NON IDEAL EFFECTS IN THERMAL CONVECTION

by

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ABSTRACT

The onset of cellular convection in an unbounded, uniformly heated horizontal layer of fluid has attracted a great deal of attention since the turn of the century. More recently, consideration has been given to the onset of convection under less restrictive circumstances. This thesis examines three such cases :

1. In the first chapter the combined influence of rigid cylindrical sidewalls and the anomalous expansion of water below 4°C are treated. A novel method is used to expand the linearised perturbation solution in terms of eigenfunctions of the unbounded problem. Asymptotic solutions for large and small aspect ratio are obtained and compared with the numerical results.

2. In chapter two we investigate the nature of convective instability in the vicinity of a hot-spot on the lower boundary of an unbounded horizontal layer. We find that convection occurs first in the form of longitudinal rolls.

3. In the third and final chapter, we study the onset of convection in an infinite horizontal annulus, where we allow the inner cylinder to rotate uniformly. We find that this problem exhibits a

strong Prandtl number dependence, even with no rotation present, with two distinct cases being noted. For $Pr \gtrsim 0{\cdot}24$ convection occurs at the top of the inner cylinder and the effect of rotation is to merely shift the site of instability away from the vertical in the direction of the rotation.

For $Pr \lesssim 0{\cdot}24$ and no rotation present convection occurs on either side of the vertical at an angle θ_c to the vertical, where θ_c increases monotonically from 0 to $\frac{\pi}{2}$ as Pr decreases from 0•24 to 0•0. In the presence of rotation convection first occurs on the "upstream" side of the vertical at a slightly modified angle.

CHAPTER 1.

PENETRATIVE CONVECTION IN A RIGID CYLINDRICAL CONTAINER.

1:1. Introduction.

In this chapter we attempt to solve the linear thermal convective stability problem for a column of water confined by a rigid cylindrical boundary, with the bottom of the column held at $0^{\circ}C$ while the top of the column is allowed to have a fixed temperature greater or equal to $4^{\circ}C$.

This represents a system where a layer of thermally unstable fluid is bounded above by a thermally stable layer, so that we can expect motions arising in the unstable lower layer to penetrate into the upper stable layer. The reason for the existence of this two tier system is due to the anamolous expansion of water, which has a maximum density at $4^{\circ}C$.

We are therefore in the ideal position of having at our disposal a simple, natural phenomenon which can be set up in the laboratory from which we can study, albeit in a limited sense, the mechanics of penetrative convection. This is important since Nature shows a predeliction for penetrative convection. Consideration of the solar convection zone reveals that it is bounded both above and below by stable regions, while convection generated at the earth's surface due to solar heating penetrates up into the stable atmosphere above. In the oceans also, evaporation at the surface causes convective motions to arise near the surface which then penetrate into the lower regions.

Around 1960, W. V. R. Malkus suggested that the water over ice experiment should be fully investigated to obtain information about penetrative convection. This lead to the experiments of Furumoto and Rooth (22), Townsend (23) and subsequently Myrup (25).

The results of Furumoto and Rooth then lead Veronis (1) to undertake a theoretical study of such a system. He did this by considering water as a Boussinesq fluid with a quadratic temperature dependence and confined it between two infinite plane parallel plates which could be either stress free or rigid. He obtained the equations apertaining to small perturbations about the mean state for that geometry from which he derived a sixth order homogeneous ordinary differential equation for the temperature perturbation, the eigenvalues of which were the Rayleigh number and wave number of the disturbance.

By finding that wave number which minimizes the Rayleigh number he calculated the critical value of the Rayleigh number at which the layer becomes unstable for various values of the temperature of the top plate.

Veronis also investigated the weakly non-linear behaviour of this system and found that with the temperature of the top plate fixed at around 6^{0}C a subcritical instability occurred.

The non-linear behaviour of the infinite plane parallel problem was further investigated by Musman (26) who sought steady state solutions numerically using the mean field approximation of Herring (27). While Moore (25) investigated the behaviour of an individual thermal impinging in a stable area.

Moore and Weiss also studied the ice-water system using a finite difference scheme on the stream function and vorticity equations, but confined themselves to a box with free boundaries top and bottom and using periodic boundary conditions on the sides.

Although these studies are of great value they have an inbuilt weakness in the sense that they did not take into account rigid side walls, a necessity forced on all laboratory experiments. This being the case it is natural to ask whether any simple theoretical means are at our disposal for taking into account rigid sidewalls and evaluating the effect they have.

This chapter is devoted to proposing and evaluating a novel technique for solving convection problems taking into account rigid side walls and in particular we apply it to the water over ice system.

A number of authors have considered the effect of rigid side walls for Bénard convection in a rectangular box. Davis (19) considered rectangular dishes with length to depth ratios varying from 0·25 to 6, and solved the linear stability problem using the Galerkin method, while Segel (17) considered a long shallow box (aspect ratio, Length/depth, large) and took the viewpoint that the solution of an unbounded layer could be 'squeezed' into such a container by allowing slow spatial changes to the amplitude of the solution. Drazin (20) showed that for a two dimensional rectangular box with free surfaces top and bottom all six solutions of the Bénard cubic $(k^2+n^2\pi^2)^3 = k^2 Ra$ were required to satisfy the boundary conditions on the side walls.

Simultaneously Hall and Walton (18) and Daniels (7) considered a rectangular two dimensional box with rigid side walls, but allowed those side walls to be slightly imperfect insulators.

In terms of cylindrical geometry Brown and Stewartson (15) considered the fluid contained in a rigid cylindrical container which was an imperfect insulator and analyzed the problem along the lines of Daniels (7). Charlson and Sani (14) solved the Bénard problem in a rigid cylindrical container using the Galerkin method, while Jones and Moore (4) considered convection in a cylindrical container which was impermeable, perfectly insulating but stress-free.

A zouni and Normand (21) undertook a numerical and experimental investigation of the water over ice system in a rigid cylindrical container. They solved the equations for the linear stability problem using the Galerkin method. However, the results they present are somewhat confusing and not very extensive.

We propose here to solve the linear stability problem for the A zouni and Normand(21) cylinder by a novel method which is similar to the Galerkin method.

In that method the solution is found by expanding the problem out in orthogonal functions which are arbitrarily chosen. We show here that it is possible by casting the vertical structure equations into a matrix form to prove that an orthogonality relationship holds between the linear eigensolutions of the infinite problem thus allowing us to solve the full problems by expanding the solution in terms of these functions.

A similar approach has been adopted by Blennerhassett and Hall (28) to the study of Taylor Vortices in a finite cylinder. Although in their case they obtained the linear eigenfunctions as a Fourier series thereby leading to a double infinite expansion, whereas we require only a single infinite expansion since we calculate the linear eigenfunctions directly from the equations. Hall (29) went on to consider the weakly non-linear problem.

The chapter is divided into six sections. In section (1:2) we state the governing equations and derive the equations for the linear stability problem, while in section (1:3) we show how to construct the solution when the aspect ratio is of order unity using the linear eigensolutions of the infinite problem. We also demonstrate an orthogonality relationship between the linear eigensolutions.

Sections (1:4) and (1:5) deal with the asymptotic formulae in the cases when the aspect ratio tends to infinity and zero respectively, while in section (1:6) we describe the numerical methods used to solve the aspect ratio order unity problem and give the results in both tabular and graphical form. Section (1:7) gives conclusions on the usefulness of the method employed.

1:2. The Governing Equations.

We start this analysis from the three dimensional Navier Stokes equations in cylindrical coordinate form :-

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r^2}\right] = -\frac{\partial p}{\partial r} +$$

$$+\rho\nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi}\right] \qquad - (1:1)$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right] = -\frac{1}{r}\frac{\partial p}{\partial \phi} +$$

$$+\rho\nu\left[\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \phi} - \frac{v}{r^2}\right] \qquad - (1:2)$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} - \rho g +$$

$$+\rho\nu\nabla^2 w \qquad - (1:3)$$

$$\rho c_p\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \phi} + w\frac{\partial T}{\partial z}\right] = \bar{k}\nabla^2 T \qquad - (1:4)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r u) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.0 \qquad - (1:5)$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

These equations are closed by specifying a Boussinesq density profile with a quadratic temperature dependence :-

$$\rho = \rho_0\left[1 - \alpha\,(T - T_0)^2\right] \qquad - (1:6)$$

In the above equations α is the coefficient of thermal expansion, ν is the coefficient of kinematic viscosity, c_p is the specific heat at constant pressure and $\bar{k}$ is the coefficient of thermal conductivity. T_0 is some reference temperature.

We will apply these equations to the inside of a rigid cylindrical container of height h and radius $d/2$, its side wall being regarded as a perfect conductor and we shall assume the ends may act as either free or rigid surfaces.

The base of the cylinder is kept at a constant 0°C and the top will be allowed to have fixed temperature λT_0 where λ is a real free parameter with range $[0, 2]$.

We will consider linear perturbations of equations (1:1) to (1:6) about a mean state of hydrostatic equilibrium where the heat transport is by conduction only. This state is given by the following equations :-

$$u \equiv v \equiv w \equiv 0.0$$

$$\frac{d\bar{p}}{dz} = \bar{\rho}\, g \qquad - (1:7)$$

$$\frac{d^2\bar{T}}{dz^2} = 0.0 \qquad - (1:8)$$

$$\bar{\rho} = \rho_0 \left[1 - \alpha (\bar{T} - T_0)^2 \right] \qquad - (1:9)$$

Equations (1:8) together with boundary conditions discussed earlier imply

$$\bar{T} = \frac{\lambda T_0}{h} z \qquad - (1:10)$$

There is no need to write down explicitly $\bar{p}$ since it does not enter our perturbation equations. Small perturbations about the base state are considered by writing

$$\left.\begin{array}{l} u = u', \quad w = w', \quad v = v' \\ p = \bar{p} + p', \quad T = \bar{T} + \theta, \quad \rho = \bar{\rho} + \rho' \end{array}\right\} \qquad - (1:11)$$

where primes denote perturbation quantities.

Substituting expressions (1:11) into (1:1→6) and ignoring quantities that are quadratic in primes we obtain

$$\frac{\partial p}{\partial r} = \nabla^2 u - \frac{u}{r^2} - \frac{2inV}{r^2} \qquad - (1:12)$$

$$\frac{inp}{r} = \nabla^2 V + \frac{2inu}{r^2} - \frac{V}{r^2} \qquad - (1:13)$$

$$\frac{\partial p}{\partial z} = Ra\,(\lambda z - 1)\,\theta + \nabla^2 w \qquad - (1:14)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{inV}{r} + \frac{\partial w}{\partial z} = 0 \qquad - (1:15)$$

$$\nabla^2 \theta = w \qquad - (1:16)$$

where now

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2}$$

In the above we have dropped the primes for the sake of convenience and placed the equations in non-dimensional form by writing

$$r,\ z \rightarrow h\,(r,\ z)$$

$$u',\ v',\ w' \rightarrow \frac{k}{h}\,(u',\ v',\ w')$$

$$\theta \rightarrow \lambda\, T_0\, \theta$$

$$p' \rightarrow \frac{\nu k}{h^2}\, \rho_0\, p'$$

where $k = \bar{k}/\rho_0 c_p$ and $Ra = \lambda \alpha g T_0^2 h^3 / \nu k$ is the Rayleigh number. Also we have allowed the density to be constant in every term except the buoyancy term, have assumed an $e^{in\phi}$ (where n is an integer) functional dependence on all the physical variables and have assumed also that the principle of exchange of stabilities is true for this sytem allowing us to set all time derivatives to zero. This last assumption is based on the fact that Veronis (1) proved that the principle of exchange of stabilities holds for the infinite plane parallel slab, for the case of FREE-FREE boundarys.

These equations are supplemented with the following boundary conditions :-

<u>Either</u>

Free - Free boundaries $\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0.0$ on $z = 0, 1$

Rigid - Rigid boundaries $\theta = \omega = u = v = 0.0$ on $z = 0, 1$

Rigid - Free boundaries $\theta = \omega = u = v = 0.0$ on $z = 0$

$$\theta = \omega = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0.0 \quad \text{on } z = 1$$

and

$$u = v = \omega = \theta = 0.0 \quad \text{on } r = L = \frac{d}{2h}$$

where L represents the aspect ratio of the cylinder.

By simple manipulation we can eliminate the horizontal velocities and the pressure from equations (1:2) - (1:15), and then using (1:16) we reduce our problem to the single sixth order equation

$$\nabla^6 \theta + Ra\,(\lambda z - 1)\,\nabla_H^2 \theta = 0.0 \qquad - (1:17)$$

where $\nabla_H^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{n^2}{r^2}$

Equation (1:17) does not however give a full description of the problem since equations (1:12) - (1:16) describe an eighth order system while (1:17) is only of sixth order. The missing equation is that for the vertical vorticity (see Chandrasekhar (2)) which in the infinite problem is satisfied identically. However Joseph (3) (see also Jones and Moore (4)) pointed out that in a finite container the vertical vorticity does not vanish on the vertical boundaries and that we need to take it into account when solving the problem. We do this by noticing that it introduces a complementary function on solving for the horizontal velocities u and V.

1:3 Solution for $L \sim 1$

We now turn our attention to constructing a solution for a rigid cylinder with aspect ratio of order unit from the solutions of equation (1:17).

The first step in doing this is to notice that separable solutions of (1:17) which are finite on the axis(+) take the form

$$\theta(r, z) = J_n(ar)\, \theta(z)$$

Where $J_n(ar)$ represents Bessel's function of the first kind, a is the horizontal wave number and $\theta(z)$ satisfies

$$(D^2 - a^2)^3 \theta(z) = a^2 Ra (\lambda z - 1)\, \theta(z) \qquad - (1:19)$$

Using (1:18) in (1:12) - (1:16) we also get that

$$W(r, z) = (D^2 - a^2)\, \theta(z)\, J_n(ar) \qquad - (1:20)$$

$$p(r, z) = \frac{1}{a^2} (D^2 - a^2) DW(z)\, J_n(ar) \qquad - (1:21)$$

$$u(r, z) = \frac{1}{a^2} \frac{d J_n(ar)}{dr} DW(z) +$$

$$+ \frac{J_n(i\gamma r)}{r} \left[A \sin \gamma z + B \cos \gamma z \right] \qquad - (1:22)$$

(+) By being able to choose the functions $J_n(z)$ we avoid any difficulties associated with the coordinate system as $r \to 0$, see Batchelor and Gill (31).

$$V(r,z) = \frac{in}{a^2 r} J_n(ar)\, DW(z) + \frac{i}{n} \frac{dJ_n}{dr}(i\gamma r)\left[A \sin\gamma z + B\cos\gamma z\right] \qquad - (1{:}23)$$

where $W(z) = (D^2 - a^2)\theta(z)$, and A , B and γ are constants to be determined. The $A \sin\gamma z + B\cos\gamma z$ terms in (1:22) and (1:23) represent the contributions from the vertical vorticity.

To specify the problem completely we need only supplement the equations with appropriate boundary condition. For example considering free boundaries top and bottom we have

$$\theta = W = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0. \qquad z = 0, 1$$

So that for these conditions to hold we must have that $\gamma = m\pi$, with m an integer, and $A = 0$. Furthermore with the aid of equations (1:19) - (1:23) the above boundary conditions can be written down as

$$\theta = D^2\theta = D^4\theta = 0. \qquad z = 0, 1$$

Similar expressions hold for rigid-rigid and rigid-free boundaries.

With the boundary conditions in this form equation (1:19) forms an eigenvalue problem for the horizontal wave number if Ra is specified (or vice versa).

In general there will be an infinite, but countable, set of eigenvalues (with positive real part) which we shall denote by a_m $(m = 1, 2, 3 \ldots)$, with the corresponding solutions being denoted by θ_m , etc. With the boundary conditions at the top and bottom of the cylinder we have only to satisfy the conditions on the side wall $r = L$, which are that

$$\theta = u = v = w = 0. \qquad r = L$$

We intend to achieve this by summing over an infinite number of horizontal modes. Naturally, for practical reasons, we can only sum over a finite number of horizontal modes and we therefore seek a solution in the form

$$\theta(r, z) = \sum_{m=1}^{NT} \beta_m \, J_n(a_m r) \, \theta_m(z) \qquad - (1:24)$$

$$W(r, z) = \sum_{1}^{NT} \beta_m \, J_n(a_m r) \, W_m(z) \qquad - (1:25)$$

$$u(r, z) = \sum_{1}^{NT} \left[\frac{\beta_m}{a_m^2} \frac{dJ_n}{dr}(a_m r) \right] DW_m(z) + $$
$$+ \sum_{p=1}^{MT} B_p \frac{J_n}{r}(ip\pi r) \cos p\pi z \qquad - (1:26)$$

$$v(r, z) = \sum_{1}^{NT} \frac{in\beta_m}{a_m^2 r} J_n(a_m r) \, DW_m + $$
$$+ \sum_{p=1}^{MT} \frac{iB_p}{n} \frac{dJ_n}{dr}(ip\pi r) \cos(p\pi z) \qquad - (1:27)$$

where the β_m and B_p's are constants to be determined and NT is the number of modes considered and $MT = NT/3$

The reason why there is only one vertical vorticity mode (i.e. a B_p term) to three β_m terms is simple. In the standard infinite Bénard problem the eigenvalue problem reduces to solving a cubic equation for a^2. It is obvious therefore that equation (1:19) will also produce wave numbers in natural sets of three. In turn three wave numbers produce three unknown constants

β_m but have four boundary conditions on $r = L$. Hence we require one B_p term for every three β_m terms.

Our side wall conditions then become :-

$$\sum_{m=1}^{NT} \beta_m' \theta_m(z) = 0.0 \qquad - (1:28)$$

$$\sum_{m=1}^{NT} \beta_m' W_m(z) = 0.0 \qquad - (1:29)$$

$$\sum_{m=1}^{NT} \beta_m' F_{1m} DW_m(z) + \\ + \sum_{p=1}^{mT} B_p' \frac{\cos(p\pi z)}{L} = 0.0 \qquad - (1:30)$$

$$\sum_{m=1}^{NT} \frac{n}{a_m^2 L} \beta_m' DW_m(z) + \\ + \sum_{p=1}^{mT} B_p' F_{2p} \cos(p\pi z) = 0.0 \qquad - (1:31).$$

Where

$$\beta_m' = \beta_m J_n(a_m L)$$

$$B_p' = B_p J_n(ip\pi L)$$

$$F_{1m} = \frac{1}{a_m^2 J_n(a_m L)} \frac{d}{dr} J_n(a_m L)$$

$$F_{2m} = \frac{1}{n J_n(ip\pi L)} \frac{d}{dr} J_n(ip\pi L)$$

By eliminating B_1' from equations (1:30) and (1:31) we can obtain the following equation

$$\sum_{m=1}^{NT} \beta_m' F_{3m}(z) = 0.0 \qquad - (1:32)$$

where $$F_{3m}(z) = \left[F_{1m} F_{21} - \frac{n}{a_m^2 L^2} \right] DW(z) -$$

$$- 2 F_{1m} \sum_{p=2}^{mT} (F_{21} - F_{2p}) I_{mp} \cos(p\pi z)$$

and

$$I_{mp} = \int_0^1 DW_m(z) \cos(p\pi z) \, dz$$

In order to determine the coefficients β_m' we need to use an orthogonality condition on the modes $\theta_m(z)$. This is constructed by first introducing the following vector notation

$$\underset{\sim}{\theta}_m = (\theta_m , W_m , \xi_m)^T \qquad - (1:33)$$

where $W_m = (D^2 - a_m^2) \theta_m$, $\xi_m = (D^2 - a_m^2)^2 \theta_m$

Equations (1:28), (1:29) and (1:32) can then be written in the more succinct form

$$\sum_{m=1}^{NT} \beta_m' (\underset{\sim}{A}_m \underset{\sim}{\theta}_m + \underset{\sim}{A1}_m) = 0.0 \qquad - (1:34)$$

where

$$\underset{\sim}{A}_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & F_{4m} D & 0 \end{bmatrix} \qquad - (1:35)$$

and

$$\underset{\sim}{A1}_m = - \begin{bmatrix} 0 \\ 0 \\ F_{5m} \end{bmatrix} \qquad - (1:36)$$

with

$$F_{4m} = \left[F_{1m} F_{21} - \frac{n}{a_m^2 L^2} \right]$$

$$F_{5m} = 2 F_{1m} \sum_{p=2}^{mT} (F_{21} - F_{2p}) I_{mp} \cos (p \pi z)$$

Then an orthogonality condition can be shown to exist between the vectors $\underset{\sim}{\theta}_m$ and their adjoints $\underset{\sim}{\theta}_m^*$ in the following way. Equation (1:19) when expressed as a first order system becomes

$$D \begin{bmatrix} \phi_m \\ \psi_m \\ \pi_m \\ \theta_m \\ W_m \\ \xi_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & a_m^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & a_m^2 & 1 \\ 0 & 0 & 0 & \sigma(z) & 0 & a_m^2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_m \\ \psi_m \\ \pi_m \\ \theta_m \\ W_m \\ \xi_m \end{bmatrix} \quad \text{- (1:37)}$$

where we have put $\sigma(z) = a_m^2 \, Ra \, (\lambda z - 1)$

Following the procedure for finding the adjoint of such a system as described by Ince (5) we find that the first order system for $\underset{\sim}{\theta}_m^*$ is

$$D \begin{bmatrix} \phi_m^* \\ \psi_m^* \\ \pi_m^* \\ \theta_m^* \\ W_m^* \\ \xi_m^* \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_m^2 & 0 & \sigma(z) & 0 & 0 & 0 \\ 1 & a_m^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_m^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_m^* \\ \psi_m^* \\ \pi_m^* \\ \theta_m^* \\ W_m^* \\ \xi_m^* \end{bmatrix} \quad \text{- (1:38)}$$

From (1:37) and (1:38) we can write the equations for $\underset{\sim}{\theta}_m$ and $\underset{\sim}{\theta}_n^{*T}$ as

$$D^2 \underset{\sim}{\theta}_m = a_m^2 \underset{\sim}{P} \underset{\sim}{\theta}_m + \underset{\sim}{Q} \underset{\sim}{\theta}_m \qquad - (1:39)$$

$$D^2 \underset{\sim}{\theta}_n^{*T} = a_n^2 \underset{\sim}{\theta}_n^{*T} \underset{\sim}{P} + \underset{\sim}{\theta}_n^{*T} \underset{\sim}{Q} \qquad - (1:40)$$

where

$$\underset{\sim}{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Ra(\lambda z - 1) & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{Q} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the boundary conditions on $\underset{\sim}{\theta}_n^* = (\phi_m^*, \psi_n^*, \pi_n^*)^T$

are, for free-free boundaries

$$\phi_m^* = \psi_n^* = \pi_n^* \qquad z = 0, 1$$

Premultiplying (1:39) by $\underset{\sim}{\theta}_n^{*T}$ and postmultiplying (1:40) by $\underset{\sim}{\theta}_m$, subtracting and integrating by z yields

$$(a_m^2 - a_n^2) \int_0^1 \underset{\sim}{\theta}_n^{*T} \underset{\sim}{P} \underset{\sim}{\theta}_m \, dz = \left[\underset{\sim}{\theta}_n^{*T} D \underset{\sim}{\theta}_m \right]_0^1 - \left[D \underset{\sim}{\theta}_n^{*T} \underset{\sim}{\theta}_m \right]_0^1 \qquad - (1:41)$$

Applying the specified boundary conditions on $\underset{\sim}{\theta}_m$ and $\underset{\sim}{\theta}_n^{*}$ shows that the right hand side of (1:41) vanishes, leaving us with the following orthogonality condition :-

$$\int_0^1 \underset{\sim}{\theta}_n^{*T} \underset{\sim}{P} \, \underset{\sim}{\theta}_m \, dz = 0.0 \qquad (m \neq n) \qquad - (1:42)$$

On using (1:42) upon equation (1:34) we get

$$\sum_{m=1}^{NT} \beta_m' \, c_{rm} = 0.0 \qquad - (1:43)$$

where

$$c_{rm} = \int_0^1 \underset{\sim}{\theta}_r^{*T} \underset{\sim}{P} \left[\underset{\sim}{A}_m \underset{\sim}{\theta}_m + \underset{\sim}{A1}_m \right] dz \qquad - (1:44)$$

In matrix notation (1:43) becomes

$$\underset{\sim}{C} \cdot \underset{\sim}{\beta}' = 0.0 \qquad - (1:45)$$

where the matrix is a function of the unknown aspect ratio . For a solution to (1:45) to exist we must have that

$$Det(\underset{\sim}{C}) = 0.0 \qquad - (1:46)$$

Thus a given Rayleigh number is the critical Rayleigh number of a finite cylinder with aspect ratio L if L is the root of equation (1:46), with the matrix $\underset{\sim}{C}$ being calculated from the linear eigen functions appropriate to that Rayleigh number.

It is obvious that the solution of (1:46) will involve an iterative process (details of the actual numerical procedure will be given later) and therefore a good initial guess is highly desirable. This we can obtain by considering two asymptotic limits of the problem, which are (i) the height of the cylinder is regarded as fixed, while the radius is allowed to become large i.e. $L \to \infty$ and (ii) the radius remains fixed and the height becomes large i.e. $L \to 0$

1:4. Asymptotic solution for $L \to \infty$

Here we are expanding about the infinite plane parallel case (see Veronis (1)) where our small parameter is $\varepsilon = 1/L$. Furthermore our task is simplified by the fact that we do not need to take into account vertical vorticity since this is a much higher order effect, see Chandrasekhar (2).

Our starting point is equation (1:19) which when using the notation of the vector (1:33) can be written in the form :

$$(D^2 - a^2)\,\theta(z) = W(z) \qquad - (1:47)$$

$$(D^2 - a^2)\,W(z) = \xi(z) \qquad - (1:48)$$

$$(D^2 - a^2)\,\xi(z) = Ra\,(\lambda z - 1)\,a^2\,\theta(z) \qquad - (1:49)$$

We expand out the dependent quantities in the following manner

$$(\theta, \omega, \xi) = (\theta_0, \omega_0, \xi_0) + \varepsilon(\theta_1, \omega_1, \xi_1) + \cdots \quad - (1:50)$$

$$(Ra, a) = (Ra_0, a_0) + \varepsilon(Ra_1, a_1) + \cdots \quad - (1:51)$$

Substituting these expansions into equations (1:47) - (1:49) and collecting together terms of like order in ε yields the following sets of equations :-

<u>0(1)</u>

$$(D^2 - a_0^2)\theta_0 = \omega_0 \quad - (1:52)$$

$$(D^2 - a_0^2)W_0 = \xi_0 \quad - (1:53)$$

$$(D^2 - a_0^2)\xi_0 = a_0^2 \, Ra_0 (\lambda z - 1)\theta_0 \quad - (1:54)$$

<u>0(ε)</u>

$$(D^2 - a_0^2)\theta_1 - \omega_1 = 2a_1 a_0 \theta_0 \quad - (1:55)$$

$$(D^2 - a_0^2)\omega_1 - \xi_1 = 2a_1 a_0 \omega_0 \quad - (1:56)$$

$$\begin{aligned}(D^2 - a_0^2)\xi_1 - a_0^2 Ra_0(\lambda z - 1)\theta_1 &= \\ a_1[2a_0\xi_0 + 2Ra_0 a_0(\lambda z - 1)\theta_0] &+ \\ Ra_1 a_0^2(\lambda z - 1)\theta_0 & \end{aligned} \quad - (1:57)$$

Equations (1:52) - (1:54) naturally are the same equations solved by Veronis (1) for the infinite plane parallel case so that (Ra_0, a_0) are the critical Rayleigh and wave number respectively, for that problem.

For equations (1:55) - (1:57) to have a solution the integral formed by mutliplying the right hand side by the adjoint vector and integrating over z must vanish (see Ince (5)) i.e.

$$2a_1 a_0 I_1 + Ra_1 a_0^2 I_2 = 0.0 \qquad \text{- (1:58)}$$

where

$$I_1 = \int_0^1 \left[\xi_0 \pi^* + Ra_0 (\lambda z - 1) \theta_0 \pi^* + W_0 \psi^* + \theta_0 \phi^* \right] \qquad \text{- (:59)}$$

$$I_2 = \int_0^1 (\lambda z - 1)\, \theta_0\, \pi^* \, dz \qquad \text{- (1:60)}$$

By solving the eigenvalue problem (1:52) - (1:54) using a fourth order Runge-Kutta method and evaluating the integrals I_1 and I_2 we find that I_1 vanishes while I_2 is non-zero. So that (1:58) gives the result that Ra_1 is identically zero, as is to be expected since a_0 is that wave number which minimizes Ra.

