

STOCHASTIC CONTROL PROBLEMS IN ENERGY MARKETS

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Stochastic control problems in energy markets

ABSTRACT

In this thesis we analyse stochastic control problems for the valuation of a natural gas storage facility and natural gas-fired power station while taking into account their operating characteristics. In the gas storage problem the underlying spot price dynamics is a mean-reverting process driven by a time-inhomogeneous Lévy process. For the power valuation both electricity and gas spot price processes exhibit mean-reverting spikes and Markov regime-switches. The Lévy regime-switching model incorporates the effects of demand-supply fluctuations in energy markets and abrupt economic disruptions or business cycles. We make use of skewed Lévy copulas to model the dependence risk of electricity and gas jumps. The corresponding HJB equations are solved by an explicit finite difference method. The numerical approach gives us both the values of the facilities and their optimal operating strategies depending on the gas and electricity prices, storage level, current temperature of the boiler and time. The surfaces of control strategies and contract values are obtained by implementing the numerical method for particular examples.

To my family.

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NATURE AND FINANCIAL MEANING OF OPTIMISATION IN ENERGY MARKETS

Securing reliable energy supply at a reasonable price while meeting environmental and climate protection targets is one of the most important objectives of the modern society. Oil, coal and natural gas are considered to be the main primary energy sources and cover 32%, 27%, and 22%, respectively, of global energy consumption (see Burger et al.²³). The share of nuclear and renewable energy is noticeably smaller. It is expected that the growing demand for energy will necessitate the increase in shares of gas and renewable energy. Note that, apart from direct consumption, primary energy sources can be used as a fuel to generate electricity.

After the liberalisation of energy markets in different regions all over the world in the 1990s, competitive markets for fossil fuels and electricity have emerged where they are traded in large volumes.

Energy commodities are also increasingly used as an important asset class

by fund managers to strengthen the portfolio²³. However, energy markets are associated with high price volatility. Therefore, it is necessary for the owners or leasers of physical assets to develop optimal control strategies to optimise their performance. In this thesis we consider two types of physical assets: natural gas storage facilities and gas-fired power plants.

The primary objective of natural gas storage facilities is to meet demand-supply fluctuations in natural gas markets. In other words, when severe weather or market related conditions contribute to sharp increase in demand for gas that cannot be covered by domestic production or imports, an energy company can release previously injected gas from the storage facility in order to make up the difference²⁷.

The use of underground gas storage facilities in the natural gas industry dates back to 1915 when the first gas storage project was completed in Ontario, Canada. The next year a similar project started near New York, US²⁷.

There are three basic types of natural storage fields: salt caverns, depleted gas/oil reservoirs, and aquifers. These storage facilities have different levels of cushion or base gas requirements. The cushion level is the amount of gas that has to be kept in the storage to provide the required pressure for gas release. Depleted oil/gas reservoirs are the most common underground gas storage facilities. They are underground rock formations that emerge naturally after the extraction of all accessible deep oil and gas from them. Unfortunately, 50% of the gas has to be maintained in the reservoir to avoid deliverability issues. The storage facilities of this type are usually used to meet seasonal demands by injecting gas during the summer season and releasing in the winter season. Aquifer is the least common reservoir. It is a permeable rock formation that functions as a natural water reservoir. The high cushion gas requirement (usually 50-80 %) makes aquifers less attractive for investments²⁷.

Salt Cavern facilities are obtained by mining out underground salt formations. The cushion gas requirement is relatively low (20-30%) and the remaining working gas can be recycled 10-12 times a year. Moreover, salt cavern facilities have high deliverability and injection capabilities. Due to all

these characteristics they are predominantly used to deal with short peak-day deliverability purposes such as fueling power generators²⁷. In this thesis we will only consider salt cavern storage facilities.

The valuation of natural gas storage has to be consistent with its uses that include supply-demand imbalance management, seasonal arbitrage and trading. The two main approaches are (intrinsic) seasonal valuation and (extrinsic) option-based valuation²⁷.

The seasonal or intrinsic value of a storage is the difference between two prices in a pair of forward prices. Although this strategy doesn't incorporate trading benefits, its financial and physical execution is quite straightforward.

The option based valuation makes use of financial derivatives pricing techniques. More precisely, it provides the storage manager the right, although not the obligation, to inject and withdraw at different times. This approach will be discussed in more detail in Chapter 3.

The second stochastic control problem investigated in this thesis is the operation of a gas-fired power station. The efficient and optimal operation and planning of power generating facilities have always been among the most challenging issues of the electric power industry. Traditionally, this problem was formulated as the minimisation of cost related to fossil-fuelled power systems. The recent climate and environmental issues made it necessary to include other objectives such as minimum carbon emissions and the use of renewable sources as much as possible⁷⁷. Furthermore, the deregulation of energy markets has introduced privately owned power generating companies in various developed countries. The economic objective for these private power producers is to maximise their profits by providing electricity to customers.

Power plant operation cannot be analysed without incorporating operational characteristics related to the electricity generation process. The fundamental concept here is a steam or boiler-turbine-generator unit. More precisely, a fuel is burned to heat a boiler. After some temperature level the boiler generates steam that drives a turbine. Finally, electricity is generated as the result of this drive. The detailed explanation of this process is beyond

the scope of this thesis and an interested reader can refer to Wood and Wollenberg⁷⁷ for more information.

The spot price dynamics of energy assets has several characteristics such as seasonality, mean-reversion, daily (for natural gas) or hourly (for electricity) instantaneous spikes etc. We can observe all of these features from Figure 1.

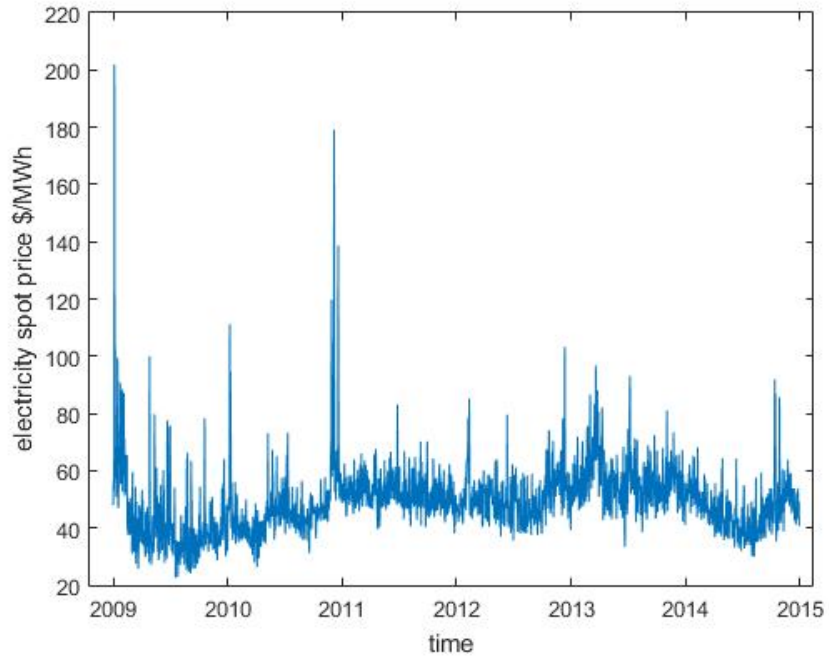
In this thesis we investigate the stochastic control problems for operation of natural gas and gas-fired power generating facilities under operational constraints. The optimisation framework is based on Thompson et al.⁷³ and Thompson et al.⁷⁴. The dynamics of underlying energy spot prices is mainly assumed to follow different Lévy driven mean-reverting processes. We adopt these models from Benth et al.¹³ and later advance them by incorporating features like Lévy copulas and regime switching. The introductory background on these concepts and their key properties is provided in Chapter 1. Note that each chapter of these thesis is self contained with its own detailed introduction and literature review. Therefore, we don't repeat that information here and move to the description of the thesis structure.

Chapter 1 gives all the necessary technical background which is later used for applications. We briefly cover the theory related to the Itô formula, Lévy processes, copulas, regime switching etc.

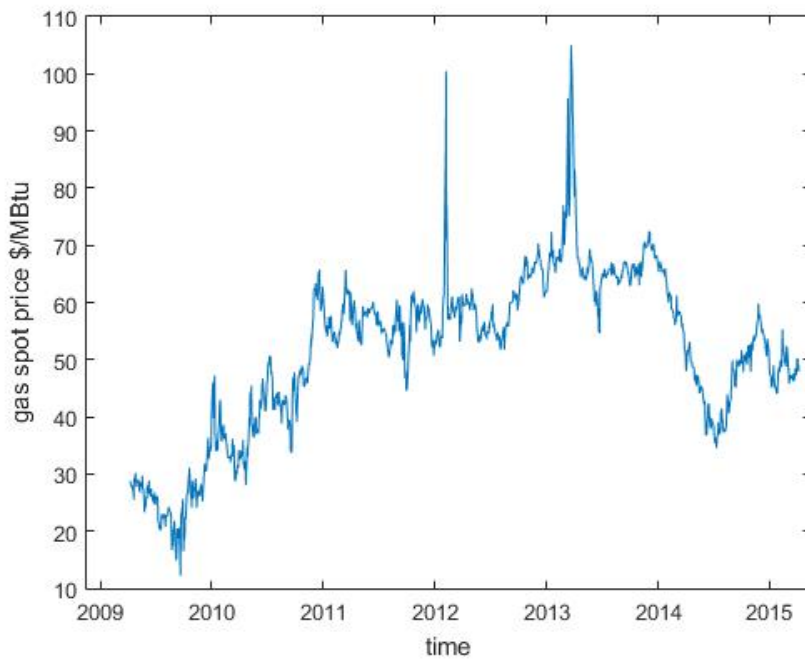
In Chapter 2 we analyse the optimisation and valuation of a natural gas storage facility under spot prices driven by time-inhomogeneous Lévy processes.

Chapter 3 presents a stochastic control problem for the operation of gas-fueled electricity generators. The underlying spot dynamics incorporates Lévy copulas and regime switching.

Note that in Chapters 2 and 3 we solve the resulting Hamilton-Jacobi-Bellman equations via finite difference methods.



(a) Electricity



(b) Natural gas

Figure 1: Day ahead UK spot prices for power and natural gas (NBP) during the time period 4 Sept 2009 - 4 July 2015 based on APX Power UK and Bloomberg.

1

TECHNICAL BACKGROUND

This chapter briefly introduces the mathematical concepts such as Itô calculus, Lévy process, copulas, optimal control, etc. Therefore, it is not directly related to the main application problems of this thesis. A reader who is familiar with the preliminaries mentioned above can skip this chapter.

1.1 BROWNIAN MOTION AND ITÔ FORMULA

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete (i.e. $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets) filtered probability space. A stochastic process $X = (X(t))_{t \geq 0}$ is a family of random variables defined on this probability space. $X(t)$ is called adapted if $X(t) \in \mathcal{F}_t$ ($\mathcal{F}_t$ measurable) for each t .

Brownian motion (BM) is probably the most important stochastic process. It is named after the botanist Robert Brown, who first observed this in 1827 while investigating the motion of a particle. Louis Bachelier introduced it into finance in 1900. BM is also called the Wiener process in honour of Norbert Wiener who proved that BM exists.

Definition 1 (Standard Brownian motion (see e.g. Karatzas and Shreve⁵⁰)).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space. A standard Brownian motion is a stochastic process $W = (W(t))_{t \geq 0}$ in $\mathbb{R}$ such that

- $W(0) = 0$ a.s.;
- $W(t)$ is $\mathcal{F}_t$ -adapted and a.s. continuous;
- for $t > s \geq 0$, the random variable $W(t) - W(s)$ is normally distributed with $\mathcal{N}(0, t - s)$ and is independent of $\mathcal{F}_s$.

For better understanding the nature of the paths of a Brownian motion and other stochastic processes we briefly introduce the concepts of (first order) variation and quadratic variation.

Definition 2 (see e.g. Shreve⁷⁰). Let $f(t)$ be a function defined for $t \in [0, T]$. Then the variation and quadratic variation of f up to time T are

$$V_T(f) = \lim_{\|\Pi\| \rightarrow 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

and

$$[f, f](T) = \lim_{\|\Pi\| \rightarrow 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^2$$

respectively, given that the limits above exist. Here $\Pi = (t_0, t_1, \dots, t_n)$ with $0 = t_0 < t_1 < \dots < t_n = T$ and $\|\Pi\| = \max_{i=1, \dots, n} |t_i - t_{i-1}|$.

Note that a Brownian motion is a process with the unbounded variation and the quadratic variation equal to $[W, W](T) = T$, almost surely. Informally, we can write it as

$$dW(t)dW(t) = dt$$

We make use of the following formula in order to differentiate functions like $f(t, W(t))$ where $W(t)$ and t are a standard BM and time, respectively (see Shreve⁷⁰ for more details).

Theorem 3 (Itô-Doeblin formula for Brownian motion). *Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous and let $W(t)$ be a standard BM. Then the differential form of the Itô-Doeblin formula for BM is*

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt \quad (1.1)$$

Itô-Doeblin formula can also be extended to more general processes.

Definition 4 (Itô process). *$X(t)$ is called an Itô process if it can be represented as*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \quad 0 \leq t \leq T \quad (1.2)$$

where the coefficients $\mu(t)$ and $\sigma(t)$ are stochastic processes such that

$$\mathbb{E} \int_0^T |\mu(t)|dt < \infty \quad \text{and} \quad \mathbb{E} \int_0^T \sigma^2(t)dt < \infty \quad (1.3)$$

We can now generalise Theorem 3 as follows (see Shreve⁷⁰).

Theorem 5 (Itô-Doeblin formula for an Itô process). *Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous, and let $X(t)$ be an Itô process defined by equation (1.2). Then the stochastic differential of $f(t, x)$ exists and is given by*

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))d[X, X](t) \quad (1.4)$$

Note that for the Itô process (1.2) we have $d[X, X](t) = \sigma^2(t)dt$. So, by substituting equation (1.2) in equation (1.4) and rearranging the terms we get

$$df(t, X(t)) = (f_t(t, X(t)) + \mu(t)f_x(t, X(t)) + \frac{1}{2}\sigma^2(t)f_{xx}(t, X(t)))dt + \sigma(t)f_x(t, X(t))dW(t) \quad (1.5)$$

Note that equations (1.1) and (1.5) can be derived by expanding the function $f(t, x)$ in its Taylor series, taking the limit $dt \rightarrow 0$ and using the properties of BM mentioned above.

1.2 RISK-NEUTRAL MEASURE

Before explaining the risk-neutral valuation we need to introduce the concept of martingales⁷⁰.

Definition 6. *An adapted stochastic process $X(t)$ is a martingale if for any $0 \leq s < t$*

$$\begin{aligned}\mathbb{E}[|X(t)|] &< \infty \\ \mathbb{E}[X(t)|\mathcal{F}_s] &= X(s) \quad a.s.\end{aligned}$$

Let $(W(t))_{t \in [0, T]}$ be a d -dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We also introduce a d -dimensional process $\lambda \in \mathbb{L}_{loc}^2$ and define the exponential martingale corresponding to λ as follows

$$Z^\lambda(t) = \exp \left(- \int_0^t \lambda(u) \cdot dW(u) - \frac{1}{2} \int_0^t |\lambda(u)|^2 du \right), \quad t \in [0, T] \quad (1.6)$$

where the symbol $\cdot$ stands for the scalar or dot product in $\mathbb{R}^d$ (see Pascucci⁶¹). Note that $Z^\lambda(t)$ is a strictly martingale if and only if $\mathbb{E}[Z^\lambda(T)] = 1$.

By using the martingale $Z^\lambda(t)$ from equation (1.6) we can define a new measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ in the following way:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z^\lambda(T) \quad (1.7)$$

(1.7) can also be written as

$$\mathbb{Q}(F) = \int_F Z^\lambda(T) d\mathbb{P}, \quad F \in \mathcal{F}$$

Here $Z^\lambda(T)$ is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Lemma 7 (see e.g. Shreve⁷⁰). *Assume that $Z^\lambda(t)$ in (1.6) is a $\mathbb{P}$ -martingale and $\mathbb{Q}$ is the probability measure defined in (1.7). Then a process $(M(t))_{t \in [0, T]}$ is a $\mathbb{Q}$ -martingale if and only if $M(t)Z^\lambda(t)$ is a $\mathbb{P}$ -martingale.*

Girsanov's theorem states that by modifying the considered probability measure and Brownian motion we can change the drift term of an Itô process, while keeping the diffusion coefficient unaltered (see e.g. Pascucci⁶¹).

Theorem 8. *Let $Z^\lambda(t)$ in (1.6) be the exponential martingale associated to the process $\lambda \in \mathbb{L}_{loc}^2$. We assume that Z^λ is a $\mathbb{P}$ -martingale and we consider the measure $\mathbb{Q}$ defined by (1.7). Then the process*

$$W^\mathbb{Q}(t) = W(t) + \int_0^t \lambda(s) ds, \quad t \in [0, T] \quad (1.8)$$

is a Brownian motion on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$.

The martingale property of $Z^\lambda(t)$ is the main assumption required for the validity of the Girsanov's theorem. For that purpose, it's usually assumed that the process $\lambda(t)$ is bounded. But the boundedness of $\lambda(t)$ isn't always easy to verify. In such cases the Novikov condition serves as a powerful tool to prove that $Z^\lambda(t)$ is a martingale (see Novikov⁵⁸).

Theorem 9 (Novikov condition). *If $\lambda(t) \in \mathbb{L}_{loc}^2$ is such that*

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T |\lambda(s)|^2 ds \right) \right] < \infty \quad (1.9)$$

then the exponential martingale $Z^\lambda(t)$ in (1.6) is a strict martingale.

Let's assume that the market consists of m risk assets

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dW_i(t), \quad i = 1, 2, \dots, m \quad (1.10)$$

and a risk-free asset B

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1 \quad (1.11)$$

where $\mu_i(t)$, $\sigma_i(t)$ and $r(t)$ are adapted processes. From equation (1.11) we easily obtain that

$$B(t) = e^{\int_0^t r(s)ds} \quad (1.12)$$

Definition 10 (see e.g. Shreve⁷⁰). *A probability measure $\mathbb{Q}$ is said to be risk-neutral if*

- $\mathbb{P}$ and $\mathbb{Q}$ are equivalent (for every $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ if and only if $\mathbb{P}(A) = 0$).
- the discounted price processes

$$\tilde{S}_i(t) = e^{\int_0^t r(s)ds} S_i(t), \quad t \in [0, T]$$

is a strict $\mathbb{Q}$ -martingale for every i . In other words, the following risk-neutral pricing formula holds for every i :

$$\tilde{S}_i(t) = \mathbb{E}[e^{\int_t^T r(s)ds} S_i(T) | \mathcal{F}_t], \quad t \in [0, T] \quad (1.13)$$

An Itô process is a martingale if and only if it has zero drift. Thus, application of Girsanov's theorem to the spot price dynamics (1.10) gives us the following drift condition:

$$\lambda_i(t) = \frac{\mu_i(t) - r(t)}{\sigma_i(t)}, \quad i = 1, 2, \dots, m \quad (1.14)$$

The process $\lambda_i(t)$ defined by equation (1.14) is called a market price of risk. We can infer that under the risk-neutral measure $\mathbb{Q}$ the underlying spot prices are given by

$$dS_i(t) = r(t)S_i(t)dt + \sigma_i(t)S_i(t)dW_i^{\mathbb{Q}}(t), \quad i = 1, 2, \dots, m \quad (1.15)$$

where $W^{\mathbb{Q}} = (W_1^{\mathbb{Q}}, W_2^{\mathbb{Q}}, \dots, W_m^{\mathbb{Q}})$ is a $\mathbb{Q}$ -Brownian motion.

An arbitrage is one of the most important concepts in asset pricing. It is the way of making risk-free money while starting with zero capital⁷⁰.

Definition 11. *An arbitrage is the portfolio value process $X(t)$ satisfying*

$$X(0) = 0 \text{ and } \mathbb{P}(X(T) \geq 0) = 1, \quad \mathbb{P}(X(T) > 0) > 0 \quad (1.16)$$

for some future time T .

The link between a risk-neutral measure and arbitrage is explained by the following theorem (see e.g. Shreve⁷⁰).

Theorem 12. *(First fundamental theorem of asset pricing) If a market model has a risk-neutral probability measure, then it does not admit arbitrage.*

If every derivative security can be replicated then the market model is complete. We refer to Shreve⁷⁰ for another key result in asset pricing.

Theorem 13. *(Second fundamental theorem of asset pricing) Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral measure is unique.*

1.3 POISSON PROCESSES

The key assumptions of the classical Black-Scholes model is that the underlying risky asset price S follows a geometric Brownian motion. This means that the asset price increments are log-normally distributed and the paths of spot prices are almost surely continuous. However, empirical evidence has shown that this model doesn't incorporate several important features of asset price dynamics. Due to demand-supply fluctuations stock price trajectories exhibit discontinuities in forms of jumps. Moreover, it is generally accepted in financial literature that the distribution of log-returns is non-Gaussian. Recall that for the normal distribution skewness and kurtosis are 0 and 3, respectively. The empirical analysis implies that large return fluctuations are more frequent than a normal distribution predicts. In other words, asset returns (in particular commodities) are usually associated with non-zero (mainly negative) skewness and excess kurtosis which can be explained by fat tails (see Pascucci⁶¹).

The Lévy processes have become a powerful tool in financial literature in order to incorporate sharp peak and heavy tail characteristics of statistical distributions. Before introducing the wide class of Lévy processes we will briefly go through the basic but fundamental concept of Poisson processes. See Cont and Tankov²⁸, Pascucci⁶¹ for further details and proofs.

Exponential random variables and their sums play an important role in defining the Poisson process.

A positive random variable Y is said to follow an exponential distribution with parameter $\lambda > 0$ if it has a probability density function of the form

$$\lambda e^{-\lambda y} 1_{y \geq 0}$$

An integer valued random variable N is said to follow a Poisson distribution with parameter λ if

$$\forall n \in \mathbb{N}, \quad \mathbb{P}(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

Definition 14 (The Poisson process). *Let $(\tau_i)_{i \geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^n \tau_i$. The process $(N(t))_{t \geq 0}$ defined by*

$$N(t) = \sum_{n \geq 1} 1_{t \geq T_n} \tag{1.17}$$

is called a Poisson process with intensity λ .

The Poisson process is therefore defined as a counting process: it counts the number of random times (T_n) which occur between 0 and t , where $(T_n - T_{n-1})_{n \geq 1}$ is an i.i.d. sequence of exponential variables. The following properties of the Poisson process can be easily deduced (see Cont and Tankov²⁸ for the full list of properties):

- For any $t > 0$, $N(t)$ follows a Poisson distribution with parameter λt .
- The increments of $N(t)$ are independent and stationary.

- The characteristic function of $N(t)$ is given by

$$\mathbb{E}[e^{iuN_t}] = \exp\{\lambda t(e^{iu} - 1)\}, \quad \forall u \in \mathbb{R}$$

Define the centered version of the Poisson process $N(t)$ by

$$\tilde{N}(t) = N(t) - \lambda t$$

One can verify that $\tilde{N}(t)$ satisfies the martingale property:

$$\forall t > s, \quad \mathbb{E}[\tilde{N}(t)|\mathcal{F}_s] = \tilde{N}(s)$$

$(\tilde{N}(t))_{t \geq 0}$ is called a *compensated* Poisson process and the deterministic expression $(\lambda t)_{t \geq 0}$ is called the compensator of $(N(t))_{t \geq 0}$: it is the quantity which has to be subtracted from N in order to obtain a martingale. In Definition 14 we defined the Poisson process $(N(t))_{t \geq 0}$ as a counting process. More precisely, $N(t)$ the number of jumps between 0 and t :

$$N(t) = \#\{i \geq 1 : T_i \in [0, t]\} \quad (1.18)$$

Based on this counting process we can define a measure M as follows:

$$M(\omega, A) = \#\{i \geq 1 : T_i(\omega) \in A\} \quad (1.19)$$

for any measurable set $A \subset \mathbb{R}^+$. ω stands for the outcome or path of a stochastic process. The measure $M(\omega, A)$ is a random measure because it depends on ω . Thus, M is called the random jump measure associated to the Poisson process N .

Poisson random measure is the extension of the measure M to more general settings. We refer to Kyprianou⁵¹ for more details.

Definition 15 (Compound Poisson process). *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $X(t)$ defined as*

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where jump sizes Y_i are i.i.d. with distribution f and $(N(t))$ is a Poisson process with intensity λ , independent from $(Y_i)_{i \geq 1}$.

Although the compound Poisson process X and Poisson process N jump at the same time, the jumps of X are of random size with distribution f , while the jump sizes of N are constant and equal to one.

Definition 16 (Compensated compound Poisson process). *Let X be a compound Poisson process with intensity λ and distribution of jumps f . The process*

$$\tilde{X}(t) = X(t) - \mathbb{E}[X(t)] = X(t) - m\lambda t$$

where

$$m = \int_{\mathbb{R}^d} x f(dx) = \mathbb{E}[Y_1]$$

is called compensated compound Poisson process.

As in the case of Poisson processes, a compensated compound Poisson process is a martingale with respect to the filtration generated by N and $(Y_i)_{i \geq 1}$, i.e.

$$\mathbb{E}[\tilde{X}(t) | \mathcal{F}_s] = \tilde{X}(s) \quad \text{a.s.}$$

1.4 LÉVY PROCESSES AND RANDOM JUMP MEASURES

Let $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be our filtered probability space, where $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions.

Definition 17. *A cadlag (right continuous with left limits) stochastic process $(X_t)_{t \geq 0}$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties:*

- *Independent increments: $X(t) - X(s)$ is independent of $\mathcal{F}_s$, $0 \leq s < t$;*

- *Stationary increments:* the law of $X(t+h) - X(t)$ doesn't depend on t ;
- *Stochastic continuity:* $\forall \epsilon > 0, \lim_{h \rightarrow 0} \mathbb{P}(|X(t+h) - X(t)| \geq \epsilon) = 0$.

Lévy driven financial models can be divided into two categories depending on the nature of jumps: finite activity and infinite activity models. If a Lévy process X has finitely many jumps in any bounded time interval, we call it a *finite activity model*, otherwise it has *infinite activity*. Finite activity models are also called *jump-diffusion models* and usually consist of a linear drift, a Brownian motion and a jump process. In these models, jumps occur rarely and the distribution of jump sizes is known. Meanwhile, infinite activity Lévy processes can have infinitely many jumps in a bounded time interval.

Derivative pricing under jump-diffusion models is mainly implemented by means of Fourier transform methods. Therefore, we denote the characteristic function of a Lévy process X as

$$\varphi_{X_t}(\xi) = \mathbb{E}[e^{i\xi X(t)}], \quad \xi \in \mathbb{R}^d, t \geq 0$$

The following theorem gives us a remarkable property of the characteristic function of a Lévy process (see Cont and Tankov²⁸).

Theorem 18. *If X is a Lévy process, then there exists a unique function ψ such that $\psi(0) = 0$ and*

$$\varphi_{X_t}(\xi) = e^{t\psi(\xi)}, \quad \xi \in \mathbb{R}^d, t \geq 0 \tag{1.20}$$

The function ψ is called the characteristic (or Lévy) exponent of X .

Therefore in order to specify the distribution of a Lévy process X , the main consequence of Theorem 18 is that to determine the distribution of $X(t)$ it is enough to determine it for a single time ($X(1)$).

The concept of Lévy processes is closely related to the property called the infinite divisibility of distributions (see Schoutens⁶⁸).

Definition 19. *A random value Y is said to be infinitely divisible if, for any*

$n \geq 2$, there exist i.i.d. random variables $Y_1^{(n)}, \dots, Y_n^{(n)}$ such that

$$Y = {}^{(d)} Y_1^{(n)} + \dots + Y_n^{(n)}$$

where $= {}^{(d)}$ means equality in distribution.

In other words, an infinitely divisible random variable can be decomposed into the sum of an arbitrary number of i.i.d. variables.

One can assign a random measure J_X to each cadlag process $(X(t))_{t \in [0, T]}$ defined in $\mathbb{R}^d$ in the following way.

$$J_X(\omega, \cdot) = \sum_{n \geq 1} \delta_{T_n, \Delta X_{T_n}} = \sum_{t \in [0, T]}^{\Delta X_t \neq 0} \delta_{t, \Delta X_t} \quad (1.21)$$

where δ is the Dirac measure such that $\delta_{t, \Delta X_t}(A) = 1$ if $\Delta X_t \in A$ and $\delta_{t, \Delta X_t}(A) = 0$ otherwise. J_X defined by (1.21) is called the jump measure of the process X . Intuitively, $J_X([0, t] \times A)$ can be assumed to be the number of jumps of X occurring between 0 and t whose amplitude belongs to A (see Cont and Tankov²⁸).

Proposition 20. *Let $(X(t))_{t \geq 0}$ be a compound Poisson process with intensity λ and jump size distribution f . Its jump measure J_X is a Poisson random measure on $\mathbb{R}^d \times [0, \infty[$ with intensity measure $\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$.*

Definition 21 (Lévy measure). *Let $(X_t)_{t \geq 0}$ and $\mathcal{B}(\mathbb{R}^d)$ be a Lévy process and a Borel σ -algebra on $\mathbb{R}^d$, respectively. The measure ν on $\mathbb{R}^d$ defined by:*

$$\nu(A) = \mathbb{E}[\#\{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}^d) \quad (1.22)$$

is called the Lévy measure of X : $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A .

In the case of a compound Poisson process with intensity λ and jump size distribution f , the Lévy measure $\nu(dx)$ becomes $\lambda f(dx)$. We can infer from

Proposition 20 that every compound Poisson process has the following representation:

$$X(t) = \sum_{s \in [0, t]} \Delta X_s = \int_{[0, t] \times \mathbb{R}^d} x J_X(ds \times dx) \quad (1.23)$$

where J_X is a Poisson random measure with intensity measure $\nu(dx)dt$. (1.23) is a special case of the Lévy-Itô decomposition. For the general representation we refer to the following proposition from Cont and Tankov²⁸.

Proposition 22 (Lévy-Itô decomposition). *Let $(X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$ and ν its Lévy measure, given by Definition 21. Then the following statements hold true:*

- *The Lévy measure ν verifies the integrability condition*

$$\int \min(|x|^2, 1) \nu(dx) < \infty \quad (1.24)$$

- *The jump measure of X , denoted by J_X , is a Poisson random measure on $[0, \infty[\times \mathbb{R}^d$ with intensity measure $\nu(dx)dt$.*
- *There exist a vector γ and a d -dimensional Brownian motion $(W(t))_{t \geq 0}$ with covariance matrix $\mathcal{C}$ such that*

$$X(t) = \gamma t + W(t) + X^l(t) + \lim_{\epsilon \downarrow 0} \tilde{X}^\epsilon(t) \quad (1.25)$$

where

$$\begin{aligned} X^l(t) &= \int_{|x| \geq 1, s \in [0, t]} x J_X(ds \times dx) \\ \tilde{X}^\epsilon(t) &= \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \tilde{J}_X(ds \times dx) \end{aligned}$$

Here $\tilde{J}_X(ds \times dx) = J_X(ds \times dx) - \nu(dx)ds$ is the compensated jump measure. The terms in equation (1.25) are independent and the convergence in the last term is almost sure and uniform in t on $[0, T]$.

The Lévy-Itô decomposition implies that every Lévy process is uniquely determined by the characteristic triplet $(\gamma, \mathcal{C}, \nu)$ where γ is the drift coefficient,

$\mathcal{C}$ is the covariance matrix of the Brownian motion and ν is the Lévy measure satisfying the condition (1.24).

By applying the Lévy-Itô decomposition we can now obtain the most general form for the characteristic exponent of a Lévy process in terms of its characteristic triplet $(\gamma, \mathcal{C}, \nu)$.

Theorem 23 (Lévy-Khinchin representation²⁸). *Let X be a Lévy process in $\mathbb{R}^d$ with characteristic triplet $(\gamma, \mathcal{C}, \nu)$. Then we have*

$$\varphi_{X_t}(\xi) = \mathbb{E}[e^{i\xi X_t}] = e^{t\psi_X(\xi)}, \quad \xi \in \mathbb{R}^d$$

where the characteristic exponent ψ_X is equal to

$$\psi_X(\xi) = -\frac{1}{2}\xi \cdot \mathcal{C}\xi + i\gamma \cdot \xi + \int_{\mathbb{R}^d} (e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{|x| \leq 1}) \nu(dx) \quad (1.26)$$

1.5 ITÔ FORMULA FOR LÉVY PROCESSES AND PIDE

Unlike martingales, especially a Brownian motion, the class of Lévy processes is not stable (i.e. doesn't preserve its defining characteristics) under stochastic integration or nonlinear transformations. The similar problem arises while working with Markov processes as well. Thus, we need to consider a larger class of stochastic processes, which is stable under the operations mentioned above and contains our objects of interest, i.e. Lévy processes. The class of semimartingales are widely used for this purpose. This class is not only stable under stochastic integration and nonlinear transformations, but it is also stable under operations such as change of measure or filtration (see Cont and Tankov²⁸).

Definition 24 (see Protter⁶⁴). *Let $(X(t))_{t \geq 0}$ be stochastic process defined on the filtered probability space $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. $X(t)$ is called a semimartingale if it can be decomposed as*

$$X(t) = M(t) + A(t) \quad (1.27)$$

where $M(t)$ is a local martingale and $A(t)$ is a càdlàg adapted process of

locally bounded variation.

Stochastic processes encountered in mathematical finance literature are almost always semimartingales. The examples include finite variation processes and square integrable martingales.

Before introducing the Itô formula for semimartingales we will go through some simple cases for better understanding.

It can be easily proved that for a deterministic function $X(t)$ with bounded variation the following holds true (see Pascucci⁶¹):

$$f(X(t)) - f(X(0)) = \int_0^t f'(X(s))dX(s) \quad (1.28)$$

where $f \in C^1(\mathbb{R})$. When X is a deterministic càdlàg function with the bounded variation the Itô formula (1.28) generalises to

$$\begin{aligned} f(X(t)) - f(X(0)) &= \int_0^t f'(X(s))dX(s) \\ &+ \sum_{s \in [0, t]} (f(X(s)) - f(X(s-)) - f'(X(s-))\Delta X(s)) \end{aligned} \quad (1.29)$$

where $X(s-)$ is the value of X just before the jump and $\Delta X(s) = X(s) - X(s-)$.

Before we introduce the Itô formula for semimartingales, the notion of quadratic variation from Definition 2 needs to be generalised. Based on Pascucci⁶¹ we can infer that if S is a semimartingale, then its quadratic variation $[S, S]$ is a càdlàg, adapted and increasing process defined by

$$[S, S](t) = S(t)^2 - \int_0^t S(s-)dS(s), \quad t \in [0, T] \quad (1.30)$$

such that $[S, S](0) = S(0)^2$ and $\Delta[S, S](t) = (\Delta S(t))^2$.

