

GLOBAL EXISTENCE FOR THE TWO-DIMENSIONAL KURAMOTO-SIVASHINSKY EQUATION WITH A SHEAR FLOW

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ABSTRACT. We consider the Kuramoto-Sivashinsky equation (KSE) on the two-dimensional torus in the presence of advection by a given background shear flow. Under the assumption that the shear has a finite number of critical points, we prove global existence of solutions with data in L^2 , using a bootstrap argument. The initial data can be taken arbitrarily large.

1. INTRODUCTION

In this article we consider the Kuramoto-Sivashinsky equation (KSE) in two-space dimension in the presence of advection by a given background shear flow. The KSE is a well-known model of large-scale instabilities, such as those arising in flame-front propagation (see e.g [26] and references therein).

The KSE comes in a scalar, potential form, and a differentiated, vectorial form. We will confine ourselves to the scalar form, since the addition of a linear transport term is meaningful for the potential:

$$(1.1) \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \Delta^2 \phi + \Delta \phi = 0.$$

We solve this equation with periodic boundary conditions on $[0, L_1] \times [0, L_2]$, that is, on a two-dimensional torus, which with slight abuse of notation we denote by $\mathbb{T}^2$. When $L_1 > 2\pi$ or $L_2 > 2\pi$, the symbol of the linear operator $\Delta^2 + \Delta$ is negative on a finite set of low frequencies. Hence there are (linearly) growing modes in the horizontal or vertical direction respectively.

We consider a modified version of (1.1), where the potential ϕ is subject to advection by a given steady shear flow, which we write without loss of generality as the horizontal shear $\mathbf{v} = (u(y), 0)$:

$$\partial_t \phi + A u(y) \partial_x \phi + \frac{1}{2} |\nabla \phi|^2 + \Delta^2 \phi + \Delta \phi = 0,$$

where the parameter $A > 0$ represents the amplitude of the flow. The KSE with general advection term has been utilized in models of turbulent premixed-combustion [15]. By a change of time, the above equation can be rewritten in an equivalent way as

$$(1.2) \quad \partial_t \phi + u(y) \partial_x \phi + \frac{\nu}{2} |\nabla \phi|^2 + \nu \Delta^2 \phi + \nu \Delta \phi = 0,$$

where $\nu = A^{-1}$ and, with slight abuse of notation, we have not relabeled the transformed variables. Because ν determines the strength of the dissipation, we will refer to ν as a viscosity coefficient. We will refer to the equation above as AKSE. We will also assume that the shear has only *isolated* critical points.

The idea of the proof is to exploit the enhanced dissipation arising from the combined action of the hyper-diffusion and the advection to control both the non-linearity as well as the destabilizing effect of the negative Laplacean at large scale. Intuitively speaking, the shear flow has no influence on purely vertical modes. For instance, the function of y obtained by averaging the solution in the x direction may grow in time. On the other hand, the mixing along streamlines of the flow moves energy from large to small scales. Therefore, the growth generated by growing horizontal modes is damped on a sufficiently large time-scale by the dissipation. The non-linearity then couples all the modes. However, growing modes in the vertical direction have no influence on a time-scale $O(\nu^{-1})$ whereas the effect of the shear is to damp horizontal perturbations on a faster time-scale $O(\lambda_\nu^{-1})$. Essentially, we converge towards a modified 1D KSE on a time-scale $O(\lambda_\nu^{-1})$ where $\nu/\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$.

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The main difficulty in dealing with both (1.1) and (1.2) is the lack of *a priori* norm estimates on the solution, which does not allow to bootstrap local existence into global existence via a standard continuation argument. The analysis of the Kuramoto-Sivashinsky equations in one space dimension is well developed by now, since in one dimension energy estimates lead to a good control on the L^2 norm of the solution [8, 9, 21–24, 32]. By contrast, there are only a handful of results concerning the well-posedness of the classical KSE (1.1) in dimension greater than one. Local well-posedness holds in L^p spaces [7, 27]. Global existence is known only under fairly restrictive assumptions, such as for thin domains and for the anisotropically reduced KSE [6, 30, 33], without growing modes [1, 17], or with only one growing mode in each direction [2], for small data. In [17], two of the authors proved global existence for AKSE for large data and any number of growing modes, when the advecting velocity field induces a sufficiently small dissipation time, e.g. if the flow is mixing, that leads to a global uniform bound on the L^2 norm of the solution. In this case, the action of the flow is to move energy from large scales to small scales in both directions, where the dissipation can efficiently damp the effect of all the growing modes.

For the case at hand of a steady shear flow, the transport operator has a large kernel, namely all the functions on the torus that are constant in the horizontal variable. One needs to project out the kernel to take advantage of the action of the flow. There is no enhanced decay of the energy on the kernel component (at a linear level), but the norm can nevertheless be controlled as they satisfy collectively a modified one-dimensional KSE. A key point is to use the fact that the linear operator

$$(1.3) \quad H_\nu := \nu \Delta^2 + u(y) \partial_x,$$

is dissipation enhancing [11, 12, 16]. More precisely, for the components of the solution orthogonal to the kernel of the transport operator, it generates an exponentially stable semigroup e^{-tH_ν} with a rate of decay of the L^2 norm of order λ_ν , where $\nu/\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$. By contrast, a standard energy estimate shows that the semigroup is contractive with rate $O(\nu)$. The improved rate in viscosity allows to control both the growing modes as well as the nonlinear terms, provided ν is small enough compared to the size of the initial data. Given $g \in L^2(\mathbb{T}^2)$, we denote

$$(1.4) \quad \langle g \rangle(y) = \frac{1}{L_1} \int_{\mathbb{T}^1} g(t, x, y) dx, \quad g_\neq(x, y) = g(x, y) - \langle g \rangle(y).$$

By Fubini-Tonelli's Theorem, $\langle g \rangle$ exists for a.e. y . We observe that $\langle g \rangle$ corresponds to the projection of g onto the kernel of the advection operator $u(y) \partial_x$, while $g_\neq$ corresponds to the projection onto the orthogonal complement in L^2 . As shown in [4], if u has a finite number of critical points of order at most $m \geq 2$, namely at most $m - 1$ derivatives vanish at the critical points, then u is mixing in the sense that for some constant $C > 0$ there holds

$$(1.5) \quad \|e^{-u \partial_x t} g_\neq\|_{H^{-1}} \leq \frac{C}{(1+t)^m} \|g_\neq\|_{H^1},$$

for every $t \geq 0$. Thanks to [12, Corollary 2.3], this translates into the enhanced dissipation estimate

$$(1.6) \quad \|e^{-H_\nu t} g_\neq\|_{L^2} \leq 5e^{-\lambda_\nu t} \|g_\neq\|_{L^2}, \quad \lambda_\nu = \varepsilon_0 \nu^{\frac{3m}{3m+2}},$$

for some $\varepsilon_0 > 0$, independent of ν , and for every $t \geq 0$. For AKSE, we use (1.6) to show that solutions are global, as stated in the next theorem.

Theorem 1.1. *Let $\phi_0 \in L^2(\mathbb{T}^2)$, and let $u : [0, L_2] \rightarrow \mathbb{R}$ be a smooth function with a finite number of critical points of order at most $m \geq 2$. Then there exists $0 < \nu_0 < 1$ depending on L_1, L_2, u and $\|\phi_0\|_{L^2}$ with the following property: for any $0 < \nu < \nu_0$, there exists a global-in-time weak solution ϕ of (1.2) with initial data ϕ_0 such that $\phi \in L^\infty([0, T], L^2) \cap L^2([0, T], H^2)$ for all $0 < T < \infty$.*

The proof is based on a bootstrap argument inspired by [5]. The main steps in this argument are as follows. For any initial data in L^2 , there exists a local-in-time mild solution of (1.2) on some interval of time $[0, t_0)$, which is also a weak solution in $L^\infty([0, t_0], L^2) \cap L^2([0, t_0], H^2)$ and satisfies the energy identity [17]. For t_0 small enough, we can make the L^2 and H^2 norms of the projected component less than a certain multiple of the size of the initial data. By using the stability of the semigroup generated by H_ν , one then shows that, for ν sufficiently small, these norms are in fact half that amount. Hence the solution can be continued for a longer time than t_0 , which allows to bootstrap existence from local to global for the projected component and then conclude using the time evolution of the kernel component of the solution.