With this result the order ε^2 equations are

$$(D^2 - a_0^2)\theta_2 - W_2 = a_1[\theta_0 + 2a_0\theta_1] + 2a_0 a_2 \theta_0 \qquad \text{- (1:61)}$$

$$(D^2 - a_0^2)W_2 - \xi_2 = a_1[W_0 + 2a_0 W_1] + 2a_0 a_2 W_0 \qquad \text{- (1:62)}$$

$$(D^2 - a_0^2)\xi_2 - a_0^2 Ra_0 (\lambda z - 1)\theta_2 = a_1 \xi_0 + 2a_0 a_2 \xi_0 + \\ + 2a_0 a_1 \xi_1 + Ra_2 (\lambda z - 1) a_0^2 \theta_0 + 2 Ra_0 a_0 a_2 (\lambda z - 1)\theta_0 + \\ + a_1 \left[Ra_0 (\lambda z - 1)\theta_0 + 2 Ra_0 a_0 (\lambda z - 1)\theta_1 \right] \qquad \text{- (1:63)}$$

Applying the same principle as before, a solution only exists if the following expression holds :-

$$a_1 I_3 + a_1 I_4 + a_2 I_5 + a_0^2 Ra_0 I_6 = 0.0 \qquad - (1:64)$$

$$I_3 = \int_0^1 [Ra_0(\lambda z-1)\theta_0 \pi^* + w_0 \psi^* + \theta_0 \phi^* + \xi_0 \pi^*]\, dz \qquad - (1:65)$$

$$I_4 = \int_0^1 [2Ra_0 a_0 (\lambda z-1)\theta_1 \pi^* + 2a_0 w_1 \psi^* + 2a_0 \theta_1 \phi^* + 2a_0 \xi_1 \pi^*]\, dz \qquad - (1:66)$$

$$I_5 = \int_0^1 [2Ra_0 a_0 (\lambda z-1)\theta_0 \pi^* + 2a_0 w_0 \psi^* + 2a_0 \theta_0 \phi^* + 2a_0 \xi_0 \pi^*]\, dz \qquad - (1:67)$$

$$I_6 = \int_0^1 (\lambda z - 1)\, \theta_0\, \pi^*\, dz \qquad - (1:68)$$

Clearly I_3 and I_5 are proportional to I_1 and therefore vanish. Furthermore since $Ra_1 = 0$ it is obvious from (1:55) - (1:57) that we can write

$$(\theta_1, w_1, \xi_1) = a_0 a_1 (\theta_1, w_1, \xi_1)$$

Then (1:64) reduces to

$$a_1^2 I_4 + Ra_2 I_6 = 0.0 \qquad - (1:69)$$

so that

$$a_1^2 = - Ra_2 \left[\frac{I_6}{I_4} \right] \qquad - (1:70)$$

Since the ratio I_6/I_4 is always negative the correction to the Rayleigh number indues a splitting of a_1 into a positive and negative value, allowing us to write the boundary conditions on the side wall to first order in ε as :-

$$(\theta_0 + \varepsilon\theta_1)\left\{A_1 J_n[(a_0+\varepsilon\alpha)L] + A_2 J_n[(a_0-\varepsilon\alpha)L]\right\} = 0.0 \qquad - (1:71)$$

$$(w_0 + \varepsilon w_1)\left\{A_1 J_n[(a_0+\varepsilon\alpha)L + A_2 J_n[(a_0-\varepsilon\alpha)L]\right\} = 0.0 \qquad - (1:72)$$

$$(Dw_0 + \varepsilon Dw_1)\left\{\frac{A_1}{(a_0+\varepsilon\alpha)^2}\left(\frac{d J_n[(a_0+\varepsilon\alpha)L]}{dr} - \frac{n}{L} J_n[(a_0+\varepsilon\alpha)L]\right) + \frac{A_2}{(a_0-\varepsilon\alpha)^2}\left(\frac{d J_n[(a_0-\varepsilon\alpha)L]}{dr} - \frac{n}{L} J_n[(a_0-\varepsilon\alpha)L]\right)\right\} = 0.0$$

- (1:73)

where we have obtained (1:73) by forming the sum $u + iv$
A_1 and A_2 are coefficients to be determined and

$$\alpha = \left(-Ra_2\frac{I_6}{I_4}\right)^{1/2}$$

For A_1 and A_2 to have non-trivial values we must have that

$$J_n[(a_0+\varepsilon\alpha)1/\varepsilon]\left\{J_n'[(a_0-\varepsilon\alpha)1/\varepsilon] - \varepsilon n J_n[(a_0-\varepsilon\alpha)1/\varepsilon]\right\}$$

$$= J_n[(a_0-\varepsilon\alpha)1/\varepsilon]\left\{J_n'[(a_0+\varepsilon\alpha)1/\varepsilon] - \varepsilon n J_n[(a_0+\varepsilon\alpha)1/\varepsilon]\right\}$$

- (1:74)

By using the asymptotic expansions of $J_n(z)$ and $J_n'(z)$ for large z (see Abramowitz and Stegun (6)) (1:74) can be reduced to

$$\frac{\sin(2\alpha)}{\cos(\gamma+\alpha)\cos(\gamma-\alpha)} = 0.0 \qquad - (1:75)$$

where $\gamma = \frac{a_0}{\varepsilon} - \frac{n\pi}{2} - \frac{\pi}{4}$

Hence $\alpha = \frac{m\pi}{2}$ $(m = 1, 2, \cdots)$ so that

$$Ra_2 = -\frac{m^2\pi^2}{4}\left(\frac{I_4}{I_6}\right) \qquad - (1:76)$$

Table 1 lists the solutions to the eigenvalue problem posed by equations (1:52) - (1:54) and their corresponding values of Ra_2 for various values of λ and the three types of boundary conditions at $z = 0, 1$.

One way of checking our results is to note that for $\lambda \geqslant 1.0$ for rigid-rigid boundaries should be identical to the critical Taylor number for flow between rotating concentric cylinders which we find to be the case (see Chandrasekhar (2) and Drazin and Reid (6)).

Also for $\lambda = 0.0$ and free-free boundaries we expect our results to give the same asymptotic formula as that given by Daniels (7) for ordinary Bénard convection (see also Brown and Stewartson (15)). From table 1 we have that

$$Ra = 657.514 + \frac{438.37}{L^2} \qquad - (1:77)$$

while Daniels gives

$$Ra = 657.514 + \frac{438.34}{L^2} \qquad - (1:78)$$

The agreement between the results is clear.

1:5. Asymptotic solution for $L \rightarrow 0$

We now turn our attention to the other asymptotic limit in which the radius remains fixed and the height of the cylinder is allowed to become large. In this case we are only able to consider free-free boundaries top and bottom since for rigid-rigid and rigid-free boundaries all the vertical modes are necessary to satisfy the boundary conditions at $z = 0$.

Since we are considering a cylinder of large vertical extent we expect that the vertical derivatives which balance the buoyancy terms will become small and therefore some form of rescaling will be required. With this in mind we non-dimensionalize the original equations (1:1) - (1:6) in the following manner :-

$$r \rightarrow a r$$

$$z \rightarrow h z$$

$$\underline{u} \rightarrow \frac{\kappa}{a} \underline{u}$$

See also Azouni and Normand. (30)

$$\theta \rightarrow \lambda T_0 \theta$$

$$p \rightarrow \frac{\nu \kappa}{a^2} \rho_0 p$$

We then find that the linear perturbation equations become

$$\delta \frac{\partial p}{\partial z} = Ra(\lambda z - 1)\theta + \nabla_H^2 \omega + \delta^2 \frac{\partial^2 \omega}{\partial z^2} \qquad - (1:79)$$

$$\delta \omega = \nabla_H^2 \theta + \delta^2 \frac{\partial^2 \theta}{\partial z^2} \qquad - (1:80)$$

The horizontal momentum equations and the equation of continuity remaining unchanged except that $\partial/\partial z$ is replaced by $\delta\, \partial/\partial z$, and that now

$$Ra = \lambda \alpha g \overline{T}_o^2 a^3/\nu k$$

Also $\delta = a/h \ll 1$, and is the small parameter about which we will expand.

It is clear from equation (1:79) that the vertical dependence of the buoyancy force (i.e. the λz term) is not in balance with the vertical derivatives. Indeed the buoyancy term is $O(1)$ in δ while the vertical derivatives are much smaller.

To achieve the necessary balance we rescale in z by writing

$$z = \delta^\alpha \bar{z} \qquad 0 < \alpha < 1$$

Substituting this expression into the equations it becomes obvious that to maintain a balance of terms we must scale ω as $\delta^{\alpha-1}$, θ as δ^α and write $\overline{Ra} = \delta Ra$. With these scalings we then must have that $\alpha = 2/3$.

Our fully scaled equations are then

$$\frac{\partial p}{\partial r} = \nabla_H^2 u + \delta^{2/3}\frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi} \qquad - (1:81)$$

$$\frac{1}{r}\frac{\partial p}{\partial \phi} = \nabla_H^2 v + \delta^{2/3}\frac{\partial^2 v}{\partial z^2} + \frac{2}{r^2}\frac{\partial u}{\partial \phi} - \frac{v}{r^2} \qquad - (1:82)$$

$$\delta^{2/3}\frac{\partial p}{\partial z} = \overline{Ra}\,(\lambda z \delta^{2/3} - 1)\theta + \nabla_H^2 w + \delta^{2/3}\frac{\partial^2 w}{\partial z^2} \qquad - (1:83)$$

$$w = \nabla_H^2 \theta + \delta^{2/3}\frac{\partial^2 \theta}{\partial z^2} \qquad - (1:84)$$

$$\frac{1}{r}\frac{\partial (r u)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0.0 \qquad - (1:85)$$

Expanding out the dependent quantities in the following manner :

$$(u, v, w) = (u_0, v_0, w_0) + \delta^{2/3}(u_1, v_1, w_1) + \cdots$$

$$\theta = \theta_0 + \delta^{2/3}\theta_1 + \cdots$$

$$p = \delta^{-2/3} p_*^{*} + p_0 + \delta^{2/3} p_1 + \cdots$$

$$\overline{Ra} = \overline{Ra}_0 + \delta^{2/3}\,\overline{Ra}_1 + \cdots$$

here the presence of the p^* term will be discussed later.

Substituting the above expansions into equations (1:81) - (1:85) yields :-

$\underline{O(\delta^{-2/3})}$

$$\frac{\partial p^*}{\partial r} = 0. \quad , \quad \frac{1}{r}\frac{\partial p^*}{\partial \phi} = 0. \qquad p^* = p^*(r) \tag{1:86}$$

$\underline{O(1)}$

$$\frac{\partial p_0}{\partial r} = \nabla_H^2 u_0 - \frac{u_0}{r^2} - \frac{2}{r^2}\frac{\partial v_0}{\partial \phi} \tag{1:87}$$

$$\frac{1}{r}\frac{\partial p_0}{\partial \phi} = \nabla_H^2 v_0 + \frac{2}{r^2}\frac{\partial u_0}{\partial \phi} - \frac{v_0}{r^2} \tag{1:88}$$

$$\frac{\partial p^*}{\partial z} = -\overline{Ra}_0\,\theta_0 + \nabla_H^2 \omega_0 \tag{1:89}$$

$$\omega_0 = \nabla_H^2 \theta_0 \tag{1:90}$$

$$\frac{1}{r}\frac{\partial (r u_0)}{\partial r} + \frac{1}{r}\frac{\partial v_0}{\partial \phi} + \frac{\partial \omega_0}{\partial z} = 0.0 \tag{1:91}$$

$\underline{O(\delta^{2/3})}$

$$\frac{\partial p_1}{\partial r} = \nabla_H^2 u_1 + \frac{\partial^2 u_0}{\partial z^2} - \frac{u_1}{r^2} - \frac{2}{r^2}\frac{\partial v_1}{\partial \phi} \qquad - (1{:}92)$$

$$\frac{1}{r}\frac{\partial p_1}{\partial \phi} = \nabla_H^2 v_1 + \frac{\partial^2 v_0}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_1}{\partial \phi} - \frac{v_1}{r^2} \qquad - (1{:}93)$$

$$\frac{\partial p_0}{\partial z} = \overline{Ra}_0 \lambda z \theta_0 - \overline{Ra}_0 \theta_1 - \overline{Ra}_1 \theta_0 + \nabla_H^2 \omega_1 + \frac{\partial^2 \omega_0}{\partial z^2} \qquad - (1{:}94)$$

$$\omega_1 = \nabla_H^2 \theta_1 + \frac{\partial^2 \theta_0}{\partial z^2} \qquad - (1{:}95)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_1) + \frac{1}{r}\frac{\partial v_1}{\partial \phi} + \frac{\partial \omega_1}{\partial z} = 0.0 \qquad - (1{:}96)$$

The boundary conditions being $\theta = \omega = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0.$ at $z = 0.$ and as $z \to \infty$

$$u = v = \omega = \theta = 0 \quad \text{and} \quad r = L$$

Solutions for θ_0 , ω_0 and p_0 which remain finite on the axis $r = 0$ are

$$\theta_0 = A(z)\left[B_1 J_n(\gamma r) + B_2 I_n(\gamma r)\right] - \frac{1}{Ra_0}\frac{dp^*}{dz} \qquad - (1:97)$$

$$\omega_0 = \gamma^2 A(z)\left[B_2 I_n(\gamma r) - B_1 J_n(\gamma r)\right] \qquad - (1:98)$$

$$p_0 = \gamma^2\left[B_1 J_n(\gamma r) - B_2 I_n(\gamma r)\right]\frac{dA}{dz} + p(z) r^n \qquad - (1:99)$$

$$(n = 0, 1, 2, \dots)$$

where B_1 , B_2 are constants and $\gamma^4 = \overline{Ra_0}$, J_n and I_n are Bessel and modified Bessel functions of the first kind respectively and $A(z)$ and $p(z)$ are functions to be determined.

In order to obtain the horizontal velocity components u_0 , v_0 we need to consider the axisymmetric and asymmetric cases separately.

(i) <u>The asymmetric case $n \neq 0$</u>

In this case the complete solutions for u_0 and v_0 are

$$u_0 = \frac{n r^{n+1}}{4n+4} p(z) + B_3 r^{n-1} u_{cF}(z)$$

$$- \frac{dA}{dz}\left[B_1 \frac{dJ_n(\gamma r)}{dz} + B_2 \frac{dI_n(\gamma r)}{dr}\right] \qquad - (1:100)$$

$$V_6 = \frac{i(n+2)}{4n+4} r^{n+1} p(z) + B_3 r^{n-1} u_{CF}(z)$$

$$- \frac{in}{r} \frac{dA}{dz} \left[B_1 J_n(\gamma r) + B_2 I_n(\gamma r) \right] \qquad - (1:101)$$

where B_3 is a constant and $u_{CF}(z)$ represents the vertical dependence of the complementary function.

Applying the boundary condition at $r=1$ we must obviously have $p(z) = B_4 \frac{dA}{dz}$, $u_{CF}(z) = \frac{dA}{dz}$ and $\frac{1}{Ra_0} \frac{dp^*}{dz} = B_5 A(z)$ with B_4 , B_5 constants, to obtain

$$B_1 J_n(\gamma) + B_2 I_n(\gamma) - B_5 = 0. \qquad - (1:102)$$

$$B_2 I_n(\gamma) - B_1 J_n(\gamma) = 0. \qquad - (1:103)$$

$$\frac{n B_4}{4n+4} + B_3 = B_1 \frac{dJ_n(\gamma)}{dr} + B_2 \frac{dI_n(\gamma)}{dr} \qquad - (1:104)$$

$$\frac{n+2}{4n+4} B_4 + B_3 = n\left[B_1 J_n(\gamma) + B_2 I_n(\gamma) \right] \qquad - (1:105)$$

Equations (1:102) - (1:105) are indeterminant as they stand since we have four equations and five unknowns. However since we introduced the $\delta^{-2/3} p^*$ term into the equations somewhat arbitrarily we can set $B_5 = 0.$.

Then for the above equations to have a solution we must have

$$2\, J_n(\gamma)\, I_n(\gamma) = 0.$$

Since $I_n(\gamma) > 0$ a solution only exists if $J_n(\gamma) = 0$ so that γ must be the first root of $J_n(r)$, and the critical Rayleigh number becomes

$$\overline{Ra}_{on} = \gamma_n^4$$

A result that agrees with that given by Gershuni and Zhukhovitski (8) for a cylinder of circular cross section. Table 2 lists the numerical values of γ_n and $\overline{Ra}_{on}$ for various values of n .

With $B_5 = 0$, and setting $B_1 = 1$ we then obtain from $(1:102) - (1:105)$

$$B_3 = \frac{(n+2)}{2} \frac{d J_n(\gamma)}{dr} \qquad - (1:106)$$

$$B_4 = -(2n+2) \frac{d J_n(\gamma)}{dr} \qquad - (1:107)$$

Summarizing the zero order solution for $n \neq 0$ we have :

$$\theta_0 = A(z)\, J_n(\gamma r) \qquad - (1:108)$$

$$\omega_0 = -\gamma^2 A(z)\, J_n(\gamma r) \qquad - (1:109)$$

$$u_0 = \left[\frac{B_4 n}{4n+4} r^{n+1} + B_3 r^{n-1} - \frac{d J_n(\gamma r)}{dr} \right] \frac{dA}{dz} \qquad \text{- (1:110)}$$

$$v_0 = i \left[\frac{B_4 (n+2)}{4n+4} r^{n+1} + B_3 r^{n-1} - \frac{n}{r} J_n(\gamma r) \right] \frac{dA}{dz} \qquad \text{- (1:111)}$$

$$p_0 = \left[\gamma^2 J_n(\gamma r) + B_4 r^n \right] \frac{dA}{dz} \qquad \text{- (1:112)}$$

The unknown function $A(z)$ can now be determined from equations (1:95) and (1:94), which become on substituting in the expressions for θ_0 , ω_0 and p_0

$$\nabla_H^2 \theta_1 - \omega_1 = - J_n(\gamma r) \frac{d^2 A}{dz^2} \qquad \text{- (1:113)}$$

$$\nabla_H^2 \omega_1 - \overline{Ra}_0 \theta_1 = \left[2\gamma^2 J_n(\gamma r) + B_4 r^n \right] \frac{d^2 A}{dz} + \overline{Ra}_0 \left[\frac{\overline{Ra}_1}{\overline{Ra}_0} - \lambda z \right] J_n(\gamma r) A(z) \qquad \text{- (1:114)}$$

As before a solution to these equations exists only if the integral formed by multiplying the right hand sides by their respective adjoints and integrating (over r) vanishes.

Following the standard procedure for finding the adjoint (see Ince (5)) we find that the appropriate multiplying factors are

$$-\gamma^2 r \, J_n(\gamma r) \qquad \text{for} \qquad - (1:113)$$

and

$$r \, J_n(\gamma r) \qquad \text{for} \qquad - (1:114)$$

so that our solvability condition is

$$\mu_{1n}(\gamma)\frac{d^2A}{dz^2} + \mu_{2n}(\gamma)\left[\frac{\overline{Ra}_1}{\overline{Ra}_0} - \lambda z\right]A(z) = 0. \qquad - (1:115)$$

where

$$\mu_{1n}(\gamma) = \int_0^1 \left\{3\gamma^2 r\left[J_n(\gamma r)\right]^2 + B_4\, r^{n+1} J_n(\gamma r)\right\} dr \qquad - (1:116)$$

$$\mu_{2n}(\gamma) = \int_0^1 \overline{Ra}_{on}\, r\left[J_n(\gamma r)\right]^2 dr \qquad - (1:117)$$

The numerical values of μ_{1n} and μ_{2n} for various values of n are tabulated in Table 2.

By writing

$$s = -\Omega_n \left[\frac{\overline{Ra}_{1n}}{\overline{Ra}_{0n}} - \lambda z \right]$$

with

$$\Omega_n^3 = \left(\frac{\mu_{2n}}{\mu_{1n}} \right) \frac{1}{\lambda^2}$$

equation (1:116) is transformed into

$$\frac{d^2 A}{d s^2} - s\, A(s) = 0. \qquad - (1:118)$$

which is Airy's equation. Since we require that $A(z) \to 0$ as $z \to \infty$ and $A(z) = \frac{d^2 A(z)}{d z^2} = 0$ at $z = 0$ we choose that solution of the Airy equation which decays to infinity, while for $A(0) = 0$ we must have that

$$2 \cdot 3381 = \Omega_n \frac{\overline{Ra}_{1n}}{\overline{Ra}_{0n}} \qquad - (1:119)$$

where $2 \cdot 3381$ is the first root of $A(s)$.

Table 3 lists the values of Ω_n and $\overline{Ra}_{1n}$ for various values of n and λ.

(ii) <u>The axisymmetric case $n = 0$</u>

In this case we have that $V \equiv 0$ and we find that

$$u_0 = -\frac{dA}{dz}\left[B_1 \frac{d J_0(\gamma r)}{dr} + B_2 \frac{d I_0(\gamma r)}{dr}\right] \qquad - (1:120)$$

Then applying the boundary conditions at $r = 1$ to equations (1:97), (1:98) and (1:120) we obtain

$$\frac{1}{\gamma^{1/4}} \frac{dp^*}{dz} = B_3 A(z) \qquad - (121)$$

$$B_2 I_0(\gamma) - B_1 J_0(\gamma) = 0. \qquad - (1:122)$$

$$B_1 \frac{d J_0(\gamma)}{dr} + B_2 \frac{d I_0(\gamma)}{dr} = 0. \qquad - (1\_123)$$

$$B_1 J_0(\gamma) + B_2 I_0(\gamma) - B_3 = 0. \qquad - (1:124)$$

(B_1, B_2, B_3 constants).

For a solution to equations (1:122) - (1:124) to exist we must have that

$$\frac{1}{J_0(\gamma)} \frac{d J_0(\gamma)}{dr} + \frac{1}{I_0(\gamma)} \frac{d I_0(\gamma)}{dr} = 0. \qquad - (1:125)$$

which is a non-linear equation for γ , which upon being solved yields a value for γ of 4·611, agreeing with the value given by Gershuni and Zhukhovitski (8) for a cylinder of circular cross section. With $\gamma = 4.611$, $\overline{Ra}_0 = 452.043$

On putting $B_1 = 1.0$ we get

$$B_2 = -0.0152$$

$$B_3 = -0.587$$

Following the same procedure as for the case $n \neq 0$ we end up with the following equation for the amplitude $A(z)$:-

$$(3\gamma^2\mu_{01} - \gamma^2\mu_{02})\frac{d^2A}{dz^2} =$$

$$-\overline{Ra}_{00}\left[\frac{\overline{Ra}_{10}}{\overline{\overline{Ra}}_{00}} - \lambda z\right](\mu_{01} + \mu_{02} - \mu_{03}) \quad \text{- (1:126)}$$

where

$$\mu_{01} = B_1\int_0^1 r\, J_0^2(\gamma r)\, dr = 0.0767 \quad \text{- (1:127)}$$

$$\mu_{02} = B_2\int_0^1 r\, J_0(\gamma r)\, I_0(\gamma r)\, dr = -0.007 \quad \text{- (1:128)}$$

$$\mu_{03} = B_3\int_0^1 r\, J_0(\gamma r)\, dr = 0.033 \quad \text{- (1:129)}$$

Using the same notation as in the case $n \neq 0$ we will have

$$2 \cdot 3381 = \Omega_0 \frac{\overline{Ra}_{10}}{\overline{Ra}_{00}} \qquad - (1:130)$$

with

$$\Omega_0^3 = \left(\frac{3 \cdot 49}{\lambda^2}\right) \qquad - (1:131)$$

Consult Table 3 for numerical values of Ω_0 and $\overline{Ra}_{10}$ for various λ.

We now have a complete description of the asymptotics for the case $L \to 0$. However it should be noted that in order to compare the results of the asymptotics with those obtained in the case $L \to \infty$ we have to take into account that Ra is scaled on the radius of the cylinder while previously it was scaled on the height of the cylinder.

To make the conversion we note that for the $L \to 0$ asymptotics we have

$$Ra = \frac{\lambda \alpha g T_0^2 a^3}{\nu \kappa} = \delta^{-1}\left[\overline{Ra}_0 + \delta^{2/3}\overline{Ra}_1 + \cdots\right] \qquad - (1:132)$$

and therefore to convert we must multiply by δ^{-3}.

Hence our formula for the aspect ratio when comparing with the $L \to \infty$ asymptotics is

$$Ra = \delta^{-4}\left[\overline{Ra}_0 + \delta^{2/3}\overline{Ra}_1 + \cdots\right]$$

which is to be compared with

$$Ra = Ra_0 + \varepsilon^2 Ra_1 + \cdots \qquad - (1:133)$$

the formula for the $L \to \infty$ asymptotics.

1:6 Numerical Solution for $L \sim O(1)$

Having obtained asymptotic formulas for the cases $L \to 0$ and $L \to \infty$ we now return to the case $L \sim O(1)$ and discuss the numerical methods employed in solving equation (1:46).

$$\mathrm{Det}(\underset{\sim}{C}) = 0.0$$

As stated previously this equation is to be regarded as an equation for determining the unknown aspect ratio for a given Rayleigh number.

The matrix $\underset{\sim}{C}$ was constructed out of the linear eigenfunctions and wave numbers of equation (1:19) as well as its adjoint. For a given Rayleigh number these eigenfunctions and wave numbers were found by solving (1:19) using a fourth order Runga-Kutta method (see (13) Conte and de Boor). The values of the wave numbers being found by calculating the zeros pf the boundary value matrix using the secant method. An initial guess to these wave numbers being provided by the solutions of the Bernard Cubic $(\pi^2 + k^2)^3 = k^2 Ra$.

All values of the Bessel, and modified Bessel, functions were obtained by using the CERN program library subroutine COMBES, program number C303.

A Simpson composite rule was used for evaluating all integrals and the value of the determinant was obtained by using the NAG subroutine, number Fϕ3AHF.

The aspect ratio was then found by making an initial guess to equation (1:46) and iterating on this guess using the secant method or in some cases, when this proved difficult, Muller's method (see (13)) until convergence was achieved.

Tables 4 - 6 give the program results for free-free boundaries and $\lambda = 0.0$, 1.0 , 1.2 for the axisymmetric and first two asymmetric modes while tables 7 and 8 show the attempts to obtain solutions in the case of rigid-rigid boundaries. These results are depicted graphically in figures 1 to 4.

For $\lambda = 0.0$, ordinary Bernard convection, rigid-rigid boundaries and $n=0$ we can compare our results against those obtained by Charlson and Sani (14) who used a variational principle. The comparison, which is found to be good, is given in Table 14.

Tables 9 - 13 give the Rayleigh numbers for given aspect ratios calculated from the asymptotic formulas (1:132) and (1:133). The results show that the asymptotics match on to the solutions from the main program, although the $L \to 0$ asymptotics only become valid at the point where the program finds it increasingly difficult to obtain solutions.

Furthermore it becomes more difficult to obtain solutions as the vertical structure becomes more complicated i.e. λ increasing and the change to rigid:rigid boundary conditions.

1:7 Conclusions

We have shown that it is possible to solve the linear stability problem for convection in a rigid cylindrical container by expanding the solution in terms of the linear eigenfunctions of the infinite problem, and that good agreement is obtained with asymptotic formulae and methods used by other authors especially when the top and bottom of the cylinder are considered as free surfaces.

In the case when the lower boundary is rigid and the upper free, or both upper and lower boundaries are rigid, a solution becomes more difficult to find and consequently a higher number of modes is required decreasing the usefulness of the method. This is probably due to the greater complexity of the vertical structure and might be removed in part by achieving a greater accuracy in calculating the linear eigenvalue solutions.

Note also that it would be feasible to apply this method to investigate the weakly non-linear behaviour of the system but the greatly increased amount of work that would be required and the resulting awkwardness would render such an approach impracticable.

Furthermore the graphs of critical Rayleigh number against aspect ratio (figs 1-4) demonstrate quite clearly the interlacing of different azimuthal modes indicating that for certain aspect ratios linear theory cannot distinguish which azimuthal mode is the most suitable, and therefore a non-linear analysis is required at these points.

FIGURE 1.

FREE-FREE BOUNDARIES.

$\lambda = 0.0$

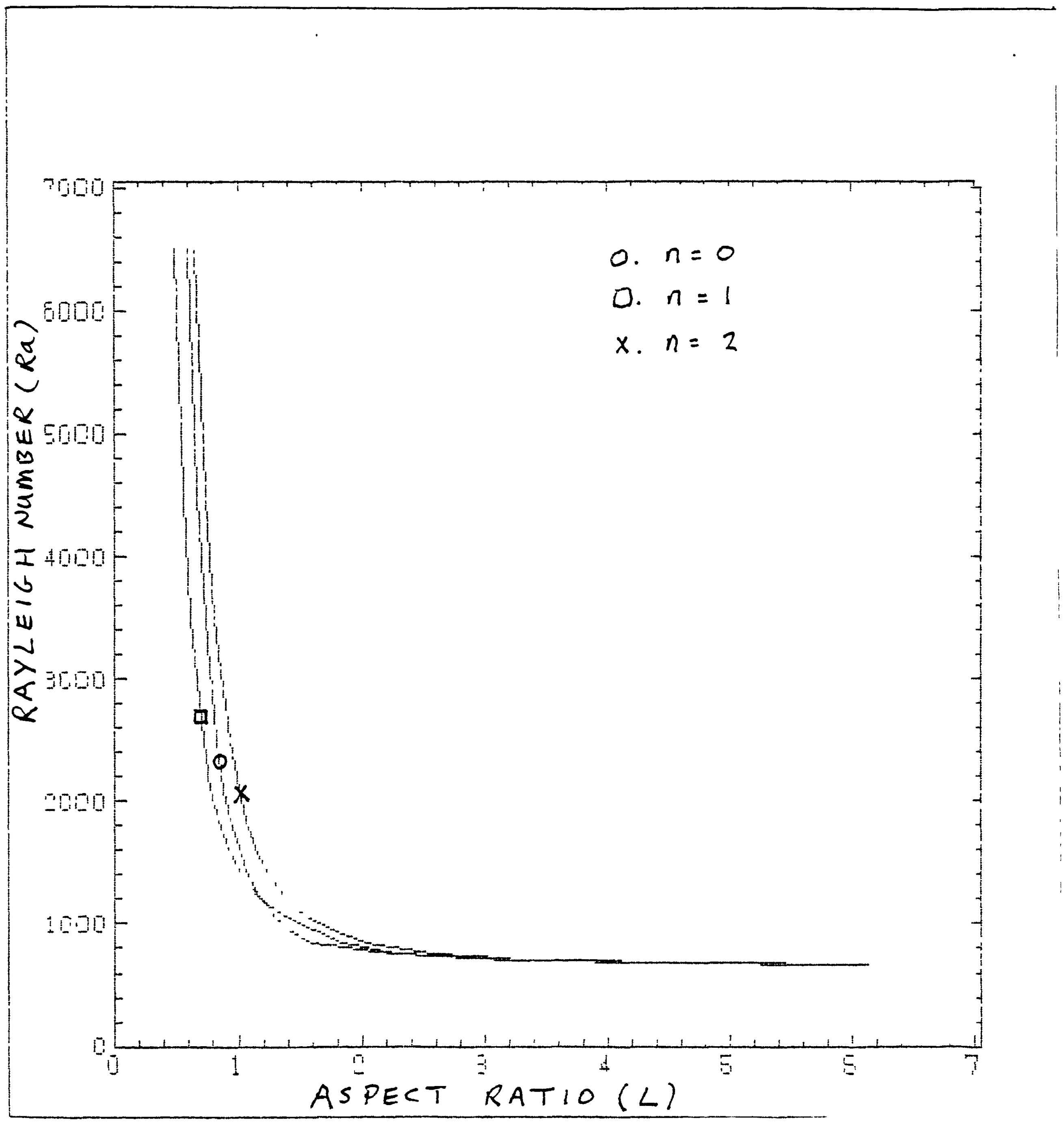

FIGURE 2.