The properties of a semimartingale S mentioned above imply that

$$\sum_{s \in [0, t]} (\Delta S(s))^2 = \sum_{s \in [0, t]} \Delta[S, S](s) \leq [S, S](t)$$

We can then denote the continuous part of $[S, S]$ as

$$[S, S]^c(t) := [S, S](t) - S(0)^2 - \sum_{s \in [0, t]} (\Delta S(s))^2 \quad (1.31)$$

Note that for a Brownian motion $W(t)$ and Poisson process $N(t)$ we have $[W, W]^c(t) = t$ and $[N, N]^c(t) = 0$, respectively. If $X(t)$ is a one dimensional Lévy process with a characteristic triplet (γ, σ, ν) then

$$[X, X]^c(t) = \sigma^2 t \quad (1.32)$$

We can now extend the Itô formula for semimartingales as follows (see Pascucci⁶¹).

Theorem 25. *Let X be a semimartingale and $f = f(t, x) \in C^{1,2}(\mathbb{R}^2)$. Then $f(t, X(t))$ is a semimartingale and we have*

$$\begin{aligned} f(t, X(t)) - f(0, X(0)) &= \int_0^t f_s(s, X(s-)) ds + \int_0^t f_x(s, X(s-)) dX(s) \\ &\quad + \frac{1}{2} \int_0^t f_{xx}(s, X(s-)) d[X, X]^c(s) \\ &\quad + \sum_{s \in [0, t]} (f(s, X(s)) - f(s, X(s-)) - f_x(s, X(s-)) \Delta X(s)) \end{aligned} \quad (1.33)$$

or in differential form

$$\begin{aligned} df(t, X(t)) &= f_t(t, X(t-)) dt + f_x(t, X(t-)) dX(t) + \frac{1}{2} f_{xx}(t, X(t-)) d[X, X]^c(t) \\ &\quad + (f(t, X(t)) - f(t, X(t-)) - f_x(t, X(t-)) \Delta X(t)) \end{aligned} \quad (1.34)$$

The Itô formula for Lévy processes can be easily derived as a special case of semimartingales (see Pascucci⁶¹).

Theorem 26. *Let X be a Lévy process with the characteristic triplet (γ, σ, ν)*

and $f = f(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$. If f is bounded then we have the following:

$$\begin{aligned} df(t, X(t)) &= \mathcal{A}f(t, X(t-)) + f_t(t, X(t-))dt + \sigma f_x(t, X(t-))dW(t) \\ &\quad + \int_{\mathbb{R}} (f(t, y + X(t-)) - f(t, X(t-))) \tilde{J}_X(dt, dy) \end{aligned} \quad (1.35)$$

where the integro-differential operator $\mathcal{A}$ is called the characteristic operator of X and has the following form:

$$\begin{aligned} \mathcal{A}f(t, x) &= \gamma f_x(t, x) + \frac{\sigma^2}{2} f_{xx}(t, x) + \\ &\quad \int_{\mathbb{R}} (f(t, x + y) - f(t, x) - f_x(t, x)y \mathbb{1}_{|y| < 1}) \nu(dy) \end{aligned} \quad (1.36)$$

1.5.1 ITÔ FORMULA FOR SDEs WITH JUMPS

In most practical applications we actually need to apply the Itô formula to general SDEs with jumps rather than just Lévy processes. We refer to Applebaum¹ for more detailed analysis of this subsection.

Let $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be our filtered probability space where W is a standard BM and J is the jump measure with intensity (Lévy) measure ν . W and J_X are independent on this probability space. We will look at the SDEs of the following form:

$$\begin{aligned} dX(t) &= \bar{b}(t, X(t-))dt + \sigma(t, X(t-))dW(t) \\ &\quad + \int_{|y| < 1} \tilde{a}(s, X(s-), y) \tilde{J}_X(dt, dy) \\ &\quad + \int_{|y| \geq 1} \bar{a}(s, X(s-), y) J_X(dt, dy) \end{aligned} \quad (1.37)$$

where $X(0) = Z$ for some $Z \in \mathbb{R}$ and the coefficients $\bar{b}(t, x)$, $\sigma(t, x)$, $\tilde{a}(s, x, y)$, and $\bar{a}(s, x, y)$ are some locally bounded and measurable mappings. As an example we can consider a Lévy process $X(t)$ with the characteristic triplet

(γ, σ, ν) . From the Lévy-Itô formula we know that

$$dX(t) = \gamma dt + \sigma dW(t) + \int_{|y|<1} y \tilde{J}_X(dt, dy) + \int_{|y|\geq 1} y J_X(dt, dy) \quad (1.38)$$

By denoting

$$\bar{b}(t, x) = \gamma, \quad \sigma(t, x) = \sigma, \quad \tilde{a}(s, x, y) = \bar{a}(s, x, y) = y$$

we can see that equation (1.38) is the SDE of type (1.37).

We can rewrite the SDE (1.37) in the following form for convenience (see Applebaum¹).

$$dX(t) = b(t, X(t-))dt + \sigma(t, X(t-))dW(t) + \int_{\mathbb{R}} a(s, X(s-), y) \tilde{J}_X(dt, dy) \quad (1.39)$$

where $X(0) = Z$ and

$$\begin{aligned} b(t, x) &= \bar{b}(t, x) + \int_{|y|\geq 1} \bar{a}(s, x, y) \nu(dy), \\ a(t, x, y) &= \tilde{a}(t, x, y) \mathbb{1}_{\{|y|<1\}} + \bar{a}(t, x, y) \mathbb{1}_{\{|y|\geq 1\}} \end{aligned}$$

Note that $X(t)$ must satisfy certain Lipschitz and growth conditions presented in Applebaum¹, Pascucci⁶¹ in order to be the solution of the SDE (1.39). We can now extend the Itô formula to SDEs with jumps as follows.

Theorem 27 (Itô formula for SDEs with jumps). *Let X be a solution of the SDE (1.39) and f be a bounded function in $C^{1,2}([0, T] \times \mathbb{R})$. Then we have*

$$\begin{aligned} df(t, X(t)) &= \mathcal{A}f(t, X(t-)) + f_t(t, X(t-))dt + \sigma(t, X(t-))f_x(t, X(t-))dW(t) \\ &\quad + \int_{\mathbb{R}} (f(t, X(t-) + a(t, X(t-), y)) - f(t, X(t-))) \tilde{J}_X(dt, dy) \end{aligned} \quad (1.40)$$

where $\mathcal{A}$ is the integro-differential operator of the following form:

$$\begin{aligned} \mathcal{A}f(t, x) = & b(t, x)f_x(t, x) + \frac{\sigma^2(t, x)}{2}f_{xx}(t, x) + \\ & \int_{\mathbb{R}} (f(t, x + a(t, x, y)) - f(t, x) - a(t, x, y)f_x(t, x))\nu(dy) \end{aligned} \quad (1.41)$$

Itô formula (1.40)-(1.41) provides a unique connection between SDEs with jumps and partial integro-differential equations (PIDE). We will consider a SDE of the form (1.39) with integro-differential operator $\mathcal{A}$ as in (1.41). We denote by $(X^{t,x}(s))_{s \in [t, T]}$ the corresponding solution with initial condition $X^{t,x}(t) = x$. Before moving to the Feynman-Kač formula, we need to introduce the notion of a classical solution (see Pascucci⁶¹).

Definition 28. Let ϕ and r be bounded continuous functions, $\phi(x) \in C_b(\mathbb{R})$ and $r(t, x) \in C_b(\mathbb{R}^2)$. A classical solution of the (backward) Cauchy problem for $\mathcal{A} + \partial_t$ with final datum ϕ , is a bounded function $f \in C^{1,2}([0, T] \times \mathbb{R})$ such that

$$\mathcal{A}f(t, x) + f_t(t, x) = r(t, x)f(t, x), \quad (t, x) \in [0, T] \times \mathbb{R} \quad (1.42)$$

with

$$f(T, x) = \phi(x) \quad (1.43)$$

We can now state the following fundamental theorem without proof (see Cont and Tankov²⁸).

Theorem 29 (Feynman-Kač representation). *If a classical solution f to the problem (1.42)-(1.43) exists and is such that $f, f_x \in L^\infty([0, T] \times \mathbb{R})$ then it has the following stochastic representation*

$$f(x, t) = \mathbb{E} \left[e^{-\int_t^T r(s, X^{t,x}(s))ds} \phi(X^{t,x}(T)) \right] \quad (1.44)$$

1.6 LÉVY COPULAS

In practice, almost all financial applications are handled by means of multivariate models. While dealing with the dependence between components, asset price jumps must be taken into account as well. Multidimensional problems are predominantly analysed via Gaussian models. In the Gaussian framework the dependence can be easily characterised via correlation matrices. If the jumps are not Gaussian then their dependence should be modelled with copulas. Although this approach is efficient for simple models with a few jump sources, it fails for higher dimensional cases. The best way to cope with higher dimensional jump sources is the use of Lévy copulas (see Cont and Tankov²⁸). Before the introduction of Lévy copulas we will briefly go through the concept of copulas for completeness (see Nelsen⁵⁷ for more details).

1.6.1 THE COPULA OF A TWO-DIMENSIONAL DISTRIBUTION

The cumulative distribution function of a two-dimensional random vector (X, Y) is defined as

$$F(x, y) = \mathbb{P}(X \leq x, Y \leq y) \quad (1.45)$$

The separate distributions of X and Y are called the margins of (X, Y) , in other words

$$F_1(x) = \mathbb{P}(X \leq x) \quad \text{and} \quad F_2(y) = \mathbb{P}(Y \leq y)$$

These distribution functions can be easily derived from the two-dimensional distribution function $F(x, y)$ in the following way:

$$F_1(x) = F(x, \infty) \quad \text{and} \quad F_2(y) = F(\infty, y)$$

A two-place function $C(x, y)$ that characterizes the dependence structure of a two-dimensional random vector (X, Y) and does not depend on the margins is called the copula of (X, Y) . More precisely, the copula C + margins F_1 and F_2 provide an alternative description of the distribution a random vector

(X, Y) :

$$F(x, y) = C(F_1(x), F_2(y)) \quad (1.46)$$

If $F_1(x)$ and $F_2(y)$ are continuous then the copula is unique and can be computed from the distribution function and the margins (see Cont and Tankov²⁸):

$$C(x, y) = F(F_1^{-1}(x), F_2^{-1}(y))$$

In order to develop mathematical formulation of the intuitive explanation stated above, we need to introduce some preliminaries from Cont and Tankov²⁸.

Definition 30. Let S_1 and S_2 be two possible infinite closed intervals of $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ and consider a function $F : S_1 \times S_2 \rightarrow \bar{\mathbb{R}}$.

- F -volume of a rectangle $B = [x_1, x_2] \times [y_1, y_2] \subset S_1 \times S_2$ is defined by

$$V_F(B) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1).$$

- F is 2-increasing if for every rectangle B in its domain, $V_F(B) \geq 0$.
- F is grounded if for every $x \in S_1$, $F(x, \min S_2) = 0$ and for every $y \in S_2$, $F(\min S_1, y) = 0$.

Note that the functions $F_1(x) = F(x, \max S_2)$ and $F_2(y) = F(\max S_1, y)$ are called margins of F . From the probabilistic perspective the following definition implies that a two dimensional copula is nothing more than a distribution function on $[0, 1]^2$ with uniform margins (see Kallsen and Tankov⁴⁹).

Definition 31 (Copula). A two-dimensional copula is a function C with domain $[0, 1]^2$ such that

- C is grounded and 2-increasing.
- C has margins $C_k, k = 1, 2$, which satisfy $C_k(u) = u$ for all $u \in [0, 1]$.

The following Sklar theorem is of paramount importance in the analysis of copulas (see Cont and Tankov²⁸).

Theorem 32 (Sklar). *Let F be a two-dimensional distribution function with margins F_1 and F_2 . Then there exists a two-dimensional copula C such that for all $x \in \bar{\mathbb{R}}^2$,*

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \quad (1.47)$$

If F_1 and F_2 are continuous then C is unique, otherwise C is uniquely determined on $\text{Ran}F_1 \times \text{Ran}F_2$ (i.e. the cartesian product of the ranges of F_1 and F_2). Conversely, if C is a copula and F_1, F_2 are distribution functions, then the function F defined by equation (1.47) is a two-dimensional distribution function with margins F_1 and F_2 .

Definitions 30-31 and Theorem 32 can be extended to n -dimensional case to generalise the copula theory presented above. We refer to Cont and Tankov²⁸ for the details. We can now introduce the concept of Lévy copulas which is used in this thesis.

1.6.2 LÉVY COPULAS

The dependence analysis of Lévy processes via ordinary copulas isn't advisable as the infinite divisibility property will be lost under strictly increasing transformations (see Cont and Tankov²⁸). The key concept defining the dependence of Lévy processes is the dependence of jumps, that needs to be investigated via the Lévy measure.

Note that the Lévy measure plays the role of the probability measure for random variables while parametrizing the dependence between jumps of Lévy processes. Therefore, we are required to construct copulas for Lévy measures in order to model the dependence of jumps. As Lévy measures are not always finite, our Lévy copulas have to be defined on infinite intervals rather than on $[0, 1]^2$.

In this thesis we will only need positive Lévy copulas which model the dependence of processes with positive jumps. We refer to Cont and Tankov²⁸, Kallsen and Tankov⁴⁹ for general Lévy copulas defined on $[-\infty, \infty]^2$.

While dealing with Lévy copulas we will need tail integral rather than dis-

tribution of random variables.

Definition 33 (see e.g. Kallsen and Tankov⁴⁹). *A d -dimensional tail integral is a d -increasing function $U : [0, \infty]^d \rightarrow [0, \infty]$ such that*

- *U is equal to zero if at least one of its arguments is equal to ∞ .*
- *U is finite everywhere except at zero and $U(0, \dots, 0) = \infty$.*
- *$U(x_1, \dots, x_d) = \nu([x_1, \infty[\times \dots \times [x_d, \infty[)$, $x_1, \dots, x_d \in [0, \infty[^d \setminus \{0\}$*

The margins of a tail integral are very similar in structure to the margins of a distribution function and are defined as follows:

$$U_k(x_k) = U(0, \dots, 0, x_k, 0, \dots, 0)$$

For simplicity, we will only look at 2-dimensional case. In this research we will need Lévy copulas while working with a gas-fired power plant. So, we are only interested in investigating the dependence of two processes. An interested reader can refer to Kallsen and Tankov⁴⁹ for the higher dimensional cases.

Definition 34 (Positive Lévy copula). *A two-dimensional Lévy copula for Lévy processes with positive jumps is a 2-increasing and grounded function $F : [0, \infty]^2 \rightarrow [0, \infty]$ with uniform margins:*

$$F(x, \infty) = F(\infty, x) = x$$

By adapting the Sklar's theorem for tail integrals and Lévy copulas we get the following (see Cont and Tankov²⁸).

Theorem 35 (Sklar's Theorem for Lévy copulas). *Let $(X(t), Y(t))$ be a two-dimensional Lévy process with positive jumps that have tail integral $U(x, y)$ and marginal tail integrals $U_1(x)$ and $U_2(y)$. Then there exists a unique positive Lévy copula F such that:*

$$U(x, y) = F(U_1(x), U_2(y)), \quad \forall x, y \in [0, \infty). \quad (1.48)$$

Conversely, if F is a positive Lévy copula, $X(t)$ and $Y(t)$ are one-dimensional Lévy processes with tail integrals U_1, U_2 then there exists a two-dimensional Lévy process such that its tail integral is given by equation (1.48).

This theorem implies that Lévy copulas do the same job with tail integrals that copulas do with distribution functions. Namely, they link multidimensional tail integrals to their margins.

When both the Lévy copula and the one-dimensional tail integrals are sufficiently smooth, we can compute the Lévy density in the following way:

$$\nu(x_1, x_2) = \frac{\partial^2 F(y_1, y_2)}{\partial y_1 \partial y_2} \Big|_{y_1=U_1(x_1), y_2=U_2(x_2)} \nu_1(x_1) \nu_2(x_2)$$

An Archimedean Lévy copula is a popular parametric Lévy copula defined as

$$F(x, y) = \phi^{-1}(\phi(x) + \phi(y)) \quad (1.49)$$

where ϕ is a strictly decreasing convex function with positive support such that $\phi(0) = \infty$ and $\phi(\infty) = 0$. In these models F always have the symmetry property $F(x, y) = F(y, x)$. We will use the skewed version of this copula later in this thesis (i.e. $F(x, y) \neq F(y, x)$).

1.7 INDEPENDENT INCREMENT PROCESSES

In this section we provide the brief background in independent increment (II) processes that we are using to model the electricity and gas markets in this research. These processes are also called time-inhomogeneous Lévy processes or additive process in different literature. For more details and proofs the reader can refer to Sato⁶⁷, Jacod and Shiryaev⁴⁸, Ikeda and Watanabe⁴⁷, Benth et al.¹³.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space.

Definition 36 (Sato⁶⁷). *An adapted càdlàg stochastic process $L(t)$ starting in $L(0) = 0$ is an II process if it satisfies the following two conditions:*

- The increments $L(t_0), L(t_1) - L(t_0), \dots, L(t_n) - L(t_{n-1})$ are independent random variables for any partition $0 \leq t_0 < t_1 < \dots < t_n$, and $n \geq 1$.
- It is continuous in probability, that is, for every $t \geq 0$ and $\epsilon > 0$, we have

$$\lim_{s \rightarrow t} \mathbb{P}(|L(s) - L(t)| > \epsilon) = 0 \quad (1.50)$$

The first property represents the fundamental characteristic of an II process. If we also add that the increments are stationary (i.e. the distribution of $L(t) - L(s)$ depends only on $t - s$ for $s < t$), we will get a Lévy process as a special case.

We can use the following Proposition to calculate the characteristic function of an II process (see Benth et al.¹³).

Proposition 37. *The characteristic function of the II process $L(t)$ is*

$$\mathbb{E} \left[\exp \left(i \int_s^t \theta(u) dL(u) \right) \right] = \exp(\psi(s, t, \theta(\cdot))) \quad (1.51)$$

with

$$\begin{aligned} \psi(s, t, \theta(\cdot)) = & i \int_s^t \theta(u) d\gamma(u) - \frac{1}{2} \int_s^t \theta^2(u) dC(u) + \\ & \int_s^t \int_{\mathbb{R}} \{ e^{i\theta(u)z} - 1 - i\theta(u)z \mathbb{1}_{|z| \leq 1} \} l(dz, du) \end{aligned} \quad (1.52)$$

where $0 \leq s < t$ and $\theta(t)$ is a real-valued continuous function on $[0, T]$. ψ is called the cumulant function of the II process $L(t)$ and $(\gamma(t), C(t), l(dt, dz))$ is the spot characteristics or generating triplet with the following properties:

- $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with $\gamma(0) = 0$.
- $C : \mathbb{R} \rightarrow \mathbb{R}$, is non-decreasing and continuous, with $C(0) = 0$.
- l is a measure on the Borel σ -algebra of $[0, \infty) \times \mathbb{R}$, which satisfies

$$l(A \times \{0\}) = 0 \quad \text{and} \quad l(t \times \mathbb{R}) = 0$$

where $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}_+)$. Meanwhile

$$\int_s^t \int_{\mathbb{R}} \min(z^2, 1) l(dz, du) < \infty$$

If we can re-write the characteristic triplet $(\gamma(t), C(t), l(dt, dz))$ as

$$\gamma(t) = \gamma t, \quad C(t) = ct, \quad \text{and} \quad l(dt, dz) = \tilde{l}(z) dt$$

where γ, c are some constants then the II process $L(t)$ is the Lévy process with the characteristic triplet $(\gamma, c, \tilde{l}(dz))$.

II processes are closely related to the concept of semimartingales introduced in Section 1.5. Note that an II process is in general not a semimartingale. According to Jacod and Shiryaev⁴⁸ II process $L(t)$ can be decomposed into the sum of a deterministic càdlàg function $\tilde{A}$ and a semimartingale II process $\tilde{L}(t)$.

$$L(t) = \tilde{A}(t) + \tilde{L}(t) \tag{1.53}$$

$L(t)$ is a semimartingale if and only if $\tilde{A}(t) = 0$ and $\gamma(t)$ is a function of finite variation.

We can define a semimartingale in a more restrictive form than in Definition 24. More precisely, we will make use of the representation from Benth et al.¹³ which was obtained by adapting the one provided by Ikeda and Watanabe⁴⁷. Let $\mathbf{X}(t) = (X_1(t), \dots, X_d(t))$ be a vector of d semimartingales each having the following dynamics:

$$\begin{aligned} X_i(t) = & X_i(0) + A_i(t) + M_i(t) \\ & + \int_0^t \int_{\mathbb{R} \setminus \{0\}} a_{1,i}(s, y) \tilde{J}_i(ds, dy) + \int_0^t \int_{\mathbb{R} \setminus \{0\}} a_{2,i}(s, y) J_i(ds, dy) \end{aligned} \tag{1.54}$$

where $X_i(0)$ is a random variable in $\mathbb{R}$, $A_i(t)$ is an adapted continuous stochastic process with finite variation paths, and $M_i(t)$ is a continuous square integrable local martingale, $a_{1,i}$ and $a_{2,i}$ are random fields satisfying the con-

ditions

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} |a_{1,i}(s, y)|^2 l_i(ds, dy) \right] < \infty \quad (1.55)$$

and

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} |a_{2,i}(s, y)| J_i(ds, dy) < \infty \quad a.s. \quad (1.56)$$

respectively. We also assume the independence of the random jump measures J_i . Let $f(t, x)$ be a function on $C^{1,2}([0, T] \times \mathbb{R}^d)$. Then $f(t, \mathbf{X}(t))$ is again a semimartingale and satisfies the following Itô formula (see Benth et al.¹³).

$$\begin{aligned} f(t, \mathbf{X}(t)) &= f(0, \mathbf{X}(0)) + \int_0^t f_t(u, \mathbf{X}(u)) du \\ &+ \sum_{i=1}^d \int_0^t f_{x_i}(u, \mathbf{X}(u)) dA_i(u) + \sum_{i=1}^d \int_0^t f_{x_i}(u, \mathbf{X}(u)) dM_i(u) \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t f_{x_i x_j}(u, \mathbf{X}(u)) d[M_i, M_j](u) \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R} \setminus \{0\}} f(u, \mathbf{X}(u-) + a_{1,i}(s, y)) - f(u, \mathbf{X}(u-)) \tilde{J}_i(ds, dy) \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R} \setminus \{0\}} f(u, \mathbf{X}(u-) + a_{2,i}(s, y)) - f(u, \mathbf{X}(u-)) J_i(ds, dy) \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R} \setminus \{0\}} f(u, \mathbf{X}(u) + a_{1,i}(s, y)) - f(u, \mathbf{X}(u)) \\ &- a_{1,i}(s, y) f_{x_i}(u, \mathbf{X}(u)) l_i(ds, dy) \end{aligned} \quad (1.57)$$

1.8 REGIME SWITCHING

The failure of the Black-Scholes model to reflect the market randomness correctly (i.e. non log-normal), necessitates the development of more advanced approaches where the key parameters of the spot price dynamics are sensitive to market movements. The regime switching framework firstly introduced by Hamilton⁴⁴ provides such environment where the parameters of the stock price change depending on the market regime (or mode) that switches among

the finite number of states (see Yao et al.⁷⁸). These regimes can represent different economic factors such as economic cycle, investor mood or general economic situation. We refer to Buffington and Elliott^{21,22}, Elliott et al.³⁸ for the following toolbox on regime switching that will be used later in the thesis.

Let us assume that the states of an economy are modelled by a finite state continuous-time Markov chain $(X(t))_{t \in [0, T]}$ on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We follow Elliott et al.³⁸ by considering the state space of $X(t)$ to be the set of unit vectors of the form $\{e_1, \dots, e_N\}$ where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$.

Let $Q = [q_{i,j}]_{i,j=1,\dots,N}$ be the generator matrix of the Markov chain process $X(t)$. Then we can infer from Elliott et al.³⁸ that the following semimartingale representation for $X(t)$ holds true.

$$X(t) = X(0) + \int_0^t QX(u)du + M(t) \quad (1.58)$$

where $M(t)$ is a martingale with respect to the filtration generated by $X(t)$.

Our financial market consists of a bank account $B(t)$ and a risky asset $S(t)$. The instantaneous interest rate, drift and volatility are defined as

$$r(t) = \langle r, X(t) \rangle, \quad \mu(t) = \langle \mu, X(t) \rangle, \quad \text{and} \quad \sigma(t) = \langle \sigma, X(t) \rangle \quad (1.59)$$

respectively. Here $r = (r_1, \dots, r_N)'$, $\mu = (\mu_1, \dots, \mu_N)'$, $\sigma = (\sigma_1, \dots, \sigma_N)' \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^N$. Hence, we further assume that the underlying risky asset price dynamics of S is represented by the following Markov-modulated SDE:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(0) = s_0 \quad (1.60)$$

where $X(t)$ and $W(t)$ are independent.

Note that this market is no longer complete because of the regime switches. The measure transform for this problem is presented by Elliott et al.³⁷ via so called Esscher transform. So, under the new measure the dynamics of $S(t)$

becomes

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)d\tilde{W}(t), \quad S(0) = s \quad (1.61)$$

where

$$\tilde{W}(t) = W(t) - \int_0^t \frac{r(u) - \mu(u)}{\sigma(u)} du$$

is the standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. Following Buffington and Elliott²¹ we can obtain that, given $S(t) = s$ and $\mathbf{X}(t) = \mathbf{x}$ the option price is evaluated by the formula

$$V(t, s, \mathbf{x}) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^t r(u) du} V(S(T)) | S(t) = s, \mathbf{X}(t) = \mathbf{x} \right] \quad (1.62)$$

By denoting $V_i := V(t, s, e_i)$ for $i = 1, \dots, N$, we can write $\mathbf{V} = (V_1, \dots, V_N)$. In other words,

$$V(t, s, \mathbf{x}) = \langle \mathbf{V}, \mathbf{x}(t) \rangle$$

The following PDE is derived by Buffington and Elliott^{21,22} for the option price $V(t, s, \mathbf{x})$.

$$V_t + r(t)sV_s + \frac{1}{2}\sigma^2(t)s^2V_{ss} - r(t)V + \langle \mathbf{V}, Q\mathbf{x}(t) \rangle = 0 \quad (1.63)$$

with terminal condition

$$V(T, s, \mathbf{x}) = V(s)$$

It's clear that equation (1.63) is equivalent to the system of N equations for the cases $X = e_i$, where $i = 1, \dots, N$. Therefore, for each $X(t) = e_i$ we have

$$r(t) = r_i, \quad \mu(t) = \mu_i, \quad \sigma(t) = \sigma_i, \quad V(t, s, \mathbf{x}) = V_i$$

and V_i s satisfy the system of coupled PDEs (see Buffington and Elliott²¹):

$$V_t^i + r_i s V_s^i + \frac{1}{2} \sigma_i^2 s^2 V_{ss}^i - r_i V^i + \langle \mathbf{V}, Q e_i \rangle = 0 \quad (1.64)$$

with

$$V(T, s, e_i) = V(s), \quad \text{for } i = 1, \dots, N$$

1.9 STOCHASTIC CONTROL AND HJB EQUATIONS

Optimal control theory is a mathematical technique that is concerned with dynamic systems and their optimization over time. Any optimisation problem consists of the following components:

- State of the system,
- Control variable,
- Performance/cost criterion.

We start from the discrete deterministic case and evolve up to stochastic control problems step by step to make sure that the concept is fully understood before moving to applications. An interested reader can refer to Evans³⁹, Bertsekas¹⁵, Fleming and Rishel⁴¹ for more details regarding different optimal control strategies.

The basic discrete-time deterministic problem can be formulated by modifying the model introduced by Bertsekas¹⁵ that includes a random parameter as well. So, a discrete-time dynamic system is given as

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, 1, \dots, N-1 \quad (1.65)$$

where the state x_k and control u_k are elements of some spaces S and U , respectively. The function f_k represents a transition from x_k to x_{k+1} . Suppose our objective is to maximise total value. We could similarly look at the minimisation problem. The total value is defined as

$$\sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N) \quad (1.66)$$

where g_k is the value obtained at $k = 0, 1, \dots, N$. The value function at each k can be denoted as

$$V_k(x_k) = \max_{u_s} \left\{ \sum_{s=k}^{N-1} g_s(x_s, u_s) + g_N(x_N) \right\} \quad (1.67)$$

By using (1.67) we can introduce Bellman's optimality principle to maximise the total value (1.66).

$$V_k(x_k) = \max_{u_k} \{g_k(x_k, u_k) + V_{k+1}(f_k(x_k, u_k,))\} \quad (1.68)$$

Bellman's principle of optimality means that from any point on an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point (see Bellman^{7,8}).

Note that $V_N(x_N) = g_N(x_N)$. Therefore, if we start from $k = N - 1$ and proceed backwards to $k = 0$ on our equation (1.68) then the last step will give us $V_0(x_0)$ which is the maximum total value of our discrete problem. This technique is known as the dynamic programming (DP) algorithm. Under the conditions $x_k = x$ and $u_k = u$ equation (1.68) can be rewritten as

$$V_k(x) = \max_u \{g_k(x, u) + V_{k+1}(f_k(x, u,))\} \quad (1.69)$$

Equation (1.69) is called Bellman equation or optimality equation. We will make use of this result later on to derive the Hamilton Jacobi Bellman (HJB) equation.

We can now introduce another fundamentally important optimisation problem, namely deterministic continuous time optimal control.

We assume that our dynamical system is defined by the following ordinary differential equation (ODE):

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{c}(t)), & t > 0 \\ \mathbf{x}(0) = x_0, \end{cases} \quad (1.70)$$

where $\mathbf{c} : [0, \infty) \rightarrow A \subset \mathbb{R}^m$ is our control function and

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(x, a) = \begin{pmatrix} f_1(x, a) \\ \vdots \\ f_n(x, a) \end{pmatrix}, \quad \mathbf{c}(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_m(t) \end{pmatrix}$$

It's intuitive to have $[0, \infty)$ domain for $\mathbf{c}(t)$ as t represents time. Note that a control strategy $\mathbf{c}(t)$ is called bang-bang if $|c_i(t)| = 1$ for $i = 1, \dots, m$. This means that the control policy can only be -1 or 1.

Our objective is to find the optimal control $c(t)$ that maximises the value over the time period $[0, T]$ (see Evans³⁹):

$$V(x(0), 0) = \max_c \left\{ \int_0^T r(\mathbf{x}(t), \mathbf{c}(t)) dt + g(x(T)) \right\} \quad (1.71)$$

Here $r : \mathbb{R}^n \times A \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are given and we call them running and terminal payoff functions, respectively. The easiest way solve the optimal control problem (1.71) is the application of Bellman's optimality principle introduced earlier for the discrete case. For that we need to redefine the control problem (1.71)-(1.71) for the time period t to T as follows.

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), \mathbf{c}(s)), & t \leq s \leq T \\ \mathbf{x}(t) = x \end{cases} \quad (1.72)$$

and

$$V(x(t), t) = \max_c \left\{ \int_t^T r(\mathbf{x}(t), \mathbf{c}(t)) dt + g(x(T)) \right\} \quad (1.73)$$

Bellman's optimality principle implies that

$$\begin{aligned} V(x(t), t) &= \max_c \left\{ \int_t^{t+dt} r(\mathbf{x}(t), \mathbf{c}(t)) dt + \int_{t+dt}^T r(\mathbf{x}(t), \mathbf{c}(t)) dt + g(x(T)) \right\} \\ &= \max_c \left\{ \int_t^{t+dt} r(\mathbf{x}(t), \mathbf{c}(t)) dt \right\} + V(x(t+dt), t+dt) \\ &= \max_c \left\{ \int_t^{t+dt} r(\mathbf{x}(t), \mathbf{c}(t)) dt + V(x(t+dt), t+dt) \right\} \end{aligned} \quad (1.74)$$

We then need the Taylor expansion of $V(x(t+dt), t+dt)$:

$$V(x(t+dt), t+dt) = V(x(t), t) + V_t(x(t), t)dt + \nabla_x V(x(t), t) \cdot \dot{\mathbf{x}}(t)dt + o(dt) \quad (1.75)$$

where ∇_x is the gradient operator. If we substitute equation (1.9) in equation (1.74), cancel $V(x(t), t)$ from both sides, divide by dt , take the limit $dt \rightarrow 0$,

and use (1.72) and the fact that

$$\lim_{dt \rightarrow 0} \frac{1}{dt} \int_t^{t+dt} r(\mathbf{x}(t), \mathbf{c}(t)) dt \approx r(\mathbf{x}(t), \mathbf{c}(t))$$

we can obtain the following PDE

$$V_t(x, t) + \max_c \{ \nabla_x V(x, t) \cdot f(x, c) + r(x, c) \} = 0 \quad (1.76)$$

subject to the terminal condition

$$V(x, T) = g(x)$$

(1.76) is called a Hamilton-Jacobi-Bellman equation. Actually, the HJB equation is the continuous-time analog to the discrete deterministic DP algorithm introduced earlier by equation (1.69).

In most financial applications the state of the dynamic system is driven by a SDE. To investigate such problems we need the toolbox of the stochastic optimal control theory. We assume that the state process $X(t)$ follows controlled version of the one dimensional Itô process introduced in Section 1.1:

$$dX(s) = \mu(s, X(s), c(s))dt + \sigma(s, X(s), c(s))dW(s), \quad s \leq s \leq T \quad (1.77)$$

with $X(0) = x$. The objective can be written as

$$V(X(t), t) = \max_c \mathbb{E} \left[\int_t^T r(X(t), c(t))dt + g(x(T)) \right] \quad (1.78)$$

We are considering the expected value here because our state process $X(t, \omega)$ is stochastic (ω here represents random outcome/path). We can derive the HJB equation for the optimal value function V as well by applying Bellman's principle of optimality. But afterwards, we need to employ the Itô formula for $V(X(t+dt), t+dt)$. After carrying out the steps similar to the deterministic case and taking the expectation of our resulting formula we get the HJB

equation for the stochastic optimal control problem.

$$V_t(x, t) + \max_c \left\{ \frac{\sigma^2(x, c, t)}{2} V_{xx}(x, t) + \mu(x, c, t) V_x(x, t) + r(x, c) \right\} = 0 \quad (1.79)$$

with the terminal condition

$$V(x, T) = g(x)$$

This HJB equation can be generalised for multidimensional case as well (see Evans³⁹). In general, HJB equations don't have a classical solution. For the introduction of more general viscosity solutions an interested reader can refer to Fleming and Soner⁴², Pham⁶².

2

NATURAL GAS STORAGE VALUATION AND OPTIMISATION UNDER TIME-INHOMOGENEOUS EXPONENTIAL LÉVY PROCESSES

*

2.1 INTRODUCTION

The demand for natural gas has grown significantly in the recent years. The development of gas-fired electricity generation further contributes to increasing demand. Gas storage provides an efficient and reliable way to deal with the demand-supply fluctuations. Gas is injected into the storage during low demand periods and released when the demand is high. The variations in gas demand are associated with increased gas consumption for heating

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and gas-fired power generators during winter and summer months. Operating characteristics and random nature of natural gas spot prices are two main factors that need to be incorporated into the valuation and optimisation problems of a gas storage facility.

