_{\text{Li}}$	Strain energy of the crossrams on the right-hand side and left-hand side of the column respectively
U_s	Strain energy of all the stays
V	Total potential energy of the system
$\tilde{V}$	Normalized total potential energy of the system
$\tilde{V}_{\text{m1}}, \tilde{V}_{\text{m2}}, \tilde{V}_{\text{int}}$	Normalized total potential energy for Modes 1, 2 and 3 respectively
$\tilde{W}$	Non-diagonalized normalized total potential energy
x	Longitudinal direction along the column
y	Transverse direction along the crossarm
z	Transverse direction out of plane
ϵ_T	Pre-existing strain of the stay introduced by the prestress
ϵ_i	Total strain of the stay
σ_c, σ_s	Total stresses in the column and the stays respectively

Structural geometries and material properties

A_a	Cross-sectional area of the column
A_e	Cross-sectional area of the edge crossarms
A_m	Cross-sectional area of the middle crossarm
A_s	Cross-sectional area of the stays
a_e	Length of the edge crossrams

a_m	Length of the middle crossram
c_0, c	Bending stiffnesses of the middle and the edge rotational springs respectively
E_a	Young's Modulus of the crossarms when $E_e = E_m$
E_c	Young's Modulus of the column
E_e	Young's Modulus of the edge crossarm
E_m	Young's Modulus of the middle crossarm
E_s	Young's Modulus of the stays
K_c	Axial stiffness of the column
K_e	Axial stiffness of the edge crossarms
K_m	Axial stiffness of the middle crossarm
K_{s2}	Axial stiffness of stay 2
K	Normalized longitudinal stiffness of the middle crossarm
k_0, k	Longitudinal stiffnesses of the edge and the middle springs respectively
L	Main column length
l_i	Original stay length
l'_i	Current stay length
R	Rotational stiffness ratio
α_e, α_m	Column aspect ratio for the edge and the middle crossarms respectively
β_e	Angle between stay 1 and the vertical direction
β_m	Angle between stay 2 and the vertical direction
γ	Crossarm length ratio a_e/a_m
$\bar{\lambda}$	Normalized slenderness of the column
ρ	Longitudinal stiffness ratio
ϕ_{co}, ϕ_{ao}	Outside diameter of the column and the crossarms respectively
ϕ_{ci}, ϕ_{ai}	Inside diameter of the column and the crossarms respectively

ϕ_s Stay diameter

Kinematic quantities and degrees of freedom

Q_i	Generalized coordinate for the main column
Q_{L1}, Q_{L2}, Q_{L3}	Tip displacements of the left-hand side crossarms at one-quarter, one-half and three-quarter length points of the column
Q_{R1}, Q_{R2}, Q_{R3}	Tip displacements of the right-hand side crossarms at one-quarter, one-half and three-quarter length points of the column
u_1, u_2, u_3	Diagonalized coordinates for Modes 1, 2 and 3 respectively
$W(x)$	Shape function of the column
$W_1(x), W_2(x)$	Shape functions of the column affine to Modes 1 and 2 respectively
$W_{Li}(y)$	Shape functions of the left-hand side crossarms
$W_{Ri}(y)$	Shape functions of the right-hand side crossarms
Δ_c	End shortening of the column caused by pure compression
Δ_e	Elongation of the edge crossarms
Δ_m	Elongation of the middle crossarm
Δ_{s2}	Change in length of stay 2
$\delta_1, \delta_2, \delta_3$	Normalized horizontal deflections of the column at one-quarter, one-half and three-quarter length points respectively
δ_m	Midspan deflection of the column
$\mathcal{E}$	Column end shortening
$\mathcal{E}_0$	First order approximation of the column end shortening
$\mathcal{E}_1, \mathcal{E}_2$	Column end shortening for Modes 1 and 2 respectively
ζ_1, ζ_3	Lateral deflections at midspan for Modes 1 and 3 respectively
η, η_s	Efficiency indicators of the column and the stays respectively

$\theta_1, \theta_2, \theta_3$	Rotations of the column at one-quarter, one-half and three-quarter length points respectively
$\theta_{a1}, \theta_{a2}, \theta_{a3}$	Rotations of the crossarms at one-quarter, one-half and three-quarter length points of the column respectively
$\theta_B, \theta_C, \theta_D$	Rotations of the pins at points B, C and D
κ	Calibration factor for the rotation of the crossarms

Imperfections

e_1, e_2	Scale factors for the imperfections comprising the symmetric and the anti-symmetric mode respectively for the discrete analytical model
δ	Normalized amplitude of the initial out-of-straightness of the column
μ_1, μ_2	Scale factors for the imperfections comprising the symmetric and the anti-symmetric mode respectively for the FE model

Chapter 1

Introduction

Prestressed stayed columns, which are usually made from tubular steel that are reinforced by external cable stays, are being increasingly used nowadays. Conventional (unstayed) columns are highly vulnerable to buckling behaviour due to their slenderness. To improve matters in terms of load carrying capacity, a system of crossarm members and prestressed cable stays can be introduced to provide significant extra axial strength without a commensurate increase in self-weight, as shown in Figure 1.1. This type of structure offers an innovative, aesthetic and practical solution to the problem of low critical buckling capacities in columns with highly slender geometries.

They have been employed in practice where a column needs to span a long distance but has to be of relatively light-weight construction. For example, during the construction of the Rock in Rio III stadium in Brazil (Andrade *et al.* , 2003a; Andrade *et al.* , 2003b), such columns were used to prop the incomplete stadium roof while the construction was completed, as shown in Figure 1.2. The columns were constructed and prestressed on site such that they eliminated the need for using expensive shoring systems that would have increased the time and cost of the construction significantly.

There are also permanent real-world applications of prestressed stayed columns with mul-

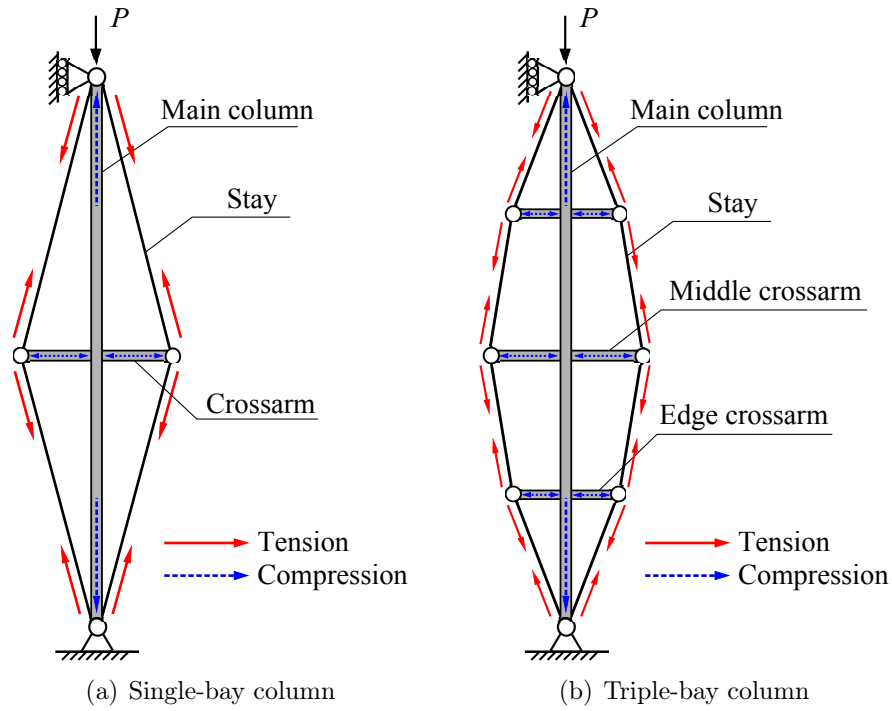


Figure 1.1: Columns reinforced by cable stays with different crossarm systems.



Figure 1.2: The single-bay prestressed stayed column used as the temporary support during the construction of the Rock in Rio III stadium (Andrade *et al.* , 2003a; Andrade *et al.* , 2003b).

multiple crossarm systems. Figures 1.3(a) and (b) show the uses of two-dimensional triple-bay



(a) Chiswick Park, London, UK



(b) London Waterloo Station, UK



(c) Parkland Mall, Dalian, China

Figure 1.3: Applications of prestressed stayed columns in the construction industry; photographs (a) and (b) were taken by Daisuke Saito; photograph (c) was taken by the author.

columns with a highly slender main column element. Figure 1.3(c) shows the use of a three-dimensional slender column with a multiple crossarm system.

Research on such structural components can be dated from the 1960s. The initial research focus was mainly on the critical buckling load (Chu & Berge, 1963; Mauch, 1967; Ellis, 1971; Pearson, 1971; Smith *et al.*, 1975; Temple, 1977; Belenya, 1977; Howson & Williams, 1980; Howson & Williams, 1984). Effects of different levels of prestress (Hafez *et al.*, 1979) and initial imperfections were then studied (Wong & Temple, 1982; Temple *et al.*, 1984). The nonlinear behaviour of prestressed stayed columns has only been studied extensively relatively recently (Saito & Wadee, 2008; Saito & Wadee, 2009a; Saito & Wadee, 2010; Osofero *et al.*, 2012). These more recent works demonstrated the significance of the post-buckling behaviour of such structures.

Most existing studies on such columns have been focused on single-bay crossarm stayed systems while multiple crossarm cases have not been satisfactorily investigated particularly

in terms of nonlinear behaviour. However, it has been shown that the load carrying capacities of such columns should theoretically be able to benefit from additional crossarms. In addition, it was indicated that for single-bay prestressed stayed columns it could be more dangerous when several buckling modes are triggered approximately simultaneously rather than separately (Saito & Wadee, 2009b; Wadee *et al.* , 2013a). Therefore, it is similarly necessary to investigate the post-buckling behaviour and to include a study of potential interactive buckling for prestressed stayed columns with multiple bays.

1.1 Research aims and objectives

The key aims of this research project are:

1. To reveal the behaviour and provide a detailed understanding of the post-buckling response, including modal interactions, of prestressed stayed columns with a multiple crossarm system.
2. To use the conclusions of the aforementioned study as a basis for determining the actual optimal configuration of these structures such that they perform safely and efficiently.

In order to study the nonlinear behaviour of stayed columns, three types of model are used in this work. A finite element model is devised using the commercial software ABAQUS (2014). The buckling and the post-buckling behaviour, including modal interactions and the sensitivities to the prestress and imperfections are studied. This is used as a basis to determine the optimal prestressing force that should be introduced in actual stayed columns with multiple crossarms.

A simplified nonlinear analytical model comprising rigid links and springs based on total potential energy principles is formulated and the governing equations are solved by the

software AUTO-07P (Doedel & Oldeman, 2011). The process of how the buckling modes interact is revealed in detail. The observations are used to validate with the conclusions from ABAQUS qualitatively.

A more sophisticated analytical model using the Rayleigh–Ritz method is also developed and the governing equations are solved by AUTO-07P, this time to validate the results from the finite element model quantitatively. Sensitivities to the prestress are also studied. This is used as the basis to determine the parametric space where mode interaction is most significant for the multiple-bay stayed column. This is followed by an optimization study where the most efficient configurations and prestress levels are evaluated.

1.2 Basic stability concepts

1.2.1 Post-buckling response

The first systematic work on determining the post-buckling behaviour of structures was conducted by Koiter (1945). He presented a nonlinear bifurcation theory for continuous elastic systems by minimizing the total potential energy of the structure. Thompson and Hunt (1973) developed the study on the post-buckling problems by introducing a series of so-called ‘generalized coordinates’. It was assumed that a general structure may have n components that describe the post-buckling geometry. Each of these components was represented by the respective generalized coordinate Q_n ; the value of Q_n indicated the amplitude of the respective post-buckling component. Hence, the total potential energy V was written as:

$$V = V(Q_1, Q_2, Q_3, \dots, Q_n, P), \quad (1.1)$$

where P is a loading component that usually represents an axial load.

Two important axioms connecting the total potential energy to equilibrium and the stability

of an equilibrium state were formally presented:

Axiom 1 A stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the equilibrium of the system.

Axiom 2 A complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state.

So long as the structure is a conservative static system, the kinetic energy is zero and the total mechanical energy is equivalent to the total potential energy. Considering a one degree of freedom system with the generalized coordinate Q , the total potential energy V is expressed thus:

$$V(Q, P) = U(Q) - P\mathcal{E}(Q). \quad (1.2)$$

where $U(Q)$ is the strain energy stored in the structural system, P is the externally applied load and $\mathcal{E}(Q)$ is the distance that the load moves in the direction of P . The quantity $P\mathcal{E}(Q)$ represents the work done by the external load.

If a small perturbation δ is introduced in the system for an equilibrium state where $Q = Q_0$ and the load P is fixed, the total potential energy $V(Q_0 + \delta, P)$ can be expressed as a power series in δ , thus:

$$V(Q_0 + \delta, P) = V(Q_0, P) + \left. \frac{dV}{dQ} \right|_{Q_0} \delta + \frac{1}{2!} \left. \frac{d^2V}{dQ^2} \right|_{Q_0} \delta^2 + \dots + \frac{1}{n!} \left. \frac{d^n V}{dQ^n} \right|_{Q_0} \delta^n + \dots \quad (1.3)$$

According to Axiom 1, the equilibrium state is defined by the stationary total potential energy with respect to the generalized coordinates, thus:

$$\left. \frac{dV}{dQ} \right|_{Q_0} = 0. \quad (1.4)$$

Substituting Equation (1.4) into (1.3) and re-arranging, the following expression is ob-

tained:

$$V(Q_0 + \delta, P) - V(Q_0, P) = \frac{1}{2!} \frac{d^2V}{dQ^2} \Big|_{Q_0} \delta^2 + \frac{1}{3!} \frac{d^3V}{dQ^3} \Big|_{Q_0} \delta^3 + \dots + \frac{1}{n!} \frac{d^nV}{dQ^n} \Big|_{Q_0} \delta^n + \dots \quad (1.5)$$

Since the change in V must be always positive for any perturbation δ for the system to be stable (i.e. from Axiom 2, V must be a minimum for stable equilibrium), this requires the first non-zero term to be positive. This implies that $\frac{d^2V}{dQ^2} > 0$ for stable equilibrium, $\frac{d^2V}{dQ^2} < 0$ for unstable equilibrium and $\frac{d^2V}{dQ^2} = 0$ for a point of instability (or bifurcation). This principle can also be extended to systems with multiple degrees of freedom (Thompson & Hunt, 1973); the matrix of second derivatives of V (the Hessian) being positive-definite for stable equilibrium and singular at a bifurcation point (Bažant & Cedolin, 1991).

For conventional columns, the post-buckling behaviour is slightly stable (Timoshenko & Gere, 1961), as represented in Figure 1.4. Regardless of the perturbation value, the applied

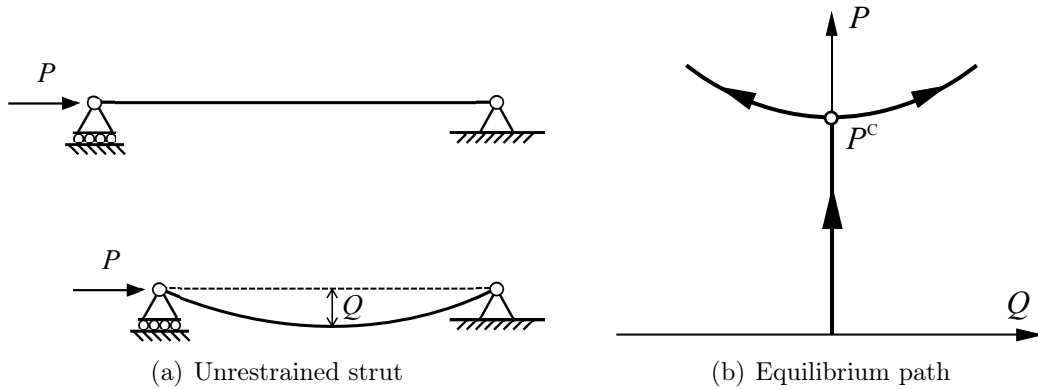


Figure 1.4: Post-buckling behaviour of a nonlinear unrestrained strut.

load P always increases, albeit marginally, for any increasing magnitude of Q . Hence, the extra attention to the post-buckling behaviour of these kind of structures is really only important in terms of structural efficiency. However, for long nonlinear columns that are continuously laterally restrained by springs (Hunt *et al.*, 1993), the post-buckling behaviour can be rather unstable, as represented in Figure 1.5. A decrease in the applied load P can be observed with the magnitude of the generalized coordinate Q increasing,

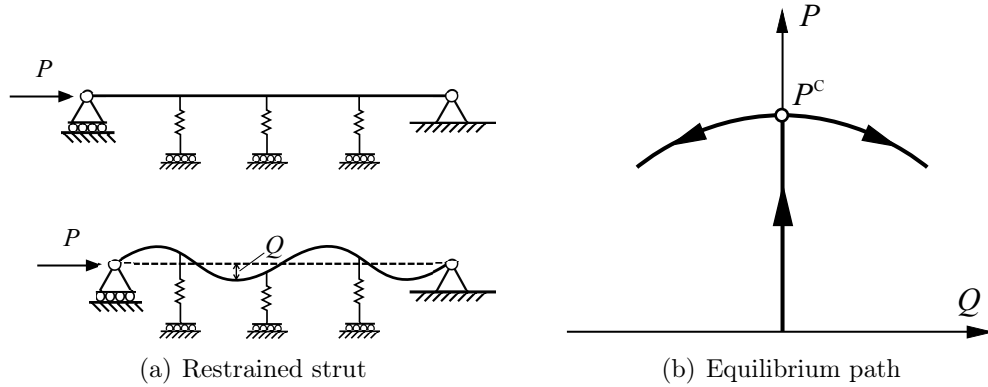


Figure 1.5: Post-buckling behaviour of a long nonlinear restrained strut on linear foundation.

indicating a potentially dangerous situation may occur once the critical buckling load P^c is reached. For the column prestressed by cable stays, the scenario is neither purely the conventional column nor the purely laterally restrained column. The intricacies of the stayed column implies that a bespoke treatment is required to evaluate the post-buckling behaviour.

1.2.2 Interactive buckling

Several buckling modes, each of which possessing a unique buckling shape, exist for structures with multiple degrees of freedom. The mode with the lowest buckling load is usually considered to be the actual buckling mode of practical interest. However, if the buckling loads are numerically close together during the post-buckling stage, the initially activated mode may transfer to another, qualitatively different, post-buckling response at a certain point. The new response may comprise the profile of more than one buckling mode, indicating it is a (not necessarily linear) combination of the relevant modes. This new response is termed as ‘mode interaction’ and the corresponding post-buckling behaviour is termed ‘interactive buckling’.

Supple (1967) presented a method that analytically revealed certain mode interaction phenomena for the conservative static system with two degrees of freedom. Two non-

dimensional generalized coordinates u_1 and u_2 were assumed to describe the buckled structural geometry. The total potential energy of the system was written as a function of the generalized coordinates u_i and the applied load p , thus:

$$V = V(u_i, p), \quad (1.6)$$

where $i = 1, 2$. The two equilibrium equations were presented by invoking Axiom 1:

$$\frac{\partial V}{\partial u_1} = 0, \quad \frac{\partial V}{\partial u_2} = 0. \quad (1.7)$$

Symmetry was imposed to the total potential energy, i.e. $V(u_1, u_2, p) = V(-u_1, u_2, p) = V(u_1, -u_2, p)$, and so the two leading orders of V were quadratic and quartic in u_i . The equilibrium equations could then be expressed by introducing a small perturbation to the load δ_p and could be simplified thus:

$$\begin{aligned} V_{11}u_1 + \frac{1}{3!} (V_{1111}u_1^3 + 3V_{1122}u_1u_2^2) + \delta_p V'_{11}u_1 &= 0, \\ V_{22}u_2 + \frac{1}{3!} (V_{2222}u_2^3 + 3V_{1122}u_1^2u_2) + \delta_p V'_{22}u_2 &= 0, \end{aligned} \quad (1.8)$$

where the prime on V denotes partial differentiation with respect to p and the subscripts on V denote partial differentiation with respect to the respective generalized coordinates, e.g. $V_{1122} = \frac{\partial^4 V}{\partial u_1^2 \partial u_2^2}$. Pure buckling modes could be studied by substituting the solutions $u_1 \neq 0, u_2 = 0$ (Mode 1) and $u_1 = 0, u_2 \neq 0$ (Mode 2) in this particular case. However, both non-trivial generalized coordinates u_1 and u_2 were taken into consideration when investigating the interactive mode. If both $u_1, u_2 \neq 0$, the relationship of u_1 and u_2 can be obtained by substituting one of the equilibrium equations in Equation (1.8) into the other while eliminating δ_p , thus:

$$(V'_{22}V_{1111} - 3V'_{11}V_{1122})u_1^2 + (3V'_{22}V_{1122} - V'_{11}V_{2222})u_2^2 = -6!V'_{11}V'_{22}\Delta_p, \quad (1.9)$$

where Δ_p is the difference in the buckling loads of Modes 1 and 2 respectively. Equation

(1.9) describes the equilibrium equation of interactive buckling that comprises a nonlinear combination of both pure modes and the possibilities are that the paths are elliptical, hyperbolic or parabolic depending on the signs of the energy coefficients and Δ_p . Figure 1.6 shows a sketch of a possible load–deflection relationship of both pure modes and an elliptical

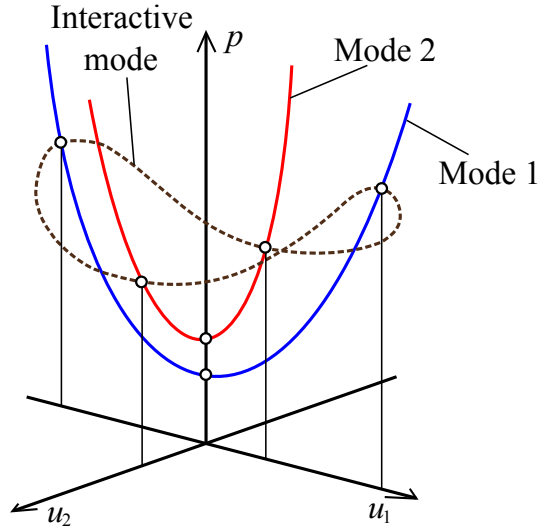


Figure 1.6: Sketch of an equilibrium diagram of the applied load p versus the generalized coordinates u_1 and u_2 for both the pure buckling modes and an interactive mode.

interactive buckling path. Certainly, for pure Mode 1 buckling $u_2 = 0$ and similarly, for pure Mode 2 buckling $u_1 = 0$. However, as expected, it can be seen that during interactive buckling both the generalized coordinates u_1 and u_2 have non-zero values, indicating that this equilibrium path comprises components of both pure buckling shapes. A potentially dangerous situation may occur during the post-buckling stage as the post-buckling stability and the resulting load carrying capacity may be qualitatively different from the response of the distinct pure modes. This is because intrinsically stable pure modes may in fact have unstable interactive buckling responses. Therefore, investigating interactive buckling behaviour is one of the most significant aspects in nonlinear stability problems since hidden instability problems may be discovered.

Interactive buckling has been reported to be observed in many structural types including reticulated columns, sandwich struts, cylindrical shells, stiffened plates, thin-walled struc-

tures and lattice materials (Hutchinson & Koiter, 1970; Thompson & Hunt, 1973; Hunt, 1986; Hunt, 1989; Gioncu, 1994; Menken *et al.*, 1997; Hunt & Wadee, 1998; Wadee, 1999; Everall, 1999; Everall & Hunt, 2000; Olhoff & Seyranian, 2008; Becque & Rasmussen, 2009b; Becque & Rasmussen, 2009a; Yiatros & Wadee, 2011; Wadee & Gardner, 2012; Wadee & Farsi, 2014b; Wadee & Bai, 2014; Martins *et al.*, 2015; Zschoernack *et al.*, 2016). It has also been investigated for the single-bay stayed column in more recent literature (Saito & Wadee, 2009b; Wadee *et al.*, 2013a). Some examples of interactive buckling are shown in Figures 1.7 and 1.8. In the current thesis, interactive buckling of the prestressed

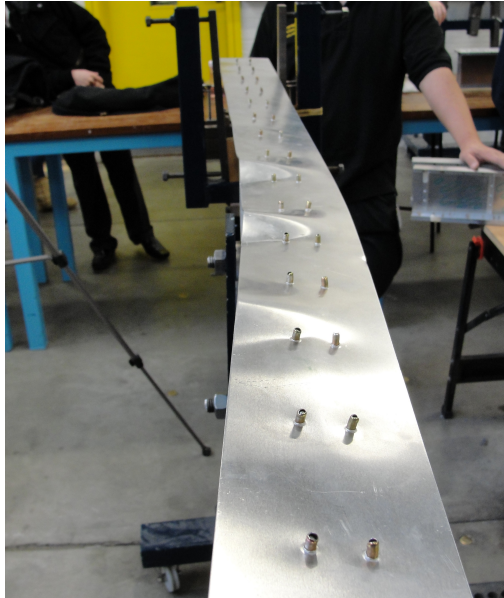


Figure 1.7: Interactive buckling of a mono-symmetric plate girder including the global lateral-torsional buckling mode along the beam length and the local buckling mode shown in the flange.

stayed column is studied and discussed in detail.

1.3 Thesis outline

The current chapter has briefly introduced the background of the prestressed stayed column and some of its industrial applications. The aims and the objectives of the research have

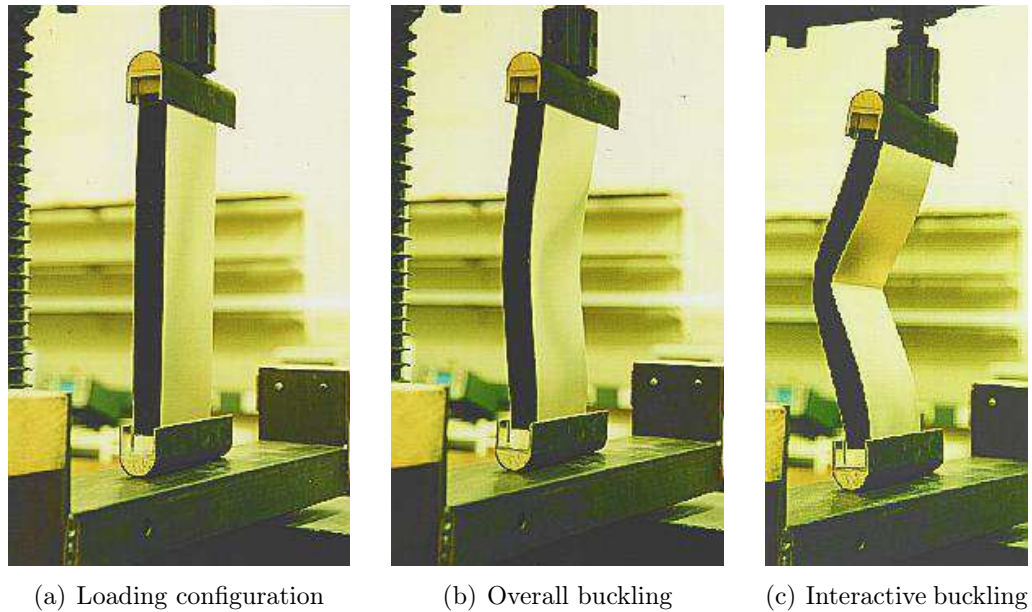


Figure 1.8: Interactive buckling of a sandwich strut (Wadee, 1999).

been presented. Two important concepts: post-buckling stability and interactive buckling, which are studied extensively in the current research, have also been introduced.

Chapter 2 presents a literature review on the previous research on the stayed column. Initial research on the critical buckling load is first introduced. Further studies on the effect of the initial imperfections and the prestressing force are then reviewed. Some of the key recent work on the post-buckling behaviour of the single-bay prestressed stayed column are also reviewed. Literature on interactive buckling, from both the analytical and the numerical models, are also introduced. In addition, a series of experimental studies on the stayed column are reviewed.

In Chapter 3, a finite element model of the triple-bay prestressed stayed column is developed and analysed using the commercial finite element software ABAQUS (2014). The numerical model is developed before the analytical models because earlier work has demonstrated that this type of modelling gives accurate results when compared to analytical and experimental studies of prestressed stayed columns (Saito & Wadee, 2008; Saito & Wadee, 2009a; Osofero *et al.*, 2012; Osofero *et al.*, 2013). Therefore, the results from Chapter 3 will be used as

a benchmark from which the accuracy of the subsequently developed models in Chapters 4 and 5 can be established. Linear buckling behaviour is first investigated. With the obtained buckling load, the theoretical optimal prestress for this structure is calculated by extending an existing method for the single-bay column presented by Hafez *et al.* (1979). Initial imperfections that allow the triggering of both pure buckling modes and interactive buckling are also introduced. The post-buckling behaviour, including interactive buckling, is investigated in detail. Further studies on the sensitivities to the initial prestress and geometric imperfections are also presented.

In Chapter 4, a highly simplified analytical model of the triple-bay column comprising rigid links and springs is developed. Instead of attempting to describe the column deflected shape and system stiffness using complicated mathematical expressions, the column is discretized into rigid links and a series of discrete springs to represent the approximate behaviour of the system. The total potential energy of the system is then formulated. Both buckling and post-buckling behaviour are investigated using the numerical continuation package AUTO-07P (2011). The mode interaction phenomenon is studied in detail and reveals how the pure modes interact in the simplified model. The results obtained in this chapter are used to validate with the FE results qualitatively.

Chapter 5 presents an enhanced analytical model. The total potential energy of the system is formulated in conjunction with the Rayleigh–Ritz method by assuming the deflected shapes of the main column and the crossarms using a multiple degrees of freedom representation. The slackening of the stays is also taken into consideration when establishing the total potential energy. Both buckling and post-buckling behaviour are investigated by the package AUTO-07P (2011). Interactive buckling behaviour is studied in depth. Parametric spaces are determined where mode interaction is most significant. Sensitivity to the levels of prestress is also studied. The results obtained are used to validate the FE model both qualitatively and quantitatively.

Chapter 6 presents an approach to define the actual optimal prestress by introducing two ef-

efficiency indicators for the triple-bay prestressed stayed column. The load carrying capacity and the required resistances of the column and the stays are both taken into consideration. More than 400 separate cases are investigated numerically. The recommended values of the actual optimal prestress are then presented.

Chapter 7 summarizes the important observations and conclusions of the entire project. Suggestions for possible future work are also presented.

Chapter 2

Literature Review

Studies on prestressed stayed columns have existed since the 1960s. A variety of literature with different methods and emphases have been presented. The initial research focus was mainly on the critical buckling load, without taking any imperfections into consideration. Later, studies on the more realistic cases with imperfections were conducted. Different structural parameters and geometrical configurations have also been varied to investigate their effect on the critical buckling behaviour. Along with the development of the finite element method (FEM), more complex models have been studied. The post-buckling behaviour of the stayed column, has also been investigated in more recent years, with a particular emphasis on potential interactive buckling. This has been subsequently demonstrated numerically and experimentally to be a crucial factor in the nonlinear stability of these structural components. Several sets of experiments have also been conducted to investigate the physical behaviour and to validate the results obtained from the aforementioned analytical and FE models. Therefore, in the current chapter, literature on the topics described above is reviewed in detail.

2.1 Critical buckling loads

The first relevant work on the prestressed stayed column was conducted by Chu and Berge (1963), who provided a general solution to determine the buckling load. An analytical model of the stayed column with multiple crossarms, as shown in Figure 2.1, was used in

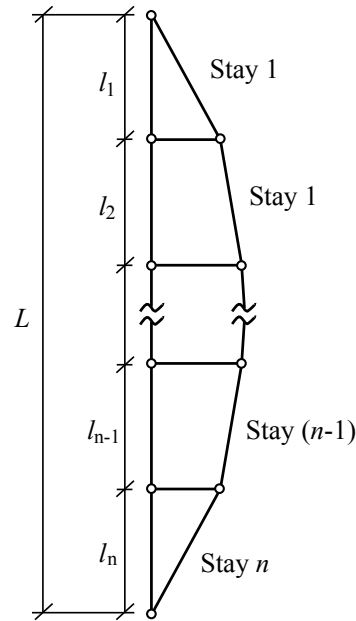


Figure 2.1: Geometry of the stayed column with multiple crossarms studied by Chu and Berge (1963). The connections between the column and the crossarms were assumed to be ideal hinges.

their study. For the first time, they revealed that the buckling load of the main column is strongly linked to the prestressing force applied to the stays. They also stated that if the crossarms are arranged symmetrically about the mid-point of the column, the largest critical load would be four times the Euler load of the unstayed column, no matter how many crossarms are used. Although this work is believed to be the very first study on the prestressed stayed column, the entire study was based on one important assumption, which is that the connections between the main column and the crossarms were ideal hinges; this made the model less realistic and not practically useful.

Later, Mauch (1967) found that under a small structural index, which was expressed as

P/L^2 with P being the buckling load and L being the main column length, the single-bay prestressed stayed column would be up to 50% lighter in terms of self-weight when compared with the simple column to obtain the same buckling load for the same length. However, owing to the fact that this was an extension of Chu and Berge's work, the identical assumption where the crossarms were pinned to the main column was made.

The work on the stayed column continued at the Royal Military College of Canada in 1970. A design, build and test project concerning the single-bay prestressed stayed column, as shown in Figure 2.2(a), was assigned to the final year civil engineering undergraduates

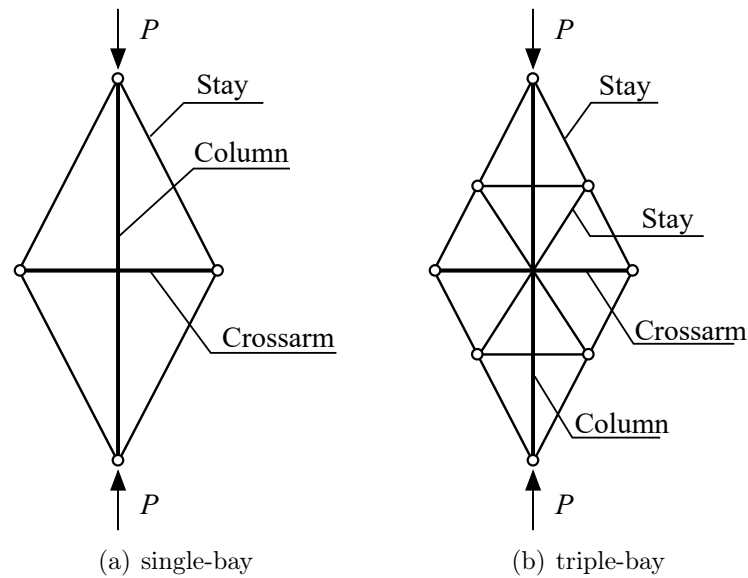


Figure 2.2: (a) The single-bay stayed column studied in the design, build, and test project at the Royal Military College of Canada in 1970; (b) the triple-bay stayed column studied by Ellis (1971). Both cases used rigid connections between the column and the crossarms.

(Ellis, 1971). Unlike the model described previously, this time the crossarms were welded to the main column, in order to provide resistance to the column bending at those locations. It was found that with the rigid connections between the column and the crossarms, the critical load of the system could be up to seven times that of the unstayed column. The work was then extended with a series of additional experiments (Pearson, 1971), which for the first time, found that the critical load of the pin-ended stayed column was significantly affected by the crossarm length and the stay diameter.

Ellis (1971) also presented a series of experiments on the triple-bay prestressed stayed column, as shown in Figure 2.2(b). It was found that with the restraints provided by the extra crossarms and stays, the critical load could be up to between a factor of 20 and 23 times that of the unstayed column with the same slenderness.

Smith *et al.* (1975) developed an analytical model of the single-bay stayed column with rigid connections between the column and the crossarms. The stiffnesses of the crossarms and the stays were represented by a lateral spring support for both the symmetric and the anti-symmetric buckling shapes, as shown in Figure 2.3. The critical loads were obtained

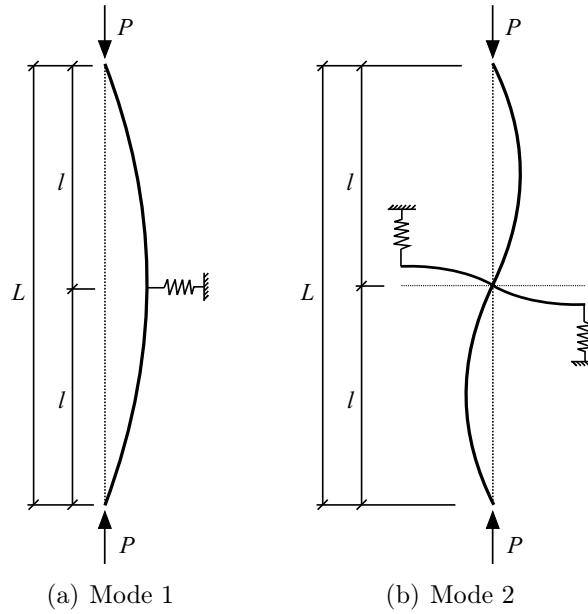


Figure 2.3: Lateral spring supports for the single-bay stayed column presented by Smith *et al.* (1975) for both the symmetric and the anti-symmetric buckling modes.

by solving the governing differential equations. They found that the maximum theoretical buckling load of the pin-ended stayed column was expressed thus:

$$P_{\max}^C = \frac{\pi^2 E_c I_c}{(0.699) l^2} = 8.18 P_E, \quad (2.1)$$

where E_c and I_c are the Young's Modulus and the second moment of area of the main column respectively, with l being the half of the column length. The Euler buckling load

P_E for the unstayed column is defined thus:

$$P_E = \frac{\pi^2 E_c I_c}{L^2}, \quad (2.2)$$

where L is the total column length (Timoshenko & Gere, 1961). This verified again that extra benefit could be obtained by adopting rigid connections between the column and the crossarms compared with the pinned connection. This research made a significant contribution since this was the first analytical study on the stayed column with rigid column–crossarm connections. However, there were two principal limitations in this work:

1. The analytical approach presented in this paper could only be used on the single-bay stayed column. Governing equations for more complex geometries with a multiple crossarm system were not presented.
2. Two relatively simple buckling shapes were assumed first, the buckling loads of the column were then calculated. If the actual buckling shape did not correspond to the assumed shape, the derived buckling load would be excessively high to be valid.

To overcome the aforementioned disadvantages, Temple (1977) presented a finite element method to study the stayed columns with single-bay, double bay and triple-bay crossarms, as shown in Figure 2.4. The structure was divided by a number of ‘finite elements’ (FE), therefore the system was treated as discretely interconnected by a series of nodal points. By minimizing the total potential energy of the system, the equilibrium relationship was given by the expression:

$$[\mathbf{K}]\{\Delta\} = \{\mathbf{F}\}, \quad (2.3)$$

where $[\mathbf{K}]$ is the stiffness matrix of the entire system, with $\{\Delta\}$ and $\{\mathbf{F}\}$ being the vectors of the nodal displacements and the loads at the nodes respectively. The system stiffness matrix $[\mathbf{K}]$ comprised two parts, the conventional elastic stiffness matrix $[\mathbf{K}_E]$ and the

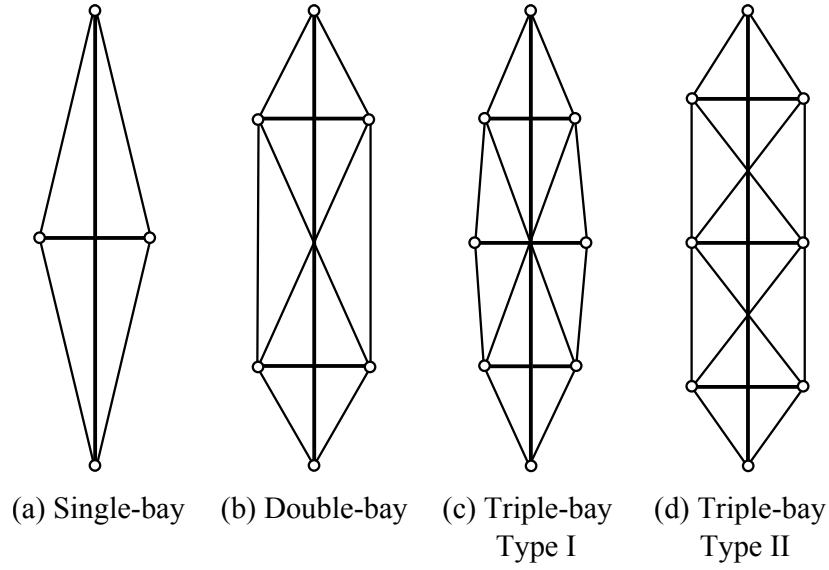


Figure 2.4: Stayed columns with different numbers of bays studied by Temple (1977).

geometric stiffness matrix $[\mathbf{K}_G]$, thus

$$[\mathbf{K}] = [\mathbf{K}_E] + [\mathbf{K}_G]. \quad (2.4)$$

The elastic stiffness matrix $[\mathbf{K}_E]$ represents the material properties and does not change with the geometry or the applied load. The geometric stiffness matrix $[\mathbf{K}_G]$ depends on not only the current geometry but also the applied load. By solving Equations (2.3) and (2.4), the buckling load and the corresponding buckling shape were obtained. The results confirmed the advantages of using rigid connections between the column and the crossarms instead of pinned connections. It was also found that the actual buckling shape of the column are often more complex as compared with the simple half-sine and the full-sine wave used by Smith *et al.* (1975). Temple's research was the first recorded that successfully used the FE method to calculate the buckling load of the stayed column.

In the same year, Belenya (1977) presented an analytical model to investigate the single-bay stayed column. A series of ordinary differential equations based on elasticity and geometrical relationships were solved in order to find the critical buckling load. The re-

sults suggested the critical buckling load could be considerably larger as compared to the unstayed column. However, the assumption that the prestress was zero was adopted, which did not reflect the real condition. Physical tests on stayed columns with and without the prestress were also conducted. It was found that the non-prestressed columns possessed a 15% to 20% smaller buckling load as compared to the prestressed column.

Howson and Williams (1980) studied the critical loads of ten different types of stayed column by using a plane frame computer programme (Howson, 1979). For the purposes of optimization, the ratio of saving in the steel weight was measured for each type of the stayed column. The minimum required steel weight for these cases were presented, which again demonstrated the significant saving in structural weight.

Howson and Williams (1984) continued their work by varying some of the structural parameters such as the crossarm length and the number of joints. By varying these parameters, the optimum structural arrangements and dimensions that minimized the required steel weight were recommended. In addition, the effect of the residual forces in the stays to the applied load was also investigated.

It is worth noting that all the aforementioned literature in this section focused on the critical buckling load of the stayed column. The post-buckling behaviour, which is also a key aspect in the stability performance of the stayed column, and the effect of the initial imperfections in the main column and the prestress in the stays, were not investigated thus far.

2.2 Prestressing levels and initial imperfections

It has been shown many times that the introduction of the prestress in the stays provides significant benefit in terms of the buckling load. However, the optimization of the prestress was first presented by Hafez *et al.* (1979) who conducted a study to evaluate the optimal

value of the prestress for the single-bay stayed column. Figure 2.5 shows a representation of

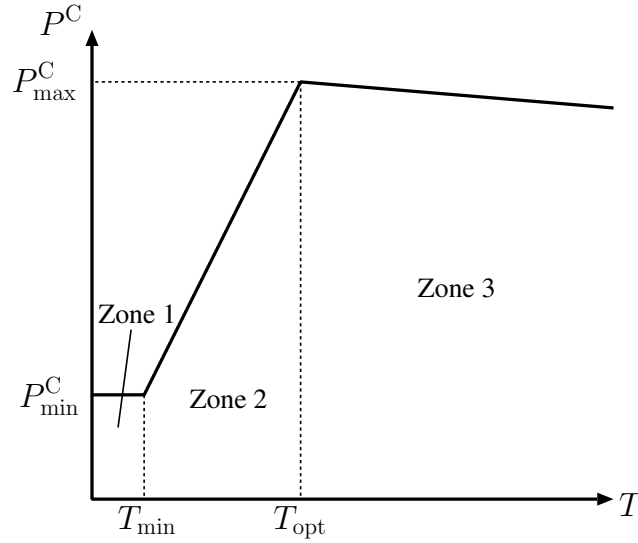


Figure 2.5: Critical buckling load P^C versus initial prestress T with the zone distinction for the single-bay prestressed stayed column from Hafez *et al.* (1979).

the relationship between the critical buckling load and the initial prestress in the column. Zone 1 represents a very small prestress that would completely disappear under increasing axial loading before or at the instant when the Euler buckling load of the unstayed column is reached. Therefore, that structure would be effectively identical to the unstayed column, which leads to the buckling load being the Euler load. The prestress in Zone 2 is sufficiently large to make the buckling load of the column larger than the Euler load, but the prestress would disappear at the instant when the column buckles. The prestress in Zone 3 is even larger, as a result there are still residual forces in the stays when the column buckles. The vertical components of the residual force in the stays effectively become a part of the applied force, which leads to a decrease in the buckling load with the increase of the initial prestress. Hafez *et al.* indicated that the prestress at the intersection of Zones 2 and 3 were the desired optimal prestress, the objective being the maximization of the linear buckling load with respect to the introduced prestressing force. The derivation of the optimal prestress for the triple-bay stayed column, which is an extension of the method used in Hafez's *et al.* work, is presented in detail in Chapter 3. It is very important to note

that the study conducted by Hafez *et al.* was purely in the linear elastic range. No lateral deflections of the column were taken into consideration, hence geometrical nonlinearities were neglected, and the post-buckling behaviour and geometric imperfections were not considered.

Wong and Temple (1982) studied the effect of the imperfection on the critical load of the single-bay stayed column for the first time. An initial out-of-straightness of the column was introduced to the model. A similar approach using the finite element method (FEM), which was first adopted by Temple (1977), was selected to calculate the critical buckling load of the column. They found that the initial imperfections in the column significantly decreased the ultimate load as compared to that of the perfect model, as shown in Figure 2.6. Experiments were also conducted to validate the theoretical results. They pointed

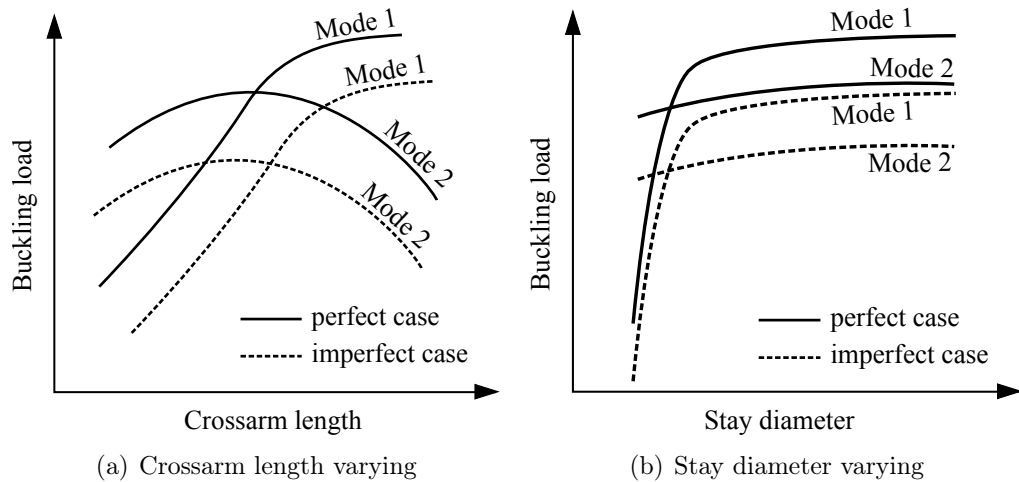


Figure 2.6: Sketches of the comparisons of the buckling loads between the perfect and the imperfect cases with the crossarm length and the stay diameter varying studied by Wong and Temple (1982).

out that the FEM was capable of predicting the strength of imperfect prestressed stayed columns accurately. The difference between the buckling loads from the experimental data and the perfect analytical model was up to 21% while the difference between the experimental data and the imperfect analytical model was less than 9%. However, owing to a lack of a parametric study, since they only used a fixed value of the initial out-of-straightness, the effect of different sizes of the imperfections was not obtained.

Temple *et al.* (1984) extended the study on the initial imperfections in the main column. In this work, three sizes of the initial out-of-straightness ($L/500$, $L/1000$ and $L/3000$) were investigated. In addition, for the case with the initial out-of-straightness being $L/1000$, five different levels of prestressing force, varying from 0 to 1.33 kN, were studied in conjunction with the FEM, as shown in Table 2.1, where L was the length of the main column. It was

Table 2.1: The adopted initial out-of-straightnesses and pretensions and the obtained buckling loads presented by Temple *et al.* (1984). Note that $1 \text{ lbf} = 4.45 \times 10^{-3} \text{ kN}$.

Initial out-of-straightness	Pretension (lbf)	Buckling load (lbf)	Buckling mode
$L/500$	150	5600	1
$L/1000$	0	5200	1
$L/1000$	40	5600	1
$L/1000$	150	5800	1
$L/1000$	225	6800	1
$L/1000$	300	8600	2
$L/3000$	150	7000	1

found that increasing the imperfection size would lead to an obvious reduction in the critical buckling load of the column. They also pointed out that, for the cases with very small prestressing forces that were similar to $T_{\min}$, as defined by Hafez *et al.* (1979), the actual buckling loads were considerably larger than the previously predicted values, as shown in Figure 2.7. In addition, the predicted maximum buckling load with the introduction of T_{opt} could not be reached for the imperfect cases.

Chan *et al.* (2002) presented a series of parametric studies on the ultimate load of the single-bay stayed column with the initial imperfections, the prestress, the stay diameters and the crossarm length varying using the FEM. Good agreements were found as compared with Temple's work (1984) while varying the initial imperfections and the prestress. Chan *et al.* also pointed out that when the symmetric mode governed the buckling behaviour, the ultimate load increased significantly with the increase of the stay diameters or the crossarm length. When the anti-symmetric mode governed, only limited benefit could be obtained by increasing the aforementioned two parameters.

It has been stated that the aforementioned theoretical optimal prestress was obtained for

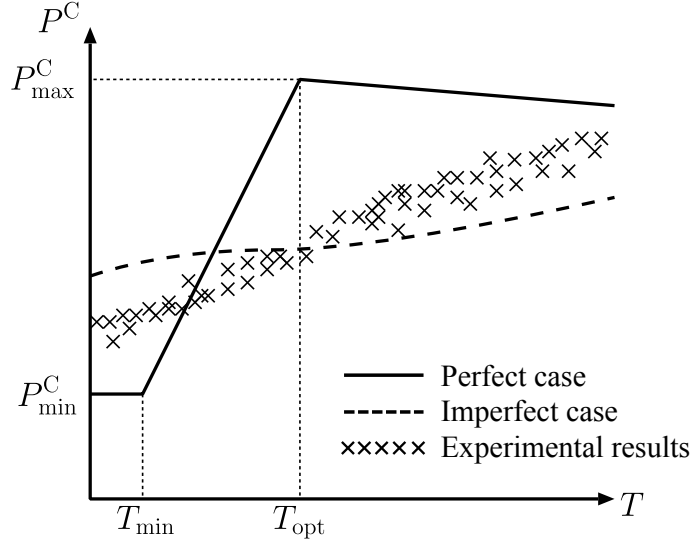


Figure 2.7: Qualitative comparisons of the relationships of the buckling load and the stay pretension between the perfect theoretical model presented by Hafez *et al.* (1979), the imperfect model presented by Temple *et al.* (1984), the latter also contained the experiment data.

the purely linear elastic range and ignored geometric nonlinearity. Therefore, it is unlikely to reflect the actual case accurately. Saito and Wadee (2010) investigated the actual optimal prestress for the single-bay prestressed stayed column with the initial imperfection taken into consideration. They pointed out that the load carrying capacity may not be the only indicator for selecting the desired prestress, since the required resistances of the column and the stays were also important. Therefore, two efficiency indicators, η and η_s , were presented:

$$\eta = \frac{P_{\max}}{A_c f_{y,\text{req}}}, \quad \eta_s = \frac{P_{\max}}{A_s f_{s,\text{req}}} \frac{L/2}{l_s}, \quad (2.5)$$

where $P_{\max}$ is the maximum load carrying capacity with $f_{y,\text{req}}$ and $f_{s,\text{req}}$ being the required column and stay resistances at the point of the maximum load respectively. Quantities A_c and A_s represented the cross-section area of the column and the stay respectively with L and l_s being the column and the stay length respectively. High values of the indicators η and η_s indicated a relatively high load carrying capacity with intrinsically low column and stay resistance requirements, which represents a more efficient system, as shown in the examples of Figure 2.8. In that figure, buckling Mode 1 represents the symmetric

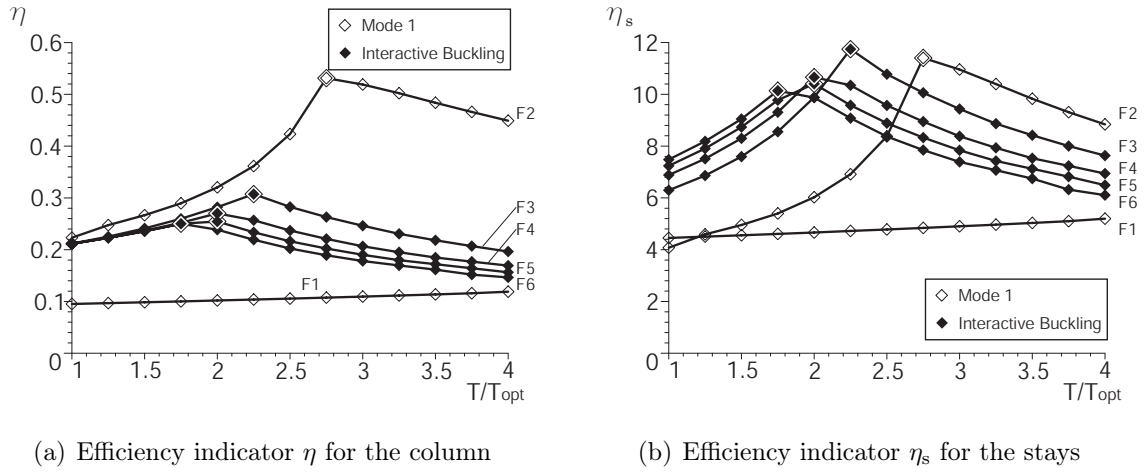


Figure 2.8: Relationships of the efficiency indicators versus the prestressing force presented by Saito and Wadee (2010). All results were obtained using a nonlinear FE model formulated in the commercial package ABAQUS (2014).

buckling mode with the lowest load, whereas interactive buckling represents a nonlinear combination of Mode 1 and Mode 2, the latter being the anti-symmetric buckling mode with the lowest load, as shown in Figure 2.9. The actual optimal prestressing forces were then calculated and presented for the column and the stays respectively. This was the first piece of research that quantitatively revealed the actual optimal prestress for realistic models with imperfections and concluded that the value T_{opt} , obtained by Hafez *et al.* (1979), was a significant underestimate on the true optimum that accounted for cost and nonlinear behaviour.

2.3 Post-buckling behaviour

Smith (1985) presented an analytical model to study the single-bay prestressed stayed column. It was not explicitly a post-buckling study, however, the concept of the ‘load capacity’ rather than the buckling load, was presented. By solving a series of differential equations, the load capacities beyond the buckling point were calculated with exclusive focus on the residual forces in the stays. An initial out-of-straightness of the column was

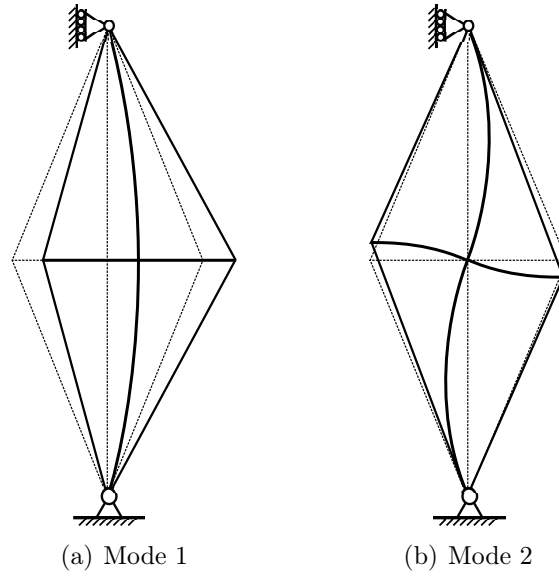


Figure 2.9: The buckling shapes of Mode 1 (symmetric) and Mode 2 (anti-symmetric) for the single-bay prestressed stayed column.

also introduced to investigate the imperfect case.

A very important study on the post-buckling behaviour of the single-bay stayed column was presented by Saito and Wadee (2008). The buckling shape of the column was assumed to be the summation of a series of sine waves accounting for both the symmetric and the anti-symmetric buckling modes respectively. Three different buckling types were identified for deformation in an anti-symmetric mode (so-called ‘Mode 2’), as shown in Figure 2.10, which comprised the following cases during post-buckling:

1. Type A: all of the stays are slack.
2. Type B: all of the stays are active.
3. Type C: only two of the four stays are active.

