

Non linear effects in plasma waves
A THEORY OF V.L.F. EMISSIONS

by

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ABSTRACT

The system first studied is that of an electrostatic Gaussian wave packet in a cold uniform plasma, undergoing excitation by a weak resonant beam. To first order in the weakness of the beam the resonant particle distribution function and resonant particle charge densities are computed. The results obtained are of considerable interest in understanding the non linear limit of the Landau problem. It is found that the power input into the wave field from the resonant particles is initially that due to linear theory, but after one trapping period the sign of the power input changes, and after many trapping periods dies away altogether. This result suggests that a homogeneous system will not exhibit triggering instability.

Next, a similar system is studied-but inhomogeneity is introduced by having a spatially dependent wave number. It is found that second order resonant particles become stably trapped in the wave, exactly as in the homogeneous case, and after a trapping period these dominate the resonant particle distribution function and charge density. The growth rate, instead of dying away, is found to increase linearly with trapping time, and thus very large growth rates may be found in a long pulse. Additionally, the reactive component of charge density is able to cause a steady change in wave frequency and wave number.

Finally, the whistler problem of the triggering of VLF emissions by morse pulses is considered, using these

concepts of second order resonant particle behaviour. Trajectories in fields approximating those which might actually be found are computed and studied analytically. It is found that second order resonant particles are again stably trapped, and these give rise to currents that are readily estimated. It is shown that the in phase component of current is able to cause very large growth rates and that the reactive component can give a steady change in wave frequency.

The growth rates and rates of change of frequency that would exist in an emission field travelling along a magnetospheric field line are computed, and using this data it is possible to give a reasonable account of the whole triggering process. Quite a number of detailed features of the process admit of an explanation in this theory—such as the offset frequency, triggering delay, formation of hooks etc. In addition the observed amplitudes agree fairly well with those predicted by the theory.

ACKNOWLEDGEMENTS

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PROLOGUE

The aim of this thesis is to provide a theoretical explanation of the triggering of VLF emissions by whistler morse pulses in the earth's magnetosphere.

It is generally agreed that VLF emissions arise as a result of resonant wave particle interactions in the vicinity of the magnetospheric equator. Such interactions are not only non linear, but take place in an inhomogeneous medium. The problem is obviously a very complex one and practically impossible to tackle without a good understanding of wave particle processes under general conditions.

It was suggested by Professor Dungey that a thorough study of the analogous but much simpler electrostatic problem would be a very valuable guide to the more complex whistler case. Accordingly the present thesis will be seen to be divided into three parts.

The first part deals with the simplest possible wave particle interaction problem-namely that of the beam excitation of an electrostatic wave packet in a homogeneous medium. This we shall see serves to elucidate the role of particle trapping in the problem, but fails to predict triggering.

The second section is a logical extension of the first and deals with the beam excitation of an electrostatic packet in the inhomogeneous case-the inhomogeneity being obtained by means of a spatially dependent wave number.

This work shows how second order resonant particles become very important, and predicts very large growth rates and changes in wave frequency.

The third section is of course an application of these ideas to the whistler problem, and takes the form of a theoretical explanation of VLF triggering by morse pulses.

SECTION ONE

WAVE PARTICLE EFFECTS IN AN ELECTROSTATIC PULSE IN A
HOMOGENEOUS MEDIUM

NON LINEAR WAVE PARTICLE INTERACTIONS IN AN ELECTROSTATIC WAVE PACKET

1. Introduction.

The problem of the non-linear behaviour of electrostatic waves due to resonant particle trapping is of considerable theoretical and experimental interest, and has been the subject of much research.

The full self consistent problem in the general case is very complex and some simplifying assumptions are usually necessary. While the question of particle trapping dynamics has been qualitatively discussed (Dungey, Sagdeev) most of the analytic work has been in connection with the so called quasi linear theory (Drummond and Pines, Bernstein and Engelmann). The latter, however, is chiefly concerned with the broad band noise problem in which complete particle trapping does not occur.

The non linear time development of an infinitely long monochromatic wave has been studied by computer, (Klozenberg, Armstrong) but these treatments are usually obliged to truncate higher harmonics and are only valid for a time of the order of a trapping period.

In this paper we choose to study the non linear development of a one dimensional electrostatic Gaussian wave packet due to the introduction of a weak resonant beam. This is a more physically realistic system than an infinite wave and differs from it in several respects.,

one of which is that resonant particles only interact with the field for a finite time. Additionally, the results are expected to provide insight into the wave particle interaction problem in general, and should help to understand the processes operating in the VLF triggered emission problem.

2. Mathematical formulation.

The ambient plasma is homogeneous, and in order to sidestep the purely linear dispersion (Houghton) is assumed cold or nearly cold. The zero order wave packet field, i.e. prior to the introduction of the resonant beam, is chosen to have the following simple form:

$$E^{(0)} = E_0 \cos(k_0 x + \pi t) e^{-x^2/d^2} \quad (1)$$

The governing equations are those of Poisson and Vlasov:

$$\partial f / \partial t + v \partial f / \partial x = e E / m \partial f / \partial v \quad (2)$$

$$\partial E / \partial x = -4\pi e \int f dv \quad (3)$$

It is convenient to deal with dimensionless quantities related to the wave packet parameters as follows;

$$R = e E_0 k_0 / m \pi^2 \quad N = k d$$

$$E' = E / E_0 \quad f' = f \pi / k_0 / \int F_c dv$$

$$\pi_e^2 = 4\pi e^2 \int F_c dv / m$$

where $F_c(v)$ is the unperturbed distribution function appropriate to the cold background plasma. We thus obtain, dropping primes

$$\partial f / \partial t + v \partial f / \partial x = R E \partial f / \partial v \quad (4)$$

$$\partial E / \partial x = -\frac{1}{R} \int f dv \quad (5)$$

The zero order field becomes;

$$E^{(0)} = \cos(x+t) e^{-x^2/N^2} \quad (6)$$

For the problem as posed to be tractable at all it is necessary to assume that $R \ll 1$. In this case it is well known that particles entering the wave packet within a narrow range of the resonant velocity $v_{res} \pm R^{1/2}$ become trapped by the wave and execute non linear oscillations in the potential well of the wave. The cold particles of the background plasma however will have a simple linear response to fields present. On account of this differing behaviour the cold and beam particles are treated separately, the corresponding distribution functions being denoted $F_c + f_{nr}$ and $F_b + f_{res}$ respectively. Here F_b is the unperturbed beam distribution function. In accordance with our weak beam assumption we suppose that the fraction of particles resonant is very small. This fraction will be of order η where η is a smallness parameter associated with the weakness of the beam. Note that η is in principle another smallness parameter independent of R . Our program is first to expand f and E in powers of η .

$$f_{nr} = \sum_{n=0}^{\infty} f_{nr}^{(n)} \quad (7)$$

$$f_{res} = \sum_{n=0}^{\infty} f_{res}^{(n)} \quad (8)$$

$$E = \sum_0^{\infty} E^{(n)} \quad (9)$$

Noting that $\int f_{res}^{(n)}$ is of order $n+1$ the Vlasov and Poisson equations may be written

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) f_{nr}^{(n)} = R \sum_{n^*=0}^n E^{(n^*)} \frac{\partial}{\partial v} f_{nr}^{(n-n^*)} + R E^{(n)} \frac{\partial F_c}{\partial v} \quad (10)$$

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) f_{us}^{(n)} = R \sum E^{(n^*)} \frac{\partial}{\partial v} f_{us}^{(n-n^*)} + R E^{(n)} \frac{\partial F_b}{\partial v} \quad (11)$$

$$\frac{\partial}{\partial x} E^{(n)} = -\frac{1}{R} \int f_{nr}^{(n)} dv - \frac{1}{R} \int f_{us}^{(n-1)} dv \quad (12)$$

where the integrals are over cold plasma and beam velocities respectively. The time for which this rather arbitrary expansion in η is valid is of the order of a Landau growth period, since then $E^{(1)} \sim E^{(0)}$. However the system can be studied for longer periods by a process of re-ordering of terms i.e. $E^{(1)}$ is included in $E^{(0)}$. Such a process turns out to require an enormous amount of computation time however.

The zero order equations of the set (11) and (12) are simply those to be satisfied by $E^{(0)}$, the wave packet field in the absence of a beam.

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) f_{nr}^{(0)} = R E^{(0)} \left(\frac{\partial F_c}{\partial v} + \frac{\partial f_{nr}^{(0)}}{\partial v} \right) \quad (13)$$

$$\frac{\partial}{\partial x} E^{(0)} = -\frac{1}{R} \int f_{nr}^{(0)} dv - \frac{1}{R} \int f_{us}^{(0)} dv$$

For cold particles we may neglect the non linear term $E^{(0)} \frac{\partial f_{nr}^{(0)}}{\partial v}$, since this is of order R , and produces effects in which we are not particularly interested. The first

approximation to the distribution function about resonance $F_{res}^{(0)}$ is given by

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} - R E^{(0)} \frac{\partial}{\partial v} \right) (F_0 + f_{res}^{(0)}) = 0 \quad (14)$$

where here the non linear term is important. The numerical computation of F_{res} and its integral the resonant particle charge density are the chief concerns of this paper. The first order field $E^{(1)}$ is the solution to the following equations

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) f_{nr}^{(1)} = R E^{(1)} \frac{\partial F_0}{\partial v} \quad (15)$$

$$\frac{\partial E^{(1)}}{\partial x} = -\frac{1}{R} \int f_{nr}^{(1)} dv - \frac{1}{R} \int f_{res}^{(0)} dv \quad (16)$$

The non linear terms $E^{(0)} \frac{\partial f_{nr}^{(1)}}{\partial v}$ and $E^{(1)} \frac{\partial f_{nr}^{(0)}}{\partial v}$ are of order R^2 for non resonant particles, and can again be neglected.

The nature of $E^{(1)}$ is better seen in Fourier Laplace transform;

$$E_{wk}^{(1)} = R \rho_{wk}^{(1)} / i k \epsilon(\omega, k) \quad (17)$$

$$\rho^{(1)} = \int f_{res}^{(0)} dv \quad \epsilon(\omega, k) = \left\{ 1 + \int \frac{\partial F_0 / \partial v}{\omega - kv} dv \right\} \quad (18)$$

Clearly then to first order in R and η , $E^{(1)}$ is the response of the cold plasma to an imposed charge density $\rho^{(1)}$.

Components of $\rho^{(1)}$ varying as $e^{i\eta t}$ will excite a resonant response and produce a time proportional electric field.

Conversely, higher order components as $e^{i2\eta t}$ will excite only small electric fields.

For completeness the next higher approximations to F_{res}

and E are given, but their computation is excessively laborious and will not be dealt with here.

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} - R E^{(0)} \frac{\partial}{\partial v} \right) f_{res}^{(1)} = R E^{(1)} \left(\frac{\partial F_b}{\partial v} + \frac{\partial f_{res}^{(0)}}{\partial v} \right)$$

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) f_{nr}^{(2)} = R E^{(2)} \frac{\partial F_c}{\partial v}$$

$$\frac{\partial}{\partial x} E^{(2)} = -\frac{1}{R} \int f_{res}^{(1)} dv - \frac{1}{R} \int f_{nr}^{(2)} dv$$

The above expressions are to order $\eta^2 R$. Terms of order $\eta^2 R^2$ for the cold plasma have been omitted.

3. The computation of $F_{res}^{(0)}$

The form of $f_{res}^{(0)}(x_0, t_0, v_i)$ for particular values of x_0, t_0 is found as follows. A sufficient number of v values v_i are chosen (in fact about 250) equispaced across the resonant width. For each particle thereby represented the equation of motion is integrated backwards numerically. The method of integration is the standard one and is as follows. At each timestep t_n, v , and x are expanded as a function of $\delta = t - t_n$ to terms of order R^2 . Substituting $\delta = t_{n+1} - t_n$ enables the entire trajectory to be followed in a stepwise fashion. If v_f is the velocity on entering the packet we may then write at once, from eq. 14

$$f_{res}^{(0)}(x_0, t_0, v_i) = F_b(v_f) - F_b(v_i)$$

Since the resonant width is very narrow, we suppose that the variation of F_b across the window is essentially linear, so we put

$$F_b \equiv C \left(1 + (v - 1)/2 \right)$$

The overall picture of the beam excitation is shown in graph 10. The quantity $f_{res}^{(0)}$ is calculated as a function of X_0 to V for various values of R and N . The reader will be spared remaining computational details, and we now proceed to discuss the results.

4. The results for $f_{res}^{(0)}$

It was found from the computations that the factor largely determining the form of the distribution function was the number of trapping oscillations ψ undergone from the time of entering the packet. Consideration will show that this is well estimated by

$$\psi \sim \int_{-3.5N}^x \sqrt{RE'(x')} dx'$$

In the computation it was found most convenient to examine the behaviour of $f_{res}^{(0)}$ behind the wave packet, where ψ is simply equal to $NR^{1/2}$.

For $\psi \ll 1$ $f_{res}^{(0)}$ has a simple linear form as follows;

$$f_{res}^{(0)} = -\frac{\partial F_b}{\partial v} R \frac{1}{V_0} \left\{ e^{i(t_0 + X_0 N)} \int_{X_0/V_0}^{\infty} e^{-V_0^2 \tilde{E}^2 / N^2 + i(V_0 - 1) \tilde{E}} d\tilde{E} \right.$$

which for $X_0 < -4N$ becomes

$$f_{res}^{(0)} = -R \frac{\partial F_b}{\partial v} \frac{\pi^{1/2} N}{V_0} e^{-(V_0 - 1)^2 N^2 / 4 V_0^2} \cos(t_0 + X_0 / V_0)$$

The most obvious qualitative feature of the non linear computation is that when ψ becomes about 1.5 and complete trapping oscillations become possible, regions of very sharp gradients, almost discontinuities, appear in $f_{res}^{(0)}$, the number of these increasing rapidly with ψ . This

behaviour is well illustrated by graphs 1-4, which show $f_{res}^{(0)}$ at $X_0 = -3.5N$ and at an arbitrary phase for various values of ψ . In order to investigate the origin of these discontinuities the program was made to count the number of particle bounces that occurred in each trajectory integration. It was found that discontinuities coincided with a change in bounce number. The trajectory at such a point must undergo a bounce actually at the crest of a wave, for in this case two closest adjacent particles can fall on opposite sides of the crest and have totally different paths.

« In order to examine the phase variation of $f_{res}^{(0)}$ it is convenient to make use of the fact that behind the packet it must satisfy

$$(\partial/\partial t + v\partial/\partial x)f_{res}^{(0)} = 0$$

$$f_{res}^{(0)}(x, v, t) = f_{res}^{(0)}(x, v, t + 2\pi)$$

The distribution function behind the packet is then expressible in the following form.

$$f_{res}^{(0)}(x_0, v_0, t) = \sum_0^{\infty} \mathcal{F}_n(v) e^{in(t_0 + x_0) - inx_0(v-1)}$$

Examples of the phase variation of $f_{res}^{(0)}$ for various v and ψ are given in graph 8. The sharp gradients indicate that the higher space time harmonics $\mathcal{F}_2 \mathcal{F}_3$ are important when ψ gets larger - there is a steady progression towards higher and higher harmonics in the distribution function. This is one reason why quasi linear theory may not be applied to narrow band signals, these harmonics being

ignored in that theory.

The physical picture of the non linear phase variation is one of a steady progression of discontinuities in v . This is shown in graph 7 which plots regions of different bounce number as a function of v , to. The graph also clearly illustrates the shearing of the pattern due to the phase mixing term

Of considerable interest is the phase average distribution function $\bar{F}_b + \overline{f_w^{(0)}} e^t$. Behind the wave packet this is $F_b + F_0$, the latter quantity being all that would remain a long way behind the packet after phase mixing and diffusion had removed all other components. $F_b + F_0$ is plotted in graph 5 for various values of ψ . Around $\psi \sim 1.3$ a plateau is formed but for $\psi > 2$ the gradient is completely reversed. This reversal takes place during the first trapping oscillation but $\bar{F}$ never returns to its previous forms during subsequent oscillations - the generation of higher harmonics in the distribution function being an irreversible process.

The narrow regions of very steep gradients at the edge of the resonant width are of interest with respect to small signal amplification, and are analagous to those in the Das theory of VLF emissions.

5. The resonant particle charge density.

As the source of the first order field $E^{(1)}$, this is of particular interest. At any point we must have

$$\rho^{(1)} = \int f_{res}^{(0)} dv = \sum_0^{\infty} \tilde{\rho}_n^{(1)} e^{int}$$

An interesting feature of the computations was that $\rho^{(1)}$ varied almost sinusoidally with time at the fundamental frequency. The quantities $\rho_2^{(1)}$, $\rho_3^{(1)}$ and higher components were generally small, in spite of $\tilde{T}_2$, $\tilde{T}_3$ etc being appreciable. This averaging out of the higher harmonics of F in the v integration has been noted by other authors, notably Klozenberg. However it does not occur if F_0 does not have a simple linear variation across the resonant window.

The quantity $\rho_0^{(1)}$ is the time independent component of density arising quite simply from the fact that the flux of resonant particles must be conserved even though the velocity distribution has been shifted towards lower velocities. The form of F (graph 5) shows that $\rho_0^{(1)} \propto R^{3/2}$ for $\psi > 2$, with a much faster dependence for $\psi < 2$. This is born out by graph 9 which shows $\rho_0^{(1)}$ behind a wave packet of $N=25$ as a function of ψ . This density will tend to be almost entirely neutralised by the cold plasma, but in a finite geometry situation long wavelength electrostatic or ion sound waves could be generated.

Graphs 6a and 6b show the form of $\rho_1^{(1)}$, the density component resonant with the ambient plasma. Its amplitude and phase are plotted as a function of x for two packets, with ψ values 1.5, 2.5. The following features of these diagrams will be noted:

- (1) The phase and amplitude of $\rho_i^{(1)}$ in the forward part of the packets are such as to give the linear Landau growth rate.
- (2) After one trapping oscillation the phase reverses sharply to give a damping region of a similar length.
- (3) Further phase reversals occur downstream, but get steadily less sharp.
- (4) With successive trapping oscillations, density amplitudes decrease, mainly because $\tilde{T}_2 \tilde{T}_3$ and higher harmonics grow at the expense of $\tilde{T}_1$. Another important factor in this decay is phase mixing, which is alone responsible for killing the $\rho_i^{(1)}$ signal downstream from the main packet in a distance of the order of N .

6. Time development of the electric field.

A qualitative picture of the time development of the electric field can be seen directly from the $\rho_i^{(1)}$ graphs. The front part of the packet undergoes Landau growth with alternate regions of growth and damping behind, but with decreasing amplitude. In the case of a warm plasma considerable diffusion of amplitude from the front to the rear of the packet would be expected.

Another general feature to be seen is the extension of the packet downstream, although this takes place in a rather complex and irregular fashion.

7. Conclusion.

The mechanism has been established for treating the weak beam excitation of an electrostatic wave packet.

even in the non linear limit of many trapping oscillations.

The results for the distribution function and fields are a valuable guide to the non linear limit of the Landau problem. Of particular interest are the overturning of $\bar{F}$, the flow of energy towards higher harmonics, and the pattern of power flow to the zero order field with successive trapping times. The overall long time effect of the generated fields is to increase the length and amplitude of the pulse, but such processes only proceed at a rate commensurate with the Landau growth rate. It is thus difficult to regard these fields as being triggered emissions in the proper sense of the word.

GRAPH 1.

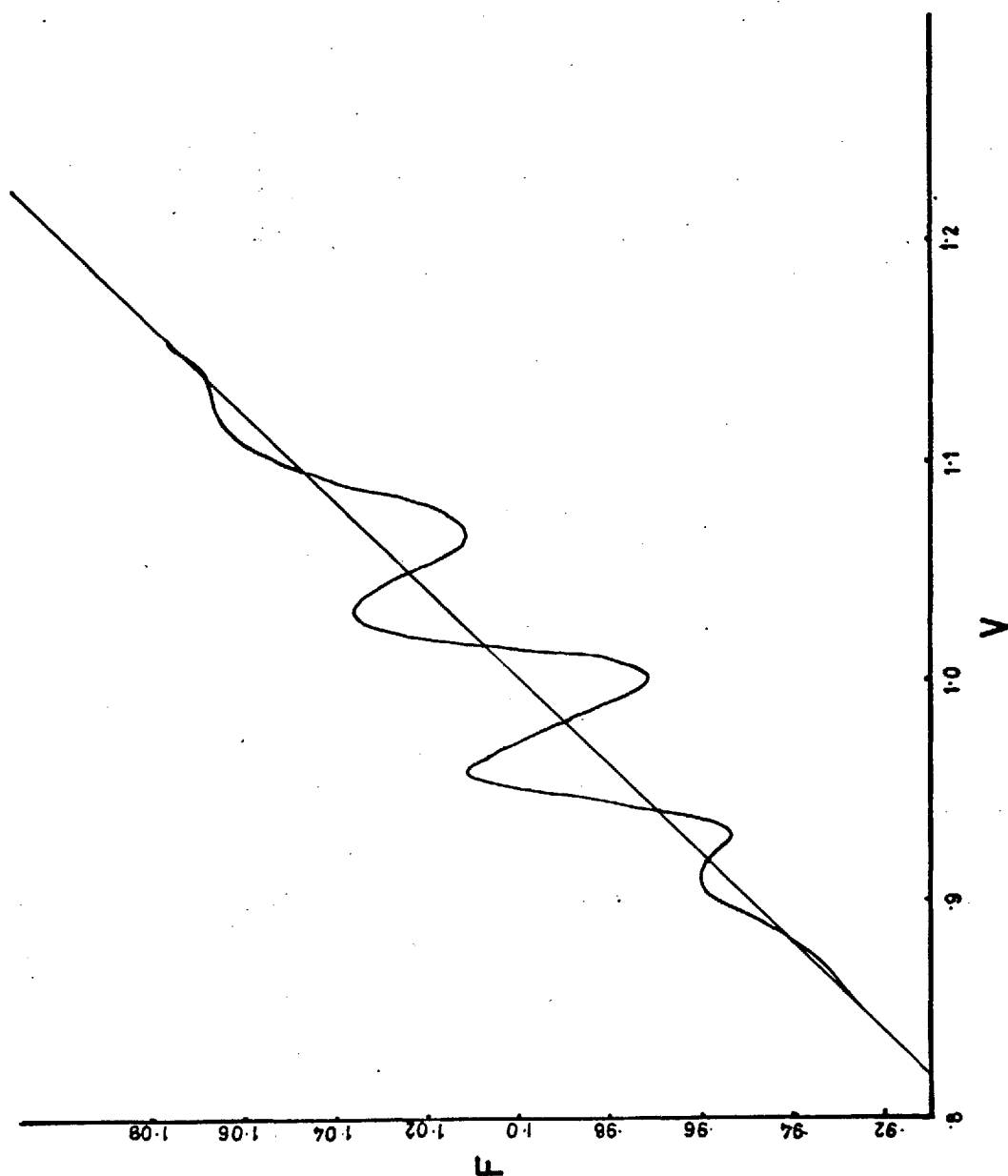


Fig. 1. Resonant particle distribution function $f_{\text{res}}^{(0)}$ at $x = -3.5N$, $t = 0$ for a Gaussian packet with $N = 25$, $\psi = 1$. Note the almost linear form.

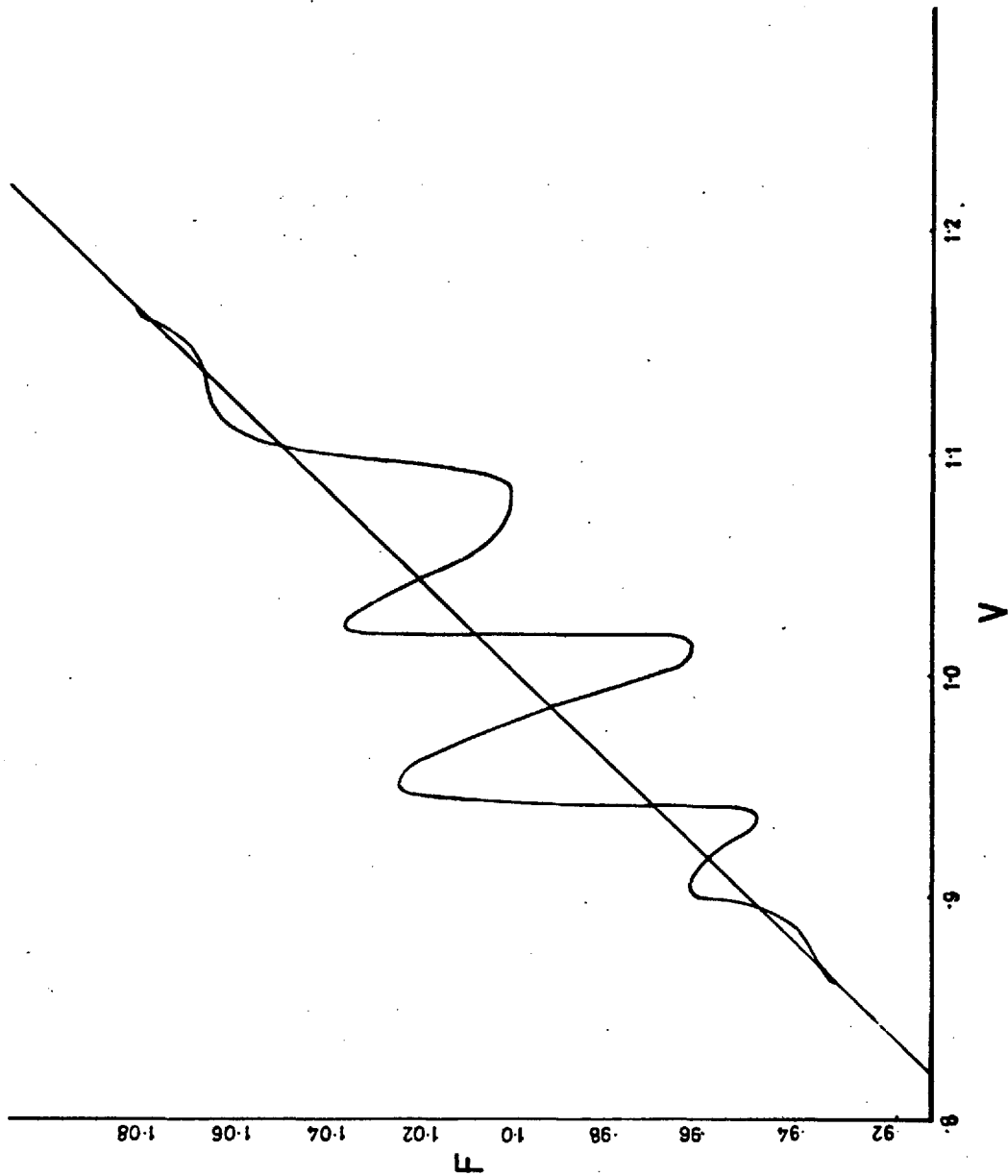
GRAPH 2.

Fig. 2. Resonant particle distribution function $f_{res}^{(0)}$ at $x = -3.5N$, $t = 0$ for a Gaussian packet with $N = 25$, $\psi = 1.25$.

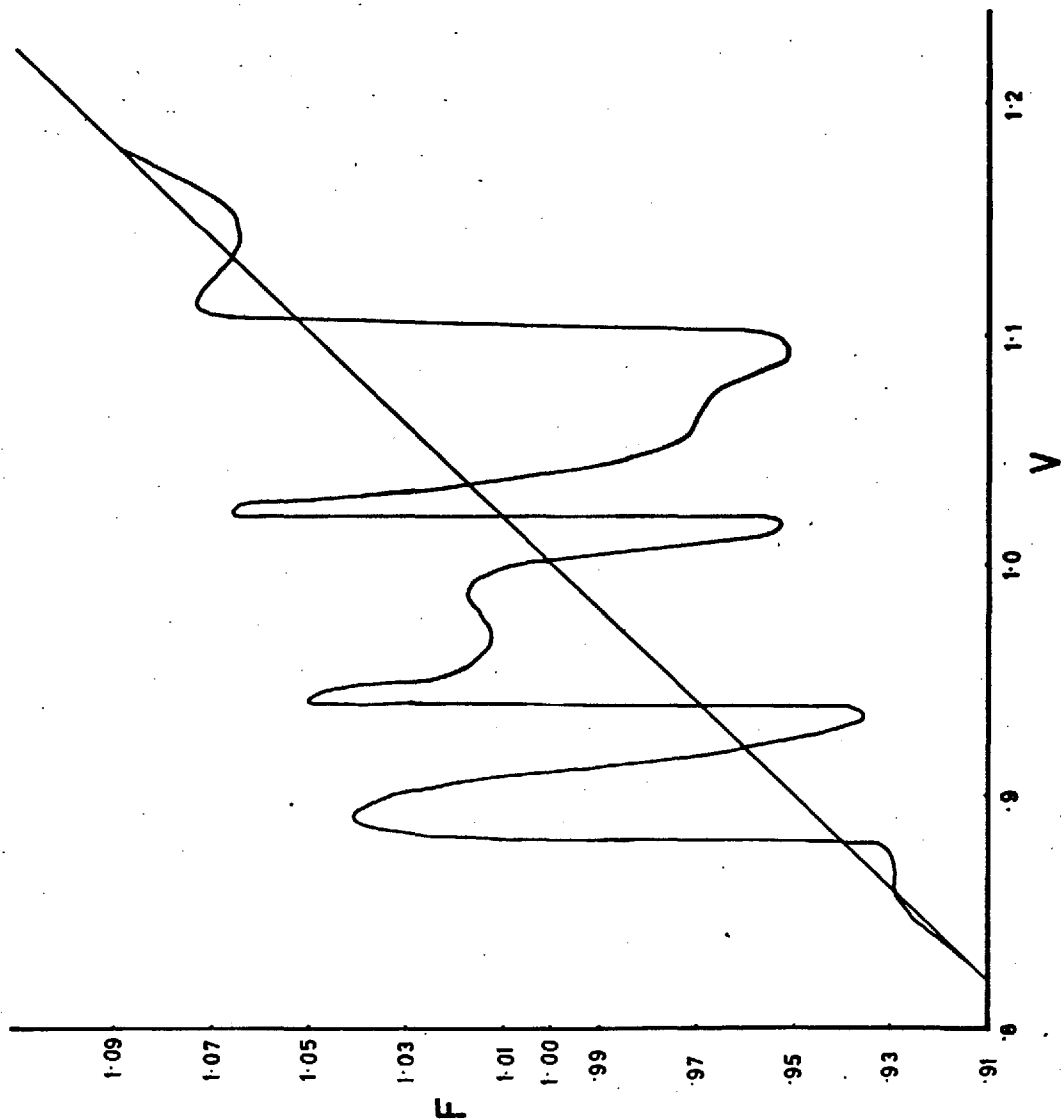
GRAPH 3.

Fig. 3. Resonant particle distribution function $f_{res}^{(0)}$ at $x = -3.5N$, $t = 0$ for a Gaussian packet with $N = 25$, $\psi = 2$. Note the "discontinuities".

GRAPH 4.

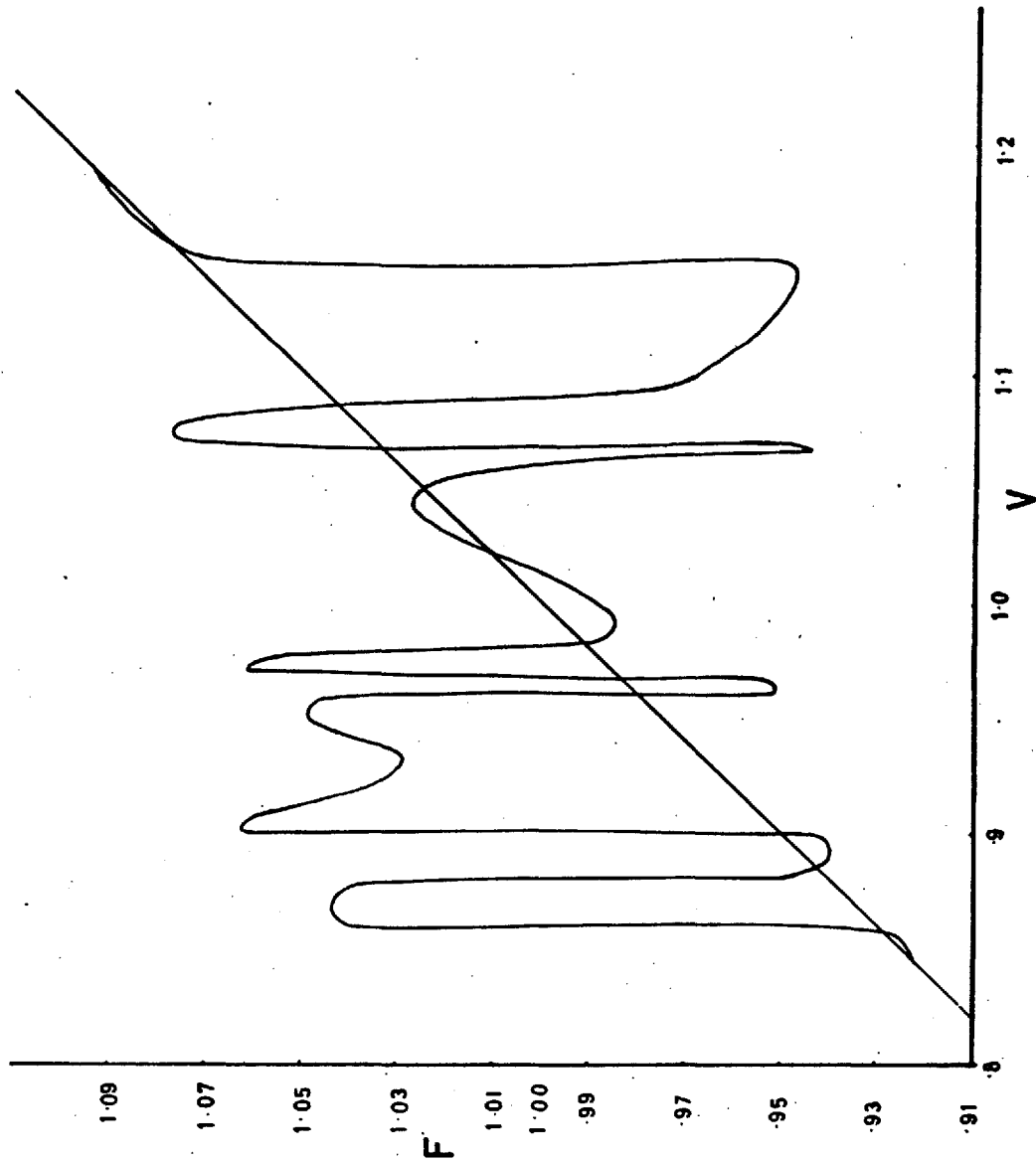


Fig. 4. Resonant particle distribution function $f_{res}^{(o)}$ at $x = -3.5N$, $t = 0$ for a Gaussian packet with $N = 25$, $\psi = 2.5$.

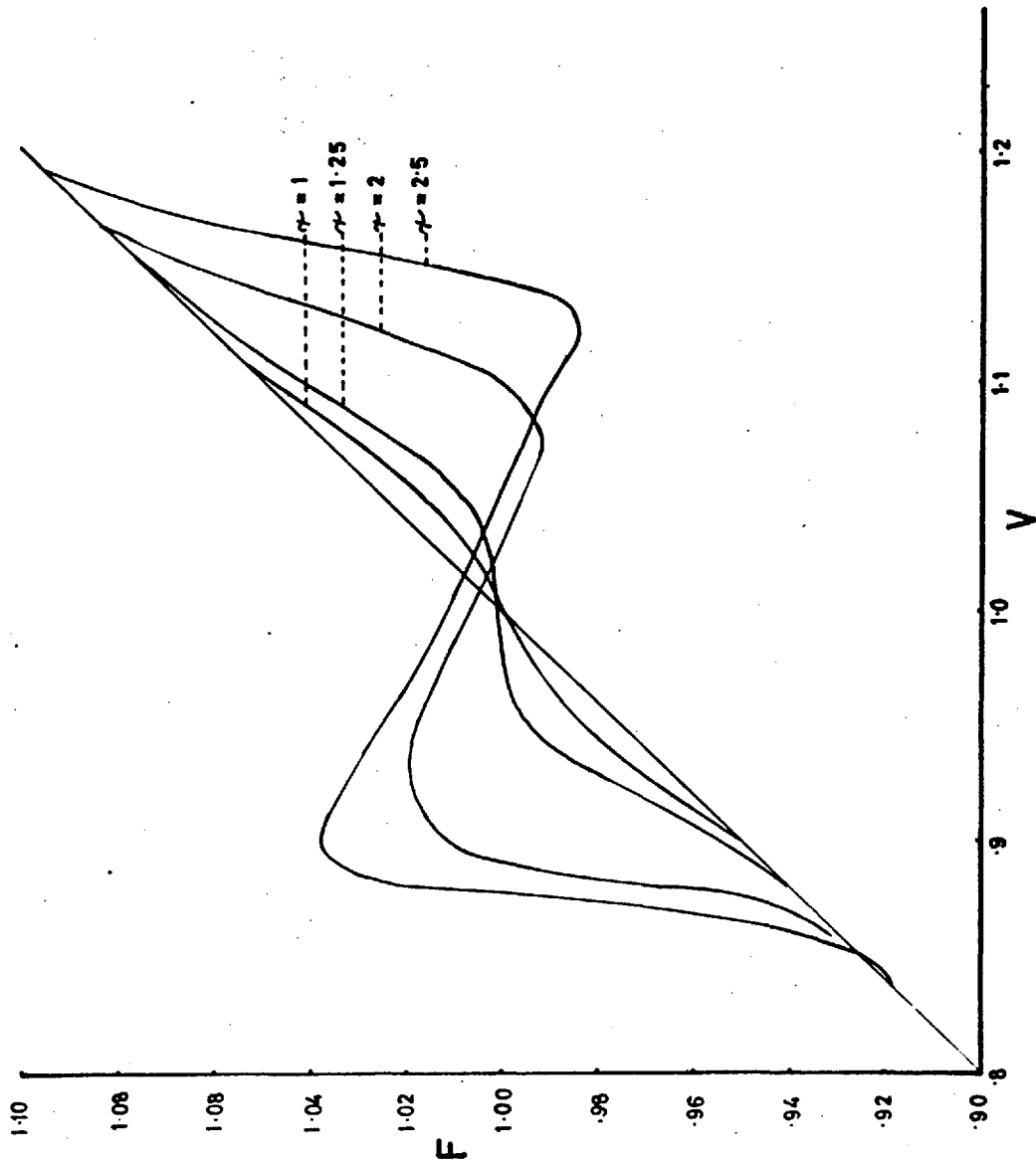
GRAPH 5.

Fig. 5. Time averaged resonant particle distribution function $F_0 + f_{res}^{(o)}$ behind a Gaussian wave packet as a function of ψ .

GRAPH 6a.

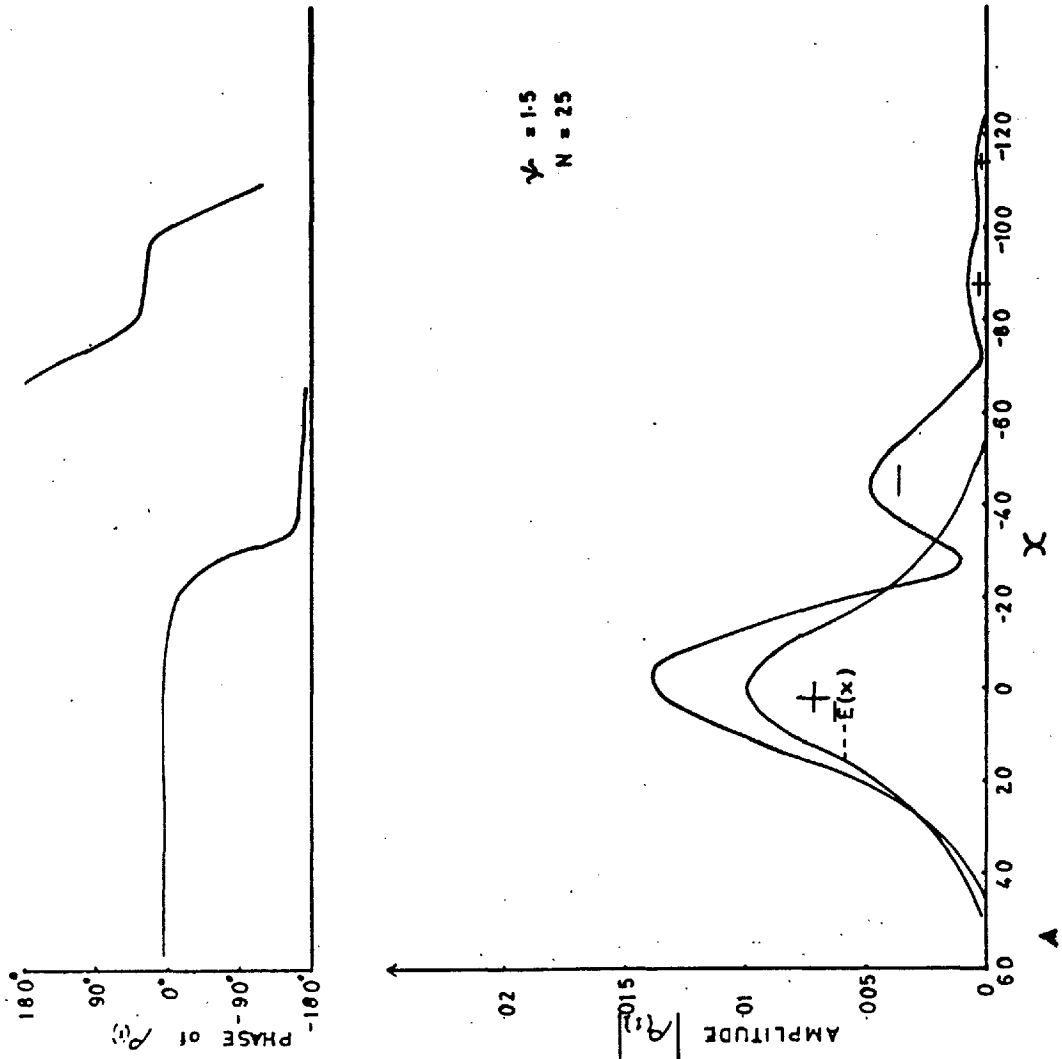


Fig. 6a. Amplitude and phase of the resonant particle charge density component $\rho_1^{(1)}$ through a Gaussian wave packet $\psi = 1.5$. Note the initial linear form, followed by successive phase reversals with decreasing amplitude.

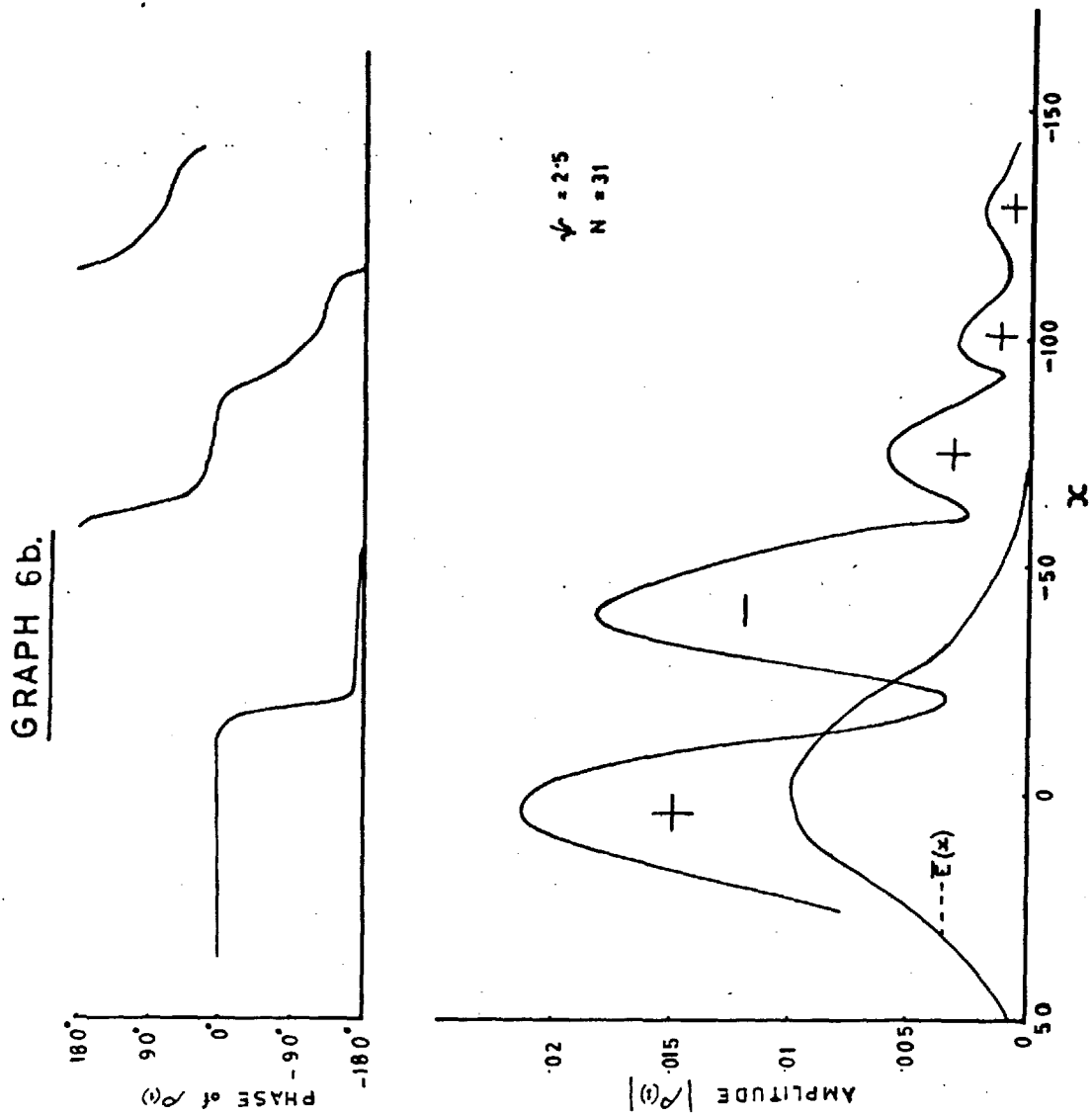


Fig. 6b. Amplitude and phase of the resonant particle charge density component $\rho_1^{(1)}$ through a Gaussian wave packet $\psi = 2.5$.

