

Dynamic fracture analysis of Kane-Mindlin plates using the dual boundary element method

Jun Li ^{a,*}, Zahra Sharif Khodaei ^a, M. H. Aliabadi ^a

^a Structural Integrity and Health Monitoring Research Group, Department of Aeronautics, Imperial College London, Exhibition Road, SW7 2AZ, London, UK.

* jun.li16@imperial.ac.uk, z.sharif-khodaei@imperial.ac.uk,
m.h.aliabadi@imperial.ac.uk

Abstract

In this paper, a new dual boundary element formulation is presented for dynamic crack problems in finite-thickness plates under in-plane loadings. The proposed formulation is based on a first-order plate theory (Kane-Mindlin theory) taking into account a coupled out-of-plane fracture mode due to in-plane shear loading and the effect of plate thickness on stress intensity factors, which cannot be considered within the framework of the two-dimensional elastic theories. The dynamic stress intensity factors are evaluated based on the crack opening displacements. Three benchmark examples are presented including the mixed-mode fracture of a finite plate. The effect of plate thickness on the dynamic stress intensity factors is investigated. The dynamic stress intensity factors obtained using the proposed formulation are shown to be in good agreement with the results from 3D finite element simulations.

Keywords: Dynamic stress intensity factor; Kane-Mindlin plates; Dual boundary element method; Thickness effect; Out-of-plane fracture mode

1. Introduction

Cracks are present to some extent in all structures and their presence increases the effort spent on maintenance and repair to avoid structural failure. Since cracks cannot be eliminated, methodologies must be developed to quantify and predict the behaviour of cracked structures under service conditions. The science of fracture mechanics has been developed to characterise cracks and their effect, and to predict if and when the structure may become unsafe.

Through-thickness cracks are commonly encountered in plate structures. Within the scope of linear elastic fracture mechanics, the through-thickness crack problems in plates are essentially three-dimensional. In order to obtain comprehensive solutions to such problems, the analysis should be conducted based on the three dimensional theory of elasticity. However, this theory is not popular in analytical studies due to mathematical difficulties and in numerical investigations

due to high computational costs. Thus, two-dimensional elasticity theories are preferred in dealing with such problems.

The plane strain and generalized plane stress theories are two classical two-dimensional theories, but the limitations of these two theories are obvious. When it comes to the fracture analysis of through-thickness crack problems, Sih [1] pointed out that the generalized plane stress solution was not a limit of the three-dimensional solution for sufficiently thin plates, while the analysis under plane strain conditions was a limiting case for the three-dimensional analysis of sufficiently thick plates. Furthermore, these two theories cannot take thickness into consideration for finite-thickness plates.

In fact, through-thickness crack problems are more complex than often thought. Although Mode II is the primary mode for the plate subjected to in-plane shear loading, a coupled out-of-plane mode is also generated due to the Poisson effect [2-5]. It is worth mentioning that such coupled mode is similar but different from Mode III because the coupled mode is symmetric with respect to the mid-plane of the plate. Thus, in order to distinguish the coupled mode from Mode III, the coupled mode is sometimes called O-mode [5, 6]. Neither plane strain theory nor generalized plane stress theories can be used to describe such coupled mode.

The Kane-Mindlin theory (K-M theory) [7] allows us to consider both the thickness effect and the O-mode, and it has been applied to through-thickness crack problems by many researchers [8-14]. In this first-order plate theory, a linear variation of the out-of-plane displacement across the thickness is assumed. It was shown that K-M solutions actually reflected an average through-thickness effect [5, 15].

Up to now, the K-M theory has been mainly applied to static fracture problems. Although dynamic fracture problems are of great interest, very few investigations are based on such theory. To the best of our knowledge, only Jin and Batra [10] conducted the dynamic fracture analysis of Kane-Mindlin plates. However, their study was limited to Mode I fracture problems in an infinite plate. In order to consider more complex fracture problems, such as the mixed mode fracture of finite plates, numerical methods should be relied on.

The prime characterisation and prediction behaviour in fracture mechanics is the stress intensity factor. Over the last four decades, many different numerical methods for evaluating the stress intensity factors have been developed. These include simple methods such as superposition [16], Green's function [17], compounding techniques [18, 19] and boundary collocation [20]. More advanced methods of finite elements [21], boundary elements [22] and mesh-free [23] are

nowadays preferred to the simpler methods as they offer possibility of investigating complex structures and loading sequences.

In this paper, the boundary element method (BEM) is used as the first numerical method for solving dynamic fracture problems in Kane-Mindlin plates. The BEM has been applied to a variety of dynamic fracture problems [24-26], and it is more attractive for plate problems [27, 28] since plate boundaries are discretized only using very simple line elements as in the two-dimensional BEM. The dual boundary element method (DBEM) [29, 30] is adopted to model cracks. The problem is formulated in the Laplace-transform domain. All the transformed fundamental solutions used for the dual boundary integral equations are derived from the fundamental solutions given in Ref. [31]. Durbin's method [32] is used to conduct numerical inversion of the Laplace transforms. The stress intensity factors are evaluated using the crack opening displacements and comparisons are made in three benchmark examples against solutions obtained using the Abaqus implicit solver.

2. Laplace transform boundary integral equations

Throughout this paper, Greek indices $(\alpha, \beta, \gamma, \eta)$ run from 1 to 2 and Roman indices (i, j, k) run from 1 to 3. Figure 1 presents a flat homogeneous isotropic plate of thickness $2h$ in the Cartesian coordinate system x_i . The x_3 -axis is normal to the mid-plane of the plate ($x_3 = 0$).

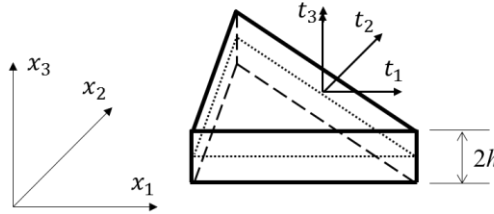


Figure 1. A flat homogeneous isotropic plate

The generalized tractions are defined as $t_i = n_\beta \int_{-h}^h \sigma_{\beta i} x_i^{\delta_{i3}} dx_3$, in which $\int_{-h}^h \sigma_{\beta i} x_i^{\delta_{i3}} dx_3$ are generalized stress resultants; n_β represent the components of a unit outward normal vector to the plate boundary; the exponent δ_{i3} denotes Kronecker delta.

In Kane-Mindlin theory [7], the displacements u_i of the plate are approximated using the expressions:

$$\begin{aligned}
u_1(x_1, x_2, x_3, t) &= v_1(x_1, x_2, t) \\
u_2(x_1, x_2, x_3, t) &= v_2(x_1, x_2, t) \quad , \\
u_3(x_1, x_2, x_3, t) &= v_3(x_1, x_2, t) x_3
\end{aligned} \tag{1}$$

in which t denotes the time.

In the absence of body forces, the governing equations of motion can be expressed in terms of generalized displacements v_i as follows:

$$2h\mu\nabla^2 v_\alpha + 2h(\lambda + \mu)v_{\beta,\beta\alpha} + 2\kappa h\lambda v_{3,\alpha} = 2\rho h \frac{\partial^2 v_\alpha}{\partial t^2} \tag{2}$$

$$\frac{2h^3\mu}{3}\nabla^2 v_3 - 2\kappa h\lambda v_{\beta,\beta} - 2\kappa^2 h(\lambda + 2\mu)v_3 = \frac{2\rho h^3}{3} \frac{\partial^2 v_3}{\partial t^2} \tag{3}$$

where $\lambda = E\nu/[(1+\nu)(1-2\nu)]$ and $\mu = E/[2(1+\nu)]$ are elastic constants; E , ν and ρ denote Young's modulus, Poisson's ratio and the density, respectively; According to Kane and Mindlin [7], the recommended value of the shear coefficient κ is $\sqrt{\pi^2/12}$.

In the Laplace domain, Eqs. (2) and (3) are written as:

$$2h\mu\nabla^2 \tilde{v}_\alpha + 2h(\lambda + \mu)\tilde{v}_{\beta,\beta\alpha} + 2\kappa h\lambda \tilde{v}_{3,\alpha} = 2\rho h p^2 \tilde{v}_\alpha \tag{4}$$

$$\frac{2h^3\mu}{3}\nabla^2 \tilde{v}_3 - 2\kappa h\lambda \tilde{v}_{\beta,\beta} - 2\kappa^2 h(\lambda + 2\mu)\tilde{v}_3 = \frac{2\rho h^3 p^2}{3} \tilde{v}_3 \tag{5}$$

where p is the Laplace transform parameter. The Laplace transform of a function $f(\mathbf{x}, t)$ is defined as:

$$\mathcal{L}[f(\mathbf{x}, t)] = \tilde{f}(\mathbf{x}, p) = \int_0^\infty f(\mathbf{x}, t) e^{-pt} dt \tag{6}$$

The displacement boundary integral equation in the Laplace domain can be written as [31]:

$$c_{ij}(\mathbf{x}') \tilde{v}_j(\mathbf{x}', p) = \int_\Gamma \tilde{U}_{ij}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma - \text{CPV} \int_\Gamma \tilde{T}_{ij}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_j(\mathbf{x}, p) d\Gamma \tag{7}$$

where $\mathbf{x}'$ and $\mathbf{x}$ denote a collocation point and a field point on the plate boundary Γ , respectively; $c_{ij}(\mathbf{x}')$ is equal to $\delta_{ij}/2$ when the collocation point $\mathbf{x}'$ is on a smooth boundary; $\tilde{U}_{ij}$ and $\tilde{T}_{ij}$ are

Laplace transform fundamental solutions, which have been given in Ref. [31]; $\text{CPV} \int_{\Gamma}$ denotes Cauchy principal value integral.