The main operating characteristics of natural gas storage facilities are: base and working gas capacities, deliverability, injection capacity, and cycling. The amount of permanent gas required to provide adequate reservoir pressure is the base gas, while working gas capacity is the amount of gas available for production and sale. The rate at which the reservoir releases gas from its reserves is deliverability and it is highest (lowest) when the reservoir is closest to its maximum capacity (nearly empty). The rate at which natural gas can be pumped into storage for later use is called injection capacity. The injection rate decreases as the reservoir approaches its maximum and reaches its highest value when the reservoir is empty. Cycling is the number of times that the working gas volumes can be pumped and released in a year. As single cycle storage facilities are not efficient, high deliverability multiple cycle (HDMC) facilities are used to meet both summer and winter demand.

Depending on their operating characteristics we can differentiate three types of natural gas storage facilities: depleted oil and gas reservoirs, salt cavern storage, and aquifer storage. All of these facilities have their own advantages and disadvantages which have been discussed earlier in our introductory chapter. In this research we will concentrate on salt cavern facilities as they are HDMC and require very low base gas. For more details regarding the other two types and their operating characteristics please refer to Thompson et al.⁷⁴.

The precise valuation and optimisation of a gas storage facility depends not only on its operating characteristics but also on the random dynamics of natural gas spot prices. In other words, investment and control decisions related to the gas storage have to be made under uncertainty of future spot prices. Real options theory can be applied to model this decision making process (see Dixit and Pindyck³⁴). The real options theory approaches investment decisions from the point of view of derivatives on financial assets. In the absence of operational constraints gas storage problem can be viewed as a series

of call and put options of different strikes. If we consider operating characteristics as well, then the storage facility becomes more complicated because of the opportunity costs. More precisely, when gas is released (injected) both the amount and the rate of future gas release (injection) are decreased. But the dependence between the opportunity costs and the amount of gas in the reservoir is nonlinear. Numerical valuation and control become reasonably complicated if we investigate these issues coupled with stochastic dynamics of natural gas prices. We will use numerical partial differential equations (PDE) technique for the reasons described in Thompson et al.⁷⁴.

In this research we are investigating the natural gas storage model presented in Thompson et al.⁷⁴. Our main contribution is the adaptation of this model to a spot price with more complicated dynamics, namely time-inhomogeneous exponential Levy process. This underlying incorporates common features of gas spot prices such as seasonality, mean-reversion and price spikes with seasonal intensities. We then extend the explicit finite difference method proposed by Cont and Tankov²⁸ to solve the resulting non-linear partial integral differential equation (PIDE). At the end, a special example for a storage facility has been numerically implemented.

2.2 LITERATURE REVIEW

In practice there are several approaches for gas storage valuation. The two most popular ones are the classical intrinsic valuation based on physical futures contracts and the extrinsic valuation, which uses spot trading strategies. Intrinsic valuation assumes that the demand for natural gas is seasonal, with peaks during winter and lows during summer. The storage strategy is determined by exploiting the predictable seasonal shape of the natural gas futures curve. In other words, the storage owner decides to buy/sell futures at the beginning of the storage contract and receives/delivers natural gas at the expiration of futures. The optimal futures positions is determined by solving a linear optimization problem. But the storage owner can't benefit from the futures curve movements as the initial optimal futures position is kept unchanged throughout the whole contract duration (see Eydeland and

Wolyniec⁴⁰). Gray and Khandelwal⁴³ extends this static approach to the rolling intrinsic in order to take advantage of the changing dynamics of the futures prices.

The fundamental idea behind these two futures-based approaches is to buy cheap summer futures and sell expensive winter futures. Therefore, the storage value depends on the seasonal spread between cold and warm periods of the year. However, the seasonal spread between winter and summer prices has declined in recent years due to the surge in shale gas supply and extensive use of natural gas in power generation. This fall diminishes intrinsic value of a gas storage contract and makes futures based valuation methods less reliable. These seasonality issues can be handled by using natural gas spot price based extrinsic valuation techniques.

Modelling of the underlying natural gas futures curve or spot price process has become an outstanding issue in the recent years. A mean-reverting process is a common model for the spot price. For example, Boogert and De Jong¹⁷ used the Least Square Monte Carlo method (initially proposed by Longstaff and Schwartz⁵³ for American options) for storage valuation, while assuming the following one factor model for the spot price, which is calibrated to the initial futures curve:

$$\frac{dS(t)}{S(t)} = \kappa(\mu(t) - \ln S(t))dt + \sigma dB(t) \quad (2.1)$$

where B is a standard Brownian motion, μ is a time-dependent parameter, calibrated to the initial futures curve $(F(0, T))_{T \geq 0}$, provided by the market; the mean reversion parameter κ and the volatility σ are positive constants. One factor models are popular because they simplify the theoretical and numerical work required to solve storage problems. As it is stated in Bjerksund et al.¹⁶, Henaff et al.⁴⁶, it's unrealistic to assume that one factor models can capture all characteristics of highly volatile natural gas prices. Parsons⁶⁰ extended (2.1) to the two-factor mean-reverting model where long-term mean is also mean-reverting process. Boogert and de Jong¹⁸ extended the application of the Least Square Monte Carlo method to multi-factor spot processes.

Another way to model the natural gas spot process is to consider it as the

limit $S(t) = \lim_{T \rightarrow t} F(t, T)$ (see Warin⁷⁶), where the futures curve is given by n-factor log-normal dynamics:

$$\frac{dF(t, T)}{F(t, T)} = \sum_{i=1}^n \sigma_i(t) e^{-a_i(T-t)} dB_t^i \quad (2.2)$$

By continuity of $F(t, T)$ we get $S(t) = F(t, t)$. The purpose of storage facilities is to mitigate seasonal demand and price fluctuations in the natural gas market by purchasing and injecting gas into storage during the low demand periods and releasing gas reserves during the high demand months. But the ability of gas storage to mitigate demand fluctuations is not perfect because of the increased use of gas as the fuel for power generation. These market imperfections can cause sudden price spikes far outside normal seasonal equilibrium levels. These spikes can be understood as the market signals for more investment in storage capacity and have to be included to our spot price models as they carry out significant arbitrage opportunities.

However, none of the models described above exhibits spiky behavior of natural gas spot prices. The first notion of jumps within gas storage valuation was proposed by Thompson et al.⁷⁴ where the risk adjusted price (P) is modelled by the following one dimensional continuous time Markov process:

$$dP = \mu_1(P, t)dt + \sigma_1(P, t)dX_1 + \sum_{k=1}^N \gamma_k(P, t, J_k)dq_k \quad (2.3)$$

where μ, σ , and the γ_k 's are any measurable functions and the J_k s are arbitrary jump sizes with distributions $Q_k(J)$. X_1 and dq_k 's denote the standard Brownian motion and Poisson processes respectively. (2.3) covers a wide range of spot price models for natural gas and is tractable for numerical calculations. However, its jump component doesn't incorporate instant and mean-reverting spikes of gas prices.

Benth et al.¹² proposes a more realistic alternative by defining energy spot dynamics as a sum of Ornstein-Uhlenbeck processes with different mean-reverting speeds. The only sources of randomness of the spot prices are positive pure jump processes. The additive structure of the model makes it

difficult to derive futures curves. Therefore, Benth⁹ considers a stochastic volatility model by Barndorff-Nielsen and Shephard⁶ for equity markets in the context of energy commodities. More precisely, we have the following non-Gaussian exponential model for the spot price $S(t)$:

$$S(t) = \Lambda(t) \exp(X(t)) \quad (2.4)$$

with

$$dX(t) = (\mu - \alpha X(t))dt + \sigma(t)dB(t) \quad (2.5)$$

where, μ and α are constants and Λ is a seasonality function. The Barndorff-Nielsen and Shephard stochastic volatility process $\sigma(t)$ is defined as

$$\sigma^2(t) = \sum_{j=1}^n \omega_j Y_j(t) \quad (2.6)$$

$$dY_j(t) = -\lambda_j Y_j(t)dt + dL_j(t) \quad (2.7)$$

where $\lambda_j > 0$ is constant and $\omega_j > 0$ with $\sum_{j=1}^n \omega_j = 1$. $L_j, j = 1, \dots, n$ are independent subordinator processes (increasing Lévy process). Benth et al.¹⁴ generalises this approach to a multivariate case to model the dynamics of cross-commodity spot prices.

The spiky behavior of energy prices can differ significantly between cold and warm months. Hence, time-inhomogeneous jump processes should be applied to capture these seasonality effects. Benth and Sgarra¹¹ describes the dynamics of electricity prices in the following way:

$$\ln S(t) = \ln \Lambda(t) + \sum_{i=1}^m X_i(t) + \sum_{j=1}^n Y_j(t) \quad (2.8)$$

where

$$dX_i(t) = (\mu_i(t) - \alpha_i(t)X_i(t))dt + \sum_{k=1}^p \sigma_{ik}(t)dB_k(t) \quad (2.9)$$

and

$$dY_j(t) = (\delta_j(t) - \beta_j(t)Y_j(t))dt + \eta_j(t)dL_j(t) \quad (2.10)$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Here B_k and L_j are independent standard Brownian motions and independent time-inhomogeneous Lévy processes, respectively. $\Lambda(t)$ is a deterministic, positive and continuously differentiable seasonality function, while the coefficients $\mu_i, \alpha_i, \delta_j, \beta_j, \eta_j$ and σ_{ik} are continuous functions of t .

In practice, gas storage contracts are valued and hedged via heuristic models due to their simplicity in dealing with high-dimensional and multi-factor price dynamics. However, derivation of the optimal injection and withdrawal strategy and inclusion of the physical constraints require the application of stochastic control framework. This is especially important for HDMC facilities that have far greater flexibility and therefore require more advanced strategies than conventional winter withdrawal and summer selling. Application of the dynamic programming theory translates the stochastic optimisation problem into the Hamilton-Jacobi-Bellman (HJB) equation (see Chapter 1). The resulting PDEs can be solved by means of standard numerical techniques.

Thompson et al.⁷⁴ developed a stochastic control model for natural gas storage facilities while assuming these spot prices follow (2.3). The HJB PIDE is then solved via an explicit finite difference method known as total variation diminishing scheme. Chen and Forsyth²⁵ implements fully implicit and Crank-Nicolson timestepping schemes based on a semi-Lagrangian approach and prove the convergence of fully implicit timestepping to the viscosity solution of the HJB equation in the presence of simple one factor spot price dynamics.

An alternative to the one factor spot price models is the regime-switching framework initially proposed by Hamilton⁴⁵. A regime-switching model consists of several regimes and the price process can randomly shift between these regimes due to demand-supply fluctuations, political instability, weather changes and other reasons. The gas price follows a distinct stochastic pro-

cess within each regime. Carmona and Ludkovski²⁴ and Chen and Forsyth²⁶ apply regime-switching approach to investigate stochastic control problems related to natural gas storage and implement the numerical calculation via Monte Carlo and finite difference methods, respectively.

2.3 NATURAL GAS SPOT PRICE

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be our complete filtered probability space. We assume that the natural gas price follows time-inhomogeneous exponential Lévy model defined by eqs. (2.7) to (2.10). For simplicity, we take $m = n = p = 1$ and

$$X(0) + Y(0) = \ln S(0) - \ln \Lambda(0)$$

Then we have the following dynamics for $S(t)$

$$S(t) = \Lambda(t) \exp(X(t) + Y(t)) \quad (2.11)$$

where

$$dX(t) = (\mu(t) - \alpha(t)X(t))dt + \sigma(t)dB(t) \quad (2.12)$$

and

$$dY(t) = (\delta(t) - \beta(t)Y(t))dt + \eta(t)dL(t) \quad (2.13)$$

We derive our optimal control model and the corresponding HJB equation assuming that $L(t)$ is a general time-inhomogeneous Lévy process (or independent increment (II) process). We will make further restrictions on $L(t)$ later in Section 2.5 in order to implement the numerical scheme.

The first fundamental theorem of asset pricing implies that the no-arbitrage requirement is equivalent to the existence of at least one equivalent probability measure which turns all the discounted tradable asset processes into local martingales. Meanwhile, according to the second fundamental theorem

of asset pricing the equivalent martingale measure is unique if the market is complete. The models introduced for energy spot prices exhibit incompleteness due to their jumps and storage problems of the commodities. Therefore, any measure $\mathbb{Q}$ equivalent to the objective probability $\mathbb{P}$ is risk-neutral, and the underlying asset process does not need to be a martingale with respect to the risk-neutral measure. Then we get the rather wide class of potential pricing measures and some restrictions must be imposed to perform calculations. Benth and Sgarra¹¹ suggests the use of the Esscher transform, which has been extensively applied in various pricing models related to Lévy processes. The reason why the Esscher transform is an interesting measure change is its structure preserving property. Although an II process loses its semimartingale characteristics after the transformation, the independence of the increments is still preserved.

We can define the Esscher transform for II processes based on Benth and Sgarra¹¹. Let $\theta(t) = (\hat{\theta}(t), \tilde{\theta}(t))$ be a two dimensional vector of real-valued continuous functions on $[0, T]$. Let's introduce the following stochastic exponential:

$$Z^\theta(t) = \hat{Z}^\theta(t) \times \tilde{Z}^\theta(t) \quad (2.14)$$

where

$$\hat{Z}^\theta(t) = \exp \left(\int_0^t \hat{\theta}(u) dB(u) - \frac{1}{2} \int_0^t \hat{\theta}^2(u) du \right) \quad (2.15)$$

and

$$\tilde{Z}^\theta(t) = \exp \left(\int_0^t \tilde{\theta}(u) dL(u) - \psi(0, t, \hat{\theta}(\cdot)) \right) \quad (2.16)$$

where ψ is the cumulant function defined by the Lévy-Kintchine representation for the II process $L(t)$ with spot characteristics $(\gamma(t), C(t), l(dt, dz))$:

$$\mathbb{E} \left[\exp \left(i \int_s^t \theta(u) dL(u) \right) \right] = \exp(\psi(s, t, \theta(\cdot))) \quad (2.17)$$

with

$$\begin{aligned} \psi(s, t, \theta(\cdot)) = & i \int_s^t \theta(u) d\gamma(u) - \frac{1}{2} \int_s^t \theta^2(u) dC(u) + \\ & \int_s^t \int_{\mathbb{R}} \{e^{i\theta(u)z} - 1 - i\theta(u)z \mathbb{1}_{|z| \leq 1}\} l(dz, du) \end{aligned} \quad (2.18)$$

Benth and Sgarra¹¹ imposes the following integrability condition on $L(t)$ in order to apply the Esscher transform:

Condition 38. *There exists a constant c such that*

$$\int_s^T \int_{\mathbb{R}} \{e^{cz} - 1\} l(dz, du) < \infty \quad (2.19)$$

If the condition

$$\sup_{0 \leq t \leq T} |\tilde{\theta}(t)| < c$$

holds together with (2.19), then the Itô formula implies that $\tilde{Z}^\theta(t)$ is a martingale process (a local martingale with $\mathbb{E}[\tilde{Z}^\theta(t)] = 1$). Meanwhile, if $\hat{\theta}(t)$ is continuous and the Novikov condition holds, then $\hat{Z}^\theta(t)$ is also a martingale. Therefore, an equivalent probability measure $\mathbb{Q}^\theta$ with the Radon-Nikodym derivative $d\mathbb{Q}^\theta/d\mathbb{P}$ can be defined as follows:

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} = Z^\theta(t) = \hat{Z}^\theta(t) \times \tilde{Z}^\theta(t) \quad (2.20)$$

According to Benth and Sgarra¹¹, the SDEs of $X(t)$ and $Y(t)$ under $\mathbb{Q}^\theta$ become:

$$dX(t) = (\mu(t) + \sigma(t)\hat{\theta}(t) - \alpha(t)X(t))dt + \sigma(t)dB^\theta(t) \quad (2.21)$$

$$dY(t) = (\delta(t) - \beta(t)Y(t))dt + \eta(t)d\Phi(t, \tilde{\theta}(t)) + \eta(t)d\tilde{L}(t) \quad (2.22)$$

where $\tilde{L}(t) = L(t) - \mathbb{E}_{\mathbb{Q}^\theta}[L(t)]$ is a $\mathbb{Q}^\theta$ -martingale II process and

$$\Phi(t, \tilde{\theta}(t)) = \frac{\partial}{\partial \theta} \phi(0, t, \tilde{\theta}(t)) = \mathbb{E}_{\mathbb{Q}^\theta}[L(t)]$$

with $\phi(s, t, \tilde{\theta}(\cdot)) = \psi(s, t, -i\tilde{\theta}(\cdot))$. We can imply from the definition of ψ (2.18) that

$$d\Phi(t, \tilde{\theta}(t)) = d\gamma(t) + \tilde{\theta}(t)dC(t) + \int_{\mathbb{R}} z\{e^{\tilde{\theta}(t)z} - \mathbb{1}_{|z|\leq 1}\}l(dz, dt) \quad (2.23)$$

If the function Φ is absolutely continuous with respect to the Lebesgue measure then there exists a function $\tilde{\Phi}$ such that

$$d\Phi(t, \tilde{\theta}(t)) = \tilde{\Phi}(t, \tilde{\theta}(t))dt$$

Then the dynamics of $Y(t)$ can be written as

$$dY(t) = (\delta(t) + \eta(t)\tilde{\Phi}(t, \tilde{\theta}(t)) - \beta(t)Y(t))dt + \int_{\mathbb{R}} \eta(t)z\tilde{J}(dz, dt) \quad (2.24)$$

We can derive the SDE for $S(t)$ by applying Itô's lemma to (2.11):

$$dS(t) = \Lambda'(t)\exp(X(t) + Y(t))dt + \Lambda(t)d\exp(X(t) + Y(t)) \quad (2.25)$$

where

$$\begin{aligned} \Lambda(t)d\exp(X(t) + Y(t)) &= S(t_-)dX(t) + S(t_-)dY(t) + \\ &\frac{1}{2}S(t_-)d[X, X]_t + \int_{\mathbb{R}} S(t_-)(e^{\eta(t)z} - 1 - \eta(t)z)J(dz, dt) = \\ &S(t_-)dX(t) + S(t_-)dY(t) + \frac{1}{2}S(t_-)d[X, X]_t + \\ &\int_{\mathbb{R}} S(t_-)(e^{\eta(t)z} - 1 - \eta(t)z)e^{\tilde{\theta}(t)z}l(dz, dt) + \\ &\int_{\mathbb{R}} S(t_-)(e^{\eta(t)z} - 1 - \eta(t)z)\tilde{J}(dz, dt) \end{aligned} \quad (2.26)$$

and

$$\Lambda'(t)\exp(X(t) + Y(t)) = \frac{\Lambda'(t)}{\Lambda(t)}S(t) \quad (2.27)$$

After substituting eqs. (2.21), (2.24), (2.26) and (2.27) in (2.25) we get

$$\begin{aligned}
\frac{dS(t)}{S(t_-)} &= \frac{\Lambda'(t)}{\Lambda(t)}dt + (\mu(t) + \sigma(t)\hat{\theta}(t) - \alpha(t)X(t))dt + \\
&\quad \sigma(t)dB^\theta(t) + \frac{1}{2}\sigma^2(t)dt + (\delta(t) - \beta(t)Y(t) + \eta(t)\tilde{\Phi}(t, \tilde{\theta}(t)))dt + \\
&\quad \int_{\mathbb{R}} \eta(t)z\tilde{J}(dt, dz) + \int_{\mathbb{R}} (e^{\eta(t)z} - 1 - \eta(t)z)e^{\tilde{\theta}(t)z}l(dz, dt) + \\
&\quad \int_{\mathbb{R}} (e^{\eta(t)z} - 1 - \eta(t)z)\tilde{J}(dz, dt) = \\
&\quad \bar{\chi}(t)dt - (\alpha(t)X(t) + \beta(t)Y(t))dt + \sigma(t)dB^\theta(t) + \\
&\quad \int_{\mathbb{R}} (e^{\eta(t)z} - 1)\tilde{J}(dz, dt) = \\
&\quad ((\beta(t) - \alpha(t))X(t) - \beta(t)(X(t) + Y(t)) + \bar{\psi}(t))dt + \\
&\quad \sigma(t)dB^\theta(t) + \int_{\mathbb{R}} (e^{\eta(t)z} - 1)\tilde{J}(dz, dt) \quad (2.28)
\end{aligned}$$

where

$$\begin{aligned}
\bar{\chi}(t) &= \frac{\Lambda'(t)}{\Lambda(t)} + \frac{1}{2}\sigma^2(t) + \mu(t) + \sigma(t)\hat{\theta}(t) + \delta(t) + \\
&\quad \eta(t)\tilde{\Phi}(t, \tilde{\theta}(t)) + \int_{\mathbb{R}} (e^{\eta(t)z} - 1 - \eta(t)z)e^{\tilde{\theta}(t)z}l(dz, dt) \quad (2.29)
\end{aligned}$$

We know from (2.11) that $X(t) + Y(t) = \ln S(t) - \ln \Lambda(t)$. Then

$$\begin{aligned}
\frac{dS(t)}{S(t_-)} &= (\bar{\chi}(t) + (\beta(t) - \alpha(t))X(t) - \beta(t)\ln S(t) + \beta(t)\ln \Lambda(t))dt + \\
&\quad \sigma(t)dB^\theta(t) + \int_{\mathbb{R}} (e^{\eta(t)z} - 1)\tilde{J}(dz, dt) \quad (2.30)
\end{aligned}$$

For simplicity, we can further denote

$$\bar{\beta}(t) = \beta(t) - \alpha(t), \quad \chi(t) = \frac{\bar{\chi}(t) + \beta(t)\ln \Lambda(t)}{\bar{\beta}(t)} \quad \text{and} \quad \xi(t) = \frac{\beta(t)}{\bar{\beta}(t)}$$

to get the following model with stochastic mean for the natural gas spot

price:

$$\frac{dS(t)}{S(t_-)} = \bar{\beta}(t)(\chi(t) + X(t) - \xi(t) \ln S(t))dt + \quad (2.31)$$

$$\sigma(t)dB^\theta(t) + \int_{\mathbb{R}} (e^{\eta(t)z} - 1)\tilde{J}(dz, dt)$$

$$dX(t) = (\mu(t) + \sigma(t)\hat{\theta}(t) - \alpha(t)X(t))dt + \sigma(t)dB^\theta(t) \quad (2.32)$$

2.4 CONTROL PROBLEM

Owners and operators of storage facilities do not have to be the owners of the gas they hold in storage. Most of the gas is held under lease with owners of the gas for some period of time defined by a contract. In this section we introduce the natural gas storage problem defined by Thompson et al.⁷⁴ and derive the PIDE for the valuation of gas storage contract. The following notation will be used throughout our calculations:

$S(t)$ - underlying spot price per unit of natural gas; T - contract expiry time; I - current amount of working gas inventory that varies within $[0, I_{max}]$; $V(S, I, t)$ - value of the natural gas storage facility at time t ; c - control variable representing the rate of producing gas from ($c > 0$) or injecting gas into ($c < 0$) the gas storage (no action is performed when $c = 0$); $c_{max}(I)$ - maximum deliverability rate of natural gas from the storage defined by

$$c_{max}(I) = k_1 \sqrt{I} \quad (2.33)$$

where k_1 is a positive constant, $c_{max}(I) > 0$ and $c_{max}(0) = 0$ (no gas can be produced if the gas storage is empty); $|c_{min}(I)|$ - maximum injection rate of natural gas into the storage given by

$$c_{min}(I) = -k_2 \sqrt{\frac{1}{I + k_3} - \frac{1}{k_4}} \quad (2.34)$$

where $c_{min}(I) < 0$ and k_2, k_3, k_4 are positive constants that satisfy $c_{min}(I_{max}) = 0$ (no gas injection when the storage is full); $a(I, c)$ - the rate that shows the amount of gas lost during release or injection while there are I units of gas

in storage.

Based on the notation given above, the change in I can be defined by the following deterministic equation:

$$dI = -(c + a(I, c))dt \quad (2.35)$$

The physical characteristics of the storage facility impose the following two conditions on the extreme values of I (see⁷⁴):

$$\begin{aligned} c + a(I, c) &\leq 0 \quad \text{for } I = 0, \\ c + a(I, c) &\geq 0 \quad \text{for } I = I_{max} \end{aligned} \quad (2.36)$$

According to these conditions gas can't be released (injected) when the reservoir is empty (full).

We need to find the optimal value $V(S, I, t)$ and optimal operating strategy $c(S, I, t)$ of the gas storage facility via the equation proposed in Thompson et al.⁷⁴:

$$V(S, I, t) = \max_{c(S, I, t)} \mathbb{E}_{\mathbb{Q}^\theta} \left[\int_t^T e^{-\rho(\tau-t)} (c - a(I, c)) S d\tau \right] \quad (2.37)$$

subject to

$$c_{min}(I) \leq c \leq c_{max}(I) \quad (2.38)$$

Rewriting (2.37) in the following way will lead to the Bellman equation:

$$\begin{aligned} V &= \max_c \mathbb{E}_{\mathbb{Q}^\theta} \left[\int_t^{t+dt} e^{-\rho(\tau-t)} (c - a(I, c)) S d\tau + \int_{t+dt}^T e^{-\rho(\tau-t)} (c - a(I, c)) S d\tau \right] \\ &= \max_c \mathbb{E}_{\mathbb{Q}^\theta} \left[\int_t^{t+dt} e^{-\rho(\tau-t)} (c - a(I, c)) S d\tau + e^{-\rho dt} \int_{t+dt}^T e^{-\rho(\tau-(t+dt))} (c - a(I, c)) S d\tau \right] \\ &= \max_c \mathbb{E}_{\mathbb{Q}^\theta} \left[\int_t^{t+dt} e^{-\rho(\tau-t)} (c - a(I, c)) S d\tau + e^{-\rho dt} V(S + dS, I + dI, t + dt) \right] \end{aligned} \quad (2.39)$$

In our case, $\log S(t)$ has a long-term stochastic mean $X(t)$. Although $X(t)$ isn't directly observable, the parameters of its SDE (2.13) can be estimated by applying Kalman filtering⁴ or particle filtering⁷¹ on gas spot price data.

Thus, we can assume that a storage manager has information about $X(t)$. We apply multi-dimensional Itô's formula to $V(S, X, I, t)$ to expand it in Taylor's series:

$$\begin{aligned} V = \max_c \mathbb{E}_{\mathbb{Q}^\theta} [(c - a(I, c))Sdt + (1 - \rho dt)[V + V_t dt - (c + a(I, c))V_I dt + \\ V_S dS + V_X dX + \frac{1}{2}V_{SS}d[S, S]_t + \frac{1}{2}V_{XX}d[X, X]_t + V_{SX}d[S, X]_t + \\ \int_{\mathbb{R}} (V(Se^{\eta(t)z}, X, I, t) - V(S, X, I, t) - S(e^{\eta(t)z} - 1)V_S)J(dz, dt)] \quad (2.40) \end{aligned}$$

where

$$d[S, S]_t = \sigma^2(t)S_t^2 dt, \quad d[X, X]_t = \sigma^2(t)dt, \quad d[S, X]_t = \sigma^2(t)S_t dt \quad (2.41)$$

If we substitute eqs. (2.31), (2.32) and (2.41) in (2.40), eliminate all terms that go to zero faster than dt , carry out some simplifications and take expectations and divide by dt then we obtain

$$\begin{aligned} \max_c [V_t + \frac{1}{2}\sigma^2(t)S^2V_{SS} + \frac{1}{2}\sigma^2(t)V_{XX} + \sigma^2(t)SV_{SX} + \bar{\beta}(t)(\chi(t) + X - \\ \xi(t) \ln S)SV_S + (\mu(t) + \sigma(t)\hat{\theta}(t) - \alpha(t)X)V_X - \rho V + \\ \lambda(t) \int_{\mathbb{R}} (V(Se^{\eta(t)z}, X, I, t) - V(S, X, I, t) - S(e^{\eta(t)z} - 1)V_S)e^{\bar{\theta}(t)z}\bar{l}(dz) + \\ (c - a(I, c))S - (c + a(I, c))V_I] = 0 \quad (2.42) \end{aligned}$$

where $l(dz, dt) = \lambda(t)\bar{l}(z)dt$.

Actually, only two terms in (2.42) contain the control variable c . So, our maximisation reduces to

$$\max_c [(c - a(I, c))S - (c + a(I, c))V_I] \quad (2.43)$$

subject to

$$c_{min}(I) \leq c \leq c_{max}(I)$$

Therefore, when we have the optimal strategy $c^*(S, I, t)$, (2.42) leads to the

following PIDE:

$$\begin{aligned}
V_t + \frac{1}{2}\sigma^2(t)S^2V_{SS} + \frac{1}{2}\sigma^2(t)V_{XX} + \sigma^2(t)SV_{SX} + \bar{\beta}(t)(\chi(t) + X - \\
\xi(t)\ln S)SV_S + (\mu(t) + \sigma(t)\hat{\theta}(t) - \alpha(t)X)V_X - \rho V + \\
\lambda(t) \int_{\mathbb{R}} (V(Se^{\eta(t)z}, X, I, t) - V(S, X, I, t) - S(e^{\eta(t)z} - 1)V_S)e^{\tilde{\theta}(t)z}\bar{l}(dz) + \\
(c^* - a(I, c^*))S - (c^* + a(I, c^*))V_I = 0 \quad (2.44)
\end{aligned}$$

We can imply from (2.37) that at time $t = T$ the PIDE (2.44) satisfies the following maturity condition (we take into account X as well):

$$V(S, X, I, T) = 0 \quad (2.45)$$

For stochastic control problems with more general optimisation equations than (2.37) the maturity condition will be

$$V(S, X, I, T) = G(S, I, X) \quad (2.46)$$

for some function G . To avoid problems with boundary conditions we assume that G is linear with respect to S and X .

2.5 SIMPLIFYING PIDE BEFORE DISCRETISATION

Before the application of numerical methods, we will employ some simplifications and transformations which do not change the nature of the general equation (2.44). Let's take $\theta = (\hat{\theta}, \tilde{\theta})$ constant. According to Benth et al.¹³, the Lévy measure $l(dz, dt)$ transforms to $e^{\tilde{\theta}z}l(dz, dt)$ under $\mathbb{Q}^\theta$. If we assume that $L(t)$ is a compound Poisson process then the jump measure l becomes

$$l(dz, dt) = \lambda(t)P_J(dz)dt$$

where $\lambda(t)$ and $P_J(dz)$ are jump intensity and jump-size distribution, respectively. So,

$$\begin{aligned} l_{\mathbb{Q}^\theta}(dz, dt) &= e^{\tilde{\theta}z} l(dz, dt) = e^{\tilde{\theta}z} \lambda(t) P_J(dz) dt = \\ &= e^{\tilde{\theta}z} \lambda(t) P_J(dz) dt \frac{\int_{\mathbb{R}} e^{\tilde{\theta}z} P_J(dz)}{\int_{\mathbb{R}} e^{\tilde{\theta}z} P_J(dz)} = \int_{\mathbb{R}} e^{\tilde{\theta}z} \lambda(t) P_J(dz) \frac{e^{\tilde{\theta}z} P_J(dz)}{\int_{\mathbb{R}} e^{\tilde{\theta}z} P_J(dz)} dt \end{aligned} \quad (2.47)$$

where denoting

$$\lambda^{\mathbb{Q}}(t) = \int_{\mathbb{R}} e^{\tilde{\theta}z} \lambda(t) P_J(dz) \quad \text{and} \quad P^{\mathbb{Q}}(dz) = \frac{e^{\tilde{\theta}z} P_J(dz)}{\int_{\mathbb{R}} e^{\tilde{\theta}z} P_J(dz)}$$

which implies

$$l_{\mathbb{Q}^\theta}(dz, dt) = \lambda^{\mathbb{Q}}(t) P^{\mathbb{Q}}(dz) dt$$

Thus, $L(t)$ is still a compound Poisson process with different intensity and jump-size distribution. The parameter θ is called a market price of risk where $\hat{\theta}$ and $\tilde{\theta}$ represent market prices of risk caused by normal price fluctuations and spikes, respectively.

After the introduction of the time to maturity $\tau = T - t$, the PIDE (2.44) becomes

$$\begin{aligned} V_\tau &= \frac{1}{2} \sigma^2 (T - \tau) S^2 V_{SS} + \frac{1}{2} \sigma^2 (T - \tau) V_{XX} + \sigma^2 (T - \tau) S V_{SX} + \bar{\beta} (T - \tau) (\psi(T - \tau) + X - \\ &\xi(T - \tau) \ln S) S V_S + (\mu(T - \tau) + \sigma(T - \tau) \hat{\theta}(T - \tau) - \alpha(T - \tau) X) V_X - \rho V + \\ &\lambda(T - \tau) \int_{\mathbb{R}} (V(S e^{\eta(T - \tau)z}, X, I, \tau) - V(S, X, I, \tau) - S(e^{\eta(T - \tau)z} - 1) V_S) e^{\tilde{\theta}z} \bar{l}(dz) + \\ &\max_{c \in C(I)} \{ (c - a(I, c)) S - (c + a(I, c)) V_I \} \end{aligned} \quad (2.48)$$

with the initial condition

$$V(S, X, I, 0) = G(S, I, X)$$

The domain of the PIDE (2.48) is $S \times X \times I = [0, \infty) \times (-\infty, \infty) \times [0, I_{\max}]$. But we will restrict our analysis to the finite domain $[0, S_{\max}) \times (X_{\min}, X_{\max}) \times [0, I_{\max}]$.

When $S \rightarrow 0$, assuming no singularities (2.48) simplifies to

$$V_\tau = \frac{1}{2}\sigma^2(T-\tau)V_{XX} + (\mu(T-\tau) + \sigma(T-\tau)\hat{\theta}(T-\tau) - \alpha(T-\tau)X)V_X - \rho V + \max_{c \in C(I)} \{-(c + a(I, c))V_I\}$$

For the limit $S \rightarrow S_{\max}$ we apply $V_{SS} \rightarrow 0$ and $V_{SX} \rightarrow 0$ to get

$$\begin{aligned} V_\tau = & \frac{1}{2}\sigma^2(T-\tau)V_{XX} + \bar{\beta}(T-\tau)(\psi(T-\tau) + X - \\ & \xi(T-\tau) \ln S_{\max})S_{\max}V_S + (\mu(T-\tau) + \sigma(T-\tau)\hat{\theta}(T-\tau) - \alpha(T-\tau)X)V_X - \rho V + \\ & \lambda(T-\tau) \int_{\mathbb{R}} (V(S_{\max}e^{\eta(T-\tau)z}, X, I, \tau) - V(S_{\max}, X, I, \tau) - S_{\max}(e^{\eta(T-\tau)z} - 1)V_S)e^{\tilde{\theta}z}\bar{l}(dz) + \\ & \max_{c \in C(I)} \{(c - a(I, c))S_{\max} - (c + a(I, c))V_I\} \end{aligned}$$

We can get the boundary conditions for the limits $X \rightarrow X_{\min}$ and $X \rightarrow X_{\max}$ in a similar way by assuming that $V_{XX} \rightarrow 0$ and $V_{SX} \rightarrow 0$.