As we shall see in Section 2, the size of ν_0 in Theorem 1.1 depends on the rate at which ν/λ_ν vanishes as $\nu \rightarrow 0$. Hence, improving the semigroup estimate (1.6) automatically implies a better global existence threshold. In Section 3, we show that imposing a possibly more restrictive condition on u , the semigroup bound can be improved. In particular, we consider as a prototypical example the case of

$$(1.7) \quad u(y) = \sin((2\pi y)/L_2)^m \quad \text{for } m \in \mathbb{N},$$

and prove the following result.

Proposition 1.2. *Let $g \in L^2(\mathbb{T}^2)$, $0 < \nu < 1$, and $u(y)$ be given as in (1.7). There exists $\varepsilon'_0 > 0$, independent of ν , such that*

$$(1.8) \quad \|e^{-tH_\nu} g_\neq\|_{L^2} \leq e^{-\lambda'_\nu t + \pi/2} \|g_\neq\|_{L^2}, \quad \lambda'_\nu = \varepsilon'_0 \nu^{\frac{\max\{2,m\}}{\max\{2,m\}+4}},$$

for every $t \geq 0$.

Notice that the role of $m \in \mathbb{N}$ here is precisely that of (1.6), as u in (1.7) has critical points of order at most $\max\{2, m\}$. Hence, a direct comparison between (1.6) and (1.8) shows that (1.8) has a much better decay rate, and in particular $\nu/\lambda'_\nu \rightarrow 0$ faster as $\nu \rightarrow 0$.

The derivation of the semigroup estimate (1.8) is carried out in Section 3 via a spectral-theoretic approach. It follows from a general Gearhart-Prüss criterion for m -accretive operators devised in [34] based on a quantitative pseudo-spectral bound. The proof is motivated by that of a similar result for the Laplace operator Δ in [20]. For the Laplace operator plus advection, decay rates akin to (1.8) were obtained in [10, 34] for a shear with infinitely many critical points, using the pseudo-spectral approach, and for shear flows with finitely many critical points in [4], using hypocoercivity. Such quantitative semigroup estimates are relevant in the investigation of enhanced diffusion for passive scalars [3, 4, 13, 34], in the study of asymptotic stability of particular solutions to the two-dimensional Navier-Stokes equations [14, 19, 31, 35], and have also applications to several other nonlinear problems [5, 25, 28, 29].

In Section 3, we prove a more general version of Proposition 1.2, namely Proposition 3.1, for shear flows satisfying a certain condition, Assumption 3.1, again inspired by [20]. This condition can be readily verified for u in (1.7). This is a main reason while we chose it as prototypical example. In fact, by refining the method of proof, we expect an analog of Proposition 1.2 to hold for any shear flow with critical points of order m .

In what follows, C denotes a generic constant that may depend on the domain, i.e., on L_1 and L_2 . We utilize standard notation to denote function spaces, e.g. $H^k(\mathbb{T}^2)$ is the usual L^2 -based Sobolev space.

Finally, the paper is organized as follows. In Section 2, we obtain the bootstrap estimates and prove Theorem 1.1. Then, in Section 3, we establish the exponential stability of the semigroup generated by H_ν with the improved decay rate, using spectral estimates.

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2. GLOBAL EXISTENCE FOR THE KSE WITH SHEAR

In this section, we establish global existence of solutions of the KSE in the presence of advection by a shear flow with a finite number of critical points. The semigroup estimate (1.6) allows to control these growing modes through a suitable decomposition of the solution and a bootstrap argument.

2.1. Decomposition of the solution and proof of the main result. In this section, we derive the system of coupled equations that describe the time evolution of the component $\langle \phi \rangle$ of the solution in the kernel of the transport operator and the time evolution of the component $\phi_\neq$ in the orthogonal complement.

We will refer informally to $\langle \phi \rangle$ and $\phi_\neq$ as the kernel and projected components, respectively. Then $\langle \phi \rangle$ satisfies

$$(2.1) \quad \partial_t \langle \phi \rangle + \frac{\nu}{2L_1} \int_{\mathbb{T}^1} |\nabla \phi_\neq + \nabla \langle \phi \rangle|^2 dx + \nu \partial_y^4 \langle \phi \rangle + \nu \partial_y^2 \langle \phi \rangle = 0,$$

while $\phi_{\neq}$ satisfies

$$\begin{aligned} \partial_t \phi_{\neq} + u(y) \partial_x \phi_{\neq} + \nu \Delta^2 \phi_{\neq} &= -\frac{\nu}{2} |\nabla \phi_{\neq} + \nabla \langle \phi \rangle|^2 + \frac{\nu}{2L_1} \int_{\mathbb{T}^1} |\nabla \phi_{\neq} + \nabla \langle \phi \rangle|^2 dx - \nu \Delta \phi_{\neq} \\ (2.2) \quad &= -\frac{\nu}{2} |\nabla \phi_{\neq}|^2 + \frac{\nu}{2} \langle |\nabla \phi_{\neq}|^2 \rangle - \nu \partial_y \phi_{\neq} \partial_y \langle \phi \rangle - \nu \Delta \phi_{\neq}. \end{aligned}$$

We remark that in the equation above the kernel component interacts with the projected ones through the term $\partial_y \langle \phi \rangle$. Denoting $\psi = \partial_y \langle \phi \rangle$ for notational ease, we have

$$(2.3) \quad \partial_t \psi + \frac{\nu}{2L_1} \int_{\mathbb{T}^1} \partial_y |\nabla \phi_{\neq}|^2 dx + \nu \psi \partial_y \psi + \nu \partial_y^4 \psi + \nu \partial_y^2 \psi = 0.$$

It was proved in [17] that the unique local mild solution to (1.2) is also a weak solution satisfying the energy identity on the time of existence of the mild solution. In particular, $\phi_{\neq} \in L^\infty((0, t_0); L^2(\mathbb{T}^2)) \cap L^2((0, t_0); H^2(\mathbb{T}^2))$, at least for a sufficiently small time $t_0 > 0$. Furthermore, it was shown in [17] that the mild and weak solution persists as long as its L^2 norm is finite, that is, if T^* is the maximal time of existence of the solution, then

$$T^* < \infty \Rightarrow \limsup_{t \rightarrow T^*} \|\phi(t)\|_{L^2} = \infty.$$

Our goal is to obtain a global bound on the L^2 norm of the solution via a bootstrap argument, from which global existence follows. We will employ both energy estimates as well as semigroup estimates to exploit enhanced dissipation arising from the addition of the advection term on $\phi_{\neq}$.

Let $\mathcal{S}_t$ be the solution operator from 0 to time $t \geq 0$ for the transport-hyperdiffusion equation:

$$\partial_t g + u(y) \partial_x g + \nu \Delta^2 g = 0,$$

that is, $\mathcal{S}_t = e^{-tH\nu}$. Then $\phi_{\neq}$ satisfies

$$(2.4) \quad \phi_{\neq}(t) = \mathcal{S}_t(\phi_{\neq}(0)) + \int_0^t \mathcal{S}_{t-s} \left(-\frac{\nu}{2} |\nabla \phi_{\neq}(s)|^2 + \frac{\nu}{2} \langle |\nabla \phi_{\neq}(s)|^2 \rangle - \nu \psi(s) \partial_y \phi_{\neq}(s) - \nu \Delta \phi_{\neq}(s) \right) ds.$$

We note that the “forcing” term under the integral sign on the right-hand side of this equation is well controlled as long as $\phi_{\neq} \in L^\infty((0, t_0); L^2(\mathbb{T}^2)) \cap L^2((0, t_0); H^2(\mathbb{T}^2))$, provided ψ is also controlled.