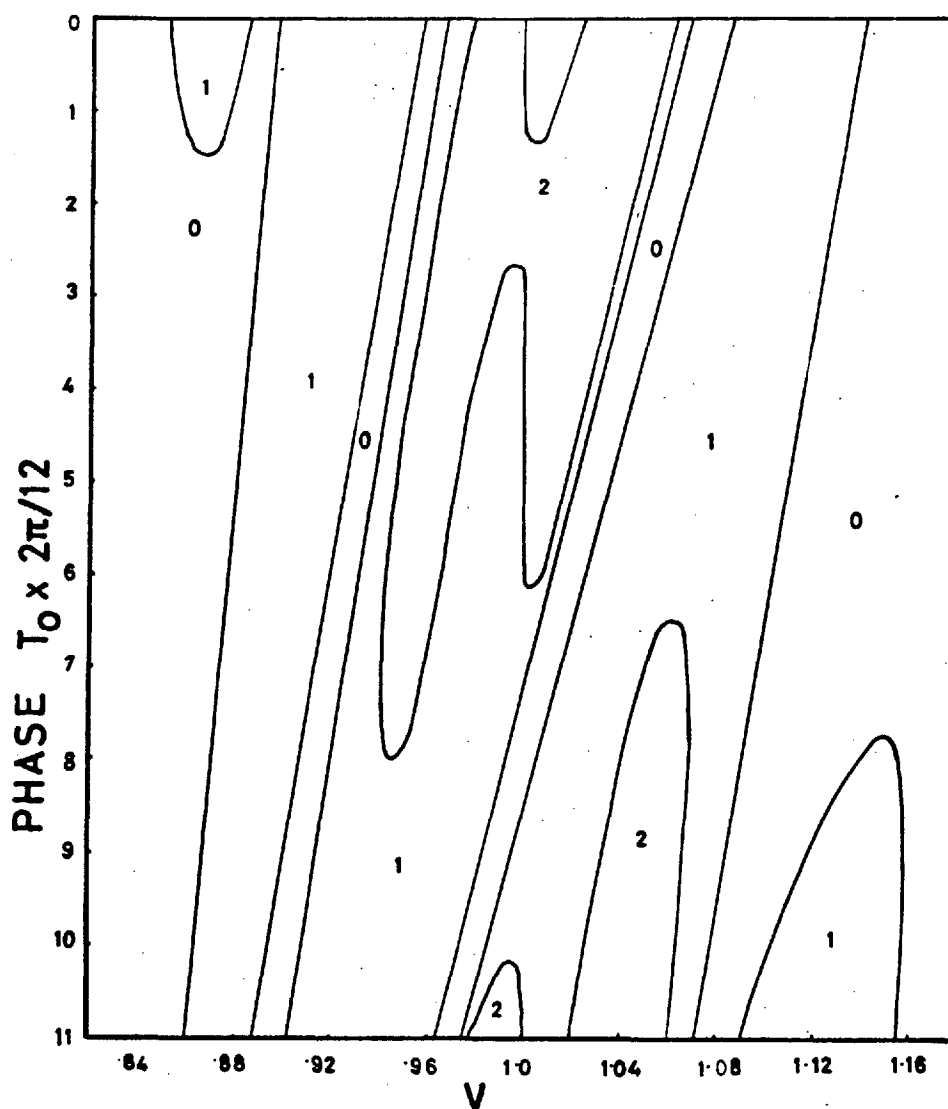
GRAPH 7

Fig. 7. Plot of the regions of different "bounce number" as a function of v and phase t , behind a Gaussian packet $\psi = 2$. Note the shearing of the pattern due to phase mixing.

GRAPH 8.

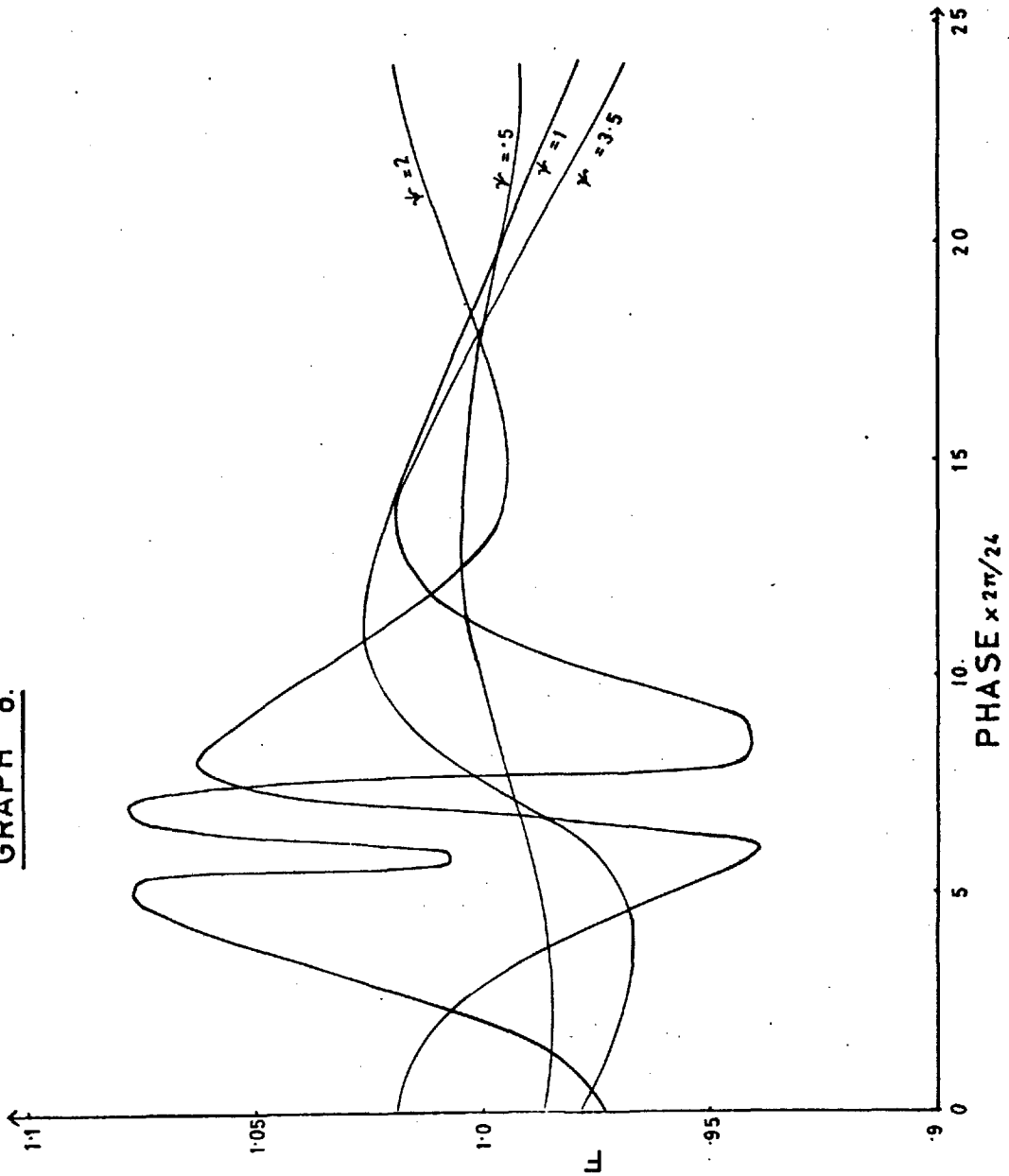


Fig. 8. Examples of the phase variation of $f_{\text{res}}^{(0)}$ for fixed v calculated behind Gaussian wave packets of varying ψ . Note the production of higher harmonics with increasing ψ .

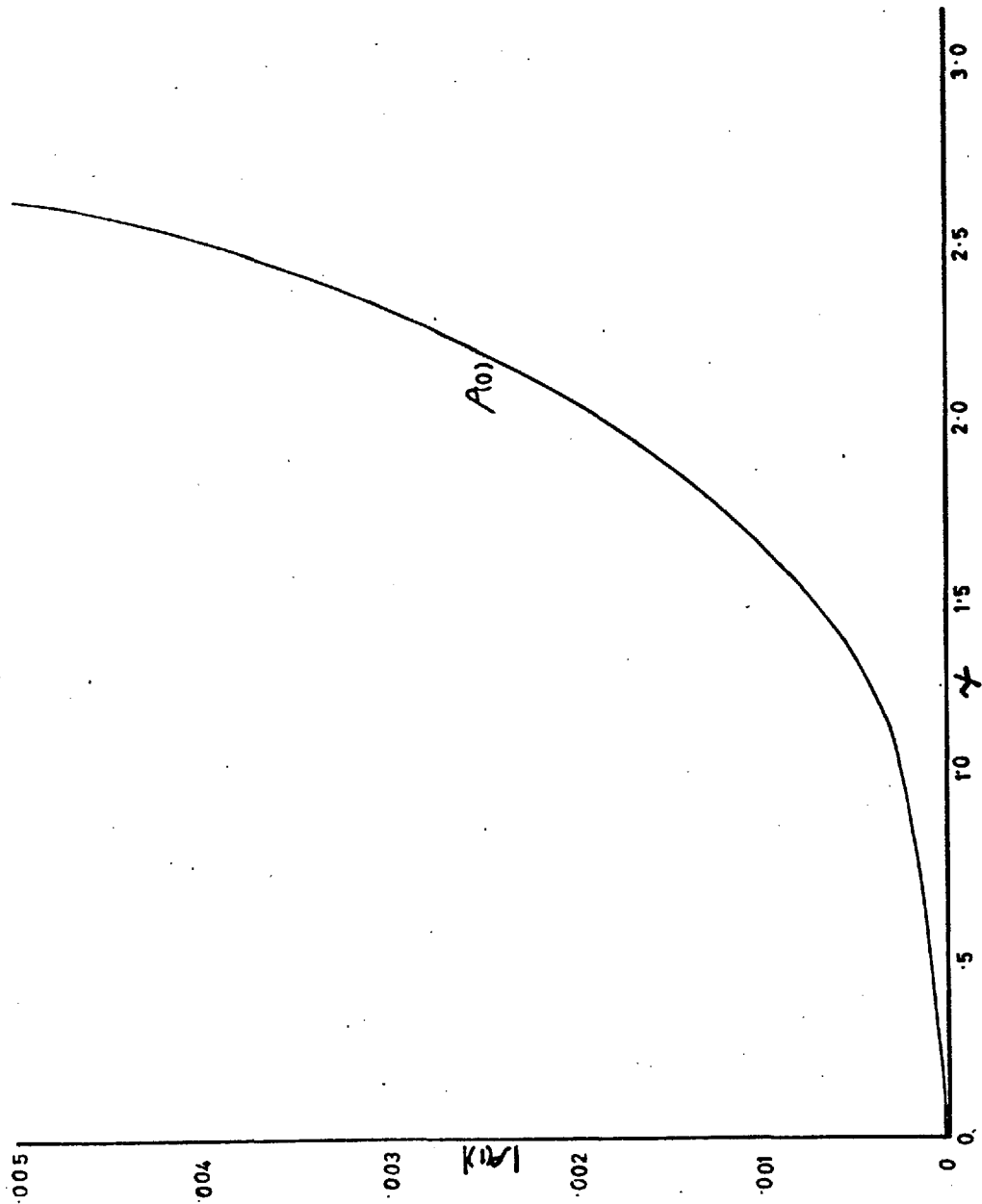
GRAPH 9.

Fig. 9. Dependence of $\rho_1^{(1)}$ on ψ behind a Gaussian wave packet.

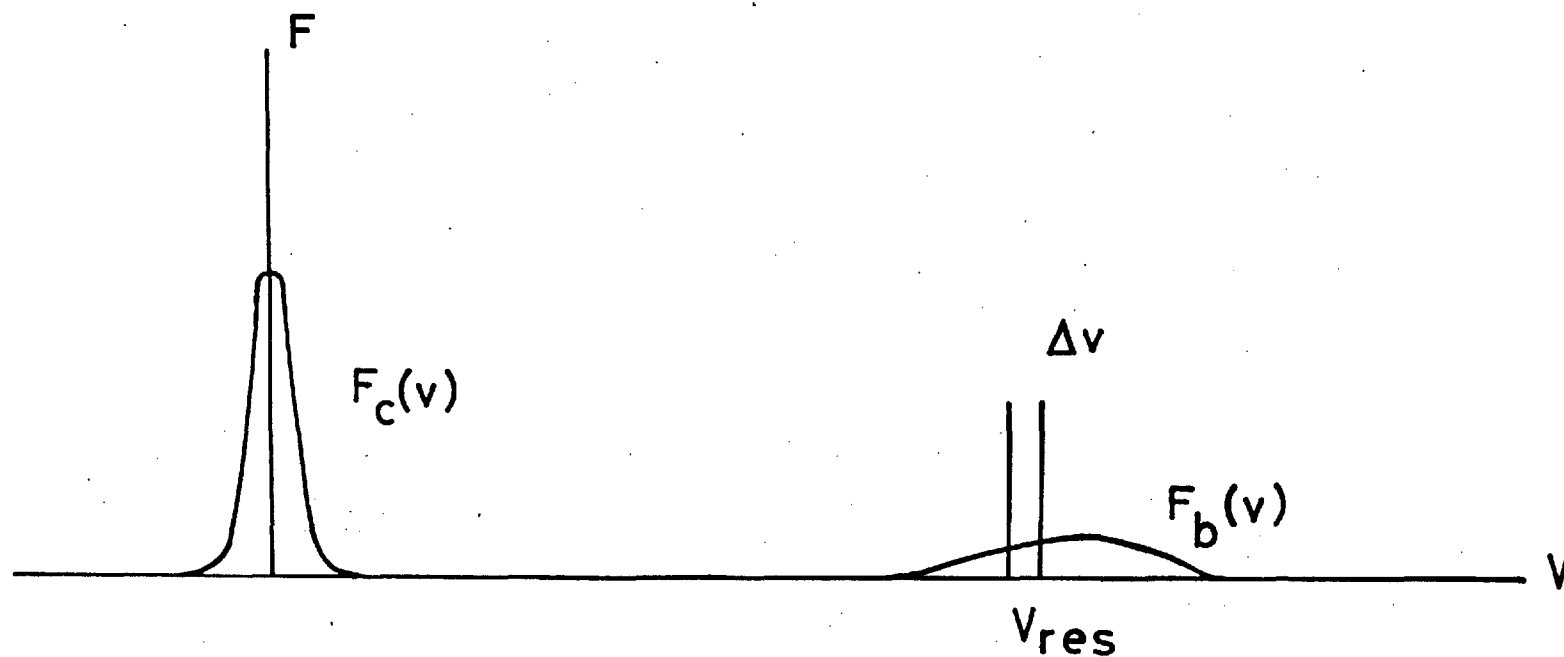


FIG. 10.

Fig. 10. Sketch of the unperturbed distribution function for cold plasma (F_c) and beam (F_b).

SECTION TWO

THE BEAM EXCITATION OF AN ELECTROSTATIC PULSE IN AN
INHOMOGENEOUS MEDIUM.

WAVE PARTICLE INTERACTIONS IN ELECTROSTATIC WAVES IN AN INHOMOGENEOUS MEDIUM

1. Introduction.

In this section we study a system very similar to that which we have just looked at, except that inhomogeneous effects are represented by having a spatially varying wave number. In this case, certain particles—termed second order resonant—closely satisfy the following condition (Swift)

$$\frac{dv_z}{dt} = \frac{dV_{res}}{dt} \quad V_z = V_{res} \quad (1)$$

These particles maintain a roughly constant phase with respect to the wave and are continuously accelerated by it. The pattern of power input from the particles to the wave is expected to be radically different from and more sustained than in the homogeneous case, and our task is to more or less compute it rigorously from the resonant particle distribution function.

The study of the above system should be of considerable help in understanding the causes of triggered emissions in the whistler mode. This also is a wave particle interaction problem in an inhomogeneous medium—though considerably more complex than the electrostatic case.

2. Formulation of the problem

We consider a one dimensional electrostatic wave packet whose electric field is taken to be initially of the following WKB form;

$$E_0 = E(x) \cos \left(\pi_e t + k_0 x - \frac{1}{2} k' x^2 \right) \quad (2)$$

The linear variation of wave number is one of the simplest ways of invoking second order resonance effects. The quantity $E(x)$ is the envelope of the pulse and is conveniently chosen to be square with Gaussian ends.

We may make varying assumptions about the ambient plasma. The above field is a perfectly valid solution to the equations appropriate to a cold homogeneous plasma, although how such a waveform would be set up in practice is difficult to see. A more physically realistic picture would be that of a one dimensional aerial transmitting a pulse of frequency π_e into a hot inhomogeneous plasma, but this involves problems with the self consistency of the ambient plasma, and from the point of view of a purely wave particle interaction study, the former option of a cold homogeneous background is more convenient.

At zero time a weak beam of resonant particles is introduced. These undergo a complex non linear interaction with the field, and we wish to investigate the resulting time development of the wave packet. The overall picture of the unperturbed distribution function of the ambient plasma (F_c) and beam (F_b) is shown in fig. 1.

3. Analysis

The problem as posed is only tractable if the resonant beam is weak—that is, the number density of resonant particles is very much less than that of the ambient

plasma. The corresponding smallness parameter is;

$$\eta \sim n_{res}/N_0 \ll 1$$

This point is discussed in more detail in the first section. To first order in η the distribution function about resonance F_{res} is given simply by

$$\left\{ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} - \frac{e}{m} E_0 \frac{\partial}{\partial v} \right\} F_{res} = 0 \quad (3)$$

$$\frac{d}{dt} F_{res} = 0$$

The quantity F_{res} may then be readily obtained by integrating resonant particle trajectories in the zero order field E_0 and applying Liouville's theorem (eq 3). Again to first order in η the wave packet field E satisfies the Vlasov and Poisson equations with an extra term inserted from the resonant particle charge density.

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) F_c' + \frac{e}{m} E \frac{\partial F_c}{\partial v} = 0 \quad (4)$$

$$\frac{\partial E}{\partial x} = -4\pi e \int F_c' dv - 4\pi e \int (F_{res} - F_b) dv \quad (5)$$

Here F_c' is the perturbation in F_c , and the non linear term has been neglected, being of second order in the wave amplitude. The two integrals are respectively over ambient plasma and beam velocities. Further discussion of the nature of solutions for E is postponed until we have discussed the dependence of resonant particle charge density upon E .

4. The numerical integration.

Since equation 3 is completely non linear, we require to determine resonant particle trajectories numerically.

It is convenient to work in dimensionless units as follows;

$$\begin{aligned} k^* &= k/k_0^2 & t' &= \pi_e t \\ x' &= k_0 x & R &= e|E_0|k_0/m\pi_e^2 \\ v' &= v k_0/\pi_e \end{aligned} \quad (6)$$

The equation of motion then becomes, dropping primes

$$\frac{d^2 x}{dt^2} = -R E(x) \cos\left(x + t - \frac{1}{2} k^* x^2\right) \quad (7)$$

The quantity $E(x)$ is the envelope function and is shown in fig. 2. To find F_{res} at a given position x and time t equally spaced velocity increments are taken (v_i), covering sufficiently thickly the velocity range which has interacted resonantly with the field up to that point. The trajectory of each particle thereby represented is integrated backwards in a stepwise fashion until clear of the packet at $x \gtrsim 250$. If the velocity at this point is v_f , then we may say, using eq. 3,

$$F_{res}(v_i) = F_b(v_f) \quad (8)$$

where for simplicity F_b is assumed to have a linear dependence upon v . In this way F_{res} and the resonant particle charge density $\int (F_{res} - F_b) dv$ are computed at regularly spaced positions along the packet, and for 12 values of phase. This is done using various values for the dimensionless numbers R and k^* .

The technique of the numerical integration is relatively straightforward and a standard one for such

problems, so the reader will be spared the computational details.

5. The dynamics of second order resonance

Before doing the actual computation we may determine a good deal about the behaviour of second order resonant particles by analytic means.

The particle equation of motion is

$$dv/dt = -R \cos(\chi + t - \frac{1}{2} k^* \chi^2) = -R \cos \phi \quad (9)$$

$$\text{whence; } V_{res} = -1/(1 - k^* \chi) = -1/\bar{k} \quad (10)$$

For exact instantaneous second order resonance we must have

$$d/dt V_{res} = \frac{dv}{dt} = -R \cos \phi_0 = k^*/\bar{k}^3 \quad (11)$$

$$\therefore \phi_0 = \cos^{-1}(k^*/R\bar{k}^3) \quad V = V_{res} \quad (12)$$

That is, a particle with $\phi = \phi_0$ will maintain a constant phase with respect to the wave. An important consideration is whether such a resonance is stable or not—in other words, does a particle with a slightly different ϕ oscillate about ϕ_0 or fall right out of resonance? This is easily investigated as follows. Define small displacements

ϕ' and V' about the second order resonance position:

$$V' = V - V_{res} \quad \phi' = \phi - \phi_0$$

$$\text{whence } d\phi'/dt = \bar{k} V' \quad (13)$$

$$\text{and } dv'/dt = R \sin \phi_0 \phi' \quad (14)$$

The equation governing V' is then

$$d^2 V'/dt^2 - \bar{k} R \sin \phi_0 V' = 0 \quad (15)$$

Clearly for $\sin \phi_0 < 0$ the solution is oscillatory in character, whereas for $\sin \phi_0 > 0$ the motion about the second order resonance point is unstable and particles fall out of resonance exponentially. We thus expect that particles with $v \sim v_{res}$ and $\phi \sim \phi_0$, where

$$\phi_0 = \cos^{-1}(-k^*/Rk^3) \quad \pi < \phi_0 < 2\pi$$

will be stably trapped second order resonant particles with a natural period of oscillation in the trap of $(\bar{k}R \sin \phi_0)^{-1/2}$, which is of the order of $R^{-1/2}$ but somewhat greater for $\phi_0 \sim \pi, 2\pi$. Furthermore we expect these trapped particles to stay in resonance with the wave for very long periods, as they are able to continuously adjust to changing resonance conditions.

These trapped particles are generally expected to occupy a well defined area in ϕ, v space, whose boundaries are readily estimated as follows. Define $x' = \phi/\bar{k}$

Whence $v' = dx'/dt$

and $\ddot{x}' = -R(\cos \phi - \cos \phi_0)$

Thus $\ddot{x}' = -R(\cos \bar{k}x' - \cos \phi_0)$

The behaviour of x' is now exactly as in a static potential problem. Particles escape from the trap when v' is such that they can just reach $\phi = -\phi_0$ (fig 3). At a phase ϕ_1 the boundaries of the trap will be at

$v = v_{res} \pm \Delta v$ where Δv is given by

$$\frac{1}{2} \Delta v^2 = R \left| \int_{\phi_1/\bar{k}}^{-\phi_0/\bar{k}} (\cos \bar{k}x' - \cos \phi_0) dx' \right|$$

$$\Delta V = \left\{ 2R/k \left(\cos \phi_0 (\phi_0 + \phi_1) - \sin \phi_0 - \sin \phi_1 \right) \right\}^{1/2}$$

The area is thus elliptically shaped, centred on V_{res} , ϕ_0 , and extending as far as $\phi = -\phi_0$.

The approximate differential in distribution function between trapped and untrapped particles will be proportional to τ , the time for which particles have been trapped. i.e.

$$\Delta F \sim -\partial F_b / \partial V R^{1/2} \cos \phi_0 (\tau / R^{-1/2})$$

Note that since $R^{1/2} \partial F_b / \partial V$ is approximately the variation in F appropriate to the homogeneous problem, the existence of second order resonance will not stand out in the distribution function unless the particles have been trapped for at least two trapping periods i.e.

$$\tau \gtrsim 2R^{-1/2}$$

Note that as $k^* \rightarrow 0$, $\phi_0 \rightarrow -\frac{\pi}{2}$ and the actual number of trapped particles becomes a maximum, but in this case $\cos \phi_0 \rightarrow 0$, $\Delta F \rightarrow 0$, and the system reverts to the homogeneous case. Conversely, when $k^* \sim Rk^3$ there are strictly no trapped particles, but for a finite length pulse particles will be found which succeed in remaining quasi trapped. That is $\phi \sim 0$ or π for the period of time they are in the wave packet.

We may now turn to the numerical results for a finite pulse and see to what extent these simple predictions are realised.

6. Numerical results for the resonant particle distribution function.

The form of the resonant particle distribution function F_{res} inside the wave packet was found to be largely determined by two parameters. The first of these is k^*/R (assuming $k=1$), which as we know controls the dynamics of second order resonance. The second is ψ , the number of trapping oscillations undergone by the particle since entering the packet.

$$\psi = \int R^{1/2} dx \quad (21)$$

It seems that ψ is the parameter that determines to what extent non linear and inhomogeneous effects appear in F_{res} . For $\psi \ll 1$ the form of F_{res} is that appropriate to the linear homogeneous case. This is born out by graph 4 which shows F_{res} at two opposing phases for $\psi \sim 1$. The curves exhibit the familiar linear form. After one or more trapping periods ($\psi > 2$) F_{res} becomes non linear and acquires a quasi discontinuous structure as in the homogeneous case. In addition second order resonance effects become discernible, trapped particles having a markedly different F from the surrounding untrapped particles. This is well illustrated in figs 5@6 which show F_{res} at the end of the packet ($z=-200$) with $R=0.0005$, $K^*=0.0002$, and for two values of phase, 0 and $-\pi$. Note the block of second order resonant particles centred about the local resonance velocity $V=0.96$ at the phase $-\pi$. Such particles are not of course seen at zero phase. The same thing is

observed in graph 7 which has $R=0.0005, K^*=-0.0005, \phi=0$. Here the block is centred around $V=1.09$ and as expected the width in velocity is much narrower. A common feature of these diagrams is the succession of spikes at regular intervals in v . These are small groups of particles that were initially trapped in the wave but became untrapped at various positions down the packet.

It is of interest to see how the computed distribution of trapped particles in v, ϕ space compares with the simple analytic predictions of the previous section. Graph 8 shows this for $R=0.0005, K^*=-0.0003$. The result is more or less as expected except for the tail of pseudo trapped particles extending into the unstable trapping zone. This is hardly surprising when one considers that the typical time for untrapping is of the order of the time spent in the wave packet. With larger γ values this tail would become steadily less significant of course, but computation in these cases would take far too long. Finally graph 9 shows the extent of trapping for $R=0.0005, K^*=-0.0005$. The shaded areas here are all pseudo trapped particles, since $\omega \phi_0 \gtrsim 1$. It can be shown however that such particles can maintain a constant phase with respect to the wave for times of the order of $(k^* + R)^{-1/2}$, which in this case is sufficiently long for them to look like stably trapped particles.

7. Results for the resonant particle charge density

The resonant particle charge density $\rho = \int (F_{res} - F_b)$ is important because it is the quantity responsible for the time variation of the wave packet field. Note that by definition

$$F_{w0}(t) = F_{res}(t + 2\pi)$$

we may express ρ in the following form

$$\rho = \mathcal{R} \sum_n \tilde{\rho}_n(x) e^{int} \quad (22)$$

When considering modification to the wave packet field and power transfer between resonant particles and the field only the $\tilde{\rho}_1$ component will be important, since this is the only one resonant with the ambient plasma.

If $\Delta\phi$ is the phase difference between $\tilde{\rho}_1$ and E, defined in the following way

$$\tilde{\rho}_1 = e^{i(x - 1/2 k^* x^2)} |\tilde{\rho}_1| e^{-i\Delta\phi} \quad (23)$$

then we shall shortly show that the rate of growth is of E is proportional to the in phase component, or $|\tilde{\rho}_1| \cos \Delta\phi$. Accordingly, in the graphical output we choose to plot $|\tilde{\rho}_1|$, $|\tilde{\rho}_2|$, $\Delta\phi$, $|\tilde{\rho}_1| \cos \Delta\phi$ as functions of x. We select $R=0.0005$ and consider values of K^* ranging from .0001 to .0008. The graphs of these quantities (figs. 10-17) will tend to speak for themselves, but we note the following important features:-

(1) The initial form of $\tilde{\rho}_1$ ($\gamma \sim 1$) resembles that found for the homogeneous problem—namely an initial peak, followed by successive phase reversals accompanied by a rapidly diminishing amplitude. Superimposed upon this pattern is a contribution from the trapped particles. This

part increases approximately linearly with ψ (and hence x), and for $\psi > 2$ is completely dominant.

(2) The second order resonant contribution to charge density gives a direction of power flow which is the same as in the corresponding linear Landau problem, and this regardless of the sign of K^* .

(3) The phase of the charge density $\tilde{\rho}_1$ settles down to a value close to ϕ_0 . i.e. $\Delta\phi \sim \phi_0$.

(4) The magnitude of the linearly increasing component of $\tilde{\rho}_1 \rightarrow 0$ as $K^* \rightarrow 0$, and reaches a maximum for $K^*/R \sim .75$

(5) For $K^* > R$ a different form prevails. The power input levels out to a constant value which is a decreasing function of K^*-R . The phase in this case slowly drifts into the unstable quadrant.

(6) As expected for small ψ $|\tilde{\rho}_2|$ is very small, but for $\psi > 2$ ρ becomes distinctly anharmonic, $\tilde{\rho}_2$ reaching a maximum when $K^* \sim R$. When K^* exceeds R , $|\tilde{\rho}_2|$ quickly drops to low levels.

(7). The time independent component of density $\tilde{\rho}_0$ was found to be consistently small, in fact usually below the computational noise level. The last two results are in curious contrast to the homogeneous problem where $\tilde{\rho}_0$ was appreciable and $\tilde{\rho}_2$ negligible.

8. Interpretation of the resonant particle charge density results.

The foregoing results can be readily understood in terms of our simple model of resonant particle trapping.

The wave in the inhomogeneous case is behaving essentially like an accelerating potential trap. At the front of the packet particles become trapped, and oscillate within the potential well, giving rise to a transitory oscillatory charge density, which dies away as the motions of the particles in the trap become dephased. For $\psi > 2$ the dominant component of density arises from the displacement in velocity of the trapped particles. This contribution may be closely estimated as follows. Let us assume for simplicity, that within the trapping region, as previously delineated, the distribution function is uniformly changed by an amount ΔF , where

$$\Delta F \sim (\tau/R^{-1/2}) R^{1/2} \cos \phi_0 \partial F_b / \partial v$$

whereas the distribution function of untrapped particles remains at F_b . Integration over v will then give the density as a function of phase. Inspection of the trapping diagrams (fig. 3) together with eq 20, immediately furnishes the qualitative features of ρ .

Clearly ρ will have a linear dependence upon $\partial F_b / \partial v$, the number of trapping periods $\tau/R^{-1/2}$, and upon amplitude R . It will also depend upon the size of the trapping region and the factor $\cos \phi_0$. These last two effects vary oppositely, and the maximum charge densities are obtained for $K^*/R \sim .7$.

Also since the trapping region is nearly symmetric about ϕ_0 , then $\tilde{\rho}_1$ is expected to have this same phase. Further the amplitude of $\tilde{\rho}_2$ will depend upon the spread

in phase of the trapping region. For $R \sim K^*$ ρ becomes very narrowly confined in phase and $|\tilde{\rho}_2|$ is consequently a maximum at this point.

Note that this simple representation of Fres will not predict $\tilde{\rho}_0$ correctly. This is because on the average there is a small change in the distribution function over the whole range of velocities which have resonated somewhere along the packet. This contribution appears to be such as to reduce $\tilde{\rho}_0$ to close to zero.

The case $K^* > R$ concerns only pseudo trapped particles, and is less easily predicted. The resonance time is now determined by the time particles are able to stay in resonance, and not by the length of packet upstream. Hence the levelling off of $|\tilde{\rho}_1|$ and its gradual decrease with increasing K^*-R . Also the phase dependence of $\tilde{\rho}_i$ in this case closely corresponds to the location of quasi trapped particles in ϕ, v space as observed in fig 9.

9. The time development of the electric field

We now wish to ascertain the manner in which the electric field develops in the presence of charge densities of the type we have found—in other words we solve eqs. 4 & 5.

We shall see that as a result of resonant particle interactions amplitude and wavelength will become slowly varying functions of both position and time. The problems of estimating the resonant particle charge density in such a more general field is rendered accessible by one important property of second order resonant charge densities. Provided

$\psi > 2$, on account of the stable nature of the trapping the phase difference between $\tilde{\rho}_1$ and E will always tend to adjust itself to ϕ_0 as defined by the purely local values of k, k^*, R . This of course will only be true provided ϕ_0 itself varies only slowly with respect to a trapping period. The amplitude of $\tilde{\rho}_1$ is less easily fixed. Clearly one should integrate over the trapping region as locally defined, but in general the quantity ΔF will be difficult to assess, depending upon the histories of all particles in the trap at that time. For example, systems where the size of the trap undergoes substantial variation will be particularly difficult.

The overall picture then is of a charge density always maintaining an approximately constant phase with respect to the electric field, and with an amplitude that is generally linearly x dependent.

In what follows considerable interest will centre on the reactive component of charge density $|\tilde{\rho}_1| \sin \phi_0$. This will continuously generate an electric field $\pi/2$ out of phase with the existing one. On account of its x dependence, this charge density will cause a combined field to have a shifted wave length. Subsequent resonant particles shift the wave length still further and so forth. Thus at first sight we might expect there to be a steady change of wavelength, and for a warm plasma, of frequency also. Thus in obtaining solutions to eqs 4 and 5 we are particularly concerned about rigorously justifying this rather contentious suggestion. In addition we would

like to know what the rate of change of k is, and under what conditions might we expect to observe it.

10. Analysis

In studying the interaction between charge density and the field we adopt a localised approach and assume that the ambient plasma is homogeneous (in other words k is independent of x)

For a cold plasma eqs 4 and 5 yield the differential equation governing the time development of E .

$$\frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial t^2} + \pi^2 \right) E = 4\pi \frac{\partial \rho}{\partial t^2} \quad (24)$$

To make the following analysis more general we insert a warm plasma term as below-so that frequency is a function of wavelength.

$$\frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial t^2} + \pi^2 - A \frac{\partial^2}{\partial x^2} \right) E = 4\pi \frac{\partial \rho}{\partial t^2} \quad (25)$$

This equation corresponds to the case of a hot ambient plasma only as a first approximation (Houghton)

We now express E as a product of an amplitude and a phase factor.

$$E \equiv E(x, t) \cos \phi(x, t)$$

We assume that only $\tilde{\rho}_i$ is important, and we work with the in phase (ρ_R) and reactive components (ρ_I) of charge density as follows

$$\rho = \rho_R \cos \phi + \rho_I \sin \phi \quad (27)$$

In accordance with our previous discussion ρ_R, ρ_I will be only slowly varying functions of x and t . We now substitute in eq 25 for E and ρ , and collect together terms in

$\cos \phi$ and $\sin \phi$. Equating each set of terms separately to zero we obtain:

$$E_x (\pi^2 - \omega^2 + 3Ak^2) - AE_{xxx} + E_{ttt} - 2\omega k E_t - 2Ek_t \omega = -4\pi\rho_R \omega^2 \quad (28)$$

$$Ek (\omega^2 - \pi^2 - Ak^2) - E_{ttt} k - 2E_{xt} \omega - 2E_{tk} t - E_x \omega_t - Ek_{tt} + 3AkE_{xx} = -4\pi\rho_I \omega^2 \quad (29)$$

Here the subscripts denote differentiations, and ω and k are defined as being: $\omega = \partial\phi/\partial t$ $k = \partial\phi/\partial x$

Terms in $\dot{\rho}_R$, $\dot{\rho}_I$ have been neglected in comparison to $\omega\rho_R$, $\omega\rho_I$. Also those involving k_x have been omitted. We now note that for a weak beam ($\eta \ll 1$) $\rho_R \rho_I k_t E_t$ are all of order η . Let us define a proper frequency

$\bar{\omega}_0$ from the dispersion relation

$$\omega_0^2 = \pi^2 + Ak^2$$

and put: $\omega = \omega_0 + \omega'$

In general ω' is of order η . Substituting these expressions and eliminating second order terms there remains

$$-2\omega_0 \omega' E_x + 2Ak^2 E_x - AE_{xxx} + E_{ttt} - 2\omega_0 k E_t - 2Ek_t \omega_0 = -4\pi\rho_R \omega_0^2 \quad (31)$$

$$2Ek\omega_0 \omega' - E_{ttt} k - 2E_{xt} \omega_0 - 2E_{tk} t - E_x \omega_t - Ek_{tt} + 3AkE_{xx} = -4\pi\rho_I \omega_0^2 \quad (32)$$

Solutions E , ϕ satisfying these equations will of course give an E field that in turn satisfies eq. 25 to first order in η . It has been found by experience with

these equations that eliminating apparently small terms is highly dangerous, and also that the initial time behaviour of a particular field configuration can be misleading indeed. One way of obtaining an exact long time solution is to assume that ρ_R, E, ω_0 and k are all independent of x , and that ρ_I varies only linearly with x . In this case equation 31 becomes

$$\frac{\partial E}{\partial t} = 2\pi \rho_R \omega_0 / k + E \partial k / \partial t / k \quad (33)$$

Equation 32 gives

$$\omega' = \frac{E_{tt}}{2E\omega_0} + \frac{E_t k_t}{Ek\omega_0} + \frac{Ek_{tt}}{2Ek\omega_0} - \frac{2\pi \rho_I \omega_0}{Ek} \quad (34)$$

Differentiating with respect to x then gives

$$\frac{\partial^2 \phi}{\partial x \partial t} = \frac{\partial k}{\partial t} = \frac{\partial \omega'}{\partial x} = -\frac{2\pi \omega_0}{Ek} \frac{\partial}{\partial x} \rho_I \quad (35)$$

also

$$\frac{\partial \omega_0}{\partial t} = \frac{\partial k}{\partial t} \frac{Ak}{\omega_0} = -\frac{2\pi A}{E} \frac{\partial \rho_I}{\partial x} \quad (36)$$

Thus in a system in which these various x dependencies may be ignored, we do in fact quite rigorously obtain a steady change of frequency and wave number proportional to the gradient of ρ_I . Noting that $\partial \rho_I / \partial x$ is of order $R^{1/2}$ we see that

$$(k_t/k) / (E_t/E) \approx R^{1/2}$$

Hence the local growth rate is to a good approximation proportional to ρ_I/E as expected.

Our equations are of little use for $E_x \neq 0$. Quite apart from the general complexity simple linear wave effects produce terms of zero order in η , as dealt with by Houghton. However, if we consider our simple picture in

which charge density generates an additional field which is continuously added on to the existing one, there seems every reason to believe that the wave number shift will still take place even though E is still x dependent. A little consideration will show, however, that the expression for $k t$ has to be modified as below;

$$\frac{\partial k}{\partial t} = -2\pi\omega_0/k \frac{\partial}{\partial x} (\rho_I/E)$$

Again, such variations in E must be slow compared to a trapping length $R^{1/2}$, in order that the amplitude and phase of the charges be predictable.

11. Development of the wave packet.

We now look at the overall time development of the wave packet itself.

The quantity k increases linearly with x so we expect to find the maximum growth rate at the downstream end of the wave packet. Note that very large growth rates can be obtained merely by trapping particles for very long periods. For instance, if the packet is 10 trapping lengths long, the growth rate at the end will be about 10 times the linear Landau value and so forth. At a given position we will have approximately

$$\frac{\partial E}{\partial t} \sim |\tilde{\rho}_1| \cos \phi_0 \sim k^*/R \cdot R^{3/2} \sim E^{1/2}$$

whence $E \approx \{At + \sqrt{E_0}\}^2$

Although the growth of the field is extremely rapid, for large times it is less than exponential. Eventually of course growth will stop, either because the resonant width exceeds the beam width, or because the wave number change

moves the resonant velocity outside the growth region.

Consulting the graphs we see that ρ_I increases linearly downstream. For $k^* > 0$ we expect a steady lengthening of wavelength and a shortening for $k^* < 0$.

Note that

$$\partial k / \partial t \sim \sin \phi_0 \partial \rho_I / \partial x / E \propto E^{1/2}$$

and the rate of change of wave number increases with E as at large amplitudes ρ_I becomes entirely reactive. An interesting case is when E is sufficiently weak so that $k^* > R$. In this case there is very little change in wave number, and the growth rate is roughly constant with distance, and only of the order of the linear value. For $k^* \gg R$ power input to the field is very small.

12. Conclusion.

Our most significant result has been that for an electrostatic pulse with spatially varying wave number second order resonant particles become stably trapped in the wave, giving rise to a steady change in wavelength and causing a power input increasing linearly with trapping time. This sustained growth rate increasing downstream is in sharp contrast to the results for the corresponding homogeneous problem, where the power input dies away after a few trapping periods.

What we seem to have then is a special kind of resonant particle instability appropriate to narrow band signals in a collision free inhomogeneous plasma. Such

instability is not predicted by linear or homogeneous analyses. Further it seems reasonable to suppose that analagous problems involving other wave types-particularly whistler and Alfven waves-should exhibit similar behaviour. It should be pointed out that there is a large variety of ways on invoking second order resonance effects. (Schram). It is only necessary to have a time dependence on at least one of the quantites determining the resonance velocity. This may be achieved by a space or time variation of frequency, magnetic field, background plasma density, or even by relativistic effects.

Having found such promising results for this simple electrostatic problem, we now turn our attention to the analagous whistler problem of triggered VLF emissions.

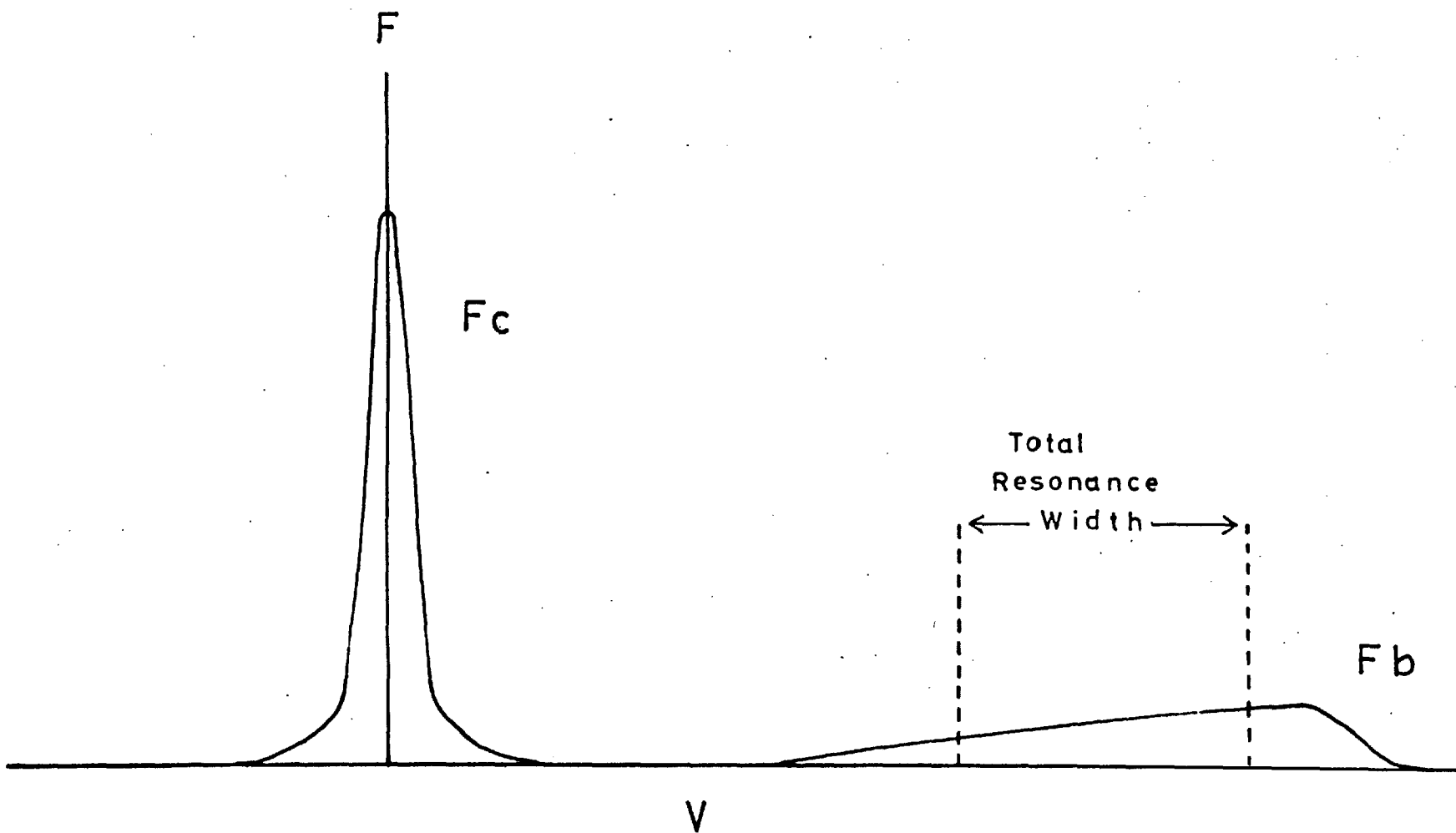


FIG. 1.

