

WAVE PROPAGATION AND INSTABILITIES IN A MAGNETIZED,
PARTIALLY IONIZED GAS UNDER NON-EQUILIBRIUM CONDITIONS

by

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TO MY

MOTHER AND FATHER

ABSTRACT.

The purpose of this thesis is to study the properties of two types of waves that can exist in the magnetized, non equilibrium plasma of closed cycle MHD generators. They are electrothermal waves and the so called magnetosonic waves. Both of these can be unstable, their amplitude growing exponentially with time from the linear theory, and can lead to a significant non-uniformity in the working fluid of a generator. The effect of this non-uniformity on the generator performance, and the significance of the instabilities in the present stage of development of MHD generators are discussed.

Using a linear, plane wave analysis the growth and movement of electrothermal waves is analysed. The dependence of the growth rate on various plasma parameters, in particular the electron temperature, is investigated. The linear dispersion relation derived here is more complete than previous dispersion relations in that it contains more of the relevant physics. Special attention is given to the description of the physical nature of the waves.

Using once again a linear, plane wave analysis a general dispersion relation for waves in an MHD generator, including both electrothermal and magnetosonic waves, is derived.

The interaction between electrothermal waves and the apparatus walls is analysed using a linearized theory. The results show that in general an initial perturbation of the plasma will be split into an infinite

number of modes. Usually only a finite number of these are unstable, and it is shown that by varying the external circuitry the plasma can, under some normally unstable circumstances, be completely stabilized.

Lastly a non-linear theory of electrothermal waves, in which second order terms are retained in the perturbation of the basic equations, is presented. From this theory the steady finite amplitude of the waves and the effect of the consequent non-uniformity on the generator characteristics are derived.

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NOMENCLATURE.

- a = sound velocity of heavy gas (Chapter 3).
 A_1, A_2 = ionization and recombination rate coefficients.
 $\text{adj}(\)$ = adjoint of matrix ()
 $\underline{B}$ = magnetic field
 β = Hall parameter
 $\text{cf}(\)$ = cofactor of element ()
 d = electrode separation (Chapter 1).
 $\underline{d}$ = wall separation (Chapter 4)
 $d\underline{s}$ = element of area
 $d\underline{l}$ = line element
 $\Delta\chi$ = range of angles for instability of electrothermal waves from the infinite plasma theory
 $\Delta\theta$ = range of angles spanned by the $\underline{K_E}$'s of the modes (Chapter 4)
 $\text{det}(\)$ = determinant of matrix ()
 $\Delta\nu_i$ = width of collision broadened resonance line of Caesium
 $\underline{E}$ = electric field
 $\underline{\mathcal{E}}$ = electron elastic energy loss to the heavy particles
 e = electric charge
 ϵ_0 = permittivity of free space
 ϵ_s = energy density of a sound wave
 ϵ_E = energy density of an electrothermal wave
 $\epsilon = \frac{n_s}{n_n}$ = seed fraction
 η_E = power extraction efficiency of an MHD generator
 $\eta_L = \frac{R_L}{R_p + R_L}$ = load factor
 G = Rosa factor (Chapter 1)

γ = ratio of specific heats of the heavy gas (Chapter 3)

g = growth rate of electrothermal wave

g_i = degeneracy of i^{th} excitation state.

g_G = degeneracy of ground state

h = Planck's constant

I_p = ionization potential of the seed

I_E = excitation energy coefficient

$\Im ()$ = imaginary part of ()

$\underline{j}$ = electrical current density

k = Boltzmann's constant

$\underline{K}$ = wave vector

κ_k = thermal conductivity of k -species

ℓ = length of electrode (Chapter 1)

= length of channel (Chapter 4)

λ_i = wavelength at line centre of i^{th} line of Caesium

Λ = wavelength of MHD wave

Λ_i = wavelength of MHD wave in i -direction

$$\Lambda_c = \frac{12\pi}{e^3} \left(\frac{\epsilon_0^3 k^3 T_{e0}}{n_{e0}} \right)^{1/2}$$

m_k = particle mass of k -species

m_ν = absorption coefficient at frequency ν

m_{pi} = absorption coefficient at line centre of i^{th} emission line

μ_0 = permeability of free space

μ = coefficient of viscosity of heavy particles (Chapter 3 and Appendix C)

n_k = number density of k -species

- $n = n_n + n_s$ (Chapter 3)
 n_{aG} = number density of Caesium atoms in the ground state
 n_{ai} = number density of Caesium atoms in the i^{th} excitation state
 $\text{num}()$ = numerator of fraction ()
 ν = photon frequency
 ν_c = electron-ion collision frequency
 ν_n = electron-noble gas collision frequency
 ν_a = electron-seed atom collision frequency (Chapter 3)
 ν_e = total electron collision frequency
 ν_i = frequency at centre of i^{th} line of Caesium
 = total ion collision frequency (Chapter 1)
 ω = complex frequency
 ω_k = cyclotron frequency of k-species
 p_k = pressure of k-species
 π_{out} = power delivered to the load of a generator
 ϕ = electrostatic potential
 ψ = current stream function
 q_{en} = electron-noble gas collision cross-section
 R = Radiation transfer
 R_L = load resistance
 R_p = plasma resistance
 R_{Ei} = energy emitted per unit volume in i^{th} line
 R_{Ai} = energy absorbed per unit volume in i^{th} line
 ρ = mass density of heavy particles, or $\rho = n_e^* / T_e^*$ (Chapter 2)
 ρ_i = impact parameter for Van der Waals collision broadening.

$\mathcal{R} () =$ real part of ()

$\underline{r} =$ position vector

$S =$ stagger length of diagonally connected segmented electrodes
(Chapter 4).

$\sigma =$ plasma electrical conductivity

$T_e =$ electron temperature

$T =$ heavy particle temperature

$\tau_e =$ electron collision time

$\tau_i =$ ion collision time (Chapter 1)

$=$ lifetime of i^{th} exciting state of Caesium (Chapter 2)

$\tau_m =$ growth time of magnetosonic wave

$\tau_E =$ growth time of electrothermal wave

$\theta = \arg (n_e^* / T_e^*)$ (Chapter 2).

$\underline{v} =$ heavy particle gas velocity

$\underline{v}_p =$ wave phase velocity

$w =$ width of electrodes

$\hat{x}, \hat{y}, \hat{z} =$ unit vectors in x,y,z directions

$Z =$ Laplace transform parameter (Chapter 4).

SUBSCRIPTS

$o =$ steady state quantities

$e =$ electrons

$i =$ ions, or i^{th} excitation state of Caesium (Chapters 2, 5 and Appendix B), or i^{th} shell of Caesium (Appendix A).

$s =$ seed particles

$a =$ seed atoms

$n =$ noble gas particles

- h = heavy particles (seed and noble gas)
 $\perp$ = component perpendicular to the wall (Chapter 4),
 $\perp$ = component perpendicular to j_0 (Chapter 5).
 $\parallel$ = component parallel to the wall (Chapter 4),
 $\parallel$ = component parallel to j_0 (Chapter 5),
 x, y, z = vector components
 es = electrostatic
 eff = effective quantity
 $\langle () \rangle$ = average over space of $()$

SUPERSCRIPTS.

- $'$ = linear perturbation quantities
 $\wedge$ = non linear perturbation quantities
 $*$ = fluctuation quantities, $()^* = \frac{()'}{()_0}$, or $\frac{\hat{()}}{()_0}$
 $-$ = Laplace transform

CHAPTER 1.

INTRODUCTION.

1.1 Waves in Partially Ionized Plasmas.

If, in a partially ionized plasma, the electron density is in Saha equilibrium at the electron temperature, then small fluctuations in the electron temperature can lead to large fluctuations in the electron density. This is due to the steep slope of the Saha function in the partially ionized regime.

For instance Fig. (1.1) shows n_e/n_s plotted against T_e for Caesium at a density of $10^{22} \frac{1}{m^3}$ and we see that at $T_e = 2000^\circ K$

$$\frac{\partial \log n_e}{\partial \log T_e} \sim 10.$$

Therefore, defining the fluctuation of a parameter to be the ratio of its perturbation to its steady state value, fluctuations in electron temperature lead to fluctuations in electron density which are larger by an order of magnitude, i.e. $\frac{n_e'}{n_{e0}} \sim 10 \frac{T_e'}{T_{e0}}$. This assumes of course that the characteristic time of the fluctuations is long compared with the characteristic Saha relaxation time determined by the ionization recombination rates.

Spatial fluctuations in the electron density and temperature will alter the local values of the plasma parameters, such as the electrical conductivity and, if there is a magnetic field present, the Hall parameter. Therefore in a current carrying plasma fluctuations in the current field will appear. This will alter the local energy and momentum equations in such a way that the spatial fluctuations will

propagate as a wave, and, under certain circumstances, grow in amplitude.

We will be concerned with the propagation and stability of these waves in this thesis.

Two types of waves can be distinguished, viz. magnetosonic and electrothermal waves. The different situations in which the two types dominate are characterized by the value of the electron-heavy particle energy transfer cross section (heavy particles include positive ions and all neutrals). When this cross section is large, so that $T_e = T$ to a good approximation, then the heavy particles must participate in the wave. We then have a magnetosonic wave which is basically a propagating sound wave distorted by fluctuating $\underline{j} \times \underline{B}$ forces and Ohmic heating.

However, when the cross section for energy transfer is small we have $T_e \neq T$ and, in the steady state, the electron Ohmic heating is locally balanced by the collision losses to the heavy particles,^{which} losing heat by convection or conduction, act as a thermal sink. Then the temperature and density can fluctuate independently of the heavy particle properties, which remain approximately constant. This is the electrothermal wave, and it is characterized by propagating fluctuations in the electrical dissipation of the plasma, rather than by dynamical oscillations of the gas as in the magnetosonic case.

The perturbed energy balance in the electrothermal wave can lead to growth as well as propagation.

Magnetosonic waves can also occur when $T_e \neq T$ and the fluctuations arising from this wave can be important when the electrothermal wave is damped.

The growth of both the magnetosonic and electrothermal waves requires not only the partially ionized nature of the plasma, but also the presence of a strong magnetic field. Consequently these waves are a possible mode of instability of the plasma in low temperature (i.e. $\sim 2000^\circ\text{K}$) magneto-hydrodynamic (MHD) devices, both power generators and accelerators, where such conditions are present. The highly nonuniform distribution of electrical conductivity and Hall parameter that results from the unstable wave is damaging to the performance of both these types of devices. Therefore the behaviour of the waves is of interest in these fields of technology, and a considerable amount of research, both theoretical and experimental, has been carried out in recent years, especially in the power generator field.

For this reason the bulk of this thesis will be written with MHD power generator technology and plasmas in mind. However the wave phenomena described may appear wherever the conditions of a partially ionized, current carrying plasma in a magnetic field are present.

In view of our present interest then, the next section gives a brief description of the philosophy and technology of MHD power generation in order to establish the significance of the magnetosonic and

electrothermal waves in this field.

1.2 Historical and Technical Background.

a) MHD devices.

In an MHD power generator electrical energy is extracted from a conducting fluid moving through a magnetic field between two electrodes, which are connected through an external load, (see Fig. (1.2)).

The electric field induced in the fluid, $\underline{v} \times \underline{B}$, drives a current round the circuit. The energy delivered to the load comes from the work done by the fluid under the action of the decelerating $\underline{j} \times \underline{B}$ force, which acts in a direction opposite to $\underline{v}$.

Conversely the same apparatus can be used as a fluid accelerator by replacing the load in the circuit by a voltage source larger than, and opposed to, the $\underline{v} \times \underline{B}$ electromotive force. This drives a current through the fluid in such a way that the $\underline{j} \times \underline{B}$ force accelerates the fluid.

These of course are the same principles as the conventional, solid armature dynamo and electric motor, and have been known since the time of Faraday. What then are the relative merits of MHD power generators and conventional steam-cycle-dynamo generators?

Firstly conventional generators have a great advantage over MHD generators in that their working substance, say copper, has a typical conductivity 10^6 to 10^7 times greater than the most optimistic estimates

for the gases available to MHD systems. This means that their specific power density, which is defined by $\pi_s = |\underline{v} \cdot \underline{j} \times \underline{B}| \sim \sigma v^2 B^2$ assuming $|\underline{j}| \sim \sigma v B$ is much higher than in MHD systems. This is true even allowing for the fact that higher velocities are attainable in the simple expanding gas flow than in the highly stressed rotor system.

The MHD system however can be made to produce a similar amount of power to the conventional systems by making it large enough in volume. This is relatively easy to do since the gas flow is not limited by the mechanical stresses of a rotor. In addition, many of the losses of an MHD system are surface losses, such as heat loss to the walls, mechanical erosion of the walls etc., hence these losses decrease in importance as the generator volume is increased, and MHD systems are most suited to large scale power production.

The MHD process is a direct conversion process where the directed energy of a flowing gas, expanding through a nozzle from a heating stage, is converted directly into electrical energy; in contrast to the indirect process of flowing steam driving a solid turbine, which in turn drives a dynamo. There is a consequent saving in conversion losses and simplicity of design. Partly due to the simplicity of design, and partly due to the ionization requirements for electrical conductivity the MHD process is most suited to the production of energy from high temperature gases. The liquid metal MHD generator may be useful for certain special applications but will not be discussed here.

The source of hot gas may be either a combustion process or heat exchange with a fission reactor. In either case the MHD system may be used to extract energy from the gas at the high temperature end, while the cooled, but still energetic, gas may be used to drive a steam cycle to obtain further electrical energy. Utilisation as a topper in this way seems the most promising application economically for MHD power generation. However it must be noted that a minimum conductivity of the order of 1 - 100 mho/m is required for the system to be economically viable. This can be seen from the following argument.

If we ignore the Hall effect the current density generated by the $\underline{v} \times \underline{B}$ electromotive force is given by

$$\underline{j} = \sigma (\underline{E}_{es} + \underline{v} \times \underline{B})$$

where $\underline{E}_{es}$ is the electric field due to the voltage drop across the load, i.e. $\underline{E}_{es} = - \frac{\underline{j}}{\sigma_L}$, where $\sigma_L = \frac{d}{R_L w l}$

hence
$$\underline{j} = \frac{\sigma \underline{v} \times \underline{B}}{1 + \frac{\sigma}{\sigma_L}}$$

and the power delivered to the load is given by,

$$\Pi_{out} = \frac{j^2 w l d}{\sigma_L} = \frac{\sigma^2 v^2 B^2 w l d}{\sigma_L (1 + \frac{\sigma}{\sigma_L})^2}$$

Maximising this with respect to σ_L , assuming σ constant, we obtain the familiar condition that the external and internal impedances should be equal for maximum power output, i.e. $\sigma = \sigma_L$. Hence $\Pi_{out}^{max} = \frac{\sigma v^2 B^2 w l d}{4}$. Defining the extraction efficiency of a generator as the ratio of Π_{out}^{max}

to the directed energy flowing into the system per second, i.e.,

$$\eta_E = \frac{\pi_{out}^{max}}{\frac{1}{2} \rho v^3 w d} ,$$

we have,
$$\eta_E = \frac{\sigma B^2 l}{2 \rho v}$$

Typically the density and velocity of the gas are 1 kg./m³ and 10³ m/sec respectively, hence for a magnetic field of 1 tesla and a generator length of 10m an extraction efficiency of 10% would be obtained from a conductivity of 10 mho/m.

The economic assessment of MHD power generators is a complex problem involving the system as a whole, i.e. capital cost of plant, fuel costs, efficiency and economics of cycling processes etc., most of which do not concern us here. However, it is sufficient for our purposes to regard 10% extraction efficiency as the absolute minimum for MHD topping to be economically viable.

Increasing the magnetic field strength would, of course, be an apparently effective way of increasing η_E , however it must be remembered that the capital cost of the magnet and the Ohmic losses in the magnet coils, which have not been considered here, become prohibitively large as B increases. It is now recognised in fact that only cryogenic magnets, with low Ohmic losses but large capital cost, will be suitable for MHD generators.

In addition there are more fundamental limitations on B which occur due to the electron and ion gyrations in the magnetic field. When $\omega_e \tau_e \left(= \frac{Be}{m_e v_e} \right)$ becomes of the order of 1 the Hall effect appears and the electrical conductivity becomes a tensor,

$$\underline{j} = \frac{\sigma}{1 + \beta^2} \left(\underline{E} - \underline{\beta} \times \underline{E} \right) \quad (1.1)$$

where $\underline{E}$ = Lorentz field = $\underline{E}_{es} + \underline{v} \times \underline{B}$, and $\underline{\beta}$ = a vector $\parallel \underline{B}$ to $\underline{B}$ of magnitude $\omega_e \tau_e$, $\beta = \omega_e \tau_e$.

The effective impedance of the system in Fig. (1.2) is increased by a factor $1 + \beta^2$. This is because $\underline{E}_x = \underline{E}_z = 0$ and hence $\underline{j}_y = \frac{\sigma}{1 + \beta^2} E_y$. For this reason the presence of the Hall effect has led to various generator types which attempt to mitigate its effects using the electrode geometry and external circuits. A comprehensive account of these has been given by Brogan (1.1) along with an account of the basic principles of power generation. The introduction of complicated electrode systems, segmentation etc., has not been entirely successful and the Hall effect is still something of an embarrassment in MHD power generation.

When the magnetic field is increased even further so that $\omega_i \tau_i \left(= \frac{Be}{m_i v_i} \right)$ becomes of the order of 1, then the ions become uncoupled from the neutrals and ion slip occurs. The flow velocity of the charged particles through the duct is then reduced below $\underline{v}$, the neutral gas velocity. A reduction in the electromotive force therefore occurs so that the power density does not keep increasing with B but levels off

at some limiting value.

In order to produce an economic value of η_E , therefore, a great deal of the research in the physics of MHD generators has been concerned with the determination and enhancement of the conductivity of partially ionized plasmas.

The first serious investigation of the possibility of MHD power generation was carried out by Karlowitz et. al. ^(1,2) from 1938 to 1947. The sort of generator that was envisaged ran at a low gas temperature, below 1000°K, sufficient ionization being produced by a beam of high energy electrons bombarding the gas before it entered the generator section. It was hoped that the electron density would stay "frozen" at the inlet level in transit through the duct. However the recombination rates proved to be larger than anticipated and a sufficiently high conductivity was not achieved.

It seemed at the time that the most likely alternative method of producing the necessary conductivity was by thermal ionization at the gas temperature. This obviously required much higher gas temperatures, and therefore serious problems of duct erosion and cycle design. Since the technology required to produce and handle gas flows of a high enough temperature were not available at the time, research in MHD power generation petered out.

However, with the advent of the jet and rocket age, this technology was quickly developed in connection with aeronautics and space flight.

New and effective sources of high temperature gases were developed, and new materials were produced to withstand the eroding blast of high temperature gas flows similar to those envisaged for MHD generators. Consequently there was a re-awakening of interest in MHD power generation around the late 1950's.

The experiments embarked upon at this time were designed to produce a uniform gas flow of sufficiently high electrical conductivity for economic generation, and to develop the necessary cycle engineering technology. The experiments can be divided into two groups, open cycle and closed cycle systems.

b) Open Cycle and Closed Cycle Generators.

As their names suggest, these two systems differ in that in the first one the gas from the heat source flows through the generator section and is exhausted from the system, while in the second the gas is recycled through the heat source after leaving the generator. In principle this should make no significant difference to the generator. However, in practice open cycle generators are almost always associated with a combustion heat source with the combustion gases as the working fluid, while the closed cycle generators are almost always associated with a fission reactor heat source, with a noble gas as the working fluid.

Although in both cases the working gas has to be seeded with a small amount of an alkali metal to obtain sufficient ionization, the cross sections for momentum and energy transfer are entirely different

due to the different atomic properties of the ambient or buffer gas.

The atomic and molecular properties of combustion gases are extremely complicated and the numerous excitation modes make the electron-heavy particle collision cross section large due to inelastic collisions. Consequently the electron and heavy particle temperatures are almost identical. This means that only thermal ionization of the alkali metal seed is possible. However this is not a serious drawback since combustion temperatures of over 3000°K can be attained giving high enough ionization for suitable conductivities even with the high inelastic momentum transfer cross section between the electrons and the combustion gas.

In the case of closed cycle MHD systems coupled to fission reactors as a heat source, the available gas temperature is determined by the working temperature of the reactors. In the foreseeable future temperatures in the range 1500°K - 2000°K will be obtainable from these sources, but not temperatures higher than 2000°K . Thermal ionization at these temperatures would be insufficient to give the electron conductivity necessary for power production. However the completed shell atomic configuration of the noble gas gives it a low cross section for inelastic collisions, and therefore, provided the seed fraction is low enough, the electrons can be partially uncoupled from the heavy particles thermally.

It was therefore suggested by Kerrebrock ^(1.3) that a suitable degree of ionization could be obtained by elevating the electron temperature above that of the ambient gas using the Ohmic heating intrinsically present in MHD generators.

That the elevation of electron temperature gives an enhanced non-equilibrium degree of ionization requires that the ionization is determined

by the Saha relation at the electron temperature. Previously the belief in this being correct has depended on the heuristic argument that the faster moving electrons determine the statistics of the ionization - recombination processes, and therefore T_e , rather than T , must appear in the Saha relation. However, there has recently been some more concrete evidence (1.4), (1.5) to show that this is indeed true above a critical value of the electron density.