All the cases were examined in detail within the study. Owing to the bending of the crossarms during anti-symmetric buckling, shape functions of the crossarms were obtained by solving the governing differential equations from the crossarm, which was modelled as a beam-column. The strain energy stored in different components of the structure were

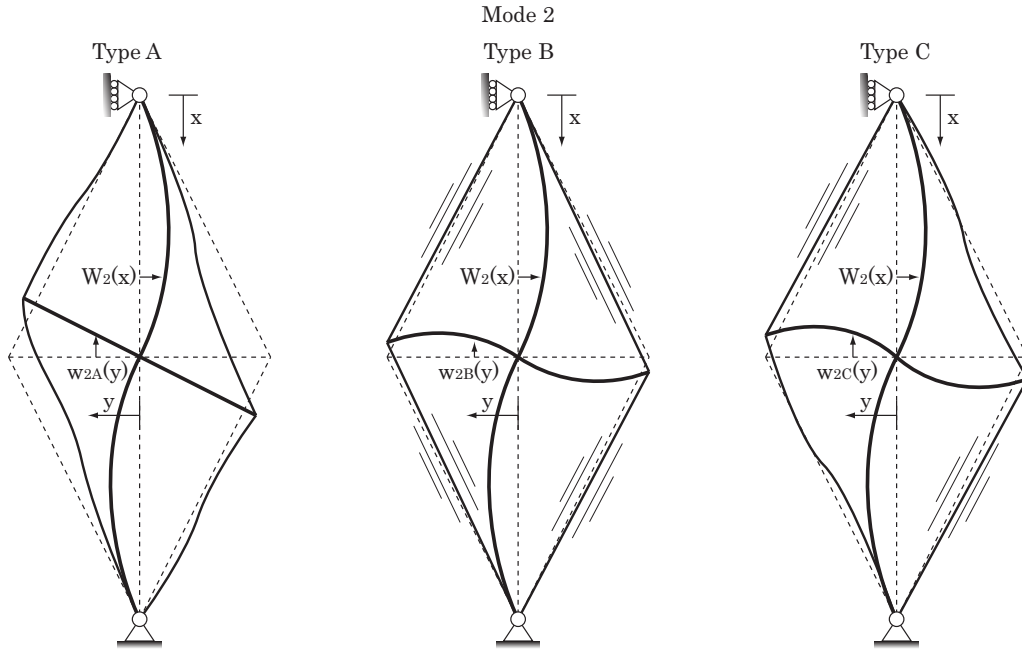


Figure 2.10: Buckling types of the anti-symmetric mode studied by Saito and Wadee (2008).

then calculated by using the Rayleigh–Ritz method. The total potential energy of the system was therefore obtained. The post-buckling behaviour was investigated by solving the subsequent equilibrium equations numerically using MAPLE (Bernardin *et al.*, 2011). Different levels of prestress located in all the three zones presented by Hafez *et al.* (1979) were examined with the verification performed using a nonlinear FE model for prestress levels within Zone 3. It was found that the post-buckling behaviour was strongly linked to the level of prestress adopted in the analysis. The Zone 1 prestress led to stable post-buckling behaviour due to the stays re-activating due to the lateral displacement of the main column. The behaviour for prestress levels within Zone 2 could be either stable or unstable, depending on the exact level of the initial prestress adopted. Weakly stable behaviour could be observed initially when the prestress level within Zone 3 was introduced, this being followed subsequently by an unstable path owing to the slackening of stays. This was the first study that investigated the post-buckling behaviour of the stayed column explicitly with the consideration of the stay slackening.

Saito and Wadee (2009a) developed their work by introducing initial imperfections and material failure criteria to the same structure. The post-buckling behaviour of the imperfect column was revealed from both the analytical and the FE models. Different levels of the prestress and imperfection sizes were examined in their work. They pointed out that an increase in the load carrying capacity of the column could be observed by increasing the prestress in the stays from T_{opt} . This indicated that the actual optimum prestress for the imperfect stayed column was located somewhere in Zone 3, rather than the boundary of Zones 2 and 3, as predicted by the theoretical linear model presented by Hafez *et al.* (1979). It was also found that for most cases, column yielding was the governing failure mode as compared with the stay fracture, with the exception that when a relatively large prestress and imperfection size were adopted, stay fracture may in fact then govern the failure mode. Either way, it was observed in several cases that the material failure occurred before the maximum load predicted from the elastic post-buckling behaviour was reached. This indicated that additional attention must be paid when studying the load carrying capacity of such structure. A detailed study on the actual optimum prestress of the single-bay prestressed stayed column that also took into account the resistance of the column and the stays was presented later by the same authors (Saito & Wadee, 2010), which confirmed the earlier findings and was mentioned in Section 2.2 (see Figure 2.8).

Wadee *et al.* (2013b) continued the aforementioned work by presenting design guidance for the single-bay column. Figure 2.11 shows one of the proposed design curves. A generic design procedure capable of evaluating the load carrying capacity of the column with different crossarm lengths, prestress levels in the stays, stay diameters and the imperfection size was presented. A checking criterion that was defined by the expression:

$$\frac{\gamma_{\text{M1}} N_{\text{Ed}}}{N_{\text{max}}} \leq 1.0 \quad (2.6)$$

was presented where N_{Ed} and N_{max} were the design load and the predicted maximum load respectively with γ_{M1} being a partial safety factor that is prevalent in Eurocode 3, the

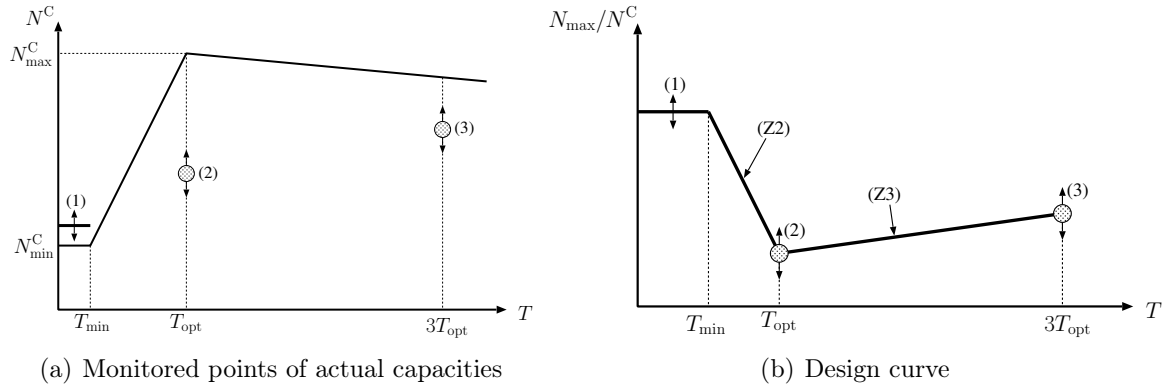


Figure 2.11: (a) Zonal prestress graph with monitored points showing the actual load capacities for imperfect stayed columns. (b) Strength graph $N_{\max}/N^C$ versus prestress level T/T_{opt} derived from analysing (a). The prestress level was limited to $3T_{\text{opt}}$ (Wadee *et al.*, 2013b).

European design code for steel structures (BS EN 1993-1-1, 2005). The most important advantage of this guidance is to simplify the design of the single-bay column by avoiding the use of the complicated analytical expressions presented previously (Saito & Wadee, 2008).

2.4 Interactive buckling

All the aforementioned studies on the post-buckling behaviour emphasized the pure buckling modes, which is either the distinct symmetric (Mode 1) or the anti-symmetric mode (Mode 2), see Figure 2.9. The first study on interactive buckling, the concept of which was defined in Chapter 1, when the buckling modes with different wavelengths are triggered simultaneously, for the single-bay prestressed stayed column was presented by Saito and Wadee (2009b). An FE model was developed to investigate the problem. When assigning the imperfection to the column, instead of using the initial out-of-straightness of a pure buckling shape as was done previously, a combined imperfection profile, which comprised both the basic shapes of Modes 1 and 2, was used to enable the triggering of the two buckling modes simultaneously. The post-buckling behaviour of three cases, pure Mode 1, pure

Mode 2 and interactive buckling was presented and compared. It was found that when Mode 1 was critical, it also governed the post-buckling behaviour. However, when Mode 2 was critical, the actual governing mode was interactive buckling, as shown in Figure 2.12. This important observation indicated the significance of interactive buckling. In practice,

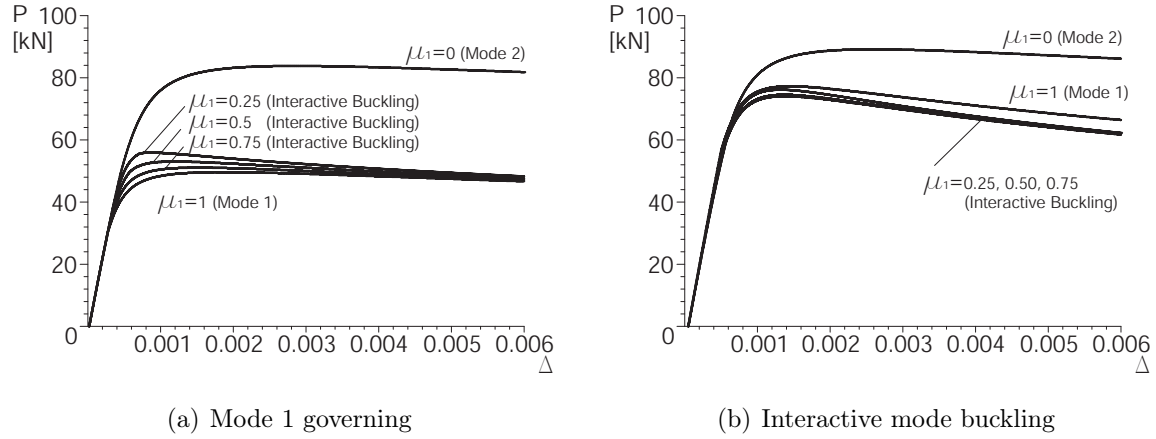


Figure 2.12: Equilibrium paths of the applied load versus the column end-shortening for Mode 1, Mode 2 and interactive mode represented by Saito and Wadee (2009b). Scale factor μ_1 represents the proportion of the Mode 1 imperfection profile; (a) shows the case where Mode 1 was critical; (b) shows the case where Mode 2 is critical.

when Mode 2 is critical, designing such structures based on the load carrying capacity of Mode 2 overestimates the system resistance; this overestimation should of course be avoided.

Wadee *et al.* (2013a) developed a highly simplified model with three degrees of freedom to study the interactive buckling of the single-bay column analytically. The column was modelled by four rigid links, with the resistances of the stays and the crossarms being represented by the longitudinal and rotational springs. The equilibrium equations were then solved to reveal the post-buckling behaviour. A loop describing interactive buckling was observed in the equilibrium path when Mode 2 is marginally critical, as shown in Figure 2.13, demonstrating how the modes interacted. This observation again demonstrated the existence of the interactive buckling mode and validated the FE results (Saito & Wadee, 2009b), at least qualitatively.

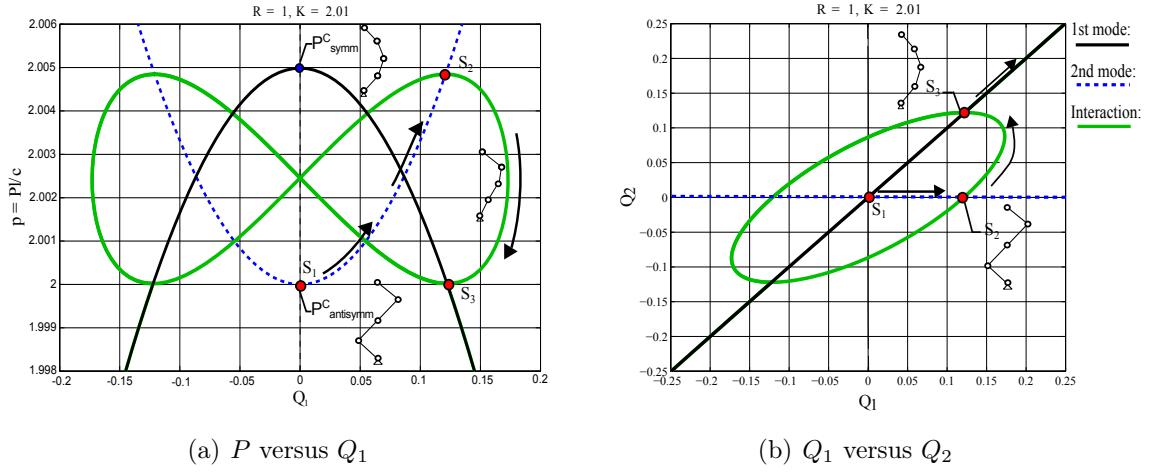


Figure 2.13: Equilibrium paths of (a) the applied load P versus the generalized coordinate Q_1 (measuring Mode 1) and (b) Q_1 versus Q_2 (measuring Mode 2) for the discrete analytical model presented by Wadee *et al.* (2013a).

2.5 Experimental studies

A number of experimental studies have been conducted on stayed columns. Jemah and Williams (1990) tested a type of stayed column that was reinforced by bipods at the mid-span, as shown in Figure 2.14. Push-fit components were used for all connections with

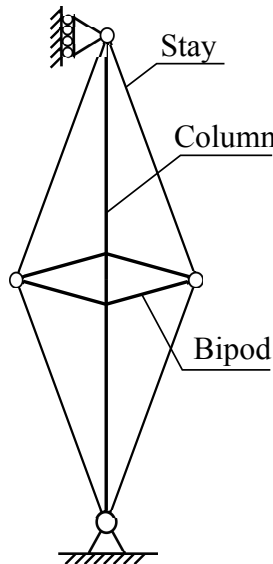


Figure 2.14: Prestressed stayed column with bipod system studied by Jemah (1990).

the exception for grub screws being used for stays in order to keep them in tension. They

found that the load capacity is generally 10% lower than the theoretical value without the consideration of the imperfection. The decrease in the load capacity was even more obvious when the pretension is nearer to T_{opt} .

Van Steirteghem *et al.* (2005) continued the study on the stayed column reinforced by bipods. Parametric studies with the stay diameter and the crossarm length varying were conducted. They found the load carrying capacity of the column was strongly linked to these parameters, which verified the results from the previous analytical and FE studies. It was also found that owing to the significant rotational resistance provided by the bipods to the column, the governing buckling mode was usually the anti-symmetric mode (Mode 2). They also pointed out that the column reinforced by the bipods was 20% more efficient in terms of the steel weight for a given load as compared with the column reinforced by the single crossarm.

Experiments on a three dimensional single-bay stayed column with a 12 m length were conducted by Araujo *et al.* (2006; 2008). It was found that the load capacity of the column is significantly larger than the unstayed column; at least a 100% increase in the load capacity could be obtained even without prestressing the stays. They also pointed out that the load carrying capacity was very sensitive to the initial imperfections and that elastic analysis was sufficient to evaluate the ultimate behaviour.

Osofero *et al.* (2012) conducted a series of experiments revealing the symmetric, the anti-symmetric and interactive buckling for the single-bay column, as shown in Figure 2.15. It was found that when the buckling loads of the symmetric and anti-symmetric mode were very close, or the anti-symmetric mode was critical, interactive buckling would be observed. This verified the conclusions obtained from the analytical (Wadee *et al.* , 2013b) and FE (Saito & Wadee, 2009b) studies, indicating the importance of paying additional attention to the load capacity of the column when the anti-symmetric mode is critical in design practice.

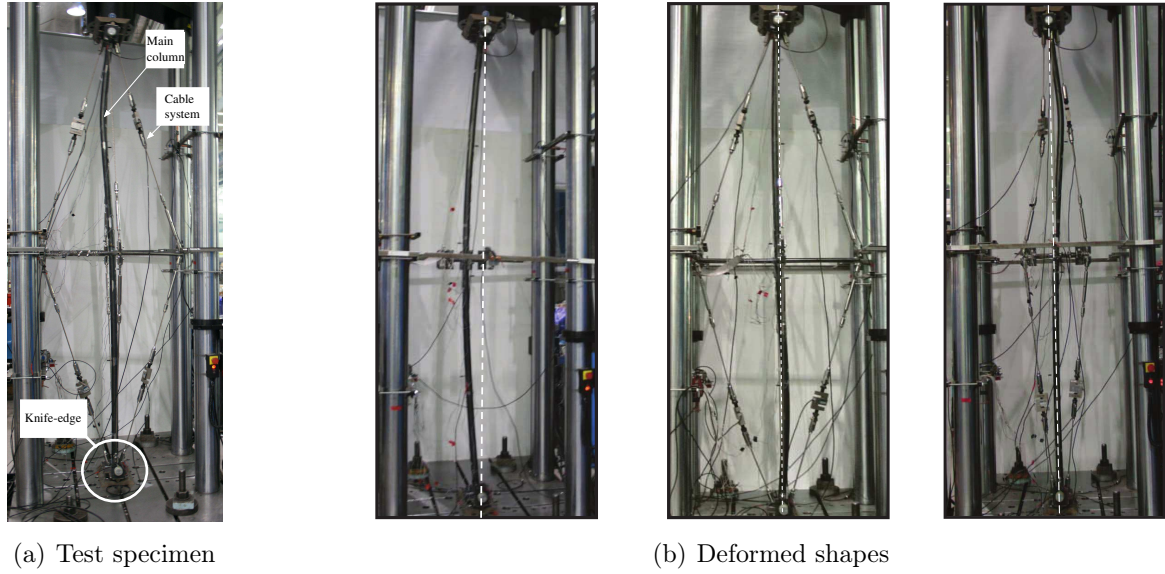


Figure 2.15: (a) The setup of the physical tests and (b) the actual deformed shapes of Mode 1, Mode 2 and interactive buckling presented by Osofero *et al.* (2012).

Serra *et al.* (2015) also conducted several full scale tests on 12 m length stayed columns with a single crossarm system. A series of parameters were varied, such as the column cross-section area, the stay diameter, the prestress and the steel grade, including high strength steel (Raoul & Günther, 2005). Apart from the similar observations as compared with the previous full scale experiments, they pointed out that the load capacity of the column was also strongly linked to the cross-section area of the column. However, they concluded that the use of high strength steel did not provide additional benefit since the main column possessed a very high slenderness; this conclusion was perhaps more connected to the highly slender geometries selected in the study.

2.6 Concluding remarks

This chapter has summarized the existing literature on the stayed column. Initial work only focused on the critical buckling load of the column. Later research took imperfections into account to investigate more realistic cases. With the development of the FEM,

parametric studies varying the geometric configurations, the level of prestress and the size of imperfection were conducted. More recent investigations were conducted on the post-buckling behaviour and interactive buckling. The significance of studying the post-buckling behaviour, especially interactive buckling, has already been indicated. However, most of the aforementioned studies were for the single-bay prestressed stayed column. Although there are some existing works on the critical buckling of the stayed column with multiple crossarm systems, the post-buckling behaviour for such structures has never been revealed in detail. Therefore, in the subsequent chapters, the post-buckling response of the multiple-bay prestressed stayed column, including interactive buckling, will be investigated in detail through both analytical and the FE modelling techniques.

Chapter 3

Finite Element Modelling

3.1 Introduction

Finite element (FE) modelling is becoming more commonly used in recent years primarily owing to the continual development of computational power. The commercial software ABAQUS (2014) is extremely powerful in handling FE models, particularly when studying nonlinear post-buckling problems with the consideration of initial imperfections and pre-stress. FE models of the single-bay prestressed stayed column have been developed and analysed relatively recently. The results of the post-buckling response have been validated with an analytical model established using the minimum total potential energy principle (Saito & Wadee, 2008) and a series of full scale experimental studies (Osofero *et al.*, 2012). It has been demonstrated that ABAQUS is capable of evaluating both critical and post-buckling behaviour of the prestressed stayed column. Therefore in the current chapter, an FE model of the triple-bay prestressed stayed column is developed and analysed in ABAQUS, and is subsequently validated both qualitatively and quantitatively with analytical models in later chapters.

3.2 Model formulation

The current investigation focuses on prestressed stayed columns with multiple bays. In the single-bay stayed column, as shown in Figure 3.1(a), a restraint is provided by the crossarm

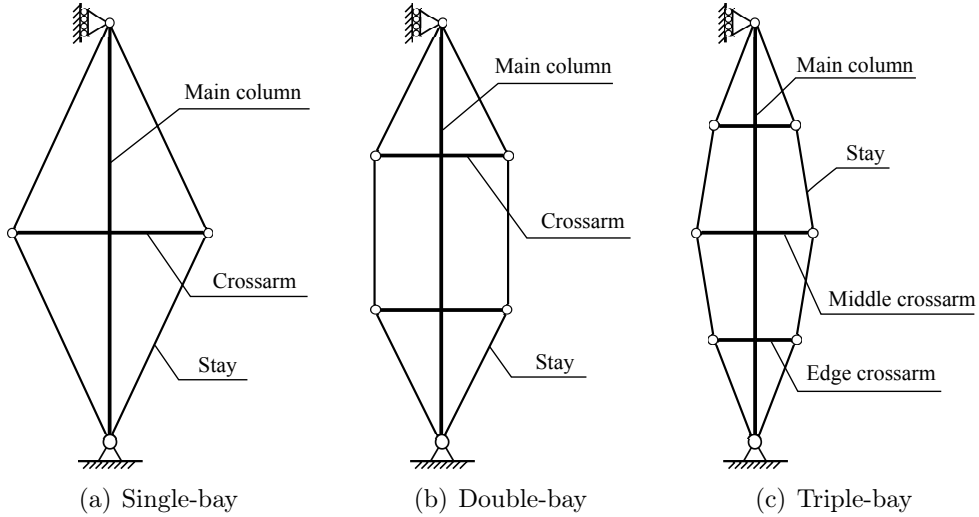


Figure 3.1: (a) The stayed column with a single crossarm system. (b)–(c) Stayed columns with multiple crossarm systems along the length. The double-bay case in (b) would be essentially identical to the particular triple-bay case shown in (c) if all the crossarms in the latter were of equal length.

system at the mid-span of the main column element. Similarly, two restraints are provided at one-thirds and two-thirds of the column length in the double-bay case, as shown in Figure 3.1(b). Owing to the lack of any effective lateral stiffness at mid-span, the double-bay column cannot resist the symmetric buckling mode as effectively when compared with the triple-bay case, as shown in Figure 3.1(c). Results obtained as part of the current study also indicate that the load carrying capacity is much lower in the triple-bay case if the edge and the middle crossarms have the same length. This is essentially similar to the double-bay case due to the horizontal component of the stay force being zero between the two edge crossarms when the lateral displacements from global buckling are small. For the particular triple-bay case with equal length crossarms and the double-bay configuration, extra struts connecting the crossarms to the main column at midspan, in a truss-like arrangement, as shown in Figure 3.2, would potentially increase their effectiveness, but this consideration

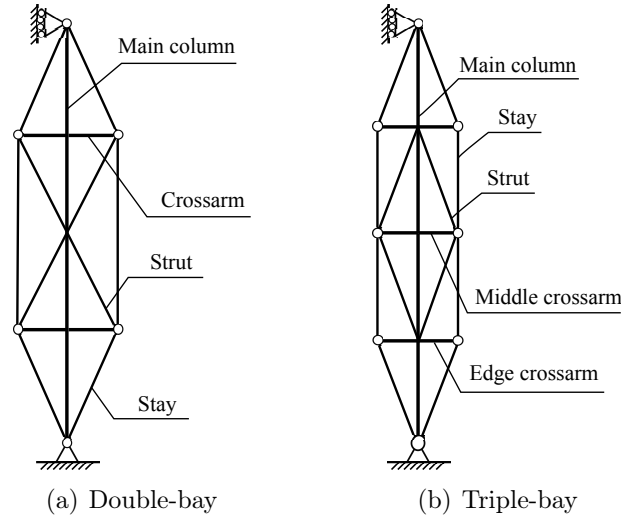


Figure 3.2: (a) The double-bay stayed column reinforced by extra struts. (b) The triple-bay stayed column with equal length crossarms also reinforced by extra struts. The additional struts provide effective lateral restraints at midspan.

is left for future work. Therefore, in the current work, the triple-bay prestressed stayed column with different length crossarms is studied exclusively.

Different arrangements of crossarms along the length in the yz -plane may be used, as shown in Figure 3.3 with (a) showing the simplest one-dimensional arrangement with both (b)

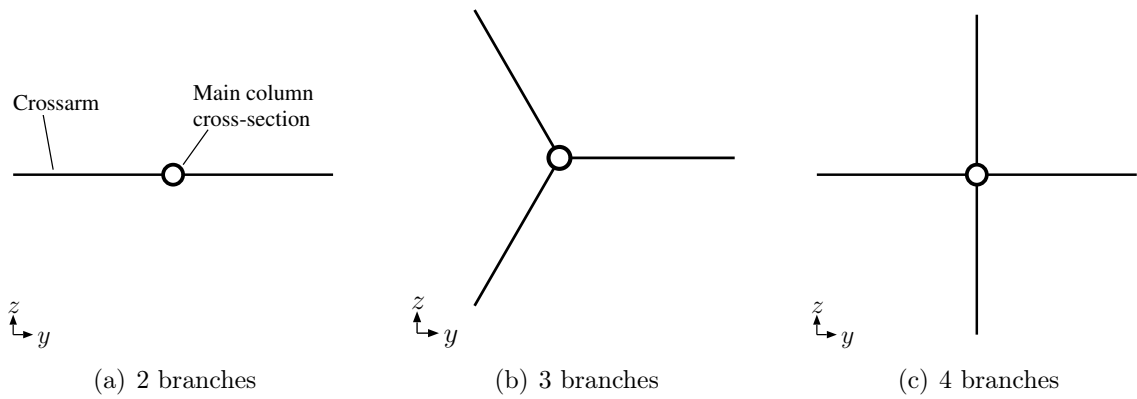


Figure 3.3: Different numbers of branches in each crossarm system along the length. Case (a) is only practical if the buckling displacements are constrained to be purely in the y -direction; cases (b) and (c) would work if the column displaces anywhere in the yz -plane.

and (c) showing two-dimensional arrangements. Note that the column in practice should be supported by at least three crossarms in the yz -plane. This would ensure that it would

benefit from the increased load carrying capacity by avoiding an obviously weaker buckling axis, assuming that the main column element is either a circular or square hollow section. The single-bay case with a three crossarm rosette, as shown in Figure 3.3(b), was studied recently (Li *et al.* , 2016). Currently, the triple-bay system is assumed to have either the case shown in Figure 3.3(a) with deflections in the z -direction completely restrained or, effectively, the case shown in Figure 3.3(c) where it has been demonstrated previously that the system has no preferred buckling direction (Saito & Wadee, 2009a). These two respective cases are also represented in the photographs shown in Figures 1.3(b) and 1.3(c) respectively. Hence, a two-dimensional model in the xy -plane, as depicted in Figure 3.4, is studied currently since it is simple yet practically important.

The FE model for the triple-bay prestressed stayed column comprises a main column, three pairs of crossarms and a series of prestressed cable stays. As shown in Figure 3.4, the bottom end of the main column is pinned to the ground, while the tip of the column is

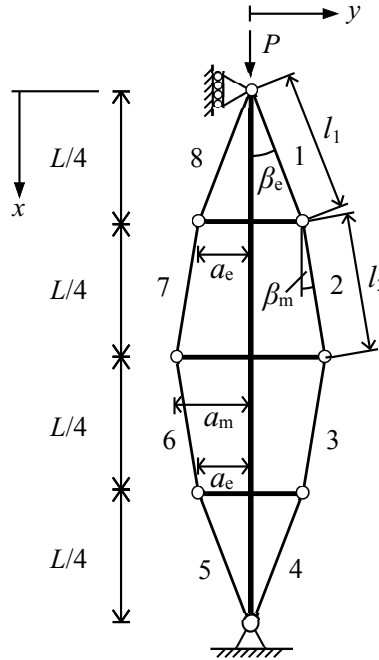


Figure 3.4: Triple-bay prestressed stayed column.

loaded axially. Three pairs of crossarms are rigidly connected to the one-quarter, one-half and three quarter length points of the main column. Prestressed cable stays are assumed to

be pinned to the ends of the column and the tips of the crossarms. The length of the middle crossarm is larger than that of the edge crossarms. The length ratio of the edge crossarm to the middle crossarm $\gamma = a_e/a_m$ is set to be between 0.5 to 1.0, otherwise at least one of the crossarms would initially be in tension and potentially lose effectiveness as buckling restraints since the welded connections between the main column and the crossarms may become vulnerable to fracture. The stay numbers are defined in a counter-clockwise sense, as shown in the diagram; these will be used later.

3.2.1 Element type section

The main column and the crossarms are modelled with beam elements within ABAQUS. Three types of beam element, B21, B22 and B23, are provided for 2-dimensional problems. Elements B21 and B22 are Timoshenko beam elements which take shear strains into consideration and use linear and quadratic interpolation functions respectively. Element B23 assumes Euler–Bernoulli bending. Owing to the fact that element B23 uses a cubic interpolation function, fewer elements are required, when compared with models using B21 and B22, to obtain accurate results. The ABAQUS manual recommends that for beams made of uniform material, typical dimensions within the cross-section should be less than approximately 1/15 of typical structural lengths for transverse shear to be negligible. These ratios, in the current study, are approximately 1/133 for the main column and between 1/3 to 1/30 for the crossarms. Therefore, in this work, the main column is formulated using the B23 (cubic Euler–Bernoulli beam element) while the crossarms are formulated using B22 (quadratic Timoshenko beam element).

As mentioned before, the stays are pinned to the ends of the column, and the crossarms and cannot take any shear force or bending moment. As a result, truss elements that support loading only along the axis of the member are used to formulate the stays. No moments or forces perpendicular to the centreline can therefore be supported. Two types

of truss element are provided within ABAQUS; the two-node straight truss element and the three-node curved truss element, as shown in Figure 3.5. The normal direction $\mathbf{n}$ is

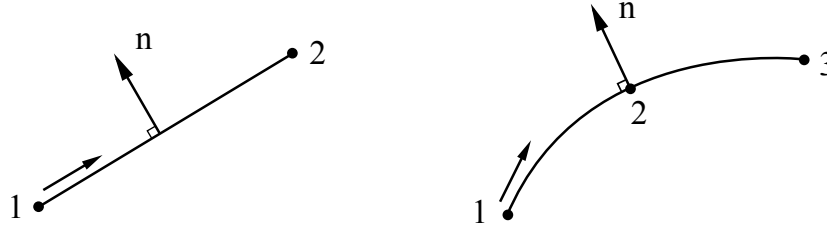


Figure 3.5: Truss element type: (a) two-node straight truss element (T2D2), (b) three-node curved truss element (T2D3).

defined by a 90° counter-clockwise rotation from the direction going from node 1 to node 2 (and to node 3, if relevant) of the element. Since the stays used in this structure are made from steel cables they are completely straight throughout the entire process (unless they go slack). Therefore the two-node straight truss element, T2D2, is selected to model the stays together with the ‘no compression’ option implemented. The FE model is very similar in essence to those developed by Saito and Wadee (2008) and Osofero *et al.* (2013), which were calibrated using analytical models and physical experiments.

3.2.2 Benchmark prestress

An initial prestress is applied to the cable stays in order to obtain benefit in terms of increasing the load carrying capacity. As mentioned in Chapter 2, the relationship between the critical buckling load and different levels of prestress for the single-bay prestressed stayed column has been found in previous work (Hafez *et al.* , 1979). In order to find the theoretical optimal prestress for the triple-bay prestressed stayed column, a similar, but extended and hence more complex method, is used to evaluate the relationship between the critical buckling load and different levels of prestress.

Initially the stays are prestressed prior to any external loads. The main column is then loaded axially and the following assumptions are made:

1. The external load is applied axially and concentrically to the column. No initial out-of-straightness exists in the column, which means that no lateral deflections occur prior to buckling.
2. The connections between the column and the crossarms are rigid. The connections between the stays and the main column and between the stays and crossarms are perfectly pinned – this is as depicted in Figure 3.4

Figure 3.6 shows the deformed shape of the triple-bay prestressed stayed column and the

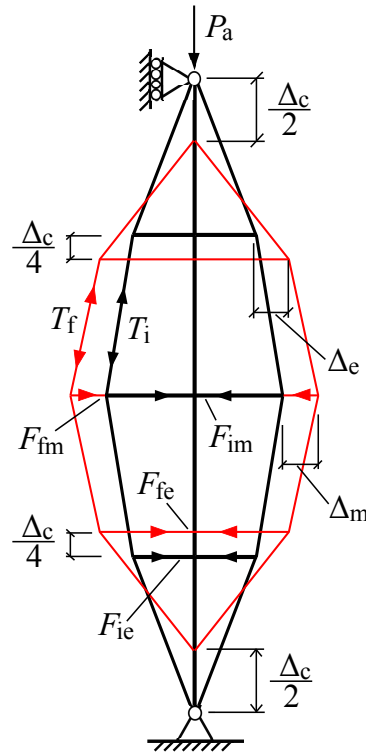


Figure 3.6: The axially deformed shape of the triple-bay prestressed stayed column and internal forces.

internal forces for the purpose of investigating the geometric and mechanical relationship of each components. Before the application of the external load, there are internal forces introduced by the prestress in the stays existing in the structures. The initial axial force P_i in the main column is provided purely by the vertical components of the prestress in

the stays:

$$P_i = 2T_i \cos \beta_e, \quad (3.1)$$

where T_i is the initial prestress in the stays and β_e is the angle between the main column and stay 1 as shown in Figure 3.4. The initial compressive forces F_{ie} and F_{im} in the edge and the middle crossarms are provided by the horizontal components of the resultant force from the prestressing force in the stays. Thus they can be expressed respectively as:

$$F_{ie} = T_i(\sin \beta_e - \sin \beta_m), \quad (3.2)$$

$$F_{im} = 2T_i \sin \beta_m, \quad (3.3)$$

where β_m is the angle between stay 2 and the vertical direction, as shown in Figure 3.4. After the application of the external load P_a , the final axial load P_f in the main column is expressed as:

$$P_f = P_a + 2T_f \cos \beta_e, \quad (3.4)$$

where T_f is the final force in the stays. Similarly, The final forces F_{fe} and F_{fm} in the edge and the middle crossarms are expressed respectively as:

$$F_{fe} = T_f(\sin \beta_e - \sin \beta_m), \quad (3.5)$$

$$F_{fm} = 2T_f \sin \beta_m. \quad (3.6)$$

From the derivation of the initial and final forces within the main column, the edge and the middle crossarms, the changes in length of these elements can be obtained. The end-shortening of the main column Δ_c is expressed as:

$$\Delta_c = \frac{P_f - P_i}{K_c}, \quad (3.7)$$

in which $K_c = E_c A_c / L$, where E_c and A_c are the Young's Modulus and cross-sectional area

of the main column respectively with L being the column length. Substituting Equations (3.1) and (3.4) into Equation (3.7):

$$\Delta_c = \frac{(P_a + 2T_f \cos \beta_e - 2T_i \cos \beta_e)L}{E_c A_c}. \quad (3.8)$$

As the main column is shortened axially, some of the prestressing forces are reduced within the stays. As a result, the edge and the middle crossarms are elongated owing to the smaller horizontal component of the prestress compared with the initial state. The elongation of the edge crossarm Δ_e is expressed as:

$$\Delta_e = \frac{F_{ie} - F_{fe}}{K_e}, \quad (3.9)$$

in which $K_e = E_e A_e / a_e$, where E_e and A_e are the Young's Modulus and cross-section area of the edge crossarm respectively with a_e being the edge crossarm length. Substituting Equations (3.2) and (3.5) into Equation (3.9), the following expression is obtained:

$$\Delta_e = \frac{(T_i - T_f)(\sin \beta_e - \sin \beta_m)a_e}{E_e A_e}. \quad (3.10)$$

Similarly, the elongation of the middle crossarm Δ_m is expressed thus:

$$\Delta_m = \frac{F_{im} - F_{fm}}{K_m}, \quad (3.11)$$

in which $K_m = E_m A_m / a_m$, where E_m and A_m are the Young's Modulus and cross-sectional area of the middle crossarm respectively with a_m being the middle crossarm length. Substituting Equations (3.3) and (3.6) into Equation (3.11), the following expression is obtained:

$$\Delta_m = \frac{2a_m(T_i - T_f) \sin \beta_m}{E_m A_m}. \quad (3.12)$$

The change in length of stay 2, Δ_{s2} , can be expressed by the elastic relationship:

$$\Delta_{s2} = \frac{T_i - T_f}{K_{s2}}, \quad (3.13)$$

in which $K_{s2} = E_s A_s / l_2$, where E_s and A_s are the Young's Modulus and cross-sectional area of the stay respectively with l_2 being the original length of stay 2. As shown in Figure 3.7, the shortening in length of stay 2 is expressed thus:

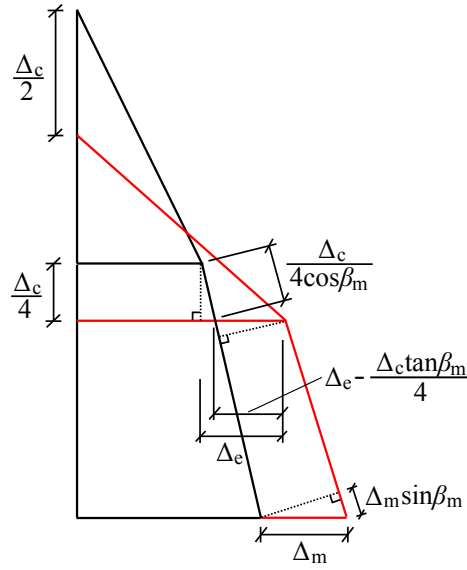


Figure 3.7: The geometrical relationship between the original and the axially deformed structure.

$$\Delta_{s2, \text{shortening}} = \frac{\Delta_c}{4 \cos \beta_m} + \left(\Delta_e - \frac{1}{4} \Delta_c \tan \beta_m \right) \sin \beta_m, \quad (3.14)$$

where the elongation of stay 2 is expressed:

$$\Delta_{s2, \text{elongation}} = \Delta_m \sin \beta_m. \quad (3.15)$$

Therefore, the change of length in stay 2 can also be expressed by the geometrical rela-

tionship given by the difference between Equations (3.14) and (3.15):

$$\Delta_{s2} = \frac{\Delta_c}{4 \cos \beta_m} + \left(\Delta_e - \frac{1}{4} \Delta_c \tan \beta_m \right) \sin \beta_m - \Delta_m \sin \beta_m. \quad (3.16)$$

Substituting Equations (3.8), (3.10) and (3.12) into (3.16) gives:

$$\begin{aligned} \frac{(T_i - T_f)l_2}{E_s A_s} = & \frac{P_a - 2(T_i - T_f) \cos \beta_e}{4E_c A_c \cos \beta_m} L - \frac{2(T_i - T_f) \sin \beta_m}{E_m A_m} a_m \sin \beta_m + \\ & \left[\frac{(T_i - T_f)(\sin \beta_e - \sin \beta_m)}{E_e A_e} a_e - \frac{P_a - 2(T_i - T_f) \cos \beta_e}{4E_c A_c} L \tan \beta_m \right] \sin \beta_m. \end{aligned} \quad (3.17)$$

Re-arranging Equation (3.17) the following expression is obtained:

$$T_i - T_f = \frac{P_a \cos \beta_m}{2K_c \left[\frac{2}{K_{s2}} + \frac{\cos^2 \beta_m}{K_c} + \frac{4 \sin^2 \beta_m + 2\gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]}, \quad (3.18)$$

where $\gamma = a_e/a_m$. Substituting Equation (3.4) into (3.8) gives:

$$\Delta_c = \frac{P_f - 2T_i \cos \beta_e}{K_c}, \quad (3.19)$$

and substituting Equation (3.19) into (3.16) gives:

$$\begin{aligned} \frac{(T_i - T_f)}{K_{s2}} = & \frac{P_f - 2T_i \cos \beta_e}{4K_c \cos \beta_m} - \frac{2(T_i - T_f) \sin^2 \beta_m}{K_m} + \\ & \left[\frac{(T_i - T_f)(\sin \beta_e - \sin \beta_m)}{K_e} - \frac{P_f - 2T_i \cos \beta_e}{4K_c} \tan \beta_m \right] \sin \beta_m. \end{aligned} \quad (3.20)$$

Re-arranging Equation (3.20) gives:

$$T_f = T_i - \frac{(P_f - 2T_i \cos \beta_e) \cos \beta_m}{4K_c \left[\frac{1}{K_{s2}} + \frac{2 \sin^2 \beta_m + \gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]}. \quad (3.21)$$

Substituting Equation (3.21) into (3.4) and re-arranging further gives:

$$P_a = (P_f - 2T_i \cos \beta_e) \left\{ 1 + \frac{\cos \beta_e \cos \beta_m}{2K_c \left[\frac{1}{K_{s2}} + \frac{2 \sin^2 \beta_m + \gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]} \right\}. \quad (3.22)$$

As described in Chapter 2, Zone 1 represents a very small prestress that would entirely vanish from the stays before the column buckles when the external axial load is smaller than or equal to Euler load. Therefore the n th buckling load for the triple-bay prestressed column is exactly the n th Euler buckling load for the unstayed column:

$$P^C = \frac{n^2 \pi^2 E_c I_c}{L^2}. \quad (3.23)$$

In Zone 2, the prestress is sufficiently large to make the critical buckling load to be larger than the Euler load, but the prestress vanishes at the instant buckling is triggered. Therefore, Equation (3.18) represents the relationship between the initial prestress and the external load when substituting $T_f = 0$, indicating that there are no residual forces in the stays when the column buckles. Hence, the initial prestressing force in Zone 2, T_i , is:

$$T_i = P_a C_{11}, \quad (3.24)$$

where:

$$C_{11} = \frac{\cos \beta_m}{2K_c \left[\frac{2}{K_{s2}} + \frac{\cos^2 \beta_m}{K_c} + \frac{4 \sin^2 \beta_m + 2\gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]}. \quad (3.25)$$

In Zone 3, the initial prestress is relatively large so that there are residual forces in the stays after the column buckles. The vertical components of the prestressing force itself on the column make a negative contribution to the column load carrying capacity. The

critical buckling load is governed by the expression:

$$P_a = (P_f - 2T_i \cos \beta_e) C_{22}, \quad (3.26)$$

where:

$$C_{22} = 1 + \frac{\cos \beta_e \cos \beta_m}{2K_c \left[\frac{1}{K_{s2}} + \frac{2 \sin^2 \beta_m + \gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]}. \quad (3.27)$$

The prestressing force T_{opt} , defined at the boundary of Zones 2 and 3, has been historically termed the ‘optimal prestressing force’, since it represents the prestressing force that maximizes the critical buckling load of the column. The expression for T_{opt} can be obtained by combining Equations (3.24) and (3.26):

$$T_{\text{opt}} = P_{\text{max}}^C C_{11}, \quad (3.28)$$

where:

$$P_{\text{max}}^C = \frac{P_{\text{Zone3}, T=0}^C}{C_{22}}. \quad (3.29)$$

The value of $P_{\text{Zone3}, T=0}^C$ is evaluated from linear eigenvalue analysis through ABAQUS without the consideration of the prestress. Note that the optimal prestress T_{opt} is used as a benchmark in the current study since it is derived from a purely elastic model that does not take any geometrically nonlinear behaviour into account. It has been found previously (Saito & Wadee, 2010) that the actual optimal prestress for the single crossarm column case, in terms of the real load carrying capacity for geometrically imperfect stayed columns, is considerably larger than the theoretical value obtained from the expression above. The evaluation of the actual optimal prestress will be discussed in later chapters.

3.2.3 Imperfections

Imperfections in the column of course exist in reality and they are thus taken into consideration in the current study. The imperfect column end-shortening $\mathcal{E}_0$ is introduced to model the size of the imperfections (Wadee, 2000). As stated earlier, the symmetric and the anti-symmetric buckling modes of the column and their combinations are considered in this study. Figure 3.8 shows a typical combination of the buckling modes of the triple-bay column. Therefore, a combination of the expressions for Modes 1 and 2, with corresponding scale factors, should be used to ensure interactive buckling is also included. The first order approximation of the end-shortening is expressed as follows to ensure that imperfections of commensurate magnitude may be compared regardless of the detailed shape of the initial imperfections (Wadee, 2000; Wadee & Simões da Silva, 2005):

$$\mathcal{E}_0 = \int_0^L \frac{1}{2} [W_1'^2(x) + W_2'^2(x)] dx, \quad (3.30)$$

where:

$$W_1(x) = \delta L \mu_1 \sin \frac{\pi x}{L}, \quad W_2(x) = \delta L \mu_2 \sin \frac{2\pi x}{L}, \quad (3.31)$$

in which $W_1(x)$ and $W_2(x)$ are shape functions affine to the symmetric mode (Mode 1) and the anti-symmetric mode (Mode 2) of the column, δ is the normalized amplitude of the initial out-of-straightness of the column; μ_1 and μ_2 are the scale factors for the imperfection comprising two modes, representing the contribution of the two distinct modes to the overall buckling shape. Substituting Equation (3.31) into Equation (3.30) gives:

$$\mathcal{E}_0 = \frac{1}{4} \pi^2 \delta^2 L_c (\mu_1^2 + 4\mu_2^2). \quad (3.32)$$

The end-shortening of the column should be the same when triggering Mode 1, Mode 2 and interactive buckling to ensure a fair comparison of the results, rather than comparing them based on the same lateral amplitude. Therefore the scale factors of the two modes

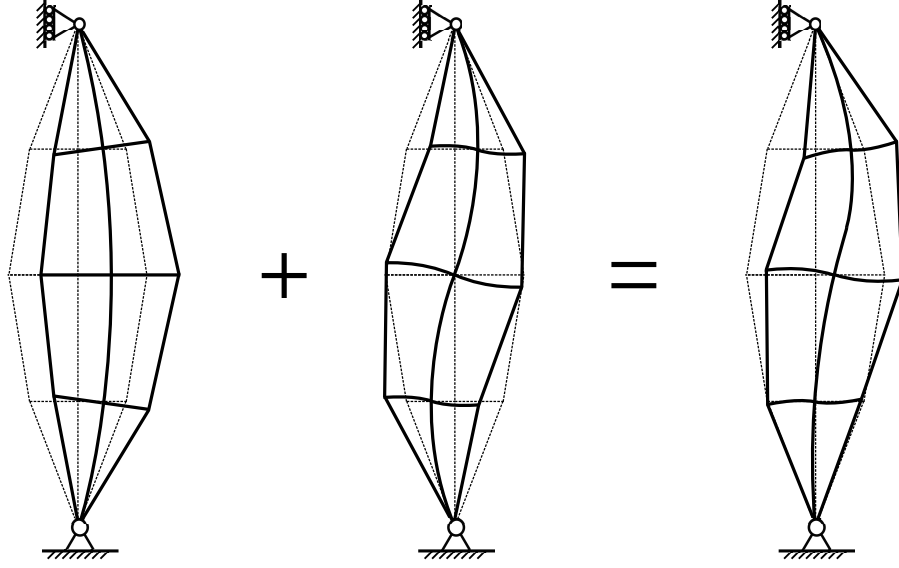


Figure 3.8: Combination of linear buckling modes of the triple-bay stayed column used to trigger interactive buckling in the FE analysis.

satisfy the following expression:

$$\mu_1^2 + 4\mu_2^2 = 1. \quad (3.33)$$

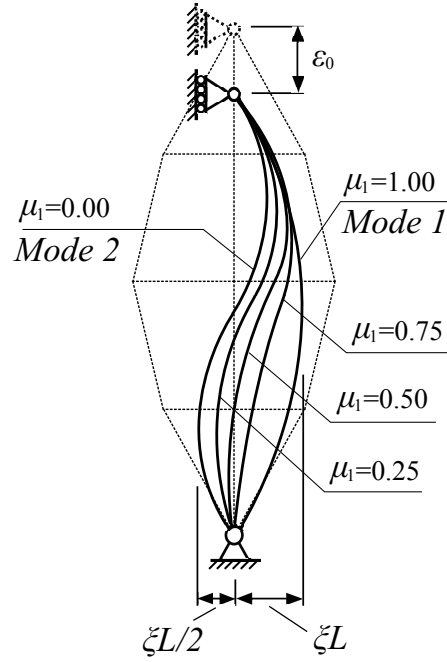
In this study, the same selection of μ_1 and μ_2 is adopted for analysis as compared with previous work (Saito & Wadee, 2009b). The selection shown in Table 3.1 represents the

Table 3.1: Selected combinations of scale factors μ_1 and μ_2 .

	Mode 1	Case 1	Case 2	Case 3	Mode 2
μ_1	1.000	0.750	0.500	0.250	0.000
μ_2	0.000	0.331	0.433	0.484	0.500

contribution that each distinct mode makes to the final buckling mode. Apart from pure Modes 1 and 2 where $\mu_2 = 0$ and $\mu_1 = 0$ respectively, three other sets of the imperfection are used to represent interactive buckling, the specified combination of the two pure modes. A transition from Mode 1 to 2, including the asymmetric interactive buckling cases, is depicted in Figure 3.9.

Different values of the initial out-of-straightness δ are also introduced in order to investigate sensitivities to imperfections in the post-buckling behaviour. Currently, values of 1/300,

Figure 3.9: Variation of the imperfection profile with μ_1 varying.

1/1500 and 1/10000, which represent the EC3 design values for hot rolled sections (BS EN 1993-1-1, 2005), experimental measurements (Wong & Temple, 1982), and a nearly perfect case (Saito & Wadee, 2009b) respectively are used (See Table 3.2). Unless stated

Table 3.2: Selected imperfection sizes for the sensitivity study.

No.	Initial out of straightness δ	
0	0	Perfect
1	1/10000	Nearly perfect
2	1/3000	
3	1/1500	Experimental measurements (Wong & Temple, 1982)
4	1/1000	
5	1/500	Manufacturing tolerance (BS EN10210-2, 2006)
6	1/300	EC3 design value for hot rolled sections (BS EN 1993-1-1, 2005)
7	1/200	EC3 design value for cold formed sections (BS EN 1993-1-1, 2005)

otherwise, the imperfection size δ adopted in the post-buckling analysis in this chapter is 1/300, which represents the EC3 design values for hot rolled sections (BS EN 1993-1-1, 2005). This value effectively includes both initial geometric deformations and residual stresses.

It is also worth noting that, in this study, the imperfections are introduced prior to applying any prestress, which replicates the case that would occur in a natural construction sequence. Therefore, the actual imperfection will be effectively amplified as soon as the prestress is introduced to the stays.

3.3 Results and discussion

To conduct the parametric study, the material and geometric properties that are fixed are presented in Table 3.3, the numerical values of which are primarily based on previous

Table 3.3: Material properties and structural dimensions of the triple-bay prestressed stayed columns.

Young's modulus of the main column and crossarms	$E_c = E_a = 201 \text{ kN/mm}^2$
Young's modulus of the stays	$E_s = 202 \text{ kN/mm}^2$
Outside diameter of the main column and crossarms	$\phi_{co} = \phi_{ao} = 38.1 \text{ mm}$
Inside diameter of the main column and crossarms	$\phi_{ci} = \phi_{ai} = 25.4 \text{ mm}$
Length of the main column	$L = 5080 \text{ mm}$
Yield stress of the main column and crossarm	$f_y = 355 \text{ N/mm}^2$
Euler buckling load of the main column	$P_E = 6.38 \text{ kN}$
Normalized slenderness of the main column	$\bar{\lambda} = 5.94$

studies (Hafez *et al.* , 1979; Saito & Wadee, 2008). The length of the main column L is fixed to 5080 mm. The prestressing force in the stays $T = T_{\text{opt}}$ is also fixed for the most part, but is varied at specific points in the investigation. The normalized slenderness of the main column element $\bar{\lambda}$ is defined by the well known expression (Trahair *et al.* , 2008):

$$\bar{\lambda} = \sqrt{P_y/P_E}, \quad (3.34)$$

with P_y being the squash load and P_E being the Euler buckling load of the main column element, where:

$$P_y = A_c f_y, \quad P_E = \frac{\pi^2 E_c I_c}{L^2}. \quad (3.35)$$

The study focuses on three parameters: the stay diameter ϕ_s , the column aspect ratio $2\alpha_m$ where $\alpha_m = a_m/L$, and the length ratio of the edge to the middle crossarm $\gamma = a_e/a_m$. When the model is investigated with ϕ_s varying, $2\alpha_m$ is fixed to 0.2 and γ is fixed to 0.8. Conversely, when varying $2\alpha_m$, ϕ_s is fixed to 4.8 mm and γ is fixed to 0.8; when varying γ , ϕ_s is fixed to 4.8 mm and $2\alpha_m$ is fixed to 0.2. In other words, $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$ are the benchmarks used when studying the effect of the other parameters. The particular cases defined from varying the aforementioned parameters are listed in Table 3.4 below.

Table 3.4: Cases from varying the stay diameter ϕ_s (when $2\alpha_m = 0.2$ and $\gamma = 0.8$), the column aspect ratio $2\alpha_m$ (when $\phi_s = 4.8$ mm and $\gamma = 0.8$) and the crossarm length ratio γ (when $\phi_s = 4.8$ mm and $2\alpha_m = 0.2$).

Case ϕ_s (mm)	F1 1.6	F2 3.2	F3 4.8	F4 6.4	F5 8.0	F6 10.0
Case $2\alpha_m$	a1 0.05	a2 0.10	a3 0.15	a4 0.20	a5 0.25	a6 0.30
Case γ	G1 0.5	G2 0.6	G3 0.7	G4 0.8	G5 0.9	G6 1.0

3.3.1 Linear buckling analysis

Buckling analysis in ABAQUS determines linear eigenvalues and eigenvectors of critical equilibrium modes. As described towards the end of §3.2.2, the stays are modelled and taken into consideration in the buckling analysis although the prestress T is not introduced at that stage. The buckling loads obtained can then be used to calculate the theoretical optimal prestress using Equations (3.28) and (3.29), while the eigenvectors are used as the initial imperfections in the main column for the post-buckling analysis.

Figure 3.10 shows the buckling loads of the column with the stay diameter ϕ_s , the column aspect ratio $2\alpha_m$ and the crossarm length ratio γ varying for the linear eigenvalue analysis. It can be seen in Figure 3.10(a) that, when the stay diameter ϕ_s is relatively small, Mode 1

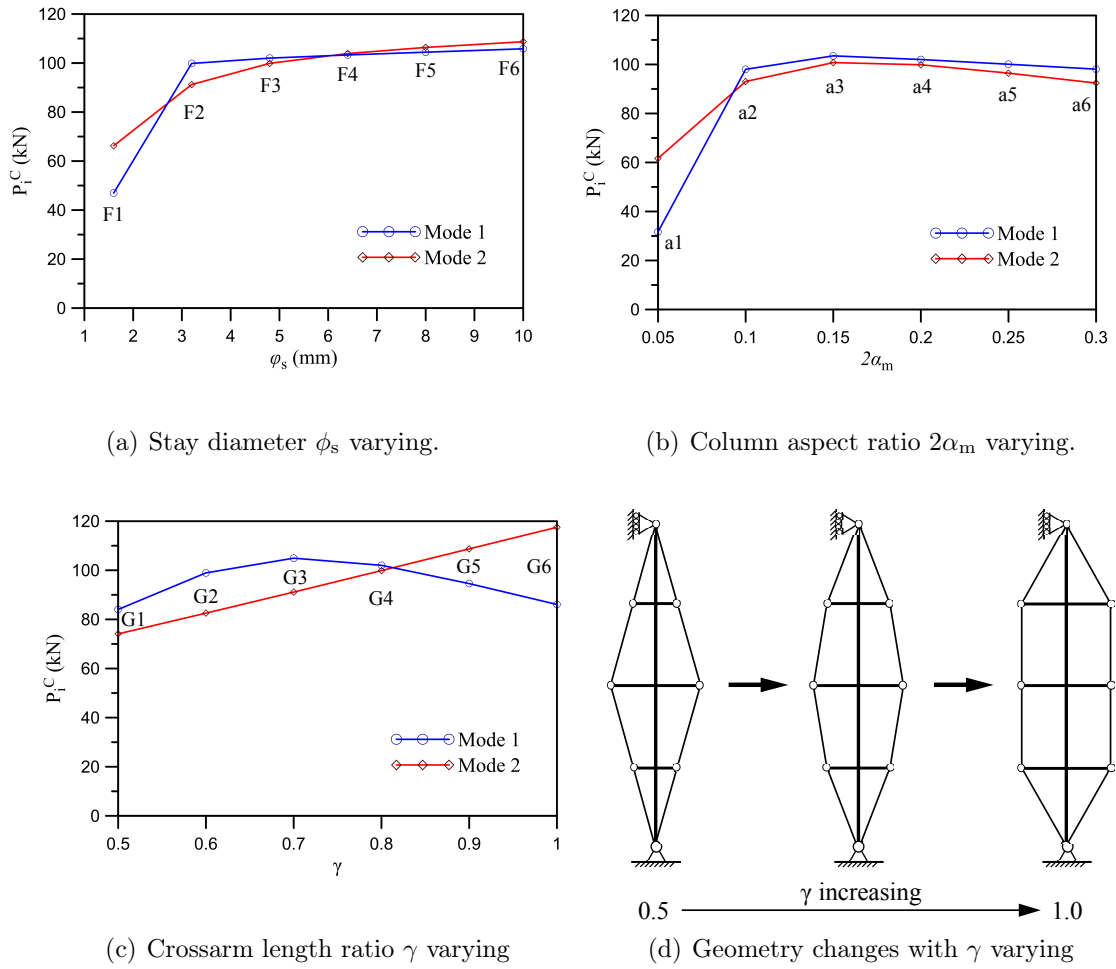


Figure 3.10: Buckling loads of the column with (a) the stay diameter ϕ_s , (b) the column aspect ratio $2\alpha_m$ and (c) the crossarm length ratio γ varying; (d) shows how the stayed column geometry changes with γ .

is critical. Along with the increase of the stay diameter, the buckling loads of Modes 1 and 2 become closer together. Mode 2 becomes critical when the stay diameter ϕ_s is increased to 3.2 mm and keeps governing until the stay diameter is 6.4 mm, when Mode 1 becomes critical once again. Similarly, Figure 3.10(b) indicates that when the column aspect ratio, which is represented by $2\alpha_m$, is relatively small, Mode 1 is critical. Mode 2 becomes critical when $2\alpha_m$ is increased to 0.1 and continues to govern beyond that point. Figure 3.10(c) shows the critical buckling loads of the column with γ varying. The value $\gamma = 0.5$ indicates that the edge crossarms are exactly half of the length of the middle crossarm, while $\gamma = 1.0$ indicates that all the crossarms have the same length. When the edge crossarm length is short, Mode 2 is critical. Along with the increase in γ , the buckling load of Mode 2 keeps increasing, while the buckling load of Mode 1 increases to a maximum value when $\gamma = 0.7$ and then begins to decrease again. As a result, Mode 1 governs again when $\gamma > 0.8$. It is also important to note that the lowest critical loads occur when $\gamma = 0.5$ and 1.0, indicating that the edge crossarms and the middle crossarm are less effective in those respective cases. The largest critical load can be found when $\gamma = 0.8$, which is the reason why this γ value is chosen as the benchmark while varying the other parameters.

