Partial slip incomplete contacts under constant normal load and subject to periodic loading

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Abstract

We present a general formulation for incomplete contact under oscillating loading, but with normal contact pressure. An asymptotic description of the contact traction *very* close to the contact edges are used. The slip zones present in the steady state in an incomplete contact subject to a constant normal load but cyclically varying bulk tension and shear force (with an arbitrary phase shift) are found. The range of the variation of the state of stress near both of the contact edges and the respective slip zone sizes are defined in terms of the loading parameters, including the phase angle, and may be used to design new experimental fretting fatigue tests.

Keywords: frictional contact; edge asymptotes; fretting

1. Introduction

The presence of small areas of slip is the dominant factor in all fretting problems. When present it is almost invariably at the edges of a contact, and it is from these points that cracks often nucleate. We know that there are mainly two forms of contact geometry which may arise in *any* engineering problem: first, there are sharp edged contacts which give rise to stick (usually) at the edges and which have to be handled by wedge asymptotic methods. The other class of contact comprises those which are incomplete (dovetails, firtrees, Hertzian contact) and where, because the contact pressure falls smoothly to zero at the edges, there is invariably some local slip. This paper is concerned exclusively with the latter type, and its function is to demonstrate that the asymptotic forms introduced in earlier articles both (a) fully control the extent of slip and (b) may be used to infer the fretting fatigue strength of problems of a very wide range of geometries. Here, it is assumed that the contact is subject to a constant normal load, so that the size of the contact (and the location of the contact edges) remain fixed.

The first study of a stationary partial slip contact problem was carried out by Cattaneo [1] who analysed the Hertz problem. Apparently unaware of this Mindlin looked at the same problem a little later and extended the solution to look at further properties [2–4]; much of Mindlin's

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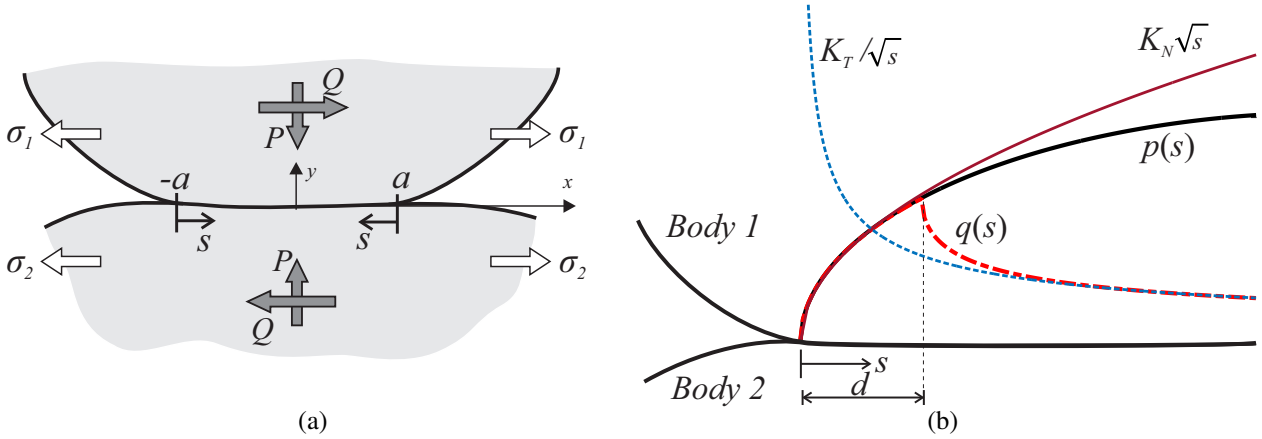


Figure 1: (a) Incomplete fretting problem under normal, shear and bulk tension loading; (b) Contact tractions and edge asymptotes.

work was on the axi-symmetric form of the contact where the Poisson effect was ignored but the solution has the additional property that it is possible to infer its absolute tangential (as well as normal) compliance. An important feature of Mindlin and Cattaneo's solutions is that they both assume that the origin of shear tractions along the interface is the exertion of an external shear force and, in most practically arising problems this is augmented by the existence of remotely applied differential surface tensions in the contacting bodies, which was first analysed in [5], for the plane case. Whereas the presence of a remote shear induces slip zones of the same sign the exertion of remote tension induces slip zones of opposite sign. The next major development in solutions came with the formulation of the Ciavarella-Jäger theorem [6, 7] which showed that the corrective shear traction in the slip zone is a scaled (and possibly shifted) form of the contact pressure. This development has proved invaluable in solving many partial slip contact problems both with and without tension, but its major drawback is that it cannot be applied where slip zones of opposite sign arise.