Elevation of the electron temperature is of course a well established phenomenon in gas discharge physics, where the discharge is produced by an external voltage, and different electron and ion temperatures are not at all unknown in plasma physics as a whole. However, the production of nonequilibrium ionization by a $\vec{v} \times \vec{B}$ induced discharge proved an elusive quarry, and it is only within the last three years that it has been demonstrated experimentally. Several laboratories have now reported the presence of nonequilibrium ionization in their experiments (1.6).

The general consensus of opinion seems to be that there is a minimum initial electron density, achieved either by a pre-ionizing electric field discharge or by a high enough initial gas temperature, necessary for the nonequilibrium ionization due to the elevation of the electron temperature in the duct to appear. This minimum value of electron density is presumably connected with the critical value above which ionization is determined by Saha at the electron temperature, though this has not yet been clearly established.

In any case the experiments of the last ten years have shown that values of the local electrical conductivity compatible with economic MHD power generation can be achieved in both open and closed cycle generators. To build an economic system it remains to remove or reduce the various phenomena which make the performance of the several generator types non-ideal.

Among these phenomena are plasma non-uniformities, electrode shorting, insulator leakage, end effects, and plasma to earth shorting. Of these, only the effects of non uniformities in the bulk of the plasma will concern us here, and a brief description of their influence on the generator performance is given in the next sub-section.

c) The Rosa Effect.

In 1962 Rosa ^(1.7) first analysed the effect of plasma non-uniformities on the electrical properties of MHD generators. A brief account of his simple and illuminating argument will be given here.

If $\underline{B}$ is in the z -dir. and $E_z = 0$, so that $\underline{j}$ is confined to the x - y plane, then the Ohm's Law with Hall effect, equation 1.1, can be written in the component form,

$$j_x = \frac{\sigma}{1+\beta^2} (E_x - \beta E_y) \quad \text{--- (1.2)}$$

$$j_y = \frac{\sigma}{1+\beta^2} (\beta E_x + E_y) \quad \text{--- (1.3)}$$

where $\beta (= w_e \tau_e)$ is the Hall parameter, and we neglect ion slip effects.

If σ and β are functions of y only, see Fig. (1.3), then the steady state field equations, i.e.

$$\nabla \cdot \underline{j} = 0 \quad , \quad \nabla \times \underline{E} = 0 \quad ,$$

require that j_y and E_x are constants. Averaging (1.3) over y (denote $\int f(y) dy$ by $\langle f(y) \rangle$), we obtain

$$\left\langle \frac{1+\beta^2}{\sigma} \right\rangle j_y = \langle \beta \rangle E_x + \langle E_y \rangle \quad - \quad (1.4)$$

Rearranging (1.2) and (1.3) we can write the x -comp of Ohm's Law in the form, $j_x = \sigma E_x - \beta j_y$ which when averaged over y , gives

$$\langle j_x \rangle = \langle \sigma \rangle E_x - \langle \beta \rangle j_y \quad (1.5)$$

It will be most illustrative if we consider a particular generator type, viz. the Faraday generator. In this type the continuous electrodes of Fig. 1.2 are replaced by segmented electrodes with the expectation that an electric field in the x -dn, will appear in such a way as to make $j_x = 0$.

Substituting $j_x = 0$ into 1.2 and 1.3, the equations for a uniform plasma, we obtain $E_x = \beta E_y$ and $j_y = \sigma E_y$. However substituting $\langle j_x \rangle = 0$ into equations (1.4) and (1.5) we obtain $j_y = \frac{\langle \sigma \rangle}{G} \langle E_y \rangle$, where

$$G = \langle \sigma \rangle \left\langle \frac{1+\beta^2}{\sigma} \right\rangle - \langle \beta \rangle^2 \quad ,$$

i.e. the effective conductivity has been reduced by a factor $\frac{1}{G}$. If there are only conductivity fluctuations in the gas, which are, say, sinusoidal around some steady state, then

$$G = 1 + \frac{(1+\beta^2)}{2} A^2$$

where A = amplitude of σ^* fluctuations. $\left(\sigma^* = \frac{\sigma'}{\sigma_0} \right)$

For large values of β the value of G can be significantly greater than one even for small values of σ^* . A similar deviation from the ideal behaviour of other generator types results from the presence of non uniformities.

The reduced output and efficiency due to the Rosa effect make non-uniformities an important factor in the economic viability of MHD generators.

1.3 Instabilities in MHD Generators.

The two wave types discussed in section (1.1) are possible instabilities of MHD generators, leading to a damaging non uniformity in the plasma parameters. As we would expect from the properties of the two types, magnetosonic waves are dominant in open cycle generators, while electro-thermal waves are dominant in closed cycle generators.

The existence of the magnetosonic instability was first suggested by Velikhov ^(1.8) in 1962, and since then it has been studied theoretically by various authors ^{(1.9) - (1.13)}. However, not much in the way of experimental observations has been published. Basically it is a sonic fluctuation distorted by the $\underline{j} \times \underline{B}$ forces and Ohmic heating, and, in a uniform atmosphere, there are three modes. In references (1.9) - (1.13) these modes and their separate properties are not clearly distinguished, however Hougen ^(1.14) has recently given an excellent analysis of the various modes.

We can obtain an estimate of the characteristic growth time of the amplitude of this type of wave by equating the rate of increase of the fluctuating momentum to the fluctuating $j \times B$ force, i.e.

$$\rho_0 \frac{\partial v'}{\partial t} \sim j' B$$

now $j^* \sim \sigma^* \sim n_e^* \sim 10 T^*$, the last step following from the slope of the Saha function.

In a sound wave $T^* \sim v^*$ hence $j^* \sim 10 v^*$, and since $j_0 \sim \sigma_0 v_0 B$, we have ,

$$j' \sim 10 v' \sigma_0 B$$

$$\text{i.e. } \rho_0 \frac{v'}{\tau_m} \sim 10 v' \sigma_0 B^2$$

where τ_m = characteristic growth time for a magnetosonic wave, and

$$\tau_m = \frac{\rho_0}{10 \sigma_0 B^2}.$$

Substituting $\rho_0 = 1 \text{ kg./m}^3$, $\sigma_0 = 100 \frac{\text{mho}}{\text{m}}$ $B = 1 \text{ tesla}$, we obtain $\tau_m = 10^{-3} \text{ secs.}$ A similar estimate would be obtained by considering the rate of change of the fluctuating temperature and the fluctuating Ohmic heating.

Note that the waves treated in this thesis have approximately zero fluctuation in the magnetic field, due to the low value of the magnetic Reynolds number in MHD generators. The validity of this approximation will be discussed in Chapter 2. It follows that B simply represents the applied field.

It is interesting to ask what relation these partially ionized waves have to the conventional magnetosonic waves, i.e. the Alfvén, fast and slow types. The canonical theory of the conventional waves assumes ∞ conductivity, and the introduction of finite conductivity is known to produce damping, not growth. However the theory of the damping of the conventional waves assumes constant conductivity in space, while the source terms of the magnetosonic and electrothermal waves come from the fluctuations of the conductivity.

These fluctuations are significant here due to the partially ionized nature of the plasmas we are considering. It may be said that the conventional magnetosonic waves concentrate on the Faraday induction effects and ignore dissipation, while in the partially ionized wave theory presented here we concentrate on the dissipation and ignore Faraday induction effects.

To return to our estimate of τ_m we see that the waves will e-fold only once in a generator of length 1 m., assuming $v = 10^3$ m/sec. Depending on the initial amplitude of the fluctuation, the magnetosonic instability may not be a serious loss mechanism in open cycle generators, except if the generator is longer than 1 metre, say 10 metres.

The conditions in closed cycle MHD generators, where every effort is made to reduce electron energy losses etc. so that the steady state electron energy equation can be written as,

$$\frac{j_e^2}{\sigma_e} = \sum_h 3 n_e k (T_e - T) \nu_{eh} \frac{m_e}{m_h},$$

are exactly the conditions required for the propagation of electrothermal waves. The existence and instability of these waves was first suggested independently by Velikhov et. al. (1.15) in the U.S.S.R. and by Kerrebrock (1.16) in the U.S.A. Since then a considerable amount of both theoretical (1.14), (1.17) - (1.20) and experimental (1.21) - (1.26) studies of the waves have been reported. A somewhat more complete bibliography of the literature on the electrothermal instability (under the name of Magnetic Striations) has recently been given by Nedaspasov (1.27).

As stated in section (1.1), electrothermal waves are characterised by fluctuations in the electron density and temperature, with the heavy particle properties remaining constant. The perturbations of the plasma parameters and the consequent perturbation of the electron energy balance cause the fluctuations to move and, under certain circumstances, grow.

The source of the instability is the enhanced local Ohmic heating in the regions of increased electron density. If the fluctuations in the energy loss mechanisms (elastic collision losses, radiation and thermal conduction) are unable to remove the excess heat, the electron temperature will rise leading, by ionization, to a further increase in the electron density, and so the wave grows.

We can obtain an estimate of the characteristic growth time by equating the energy required for ionization to the fluctuating Ohmic

heating

$$\text{i.e.} \quad I_p \frac{\partial n_e'}{\partial t} \sim \frac{2 j_o j'}{\sigma_o}$$

$$\text{now} \quad j^* \sim n_e^*$$

$$\text{hence} \quad \tau_E \sim \frac{\sigma_o I_p n_{eo}}{2 j_o^2}$$

for Caesium $I_p \approx 5 \times 10^{-19}$ joules and typical values are $\frac{j_o^2}{\sigma_o} \sim 10$ megawatts, $n_{eo} \sim 10^{20} \frac{1}{m^3}$, hence $\tau_E \sim 10^{-5}$ secs.

Hence, if the transit time of the gas in the generator is 10^{-3} secs, the wave may e-fold about 100 times.

The growth rate of both the magnetosonic and electrothermal instabilities are dependent on the direction of the wave vector relative to $\underline{B}$ and $\underline{j}_o$. They grow only in some well defined range of angles.

However the magnetosonic instability grows for all non zero values of B , within the appropriate range of angles, while the electrothermal instability will grow only for values of B above a certain value determined by a critical Hall parameter. This is because below the critical Hall parameter the fluctuations of the energy losses, collisions etc. are greater than the fluctuations in the Ohmic heating. The critical Hall parameter usually has a value of the order of 1.

It should be noted that the conditions for a large elevation of the electron temperature induced by the $\underline{v} \times \underline{B}$ field require a value of the Hall parameter of this order of magnitude. This can be seen by writing

the energy balance in the approximate form,

$$\frac{j^2}{\sigma} \sim 3 n_e k (T_e - T) \frac{m_e}{m_h} v_{eh}$$

where h represents the most dominant heavy particle for collisions. Now,

$$j \sim \sigma v B$$

hence

$$\sigma v^2 B^2 \sim 3 n_e k (T_e - T) \frac{m_e}{m_h} v_{eh}$$

and

$$\sigma \sim \frac{n_e e^2}{m_e v_{eh}}$$

therefore

$$\frac{B^2 e^2}{m_e v_{eh}} v^2 \sim 3 k (T_e - T) \frac{m_e v_{eh}}{m_h}$$

The Hall parameter is given by $\beta \sim \frac{B e}{m_e v_{eh}}$ hence we have,

$$\beta^2 \sim \frac{3 k (T_e - T)}{m_h v^2}$$

for a flow in the region of Mach No. 1, $m_h v^2 \sim k T$ hence,

$$\beta^2 \sim 3 \left(\frac{T_e}{T} - 1 \right)$$

For a temperature elevation $\frac{T_e}{T} \approx 2$ we therefore require $\beta \sim 1$.

In view of the high value of the electrothermal growth rate under the conditions encountered in closed cycle MHD power generators, and the damaging effect of plasma non-uniformities as demonstrated by Rosa, the electrothermal instability may have a serious effect on the economics of this type of generator. For this reason the behaviour of the waves and any possible means of stabilizing them are of great interest.

1.4 Synopsis of Chapters 2-5.

In Chapter 2 a dispersion relation for electrothermal waves in an alkali seeded noble gas is derived from a simple linear, infinite plasma analysis. The dispersion relation differs from previous dispersion relations in that it is more complete containing the effects of finite ionization - recombination rates, finite degree of ionization, radiation transfer, electron thermal conduction, and the combination of both neutral and Coulomb collisions.

The expressions for the complex frequency derived previously, although usually of a simplified form, are nevertheless fairly complicated and it is difficult to obtain a clear picture of how the growth rate varies with different plasma situations. In Chapter 2 the dispersion relation is solved numerically, and the complex frequency is plotted as a function of various plasma parameters. By doing this it is hoped that the properties of the wave are clarified. In addition the nature of the waves, both their motion and the source of their growth, are investigated in detail.

The influence of magnetosonic waves in non-equilibrium plasmas is discussed and analysed in Chapter 3. The various magnetosonic modes are described and their interaction with the electrothermal modes is investigated. A unified dispersion relation involving linear, plane-wave fluctuations of all the gas parameters, i.e. including both magnetosonic and electrothermal effects, is derived.

In chapter 4 the infinite plasma assumption for the analysis of

electrothermal waves is dropped. The interaction of the waves with various types of boundaries is investigated, with particular interest in the influence of the various boundaries on the linear growth rate.

Non-linear effects in electrothermal waves are investigated in Chapter 5. Experimentally the waves are observed to attain a steady non-linear amplitude, and the value of this amplitude is derived from a second order theory. The existence of the steady state is closely related to the Rosa effect, which is also second order, and the effective conductivity resulting from the instability is derived in a straightforward way from the non-linear theory.

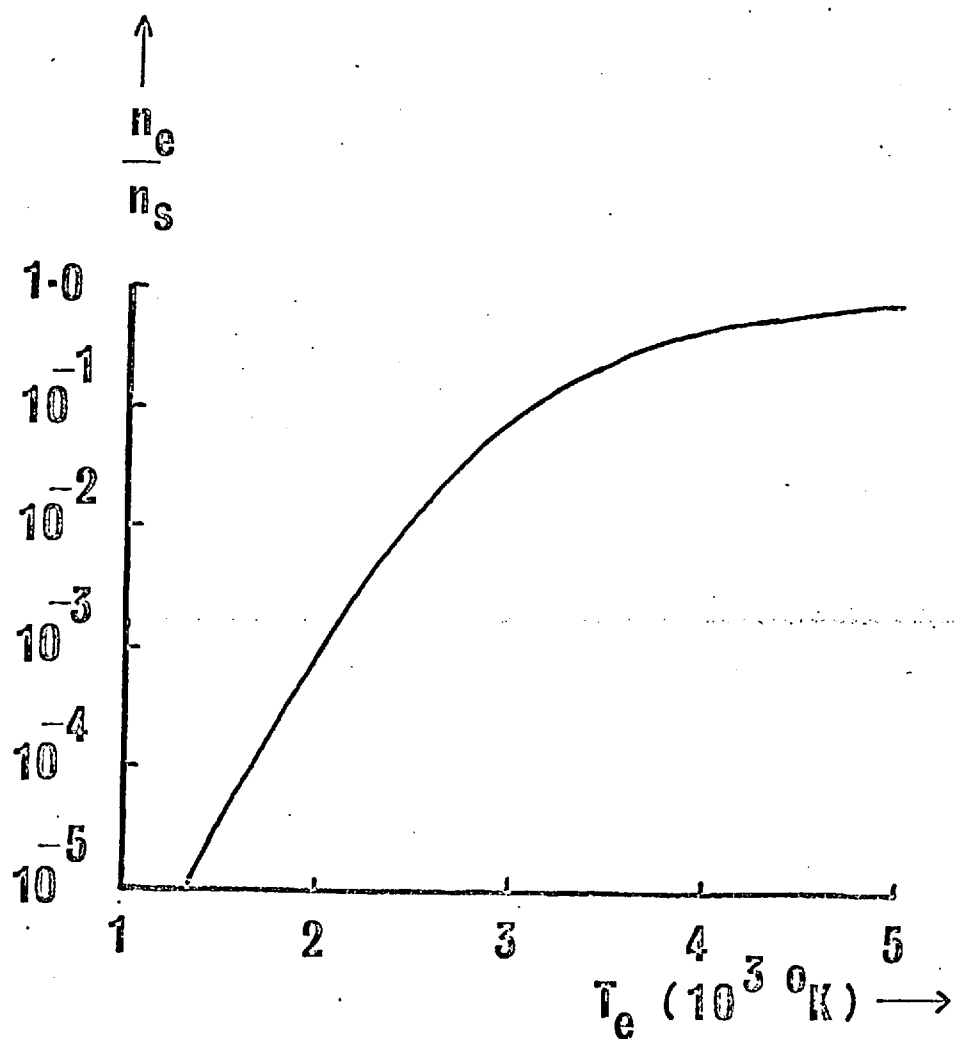


Fig. (1.1) Degree of ionization as a function of electron temperature for Saha equilibrium.

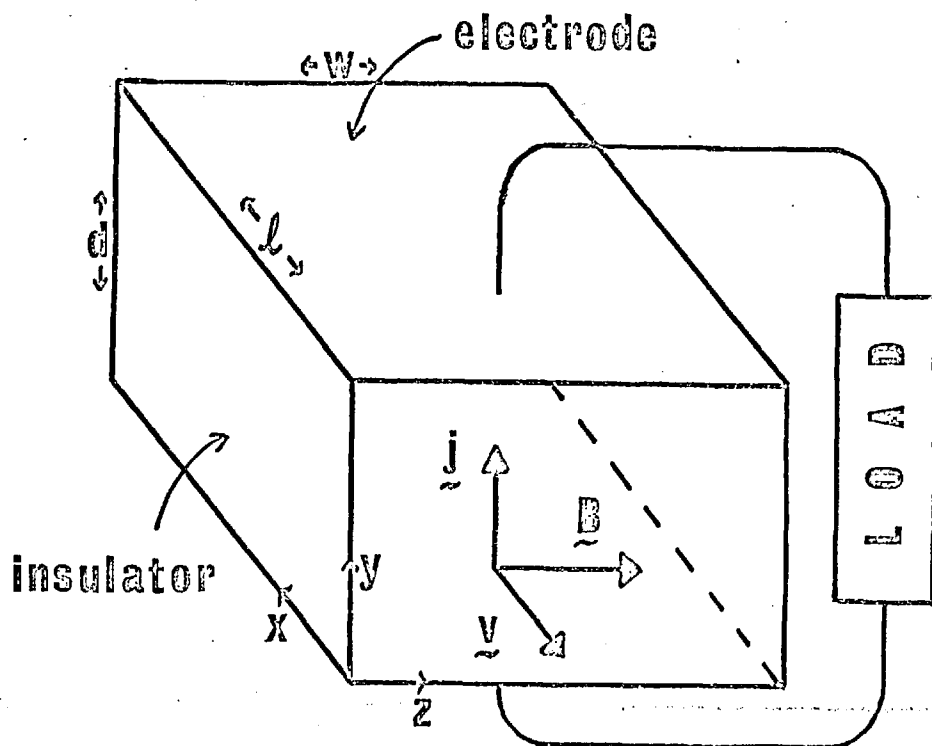


Fig.(1.2) Geometry of a simple MHD generator.

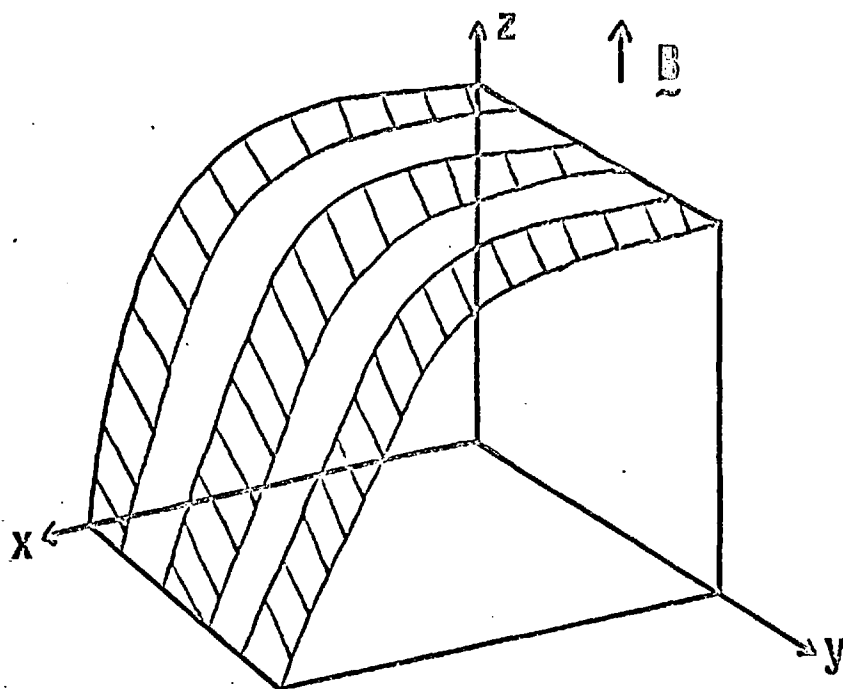


Fig.(1.3) Topology of plasma nonuniformity for Rosa effect (shaded layers represent regions of higher electron density, blank areas of lower electron density).

CHAPTER 2.

ANALYSIS OF THE NATURE AND GROWTH OF ELECTROTHERMAL WAVES.

