Semiclassical limit problems with concurrent use of several clocks in quantum cosmology

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We revisit a recent proposal for a definition of time in quantum cosmology, to investigate the effects of having more than one possible type of clock "at the same time." We use as a test tube an extension of Einstein gravity with a massless scalar field in which the gravitational coupling G_N is only a constant on-shell, mimicking the procedure for Λ in unimodular gravity. Hence we have two "simultaneous" clocks in the theory: a scalar field clock, and the conjugate of G_N. We find that attempts to use two coherent clocks concurrently are disastrous for recovering the classical limit. The Heisenberg relations, instead of being saturated, are always realized abundantly above their bound, with strong quantum effects expected at least in parts of the trajectory. Semiclassical states always result from situations where we effectively impose a single clock, either by making the other clock a failed clock (i.e., by choosing a state where its conjugate constant is infinitely sharp) or by choosing a basis of constants where all clocks but one are redundant; i.e., motion or change in phase space does not occur with the passing of their "times." We show how this conclusion generalizes to fluids with any equation of state. It also applies to systems where "subclocks" of the same type could be used, for example, in mixtures of different fluids with the same equation of state.

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I. INTRODUCTION

The idea that there might be more than one time strand running in parallel is either anathema (in physics) or self-evident (in human experience, in literature, etc.). Yet, the conflict between general relativity and quantum mechanics, in its implications for the meaning of time [1–6], has created a situation where the universe is either timeless or plagued by the appearance of several relational physical times. This is far from innocuous, e.g., for the discussion of whether there is a physical primordial singularity in our Universe [7,8].

In recent work [9] it was proposed that the constants of nature might provide excellent relational quantum clocks in the form of their conjugate momenta in theories where they are only constant on-shell. In this context the ubiquitous embarrassment of riches follows from the plethora of constants [10] that can act as progenitors of time. In [9,11] this was embraced, rather than regretted, following the view that different clocks should be used at different stages in the life of the Universe, with the adjustment of clocks across "time zones" to be seen as a feature of our Universe. "Minority" clocks always exist but should be abandoned to give way to the dominant clocks. Arguments related to unitary evolution and preservation of a semiclassical limit were provided in favor of this view.

In [11] we did not consider the possibility that several dominant clocks might concur in the same epoch. What would be the implications? In this paper we start to answer this question availing ourselves of a few simple toy models. Our main finding is that the more clocks we use concurrently, the more difficult it becomes to find a semiclassical limit. We examine the Heisenberg uncertainty bound, usually saturated by coherent states, and show how in all but one case balanced saturation of the bound becomes elusive once several dominant clocks are at play.

The plan of this paper is as follows. In Sec. II we start by presenting our simplest illustrative model, where the gravitational coupling G_N is deconstantized (following a procedure similar to unimodular gravity and the deconstantization of Λ). Thus, the momentum of G_N provides a dominant clock in addition to that supplied by the massless scalar field. We then subject the model to the treatment proposed in [9,11], identifying the "two-time" Schrödinger equation and the most general wave functions (Sec. III). These are superpositions of monochromatic partial waves in two frequencies and times, with amplitudes in the two "de-constants" (G_N and the momentum π_ϕ of the field).

The core of our paper is in Sec. IV, where we examine how concrete solutions interact with the Heisenberg uncertainty bound and its possible saturation. We can of course consider an infinitely sharp wave function in one constant (so that its conjugate clock becomes uniformly distributed,

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and so it is a failed clock), but this effectively reduces the two-clock system to a single one. Using two coherent states results in disaster, with a surplus uncertainty over the bound for each clock, which blows up in some regions of connection space. Any attempts to bypass this disaster always reduce the two-clock system to a single one (for example considering coherent states in the frequency conjugate to the wave number; see Sec. IV C for details). One clock is better than two.

The rest of the paper is spent generalizing this result and encountering it in different settings. In Sec. V, we consider scalar fields with nonquadratic kinetic terms (i.e., K-essence [12]), with equations of state w other than $w = 1$. The dependence on G_N is the same as in other perfect fluid models (e.g., [7,8,13]). In Sec. VI we consider a mixture of fluids with the same w, so that each can provide a dominating clock. The same difficulties in finding coherent states are found in all cases, with the exception of a multiradiation fluid, where the wave function miraculously factors into semiclassical states in each of the subclocks.

In two concluding sections we examine the relation between this paper and other approaches based on the WKB limit, and we digress on the implications of our result, in particular regarding the cosmological constant problem.