Assuming the collocation point $\mathbf{x}'$ is located on a smooth boundary, differentiating Eq. (7) with respect to $\mathbf{x}'$, and using the relations between stress resultants and displacements, the traction boundary integral equations in the Laplace domain are obtained as follows:

$$\frac{1}{2} \tilde{t}_{\alpha}(\mathbf{x}', p) = n_{\beta}(\mathbf{x}') \left[\text{CPV} \int_{\Gamma} \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_{\gamma}(\mathbf{x}, p) d\Gamma(\mathbf{x}) + \int_{\Gamma} \tilde{D}_{\alpha\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right. \\ \left. - \text{HPV} \int_{\Gamma} \tilde{S}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_{\gamma}(\mathbf{x}, p) d\Gamma(\mathbf{x}) - \text{CPV} \int_{\Gamma} \tilde{S}_{\alpha\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right] \quad (8)$$

$$\frac{1}{2} \tilde{t}_3(\mathbf{x}', p) = n_{\beta}(\mathbf{x}') \left[\int_{\Gamma} \tilde{D}_{3\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_{\gamma}(\mathbf{x}, p) d\Gamma(\mathbf{x}) + \text{CPV} \int_{\Gamma} \tilde{D}_{3\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right. \\ \left. - \text{CPV} \int_{\Gamma} \tilde{S}_{3\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_{\gamma}(\mathbf{x}, p) d\Gamma(\mathbf{x}) - \text{HPV} \int_{\Gamma} \tilde{S}_{3\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right] \quad (9)$$

where $\text{HPV} \int_{\Gamma}$ denotes Hadamard principal value integral; $\tilde{D}_{\alpha\beta\gamma}$, $\tilde{D}_{\alpha\beta 3}$, $\tilde{D}_{3\beta 3}$, $\tilde{S}_{\alpha\beta\gamma}$, $\tilde{S}_{\alpha\beta 3}$ and $\tilde{S}_{3\beta 3}$ are Laplace transform fundamental solutions which can be obtained using the relationships as follows:

$$\begin{aligned} \tilde{X}_{\alpha\beta\gamma} &= 2h \left[\lambda \left(\tilde{Y}_{\eta\gamma, \eta} + \kappa \tilde{Y}_{3\gamma} \right) \delta_{\alpha\beta} + \mu \left(\tilde{Y}_{\alpha\gamma, \beta} + \tilde{Y}_{\beta\gamma, \alpha} \right) \right] \\ \tilde{X}_{\alpha\beta 3} &= 2h \left[\lambda \left(\tilde{Y}_{\gamma 3, \gamma} + \kappa \tilde{Y}_{33} \right) \delta_{\alpha\beta} + \mu \left(\tilde{Y}_{\alpha 3, \beta} + \tilde{Y}_{\beta 3, \alpha} \right) \right] \\ \tilde{X}_{3\beta\gamma} &= \frac{2h^3 \mu}{3} \tilde{Y}_{3\gamma, \beta} \\ \tilde{X}_{3\beta 3} &= \frac{2h^3 \mu}{3} \tilde{Y}_{33, \beta} \end{aligned} \quad , \quad (10)$$

in which $\tilde{X}$ denotes $\tilde{D}$ and $\tilde{S}$ when $\tilde{Y}$ is $\tilde{U}$ and $\tilde{T}$, respectively. All the transformed fundamental solutions mentioned above are listed in Appendix A.

The displacement integral equation (7) is used for the collocation points on plate edges. When it comes to the modelling strategy for cracks, the dual boundary element method (DBEM) is employed. In the DBEM, the displacement and traction boundary integral equations are applied for the collocation on upper and lower crack surfaces, respectively. The detailed implementation of the DBEM can be found in Refs. [26, 33].

3. Treatment of singularities

The behaviour of the dynamic fundamental solutions in the neighbourhood of a collocation point determines how to compute the singular integrals in the boundary integral equations. It has been found in both isotropic problems [25, 26, 34] and anisotropic problems [35, 36] that the dominant singularities in the dynamic fundamental solutions have the same expressions as those in the static fundamental solutions. This is also true for the K-M problems, and the detailed singular behaviour of the kernels is given in Appendix B.

It is convenient to write the transformed fundamental solution into two parts, the static fundamental solution and a modified term:

$$\tilde{U}_{ij}(\mathbf{x}', \mathbf{x}, p) = U_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{U}_{ij}^{\text{m}}(\mathbf{x}', \mathbf{x}, p) \quad (11)$$

$$\tilde{T}_{ij}(\mathbf{x}', \mathbf{x}, p) = T_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{T}_{ij}^{\text{m}}(\mathbf{x}', \mathbf{x}, p) \quad (12)$$

$$\tilde{D}_{k\beta j}(\mathbf{x}', \mathbf{x}, p) = D_{k\beta j}^{\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{D}_{k\beta j}^{\text{m}}(\mathbf{x}', \mathbf{x}, p) \quad (13)$$

$$\tilde{S}_{k\beta j}(\mathbf{x}', \mathbf{x}, p) = S_{k\beta j}^{\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{S}_{k\beta j}^{\text{m}}(\mathbf{x}', \mathbf{x}, p). \quad (14)$$

There are three types of singularities in the static fundamental solutions. On non-crack plate boundaries, only the displacement boundary integral equation (7) is used and only the strong singularity of order $1/r$ exists. The methods adopted for computing corresponding singular integrals have been given in Ref. [31].

The displacement boundary integral equation (7) and the traction boundary integral equations (8) and (9) are used for the collocation on the upper and lower crack surfaces, respectively. The singular integrals involving the strong singularity of order $1/r$ and the hyper singularity of order $1/r^2$ are evaluated analytically, and the implementation method has been given in detail in Ref. [33].

The modified terms do not contain any singularities, except for $\tilde{S}_{\alpha\beta\gamma}^{\text{m}}$ and $\tilde{S}_{3\beta 3}^{\text{m}}$ containing the weak singularities of order $\ln(r)$. A bi-cubic transformation of variable technique proposed by Telles [37] is able to be used to calculate the integrals with such logarithmic singularities.

4. Numerical implementation

The modelling strategy for cracked plates in Ref. [38] is adopted in our study. The plate edges and crack surfaces are discretized using continuous and discontinuous quadratic elements, respectively.

The discretized displacement and traction boundary integral equations are given by:

$$c_{ij}^l \tilde{v}_j^l(p) = \sum_{n=1}^N \sum_{m=1}^3 \left[\tilde{t}_j^{nm}(p) \int_{-1}^1 \tilde{U}_{ij}(\xi, p) N_m(\xi) J_n(\xi) d\xi - \tilde{v}_j^{nm}(p) \text{CPV} \int_{-1}^1 \tilde{T}_{ij}(\xi, p) N_m(\xi) J_n(\xi) d\xi \right] \quad (15)$$

$$\begin{aligned} \frac{1}{2} \tilde{t}_\alpha^l(p) = & \sum_{n=1}^N \sum_{m=1}^3 n_\beta^l \left[\tilde{t}_\gamma^{nm}(p) \text{CPV} \int_{-1}^1 \tilde{D}_{\alpha\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi + \right. \\ & \tilde{t}_3^{nm}(p) \int_{-1}^1 \tilde{D}_{\alpha\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi - \tilde{v}_\gamma^{nm}(p) \text{HPV} \int_{-1}^1 \tilde{S}_{\alpha\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi \\ & \left. - \tilde{v}_3^{nm}(p) \text{CPV} \int_{-1}^1 \tilde{S}_{\alpha\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi \right] \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{1}{2} \tilde{t}_3^l(p) = & \sum_{n=1}^N \sum_{m=1}^3 n_\beta^l \left[\tilde{t}_\gamma^{nm}(p) \int_{-1}^1 \tilde{D}_{3\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi + \right. \\ & \tilde{t}_3^{nm}(p) \text{CPV} \int_{-1}^1 \tilde{D}_{3\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi - \tilde{v}_\gamma^{nm}(p) \text{CPV} \int_{-1}^1 \tilde{S}_{3\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi \\ & \left. - \tilde{v}_3^{nm}(p) \text{HPV} \int_{-1}^1 \tilde{S}_{3\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi \right] \end{aligned} \quad (17)$$

where l denotes the collocation node number; N represents the total number of boundary elements; $N_m(\xi)$ are the shape functions and $J_n(\xi)$ are the Jacobians.

Durbin's method [32] is used to convert the solutions in the Laplace domain back to those in the time domain. The inversion equation has the following form:

$$\begin{aligned} f(t) = & 2 \frac{e^{at}}{T} \left\{ \frac{1}{2} \text{Re}[\tilde{f}(a)] + \sum_{k=1}^K \text{Re}[\tilde{f}(p)] \cos k \frac{2\pi}{T} t - \sum_{k=1}^K \text{Im}[\tilde{f}(p)] \sin k \frac{2\pi}{T} t \right\} \\ & , p = a + ik \frac{2\pi}{T}, \end{aligned} \quad (18)$$

where p , a , K , and T are the Laplace transform parameter, the real part of p , the maximum number of Laplace term, and the time period of interest, respectively. It was found that when aT was equal to 5, satisfactory results could be obtained.

5. Fracture modes and dynamic stress intensity factors

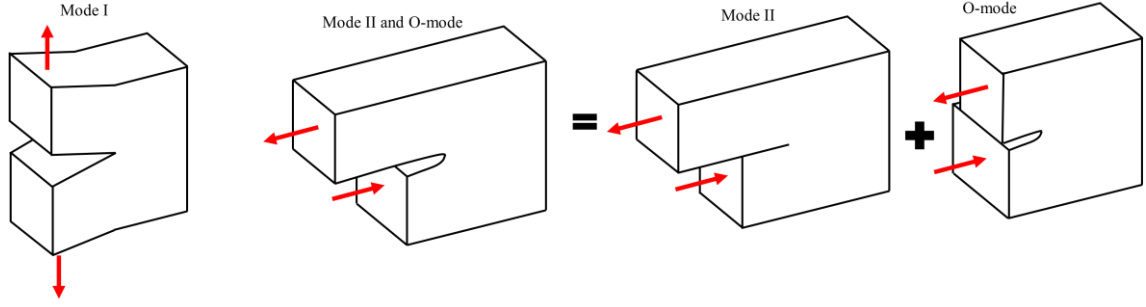


Figure 2. Crack deformation modes under in-plane loadings

The basic fracture modes of the cracked plate under in-plane loads are shown in Figure 2. Mode II is the primary mode for the plate subjected to shear loading, and a coupled out-of-plane mode (O-mode [5]) is generated due to the Poisson effect. The capability to capture the O-mode is one advantage of the K-M theory over the classical two-dimensional elasticity theories.