FREE-FREE BOUNDARIES.

$\lambda = 1.0$

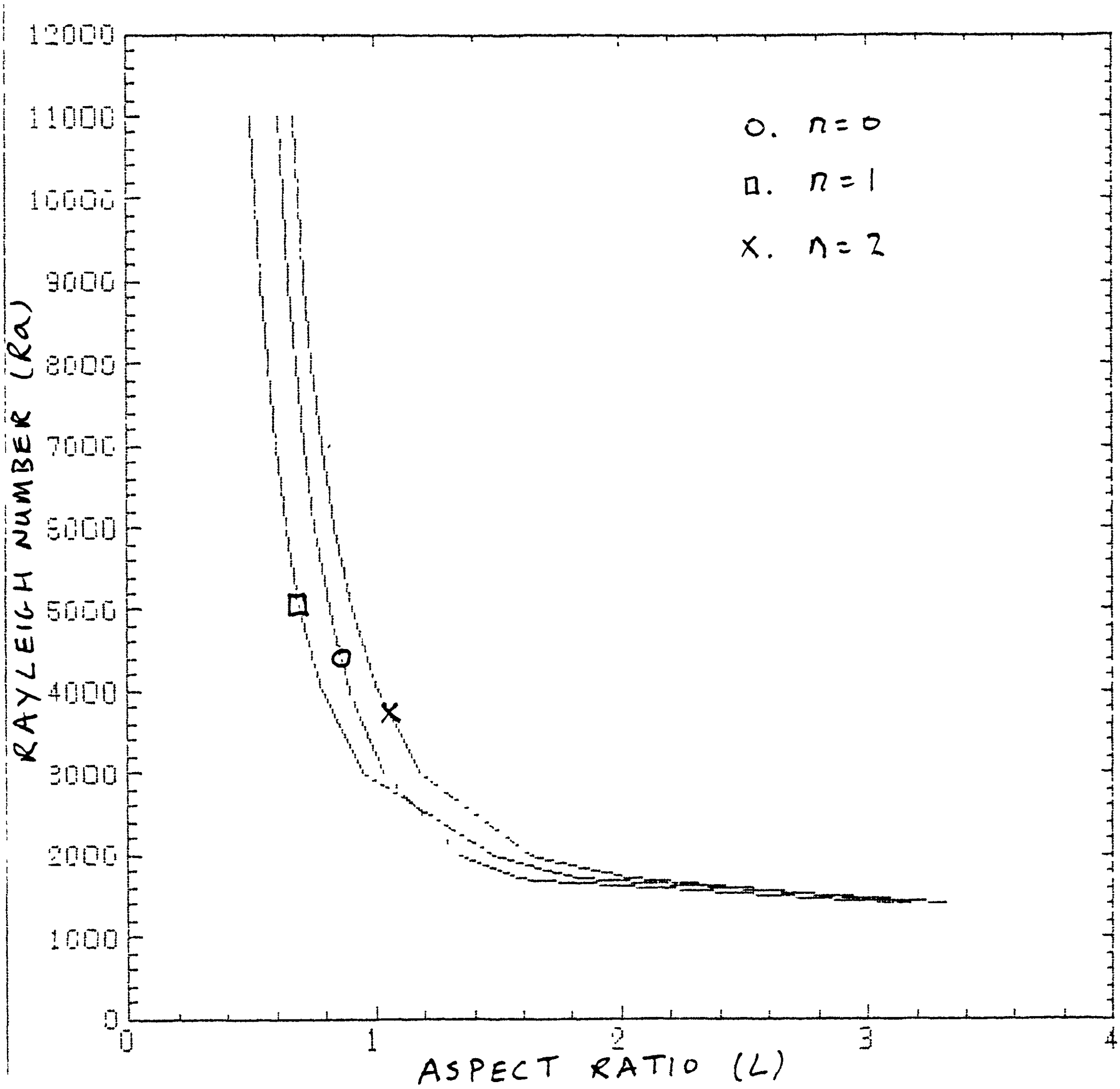

FIGURE 3.

FREE-FREE BOUNDARIES.

$\lambda = 1.2$

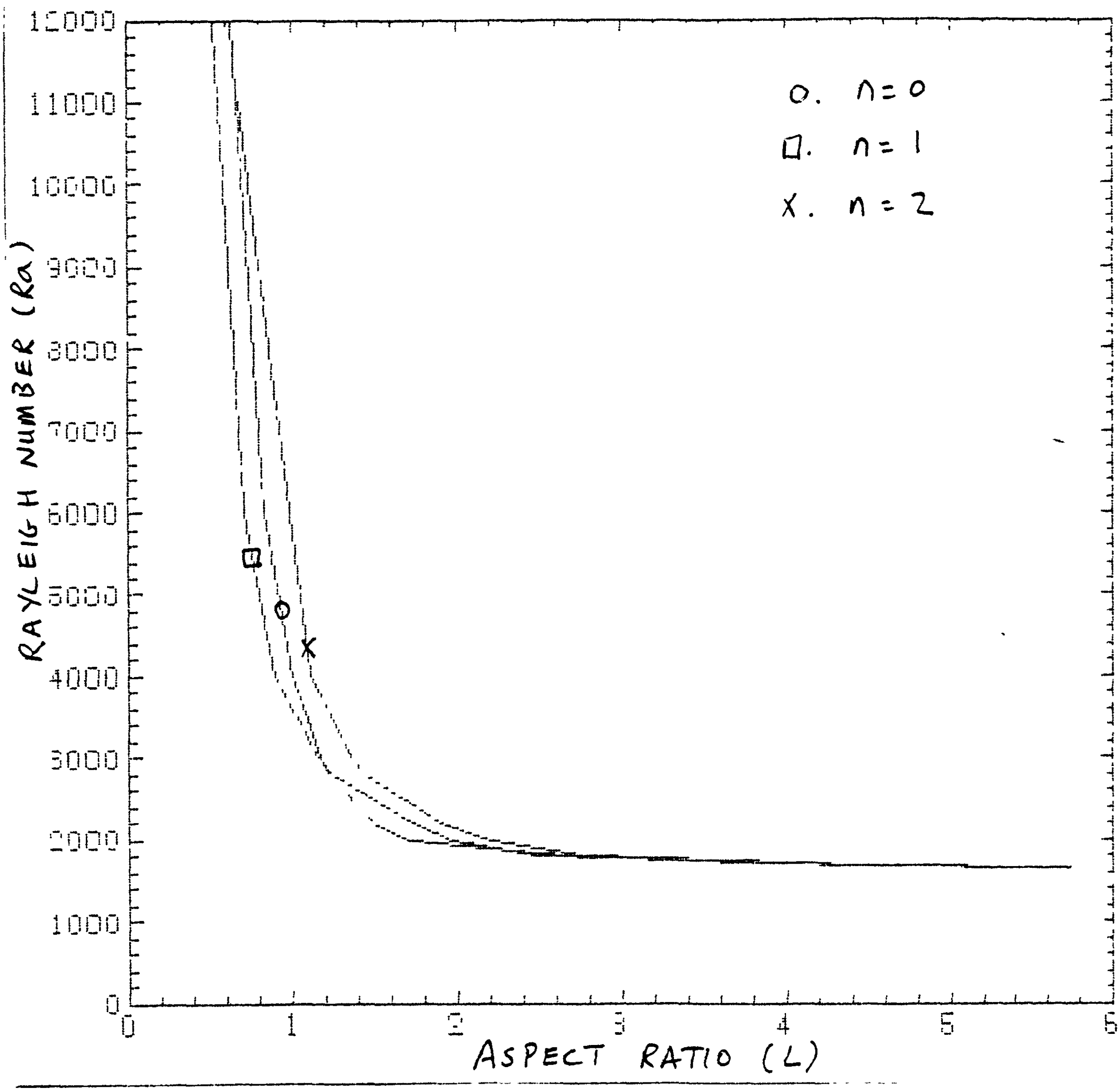

FIGURE 4.

RIGID-RIGID BOUNDARIES.

$\lambda = 0.0$

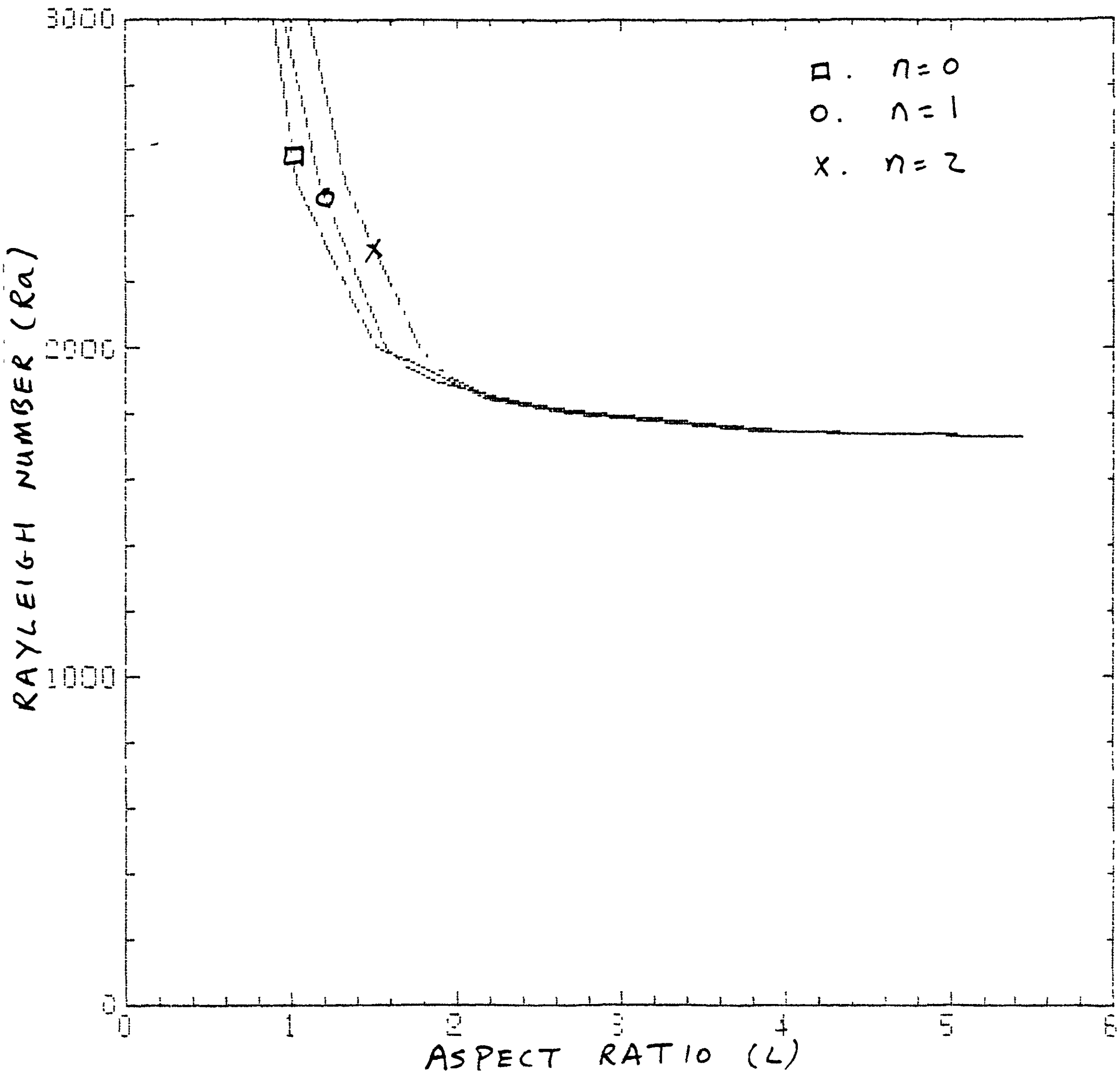

TABLE 1.

λ	RIGID-RIGID BOUNDARIES			RIGID-FREE BOUNDARIES			FREE-FREE BOUNDARIES		
	a_0	Ra_0	Ra_2	a_0	Ra_0	Ra_2	a_0	Ra_0	Ra_2
0·0	3·116	1707·78	623·72	2·683	1100·66	532·27	2·221	657·514	438·37
1·0	3·126	3389·94	1223·40	2·703	2380·52	1121·70	2·227	1308·517	863·78
1·2	3·138	4198·57	1493·80	2·722	3081·22	1415·14	2·234	1625·662	1060·28
1·4	3·168	5473·02	1884·0	2·773	4314·48	1858·80	2·251	2134·73	1352·31
1·6	3·250	7681·29	2431·88	2·942	6807·42	2441·75	2·302	3055·643	1784·16

TABLE 2.

CRITICAL RAYLEIGH NUMBERS FOR $L \to 0$ ASYMPTOTICS

n	γ_n	$\overline{Ra}_{on}$
0	4·611	452·043
1	3·832	215·63
2	5·136	685·82
3	6·380	1656·85

TABLE 3.

VALUES OF Ω_n AND $\overline{Ra}_{1n}$ FOR $L \to 0$ ASYMPTOTICS.

λ	1·0		1·2		1·5		2·0		*	*
n	Ω_n	$\overline{Ra}_{1n}$	Ω_n	$\overline{Ra}_{1n}$	Ω_n	$\overline{Ra}_{1n}$	Ω_n	$\overline{Ra}_{1n}$	μ_{1n}	μ_{2n}
0	1·517	696·718	1·343	786·99	1·157	913·50	0·955	1106·72	0·0767	-0·007
1	1·711	294·66	1·515	332·78	1·306	386·04	1·078	476·68	3·49	17·47
2	2·070	785·92	1·833	887·56	1·580	1029·68	1·304	1247·62	4·55	40·36
3	2·388	1621·89	2·115	1831·62	1·822	2126·17	1·504	2575·72	5·35	72·90

*μ_{1n} and μ_{2n} do not depend on λ .

TABLE 4.

FREE-FREE BOUNDARIES.

$\lambda = 0.0$

RAYLEIGH NUMBER (RA)	ASPECT RATIO (L)			
RA	n=0	n=1	n=2	
670·0	5·926 5·926 5·926	6·1184 6·1185 6·1185	 5·974 5·974	(3) (6) (9)
700·0	3·199 3·199 3·199	3·4469 3·4471 3·4471	3·3188 3·3189 3·3189	
750 0	2·257 2·257 2·257	2·2871 2·2871 2·2871	2·5732 2·5732 2·2732	
850·0	1·593 1·593 1·593	1·846 1·846 1·846	2·0551 2·0554 2·0554	
950·0	1·412 1·412 1·412	1·5973 1·5970 1·5970	1·7601 1·7602 1·7602	
1000·0	1·350 1·350 1·350	1·4956 1·4957 1·4957	1·6575 1·6573 1·6573	
1050·0	1·299 1·299 1·299	1·4045 1·4047 1·4047	1·5741 1·5741 1·4473	
1150·0	1·218 1·218 1·218	1·2528 1·2528 1·2528	1·4472 1·4472 1·4473	
1230·0	1·167 1·167 1·157	1·1595 1·1595 1·1595	1·3708 1·3708 1·3708	
1430·0	1·0703 1·07026 1·07026	1·004 1·004 1·004	1·2342 1·2342 1·2342	
1630·0	1·000 1·000 1·000	0·9084 0·9084 0·9084	1·1413 1·1414 1·1414	
1850·0	0·9420 0·9419 0·9419	0·8370 0·8369 0·9069	1·0665 1·0665 1·0665	

Table 4 ... continued

2100·0	0·890	0·7785	1·002	
	0·890	0·7785	1·002	
	0·890	0·7785	1·002	
2500·0	0·8274	0·7125	-	
	0·8273	0·7126	0·9263	
	0·8273	0·7126	0·9263	
3500·0	0·726	0·615	0·8083	
	0·727	0·6149	0·8084	
	0·727	0·6149	0·8084	
4500·0	0·664	0·5578	0·7371	(6)
	0·664	0·5578	0·7371	(9)
	0·664	0·5578	0·7371	(12)
6500·0	0·587	0·4897	0·6502	
	0·587	0·4897	0·6502	
	0·587	0·4897	0·6502	

TABLE 5.

FREE-FREE BOUNDARIES.

$\lambda = 1.0$

RAYLEIGH NUMBER (RA)	ASPECT RATIO (L)			
RA	$n=0$	$n=1$	$n=2$	
1400·0	3·1052 3·1049 3·1049	3·3276 3·3248 3·3249	3·1714 3·1712 3·1704	(6) (9) (12)
1700·0	1·5745 1·5764 1·5766	1·8254 1·8252 1·8254	2·028 2·029 2·028	
2000·0	1·3462 1·3487 1·3391	1·4762 1·4770 1·4761	1·623 1·636 1·633	
3000·0	1·0199 1·0342 1·0343	0·8615 0·9526 0·9506	1·135 1·181 1·178	
4000·0	0·9002 0·8999 0·8998	0·8666 0·7880 0·7864	- 1·01 1·006	
5000·0	- 0·8177 0·8177	- 0·7011 0·6995	- 0·9105 0·9071	
6000·0	0·7599 0·7603 0·7599	0·6444 0·6427 0·6430	0·8428 0·8392 0·8391	(9) (12) (15)
7000·0	0·7171 0·7171 0·7165	0·6032 0·6014 0·6015	0·7922 0·7887 0·7884	
8000·0	0·6844 0·6831 0·6820	0·5713 0·5695 0·5695	0·7522 0·7490 0·7486	
9000·0	0·6589 0·6555 0·6537	0·5455 0·5438 0·5436	0·7194 0·7167 0·7162	
10000·0	0·6385 0·6325 0·6298	0·5239 0·5224 0·5222	- - -	
11000·0	0·6216 0·6132 0·6094	0·5054 0·5042 0·5039	0·6672 0·6666 0·6666	

TABLE 6.

FREE-FREE BOUNDARIES.

$\lambda = 1.2$

RAYLEIGH NUMBER (RA)	ASPECT RATIO (L)			
RA	n=0	n=1	n=2	
1660·0	5·7241	-	5·7252	(6)
	5·7052	5·5071	5·7253	(9)
	5·7057	5·5051	5·7259	(12)
	5·7051	5·5143	5·7256	(15)
1680·0	4·4773	4·6251	-	
	4·4713	4·6100	4·4960	
	4·4717	4·6086	4·4955	
	4·4713	4·6100	4·4963	
1700·0	3·8136	-	4·0794	
	3·9658	3·8137	4·0744	
	3·9642	3·8137	4·0762	
	3·9655	3·8139	4·0747	
1800·0	2·6999	2·4796	2·7482	
	2·6901	2·4822	2·7472	
	2·6907	2·4813	2·7491	
	2·6899	2·4822	2·7475	
2000·0	1·6961	1·9662	2·2134	
	1·7039	1·0652	2·2048	
	1·7036	1·9665	2·2034	
	-	-	2·2049	
2200·0	1·5043	1·7225	1·8984	
	1·4941	1·7178	1·8953	
	1·4947	1·7184	1·8768	
	1·4941	1·7181	1·8950	
2800·0	1·2227	1·2381	1·4130	
	1·2175	1·2546	1·4379	
	1·2181	1·2418	1·4599	
	1·2174	1·2539	1·4368	
3000·0	1·1537	1·1276	1·3269	
	1·1634	1·1552	1·3565	
	1·1638	1·1384	1·3687	
	1·1631	1·1545	1·3551	

Table 6 ..... continued

4000·0		-	-	-	
		0·9894	0·8921	1·1179	
		0·9894	0·8463	1·1183	
		0·9890	0·8916	1·1140	
6000·0		0·7993	0·7020	-	(9)
		0·8207	0·6939	-	(12)
		0·8203	0·6999	-	(15)
8000·0		0·4335	0·6153	-	
		0·7305	0·6109	-	
		0·7298	0·6105	-	
10000·0		-	-	-	
		-	0·5560	0·7289	
		-	0·5548	0·7280	
12000·0		-	-	-	
		0·6293	0·5168	0·6799	
		0·6267	0·5156	0·6790	

Table 7.

RIGID-RIGID BOUNDARIES.

$\lambda = 0.0$

RAYLEIGH NUMBER (RA)	ASPECT RATIO (L)			
Ra	$n=0$	$n=1$	$n=2$	
1730·0	5·401	5·4703	5·340	(6)
	5·298	5·4405	5·324	(9)
	5·298	5·4410	5·318	(12)
	5·298	5·4368	-	(15)
1750·0	4·1077	3·8514	4·0363	
	3·9896	3·8461	5·0017	
	3·9750	3·8399	4·0059	
	3·9790	3·8516	3·9977	
1800·0	3·0214	2·664	2·908	
	2·7640	2·663	2·833	
	2·7050	2·665	2·833	
	2·7160	2·663	2·829	
1850·0	2·1710	2·326	2·209	
	2·1540	2·279	2·199	
	2·1530	2·271	2·195	
	2·1540	2·282	2·206	
1900·0	2·0570	1·851	1·9998	
	1·9620	1·838	1·986	
	1·9510	1·826	1·990	
	1·9550	1·846	1·983	
1950·0	1·9780	1·669	1·894	
	1·7950	1·670	1·861	
	1·7590	1·671	1·865	
	1·7690	1·671	1·865	
2000·0	1·9090	1·574	1·814	
	1·5500	1·572	1·768	
	1·5010	1·575	1·770	
	1·5090	1·571	1·764	

Table 7 ..... continued

2500·0	0·530	1·107	1·146	
	1·0120	1·160	1·287	
	1·0160	1·149	1·261	
	1·0190	1·166	1·321	
3000·0	-	-	-	(6)
	0·7340	0·9450	1·067	(9)
	0·8699	0·8990	1·043	(12)
	0·8890	0·9540	1·066	(18)
	0·8880			
4000·0	-	-	-	
	-	-	-	
	0·4230	-	-	
	0·7550	0·7550	-	
	0·7560	0·7200		

TABLE 8.

RIGID-RIGID BOUNDARIES.

$\lambda = 1.0$

RAYLEIGH NUMBER (RA)	ASPECT RATIO (L)	
RA	$n = 0$	
3450·0	4·9365	(6)
	4·423	(9)
	4·515	(12)
	4·532	(15)
3500·0	3·311	(9)
	3·322	(12)
	3·329	(15)
3600·0	2·361	
	2·404	
	2·418	
3700·0	2·042	(9)
	2·090	(12)
	2·080	(15)
	2·075	(18)
	2·077	(21)
3800·0	1·622	
	1·919	
	1·896	
	1·901	
	1·893	

TABLE 9.

ASPECT RATIOS FROM $L \to \infty$ ASYMPTOTICS FREE-FREE BOUNDARIES $\lambda = 0.0$	
RA	L
670·0	5·925
700·0	3·212
750·0	2·177
850·0	1·509
150·0	1·224

TABLE 10.

ASPECT RATIOS FROM $L \to \infty$ ASYMPTOTICS FREE-FREE BOUNDARIES $\lambda = 1.0$	
RA	L
1400·0	3·073
1700·0	1·485
2000·0	1·118
3000·0	0·715
4000·0	0·566

TABLE 11.

ASPECT RATIOS FROM $L \to \infty$ ASYMPTOTICS FREE-FREE BOUNDARIES $\lambda = 1 \cdot 2$	
RA	L
1660·0	5·557
1680·0	4·417
1700·0	3·777
1800·0	2·466
2000·0	1·683

TABLE 12.

ASPECT RATIOS FROM $L \to 0$ ASYMPTOTICS FREE-FREE BOUNDARIES $\lambda = 1 \cdot 0$		
n	RA	L
0	14255·165	0·5
	32432·801	0·4
	94354·387	0·3
1	6420·067	0·5
	14671·721	0·4
	42923·342	0·3
2	19054·697	0·5
	43846·993	0·4
	129385·501	0·3

TABLE 13.

ASPECT RATIOS FROM $L \to 0$ ASYMPTOTICS FREE-FREE BOUNDARIES $\lambda = 1 \cdot 2$		
n	RA	L
0	15165•01	0•5
	34347•06	0•4
	75239•53	0•3
1	6804•29	0•5
	15480•11	0•4
	45032•37	0•3
2	19054•70	0•5
	43846•99	0•4
	129385•50	0•3

TABLE 14.

COMPARISON OF RESULTS WITH CHARLSON & SANI.

CHARLSON & SANI		TABLE 7	
RA	L	RA	L
1801.18	2.71	1800.0	2.72
1968.0	1.70	1950.0	1.78
2009.61	1.51	2000.0	1.51

REFERENCES

(1) VERONIS, G. ASTROPHYSICS J. 137, 641. 1963.

(2) CHANDRASEKHAR, S. HYDRODYNAMIC AND HYDROMAGNETIC STABILITY. CLARENDON PRESS. 1961.

(3) JOSEPH, D.D. J.F.M. 47 (257). 1971.

(4) JONES, C.A. & MOORE, D.R. GEO. ASTRO FLUID DYNAMICA VOL 11.245. 1979.

(5) INCE, E.L. ORDINARY DIFFERENTIAL EQUATIONS. DOVER 1944.

(6) DRAZIN, P.G. & REID, W.H. HYDRODYNAMIC STABILITY CAMBRIDGE 1981.

(7) DANIELS, P.G. PROC.ROY.SOC. A358 173-197 1977.

(8) GERSHUNI, G.Z. & ZHUKHOVITSKI, I. CONVECTIVE STABILITY OF INCOMPRESSIBLE FLUIDS. ISRAEL PROGRAM FOR SCIENTIFIC TRANSLATIONS, JERUSALEM 1976.

(9) ABRAMOWITZ, M. & STEGUN, I.A. HANDBOOK OF MATHEMATICAL FUNCTIONS. DOVER 1972.

(10) ERDÉLYI, A. et al. HIGHER TRANSCENDENTAL FUNCTIONS VOL. 2. McGRAW HILL 1953.

(11) LUKE, Y.L. INTEGRALS OF BESSEL FUNCTIONS. McGRAW HILL 1962.

(12) WATSON, G.N. A TREATISE ON THE THEORY OF BESSEL FUNCTIONS. CAMBRIDGE 1958.

(13) CONTE, S.D. & de BOOR, C. ELEMENTARY NUMERICAL ANALYSIS. McGRAW HILL 1972.

(14) CHARLSON, S.G. & SANI, R.L. INT.J. HEAT TRANSFER 13 p. 1479. 1970.

(15) BROWN, S.N. & STEWARTSON, K. PROC. ROY. SOC. A360 1978.

(16) MOORE, D.R. & WEISS, N.O. J.F.M. 61 p.553. 1973.

(17) SEGEL, L.A. J.F.M. 38 p. 203. 1969.

(18) HALL, P. & WALTON, I.C. PROC. ROY. SOC. A358 p. 199 1977.

(19) DAVIS, S. J.F.M. 30 p. 465. 1967.

(20) DRAZIN, P.G. Z. ANFEW. MATH. PHYS. 26 p. 239. 1975.

(21) A ZOUNI, M.A. & NORMAND, C. PREPRINT. COMMISSARIAT A L'ENERGIE ATOMIQUE. DIVISION DE LA PHYSIQUE. 1979.

(22) FURUMOTO, A. & ROOTH, C. GEOPHYSICAL FLUID DYNAMICS. WOODS HOLE OCEANOGRAPHIC Inst. Rep. 1961.

(23) TOWNSEND, A.A. J.F.M. <u>24</u> p.307. 1966.

(24) MYRUP, L. et al. WEATHER <u>24</u> p.307. 1966.

(25) MOORE, D.W. in AERODYNAMIC PHENOMENA IN STELLA ATMOSPHERES (ed. R.N. THOMAS) p.405. ACADEMIC 1967.

(26) MUSMAN, S. J.F.M. <u>31</u> p. 343. 1968.

(27) HERRING, J.R. J. ATMOS. SCI. <u>20</u> p.325. 1963.

(28) BLENNERHASSET, P. & HALL, P. PROC. ROY. SOC. A<u>365</u> p. 191 1979.

(29) HALL, P. PROC. ROY. SOC. A372 p.317. 1980.

(30). AZOUNI, M.A. & NORMAND, C. GEO. ASTRO. FLUID DYNAMICS Vol. <u>23</u> 1983 p. 223

(31) BATCHELOR, G.K & GILL, A.E, J. F.M. 14, 1962, p.529

CHAPTER 2.

ON THE NATURE OF STEADY BÉNARD CONVECTION IN THE VICINITY OF A TWO DIMENSIONAL HOT SPOT.

1. INTRODUCTION

In this chapter we consider the effect of non-uniform heating on classical Rayleigh-Bénard convection.

In the classical Rayleigh-Bénard problem convection takes place between two infinite plane parallel plates held at constant but unequal temperatures, and the plates themselves are assumed to be perfectly smooth i.e. containing no bumps or ridges. This is of course an idealization that is not only realistically beyond experimental technique but also physically dubious for engineering, astrophysical and oceanic dynamics to name but a few.

As a first step then to understanding the possible effects of non-uniformity to the boundaries in Rayleigh-Bénard convection various authors have considered two types of models. The first of these is to let the height between the two horizontal plates vary with distance along the horizontal. This approach has been studied by Walton (1, 2) and Eagles (3).

The second method is to suppose that the temperature profiles of the two plates varies along the plates. In particular it is convenient to consider that the upper plate has a fixed constant temperature, while the lower plate has some imposed profile. Both approaches immediately give rise to a difference with the classical problem since now a mean state of hydrostatic equilibrium is no longer possible - the variation of temperature along the lower boundary or of distance between the boundaries generating a large scale circulation throughout the fluid. (see Walton (9)).

This in turn gives rise to the problem of how to separate the onset of convection in the classical sense from that of the mean flow convection. There are two situations in which progress may be made analytically.

The first is to assume that the amplitude (δ) of the non-uniformity in the temperature profile of the bottom plate is small and we use this as an expansion parameter. Such an approach has been adopted by Kelly and Pal in a series of papers (4, 5, 6, 7).

The second approach is to assume that the horizontal length scale over which the temperature of the bottom boundary changes is much larger than the distance between the top and bottom plates while δ remains order unity. This leads to the small parameter $\varepsilon = h/L$ where L is the horizontal length scale and h is the gap width. Such a model was used by Richter (8) to study the earth's mantle, while Walton (9) did a much more thorough study of a similar problem but considered only the possibility of transverse rolls.