The proposal here is to model everything at the contact edge by two asymptotic forms [8, 9] and to show how those solutions may be used to infer everything about the local edge slip region. The method is very simple to apply, but the price one pays for the simplicity is that it may only be used when the slip region is a relatively small fraction of the contact half-width; that is, it will not work when the problem approaches the sliding condition. On the other hand, that does not often arise in practice, and the major advantage is that we may infer material properties from a simple laboratory test [10] which are transferable between a wide range of prototypical geometries.

2. Basic asymptotes and their calibration

2.1. Shear contact traction

Consider the incomplete contact shown in schematic form in Figure 1a, but note that it need not be Hertzian. We assume, initially, that the coefficient of friction is sufficiently high for all slip to be prevented. If the contact half-width is a, the shearing traction, $q(x)$, induced by a shear force,

Q, is given by

$$q(x) = \frac{Q}{\pi\sqrt{a^2 - x^2}}. \tag{1}$$

If, on the other hand, remote bulk tensions, σ_1 in the upper component and σ_2 in the lower component are applied in the two bodies, a shear traction distribution of the form

$$q(x) = -\frac{\sigma_0 x}{4\sqrt{a^2 - x^2}}, \tag{2}$$

arises, where $\sigma_0 = \sigma_2 - \sigma_1$. A shift of coordinates to the left hand edge ($x = -a$) is effected by setting $s = a + x$, and we may write the shear traction in the neighbourhood of this point, $s \to 0$, and neglecting the higher order terms, in the form

$$q(s) = \frac{K_T}{\sqrt{s}}, \tag{3}$$

where

$$K_T = \frac{Q}{\pi\sqrt{2a}} + \frac{\sigma_0}{4}\sqrt{\frac{a}{2}}. \tag{4}$$

It should be emphasized that this result is universal in the sense that it applies to contacts of any geometric form, subject only to the requirement that each body is capable of idealization by a half-plane. In some cases, the half-plane idealisation is applicable *only* near the edge of the contact (e.g. elastically similar flat and rounded punch) but, even if the centre of contact does not behave as a half-plane, the approach can still be implemented [11]. We adopt here a *local* convention that a positive value of K_T implies, at each contact edge in body 2, a shear traction which is directed inwards, away from the contact corner. In body 1, the opposite occurs and K_T is positive when the shear traction acts outwards. Note that when a positive shear force, Q, is applied as in Figure 1a, the shear traction in body 2 is directed inwards on the left hand edge, but outwards on the right hand edge. On the other hand, when a positive bulk tension, σ_0, is applied, the shear traction is directed inwards in body 2 on both the left and right hand edges. At the right hand contact edge, with the same sign convention on local tractions and with the same applied shear force and bulk tension, we find that

$$K_T = -\frac{Q}{\pi\sqrt{2a}} + \frac{\sigma_0}{4}\sqrt{\frac{a}{2}}. \tag{5}$$

The tractions on the left hand edge act inwards on body 2, i.e. K_T is *positive*, if

$$\frac{\sigma_0 a}{Q} < -\frac{4}{\pi}, \tag{6}$$

and on the right hand edge if

$$\frac{\sigma_0 a}{Q} > \frac{4}{\pi}. \tag{7}$$

2.2. *Normal contact pressure*

We turn, now, to a description of the contact pressure, $p(s)$, in the neighbourhood of the contact edge. A single term is used and, for half-plane problems, we know from basic Riemann-Hilbert theory that it must decay in a square root bounded manner. Hence, we define the pressure near the contact edge in terms of the normal edge scaling factor, K_N, as

$$p(s) = K_N \sqrt{s}. \tag{8}$$

This *is* geometry dependent and, generally, where the contact problem itself is solved by, for example, the finite element method, we can find the value by plotting $p(s)/\sqrt{s}$, and inferring the value as $s \to 0$. Note that results very close to the contact edge may need to be discarded if convergence of the solution is imperfect. In the case of contacts having a simple geometry we can find the calibration for K_N in closed form, and this is done in Appendix A for a Hertzian contact [8], for contact of shallow wedges [14], and for the cases of slightly rounded contact [11–13].

