

Pricing Contingent Claims on Credit and Carbon Single and Multiple Underlying Assets

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by

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To my wife Leticia and my children Enrique and Natalia.

Abstract

This thesis proposes alternative ways to price contingent claims written on portfolios of credit instruments as well as on carbon underlying assets. On the first topic of this research we tackle the pricing of Collateralized Debt Obligations (CDOs) by introducing two different approaches through the application of respectively Johnson SB distributions and entropy optimization principles, in contrast to market standard pricing approaches based on variations of the Gaussian copula model. The relevance of this topic is in line with the events that unfolded during the “credit crunch” of mid-2007 to early 2009, when CDOs made headlines as being responsible for more than \$542 billion in losses through write-downs by financial institutions. On the second topic we propose a pricing methodology for Emission Reduction Purchase Agreement (ERPA) contracts. These are instruments based on carbon as an asset class and created by the emergence of an international carbon market that followed the adoption of the Kyoto Protocol (KP) to the United Nations Framework Convention on Climate Change (UNFCCC) in December 1997. ERPAs are of vital importance to the function of KP’s market mechanisms and the carbon markets at large as they formalize transactions of emissions reduction offsets between sellers and buyers, more specifically transactions involving Certified Emission Reductions (CERs). We propose a pricing methodology based on stochastic modeling of CER volume delivery risk and carbon prices as the two main drivers underlying ERPAs, and apply it to a case study on a run-of-river hydro power CDM project activity in China.

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Publication

One of the topics the research presented in this thesis is coming up in the following publication:

M. Labre and C. Atkinson. On the pricing of emission reduction purchase agreement contracts, *Journal of Energy Markets*, Volume 3, Number 2, Summer 2010.

Abstract

The vital importance of Emission Reduction Purchase Agreement (ERPA) contracts to the carbon market motivates the development of accurate valuation techniques. We propose a pricing methodology based on the formulation of a generic payoff function, with an analyses of underlying drivers (emissions reduction delivery and carbon prices) leading to their stochastic model representation. Although applicable to any emissions reduction markets and standards, the rationale is developed around ERPAs based on delivery of Certified Emission Reduction (CER) certificates from Clean Development Mechanism (CDM) project activities. We apply the methodology to the pricing of ERPAs on run-of-river hydro power CDM project activities in China.

Introduction

This thesis proposes alternative ways to price contingent claims written on portfolios of credit instruments as well as on carbon underlying assets. We start in chapter 1 with an overview of the mathematical tools used in the development of the research on these two topics. More specifically, in section 1.1 we introduce the Johnson family of distributions, which are parametric distributions based on transformations of observed random variables into normally distributed ones. This transformation aspect allows Johnson distributions to be reasonably tractable while being able to handle skewness and fat-tails. Then in section 1.2 we discuss the powerful idea of entropy in the context of uncertainty measure, and introduce optimization principles and techniques widely applied in this research. Chapter 1 concludes with a brief introduction of Goodwin's quadrature formula in section 1.3.

The first topic of this thesis is focused on the pricing of Collateralized Debt Obligations (CDOs) and is described in chapter 2. With a market size reaching over \$2 trillion by the end of 2006, there is no debate about the importance that CDOs have had to the financial industry over the past 6 years. However, CDOs only became known by the general public once they made headlines during the unfolding of the credit crisis from mid-2007 to early 2009, when CDOs were blamed as one of the culprits for the worst financial crisis since the Great Depression of the 1930's. It is estimated that CDOs were responsible for \$542 billion in losses due to write-downs suffered by financial institutions in 2007 alone. Among the many factors that caused this were the pricing and risk assessment involved in CDO positions, clearly observed by the fact that most of the securities marked down were initially given a rating of AAA by one or more of the three internationally recognized credit rating

agencies (Moody's, Standard & Poors and and Fitch).

In this context, in chapter 2 we challenge the standard model paradigm for CDOs based on variations of the application of Gaussian copulas by proposing two alternative approaches to price CDOs. We start chapter 2 in section 2.1 with an explanation of the CDO payoff structure and an overview of the Gaussian copula and its application on pricing CDOs, concluding the section with the formulation of the CDO payoff function. Then in section 2.2 we detail the first alternative approach based on modeling the loss distribution of a CDO's underlying portfolio with JSB distributions, including formulation and calibration of the model. And in section 2.3 we elaborate on the formulation and calibration of CDO pricing with our second approach, based on entropy optimization principles and techniques. Finally, in section 2.4 we show the application of both approaches and compare their results to market prices.

In chapter 3 we discuss the second topic of this thesis based on carbon as an asset class, created by the emergence of an international carbon market following the adoption of the Kyoto Protocol (KP) to the United Nations Framework Convention on Climate Change (UNFCCC) in December 1997. The KP represents the first time governments accepted legally-binding constraints on their greenhouse gas (GHG) emissions in order to mitigate climate change. Under the hypothesis that where emissions reductions are achieved is irrelevant to the planet's climate system, the KP created innovative market-based mechanisms aimed at achieving these reductions where they are least costly. One such mechanism is the Clean Development Mechanism (CDM), with the premise that emissions reduction can be achieved at a lower cost in developing countries than it could be done domestically by a developed country with caps on emissions.

Based on this, chapter 3 proposes a methodology for pricing Emission Reduction Purchase Agreement (ERPA) contracts, which are vital to the function of the CDM and the carbon market at large. With the CDM, participants may claim Certified Emission Reductions (CERs) through the implementation of projects that reduce emissions below a certain baseline scenario. These CERs are issued by the UNFCCC to project participants through a thorough process that starts with the registration of emissions reduction projects and ends

with emissions reduction monitoring, verification and certification. Once issued, CERs can be sold to governments of countries with binding targets on emissions under the KP or to industry participants under an emissions cap-and-trade system established by their respective governments (e.g., European Union Emissions Trading Scheme). These transactions involving CERs are formalized by an ERPA.

Thus, we start chapter 3 in section 3.1 with a detailed explanation of the CDM project cycle and draw on a probability tree representation of the risks in CER delivery through the various CDM project stages. In section 3.2 we propose a pricing methodology based on the formulation of a generic payoff function and the analysis of underlying drivers such as delivery risk and carbon prices. More specifically, in section 3.2.3 we model the delivery dynamics with JSB distributions and calibrate it by applying entropy optimization techniques, and in section 3.2.4 a Bounded Geometric Brownian Motion (GBGM) is proposed for carbon prices dynamics. Finally, in section 3.3 we establish a pricing algorithm and apply the models and techniques proposed with a case study on a run-of-river hydro power CDM project activities in China.

We conclude by revisiting the two topics of this thesis through an overview of the research performed and results achieved. In addition, we propose future research that builds on the foundations established on these topics.

Chapter 1

Mathematical Tools

Introduction

In this chapter we elaborate on the mathematical tools used on the pricing of the contingent claims discussed in the chapters that follow. We start by introducing the Johnson family of distributions in section 1.1, followed by a thorough discussion of entropy optimization principles and techniques in section 1.2. We conclude with a brief exposition of Goodwin's quadrature formula in section 1.3.

1.1 The Johnson family of distributions

Johnson distributions originally come from the field of statistics and over time have been applied to a wide variety of applications in analysis and modeling of sample data in various disciplines. As pointed out by Johnson [25], although the normal distribution has always played a central role in statistical analysis, it is apparent that it cannot adequately represent many of the distribution laws observed in random numbers from natural and economic processes. Given the properties of the normal distribution and its mathematical tractability, it became natural to consider transformations of certain probability distributions into

the normal distribution. The Johnson system appears in this context, where parametric transformations of an observed random variable into a normally distributed one allows for probability density functions that are reasonably tractable while able to handle skewness and fat-tails. Johnson [25] first illustrated the transformation of a random variable x to normality in the form

$$z = g(x) \sim \Phi(0, 1),$$

where Φ is the cumulative normal distribution function, z is standard normally distributed and g a transformation function.

In principle it is possible to find one single function that can exactly match that condition. However, as pointed out by Elderton and Johnson [10], this function would be so complicated that would defeat the purpose of the transformation. Elderton and Johnson [10] mentioned that this should be a simple function expressed in terms of a few arbitrary parameters as well as enable a wide variety of distributions to be represented. Hahn and Shapiro [18] show the general form of this transformation with four parameters to be

$$z = a + bg(c, d, x) \sim \Phi(0, 1), \quad (1.1)$$

where a and b are parameters defining the shape of the transformation, while g should be a monotonic non-decreasing function of x and must be $(-\infty, +\infty)$ given the domain of z . As described in Johnson [25], three alternative forms of families for g were proposed as

$$g_{SL}(c, d, x) = \log \left(\frac{x - c}{d} \right), \quad (1.2a)$$

$$g_{SU}(c, d, x) = \sinh^{-1} \left(\frac{x - c}{d} \right), \quad (1.2b)$$

$$g_{SB}(c, d, x) = \log \left(\frac{x - c}{c + d - x} \right), \quad (1.2c)$$

where c and d are respectively location and scale parameters. Elderton and Johnson [10] later showed that each of these transformations translate respectively into the probability density functions

$$f_{SL}(x) = \frac{b}{\sqrt{2\pi}(x-c)} \exp \left\{ -\frac{1}{2} \left[a + b \log \left(\frac{x-c}{d} \right) \right]^2 \right\}, \quad (1.3a)$$

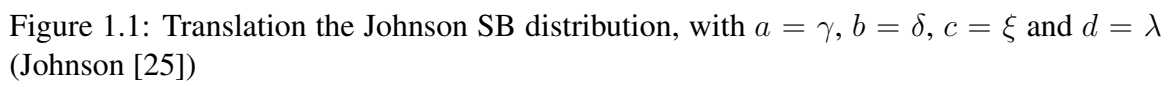
$$f_{SU}(x) = \frac{b}{d\sqrt{2\pi}} \left[1 + \left(\frac{x-c}{d} \right)^2 \right]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \left[a + b \sinh^{-1} \left(\frac{x-c}{d} \right) \right]^2 \right\}, \quad (1.3b)$$

$$f_{SB}(x) = \frac{b}{\sqrt{2\pi}} \frac{d}{(x-c)(c+d-x)} \exp \left\{ -\frac{1}{2} \left[a + b \log \left(\frac{x-c}{c+d-x} \right) \right]^2 \right\}, \quad (1.3c)$$

where the translation caused by applying one of the relationships in (1.2) can be seen in Figure 1.1.

The transformation defined by (1.2a) resulting in the probability density (1.3a) is known as the *Johnson SL* (JSL) system which represents a shifted log-normal distribution, where the parameter domain is $c \leq x < +\infty$, $-\infty < a < +\infty$, $0 < b < +\infty$, $-\infty < c < +\infty$, $0 < d < +\infty$. On the other hand, the transformation in (1.2b) is known as *Johnson SU* (JSU) system as x is unbounded and the probability density of (1.3b) departs from log-normality shapes depending on the values of its parameters, with domain defined as $-\infty < x < +\infty$, $-\infty < a < +\infty$, $0 < b < +\infty$, $-\infty < c < +\infty$, $0 < d < +\infty$. Finally in the *Johnson SB* (JSB) transformation, x is bounded and defined within the domain $c \leq x \leq c+d$, with $-\infty < c < +\infty$, $0 < d < +\infty$, $-\infty < a < +\infty$ and $0 < b < +\infty$, making it a powerful choice for representing a variety of bounded random variables.

Note that the probability density for x maps into the normal distribution given the transformation in (1.1). This yields the cumulative distribution function of the functions in (1.3) as



simply

$$\mathcal{J}_{SL}(x) = \Phi \left[a + b \log \left(\frac{x - c}{d} \right) \right], \quad (1.4a)$$

$$\mathcal{J}_{SU}(x) = \Phi \left[a + b \sinh^{-1} \left(\frac{x - c}{d} \right) \right] \text{ and} \quad (1.4b)$$

$$\mathcal{J}_{SB}(x) = \Phi \left[a + b \log \left(\frac{x - c}{c + d - x} \right) \right]. \quad (1.4c)$$

Using one of the three Johnson families to fit the behavior of random variables implies an approximation of the real probability distribution by its first four moments, due to only four parameters (or degrees of freedom) defining each of those families. This is an approximation with higher moments (skewness and kurtosis) as compared to assuming pure normality, while still keeping tractability for analytical solutions. If β_1 and β_2 are defined as

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} \quad (1.5a)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \quad (1.5b)$$

where μ_2 , μ_3 and μ_4 are respectively the second, third and fourth moments about the mean of a probability distribution, Hahn and Shapiro [18] show that the three Johnson transformations accommodate a broad scope of possible distribution shapes, as illustrated graphically in Figure 1.2.

Another well known system of transformation was proposed early on by Pearson [40], where a Pearson density $f(x)$ is defined to be any valid solution to the differential equation

$$\frac{df(x)}{dx} = \frac{x - \alpha_3}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}. \quad (1.6)$$

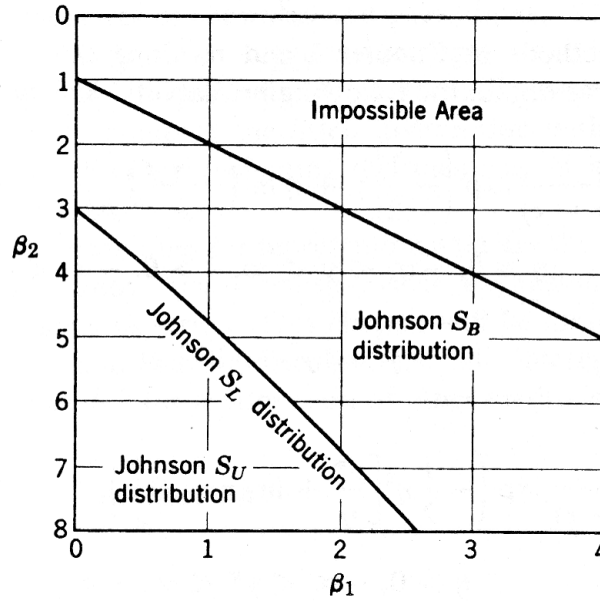


Figure 1.2: Space of distributions that can be represented by the Johnson family (Hahn and Shapiro [18])

This is also able to represent a large number of distribution families as showed in Figure 1.3. However, As pointed out by Hahn and Shapiro [18], the procedures for fitting Pearson distributions are lengthy since each type requires solution of a different set of differential equations. From this perspective we favor the use of the Johnson family due to its simplicity, tractability and ease of fitting extensively demonstrated in the literature. We are particularly interested in the transformation JSB defined in (1.2c), given the highly non-normal and bounded nature of the random variables in the applications of this research.

1.2 Entropy optimization

Although originated in the literature on thermodynamics, the term *entropy* has found its way into almost all disciplines. This is mainly due to its association with the concept of information introduced in an influential article by Shannon [46], which established the foundations of the field of information theory. In other to give clarity to the concept of entropy, it is useful to make a parallel to the intuitive understanding of entropy as a measure

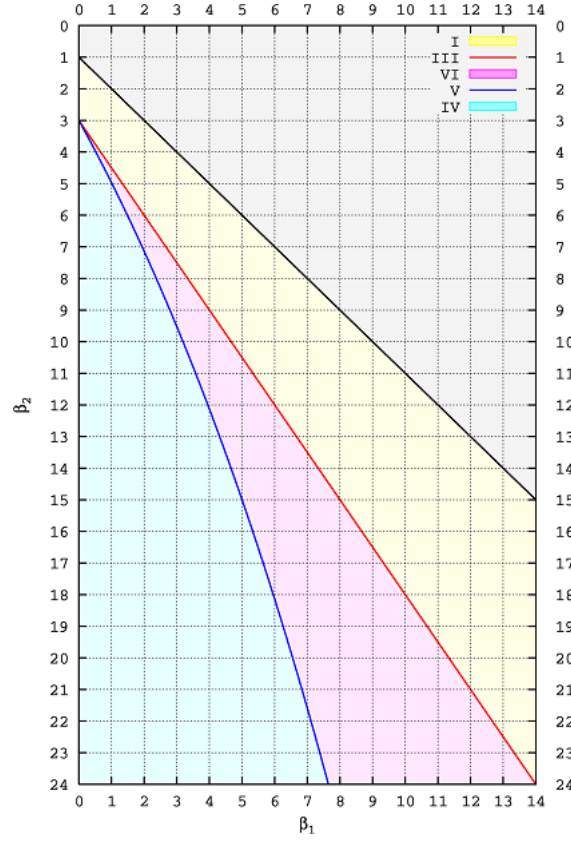


Figure 1.3: Space of distributions that can be represented by the Pearson family

of uncertainty. In our case we are interested in probabilistic uncertainty, i.e., from an experiment with n possible outcomes with probabilities $p_1, p_2, \dots, p_n$ where $p_1 \geq 0, p_2 \geq 0, \dots, p_n \geq 0$ and

$$\sum_{i=1}^n p_i = 1. \quad (1.7)$$

A classic example given in Kapur and Kesavan [29] illustrates the idea that the uncertainty of the probability distribution (0.5,0.5) for a coin tossing is much higher than the one of the probability distribution (0.00001,0.99999) of winning or not winning a lottery, and therefore the entropy in the former is much higher than in the latter. This example reveals the fact that the distribution of tossing a coin has the highest possible entropy among all distributions defined by only two outcomes, given that any other distribution different

from (0.5,0.5) would decrease the uncertainty by increasing the likelihood of one of the outcomes by meeting condition (1.7). It generally points out that knowledge of the level of uncertainty about the underlying event is crucial in order to assess its outcomes. The most well known measure of uncertainty was introduced by Shannon [46], who proposed a measure of uncertainty of a probability distribution given by

$$\mathcal{S} = - \sum_{i=1}^n p_i \log p_i \quad (1.8)$$

and called it *entropy* as it is the same expression for entropy in thermodynamics.

An important idea involving this entropy measure is about estimating probability distributions by reducing the uncertainty under the available information. For instance, consider two matches have already been played in a five-match playoff for the title of a basketball tournament, where we wish to assess the odds of teams **A** or **B** to be winners in the contest. If we have absolutely no information about the two teams and the results of the two previous playoff matches, the best estimate we can possibly make is 50%-50% which happens to yield the highest entropy if no available information. Any odds different than this represent a bias of possible outcomes, and one would feel more comfortable placing a bet on these odds as there is no justifiable reason to choose any other. On the other hand, if we were aware that team **A** was the winner in the first two matches, the odds would change to 75%-25% given **A** becomes champion by another single victory in any of the next 3 matches while **B** must win the next three matches. Conditional on this new information, 75%-25% seems to offer the highest entropy level given higher entropic 50%-50% odds would represent a biased choice. Now consider that we know that three of the best players of team **A** are either injured or suspended and unable to play the next three matches, two of which will be played on team **B**'s home ground. In this case it is unlikely 75%-25% is right and unbiased odds might be back to 50%-50%.

This example shows how the arrival of new information changes perception of the probabilities that represent the outcome of certain events. Furthermore, the example also introduces the idea of *entropy maximization* by demonstrating that the fair or unbiased choice for these

probabilities is the one that yields the highest entropy conditional on the available information. It may sound paradoxical that we achieve our goal of assessing appropriate probability distributions by maximizing uncertainty. The idea has parallel with *Laplace's principle of insufficient reason*, which argues that if we have no information except that each $p_i \geq 0$ and the sum of the probabilities is unity, we should choose the uniform distribution since we have no reason to choose any other. In this sense, being most uncertain means being nearest to the uniform distribution (the one with the highest entropy) but this is exactly the path to a least biased assessment of a probability distribution. Out of an infinite number of possibilities, maximizing the entropy enables us to choose one for the simple reason that the use of any other implies assuming information that has not been given.

These arguments provide the background for introducing the *Maximum Entropy Principle* (MaxEnt) originally proposed in Jaynes [23] and [24], which provided a quantitative technique for assigning probabilities based on the commonsensical concept of using only the information provided. In his two seminal articles, Jaynes proposed that a least biased probability distribution of a random variable x_i for $\{i = 1, \dots, n\}$ is achieved by maximizing Shannon's measure (1.8) conditional to a normalization constraint (1.7) and to a set of functional form constraints written as

$$\sum_{i=1}^n h_k(x_i)p_i - c_k = 0, \{k = 1, 2, \dots, m\}, \quad (1.9)$$

where $h_k(x_i)$ are constraint functions and c_k are constants for $\{k = 1, 2, \dots, m\}$.

As shown by Kapur and Kesavan [29], Shannon's measure is a concave function of $\{p_1, \dots, p_n\}$, which guarantees the existence and uniqueness of a probability distribution that maximizes the entropy. In order to find the MaxEnt probability distribution, we first write the Lagrangian equation as

$$\mathcal{L} = - \sum_{i=1}^n p_i \log p_i - (\lambda_0 - 1) \left(\sum_{i=1}^n p_i - 1 \right) - \sum_{k=1}^m \lambda_k \left(\sum_{i=1}^n h_k(x_i)p_i - c_k \right), \quad (1.10)$$

where $\{\lambda_0, \lambda_1, \dots, \lambda_m\}$ are $m + 1$ Lagrange multipliers corresponding to the $m + 1$ constraints of (1.7) and (1.9). Now we make

$$\frac{\partial \mathcal{L}}{\partial p_i} = 0 \Rightarrow -\log p_i - \lambda_0 - \sum_{k=1}^m \lambda_k h_k(x_i) = 0,$$

which results in

$$p_i = \exp \left[-\lambda_0 - \sum_{k=1}^m \lambda_k h_k(x_i) \right].$$

The multipliers are determined by substituting p_i into (1.7) and (1.9), i.e.,

$$\sum_{i=1}^n \exp \left[-\lambda_0 - \sum_{k=1}^m \lambda_k h_k(x_i) \right] = 1,$$

$$\sum_{i=1}^n h_k(x_i) \exp \left[-\lambda_0 - \sum_{k=1}^m \lambda_k h_k(x_i) \right] - c_k = 0, \{k = 1, 2, \dots, m\}.$$

The former equation gives

$$\exp(\lambda_0) = \sum_{i=1}^n \exp \left[-\sum_{k=1}^m \lambda_k h_k(x_i) \right],$$

which is substituted in the latter equation to yield

$$\sum_{i=1}^n (h_k(x_i) - c_k) \exp \left[-\sum_{j=1}^m \lambda_j h_j(x_i) \right] = 0, \{k = 1, 2, \dots, m\}. \quad (1.11)$$

The relation above represents a system of m nonlinear equations that can be solved for m Lagrange multipliers $\{\lambda_1, \dots, \lambda_m\}$, where we can write p_i as

$$p_i = \frac{\exp \left[- \sum_{k=1}^m \lambda_k h_k(x_i) \right]}{\sum_{j=1}^n \exp \left[- \sum_{k=1}^m \lambda_k h_k(x_j) \right]}. \quad (1.12)$$

Now suppose we have information in the form of a *prior* probability distribution q_i , which may be based on intuition or previous experience with random variable x_i . Kullback and Leibler [31] introduced the measure

$$\mathcal{K} = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \quad (1.13)$$

which can reasonably be used as a measure of divergence of p_i in respect to q_i , and for that reason called relative entropy measure. We verified before that from MaxEnt and in the absence of (1.9) one chooses the uniform distribution as the least biased distribution. This principle also applies to a given prior distribution, as there would no reason to choose any other distribution (including the uniform distribution) for an estimate of the least biased distribution of x_i in the absence of any other information. In addition to further available information in the form of functional constraints, Kullback and Leibler [31] proposed the *Minimum Entropy Principle* (MinEnt) which consists in finding the unbiased distribution of x_i by minimizing (1.13) given a prior distribution q_i and conditional to the normalization (1.7) and the functional constraints (1.9). As before, Kapur and Kesavan [29] showed that (1.13) is a convex function, therefore having a global minimum and guaranteeing that p_i exists and is unique. In this case, we write the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \sum_{i=1}^n p_i \log p_i - \sum_{i=1}^n p_i \log q_i - (\lambda_0 - 1) \left(\sum_{i=1}^n p_i - 1 \right) \\ & - \sum_{k=1}^m \lambda_k \left(\sum_{i=1}^n h_k(x_i) p_i - c_k \right), \end{aligned} \quad (1.14)$$

which gives

$$\frac{\partial \mathcal{L}}{\partial p_i} = 0 \Rightarrow \log p_i - \log q_i - \lambda_0 - \sum_{k=1}^m \lambda_k h_k(x_i) = 0.$$

The same steps are followed as with MaxEnt to yield

$$\sum_{i=1}^n q_i (h_k(x_i) - c_k) \exp \left[- \sum_{k=1}^m \lambda_k h_k(x_i) \right] = 0, \{k = 1, 2, \dots, m\} \quad (1.15)$$

as the system of m nonlinear equations that can be solved for the Lagrange multipliers, and

$$p_i = q_i \frac{\exp \left[- \sum_{k=1}^m \lambda_k h_k(x_i) \right]}{\sum_{j=1}^n q_j \exp \left[- \sum_{k=1}^m \lambda_k h_k(x_j) \right]} \quad (1.16)$$

as the unbiased probability distribution with $\{\lambda_1, \dots, \lambda_m\}$. Note that MinEnt can be interpreted as a generalization of the MaxEnt Principle, where minimizing the relative entropy (1.13) for the same set of constraints is equivalent to maximizing the entropy in (1.8). This is easily observed in the case of the prior distribution being the uniform distribution, i.e.,

$$\begin{aligned} \mathcal{K} &= \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \\ &= \sum_{i=1}^n p_i \log \frac{p_i}{(1/n)} \\ &= \log n - \left(- \sum_{i=1}^n p_i \log p_i \right) \\ &= \log n - \mathcal{S}. \end{aligned}$$

As detailed by Kapur and Kesavan [29], the MaxEnt and MinEnt principles can be written in continuous form, which is very useful for applications in this research. First, Jayne's MaxEnt can be changed to maximizing

$$\mathcal{S} = - \int_{X_A}^{X_B} f(x) \log f(x) dx, \quad (1.17)$$

conditional to the normalization constraint

$$\int_{X_A}^{X_B} f(x) dx = 1 \quad (1.18)$$

and functional constraints

$$\int_{X_A}^{X_B} h_k(x) f(x) dx - c_k = 0, \{k = 1 \dots N_k\}. \quad (1.19)$$

Similarly, the MinEnt by Kullback and Leibler then consists in minimizing

$$\mathcal{K} = \int_{X_A}^{X_B} f(x) \log \frac{f(x)}{g(x)} dx \quad (1.20)$$

conditional to (1.18) and (1.19).

1.3 Goodwin's quadrature formula

A useful quadrature formula is introduced by Goodwin [16], which proposes a method to evaluate integrals of the form

$$\int_{-\infty}^{+\infty} f(x) e^{-x^2} dx = h \sum_{n=-\infty}^{+\infty} f(nh) e^{-n^2 h^2} - \epsilon(h). \quad (1.21)$$

From this equation, n intervals of size h define the discretization of the domain of x and $\epsilon(h)$ is the error term. Goodwin indicates in his article that the error is only likely to be underestimated when $f(x)$ tends rapidly to infinity with x . He shows that using this

formula with functions such as $f(x) = x^2$ is still quite accurate, where in such cases the error term tends rapidly to zero with h . We observed negligible values for the error with $n \geq 100$ and $h \leq 0.03$, an inexpensive parameter domain in terms of computational effort. Given the type of integrals involved throughout this research, we make extensive use of Goodwin's quadrature formula.

Conclusion

In this chapter we discussed the mathematical tools used on the pricing of the contingent claims discussed in the chapters that follow. In section 1.1 we introduced the Johnson family of distributions, followed by the discussion of entropy optimization principles and techniques in section 1.2. We concluded with a brief introduction to Goodwin's quadrature formula in section 1.3. The application of these tools become clear in the chapters that follow.

Chapter 2

Pricing Collateralized Debt Obligations (CDOs)

Introduction

The importance that Collateralized Debt Obligations (CDOs) have had to the financial industry over the past 6 years is unquestionable. By 2005 the CDO market had reached over \$1.5 trillion and is estimated to have topped \$2 trillion by the end of 2006,* after when it started its decline due to the unfolding of the credit crisis from mid-2007 to early 2009. Once a money making machine of the investment banking industry, CDOs are one of the main culprits for the worst financial crisis since the Great Depression of the 1930's, being responsible for \$542 billion in losses due to write-downs suffered by financial institutions in 2007 alone.† Among the many factors involved in its failure is pricing and risk assessment, clearly observed through the fact that most of the securities marked down were initially given a rating of AAA by one or more of the three internationally recognized credit rating

*Report Published by research firm Celent available at <http://reports.celent.com/PressReleases/20051031/CDOMarket.htm>.

†CreditFlux Newsletter of January 2008, available at <http://www.creditflux.com/Newsletter/January-2008/>.

agencies (Moody's, Standard & Poors and and Fitch).

Although new issuances of CDOs are nowadays limited, the pricing and risk management of legacy securities and agreements remain a topic of central importance to the investment banking industry as well as the governments that ultimately intervened with bail out packages. From this perspective, this chapter offers viable alternatives to the standard market models through two top-down approaches for pricing CDOs, challenging main stream research and market practices based on bottom-up approaches such as the one-factor Gaussian copula. First we introduce the reader to the different types of CDOs as well as their payoff structure and features. This is followed by a review of the one-factor Gaussian copula and its adjustment by practitioners via base correlations, widely recognized as the market standard for pricing CDOs. We then elaborate on the CDO payoff function and apply it in both the Johnson SB and the Entropy approaches that follow. We formulate the model and discuss calibration techniques in each of these approaches, after which we present results through a practical application.

2.1 The Collateralized Debt Obligation (CDO)

2.1.1 Introduction

The core principle behind CDOs revolves around diversification through portfolios. These are structures written on the credit worthiness of obligors in an underlying portfolio, where spreading the risk of credit default across a large number of names significantly reduces the risk of an investment by lending to be wiped out. In theory, this should offer a more stable risk versus return profile, where a typical cash CDO* involves the packaging of various fixed income securities in a Special Purpose Vehicle (SPV) or Structured Investment Vehicle (SIV). These securities are typically corporate and sovereign bonds and loans in case of institutional portfolios, or individual loans and commercial obligations for CDOs written on Mortgage-Backed and Commercial-Backed Securities (respectively MBS and

*Cash CDOs represent the majority of the transactions involved in the recent credit crisis.

CBS). The SPE or SIV buys these underlying securities for the purpose of using their cash flows to back the issuance of specific notes or bonds, which are divided into a number of tranches with different claims on the principal and interest generated by the portfolio.

A more detailed discussion about CDO's different types, structures and characteristics can be found in J.P. Morgan Securities [27], Bank of America Securities [4] and The Royal Bank of Scotland [49]. For the purpose of our research, we are particularly more interested in synthetic CDOs written on portfolios of institutional obligors. A synthetic CDO behaves mathematically in exactly the same way as a cash CDO, but its underlying portfolio is based on credit default swaps (CDSs)* instead of corporate or sovereign bonds. Synthetic CDOs enjoyed high transactional volume during the credit derivatives boom due to their CDS portfolios, given CDSs on the larger obligors were extremely liquid and provided a high degree of price discovery. In addition, CDS prices reflect a clean measure of default expectations as viewed by the market, while bonds and loans are still shaded by interest rates movements.

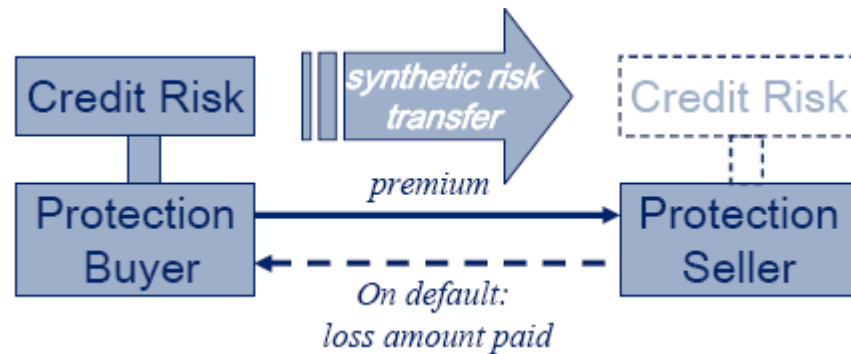


Figure 2.1: Credit Default Swap (CDS) diagram (The Royal Bank of Scotland [49])

In brief, a typical CDS is a swap composed of a *premium* and a *loss* leg where a protection buyer normally holds a defaultable security (e.g., a corporate bond) and buys protection against credit default by paying a regular fee to a protection seller, as illustrated by the diagram in Figure 2.1. In the case of a credit event, the loss leg is triggered and the protection seller pays the protection buyer the notional of the deal in exchange for the defaulted

*The reader is referred to The Royal Bank of Scotland [49] and J.P. Morgan Securities [26] for a detailed discussion of how credit default swaps work.

security, after which the agreement ceases to exist. The protection seller is left with the underlying defaulted security, from which part of the face value can be recovered after all obligations with equity holders and senior debt holders are met by the obligor. Note that the definition of credit events can be different in each CDS agreement, but typically follows the ISDA (International Swaps and Derivatives Association, Inc.) standards. Also, the *fair* or *par* spread* of a CDS is computed as the spread that equalizes the present value of the loss leg (expected cash flows generated by a credit event) to the present value of the cash flows from the premium leg (regular spread payments). Market quotes of spreads of CDSs for various obligors are widely available at par for a number of maturities, and for that reason CDSs have become basic building blocks for the pricing of all credit derivatives through the bootstrapping of probabilities of survival (or default) from market prices.

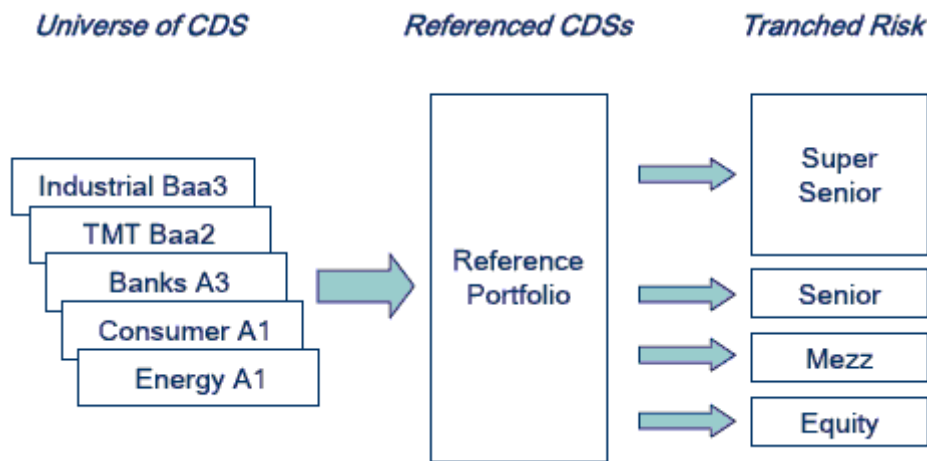


Figure 2.2: Collateralized Debt Obligation (CDO) diagram (The Royal Bank of Scotland [49])

Similarly to a CDS, the payoff agreement of a synthetic CDO provides protection against credit events in the underlying portfolio in exchange for a regular fee payment. A CDO in its synthetic form is also a swap agreement, as showed by the diagram in Figure 2.2. However, the main difference is that synthetic CDOs consist of a waterfall structure to divide the potential losses in the portfolio into specific levels of risk. It is divided into swaps with different subordination levels, where the protection leg is triggered only if the

*Note that the term *spread* is the market standard, but it may be misleading as in the case of CDSs it represents a fixed fee payment as opposed to an actual spread over the prevailing interest rates.

cumulative loss given default in the underlying portfolio reaches the *attachment* while being limited by the *detachment* points of the structure. The attachment point defines the level of cumulative loss from which the specific CDO tranche protection leg starts paying off protection buyers for defaults in the portfolio. The notional of the CDO is dynamically reduced by the amount of the loss as defaults continue beyond the attachment point and until they reach the detachment point. The detachment point is therefore the level of the cumulative loss in the underlying portfolio in which the notional of the CDO is zero, and therefore the agreement ceases to exist.