2.1 Introduction.

The object of this chapter is to derive a linear dispersion relation from the basic electrothermal equations, including as much of the relevant physics as possible. The dispersion relation will then be solved for the complex frequency, and the variation of this frequency with various plasma and wave parameters examined. From this analysis we hope to find the ranges of plasma and wave parameters, especially electron temperature, for which the waves are stable.

Among the physical mechanisms included are finite ionization - recombination rates, finite degree of ionization, radiation transfer, electron thermal conduction, and the combination of both neutral and Coulomb collisions.

It is expected that the effect of finite ionization rates will be to damp the waves at low temperatures. If we look at the equation for the rate of change of electron density, in the gas frame of reference, using the collisional-radiative theory (2.1) - (2.3), for the recombination coefficient and assuming that the equilibrium state is Saha equilibrium at the electron temperature we have,

$$\frac{\partial n_e}{\partial t} = A_1 n_e (n_s - n_e) - A_2 n_e^3 \quad (2.1)$$

where $A_2 = 1.1 \times 10^{-20} T_e^{-9/2} \frac{m^6}{\text{sec.}}$

and $A_1 = \left(\frac{2\pi m_e k T_e}{h^2} \right)^{3/2} e^{-\frac{I_p}{k T_e}} A_2$

(In view of the importance of the ionization-recombination mechanism to electrothermal and magnetosonic waves, a brief outline of the collisional radiative theory is given in Appendix A).

Suppose we have a perturbation, n'_e , in the electron density, we can then find the characteristic time it takes for equilibrium to be re-established in the absence of destabilising effects. Linearizing with respect to n_e equation (2.1) becomes,

$$\frac{\partial n'_e}{\partial t} = -n'_e A_{20} n_{e0}^2 \left(\frac{2n_s - n_{e0}}{n_s - n_{e0}} \right)$$

Hence, for $n_s \gg n_e$, we have

$$\frac{\partial n'_e}{\partial t} = -\frac{n'_e}{\tau_s}$$

where τ_s , the Saha relaxation time, is given by $\tau_s = \frac{1}{2n_{e0}^2 A_{20}}$

Physically it seems likely that the relaxation rate $1/\tau_s$ will be an upper limit to the growth rate of an electrothermal instability, since the wave grows by ionization in the peaks and recombination in the troughs. Since $1/\tau_s$ decreases rapidly with electron temperature (see Fig. 2.1), there ought to be a significant damping effect on

the waves at low temperatures, but a negligible effect at high temperatures.

The recent work, (1.4), (1.5), on the validity of the Saha relation for the steady state in a current carrying plasma has shown that there is a critical value of electron density above which Saha holds. With no other analytical expression for the steady state available, the existence of this critical density has been ignored and the Saha relation is used for the steady state over the whole range considered. The effect of a reduced electron density at equilibrium is equivalent to reducing the ionization rate, and hence will tend to stabilize the electro-thermal waves. Using Saha equilibrium for the steady state therefore gives an upper limit for the growth rate of the waves.

Since the instabilities are dependent on the partially ionized nature of the caesium seed it is expected that when the seed becomes fully ionized the instabilities will die away. The important parameter here is $\frac{\partial \log n_e}{\partial \log T_e}$ (Fig. 1.1) which we have seen is of the order 10 at 2000°K for Caesium, but decreases to below 1 above 5000°K. We expect therefore that the growth rate of the electro-thermal instability will have a maximum as a function of electron temperature, falling off at low and high temperatures due to the effects of finite ionization- recombination rates and finite degree

of ionization respectively.

The effects of radiation and thermal conduction will be to damp the wave by transferring energy from the peaks to the troughs. The radiation term used in the calculations is that derived by Lutz^(2.4). The basic assumption employed is that the first doublet excitation level of the caesium seed atoms, which dominates the radiation transfer is populated at L.T.E. with respect to the electron temperature; i.e. we assume that the level is populated by inelastic and super-elastic collisions with electrons, and radiative de-excitations play only a small role. Due to the large self absorption which increases the effective lifetime of an excited state, and the high electron densities this is a good approximation in the one atmosphere plasmas with .1% seed fraction that we consider here. We will discuss the possibility of enhancing the radiation transfer to damp the waves.

At the low temperature end of the range considered, neutral collisions dominate the electron collision frequency, while at the high temperature end, due to the large increase in ionization, Coulomb collisions dominate. To examine the temperature dependence of the wave it is therefore necessary to include both types in evaluating the collision frequency.

Some, but not all, of these physical mechanisms have been

considered by previous authors. For instance Kerrebrock^(1.16) has a section with finite ionization rates, Nedaspasov^(1.17) includes finite degree of ionization, and Hougen^(1.14) includes all the physical effects used here except finite degree of ionization. A more complete range of the relevant physics is therefore included here to obtain a more complete picture of the properties of the waves. In addition, the physical nature of the waves, how they move and grow, was previously unclear, and this is examined here in detail. A comparison of the results obtained here and previous theoretical and experimental results is given in section 7 of this chapter.

2.2 Basic Assumptions of Electrothermal Theory.

The plasmas we will consider throughout this thesis have four components, viz. electrons, seed ions, seed atoms and neutral buffer atoms. The seed is considered to be an alkali metal (usually Caesium), and the buffer a noble gas (usually Argon). Under the conditions of 1 atmosphere pressure, seed fraction of the order of $\cdot 1\%$, and temperatures of the order of thousands of degrees Kelvin we make the following assumptions:-

- 1) The electrons have a Maxwellian distribution in velocity space.

This is an implicit assumption in the evaluation of the collision frequencies and is of course intimately connected with the assumption of Saha equilibrium, and the use of the ideal gas laws

for the electron gas. The calculations of refs. (1.4) and (1.5) show that this is not a good approximation at low temperatures and densities. However it is a good approximation in the electron temperature range of most interest to MHD generators, viz. 2500°K to 3000°K.

2) The electron number density n_e and the ion number density n_i are equal.

Essentially we are removing high frequency plasma oscillations from our equations. The assumption is not strictly true since space charge electric fields exist in the fluctuations. However the difference in electron and ion number densities required to produce these fields is very small compared to the fluctuation amplitudes of the two densities. This can be seen as follows.

Poisson's equation for the perturbed state gives us,

$$\nabla \cdot \tilde{E}' \sim \frac{2\pi \tilde{E}'}{\lambda} \sim \frac{(n_i' - n_e')e}{\epsilon_0}$$

where λ = fluctuation wavelength.

From the linearized Ohm's law, which will be described in section (2.5), we have,

$$E^* \sim n_e^* \quad \text{hence,}$$

$$(n_i' - n_e') \sim \left(\frac{2\pi \epsilon_0 E_0}{e \lambda n_{e0}} \right) n_e'$$

Typical values are $E_0 = 10^3$ volts/m, $n_{e0} = 10^{20} \frac{1}{m^3}$ and $\Lambda = 10^{-2} m.$,

therefore we have

$$(n_i' - n_e') \sim 10^{-5} n_e'$$

3) The heavy particles have the same temperature, i.e.

$$T_i = T_a = T_n = T, \text{ but } T_e \neq T.$$

The equality of the ion, seed and neutral temperatures follows from their approximately equal mass. Thermal coupling between them is strong, while the much lighter electrons are partially uncoupled thermally from the other components.

4) The heavy particles have the same centre of mass velocity,

$$v_i = v_a = v_n = v, \quad \text{i.e. no ion or seed slip.}$$

5) T, v, n_n, n_s are all constant in space and time.

We assume therefore that the steady state is "uniform", i.e. that $L \gg \Lambda$ where L = characteristic length over which the steady state variables change. Furthermore we assume that the heavy particles do not participate in the waves, apart from variations in the degree of ionization. This is equivalent to assuming that the inertia and thermal capacity of the waves is so large that their properties under the influence of the fluctuating $\underline{j} \times \underline{B}$ forces and collision energy transfer do not have time

to change within the characteristic times of the wave (period and growth time). The validity of this assumption and the interaction between electrothermal and sonic effects are discussed in Chapter 3.

6) The magnetic field, $\underline{B} = (0,0,B)$, is constant in space and time, i.e. zero magnetic Reynolds number.

This is a good approximation for the characteristic conductivities, dimensions and velocities in MHD generators. To compare B^* with n_e^* , we use Ampere's law, i.e.,

$$\frac{\nabla \times \underline{B}'}{\mu_0} \sim \frac{2\pi \underline{B}'}{\mu_0 \lambda} \sim j'.$$

$$\text{Now } j^* \sim n_e^*$$

and hence $j' \sim \sigma_0 v B n_e^*$ since $j_0 \sim \sigma_0 v B$

$$\text{Therefore, } B^* \sim \left(\frac{\sigma_0 v \mu_0 \lambda}{2\pi} \right) n_e^*$$

The expression in the brackets is the usual magnetic Reynolds number with the reciprocal of the wave number substituted for the characteristic length.

Using $\sigma_0 = 100 \text{ mho/m}$, $v = 10^3 \text{ m/sec}$ and $\lambda = 10^{-2} \text{ m}$, we have $B^* \sim 10^{-4} n_e^*$, hence fluctuations in B are negligible.

All these assumptions are made throughout this thesis, with the exception of assumption (5) which is dropped in chapter 3.

2.3 Basic Equations of the Electrothermal Theory.

Under the foregoing approximations the plasma state can be described by three electron equations (density, momentum, and energy), and two field equations. In the frame of reference of the heavy particles, i.e. $\underline{v} = 0$, these equations have the following form:-

Electron density.

This is equation (2.1) given in section (2.1).

Electron momentum (Ohm's law)

$$j_x = \frac{\sigma}{1 + \beta^2} (F_x - \beta F_y) \quad \text{--- (2.2)}$$

$$j_y = \frac{\sigma}{1 + \beta^2} (\beta F_x + F_y) \quad \text{--- (2.3)}$$

(note we assume $E_z = 0$, $\frac{\partial}{\partial z} = 0$)

where $\underline{F} = \underline{E} + \frac{\nabla p_e}{n_e e}$, $\underline{E} = \underline{E}_{es} + \underline{v} \times \underline{B}$,

$$p_e = n_e k T_e ,$$

$$\sigma = \frac{n_e e^2}{m_e \nu_e} ,$$

$$\beta = \frac{B e}{m_e \nu_e} ,$$

the electron collision frequency is the sum of the Coulomb and neutral collisions,

i.e. $\mathcal{V}_e = \mathcal{V}_c + \mathcal{V}_n$

where $\mathcal{V}_c = \frac{n_e}{6} (\pi k T_e)^{-3/2} \frac{e^4}{\epsilon_0^2 \sqrt{2 m_e}} \log \Lambda_c$

$$\Lambda_c = \frac{12\pi}{e^3} \left(\frac{\epsilon_0^3 k^3 T_e^3}{n_e} \right)^{1/2}$$

$$\mathcal{V}_n = n_n \left(\frac{8}{\pi} \frac{k T_e}{m_n} \right)^{1/2} q_{en}$$

and (2.5) $q_{en} = [6.0 + 5.5 \times 10^{-3} (T_e - 2000)] \times 10^{-21} \text{ m}^2$

Electron energy equation

$$\begin{aligned} \frac{\partial U_e}{\partial t} - \nabla \cdot \left(\frac{U_e \underline{j}}{n_e e} \right) - p_e \nabla \cdot \left(\frac{\underline{j}}{n_e e} \right) = \\ \frac{\underline{j}^2}{\sigma} - 3 n_e k (T_e - T) \left(\mathcal{V}_c \frac{m_e}{m_s} + \mathcal{V}_n \frac{m_e}{m_n} \right) - \frac{\partial n_e}{\partial t} I_p \\ - R + \nabla \cdot (\kappa_e \nabla T_e) \end{aligned} \quad (2.4)$$

where the internal energy is given by $U_e = \frac{3}{2} n_e k T_e$ and the thermal conductivity by,

$$\kappa_e = \frac{n_e k^2 T_e}{m_e \mathcal{V}_e} \frac{5/2}{(1 + \beta^2)}$$

The radiation in the plasma is dominated by the lowest order Caesium doublet ($6^2 S_{1/2} - 6^2 P_{1/2}$, $6^2 S_{1/2} - 6^2 P_{3/2}$).

Since the absorption length of this resonance radiation is small

compared with the plasma dimensions the radiation is largely trapped. Using the expression derived by Lutz ^(2.4) for an infinite plasma we obtain, assuming the plasma to be uniform in the y - direction

$$R(y) = \sum_i \left\{ 2\pi B_i(y) \Delta\nu_i m_{pi}^{1/2} \int_0^{m_{pi}} m_\nu^{-1/2} \left(1 + \frac{m_\nu}{2m_{pi}}\right) dm_\nu \right. \\ \left. - \pi \Delta\nu_i m_{pi}^{1/2} \int_0^{m_{pi}} m_\nu^{1/2} \left(1 + \frac{m_\nu}{2m_{pi}}\right) dm_\nu \times \right. \\ \left. \int_1^\infty \frac{dz}{z} \left[\int_y^\infty B_i(t) e^{-m_\nu z(t-y)} dt + \int_{-\infty}^y B_i(t) e^{-m_\nu z(y-t)} dt \right] \right\}$$

The summation is over the two resonance lines, and,

$$B_i(y) = \frac{2h\nu_i^3}{c^2} \frac{1}{\exp\left(\frac{h\nu_i}{kT_e(y)}\right) - 1},$$

$$m_{pi} = \frac{n_{aG}}{4\pi^2} \lambda_i^2 \frac{g_i}{g_G} \frac{1}{\Delta\nu_i \tau_i}$$

$$\Delta\nu_i = \pi \rho_i^2 n_n \sqrt{\frac{8kT}{\pi m_n}}$$

The first term represents emission from the volume element and the other two terms represent absorption from the rest of the plasma. It is assumed that the population density n_{ai} of each excited state is dominated by electron collisional excitation

and de-excitation, so that,

$$n_{ai} = n_{ag} \frac{g_i}{g_g} \exp\left(-\frac{h\nu_i}{kT_e}\right)$$

g_i = state degeneracy

and
$$n_{ag} + \sum_i n_{ai} = n_a$$

i.e.
$$n_{ag} = \frac{n_a}{1 + \sum_i \frac{g_i}{g_g} \exp\left(-\frac{h\nu_i}{kT_e}\right)}$$

The dominant absorption line broadening mechanism is Van der Waals broadening by the neutral atoms. The values of λ_i and ν_i used are those given by Corliss and Ozman (2.6) and ρ_i is calculated from a formula given by Griem (2.7).

Charge Conservation

$$\nabla \cdot \underline{\underline{j}} = 0 \quad - (2.5)$$

This follows from assumption (2)

Faraday's Law.

$$\nabla \times \underline{\underline{E}} = 0 \quad - (2.6)$$

This follows from assumption (6).

We have then a set of six equations in the six variables n_e, T_e, j_x, j_y, E_x and E_y . We can linearize these to obtain equations for electrothermal fluctuations about a steady state. In chapter 4 these equations are solved under boundary conditions on $\underline{\underline{j}}$ and $\underline{\underline{E}}$. In this chapter we assume that

the plasma is infinite and apply plane wave solutions. The justification for this is that often nearly plane wave fluctuations are observed experimentally, and also the relatively simple plane-wave approach enables us to examine in detail the mechanisms of movement and growth without boundary complications.

2.4 The Steady State

The steady state of the plasma is assumed to be uniform and the equations (2.1) - (2.4) reduce to,

$$(2.1) \quad \frac{n_{e0}^2}{n_s - n_{e0}} = \frac{A_{10}}{A_{20}} \quad \text{i.e. Saha}$$

Equilibrium,

$$(2.2) \quad j_{0x} = \frac{\sigma_0}{1 + \beta_0^2} (E_{0x} - \beta_0 E_{0y})$$

$$(2.3) \quad j_{0y} = \frac{\sigma_0}{1 + \beta_0^2} (\beta_0 E_{0x} + E_{0y})$$

$$(2.4) \quad \frac{j_{0x}^2}{\sigma_0} = 3 n_{e0} k (T_{e0} - T) \left(\nu_{co} \frac{m_e}{m_s} + \nu_{no} \frac{m_e}{m_n} \right)$$

The last of these of course represents the nonequilibrium elevation of the electron temperature. The radiation transfer term is zero in the steady state (evaluation of the integrals in R for a uniform plasma easily demonstrates this), since each part of the plasma is absorbing as much as it emits.

For a finite plasma, however, the limits of the integral would be different and R would be non-zero. Physically this means

we have radiation escaping from a finite plasma; however, for the resonance radiation the absorption length (typically 10^{-5} m in the centre of the line) is very much smaller than the dimensions of typical laboratory MHD plasmas, and the radiation escape is small compared to the elastic loss to the heavy particles.

Similarly the short collision length ($\sim 10^{-5}$ m) means that the electron thermal conduction losses from the bulk of a typical laboratory MHD plasma are small compared to the elastic losses.

We assume therefore that the steady state of our infinite plasma model is a good approximation to that in a finite plasma. Unfortunately this also means that radiation and thermal conduction will be ineffective in damping wavelengths of the order of the dimensions of a typical apparatus, however they will have an important damping effect on short wavelength modes.

Note that strictly speaking our assumptions of a uniform steady state and Ohmic heating of the heavy particles via the equipartition elastic loss term are incompatible. The heavy particles must lose the energy either by convection or conduction, or both. The implicit assumption therefore is that either the velocity of convection or the heavy particle thermal conductivity is large enough for the condition $L \gg \Lambda$ to hold.

2.5 First order perturbation of equations, and the Dispersion Relation.

We perturb the set of equations (2.1) - (2.6) about the steady state defined in section 4. Neglecting terms of higher than first order, we take into account the functional dependence of all the plasma parameters on n_e and T_e in the perturbation. The linearization is straightforward, but algebraically tedious.

Equation (2.1) linearizes to

$$\frac{\partial n_e^*}{\partial t} = a_0 T_e^* + b_0 n_e^* \quad - (2.7)$$

where
$$a_0 = A_{20} n_{e0}^2 \left(\frac{3}{2} + \frac{I_p}{k T_{e0}} \right)$$

and
$$b_0 = A_{20} \frac{n_{e0}^2 (n_{e0} - 2n_s)}{n_s - n_{ec}}$$

The linearization of equations (2.2) and (2.3) can be facilitated by applying the plane wave solution for the perturbations to equations (2.5) and (2.6). If the y -axis defines the direction of the wave vector (i.e. all perturbed quantities are proportional to $\exp(i\omega t - iKy)$), we have

$$\frac{\partial}{\partial x} \equiv 0,$$

$$\therefore \frac{\partial j_y'}{\partial y} = 0 \quad \text{and} \quad \frac{\partial E_x'}{\partial y} = 0, \quad \text{from}$$

(2.5) and (2.6) respectively. That is,

$$i K j_y' = 0 \quad \text{and} \quad i K E_x' = 0$$

$$\text{therefore } j_y' = 0 \quad \text{and} \quad E_x' = 0$$

$$\text{Since } \frac{\partial}{\partial x} \equiv 0 \quad \text{it follows from } E_x' = 0 \quad \text{that } F_x' = 0.$$

Rearranging (2.2) and (2.3) to obtain

$$j_x = \sigma F_x - \beta j_y, \quad \text{we can linearize this simply}$$

to get,

$$j_x' = \sigma' E_{0x} - \beta' j_{0y}$$

Linearization of equation (2.3) simply gives us the F_y' that must exist in the wave to make $j_y' = 0$ and is not necessary to obtain the dispersion relation.

From the functional dependence of σ and β on T_e and n_e , and assuming that $\underline{j}_0$ makes an angle χ with $\underline{K}$ (more specifically,

$$\chi = \sin^{-1} \left(\frac{\underline{j}_0 \times \underline{K} \cdot \underline{B}}{\underline{j}_0 \cdot \underline{K} B} \right)$$

we can write j_x' as,

$$j_x' = n_e^* \left[(1 - r(1 - u))(\sin \chi + \beta_0 \cos \chi) + r(1 - u)\beta_0 \cos \chi \right] - T_e^* \left[\frac{3}{2} \sin \chi (2r(u - 1) + 1) \right]$$

$$\text{where } r = \frac{\nu_{co}}{\nu_0}, \quad u = \frac{1}{2 \log \Lambda_{co}}$$

Linearizing the energy equation we get,

$$n_{e0} \left(\frac{3}{2} k T_{e0} + I_p \right) \frac{\partial n_e^*}{\partial t} + \frac{3}{2} n_{e0} k T_{e0} \frac{\partial T_e^*}{\partial t} +$$

$$i K \frac{k T_{e0} j_{0y}}{e} \left(\frac{3}{2} T_e^* - n_e^* \right) = \frac{2 j_0 \cdot j'}{\sigma_0} - \frac{j_0^2}{\sigma_0^2} \sigma^*$$

$$- \mathcal{E}' - R' - K^2 \kappa_e T_{e0} T_e^*$$

Where $\mathcal{E}$ = equipartition elastic loss term. Note that $-iK$ has been substituted for $\partial/\partial y$. The perturbation of the radiation and the equipartition terms are described in Appendix B.