In our study, since crack surfaces were discretized by discontinuous elements, singular behaviour of stress fields was represented using the discontinuous quarter-point elements [39]. The crack opening displacement (COD) at the node pair closest to the crack tip was used for evaluating the DSIFs. On the basis of the COD Δu_i^{ct} at such node pair [5, 40], the DSIFs can be evaluated using the expressions:

$$K_I = \frac{E}{4(1-\nu^2)} \sqrt{\frac{\pi}{2d}} \Delta u_2^{\text{ct}}, \quad (19)$$

$$K_{II} = \frac{E}{4(1-\nu^2)} \sqrt{\frac{\pi}{2d}} \Delta u_1^{\text{ct}}, \quad (20)$$

$$K_O = \frac{E}{4(1+\nu)} \sqrt{\frac{\pi}{2d}} \Delta u_3^{\text{ct}}, \quad (21)$$

where d denotes the distance between the node pair and the crack tip.

In our study, the parameters of discontinuous quarter-point elements in Ref. [39] were adopted. Thus, d is equal to $l_e/36$, in which l_e is the length of the crack-tip element, and Eqs. (19)-(21) become:

$$K_I = \frac{3E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_2^{\text{ct}}, \quad K_{II} = \frac{3E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_1^{\text{ct}}, \quad K_O = \frac{3E}{2(1+\nu)} \sqrt{\frac{\pi}{2l_e}} \Delta u_3^{\text{ct}}. \quad (22)$$

In order to validate the BEM results, the DSIFs were also evaluated using the FEM, in which wedge quarter-point elements were used. Since these elements are not supported in the Abaqus explicit solver, the implicit solver was used in our study. These elements are continuous elements, so d is set to $l_e/4$ and the DSIFs can be expressed as:

$$K_I = \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_2^{\text{ct}}, \quad K_{II} = \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_1^{\text{ct}}, \quad K_O = \frac{E}{2(1+\nu)} \sqrt{\frac{\pi}{2l_e}} \Delta u_3^{\text{ct}}. \quad (23)$$

It has been pointed out in Refs. [5, 15] that the solutions based on the K-M theory agreed very well with the thickness-averaged results from the 3D FEM simulation. Thus, in the following numerical examples, the comparisons between the BEM and FEM results are based on the thickness-averaged DSIFs, $K_I^a = \frac{1}{h} \int_0^h K_I dx_3$, $K_{II}^a = \frac{1}{h} \int_0^h K_{II} dx_3$ and $K_O^a = \frac{1}{h} \int_0^h K_O dx_3$.

It should be mentioned here that the so-called corner singularity [2, 41-43], which is limited to a very small zone near the intersection between the crack front and free surface, is complex and cannot be described by using the inverse square-root singularity. However, since the corner singularity has been found to have a minor influence on the through-thickness averaged results [44] and the 3D FEM simulation aimed to validate the K-M based BEM results, there was no need to introduce such complex singularity into the finite element model in our study.

6. Numerical examples

The FEM simulation was carried out using the Abaqus implicit solver. The cracked plates were discretized with the elements of type C3D20R, which can be modified to generate wedge quarter-point crack-tip elements easily. Since the field quantities near the intersection between the crack front and the free surface exhibit strong localized gradient along the x_3 -axis, the thickness of element layers was gradually decreased toward the free surface.

The time t is normalized with respect to l_c / c_2 , where $c_2 = \sqrt{\mu/\rho}$ is the shear wave velocity and l_c is a characteristic length, such as half crack length. The DSIFs are normalized with respect to $K_0 = \sigma_A \sqrt{\pi a}$, in which σ_A is the amplitude of the load and a denotes half crack length.

6.1. A crack under opening loading

An infinite plate with a through-thickness crack, as shown in Figure 3, is considered here. The thickness of the plate and the length of the crack are denoted by $2h$ and $2a$, respectively. The crack surfaces are subjected to uniform opening loading $F_N H(t)$, where F_N is the load amplitude and $H(t)$ is the Heaviside function. The material properties are the density, $\rho = 290 \text{ kg/m}^3$, Young's modulus, $E = 69 \text{ Gpa}$, and the Poisson's ratio, $\nu = 0.3$.

Only the crack was considered and each crack surface was discretized with 30 boundary elements. In the FEM simulation, a large square plate with an edge length of $20a$ was chosen to approximate the infinite plate, which ensured that the DSIF during a specified time period was not influenced by the reflected waves from the plate edges. Due to the geometric and loading symmetry properties, only one-eighth of the plate was modelled in the finite element analysis.

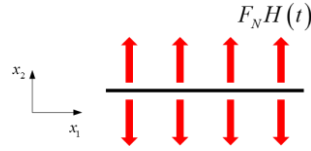


Figure 3. A crack subjected to opening loading

It should be mentioned that the dynamic fracture analysis of this example was first carried out by Jin and Batra [10]. However, the DSIF obtained in their study does not seem to be accurate because their DSIF curves are very smooth and do not contain any kinks which are caused due to the propagation of Rayleigh-type surface waves along the crack surfaces.

First, a very thin plate ($h/a = 0.025$) was taken into consideration. Sih [1] pointed out that the generalized plane stress theory could not serve as a limiting thin-plate case for the three-dimensional analysis and it could only give approximate results for such thin plate. Thus, the fracture analysis of the thin plate should be carried out using higher-order plate theories, such as the K-M theory. In spite of this, for an infinitely thin plate ($h/a \rightarrow 0$), it has been found in Refs. [8, 10] that the Mode I stress intensity factor obtained using the K-M theory, K_I^{K-M} , is equal to $K_I^{PS} / \sqrt{1-\nu^2}$, in which K_I^{PS} is the solution based on the generalized plane stress theory. This relationship can be used to modify the DSIF K_I^{PS} from the BEM codes [45] for the generalized plane stress problems, and then validate K_I^{K-M} . In the 3D FEM simulation, 204,880 elements

were used in which there were ten layers of elements along the thickness direction and the radial size of the crack-tip singular elements was equal to $0.001a$.

Figure 4 compares the normalized DSIF K_I^a obtained using different numerical methods. The BEM results shown in this figure were obtained using 50 Laplace terms. It is evident that the DSIF obtained using the K-M based BEM is in excellent agreement with both the modified DSIF from 2D boundary element analysis and the 3D FEM result. The influence of the number of Laplace terms on the normalized DSIF K_I^a is shown in Figure 5. It can be seen that the DSIF reaches the maximum value later when fewer Laplace terms are used. However, the maximum values of the DSIF obtained using 25 and 50 Laplace terms are very close (relative error $<1\%$). Thus, if the maximum value of the DSIF is the main focus, 25 Laplace terms can be used to increase the computational efficiency. If more details of the DSIF are required, 50 Laplace terms should be used.

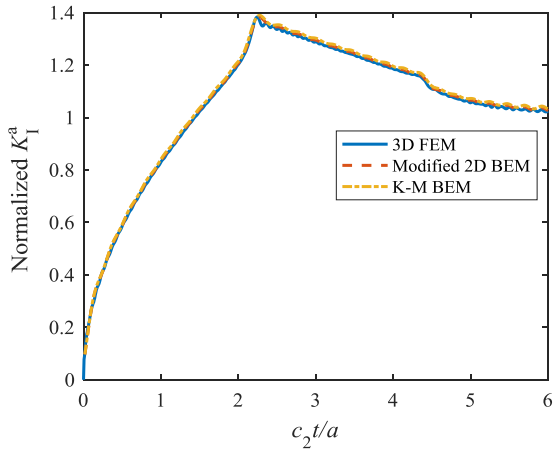


Figure 4. Normalized DSIF for the crack under opening loading ($h/a = 0.025$)

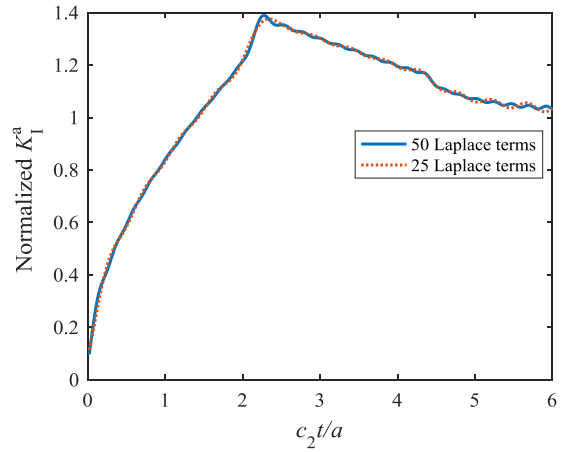


Figure 5. The influence of the number of Laplace terms on the normalized DSIF obtained using the K-M BEM for the crack under opening loading ($h/a = 0.025$)

Next, a very thick plate ($h/a = 100$) was considered to approximate an infinitely thick plate. Sih [1] demonstrated that the plane-strain crack problem was exactly the limiting case for such thick plate. Thus, the finite element analysis using 58,000 plane-strain (CPE8R) elements, and the boundary element analysis for the plane strain problem [45], were conducted to validate the K-M based BEM results. In the boundary element simulations, 50 Laplace terms were used. As shown in Figure 6, the K-M based DSIF K_I^a is in excellent agreement with the plane-strain

solutions. This conclusion is consistent with that given by Kotousov and Wang in the static fracture analysis of the K-M plates [12].