In order to give an example, let the attachment point $X_L = 2\%$ and the detachment point $X_U = 5\%$ define a synthetic CDO tranche based on a portfolio of 100 equally weighted CDSs, with notional of €400 thousand each. Based on this, the total CDO portfolio notional is €40 million, while the the notional of the tranche is $0.05 - 0.02 = 0.03$ times €40 million, or €1.2 million. As the protection starts at 2% and assuming a recovery rate of 50%, it takes exactly 4 credit events in the portfolio for the attachment point to be reached. On the fifth credit event, the notional of the tranche is reduced to €1 million (given €200 thousand is lost and another €200 thousand is recovered) while the defaulted obligor is written off from the portfolio. The protection buyer then receives the full notional of the defaulted CDS (i.e., €400 thousand) while the protection seller is left with whatever is recoverable (i.e., €200 thousand) from the defaulted security*. This process goes on for each credit event, until five extra defaults wipe out the tranche.

Note that a portfolio of CDSs underlying a CDO structure resembles the assets of a balance sheet, while the tranching structure replicates the liabilities. Thus, by convention CDO tranches with subordination points between 0% and a certain $x\%$ are called *equity* tranches, just as in the case of the shareholder's equity piece of a corporate balance sheet. Any CDO tranche defined between $x\%$ and a detachment point lower than 100% is called *mezzanine* tranche, while *senior* or *super senior* tranches are the one defined from any point in the structure up to 100%.

*This is the case for an agreement with physical settlement, while cash settled agreements are also common.

Tranche	CDX Investment Grade North America Series 3		iTraxx Europe Series 2	
	Lower- Upper strike	Upfront / Running Spread (bp)	Lower-Upper Strike	Upfront / Running Spread (bp)
Equity	0-3%	37.125% + 500	0-3%	24.25% + 500
Junior Mezzanine	3-7%	259.5	3-6%	137.5
Senior Mezzanine	7-10%	101.0	6-9%	47.5
Senior	10-15%	38.5	9-12%	34.5
Super Senior	15-30%	11.5	12-22%	15.5

Figure 2.3: Standard tranches for iTraxx and CDX indices

The markets for synthetic CDOs have developed in such a way that standard portfolios and tranches have emerged and are actively quoted by market participants. There are two main standard portfolios or indices, the iTraxx Europe and the CDX North America. The iTraxx is composed of 125 investment grade equally weighted European obligors while the CDX is based on an equal number of North American obligors. Both iTraxx and CDX indices have five standard tranches quoted in the market, as showed by the sample in Figure 2.3. The pricing of such standard tranches is currently based on the Gaussian copula with base correlations, as per discussion that follows.

2.1.2 The Gaussian copula model

The *Gaussian copula* model emerged in late 1990's as the market standard for pricing portfolio credit derivatives, having become well known through the articles [34] and [35] by David Li. The model is not new, it is in fact a simplified form of earlier one-factor models by Vasicek [50] and almost identical to the original CreditMetrics model introduced by Gupton et al [17]. However its breakthrough was the analogy established between credit risk management techniques and portfolios of credit securities underlying market instruments, with its consequent application to price portfolio credit derivatives.

The idea consists of a random variable time-until-default τ , which defines the expected length of time it takes for an obligor to default. Each obligor m is then defined by its marginal cumulative survival probability distribution $F_m(T) = 1 - P[\tau_m \leq T]$, where T

is the instrument's horizon (typically the expiry of the agreement). Finally, a copula function is introduced to establish the dependence between each marginal distribution $F_m(T)$, creating a full multivariate or joint distribution of the portfolio. This is represented from Li [34] by

$$P[\tau_1 \leq T, \tau_2 \leq T, \dots, \tau_M \leq T] = C(u_1, u_2, \dots, u_M), \quad (2.1)$$

where

$$u_m = 1 - F_m(T) = P[\tau_m \leq T] \quad (2.2)$$

are uniformly distributed random variables, resulting in

$$P[\tau_1 \leq T, \tau_2 \leq T, \dots, \tau_M \leq T] = C(F_1(T), F_2(T), \dots, F_M(T)). \quad (2.3)$$

The copula function chosen by Li was the Gaussian copula which is essentially equivalent to the cumulative normal multivariate probability distribution function, i.e.,

$$C(F_1(T), F_2(T), \dots, F_M(T)) = \int_{-\infty}^{A_1} \int_{-\infty}^{A_2} \dots \int_{-\infty}^{A_M} \frac{1}{(2\pi)^{M/2} |\rho|^{1/2}} e^{-\mathbf{v} \rho^{-1} \mathbf{v}} dv_1 dv_2 \dots dv_M, \quad (2.4)$$

where

$$A_m = \Phi^{-1}(1 - F_m(T))$$

and ρ is the correlation matrix and Φ is the cumulative normal probability distribution function. A typical implementation of (2.4) is through a Monte Carlo simulation with A_m sampled by

$$A = RZ \sim \Phi(0, \rho),$$

where Z are standard normal random variables and R is the Cholesky decomposition of the correlation matrix (i.e., $\rho = R^t R$). The time until default $\tau_m = F^{-1}(1 - \Phi(A_m))$ is computed for each simulation path and if $\tau_m \leq T$, the obligor m is counted as defaulted on that specific path.* Li [34] also proposed that the correlation between each obligor's time until default is equivalent to the asset correlation, i.e., the correlation between the balance sheet assets of each obligor. As dynamic asset values of institutions are unobservable directly, computing these correlations from time-series of assets derived from equity prices became a common early practice as well as the use of equity prices correlation as proxy for asset correlation.

Li's approach is a general case of the application of copulas to credit portfolios and, as pointed out by Laurent and Gregory [32], it is the dependence structure underlying Credit-Metrics as well as the new Basel agreement. A particular case of this approach is associated with a one factor representation introduced by Vasicek [50], which opened up implementation possibilities particularly interesting in the case of CDOs. Vasicek's based his idea on Merton's model for the capital structure of firms, where the assets of an obligor are assumed to follow a Geometric Brownian Motion. Given a credit portfolio of M obligors, the approach states that an obligor m defaults if the value of its assets A_m falls below a certain barrier H_m , or $A_m < H_m$. In this case the probability of default for that obligor is given by

$$p_m = \Phi(H_m), \tag{2.5}$$

while the central argument is that the value of the assets A_m of each obligor m is given by

*Schönbucher [43] shows in detail the generation of sample paths for copulas in a general case.

$$A_m = y\sqrt{\beta_m} + z_m\sqrt{1 - \beta_m} \quad (2.6)$$

where the random variable y is a common portfolio factor interpreted as the systematic risk of the economic environment in which the obligors are inserted. On the other hand, each z_m is interpreted as the idiosyncratic or specific risk of each obligor m . The coefficient β_m is the correlation between each obligor's z_m and the common portfolio factor y , where y and all z_m are mutually independent normal standard random variables. Therefore, conditional on the realization of the common factor y , each obligor's assets as well as defaults are independent. If (2.5) provides the unconditional probability of default for obligor m , applying it to (2.6) for $A_m = H_m$ results in the conditional probability of default given as

$$p_m(y) = \Phi \left(\frac{\Phi^{-1}(p_m) - y\sqrt{\beta_m}}{\sqrt{1 - \beta_m}} \right). \quad (2.7)$$

From this point on, Vasicek's approach departs from the application of pricing credit portfolio derivatives as market instruments. In Vasicek [51], the law of large numbers is applied in order to derive the total portfolio loss conditional on y for $n \rightarrow \infty$. Vasicek [52] shows both cumulative and density loss probability distributions resulting from this large pool assumption, only relevant for assessing large portfolios of loans and credit obligations in financial institutions. However, (2.6) and (2.7) are still central to the development of the various copula approaches and many structural models for portfolio credit derivatives. Following the argument that default events are independent conditional on the common factor y , Laurent and Gregory [32] show that integrating over the common factor y in (2.7) results in the cumulative joint loss distribution of the portfolio, given by

$$\begin{aligned} F(\tau_1, \tau_2, \dots, \tau_M) &= \int_{-\infty}^{+\infty} \prod_{i=1}^M p_i(y) f(y) dy \\ &= \int_{-\infty}^{+\infty} \prod_{m=1}^M \Phi \left(\frac{\Phi^{-1}(F_m(\tau_m)) - \sqrt{\beta_m} y}{\sqrt{1 - \beta_m}} \right) f(y) dy, \end{aligned} \quad (2.8)$$

where $f(y)$ is the standard normal density function. It can be showed that (2.8) is equivalent to (2.4), a realization that has led to various implementation possibilities.

There are numerous standard convolution techniques that can be used to solve (2.8), e.g. Fourier transforms, Laplace transforms, moment generating functions, etc. For instance, NumeriX LLC [39] shows that (2.8) can be evaluated through simple convolution of the marginal distributions, or by applying Fourier transforms subsequently solved through quadrature techniques. Furthermore, NumeriX LLC [38] evaluates (2.8) analytically through Laplace transforms applying the saddle point approximation on the cumulant generating function, while Andersen et al [1] proposed a semi-analytical implementation that became widely used in the industry. We now discuss how the Gaussian copula model is actually applied by practitioners through the use of base correlations.

2.1.3 Gaussian copula and base correlations

Early applications of (2.8) for CDO pricing adopted the correlation approach previously mentioned, i.e., asset correlations from time-series of equity prices or equity correlations as proxies. However, with the standard iTraxx and CDX tranches becoming increasingly liquid, it was evident that prices of CDOs computed with the Gaussian copula under these correlation assumptions were not in line with quoted market prices. The zero tail dependence from the Gaussian copula was immediately pointed out as one of the limitations for applications of the Gaussian copula for portfolio credit derivatives. In order to improve on the tail dependence, Mashal and Naldi [36] as well as Andersen et al [1] tried the t-copula, while Schönbucher [43] applied Archimedean copulas. However, none of these approaches have significantly improved the matching of market quotes by the model. Burtschell et al [7] show this by comparing market quotes of iTraxx tranches with prices computed by the Gaussian copula, the t-copula and the Clayton copula (one of the most well known Archimedean copula for credit portfolio applications).

It had become clear that copula functions based on historical calibration of dependence parameters failed to provide a viable solution. Then market practitioners gradually adopted

the practice of restricting the correlation in (2.6) to a single value across all obligors in the portfolio as opposed to using pairwise correlation matrices, i.e., $\beta_m = \beta$. Although this assumption limits the range of possible correlation structures, it has the main benefit of reducing the number of parameters required by the model. This resulted in more efficient calibration and computation while giving a more intuitive meaning to the Gaussian copula correlation as a representation of market prices. However, Burtschell et al [7] showed that the single-correlation Gaussian copula also fails to match observed prices of tranches for the iTraxx index. More specifically, they demonstrated that market quotes of iTraxx tranches are consistent with a *correlation smile*. Computing the correlation parameter that reproduces market quotes for each iTraxx tranche leads to the so called *implied correlation smile* or *compound correlation skew*, in a similar fashion to the volatility smile concept from option pricing where the parameter itself loses its original statistical meaning and becomes simply a representation of market prices. Note the implied correlation showed by the right chart of Figure 2.4.

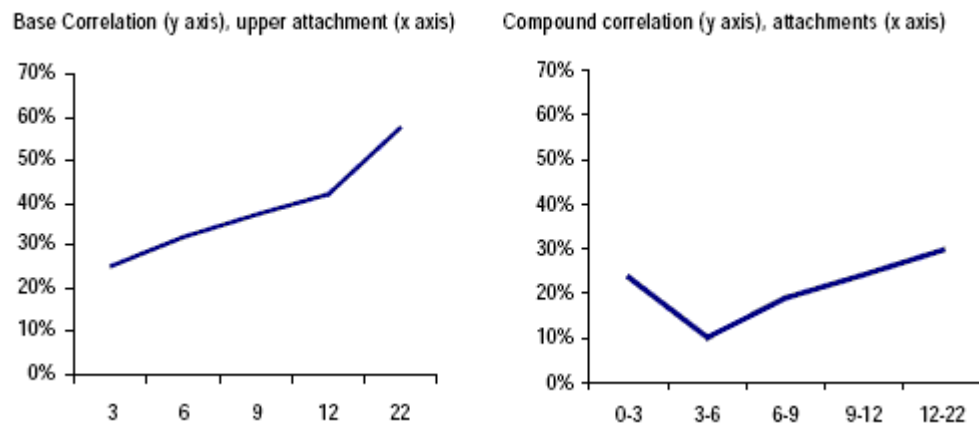


Figure 2.4: Base and compound correlations (J.P. Morgan Securities [28])

The application of a correlation smile seemed to solve the market-model matching issue for a single correlation Gaussian copula framework. However, a common problem with that is the difficulty to imply correlations for non-standard tranches. Taking the iTraxx index as an example, the correlation for a 5-7 tranche is not directly an interpolation of the correlation curve from the standard tranches 3-6 and 6-9 in Figure 2.4. Furthermore, Finger [12] pointed out that the implied correlation fails when applied to mezzanine tranches because

either “...there are two correlation levels that reproduce the tranche pricing, or, more seriously, that no correlation level reproduces the tranche pricing.” An example of this is showed in J.P. Morgan Securities [28], where the iTraxx 3-6 tranche spread can be mapped into two different correlation levels, as illustrated in Figure 2.5. This is due to the fact that a mezzanine tranche spread is not a monotonic function of the correlation parameter. This indicates that “...we should utilize the constant correlation framework differently,” as indicated by Finger [12].

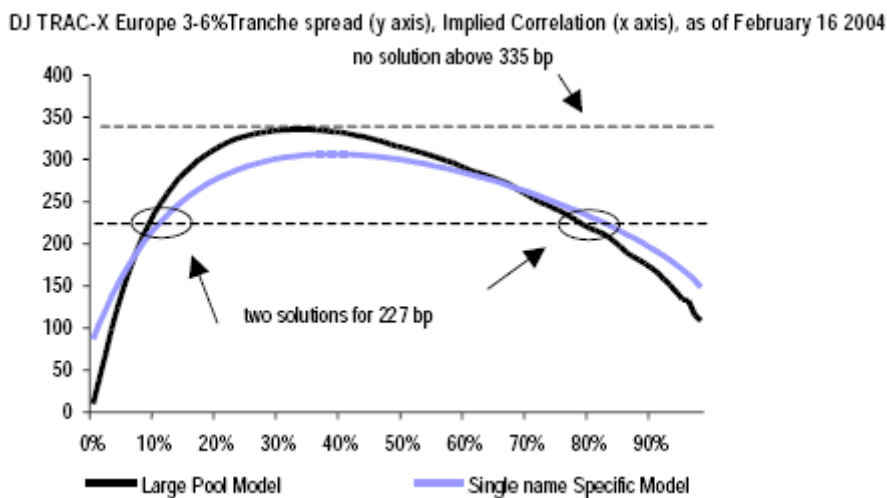


Figure 2.5: Two solutions for mezzanine compound correlation (J.P. Morgan Securities [28])

The solution came with *base correlations*, which effectively map equivalent equity tranche spreads directly into correlations in a monotonic relationship. The key is the idea that any specific tranche can be represented as the difference between two equity tranches. For instance, selling protection on a 3-6 iTraxx tranche is equivalent to buying protection in the 0-3 and then selling protection on a fictitious 0-6 tranche. In this case, implying the correlation from the 0-3 tranche leads to the one for the 0-6 tranche, and subsequently for higher detachment points. A typical base correlation curve for the iTraxx index is shown by the left chart in Figure 2.4. Note that besides solving the calibration issue with mezzanine tranches, base correlations also provide a direct translation of correlations into spreads of non-standard index tranches. For instance, in order to compute the par spread of a 2-5 tranche in the iTraxx, one is able to interpolate the base correlation curve in Figure 2.4 to

find the correlations for 0-2 and 0-5 tranches, which can in turn be used to price the 2-5 tranche.

One clear flaw of the base correlation approach is the assumption made by the interpolation technique. Linear interpolation is typically used, but other methods like polynomial and cubic spline are also applied and provide different results. In addition to this, RISK [42] pointed out that the base correlation approach “...is fundamentally flawed because in arbitrage pricing theory there is a link between price and the unique dynamics imposed in the model”, while base correlation “...cannot be linked to a unique choice of dynamics and therefore the concept of price is weakened.” As indicated in that article, there is still a lot of discomfort among academics and market players with the standard market model. We try to address this issue by suggesting viable alternatives, but before that we move forward to the next section where the CDO is represented mathematically through its payoff function.

2.1.4 The CDO payoff function

The previously discussed Gaussian copula model is one type of bottom-up approach for pricing CDOs, where the loss probability distribution of a credit portfolio is constructed from marginal loss probabilities of each obligor in the portfolio to form a multinomial loss distribution through an assumed dependence structure. In this research we apply top-down approaches for the pricing of CDOs, where the portfolio loss probability distribution is modeled directly and calibrated through market prices. For that we define a random variable x_t which represents the cumulative loss of the underlying portfolio at a certain horizon t , written as

$$x_t = \sum_{m=1}^M (1 - r_m) \omega_m y_m(t), \quad (2.9)$$

where M is the number of obligors, r_m is the recovery rate and ω_m is the weight of each obligor m in the portfolio. Note that x_t is in reality a discrete variable for every time t from inception until expiry of the CDO agreement, as $y_m(t)$ is a Bernoulli variable representing

the credit event by each obligor given by

$$y_m(t) = \begin{cases} 1 & \text{if } \tau_m > t \\ 0 & \text{if } \tau_m \leq t \end{cases}, \quad (2.10)$$

where τ_m is the time until default of obligor m . For every horizon t we assume that the expected value of $y_m(t)$ is known. In practice, this is accomplished by implying the expected survival (or default) probabilities from the market quotes of CDSs for each obligor, where $\mathbf{E}[y_m(t)] = [1 - F_t(\tau_m)]$ with $F_t(\tau_m)$ from (2.2) or $y_m(t) = \Phi(H_m)$ in (2.5).^{*} Equation (2.9) shows that assuming r_m and ω_m as constants, x_t is defined for every horizon t and

$$\mathbf{E}[x_t] = \sum_{m=1}^M (1 - r_m) \omega_m \mathbf{E}[y_m(t)] = \sum_{m=1}^M (1 - r_m) \omega_m [1 - F_t(\tau_m)]. \quad (2.11)$$

Now consider a CDO with attachment point X_U and detachment point X_L . The cumulative expected loss due to the subordination of such a tranche for every horizon t can be written as

$$\varepsilon(t) = \mathbf{E} [(\min(x_t, X_U) - X_L)^+], \quad (2.12)$$

which is equivalent to

$$\varepsilon(t) = \mathbf{E} [(x_t - X_L)^+ - (x_t - X_U)^+], \quad (2.13)$$

where $(A)^+ = \max(A, 0)$.

We now refer to the CDO diagram of Figure 2.6, which indicates that the premium payments of a CDO agreement is divided into N_j payment dates such as $\{j = 1, \dots, N_j\}$ for

^{*}The reader is referred to Schönbucher [44] and Li [33] for details on the calibration of survival probabilities from market quotes of CDSs.

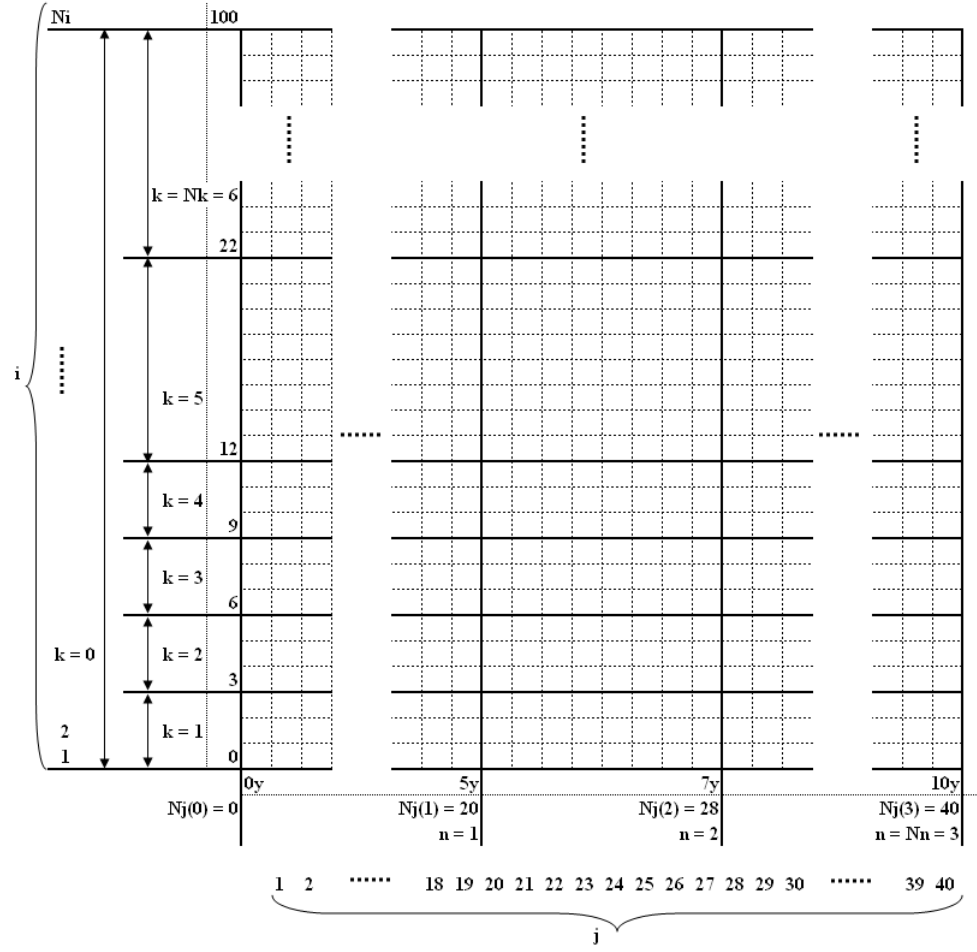


Figure 2.6: CDO discretization diagram for Entropy formulation and calibration

each of N_k possible tranches from zero to total loss in the portfolio. Although defaults in the portfolio can happen at any point in time, it is useful to define the loss leg of the CDO tranche in order to match the payment dates from the premium leg. Given (2.13), the loss leg of the CDO can be written as

$$\sum_{j=1}^{N_j} B_j (\varepsilon_{kj} - \varepsilon_{k(j-1)}) , \quad (2.14)$$

where B_j is the discount factor at time t_j and ε_{kj} is such that $X_U = X_k$ and $X_L = X_{k-1}$ in (2.12) and (2.13). Similarly, the premium leg can be defined by

$$S_k \sum_{j=1}^{N_j} (t_j - t_{j-1}) B_j \mathbf{E} [(1 - \alpha) A_{k(j-1)} - \alpha A_{kj}] + U_k (X_k - X_{k-1}),$$

where S_k is the spread paid on regular dates t_j , U_k is an upfront premium payment, and A_{kj} is the notional of the tranche k at a certain point in time $t_j \geq 0$. Note that $[t_j - t_{j-1}]$ represents the time interval between two spread payment dates, and the parameter α takes into account the accrued interest depending at which specific point in time between t_j and t_{j-1} a credit event happens. For instance, if $\alpha = 0.5$ then loss is assumed to happen in

$$t = \frac{t_j - t_{j-1}}{2} + t_{j-1}.$$

Given that

$$A_j = A_{k0} - \varepsilon_{kj} = X_k - X_{k-1} - \varepsilon_{kj} \text{ and}$$

$$A_{k(j-1)} = A_{k0} - \varepsilon_{k(j-1)} = X_k - X_{k-1} - \varepsilon_{k(j-1)}$$

where A_{k0} is the notional of the tranche at inception of the agreement, the premium leg can be rewritten as

$$S_k \sum_{j=1}^{N_j} (t_j - t_{j-1}) B_j [X_k - X_{k-1} - (1 - \alpha) \varepsilon_{k(j-1)} - \alpha \varepsilon_{kj}] + U_k (X_k - X_{k-1}). \quad (2.15)$$

The majority of synthetic CDO transactions happen on iTraxx and CDX indices, while market quotes of tranches on these indices are given in pairs of par spread S_k and upfront payment U_k such that

$$\begin{aligned}
S_k \sum_{j=1}^{N_j} (t_j - t_{j-1}) B_j [X_k - X_{k-1} - (1 - \alpha) \varepsilon_{k(j-1)} - \alpha \varepsilon_{kj}] + U_k (X_k - X_{k-1}) \\
- \sum_{j=1}^{N_j} B_j [\varepsilon_{kj} - \varepsilon_{k(j-1)}] = 0,
\end{aligned} \tag{2.16}$$

where U_k is typically quoted as zero for mezzanine and senior tranches as shown in Figure 2.3. A further application of (2.16) is for CDOs on bespoke portfolios, i.e., CDOs with CDS portfolios different from iTraxx and CDX. It becomes clear that the key to pricing CDO tranche agreements for any attachment and detachment points relies on modeling ε and in evaluating the expectation in (2.13). These are the topics of the two sections that follow.

2.2 Johnson SB (JSB) approach

2.2.1 Introduction

In this section we propose the application of the JSB distribution discussed in section 1.1 for the modeling of a default loss random variable from a portfolio of institutional CDSs. Due to the high degree of tractability of JSB distributions, we are able to achieve a semi-analytical solution for the distribution of cumulative losses for each payment j from (2.14) and (2.15). The hypothesis behind this approach is that JSB distributions should reasonably match the actual loss distribution of a credit portfolio as the one showed by J.P. Morgan Securities [28] and illustrated in Figure 2.7.

The idea is to adopt a top-down approach by modeling directly the distribution of the the total portfolio losses. This is accomplished by directly fitting the distribution of x_t from (2.9) through a JSB distribution. The development of this hypothesis into an analytical formula for pricing CDOs is performed in this section. We start by matching the characteristics of

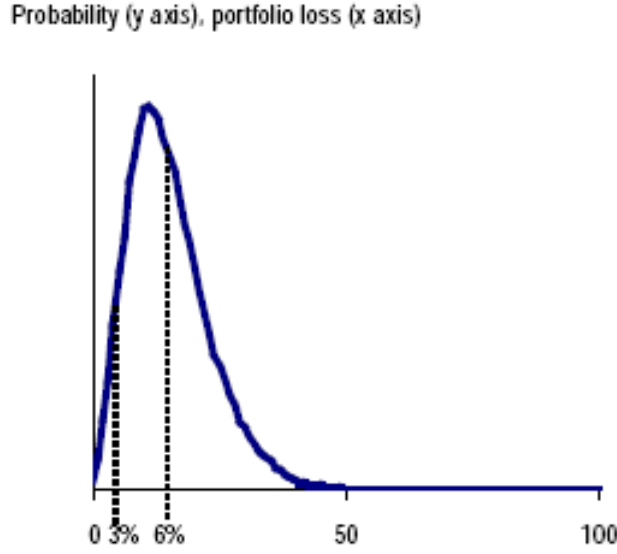


Figure 2.7: Implied loss distribution of a CDO's underlying portfolio (J.P. Morgan Securities [28])

the random variable x_t and show it can be represented by a JSB distribution, followed by the application of the CDO payoff.

2.2.2 Formulation

The idea is to have an analytic or semi-analytic solution for the pricing of CDOs by applying one of the Johnson transformations to x_t . Note that the cumulative loss x_t is a bounded random variable, which makes the JSB system an obvious alternative to fit a parametric distribution for x_t . In this case, each slice in time t will have a different set of parameters that will fit (1.2c), i.e.,

$$z_t = a_t + b_t \log \frac{x_t - c_t}{d_t + c_t - x_t}. \quad (2.17)$$

For a portfolio of M names, the lower bound is zero while the upper bound is given by $X_M \leq 1$ given the recovery rate is $r_m \geq 0$ for each obligor m in the portfolio. Note that $X_M = 1$ only if all $r_m = 0$, and $x_t = X_M$ when all $y_m(t) = 1$ in (2.10), establishing the

relation

$$X_M = \sum_{m=1}^M (1 - r_m) \omega_m. \quad (2.18)$$

As a result, both $c_t = 0$ and $d_t = X_M$ for all t in (2.17) given $c_t \leq x_t \leq c_t + d_t$, and that the minimum possible portfolio loss is zero. Thus,

$$z_t = a_t + b_t \log \frac{x_t}{X_M - x_t}. \quad (2.19)$$

It follows from (2.12) that ε_t can be decomposed as*

$$\begin{aligned} \varepsilon(t) = & \int_0^{X_M} (x_t - X_L) f(x_t) dx_t \\ & - \int_0^{X_L} (x_t - X_L) f(x_t) dx_t - \int_{X_U}^{X_M} (x_t - X_U) f(x_t) dx_t, \end{aligned}$$

where $f(x_t)$ is the probability density distribution of x_t . Now make $\varepsilon(t) = \mathcal{A} - \mathcal{B} - \mathcal{C}$ where[†]

$$\mathcal{A} = \int_0^{X_M} (x_t - X_L) f(x_t) dx_t = \mathbf{E}[x_t] - X_L,$$

$$\mathcal{B} = \int_0^{X_L} (x_t - X_L) f(x_t) dx_t = - \int_0^{X_L} \mathcal{J}_S B(x_t) dx_t$$

and

*Proof in Appendix A.3.

†Proof in Appendix A.4.

$$\mathcal{C} = \int_{X_U}^{X_M} (x_t - X_U) f(x_t) dx_t = X_M - X_U - \int_{X_U}^{X_M} \mathcal{J}_{SB}(x_t) dx_t,$$

with

$$\mathcal{J}_{SB}(x_t) = \int_0^{x_t} f(u) du.$$

This result leads to

$$\varepsilon(t) = \mathbf{E}[x_t] - X_M + X_U - X_L + \int_0^{X_L} \mathcal{J}_{SB}(x_t) dx_t + \int_{X_U}^{X_M} \mathcal{J}_{SB}(x_t) dx_t. \quad (2.20)$$

Key now is to solve for $\mathcal{J}_{SB}(x_t)$, where we define

$$\psi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \frac{e^{-\frac{v^2}{2}}}{1 + e^{\frac{a_t - v}{b_t}}} dv, \quad (2.21)$$

and arrive at a solution given in Appendix A.5 as

$$\begin{aligned} \varepsilon(t) = & \mathbf{E}[x_t] + X_U[1 - \Phi(K(X_U))] - X_L[1 - \Phi(K(X_L))] \\ & + X_M[\psi(K(X_U)) - \psi(K(X_L)) - \psi(K(X_M))]. \end{aligned} \quad (2.22)$$

The final solution for $\varepsilon(t)$ can be achieved by applying some boundary conditions. Note that if $X_L = 0$ and $X_U \leq X_M$, the CDO tranche is equivalent to the whole underlying portfolio. In this case, $\varepsilon(t)$ must equal the expected value of x_t . This can be better understood by testing it with (2.12), as follows

$$\varepsilon(t) = \mathbf{E}[(\min(x_t, X_U) - X_L)^+] = \mathbf{E}[(\min(x_t, X_M) - 0)^+] \Rightarrow$$

$$\varepsilon(t) = \int_0^{X_M} x_t f(x_t) dx_t = \mathbf{E}[x_t].$$

In this case we have

$$X_L = 0 \Rightarrow K(0) = -\infty$$

and

$$X_U = X_M \Rightarrow K(X_M) = +\infty.$$

We can test it now by applying it back into (2.22), which results in

$$\begin{aligned} \varepsilon(t) &= \mathbf{E}[x_t] + X_M[1 - \Phi(K(X_M))] - 0[1 - \Phi(K(0))] \\ &\quad + X_M[\psi(K(X_M)) - \psi(K(0)) - \psi(K(X_M))] \Rightarrow \end{aligned}$$

$$\begin{aligned} \varepsilon(t) &= \mathbf{E}[x_t] + X_M[1 - \Phi(+\infty)] - 0[1 - \Phi(-\infty)] \\ &\quad + X_M[\psi(+\infty) - \psi(-\infty) - \psi(+\infty)] \Rightarrow \end{aligned}$$

$$\varepsilon(t) = \mathbf{E}[x_t]$$

given that $\Phi(+\infty) = 1$, $\Phi(-\infty) = 0$ and $\psi(-\infty) = 0$.

Thus, the solution in (2.22) has passed the first boundary condition. The second boundary condition is the case where the CDO has zero notional, i.e., the attachment and detachment points are the same. We assess intuitively that if $X_L = X_U = X_A$ then $\varepsilon(t) = 0$ for all t as the CDO is not exposed to any risk of a credit default. This fictitious case can be seen by testing it with (2.12) as

$$\varepsilon(t) = \mathbf{E}[(\min(x_t, X_U) - X_L)^+] = \mathbf{E}[(\min(x_t, X_A) - X_A)^+] = 0.$$

Once again we test it by applying it back into (2.22), i.e.,

$$\begin{aligned} \varepsilon(t) &= \mathbf{E}[x_t] + X_A[1 - \Phi(K(X_A))] - X_A[1 - \Phi(K(X_A))] \\ &\quad + X_M[\psi(K(X_A)) - \psi(K(X_A)) - \psi(K(X_M))] \Rightarrow \\ \varepsilon(t) &= \mathbf{E}[x_t] - X_M[\psi(K(X_M))]. \end{aligned}$$

The only possible situation for $\varepsilon(t)$ is if $\mathbf{E}[x_t] = X_M[\psi(K(X_M))]$. From the Johnson transformation (1.2c) we have that

$$\frac{x_t}{X_M} = \left(1 + e^{\frac{a_t - z}{b_t}}\right)^{-1},$$

and thus in the case of $K(X_M)$

$$\psi(K(X_M)) = \mathbf{E}\left[\frac{x_t}{X_M}\right] = \frac{1}{X_M}\mathbf{E}[x_t].$$

Substituting this result back into $\varepsilon(t)$ yields

$$\varepsilon(t) = \mathbf{E}[x_t] - \frac{X_M}{X_M}\mathbf{E}[x_t] = 0,$$

which proves that the solution respects the second boundary condition.