Here we study Walton's (9) model but consider the possibility of convection taking the form of longitudinal rolls.

We derive a fourth order ordinary differential equation for the amplitude of the motion with the correction to the Rayleigh number being an eigenvalue to this equation. The equation is solved and the first two odd and even modes found. It is found that the longitudinal mode appears at a Rayleigh number below that for transverse rolls. We further consider the case when δ becomes of order ε^4.

2. GOVERNING EQUATIONS

We shall consider the onset of convection between a layer of fluid bounded horizontally by $z = 0$, $z = h$ and heated to temperatures $T = T_u$ at $z = h$, $T = F(\epsilon x, \delta)$ at $z = 0$

Taking rectangular coordinates with z positive in the vertical direction the governing equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u \qquad - (2:1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + v\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v \qquad - (2:2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g + \nu\nabla^2 w \qquad - (2:3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad - (2:4)$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = k\nabla^2 T \qquad - (2:5)$$

$$\rho = \rho_0\left[1 - \alpha\,(T - T_0)\right] \qquad - (2:6)$$

where as usual α denotes the coefficient of thermal expansion, ν the coefficient of kinematic viscosity and k the coefficient of thermal conductivity.

Following Walton (9) we shall assume that the mean flow is steady and two dimensional, and also that the principle of the exchange of stability is valid so that if we restrict attention to the onset of instability we can ignore all time derivatives i.e. $\partial/\partial t \equiv 0$.

As a first step, we cast the above system of equations into dimensionless form with the following transformations

$$(x, y, z) \rightarrow h(x, y, z)$$
$$(u, v, w) \rightarrow \frac{k}{h}(u, v, w)$$
$$p \rightarrow \rho_0 \frac{\nu k}{h^2} p$$
$$T \rightarrow T_u + (\Delta T) T$$

where h is the distance between the upper and lower boundaries, T_u is the temperature of the upper boundary and ΔT is the maximum temperature difference between the two horizontal boundaries. Equations (2:1) - (2:6) therefore take the form

$$\frac{1}{\sigma}\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right\} = -\frac{1}{\sigma}\frac{\partial p}{\partial x} + \nabla^2 u \qquad - (2:7)$$

$$\frac{1}{\sigma}\left\{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right\} = -\frac{1}{\sigma}\frac{\partial p}{\partial y} + \nabla^2 v \qquad - (2:8)$$

$$\frac{1}{\sigma}\left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\} = -\frac{1}{\sigma}\frac{\partial p}{\partial z} + Ra\,T + \nabla^2 w - g\left[1 + \alpha(T_0 - T_u)\right]\frac{h^3}{\nu k} \qquad - (2:9)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad - (2:10)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \nabla^2 T \qquad - (2:11)$$

where $Ra = \frac{\alpha g \Delta T h^3}{\nu \kappa}$ is the Rayleigh number and $\sigma = \nu / \kappa$ is the Prandtl number. The constant term on the right hand side of the equation (2:9) is removed by absorbing it into the pressure i.e. we define a new pressure

$$p = p + g\left[1 + \alpha\,(T_0 - T_u)\right]\frac{h^3}{\nu\kappa}\,z$$

so that (2:9) becomes

$$\frac{1}{\sigma}\left\{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right\} = -\frac{1}{\sigma}\frac{\partial p}{\partial z} + Ra\,T + \nabla^2 w \quad \text{- (2:12)}$$

We will consider three types of conditions at $z = 0, 1$ on the components of velocity.

(i) <u>Free-free boundaries</u>

$$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

(ii) <u>Rigid-rigid boundaries</u>

$$u = v = w = 0$$

(iii) <u>Rigid-free boundaries</u>

$$u = v = w = 0 \qquad z = 0$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \qquad z = 1$$

while the temperature field will satisfy the conditions for perfectly conducting boundaries

$$T = 0 \;,\; z = 1 . \qquad T = F(\varepsilon x, \delta) \qquad z = 0$$

where $F(\varepsilon x, \delta)$ is some function with a local maximum at $X = 0$ which is modulated along the horizontal, ε being a small parameter which is proportional to the length scale of the modulation.

In particular we follow Walton (9) and consider

$$F(\varepsilon x, \delta) = 1 - \delta \sin^2 \varepsilon x$$

and

$$F(\varepsilon x, \delta) = 1 - \delta \tanh^2 \varepsilon x$$

Because of this lower boundary temperature profile a state of hydrostatic equilibrium is not possible and we assume that the resulting mean state consists of a steady two dimensional circulation.

3. THE MEAN CIRCULATION

On the assumption that the mean circulation takes place in the x-z plane we put $\frac{\partial}{\partial y} \equiv 0$ in equations (2:7, 8, 10, 11 and 12) to get

$$\frac{1}{\sigma}\left\{ u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right\} = -\frac{1}{\sigma}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \qquad - (3:1)$$

$$\frac{1}{\sigma}\left\{ u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right\} = -\frac{1}{\sigma}\frac{\partial p}{\partial z} + Ra\,T + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \qquad - (3:2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \qquad - (3:3)$$

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \qquad - (3:4)$$

Since the circulation is generated by a non-uniformity with horizontal scale $\alpha(\varepsilon)$ we expect that the amplitude of the motion it generates will be of similar structure. This leads us to write $X = \varepsilon x$ and in order to preserve continuity $w = \varepsilon \bar{w}$ the equations then become

$$\frac{\varepsilon}{\sigma}\left\{u\frac{\partial u}{\partial x}+\bar{\omega}\frac{\partial u}{\partial z}\right\}=-\frac{\varepsilon}{\sigma}\frac{\partial p}{\partial x}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2} \qquad -(3:5)$$

$$\frac{\varepsilon^2}{\sigma}\left\{u\frac{\partial \bar{\omega}}{\partial x}+\bar{\omega}\frac{\partial \bar{\omega}}{\partial z}\right\}=-\frac{1}{\sigma}\frac{\partial p}{\partial z}+RaT+\varepsilon^3\frac{\partial^2 \bar{\omega}}{\partial x^2}+\varepsilon\frac{\partial^2 \bar{\omega}}{\partial z^2} \qquad -(3:6)$$

$$\varepsilon\left\{u\frac{\partial T}{\partial x}+\bar{\omega}\frac{\partial T}{\partial z}\right\}=\varepsilon^2\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2} \qquad -(3:7)$$

$$\frac{\partial u}{\partial x}+\frac{\partial \bar{\omega}}{\partial z}=0. \qquad -(3:8)$$

Equations (3:5) - (3:8) lead us to an expansion of the following form

$$T=T_0+\varepsilon^2T_1+\ldots$$
$$p=p_0+\varepsilon^2p_1+\ldots$$
$$u=\varepsilon u_0+\varepsilon^3u_1+\ldots$$
$$\bar{\omega}=\varepsilon\bar{\omega}_0+\varepsilon^3\bar{\omega}_1+\ldots$$

Substituting these expansions into the above equations and equating like powers of ε yields the following sets of equations :-

<u>O(1)</u>

$$\frac{1}{\sigma}\frac{\partial p_0}{\partial z}=Ra\,T_0 \qquad -(3:9)$$

$$\frac{\partial^2 T_0}{\partial z^2}=0. \qquad -(3:10)$$

with the specified boundary conditions we get

$$T_0 = (1 - z)\, F(X, \delta) \qquad - (3:11)$$

$$p_0 = \sigma Ra \left(z - \frac{z^2}{2}\right) F(X, \delta) + f_0(X) \qquad - (3:12)$$

where the unknown function $f_0(X)$ will be determined at the next order.

$O(\varepsilon^2)$

$$\frac{1}{\sigma}\frac{\partial p_0}{\partial X} = \frac{\partial^2 u_0}{\partial z^2} \qquad - (3:13)$$

$$\frac{1}{\sigma}\frac{\partial p_1}{\partial z} = Ra\, T_1 + \frac{\partial^2 \overline{\omega}_0}{\partial z^2} \qquad - (3:14)$$

$$u_0 \frac{\partial T_0}{\partial X} + \overline{\omega}_0 \frac{\partial T_0}{\partial z} = \frac{\partial^2 T_0}{\partial X^2} + \frac{\partial^2 T_1}{\partial z^2} \qquad - (3:15)$$

$$\frac{\partial u_0}{\partial X} + \frac{\partial \overline{\omega}_0}{\partial z} = 0. \qquad - (3:16)$$

Since we require the first terms in each expansion we need only solve for $u_0, \overline{\omega}_0$. Using (3:12) equation (3:13) gives

$$\frac{\partial^2 u_0}{\partial z^2} = Ra \left(z - \frac{z^2}{2}\right) F_X(X, \delta) + f_{0X}(X) \qquad - (3:17)$$

where $F_X(X, \delta) = \dfrac{d}{dX} F(X, \delta)$ etc.

Integrating (3:17) gives

$$u_0 = Ra\left[\frac{z^3}{6} - \frac{z^4}{24}\right]F_X(X,\delta) + f_{0X}(X)\frac{z^2}{2} + f_1(X)z + f_2(X) \qquad \text{- (3:18)}$$

Using this directly in (3:16) and integrating with respect to z gives

$$\bar{\omega}_0 = -Ra\left[\frac{z^4}{24} - \frac{z^5}{120}\right]F_{XX}(X,\delta) - f_{0XX}\frac{z^3}{6} - f_{1X}\frac{z^2}{2} - f_2 z - f_3 \qquad \text{- (3:19)}$$

We are now in a position to determine the unknown functions $f_i(X)$ $(i = 0, .. 3)$.

(i) <u>Free:free boundaries</u>

Here we have $\frac{\partial u_0}{\partial z} = \bar{\omega}_0 = 0 \qquad z = 0, 1 \quad \forall X$

Hence $$f_1(X) = 0. \qquad \text{- (3:20)}$$

$$Ra\left(\frac{1}{2} - \frac{1}{6}\right)F_X(X,\delta) + f_{0X} = 0 \qquad \text{- (3:21)}$$

$$f_3(X) = 0. \qquad \text{- (3:22)}$$

$$-Ra\left(\frac{1}{24} - \frac{1}{120}\right)F_{XX} - f_{0XX}\frac{1}{6} - f_{1X}\frac{1}{2} - f_{2X} = 0. \qquad \text{- (3:23)}$$

$$-Ra\left(\frac{1}{24} - \frac{1}{120}\right)F_{XX} - f_{0XX}\frac{1}{6} - f_{2X} = 0. \qquad \text{- (3:24)}$$

Equation (3:21) yields

$$f_{0X}(X) = -\frac{Ra}{3} F_X(X, \delta) \qquad - (3:25)$$

Equations (3:25) and (3:24) then give

$$f_{2X}(X) = \frac{Ra}{45} F_{XX}(X, \delta) \qquad - (3:26)$$

summarising we have

$$u_0 = Ra\, F_X(X,\delta)\left[\frac{z^3}{6} - \frac{z^4}{24} - \frac{z^2}{6} + \frac{1}{45}\right] \qquad - (3:27)$$

$$\overline{\omega}_0 = -Ra\, F_{XX}(X,\delta)\left[\frac{z^4}{24} - \frac{z^5}{120} - \frac{z^3}{18} + \frac{z}{45}\right] \qquad - (3:28)$$

which is identical to that obtained by Walton (9).

(ii) <u>Rigid-rigid boundaries</u>

Here

$$u_0 = \overline{\omega}_0 = 0 \qquad z = 0, 1 \quad \forall X$$

Then (3:18) and (3:19) give

$$f_2(X) = 0. \qquad f_3(X) = 0.$$

$$Ra\left[\frac{1}{6} - \frac{1}{24}\right] F_X(X,\delta) + f_{0X}\frac{1}{2} + f_1(X) = 0.$$

$$-Ra\left[\frac{1}{24} - \frac{1}{120}\right] F_{XX}(X,\delta) - f_{0XX}\frac{1}{6} - f_{1X}\frac{1}{2} = 0.$$

$$\frac{3}{24} Ra\ F_X(X,\delta) + f_{0X}\frac{1}{2} + f_1(X) = 0. \qquad - (3:29)$$

$$\frac{Ra}{30} F_{XX}(X,\delta) + f_{0XX}\frac{1}{6} + f_{1X}\frac{1}{2} = 0. \qquad - (3:30)$$

Clearly we can put

$$f_{0X} = \alpha_1 F_X(X,\delta)$$

$$f_1 = \alpha_2\ Ra\ F_X(X,\delta)$$

where α_1, α_2 are constants determined from the following simultaneous equations

$$\alpha_2 + \frac{\alpha_1}{2} + \frac{1}{8} = 0$$

$$\frac{\alpha_2}{2} + \frac{\alpha_1}{6} + \frac{1}{30} = 0$$

From which we get

$$\alpha_1 = -\frac{7}{20}$$

$$\alpha_2 = \frac{1}{20}$$

and we therefore have

$$u_0 = Ra\ F_X(X,\delta)\left[\frac{z^3}{6} - \frac{z^4}{24} - \frac{7z^2}{40} + \frac{z}{20}\right] \qquad - (3:31)$$

$$\overline{w}_0 = -Ra\ F_{XX}(X,\delta)\left[\frac{z^4}{24} - \frac{z^5}{120} - \frac{7z^3}{120} + \frac{z^2}{40}\right] \qquad - (3:32)$$

(iii) Rigid-free boundaries

Here we must have

$$u_0 = \bar{\omega}_0 = 0 \qquad z = 0$$
$$\frac{\partial u_0}{\partial z} = \bar{\omega}_0 = 0 \qquad z = 1 \qquad \forall X$$

so that (3:18) and (3:19) yield

$$f_2(X) = f_3(X) = 0$$

and

$$\frac{1}{6} Ra\, F_X(X,\delta) + f_{0X} + f_1(X) = 0.$$

$$\frac{1}{30} Ra\, F_{XX}(X,\delta) + f_{0XX}\frac{1}{6} + f_{1X}\frac{1}{2} = 0.$$

Again we put $f_0 = \alpha_1 F(X,\delta) Ra$ and $f_1 = \alpha_2 Ra\, F_X(X,\delta)$ to obtain the following two equations

$$\alpha_1 + \alpha_2 + \frac{1}{3} = 0.$$

$$\frac{\alpha_1}{6} + \frac{\alpha_2}{2} + \frac{1}{30} = 0.$$

From which we get

$$\alpha_1 = -\frac{2}{5}$$

$$\alpha_2 = \frac{1}{15}$$

and so

$$u_0 = Ra\, F_X(X,\delta)\left[\frac{z^3}{6} - \frac{z^4}{24} - \frac{z^2}{5} + \frac{z}{15}\right] \qquad \text{- (3:33)}$$

$$\overline{\omega}_0 = -Ra\, F_{XX}(X,\delta)\left[\frac{z^4}{24} - \frac{z^5}{120} - \frac{z^3}{15} + \frac{z^2}{30}\right] \qquad - (3:34)$$

Figure one shows the isotherms for the profile $F(X,\delta) = 1 - \delta \sin^2 X$; while figures two, three and four show the stream lines for free-free, rigid-rigid and rigid-free boundaries respectively.

We can see from these figures that the mean flow generated by the $\sin^2 X$ profile consists of two convective cells on either side of the point of maximum temperature. We can also see that a change in boundary conditions from free-free to rigid-rigid tends to reduce the aspect ratio of the cells.

4. THE LINEAR STABILITY PROBLEM

We now proceed to investigate the stability of the above mean flows to small perturbations. In particular we will confine ourselves to linear perturbation theory and longitudinal modes only. For these, typically, the z and y scales are O(1) while the X -scale is that of the boundary modulation i.e. $O(\varepsilon)$.

We therefore introduce the slow variable $X = \varepsilon x$ with equations (2:7, 8, 10, 11 and 12) and write $\omega = \varepsilon\overline{\omega}$, $V = \varepsilon\overline{V}$ to get

$$\frac{\varepsilon}{\sigma}\left\{u\frac{\partial u}{\partial x}+\bar{v}\frac{\partial u}{\partial y}+\bar{w}\frac{\partial u}{\partial z}\right\}=-\frac{\varepsilon}{\sigma}\frac{\partial p}{\partial x}+$$

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} \qquad \text{- (4:1)}$$

$$\frac{\varepsilon^2}{\sigma}\left\{u\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right\}=-\frac{1}{\sigma}\frac{\partial p}{\partial y}+$$

$$\varepsilon^3\frac{\partial^2 \bar{v}}{\partial x^2}+\varepsilon\frac{\partial \bar{v}}{\partial y^2}+\varepsilon\frac{\partial^2 \bar{v}}{\partial z} \qquad \text{- (4:2)}$$

$$\frac{\varepsilon^2}{\sigma}\left\{u\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right\}=-\frac{1}{\sigma}\frac{\partial p}{\partial z}+RaT+ \qquad \text{- (4:3)}$$

$$\varepsilon^3\frac{\partial^2 \bar{w}}{\partial x^2}+\varepsilon\frac{\partial^2 \bar{w}}{\partial y^2}+\varepsilon\frac{\partial^2 \bar{w}}{\partial z^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z} \qquad \text{- (4:4)}$$

$$\varepsilon\left\{u\frac{\partial T}{\partial x}+\bar{v}\frac{\partial T}{\partial y}+\bar{w}\frac{\partial T}{\partial z}\right\}=\varepsilon^2\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2} \qquad \text{- (4:5)}$$

We now seek solutions to these equations by perturbing about the mean state i.e.

$$u = u_B + \Delta u' \qquad \text{- (4:6)}$$

$$\bar{v} = \Delta \bar{v}' \qquad \text{- (4:7)}$$

$$\bar{w} = \bar{w}_B + \Delta \bar{w}' \qquad \text{- (4:8)}$$

$$T = T_B + \Delta \theta \qquad \text{- (4:9)}$$

$$p = p_B + \Delta p' \qquad \text{- (4:10)}$$

where Δ represents the amplitude of the perturbation assumed to satisfy $\Delta << \varepsilon$. Then substituting (4:6-4:10) into (4:1-4:5) dropping the primes and ignoring quantities $O(\Delta^2)$ we get

$$\varepsilon \left\{ u_B \frac{\partial u}{\partial x} + \bar{w}_B \frac{\partial u}{\partial z} + u \frac{\partial u_B}{\partial x} + \bar{w} \frac{\partial \bar{w}_B}{\partial z} \right\} = -\frac{\varepsilon}{\sigma} \frac{\partial p}{\partial x} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad - (4:11)$$

$$\frac{\varepsilon^2}{\sigma} \left\{ u_B \frac{\partial \bar{v}}{\partial x} + \bar{w}_B \frac{\partial \bar{v}}{\partial z} \right\} = -\frac{1}{\sigma} \frac{\partial p}{\partial y} + \varepsilon^3 \frac{\partial^2 \bar{v}}{\partial x^2} + \varepsilon \frac{\partial^2 \bar{v}}{\partial y^2} + \varepsilon \frac{\partial^2 \bar{v}}{\partial z^2} \qquad - (4:12)$$

$$\frac{\varepsilon^2}{\sigma} \left\{ u_B \frac{\partial \bar{w}}{\partial x} + \bar{w}_B \frac{\partial \bar{w}}{\partial z} + u \frac{\partial \bar{w}_B}{\partial x} + \bar{w} \frac{\partial \bar{w}_B}{\partial z} \right\} = -\frac{1}{\sigma} \frac{\partial p}{\partial z} + Ra\, \theta + \varepsilon^3 \frac{\partial^2 \bar{w}}{\partial x^2} + \varepsilon \frac{\partial^2 \bar{w}}{\partial y^2} + \varepsilon \frac{\partial^2 \bar{w}}{\partial z^2} \qquad - (4:13)$$

$$\frac{\partial u}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \qquad - (4:14)$$

$$\varepsilon \left\{ u_B \frac{\partial \theta}{\partial x} + \bar{w}_B \frac{\partial \theta}{\partial z} + u \frac{\partial T_B}{\partial x} + w \frac{\partial T_B}{\partial z} \right\} = \varepsilon^2 \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \qquad - (4:15)$$

We see that to balance the vertical and y diffusion out of the temperature perturbation in the heat equation with advection of the mean temperature field we need to rescale θ by writing $\theta = \varepsilon \bar{\theta}$, while to maintain the balance between the buoyancy fluctuation, and the pressure fluctuation we need to rescale p by writing $p = \varepsilon \bar{p}$. This leads to the following set of equations where we have dropped the overbars for convenience.

$$\frac{\varepsilon}{\sigma}\left\{ u_B \frac{\partial u}{\partial x} + w_B \frac{\partial u}{\partial z} + u \frac{\partial u_B}{\partial x} + w \frac{\partial u_B}{\partial z} \right] =$$

$$- \frac{\varepsilon^2}{\sigma} \frac{\partial p}{\partial x} + \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{- (4:16)}$$

$$\frac{\varepsilon}{\sigma}\left\{ u_B \frac{\partial v}{\partial x} + w_B \frac{\partial v}{\partial z} \right\} = -\frac{1}{\sigma} \frac{\partial p}{\partial y} +$$

$$\varepsilon^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \qquad \text{- (4:17)}$$

$$\frac{\varepsilon}{\sigma}\left\{ u_B \frac{\partial w}{\partial x} + w_B \frac{\partial w}{\partial z} + u \frac{\partial w_B}{\partial x} + w \frac{\partial w_B}{\partial z} \right\} =$$

$$-\frac{1}{\sigma} \frac{\partial p}{\partial z} + Ra\,\theta + \varepsilon^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \qquad \text{- (4:18)}$$

$$\varepsilon \left\{ u_B \frac{\partial \theta}{\partial x} + w_B \frac{\partial \theta}{\partial z} \right\} + u \frac{\partial T_B}{\partial x} + w \frac{\partial T_B}{\partial z} =$$

$$\varepsilon^2 \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \qquad \text{- (4:19)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad \text{- (4:20)}$$

Our problem has been so formulated as to allow us to consider it as a perburbation about the classical Bénard problem. We therefore consider the solution to have the form of a modulated Bénard solution to the zeroth order where the amplitude of modulation $A(X)$ is determined at a higher order. This being the case we write

$$Ra = Ra_0 + \varepsilon^2 Ra_1 + \cdots$$

$$w = e^{iky}\left\{ A(X) f_0(z) + \varepsilon^2 f_1(X,z) + O(\varepsilon^4) \ldots \right\}$$

$$\theta = e^{iky}\left\{ A(X)\, g_0(z) + \varepsilon^2 g_1(X,z) + O(\varepsilon^4) + \cdots \right\}$$

$$u = e^{iky}\left\{ u_0(X,z) + \varepsilon^2 u_1(X,z) + O(\varepsilon^4) + \cdots \right\}$$

and likewise for v and p .

Substituting these expansions into the equations and equating like powers of ε leads to :

$\underline{O(1)}$

$$\frac{\partial^2 u_0}{\partial z^2} - k^2 u_0 = 0. \qquad - (4:21)$$

$$\frac{\partial^2 v_0}{\partial z^2} - k^2 v_0 = \frac{i p_0 k}{\sigma} \qquad - (4:22)$$

$$A(X)\left\{ \frac{d^2 f_0}{dz^2} - k^2 f_0 \right\} + A(X) Ra_0\, g_0$$

$$= \frac{1}{\sigma} \frac{\partial p_0}{\partial z} \qquad - (4:23)$$

$$A(X)\left\{\frac{d^2 g_0}{dz^2} - k^2 g_0\right\} = u_0 (1-z) F_X (X, \delta)$$

$$- A(X) f_0(z) F(X, \delta)$$

- (4:24)

$$\frac{\partial u_0}{\partial X} + A(X)\frac{d f_0}{dz} + ik V_0 = 0.$$

- (4:25)

These equations are supplemented by boundary conditions (i), (ii) or (iii) for u_0, V_0, ω_0 with $\theta_0 = 0$ $z = 0, 1$.

Equation (4:21) immediately tells us that u_0 is identically zero, and by some standard manipulation the remaining equations can be reduced to

$$A(X)\left\{\left(\frac{d^2}{dz^2} - k^2\right)^2 f_0 - k^2 Ra_0\, g_0\right\} = 0$$

- (4:26)

$$A(X)\left\{\left(\frac{d^2}{dz^2} - k^2\right) g_0 + f_0(z) F(X, \delta)\right\} = 0.$$

- (4:27)

Obviously, since g_0 and f_0 are only functions of z and and Ra are constants, no solution to these equations can exist because of the term $F(X, \delta)$. However we are concerned here with functions F(X) which have a local maximum of 1 at $X = 0$ and therefore solutions of the type sought above can be obtained for $|X| << 1$. To achieve this we write $X = \varepsilon^{\alpha} \bar{X}$ $(\alpha > 0)$ and seek solutions in the form

$$w = e^{iky}\left(A(\bar{X}) f_0(z) + \ldots \right)$$

where the value of α is obtained by balancing terms at higher order.

As in Walton's (9) analysis for transverse rolls we do not expect that the mean flow has any effect to the order considered on the stability problem. Ignoring then for the moment terms generated by the mean circulation we arrive at the following equations.

$$-\left[1 - \delta\left(\varepsilon^{2\alpha}\bar{X}^2 + \ldots\right)\right] w = \frac{\partial^2\theta}{\partial z^2} + \frac{\partial^2\theta}{\partial y^2} + \varepsilon^{2-2\alpha}\frac{\partial^2\theta}{\partial \bar{X}^2} \qquad - (4:28)$$

$$\frac{\varepsilon^{2-\alpha}}{\sigma}\frac{\partial p}{\partial \bar{X}} = \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} + \varepsilon^{2-2\alpha}\frac{\partial^2 u}{\partial \bar{X}^2} \qquad - (4:29)$$

$$\frac{1}{\sigma}\frac{\partial p}{\partial z} = Ra\,\theta + \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial y^2} + \varepsilon^{2-2\alpha}\frac{\partial^2 w}{\partial \bar{X}^2} \qquad - (4:30)$$

$$\frac{1}{\sigma}\frac{\partial p}{\partial y} = \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial y^2} + \varepsilon^{2-2\alpha}\frac{\partial^2 v}{\partial \bar{X}^2} \qquad - (4:31)$$

$$\varepsilon^{-\alpha}\frac{\partial u}{\partial \bar{X}} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad - (4:32)$$

At zero order we expect the flow to be the same as a longitudinal roll in standard Bénard convection, but at the next, and higher orders, the horizontal pressure gradient along the X- axis must be balanced by a non-vanishing u velocity. We therefore seek an expansion in the form

$$u = \varepsilon^{2-\alpha}\left[u_0 + \varepsilon^{\alpha} u_1 + \varepsilon^{2\alpha} u_2 + \dots\right] \quad - (4:33)$$

$$v = v_0 + \varepsilon^{\alpha} v_1 + \dots \quad - (4:34)$$

$$w = w_0 + \varepsilon^{\alpha} w_1 + \dots \quad - (4:35)$$

$$\theta = \theta_0 + \varepsilon^{\alpha} \theta_1 + \dots \quad - (4:36)$$

$$p = p_0 + \varepsilon^{\alpha} p_1 + \dots \quad - (4:37)$$

$$Ra = Ra_0 + \varepsilon^{\alpha} Ra_1 + \varepsilon^{2\alpha} Ra_2 + \dots \quad - (4\_38)$$

The question now arises at which order we should balance the non-linearity of the mean temperature gradient against the change in Rayleigh number since $Ra = Ra_0$ is the minimum value of Ra for which solutions of the zero order problem exist we expect that $Ra - Ra_0$ will be proportional to the square of the departure from the zero order solution in the governing equations.

We therefore expect that

$$\alpha = 2 - 2\alpha$$
$$\alpha = 2/3$$

will lead to the correct balance of terms. With this choice of α and expanding the X dependence of equations (3:27) and (3:28) (or the equivalent expressions for rigid-rigid and rigid-free boundaries) in equations (4:16) - (4:19) we find that the lowest order term induced by the mean circulation is an $\mathcal{O}(\varepsilon)$ term, which is small compared with the horizontal diffusion terms, thereby justifying our neglect of such terms.

Choosing $\alpha = 2/3$ and replacing $\frac{\partial}{\partial y}$ by ik we get the following sets of equations :

<u>$O(1)$</u>

$$\frac{\partial^2 \theta_0}{\partial z^2} - k^2 \theta_0 = - w_0 \qquad - (4:39)$$

$$\frac{\partial^2 w_0}{\partial z^2} - k^2 w_0 + Ra_0\, \theta_0 = \frac{1}{\sigma} \frac{\partial p_0}{\partial z} \qquad - (4:40)$$

$$\frac{\partial^2 v_0}{\partial z^2} - k^2 v_0 = \frac{ik}{\sigma} p_0 \qquad - (4:41)$$

$$ik\, v_0 + \frac{\partial w_0}{\partial z} \qquad - (4:42)$$

Putting

$$\theta_0 = A(\bar{X})\, g_0$$

$$w_0 = A(\bar{X})\, f_0$$

we see that f_0 and g_0 satisfy

$$\left(\frac{d^2}{dz^2} - k^2\right) g_0 = -f_0 \qquad - (4:43)$$

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 f_0 = k^2 Ra_0\, g_0 \qquad - (4:44)$$

and

$$V_0 = -\frac{A(\bar{X})}{ik} \frac{df_0}{dz} \qquad - (4:45)$$

$$\frac{p_0}{\sigma} = \frac{A(\bar{X})}{k^2} \frac{d}{dz}\left[\frac{d^2}{dz^2} - k^2\right] f_0 \qquad - (4:46)$$

For free-free boundaries we have the simple analytic solution

$$g_0 = \sin(\pi z) \qquad - (4:47)$$

$$f_0 = (\pi^2 + k^2)\sin(\pi z) \qquad - (4:48)$$

and

$$(\pi^2 + k^2)^3 = k^2 Ra_o \qquad - (4:49)$$

Equation (4:49) is of course just the standard Bénard cubic. The minimum value of Ra_o for which a real solution exists for k is

$$Ra_o = Ra_{oc} = \frac{27\pi^4}{4} \qquad - (4:50)$$

$$k = k_c = \frac{\pi}{\sqrt{2}} \qquad - (4:51)$$

$\underline{O(2/3)}$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\theta_1 + \frac{\partial^2 \theta_0}{\partial X^2} = -\omega_1 \qquad - (4:52)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\omega_1 + Ra_0\,\theta_1 = \frac{1}{\sigma}\frac{\partial p_1}{\partial z} - Ra_1\,\theta_0 +$$

$$+ \frac{\partial^2 \omega_0}{\partial x^2}$$

- (4:53)

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) v_1 + \frac{\partial^2 v_0}{\partial x^2} = \frac{ik}{\sigma} p_1$$

- (4:54)

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right) u_0 = \frac{1}{\sigma}\frac{\partial p_0}{\partial x}$$

- (4:55)

$$\frac{\partial u_0}{\partial x} + ikv_1 + \frac{\partial \omega_1}{\partial z} = 0.$$

- (4:56)

Using (4:46) we have

$$u_0 = \frac{1}{k^2}\frac{dA}{dx}\frac{df_0}{dz}$$

- (4:57)

After some standard manipulation we get

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)^2 \omega_1 - k^2 Ra_0\, \theta_1 = k^2 Ra_1\, g_0\, A(\bar{X}) + $$
$$+\frac{1}{k^2}\frac{d^2A}{d\bar{X}^2}\left[k^4 f_0 - \frac{d^4 f_0}{dz^4}\right] \qquad - (4:58)$$

$$\left(\frac{\partial^2}{\partial z^2}-k^2\right)\theta_1 + \omega_1 = -g_0\frac{d^2A}{d\bar{X}^2} \qquad - (4:59)$$

If we denote by g_0^+, f_0^+ the appropriate adjoint multipliers for (4:58) and (4:59) (obtained from (4:43) and (4:44)) then a solution exists only if

$$k^2 Ra_1\, A(\bar{X})\, I_1 + \frac{d^2A}{d\bar{X}^2} I_2 = 0. \qquad - (4:60)$$

where

$$I_1 = \int_0^1 g_0 f_0^+\, dz$$

$$I_2 = \int_0^1 \left[k^2 f_0 f_0^+ - \frac{1}{k^2}\frac{d^4 f_0}{dz^4} f_0^+ - g_0 g_0^+\right] dz$$

For the three types of boundary conditions that we consider we find that I_2 is zero. So that (4:60) yields that Ra_1 must be identically zero, vindicating our choice of $\alpha = 2/3$.