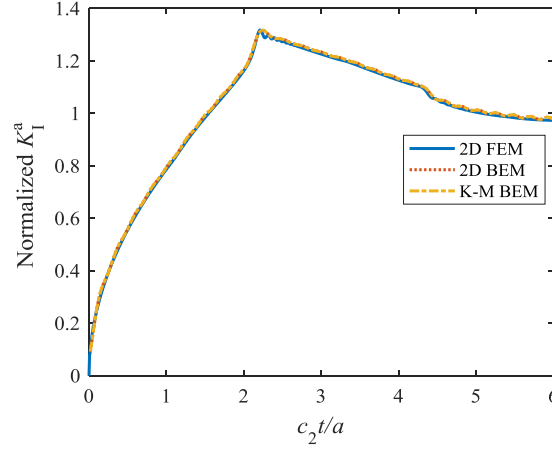


Figure 6. Normalized DSIF for the crack ($h/a = 100$) under opening loading obtained using the K-M BEM and the DSIFs from the plane-strain based FEM and BEM simulations

Finally, two moderately thick plates ($h/a = 1, 2$) were taken into consideration. The K-M BEM results were obtained using 50 Laplace terms. In the finite element analysis, these two plates were discretized using 191,200 and 267,680 elements with 25 and 35 layers of elements along the thickness direction, respectively. In both cases, the radial sizes of the crack-tip wedge singular elements were the same and equal to $0.005a$. Figure 7 and Figure 8 compare the K-M based BEM DSIFs and those from the finite element simulation for the two moderately thick plates ($h/a = 1, 2$), respectively. It can be seen from these two figures that the BEM results agree well with the FEM solutions. The slight discrepancy results from the linear assumption for the displacement u_3 in the K-M theory. This assumption cannot represent the real variation accurately for moderately thick plates.

Since the K-M theory considers the plate thickness effect, this theory can be used to analyse the influence of the plate thickness on the DSIF K_I^a , which is shown in Figure 9. Generally speaking, it can be seen that the plate thickness does not have a significant influence on the mode I DSIF, especially on the value and location of the maximum DSIF. The DSIF for the infinitely thin plate is larger than that for the infinitely thick plate. However, the maximum value of the DSIF does not decrease monotonically with increasing plate thickness.

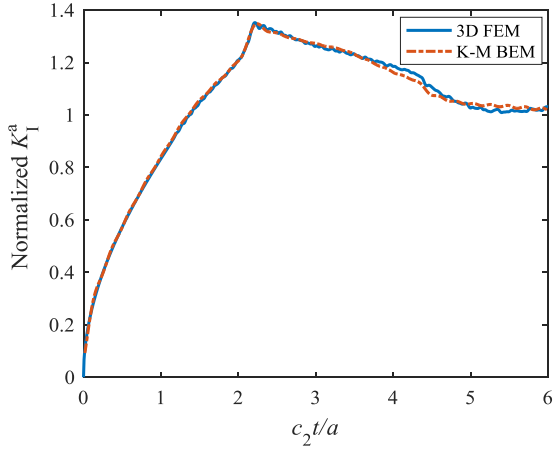


Figure 7. Normalized DSIF for the crack under opening loading ($h/a = 1$)

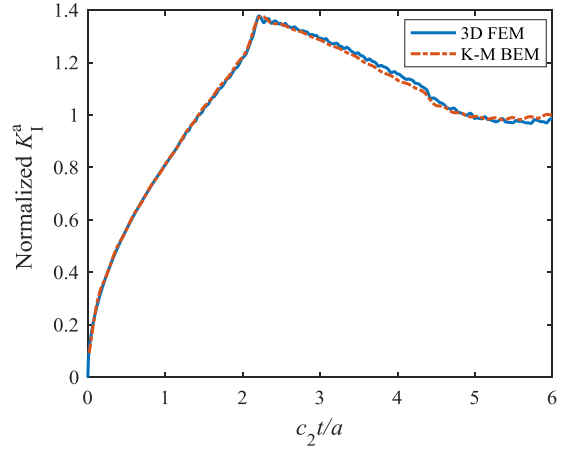


Figure 8. Normalized DSIF for the crack under opening loading ($h/a = 2$)

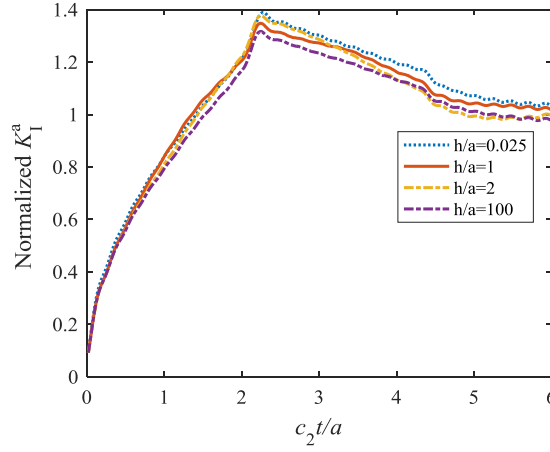


Figure 9. Influence of plate thickness on normalized DSIF for the crack under opening loading

6.2. A crack under shear loading

Figure 10 shows an infinite plate with a central crack loaded by shear loading $F_s H(t)$, where F_s is the load amplitude. As mentioned earlier, under such shear loading, Model II (primary mode) and O-mode (secondary mode) are coupled. Since the geometry and material properties are the same as those in the previous example, the same BEM models and FEM models, including the meshes, used in the first example were adopted in this example. The only things changed were the loads and boundary conditions.



Figure 10. A crack subjected to shear loading

As in the previous example, a very thin plate ($h/a = 0.025$) was considered first. Figure 11 shows the DSIFs K_{II} and K_O obtained using both the K-M based BEM and the 3D FEM. It can be seen that the BEM results agree very well with the FEM results. The maximum value of K_O^a is only around 15 percent of that value of K_{II}^a .

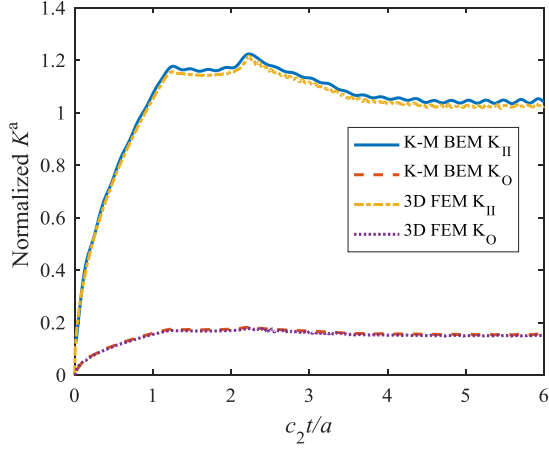


Figure 11. Normalized DSIFs for the crack under shear loading ($h/a = 0.025$)

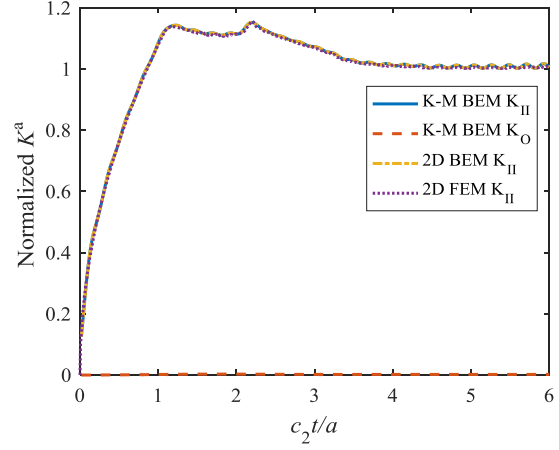


Figure 12. Normalized DSIFs for the crack under shear loading ($h/a = 100$)

Next, a very thick plate ($h/a = 100$) was analysed, for which the plane strain conditions prevail. It means that the coupled O-mode stress intensity factor should be negligible, which is reflected correctly in the K-M based BEM results, as shown in Figure 12. The DSIF K_{II}^a obtained using the K-M based BEM is in excellent agreement with both the BEM and FEM results on the basis of the plane strain theory.

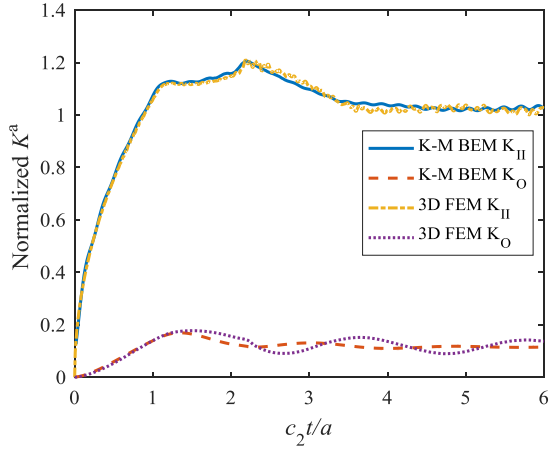


Figure 13. Normalized DSIFs for the crack under shear loading ($h/a = 1$)

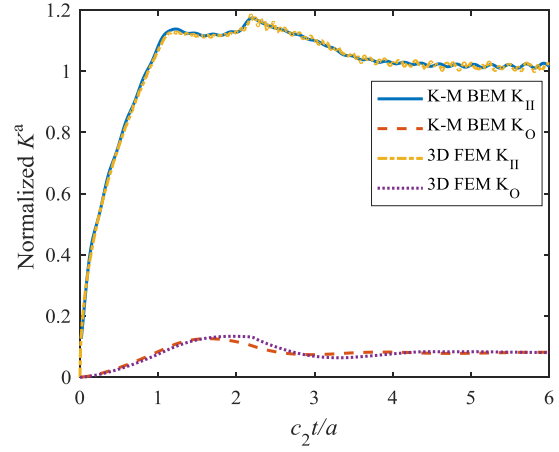


Figure 14. Normalized DSIFs for the crack under shear loading ($h/a = 2$)

When it comes to moderately thick plates ($h/a = 1, 2$), the comparisons were carried out between the results from the K-M based boundary element analysis and the 3D finite element simulation. As shown in Figure 13 and Figure 14, the DSIFs from the K-M based BEM match well with those from the 3D FEM. Furthermore, compared with the coupled DSIF K_O^a , better agreement is shown in terms of the DSIF K_{II}^a . As mentioned earlier, the limitation of the linear assumption in the K-M theory results in the discrepancy between the K-M results and the thickness-averaged 3D results.