Fig. 1. The unperturbed distribution function of the ambient plasma (F_c) and beam (F_b). Note the linear variation of F_b over the range of resonant velocities present.

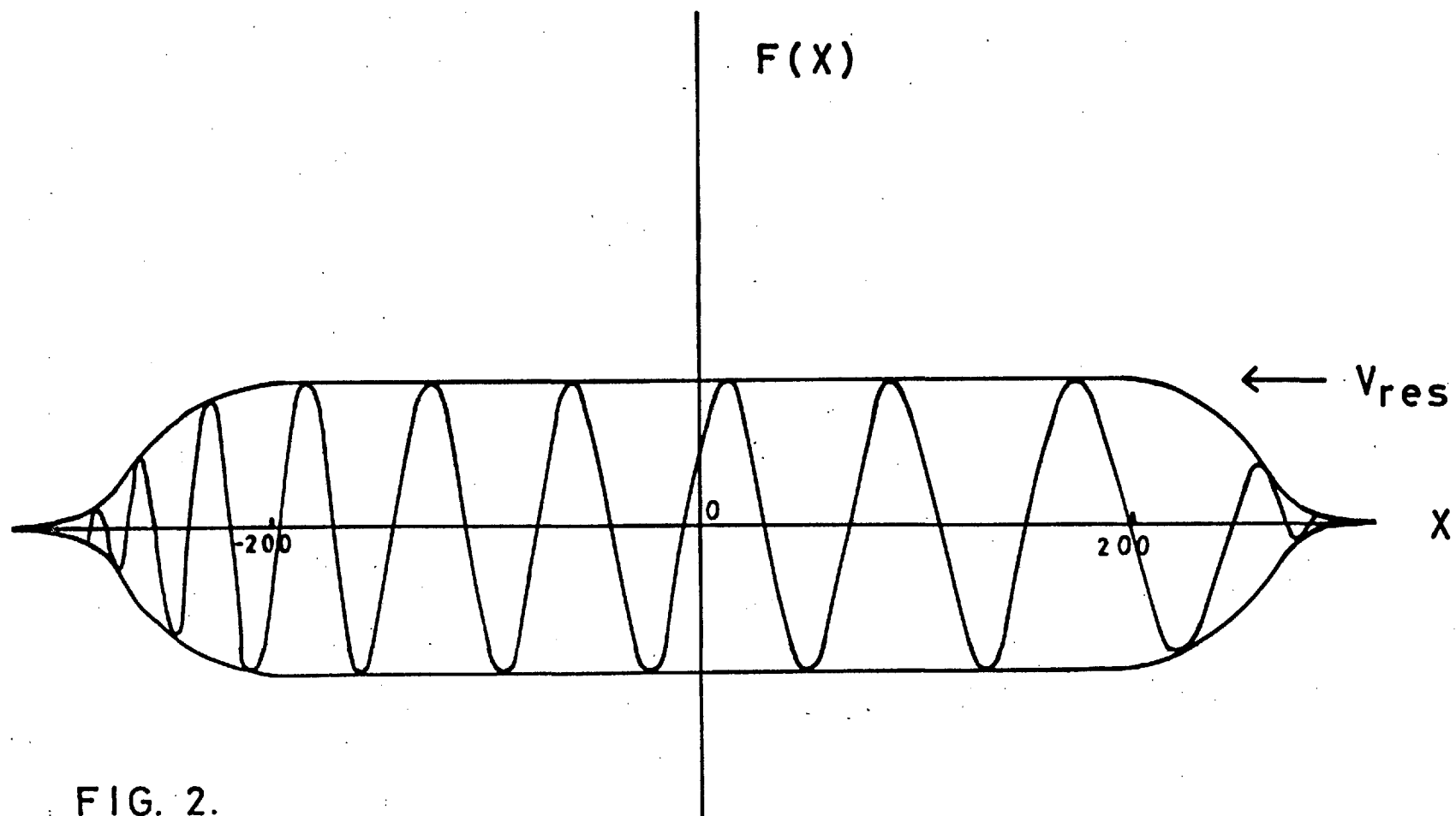


FIG. 2.

Fig. 2. Sketch of the wave packet envelope $E(x)$.

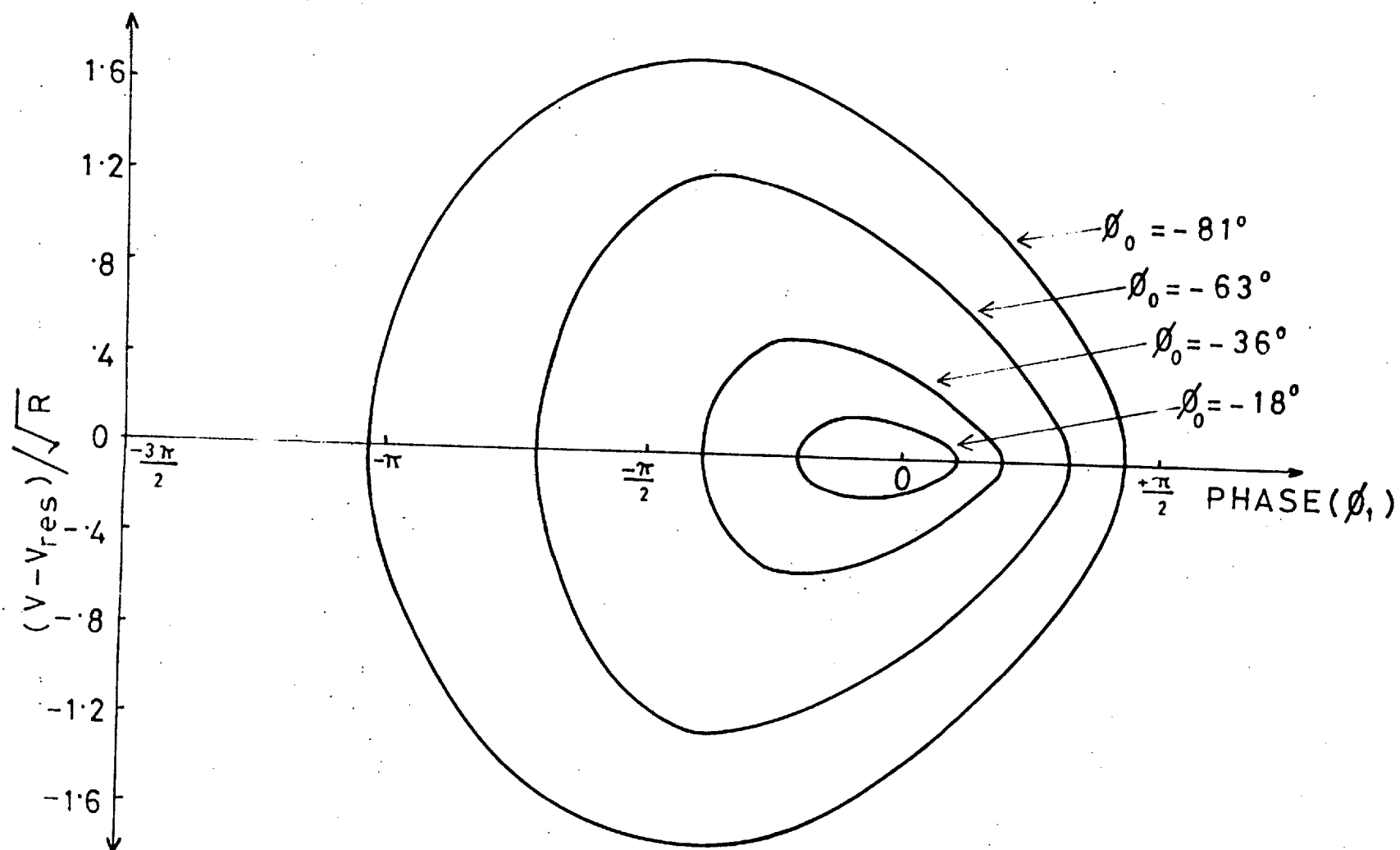


FIG. 3.

Fig. 3. The theoretical trapping regions $\Delta v(\phi)$ in ϕ, V space for various ϕ_0 .

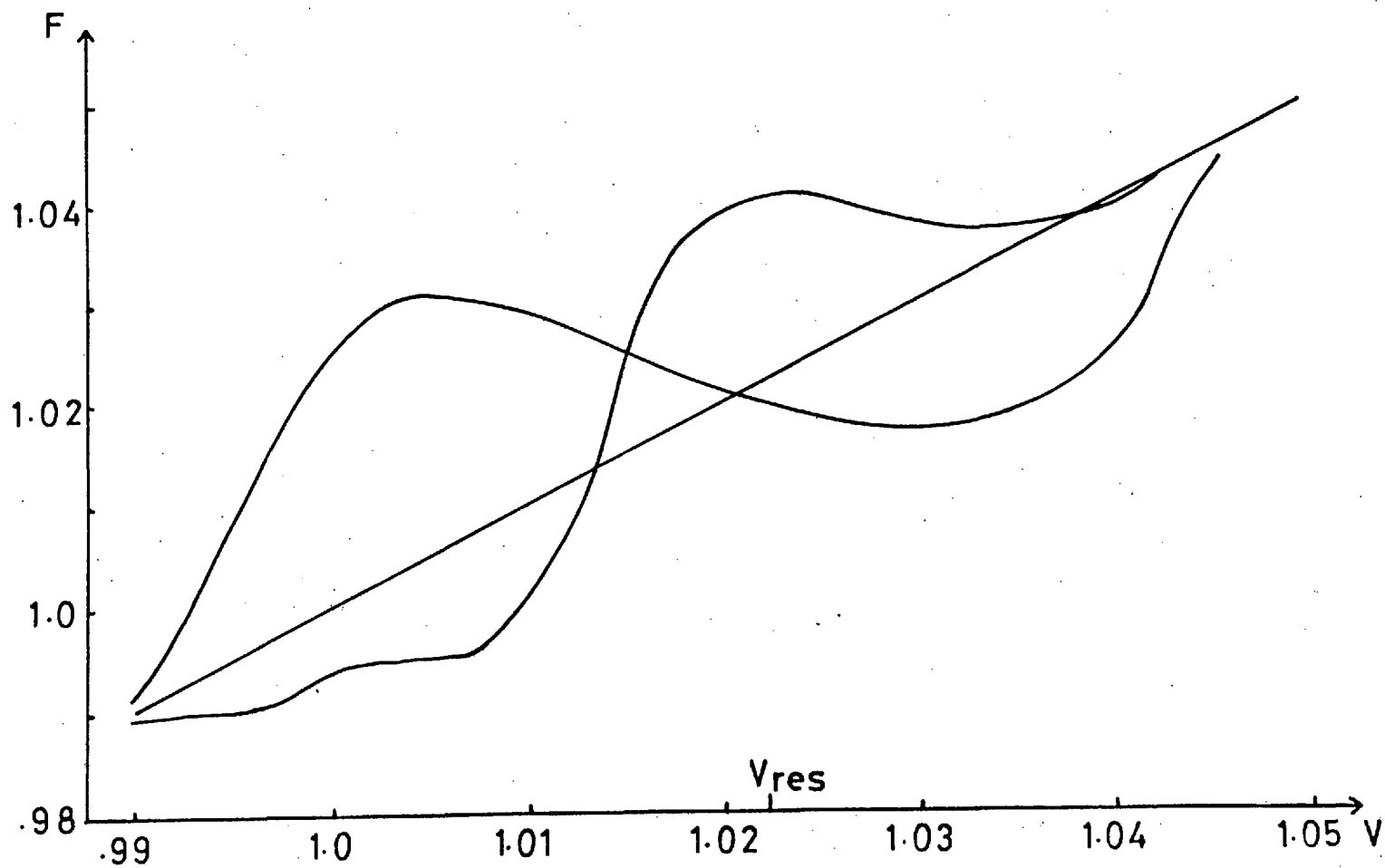


FIG. 4.

Fig 4. Graph of F_{res} for $\psi \sim 1$ at two phases $\phi = 0, \pi$.

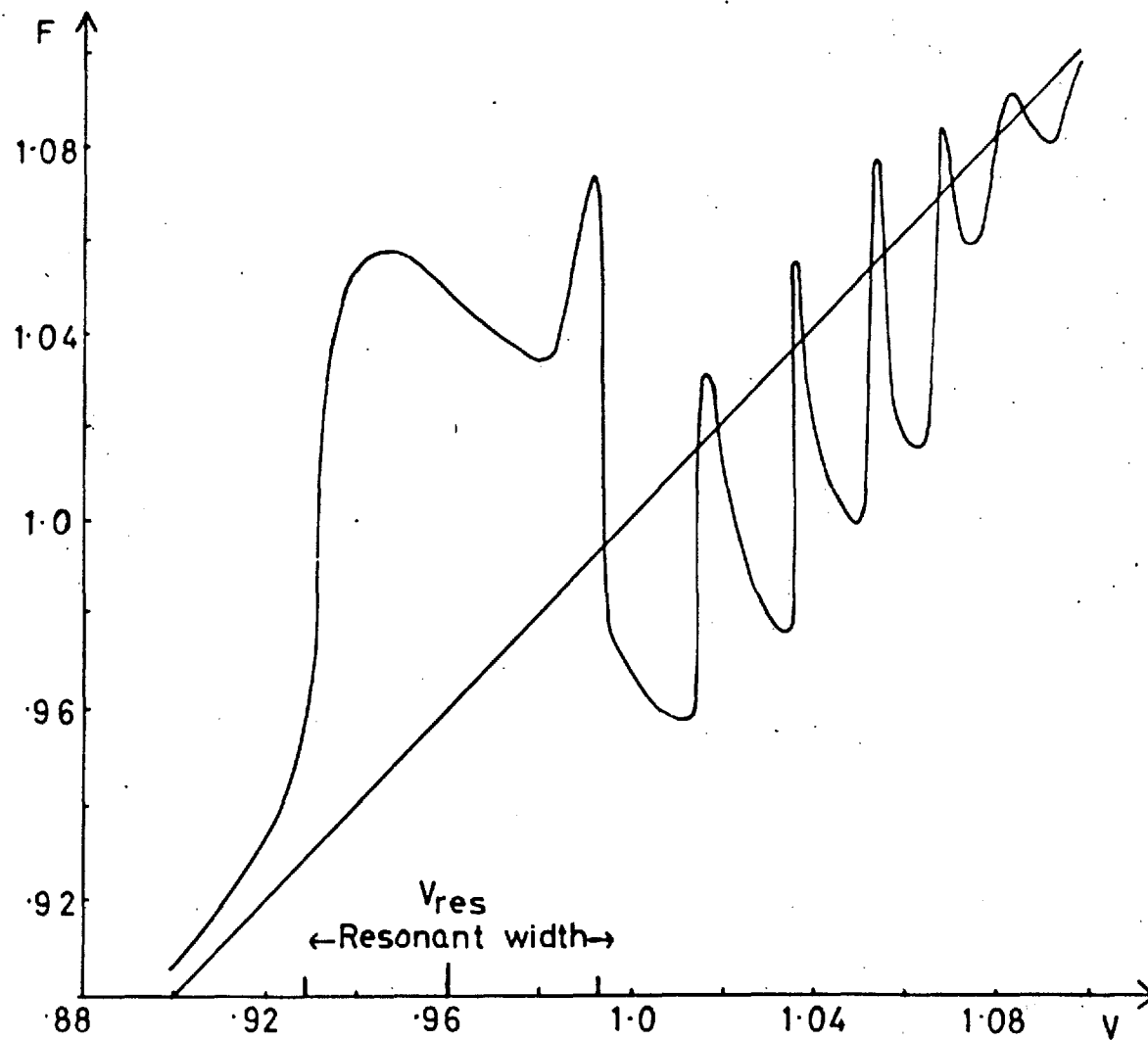


FIG 5

Fig. 5. Graph of F_{res} at $X = -200$ for $R = 0.0005, k^* = 0.0002$
 $\phi = -\pi$.

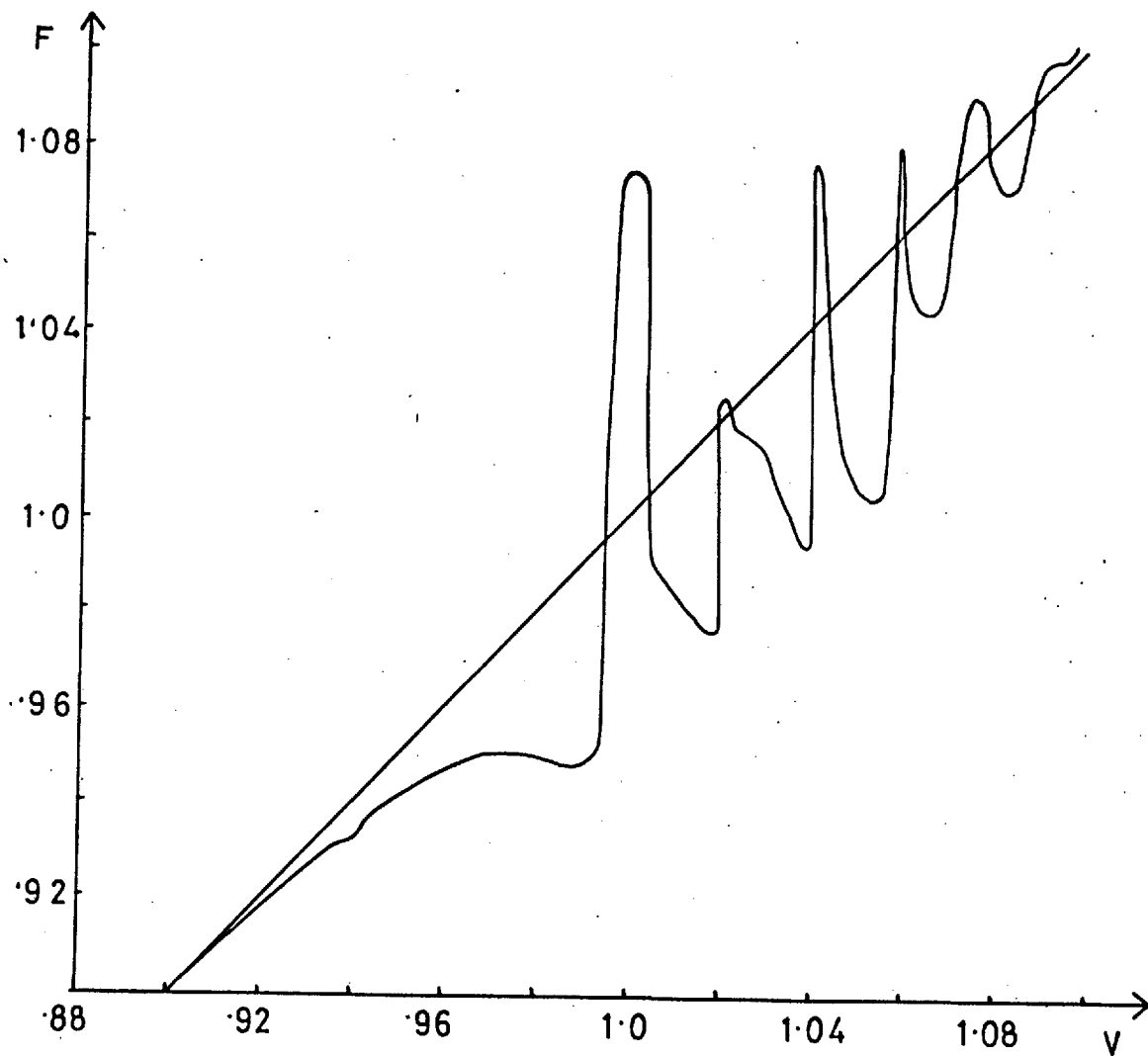


FIG. 6.

Fig 6. Graph of F_{res} at $X = -200$ for $R = .0005$, $k^* = .0002$
 $\phi = -\pi$.

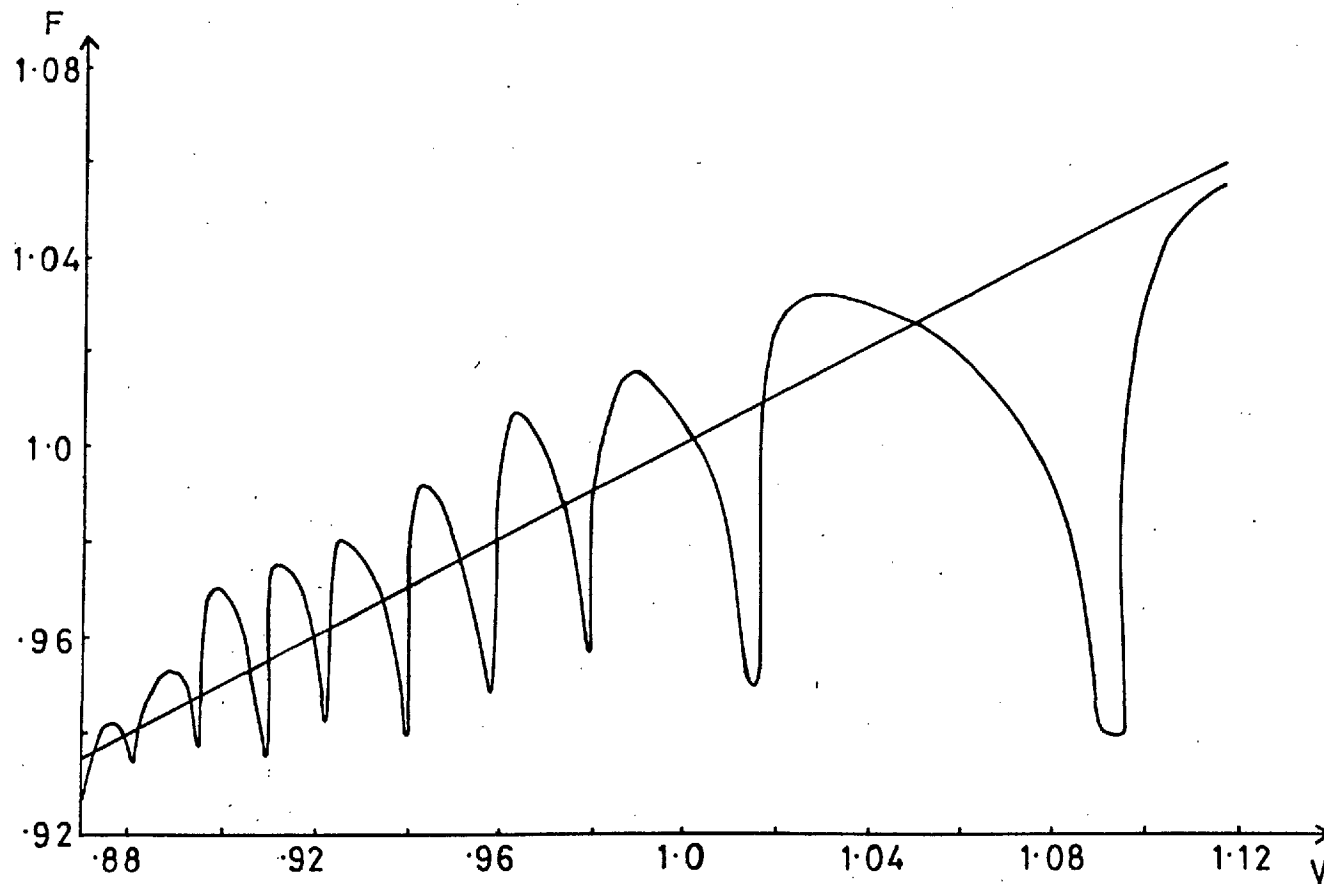


FIG. 7.

Fig 7. Graph of F_{res} with $R = .0005$ $k^* = -.0005$
 $\phi = 0$.

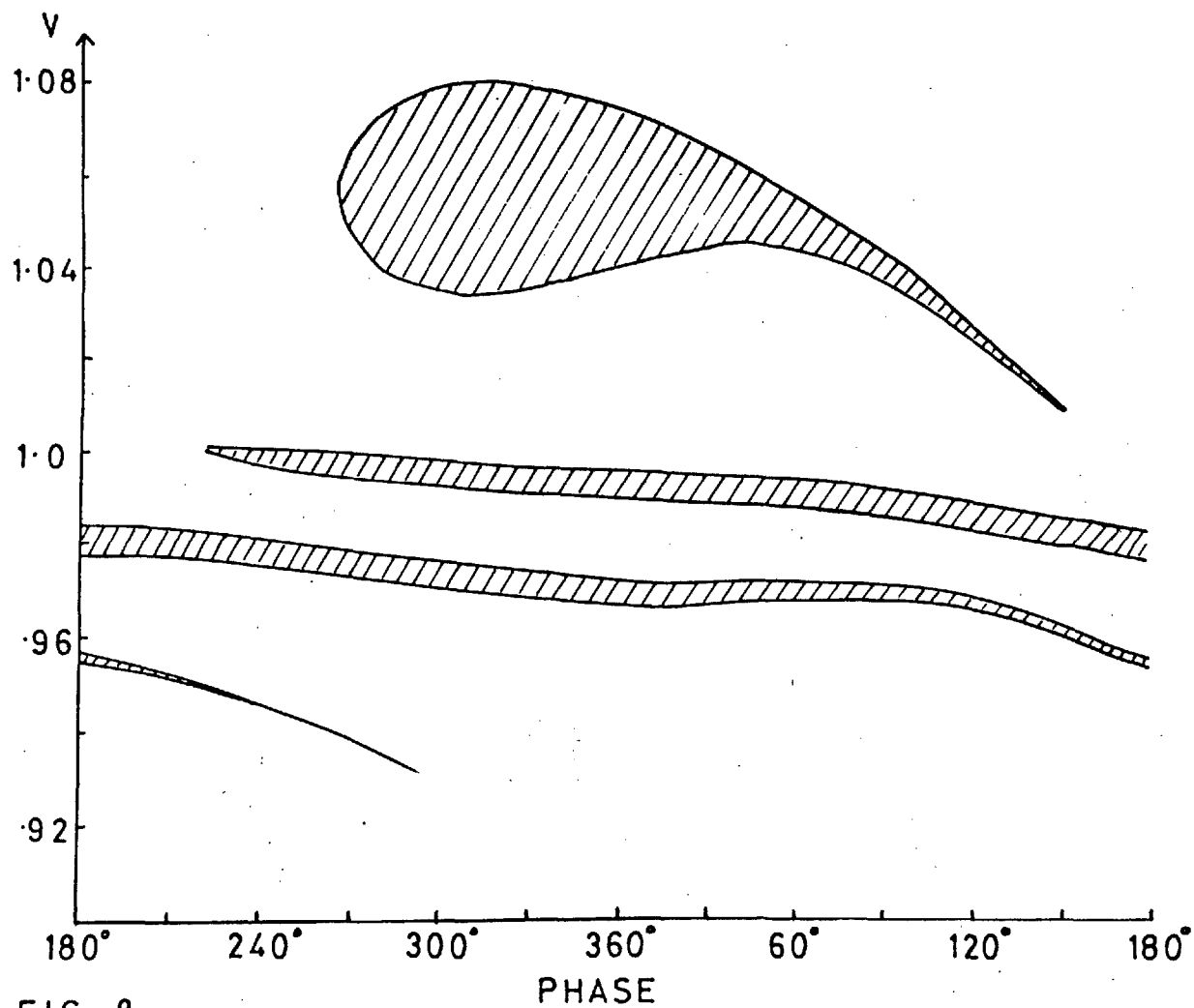


FIG. 8.

Fig 8. Plot of the actual trapping region(shaded) for $R = .0005$ $k^* = -.0003$. Note the resemblance to the predicted form.

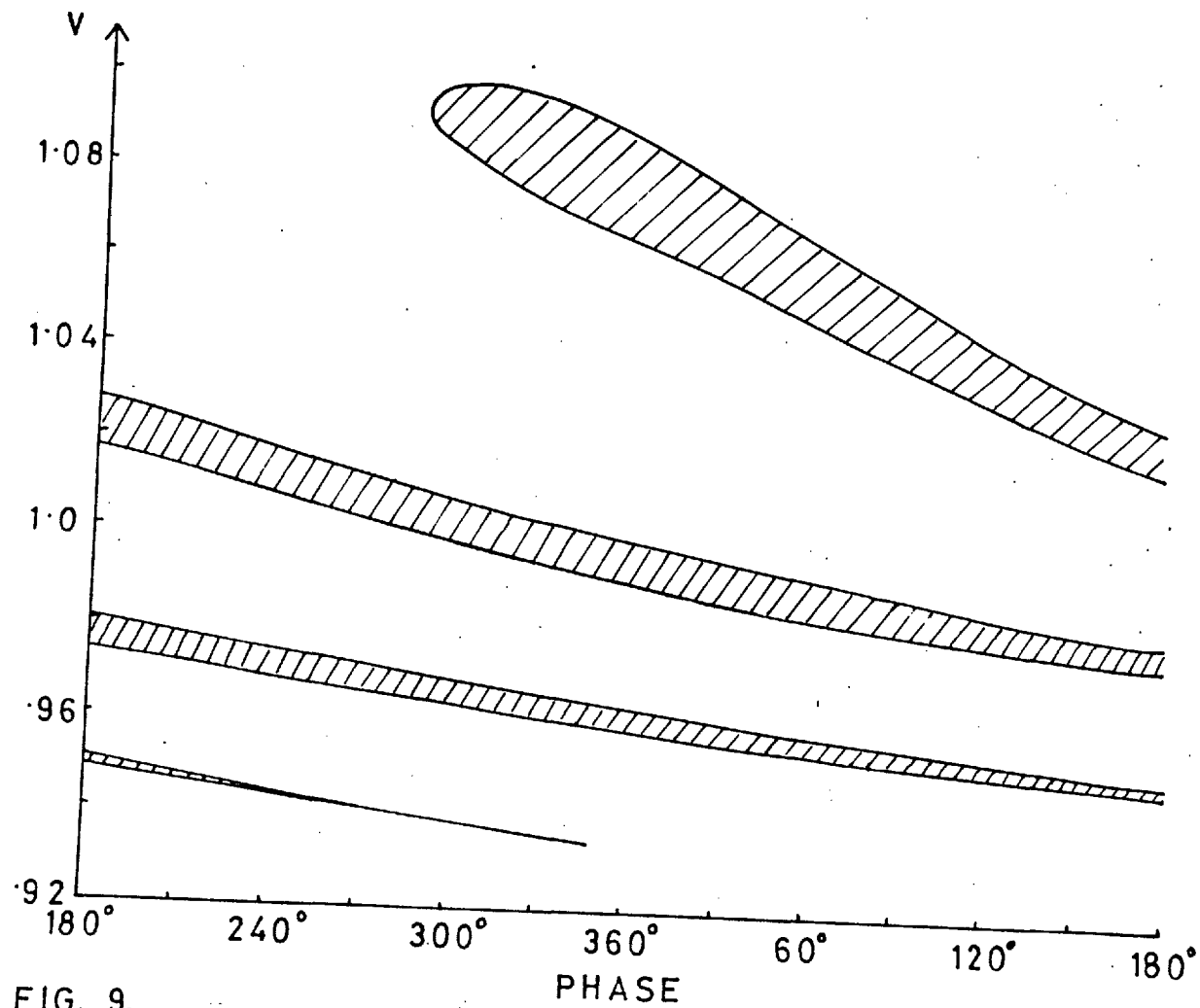


FIG. 9.

Fig 9. Graph of the location of pseudo-trapped particles
for $R = -0.0005$ $k^* = -0.0005$.

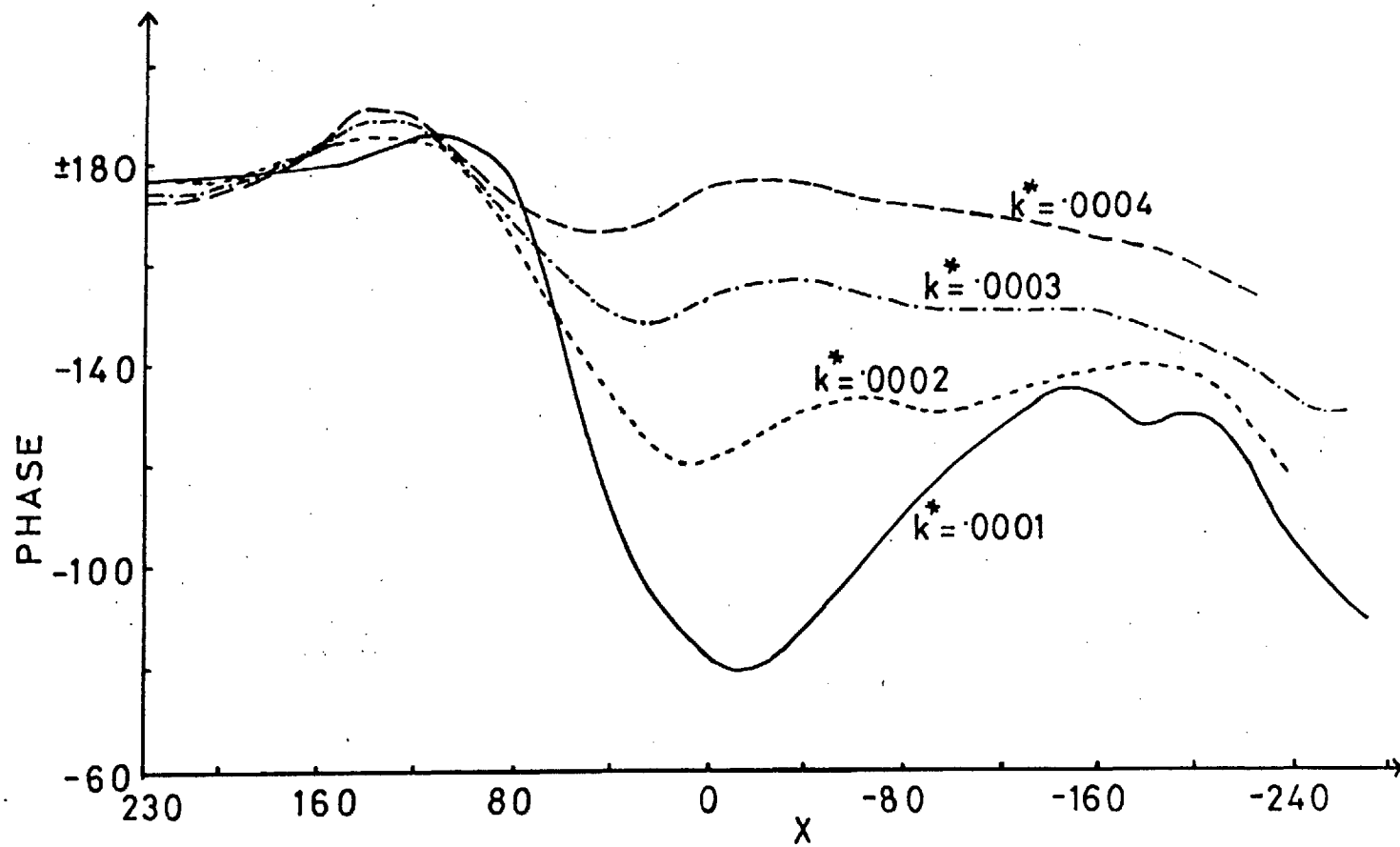


FIG. 10.

Fig 10. The phase of $\tilde{\rho}_i$ for $k^* = .0001, 2, 3, 4$.

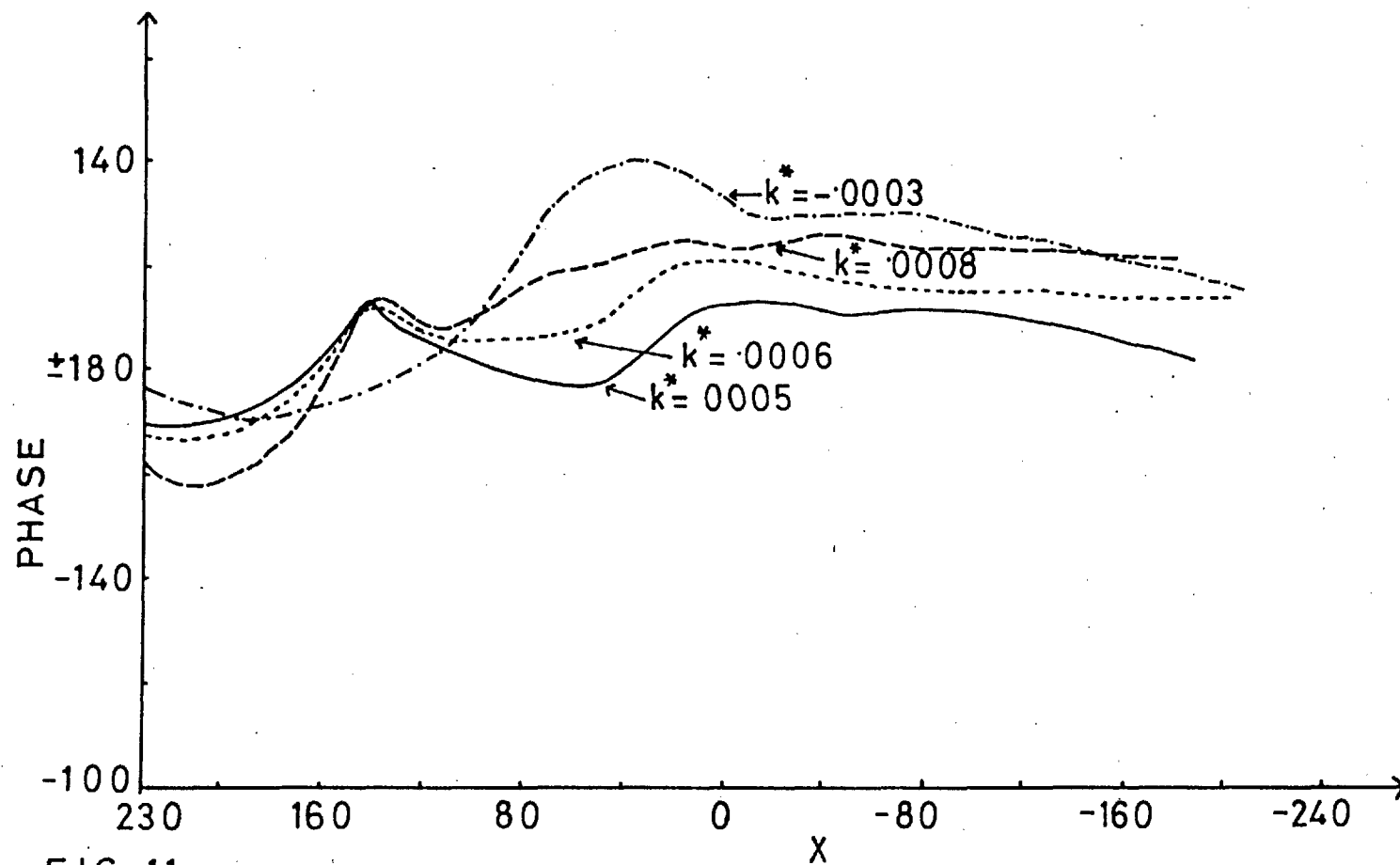


FIG. 11.

Fig 11. The phase of $\tilde{\rho}_1$ for $k^* = -0.0005, 6, 8, -3$.

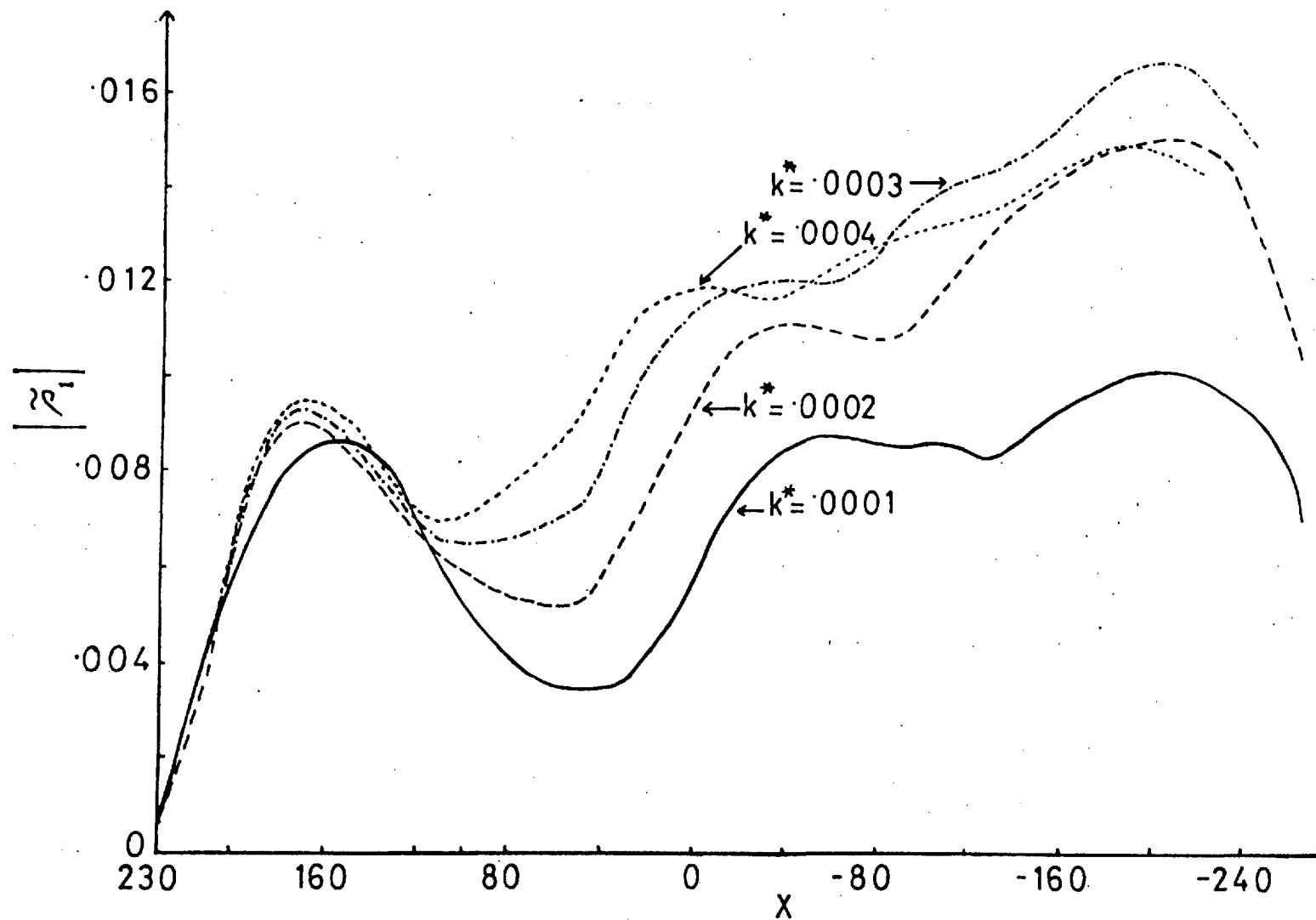


FIG 12.