Using the decay of $\mathcal{S}_t$ on the projected component given by (1.6), it follows from (2.4) that

$$\begin{aligned} \|\phi_{\neq}(t)\|_{L^2} &\leq \|\mathcal{S}_t(\phi_{\neq}(0))\|_{L^2} + C\nu \int_0^t (\|\nabla \phi_{\neq}(s)\|_{L^4}^2 + \|\psi\|_{L_y^4} \|\nabla \phi_{\neq}\|_{L^4} + \|\Delta \phi_{\neq}(s)\|_{L^2}) ds \\ &\leq \|\mathcal{S}_t(\phi_{\neq}(0))\|_{L^2} + C\nu \int_0^t (\|\phi_{\neq}\|_{L^2}^{1/2} \|\Delta \phi_{\neq}\|_{L^2}^{3/2} + \|\Delta \phi_{\neq}\|_{L^2} \\ &\quad + \|\phi_{\neq}\|_{L^2}^{1/4} \|\Delta \phi_{\neq}\|_{L^2}^{3/4} \|\psi\|_{L_y^2}^{7/8} \|\partial_y^2 \psi\|_{L_y^2}^{1/8}) ds \\ (2.5) \quad &\leq \|\mathcal{S}_t(\phi_{\neq}(0))\|_{L^2} + C\nu \int_0^t (\|\phi_{\neq}\|_{L^2}^{1/2} \|\Delta \phi_{\neq}\|_{L^2}^{3/2} + \|\Delta \phi_{\neq}\|_{L^2} + \|\phi_{\neq}\|_{L^2}^{1/4} \|\Delta \phi_{\neq}\|_{L^2}^{3/4} \|\partial_y^2 \psi\|_{L_y^2}) ds, \end{aligned}$$

where in the above estimate we used the fact $\|\psi\|_{L_y^2} \leq C \|\partial_y^2 \psi\|_{L_y^2}$ and the following Gagliardo-Nirenberg interpolation inequalities:

$$(2.6) \quad \|\nabla \phi_{\neq}\|_{L^4} \leq C \|\phi_{\neq}\|_{L^2}^{1/4} \|\Delta \phi_{\neq}\|_{L^2}^{3/4}, \quad \|\psi\|_{L_y^4} \leq C \|\psi\|_{L_y^2}^{7/8} \|\partial_y^2 \psi\|_{L_y^2}^{1/8}.$$

We next derive some energy estimates that will be needed for the bootstrap argument. Multiplying (2.2) by $\phi_{\neq}$ and integrating by part, using the periodic boundary conditions, yields:

$$(2.7) \quad \frac{1}{2} \frac{d}{dt} \|\phi_{\neq}\|_{L^2}^2 + \nu \|\Delta \phi_{\neq}\|_{L^2}^2 = -\frac{\nu}{2} \int_{\mathbb{T}^2} |\nabla \phi_{\neq}|^2 \phi_{\neq} dx dy + \frac{\nu}{2L_1} \int_{\mathbb{T}^1} \left(\int_{\mathbb{T}^1} |\nabla \phi_{\neq}|^2 dx \right) \phi_{\neq} dx dy$$

$$(2.8) \quad -\nu \int_{\mathbb{T}^2} \psi \partial_y \phi_{\neq} \phi_{\neq} dx dy + \nu \|\nabla \phi_{\neq}\|_{L^2}^2$$

$$(2.9) \quad \leq C\nu \|\nabla \phi_{\neq}\|_{L^4}^2 \|\phi_{\neq}\|_{L^2} + C\nu \|\psi\|_{L_y^2} \|\nabla \phi_{\neq}\|_{L^4} \|\phi_{\neq}\|_{L^4} + \nu \|\nabla \phi_{\neq}\|_{L^2}^2.$$

We recall the Gagliardo-Nirenberg interpolation inequalities in (2.6) and

$$(2.10) \quad \|\phi_{\neq}\|_{L^4} \leq C \|\phi_{\neq}\|_{L^2}^{3/4} \|\Delta\phi_{\neq}\|_{L^2}^{1/4}.$$

These estimate imply:

$$(2.11) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi_{\neq}\|_{L^2}^2 + \nu \|\Delta\phi_{\neq}\|_{L^2}^2 &\leq C\nu \|\phi_{\neq}\|_{L^2}^{3/2} \|\Delta\phi_{\neq}\|_{L^2}^{3/2} + C\nu \|\psi\|_{L_y^2} \|\phi_{\neq}\|_{L^2} \|\Delta\phi_{\neq}\|_{L^2} \\ &\quad + \nu \|\phi_{\neq}\|_{L^2} \|\Delta\phi_{\neq}\|_{L^2}, \end{aligned}$$

where we have integrated by parts in the last term in (2.7). Applying Young's inequality, we further get

$$(2.12) \quad \frac{d}{dt} \|\phi_{\neq}\|_{L^2}^2 + \nu \|\Delta\phi_{\neq}\|_{L^2}^2 \leq C\nu \|\phi_{\neq}\|_{L^2}^6 + C\nu \|\phi_{\neq}\|_{L^2}^2 + C\nu \|\phi_{\neq}\|_{L^2}^2 \|\psi\|_{L_y^2}^2,$$

using also the 1D Poincaré's inequality twice to bound both the L^2 norm of ψ as well as that of $\partial_y\psi$ (we exploit here that ψ and, hence, all its derivatives have zero average by definition).

We also recall that the enhanced diffusion estimate (1.6) for S_t :

$$\|\mathcal{S}_t g\|_{L^2} \leq 5 e^{-\lambda_\nu t} \|g\|_{L^2},$$

for any $g \in L^2(\mathbb{T}^2)$ with $\int_{\mathbb{T}^1} g(x, y) dx = 0$. Above λ_ν satisfies

$$(2.13) \quad \frac{\nu}{\lambda_\nu} \rightarrow 0, \text{ as } \nu \rightarrow 0.$$

In view of (2.4), the regularity of the mild and weak solution and the continuation principle, for all sufficiently small times $t > 0$ we can assume that

$$\begin{aligned} \text{(H1)} \quad &\|\phi_{\neq}(t)\|_{L^2} \leq 8e^{-\lambda_\nu t/4} \|\phi_{\neq}(0)\|_{L^2}, \\ \text{(H2)} \quad &\nu \int_0^t \|\Delta\phi_{\neq}(s)\|_{L^2}^2 ds \leq 4 \|\phi_{\neq}(0)\|_{L^2}^2. \end{aligned}$$

Let $t_0 > 0$ be the maximal time such that the estimates above hold on $[0, t_0]$. Following [7], we refer to (H1) - (H2) with $t \in [0, t_0]$ as the *bootstrap assumptions*. The next lemma ensures suitable bounds on ψ once the bootstrap assumptions (H1) and (H2) hold.

Lemma 2.1. *Assume the bootstrap assumptions (H1) and (H2). There exists a ν -independent constant $C_1 = C_1(\|\phi_{\neq}(0)\|_{L^2}, \|\psi(0)\|_{L_y^2})$, which can be explicitly computed, such that*

$$(2.14) \quad \|\psi(t)\|_{L_y^2}^2 + \nu \int_0^t \|\partial_y^2 \psi(s)\|_{L_y^2}^2 ds \leq C_1 \left(\|\phi_{\neq}(0)\|_{L^2}, \|\psi(0)\|_{L_y^2} \right) e^{C\nu t},$$

for all $t \in [0, t_0]$.