We can then put

$$W_1 = \frac{d^2A}{d\bar{X}^2} f_1 + B(\bar{X}) f_0 \qquad - (4:61)$$

$$\theta_1 = \frac{d^2A}{d\bar{X}^2} g_1 + B(\bar{X}) g_0 \qquad - (4:62)$$

where

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 f_1 - k^2 Ra_0 g_1 = \frac{1}{k^2}\left[k^4 f_0 - \frac{d^4 f_0}{dz^4}\right] \qquad - (4:63)$$

$$\left(\frac{d^2}{dz^2} - k^2\right) g_1 + f_1 = -g_0 \qquad - (4:64)$$

and $B(\bar{X})$ is another undetermined function of $\bar{X}$.

For free-free boundaries equations (4:63) and (4:64) become

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 f_1 - \frac{27\pi^6}{8} g_1 = -\frac{9\pi^4}{4} \sin(\pi z) \qquad - (4:65)$$

$$\left(\frac{d^2}{dz^2} - k^2\right) g_1 + f_1 = -\sin(\pi z) \qquad - (4:66)$$

where we have made use of (4:49) and (4:51).

An obvious solution to these equations is then

$$f_1 = 0 \qquad - (4:67)$$

$$g_1 = \frac{2}{3\pi^2} \sin(\pi z) \qquad - (4:68)$$

So that for free-free boundaries we have that

$$\omega_1 = B(\bar{X})\,(\pi^2 + k^2)\sin(\pi z) \qquad - (4:69)$$

$$\theta_1 = \frac{d^2 A}{d\bar{X}^2}\,\frac{2}{3\pi^2}\sin(\pi z) + B(\bar{X})\sin(\pi z) \qquad - (4:70)$$

Furthermore from (4:54) and (4:56) we have

$$ikv_1 = -\frac{d^2 A}{d\bar{X}^2}\,\frac{df_1}{dz} - \frac{df_0}{dz}\left[B(\bar{X}) + \frac{1}{k^2}\frac{d^2 A}{d\bar{X}^2}\right] \qquad - (4:71)$$

$$\frac{p_1}{\sigma} = \frac{1}{k^2}\,\frac{d^2 A}{d\bar{X}^2}\left[\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{df_1}{dz} + \frac{1}{k^2}\frac{df_0}{dz}\right) + \frac{df_0}{dz}\right]$$

$$+ \frac{B(\bar{X})}{k^2}\left[\frac{d^2}{dk^2} - k^2\right]\frac{df_0}{dz} \qquad - (4:72)$$

$O(\varepsilon^{4/3})$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\theta_2 + \omega_2 = -\frac{\partial^2\theta_1}{\partial \bar{X}^2} + \delta\beta_1 \bar{X}^2 \omega_0 \qquad - (4:73)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\omega_2 + Ra_0\,\theta_2 = \frac{1}{\sigma}\frac{\partial p_2}{\partial z} - Ra_2\theta_0 - \frac{\partial^2\omega_1}{\partial \bar{X}^2} \qquad - (4:74)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)V_2 + \frac{\partial^2 V_1}{\partial \bar{X}^2} = \frac{ik}{\sigma}p_2 \qquad - (4:75)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)u_1 + \frac{\partial^2 u_0}{\partial \bar{X}^2} = \frac{1}{\sigma}\frac{\partial p_1}{\partial \bar{X}} \qquad - (4:76)$$

$$\frac{\partial u_1}{\partial \bar{X}} + ikV_2 + \frac{\partial \omega_2}{\partial z} = 0. \qquad - (4:77)$$

Following the same manipulative procedure as before we get

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2\omega_2 - k^2Ra_0\,\theta_2 = k^2 Ra_2\, g_0\, A(\bar{X}) +$$

$$+ \frac{1}{k^2}\frac{d^4A}{d\bar{X}^4}\left[k^4 f_1 - \frac{d^4 f_1}{dz^4} - \frac{1}{k^2}\frac{d^4 f_0}{dz^4}\right] +$$

$$+ \frac{1}{k^2}\frac{d^2B}{d\bar{X}^2}\left[k^4 f_0 - \frac{d^4 f_0}{dz^4}\right] \qquad - (4:78)$$

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\theta_2 + \omega_2 = -\frac{d^4A}{d\bar{x}^4} g_1 -$$

$$- \frac{d^2B}{d\bar{x}^2} g_0 + \delta\beta_1 \bar{x}^2 A(\bar{x}) f_0 \qquad - (4:79)$$

As before for a solution to (4:78) and (4:79) to exist we must have

$$I_1 k^2 Ra_2 A(\bar{x}) + I_3 \frac{d^4A}{d\bar{x}^4} + \delta\beta_1 \bar{x}^2 A(\bar{x}) I_4 = 0. \qquad - (4:80)$$

where

$$I_3 = \int_0^1 \Big[k^2 f_1 f_0^+ - \frac{1}{k^2} f_0^+ \frac{d^4 f_1}{dz^4} - \frac{1}{k^4} \frac{d^4 f_0}{dz^4} f_0^+$$

$$- g_1 g_0^+ \Big] dz \qquad - (4:81)$$

$$I_4 = \int_0^1 f_0 g_0^+ \, dz \qquad - (4:82)$$

So that our amplitude equation is

$$\frac{d^4A}{d\bar{x}^4} + \frac{1}{I_3}\left[I_1 k^2 Ra_2 + \delta\beta_1 \bar{x}^2 I_4 \right] A(\bar{x}) = 0. \qquad - (4:83)$$

Notice that the function $B(\bar{x})$ is eliminated since it is multiplied by I_2 which we know vanishes. It should be noted that Newell and Whitehead (11) obtained a fourth order amplitude equation similar to the above for finite band convection.

Numerical values of the integrals I_1, I_3, I_4 for all three types of boundary conditions are set out in Table 1.

By simple manipulation equation (4:83) can be re-written in the form

$$\frac{d^4A}{d\phi^4} + (\alpha^* + \phi^2)A(\phi) = 0. \qquad - (4:84)$$

where

$$\alpha^* = \alpha/\gamma^4 \qquad - (4:85)$$

$$\gamma^6 = \pi \qquad - (4:86)$$

$$\pi = \delta\,\beta_1\, I_4/I_3 \qquad - (4:87)$$

$$\alpha = k^2 Ra_2\, I_1/I_3 \qquad - (4:88)$$

$$\phi = \gamma\,\bar{X} \qquad - (4:89)$$

We solve equation (4:84) subject to the following boundary conditions :

<u>ODD SOLUTIONS</u>

The solution is normalized by setting $A(0)$ to some number

$$A'(0) = 0.$$

$$\left.\begin{matrix} A(\phi) \\ A'(\phi) \end{matrix}\right\} \rightarrow 0 \qquad |\phi| \rightarrow \infty$$

EVEN SOLUTIONS

The solution is normalized by setting $A'(o)$ to some number

$$A(0) = 0$$

$$\left.\begin{array}{l} A(\phi) \\ A'(\phi) \end{array}\right\} \to 0 \qquad |\phi| \to \infty$$

Figures 5 to 8 show four solutions to equation (4:84), two odd modes and two even modes, obtained by using an algorithm developed by Moore & Cash (10) and a program written by members of Imperial College Mathematics Department. The first two odd and even eigenvalues for α^* are given in Table 2.

Using Tables 1 and 2, and noting that for the profile $\sin^2 X$ we find that for free-free boundaries

$$Ra_2 = -157\cdot734\ \delta^{2/3}\alpha^*$$

which for the first even mode, giving the lowest value of Ra_2 gives

$$Ra_2 = 166\cdot933\ \delta^{2/3} \qquad - (4:90)$$

and therefore

$$Ra = \frac{27\pi^4}{4} + \varepsilon^{4/3}(166\cdot933)\,\delta^{2/3} \qquad - (4:91)$$

Equation (4:91) is to be compared with

$$Ra = \frac{27\pi^4}{4} + 18\pi^3\left(\frac{3}{8}\right)^{1/2}\delta^{1/2}\varepsilon + O(\varepsilon^2) \qquad - (4:92)$$

which is the result obtained by Walton (9) for transverse rolls.

Remembering that the above analysis was constructed under the assumptions that $\delta \sim O(1)$ and $0 < \varepsilon << 1$ we must have that for longitudinal rolls to be more unstable than transverse rolls that

$$2.05\,\varepsilon > \varepsilon^{4/3}$$

(this condition being obtained by subtracting (4:91) from (4:92) which is clearly true for all $\varepsilon \leq 1$

5. $\underline{\delta \sim \varepsilon^4}$

We now turn our attention to the case where δ is no longer O(1) but is $O(\varepsilon^4)$, and consider only a single hot spot where

$$F(X, \delta) = 1 - \delta \tanh^2 X$$

Proceeding as before we find that there is no need to rescale and can expand in powers of ε^2 to obtain the somewhat obvious amplitude equation

$$\frac{d^4 A}{d\bar{X}^4} + \frac{1}{I_3}\left[I_1 k^2 Ra_2 + I_4 \bar{\delta} \tanh^2 \bar{X}\right] A(\bar{X}) = 0. \qquad - (5:1)$$

where we have written $\delta = \bar{\delta}\,\varepsilon^4$, with $\bar{\delta} \sim O(1)$.

Equation (4:83) can be recovered from (5:1) by simply letting $\bar{\delta} \to \infty$. To do this we write

$$\bar{X} = \bar{\delta}^{-p} X^* \qquad - (5:2)$$

to get

$$\bar{\delta}^{4p} \frac{d^4 A}{dx^*} + \frac{1}{I_3}\left[I_1 k^2 Ra_2 + I_4 \bar{\delta}^{1-2p} x^{*2} \right] A(x^*) = 0.$$

(5:3)

where we have expanded out the $\tanh^2(\bar{\delta}^{-p} x^*)$ term and ignored higher powers. We see that to achieve the necessary balance we must write

$$Ra_2 = \bar{\delta}^{1-2p} \overline{Ra}_2 \qquad - (5:4)$$

$$p = 1/6 \qquad - (5:5)$$

where now $\overline{Ra}_2$ is the Ra_2 of equation (4:83).

We solve equation (5:1) numerically in the form

$$\frac{d^4 A}{d\phi^4} + \left[\alpha^* + \tanh^2(\phi/\gamma) \right] A(\phi) = 0. \qquad - (5:6)$$

where now

$$\alpha^* = \frac{I_1 k^2 Ra_2}{I_3 \gamma^4} \qquad - (5:7)$$

$$\gamma^4 = \frac{I_4}{I_3} \bar{\delta} \qquad - (5:8)$$

The boundary conditions being the same as for equation (4:84).

Table 3 gives the values of α^{*} for the first even mode of equation (5:6) for various values of $\bar{S}$, and these solutions are depicted in figures 9-11.

For $\bar{S} = 100.0$ and free-free boundaries we get from (5:7), (5:8) and (5:4) that

$$Ra_2 = 3469 \cdot 572 \qquad - (5:9)$$

$$\overline{Ra}_2 = 161 \cdot 04 \qquad - (5:10)$$

The value for $\overline{Ra}_2$ is in reasonable agreement with that of equation (4:90).

CONCLUSIONS

We have shown that for convection confined between two infinite plane parallel plates with a localized non-uniform temperature distribution along the lower plate that a steady two dimensional circulation can be constructed which has no effect on the onset of convection when the length scale of the non-uniformity is much greater than the distance between the plates.

Further, and more importantly, the convective instability occurring in the vicinity of the hot spot (or more exactly, a hot ridge, since the configure is two-dimensional) takes on a preferred mode unlike the classical Rayleigh-Bernard situation, and that this mode is the longitudinal mode in which the axes of the cells lie across the ridge.

Although we have dealt with an idealized situation it is not unrealistic to extrapolate our results to laboratory experiments and state that even small imperfections in the heated boundary either in its smoothness or its thermal conductivity will dictate the form of convective instability arising.

FIGURE 1.

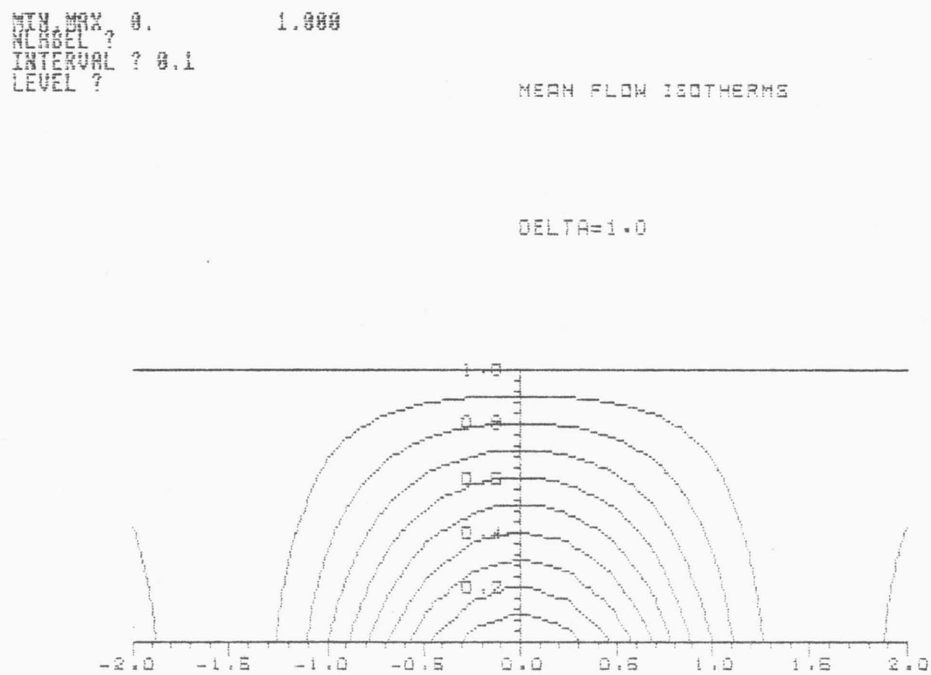
MIN,MAX 0. 1.000
NLABEL ?
INTERVAL ? 0.1
LEVEL ?
MEAN FLOW ISOTHERMS
DELTA=1.0
1.0
0.8
0.6
0.4
0.2
-2.0 -1.5 -1.0 -0.5 0.0 0.5 1.0 1.5 2.0

FIGURE 2. MEAN FLOW STREAMLINES FREE-FREE BOUNDARIES

DELTA = 1.

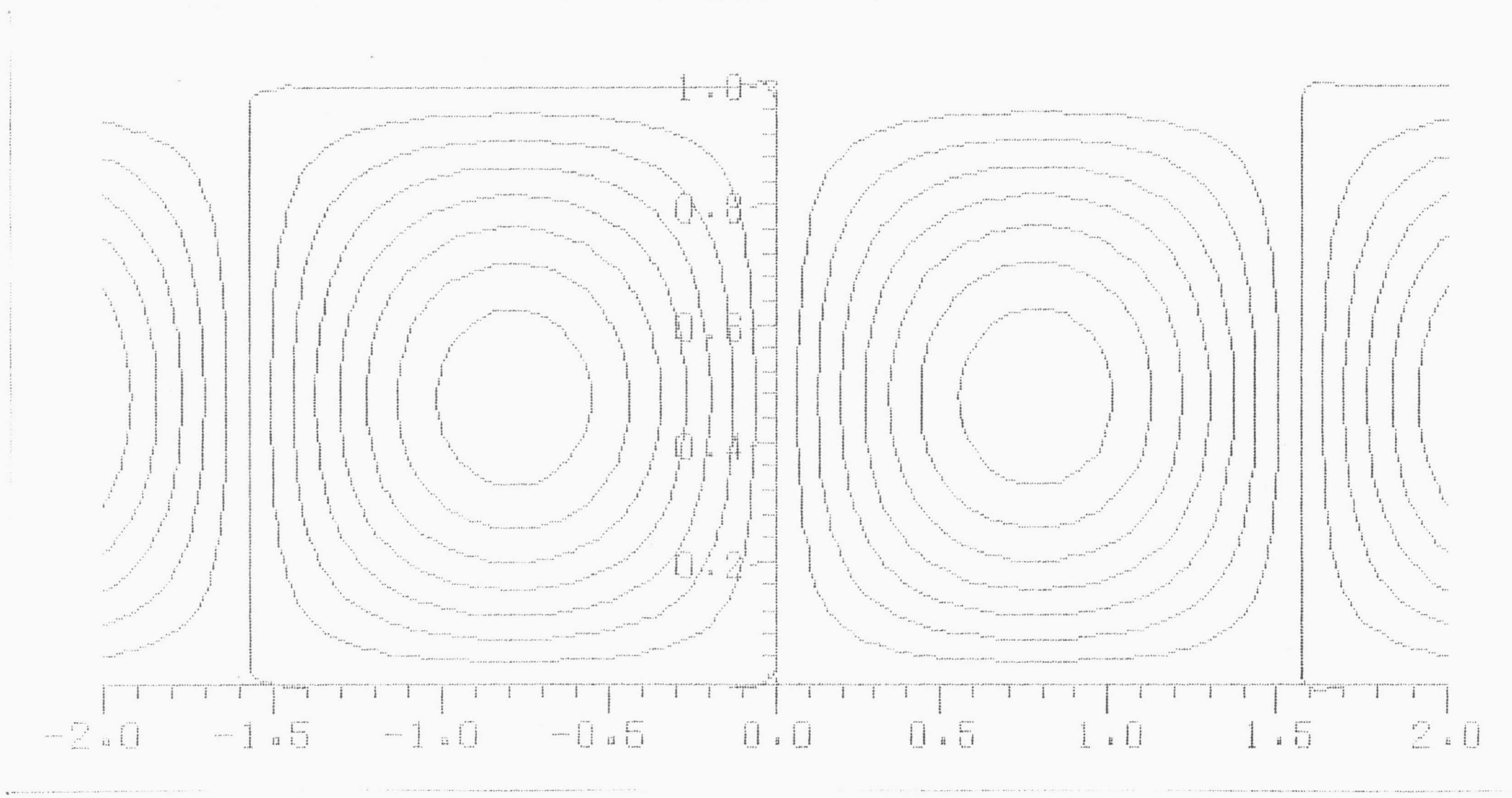

FIGURE 3. MEAN FLOW STREAMLINES RIGID-FREE BOUNDARIES.

DELTA = 1.

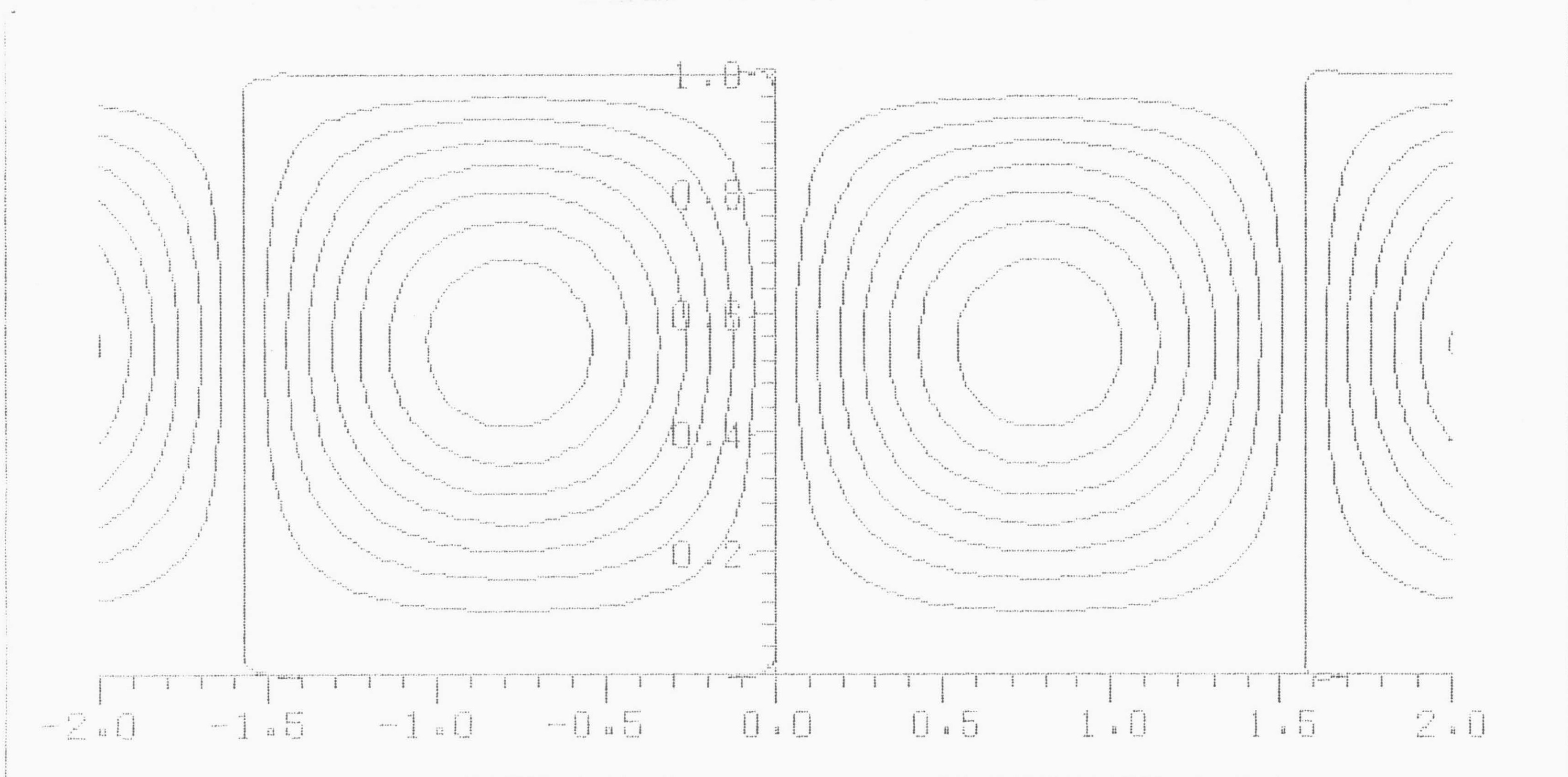

FIGURE 4. MEAN FLOW STREAMLINES RIGID-RIGID BOUNDARIES

DELTA = 1.

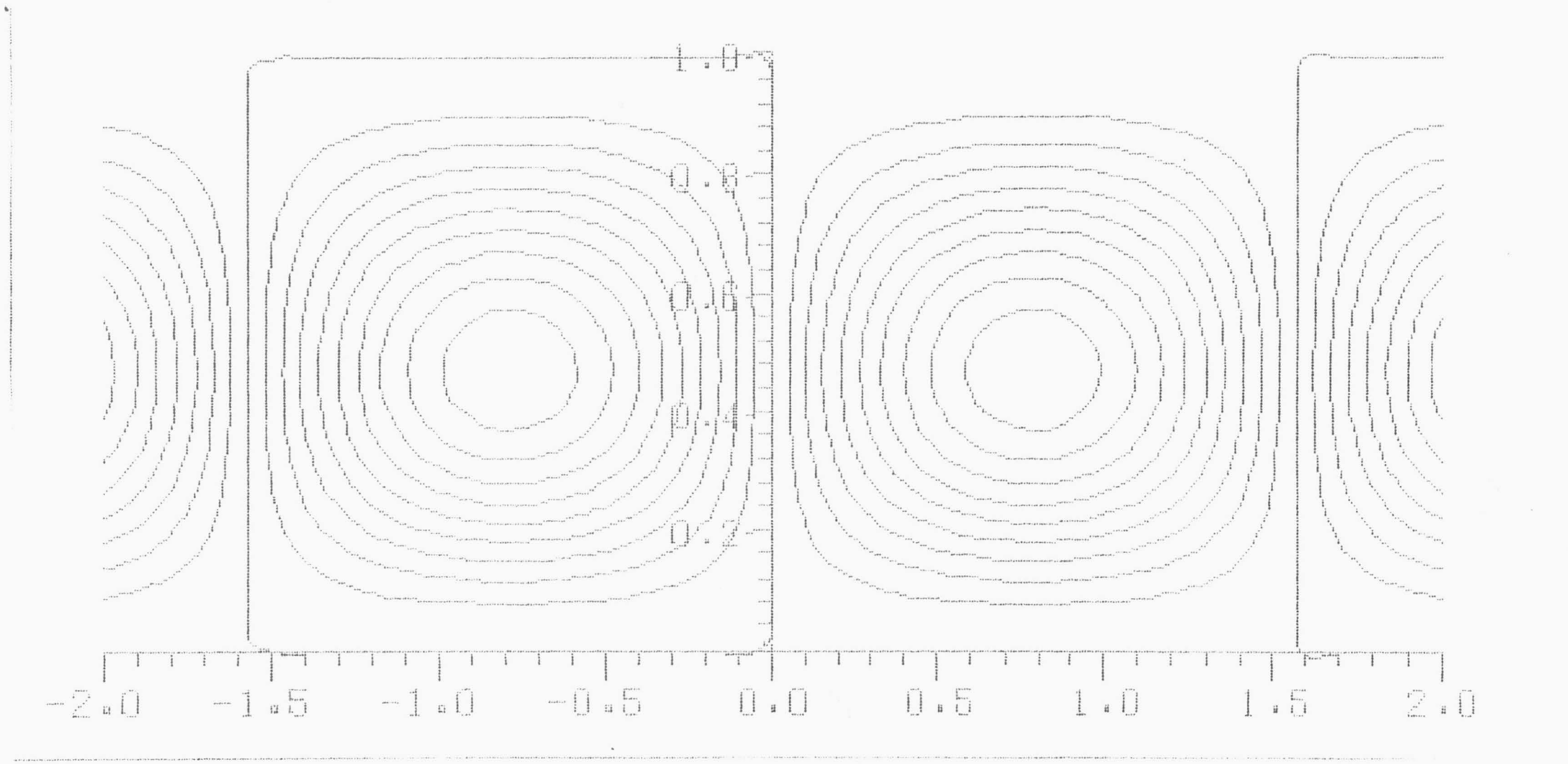

FIGURE 5.

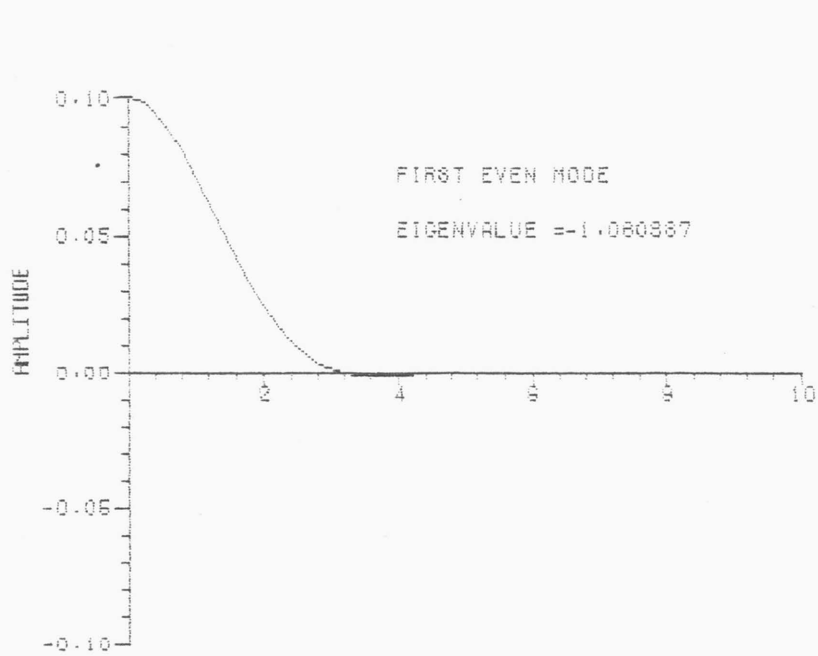

FIGURE 6.