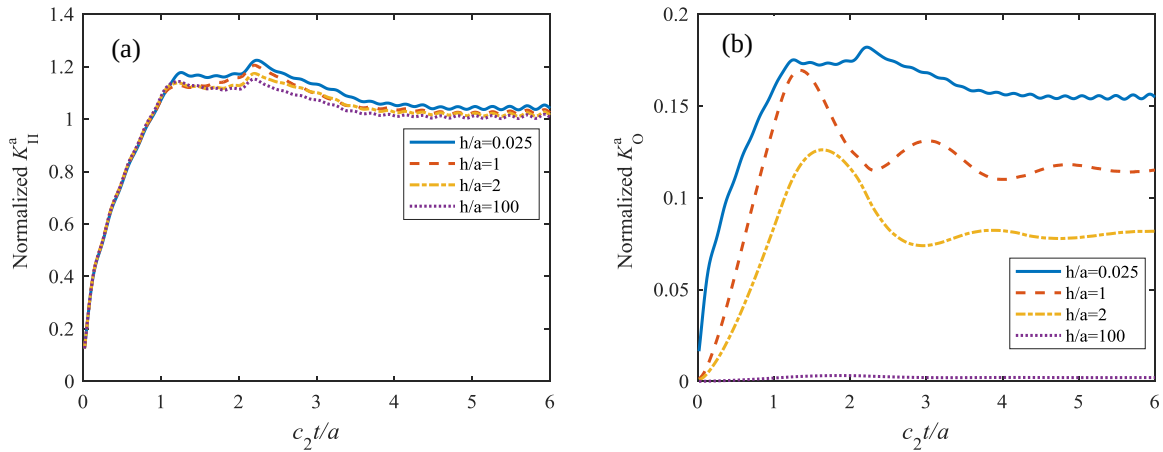


Figure 15. Influence of plate thickness on normalized DSIFs for the crack under shear loading: (a) Mode II, (b) O-mode

The influences of plate thickness on the mode II and the O-mode DSIFs are presented in Figure 15. It is shown that the plate thickness has a minor influence on the mode II DSIF, but it affects

O-mode DSIF significantly. It seems that the maximum values of the normalized mode II and O-mode DSIFs decrease as the plate becomes thicker.

6.3. A square plate with a slant central crack

A mixed-mode crack problem is considered in this example. A cracked square plate loaded in tension $\sigma_0 H(t)$ is shown in Figure 16. The edge and crack lengths of this plate are $2b = 15$ mm and $2a = 4$ mm, respectively. The crack makes an angle of $\theta = \pi/4$ with the direction of the x_2 axis. This aluminium plate has a Young's modulus of 70 GPa, a Poisson's ratio of 0.33 and a density of 2700 kg/m³.

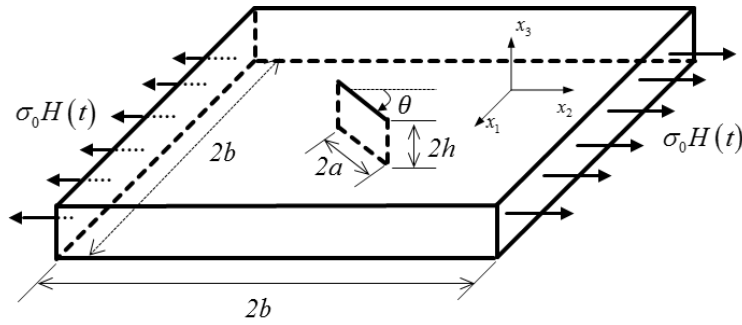


Figure 16. A square plate with a slant central crack

In the following dual boundary element analyses, each plate edge and crack surface were discretized using 10 and 20 elements, respectively. The numerical inversion of Laplace transforms was conducted using 50 Laplace terms.

First, a thin plate ($h/a = 0.05$) was considered. Owing to symmetry conditions with respect to the mid-plane ($x_3 = 0$), in the 3D FEM simulation, only half of the plate was modelled with 392,480 elements, including 10 layers of elements along the x_3 direction and sufficiently small crack-tip elements with a radial size of $0.005a$.

Since this example is a mixed-mode problem, all the three fracture modes, including the coupled O-mode, can be observed. Figure 17 compares the DSIFs for all three fracture modes obtained in the K-M BEM simulation and the 3D FEM simulation. The comparison shows excellent agreement between the BEM results and the FEM results. In addition, it can be seen that the maximum values of Mode I and Mode II DSIFs are nearly the same, and the maximum value of the O-mode DSIF is around 16% of that of Mode II DSIF.

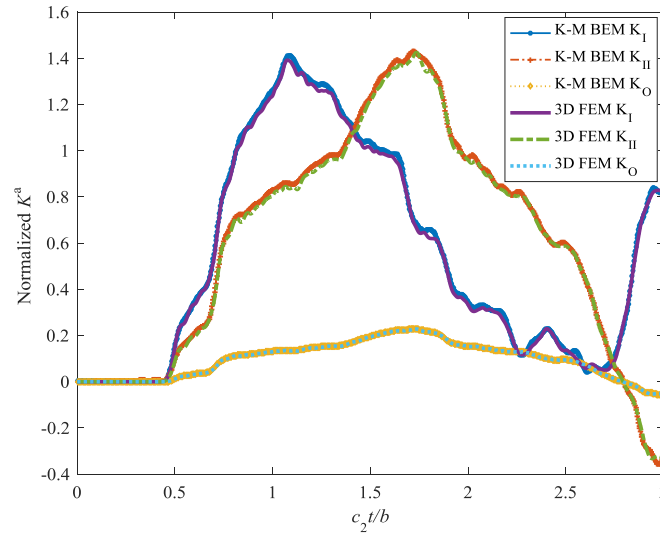


Figure 17. Normalized DSIFs for the square plate with a slant central crack ($h/a = 0.05$)

Next, a very thick plate ($h/a = 100$) was taken into consideration. For such a thick plate, plane strain conditions should prevail. Figure 18 compares the DSIFs obtained using the K-M based BEM and the 2D BEM based on the plane strain theory. The excellent agreement in terms of K_I^a and K_{II}^a , as well as the negligible K_O^a , demonstrates that this limiting case (plane strain) can be represented accurately using the K-M theory.

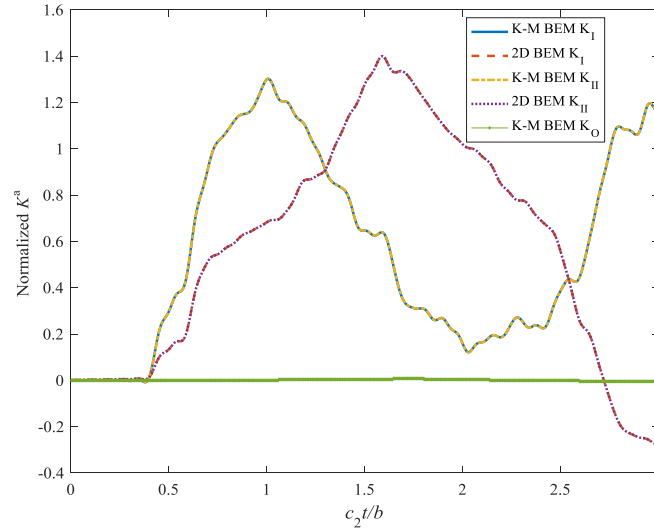


Figure 18. Normalized DSIFs for the square plate with a slant central crack ($h/a = 100$)

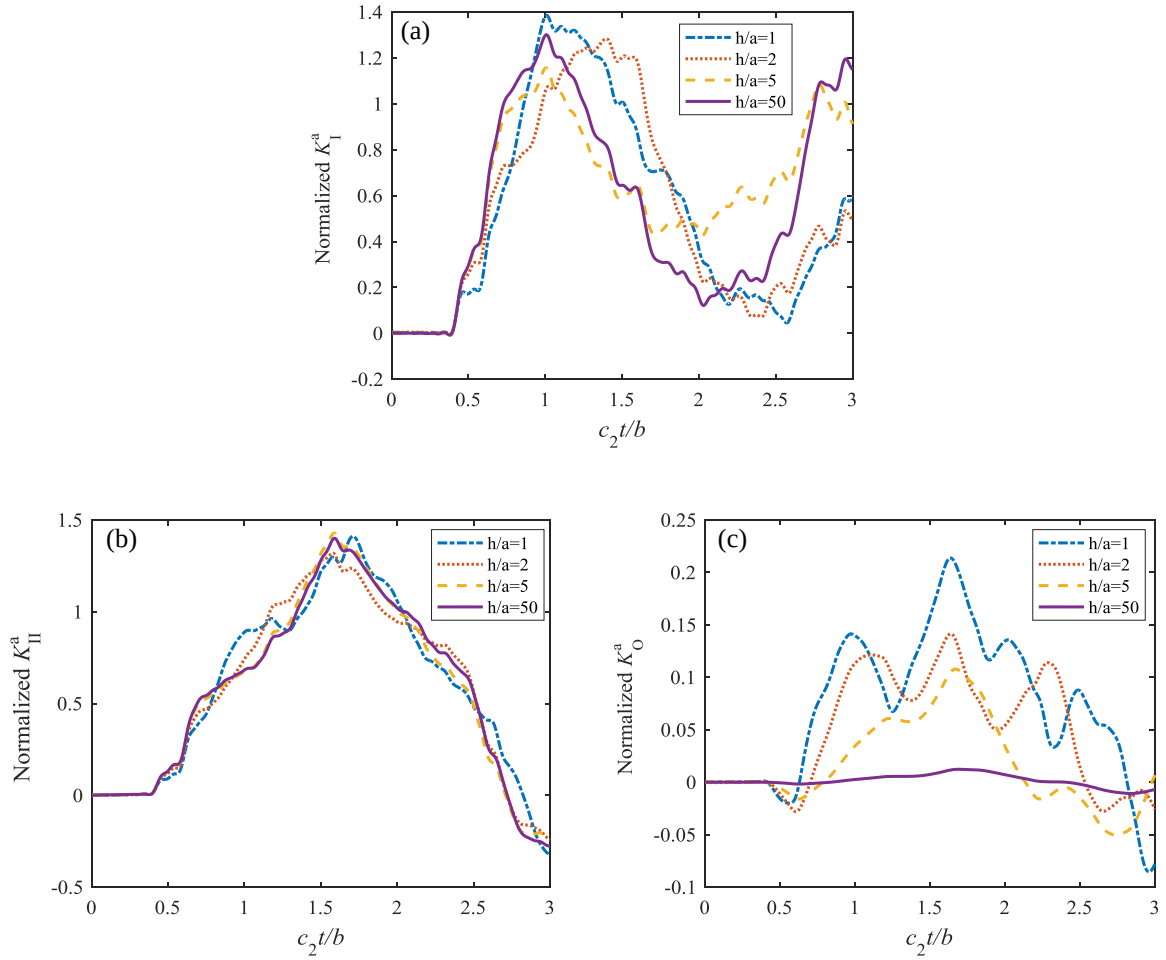


Figure 19. Influence of plate thickness on normalized DSIFs for the square plate with a slant central crack: (a) Mode I, (b) Mode II, (c) O-mode

Lastly, moderately thick plates were considered using the K-M based BEM. Figure 19 presents the influence of the plate thickness on the DSIFs for all three fracture modes. Generally speaking, the plate thickness has a significant influence on the O-mode and Mode I DSIFs in terms of the maximum value, but its influence on Mode II DSIF is limited. There is no monotonic relationship between the plate thickness and the maximum values of Mode I and Mode II. It seems that the maximum value of the O-mode DSIF decreases with increasing plate thickness.