With the derivation of the critical loads for different cases, the theoretical optimal prestressing force can be calculated by using the expressions stated above. Tables 3.5, 3.6 and 3.7 show the values of the calculated theoretical optimal prestress T_{opt} for the triple-bay

Table 3.5: The optimal prestressing force T_{opt} obtained with the stay diameter ϕ_s varying.

Case	ϕ_s (mm)	P_1^C (kN)	P_2^C (kN)	Critical mode	P^C (kN)	T_{opt} (kN)
F1	1.6	46.93	66.24	1	46.93	0.15
F2	3.2	99.85	91.24	2	91.24	1.10
F3	4.8	102.02	99.87	2	99.87	2.56
F4	6.4	103.31	103.88	1	103.31	4.36
F5	8.0	104.44	106.38	1	104.44	6.25
F6	10.0	105.81	108.70	1	105.81	8.58

prestressed stayed column with the stay diameter ϕ_s , the column aspect ratio $2\alpha_m$ and the crossarm length ratio γ varying respectively.

Table 3.6: The optimal prestressing force T_{opt} obtained with the column aspect ratio $2\alpha_m$ varying.

Case	$2\alpha_m$	P_1^C (kN)	P_2^C (kN)	Critical mode	P^C (kN)	T_{opt} (kN)
a1	0.05	31.71	61.68	1	31.71	0.81
a2	0.10	98.01	93.05	2	93.05	2.39
a3	0.15	103.51	100.76	2	100.76	2.59
a4	0.20	102.02	99.87	2	99.87	2.56
a5	0.25	100.09	96.45	2	96.45	2.48
a6	0.30	98.08	92.41	2	92.41	2.37

Table 3.7: The optimal prestressing force T_{opt} obtained with the crossarm length ratio γ varying.

Case	γ	P_1^C (kN)	P_2^C (kN)	Critical mode	P^C (kN)	T_{opt} (kN)
G1	0.5	84.05	74.11	2	74.11	1.85
G2	0.6	98.94	82.55	2	82.55	2.09
G3	0.7	104.93	91.12	2	91.12	2.32
G4	0.8	102.02	99.87	2	99.87	2.56
G5	0.9	94.57	108.73	1	94.57	2.44
G6	1.0	86.08	117.51	1	86.08	2.22

3.3.2 Post-buckling analysis

The post-buckling analysis of the model is conducted using the Riks arclength method (Riks, 1979) within ABAQUS such that nonlinear force versus deflection relationships can be evaluated. The advantage of the Riks arclength method over purely load or displacement control is that paths including instabilities such as limit points and snap-backs can be traced so long as the equilibrium is continuous.

Unlike the previously described buckling analysis, Riks analysis takes the prestress in the stays and initial imperfections of the column into consideration. The introduced imperfections are generated by the previously described buckling analysis. As mentioned above, in order to investigate the behaviour of interactive buckling, rather than only the two distinct buckling modes, different combinations of buckling modes are introduced to trigger the practically important post-buckling responses.

Figure 3.11 shows the equilibrium paths presented as plots of the axial load P and the

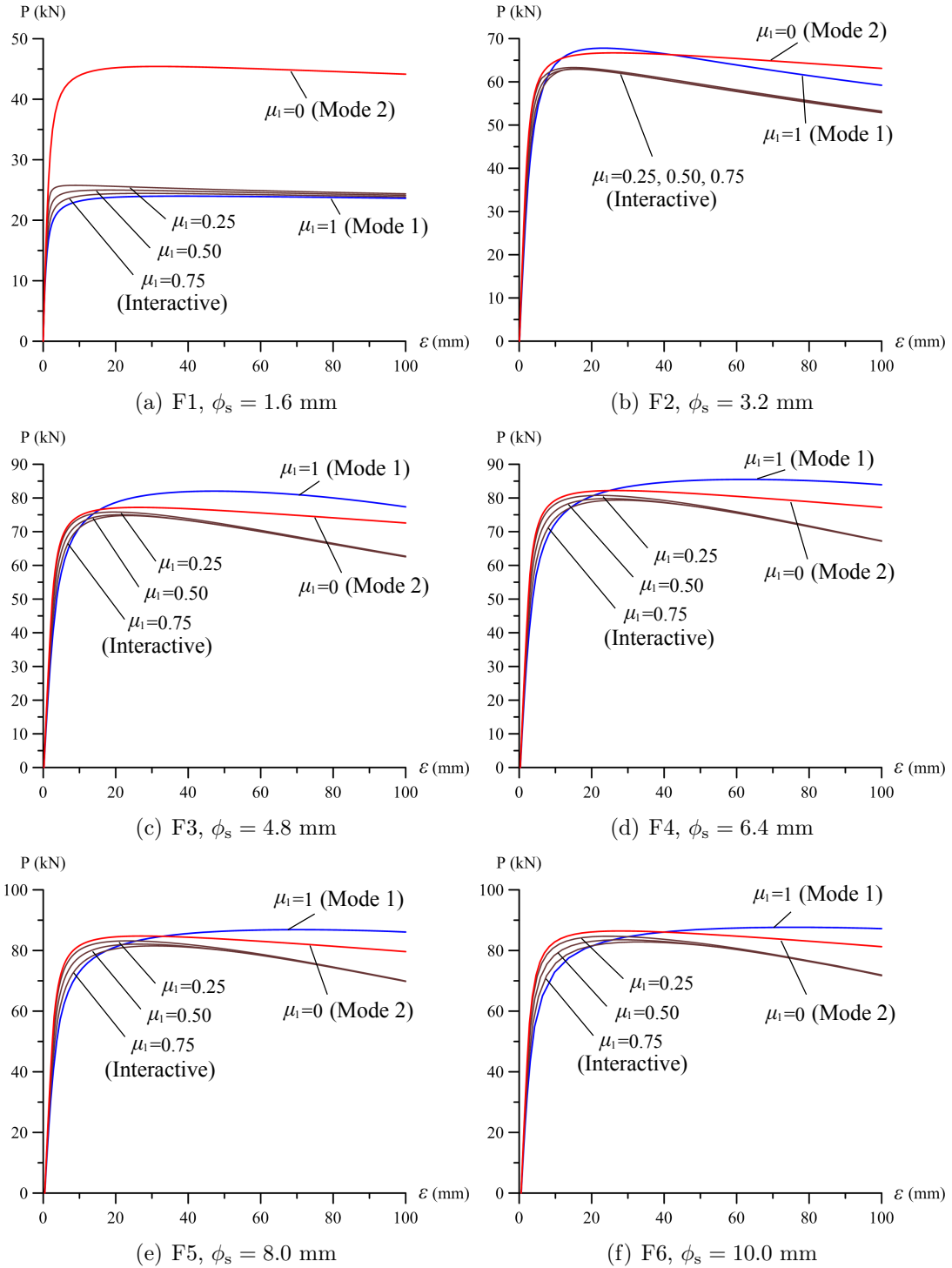


Figure 3.11: Equilibrium paths of the axial load P versus the end-shortening of the main column ε with the stay diameter ϕ_s varying.

column end-shortening $\mathcal{E}$ with the stay diameter ϕ_s varying. A series of μ_1 values are used to model the triggering of different behaviours. It can be seen that Mode 1 governs when the stay diameter ϕ_s is small. The load capacity of Mode 2 is significantly larger than that of Mode 1 while interactive buckling gives a slightly higher load carrying capacity. However, as ϕ_s is increased the buckling loads of Modes 1 and 2 become closer together. When the stay diameter $\phi_s = 3.2$ mm, as shown in Figure 3.11(b), the load capacity of Mode 1 only slightly exceeds that of Mode 2. It can be seen clearly that although Mode 2 is critical, the responses from the cases with imperfections comprising a combination of Modes 1 and 2, thereby exhibiting interactive buckling, are effectively dominant. When the stay diameter ϕ_s keeps increasing, as shown in Figures 3.11(c)–(f), interactive buckling remains as the dominant post-buckling response so long as the ultimate load for the pure Mode 2 imperfect response is lower than that of the pure Mode 1 imperfect response; the mismatch between the maximum loads in Figure 3.11(d)–(f) and Table 3.5 arises from the introduction of the relatively large $L/300$ imperfection.

Figure 3.12 shows the equilibrium paths presented similarly to Figure 3.11 but with the column aspect ratio $2\alpha_m$ varying instead. When the column aspect ratio $2\alpha_m$ is small, for example when $2\alpha_m = 0.05$ and 0.10 , Mode 1 is critical. However, along with the increase of $2\alpha_m$, the load capacities of Modes 1 and 2 become close together. When $2\alpha_m = 0.15$, as shown in Figure 3.12(c), interactive buckling becomes the dominant post-buckling behaviour when the buckling capacity of Mode 1 exceeds that of Mode 2. As $2\alpha_m$ increases, interactive cases are always dominant although Mode 2 is critical.

Figure 3.13 shows the equilibrium paths of the axial load P versus the column end-shortening $\mathcal{E}$ with the crossarm length ratio γ varying. When $\gamma = 0.5$, Mode 1 is critical, but the Mode 2 buckling load is relatively close. It can be seen that interactive buckling with $\mu_1 = 0.75$ is the actual dominant imperfection case. Identical phenomena can also be found in Figures 3.13(b) and 3.13(c), when $\gamma = 0.6$ and 0.7 . Mode 1 is critical compared with Mode 2 while their load capacities are even closer. It is even more obvious that all

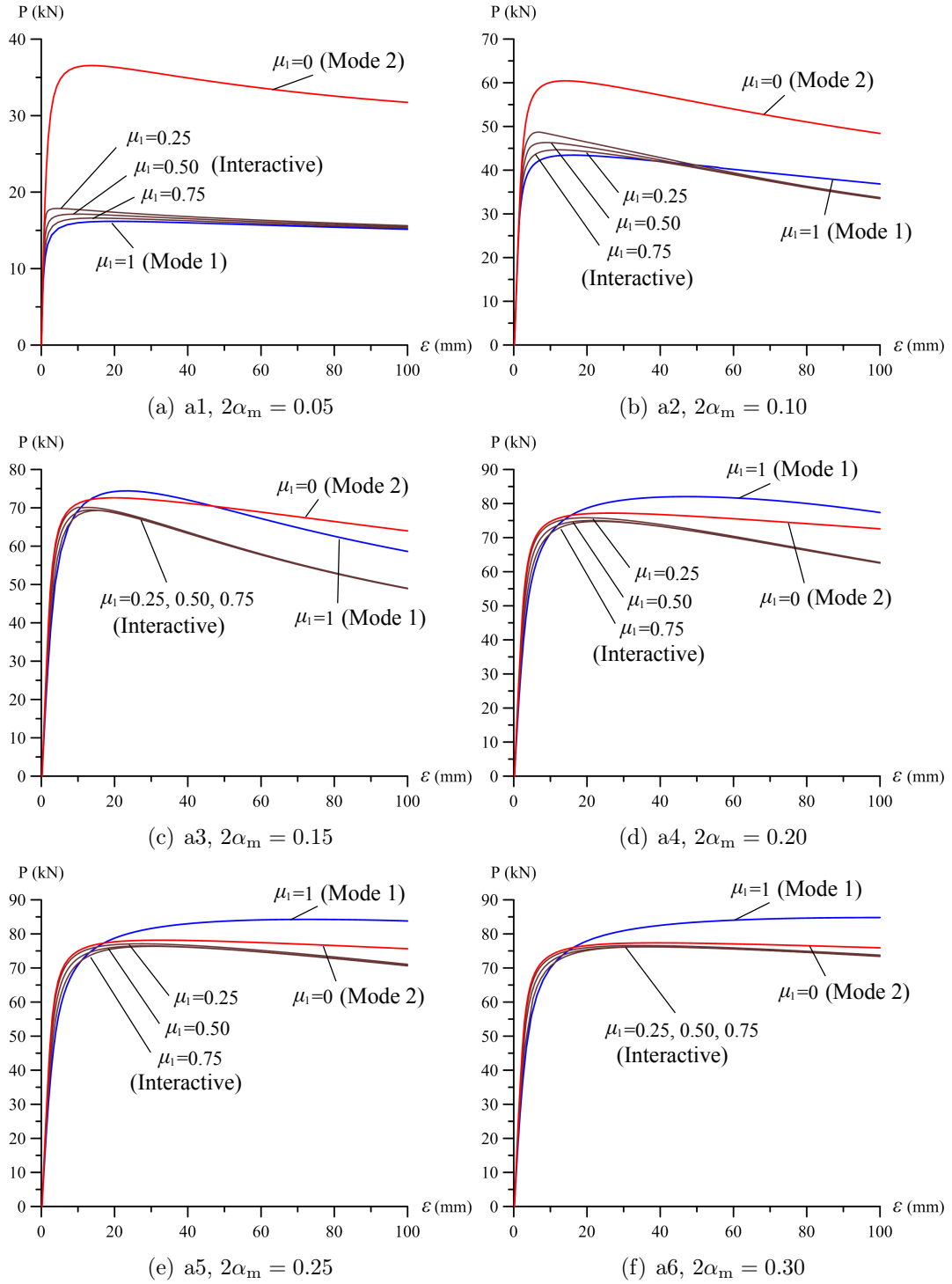


Figure 3.12: Equilibrium paths of the axial load P versus the end-shortening of the main column ε with the column aspect ratio $2\alpha_m$ varying.

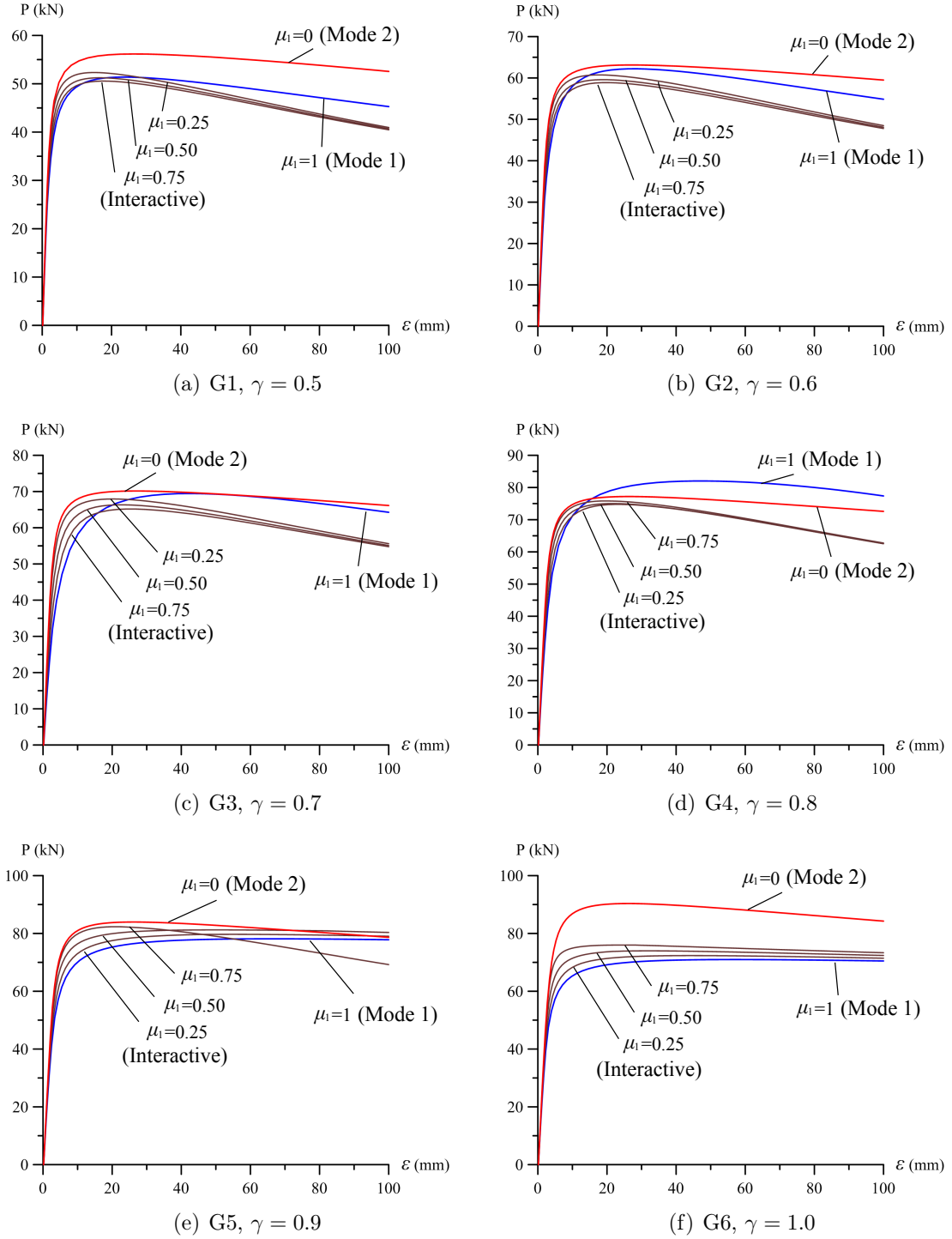


Figure 3.13: Equilibrium paths of the axial load P versus the end-shortening of the main column ε with the crossarm length ratio γ varying.

three interactive cases dominate. Mode 2 becomes critical when $\gamma = 0.8$, while interactive buckling in fact dominates. When γ increases further to 0.9 and 1.0, Mode 1 becomes critical again and also dominates the post-buckling response.

The same conclusion can be reached from the sensitivity study of the mode combination shown in Figures 3.14, 3.15 and 3.16. It can be seen that when Mode 1 is critical, the lowest

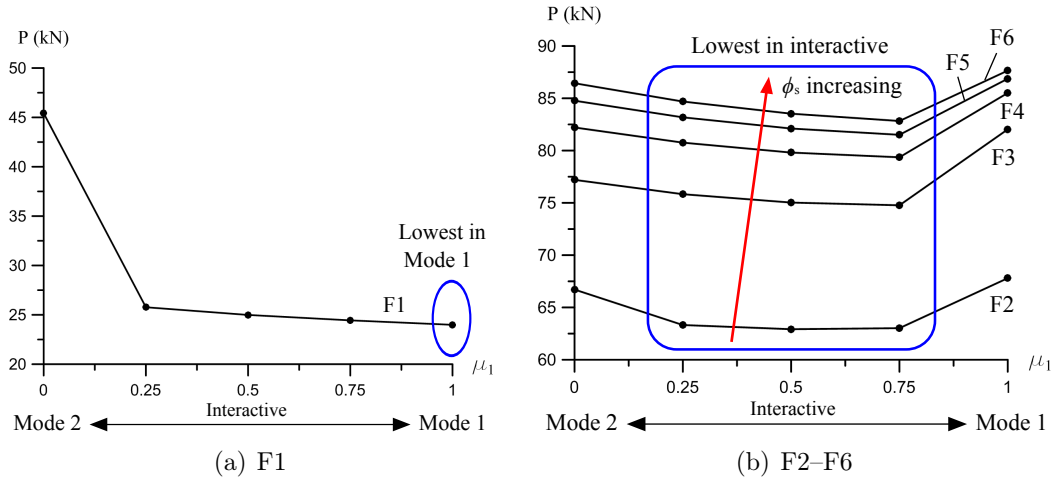


Figure 3.14: Sensitivity to the imperfection combination with the stay diameter ϕ_s varying.

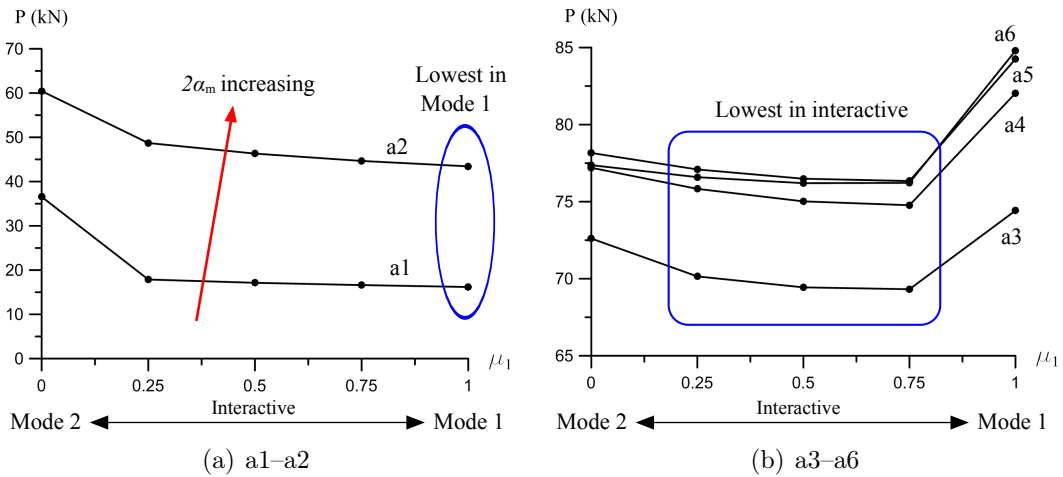


Figure 3.15: Sensitivity to the imperfection combination with the column aspect ratio $2\alpha_m$ varying.

load can also be found with a pure Mode 1 imperfection, as shown in Figures 3.14(a), 3.15(a) and 3.16(a). However, when Mode 2 is critical, the lowest load is always found where the

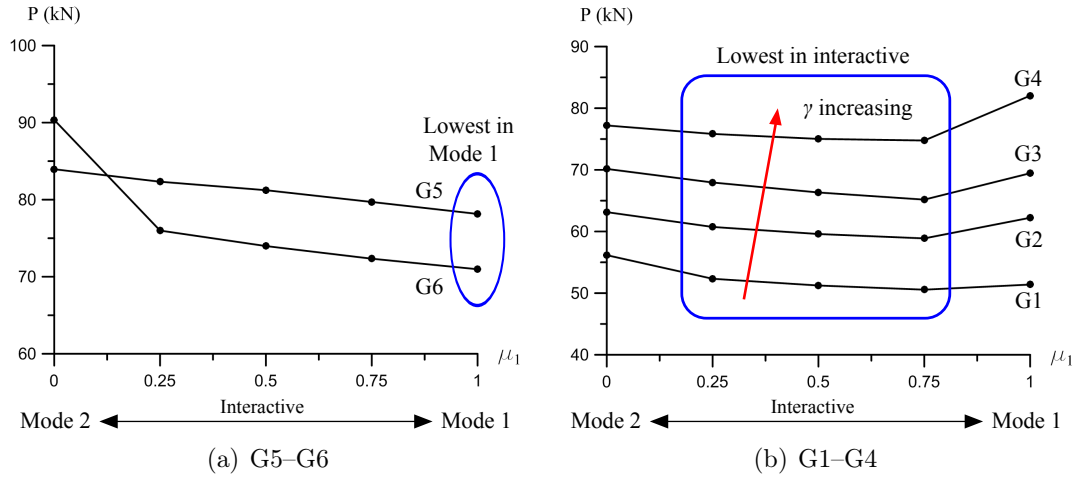


Figure 3.16: Sensitivity to the imperfection combination with the crossarm length ratio γ varying.

non-symmetric imperfections are introduced as shown in Figures 3.14(b) and 3.15(b). In addition, when the buckling loads of Modes 1 and 2 are close, interactive buckling may still be the governing post-buckling behaviour even when Mode 1 is marginally critical, as shown in plots for F4–F6 and G1–G2 in Figure 3.16(b). The observations above indicate that in the post-buckling analysis, when Mode 2 is critical or the buckling loads of the two distinct modes are similar, it is important to investigate the potential of interactive buckling behaviour since it may be the actual governing post-buckling response. This is subtly different from the findings for single-bay stayed columns (Saito & Wadee, 2009b) and may be attributed to the bending of the edge crossarms when Mode 1 buckling occurs.

Figure 3.17 shows the comparison of the critical loads between the linear buckling and the post-buckling modes with the stay diameter ϕ_s , the column aspect ratio $2\alpha_m$ and the crossarm length ratio γ varying. It can be observed that the shapes and trends of the plots from the linear buckling and the post-buckling analysis are very similar for the two distinct buckling modes. However, the capacities derived from the buckling analysis are larger than those from the post-buckling analysis. It is reasonable to assume that this is because of the consideration of imperfections in the post-buckling analysis. It can also be seen that in the post-buckling analysis, when Mode 1 is critical the eigenmode in fact resembles the actual

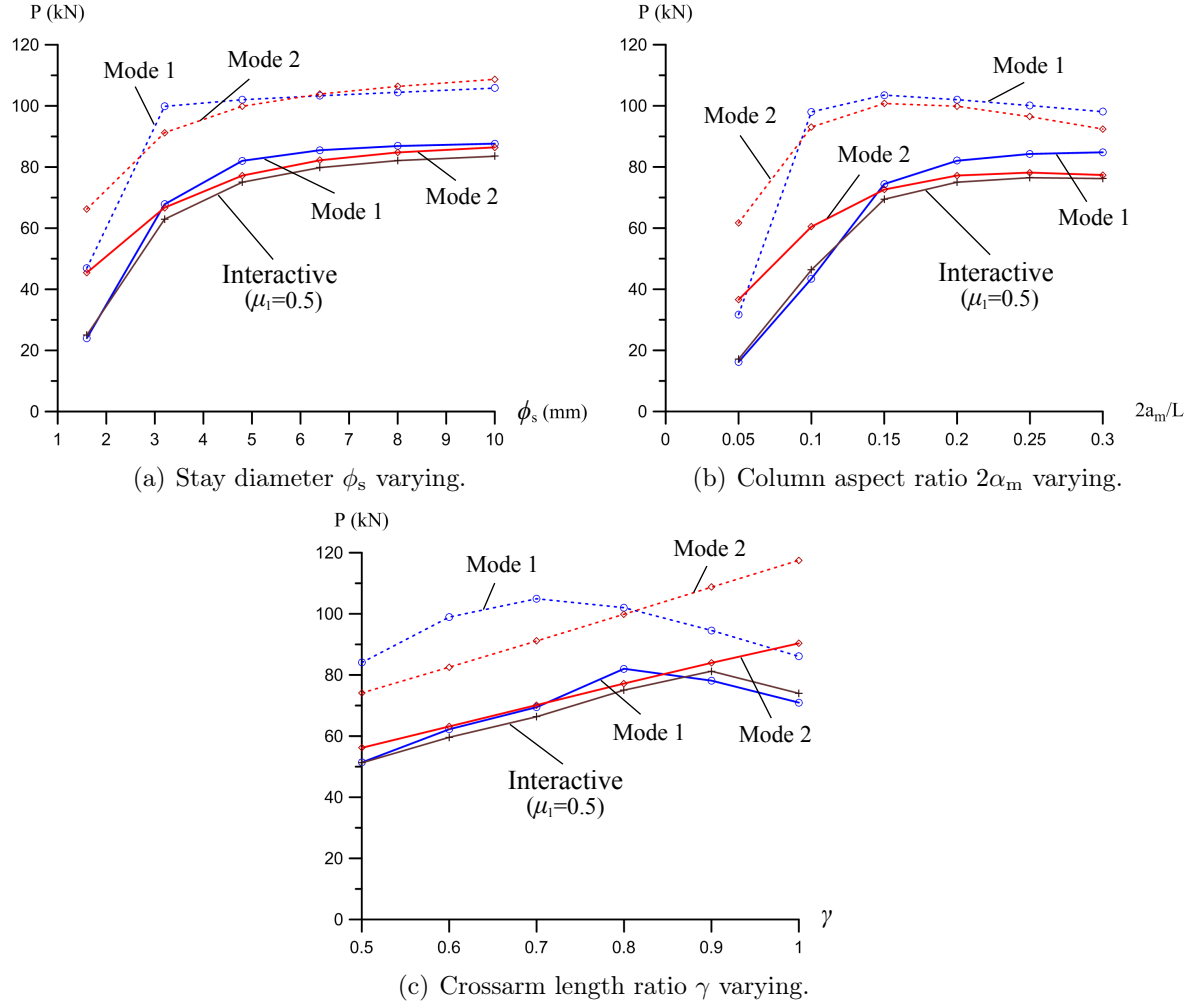


Figure 3.17: Linear and nonlinear load capacities with (a) the stay diameter ϕ_s , (b) the column aspect ratio $2\alpha_m$ and (c) the crossarm length ratio γ varying.

post-buckling mode even taking interactive buckling into consideration. When Mode 2 is critical, however, asymmetric interactive buckling is always the dominant post-buckling behaviour. Moreover, when the loads of Modes 1 and 2 are close, it is still necessary to investigate interactive buckling since it may become the dominant post-buckling response under such circumstances.

3.3.3 Prestress sensitivity

As stated above, the derivation of the optimal prestress T_{opt} in the stay is from a perfect elastic model that does not take any nonlinear behaviour into consideration. In reality, owing to the existence of initial imperfections, lateral displacements would occur as soon as the main column is loaded axially. A previous study (Saito & Wadee, 2010) determined that T_{opt} is not the true optimal prestress for the single-bay prestressed stayed column and that the prestress should be increased from T_{opt} to obtain a higher load carrying capacity, indicating that geometrical nonlinearity plays an important role. The effect of the initial prestress for the triple-bay prestressed stayed column is hence investigated currently.

Figure 3.18 shows the governing equilibrium paths presented by the axial load P versus the end-shortening of the main column $\mathcal{E}$ for the prestress levels T_{opt} and $2T_{\text{opt}}$ for cases with the stay diameter ϕ_s , the column aspect ratio $2\alpha_m$ and the crossarm length ratio γ varying. It can be seen that increasing the initial prestress from T_{opt} to $2T_{\text{opt}}$ indeed provides a higher load capacity to the main column. The benefits in the load capacity vary with different geometric configurations. However, when adopting $2T_{\text{opt}}$ as the initial prestress, the post-buckling stiffness becomes slightly more negative after the peak load is reached in each equilibrium path.

Figure 3.19 shows the sensitivity to the imperfection combination when $T = T_{\text{opt}}$ and $T = 2T_{\text{opt}}$ with the stay diameter ϕ_s , the column aspect ratio $2\alpha_m$ and the crossarm length ratio γ varying. As expected, for all the cases the load carrying capacity increases

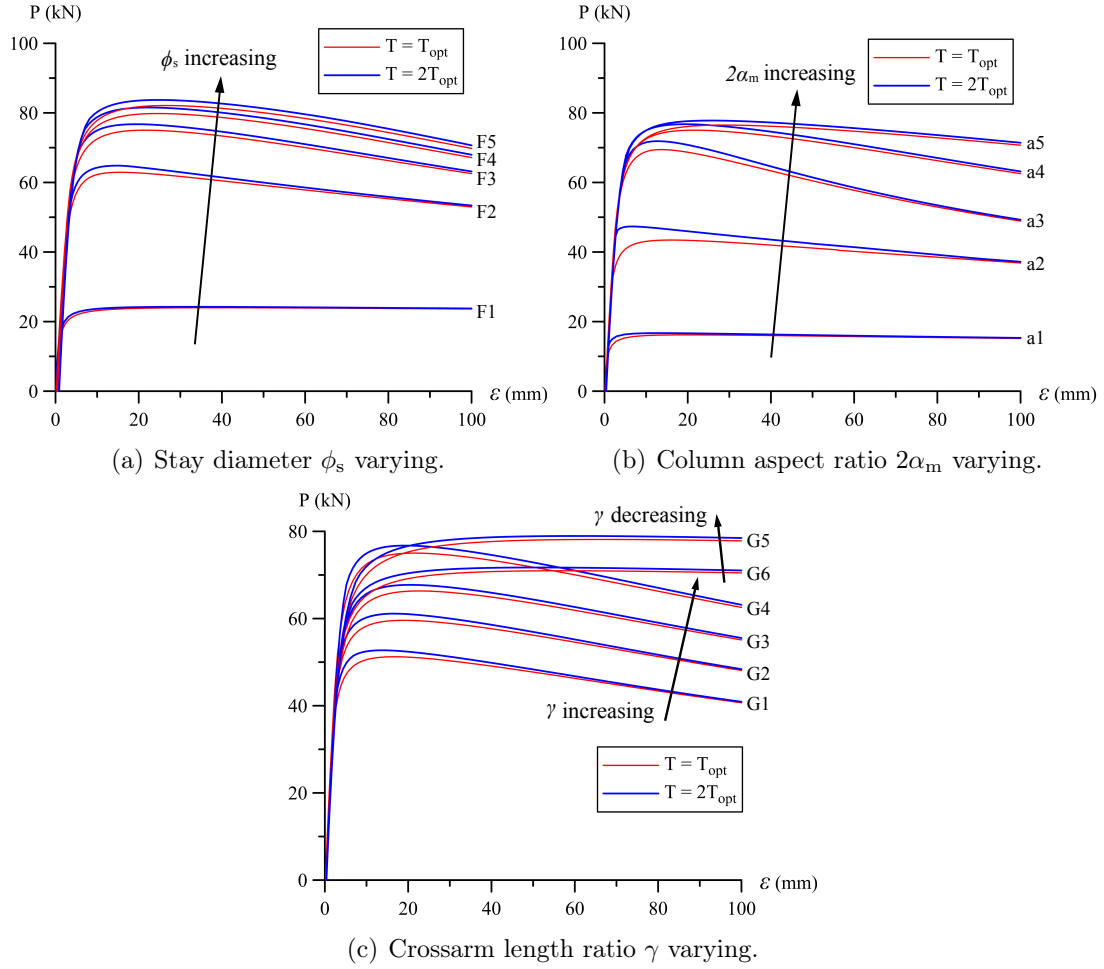


Figure 3.18: Equilibrium paths of the axial load P versus the end-shortening of the main column ε with (a) the stay diameter ϕ_s , (b) the column aspect ratio $2\alpha_m$ and (c) the crossarm length ratio γ varying with the initial prestressing force being $T = T_{\text{opt}}$ and $T = 2T_{\text{opt}}$ with $\mu_1 = 0.5$. Note that in (a) and (b) the cases F6 and a6 are not presented for the sake of clarity since they are very similar to cases F5 and a5 respectively.

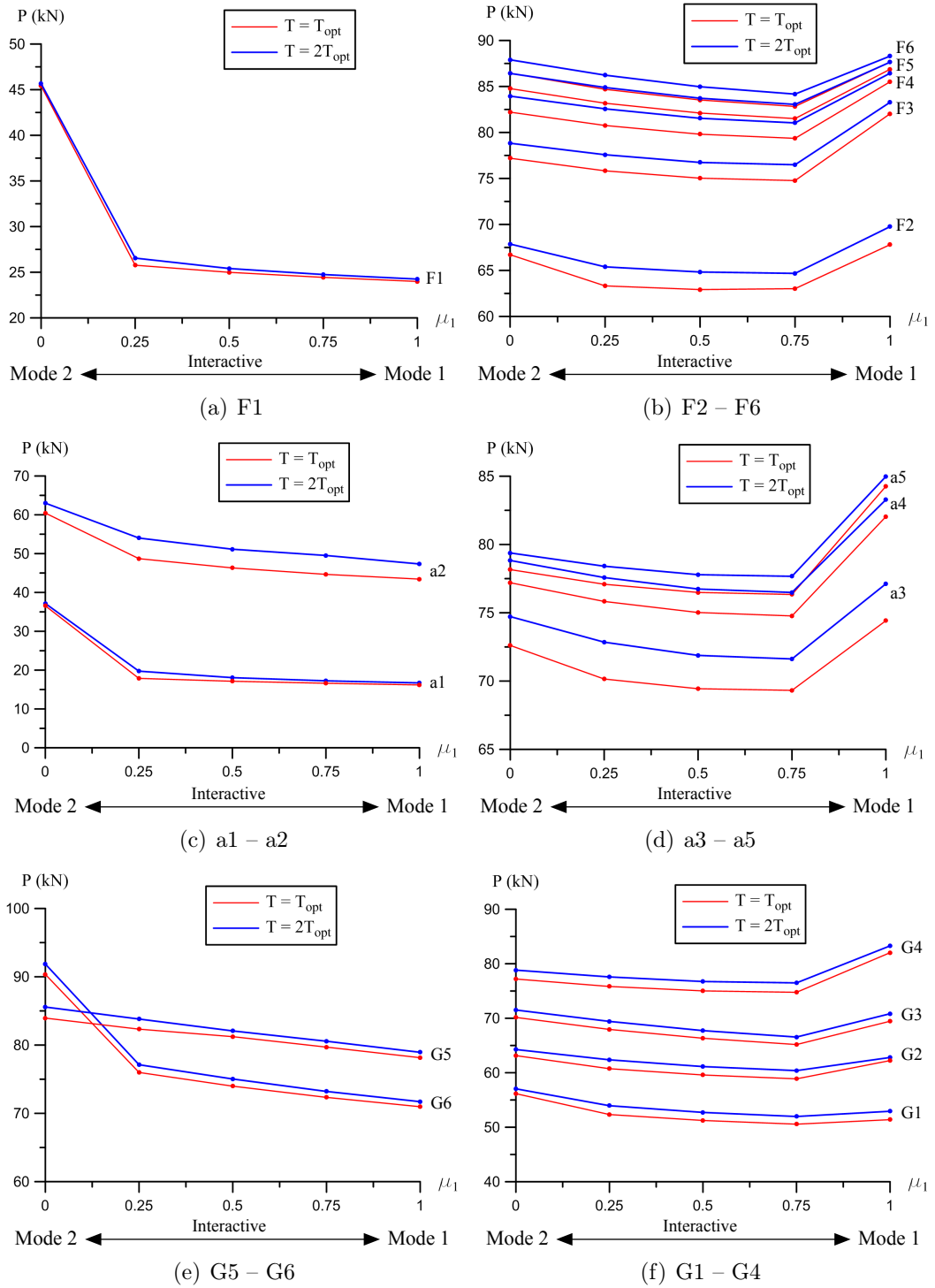


Figure 3.19: Sensitivity to the imperfection combination with the prestressing force T varying from T_{opt} to $2T_{\text{opt}}$. Note that in (d) the case a6 is not presented for the sake of clarity since it is very similar to case a5.

while increasing the prestress from T_{opt} to $2T_{\text{opt}}$. It can also be seen that increasing the prestressing force does not affect the sensitivity to the imperfection combination, since the most severe buckling response with $2T_{\text{opt}}$ can be found in exactly the same regions as those cases with T_{opt} .

A more detailed study of varying the prestress is carried out for cases when the column aspect ratio $2\alpha_m = 0.1$ and 0.2 , when Mode 1 and interactive buckling are the dominant post-buckling responses respectively, as shown in Figure 3.20. Four prestressing force levels, ranging from T_{opt} to $4T_{\text{opt}}$, are used to compare the effect in terms of the post-buckling behaviour; $\mu_1 = 0.5$ is adopted for the triggering of interactive mode. It can be seen that extra load capacity can always be obtained by increasing the prestressing force. Similarly, the equilibrium paths become increasingly unstable with the increase of the prestressing force, since the applied force reduces more dramatically beyond the ultimate load.

It is very important to note that, it is potentially dangerous to apply an extremely large prestress in order to seek a large load capacity for the following reasons:

1. The prestress cannot be increased forever since its vertical components also make a contribution to the axial load applied to the main column. Too large a prestress would decrease the load carrying capacity of the column system, as shown in Zone 3 in Figure 2.5.
2. The main column and the cable stays of course have strength limits. Too large a prestress could result in the axial stress of the stays and the column exceeding their yield strength before the column buckles. Thus the stayed column would lose stiffness and hence its effectiveness.
3. As shown in Figure 3.20, the equilibrium paths of some cases with $4T_{\text{opt}}$ become very unstable immediately after reaching the ultimate load. Larger prestresses could lead to even more unstable responses.

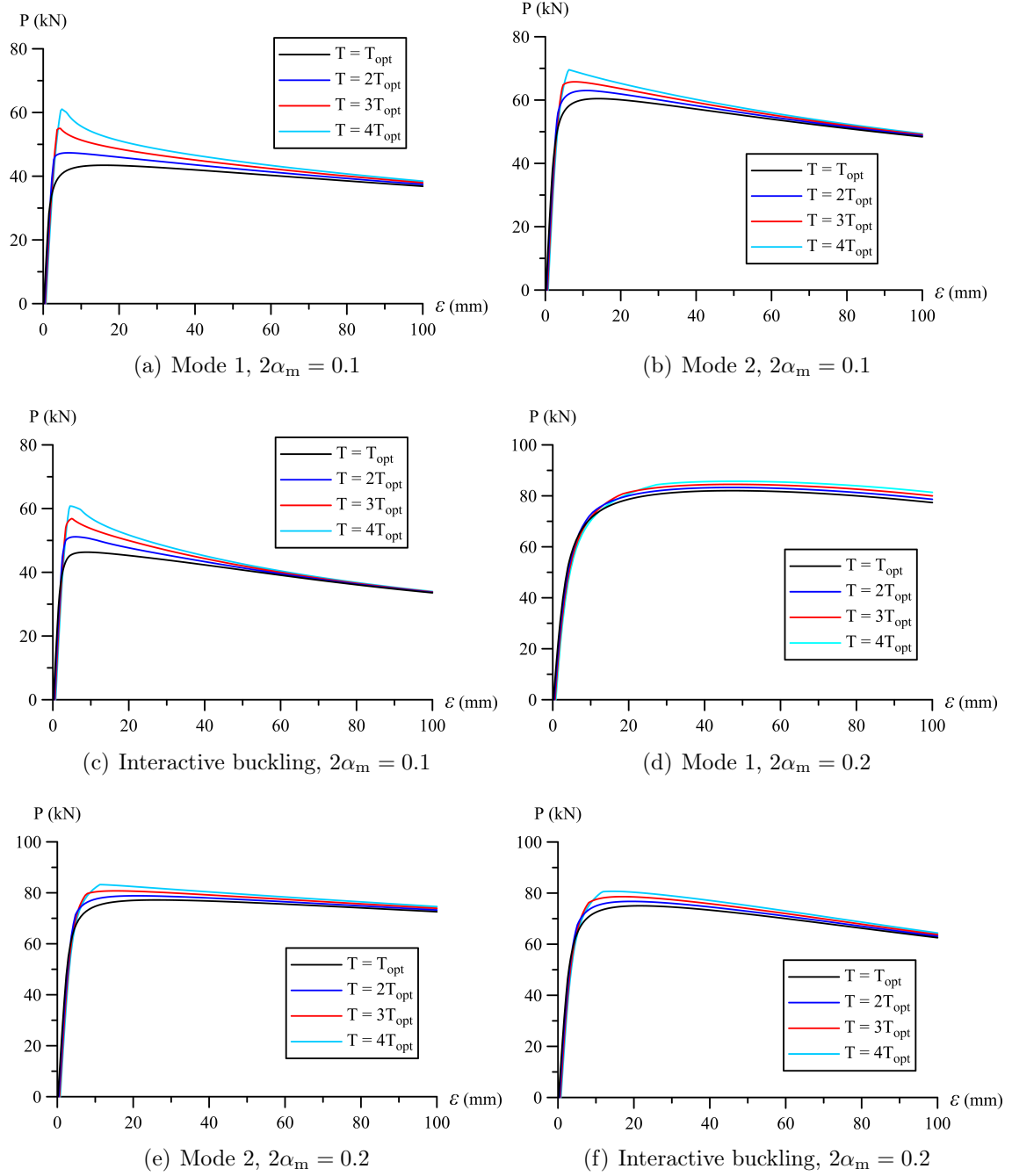


Figure 3.20: Equilibrium paths of the axial load P versus the end-shortening of the main column ε with the initial prestressing force T varying for cases when (a)–(c): Mode 1 is critical ($2\alpha_m = 0.1$) and (d)–(f): interactive buckling dominates ($2\alpha_m = 0.2$).

Therefore it is crucial to find an appropriate actual optimal prestress for the triple-bay prestress column. The definition and derivation of such a prestress is discussed in Chapter 6.

3.3.4 Imperfection sensitivity

Previous studies demonstrated that the post-buckling behaviour can be affected significantly by introducing different sizes of imperfection in the stayed column. In the current study, the imperfection size $\delta = 1/300$ was used in the nonlinear FE analysis. In order to determine the effect of different sizes of the imperfection, a more detailed study is conducted by using a series of selected imperfections that are presented in Table 3.2.

Figure 3.21 shows the equilibrium paths of the axial load P versus the end-shortening of the main column $\mathcal{E}$ with the imperfection size δ varying for cases when Mode 1 and the interactive responses are dominant in the post-buckling range respectively. It can be seen that, similar to the cases where T was increased from T_{opt} , decreasing the imperfection size provides a higher load capacity but also leads to a more unstable path once the maximum load is reached. It also demonstrates that the assumption made earlier that the gaps between the load capacities between the linear buckling and post-buckling are generated by the adoption of the larger imperfections ($\delta = 1/300$) in the post-buckling analysis.

A more detailed study in Figure 3.22 shows the equilibrium paths of the axial load P versus the end-shortening of the main column $\mathcal{E}$ with the imperfection δ varying under different levels of the prestressing force T . The benchmark parameters of $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$ are used to configure the stayed column geometry. The governing post-buckling responses for this column are interactive. An imperfection scale factor $\mu_1 = 0.5$ is chosen since the buckling loads are not very sensitive with μ_1 varying within the interactive buckling range. It can be seen that, as expected, smaller imperfections provide larger buckling loads for all the four prestress cases. As shown earlier, increasing

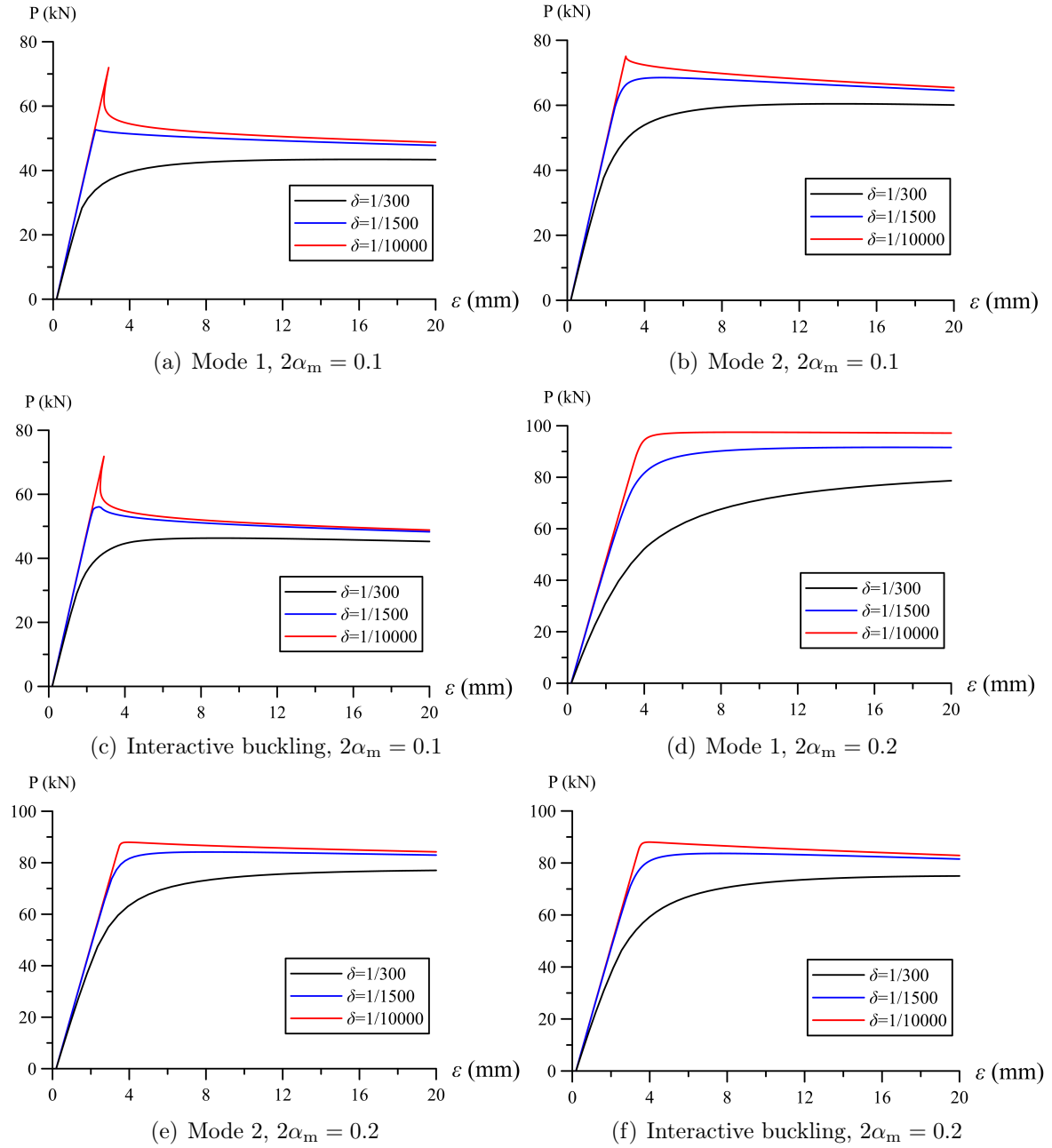


Figure 3.21: Equilibrium paths of the axial load P versus the end-shortening of the main column ε with the imperfection size δ varying for cases when (a)–(c): Mode 1 is critical ($2\alpha_m = 0.1$) and (d)–(f): interactive buckling dominates ($2\alpha_m = 0.2$).

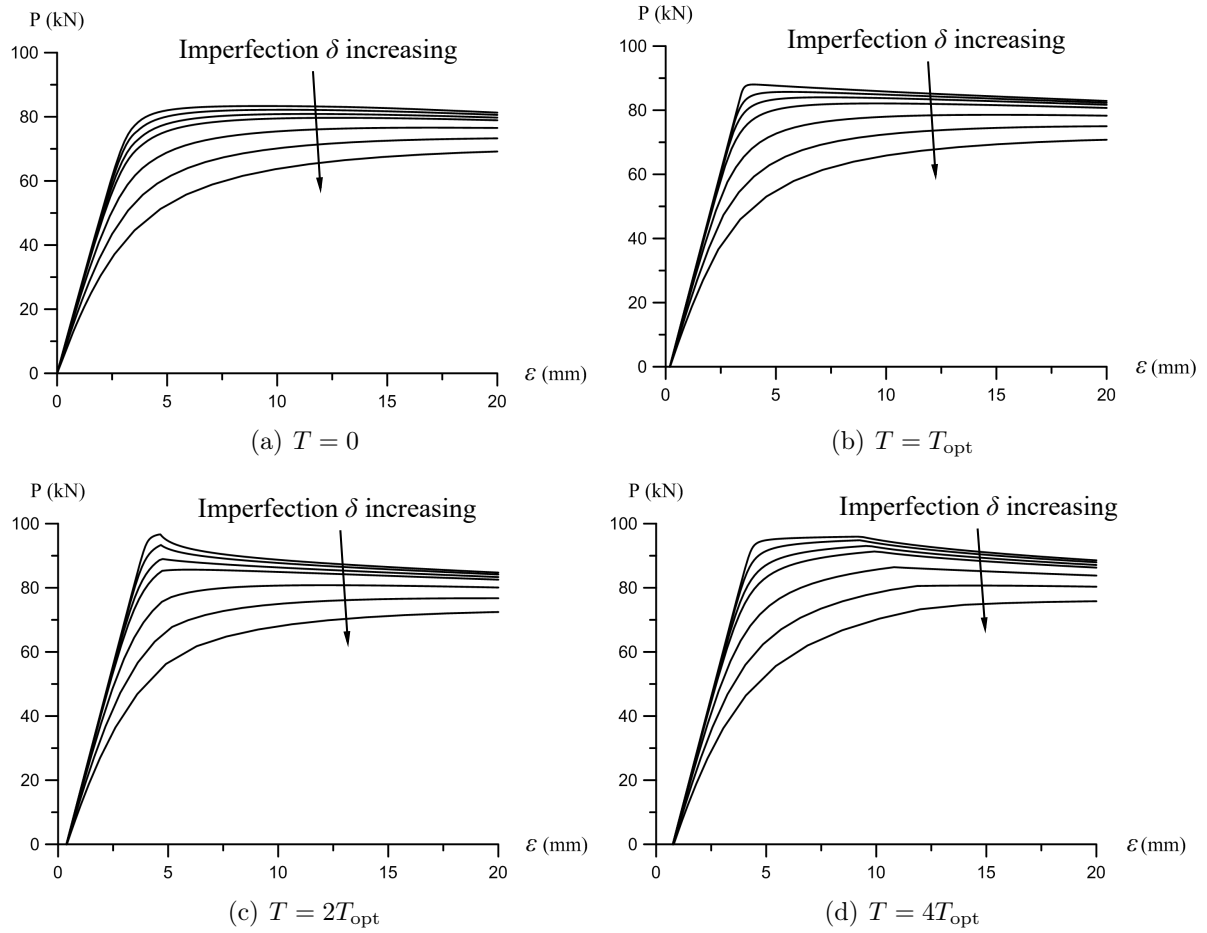


Figure 3.22: Equilibrium paths of the axial load P versus the column end-shortening ε with the imperfection δ varying under different levels of prestress when $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

the prestress can also provide a rise in the load carrying capacity. However, although T_{opt} is not the actual optimal prestress since the buckling loads when the prestress is $2T_{\text{opt}}$ are larger than those for prestress levels below $2T_{\text{opt}}$, the buckling load cannot benefit from increasing the prestress in perpetuity since the vertical component of the prestress contributes increasingly to the column compressive stress. An example that illustrates this point is that when $\delta = 1/10000$, the load capacity with a prestress of $4T_{\text{opt}}$ is smaller than the column with a prestress of $2T_{\text{opt}}$, as shown in Figure 3.23, which shows the maximum

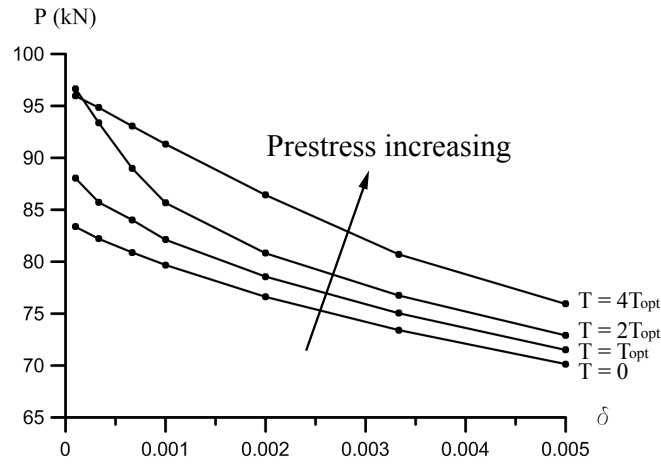


Figure 3.23: Maximum load capacities with the imperfection δ varying under different levels of prestress when $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

load capacities with the imperfection δ varying under different levels of prestress. The figure also indicates that the system is more sensitive to imperfections for smaller imperfection sizes since the most dramatic drops in the buckling loads are observed in the neighbourhood of $\delta = 1/10000$ under all four prestress levels shown.

3.4 Concluding remarks

A finite element model has been developed to investigate the buckling and post-buckling behaviour of the triple-bay prestressed stayed column in this chapter. Methods to calculate the theoretically optimal prestressing force T_{opt} and the derivation of the imperfection

profile are also introduced. It is shown that the critical buckling load and the post-buckling response are strongly related to the structural geometries. The prestressing level and imperfection sizes also play important roles in the mechanical response.

It has been found that interactive buckling is potentially the most dangerous situation, when Mode 2 is the dominant buckling mode, or Mode 1 is critical but the buckling loads of Modes 1 and 2 are very close. This indicates that it is not necessarily accurate to predict the load carrying capacity of the triple-bay prestressed column only using linear buckling analysis. It has also been demonstrated that increasing the prestress level from T_{opt} leads to a rise in the load carrying capacity, although a less stiff behaviour may be observed once the ultimate load is reached. A methodology to determine the actual optimal prestress will be covered in a later chapter.

Hitherto, the conclusions derived above are purely from the FE model. In the following chapters, different kinds of analytical model are developed to validate the results obtained from the FE analysis both qualitatively and quantitatively.

Chapter 4

Formulation of Discrete Analytical Model

4.1 Introduction

In the previous chapter, a finite element model of the triple-bay prestressed stayed column was developed with the buckling and the post-buckling responses being studied. In the current chapter, a simplified analytical model comprising rigid links and springs with multiple degrees of freedom is devised to validate the conclusions obtained from the FE model at least qualitatively. Such a model with a simpler configuration using minimum total potential energy principles has been previously used to study the single-bay prestressed stayed column (Wadee *et al.* , 2013a). Good agreement was found when comparing the results qualitatively with the FE model (Saito & Wadee, 2009b). The advantage of this analytical model is that, instead of attempting to describe the column deflected shape and system stiffness using complicated mathematical expressions, the column is discretized and a series of discrete stiffnesses are used to represent the approximate behaviour of the system. As a result, the energy terms can be written in terms of exact geometric expressions and the

total potential energy can therefore be obtained. The resulting equilibrium equations are solved by the software AUTO-07P (Doedel & Oldeman, 2011).

Currently, a discrete analytical model is developed to study the buckling and post-buckling behaviour of the triple-bay prestressed stayed column. Equilibrium paths, starting from the fundamental path to the first bifurcation point and then to subsequent bifurcation points triggering interactive buckling, are exhibited and analysed. The effect of initial imperfections is also taken into consideration.