Fig 12. Graph of $|\tilde{p}_1|$ as a function of x for $k^* = .0001 - 4$

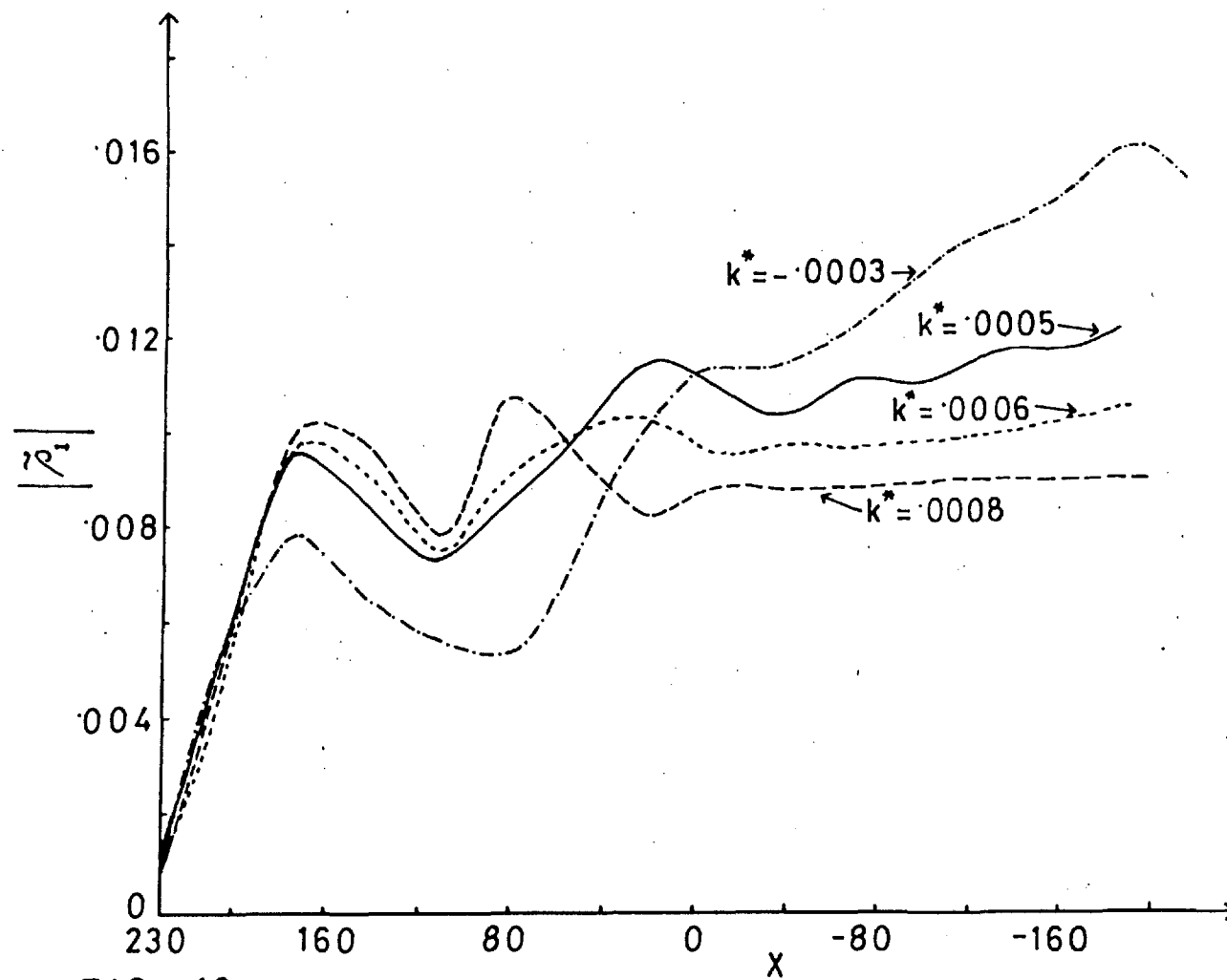


FIG. 13.

Fig 13. Graph of $|\tilde{p}|$ as a function of x for $k^* = -0.0005, 6, 8, -3$

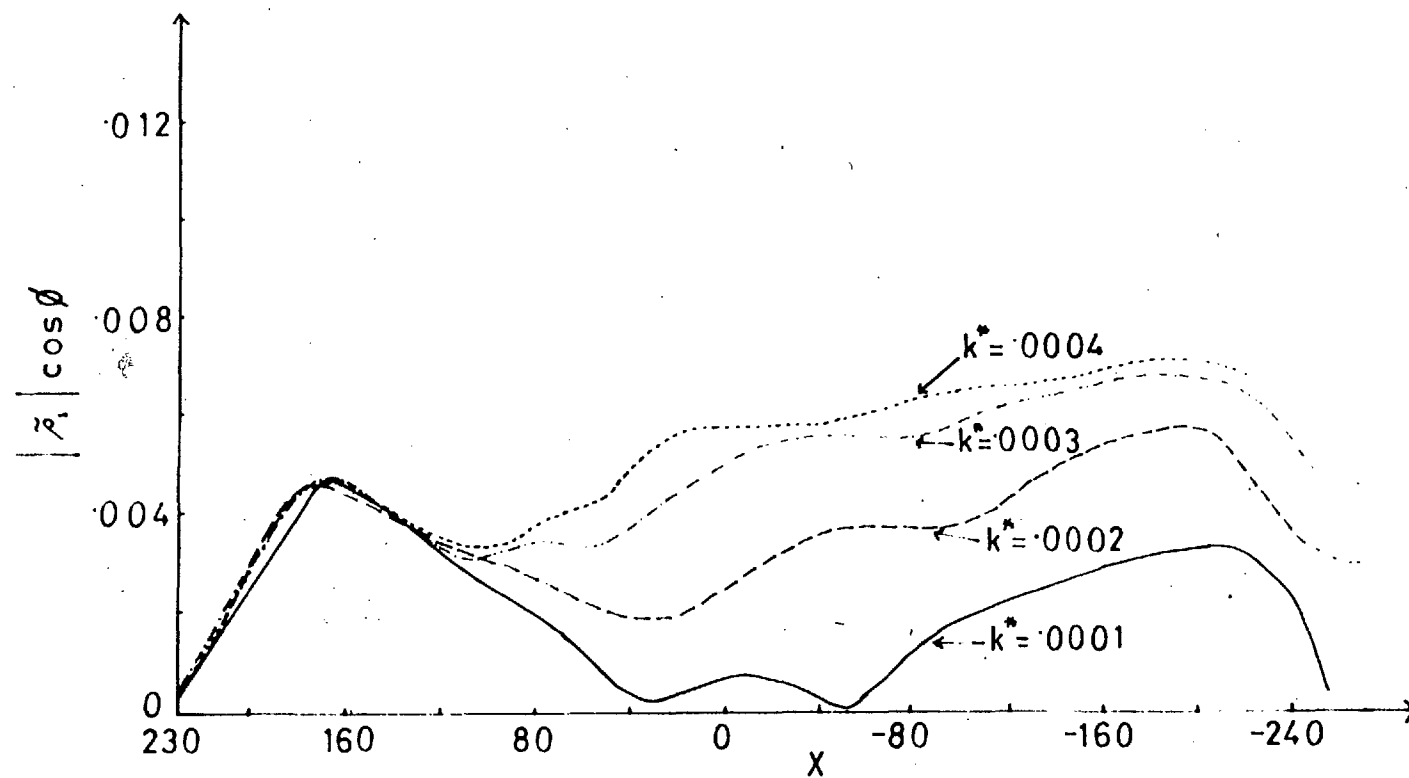


FIG. 14.

Fig 14. Graph of p_R as a function of x for $k^* = .0001-4$

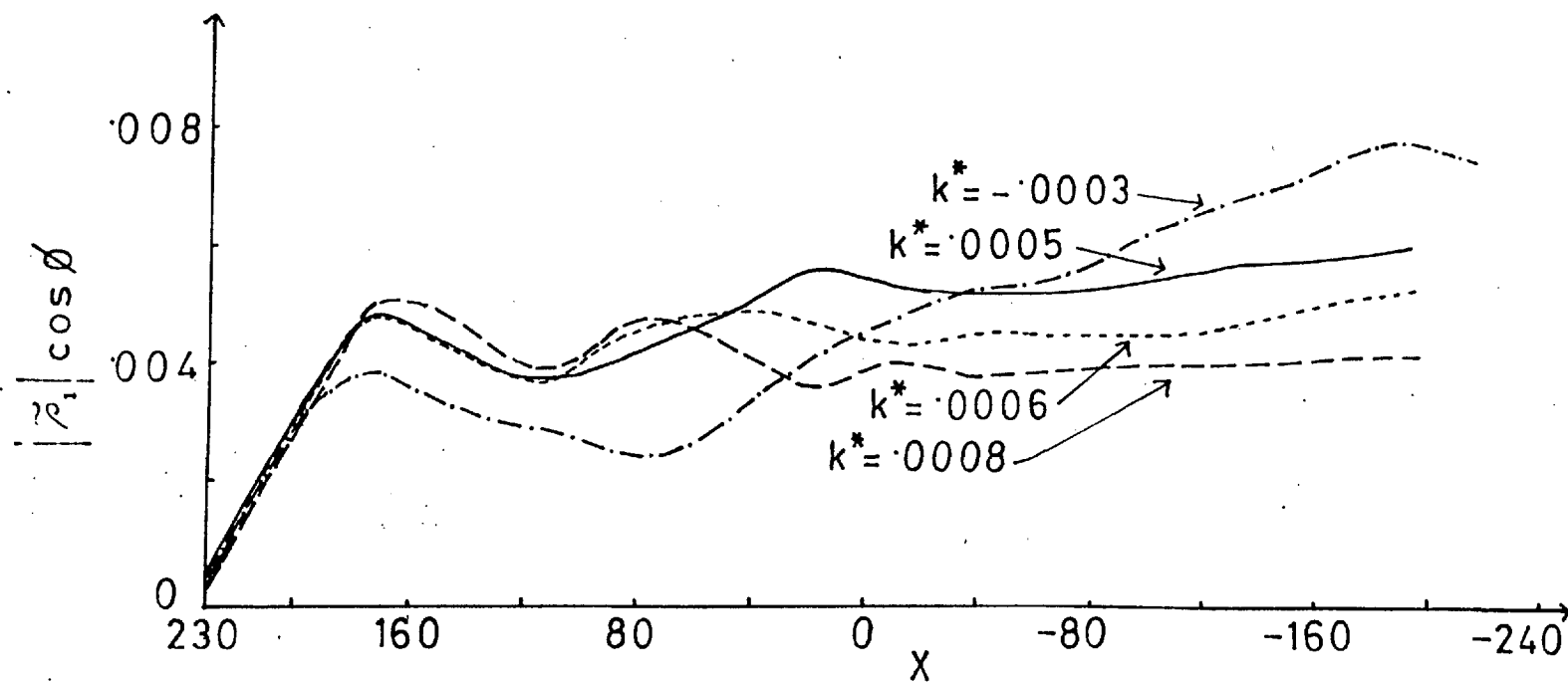


FIG. 15.

Fig 15. Graph of p_r as a function of x for $k^* = 0.0005, 6, 8, -3$.

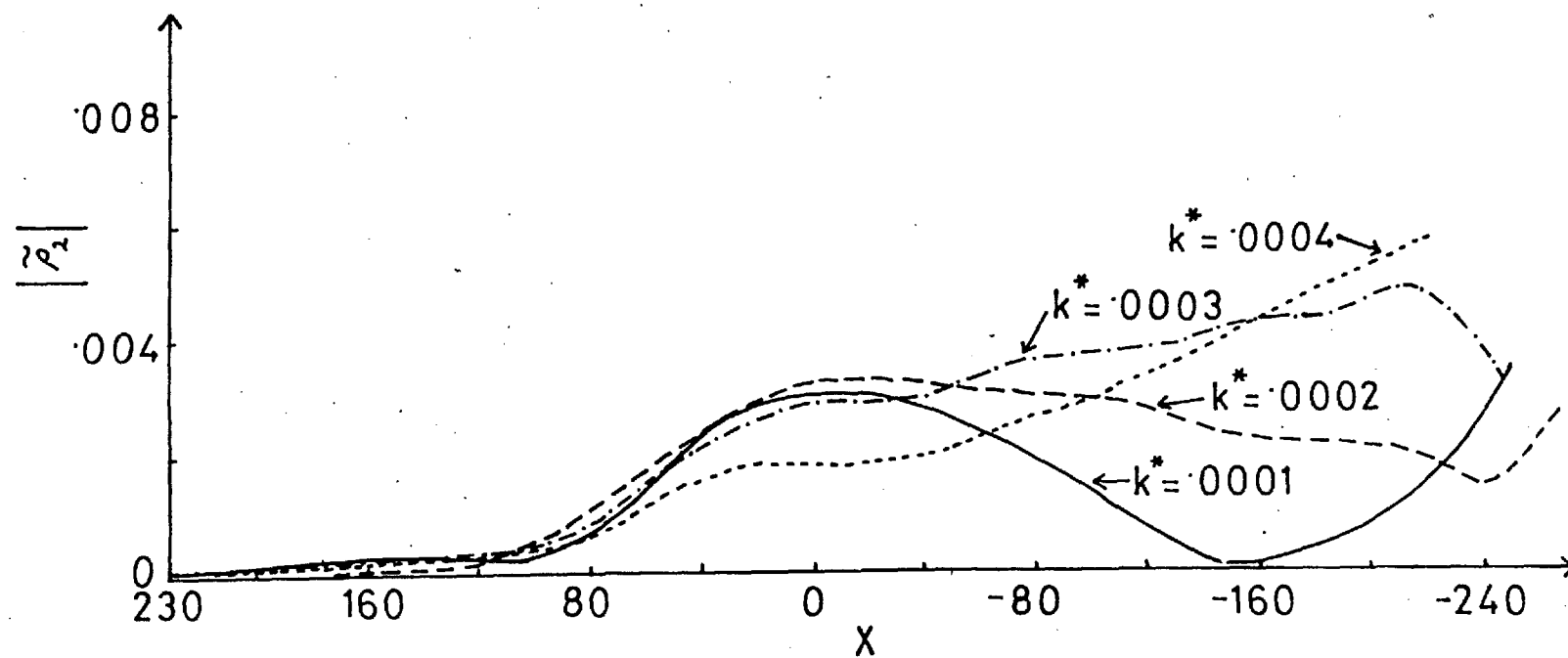


FIG. 16.

Fig 16. Graph of $|\tilde{p}_2|$ as a function of x for $k^* = .0001 - 4$

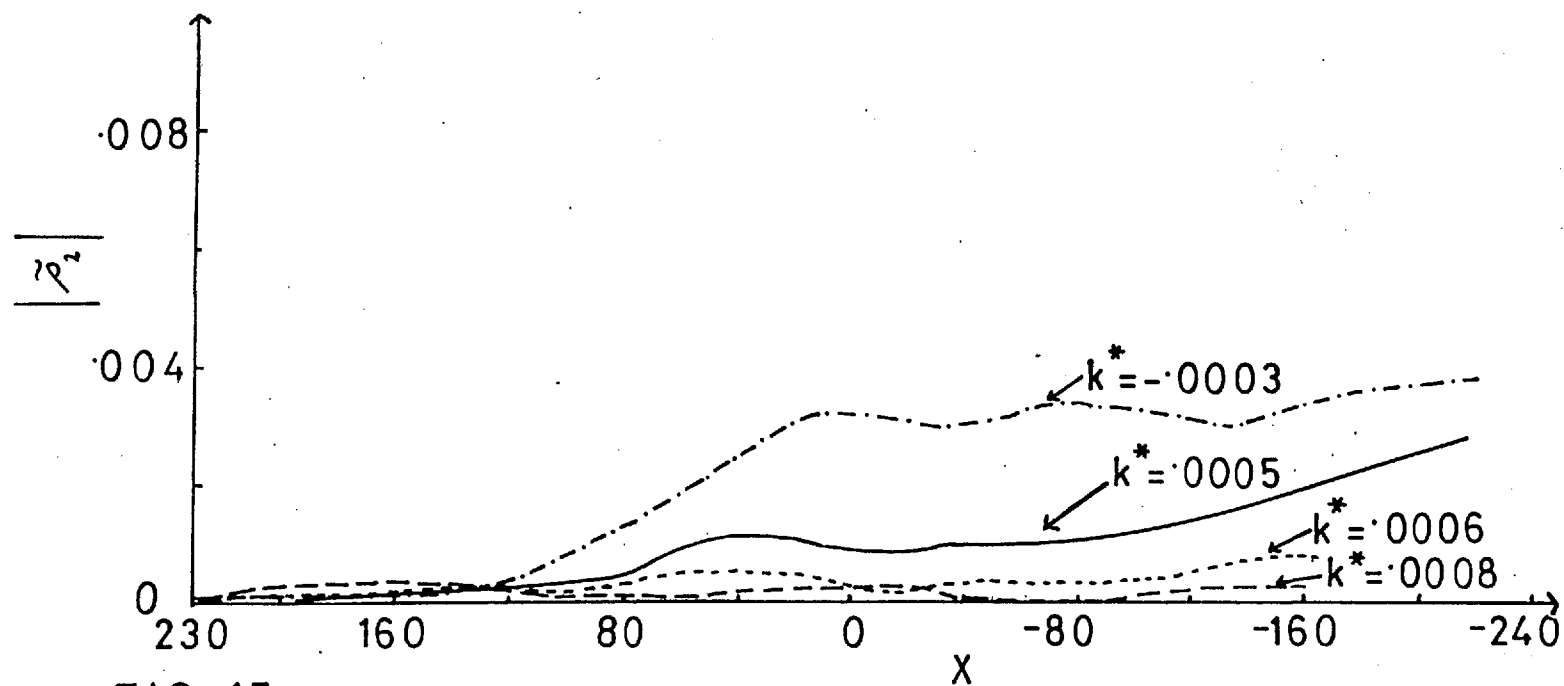


FIG. 17.

Fig 17. Graph of $|\tilde{p}_2|$ as a function of x for $k^* = .0005, 6, 8, -3$

SECTION THREE

A THEORY OF THE TRIGGERING OF VLF EMISSIONS
IN THE WHISTLER MODE.

CHAPTER 1.

INTRODUCTION.

1. Description of the process.

In this work we shall be concerned with the propagation of whistler mode waves through the earth's magnetosphere. The question of how such propagation takes place is a difficult one in its own right, and has been extensively studied, particularly by the Stanford Group. The interested reader is referred to the extensive literature on the subject. Here we shall confine our attention to the so called ducted mode in which field aligned density variations or ducts channel the signal along a field line in such a way that the wave number vector is always closely parallel to the magnetic field. We shall not consider the recently discovered unducted mode (Walter, Scarabucci et al.) in which the k vector is not parallel to the magnetic field, and which seems to rely on natural focussing effects.

Ducted propagation makes it possible to transmit whistler signals such as morse code from one point on the earth's surface to its magnetically conjugate point. During such transmissions "triggering" often occurs, in which the whistler pulse will stimulate the production of a large amplitude, narrow band signal of a rising or falling tone.

Thanks to U.S. military morse transmissions, a great deal of data on this phenomenon has been collected in recent years. High power V.L.F. transmitters are located at Jim

Creek, Wash. (NPG) at 18.6 Kcs and 2.10^5 Watts , and at Cutler, Maine (NAA) at 14.7 Kcs and 10^6 Watts . The magnetospherically transmitted signals are picked up in the southern hemisphere at Eights, Antarctica. The full details of such data will be found in the literature (Helliwell, Smith, Carpenter etc), but here we will draw attention to the important features of morse triggered emissions which any theory of the process should be able to account for.

1. The emission usually has a considerably larger amplitude than that of the triggering signal, and once started is a self sustaining entity which may continue for some time.

2. The occurrence of triggering is closely confined to cases where the morse frequency is just below one half the equatorial gyrofrequency on the path of propagation.

3. On a time scale of minutes triggering is a reproducible process. However there is a considerable variation in behaviour throughout the day and from day to day. Triggering occurs for only a fraction of transmissions actually made, but on some occasions the original morse is completely swamped by the additional signals.

4. The signal at the receiver is usually analysed in the form of a frequency-time diagram. Some typical examples of the appearance of emissions on such a diagram are shown in graphs 1 and 2. They usually have rising or falling tones, emissions of a constant frequency being rather rare. The frequency at the start of the emission is usually that of the morse, but it is often displaced upwards by a small amount $\sim 300 \text{ Hz}$, known as the offset frequency. (Helliwell)

5. The rising tones frequently go to high frequencies

quite near the equatorial gyrofrequency. The frequency gradient and amplitude may vary considerably on route, and some risers terminate with a downward hook. The falling tones are usually shorter in duration, and very often turn round to become full length risers.

6. While some emissions seem to be very sharp or narrow band in character (with a bandwidth of say 50-200 Hz), on some occasions they are very broad. Detailed examination of these cases however, shows that several emissions may be present, very close together. Risers are in fact sometimes observed to split up into two during their development.

7. It seems that a definite time-or length of pulse- is required in order to trigger an emission. This delay time-as it is called- becomes shorter as the triggering pulse is made stronger. This is born out by the NAA transmissions where morse dashes are able to trigger, but the dots are too short. In this connection a fascinating result is that of the Omega transmissions (Kimura). The station at Forest Port N.Y. of only 100 Watts (10.2KHz) has been observed to cause triggering from pulses of one second duration.

8. At this point we would draw attention to various phenomena closely allied to artificially triggered emissions. These are triggering by natural whistlers and the innumerable varieties of spontaneous VLF emissions. We shall not deal with these in this work, but the general theory developed should be applicable to these cases.

The aim of this work then is to give a theoretical explanation of morse triggered emissions, and to account for the various detailed features of the process.

2. Previous theories.

There is general agreement among researchers that the cause of VLF emissions most likely lies in the resonant interaction between energetic radiation belt particles and the wave, and that such interaction is most significant at the magnetospheric equator where the resonant velocity is the lowest. It is of course possible that the observed emissions originate in the ionosphere, but more detailed satellite observations are expected to confirm that this is in fact not the case.

An early theory was that of Dowden (qv) which was designed to explain the spontaneous VLF emissions. He supposed that they were caused by bunches of high energy particles oscillating up and down the field line. These could give rise to a linear cyclotron instability, and generate turbulence localised with respect to both position and frequency. It is easily seen that such bunches could generate almost any emission form. The idea is ingenious and there is little doubt that time variation in the zero order distribution function is important in considering the origin of spontaneous VLF emissions.

Das (q.v.) adopted Dowden's idea that the emission consisted of selectively amplified turbulence, and addressed himself to the more closed problem of artificially stimulated emissions. Using the familiar loss cone distribution function, Das was able to show that the triggering pulse distorted it in such a way as to cause a much enhanced linear growth rate in two narrow frequency bands on either side of the pulse frequency, and it was argued that these would

generate emissions of the type observed. The snag with this theory however, is that the pulse and emission together form a continuous narrow band wavetrain, and in this case it has been recently shown (Abdalla) that after one trapping period the power transfer from the particles to the field actually reverses sign.

An important aspect of the problem was recognised by Helliwell (qv) in his well known phenomenological theory of VLF emissions. He pointed out that on account of the variation of both wave frequency and background magnetic field, only a limited number of particles- termed second order resonant- would be able to remain resonant with the wave for an appreciable period. It was suggested that such enhanced resonance was confined to a narrow zone called the generation region, and that this moved back and forth across the equator in such a way as to give rise to the various observed emission forms. The present theory is based on this general concept, but in detail is quite different from Helliwell's.

CHAPTER 2

RESONANT PARTICLE BEHAVIOUR IN AN INHOMOGENEOUS
MAGNETIC FIELD

1. Basis of treatment

Our task is clearly to examine the behaviour of resonant particle trajectories in fields which realistically approximate those which might be found when an emission of rising or falling frequency propagates along a magnetospheric field line. A resonant particle travelling through such a field will see wave frequency ω , wave number k , and gyro frequency Ω all steadily changing with time. We shall be particularly interested in second order resonant particles which closely satisfy the following conditions.

$$\frac{dV_z}{dt} = \frac{dV_{res}}{dt} = \frac{d}{dt} \left(\frac{\omega - \Omega}{k} \right) = \frac{\dot{\omega} - \dot{\Omega}}{k} - \frac{\omega - \Omega}{k^2} \dot{k} \quad (1)$$

$$V_z = V_{res}$$

where V_{res} is the cyclotron resonance velocity, and V_z the velocity component along the magnetic field.

This interest in second order resonance is inspired by two recent papers on the simpler but closely analagous problem of an electrostatic wave with spatially varying wave number. It has been shown (Swift) that particles in such a wave can undergo very large changes in energy, and we have shown in section two that second order resonant particles in this case become stably trapped in the wave, and that the beam excitation of such a system can result in very large growth rates.

Noting that the condition 1 will involve the gradients of B_0 , ω , and k , the simplest approximation to the whistler emission field that will best illustrate resonant particle behaviour is one in which B_0 , ω , and k are taken as linear functions of position. Our initial program of investigation is to thoroughly examine the nature of resonant particle trajectories in such fields, by both numerical and analytic means, in the hope of finding stably trapped second order resonant particles.

2. The fields

We require a constant amplitude whistler with k parallel to B_0 , and with linearly varying B_0 and ω . For small values of the gradients i.e.

$$\frac{\partial \omega}{\partial z} \ll k\omega \quad \frac{\partial B_0}{\partial z} \ll kB_0$$

we may write the fields as;

$$\begin{aligned} E_x &= E \cos \phi \\ E_y &= E \sin \phi \\ E_z &= 0 \\ B_x &= -Ek(z,t) \sin \phi / \omega(z,t) \\ B_y &= Ek \cos \phi / \omega \\ B_z &= B_0 \beta(z) \end{aligned} \tag{2}$$

The quantity ϕ is the phase of the wave. For $\partial \omega / \partial z \neq 0$ it is clearly going to be difficult to give a satisfactory expression for ϕ as a general function of z and t . However if we regard ϕ as being just the phase of the wave as seen by a particular particle, we do know that in some average sense the following will be true.

$$\frac{d\phi}{dt} = -k(z,t)v_z + \omega(z,t)$$

In reality $d\phi/dt$ will have an irregular variation about this value proportional to the frequency band width of the emission at that particular point. However in what follows we shall see that we are concerned with phase stable particles, and small irregularities in phase are not expected to be significant. Thus we may represent the phase as seen by a particular particle as being the following trajectory integral.

$$\phi = \int (\omega - kv_z) dt \quad (3)$$

For simplicity we shall assume that the cold plasma density is proportional to B_0 , which gives the following dispersion relation;

$$k^2 = A^2 \omega \Omega / (\Omega - \omega) \quad (4)$$

The group velocity simply becomes

$$V_g = 2/A (1 - \omega/\Omega)^{3/2} \omega^{1/2} \quad (5)$$

3. The particle equations of motion

Provided that the electron Larmor radius is much less than the scale length of the magnetic field, the equations of motion of a particle can be shown to be:

$$m \frac{dv_x}{dt} = -e \left[E_x + \frac{1}{c} (v_y B_z - v_z B_y) \right] + \frac{m}{2} \frac{v_x v_z}{B_z} \frac{\partial B_z}{\partial z}$$

$$m \frac{dv_y}{dt} = -e \left[E_y + \frac{1}{c} (v_z B_x - v_x B_z) \right] + \frac{m}{2} \frac{v_y v_z}{B_z} \frac{\partial B_z}{\partial z} \quad (6)$$

$$m \frac{dv_z}{dt} = -\frac{e}{c} (v_x B_y - v_y B_x) - \frac{m}{2 B_z} (v_x^2 + v_y^2) \frac{\partial B_z}{\partial z}$$

It is convenient to dedimensionalize these equations using the values of ω , k , and B_0 appertaining to the point $z=0$, which is where the trajectory integrations will start. Assuming that at this point $\omega = \bar{\omega}$, $k = \bar{k}$,

and $\beta = 1$, we define dimensionless variables as follows

$$\begin{aligned} t' &= \bar{\omega} t & v_x' &= v_x \bar{k} / \bar{\omega} \\ z' &= \bar{k} z & R &= e E \bar{k} / m \bar{\omega}^2 \\ k' &= k / \bar{k} & \alpha &= \frac{\bar{\omega}}{\Omega_0} = \frac{\bar{\omega} m c}{e B_0} \\ \omega' &= \omega / \bar{\omega} \end{aligned} \quad (7)$$

Substituting from eqs 2 and 7 into equation 6 we obtain dropping primes;

$$\begin{aligned} \frac{dv_x}{dt} &= -R \cos \phi \left(1 - \frac{k v_z}{\omega}\right) + \frac{1}{2} v_x v_z \frac{\partial \beta / \partial z}{\beta} - \beta / \alpha v_y \\ \frac{dv_y}{dt} &= -R \sin \phi \left(1 - \frac{k v_z}{\omega}\right) + \frac{1}{2} v_y v_z \frac{\partial \beta / \partial z}{\beta} + \beta / \alpha v_x \\ \frac{dv_z}{dt} &= -\frac{R k}{\omega} (v_x \cos \phi + v_y \sin \phi) - (v_x^2 + v_y^2) \frac{\partial \beta / \partial z}{2 \beta} \end{aligned} \quad (8)$$

Defining $\gamma = v_x + i v_y$ this reduces to

$$\begin{aligned} \frac{d\gamma}{dt} &= -R \left(1 - \frac{k v_z}{\omega}\right) e^{i\phi} + \frac{1}{2} \frac{\gamma v_z}{\beta} \frac{\partial \beta}{\partial z} + \frac{i \beta \gamma}{\alpha} \\ \frac{dv_z}{dt} &= -\frac{R k}{\omega} \Re \{ \gamma e^{-i\phi} \} - \frac{\gamma \gamma^*}{2 \beta} \frac{\partial \beta}{\partial z} \end{aligned} \quad (9)$$

We suppose a spatial variation of frequency of the form

$$\omega(z, t=0) \equiv 1 - \xi z$$

The entire emission field is advanced along the field line at the initial group velocity $2(1-\alpha)$. Thus

$$\omega(z, t) = 1 - \xi(z - 2(1-\alpha)t) \quad (10)$$

Note that in the generation region of an emission there will also be an absolute rate of change of frequency $d\omega/dt$. As far as resonant particle behaviour is concerned, however, the system is exactly as here if we define an effective ξ as follows;

$$\xi_{\text{eff}} = -\left(\frac{\partial\omega}{\partial z}\right)_t + \frac{d\omega}{dt} / \{ |V_g| + |V_{\text{res}}| \} \quad (11)$$

$$\frac{d\omega}{dt} = \left(\frac{\partial\omega}{\partial t}\right)_z + V_g \left(\frac{\partial\omega}{\partial z}\right)_t$$

The dispersion relation now becomes

$$k^2 = \omega(1-\alpha)/(1-\omega\alpha/\beta) \quad (12)$$

We also specify the z dependence of magnetic field,

$$\beta = 1 + \eta z \quad (13)$$

and the rate of change of wave phase is given by

$$\frac{d\phi}{dt} = \omega - kV_z \quad (14)$$

Equations 9-14 are those requiring to be solved to determine resonant particle trajectories. Before discussing their numerical integration, we shall investigate analytically some of the properties of second order resonant particles.

4. The condition for exact second order resonance

We now enquire which particles are able to satisfy instantaneously the exact second order resonance conditions:

$$V_z = V_{\text{res}} \quad dV_z/dt = dV_{\text{res}}/dt$$

This is quite easily found as follows. Using the same dimensionless units as before, the resonance velocity is given by

$$V_{res} = (-\beta/\alpha + \omega)/k$$

and its rate of change is

$$\dot{V}_{res} = (-\dot{\beta}/\alpha + \dot{\omega})/k - (-\beta/\alpha + \omega)\dot{k}/k^2$$

From equations 10, 12, and 13 we have that

$$\dot{\beta} = \eta V_z$$

$$\dot{\omega} = -\xi V_z + 2\xi(1-\alpha)$$

$$2k\dot{k} = \dot{\omega}(1-\alpha)/(1-\frac{\omega\alpha}{\beta}) + \frac{\omega(1-\alpha)}{(1-\frac{\omega\alpha}{\beta})^2} \left(\frac{\dot{\omega}\alpha}{\beta} - \frac{\omega\alpha}{\beta^2} \dot{\beta} \right)$$

Thus:

$$\dot{V}_{res} = -\frac{\eta V_z}{\alpha k} - \frac{\eta V_z \omega}{2k\beta} + \left\{ [2(1-\alpha) - V_z] \left(1 + \frac{\beta}{2\alpha\omega} \right) \right\}$$

The equation of motion 9 furnishes the rate of change of V_z .

$$\dot{V}_z = -\frac{Rk}{\omega} |\chi| \cos \mathcal{P} - |\chi|^2 \eta / 2\beta$$

$$\text{where; } \mathcal{P} = \tan^{-1}(V_y/V_x) - \phi$$

is the phase difference between the perpendicular velocity χ and the electric field. The second order resonant

conditions thus reduce to

$$A + \tan |\alpha_p| \cos \mathcal{P} + C \tan^2 |\alpha_p| = 0$$

$$V_z = (-\beta/\alpha + \omega)/k$$

where α_p is the pitch angle and the coefficients A and C are given by

$$A = \frac{\left\{ -\eta V_z \left(\frac{1}{\alpha k} + \frac{\omega}{2\beta k} \right) + \left\{ (2(1-\alpha) - V_z) \left(1 + \frac{\beta}{2\alpha\omega} \right) \right\} / k \right\}}{RkV_z/\omega}$$

$$C = |V_z| \eta \omega / 2 R \beta k$$

Specializing to the point $z=0$ where $\omega = k = \beta = 1$, this reduces to

$$\cos P = \frac{-C \tan^2 |\alpha_p| - A}{\tan |\alpha_p|} \quad (15)$$

$$C = \left\{ \frac{1-\alpha}{2\alpha} \eta / R \right\} \quad (16)$$

$$A = \left\{ \eta / R \left(\frac{\alpha+2}{2\alpha} \right) + \xi / R \left(\frac{2\alpha+1}{2\alpha} \right)^2 \right\} \quad (17)$$

Thus exact second order resonance is found along a line in the $V_z = V_{res}$ plane, whose form is determined by the three quantities η/R , ξ/R and α . Examples of some of these resonance lines for various values of these parameters are given in figs 4 and 5. They are plotted on polar graphs with P as phase and pitch angle as the radial coordinate. Clearly the curves are of two types depending upon whether A and C are of the same or opposite sign. The first case is operative when the magnetic field gradient is strong relative to the frequency gradient, or when the latter are in opposite directions. Note that the curve will disappear altogether when $\sqrt{AC} > 0.5$, and in this case second order resonance is impossible. The second category of curve encloses the origin and is only found when the field and frequency gradients are in the same direction and also

$$|\xi| > |\eta| (\alpha+2) / (2\alpha+1)^2$$

Note that in this case no matter how small R is made the curve does not disappear but becomes circular at a pitch

angle of

$$\tan^{-1} \left\{ \sqrt{\frac{-A}{c}} \right\} = \tan^{-1} \left\{ \sqrt{\frac{\alpha+2}{\alpha-1} - \frac{\xi}{\eta} \frac{(2\alpha+1)^2}{1-\alpha}} \right\}$$

5. Phase stability and particle trapping.

As in the electrostatic case, it is not enough that a particle be initially second order resonant. We must enquire in some detail of its subsequent motion. We are particularly interested in whether particles are phase stable—in other words, if they are displaced from the resonance line, do they oscillate about it or do they fall right out of resonance? Of course, the exact behaviour of resonant particles will be very complicated, particularly as the resonance line is itself a function of time, but it is not difficult to extract from the equations of motion the dominant terms that impart overall stability or instability.

We proceed as follows. The equations of motion may be written:

$$\frac{dP}{dt} = k(V - V_{res}) + \frac{R}{|\gamma|} \sin P \left(1 - kV_z/\omega \right) \quad (18)$$

$$\frac{d|\gamma|}{dt} = -R \left(1 - \frac{kV_z}{\omega} \right) \cos P + \frac{1}{2} \frac{|\gamma| V_z}{\beta} \frac{\partial \beta}{\partial z} \quad (19)$$

$$\frac{dV_z}{dt} = -\frac{Rk}{\omega} |\gamma| \cos P - \frac{|\gamma|^2}{2\beta} \frac{\partial \beta}{\partial z} \quad (20)$$

We assume throughout this paper that $R \ll 1$, or that the resonant width is very narrow. ($\sqrt{R} \ll 1$) Let us now consider a particle whose z component of velocity is roughly within the limits $V_{res} \pm R^{1/2}$. We deal with a quantity $P_0(t)$, which is the phase of the second order resonance line at

the particle's perpendicular velocity $|\gamma|$, and at the particle's instantaneous position. Note that P_0 can be imaginary if the resonance line does not exist at that particular value of $|\gamma|$. Neglecting some small terms in $R^{1/2}$, we may write;

$$\frac{dV_{res}}{dt} = -\frac{Rk}{\omega} |\gamma| \cos P_0 - |\gamma|^2 \partial \beta / \partial z / 2\beta$$

Defining $V' = V_z - V_{res}$, we then get

$$\frac{dV'}{dt} = -\frac{Rk}{\omega} |\gamma| [\cos P - \cos P_0]$$

Since for trapped particles $V' \sim R^{1/2}$, equation 18 may be written, to the same order of approximation, as

$$\frac{dP}{dt} = kV'$$

On eliminating V' , and dropping a small term in dk/dt ,

we obtain the following differential equation for P

$$\frac{d^2 P}{dt^2} + \frac{Rk^2}{\omega} |\gamma| \{\cos P - \cos P_0\} = 0 \quad (21)$$

A first approximation to the detailed motion may be obtained by noting that $R, k, \omega, |\gamma|$, and P_0 are all slowly varying functions of time with a characteristic time scale of $1/R$. (This assumes that R, η, ξ are all of the same order of magnitude). Neglecting the time dependence of these quantities, and confining our attention to small amplitude oscillations about the resonance line, there results

$$\frac{d^2 P'}{dt^2} - \frac{Rk^2}{\omega} |\gamma| (\sin P_0) P' = 0$$

$$P' = P - P_0$$

For $0 < P_0 < \pi$ the solution will be exponential in character, and particles lying initially on the right

hand side of the resonance line, will fall away from it and become non resonant. On the other hand for $0 < P_0 < -\pi$ the solution is oscillatory in character, and second order resonant particles oscillate about the line and are thus STABLY TRAPPED. The nature of particle motion within the trap is illustrated in fig. 6. Particles maintain a nearly constant $|\chi|$ and describe elliptical trajectories in the coordinates P', V' around the resonance line. The frequency of these oscillations is seen to be

$$\omega_{\text{trap}} = R^{1/2} K \sqrt{\frac{|\chi| |\sin P_0|}{\omega}} \quad (22)$$

which is of the order of the "trapping frequency" used in the homogeneous problem. The amplitude of the oscillation in V' is given by

$$\Delta V' \sim R^{1/2} \sqrt{\frac{|\chi| |\sin P_0|}{\omega}} \Delta P'$$

which is $\sim R^{1/2}$ as expected. For large amplitude

oscillations the motion becomes anharmonic, and the path of the particle will be fixed by the relationship:

$$\frac{dv'}{dP} = -\frac{R|\chi|}{\omega V'} (\cos P - \cos P_0) \quad (23)$$

It is not difficult to see that if the particle trajectory goes beyond the unstable branch of the resonance line $P = -P_0$, it will escape from the trap and fall out of resonance. It is now possible to construct the region in velocity space occupied by trapped second order resonant particles. One takes the instantaneous resonance line, and for each value of $|\chi|$ constructs (from eq 23) the limiting trajectory that just reaches $P = -P_0$. This is illustrated in fig 6.

The resultant surface is the outer boundary of the trap.

A more precise picture of particle motions is obtained by noting that the resonance line varies continuously. There is also a steady change in $|X|$ due to both the zero order magnetic field gradient and the resonance with the wave (eq. 19) The problem is analagous to that of a pendulum whose point of support undergoes movements. Provided that the resonance line and the perpendicular velocity do not change much within one trapping period ($R^{-1/2}$), the particles will be able to continuously circulate around the line without difficulty. Some computed examples of the variation of pitch angle and phase P of trapped particles are shown in fig. 7. Note the oscillations of the type we have described, and the slow pitch angle drift superimposed.

6. Untrapping

It might be thought that in view of this phase stability a second order resonant particle in an extended wave train could stay trapped almost indefinitely. It will be of some use to examine ways in which particles can escape from the trap.

1. Irregularities in amplitude or phase of the wave may increase the amplitude of oscillation within the trap until the particle spills out at $P = -P_0$. A similar kind of effect can be caused by the movement of the resonance line. For instance, when this curve moves towards the $P = 0$ axis, the "size" of the trap decreases and particles are lost accordingly.

2. Inspection of equation 15 will reveal that at any point trapping is confined to a range of pitch angles between:

$$\tan^{-1} \left\{ (1 + \sqrt{1 - 4AC}) / 2|C| \right\}$$

and $\tan^{-1} \left\{ |(1 - \sqrt{1 - 4AC}) / 2C| \right\}$

Particles with pitch angles too near these limits may be inexorably forced out of resonance if they should fall outside this range. This may be due to the change of the particle's pitch angle and or movement in the resonance line itself. An interesting example of this effect is the case of low pitch angle particles. These will always tend to be lost, for the condition that the resonance line reach zero pitch angle is that $A=0$, and this condition is only transiently satisfied.

7. The numerical integration

By way of confirmation of the foregoing ideas we now turn to exact numerical integrations in an actual wave train using the fields already outlined. To give the trapping a thorough test we choose a fairly large amplitude of $R=0.0004$ and assume that the pulse extends from $z=0$ to $z=2000$. This means that a resonant particle spends about ten trapping periods in the pulse. The pulse is sketched in fig. 3.

Our task is to numerically integrate equations 9-13. The computational technique is the standard one for such problems, and is discussed in the appendix. The integrations start at $z=0$ and follow the trajectories backwards in a stepwise fashion until $z=2000$ is reached. A network of

particles is selected in the χ plane and for 5 different values of V_z spaced across the resonant width. All this is done for various values of the coefficients α, β, η, ξ . Noting that the rate of change of particle energy goes as

$$\dot{W} \propto |V_{\perp}| \cos P$$

a suitable indicator of the trapping history of a particle would obviously be

$$\Delta W^* = \Delta W / |V_{\perp}|_{z=0}$$

where ΔW is the total change in energy as a result of passing through the pulse. The quantity ΔW^* is in the nature of a trajectory integral of $\cos P$, and the numerical output of the program takes the form of a plot of this quantity over velocity space in the neighbourhood of resonance.

8. Results of trajectory integrals

The results are readily understood in the light of our analytic study of resonant particle behaviour. A typical plot of ΔW^* in the resonance plane is shown in fig. 8. Outside the instantaneous particle trap as previously defined this quantity is everywhere small and irregular. Inside it is much larger by a factor in this case of about 20 (of the order of the number of trapping periods spent within the pulse). The variation within the trap of ΔW^* is roughly one with pitch angle as $\cos P_0$. This is as expected as P_0 is the average phase about which trapped particles oscillate. In addition there is considerable modification to ΔW^* due to particles escaping from the trap prematurely (in the sense of the backward integration). Generally ΔW^*

is greatest in the innermost part of the trap and falls off towards the outer edges, and also at high and low pitch angles. In most cases it was found possible to delineate quite clearly a region in velocity space where trapping was in practice fairly effective, and the remainder of the results are conveniently presented in the form of plots of these regions for a wide variety of cases. In the diagrams (figs 9-19) heavy shading represents regions of weak trapping, and regions where $P \sim -\pi/2$ $\Delta W \sim 0$, are lightly hatched.

On the whole these diagrams will speak for themselves. Predictably, and without exception, trapped particles are found inside the trapping volume as defined by the local second order resonance curve. A few additional features of the diagrams are as follows.

1. When η and ξ are small the resonance line changes little along the pulse, and the effective trapping region is close to the maximum theoretical size.
2. For larger ξ , η the resonance line changes noticeably along the pulse, and particle loss becomes more extensive. The extent to which this happens is rather variable, and is not readily predicted without examining the resonance line right along the packet. In some cases effective trapping only reached as far as the $P=0$ axis, though $P=-P_0/2$ was a more typical extent. Note that the particle loss at the upper and lower pitch angle limits of the resonance line was frequently in evidence.
3. In some cases the trapping region is markedly unsymmetric

with respect to V_z about $V=V_{res}$, which suggests that the wavelength of the resonant particle current is shifted from that of the wave. We shall return to this point later in discussing the offset frequency effect.

CHAPTER 3

RESONANT PARTICLE DISTRIBUTION FUNCTION AND
RESONANT PARTICLE CURRENTS

1. The resonant particle distribution function

Let us now consider the general case of a long wave pulse in which amplitude, frequency, and zero order magnetic field are slowly varying functions of position and time. Using the same system of dedimensionalization as before we have

$$\beta \equiv \beta(z) \quad R \equiv R(z, t) \quad \omega \equiv \omega(z, t)$$

We now enquire as to the distribution function about resonance F_{res} at some arbitrary point, say $z=0$.

We consider a time independent zero order distribution function F_0 that is a function of energy and magnetic moment only.

$$F_0(W, \mu) \quad W = \frac{1}{2}(|\delta|^2 + v_z^2) \quad \mu = |\delta|^2 / 2\beta$$

From Liouville's theorem we may write at once:

$$\begin{aligned} F_{res}(|\delta|, P, v_z) &= F_0(W - \delta W, \mu - \delta\mu) \\ &= F_0(W, \mu) - \delta W \frac{\partial F_0}{\partial W} - \delta\mu \frac{\partial F_0}{\partial \mu} + O(\delta\mu^2) + O(\delta W^2) \end{aligned} \quad (26)$$

where $\delta W, \delta\mu$ are the total changes in energy and magnetic moment undergone by a particle up to that point as a result of interacting with the wave. Now if particles are trapped for times of the order of $1/R$ $\delta\mu, \delta W$, will be large, but this case normally represents a very powerful wave. In the VLF emission problem we are justified in neglecting $(\delta\mu)^2, (\delta W)^2$ and higher order terms. From the equations of motion (eq 9) we quickly derive:-

$$\begin{aligned}\dot{W} &= -R \cos P \\ \dot{\mu} &= -R (1 - kV_z/\omega) \cos P/\beta\end{aligned}\quad (27)$$

$$\begin{aligned}\text{Whence; } \delta W &= - \int R \cos P dt' \\ \delta \mu &= - \int R (1 - kV_z/\omega) \cos P/\beta dt'\end{aligned}\quad (28)$$

In the theory to be presented we must assume that the wave amplitude is sufficiently large such that the typical resonant particle remains trapped for quite a number (say at least three) of "trapping periods" ($R^{-1/2}$). As in the inhomogeneous electrostatic problem (Nunn) trapped second order resonant particles will not dominate the distribution function unless this is the case.