3. The partial slip problem.

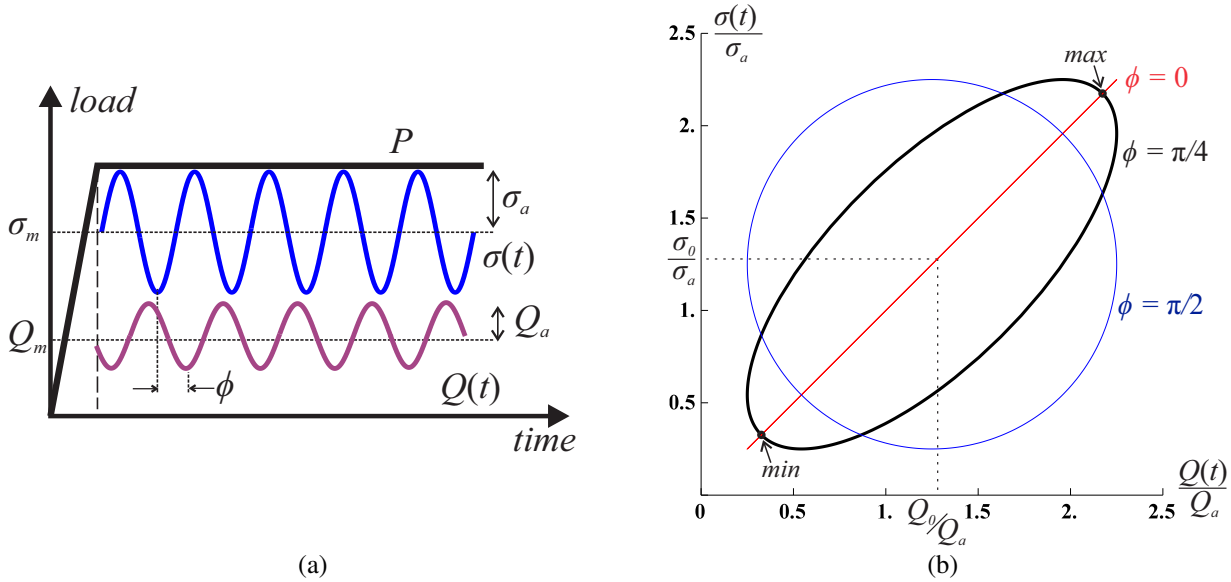
3.1. *Size of the slip zone in the steady state loading regime*

The contact pressure in the slip regions (if small) is encapsulated in the value of K_N, and so too is the magnitude of the shear tractions if the coefficient of friction, f, is specified. What remains to be determined is simply the extent of the slip zone, d. In any cyclic loading problem, unless the applied loading is fully reversing, some frictional shakedown will always occur within the first cycle so that the steady state extent of slip, d, is independent of the mean value of Q or σ_0 or, here, K_T. It is only the *range,* ΔK_T, which matters. So, for any closed loop trajectory in $Q - \sigma_0$ space we need only find how this maps into K_T in order to find the range itself. The steady state slip length (and of course the forwards and backwards slip lengths must be of the same extent, for conservation of material) is given by [9]

$$fd = \frac{\Delta K_T}{K_N}. \tag{9}$$

In fact, the mean value of K_T will affect the solution only in so far as it controls the residual interfacial shearing tractions (or residual interfacial slip displacement distribution) in the stick region. But there is no reason to suppose that this would have any bearing on the fatigue strength of the contact.

Equation (9) is different from the slip zone size under monotonically increasing loading. The slip zone for the initial loading phase was derived by Dini *et al.* [8] and in the presence of an initial bounded shear traction by Fleury *et al.* [12]. However, once the load is reversed, taken to its minimum value, and then increased back to its maximum value, some shakedown takes place. Hence, the slip zone size in the "steady" state periodic loading is, therefore, described by equation (9).



(a) (b)

Figure 2: (a) Constant normal load and periodic shear load and bulk tension; (b) Load history in normalised $Q-\sigma$ space for different phase angles, ϕ.

3.2. Periodic loading and the range ΔK_T

Suppose we subject a Hertzian contact to pulsating tension and shear, imposed at an angular frequency ω and with arbitrary phase shift, ϕ, (Figure 2a) so that

$$Q(t) = Q_m + Q_a \sin \omega t \tag{10}$$

$$\sigma(t) = \sigma_m + \sigma_a \sin(\omega t - \phi), \tag{11}$$

and hence, in normalised $Q-\sigma$ space, the loading trajectory appears as ellipse, Figure 2b (if $\phi = 0$ the loading trajectory is a straight line and a circle if $\phi = \pm\pi/2$).

We enquire what the range of ΔK_T generated is. Combining equations (10) and (11) with (4), the singular scaling factor K_T at the left hand side (L.H.S.) of the contact is rewritten as

$$K_T(t) = \frac{Q_m + Q_a \sin \omega t}{\pi\sqrt{2a}} + \frac{\sigma_m + \sigma_a \sin(\omega t - \phi)}{4}\sqrt{\frac{a}{2}} \tag{12}$$

and

$$\frac{dK_T(t)}{dt} = Q_N \omega \cos \omega t + \sigma_N \omega \cos(\omega t - \phi) \tag{13}$$

where

$$Q_N = \frac{Q_a}{\pi\sqrt{2a}} \qquad \text{and} \qquad \sigma_N = \frac{\sigma_a}{4}\sqrt{\frac{a}{2}} \tag{14}$$