II. A SIMPLE MODEL

It is well known that a massless scalar field can provide an excellent relational clock, dual to a constant conjugate momentum (e.g., [7,8,14,15]). We apply the prescription in [9] to a toy model with Einstein gravity and a massless scalar field, in which the gravitational coupling to matter is deconstantized according to the prescription used for Λ in unimodular gravity [16–21]. That is, we take as base action

$$S_0 = \frac{1}{16\pi G_0}\int d^4x\sqrt{-g}(R - 8\pi G_N \partial_\mu\phi\partial^\mu\phi), \quad (1)$$

where the Planck length (determined by G_0) is kept fixed, but the gravitational coupling to matter, determined by G_N is subject to deconstantization according to

$$S_0 \to S = S_0 - \int d^4x\alpha\partial_\mu T^\mu = S_0 + \int d^4x T^\mu\partial_\mu\alpha, \quad (2)$$

where $\alpha_G = \alpha_G(G_N)$ is an arbitrary function of G_N at this stage (and the last identity is valid up to a boundary term which we shall ignore). Different choices of $\alpha_G(G_N)$ and associated T are related by a canonical transformation, leading to classically (but not quantum mechanically) equivalent theories.

We will examine this model in mini-superspace, so that

$$S_0 = \frac{3V_c}{8\pi G_0}\int dt\left[\dot{b}a^2 + \dot{\phi}\pi_\phi - Na\left(-(b^2+k) + \frac{\pi_\phi^2}{2\tilde{G}_N a^4}\right)\right],$$

where $k = 0, \pm 1$ is the normalized spatial curvature, a is the expansion factor, the connection variable b is the off-shell version of the Hubble parameter (since $b \approx \dot{a}$ if there is no torsion), the Lagrange multiplier N is the lapse function, and $V_c = \int d^3x$ is the comoving volume of the region under study (which could be the whole manifold, should this be compact). In addition,

$$\pi_\phi = \tilde{G}_N\dot{\phi}\frac{a^3}{N}, \quad (3)$$

$$\tilde{G}_N \equiv \frac{8\pi G_N}{3}. \quad (4)$$

This implies the Poisson brackets with normalization:

$$\{b, a^2\} = \{\phi, \pi_\phi\} = \frac{8\pi G_0}{3V_c}. \quad (5)$$

The Hamiltonian constraint is

$$H = Na\left(-(b^2+k) + \frac{\pi_\phi^2}{2a^4\tilde{G}_N}\right) = 0, \quad (6)$$

which can be placed in the form proposed in [9],

$$h(b)a^2 - \alpha_1\alpha_2 = 0, \quad (7)$$

with

$$h(b) = \sqrt{b^2+k}, \quad (8)$$

$$\alpha_1 = \frac{1}{\sqrt{2\tilde{G}_N}}, \quad (9)$$

$$\alpha_2 = \pi_\phi, \quad (10)$$

corresponding to a fluid with $w = 1$. The $\alpha(G_N)$ in (2) should therefore be proportional to α_1, with the proportionality constant adjusted so that in mini-superspace

$$S_0 \to S = S_0 + \frac{3V_c}{8\pi G_0}\int dt\dot{\alpha}_1 T_1, \quad (11)$$

leading to the new Poisson bracket:

$$\{\alpha_1, T_1\} = \frac{8\pi G_0}{3V_c}. \quad (12)$$

Hence we have a theory with two relational times, T_1 which is a conjugate of a function of G_N, and $T_2 = \phi$, conjugate to the on-shell constant π_ϕ.

III. QUANTIZATION AND SOLUTIONS

Quantization can now proceed just as in [11], with the Poisson brackets becoming commutators

$$[\hat{b}, \hat{a^2}] = [\hat{\alpha_1}, \hat{T_1}] = [\hat{\alpha_2}, \hat{T_2}] = i\mathfrak{h}, \tag{13}$$

where $\mathfrak{h} = l_P^2/3V_c$, with the reduced Planck l_P determined by the fixed G_0. In the representation diagonalizing b, T_1, and T_2 we therefore transform the Hamiltonian constraint, not in a timeless Wheeler-DeWitt equation, but in a double-time Schrödinger equation:

$$\left(-i\mathfrak{h}h(b)\frac{\partial}{\partial b} + \mathfrak{h}^2\frac{\partial^2}{\partial T_1 \partial T_2}\right)\psi = 0. \tag{14}$$

This is only a minor extension of the results in [11], the upshot of which is that the general wave function takes the form

$$\begin{aligned}\psi(b; T_1, T_2) &= \int d\alpha_1 d\alpha_2 A(\alpha_1, \alpha_2) e^{-\frac{i}{\mathfrak{h}}[\alpha_1 T_1 + \alpha_2 T_2]} \psi_s(b, \alpha_1, \alpha_2) \\ &= \int \frac{d\alpha_1 d\alpha_2}{2\pi\mathfrak{h}} A(\alpha_1, \alpha_2) e^{\frac{i}{\mathfrak{h}}[X(b)\alpha_1\alpha_2 - \alpha_1 T_1 - \alpha_2 T_2]},\end{aligned} \tag{15}$$