4.2 Analytical modelling

4.2.1 Model formulation

The column comprises four equal length rigid links with length l , as shown in Figure 4.1. The stiffnesses provided by the stays and the crossarms are represented by three lateral

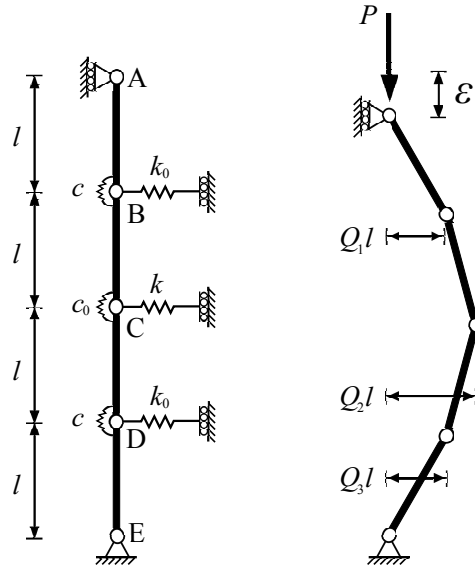


Figure 4.1: The discrete analytical model with rigid links and elastic longitudinal and rotational springs.

springs connected to pins between the discrete links. The stiffnesses of the edge lateral

springs are k_0 while the middle one has stiffness k . Similarly, the bending stiffness of the column is represented by three rotational springs located at every pinned joint remote from the boundaries of the column with the value c at the edge pins and c_0 at the middle pin. The base of the column is pinned while the top is free to translate vertically. Once the column is loaded externally and it then begins to buckle, the lateral deflections at points B, C and D are Q_1l , Q_2l and Q_3l respectively, where the generalized coordinate Q_i represents the proportion of lateral deflection normalized to the link length l , with $i = 1, 2$ and 3 , and the column end-shortening being $\mathcal{E}$.

The following assumptions are made:

1. The column is loaded concentrically through the cross-section and lateral deflections exist before buckling occurs.
2. The links that comprise the column structure are assumed to be rigid, hence no axial compression is allowed.
3. The links are ideally pinned to each other; both the longitudinal and rotational springs are assumed to be frictionless.

4.2.2 Potential energy formulation

The total potential energy V of the system is given by the strain energy stored in the system minus the work done by the load, thus:

$$V = U - P\mathcal{E}, \quad (4.1)$$

where U is the strain energy stored in the system, P is the external load applied axially at the top of the column, and $\mathcal{E}$ is the end-shortening of the column introduced by the external loading. The strain energy U is provided by the lateral and rotational springs and

can be expressed as:

$$U = \frac{1}{2}k_0l^2Q_1^2 + \frac{1}{2}kl^2Q_2^2 + \frac{1}{2}k_0l^2Q_3^2 + \frac{1}{2}c\theta_B^2 + \frac{1}{2}c_0\theta_C^2 + \frac{1}{2}c\theta_D^2, \quad (4.2)$$

where θ_B , θ_C and θ_D represent rotations of the pins at points B, C and D respectively, where:

$$\theta_B = \arcsin(Q_1) - \arcsin(Q_2 - Q_1), \quad (4.3)$$

$$\theta_C = \arcsin(Q_2 - Q_1) - \arcsin(Q_3 - Q_2), \quad (4.4)$$

$$\theta_D = \arcsin(Q_3) - \arcsin(Q_2 - Q_3). \quad (4.5)$$

Approximate expressions for θ_B , θ_C and θ_D can be obtained from taking the dominating order of the Taylor expansion of $\arcsin x$:

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1}, \quad |x| \leq 1. \quad (4.6)$$

This gives:

$$\theta_B = 2Q_1 - Q_2, \quad \theta_C = 2Q_2 - Q_1 - Q_3, \quad \theta_D = 2Q_3 - Q_2, \quad (4.7)$$

to leading order and the expressions are then sufficient for critical buckling analysis. Substituting the relationships in Equation (4.7) into (4.2) gives:

$$\begin{aligned} U &= \frac{1}{2}k_0l^2Q_1^2 + \frac{1}{2}kl^2Q_2^2 + \frac{1}{2}k_0l^2Q_3^2 + \frac{1}{2}c(2Q_1 - Q_2)^2 + \frac{1}{2}c_0(2Q_2 - Q_1 - Q_3)^2 \\ &\quad + \frac{1}{2}c(2Q_3 - Q_2)^2 \\ &= \left(\frac{1}{2}k_0l^2 + 2c + \frac{1}{2}c_0\right)Q_1^2 + \left(\frac{1}{2}kl^2 + c + 2c_0\right)Q_2^2 + \left(\frac{1}{2}k_0l^2 + 2c + \frac{1}{2}c_0\right)Q_3^2 \\ &\quad - 2(c + c_0)Q_1Q_2 - 2(c + c_0)Q_2Q_3 + c_0Q_1Q_3. \end{aligned} \quad (4.8)$$

The end-shortening $\mathcal{E}$ of the column is given by the following expression since the column is assumed to be incompressible:

$$\mathcal{E} = \left[4 - \sqrt{1 - Q_1^2} - \sqrt{1 - (Q_2 - Q_1)^2} - \sqrt{1 - (Q_2 - Q_3)^2} - \sqrt{1 - Q_3^2} \right] l. \quad (4.9)$$

Equation (4.9) can be simplified by taking the leading order of the binomial expansion of $\sqrt{1+x}$:

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!) (1-2n)} x^n, \quad |x| \leq 1. \quad (4.10)$$

Therefore, the column end-shortening $\mathcal{E}$ is expressed to leading order in Q_i thus:

$$\begin{aligned} \mathcal{E} &= \left\{ 4 - \left(1 - \frac{Q_1^2}{2} \right) - \left[1 - \frac{(Q_2 - Q_1)^2}{2} \right] - \left[1 - \frac{(Q_2 - Q_3)^2}{2} \right] - \left(1 - \frac{Q_3^2}{2} \right) \right\} l \\ &= (Q_1^2 + Q_2^2 + Q_3^2 - Q_1 Q_2 - Q_2 Q_3) l. \end{aligned} \quad (4.11)$$

The expression of the total potential energy V for critical buckling analysis can be obtained by substituting Equations (4.8) and (4.11) into Equation (4.1), thus:

$$\begin{aligned} V &= \left(\frac{1}{2} k_0 l^2 + 2c + \frac{1}{2} c_0 \right) Q_1^2 + \left(\frac{1}{2} k l^2 + c + 2c_0 \right) Q_2^2 + \left(\frac{1}{2} k_0 l^2 + 2c + \frac{1}{2} c_0 \right) Q_3^2 \\ &\quad - 2(c + c_0) Q_1 Q_2 - 2(c + c_0) Q_2 Q_3 + c_0 Q_1 Q_3 - Pl (Q_1^2 + Q_2^2 + Q_3^2 - Q_1 Q_2 - Q_2 Q_3). \end{aligned} \quad (4.12)$$

Equation (4.12) can be further simplified by normalizing the total potential energy V with respect to the rotational spring stiffness c . Hence, $\tilde{V} = V/c$, and defining $\rho = k_0/k$, $K = kl^2/c$, $p = Pl/c$, $R = c_0/c$, the expression can be re-written thus:

$$\begin{aligned} \tilde{V} &= \left(\frac{1}{2} \rho K + 2 + \frac{1}{2} R \right) Q_1^2 + \left(\frac{1}{2} K + 1 + 2R \right) Q_2^2 + \left(\frac{1}{2} \rho K + 2 + \frac{1}{2} R \right) Q_3^2 \\ &\quad - 2(1 + R) Q_1 Q_2 - 2(1 + R) Q_2 Q_3 + R Q_1 Q_3 - p (Q_1^2 + Q_2^2 + Q_3^2 - Q_1 Q_2 - Q_2 Q_3). \end{aligned} \quad (4.13)$$

It can be seen that the normalized total potential energy $\tilde{V}$ is dependent on the normalized lateral stiffness K , the ratio of the lateral stiffnesses ρ and the ratio of the rotational stiffnesses R . Moreover, it can be seen that there are quadratic cross-terms in the expression – the system is hence non-diagonalized and $\tilde{V}(Q_i, p)$ is written as $\tilde{W}(Q_i, p)$ using the notation of Thompson & Hunt (1973). It is useful to diagonalize $\tilde{W}$ to obtain the individual buckling modes.

4.2.3 Diagonalization

Since there are only three degrees of freedom, Figure 4.2 shows the three buckling modes of

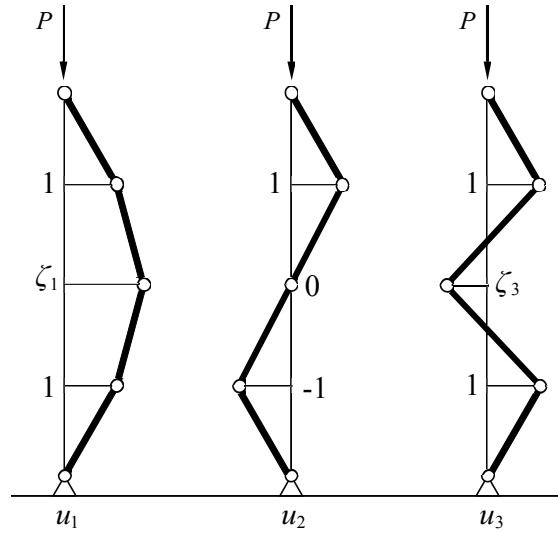


Figure 4.2: The three buckling modes of the link model.

the link model which were first shown in Wadee *et al* (2013a). The two edge pins have unit lateral deflections for all the modes; ζ_1 and ζ_3 are the lateral deflections at the mid-span of the column for Modes 1 and 3 respectively. The lateral deflection at the mid-span of the column is zero for Mode 2 since its buckling shape is anti-symmetric. The generalized

coordinates Q_1 , Q_2 and Q_3 can be replaced by a new set of the coordinates:

$$\begin{aligned} Q_1 &= u_1 + u_2 + u_3, \\ Q_2 &= u_1 \zeta_1 - u_3 \zeta_3, \\ Q_3 &= u_1 - u_2 + u_3, \end{aligned} \tag{4.14}$$

where u_1 , u_2 and u_3 are the scale factors of the first symmetric mode (Mode 1), the first anti-symmetric mode (Mode 2) and the second symmetric mode (Mode 3) respectively. Substituting Equation (4.14) into (4.13) gives:

$$\begin{aligned} \tilde{W} &= \left(\frac{1}{2} \rho k + 2 + \frac{1}{2} R \right) (u_1 + u_2 + u_3)^2 + \left(\frac{1}{2} k + 1 + 2R \right) (u_1 \zeta_1 - u_3 \zeta_3)^2 \\ &\quad + \left(\frac{1}{2} \rho k + 2 + \frac{1}{2} R \right) (u_1 - u_2 + u_3)^2 - 2(1+R)(u_1 + u_2 + u_3)(u_1 \zeta_1 - u_3 \zeta_3) \\ &\quad - 2(1+R)(u_1 \zeta_1 - u_3 \zeta_3)(u_1 - u_2 + u_3) + R(u_1 + u_2 + u_3)(u_1 - u_2 + u_3) \\ &\quad - p \left[(u_1 + u_2 + u_3)^2 + (u_1 \zeta_1 - u_3 \zeta_3)^2 + (u_1 - u_2 + u_3)^2 \right. \\ &\quad \left. - (u_1 + u_2 + u_3)(u_1 \zeta_1 - u_3 \zeta_3) - (u_1 \zeta_1 - u_3 \zeta_3)(u_1 - u_2 + u_3) \right] \\ &= \frac{1}{2} \tilde{W}_{11}^O u_1^2 + \frac{1}{2} \tilde{W}_{22}^O u_2^2 + \frac{1}{2} \tilde{W}_{33}^O u_3^2 + \tilde{W}_{13}^O u_1 u_3 \\ &\quad - p \left(\frac{1}{2} \tilde{W}_{11}'^O u_1^2 + \frac{1}{2} \tilde{W}_{22}'^O u_2^2 + \frac{1}{2} \tilde{W}_{33}'^O u_3^2 + \tilde{W}_{13}'^O u_1 u_3 \right), \end{aligned} \tag{4.15}$$

where:

$$\begin{aligned} \frac{1}{2} \tilde{W}_{11}^O &= \rho K + 2R + 4 + \frac{1}{2} K \zeta_1^2 + 2R \zeta_1^2 + \zeta_1^2 - 4\zeta_1 - 4R\zeta_1, & \frac{1}{2} \tilde{W}_{22}^O &= \rho K + 4 \\ \frac{1}{2} \tilde{W}_{33}^O &= \rho K + 2R + 4 + \frac{1}{2} K \zeta_3^2 + 2R \zeta_3^2 + \zeta_3^2 + 4\zeta_3 + 4R\zeta_3, \\ \frac{1}{2} \tilde{W}_{11}'^O &= 2 + \zeta_1^2 - 2\zeta_1, & \frac{1}{2} \tilde{W}_{22}'^O &= 2, & \frac{1}{2} \tilde{W}_{33}'^O &= 2 + \zeta_3^2 + 2\zeta_3 \\ \tilde{W}_{13}^O &= 2\rho K + 4R + 8 - K\zeta_1\zeta_3 - 4R\zeta_1\zeta_3 - 2\zeta_1\zeta_3 - 4\zeta_1 + 4\zeta_3 - 4R\zeta_1 + 4R\zeta_3, \\ \tilde{W}_{13}'^O &= 4 - 2\zeta_1\zeta_3 - 2\zeta_1 + 2\zeta_3, \end{aligned} \tag{4.16}$$

with the superscript O representing the term is evaluated at the unloaded state. Eliminating $u_1 u_3$ cross-terms $\tilde{W}_{13}^O$ and $\tilde{W}_{13}'^O$ gives the following equations in ζ_1 and ζ_3 :

$$\begin{cases} 2\rho K + 4R + 8 - K\zeta_1\zeta_3 - 4R\zeta_1\zeta_3 - 2\zeta_1\zeta_3 - 4\zeta_1 + 4\zeta_3 - 4R\zeta_1 + 4R\zeta_3 = 0, \\ 4 - 2\zeta_1\zeta_3 - 2\zeta_1 + 2\zeta_3 = 0. \end{cases} \quad (4.17)$$

Solving the equations in (4.17) simultaneously, the following expressions are obtained:

$$\begin{cases} \zeta_1 = \frac{-(\rho K - K - 2R + 2) - X}{K - 2}, \\ \zeta_3 = \frac{(\rho K - K - 2R + 2) - X}{K - 2}, \end{cases} \quad (4.18)$$

where:

$$X = \sqrt{\rho^2 K^2 + K^2 + 4R^2 - 4\rho KR - 4K + 4}. \quad (4.19)$$

Since $\tilde{W}(Q_i, p)$ has been diagonalized as $\tilde{W}(u_i, p)$ it will now be expressed with $\tilde{A}(u_i, p)$ in recognition of this fact, again this is using the notation from Thompson & Hunt (1973). Therefore, the diagonalized total potential energy $\tilde{A}$ of the system is expressed by:

$$\tilde{A} = \frac{1}{2}\tilde{A}_{11}^O u_1^2 + \frac{1}{2}\tilde{A}_{22}^O u_2^2 + \frac{1}{2}\tilde{A}_{33}^O u_3^2 - p \left(\frac{1}{2}\tilde{A}_{11}'^O u_1^2 + \frac{1}{2}\tilde{A}_{22}'^O u_2^2 + \frac{1}{2}\tilde{A}_{33}'^O u_3^2 \right). \quad (4.20)$$

Equation (4.20) can be re-written without any approximation, thus:

$$\begin{aligned} \tilde{A} &= \frac{1}{2}\tilde{A}_{11}^O u_1^2 + \frac{1}{2}\tilde{A}_{22}^O u_2^2 + \frac{1}{2}\tilde{A}_{33}^O u_3^2 - p^F \left(\frac{1}{2}\tilde{A}_{11}'^O u_1^2 + \frac{1}{2}\tilde{A}_{22}'^O u_2^2 + \frac{1}{2}\tilde{A}_{33}'^O u_3^2 \right) \\ &\quad - (p - p^F) \left(\frac{1}{2}\tilde{A}_{11}'^O u_1^2 + \frac{1}{2}\tilde{A}_{22}'^O u_2^2 + \frac{1}{2}\tilde{A}_{33}'^O u_3^2 \right) \\ &= \frac{1}{2} \left(\tilde{A}_{11}^O - p^F \tilde{A}_{11}'^O \right) u_1^2 + \frac{1}{2} \left(\tilde{A}_{22}^O - p^F \tilde{A}_{22}'^O \right) u_2^2 + \frac{1}{2} \left(\tilde{A}_{33}^O - p^F \tilde{A}_{33}'^O \right) u_3^2 \\ &\quad - (p - p^F) \left(\frac{1}{2}\tilde{A}_{11}'^O u_1^2 + \frac{1}{2}\tilde{A}_{22}'^O u_2^2 + \frac{1}{2}\tilde{A}_{33}'^O u_3^2 \right) \\ &= \frac{1}{2}\tilde{A}_{11}^F u_1^2 + \frac{1}{2}\tilde{A}_{22}^F u_2^2 + \frac{1}{2}\tilde{A}_{33}^F u_3^2 - (p - p^F) \left(\frac{1}{2}\tilde{A}_{11}'^F u_1^2 + \frac{1}{2}\tilde{A}_{22}'^F u_2^2 + \frac{1}{2}\tilde{A}_{33}'^F u_3^2 \right), \end{aligned} \quad (4.21)$$

where the superscript F representing the term is evaluated on the fundamental path. The buckling loads of the column are obtained by the condition:

$$\det(\tilde{\mathbf{A}}_{ij}^F) = 0, \quad (4.22)$$

where:

$$\tilde{\mathbf{A}}_{ij}^F = \begin{pmatrix} \tilde{A}_{11}^F & 0 & 0 \\ 0 & \tilde{A}_{22}^F & 0 \\ 0 & 0 & \tilde{A}_{33}^F \end{pmatrix}. \quad (4.23)$$

At the critical point, the superscript F becomes C:

$$\begin{aligned} \tilde{A}_{11}^F &= \tilde{A}_{11}^O - p^C \tilde{A}_{11}'^O = 0, \\ \tilde{A}_{22}^F &= \tilde{A}_{22}^O - p^C \tilde{A}_{22}'^O = 0, \\ \tilde{A}_{33}^F &= \tilde{A}_{33}^O - p^C \tilde{A}_{33}'^O = 0. \end{aligned} \quad (4.24)$$

Substituting Equations (4.16) and (4.18) into (4.24) gives the normalized buckling loads:

$$\begin{aligned} p_1^C &= \frac{\rho^2 + 1}{2X(X + \rho K - 2R)} K^3 + \frac{2\rho^2 - 2 - X + \rho X - 4\rho R + 4\rho^2 R + 4R}{2X(X + \rho K - 2R)} K^2 \\ &+ \frac{4\rho R X - 4 - 8\rho R + 4X - 2RX - 16\rho R^2 + 2\rho X - 16R + 4R^2}{2X(X + \rho K - 2R)} K \\ &+ \frac{8 + 16R + 8R^2 - 4X - 4RX - 8R^2 X + 16R^3}{2X(X + \rho K - 2R)}, \end{aligned} \quad (4.25)$$

$$p_2^C = \frac{1}{2}\rho K + 2, \quad (4.26)$$

$$\begin{aligned} p_3^C &= \frac{\rho^2 + 1}{2X(X - \rho K + 2R)} K^3 + \frac{2\rho^2 - 2 + X - \rho X - 4\rho R + 4\rho^2 R + 4R}{2X(X - \rho K + 2R)} K^2 \\ &+ \frac{-4\rho R X - 4 - 8\rho R - 4X + 2RX - 16\rho R^2 - 2\rho X - 16R + 4R^2}{2X(X - \rho K + 2R)} K \\ &+ \frac{8 + 16R + 8R^2 + 4X + 4RX - 8R^2 X + 16R^3}{2X(X - \rho K + 2R)}. \end{aligned} \quad (4.27)$$

It can be seen that the critical loads p_1^C , p_2^C and p_3^C are dependent on the normalized lateral stiffness K , the ratio of the lateral stiffnesses ρ and the ratio of the rotational stiffnesses R

(the latter, however, is not applicable for p_2^C). However, the critical load p_i^C is normalized from $p_i^C = P_i^C l / c$. Therefore the magnitude of the actual critical loads also depend on the link length l and the rotational stiffness c at the edge joints of the column.

4.2.4 Equilibrium equations

The post-buckling response of the model is studied by using the principle of stationary potential energy giving the condition of equilibrium. This is achieved by equating the first partial derivatives of the normalized total potential energy $\tilde{V}$ now in its geometrically exact form with respect to the original generalized coordinates Q_i to zero and are then solved numerically in AUTO-07P (Doedel & Oldeman, 2011):

$$\frac{\partial \tilde{V}}{\partial Q_i} = 0, \quad (4.28)$$

where $i = 1, 2$ and 3 with $\tilde{V}$ being expressed by:

$$\begin{aligned} \tilde{V} = & \frac{1}{2} \rho K Q_1^2 + \frac{1}{2} K Q_2^2 + \frac{1}{2} \rho K Q_3^2 + \frac{1}{2} [\arcsin Q_1 - \arcsin(Q_2 - Q_1)]^2 \\ & + \frac{1}{2} R [\arcsin(Q_2 - Q_1) + \arcsin(Q_2 - Q_3)]^2 + \frac{1}{2} [\arcsin Q_3 - \arcsin(Q_2 - Q_3)]^2 \\ & - p \left[4 - \sqrt{1 - Q_1^2} - \sqrt{1 - (Q_2 - Q_1)^2} - \sqrt{1 - (Q_2 - Q_3)^2} - \sqrt{1 - Q_3^2} \right]. \end{aligned} \quad (4.29)$$

Therefore the equilibrium equations are thus:

$$\begin{aligned} \frac{\partial \tilde{V}}{\partial Q_1} = & \rho K Q_1 + [\arcsin Q_1 + \arcsin(Q_1 - Q_2)] \left[\frac{1}{\sqrt{1 - Q_1^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] \\ & - \frac{R [-\arcsin(Q_1 - Q_2) + \arcsin(Q_2 - Q_3)]}{\sqrt{1 - (Q_2 - Q_1)^2}} \\ & - p \left[\frac{Q_1}{\sqrt{1 - Q_1^2}} - \frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] = 0, \end{aligned} \quad (4.30)$$

$$\begin{aligned}
\frac{\partial \tilde{V}}{\partial Q_2} = & KQ_2 - \frac{\arcsin Q_1 + \arcsin(Q_1 - Q_2)}{\sqrt{1 - (Q_2 - Q_1)^2}} - \frac{\arcsin Q_3 + \arcsin(Q_2 - Q_3)}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& + R[\arcsin(Q_2 - Q_3) - \arcsin(Q_1 - Q_2)] \left[\frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] \\
& - p \left[\frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0,
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
\frac{\partial \tilde{V}}{\partial Q_3} = & \rho KQ_3 + [\arcsin Q_3 + \arcsin(Q_2 - Q_3)] \left[\frac{1}{\sqrt{1 - Q_3^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] \\
& - \frac{R[\arcsin(Q_2 - Q_3) - \arcsin(Q_1 - Q_2)]}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& - p \left[\frac{Q_3}{\sqrt{1 - Q_3^2}} - \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0.
\end{aligned} \tag{4.32}$$

Note that in the normalized total potential energy $\tilde{V}$, the cross-terms of Q_1 , Q_2 and Q_3 are not eliminated, nor are the cubic and higher order terms neglected owing to the fact that the energy terms are written in their geometrically exact forms. Therefore, the equilibrium equations listed above are capable of describing interactive buckling, the nonlinear coupling of pure modes.

4.2.5 Imperfections

The analytical link model developed above assumes the column is perfectly straight and no initial imperfections exist. In this section, an initial imperfection is incorporated to investigate their important effect. As shown in Figure 4.3, the end-shortening of the column in Mode 1 buckling is expressed by:

$$\mathcal{E}_1 = 4l - 2l \left(\sqrt{1 - e_1^2 \delta_0^2} + \sqrt{1 - e_1^2 \delta_0^2 (\zeta_1 - 1)^2} \right), \tag{4.33}$$

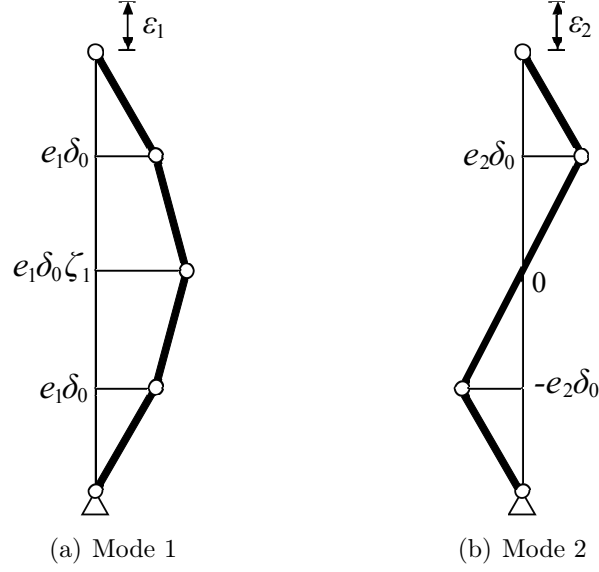


Figure 4.3: Buckling shapes of Modes 1 and 2 in the discrete analytical model.

where δ_0 is the proportion of the lateral deflection to the link length l . Similarly, the end-shortening of the column in Mode 2 buckling is given by:

$$\mathcal{E}_2 = 4l \left(1 - \sqrt{1 - e_2^2 \delta_0^2} \right). \quad (4.34)$$

Equations (4.33) and (4.34) can be simplified by using the binomial expansion of $\sqrt{1+x}$ that is stated in (4.10). Therefore, the simplified end-shortenings are expressed thus:

$$\mathcal{E}_1 = 4l - 2l \left[1 - \frac{1}{2} e_1^2 \delta_0^2 + 1 - \frac{1}{2} e_1^2 \delta_0^2 (\zeta_1 - 1)^2 \right] = e_1^2 \delta_0^2 l (\zeta_1^2 - 2\zeta_1 + 2), \quad (4.35)$$

$$\mathcal{E}_2 = 4l \left[1 - \left(1 - \frac{1}{2} e_2^2 \delta_0^2 \right) \right] = 2e_2^2 \delta_0^2 l. \quad (4.36)$$

The overall end-shortening of the column is given by the following expression:

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \delta_0^2 l \left[(\zeta_1^2 - 2\zeta_1 + 2) e_1^2 + 2e_2^2 \right] \quad (4.37)$$

In a similar way to the imperfection scale factor selection in Chapter 3, the end-shortening of the column should be the same when triggering Mode 1, Mode 2 and interactive buckling

to ensure a fair comparison of the results, rather than comparing them based purely on the same lateral amplitude. Therefore the scale factors of the two modes satisfy the following expression:

$$(\zeta_1^2 - 2\zeta_1 + 2) e_1^2 + 2e_2^2 = 1, \quad (4.38)$$

where ζ_1 is defined in Equation (4.18).

Symmetric imperfection

Figure 4.4(a) shows the initial imperfection profile that is used to trigger Mode 1 buckling.

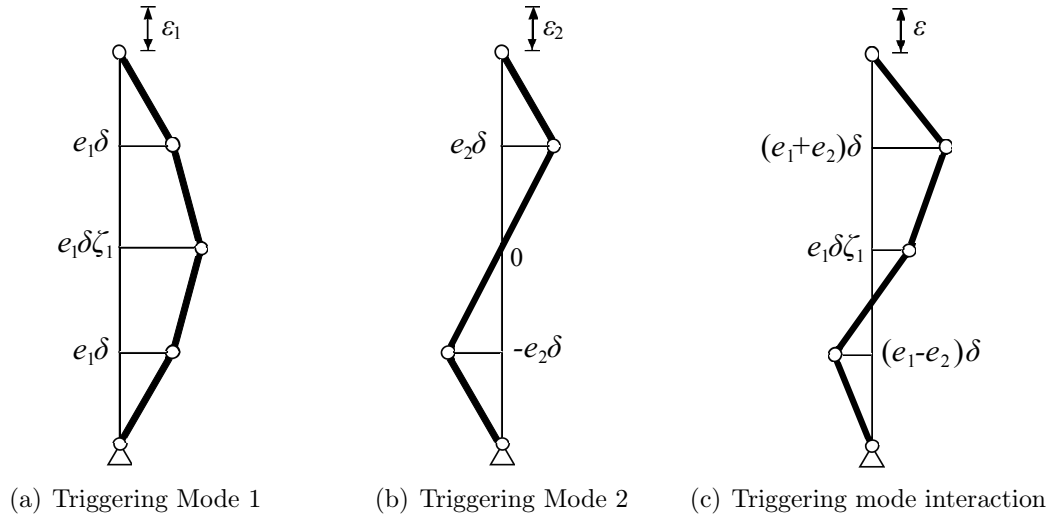


Figure 4.4: Different profiles of initial imperfections used in the imperfect cases.

The quantity δ represents the normalized initial out-of-straightness of the column with respect to the link length l . It can be seen that the generalized coordinates Q_1 , Q_2 and Q_3 and the rotations of the joints are no longer zero at the initial state. Therefore, the effect of the imperfect initial state provides changes to both the strain energy stored within the longitudinal and rotational springs. However, this does not change the work done by the external load, since the initial end-shortening of the column is entirely independent from the coordinates Q_i and will be eliminated on differentiating $\tilde{V}$ to obtain the equilibrium equations. Hence the new total potential energy with the introduction of the Mode 1

imperfection profile is given by:

$$\begin{aligned}
\tilde{V}_{m1} = & \frac{1}{2}\rho K (Q_1 - e_1\delta)^2 + \frac{1}{2}K (Q_2 - e_1\delta\zeta_1)^2 + \frac{1}{2}\rho K (Q_3 - e_1\delta)^2 \\
& + \frac{1}{2}\left\{ \arcsin Q_1 - \arcsin(Q_2 - Q_1) - \arcsin(e_1\delta) + \arcsin[e_1\delta(\zeta_1 - 1)] \right\}^2 \\
& + \frac{1}{2}R\left\{ \arcsin(Q_2 - Q_1) + \arcsin(Q_2 - Q_3) - 2\arcsin[e_1\delta(\zeta_1 - 1)] \right\}^2 \\
& + \frac{1}{2}\left\{ \arcsin Q_3 - \arcsin(Q_2 - Q_3) - \arcsin(e_1\delta) + \arcsin[e_1\delta(\zeta_1 - 1)] \right\}^2 \\
& - p\left[4 - \sqrt{1 - Q_1^2} - \sqrt{1 - (Q_2 - Q_1)^2} - \sqrt{1 - (Q_2 - Q_3)^2} - \sqrt{1 - Q_3^2} \right].
\end{aligned} \tag{4.39}$$

The respective equilibrium equations for Modes 1, 2 and 3 are expressed by:

$$\begin{aligned}
\frac{\partial \tilde{V}_{m1}}{\partial Q_1} = & \rho K (Q_1 - e_1\delta) + \left\{ \arcsin Q_1 + \arcsin(Q_1 - Q_2) - \arcsin(e_1\delta) \right. \\
& \left. + \arcsin[e_1\delta(\zeta_1 - 1)] \right\} \left[\frac{1}{\sqrt{1 - Q_1^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] \\
& - \frac{R\left\{ -\arcsin(Q_1 - Q_2) + \arcsin(Q_2 - Q_3) - 2\arcsin[e_1\delta(\zeta_1 - 1)] \right\}}{\sqrt{1 - (Q_2 - Q_1)^2}} \\
& - p\left[\frac{Q_1}{\sqrt{1 - Q_1^2}} - \frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] = 0,
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
\frac{\partial \tilde{V}_{m1}}{\partial Q_2} = & K (Q_2 - e_1\delta\zeta_1) - p\left[\frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] \\
& - \frac{\arcsin Q_1 + \arcsin(Q_1 - Q_2) - \arcsin(e_1\delta) + \arcsin[e_1\delta(\zeta_1 - 1)]}{\sqrt{1 - (Q_2 - Q_1)^2}} \\
& - \frac{\arcsin Q_3 + \arcsin(Q_2 - Q_3) - \arcsin(e_1\delta) + \arcsin[e_1\delta(\zeta_1 - 1)]}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& + R\left\{ \arcsin(Q_2 - Q_3) - \arcsin(Q_1 - Q_2) - 2\arcsin[e_1\delta(\zeta_1 - 1)] \right\} \\
& \times \left[\frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0,
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\frac{\partial \tilde{V}_{m1}}{\partial Q_3} = & \rho K (Q_3 - e_1 \delta) + \left\{ \arcsin Q_3 - \arcsin(Q_2 - Q_3) - \arcsin(e_1 \delta) \right. \\
& + \arcsin [e_1 \delta (\zeta_1 - 1)] \left. \right\} \left[\frac{1}{\sqrt{1 - Q_3^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] \\
& - \frac{R \left\{ -\arcsin(Q_1 - Q_2) + \arcsin(Q_2 - Q_3) - 2 \arcsin [e_1 \delta (\zeta_1 - 1)] \right\}}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& - p \left[\frac{Q_3}{\sqrt{1 - Q_3^2}} - \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0,
\end{aligned} \tag{4.42}$$

where:

$$e_1 = \sqrt{\frac{1}{\zeta_1^2 - 2\zeta_1 + 2}}. \tag{4.43}$$

Note that in this case e_1 is a non-zero value and is dependent on the lateral and rotational stiffnesses represented by ζ_1 , while $e_2 = 0$.

Anti-symmetric imperfection

In a similar way to the symmetric imperfection case, owing to the non-zero initial lateral deflections and rotations, the energy terms are different from those of the perfect case, apart from those describing the mid-span, since there is no lateral deflection or rotation in this location at all times, as shown in Figure 4.4(b). Therefore the new total potential energy with the introduction of the Mode 2 imperfection profile is given by:

$$\begin{aligned}
\tilde{V}_{m2} = & \frac{1}{2} \rho K (Q_1 - e_2 \delta)^2 + \frac{1}{2} K Q_2^2 + \frac{1}{2} \rho K (Q_3 + e_2 \delta)^2 \\
& + \frac{1}{2} \left[\arcsin Q_1 - \arcsin (Q_2 - Q_1) - 2 \arcsin(e_2 \delta) \right]^2 \\
& + \frac{1}{2} R \left[\arcsin(Q_2 - Q_1) + \arcsin(Q_2 - Q_3) \right]^2 \\
& + \frac{1}{2} \left[\arcsin Q_3 - \arcsin(Q_2 - Q_3) + 2 \arcsin(e_2 \delta) \right]^2 \\
& - p \left[4 - \sqrt{1 - Q_1^2} - \sqrt{1 - (Q_2 - Q_1)^2} - \sqrt{1 - (Q_2 - Q_3)^2} - \sqrt{1 - Q_3^2} \right].
\end{aligned} \tag{4.44}$$

The respective equilibrium equations for Modes 1, 2 and 3 are expressed by:

$$\begin{aligned}
\frac{\partial \tilde{V}_{m2}}{\partial Q_1} = & \rho K (Q_1 - e_2 \delta) - \frac{R[-\arcsin(Q_1 - Q_2) + \arcsin(Q_2 - Q_3)]}{\sqrt{1 - (Q_2 - Q_1)^2}} \\
& + [\arcsin Q_1 + \arcsin(Q_1 - Q_2) - 2 \arcsin(e_2 \delta)] \left[\frac{1}{\sqrt{1 - Q_1^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] \\
& - p \left[\frac{Q_1}{\sqrt{1 - Q_1^2}} - \frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] = 0,
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
\frac{\partial \tilde{V}_{m2}}{\partial Q_2} = & K Q_2 - \frac{\arcsin Q_1 + \arcsin(Q_1 - Q_2) - 2 \arcsin(e_2 \delta)}{\sqrt{1 - (Q_2 - Q_1)^2}} \\
& - \frac{\arcsin Q_3 + \arcsin(Q_2 - Q_3) + 2 \arcsin(e_2 \delta)}{\sqrt{1 - (Q_2 - Q_3)^2}} + R[\arcsin(Q_2 - Q_3) \\
& - \arcsin(Q_1 - Q_2)] \left[\frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] \\
& - p \left[\frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0,
\end{aligned} \tag{4.46}$$

$$\begin{aligned}
\frac{\partial \tilde{V}_{m2}}{\partial Q_3} = & \rho K (Q_3 + e_2 \delta) - \frac{R[\arcsin(Q_2 - Q_3) - \arcsin(Q_1 - Q_2)]}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& + [\arcsin Q_3 + \arcsin(Q_2 - Q_3) + 2 \arcsin(e_2 \delta)] \left[\frac{1}{\sqrt{1 - Q_3^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] \\
& - p \left[\frac{Q_3}{\sqrt{1 - Q_3^2}} - \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0,
\end{aligned} \tag{4.47}$$

where:

$$e_2 = \frac{1}{\sqrt{2}}. \tag{4.48}$$

Note that in this case e_2 is a non-zero value and is independent of the lateral and rotational stiffnesses, while $e_1 = 0$.

Asymmetric imperfection

Unlike imperfect cases for Modes 1 and 2, where imperfection profiles only triggering pure modes are introduced, a combination of the imperfection profiles is also used to trigger the possibility of interactive buckling, as shown in Figure 4.4(c). Initial lateral deflections and rotations are non-zero values at all three joints of the column. Again, the work done by the external load remains unchanged. Therefore the new total potential energy with the introduction of the asymmetric imperfection profile is given by:

$$\begin{aligned}
\tilde{V}_{\text{int}} = & \frac{1}{2}\rho K [Q_1 - \delta(e_1 + e_2)]^2 + \frac{1}{2}K(Q_2 - e_1\delta\zeta_1)^2 + \frac{1}{2}\rho K [Q_3 - \delta(e_1 - e_2)]^2 \\
& + \frac{1}{2}\left\{ \arcsin Q_1 - \arcsin(Q_2 - Q_1) - \arcsin[\delta(e_1 + e_2)] \right. \\
& \left. - \arcsin[\delta(e_1 + e_2) - e_1\delta\zeta_1] \right\}^2 + \frac{1}{2}R\left\{ \arcsin(Q_2 - Q_1) + \arcsin(Q_2 - Q_3) \right. \\
& \left. - \arcsin[e_1\delta\zeta_1 - \delta(e_1 - e_2)] + \arcsin[\delta(e_1 + e_2) - e_1\delta\zeta_1] \right\}^2 \\
& + \frac{1}{2}\left\{ \arcsin Q_3 - \arcsin(Q_2 - Q_3) - \arcsin[\delta(e_1 - e_2)] \right. \\
& \left. + \arcsin[e_1\delta\zeta_1 - \delta(e_1 - e_2)] \right\}^2 - p\left[4 - \sqrt{1 - Q_1^2} - \sqrt{1 - (Q_2 - Q_1)^2} \right. \\
& \left. - \sqrt{1 - (Q_2 - Q_3)^2} - \sqrt{1 - Q_3^2} \right].
\end{aligned} \tag{4.49}$$

The respective equilibrium equations for Modes 1, 2 and 3 are expressed by:

$$\begin{aligned}
\frac{\partial \tilde{V}_{\text{int}}}{\partial Q_1} = & \rho K [Q_1 - \delta(e_1 + e_2)] - p\left[\frac{Q_1}{\sqrt{1 - Q_1^2}} - \frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] + \left\{ \arcsin Q_1 \right. \\
& \left. - \arcsin(Q_1 - Q_2) - \arcsin[\delta(e_1 + e_2)] - \arcsin[\delta(e_1 + e_2) - e_1\delta\zeta_1] \right\} \\
& \times \left[\frac{1}{\sqrt{1 - Q_1^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right] - \frac{R[-\arcsin(Q_1 - Q_2) + \arcsin(Q_2 - Q_3)]}{\sqrt{1 - (Q_2 - Q_1)^2}} \\
& - \frac{R\left\{ \arcsin[e_1\delta\zeta_1 - \delta(e_1 - e_2)] - \arcsin[\delta(e_1 + e_2) - e_1\delta\zeta_1] \right\}}{\sqrt{1 - (Q_2 - Q_1)^2}} = 0,
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
\frac{\partial \tilde{V}_{\text{int}}}{\partial Q_2} = & K(Q_2 - e_1 \delta \zeta_1) \\
& - \frac{\arcsin Q_1 + \arcsin(Q_1 - Q_2) - \arcsin [\delta(e_1 + e_2)] - \arcsin [\delta(e_1 + e_2) - e_1 \delta \zeta_1]}{\sqrt{1 - (Q_2 - Q_1)^2}} \\
& - \frac{\arcsin Q_3 + \arcsin(Q_2 - Q_3) - \arcsin [\delta(e_1 - e_2)] + \arcsin [e_1 \delta \zeta_1 - \delta(e_1 - e_2)]}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& + R \left[\arcsin(Q_2 - Q_3) - \arcsin(Q_1 - Q_2) \right] \left[\frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right. \\
& + \left. \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] - R \left\{ \arcsin [e_1 \delta \zeta_1 - \delta(e_1 - e_2)] - \arcsin [\delta(e_1 + e_2) \right. \\
& - \left. e_1 \delta \zeta_1] \right\} \left[\frac{1}{\sqrt{1 - (Q_2 - Q_1)^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] - p \left[\frac{Q_2 - Q_1}{\sqrt{1 - (Q_2 - Q_1)^2}} \right. \\
& + \left. \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] = 0,
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
\frac{\partial \tilde{V}_{\text{int}}}{\partial Q_3} = & \rho K [Q_3 - \delta(e_1 - e_2)] - p \left[\frac{Q_3}{\sqrt{1 - Q_3^2}} - \frac{Q_2 - Q_3}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] + \left\{ \arcsin Q_3 \right. \\
& - \arcsin(Q_2 - Q_3) + \arcsin [e_1 \delta \zeta_1 - \delta(e_1 - e_2)] - \arcsin [\delta(e_1 - e_2)] \left. \right\} \\
& \times \left[\frac{1}{\sqrt{1 - Q_3^2}} + \frac{1}{\sqrt{1 - (Q_2 - Q_3)^2}} \right] - \frac{R [-\arcsin(Q_1 - Q_2) + \arcsin(Q_2 - Q_3)]}{\sqrt{1 - (Q_2 - Q_3)^2}} \\
& - \frac{R \left\{ \arcsin [e_1 \delta \zeta_1 - \delta(e_1 - e_2)] - \arcsin [\delta(e_1 + e_2) - e_1 \delta \zeta_1] \right\}}{\sqrt{1 - (Q_2 - Q_3)^2}} = 0.
\end{aligned} \tag{4.52}$$

Note that both e_1 and e_2 are non-zero values and satisfy the relationship described in Equation (4.38).

4.3 Results and discussion

4.3.1 Buckling loads

The expressions of the critical loads for Modes 1, 2 and 3 have been stated already in Equations (4.25), (4.26) and (4.27) respectively. Figure 4.5 shows the distributions of the

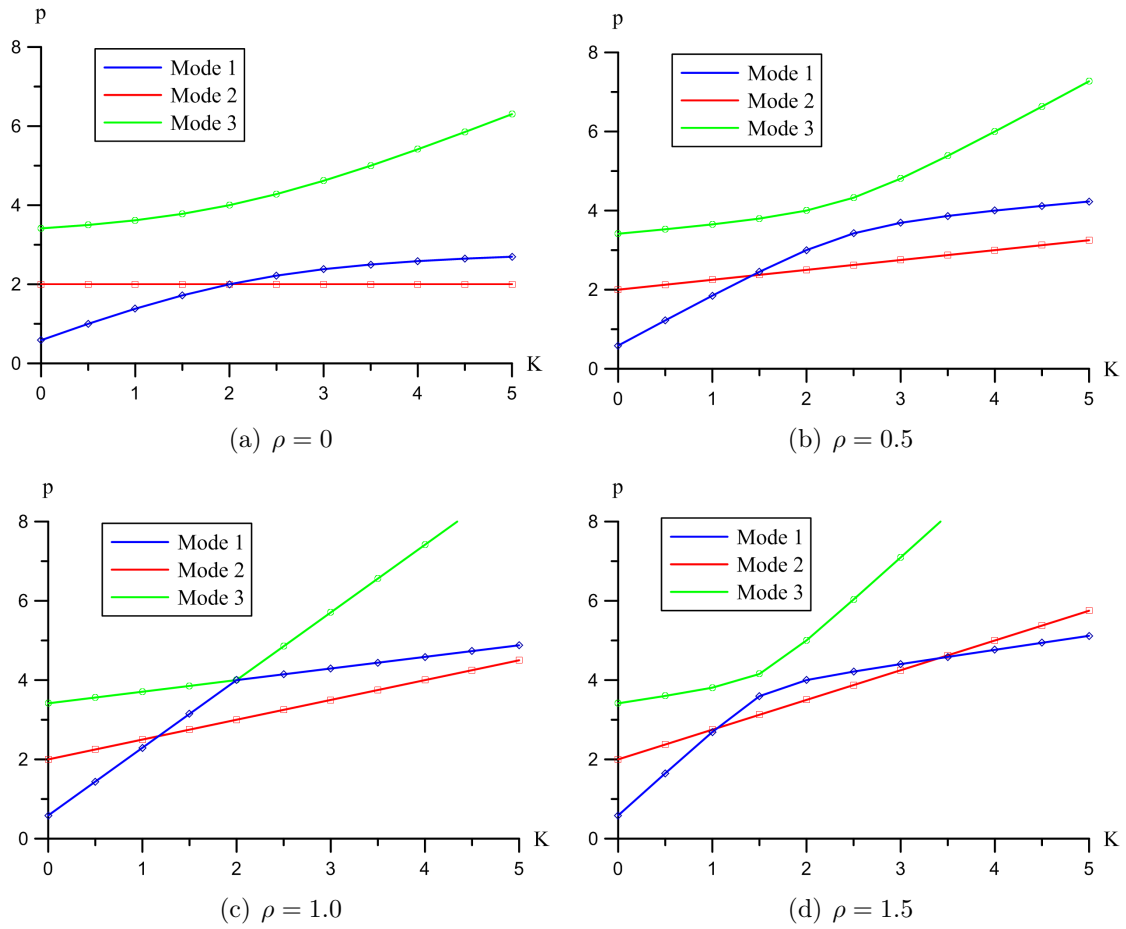


Figure 4.5: Buckling loads of Modes 1, 2 and 3 with the normalized lateral stiffness K varying when the rotational stiffness ratio $R = 1$.

respective normalized critical load p with the normalized lateral stiffness K varying when the rotational stiffness ratio $R = c_0/c = 1$, which indicates that the rotational stiffnesses of the edge and the middle joint of the column are the same. As shown in Figure 4.5(a), when $\rho = k_0/k = 0$, which indicates there are no lateral springs at the two edge joints

of the column, the system effectively becomes a single-bay prestressed stayed column. It can be seen that Mode 1 is critical initially, until K reaches 2 when the buckling loads of Modes 1 and 2 coincide with Mode 2 becoming critical when $K > 2$. It is worth noting that Mode 3 never becomes critical among the three modes. It is also important to note that the normalized buckling load of Mode 2 is fixed to 2 regardless of K values. This is because that there is no lateral deflection in Mode 2 buckling, hence the stiffness of the only lateral spring does not affect the buckling load. This is demonstrated directly when substituting $\rho = 0$ into Equation (4.26).

As shown in Figure 4.5(b), when $\rho = 0.5$, which indicates the stiffness at the edge joint longitudinal springs of the column is one-half of the stiffness at the middle joint longitudinal spring, Mode 1 is critical initially, and then replaced by Mode 2 when K is approximately 1.5. Mode 3 still possesses the largest buckling load among the three eigenmodes. Unlike the effective single-bay case shown in Figure 4.5(a), the critical load of Mode 2 increases along with the increase of K . This is because the longitudinal springs do exist at the two edge joints of the column, therefore the lateral deflections at these two locations are now affected by the corresponding springs.

In Figure 4.5(c), when $\rho = 1.0$, which indicates that all the longitudinal springs have the same stiffness, Mode 1 is critical initially, and then replaced by Mode 2 when K is approximately 1.2. Mode 3 is never critical, however, when $K = 2$ its buckling load coincides with that of Mode 1.

Lastly, Figure 4.5(d) shows the case when $\rho = 1.5$, which indicates that the lateral stiffness at the edge joints of the column is larger than that at the middle joint, which may be more akin to a double-bay case with a smaller lateral stiffness at midspan. Mode 1 is critical until K reaches at approximately 1, when Mode 2 becomes critical. Mode 1 becomes critical once again when K exceeds at approximately 3.5 and remains critical for higher K values.

The phenomenon observed above agrees qualitatively with the findings obtained from the

FE model, where Mode 1 is critical initially and Mode 2 becomes critical after their buckling loads coincide while increasing the stay diameter ϕ_s and the column aspect ratio $2\alpha_m$. It can also be seen that for triple-bay cases shown effectively in Figures 4.5(b)–(d), the Mode 1 buckling load increases relatively rapidly along with the increase of the normalized lateral stiffness K initially. The rates of increase subsequently reduces at a certain point particularly when Mode 2 is critical. The buckling load of Mode 2 increases at a constant rate in each case, while the rate is higher when ρ is larger. As observed, Mode 3 never becomes critical; in contrast with Mode 1, the Mode 3 buckling load increases slowly initially with K , and subsequently the rate of increase becomes faster at the instant when the increase of the buckling load of Mode 1 reduces.

4.3.2 Post-buckling analysis

The post-buckling analysis is conducted by solving the derived nonlinear equilibrium equations by using the package AUTO-07P (Doedel & Oldeman, 2011). This is a sophisticated and very powerful package designed for solving nonlinear algebraic equations and ordinary differential equations. The principal feature of AUTO-07P is that is a numerical continuation package (Seydel, 1994; Seydel, 1997; Krauskopf *et al.*, 2007), which implies that it can solve nonlinear equations and trace solution paths as free parameters are varied. In the current project, the free parameter is the external load p , the solution variables are Q_i and the solution paths are the equilibrium paths. The software AUTO-07P is very well documented and tested (Doedel, 1997; Doedel, 2007) and it has been in existence since the 1980s. It is based on a combination of orthogonal collocation techniques to discretize the equations (De Boor & Swartz, 1973; Russell & Christiansen, 1978) and pseudo-arclength continuation (Keller, 1977) to find the solutions for a changing parameter – this is essentially a variant of the Riks method that was mentioned previously and is implemented within ABAQUS.

A key feature of AUTO-07P is that it is capable of numerically pinpointing different classes of bifurcations including limit points alongside pitchfork, transcritical, period-doubling, torus and Hopf bifurcations (Glendinning, 1994), the latter three bifurcation types being beyond the scope of the current work. Moreover, once these bifurcations are determined, it also has the important capability of switching between solution branches such that a complete solution structure can be established by tracing stable and unstable paths. Hence, unlike ABAQUS, AUTO-07P is quite capable of solving for the system of equations which describe the perfect geometry in the nonlinear range without the requirement for introducing small perturbations to trigger the post-buckling response. Indeed, it is also capable of determining the solutions for the system with the different initial imperfections.

For post-buckling stability problems of the column, the number of the equilibrium equations is directly linked to the number of degrees of freedom adopted within the model. Therefore, for the current model, three equilibrium equations given in Equations (4.30), (4.31) and (4.32) are solved numerically in AUTO-07P to study the post-buckling behaviour of the triple-bay prestressed stayed column. The applied load, p , is the only free parameter in the analysis. Note that unless stated otherwise, the lateral stiffness ratio $\rho = 0.5$ and the rotational stiffness ratio $R = 1$ are used as the benchmark values in the current study.

Figure 4.6 shows the equilibrium paths of the normalized buckling load p versus Q_1 and Q_2 versus Q_1 with the lateral stiffness K varying when Mode 1 is critical. It can be seen that Mode 1 is critical and it forms a slightly unstable equilibrium state while both the post-buckling responses for Modes 2 and 3 are stable for all the three K values. Along with the increase of K , the critical loads of Modes 1 and 2 become closer together. It can also be seen from the relationships between Q_2 and Q_1 that, unlike Modes 1 and 3, Q_2 remains at zero for pure Mode 2 buckling, which of course reflects the reality since there is no lateral deflection at the mid-span in that mode. It is worth mentioning that, although the equilibrium equations used in the post-buckling analysis are capable of capturing interactive buckling since the cross terms of the generalized coordinates Q_i are included, no modal

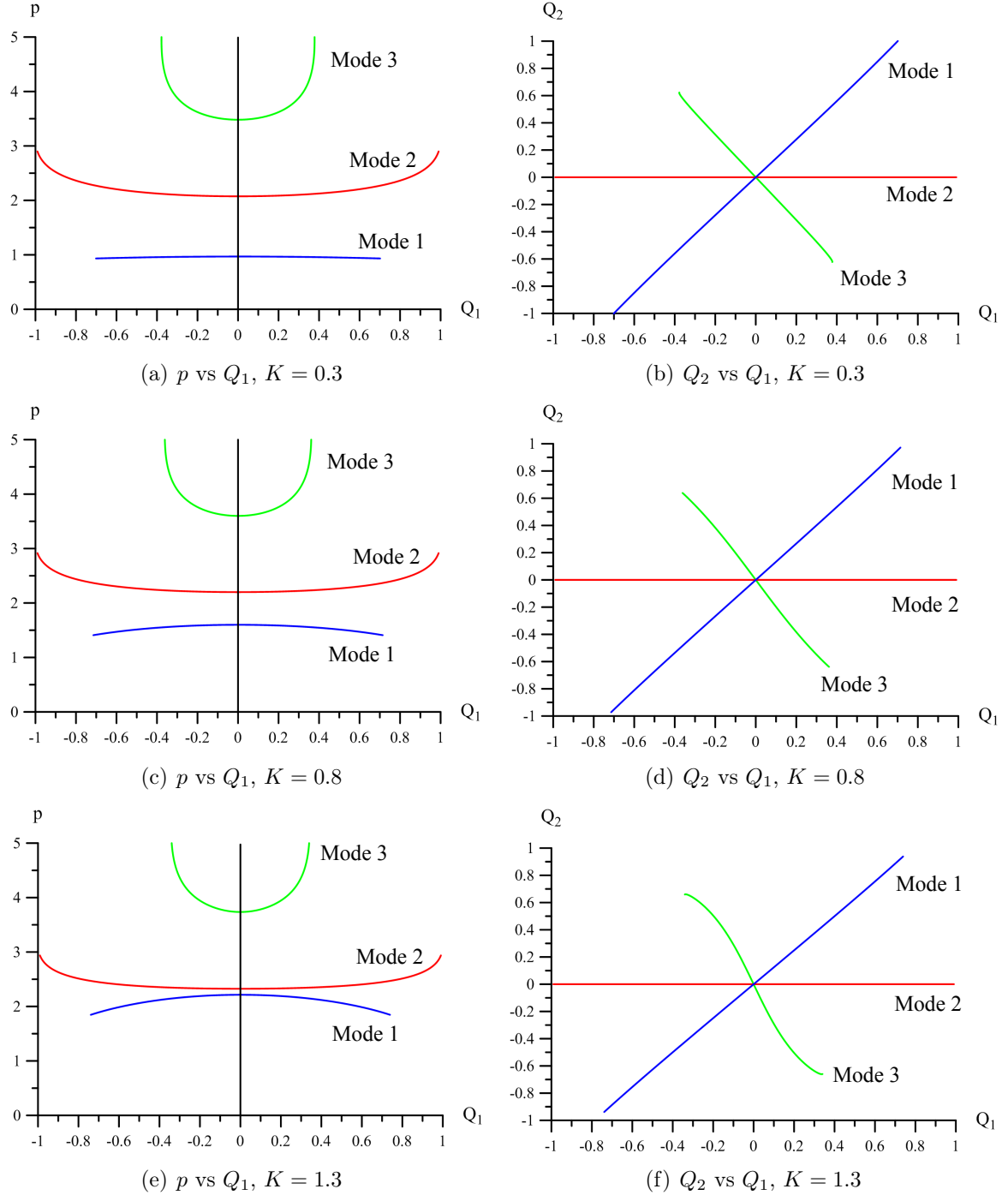


Figure 4.6: Equilibrium paths of the normalized buckling load p versus Q_1 and Q_2 versus Q_1 with the lateral stiffness K varying when Mode 1 is critical.

interactions are observed when Mode 1 is critical. This also agrees with the conclusion obtained in the FE results.