The maximum and minimum values of ΔK_T may be obtained from the points of zero derivative,

$$\frac{dK_T(t)}{dt} = 0 \quad \rightarrow \quad Q_N \cos\omega t + \sigma_N (\cos\omega t \cos\phi + \sin\omega t \sin\phi) = 0, \qquad \omega \neq 0, \tag{15}$$

so the turning points of the function occur when

$$\tan\omega t = \frac{-(Q_N + \sigma_N \cos\phi)}{\sigma_N \sin\phi}. \tag{16}$$

Let us assume, for the time being, that $\tan\omega t > 0$, so that the values of ωt satisfying equation (16) fall into the first and third quadrants. It follows that there is one solution where both $\sin\omega t, \cos\omega t > 0$, and one where $\sin\omega t, \cos\omega t < 0$, so that the range of K_T experienced, ΔK_T, is

$$\Delta K_T = 2Q_N \sin\omega t + 2\sigma_N \sin(\omega t - \phi) = 2Q_N \sin\omega t + 2\sigma_N (\sin\omega t \cos\phi - \cos\omega t \sin\phi). \tag{17}$$

The range of stress intensity experienced is therefore given by

$$\Delta K_T = 2\sqrt{(Q_N + \sigma_N \cos\phi)^2 + \sigma_N{}^2 \sin^2\phi} \tag{18}$$

$$= \sqrt{\frac{2Q_a{}^2}{\pi^2 a} + \frac{\sigma_a{}^2 a}{8} + \frac{Q_a \sigma_a}{\pi}\cos\phi} \tag{19}$$

Equation (19) may also be represented in a dimensionless form as

$$\frac{\Delta K_T \sqrt{a}}{Q_a} = \sqrt{\frac{2}{\pi^2} + \frac{1}{8}\left(\frac{\sigma_a a}{Q_a}\right)^2 + \frac{\cos\phi}{\pi}\frac{\sigma_a a}{Q_a}} \qquad \text{or} \tag{20}$$

$$\frac{\Delta K_T}{\sigma_a \sqrt{a}} = \sqrt{\frac{2}{\pi^2}\left(\frac{Q_a}{\sigma_a a}\right)^2 + \frac{1}{8} + \frac{\cos\phi}{\pi}\frac{Q_a}{\sigma_a a}}, \tag{21}$$

where we must choose the first form if there is no bulk tension present and the second form if there is no shear force present. This result will apply for any geometry if the contact is incomplete.

One may now plot the normalised form of ΔK_T of equation (20) as a function of the dimensionless loading variable, $\sigma_a a/Q_a$, and the phase angle, ϕ. The results are displayed in Figure 3. Note that the minimum value, $\Delta K_T \sqrt{a}/Q_a = 0$, will occur when $\phi = \pm\pi$ and $\sigma_a a/Q_a = 4/\pi$.

At the right hand side (R.H.S.) of the punch, K_T is given by equation (5) and the normalised range $\Delta K_T \sqrt{a}/Q_a$ at that edge is given by

$$\frac{\Delta K_T \sqrt{a}}{Q_a} = \sqrt{\frac{2}{\pi^2} + \frac{1}{8}\left(\frac{\sigma_a a}{Q_a}\right)^2 - \frac{\cos\phi}{\pi}\frac{\sigma_a a}{Q_a}}. \tag{22}$$

By comparing equations (20) and (22), we may observe that the two edges of the contact are shifted by an angle π; i.e. at the right hand side of the contact one may replace $\Delta K_T(\phi)$ by

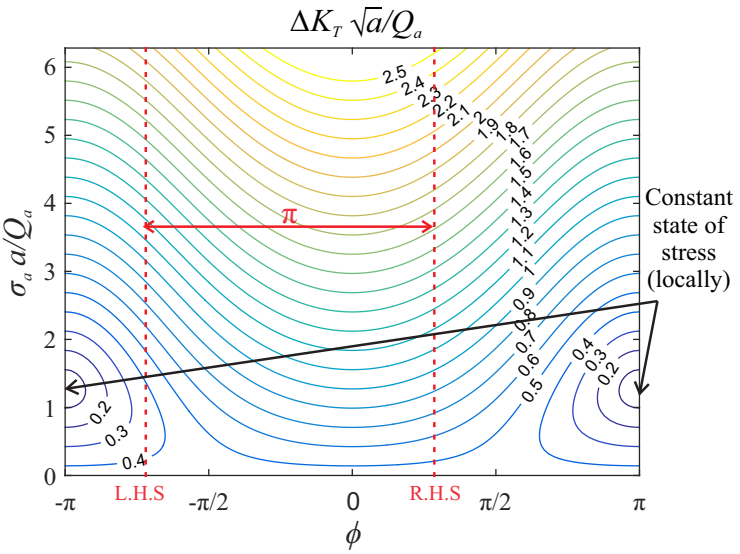


Figure 3: Contour lines of constant ΔK_T at the L.H.S. edge of contact as a function of the dimensionless load parameter $\sigma_a a/Q_a$ and the phase angle ϕ.