Substituting for j'_x the final form of the linearized energy equation can be written as

$$\mathcal{C}_0 \frac{\partial n_e^*}{\partial t} + h_0 \frac{\partial T_e^*}{\partial t} = d_0 n_e^* + f_0 T_e^* \quad (2.8)$$

where,

$$\mathcal{C}_0 = \left(\frac{3}{2} k T_{e0} + I_p \right) n_{e0} ,$$

$$h_0 = \frac{3}{2} n_{e0} k T_{e0} ,$$

$$d_0 = \frac{i k T_{e0} K j_0 \cos \chi}{e} + \frac{2 j_0^2}{\sigma_0} \left\{ \sin \chi \left[(1 - r(1-u)) \sin \chi \right. \right.$$

$$\left. \left. + \beta_0 \cos \chi \right] - 1 + \frac{1}{2} (1-u)(r-s) \right\}$$

where,
$$s = \left(1 + \frac{\gamma_{m0}}{\gamma_{c0}} \frac{m_s}{m_n} \right)^{-1}$$

$$\text{and, } f_6 = -i \frac{\frac{3}{2} k T_{e0} K j_0 \cos \chi}{e} + \frac{j_0^2}{\sigma_0} \left\{ \left(\frac{1}{2} - \sin^2 \chi \right) \times \right. \\ \left. (6 \tau (u-1) + 3) - \frac{3}{2} (1 + 2 (3u-1) S) - \frac{T_{e0}}{T_{e0} - T} \right\} \\ + \sum 3 \pi \left(\frac{h \nu_i}{k T_{e0}} \right)^2 \frac{k T_{e0}}{\lambda_i^2} (m_{opi} K)^{1/2} \Delta \nu_i e^{-\frac{h \nu_i}{k T_{e0}}} + K_e K^2$$

where we have assumed that $\frac{T_{e0}}{q_{e0}} \frac{\partial q_{en}}{\partial T_e} \approx 1$.

Substituting $i\omega$ for $\partial/\partial t$, (2.7) and (2.8) can be written in the form

$$\begin{pmatrix} i\omega - b_0 & -a_0 \\ i\omega \tau_0 - d_0 & i\omega h_0 - f_0 \end{pmatrix} \cdot \begin{pmatrix} n_e^* \\ T_e^* \end{pmatrix} = 0$$

Hence, using the condition for a non trivial solution for n_e^* and T_e^* , we get the dispersion relation

$$\det \begin{pmatrix} i\omega - b_0 & -a_0 \\ i\omega \tau_0 - d_0 & i\omega h_0 - f_0 \end{pmatrix} = 0$$

Thus we have a quadratic for the complex frequency ω , with coefficients which are complicated functions of T_{e0} , χ , β_0 and K . The variation of the solution of this quadratic as a function of the four variables will be described in the next section.

2.6 Results of the Solution of the Dispersion Relation.

a) Procedure of the Calculations.

In most of the numerical evaluations of w the plasma considered was an Argon gas seeded with Caesium; however some calculations were also carried out for Argon and Potassium, Helium and Potassium, and Helium and Caesium. In all cases the neutral gas has a number density of $10^{25} \frac{1}{m^3}$ and the seed fraction is .1%.

The calculations were usually carried out with the heavy particle temperature fixed at 1500°K, and the variations of T_{eo} and B were made to correspond as closely as possible to experimentally realisable situations. For instance T_{eo} was varied for fixed B . This corresponds to increasing the internal Ohmic heating by decreasing the external load in the generator circuitry. Of course in the real situation this would also cause T to rise slightly, the magnitude of the rise depending on the thermal capacity and energy loss mechanisms of the heavy gas. Assuming the consequent rise in T to be small its effect on the growth rate will be small; and even if it is not small it will not qualitatively affect the way in which the growth varies with T_{eo} .

Once T_{eo} is arbitrarily fixed, n_{eo} and the plasma parameters, such as collision frequency, conductivity and Hall parameter, can be calculated and the magnitude of $\underline{j}_o$ deduced from the steady state energy balance. Fixing χ and K then determines all the

coefficients of the quadratic dispersion relation, which can then be solved.

T_{eo} was also varied by varying T , keeping $(T_{eo}-T)$ and B constant. This corresponds to varying the incoming gas temperature as well as the external load of a generator. This variation was employed to examine the wave behaviour at electron temperatures approaching 1500°K without $\frac{j_c^2}{\sigma_c}$ going to zero.

These two methods of varying T_{eo} keep B constant, and for completeness another method of varying T_{eo} was tried. This consists of keeping the load factor $(= \frac{R_L}{R_p + R_L})$ and T constant but varying the $\underline{v} \times \underline{B}$ induced field by varying B . This varies the Ohmic heating, and a load factor of 0.75 along with a velocity of 10^3 m/sec . were used. Because of the difficulty of solving the steady state energy balance for T_{eo} given the $\underline{v} \times \underline{B}$ field and load factor, the procedure was inverted, T_{eo} fixed and the required B derived.

All the parameters χ , β_o and K were also varied keeping the others constant. Each time the plasma parameters etc. had been fixed the coefficients of the quadratic dispersion relation were calculated and used as inputs to a computer subroutine to obtain w . In addition, the ratio of n_e^* to T_e^* in the wave was calculated for each solution w .

b) Wave Modes.

Since the dispersion relation is a quadratic its solution

gives two independent modes for electrothermal waves. Essentially they are a high frequency mode, which is always severely damped, and a low frequency mode which is unstable under certain conditions (See Figs. (2.2) and (2.3). Note that the complex frequency is written as $\omega = \omega_r - ig$ so that ω_r = real frequency, and g = growth rate). In what follows the high and low frequency modes are referred to as the fast thermal mode (F.T.M.) and the ionization mode (I.M) respectively.

If we define ρ and θ by

$$\rho e^{i\theta} = n_e^* / T_e^*$$

we can see from Fig. (2.4) that, at $T_{e0} = 2500^\circ\text{K}$, $\rho \ll 1$ for the F.T.M., while $\rho \sim 10$ for the I.M; also from Fig. (2.5) we see that $\theta \approx \pi$ for the F.T.M. and $\theta \approx 0$ for the I.M. Therefore the F.T.M. mode has small fluctuations in n_e^* compared to T_e^* , with the two out of phase, while the I.M. has the expected large fluctuations of n_e^* compared to T_e^* , with the two in phase. It is not surprising therefore that the I.M. is the unstable mode.

For both modes the values of all the wave parameters are identical for χ and $\chi + \pi$, except for the sign of the real frequency and θ . That is, given an orientation of the wave front w.r.t. $\underline{j}_0$, the waves travel in one direction only. For this reason quantities plotted as a function of χ are plotted in the range $|\chi| < \pi/2$ only.

At temperatures below 4600°K the F.T.M. and the I.M. travel

in opposite directions, the former has a wave vector such that

$$\underline{j}_0 \cdot \underline{v}_p < 0 \quad (\underline{v}_p = \hat{k} \frac{\omega_r}{k}) \text{ and the latter has } \underline{j}_0 \cdot \underline{v}_p > 0 .$$

That is the F.T.M. travels in the same sense as the electrons

while the ionization mode travels in the opposite sense. However,

above 4600°K the phase velocity of the ionization mode reverses.

The physical reason for this wave kinematics will be explained in section (2.6g).

c) Dependence of the instability on χ .

The wave quantities g , ρ , θ and v_p for the I.M. are plotted as functions of χ in the interval $-\pi/2 \rightarrow \pi/2$ for various values of Te_0 in Figs. (2.6) - (2.9) respectively.

These curves have constant magnetic field (5 tesla) and it can be seen from (2.6) that for low temperatures ($\lesssim 3000^\circ K$) there is growth in a range of angles centred around $\chi = \frac{\pi}{4}$. The value of χ where g maximises, χ_m , is approximately given by,

$$\chi_m = \frac{\pi}{4} + \frac{(1 - r(1 - u))}{2\beta_0}$$

This is obtained by finding the value at which the heating term proportional to n_e^* (i.e. d_0) maximises with respect to χ .

We see that $\chi_m = \frac{\pi}{4}$ for large values of β_0 only.

From Fig. (2.7) we see that ρ is not a constant with respect to χ as it should be if n_e^* and Te^* were related to each other through instantaneous Saha equilibrium. Above 2500°K the approximation of instantaneous Saha equilibrium is a good one, but below

2500°K it is not. At 2500°K the value of $\frac{\partial \log n_{eo}}{\partial \log T_{eo}}$ from the Saha equation is approximately 10.5 and we see that ρ equals this where $g = 0$. This we would expect since the period of the ionization mode is $\sim 10^{-3}$ secs for $\Lambda = 10^{-2}$ m. and the Saha relaxation time is much less than this at $T_{eo} = 2500^\circ\text{K}$.

However when the wave is growing or decaying the magnitude of the growth rate is of the order of the Saha relaxation rate and the rate of change of the electron density fluctuation amplitude is limited by the finite ionization - recombination rates. Hence Te^* gets ahead of n_e^* , so that when the wave is growing ρ is less than $\frac{\partial \log n_{eo}}{\partial \log T_{eo}}$, and when it is decaying ρ is greater than $\frac{\partial \log n_{eo}}{\partial \log T_{eo}}$.

At lower temperatures the rise and fall of ρ above and below $\frac{\partial \log n_{eo}}{\partial \log T_{eo}}$ is increased because of decreasing ionization and recombination rates, but in the stable range of χ a new effect appears. The I.M. becomes similar to the F.T.M. in that θ falls to $-\pi$ (see Fig. (2.8)) and the effect of this is to take a large "bite" out of the rise in ρ leaving two spikes on the edge of the stable range. The Figs. clearly show that as the temperature rises the range of χ over which $\theta = -\pi$ decreases, the size of the "bite" decreases and the spikes converge.

From Fig. (2.9) we see that the phase speed $|\omega_r/K|$ maximises at $\chi=0$, except at lower temperatures, and at all temperatures

$$\omega_r/K = 0 \text{ at } \chi = \pm \frac{\pi}{2}.$$

d) Dependence of the instability on the Hall parameter.

The Hall parameter was varied by varying B , keeping everything else constant. This assumes that the gas velocity also varies in order to keep $\mathcal{U} \times \underline{B}$ and the steady state Ohmic heating constant.

The dependence of the instability on β_o is now well known from experimental observations (1.21), (1.22). The results derived here confirm that the instability has a critical Hall parameter, β_{ocrit} , of the order of 1. Figure (2.10) shows that g_m varies linearly with β_o , where g_m is the growth rate at χ_m . The temperature dependence of β_{ocrit} , shown in Fig. (2.11), is similar to that previously reported. (1.21), (2.8)

The range of angles over which the wave is unstable, $\Delta\chi$, is plotted in Fig. (2.12) as a function of β_o at $T_{eo} = 2500^\circ\text{K}$. We see that $\Delta\chi$ increases with β_o . In some experiments (1.21) it is observed that the instability has a plane wave structure at low values of β_o , and this breaks down into apparently random turbulence as the Hall parameter is increased. It seems likely that this is due to different Fourier components of the instability becoming successively destabilized by the increasing range $\Delta\chi$.

This phenomenon follows naturally from the analysis of boundary effects on the wave, and will be demonstrated in Chapter 4.

e) Dependence of the instability on T_{eo} .

The growth rate plotted as a function of the steady state

electron temperature is shown in Fig. (2.13) curve 1, for $T = 1500^\circ\text{K}$, $B = 5$ tesla, $\lambda = 10^{-2}$ m and $\chi = \chi_m$.

The growth rate shows a maximum with electron temperature at $T_{eo} = 2500^\circ\text{K}$ and at higher temperatures, above approximately 3000°K , the waves are damped.

The reason for the decrease of g towards lower temperatures is partly due to the decreasing value of $T_{eo} - T$. Since we calculate $\frac{j_o^2}{\sigma_o}$ from the steady state energy balance we have, at low temperatures, $\frac{j_o^2}{\sigma_o} \propto n_{eo} (T_{eo} - T) T_{eo}^{1/2}$. Assuming that neutral collisions are to be dominant at low temperatures, then the constant of proportionality in this relation is independent of either n_{eo} or T_{eo} . In section (1.3) we saw that

$$g = \frac{1}{\tau_E} \sim \frac{2 \cdot j_o^2}{\sigma_o^2 n_{eo} I_p} \beta_o$$

hence,

$$g \propto (T_{eo} - T) T_{eo}^{1/2}$$

at the low temperatures. Keeping $T_{eo} - T$ constant, however, g still decreases in a very similar manner as T_{eo} decreases (see Fig. 2.14, curve 1). This must be due to the finite ionization recombination rates, and this is confirmed by arbitrarily increasing these rates by a factor 10^5 and repeating the calculations. Fig. (2.14), curve 2, shows the result. This time the growth rate does not start decreasing until 1800°K .

We conclude therefore that the infinite ionization - recombination

rate approximation over-estimates the growth rate for temperatures below 2500°K, but gives a good approximation above this temperature.

The growth rate decreases as T_{eo} increases past 2500°K in Fig. (2.13) curve 1, not because of the increasing degree of ionization, but because β_o decreases with fixed B due to ν_{eo} increasing with T_{eo} . This can be seen from curves 2 and 4 in Fig. (2.13); curve 4 is β_o versus T_{eo} , and we see that $\beta_o \approx 2$ when the wave becomes stable with increasing T_{eo} ; curve 2 shows g versus T_{eo} with the Hall parameter fixed at 5, i.e. B increasing with T_{eo} , and here g does not have a maximum as a function of T_{eo} in the plotted range. However if we continue curve 2 to higher values of T_{eo} (see Fig. 2.15) we find that g does in fact start to decrease at about 5000°K, and quickly becomes stable, due to the finite degree of ionization.

If instead we increase T_{eo} by increasing B to produce more magnetically induced nonequilibrium ionization with fixed load factor then g as a function of T_{eo} is Fig. (2.13) curve 3. The growth rate increases monotonically for the values of T_{eo} plotted. This is because, for fixed load factor, η_L , we have, from the argument in section (1.3),

$$\beta^2 \sim \frac{3 k (T_e - T)}{m_h v^2 (1 - \eta_L)^2}$$

The extra factor $(1 - \eta_L)^2$ simply takes account of the external circuitry. Hence β_0 increases with T_{e0} . However, once again, at high temperatures β_0 goes through a maximum and goes to zero (see Fig. (2.15), curve 2).

(Note that the range of electron temperatures over which the plasma is unstable corresponds to the envisaged range of temperatures for closed-cycle MHD generators)

f) Dependence of the instability on K.

The dependence of ω_r on K was given in Fig. (2.2), for $\chi = \chi_m$, $T_{e0} = 2500^\circ\text{K}$, and $B = 5$ tesla. The dispersion curve is a straight line through the origin therefore the phase velocity (ω_r/K) and the group velocity ($d\omega_r/dK$) are identical and independent of K.

However the growth rate varies with K, and we see from Fig. (2.16) that, for various temperatures, the growth rate decreases, as expected, when the wavelength, λ , decreases.

The stabilization at short wavelengths is due mainly to the thermal conduction, which varies as $\frac{1}{\lambda^2}$, and not so much to the radiation transfer which varies only as $\frac{1}{\sqrt{\lambda}}$.

However, for wavelengths of the order of 10^{-2} m, at around 2500°K , the radiation transfer in the wave is about 10% of the elastic losses, while the thermal conduction is less than 1%.

For this reason it may be possible to stabilize to a significant degree the waves observed in the laboratory, which have wavelengths of the order of 10^{-2} m, by increasing the radiation. The possibility of doing this will be discussed in section (2.6h).

We note that, according to Fig. (2.16), the nonequilibrium plasma is unstable to perturbations of infinite wavelength, i.e. uniform perturbations. However this is consequence of our plane wave periodicity conditions which demands that $\underline{j}'$ be perpendicular to $\underline{K}$. This condition persists even if we let $|\underline{K}| \rightarrow 0$. For a uniform perturbation the direction of $\underline{j}'$ would be imposed by the boundary conditions appropriate to the finite apparatus and the stability of a uniform perturbation is demonstrated in Chapter 5.

g) Physical Nature of the Waves.

The development of a physical picture of the waves, how they move and how they grow is of considerable interest. Unfortunately the interplay of the physical mechanisms in the wave is obscured by the number and complexity of the terms, especially in the energy equation. However, if we assume a sinusoidal fluctuation in n_e^* and T_e^* of the form shown in Fig. (2.17), we can say something of the way in which the electrons move in the fluctuations, and of the mechanism by which the fluctuations themselves move.

It follows from $j_y' = 0$ that $|v_e|^* = -n_e^*$. Therefore the electrons stream through the fluctuation with their velocity perpendicular to the wave fronts decreased in the peaks of n_e^* , and increased in the troughs. The fluctuation itself does not tend to convect with the electrons, since the assumption $n_i = n_e$ would require the ions to move with the wave speed which is incompatible with our assumption of stationary ions. In fact the wave moves by enhanced ionisation on one slope and enhanced recombination on the other.

The terms contributing to the movement are the terms in the energy equation which are $\pi/2$ out of phase with n_e^* and Te^* viz. the gradient terms. (Remember that the phase difference between n_e^* and Te^* in the I.M. is generally very small). In the linearized energy equation these are $-\frac{3}{2} \frac{kT_{e0} j_{oy}}{e} \frac{\partial T_e^*}{\partial y}$, viz. the convection term, and $\frac{kT_{e0} j_{oy}}{e} \frac{\partial n_e^*}{\partial y}$, viz. compressional heating. At lower temperatures we have $n_e^* \gg Te^*$ for the I.M. and hence compressional heating dominates. Hence we have a situation as in

Fig. 2.17a, where the electrons are heated as they move into a peak in n_e^* and cooled as they move out. The wave then moves in a direction opposite to the electron drift across the wave fronts. In contrast at higher temperatures where $Te^* > \frac{2}{3} n_e^*$ (i.e. above 4600°K) the convection of the electrons dominates and the phase velocity reverses due to reversal of the relative heating on the slopes. The waves now move in the same direction as the electrons (see Fig. 2.17b).

When $\chi = \pm \frac{\pi}{2}$ the electrons are moving parallel to the wave fronts and both the compressional heating and the convection are zero, therefore the waves are stationary (see Fig. 2.9). The stabilizing and destabilizing terms are in phase with n_e^* and Te^* and do not contribute to the motion of the waves.

The small phase difference between n_e^* and Te^* is of course due to the finite ionisation - recombination rates, and is such that Te^* always leads n_e^* in the I.M. Neglecting this phase difference we can write down an expression for the phase speed from the imaginary part of the energy equation, viz.,

$$|\underline{v}_p| = \left| \frac{j_0 \cos \chi (\rho - \frac{3}{2})}{n_{e0} (\frac{3}{2} + (\frac{3}{2} + \frac{I_p}{k T_{e0}}) \rho)} \right|$$

The phase velocity (with sign defined by $\frac{\underline{v}_p \cdot \underline{j}_0}{\underline{v}_p j_0}$) is plotted as a function of Te_0 in Fig. (2.18).

Since $Te^* \gg n_e^*$ always in the F.T.M. this mode moves in the same direction as the electrons.

Physically the source of the instability is the enhanced Ohmic heating in the peaks of n_e^* and Te^* . This is obvious from the fact that the perturbed Ohmic heating gives the only positive contribution to the growth rate. However the fact that the waves require a magnetic field to grow and the χ dependence of the growth rate remain to be explained physically.

For zero magnetic field it is easily verified by perturbing

equations (2.2), (2.3) and (2.4) that the damping elastic losses dominate the Ohmic heating; i.e. although the rate at which electrons gain energy from the electric field is, for some values of χ , increased in the peaks of n_e^* and decreased in the troughs, the rate at which the electrons exchange energy with the heavy particles is perturbed even more. The effect of a magnetic field is to increase the fluctuation in the current, and hence the Ohmic heating, for a given n_e^* . So that, for $\beta_0 \gg 1$ the perturbed Ohmic heating becomes larger than the sum of all the loss mechanisms, and therefore the wave grows.

In section (2.5) we saw that,

$$j'_x = \sigma' E_{0x} - \beta' j_{0y} ,$$

using the steady state Ohm's Law of section (2.4) and the definition of χ , we can write this as,

$$j'_x = j_0 (\sigma^* \sin \chi + (\sigma^* - \beta^*) \beta_0 \cos \chi)$$

This includes the effects of $\underline{E}'$ implicitly, as well as the effects of σ' and β' . More explicitly we can write $\underline{j}'$ as the sum of the five following components (see Fig. 2.19),

$$\begin{aligned} \underline{j}'_1 &= \frac{\sigma_0 \underline{E}'}{1 + \beta_0^2} &= \text{current induced parallel to } \underline{E}' \\ \underline{j}'_2 &= \frac{\sigma' \underline{E}_0}{1 + \beta_0^2} &= \text{current induced parallel to } \underline{E}_0 \\ \underline{j}'_3 &= - \frac{\sigma_0 \beta_0}{1 + \beta_0^2} \underline{E}' \times \hat{z} &= \text{current induced perpendicular to } \underline{E}' \end{aligned}$$

$$\underline{j}'_4 = - \frac{\sigma' \beta_0}{1 + \beta_0^2} \underline{F}_0 \times \underline{\hat{z}} = \text{current induced perpendicular to } \underline{F}_0$$

$$\text{and } \underline{j}'_5 = - \frac{\sigma_0 F_0}{1 + \beta_0^2} \beta' \underline{\hat{u}} = \text{current induced due to the change in } \beta.$$

Here $\underline{\hat{z}}$ = unit vector in the z-direction and $\underline{\hat{u}}$ = unit vector at an angle $\tan^{-1}(\beta) - \chi$ to the x-axis in the 1st quadrant.