We can now write $\varepsilon(t)$ in the form

$$K_U = K(a_t, b_t, X_U) = a_t + b_t \log \frac{X_U}{X_M - X_U}, \quad (2.23a)$$

$$K_L = K(a_t, b_t, X_L) = a_t + b_t \log \frac{X_L}{X_M - X_L}, \quad (2.23b)$$

$$\varepsilon(t) = X_U[1 - \Phi(K_U)] - L[1 - \Phi(K_L)] + X_M[\psi(K_U) - \psi(K_L)], \quad (2.23c)$$

$$\psi(K_U) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_U} \frac{e^{-\frac{v^2}{2}}}{1 + e^{\frac{a_t - v}{b_t}}} dv, \quad (2.23d)$$

$$\psi(K_L) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{K_L} \frac{e^{-\frac{v^2}{2}}}{1 + e^{\frac{a_t - v}{b_t}}} dv. \quad (2.23e)$$

One note remains on evaluating (2.21), which must be performed numerically. We can apply Goodwin's quadrature formula from section 1.3, where from (2.21) we identify $f(x)$ in (1.21) as

$$f(x)dx = \frac{1}{\sqrt{2\pi}} \left(1 + e^{\frac{a_t - v}{b_t}}\right)^{-1} dv,$$

given from section 1.1 that (1.3c) maps into the standard normal density and that

$$x_t = \left[c + d \left(1 + e^{\frac{a_t - v}{b_t}}\right) \right]^{-1}$$

from the JSB transformation (1.2c) for $z = v$. Note that $f(x) \rightarrow 1$ as $x \rightarrow +\infty$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$ given a is finite and $b > 0$, thus a suitable function for application of Goodwin's method. By neglecting the error term and making $x = \sqrt{2}u$ with $u = nh$, we change ψ to

$$\psi(x) = \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{N_x} \left(1 + e^{\frac{a_t - \sqrt{2}nh}{b_t}}\right)^{-1} e^{-n^2 h^2}, \quad (2.24)$$

where N_x can be chosen accordingly so that $N_x h$ matches K_x as the upper bound of the

integral.

Goodwin's method applies well to the case of $\psi(x)$, but one needs to be cautious to use it in a production environment due to performance issues. Appendix A.1 compares the application of Goodwin's formula to $\psi(x)$ with a the Gauss-Kronrod quadrature. While the result of the actual integral is satisfactory, clearly the application of Goodwin's method may be only be justified for research purposes. With that in mind, we propose the final solution of our formulation, given by

$$K_U = K(a_t, b_t, X_U) = a_t + b_t \log \frac{X_U}{X_M - X_U}, \quad (2.25a)$$

$$K_L = K(a_t, b_t, X_L) = a_t + b_t \log \frac{X_L}{X_M - X_L}, \quad (2.25b)$$

$$\varepsilon(t) = X_U[1 - \Phi(K_U)] - L[1 - \Phi(K_L)] + X_M[\psi(K_U) - \psi(K_L)], \quad (2.25c)$$

$$\psi(a_t, b_t, X_U) = \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{N_U} \left(1 + e^{\frac{a_t - \sqrt{2}nh}{b_t}} \right)^{-1} e^{-n^2 h^2}, \quad K_U = N_U h, \quad (2.25d)$$

$$\psi(a_t, b_t, X_L) = \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{N_L} \left(1 + e^{\frac{a_t - \sqrt{2}nh}{b_t}} \right)^{-1} e^{-n^2 h^2}, \quad K_L = N_L h. \quad (2.25e)$$

We now apply the concepts and equations from this section in the discussion that follows on model parameter calibration.

2.2.3 Calibration

The first thing to note from the previous formulation is that

$$\psi(X_M) = \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \left(1 + e^{\frac{a_t - \sqrt{2}nh}{b_t}} \right)^{-1} e^{-n^2 h^2} = \frac{1}{X_M} \mathbf{E}[x_t], \quad (2.26)$$

as determined in the test of the second boundary conditions. From the CDO discretiza-

tion shown in Figure 2.6, (2.26) provides a set of $\{j = 1, \dots, N_j\}$ equations to be used in the calibration, where $\psi_j(X_M) = \mathbf{E}[x_j]/X_M$ is the expected value of the cumulative portfolio loss at t_j per unit of the total possible loss X_M . The expectation $\mathbf{E}[x_j]$ can be implied from market quotes of CDS spreads for each obligor as discussed in section 2.1.2, which together with $\psi_j(X_M)$ provides the first minimization condition of the calibration. If $F_{mj} = 1 - P[\tau_m \leq t_j]$ is the probability of survival* with τ_m as the time until default, r_m is the recovery rate and $\mathcal{N}_m$ is the notional amount for the CDS of obligor m , the expected cumulative portfolio loss at time t_j in proportion to the total portfolio notional $\mathcal{N}_M$ is defined as

$$\mathbf{E}[x_j] = \sum_{m=1}^M (1 - r_m) \frac{\mathcal{N}_m}{\mathcal{N}_M} (1 - F_{mj}). \quad (2.27)$$

This leads to the set of objective functions

$$\mathcal{F}_1(a_j, b_j) = \left| \frac{X_M h}{\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \left(1 + e^{\frac{a_j - \sqrt{2} n h}{b_j}} \right)^{-1} e^{-n^2 h^2} - \sum_{m=1}^M (1 - r_m) \frac{\mathcal{N}_m}{\mathcal{N}_M} (1 - F_{mj}) \right|^2 \quad (2.28)$$

to be applied for each j in a least-square minimization method.

The other set of equations come from the tranche payoff introduced in section 2.1.4, where parameters are typically calibrated against CDX and iTraxx tranche prices. In the case of the JSB approach, we leave out the basket spread defined for subordination points 0-100 as this quantifies the expected loss in the whole portfolio which is already taken into account by (2.28). However, there are N_k tranche equations plus N_j expected loss equations (2.28), while the problem contains $2N_j$ variables (a_j and b_j for $\{j = 1, \dots, N_j\}$). Given $2N_j > N_j + N_k$ and in order to avoid working with an underdetermined system of equations, it

*More details on bootstrapping probabilities of survival from CDS quotes can be found in Schönbucher [44] and Li [33].

is useful to establish a functional form for either a_j or b_j . Given a_j acts like the mean parameter of a probability distribution in the JSB transformation of (1.2c), we choose to define b_j as a polynomial function of time*, i.e.,

$$b_j = \sum_{k=1}^{N_k} \beta_{k-1} t_j^{k-1}. \quad (2.29)$$

We can now write the final set of nonlinear equations from (2.16), given by

$$\begin{aligned} g(k, a_j, b_j) = & \\ & S_k \sum_{j=1}^{N_j} (t_j - t_{j-1}) B_j [X_k - X_{k-1} \\ & - (1 - \alpha) \varepsilon_{k(j-1)}(a_{j-1}, b_{j-1}) - \alpha \varepsilon_{kj}(a_j, b_j)] + U_k(X_k - X_{k-1}) \\ & - \sum_{j=1}^{N_j} B_j [\varepsilon_{kj}(a_j, b_j) - \varepsilon_{k(j-1)}(a_{j-1}, b_{j-1})], \\ \mathcal{F}_2(a_j, b_j) = & \sum_{k=1}^{N_k} g(k, a_j, b_j)^2. \end{aligned} \quad (2.30)$$

where $\varepsilon_{kj}(a_j, b_j)$ is defined in (2.25c).

Numerical methods such as Newton-Raphson can be used to minimize (2.28) and (2.30), while the calibration procedure can be established as follows:

1. Start with an initial guess on the values of β_{k-1} from (2.29).
2. For each j , compute b_j through (2.29).
3. For each j , minimize $\mathcal{F}_1(a_j, b_j)$ in (2.28) for a_j .
4. Compute (2.30) given a_j and b_j for $\{j = 1, \dots, N_j\}$.

*See Appendix A.2 for an alternative method.

5. If $\mathcal{F}_2(a_j, b_j) \leq \text{TOL}$, calibration is complete. Otherwise, change β_{k-1} and go back to step 2.

Once the calibration with index tranches (iTraxx or CDX) is performed, a useful further application is to calibrate the model to bespoke portfolios. However, the only information available for this task is the cumulative expected loss $\mathbf{E}[x_j]$ for the new portfolio, bootstrapped from CDS spreads as previously discussed in the cases of iTraxx and CDX. Thus, we need to make the assumption of keeping b_j and recalibrating a_j for every j given $\mathbf{E}[x_j]$ and (2.28). In this case the calibration steps are as follows:

1. Calibrate a_j and b_j according to previous steps for iTraxx or CDX tranches.
2. Make the previous a_j as initial guess and keep the values of b_j .
3. For each j , minimize (2.28) for a_j with $\{j = 1, \dots, N_j\}$, where the calibration is complete after $j = N_j$.

2.3 Entropy optimization approach

2.3.1 Introduction

The application of entropy optimization techniques in finance is not new. Avellaneda et al [2] proposed the application of the relative entropy minimization to calibrate volatility surfaces in Dollar-Deutschemark options, while later on Goldman Sachs[15] applied the concept to estimate a fair volatility smile of equity options from the time-series of the underlying stock's historical returns. Here we propose a similar application of entropy optimization for modeling the loss due to default from an underlying portfolio of CDSs, and consequently use calibrated Langrange multipliers and probabilities to price CDO tranches. We start by formulating the approach, followed by a description of how we use it in the calibration process.

2.3.2 Formulation

We now apply the entropy optimization concepts of section 1.2 in the pricing of CDOs. Recognizing that loss given default in the underlying credit portfolio is the driver behind CDO prices and that it is in reality a discrete variable, we take particular interest in Kullback's relative entropy definition of (1.13). Then the very first step in formulating this approach is to make a suitable representation of the problem by means of discretizing the underlying variable and events. This is shown graphically by Figure 2.6, where x_i represents the discretization of the loss given default in the portfolio with $\{x_i = 0, \dots, x_{N_i}\}$ and $x_{N_i} = 1$. Each default in the portfolio leads to a loss that can be realized at any point in time until expiry of the CDO agreement. In that case, time is discretized by $\{j = 1, \dots, N_j\}$, where j defines the spread payment dates on the synthetic CDO swap. The problem is then reduced to minimizing the relative entropy of (1.13) conditional to a set of underlying constraints, i.e.,

$$\min \left\{ \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} p_{ji} \log \frac{p_{ji}}{q_{ji}} \right\}, \quad (2.31)$$

conditional on the normalization constraint

$$\sum_{i=1}^{N_i} p_{ji} - 1 = 0, \{j = 1, \dots, N_j\} \quad (2.32)$$

and to the functional constraints

$$\begin{aligned} S_k \sum_{j=1}^{N_j} B_j \Delta_j [X_k - X_{k-1} - \alpha \varepsilon_{kj} - (1 - \alpha) \varepsilon_{k(j-1)}] \\ - \sum_{j=1}^{N_j} B_j [\varepsilon_{kj} - \varepsilon_{k(j-1)}] - U_k (X_k - X_{k-1}) = 0, \\ \{k = 0, \dots, N_k\}, X_{-1} = 0, \end{aligned} \quad (2.33)$$

where $\Delta_t = t_j - t_{j-1}$ and

$$\varepsilon_{kj} = \sum_{i=1}^{N_i} (x_i - X_{k-1})^+ p_{ji} - \sum_{i=1}^{N_i} (x_i - X_k)^+ p_{ji}, \quad (2.34)$$

with the functional constraints coming from the CDO tranche payoff in (2.16). In the case of an entropy optimization problem for pricing CDOs, these constraints effectively define the information available given CDO tranche market quotes on credit indices as discussed in section 2.1.1. In this case, S_k and U_k are respectively the market quoted spread and upfront payment required for tranche k , while X_k and X_{k-1} are respectively the detachment and attachment points. In addition, (2.34) is simply a discretized version of (2.13) also discussed in section 2.1.1, while again α defines the point in time in which default happens between payment dates (typically assumed as 0.5).

We can now apply the steps described in section 1.2, by first writing the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} p_{ji} \log p_{ji} - \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} p_{ji} \log q_{ji} \\ & + \sum_{j=1}^{N_j} (\mu_j - 1) \left(\sum_{i=1}^{N_i} p_{ji} - 1 \right) + \sum_{k=1}^{N_k} \lambda_k f_k, \end{aligned} \quad (2.35)$$

where f_k is given by

$$\begin{aligned} f_k = & S_k \sum_{j=1}^{N_j} B_j \Delta_j [X_k - X_{k-1} - \alpha \varepsilon_{kj} - (1 - \alpha) \varepsilon_{k(j-1)}] \\ & - \sum_{j=1}^{n_{kj}} B_j [\varepsilon_{kj} - \varepsilon_{k(j-1)}] - U_k (X_k - X_{k-1}), \end{aligned} \quad (2.36)$$

$$\{k = 0, \dots, N_k\}, X_{-1} = 0,$$

and ε_{kj} defined by (2.34) as a function of p_{ji} . Differentiating $\mathcal{L}$ with respect to p_{ji} and setting it to zero results in

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial p_{ba}} = 0 = & \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} \delta_{ia} \delta_{jb} \log p_{ji} + \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} \delta_{ia} \delta_{jb} - \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} \delta_{ia} \delta_{jb} \log q_{ji} \\
& + \sum_{j=1}^{N_j} (\mu_j - 1) \sum_{i=1}^{N_i} \delta_{ia} \delta_{jb} + \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{ba}},
\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}
& \frac{\partial f_k}{\partial p_{ba}} = \\
& -S_k \sum_{j=1}^{N_j} B_j \Delta_j \left[\alpha \frac{\partial \varepsilon_{kj}}{\partial p_{ba}} + (1 - \alpha) \frac{\partial \varepsilon_{k(j-1)}}{\partial p_{ba}} \right] - \sum_{j=1}^{N_j} B_j \left[\frac{\partial \varepsilon_{kj}}{\partial p_{ba}} - \frac{\partial \varepsilon_{k(j-1)}}{\partial p_{ba}} \right],
\end{aligned} \tag{2.38}$$

$$\frac{\partial \varepsilon_{kj}}{\partial p_{ba}} = \sum_{i=1}^{N_i} (x_i - X_{k-1})^+ \delta_{ia} \delta_{jb} - \sum_{i=1}^{N_i} (x_i - X_k)^+ \delta_{ia} \delta_{jb},$$

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

Note that the filtering function $\delta_{(i-1)a} = 1$ only if $i - 1 = a$ or $i = a + 1$, while $\delta_{ia} = 1$ for $i = a$. Similarly, $\delta_{(j-1)b} = 1$ only if $j = b + 1$ while $\delta_{jb} = 1$ for $j = b$. We can now rearrange (2.38) to

$$\begin{aligned}
\frac{\partial f_k}{\partial p_{ba}} = & -S_k \sum_{j=1}^{N_j} B_b \Delta_b \alpha [(x_a - X_{k-1})^+ - (x_a - X_k)^+] \delta_{ia} \delta_{jb} \\
& -S_k \sum_{j=1}^{N_j} B_b \Delta_b (1 - \alpha) [(x_a - X_{k-1})^+ - (x_a - X_k)^+] \delta_{ia} \delta_{(j-1)b} \\
& - \sum_{j=1}^{N_j} B_b [(x_a - X_{k-1})^+ - (x_a - X_k)^+] \delta_{ia} \delta_{jb} \\
& + \sum_{j=1}^{N_j} B_b [(x_a - X_{k-1})^+ - (x_a - X_k)^+] \delta_{ia} \delta_{(j-1)b},
\end{aligned}$$

and apply the filtering property of the δ function to obtain

$$\begin{aligned}
\frac{\partial f_k}{\partial p_{ji}} = & [(x_i - X_{k-1})^+ - (x_i - X_k)^+] \\
& \times \{B_{j+1}[1 - (1 - \alpha)S_k \Delta_{j+1}] - B_j(1 + \alpha S_k \Delta_j)\}.
\end{aligned} \tag{2.39}$$

Similarly, we can rewrite (2.37) as

$$\log p_{ji} - \log q_{ji} + 1 + (\mu_j - 1) + \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{ji}} = 0,$$

resulting in p_{ji} defined by

$$p_{ji} = q_{ji} \exp \left(-\mu_j - \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{ji}} \right).$$

The next step is to extract μ_j by applying this equation to (2.32), where we write

$$\exp(\mu_j) = \sum_{m=1}^{N_i} q_{jm} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{jm}} \right), \tag{2.40}$$

leading to the solution for p_{ji} as

$$p_{ji} = \frac{q_{ji} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{ji}} \right)}{\sum_{m=1}^{N_i} q_{jm} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{jm}} \right)}. \quad (2.41)$$

Finally, applying (2.41) to (2.34) results in

$$\begin{aligned} \varepsilon_{kj} = & \sum_{i=1}^{N_i} (x_i - X_{k-1})^+ \frac{q_{ji} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{ji}} \right)}{\sum_{m=1}^{N_i} q_{jm} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{jm}} \right)} \\ & - \sum_{i=1}^{N_i} (x_i - X_k)^+ \frac{q_{ji} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{ji}} \right)}{\sum_{m=1}^{N_i} q_{jm} \exp \left(- \sum_{k=1}^{N_k} \lambda_k \frac{\partial f_k}{\partial p_{jm}} \right)}. \end{aligned} \quad (2.42)$$

This equation is then used to substitute ε_{kj} in the constraint equations (2.33), resulting in a system of N_k nonlinear equations to be solved for N_k Lagrange multipliers λ_k .

The above steps are useful to compute Lagrange multipliers and consequently the probability surface defined in N_i by N_j points, specifically for the portfolio underlying the CDO market quotes given as constraints. For instance, in the case of the iTraxx index discussed in section 2.1.1, the spreads and upfront payments quotes of six tranches plus the basket swap in the three standard expiries of 5y, 7y and 10y should be exactly matched by applying the probability surface computed by (2.41) to the constraints (2.33). Depending on the discretization adopted for x_i , a useful application of this probability surface is in computing fair prices of CDO agreements in bespoke tranches for any maturity up to 10y based on the iTraxx portfolio.

However, a common and even more useful application of this method is for CDO agreements on bespoke underlying portfolios, as discussed previously in section 2.2.3. The premise is to achieve a new set of Lagrange multipliers and consequently a new probability surface p_{ji} given the cumulative expected loss of the portfolio for each $\{j = 1, \dots, N_j\}$ as the only information available about the new portfolio. Fortunately in the case of the

entropy approach, the p_{ji} found for iTraxx and CDX provide a reference measure regarding the dependence structure of credit portfolios and can be used as prior distribution in an entropy minimization problem. We achieve this by simply defining a new constraint as

$$\sum_{i=1}^{N_i} x_i p_{ji} - c_j = 0, \quad (2.43)$$

$$\{j = 1, \dots, N_j\},$$

where c_j is the portfolio cumulative expected loss defined as in (2.27) for a portfolio with M obligors. The new Lagrangian is then given as

$$\begin{aligned} \mathcal{L} = & \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} p_{ji} \log p_{ji} - \sum_{j=1}^{N_j} \sum_{i=1}^{N_i} p_{ji} \log q_{ji} \\ & + \sum_{j=1}^{N_j} \gamma_j \left(\sum_{i=1}^{N_i} x_i p_{ji} - c_j \right), \end{aligned} \quad (2.44)$$

where q_{ji} is the probability surface found for iTraxx and CDX used here as a prior distribution. We follow the same previous steps to arrive at the p_{ji} equation for the new portfolio, given by

$$p_{ji} = \frac{q_{ji} \exp(-\gamma_j x_i)}{\sum_{m=1}^{N_i} q_{jm} \exp(-\gamma_j x_m)}. \quad (2.45)$$

This can now be applied to (2.43) to generate a nonlinear system of N_j equations with N_j unknowns given as

$$\sum_{i=1}^{N_i} \frac{x_i q_{ji} \exp(-\gamma_j x_i)}{\sum_{m=1}^{N_i} q_{jm} \exp(-\gamma_j x_m)} - c_j = 0 \quad (2.46)$$

$$\{j = 1, \dots, N_j\}.$$

By integrating (2.46) in respect to γ_j , we can alternatively define a potential function

$$\mathcal{G}(\gamma_j) = \sum_{j=1}^{N_j} \log \left(\sum_{i=1}^{N_i} q_{ji} \exp(-\gamma_j x_i) \right) + \sum_{j=1}^{N_j} \gamma_j c_j, \quad (2.47)$$

that can be minimized for γ_j .

2.3.3 Calibration

Following the discretization defined by Figure 2.6, there are $N_k + 1$ tranche equations for N_n different expiries. In the case of iTraxx or CDX, there are six tranches (i.e., $N_k = 6$) plus the basket spread (i.e., $k = 0$ for 0-100 subordination) for three different expiries (i.e., $N_n = 3$). Spread payments are made at specific dates defined by j , where $\{j = 1, \dots, N_j(n)\}$ and $N_j(n)$ is a function of the tranche's expiry defined by $\{n = 1, \dots, N_n\}$. This provides the background to our first calibration approach, where we establish

$$\begin{aligned} S_k \sum_{j=1}^{N_j(n)} B_j \Delta_j (X_k - X_{k-1} - \alpha \varepsilon_{kj} - (1 - \alpha) \varepsilon_{k(j-1)}) \\ - \sum_{j=1}^{N_j(n)} B_j (\varepsilon_{kj} - \varepsilon_{k(j-1)}) - U_k(X_k - X_{k-1}) = 0, \end{aligned} \quad (2.48)$$

$$\{k = 0, \dots, N_k\}, \{n = 1, \dots, N_n\},$$

as the functional constraints previously discussed in section 2.3.2. In this approach, a set of $\lambda_k(n)$ parameters are calibrated sequentially for each of the N_n expiries in (2.48). This consists of an iterative procedure where we apply $p_{ji}(n-1)$ and $\lambda_k(n-1)$ calibrated from the $n-1$ expiry in order to compute $\lambda_k(n)$ and calculate the probabilities for the n expiry (i.e., $N_j(n-1) + 1 \leq j \leq N_j(n)$). In this case, by making

$$q_{ji} = \frac{1}{N_i}$$

as the uniform distribution, we change (2.41) to

$$p_{ji}(n) = \frac{\exp\left(-\sum_{k=1}^{N_k} \lambda_k(n) \frac{\partial f_k}{\partial p_{ji}}\right)}{\sum_{m=1}^{N_i} \exp\left(-\sum_{k=1}^{N_k} \lambda_k(n) \frac{\partial f_k}{\partial p_{jm}}\right)}. \quad (2.49)$$

In each iteration n , the N_k constraint equations become

$$\begin{aligned} & S_k \mathcal{A}_{n-1} - \mathcal{B}_{n-1} \\ & + S_k \sum_{j=N_j(n-1)+1}^{N_j(n)} B_j \Delta_j [X_k - X_{k-1} - \alpha \varepsilon_{kj}(n) - (1-\alpha) \varepsilon_{k(j-1)}(n)] \\ & - \sum_{j=N_j(n-1)+1}^{N_j(n)} B_j [\varepsilon_{kj}(n) - \varepsilon_{k(j-1)}(n)] - U_k(X_k - X_{k-1}) = 0, \\ & \{k = 0, \dots, N_k\}, \{n = 1, \dots, N_n\}, \mathcal{A}_0 = \mathcal{B}_0 = N_j(0) = 0, \end{aligned} \quad (2.50)$$

where $\mathcal{A}_{n-1}$ and $\mathcal{B}_{n-1}$ are known from iteration $n-1$ as

$$\begin{aligned} \mathcal{A}_{n-1} &= \sum_{j=N_j(n-2)+1}^{N_j(n-1)} B_j \Delta_j \\ & [X_k - X_{k-1} - \alpha \varepsilon_{kj}(p_{ji}(n-1)) - (1-\alpha) \varepsilon_{k(j-1)}(p_{ji}(n-1))], \\ \mathcal{B}_{n-1} &= \sum_{j=N_j(n-2)}^{N_j(n-1)} B_j [\varepsilon_{kj}(p_{ji}(n-1)) - \varepsilon_{k(j-1)}(p_{ji}(n-1))], \\ & \{n = 2, \dots, N_n\}, N_j(0) = 0, \end{aligned} \quad (2.51)$$

and $\varepsilon_{kj}(n)$ is given by substituting (2.49) into (2.34) as

$$\begin{aligned} \varepsilon_{kj}(n) = & \sum_{i=1}^{N_i} (x_i - X_{k-1})^+ \frac{\exp\left(-\sum_{k=1}^{N_k} \lambda_k(n) \frac{\partial f_k}{\partial p_{ji}}\right)}{\sum_{m=1}^{N_i} \exp\left(-\sum_{k=1}^{N_k} \lambda_k(n) \frac{\partial f_k}{\partial p_{jm}}\right)} \\ & - \sum_{i=1}^{N_i} (x_i - X_k)^+ \frac{\exp\left(-\sum_{k=1}^{N_k} \lambda_k(n) \frac{\partial f_k}{\partial p_{ji}}\right)}{\sum_{m=1}^{N_i} \exp\left(-\sum_{k=1}^{N_k} \lambda_k(n) \frac{\partial f_k}{\partial p_{jm}}\right)}. \end{aligned} \quad (2.52)$$

A second possible approach involves a recursive procedure based on the entropy minimization and the concept of new information arrival discussed in section 1.2. We start just as in the previous approach, by calibrating N_k parameters λ_k for step $n = 1$. But then in the next step $n = 2$ we use $p_{ji}(1)$ from the previous step as a prior distribution q_{ji} in the relative entropy minimization equation of (2.31), and calibrate another $2N_k$ parameters given now we have N_k equations in respect of expiry $n = 1$ plus N_k equations for expiry $n = 2$. This process is subsequently repeated until $n = N_n$, changing (2.41) to

$$\begin{aligned} p_{ji}(n) = & \frac{p_{ji}(n-1) \exp\left(-\sum_{s=1}^n \sum_{k=1}^{N_k} \lambda_k(s) \frac{\partial f_k}{\partial p_{ji}}\right)}{\sum_{m=1}^{N_i} p_{jm}(n-1) \exp\left(-\sum_{s=1}^n \sum_{k=1}^{N_k} \lambda_k(s) \frac{\partial f_k}{\partial p_{jm}}\right)} \\ & \{n = 1, \dots, N_n\}, p_{ji}(0) = \frac{1}{N_i}. \end{aligned} \quad (2.53)$$

In this case the constraints equations are still given by (2.50) with $\mathcal{A}_{n-1}$ and $\mathcal{B}_{n-1}$ from (2.51), while $\varepsilon_{kj}(n)$ changes to

$$\begin{aligned} \varepsilon_{kj}(n) = & \sum_{i=1}^{N_i} (x_i - X_{k-1})^+ \frac{p_{ji}(n-1) \exp\left(-\sum_{s=1}^n \sum_{k=1}^{N_k} \lambda_k(s) \frac{\partial f_k}{\partial p_{ji}}\right)}{\sum_{m=1}^{N_i} p_{jm}(n-1) \exp\left(-\sum_{s=1}^n \sum_{k=1}^{N_k} \lambda_k(s) \frac{\partial f_k}{\partial p_{jm}}\right)} \\ & - \sum_{i=1}^{N_i} (x_i - X_k)^+ \frac{p_{ji}(n-1) \exp\left(-\sum_{s=1}^n \sum_{k=1}^{N_k} \lambda_k(s) \frac{\partial f_k}{\partial p_{ji}}\right)}{\sum_{m=1}^{N_i} p_{jm}(n-1) \exp\left(-\sum_{s=1}^n \sum_{k=1}^{N_k} \lambda_k(s) \frac{\partial f_k}{\partial p_{jm}}\right)}. \end{aligned} \quad (2.54)$$

These two methods are equivalent and must converge to the solution. However, the second

method is computationally more expensive and should be used in the cases of arbitrageable and incoherent market quotes where computations for one tranche equation might contradict another. We will adopt the first method given its simplicity and the high level of liquidity of CDO index tranches in the market. In this case the calibration procedure is given as follows:

1. Start with iteration $n = 1$, define N_i discretization and $p_{ji}(0) = 1/N_i$.
2. Solve (2.50) for $\lambda_k(n)$ by computing p_{ji} through (2.49) for $\{j = N_j(n-1)+1, \dots, N_j(n)\}$ and using $p_{ji}(n-1)$ defined up to $N_j(n-1)$.
3. If $n = N_n$, calibration is complete. Otherwise, make $n = n + 1$ and go back to step 2.

We also define the calibration steps for bespoke portfolios according to the discussion in section 2.3.2, which consists in simply solving the system of N_j nonlinear equations in (2.46) given p_{ji} from the previous steps for CDO index tranches as prior distribution. In this case, the calibration steps are:

1. Compute p_{ji} according to previous steps for iTraxx or CDX tranches.
2. Compute c_j through (2.27).
3. Either solve (2.46) or minimize (2.47) for γ_j where $\{j = 1, \dots, N_j(N_n)\}$, after which calibration is complete.

2.4 Application

In this section we apply the JSB and Entropy approaches formulated in sections 2.2 and 2.3 to a practical CDO pricing case. The first step is to calibrate the cumulative loss probabilities that are used for evaluating the CDO payoff of (2.16). For that we choose the

iTraxx Europe Series 6 portfolio*, where market quotes as of 15-Mar-2007 are used in the calibration and are shown by Table 2.1. The iTraxx quotes are based on quarterly payment schedules according to the IMM maturities† and are provided for 5y, 7y and 10y expiries. Also, they come in six tranche subordinations 0-3, 3-6, 6-9, 9-12, 12-22 and 22-100 plus one quote for the basket spread representing subordination 0-100. Given that the possibility of defaults in the portfolio increases with time, spreads and upfront payments demanded by the market are higher for longer expiry dates. In addition, note that due to its higher probability of being wiped out, the market demands for equity tranche 0-3 an upfront payment as a percentage of its notional amount while quoting its spread at a fixed 500 bps (basis points).

In the calibration process for both JSB and Entropy approaches, the time dimension j in the diagram of Figure 2.6 is discretized according to iTraxx's quarterly payment schedules, where $t_j - t_{j-1} \approx 0.25$ years depending on each specific IMM maturity date. We define $N_n = 3$ with $n = 1$ for expiry 5y, $n = 2$ for 7y and $n = 3$ for 10y, while $N_j(1) = 20$, $N_j(2) = 28$ and $N_j(3) = 40$ given quarterly payment schedules. From the available iTraxx information, we define $N_k = 6$ where $k = 0$ represents the 0-100 subordination. In the case of the JSB approach, the iTraxx Europe Series 6 cumulative expected loss for each j is calculated using (2.27), where probabilities of survival are extracted from the CDS quotes of the obligors in the portfolio available from Appendix A.6. In the case of the Entropy approach, the portfolio loss dimension i is discretized in steps of 1%. For the minimization of objective functions (2.28) and (2.30) in the JSB approach we use the function *fmincon* from Matlab, which implements the sequential quadratic programming method described by Fletcher and Powell [13] and Goldfarb [14]. In the case of solving the nonlinear system of equations (2.50) and (2.46) in the Entropy approach we use the function *fsolve* from Matlab, which implements a Gauss-Newton nonlinear least-squares algorithm as described by Dennis [8].

*Appendix A.6 shows the composition of the iTraxx Series 6 portfolio with the respective CDS spread curves of its constituents as of 15-Mar-2007.

†The IMM maturities define the cycle of dates used by the International Monetary Market at the Chicago Mercantile Exchange, i.e., 20-Mar, 20-Jun, 20-Sep and 20-Dec.