$\Delta K_T(\phi+\pi)$ in Figure 3. We can also establish the phase angle, ϕ, that will result in the same range ΔK_T on both edges of the punch. This will occur when the phase angle is $\phi = \pm\pi/2$. Furthermore, if $-\pi < \phi < -\pi/2$ or $\pi/2 > \phi > \pi$ the range ΔK_T at the left hand side will be smaller than the range at the right hand side, $\Delta K_T{}^{\text{LHS}} < \Delta K_T{}^{\text{RHS}}$. On the other hand, if $-\pi/2 < \phi < \pi/2$ we find that $\Delta K_T{}^{\text{LHS}} > \Delta K_T{}^{\text{RHS}}$. Also, if $\phi = \pm\pi$ (or $\phi = 0$) and $\sigma_a a/Q_a = 4/\pi$ we have a special case where the state of stress *locally* near the edge is constant on the left (or right) hand side of the contact while the other edge experiences an oscillating local stress.

3.3. Size of the slip zone - Hertzian problem

If we specialise to the case of a Hertzian contact, where the normal edge scaling factor is given by (see Appendix A)

$$K_N = \frac{P}{\pi}\sqrt{\frac{8}{a^3}}\,. \tag{23}$$

For the specific case of a two dimensional Hertzian contact, the maximum size of slip zone in the periodic steady state loading is given by

$$\frac{d}{a} = \frac{\Delta K_T}{afK_N} = \frac{1}{2}\sqrt{\left(\frac{Q_a}{fP}\right)^2 + \left(\frac{\pi}{4}\frac{\sigma_a a}{fP}\right)^2 + 2\frac{\pi\sigma_a a}{4fP}\frac{Q_a}{fP}\cos\phi}\,. \tag{24}$$

There are clearly many other ways that this could be presented, but this gives the general result in terms of the two independent dimensionless variables Q_a/fP and $\pi\sigma_a a/4fP$. The contour lines of constant slip zone size given by equation (24) are presented in Figure 4 for $\phi = 0,\ \pi/2$ and π. For the special case of $\phi = \pm\pi$ (Figure 4c), the size of the slip zone on the left hand side is zero when $\sigma_a a/Q_a = 4/\pi$. On the right hand side, this will occur when $\phi = 0$.

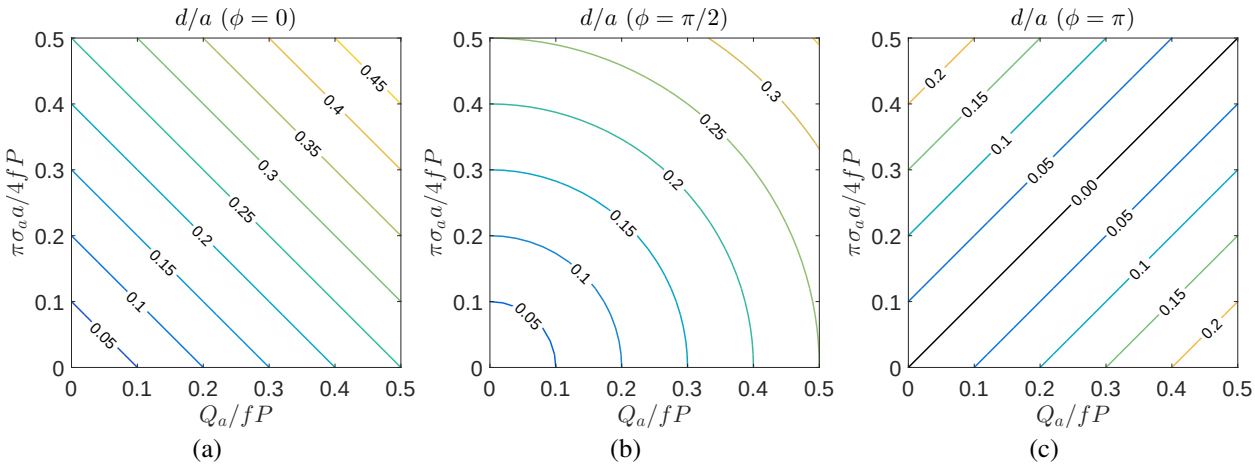


Figure 4: Size of the steady state slip zone at the left hand side of a Hertzian contact for a phase angle of (a) $\phi = 0$; (b) $\phi = \pi/2$; (c) $\phi = \pi$.

4. Quality of the approximation

Various approximations have been made in the solution presented here. In this section we will look at the nature of some of these approximations and the range of applicability of the asymptotic approach to incomplete contacts.