We also need to look at the joint limits $(S, X) \rightarrow (0, X_{\min})$, $(S, X) \rightarrow (S_{\max}, X_{\max})$, $(S, X) \rightarrow (0, X_{\max})$ and $(S, X) \rightarrow (S_{\max}, X_{\min})$

$$V_\tau = (\mu(T-\tau) + \sigma(T-\tau)\hat{\theta}(T-\tau) - \alpha(T-\tau)X_{\min})V_X - \rho V + \max_{c \in C(I)} \{-(c + a(I, c))V_I\}$$

$$\begin{aligned} V_\tau = & \bar{\beta}(T-\tau)(\psi(T-\tau) + X_{\max} - \xi(T-\tau) \ln S_{\max})S_{\max}V_S + \\ & (\mu(T-\tau) + \sigma(T-\tau)\hat{\theta}(T-\tau) - \alpha(T-\tau)X_{\max})V_X - \rho V + \\ & \lambda(T-\tau) \int_{\mathbb{R}} (V(S_{\max}e^{\eta(T-\tau)z}, X_{\max}, I, \tau) - V(S_{\max}, X_{\max}, I, \tau) - S_{\max}(e^{\eta(T-\tau)z} - 1)V_S)e^{\tilde{\theta}z}\bar{l}(dz) + \\ & \max_{c \in C(I)} \{(c - a(I, c))S_{\max} - (c + a(I, c))V_I\} \end{aligned}$$

$$V_\tau = (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)X_{\max})V_X - \rho V + \max_{c \in C(I)} \{-(c + a(I, c))V_I\}$$

$$\begin{aligned} V_\tau = & \bar{\beta}(T - \tau)(\psi(T - \tau) + X_{\min} - \xi(T - \tau) \ln S_{\max})S_{\max}V_S + \\ & (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)X_{\min})V_X - \rho V + \\ & \lambda(T - \tau) \int_{\mathbb{R}} (V(S_{\max}e^{\eta(T - \tau)z}, X_{\min}, I, \tau) - V(S_{\max}, X_{\min}, I, \tau) - S_{\max}(e^{\eta(T - \tau)z} - 1)V_S)e^{\tilde{\theta}z}\bar{l}(dz) + \\ & \max_{c \in C(I)} \{(c - a(I, c))S_{\max} - (c + a(I, c))V_I\} \end{aligned}$$

respectively, in order to extend our boundary conditions to the corner points.

Now, let's take the change of variables $y = \ln S$, $x = X$ and define:

$$V(e^y, X, I, t) = \bar{V}(y, X, I, t) \quad \text{and} \quad G(e^y, X, I) = g(y, X, I). \quad (2.49)$$

Then our PIDE simplifies to

$$\begin{aligned} \bar{V}_\tau = & \frac{1}{2}\sigma^2(T - \tau)(\bar{V}_{yy} + \bar{V}_{xx}) + \sigma^2(T - \tau)\bar{V}_{xy} + (-\frac{1}{2}\sigma^2(T - \tau) + \bar{\beta}(T - \tau)(\chi(T - \tau) + x - \\ & \xi(T - \tau)y))\bar{V}_y + (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)x)\bar{V}_x - \rho\bar{V} + \\ & \int_{\mathbb{R}} (\bar{V}(y + \eta(T - \tau)z, x, I, \tau) - \bar{V}(y, x, I, \tau) - (e^{\eta(T - \tau)z} - 1)\bar{V}_y)\lambda^{\mathbb{Q}}(T - \tau)P_J^{\mathbb{Q}}(dz) + \\ & \max_{c \in C(I)} \{(c - a(I, c))e^y - (c + a(I, c))\bar{V}_I\} \quad (2.50) \end{aligned}$$

with the initial condition

$$\bar{V}(y, x, I, 0) = g(y, x, I)$$

Denote

$$\bar{\gamma}(\tau) = \lambda^{\mathbb{Q}}(T - \tau) \int_{\mathbb{R}} (e^{\eta(T - \tau)z} - 1)P_J^{\mathbb{Q}}(dz)$$

and

$$\gamma(\tau) = \chi(T - \tau) - \frac{\bar{\gamma}(\tau)}{\bar{\beta}(T - \tau)}$$

Then (2.50) becomes

$$\begin{aligned} \bar{V}_\tau = & \frac{1}{2}\sigma^2(T - \tau)(\bar{V}_{yy} + \bar{V}_{xx}) + \sigma^2(T - \tau)\bar{V}_{xy} + (\bar{\beta}(T - \tau)(\gamma(\tau) + x - \\ & \xi(T - \tau)y) - \frac{1}{2}\sigma^2(T - \tau))\bar{V}_y + (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)x)\bar{V}_x - \\ & (\rho + \lambda^{\mathbb{Q}}(T - \tau))\bar{V} + \lambda^{\mathbb{Q}}(T - \tau) \int_{\mathbb{R}} (\bar{V}(y + \eta(T - \tau)z, x, I, \tau)P_J^{\mathbb{Q}}(dz) + \\ & \max_{c \in C(I)} \{(c - a(I, c))e^y - (c + a(I, c))\bar{V}_I\} \quad (2.51) \end{aligned}$$

with the initial condition

$$\bar{V}(y, x, I, 0) = g(y, x, I) \quad \text{for } \forall x, y \in \mathbb{R}$$

Now, we need to transform our boundary conditions defined earlier in this section. The bounded domain of the equation (2.51) is $y \times x \times I = [y_{\min}, y_{\max}] \times [x_{\min}, x_{\max}] \times [0, I_{\max}]$. So, the boundary conditions for $S \rightarrow 0$ and $S \rightarrow S_{\max}$ become

$$\begin{aligned} \bar{V}_\tau = & \frac{1}{2}\sigma^2(T - \tau)\bar{V}_{xx} + (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)x)\bar{V}_x \\ & - (\rho + \lambda^{\mathbb{Q}}(T - \tau))\bar{V} + \max_{c \in C(I)} \{-(c + a(I, c))\bar{V}_I\} \quad (2.52) \end{aligned}$$

and

$$\begin{aligned} \bar{V}_\tau = & \frac{1}{2}\sigma^2(T - \tau)\bar{V}_{xx} + (\bar{\beta}(T - \tau)(\gamma(\tau) + x - \xi(T - \tau)y_{\max}) - \frac{1}{2}\sigma^2(T - \tau))\bar{V}_y + \\ & (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)x)\bar{V}_x - (\rho + \lambda^{\mathbb{Q}}(T - \tau))\bar{V} + \\ & \lambda^{\mathbb{Q}}(T - \tau) \int_{\mathbb{R}} (\bar{V}(y_{\max} + \eta(T - \tau)z, x, I, \tau)P_J^{\mathbb{Q}}(dz) + \\ & \max_{c \in C(I)} \{(c - a(I, c))e^{y_{\max}} - (c + a(I, c))\bar{V}_I\} \quad (2.53) \end{aligned}$$

when $y \rightarrow y_{min}$ and $y \rightarrow y_{max}$, respectively. The other boundary equations transform in a similar way. We will discuss the requirements on the coefficients for each boundary condition in the following section.

2.6 DISCRETISATION

The PIDE (2.51) can be written as

$$\bar{V}_\tau = \mathcal{D}\bar{V} + \mathcal{H}\bar{V} + \max_{c \in C(I)} \{(c - a(I, c))e^y - (c + a(I, c))\bar{V}_I\} \quad (2.54)$$

where

$$\begin{aligned} \mathcal{D}\bar{V} = & \frac{1}{2}\sigma^2(T - \tau)(\bar{V}_{yy} + \bar{V}_{xx}) + \sigma^2(T - \tau)\bar{V}_{xy} + (\bar{\beta}(T - \tau)(\gamma(\tau) + x - \\ & \xi(T - \tau)y - \frac{1}{2}\sigma^2(T - \tau)))\bar{V}_y + (\mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \\ & \alpha(T - \tau)x)\bar{V}_x - (\rho + \lambda^Q(T - \tau))\bar{V} \end{aligned}$$

and

$$\mathcal{H}\bar{V} = \lambda^Q(T - \tau) \int_{\mathbb{R}} (\bar{V}(y + \eta(T - \tau)z, x, I, \tau) P_J^Q(dz))$$

We will extend the explicit finite difference method proposed in Cont and Tankov²⁸ to solve our non-linear PIDE (2.54) with two state variables, time-dependent coefficients and seasonal jump intensity. First of all, we need to restrict our PIDE from $\mathbb{R}$ to bounded domain:

$$\mathbb{R} \rightarrow]-A, A[, \quad \text{where } A > 0$$

and impose an artificial boundary condition with respect to y at all points outside $]-A, A[$:

$$\bar{V}(y, x, I, \tau) = g(y, x, I), \quad \text{for } \forall y \notin]-A, A[\quad (2.55)$$

We then replace the domain $[-A, A] \times [-A, A] \times [0, I_{max}] \times [0, T]$ by the

following discrete grid:

$$\begin{aligned}\tau_n &= n\Delta\tau, \quad n = 0, \dots, M, \quad \Delta\tau = \frac{T}{M} \\ y_i &= -A + i\Delta y, \quad i = 0, \dots, N, \quad \Delta y = \frac{2A}{N} \\ x_u &= -A + u\Delta x, \quad u = 0, \dots, R, \quad \Delta x = \frac{2A}{R} \\ I_j &= j\Delta I, \quad j = 0, \dots, L, \quad \Delta I = \frac{I_{max}}{L}\end{aligned}$$

So, $\bar{V}(y_i, x_u, I_j, \tau_n) = \bar{V}(i\Delta y, u\Delta x, j\Delta I, n\Delta\tau)$ is the exact solution of the PIDE at the node (y_i, x_u, I_j, τ_n) , while $\bar{V}_{i,u,j}^n$ is the approximate solution at the same node.

For the integral term we need some finite upper/lower bounds B_u, B_l to truncate the large jumps. So, the integral terms become

$$\int_{B_l}^{B_u} (\bar{V}(y + \eta(T - \tau)z, x, I, \tau) P_J^{\mathbb{Q}}(dz)$$

and

$$\bar{\gamma}(\tau) = \lambda^{\mathbb{Q}}(T - \tau) \int_{B_l}^{B_u} (e^{\eta(T - \tau)z} - 1) P_J^{\mathbb{Q}}(dz)$$

To approximate the integral terms we can use the trapezoidal quadrature rule where $\Delta z = \Delta y$. To avoid any interpolation issues we introduce the following change of variable:

$$z' = y + \eta(T - \tau)z \implies z = \frac{z' - y}{\eta(T - \tau)} \implies dz = \frac{dz'}{\eta(T - \tau)}$$

Let take K_l, K_u such that $[B_l, B_u] \subset [(K_l - \frac{1}{2})\Delta y, (K_u + \frac{1}{2})\Delta y]$. Then we

have

$$\begin{aligned} \int_{B_l}^{B_u} (\bar{V}(y + \eta(T - \tau)z, x, I, \tau) P_J^Q(dz) = \\ \int_{y_i + \eta(T - \tau_n)B_l}^{y_i + \eta(T - \tau_n)B_u} (\bar{V}(z', x_u, I_j, \tau_n) P_J^Q\left(\frac{dz'}{\eta(T - \tau)}\right) \approx \sum_{s=K_l}^{K_u} \bar{V}_{s,u,j}^n P_s \end{aligned} \quad (2.56)$$

and

$$\bar{\gamma}(\tau_n) \approx \hat{\gamma}(\tau_n) = \lambda^Q(T - \tau_n) \sum_{s=K_l}^{K_u} (e^{\eta(T - \tau_n)z} - 1) P_s(dz) \quad (2.57)$$

where

$$P_s = \int_{(s - \frac{1}{2})\Delta y}^{(s + \frac{1}{2})\Delta y} P_J^Q\left(\frac{dz'}{\eta(T - \tau)}\right)$$

Contrary to Cont and Tankov²⁸, we assume that λ (we can derive λ^Q from λ) is a given deterministic function of t and therefore, we don't need to derive it numerically. We need to enlarge our domain to $[-A + B_l, A + B_u]$ in order to compute the integral terms. Thus, the discretised grid with respect to y extends from $i = K_l$ to $i = K_u + N$, where $\bar{V}_{i,u,j}^n = h(y_i, x_u, I_j)$ for $i \notin [0, N]$. Let us denote

$$\begin{aligned} A(\tau, y, x) &= \bar{\beta}(T - \tau)(\gamma(\tau) + x - \xi(T - \tau)y) - \frac{1}{2}\sigma^2(T - \tau) \\ B(\tau, x) &= \mu(T - \tau) + \sigma(T - \tau)\hat{\theta}(T - \tau) - \alpha(T - \tau)x \end{aligned}$$

and therefore

$$\begin{aligned} A_{i,u}^n &= A(\tau_n, y_i, x_u) \\ B_u^n &= B(\tau_n, x_u) \end{aligned}$$

to simplify the ensuing numerical work. Note that we need to choose the coefficients in such a way that $\lim_{y \rightarrow y_{min}} A(\tau, x, y) \geq 0$, $\lim_{y \rightarrow y_{max}} A(\tau, x, y) \leq 0$, $\lim_{x \rightarrow x_{min}} B(\tau, x,) \geq 0$ and $\lim_{x \rightarrow x_{max}} B(\tau, x,) \leq 0$. If we have these conditions then no other information is required from outside the domain to solve the equation on the corresponding boundary²⁵. We can now apply

finite difference formulas to approximate the partial derivatives appearing in the PIDE and boundary conditions:

$$\begin{aligned}
(\bar{V}_y)_{i,u,j}^n &\approx \begin{cases} \frac{\bar{V}_{i+1,u,j}^n - \bar{V}_{i,u,j}^n}{\Delta y}, & \text{if } A_{i,u}^n \geq 0 \\ \frac{\bar{V}_{i,u,j}^n - \bar{V}_{i-1,u,j}^n}{\Delta y}, & \text{if } A_{i,u}^n < 0 \end{cases} \\
(\bar{V}_x)_{i,u,j}^n &\approx \begin{cases} \frac{\bar{V}_{i,u+1,j}^n - \bar{V}_{i,u,j}^n}{\Delta x}, & \text{if } B_u^n \geq 0 \\ \frac{\bar{V}_{i,u,j}^n - \bar{V}_{i,u-1,j}^n}{\Delta x}, & \text{if } B_u^n < 0 \end{cases} \\
(\bar{V}_{yy})_{i,u,j}^n &\approx \frac{\bar{V}_{i+1,u,j}^n - 2\bar{V}_{i,u,j}^n + \bar{V}_{i-1,u,j}^n}{\Delta y^2} \\
(\bar{V}_{xx})_{i,u,j}^n &\approx \frac{\bar{V}_{i,u+1,j}^n - 2\bar{V}_{i,u,j}^n + \bar{V}_{i,u-1,j}^n}{\Delta x^2} \\
(\bar{V}_{xy})_{i,u,j}^n &\approx \frac{\bar{V}_{i+1,u+1,j}^n - \bar{V}_{i+1,u-1,j}^n - \bar{V}_{i-1,u+1,j}^n + \bar{V}_{i-1,u-1,j}^n}{4\Delta x \Delta y} \\
(\bar{V}_\tau)_{i,u,j}^n &\approx \frac{\bar{V}_{i,u,j}^{n+1} - \bar{V}_{i,u,j}^n}{\Delta \tau}
\end{aligned}$$

Then our diffusion part becomes

$$\begin{aligned}
(D\bar{V})_{i,u,j}^n &\approx \frac{1}{2}\sigma^2(T-\tau_n)\left(\frac{\bar{V}_{i+1,u,j}^n - 2\bar{V}_{i,u,j}^n + \bar{V}_{i-1,u,j}^n}{\Delta y^2} + \frac{\bar{V}_{i,u+1,j}^n - 2\bar{V}_{i,u,j}^n + \bar{V}_{i,u-1,j}^n}{\Delta x^2}\right) + \\
&\quad \sigma^2(T-\tau_n)\frac{\bar{V}_{i+1,u+1,j}^n - \bar{V}_{i+1,u-1,j}^n - \bar{V}_{i-1,u+1,j}^n + \bar{V}_{i-1,u-1,j}^n}{4\Delta x \Delta y} + \\
&\quad A_{i,u}^n \frac{\bar{V}_{i+1,u,j}^n - \bar{V}_{i,u,j}^n}{\Delta y} + B_u^n \frac{\bar{V}_{i,u+1,j}^n - \bar{V}_{i,u,j}^n}{\Delta x} - (\rho + \lambda^Q(T-\tau_n)\bar{V}_{i,u,j}^n) \quad (2.58)
\end{aligned}$$

where we have assumed that $A_{i,u}^n \geq 0$ and $B_u^n \geq 0$.

Convergence of a finite difference scheme is impossible unless it's monotone. A scheme is monotone if it's decreasing with respect to its terms other than $\bar{V}_{i,u,j}^{n+1}$. The terms $-\bar{V}_{i+1,u-1,j}^n$ and $-\bar{V}_{i-1,u+1,j}^n$ appear will have plus signs once moved to the LHS of the equation. So, the scheme is unconditionally increasing with respect to them which is damaging the monotonicity property. Bouchut and Frid¹⁹ suggests adding a corrector to the above diffusion part

in the following way to solve this issue:

$$(D\bar{V})_{i,u,j}^n + \frac{\sigma^2(T - \tau_n)}{4\Delta x \Delta y} \Delta_y^+ \Delta_y^- \Delta_x^+ \Delta_x^- \bar{V}_{i,u,j}^n \quad (2.59)$$

where

$$\begin{aligned} \Delta_y^+ \bar{V}_{i,u} &= \bar{V}_{i+1,u} - \bar{V}_{i,u} & \Delta_x^+ \bar{V}_{i,u} &= \bar{V}_{i,u+1} - \bar{V}_{i,u} \\ \Delta_y^- \bar{V}_{i,u} &= \bar{V}_{i,u} - \bar{V}_{i-1,u} & \Delta_x^- \bar{V}_{i,u} &= \bar{V}_{i,u} - \bar{V}_{i,u-1} \end{aligned}$$

If we omit j just for now and take $\Delta x = \Delta y$ then (2.59) can be rewritten as:

$$\begin{aligned} (D\bar{V})_{i,u}^n &+ \frac{\sigma^2(T - \tau_n)}{4\Delta y^2} (\bar{V}_{i+1,u+1}^n - 2\bar{V}_{i,u+1}^n - 2\bar{V}_{i+1,u}^n + 4\bar{V}_{i,u}^n + \\ &\bar{V}_{i+1,u-1}^n + \bar{V}_{i-1,u+1}^n - 2\bar{V}_{i-1,u}^n - 2\bar{V}_{i,u-1}^n + \bar{V}_{i-1,u-1}^n) = \\ &\frac{\sigma^2(T - \tau_n)}{4\Delta y^2} (\bar{V}_{i+1,u}^n - 4\bar{V}_{i,u}^n + \bar{V}_{i-1,u}^n + \bar{V}_{i,u+1}^n + \bar{V}_{i,u-1}^n) + \\ &\frac{\sigma^2(T - \tau_n)}{2\Delta y^2} (\bar{V}_{i+1,u+1}^n - \bar{V}_{i,u+1}^n - \bar{V}_{i+1,u}^n + 2\bar{V}_{i,u}^n - \\ &- \bar{V}_{i-1,u}^n - \bar{V}_{i,u-1}^n + \bar{V}_{i-1,u-1}^n) + A_{i,u}^n \frac{\bar{V}_{i+1,u}^n - \bar{V}_{i,u}^n}{\Delta y} + \\ &B_u^n \frac{\bar{V}_{i,u+1}^n - \bar{V}_{i,u}^n}{\Delta y} - (\rho + \lambda^Q(T - \tau_n)) \bar{V}_{i,u}^n = \\ &\frac{\sigma^2(T - \tau_n)}{2\Delta y^2} (\bar{V}_{i+1,u+1}^n - 2\bar{V}_{i,u}^n + \bar{V}_{i-1,u-1}^n) + \\ &\frac{A_{i,u}^n}{\Delta y} (\bar{V}_{i+1,u}^n - \bar{V}_{i,u}^n) + \frac{B_u^n}{\Delta y} (\bar{V}_{i,u+1}^n - \bar{V}_{i,u}^n) - (\rho + \lambda^Q(T - \tau_n)) \bar{V}_{i,u}^n \quad (2.60) \end{aligned}$$

where $n = 1, \dots, M$, $i = K_l + 1, \dots, K_u - 1$ and $u = 1, \dots, N - 1$. The corrected scheme (2.60) is now monotone.

After denoting the approximate value of $c(y_i, I_j, \tau_n)$ by $\omega_{i,j}^n$ our explicit scheme

becomes

$$\begin{aligned} \frac{\bar{V}_{i,u,j}^{n+1} - \bar{V}_{i,u,j}^n}{\Delta\tau} = D\bar{V}_{i,u,j}^n + H\bar{V}_{i,u,j}^n + \\ \max_{\omega_{i,j}^n \in C(I)} \left\{ (\omega_{i,j}^n - a(I_j, \omega_{i,j}^n))e^{y_i} - (\omega_{i,j}^n + a(I_j, \omega_{i,j}^n)) \frac{\bar{V}_{i,u,j+1}^n - \bar{V}_{i,u,j}^n}{\Delta I} \right\} \end{aligned} \quad (2.61)$$

where

$$\begin{aligned} D\bar{V}_{i,u,j}^n &= \frac{\sigma^2(T - \tau_n)}{2\Delta y^2} (\bar{V}_{i+1,u+1}^n - 2\bar{V}_{i,u}^n + \bar{V}_{i-1,u-1}^n) + \\ \frac{A_{i,u}^n}{\Delta y} (\bar{V}_{i+1,u}^n - \bar{V}_{i,u}^n) &+ \frac{B_u^n}{\Delta y} (\bar{V}_{i,u+1}^n - \bar{V}_{i,u}^n) - (\rho + \lambda^Q(T - \tau_n)\bar{V}_{i,u}^n, \\ H\bar{V}_{i,u,j}^n &= \lambda^Q(T - \tau_n) \sum_{s=K_l}^{K_u} \bar{V}_{s,u,j}^n P_s \end{aligned}$$

When $((\omega_{i,j}^n + a(I_j, \omega_{i,j}^n)) > 0$ we will discretise $\bar{V}_I$ with $\frac{\bar{V}_{i,u,j}^n - \bar{V}_{i,u,j-1}^n}{\Delta I}$. We know from the initial condition that $\bar{V}_{i,u,j}^0 = h(y_i, x_u, I_j)$. So, starting from time $\tau_0 = 0$ we first solve the maximization problem

$$\hat{\omega}_{i,j}^n = \arg \max_{\omega_{i,j}^n \in C(I)} \left\{ (\omega_{i,j}^n - a(I_j, \omega_{i,j}^n))e^{y_i} - (\omega_{i,j}^n + a(I_j, \omega_{i,j}^n)) \frac{\bar{V}_{i,u,j+1}^n - \bar{V}_{i,u,j}^n}{\Delta I} \right\} \quad (2.62)$$

then substitute this optimal strategy $\hat{\omega}_{i,j}^n$ in (2.62) to get

$$\begin{aligned} \bar{V}_{i,u,j}^{n+1} = \bar{V}_{i,u,j}^n + \Delta\tau D\bar{V}_{i,u,j}^n + \Delta\tau H\bar{V}_{i,u,j}^n + \\ \Delta\tau \left\{ (\hat{\omega}_{i,j}^n - a(I_j, \hat{\omega}_{i,j}^n))e^{y_i} - (\hat{\omega}_{i,j}^n + a(I_j, \hat{\omega}_{i,j}^n)) \frac{\bar{V}_{i,u,j+1}^n - \bar{V}_{i,u,j}^n}{\Delta I} \right\} \end{aligned} \quad (2.63)$$

where $n = 0, 1, \dots, M - 1$.

2.7 NUMERICAL RESULTS

In this section, we conduct numerical experiments based on the discretisation scheme presented in the previous sections. As the default units for gas spot

price S and gas inventory I , we will use "dollars per million British thermal unit" (\$/mmBtu) and million cubic feet (MMcf), respectively. Note that 1000 mmBtus are roughly equal to 1 MMcf. Therefore, we need to multiply gas spot price by 1000 when computing payoffs or revenues in order to adjust the units.

For better understanding of the problem and tracking of the results, we will start the section by the analysis of the simple case without jumps and seasonality. In other words, our underlying model is given by one factor geometric mean-reverting model. We will then apply the spikes and seasonality of the spot price and jump-size. We will assume that other parameters of the underlying spot price dynamics are constant as it will not affect the generality of the results. Meanwhile, we will refer to Thompson et al.⁷⁴ for the definition of the gas loss rate $a(I, c)$:

$$a(I, c) = a(c) = \begin{cases} 0 & \text{for } c \geq 0 \\ 1.7365 & \text{for } c < 0 \end{cases} \quad (2.64)$$

which means that gas loss occurs on a fixed rate 1.7 MMcf/day only when we inject gas into the storage. In this case we can explicitly solve the optimisation part of the equation (2.54):

$$c = \begin{cases} c_{max}(I) & \text{for } 1000P > V_I \\ c_{min}(I) & \text{for } (c_{min}(I) - 1.7365)1000P > (c_{min}(I) + 1.7365)V_I \\ 0 & \text{otherwise} \end{cases} \quad (2.65)$$

Note that there was a mistake in this formula in the original paper which has been corrected here. The payoff at maturity can be defined in different ways. Based on Thompson et al.⁷⁴ and Chen and Forsyth²⁵ we can use $V(S, X, I, T) = 0$ or $V(S, X, I, T) = -2S \max\{1000 - I, 0\}$, respectively. In this work, we look at the second case in order to compare our results with Chen and Forsyth²⁵. Figures 2.1 to 2.7 show the numerical results obtained after implementing our explicit scheme.

Parameter	Value	Parameter	Value
I_{max}	2000 MMcf	k_2	730000
T	3 years	k_3	500
k_1	2040.41	k_4	2500

Table 2.1: Input parameters used to evaluate the gas storage contract

Let's first look at the following model without jumps and seasonality:

$$\frac{dS(t)}{S(t)} = \beta(\chi - \xi \ln S(t))dt + \sigma dB^\theta(t) \quad (2.66)$$

By taking the parameters as $\beta = 3.4$, $\chi = \log 6$, $\xi = 1$, $\sigma = 0.59$ and $\rho = 0.1$ our PIDE (2.48) transforms to the following PDE:

$$V_\tau = \frac{1}{2}0.59^2 S^2 V_{SS} + 3.4(\log 6 - \ln S)SV_S - 0.1V + \max_{c \in C(I)} \{1000(c - a(c))S - (c + a(c))V_I\} \quad (2.67)$$

The rest of the parameters required to price the gas storage contract is given in Table 2.1. The maximum value that the spot price can achieve is $S_{max} = 2000\$/mmBtu$. As we are using an explicit scheme, the number of time steps needs to be large enough to get stable results. So, we will take 8000, 40 and 20 partition points for the variables τ , S and I , respectively.

The control strategies for operating the gas storage facility are presented in Figure 2.1. We can distinguish three visible regions: negative, zero and positive. We do nothing in the zero region, while negative and positive areas correspond to injecting gas into and releasing it from the storage, respectively. Figure 2.1 implies that gas can be pumped into the reservoir very quickly when gas inventory levels are low. The injectivity rate gets less deeper and approaches to 0 as working gas in the storage increases. In contrast, the deliverability rate increases as the level of gas gets higher. We can also observe that the zero region is thicker in high inventory levels as it gets more expensive to inject gas when the reservoir is almost full.

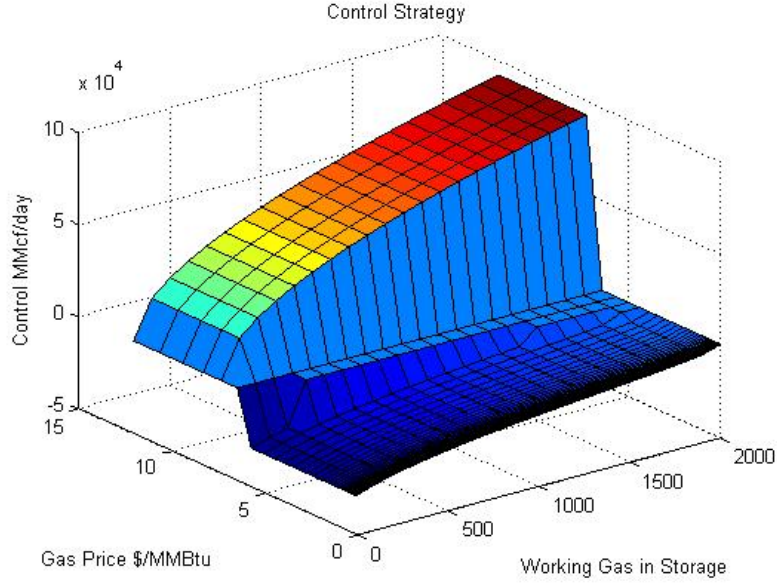


Figure 2.1: Control strategy surface for the one factor model (2.66).

Figure 2.2 depicts the value surface of the contract. We can see that the facility gets its lowest value when prices are high and the reservoir is empty and approaches to its highest value when the storage is full and the prices are high.

If we replace the constant long-term mean by the log of a seasonality function

$$\Lambda(t) = 6 + \sin(4\pi t)$$

then the zero region of the control strategy will become larger as it is shown in Figure 2.3a. The intuition behind this is that the manager prefers to wait rather than release gas while the prices aren't high enough during a low season. If we further assume that gas prices may jump with constant jump intensity $\lambda = 3$ and normal jump-size distribution with mean 0.3 and variance 0.2 then the control surface transforms to Figure 2.3. As we see the positive region is considerably wider now. This behavior can be explained by possibility of negative jumps. In other words, as the prices get higher, the manager becomes reluctant to keep gas in the storage to avoid loss from the downside spike. Making the jump intensity seasonal, $\lambda(t) = 2\sin(4\pi t)$

will have a minor change in the control strategy (see Figure 2.3c). However, the seasonal jump intensity will have its effect on contract value. Hence, the value surface of a natural gas storage in the presence of seasonality and jumps with seasonal jump-size intensity is given by Figure 2.4. We can see that the value reaches to its minimum faster as S increases. Meanwhile, the value surface moved downwards compared to the Figure 2.2.

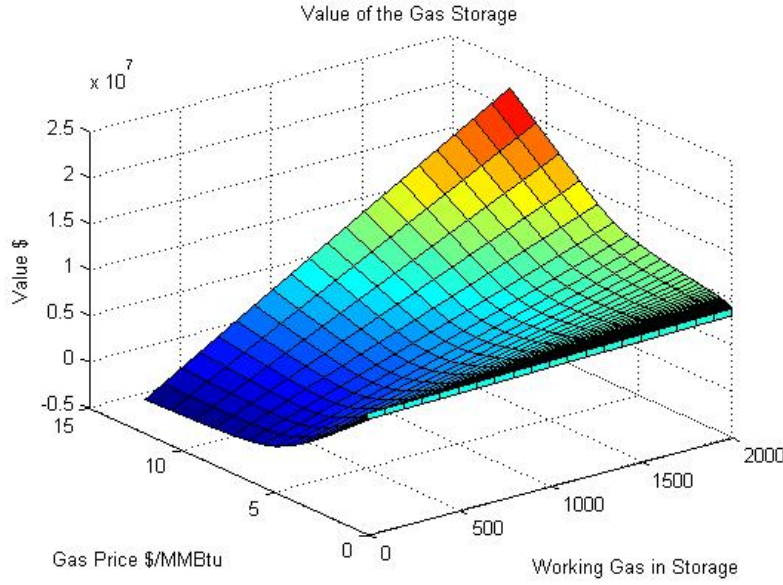
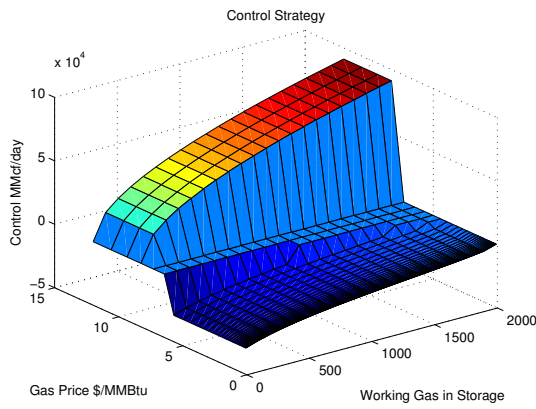


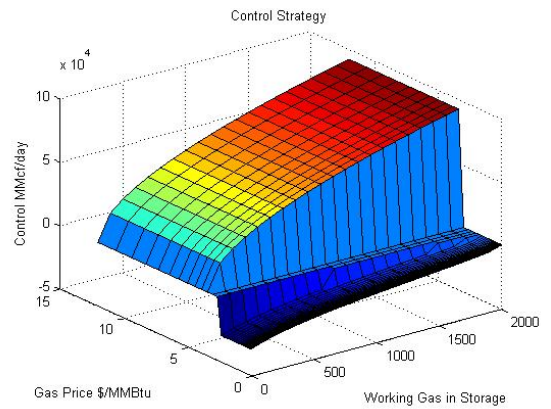
Figure 2.2: Value surface for the one factor model (2.66).

After investigating our numerical approach for the one factor model, we can move to solving difference equations (2.63). We will keep parameters presented for the one factor model the same. The additional parameters are $\mu = \log 5$ and $\alpha = 1$. We will have 8000, 20, 20 and 10 steps for the variables τ, y, x and I . The surfaces of control strategy and contract value obtained from the numerical implementation for a fixed $x = -1.86$ are shown in Figure 2.5. According to Figure 2.5a the zero region of no action is thinner than in Figures 2.1 and 2.3. That is why, the value goes up to its maximum more rapidly as S increases (Figure 2.5b).

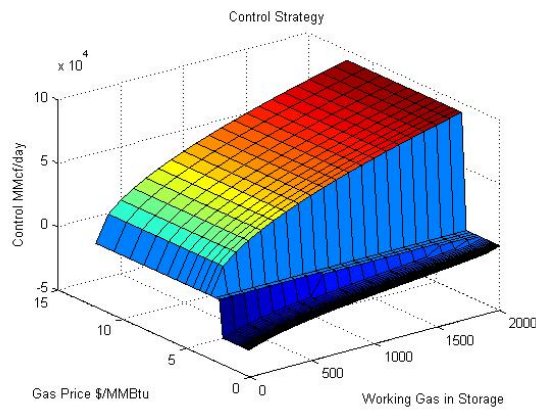
Note that these are the results for a fixed value of long-run mean x . Therefore, Figure 2.6 shows the behavior of the contract value with respect to S and X for different fixed values of I . Figure 2.6 is the extended version of Figure 4



(a)



(b)



(c)

Figure 2.3: Control strategy surfaces for models with seasonality and (a) no jumps; (b) jumps with constant intensity; (c) jumps with seasonal intensity

Time steps	State steps	Value \$
400	5	271010000
1600	10	199270000
6400	20	177080000
15600	40	173360000
102400	80	173350000

Table 2.2: Convergence of the gas storage scheme at the spot price $S = 12\$/MMBtu$ and working gas level $I = 1200MMcf$.

from Thompson et al.⁷⁴.

Finally, Figure 2.7 depicts the dependence of the contract value on X and I for a fixed spot price $S = 1MMBtu$. As we see, the value of contract increases with respect to I at all levels of long-run mean. The higher is the mean, the faster the value goes up.