Proof. First from the energy estimate, we have

$$(2.15) \quad \frac{1}{2} \frac{d}{dt} \|\psi\|_{L_y^2}^2 + \nu \|\partial_y^2 \psi\|_{L_y^2}^2 = \nu \|\partial_y \psi\|_{L_y^2}^2 + \frac{\nu}{2L_1} \int_{\mathbb{T}^1} |\nabla \phi_{\neq}|^2 \partial_y \psi dx \leq \nu \|\partial_y \psi\|_{L_y^2}^2 + \frac{\nu}{2L_1} \|\nabla \phi_{\neq}\|_{L^4}^2 \|\partial_y \psi\|_{L_y^2}.$$

It follows from the Gagliardo-Nirenberg inequality that

$$(2.16) \quad \|\partial_y \psi\|_{L_y^2}^2 \leq C \|\psi\|_{L_y^2} \|\partial_y^2 \psi\|_{L_y^2} \leq \frac{C}{2} \|\psi\|_{L_y^2}^2 + \frac{1}{2} \|\partial_y^2 \psi\|_{L_y^2}^2,$$

$$(2.17) \quad \|\nabla \phi_{\neq}\|_{L^4} \leq C \|\phi_{\neq}\|_{L^2}^{1/4} \|\Delta\phi_{\neq}\|_{L^2}^{3/4}.$$

Appealing to the two bounds above, estimate (2.15) becomes

$$\frac{1}{2} \frac{d}{dt} \|\psi\|_{L_y^2}^2 + \frac{\nu}{2} \|\partial_y^2 \psi\|_{L_y^2}^2 \leq C\nu \|\phi_{\neq}\|_{L^2}^{1/2} \|\Delta\phi_{\neq}\|_{L^2}^{3/2} \|\psi\|_{L_y^2}^{1/2} \|\partial_y^2 \psi\|_{L_y^2}^{1/2} + \frac{C\nu}{2} \|\psi\|_{L_y^2}^2.$$

It then follows by Young's inequality that

$$(2.18) \quad \frac{d}{dt} \|\psi\|_{L_y^2}^2 + \nu \|\partial_y^2 \psi\|_{L_y^2}^2 \leq C\nu \|\phi_{\neq}\|_{L^2}^{2/3} \|\Delta\phi_{\neq}\|_{L^2}^2 \|\psi\|_{L_y^2}^{2/3} + C\nu \|\psi\|_{L_y^2}^2$$

$$(2.19) \quad \leq C\nu \|\phi_{\neq}\|_{L^2}^{2/3} \|\Delta\phi_{\neq}\|_{L^2}^2 + C\nu \|\phi_{\neq}\|_{L^2}^{2/3} \|\Delta\phi_{\neq}\|_{L^2}^2 \|\psi\|_{L_y^2}^2 + C\nu \|\psi\|_{L_y^2}^2.$$

We define an integrating factor $\mu = \exp(-C\nu t - C\nu \int_0^t \|\phi_\neq\|_{L^2}^{2/3} \|\Delta\phi_\neq\|_{L^2}^2 ds)$. Then solving (2.19) gives

$$\begin{aligned} \|\psi(t)\|_{L_y^2}^2 &\leq C\nu\mu^{-1} \int_0^t \|\phi_\neq(s)\|_{L^2}^{2/3} \|\Delta\phi_\neq(s)\|_{L^2}^2 ds + \mu^{-1} \|\psi(0)\|_{L_y^2}^2 \\ (2.20) \quad &\leq \left(16Ce^{16C\|\phi_\neq(0)\|_{L^2}^{8/3}} \|\phi_\neq(0)\|_{L^2}^{8/3} + e^{16C\|\phi_\neq(0)\|_{L^2}^{8/3}} \|\psi(0)\|_{L_y^2}^2\right) e^{C\nu t}, \end{aligned}$$

where the last inequality followed by the bootstrap assumptions (H1) and (H2). By using (2.20) in (2.18), we get the claimed estimate (2.14). $\square$

We show below in Subsection 2.2 that, in fact, there exists $\nu_0 > 0$ small enough such that, if $\nu < \nu_0$, then

$$\begin{aligned} \text{(B1)} \quad &\|\phi_\neq(t)\|_{L^2} \leq 4e^{-\lambda_\nu t/4} \|\phi_\neq(0)\|_{L^2}, \\ \text{(B2)} \quad &\nu \int_0^t \|\Delta\phi_\neq(s)\|_{L^2}^2 ds \leq 2 \|\phi_\neq(0)\|_{L^2}^2, \end{aligned}$$

for all $t \in [0, t_0]$. We refer to (B1)-(B2) as the *bootstrap estimates*. Assuming temporarily this fact, we proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.2 and Lemma 2.5 and the definition of t_0 , we must have $t_0 = \infty$. In particular, $\phi_\neq \in L^\infty([0, \infty); L^2(\mathbb{T}^2)) \cap L^2([0, \infty); H^2(\mathbb{T}^2))$. By Lemma 2.1, we have $\psi \in L_{loc}^\infty([0, \infty); L^2(\mathbb{T}^1)) \cap L_{loc}^2([0, \infty); H^2(\mathbb{T}^1))$. By the triangle and Poincaré's inequalities, it then follows $\Delta\phi \in L_{loc}^\infty([0, \infty); L^2(\mathbb{T}^2))$. If we further denote

$$(2.21) \quad \bar{\phi} = \frac{1}{L_1 L_2} \int_{\mathbb{T}^2} \phi(x, y) dx dy = \frac{1}{L_2} \int_{\mathbb{T}^1} \langle \phi \rangle dy,$$

then from (2.1) we have

$$(2.22) \quad \partial_t \bar{\phi} = -\frac{\nu}{2L_1 L_2} \int_{\mathbb{T}^2} |\nabla \phi_\neq + \nabla \langle \phi \rangle|^2 dx dy = -\frac{\nu}{2L_1 L_2} \int_{\mathbb{T}^2} |\nabla \phi_\neq|^2 dx dy - \frac{\nu}{2L_2} \int_{\mathbb{T}^1} |\psi|^2 dy.$$

By integrating the above equation and applying estimate (2.14) and (B2), we obtain $\bar{\phi} \in L_{loc}^\infty([0, \infty))$. By Lemma 2.1, it follows that $\psi \in L_{loc}^\infty([0, \infty); L^2(\mathbb{T}^1))$. Hence, (2.21) and the Poincaré inequality imply that $\langle \phi \rangle \in L_{loc}^\infty([0, \infty); L^2(\mathbb{T}^1))$. As a consequence, $\phi \in L_{loc}^\infty([0, \infty); L^2(\mathbb{T}^2))$. Finally, we note that $\nabla^2 \phi = \nabla^2 \phi_\neq + \nabla \psi$ so that $\phi \in L_{loc}^2([0, \infty); H^2(\mathbb{T}^2))$. This concludes the proof. $\square$

2.2. Bootstrap estimates. It remains to establish the bootstrap estimates (B1)-(B2), which we accomplish through a series of lemmas. We address (B2) first.

Lemma 2.2. *Assume the bootstrap assumptions (H1) and (H2). There exists $\nu_0 = \nu_0(\|\phi_\neq(0)\|_{L^2})$, explicitly computable, with the following property: for any $t \in [0, t_0]$ and for any $\nu \leq \nu_0$, it holds that*

$$(2.23) \quad \nu \int_0^t \|\Delta\phi_\neq(s)\|_{L^2}^2 ds \leq 2 \|\phi_\neq(0)\|_{L^2}^2.$$

In particular, (B2) holds.