4.1. Single term contact pressure description

The contact pressure distribution given in equation (8) corresponds to one of the eigensolutions to the problem of a wedge (here a wedge of internal angle 180°) where one face is traction free and the other is given mixed boundary conditions of being free of shear tractions but with a constant displacement u_θ. In other words, the pressure distribution applied to a half-plane implies a constant normal displacement beneath it. It is therefore equally applicable to any contact geometry.

4.2. Partial slip solution

A reasonable question to ask is 'up to what values of shear load is this procedure a good approximation?' and this will be, to some extent, geometry specific. If we take the Hertzian contact problem, with an oscillatory shear force of magnitude ΔQ (only) exerted as an example, the size of the slip zone calculated from the traditional full Cattaneo-Mindlin procedure is given by

$$\frac{d}{a} = 1 - \sqrt{1 - \frac{Q_a}{fP}}, \tag{25}$$

where Q_a is the amplitude of the oscillating shear force. Equation (25) may easily be written in a series form using the binomial expansion to give

$$\frac{d}{a} \simeq \frac{1}{2}\frac{Q_a}{fP} + \frac{1}{8}\left(\frac{Q_a}{fP}\right)^2 + \frac{1}{16}\left(\frac{Q_a}{fP}\right)^3 + \ldots \ . \tag{26}$$

The first term is precisely the same as that implied by equation (9) and replacing ΔK_T and K_N by their definitions. So, the quality of the approximation may be judged by considering the contribution from the next term. The size of the steady state slip zone will differ by 10% if $Q_a/fP > 0.33$ when comparing with equation (25). The restriction is then that the shear force may not exceed $1/3$ of the value needed for sliding

If we apply *just* bulk tension to a Hertzian contact it is not possible to solve for the slip zone size without recourse to numerical methods [5], but it is easy to find the extent of small slip using the asymptote with the result

$$\frac{d}{a} = \frac{\pi\sigma_a a}{8fP} + \ldots \ . \tag{27}$$

The error of the single term asymptotic solution of equation (27) when compared with the numerical solution in [5] is less than 10% when $\sigma_a a/fP < 0.75$.

When both shear traction and bulk tension are applied, the analytical solution of the slip zone size can be split into two cases: i) when $\pi\sigma_a a/8fP \leq (1-\sqrt{1-Q_a/fP})$ and both slip zones slip in the same direction; and ii) when $\pi\sigma_a a/8fP > (1-\sqrt{1-Q_a/fP})$ and reverse slip occurs at the right hand edge (if $\phi = 0$) [5]. The first have an analytical solution, whereas the latter need to be solved numerically. Using the solutions given by Nowell and Hills (1987), the relative error of the slip zone size on the left hand edge obtained by the asymptotic solution is displayed in Figure 5 for $\phi = 0$ and $\phi = \pm\pi$ (on the right hand side Figure 5a occurs when $\phi = \pm\pi$ and Figure 5b when $\phi = 0$). It is important to notice that Nowell and Hills' solution predicts zero slip on the right hand edge when $\pi\sigma_a a/4fP = 2(1-(\sqrt{1-Q_a/fP}))$ and in the asymptotic solution this occurs when $\pi\sigma_a a/4fP = Q_a/fP$. The relative error of the asymptotic approach becomes greater than 10% if $Q_a/fP > 0.36$. The areas in blue in Figures 5a and 5b represent the combination of bulk tension and shear force amplitudes in which the error of the size of the slip zone predicted with the asymptotic approach is below 10% with respect to the actual size of the slip zone. Note that when $\phi = \pm\pi$ the large relative error observed when Q/fP is approximately $\pi\sigma a/4fP$ (diagonal line of Figure 5b) is due to the increasingly small slip zone size, and in fact, provided that $Q/fP \leq 0.33$ and $\pi\sigma a/4fP \leq 0.75$ the difference between the predicted and actual size of the slip zone remains below about 2% of the contact length, which is still an acceptable approximation for the solution of the problem.

4.3. The Ciavarella-Jäger theorem

It is perhaps worth emphasizing, here, that there is another approximation inherent in the extent of the partial slip solution beyond that implied by the use of a single term traction representation. This concerns the way in which we have employed the Ciavarella-Jäger theorem. If the load on a finite contact whose edge behaviour we are representing by the asymptotic form is currently P_2 and half-width a, and the expected physical extent of the slip zone is d, the shear traction in the adjacent stick region is given, according to Ciavarella [6] and Jäger [7], by