The first two components are present without a magnetic field, but the last three require a magnetic field. Components $\underline{j}'_3$ and $\underline{j}'_4$ are $\underline{E}/B$ drift currents, while component $\underline{j}'_5$ is due to the decrease in the Hall parameter for a positive perturbation of n_e (Remember $n_e^* \gg Te^*$ in the I.M. and Te' can be neglected in this simplified discussion). Since the angle between $\underline{j}$ and $\underline{F}$ is $\tan^{-1}(\beta)$, a decrease in β causes $\underline{j}$ to swing toward $\underline{F}$. Therefore a decrease in the Hall parameter will increase $\underline{F}_{\parallel}$, where $\underline{F}_{\parallel}$ is the component of $\underline{F}$ parallel to $\underline{j}$ and, since $\underline{j} = \sigma \underline{E}_{\parallel}$ in general, this will consequently increase the magnitude of $\underline{j}$.

From our assumption of plane wave solutions $\underline{E}'$ is of course determined by,

$$(\underline{j}'_1 + \underline{j}'_2 + \underline{j}'_3 + \underline{j}'_4 + \underline{j}'_5)_y = 0, \text{ and } F'_x = 0.$$

For large β_0 the drift currents and the current due to the change in Hall parameter, which together give the $\cos \chi$ term in $\underline{j}'_x$ dominate $\underline{j}'_x$, so that $\underline{j}'_x \approx j_0 (\sigma^* - \beta^*) \beta_0 \cos \chi$. This of course maximizes at $\chi = 0$. However the only significant part

of $\underline{j}'$ as far as growth is concerned is the component parallel to $\underline{j}_0$, since,

$$\begin{aligned} \left(\frac{j^2}{\sigma}\right)' &= 2 \frac{j_0 \cdot j'_x}{\sigma_0} - \frac{j_0^2}{\sigma_0} \sigma^* \\ &= \frac{2 j_0 j'_x \sin \chi}{\sigma_0} - \frac{j_0^2}{\sigma_0} \sigma^* \end{aligned}$$

Substituting for j'_x we have,

$$\left(\frac{j^2}{\sigma}\right)' = \frac{j_0^2}{\sigma_0} [(\sigma^* - \beta^*) \beta_0 \sin 2\chi - \sigma^*]$$

The positive contribution to $\left(\frac{j^2}{\sigma}\right)'$ therefore varies as $\sin 2\chi$ and maximises at $\chi = \frac{\pi}{4}$. Consequently, for high values of β_0 , the growth rate maximises at $\chi = \frac{\pi}{4}$.

When n_e^* becomes less than Te^* due to the finite degree of ionization, σ^* tends to cancel β^* in the $\cos \chi$ term since the dependence of σ^* and β^* on Te^* is identical. Hence at high temperatures the $\sin \chi$ term in j'_x becomes more dominant and substituting this term for j'_x into the Ohmic heating we obtain a $\sin^2 \chi$ dependence for the positive term, i.e. the Ohmic heating maximizes at $\chi = \frac{\pi}{2}$.

Due to the finite degree of ionization we have seen that the wave is damped at high temperatures ($> 5000^\circ K$). Nevertheless, the absolute maximum of g can clearly be seen to move from $\frac{\pi}{4}$ to $\frac{\pi}{2}$

as T_{eo} increases in Fig. (2.6).

h) Radiation Transfer.

Due to the fact that the waves are damped for $\beta_o < \beta_{ocrit}$, the undesirable effects of the instability could be avoided for instance by having a small enough magnetic field for this condition to hold. However, we have seen that a high elevation temperature requires $\beta_o \gg 1$; also, from section (1.2a), the specific power density is proportional to B^2 for fixed conductivity, and therefore, provided the generator design takes the Hall effect into account, and B is not big enough for ion slip effects be important, the bigger B is the higher will be the generator output.

Imposing the condition $\beta_o < \beta_{ocrit}$ is therefore an unsatisfactory way of avoiding fluctuations. We must therefore consider other ways of damping the waves.

One method would be to increase the radiative energy transfer in the plasma to cancel the Ohmic heating. For wavelengths of the order of 1cm. the radiation transfer is of the order of 10% of the elastic losses in the plasma (which is of the same order as the Ohmic heating). So that if we arbitrarily increase the radiation transfer by a factor of 10 then the dependence of g on T_{eo} looks like curve 2, Fig. (2.20). Comparing this with g versus T_{eo} for the unenhanced radiation transfer (Fig. (2.20), curve 1), we see that a substantial damping effect would be produced.

In most laboratory MHD plasmas, which have dimensions of the order of centimetres, the instability wavelength observed is of the order of the apparatus dimensions. It seems likely that if the dimensions of the device were to be increased, i.e. to a size compatible with economic power generation, that instabilities of the order of the apparatus dimensions would still be present.

In this case stabilization of the waves by radiation transfer implies a considerable escape of energy from the plasma by radiation. Therefore, an energy escape term of the order of the other two terms, would appear in the steady state energy equation, with a consequent decrease in the steady state electron temperature and density.

However, since the elastic losses are proportional to $n_{eo} (T_{eo} - T) T_{eo}^{\frac{1}{2}}$ in the case of neutral collisions dominant, and $n_{eo}^2 (T_{eo} - T) T_{eo}^{-\frac{3}{2}}$ in the case of Coulomb collisions dominant, the reduction in n_{eo} , though significant, is such that n_{eo} has still the same order of magnitude. While the reduction in T_{eo} is small.

In addition the enhancement of the radiation transfer will involve adding further complicated atoms to the plasma, with high electron inelastic collision cross-sections, and possibly with chemical properties which are undesirable as far as the various components of the MHD cycle are concerned. Even if radiation enhancement can be achieved, therefore, it may be costly in terms of generator performance.

To consider how such an enhancement could be achieved we look at

the perturbed radiation term. From Appendix 2 the perturbation of the radiation from the i^{th} line is given by

$$R_i' = \left\{ \frac{3k^2 T_{eo}}{2h\tau} \left(\frac{\kappa n_{ag} \Delta\nu_i g_i}{\tau_i g_g} \right)^{1/2} \right\} \left(\frac{h\nu_i}{kT_{eo}} \right)^3 \frac{e^{\frac{h\nu_i}{kT_{eo}}}}{\left(e^{\frac{h\nu_i}{kT_{eo}}} - 1 \right)^2} T_e'$$

where
$$n_{ag} = n_a / \left(1 + \sum_j \frac{g_j}{g_g} e^{-\frac{h\nu_j}{kT_{eo}}} \right)$$

$$\therefore R_i' \propto \frac{\left(\frac{h\nu_i}{kT_{eo}} \right)^3 e^{\frac{h\nu_i}{kT_{eo}}}}{\left(e^{\frac{h\nu_i}{kT_{eo}}} - 1 \right)^2 \left(1 + \sum_j \frac{g_j}{g_g} e^{-\frac{h\nu_j}{kT_{eo}}} \right)^{1/2}}$$

The R.H.S. of this proportionality relation is plotted as a function of $h\nu_i/kT_{eo}$ in Fig. (2.21), assuming $j = 1, 2$ for the Caesium resonance doublet.

We see that the R.H.S. maximises at $h\nu_i/kT_{eo} \approx 3$ and the maximum value is approximately 1.3. At $T_{eo} = 2500^\circ\text{K}$ the value of $h\nu_i/kT_{eo}$ is ≈ 7 for the Caesium doublet and the R.H.S. has a value .25. We could at least enhance the radiation by a factor of 5 by doping the plasma with an element whose resonance line corresponds to $h\nu_i/kT_{eo} = 3$, with a "dope fraction" of .1%. (Note that the values of τ_i , $\Delta\nu_i$ and g_i do not change a great deal for atomic resonance levels under these conditions, and they appear under a square root sign. Therefore maximising w.r.t. ν_i is equivalent to maximising with respect to the identity of the radiating element). However, at $T_{eo} = 2500^\circ\text{K}$, this corresponds to a wavelength of the

order of $16000 \text{ \AA}$, i.e. infra red radiation. Since resonance lines have much shorter wavelengths than this, it does not seem possible to substantially increase the radiation by optimising with respect to λ_i .

Simply increasing the number of radiating atoms is an alternative method of increasing the radiation transfer. In view of the undesirable effects of such atoms, however, it is necessary to keep their number to a minimum.

If we increase the Caesium density by a factor N , say, then the radiation increases by only a factor $\sqrt{N}$. This is because absorption is increased as well as the emission. However suppose we add $N - 1$ other elements, all distinct, with doublet resonance lines that have λ_i of the order of that for Caesium, but which do not overlap even after collision broadening. Then the radiation will be increased by a factor N if the "dope fraction" of each new element equals the seed fraction.

Mathematically this is because, in the case of added dopes, the radiation term is the sum of that of all the different components, whereas in the first case, adding only Caesium, the new radiation term is obtained by substituting the new Caesium density into the term under a square root sign. Physically the greater increase in the radiation transfer in the case with added dopes is due to the fact that the absorption for each wavelength has approximately the

value for the Caesium doublet which is unchanged, while the total energy emitted is increased.

It may be that by using this technique the impedance of an MHD plasma may be minimized at some optimum dope density, with the optimization involving a compromise between the Rosa effect and the damaging radiative, collisional and chemical effects of dopes.

i) The Growth Rate in Different Gas Mixtures.

The growth rate of the I.M. was calculated as a function of temperature for the mixtures Argon and Potassium, Helium and Caesium, and Helium and Potassium as well as the Argon and Caesium mixture previously considered in detail. The results are shown in Fig. (2.22).

We see that for fixed magnetic field the general form of the curves is the same. The differences in g between the different gas mixtures are due to the different heavy particle masses, collision cross sections, and seed ionization potentials. A detailed description of all the differences would be rather complicated, however a brief discussion of the four main differences will be given.

First of all, for a given seed, g is always greater in Helium than in Argon for the same T_{eo} . Due to the higher cross section for momentum transfer in Helium (for Helium ^(2.5) $q_{en} = 9.7 \times 10^{-20} \text{ m}^2$) the value of β_0 is lower and this tends to damp the wave.

However, the decrease in β_0 is accompanied by an increase in the Ohmic heating required to produce a given value of $(T_{e0} - T)$ in the lighter Helium, since the thermal coupling between the neutrals and the electrons is proportional to $\frac{m_e}{m_n}$. Therefore, because the amplifying term in the energy equation is proportional to $\frac{j_e^2}{\sigma_0} \beta_0$ (for $\beta_0 \gg \beta_{crit}$) and the increase in j_e^2/σ_0 exceeds the decrease in β_0 , the growth rate for Helium is higher than for Argon.

Secondly, for a given neutral gas, Potassium has a larger maximum of growth rate than Caesium. This is similar to the first point since it is partly due to the fact that, when Coulomb collisions are important, higher Ohmic heating is required in Potassium to produce a given difference in the gas and electron temperatures than in Caesium. Although Potassium and Caesium have different ionization potentials, the peaks of g occur at different temperatures where n_e is approximately equal for both gases, therefore β_0 is the same. The reason for the increased j_e^2/σ_0 is then due to the smaller mass of the Potassium compared with Caesium, and the higher value of $(T_{e0} - T)$ at the maximum of g .

But we see that at low temperatures the growth rate of Caesium can exceed that for Potassium, and this brings us to the third point, viz. the maximum of g for potassium is shifted towards higher temperatures compared to that for Caesium. The reason for this is simply the higher ionization potential of Potassium compared to Caesium which gives a lower electron density for a given electron temperature.

At lower temperatures the growth is limited by the finite ionization and recombination rates, and, since these are proportional to n_{eo}^2 , their damping effect is greater in potassium than in Caesium, hence the relative values of the growth rate. At high temperatures the ionization and recombination rates cease to be limiting, and the increasing importance of Coulomb collisions causes the Hall parameter to fall in both cases. However, the lower electron density in Potassium gives a lower collision frequency and hence a higher Hall parameter, therefore the growth rate for potassium is greater in this region. Thus, for Potassium, the maximum of g is shifted towards higher temperatures compared with that of Caesium.

Fourthly, for a given seed, the maximum of g for Helium is shifted towards lower temperatures compared with that of Argon. The reason for this is the higher cross-section for Helium compared to Argon, which makes the Hall parameter at a given temperature lower. This does not give us a lower growth rate for the reasons previously stated. However it does mean that the decrease in g due to the decreasing Hall parameter at high temperatures, and the consequently decreasing relative importance of the $\beta_o \sin 2X$ term in d_o , starts at a lower temperature for Helium than for Argon. Hence the maximum of g is shifted towards lower temperatures.

2.7 Comparison of Results with Experiments and Previous Theoretical Studies.

The dependence of the electrothermal instability on the steady state Hall parameter that is derived here is in agreement with the behaviour of the instability observed in the experiments of Shipuk and Pashkin (1.21), Zukoski and Gilpin (1.22), Louis (1.23), and Kerrebrock and Dethlefsen (1.24). The existence of a critical value of the Hall parameter is therefore a well established experimental fact. The variation of β_{ocrit} with Te_0 given here is similar to some theoretical results by Shipuk and Pashkin (1.21), and Angrolov et. al. (2.8).

Experimentally the instability observed in all experiments has a near plane wave structure for values of β_0 just above β_{0crit} . This is in agreement with the results presented here, since the fact that the waves are unstable only in a narrow range $\Delta\chi$ would lead to a plane wave structure. Experimentally the value of χ at which the waves appear is near the expected value of approximately $\pi/4$. This is also in agreement with some remarks by Zeltwoog (2.9) and a recent result given by Kerrebrock and Dethlefsen (1.24).

The breakdown of the structure into turbulence as β_0 increases is not fully explained by the simple plane wave theory. However, the increase of $\Delta\chi$ with β_0 suggests a possible explanation, which is confirmed by the results of Chapter 4.

The reversal of the phase velocity discussed in sub-section (2.6g)

was predicted by Nedaspasov (1.17).

Close comparison of the variation of the growth rate with electron temperature given here and experimental results is not possible, since no experiment has measured g . The waves grow to a non-linear steady amplitude before they are actually observed due to the large magnitude of g . However Kerrebrock and Dethlefsen (1.24) observed that at low values of Teo the fluctuations which they identify as electrothermal waves disappear, giving way to lower amplitude fluctuations which they identify as magnetosonic waves. They differentiate the two using the β_o dependence of the fluctuations. Since magnetosonic waves grow for all values of β_o , $\beta_{o\text{crit}}$ does not exist, and they deduce from the absence of a $\beta_{o\text{crit}}$ for the fluctuations at low temperatures that they are magnetosonic waves.

In addition Shipuk and Pashkin (1.21) have observed that at high temperatures, near complete ionization, the electrothermal waves disappear, giving way again to magnetosonic waves. This they deduce from the orientation of the waves with respect to $\vec{j}_o$. Louis (1.6) has given some results that show that the amplitude of the fluctuations he observes decreases with increasing T , and therefore, presumably, with increasing Teo . There is some experimental evidence, therefore, for the damping of the electrothermal waves at high and low temperatures.

Some of the results given by Hougen (1.14) and his interpretation of the physical nature of the waves, closely parallel the results

given here. However, he obtains the result that, although g decreases at low temperatures due to the effects of finite ionization recombination rates, g never falls to zero with decreasing T_{eo} . This is in contrast to the results presented here, where, due to the radiation and conduction losses, g does go to zero as T_{eo} decreases.

CORRIGENDUM

The calculations in this chapter were carried out using a factor $(1-r)$ in the $\sin^2 \chi$ term of d_0 , instead of $(1-r(1-u))$ (see section (2.5)). The $\sin \chi \cos \chi$ term is dominant when the electrothermal wave is unstable, hence the consequent error in the results for the instability, except for β_{ocrit} , is small. The value of β_{ocrit} is some 10 to 20 per cent too high.

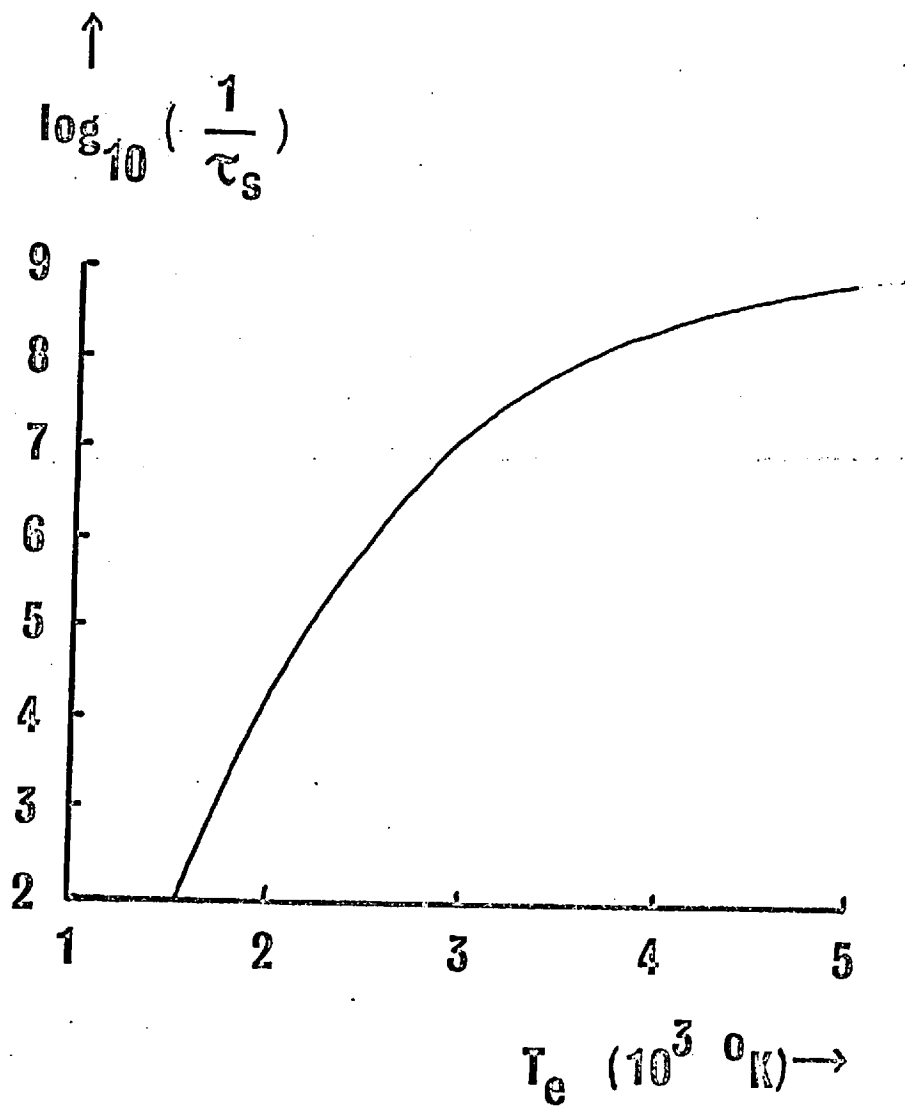


Fig.(2.1) $\frac{1}{\tau_s}$ versus T_e (τ_s is in seconds).

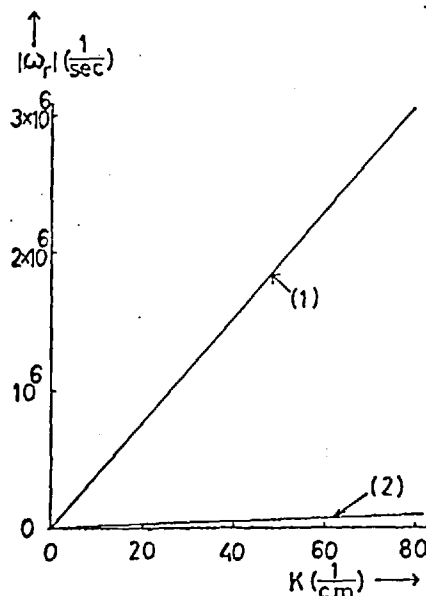


Fig. (2.2) Graph of the modulus of the real part of ω vs. wave number. (1) Fast thermal mode; (2) Ionization mode. For both curves: $T_{e0} = 2500^\circ\text{K}$, $T = 1500^\circ\text{K}$, $\chi = \pi/4$, $B = 5.0$ tesla.

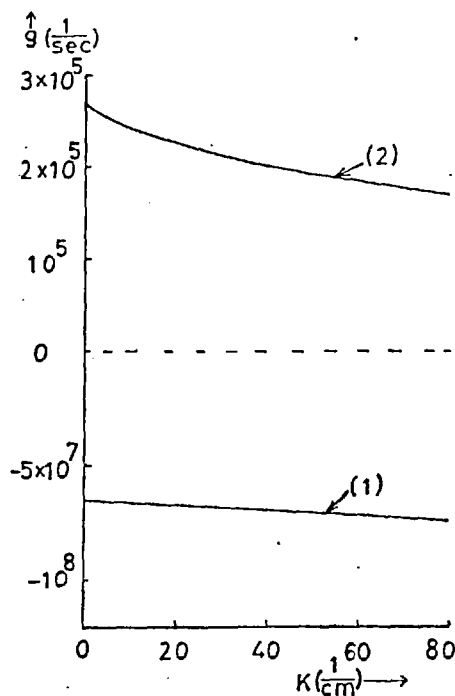


Fig. (2.3) Graph of growth rate vs. wave number. (1) Fast thermal mode (2) ionization mode. For both curves: $T_{e0} = 2500^\circ\text{K}$, $T = 1500^\circ\text{K}$, $\chi = \pi/4$, $B = 5.0$ tesla. (Note difference in scale above and below the zero axis.)