When K is increased to 1.45, as shown in Figure 4.7, which shows the equilibrium paths of

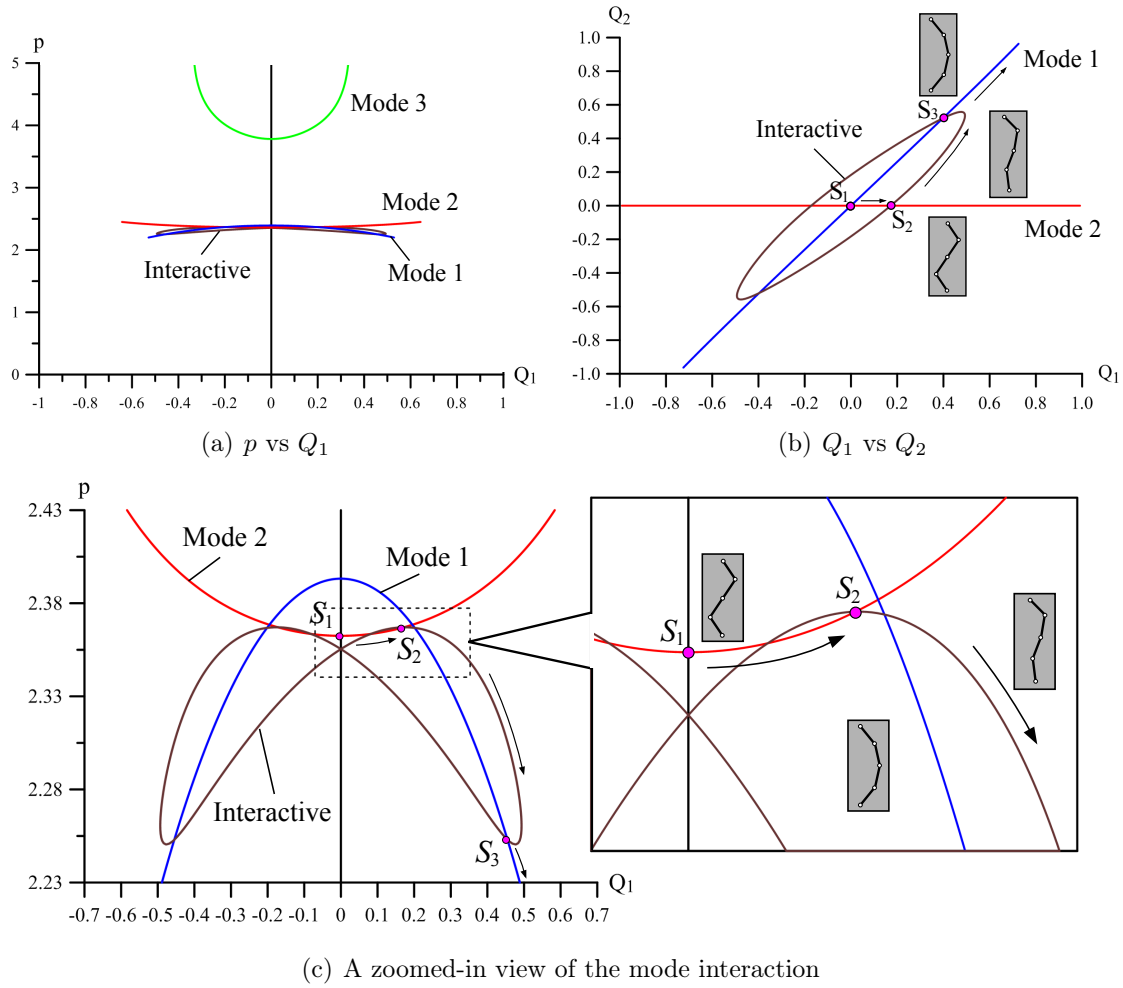


Figure 4.7: Equilibrium paths of the normalized buckling load p versus Q_1 with Q_1 versus Q_2 when the lateral stiffness $K = 1.45$ and Mode 2 is marginally critical.

the normalized buckling load p versus Q_1 and Q_2 versus Q_1 when Mode 2 is critical but the buckling loads of Modes 1 and 2 are very close. In this particular case, interactive buckling is observed. It can be seen that in the closeup of Figure 4.7(a), which is shown in Figure 4.7(c), the equilibrium path of Mode 2 starts from the first bifurcation point S_1 , and then transfers to the path of an interactive mode after a second bifurcation point S_2 . The path subsequently transfers back to the distinct Mode 1, through a third bifurcation point S_3 , as

shown in Figure 4.7(c). The same observation can also be made from Figure 4.7(b), where the equilibrium path of Q_1 versus Q_2 forms an elliptical interactive loop. The phenomenon where the equilibrium path transfers from one distinct buckling mode to another through an interactive mode has been appropriately termed as ‘mode jumping’ in the literature (Everall & Hunt, 2000). A very similar interactive buckling phenomenon was also found for the simplified analytical model of the single-bay prestressed stayed column (Wadee *et al.*, 2013a). It can also be seen that although Mode 2 is critical, interactive buckling actually governs the post-buckling behaviour, which is also found in the FE analysis.

When K is increased further, as shown in Figure 4.8, which shows the equilibrium paths of the normalized buckling load p versus Q_1 and Q_2 versus Q_1 when Mode 2 is critical. The buckling loads of Modes 1 and 2 are further away from each other. Interactive buckling can still be observed in these cases. However, both interactive loops, as exhibited in Figures 4.7(b) and (c), break in the equilibrium paths of p versus Q_1 and Q_1 versus Q_2 for the higher K values. Although interactive buckling still exists and governs the post-buckling response, it can be seen that the equilibrium path never transfers back to Mode 1 after it transfers from Mode 2 to the interactive mode once the loop breaks, as shown in Figures 4.8(b), (d) and (f). This, again, is in accord with the FE model where interactive buckling governs throughout the post-buckling response when Mode 2 is critical.

Figure 4.9 provides a more detailed picture for how the interactive loop breaks. When $K = 1.45$, as shown in Figure 4.9(a), the equilibrium path of Q_2 versus Q_1 for interactive buckling forms a closed loop. The equilibrium behaviour transfers from interactive buckling to Mode 1 at the left and right hand edges of the loop where interactive buckling and the Mode 1 paths intersect. So long as K is increased to 1.46, as shown in Figure 4.9(b), the left and right hand edges of the loop break and no intersections are found in these two locations. As K is increased further to 1.48 and 1.6, as shown in Figures 4.9(c) and (d), the gap between the top and the bottom edges of the interactive buckling path becomes larger and the path could no longer close. Note also that $Q_1 = 1$, $(Q_2 - Q_1)^2 = 1$, $(Q_2 - Q_3)^2 = 1$

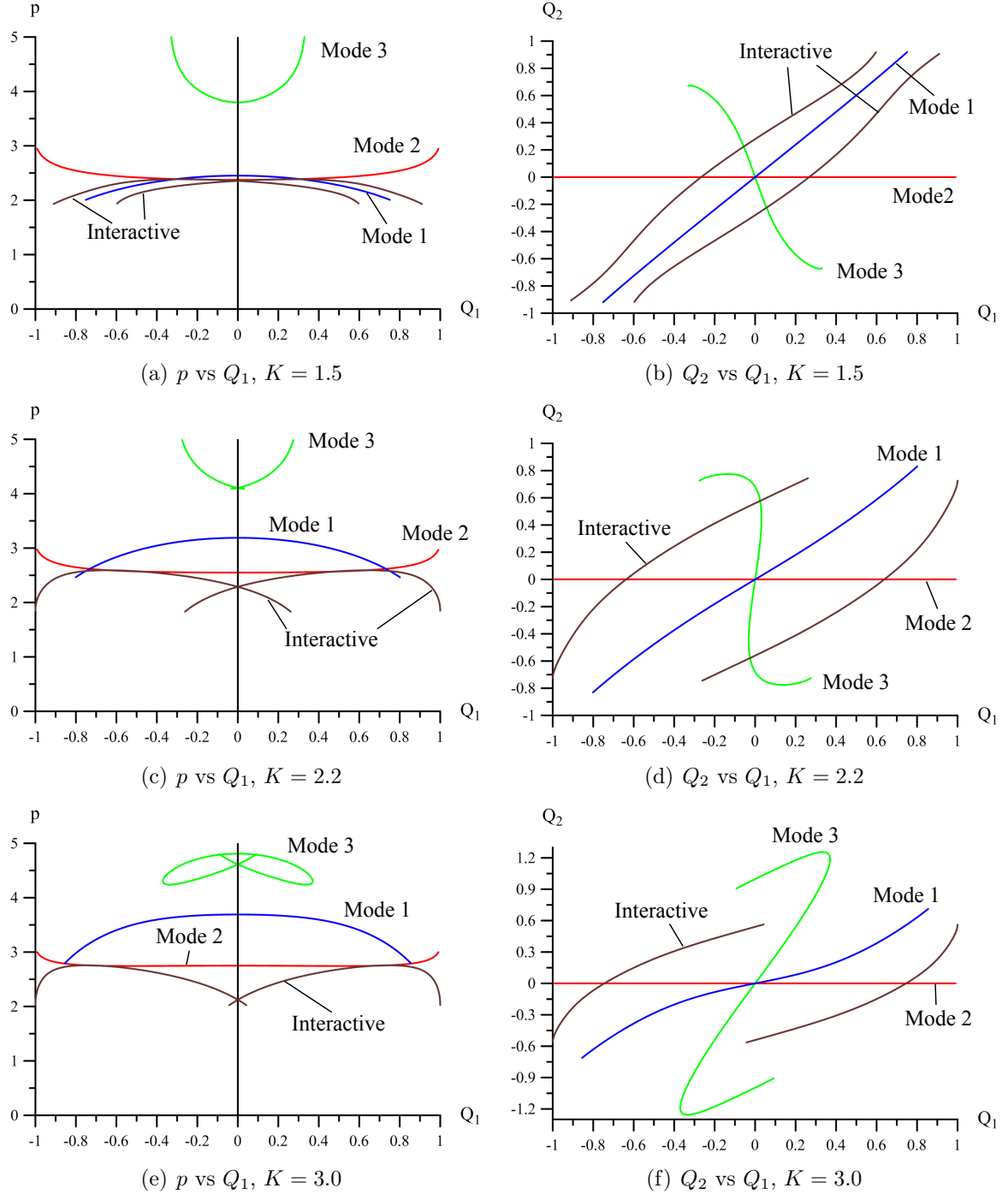


Figure 4.8: Equilibrium paths of the normalized buckling load p versus Q_1 with Q_2 versus Q_1 when Mode 2 is clearly critical.

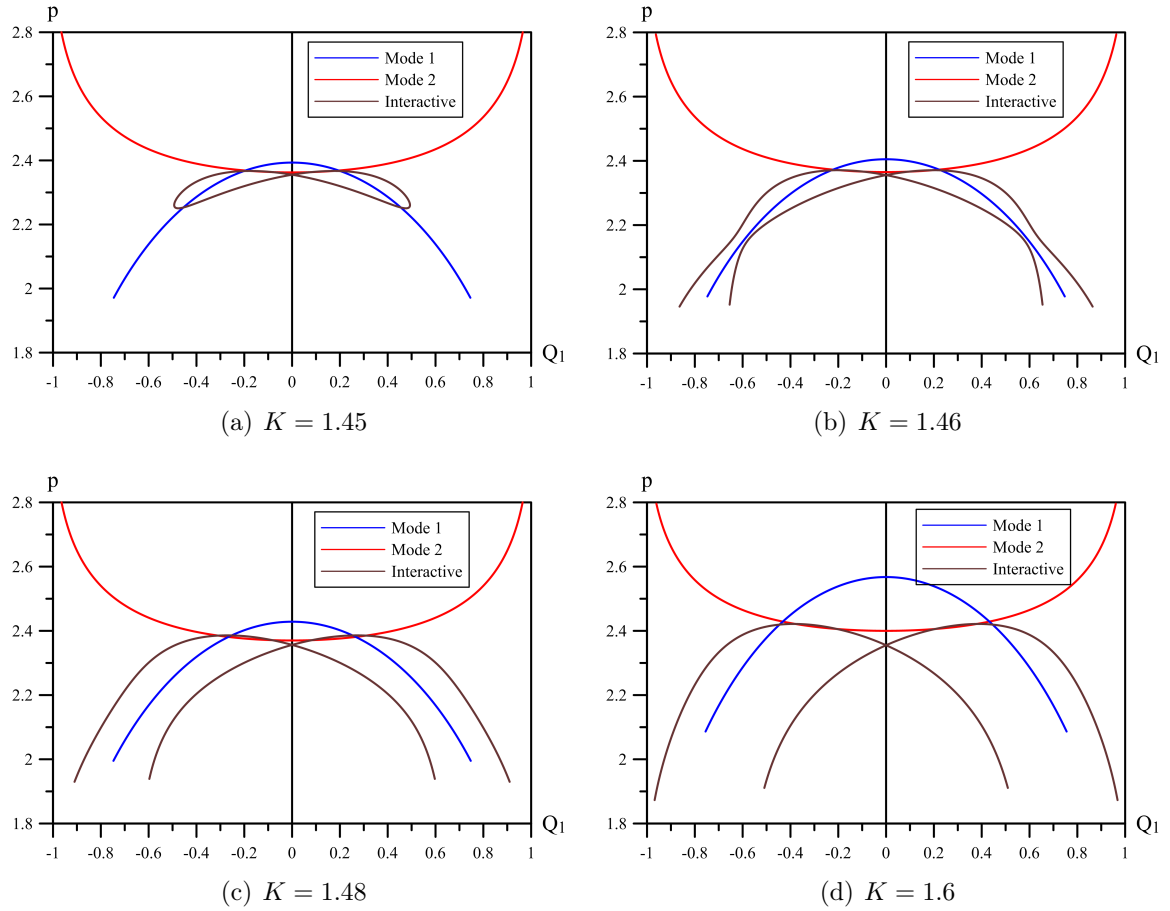


Figure 4.9: Equilibrium paths of the normalized buckling load p versus Q_1 with the normalized lateral stiffness K varying from 1.45 to 1.6. Note the interactive loop breaking as the difference in the Modes 1 and 2 buckling loads increase with K .

and $Q_3 = 1$ are also asymptotic limits for the model with singularities occurring in the equilibrium equations when these levels of deflection are reached. Hence, for the interactive loops to close the maximum deflections have to be significantly less than these levels.

In addition, when the normalized lateral stiffness K is relatively small, the equilibrium path of Mode 1 is very slightly unstable and nearly flat. With the increase of K , the post-buckling response of pure Mode 1 becomes increasingly unstable, as shown in Figure 4.10. On the contrary, the post-buckling response of Mode 2 is stable when K is relatively

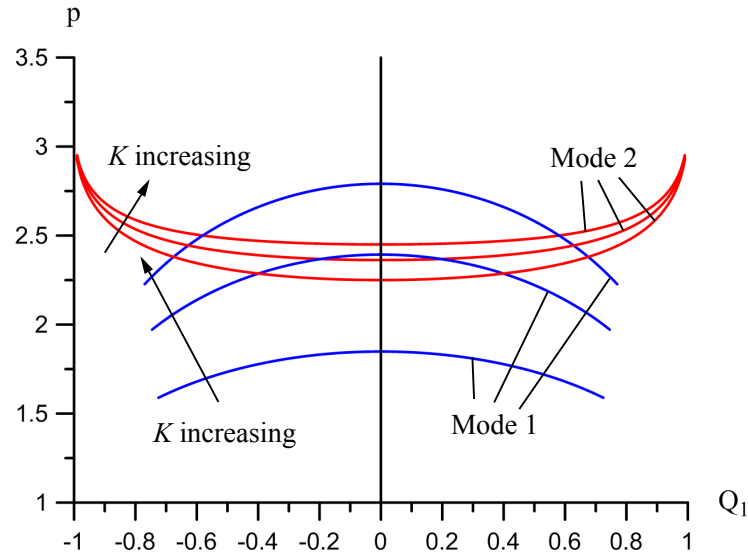


Figure 4.10: Equilibrium paths of the normalized buckling load p versus Q_1 for Modes 1 and 2 when $K = 1, 1.45$ and 1.8 .

small, and becomes less stable when K increases with the post-buckling stiffness reducing.

4.3.3 Imperfect cases

The models devised above are perfect cases without the consideration of any initial imperfections. Total potential energies described in Equations (4.39), (4.44) and (4.49) are used to trigger Mode 1, Mode 2 and interactive buckling respectively. Unless stated otherwise, the normalized initial out-of-straightness $\delta = 1/500$ is used for all the imperfect

cases, which represents the level of imperfection used in steel design to Eurocode 3 (BS EN 1993-1-1, 2005).

Firstly, cases for $\rho = 0$, which represent the single-bay prestressed stayed column, are investigated, since the imperfect model has not been studied for the single-bay case previously.

Figure 4.11 shows the equilibrium paths of the normalized buckling load p versus Q_1 with

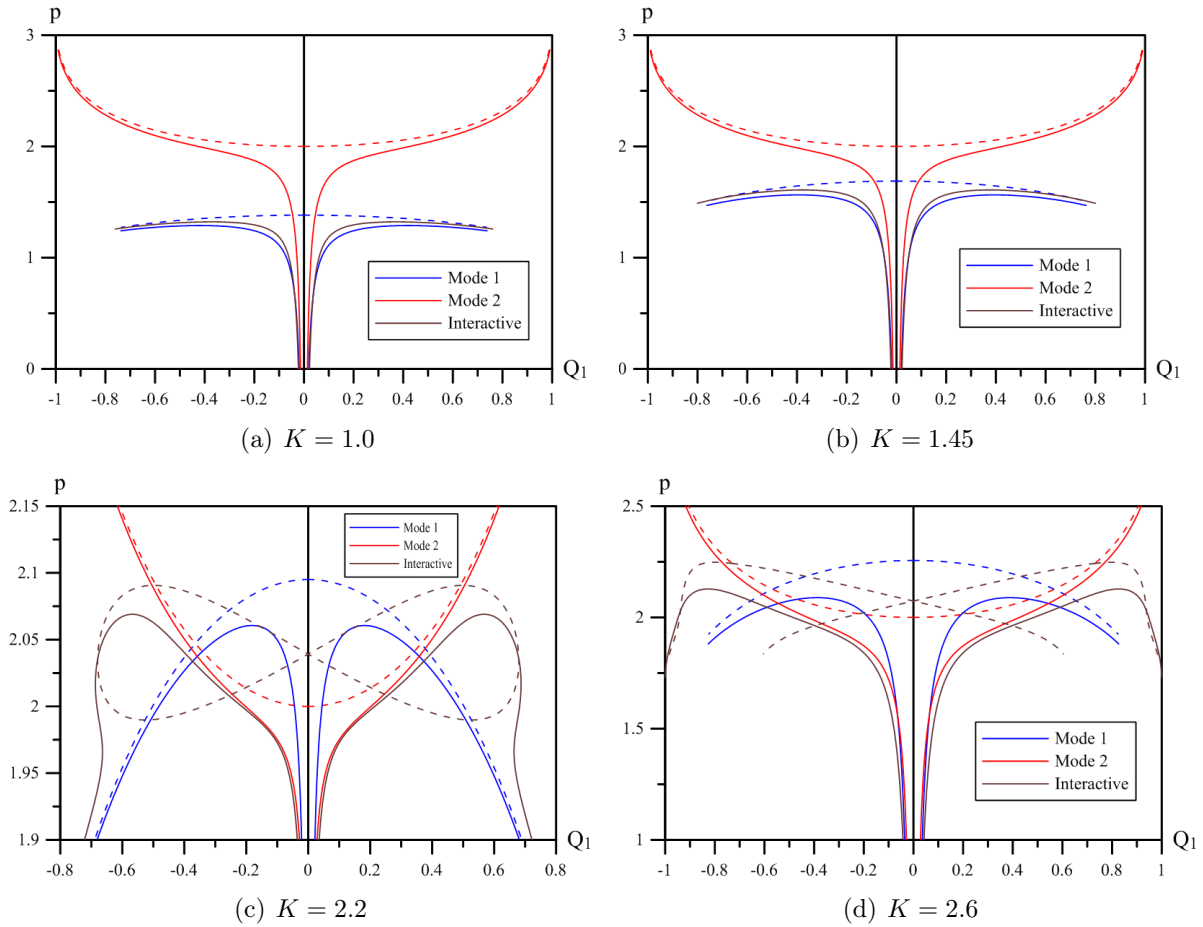


Figure 4.11: Equilibrium paths of the normalized buckling load p versus Q_1 with the normalized lateral stiffness K varying for the effective single-bay case ($\rho = 0$).

the normalized lateral stiffness K varying for the single-bay case. The solid lines represent the imperfect paths while the dashed ones represent the perfect paths. Owing to the existence of initial imperfections, the equilibrium paths for the imperfect case are asymptotic to the corresponding perfect paths, both fundamental and post-buckling. Figures 4.11(a) and (b) show the cases when Mode 1 is critical. It can be seen that although no interactive

buckling is observed in the perfect case, it can still be triggered by introducing asymmetric imperfections in the imperfect case even when Mode 1 is critical. However, it is noteworthy that Mode 1 buckling governs the actual post-buckling response compared with interactive buckling since it gives the lowest ultimate load. Figure 4.11(c) shows the case when mode jumping is observed in the perfect case. It can be seen that the equilibrium path of the imperfect interactive mode does not form a loop any more. It is of course asymptotic to the imperfect path of Mode 2 initially, and then to the perfect interactive loop but never exceeds the imperfect path of Mode 2. This indicates that interactive buckling governs the post-buckling behaviour in the imperfect case. Similarly in Figure 4.11(d), when the interactive loop breaks in the perfect case, interactive buckling still governs the post-buckling behaviour in the imperfect case.

Figure 4.12 shows the equilibrium paths of the normalized buckling load p versus Q_1 and Q_2 versus Q_1 with the normalized lateral stiffness K varying for the triple-bay case. In a similar way to the single-bay case, as shown in Figure 4.12(a), interactive buckling can be triggered by introducing asymmetric imperfections in the imperfect case even when Mode 1 is critical. However, the actual governing post-buckling mode is still Mode 1, as expected. When $K = 1.45$, as shown in Figure 4.12(c), the equilibrium path of interactive buckling does not form as a loop. Instead, it is asymptotic to the imperfect path of Mode 2 initially, and then to the perfect interactive loop but never exceeds the imperfect path of Mode 2, which indicates that interactive buckling is the actually governs the post-buckling behaviour. Similarly in Figure 4.12(e), when the interactive loop breaks in the perfect case, interactive buckling still governs the post-buckling behaviour in the imperfect case.

Figure 4.13 shows the comparison of the buckling load capacities between the perfect and imperfect cases with the normalized lateral stiffness K varying. It can be seen that the plots of the imperfect Modes 1 and 2 are slightly below the plots of the perfect cases. This is as expected since the initial imperfections are introduced. It is very important to note that when Mode 1 is critical, it is the actual governing post-buckling response. However,

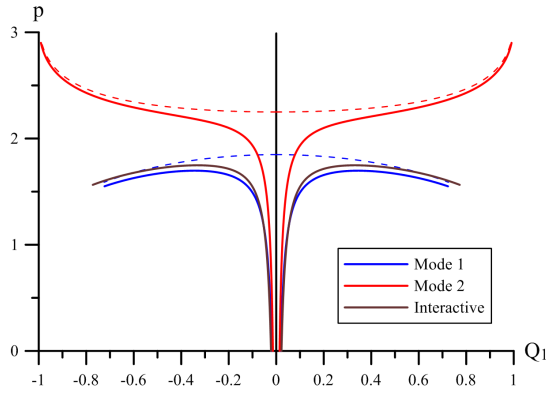
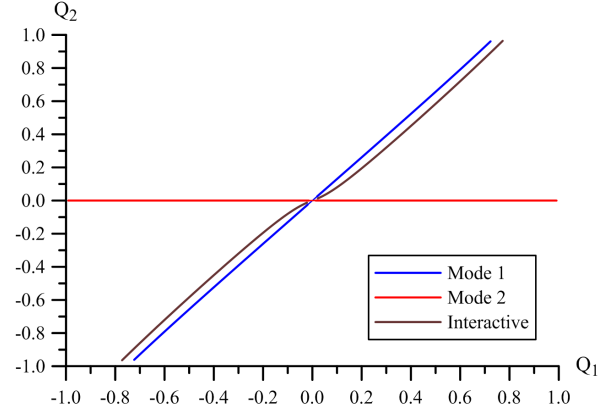
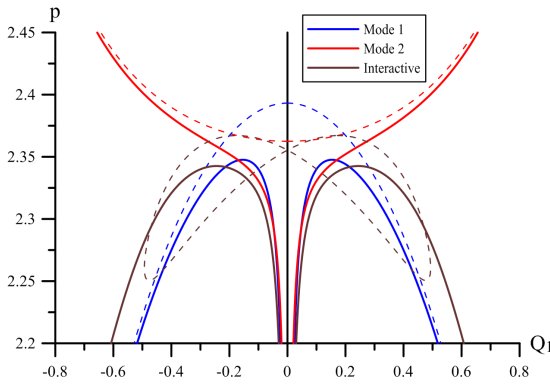
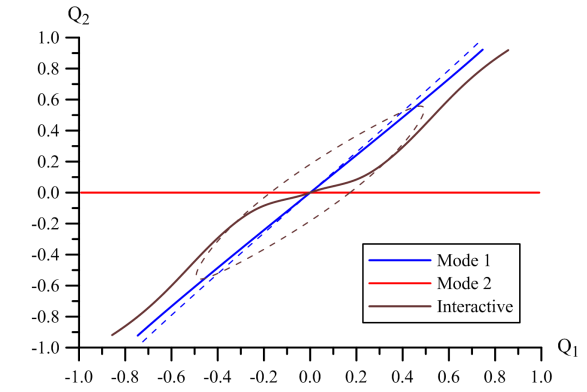
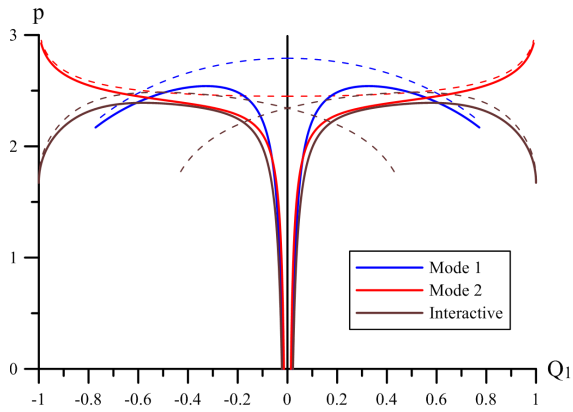
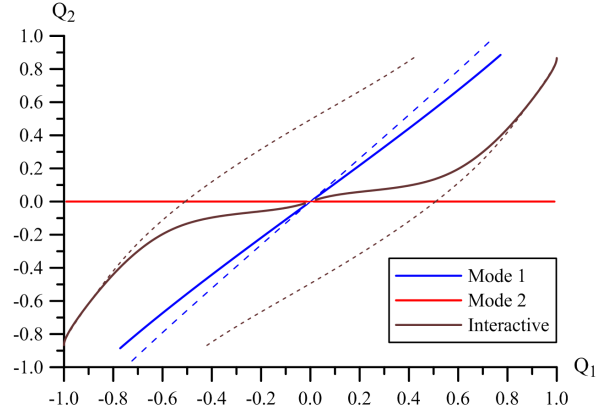
(a) p vs Q_1 , $K = 1.0$ (b) Q_2 vs Q_1 , $K = 1.0$ (c) p vs Q_1 , $K = 1.45$ (d) Q_2 vs Q_1 , $K = 1.45$ (e) p vs Q_1 , $K = 1.8$ (f) Q_2 vs Q_1 , $K = 1.8$

Figure 4.12: Equilibrium paths of the normalized buckling load p versus Q_1 and Q_2 versus Q_1 with the normalized lateral stiffness K varying for the triple-bay case ($\rho = 0.5$).

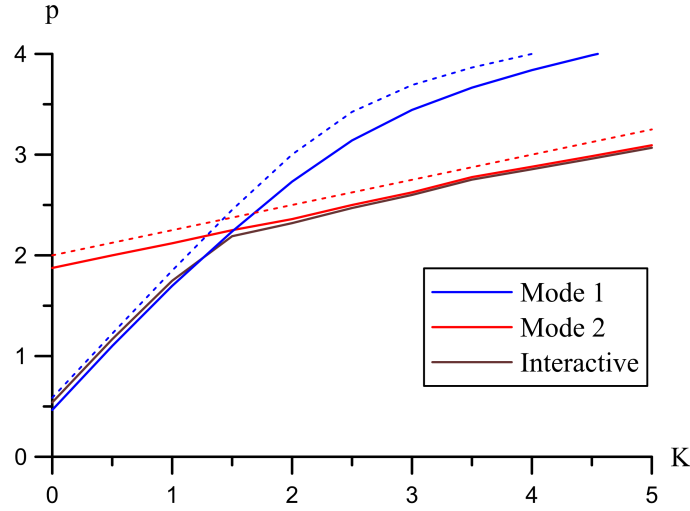


Figure 4.13: Comparison of the buckling load capacities between the perfect and imperfect cases with the normalized lateral stiffness K varying ($\rho = 0.5$).

when Mode 2 is critical, the actual governing post-buckling behaviour is interactive.

Figure 4.14 shows the equilibrium paths of the normalized buckling load p versus Q_1 with the initial out-of-straightness δ varying when the normalized lateral stiffness $K = 1$. It can be seen that with the increase of the imperfection size, the equilibrium paths of the imperfect case are more remote from the perfect paths, hence the buckling load capacities decrease. This is in complete accordance with the imperfection study in the FE model.

4.4 Concluding remarks

In the present chapter, a highly simplified discrete rigid link and spring model was developed to study the triple-bay prestressed stayed column quantitatively. The system stiffnesses in the actual structure were represented by the a series of longitudinal and rotational springs. The total potential energy was then formulated and diagonalized to calculate the buckling modes and loads. Equilibrium equations for both the perfect and imperfect cases were also obtained to study the post-buckling behaviour of the simplified model.

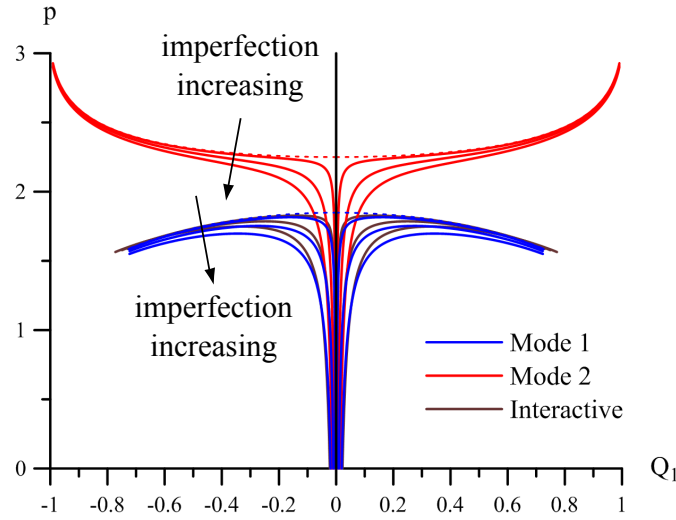


Figure 4.14: Equilibrium paths of the normalized buckling load p versus Q_1 with the initial out-of-straightness δ being $1/100$, $1/500$ and $1/10000$ when the normalized lateral stiffness $K = 1$ and $\rho = 0.5$.

In the perfect cases, when Mode 1 is critical, no mode interaction occurs and therefore the post-buckling response remains affine to Mode 1. When Mode 2 is critical but the buckling loads of the two modes are very close, the mode jumping phenomenon is observed and the post-buckling behaviour transfers from Mode 2 to Mode 1 through an interactive path. When Mode 2 is obviously critical, compared with Mode 1, the actual post-buckling response transfers from Mode 2 to the interactive path and remains interactive. A similar conclusion is also drawn from the imperfect case, where Mode 1 is the actual governing post-buckling mode when it is critical, even though interactive buckling can be triggered by introducing an asymmetric imperfection. However, interactive buckling governs the post-buckling response so long as Mode 2 is critical. This agrees with the results obtained from the FE model in Chapter 3, which indicates that when Mode 2 is the critical distinct mode, interactive buckling would be the most dangerous case in terms of the load carrying capacity.

Owing to the fact that the model developed in this chapter uses discrete springs to simulate the behaviour of the main column, the crossarms and the stays, the results from it can

only be compared with the FE results qualitatively. It is difficult to distinguish the effect introduced by the diameter of the stay, the length of the crossarm, and the levels of prestress in the actual structure directly. In addition, stay slackening is not taken into consideration in the discrete model, which makes the post-buckling behaviour less realistic when the end-shortening of the column reaches a certain value. Therefore, a more sophisticated analytical treatment is necessary, which models the actual structure more completely; this is developed in the following chapter and allows the analytical model to be compared quantitatively with the FE model. Nevertheless, the model presented in the current chapter gives strong evidence that formulating an analytical model is feasible and should be able to explain the mechanism behind the actual behaviour of prestressed stayed columns.

Chapter 5

Formulation of Continuous Analytical Model

5.1 Introduction

In previous chapters, a finite element (FE) model and a discrete analytical model were developed to investigate the buckling and the post-buckling behaviour of the triple-bay prestressed stayed column. However, owing to the discrete stiffnesses used in the analytical model presented in Chapter 4, its results could only be realistically compared with those from the FE model qualitatively. In the current chapter, an enhanced analytical model is presented that implements the Rayleigh–Ritz method to model the nonlinear elastic behaviour of the column as a multiple degree-of-freedom system. The bending of the main column element and the crossarms are included, as is the explicit response of the prestressed cable stays. Column arrangements with three sets of crossarms along the column span, as shown in Figure 3.1(c), are investigated to study the enhancement in the critical buckling load followed by the detailed post-buckling behaviour. Numerical solutions of the analytically derived equilibrium equations are obtained from the continuation and

bifurcation software AUTO-07P (Doedel & Oldeman, 2011).

The analytical model is then validated with the nonlinear finite element (FE) model developed in Chapter 3 within the commercial software ABAQUS (2014), which has been shown in the literature to provide an excellent benchmark when compared to actual physical experiments (Osofero *et al.* , 2012). Moreover, it has been demonstrated that for highly slender systems, elastic FE models provide an excellent prediction of the load carrying capacity (De Araujo *et al.* , 2008) since the stresses within the system generally do not approach the material yield stress.

5.2 Analytical modelling

The current investigation focuses on prestressed stayed columns with three sets of crossarms, which is the same as the model studied in the FE and the discrete analytical models in the previous two chapters. As shown in Figure 5.1(a), three sets of crossarms are rigidly

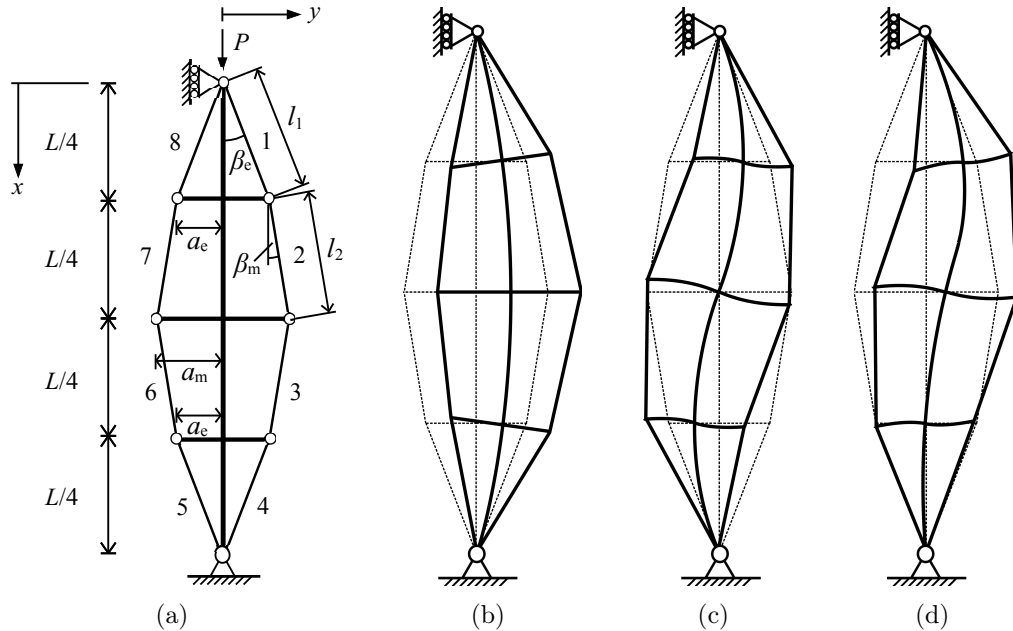


Figure 5.1: (a) The triple-bay prestressed stayed column with geometric and stay number definitions (1–8 inclusive); (b) symmetric buckling Mode 1; (c) anti-symmetric buckling Mode 2; (d) interactive asymmetric post-buckling profile.

connected to the one-quarter, one-half and three-quarter length points of the simply supported column. Prestressed cable stays are assumed to be pinned to the ends of the column and the tips of the crossarms; identical assumptions, as described in Chapter 3, are made in the current model. Unlike the discrete analytical model presented in Chapter 4, stay slackening is explicitly included.

Shape functions of the column and the crossarms are introduced such that the Rayleigh–Ritz method may be implemented to find the equilibrium equations from minimizing the total potential energy V ; it is obtained by summing the strain energy stored in the entire system minus the work done by the external load. The resulting nonlinear equilibrium equations obtained from the principle of stationary potential energy can then be solved in the continuation and bifurcation software AUTO-07P (Doedel & Oldeman, 2011) to reveal the critical and post-buckling behaviour of the column. Moreover, the bifurcational structure of the mechanical system is maintained, which provides key information regarding modal interactions and any potential sensitivities to initial geometric imperfections.

5.2.1 Benchmark prestressing force

The expression of the benchmark prestressing force was obtained in Chapter 3:

$$T_{\text{opt}} = P^C \frac{C_{11}}{C_{22}}, \quad (5.1)$$

where:

$$\begin{aligned} C_{11} &= \frac{\cos \beta_m}{2K_c \left[\frac{2}{K_{s2}} + \frac{\cos^2 \beta_m}{K_c} + \frac{4 \sin^2 \beta_m + 2\gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]}, \\ C_{22} &= 1 + \frac{\cos \beta_e \cos \beta_m}{2K_c \left[\frac{1}{K_{s2}} + \frac{2 \sin^2 \beta_m + \gamma(\sin \beta_e \sin \beta_m - \sin^2 \beta_m)}{K_m} \right]}, \end{aligned} \quad (5.2)$$

with $\gamma = a_e/a_m$ and:

$$\beta_e = \arctan(4\alpha_e), \quad \beta_m = \arctan[4(\alpha_m - \alpha_e)], \quad (5.3)$$

where $\alpha_e = a_e/L$ and $\alpha_m = a_m/L$ with K_c , K_m and K_{s2} being the axial stiffness of the main column, the middle crossarm and stay number 2:

$$K_c = \frac{E_c A_c}{L}, \quad K_m = \frac{E_a A_a}{a_m}, \quad K_{s2} = \frac{E_s A_s}{l_2}, \quad (5.4)$$

with E_c , E_a and E_s being the Young's Moduli alongside A_c , A_a and A_s being the cross-sectional areas of the main column, the crossarm and the stays respectively. The length of stay 2, l_2 , is $L\sqrt{(\alpha_m - \alpha_e)^2 + 1/16}$. The critical buckling load P^C is evaluated from linear eigenvalue analysis of the stayed column system by the commercial FE package ABAQUS (2014) without the consideration of the prestressing force, a procedure that has been successfully applied previously (Saito & Wadee, 2009b).

5.2.2 Shape function of the main column

The buckling profile of the main column element can be expressed as a combination of a symmetric shape W_1 (Mode 1, as shown in Figure 5.1(b)) and an anti-symmetric shape W_2 (Mode 2, as shown in Figure 5.1(c)) thus:

$$W_1(x) = \sum_{i=1}^n Q_{2i-1} L \sin \left[\frac{(2i-1)\pi x}{L} \right], \quad W_2(x) = \sum_{i=1}^n Q_{2i} L \sin \left(\frac{2i\pi x}{L} \right), \quad (5.5)$$

where Q_j is the generalized coordinate for the buckling mode shape contribution that contains j half-sine waves with L being the total length of the main column. The adoption of a larger j value allows the expressions above to describe more sophisticated and hence more accurate deformation profiles, but of course results in more equations to be solved, thereby increasing the complexity of the model. In the current work, $j = 8$ is deemed to be

an appropriate value to meet both requirements of accuracy and efficiency. The generalized shape of the column, as shown in Figure 5.1(d), may be obtained by combining the two distinct buckling shapes that would break both the reflective and rotational symmetries if modal interactions are triggered, as shown in Figure 5.1(d):

$$W(x) = W_1(x) + W_2(x) = \sum_{i=1}^{2n} Q_i L \sin \frac{i\pi x}{L}. \quad (5.6)$$

The normalized horizontal deflections of the main column element δ_1 , δ_2 and δ_3 at the one-quarter, one-half and three-quarter length points respectively can be expressed thus:

$$\delta_1 L = W(L/4), \quad \delta_2 L = W(L/2), \quad \delta_3 L = W(3L/4). \quad (5.7)$$

The rotations of the column about the vertical axis θ_1 , θ_2 and θ_3 at the one-quarter, one-half and three-quarter length points can be expressed to leading order as:

$$\theta_1 = -W'(L/4), \quad \theta_2 = -W'(L/2), \quad \theta_3 = -W'(3L/4), \quad (5.8)$$

where primes denote differentiation with respect to x . However, owing to the restraints provided by the stays to the crossarms, the actual rotations transferred to the crossarms may be expected to be different than those obtained from the expressions in Equation (5.8). Therefore the relationship between the main column and the crossarm rotations is expressed by introducing a calibration factor κ :

$$\theta_{ai} = \kappa \theta_i, \quad (5.9)$$

where $i = \{1, 2, 3\}$ and the value of κ is evaluated with assistance from the FE model and is presented later.

5.2.3 Shape functions of the crossarms

There are three sets of crossarms in the triple-bay prestressed stayed column. Firstly, since a lateral point load is applied to each crossarm from the stays, a cubic shape function $W_{R1}(y)$ is assumed to describe the top right crossarm, thus:

$$W_{R1}(y) = C_0 + C_1y + C_2y^2 + C_3y^3, \quad (5.10)$$

as shown in Figure 5.2. Four boundary conditions need to be evaluated for this crossarm:

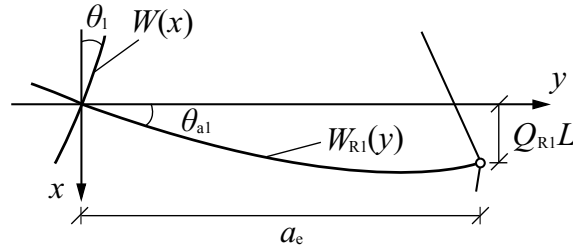


Figure 5.2: The deflected shape of the top right crossarm.

1. The deflection at the left-hand end of the crossarm is zero, hence $W_{R1}(0) = 0$.
2. The deflection at the right-hand end of the crossarm is defined as $Q_{R1}L$, hence $W_{R1}(a_e) = Q_{R1}L$ with Q_{R1} being a new generalized coordinate.
3. The left-hand side of the crossarm is rigidly connected to the main column, therefore the rotation of the crossarm at the connection is related to the rotation of the main column at the one-quarter length, hence $\dot{W}_{R1}(0) = \theta_{a1}$ as discussed above.
4. The right-hand end of the crossarm is pinned to the stays, therefore there is no bending moment at the end, hence $\ddot{W}_{R1}(a_e) = 0$.

Note that dots represent differentiation with respect to y and that the compression in the crossarm from the stay is neglected. The latter assumption simplifies the analysis since

iteration is not required to determine the shape function of the crossarm; this marks a departure from earlier work (Saito & Wadee, 2008). The shape function of the top right-hand crossarm $W_{R1}(y)$ can be obtained by solving for the constants C_0 , C_1 , C_2 and C_3 using the aforementioned boundary conditions. The shape functions of the other five crossarms can also be obtained by assuming similar cubic functions and boundary conditions with new generalized coordinates Q_{Ri} and Q_{Li} ; the detailed expressions for all the crossarms are thus:

$$\begin{aligned} W_{Ri}(y) &= \kappa\theta_i y - \frac{(a_i\kappa\theta_i - Q_{Ri}L)(3a_i - y)y^2}{2a_i^3}, \\ W_{Li}(y) &= \kappa\theta_i y - \frac{(a_i\kappa\theta_i - Q_{Li}L)(3a_i - y)y^2}{2a_i^3}, \end{aligned} \quad (5.11)$$

where $i = \{1, 2, 3\}$ with $a_1 = a_3 = a_e$ and $a_2 = a_m$. Note that in the pure symmetric buckling case (Mode 1): $|W_{R1}(y)| = |W_{R3}(y)|$ and $|W_{L1}(y)| = |W_{L3}(y)|$, while $W_{R2}(y) = W_{L2}(y) = 0$ because the crossarms at the middle section do not bend. Similarly in the pure anti-symmetric buckling case (Mode 2): $|W_{R1}(y)| = |W_{L3}(y)|$, $|Q_{L1}| = |Q_{R3}|$ and $|W_{R2}(y)| = |W_{L2}(y)|$. However, since the general column shape function in this study is capable of describing an asymmetric buckling shape, which is a key nonlinear deformation profile describing interactive buckling, the shape functions of the three sets of crossarms could all in fact be different to each other; this is of course allowable within the current model framework. Finally, it is worth noting that the calibration factor κ , introduced in Equation (5.9), is purely an instrument to correct the bending energy of the crossarms due to the neglect of the axial force in the crossarms from the stays.

5.2.4 Strain energy stored within the stays

Figure 5.3 shows the generalized deformed shape of the column. The strains in each stay can be determined, which in turn allows the evaluation of their stored axial strain energy. Before applying any prestressing force and external load, the original lengths of stays 1, 4,

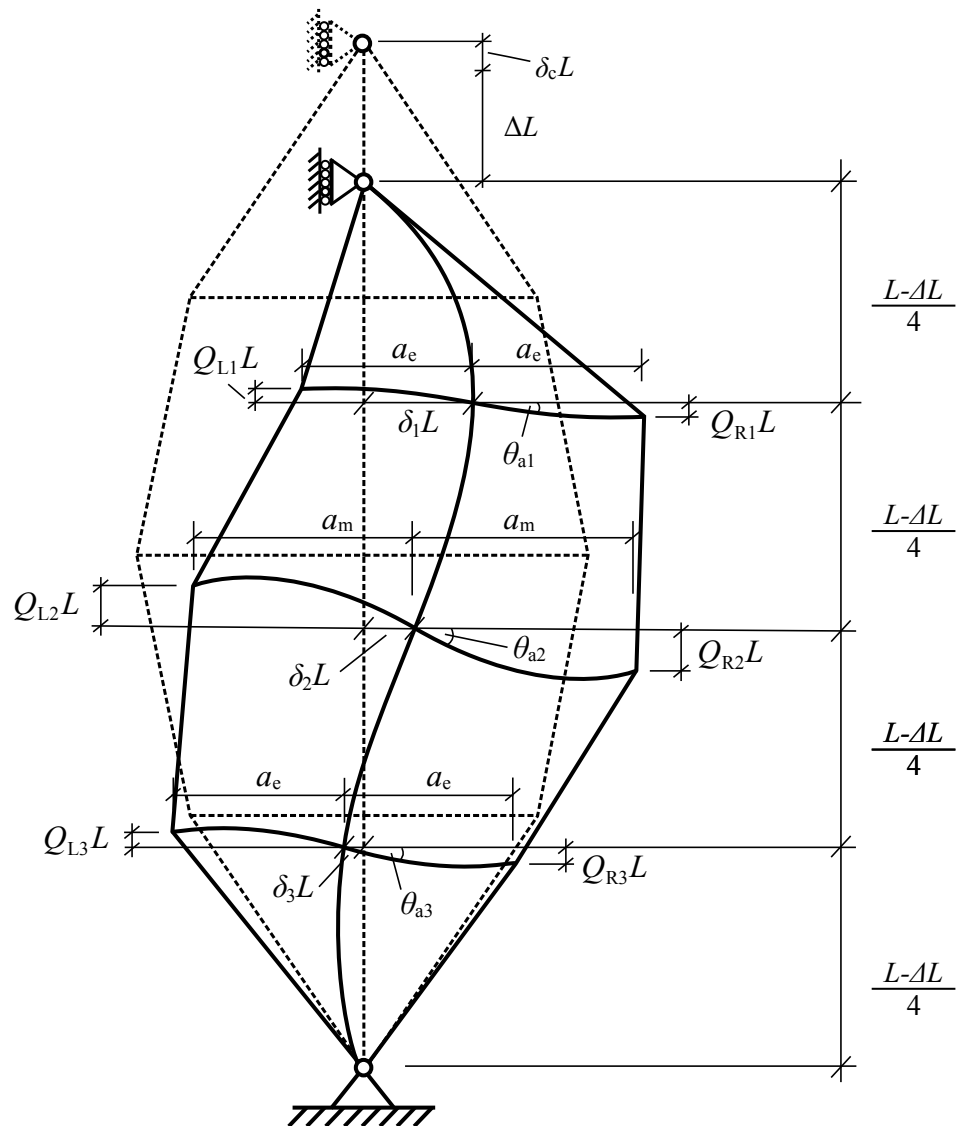


Figure 5.3: The generalized deflected shape of the prestressed stayed column with the defined variables.

5 and 8 are:

$$l_1 = l_4 = l_5 = l_8 = L\sqrt{\alpha_e^2 + 1/16}, \quad (5.12)$$

while the original lengths of stays 2, 3, 6 and 7 are:

$$l_2 = l_3 = l_6 = l_7 = L\sqrt{(\alpha_m - \alpha_e)^2 + 1/16}, \quad (5.13)$$

with the stay numbering system being defined in Figure 5.1(a). As the external load P is introduced, the column would compress purely and this is denoted by the generalized coordinate δ_c . Since the column is assumed to be very slender, δ_c is assumed to be sufficiently small to have no significant effect on the stay lengths, but it does facilitate the definition of the pre-buckling equilibrium path. When the main column begins to deflect laterally by $W(x)$, however, the stays would be affected significantly and an additional variable Δ is introduced that is dependent on the deflected shape and is therefore purely a function of Q_i . Hence, the expressions for the new lengths of each stay l'_i are:

$$\begin{aligned} l'_1 &= L\sqrt{(\Lambda + Q_{R1})^2 + (\alpha_e + \delta_1)^2}, \\ l'_2 &= L\sqrt{(\Lambda - Q_{R1} + Q_{R2})^2 + [(\alpha_m + \delta_2) - (\alpha_e + \delta_1)]^2}, \\ l'_3 &= L\sqrt{(\Lambda - Q_{R2} + Q_{R3})^2 + [(\alpha_m + \delta_2) - (\alpha_e + \delta_3)]^2}, \\ l'_4 &= L\sqrt{(\Lambda - Q_{R3})^2 + (\alpha_e + \delta_3)^2}, \\ l'_5 &= L\sqrt{(\Lambda + Q_{L3})^2 + (\alpha_e - \delta_3)^2}, \\ l'_6 &= L\sqrt{(\Lambda - Q_{L3} + Q_{L2})^2 + [(\alpha_m - \delta_2) - (\alpha_e - \delta_3)]^2}, \\ l'_7 &= L\sqrt{(\Lambda - Q_{L2} + Q_{L1})^2 + [(\alpha_m - \delta_2) - (\alpha_e - \delta_1)]^2}, \\ l'_8 &= L\sqrt{(\Lambda - Q_{L1})^2 + (\alpha_e - \delta_1)^2}, \end{aligned} \quad (5.14)$$

where $\Lambda = (1 - \Delta)/4$. Apart from the strains generated by changing the stay lengths, there is also a pre-existing strain introduced by the initial prestressing force that can be

expressed thus:

$$\epsilon_T = \frac{T}{E_s A_s}, \quad (5.15)$$

where T is the initial prestressing force applied within the stay. Therefore, the total strain in each stay can be expressed as:

$$\epsilon_i = l'_i/l_i + \epsilon_T - 1, \quad (5.16)$$

where i represents the stay number, as shown in Figure 5.1(a), and ranges from 1 to 8 inclusive. Of course the expression implies that tensile strain is assumed to be positive. The strain energy of a single stay can then be expressed thus:

$$U_{si} = \frac{1}{2} E_s A_s l_i \epsilon_i^2, \quad (5.17)$$

so long as the strain ϵ_i remains positive. It can be observed that when the column begins to deflect laterally, the strain energy within the stays on the convex side would increase as the magnitude of the bending curvature in the main column element becomes larger, while the strain energy within the stays on the concave side would decrease commensurately. Eventually, one or more stays on the concave side of the column go slack when the column end displacement reaches certain values. Since the stays cannot resist compression or bending it implies that their components of the strain energy should effectively vanish when the stay strains decrease to zero and below. To model this discontinuous behaviour in terms of the strain energy, a series of switches S_i is introduced into the energy formulation, hence:

$$U_s = \frac{1}{2} E_s A_s \sum_{i=1}^8 S_i l_i \epsilon_i^2, \quad (5.18)$$

where:

$$S_i = \begin{cases} 1 & \epsilon_i \geq 0, \\ 0 & \epsilon_i < 0. \end{cases} \quad (5.19)$$

Note that during the interactive buckling process, the stays on the concave side may become slack at various different times owing to the broken symmetry introduced as the symmetric Mode 1 and the anti-symmetric Mode 2 couple nonlinearly.

5.2.5 Strain energy from bending

Strain energy from bending exists in the column and the crossarm elements. With the introduction of the respective shape functions, the bending energy of the main column U_{cb} is obtained by the expression:

$$U_{cb} = \frac{1}{2} E_c I_c \int_0^L W'^2 dx, \quad (5.20)$$

where I_c is the second moment of area of the main column element. The bending energy stored in the three sets of crossarms on the right-hand side and the left-hand side of the main column are obtained from the respective expressions:

$$U_{Ri} = \frac{1}{2} E_a I_a \int_0^{a_i} \ddot{W}_{Ri}^2 dy, \quad U_{Li} = \frac{1}{2} E_a I_a \int_0^{a_i} \ddot{W}_{Li}^2 dy, \quad (5.21)$$

where I_a is the second moment of area of each crossarm element, which are all assumed to have the same value currently. In these two expressions $i = \{1, 2, 3\}$ with $a_1 = a_3 = a_e$ and $a_2 = a_m$, as before.

5.2.6 Strain energy from axial compression

As shown in Figure 5.3, the main column is assumed to have a purely compressive displacement $\delta_c L$ and due to the high slenderness of the main column, it is assumed to be independent of $W(x)$. Hence the contribution to the strain energy U_{ca} is given by the

expression:

$$U_{ca} = \frac{1}{2} E_c A_c \int_0^L \delta_c^2 dx. \quad (5.22)$$

The crossarms, in turn, are assumed to be inextensional and so the model is only in fact valid for the prestressed column system with an initial prestressing force in Zone 3 (see Figure 2.5), i.e. $T \geq T_{opt}$, where all stays are in tension at the point of initial instability. The purpose for these assumptions is to simplify a highly complex system to one that is feasible to solve numerically. However, previous work has demonstrated conclusively that the system strength is maximized where $T > T_{opt}$, as shown in Figure 2.7. For lower prestressing levels, i.e. where $T < T_{opt}$, the column has been deemed to be suboptimal, either in terms of the load carrying capacity if T were very small, or in terms of the post-buckling stability if T were less than, but closer to, T_{opt} , where the response was generally found to be more sensitive to imperfections without having the maximum increase in the critical buckling load (Saito & Wadee, 2008; Saito & Wadee, 2009a; Wadee *et al.*, 2013b).

5.2.7 Total potential energy

The total potential energy of the system comprises the strain energy stored minus the work done by the external load P . The strain energy comprises the axial energy of the main column, the bending energies of the main column and the crossarms alongside the strain energy stored from stretching the stays. Therefore the total potential energy V is obtained thus:

$$V = U_{ca} + U_{cb} + U_s + \sum_{i=1}^3 (U_{Ri} + U_{Li}) - P\mathcal{E}, \quad (5.23)$$

where $\mathcal{E}$ is the column end displacement, thus:

$$\mathcal{E} = L (\delta_c + \Delta) \quad (5.24)$$

and ΔL is given by the well-known expression for the inextensional end-shortening of a strut (Thompson & Hunt, 1973), thus:

$$\Delta L = \frac{1}{2} \int_0^L W'^2 dx. \quad (5.25)$$

Equilibrium states of the system are found by invoking the stationary potential energy criterion, hence:

$$\frac{\partial V}{\partial \delta_c} = 0, \quad \frac{\partial V}{\partial Q_j} = 0, \quad \frac{\partial V}{\partial Q_{Rk}} = 0, \quad \frac{\partial V}{\partial Q_{Lk}} = 0, \quad (5.26)$$

where the subscript j ranges from 1 to $2n$ (inclusive) and the subscript k ranges from 1 to 3 (inclusive). The first equation gives the pre-buckling path:

$$\delta_c = \frac{P}{E_c A_c} \quad (5.27)$$

and the effective elimination of the passive coordinate δ_c . The remaining expressions lead to a system of $2n+6$ simultaneous nonlinear algebraic equations that are formulated within the symbolic computation package Maple (2015) and are subsequently solved numerically using the continuation and bifurcation software AUTO-07P (Doedel & Oldeman, 2011), which was introduced in Chapter 4.

5.3 Finite element modelling

The finite element (FE) model, as introduced in Chapter 3, is used in the current chapter for two purposes, firstly to calibrate the crossarm shape functions used in the continuous analytical model for linear buckling analysis and, secondly, to validate the subsequent nonlinear post-buckling results from the continuous analytical model. The buckling loads are used to evaluate the theoretical optimal prestressing force T_{opt} , as described in §5.2.1; they are also used to calibrate the parameter κ that was introduced in Equation (5.9). Comparisons are then made with the nonlinear analytical model for a variety of different

cases for validation purposes.

5.4 Results and discussion

To conduct the parametric study, the material and geometric properties that are fixed were presented in Table 3.3. The crossarm length ratio $\gamma = a_e/a_m = 0.8$ and the prestressing force in the stays $T = T_{\text{opt}}$ are also fixed for the most part, but are varied at specific points in the investigation. Two parameters, the stay diameter ϕ_s and the middle crossarm length a_m , are varied extensively in the current study. While investigating the effect of varying ϕ_s , the ratio $2\alpha_m$ is fixed to 0.2; whereas while $2\alpha_m$ is varied ϕ_s is fixed to 3 mm.

5.4.1 Buckling loads

In the current study, eight generalized coordinates are introduced to describe the main column shape to meet both the requirements of computational efficiency and model accuracy (from Equation (5.5): $n = 4$); the shape therefore comprises four symmetric alongside four anti-symmetric component waves. The model has therefore a total of 14 degrees of freedom plus the pure compressive displacement component δ_c .

In previous work (Saito & Wadee, 2008), the analytical prediction of the anti-symmetric buckling load was relatively poor as compared to the corresponding symmetric buckling load. Hence the parameter κ , which was introduced in Equation (5.9), is parametrized and compared numerically against the FE model formulated within ABAQUS. For the current geometry, it was determined that a good prediction was found for $\kappa = 0.55$, where the analytical and FE buckling loads for Mode 2 first coincide with Mode 1 at the same values of ϕ_s and $2\alpha_m$. Hence this value is used in all the subsequent analytical results presented. However, if the size of the main column cross-section were to be changed relative to those of the crossarms, it is likely that κ would require re-calibration.