$$\frac{q(s)}{f} = K_N(P_2)\ \sqrt{s} - K_N(P_1)\ \sqrt{s-d}, \tag{28}$$

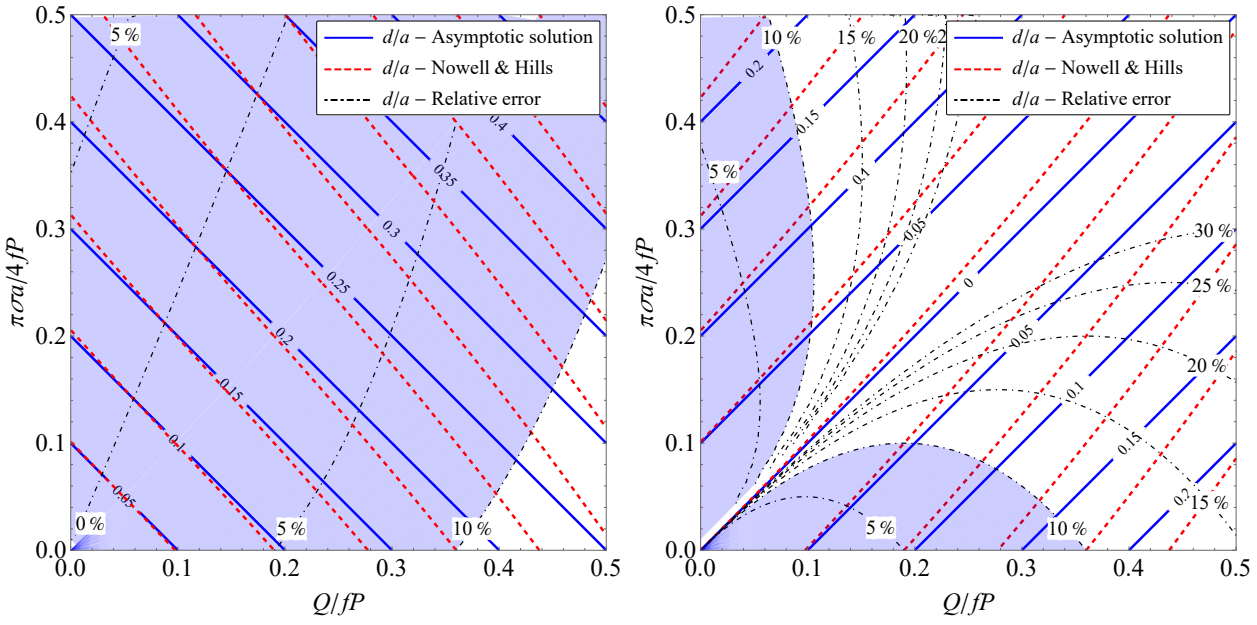


Figure 5: Slip zone size, d/a, comparison between the asymptotic solution and the Nowell and Hills (1987) solution at the (a) L.H.S ($\phi = 0$); and (b) R.H.S ($\phi = \pi$).

where P_1 is the contact load which corresponds to a contact of half-width $a-d$. So by a series expansion of the last term we obtain

$$\frac{q(s)}{f} \simeq K_N(P_2)\,\sqrt{s} - K_N(P_1)\,\sqrt{s}\left(1 - \frac{d}{2s} - \ldots.\right). \tag{29}$$

If we say that $P_1 \simeq P_2$ this enables us to write

$$\frac{q(s)}{f} \simeq \frac{K_N(P_2)\,d}{2\sqrt{s}}, \tag{30}$$

giving equation (9) (but with a factor of 2 because no residual shear tractions are present here, i.e. this relates to the first cycle of loading, no shakedown having occurred). So, the question arises of when the approximation $P_1 \simeq P_2$ is reasonable. If we restrict consideration to the Hertz problem, and use the result

$$\frac{d}{a} \simeq \frac{1}{4}\left(1 - \frac{P_1}{P_2}\right) + \frac{1}{32}\left(1 - \frac{P_1}{P_2}\right)^2 + \frac{1}{128}\left(1 - \frac{P_1}{P_2}\right)^3 \ldots, \tag{31}$$

we are implicitly using just the first term of this series, and the solution is appropriate only when $1 \simeq P_1/P_2$. So, there is an additional reason for the model of the size of the slip zone to be accurate only when the size of the slip zone is relatively small compared with the contact width overall.

5. Nucleation, propagation and the design of new experiments

Nucleation of cracks is controlled by the state of stress and slip displacement in the immediate neighbourhood of the contact edge. This is controlled by just the magnitude of ΔK_T, and the relative contributions of the shear force and bulk tension do not matter. On the other hand, it is well known that, in the absence of bulk tension, a nucleated crack will not propagate. We therefore have the potential to discriminate between crack nucleation forces and crack propagation forces. This is potentially a very useful result which may allow us to design tests, for example, in which the stresses driving the crack initiation are identical on both edges. Note that the crack initiation is controlled by the stresses at the edge of the contact and if both edges have the same ΔK_T the probability of crack initiation is identical on both sides. Propagation, however, occurs far from the contact zone and is driven by the applied tension amplitude σ_a (we neglect here any effect of the mean tension on the propagation).