(a) Market quotes for 5y iTraxx CDO tranches

Tranche (%)	Bid		Ask		Mid	
	Spread (bps)	Upfront (%)	Spread (bps)	Upfront (%)	Spread (bps)	Upfront (%)
0-3	500.0000	11.7500	500.0000	12.0000	500.0000	11.8750
3-6	53.7500	-	55.2500	-	54.5000	-
6-9	14.0000	-	15.5000	-	14.7500	-
9-12	5.7500	-	6.7500	-	6.2500	-
12-22	2.1250	-	2.8750	-	2.5000	-
22-100	0.9000	-	1.4000	-	1.1500	-
0-100	25.0000	-	-	-	-	-

(b) Market quotes for 7y iTraxx CDO tranches

Tranche (%)	Bid		Ask		Mid	
	Spread (bps)	Upfront (%)	Spread (bps)	Upfront (%)	Spread (bps)	Upfront (%)
0-3	500.0000	26.7500	500.0000	27.0000	500.0000	26.8750
3-6	130.5000	-	132.5000	-	131.5000	-
6-9	36.7500	-	38.2500	-	37.5000	-
9-12	16.5000	-	18.0000	-	17.2500	-
12-22	5.5000	-	6.5000	-	6.0000	-
22-100	2.6000	-	3.1000	-	2.8500	-
0-100	34.0000	-	-	-	-	-

(c) Market quotes for 10y iTraxx CDO tranches

Tranche (%)	Bid		Ask		Mid	
	Spread (bps)	Upfront (%)	Spread (bps)	Upfront (%)	Spread (bps)	Upfront (%)
0-3	500.0000	41.8750	500.0000	42.1250	500.0000	42.0000
3-6	347.0000	-	352.0000	-	349.5000	-
6-9	93.0000	-	95.0000	-	94.0000	-
9-12	40.0000	-	42.0000	-	41.0000	-
12-22	13.2500	-	14.2500	-	13.7500	-
22-100	4.4000	-	4.9000	-	4.6500	-
0-100	45.0000	-	-	-	-	-

Table 2.1: Market quotes for iTraxx CDO tranches

(a) 5y

Tranche (%)	Spread (bps)		
	Market	Entropy	JSB
0-3	500.0000	500.0002	686.4862
3-6	54.5000	54.5001	54.4702
6-9	14.7500	14.7500	14.8244
9-12	6.2500	6.2500	6.1877
12-22	2.5000	2.5000	3.1182
22-100	1.1500	1.1500	0.5728
0-100	25.0000	25.0000	25.6513

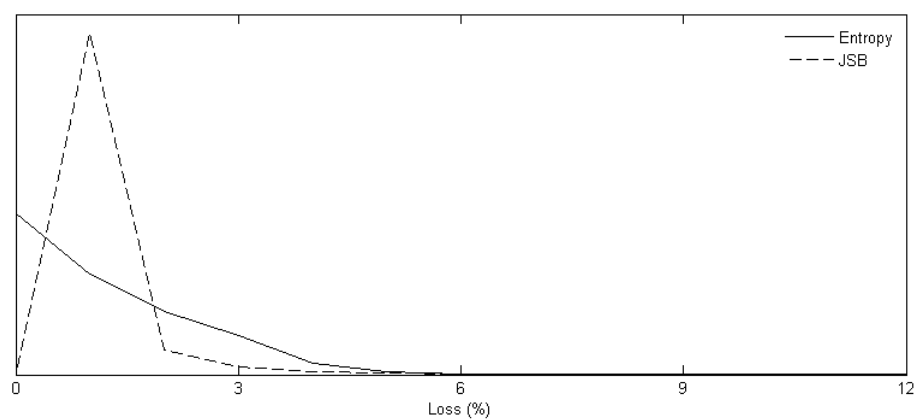
(b) 7y

Tranche (%)	Spread (bps)		
	Market	Entropy	JSB
0-3	500.0000	500.0000	867.3606
3-6	131.5000	131.5000	131.4007
6-9	37.5000	37.5000	38.1782
9-12	17.2500	17.2500	16.0135
12-22	6.0000	6.0000	7.9422
22-100	2.8500	2.8500	1.3184
0-100	34.0000	34.0000	34.8828

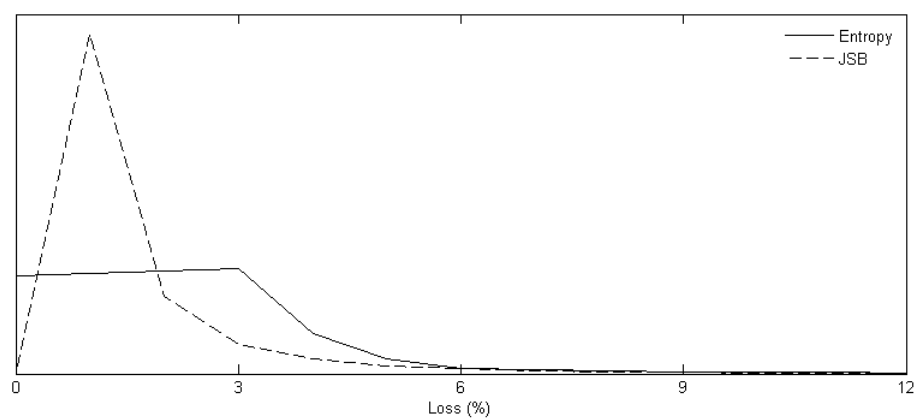
(c) 10y

Tranche (%)	Spread (bps)		
	Market	Entropy	JSB
0-3	500.0000	499.9985	1,046.4090
3-6	349.5000	349.4997	350.0946
6-9	94.0000	94.0000	96.1824
9-12	41.0000	41.0000	35.0908
12-22	13.7500	13.7500	15.0462
22-100	4.6500	4.6500	1.6874
0-100	45.0000	44.9999	45.2162

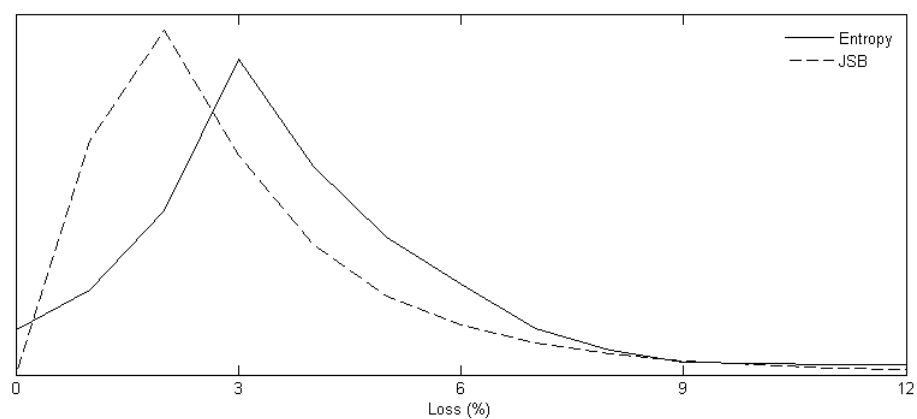
Table 2.2: Results for iTraxx Europe Series 6 CDO tranches as of 15-Mar-2007



(a) 5y

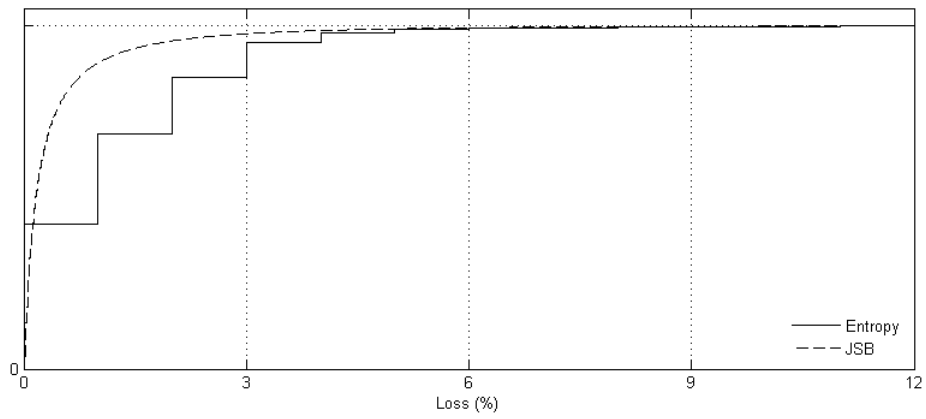


(b) 7y

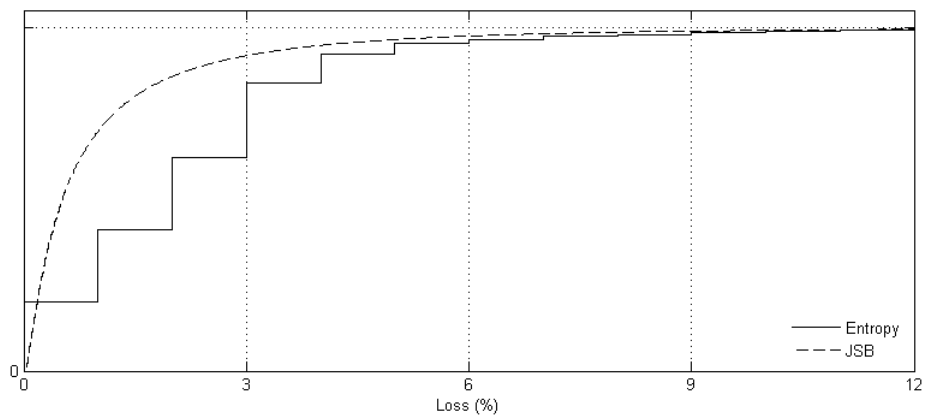


(c) 10y

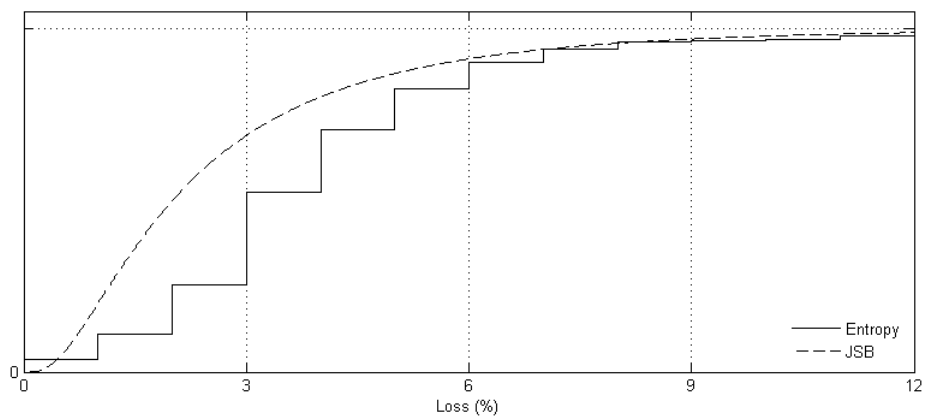
Figure 2.8: Calibrated loss density probability distributions for 5y, 7y and 10y



(a) 5y



(b) 7y



(c) 10y

Figure 2.9: Calibrated loss cumulative probability distributions for 5y, 7y and 10y under Entropy and JSB

With all input parameters and methods defined, we then apply the procedures described in section 2.2.3 for the JSB approach and in section 2.3.3 for the Entropy approach. We perform the calibration and then reprice the iTraxx tranches in order to compare the calculated results against market quotes, as shown by Table 2.2. While the JSB calibration is straightforward, the entropy calibration is not as easy and it took major effort to get it right. Given the size of the optimization problem and limitations of the Gauss-Newton nonlinear least-squares algorithm in Matlab's *fsolve* function, it took several trials to reach a version of the calibration procedure that would converge and not crash the program. By splitting the calibration procedure into three smaller procedures (for the three CDO tenors), the method described in section 2.3.3 was the one that converged to a satisfactory solution.

Given that, the resulting calibrated density loss probabilities are plotted in steps of 1% in Figure 2.8, while cumulative loss probabilities are shown in Figure 2.9 for expiries 5y, 7y and 10y and for both JSB and Entropy approaches. The first observation from Table 2.2 is that the Entropy approach matches all market quotes almost exactly while the JSB approach is relatively close for mezzanine tranches but performs poorly for the equity tranche 0-3 and the super senior tranche 22-100.

The result for the equity tranche points to one limitation of the JSB approach for pricing CDOs, especially evident through observing the densities in Figure 2.8 where JSB distributions yield zero probability to a zero portfolio loss event. However, given that defaults in a credit portfolio are supposed to be rare events, a zero portfolio loss is clearly an event with probability higher than zero. This is well captured by the Entropy approach for all expiries as observed through the non-zero probabilities for zero portfolio loss in Figure 2.8. This can also be explained through the cumulative loss distributions of Figure 2.9. Note that at subordination 3% the JSB approach consistently place a higher probability of the tranche 0-3 being wiped out when compared to the Entropy approach, a fact that explains the resulting higher spreads. On the other hand, from subordination 6% onward the cumulative loss probabilities of both approaches are reasonably close, also explaining why the spreads for mezzanine tranches from both approaches are not too far apart. Finally, the poor results of the JSB approach on the super senior tranche can be explained by the rapid decay of the distribution to zero for high subordinations.

Tranche (%)	Base correlations (%)		
	5y	7y	10y
0-3	15.50	13.14	11.74
0-6	24.59	22.97	17.41
0-9	31.79	31.03	25.31
0-12	38.10	37.99	32.55
0-22	55.05	57.56	53.96

Table 2.3: Market quoted base correlations for iTraxx Europe Series 6 CDO tranches as of 15-Mar-2007

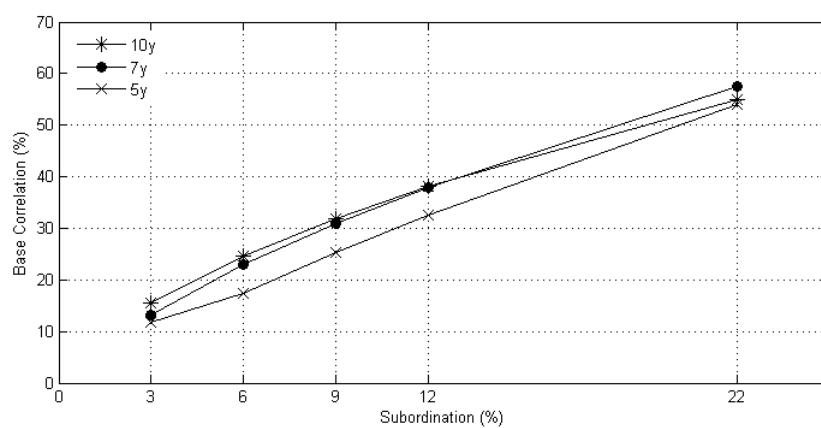
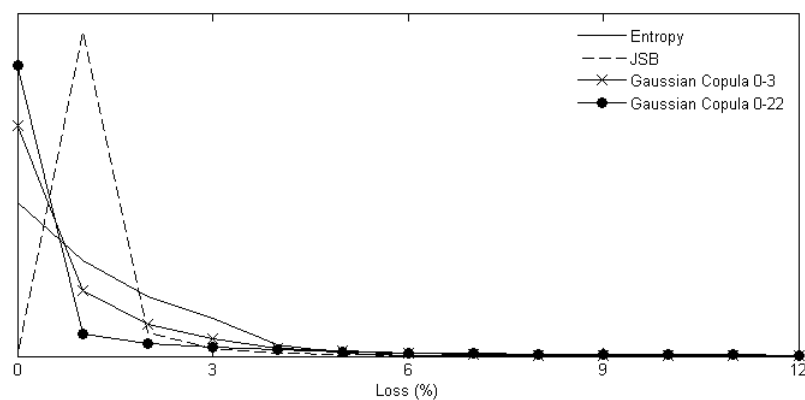
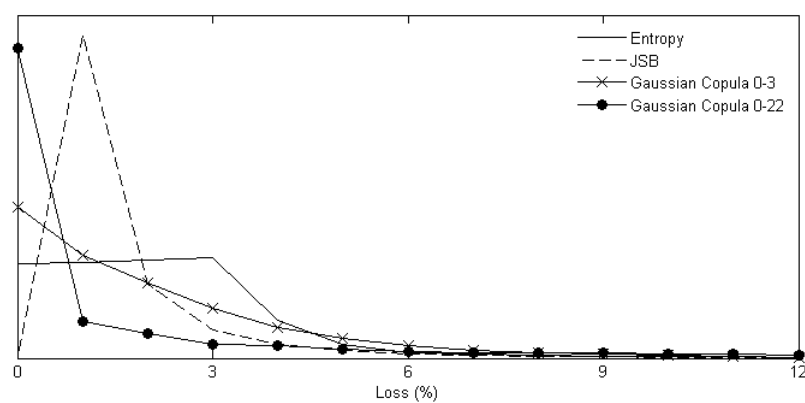


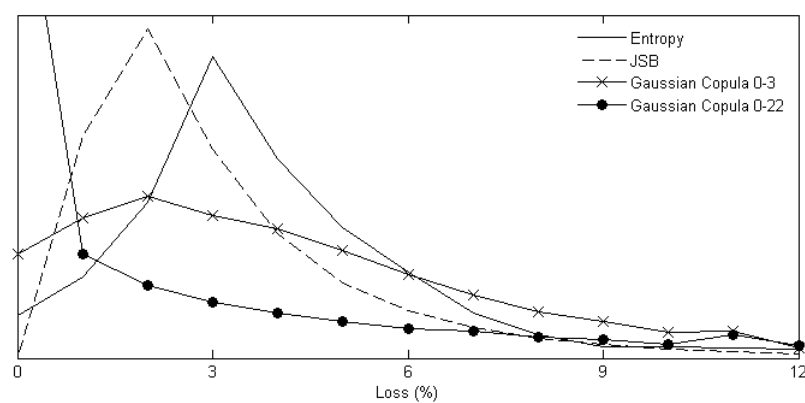
Figure 2.10: Plot of market quoted base correlations for iTraxx Europe Series 6 CDO tranches as of 15-Mar-2007



(a) 5y

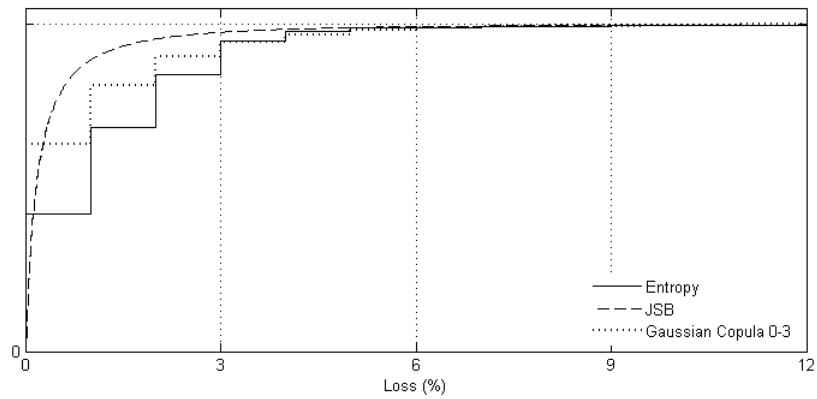


(b) 7y

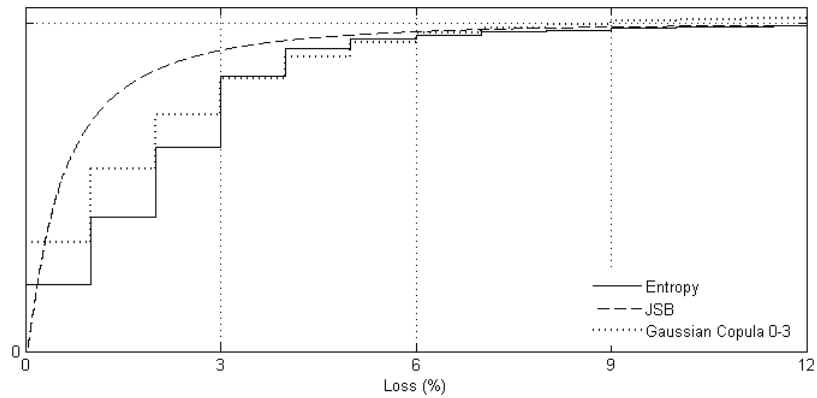


(c) 10y

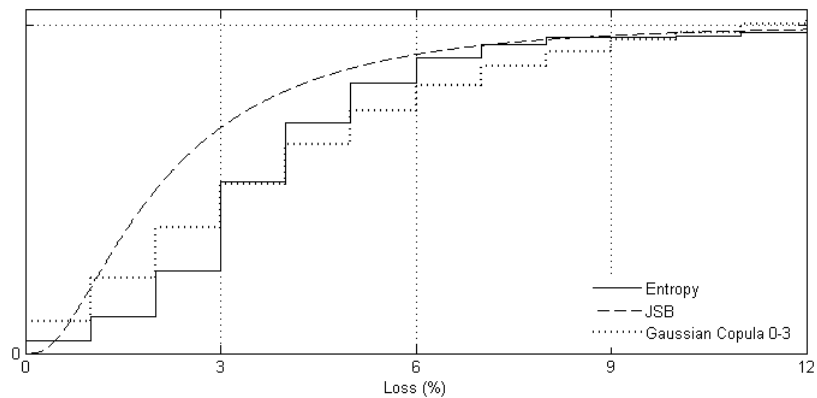
Figure 2.11: Calibrated loss density probability distributions for 5y, 7y and 10y of JSB and Entropy approaches in comparison to the Gaussian copula model with base correlations



(a) 5y

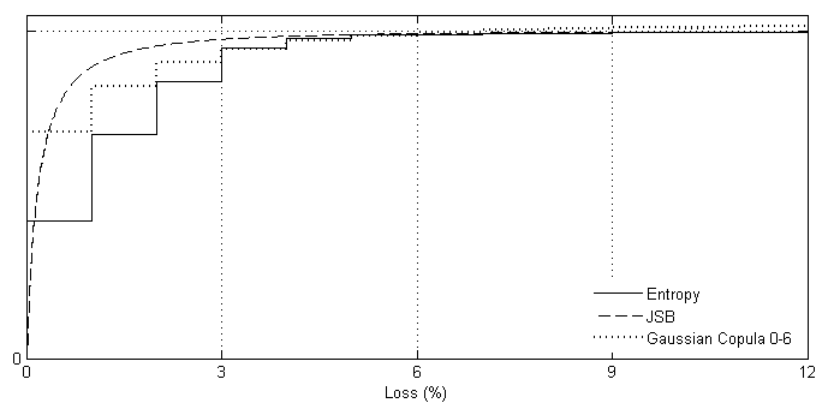


(b) 7y

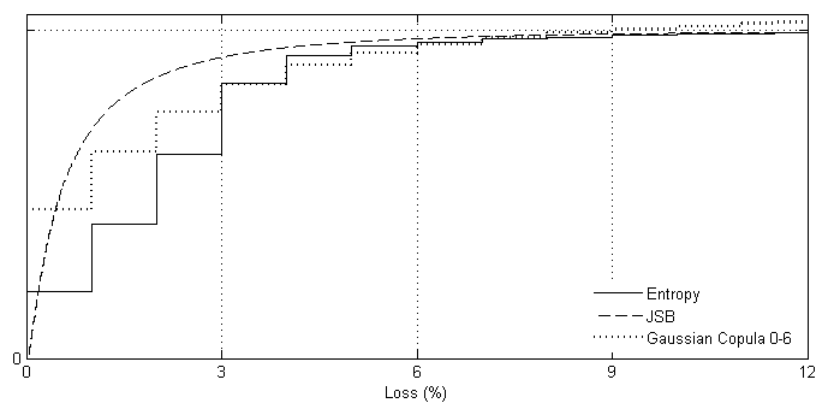


(c) 10y

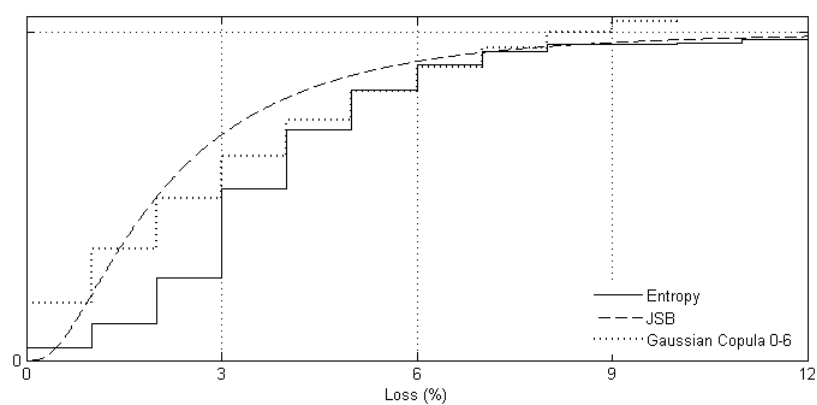
Figure 2.12: Calibrated loss cumulative probability distributions for 5y, 7y and 10y of JSB and Entropy approaches in comparison to the Gaussian copula model with 0-3 base correlation



(a) 5y



(b) 7y



(c) 10y

Figure 2.13: Calibrated loss cumulative probability distributions for 5y, 7y and 10y of JSB and Entropy approaches in comparison to the Gaussian copula model with 0-6 base correlation

A useful comparison can be made with the standard market model which applies the Gaussian copula* with base correlations. Table 2.3 shows the market quoted base correlations for the iTraxx Europe Series 6 CDO tranches as of 15-Mar-2007 while Figure 2.10 shows the well-known plot of these figures. These correlations map directly into the quotes from Table 2.1 in similar fashion to how implied volatilities map into vanilla option prices through the Black-Scholes model.† Figure 2.11 shows the previously calibrated JSB and Entropy probability densities for 5y, 7y and 10y in comparison to 0-3 and 0-22 densities from the Gaussian copula model respectively using 0-3 and 0-22 base correlations.‡ It is immediately observed that also the Gaussian copula model with base correlations prescribes a non-zero probability of no default, in line the results from the Entropy approach and further stressing the previously discussed limitations of the JSB approach. The market price matching and consistence of the results under the Entropy method are better observed through the cumulative probabilities in Figures 2.12 and 2.13. This result lends credibility to the Entropy method as it shows that its cumulative probabilities on subordination points 3% and 6% match the ones achieved by the Gaussian copula for each of the two respective base correlations.

A typical application of the parameters and probabilities calibrated previously is on pricing tranches with subordinations different from the standard ones. For instance, Table 2.4 shows the results for tranche subordination from the CDX index on the iTraxx Europe Series 6 portfolio. However, the real usefulness of a pricing method for CDOs relies on the ability to appropriately handle bespoke portfolios. These are portfolios that differ from the standard iTraxx or CDX indices, in which the only information available is the cumulative expected loss implied from CDS quotes of the underlying obligors, as previously discussed in sections 2.2 and 2.3. In this case, the strategy relies on partially using parameters and probabilities calibrated from the standard tranches and recalibrating them with the cumu-

*Please refer to Appendix A.7 for details on a Monte Carlo implementation of the Gaussian copula model.

†Please refer to J.P. Morgan Securities [28] for a description of the widely used base correlations to price CDOs with the Gaussian copula model.

‡Note that under the Gaussian copula with base correlations, each subordination point implies a distinct dummy probability density for which market prices of tranches under the respective subordinations are matched. Please refer to J.P. Morgan Securities [28] for more details on the base correlation method.

lative expected loss of the new portfolio. Societe Generale [48] shows the various methods that are applied in the industry in order to interpolate the Gaussian copula base correlation curve discussed in section 2.1.3 where, depending on the subordination level, each of those methods perform quite differently while some exhibit critical limitations such as computed negative spreads.

For the purpose of our application, a fictitious portfolio of 100 obligors is selected with 50 obligors assigned spreads from Cadbury Schweppes PLC and another 50 with spreads from Carrefour.* These spreads are the source to the only information available on this new portfolio, i.e., its cumulative expected loss shown in Figure 2.14. Table 2.5 shows the results after applying the steps for bespoke portfolios described in sections 2.2.3 and section 2.3.3 for the JSB and Entropy approaches, with the new portfolio cumulative expected loss and upfront payment of 15% for the equity tranche in all expiries. Apart from the 0-100 subordination, the results for JSB and Entropy approaches could not be more different. One immediate point is that keeping the b_j parameters from the iTraxx calibration as proposed in section 2.2.3 for the JSB approach is an assumption that bears no reasonable connection with the characteristics of the new portfolio. In the case of the Entropy approach, the spread are calculated with probabilities that are unbiased, achieved through the minimization of the relative entropy given prior probabilities (i.e., from the iTraxx calibration) and conditional to the arrival of new information (i.e., cumulative expected loss of the new portfolio). On this observation, there is no reason why the spreads from the Entropy approach should not represent the real spreads for the tranches on the bespoke portfolio, as they are unbiased estimates given available market information.

In conclusion, the Entropy approach clearly shows superior performance to the JSB approach on the calibration of standard indices tranches such as the iTraxx. However, it is with tranches on bespoke portfolios that the real advantage of the Entropy approach is revealed not only over the JSB approach, but also over the Gaussian copula with base correlations. Given any information set, the Entropy approach implicitly takes into account the dependence structure of the underlying portfolio as opposed to rely on parametric rep-

*The spread curves of Cadbury Schweppes PLC and Carrefour can be found in Appendix A.6.

Tranche (%)	Spread (bps)					
	5y		7y		10y	
	Entropy	JSB	Entropy	JSB	Entropy	JSB
3-7	45.54	44.45	110.47	107.45	293.06	282.68
7-10	11.19	11.94	28.76	30.75	68.66	75.50
10-15	4.10	4.36	10.87	11.17	24.46	23.02
15-30	2.03	1.76	4.79	4.41	11.00	7.89
30-100	1.06	0.57	2.66	1.32	4.05	1.69

Table 2.4: Results for CDX tranche subordinations on iTraxx Europe Series 6 portfolio

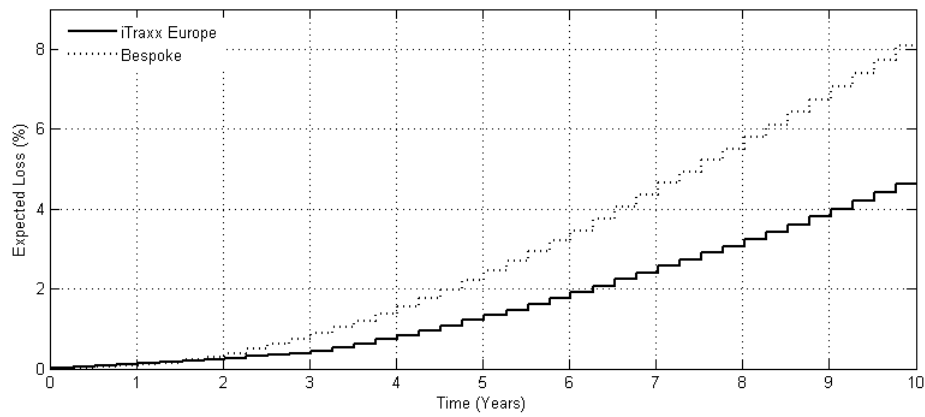


Figure 2.14: Expected loss for iTraxx Europe and bespoke portfolios

Tranche (%)	Spread (bps)					
	5y		7y		10y	
	Entropy	JSB	Entropy	JSB	Entropy	JSB
0-3	446.85	1007.96	780.84	1141.65	1131.07	1584.29
3-6	88.23	223.81	179.68	247.49	488.72	589.50
6-9	43.20	190.82	78.77	214.34	185.18	240.59
9-12	32.04	170.50	54.62	193.70	114.69	145.85
12-22	26.39	139.82	39.70	162.95	77.69	112.69
22-100	19.00	30.03	28.30	40.73	49.61	37.86
0-100	45.15	45.53	62.16	63.26	99.44	100.58

Table 2.5: Results for bespoke portfolio on iTraxx tranche subordinations and expiries

representations of dependence (e.g., correlation). In addition, the Entropy approach achieves an unbiased estimate of the spread for tranches in bespoke portfolios, far from true from the JSB approach and from the adopted market standard model Gaussian copula with base correlations as demonstrated by Societe Generale [48]. For these reasons, we favor the application of the Entropy approach described in section 2.3 as a more suitable method for pricing CDO tranches.

Conclusion

In this chapter the Collateralized Debt Obligation (CDO) was introduced in section 2.1.1 with a discussion of its characteristics, underlying securities, payoff structures and market indices. This was followed by a review of the Gaussian copula model in section 2.1.2 and its base correlation form in section 2.1.3 adopted by market participants as the standard model for pricing CDOs. After representing the CDO mathematically through a payoff function in section 2.1.4, the Johnson SB (JSB) approach for evaluating this payoff function was formulated in section 2.2.2 followed by its calibration procedure in section 2.2.3. Similarly, model formulation and calibration procedure were proposed in sections 2.3.2 and 2.3.3 for the approach applying entropy optimization techniques on the pricing of CDOs. Finally the two proposed approaches were applied to a real case pricing in section 2.4, where the Entropy approach stood out as the most suitable for pricing index and especially bespoke CDOs.

Chapter 3

Pricing Emission Reduction Purchase Agreements (ERPAs)

Introduction

Emission Reduction Purchase Agreement (ERPA) contracts are instruments linked to the existence of a carbon market, more specifically to project activities aimed at reducing greenhouse gas (GHG) emissions to the Earth's atmosphere. The implementation of such projects is motivated by the impact that humans have made on the climate of the planet. According to the scientific consensus,* the warming of the climate system is evident from observations of increases in global average air and ocean surface temperatures as well as through the melting of snow and ice causing sea levels to rise. With a high degree of confidence, the cause for this has been associated with the enormous growth in GHG emissions due to human activities since pre-industrial times. One immediate action that must follow is *adaptation*, since some of the effects of climate change are already felt in various parts of the world. Another crucial step is establishing alternatives that promote *mitigation* of GHG emissions, with the objective of achieving “stabilization of greenhouse gas concentrations

*For a comprehensive overview of climate science refer to Intergovernmental Panel on Climate Change (IPCC) [22].

in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system.”*

Key to both adaptation and mitigation is international cooperation, where governments should avail themselves of a variety of policies and instruments to create the incentives for action. The international carbon market is one such remarkable initiative created by the Kyoto Protocol (KP) to the United Nations Framework Convention on Climate Change (UNFCCC), adopted in December 1997 at the Conference of Parties 3 (COP3) in Kyoto, Japan. The KP represented a significant milestone as for the first time governments accepted legally-binding constraints on their GHG emissions, with industrialized countries (called Annex I parties) agreeing to reduce their collective GHG emissions on average to 5.2% below 1990 levels by 2012.[†] Under the hypothesis that where emissions reductions are achieved is irrelevant to the planet’s climate system, the KP created innovative market-based mechanisms aimed at achieving these reductions where they are least costly. The *International Emissions Trading* (IET), *Joint Implementation* (JI) and *Clean Development Mechanism* (CDM) are the three KP market-based mechanisms that have effectively established global markets for GHG emissions reductions and permits. These mechanisms give the parties of the KP as well as private sector participants the ability to reduce emissions in any part of the world where the cost of doing so is lowest in addition to claiming these reductions towards their own emissions targets.

The KP established GHG emissions caps where Annex I parties are allocated Assigned Amount Units (AAUs) in a quantity equal to their respective emissions target. At the end of the compliance period (2012 for the 1st commitment period of the KP), parties must surrender a quantity of AAUs equal to their actual emissions in the period. For parties expecting emissions to exceed their original AAU allocation, one obvious solution is mitigation via domestic government policies and initiatives and/or by devolving national level

*Article 2 of the The United Nations Framework Convention on Climate Change (UNFCCC) available at http://unfccc.int/essential_background/convention/background/items/1353.php.

[†]The reader is referred to http://unfccc.int/kyoto_protocol/items/3145.php for more details and current status of the KP.

emissions caps to individual emitting installations such as power plants. An example of the latter is the European Union Emissions Trading Scheme (EU ETS), where installations in the EU are imposed emissions caps in the form of European Union Allowances (EUAs) allocations. Another alternative through the IET (KP Article 17) is by trading emissions, where a party is able to meet its AAU shortage by purchasing the necessary amount from another willing party that has a surplus. And yet another possibility is importing emissions offset certificates generated through the other two KP mechanisms, i.e. CDM and JI.

The JI is defined under KP Article 6, where one Annex I party may invest in an emissions reduction project activity in another Annex I party and by doing so generate Emission Reduction Units (ERUs) that will count towards its own KP target. This mechanism was originally devised to encourage investment in emission reduction activities from industrialized countries to Annex I parties characterized as countries “in transition to a market economy,” typically Eastern European countries from the former communist block. In contrast to the IET, the JI is a baseline-and-credit system where project participants earn emissions reduction offsets (i.e., ERUs) through the implementation of a project activity that reduces emissions below a certain baseline scenario representing the level of emissions that would have happen in the absence of the project.

The CDM (KP Article 12) is also a baseline-and-credit system where project participants may claim Certified Emission Reductions (CERs) through the implementation of project activities that reduce emissions below a certain baseline scenario.* CDM offers Annex I parties the possibility of importing CERs as a flexible alternative to how they meet their emissions reduction targets, which in this case can be achieved at a lower cost in a developing country (KP non-Annex I) than it could be done domestically. By catalyzing green investment efforts in developing countries, CDM also aims to achieve sustainable development in those parts of the world that otherwise would be absorbed by immediate economic and social needs.

Whether these mechanisms are effective in promoting a shift in long-term investment pat-

*See UNEP’s publication “The Clean Development Mechanism (CDM)” <http://www.cd4cdm.org/Publications/CDMintro.pdf> for an introduction to the concept and benefits of the CDM.

terns toward low carbon initiatives is a heated debate, and highly dependent on carbon price signals coming from international and regional climate policy developments. However, it is important to note that the KP market-base mechanisms have at least established the foundations of future more successful mechanisms, a fact unquestionably verified by the notable response to climate change efforts globally through increased levels of accountability in emissions monitoring and management. And in the particular case of the CDM, a vibrant entrepreneurial contingent has been unleashed in both developed and developing countries, ranging from project developers to service providers, investors and financial institutions.