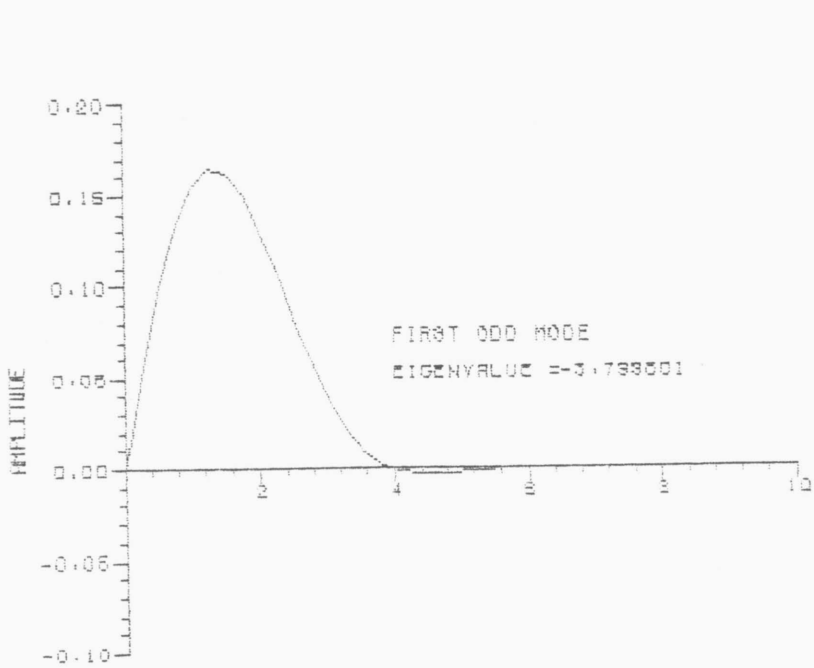

FIGURE 7.

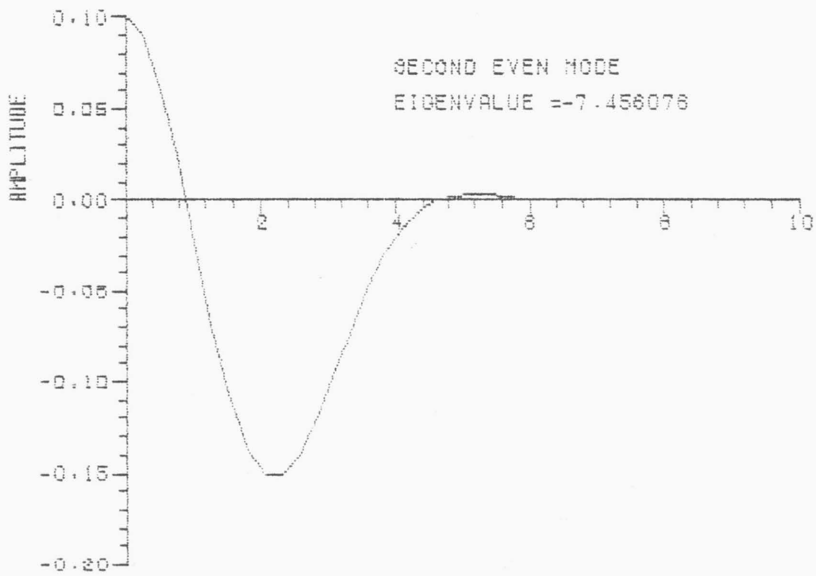

FIGURE 8.

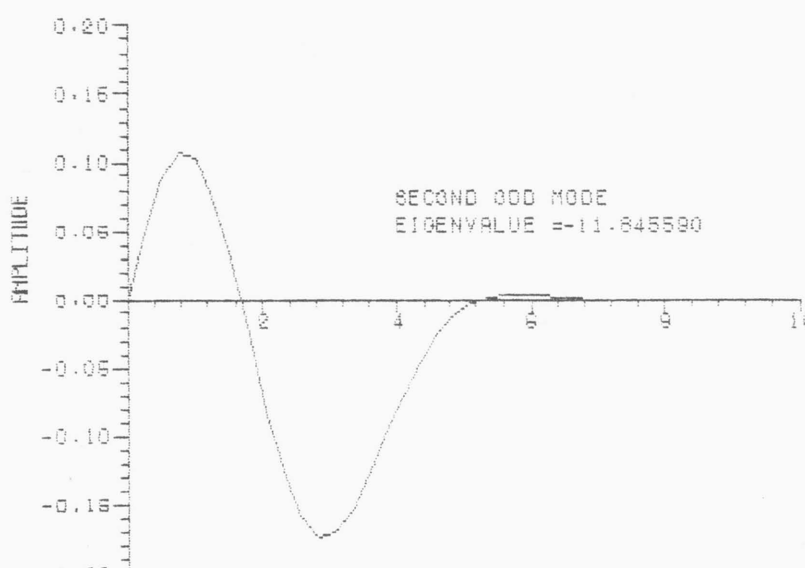

FIGURE 9.

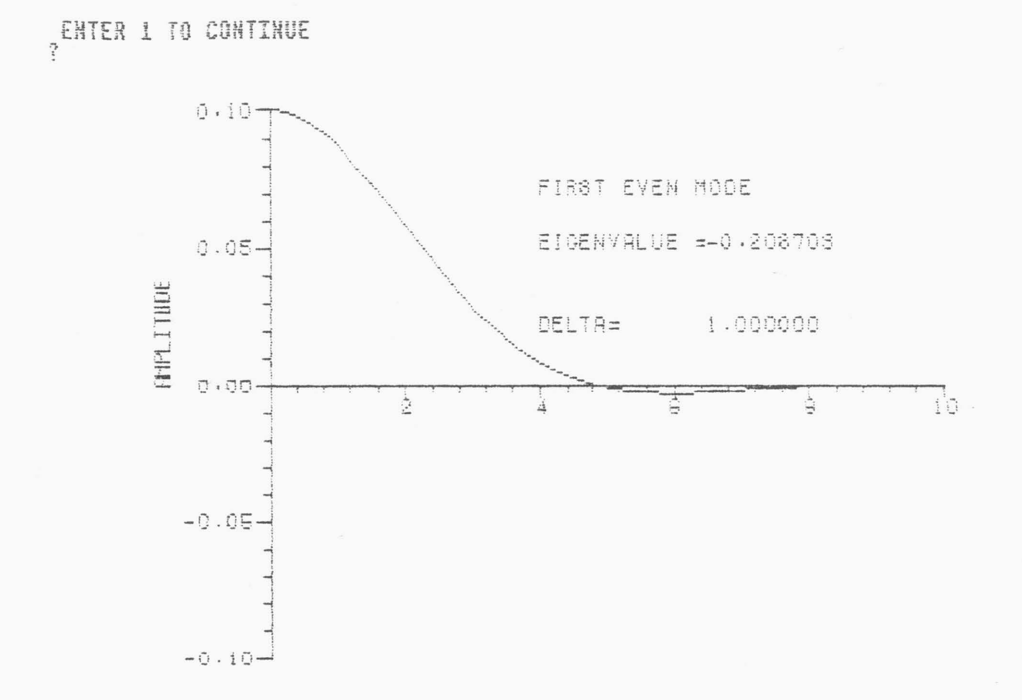

FIGURE 10.

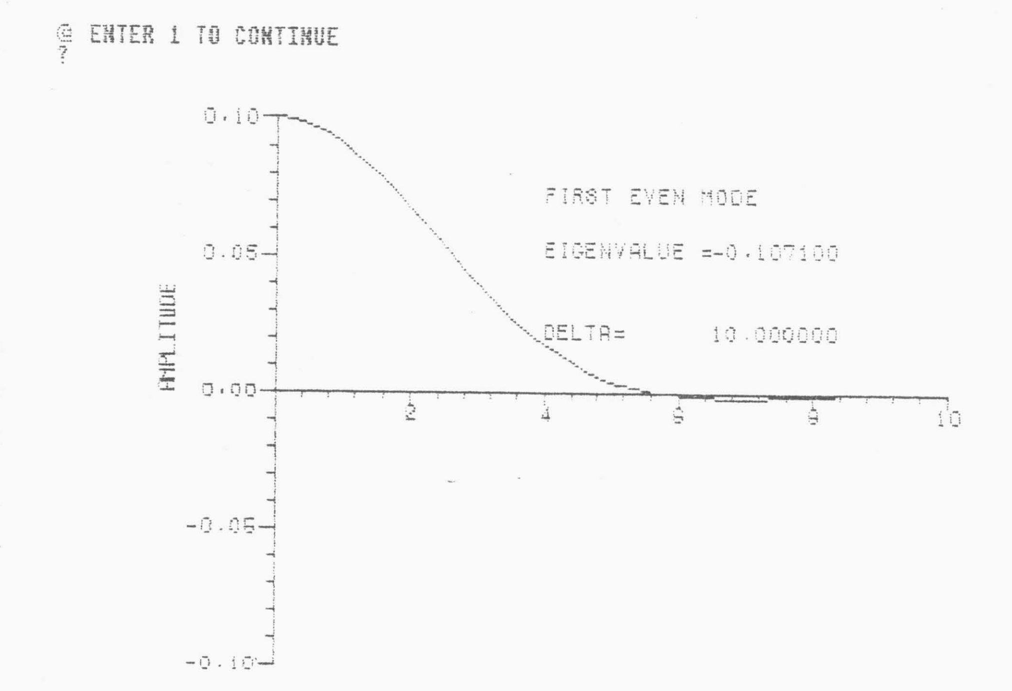

FIGURE 11.

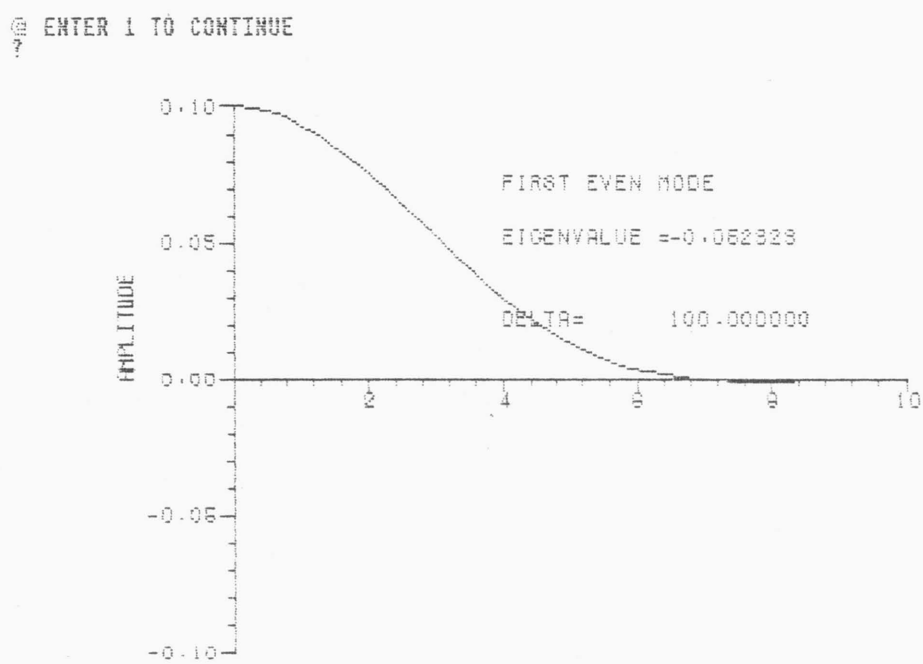
@ ENTER 1 TO CONTINUE
?
0.10
0.05
0.00
-0.05
-0.10
AMPLITUDE
2
4
6
8
10
FIRST EVEN MODE
EIGENVALUE =-0.062328
DELTA= 100.000000

TABLE 1.

	Free-Free	Rigid-Rigid	Rigid-Free
I_1	-0·0038	-0·0009923	-0·0002975
I_3	0·1688	0·06269	0·01603
I_4	12·334	16·4569	2·3557

TABLE 2.

Mode	α^*
odd	-3·788 -11·846
even	-1·061 -7·456

TABLE 3.

α^*	$\bar{\delta}$
-0·2087	1·0
-0·1071	10·0
-0·0528	100·0

REFERENCES

(1) WALTON, I.C. STUDIES IN APPLIED MATHS 67: 199. 1982.

(2) WALTON, I.C. J.F.M. 1983 (June)

(3) EAGLES, P.M. PROC. ROC. SOC. A371 1980.

(4) KELLY, R.E. & Pal, D. PROC. HEAT TRANSFER AND FLUID MECHANICS INST. pp 1-17. STANFORD UNIVERSITY PRESS. 1976.

(5) KELLY, R.E. & PAL, D. J.F.M. 86 1978.

(6) KELLY, R.E. & PAL, D. PROC. SIXTH INTERNATIONAL HEAT TRANSFER CONFERENCE. NATIONAL RESEARCH COUNCIL OF CANADA. Vol. 2.

(7) KELLY, R.E. & PAL, D. ASME/AICHE NATIONAL HEAT TRANSFER CONFERENCE ASME PAPER 79-HT-109 1979.

(8) RICHTER, F.M. REV. GEO. SPACE PHYSICS 11 1973.

(9) WALTON, I.C. Q.J.M.A.M. vol. XXXV PART 1.

(10) CASH, J.R. & MOORE, D.R.

(11). NEWELL, A C. & WHITEHEAD, J.A. J.F.M. 38 1969 p. 279.

CHAPTER 3.

THE STABILITY OF FREE CONVECTION IN A HORIZONTAL CYLINDRICAL ANNULUS WITH THE INNER BOUNDARY ROTATING.

Introduction

In this chapter we consider the onset of thermal convection in a Boussinesq fluid confined between two infinite horizontal cylindrical annuli, with the inner cylinder being the hotter. Also we allow the inner cylinder to rotate uniformly

Such a geometry is of course a radical departure from that of the standard infinite plane parallel plate problem (see Chandrasekhar (1)) and we can anticipate that the nature of the instability in such a geometry will show marked differences to that of the standard problem.

A fairly extensive literature exists, both theoretical and experimental, on heat transfer in long cylindrical annuli in the case of no rotation. (+)

On the experimental side Liu, Mueller and Landis (2), Grigull and Hauf (3), Lin (4), Bishop, Carley and Powe (5) and Powe, Carley and Bishop (6) have published results on heat transfer in long cylindrical annuli for various gap widths and using fluids with different Prandtl numbers. The essential features of their results are as follows. Firstly no state of hydrostatic equilibrium exists within the annuli and that for moderate Rayleigh numbers flow takes the form of two stable two-dimensional crescent shaped eddies, symmetrical about the vertical, where fluid rises near the inner hotter cylinder and sinks near the outer colder one. As the Rayleigh number is increased it is found that these eddies become unstable.

(+) Becker and Kaye (16) studied the problem with the inner cylinder rotating but ignored gravity i.e. Re large while Görtler (17) considered the stability of a fluid flowing over a concave boundary.

Korpella (7) and Birikh et al (8) studied the flow perturbation equations for flow in an inclined channel. These equations are identical to those for the above geometry, to leading order, when the ratio of gap width to inner radius ε is small.

They found that there are two distinct cases. One for fluids with Prandtl number greater than about 0·24 where the instability occurs at the top of the inner cylinder as a purely convective mode, while for fluids with a Prandtl number less than 0·24 the instability occurs at a point 65^0 away from the vertical and is of mixed hydrodynamic convective mode. Walton (9) undertook a more detailed study of the local structure of the instability in the case of the gap width being small and deduced that the instability had a parabolic cylinder function form in both cases.

In this chapter we repeat Walton's analysis allowing for the inner cylinder to rotate uniformly.

In section (1:2) we state the equations and derive the solution of the mean flow, the solution being expressed in a power series in terms of the gap width to inner radius ratio. Section (1:3) deals with the linear perturbation equations and investigates the local structure of the instability when the Reynolds number, Re, associated with the rotation of the inner cylinder is $O(\varepsilon^{1/2})$. Section (1:4) deals with the solution of the leading order linear perturbation equations when Re $\sim O(1)$ and compares the results with those obtained in Section (1:3). Section (1:5) contains the conclusions.

1:2

We start first by discussing the basic flow that exists between the two cylinders and on which we will base our linear stability analysis. Even if the inner cylinder was not rotating we know that a state of hydrostatic equilibrium is not possible and experiments of Liu, Muller and Landis (2), Powe, Carley and Bishop (6) and Kuehn and Goldstein (10) have shown that for moderate values of the Rayleigh number a crescent shaped eddy forms with hot fluid rising from the hotter inner cylinder and cool fluid descending from the outer cylinder.

In our case the rotation from the inner cylinder causes a shear flow to be imposed upon that due to thermal conduction and we shall suppose that the Reynolds number, Re, of this shear flow is sufficiently low as to rule out any shear instabilities.

This being the case we seek a steady two dimensional basic flow. We let the inner cylinder have radius L and the outer cylinder have radius $L+h$, denote $h/L = \varepsilon$ and assume that ε is small. The equations for describing such a flow are those obtained by Walton (9).

$$\nabla^4 \Psi = A\left(\sin\theta \frac{\partial T}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial T}{\partial \theta}\right) - \frac{1}{P_r\, r}\frac{\partial(\nabla^2\Psi, \Psi)}{\partial(r, \theta)} \qquad - (1:1)$$

$$\nabla^2 T = -\frac{1}{r}\frac{\partial(\Psi, T)}{\partial(r, \theta)} \qquad - (1:2)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad - (1:3)$$

and Ψ , T are dimensionless stream functions and temperature respectively. The radius has been made dimensionless on L , the radius of our inner cylinder. The Rayleigh number A is based on L and is given by

$$A = (g\beta \, \Delta T L^3)/\nu\alpha \qquad - (1:4)$$

where g is the acceleration due to gravity, β is the coefficient of thermal expansion, ΔT is the difference in temperature of the two cylinders, ν the kinematic viscosity and α the coefficient of thermal diffusivity. The Prandtl number, Pr , is defined by $Pr = \nu/\alpha$

The boundary conditions that we then need to apply to these equations are

$$\begin{aligned} &T = 1 \ \text{ at } r = 1+\varepsilon \ , \quad T = 0 \ \text{ at } r = 1 \\ &\Psi = 0 \ \text{ at } \ r = 1+\varepsilon \ , \quad r = 1 \\ &\frac{\partial \Psi}{\partial r} = 0 \ \text{ at } r = 1+\varepsilon \ , \quad \frac{\partial \Psi}{\partial r} = \frac{\Omega L^2}{\nu} \ \text{ at } r = 1 \end{aligned} \qquad - (1:5)$$

Following Walton (9) equations (1:1) and (1:2) have been written down with the following co-ordinate convention in mind. The angle θ being measured positive clock-wise from the vertical.

Again following Walton (9) we redefine the following variables, and assume that $Pr \sim O(1)$

$$Ra = \varepsilon^3 A \qquad - (1:6)$$

$$\Psi = -Ra\,\psi^* \qquad - (1:7)$$

$$r = 1 + \varepsilon x \qquad - (1:8)$$

equations (1:1), (1:3) and (1:5) then become

$$\nabla^4 \psi^* = -\left[\sin\theta \frac{\partial T}{\partial x} + \frac{\varepsilon \cos\theta}{1+\varepsilon x} \frac{\partial T}{\partial \theta} \right] + \qquad - (1:9)$$

$$+ \frac{\varepsilon\, Ra}{Pr\,(1+\varepsilon x)} \frac{\partial(\nabla^2 \psi^*, \psi^*)}{\partial(x, \theta)}$$

$$\nabla^2 T = \frac{\varepsilon\, Ra}{(1+\varepsilon x)} \frac{\partial(T, \psi^*)}{\partial(x, \theta)} \qquad - (1:10)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\varepsilon}{1+\varepsilon x} \frac{\partial}{\partial x} + \frac{\varepsilon^2}{(1+\varepsilon x)^2} \frac{\partial^2}{\partial \theta^2}$$

and the boundary conditions become

$$T = 1 \quad at \quad x = 1 \qquad T = 0 \quad at \quad x = 0$$

$$\psi^* = 0 \quad at \quad x = 1 \qquad - (1:11)$$

$$\frac{\partial \psi^*}{\partial x} = 0 \quad at \quad x = 1 \qquad \frac{\partial \psi^*}{\partial x} = -\frac{Re}{Ra} \quad at \quad x = 0$$

where $Re = \frac{\Omega L h}{\nu}$ is the Reynolds number, assumed to be $O(1)$ or smaller.

We now seek a solution to equations (1:9), (1:10) in the form

$$(\psi^*, T) = (\psi_0^*, T_0) + \varepsilon(\psi_1^*, T_1) + O(\varepsilon^2) + \cdots \qquad - (1:12)$$

<u>$O(1)$ equations</u>

$$\frac{\partial^2 T_0}{\partial x^2} = 0 \qquad - (1:13)$$

$$\frac{\partial^4 \psi_0^*}{\partial x^4} = -\sin\theta \frac{\partial T_0}{\partial x} \qquad - (1:14)$$

From (1:13) and (1:11) we get that

$$T_0 = 1 - x \qquad - (1:15)$$

Using (1:15) in (1:14) we arrive at

$$\psi_0^* = \frac{x^2}{24}(1-x)^2 \sin\theta - \frac{Re}{Ra} x\left(1 - \frac{x}{2}\right)$$

$$= a_0(x)\sin\theta - \frac{Re}{Ra} c_0(x) \qquad - (1:16)$$

$O(\varepsilon)$ equations

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial T_0}{\partial x} = Ra\left[\frac{\partial T_0}{\partial x}\frac{\partial \psi_0^*}{\partial \theta} - \frac{\partial T_0}{\partial \theta}\frac{\partial \psi_0^*}{\partial x}\right] \qquad - (1{:}17)$$

$$\frac{\partial^4 \psi_1^*}{\partial x^4} + 2\frac{\partial^3 \psi_0^*}{\partial x^3} = -\sin\theta\frac{\partial T_1}{\partial x} - \cos\theta\frac{\partial T_0}{\partial \theta} +$$

$$+ \frac{Ra}{Pr}\left[\frac{\partial^3 \psi_0^*}{\partial x^3}\frac{\partial \psi_0^*}{\partial \theta} - \frac{\partial^3 \psi_0^*}{\partial x^2 \partial \theta}\frac{\partial \psi_0^*}{\partial x}\right] \qquad - (1{:}18)$$

Substituting (1:15) and (1:16) into (1:17) we get

$$\frac{\partial^2 T_1}{\partial x^2} = 1 - Ra\, a_0(x)\cos\theta \qquad - (1{:}19)$$

the solution to which is

$$T_1 = \frac{x}{2}(x-1) - \frac{Ra\cos\theta}{1440}\, x\,(2x^5 - 6x^4 + 5x^3 - 1)$$

$$= b_0(x) - \frac{Ra\cos\theta}{1440}\, b_1(x) \qquad - (1{:}20)$$

After a considerable amount of algebra we find that the solution of (1:18) is

$$\psi_1^* = -\frac{\sin\theta}{80}\, a_1(x) + \frac{Ra \sin 2\theta}{2880}\, a_2(x) + \qquad \text{- (1:21)}$$

$$+ \frac{Ra}{Pr}\,\frac{\sin 2\theta}{288}\, a_3(x) + \frac{Re}{Pr}\,\frac{\cos\theta}{24}\, a_4(x)$$

where

$$a_1(x) = x^2(2x^3 - 5x^2 + 4x - 1) \qquad \text{- (1:22)}$$

$$a_2(x) = \frac{x^9}{252} - \frac{x^8}{56} + \frac{x^7}{42} - \frac{x^4}{24} + \frac{11x^3}{252} - \frac{x^2}{84} \qquad \text{- (1:23)}$$

$$a_3(x) = -\frac{x^9}{504} + \frac{x^8}{112} - \frac{x^7}{60} + \frac{x^6}{60} - \frac{x^5}{120} + \frac{x^3}{50} - \qquad \text{- (1:24)}$$

$$- \frac{x^2}{1680}$$

$$a_4(x) = -\frac{x^7}{70} + \frac{x^6}{15} - \frac{7x^5}{60} + \frac{x^4}{12} - \frac{x^3}{84} - \frac{x^2}{140} \qquad \text{- (1:25)}$$

Figures 1-4 show the streamlines $\bar{\Psi} =$ constant plotted in $\chi - \theta$ space for various values of Re. Due to the smallness of most of the terms in ψ_1^* no visible Prandtl number dependence is apparent and therefore all diagrams are plotted with $Pr = 1.0$.

The first obvious effect of rotation is the breaking of the symmetry about the $\theta = 0$ line. As the Reynolds number Re is increased the strength of the left hand side eddy is increased while that of the right hand side is decreased proportionally. That is to say for a clockwise rotation the effect of increasing Re is to strengthen the mean flow eddy on the left of the vertical. We find this phenomena is important to the understanding of the results obtained in section (1:4).

1:3 The Perturbation Equations

Following Walton (9) we now consider small perturbation to the base flow by letting

$$\psi^* \rightarrow \psi^* + \Omega \qquad - (2:1)$$

$$T \rightarrow T + \phi \qquad - (2:2)$$

where Ω and ϕ are small perturbations to the mean stream function and temperature profile respectively.

Substituting (2:1) and (2:2) into (1:9), (1:10) noting that time derivatives of perturbation quantities must be added in, and ignoring terms quadratic in the perturbation quantities we arrive at Walton's (9) equations

$$\nabla^2\left[\nabla^2 - \frac{\partial}{\partial t}\right]\Omega = -\left[\sin\theta\frac{\partial\phi}{\partial x} + \frac{\varepsilon\cos\theta}{1+\varepsilon x}\frac{\partial\phi}{\partial\theta}\right] +$$

$$+\frac{\varepsilon Ra}{Pr\,(1+\varepsilon x)}\left[\frac{\partial(\nabla^2\psi^*, \Omega)}{\partial(x,\theta)} + \frac{\partial(\nabla^2\Omega, \psi^*)}{\partial(x,\theta)}\right] \qquad - (2:3)$$

$$\left[\nabla^2 - Pr\frac{\partial}{\partial t}\right]\phi =$$

$$\frac{\varepsilon Ra}{(1+\varepsilon x)}\left[\frac{\partial(T,\Omega)}{\partial(x,\theta)} + \frac{\partial(\phi, \psi^*)}{\partial(x,\theta)}\right] \qquad - (2:4)$$

and where Ω and ϕ must satisfy the following boundary conditions

$$\Omega = \frac{\partial\Omega}{\partial x} = \phi = 0 \quad at \quad x = 0, 1 \qquad - (2:5)$$

As discussed in Walton's (9) paper at $\theta = 0, \pi$ there is no mean flow and with the mean temperature gradient positive at $\theta = \pi$ the flow is convectively stable there and for $\frac{\pi}{2} \leq \theta \leq \pi$. While at $\theta = 0$ the mean temperature gradient is negative and therefore the flow is potentially unstable at $\theta = 0$ and for $0 \leq \theta \leq \pi/2$. At $\theta = \pi/2$ the flow is equivalent to that in a vertical slot (see Korpela, Gozum and Baxi (12)) and it is known that in such cases the source of the instability is of a hydrodynamic nature rather than thermal. Between the angles 0 and $\pi/2$ both mechanisms are active and it is impossible to tell a priori at which angle the flow is most unstable for a given Prandtl number.

We therefore seek a WKB type of solution, centred about some critical angle θ_c, in the form

$$(\Omega, \phi) = A(\theta) e^{\sigma t} \exp\left\{ \frac{i}{\varepsilon} \int_0^{\theta_c} k(\theta)\, d\theta \right\} \left[(f_0, g_0) + \varepsilon (f_1, g_1) + \cdots \right] \qquad - (2:6)$$

$$Ra = Ra_0 + \varepsilon Ra_1 + \cdots \quad ; \quad \sigma = \sigma_0 + \varepsilon \sigma_1 + \cdots \qquad - (2:7)$$

where (f_i, g_i) $(i = 0, 1, \ldots)$ are functions of x and θ

Substituting the expansions (2:6), (2:7) into (2:4), (2:5) and using (1:16) we find that the order unity equations are :-

$$(D^2 - k^2 - \sigma_0)(D^2 - k^2) f_0 + \frac{ik}{Pr} Ra_0 \sin\theta \left[a_0'(x)(D^2 - k^2) f_0 - a_0'''(x) f_0 \right]$$

$$- \frac{ik\,Re}{Pr} c_0'(x)(D^2 - k^2) f_0 + \sin\theta\, D g_0 + ik\cos\theta\, g_0 = 0 \qquad - (2:8)$$

and

$$(D^2-k^2-Pr\,\sigma_0)\,g_0 + ik\,Ra_0\,a_0'(x)\sin\theta\,g_0$$

$$-ik\,Re\,c_0'(x)\,g_0 + ik\,Ra\,f_0 = 0$$

- (2:9)

where the $a_0'(x)$ represents $\dfrac{da_0(x)}{dx}$ etc, and $D \equiv \dfrac{\partial}{\partial x}$

From (2:5) the boundary conditions for (2:8) and (2:9) are obviously

$$f_0 = Df_0 = g_0 = 0 \quad at \quad x = 0, 1$$

- (2:10)

For a given value of θ equations (2:8) and (2:9) combined with the boundary conditions (2:10) form an eigenvalue problem for Ra_0 as a function of k, σ_0, Pr and Re.

Instability of the mean flow occurs if there exists a value of Ra_0 for which k is real and Real$(\sigma_0) \geqslant 0$, for given values of Pr and Re. We therefore define the critical value of the Rayleigh number for a given angle θ as the minimum value of Ra_0 at that angle for which k is real and Real$(\sigma_0) \geqslant 0$ (having specified Pr and Re).

This leads us to the obvious definition of the global critical Rayleigh number of the flow as being the minimum critical Rayleigh number over all angles i.e. $\partial Ra_0/\partial\theta = 0$ and $\partial Ra_0/\partial k = 0$. — (†)

The angle at which this occurs we will call the critical angle θ_c and will denote the global critical Rayleigh number by Ra_{oc}.

Before proceeding to solve equations (2:8) and (2:9) numerically for values of $Re \sim O(1)$ we can make analytical progress in the case Re small and determine what kind of results to expect. We first summarize Walton's (9) results in the case when $Re = 0$.

With $Re = 0$ equations (2:8) and (2:9) represent perturbations to a flow in an inclined channel, see Korpela (7) and Birikh et al (8). Birikh et al (8) calculated the critical value of the Grashof number $Gr_c = Ra_{oc} Pr$ for which instability first occurs for $0 \leq \theta \leq \pi$ (remembering that in the case of $Re = 0$ the base flow is symmetrical about $\theta = 0$) and $Pr = 0.2, 1.0$ and 5.0 They found that for $Pr = 1.0$ and 5.0, $\theta_c = 0$ and $Gr_c = 1708./Pr$. While for $Pr = 0.2$ they found that $\theta_c \sim 65^\circ$ and $Ra_{oc} \sim 1400$. So that for $Pr = 1.0$ and 5.0 the most unstable mode is a purely convective mode centred about the vertical $\theta = 0.$, while for $Pr = 0.2$ the instability occurs as a mixed hydrodynamic convective mode away from the vertical.