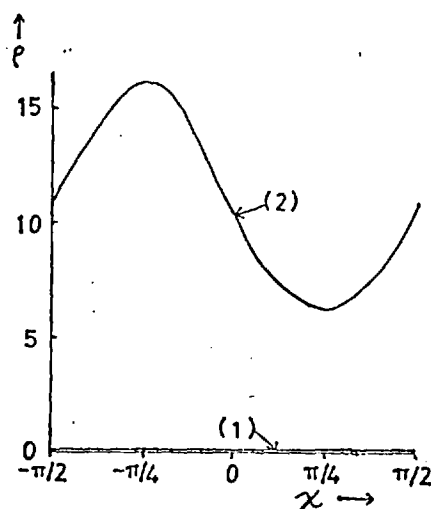


Fig.(2.4) Graph of mod (n_e^*/T_e^*) vs. χ (1) Fast thermal mode; (2) Ionization mode.
For both curves:— $T_{e0} = 2500^\circ\text{K}$, $T = 1500^\circ\text{K}$, $B = 5.0$ tesla, $\lambda = 1$ cm.

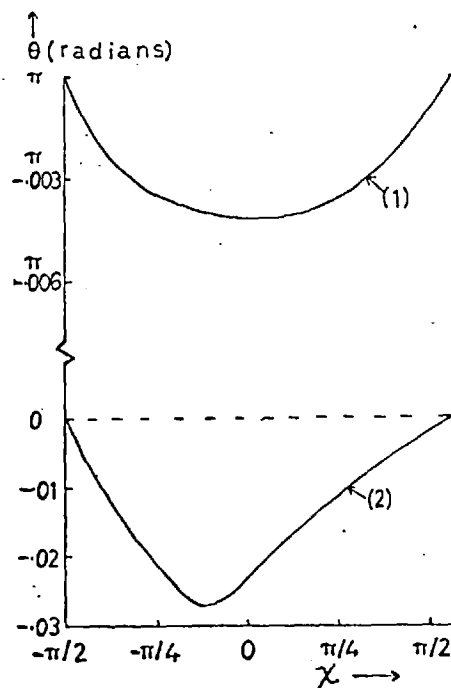


Fig.(2.5) Graph of arg (n_e^*/T_e^*) vs. χ (1) Fast thermal mode; (2) ionization mode.
For both curves:— $T_{e0} = 2500^\circ\text{K}$, $T = 1500^\circ\text{K}$, $B = 5.0$ tesla, $\lambda = 1$ cm. (Note θ scale.)

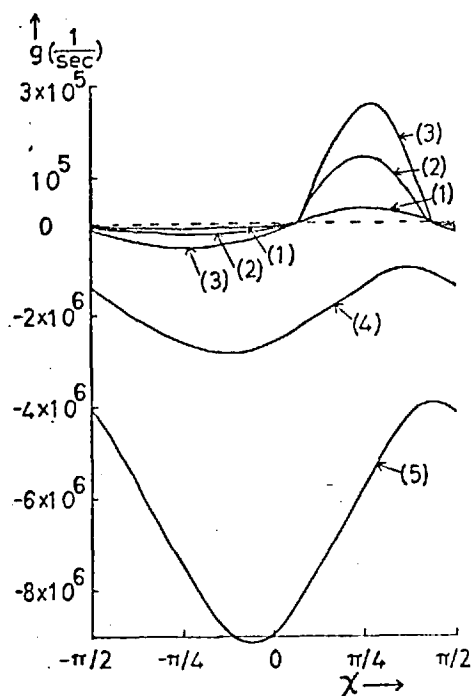


Fig. (2.6) Graph of growth rate vs. χ (ionization mode). (1) $T_0 = 2000^\circ\text{K}$; (2) $T_0 = 2250^\circ\text{K}$; (3) $T_0 = 2500^\circ\text{K}$; (4) $T_0 = 4000^\circ\text{K}$; (5) $T_0 = 5500^\circ\text{K}$. For all curves: $T = 1500^\circ\text{K}$, $\lambda = 1 \text{ cm}$, $B = 5.0 \text{ tesla}$ (note different scale above and below zero axis).

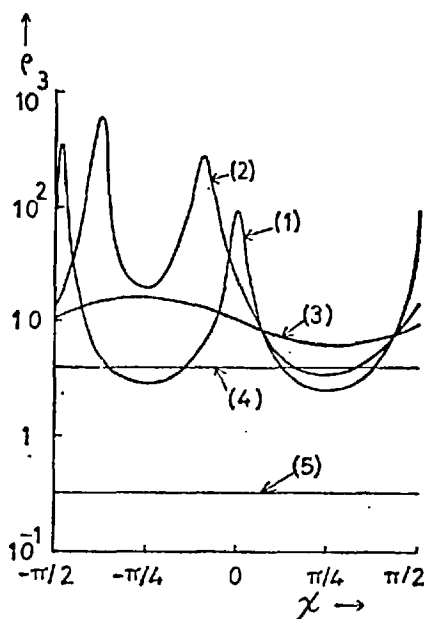


Fig. (2.7) Graph of $\text{mod}(n_e^*/T_e^*)$ vs. χ (ionization mode). (1) $T_0 = 2000^\circ\text{K}$; (2) $T_0 = 2250^\circ\text{K}$; (3) $T_0 = 2500^\circ\text{K}$; (4) $T_0 = 4000^\circ\text{K}$; (5) $T_0 = 5500^\circ\text{K}$. For all curves: $T = 1500^\circ\text{K}$, $\lambda = 1 \text{ cm}$, $B = 5.0 \text{ tesla}$.

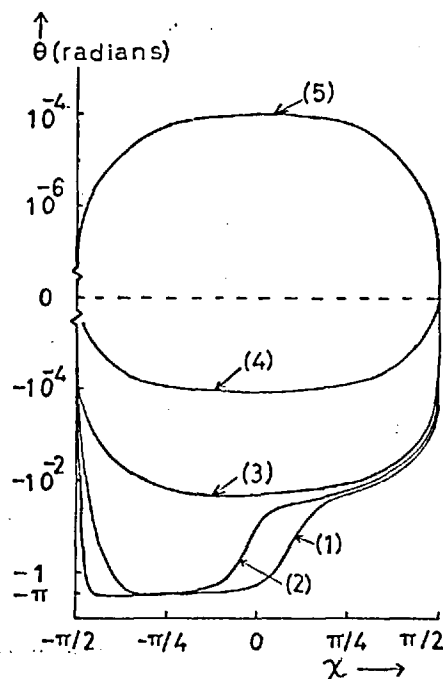


Fig.(2.8) Graph of $\arg(n_e^*/T_e^*)$ vs. χ (ionization mode). (1) $T_{e0} = 2000^\circ\text{K}$; (2) $T_{e0} = 2250^\circ\text{K}$; (3) $T_{e0} = 2500^\circ\text{K}$; (4) $T_{e0} = 4000^\circ\text{K}$; (5) $T_{e0} = 5500^\circ\text{K}$. For all curves:— $T = 1500^\circ\text{K}$, $\lambda = 1$ cm, $B = 5.0$ tesla.

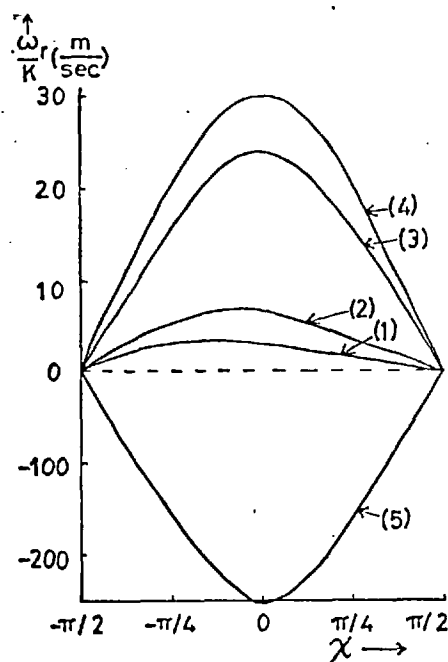
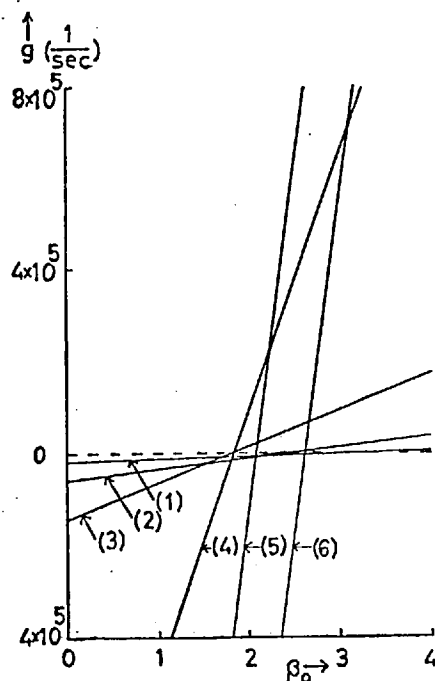


Fig.(2.9) Graph of phase velocity (ω_R/K) vs. χ (ionization mode). (1) $T_{e0} = 2000^\circ\text{K}$; (2) $T_{e0} = 2250^\circ\text{K}$; (3) $T_{e0} = 2500^\circ\text{K}$; (4) $T_{e0} = 4000^\circ\text{K}$; (5) $T_{e0} = 5500^\circ\text{K}$. For all curves:— $T = 1500^\circ\text{K}$, $\lambda = 1$ cm, $B = 5.0$ tesla. (Note different scale above and below zero axis.)



Fig(2.10) Graph of growth rate vs. Hall parameter (ionization mode). (1) $T_{e0} = 2000^\circ\text{K}$; (2) $T_{e0} = 2250^\circ\text{K}$; (3) $T_{e0} = 2500^\circ\text{K}$; (4) $T_{e0} = 3500^\circ\text{K}$; (5) $T_{e0} = 4500^\circ\text{K}$; (6) $T_{e0} = 5000^\circ\text{K}$. For all curves $T = 1500^\circ\text{K}$, $\lambda = 1 \text{ cm}$, $\chi = \chi_m$

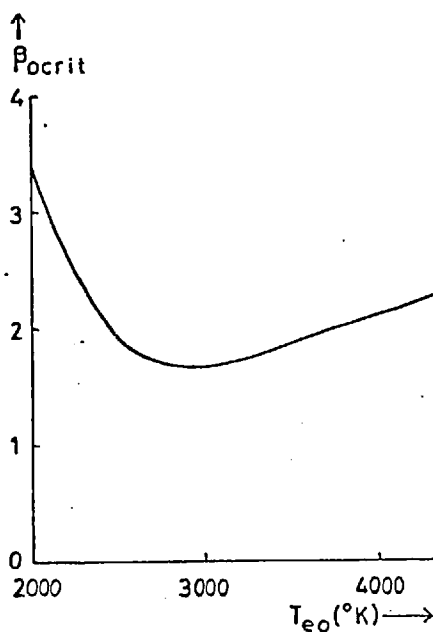


Fig.(2.11) Graph of critical Hall parameter for stability vs. electron temperature (ionization mode). $T = 1500^\circ\text{K}$, $\lambda = 1 \text{ cm}$, $\chi = \chi_m$

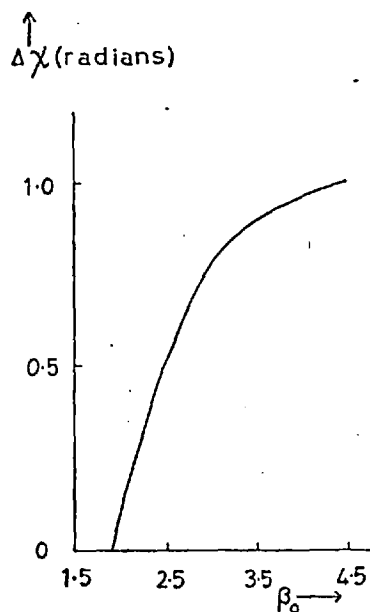


Fig. (2.12) Graph of range of angles for instability vs. Hall parameter (ionization mode).
 $T_{e0} = 2500^\circ\text{K}$, $T = 1500^\circ\text{K}$, $\lambda = 1$ cm.

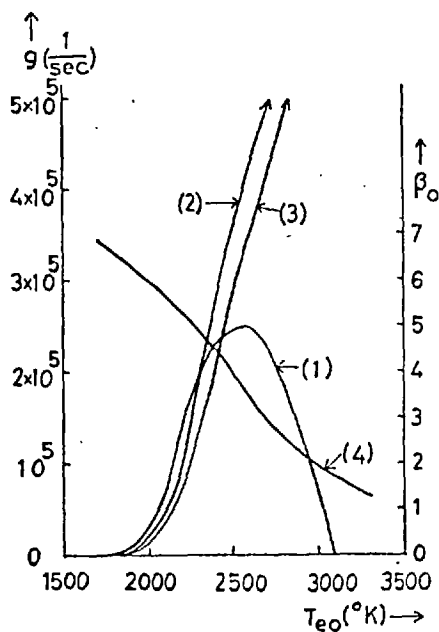


Fig. (2.13) (A) Graph of growth rate vs. electron temperature (ionization mode).
 (1) $B = 5.0$ tesla; (2) $\beta_0 = 5.0$; (3) Magnetically induced elevated electron temperature
 ($\eta = 0.75$, $V = 10^3$ m/sec). For all three curves: $T = 1500^\circ\text{K}$, $\lambda = \lambda_m$, $\lambda = 1$ cm.
 (B) Graph of Hall parameter vs. electron temperature. (4) $B = 5.0$ tesla (right hand
 scale).

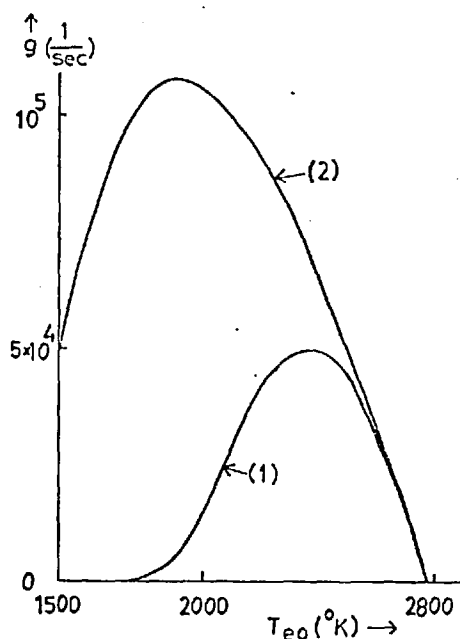


Fig.(2.14)

Graph of growth rate vs. electron temperature with constant (ionization mode).
 (1) $b = 1.1 \times 10^{-20}$; (2) $b = 10^{-15}$. For both curves: $-T_{eo} - T = 300^{\circ}\text{K}$, $\chi = \chi_m$,
 $\lambda = 1 \text{ cm}$, $B = 5.0 \text{ tesla}$.

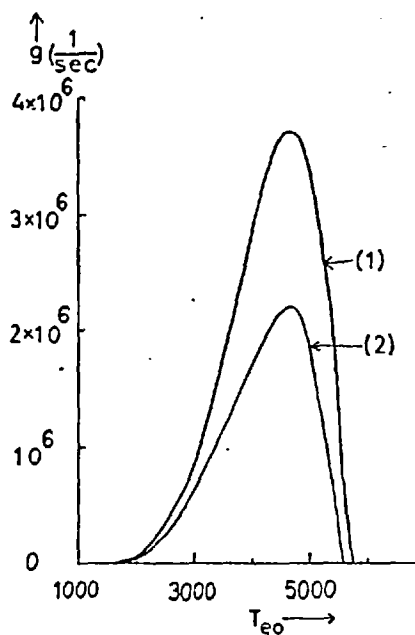


Fig.(2.15)

Growth rate vs. electron temperature (ionization mode). (1) $\beta_0 = 5.0$;
 (2) Magnetically induced elevated electron temperature ($\eta = 0.75$, $V = 10^3 \text{ m/sec}$).
 For both curves: $-T = 1500^{\circ}\text{K}$, $\lambda = 1 \text{ cm}$, $\chi = \chi_m$

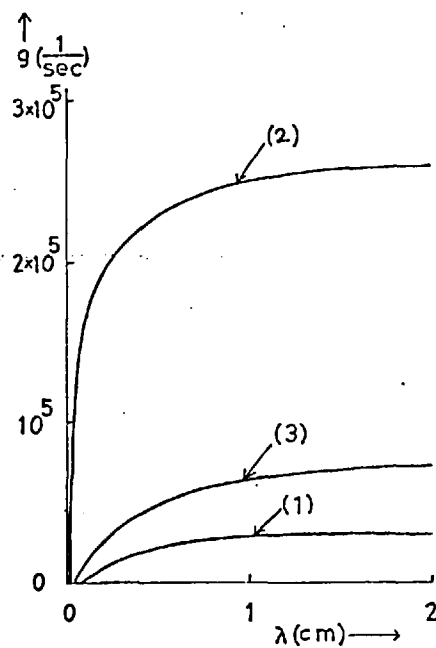


Fig.(2.16) Growth rate vs. wavelength (ionization mode). (1) $T_{e0} = 2000^\circ\text{K}$; (2) $T_{e0} = 2500^\circ\text{K}$; (3) $T_{e0} = 3000^\circ\text{K}$. For all curves: $T = 1500^\circ\text{K}$, $\lambda = \lambda_m$, $B = 5.0$ tesla.

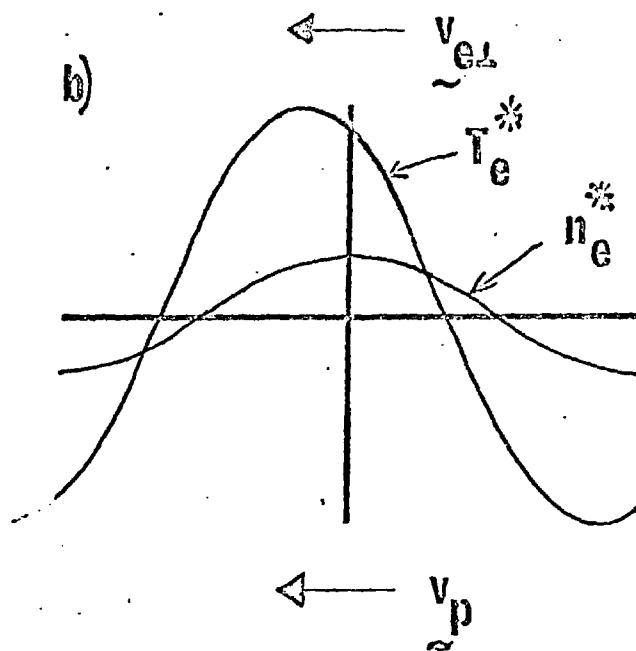
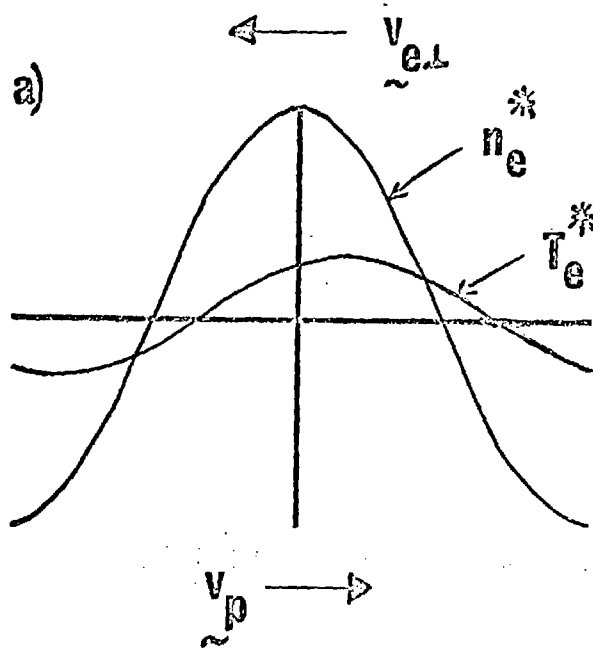


Fig.(2.17) Profiles of n_e^* and T_e^* in space
(phase difference is exaggerated).

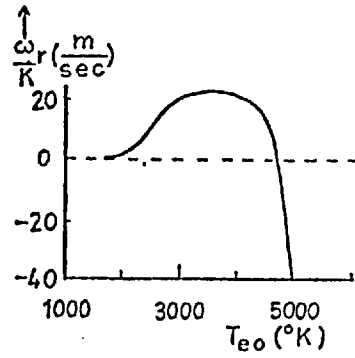


Fig.(2.18) Graph of phase velocity vs. electron temperature (ionization mode). $T = 1500^{\circ}K$, $\lambda = 1$ cm, $\chi = \chi_m$, $B = 5.0$ tesla.