The administrative tasks of approving CDM project activities and issuing CERs are coordinated by the UNFCCC, with approval of eligible projects going back as early as 2004. Since then the number of projects in the pipeline has grown to about 2050 at the time of writing,^{*} and the resulting CERs have been an important source of emissions offsets assisting Annex I parties and capped domestic installations meeting their emissions targets. One example is the fact that Japan has relied heavily on CERs, although plans for a domestic emissions trading scheme (ETS) are already moving forward[†]. In addition, CERs are closely linked to the EU ETS[‡] and its allowance instrument the EUA, where installations covered by the scheme take advantage of cheaper CERs to achieve lower compliance costs.

The consequence of the importance and relative success of the CDM has been the increased liquidity in the market for secondary CERs, i.e., those CERs that have been already issued and are therefore available for delivery. In addition, CER Futures contracts were already launched by the European Climate Exchange (ECX) for trading in its ICE Futures electronic platform in late 2007, with a significant increase in trading volumes since then. Thus, it is not surprising that the CER is the underlying asset for an absolute majority of all ERPAs signed to date. In the context of the carbon market, ERPAs define the terms of generation, ownership and transfer of CERs from CDM projects while establishing the

^{*}Refer to project activities at <http://cdm.unfccc.int/Projects/index.html>.

[†]See Point Carbon [41] for an overview of emissions trading schemes around the world.

[‡]The EU ETS is currently the main ETS in the world, adopting CERs via the EU Linking Directive which allows installations to import CERs for meeting their emissions obligations.

allocation of risks among the all parties involved. They are vital instruments to the function of the CDM market as they formalize the terms of the offtake of CERs by the buyers, while ensuring that project participants are remunerated for their efforts in implementing emissions reduction project activities. In some cases ERPAs may also play a crucial role in securing early debt and/or equity financing for the project activities.

We limit the scope of this research to ERPA terms based on CER delivery, although in the presence of appropriate market information the concepts and techniques presented can be extended to other emissions offset instruments such as ERUs from JI projects and Voluntary Emission Reduction (VER) certificates from the various voluntary markets standards. We continue in section 3.1 with a detailed analysis of the CDM, where its project cycle is dissected for a subsequent mathematical representation. Then in section 3.2 we formulate the payoff function of an ERPA, provide analyses of underlying drivers and mathematical meaning in the form of stochastic models, and expound on the calibration of model parameters. Finally, in section 3.3 we establish a pricing algorithm and apply the models and techniques proposed with a case study on a run-of-river hydro power CDM project activities in China. We then close with our concluding remarks.

3.1 The CDM project cycle

In this section we make a detailed analysis of the CDM project cycle in order to derive an accurate representation of the steps to be modeled in the formulation of the ERPA pricing methodology. We start by introducing the diagram of Figure 3.1(a) which shows the CDM project cycle in a workflow form, dividing the CDM project cycle into two major blocks (*Survival* and *Performance*) and six intermediate process blocks.* In Figure 3.1(b) we translate these processes into an accurate probability tree with respect to CER delivery volume possibilities for each stage of a project. And finally Figure 3.1(c) illustrates a simplified probability tree which allows for a tractable mathematical representation of the

*A more detailed workflow of the CDM project cycle can be found in Institute for Global Environmental Strategies (IGES) [20], while a comprehensive description of each intermediate process is given by the CDM Rulebook at <http://cdmrulebook.org/>.

CDM project cycle. Below we discuss each process in the workflow of Figure 3.1(a), how it translates into CER delivery according to the probability tree of Figure 3.1(b), and the rationale for the simplifications leading to the probability tree of Figure 3.1(c).

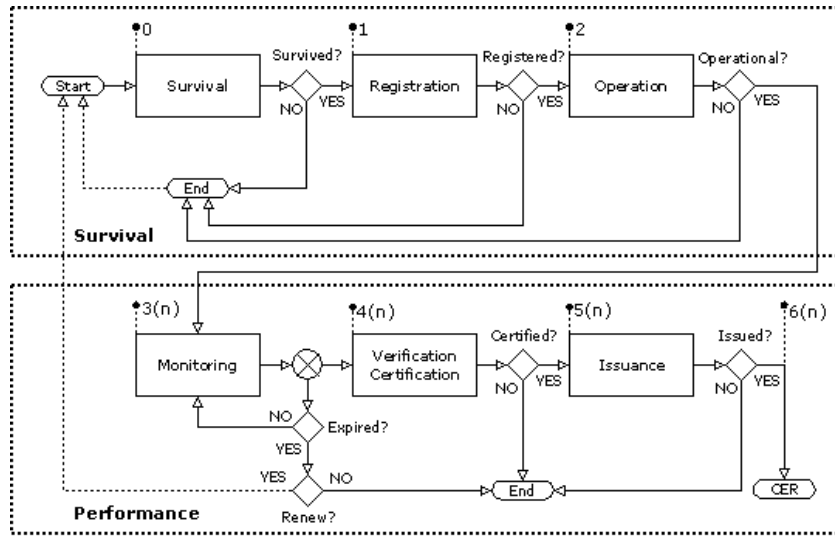
- **Survival**

This is the first process in the workflow of Figure 3.1(a), where a great number of steps must be successfully taken before a CDM project idea becomes reality. We call it survival precisely because failure in any of these steps mean that the project will not materialize. The very first step is the development of a baseline and monitoring methodology, which provides guidance on the calculation of the amount of emission reduction to be achieved by the project as well as how it should be measured. Assuming baseline and monitoring methodologies have been previously developed and approved*, the process really starts with a Project Idea Note (PIN). The PIN is a brief document (5-10 pages) containing information about the project (type, location, anticipated emission reduction, etc) as well as preliminary financial structure (indicative primary CER price, equity and debt financing needed, etc). This is ideally the step where project participants secure either financing or at least funds to cover for the costs of the CDM process.

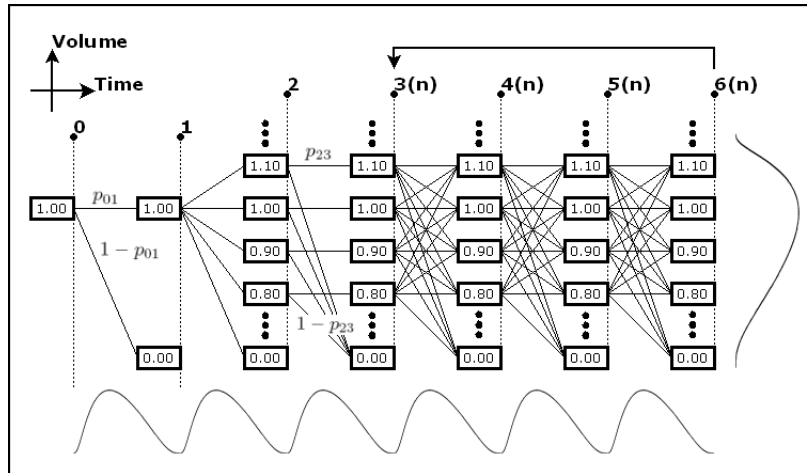
The next step is the preparation of a Project Design Document (PDD)[†], a complex task usually undertaken by consultants hired by the project participants. The PDD is such an important document in the CDM project cycle that no project can earn CERs without one being developed, validated and accepted by a CDM Executive Board (CDM-EB). A PDD contains more granular information on the project as compared to a PIN, including thorough planning with timetables and milestones as well as technical description including the technologies involved. However, two of the key components of a PDD have

*Usually the case for a wide variety of emissions reduction activities See currently approved methodologies at <http://cdm.unfccc.int/methodologies/index.html>

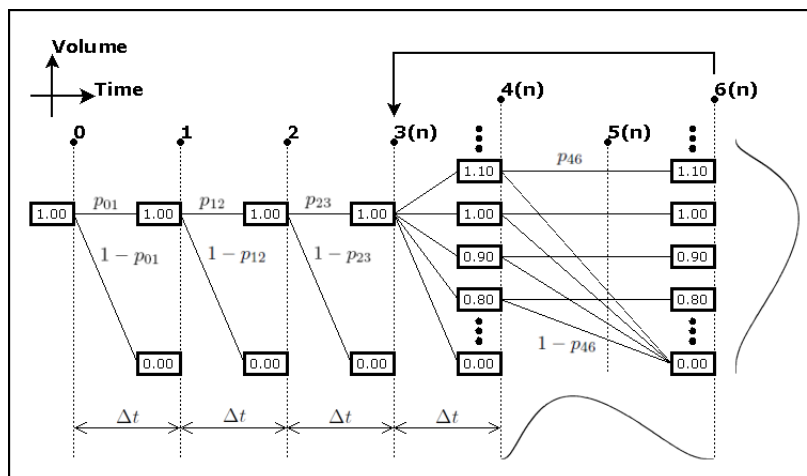
[†]PDD templates can be found at http://cdm.unfccc.int/Reference/PDDs_Forms/PDDs/index.html.



(a) CDM workflow



(b) Complete CDM probability tree



(c) Simplified CDM probability tree

Figure 3.1: CDM workflow and probability trees

to do with the core goals of CDM: additionality and sustainable development.

According to Article 12.5c of the Kyoto Protocol, CDM activities must generate emissions reductions that are additional to any that would have occurred in the absence of the project activity. This must be demonstrated by the PDD, implying that the proposed project activity requires financial support in the form of CERs to become viable. The challenge lies in performing a baseline analysis (by applying the baseline methodology) which estimates the hypothetical reference level of emissions that would have occurred without implementation of the project, the so called business as usual (BAU) scenario. In addition to that, the PDD must also demonstrate that the project activity will support the host country's sustainable development path. Finally, emissions reduction must be measured and therefore the PDD must include a monitoring plan based on a monitoring methodology.

Once ready, the PDD must be submitted to a Designated Operational Entity (DOE) for validation. This is the process of confirming that the proposed project activity meets the requirements of the CDM and therefore is eligible to be registered as a CDM project. Although project participants pay for validation services, the DOE is an independent entity acting on behalf of the CDM-EB in order to perform the pre-registration background work, which also includes consultation with stakeholders. The DOE is also the entity submitting the request for registration which broadly contains a PDD, a validation report and a Letter of Approval (LoA) by a Designated National Authority (DNA). A LoA can be requested from the DNA anytime while the above previous steps are being undertaken. The purpose of the LoA is for the DNA to confirm that it has ratified the Kyoto Protocol, that it approves the participation of the proponents in the project activity, and in the case of the host party, that the project will contribute to sustainable development within its borders.

Referring to Figure 3.1(a), the survival stage starts at node 0. Throughout the steps above until node 1, the issue lies on whether the project is able to survive from one step to the

next. In reality this constitutes a binary process which is widely represented in finance by probabilities of survival. This is illustrated in Figure 3.1(b) and Figure 3.1(c), where p_{01} is the probability of survival from node 0 to 1 while $1 - p_{01}$ quantifies the probability of failure. The quantity 1.00 represents the reference amount of 100% of the emissions reduction proposed in the PDD.

- **Registration**

The next step in the workflow of Figure 3.1(a) is registration, which represents the formal acceptance by the CDM-EB that makes a project activity eligible to receive CERs from its emissions reduction. The process starts with the submission of a request for registration by the DOE that validates the project. Then the UNFCCC secretariat processes the request, leading to the appointment by the CDM-EB of a CDM Registration and Issuance Team (CDM-RIT) with the responsibility to appraise the request. Based on the conclusions of the CDM-RIT, the CDM-EB may approve or reject the project activity, or request a review in case at least three board members raise concerns with the proposal. Causes for request for review vary from minor breach of existing guidance on PDD preparation and documentation to major ones involving additionality and the estimated emissions reduction by the project. DOE and project participants may have to provide further inputs and the outcome of this process may be approval, rejection or approval with a reviewed estimated emissions reduction for the project activity.

Shown by Figure 3.1(a) between nodes 1 to 2, this is now a process that has a more granular probabilistic profile as compared to the survival stage. The fact that PDD amounts can be amended through the review process makes this granular approach a more accurate representation of what happens during registration, as illustrated by Figure 3.1(b). However, the distribution is in practice highly concentrated around 0.00 and 1.00 due to higher likelihood of a project being either approved or rejected. Because this fact resembles the behavior of a survival stage and in addition to difficulties in identifying reviewed PDD amounts in the data, we choose to simplify its representation to the one in Figure 3.1(c). Now we have a single probability of survival p_{12} and any granular volumetric

variation missed by this approximation can be captured at later stages by issuance and delivery probabilities with reference to original PDD amounts.

- **Operation**

Figure 3.1(a) illustrates this post-registration stage which might not even exist in case the project is already operational and the monitoring procedures are implemented. In this case, generation of emission reduction eligible for CERs starts immediately after registration is granted. Otherwise there will inevitably be a delay between the registration date and the official emission reduction start date. And despite registration, the project could still be going through operational and financial issues that might prevent it from ever generating any CERs.

This stage is showed in Figure 3.1(a) between nodes 2 to 3, while its contribution toward delivery risk is represented by a single probability of survival p_{23} in both Figure 3.1(b) and Figure 3.1(c). This is reasonable given the only impact this stage may have on CER delivery is through operational failure, as previously discussed.

- **Monitoring**

The first monitoring activity effectively starts with the emissions reduction start date, i.e., once the project is both registered and operational (Figure 3.1(a)). It involves the measurement and analysis of emissions to determine the volume of emissions reduction achieved by the project activity in a certain time interval. The monitoring activity must follow the monitoring plan defined in the PDD, which provides for the collection and analysis of all data relevant to the calculation of emissions reduction by the project. At the end of every monitoring period, project participants must provide a monitoring report to the DOE engaged to conduct verification and certification of the achieved emissions reduction in order to then be able to receive CERs.

The monitoring process is located between nodes $3(n)$ to $4(n)$ in Figure 3.1(a) and this is

the stage where actual emissions reduction performance occurs. The n between brackets represents the fact that a project activity may have a number of monitoring periods until expiry. This stage is the driver for the probabilistic behavior of emissions reduction by CDM projects, illustrated in Figure 3.1(b) by the entangled set of possibilities. In this case Figure 3.1(c) confirms that no simplification can be made given that in reality a project can underperform or overperform its PDD emission reduction in each monitoring period.

- **Verification and Certification**

This is another stage where DOEs play a central role, this time following a monitoring activity as shown by the diagram in Figure 3.1(a). In the verification process, a DOE confirms the authenticity of emissions reduction by the project over the monitoring period indicated in the monitoring report. This involves documentation and procedural checks, on-site inspections and even use of additional data from independent sources. Project participants in the end receive a verification report with the verified amount of emissions reduction, which might differ from the one in the monitoring report. This is due to the complexities of data processing and calculations in monitoring methodologies as well as conservatism by the DOE in case of any deviations in the carried out monitoring activity when compared to the registered monitoring plan. The next step in the process is called certification, where the DOE formally certifies the emissions reduction achieved by the project by means of written notice to the project participants, parties involved and the CDM-EB.

This process is represented through nodes $4(n)$ to $5(n)$ of Figure 3.1(a) and is similar to the case of registration, where Figure 3.1(b) illustrates a more accurate granular representation. From the process description above, the appraisal by a DOE may conclude on the certification of any level of emissions reduction in reference to the PDD. However, the same issues observed with registration apply here, which have to do with the higher likelihood that a DOE will either approve or reject the monitored emissions reduction and the fact that verified versus monitored amounts are not easy to capture from

the CDM data. For these reasons we make a simplification from a granular approach to a survival process, and combine this stage's survival probability with the one from the issuance process (Figure 3.1(c)) discussed below.

- **Issuance**

Issuance is the final stage in the workflow of Figure 3.1(a), and represents the creation of CERs equivalent to the amount of emissions reduction that have been generated, verified and certified in respect of a CDM project activity. Issuance concludes with the crediting of the CERs into the pending account of the CDM-EB in the CDM Registry, which can then be forwarded to the accounts of parties and project participants. Similarly to the registration process, a CDM-RIT is appointed to appraise the request. Based on its conclusions, the CDM-EB may issue, reject or issue a reviewed amount. Note that a levy of 2% is applied to any volume issued, and subsequently credited to a Share of Proceeds (SOP) account which currently provides the only source of funding for adaptation projects.

Figure 3.1(a) shows this stage as the process flow from node $5(n)$ to $6(n)$. Again, while a granular probabilistic approach is a more accurate representation as showed in Figure 3.1(b), we reduce it to a survival process as illustrated by Figure 3.1(c) due to similarities with the registration and verification processes. We combine survival probabilities from $4(n)$ to $6(n)$ into a single probability p_{46} given all issuance events will have to go through verification, certification and issuance regardless of whether the project is at an early stage or already monitoring emissions reduction.

3.2 Formulation

3.2.1 Introduction

ERPA contracts define the terms of generation, ownership and transfer of CERs from CDM projects, while establishing the allocation of risks among the parties involved. An ERPA effectively formalizes the terms of CER offtake from a buyer, while for the seller it ensures that the effort of implementing a CDM project translates into revenue through delivery of CERs. In some cases ERPAs may also play a crucial role in securing early debt and/or equity financing for the project.

In an ERPA the buyer (emitter of a cap-and-trade scheme, financial intermediary, etc) pays a certain price structure to the seller (project participants) in exchange for delivery of issued CERs under a certain quantity or volume structure, as illustrated in Figure 3.2. The major elements of ERPAs are:

- Volume
 - Delivery structure: full volume, capped and/or floored volume, etc
 - Delivery schedule: as CERs are issued by the CDM EB, at pre-defined future dates, etc
- Price
 - Payment structure: prepaid, spot price on delivery, capped and/or floored price, etc
 - Payment schedule: at inception of contract, upon delivery, at pre-defined future dates, etc
- Legal
 - Buyers and sellers rights, obligations, shares, etc
 - Provisions for event of non-delivery
 - Penalties for delays on delivery

- Counterpart default events
- Other legal provisions and penalties

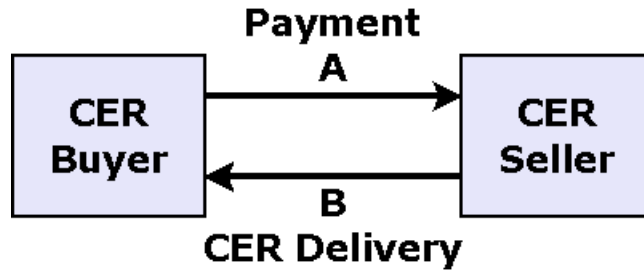


Figure 3.2: ERPA diagram

Legal aspects of ERPAs must not be underestimated and we encourage the reader to refer to Baker & McKenzie [3] for more details on legal issues in the CDM market. However, in this research we focus on the pricing elements of ERPAs which involve volume and price. We continue with section 3.2.2 where we formulate the ERPA payoff function and propose a risk-neutral pricing scheme. This is followed by the mathematical development of the dynamics (including calibration) for the major drivers of ERPA contracts, respectively delivery time and volume in section 3.2.3 and carbon prices in section 3.2.4.

3.2.2 Payoff function

In the context previously described, ERPAs are effectively contingent claims on carbon prices and delivery time and volume. Furthermore, as per diagram in Figure 3.2, an ERPA can be mathematically represented by a swap with a payment leg $\mathcal{A}$ and a CER delivery leg $\mathcal{B}$. In financial terms, a fair-valued ERPA is achieved when the present value of all expected future cash flows of $\mathcal{A}$ and $\mathcal{B}$ match. This is mathematically expressed as

$$\mathcal{A} - \mathcal{B} = 0,$$

$$\mathcal{A} = \sum_{n=1}^{N_n} \mathbf{E} [f_n(B_n, C_n, E_n, V_n, t_n, y_n(t_n))] + \sum_{j=1}^{N_j} U_j \mathbf{E} [B_j], \quad (3.1)$$

$$\mathcal{B} = \sum_{n=1}^{N_n} \mathbf{E} [B_n(t_n) C_n(t_n) g_n(y_n(t_n))],$$

where:

- each n represents a payment event by the CER buyer under the payoff structure f_n defined by the ERPA, which matches a delivery event (illustrated by n between brackets in Figure 3.1(c)) as per schedule and volume structure g_n defined by the ERPA;
- each j represents a scheduled lump sum payment U_j by the CER buyer;
- V_i and B_i are respectively PDD volume and discount factor for a certain schedule i ;
- C_i and E_i are dynamic drivers representing carbon prices for a certain schedule i , respectively CER prices and any other carbon price driver that may be considered in the ERPA (e.g. EUA or ERU);
- t_n and y_n are the dynamic drivers linked to delivery event n , respectively the delivery time and volume;
- and $\mathbf{E}$ is the expectation taken under the risk-neutral measure.

Equation (3.1) is a general representation that is able to accommodate a wide range of payoff structures, and implicitly takes into account interdependencies between the dynamic variables. However, it is still not tractable, and it must be simplified in order to reach its usable form. We do this by proposing the set of assumptions that follows.

Assumption 1. B_i is deterministic and independent of dynamic variables C_i , E_i , and y_n .

Assumption 2. y_n is independent of carbon prices C_i and E_i .

Assumption 3. y_n is independent of delivery time t_n .

Assumption 4. t_n is independent of t_{n-1} .

Assumption 5. y_n is independent of y_{n-1} .

Assumption 1 is common in the mathematical finance literature when payoff functions are not directly written on interest rates. In our formulation the discount factor B_i is a time-dependent function of deterministic interest rates, interpolated according to t_n or specific schedules n and j as will become clear in section 3.2.4.

On Assumption 2, in theory low carbon prices should have the effect of diminishing the incentive to develop emissions reduction projects that would earn CERs, while the opposite is true for high carbon prices. On the other hand, this should be observed only as a long term market-wide dependence, realized mainly when carbon prices are kept in certain extreme low or high levels for a long period of time. Such circumstances are more likely to affect long-term planning of future projects, having only a marginal effect on currently ongoing initiatives. Furthermore, the design of cap-and-trade schemes (where caps are tightened over time) coupled with expected higher future abatement costs should guarantee upward trend in prices over time.

Assumption 3 is not exactly true from a post-2012 point of view. The capacity of projects to produce CERs should in theory be affected when delivery happens in relation to post-2012 cutoff dates*. Early stage projects might struggle securing financing given that the majority of expected CERs from such activities would happen in post-2012 schedules. And in already advanced stage projects, monitoring and verification costs might exceed post-2012 CER revenues, leading project participants to halt CER-related operations. Arguably market participants already incorporate these considerations in ERPA transactions by focusing on pre-2012 ERPA schedules and adjusting survival expectations of early stage projects through due diligence.

*There exist a great level of uncertainty on demand for CERs post-2012, given the KP 1st commitment period expires then and that there is no international agreement in place to define post-2012 emissions targets. As a result ERPAs generally impose a cutoff with dates up to Apr-2013, when delivered CERs may still find their way into Phase 2 of the EU ETS.

Assumptions 4 and 5 propose that delivery of CERs in a delivery event n are not dependent on the outcome of delivery event $n - 1$. This is again not exactly true as indirect interdependencies may emerge in practice during the operational life of the project. Besides, issues with equipment performance and malfunctioning, fires and other accidental events in the project's installations affecting generation of CERs in $n - 1$ may still persist in the next monitoring period. We assume that most disruptions are unlikely to linger in the interest of project participants to maximize CER revenues, while the more extreme events are expected to be covered by insurance and therefore beyond the scope our analysis.

We now apply the above assumptions to (3.1) to obtain

$$\begin{aligned}\mathcal{A} - \mathcal{B} &= 0, \\ \mathcal{A} &= \sum_{n=1}^{N_n} \mathbf{E} [f_n(B_n, C_n, E_n, V_n, t_n, y_n(t_n))] + \sum_{j=1}^{N_j} B_j U_j, \\ \mathcal{B} &= \sum_{n=1}^{N_n} B(t_n) \mathbf{E} [C_n(t_n)] \mathbf{E} [g_n(y_n)].\end{aligned}$$

Note that $\mathbf{E} [C_n(t_n)]$ under the risk-neutral measure is the CER Forward price which we write as $C_n(t_n)$, yielding

$$\begin{aligned}\mathcal{A} - \mathcal{B} &= 0, \\ \mathcal{A} &= \sum_{n=1}^{N_n} \mathbf{E} [f_n(B_n, C_n, E_n, V_n, t_n, y_n(t_n))] + \sum_{j=1}^{N_j} B_j U_j, \\ \mathcal{B} &= \sum_{n=1}^{N_n} B_n(t_n) C_n(t_n) \mathbf{E} [g_n(y_n)].\end{aligned}\tag{3.2}$$

Equation (3.2) effectively reduces $\mathcal{B}$ to a sum of N_n independent delivery events, and can now be used to formulate actual ERPA contracts. One typical payoff is defined by an outright primary CER purchase price with respect to the PDD volume, where the buyer agrees to pay a fixed price in exchange for all issued CERs as they are delivered. In

this case $U_j = 0$ and schedules m and n are equal, where the risk-neutral expectation $\mathbf{E}[f_n(B_n, C_n, E_n, V_n, t_n, y_n(t_n))]$ becomes $B_n(t_n)V_nP$. Note that P is the fixed primary CER price, calculated as

$$P = \frac{\sum_{n=1}^{N_n} B_n(t_n)C_n(t_n)\mathbf{E}[y_n]}{\sum_{n=1}^{N_n} B_n(t_n)V_n}. \quad (3.3)$$

where $\mathbf{E}[y_n]$ represents the expected CER delivery in each event n .

In a similar way, another common payoff is one where the buyer pays a floating price based on a percentage of the secondary CER price. In this case $\mathbf{E}[f_n(B_n, C_n, E_n, V_n, t_n, y_n(t_n))] = B_n(t_n)C_n(t_n)V_nP$, which results in

$$P = \frac{\sum_{n=1}^{N_n} B_n(t_n)C_n(t_n)\mathbf{E}[y_n]}{\sum_{n=1}^{N_n} B_n(t_n)C_n(t_n)V_n}. \quad (3.4)$$

A pattern quickly emerges from (3.2), as the key to pricing ERPA contracts depends on the ability to formulate and evaluate the expectations $\mathbf{E}[f_n(B_n, C_n, E_n, V_n, t_n, y_n(t_n))]$ and $\mathbf{E}[g_n(y_n)]$ under the risk-neutral measure. Although the payoff examples discussed above are straightforward, this task becomes challenging as in reality f_n and g_n may be any functional form. It is not uncommon to find collar price structures with capped and floored floating prices, as well as volume delivery tranche structures where sellers commit to deliver up to a certain percentage of the PDD volume while keeping the excess as performance upside.

Thus, depending on the form of f_n and g_n , closed-form solutions may not be available and (3.2) must be evaluated numerically. One alternative is simulating the underlying drivers in a Monte Carlo algorithm, evaluating the f_n and g_n for each path and taking their respective expectations. What follows in the next two sections is the definition of stochastic representations for the dynamic drivers in ERPAs as well as the calibration of their model parameters.

3.2.3 Emissions reduction delivery dynamics

Delivery dynamics and Johnson SB distributions

In this section we establish the dynamic behavior of t_n and y_n in terms of stochastic models. First define

$$\begin{aligned} w_n &= T_n + \alpha_n, \\ t_n &= w_n + h(w_n) \end{aligned} \tag{3.5}$$

as the model for t_n . In (3.5), T_n is a deterministic variable representing the scheduled end time of the n th monitoring period, taking into account all time delays due to early stages in the CDM probability tree of Figure 3.1(c). On the other hand, α_n is the actual independent random variable representing the elapsed time taken by project participants to complete and submit the monitoring report, by the DOE to verify and certify the emissions reduction, and by the CDM-EB to issue the requested CERs. Finally, $h(w_n)$ is a function defined to accommodate any specific delivery schedule in relation to issuance dates. For instance, for actual issuance on 26-Mar-2010 with an ERPA agreed delivery date of 20-Dec-2010, the value of $h(w_n)$ in the n th delivery event is a constant equal to 311 days.

Now define the model for y_n as

$$y_n = V_n \beta_n \tag{3.6}$$

where V_n is the PDD volume and β_n is an independent random variable quantifying the volume to be delivered in t_n .

The key now is to determine the probabilistic law which governs the behavior of the random numbers α_n and β_n . Observation of the CDM historical data indicates that these variables are highly skewed and fat-tailed, thus not obvious candidates for the widely applied normal distribution paradigm. Regarding their domain, it is quite intuitive that these variables must

be non-negative and are capped by a maximum value. Time cannot go backwards while the minimum possible CER volume delivered is zero. On the other hand, time is limited at the far end by the expiry date of the project activity while volume is restricted by the fact that it is physically impossible for a project activity to produce an infinite amount of emissions reduction.

We found that the Johnson SB (JSB) distribution introduced in section 1.1 offers a suitable probability density function to represent both α_n and β_n . Due to its reasonable tractability, closed-form and semi-analytical solutions may be achievable in some particular cases of payoff functions with JSB underlying variables as shown in the CDO pricing formulation of section 2.2.2. However, the general case requires sampling of random numbers with (1.4c) and application of Monte Carlo simulation. This is the path we follow in this research, where we calibrate α_n and β_n as JSB variables, simulate t_n and y_n respectively with (3.5) and (3.6) through sampling α_n and β_n , and finally apply t_n and y_n to the pricing of ERPAs with (3.2).

Calibration of JSB parameters

The aim of this section is to formulate a method for the calibration of JSB parameters of the random variables α_n and β_n discussed previously. In finance, the formulation of appropriate parameter calibration methods are well known in the literature and in practice to be key to the applicability of the model. The *Fundamental Theorem of Asset Pricing** states that if markets are arbitrage-free and complete, there exist an equivalent unique risk-neutral or martingale measure through which prices of all assets are determined and have the same expected rate of return (the risk-free rate).

In a practical model calibration exercise, the arbitrage-free markets assumption is absorbed implicitly while a non-linear optimization routine applies the complete markets postulate by matching the number of asset prices (equations) to the number of parameters in the model (unknowns). However, complete markets are rarely the case in practice, which implies the

*See Harrison and Pliska [19].

existence of infinitely many equivalent risk-neutral measures. This usually translates into an underdetermined nonlinear system where the number of unknowns exceed the number of equations, clearly a case where the value of at least one parameter must be statistically estimated or arbitrarily assigned.

In the case of carbon, instruments such as CER shortfall swaps (where a buyer pays a seller to cover for CER delivery shortfall from ERPAs) or even plain volumetric options on CER delivery would be ideal inputs to the calibration of JSB parameters for processes (3.5) and (3.6). Unfortunately the CDM primary market has not yet reached this level of sophistication, which makes the issue with incomplete markets on CER delivery quite pronounced. In the absence of observable primary CER prices or other contingent claims on CER delivery, the only available information that can be used in model calibration is CDM historical delivery data.

However, even in incomplete markets we can still achieve unique equivalent risk-neutral probabilities for CER delivery by applying the entropy optimization techniques introduced in section 1.2. As discussed in that section, according to the minimum relative entropy (MinEnt) principle by Kullback and Leibler [31], an unbiased probability density is achieved by minimizing the relative entropy in (1.20) given a prior distribution and conditional to one normalization condition and a set of functional constraints respectively in (1.18) and (1.19). Kapur and Kesavan [29] showed that the density $f(x)$ found this way exists and is unique because the relative entropy is a convex function.

This result is crucial to the underlying idea of our approach, that in the absence of further market-based information, a least-biased probability density estimated through the minimization of the relative entropy in respect to a prior distribution and a set of constraints is the suitable choice for an equivalent risk-neutral measure. The reasoning is that among an infinite number of equivalent risk-neutral densities and given the MinEnt principle, any other density than the one that minimizes the relative entropy is biased and therefore must not prevail the unbiased alternative. This idea is not new and has a parallel with the concept of Minimal Entropy Martingale Measure (MEMM) introduced by Miyahara [37], which later appeared in the literature on option pricing applications in incomplete markets.

We now move to define $f(x)$ as the JSB density function from (1.3c), which directly satisfies the normalization constraint. Given the available CDM historical data, the functional constraints are moment functions defined by $h_k(x) = x^k$ for the k th moment of x about zero, while $c_k = \overline{m}_k$ is the k th sample moment about zero. For calibration straight from the CDM data, the domain $[X_A, X_B]$ can be discretized as a histogram in intervals $[x_{i-1}, x_i)$ such that $g(x_i) = p_i > 0$ is constant for every i . This ensures that no assumptions are made on the prior distribution and that actual prior information derived from the data is effectively used, thus preserving the goal toward an unbiased solution.

In order to reduce the relative entropy from (1.20) to a usable form, we start by splitting it into two integrals $\mathcal{A}$ and $\mathcal{B}$ such that

$$\begin{aligned}\mathcal{K} &= \mathcal{A} - \mathcal{B}, \\ \mathcal{A} &= \int_{X_A}^{X_B} f(x) \log f(x) dx, \\ \mathcal{B} &= \int_{X_A}^{X_B} f(x) \log g(x) dx.\end{aligned}$$

Because $g(x_i) = p_i$ is constant for every i and given $\mathcal{J}_{SB}(x)$ as the JSB cumulative probability distribution from (1.4c), the solution for $\mathcal{B}$ immediately follows as

$$\begin{aligned}\mathcal{B} &= \sum_{i=1}^{N_i} \log p_i [\mathcal{J}_{SB}(x_i) - \mathcal{J}_{SB}(x_{i-1})] \Rightarrow \\ \mathcal{B} &= \sum_{i=1}^{N_i} \log p_i \left\{ \Phi \left[a + b \log \left(\frac{x_i - c}{c + d - x_i} \right) \right] - \Phi \left[a + b \log \left(\frac{x_{i-1} - c}{c + d - x_{i-1}} \right) \right] \right\}, x_0 = c.\end{aligned}$$

Now given the JSB density in (1.3c), we split $\log f(x)$ such that

$$\log f(x) = \log \left[\frac{b}{\sqrt{2\pi}} \right] + \log \left[\frac{d}{(x-c)(c+d-x)} \right] - \frac{1}{2} \left[a + b \log \left(\frac{x-c}{c+d-x} \right) \right]^2,$$

and rewrite $\mathcal{A}$ as

$$\begin{aligned}
\int_c^{c+d} f(x) \log f(x) dx &= \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3, \\
\mathcal{A}_1 &= \log \left(\frac{b}{\sqrt{2\pi}} \right) [\mathcal{J}_{SB}(c+d) - \mathcal{J}_{SB}(c)], \\
\mathcal{A}_2 &= \int_c^{c+d} \log \left[\frac{d}{(x-c)(c+d-x)} \right] \frac{b}{\sqrt{2\pi}} \frac{d}{(x-c)(c+d-x)} \\
&\quad \exp \left\{ -\frac{1}{2} \left[a + b \log \left(\frac{x-c}{c+d-x} \right) \right]^2 \right\} dx, \\
\mathcal{A}_3 &= -\frac{1}{2} [\mathcal{J}_{SB}(c+d) - \mathcal{J}_{SB}(c)].
\end{aligned}$$

Integrals $\mathcal{A}_1$ and $\mathcal{A}_3$ have closed-form solutions given by*

$$\mathcal{A}_1 = \log \left(\frac{b}{\sqrt{2\pi}} \right) \text{ and } \mathcal{A}_3 = -\frac{1}{2},$$

while $\mathcal{A}_2$ can be reduced to†

$$\mathcal{A}_2 = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \log \left(1 + e^{\frac{z-a}{b}} \right) e^{-\frac{1}{2}z^2} dz - \frac{bd+a}{b}.$$

where z is the standard normal variable from the JSB transformation in (1.1) and (1.2c). The remaining integral portion of $\mathcal{A}_2$ can be solved numerically through the quadrature formula by Goodwin [16] discussed in section 1.3. We identify $f(x)$ in (1.21) from the integral in $\mathcal{A}_2$ as

$$f(x) = \frac{2}{\sqrt{2\pi}} \log \left(1 + e^{\frac{z-a}{b}} \right).$$

*Proof in Appendix B.1.