Korpela (7) presented more extensive results and showed that for $Pr = 0.$, $\theta_c = \pi/2$ and that therefore the instability is of a purely hydrodynamic form. As he increased the Prandtl number the critical angle decreases the instability becoming a mixed mode, until Pr reached about 0·24 when the most unstable mode became a purely convective one centred about $\theta = 0$, and the critical Rayleigh number became independent of the Prandtl number. All these calculations were done with $\sigma_0 = 0.$ i.e. only stationary modes were considered.

In an attempt to obtain further information about the instability Walton (9) considered the case $\varepsilon \ll 1$ and considered higher order terms in the expansions (2:6) and (2:7) thereby taking into account the curvature of the boundaries. As he pointed out in this case it is not possible to set the imaginary part of σ_1 equal to zero. By writing $\varrho = \theta - \theta_c$ and expanding about the critical angle he found that the equation governing the amplitude $A(\theta)$ determined at order ε broke down when ϱ became of order $\sqrt{\varepsilon}$. To resolve this difficulty he then sought a solution of the form

$$(\Omega, \phi) = e^{\varepsilon \sigma_1 t} \exp\left\{ \frac{i k_c}{\sqrt{\varepsilon}} y \right\} \left[B_0(y)(f_0, g_0) + O(\sqrt{\varepsilon}) + \dots \right] \qquad - (2:11)$$

On substituting (2:11) into the equations he found that the leading order equations corresponded to those for flow in an inclined channel, the $O(\varepsilon^{1/2})$ equations being satisfied automatically by the choice of θ_c and Ra_{oc} while the $O(\varepsilon)$ equations showed that $B_0(y)$ was related to a parabolic cylinder function by the transformation

$$B_0(y) = e^{\frac{1}{2} i \alpha_0 y^2} \, \bar{B}_0(y) \qquad - (2:12)$$

where $\bar{B}_0(y)$ satisfies

$$\frac{d^2 \bar{B}_0(y)}{dy^2} - (\alpha_1^2 y^2 + 2\alpha_2 \alpha_1) \bar{B}_0(y) = 0. \qquad - (2:13)$$

and $\varepsilon^{1/2} y = \varphi$, α_0, α_1 and α_2 being numerical constants.

Since α_2 contained Ra_1 , he was able to calculate the correction to the Rayleigh number Ra_{oc} .

From the discussion of the case $Re = 0$. it is clear that for the shear flow to effect the structure of the instability at the same order as that due to the curvature of the boundaries we must write $Re = \varepsilon^{1/2} \overline{Re}$, where $\overline{Re} \sim O(1)$. Notice that if we put $Re = \varepsilon \overline{Re}$ then the shear flow would have no effect on the previous analysis.

With $Re = \varepsilon^{1/2} \overline{Re}$ we then seek a solution in the form

$$(\Omega, \phi) = e^{\sigma t} \exp\left\{ \frac{i k_c}{\sqrt{\varepsilon}} y \right\} \left[B_0(y) (f_{oo}, g_{oo}) + O(\varepsilon^{1/2}) + \ldots \right] \qquad - (2:14)$$

with

$$\left. \begin{aligned} Ra &= Ra_{oc} + \varepsilon Ra_1 + \cdots \\ \sigma &= \varepsilon^{1/2} \sigma_0 + \varepsilon \sigma_1 + \cdots \end{aligned} \right\} \qquad - (2:15)$$

Substituting these expansions into equations (2:3) and (2:4) we get

<u>$O(1)$ equations</u>

$$(D^2 - k_c^2)^2 f_{oo} + \sin\theta_c \, D g_{oo} + i k_c \cos\theta_c \, g_{oo}$$

$$+ \frac{i k_c}{P_r} Ra_{oc} \sin\theta_c \left[a_0'(x) (D^2 - k_c^2) f_{oo} - a_0'''(x) f_{oo} \right] = 0. \qquad - (2:16)$$

and

$$(D^2-k_c^2)g_{00}+ik_c Ra_{oc}\sin\theta_c a_0'(x)g_{00}+ik_c Ra_{oc}f_{00}=0.$$

- (2:17)

which are identical to those given by Walton (9).

Equations (2:16) and (2:17) are identical to (2:8) and (2:9) with σ_0 and Re equal to zero.

Solving equations (2:16) and (2:17) for various values of θ and Pr we obtain the distribution of minimum Rayleigh number against angle for various values of the Prandtl number. These results are tabulated in Table One and displayed graphically in figures 5, 6 and 9. A cursory glance at these diagrmas reveals the behaviour pointed out by Birikh et al (8) in that the critical angle is $\theta_c=0$. with $Ra_{oc}=1707{\cdot}762$ and $k_c=3{\cdot}117$ for $Pr=1{\cdot}0,\ 0{\cdot}5$ while for $Pr=0{\cdot}2$, $\theta_c=1{\cdot}133$, $Ra_{oc}=1403{\cdot}956$ and $k_c=2{\cdot}757$. We also found that the nature of the instability changes from one of a purely convective mode to a mixed hydrodynamic-convective mode at around $Pr=0{\cdot}24$.

$O(\varepsilon^{1/2})$ Terms

By expanding out ψ^* and T in powers of $\varepsilon^{1/2}$ we get from (1:15), (1:16), (1:20) and (1:21)

$$T=(1-x)+\varepsilon\left\{b_0(x)+\frac{Ra_{oc}\cos\theta_c}{1440}b_1(x)\right\}$$

- (2:18)

and

$$\psi^* = a_0(x)\sin\theta_c + \sqrt{\varepsilon}\left\{a_0(x)\,y\cos\theta_c - \frac{\overline{Re}}{Ra_{oc}}c_0(x)\right\}$$

$$+\varepsilon\left\{-\frac{a_0}{2}(x)\,y^2\sin\theta_c - \frac{\sin\theta_c}{80}a_1(x) + \frac{Ra_{oc}\sin 2\theta_c}{2880}a_2(x)\right.$$

$$\left.+\frac{Ra_{oc}}{Pr}\frac{\sin 2\theta_c}{288}a_3(x)\right\}$$

- (2:19)

Using (2:18) and (2:19) in the expansion we get for the order $\sqrt{\varepsilon}$ equations :

$$(D^2-k_c^2)g_1 + ik_c Ra_{oc}\,a_0'(x)\sin\theta_c\,g_1 + ik_c Ra_{oc}f_1 =$$

$$\left[Pr\,\sigma_0 + ik_c\overline{Re}\,c_0'(x)\right]g_{oo}B(y) - ik_c Ra_{oc}\,a_0'(x)g_{oo}\,y\,B(y)$$

$$-\left[Ra_{oc}\,a_0'(x)\sin\theta_c\,g_{oo} + 2ik_c\,g_{oo} + Ra_{oc}f_{oo}\right]\frac{dB_o}{dy}$$

- (2:20)

$$(D^2 - k_c^2) f_1 + \sin\theta_c \, D g_1 + i k_c \cos\theta_c \, g_1 + \frac{i k_c}{P_r} Ra_{oc} \sin\theta_c \Big[a_0'(x) (D^2 - k_c^2) f_1$$

$$- a_0'''(x) f_1 \Big] = Q_{01} B_0(y) + Q_{02} \frac{dB_0}{dy} + Q_{03} \, y \, B(y)$$

- (2:21)

where

$$Q_{01} = \sigma_0 (D^2 - k_c^2) f_{00} +$$

$$+ \frac{i k_c \overline{Re}}{P_r} \Big[c_0'(x) (D^2 - k_c^2) f_{00} - c_0'''(x) f_{00} \Big] \qquad \text{- (2:22)}$$

$$Q_{02} = - 4 i k_c (D^2 - k_c^2) f_{00} - \frac{Ra_{oc}}{P_r} \sin\theta_c \Big[a_0'(x) (D^2 - 3 k_c^2) f_{00}$$

$$- a_0'''(x) f_{00} \Big] - \cos\theta_c \, g_{00} \qquad \text{- (2:23)}$$

and

$$Q_{03} = - \frac{i k_c}{P_r} Ra_{oc} \cos\theta_c \Big[a_0'(x) (D^2 - k_c^2) f_{00} - a_0'''(x) f_{00} \Big]$$

- (2:24)

$$- \cos\theta_c \, D g_0 + i k \sin\theta_c \, g_0$$

From equations (2:20) and (2:21) it is obvious that we can write g_1 and f_1 as

$$g_1 = B_0(y)\, g_{10} + \frac{dB_0}{dy}\, g_{11} + y\, B_0(y)\, g_{12} + + H_0(y)\, g_{00} \qquad - (2:25)$$

$$f_1 = B_0(y) f_{10} + \frac{dB_0(y)}{dy} f_{11} + y\, B_0(y) f_{12} + + H_0(y) f_{00} \qquad - (2:26)$$

where now

$$G(g_{10}, f_{10}) = \left[P_r\, \sigma_0 + i k_c\, \bar{R}_e\, c_0'(x) \right] g_{00} \qquad - (2:27)$$

$$F(f_{10}, g_{10}) = \varphi_{01} \qquad - (2:28)$$

$$G(g_{11}, f_{11}) = - Ra_{oc}\, a_0'(x) \sin\theta_c\, g_{00} - 2 i k_c\, g_{00} + Ra_{oc}\, f_{00} \qquad - (2:29)$$

$$F(f_{11}, g_{11}) = Q_{02} \qquad \text{- (2:30)}$$

$$G(g_{12}, f_{12}) = -i k_c \, Ra_{oc} \, a_0'(x) \, g_{00} \qquad \text{- (2:31)}$$

$$F(f_{12}, g_{12}) = Q_{03} \qquad \text{- (2:32)}$$

where

$$G(g,f) \equiv (D^2 - k_c^2) g + i k_c \, Ra_{oc} \, a_0'(x) \sin\theta_c \, g + i k_c \, Ra_{oc} \, f \qquad \text{- (2:33)}$$

and

$$F(f,g) \equiv (D^2 - k_c^2)^2 f + \sin\theta_c \, Dg + i k_c \cos\theta_c \, g + \frac{i k_c}{P_r} Ra_{oc} \sin\theta_c \left[a_0'(x)(D^2 - k_c^2) f - a_0'''(x) f \right] \qquad \text{- (2:34)}$$

Now for a solution to equations (2:20) and (2:21) to exist we must have that the integral formed by multiplying the right hand sides by their respective adjoint components adding and integrating over x must vanish. This leads to the following condition

$$I_1 B_0(y) + I_2 \frac{dB_0}{dy} + I_3 y B_0(y) = 0. \qquad - (2:35)$$

where

$$I_1 = \int_0^1 \left\{ [P_r \sigma_0 + i k_c \bar{R}_e C_0'(x)] g_{00} g_0^+ + Q_{01} f_{00}^+ \right\} dx \qquad - (2:36)$$

$$I_2 = \int_0^1 [P_{01} g_0^+ + Q_{02} f_0^+] dx \qquad - (2:37)$$

$$I_3 = \int_0^1 [P_{02} g_0^+ + Q_{03} f_0^+] dx \qquad - (2:38)$$

where g_0^+ and f_0^+ are the adjoint multiplier for (2:20) and (2:21) respectively, and P_{01} , P_{02} are the right hand sides of (2:29) and (2:31) respectively.

On evaluating these integrals numerically we find that I_2 and I_3 vanish, so that I_1 must vanish thereby determining the value of σ_0 .

By writing $\sigma_0 = \dfrac{i k_c \bar{Re}}{Pr}\beta$ we find from (2:36), (2:22) and (1:16) that

$$\beta = \frac{\int_0^1 [Pr\, g_{00}\, g_0^+ + (D^2 - k_c^2) f_{00}\, f_0^+](x-1)\, dx}{\int_0^1 [Pr\, g_{00}\, g_0^+ + (D^2 - k_c^2) f_{00}\, f_0^+]\, dx} \qquad - (2:39)$$

which when evaluated numerically yields a value of -0.5, which is a well known result for symmetric Couette flow (see Walton (11)).

$O(\varepsilon)$ Terms

The $O(\varepsilon)$ equations are

$$G(g_2, f_2) = (P_{10} + Ra_1 P_{11} + \sigma_1 P_{12} + y P_{13} + y^2 P_{14}) B(y)$$
$$+ (P_{15} + y P_{16}) \frac{dB}{dy} + P_{17} \frac{d^2B}{dy^2} + P_{18} \frac{dH}{dy} + P_{19} H(y) \qquad - (2:40)$$

$$F(g_2, f_2) = (Q_{10} + Ra_1 Q_{11} + \sigma_1 Q_{12} + y Q_{13} + y^2 Q_{14}) B(y)$$
$$+ (Q_{15} + y Q_{16}) \frac{dB}{dy} + Q_{17} \frac{d^2B}{dy^2} + Q_{18} \frac{dH(y)}{dy} + Q_{19} H(y) \qquad - (2:41)$$

Where the coefficients Q_{ij} , P_{ij} are functions of $f_{00}, f_{10}, f_{11}, f_{12}$ $g_{00}, g_{10}, g_{12},$ and their explicit form is given in Appendix One.

As before a solution to equations (2:40), (2:41) only exists if the integral formed by mutliplying the right hand sides by their respective adjoint functions, adding and integrating over x vanishes. This leads to the following equation for $B(y)$:-

$$I_{11}\frac{d^2B}{dy^2} + (yI_{10} + I_9)\frac{dB}{dy} + (y^2I_8 + yI_7 + \sigma_1 I_6 + Ra_1 I_5 + I_4)B(y) = 0. \qquad - (2:42)$$

where

$$I_{j+4} = \int_0^1 \left[Q_{1j} f_0^+ + P_{1j} g_0^+ \right] dx \qquad - (2:43)$$

The function $H(y)$ does not appear in equation (2_42) since the integrals I_{12} and I_{13} are the same as I_2 and I_3 respectively which we know vanish from the $O(\varepsilon^{1/2})$ equations.

Equation (2:42) can be transformed into a simpler form by writing

$$B(y) = e^{\frac{i\nu}{2}(y-\alpha)^2}\bar{B}(y) \qquad - (2:44)$$

where

$$\nu = \frac{i}{2}\left(\frac{I_{10}}{I_{11}}\right), \quad \alpha = -\left(\frac{I_9}{I_{10}}\right)$$

which gives

$$\frac{d^2\bar{B}(y)}{dy^2} + a\left[(y+b)^2 - c\right]\bar{B}(y) = 0. \qquad - (2:45)$$

where

$$a = \frac{I_8}{I_{11}} - \frac{1}{4}\left(\frac{I_{10}}{I_{11}}\right)^2 \qquad - (2:46)$$

$$b = \frac{1}{2a}\left[\left(\frac{I_7}{I_{11}}\right) + \frac{1}{2}\left(\frac{I_9\, I_{10}}{I_{11}}\right)\right] \qquad - (2:47)$$

$$c = -\left\{\sigma_1\left(\frac{I_6}{I_{11}}\right) + Ra_1\left(\frac{I_5}{I_{11}}\right) + \left(\frac{I_4}{I_{11}}\right) - \frac{1}{2}\left(\frac{I_{10}}{I_{11}}\right) - \frac{1}{4}\left(\frac{I_9}{I_{10}}\right)^2 - a\,b^2\right\} \qquad - (2:48)$$

Table 2 gives the values of I_i / I_{11} $(i: 4, .., 16)$, a, b, σ_1, Ra_1 for $Pr = 1.0, 0.2$ and $Re = 0.$ and $\overline{Re} = 1.0$. For $Re = 0.$ and $Pr = 1.0, 0.2$ we find that a is negative and b equals zero so that equation (2:45) can be written

$$\frac{d^2\bar{B}}{dy^2} - \left[|a|\, y^2 + c\right]\bar{B}(y) = 0. \qquad - (2:49)$$

$\bar{B}(y)$ is then just the parabolic cylinder function (see Abramovitz and Stegun (12)) of the first kind which is regular at $y = 0.$ and is bounded as $y \to \pm\infty$ only if

$$\frac{c}{2\sqrt{|a|}} = -\left(n + \frac{1}{2}\right) \qquad - (2:50)$$

In this case $\bar{B}(y)$ can be written as

$$\bar{B}(y) = \bar{B}_{oo}\, e^{-\frac{1}{2}\sqrt{|a|}\, y^2} H_n(\sqrt{|a|}\, y)$$

where $\bar{B}_{oo}$ is an arbitrary constant and $H_n(z)$ is a Hermite polynomial. We then determine Ra_1 and σ_1 by considering the real and imaginary parts of (2:50) using (2:48) and noting that σ_1 is pure imaginary.

From Table 2 we can see that for $Re = 0.0$, I_4/I_{11} , I_{10}/I_{11} are pure imaginary and I_6/I_{11}, I_5/I_{11} are pure real. So that on taking the real and imaginary parts of (2:50) we obtain

$$Ra_1 = 2\sqrt{|a|}\,(n+\tfrac{1}{2})\left(\frac{I_5}{I_{11}}\right)^{-1} \qquad - (2:51)$$

$$\sigma_1 = \left[\frac{1}{2}\left(\frac{I_{10}}{I_{11}}\right) - \left(\frac{I_4}{I_{1p}}\right)\right]\left(\frac{I_6}{I_{11}}\right)^{-1} \qquad - (2:52)$$

These results agree with those of Walton (9). The lowest value of Ra_1 occurring when $n=0$.

With $\overline{Re}=1.0$ we find that b is non-zero and that a is unchanged. So that the local structure is still that of a parabolic cylinder function but the effect of the shear flow is to just shift the site of the instability. Furthermore it is easily seen using the appendix that I_7 and I_9 are proportional to $\overline{Re}$ so that b can be written

$$b = \overline{Re}\ \overline{b} \qquad - (2:53)$$

and hence the shift in origin is linearly proportional to the Reynolds number.

Taking the real and imaginary parts of equation (2:50) we obtain

$$Ra_1 = \left\{2\sqrt{|a|}\,(n+\tfrac{1}{2}) + a\,b^2 + \frac{1}{4}\left(\frac{I_9}{I_{11}}\right)^2 - Real\left(\frac{I_4}{I_{11}}\right)\right\}\left(\frac{I_5}{I_{11}}\right)^{-1} \qquad - (2:54)$$

$$\sigma_1 = \left[\frac{1}{2}\left(\frac{I_{10}}{I_{11}}\right) - i\,I_m\left(\frac{I_4}{I_{11}}\right)\right]\left(\frac{I_6}{I_{11}}\right)^{-1} \qquad - (2:55)$$

Since I_9/I_{11} and I_7/I_{11} are proportional $\overline{Re}$ and from the appendix the real part of I_4/I_{11} we can write (2:54) in the form

$$Ra_1 = \left\{ 2\sqrt{|a|}\,(n+\tfrac{1}{2}) + \overline{Re}^2 \left[\overline{a\,b}^2 + \frac{1}{4\overline{Re}^2}\left(\frac{I_9}{I_{10}}\right) - \frac{Real}{\overline{Re}^2}\left(\frac{I_4}{I_{11}}\right)\right]\right\} \left(\frac{I_5}{I_{11}}\right)^{-1} \qquad - (2:56)$$

Equation (2:56) shows us that the effect of rotation is to add on a contribution proportional to $\overline{Re}^2$ to the value of Ra_1 in case of no rotation.

1:4

We now return to equations (2:8) and (2:9) and solve the eigenvalue problem with $Re \sim O(1)$. We note that from the preceeding section we must put

$$\sigma = -\frac{1}{2}\ \frac{i\,k_c\,Re}{Pr}$$

Also with $Re \neq 0$ we have that the symmetry about $\theta = 0$ is broken and we must therefore investigate the behaviour of the equations over the range $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Figures 8 - 13 show curves of Rayleigh number against angle for various values of Re and Pr. For $Pr \sim 1.0$ we see that as predicted by the asymptotics the curve is deformed slightly to the right near $\theta = 0$. In the case of $Pr \lesssim 0.25$ a more dramatic change is seen, however, as Re becomes larger the site of the instability is on the left hand side of the cylinder. In fact with $Pr \lesssim 0.24$ even with $Re \ll 1$. the instability is sited on the left hand side of the inner cylinder.

Table 3 lists critical angles and Rayleigh numbers for various values of Re and Pr. The results of Table 3 clearly demonstrate that a different mechanism is at work when $Pr \lesssim 0.24$ than when $Pr \sim 1.0$. A possible explanation of this is revealed if we consider the stream function of the mean flow (see figs 5 - 7). With rotation present, the inner cylinder is rotating clockwise, the shear flow reinforces the strength of the eddy due to the thermal gradient on the left hand side of the cylinder while decreasing it on the right hand side. Since in the case of no rotation and $Pr \leq 0.24$ the instability is a mixed hydrodynamic convection mode we can expect that the imposed shear flow will push the instability into a more hydrodynamic type which will occur where the shear flow has the greatest effect.

In order to compare the resulsts in Table 3 with those from the asymptotics we must remember to subtract the contribution to the Rayleigh number due to the curvature of the boundaries. So that for $Pr = 1.0$ the correct asymptotic formula for using in comparing results in Table 3 is

$$Ra_c = 1707.762 + \varepsilon\,(847.777 - 847.725)$$

$$= 1707.762 + \varepsilon\; 0.052$$

and

$$\theta_c = \sqrt{\varepsilon}\; 0.022$$

For $\varepsilon = (0.05)^2$ this gives

$$Ra_c = 1707.7621$$

$$\theta_c = 0.001$$

Table 3 gives $Ra_c = 1707.762$ and $\theta_c \simeq 6.5 \times 10^{-4}$ which gives a reasonably good agreement.

1:5 Conclusions

We have studied the onset of thermal convection in an infinite horizontal cylindrical annulus and have allowed the inner cylinder to rotate uniformly.

It is found that the system exhibits a strong Prandtl number dependence with two distinct cases :

(i) $Pr \sim 1$. where the onset of convection occurs at the top of the cylinder and exhibits a parabolic cylinder function structure. The effect of the rotation is to displace the site of convection away from the vertical leaving the local structure the same.

(ii) $Pr \lesssim 0{\cdot}24$ Here there is a sharp distinction between the case of rotation and no rotation. With no rotation the instability is a mixed hydrodynamic - convection mode centred about an angle 65^0 from the vertical. With rotation, however small, the site of instability occurs to the left of the vertical and as the strength of the rotation increases the site of instability moves closer to $\theta = -\pi/2$

The asymptotics again yield a parabolic cylinder function dependence.

FIGURE 1.

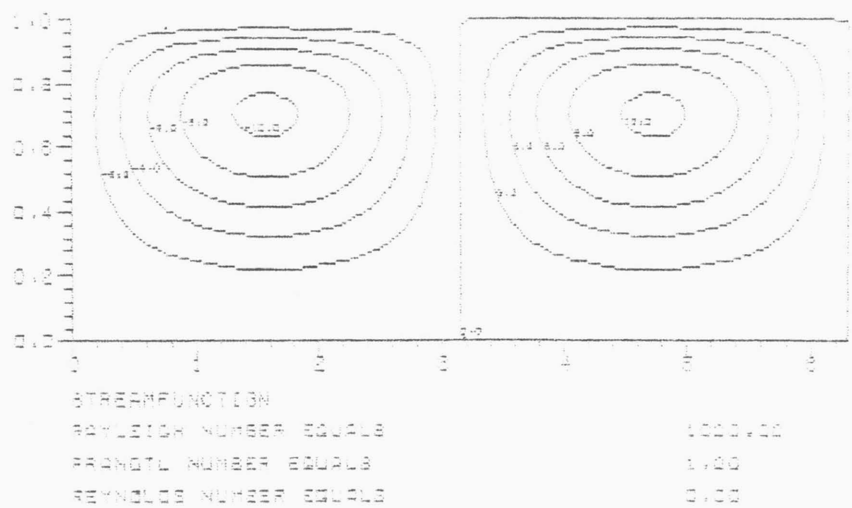

FIGURE 2.

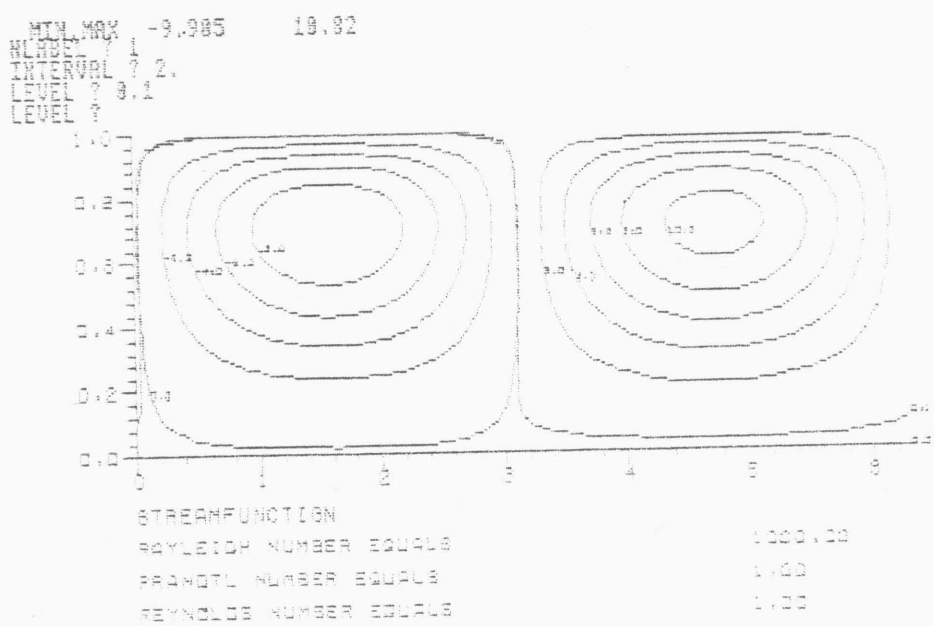

FIGURE 3.

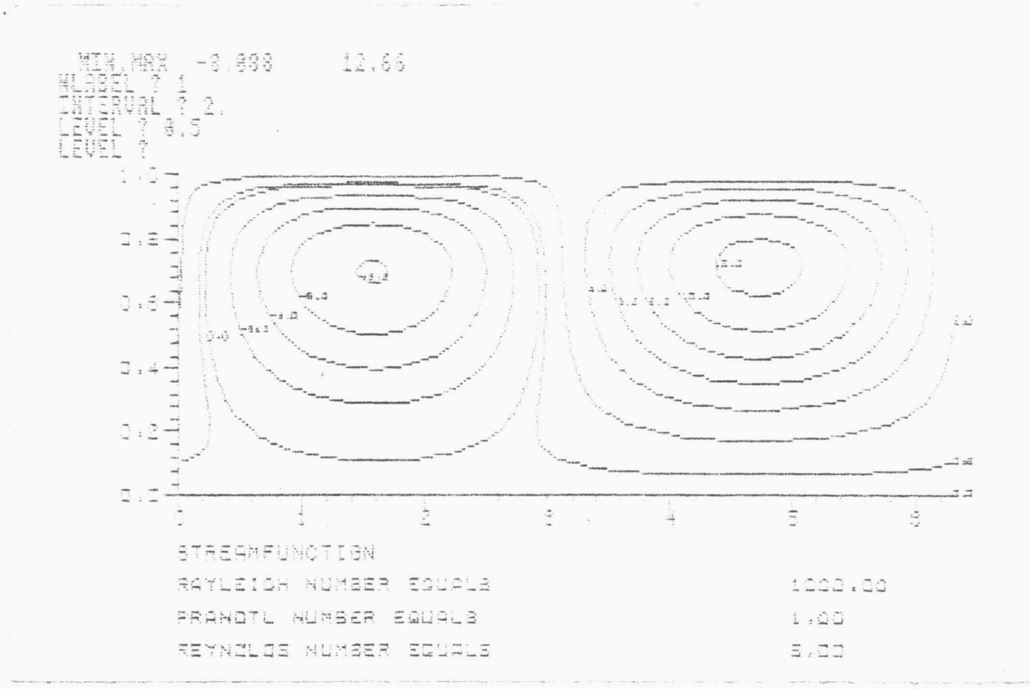

FIGURE 4.

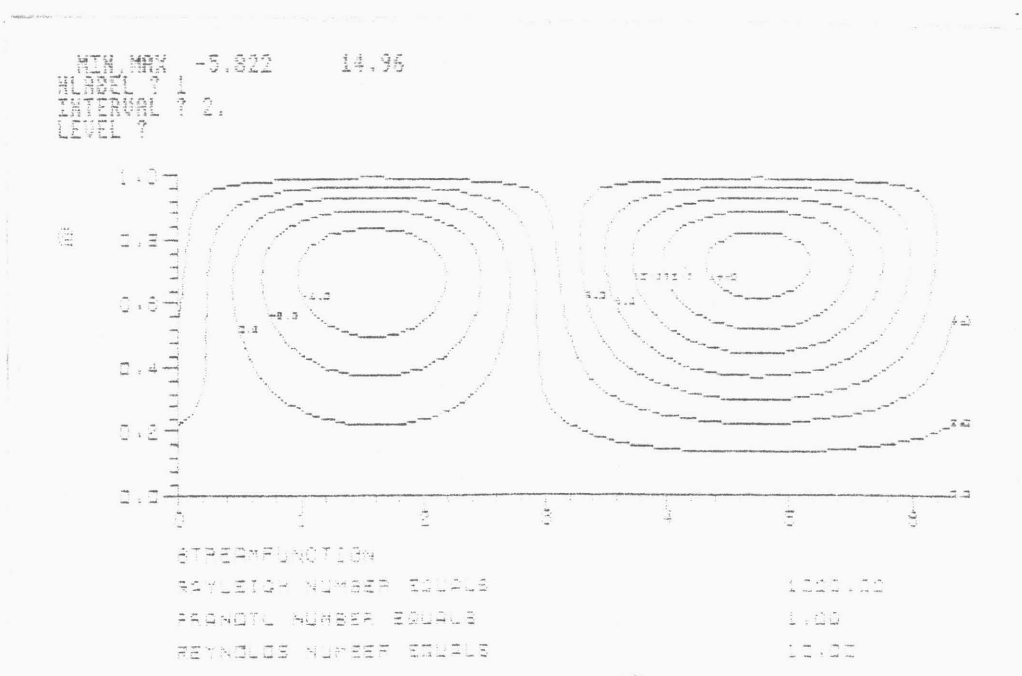

FIGURE 5.