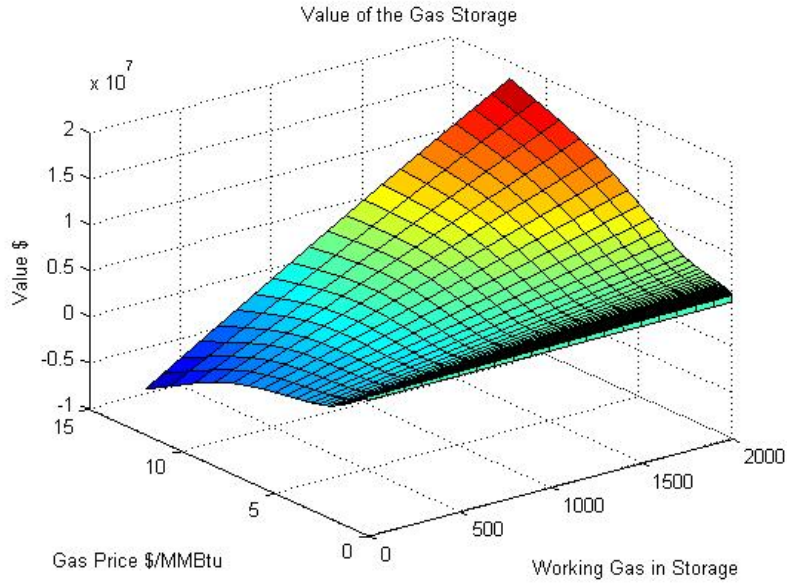
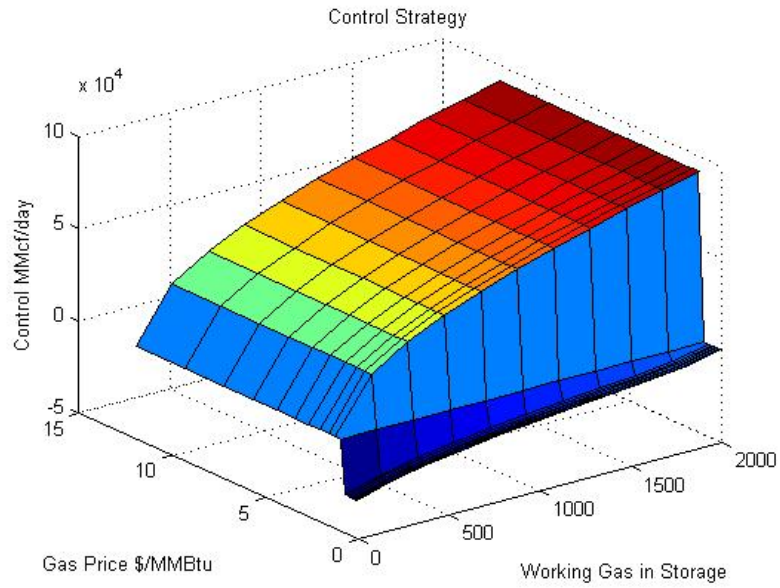
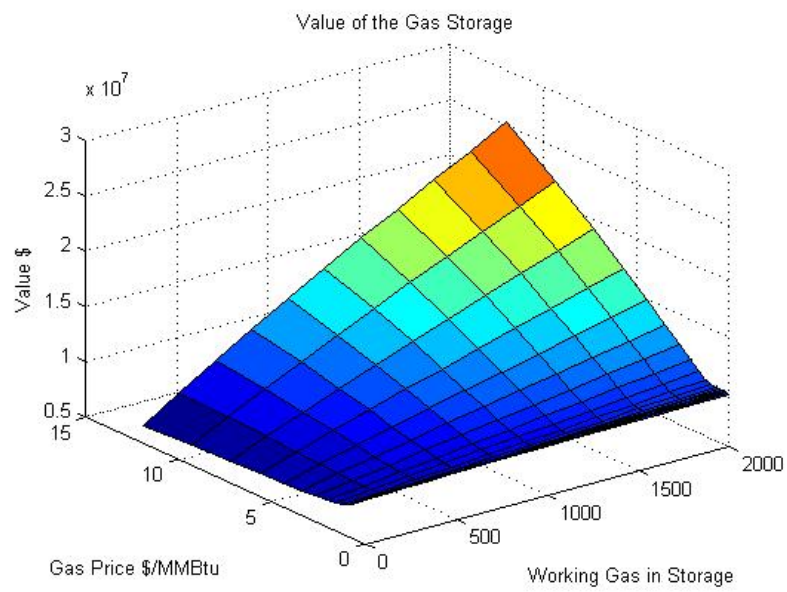


Figure 2.4: Value surface for a jump diffusion model with seasonality and seasonal jump intensity.

We can now investigate the convergence of our numerical scheme. To do this we first consider a problem with 5, 5, and 400 number of grid points for the variables y, x and τ , respectively. The number of steps for I will be fixed as

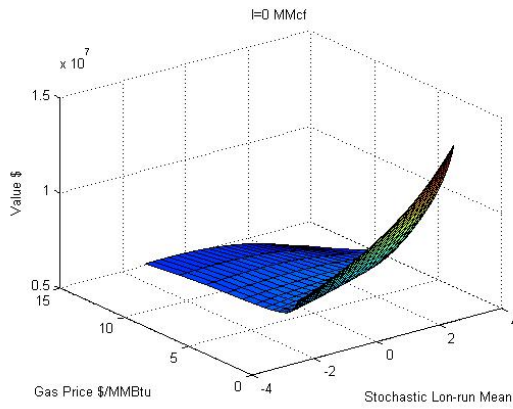


(a) Control strategy

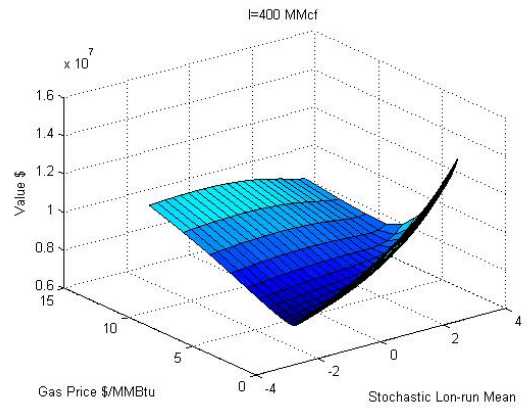


(b) Value surface

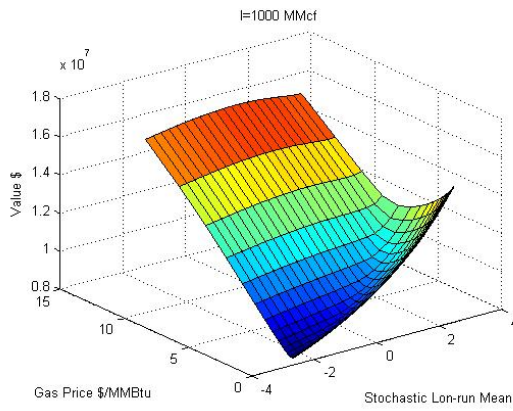
Figure 2.5: Control strategy and value surfaces for the PIDE (2.54) at the fixed $X = -1.86$.



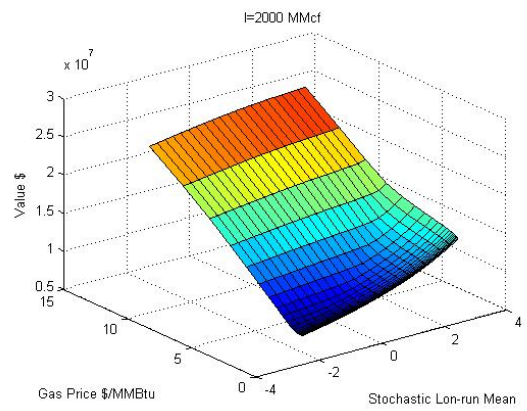
(a) $I=0$



(b) $I=400$



(c) $I=1000$



(d) $I=2000$

Figure 2.6: Value of the storage contract with respect to spot price S and long-run mean X .

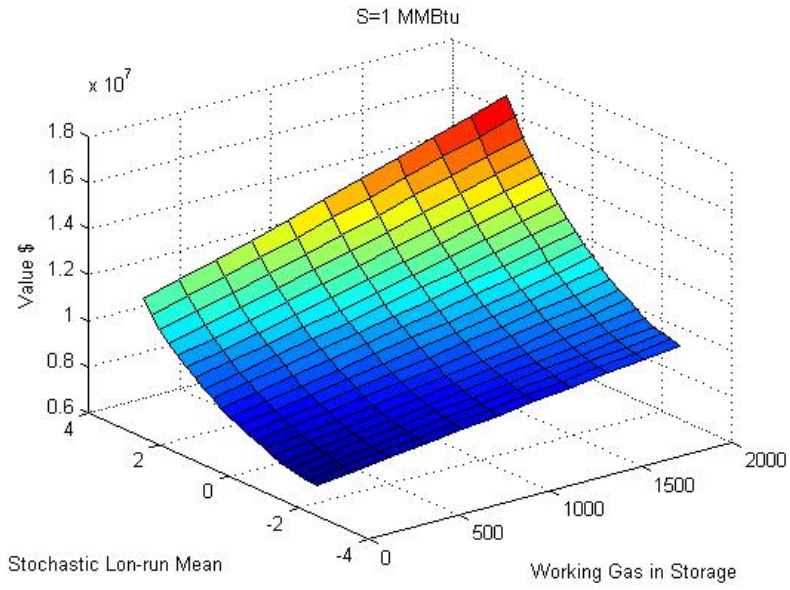


Figure 2.7: Value of the contract with respect to long-run mean X and working gas level I .

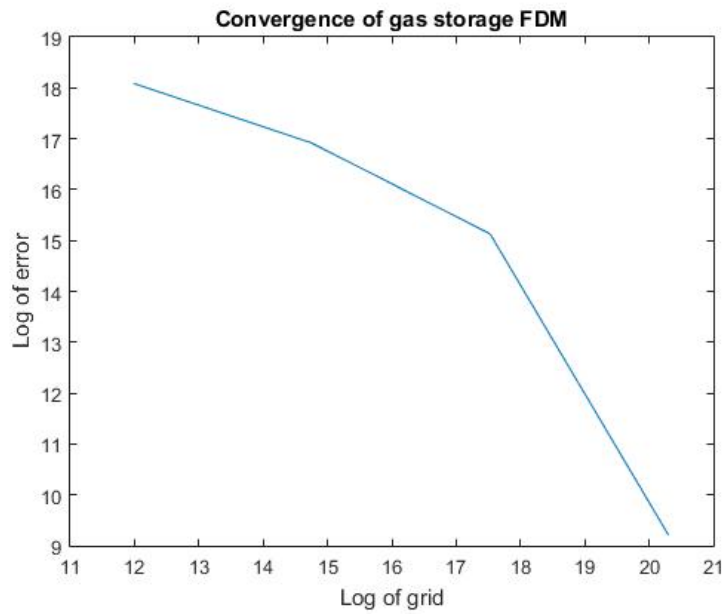


Figure 2.8: Convergence of FDM for gas storage. Log of errors decreases as we increase the number of grid points.

10. We implement the scheme and take the value V when $y = x = \log(12)$

and $I = 1200$. We then keep increasing the number of state and time variable partition points 2 and 4 times, calculate the value V and compare with its value for the previous grid size. These values are presented in Table 2.2. The plot of log errors with respect to log mesh size confirms the convergence of our scheme (see Figure 2.8).

2.8 EMPIRICAL ANALYSIS OF GAS SPOT PRICES

Empirical analysis of energy spot prices is a rather complicated task and requires a comprehensive investigation. Unfortunately, the time constrain of this research didn't allow us to carry out rigorous parameter estimation. Meanwhile, this section gives an insight to the problem by following the approach presented in Benth et al.¹³. We investigate natural gas market by analysing Bloomberg Day Ahead Spot Price, namely UK National Balancing Point (NBP) during the time period 4/9/2009 - 4/7/2015. Therefore, the underlying spot price dynamics is given by

$$S(t) = \Lambda(t) \exp(X(t) + Y(t))$$

where

$$\begin{aligned} dX(t) &= -\alpha X(t)dt + \sigma dB(t) \\ dY(t) &= -\alpha Y(t)dt + dL(t) \end{aligned}$$

which is the special case of (2.11) with constant coefficients and a Lévy process $L(t)$. As the mean-reverting coefficients of $X(t)$ and $Y(t)$ are the same α , we can derive the following SDE for $\ln S(t)$:

$$d \ln S(t) = d \ln \Lambda(t) - \alpha(\ln S(t) - \ln \Lambda(t))dt + \sigma dB(t) + dL(t) \quad (2.68)$$

After substituting the missing observations by the closest two data points the logarithm of spot prices is plotted in Figure 2.9. We can notice from

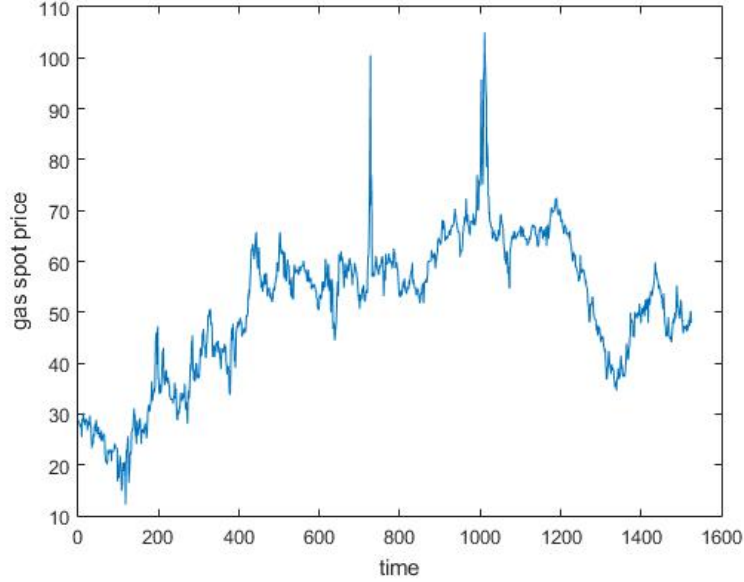


Figure 2.9: The logarithm of gas spot prices from the UK NBP during 4 September 2009-4 July 2015.

Figure 2.9 the presence of outliers which represent large positive or negative price spikes. The histogram in Figure 2.10 shows that these outliers are not normally distributed. Before proceeding to the estimation of trend and seasonality parameters we can exclude the outliers from the data series and replace them with the mean of two closest points (see Benth et al.¹³). The trend and seasonal component is modelled by the function

$$\ln \Lambda(t) = a_0 + a_1 t + a_2 \cos(2\pi(t - a_3)/250) \quad (2.69)$$

where 250 stands for the number of trading days in a year. By implementing the least squares approach we can estimate the parameters of $\ln \Lambda(t)$ as

$$(a_0, a_1, a_2, a_3) = (3.5713, 0.0004, -0.0889, 338.5201), \quad RMSE = 7.2$$

where RMSE is the root mean-square error. If we subtract $\ln \Lambda(t)$ from the original data set (with outliers) then we get the detrended and deseasonalised logarithmic prices depicted in Figure 2.11. The empirical autocorrelation function (ACF) and partial ACF (PACF) are presented in Figures 2.12a

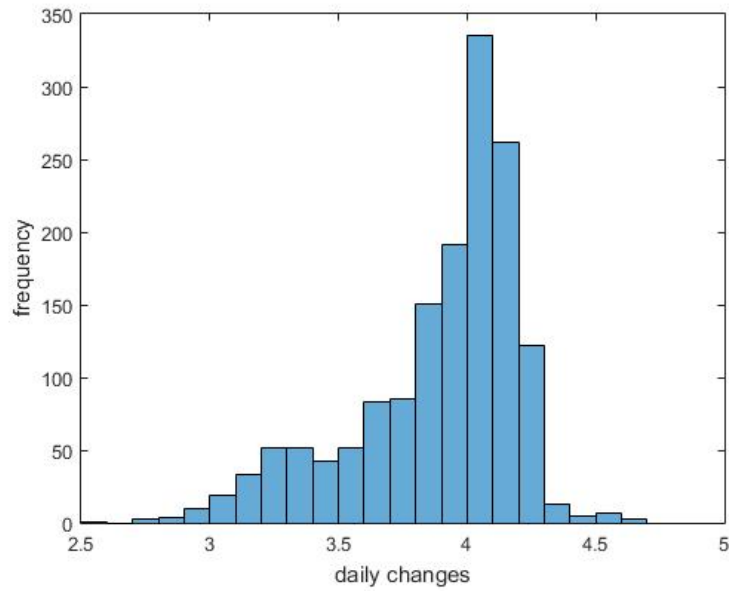


Figure 2.10: Histogram of daily changes in the logarithmic gas spot prices.

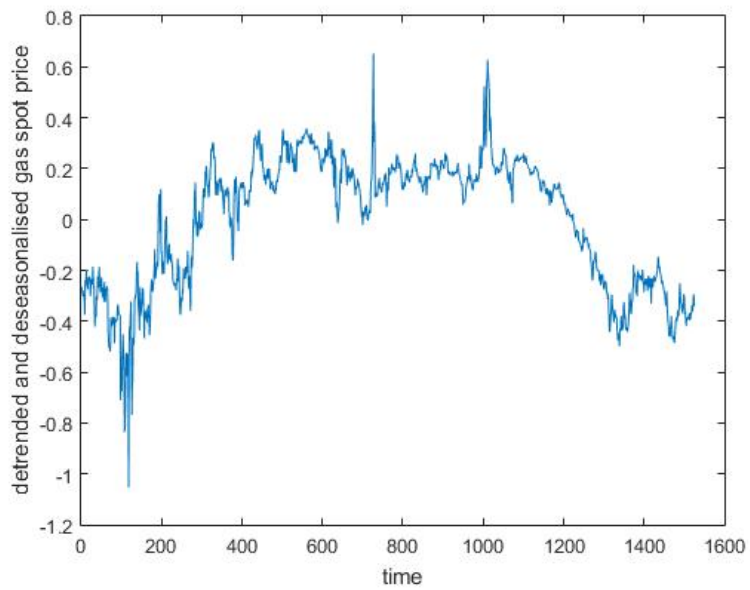


Figure 2.11: Detrended and deseasonalized logarithm of gas spot prices.

and 2.12b. There is a strong enough memory effect in detrended and deseasonalised data according to the PACF plot. Hence, we can make use of

AR(1) process

$$z(t+1) = \gamma z(t) + \epsilon(t) \quad (2.70)$$

to model our time series (see Benth et al.¹³). We assume that $\epsilon(t)$ is an iid process and the autoregression coefficient is constant. We denote

$$Z(t) \triangleq X(t) + Y(t) = \ln S(t) = \ln \Lambda(t)$$

in order to connect the discrete time series to our continuous-time model (2.68). In other words, we have

$$dZ(t) = d \ln S(t) = -\alpha Z(t)dt + \sigma dB(t) + dL(t)$$

After the discretisation of differentials with increments $\Delta Z(t) = Z(t+1) - Z(t)$, $\Delta B(t) = B(t+1) - B(t)$ and $\Delta L(t) = L(t+1) - L(t)$ we obtain

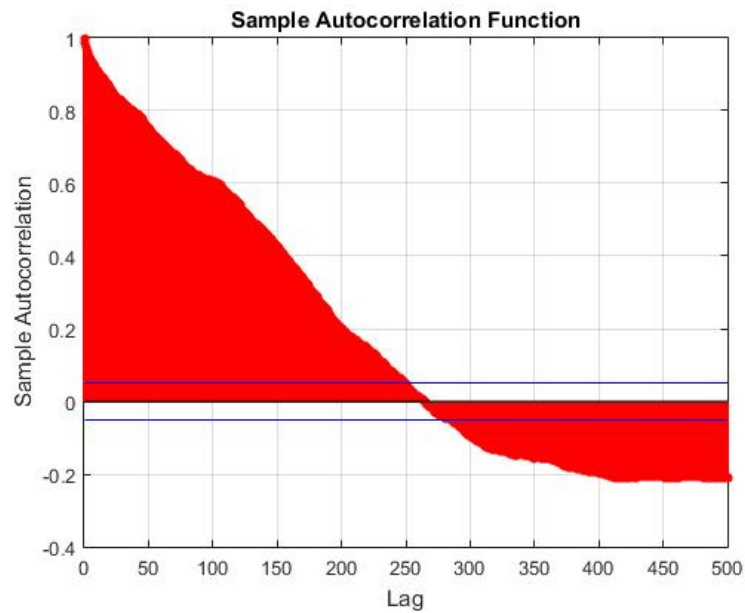
$$Z(t+1) = (1 - \alpha)Z(t) + \sigma \Delta B(t) + \Delta L(t)$$

where $\sigma \Delta B(t) + \Delta L(t)$ is an i.i.d. sequence of random variables for $t = 1, 2, \dots$. So, if we take $\gamma = 1 - \alpha$ and $\epsilon(t) = \sigma \Delta B(t) + \Delta L(t)$, then $Z(t)$ and $z(t)$ coincide. Implementation of autoregressive function `ar(y,n)` in MATLAB will give us the estimate for γ :

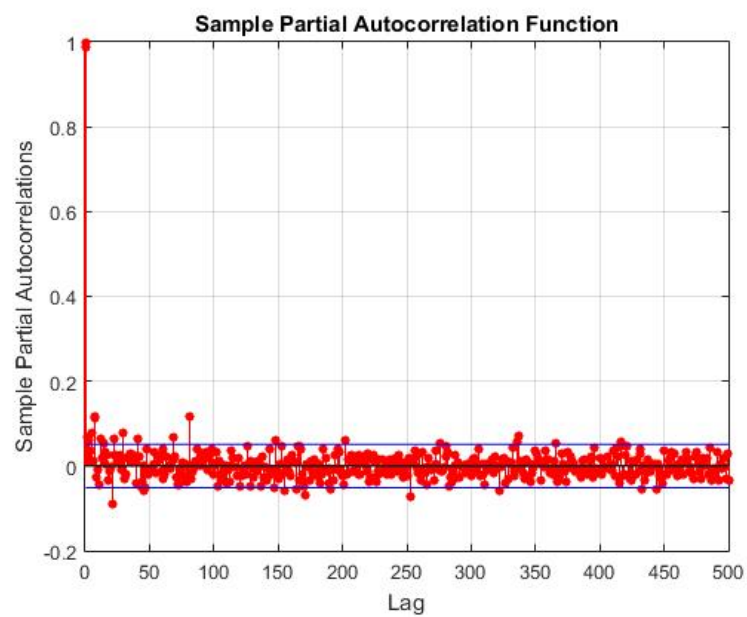
$$\hat{\gamma} = 0.9862, \quad RMSE = 4.22$$

Then we can imply that $\hat{\alpha} = 0.0138$

The plot of the residuals after the removal of AR(1) process is presented in Figure 2.13. Meanwhile, Figure 2.14 plots the histogram of the residuals with the fitted normal density curve. It's obvious that the residuals are not normally distributed. We model the time series of the residuals by means of a normal inverse Gaussian (NIG) Lévy process.



(a) ACF



(b) PACF

Figure 2.12: ACF and PACF of detrended and deseasonalized logarithm of gas spot prices with 95% condence interval.

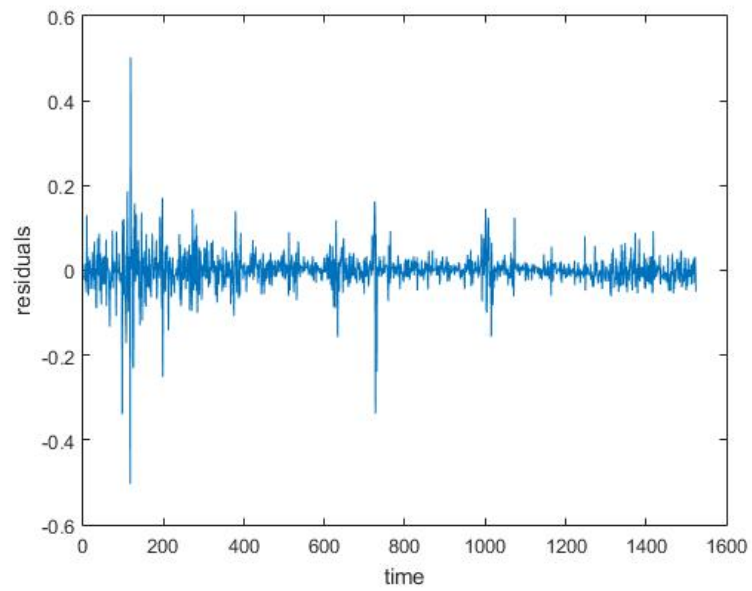


Figure 2.13: Residuals after the elimination of trend, seasonal component and AR(1) process.

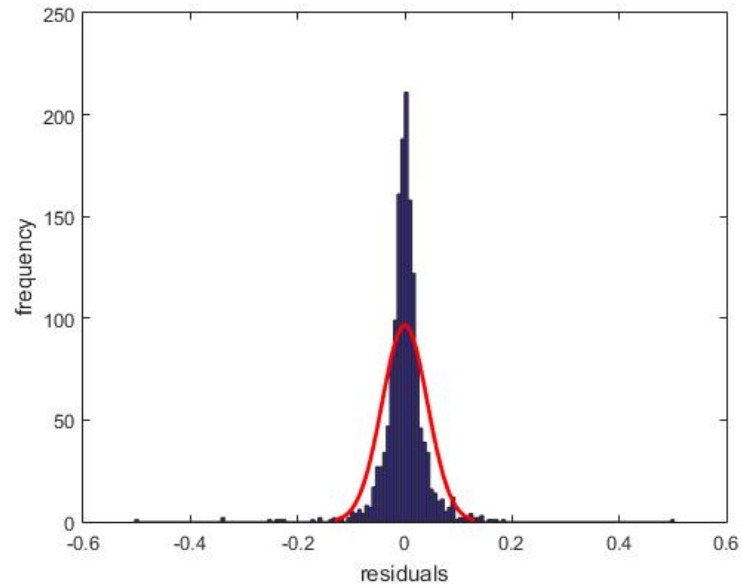


Figure 2.14: Histogram of the residuals with normal density curve after trend, seasonal component and AR(1) process were eliminated.

2.8.1 MODELLING THE RESIDUALS WITH THE NIG DISTRIBUTION

The NIG Lévy process was first applied in a financial framework by⁵. A process $L(t)$ is called a NIG Lévy process if $L(1) \sim NIG(\zeta, \beta, \delta, \mu)$. The NIG distribution is a mean-variance mixture model with the following density function:

$$f_{NIG}(x) = \frac{\zeta}{\pi} \exp(\delta \sqrt{\zeta^2 - \beta^2} + \beta(x - \mu)) \frac{K_1 \left(\zeta \delta \sqrt{1 + \frac{(x-\mu)^2}{\delta}} \right)}{\sqrt{1 + \frac{(x-\mu)^2}{\delta}}} \quad (2.71)$$

μ and β are location and skewness parameters respectively. $\zeta \geq |\beta|$ measures the heaviness of the tails while $\delta > 0$ is the scale parameter. We can estimate these parameters by using Expectation-Maximization (EM) algorithm for NIG⁵⁹. This is a algorithm which computes the parameters iteratively until the absolute difference of two subsequent iterations is less than a tolerance criteria. If we set the criteria to be 0.0000001, then the parameter estimates are

$$(\zeta, \beta, \delta, \mu) = (11.3458, -0.1533, 0.0179, 0.0002)$$

2.9 CONCLUSION AND FUTURE RESEARCH PLANS

In this chapter we analysed a stochastic control problem for natural gas storage facilities while taking into account operating characteristics. The real option based valuation approach by Thompson et al.⁷⁴ was combined with the spot price dynamics presented by Benth and Sgarra¹¹. The key differences of this two-factor spot price model from the original one are instant spikes with a time-dependent jump intensity. The resulting non-linear HJB equation was solved by an explicit finite difference method. The numerical approach gives us both the value of the storage contract and optimal operating strategy depending on the gas price, current level of gas in the storage and time. The monotonicity issue of the scheme was solved by using the corrector of the diffusion term of the PIDE. The numerical method was implemented in MATLAB for different underlying models. The effects of

seasonality and seasonal jump intensity were analysed by observing the surfaces of control strategies and contract values. It was revealed that control strategies with time-dependent and constant jump intensities were almost the same. The convergence of the scheme was confirmed by obtaining close results while increasing the number of time steps. We also looked at value surfaces depending on spot prices and long-run mean at different levels of gas in the reservoir. Finally, the value surface with respect to long-run mean and inventory level was investigated.

3

NATURAL GAS-FIRED POWER PLANTS VALUATION AND OPTIMISATION UNDER LÉVY COPULAS AND REGIME-SWITCHING

*

3.1 INTRODUCTION

The liberalisation of electricity and other energy markets, such as PJM in the United State, UKPX in the United Kingdom, Nord Pool in North Europe, and JEPX in Japan makes it necessary to incorporate highly volatile spot price dynamics into the optimisation problems for power generation.

During the traditional regulatory regime, the regulators used to set electricity

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prices based on cost of service. Investments in energy generating facilities were allowed to earn a fixed return through electricity tariffs upon approval by the regulators. The economic viability of such investments and value of power plants could be calculated via a discounted cash flow (DCF) method. But it has been shown that the DCF approach doesn't give correct results as it ignores the opportunity costs (see Dixit and Pindyck³⁴). Meanwhile, the existence of competitive electricity markets necessitates the valuation of power plants by means of financial instruments. This kind of real option approach to electricity generation was proposed by Deng et al.³³. Deng and Oren³² further extend this work by including some physical operating constraints. Note that the omission of physical characteristics usually results in over-valuation.

The real options approach to power plant investment and optimisation problems has become an outstanding research area of different scientific groups. Pindyck⁶³ applies this method to investigate the investment decisions to build a nuclear power plant. Tseng and Barz⁷⁵ propose forward-moving Monte Carlo simulation with backward-moving dynamic programming technique to solve a multistage stochastic model used to evaluate a power plant with unit commitment constraints. Näsäkkälä and Fleten⁵⁶ obtain a method to calculate thresholds for building a base load plant and upgrading it to a peak load plant when spark spread covers emission related costs. Takashima et al.⁷² investigate the investment problem for a power plant construction under the assumption that electricity prices are determined by the supply function and the equilibrium quantity.

Power generating assets can be classified as (a) nuclear, (b) hydroelectric, (c) fossil, or (d) renewable. In this work we will only consider fossil power stations. In general, a thermal power plant is a broader concept than fossil-fueled power generator. But we will use them interchangeably in this chapter for simplicity purposes. Fossil power plants convert heat energy to electric power. More precisely, chemical energy in fossil fuels is first used to heat the water in the boiler. The resulting steam then spins the turbine which drives an electrical generator. The reader can refer to Wood and Wollenberg⁷⁷ for more details regarding the operation of thermal power plants. Coal, natural

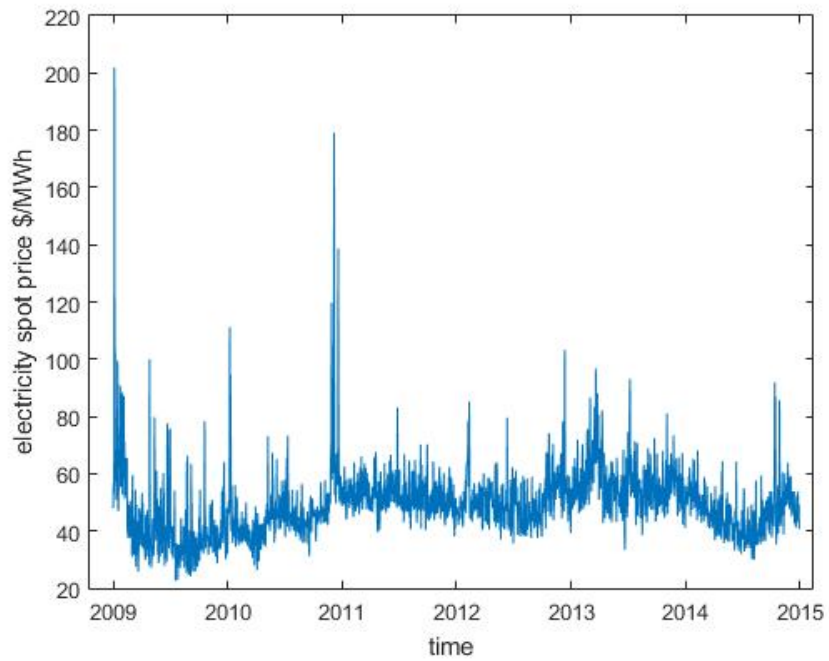
gas and heating oil are the most commonly used fuels for power generation. However, natural gas has become a primary fuel source for thermal power generators because of its low CO_2 emissions and flexibility during peak hours. The main objective in operating a power plant is to maximise the profit by utilising the difference (or spread) between electricity and natural gas prices. So, the valuation of a gas-fired power plant is closely related to spark spread, which is the most important cross-commodity transaction in power markets and is presented by Deng et al.³³ as

$$S(t) = S_e(t) - cS_g(t) \quad (3.1)$$

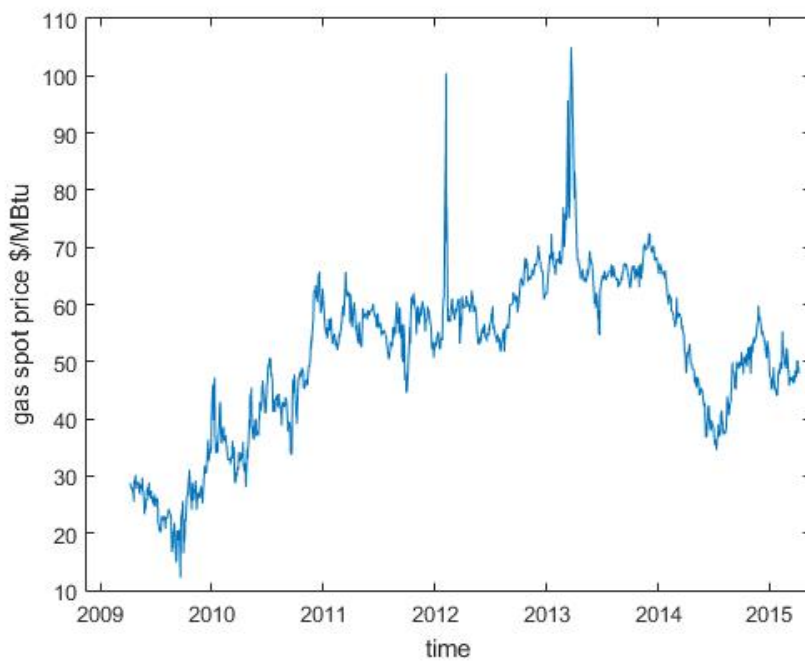
where S_e and S_g are electricity and gas spot prices, respectively. Heat rate is the amount of gas required to generate 1 MWh of electricity which is defined by heat rate c . It is a measure of efficiency. More precisely, the higher is the heat rate, the less efficient is the electricity generation. The heat rate has been usually assumed to be constant for convenience purposes (see Deng et al.³³, Benth and Kettler¹⁰, Meyer-Brandis and Morgan⁵⁵). Deng and Oren³² generalise this approach by introducing an operating heat rate which changes depending on the discrete output levels. Thompson et al.⁷³ consider the case where c is the amount of natural gas that needs to be burned to maximise the total revenue from electricity generation. In this model c depends continuously on time, boiler temperature, gas and electricity prices and is calculated via dynamic optimisation techniques.