Proof. The bootstrap assumptions, Lemma 2.1 and the energy estimate (2.12) give

$$\begin{aligned} \nu \int_0^t \|\Delta\phi_\neq(s)\|_{L^2}^2 ds &\leq \|\phi_\neq(0)\|_{L^2}^2 + C\nu \int_0^t \|\phi_\neq(s)\|_{L^2}^6 ds + (1 + C_1 e^{C\nu s}) \|\phi_\neq(s)\|_{L^2}^2 ds \\ &\leq \|\phi_\neq(0)\|_{L^2}^2 + \frac{\nu}{\lambda_\nu} C (\|\phi_\neq(0)\|_{L^2}^4 + 1) \|\phi_\neq(0)\|_{L^2}^2 \\ (2.24) \quad &\quad + C_1 C \nu \int_0^t e^{-\lambda_\nu s(1-4C\nu/\lambda_\nu)/4} e^{-\lambda_\nu s/4} \|\phi_\neq(0)\|_{L^2}^2 ds. \end{aligned}$$

We observe that, since $\nu/\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$, there exists ν_0 such that

$$\frac{\nu_0}{\lambda_{\nu_0}} \leq \min \left\{ \frac{1}{C(1+C_1)(\|\phi_\neq(0)\|_{L^2}^4 + 1)}, \frac{1}{4C} \right\}.$$

Hence, by combining the choice ν_0 with (2.24) we conclude the proof of the Lemma. $\square$

It remains to prove (B1), which we accomplish in different steps. The next lemma states that within a fixed length of time, the quantity $\|\phi_\neq\|_{L^2}$ will never grow too fast.

Lemma 2.3. *Assume the bootstrap assumptions (H1) and (H2), and fix $\tau^* = 4/\lambda_\nu$. For any $0 \leq t_1 < t_0$, there exists $\nu_0 = \nu_0(\|\phi_\neq(0)\|_{L^2})$ such that for any $\nu \leq \nu_0$ there holds*

$$(2.25) \quad \|\phi_\neq(t)\|_{L^2} \leq \sqrt{2} \|\phi_\neq(t_1)\|_{L^2} .$$

for all $t \in [t_1, t_1 + \tau^*] \cap [0, t_0]$.

Proof. We will assume that ν_0 is small enough so that Lemma 2.2 applies. Again by Lemma 2.1, the energy estimate (2.12), and the bootstrap assumption (H1), for some positive $C_2 = C_2(\|\phi_\neq(0)\|_{L^2}, \|\psi(0)\|_{L_y^2})$ we have that

$$(2.26) \quad \begin{aligned} \frac{d}{dt} \|\phi_\neq\|_{L^2}^2 &\leq C\nu \|\phi_\neq\|_{L^2}^6 + CC_1\nu e^{C\nu t} \|\phi_\neq\|_{L^2}^2 \\ &\leq CC_1\nu e^{-\lambda_\nu t(1-2C\nu/\lambda_\nu)/2} \|\phi_\neq(0)\|_{L^2}^2 (\|\phi_\neq\|_{L^2}^4 + 1) \\ &\leq \nu C_2 \left(\|\phi_\neq\|_{L^2}^4 + 1 \right), \end{aligned}$$

where we used the fact that $2C\nu/\lambda_\nu \leq 1$ for $\nu < \nu_0$ sufficiently small. Now, we define $T(B)$ as

$$(2.27) \quad T(B) = \frac{1}{\nu C_2} \int_{B^2}^{2B^2} \frac{dy}{y^2 + 1}.$$

It is easy to see that $T(\cdot)$ is a decreasing function, and since by the bootstrap assumption (H1) we have that $\|\phi_\neq(t_1)\|_{L^2} \leq 8\|\phi_\neq(0)\|_{L^2}$, it follows that $T(\|\phi_\neq(t_1)\|_{L^2}) \geq T(8\|\phi_\neq(0)\|_{L^2})$. In light of (2.26), by the definition of $T(B)$ we have that for any $t \in [t_1, t_1 + T(8\|\phi_\neq(0)\|_{L^2})] \cap [0, t_0]$, the estimate $\|\phi_\neq(t)\|_{L^2} \leq \sqrt{2} \|\phi_\neq(t_1)\|_{L^2}$ holds. The lemma is now proved if we choose $\tau^* \leq T(8\|\phi_\neq(0)\|_{L^2})$, which is equivalent to asking for

$$(2.28) \quad \frac{\nu}{\lambda_\nu} \leq \frac{T(8\|\phi_\neq(0)\|_{L^2})}{4}.$$

Finally, $\nu/\lambda_\nu \rightarrow 0$ as $\nu \rightarrow 0$, so it is enough to satisfy (2.28) for ν_0 . This concludes the proof. $\square$

The next lemma tells us that by selecting ν sufficiently small, we will have a constant fraction decay of $\|\phi_\neq\|_{L^2}$ after a fixed length of time.

Lemma 2.4. *Assume the bootstrap assumptions (H1) and (H2), and fix again $\tau^* = 4/\lambda_\nu$. There exists $\nu_0 = \nu_0(\|\phi_\neq(0)\|_{L^2})$, explicitly computable, with the following property: for any $t \in [0, t_0]$ and for any $\nu \leq \nu_0$,*

$$(2.29) \quad \|\phi_\neq(\tau^*)\|_{L^2} \leq \frac{1}{e} \|\phi_\neq(0)\|_{L^2} .$$

Proof. In the course of this proof, we assume that ν_0 is small enough so that Lemma 2.2 can be applied. By the definition of τ^* , we have

$$(2.30) \quad \|\mathcal{S}_{\tau^*}(\phi_\neq(0))\|_{L^2} \leq \frac{5}{e^4} \|\phi_\neq(0)\|_{L^2} \leq \frac{1}{e^2} \|\phi_\neq(0)\|_{L^2} .$$

Using this inequality in (2.5) yields

$$(2.31) \quad \begin{aligned} \|\phi_\neq(\tau^*)\|_{L^2} &\leq \frac{1}{e^2} \|\phi_\neq(0)\|_{L^2} + C\nu \int_0^{\tau^*} \left(\|\phi_\neq\|_{L^2}^{1/2} \|\Delta\phi_\neq\|_{L^2}^{3/2} + \|\Delta\phi_\neq\|_{L^2} + \|\phi_\neq\|_{L^2}^{1/4} \|\Delta\phi_\neq\|_{L^2}^{3/4} \|\partial_y^2 \psi\|_{L_y^2} \right) ds \\ &\leq \frac{1}{e^2} \|\phi_\neq(0)\|_{L^2} + C \left(\nu \int_0^{\tau^*} \|\phi_\neq(s)\|_{L^2}^2 ds \right)^{1/4} \left(\nu \int_0^{\tau^*} \|\Delta\phi_\neq(s)\|_{L^2}^2 ds \right)^{3/4} \\ &\quad + C \left(\nu \int_0^{\tau^*} \|\Delta\phi_\neq(s)\|_{L^2}^2 ds \right)^{1/2} (\nu\tau^*)^{1/2} \\ &\quad + C \left(\nu \int_0^{\tau^*} \|\Delta\phi_\neq(s)\|_{L^2}^2 ds \right)^{3/8} \left(\nu \int_0^{\tau^*} \|\partial_y^2 \psi(s)\|_{L_y^2}^2 ds \right)^{1/2} \left(\nu \int_0^{\tau^*} \|\phi_\neq(s)\|_{L^2}^2 ds \right)^{1/8}. \end{aligned}$$