Another requirement is as follows. It takes a definite time-of the order of a trapping period-for a given second order resonance line to establish control over the resonant particles-in other words for the particles within the trap to be uniformly stirred, and for those outside it to be dispersed to non resonant velocities. We thus demand that the resonance line, and hence β, ω, R , vary slowly with respect to a trapping period, in order that a pseudo equilibrium be maintained, in which the local trapping dynamics are important. Given these conditions, the results of the numerical computation strongly suggest that at any point the resonant particle distribution function will be dominated by a core of strongly trapped particles centred about the instantaneous resonance line. In this limit F_{res} may be approximated as follows. For particles outside the trap $\delta \mu, \delta W$ will be small, and we may put $F_{res} = F_0$. For

those inside the trap we need only integrate over the time for which they have been trapped, replacing $\cos P$ by $\cos P_0$, and V_z by V_{res} . Hence

$$F_{res} = F_0(W, \mu) + \int_{\text{trap}} R \cos P_0 \left\{ \frac{\partial F_0}{\partial W} + \left(1 - \frac{k V_{res}}{\omega} \right) \frac{\partial F_0}{\partial \mu} / \beta \right\} dt''$$

We may write this as

$$F_{res} = F_0(W, \mu) + R \cos P_0 \left\{ \frac{\partial F_0}{\partial W} + \left(1 - \frac{k V_{res}}{\omega} \right) \frac{\partial F_0}{\partial \mu} / \beta \right\} \tau_{eff} \quad (29)$$

which defines an "effective trapping time" $\tau_{eff}(|\mathbf{x}|, P, V_z)$

Except in simple cases τ_{eff} may be difficult to estimate accurately, without resorting to trajectory integrations.

However on account of the continual stirring of particles within the trap, it will possess a certain symmetry, and should be mainly a function of pitch angle, and amplitude of oscillation within the trap.

2. Resonant particle currents.

Particles with their parallel velocity close to the local resonance velocity V_{res} can give rise to transverse currents whose frequency and wavelength are very close to those of the wave at that point. Such currents will thus be resonant with the ambient plasma and will be able to cause both growth, and as we shall see frequency shift, in the wave field.

The current due to trapped second order resonant particles is well estimated by integrating over the volume of the trap in velocity space. There will also be a contribution from non resonant particles, but owing to dephasing, this will be parallel to B_0 .

$$\begin{aligned}
 \tilde{J} &= j_R + i j_I = -e \iiint \gamma |\gamma| (F_{res} - F_0) d|\gamma| dv_z dP \\
 &= -e R \iiint_{\text{trap}} \gamma |\gamma| \cos P_0 \left\{ \frac{\partial F_0}{\partial W} + \left(\frac{1 - \frac{k v_{res}}{\omega}}{\beta} \right) \frac{\partial F_0}{\partial \mu} \right\} \tau_{eff} d|\gamma| dP dv_z \quad (30)
 \end{aligned}$$

The current has been divided into two—the in phase component J_r , and the reactive component J_i . Clearly $\tilde{J}$ will depend upon F_0 and the resonance line in a rather complex way, and this will be the concern of chapter 5. At this juncture we will discuss two very important general properties of the resonant particle current $\tilde{J}$.

First there is a general proportionality to the trapping time τ_{eff} . Other things being equal, both J_r and J_i will tend to increase steadily downstream from the front of the wave packet (in the direction of V_{res}). Superimposed upon this linear spatial dependence will of course be a variation due to the changing trap geometry along the pulse, and to other complex effects.

Another significant feature is as follows. The current $\tilde{J}$ is caused by a "jet" of phase stable particles. The phase difference between the electric field of the wave and the particle current will be closely determined by the second order resonant conditions, which vary only slowly over a trapping length. Thus, provided the wave field remains sufficiently narrow band, this phase difference will remain nearly constant, and be maintained under fairly general conditions. This way in which the current remains as if

"glued" onto the electric field is a rather remarkable feature of wave particle interactions in an inhomogeneous medium, and does not occur in the corresponding homogeneous problem.

CHAPTER 4

TIME DEVELOPMENT OF THE WAVE FIELD- GENERAL THEORY

1. Descriptive account of the process

Being resonant with the ambient plasma the current J will generate additional fields which will add on to the existing one in a continuous fashion-the current J of course being all the time determined by the instantaneous total field in the manner already discussed.

The component J_r will generate an in-phase field and will give rise to growth or damping. The reactive J_i will generate a field $\pi/2$ out of phase with the existing one. Let us consider a pulse of constant amplitude and wavelength, and then look at it a very short time later after the addition of the field generated by J_i . If, as we have already suggested J_i varies linearly with z , a little consideration will reveal that the combined field will have a slightly shifted wavelength. This is illustrated in fig. 20. Subsequent resonant particles will of course respond to the new wavelength, and will shift it still further. At first sight then we might expect there to be a steady change in the wavelength and frequency of the pulse. The reader will probably find this descriptive account of the process hopelessly inadequate, particularly when such a surprising result is being proposed. Accordingly we now provide a rigorous analysis, not only to justify this suggestion, but also to find an expression for the rate of change of frequency, and to gain some insight into

what conditions are necessary for it to take place.

2. Analytic derivation of the growth rate and frequency shift.

We choose to look at a limited section of the wave train-of the order of a trapping length say. One assumes that although the inhomogeneity is important in determining the resonant particle currents, for the purpose of analysing the effect of the currents upon the wave field, one may assume the medium to be locally homogeneous. We assume the background plasma is cold and with a gyrofrequency Ω , and plasma frequency Π .

The electric field is represented in terms of an amplitude and phase thus;

$$\tilde{E} = E_x + i E_y = E(z, t) e^{i\phi(z, t)} \quad (31)$$

and; $\tilde{j}_0 = j_x + i j_y$

A single differential equation for the time development of $\tilde{E}$ can be obtained from Maxwell's equations and from the equation of motion of the cold particles. (See appendix 2)

$$\left[\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial}{\partial t} - i\Omega \right) - \frac{\Pi^2}{c^2} \frac{\partial}{\partial t} \right] \tilde{E} = \frac{4\pi}{c^2} \left(\frac{\partial}{\partial t} - i\Omega \right) \frac{\partial}{\partial t} \tilde{j}_0^{**} \quad (32)$$

The reactive and in-phase components of current are defined by $\tilde{j}_0 = e^{i\phi} (j_R + i j_I)$ (33)

In accordance with our previous discussion we expect j_R and j_I to increase linearly in the z direction, but to be otherwise only weakly dependent upon z or t . This holds ONLY if E remains narrow band and continues to satisfy the "slowly varying" criteria of the previous chapter. We now

**This analysis is conveniently done in dimensioned variables

substitute for $\tilde{E}$ and $\tilde{J}_0$ in eq. 32, and neglecting small terms in $1/c^2$, there results two simultaneous differential equations for E and ϕ .

$$\frac{\partial E}{\partial t} \left(k^2 + \pi^2/c^2 \right) + 2kE \frac{\partial k}{\partial t} - 2k(\omega - \Omega) \frac{\partial E}{\partial z} + \frac{\partial k}{\partial z} (\omega - \Omega) E - \frac{\partial^3 E}{\partial z^2 \partial t} = \frac{4\pi}{c^2} \omega(\omega - \Omega) j_R \quad (34)$$

$$E \left\{ k^2 (\Omega - \omega) - \omega \pi^2/c^2 \right\} + (\omega - \Omega) \frac{\partial^2 E}{\partial z^2} - \frac{\partial^2 k}{\partial z \partial t} E - \frac{\partial k}{\partial z} \frac{\partial E}{\partial t} - 2 \frac{\partial k}{\partial t} \frac{\partial E}{\partial z} - 2k \frac{\partial^2 E}{\partial z \partial t} = -\frac{4\pi}{c^2} \omega(\omega - \Omega) j_I \quad (35)$$

Here ω and k are simply defined as being,

$$\omega = \partial \phi / \partial t \quad k = -\partial \phi / \partial z$$

Terms in $\dot{j}_R$, $\dot{j}_I$ have been neglected in comparison with ωj_R , ωj_I . In seeking solutions to these equations $E(z, t)$, $\phi(z, t)$ for the time development of the field, it was found by experience that eliminating apparently small higher derivatives was highly dangerous, and also that the initial time behaviour of a particular field configuration could be completely misleading. The only really safe procedure is to find exact long time solutions, though these may be under simpler conditions than we would really like. To do this we have to remove the z dependence of E . We assume that E, k, j_R are all independent of z , and that j_I varies only linearly with z . In this case equations 34, 35 reduce to:

$$\frac{\partial E}{\partial t} \left(k^2 + \pi^2/c^2 \right) + 2kE \frac{\partial k}{\partial t} = \frac{4\pi}{c^2} \omega(\omega - \Omega) j_R \quad (36)$$

$$E(k^2(\Omega - \omega) - \frac{\omega \Pi^2}{c^2}) = -\frac{4\pi}{c^2} \omega(\omega - \Omega) j_I \quad (37)$$

We now assume that the resonant particle beam is weak. If

$\mathcal{E}$ is the smallness parameter associated with this, then we see that the quantities E, k, j_R, j_I are all of first order in $\mathcal{E}$. Let us define a proper frequency from the dispersion relation

$$k^2(\Omega - \bar{\omega}) = \bar{\omega} \Pi^2 / c^2 \quad (38)$$

and put; $\omega = \bar{\omega} + \omega'$

Noting that ω' is also of order $\mathcal{E}$, we may substitute into equations 36 and 37, and eliminate terms in $\mathcal{E}^2$. From eq. 37 there results

$$\omega' = \frac{4\pi}{c^2} \frac{\bar{\omega}(\bar{\omega} - \Omega) j_I}{E(k^2 + \Pi^2/c^2)} \quad (39)$$

Differentiating with respect to z yields the time dependence of k

$$\frac{\partial k}{\partial t} = -\frac{\partial \omega'}{\partial z} = \frac{4\pi}{\Pi^2} \frac{\bar{\omega}(\bar{\omega} - \Omega)^2}{\Omega E} \frac{\partial}{\partial z} j_I \quad (40)$$

Note that the z independence of k is preserved, as is essential for a long time solution. The rate of change of frequency $\bar{\omega}$ is obtained at once from its defining

equation (38):

$$\frac{\partial \bar{\omega}}{\partial t} = + \frac{8\pi \bar{\omega}^2 (\Omega - \bar{\omega})^3}{k \Omega^2 E \Pi^2} \frac{\partial}{\partial z} j_I \quad (41)$$

Equation 36 becomes;

$$\frac{\partial E}{\partial t} = \frac{4\pi \bar{\omega}(\bar{\omega} - \Omega) j_R}{\Pi^2 + k^2 c^2} - \frac{2kE \partial k / \partial t}{k^2 + \Pi^2/c^2} \quad (42)$$

where again we note that $\bar{E}$ is not a function of z . Some consideration will reveal that

$$\partial k / \partial t / k \ll \partial E / \partial t / E$$

which reduces the growth rate to

$$\frac{\partial E}{\partial t} = -\frac{4\pi}{\pi^2} \frac{\bar{\omega}(\bar{\omega} - \bar{\Omega})^2}{\bar{\Omega}} j_R \quad (43)$$

We have thus found a long time solution as follows.

A wave with E and K independent of position in a locally homogeneous medium is excited by a constant in phase current and a linearly z dependent reactive current, Quite rigorously to first order in ξ , the wave undergoes a steady change in amplitude, wavelength and frequency as given by the above expressions.

Having found such an exact solution certainly lends support to our "pictorial" argument for a shifting frequency. Perhaps the greatest shortcoming of the above model is the lack of a z dependence in E and J_r , which one expects to be quite sharp in a generation region. A little tampering with equations 34 and 35 soon convinces one that useful results will never be obtained in this case, partly because all sorts of wave effects of zero order in ξ become embroiled in the equations-like advection of the packet envelope along the field line, diffusion of the packet envelope etc etc. The only possibility is to return to our simple physical picture of the process.

We now consider the case of a linear z dependence in both J_i and E. For a weak beam the addition of fields concept will be a valid one. Let us consider a time independent amplitude, maintained by a constant in phase current J_r . This is the kind of situation one expects in a "generation region" This may be written as;

$$\begin{aligned}\hat{E}' &= (E_1 + E_2 z) e^{i(\bar{\omega}_0 t - k_0 z)} \\ k_0^2 (\Omega - \bar{\omega}_0) &= \bar{\omega}_0 \pi^2 / c^2 \\ 2k_0 (\bar{\omega}_0 - \Omega) E_2 &= \frac{4\pi}{c^2} \bar{\omega}_0 (\bar{\omega}_0 - \Omega) j_R\end{aligned}$$

The field $\tilde{E}'$ will be seen to be a solution of equations 34 and 35, provided the gradient in amplitude is sufficiently weak as to remove the point $E=0$ a very large number of wave lengths away. Next we introduce the linearly z dependent J_1 . Provided the gradient of it is weak, it will generate an electric field everywhere proportional to itself. After a small element of time the total field will be obtained by a summation of the form: (see fig 20)

$$\begin{aligned}E_{TOT} &= \{ (E_1 + E_2 z) + iA(j_{I1} + j_{I2} z) \} e^{-ik_0 z} \\ &\sim (E_1 + E_2 z) e^{-ik_0 z} e^{i\psi} \\ \psi &= \tan^{-1} \left\{ \frac{j_{I1} + j_{I2} z}{E_1 + E_2 z} \right\} \sim \omega \text{nst} + A \frac{\partial}{\partial z} \left(\frac{j_{I1}}{E} \right) z + \dots\end{aligned}$$

A wave length shift proportional to $\partial/\partial z (j_{I1}/E)$ is thus predicted, and accordingly we modify equation 41 to

$$\frac{\partial \bar{\omega}}{\partial t} = \frac{8\pi \bar{\omega}^2 (\Omega - \bar{\omega})^3}{k \Omega^2 \pi^2} \frac{\partial}{\partial z} \left(\frac{j_{I1}}{E} \right) \quad (44)$$

The reader will probably like to form his own opinion as to exactly how valid this expression will be in more general situations where $\frac{\partial E}{\partial t}$, $\frac{\partial^2 E}{\partial z^2}$, $\frac{\partial k}{\partial z}$, are all non zero. In the author's opinion, it will be substantially correct where all the relevant quantities remain slowly varying with respect to a trapping period.

CHAPTER 5

COMPUTATION OF GROWTH RATES AND RATES OF CHANGE OF
FREQUENCY IN A MAGNETOSPHERIC WHISTLER PULSE1. Choice of units-expressions for growth rates.

Having discovered that second order resonant particles are able to cause growth of the field and also to continuously change its frequency, the way ahead is now quite clear. We model a magnetospheric field line and select a realistic distribution function F_0 . One assumes that the morse pulse and emission together form a continuous narrow band wavetrain propagating along the line in a ducted mode. Then using the expressions already derived we endeavour to compute the growth rate and rate of change of frequency as a function of position along the field line, wave amplitude, frequency, and also frequency gradient. These results should enable us to provide a reasonable description of the triggering process.

In what follows we really require constant units for velocity, time etc., so we now dedimensionalize with respect to ω_0 , one half the equatorial gyrofrequency, and K_0 , the corresponding wave number. All subsequent expressions will be in terms of these new units.

The field line is modelled in the familiar way

$$\beta = 1 + \frac{1}{2} \chi z^2 \quad (45)$$

where χ is taken to be $2 \cdot 10^{-9}$.

In presenting the results we need independent variables related to the invariant properties of a wave pulse travelling along a magnetospheric field line in the

absence of excitation. Thus frequency is referred to the equatorial gyrofrequency as follows;

$$\alpha^* = \omega / \omega_{EQ} \quad (46)$$

The flux of wave energy down a duct is proportional to

$$E^2 k / \omega \beta$$

Accordingly we construct a dimensionless amplitude factor R^* depending only on the power flux.

$$R^* = \frac{eE}{m} \sqrt{\frac{k k_0}{\omega \beta \omega_0^3}} \quad (47)$$

The frequency gradient is referred to ω_0, k_0 as follows;

$$\xi^* = -\partial \omega / \partial z / k_0 \omega_0 \quad (48)$$

We numerically compute the growth rate as a function of R^*, α^*, z^* using equation 43 and substituting for J_r with equation 30. It is apparent that some kind of simple approximation for τ_{eff} is going to be required. We make a rather arbitrary assumption about the average extent of particle loss, and suppose that particles whose amplitude of oscillation within the trap takes them beyond $P = -P_0/2$ will not be effectively trapped. Using equation 23 we may integrate the limiting trajectories, and in any case obtain a "reduced" trapping volume. (see fig 21). Inside this volume we suppose that τ_{eff} is constant with a value of $R^{*-1/2} / \sqrt{v_{res}}$. We are thus computing a growth rate per "trapping length" and the actual growth rate will be estimated by multiplying by $\Delta z^* / R^{*-1/2}$, where Δz^* is the distance over which effective trapping takes place. The growth rate is going to depend upon resonant particle density, and it is best expressed as a multiple of a quantity γ_0 , which estimates the order of magnitude of the linear cyclotron

growth rate.

$$\gamma_0 = \frac{\pi^2 \omega_0^2}{k_0 N_0} \int_0^\infty v_\perp dv_\perp F_0(v_z = \beta = 1)$$

The reader will be spared the extremely tedious manipulation which now follows, and we shall merely quote the final expression for the relative growth rate γ^* .

$$\gamma^* = \frac{\partial E / \partial t}{E \gamma_0} = \left\{ \frac{\bar{\omega}^3}{\beta^3 \bar{k}} \right\}^{1/4} \frac{(2\beta - \bar{\omega})^2}{|V_{res}| \sqrt{2} \pi^2} \cdot \iint |\delta|^{7/2} \cdot \cos P \left[\frac{\partial F_0}{\partial W} + \frac{1}{\beta} \frac{\partial F_0}{\partial \mu} \left(1 + \frac{|V_{res}| \bar{k}}{\bar{\omega}} \right) \right] \cos P \tilde{F}(P, P_0)_{V_z=V_{res}} \quad (49)$$

$$\cdot d|\delta| dP / \int_0^\infty |\delta| d|\delta| F_0(v_z = 1, \beta = 1)$$

where

$$\bar{\omega} = \omega / \omega_0 = 2\alpha^*$$

$$\bar{k} = k / k_0 = \sqrt{\left\{ \frac{\beta \alpha^*}{\beta - \alpha^*} \right\}}$$

$$|V_{res}| = (2\beta - \bar{\omega}) / \bar{k}$$

The velocities are all in units of ω_0/k_0 . The quantity $\tilde{F}$ is the thickness of the trapping zone in V_z , and is only non zero for the appropriate range around $P=P_0$.

$$\tilde{F}(P, P_0) = \sqrt{(\cos P_0 (P_0/2 + P) - \sin P_0/2 - \sin P)}$$

$$\cos P = \int \cos P \tilde{F} dP / \int \tilde{F} dP$$

The rate of change of frequency is more difficult to

deal with being proportional to $\partial/\partial z (J_i/E)$, and in a narrow generation region $\partial E/\partial z$ will be quite large. During this research a good deal of work was done trying to construct self-consistent, quasi-static generation regions. Although rather fruitless, it did show that on the average, at constant τ_{eff} , J_i was roughly proportional to E . Accordingly we expect that the consistent contribution to dw/dt should come from the z dependence of τ_{eff} . Under certain circumstances, however, the variation with z of the resonance line can cause an appreciable contribution via the term $\partial J_i/\partial z$, but we shall deal with these cases as they arrive. Provided loss of particles from the trap is not significant, we may put, in some average sense;

$$\partial/\partial z \tau_{eff} = -1/|V_{res}|$$

Using this result enables us to work out an expression for dw/dt as follows; (using eqs 30, 41)

$$\gamma_{\omega}^* = \frac{dw/dt}{\omega_0 \gamma_0} = \frac{R^{*1/2} (2\beta - \bar{\omega})^3}{4\pi^2 |V_{res}| \beta^2} \left\{ \frac{\beta \bar{\omega}^7}{\bar{k}^5} \right\}^{1/4} \cdot$$

$$\iint |\chi|^{7/2} \cos P \sin P \cdot \tilde{F}(P, P_0) \left[\frac{\partial F_0}{\partial W} + \left(1 + \frac{\bar{k} |V_{res}|}{\bar{\omega}} \right) \right] \cdot$$

$$\frac{1}{\beta} \frac{\partial F_0}{\partial \mu} \Big|_{V_z = V_{res}} d|\chi| dP \Big/ \int |\chi| d|\chi| F_0 (V_z = \beta = 1) \quad (50)$$

Equations 49 and 50 are the ones to be numerically integrated on the computer. The numerical techniques are straightforward and need not be discussed.

2. The zero order distribution function F_0 .

The choice of a suitable distribution function F_0 for these calculations is no easy matter. Clearly there will be as many different types of emission behaviour as there are distribution functions. The form of F_0 finally chosen is shown in fig 20. Factors guiding this choice are as follows.

Following Kennel and Petschek we suppose that F_0 consists of a quasi static equilibrium in which high energy particles are continuously injected-probably by some convective mechanism-and diffuse towards the loss cone under the influence of whistler turbulence. The cyclotron resonance mode tends to give a pure pitch angle diffusion, (Kennel and Petschek), and gives rise in the weak diffusion limit to a pitch angle dependence of the form $\log(\sin\alpha/\sin\alpha_0)$ where α_0 is the loss cone angle. We adopt this form even though the strong diffusion limit-with appreciable particle densities inside the loss cone-is often more appropriate.

Another possibly important source of diffusion is electrostatic resonance with whistler components whose k vectors are not exactly parallel to B , but lie within a narrow cone centred about that direction. At any point it can be shown that

$$\partial/\partial k (V_{res}) \equiv \partial/\partial k (\omega/k) = 0$$

at $\omega/k = 1/2$, and we expect electrostatic diffusion to be particularly intense at the resonance velocity corresponding to this frequency. Thus the electrostatic diffusion in the V_z direction will show a sharp peak at an equatorial V_z of

unity, tailing off towards higher velocities. At present it is not very clear what effect such an intense band of diffusion will have upon F_0 , except that some kind of discontinuous behaviour is to be expected. Here we construct a rather arbitrary model.

For equatorial V_z above unity we impose a realistic fall off with energy as $1/V_z^4$. Discontinuous behaviour is introduced by having a weaker exponent at lower velocities - in fact $1/V_z^{1.4}$. There is also a fall off as $1/V_\perp^4$ when

$\mu > 1.5$. Onto this high energy particle distribution we add a thermal type component varying just as $1/W^3$. The

final form of F_0 , expressed as a function of μ and W is;

$$F_0(W, \mu) = \log\left(\frac{30\mu}{W}\right) \left(\frac{\mu}{1.5}\right)^{-m} \left\{2(W-\mu)\right\}^{-n} + \frac{.16}{W^3} \quad (51)$$

$$\begin{array}{llll} \mu > 1.5 & m = 2 & 2(W-\mu) > 1 & n = 2 \\ \mu < 1.5 & m = 0 & 2(W-\mu) < 1 & n = 0.7 \end{array}$$

No attempt will be made to disguise the arbitrary nature of the above choice for F_0 . However I am sure the reader will appreciate the difficulty of constructing a distribution function F_0 that is (a) Analytic (b) Bounded in velocity space (c) Incorporates an overall fall off with energy as $1/W^2$ and (d) Nonetheless has sizeable volumes in velocity space in which the gradient is of a pitch-angle character.

3. Computed results for the growth rates γ^* , γ_w^*

The results for γ^* , γ_w^* are illustrated in graphs 23-29, and on the whole these will speak for themselves. A number of important comments, however, should be made. Referring to equation 49 we see that instability comes from a positive value for $\left(\frac{\partial F_0}{\partial \mu}\right)_w$, whereas the usual fall off

with energy $\partial F_0 / \partial W$ will tend to give wave damping. Thus the phenomenon of triggering instability is due entirely to the presence of a loss cone, and the sharper the pitch angle gradient the more unstable the system will be. Even so though, if the fall off with energy $\partial F_0 / \partial W$ is too rapid, the instability may be quenched. For particles of low pitch angle $\frac{\partial F_0}{\partial \mu}$ will almost certainly be the dominant term, whereas for high pitch angles $\frac{\partial F_0}{\partial W}$ will tend to be more important. Much of the variation of γ^* with z^* , ξ^* , is due to the balance between these opposing contributions.

An interesting point is the sign of the contribution from gradients in V_z , such as might result from electrostatic effects. Inspection will show that a decrease in F_0 with V_z will give wave growth, hence a sharp drop in F_0 at $V_z=1$ could give enormous growth rates. This could be important in the triggering phase.

Typical graphs of the variation of γ^* with z^* , ξ^* are shown in figs. 22-26., for two values of the power flux $R^*=0.00002, 0.000005$ and for three frequencies $\alpha^*=$

0.4, 0.5, 0.6. Note that the pattern of γ^* is hardly altered by changing R^* , but that its extent along the axes is directly proportional to this quantity. Thus reducing R^* tends merely to shrink the diagram towards the origin. Also in surprising contrast to Helliwell's predictions the region of maximum growth is found close to the equator and for rising and falling frequencies.

It was found that the general pattern of the dependence upon z^* and ξ^* was not critically dependent upon F_0 , and always had much the same form. In contrast the

"amplitude" of the pattern, and in particular its dependence upon α^* , were very sensitive. The variation with α^* for the particular F_0 we have used is shown in fig. 27. A feature of almost any reasonable F_0 is the rapid fall off in growth rate with decreasing frequency. This is because the resonance velocity becomes larger and the supply of resonant particles diminishes accordingly. Also anticipated is a fall off of growth rate at high frequencies, $\alpha^* > .75$. This is due to the increasingly thermal character of the distribution function at low energies.

The graphs show that γ^* is typically of order unity. In other words the second order resonant growth rates are of the order of the linear growth rates when T_{eff} is one trapping period. The procurement of very large growth rates from second order resonant particle trapping is thus entirely dependent upon keeping particles trapped for many trapping periods.

The results for γ_{ω}^* are very similar in their dependence upon $z^*, R^*, \xi^*, \alpha^*$, except for an overall dependence on R^* . A sample plot of γ_{ω}^* is shown in fig 26 for $R^* = 0.00002, \alpha^* = 0.5$. Note the two symmetrical zones in which the wave frequency is made either to rise or fall. The rising zone is on the receiver side of the equator, at least in the initial stages when ξ^* is small.

At this point we note that the entire description of VLF emissions remains the same if the direction of the static magnetic field is reversed. This merely changes the direction of rotation from right to left.

CHAPTER 6

THE TRIGGERING PHASE

1. The initial growth rate.

We now turn our attention to the triggering phase of the process-or in other words we examine the growth rates and rates of change of frequency to which the morse pulse is first subjected on entering the equatorial zone. To examine this initial phase we may put the power flux constant at some value $R^*=R^*_{in}$, the frequency constant ($\alpha^* = .5$), and suppose that ξ^* is initially very small or zero.

Inspection of the graphs points to two main regions of growth, one on each side of the equator. The actual growth rate will be proportional to T_{eff} , and will thus reach a maximum on the transmitter side of each zone. This value is estimated by

$$\gamma \sim \gamma_0 \gamma_m^* \Delta Z^* / R^{*-1/2}$$

where ΔZ^* is the maximum effective trapping length, which will be the characteristic width of the growth region. We have already noted that the latter quantity varies as R^* , and so the maximum initial growth rate will vary as $R^{*1.5}$. Now when R^* is sufficiently low ($R^*=R^*_0 \sim 0.000001$) we see that the permitted interaction region is only one trapping length long. This amplitude represents the lower limit of validity of the present theory, and is where the growth rate is of the order of the linear value. It may be that R^*_0 is the lowest amplitude that can successfully trigger, though it is possible that linear amplification mechanisms could

could first raise a weaker signal up to this level.

The maximum initial growth rate is conveniently expressed as follows

$$\frac{dR^*}{dt} \sim \gamma_0 \gamma_m^* \left(R^*/R_0^* \right)^{5/2} \quad (52)$$

where γ_m^* is defined as being the maximum value of γ^* along the z^* axis. Clearly equation 52 prophesies an almost explosive kind of instability, but the situation is rather more complicated than that. As the morse pulse passes the equator amplification occurs, but the group velocity tends to advect away the larger signal, and reduce R^* to that of the incident wave. The equation governing the time development of R^* in the triggering zone would be more likely to be of the following rough form.

$$\dot{R}^* \approx \gamma_0 \gamma_m^* \left(R^*/R_0^* \right)^{5/2} - \frac{V_g (R^* - R_{IN}^*)}{500 R^*} R_0^* \quad (53)$$

Once R^* reaches a certain critical value the growth rate will be sufficiently large to form a self sustaining generation region of an emission. The growth on the receiver side of the equator will develop into a riser, with a faller on the other side. A given morse pulse can excite one or both of these, and once the frequencies have separated, they will be completely independent entities-uninfluenced by each other or by subsequent incident waves from the transmitter.

2. The triggering delay.

The building up of the amplitude R^* to a self sustaining value before the end of the incident pulse is obviously a complex process, and it would be very difficult to estimate the time it takes for this to happen-or the delay

time as it is called. However we expect it to be mainly dependent upon morse amplitude R^*_{in} , and also $\gamma_0 \gamma_m^*$.

Clearly, the smaller is R^*_{in} , the longer will the delay time be. Although this remark is only qualitative, we can now get a physical picture of certain observed aspects of triggering. The fact that morse dots often fail to trigger, where the dashes do, is simply due to the fact that the time delay lies in between the duration of a dot and a dash. The triggering by the very weak, one second long Omega pulses can be viewed as a very gradual build up of wave energy in the equatorial zone, probably involving the magnetospheric reflection of wave energy.

One also expects the delay time to be strongly dependent upon the product $\gamma_0 \gamma_m^*$. If this is too small the second term in equation 53 will dominate, and there will be no build up of amplitude. On the other hand if it is large we would expect tremendous emissions to be produced in a very short time. However extensive analysis of delay times for a large number of cases (Helliwell) shows that this quantity is far more consistent in its behaviour than we would expect from the above considerations. The answer to this point may be as follows.

Owing to the loss cone distribution function, the system will be linearly unstable and generate broad band turbulence (Kennel & Petschek). The linear growth rate will depend upon the resonant particle density and also on the pitch angle anisotropy, and will vary with F_0 in very much the same way as $\gamma_0 \gamma_m^*$. Concentrating for a moment on

on turbulence with $\omega \sim \Omega_{eg}/2$, we see that if the integrated linear growth along the field line more than replaces the loss of wave energy on reflection at the ionosphere, the system will be absolutely unstable at that frequency. In such a case a burst of turbulence would result, which would reduce the resonant particle population to a level where stability was attained again. We thus expect an approximate upper limit to the quantity $\gamma_0 \gamma_m^*$, for a given R^* , and under active conditions it may remain almost constant at this value. Such considerations do seem to explain the consistent behaviour of the time delay at a given R^* , and also what seems to be a rough lower limit on this quantity. In fact, anomalously short delay times are never observed.

3. The frequency dependence of triggering

The frequency dependence of the initial power input is shown in fig 28, which plots approximate maximum growth rate in the trigger zone at a fixed power flux and as a function of frequency. The distribution function we have chosen peaks this around $\alpha^* = .5$, but in practice it is found that triggering is consistently and sharply confined to frequencies around $\alpha^* = .48$. This suggests the operation of some specific factor other than the general form of F_0 . Three possibilities might be suggested:-

1. The most likely explanation is as follows. The frequency $\alpha^* = .5$ is a limiting one for ducting (Helliwell), and above this value a pulse will become unducted in the equatorial zone, giving a considerable loss of wave energy. Below

$\alpha^* = .5$ we have already seen that for most distribution functions the initial power input to the field falls off rapidly with decreasing frequency. Thus we might expect triggering to be most likely just below the upper limit for ducting, or at about $\alpha^* = .48$. One may ask how it is that the mature emission is able to rise to frequencies well above this limit. The answer to this may be twofold. In the first place there is a very large power input to the field in a generation region, more than enough to compensate for wave energy loss due to unducting. Secondly, it is very likely that the generation region, unlike the incident signal, is not confined to the duct, and probably acquires considerable lateral extent during its development.

Some supporting evidence for this view of the situation may be obtained from whistler data (Helliwell). Sometimes trains of whistlers are observed to have a distinct cutoff at the ducting limit $\alpha^* = .5$, but show signs of considerable growth immediately below this level.

2. Another effect which may be operative is as follows. We have already discussed the intense electrostatic diffusion in the neighbourhood of $V_z = \bar{v}_1$ at the equator. This could cause a step in the distribution function at this point and give enormous growth rates around $\alpha^* = .5$. In fact the peak in the graph here results from the discontinuous character of the F_0 chosen.

3. A third aspect of the problem is the long term effect of the previous morse pulses upon the distribution function. A theory of triggering based upon such an effect is at present

being developed by Dungey and Abdalla.

4. The offset frequency and the initial frequency behaviour

One feature of emissions often observed is the frequency gap between the morse pulse and the start of the emission. This is easily explained as follows. Let us consider the resonant particle current that first appears in the morse pulse in the equatorial zone. This is sketched in fig. 29 for the case $R^*=0.000005$, $\alpha^*=0.5$. At $z^*=1000$ the current is in antiphase to the electric field, but steadily rotates to become almost entirely reactive at the equator. The direction of rotation is the same on the other side of the equator, and at $z^*=-1000$ the current is again in antiphase. On average then this current will have a fractionally shifted wavelength of

$$\frac{\delta k}{k} \sim \frac{\pi/2}{\Delta z^*} \sim \frac{\pi}{400} \left(\frac{R_o^*}{R_{IN}^*} \right)$$

This shift is inversely proportional to R_{IN}^* and for small amplitudes may be quite large. For example, for $R^*=0.0000025$ we get

$$\delta k/k \sim .003$$

which corresponds to a frequency shift of about 100Hz., which is about the right value for the offset frequency.

The first extra fields generated by this current will have this shifted wavelength. If δk is small, the net result will be a steady change in the frequency and wavelength of the total field, this being proportional to $\partial \omega / \partial z$ and taking place in the manner described in Chapter 4.

However, if δk is sufficiently large, the original morse field and the fields generated by the currents will become separate entities as far as their subsequent development by resonant particle currents is concerned. The condition that this takes place is that the resonant velocity widths of the two waves become clearly separated in V_z . In this case each wave has its own set of trapped particles, and one wave does not appreciably interfere with particles trapped in the other. Such separation occurs when $R^{1/2} \gtrsim \delta k/k$ a condition satisfied for $\delta\omega > 50$ approximately. We also notice from fig 29 that $\delta\omega, \delta k$ are of the same sign on both sides of the equator. Hence, if separation does take place, both riser and faller will start from a frequency slightly higher than that of the morse pulse by an amount 50-400 Hz roughly.

As the amplitude of the current generated fields overtakes that of the morse, these start to develop independently, and of course undergo a steady change in frequency proportional to $\partial/\partial z (\partial x/E)$. The frequency of the riser will climb steeply away with a gradient $\propto \sqrt{R^*}$, because in this case the contribution to $\frac{\partial \omega}{\partial z}$ from the τ_{eff} dependence is reinforced by that due to the phase variation of the total current with z^* . In the case of a faller though these effects are of opposite sign, and in the early stages $\partial x/\partial z$ may in some average sense hover around zero. As the amplitude increases however, the contribution from τ_{eff} becomes dominant, This explains why offset fallers often

have constant or rising frequencies in their early stages.

CHAPTER 7

THE MAIN PHASE

1. The generation region

After triggering has taken place, the wave train settles down to a semi equilibrium condition characteristic of the mature emission phase. This is the kind of situation described by Helliwell, and is illustrated for a riser and a faller in fig. 30. The head of the wave train is known as the generation region, and is where resonant particle interaction is most energetic and effective. This region is characterised by a very large power input, which must be sufficient to maintain the wave envelope in a constant position. In addition the frequency in the generation region is steadily changing. Downstream the resonant particle interaction is weak or non existent, and the generated waves merely stream down the field line to be picked up by the receiver as a rising or falling frequency.

The generating region is a highly complex affair and it turns out to be practically impossible to describe its detailed structure using the γ^* , and γ_w^* diagrams. There are two main reasons for this. The first is that a condition of static equilibrium with a stationary wave envelope is never really achieved on account of the steadily changing frequency. The situation is more fluid with the wave field constantly adjusting itself to changing conditions. Secondly, inspection of the γ^* graphs shows that the generation region should be quite narrow-typically about 2000 units or 300 wavelengths. Thus we expect to find a

fairly large value for the quantity:

$$\frac{\partial E}{\partial z} \sim \gamma E / v_g \approx \frac{\gamma_c \gamma^*}{\omega_0} \left(\frac{\tau_{eff}}{R^{*-1/2}} \right) E \approx .005 E$$

However we recall that the calculations of $\gamma^* \gamma_\omega^*$ were made under the assumption that E, ω, k were all slowly varying with respect to a trapping period (~ 300), and they are thus not strictly valid in the present case. This breakdown is particularly noticeable at the tip where $E \sim 0$, as here the field is determined by the resonant particles rather than the other way around.

In fact the generation region will tend to behave like a single cohesive entity with a common frequency and rate of change of frequency. The current caused by the jet of second order resonant particles will have a relatively slow time response to the field ($\sim R^{*-1/2}$) and consequently its phase will not necessarily be that predicted from the instantaneous second order resonance conditions. It is this slow response of the current that imparts to the generation region its cohesiveness, since it will tend not to respond to sudden departures from the common frequency and wavelength. If however, a generation region should become too long, this argument may break down, and it may split up into two or more separate units and give rise to multiple emissions.

Although not much help with regard to structure, nevertheless, the $\gamma^* \gamma_\omega^*$ diagrams can still tell us a great deal about the generation region. Firstly, we expect the location of the generation region to be somewhere in the vicinity of the well defined maximum in γ^* , both for z^*

and ξ^* , in order that the power supply be sufficient.

Hence both riser and faller must be generated close to the equator—a result in surprising contrast to Helliwell's.

The rate of change of frequency ξ_{eff}^* of a generation region is difficult to estimate. As a first approximation we may write

$$\xi_{eff}^* \approx \gamma_0 \overline{\gamma_w^*} / \omega_0 (|V_g| + |V_{res}|)$$

where $\overline{\gamma_w^*}$ is some kind of average of γ_w^* over the generating region. It is convenient to introduce here the concept of the core of the g.r., which is the central section, of amplitude R^*c which may be regarded as controlling the frequency behaviour of the entire field. If this idea has any substance, we may put ξ_{eff}^* equal to;

$$\xi_{eff}^* \approx \frac{\gamma_0 \gamma_w^* (z_c^*, R_c^*, \xi_{eff}^*)}{\omega_0 (|V_g| + |V_{res}|)}$$

Note that if this relation is satisfied, then we must have a self consistent wave particle picture, which is really the basis of the present theory. The wave field with a changing frequency organizes resonant particles in a particular way, and generates the currents which cause that frequency to change.

The growth rate within the generation region is more readily estimated. We have approximately

$$\gamma \sim \gamma_0 \gamma^* \tau_{eff} R^{*1/2} \sim \gamma_0 \gamma^* \int R^* dz / R^{*1/2} \quad (54)$$

where the integration is from the given position to the high amplitude end of the g.r., and $\gamma^*(\xi_{eff}^*, z^*, R^*)$ is the instantaneous value. However, the jet of second order resonant particles

is actually driving into a region of rapidly diminishing amplitude R^* , and consequently sees a particle trap steadily shrinking in size, in both pitch angle extent and width in V_z . Owing to the slow response time of the current, it will tend to retain the magnitude it had one trapping period previously, and thus the growth rate will tend to be somewhat bigger than that predicted by eq 54., approaching infinity at the tip of the g.r. We again recall that γ must be sufficiently large to satisfy

$$V_g \partial E / \partial z \sim \gamma E$$

in order for a g.r. to be established.

2. Stability of the generation region.

In the previous section we have endeavoured to describe the generation region. We now consider the very important question of its stability. For instance, if there are small changes in overall amplitude, position, or ξ_{eff}^* , do these quantities tend to return to an "equilibrium" value? We may put this question in a more urgent way. We have already noted the explosive type of instability in the triggering phase, where $\dot{R}^* \propto R^{*5/2}$. Once triggering has been achieved, what factors intervene to limit this explosive growth and cause the amplitude to level off? This is not at all obvious, and I put forward the following tentative explanation.

A signal such as an emission will have a certain minimum bandwidth-i.e. E and ϕ will have a random variation about their mean. In practice it is observed that VLF emissions usually have a bandwidth largely in excess of

the theoretical minimum-typically of the order of 50-150Hz, but on occasions somewhat greater. The dominant cause of this broadening probably lies in irregularities in the cold plasma and the duct, and in time variation in the zero order distribution function F_0 . In addition the output of even an ideal generation region will have a distinct bandwidth related to the scale length of its internal structure. Whatever the origin of the bandwidth-and it is difficult enough to specify-it has a very adverse effect upon particle trapping. The randomness in E and ω will tend to shake particles out of their trap in a characteristic time which we call $T_{\max}$. When the overall amplitude is such that the time spent by a resonant particle in the generation region is of the order of $T_{\max}$, two things happen. Since T_{eff} has reached its maximum permitted value, the growth rate levels off. Secondly, since the rate of change of frequency is mainly derived from $\partial \omega / \partial T_{\text{eff}}$, there is a rapid fall in ξ_{eff}^* . Now, if for the value of R^* in the core of the generation region, R^*c , the value of ξ_{eff}^* lies below that of the maximum in $\gamma^*, \gamma_{\omega}^*$, a decrease in ξ_{eff}^* will result in a decrease in growth rate. This being so, the generation region can achieve a situation in which a decrease in amplitude causes an increase in growth rate-which is an amplitude stable situation. If the power input into the system is sufficiently reduced, such a stable configuration will not be possible, and the system becomes unstable and liable to spontaneous changes to other configurations.

3. Termination of emissions-hooks.

Consulting the graph (fig 27) showing the dependence of γ^* upon frequency, we see that around $\omega^* \sim .4$ the faller will suffer a drastic fall off in power due to the decreased supply of resonant particles. A similar thing will happen to the riser after a longer time - say at $\omega^* \sim .7$. The normal mode of behaviour in these cases is that the wave train escapes from the equatorial zone and the emission terminates. Sometimes the faller will slowly turn round and develop into a full length riser. Helliwell first pointed out how this happens. As the power input slowly weakens the generation region falls back towards the receiver and into the rising frequency zone. If ξ_{eff}^* has fallen sufficiently by this time the power input will be sufficient to hold the wave train there. As the frequency then starts to rise, power input revives, and a fully developed riser comes into being.

The converse process of the downturning of a riser is clearly less likely as the rising zone is already the downstream one. A suggestion for how a downward hook might be produced is as follows. Let us inspect the region between the two maxima in the γ_w^* diagram (fig. 26)

We notice that in the rising zone there is a sharp decrease towards the left in γ_w^* , and thus in J_i . This variation opposes that due to T_{eff} and will drastically reduce ξ_{eff}^* - conversely, in the falling zone ξ_{eff}^* is reinforced. These contributions to ξ_{eff}^* from the change in the resonance line

with Z^* increase with decreasing R^* , and it seems quite likely that a weak, unstabilised generating region will rapidly transfer from the rising to the falling zone. Both these processes are tentatively illustrated in fig 31. Helliwell suggested that risers turn into fallers as a result of a superfluity of power, which causes the generating region to spread upstream. It is possible that this is the reason, though if so it is difficult to see why downward hooks usually are so short.

An interesting point is that the concept of an upper limit on the time for which a particle may be trapped makes possible the production of emissions of zero or very small frequency gradient, since $\frac{\partial}{\partial Z} T_{eff} \sim 0$ in the body of the generating region. The appropriate configuration, shown in fig 30, will tend to be rather difficult of access, and is thus not often observed.

4. Discussion of numerical values.

At this point we see how observed numerical values for the frequency gradients of emissions fit in with the foregoing considerations. Before doing this we should point out that attempts to make accurate numerical predictions from this theory are not really possible, owing to the uncertainties about (1) The cold plasma density at the equator, (2) The distribution function F_0 , (3) The detailed structure of the generating region and (4) The amplitude of the morse pulse. However a rough check on numbers can be made, and they seem to fit

fairly well together.

The morse amplitude is well estimated by the offset frequency, which suggests $R^* \sim 0.000001-2$ roughly. The NAA pulses often do not exhibit offsetting, and their amplitude may be a bit higher at say $R^* = 0.000004$. The emission amplitude is observed to be about 10 times greater than that of the triggering pulse, so we suggest that R^* for the mature emission to be of the order $0.00001-2$. This will give a generating region of about 2000 units long, or 300 wavelengths. In so far as the quantity $T_{\max}$ exists, it is very much an unknown quantity, but let us suppose that it is of the order of 2000. The corresponding value of ξ_{eff}^* for a stabilised generation region is around $0.000005-0.00001$, which is in accord with observed frequency gradients for long time risers. Some short emissions have gradients rather steeper than this. Either their amplitude is very large, or else in this case the stabilisation we have discussed is neither present or necessary. Knowledge of ξ_{eff}^* permits an estimate of γ_0 .

$$\xi_{\text{eff}}^* \sim \gamma_0 \gamma_{\omega}^* \sim \gamma_0 / 1000 \quad \gamma_0 \sim .005$$

This is quite a large linear growth rate, but is not unexpected since for these frequencies the ionospheric reflectivity is quite small. For $R^* \sim 0.00002$, a resonant particle will spend about 5-10 trapping periods in the g.r., and the typical growth rate therein will be about $10 \gamma_0$ or 0.05. This represents a sufficient power input to maintain a generating region of 300 wavelengths in position.