†Proof in Appendix B.1.

In the case of $\mathcal{A}_2$ Goodwin's method is appealing given its simplicity and suitability as

$$\log \left[1 + e^{\frac{z-a}{b}} \right] \rightarrow \frac{z-a}{b}$$

for large values of z while a is finite and $b > 0$, leading to negligible error as explained in section 1.3. We now make $z = \sqrt{2}u$ in $\mathcal{A}_2$ and apply Goodwin's formula to yield

$$\mathcal{A}_2 \approx \frac{2h}{\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \log \left(1 + e^{\frac{\sqrt{2}nh-a}{b}} \right) e^{-n^2 h^2} - \frac{bd-a}{b}.$$

Combining $\mathcal{A}$ and $\mathcal{B}$ leads to the final form of our objective function, defined by

$$\begin{aligned} \mathcal{K} = & \log \left(\frac{b}{\sqrt{2\pi}} \right) + \frac{2}{\sqrt{\pi}} h \sum_{n=-\infty}^{+\infty} \log \left(1 + e^{\frac{\sqrt{2}nh-a}{b}} \right) e^{-n^2 h^2} - \frac{bd-a}{b} - \frac{1}{2} \\ & - \sum_{i=1}^{N_i} \log p_i \left\{ \Phi \left[a + b \log \left(\frac{x_i - c}{c + d - x_i} \right) \right] - \Phi \left[a + b \log \left(\frac{x_{i-1} - c}{c + d - x_{i-1}} \right) \right] \right\}, x_0 = c. \end{aligned} \quad (3.7)$$

As for the moment constraint equations, we first write

$$\mu_k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^k f(x) dx.$$

as the k th moment of variable x about zero. We know from the discussion in section 3.2.3 that the density of a JSB variable x maps exactly into the standard normal distribution function through the transformation (1.1) and (1.2c). Thus, writing

$$x = \left[c + d \left(1 + e^{\frac{a-z}{b}} \right) \right]^{-1}$$

results in

$$\mu_k(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[c + d \left(1 + e^{-\frac{a-z}{b}} \right) \right]^{-k} e^{-\frac{1}{2}z^2} dz,$$

where z is a standard normal variable. Once again we apply Goodwin's formula, this time to the moment equation of a JSB variable* where we make $z = \sqrt{2}u$ in $\mu_k(x)$ and rewrite it as

$$\mu_k(x) \approx \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \left[c + d \left(1 + e^{\frac{a-\sqrt{2}nh}{b}} \right) \right]^{-k} e^{-n^2 h^2}. \quad (3.8)$$

by omitting the error term $\varepsilon_k(h)$. Equation (3.8) can be directly used in the calibration of delivery time probability densities. As for delivery volume, it is applicable only to probability densities in the transition from $3(n)$ to $4(n)$ in Figure 3.1(c), i.e., the monitoring stage where emissions reduction performance effectively occurs. In the general case, delivery volume in any other stage of the CDM probability tree must be corrected by a rate of survival, as per discussion in section 3.1. For instance, if delivery volume performance in node $4(n)$ is given by $y_{4(n)}$, then in node $6(n)$ we must have $y_{4(n)}p_{46}$ where p_{46} is the verification and issuance rate of survival discussed in section 3.1. We incorporate this rate of survival $r \leq 1$ by making $x \rightarrow rx$, changing (3.8) to its more general form[†]

$$m_k(x) \approx \frac{h}{\sqrt{\pi}} \sum_{n=-\infty}^{+\infty} \left\{ r \left[c + d \left(1 + e^{\frac{a-\sqrt{2}nh}{b}} \right) \right] \right\}^{-k} e^{-n^2 h^2}. \quad (3.9)$$

The constraint equations in our formulation then become

$$m_k(x) - \overline{m}_k(x) = 0, \{k = 1 \dots N_k\}, \quad (3.10)$$

*The application of Goodwin's formula to the moment equation of JSB variables has been previously performed by Draper [9].

[†]Interesting to note that (3.9) implies $m_k(x) \approx \mu_k(x)r^{-k}$.

with $m_k(x)$ given by (3.9) and the sample moments $\overline{m}_k(x)$ computed by

$$\overline{m}_k(x) = \frac{1}{N_i} \sum_{i=1}^{N_i} x_i^k.$$

Under this formulation, the calibration problem is reduced to minimizing (3.7) conditional on the moment constraints in (3.10), with the goal to estimate JSB parameters for the stochastic drivers α and β in (3.5) and (3.6). While all JSB parameters may be estimated via this method, in practice we make only the distribution shape parameters a and b unknowns. The reason for this relies on the fact that c and d define the bounds of JSB variables, with c and $c + d$ being respectively the lower and upper bounds. Concerning delivery volume, we know that the minimum amount of CERs a project can deliver is zero, and therefore by default $c = 0$. On the other hand, the maximum possible volume $c + d$ is restricted by intrinsic natural (e.g. hydraulic characteristics of the river in a run-of-river hydro power project) and operational (e.g. capacity of wind turbines in a wind power installation) constraints from a specific project. A similar argument applies to delivery time.

Figure 3.3(a) shows an example of a JSB density calibration (dotted line) and sample data (histogram) for delivery volume of CDM wind power projects in China. Figure 3.3(b) illustrates that the calibration converges to the final solution with about 4 moment constraints. And Figure 3.3(c) shows the JSB calibration for survival rates in different nodes of the simplified CDM probability tree, where survival rates lower than one have the effect of drifting the probability mass to the left of the chart. Although the profiles in Figures 3.3(b) and Figure 3.3(c) make sense from a calibration methodology point of view, one needs to be cautious about the calibration process when using raw CDM data such as the one in Figure 3.3(a). For instance, although the entropy calibration method leads to unbiased probability distributions, the raw data used as prior probabilities themselves may not lend itself suitable to generate meaningful JSB distributions. Figure 3.3(a) may be seen as one example of data points are too dispersed while the data points in Figures 3.8(a) and 3.8(b) are more suitable for JSB calibration. This limitation might be overcome through a pure

entropy optimization approach in the same way as performed with CDOs, but this is a topic for further research.

Another important issue is regarding the convergence of the calibration shown by Figure 3.3(b). Functions such as *fmincon* from Matlab can be used to solve the constrained minimization problem posed by (3.7) and (3.10). However, the application of penalty functions may provide a more efficient method to achieve convergence. A penalty method replaces a constrained optimization problem by an unconstrained problem whose solution ideally converges to the solution of the original constrained problem. The unconstrained problem is formed by adding a penalty term to the objective function consisting of a penalty function which acts as a measure of violation of the constraints. In this case, a constrained optimization problem

$$\min [\mathcal{K}(a, b)]$$

conditional to

$$c_k(a, b) = m_k(a, b) - \bar{m}_k = 0, \{k = 1 \dots N_k\},$$

can be changed to an unconstrained problem defined by

$$\min [\mathcal{K}(a, b) + \mathcal{F}(c_k(a, b))]$$

Various families of functions $\mathcal{F}(c_k(a, b))$ have been studied in the literature, where some of the more notable examples including static and adaptive methods are described by Smith and Coit [47]. In our case, a typical static method could be proposed by defining

$$\mathcal{F}(c_k(a, b)) = \sum_{k=1}^{N_k} a_k c_k^2(a, b)$$

with the a penalty coefficient a_k large for each constraint k . However, Smith and Coit [47] point out that it can be difficult to find an effective and efficient penalty function and different approaches have been attempted with different levels of success depending on the problem. Thus, this in itself is a topic for further research.

The next challenge in the calibration scheme is to work out the data that are used as input to the formulation above. For this, Figure 3.4 shows a snapshot of some basic data series. With reference to the CDM probability tree of Figure 3.1(c), Figure 3.4(a) shows project activity data that can be used for generating historical delivery time data series of nodes 1, 2 and 3(n) as well as survival rates for nodes up to 3.* Note also that proper due diligence on the specific project underlying the ERPA is an essential part of the process, as it helps improving the information from historical data to better assess time delays and volumetric risks in early stages. Regarding performance data, Figure 3.4(b) exhibits information that can be used to generate volumetric data series for nodes 3(n) to 4(n) and time data series for nodes 4(n) to 6(n). Volumetric data sample points are computed by dividing the issued volume by the PDD volume, while time data sample points effectively quantify the time difference from the end of the n th monitoring period until the date in which CERs are credited into the registry account.

Another important aspect of the data in Figure 3.4(b) is the monitoring frequency. This is observed as the time interval between nodes 3(n) and 4(n), with historical sample points plotted in Figure 3.4(c) for the universe of projects in the CDM pipeline. Given an expiry date or a certain cutoff date defined by the terms of the ERPA, knowledge of the monitoring frequency is crucial to compute the number of delivery events N as an input into the ERPA payoff equation (3.2). Monitoring cannot be represented as random variables but should instead be either defined in the ERPA terms or worked out informally with project participants. In theory there should be an incentive for project participants to shorten the monitoring period (by increasing the frequency) for large projects in order to be able to monetize CERs early enough. However, only a marginal inverse proportionality between

*Survival rate information available from the Carbon Valuation Tool by Point Carbon can be used as a starting point for early stage rates and combined with CDM historical data for later stages.

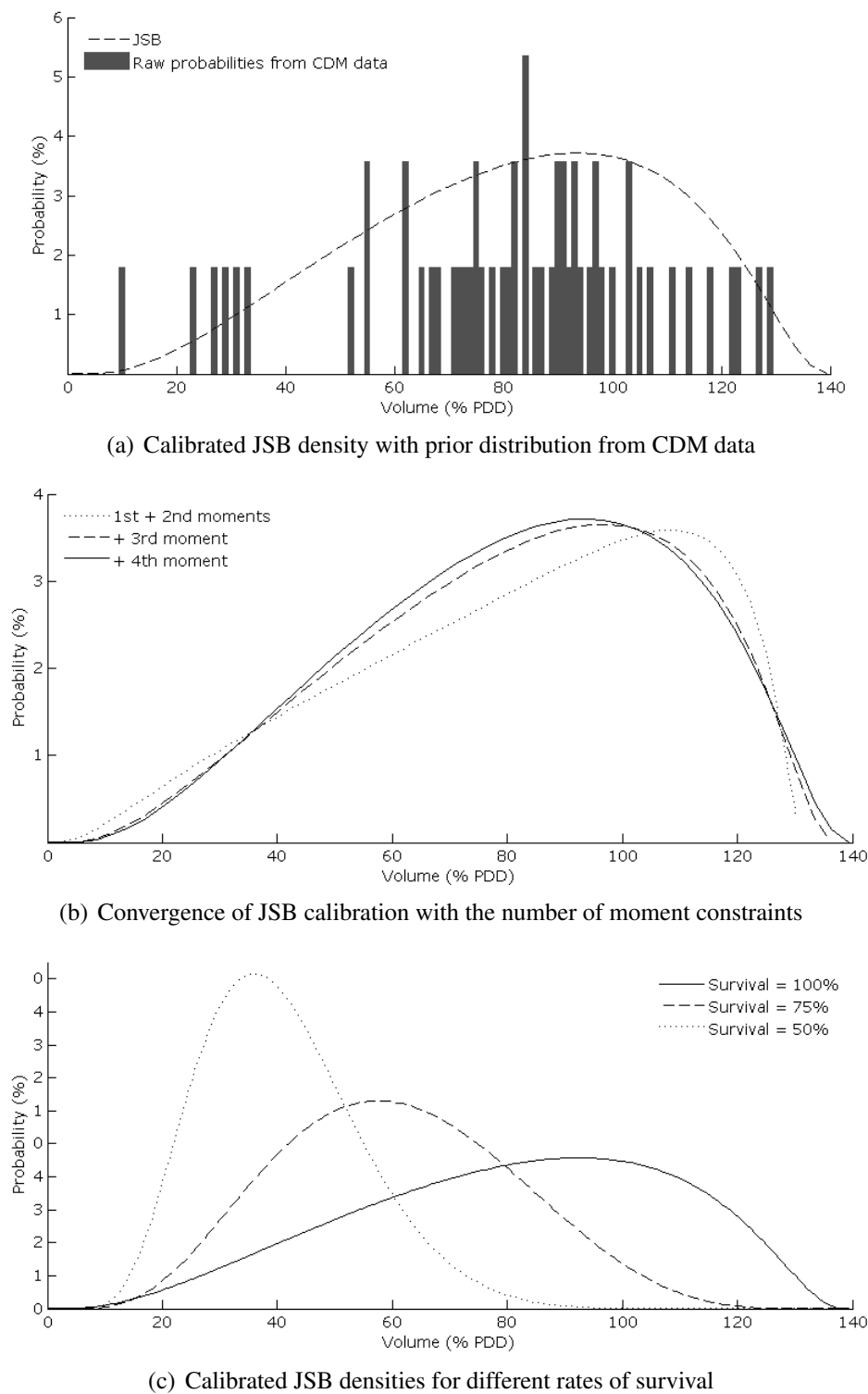


Figure 3.3: Calibration

CER volume and monitoring period is observed in the data. This may be explained by an increase in verification and certification fees over time, as well as less attractive secondary CER market prices during the latest economic downturn.

One concern remains on the actual CDM data published online by the UNFCCC.^{*} The fact that the raw data is currently only available through a web browser interface and not open to direct database queries is extremely disappointing to market participants, and effectively unusable in any form of quantitative analysis. Various market players and information providers have worked around this problem by laboriously building their own databases, with continuous extraction of data from the UNFCCC web pages as they are made available. In some instances these proprietary databases are shared online, as it is the case of the outstanding spreadsheets by the Institute for Global Environmental Strategies (IGES).[†] Only from sources like this or by constructing proprietary databases one able to filter the necessary data series and use it in practical applications such as the one of this research.

3.2.4 Carbon price dynamics

Carbon price drivers

The typical starting point for understanding and mathematically representing stochastic variables is the observation of their historical time-series. Although a necessary step, this is far from sufficient in the case of carbon. This is due to the fairly recent emergence of carbon as an asset class, and mainly because of its link to some major fundamental drivers. Before jumping to an assumption on the stochastic behavior of carbon prices, it is paramount to understand these underlying drivers ranging from macroeconomic and technological factors to elements in the market microstructure created by scope and design of cap-and-trade systems. Keeping in mind that market prices are ultimately determined by the balance between supply and demand, the goal of the analysis that follows is to

^{*}Available at <http://cdm.unfccc.int/index.html>.

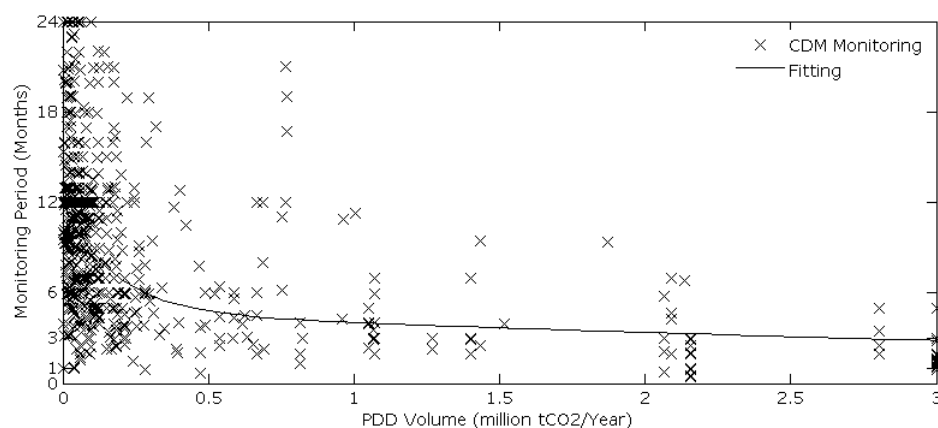
[†]Available at <http://hydmez001.iges.or.jp/en/cdm/>.

Time							Volume			
UNFCCC CDM ID	Host Party	Project Type	Date Node 0	Date Node 1	Date Node 2	Date Node 3	Country	Project Type	Node	Survival Rate
0872	India	Waste gas heat utilization	04-May-06	26-Jan-07	19-Oct-07	19-Oct-07	China	Hydro reservoir	0	55%
1105	China	HFC	25-Jan-07	24-Aug-07	15-Feb-08	01-May-08	China	Hydro run-of-river	0	37%
1222	China	Hydro power reservoir	02-Aug-06	03-Jul-07	11-Oct-07	01-Jan-08	China	Waste gas heat utilization	0	25%
1226	China	Hydro power reservoir	02-Aug-06	03-Jul-07	12-Oct-07	12-Oct-07	China	Wind	0	46%
1229	Chile	N2O decomposition	23-Dec-06	05-Jul-07	13-Oct-07	26-Jun-08	India	Biomass	0	19%
1231	China	Hydro power run of river	14-Apr-07	10-Jul-07	20-Sep-07	25-Dec-07	China	Hydro reservoir	1	83%
1236	India	Wind power	01-Dec-06	07-Jun-07	22-Sep-07	22-Sep-07	China	Hydro run-of-river	1	88%
1238	China	N2O	30-Dec-06	29-Aug-07	30-Nov-07	14-Mar-08	China	Waste gas heat utilization	1	76%
1249	Peru	Biogas wastewater treatment	14-Feb-07	24-Jul-07	23-Sep-07	23-Sep-07	China	Wind	1	92%
1352	India	Fuel switch	07-Dec-06	19-Jul-07	04-Feb-08	10-Feb-08	India	Biomass	1	79%
2375	Brazil	Hydro power run of river	13-Jun-08	19-Jan-09			China	Hydro reservoir	2	80%
2548	Brazil	Landfill methane recovery and utilization	03-Sep-08	29-Apr-09			China	Hydro run-of-river	2	80%
2602	Malaysia	Biogas wastewater treatment	07-Jun-08	26-May-09			China	Waste gas heat utilization	2	80%
2712	Ethiopia	Afforestation and reforestation	17-Oct-08	23-Jun-09			China	Wind	2	80%
2772	Sri Lanka	Biomass	26-Aug-08	20-Jul-09			India	Biomass	2	80%
2777	China	Wind power	15-Oct-08	22-Aug-09			China	Hydro reservoir	4	97%
2834	South Korea	Landfill methane recovery and utilization	23-Jan-09	05-Aug-09			China	Hydro run-of-river	4	98%
2841	China	Hydro power reservoir	25-Sep-08	03-Aug-09			China	Waste gas heat utilization	4	97%
2876	Indonesia	Geothermal	04-Jul-07	04-Aug-09			China	Wind	4	97%
2895	India	Biomass baasse	07-Aug-08	06-Aug-09			India	Biomass	4	97%

(a) Survival CDM data

UNFCCC CDM ID	Host Party	Project Type	Time			Volume		
			Date Node 3	Date Node 4	Date Node 6	Volume Issued	Volume PDD	Volume Annual
0542	Mexico	Biogas animal waste	16-Dec-06	30-Sep-07	27-Jun-08	5,794	14,817	18,779
0577	India	Biomass	12-Apr-02	31-Mar-07	14-Dec-07	170,103	318,489	64,084
0673	Indonesia	Geothermal power	14-Jun-07	31-Aug-07	11-Jun-09	90,804	139,368	652,173
0696	India	Waste gas heat utilization	01-Dec-06	30-Nov-07	23-Apr-08	32,066	54,191	54,340
0726	India	Biomass rice husk	03-May-07	31-Dec-07	01-Dec-08	25,848	14,247	21,488
0751	Sri Lanka	Hydro power	01-Jan-04	30-Apr-07	26-Sep-08	8,932	18,272	5,489
0752	South Africa	N2O	18-Mar-08	08-Jul-08	19-Jan-09	157,352	145,243	473,338
0767	China	HFC	01-May-07	25-May-07	21-Sep-07	136,661	135,816	2,065,533
0887	China	Landfill methane recovery and utilization	01-Jul-07	01-Sep-07	02-Mar-09	19,863	80,111	471,619
0906	China	Wind power	11-May-07	14-Oct-07	06-Mar-08	14,718	32,757	76,644
0922	South Korea	N2O	27-Jun-07	31-Dec-07	26-Sep-08	123,273	144,104	281,272
0939	China	Hydro power reservoir	03-May-07	25-Oct-07	25-Apr-08	21,138	17,367	36,222
0961	South Africa	N2O	25-May-07	01-Oct-07	07-Aug-08	259,537	339,401	960,322
0969	China	Wind power	18-May-07	31-Jul-07	16-Nov-07	9,757	12,563	61,966
0998	India	Wind power	01-Apr-01	31-Mar-07	19-May-08	270,512	343,488	57,248
1009	India	Wind power	01-Jun-07	31-May-08	15-Jun-09	4,829	7,246	7,246
1037	Brazil	Fuel switch	01-Jan-03	30-Apr-07	03-Dec-07	154,570	174,588	40,332
1039	India	Fuel switch	01-Aug-04	31-Jul-07	05-May-08	6,344	6,867	2,291
1258	The Philippines	Landfill methane recovery and utilization	01-Feb-08	31-Aug-08	25-Jun-09	30,860	67,572	116,339
1758	Jordan	Fuel switch	30-Sep-08	31-Jan-09	20-Aug-09	215,513	133,838	397,163

(b) Performance CDM data



(c) Monitoring period versus PDD volume

Figure 3.4: Snapshot of CDM data

identify clear signals of short and long-term direction for carbon prices. We leave climate and carbon market policy purposely out of the discussion, given they represent the very reason for the existence of carbon as an asset class as opposed to being simply a price driver. On this note, the analysis assumes that the EU ETS goes forward with Phase 3 and that demand for CERs into the scheme continues into the post-2012 period.

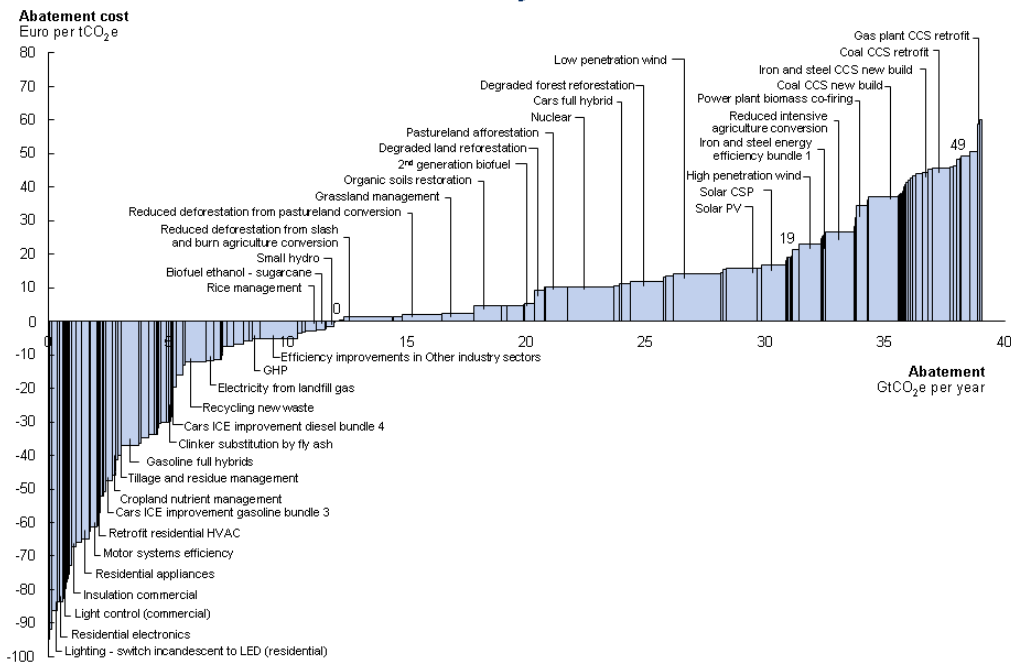
- GDP growth

The fact that current economic activity is primarily dependent on fossil fuels explains the strong influence GDP growth has on carbon prices. The rationale is that a growing economy causes an increase in emissions through intensifying fossil fuel combustion and other emissions-related activities. However, in a carbon constrained economy, this has the effect of increasing the demand for carbon allowances or permits (the currency of cap-and-trade systems) such as the EUA, in addition to emissions reduction offsets such as the CER. A demand increase for carbon instruments in an environment with either marginally decreasing or stable supply causes an upward trend in future expected prices. This dependence is noticeable in the high correlation exhibited between carbon prices and equities and oil, commonly understood as proxies for market sentiment on GDP growth. Under the paradigm of economic growth versus prosperity, and assuming a future thriving international carbon market, this driver sends a clear upward signal for middle to long-term carbon prices.

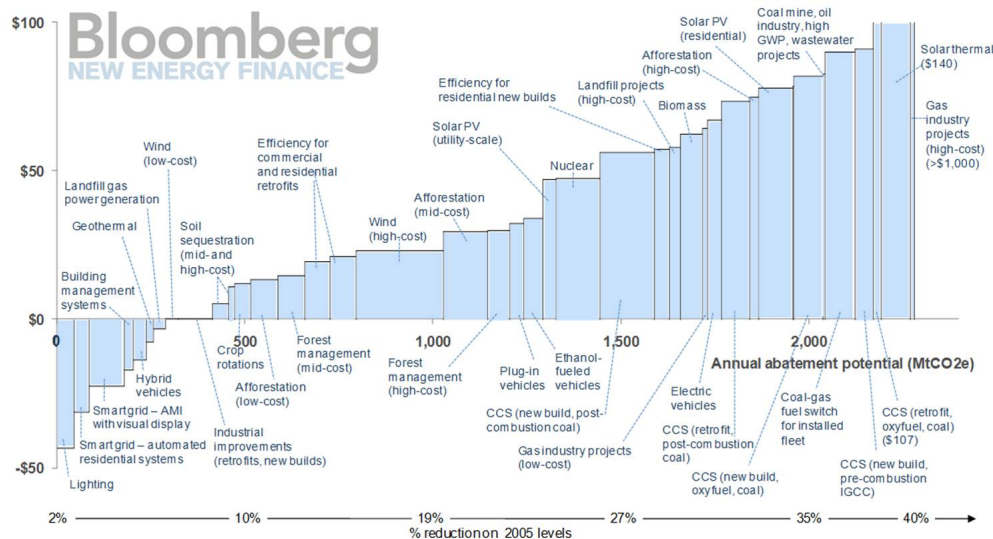
- Marginal Abatement Cost (MAC) curve

The MAC curve quantifies the effort for incremental additional abatement of emissions. Figure 3.5(a) shows the well known McKinsey global abatement cost curve which originally appeared in Enkvist, Nauclér and Rosander [11], while Figure 3.5(b) illustrates a MAC curve constructed by Bloomberg New Carbon Finance. These curves are categorized by technology and read as estimates of the prospective annual cost per additional ton of avoided CO₂-equivalent emission. Despite the industry-wide criticism regarding the use of such curves and even about their relevance, they do provide an indication of

Global GHG abatement cost curve beyond 2030 BAU



(a) McKinsey global MAC curve version 2.0



(b) Bloomberg New Carbon Finance MAC curve

Figure 3.5: Marginal Abatement Curve (MAC)

the trend in future prices of carbon. In a carbon-constrained economy and under the assumption of such a snapshot in the current state of low carbon technologies, carbon prices are expected to fundamentally rise in order to match abatement costs in the long run. Again, this sends an upward signal regarding long-term prices.

- Energy prices

Energy is one of the major contributors to emissions globally through its cycle of exploration and extraction, production and consumption. Particularly in power generation, the combustion of fossil fuels for electricity generation is a widespread activity in both developed and developing economies. For instance in the countries under the EU ETS, the majority of the combustion installations use coal and gas and not surprisingly EUA and CER prices have a close link with prices of these fuels. Key to this relationship is the fact that coal has a higher emissions factor as compared to gas, i.e., generating electricity with coal emits significantly more CO₂ per MWh than the same activity using gas as fuel.

Thus, conditional to fuel switching capacity and under a cap-and-trade regime, the price of carbon is expected to reflect the price differential between coal and gas as measured in terms of dark and spark spreads. For instance, if coal prices become relatively cheap when compared to gas, installations would have an incentive to switch to coal. In order to compensate for the increase in emissions and thus keeping emissions commitments on target, those same installations then demand more carbon allowances and/or offsets. This increased demand for carbon instruments tends to drive carbon prices up and at least partially diminishes the original attractiveness of coal as fuel. The converse is true when gas becomes relatively cheap to coal.

Although generally accepted, this pattern is intermittent given its high dependence on the currently limited fuel switching capacity in the EU, which can be exhausted under certain circumstances. For instance, gas prices in the EU have plummeted over the course of the recession in 2009 while coal has been more stable given its status as a global commodity. In fact Asia has significantly driven coal demand in 2009 for restocking purposes due

to its shorter recessionary period as compared to Western economies. This caused an economic incentive to switch from coal to gas in the EU, leading to a relative disconnect between carbon and energy prices in early to middle 2009, reflecting a system close to its switching capacity. It is then logical to assess that energy as a driver points to a marginal influence in carbon prices while levels are kept low in the short-term, but with a middle-term upward trend due to the return of high demand for gas in EU economies emerging from recession.

- EUA-CER spread

Despite the fact that EUA and CER represent the same one ton of CO₂-equivalent emission, they are set apart by the design of the EU ETS through quantitative and qualitative restrictions on imports of CERs. While the amount of CERs that industrial installations can use is capped in Phase 2, CERs from large hydro power projects (usually but not necessarily projects involving reservoir) and from land use, land use change and forestry (LULUCF) are banned from entering the ETS. These restrictions have implicitly created a distinction in the applicability of EUAs and CERs as compliance instruments, which has led to a price differential between the two assets called EUA-CER spread.

Although the EU ETS does not impose EUA and CER cap or floor prices, the notorious Chinese primary CER floor price has significantly impacted the way carbon prices behave. This should be expected given that China currently represents close to 60% of the total expected average annual supply of CERs from registered projects.* This dependence is apparent in historical prices through the observation of high correlation between absolute prices and their respective spread (Figure 3.6). This can only be explained by the hypothesis of an implicit carbon floor price, which intuitively causes the tightening of the spread when absolute prices get close. A similar conclusion may be drawn for prices approaching an upper bound interpreted in the EU ETS as the cost of carbon capture and storage (CCS). We conclude that while the EUA-CER spread as a driver does

*According to the CDM pipeline statistics as of 10-Feb-2010, available at <http://cdm.unfccc.int/Statistics/index.html>.

not provide obvious signals on the absolute levels of prices, it does indicate that carbon prices are effectively bounded at fundamental floor and cap levels.

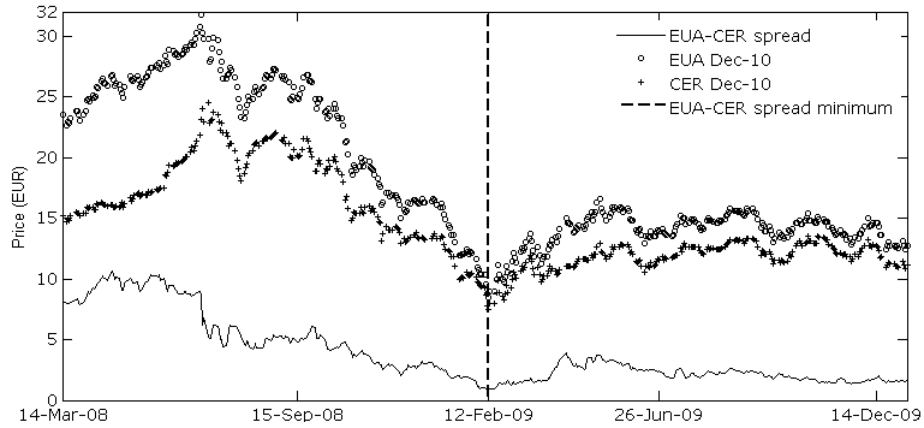


Figure 3.6: EUA-CER spread time-series

Price dynamics and calibration

The objective of this section is to define a mathematical representation for carbon prices in the form of a stochastic process. Although none of the price drivers previously discussed exhibit an exact functional relationship with carbon prices, they do offer clues of an upward future middle and long-term trend with expected low levels in the short-term. This observation has support in a study by Bole [6] which provides a literature review of scientific models used to calculate fundamental global carbon prices. Many of these models incorporate in one way or another the drivers previously discussed, in addition to scenarios on climate policy-making. Based on a stabilization level of 450 ppm (parts per million) for the atmospheric CO₂ concentration*, the results point out to a carbon price range of €31.4 to €76.6 by 2020.

*This is the CO₂ concentration level which is widely accepted by the scientific community and policy-makers to be economically achievable, while yielding a high probability of keeping the planet's surface temperature below 2° by 2100. More details on climate scenarios and science can be found in IPCC [22].

On these observations, we assess that carbon prices exhibit a clear drift that is not mean-reverting but instead increases as a function of time. We now make an assumption common from the literature, that the stochastic component follows a Brownian motion and that prices are non-negative. This leads to

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma_t dW_t, \quad (3.11)$$

which is the well known Geometric Brownian Motion (GBM). In (3.11), S_t represents the spot carbon price, r_t and q_t are respectively the risk-free interest rate and the carry rate (more broadly known in commodities as convenience yield), and σ_t is the volatility parameter. In this setup we assume r_t , q_t and σ_t to be deterministic variables and functions of time as illustrated by the subscript t . Given (3.11) and a certain horizon T , the solution for S_T can be written as

$$S_T = S_0 \exp \left[\int_0^T (r_t - q_t - \frac{1}{2}\sigma_t^2)dt + \int_0^T \sigma_t dW_t \right],$$

where $\int_0^T \sigma_t dW_t$ is an Itô integral*. We rewrite

$$S_T = \frac{Q_T}{B_T} \exp \left(- \int_0^T \frac{1}{2}\sigma_t^2 dt + \int_{t=0}^T \sigma_t dW_t \right),$$

where

$$B_T = e^{-\int_0^T r_t dt}$$

and

*See Kloeden and Platen [30].

$$Q_T = e^{-\int_0^T q_t dt}$$

are discount factors associated with r_t and q_t , resulting in

$$F_T = \frac{Q_T}{B_T} S_0 \quad (3.12)$$

as the expected value of S_T under the risk-neutral measure.