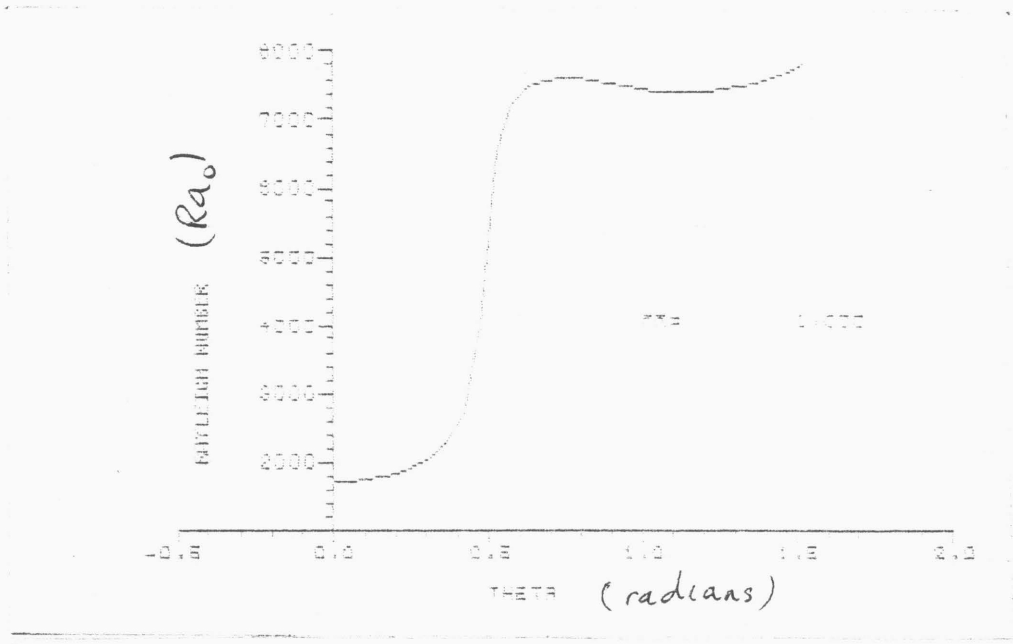

FIGURE 6.

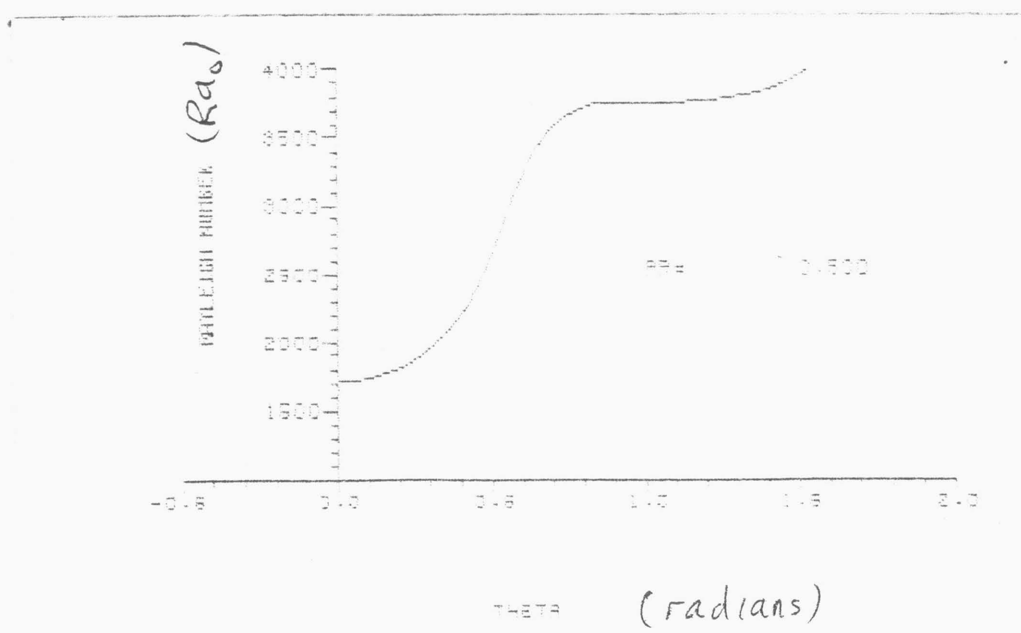

FIGURE 7.

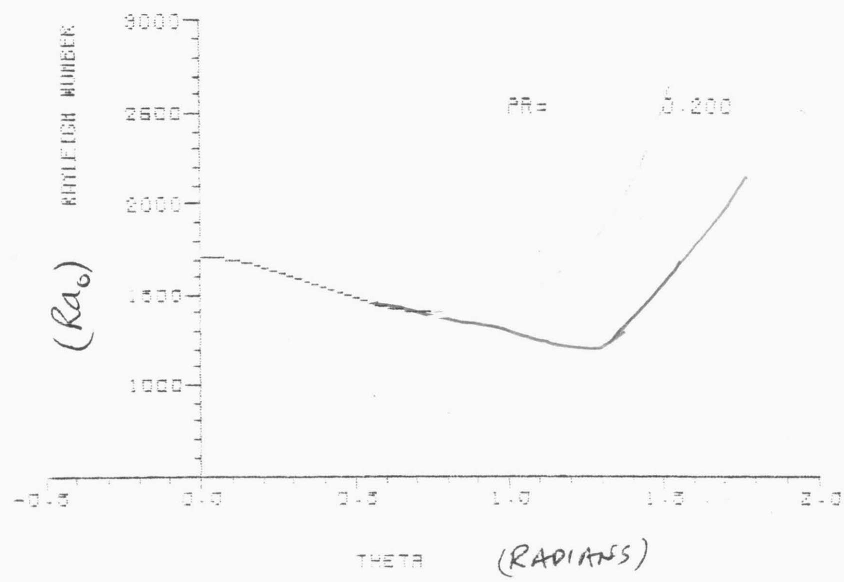

FIGURE 8.

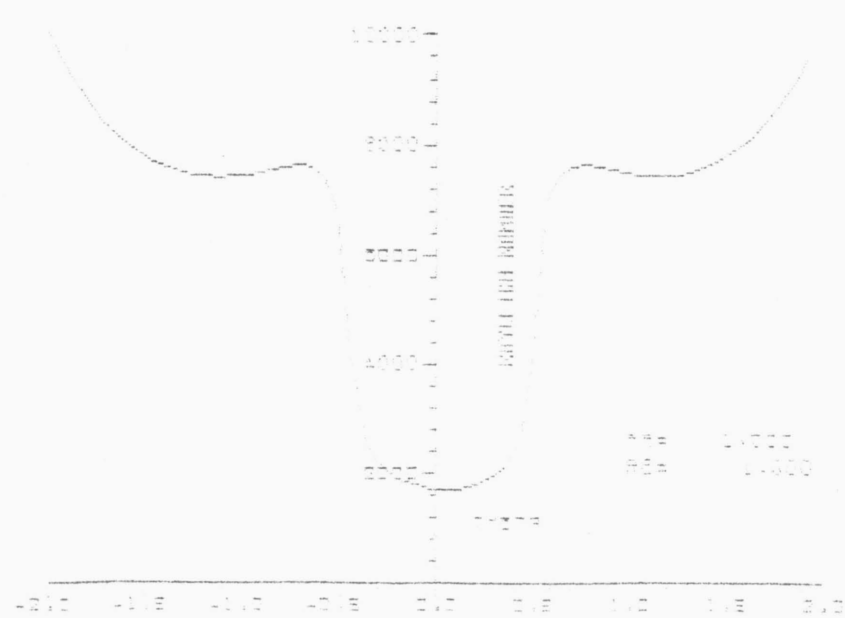

FIGURE 9.

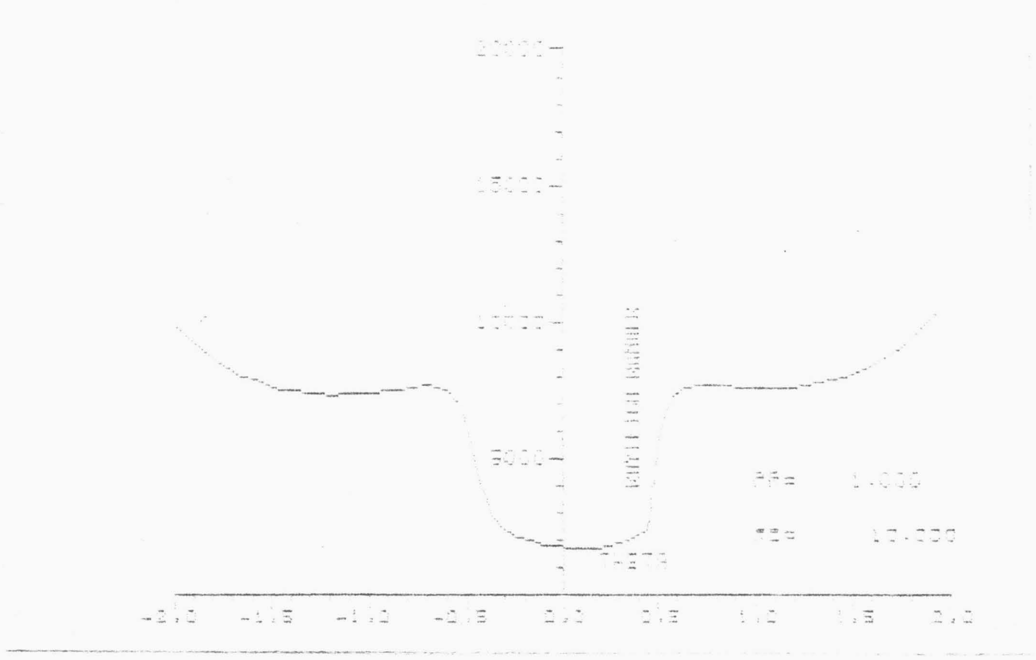

FIGURE 10.

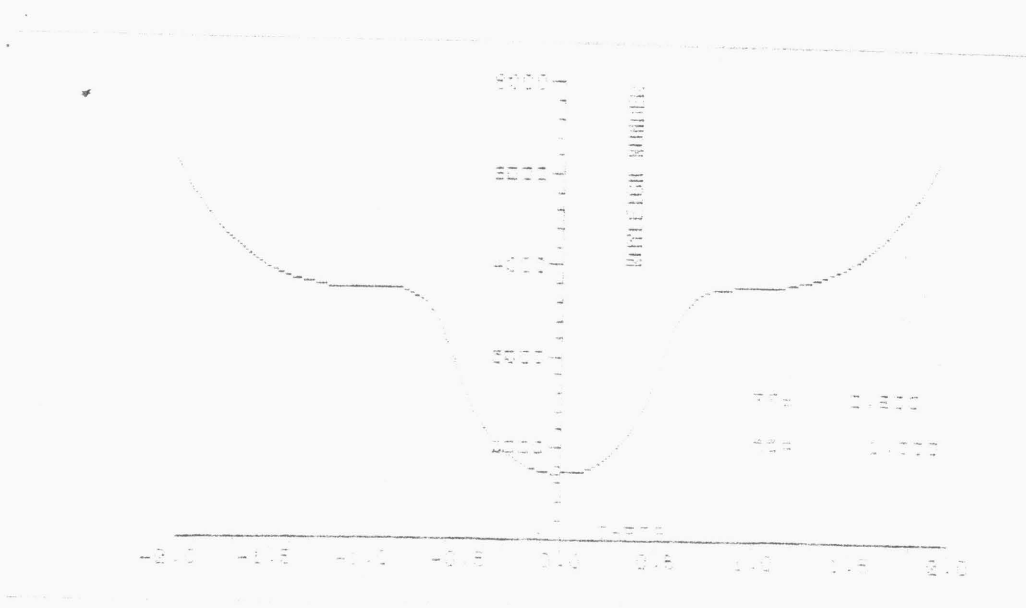

FIGURE 11.

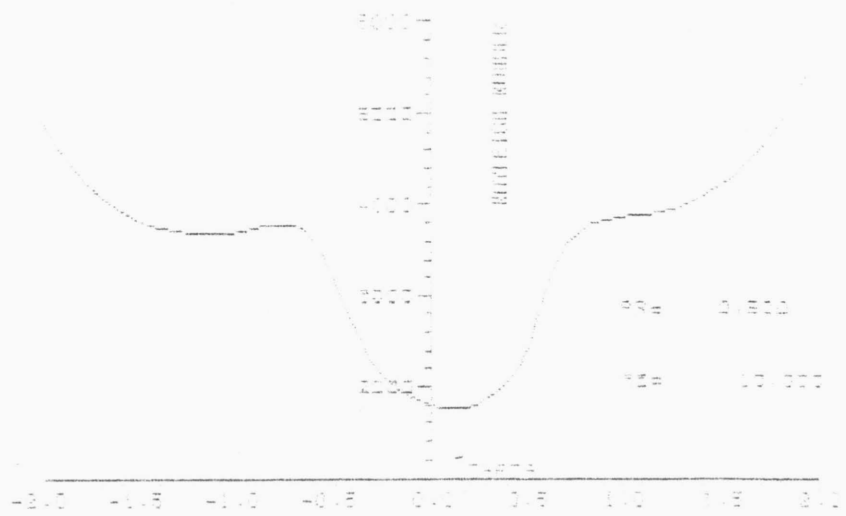

FIGURE 12.

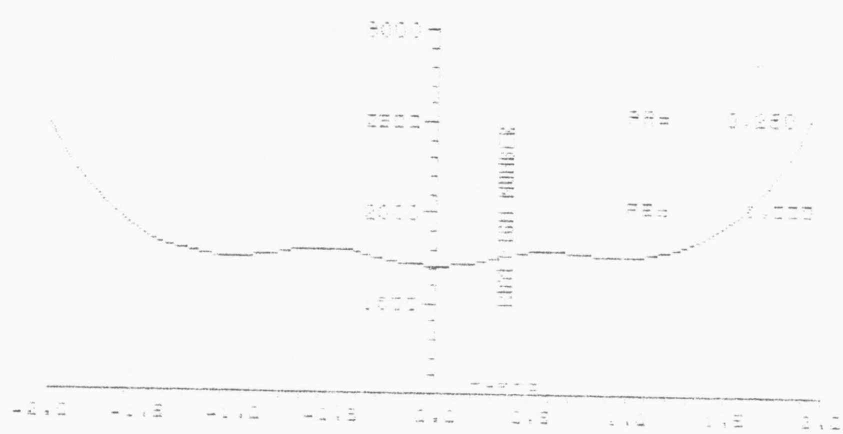

FIGURE 13.

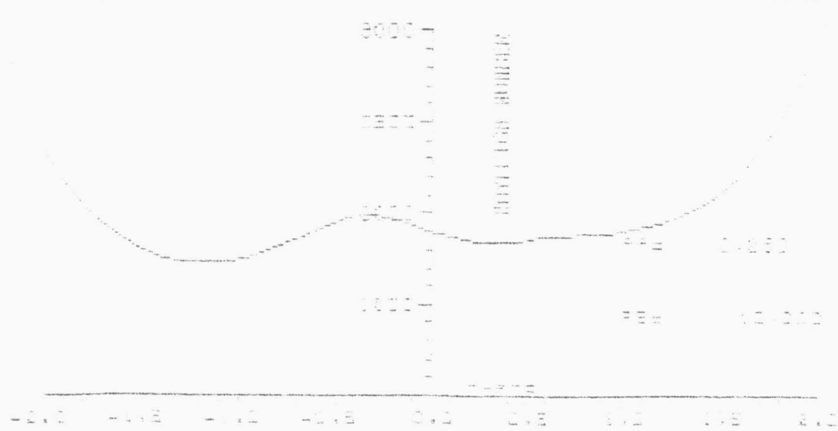

TABLE 1.

θ	Ra		
	$Pr = 1{\cdot}0$	$Pr = 0{\cdot}5$	$Pr = 0{\cdot}2$
0·0	1707·762	1707·762	1707·762
0·174	1801·518	1782·253	1691·319
0·349	2222·304	2063·519	1640·182
0·524	6549·218	2808·120	1565·695
0·689	7615·672	3595·406	1492·064
0·873	7559·846	3752·529	1436·767
1·047	7449·780	3758·689	1407·606
1·396	7595·183	3867·197	1441·679
1·571	7940·771	4047·136	1514·272

TABLE 2.

	Pr = 1·0		Pr = 0·2	
	Re = 0·0	Re = 1·0	Re = 0·0	Re = 1·0
I_4/I_{11}	(0·0, 0·0)	$(-6{\cdot}4,\ 7{\cdot}9)\times 10^{-3}$	(0·0, -1·255)	(-3·853, -1·321)
I_5/I_{11}	$(3{\cdot}958,\ 0{\cdot}0)\times 10^{-3}$	$(3{\cdot}958,\ 0{\cdot}0)\times 10^{-3}$	$(2{\cdot}898,\ 0{\cdot}0)\times 10^{-3}$	$(2{\cdot}898,\ 0{\cdot}0)\times 10^{-3}$
I_6/I_{11}	(-0·520, 0·0)	(-0·520, 0·0)	(-0·116, 0·0)	(-0·116, 0·0)
I_7/I_{11}	(0·0, 0·0)	(0·510, 0·0)	(0·0, 0·0)	(1·571, 0·0)
I_8/I_{11}	(-11·258, 0·0)	(-11·528, 0·0)	(-1·483, 0·0)	(-1·483, 0·0)
I_9/I_{11}	(0·0, 0·0)	$(0{\cdot}0,\ -1{\cdot}281)\times 10^{-3}$	(0·0, 0·0)	(0·0, -3·463)
I_{10}/I_{11}	(0·0, 0·0)	(0·0, 0·0)	(0·0, 0·292)	(0·0, 0·292)
a	-11·258	-11·528	-1·462	-1·462
b	(0·0, 0·0)	(-0·022, 0·0)	(0·0, 0·0)	(-0·364, 0·0)
Ra_1	847·725	847·777	417·230	645·387
σ_1	(0·0, 0·0)	(0·0, 0·0152)	(0·0, -12·077)	(0·0, -12·646)

TABLE 3.

Re	$Pr = 0.2$			$Pr = 1.0$		
	θ_c	Ra_c	k_c	θ_c	Ra_c	k_c
0·05	-1·1330	1403·444	2·755	$6{\cdot}527\times10^{-4}$	1707·762	3·1163
0·1	-1·1334	1402·936	2·753	$1{\cdot}305\times10^{-3}$	1707·7644	3·1163
0·25	-1·1346	1401·429	2·747	$3{\cdot}26\times10^{-3}$	1707·778	3·1163
0·5	-1·1366	1398·984	2·737	$6{\cdot}53\times10^{-3}$	1707·828	3·1163
1·0	-1·1408	1394·335	2·717	0·1305	1708·028	3·1162

APPENDIX

$$P_{10} = -2x k_c^2 g_{00} - Dg_{00} - Ra_{oc} f_{12} - 2ik_c g_{12}$$

$$+ \frac{ik_c Ra_{oc}^2}{1440} \cos\theta_c\, b_1(x) f_{00} + ik_c Ra_{oc} b_0'(x) f_{00}$$

$$+ ik_c Ra_{oc}\, x f_{00} - Ra_{oc} a_0(x) \cos\theta_c\, Dg_{00}$$

$$+ Ra_{oc} a_0'(x) \sin\theta_c\, g_{12} + ik_c Ra_{oc} a_0'(x) \sin\theta_c\, g_{00}$$

$$+ \frac{ik_c Ra_{oc}^2}{Pr}\left[\frac{Pr\, a_2(x)}{2880} + \frac{a_3(x)}{288}\right] \sin 2\theta_c\, g_{00}$$

$$+ ik_c Ra_{oc} a_0'(x) \sin\theta_c\, x g_{00} + ik_c \overline{Re}\, g_{10}\left[c_0'(x) - 0.5\right]$$

$$Q_{10} = -\frac{Ra_{oc}}{Pr} \sin\theta_c \left[a_0'(x)(D^2 - 3k_c^2) f_{12} - a_0'''(x) f_{12}\right]$$

$$- \frac{ik_c}{Pr} Ra_{oc} \cos\theta_c \left[a_0'(x)(D^2 - k_c^2) f_{10} - a_0'''(x) f_{10}\right]$$

$$+ \frac{ik_c}{80\, Pr} Ra_{oc} \sin\theta_c \left[a_1'(x)(D^2 - k_c^2) f_0 - a_1'''(x) f_0\right]$$

$$+ \frac{ik_c}{Pr} Ra_0^2 \sin 2\theta_c \left\{\left[\frac{Pr\, a_2'(x)}{2880} + \frac{a_3'(x)}{288}\right](D^2 - k_c^2) f_0\right.$$

$$\left. - \left[\frac{Pr\, a_2'''(x)}{2880} + \frac{a_3'''(x)}{288}\right] f_0 \right\} + \frac{Ra_{oc}}{Pr} \cos\theta_c \left[a_0(x)(D^3 - k_c D) f_0\right.$$

$$\left. - a_0''(x) D f_0\right] + i\frac{k_c}{Pr} Ra_{oc}\, x \sin\theta_c \left[a_0'(x)(D^2 - 3k_c^2) f_0 - a_0'''(x) f_0\right]$$

$$- \frac{i k_c}{P_r} Ra_{oc} \sin\theta_c \, a_0'(x) \, D f_{00} + \frac{i k_c}{P_r} Ra_{oc} \sin\theta_c \, a_0''(x) f_{00}$$

$$- \cos\theta_c \, g_{12} + i k_c \, x \cos\theta_c \, g_{00} - i k_c (D^2 - k_c^2) f_{12}$$

$$+ 2 (D^2 + k_c^2) D f_{00} - 4 k^2 x (D^2 - k_c^2) f_{00}$$

$$+ \frac{i k_c}{P_r} \overline{R}e \, (D^2 - k_c^2) f_{10} \left[\cos\theta_c \, c_0'(x) - 0.5 \right]$$

$$P_{11} = i k_c \left[a_0'(x) \sin\theta_c \, g_{00} - f_{00} \right]$$

$$Q_{11} = \frac{i k_c}{P_r} \sin\theta_c \left[a_0'(x) (D^2 - k_c^2) f_{00} - a_0'''(x) f_{00} \right]$$

$$P_{12} = P_r \, g_{00}$$

$$Q_{12} = (D^2 - k_c^2) f_{00}$$

$$P_{13} = i k_c \, Ra_0 \, a_0'(x) \cos\theta_c \, g_{10} + i k_c \overline{R}e \, g_{12} \left[c_0'(x) - 0.5 \right]$$

$$Q_{13} = - \cos\theta_c \, D g_{10} + i k_c \sin\theta_c \, g_{10} + \frac{i k_c}{P_r} \overline{R}e \, (D^2 - k_c^2) f_{12} \left[c_0'(x) - 0.5 \right]$$

$$P_{14} = i k_c \, Ra_{oc} \, a_0'(x) \cos\theta_c \, g_{10} - \frac{i k_c}{2} Ra_{oc} \, a_0'(x) \sin\theta_c \, g_0$$

$$Q_{14} = - \frac{i k_c}{P_r} Ra_{oc} \cos\theta_c \left[a_0'(x) (D^2 - k_c^2) f_{12} - a_0'''(x) f_{12} \right]$$

$$+ \frac{1}{2} \frac{i k_c}{P_r} Ra_{oc} \sin\theta_c \left[a_0'(x) (D^2 - k_c^2) f_0 - a_0'''(x) f_0 \right]$$

$$- \cos\theta_c\, D g_{12} + \frac{1}{2} \sin\theta_c\, D g_{00} + i k_c \sin\theta_c\, g_{12}$$

$$+ \frac{1}{2} i k_c \cos\theta_c\, g_{00}$$

$$P_{15} = 2 i k_c g_{13} - Ra_{oc} f_{13} - Ra_{oc}\, a_o'(x) \sin\theta_c\, g_{13}$$

$$+ \overline{Re} \left\{ i k_c g_{11} \left[c_o'(x) - 0.5 \right] + c_o'(x)\, g_{00} \right\}$$

$$Q_{15} = - \frac{Ra_{oc}}{Pr} \sin\theta_c \left[a_o'(x)\,(D^2 - k_c^2) f_{10} - a_o'''(x) f_{10} \right]$$

$$- \cos\theta_c\, g_{10} - 4 i k_c\, (D^2 - k_c^2) f_{10}$$

$$+ \overline{Re} \left\{ \frac{c_o'(x)}{Pr} (D^2 - 3k_c^2) f_{00} - \frac{i k_c}{2Pr} (D^2 - k_c^2) f_{11} + \frac{k_c^2}{Pr} f_o \right.$$

$$\left. + i \frac{k_c}{Pr} c_o'(x)\, (D^2 - k_c^2) f_{11} \right\}$$

$$P_{16} = - 2 i k_c g_{12} - Ra_{oc} f_{12} - Ra_{oc}\, a_o'(x) \sin\theta_c\, g_{12}$$

$$- i k_c Ra_{oc}\, a_o'(x) \cos\theta_c\, g_{11} - Ra_{oc}\, a_o'(x) \cos\theta_c\, g_{00}$$

$$Q_{16} = - \frac{Ra_{oc}}{Pr} \sin\theta_c \left[a_o'(x)\,(D^2 - 3k_c^2) f_{12} - a_o'''(x) f_{12} \right]$$

$$- i \frac{k_c}{Pr} Ra_{oc} \cos\theta_c \left[a_o'(x)\,(D^2 - k_c^2) f_{11} - a_o'''(x) f_{11} \right]$$

$$- \frac{Ra_{oc}}{Pr} \cos\theta_c \left[a_o'(x)\,(D^2 - 3k_c^2) f_{00} - a_o'''(x) f_{00} \right] - \cos\theta_c\, D g_{11}$$

$$- \cos\theta_c\, g_{12} + k_c \operatorname{sen}\theta_c\, g_{11} + \operatorname{sen}\theta_c\, g_{00}$$

$$- i k_c\, (D^2 - k_c^2)\, f_{12}\,.$$

$$P_{17} = -g_{00} - 2ik_c\, g_{11} - Ra_{oc}\, f_{11} - Ra_{oc}\, a_0'(x) \operatorname{sen}\theta_c\, g_{11}$$

$$Q_{17} = -\frac{3ik_c}{Pr} Ra_{oc} \sin\theta_c\, a_0'(x)\, f_{00}$$

$$- \frac{Ra_{oc}}{Pr} \operatorname{sen}\theta_c \left[a_0'(x)(D^2 - 3k_c^2) f_{11} - a_0'''(x) f_{11} \right]$$

$$- \cos\theta_c\, g_{11} - 4ik_c\, (D^2 - k_c^2) f_{11} - 2\, (D^2 - 3k_c^2) f_{00}$$

$$P_{18} = -2ik_c\, g_{00} - Ra_{oc}\, f_{00} + Ra_{oc}\, a_0'(x) \operatorname{sen}\theta_c\, g_{00}$$

$$Q_{18} = -\frac{Ra_{oc}}{Pr} \operatorname{sen}\theta_c \left[a_0'(x)\, (D^2 - 3k_c^2) f_{00} - a_0'''(x) f_0 \right]$$

$$- \cos\theta_c\, g_{00} - 4ik_c\, (D^2 - k_c^2) f_{00}$$

$$P_{19} = Pr\, \sigma_0\, g_0 + 4ik_c\, Ra_{oc}\, a_0'(x) \cos\theta_c\, g_{00}$$

$$+ ik_c\, \overline{Re}\, c_0'(x)\, g_{00}$$

$$q_{19} = -y \frac{ik_c}{Pr} Ra_{oc} \cos\theta_c \left[a_0'(x)(D^2-k_c^2) f_{oo} - a_0'''(x) f_{oo} \right]$$

$$+ \frac{ik_c \overline{Re}}{Pr} c_0'(x)(D^2-k_c^2) f_0 - y \cos\theta_c \, D g_{oo}$$

$$+ y \, ik_c \sin\theta_c \, g_{oo} + \sigma_0 (D^2-k_c^2) f_{oo}$$

REFERENCES

1. CHANDRASEKHAR, S. HYDRODYNAMIC AND HYDROMAGNETIC STABILITY Clarendon Press 1961.

2. LIU, C-Y., MUELLER, W.K., LANDIS, F. INTERNATIONAL DEVELOPMENTS IN HEAT TRANSFER PART V (American Soc. Mech. Engrs., New York 1961).

3. GRIGULL, U., HAUF, W. PROC. 3rd INT. HEAT TRANSFER CONFERENCE, CHICAGO 1966.

4. LIS, J. Ibid.

5. BISHOP, E.H., CARLEY, C.T. and POWE, R.E. INTERNATIONAL JOURNAL OF HEAT AND MASS TRANSFER Vol II 1968.

6. POWE, R.E., CARLEY, C.T. and BISHOP, E.H. J. HEAT TRANSFER, TRANS ASMEC <u>91</u> 1969.

7. KORPELA, S.A. INT. J. HEAT MASS TRANSFER <u>17</u> 1974.

8. BIRIKH, R.V., GERSHUNI, G.Z., ZHUKOVITSHII, E.M. and RUDAKOV, R.N. PRIKL. MAT. i MEKH. <u>32</u> 1968.

9. WALTON, I.C. Q.J.M.A.M. <u>33</u> 1980.

10. KUEHN, T.H. and GOLDSTEIN, R.J. J.F.M. <u>74</u> 1976.

11. WALTON, I.C. J.F.M. <u>154</u> 1985.

12. ABRAMOWITZ, M. and STEGUN, I.A. HANDBOOK OF MATHEMATICAL FUNCTIONS Dover 1972.

13. WALTON, I.C. J.F.M. <u>86</u> 1978 p. 673

14. HOCKING, L.M. & Q.J.M.A.M. <u>34</u> 1981 p. 475.

15. SOWARD, A.M. & JONES, C.A. Q.J.M.A.M. <u>36</u> 1983 p 19.

16. BECKER, K.M. & KAYE, J. ASME TRANS. MAY 1962 p. 106.

17. GÖRTLER, H. ING. ARCH. <u>28</u> 1959 p 71.