CHAPTER 8

CONCLUSIONS

We shall conclude briefly by mentioning some of the possible flaws in the theory, and with suggestions for alternative or additional work.

I think the conclusion overall must be that a theory based upon second order resonance is able to give a reasonable account of VLF triggering and seems to explain quite a number of detailed features-though some of the explanations are a little tongue-in-cheek.

The theory rests upon a number of crucial assumptions, and if any of these prove untenable the theory may require substantial modification. The first is the representation of the wave field by a wave with k parallel to B_0 . It is well known however (Helliwell, Walter etc) that whistler propagation is a complex problem in its own right, and it may be that a proper theory will require more details of a waveform in a duct-in particular it may be necessary to have a spread of k vectors about the B_0 direction.

A crucial test of the theory will be the accurate measurement of the electric fields at the equator in a morse pulse and in the emission. The present theory demands that the amplitude be sufficiently high such that resonant particles remain trapped for several trapping periods. The fields required are about 10^{-5} V/m for the morse and 10^{-4} V/m for the emission. If observed fields are much below

these levels a low amplitude theory will have to be developed, but it is difficult to believe that this would give sufficiently large growth rates to maintain a generating region.

Certain aspects of the theory remain obscure. The sharpness of the triggering around $\alpha^*_{\sim .48}$ is puzzling, although several vague explanations for this were advanced. The chief difficulty however is in giving a proper description of the generating region, and in understanding its behaviour. Some of the ideas introduced in chapter 8 were highly speculative in character and may not stand up to detailed examination.

A great deal of additional work is really required on the nature of wave particle interactions in a narrow region in which amplitude varies rapidly. Another problem is that of the zero order distribution function, which warrants a good deal of theoretical and observational study.

Whatever the deficiencies this theory may have it is hoped that it may shed a little light upon a very complex phenomenon, and perhaps pave the way for more sophisticated analyses.

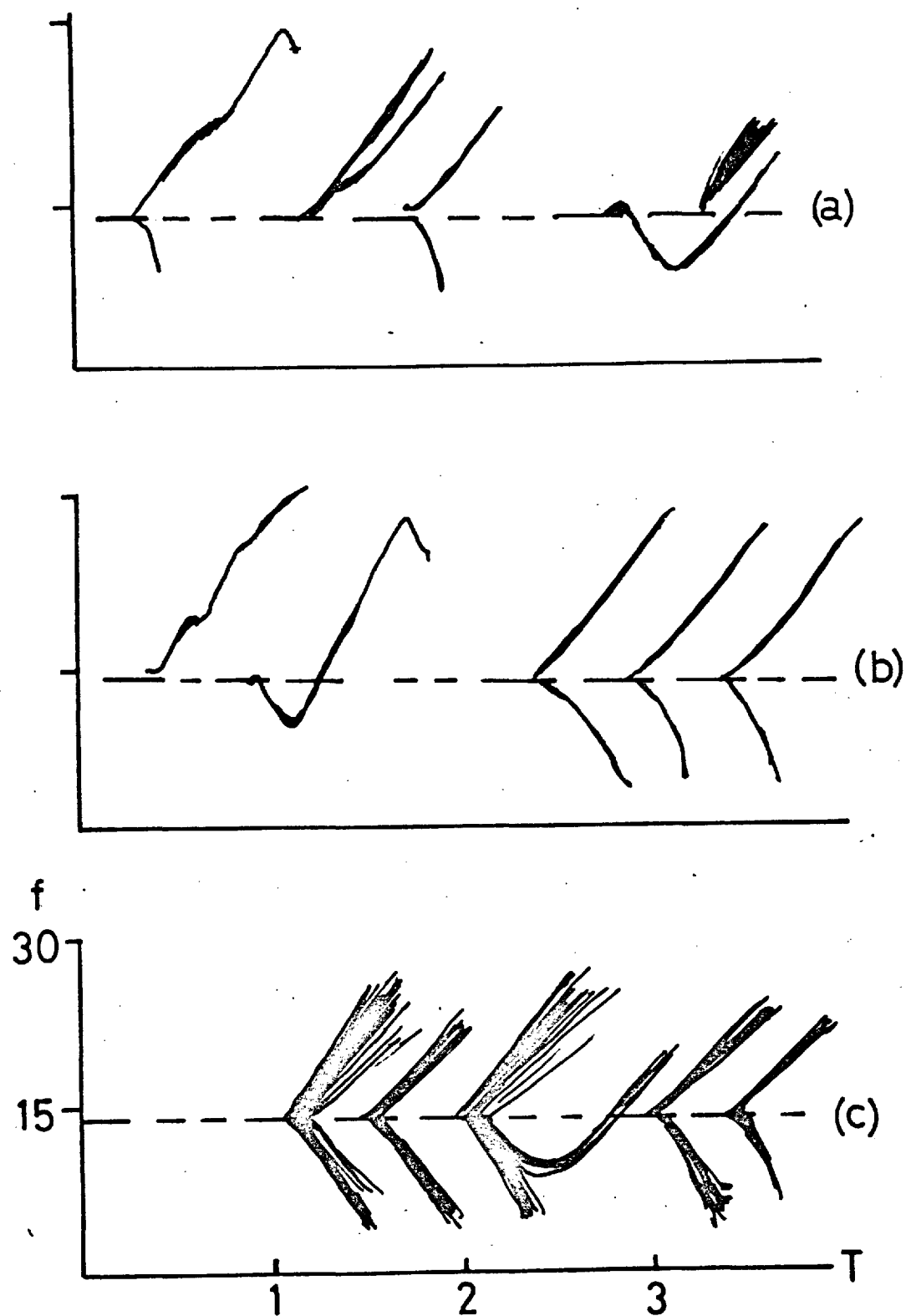


FIG 1

Fig. 1. Some examples of morse triggered VLF emissions on a sonogram, or frequency-time diagram.

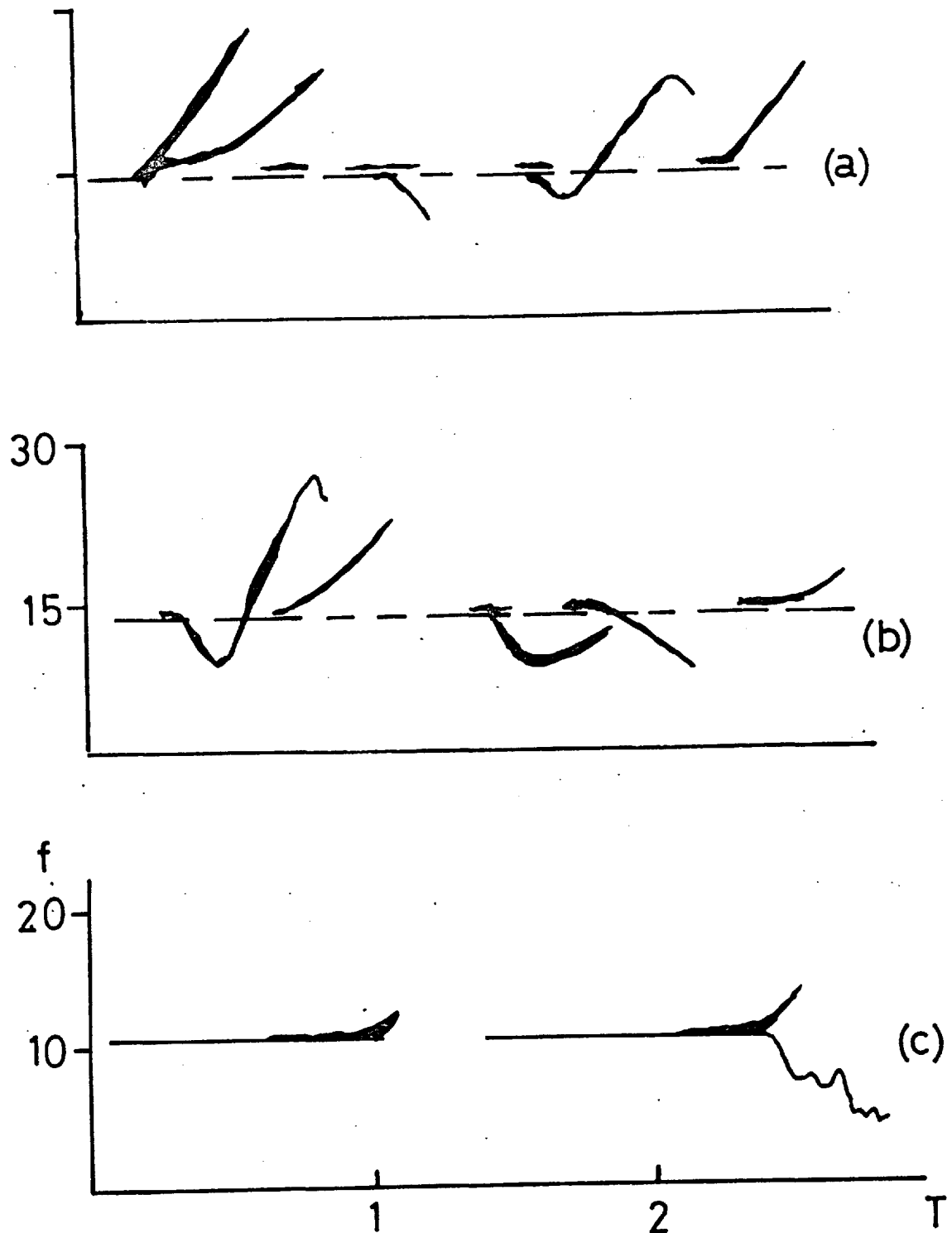


FIG 2

Fig. 2. Further examples of artificially triggered VLF emissions. Example (c) shows triggering by the 1 sec long Omega pulses.

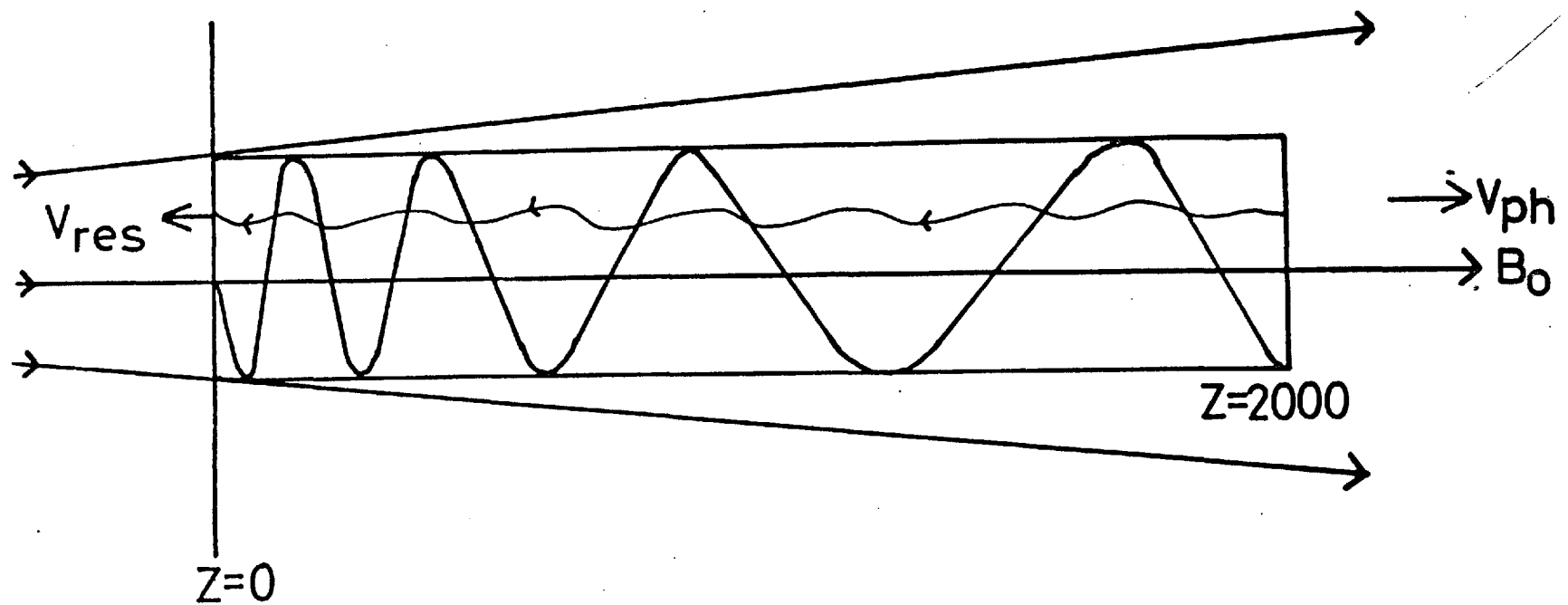


FIG 3

Fig. 3. A sketch of the wave packet used for the trajectory integrations.

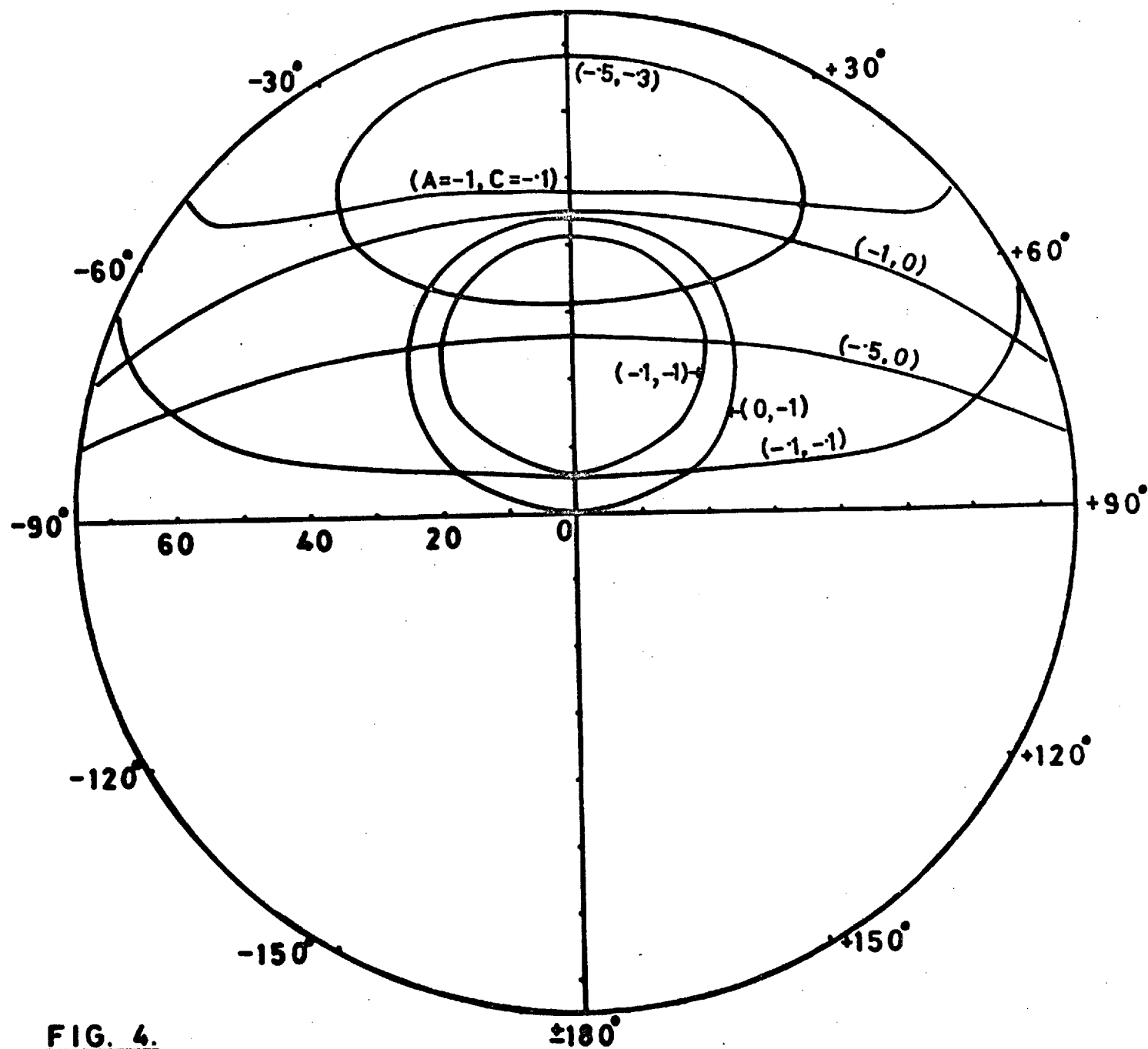


FIG. 4.

Fig. 4. Resonance lines in the Vres plane, for various values of the coefficients A and C.

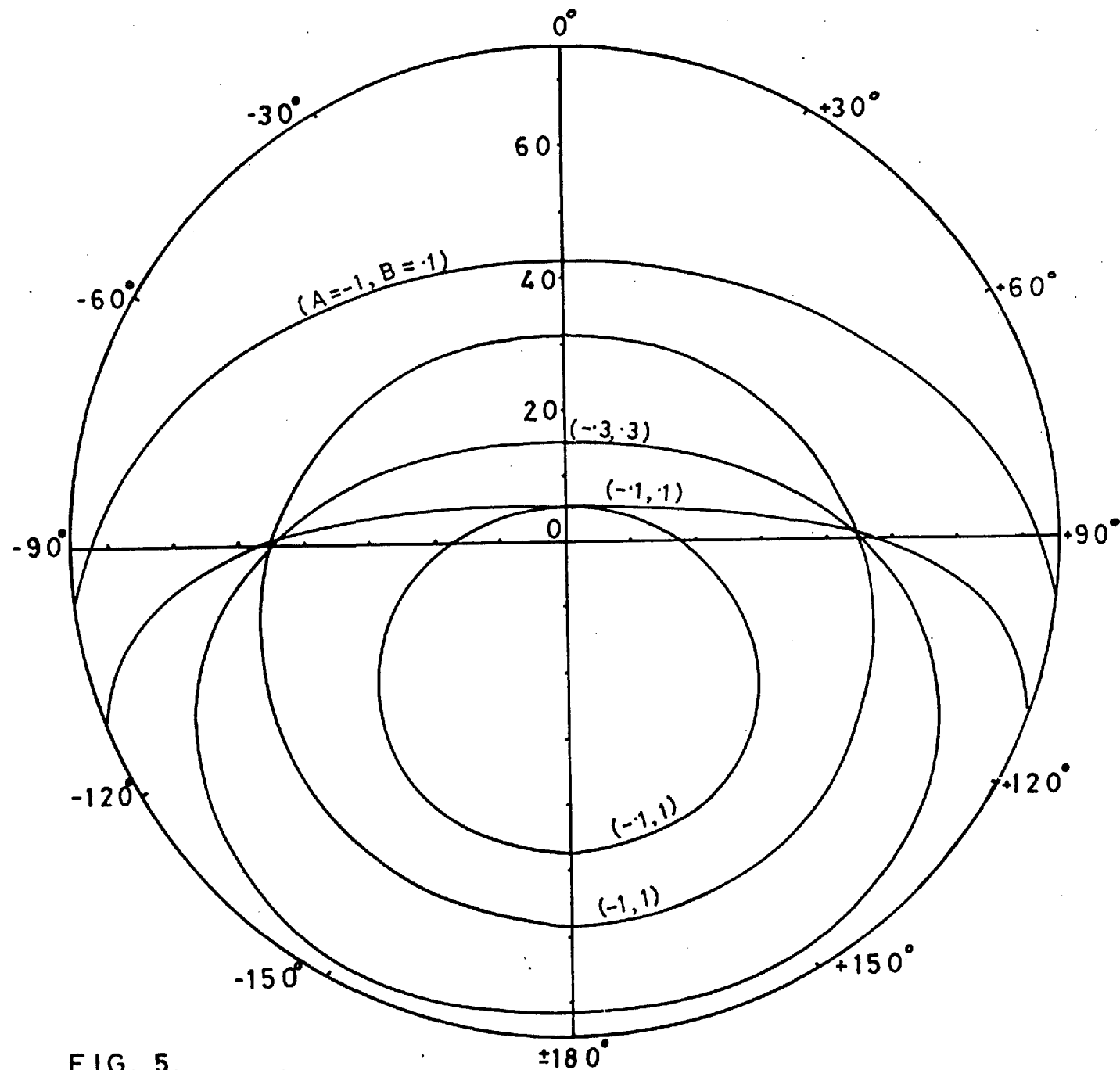


FIG. 5.

Fig. 5. Resonance lines for various A and C , where these are of opposite sign.

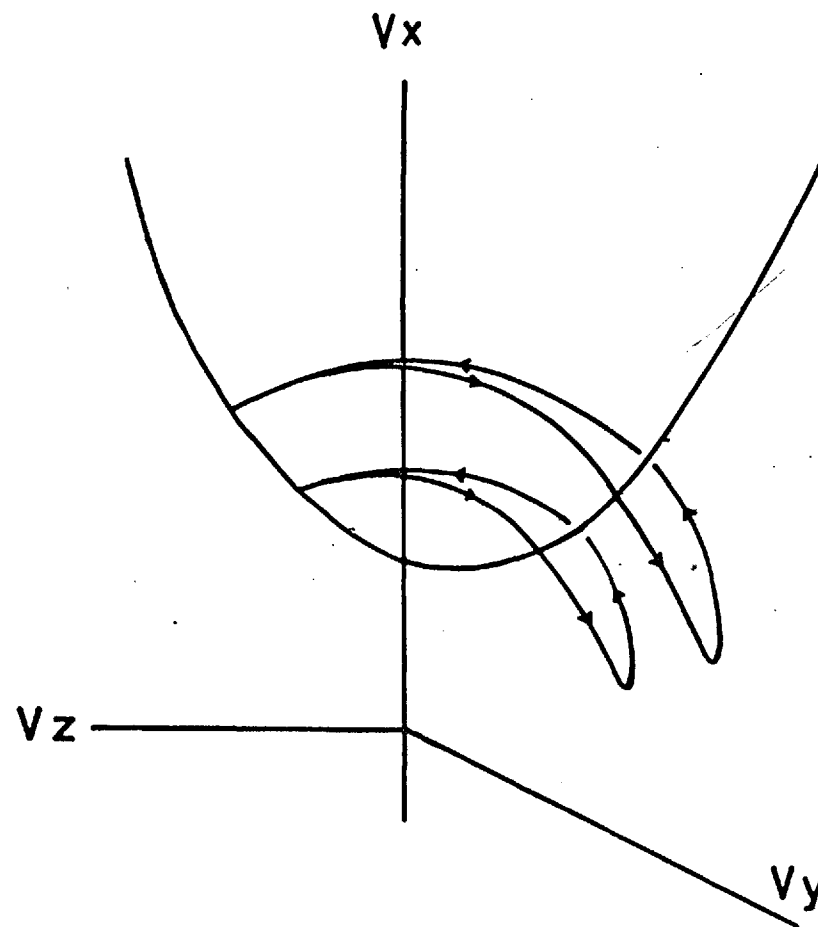
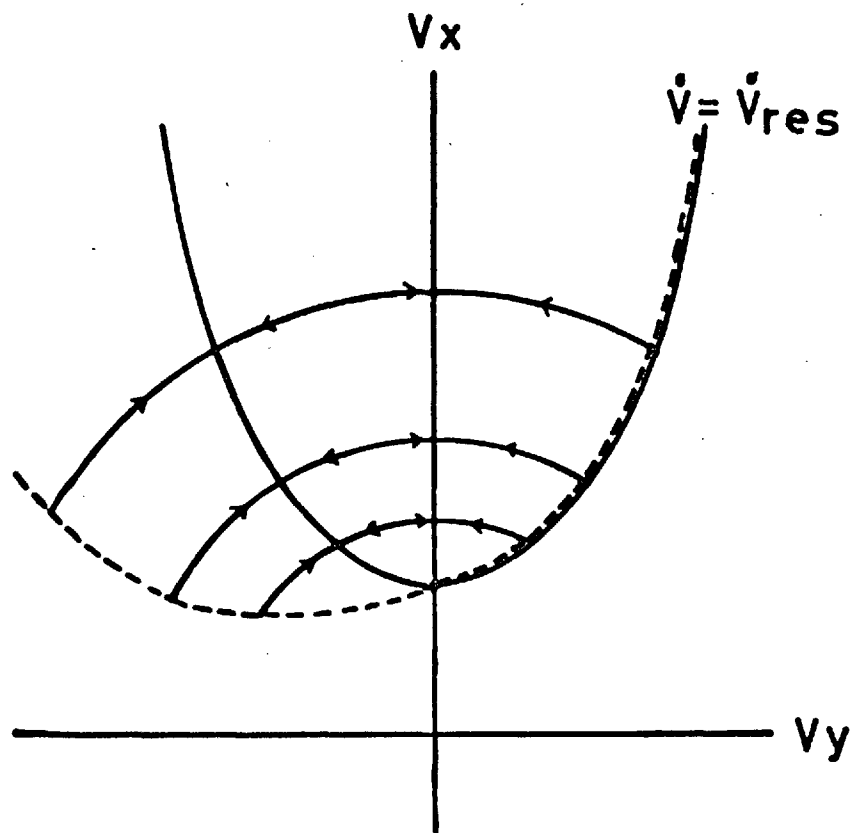


FIG. 6.

Fig. 6. Trajectories of stably trapped second order resonant particles, showing the method of constructing the trap in velocity space.

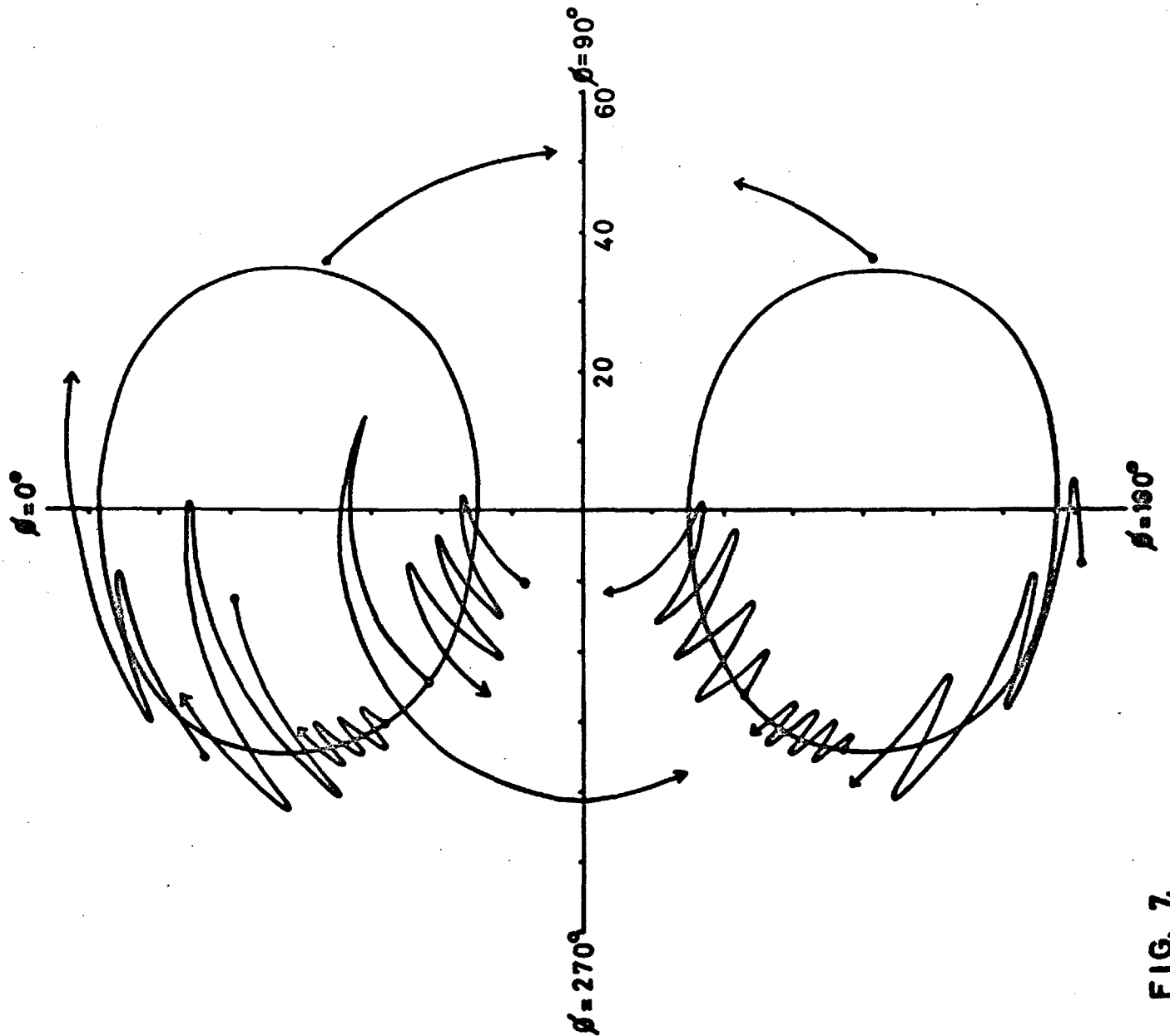


FIG. 7.

Fig. 7. Computed examples of the time variation of phase P and pitch angle α_p , for trapped resonant particles. The curves are two separate examples of the resonance line.

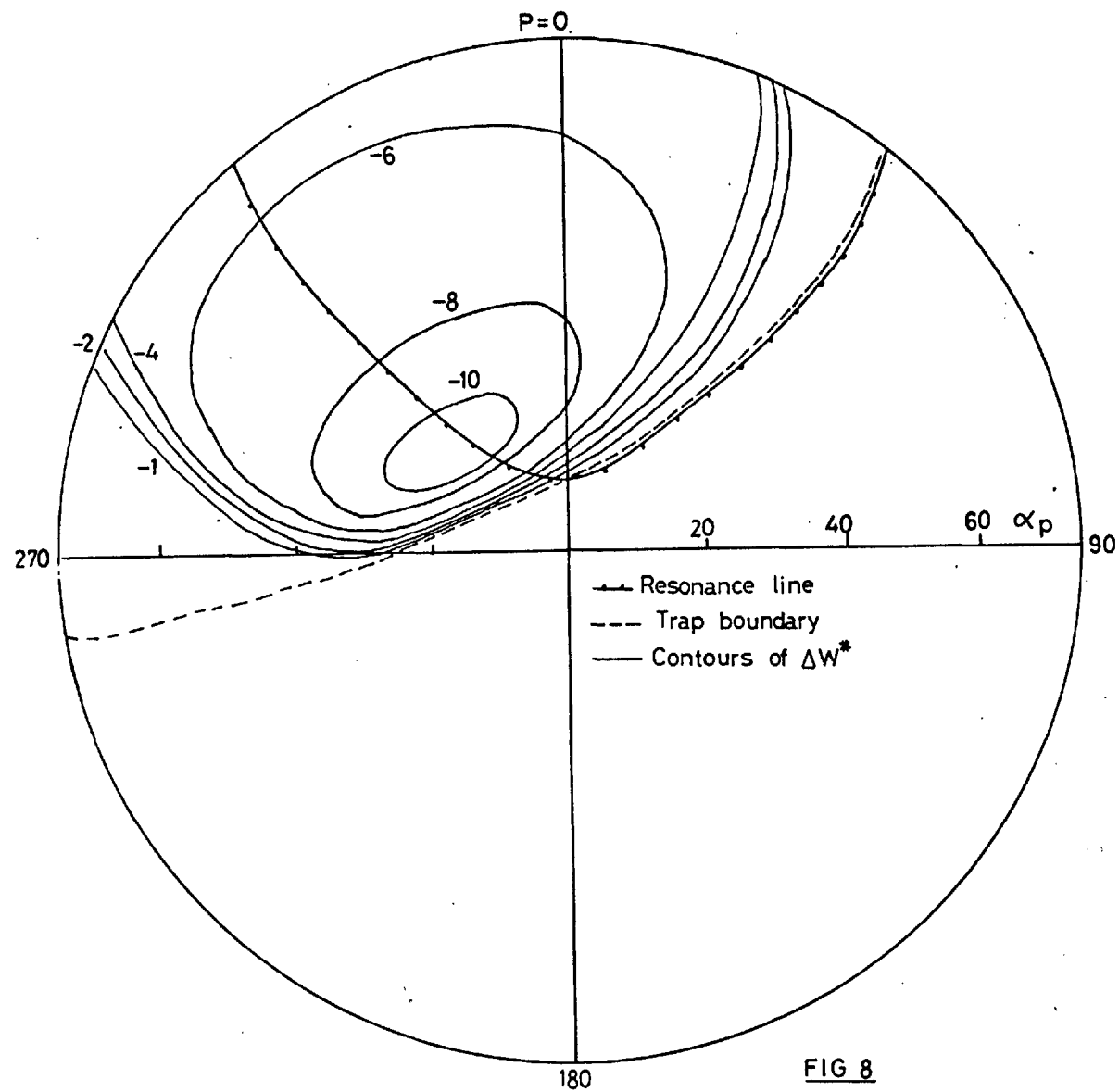


FIG 8

Fig. 8. Typical plot of ΔW^* in the resonance plane.

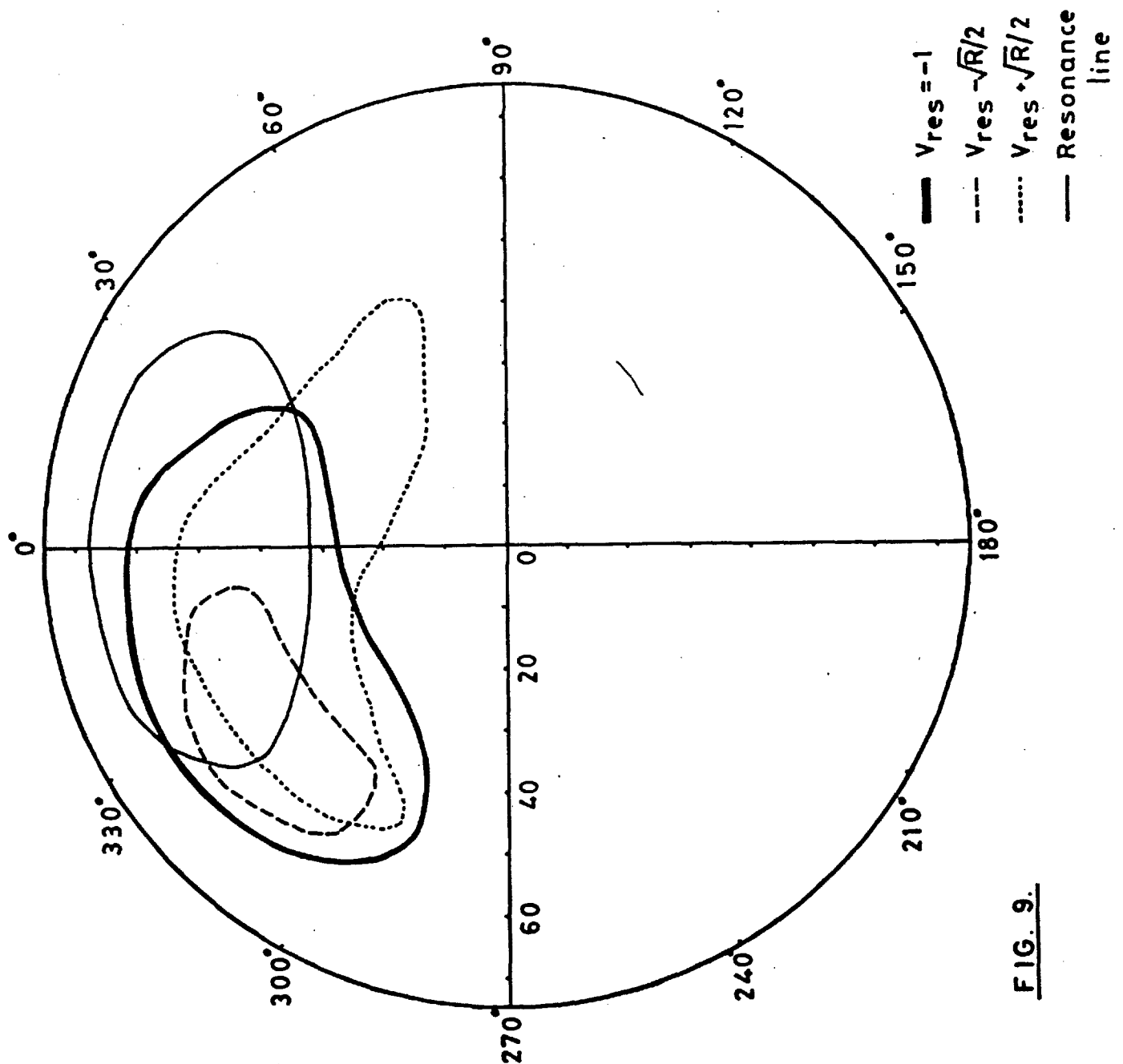


FIG. 9.

Fig. 9. Region in velocity space of effective particle trapping (large ΔW^*). The graph shows the boundary of the region in the $V_{\perp}$ plane for 3 values of V_z .

$$\eta/R = -0.6, \quad \xi/R = 0.25, \quad \alpha = 0.5$$

Fig. 10. Trapping region for $\eta/R=0$, $\xi/R=-0.25$, $\alpha=0.5$

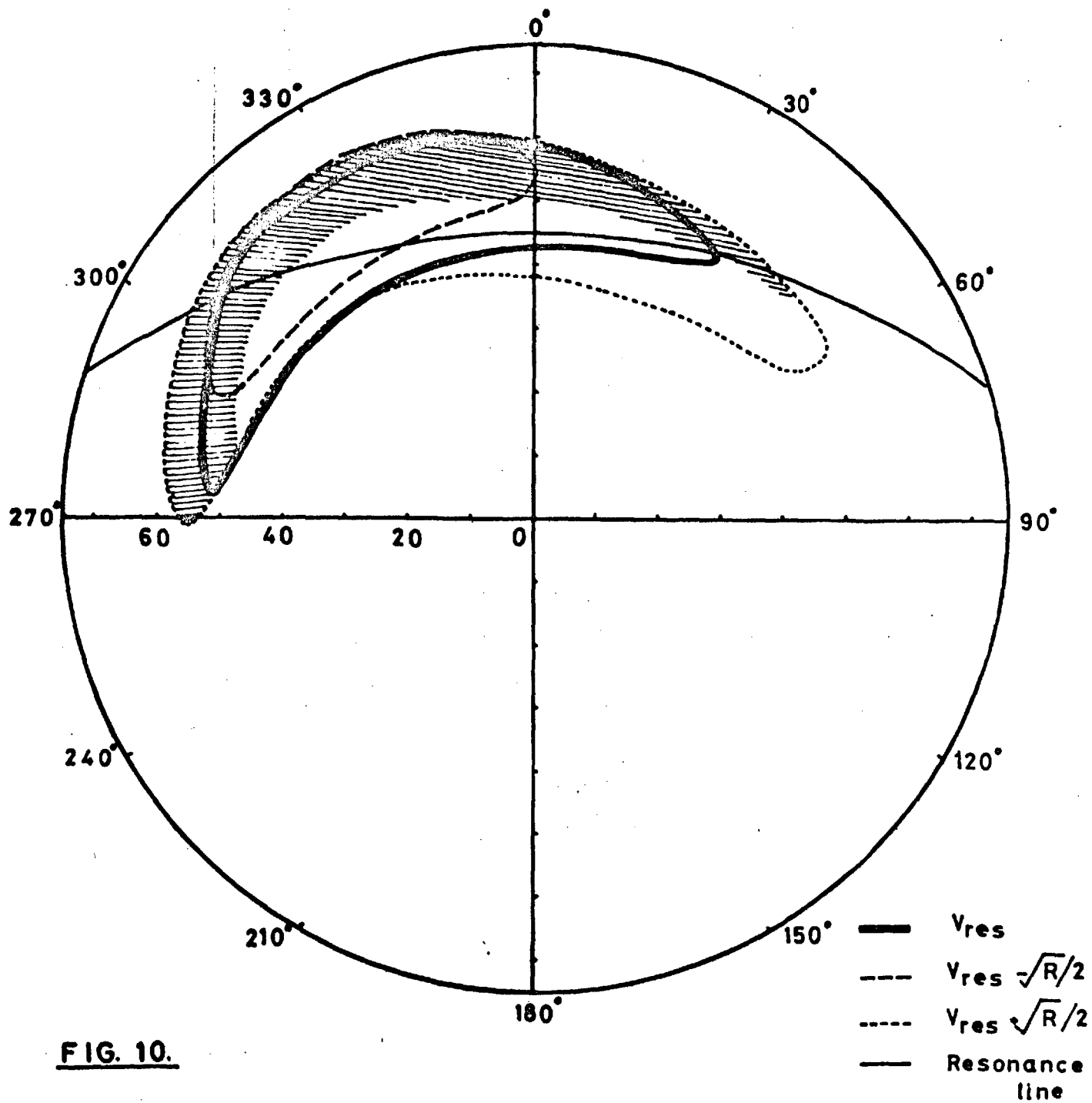


FIG. 10.

Fig. 11. Trapping region for $\eta/R=2$, $\xi/R=-1.275$, $\alpha=0.5$

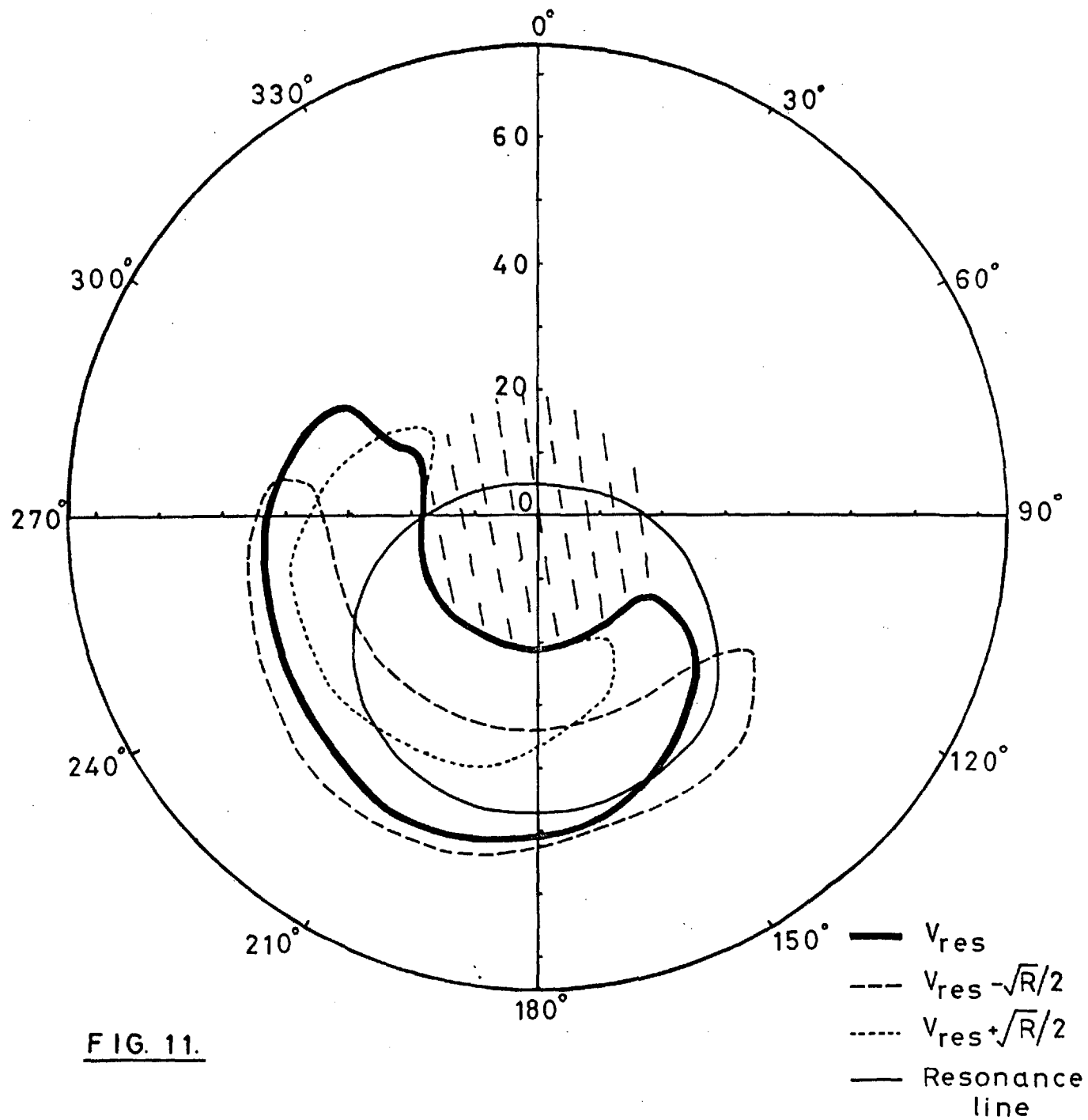


FIG. 11.