where the spatial ψ_s is a suitably normalized solution to the timeless Wheeler-DeWitt equation

$$\psi_s(b, \alpha_1, \alpha_2) = \frac{e^{\frac{i}{\mathfrak{h}}\alpha_1\alpha_2 X(b)}}{2\pi\mathfrak{h}} \tag{16}$$

and is a plane wave in the linearizing variable [22],

$$\begin{aligned}X(b) = \int \frac{db}{h(b)} &= \operatorname{arctanh}\frac{b}{\sqrt{b^2 + k}}, \qquad k > 0 \\ &= \log b, \qquad k = 0 \\ &= \operatorname{arctanh}\frac{b}{\sqrt{b^2 - |k|}}, \quad k < 0,\end{aligned} \tag{17}$$

associated with $w = 1$.

IV. A TWO-TIME HEISENBERG UNCERTAINTY PRINCIPLE

The Heisenberg uncertainty principle is a purely kinematical result involving any canonical pair, and as such we have for a two-time, two-constant setting

$$\sigma_{T_i}\sigma_{\alpha_i} \geq \frac{\mathfrak{h}}{2}. \tag{18}$$

The question, however, is how the dynamical solutions to the theory make these two sets of uncertainties interact: Can we saturate them simultaneously? Or do multiple clocks get in each other's way? Rather than providing a formal analysis we present a number of physically obvious solutions and examine how they relate to the Heisenberg uncertainty bounds.

A. One failed and one coherent clock

As a warm-up we consider the product of a coherent (semiclassical) state in one clock/constant and an infinitely sharp constant (i.e., a failed clock) in the other:

$$A(\alpha_1, \alpha_2) = \delta(\alpha_1 - \alpha_{10})\sqrt{\boldsymbol{N}(\alpha_{20}, \sigma)}. \tag{19}$$

Unsurprisingly, integrating (15) leads to the usual coherent state in the second variable, i.e., the product of a plane wave centered on $\boldsymbol{\alpha}_0$ and a Gaussian envelope in the coherent variable:

$$\begin{aligned}\psi(b, T_1, T_2) &= Ne^{-\frac{\sigma^2}{\mathfrak{h}^2}(\alpha_{10}X - T_2)^2 + \frac{i}{\mathfrak{h}}[\alpha_{20}\alpha_{10}X - \alpha_{10}T_1 - \alpha_{20}T_2]} \\ &= \psi(b, T_1, T_2; \boldsymbol{\alpha}_0)e^{-\frac{\sigma^2}{\mathfrak{h}^2}(\alpha_{10}X - T_2)^2}.\end{aligned} \tag{20}$$

The only novelty is that the fixed constant α_{10} (i.e., the failed clock) now acts as the inverse of the speed of propagation of the wave packet in the semiclassical time and space. This solution saturates, with equally spread uncertainties, the standard uncertainty principle in the variable with the coherent amplitude:

$$\sigma_{T_2}^2 = \sigma_{\alpha_2}^2 = \frac{\mathfrak{h}}{2}. \tag{21}$$

All of this is unsurprising because having an infinitely sharp constant is the same as not having deconstantized that constant all, so that its conjugate is a "failed" or uniformly distributed clock. Thus, we are left with a single clock, and so we recover the usual result [9].

We could of course have chosen either of the two clocks as the failed clock. Note that the classical equations of motion can be written either as

$$\alpha_1\dot{X} = T_2 \tag{22}$$

or

$$\alpha_2 \dot{X} = T_1 \tag{23}$$

so both options reproduce the classical limit. Indeed, the Hamilton equations for the system are

$$\dot{b} = \{b, H\} = -2N\frac{\alpha_1^2\alpha_2^2}{a^5}, \tag{24}$$

$$\dot{T}_1 = \{T_1, H\} = -2N\frac{\alpha_1\alpha_2^2}{a^3}, \tag{25}$$

$$\dot{T}_2 = \{T_2, H\} = -2N\frac{\alpha_1^2\alpha_2}{a^3}, \tag{26}$$

with the (Hamiltonian) constraint

$$b^2 + k = \frac{\alpha_1^2\alpha_2^2}{a^4}. \tag{27}$$

Since, from the definitions of X and $h(b)$,

$$\dot{X} = \frac{\dot{b}}{\sqrt{b^2+k}}, \tag{28}$$

we can easily prove that the motion of the peak of either coherent packet [i.e., Eqs. (22) or (23)] represents (24), once we use the fact that for a coherent state (25) or (26) has minimal uncertainty (beyond the obvious fact that they must be true on average, as per Ehrenfest's theorem [9]).