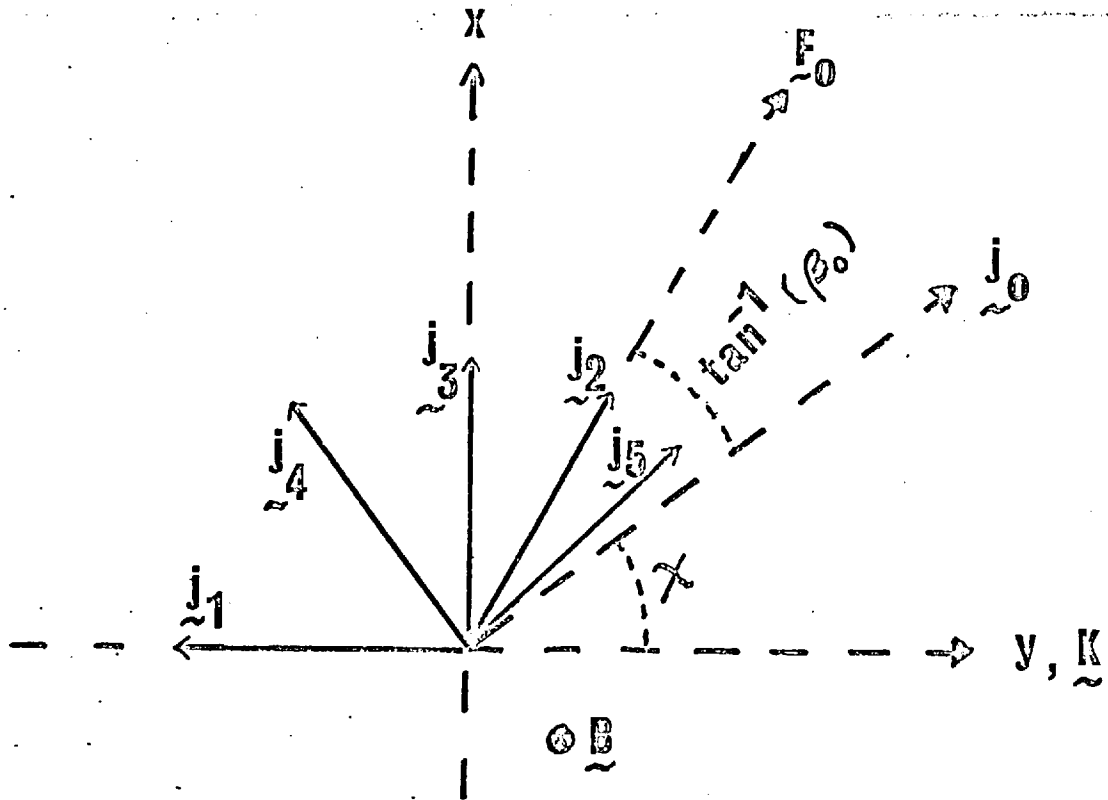


Fig.(2.19) Components of the perturbed current.

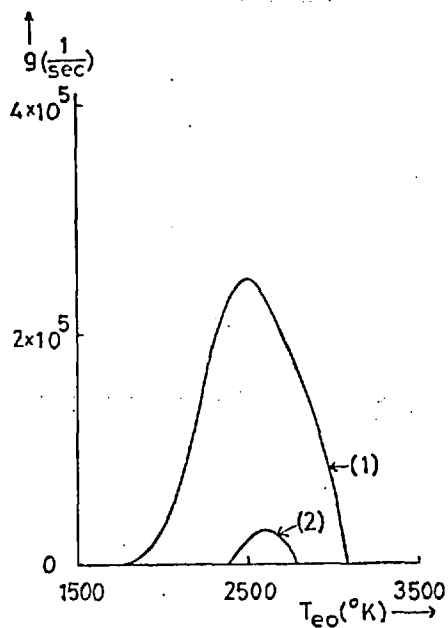


Fig.(2.20)

Graph of growth rate vs. electron temperature (ionization mode). (1) with normal radiation term for the caesium resonance lines; (2) with radiation term = 10 times caesium radiation. For both curves $T = 1500^{\circ}\text{K}$, $\chi = \chi_m$, $B = 5.0$ tesla, $\lambda = 1$ cm.

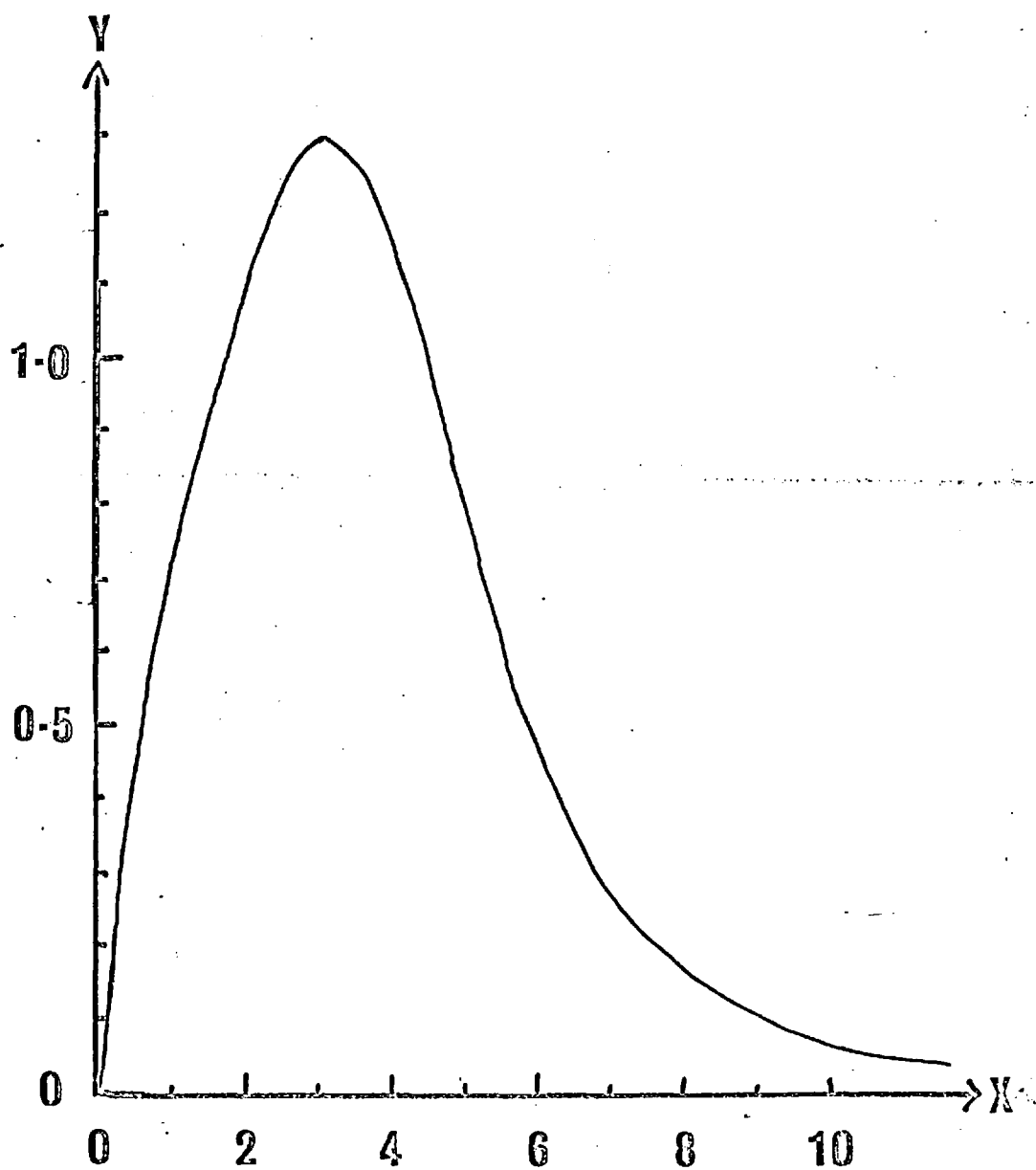


Fig.(2.21) Y versus X, where, $Y \approx \frac{x^3 \exp X}{(\exp X - 1)^2 (1 + 2 \exp(-X))^{1/2}}$
 (Note that ν_i is nearly the same for the resonance doublet lines).

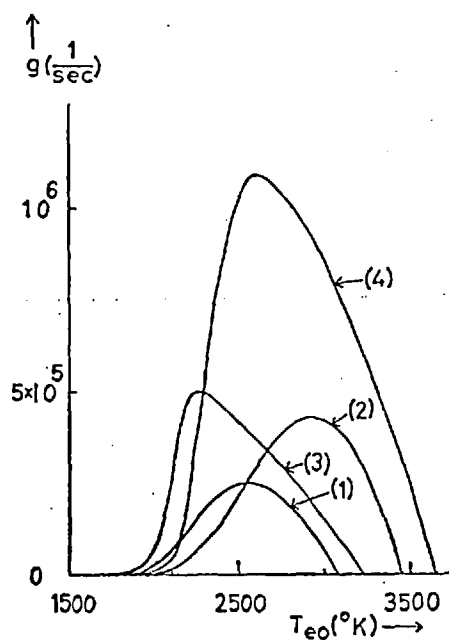


Fig.(2.22) Graph of growth rate vs. electron temperature for various gas mixtures (ionization mode). (1) Argon and caesium; (2) argon and potassium; (3) helium and caesium; (4) helium and potassium. For all curves: $\bar{T} = 1500^{\circ}\text{K}$, $\lambda = 1 \text{ cm}$, $\chi = \chi_m$, $B = 5.0 \text{ tesla}$, and the seed fraction = 0.001.

CHAPTER 3.

MAGNETOSONIC WAVES IN NON - EQUILIBRIUM GASES

3.1 Introduction.

As stated in Chapter 1 the magnetosonic waves considered in this thesis are sonic fluctuations of the heavy particle properties, which, coupled with fluctuations in the electron properties, are distorted by the fluctuating $\underline{j} \times \underline{B}$ forces and Ohmic heating. When the energy coupling between the heavy particles and the electrons is strong, and therefore $T_e = T$, only magnetosonic waves exist. In the absence of gradients, there are three modes; two sonic modes (travelling in opposite directions) and a third mode called a thermal mode.

When a nonequilibrium situation exists, with $T_e \neq T$, we can have electrothermal waves. These are dominant over a wide range of plasma parameters and, as we saw in Chapter 2, two modes exist, i.e. the ionization and the fast thermal modes.

However, fluctuations in the heavy particle properties can still exist and grow even though T_e and T are only weakly coupled. These fluctuations are analogous to the magnetosonic waves of the one-temperature plasma, and their properties and the extent to which they interact with electrothermal waves will concern us in this chapter.

The procedure we will follow is to take the full set of MHD equations describing the nonequilibrium gas in a closed cycle MHD generator, linearize these with respect to all the gas properties, i.e. of both heavy

particles and electrons, and hence obtain a dispersion relation by applying a plane wave solution for the fluctuation quantities. The general wave involving fluctuations of all quantities we will refer to as an electrosonic wave. We would expect therefore that pure electrothermal waves would be represented by electrosonic modes where the amplitudes of n_e^* and T_e^* are much larger than the amplitudes of the fluctuations in the heavy particle properties. While in the other modes there will be significant fluctuations in the heavy particle properties and these will correspond to the magnetosonic modes.

An analysis similar to this has been carried out and some interesting results obtained by Hougen^(1.14). The work presented in this chapter was begun before Hougen's thesis came to the notice of the author, and for two reasons it was decided that it was worthwhile to continue.

Firstly the experiments of Kerrebrock and Dethlefsen^(1.24) showed that at low electron densities, where electrothermal waves are damped due to finite ionization recombination rates (see Chapter 2), substantial fluctuations, whose amplitude increased with β_0 but showed no β_{ocrit} still existed. These fluctuations were tentatively identified with magnetosonic waves. However the growth rate for these waves predicted by Hougen and by other magnetosonic theories is too low for the amplitude to grow appreciably during the residence time of the gas in the apparatus. Kerrebrock and Dethlefsen suggest that this discrepancy may be due to the neglect by the theories of gradients in the steady state. In the dispersion relation developed in this chapter gradient effects are included.

Secondly Hougen's results show that the electrothermal and magnetosonic modes are quite distinct in a seeded Argon gas. However it seems likely that this may not be the case when electrothermal waves are only weakly amplified. The electrothermal assumption of zero fluctuation in the heavy particle properties is based on the idea that during the short time the electrothermal wave takes to grow the fluctuating $\underline{j} \times \underline{B}$ and elastic losses do not have time to affect the heavy particles. If the electrothermal growth time is long this assumption may not hold.

In addition, if the neutral gas is the lighter element Helium, the generation of sonic effects by electrothermal waves should be much larger than for Argon. Hence we would like to establish to what extent sonic fluctuations are generated by and to what extent they affect electrothermal waves, especially in the regime of low growth rate and in the case of Helium as the neutral gas.

In the next section we will give a brief description of purely magnetosonic modes; section (3.3) then discusses to what extent electrothermal and sonic effects are expected to interact; and finally we derive the electrosonic dispersion relation in section (3.4).

3.2 Magnetosonic Modes.

In this section we will consider sonic waves in a one temperature plasma, i.e. $T_e = T$, which are distorted by extra terms in the momentum and energy equations. We do this in order to demonstrate the general

character and growth of magnetosonic waves, without deriving in detail the exact form of the extra terms.

Several linear theories for magnetosonic waves have been reported in the literature (1.8)- (1.13). However, they all derive a complicated dispersion relation, which is then simplified using some ordering scheme to reduce the order of the dispersion relation as a polynomial in w . The identity of the various modes get's lost in the simplification procedures, and it is therefore useful conceptually to take sets of simpler equations whose dispersion relation can be solved without simplification and thereby construct a picture of the modes.

The basic equations we consider are the sound wave equations in the frame of the moving gas, i.e.

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_y'}{\partial y} = 0 \quad \text{---} \quad (3.1)$$

$$\rho_0 \frac{\partial v_y'}{\partial t} + \frac{\partial p'}{\partial y} = 0 \quad \text{---} \quad (3.2)$$

$$\frac{\partial p'}{\partial t} - a_0^2 \frac{\partial \rho'}{\partial t} = 0 \quad \text{---} \quad (3.3)$$

where $a_0^2 = \frac{\partial p_0}{\partial \rho_0}$

Equation (3.1), the continuity equation, is unchanged by MHD effects.

Equations (3.2) and (3.3), the momentum and energy equations, are altered by the $\underline{j} \times \underline{B}$ forces and Ohmic heating respectively.

We will first of all neglect the Ohmic heating and assume that the

fluctuations are adiabatic (We are also of course neglecting thermal conduction and viscous heating). Only the current density fluctuates in the $\underline{j} \times \underline{B}$ forces, due to the low Magnetic Reynolds number, and the fluctuations are proportional to σ' , β' and $\underline{E}'$. The σ' , β' and $\underline{E}'$ fluctuations are in turn determined by ρ' , p' , and $\underline{v}'$ in a one temperature plasma. For an adiabatic wave p' can always be written as $a_0^2 \rho'$ hence we can write $\underline{j}' \times \underline{B}$ as $M_1 \underline{v}'_y + M_2 \rho'$ and the equations become,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'_y}{\partial y} = 0$$

$$\rho_0 \frac{\partial v'_y}{\partial t} + a_0^2 \frac{\partial \rho'}{\partial y} = M_1 v'_y + M_2 \rho'$$

Applying a plane wave solution, i.e. $\exp(i\omega t - ik_y)$, we obtain a dispersion relation,

$$\det \begin{pmatrix} i\omega & -\rho_0 i k \\ -a_0^2 i k - M_2 & i\omega \rho_0 - M_1 \end{pmatrix} = 0$$

$$\text{i.e. } \omega^2 \rho_0 + i\omega M_1 - \rho_0 a_0^2 k^2 + \rho_0 i k M_2 = 0,$$

$$\therefore \omega = -\frac{i M_1}{2 \rho_0} \pm \sqrt{a_0^2 k^2 - \left(\frac{M_1}{2 \rho_0}\right)^2 - i k M_2}$$

We therefore have two modes and since the real part of ω comes from the square root term these are simply sound waves travelling in opposite direction with a modified sound speed. The imaginary part of ω , leading to growth or decay, is different for the two directions

of propagation. The growth rate depends on M_1 and M_2 , which in turn depend on χ , the angle between $\underline{k}$ & $\underline{j}_0$ as before, and the dependence of σ , β and $\underline{E}$ on ρ , $\underline{v}$ and p (i.e. T).

For instance if we take $M_2 = 0$, i.e. $\sigma' = \beta' = 0$, and note that the contribution from $\underline{v}'$ is due to the perturbation of the $\underline{v} \times \underline{B}$ e.m.f. we have, from the perturbed Ohm's Law,

$$j'_x = \frac{\sigma}{1+\beta^2} (v'_y B - \beta F'_y) \quad (x\text{-compt.})$$

$$0 = \frac{\sigma}{1+\beta^2} (\beta v'_y B + F'_y) \quad (y\text{-compt.})$$

where $F'_y = E'_y + \frac{1}{n_{eo} e} \frac{\partial p}{\partial y}$,
eliminating F'_y , we get, $j'_x = \sigma v'_y B$

Hence $M_1 = -\sigma B^2$ and with $M_2 = 0$,

$$\omega = -\frac{i M_1}{2 \rho_0} \pm \sqrt{a_0^2 K^2 - \left(\frac{M_1}{2 \rho_0}\right)^2}$$

since $a_0 K \gg \frac{M_1}{2 \rho_0}$ for centimetre wavelengths, we have

$$\omega_r = \pm a_0 K \left(1 - \frac{1}{8} \left(\frac{M_1}{\rho_0 a_0 K}\right)^2\right)$$

and $\omega_i = -\frac{M_1}{2 \rho_0} = \frac{\sigma B^2}{2 \rho_0}$

Hence we obtain the familiar result that, for constant conductivity and Hall parameter, the Magnetic field damps sound waves. This is familiar in the form that magnetic fields increase the critical Reynolds number

for the transition from laminar to turbulent flow.

However, the variations of σ and β are of great importance to us in this thesis, and they can lead to wave amplification rather than damping. For instance if we set $M_1 = 0$ and neglect β' variations we have, from section (2.6 g),

$$j'_x = j_0 \sigma^* (\sin \chi + \beta_0 \cos \chi)$$

for large β_0 we then have,

$$j'_x \approx \sigma^* \beta_0 \cos \chi$$

now the relation between ρ^* and σ^* has a positive constant of proportionality when the wave is adiabatic. This is because $\rho^* = \frac{T^*}{\delta - 1}$ and T^* gives rise to large fluctuations in n_e ($T_e = T$), which is the dominant contribution to σ^* .

Hence we can write $M_2 = M_3 \cos \chi$, where M_3 is positive. We have therefore,

$$w = \pm \sqrt{a_0^2 K^2 - i K M_3 \cos \chi}$$

if we let χ vary from $-\pi$ to π , then for $|\chi| < \pi/2$

$$w_r = \pm (a_0^4 K^4 + K^2 M_3^2 \cos^2 \chi)^{1/4} \cos \theta$$

$$w_i = \mp (a_0^4 K^4 + K^2 M_3^2 \cos^2 \chi)^{1/4} \sin \theta$$

$$\text{where } \theta = \frac{1}{2} \tan^{-1} \left| \frac{M_3 \cos \chi}{a_0^2 K} \right|,$$

and if $|\chi| > \frac{\pi}{2}$

$$w_r = \pm (a_0^4 K^4 + K^2 M_3^2 \cos^2 \chi)^{1/4} \cos \theta$$

$$w_i = \pm (a_0^4 K^4 + K^2 M_3^2 \cos^2 \chi)^{1/4} \sin \theta$$

Hence we see that waves with $\underline{v}_p$ parallel to $\underline{j}_0$ will tend to grow, while waves with $\underline{v}_p$ antiparallel to $\underline{j}_0$ will tend to decay.

These remarks of course are very rough in that we arbitrarily drop terms from the equations in order to obtain a solution for w . However, they serve to illustrate the possibility of wave growth or decay when $\underline{j}' \times \underline{B}$ is introduced into the sound wave equations.

When we drop the assumption of adiabatic fluctuations and introduce the Ohmic heating to the energy equation the possibility of wave growth or decay again arises. In addition a third mode appears, i.e. the dispersion relation becomes a cubic. Neglecting the $\underline{j}' \times \underline{B}$ term in the momentum equation, we can write the modified equations as

$$\frac{\partial p'}{\partial t} + \rho_0 \frac{\partial v_y'}{\partial y} = 0$$

$$\rho_0 \frac{\partial v_y'}{\partial t} + \frac{\partial p'}{\partial y} = 0$$

and
$$\frac{\partial p'}{\partial t} - a_0^2 \frac{\partial p'}{\partial t} = \left(\frac{j^2}{\sigma} \right)'$$

Taking only the p' dependence of $\left(\frac{j^2}{\sigma} \right)'$ we write the last equation as,

$$\frac{\partial p'}{\partial t} - a_0^2 \frac{\partial p'}{\partial t} = M_4 p'$$

Hence applying plane wave solutions we get a dispersion relation

$$\det \begin{pmatrix} i\omega & -\rho_0 i k & 0 \\ 0 & i\omega \rho_0 & -i k \\ -a_0^2 i\omega - M_4 & 0 & i\omega \end{pmatrix} = 0$$

$$\text{i.e. } \omega^3 - K^2 a_o^2 \omega + i M_4 K^2 = 0$$

The solution of a cubic can be written explicitly (3.1) and when $\frac{M_4}{K a_o^3} \ll 1$, the solution to this cubic can be written as

$$\omega = \pm K a_o - \frac{i M_4}{2 a_o^2}, \text{ or } \frac{i M_4}{a_o^2}.$$

Hence we have the two sound waves travelling in opposite directions which are damped or amplified depending on whether M_4 is negative or positive. In addition we have a non propagating mode which grows when M_4 is negative and decays when it is positive. Therefore the ρ' term in $\left(\frac{j^2}{\sigma}\right)'$ always tends to give instability (except when χ is such that $M_4 = 0$). The sound wave is unstable when $M_4 > 0$, and the third or thermal mode is unstable when $M_4 < 0$.