7. Conclusions

In this paper, a new dual boundary element formulation in the Laplace domain has been developed for the dynamic fracture analyses of Kane-Mindlin plates subjected to in-plane loads. The dynamic stress intensity factors (DSIFs) based on the K-M theory represent average through-thickness stress intensity factors, which is consistent with the findings in the corresponding static problems. The DSIFs obtained using the K-M based BEM are in good agreement with the results

from the 3D finite element calculations. Compared with Mode I and Mode II DSIFs, plate thickness has a more significant influence on the coupled O-mode DSIF. The proposed DBEM was shown to be a successful method which is capable of considering the thickness effect in the dynamic fracture analyses of plates. This is especially valuable for the numerical analyses of plates under in-plane shear loading and of mixed mode crack problems. For such cases, the classical two-dimensional theories of elasticity (plane strain and generalized plane stress) fail to capture the coupled O-mode, and the numerical simulations based on the three-dimensional elasticity theory are computationally inefficient.

Acknowledgments

This research was supported by a grant provided by the China Scholarship Council (CSC) (Grant No. [2016]3100).

Appendix A

The transformed fundamental solutions in the displacement boundary integral equation can be arranged as follows:

$$\tilde{U}_{\alpha\beta} = \frac{1}{4\pi h\mu(e_2 - e_1)\zeta_3^2} (\psi\delta_{\alpha\beta} - \chi r_{,\alpha} r_{,\beta}), \quad (\text{A.1})$$

in which

$$\begin{aligned} \psi &= \frac{(e_2 - e_1)\zeta_3 [K_1(z_3) + z_3 K_0(z_3)] - e_2 \zeta_1 K_1(z_1) + e_1 \zeta_2 K_1(z_2)}{r}, \\ \chi &= \frac{(e_2 - e_1)\zeta_3 [2K_1(z_3) + z_3 K_0(z_3)] - e_2 \zeta_1 [2K_1(z_1) + z_1 K_0(z_1)] + e_1 \zeta_2 [2K_1(z_2) + z_2 K_0(z_2)]}{r}; \end{aligned}$$

$K_0(z_i)$ and $K_1(z_i)$ are modified zero- and first-order Bessel functions of the second kind, respectively; $z_i = \zeta_i r$. The coefficients ζ_i and e_β in the above equations are given by:

$$\zeta_\beta^2 = \frac{3\kappa^2}{2\alpha_2 h^2} \left\{ (\alpha_1 + \alpha_2) \frac{p^2}{\varpi^2} + 1 + (-1)^\beta \sqrt{\left[(\alpha_1 + \alpha_2) \frac{p^2}{\varpi^2} + 1 \right]^2 - 4\alpha_1 \alpha_2 \frac{p^2}{\varpi^2} \left(1 + \frac{p^2}{\varpi^2} \right)} \right\} \quad (\text{A.2})$$

$$\zeta_3^2 = \frac{p^2}{c_2^2} \quad (\text{A.3})$$

$$e_\beta = -\frac{\lambda + 2\mu}{\kappa\lambda} \left(\zeta_\beta^2 - \frac{p^2}{c_1^2} \right), \quad (\text{A.4})$$

where ϖ denotes the cut-off frequency $\sqrt{3}\kappa c_1/h$ and the constants α_β are equal to c_β^2/c_3^2 ; $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$ and $c_3^2 = E/[\rho(1-\nu^2)]$ are three different wave velocities.

$$\tilde{U}_{\alpha 3} = -\tilde{U}_{3\alpha} = -\frac{3}{4\pi h^3 \mu (e_2 - e_1)} [\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)] r_{,\alpha} \quad (\text{A.5})$$

$$\tilde{U}_{33} = \frac{3}{4\pi h^3 \mu (e_2 - e_1)} [-e_1 K_0(z_1) + e_2 K_0(z_2)] \quad (\text{A.6})$$

$$\begin{aligned} \tilde{T}_{\alpha\beta} = \frac{1}{2\pi(e_2 - e_1)\zeta_3^2} & \left\{ \left(\psi_{,r} - \frac{\chi}{r} \right) (\delta_{\alpha\beta} r_{,n} + n_\alpha r_{,\beta}) - 2\frac{\chi}{r} (n_\beta r_{,\alpha} - 2r_{,\alpha} r_{,\beta} r_{,n}) - 2\chi_{,r} r_{,\alpha} r_{,\beta} r_{,n} \right. \\ & \left. + \frac{\lambda}{\mu} \left[\psi_{,r} - \frac{\chi}{r} - \chi_{,r} - \frac{3\kappa\zeta_3^2}{h^2} (\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)) \right] r_{,\alpha} n_\beta \right\}, \end{aligned} \quad (\text{A.7})$$

where

$$\begin{aligned} \psi_{,r} &= -\left[(e_2 - e_1) \zeta_3^3 K_1(z_3) + \frac{\chi}{r} \right]; \\ \chi_{,r} &= -\left[2\frac{\chi}{r} + (e_2 - e_1) \zeta_3^3 K_1(z_3) - e_2 \zeta_1^3 K_1(z_1) + e_1 \zeta_2^3 K_1(z_2) \right]; \\ r_{,n} &= r_{,\alpha} n_\alpha. \end{aligned}$$

$$\tilde{T}_{\alpha 3} = \frac{1}{2\pi(e_2 - e_1)} \left\{ [\zeta_1^2 A(z_1) - \zeta_2^2 A(z_2)] r_{,\alpha} r_{,n} - [\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)] \frac{n_\alpha}{r} \right\} \quad (\text{A.8})$$

$$\begin{aligned} \tilde{T}_{3\alpha} = -\frac{3}{2\pi h^2 (e_2 - e_1)} & \left\{ \frac{\lambda}{\mu} [\zeta_1^2 K_0(z_1) - \zeta_2^2 K_0(z_2) + \kappa(e_1 K_0(z_1) - e_2 K_0(z_2))] \right. \\ & \left. - \frac{2\mu}{\lambda r} (\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)) \right] n_\alpha + 2[\zeta_1^2 A(z_1) - \zeta_2^2 A(z_2)] r_{,\alpha} r_{,n} \Big\} \end{aligned} \quad (\text{A.9})$$

$$\tilde{T}_{33} = \frac{1}{2\pi(e_2 - e_1)} [e_1 \zeta_1 K_1(z_1) - e_2 \zeta_2 K_1(z_2)] r_{,n}, \quad (\text{A.10})$$

where $A(z_i) = K_0(z_i) + \frac{2}{z_i} \left[K_1(z_i) - \frac{1}{z_i} \right]$.

The fundamental solutions in the traction boundary integral equation can be arranged as follows:

$$\begin{aligned} \tilde{D}_{\alpha\beta\gamma} = \frac{1}{2\pi(e_2 - e_1)\zeta_3^2} & \left\{ 2\left(\chi_{,r} - 2\frac{\chi}{r}\right)r_{,\alpha}r_{,\beta}r_{,\gamma} - \left(\psi_{,r} - \frac{\chi}{r}\right)(\delta_{\alpha\gamma}r_{,\beta} + \delta_{\beta\gamma}r_{,\alpha}) \right. \\ & \left. + \left[2\frac{\chi}{r} - \frac{\lambda}{\mu}\left(\psi_{,r} - \chi_{,r} - \frac{\chi}{r} - \frac{3\kappa\zeta_3^2}{h^2}(\zeta_1K_1(z_1) - \zeta_2K_1(z_2))\right) \right] \delta_{\alpha\beta}r_{,\gamma} \right\} \end{aligned} \quad (A.11)$$

$$\begin{aligned} \tilde{D}_{\alpha\beta 3} = -\frac{3}{2\pi h^2(e_2 - e_1)} & \left\{ 2[\zeta_1^2A(z_1) - \zeta_2^2A(z_2)]r_{,\alpha}r_{,\beta} + \frac{\lambda}{\mu}[\zeta_1^2K_0(z_1) - \zeta_2^2K_0(z_2) \right. \\ & \left. + \kappa(e_1K_0(z_1) - e_2K_0(z_2)) + 2\frac{\mu}{\lambda r}(-\zeta_1K_1(z_1) + \zeta_2K_1(z_2))] \delta_{\alpha\beta} \right\} \end{aligned} \quad (A.12)$$

$$\tilde{D}_{3\beta\gamma} = \frac{1}{2\pi(e_2 - e_1)} \left\{ [\zeta_1^2A(z_1) - \zeta_2^2A(z_2)]r_{,\beta}r_{,\gamma} - [\zeta_1K_1(z_1) - \zeta_2K_1(z_2)]\frac{\delta_{\beta\gamma}}{r} \right\} \quad (A.13)$$

$$\tilde{D}_{3\beta 3} = -\frac{1}{2\pi(e_2 - e_1)} [e_1\zeta_1K_1(z_1) - e_2\zeta_2K_1(z_2)]r_{,\beta} \quad (A.14)$$