B. Two independent coherent clocks

But what if we attempt to use the two semiclassical clocks concurrently? Choosing a factorizable coherent state in the two constants,

$$A(\alpha_1, \alpha_2) = \sqrt{N(\alpha_{10}, \sigma_1)}\sqrt{N(\alpha_{20}, \sigma_2)} \tag{29}$$

[so that $\sigma^2(\alpha_i) = \sigma_i^2$], and using

$$\int d^n x e^{-\frac{1}{2}A_{ij}x^i x^j + B_i x^i} = \sqrt{\frac{(2\pi)^n}{\det(A)}} e^{\frac{1}{2}B^i (A^{-1})_{ij} B^j} \tag{30}$$

with

$$A_{ij} = \begin{bmatrix} \frac{1}{2\sigma_1^2} & \frac{-iX}{\hbar} \\ \frac{-iX}{\hbar} & \frac{1}{2\sigma_2^2} \end{bmatrix} \tag{31}$$

and

$$B_i = \frac{\alpha_{i0}}{2\sigma_i^2} - \frac{i}{\hbar}T_i \tag{32}$$

we can integrate (15) into

$$\begin{aligned}\psi(b, T_1, T_2) = &\sqrt{\frac{2\sigma_1\sigma_2}{\pi(\hbar^2 + 4\sigma_1^2\sigma_2^2X^2)}}\exp\left[-\sum_{i=1}^{2}\frac{\alpha_{i0}^2}{4\sigma_i^2}\right] \\ &\cdot \exp\left[\frac{1}{2\det(A)}\left(\frac{1}{2\sigma_2^2}\left(\frac{\alpha_{10}}{2\sigma_1^2} - \frac{iT_1}{\hbar}\right)^2 \right.\right. \\ &\left.\left. + \frac{1}{2\sigma_1^2}\left(\frac{\alpha_{20}}{2\sigma_2^2} - \frac{iT_2}{\hbar}\right)^2\right)\right] \\ &\cdot \exp\left[\frac{iX}{\hbar\det(A)}\left(\frac{\alpha_{10}}{2\sigma_1^2} - \frac{iT_1}{\hbar}\right)\left(\frac{\alpha_{20}}{2\sigma_2^2} - \frac{iT_2}{\hbar}\right)\right],\end{aligned} \tag{33}$$

with $\det(A) = \frac{1}{4\sigma_1^2\sigma_2^2} + \frac{X^2}{\hbar^2}$. This can be repackaged as

$$\begin{aligned}\psi(b, T_1, T_2) = &\sqrt{\frac{2\sigma_1\sigma_2}{\pi(\hbar^2 + 4\sigma_1^2\sigma_2^2X^2)}}\exp\left[-\frac{\sigma_1^2(T_1 - \alpha_{20}X)^2 + \sigma_2^2(T_2 - \alpha_{10}X)^2}{\hbar^2 + 4\sigma_1^2\sigma_2^2X^2}\right] \\ &\cdot \exp\left[\frac{i\hbar}{\hbar^2 + 4\sigma_1^2\sigma_2^2X^2}\left(\alpha_{10}\alpha_{20}X - \alpha_{10}T_1 - \alpha_{20}T_2 - \frac{4\sigma_1^2\sigma_2^2}{\hbar^2}T_1T_2X\right)\right]\end{aligned} \tag{34}$$

with the interesting result that although the wave function does not factor, the cross terms are all in its phase (suggesting interesting entanglement effects), so that the probabilities do factor.

Thus, we arrive at a modified saturated Heisenberg uncertainty identity:

$$\sigma_{Ti}^2\sigma_{\alpha i}^2 = \frac{\hbar^2}{4} + \sigma_{\alpha 1}^2\sigma_{\alpha 2}^2X^2. \tag{35}$$

It reduces to the usual one [i.e., Eq. (18)] should one of the $\sigma_{\alpha i}$ vanish (i.e., one of the clocks be a failed clock), as expected.[1] However, the situation is totally different when two clocks are at work: then there is an excess uncertainty in the uncertainty relations of both clocks/constants due to their simultaneous use, which only vanishes when $X = 0$.

[1]Note that a failed clock can be a coherent squeezed state: infinitely squeezed toward zero σ_α.

For $X \gg \hbar/(\sigma_1\sigma_2)$ the situation is very different. The peak of the distributions is still on the classical trajectory, but the probability becomes Gaussian in T_i/X, rather than in T_i and X. The limiting form of the probability distribution is now

$$P(b,T_1,T_2) \propto \exp\left[-\frac{\left(\frac{T_1}{X}-\alpha_{20}\right)^2}{2\sigma_2^2}\right]\exp\left[-\frac{\left(\frac{T_2}{X}-\alpha_{10}\right)^2}{2\sigma_1^2}\right].$$

The simplicity of this form cannot hide the fact that strong distortions from Gaussianity are now present in terms of the original measures X and T. It is as if in this limit the splitting into space and time has to be done differently. Whether this is general or a peculiarity of this system is not clear to us.