The evaluation of a power plant cannot deliver satisfactory results without realistic stochastic models for underlying spot prices (see Fig 3.1). The detailed discussion of the related literature was presented in Safarov and Atkinson⁶⁶. For completeness purposes we will briefly repeat them here.

A mean-reverting process is a common model for the energy spot price. For example, Schwartz⁶⁹ presented three mean reverting models for commodity spot prices while assuming that stochastic convenience yield and interest rates also mean revert. Boogert and De Jong¹⁷ used the Least Square Monte Carlo method (initially proposed by Longstaff and Schwartz⁵³ for American options) for gas storage valuation, while assuming the following one factor



(a) Electricity



(b) Natural gas

Figure 3.1: Day ahead UK spot prices for power and natural gas (NBP) during the time period 4 Sept 2009 - 4 July 2015 based on APX Power UK and Bloomberg.

model for the spot price, which is calibrated to the initial futures curve:

$$\frac{dS(t)}{S(t)} = \kappa(\mu(t) - \ln S(t))dt + \sigma dB(t) \quad (3.2)$$

where B is a standard Brownian motion, μ is a time-dependent parameter, calibrated to the initial futures curve $(F(0, T))_{T \geq 0}$, provided by the market; the mean reversion parameter κ and the volatility σ are positive constants. One factor models are popular because of their simplicity. As is stated in Bjerk Sund et al.¹⁶, Henaff et al.⁴⁶, it's unrealistic to assume that one factor models can capture all characteristics of highly volatile power or natural gas prices. Parsons⁶⁰ extended equation (3.2) to the two-factor mean-reverting model where the long-term mean is also a mean-reverting process. Boogert and de Jong¹⁸ extended the application of the Least Square Monte Carlo method to multi-factor spot processes.

The unstorability of electricity in large quantities and storage complexities of natural gas create frequent demand-supply fluctuations in the corresponding commodity markets. These disruptions lead to the spiky behavior of energy spot prices. Therefore, price spikes have to be included in our spot price models as they carry out significant arbitrage opportunities.

However, none of the models described above exhibit spiky instantaneous jumps of energy prices. Thompson et al.^{73, 74} propose to model the risk adjusted power and natural spot prices, respectively by the following one dimensional continuous time Markov process:

$$dP = \mu_1(P, t)dt + \sigma_1(P, t)dX_1 + \sum_{k=1}^N \gamma_k(P, t, J_k)dq_k \quad (3.3)$$

where μ, σ , and the γ_k 's are any arbitrary functions and the J_k s are arbitrary jump sizes with distributions $Q_k(J)$. X_1 and dq_k 's denote the standard Brownian motion and Poisson processes respectively. Eq. (3.3) covers a wide range of spot price models for natural gas and is tractable for numerical calculations. However, its jump component isn't appropriate to represent instant and mean-reverting spikes.

Benth et al.¹² propose a more realistic alternative by defining energy spot dynamics as a sum of Ornstein-Uhlenbeck (OU) processes with different mean-reverting speeds. The only sources of randomness of the spot prices are positive pure jump processes. The additive structure of the model makes it difficult to derive futures curves. Therefore, Benth⁹ considers a stochastic volatility model by Barndorff-Nielsen and Shephard⁶ for equity markets in the context of energy commodities. Benth et al.¹⁴ generalise this non-Gaussian approach to a multivariate case to model the dynamics of cross-commodity spot prices.

The volatile behaviour of energy prices can differ significantly between cold and warm months. Benth et al.¹³ make use of time-inhomogeneous jump processes to capture these seasonality effects by introducing geometric and arithmetic spot price dynamics in the following way:

$$\ln S(t) = \ln \Lambda(t) + \sum_{i=1}^m X_i(t) + \sum_{j=1}^n Y_j(t) \quad (\text{geometric}) \quad (3.4)$$

and

$$S(t) = \Lambda(t) + \sum_{i=1}^m X_i(t) + \sum_{j=1}^n Y_j(t) \quad (\text{arithmetic}) \quad (3.5)$$

where

$$dX_i(t) = (\mu_i(t) - \alpha_i(t)X_i(t))dt + \sum_{k=1}^p \sigma_{ik}(t)dB_k(t) \quad (3.6)$$

$$dY_j(t) = (\delta_j(t) - \beta_j(t)Y_j(t))dt + \eta_j(t)dU_j(t) \quad (3.7)$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Here B_k and U_j are independent standard Brownian motions and independent time-inhomogeneous Lévy processes, respectively. $\Lambda(t)$ is a deterministic, positive and continuously differentiable seasonality function, while the coefficients $\mu_i, \alpha_i, \delta_j, \beta_j, \eta_j$ and σ_{ik} are continuous functions of t .

In Safarov and Atkinson⁶⁶ the optimal operation problem for gas storage was investigated based on the geometric model. But for the optimisation of

gas-fired power plant we will follow the arithmetic dynamics based on the justification presented in Benth et al.¹³, Meyer-Brandis and Morgan⁵⁵.

Accurate valuation and optimisation of power plants is impossible without the analysis of the multivariate dependence between electricity and gas prices. For Gaussian OU components it suffices to specify the appropriate covariance matrix. This approach is not useful when we consider non-Gaussian spot price models. Meanwhile, Meyer-Brandis and Morgan⁵⁵ proposed to make use of Lévy copulas to model the dependence between the non-Gaussian OU components of electricity and gas spot prices. Although Thompson et al.⁷³ develop a comprehensive stochastic control model which incorporates all the necessary operational constraints, the dependence of non-Gaussian jump components is assumed to be independent. In the actual calculations the gas prices are even assumed to be constant which is unlikely to be true under unregulated markets.

An alternative approach is the regime-switching framework initially proposed by Hamilton⁴⁵. In a regime-switching model the price process can randomly shift between several regimes due to long-term demand-supply fluctuations, political instability, weather changes and other reasons. The spot price follows a distinct stochastic process within each regime. Carmona and Ludkovski²⁴ and Chen and Forsyth²⁶ apply a regime-switching approach to investigate stochastic control problems related to natural gas storage and implement the numerical calculation via Monte Carlo and finite difference methods, respectively. However, there is a lack of research related to the valuation of a power plant or spark-spread option under regime-switching models.

In this chapter we investigate the valuation and optimal operation of gas-fired power generating facilities and refer to Thompson et al.⁷³ for the physical operational constraints. But we assume that the underlying electricity and gas spot prices follow the Lévy regime-switching model. More precisely, at each regime the spot dynamics is defined by a special case of an arithmetic model (3.7). For the dependence of gas and electricity spikes we make use of skewed Lévy copulas for the reasons stated in Meyer-Brandis and Morgan⁵⁵. Therefore, our approach not only incorporates the most important physi-

cal characteristics of a power plant but also takes into account short-term demand-supply variations and long-term economic disruptions.

We assume that the reader is familiar with fundamentals of the theory of Lévy processes, as briefly presented in Chapter 1. Cont and Tankov²⁸, Schoutens⁶⁸ and Kyprianou⁵¹ provide more detailed background.

To solve this stochastic control problem we employ the dynamic programming theory which gives us coupled Hamilton-Jacobi-Bellman (HJB) equations. We then solve this system of nonlinear partial integro-differential equations (PIDE) numerically.

3.2 NATURAL GAS AND ELECTRICITY SPOT PRICES WITHOUT REGIME SWITCHING

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space. We assume that the underlying spot prices follow the arithmetic model defined by eqs. (3.5) to (3.7). For simplicity, we take $m = n = p = 1$. We also define the initial condition

$$X(0) + Y(0) = S(0) - \Lambda(0) \quad (3.8)$$

Then we have the following dynamics for $S(t)$:

$$S(t) = \Lambda(t) + X(t) + Y(t) \quad (3.9)$$

where

$$dX(t) = (\mu(t) - \alpha(t)X(t))dt + \sigma(t)dB(t) \quad (3.10)$$

$$dY(t) = (\delta(t) - \beta(t)Y(t))dt + \eta(t)dU(t) \quad (3.11)$$

We further assume that the coefficients of eqs. (3.10) and (3.11) are constants such that $\mu(t) = \delta(t) = 0, \alpha(t) = \beta(t) = \alpha$ (to reduce the number of state variables of the corresponding PDE), $\sigma(t) = \sigma$ and $\eta(t) = 1$. Then the

dynamics of electricity spot prices is given by

$$S^e(t) = \Lambda_e(t) + X^e(t) + Y^e(t) \quad (3.12)$$

where

$$dX^e(t) = -\alpha_e X^e(t)dt + \sigma_e dB^e(t) \quad (3.13)$$

$$dY^e(t) = -\alpha_e Y^e(t)dt + dU^e(t) \quad (3.14)$$

The underlying natural gas spot price follows the similar model:

$$S^g(t) = \Lambda_g(t) + X^g(t) + Y^g(t) \quad (3.15)$$

where

$$dX^g(t) = -\alpha_g X^g(t)dt + \sigma_g dB^g(t) \quad (3.16)$$

$$dY^g(t) = -\alpha_g Y^g(t)dt + dU^g(t) \quad (3.17)$$

Note that we could take unequal mean-reversion rates for normal variations and jumps in our spot price models following Safarov and Atkinson⁶⁶. But that would increase the dimension of our HJB equation (state variables) which is derived in Section 3.3. We keep them equal to reduce the computational time.

In general, U^e and U^g can be subordinator Lévy processes

$$U^e(t) = \int_0^t \int_{\mathbb{R}_+} z J^e(dz, ds) \quad \text{and} \quad U^g(t) = \int_0^t \int_{\mathbb{R}_+} z J^g(dz, ds) \quad (3.18)$$

where $J^e(dz, ds)$ and $J^g(dz, ds)$ are Poisson random measures with the corresponding Lévy intensity measures $\nu^e(dz, ds)$ and $\nu^g(dz, ds)$ (see Jacod and Shiryaev⁴⁸). These jump measures have positive supports because we only consider upward price spikes. Following Meyer-Brandis and Morgan⁵⁵ we further assume that U^e and U^g are compound Poisson processes with intensities λ_e and λ_g and jump size distributions D_e and D_g , respectively. In other

words,

$$\nu^e(dz, ds) = \lambda_e D_e(dz) ds = \nu^e(dz) ds \quad (3.19)$$

$$\nu^g(dz, ds) = \lambda_g D_g(dz) ds = \nu^g(dz) ds \quad (3.20)$$

The complete characterisation of the two-dimensional model $(S^e(t), S^g(t))$ requires the specification of the dependence risk between the spot prices S^e and S^g . We can separate this problem in defining the multivariate distributions of $(B^e(t), B^g(t))$ and $(U^e(t), U^g(t))$. The former issue can be easily resolved by determining the correlation parameter ρ between $B^e(1)$ and $B^g(1)$

$$dB^e(t)dB^g(t) = \rho dt \quad (3.21)$$

To specify the dependence between power and gas spot prices we will refer to Meyer-Brandis and Morgan⁵⁵ for using positive Lévy copulas. This concept will be analysed in Section 3.4.

Before moving to the optimisation problem of a power plant we can write the electricity and gas spot price models in a more convenient SDE form as follows:

$$\begin{aligned} dS^e(t) &= d\Lambda_e(t) + dX^e(t) + dY^e(t) \\ &= \Lambda'_e(t)dt - \alpha_e(X^e(t) + Y^e(t))dt + \sigma_e dB^e(t) + UL^e(t) \\ &= \Lambda'_e(t)dt - \alpha_e(S^e(t) - \Lambda_e(t))dt + \sigma_e dB^e(t) + dU^e(t) \\ &= (\bar{\Lambda}_e(t) - \alpha_e S^e(t))dt + \sigma_e dB^e(t) + dU^e(t) \end{aligned} \quad (3.22)$$

where $\bar{\Lambda}_e(t) = \Lambda'_e(t) + \alpha_e \Lambda_e(t)$. Similarly, we can derive the SDE for the gas spot price

$$dS^g(t) = (\bar{\Lambda}_g(t) - \alpha_g S^g(t))dt + \sigma_g dB^g(t) + dU^g(t) \quad (3.23)$$

Note that we can get the compensated compound Poisson processes $\tilde{U}^e(t)$ and $\tilde{U}^g(t)$ as follows:

$$\tilde{U}^e(t) = \int_0^t \int_{\mathbb{R}_+} z(J^e(dz, ds) - \nu^e(dz, ds)) = \int_0^t \int_{\mathbb{R}_+} z\tilde{J}^e(dz, ds) \quad (3.24)$$

$$\tilde{U}^g(t) = \int_0^t \int_{\mathbb{R}_+} z(J^g(dz, ds) - \nu^g(dz, ds)) = \int_0^t \int_{\mathbb{R}_+} z\tilde{J}^g(dz, ds) \quad (3.25)$$

where $\tilde{J}^e$ and $\tilde{J}^g$ are compensated Poisson random measures. Then eqs. (3.22) and (3.23) transform to

$$dS^e(t) = (\tilde{\Lambda}_e(t) - \alpha_e S^e(t))dt + \sigma_e dB^e(t) + \int_{\mathbb{R}_+} z\tilde{J}^e(dz, dt) \quad (3.26)$$

$$dS^g(t) = (\tilde{\Lambda}_g(t) - \alpha_g S^g(t))dt + \sigma_g dB^g(t) + \int_{\mathbb{R}_+} z\tilde{J}^g(dz, dt) \quad (3.27)$$

where $\tilde{\Lambda}_e(t) = \bar{\Lambda}_e(t) + \int_{\mathbb{R}_+} z\nu^e(dz)$ and $\tilde{\Lambda}_g(t) = \bar{\Lambda}_g(t) + \int_{\mathbb{R}_+} z\nu^g(dz)$

3.3 OPTIMAL CONTROL OF POWER GENERATION

The theoretical framework and physical constraints for the optimal operation of a gas-fired power generator will be based on Thompson et al.⁷³. Therefore, the following physical characteristics need to be incorporated in our optimisation model:

- The minimum generation temperature: The plant cannot operate below a certain temperature.
- Variable start-up times: The time required to heat the boiler to its minimum generation level depends on the heat rate of natural gas and the current temperature.
- The variable output rates: The efficiency of the plant varies non-linearly with the amount of generated electricity.
- The variable start-up and production costs. Increase of electricity output is achieved by raising the boiler temperature. The fuel used for

this heat increase does not generate any additional power until the required temperature is reached.

- Control response time lags: Changing boiler temperature takes some time. Therefore, any decision to alter the power output will take effect after a reasonable amount of time.
- The ramp rate limit: This is the minimum amount of time required for switching a unit on/off to avoid thermal stress and fatigue.

Following Thompson et al.⁷³ we model the boiler temperature L as a state variable and incorporate all these operational constraints in its mean-reverting equation

$$dL = \eta(L, c)(\bar{L}(c) - L)dt \quad (3.28)$$

where c is the variable controlling the amount of gas consumed per instant in time; $\bar{L}(c)$ is the equilibrium temperature of the generator depending on c ; η is a mean-reverting speed function specific to each power plant. More precisely, η is the speed at which L will revert back to $\bar{L}(c)$ when $L \neq \bar{L}(c)$. The non-linear dependence of the amount of produced electricity on the boiler temperature is given by the output function $H(L)$. We can now move to our stochastic control problem for the power plant.

The objective of the optimisation is to find a control variable $c(S_e, S_g, L, t)$ that maximises the expected future cash flow up to maturity time T and discounted at a rate r

$$V(S_e, S_g, L, t) = \max_{c(S_e, S_g, L, t)} \mathbb{E} \left[\int_t^T e^{-r(\tau-t)} (H(L)S_e - S_g c) d\tau \right] \quad (3.29)$$

subject to

$$c_{min}(L) \leq c \leq c_{max}(L) \quad (3.30)$$

The minimum and maximum bounds on c are necessary to avoid thermal stresses.

Rewriting (3.29) in a similar way to Thompson et al.⁷³ will lead to the Bellman equation:

$$\begin{aligned}
V &= \max_c \mathbb{E} \left[\int_t^{t+dt} e^{-r(\tau-t)} (H(L)S_e - S_g c) d\tau + \int_{t+dt}^T e^{-r(\tau-t)} (H(L)S_e - S_g c) d\tau \right] \\
&= \max_c \mathbb{E} \left[\int_t^{t+dt} e^{-r(\tau-t)} (H(L)S_e - S_g c) d\tau + e^{-r dt} \int_{t+dt}^T e^{-\rho(\tau-(t+dt))} (H(L)S_e - S_g c) d\tau \right] \\
&= \max_c \mathbb{E} \left[\int_t^{t+dt} e^{-r(\tau-t)} (H(L)S_e - S_g c) d\tau + e^{-r dt} V(S_e + dS_e, S_g + dS_g, L + dL, t + dt) \right]
\end{aligned} \tag{3.31}$$

We can now apply the multidimensional Itô's formula (see Cont and Tankov²⁸) to $V(S_e, S_g, L, t)$ to expand it above in Taylor's series:

$$\begin{aligned}
V &= \max_c \mathbb{E} [(H(L)S_e - S_g c)dt + (1 - rdt)[V + V_t dt + V_L dL \\
&\quad + V_{S_e} dS_e + V_{S_g} dS_g + \frac{1}{2} V_{S_e S_e} d[S_e, S_e]_t^c + \frac{1}{2} V_{S_g S_g} d[S_g, S_g]_t^c + V_{S_e S_g} d[S_e, S_g]_t^c \\
&\quad + \iint_{\mathbb{R}_+^2} (V(S_e + z_e, S_g + z_g, L, t) - V(S_e, S_g, L, t) - z_e V_{S_e} - z_g V_{S_g}) J(dz, dt)]
\end{aligned} \tag{3.32}$$

where $J(dz, dt) = J(dz_e, dz_g, dt)$ is a two-dimensional Poisson random measure with Lévy intensity measure $\nu(dz, dt)$ and

$$d[S_e, S_e]_t^c = \sigma_e^2 dt, \quad d[S_g, S_g]_t^c = \sigma_g^2 dt \quad \text{and} \quad d[S_e, S_g]_t^c = \rho \sigma_e \sigma_g dt \tag{3.33}$$

We then denote

$$\mu_e(S^e, t) = \tilde{\Lambda}_e(t) - \alpha_e S^e(t) \tag{3.34}$$

and

$$\mu_g(S^g, t) = \tilde{\Lambda}_g(t) - \alpha_g S^g(t) \tag{3.35}$$

for clarity purposes. After substituting eqs. (3.26) to (3.28) and (3.33) in equation (3.32) we obtain

$$\begin{aligned}
V = \max_c \mathbb{E}[(H(L)S_e - S_g c)dt + (1 - rdt)(V + V_t + \\
\eta(L, c)(\bar{L}(c) - L)V_L + \mu_e(S^e, t)V_{S_e} + \mu_g(S^g, t)V_{S_g} + \\
\frac{1}{2}\sigma_e^2 V_{S_e S_e} + \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \rho\sigma_e\sigma_g V_{S_e S_g})dt + \sigma_e dB^e(t) + \sigma_g dB^g(t) \\
\iint_{\mathbb{R}_+^2} (V(S_e + z_e, S_g + z_g, L, t) - V(S_e, S_g, L, t) - z_e V_{S_e} - z_g V_{S_g})\nu(dz, dt) + \\
\iint_{\mathbb{R}_+^2} (V(S_e + z_e, S_g + z_g, L, t) - V(S_e, S_g, L, t))\tilde{J}(dz, dt)] \quad (3.36)
\end{aligned}$$

Elimination of all terms that go to zero faster than dt and some further simplifications similar to Safarov and Atkinson⁶⁶ results in the following HJB equation

$$\begin{aligned}
\max_c [V_t + \frac{1}{2}\sigma_e^2 V_{S_e S_e} + \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \rho\sigma_e\sigma_g V_{S_e S_g} + \mu_e(S^e, t)V_{S_e} + \mu_g(S^g, t)V_{S_g} - rV + \\
\iint_{\mathbb{R}_+^2} (V(S_e + z_e, S_g + z_g, L, t) - V(S_e, S_g, L, t) - z_e V_{S_e} - z_g V_{S_g})\nu(dz, dt) + \\
H(L)S_e - S_g c + \eta(L, c)(\bar{L}(c) - L)V_L] = 0 \quad (3.37)
\end{aligned}$$

Note that the control variable c only appears in the last two terms. We also need to introduce time to maturity $\tau = T - t$. Then HJB equation above becomes

$$\begin{aligned}
V_\tau = \frac{1}{2}\sigma_e^2 V_{S_e S_e} + \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \rho\sigma_e\sigma_g V_{S_e S_g} + \mu_e(S^e, T - \tau)V_{S_e} + \mu_g(S^g, T - \tau)V_{S_g} - rV + \\
\iint_{\mathbb{R}_+^2} (V(S_e + z_e, S_g + z_g, L, \tau) - V(S_e, S_g, L, \tau) - z_e V_{S_e} - z_g V_{S_g})\nu(dz, d\tau) + \\
\max_c [H(L)S_e - S_g c + \eta(L, c)(\bar{L}(c) - L)V_L] = 0 \quad (3.38)
\end{aligned}$$

where

$$c_{min}(L) \leq c \leq c_{max}(L) \quad (3.39)$$

In other words, the maximisation problem reduces to

$$c^*(S_e, S_g, L, \tau) = \arg \max_c [H(L)S_e - S_g c + \eta(L, c)(\bar{L}(c) - L)V_L] \quad (3.40)$$

After the calculation of the optimal strategy c^* we substitute it in equation (3.38) to get

$$\begin{aligned} V_\tau = & \frac{1}{2}\sigma_e^2 V_{S_e S_e} + \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \rho\sigma_e\sigma_g V_{S_e S_g} + \mu_e(S^e, T-\tau)V_{S_e} + \mu_g(S^g, T-\tau)V_{S_g} - rV + \\ & \int \int_{\mathbb{R}_+^2} (V(S_e + z_e, S_g + z_g, L, \tau) - V(S_e, S_g, L, \tau) - z_e V_{S_e} - z_g V_{S_g}) \nu(dz, d\tau) + \\ & H(L)S_e - S_g c^* + \eta(L, c^*)(\bar{L}(c^*) - L)V_L] \quad (3.41) \end{aligned}$$

Thus, once the optimal control c^* is known, the nonlinear HJB equation simplifies to linear PIDE. Before starting our numerical approach to solve equation (3.41), we analyse the two-dimensional Lévy measure ν by means of Lévy copulas in the next section.

3.4 LÉVY COPULA

We have already introduced Lévy copulas in Chapter 1. Before starting the analysis of the dependence of $(L^e(t), L^g(t))$ we repeat some of those concepts below for convenience.

Theorem 39. (*Sklar's Theorem for Lévy copulas*) *Let $(X(t), Y(t))$ be a two-dimensional Lévy process with positive jumps that have tail integral $U(x, y)$ and marginal tail integrals $U_1(x)$ and $U_2(y)$. Then there exists a unique positive Lévy copula F such that:*

$$U(x, y) = F(U_1(x), U_2(y)), \quad \forall x, y \in [0, \infty) \quad (3.42)$$

Conversely, if F is a positive Lévy copula, $X(t)$ and $Y(t)$ are one-dimensional Lévy processes with tail integrals U_1, U_2 then there exists a two-dimensional Lévy process such that its tail integral is given by equation (3.42).

An Archimedean Lévy copula is a popular parametric Lévy copula defined

as

$$F(x, y) = \phi^{-1}(\phi(x) + \phi(y)) \quad (3.43)$$

where ϕ is a strictly decreasing convex function with positive support such that $\phi(0) = \infty$ and $\phi(\infty) = 0$. In these models F always has the symmetry property $F(x, y) = F(y, x)$. However Meyer-Brandis and Morgan⁵⁵ argue that this symmetry property is not supported by the data. Therefore they introduce skewed Archimedean Lévy copulas as follows

$$F(x, y) = \phi^{-1}(\psi_1(y)\phi(x) + \psi_2(x)\phi(y)) \quad (3.44)$$

where ψ_1 and ψ_2 are decreasing functions satisfying $\psi_1(\infty) = \psi_2(\infty) = 1$, while ϕ is the same as above. In our calculations we will use the skewed Clayton- Lévy copula where

$$\phi(x) = x^{-\theta}, \quad \psi_1(x) = (\alpha x^{-\beta} + 1), \quad \text{and} \quad \psi_2 \equiv 1 \quad (3.45)$$

for $\alpha > 0, \theta > 0$ and $0 < \beta \leq \theta + 1$. More precisely,

$$F(x, y) = ((\alpha y^{-\beta} + 1)x^{-\theta} + y^{-\theta})^{-1/\theta} \quad (3.46)$$

We can construct multivariate jump densities from univariate ones by using Lévy copulas. The following is the two-dimensional version of the Lemma proposed by Reich et al.⁶⁵.

Lemma 40. *Let $f \in C^2(\mathbb{R}^2)$ be a bounded function vanishing on a neighbourhood of the origin. Moreover, let $(X(t), Y(t))$ be a two-dimensional Lévy process with Lévy measure ν , Lévy copula F , and marginal Lévy measures ν_1 and ν_2 . Then*

$$\begin{aligned} \int_{\mathbb{R}^2} f(x, y) \nu(dx, dy) &= \int_{\mathbb{R}} \frac{\partial f}{\partial x}(x, 0) \nu_1(dx) + \int_{\mathbb{R}} \frac{\partial f}{\partial y}(0, y) \nu_2(dy) \\ &\quad + \int_{\mathbb{R}^2} \frac{\partial^2 f}{\partial x \partial y}(x, y) F(U_1(x), U_2(y)) dx dy \end{aligned} \quad (3.47)$$

We can now make use of this Lévy copula techniques to break the double integral term in equation (3.41) into parts that can be calculated via standard

numerical integration techniques. First of all, let's denote the integrand as follows to reduce the size of calculations

$$G(z_e, z_g) = V(S_e + z_e, S_g + z_g, L, \tau) - V(S_e, S_g, L, \tau) - z_e V_{S_e} - z_g V_{S_g} \quad (3.48)$$

for fixed $S = (S_e, S_g)$ and L . The application of Lemma 40 to the integral term in equation (3.41) results in

$$\begin{aligned} \int_{\mathbb{R}_+^2} G(z_e, z_g) \nu((dz_e \times dz_g)) &= \int_{\mathbb{R}_+} \frac{\partial G}{\partial z_e}(z_e, 0) \nu_1((dz_e)) + \int_{\mathbb{R}_+} \frac{\partial G}{\partial z_g}(0, z_g) \nu_2(dz_g) \\ &\quad + \int_{\mathbb{R}_+^2} \frac{\partial^2 G}{\partial z_e \partial z_g}(z_e, z_g) F(U_1(z_e), U_2(z_g)) dz_e dz_g \end{aligned} \quad (3.49)$$

By taking the partial derivatives of G in equation (3.48) we obtain

$$\begin{aligned} \frac{\partial G}{\partial z_e}(z_e, 0) &= V_{S_e}(S_e + z_e, S_g, L, \tau) - V_{S_e}(S_e, S_g, L, \tau) \\ \frac{\partial G}{\partial z_g}(0, z_g) &= V_{S_g}(S_e, S_g + z_g, L, \tau) - V_{S_g}(S_e, S_g, L, \tau) \\ \frac{\partial^2 G}{\partial z_e \partial z_g}(z_e, z_g) &= V_{S_e S_g}(S_e + z_e, S_g + z_g, L, \tau) \end{aligned} \quad (3.50)$$

So, after the substitution of eqs. (3.49) and (3.50) in equation (3.41) our HJB PIDE becomes

$$\begin{aligned} V_\tau &= \frac{1}{2} \sigma_e^2 V_{S_e S_e} + \frac{1}{2} \sigma_g^2 V_{S_g S_g} + \rho \sigma_e \sigma_g V_{S_e S_g} + \mu_e(S^e, T - \tau) V_{S_e} + \mu_g(S^g, T - \tau) V_{S_g} - rV + \\ &\quad \int_{\mathbb{R}_+} (V_{S_e}(S_e + z_e, S_g, L, \tau) - V_{S_e}(S_e, S_g, L, \tau)) \nu_e((dz_e)) \\ &\quad + \int_{\mathbb{R}_+} (V_{S_g}(S_e, S_g + z_g, L, \tau) - V_{S_g}(S_e, S_g, L, \tau)) \nu_g(dz_g) \\ &\quad + \int_{\mathbb{R}_+^2} V_{S_e S_g}(S_e + z_e, S_g + z_g, L, \tau) F(U_1(z_e), U_2(z_g)) dz_e dz_g \\ &\quad \max_c [H(L) S_e - S_g c + \eta(L, c)(\bar{L}(c) - L) V_L] \end{aligned} \quad (3.51)$$

We assumed in Section 3.2 that $L_e(t)$ and $L_g(t)$ were compound Poisson processes with intensities λ_e and λ_g and jump size distributions D_e and D_g . Then the tail integrals of $L_e(t)$ and $L_g(t)$ are

$$U_1(x) = \lambda_e(1 - D_e(x)) \quad (3.52)$$

$$U_2(x) = \lambda_g(1 - D_g(x)) \quad (3.53)$$

respectively. If we substitute U_1 and U_2 in the formula (3.46) for the skewed Clayton- Lévy copula we get

$$F(U_1(z_e), U_2(z_g)) = ((\alpha U_2^{-\beta} + 1)U_1^{-\theta} + U_2^{-\theta})^{-1/\theta} = \left(\left(\frac{\alpha}{\lambda_g^\beta(1 - D_g(z_g))^\beta} + 1 \right) \frac{1}{\lambda_e^\theta(1 - D_e(z_e))^\theta} + \frac{1}{\lambda_g^\theta(1 - D_g(z_g))^\theta} \right)^{-\frac{1}{\theta}} \quad (3.54)$$

which is a two-dimensional function depending on z_e and z_g . We will make use of this formula later in Section 3.6 in our numerical approximations.

3.5 BOUNDARY CONDITIONS

The terminal condition for our power plant valuation is assumed to be

$$V(S_e, S_g, L, \tau = 0) = 0 \quad (3.55)$$

where $\tau = T - t$. Based on Thompson et al.⁷³, Safarov and Atkinson⁶⁶ we choose the following limits as the boundary conditions:

$$\begin{aligned} S_e \rightarrow 0 \quad \text{or} \quad S_e \rightarrow \infty &\implies V_{S_e S_e} \rightarrow 0 \quad \text{and} \quad V_{S_e S_g} \rightarrow 0 \\ S_g \rightarrow 0 \quad \text{or} \quad S_g \rightarrow \infty &\implies V_{S_g S_g} \rightarrow 0 \quad \text{and} \quad V_{S_e S_g} \rightarrow 0 \end{aligned} \quad (3.56)$$

The domain of the HJB equation (3.51) is $S_e \times S_g \times L = [0, \infty) \times [0, \infty) \times [L_{\min}, L_{\max}]$. But we need bounded domain for numerical calculations. So we need to restrict this unbounded domain to $[0, S_e^{\max}, [\times [0, S_g^{\max}, [\times [L_{\min}, L_{\max}]$.

Hence, for the case $S_e \rightarrow 0$ we apply $V_{S_e S_e} \rightarrow 0$ and $V_{S_e S_g} \rightarrow 0$ to simplify equation (3.51) to

$$V_\tau = \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \mu_e(0, T - \tau)V_{S_e} + \mu_g(S^g, T - \tau)V_{S_g} - rV + \int_{\mathbb{R}} (V_{S_g}(0, S_g + z_g, L, \tau) - V_{S_g}(0, S_g, L, \tau))\nu_g(dz_g) + \max_c [-S_g c + \eta(L, c)(\bar{L}(c) - L)V_L] \quad (3.57)$$

For the case $S_e \rightarrow S_e^{\max}$ we employ the first line of (3.56) again to get

$$V_\tau = \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \mu_e(S_e^{\max}, T - \tau)V_{S_e} + \mu_g(S^g, T - \tau)V_{S_g} - rV + \int_{\mathbb{R}} (V_{S_g}(S_e^{\max}, S_g + z_g, L, \tau) - V_{S_g}(S_e^{\max}, S_g, L, \tau))\nu_g(dz_g) + \max_c [H(L)S_e^{\max} - S_g c + \eta(L, c)(\bar{L}(c) - L)V_L] \quad (3.58)$$

We can obtain the boundary conditions for the limits $S_g \rightarrow 0$ and $S_g \rightarrow S_g^{\max}$ in a similar way. To conclude our analysis of boundary conditions we need to construct the equations corresponding to 4 corner points: $(S_e, S_g) \rightarrow (0, 0)$, $(S_e, S_g) \rightarrow (0, S_g^{\max})$, $(S_e, S_g) \rightarrow (S_e^{\max}, 0)$ and $(S_e, S_g) \rightarrow (S_e^{\max}, S_g^{\max})$ as follows

$$V_\tau = \mu_e(0, T - \tau)V_{S_e} + \mu_g(0, T - \tau)V_{S_g} - rV + \max_c [\eta(L, c)(\bar{L}(c) - L)V_L]$$

and

$$V_\tau = \mu_e(0, T - \tau)V_{S_e} + \mu_g(S_g^{\max}, T - \tau)V_{S_g} - rV + \max_c [-S_g^{\max} c + \eta(L, c)(\bar{L}(c) - L)V_L]$$

$$\begin{aligned}
V_\tau = & \mu_e(S_e^{\max}, T - \tau)V_{S_e} + \mu_g(0, T - \tau)V_{S_g} - rV + \\
& \int_{\mathbb{R}} (V_{S_g}(S_e^{\max}, z_g, L, \tau) - V_{S_g}(S_e^{\max}, S_g, L, \tau))\nu_g(dz_g) + \\
& \max_c [H(L)S_e^{\max} + \eta(L, c)(\bar{L}(c) - L)V_L]
\end{aligned}$$

$$\begin{aligned}
V_\tau = & \mu_e(S_e^{\max}, T - \tau)V_{S_e} + \mu_g(S_g^{\max}, T - \tau)V_{S_g} - rV + \\
& \int_{\mathbb{R}} (V_{S_g}(S_e^{\max}, S_g^{\max} + z_g, L, \tau) - V_{S_g}(S_e^{\max}, S_g^{\max}, L, \tau))\nu_g(dz_g) + \\
& \max_c [H(L)S_e^{\max} - S_g^{\max}c + \eta(L, c)(\bar{L}(c) - L)V_L]
\end{aligned}$$

3.6 DISCRETISATION SCHEME

U^e and U^g are finite activity Lévy processes where $\nu_e(\mathbb{R}_+) = \lambda_e$ and $\nu_g(\mathbb{R}_+) = \lambda_g$. Therefore, we can rewrite the single integral terms of equation (3.51) as

$$\begin{aligned}
& \int_{\mathbb{R}_+} (V_{S_e}(S_e + z_e, S_g, L, \tau) - V_{S_e}(S_e, S_g, L, \tau))\nu_e(dz_e) = \\
& \int_{\mathbb{R}_+} V_{S_e}(S_e + z_e, S_g, L, \tau)\nu_e(dz_e) - \lambda_e V_{S_e}(S_e, S_g, L, \tau)
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}_+} (V_{S_g}(S_e, S_g + z_g, L, \tau) - V_{S_g}(S_e, S_g, L, \tau))\nu_g(dz_g) = \\
& \int_{\mathbb{R}_+} V_{S_g}(S_e, S_g + z_g, L, \tau)\nu_g(dz_g) - \lambda_g V_{S_g}(S_e, S_g, L, \tau)
\end{aligned} \tag{3.60}$$

If we substitute eqs. (3.59) and (3.60) in equation (3.51) the coefficients of partial derivatives V_{S_e} and V_{S_g} modify to

$$\bar{\mu}_e(S^e, t) = \bar{\Lambda}_e(t) - \alpha_e S^e(t) - \lambda_e \tag{3.61}$$

$$\bar{\mu}_g(S^g, t) = \bar{\Lambda}_g(t) - \alpha_g S^g(t) - \lambda_g \tag{3.62}$$

and the HJB equation (3.51) can be written as

$$V_\tau = \mathcal{D}V + \mathcal{H}_e V + \mathcal{H}_g V + \mathcal{H}_{eg} V + \max_c [H(L)S_e - S_g c + \eta(L, c)(\bar{L}(c) - L)V_L] \quad (3.63)$$

where

$$\begin{aligned} \mathcal{D}V = & \frac{1}{2}\sigma_e^2 V_{S_e S_e} + \frac{1}{2}\sigma_g^2 V_{S_g S_g} + \rho\sigma_e\sigma_g V_{S_e S_g} + \\ & \mu_e(S_e, T - \tau)V_{S_e} + \mu_g(S_g, T - \tau)V_{S_g} - rV \end{aligned} \quad (3.64)$$

$$\mathcal{H}_e V = \int_{\mathbb{R}_+} V_{S_e}(S_e + z_e, S_g, L, \tau) \nu_e(dz_e) \quad (3.65)$$

$$\mathcal{H}_g V = \int_{\mathbb{R}_+} V_{S_g}(S_e, S_g + z_g, L, \tau) \nu_g(dz_g) \quad (3.66)$$

and

$$\mathcal{H}_{eg} V = \int_{\mathbb{R}_+^2} V_{S_e S_g}(S_e + z_e, S_g + z_g, L, \tau) F(U_1(z_e), U_2(z_g)) dz_e dz_g \quad (3.67)$$

Note that we dropped the bar signs from our coefficients $\bar{\mu}_e$ and $\bar{\mu}$ for simplicity. We will apply the explicit finite difference scheme proposed in Cont and Voltchkova^{29, 30} to deal with the diffusion $\mathcal{D}V$ and single integral terms $\mathcal{H}_e V$ and $\mathcal{H}_g V$. The discretisation of $\mathcal{H}_{eg} V$ and optimisation terms requires further analysis.