$$\begin{aligned} \tilde{S}_{\alpha\beta\gamma} = \frac{h\mu}{\pi(e_2 - e_1)\zeta_3^2} & \times \\ & \left\{ \left[4\left(\chi_{,rr} - 5\frac{\chi_{,r}}{r} + 8\frac{\chi}{r^2}\right)r_{,\alpha}r_{,\beta}r_{,\gamma} - \left(\psi_{,rr} - 3\frac{\chi_{,r}}{r} - \frac{\psi_{,r}}{r} + 6\frac{\chi}{r^2}\right)(\delta_{\beta\gamma}r_{,\alpha} + \delta_{\alpha\gamma}r_{,\beta}) \right. \right. \\ & + 2\left(2\frac{\chi_{,r}}{r} - 4\frac{\chi}{r^2} + \frac{\lambda}{\mu}\left(\chi_{,rr} - 2\frac{\chi}{r^2} - \psi_{,rr} + \frac{\psi_{,r}}{r} - \frac{3\kappa\zeta_3^2}{h^2}(\zeta_1^2A(z_1) - \zeta_2^2A(z_2))\right) \right) \delta_{\alpha\beta}r_{,\gamma} \left. \right] r_{,n} \\ & + 2\left[2\frac{\chi_{,r}}{r} - 4\frac{\chi}{r^2} + \frac{\lambda}{\mu}\left(\chi_{,rr} - 2\frac{\chi}{r^2} - \psi_{,rr} + \frac{\psi_{,r}}{r} - \frac{3\kappa\zeta_3^2}{h^2}(\zeta_1^2A(z_1) - \zeta_2^2A(z_2))\right) \right] n_{\gamma}r_{,\alpha}r_{,\beta} \\ & - \left(\psi_{,rr} - \frac{\psi_{,r}}{r} - 3\frac{\chi_{,r}}{r} + 6\frac{\chi}{r^2}\right)(n_{\alpha}r_{,\beta} + n_{\beta}r_{,\alpha})r_{,\gamma} - 2\left(\frac{\psi_{,r}}{r} - \frac{\chi}{r^2}\right)(\delta_{\alpha\gamma}n_{\beta} + \delta_{\beta\gamma}n_{\alpha}) \\ & + \left[4\frac{\chi}{r^2} + 4\frac{\lambda}{\mu}\left(\frac{\chi_{,r}}{r} + \frac{\chi}{r^2} - \frac{\psi_{,r}}{r} + \frac{3\kappa\zeta_3^2}{h^2}(\zeta_1K_1(z_1) - \zeta_2K_1(z_2))\frac{1}{r}\right) + \frac{\lambda^2}{\mu^2}\left(\chi_{,rr} + 2\frac{\chi_{,r}}{r} \right. \right. \\ & \left. \left. - \psi_{,rr} - \frac{\psi_{,r}}{r} - \frac{3\kappa\zeta_3^2}{h^2}\left(2(\zeta_1^2K_0(z_1) - \zeta_2^2K_0(z_2)) + \kappa(e_1K_0(z_1) - e_2K_0(z_2))\right)\right) \right] \delta_{\alpha\beta}n_{\gamma} \left. \right\} \end{aligned} \quad (A.15)$$

in which

$$\begin{aligned}
\psi_{,rr} &= -\left[\frac{\psi_{,r}}{r} + \frac{\chi_{,r}}{r} - (e_2 - e_1) \zeta_3^4 K_0(z_3) \right] \\
\chi_{,rr} &= -3 \frac{\chi_{,r}}{r} + (e_2 - e_1) \zeta_3^4 K_0(z_3) - e_2 \zeta_1^4 K_0(z_1) + e_1 \zeta_2^4 K_0(z_2) \\
\tilde{S}_{\alpha\beta 3} &= \frac{h\mu}{\pi(e_2 - e_1)} \left\{ \left[2 \left(\zeta_1^3 \left(K_1(z_1) + \frac{4}{z_1} A(z_1) \right) - \zeta_2^3 \left(K_1(z_2) + \frac{4}{z_2} A(z_2) \right) \right) r_{,\alpha} r_{,\beta} \right. \right. \\
&\quad \left. \left. + \left(\frac{\lambda}{\mu} \left(\zeta_1^3 K_1(z_1) - \zeta_2^3 K_1(z_2) + \kappa (e_1 \zeta_1 K_1(z_1) - e_2 \zeta_2 K_1(z_2)) \right) \right. \right. \right. \\
&\quad \left. \left. - 2 \frac{\zeta_1^3}{z_1} A(z_1) + 2 \frac{\zeta_2^3}{z_2} A(z_2) \right) \delta_{\alpha\beta} \right] r_{,n} - 2 \left[\frac{\zeta_1^3}{z_1} A(z_1) - \frac{\zeta_2^3}{z_2} A(z_2) \right] (r_{,\alpha} n_{\beta} + r_{,\beta} n_{\alpha}) \right\} \quad (A.16)
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{3\beta\gamma} &= \frac{h\mu}{\pi(e_2 - e_1)} \left\{ 2 \left[-\zeta_1^3 \left(K_1(z_1) + \frac{4}{z_1} A(z_1) \right) + \zeta_2^3 \left(K_1(z_2) + \frac{4}{z_2} A(z_2) \right) \right] r_{,\beta} r_{,\gamma} r_{,n} \right. \\
&\quad \left. + 2 \left[\zeta_1^2 A(z_1) - \zeta_2^2 A(z_2) \right] \frac{1}{r} (\delta_{\gamma\beta} r_{,n} + r_{,\gamma} n_{\beta} + r_{,\beta} n_{\gamma}) \right. \\
&\quad \left. - \frac{\lambda}{\mu} \left[\zeta_1^3 K_1(z_1) - \zeta_2^3 K_1(z_2) + \kappa (e_1 \zeta_1 K_1(z_1) - e_2 \zeta_2 K_1(z_2)) \right] r_{,\beta} n_{\gamma} \right\} \quad (A.17)
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_{3\beta 3} &= \frac{h^3\mu}{3\pi(e_2 - e_1)} \left\{ \left[\frac{2}{r^2} (e_1 - e_2) + e_1 \zeta_1^2 A(z_1) - e_2 \zeta_2^2 A(z_2) \right] r_{,\beta} r_{,n} \right. \\
&\quad \left. - \frac{1}{r} [e_1 \zeta_1 K_1(z_1) - e_2 \zeta_2 K_1(z_2)] n_{\beta} \right\} \quad (A.18)
\end{aligned}$$

Appendix B

When the distance r between the collocation point $\mathbf{x}'$ and the field point $\mathbf{x}$ tends to zero, the limits of the transformed fundamental solutions have the following forms:

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha\beta} = -\frac{3-4\nu}{16\pi h\mu(1-\nu)} \ln r \delta_{\alpha\beta} = \lim_{r \rightarrow 0} U_{\alpha\beta}^{\text{static}} \quad (B.1)$$

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha 3} = -\lim_{r \rightarrow 0} \tilde{U}_{3\alpha} = 0 = \lim_{r \rightarrow 0} U_{\alpha 3}^{\text{static}} = -\lim_{r \rightarrow 0} U_{3\alpha}^{\text{static}} \quad (B.2)$$

$$\lim_{r \rightarrow 0} \tilde{U}_{33} = -\frac{3}{4\pi h^3\mu} \ln r = \lim_{r \rightarrow 0} U_{33}^{\text{static}} \quad (B.3)$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha\beta} = -\frac{1}{4\pi(1-\nu)r} \left[(1-2\nu) (\delta_{\alpha\beta} r_{,n} + n_{\alpha} r_{,\beta} - n_{\beta} r_{,\alpha}) + 2r_{,\alpha} r_{,\beta} r_{,n} \right] = \lim_{r \rightarrow 0} T_{\alpha\beta}^{\text{static}} \quad (\text{B. 4})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha 3} = -\frac{\kappa\nu}{4\pi(1-\nu)} n_{\alpha} \ln r = \lim_{r \rightarrow 0} T_{\alpha 3}^{\text{static}} \quad (\text{B. 5})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{3\alpha} = -\frac{3\kappa\nu}{2\pi h^2(1-\nu)} n_{\alpha} \ln r = \lim_{r \rightarrow 0} T_{3\alpha}^{\text{static}} \quad (\text{B. 6})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{33} = -\frac{1}{2\pi r} r_{,n} = \lim_{r \rightarrow 0} T_{33}^{\text{static}} \quad (\text{B. 7})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta\gamma} = \frac{1}{4\pi(1-\nu)r} \left[(1-2\nu) (\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha} - \delta_{\alpha\beta} r_{,\gamma}) + 2r_{,\alpha} r_{,\beta} r_{,\gamma} \right] = \lim_{r \rightarrow 0} D_{\alpha\beta\gamma}^{\text{static}} \quad (\text{B. 8})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta 3} = -\frac{3\kappa\nu}{2\pi h^2(1-\nu)} \delta_{\alpha\beta} \ln r = \lim_{r \rightarrow 0} D_{\alpha\beta 3}^{\text{static}} \quad (\text{B. 9})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta\gamma} = -\frac{\kappa\nu}{4\pi(1-\nu)} \delta_{\beta\gamma} \ln r = \lim_{r \rightarrow 0} D_{3\beta\gamma}^{\text{static}} \quad (\text{B. 10})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta 3} = \frac{1}{2\pi r} r_{,\beta} = \lim_{r \rightarrow 0} D_{3\beta 3}^{\text{static}} \quad (\text{B. 11})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta\gamma} = \frac{\kappa\nu h\mu}{\pi(1-\nu)r} \left[(2r_{,\beta} r_{,\gamma} - \delta_{\gamma\beta}) r_{,n} - r_{,\gamma} n_{\beta} + r_{,\beta} n_{\gamma} \right] = \lim_{r \rightarrow 0} S_{3\beta\gamma}^{\text{static}} \quad (\text{B. 12})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta 3} = \frac{\kappa\nu h\mu}{\pi(1-\nu)r} \left[-(2r_{,\alpha} r_{,\beta} + \delta_{\alpha\beta}) r_{,n} + r_{,\alpha} n_{\beta} + r_{,\beta} n_{\alpha} \right] = \lim_{r \rightarrow 0} S_{\alpha\beta 3}^{\text{static}} \quad (\text{B. 13})$$