C. Other semiclassical states

There are other semiclassical states but they all end up reducing the clocking system to a single clock. A notable example is the theory resulting from absorbing G_N in a field redefinition,

$$\tilde{\phi} = \sqrt{G_N}\phi, \tag{36}$$

so that it looks as if the G_N disappears. Ignoring the issue of other fields, and of their possible interactions and self-interactions, it should be obvious that such an operation leads to a different quantum theory, the states of which are a subspace of the states we started from, the $A(G_N,\pi_\phi)$ forced to take the form $F(\pi_\phi/\sqrt{G_N})$, as we now explain in more detail.

Indeed, a field redefinition is an example of a general change of "constant" basis:

$$\boldsymbol{\alpha} \to \boldsymbol{\beta}(\boldsymbol{\alpha}) \tag{37}$$

in the prescription

$$S_0 \to S = S_0 + \int d^4x \boldsymbol{\alpha}\cdot\partial_\mu \boldsymbol{T}^\mu, \tag{38}$$

where the function $\boldsymbol{\beta}$ can be degenerate. All theories resulting from such changes, albeit classically equivalent, lead to different quantum theories. Their measures and inner products are different, and states which are coherent in one basis will not be coherent in the other. Their spatial solutions, ψ_s, however, are still all the same.

Hence we can choose a degenerate transformation reducing our two times to a single one, such that its conjugate matches the wave number of ψ_s [as in (16)], that is, choose a single β_1:

$$\beta_1 = \alpha_1\alpha_2 = \frac{\pi_\phi}{\sqrt{2\tilde{G}_N}}. \tag{39}$$

It is not difficult to see that the time or momentum conjugate to β_1 is nothing but the redefined field $\tilde{\phi}$. These are the natural constant and time variable of the system, since the general solutions then become

$$\psi(b,T_{\beta 1}) = \int d\beta_1 A(\beta_1) e^{\frac{i}{\hbar}\beta_1[X(b)-T_{\beta 1}]}. \tag{40}$$

The standard semiclassical results for one clock follow from the choice $A(\beta_1) = \sqrt{\boldsymbol{N}(\beta_{10},\sigma_1)}$. The classical limit is reproduced by $\dot{X}(b) = \dot{T}_{\beta 1}$ in the usual way.

In fact, we do not even need to make the transformation degenerate. We could choose any other independent β_2, and it would drop out of ψ_s. Then the second clock (conjugate to β_2) is a bad clock, not because it is failed (indeed, we could choose a coherent state for it), but because it becomes redundant. Its wave function does not engage with X. The passing of its time is not associated with any change in X, and the clock would simply factorize in

$$\psi(b,T_{\beta 1},T_{\beta 2}) = \int d\beta_1 d\beta_2 A(\beta_1,\beta_2) e^{\frac{i}{\hbar}\beta_1[X(b)-T_{\beta 1}]} \tag{41}$$

[assuming that $A(\beta_1,\beta_2)$ factorizes].

V. MORE GENERAL FLUIDS

The situation described here for a deconstantized G_N and a massless scalar field (for which $w = 1$) can be generalized to a perfect fluid with any w. This can be seen in two ways, which although not equivalent, leads to the same result.

For example, we can stay close to the scalar field description and model such a fluid by the K-essence model [12], in which we take a power of

$$X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{42}$$

as the kinetic term $K = K(X)$ in the action. Choosing

$$K = \frac{w}{1+w}X^{\frac{1+w}{2w}}, \tag{43}$$

we end up with the ϕ action in mini-superspace:

$$\begin{aligned} S &= \frac{3V_c}{8\pi G_0}\int dt a^3 \tilde{G}_N K \\ &= \frac{3V_c}{8\pi G_0}\int dt\left[\dot{\phi}\pi_\phi - Na\left(C_w\frac{\pi_\phi^{1+w}}{2\tilde{G}_N^w a^{1+3w}}\right)\right] \end{aligned}$$

with

$$\pi_\phi = \tilde{G}_N a^3\left(\frac{\dot{\phi}}{N}\right)^{\frac{1}{w}} \tag{44}$$

(where C_w is an irrelevant coefficient) instead of (3). We would therefore obtain the usual solutions (40), with $\beta_1 = m$ where

$$m = \frac{\pi_\phi^{1+w}}{G_N^w}. \tag{45}$$

The only case in which a G_N clock could not be used is that of a dust fluid, with $w = 0$.