Figure 5.4 shows the linear buckling loads with both the stay diameter ϕ_s and the middle

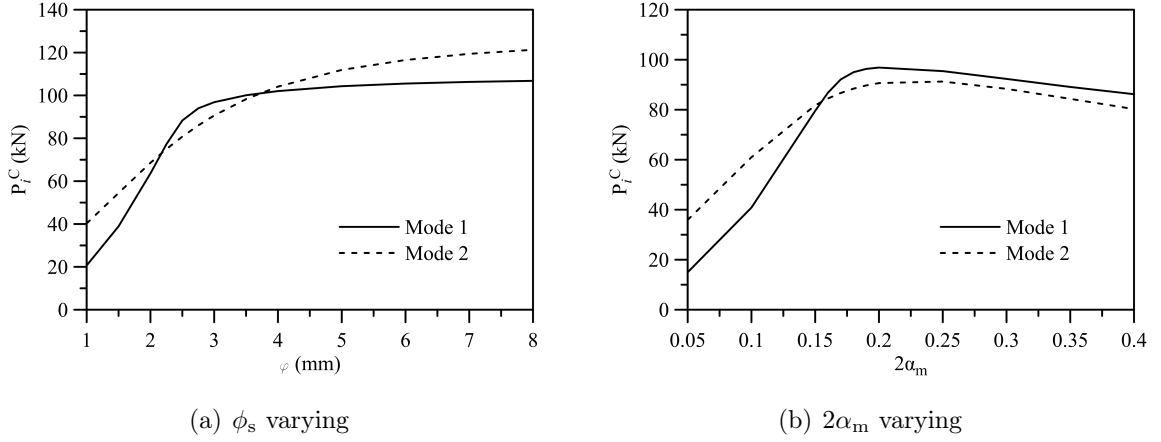


Figure 5.4: Buckling loads P_i^C for Mode i obtained from the analytical model with (a) the stay diameter ϕ_s and (b) the middle crossarm length $2\alpha_m$ varying.

crossarm length $2\alpha_m$ varying. While ϕ_s varies, Mode 1 is critical when ϕ_s is small. Along with the increase of the stay diameter ϕ_s , the buckling loads of Modes 1 and 2 become closer and intersect at approximately $\phi_s = 2.5$ mm. Mode 2 then becomes critical until ϕ_s reaches 3.8 mm, where Mode 1 becomes critical once again. Similarly, while $2\alpha_m$ varies Mode 1 governs initially until $2\alpha_m$ is approximately 0.15; subsequently Mode 2 is critical for higher values of $2\alpha_m$.

Figure 5.5 shows the comparisons of the buckling loads between the analytical and FE models with both ϕ_s and $2\alpha_m$ varying. The comparisons show excellent agreement especially for Mode 1, owing to the fact that the middle crossarms during Mode 1 buckling do not bend. The results justify the numerical selection of κ . It is also noteworthy that the stayed system increases the critical buckling load by up to a factor of approximately 15 for the considered configuration. However, given that the normalized slenderness $\bar{\lambda}$ of the main column is 5.94, the critical buckling load of the prestressed stayed system only reaches up to approximately 50% of the squash load P_y of the main column element, which in the current case is 225 kN. Hence the assumption of purely elastic behaviour is justified as in previous work (Serra *et al.* , 2015; Osofero *et al.* , 2012; Osofero *et al.* , 2013; De Araujo *et al.* , 2008), and the investigation of the effect of material yielding is left for future work.

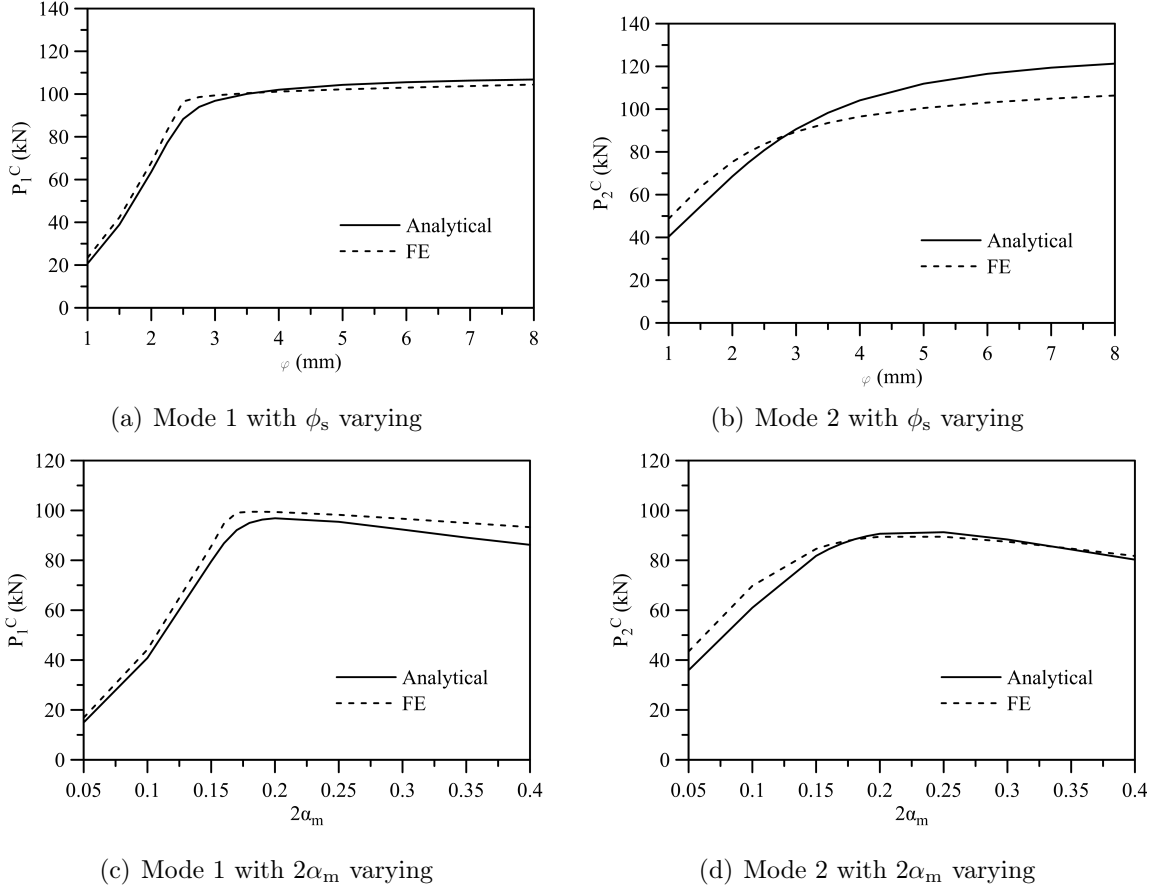


Figure 5.5: Comparisons of the obtained buckling loads evaluated using the analytical and FE models for Mode 1 (P_1^C) and Mode 2 (P_2^C) with the stay diameter ϕ_s and middle crossarm length $2\alpha_m$ varying. Note that the calibration factor $\kappa = 0.55$ throughout.

5.4.2 Nonlinear buckling and mode interaction

The principal focus of the current study concerns the post-buckling behaviour. Apart from buckling with purely Mode 1 or Mode 2 being triggered, interactive buckling is also observed in the equilibrium path especially when the buckling loads of Modes 1 and 2 are very close. Figure 5.6 shows the equilibrium paths of the applied axial load P versus the column end displacement $\mathcal{E}$ with $\phi_s = 3$ mm while $2\alpha_m$ is varied. When $2\alpha_m$ is relatively small, such as when $2\alpha_m = 0.1$, Mode 1 governs throughout the post-buckling stage. As $2\alpha_m$ is increased, the paths of Modes 1 and 2 become closer and both ultimate load capacities increase. When $2\alpha_m = 0.175$, where Mode 2 is marginally critical and the ultimate load capacities of both pure modes are very close, interactive buckling can be

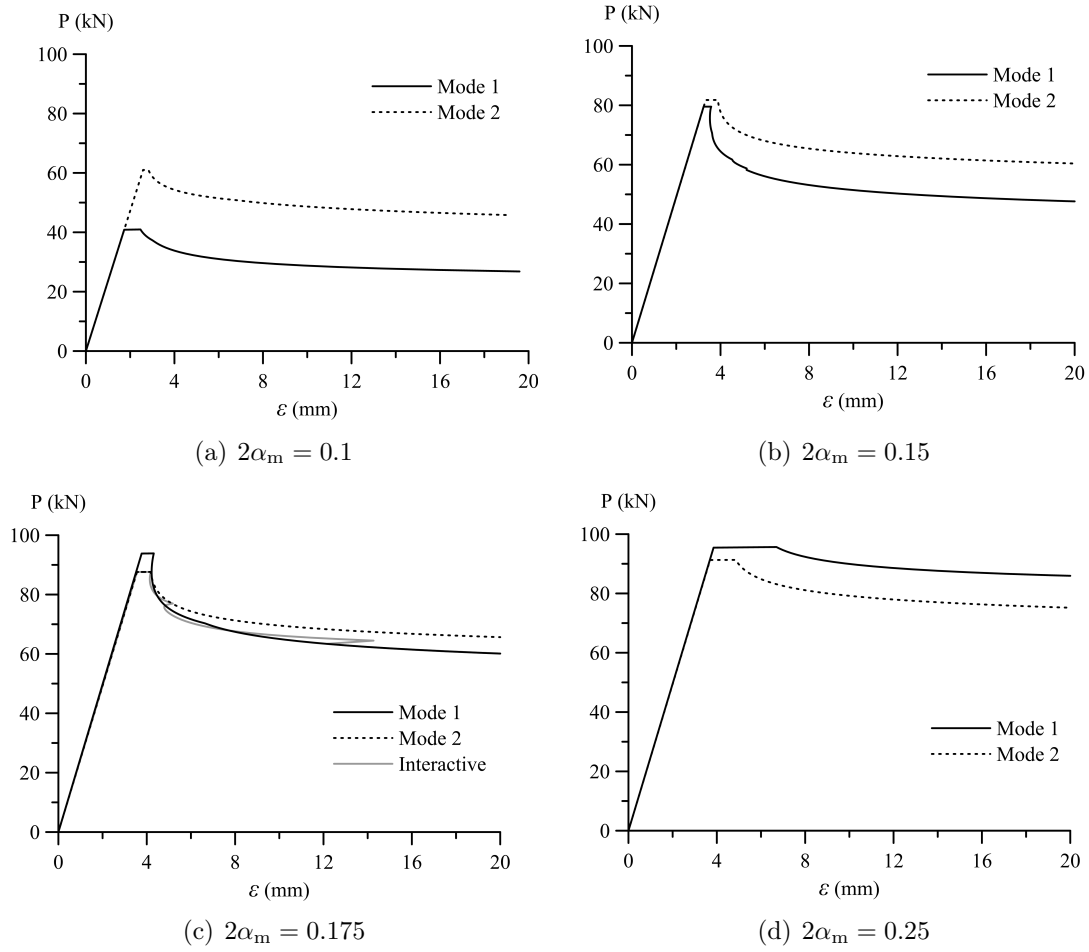


Figure 5.6: Equilibrium paths of the applied load P versus the column end displacement ε with $2\alpha_m$ varying while $\phi_s = 3$ mm and $\gamma = 0.8$. All results are obtained from the analytical model.

observed, see Figure 5.6(c), which bifurcates from Mode 2, then becomes dominant before the equilibrium path intersects the path of pure Mode 1. As $2\alpha_m$ is increased further, Mode 2 still governs the critical buckling but the interactive path leading eventually to Mode 1 is not observed.

Similarly, Figure 5.7 shows the equilibrium paths of the axial load P versus the column end displacement ε with $2\alpha_m = 0.2$ while ϕ_s is varied. When the stay diameter ϕ_s is small, Mode 1 governs throughout the post-buckling stage. As ϕ_s is increased, the ultimate load capacities of Modes 1 and 2 become closer and eventually Mode 2 begins to govern. Again, this time when $\phi_s = 2.5$ mm, interactive buckling is observed; it dominates the behaviour

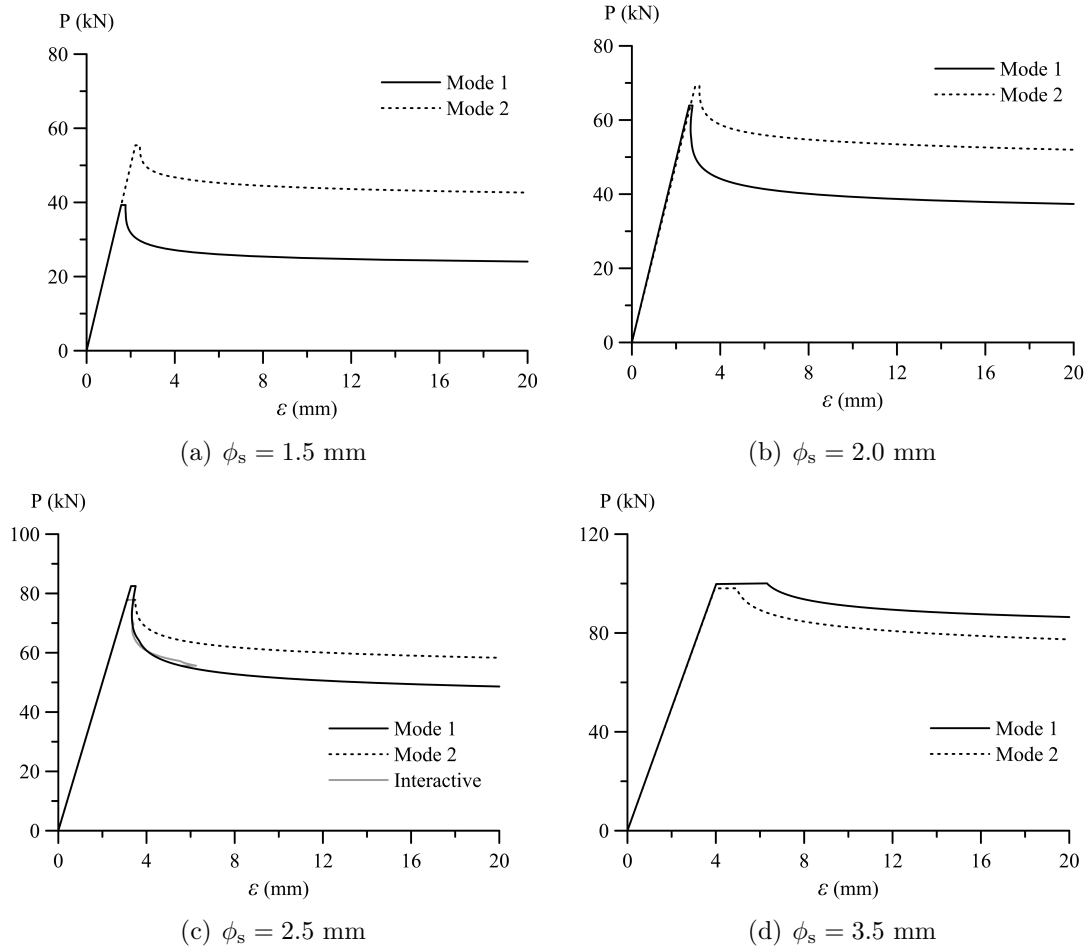


Figure 5.7: Equilibrium paths of the applied load P versus the column end displacement ε with ϕ_s varying while $2\alpha_m = 0.2$ and $\gamma = 0.8$. All results are obtained from the analytical model.

for a while before the path eventually returns to Mode 1. Cases with $2\alpha_m$ and ϕ_s varying imply that when the critical loads of Modes 1 and 2 are very close and Mode 2 is critical, the actual post-buckling behaviour is governed by mode interaction. This observation concurs precisely with previous FE studies for the single-bay (Saito & Wadee, 2009b) and the triple-bay cases (Yu & Wadee, 2015).

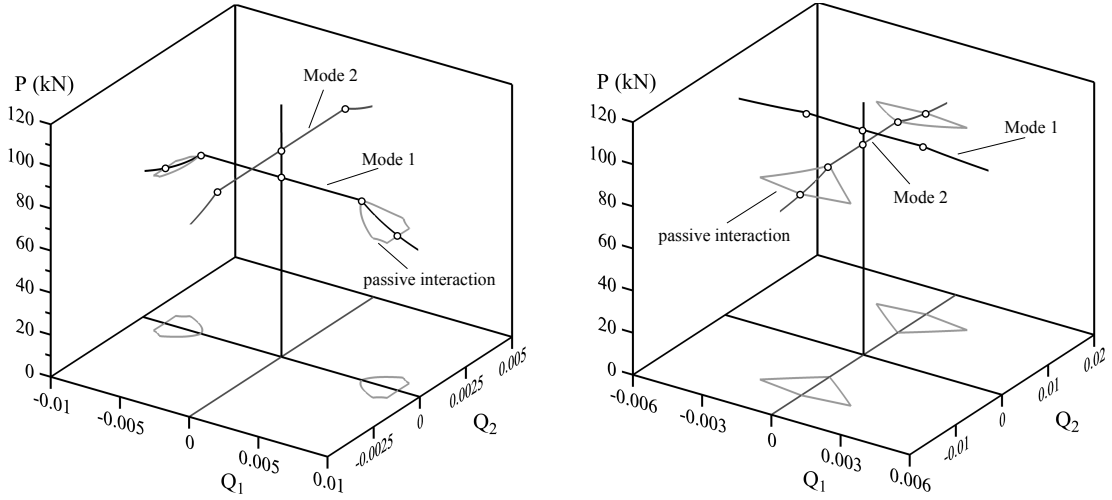
Passive interaction and mode jumping

Figures 5.6 and 5.7 demonstrate that interactive buckling is observed when the buckling loads of Modes 1 and 2 are close and is considered to be the most dangerous scenario.

The process for how the two modes interact is now discussed. Initially, when ϕ_s or $2\alpha_m$ is small, the ultimate load capacity of Mode 2 is significantly larger than that of Mode 1 and only pure modes are observed during post-buckling. Along with the increase of ϕ_s or $2\alpha_m$, when Mode 1 is still critical but the buckling loads of the two pure modes are close, a pair of passive interaction loops can be observed orbiting the pure Mode 1 branch, as shown in Figure 5.8(a).

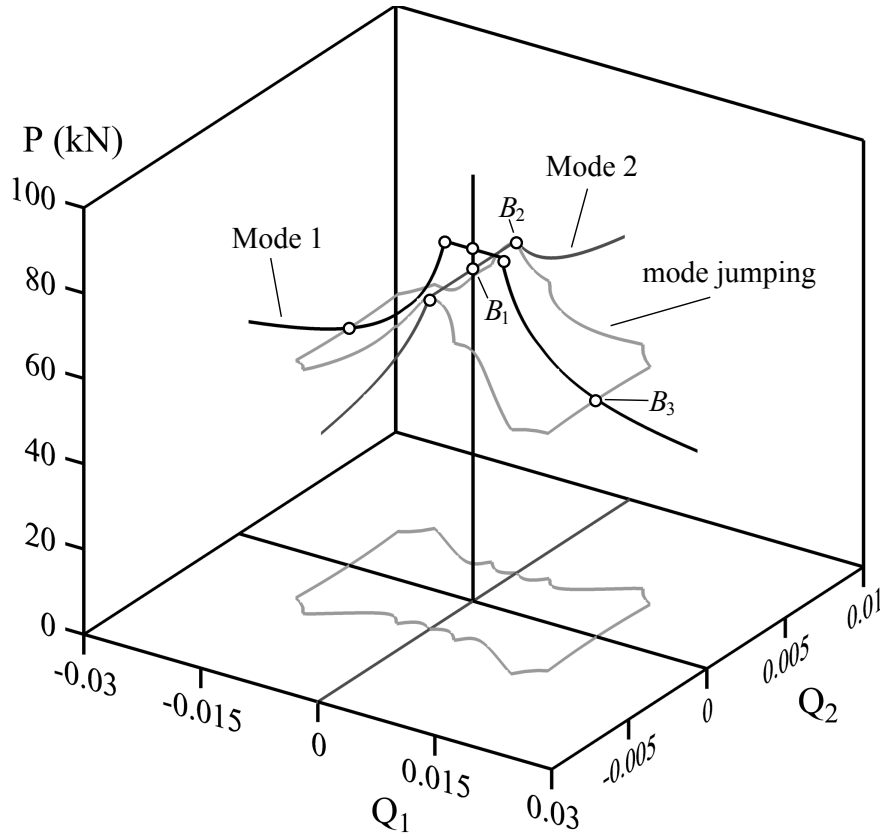
At the instance when the buckling load of Mode 1 exceeds that of Mode 2, the passive interaction loop orbiting the pure Mode 1 post-buckling path is replaced with a fully interactive buckling path with a characteristic loop that connects the post-buckling paths of both pure Modes 1 and 2, see Figure 5.8(c), which was also predicted in a much more simplified model comprising rigid links and discrete springs in previous work on single-bay crossarm systems (Wadee *et al.* , 2013a) and in the simplified discrete model presented in Chapter 4 for triple-bay systems.

The process begins by triggering Mode 2 at the first bifurcation point B_1 , then coordinate Q_2 grows with $Q_1 = 0$ until the second bifurcation point B_2 beyond which interactive buckling is triggered and Q_1 grows alongside Q_2 . The path eventually returns to Mode 1 at the third bifurcation point B_3 . The complete process is also known as ‘mode jumping’ (Everall & Hunt, 2000). This interactive post-buckling response is in fact the dominant behaviour, even though Mode 2 is critical, and the load carrying capacity is seen to reduce substantially during this process. Moreover, there is a sequence of potentially explosive snap-backs in the mechanical response where there are discontinuities in slope; these are signatures of the stays slackening sequentially on the concave side of the buckling column and may be considered to produce a ‘cellular buckling’ (Hunt *et al.* , 2000; Wadee & Bai, 2014; Wadee & Farsi, 2014a) or ‘snaking’ response (Chapman & Kozyreff, 2009). Cellular buckling is strongly associated with system that have contacting destabilizing and restabilizing characteristics; currently, the destabilization is caused by stay slackening on the concave side and the restabilization is caused by the stay forces increasing on the



(a) Passive interaction orbiting Mode 1

(b) Passive interaction orbiting Mode 2



(c) Full interaction: mode jumping from Mode 2 to Mode 1

Figure 5.8: Three-dimensional plots of the equilibrium paths of P versus Q_1 and Q_2 with $2\alpha_m = 0.2$, $\gamma = 0.8$ and (a) $\phi_s = 1.5$ mm, (b) $\phi_s = 3.0$ mm and (c) $\phi_s = 2.5$ mm; (a) and (b) show passive interactions orbiting individual Modes 1 and 2 respectively, whereas (c) shows a full interaction loop connecting both Modes 1 and 2 with the mode jumping from Mode 2 to Mode 1 in a single process.

convex side.

If ϕ_s or $2\alpha_m$ are increased further and Mode 2 remains critical, but the difference in the load capacities of the two distinct modes is larger, the full mode jumping interaction between Modes 1 and 2 is no longer observed. Instead, a pair of passive interaction loops can be observed orbiting the pure Mode 2 post-buckling path, see Figure 5.8(b), in precisely the same way as they had existed in Figure 5.8(a) orbiting the pure Mode 1 post-buckling path.

Figure 5.9 shows the equilibrium paths of Q_1 versus Q_2 when $\phi_s = 3$ mm and $\gamma = 0.8$

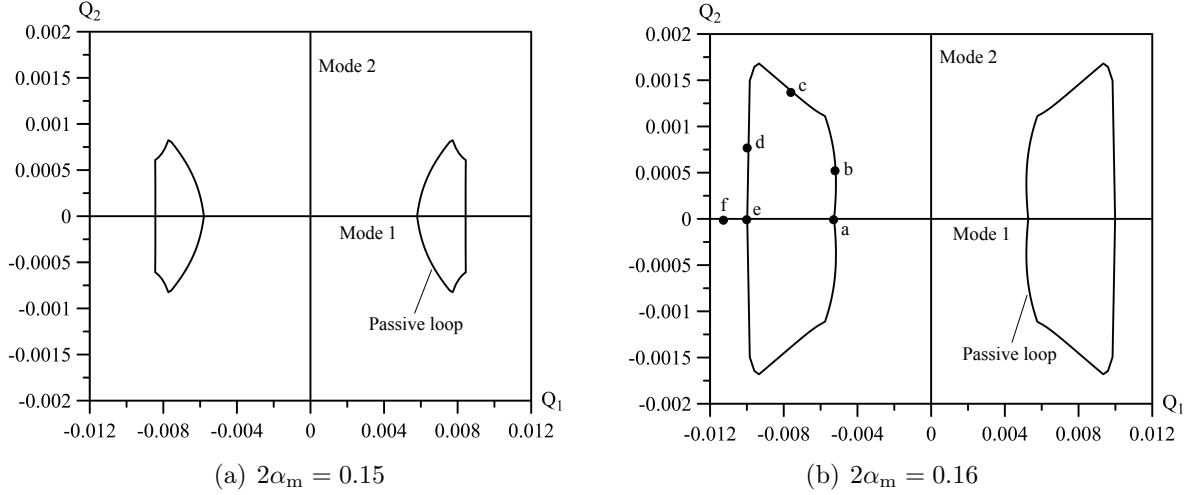


Figure 5.9: Equilibrium paths of Q_1 versus Q_2 when $\phi_s = 3$ mm and $\gamma = 0.8$ with increasing $2\alpha_m$ showing different sizes of the passive interactive loops orbiting Mode 1.

with increasing $2\alpha_m$ showing different sizes of the passive interactive loops for Mode 1. It can be seen that with the increase of $2\alpha_m$, the size of the passive interactive loop becomes larger. The strains of the eight stays are monitored at different locations on the loop, as shown in Table 5.1. The monitored points are defined in Figure 5.9(b). It can be seen that, initially, at the first bifurcation point, stays 2 and 3 go slack simultaneously and the post-buckling profile is symmetric, since $\epsilon_1 = \epsilon_4$, $\epsilon_5 = \epsilon_8$ and $\epsilon_6 = \epsilon_7$. Once the path is on the passive loop, the buckling shape of the column becomes asymmetric. Eventually when the path goes back to pure Mode 1, the post-buckling behaviour becomes symmetric once

Table 5.1: The strains of the eight stays at the selected points when $\phi_s = 3$ mm, $2\alpha_m = 0.16$ and $\gamma = 0.8$. Note that the stay goes slack when the strain is zero or negative.

Point	ϵ_1 (%)	ϵ_2 (%)	ϵ_3 (%)	ϵ_4 (%)	ϵ_5 (%)	ϵ_6 (%)	ϵ_7 (%)	ϵ_8 (%)	Mode
a	0.002	slack	slack	0.002	0.149	0.139	0.139	0.149	sym
b	slack	slack	0.002	0.005	0.143	0.142	0.150	0.160	asym
c	slack	slack	slack	0.002	0.154	0.164	0.180	0.188	asym
d	slack	slack	slack	slack	0.166	0.173	0.181	0.183	asym
e	slack	slack	slack	slack	0.174	0.177	0.177	0.174	sym
f	slack	slack	slack	slack	0.199	0.205	0.205	0.199	sym

again with two more stays being slack.

Figure 5.10 shows the equilibrium paths of Q_1 versus Q_2 when $\phi_s = 3$ mm and $\gamma = 0.8$

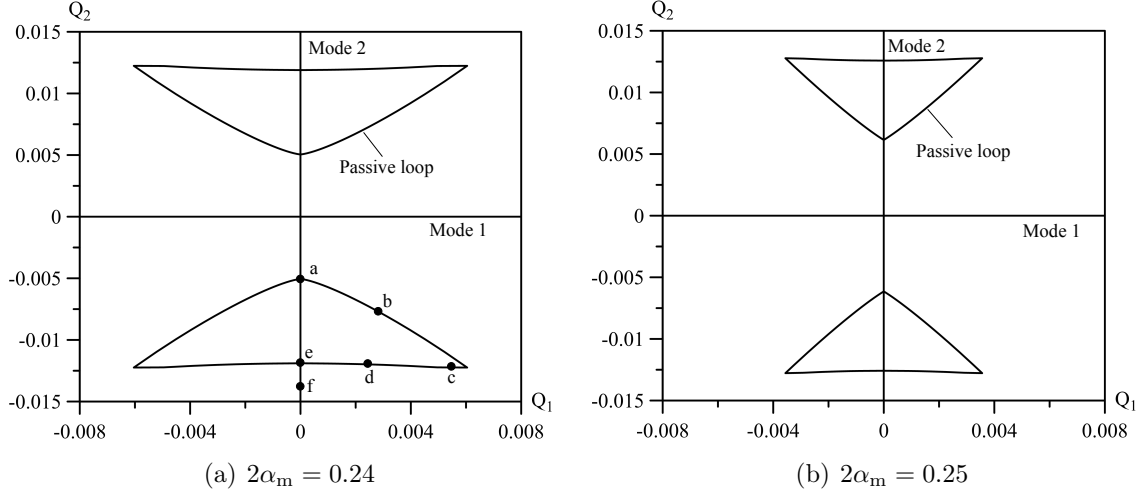


Figure 5.10: Equilibrium paths of Q_1 versus Q_2 when $\phi_s = 3$ mm and $\gamma = 0.8$ with increasing $2\alpha_m$ showing different sizes of the passive interactive loops orbiting Mode 2.

with increasing $2\alpha_m$ showing different sizes of the passive interactive loops for Mode 2. It can be seen that, unlike the passive interactive loop for Mode 1, the loop size becomes smaller along with the increase of $2\alpha_m$. The strains of the eight stays for this case are also monitored at different locations of the loop, as shown in Table 5.2. The monitored points are defined in Figure 5.10(a). It can be seen that initially, at the first bifurcation point, stays 1 and 5 go slack simultaneously and the post-buckling profile is anti-symmetric, since $\epsilon_2 = \epsilon_6$, $\epsilon_3 = \epsilon_7$ and $\epsilon_4 = \epsilon_8$. Once the path is on the passive loop, the buckling shape of the column becomes asymmetric. Eventually when the path returns to pure Mode 2,

Table 5.2: The strains of the eight stays at the selected points when $\phi_s = 3$ mm, $2\alpha_m = 0.24$ and $\gamma = 0.8$. Note that the stay goes slack when the strain is zero or negative.

Point	ϵ_1 (%)	ϵ_2 (%)	ϵ_3 (%)	ϵ_4 (%)	ϵ_5 (%)	ϵ_6 (%)	ϵ_7 (%)	ϵ_8 (%)	Mode
a	slack	0.017	0.042	0.063	slack	0.017	0.042	0.063	anti-sym
b	slack	0.016	0.048	0.065	0.008	0.030	0.071	0.125	asym
c	slack	0.015	0.054	0.074	0.004	0.035	0.092	0.172	asym
d	slack	0.016	0.061	0.088	slack	0.034	0.096	0.180	asym
e	slack	0.025	0.078	0.131	slack	0.025	0.078	0.131	anti-sym
f	slack	0.030	0.096	0.167	slack	0.030	0.096	0.167	anti-sym

the post-buckling behaviour becomes anti-symmetric once again with larger strains in the active stays compared with the initial stage.

Figure 5.11 shows the distribution of Q_1 versus Q_2 with $2\alpha_m$ varying from 0.175 to 0.2, where the value of the Mode 1 buckling load just exceeds that of Mode 2, but is still very close. It can be seen that the size of the mode jumping interactive loop becomes larger, along with the increase of $2\alpha_m$, as the practical effect of mode interaction becomes less significant since the $2\alpha_m$ values are further away from the point where the critical loads coincide, as shown in Figure 5.4(b). The loop in fact eventually breaks at a certain point, after which the fully interactive behaviour is not observed in the analytical model and only passive interactions with the paths orbiting individual pure modes are observed, as shown in Figures 5.8(a) and (b).

Table 5.3 shows the strains of the eight stays at the selected points when $\phi_s = 3$ mm,

Table 5.3: The strains of the eight stays at the selected points when $\phi_s = 3$ mm, $2\alpha_m = 0.175$ and $\gamma = 0.8$. Note that the stay goes slack when the strain is zero or negative.

Point	ϵ_1 (%)	ϵ_2 (%)	ϵ_3 (%)	ϵ_4 (%)	ϵ_5 (%)	ϵ_6 (%)	ϵ_7 (%)	ϵ_8 (%)	Mode
a	0.085	0.068	0.045	0.031	0.085	0.068	0.045	0.031	anti-sym
b	0.115	0.081	0.032	slack	0.115	0.081	0.032	slack	anti-sym
c	0.118	0.093	0.065	0.052	0.063	0.048	0.021	slack	asym
d	0.225	0.188	0.144	0.119	0.013	0.009	slack	slack	asym
e	0.281	0.246	0.187	0.148	0.001	slack	slack	slack	asym
f	0.232	0.223	0.206	0.194	slack	slack	slack	slack	asym
g	0.213	0.214	0.214	0.213	slack	slack	slack	slack	sym
h	0.238	0.242	0.242	0.238	slack	slack	slack	slack	sym

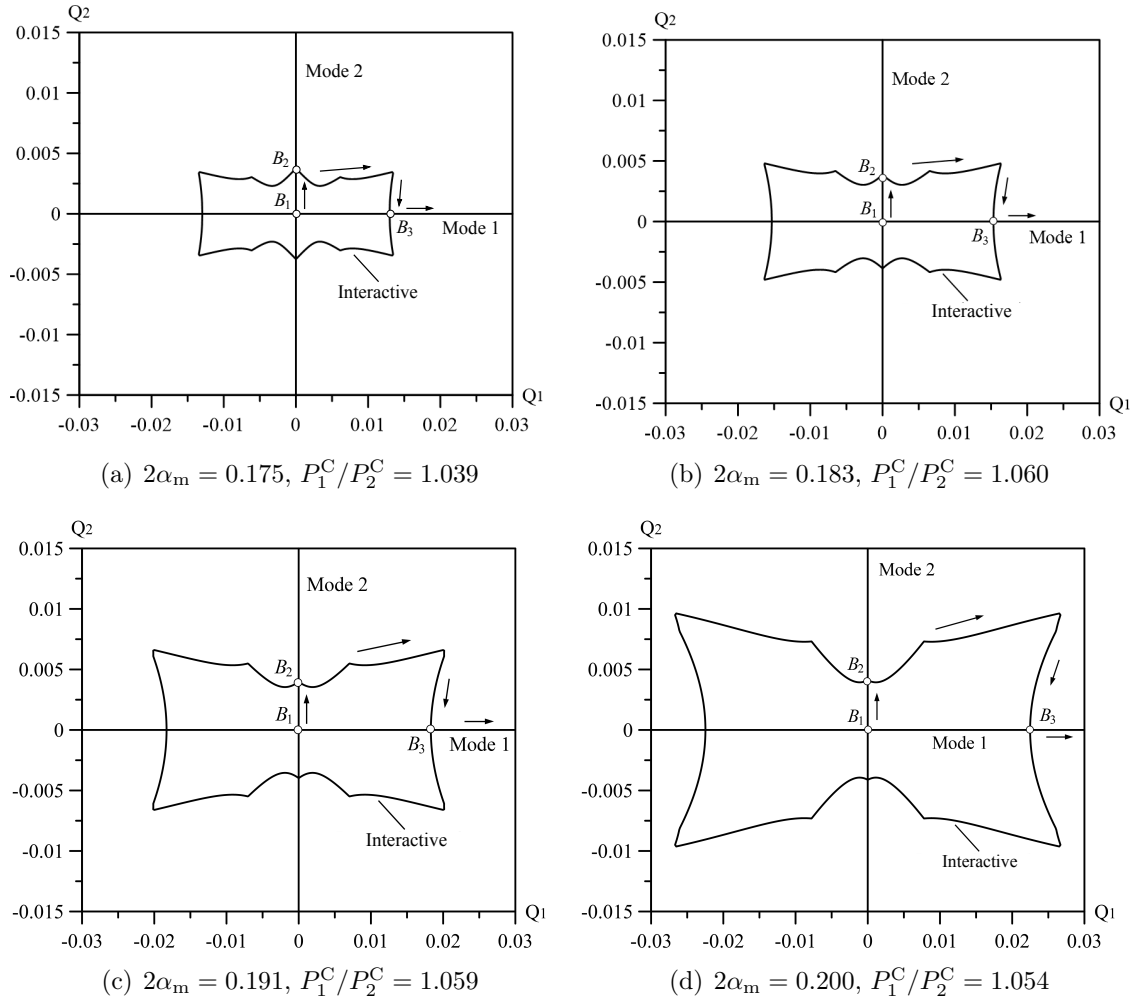


Figure 5.11: Equilibrium paths of Q_1 versus Q_2 when $\phi_s = 3$ mm and $\gamma = 0.8$ with increasing $2\alpha_m$ showing different sizes of interactive buckling loops and paths where the mode jumping process occurs.

$2\alpha_m = 0.175$ and $\gamma = 0.8$. The selected points are defined in Figure 5.12. It can be seen

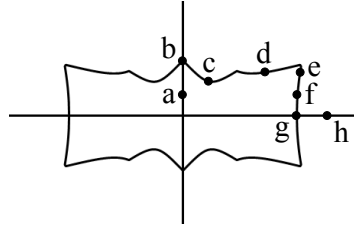


Figure 5.12: The selected points for monitoring the strains in the stays for interactive buckling that exhibits mode jumping from the anti-symmetric to the symmetric buckling.

that before the path hits the secondary bifurcation point at ‘b’, the post-buckling response is anti-symmetric and all the stays are active. At the point of the secondary bifurcation, stays 1 and 5 go slack simultaneously and the column remains in an anti-symmetric profile. As the path moves further on the interactive loop, the column shape becomes asymmetric and the stays go slack in a certain sequence. Eventually when the path moves to the pure Mode 1 path, the transformation from the anti-symmetric to the symmetric buckling is complete.

Figure 5.13 shows the sequence of behaviour that depicts the stage-by-stage transformation of the buckling shape in the mode jumping process. Initially, the system hits the first bifurcation point B_1 and then Q_2 grows towards the second bifurcation point B_2 , see Figure 5.13(a). Since $Q_1 = 0$ and all of the stays are in tension, the buckling forms as the purely anti-symmetric shape (Mode 2) at this stage. When the path reaches the bifurcation point B_2 , the strain in the first stays on the concave side (stays 4 and 8 as depicted – see Figure 5.1(a) for stay label definitions) just decreases to zero simultaneously, it becomes slack and mode interaction is first triggered, see Figure 5.13(b). However, as $Q_1 > 0$, stay 4 immediately reactivates. When the path passes the next slope discontinuity, another stay goes slack (stay 7 as depicted), as shown in Figure 5.13(c). Once the path reaches the top-right corner of the interactive loop, the third stay becomes slack (stay 6 as depicted) and from this point, the buckling shape more or less resembles Mode 1 rather than Mode 2, as shown in Figure 5.13(d). Finally, when the path eventually arrives at B_3 , the mode

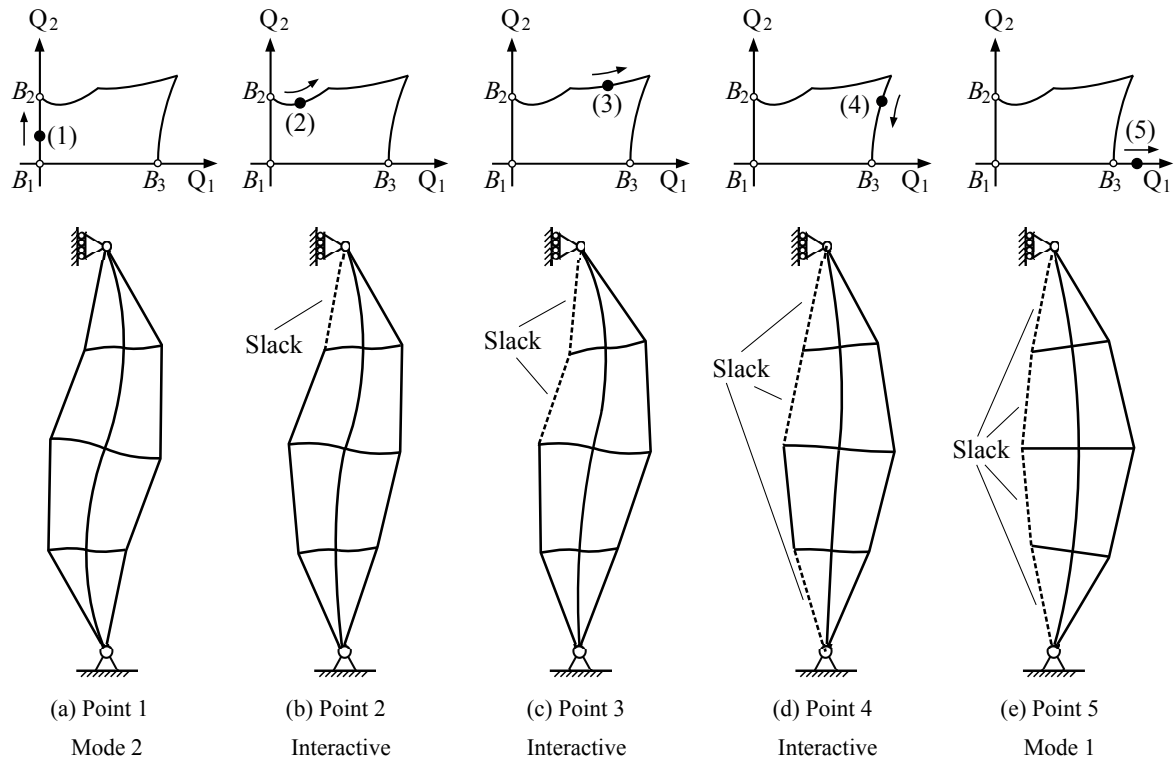


Figure 5.13: The detailed transformation of the post-buckling profile in the nonlinear mode jumping process. The solid circle shows the combination of Q_1 and Q_2 for the corresponding deformation profile shown in (a–e).

jumping process finishes and the transformation from pure Mode 2 to pure Mode 1 is complete, as shown in Figure 5.13(e) with the final stay going slack (stay 5 as depicted).

Owing to the fact that when interactive buckling leads to mode jumping the structural component has the most severely unstable response, it is of paramount practical importance to determine for which combinations of system parameters this occurs. Passive Mode 1 interaction occurs when Mode 1 is critical but the buckling loads of Modes 1 and 2 are close, see Figure 5.8(a). The size of the passive loop becomes larger along with the increase of ϕ_s or $2\alpha_m$ and the passive interaction is eventually replaced by a full interaction with mode jumping when Mode 2 just becomes critical. The mode jumping interaction loop becomes larger along with the increase of ϕ_s or $2\alpha_m$, as shown in Figure 5.11, eventually breaking where it is replaced by a passive Mode 2 interaction, see Figure 5.8(b). Unlike the passive Mode 1 interaction and the full mode jumping interaction, the passive Mode 2 interaction loop becomes smaller along with further increase of ϕ_s or $2\alpha_m$. In the current study, when the amplitude of Q_1 within the passive Mode 2 interaction loop is smaller than 0.0025, it is assumed that the contribution of the symmetric mode is very small, the passive interaction is deemed to be negligible and the buckling mode is considered to be purely Mode 2. This size of lateral midspan deflection, measured as $Q_1 L$, defined as $L/400$, is in accord with manufacturing tolerances for the design of structural steel components (BS EN10210-2, 2006).

Figure 5.14 shows the configurations where mode jumping and the passive interactions are significant with the normalized stay diameter Φ_s , where:

$$\Phi_s = \frac{\phi_s}{\sqrt{I_c/A_c}}, \quad (5.28)$$

the ratio $2\alpha_m$ and the crossarm length ratio γ varying. The denominator in the expression of Equation (5.28) is the radius of gyration of the main column. Hence Φ_s is effectively the ratio of the length scale defining the axial stiffness of the stay to the length scale defining

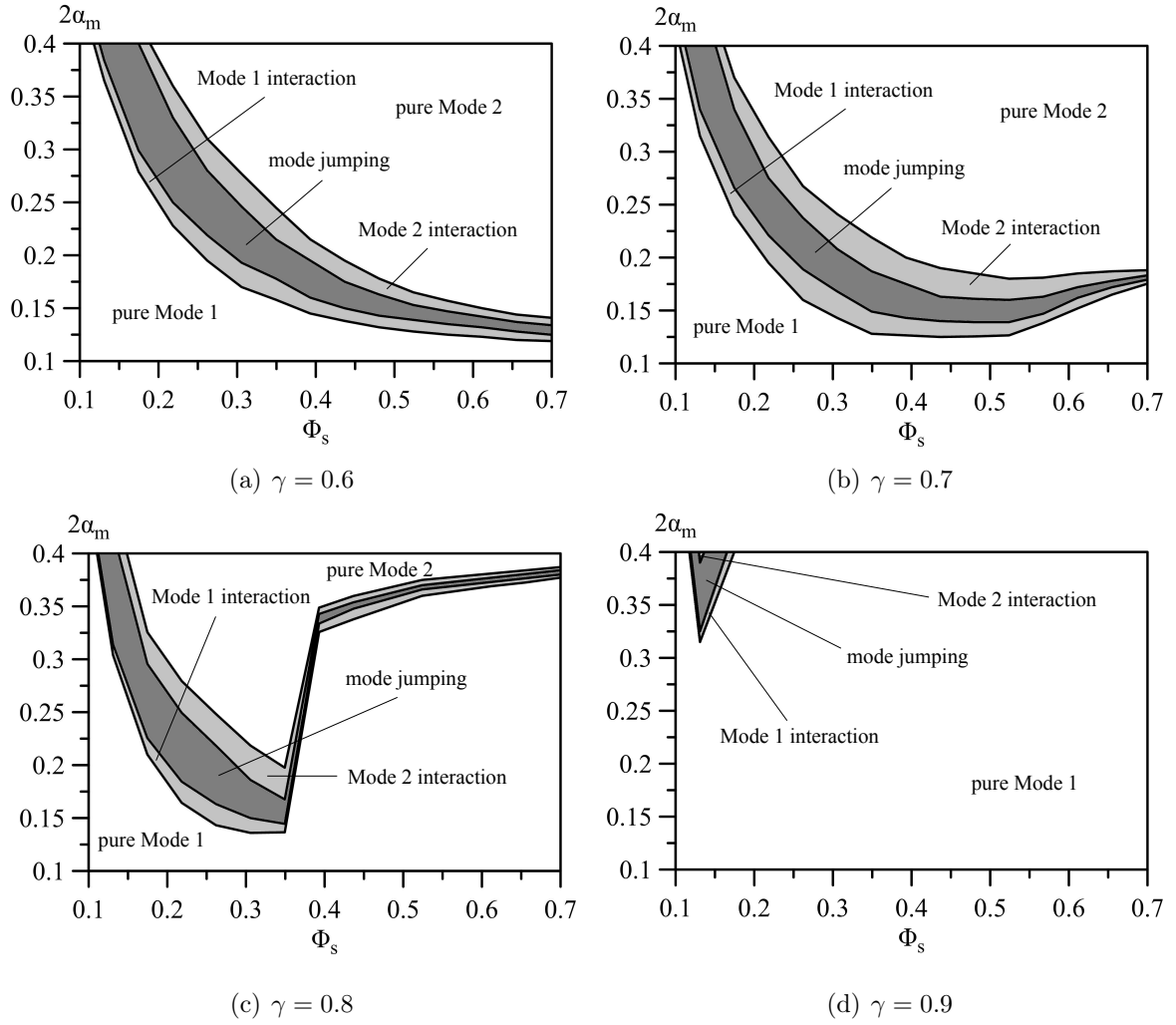


Figure 5.14: Parametric spaces of $2\alpha_m$, Φ_s and γ showing regions where mode interactions that cause mode jumping (i.e. full interaction between Modes 1 and 2 with cellular buckling) and passive interactions.

the relative magnitudes of the bending and axial stiffnesses of the main column. It can be seen that there is a wider range of configurations where interactive buckling occurs with relatively small Φ_s and large $2\alpha_m$ values. When $\gamma = 0.6$ and 0.7 , the parametric space of mode jumping and passive interaction ranges from the region with large $2\alpha_m$ and small Φ_s values to the region with small $2\alpha_m$ and large Φ_s values relatively smoothly. When $\gamma = 0.8$, which is the value used for most of the presented results in the current work, it can be seen from Figure 5.15 that the point of coincidence for buckling loads of Modes 1 and 2 changes dramatically from $2\alpha_m$ being approximately 0.15 to approximately 0.34 while increasing ϕ_s from 3 mm to 4.5 mm. Therefore a significant change in the parametric space of mode jumping and passive pure mode interaction is observed in this range. This change can be attributed to the sudden change in the position of the intersection of the linear buckling loads, as seen in Figures 5.15(b) and 5.15(d). It can also be seen that when $\gamma = 0.9$, interactive buckling only occurs at very small Φ_s and very large $2\alpha_m$ values, owing to the fact that Mode 2 is only critical for very large crossarm lengths. The reason for this is that the stay force component acting as a restraint on the main column through the midspan crossarms is relatively much lower in magnitude than for smaller values of γ presented in the graphs in Figure 5.14(a–c); this is simply related to the commensurate reduction in the angle β_m as shown in Figure 5.1(a) and may be considered to becoming similar to a double-bay column case.

5.4.3 Sensitivity to initial prestressing force

The benchmark prestressing force T_{opt} , used in all the results presented in the study hitherto, was evaluated through Equation (5.1). As mentioned before, T_{opt} is derived purely from an elastic model based on linear eigenvalue analysis, which may not be the actual optimal prestressing force for the stayed column in terms of the true load carrying capacity and post-buckling stability. Figure 5.16 shows the equilibrium paths of the applied load P versus the end displacement $\mathcal{E}$ with the prestressing force T varying while $\phi_s = 2$ mm,

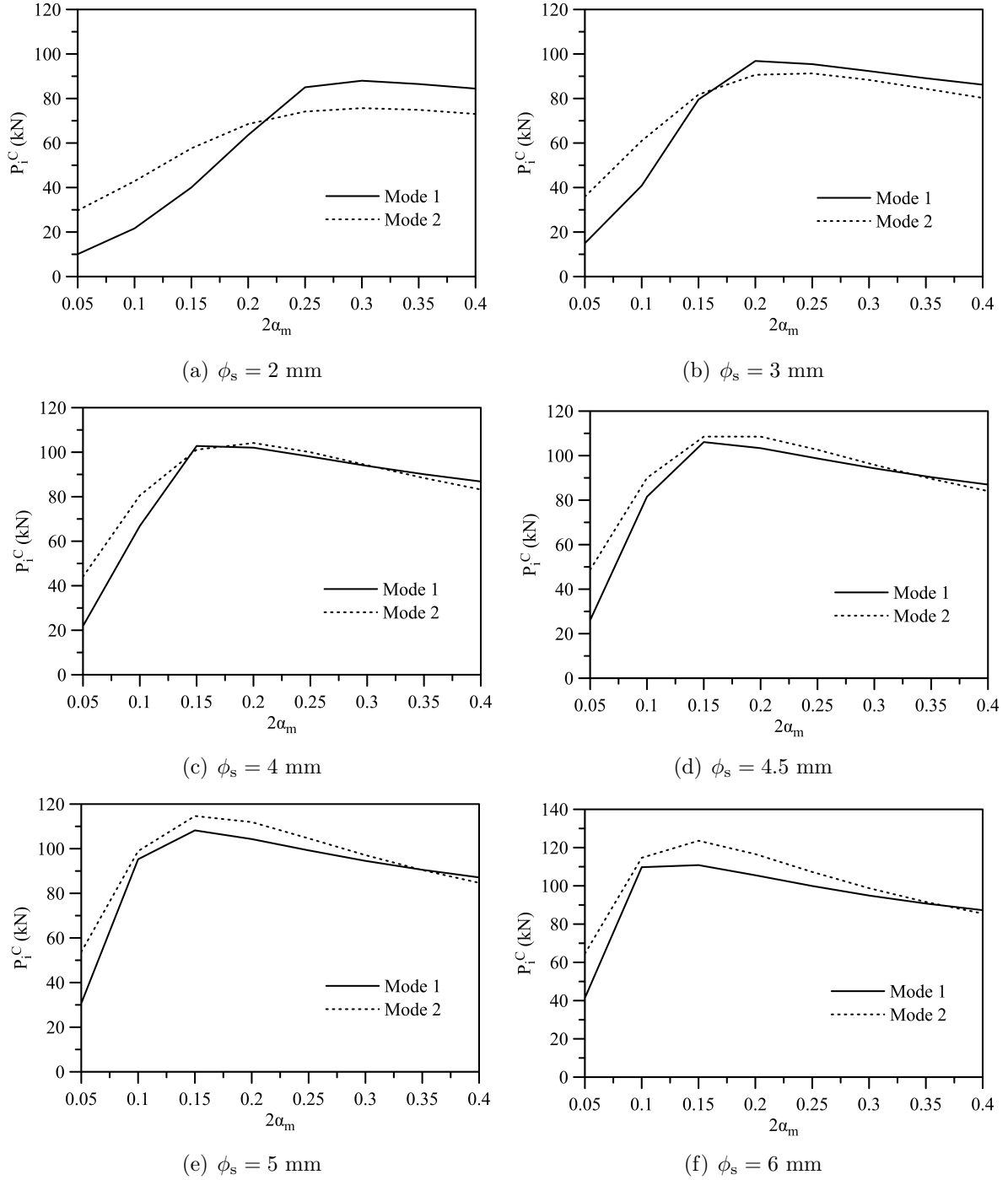


Figure 5.15: Buckling loads of Modes 1 and 2 with $2\alpha_m$ and ϕ_s varying while $\gamma = 0.8$.

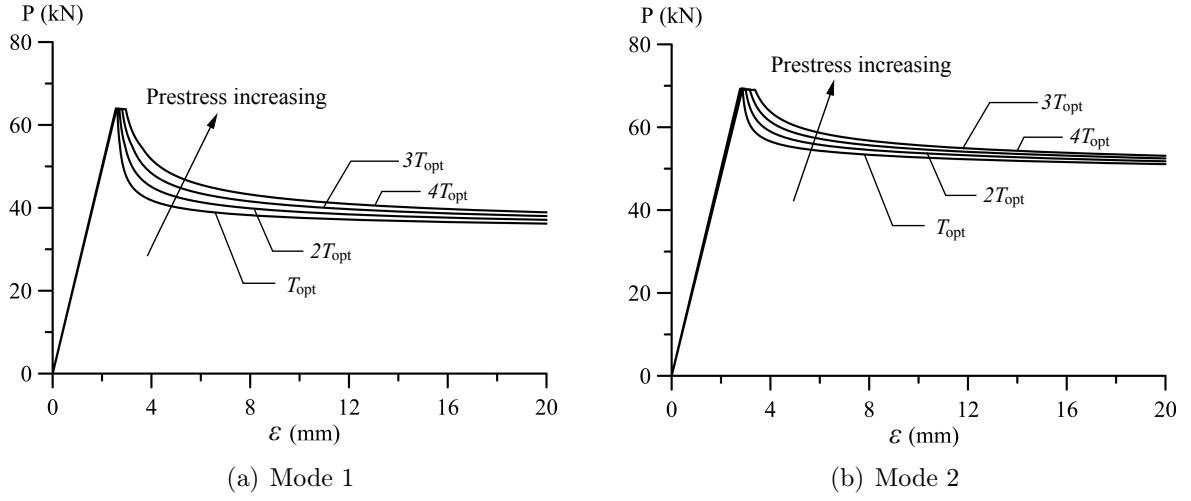


Figure 5.16: Equilibrium paths of the applied load P versus ϵ with the prestress T varying while $\phi_s = 2$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$. Note the increasing plateau size and the higher residual load in the initial and advanced post-buckling stages respectively with increasing T .

$2\alpha_m = 0.2$ and $\gamma = 0.8$. It can be seen that in both cases for Modes 1 and 2, increasing the prestressing force from T_{opt} results in a very slight decrease in the load capacity but makes the post-buckling behaviour significantly more stable. This implies that the prestressed stayed column would benefit more from the prestressing force being larger than T_{opt} . The investigation of determining the actual optimal prestressing force is presented in Chapter 6, but it is noted presently that it would be higher than T_{opt} , as found for the single-bay case (Saito & Wadee, 2010). It is worth recalling that columns with prestressing forces less than T_{opt} would have significantly less buckling load capacities, as depicted in Figure 2.5, but this is also discussed in Chapter 6.

Figure 5.17 shows the equilibrium paths of the interactive buckling case with the prestressing force T varying while $\phi_s = 2.5$ mm and $2\alpha_m = 0.2$. It can be seen that the interactive buckling loops become larger and the snap-backs become more pronounced along with the increasing of the prestressing force T . When the column is loaded axially and begins to deflect laterally, the stays on the concave side of the column need more deformation to eliminate the initial prestress and go slack since the initial strain is larger. As a result, the bifurcation points, where the stays go slack, are further away from the initial critical

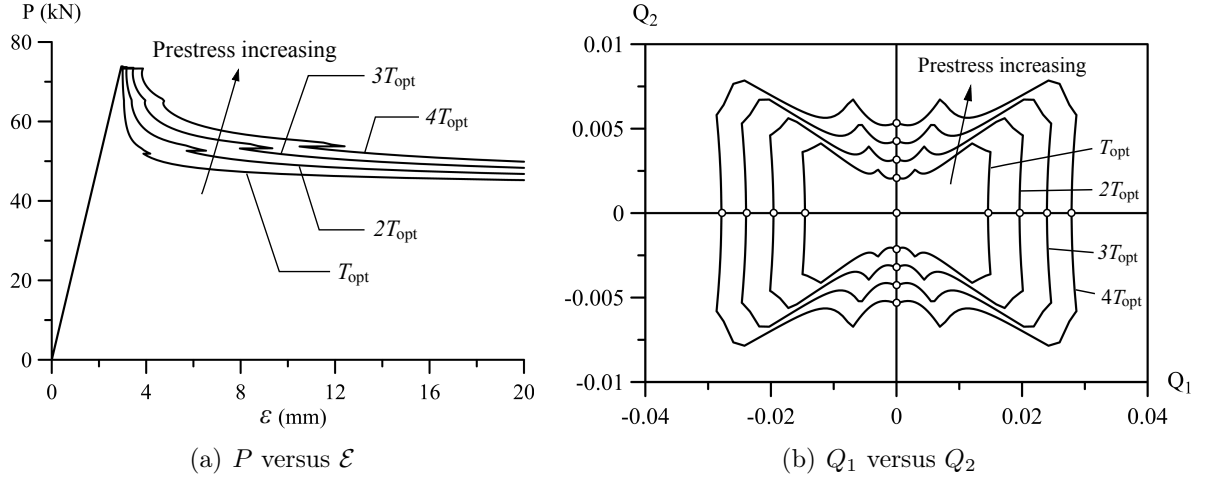


Figure 5.17: Equilibrium paths of the interactive buckling case with the prestressing force T varying while $\phi_s = 2.5$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$. Note the increasing plateau size and the higher residual load in the initial and advanced post-buckling stages respectively with increasing T in (a). Also, note the increasing size of the interaction loop in (b).

bifurcation point. The identical phenomenon can also be observed in Figure 5.16, where the distribution of the peak load plateau is extended by increasing the prestressing force from T_{opt} . Of course, there is a limit on the level of the prestressing force T , which relates directly to the intrinsic compressive capacity of the main column element, either from elastic instability or material failure – this is also investigated in more detail in Chapter 6.