First of all, we need to discretise the domain of the HJB equation by the following grid:

$$\begin{aligned} \tau_n &= n\Delta\tau, \quad n = 0, \dots, M, \quad \Delta\tau = \frac{T}{M} \\ S_i^e &= i\Delta S^e, \quad i = 0, \dots, N_e, \quad \Delta S^e = \frac{S_{\max}^e}{N_e} \\ S_j^g &= j\Delta S^g, \quad j = 0, \dots, N_g, \quad \Delta S^g = \frac{S_{\max}^g}{N_g} \\ L_u &= L_{\min} + u\Delta L, \quad u = 0, \dots, L, \quad \Delta L = \frac{L_{\max} - L_{\min}}{N_L} \end{aligned} \quad (3.68)$$

So, $V(S_i^e, S_j^g, L_u, \tau_n) = V(i\Delta S^e, j\Delta S^g, u\Delta L, n\Delta\tau)$ is the exact solution of

equation (3.51) at the node $(S_i^e, S_j^g, L_u, \tau_n)$, while $V_{i,j,u}^n$ stands for the approximate solution at the same node.

For the diffusion/PDE part we follow the steps discussed in Safarov and Atkinson⁶⁶, use the explicit method discussed in Safarov and Atkinson⁶⁶ and apply similar corrector term to get rid off the monotonicity issue. In other words,

$$\begin{aligned} \mathcal{D}V_{i,j,u}^n = & \frac{\sigma_e^2}{2\Delta S_e^2}(V_{i+1,j,u}^n - 2V_{i,j,u}^n + V_{i-1,j,u}^n) + \frac{\sigma_g^2}{2\Delta S_g^2}(V_{i,j+1,u}^n - 2V_{i,j,u}^n + V_{i,j-1,u}^n) + \\ & \frac{\rho\sigma_e\sigma_g}{4\Delta S_e\Delta S_g}(V_{i+1,j+1,u}^n + V_{i-1,j-1,u}^n - V_{i,j+1,u}^n - V_{i,j-1,u}^n - V_{i+1,j,u}^n - V_{i-1,j,u}^n + 2V_{i,j,u}^n) + \\ & \frac{\mu_e(S_i^e, T - \tau_n)}{\Delta S_e}(V_{i+1,j,u}^n - V_{i,j,u}^n) + \frac{\mu_g(S_j^g, T - \tau_n)}{\Delta S_g}(V_{i,j+1,u}^n - V_{i,j,u}^n) - rV_{i,j,u}^n \end{aligned} \quad (3.69)$$

assuming that $\rho > 0$, $\mu_e \geq 0$ and $\mu_g \geq 0$. We will use the downwind scheme for the cases $\mu_e \leq 0$ and $\mu_g \leq 0$.

For the integral terms $\mathcal{H}_e V$ and $\mathcal{H}_g V$ we first need to restrict the unbounded integration domains by setting finite upper bounds B_e and B_g to truncate the larger jumps. Alternatively, we could choose a distribution for jumps with a compact support. We employ the trapezoidal quadrature method in order to get the numerical approximations for these integrals and take $\Delta z_e = \Delta S_e$ and $\Delta z_g = \Delta S_g$. We also choose positive integers K_e and K_g such that $[0, B_e] \subset [0, (K_e + \frac{1}{2})\Delta z_e]$ and $[0, B_g] \subset [0, (K_g + \frac{1}{2})\Delta z_g]$, respectively. By using the central difference discretisation for the partial derivatives V_{S_e} and V_{S_g} we get

$$\begin{aligned} \mathcal{H}_e V_{i,j,u}^n &= \int_0^{B_e} V_{S_e}(S_i^e + z_e, S_j^g, L_u, \tau_n) \nu_e((dz_e)) \\ &= \frac{1}{2\Delta S_e} \int_0^{B_e} (V(S_{i+1}^e + z_e, S_j^g, L_u, \tau_n) - V(S_{i-1}^e + z_e, S_j^g, L_u, \tau_n)) \nu_e((dz_e)) \\ &\approx \frac{1}{2\Delta S_e} \left(\sum_{k=0}^{K_e} \nu_k^e V_{i+k+1,j,u}^n - \sum_{k=0}^{K_e} \nu_k^e V_{i+k-1,j,u}^n \right) \end{aligned} \quad (3.70)$$

and

$$\begin{aligned}
\mathcal{H}_g V_{i,j,u}^n &= \int_0^{B_g} V_{S_g}(S_i^e, S_j^g + z_g, L_u, \tau_n) \nu_g((dz_g) \\
&= \frac{1}{2\Delta S_g} \int_0^{B_g} (V(S_i^e, S_{j+1}^g + z_g, L_u, \tau_n) - V(S_i^e, S_{j-1}^g + z_g, L_u, \tau_n)) \nu_g((dz_e) \\
&\approx \frac{1}{2\Delta S_g} \left(\sum_{k=0}^{K_g} \nu_k^g V_{i,j+k+1,u}^n - \sum_{k=0}^{K_g} \nu_k^g V_{i,j+k-1,u}^n \right) \quad (3.71)
\end{aligned}$$

where

$$\nu_k^e = \int_{(k-\frac{1}{2})\Delta S_e}^{(k+\frac{1}{2})\Delta S_e} \nu_e(dz_e) \quad \text{and} \quad \nu_k^g = \int_{(k-\frac{1}{2})\Delta S_g}^{(k+\frac{1}{2})\Delta S_g} \nu_g(dz_g) \quad (3.72)$$

We can now go through the numerical calculation of the double integral

$$\mathcal{H}_{eg} V = \int_0^{B_e} \int_0^{B_g} V_{S_e S_g}(S_e + z_e, S_g + z_g, L, \tau) F(U_1(z_e), U_2(z_g)) dz_e dz_g \quad (3.73)$$

First of all, denote $\bar{F}(z_e, z_g) = F(U_1(z_e), U_2(z_g))$ for clarity purposes. Then apply the central difference formula to approximate the cross-derivative $V_{S_e S_g}$ at the node $(S_i^e + z_e, S_j^g + z_g, L_u, \tau_n)$:

$$\begin{aligned}
V_{S_e S_g}(S_i^e + z_e, S_j^g + z_g, L_u, \tau_n) &\approx \frac{1}{4\Delta S_e \Delta S_g} (V(S_{i+1}^e + z_e, S_{j+1}^g + z_g, L_u, \tau_n) - \\
&V(S_{i+1}^e + z_e, S_{j-1}^g + z_g, L_u, \tau_n) - V(S_{i-1}^e + z_e, S_{j+1}^g + z_g, L_u, \tau_n) \\
&+ V(S_{i-1}^e + z_e, S_{j-1}^g + z_g, L_u, \tau_n)) \quad (3.74)
\end{aligned}$$

By taking the double integral of both sides above we get

$$\begin{aligned}
\mathcal{H}_{eg} V &\approx \frac{1}{4\Delta S_e \Delta S_g} \int_0^{B_e} \int_0^{B_g} (V(S_{i+1}^e + z_e, S_{j+1}^g + z_g, L_u, \tau_n) - \\
&V(S_{i+1}^e + z_e, S_{j-1}^g + z_g, L_u, \tau_n) - V(S_{i-1}^e + z_e, S_{j+1}^g + z_g, L_u, \tau_n) + \\
&V(S_{i-1}^e + z_e, S_{j-1}^g + z_g, L_u, \tau_n)) \bar{F}(z_e, z_g) dz_e dz_g \quad (3.75)
\end{aligned}$$

Thus, we need to integrate 4 terms over $[0, B) \times [0, B)$ in equation (3.75) in

order to evaluate $\mathcal{H}_{eg}V$. For this purpose we will make use of two-dimensional trapezoidal rule explained below (see Davis and Rabinowitz³¹).

Let $f(x, y)$ be given over the rectangle $\{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$. Assume that the intervals $[0, a]$ and $[0, b]$ are equally subdivided into K_1 and K_2 subintervals with widths $h_1 = \frac{a}{K_1}$ and $h_2 = \frac{b}{K_2}$, respectively. The sample points x_i and y_j are defined as $x_i = ih_1$ and $y_j = jh_2$ where $i = 0, \dots, K_1$ and $j = 0, \dots, K_2$. Then the two-dimensional trapezoidal rule can be formulated as follows

$$\int_0^a \int_0^b f(x, y) dx dy = \sum_{i=0}^{K_1} \sum_{j=0}^{K_2} \omega_{ij} f(x_i, y_j) h_1 h_2 + O(h_1^2) + O(h_2^2) \quad (3.76)$$

where

$$\omega_{ij} = \begin{cases} 1 & \text{for } 0 < i < K_1 \text{ and } 0 < j < K_2 & \text{interior} \\ \frac{1}{2} & \text{for } (0, j) \text{ or } (i, 0) & \text{exterior} \\ \frac{1}{4} & \text{for } (i, j) \in \{(0, 0), (N_B, N_B)\} & \text{corners.} \end{cases} \quad (3.77)$$

We apply this integration rule to approximate the four terms on the right hand side (RHS) of equation (3.75). Let's first consider the first term

$$f(z_e, z_g) = V(S_{i+1}^e + z_e, S_{j+1}^g + z_g, L_u, \tau_n) \bar{F}(z_e, z_g) \quad (3.78)$$

This is a two-dimensional function depending on z_e and z_g . So, we can apply the 2d trapezoidal rule to get

$$\begin{aligned} \int_0^{B_1} \int_0^{B_2} V(S_{i+1}^e + z_e, S_{j+1}^g + z_g, L_u, \tau_n) \bar{F}(z_e, z_g) dz_e dz_g \approx \\ \sum_{k_1=0}^{K_1} \sum_{k_2=0}^{K_2} V(S_{i+k_1+1}^e, S_{j+k_2+1}^g, L_u, \tau_n) \bar{F}(S_{k_1}^e, S_{k_2}^g) h_1 h_2 \omega_{k_1 k_2} \end{aligned} \quad (3.79)$$

We can similarly calculate the other three terms.

Hence, after denoting the approximate value of $c(S_i^e, S_j^g, L_u, \tau_n)$ by $\Omega_{i,j,u}^n$ our

explicit scheme becomes

$$\begin{aligned} \frac{V_{i,j,u}^{n+1} - V_{i,j,u}^n}{\Delta\tau} = & DV_{i,j,u}^n + H_e V_{i,j,u}^n + H_g V_{i,j,u}^n + H_{eg} V_{i,j,u}^n + \\ & \max_{\Omega_{i,j,u}^n \in C(I)} \left\{ H(L_u) S_i^e - S_j^g \Omega_{i,j,u}^n + \eta(L_u, \Omega_{i,j,u}^n) (\bar{L}(\Omega_{i,j,u}^n) - L_u) \frac{V_{i,j,u+1}^n - V_{i,j,u}^n}{\Delta L} \right\} \end{aligned} \quad (3.80)$$

where D , H_e , H_g and H_{eg} are discretisation operators corresponding to $\mathcal{D}$, $\mathcal{H}_e$, $\mathcal{H}_g$ and $\mathcal{H}_{eg}$, respectively. Note that when $\eta(L_u, \Omega_{i,j,u}^n) (\bar{L}(\Omega_{i,j,u}^n) - L_u) < 0$ we will discretise V_L by $(V_{i,j,u}^n - V_{i,j,u-1}^n)/\Delta L$.

We know from the initial condition that $V_{i,j,u}^0 = 0$. So, starting from time $\tau_0 = 0$ we first solve the maximization problem

$$\hat{\Omega}_{i,j}^n = \arg \max_{\Omega_{i,j,u}^n \in C(I)} \left\{ H(L_u) S_i^e - S_j^g \Omega_{i,j,u}^n + \eta(L_u, \Omega_{i,j,u}^n) (\bar{L}(\Omega_{i,j,u}^n) - L_u) \frac{V_{i,j,u+1}^n - V_{i,j,u}^n}{\Delta L} \right\} \quad (3.81)$$

then substitute this optimal strategy $\hat{\Omega}_{i,j}^n$ in (3.80) to get

$$\begin{aligned} V_{i,j,u}^{n+1} = & V_{i,j,u}^n + \Delta\tau (DV_{i,j,u}^n + H_e V_{i,j,u}^n + H_g V_{i,j,u}^n + H_{eg} V_{i,j,u}^n) + \\ & \Delta\tau \left\{ H(L_u) S_i^e - S_j^g \hat{\Omega}_{i,j,u}^n + \eta(L_u, \hat{\Omega}_{i,j,u}^n) (\bar{L}(\hat{\Omega}_{i,j,u}^n) - L_u) \frac{V_{i,j,u+1}^n - V_{i,j,u}^n}{\Delta L} \right\} \end{aligned} \quad (3.82)$$

where $n = 0, 1, \dots, M - 1$.

3.7 HYPOTHETICAL GAS-FIRED POWER PLANT EXAMPLE

For this example we specifically refer to Thompson et al.⁷³ for all assumptions regarding operational characteristics. In particular, to model the power

output as a function of heat L we use

$$H(L) = \begin{cases} 0 & \text{for } L < 300 \\ \frac{5}{6}L - 100 & \text{for } 300 \leq L \leq 600 \end{cases} \quad (3.83)$$

Hence, the minimum and maximum operating temperatures for electricity generation are 300°C and 600°C, respectively which correspond to 150 MW and 400 MW output levels. As we are dealing with a dynamic model, we also need the equilibrium temperature for the boiler that depends on the amount of burned gas c :

$$\bar{L}(c) = b_0 - b_1(c - b_2)^2 \quad (3.84)$$

where $(b_0, b_1, b_2) = (650, 0.00003571, 4200)$. By assuming $\eta(L, c) = \eta = 0.1$ in equation (3.28) we get the following dynamics for the temperature

$$\frac{dL}{dt} = 0.1(\bar{L}(c) - L) \quad (3.85)$$

where $0 \leq c \leq 3017$. We then impose a ramp rate restriction

$$\left| \frac{dL}{dt} \right| \leq 15 \quad (3.86)$$

proposed by Thompson et al.⁷³. If we combine this restriction with (3.85), we can update the lower and upper boundaries for the control variable c in the following way

$$\bar{c}_{\min}(L) \leq c \leq \bar{c}_{\max}(L), \quad 20 \leq L \leq 600 \quad (3.87)$$

where

$$\bar{c}_{\min}(L) = \max \left(0, 4200 - 100\sqrt{\frac{800 - L}{0.3571}} \right) \quad (3.88)$$

$$\bar{c}_{\max}(L) = \min \left(3017, 4200 - 100\sqrt{\frac{500 - L}{0.3571}} \right) \quad (3.89)$$

Note the discontinuity of $H(L)$ at $L = 300$ causes some oscillation problems related to the discretisation of the partial derivative V_L with the smoothness of solutions. We will follow Thompson et al.⁷³ in making use of minmod slope limiters to get rid off this smoothness issue. The detailed explanation regarding slope or flux limiters is provided by LeVeque and Leveque⁵².

3.7.1 SLOPE LIMITER

The following calculations do not affect the diffusion and integration parts of equation (3.51). Therefore, we can omit them in order to simplify our PIDE to

$$V_\tau = H(L)S_e - S_g c^* + \eta(L, c^*)(\bar{L}(c^*) - L)V_L] \quad (3.90)$$

If we denote $a(L) = \eta(L, c^*)(\bar{L}(c^*) - L)$ in equation (3.90) then we can see that

$$V_\tau = H(L)S_e - S_g c^* + a(L)V_L] \quad (3.91)$$

is an advection equation with variable speed $a(L)$. According to equation (3.82), we find c^* before applying explicit scheme to discretise (3.91). Therefore, we can extend the MUSCL scheme⁵² (Monotonic Upwind-Centered Scheme for Conservation Laws) for advection equations with constant speed to cover equation (3.91) as well. So,

$$V_u^{n+1} = V_u^n + \Delta\tau(H(L_u)S_i^e - S_j^g \hat{\Omega}_u^n) - \zeta(V_{u_1}^n - V_{u_1-1}^n) - \frac{1}{2}\zeta(\text{sign}(\zeta) - \zeta)\Delta L(\sigma_{u_1} - \sigma_{i_1-1}) \quad (3.92)$$

where $\zeta = \frac{\Delta\tau a(L_u)}{\Delta L}$ and

$$u_1 = \begin{cases} u & \text{if } \zeta > 0 \\ u + 1 & \text{if } \zeta \leq 0 \end{cases} \quad (3.93)$$

σ_u^n is some slope that needs to be defined in order to reduce the numerical oscillations. The oscillations in a solution are measured by the total variation

(TV) of V :

$$TV(V) = \sum_{u=1}^N |V_u - V_{u-1}| \quad (3.94)$$

The increase of TV leads to the development of oscillations in a solution. Therefore, a numerical method is total variation diminishing (TVD) if

$$TV(V^{n+1}) \leq TV(V^n) \quad (3.95)$$

We need to limit the slope σ_u^n of the MUSCL scheme to make it TVD. One of such ways is the minmod slope limiter:

$$\sigma_u^n = \text{minmod} \left(\frac{V_u^n - V_{u-1}^n}{\Delta L}, \frac{V_{u+1}^n - V_u^n}{\Delta L} \right) \quad (3.96)$$

where

$$\begin{aligned} \text{minmod}(a, b) &= \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0 \\ b & \text{if } |a| > |b| \text{ and } ab > 0 \\ 0 & \text{if } ab \leq 0 \end{cases} \\ &= \frac{1}{2}(\text{sign}(a) + \text{sign}(b)) \min(|a|, |b|) \end{aligned} \quad (3.97)$$

Thus, after improving the numerical algorithm (3.82) via the minmod slope limiter technique, we can get benefit from the accuracy of second order differencing scheme while avoiding numerical oscillations.

3.7.2 NUMERICAL RESULTS

We first consider the framework similar to the one presented in Thompson et al.⁷³. More precisely, we assume that the natural gas spot price is constant $S_g = 3.5 \text{MMBtu}$ and the input parameters are

$$(\alpha_e, \sigma_e, \lambda_e, T, r) = (.1, .12, .1, 200 \text{ hours}, .05) \quad (3.98)$$

Moreover, power spot price jump sizes are normally distributed, $\mathcal{N}(700, 100^2)$. The deterministic seasonality function Λ_e is given as

$$\Lambda_e(t) = 15 \sin((2\pi t - 15.4\pi)/24) + 27 \quad (3.99)$$

The grid size for τ, S_e, L is $(2000, 20, 20)$. The optimal operating strategy surface for the gas-fired power plant is presented in Figure 3.2. For boiler temperatures below 300°C the optimal strategy is to burn maximum amount of fuel within physical limitations in order to keep the plant on line. For low electricity spot prices gas consumption decreases when the temperature is above the minimum operation level to minimise the cost. But for higher power prices the optimal control of gas consumption increases nonlinearly with respect to spot prices and temperature. We can see from Figure 3.2 that in the case of low electricity spot prices the power plant burns more fuel for very high temperatures. This strategy is explained by the ramp rate restriction. In other words, we can't shut down the plant immediately when the boiler is close to its maximum temperature limit. It will take some time for the unit to cool and we have to burn the required amount of gas during that time period.

The Figure 3.3 shows the value or expected cash flow surface of power generating facility. We can see that the value surface does not depend on electricity spot prices when the boiler temperature is below the operating level. The plant is not reacting to spot price changes because there is no guarantee that by the time we increase the temperature to 300°C spot prices will remain the same. Beyond 300°C the plant starts to generate electricity which explains the non-smooth behaviour of the value surface at $L = 300$. Beyond this line both higher power prices and boiler temperatures contribute to larger values of the power generator. Thus, our optimal control and value surfaces look similar to the figures given by Thompson et al.⁷³. Unfortunately, we can't conduct more accurate comparison due to some missing parameter specifications in the original paper.

We can now investigate the numerical calculations for the underlying spot price models (3.12) and (3.15). The parameters for our single regime model

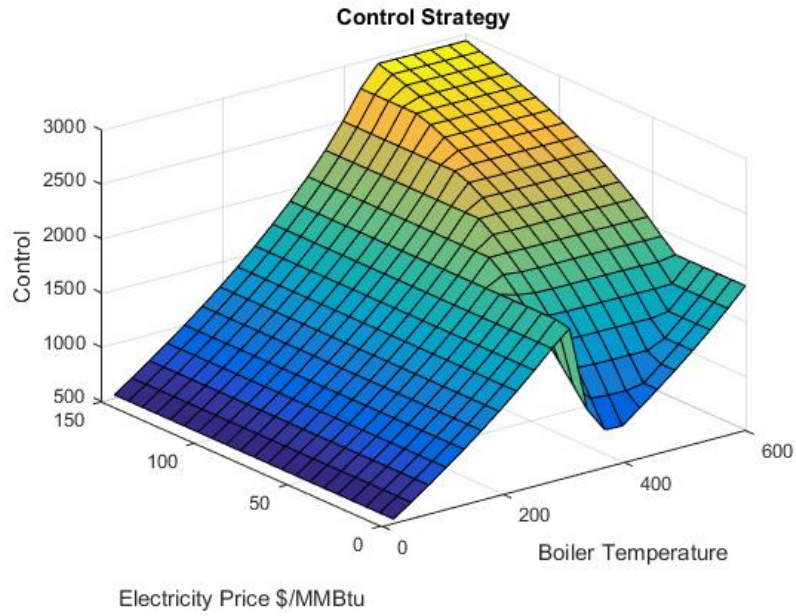


Figure 3.2: The optimal operating strategy surface for the gas-fired power plant for fixed gas price $S_g = 3.5$.

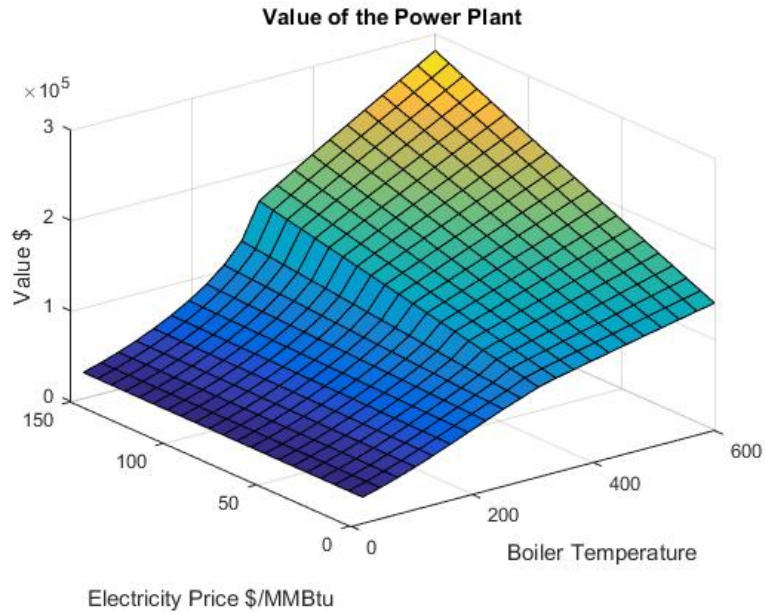


Figure 3.3: The value surface of power generating facility for fixed gas price $S_g = 3.5$.

are given in Table 3.1. We assume that the risk-free interest rate is the same, $r = 0.05$. As the jump size distributions D_e and D_g we take inverse

Parameter	Value	Parameter	Value
α_e	.1	λ_e	.1
α_g	.23	λ_g	.4
σ_e	.11	T	200 hours
σ_g	.09	ρ	.15

Table 3.1: Input parameters for the single regime power plant model.

Gaussian distributions $IG(\mu, \lambda)$ where $\mu > 0$ and $\lambda > 0$ are mean and shape parameters, respectively. We can use the parameter values (.60, .56) and (.54, .32) for the corresponding distributions of power and gas price jumps.

Meanwhile, the deterministic seasonality terms Λ_e and Λ_g will be defined as follows

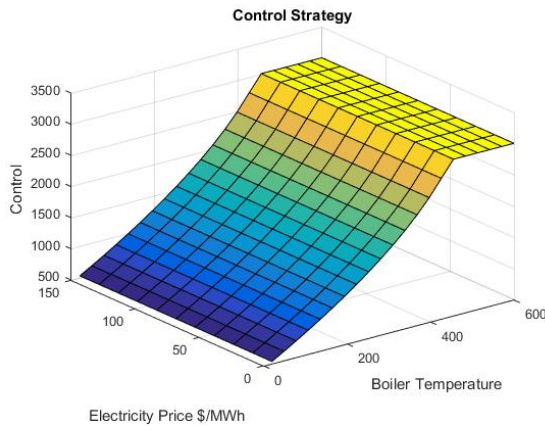
$$\Lambda_e(t) = 15 \sin((2\pi t - 15.4\pi)/24) + 27 \quad (3.100)$$

$$\Lambda_g(t) = 0.6 \cos(2\pi(t - 18\pi)/24) + 2.7 \quad (3.101)$$

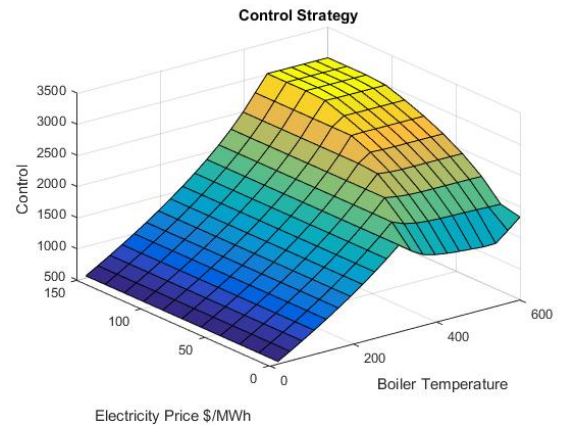
Figure 3.4 compares optimal operating surfaces with respect to power prices and boiler temperature for various gas spot prices. When $S_g \rightarrow 0$, the generation costs are so small that the plant burns gas as much as possible at all levels of power prices and temperatures (Figure 3.4a). Figure 3.4b depicts the case when $S_g = 10\$/\text{MMBtu}$. We can see that the surface looks more like in 3.2. For higher levels of natural gas spot prices the region of intensive fuel consumption diminishes (Figure 3.4c and 3.4d).

Value surfaces with respect to power spot price and boiler temperature for different gas prices are depicted in Figure 3.5. The shapes of these surfaces are almost the same. As gas prices increase, the surfaces shift downwards. But when S_g is too high, the power plant value becomes dependent on power spot prices even for low temperature levels (Figure 3.5c).

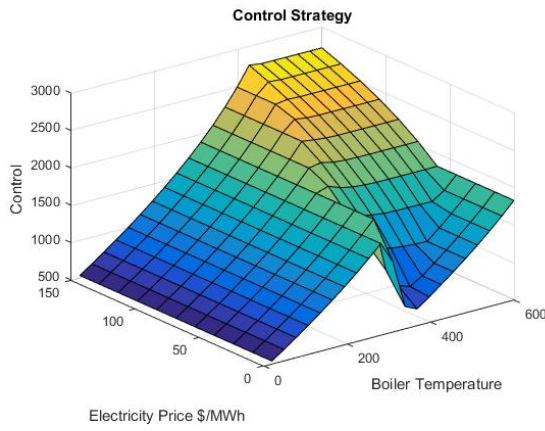
In Figure 3.6 we can observe optimal control and corresponding value surfaces with respect to gas spot price and boiler temperature for different electricity prices. Figures (a)-(c) imply that the higher is S_e , the larger is the maximum consumption region. For very low values of electricity the plant minimises



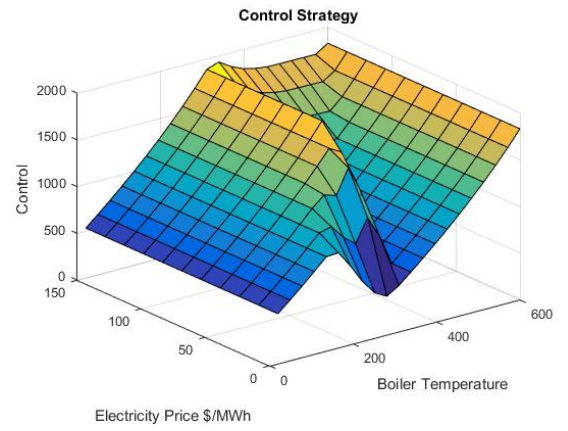
(a) $S_g = 0$



(b) $S_g = 10$

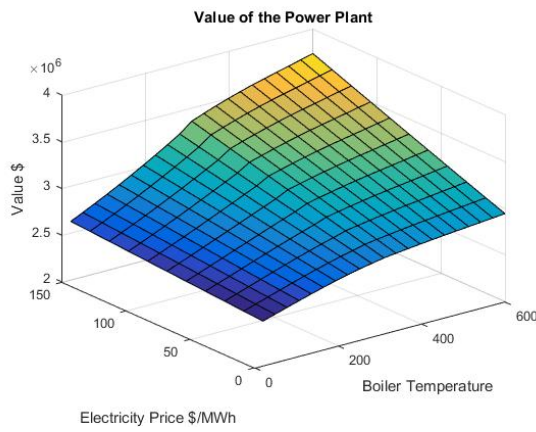


(c) $S_g = 14$

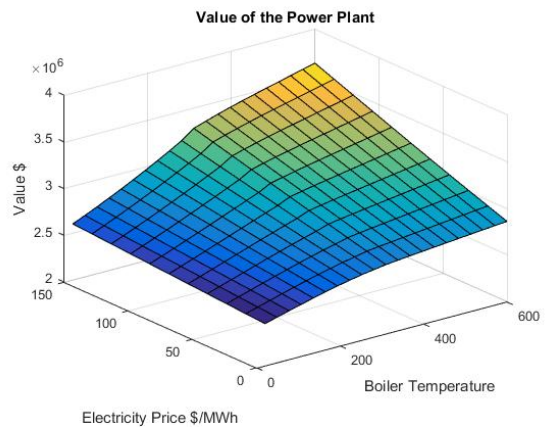


(d) $S_g = 20$

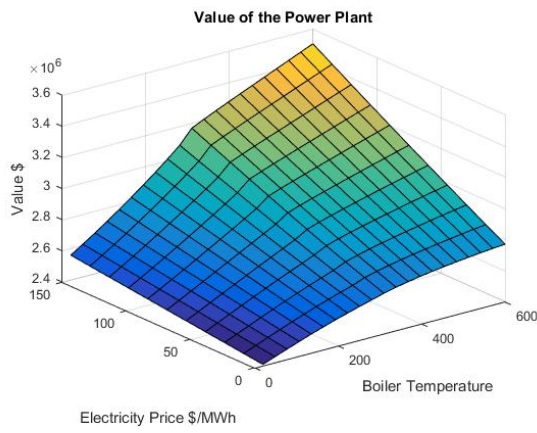
Figure 3.4: Optimal operating strategy for the single regime model with respect to electricity spot price and boiler temperature for different gas prices.



(a) $S_g = 0$



(b) $S_g = 10$



(c) $S_g = 20$

Figure 3.5: Value surface of the power plant for the single regime model with respect to electricity spot price and boiler temperature for different gas prices.

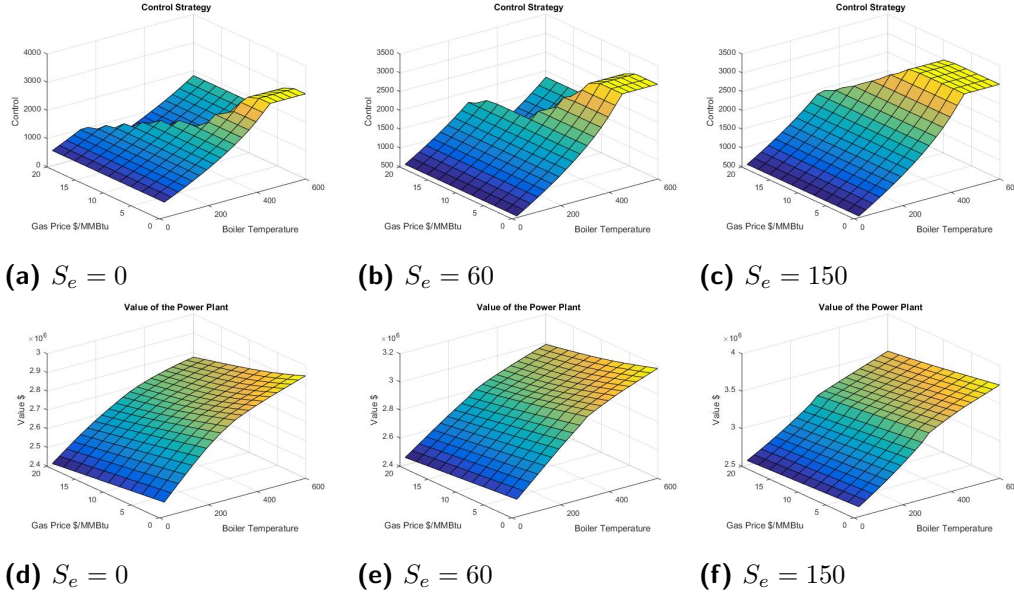
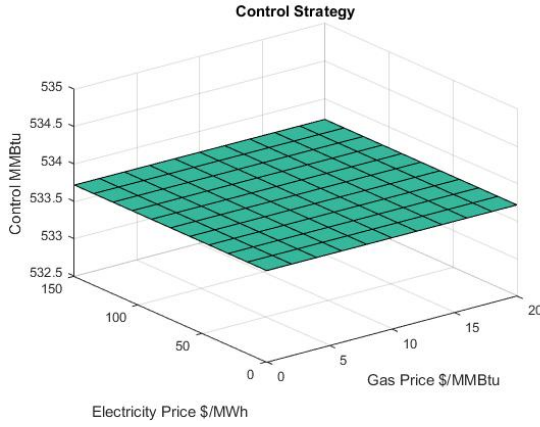


Figure 3.6: Operating strategy (a)-(c) and value surfaces (d)-(f) of the power generating facility for a single regime model with respect to gas spot price and boiler temperature for different electricity prices.