We get the same result using the description employed in [7,8], itself taken from [13]. Then the general action [8]

$$S_{\text{isent}} = \frac{1}{G_0}\int d^4x\left[-G_N\sqrt{-g}\rho\left(\frac{|J|}{\sqrt{-g}}\right) + G_N J^\mu(\partial_\mu\theta + \beta_A\partial_\mu\alpha^A)\right], \tag{46}$$

where J^μ is the densitized particle number flux and θ, β_A, and α^A are Lagrange multipliers. ρ is the energy density of the fluid and is a function of $|J| = \sqrt{-g_{\mu\nu}J^\mu J^\nu}$ and $\sqrt{-g}$. In mini-superspace $J^\mu = a^3 n\delta^\mu_t$ where $n = n(t)$ is the particle number density [23]. Thus, the action (46) reduces to

$$S_{\text{isent}} = \frac{3V_c}{8\pi G_0}\int dt[\tilde{G}_N a^3 n\dot{\theta} - Na^3\rho(n)\tilde{G}_N] \tag{47}$$

implying the conserved variable

$$C = \tilde{G}_N a^3 n, \tag{48}$$

and we dropped the last term involving the other Lagrange multipliers. For a perfect fluid with $p = w\rho$ we have $\rho = \rho_0 n^{1+w}$, for some constant ρ_0. Hence, the Hamiltonian can be written

$$H_{pf} = a^3\rho\tilde{G}_N = \frac{\rho_0}{a^{3w}\tilde{G}_N^w} \tag{49}$$

so that defining

$$m = \rho_0 C^{1+w} \tag{50}$$

we finally have, for some χ,

$$S_{pf} = \frac{3V_c}{8\pi G_0}\int dt\left[\dot{m}\chi - Na\frac{m}{\tilde{G}_N^w a^{1+3w}}\right] \tag{51}$$

in agreement with (45).

One may object to both actions on different grounds, but it is interesting that for our purposes it does not make any difference. As we see the discussion for a scalar field then generalizes for any fluid, with

$$\alpha_1 \propto G_N^{-w/2} \tag{52}$$

and α_2 a suitable power of the fluid invariant variables, π_ϕ or m depending on the description.

VI. SUBCLOCKS AND MULTICOMPONENTS WITH THE SAME w

A final example is a universe filled with different components with the same w, so that they all dominate at the same epoch. In principle, each one of them could be used to produce a separate quantum clock (all of them proportional to each other on-shell). A simple illustration is provided by a set of N massless noninteracting scalar fields, ϕ_i, with conjugate momenta π_i (keeping $G_N = G_0$ fixed, for simplicity). The mini-superspace action has the form

$$S = \frac{3V_c}{8\pi G_0}\int dt\left[\dot{b}a^2 + \sum_i\dot{\phi}_i\pi_i - H\right],$$

where each ϕ_i can be seen as a relational clock, conjugate to "constant" π_i. The Hamiltonian constraint

$$H = Na\left(-(b^2+k) + \frac{1}{2a^4\tilde{G}_0}\sum_i\pi_i^2\right) = 0 \tag{53}$$

can be placed in the form $h(b)a^2 - \alpha = 0$ with the same $h(b)$ and $X(b)$ used before [cf. Eqs. (8) and (17)] and

$$\alpha = \sqrt{\sum_i\alpha_i^2} = \sqrt{\sum_i\frac{\pi_i^2}{2\tilde{G}_0}}. \tag{54}$$

Thus, the spatial wave function ψ_s will be the usual plane wave in $X(b)$ for scalar fields, but with a "wave number" α which does not match any of the frequencies α_i associated with each partial clock.

As a result, the most general wave function (complete with time/frequency factors) is the superposition

$$\psi(b, T_1, \ldots, T_N) = \int d\alpha_1\cdots d\alpha_N A(\alpha_1, \ldots, \alpha_N)\exp\left[i\frac{3V_c}{l_P^2}\left(X(b)\sqrt{\sum_i\alpha_i^2} - \sum_i\alpha_i T_i\right)\right], \tag{55}$$

and the choice

$$A(\alpha_1,\ldots,\alpha_N)=\prod_i \frac{1}{(\sqrt{2\pi}\sigma_i)^{1/2}}\exp\left[-\frac{(\alpha_i-\alpha_{i0})^2}{4\sigma_i^2}\right] \tag{56}$$

leads to a non-Gaussian state

$$\psi(b,T_1,\ldots,T_N)\propto\int d\alpha_1\cdots d\alpha_N \exp\left[B\sqrt{\sum_i \alpha_i^2}-\sum_i C_i\alpha_i^2+\sum_i D_i\alpha_i\right], \tag{57}$$

where the coefficients B, C_i, and D_i depend on α_{0i}, σ_i, T_i, and $X(b)$, with the problems we have already highlighted.