Fig. 12. Trapping region for $\eta/R = .2$, $\xi/R = .375$, $\alpha = 0.5$

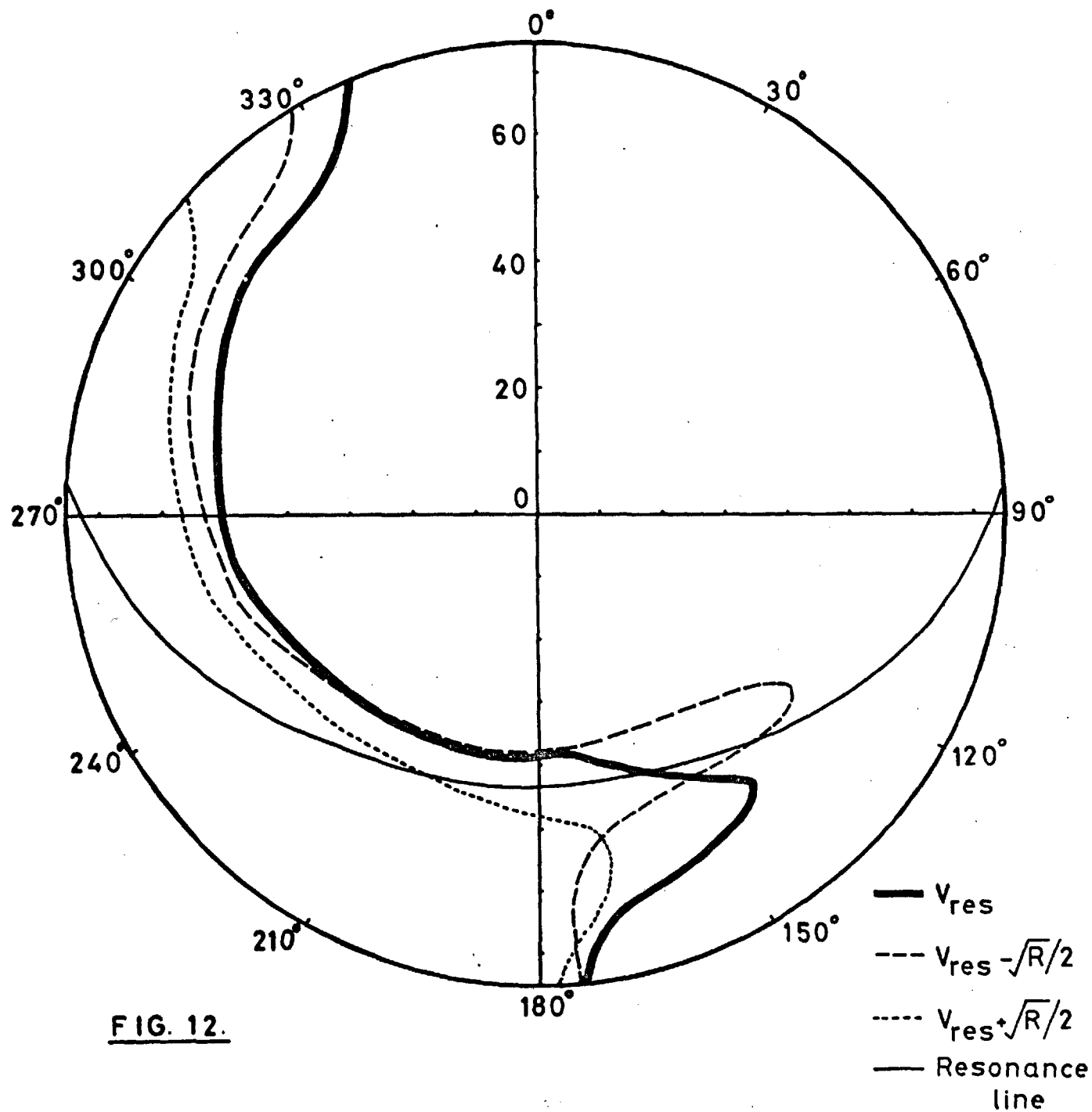


Fig. 13. Trapping region for $\eta/R = -.2$, $\xi/R = .1$, $\alpha = 0.5$

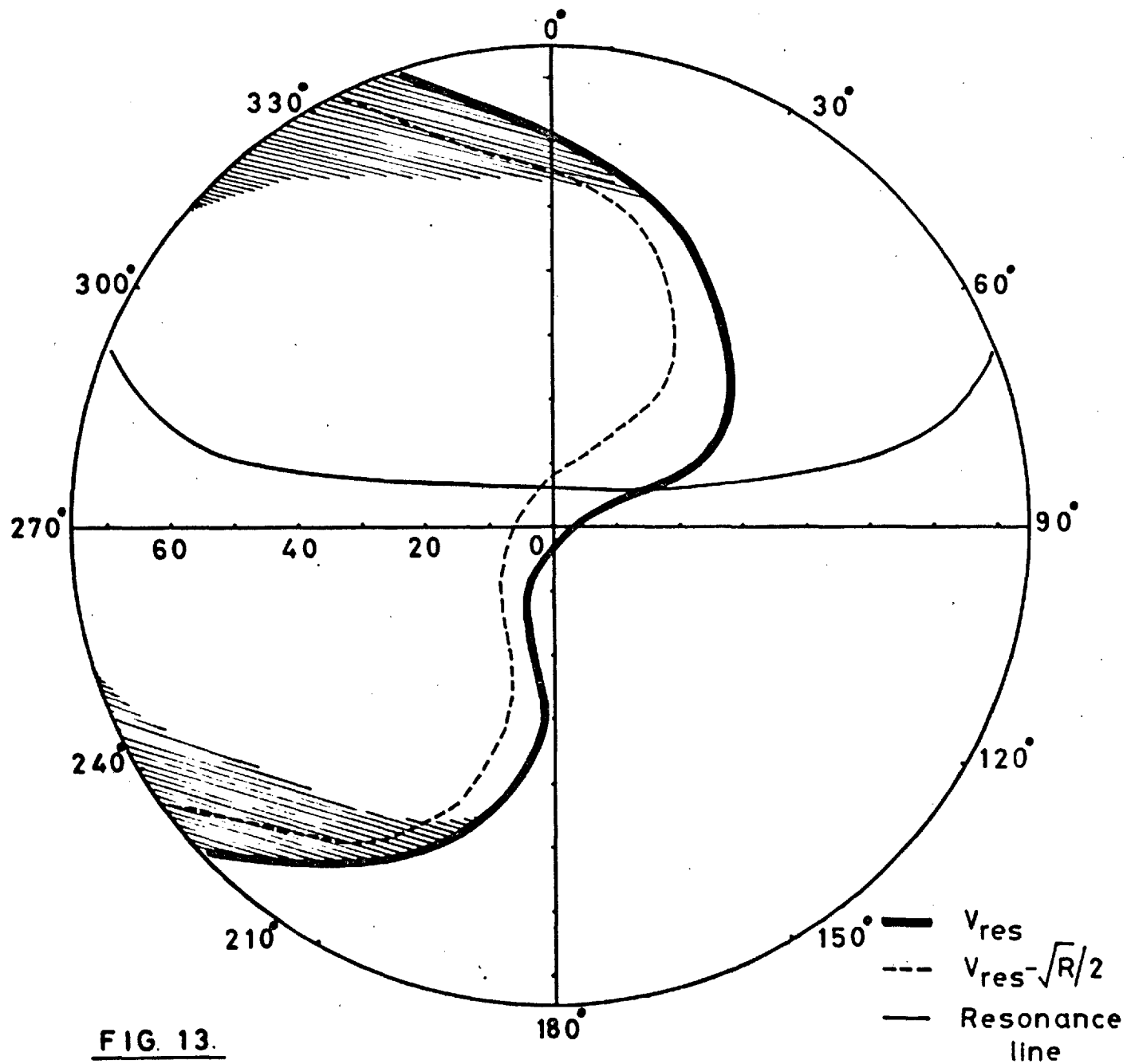


FIG. 13.

Fig. 14. Trapping region for $\xi/R = -.45$, $\eta/R = 0.6$, $\alpha = 0.5$

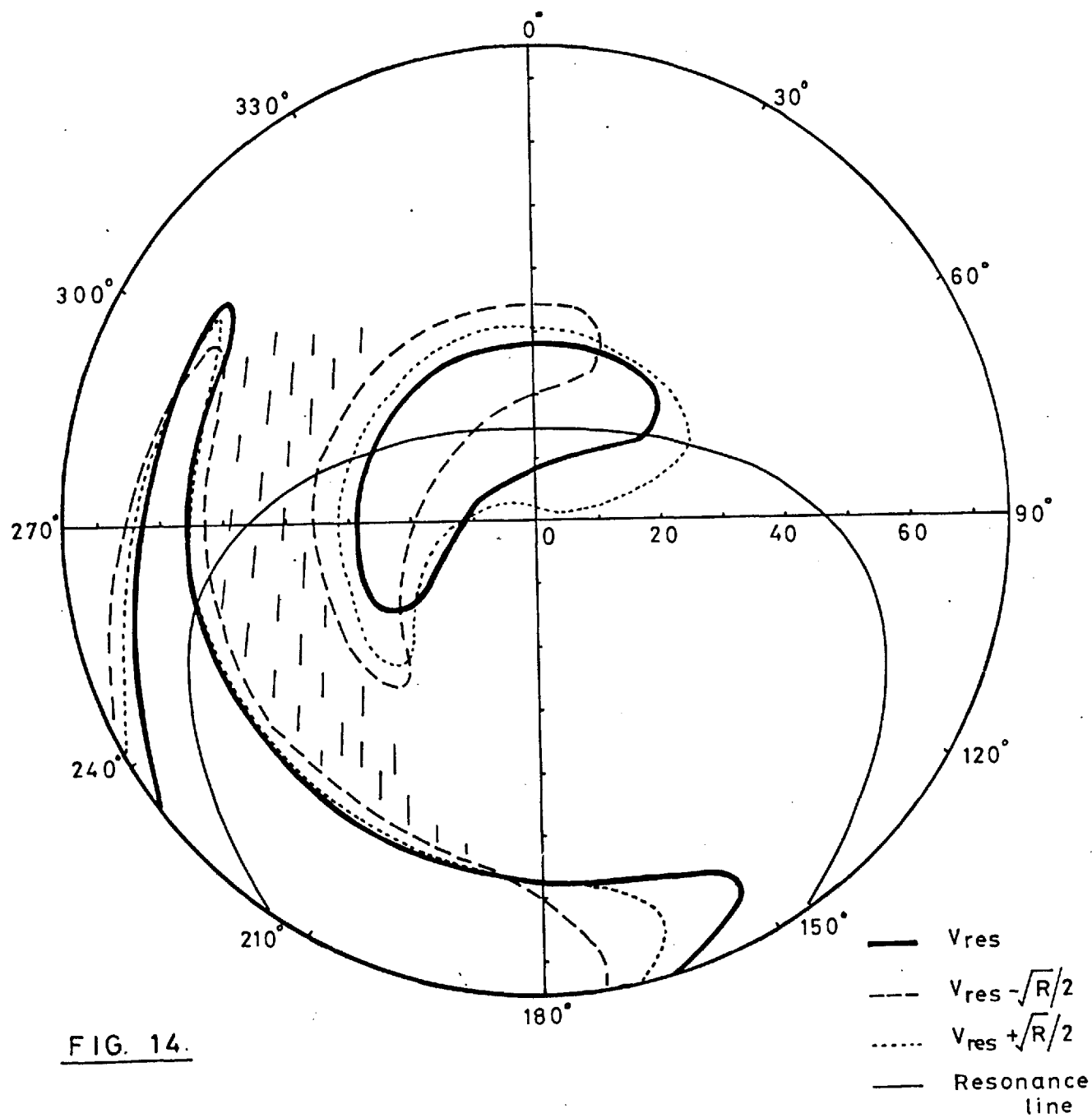


FIG. 14.

Fig. 15. Trapping region for $\eta/R=-0.066$, $\xi/R=0.0277$
 $\alpha=0.1$

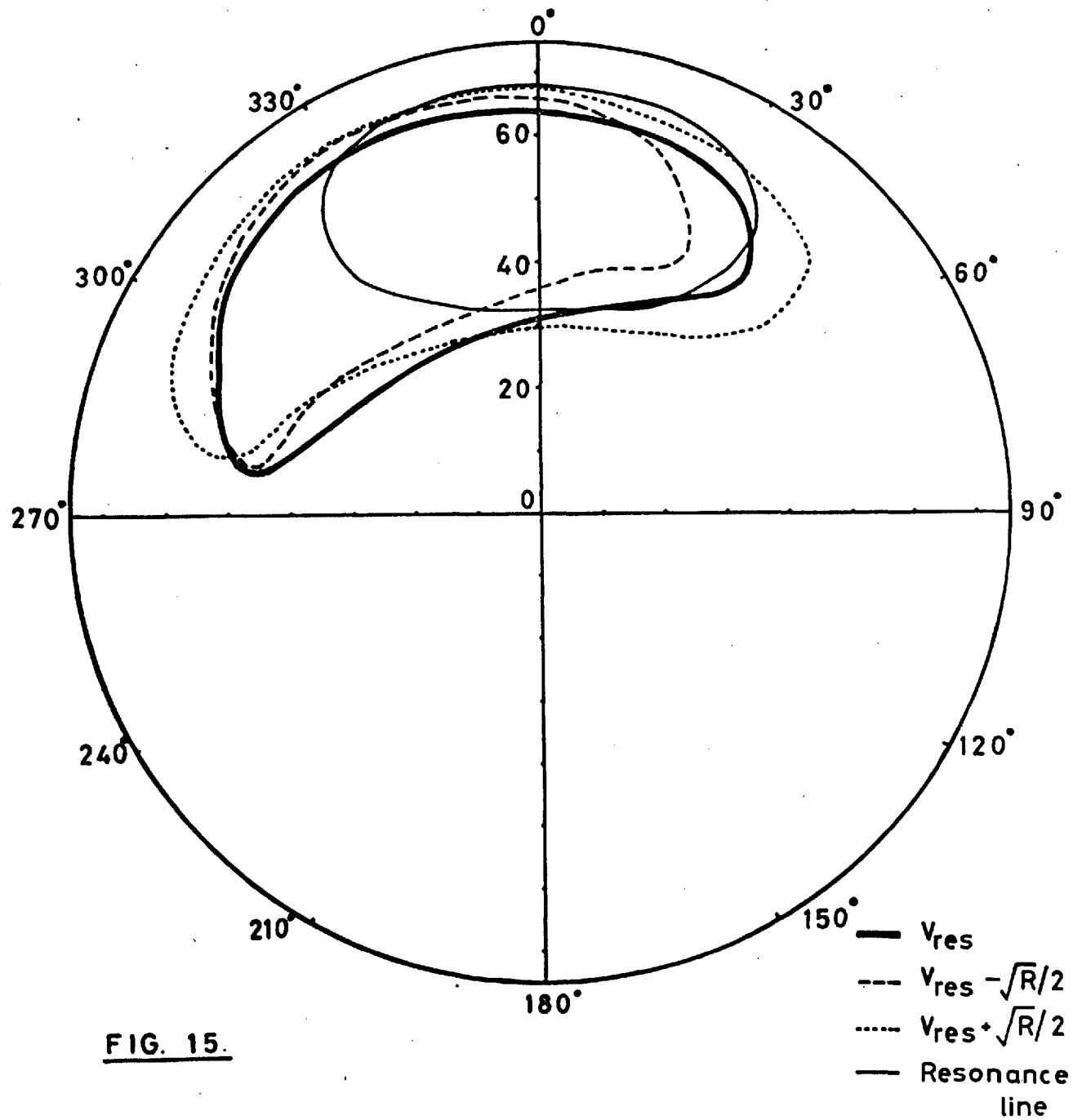


Fig. 16. Trapping region for $\eta/R=-2$, $\xi/R=1.25$, $\alpha=0.5$

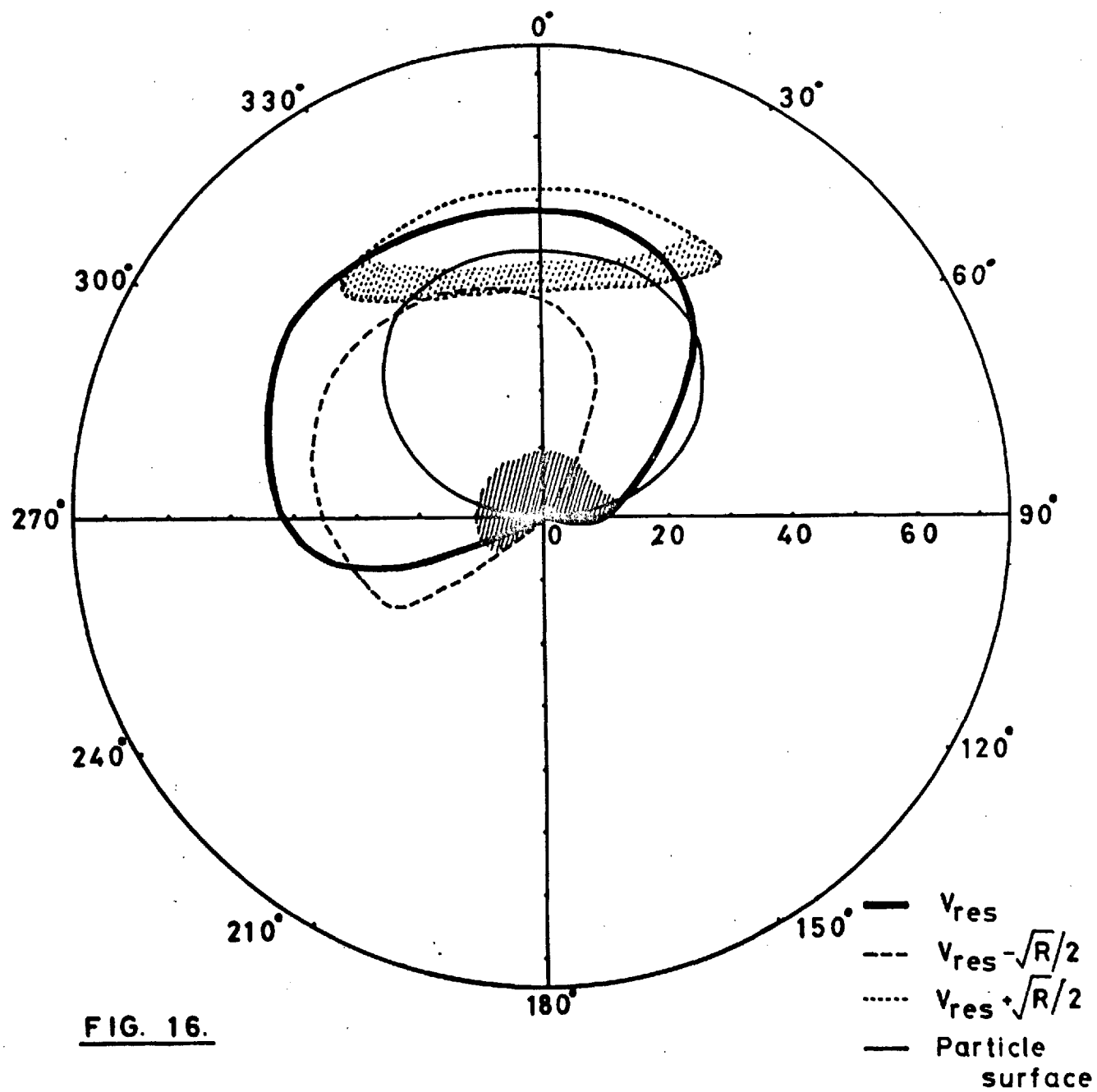
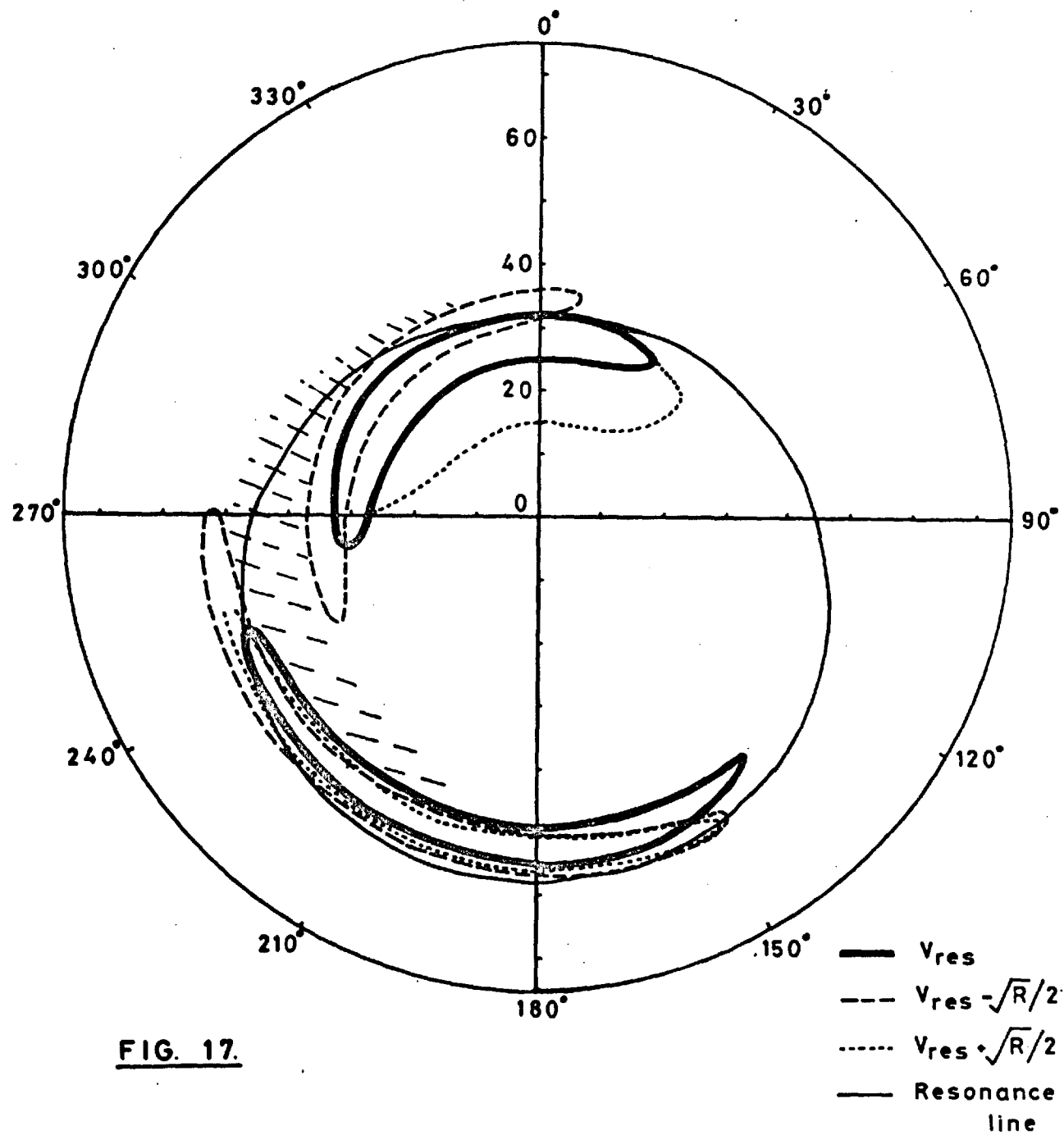


Fig. 17. Trapping region for $\eta/R=2$, $\xi/R=-1.5$, $\alpha=0.5$



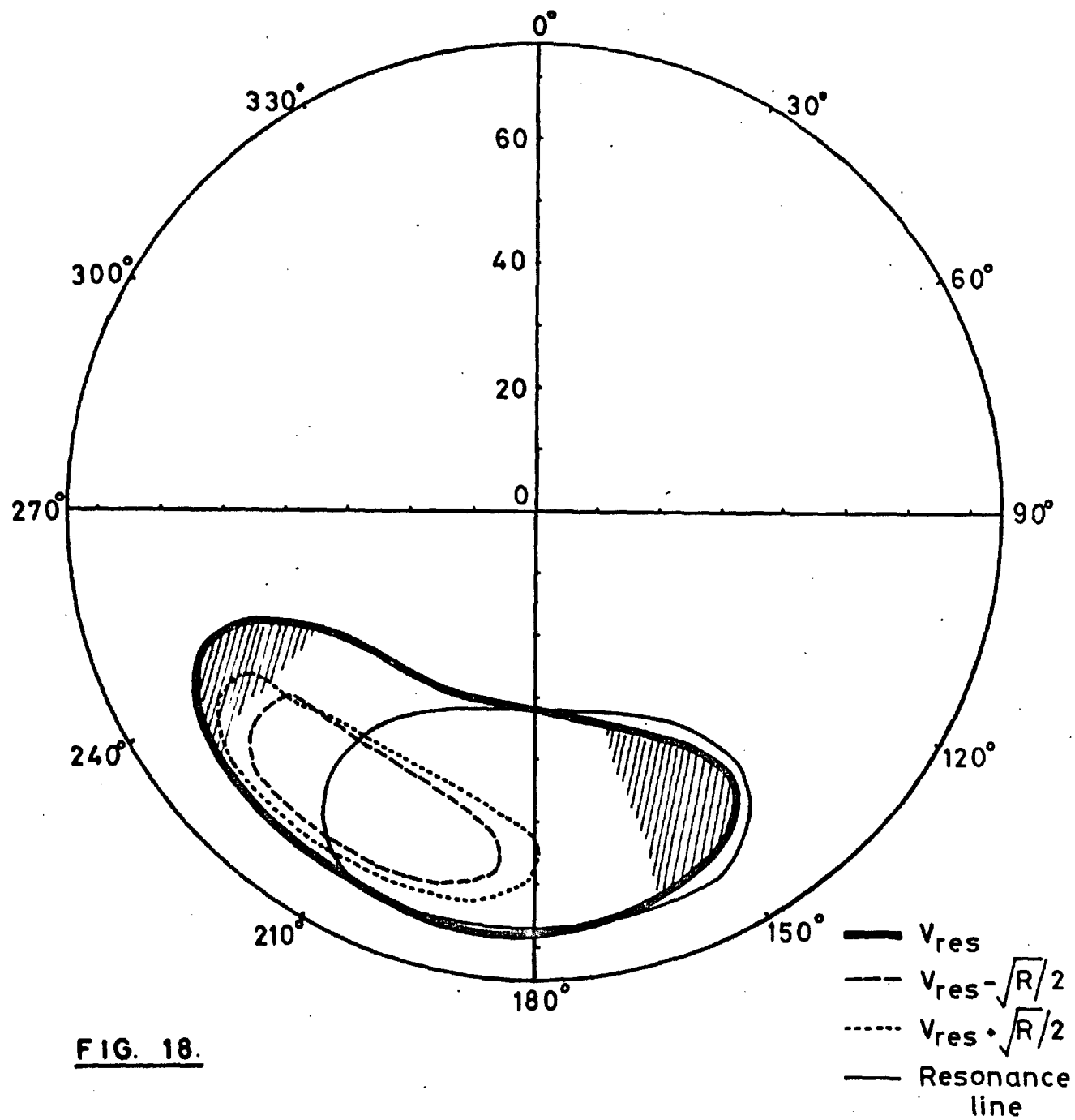


Fig. 18. Trapping region for $\eta/R=0.6$, $\xi/R=-0.25$ $\alpha=0.5$

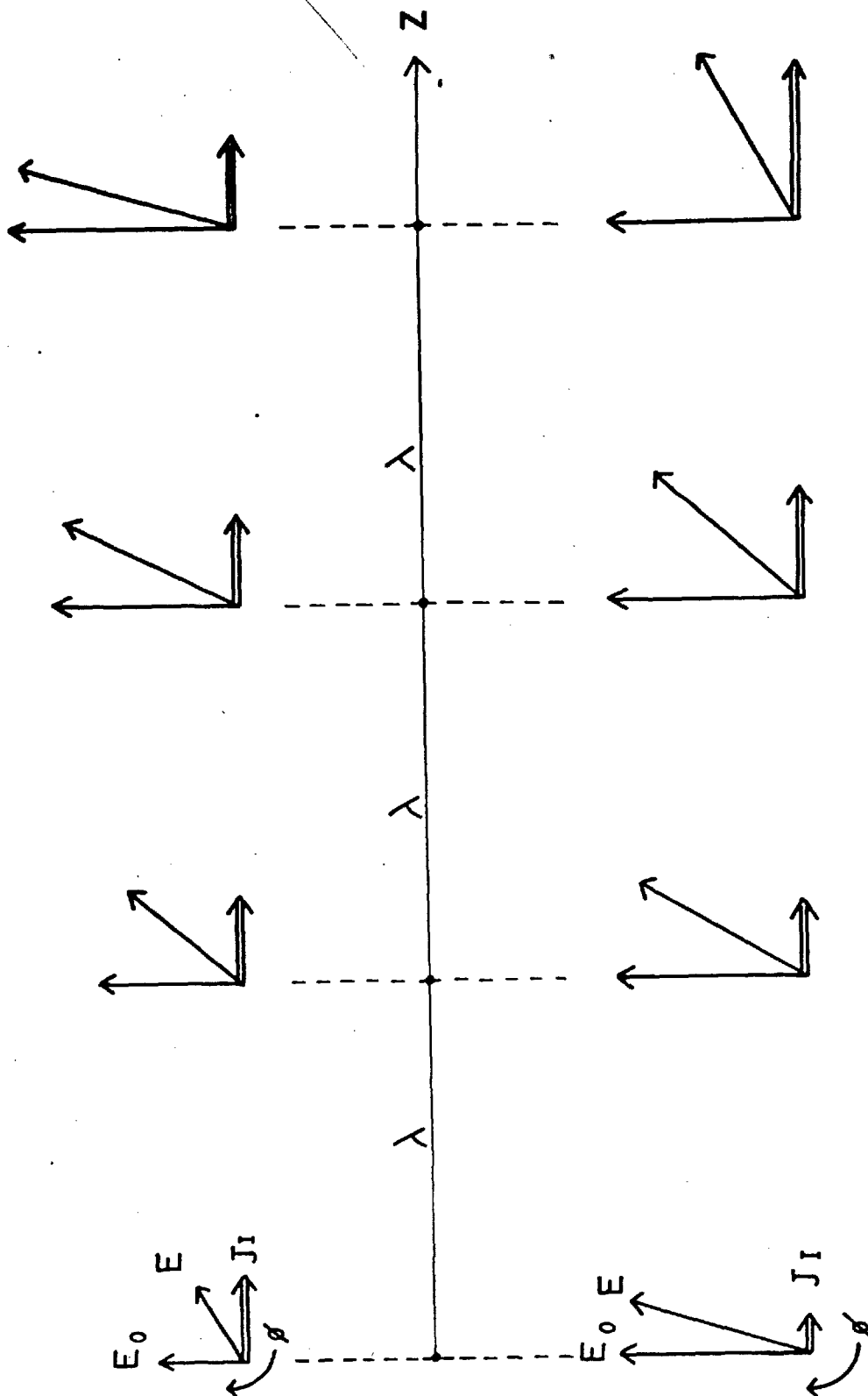


FIG 19

Fig. 19. Sketch showing addition of current generated fields, labelled J_1 , to the existing field, labelled E_0 , to give a net field E with a shifted wave length. The wave length shift can be seen to be proportional to $\frac{\partial}{\partial z} (J_1/E)$

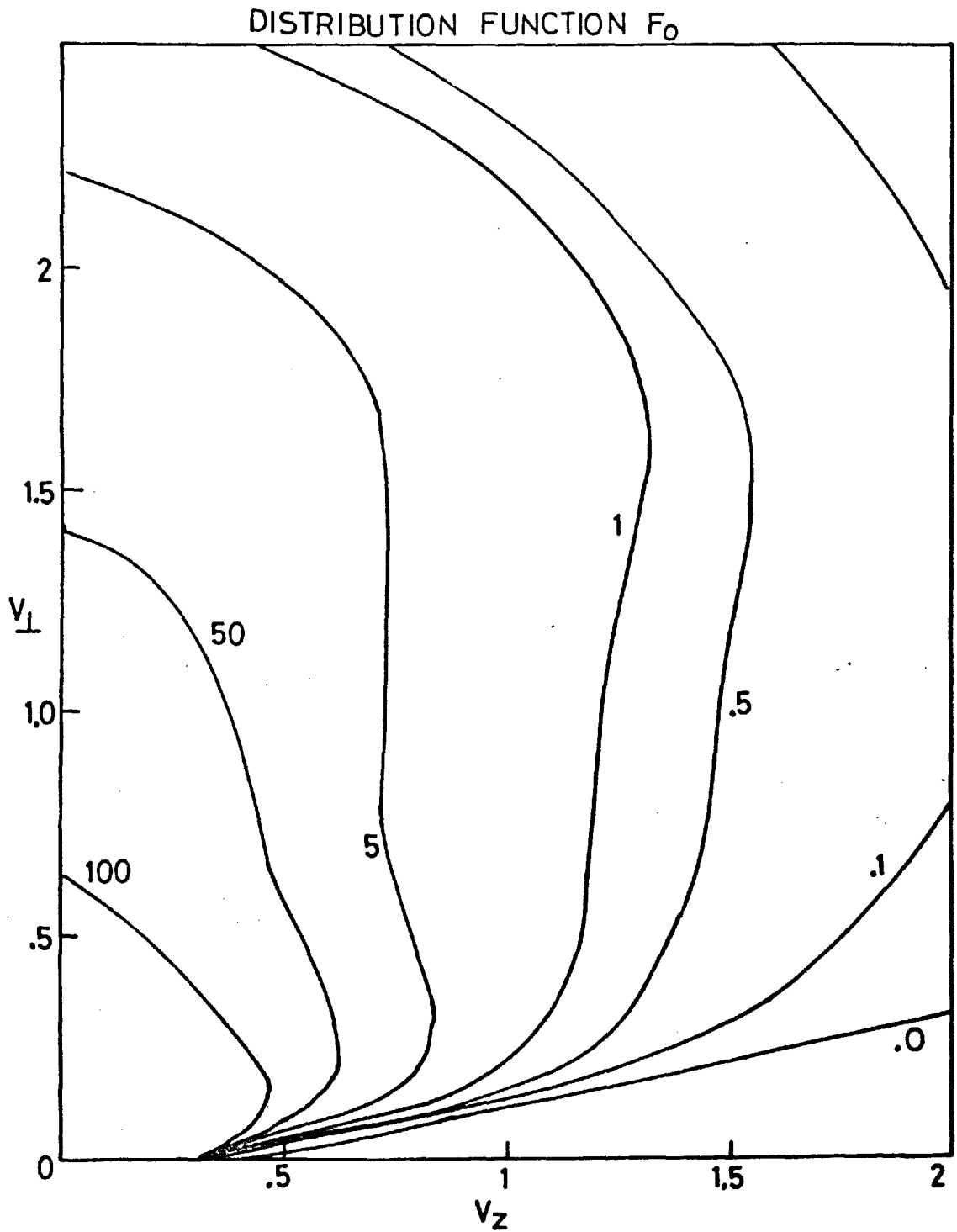


FIG. 20

Fig. 20. The zero order distribution function F_0 as a function of equatorial V_z and $V_{\perp}$.

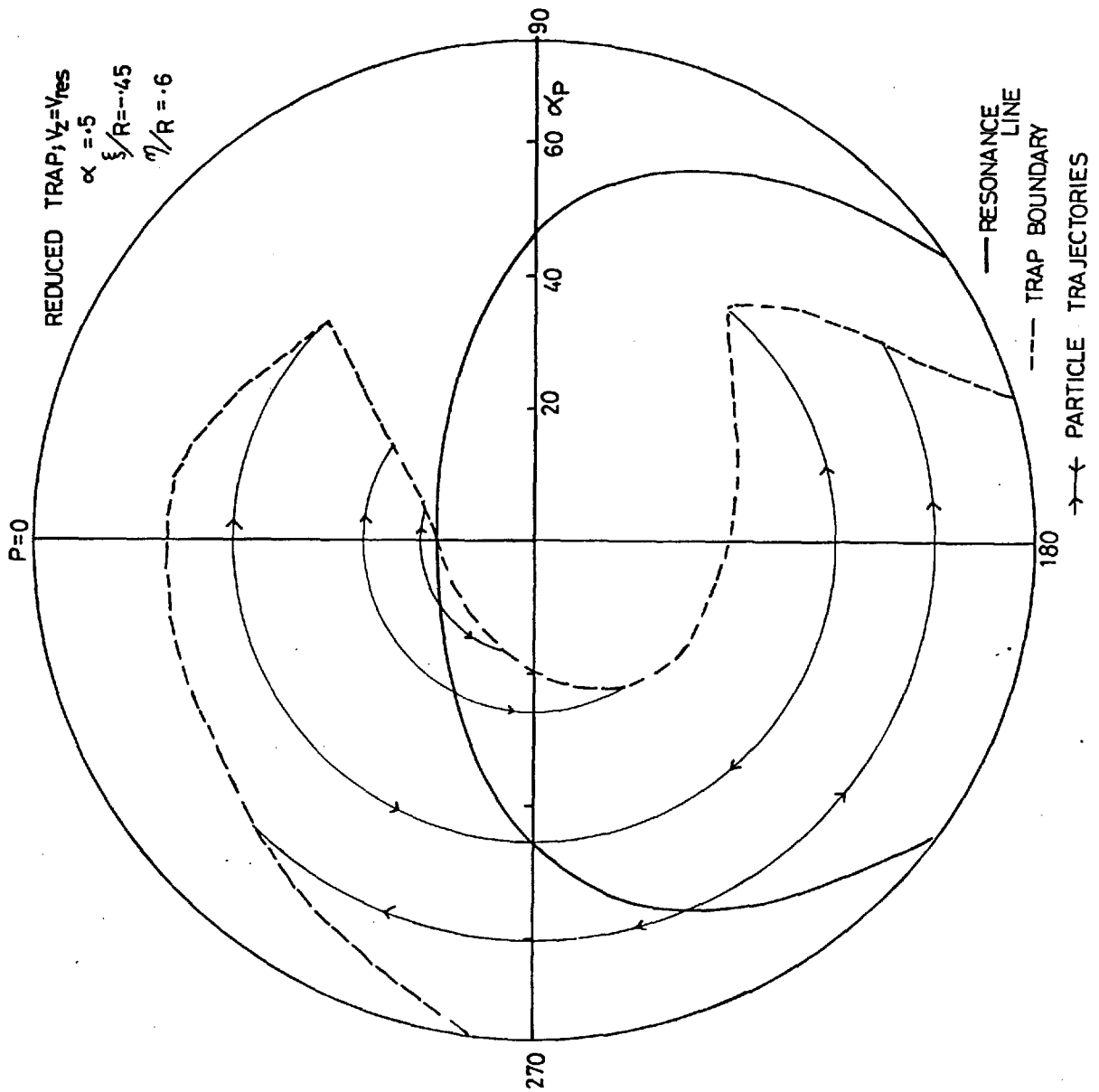


FIG 21

Fig. 21. Example of the "reduced trap" used to compute the growth rate γ^* (cross section in the resonance plane)

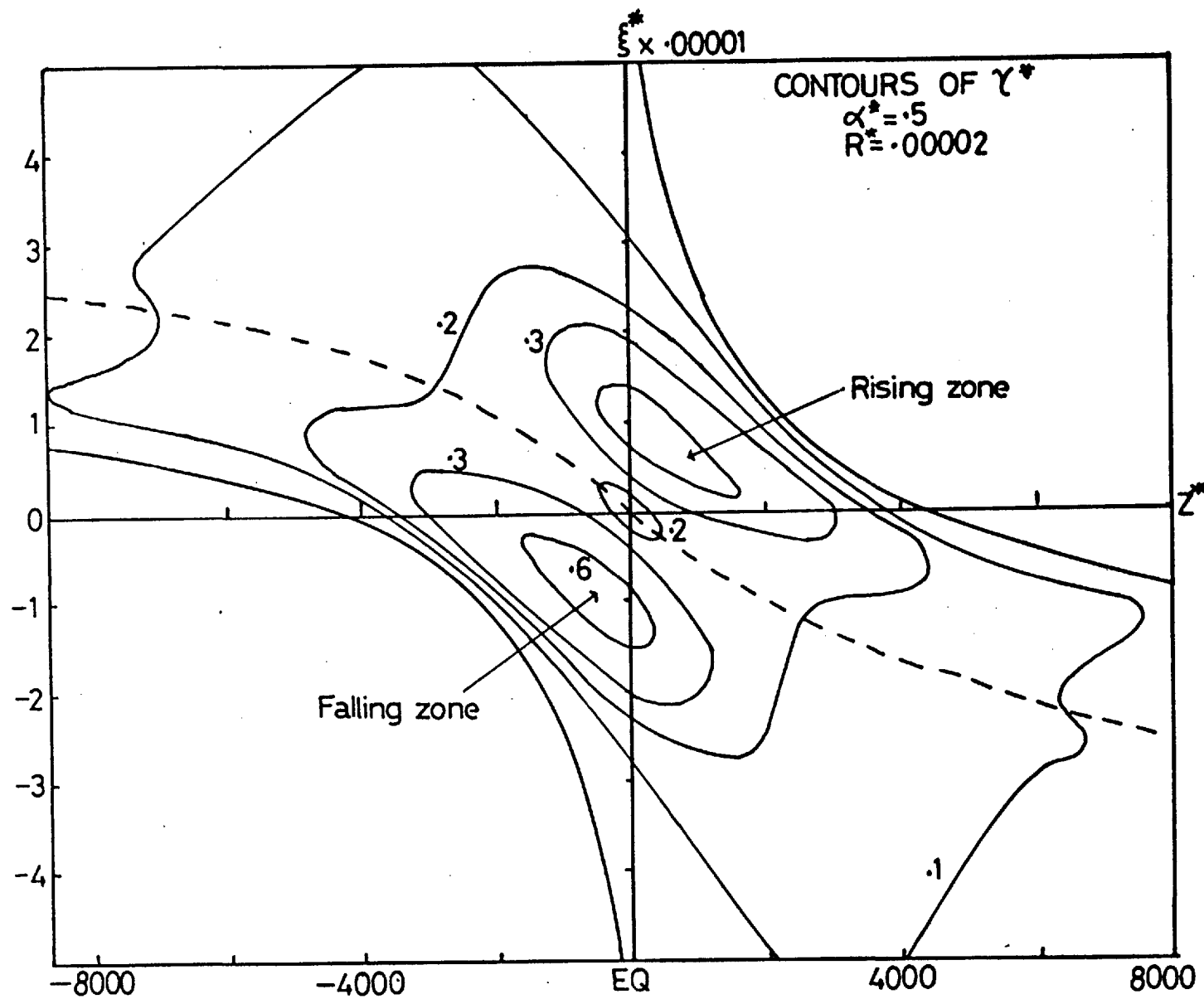


FIG 22

Fig. 22. Plot of γ^* as a function of z^* , ξ^* , for $\alpha^* = 0.5$,
 $R^* = 0.00002$.

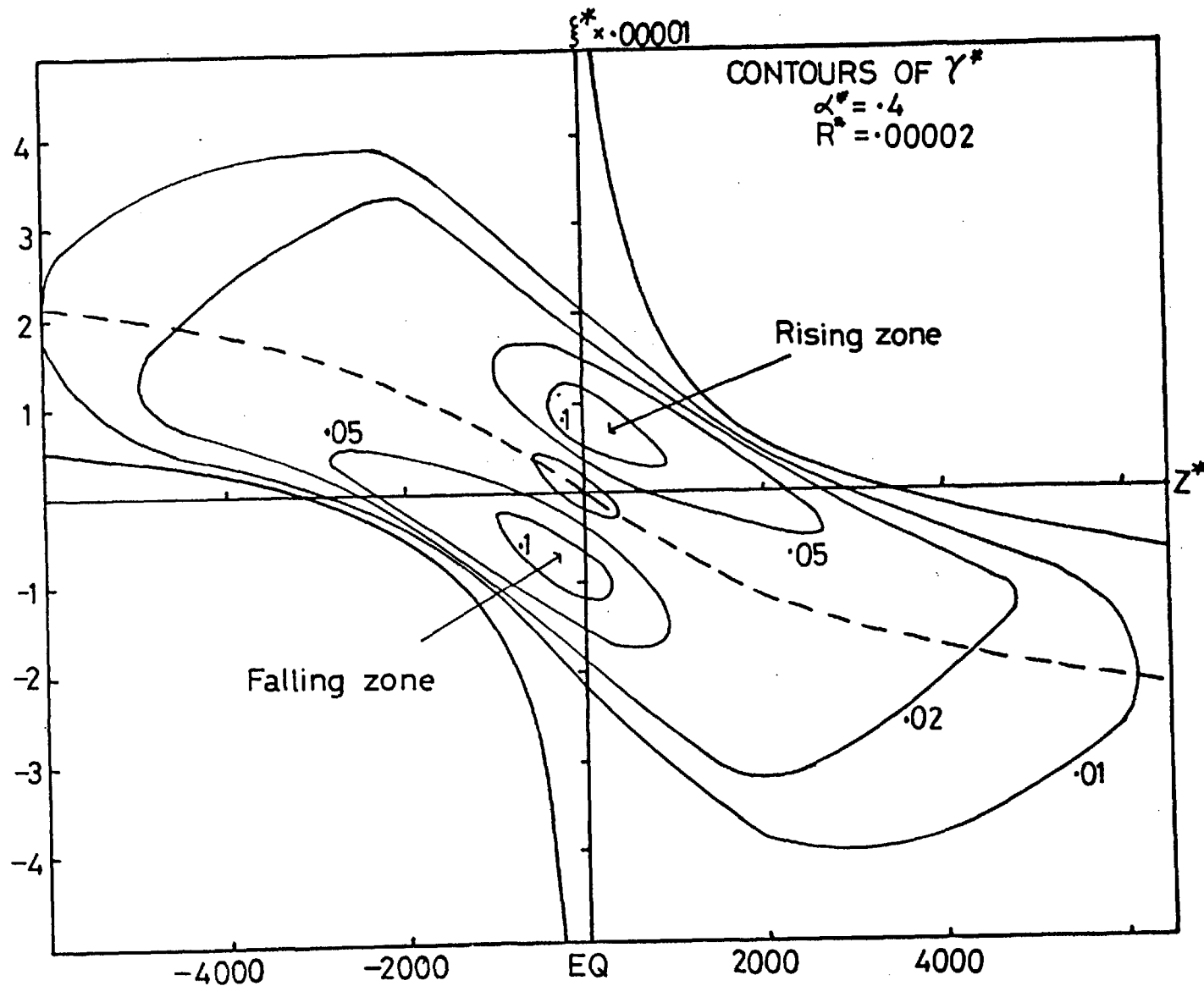


FIG 23

Fig. 23. Plot of γ^* for $\alpha^* = 0.4$, $R^* = 0.00002$.