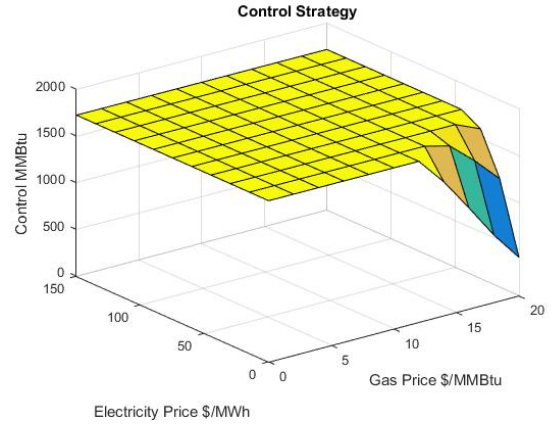
its fuel cost even before reaching 300°C . But when the power prices are too high the plant can keep burning gas at full capacity beyond 300°C as well. Running the plant in the presence of extremely expensive fuel prices when the boiler temperature is too high becomes unbearable. Therefore, the optimal strategy is to reduce gas consumption to minimum.

We can see from Figures 3.6(d)-(f) that value surfaces shift upwards when S_e increases. Meanwhile, the value surface doesn't exhibit a sharp change in $L = 300^\circ\text{C}$ when power prices are too low. In this case the surface becomes dependent on gas prices for boiler temperatures below the operating level. This dependence vanishes away for higher values of S_e .

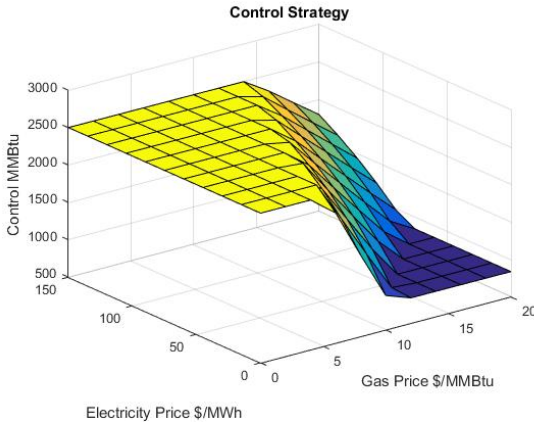
The operating strategies with respect to electricity and gas spot prices for different boiler temperatures are depicted in Figure 3.7. For very low boiler temperatures the control strategy surface is flat. In other words, it doesn't change depending on underlying spot prices. Starting from the case $L = 300^\circ\text{C}$ we notice a small region of dependence on power and gas spot prices. For higher temperature levels we can see that dependence more clearly (Figures 3.7c



(a) $L=20$



(b) $L=300$



(c) $L=420$



(d) $L=600$

Figure 3.7: Operating strategy for a single regime model with respect to electricity and gas spot prices for different boiler temperatures.

and 3.7d).

We can differentiate 3 different regions here: $c_{\min}$ (dark blue), $c_{\max}$ (yellow), and $c_{\min} \leq c \leq c_{\max}$ (in between).

Figure 3.8 describes the power generator value surface with respect to electricity and gas spot prices when $L = 320^\circ\text{C}$. The value surface exhibits increasing and decreasing dependence on power and natural gas spot prices, respectively. The overall shape of the surface is the same for other temperature levels as well.

We can now investigate the convergence of our numerical scheme. To do this

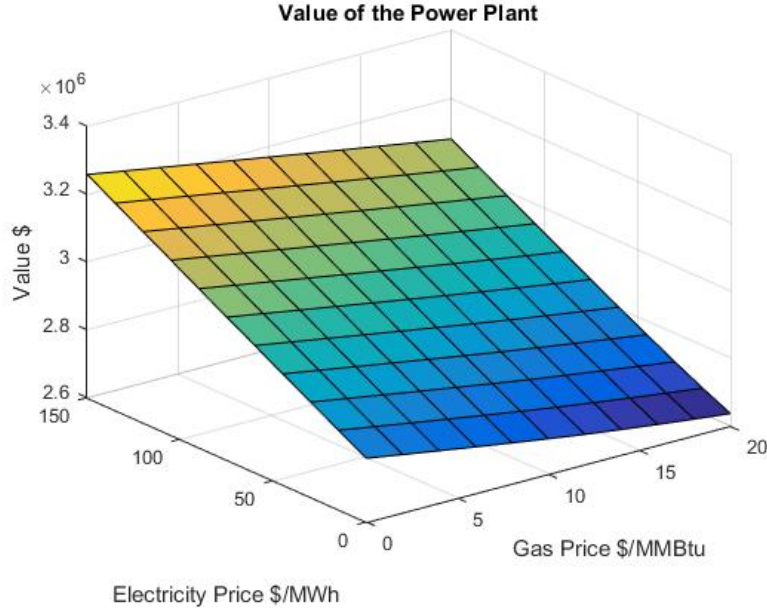


Figure 3.8: Value surface of the power generator value surface with respect to electricity and gas spot prices when $L = 320^\circ\text{C}$. The value surface exhibits increasing and decreasing dependence on power and natural gas spot prices, respectively.

we first consider a problem with 5, 5, and 200 number of grid points for the variables S_e, S_g and τ , respectively. The number of steps for L will be fixed as 10. We implement the scheme and take the value V when $S_e = 150$ $\$/\text{MWh}$, $S_g = 20$ $\$/\text{MMBtu}$ and $L = 484^\circ\text{C}$. We then keep increasing the number of state and time variable partition points 2 and 4 times, calculate the value V and compare with its value for the previous grid size. These values are presented in Table 3.2. The plot of log errors with respect to log mesh size confirms the convergence of our scheme (see Figure 3.9).

Time steps	State steps	Value \$
200	5	3369400
800	10	3371700
3200	20	3369700
12800	40	3369300
51200	80	3369200

Table 3.2: Convergence of the power plant scheme.

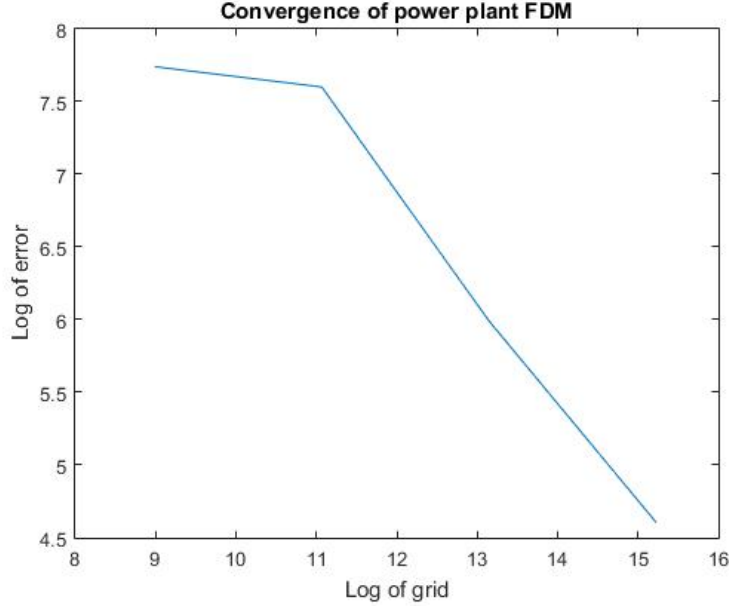


Figure 3.9: Convergence of FDM for power plant. Log of errors decreases as we increase the number of grid points.

3.8 REGIME SWITCHING

Let's assume the economy has two phases/regimes (0 and 1) and the switch has some finite probabilities: $\lambda^{0 \rightarrow 1} dt$ and $\lambda^{1 \rightarrow 0} dt$. We will model this regime switching by the following two state continuous-time Markov chain $m(t)$ presented by Chen and Forsyth²⁶:

$$dm(t) = (1 - m(t-))dN^{0 \rightarrow 1} - m(t-)dN^{1 \rightarrow 0} \quad (3.102)$$

where $t-$ is the time infinitesimally before t , $N^{0 \rightarrow 1}$ and $N^{1 \rightarrow 0}$ are independent Poisson processes with intensities $\lambda^{0 \rightarrow 1}$ and $\lambda^{1 \rightarrow 0}$, respectively. So, within a regime $l = m(t-)$ the underlying electricity and gas spot prices follow:

$$S^e(t) = \Lambda_e^l(t) + X^e(t) + Y^e(t) \quad (3.103)$$

where

$$dX^e(t) = -\alpha_e^l X^e(t)dt + \sigma_e^l dB^e(t) \quad (3.104)$$

$$dY^e(t) = -\alpha_e^l Y^e(t)dt + dL_e^l(t) \quad (3.105)$$

and

$$S^g(t) = \Lambda_g^l(t) + X^g(t) + Y^g(t) \quad (3.106)$$

where

$$dX^g(t) = -\alpha_g^l X^g(t)dt + \sigma_g^l dB^g(t) \quad (3.107)$$

$$dY^g(t) = -\alpha_g^l Y^g(t)dt + dL_g^l(t) \quad (3.108)$$

where L_e^l and L_g^l are Lévy processes with intensity measures ν_e^l and ν_g^l , respectively. In our compound Poisson case we have

$$\nu_e^l(dz_e) = \lambda_e^l D_e^l(dz_e) \quad \text{and} \quad \nu_g^l(dz_g) = \lambda_g^l D_g^l(dz_g)$$

We can rewrite our power plant optimisation objective (3.29) for each regime l as follows

$$V^l(S_e, S_g, L, t) = \max_{c^l(S_e, S_g, L, t)} \mathbb{E} \left[\int_t^T e^{-r(\tau-t)} (H(L)S_e - S_g c) d\tau \mid m(t) = l \right] \quad (3.109)$$

subject to

$$c_{min}(L) \leq c^l \leq c_{max}(L) \quad (3.110)$$

The application of previously used techniques, such as Bellman's Principle of

Parameter	Value	Parameter	Value
α_e	.6	λ_e	.2
α_g	1	λ_g	.6
σ_e	.2	T	200 hours
σ_g	.3	ρ	.15

Table 3.3: Input parameters for the regime 1 of regime-switching model.

Optimality and Itô's Lemma will give us the following coupled HJB equations

$$\begin{aligned}
V_\tau^l = & \frac{1}{2}(\sigma_e^l)^2 V_{S_e S_e}^l + \frac{1}{2}(\sigma_g^l)^2 V_{S_g S_g}^l + \rho \sigma_e^l \sigma_g^l V_{S_e S_g}^l + \mu_e^l(S^e, T - \tau) V_{S_e}^l + \\
& \mu_g^l(S^g, T - \tau) V_{S_g}^l - r V^l + \lambda^{l \rightarrow (1-l)} (V^{(1-l)} - V^l) \\
& \int_{\mathbb{R}_+} (V_{S_e}^l(S_e + z_e, S_g, L, \tau) - V_{S_e}^l(S_e, S_g, L, \tau)) \nu_e^l(dz_e) \\
& + \int_{\mathbb{R}_+} (V_{S_g}^l(S_e, S_g + z_g, L, \tau) - V_{S_g}^l(S_e, S_g, L, \tau)) \nu_g^l(dz_g) \\
& + \int_{\mathbb{R}_+^2} V_{S_e S_g}^l(S_e + z_e, S_g + z_g, L, \tau) F(U_1^l(z_e), U_2^l(z_g)) dz_e dz_g \\
& \max_c [H(L) S_e - S_g c + \eta(L, c)(\bar{L}(c) - L) V_L^l] \quad (3.111)
\end{aligned}$$

with the terminal condition

$$V^l(S_e, S_g, L, \tau = 0) = 0 \quad (3.112)$$

The boundary conditions at each regime will be the same as in Section 3.5.

3.8.1 NUMERICAL RESULTS

We first need to define our input parameters for the regime switching model. We adopt the parameter values introduced in Section 3.7.2 for regime 0. Analogously, we make the following assumptions for regime 1 (see Table 3.3). We assume that the risk-free interest rate is the same, $r = 0.05$. As the jump size distributions D_e^1 and D_g^1 we again take inverse Gaussian distributions with parameters $(.30, .46)$ and $(.42, .28)$, respectively.

The seasonality terms Λ_e^1 and Λ_g^1 will be assumed to be

$$\Lambda_e^1(t) = 5 \sin((2\pi t - 15.4\pi)/24) + 10 \quad (3.113)$$

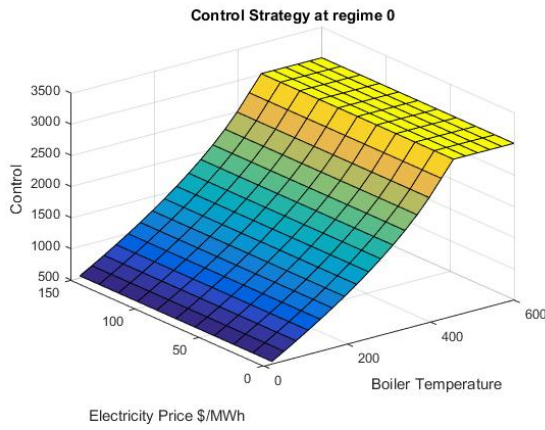
$$\Lambda_g^1(t) = 0.3 \cos(2\pi(t - 18\pi)/24) + 1.4 \quad (3.114)$$

Figures 3.10 and 3.11 compare operating strategies for the regimes 0 and 1, respectively with respect to power prices and boiler temperature for different gas spot prices. We can see that under both of these regimes the optimal strategy for $S_g \rightarrow 0$ is to burn maximum amount of gas all the time. However, for all other non-zero natural gas prices the region of maximum fuel consumption is larger for the regime 0 than for the regime 1. When gas prices are too high the plant no longer consumes fuel when electricity prices are low and $L < 300^\circ\text{C}$. In other words, the plant manager becomes more pessimistic (or risk-averse) under the regime-switching model.

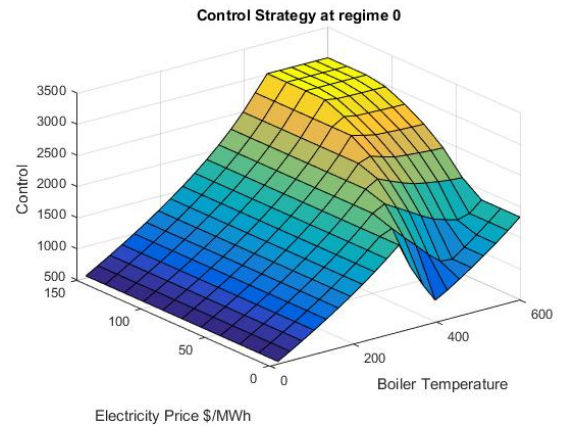
Value surfaces for the regime-switching model with respect to power prices and boiler temperature for the fixed gas price are depicted in Figure 3.12. The surfaces look similar in their shapes but in regime 1 we have a downside shift which reflects an unfavorable economic situation.

Control strategies for regime-switching model with respect to power and gas spot prices for different boiler temperatures are presented in Figure 3.13. When the temperature is above the operating level $L = 300^\circ\text{C}$, optimal operation surfaces are more risk-averse in regime 1. As the plant doesn't produce electricity below 300°C , the change in power spot prices does not affect our decision making. Hence, in regime 1 the optimal control surface for $L = 20^\circ\text{C}$ is more optimistic than in regime 0 due to fall in gas spot prices (see Figures 3.13a and 3.13d). Moreover, in contrast to Figure 3.7, the surface isn't flat when the temperature is close to its minimum.

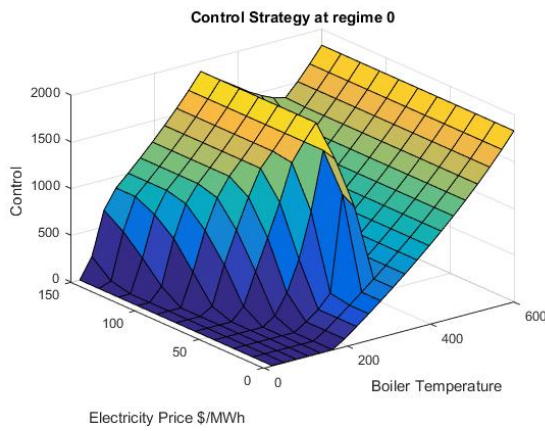
In Figure 3.14 we can observe operating strategy for regime-switching model with respect to gas spot prices and boiler temperature for different electricity spot prices. In general, these figures show less appealing results than in single regime case (see Figure 3.6). When the power prices are extremely low or high, the surfaces in regimes 0 and 1 look almost the same. However, for moderate electricity prices we have a larger region of minimal fuel



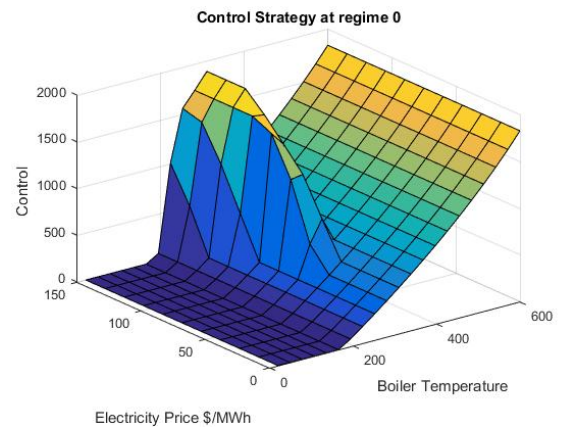
(a) $S_g = 0$



(b) $S_g = 10$

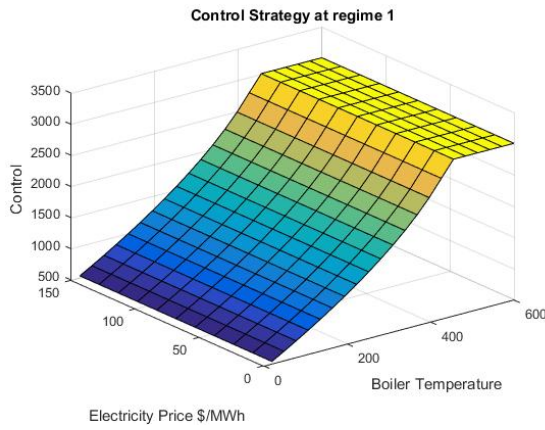


(c) $S_g = 14$

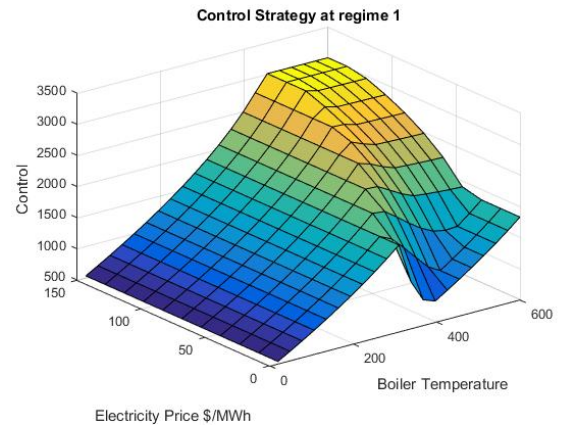


(d) $S_g = 20$

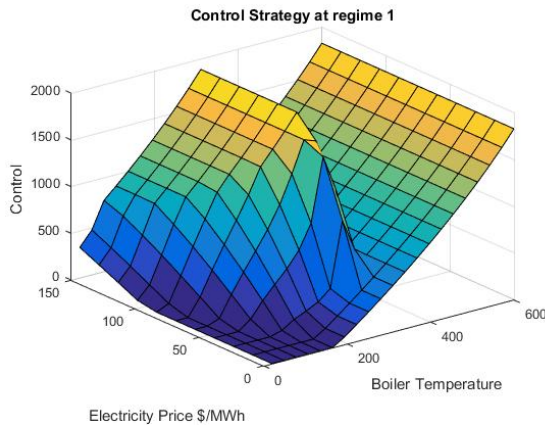
Figure 3.10: Operating strategy for the regime 0 of the regime-switching model with respect to electricity spot price and boiler temperature for different gas prices.



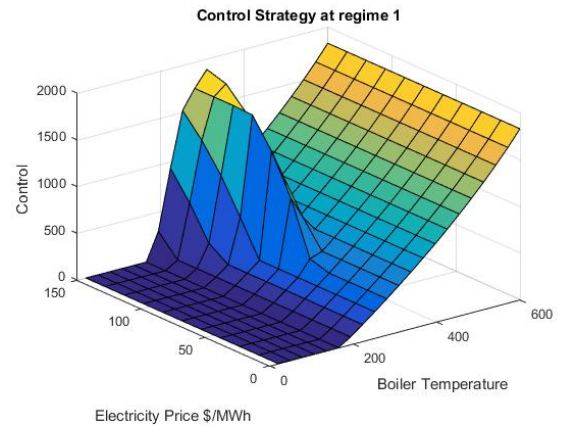
(a) $S_g = 0$



(b) $S_g = 10$



(c) $S_g = 14$



(d) $S_g = 20$

Figure 3.11: Operating strategy for the regime 1 of the regime-switching model with respect to electricity spot price and boiler temperature for different gas prices.

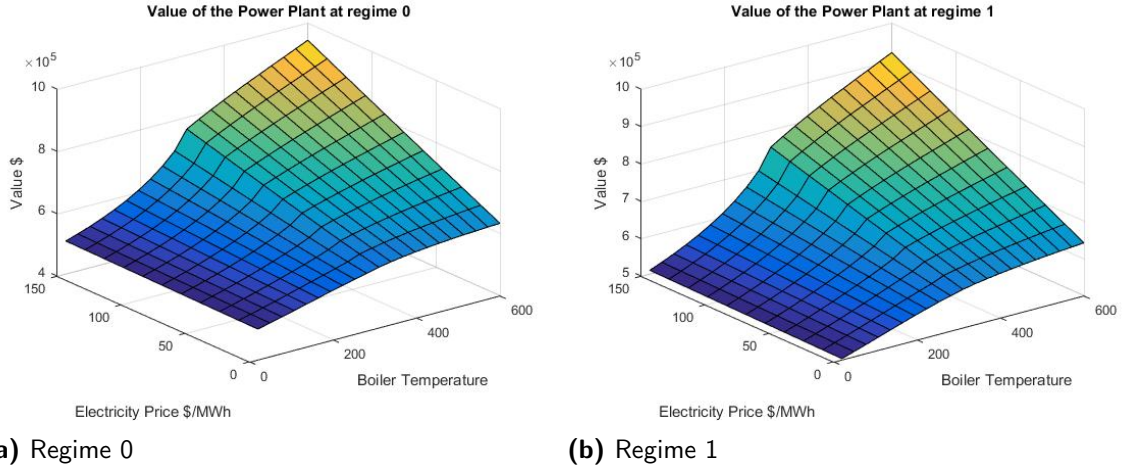


Figure 3.12: Value surfaces for the regime-switching model with respect to power prices and boiler temperature for fixed gas price. Although the surfaces look similar in their shapes, there is a downside shift in regime 1 which reflects an unfavorable economic situation.

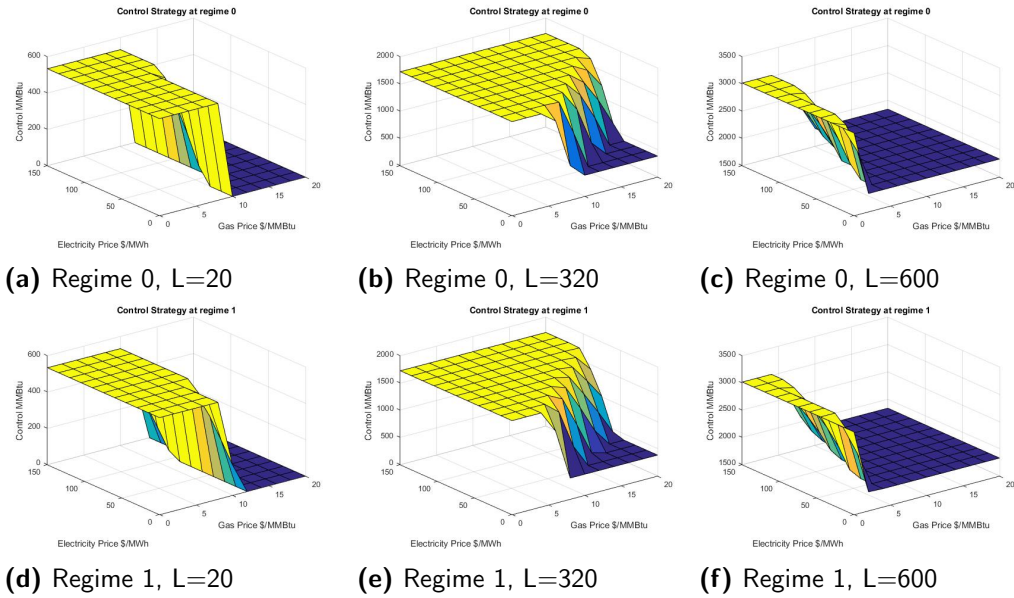


Figure 3.13: Operating strategy for regime-switching model with respect to electricity and gas spot prices for different boiler temperatures.

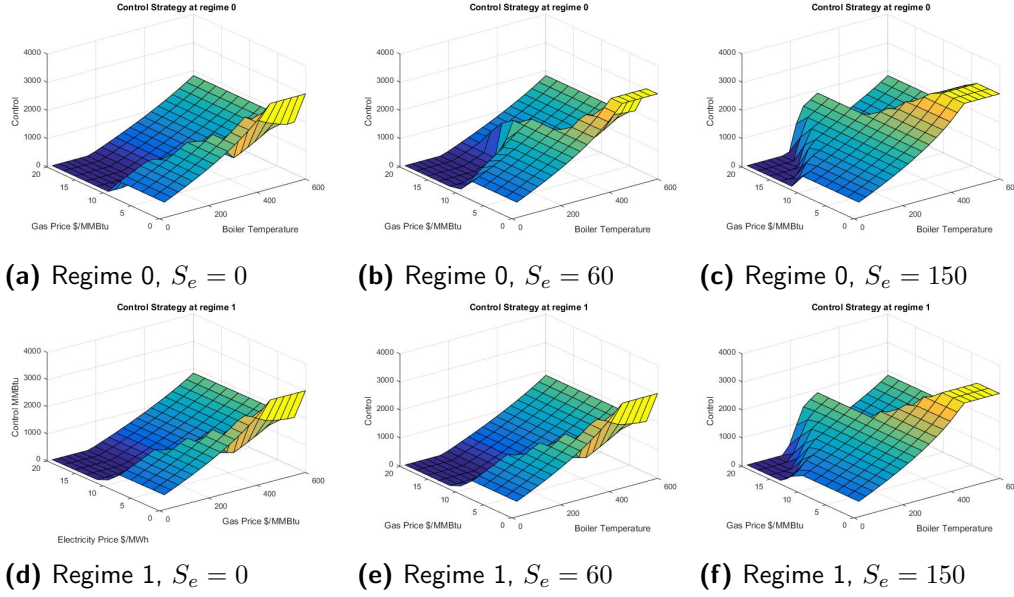


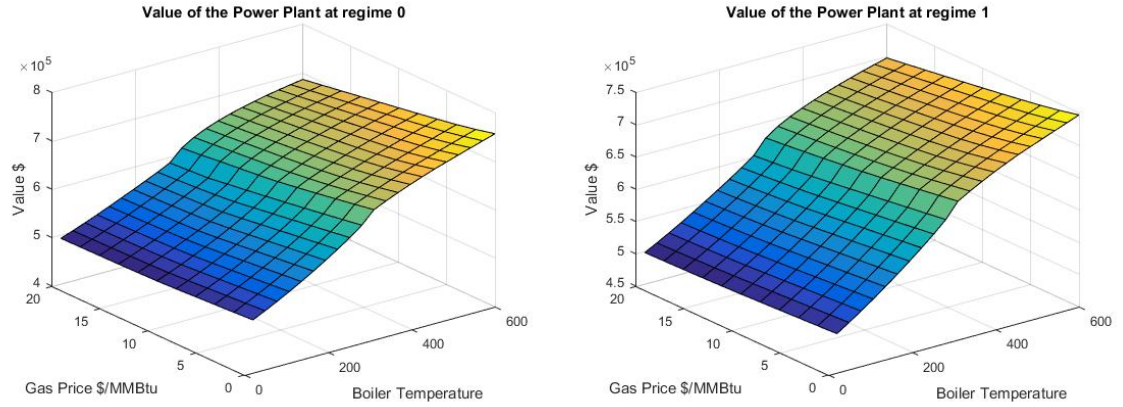
Figure 3.14: Operating strategy for regime-switching model with respect to gas spot prices and boiler temperature for different power prices. When the power prices For extremely low or high power prices the surfaces in regimes 0 and 1 look very similar. However, we have a larger region of minimal fuel consumption in regime 1 for moderate electricity prices.

consumption in regime 1.

Value surfaces for the regime-switching case with respect to gas spot prices and boiler temperature for fixed electricity price are provided in Figure 3.15, while the corresponding surfaces with respect to electricity and gas spot prices are given in Figure 3.16. We can infer from these figures that the value surfaces are shifted down in regime 1 due to economic disruptions.

3.9 CONCLUSION

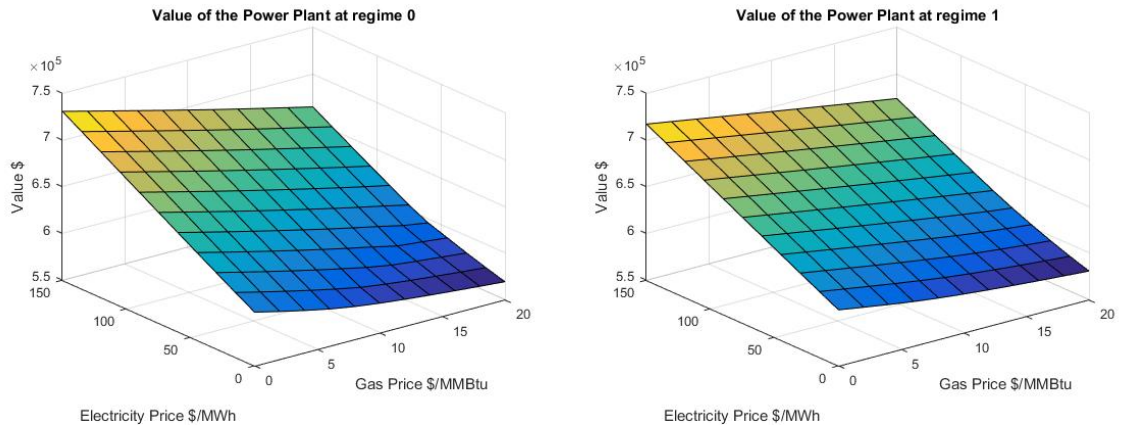
In this chapter we have analysed a stochastic control problem for gas-fired power generators while taking into account operating characteristics. The optimisation method introduced by Thompson et al.⁷³ has been elaborated by incorporating electricity and natural gas spot prices that exhibit regime-switching and interdependent spikes. The dependence of the price jumps has been modeled by means of skewed Lévy copulas. We have combined different numerical techniques to solve the resulting coupled non-linear HJB equations.



(a) Regime 0

(b) Regime 1

Figure 3.15: Value surfaces for the regime-switching model with respect to gas spot prices and boiler temperature for fixed electricity price. The value surface is shifted down in regime 1 due to economic disruptions.



(a) Regime 0

(b) Regime 1

Figure 3.16: Value surfaces for the regime-switching model with respect to electricity and gas spot prices for fixed boiler temperature. Less favourable economic situation resulted in downward shift in regime 1.

We have also used a minmod slope limiter to eliminate numerical oscillations from the optimal solution. The numerical approach gives us both the value of the power plant and optimal control strategy depending on the electricity and gas price, boiler temperature and time. The numerical approach has been implemented in MATLAB for a single regime and regime-switching cases. We have investigated the differences in those models by observing the surfaces of optimal operation and value surfaces.

4

CONCLUSION

In this thesis we have analysed stochastic control problems related to two physical assets: natural gas storage and gas-fired power plant. The main operating characteristics of these facilities have been taken into account as well. For the gas storage problem the real option based valuation approach by Thompson et al.⁷⁴ has been combined with the spot price dynamics presented by Benth and Sgarra¹¹. This two-factor spot price model is different from the original one by its instant spikes with a time-dependant jump intensity. The resulting non-linear HJB equation has been solved by an explicit finite difference method. The numerical approach gives us both the value of the storage contract and optimal operating strategy depending on the gas price, current level of gas in the storage and time. The numerical method was implemented in MATLAB for different underlying SDE models and the effects of seasonality and seasonal jump intensity have been analysed by observing the surfaces of control strategies and contract values. It has been revealed that except the boundary $I = I_{max}$, control strategies with time-dependant and constant jump intensities are almost the same.

In the case of gas-fired power generation, the optimisation method introduced by Thompson et al.⁷³ has been advanced by incorporating mean-reverting

electricity and natural gas spot prices that exhibit regime-switching and interdependent spikes. We have modeled the dependence of the gas and electricity spot price jumps by means of skewed Lévy copulas. Different numerical techniques have been combined to solve the resulting coupled non-linear HJB equations. The numerical approach gives us both the value of the power plant and optimal control strategy depending on the electricity and gas price, boiler temperature and time. The numerical approach has been implemented in MATLAB for a single regime and regime-switching cases. We have investigated the differences in those models by observing the surfaces of optimal operation and value surfaces.

As the next step, we plan to extend our numerical examples to the cases when underlying power and natural gas spot prices follow interdependent generalised hyperbolic (see Eberlein³⁵ and Eberlein and Prause³⁶) and variance-gamma (see Madan et al.⁵⁴ or Brody et al.²⁰) Lévy models. The application of this optimisation approach into more complicated problem such as joint optimisation of gas storage and power plant facilities could also be investigated. Another generalisation could be the optimal control of a multi-fuel power generating asset. In this case carbon emission and transaction costs (see Atkinson and Mokkhavesa³) of each fuel should also be incorporated to the model.

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