To produce a coherent state we proceed as in Sec. IV C and seek a canonical transformation into a time variable that has conjugate α. This could be obtained by writing the α_i in polar coordinates and absorbing the radial part of the Jacobian into a redefinition of the amplitude. Then, all the times associated with the angles can either be discarded (with a degenerate transformation) or, if still present, become irrelevant, since they factor out of the wave function, just as in Sec. IV C. The wave function (or its relevant part that is not a phase factor) becomes

$$\psi(b,T_\alpha)=\int d\alpha A(\alpha)e^{\frac{i}{\hbar}\alpha(X(b)-T_\alpha)} \tag{58}$$

and now a Gaussian $A(\alpha)$ does produce a coherent state. Using the standard expression for the transformation of the momenta for this canonical transformation [24]

$$\boldsymbol{T}^\mu_{\boldsymbol{\beta}}=\frac{\delta\boldsymbol{\alpha}}{\delta\boldsymbol{\beta}}\boldsymbol{T}^\mu_{\boldsymbol{\alpha}}, \tag{59}$$

we find that the clock conjugate to α is

$$\Phi=\sum_i\frac{\alpha_i}{\alpha}\phi_i. \tag{60}$$

The situation we have illustrated for scalar fields applies to any fluid for which the linearizing wave numbers α_i are not additive. This is the general rule with the exception of radiation. Indeed, as in [9]

$$-(b^2+k)+\frac{1}{a^{1+3w}}\sum_i m_i=0 \tag{61}$$

$$\Rightarrow(b^2+k)^{\frac{2}{1+3w}}a^2-\left(\sum\nolimits_i m_i\right)^{\frac{2}{1+3w}}=0. \tag{62}$$

Radiation is exceptional in that the general wave function does factorize into wave functions for the different components,

$$\psi(b,T_1,\ldots,T_N)=\int dm_1\ldots dm_N A(m_1,\ldots,m_N)\exp\left[\frac{i}{\hbar}\sum_i m_i(X_r(b)-T_i)\right], \tag{63}$$

so that a coherent state in each leads to

$$\psi(b,T_1,\ldots,T_N)=\prod_i(8\pi\sigma_i^2)^{1/4}\exp\left[\frac{i}{\hbar}(X_r-T_i)m_{i0}-\frac{\sigma_i^2}{\hbar^2}(X_r-T_i)^2\right], \tag{64}$$

where $X_r=\int\frac{db}{b^2+k}$. For any other w, *including the cosmological constant*, we have a problem using partial clocks, and we should use the full clock matching all the components appearing in the Hamiltonian.

We finally note that a problem inevitably occurs if not all components with the same w appearing in the Hamiltonian are clocks. Then, we can never find a frequency matching the spatial wave number and (unless the m are additive, as in the case of radiation) we have a problem.

VII. RELATION BETWEEN OUR COHERENT STATES AND THE WKB EXPANSION

In this paper we used the Heisenberg bound as a condition for classicality, as well as coherent states as a way to probe this bound. Clearly when a solution vastly exceeds this bound, one cannot have a classical state. But how does this method relate to the WKB criterium for classicality?

Historically, coherent states have rarely been used in standard quantum cosmology because, with the most obvious definition (a state which is an eigenstate of the annihilation operators for each pair of noncommuting variables), they do not solve the Wheeler-DeWitt equation.[2] For example, for the pair b and a^2, the eigenstates of $\hat{z} = \hat{b} + i\hat{a}^2$ (which are kinematically defined, and so do not have to satisfy the constraints) are not solutions of the Wheeler-DeWitt equation. The fact that such states are solutions for the unimodularlike pairs $\{\alpha, T_\alpha\}$ results from the fact that T_α does not appear anywhere in the dynamics.

Hence we use coherent states in this context because we can, and without prejudice to the good work on assessing quantum vs semiclassical behavior based on the WKB approximation (e.g., [27–29]) in the context of standard General Relativity (GR). But how does the WKB approach fare in the context of constant-time pairs? We examine the example in Sec. III, but what follows applies to any other unimodularlike situation. Let us consider the Schrödinger equation (14) and try out the WKB ansatz:

$$\psi = \mathcal{N} \exp\left(\frac{1}{\hbar}\sum_n \hbar^n S_n\right). \tag{65}$$

To order $n = 0$ we have

$$-i\frac{\partial S_0}{\partial X} + \frac{\partial S_0}{\partial T_1}\frac{\partial S_0}{\partial T_2} = 0 \tag{66}$$

so that

$$S_0 = i[X(b)\alpha_{10}\alpha_{20} - \alpha_{10}T_1 - \alpha_{20}T_2], \tag{67}$$

where $\alpha_1 = \alpha_{10}$ and $\alpha_2 = \alpha_{20}$ are fixed. Hence the monochromatic partial waves in (15) are WKB zeroth order solutions and these are exact. This is unusual. It is due to the use of the connection representation. Even for a pure Lambda we know that the real Chern-Simons state is an exact sine wave [30]; the Airy function solutions obtained in the metric representation are not, and hence the need of a WKB approximation [31]. From the point of view of the b representation, the approximation is in the Fourier transform into a^2 [32].