$$\begin{aligned} \lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta\gamma} &= \frac{h\mu}{\pi(1-\nu)r^2} \left\{ 2 \left[-4r_{,\alpha} r_{,\beta} r_{,\gamma} + \nu (\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) + (1-2\nu) \delta_{\alpha\beta} r_{,\gamma} \right] r_{,n} \right. \\ &\quad \left. + 2\nu (n_{\alpha} r_{,\beta} + n_{\beta} r_{,\alpha}) r_{,\gamma} + (1-2\nu) (\delta_{\alpha\gamma} n_{\beta} + \delta_{\beta\gamma} n_{\alpha} + 2n_{\gamma} r_{,\alpha} r_{,\beta}) - (1-4\nu) \delta_{\alpha\beta} n_{\gamma} \right\} \\ &= \lim_{r \rightarrow 0} S_{\alpha\beta\gamma}^{\text{static}} \end{aligned} \quad (\text{B. 14})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta 3} = \frac{h^3\mu}{3\pi r^2} (n_{\beta} - 2r_{,\beta} r_{,n}) = \lim_{r \rightarrow 0} S_{3\beta 3}^{\text{static}} \quad (\text{B. 15})$$

where U_{ij}^{static} , T_{ij}^{static} , $D_{k\beta j}^{static}$ and $S_{k\beta j}^{static}$ are static fundamental solutions for the Kane-Mindlin theory.

References

- [1] Sih GC. A review of the three-dimensional stress problem for a cracked plate. *International Journal of Fracture Mechanics* 1971;7(1): 39-61.
- [2] Nakamura T, Parks D. Antisymmetrical 3-D stress field near the crack front of a thin elastic plate. *International Journal of Solids and Structures* 1989;25(12): 1411-26.
- [3] Sih GC. *Plates and shells with cracks: a collection of stress intensity factor solutions for cracks in plates and shells*: Springer Science & Business Media; 2012.
- [4] Pook LP, Berto F, Campagnolo A, Lazzarin P. Coupled fracture mode of a cracked disc under anti-plane loading. *Engineering Fracture Mechanics* 2014;128: 22-36.
- [5] Kotousov A. Fracture in plates of finite thickness. *International Journal of Solids and Structures* 2007;44(25-26): 8259-73.
- [6] He Z, Kotousov A, Berto F. Effect of vertex singularities on stress intensities near plate free surfaces. *Fatigue & Fracture of Engineering Materials & Structures* 2015;38(7): 860-69.
- [7] Kane TR, Mindlin RD. High-frequency extensional vibrations of plates. *American Society of Mechanical Engineers -- Transactions -- Journal of Applied Mechanics* 1956;23(2): 277-83.
- [8] Yang W, Freund L. Transverse shear effects for through-cracks in an elastic plate. *International Journal of Solids and Structures* 1985;21(9): 977-94.
- [9] Zhi-He J, Noda N. A quasi-three-dimensional crack problem of an elastic plate subjected to in-plane shear loading. *Engineering fracture mechanics* 1994;49(5): 773-79.
- [10] Jin Z, Batra R. Dynamic fracture of a Kane-Mindlin plate. *Theoretical and Applied Fracture Mechanics* 1997;26(3): 199-209.
- [11] Li Z, Guo W, Kuang Z. Three-dimensional elastic stress fields near notches in finite thickness plates. *International Journal of Solids and Structures* 2000;37(51): 7617-32.
- [12] Kotousov A, Wang CH. Fundamental solutions for the generalised plane strain theory. *International Journal of Engineering Science* 2002;40(15): 1775-90.
- [13] Berto F, Lazzarin P, Wang CH. Three-dimensional linear elastic distributions of stress and strain energy density ahead of V-shaped notches in plates of arbitrary thickness. *International Journal of Fracture* 2004;127(3): 265-82.
- [14] Lazzarin P, Zappalorto M. A three - dimensional stress field solution for pointed and sharply radiused V - notches in plates of finite thickness. *Fatigue & Fracture of Engineering Materials & Structures* 2012;35(12): 1105-19.
- [15] Chang T, Guo W, Dong H. Three-dimensional effects for through-thickness cylindrical inclusions in an elastic plate. *The Journal of Strain Analysis for Engineering Design* 2001;36(3): 277-86.
- [16] Rooke D, Baratta F, Cartwright D. Simple methods of determining stress intensity factors. *Engineering Fracture Mechanics* 1981;14(2): 397-426.
- [17] Tweed J, Rooke D. The elastic problem for an infinite solid containing a circular hole with a pair of radial edge cracks of different lengths. *International journal of engineering science* 1976;14(10): 925-33.
- [18] Rooke D, Hutchins S. Stress intensity factors for cracks at loaded holes-effect of load distribution. *The Journal of Strain Analysis for Engineering Design* 1984;19(2): 81-96.
- [19] Rooke DP. An improved compounding method for calculating stress-intensity factors. *Engineering Fracture Mechanics* 1986;23(5): 783-92.
- [20] Rooke DP. The solution of integral equation in the determination of stress intensity factors. *Proc1stIntConf on Numerical methods in fracture mechanics*, edited by Luxmoore, AR and Owen,DRJ University College Swansea 1978: 105-14.
- [21] Wang Z, Yu T, Bui TQ, Tanaka S, Zhang C, Hirose S, et al. 3-D local mesh refinement XFEM with variable-node hexahedron elements for extraction of stress intensity factors of straight and curved planar cracks. *Computer Methods in Applied Mechanics and Engineering* 2017;313: 375-405.

- [22] Aliabadi M, Rooke D, Cartwright D. An improved boundary element formulation for calculating stress intensity factors: application to aerospace structures. *The Journal of Strain Analysis for Engineering Design* 1987;22(4): 203-07.
- [23] Wen P, Aliabadi M. Evaluation of mixed-mode stress intensity factors by the mesh-free Galerkin method: static and dynamic. *The Journal of Strain Analysis for Engineering Design* 2009;44(4): 273-86.
- [24] Chirino F, Dominguez J. Dynamic analysis of cracks using boundary element method. *Engineering Fracture Mechanics* 1989;34(5-6): 1051-61.
- [25] Wen P, Aliabadi MH, Rooke D. Cracks in three dimensions: A dynamic dual boundary element analysis. *Computer methods in applied mechanics and engineering* 1998;167(1): 139-51.
- [26] Li J, Khodaei ZS, Aliabadi M. Dynamic Dual Boundary Element Analyses for Cracked Mindlin Plates. *International Journal of Solids and Structures* 2018;152-153: 248-60.
- [27] Beskos DE. *Boundary element analysis of plates and shells*: Springer Science & Business Media; 2012.
- [28] Providakis CP, Beskos DE. Dynamic analysis of plates by boundary elements. *Applied Mechanics Reviews* 1999;52(7): 213-36.
- [29] Mi Y, Aliabadi MH. Dual boundary element method for three-dimensional fracture mechanics analysis. *Engineering Analysis with Boundary Elements* 1992;10(2): 161-71.
- [30] Dirgantara T, Aliabadi MH. Dual boundary element formulation for fracture mechanics analysis of shear deformable shells. *International Journal of Solids and Structures* 2001;38(44): 7769-800.
- [31] Li J, Khodaei ZS, Aliabadi MH. Modelling of the high-frequency fundamental symmetric Lamb wave using a new boundary element formulation. *International Journal of Mechanical Sciences* 2019;155: 235-47.
- [32] Durbin F. Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method. *The Computer Journal* 1974;17(4): 371-76.
- [33] Portela A, Aliabadi MH, Rooke D. The dual boundary element method: effective implementation for crack problems. *International Journal for Numerical Methods in Engineering* 1992;33(6): 1269-87.
- [34] Fedelinski P, Aliabadi MH, Rooke D. The Laplace transform DBEM for mixed-mode dynamic crack analysis. *Computers & structures* 1996;59(6): 1021-31.
- [35] Wang C-Y, Zhang C. 3-D and 2-D Dynamic Green's functions and time-domain BIEs for piezoelectric solids. *Engineering Analysis with Boundary Elements* 2005;29(5): 454-65.
- [36] Rojas-Díaz R, Sáez A, García-Sánchez F, Zhang C. Time-harmonic Green's functions for anisotropic magneto-electroelasticity. *International Journal of Solids and Structures* 2008;45(1): 144-58.
- [37] Telles J. A self - adaptive co - ordinate transformation for efficient numerical evaluation of general boundary element integrals. *International Journal for Numerical Methods in Engineering* 1987;24(5): 959-73.
- [38] Dirgantara T, Aliabadi MH. Crack growth analysis of plates loaded by bending and tension using dual boundary element method. *International journal of fracture* 2000;105(1): 27-47.
- [39] Fedelinski P, Aliabadi MH, Rooke D. A single-region time domain BEM for dynamic crack problems. *International Journal of Solids and Structures* 1995;32(24): 3555-71.
- [40] Aliabadi MH. *The Boundary Element Method: Applications in Solids and Structures*, Vol. 2. Chichester: Wiley 2002.
- [41] Bazant ZP, Estenssoro LF. Surface singularity and crack propagation. *International Journal of Solids and Structures* 1979;15(5): 405-26.
- [42] Nakamura T, Parks D. Three-dimensional stress field near the crack front of a thin elastic plate. *Journal of applied Mechanics* 1988;55(4): 805-13.
- [43] Benthem J. State of stress at the vertex of a quarter-infinite crack in a half-space. *International Journal of Solids and Structures* 1977;13(5): 479-92.
- [44] She C, Guo W. The out-of-plane constraint of mixed-mode cracks in thin elastic plates. *International journal of solids and structures* 2007;44(9): 3021-34.
- [45] Li J, Khodaei ZS, Aliabadi MH. Spectral BEM for the Analysis of Wave Propagation and Fracture Mechanics. *Journal of Multiscale Modelling* 2017;8(03n04): 1740007.