5.4.4 Validation of analytical model from FE

The same structural geometries and material properties are used in the FE model to conduct both linear and nonlinear analyses. Owing to the requirement of introducing imperfections as perturbations to obtain the near-perfect post-buckling response, an amplitude of $L/10000$ is selected to ensure a fair comparison with the analytical model. A mixture of the perturbations from the first two pure modes are used to trigger interactive buckling in the FE model. Figures 5.18 and 5.19 show comparisons between the equilibrium paths of the applied load P versus the column end displacement ε for both $2\alpha_m$ and ϕ_s varying respectively. It is seen that the maximum load carrying capacity and the nonlinear

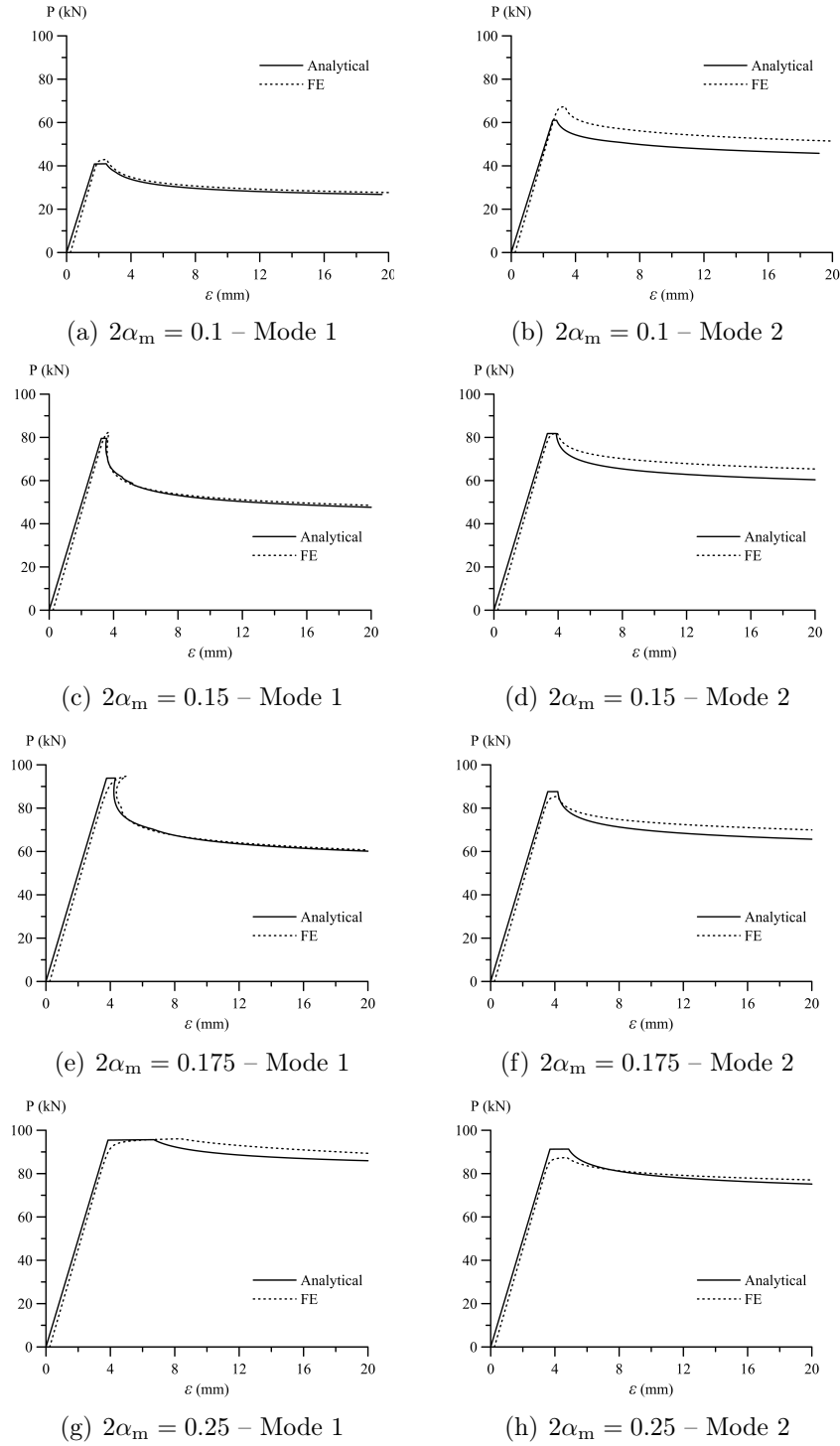


Figure 5.18: Comparisons of the equilibrium paths of the applied load P versus the column end displacement ε with $2\alpha_m$ varying while $\phi_s = 3$ mm and $\gamma = 0.8$. The graphs show the cases where (a, c, e and g) pure Mode 1 or (b, d, f and h) pure Mode 2 perturbations are introduced into the FE model.

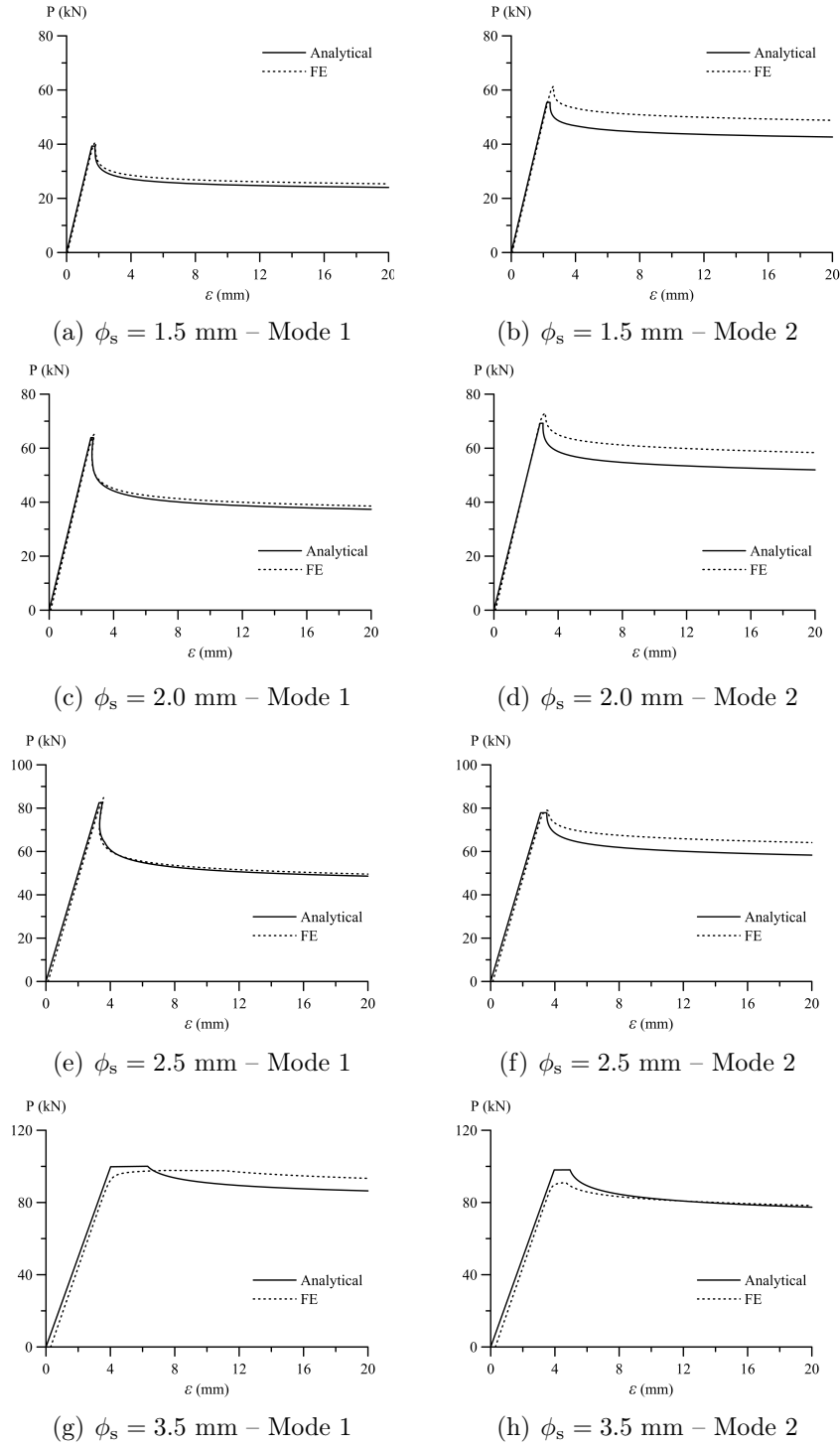


Figure 5.19: Comparisons of the equilibrium paths of the applied load P versus the column end displacement ε with ϕ_s varying while $2\alpha_m = 0.2$ and $\gamma = 0.8$. The graphs show the cases where (a, c, e and g) pure Mode 1 or (b, d, f and h) pure Mode 2 perturbations are introduced into the FE model.

post-buckling response compare excellently for all cases, especially for Mode 1 since the middle crossarms do not bend during symmetric buckling, hence there is less potential error introduced into the model from the shape functions of the crossarms.

Figure 5.20 shows the comparisons of two interactive buckling cases with mode jumping.

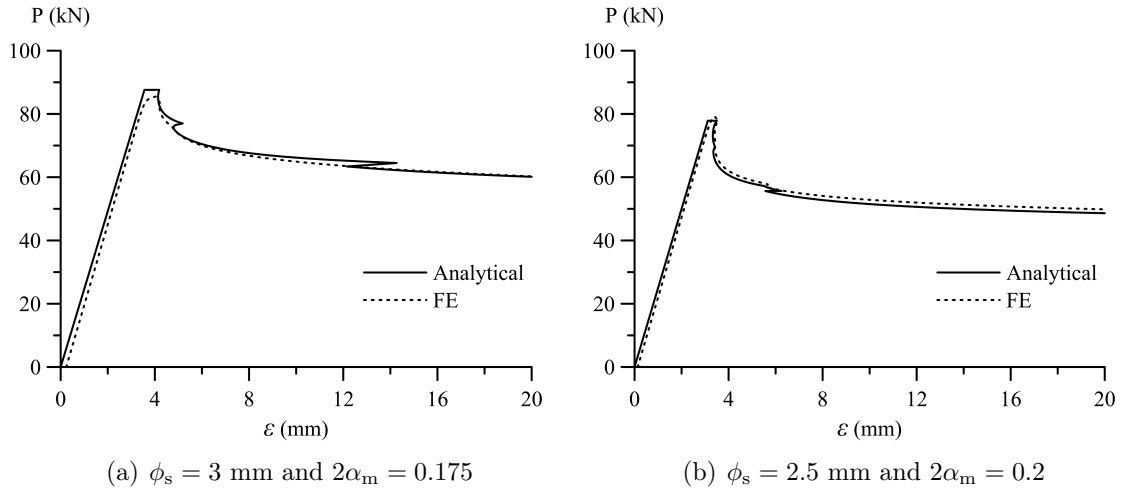


Figure 5.20: Comparisons of the equilibrium paths of the applied load P versus the column end displacement $\mathcal{E}$ for the cases exhibiting interactive buckling. In these cases, the FE model has a combination of Modes 1 and 2 for the initially perturbed shape and $\gamma = 0.8$.

The results show excellent agreement between the analytical and FE models, which implies that the current analytical approach captures the interactive buckling behaviour both qualitatively and quantitatively. Figures 5.21 and 5.22 show the comparisons of the post-buckling profiles between the analytical and the FE model at the peak load and when $\mathcal{E} = 20 \text{ mm}$ for $2\alpha_m = 0.1$ and 0.25 while $\phi_s = 3 \text{ mm}$ for pure Modes 1 and 2. The comparisons show excellent visual agreement in terms of buckling profile between the two models. Figures 5.23 and 5.24 show the corresponding comparisons of post-buckling profiles between the analytical and the FE model at the peak load and the interactive buckling region with the following cases: (1) $2\alpha_m = 0.2$ and $\phi_s = 2.5 \text{ mm}$; (2) $2\alpha_m = 0.175$ and $\phi_s = 3 \text{ mm}$. The comparisons also show excellent visual agreement in terms of the deformation profile at the peak load and when the column end displacement is increased in magnitude. This, again, demonstrates that the analytical model established currently

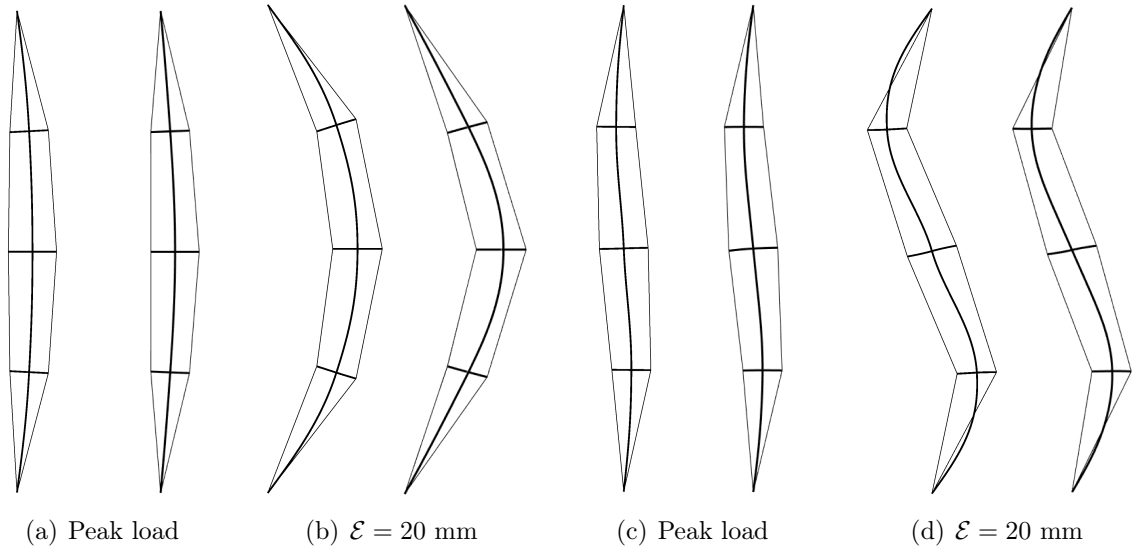


Figure 5.21: Comparisons of the post-buckling profiles between the analytical and the FE model at the peak load and when $\mathcal{E} = 20$ mm with $2\alpha_m = 0.1$, $\phi_s = 3$ mm and $\gamma = 0.8$ (left: FE; right: analytical). The amplification factor is 5 for visualization purposes; (a–b) show Mode 1; (c–d) show Mode 2.

is quite capable of predicting the nonlinear behaviour accurately both qualitatively and quantitatively.

5.5 Concluding remarks

A nonlinear analytical model formulated with potential energy principles in conjunction with the Rayleigh–Ritz method has been presented. Shape functions for each part of the structure were used to derive the total potential energy stored in the entire system. Both linear and nonlinear analyses were conducted with parametric variations. The model results clearly revealed, for the first time, the detailed process explaining how the two principal buckling modes interact when their corresponding loads are in relatively close proximity. It was also clearly identified that mode interaction in prestressed stayed columns is intimately linked to the stay slackening and crossarm bending phenomena, which were successfully modelled in the current study. Moreover, it is found that if the anti-symmetric Mode 2 is marginally critical, the actual post-buckling response is dominated by nonlinear modal

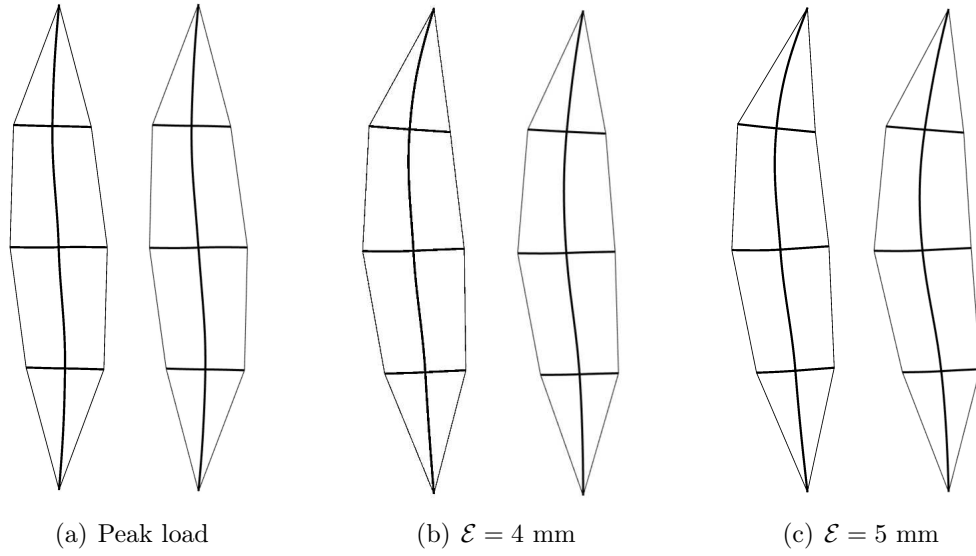


Figure 5.23: Comparisons of the post-buckling profiles between the analytical and the FE model at the peak load and the interactive buckling region where $2\alpha_m = 0.2$, $\phi_s = 2.5$ mm and $\gamma = 0.8$ (left: FE; right: analytical). The amplification factor is 5 for visualization purposes.

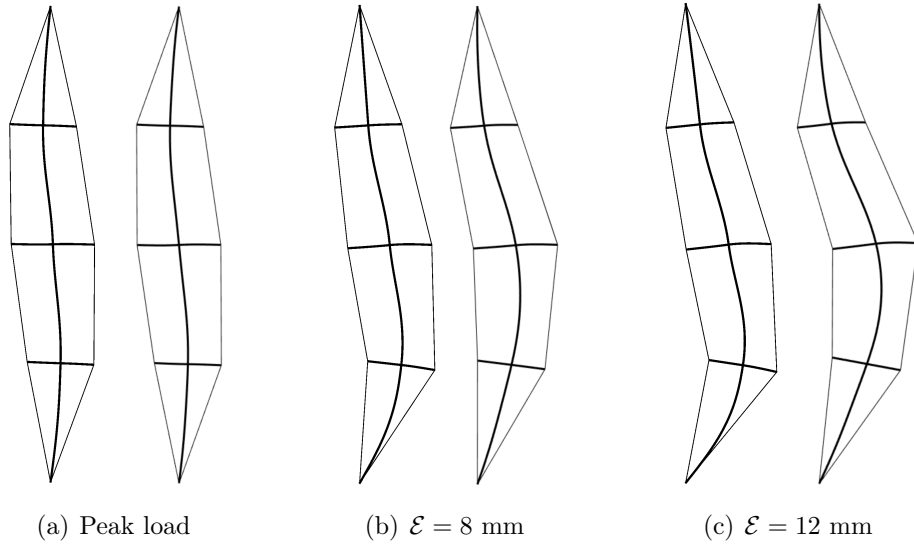


Figure 5.24: Comparisons of the post-buckling profiles between the analytical and the FE model at the peak load and the interactive buckling region where $2\alpha_m = 0.175$, $\phi_s = 3$ mm and $\gamma = 0.8$ (left: FE; right: analytical). The amplification factor is 5 for visualization purposes.

interactions that feature mode jumping and, to some extent, cellular buckling. Therefore there is a potential danger in overestimating the actual load carrying capacity if Mode 2 is critical. The comparisons between the analytical and the FE models have been shown to be excellent both qualitatively and quantitatively.

The ranges where interactive buckling, including mode jumping and cellular buckling, is most significant were also presented for different structural configurations. these configurations would require careful attention in practice since they also exhibit effects synonymous with cellular buckling (Hunt *et al.* , 2000; Budd *et al.* , 2001; Wadee & Gardner, 2012; Wadee & Bai, 2014), which is usually associated with dangerously unstable responses. Hence, the design loads for such cases should be set well below the ultimate load to avoid the dangerously unstable behaviour. It was not only demonstrated that the benchmark prestressing force T_{opt} promotes the most undesirable behaviour in terms of post-buckling instability, but also that this may be mitigated by simply increasing the initial prestressing force; this is developed further in the next chapter.

Chapter 6

Actual optimal prestress and parametric studies

6.1 Introduction

For the triple-bay stayed column studied in this thesis, initial prestressing forces are applied to the cable stays. The stays in the similar, but less complicated structure, the single-bay stayed column, are also pretensioned. In Chapter 3, an analytical method was developed to obtain the theoretical optimal prestress that ideally could maximize the column load carrying capacity. This method was extended from a similar approach (Hafez *et al.* , 1979) of calculating the optimal prestress for the single-bay prestressed stayed column. However, both approaches are based on the same assumption, which is that the main column is perfectly straight and no lateral deflections are allowed prior to critical buckling. This assumption ensures both the single-bay and the triple-bay models are studied purely in the linear range, which makes the investigation of the optimal prestress feasible, but less realistic, as has been previously demonstrated (Saito & Wadee, 2008; Saito & Wadee, 2010). Owing to the existence of initial imperfections in the column, such as lateral deflections in

the column prior to loading, this brings geometric nonlinearities into the problem, which may completely alter the level of the theoretical optimal prestress obtained in Chapter 3 for the triple-bay column and by Hafez *et al.* (1979) for the single-bay case.

It can be seen that when increasing the prestressing force from T_{opt} in both the FE model, described in Chapter 3, and the analytical model, described in Chapter 5, extra benefit can still be obtained in terms of the load carrying capacity. This means that the theoretical optimal prestress for the triple-bay column is good to work as a benchmark when using different levels of prestress, but the actual optimal prestress needs to be investigated further. As stated in Chapter 2, Saito and Wadee (2010) presented an approach to obtain the actual optimal prestress for the single-bay prestressed stayed column using FE modelling. Unlike earlier research on the magnitude of the prestress (Smith, 1985), the maximum load carrying capacity is not necessarily the only indicator for choosing the preferred prestress. The required resistances of the column and the stays need also to be taken into consideration. This allows the designer to choose the appropriate prestress more wisely when designing the stayed column, since the strength of the materials and their relative costs are also important factors. A similar approach is also used in this chapter to investigate the actual optimal prestress for the triple-bay prestressed stayed column. Recommendations for the actual optimal prestress are provided for the structures with different geometric arrangements.

6.2 Methodology

6.2.1 Levels of prestressing forces

The theoretical optimal prestressing force T_{opt} , the expression for which was derived in Chapter 3, is used as a benchmark for both the analytical and FE models in this thesis. As described earlier, expressions for the prestressing force smaller than T_{opt} are also obtained.

In Zone 1, described in Figure 2.5, which indicates that a very small prestress is applied to the stays, owing to the axial shortening of the main column, all the stays go slack before the Euler load is reached. In this particular case, the system is effectively identical to the unstayed column with the buckling load being the Euler load. In order to mimic the behaviour of the perfect case, a very small column initial out-of-straightness, $\delta = 1/10000$, is used to represent a nearly perfect system. A small prestress in Zone 1, $\sigma_s = 5 \text{ N/mm}^2$, is applied to the triple-bay stayed column with the stay diameter, the column aspect ratio and the crossarm length ratio being $\phi_s = 4.8 \text{ mm}$, $2\alpha_m = 0.2$ and $\gamma = 0.8$ respectively. The remaining parameters are identical to the values used in Chapter 3. Figure 6.1 shows the equilibrium path of the column axial load P versus the column mid-

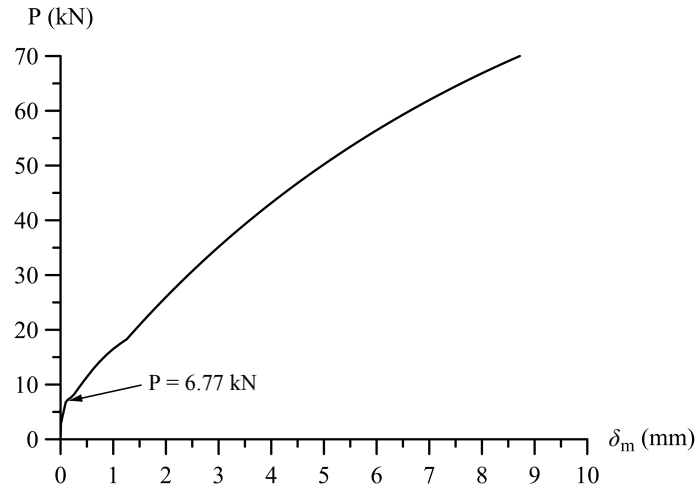


Figure 6.1: Equilibrium path of the column axial load P versus the column mid-span lateral displacement ϵ_m in the post-buckling range when $\sigma_s = 5 \text{ N/mm}^2$, $\phi_s = 4.8 \text{ mm}$, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

span lateral displacement ϵ_m in the post-buckling range for the triple-bay prestressed stayed column obtained using the FE model. It can be seen clearly that the load carrying capacity is definitely larger than the Euler load ($P_E = 6.38 \text{ kN}$) even if a Zone 1 level of prestress is adopted. However, a significantly smaller slope is observed in the path when the load reaches $P = 6.77 \text{ kN}$ which is slightly larger than the Euler load. Later on, the slope of the

path becomes larger and the axial load continues to increase. This observation replicates the behaviour of the theoretical model, where the buckling load in Zone 1 is the Euler load of the main column element.

The mechanism when the Zone 1 level of prestress is adopted in the imperfect system is even more clearly revealed in Figure 6.2, which shows the equilibrium paths of the stay stresses σ_s and the column axial load P versus the column mid-span lateral displacement δ_m simultaneously for the nonlinear buckling response. It can be seen that initially the stresses in all the eight stays drop dramatically from 5 N/mm² to approximately 0.7 N/mm² and the axial force in the column increases from zero to approximately 3 kN with the mid-span deflection almost remaining at zero. This indicates that at the very early loading stage, the axial compression of the column is dominant while the lateral deflection is negligible. This is in accordance with the fundamental paths of the perfect analytical model presented in Chapters 4 and 5. Along with the increase of the external axial load, the mid-span deflection of the column starts to increase gradually, which means that the lateral deflection of the column becomes increasingly important. However, it can be seen that the stresses in all the eight stays are still decreasing, indicating that the axial shortening of the column still dominates when compared with the lateral deflection. This is the stage when all stays are de-stressing until a temporary buckling load that is almost equal to the Euler load is observed while only one of the stays is marginally active. However, this approximate Euler buckling does not become the eventual buckling load of the column as predicted for the theoretically perfect case. While the mid-span deflection of the column increases further, stay 3 on the convex side of the slightly asymmetric column, which almost loses all its stiffness, re-obtains its tensile force gradually. Therefore the axial force in the column increases gradually with the re-stressing of stay 3. When the mid-span deflection is increased further, stay 2, which went slack completely in the previous stage, also starts to re-obtain its tensile force. As a result, the axial force in the column increases at a larger rate owing to the re-stressing of one more stay, in this case, stay 2.

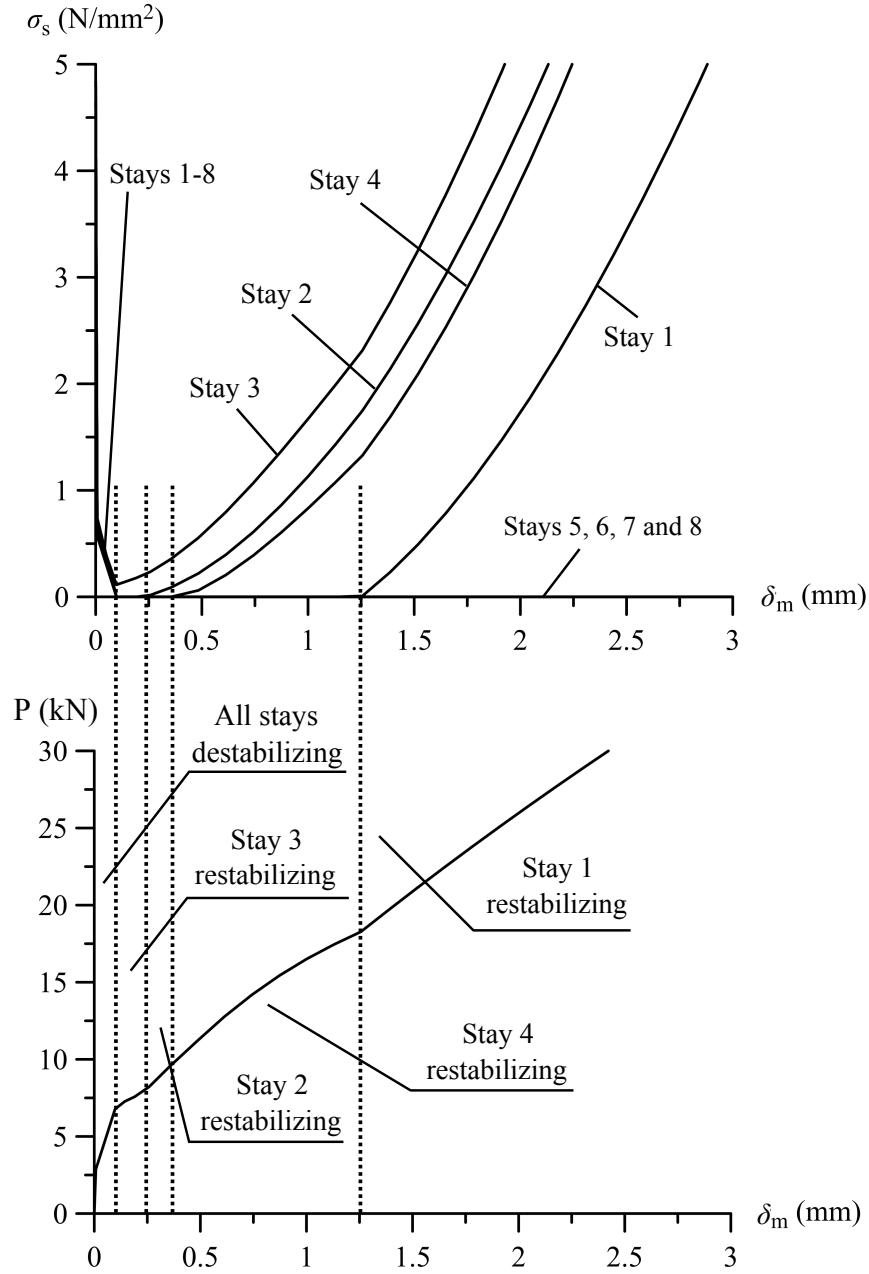


Figure 6.2: Equilibrium paths of the stay stresses σ_s and the column axial load P versus the column mid-span lateral displacement δ_m for interactive buckling when $\sigma_s = 5$ N/mm² initially, $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

As expected, along with the increase of the column mid-span deflection, the stays on the convex side of the column regain their tensile force one by one. Therefore the axial force in the column keeps increasing as well until the actual load carrying capacity is reached, as stated in the previous chapters. It can be seen that owing to the mechanism of de-stressing and re-stressing of the stays, the Euler buckling load in Zone 1 is not in fact practically applicable for the imperfect system. Even with such a small imperfection which nearly mimics the perfect case, the actual Euler buckling instability can only be observed for a very small deformation range and is immediately superceded by the effect of stay re-stressing which increases the axial load. Indeed, in more realistic cases, for example when the initial out-of-straightness δ of the column is $1/300$, which is the EC3 design value for hot rolled sections (BS EN 1993-1-1, 2005), the lateral deflection of the column becomes dominant compared with the axial compression much earlier than the nearly perfect case presented above. None of the stays on the convex side of the column go slack; the stresses in these stays decrease by a small amount at the very loading stage and immediately start to increase when these stays reactivate. Therefore, the loss of stiffness at or near the Euler buckling load cannot be practically observed in these more realistic cases.

Similarly, for the Zone 2 prestress, even if a nearly perfect case is investigated, owing to the fact that the lateral deflection is sufficiently large to dominate when compared with the axial displacement of the column, the situation where the prestress in the stay disappears at the instant when the column buckles cannot be practically observed. The stays on the convex side of the column are already in the re-stressing stage long before the buckling load is reached. Therefore the prediction of the relationship between the Zone 2 prestress and the buckling load is not suitable for the imperfect cases also.

Figure 6.3 shows the comparison of the relationships of the load carrying capacity and the initial prestressing force from the FE and the linear theoretical models. The theoretical relationship, which is represented by the dashed lines, is obtained from the expressions of the prestressing force for the different zones described in Chapter 3. It can be seen that

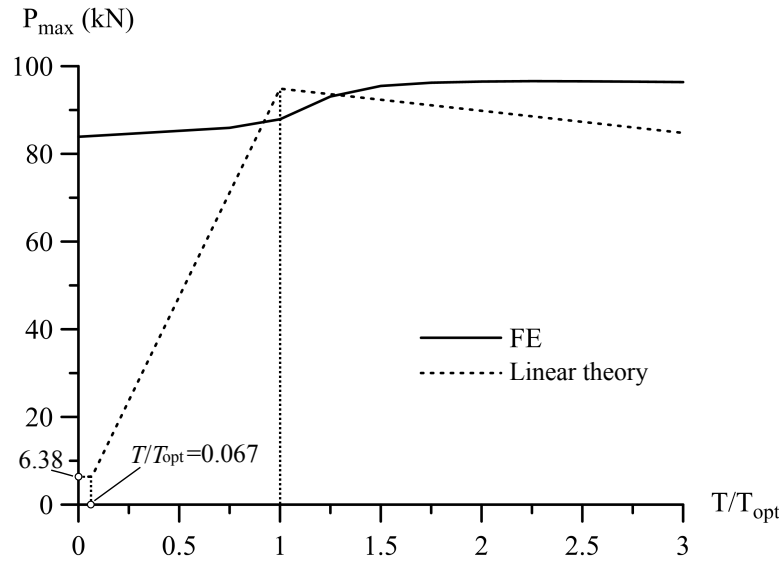


Figure 6.3: Comparison of the relationships of the load carrying capacity and the initial prestressing force from the FE and the linear theoretical models for the triple-bay stayed column where $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

there are very significant differences when comparing Zones 1 and 2 between the FE and theoretical results from the linear model, even though the initial out-of-straightness of the column $\delta = 1/10000$ is adopted. The FE relationship is qualitatively in accordance with experimental data (Osofero *et al.* , 2012) obtained for the single-bay prestressed stayed column. It can also be seen that increasing the prestressing force from T_{opt} could still provide benefit in terms of the load carrying capacity, which is also observed in the FE and the analytical models studied in Chapters 3 and 5 respectively. Therefore, prestressing forces larger than or equal to T_{opt} are investigated in order to find the actual optimal prestressing force of the triple-bay prestressed stayed column.

6.2.2 Efficiency indicators

It has been observed on several occasions in both the analytical and the FE models presented within the current work that a larger load carrying capacity of the triple-bay prestressed stayed column can be achieved by increasing the prestressing force in the stays

from T_{opt} . However, the benefit in the load carrying capacity cannot be obtained in perpetuity by increasing the prestress. The larger the prestress in the stays is, the larger will be the vertical component of the induced axial stress within the column before external loading is commenced. In the cases that have been studied thus far, although the prestress introduces an extra axial component to the column stress that very slightly reduces the external load carrying capacity, the extra lateral stiffness that is introduced to the system plays a significantly more important role in resisting the buckling deformation, which essentially neutralizes the potential disadvantage. Therefore, an increase in the load carrying capacity is usually observed. However, if the prestress in the stays is sufficiently large, the effect of the initial axial stress in the column might neutralize the benefit of the extra stiffness in the stays instead; if this is the case, the load carrying capacity would begin to decrease for larger prestressing forces.

Figure 6.4 shows the relationship of the load carrying capacity P_{max} obtained from the FE

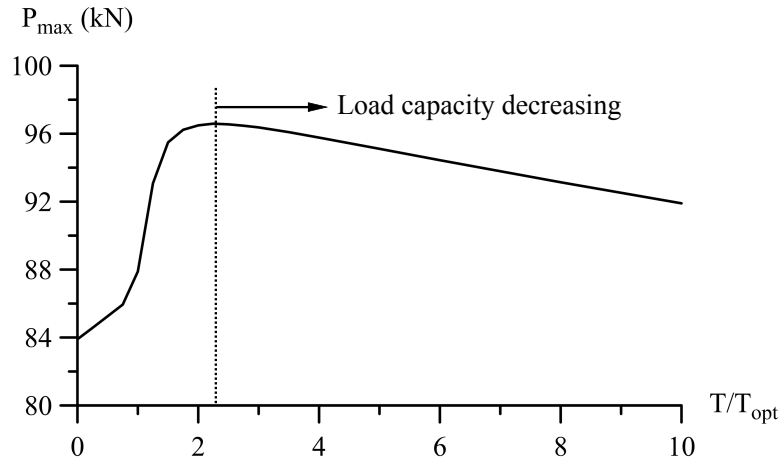


Figure 6.4: Relationship of the load carrying capacity P_{max} obtained from the FE model versus the normalized initial prestressing force in the stays T/T_{opt} for the triple-bay prestressed stayed column when $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

model versus the normalized initial prestressing force in the stays T/T_{opt} for the triple-bay prestressed stayed column. It can be seen that the load carrying capacity reaches a plateau

when the initial prestressing force is approximately $2T_{\text{opt}}$. As expected, by increasing the prestressing force further from approximately $2.5T_{\text{opt}}$, a decrease is observed in the column load carrying capacity. Therefore, a larger prestress may not always provide a larger load carrying capacity. It is therefore unwise to use a very large prestress without further investigation.

In addition, excessive prestress may also result in the stress in the stays exceeding the yield stress of the column or the stays. In the imperfect system, particularly with a realistic column initial out-of-straightness value, geometric nonlinearity becomes important as soon as the column is axially loaded. Owing to the lateral deflection of the column, the stays on the convex side of the column will be stretched further. Therefore, before, or at the point of the maximum load, the stresses in these stays may already be larger than the material yield stress. If this is the case, material damage in the stays would occur prior to the failure of the entire system, which implies that the design load probably will not be achieved. Therefore, apart from the load carrying capacity, the required stay resistance is also an essential factor when investigating an optimal prestress level.

The stresses within the column also need to be monitored. As presented in Figure 6.5, the

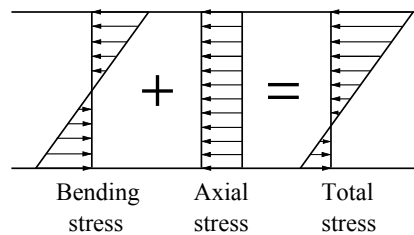
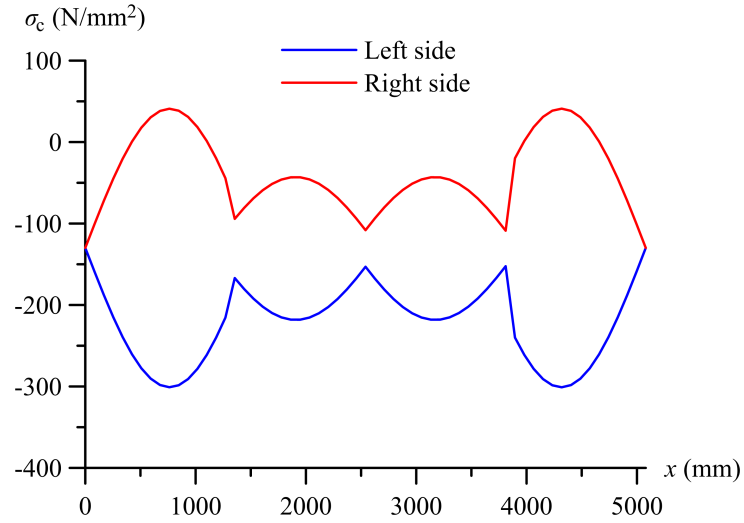


Figure 6.5: Components of the total direct stresses in the column element.

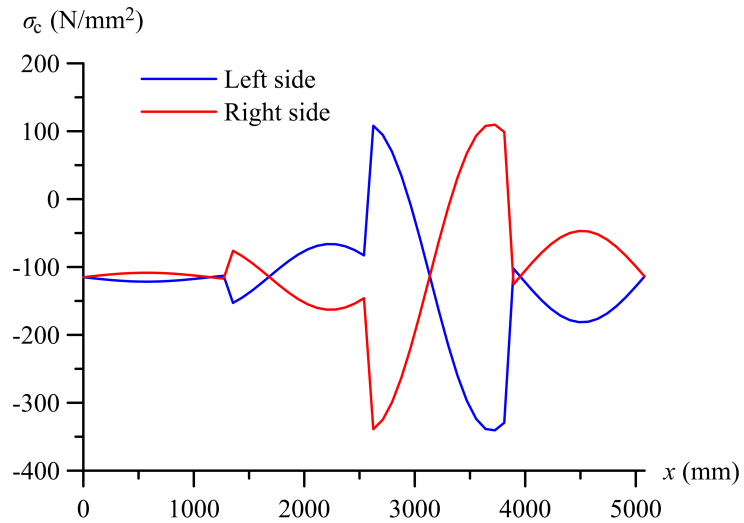
total stress in the column element is a combination of the axial stress due to axial compression and the bending stress due to column bending. Although it has been demonstrated that the maximum load carrying capacity is always less than one-half of the column squash load, the effect of the bending stress has not been studied explicitly. Therefore the effect of both the axial and the bending stresses are investigated currently to ensure that the total

direct stress remains within the elastic range.

Figure 6.6 shows the stress distribution along the column length at the point of the maxi-



(a) Mode 1 buckling



(b) Interactive buckling

Figure 6.6: The direct stress distribution σ_c within the column along its length during the post-buckling range for (a) Mode 1 and (b) interactive buckling when $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$.

mum load for Mode 1 and interactive buckling when $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$, where a positive stress value represents tension. As shown in Figure 6.6(a), in the symmetric buckling (Mode 1) case, the maximum total stress in the column is approximately 300 N/mm², while the component from the axial load is only approximately 130 N/mm².

This means that the bending stress in this particular location is approximately 170 N/mm^2 , which is significantly larger than the axial stress. A similar observation can be made from Figure 6.6(b), which represents interactive buckling; the maximum total stress in the column is approximately 340 N/mm^2 while the axial stress is only approximately 115 N/mm^2 . Again, the bending stress, which is approximately 225 N/mm^2 , is significantly larger than the axial stress. Therefore, judging whether the column strength is sufficient cannot be accomplished simply by only comparing the axial load capacity with the column squash load. Therefore, the actual resistance of the column is another important factor when studying the actual optimal prestress (Saito & Wadee, 2009a).

A higher load carrying capacity is almost always preferable. However, extra caution needs to be exercised such that material failure does not occur prior to the failure due to the geometry. Therefore, similar ‘efficiency indicators’, first adopted by Saito and Wadee (2010), are used in the current study. Firstly, the efficiency indicator for the column is defined thus:

$$\eta = \frac{P_{\max}}{A_c f_{y,\text{req}}}, \quad (6.1)$$

where $P_{\max}$ is the maximum load carrying capacity with $f_{y,\text{req}}$ being the required column resistance at the point of the maximum load. A high value of this indicator η means a relatively high load carrying capacity with an intrinsically low column resistance requirement, which represents a more efficient system. Similarly, the efficiency indicator for the stays is defined thus:

$$\eta_s = \frac{P_{\max}}{A_s f_{s,\text{req}}} \frac{L/4}{l_1}, \quad (6.2)$$

where $f_{s,\text{req}}$ is the required stay resistance at the point of the maximum load; the ratio $L/(4l_1)$ is a correction factor, since l_1 changes with different values of the column aspect ratio $2\alpha_m$ and the crossarm length ratio γ . Again, a high value of this indicator η_s implies a relatively high load carrying capacity with a relatively low stay resistance requirement, which represents a more efficient stay system. Therefore, aiming for large values of the two efficiency indicators, η and η_s for the column and the stays respectively, is the objective

when investigating the actual optimal prestress currently.

6.2.3 Model formulation

An identical FE model, which was first developed in Chapter 3, is used to conduct the study since the analytical models have demonstrated that it is physically accurate. Buckling analysis is conducted first to obtain the buckling loads and the eigenvectors. The theoretical optimal prestress T_{opt} is calculated using Equation (3.28). Riks analysis is then conducted to investigate the post-buckling behaviour of the system. The respective eigenvectors are introduced as the initial imperfections in the column according to the buckling mode, while different levels of the prestress are applied parametrically to the stays in order to find the actual optimal prestress.

6.3 Results and discussion

Three parameters are varied in this section, the stay diameter ϕ_s ranging from 1.6 mm to 10 mm, the column aspect ratio $2\alpha_m$ ranging from 0.05 to 0.3 and the crossarm length ratio γ ranging from 0.5 to 1.0. The critical loads and the buckling modes for these configuration have already been presented in Tables 3.5, 3.6 and 3.7. The column initial out-of-straightness value, $\delta = 1/300$ (BS EN 1993-1-1, 2005), is used in the post-buckling analysis to reflect the behaviour of practically realistic cases. Unless stated otherwise, all other structural dimensions and material properties are exactly the same as the values adopted in Chapter 3.

Nine different levels of the prestressing force, with an increment of $0.25T_{\text{opt}}$ ranging from T_{opt} to $3T_{\text{opt}}$, are studied for each case while focusing on the efficiency indicators for the column η and for the stays η_s . Therefore, 162 separate cases in total are studied.

6.3.1 Varying the stay diameter ϕ_s

Figure 6.7 shows the relationships of the maximum load capacities $P_{\max}$ versus the nor-

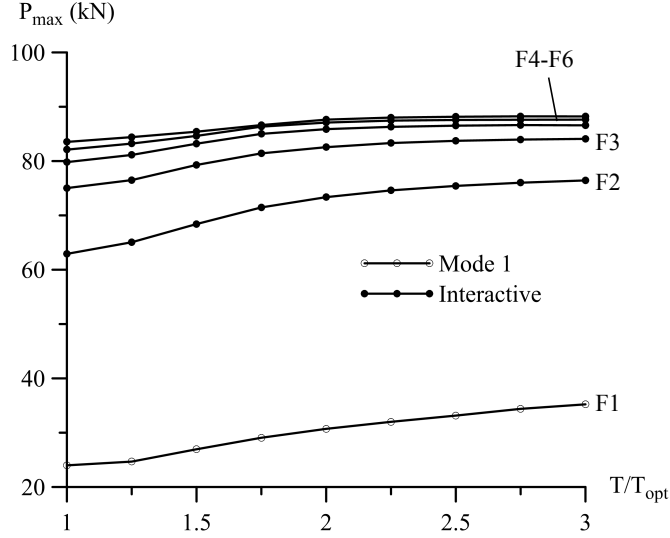


Figure 6.7: Relationships of the maximum load capacities $P_{\max}$ versus the normalized prestressing forces in the stays T/T_{opt} with the stay diameter ϕ_s varying.

malized prestressing forces in the stays T/T_{opt} with the stay diameter ϕ_s varying. Owing to the small stay diameter (1.6 mm) adopted in case F1, its load carrying capacities are much lower than the other cases. It can be seen that for all the six cases, increasing the prestressing force from T_{opt} provides additional load carrying capacity. However, for cases F2–F6, this benefit is not obvious once the prestressing force reaches approximately $2T_{\text{opt}}$. It can also be seen that the increase in the load carrying capacity stops if the prestressing force is increased further for cases F3–F6.

Figure 6.8 shows the relationships of the required column resistance $f_{y,\text{req}}$ and the corresponding efficiency indicator η versus the normalized prestressing forces in the stays T/T_{opt} with the stay diameter ϕ_s varying at the point of the maximum load. As shown in Figure 6.8(a), the required resistance of the column for F1 is basically constant, while for F2–F6 the required resistance decreases first and then increases once the prestressing force reaches $1.5T_{\text{opt}}$ (F2–F5) or $1.75T_{\text{opt}}$ (F6). It can also be seen that if the yield stress of the column

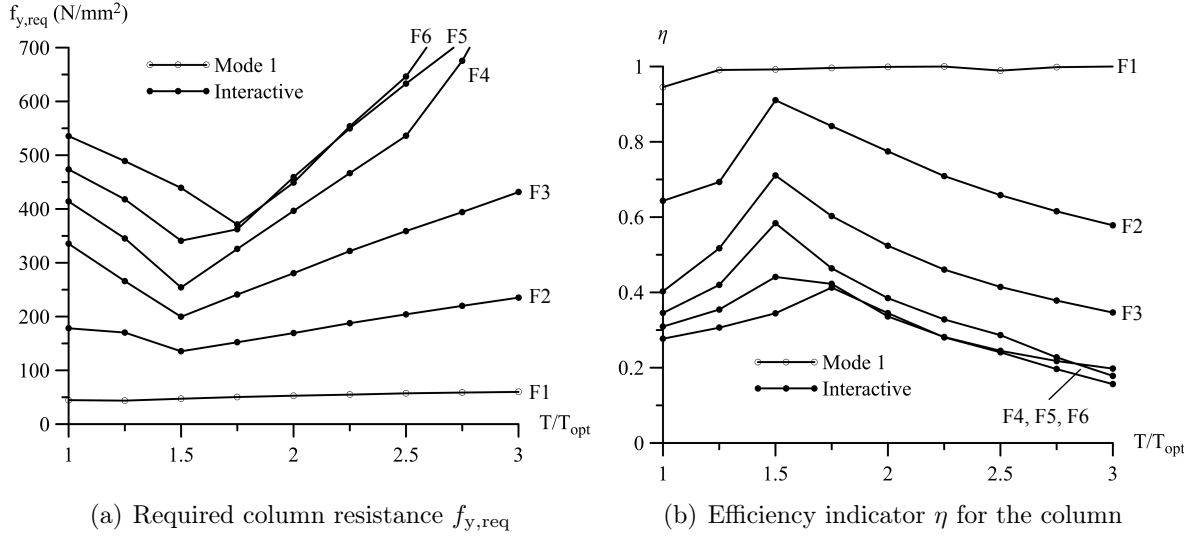


Figure 6.8: Relationships of the required column resistance $f_{y,req}$ and the corresponding efficiency indicator η versus the normalized prestressing forces in the stays T/T_{opt} with the stay diameter ϕ_s varying.

is set to 355 N/mm², the required resistance for cases with large stay diameters requires a higher grade of steel for the column, which puts it in the range of high strength steel (Raoul & Günther, 2005). This is particularly obvious when adopting a very small or a very large prestress. As stated above, higher values of the efficiency indicator for the column are preferred. It can be seen in Figure 6.8(b) that for cases F2–F6, the highest indicator values are found when $T = 1.5T_{opt}$ (F2–F5) or $T = 1.75T_{opt}$ (F6). It is very important to note that this is just the range where the required column resistance is found to be a minimum.

Similarly, Figure 6.9 shows the relationships of the required stay resistance $f_{s,req}$ and the corresponding efficiency indicator η_s versus the normalized prestressing forces in the stays T/T_{opt} with the stay diameter ϕ_s varying at the point of the maximum load. As shown in Figure 6.9(a), the minimum required stay resistance can be found when $T = 1.5T_{opt}$ for F2–F4, $T = 1.25T_{opt}$ for F1 and $T = 1.75T_{opt}$ for F5 and F6. Again, the best values of the efficiency indicator are found mostly when $T = 1.5T_{opt}$ and $1.75T_{opt}$, as shown in Figure 6.9(b). This prestressing force range is in accordance with the finding obtained in the cases studying the efficiency indicator for the column. Therefore, the actual optimal

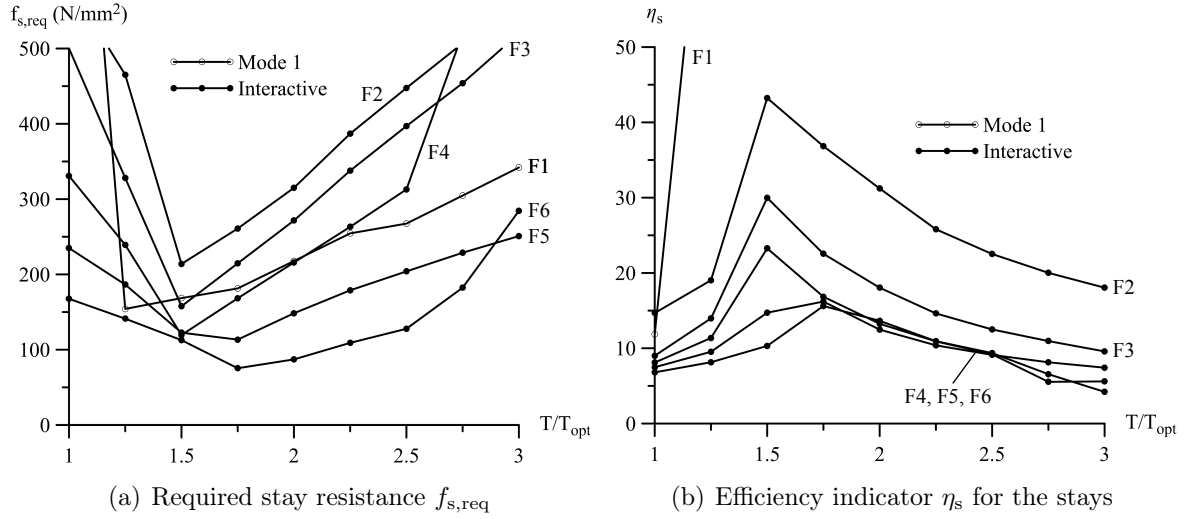


Figure 6.9: Relationships of the required stay resistance $f_{s,req}$ and the corresponding efficiency indicator η_s versus the normalized prestressing forces in the stays T/T_{opt} with the stay diameter ϕ_s varying.

prestress selected on the basis of the efficiency indicators of the column and the stays are shown in Table 6.1.

Table 6.1: The actual optimal prestressing force for the column $T_{opt,c}$ and for the stays $T_{opt,s}$ with the stay diameter ϕ_s varying.

Case	ϕ_s (mm)	$T_{opt,c}$	$T_{opt,s}$	Buckling mode
F1	1.6	—	—	Mode 1
F2	3.2	$1.5T_{opt}$	$1.5T_{opt}$	Interactive
F3	4.8	$1.5T_{opt}$	$1.5T_{opt}$	Interactive
F4	6.4	$1.5T_{opt}$	$1.5T_{opt}$	Interactive
F5	8.0	$1.5T_{opt}$	$1.75T_{opt}$	Interactive
F6	10.0	$1.75T_{opt}$	$1.75T_{opt}$	Interactive

6.3.2 Varying the column aspect ratio $2\alpha_m$

Figure 6.10 shows the relationships of the maximum load capacities P_{max} versus the normalized prestressing forces in the stays T/T_{opt} with the column aspect ratio $2\alpha_m$ varying. It can be seen that additional load carrying capacities can be obtained initially by increasing the prestressing force from T_{opt} . However, the load carrying capacity trends become rather

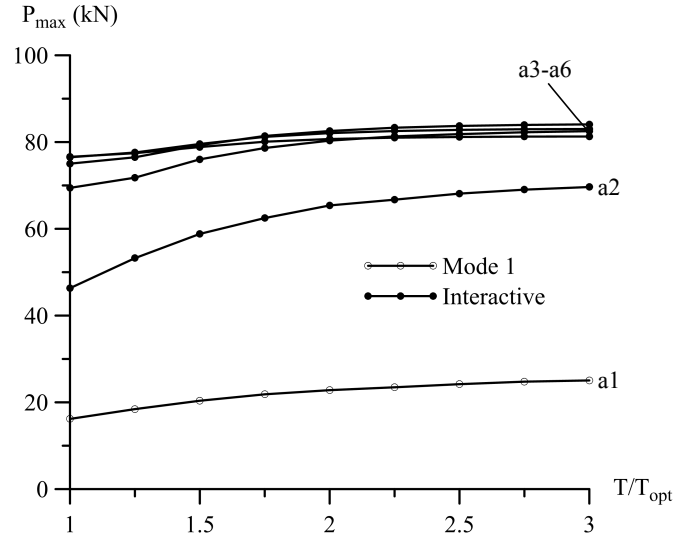


Figure 6.10: Relationships of the maximum load capacities $P_{\max}$ versus the normalized prestressing forces in the stays T/T_{opt} with the column aspect ratio $2\alpha_m$ varying.

flat when the prestressing force reaches approximately $2T_{\text{opt}}$, indicating that not too much benefit is obtained by increasing the prestressing force further. It can also be seen that cases a3–a6 provide higher load carrying capacities.

Figure 6.11 shows the relationships of the required column resistance $f_{y,\text{req}}$ and the corresponding efficiency indicator η versus the normalized prestressing forces in the stays T/T_{opt} with the column aspect ratio $2\alpha_m$ varying at the point of the maximum load. It can be seen in Figure 6.11 that for cases a1 and a2, the respective required column resistances increase with the prestressing force, with the latter possessing a higher rate of increase with T/T_{opt} . For cases a3–a6, the required column resistances decrease first and then increase. The minimum required column resistance can be found when $T = 1.5T_{\text{opt}}$. Note that for larger values of the column aspect ratio, such as a4–a6, the required respective column resistances only within the range from $1.25T_{\text{opt}}$ to $2.5T_{\text{opt}}$ are less than the assumed column yield stress, 355 N/mm^2 ; otherwise high strength steel (HSS) columns would need to be used.