Using the bootstrap assumptions (H1)-(H2) and Lemma 2.1, since $\exp(C\nu\tau^*) \leq C$, it then follows that

$$\begin{aligned} \|\phi_{\neq}(\tau^*)\|_{L^2} &\leq \frac{1}{e^2} \|\phi_{\neq}(0)\|_{L^2} + C(\nu\tau^*)^{1/4} \|\phi_{\neq}(0)\|_{L^2}^2 + C(\nu\tau^*)^{1/2} \|\phi_{\neq}(0)\|_{L^2} + C\sqrt{C_1}(\nu\tau^*)^{1/8} \|\phi_{\neq}(0)\|_{L^2} \\ (2.32) \quad &\leq \frac{1}{e^2} \|\phi_{\neq}(0)\|_{L^2} + C(\nu\tau^*)^{1/8} (\|\phi_{\neq}(0)\|_{L^2} + \sqrt{C_1}) \|\phi_{\neq}(0)\|_{L^2}, \end{aligned}$$

where we used the fact that $\nu\tau^* \ll 1$ when ν_0 is small enough. By further restring ν_0 so that

$$(2.33) \quad \frac{1}{e^2} + C(4\nu_0\lambda_{\nu_0}^{-1})^{1/8} (\|\phi_{\neq}(0)\|_{L^2} + \sqrt{C_1}) \leq \frac{1}{e},$$

the desired result follows from (2.32). $\square$

Now we are ready to show that the bootstrap assumption (H1) can be refined.

Lemma 2.5. *Assume the bootstrap assumptions (H1) and (H2). There exists $\nu_0 = \nu_0(\|\phi_{\neq}(0)\|_{L^2})$, explicitly computable, with the following property: for any $t \in [0, t_0]$ and for any $\nu \leq \nu_0$, it holds that*

$$(2.34) \quad \|\phi_{\neq}(t)\|_{L^2} \leq 4e^{-\lambda_\nu t/4} \|\phi_{\neq}(0)\|_{L^2}.$$

In particular, (B1) holds.

Proof. We fix $\nu_0 = \nu_0(\|\phi_{\neq}(0)\|_{L^2})$ so that all the restrictions in Lemmata 2.2-2.4 are fulfilled. By Lemma 2.4, we have

$$(2.35) \quad \|\phi_{\neq}(n\tau^*)\|_{L^2} \leq e^{-n} \|\phi_{\neq}(0)\|_{L^2} \leq \|\phi_{\neq}(0)\|_{L^2}, \quad \text{for any } n \in \mathbb{Z}_+.$$

For any $t \in [0, t_0]$, there exists n such that $t \in [n\tau^*, (n+1)\tau^*)$. From Lemma 2.3 with $t_1 = n\tau^*$, it follows that

$$(2.36) \quad \|\phi_{\neq}(t)\|_{L^2} \leq \sqrt{2} \|\phi_{\neq}(n\tau^*)\|_{L^2} \leq \sqrt{2} e^{-n} \|\phi_{\neq}(0)\|_{L^2} \leq \sqrt{2} e^{1-t/\tau^*} \|\phi_{\neq}(0)\|_{L^2} \leq 4e^{-\lambda_\nu t/4} \|\phi_{\neq}(0)\|_{L^2}.$$

This concludes the proof of the lemma. $\square$

3. SEMIGROUP ESTIMATES

In this section, we prove Proposition 1.2, namely an improved decay estimate for the semigroup generated by H_ν in L^2 , under a general condition on the shear velocity profile u .

We denote by $\dot{L}^2(\mathbb{T}^2)$ the closed subspace of $L^2(\mathbb{T}^2)$ of elements for which $\langle g \rangle = 0$. By Fubini-Tonelli's Theorem, such elements are also mean-zero on the torus. We will be concerned with the restriction of the operator H_ν to $\dot{L}^2(\mathbb{T}^2)$ viewed as an unbounded operator. By slight abuse of notation, we denote the restriction also by H_ν . It is straightforward to check that the projection onto $\dot{L}^2(\mathbb{T}^2)$ commutes with the semigroup e^{-tH_ν} generated by H_ν .

Let $(X, \|\cdot\|)$ be a complex Hilbert space and let H be a closed, densely defined operator on X . As shown in [34], if H is an m -accretive operator on X , then the decay properties of the semigroup e^{-tH} can be understood in terms of the following quantity:

$$(3.1) \quad \Psi(H) = \inf \{ \|(H - i\lambda)g\| : g \in D(H), \lambda \in \mathbb{R}, \|g\| = 1 \},$$

is related to the *pseudospectral* properties of the operator [18]. Following [34], for $\frac{L_1 k}{2\pi} \in \mathbb{Z}_*$ and $\nu \in (0, 1]$, we consider the operator H_ν localized to the k th Fourier mode in the direction of the shear, namely, the operator

$$(3.2) \quad H_{\nu,k} = \nu\Delta_k^2 + iku(y), \quad \Delta_k := -k^2 + \partial_{yy}.$$

Following the arguments in [34] for the Laplace operator, it can be shown that $H_{\nu,k}$ is an m -accretive operator on $L^2(\mathbb{T}^1)$ with domain $H^4(\mathbb{T}^1)$. Here, L^2 is a space of complex-valued functions. Then, as a consequence of [34, Theorem 1.3],

$$(3.3) \quad \|e^{-H_{\nu,k}t}\|_{\text{op}} \leq e^{-t\Psi(H_{\nu,k})+\pi/2}, \quad \forall t \geq 0,$$

where $\|\cdot\|_{\text{op}}$ denotes the operator norm. To establish lower bounds on $\Psi(H_{\nu,k})$, we assume the following condition on the shear flow.

Assumption 3.1. There exist $m, N \in \mathbb{N}$, $c_1 > 0$ and $\delta_0 \in (0, L_2)$ with the property that, for any $\lambda \in \mathbb{R}$ and any $\delta \in (0, \delta_0)$, there exist $n \leq N$ and points $y_1, \dots, y_n \in [0, L_2]$ such that

$$(3.4) \quad |u(y) - \lambda| \geq c_1 \left(\frac{\delta}{L_2} \right)^m, \quad \forall |y - y_j| \geq \delta, \quad \forall j \in \{1, \dots, n\}.$$

Remark 3.1. Assumption 3.1 is heavily inspired by a similar property of the velocity field associated to the Oseen's vortex [31]. In [20], Gallay previously observed that the method of proof in [31] can be extended to more general shear flows assuming a condition similar to (3.4).

The following is the main result of this section.

Proposition 3.1. *Let u satisfy Assumption 3.1. Assume $k \neq 0$ and $\nu|k|^{-1} \leq 1$. There exists a constant $\varepsilon'_0 > 0$, independent of ν and k , such that*

$$(3.5) \quad \Psi(H_{\nu,k}) \geq \varepsilon'_0 \nu^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}}.$$

We state next a direct consequence of the theorem.

Corollary 3.2. *In the hypotheses of Proposition 3.1, let P_k denotes the L^2 projection onto the k -th Fourier mode in the horizontal direction. Then, for every $t \geq 0$,*

$$(3.6) \quad \|e^{-H_\nu t} P_k\|_{op} \leq e^{-\varepsilon'_0 \nu^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}} t + \pi/2}.$$

In particular, H_ν generates an exponentially stable semigroup in $\dot{L}^2(\mathbb{T}^2)$ with rate:

$$(3.7) \quad \|e^{-H_\nu t}\|_{op} \leq e^{-\lambda'_\nu t + \pi/2}, \quad t > 0,$$

where $\lambda'_\nu = \varepsilon'_0 \nu^{\frac{m}{m+4}}$ for some $\varepsilon'_0 > 0$.

Before proving Proposition 3.1, we show that the Assumption 3.1 is satisfied for u as in (1.7).