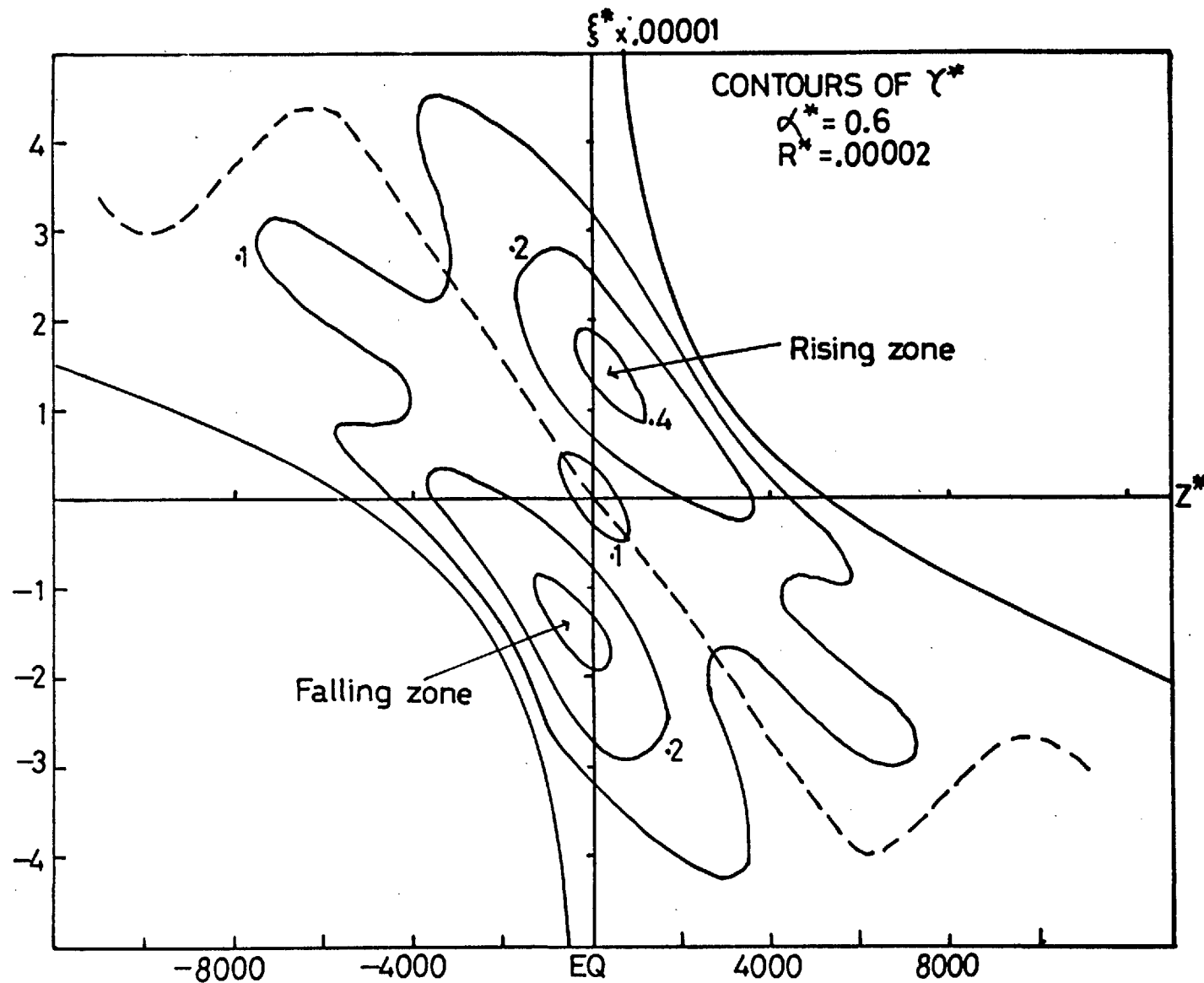


FIG 24

Fig. 24. Plot of γ^* for $\alpha^* = 0.6$, $R^* = 0.00002$.

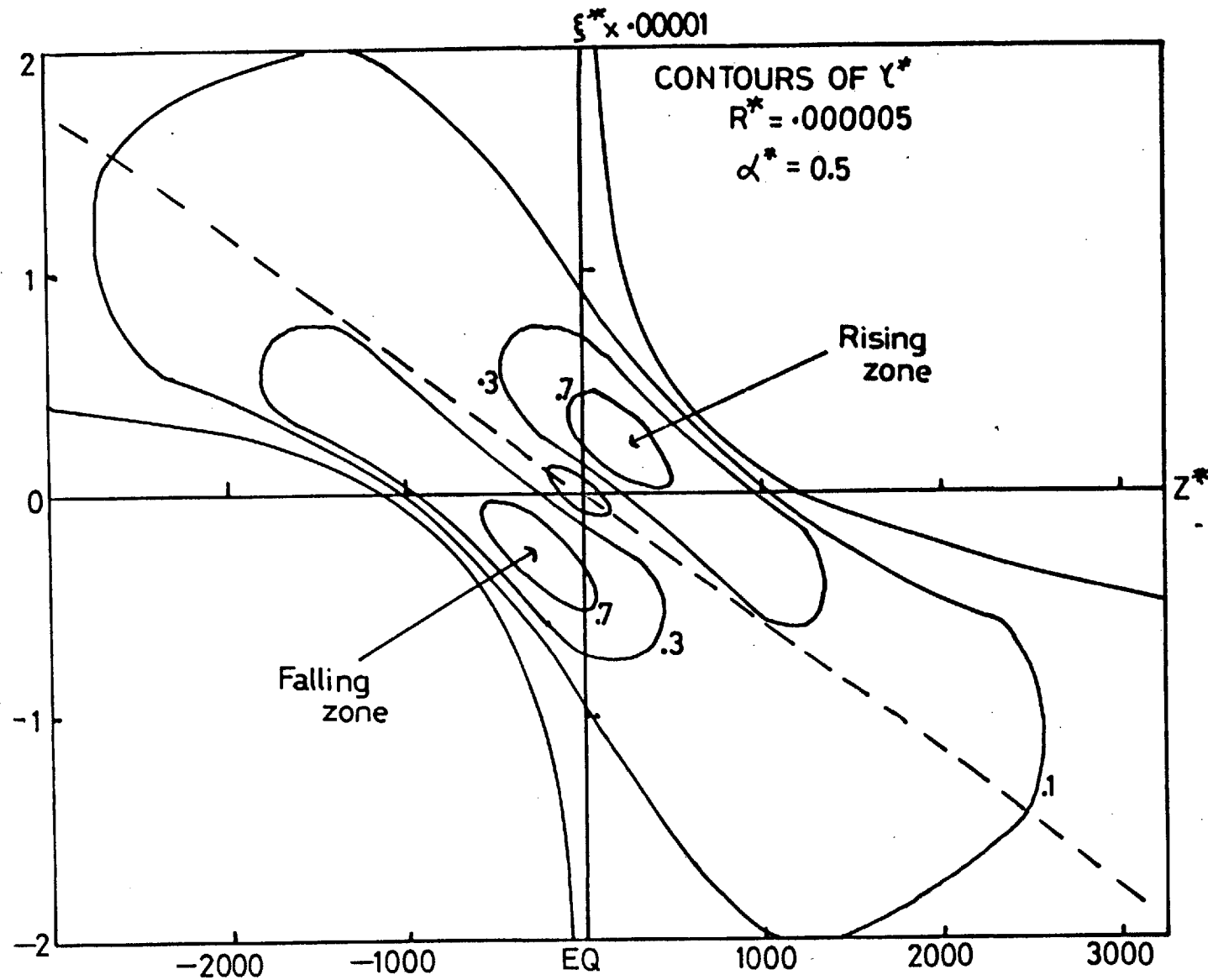


FIG 25

Fig. 25. Plot of Υ^* for $\alpha^* = 0.5, R^* = 0.000005$.

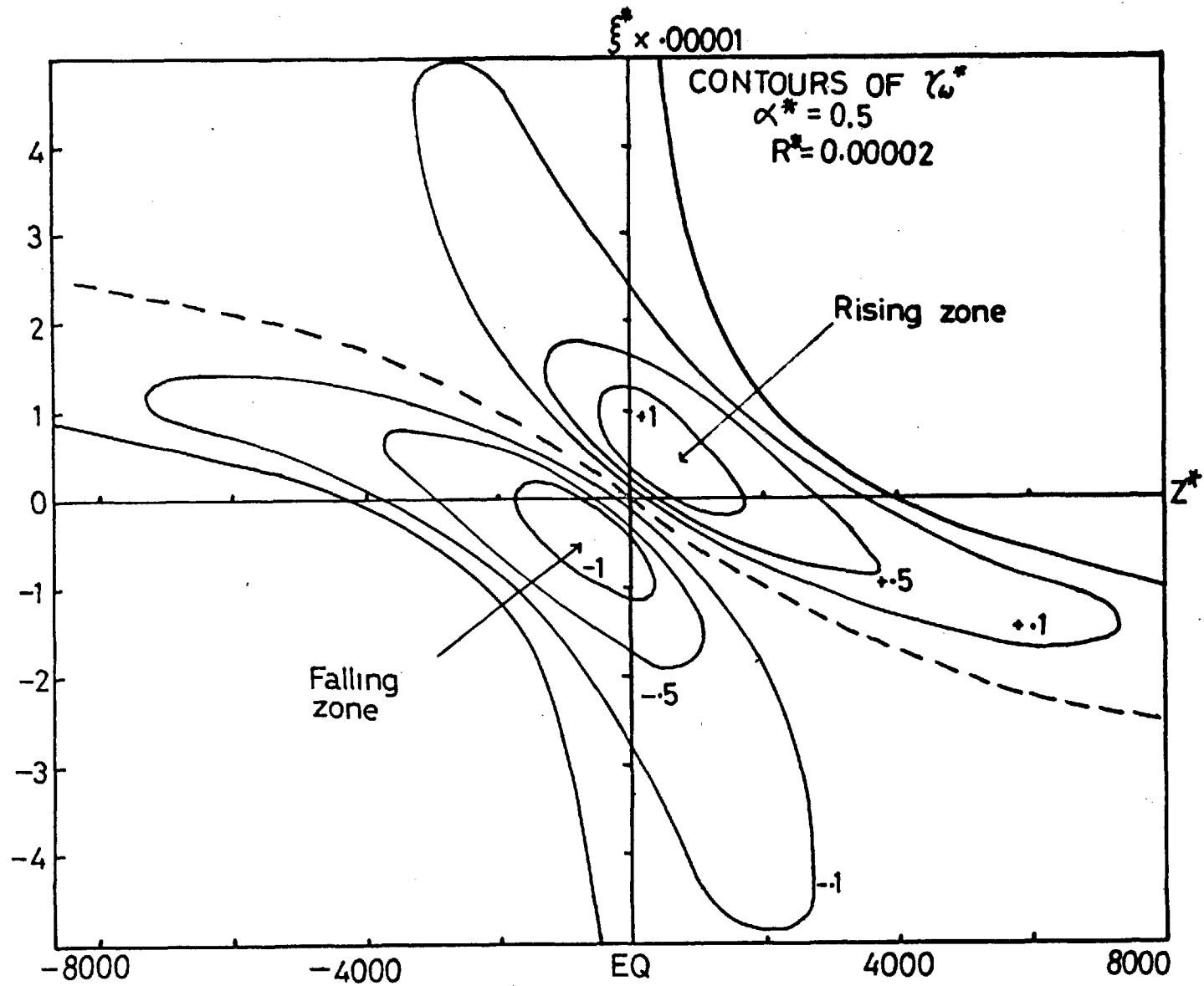


FIG 26

Fig. 26. Plot of γ_w^* (times 1000) as a function of Z^* ,
 ξ^* , for $\alpha^* = 0.5$, $R^* = 0.00002$.

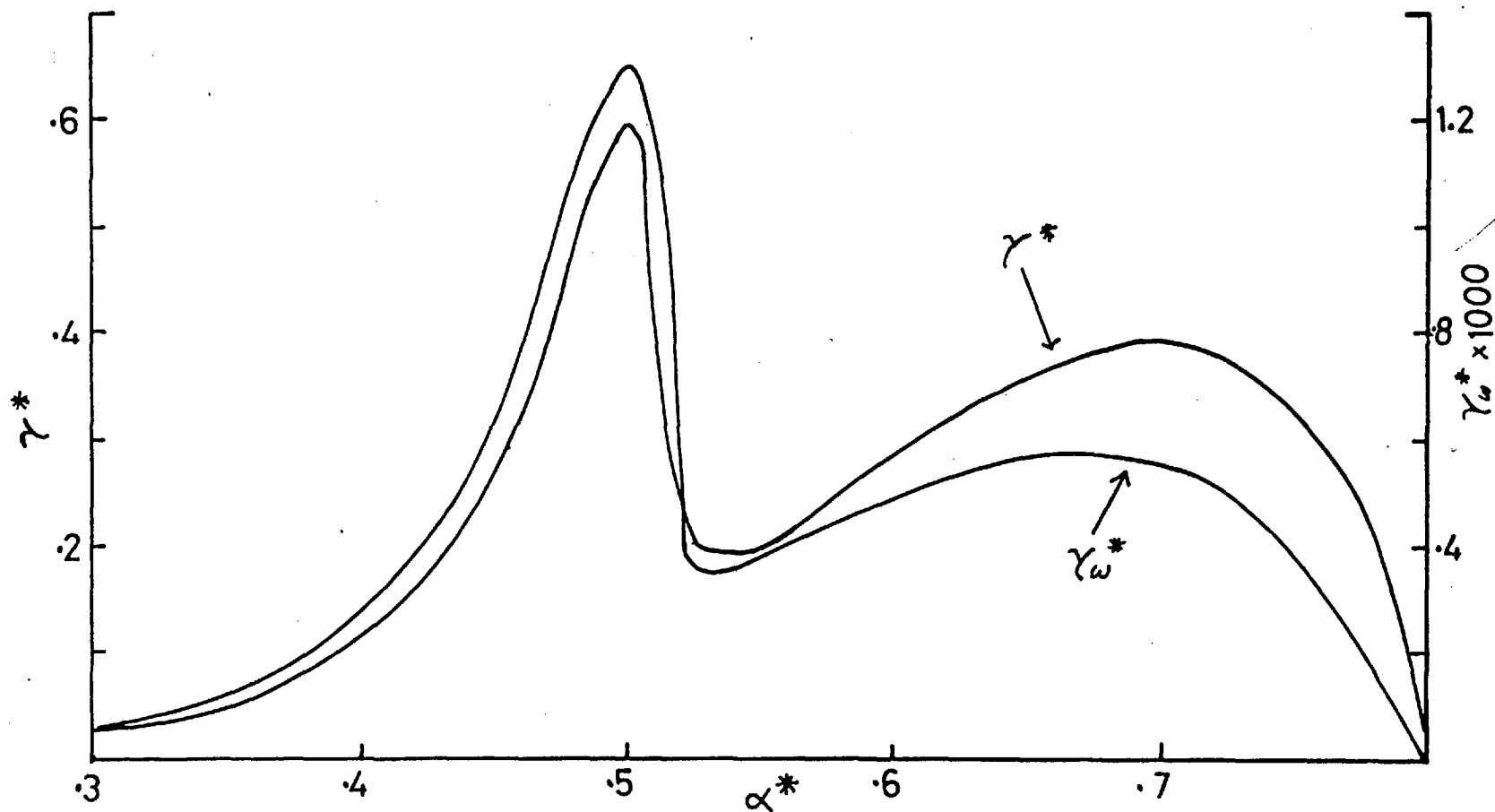


FIG 27

Fig. 27. Graph of the maximum value of γ^* , γ_w^* , along the $\xi^* = 0$ axis, as a function of α^* , and for $R^* = 0.000005$.

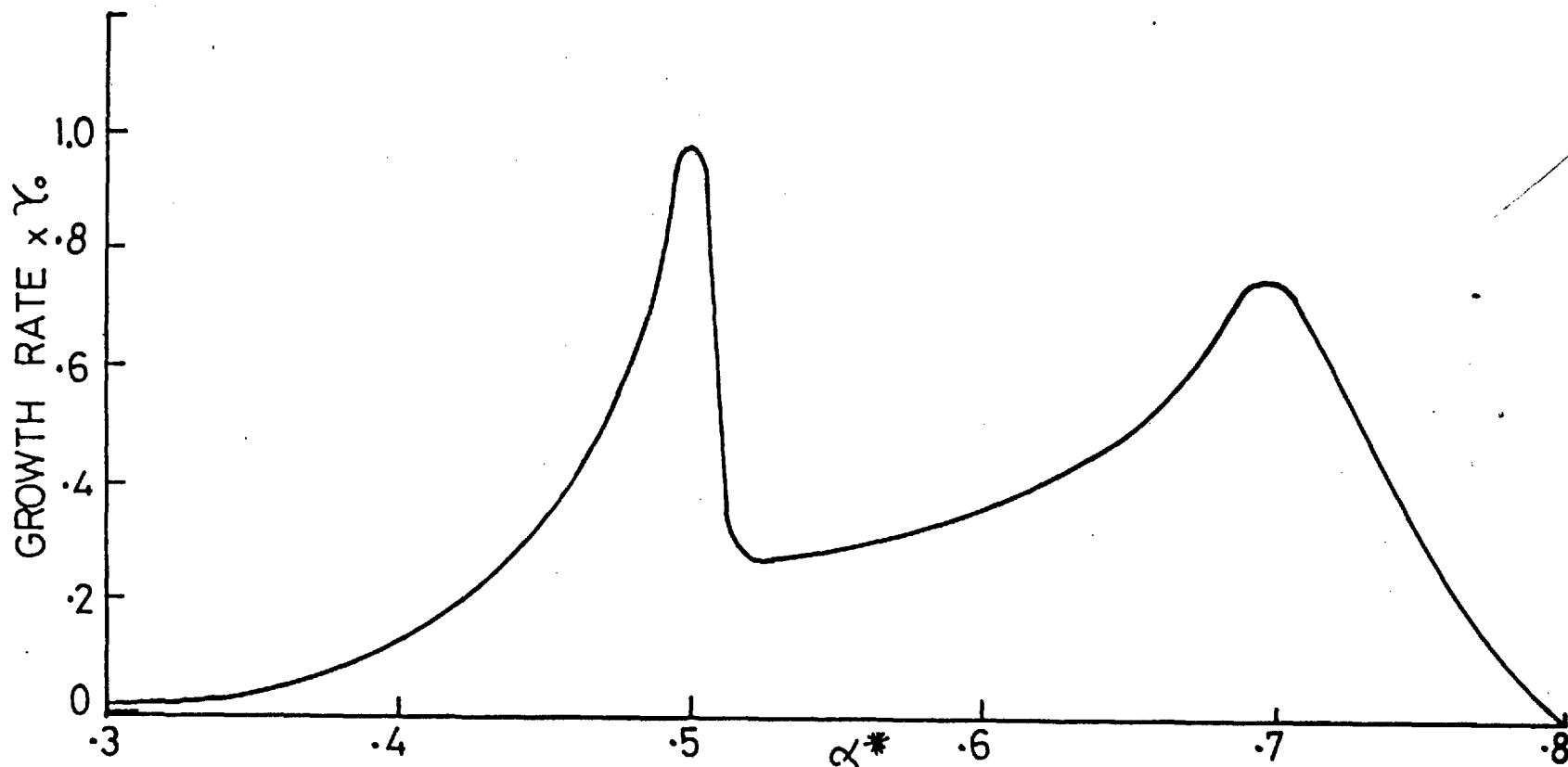


FIG 28

Fig. 28. Estimate of the maximum initial growth rate (as a multiple of χ_0) occurring in a morse pulse of constant amplitude $R^*=0.000005$. This is shown as a function of α^* .

INITIAL CURRENTS IN PULSE

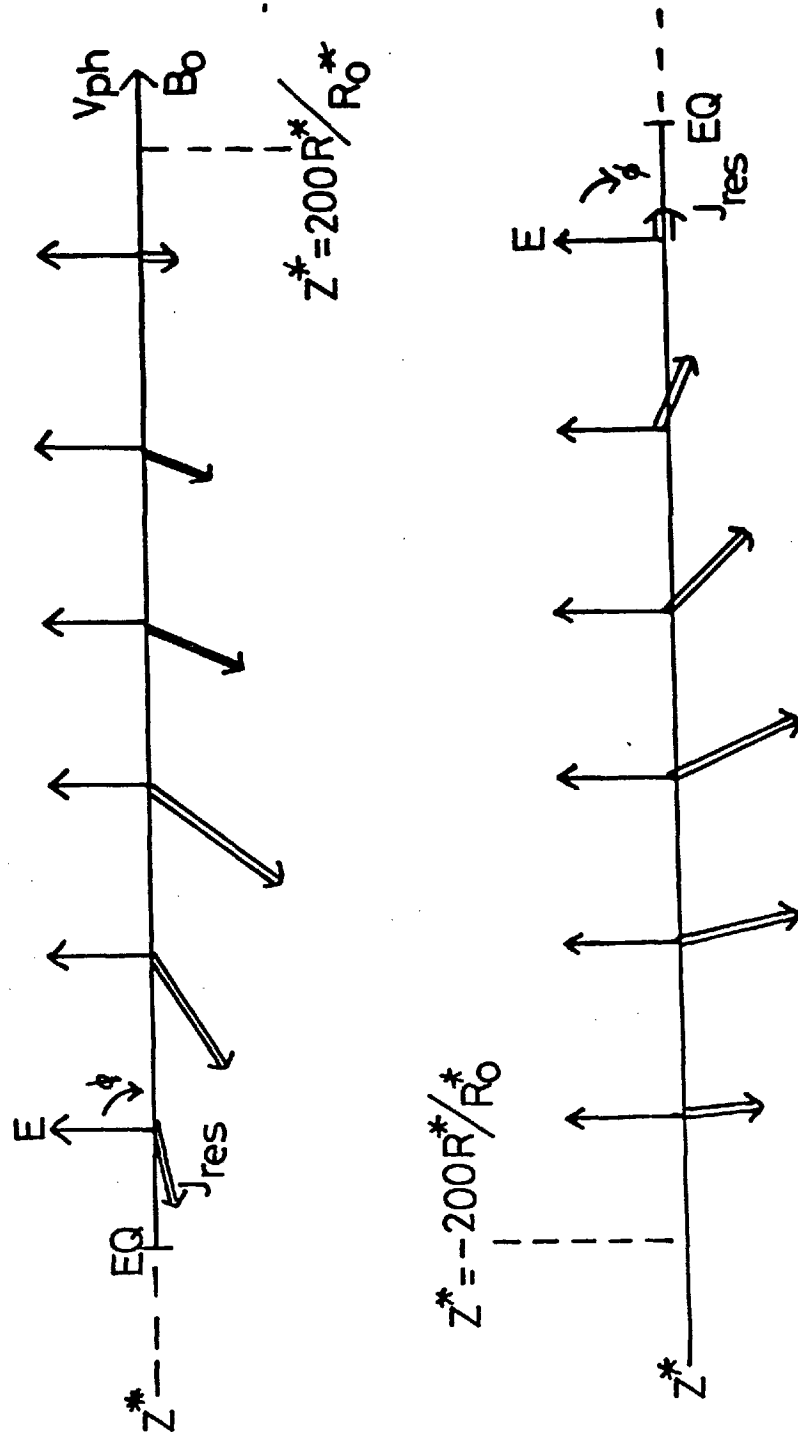
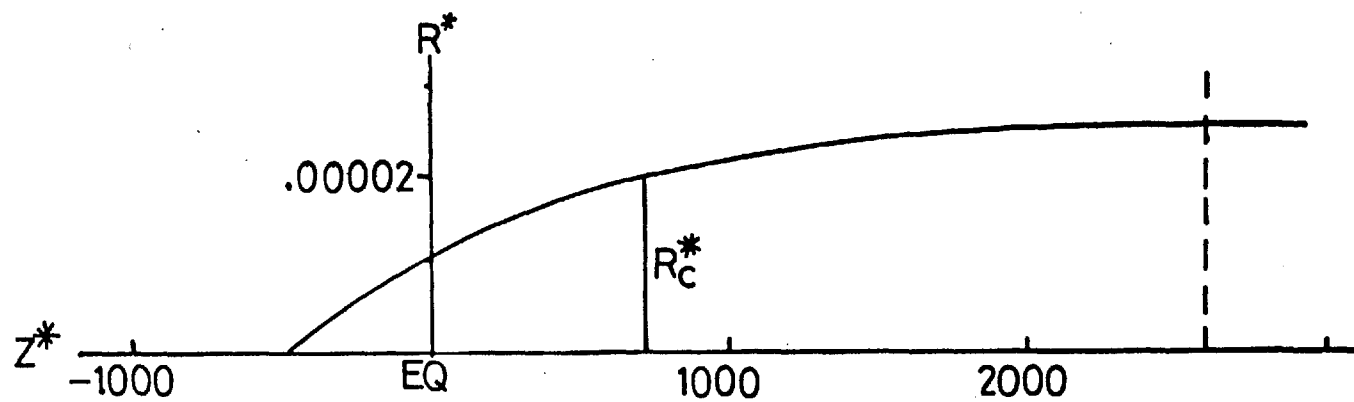
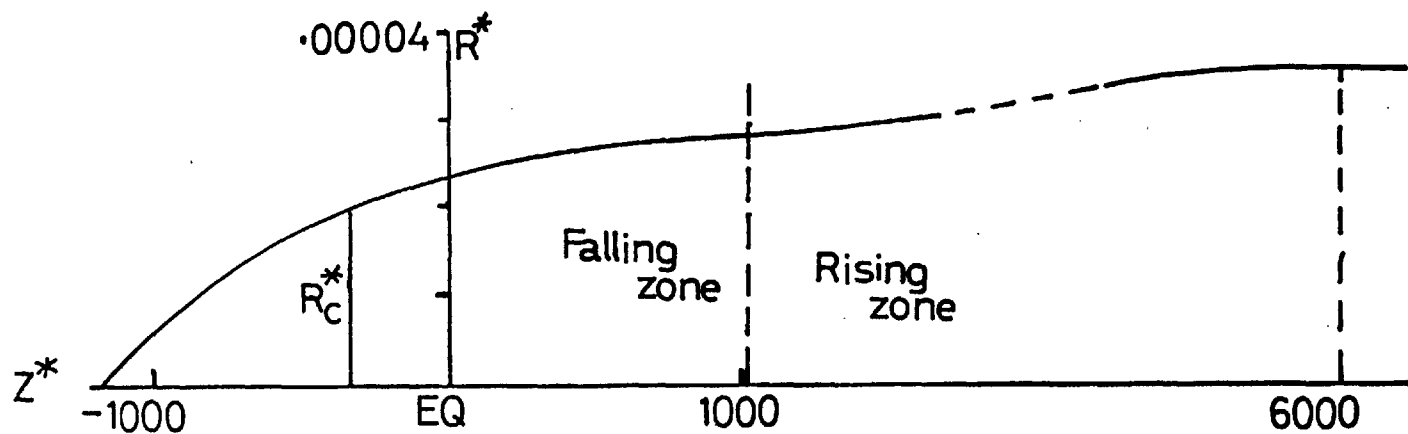


FIG 29

Fig. 29. Sketch of the amplitude and phase of second order resonant particle currents that first appear in a morse pulse in the equatorial zone. These currents have on average a shorter wave length than the field-thus giving rise to frequency offsets.



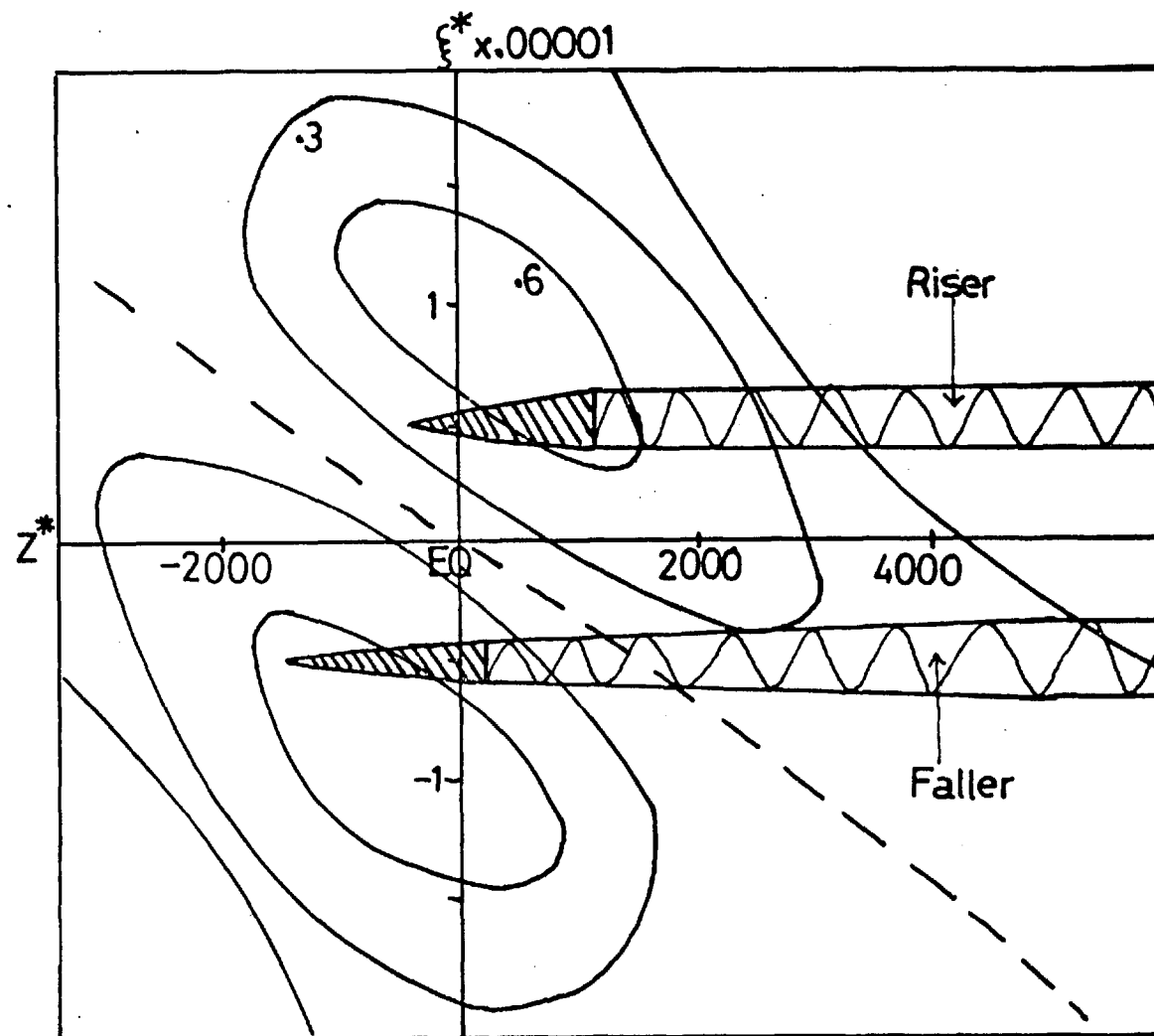
(a) Generation region of riser $\alpha^* = 0.5$ $\xi^* = .000005$



(b) Generation region of faller $\alpha^* = 0.5$ $\xi^* = -.000005$

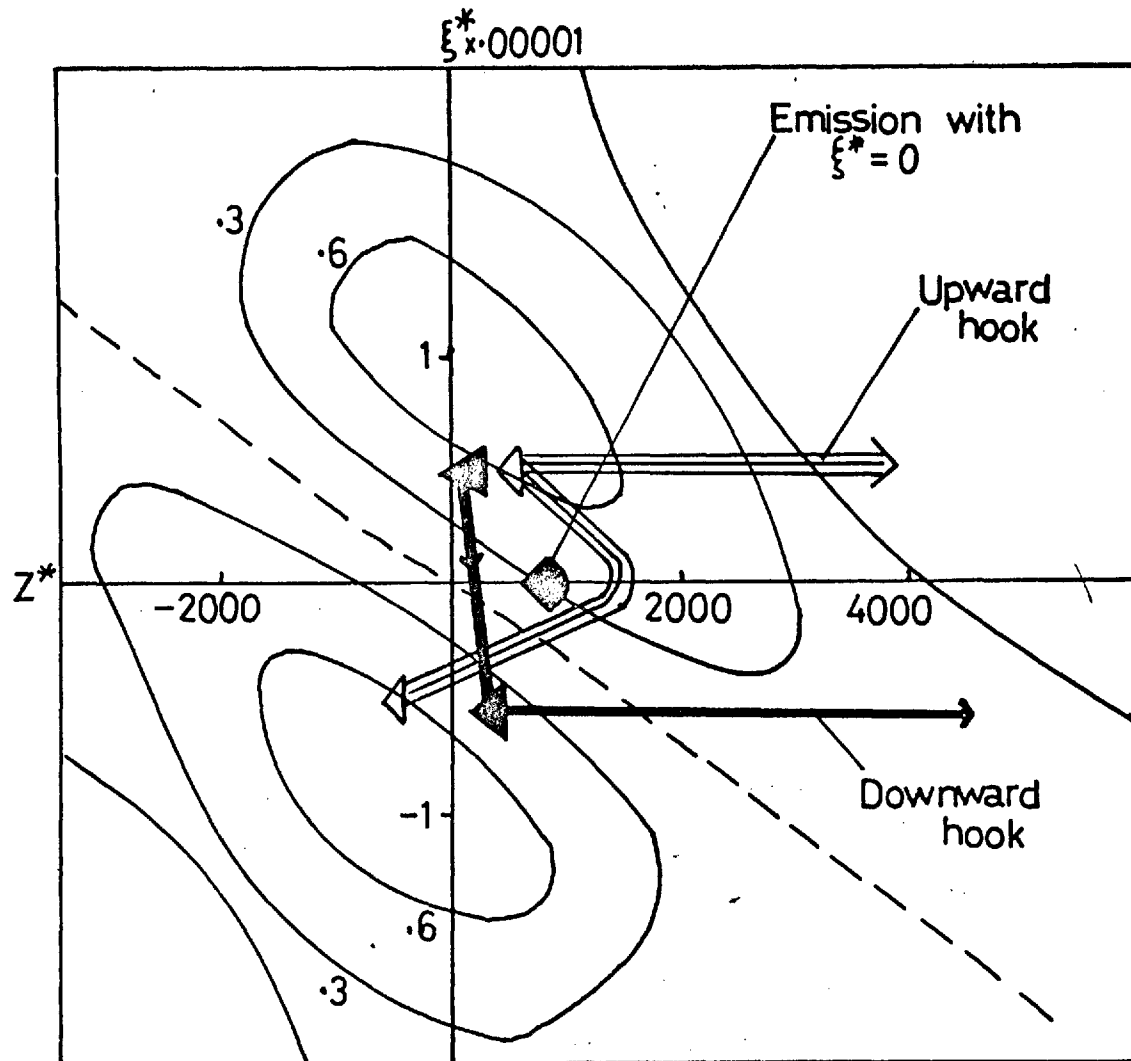
FIG 30(a)

Fig. 30(a). Anticipated forms of the generating regions of a riser and a faller.



POSITION OF G.R.'s ON A γ^* DIAGRAM FOR R_c^*
FIG 30(b)

Fig. 30(b). The generating regions of fig. 30(a) shown on the appropriate γ^* diagram.



Movement of G.R. in hook production — contours are of γ^* with $R_c^* = R_c^*$

FIG 31

Fig. 31. Suggested movements of the generating region in Z^* , ξ^* , during the production of upward and downward hooks.

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APPENDIX 1.

The Numerical Integration

We require to numerically integrate equations 9-14. This is done in a stepwise fashion, using a timestep Δ . Let us consider the progression from the n th to the $(n+1)$ th timestep. There is no need to integrate over the very rapid cyclotron motion, so we factor this out of the equations by defining a quantity

$$\psi = \gamma e^{-i\beta_n \delta / \alpha} ; \quad \delta = t - t_n$$

where β_n, t_n are the values at the n th timestep. The equations of motion now become;

$$\begin{aligned} \frac{d\psi}{dt} &= -R \left(1 - \frac{kV_z}{\omega}\right) e^{i(\phi - \delta\beta_n/\alpha)} + \frac{i\psi}{\alpha} (\beta - \beta_n) \\ &\quad + \frac{1}{2} \frac{\psi V_z}{\beta} \frac{\partial \beta}{\partial z} \\ \frac{dV_z}{dt} &= -\frac{Rk}{\omega} \rho \left(\psi e^{+i(\delta\beta_n/\alpha - \phi)} \right) - \frac{\psi \bar{\psi}}{2\beta} \frac{\partial \beta}{\partial z} \\ \frac{d\phi}{dt} &= \omega - kV_z \end{aligned}$$

Expanding about the n th time step as a function of δ , and retaining only terms of order R , we obtain;

$$\begin{aligned} \frac{d\psi}{d\delta} &= -R \left(1 - \frac{k_n V_{zn}}{\omega_n}\right) e^{i\phi_n} e^{iG_n \delta} + \frac{1}{2} \frac{\psi_n V_n}{\beta_n} \left(\frac{\partial \beta}{\partial z} \right)_n \\ &\quad + \frac{i}{\alpha} \psi_n \left(\frac{\partial \beta}{\partial z} \right)_n V_n \delta \end{aligned}$$

$$\frac{dV_z}{d\delta} = -\frac{Rk_n}{\omega_n} \Re \left\{ e^{-i\phi_n} e^{-iG_n\delta} \psi_n \right\} - \frac{\psi_n \bar{\psi}_n}{2\beta_n} \left(\frac{\partial \beta}{\partial z} \right)_n$$

$$\frac{d\phi}{d\delta} = \omega_n - k_n V_{zn} + \dot{\omega}_n \delta - \dot{k}_n V_{zn} \delta - k_n \dot{V}_{zn} \delta$$

$$G_n = -\beta_n/\alpha - k_n V_{zn} + \omega_n \sim -k_n (V_{zn} - V_{res n})$$

where

$$\dot{V}_{zn} = -\frac{Rk_n}{\omega_n} \Re \left\{ \psi_n e^{-i\phi_n} \right\} - \frac{\psi_n \bar{\psi}_n}{2\beta} \frac{\partial \beta}{\partial z}$$

$$\dot{\omega}_n = -\xi (V_{zn} - 2(1-\alpha))$$

$$\dot{k}_n = \frac{k_n}{2(1-\omega_n\alpha/\beta_n)} \left[\frac{\dot{\omega}_n}{\omega} - \frac{\eta V_n \omega_n \alpha}{\beta_n^2} \right]$$

Note that in this problem G_n is not necessarily small as particles may fall out of resonance during the integration. Integration of the above equations from $\delta = 0 \rightarrow \Delta$ gives the "recipe" for advancing from one step to the next.

$$\begin{aligned} \psi_{n+1} = & \left[\psi_n - R \left(1 - \frac{k_n V_n}{\omega_n} \right) e^{i\phi_n} \left\{ \frac{e^{iG_n\Delta} - 1}{iG_n} \right\} \right. \\ & \left. + \frac{1}{2} \frac{\psi_n V_n}{\beta_n} \left(\frac{\partial \beta}{\partial z} \right)_n \Delta + \frac{1}{2} \frac{i\psi_n}{\alpha} \left(\frac{\partial \beta}{\partial z} \right)_n V_n \Delta^2 \right] e^{i\beta_n \Delta / \alpha} \\ V_{n+1} = & V_n - \frac{Rk_n}{\omega_n} \Re \left\{ \psi_n e^{-i\phi_n} \left(\frac{e^{-iG_n\Delta} - 1}{-iG_n} \right) \right\} \\ & - \frac{\psi_n \bar{\psi}_n}{2\beta_n} \left(\frac{\partial \beta}{\partial z} \right)_n \Delta \end{aligned}$$

$$\begin{aligned}
z_{n+1} &= z_n + v_n \Delta - \frac{R k_n}{\omega_n} \Re \left\{ \psi_n e^{-i\phi_n} \left(\frac{e^{-iG_n \Delta} - 1}{-G_n^2} \right. \right. \\
&\quad \left. \left. + \frac{\Delta}{iG_n} \right) \right\} - \frac{\psi_n \bar{\psi}_n}{4\beta_n} \left(\frac{\partial \beta}{\partial z} \right)_n \Delta^2 \\
\phi_{n+1} &= \phi_n + (\omega_n - k_n v_n) \Delta + \frac{\Delta^2}{2} (\dot{\omega}_n - \dot{k}_n v_n - k_n \dot{v}_n)
\end{aligned}$$

Also we have;

$$k_n^2 = \omega_n(1-\alpha)/(1-\omega_n\alpha/\beta_n)$$

$$\omega_n = 1 - \xi(z_n - 2(1-\alpha)t_n)$$

$$t_{n+1} = t_n + \Delta$$

$$\beta_n = 1 + \eta z_n$$

These expressions form the basis of the computer program.

The time step Δ is of course negative, and is chosen so that the percentage error at each step is less than a certain amount (say 1%). The leading terms neglected are

$$\text{in } \frac{\partial k}{\partial z} \delta, \frac{\partial \beta}{\partial z} \delta, R\delta, \frac{\delta^2}{2} R, \frac{\delta^2}{2} \frac{\partial k}{\partial z}, \frac{\delta^2}{2} \frac{\partial \beta}{\partial z}$$

relative to those retained. A convenient estimate of the required time step is then;

$$|\Delta| = \frac{C}{\sqrt{\xi^2 + \eta^2 + R^2}} + \sqrt{\frac{2C}{\sqrt{\xi^2 + \eta^2 + R^2}}}$$

APPENDIX 2

In this appendix we derive equation 22. We take a cold homogeneous electron plasma with a rigid ion background, immersed in a uniform magnetic field B_0 in the z direction. We consider its excitation by a small transverse resonant particle current J_{res} , that depends only upon the coordinate z .

The governing equations are those of Maxwell, together with the linearised equation of motion of the cold particles. Thus:-

$$\nabla \times \underline{B} = \frac{4\pi}{c} (\underline{j}_{res} + \underline{j}_c) + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

$$\frac{\partial \underline{v}}{\partial t} = -\frac{e}{m} \left\{ \underline{E} + \frac{1}{c} \underline{v} \times B_0 \right\}$$

$$\underline{j}_c = -N_0 e \underline{v}$$

where J_c is the current due to cold particles of number density N_0 . For transverse $\underline{j}_{res}$, $\underline{E}$, $\underline{B}'$, and $\underline{v}$, will also be transverse. We may now write;

$$i \partial / \partial z \tilde{B} = \frac{4\pi}{c} (\tilde{j}_c + \tilde{j}_{res}) + \frac{1}{c} \frac{\partial \tilde{E}}{\partial t}$$

$$i \partial / \partial z \tilde{E} = -\frac{1}{c} \frac{\partial \tilde{B}}{\partial t}$$

$$(\partial / \partial t - i\Omega) \tilde{j}_c = \frac{N_0 e^2}{m} \tilde{E}$$

where $\Omega = eB_0/mc$ $\tilde{j}_c = j_{cx} + i j_{cy}$

$$\tilde{E} = E_x + i E_y$$

$$\tilde{B} = B_x + i B_y$$

$$\tilde{j}_{res} = j_{res_x} + i j_{res_y}$$

Elimination of $\tilde{B}$, and $\tilde{j}_c$, gives the final differential equation for E .

$$\left[\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial}{\partial t} - i\Omega \right) - \frac{\Omega^2}{c^2} \frac{\partial}{\partial t} \right] \tilde{E} = \frac{4\pi}{c^2} \left(\frac{\partial}{\partial t} - i\Omega \right) \frac{\partial \tilde{j}_{res}}{\partial t}$$