From the observation regarding carbon price bounds in the EUA-CER spread discussion, we now add a collar feature to the process in (3.11) and define

$$X_t = \min(\max(S_t, K_a), K_b) = S_t + \max(K_a - S_t, 0) - \max(S_t - K_b, 0). \quad (3.13)$$

We call this process a *Bounded Geometrical Brownian Motion*, where K_a and K_b are respectively the lower and upper bounds which translate into the fundamental floor and cap carbon prices. Once again we take the expectation under the risk-neutral measure in order to obtain the actual Forward price of carbon, given by

$$\mathbf{E}[X_T] = \mathbf{E}[S_T] + \mathbf{E}[\max(K_a - S_T, 0)] - \mathbf{E}[\max(S_T - K_b, 0)] \Rightarrow$$

$$\begin{aligned} C(S_0, K_a, K_b, T, B_T, Q_T, \sigma_{T, K_a, K_b}) &= F(S_0, B_T, Q_T) \\ &+ G(F_T, K_a, T, \sigma_{T, K_a}, -1) - G(F_T, K_b, T, \sigma_{T, K_b}, +1), \end{aligned} \quad (3.14)$$

where F comes from (3.12) and G is the widely known Black model by Black [5] with flags -1 and $+1$ representing respectively put and call option pricing functions.

This result is particularly useful for the calibration of Q_T and the well known volatility

smile (or surface) $\sigma_{T,K}$, the only two unobserved quantities in the model. B_T can be bootstrapped from interest rate curves, S_0 is the spot carbon price quoted in the market, T is defined by the schedule in the contract terms, and K_a and K_b must be either assumed or assessed through market analysis and fundamental research. Because r_t and q_t are deterministic and ERPA payoffs are normally not written on these variables, we assume that $\log B_T$ and $\log Q_T$ are linear functions of the form $a_0 + a_1x$ in order to simplify model calibration and application. Thus, in formulating a non-linear least-square minimization scheme we define

$$\sum_{i=1}^{n_T} [\bar{C}_{T_i} - C(S_0, K_a, K_b, T_i, B_{T_i}, Q_{T_i}, \sigma_{T_i,0})]^2 \quad (3.15)$$

as one of the minimization functions given n_T carbon Futures price quotes $\bar{C}_{T_i}$. Depending on the existence of option quotes, another minimization function is

$$\sum_{i=1}^{n_T} \sum_{j=1}^{n_K} [\bar{G}_{T_i,K_j} - G(\bar{C}_{T_i}, K_j, T_i, B_{T_i}, Q_{T_i}, \sigma_{T_i,K_j})]^2. \quad (3.16)$$

Here $\bar{G}_{T_i,K_j}$ represents option quotes for n_T maturities and n_K strikes. In Figure 3.7 we show a simulated probability density for the CER Dec-09 Future price calibrated with the above formulation. We assumed $K_a = \text{€}7$ (slightly below the Chinese floor prices) and $K_b = \text{€}40$ (a low estimate for the cost of CCS in the EU). In this case only one Future and one option market quote were used, allowing for the calibrating of only one Q_T and a constant implied volatility σ_T . Note that the shape of the probability density in Figure 3.7 indicates the presence of a skew (better noticeable by the distance difference between the median and its 1% tail confidence intervals) while fat-tails are also explicit. This is despite the lack of enough option quotes to calibrate a volatility smile. This result is supporting evidence that a market participant with a proper assessment of the fundamental floor and cap prices would have at least a marginal temporary commercial advantage over other participants in the carbon derivatives market. It is possible to improve the calibration methodology to bootstrap K_a and K_b as the derivatives market becomes more mature and more

information on prices of options and other instruments is made available.

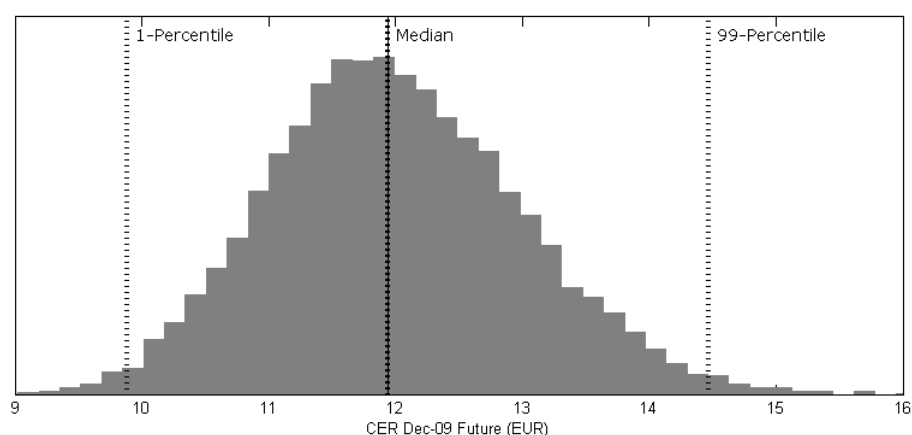


Figure 3.7: Simulated probability density for CER Forward Dec-09

3.3 Application

3.3.1 Pricing algorithm

We now bring all previously discussed elements together to propose a practical set of steps to be applied in the pricing of ERPAs, described below.

1. Formulate the payoff in line with (3.2) and according to ERPA contract terms.
2. Identify the stage of the project in the CDM probability tree of Figure 3.1(c), as well as country and technology category or methodology.
3. Extract delivery data series for time and volume from CDM historical data and collect appropriate market quotes of Futures and options from secondary CER markets and other relevant carbon price instruments (EUA, ERU, etc).
4. Quantify appropriate survival rates of the project according to its stage and any other risks identified during due diligence (operational, financial, country, etc).

5. Compute sample moments for delivery time and volume from the delivery data series.
6. Calibrate the delivery time stochastic process (3.5) through the optimization scheme of (3.7) and (3.10), using the previously calculated sample moments and applying a survival rate of 1.
7. Calibrate the delivery volume stochastic model (3.6) through the optimization scheme of (3.7) and (3.10), using the previously calculated sample moments and applying the previously defined survival rates.
8. Calibrate carbon price stochastic model (3.13) through the formulation of (3.15) and (3.16).
9. Define payment, lump sum and monitoring schedules m , j and n in (3.2).
10. Sample M by N_n independent random numbers for stochastic processes (3.5) and (3.6), where M is the number of paths in the Monte Carlo simulation and N_n is the number of delivery events.
11. Sample M by N_m independent random numbers for stochastic process (3.13), where M is the number of paths in the Monte Carlo simulation and N_m is the number of payment events defined in the schedule of the ERPA, which may be equal to N_n for agreements with payment and delivery upon CER issuance.
12. Apply sample random numbers to the payoff and compute price by taking the average over the M paths of the Monte Carlo simulation.

The steps above can also be iteratively applied through an optimization scheme in order to compute other parameters. For instance, given a certain ERPA structure and price, one can bootstrap the break-even point for expected CER delivery volume or future carbon price levels. This is particularly useful for simulation of future scenarios when managing and hedging existent positions, or in stress test calculations to define necessary levels of required capital buffer in the case of banks and other financial institutions. What follows now is the discussion of a practical case applying the steps above to price ERPAs.

3.3.2 Case study: run-of-river hydro power project in China

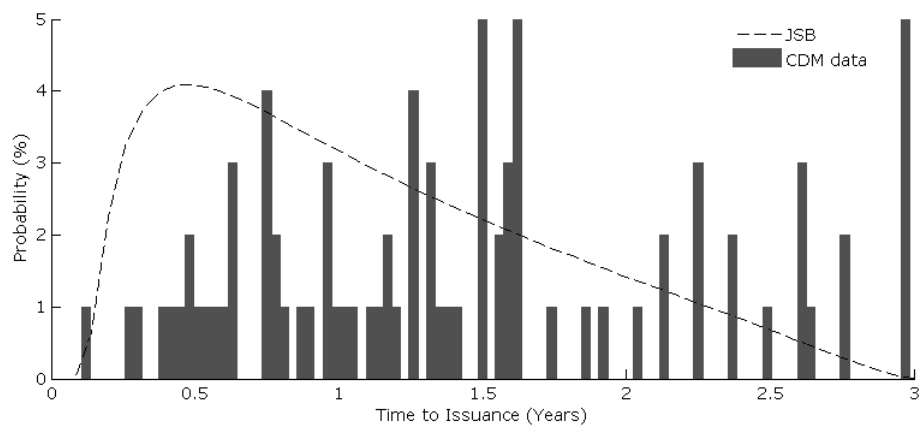
We now present a case study that illustrates the application of the methodology previously described for a run-of-river hydro power project in China. These projects fall in the category of renewable energy given they use a resource that is naturally replenished (water flow) to generate electricity. When connected to national or regional power grids, their supplied electricity leads to a reduction in future CO₂ emissions that would have occurred through electricity generation with fossil fuels (e.g. coal). In China this effect is clearly observed through the emissions factor* of electricity grids, where the average figure for build margin (i.e., emissions of new installations) and operating margin (i.e., emissions of existent installations) across all regional grids has consistently declined from 2007.[†] This result is at least in part due to a highly successful CDM market fueled by government policies geared toward promoting a shift to renewables.

In our particular case, run-of-the-river hydro power projects are types of hydroelectric generation whereby the kinetic energy carried by water flow is captured through the natural downward flow and elevation drop of rivers. Typically water is diverted from the river at a high point and fed down through a pipe called penstock to the powerhouse located down-river, where the kinetic energy of the flow is transformed into electrical energy through the turbines and the diverted water is returned to the stream.[‡] Implementation of run-of-river projects is relatively cheap and these projects offer long operational lifetimes with minimal maintenance. In addition, run-of-river projects are inherently more environmentally friendly than reservoir projects which require the construction of dams, involving the flood-

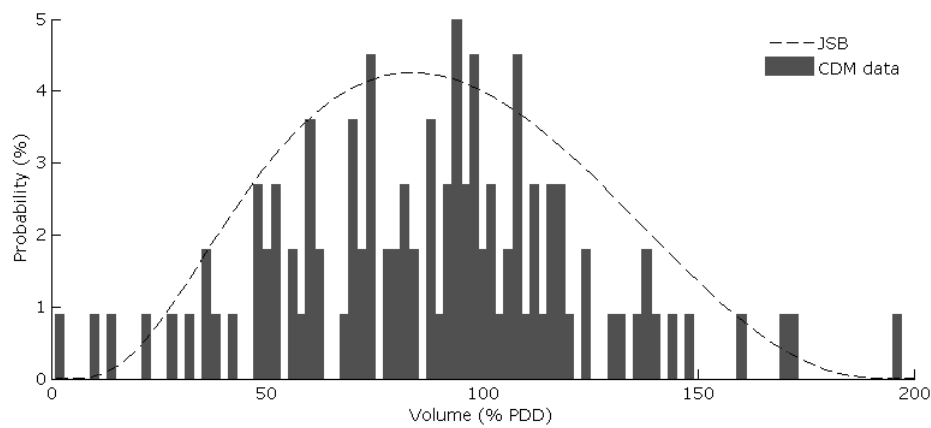
*The emissions factor measures the amount of emissions produced in tCO₂ per MWh of electricity generated. Figure B.1 in Appendix B.2 shows estimates of emissions factors for different fuels.

[†]Chinese emissions factors are published by the of Climate Change, National Development and Reform Commission of the People's Republic of China at <http://cdm.ccchina.gov.cn/english/>, summarized in Appendix B.3.

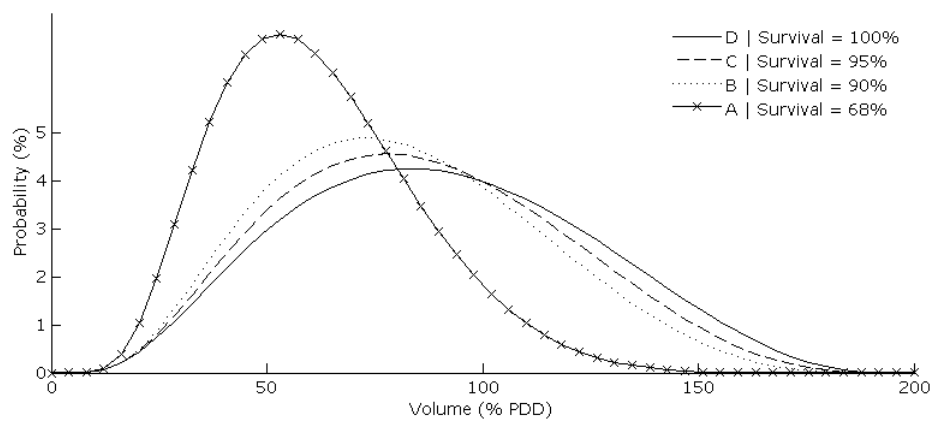
[‡]A diagram is available in Appendix B.4 from Figure B.3(a).



(a) Calibrated JSB delivery time against CDM data



(b) Calibrated JSB delivery volume against CDM data



(c) Calibrated JSB delivery volume for different survival rates

Figure 3.8: Delivery distributions for a run-of-river hydro power project

ing of large areas of land.* This causes the destruction of existing forests that serve as CO₂ sinks and a disruption in the local ecosystem, with the ultimate social impact of dislocating communities that live in the flooded areas.†

For the purpose of our case study we define four fictitious Chinese run-of-the-river hydro power projects, all with estimated PDD emissions reduction of 120 tCO₂/year and different only by their respective stage in the CDM probability tree of Figure 3.1(c):

- **A** → Node 0. This project is in the stage between submission of a complete PDD and start of the validation, with 01-Jun-2010 as the expected emissions reduction start date according to the PDD.
- **B** → Node 1. A request for registration has been submitted for this project, with an expected emissions reduction start date following CDM-EB meeting 53 (EB53) on 26-Mar-2010.
- **C** → Node 3(1). This project activity has been registered following EB52 and has started its first monitoring period on 12-Feb-2010.
- **D** → Node 3(2). Registration was granted for this project on EB48 with an emissions reduction start date of 17-Jul-2009, while it is currently undergoing a second monitoring period with one previous issuance of CERs in relation to its first monitoring period.

Following the steps in our pricing algorithm, we leave the ERPA payoff formulation as a subject of the subsequent sections. We then calibrate JSB distributions for delivery time and volume from the CDM data for Chinese run-of-river hydro power projects. Results of this calibration are shown in Figure 3.8, where in Figures 3.8(a) and 3.8(b) we plot the calibrated JSB distributions for delivery time and volume respectively. In Figure 3.8(c) we

*In order to harness a river's full potential for electricity generation, some hydro projects involve a small reservoir behind a dam constructed across the full width of the river. However, since flooding is minimal these projects are still considered run-of-river. A diagram of such installations is available in Appendix B.4 from Figure B.3(b).

†This is primarily the reason for the EU ETS ban on imports of CERs from large hydro power projects, which usually involve the construction of reservoirs.

show JSB distributions for each project with their respectively assigned survival probabilities, where we assume a survival probability of 100% for CER delivery from project **D** given its proven success in previously achieving CER issuance. Regarding carbon prices, we use the closing Futures prices for CER and EUA as well as one market implied volatility per Future contract horizon. We also assume €6 and €40 respectively for fundamental floor and cap prices.

Finally, we construct the monitoring schedules by assuming subsequent 4-month monitoring periods from the emissions reduction start date. No further delays are added to projects **C** and **D** given they are already registered and operational. However, we add an extra 3-month delay for project **B** given the possibility of registration review, and a 6-month delay for project **A** due to an unrealistic PDD emissions reduction start date given its early stage in the CDM process. In addition we define $m = n$ in (3.2) (i.e., CERs are delivered to buyers upon issuance) and establish the cutoff date for delivery as 01-Apr-2013. We proceed with the formulation of three different types of ERPA payoff and apply the parameters above to assess the fair value of these contracts.

Fixed primary CER price

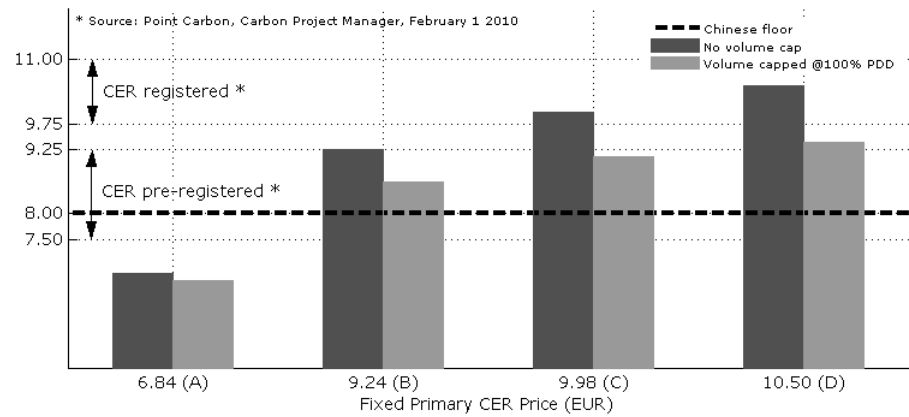
A typical fixed primary CER price payoff was introduced in (3.3). A more general formulation with tranches on CER delivery volume can be achieved by defining a cumulative volume variable

$$x_n = \sum_{i=1}^n y_i$$

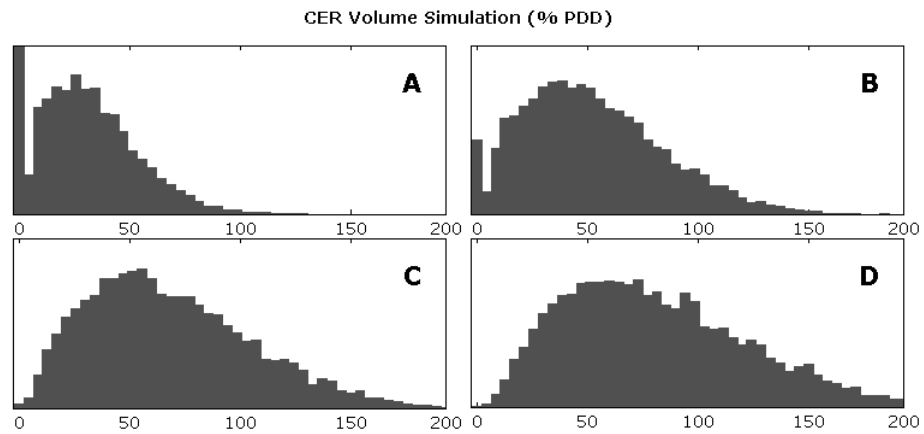
and rewriting the delivery function as

$$\begin{aligned} g_n &= \max(\min(x_n, X_k) - X_{k-1}, 0) \Rightarrow \\ g_n &= \max(x_n - X_{k-1}, 0) - \max(x_n - X_k, 0), \end{aligned}$$

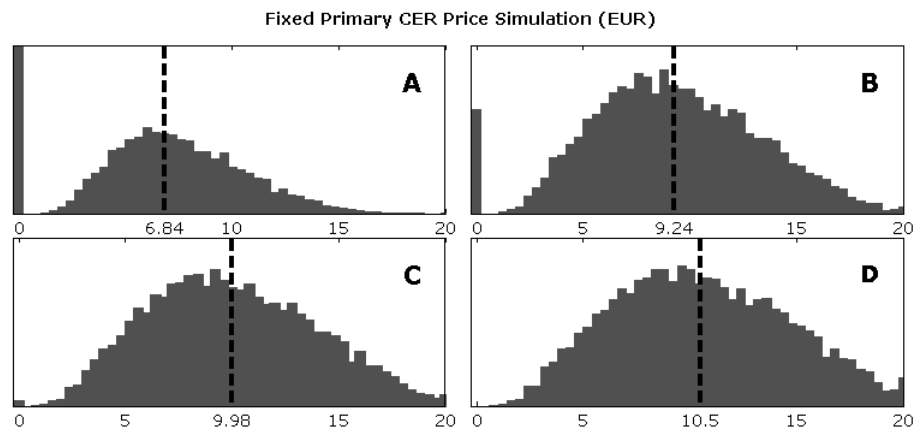
bounding the total volume between subordination levels X_{k-1} and X_k with $X_k > X_{k-1}$.



(a) Results



(b) Volume simulation



(c) Price simulation

Figure 3.9: Fixed primary CER price results and simulation

For instance, projects with delivery volume capped at 100% of the PDD implicitly take $X_1 = V_n$ and $X_0 = 0$. Another example is from ERPAs that split delivery of CERs into multiple tranches assigned to different buyers, e.g. $X_3 = 1.2V_n$, $X_2 = 0.8V_n$, $X_1 = 0.4V_n$ and $X_0 = 0$. Under this rationale, the fixed primary CER price can be generalized as

$$P = \frac{\sum_{n=1}^{N_n} B_n(t_n) C_n(t_n) \mathbf{E}[g_n - g_{n-1}]}{\sum_{n=1}^{N_n} B_n(t_n) V_n}, \quad (3.17)$$

$$\{k = 1 \cdots N_k\},$$

where (3.3) is simply a particular case with $X_{k-1} = X_{\min} = 0$ and $X_k = X_{\max} = dN_n V_n$ given $c = 0$ under a JSB distribution as the one in Figure 3.8(b).

In Figure 3.9(a) we present the results of a fixed primary CER price for each of the projects in scope and for the non-capped and capped delivery volume cases. We also illustrate the level of €8 as the unofficial Chinese price floor for hydro power projects, as well as pre-registered and registered benchmark price ranges from indicative primary CER market transactions*. Note that in the case of no volume cap, projects **B**, **C**, and **D** validate quite well the benchmark price ranges according to their respective stages. On the other hand, the price for project **A** falls below both the pre-registration reference price range and the Chinese price floor, and in this case project participants and buyers would struggle to get such ERPA price approved by the Chinese government. This seems to be increasingly the case with current low levels of secondary CER prices, giving clues to the difficulties that participants in early stage Chinese CDM projects have to endure as well as the level of concessions that willing buyers must make to gain access to these primary CERs.

In the case of capped volume at 100% of the PDD, note that the only result that reaches its respective price band is the price for project **B**. However the only concern is still project **A** due to the Chinese floor, while the other lower prices are justified since project participants are remunerated by the possibility of upside from the project's volume performance. And finally, for completion we present in Figures 3.9(b) and 3.9(c) histograms with the simu-

*Source: Point Carbon, Carbon Project Manager, February 1 2010.

lation results of respectively volume and fixed primary CER price. As expected, note the delivery volume distribution of project **A** is skewed toward zero delivery while the one for project **D** shows several events of delivery in excess of 100% of the PDD. Also note the wide distribution for the fixed primary CER price due to high market implied volatility of secondary CERs.

Collared floating CER structure

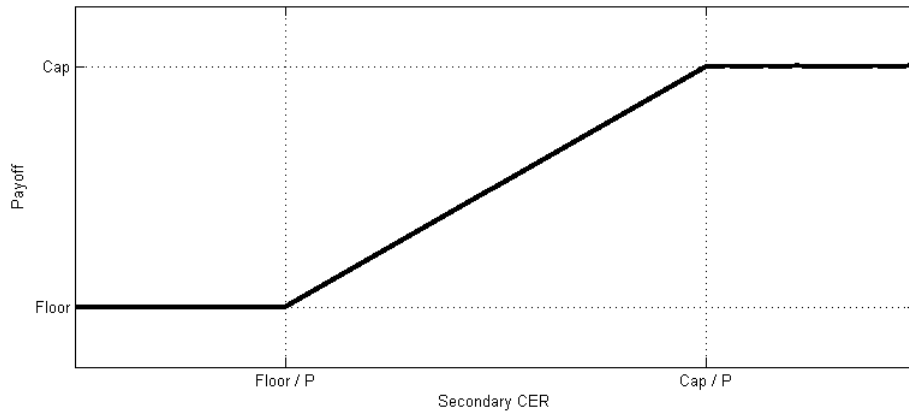


Figure 3.10: Collar payoff diagram

One alternative to the fixed primary CER involves floating structures where the ERPA price is defined as a percentage over the prevailing secondary CER market price. While the most common of these structures is defined in (3.4), a more general payoff involves collars where the ERPA price can float between a floor and cap price as shown in Figure 3.10. Writing the payoff function as $f_n = B_n(t_n) \mathbf{E} [\min(\max(C_n(t_n), C_a), C_b)] V_n$, or

$$f_n = \sum_{n=1}^{N_n} B_n(t_n) \mathbf{E} \left[PC_n(t_n) + P \max \left(\frac{C_a}{P} - C_n(t_n), 0 \right) - P \max \left(C_n(t_n) - \frac{C_b}{P}, 0 \right) \right] V_n, \quad (3.18)$$

where C_a and C_b are respectively the ERPA floor and cap prices, and P is the floating price.

Applying this equation to (3.2) with $g_n = \mathbf{E}[y_n]$, we chose to define the ERPA floating price and the floor level in order to solve for a fair value of the cap price. We show this in Figure 3.11, where floor and cap levels are plotted on the top of the price scenarios simulated in the fixed primary CER case. Note that ERPAs on projects **B**, **C** and **D** would easily pass the Chinese floor price hurdle due to their agreed minimum CER price of €8. These ERPAs shift a portion of the secondary CER price risk from buyers to project participants, who are in turn compensated by the possibility of upside in future market prices.

Once again project **A** suffers from its low CER delivery expectation, where an ERPA with agreed floor of €7 would be unlikely to go through. A floating price at 40% (as well as lower percentages) of the secondary CER price combined with a floor price of €8 is not achievable as it requires a decrease in the cap that eventually leads to it reaching the floor. This would configure a structure without any payoff scenarios, as intuitively demonstrated in Figure 3.11 by the floor and cap lines meeting. On the other hand, this result also shows how a different Chinese floor price policy could be enough to make ERPAs like this turn into reality. Contrary to the original concerns that motivated the Chinese government to establish price floors, in the case of floating price ERPAs project participants have an upside which is paid for through a compromise in the agreed floor and cap prices.

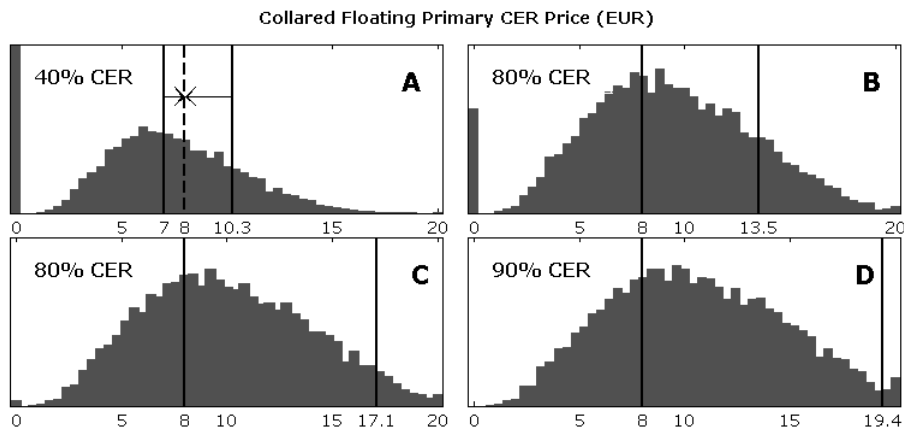


Figure 3.11: Collar floating primary CER price results

Bespoke EUA structure

The other ERPA payoff alternative we present is what we call a bespoke EUA structure. These structures were more common during the early days of the CDM, more specifically in the pre-2008 period when CER Futures contracts were either not traded or illiquid.* In the absence of an accurate CER Forward curve, market participants had no choice but to use the EUA Forward curve as a proxy for guidance on the pricing terms of ERPAs. In some cases complex structures linked to EUA prices were established. One such structure can be written as

$$f_n = B_n \mathbf{E} \left[\begin{cases} P & E_n(t_n) < 10.0 \\ P - 4.0 + 0.4E_n(t_n) & 10.0 \leq E_n(t_n) < 15.0 \\ P + 2.0 & 15.0 \leq E_n(t_n) < 20.0 \\ P - 6.0 + 0.4E_n(t_n) & 20.0 \leq E_n(t_n) < 25.0 \\ P + 4.0 & E_n(t_n) \geq 25.0 \end{cases} \right] V_n, \quad (3.19)$$

where E_n is the EUA price on delivery event n , f_n is the payoff function and P is a fixed floor price.

The payoff diagram for (3.19) is plotted in Figure 3.12 for different EUA prices. The results shown there are achieved by applying f_n to (3.2) and minimizing the difference between the swap legs $\mathcal{A}$ and $\mathcal{B}$ for P , where $g_n = \mathbf{E}[y_n]$. Note from these results that projects **C** and **D** immediately pass the Chinese floor restriction, while reaching an agreement for project **B** is also a quite realistic prospect. Also note that although based on fixed price rules, this structure offers significant upside to project participants but it does not improve the prospects of an ERPA for project **A**. Although bespoke EUA structures are probably only observed in legacy contracts, their impact in terms of risk and return still need to be properly assessed and managed. Thus, this particular case demonstrates the ability of the proposed approach to accommodate a wide variety of ERPA payoff functions.

*Figure B.4 from Appendix B.5 shows that CER Futures started in 2007 with fairly low volumes, which increased significantly from Mar-2008 onwards on the ICE ECX. Only then CERs started showing reasonable liquidity and the price discovery essential to ERPA negotiations.

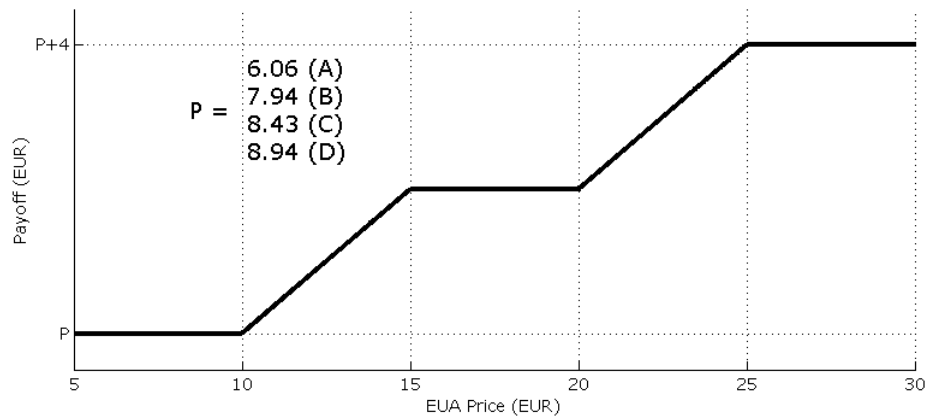


Figure 3.12: EUA structure price results

Conclusion

A methodology for the pricing of Emission Reduction Purchase Agreement (ERPA) contracts has been proposed in section 3.2 with the formulation of a general payoff function, followed by detailed analyses of CER delivery and carbon prices as underlying drivers in addition to their respective stochastic modeling and calibration. In section 3.3 we have shown the application of this methodology to run-of-river hydro power project activities in China for three different payoff structures. The results for a fixed primary CER price structure validate well to benchmark prices from actual market transactions except in the case of early stage projects, which in practice do experience difficulties in closing ERPAs given the high level of uncertainties involved. We have also argued that the results observed for a collared floating CER structure and a bespoke EUA structure are quite sensible given their respective intrinsic payoff characteristics. We consider this to be a demonstration of the level of flexibility and accuracy of the proposed methodology, awarding it an extra degree of relevance for applications on the pricing of ERPAs.

Conclusion

This thesis proposed alternative ways to price contingent claims written on portfolios of credit instruments as well as on carbon underlying assets. We started in chapter 1 by introducing the mathematical tools used in the development of these topics. Then in chapter 2 we elaborated on the first topic involving the pricing of Collateralized Debt Obligations (CDOs), where two alternative pricing approaches based on JSB distributions and entropy optimization were proposed followed by results and comparison with market prices. After that, we discussed the pricing of Emission Reduction Purchase Agreement (ERPA) contracts in chapter 3, where a pricing methodology was formulated and an application shown through a case study of run-of-river hydro power CDM project activities in China.

Given the feasibility of the proposed approaches, the research presented has the potential to be extended beyond the scope of this thesis. On CDO pricing, for instance, the entropy pricing approach can be improved to include a hybrid calibration combining both iTraxx Europe and CDX North America market quotes. Especially on the pricing of ERPAs, the application of entropy optimization techniques can be used to generate delivery probability distributions as an alternative to relying on the JSB assumption proposed in section 3.2.3. Furthermore, the carbon market is starting to transact on portfolios of CDM projects, and the pricing of ERPA portfolios with price and delivery volume structured payoffs can be explored with both JSB distributions and entropy optimization principles.

The results achieved were satisfactory and the learning experience rewarding. We conclude with the hope for having planted the seed for further work on these topics and with the motivation to continue forward with the future research opportunities mentioned.

Appendix A

Appendix: CDOs

A.1 Analysis of Goodwin versus Gauss-Kronrod quadrature

In order to check appropriateness and efficiency of applying Goodwin's formula for integrals of the form (2.21), we tested it against more modern methods such as Gauss-Kronrod quadrature as described by Shampine [45]. According to Shampine [45], the Matlab implementation `quadgk` of the Kronrod extension brings the Gauss formula up to 15 points, while we set the number of points of the Goodwin quadrature to 100. Figure A.1 shows the a sample JSB distribution with $a = 1.1630$, $b = 1.5063$, $c = 0.0000$ and $d = 2.0839$ for which this test has been applied. The Goodwin formula was hard-coded in Matlab while for the Gauss-Kronrod quadrature we use the function `quadgk` in Matlab both to solve the function:

$$\psi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{1 + e^{\frac{a_t - v}{b_t}}} dv$$

This integral results in 0.3310 for both Goodwin and Gauss-Kronrod, with the a negligible error term below 10^{-12} . The difference appears on the processing time, where it takes 1.5 milliseconds for the Goodwin method with 0.3 milliseconds for Gauss-Kronrod. The

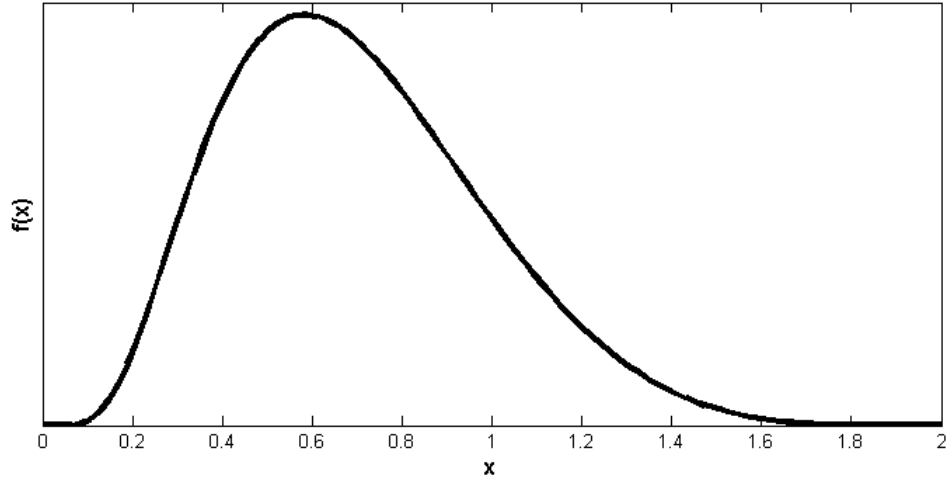


Figure A.1: Sample JSB distribution with $a = 1.1630$, $b = 1.5063$, $c = 0.0000$ and $d = 2.0839$

difference in processing time of about 5 times is negligible for experimental and research purposes, but may be significant to professional applications such as running large books of derivatives in banks. This was expected given the significant lower number of points used by the Gauss-Kronrod quadrature, and this alone might justify to switch from the Goodwin method in a production environment.