Example 3.1. We consider the case of $u(y) = (\sin(y))^m$, defined on $[0, 2\pi)$. For a general period L_2 , the result follows by a standard rescaling argument. Without loss of generality, we may assume that $\delta_0 > 0$ is small enough so that $\cos(\delta_0) \geq 1/2$. In particular, for every $C \geq 1$,

$$(3.8) \quad \sin(\delta/C) \geq \frac{\delta}{2C}, \quad \forall \delta \in (0, \delta_0).$$

Given $\lambda \in \mathbb{R}$, we choose the set of points $y_1, \dots, y_n$ to be the union of the set of the critical points of u with the set $u^{-1}(\{\lambda\})$ (possibly empty). More precisely, we consider

$$(3.9) \quad Y := \begin{cases} \{\pi/2, 3\pi/2\} \cup u^{-1}(\{\lambda\}), & m = 1, \\ \{0, \pi/2, \pi, 3\pi/2\} \cup u^{-1}(\{\lambda\}), & m \geq 2, \end{cases} \quad n := |Y|.$$

There are at most $N = 8$ points in Y . Observe that the critical points at $y = \pi/2, 3\pi/2$ are such that $u''(\pi/2), u''(3\pi/2) \neq 0$ for any $m \geq 1$. On the other hand, for $m \geq 2$ the critical points at $y = 0, \pi$ are such that $u^{(j)}(0) = u^{(j)}(\pi) = 0$ for every $j = 1, \dots, m-1$ and $u^{(m)}(0), u^{(m)}(\pi) \neq 0$. We denote $Y = \{y_i\}_{i=1}^n$ and we order the points in such a way that $0 \leq y_i \leq y_{i+1} \leq 2\pi$ for $i = 1, \dots, N$. We fix $\delta \in (0, \delta_0)$ and consider different cases.

Case $\lambda \in [0, 1]$. Denote by y_λ the smallest element of $u^{-1}(\{\lambda\})$. Due to the symmetries of u , we know that $y_\lambda \in [0, \pi/2]$. We consider three situations.

- $|y_\lambda| < \delta/4$: In this case, thanks to (3.8), we have

$$(3.10) \quad \begin{aligned} |u(y) - \lambda| &= |(\sin(y))^m - (\sin(y_\lambda))^m| \geq |\sin(\delta)|^m - |\sin(y_\lambda)|^m \\ &\geq \frac{\delta^m}{2^m} - |\sin(\delta/4)|^m \geq \frac{\delta^m}{2^m} - \frac{\delta^m}{4^m} \geq \frac{\delta^m}{2^{m+1}}, \end{aligned}$$

where we used that $|\sin(y)| \leq |y|$.

- $|y_\lambda - \pi/2| < \delta/4$: In this case we have

$$(3.11) \quad \begin{aligned} |u(y) - \lambda| &= |(\sin(y))^m - (\sin(y_\lambda))^m| \geq |(\sin(\pi/2 - \delta))^m - (\sin(y_\lambda))^m| \\ &\geq |(\sin(\pi/2 - \delta))^m - (\sin(\pi/2))^m| - |(\sin(\pi/2))^m - (\sin(\pi/2 - \delta/4))^m|. \end{aligned}$$

Since $\partial_y(\sin(y))^m = m(\sin(y))^{m-1} \cos(y)$ and $\partial_y^2(\sin(y))^m = m(m-1)(\sin(y))^{m-2} \cos^2(y) - m(\sin(y))^m$,

$$(3.12) \quad |(\sin(\pi/2 - \delta))^m - (\sin(\pi/2))^m| = \left| \left(1 - \frac{\delta^2}{2} + O(\delta^4) \right)^m - 1 \right| = \frac{m}{2} \delta^2 + O(\delta^4).$$

so that, by possibly restricting the size of δ_0 further (depending on m), we also have

$$(3.13) \quad |(\sin(\pi/2 - \delta))^m - (\sin(\pi/2))^m| \geq \frac{m}{4} \delta^2.$$

On the other hand,

$$(3.14) \quad |(\sin(\pi/2))^m - (\sin(\pi/2 - \delta/4))^m| \leq \frac{m}{8} \delta^2.$$

Consequently,

$$(3.15) \quad |u(y) - \lambda| \geq \frac{m}{8} \delta^2.$$

- $|y_\lambda| \geq \delta/4$ and $|y_\lambda - \pi/2| \geq \delta/4$: In this case, we have

$$(3.16) \quad \begin{aligned} |u(y) - \lambda| &= |(\sin(y))^m - (\sin(y_\lambda))^m| \\ &\geq \min \{ |(\sin(y_\lambda + \delta/8))^m - (\sin(y_\lambda))^m|, |(\sin(y_\lambda - \delta/8))^m - (\sin(y_\lambda))^m| \}. \end{aligned}$$

We observe that $y_\lambda + \delta/8 < \pi/2 - \delta/8$ and $y_\lambda - \delta/8 > \delta/8$. Then if $y_\lambda \geq \pi/4$, it follows by the Mean Value Theorem that for some $\xi \in (y_\lambda, y_\lambda + \delta/8)$,

$$(3.17) \quad \begin{aligned} |(\sin(y_\lambda + \delta/8))^m - (\sin(y_\lambda))^m| &= m |(\sin(\xi))^{m-1} \cos(\xi)| \frac{\delta}{8} \\ &\geq m |(\sin(\pi/4))^{m-1} \cos(\pi/2 - \delta/8)| \frac{\delta}{8} \\ &\geq \frac{m}{2^{\frac{m-1}{2}}} |\sin(\delta/8)| \frac{\delta}{8} \geq \frac{m}{2^{\frac{m-1}{2}}} \frac{\delta^2}{128}, \end{aligned}$$

also using (3.8). Otherwise, if $y_\lambda \leq \pi/4$, we have

$$(3.18) \quad \begin{aligned} |(\sin(y_\lambda + \delta/8))^m - (\sin(y_\lambda))^m| &= m |(\sin(\xi))^{m-1} \cos(\xi)| \frac{\delta}{8} \\ &\geq m |(\sin(\delta/4))^{m-1} \cos(\pi/4 + \delta/8)| \frac{\delta}{8} \\ &\geq m |(\sin(\delta/4))^{m-1} \cos(\pi/3)| \frac{\delta}{8} \geq \frac{m}{8^{m+1}} \delta^m. \end{aligned}$$

A lower bound for $|(\sin(y_\lambda - \delta/8))^m - (\sin(y_\lambda))^m|$ can be proved in a similar way.

Case $\lambda > 1$. In this case $u^{-1}(\{\lambda\}) = \emptyset$. Hence, since $\text{dist}(y, Y) \geq \delta$, we simply have

$$(3.19) \quad \begin{aligned} |u(y) - \lambda| &\geq |(\sin(y))^m - (\sin(\pi/2))^m| \geq |(\sin(\pi/2 - \delta))^m - (\sin(\pi/2))^m| \\ &\geq m |(\sin(\xi))^{m-1} \cos(\xi)| \delta \geq m |(\sin(\pi/6))^{m-1} \cos(\pi/2 - \delta)| \delta \geq \frac{m}{2^m} \delta^2, \end{aligned}$$

where we used (3.8) in the last inequality.

Case $\lambda < 0$. This case is only relevant for m even. In fact, for m odd we proceed as for the case $\lambda \in [0, 1]$ by symmetry. Here, again $u^{-1}(\{\lambda\}) = \emptyset$. Hence, since $\text{dist}(y, Y) \geq \delta$, we have

$$(3.20) \quad |u(y) - \lambda| \geq |(\sin(y))^m| \geq (\sin(\delta))^m \geq \frac{\delta^m}{2^m},$$

by (3.8).

Hence, there exists a constant $c_m > 0$ such that

$$(3.21) \quad |u(y) - \lambda| \geq c_m \delta^{\max\{m, 2\}},$$

as we wanted.