Alternatively, an experiment may be designed where the stresses driving the crack propagation, σ_a, is the same and ΔK_T, which drives the initiation, is different, by controlling only the phase angle. A test with those properties may be used to quantify the crack initiation time as a function of ΔK_T and, potentially, significantly improve the life prediction capability of fretting experiments. The correlation of ΔK_T and initiation life would be applicable to any geometrical form, provided the contact is incomplete (i.e. bounded contact pressure at the edges). Figure 3 may be used, for example, to design a sequence of tests where the ΔK_T at one of the edges is increased between tests, but while maintaining σ_a constant, i.e. a moving along a horizontal line in Figure 3.

6. Conclusions

We have presented here a general asymptotic approach to finding the contact parameters of a problem with periodic tangential force and bulk tension, but with constant normal force. The contact tractions are described, *locally* near the contact edges, in terms of edge asymptotes: K_N for the normal pressure and K_T for singular shear traction, which describe. The extent of the slip zone size in incomplete contacts during the 'steady state' loading phase may be described in terms of the edge asymptotes and the phase angle between the tangential force and bulk tension applied. The results obtained here may be used to design a wide range of experiments in which the local state of stress is controlled by a combination of the phase angle, while keeping the remote bulk tension loading, which drives the crack propagation time, constant. These experiments may be used, among other things, to quantify the crack initiation time in incomplete contacts.

It is important to realise that *all* the local features of the contact problem - the tractions, extent of slip, slip displacement, frictional energy expenditure - are encapsulated in just three parameters, viz. $f, K_N, \Delta K_T$ although to evaluate some of these further material properties such as the plane strain modulus are needed. Furthermore, the K_T is geometry independent and will depend only on the loading applied to the problem, while K_N is geometry dependent and f depends on the material pair in the contact.

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Appendix A. Calibrations for multipliers

Appendix A.1. Hertzian contact

If two elastically similar bodies having plane strain elastic modulus, E^*, and relative radius of curvature, R, are pressed together a contact of half-width, a, and peak contact pressure, p_0, develops where

$$p_0 = \sqrt{\frac{PE^*}{\pi R}} \qquad \text{and} \qquad a^2 = \frac{4PR}{\pi E^*}, \tag{A.1}$$

and where the contact pressure distribution is semi-elliptical and of the form

$$p(x) = p_0 \sqrt{1 - \left(\frac{x}{a}\right)^2}. \tag{A.2}$$

Shifting the origin to the left hand edge and taking the lead term in a series expansion means that we can infer the value of K_N in various forms, for example

$$K_N = p_0 \sqrt{\frac{2}{a}} \equiv \frac{P}{\pi} \sqrt{\frac{8}{a^3}} \equiv \sqrt[4]{\frac{P}{\pi}\left(\frac{E^*}{R}\right)^3}. \tag{A.3}$$

The last form includes no dependent variables.

Appendix A.2. Shallow wedge

Contact between two elastically similar bodes of plane strain elastic modulus, E^*, and where the (small) external wedge angle is ϕ gives rise to a contact pressure distribution [14],

$$p(x) = \frac{\phi E^*}{\pi} \ln \left| \frac{1 - \sqrt{(1-x/a)\,(1+x/a)}}{1 + \sqrt{(1-x/a)\,(1+x/a)}} \right|, \tag{A.4}$$

and where the contact law is given by

$$P = \phi a E^*. \tag{A.5}$$

Then

$$K_N = \frac{8}{\pi} \sqrt{\frac{\phi^3 E^{*3}}{P}}. \tag{A.6}$$

It may seem rather odd that the value of K_N goes down as the load is increased, but we should note that the contact size increases linearly with P, and that K_N has dimensions of $[FL^{-5/2}]$, implicitly incorporating information about the contact *size*.

Appendix A.3. Flat and rounded punch

The two example contacts looked at have simple closed formed calibrations for K_N. But the concept may be applied to any contact which in the neighbourhood of the contact edge (at least) behaves like a half-plane. So, for example, for a slightly rounded rigid wedge pressed onto an incompressible half plane, we can, again, obtain a closed form solution. In that problem, remote from the contact edge, the pressure falls off in a square root manner , so if we write $p(x) = K_R x^{-1/2}$ $x \gg \rho$, where ρ is the edge radius, we can show that [13]

$$K_N = \sqrt[3]{\frac{12 E^{*2} K_R}{\pi^2 \rho^2}}. \tag{A.7}$$

We have very recently extended the solution just described to the case where both bodies have the same elastic constants, and this enables the influence of small amounts of rounding to be taken into account very precisely [11, 12]. The calibrated semi-analytical formulation for the edge scaling factor K_N is given by

$$K_N \simeq 0.629 \frac{E^*}{\rho} \sqrt{l}, \tag{A.8}$$

where l is the size of the rounded edge in contact, and whose semi-analytical formulation in terms of the applied loads P and Q is found in [12].