A.2 Alternative method for time-dependent b_j

The polynomial method proposed for b_j is one way to turn it into a time-dependent variable. Another way is to use the normal transformation ability of the JSB distribution to use a time-dependent Brownian motion. In this case we can turn z_t from (2.19) into

$$z_t = \frac{W_t}{\sqrt{t}},$$

where W_t is a Brownian motion. The calibration can then be achieved via a two-dimension Monte Carlo simulation by sampling W_t for each t across N sample paths.

A.3 Decomposition of ε_t from section 2.2.2

$$\varepsilon(t) = \int_{X_L}^{X_M} \min(x_t - X_L, X_U - X_L) f(x_t) dx_t \Rightarrow$$

$$\varepsilon(t) = \int_{X_L}^{X_U} (x_t - X_L) f(x_t) dx_t + \int_{X_U}^{X_M} (X_U - X_L) f(x_t) dx_t \Rightarrow$$

$$\begin{aligned} \varepsilon(t) &= \int_0^{X_M} (x_t - X_L) f(x_t) dx_t - \int_0^{X_L} (x_t - X_L) f(x_t) dx_t \\ &\quad - \int_{X_U}^{X_M} (x_t - X_L) f(x_t) dx_t + \int_{X_U}^{X_M} (X_U - X_L) f(x_t) dx_t \Rightarrow \end{aligned}$$

$$\begin{aligned} \varepsilon(t) &= \int_0^{X_M} (x_t - X_L) f(x_t) dx_t \\ &\quad - \int_0^{X_L} (x_t - X_L) f(x_t) dx_t - \int_{X_U}^{X_M} (x_t - X_U) f(x_t) dx_t. \end{aligned}$$

A.4 Solution for ε_t integrals $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ from section 2.2.2

First solve for $\mathcal{A}$ as

$$\mathcal{A} = \int_0^{X_M} (x_t - X_L) f(x_t) dx_t = \mathbf{E}[x_t] - X_L,$$

Now let $\mathcal{B}$ be

$$\mathcal{B} = \int_0^{X_L} (x_t - X_L) f(x_t) dx_t,$$

which can be solved with integration by parts where

$$g = (x_t - X_L) \Rightarrow g' = 1 \text{ and } h' = f(x_t) \Rightarrow h = \int_0^{x_t} f(u)du,$$

$$\int_0^{X_L} gh' dx_t = [gh]_0^{X_L} - \int_0^{X_L} g' h dx_t,$$

$$\begin{aligned} \mathcal{B} &= \int_0^{X_L} (x_t - X_L) f(x_t) dx_t \\ &= \left[(x_t - X_L) \int_0^{x_t} f(u) du \right]_0^{X_L} - \int_0^{X_L} \left[\int_0^{x_t} f(u) du \right] dx_t \Rightarrow \end{aligned}$$

$$\mathcal{B} = - \int_0^{X_L} \mathcal{J}_{SB}(x_t) dx_t \Rightarrow$$

$$\mathcal{B} = - \int_0^{X_L} \mathcal{J}_{SB}(x_t) dx_t,$$

where

$$\mathcal{J}_{SB}(x_t) = \int_0^{x_t} f(u) du.$$

Finally,

$$\mathcal{C} = \int_{X_U}^{X_M} (x_t - X_U) f(x_t) dx_t$$

which again can be solved with integration by parts, as

$$g = (x_t - X_U) \Rightarrow g' = 1 \text{ and } h' = f(x_t) \Rightarrow h = \int_0^{x_t} f(u) du,$$

$$\int_{X_U}^{X_M} gh' dx_t = [gh]_{X_U}^{X_M} - \int_{X_U}^{X_M} g' h dx_t,$$

$$\begin{aligned} \mathcal{C} &= \int_{X_U}^{X_M} (x_t - X_U) f(x_t) dx_t \\ &= \left[(x_t - X_U) \int_0^{x_t} f(u) du \right]_{X_U}^{X_M} - \int_{X_U}^{X_M} \left[\int_0^{x_t} f(u) du \right] dx_t \Rightarrow \end{aligned}$$

$$\mathcal{C} = (X_M - X_U) \int_0^{X_M} f(u) du - (X_U - X_U) \int_0^{X_U} f(u) du - \int_{X_U}^{X_M} \mathcal{J}_{SB}(x_t) dx_t \Rightarrow$$

$$\mathcal{C} = X_M - X_U - \int_{X_U}^{X_M} \mathcal{J}_{SB}(x_t) dx_t,$$

where once again

$$\mathcal{J}_{SB}(x_t) = \int_0^{x_t} f(u) du.$$

A.5 Solution for $\mathcal{J}_{SB}(x_t)$ from section 2.2.2

We first rewrite $\mathcal{J}_{SB}(x_t)$ in the general case as

$$\int_{X_A}^{X_B} \mathcal{J}_{SB}(x_t) dx_t = \int_{X_A}^{X_B} P[z \leq a + b \log \frac{x_t}{X_M - x_t}] dx_t. \quad (\text{A.1})$$

By substituting variables, make

$$u = a_t + b_t \log \frac{x_t}{X_M - x_t} \Rightarrow e^{\frac{u-a_t}{b_t}} = \frac{x_t}{X_M - x_t} \Rightarrow X_M - x_t = x_t e^{\frac{a_t-u}{b_t}} \Rightarrow$$

$$x_t = \frac{X_M}{1 + e^{\frac{a_t-u}{b_t}}},$$

or,

$$dx_t = \frac{1}{b_t} \frac{X_M e^{\frac{a_t-u}{b_t}}}{\left(1 + e^{\frac{a_t-u}{b_t}}\right)^2} du.$$

Applying this result to (A.1) results in

$$\int_{X_A}^{X_B} \mathcal{J}_{SB}(x_t) dx_t = \int_{K(X_A)}^{K(X_B)} P[z \leq u] \frac{1}{b_t} \frac{X_M e^{\frac{a_t-u}{b_t}}}{\left(1 + e^{\frac{a_t-u}{b_t}}\right)^2} du,$$

where

$$K(y) = a_t + b_t \log \frac{y}{X_M - y}.$$

But $P[z \leq u]$ is the cumulative normal distribution, or

$$P[z \leq u] = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv.$$

This results in

$$\int_{X_A}^{X_B} \mathcal{J}_{SB}(x_t) dx_t = \int_{K(X_A)}^{K(X_B)} \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \frac{1}{b_t} \frac{X_M e^{\frac{a_t-u}{b_t}}}{\left(1 + e^{\frac{a_t-u}{b_t}}\right)^2} dv du.$$

This integral can be solved by inverting the order of integration, which gives

$$\int_{X_A}^{X_B} \mathcal{J}_{SB}(x_t) dx_t = \mathcal{A} + \mathcal{B},$$

where

$$\mathcal{A} = \int_{K(X_A)}^{K(X_B)} \int_v^{K(X_B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \frac{1}{b_t} \frac{X_M e^{\frac{a_t-u}{b_t}}}{\left(1 + e^{\frac{a_t-u}{b_t}}\right)^2} du dv$$

and

$$\mathcal{B} = \int_{-\infty}^{K(X_A)} \int_{K(X_A)}^{K(X_B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \frac{1}{b_t} \frac{X_M e^{\frac{a_t-u}{b_t}}}{\left(1 + e^{\frac{a_t-u}{b_t}}\right)^2} du dv.$$

First we solve for $\mathcal{A}$, where

$$\mathcal{A} = \int_{K(X_A)}^{K(X_B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \left[\frac{X_M}{1 + e^{\frac{a_t-u}{b_t}}} \right]_v^{K(X_B)} dv \Rightarrow$$

$$\mathcal{A} = \int_{K(X_A)}^{K(X_B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \left[\frac{X_M}{1 + e^{\frac{a_t-K(X_B)}{b_t}}} - \frac{X_M}{1 + e^{\frac{a_t-v}{b_t}}} \right] dv \Rightarrow$$

$$\mathcal{A} = \frac{X_M}{1 + e^{\frac{a_t-K(X_B)}{b_t}}} [\Phi(K(X_B)) - \Phi(K(X_A))] - X_M \int_{K(X_A)}^{K(X_B)} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{v^2}{2}}}{1 + e^{\frac{a_t-v}{b_t}}} dv.$$

Given that

$$X_B = \frac{X_M}{1 + e^{\frac{a_t - K(X_B)}{b_t}}},$$

then

$$\mathcal{A} = X_B[\Phi(K(X_B)) - \Phi(K(X_A))] - X_M[\psi(K(X_B)) - \psi(K(X_A))],$$

where

$$\psi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \frac{e^{-\frac{v^2}{2}}}{1 + e^{\frac{a_t - v}{b_t}}} dv. \quad (\text{A.2})$$

and Φ is the cumulative normal probability distribution function. Now, we solve for $\mathcal{B}$, given

$$\begin{aligned} \mathcal{B} &= \int_{-\infty}^{K(X_A)} \int_{K(X_A)}^{K(X_B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \frac{1}{b_t} \frac{X_M e^{\frac{a_t - u}{b_t}}}{\left(1 + e^{\frac{a_t - u}{b_t}}\right)^2} du dv \Rightarrow \\ \mathcal{B} &= \left[\frac{X_M}{1 + e^{\frac{a_t - u}{b_t}}} \right]_{K(X_A)}^{K(X_B)} \int_{-\infty}^{K(X_A)} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv, \end{aligned}$$

which results in

$$\mathcal{B} = (X_B - X_A)\Phi(K(X_A))$$

given that

$$X_A = \frac{X_M}{1 + e^{\frac{a_t - K(X_A)}{b_t}}}.$$

Substituting back $\mathcal{A}$ and $\mathcal{B}$ gives

$$\begin{aligned} & \int_{X_A}^{X_B} \mathcal{J}_{SB}(x_t) dx_t \\ &= X_B \Phi(K(X_B)) - X_A \Phi(K(X_A)) - X_M [\psi(K(X_B)) - \psi(K(X_A))]. \end{aligned}$$

Finally we replace X_A and X_B by zero, X_L , X_U or X_M according to (A.1) to result in

$$\int_0^{X_L} \mathcal{J}_{SB}(x_t) dx_t = X_L \Phi(K(X_L)) - X_M \psi(K(X_L)),$$

$$\int_{X_U}^{X_M} \mathcal{J}_{SB}(x_t) dx_t = X_M - X_U \Phi(K(X_U)) - X_M \psi(K(X_M)) + X_M \psi(K(X_U)),$$

Then from (2.20)

$$\begin{aligned} \varepsilon(t) &= \mathbf{E}[x_t] + X_U [1 - \Phi(K(X_U))] - X_L [1 - \Phi(K(X_L))] \\ &\quad + X_M [\psi(K(X_U)) - \psi(K(X_L)) - \psi(K(X_M))]. \end{aligned} \tag{A.3}$$

A.6 iTraxx Europe portfolio and CDS spreads as of 15-Mar-2007

	CDS Spreads (bps)						CDS Spreads (bps)				
	1Y	3Y	5Y	7Y	10Y		1Y	3Y	5Y	7Y	10Y
AB Electrolux	7.43	22.29	38.39	52.10	67.29	Hellenic Telecom Org SA	6.92	18.29	31.68	43.73	59.39
AB Volvo	5.22	14.51	24.58	34.13	44.82	Henkel KG auf Aktien	2.62	6.56	11.28	17.83	25.74
ABN AMRO Bk NV	2.27	4.63	7.56	9.96	14.01	ITV PLC	5.09	11.62	18.03	24.60	31.91
ACCOR	8.11	21.61	37.59	52.35	64.85	Iberdrola SA	6.48	19.76	34.22	48.25	62.89
AKZO Nobel NV	8.66	22.24	38.77	51.65	65.54	Imperial Chem Inds PLC	8.43	24.45	42.05	59.07	77.26
ALLIANCE BOOTS PLC	2.98	6.82	11.03	14.20	18.29	Imperial Tob Gp PLC	2.35	4.86	7.91	9.97	14.09
ALTABIS SA	6.12	17.45	29.71	42.38	54.66	Intesa Sanpaolo SpA	15.47	46.00	80.12	109.21	130.24
ARCELOR Fin SCA	23.55	77.86	146.19	193.68	232.63	Kingfisher PLC	7.70	16.54	28.80	41.31	54.12
AXA	3.29	6.25	9.69	12.50	16.64	Koninklijke DSM NV	9.72	28.85	52.17	70.20	98.68
Adecco SA	9.46	22.43	42.16	61.04	79.23	Koninklijke KPN NV	6.03	15.78	27.84	39.67	50.92
Aegon NV	5.25	16.32	30.79	41.54	53.34	Koninklijke Philips Electre NV	5.65	14.48	26.03	37.87	50.91
Allianz AG	3.27	5.55	8.86	11.84	15.29	LVMH Moet Hennessy Louis Vuitton	5.17	13.82	24.13	34.84	46.50
Assicurazioni Generali SpA	3.46	6.19	9.41	12.29	16.10	Lafarge	3.78	10.89	19.56	28.02	37.73
Aviva PLC	3.61	7.46	11.57	14.90	18.89	Linde AG	8.79	22.95	40.93	58.46	74.27
BAE Sys PLC	2.89	5.03	7.81	10.46	14.04	METRO AG	8.05	20.46	36.18	52.64	68.66
Barclays Bk PLC	2.26	5.34	8.81	13.16	17.50	Marks & Spencer PLC	3.38	6.39	10.19	13.14	16.94
Bay Motoren Werke AG	3.96	11.63	20.88	28.37	37.79	Munich Re	5.25	14.83	24.07	32.51	42.95
Bayer AG	2.52	5.07	7.98	10.77	15.16	Natl Grid PLC	10.72	26.76	47.59	64.17	85.82
Bca Monte dei Paschi di Siena SpA	5.58	9.45	13.22	16.52	21.34	Nokia Oyj	5.55	13.43	23.87	33.07	44.45
Bca Pop Italiana	2.73	5.89	9.43	11.70	14.99	PPR	8.19	25.49	44.10	60.23	76.07
Bco Bilbao Vizcaya Argentaria SA	4.05	6.96	10.45	12.77	16.18	Pearson PLC	5.74	16.64	27.44	39.26	52.70
Bco Comercial Portugues SA	3.29	7.28	11.20	13.63	16.84	Peugeot SA	5.46	15.54	25.95	37.02	51.34
Bco Espirito Santo SA	3.44	6.18	9.55	11.96	15.11	RENTOKIL INITIAL PLC	5.08	14.54	24.60	34.06	46.82
Bco Santander Cen Hispano SA	4.75	13.87	23.09	33.37	46.74	RWE AG	6.22	14.78	25.19	35.69	46.20
Bertelsmann AG	2.60	4.64	6.84	9.05	12.11	Reed Elsevier PLC	4.90	13.38	23.32	32.52	43.31
Brit Amern Tob PLC	5.79	14.94	26.74	38.87	50.30	Renault	3.78	7.73	12.08	15.34	20.19
Brit Telecom PLC	8.54	23.20	40.34	54.54	75.42	Repsol YPF SA	2.39	4.73	7.89	10.54	13.74
CAPITALIA S PER AZIONI	10.38	25.88	46.02	62.38	79.84	Reuters Gp PLC	2.92	7.22	11.64	16.33	21.86
CIE Fin Michelin	3.21	7.01	11.22	13.78	17.73	Royal & Sun Alliance Ins PLC	10.67	33.25	57.88	79.22	100.52
Cadbury Schweppes PLC	6.92	22.43	36.53	51.18	67.78	Royal Bk Scotland PLC	2.36	6.96	12.51	17.00	22.98
Carrefour	11.56	34.78	59.13	78.85	99.03	SAFEMWAY LTD	2.54	6.78	11.80	18.01	24.53
Casino Guichard Perrachon	4.57	12.11	20.09	28.64	40.04	SANOFI AVENTIS	4.83	13.78	24.44	35.83	47.65
Centrica PLC	9.95	28.46	48.05	63.42	79.97	SiemensAG	2.77	7.21	13.66	19.48	25.73
Ciba Specialty Chems Hldg Inc	5.42	14.59	26.23	36.89	50.61	Sodexo Alliance	4.42	16.51	28.96	40.04	53.96
Cie de St Gobain	5.14	14.49	25.10	35.01	47.26	Solvay SA	8.79	23.31	41.66	57.65	75.25
Commerzbank AG	3.51	6.39	9.94	13.43	17.50	Stora Enso CORP	3.04	8.69	13.70	18.33	25.41
Compass Gp PLC	8.81	22.96	40.55	55.18	68.81	Suez	4.51	13.89	24.76	34.96	46.99
Conti AG	7.88	23.73	41.78	58.15	78.53	Svenska Cellulosa AB SCA	3.74	7.11	11.08	14.12	17.86
DSG Intl PLC	13.22	28.88	41.71	53.87	67.29	Swiss Reins Co	11.26	31.20	51.88	71.56	90.73
DaimlerChrysler AG	3.69	8.59	14.28	17.45	21.22	THOMSON	11.58	31.02	54.16	70.86	96.68
Deutsche Bk AG	8.74	24.81	42.80	57.43	71.62	Tate & Lyle PLC	9.16	20.52	30.78	41.51	55.41
Deutsche Lufthansa AG	6.65	17.65	28.99	40.53	55.42	Telecom Italia SpA	5.64	16.35	28.52	39.39	53.25
Deutsche Telekom AG	4.46	10.32	16.82	24.55	32.30	Telefonica SA	5.98	13.27	23.00	32.65	44.00
Diageo PLC	13.31	28.10	50.08	68.18	84.52	Telekom Austria AG	7.21	18.74	32.28	45.81	63.48
E.ON AG	6.25	9.92	14.00	18.32	25.24	Telenor ASA	3.24	6.87	10.74	16.20	22.32
EDP Energias de Portugal SA	3.56	9.10	15.58	20.55	27.05	TeliaSonera AB	11.48	36.25	64.26	86.23	108.96
ELECTRICITE DE FRANCE	3.80	7.79	12.33	16.88	23.32	Tesco PLC	5.11	17.57	31.08	43.56	57.08
ENEL SpA	2.39	5.72	8.70	11.76	15.68	ThyssenKrupp AG	4.55	10.09	17.36	23.32	30.25
EXPERIAN Fin PLC	2.95	6.96	11.21	15.01	22.24	UPM Kymmene CORP	3.60	7.13	10.92	13.65	17.32
Edison SpA	5.54	10.07	14.63	20.00	25.75	Un Fenosa SA	4.50	11.12	20.31	31.17	41.40
EnBW Energie Baden Wuerttemberg AG	3.49	7.73	13.12	17.76	24.62	UniCredito Italiano SpA	8.52	24.26	42.02	57.59	74.95
Endesa SA	3.75	10.94	18.41	26.78	36.87	Unilever NV	5.12	15.15	24.67	34.11	46.12
Eurpn Aero Defence & Space Co Fads NV	12.71	30.02	49.79	68.75	85.76	Utd Utils PLC	18.96	52.64	88.89	123.55	145.37
Fortum CORP	3.37	7.61	12.40	17.15	22.83	Valeo	2.94	7.61	12.63	16.90	22.84
France Telecom	5.77	14.07	24.47	33.62	46.42	Vattenfall AB	4.88	13.44	24.11	33.22	45.59
GAZ DE FRANCE	2.09	5.64	9.52	13.36	17.54	Veolia Environnement	7.03	18.42	32.23	43.74	57.04
GN Hldgs PLC	3.85	8.67	14.31	19.03	25.07	Viendi	9.62	27.56	47.80	63.48	83.07
GROUPE AUCHAN	2.52	5.69	8.78	14.02	18.10	Vodafone Gp PLC	5.42	14.05	24.08	33.53	46.70
GUS PLC	8.51	26.92	47.49	64.45	82.36	Volkswagen AG	3.64	11.68	19.20	27.12	35.77
Gallaher Gp PLC	28.81	50.79	78.09	103.83	124.98	WPP 2005 Ltd	10.13	26.62	44.66	59.49	79.37
Gas Nat SDG SA	4.63	11.66	19.62	27.50	38.04	Wolters Kluwer NV	6.83	16.76	27.57	39.25	53.00
Glencore Intl AG	4.32	8.60	12.76	16.12	21.61	Zurich Ins Co	3.91	7.47	12.08	15.61	19.89
Hannover Ruck AG	5.79	18.58	32.53	45.90	61.51						

Figure A.2: iTraxx Europe portfolio and CDS spreads as of 15-Mar-2007

A.7 Monte Carlo implementation of the Gaussian copula model

The steps for implementing a Gaussian copula model under Monte Carlo simulation are as follows.

1. Calibrate $F_m(\tau_m)$ as the survival probability function from each individual obligor's credit curve m according to Li [33] or any other technique.
2. Sample u_{nm} independent uniformly distributed random numbers for N Monte Carlo paths by M obligors.
3. Generate $z_{nm} = \Phi^{-1}(u_{nm})$ independent standard normally distributed random numbers.
4. Given M by M correlation matrix C^* , generate the matrix R as the Cholesky decomposition of $C = R^T R$.
5. Compute the set of correlated normally distributed random numbers $f_{nm} = R z_{nm}$.
6. Transform f_{nm} into a set of correlated uniform numbers $\bar{u}_{nm}$ such that $\bar{u}_{nm} = \Phi(f_{nm})$.
7. For each path n and obligor m , imply the time-until-default $\tau_{nm} = F_m^{-1}(\bar{u}_{nm})$. For expiry T , if $\tau_{nm} \leq T$ then obligor m defaults on path n and $y_{nm} = 1$; otherwise it survives and $y_{nm} = 0$.
8. Default losses are accumulated for each path n as $x_n = \sum_{m=1}^M (1 - r_m) \omega_m y_{nm}$.
9. Density and cumulative loss probability distributions can be constructed and applied to any payoff function including the CDO payoff of (2.16).

*See J.P. Morgan Securities [28] for details on the base correlation method.

Appendix B

Appendix: ERPAs

B.1 Solution for $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ in section 3.2.3

Let

$$\begin{aligned}\mathcal{A}_1 &= \log \left(\frac{b}{\sqrt{2\pi}} \right) \int_c^{c+d} f_{SB}(x) dx, \\ \mathcal{A}_3 &= -\frac{1}{2} \int_c^{c+d} f_{SB}(x) dx,\end{aligned}$$

where $f_{SB}(x)$ is the JSB density given by (1.3c). Thus,

$$\begin{aligned}\mathcal{A}_1 &= \log \left(\frac{b}{\sqrt{2\pi}} \right) [\mathcal{J}_{SB}(c+d) - \mathcal{J}_{SB}(c)], \\ \mathcal{A}_3 &= -\frac{1}{2} [\mathcal{J}_{SB}(c+d) - \mathcal{J}_{SB}(c)].\end{aligned}$$

where $\mathcal{J}_{SB}$ is the JSB cumulative distribution function given by (1.4c). According to the JSB transformation of (1.2c), the domain of x is defined as $c \leq x \leq c+d$. Therefore, $\mathcal{J}_{SB}(c+d) = 1$ and $\mathcal{J}_{SB}(c) = 0$, and

$$\mathcal{A}_1 = \log \left(\frac{b}{\sqrt{2\pi}} \right),$$

$$\mathcal{A}_3 = -\frac{1}{2}.$$

Now let

$$\mathcal{A}_2 = \int_c^{c+d} \log \left[\frac{d}{(x-c)(c+d-x)} \right] \frac{b}{\sqrt{2\pi}} \frac{d}{(x-c)(c+d-x)} \exp \left\{ -\frac{1}{2} \left[a + b \log \left(\frac{x-c}{c+d-x} \right) \right]^2 \right\} dx.$$

Make

$$u = \log \left(\frac{x-c}{c+d-x} \right) \Rightarrow \frac{du}{dx} = \frac{d}{(x-c)(c+d-x)},$$

and

$$e^u = \frac{x-c}{c+d-x} \Rightarrow x = \frac{(c+d)e^u + c}{1+e^u} \Rightarrow$$

$$x-c = \frac{de^u}{(1+e^u)}, c+d-x = \frac{d}{(1+e^u)} \Rightarrow$$

$$\frac{d}{(x-c)(c+d-x)} = \frac{(1+e^u)^2}{de^u},$$

Applying these results to $\mathcal{A}_2$ yields

$$\mathcal{A}_2 = \int_{-\infty}^{+\infty} [2 \log(1+e^u) - d - u] \frac{b}{\sqrt{2\pi}} \exp \left[-\frac{1}{2}(a+bu)^2 \right] du.$$

Now

$$z = a + bu \Rightarrow$$

$$\frac{dz}{du} = b, d + u = d + \frac{z - a}{b} = \frac{bd - a}{b} + \frac{z}{b}.$$

This result leads to

$$\begin{aligned} & \int_{-\infty}^{+\infty} (d + u) \frac{b}{\sqrt{2\pi}} \exp \left[-\frac{1}{2}(a + bu)^2 \right] du \\ &= \int_{-\infty}^{+\infty} \frac{bd - a}{b} \frac{b}{\sqrt{2\pi}} \frac{1}{b} e^{-\frac{1}{2}z^2} dz \\ &= \frac{bd - a}{b} \Rightarrow \end{aligned}$$

$$\mathcal{A}_2 = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \log \left(1 + e^{\frac{z-a}{b}} \right) e^{-\frac{1}{2}z^2} dz - \frac{bd - a}{b}.$$

B.2 Estimates of emissions factors for different fuels

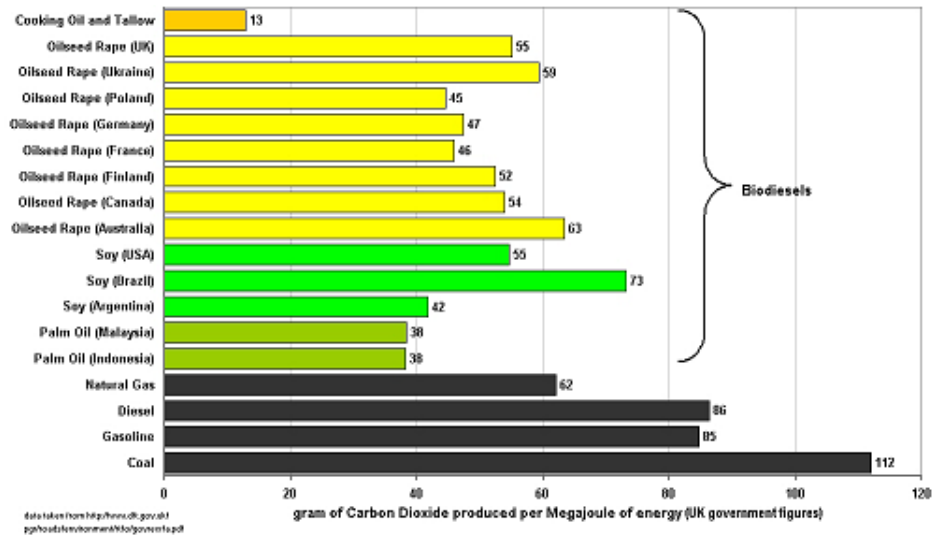


Figure B.1: Estimates of emissions factors for different fuels

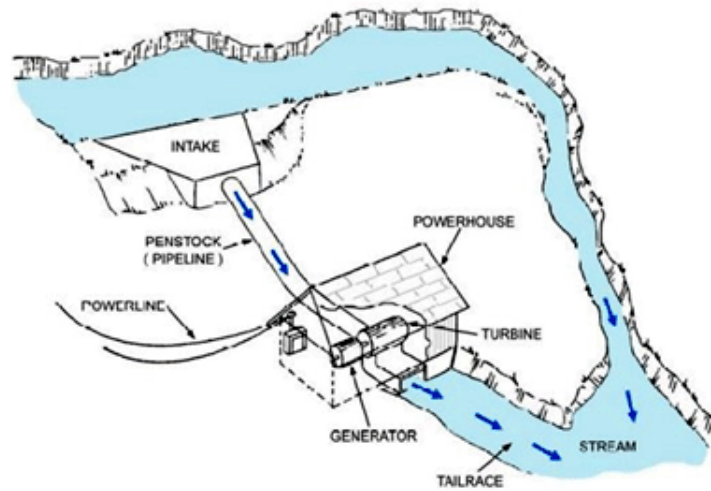
B.3 Chinese emissions factors 2009

Regional Grid	Covered Region	OM* (t-CO ₂ /MWh)	2008 OM	2007 OM	BM* (t-CO ₂ /MWh)	2008 BM	2007 BM
North China Grid	Beijing, Tianjin, Hebei, Shanxi, Shandong, Inner-Mongolia	1.0069	1.169	1.2172	0.7802	0.8687	0.9397
Northeast China Power Grid	Liaoning, Jilin, Heilongjiang	1.1293	1.2561	1.2404	0.7242	0.8068	0.8631
East China Grid	Shanghai, Jiangsu, Zhejiang, Anhui, Fujian	0.8825	0.9540	0.9421	0.6826	0.8236	0.8672
Central China Power Grid	Henan, Hubei, Hunan, Jiangxi, Sichuan, Chongqing	1.1255	1.2783	1.2899	0.5802	0.6687	0.6592
Northwest China Power Grid	Shaanxi, Gansu, Qinghai, Ningxia, Xinjiang Uyghur	1.0246	1.1225	1.1257	0.6433	0.6199	0.5739
Southern China Power Grid	Guangdong, Jiangxi, Yunnan, Guizhou	0.9987	1.0608	1.0119	0.5772	0.6816	0.6748
Hainan Provincial Power Grid	Hainan	0.8154	0.8944	0.9209	0.7297	0.7523	0.7517

*OM: Operating Margin from 2005-2007 data; BM: Build Margin from 2007 data
 Source: CDM in CHINA, 2009. <<http://cdm.cchina.gov.cn/english/>> Recent data was announced in 3 September 2009.

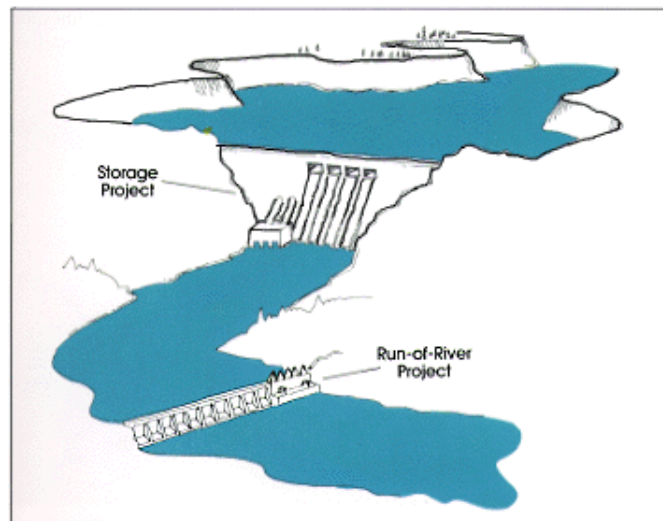
Figure B.2: Chinese emissions factors 2009 as published in Institute for Global Environmental Strategies (IGES) [21]

B.4 Run-of-river hydro power installations



(a) Water through penstock to the powerhouse downriver

Storage and Run-of-River Projects



(b) Water stored in a small reservoir behind a dam constructed across the full width of the river

Figure B.3: Diagrams of two types of run-of-river hydro power installations

B.5 CER exchange trading in 2009

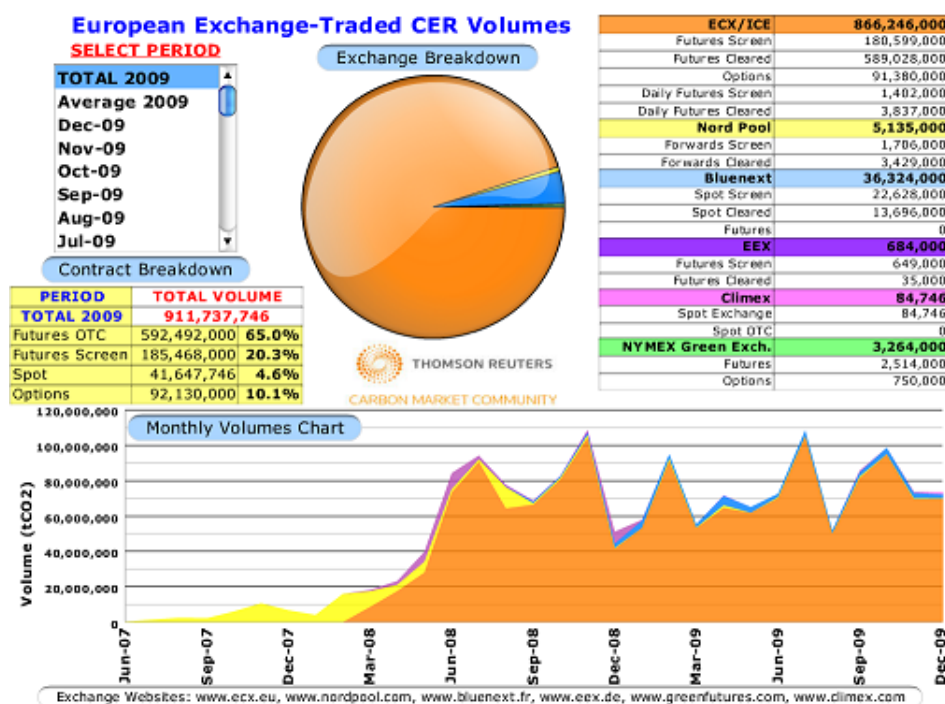


Figure B.4: CER exchange trading in 2009

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