Similarly, Figure 6.12 shows the relationships of the required stay resistance $f_{s,\text{req}}$ and

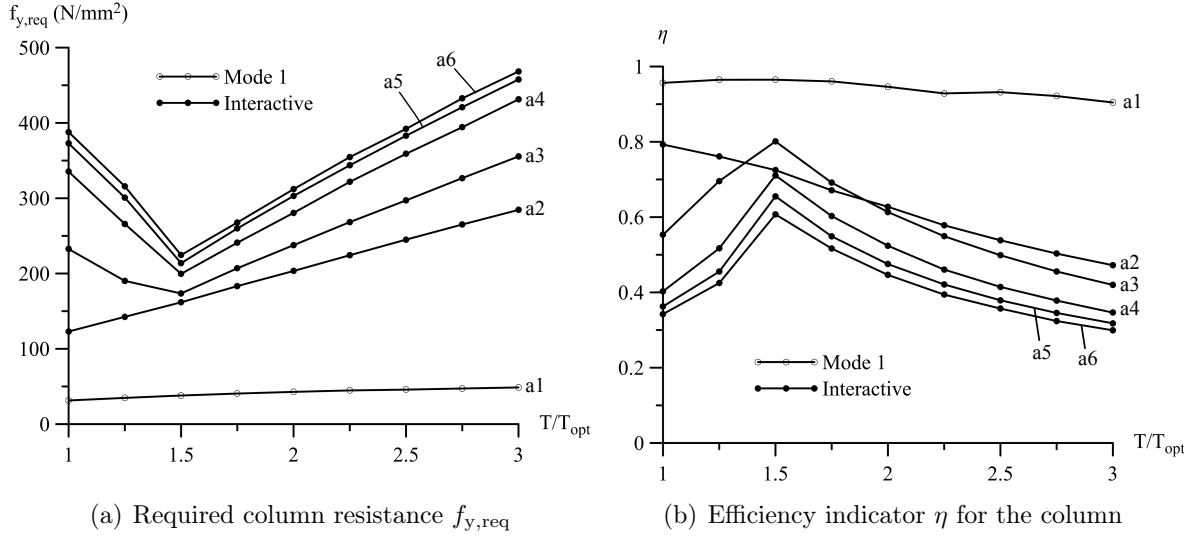


Figure 6.11: Relationships of the required column resistance $f_{y,req}$ and the corresponding efficiency indicator η versus the normalized prestressing forces in the stays T/T_{opt} with the column aspect ratio $2\alpha_m$ varying.

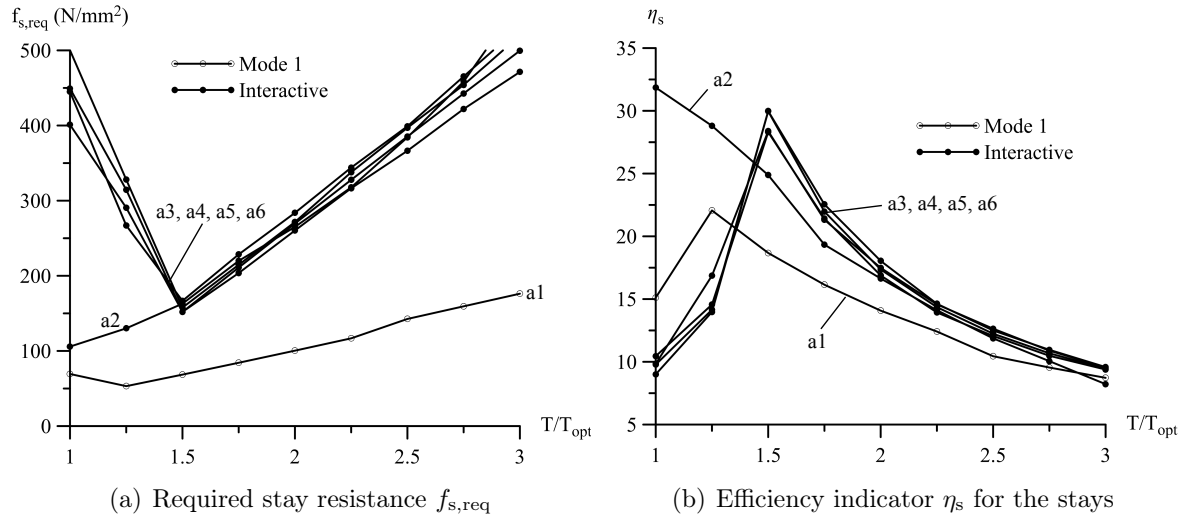


Figure 6.12: Relationships of the required stay resistance $f_{s,req}$ and the corresponding efficiency indicator η_s versus the normalized prestressing forces in the stays T/T_{opt} with the column aspect ratio $2\alpha_m$ varying.

the corresponding efficiency indicator η_s versus the normalized prestressing forces in the stays T/T_{opt} with the column aspect ratio $2\alpha_m$ varying at the point of the maximum load. As shown in Figure 6.12(a), the minimum required stay resistances can be found when $T = 1.5T_{\text{opt}}$ for cases a3–a6, while they are $1.25T_{\text{opt}}$ for case a1 and T_{opt} for case a2. Again, as can be seen in Figure 6.12(b), the preferred efficiency indicator values are mostly found when $T = 1.5T_{\text{opt}}$. Therefore, the actual optimal prestress selected on the basis of the efficiency indicators of the column and the stays are presented in Table 6.2.

Table 6.2: The actual optimal prestressing force for the column $T_{\text{opt,c}}$ and for the stays $T_{\text{opt,s}}$ with the column aspect ratio $2\alpha_m$ varying.

Case	$2\alpha_m$	$T_{\text{opt,c}}$	$T_{\text{opt,s}}$	Buckling mode
a1	0.05	$1.5T_{\text{opt}}$	$1.25T_{\text{opt}}$	Mode 1
a2	0.10	$1.0T_{\text{opt}}$	$1.0T_{\text{opt}}$	Interactive
a3	0.15	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Interactive
a4	0.20	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Interactive
a5	0.25	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Interactive
a6	0.30	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Interactive

6.3.3 Varying the crossarm length ratio γ

Figure 6.13 shows the relationships of the maximum load capacities P_{max} versus the normalized prestressing forces in the stays T/T_{opt} with the crossarm length ratio γ varying. It can be seen that the additional load carrying capacities can be obtained by increasing the prestressing force from T_{opt} , but this benefit is not that obvious when the prestressing force goes much beyond $2T_{\text{opt}}$. It can also be seen that increasing γ values from 0.5 to 0.8 significantly increases the load carrying capacity of the column. However, when γ is increased further to 0.9 and 1.0, a decrease in the load carrying capacity can be observed when the prestressing force is larger than $1.75T_{\text{opt}}$, owing to the fact that the middle crossarm becomes inherently less effective.

Figure 6.14 shows the relationships of the required column resistance $f_{y,\text{req}}$ and the corresponding efficiency indicator η versus the normalized prestressing forces in the stays T/T_{opt}

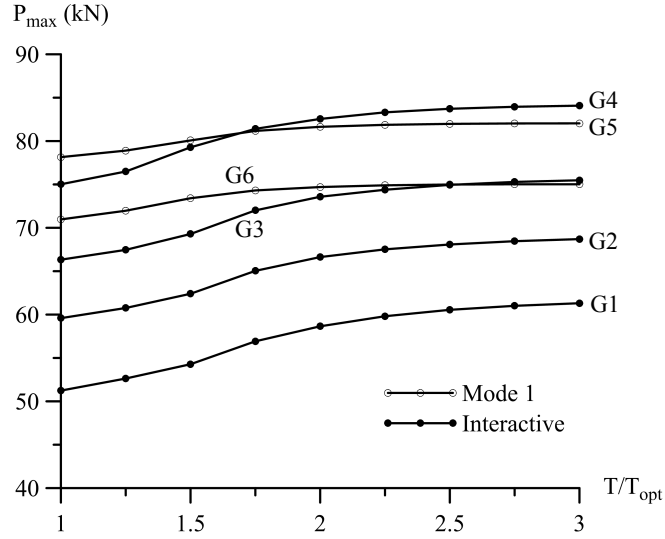


Figure 6.13: Relationships of the maximum load capacities $P_{\max}$ versus the normalized prestressing forces in the stays T/T_{opt} with the crossarm length ratio γ varying.

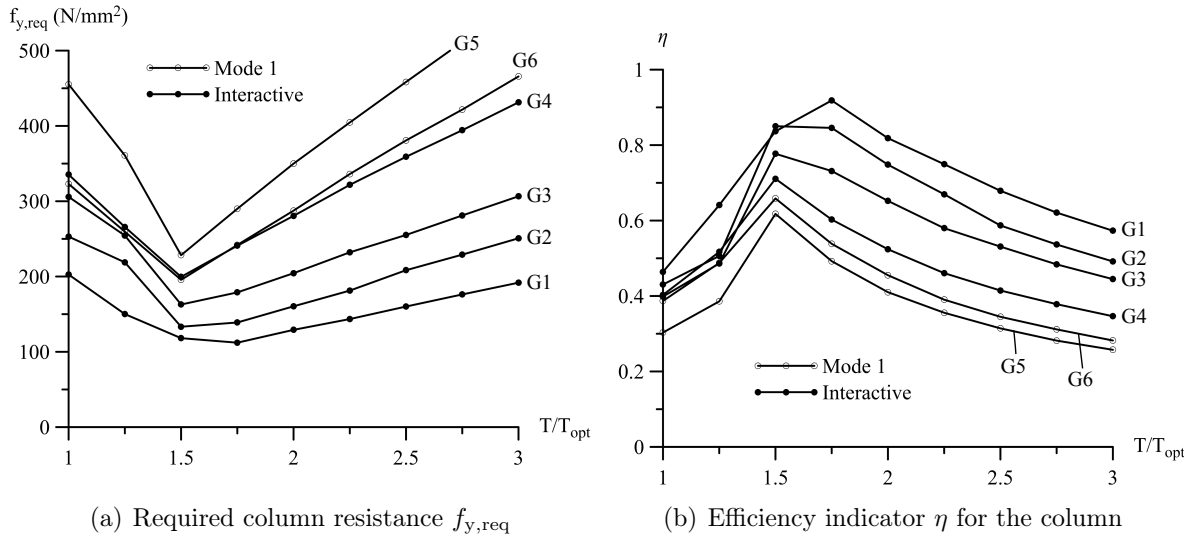


Figure 6.14: Relationships of the required column resistance $f_{y,\text{req}}$ and the corresponding efficiency indicator η versus the normalized prestressing forces in the stays T/T_{opt} with the crossarm length ratio γ varying.

with the crossarm length ratio γ varying at the point of the maximum load. As shown in Figure 6.14(a), the required column resistances decrease first and then increase with the prestressing force. The lowest required column resistance can be found when $T = 1.5T_{\text{opt}}$ for cases G2–G6 and $T = 1.75T_{\text{opt}}$ for case G1. As expected, it can be seen in Figure 6.14(b) that the preferred efficiency indicator values of the column can be mostly found when $T = 1.5T_{\text{opt}}$.

Similiarly, Figure 6.15 shows the relationships of the required stay resistance $f_{s,\text{req}}$ and the

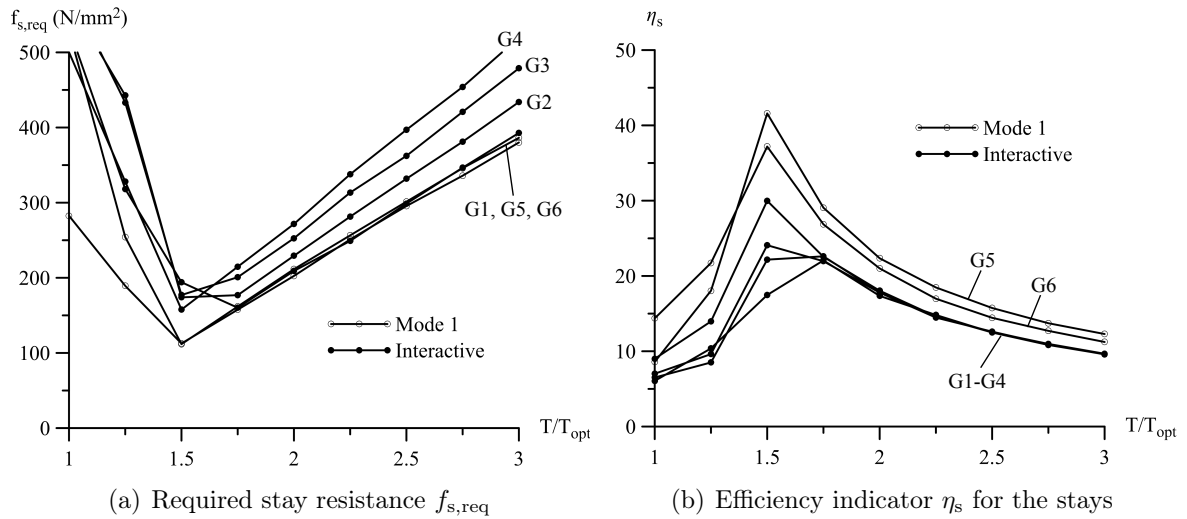


Figure 6.15: Relationships of the required stay resistance $f_{s,\text{req}}$ and the corresponding efficiency indicator η_s versus the normalized prestressing forces in the stays T/T_{opt} with the crossarm length ratio γ varying.

corresponding efficiency indicator η_s versus the normalized prestressing forces in the stays T/T_{opt} with the crossarm length ratio γ varying at the point of the maximum load. It can be seen in Figure 6.15(a) that the required stay resistances decrease initially and then increase with the prestressing force. The lowest required stay resistance can be found when the prestressing force is $1.5T_{\text{opt}}$ and $1.75T_{\text{opt}}$. As shown in Figure 6.15(b), the highest values of the efficiency indicator for the stays can be found when the prestressing force ranges from $1.5T_{\text{opt}}$ to $1.75T_{\text{opt}}$. Therefore, the actual optimal prestress selected on the basis of the efficiency indicators of the column and the stays are presented in Table 6.3.

Table 6.3: The actual optimal prestressing force for the column $T_{\text{opt},c}$ and for the stays $T_{\text{opt},s}$ with the crossarm length ratio γ varying.

Case	γ	$T_{\text{opt},c}$	$T_{\text{opt},s}$	Buckling mode
G1	0.5	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	Interactive
G2	0.6	$1.5T_{\text{opt}}$	$1.75T_{\text{opt}}$	Interactive
G3	0.7	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Interactive
G4	0.8	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Interactive
G5	0.9	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Mode 1
G6	1.0	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	Mode 1

6.3.4 Further parametric studies

In the previous section, three parameters were varied. The stay diameter, the column aspect ratio and the crossarm length ratio are fixed to $\phi_s = 4.8$ mm, $2\alpha_m = 0.2$ and $\gamma = 0.8$ respectively when varying the other two parameters. In this section, cases with a wider range of the geometric configurations are studied in order to investigate the actual optimal prestress. It can be seen in Figure 6.14 that the crossarm length ratio $\gamma = 0.8$ provides the highest load carrying capacity, which indicates that both the middle and the edge crossarms are sufficiently effective in this ratio. Therefore, a series of cases that vary the stay diameter and the column aspect ratio simultaneously with the crossarm length ratio γ being fixed to 0.8 are studied. Nine different levels of prestressing force ranging from T_{opt} to $3T_{\text{opt}}$ with an increment of $0.25T_{\text{opt}}$ are adopted for six different stay diameters and six different column aspect ratios. In total, 324 cases are investigated in this section.

The buckling analysis of the 36 cases with the different stay diameters and the column aspect ratios is first conducted. The results are shown in Figure A.1 in Appendix A. With the obtained buckling loads, the theoretically optimal prestress T_{opt} can be calculated. Similar to the previous section, post-buckling analysis is then conducted with the adoption of nine different levels of prestressing force. The results of the column load carrying capacity and the required resistances of the column and stays are respectively presented in Figures A.2, A.3, A.4, A.5 and A.6 along with the two efficiency indicators.

Tables 6.4 and 6.5 summarize the actual optimal prestress based on the preferred values

Table 6.4: The actual optimal prestressing force based on the efficiency indicator of the column η with the stay diameter ϕ_s and the column aspect ratio $2\alpha_m$ varying simultaneously when the crossarm length ratio $\gamma = 0.8$.

Case	a1	a2	a3	a4	a5	a6
F1	—	—	—	—	—	$1.25T_{\text{opt}}$
F2	—	—	—	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F3	—	—	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F4	—	$1.25T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F5	—	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F6	—	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$

Table 6.5: The actual optimal prestressing force based on the efficiency indicator of the stay η_s with the stay diameter ϕ_s and the column aspect ratio $2\alpha_m$ varying simultaneously when the crossarm length ratio $\gamma = 0.8$.

Case	a1	a2	a3	a4	a5	a6
F1	—	—	—	—	—	—
F2	—	—	—	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F3	—	—	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F4	—	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F5	—	$1.5T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.5T_{\text{opt}}$	$1.5T_{\text{opt}}$
F6	$1.5T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$	$1.75T_{\text{opt}}$

of the efficiency indicators of the column and the stays respectively. It can be seen that for most of the cases in both tables, the actual optimal prestressing force is $1.5T_{\text{opt}}$, while for cases with larger stay diameters the preferred prestressing force becomes $1.75T_{\text{opt}}$. This is significantly different to the results for the single-bay prestressed stay column, where the actual optimal prestress is mostly ranging between $2T_{\text{opt}}$ to $3T_{\text{opt}}$ (Saito & Wadee, 2010). Note that the actual optimal prestress in terms of the efficiency indicators cannot satisfactorily be found for configurations with both a small stay diameter and a small column aspect ratio. However, a suitable prestress is still able to be selected by checking the column load carrying capacity shown and the required resistances of the column and the stays when designing structures with the configurations within this range.

Since the actual optimal prestress for the triple-bay stayed column is recommended, the parametric spaces, which indicate where mode jumping and passive interactions are signif-

icant with the normalized stay diameter Φ_s and the ratio $2\alpha_m$ varying, are presented with the adoption of the new prestress levels. Figure 6.16 shows the cases with the two desired

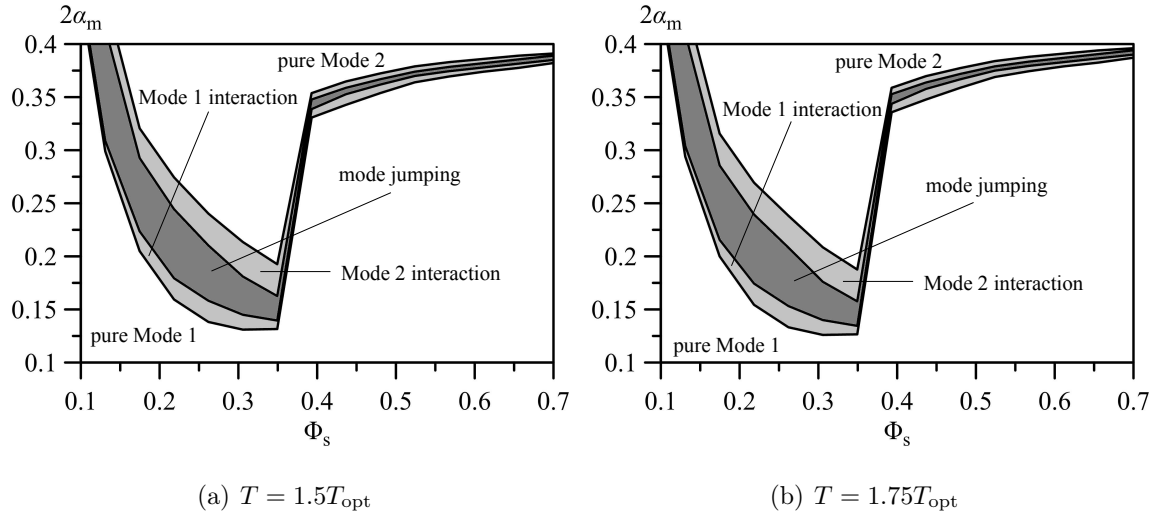


Figure 6.16: Parametric spaces of $2\alpha_m$ and Φ_s showing regions where mode interactions that cause mode jumping (i.e. full interaction between Modes 1 and 2 with cellular buckling) and passive interactions with (a) $T = 1.5T_{\text{opt}}$ and (b) $T = 1.75T_{\text{opt}}$. Results obtained from the analytical model developed in Chapter 5.

prestress levels obtained from the analytical model presented in Chapter 5. Very little difference can be found among the cases of $T = T_{\text{opt}}$ (presented in Chapter 5), $T = 1.5T_{\text{opt}}$ and $T = 1.75T_{\text{opt}}$. It is reasonable therefore to assert that changing the prestress levels mainly affects the load carrying capacity of the column, while it has little effect on the conditions triggering the modal interactions.

6.4 Concluding remarks

In this chapter, an FE model for the near-perfect case was used to compare with the linear theoretical model that provides the so-called optimal prestress T_{opt} . The stresses in the stays were investigated in detail to determine the difference between the linear theoretical model and the nonlinear FE model. It was found that even for the nearly perfect model, the three-zone prestress diagram is not suitable to predict the buckling load

and the optimal prestress. The reactivation of the cable stays in the post-buckling range enables the structure to carry loads significantly higher than those predicted in Zones 1 and 2 from linear analysis. Therefore, the approach to obtain the actual optimal prestress was investigated for the imperfect case.

It was demonstrated that although extra load carrying capacity can be obtained by increasing the prestressing force from T_{opt} , but introducing excessively large prestress levels can decrease the maximum load of the column. In addition, although the maximum axial load in the column is significantly less than the squash load, it has been shown that once the column starts to deflect laterally, the bending stress makes a very significant contribution to the total stress of the column. These observations indicate that nominating the actual optimal prestress by only considering the load carrying capacity is neither necessarily efficient nor safe. Therefore, a pair of efficiency indicators, which take into account the load carrying capacity together with the required resistances of the column and the stays simultaneously, are used as guidance to determine the actual optimal prestress.

The actual optimal prestresses were presented while varying the stay diameter, the column aspect ratio or the crossarm length ratio when two of these parameters were fixed. The actual optimal prestresses were also presented for cases while varying the stay diameter and the column aspect ratio simultaneously. In total, more than 400 cases have been investigated. It is demonstrated that for most cases the actual optimal prestressing force is $1.5T_{\text{opt}}$, while for cases with larger stay diameters the actual optimal prestressing force may become $1.75T_{\text{opt}}$. Note that the nomination of the optimal prestress is based on the preferred values of the aforementioned efficiency indicators. So long as the required resistances of the column and the stays are within the elastic range, the load carrying capacity of the main column element may also be a factor in determining the actual optimal prestress if a high load carrying capacity is paramount; this consideration, however, is left for future work.

Chapter 7

Conclusions

Prestressed stayed columns have become increasingly popular in the construction industry due to their high buckling resistance and low self-weight. Most of the existing research was on the critical buckling load of the stayed column. The post-buckling behaviour had not been satisfactorily explored until relatively recently (Saito & Wadee, 2008; Saito & Wadee, 2009a; Saito & Wadee, 2009b). It has been shown that extra benefit in terms of the load carrying capacity could be obtained by increasing the number of the crossarm systems along the column length. However, most of the aforementioned work was focused on the single-bay stayed column while the post-buckling behaviour of the stayed column with multiple-bays had not been revealed hitherto. Therefore, the key aim of this thesis was to investigate and reveal the nonlinear response of the stayed column with multiple crossarm systems.

It was discussed in Chapter 3 that owing to the lack of restraint at the mid-span in the double-bay column, as shown in Figure 3.1(b), the system cannot provide sufficient resistance to the symmetric buckling mode. Therefore, the triple-bay prestressed stayed column, as shown in Figure 3.1(c), is focused upon exclusively in this thesis. The advantage of this structural configuration is that the middle crossarm provides the resistance to the

symmetric buckling mode while the two edge crossarms provide additional resistance to the anti-symmetric buckling mode. The actual load carrying capacity would be expected to be considerably higher as compared to the single-bay and double-bay stayed columns with the same length due to a significant reduction in the buckling length.

Firstly, a finite element model was developed by using the commercial software ABAQUS. Three principal parameters: the stay diameter, the column aspect ratio and the crossarm length ratio, were varied throughout the analysis. The first two buckling modes, comprising a symmetric and an anti-symmetric shape respectively were studied exclusively. Linear buckling analysis was first conducted to provide the buckling load and the corresponding shape for each mode. Prestress was applied to the cable stays prior to the application of the external load to increase the load capacity. The theoretical optimal prestress for the single-bay stayed column was presented by Hafez *et al.* (1979). A similar but extended approach was developed for the triple-bay column and the calculated optimal prestress, which maximizes the linear critical buckling load, was used as the benchmark for later studies. Initial imperfections were also introduced to the system. In order to be able to trigger interactive buckling, a combination of the imperfections from Modes 1 and 2 was used, allowing the deflections of both pure modes to occur simultaneously.

With the introduction of the benchmark prestress and initial imperfections, the post-buckling behaviour was investigated. It was shown that both the buckling and the post-buckling responses are strongly linked to the aforementioned parameters, indicating the geometrical configuration of the structure should be selected carefully. It was found that when Mode 1 is critical, it also governs the response during the post-buckling stage. However, when Mode 2 is critical, or the buckling loads of Modes 1 and 2 are very close, the governing post-buckling response is interactive since it possesses the lowest load carrying capacity as compared to the two pure modes. In earlier work, the predicted capacity of the column was set to be in accordance with the ultimate load of Mode 2 when this mode is found to be critical. However, catastrophic failure potentially may occur in this scenario

owing to the lack of attention paid to interactive buckling. Therefore the significance of investigating interactive buckling for such structures was demonstrated.

Sensitivities to the sizes of imperfection and levels of prestress were also studied in Chapter 3. The benchmark value of the initial out-of-straightness of the column was set to the EC3 design values for hot rolled sections (BS EN 1993-1-1, 2005). Smaller sizes of imperfections were also studied. It was found that reducing the imperfection sizes provides larger ultimate loads, although the behaviour would be considerably more unstable during the initial post-buckling stage. It was also observed that additional benefit in terms of the load carrying capacity could still be obtained by increasing the prestress from the benchmark value, which is the theoretically optimal prestress. This indicated that the predicted optimal prestress from the linear elastic analysis was not suitable for application to maximize the true ultimate load of the column. Therefore, a detailed method to find the actual optimal prestress for the nonlinear imperfect case was presented in a subsequent study.

Interactive buckling studied in Chapter 3 was manually triggered by adopting combined imperfection shapes affine to both Modes 1 and 2. The mechanism showing how the modes interact naturally was not clearly revealed by the FE model. Therefore, in Chapter 4, a simplified analytical model comprising rigid links and springs was developed. The restraints provided by the crossarms and the cable stays were represented by a series of discrete stiffnesses from the longitudinal and rotational springs. The total potential energy of the system was then formulated and the equilibrium equations were evaluated with respect to the generalized coordinates by using the numerical continuation software AUTO-07P. Again, it was found that when Mode 1 was critical, it also governed the post-buckling response. Once the buckling load of Mode 1 exceeded that of Mode 2, with Mode 2 being marginally critical, interactive buckling, which manifested itself as a closed loop equilibrium path that connects the equilibrium paths of the two pure modes through a sequence of secondary bifurcation points, could be observed. The transition, where the equilibrium path transfers from the first pure mode to the interactive mode through a

secondary bifurcation point and then transfers back to the other pure mode through a third bifurcation point, revealed how the modes interact in detail. Initial imperfections were then introduced to the discrete model. The more realistic post-buckling behaviour still demonstrated that interactive buckling governed the response when Mode 2 is critical. This observation agreed with the conclusion obtained from the FE model qualitatively.

It was observed that although the discrete model developed in Chapter 4 was proficient at revealing the mode interaction, the results from it could not be used to compare with the FE quantitatively. Owing to the use of the discrete stiffnesses, adopting different sizes of the stay diameter, the column aspect ratio and the crossarm length ratio was very difficult. Most importantly, the discontinuous effect of the stay slackening was not taken into consideration, which in reality would occur on the concave side of the column during post-buckling.

To resolve matters, a more sophisticated analytical model was developed and presented in Chapter 5. Shape functions of the column were assumed that comprised components of different wavelengths describing the symmetric and anti-symmetric buckling shapes. Shape functions for the crossarms were also developed. The total potential energy of the system was then formulated in conjunction with the Rayleigh–Ritz method. Eight switches that imposed the condition where the respective energy terms were eliminated when the respective strain was less than zero were implemented. This successfully simulated the stay slackening behaviour since the stays cannot take any axial compression or bending moment.

The results for both the linear buckling and post-buckling behaviour from the FE and the more sophisticated analytical model showed excellent comparisons, indicating that both models were capable of studying the triple-bay prestressed stayed column accurately. As expected, interactive buckling was observed when Mode 2 was marginally critical. The strains at several key points on the interactive buckling path were monitored for the stays. The process of how the modes interact along with the slackening of the stays in sequence

was demonstrated. A parametric study varying the diameter, the column aspect ratio and the crossarm length ratio was then conducted; the geometrical configurations where interactive buckling is most significant were revealed. It is potentially dangerous when designing such structures with the geometries in the identified parametric space; additional attention must be paid to ensure that the design strengths are below the level where the highly unstable post-buckling response would occur. Different levels of prestress were also tested, but these were shown to stabilize the behaviour when the prestress was increased from the benchmark prestress identified in Chapter 3.

It was mentioned that the calculated theoretical optimal prestress was not the actual optimal value for the imperfect triple-bay column. In Chapter 6, the most desirable prestress value was determined for individual arrangements. Previously, the only criteria to select the prestress was to maximize the column load carrying capacity. However, the yield strength of the column and the stays were not considered in conjunction with optimizing the prestress. However, if the required material strength is too high to reach the desired ultimate load, the structure would be inherently more expensive and hence less economically efficient. Therefore, by a similar method employed for the single-bay column (Saito & Wadee, 2010), two efficiency indicators that take the both the load carrying capacity and the required resistance of the column and the stays taken into consideration were presented. Structures with different stay diameters and column aspect ratios were examined with nine separate levels of prestress. It was determined that the actual optimal prestress is approximately 1.5 times the previously obtained theoretical value for most cases, while the ratio rises to 1.75 for cases with larger stay diameters.

To conclude, the post-buckling behaviour of the triple-bay prestressed stayed column was investigated using three different types of model. Results from the FE and the analytical models show excellent comparisons both qualitatively and quantitatively. Interactive buckling is found indeed to be important, especially when the anti-symmetric mode is critical. Finally, the recommended levels of the actual prestress were determined for different

structural configurations.

7.1 Suggestions for future work

In the current thesis, the triple-bay prestressed stayed column was investigated on a two-dimensional (2-D) basis. It has been sufficient to study the fundamentals of the post-buckling behaviour, modal interactions and the effect of the stay slackening. It is also suitable for the realistic cases where the column is supported horizontally from the other out-of-plane dimension, as shown in Figure 1.3(a). The practical deflections of the structure only exist in the 2-D plane generated by the column and the crossarms, therefore the entire post-buckling behaviour only needs to be considered within the same plane. However, in some scenarios, there are no external restraints apart from the two end points, as shown in Figure 1.3(c). In cases such as these, it is recommended to develop a three-dimensional model that takes the deflections in both axes into account. It would be worth investigating to see whether there is interaction between the buckling modes in two perpendicular axes. Moreover, non-symmetric crossarm configurations in three-dimensional could be studied. A case with a rotationally symmetric three-branch crossarm system was studied recently (Li *et al.* , 2016); this could be extended for cases with multiple bays.

Three different models were developed in the current work, two of which can be compared quantitatively with excellent agreement. However, the actual optimal prestress is only examined by using the FE model. It would be worth conducting experiments to investigate the post-buckling behaviour of the triple-bay column, especially to confirm whether the same recommended actual optimal prestress is found. This would enable the formulation of guidance for designers in a similar fashion to that developed in Wadee *et al.* (2013b). In addition, some other factors that may affect the post-buckling response, such as the relaxation of the stays in the longer term and the imperfections of the crossarms, both of which were not studied currently. Although the relaxation of the prestress is deemed to be

less significant in the short-term, it would affect the whole-life behaviour and maintenance costs of the component.

Currently the effect of the stay diameter, the column aspect ratio and the crossarm length ratio were focused upon exclusively. More geometrical parameters can be investigated, such as the column length along with the cross-section type of the column and the crossarm. The cable stays connecting the tips of the column and the crossarms can be replaced by rods to seek further stiffness since the rods will always be active owing to the fact that they can undertake axial compressions. The crossarms can also be placed unevenly along the column length in an attempt to equalize the individual buckling loads of the main column element between the crossarms. To model the latter analytically would require more degrees of freedom since the buckling wavelengths would necessarily vary along the column span.

It is expected that higher buckling resistances can be obtained by increasing the number of the crossarm systems along the column length, such as quadruple-bay and quintuple-bay stayed columns. Connecting the tips of the column and the crossarms diagonally using the stays, as shown in Figures 2.4(c) and 2.4(d), may also help in increasing the ultimate load. Investigations on such configurations should also be conducted in future. However, the extra cost in self-weight and the efficiency in terms of the required resistances of the structural components should also be examined carefully. The current work has highlighted the potential benefits of the stayed column system with multiple bays with a focus on the triple-bay case and there are certainly several areas of work that would provide practically and technically challenging research work in the future.

Appendix A

Results from parametric studies

In order to determine the actual optimal prestress, separate tests with different stay diameters ϕ_s and column aspect ratio $2\alpha_m$ are also conducted. Figure A.1 shows the linear buckling analysis of the triple-bay prestressed column from FE with the stay diameter ϕ_s and the column aspect ratio varying simultaneously. With the obtained buckling loads, the theoretical optimal prestresses can be calculated for each case.

The post-buckling analysis is then conducted while the initial prestress is varied from T_{opt} to $3T_{\text{opt}}$ with an increment of $0.25T_{\text{opt}}$, as shown in Figures A.2 to A.6. The values of the two efficiency indicators introduced in Chapter 6, η and η_s , are presented. The actual optimal prestresses selected based on the desired values of the indicators are summarized in Tables 6.4 and 6.5.

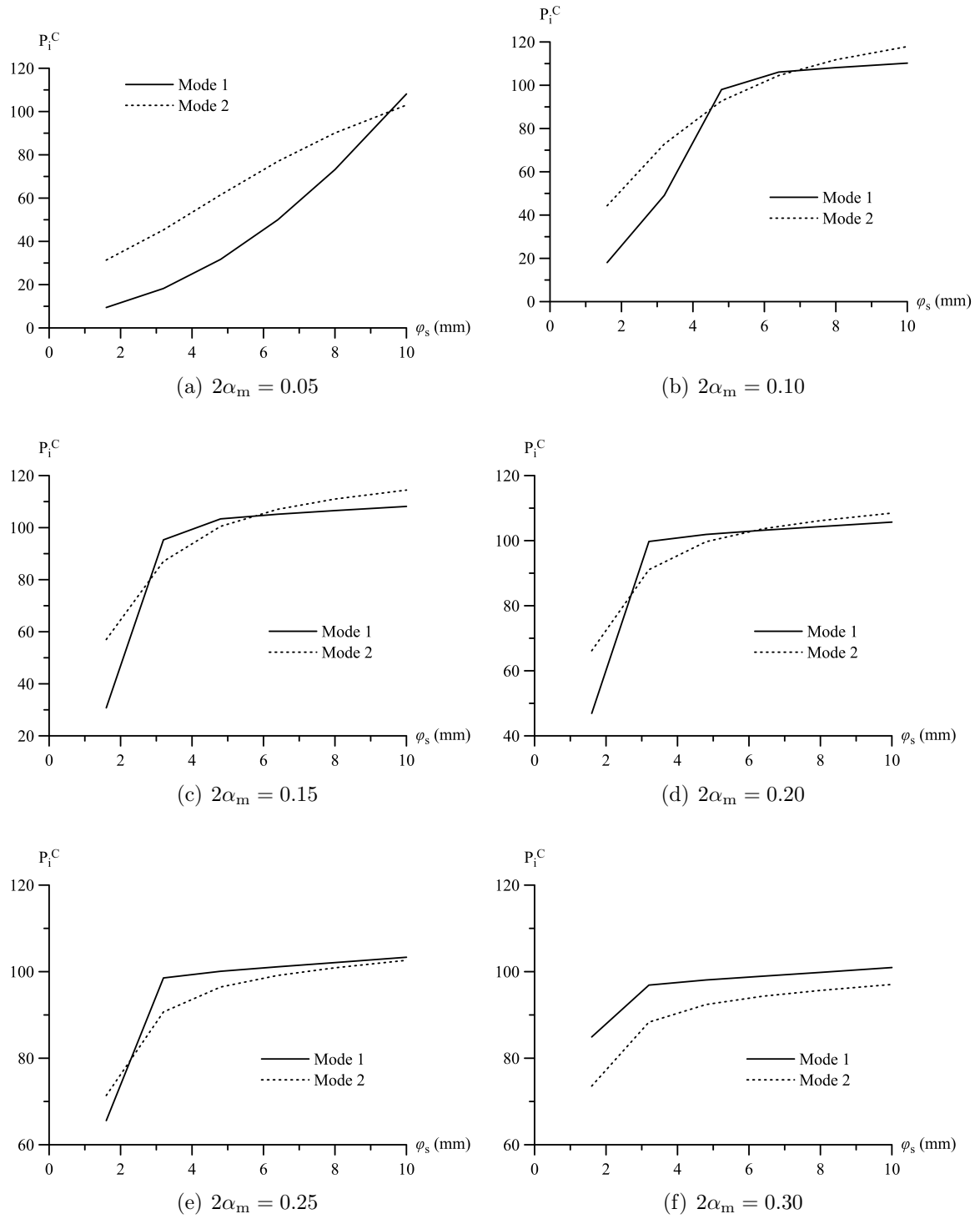


Figure A.1: Buckling analysis of the triple-bay prestressed column from FE with the stay diameter ϕ_s and the column aspect ratio varying simultaneously when the crossarm length ratio $\gamma = 0.8$.

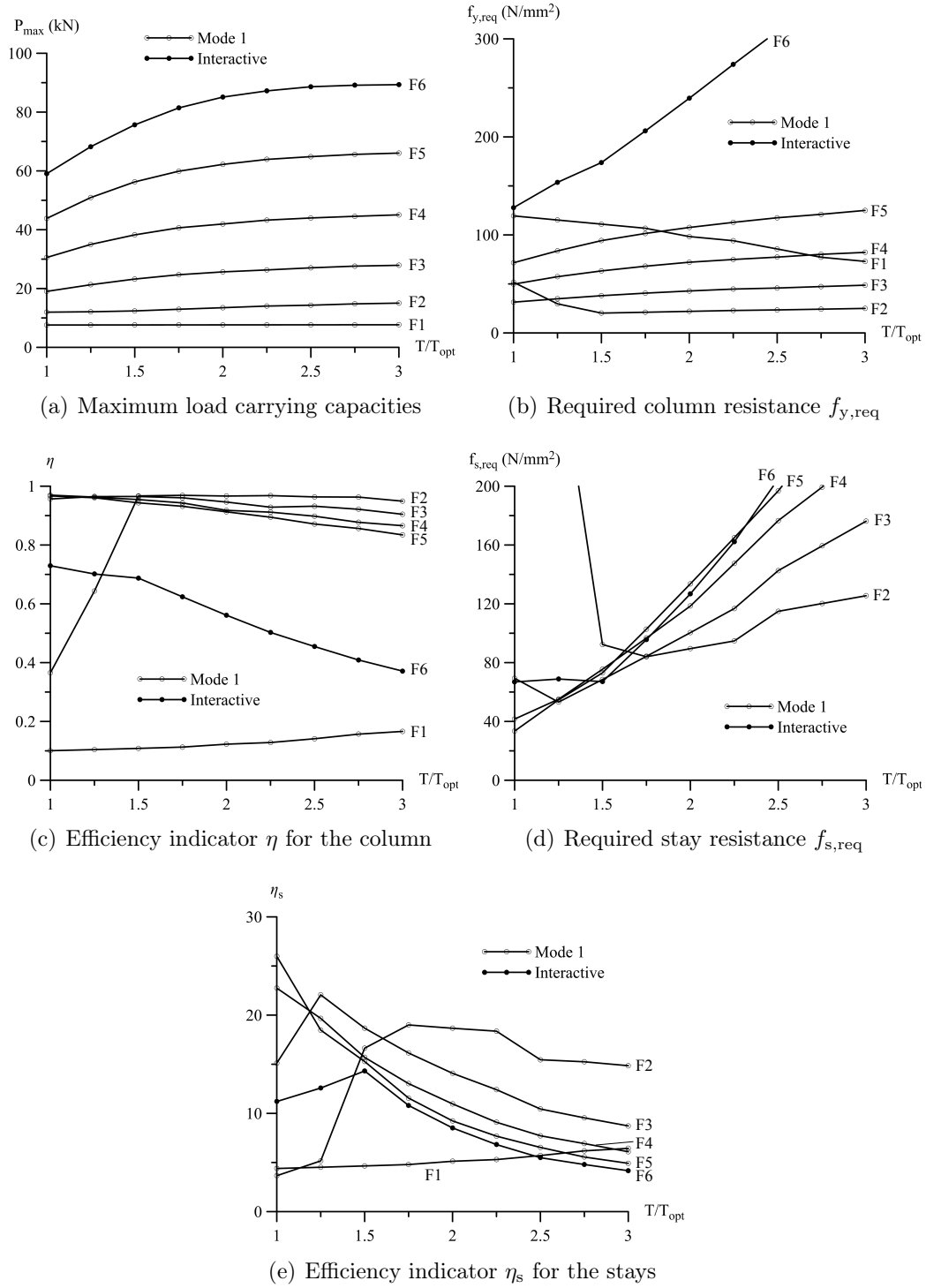


Figure A.2: The maximum load carrying capacities, the required column and stay resistances and the efficiency indicators with the stay diameter ϕ_s varying when the column aspect ratio $2\alpha_m = 0.05$. All results were obtained using the FE model.

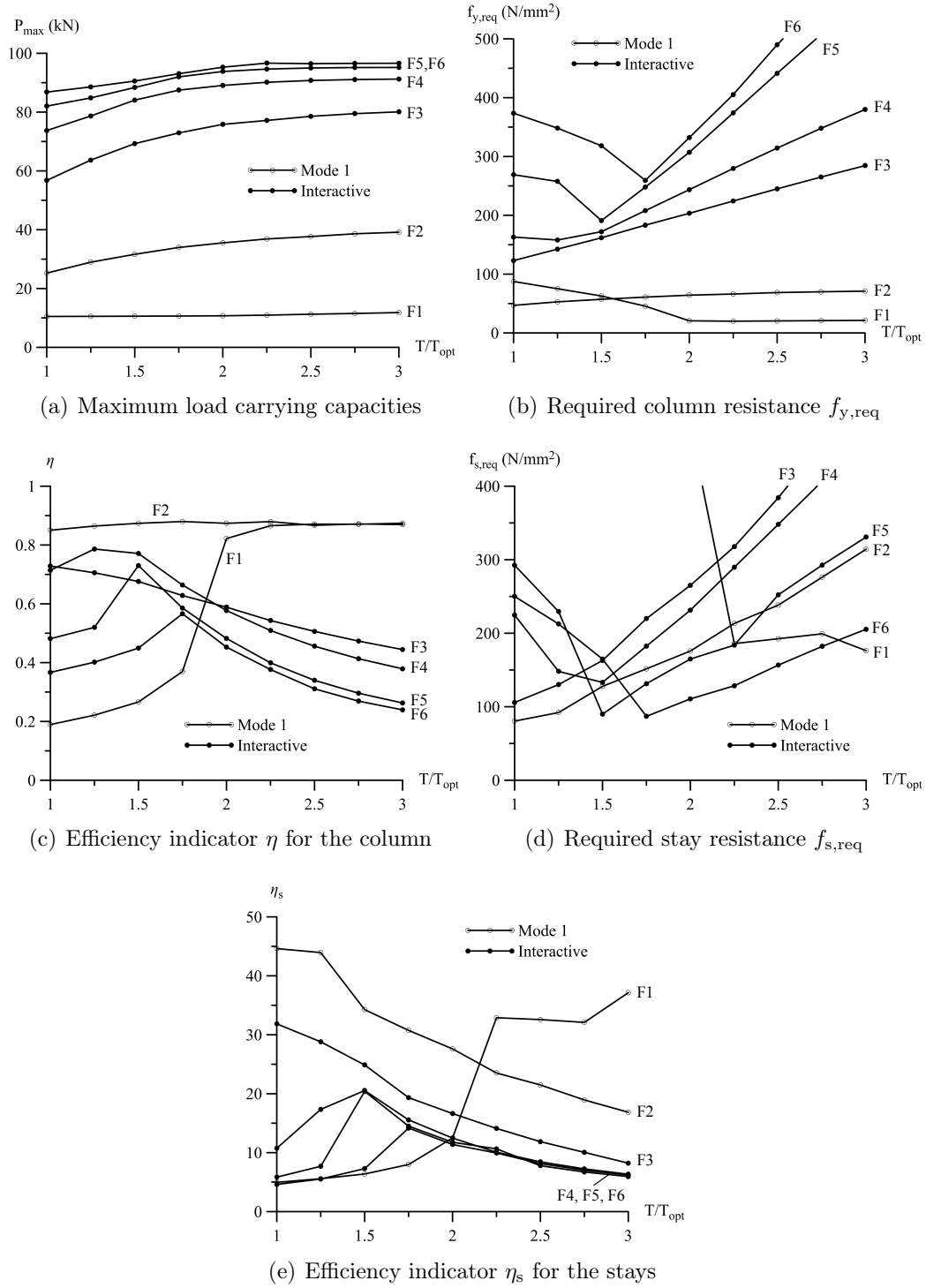


Figure A.3: The maximum load carrying capacities, the required column and stay resistances and the efficiency indicators with the stay diameter ϕ_s varying when the column aspect ratio $2\alpha_m = 0.10$. All results were obtained using the FE model.

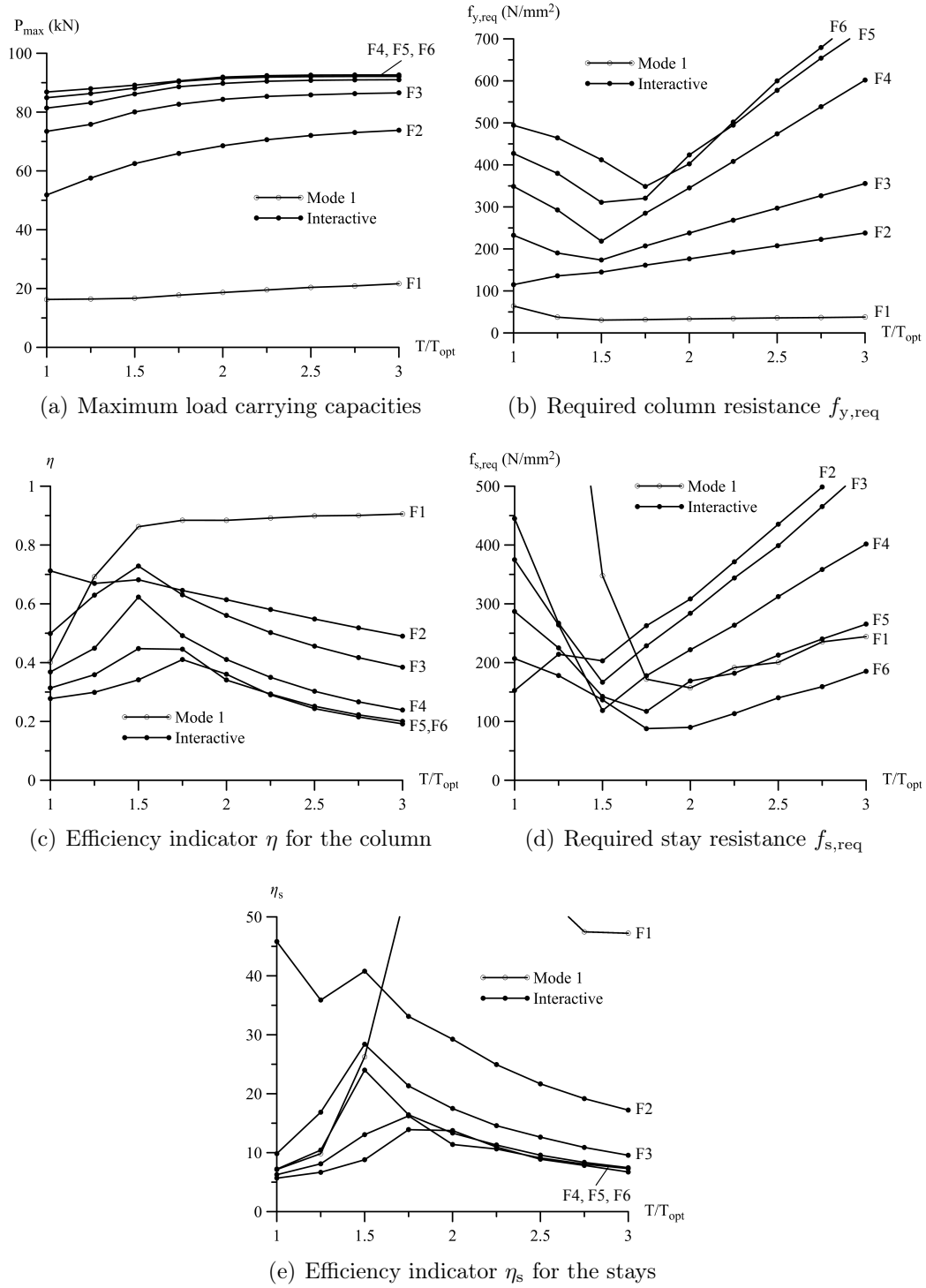


Figure A.4: The maximum load carrying capacities, the required column and stay resistances and the efficiency indicators with the stay diameter ϕ_s varying when the column aspect ratio $2\alpha_m = 0.15$. All results were obtained using the FE model.

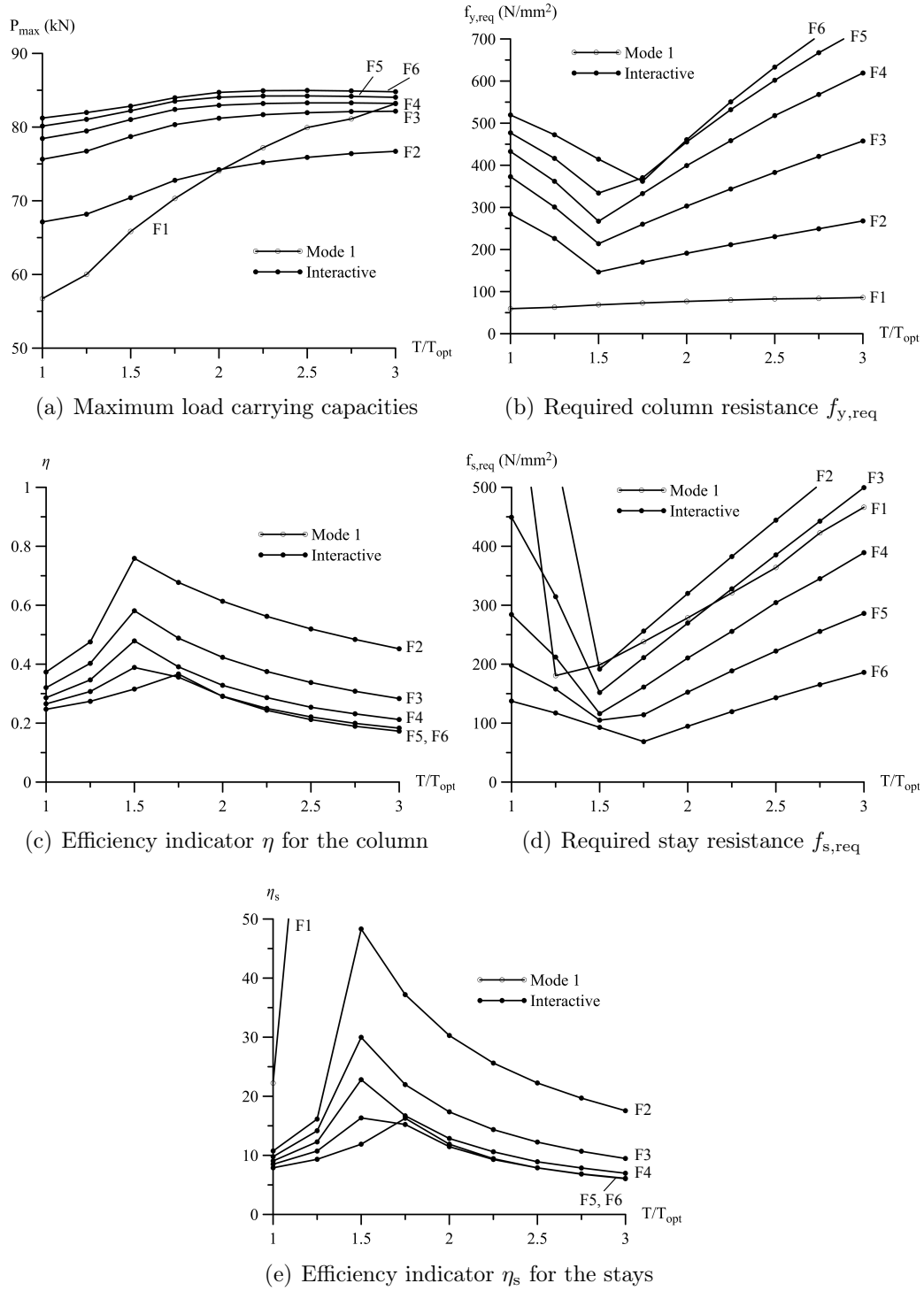


Figure A.5: The maximum load carrying capacities, the required column and stay resistances and the efficiency indicators with the stay diameter ϕ_s varying when the column aspect ratio $2\alpha_m = 0.25$. All results were obtained using the FE model.

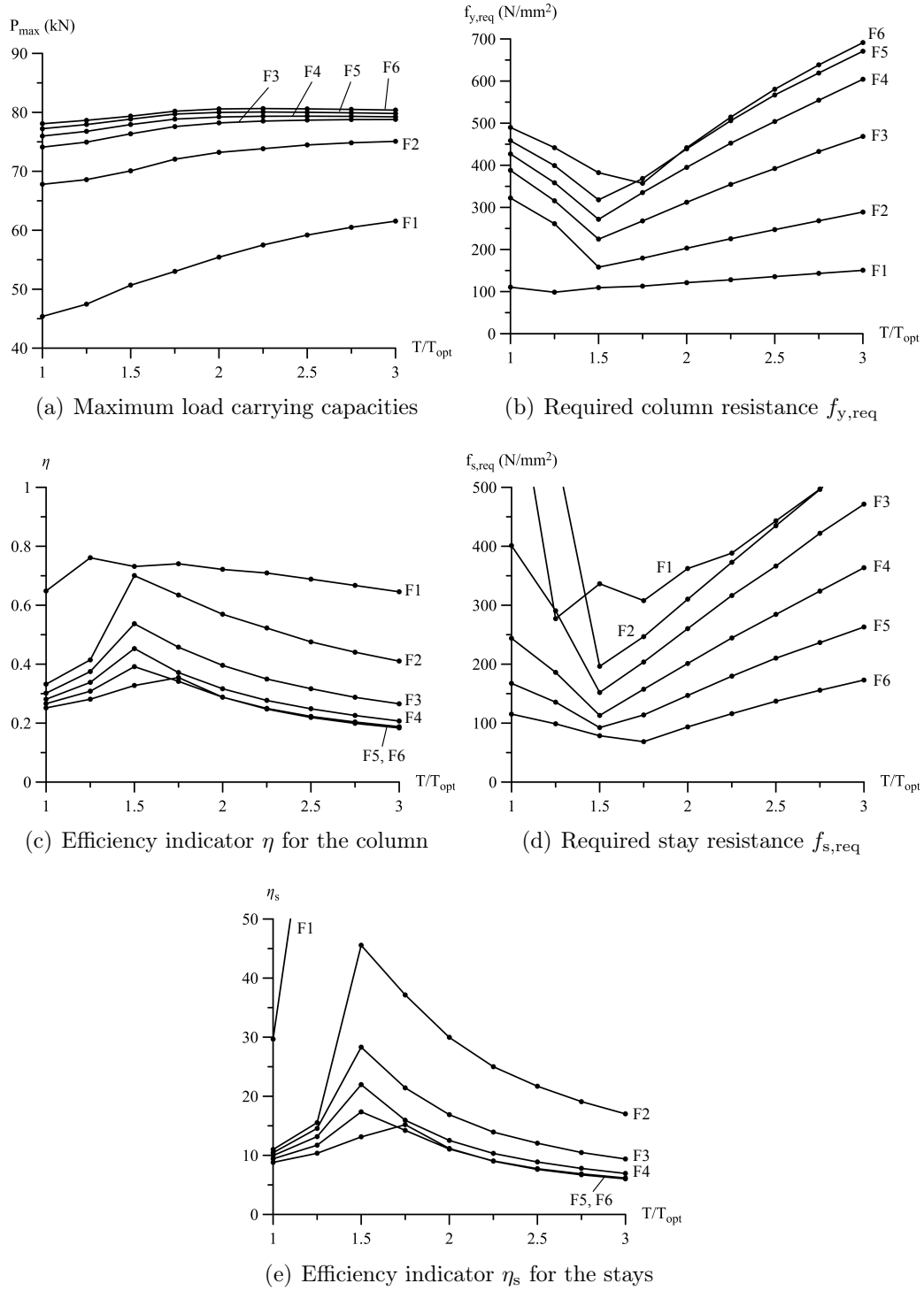


Figure A.6: The maximum load carrying capacities, the required column and stay resistances and the efficiency indicators with the stay diameter ϕ_s varying when the column aspect ratio $2\alpha_m = 0.30$. All results were obtained using the FE model.

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