Now the problem is that the most general solution in a deconstantized setting (but not in standard GR) is a superposition of $n = 0$ solutions and such superpositions must be considered to obtain a normalizable solution. So, even though $S_n = 0$ for $n > 0$ provides an unexpected exact solution (and this is all there is in the standard GR, nondeconstantized setting), we should seek other solutions for $n \geq 1$. To order $n = 1$ we find

$$-i\frac{\partial S_1}{\partial X} + \frac{\partial S_0}{\partial T_1}\frac{\partial S_1}{\partial T_2} + \frac{\partial S_0}{\partial T_2}\frac{\partial S_1}{\partial T_1} + \frac{\partial^2 S_0}{\partial T_1 \partial T_2} = 0, \tag{68}$$

that is,

$$\frac{\partial S_1}{\partial X} + \alpha_{10}\frac{\partial S_1}{\partial T_2} + \alpha_{20}\frac{\partial S_1}{\partial T_1} = 0. \tag{69}$$

Besides the trivial $S_1 = 0$, we can have any solution of the form

$$S_1 = F_1(\alpha_{10}X - T_2)F_2(\alpha_{20}X - T_1), \tag{70}$$

where F_1 and F_2 are general functions. Nothing forces these functions to be quadratic, and if quadratic nothing is said about squeezing. The only condition asked is that the amplitudes dressing the plane wave be *any* function peaked on the classical trajectory [that is, Eqs. (22) and (23), assuming the T_i satisfies the classical equations of motion]. The WKB solution could be very nonclassical indeed.

This is to be contrasted with our exact solution (34), which is a special case of (70) and so a WKB state too. Again, with this choice we have that $S_n = 0$ for $n \geq 2$ in the WKB expansion. Our treatment is therefore more exact and more discerning regarding the quantum uncertainties of the system.

VIII. CONCLUSIONS

In this paper we examined the implications of using more than one cosmological clock in the same epoch, using a simple model based on a massless scalar field coupled to standard Einstein gravity, augmented by a "deconstantized" gravitational constant G_N mimicking the prescription used in unimodular gravity to deconstantize Lambda. This leads to two distinct clocks, one the scalar field itself and the other the conjugate momentum to G_N.

We then explored solutions to the double-time Schrödinger equation, with the view to drawing conclusions on the pros and cons of using multiple clocks simultaneously. Considering one constant to be infinitely sharp and the other a coherent state turned out to be trivial as this is equivalent to using just one clock (the other being a failed clock). Considering the product of two semiclassical states, instead, leads to a modified saturated Heisenberg uncertainty relation, dependent on the uncertainty of both constants and on the linearizing [22] spatial variable $X(b)$. Surprisingly, working with two clocks at the same time does not make our lives easier, since this increases the uncertainty associated with each clock working with a single clock is advisable. There are other well behaved semiclassical solutions, but they all reduce to a single effective clock, the remaining degrees of freedom corresponding to redundant clocks, the ticking of which does not correlate with changes in space.

[2]Less rigid definitions [25,26] do exist and have been used.

How general this might be is unclear, but we certainly were able to generalize our result in the most obvious directions. In particular, we explored a deconstantized G_N faced with matter with other equations of state. We also considered universes dominated by different fluids with the same equation of state, each supplying a clock. Again (with the notable exception of multiradiation fluids) a single collective clock is preferable. The implications for the cosmological constant problem are under active investigation [19,33]. The most obvious conclusion to draw would be the statement that quantum cosmology requires the existence of a total Lambda field, conjugate to a single total momentum, with obvious implications for the fine-tuning of the partial Lambdas.

One might wonder whether clock independence would not be achievable by "decoherence" of the clocks. But the analogy with decoherence is not quite applicable in the context of unimodularlike clocks, where classicality is achieved when the wave functions for the constants factorize and are suitably peaked around each "constant," but not so much so that the complementary clock/time cannot also be sharply peaked. The *relative* phase information between the two clocks is actually irrelevant for the matter. Rather than decoherence, nonentanglement is what must be achieved for classicality and independence. A nonfactorizable amplitude for the constants would lead to a very nonclassical state. The different clocks, however, still share the same "space" (the same b) which ultimately is why we obtained the results we did.

In closing, we note that the situation explored here differs from that in which different clocks are used at different epochs, "one at a time," as it were. As explained in [11], for the sake of a simple definition of unitarity and measure, one should in fact use different clocks at different epochs. What we seem to have shown here is that it is advisable to use one and only one clock in each epoch. There are of course situations where a clock seems handy throughout several epochs, for example the clock conjugate to G_N. But even then, we should only use it once, in one epoch. As the discussion in Sec. V shows, different powers of G_N would appear in the definition of coherent states in different epochs [cf. Eq. (52)]. The state of G_N could only be coherent in one of them.

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