We now turn our attention to the proof of Proposition 3.1.

Proof of Proposition 3.1. The theorem follows by establishing a lower bound on $\Psi(H_{\nu,k})$. In the following, $\|\cdot\|$ denotes the L^2 norm and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product in L^2 .

We fix $\lambda \in \mathbb{R}$ and pick $g \in D(H_{\nu,k})$ with $\|g\| = 1$. For notational convenience, we set

$$H := H_{\nu,k} - i\lambda = \nu\Delta_k^2 + ik(u(y) - \lambda).$$

Let $\chi : [0, L_2] \rightarrow [-1, 1]$ be a smooth approximation of $\text{sign}(u(y) - \lambda)$ such that $\|\chi'\|_{L^\infty} \leq c_2\delta^{-1}$, $\|\chi''\|_{L^\infty} \leq c_2\delta^{-2}$, $\chi(u - \lambda) \geq 0$ and

$$(3.22) \quad \chi(y)(u(y) - \lambda) = |u(y) - \lambda|, \quad \text{whenever } |y - y_j| \geq \delta, \quad \forall j \in \{1, \dots, n\},$$

where y_j are the points in Assumption 3.1. The function χ can be constructed via a standard mollification argument. A double integration by parts in y yields the identity

$$(3.23) \quad \text{Re}\langle Hg, g \rangle = \nu\|\Delta_k g\|^2,$$

which implies that

$$(3.24) \quad \|\Delta_k g\|^2 \leq \frac{1}{\nu}\|Hg\|\|g\|.$$

On the other hand, we have

$$(3.25) \quad \begin{aligned} \langle Hg, \chi g \rangle &= \nu\langle \Delta_k^2 g, \chi g \rangle + ik\langle (u(y) - \lambda)g, \chi g \rangle \\ &= \nu\langle \Delta_k g, \chi'' g \rangle + 2\nu\langle \Delta_k g, \chi' \partial_y g \rangle + \nu\langle \Delta_k g, \chi \Delta_k g \rangle + ik\langle (u(y) - \lambda)g, \chi g \rangle \end{aligned}$$

so that

$$(3.26) \quad \text{Im}\langle Hg, \chi g \rangle = \nu\text{Im}\langle \Delta_k g, \chi'' g \rangle + 2\nu\text{Im}\langle \Delta_k g, \chi' \partial_y g \rangle + k\langle (u(y) - \lambda)g, \chi g \rangle.$$

In particular, from the properties of the function χ and the interpolation inequality $\|\partial_y g\|^2 \leq \|\Delta_k g\|\|g\|$, it follows that

$$(3.27) \quad |k|\langle (u(y) - \lambda)g, \chi g \rangle \leq \|Hg\|\|g\| + \frac{c_2\nu}{\delta^2}\|\Delta_k g\|\|g\| + \frac{c_2\nu}{\delta}\|\Delta_k g\|^{3/2}\|g\|^{1/2}.$$

We denote

$$(3.28) \quad E := \{y \in [0, L_2] : |y - y_j| \geq \delta, \quad \text{for } j = 1, \dots, n\},$$

where y_j are the points in Assumption 3.1. By (3.4) we have

$$(3.29) \quad \langle (u(y) - \lambda)g, \chi g \rangle \geq \int_E |u(y) - \lambda||g(y)|^2 dy \geq c_1 \left(\frac{\delta}{L_2}\right)^m \int_E |g(y)|^2 dy.$$

Utilizing (3.24) and (3.27), we find that there exists a positive constant $\tilde{c}_2$ such that

$$(3.30) \quad \begin{aligned} \int_E |g(y)|^2 dy &\leq \frac{1}{c_1|k|} \left(\frac{L_2}{\delta}\right)^m \left[\|Hg\|\|g\| + \frac{c_2\nu}{\delta^2}\|\Delta_k g\|\|g\| + \frac{c_2\nu}{\delta}\|\Delta_k g\|^{3/2}\|g\|^{1/2} \right] \\ &\leq \frac{1}{c_1|k|} \left(\frac{L_2}{\delta}\right)^m \|Hg\|\|g\| + \tilde{c}_2 \left(\left(\frac{\nu}{|k|\delta^2}\right)^2 \left(\frac{L_2}{\delta}\right)^{2m} + \left(\frac{\nu}{|k|\delta}\right)^{4/3} \left(\frac{L_2}{\delta}\right)^{\frac{4m}{3}} \right) \|\Delta_k g\|^2 + \frac{1}{4}\|g\|^2 \\ &\leq \left(\frac{1}{c_1|k|} \left(\frac{L_2}{\delta}\right)^m + \frac{\tilde{c}_2}{\nu} \left(\left(\frac{\nu}{|k|L_2^2}\right)^2 \left(\frac{L_2}{\delta}\right)^{2m+4} + \left(\frac{\nu}{|k|L_2}\right)^{4/3} \left(\frac{L_2}{\delta}\right)^{\frac{4}{3}(m+1)} \right) \right) \|Hg\|\|g\| + \frac{1}{4}\|g\|^2. \end{aligned}$$

On the other hand, since E^c is of size at most $N\delta$, we have

$$(3.31) \quad \begin{aligned} \int_{E^c} |g(y)|^2 dy &\leq N\delta\|g\|_{L^\infty}^2 \leq N\delta \left(2\|g\|\|\partial_y g\| + \frac{1}{L_2}\|g\|^2 \right) \\ &\leq N\delta \left(2\|g\|^{3/2}\|\Delta_k g\|^{1/2} + \frac{1}{L_2}\|g\|^2 \right) \\ &\leq \frac{(6N\delta)^4}{12}\|\Delta_k g\|^2 + \left(\frac{N\delta}{L_2} + \frac{1}{4} \right) \|g\|^2 \\ &\leq \frac{(6N\delta)^4}{12\nu}\|Hg\|\|g\| + \left(\frac{N\delta}{L_2} + \frac{1}{4} \right) \|g\|^2, \end{aligned}$$

where we made use of (3.24) in the last inequality. Without loss of generality, we can assume that δ_0 in Assumption 3.1 is small enough so that $\delta \leq L_2/(4N)$ for all $\delta \in (0, \delta_0)$. Thus we can add (3.30) and (3.31) to conclude that

$$(3.32) \quad \|g\| \leq 4 \left(\frac{1}{c_1|k|} \left(\frac{L_2}{\delta} \right)^m + \tilde{c}_2 \frac{\nu}{(|k|L_2^2)^2} \left(\frac{L_2}{\delta} \right)^{2m+4} + \tilde{c}_2 \frac{\nu^{1/3}}{(|k|L_2)^{4/3}} \left(\frac{L_2}{\delta} \right)^{\frac{4}{3}(m+1)} + \frac{(6N\delta)^4}{12\nu} \right) \|Hg\|.$$

We now take δ satisfying

$$(3.33) \quad \frac{\delta}{L_2} = c_3 \left(\frac{\nu}{|k|} \right)^{\frac{1}{m+4}}$$

for some sufficiently small constant $c_3 > 0$. We conclude that

$$(3.34) \quad \nu^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}} = \nu^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}} \|g\| \leq c_4 \|Hg\|,$$

for a large enough constant $c_4 > 0$ independent of ν and k . By definition then

$$\Psi(H) \geq \varepsilon'_0 \nu^{\frac{m}{m+4}} |k|^{\frac{4}{m+4}},$$

for some constant $\varepsilon'_0 > 0$ independent of ν and k , whence proving the proposition. $\square$

Remark 3.2. The proof of Proposition 3.1 carries over to the slightly more general case of the semigroup generated by the hypoelliptic operator

$$(3.35) \quad \tilde{H}_\nu = u(y)\partial_x + \nu\partial_y^4,$$

for which the same semigroup estimate (3.7) holds.

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