These three modes, the two sound waves and the thermal mode, together with the two electrothermal modes comprise the complete set of electrosonic fluctuation modes of a uniform MHD plasma. The general electrosonic dispersion relation for a uniform plasma is therefore a fifth power polynomial and it was this dispersion relation that was solved by Hougen.

However when gradients in the steady state are introduced a sixth mode of fluctuation appears. This is due to the appearance of a transverse fluctuation in the gas velocity. In a uniform plasma we have only longitudinal fluctuations v_y' (fluctuations $\propto \exp(i\omega t - iKy)$) since the x-component of the momentum equation gives simply $\frac{\partial v_x'}{\partial t} = 0$.

However when gradients are present terms such as $\rho_o v_y' \frac{\partial v_{ox}}{\partial y}$,

$\rho'(\underline{v}_0 \cdot \nabla) v_{0x}$ appear in the x-component of the momentum equation and v_x' becomes non zero. Since this extra equation with a $\frac{\partial}{\partial t}$ term must therefore be included, the power of the dispersion relation increases by one, and a sixth mode appears.

If the magnetosonic and electrothermal modes are damped or have too low a growth rate to account for the fluctuations observed by Kerrebrock and Dethlefsen at low electron densities, then it may be that this sixth mode is the source of these fluctuations.

3.3 Discussion of the Generation of Sonic Fluctuations by Electrothermal Waves.

Hougen (1.14) argues that the large difference in the electrothermal and sound phase velocities ($\lesssim 10 \frac{\text{m}}{\text{sec}}$ and $\sim 10^3 \frac{\text{m}}{\text{sec}}$ respectively) means that the two waves will not interact. This seems a reasonable argument, however it must be noted that the thermal magnetosonic mode has zero phase velocity, not very far from that of electrothermal waves; and also the possibility of the generation by electrothermal waves of fluctuations in the heavy particles travelling with the same phase velocity must be considered.

In short the flowing gas passes through fluctuations in $\underline{j}$ associated with the electrothermal wave which must give rise to synchronized fluctuations in the flow due to $\underline{j}' \times \underline{B}$ and $\underline{g}'$ (the fluctuating elastic losses in a two temperature plasma). How big are these flow fluctuations and how might they affect the electrothermal waves?

We can estimate the $\underline{j}' \times \underline{B}$ effects by equating the fluctuating force to $\frac{\rho_0 v'}{\tau_E}$ where τ_E = electrothermal e-folding time, and $\underline{j}'$ and $\underline{v}'$ are the perturbations in $\underline{j}$ and $\underline{v}$ after one e-folding time. Therefore we have,

$$\rho_0 \frac{v'}{\tau_E} \sim j' B$$

and since $j' \sim n_e^* j_0$

we have $v^* \sim \tau_E \frac{j_0 B}{\rho_0 v_0} n_e^*$

now $\frac{j_0 B}{\rho_0 v_0} \sim 10^2$ using typical values, therefore we have

$$v^* \sim \tau_E 10^2 n_e^*$$

Hence, when the electrothermal wave growth is a maximum

$$v^* \sim 10^{-3} n_e^*$$

Similarly we can estimate the $\underline{\mathcal{E}}'$ effects from the relation,

$$\frac{n k T'}{\tau_E} \sim \underline{\mathcal{E}}' \sim \left(\frac{j^2}{\sigma} \right)' \sim \frac{2 j_0^2}{\sigma_e} n_e^*$$

$$\text{i.e. } T^* \sim \tau_E \left(\frac{2 j_0^2}{\sigma_e n k T_0} \right) n_e^*$$

$$\sim \tau_E 10^2 n_e^*$$

$$\sim 10^{-3} n_e^*$$

at peak growth.

It seems therefore that only small fluctuations in the heavy particle properties will be generated by electrothermal waves. However it must be noted that for fluctuations of equal amplitude the energy of a sonic wave is much larger than the energy of an electrothermal wave.

For a sound wave the energy density is given by,

$$\begin{aligned} E_s &\sim n_0 k T_e T^* \\ &\sim 10^5 T^* \quad \frac{\text{joules}}{\text{m}^3} \end{aligned}$$

while for an electrothermal wave,

$$\begin{aligned} E_E &\sim I_p n_{e0} n_e^* \\ &\sim 10^2 n_e^* \quad \frac{\text{joules}}{\text{m}^3} \end{aligned}$$

Hence the heavy particle fluctuations generated by the electrothermal wave could affect significantly the energy input to the growing instability. Two mechanisms by which the electrothermal^{wave} may be affected are the following.

The fluctuating $\tilde{j} \times \underline{B}$ forces generate $\tilde{y}'$, which gives rise to an extra component of E_y' , this in turn produces an extra component of j_x' . The direction of this extra component will be given by,

$$\sigma (\tilde{j}_E' \times \underline{B}) \times \underline{B} = -\sigma B^2 \tilde{j}_{Ex}'$$

where $\tilde{j}_E'$ is the fluctuating current from electrothermal effects alone. We see that this extra component will therefore be in opposition to $\tilde{j}_E'$ and the current fluctuations will be reduced. The growth rate of the electrothermal instability will therefore be reduced.

This effect is analogous to the damping of sound waves by the $\tilde{v} \times \underline{B}$ generated currents described in section (3.2). In that case fluctuating velocities generate fluctuating currents which interact with $\underline{B}$ to produce a force which opposes the movement; in this case fluctuating currents

give rise to fluctuating velocities which interact with $\underline{B}$ to produce an electric field opposing the currents.

Although the generation of heavy particle fluctuations by $\underline{j} \times \underline{B}$ forces may damp electrothermal waves, the heating of the heavy particles by $\underline{E}'$ may make the waves more unstable. This is because the heat loss to the heavy particles is included in the electrothermal theory (see section (2.2)), under the assumption that the heavy particles constitute an isothermal heat sink, hence T never varies. If we take into account the finite thermal capacity and the finite thermal losses of the heavy particles, then T must increase at local points in the plasma where T_e and n_e increase above their steady state values. This means that the elastic losses are reduced at this local point since $\xi \propto (T_{eo} - T)$. Therefore the growth rate of the electrothermal wave will tend to be increased, because the elastic losses are an important damping factor in the wave.

The two effects postulated here oppose each other, and it will be of interest to examine to what extent and in what direction the electrothermal instability is affected. The amplitude of the heavy particle fluctuations generated increase with increasing τ_E , and hence the effects should be more marked in the regimes where the electrothermal wave is slowly growing. In addition the effects of $\underline{j}' \times \underline{B}$ on the heavy gas will be stronger in Helium since $\underline{v}'$ is proportional to $\frac{1}{\rho_0}$.

3.4 Equations and Dispersion Relation for Electrosonic Waves.

The two fluid MHD equations presented here are similar to those derived by Appleton and Bray (3.2). The equations are derived by taking the moments of the Boltzmann equation for the four components of the plasma, viz. noble gas, seed atoms, seed ions and electrons. We then obtain two fluid equations by combining the equations for the first three particle types to obtain a single set of equations for the heavy particles, together with the electron equations.

In the laboratory frame, the equations are:-

Density equations

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \underline{v}_e) = A_1 n_e \left(\frac{\epsilon}{1+\epsilon} n - n_e \right) - A_2 n_e^3 \quad \text{---(3.4)}$$

$$\text{and} \quad \frac{\partial n}{\partial t} + \nabla \cdot (n \underline{v}) = 0 \quad \text{---(3.5)}$$

$$\text{where} \quad n = n_n + n_s$$

$$n_n = \text{noble gas density}$$

$$\text{and } n_s = \text{seed particle density (ions plus atoms)}$$

$$\epsilon = \text{seed fraction, i.e. } n_s = \epsilon n_n$$

We once again neglect ion slip and assume that all the heavy particles have velocity $\underline{v}$.

Momentum equations

$$j_x = \frac{\sigma}{1+\beta^2} (F_x - \beta F_y) \quad \text{--- (3.6)}$$

$$j_y = \frac{\sigma}{1+\beta^2} (\beta F_x + F_y) \quad \text{--- (3.7)}$$

where $\underline{F} = \underline{E} + \frac{\nabla p_e}{n_e e}$, $\underline{E} = \underline{E}_{es} + \underline{v} \times \underline{B}$,

$$\underline{j} = n_e e (\underline{v} - \underline{v}_e)$$

$$\rho \frac{\partial v_x}{\partial t} + \rho \underline{v} \cdot \nabla v_x + \frac{\partial p}{\partial x} = j_y B + \mu \nabla^2 v_x + \frac{\mu}{3} \frac{\partial}{\partial x} (\nabla \cdot \underline{v}) \quad (3.8)$$

$$\rho \frac{\partial v_y}{\partial t} + \rho \underline{v} \cdot \nabla v_y + \frac{\partial p}{\partial y} = -j_x B + \mu \nabla^2 v_y + \frac{\mu}{3} \frac{\partial}{\partial y} (\nabla \cdot \underline{v}) \quad (3.9)$$

In the electron momentum equation, i.e. the Ohm's law ((3.6) & (3.7)), we neglect the electron inertia. In other words we are looking at phenomena which vary slowly enough in time for the electrons to effectively take up instantaneously the velocity determined by the fields and collisional forces at any given instant of time.

The coefficients σ and β are as defined in section (2.2), except that we include collisions with the neutral seed atoms in the collision frequency, i.e.

$$\nu = \nu_n + \nu_c + \nu_a, \quad \text{where}$$

$$\nu_a = n_a \left(\frac{8}{\pi} \frac{k T_e}{m_e} \right)^{1/2} q_{ea}$$

q_{ea} is between 10 and 100 times the magnitude of q_{en} . When $\epsilon = 10^{-3}$, as in Chapter 2, the effect of these collisions is negligible. However it would be of interest to consider how the waves vary with varying seed fraction and for this purpose it is necessary to include ν_a .

$$\rho \text{ is defined by, } \rho = n_n m_n + n_s m_s = \frac{n_n + \epsilon m_s}{1 + \epsilon} n \quad \text{and}$$

$$p = p_n + p_s + p_e$$

the coefficient of viscosity is taken to be

$$\mu = \frac{1}{2} \frac{n_n k T}{\nu_{nn}}$$

where ν_{nn} = collision frequency for noble gas - noble gas collisions

The viscous terms are usually small due to the short collision length, however they are included here to examine their influence on short wavelengths.

Energy equations

$$\begin{aligned} \frac{\partial}{\partial t}(U_e) + \nabla \cdot (\underline{v}_e U_e) + p_e \nabla \cdot \underline{v}_e &= \frac{j^2}{\sigma} - \mathcal{E} \\ + \nabla \cdot (K_e \nabla T_e) - R - I_P \dot{n}_e - I_E \frac{\partial T_e}{\partial t} & \quad - (3.10) \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right) p_h - a^2 \left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right) \rho &= (\gamma - 1) \left[\mathcal{E} \right. \\ \left. + \nabla \cdot (K_h \nabla T) - \underline{\tau} : \nabla \underline{v} \right] & \quad - (3.11) \end{aligned}$$

where $\dot{n}_e$ = rate of change of n_e due to ionization and recombination, and

$$I_E = \eta_{a1} \sum_{i=2}^{i_{\max}} \left(\frac{h^2 i^2}{k T_e} \right)^2 k(i)^2 \exp\left(-\frac{h^2 i^2}{k T_e}\right)$$

this is the Caesium excitation energy and is discussed in Appendix A.

(i here is the principal quantum number).

$$p_h = p_n + p_s$$

$$a^2 = \frac{\gamma p_h}{\rho}, \quad \gamma = \frac{5}{3}$$

K_e, K_h = electron and heavy particle thermal conduction

respectively. K_e is defined in section (2.2) and

$$K_h = \frac{3}{4} \frac{n_n k^2 T}{m_n \nu_{nn}}.$$

$$\underline{V} = \nabla \cdot \underline{\tau} = \text{viscous force} = \mu \nabla^2 \underline{v} + \frac{\mu}{3} \nabla (\nabla \cdot \underline{u})$$

Field Equations.

$$\nabla \cdot \underline{j} = 0 \quad - (3.12)$$

$$\nabla \times \underline{E} = 0 \quad - (3.13)$$

We have then 10 equations in the 10 variables n_e , T_e , v_{ex} , v_{ey} , n , T , v_x , v_y , E_x and E_y . These equations can then be linearized, assuming small perturbations about some steady state. The steady state is allowed to have gradients, but second order derivatives of the steady state variables are assumed to be zero.

Using equation (3.13) we put $E_x = 0$ for a plane wave travelling in the y-direction, and the linearized set of equations, now reduced to nine, can be written as,

$$M \xi = 0$$

where M is a 9 x 9 matrix, whose elements are given in Appendix C, and

$$\xi = \begin{pmatrix} n_e^* \\ T_e^* \\ v_{ex}' \\ v_{ey}' \\ n^* \\ T^* \\ v_x' \\ v_y' \\ E_y \end{pmatrix}$$

The dispersion relation is therefore given by

$$\det M = 0$$

Since $\frac{\partial}{\partial t}$ appears in six of the equations this is a sixth power polynomial in w . The elements of M are functions of the gradients of the steady state variables as well as the steady state variables themselves.

To obtain estimates of the gradients consistent with the steady state equations we substitute an arbitrary set of steady state variables into equations (3.4) - (3.13) with $\frac{\partial}{\partial t} = 0$, taking the spatial gradients to be undetermined. This gives us a set of ten equations in the twenty unknown gradients ($\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ for each of the ten variables). Arbitrarily fixing ten of these, all the $\frac{\partial}{\partial c}$'s say, enables us to evaluate the other ten variables in such a way as to give us a steady state locally consistent with the equations, under the assumption of small second order spatial gradients.

In order that the plane wave assumption may be applicable the characteristic variation lengths determined by these gradients must be much greater than the wavelengths investigated.

The method being used to solve the dispersion relation is somewhat crude but, it is hoped, effective. The method is simply to plot the real and imaginary parts of $\det M$ in a wide region of the w plane. The regions where the lines determined by $\Re(\det M) = 0$ and $\Im(\det M) = 0$ cross are then plotted in greater detail. The root is then located to a greater degree of accuracy again by the crossing of these two lines. This method can be taken to any degree of accuracy required, and has the advantage over iterative procedures for solving polynomials that no initial estimates of the roots are required.

Unfortunately no results are yet available from this calculation.

CHAPTER 4.

THE DEVELOPMENT OF THE ELECTROTHERMAL INSTABILITY

WITH BOUNDARY EFFECTS.

4.1 Introduction.

In Chapters 2 and 3 of this thesis we have assumed that the plasma under consideration was infinite in all directions. In this chapter we shall not make this assumption, but will examine the effects of electrode and insulator boundaries on the behaviour of electrothermal waves.

The plane wave, infinite plasma analyses of electrothermal waves have had considerable success in predicting and explaining the experimental observations. In particular the movement of the waves, and the two major stability properties, viz 1) that the plasma is unstable if the steady state Hall parameter, β_0 , exceeds a certain critical value, and 2) that the wave vector $\underline{K}$ for maximum growth satisfies $\frac{\underline{J}_0 \times \underline{K} \cdot \underline{B}}{\underline{J}_0 \cdot \underline{K} B} = 1$ for large β_0 , are all well understood.

However experimentally the wavelengths observed have been of the order of the apparatus dimensions; also a property of the instability not fully explained by the plane wave theories is that the structure of the instability deteriorates from an approximate plane wave to apparently random turbulence as the Hall parameter is increased (1.21). It would seem likely therefore that the boundaries will considerably affect the wave. In addition, the interaction between the wave and the boundaries may be such as to give us some form of control

over the instability. In other words it is of interest to consider if it is possible, say, to increase the critical Hall parameter by varying the boundaries and/ or the external circuitry.

Recently Velikhov et. al. ^(1.19) and Lengyel ^(1.20) have computed numerically the development of the waves to non-linear levels, applying simple boundary conditions. In both of these treatments restricting assumptions, are made in order to implement the computations. Lengyel assumes constant current in the external circuits, i.e. he assumes that there is a high resistance and inductance in the circuit to impede current changes. Velikhov et. al. assume both constant external current and constant conductivity in the plasma. The latter assumption is a good approximation when Coulomb collisions are dominant in view of the small magnitude of the electron temperature fluctuations. These conditions do not of course hold in general and the results of these calculations to date do not seem to advance significantly our knowledge of the wave boundary interaction.

In this Chapter we present^a/more generally applicable analytic treatment of the linear development of the instability under simple boundary conditions. The set of partial differential equations in space and time, obtained by linearizing the pertinent electron and field equations, are solved under boundary conditions on the current density and the electric field. We do not require to make the assumptions of references (4.1) and (4.2). However, to make the problem mathematically tractable, it is necessary to assume that the plasma

is bounded in only one direction perpendicular to the magnetic field. We also make the usual assumption that $\frac{\partial}{\partial z}$ is identically zero for all quantities (where the z-direction is parallel to the magnetic field).

The apparatus walls which give rise to the boundary conditions are assumed to lie at $x = 0$ and $x = d$, while the plasma is assumed to be infinite in the y-direction. Three cases are examined, 1) insulator walls, 2) continuous electrode walls, 3) infinitely finely segmented electrode walls.

The most severe limitation of this theory is of course that it is linear, while the non-linear interaction of different modes of the instability with each other and with the walls will be important in determining the instability structure. With increasing availability of more sophisticated numerical techniques it seems likely that more realistic electrothermal wave calculations, to determine this structure for different plasma and boundary situations, will be implemented, without the restricting assumptions of references (4.1) and (4.2). It is hoped that the results of the analysis presented here will at least give insight and direction to such calculations; and at any rate the linear analysis will give a strong indication of the effect of the boundaries on whether wave growth takes place or not.

The method of solution employed here is to Laplace transform the equations and boundary conditions in time, and solve the resulting spatial differential equations under the transformed boundary conditions.

The Laplace transform method is of course admirably suited to initial value problems, since the initial perturbation enters the calculation via the transform of the time derivative. Applying the inverse Laplace transform to the solution of the transformed equations, we obtain, as a function of space and time, the waves which result from an instantaneous perturbation.

The complex integration which constitutes the inverse Laplace transformation is the most difficult part of the calculation. To make this step mathematically tractable we have, for the cases of continuous and segmented electrodes, to neglect the gradient terms in the energy equation. These terms represent compression and convection of the electrons and contribute only to the movement of the waves as we saw in Chapter 2; also they are usually more than an order of magnitude smaller than the other terms in the energy equation, since $w_r \ll g$ from Chapter 2. Hence it is valid to neglect them in a stability analysis.

In section (4.2) the basic electrothermal equations used in this Chapter are described, followed by the presentation of the formal details of the calculation to solve them under any boundary conditions which have the specific topology used here.

The results of the calculations show that in general an initial plane wave perturbation of electron density is split into an infinite number of electrothermal modes. The amplitudes of the individual

modes vary with time in a characteristic way. In general some grow and some decay, and the structure and properties of the modes for specific geometries are described in sections (4.3), (4.4) and (4.5).

4.2 Basic Equations and Analysis.

We use the basic Electrothermal set, viz. (2.1) - (2.6), except that the rate equation (2.1) is replaced by the Saha equilibrium equation. We assume therefore that $T_{e0} \gg 2500^\circ\text{K}$ so that the effect of the finite ionization rates is negligible. In addition we neglect the effects of radiation transfer and thermal conduction since we have seen in Chapter 2 that these are important only for wavelengths $\lesssim 1\text{mm}$. The wavelengths considered in this chapter will be of the order of, or greater than, 1 cm. Neglecting these effects simplifies the analysis in that T_e drops out of the calculation explicitly, T_e' being replaced by $\frac{\partial T_e}{\partial n_e} \times n_e'$, and we do not require a boundary condition for T_e . Under these assumptions the effect of the T_e fluctuations on the behaviour of the wave (since $n_e^* \gg T_e^*$) is relatively unimportant but will be included here for completeness.

The boundary conditions to be satisfied by the vector fields $\underline{j}$ and $\underline{E}$ are

- a) $\underline{j}_\perp = 0$ on insulator walls
- b) $\underline{E}_\parallel = 0$ on electrode walls
- c) if electrode $\epsilon 1$ and electrode $\epsilon 2$ are connected externally by a load R_L , then,

$$R_L \int_{E1} \underline{j} \cdot d\underline{s} = -R_L \int_{E2} \underline{j} \cdot d\underline{s} = \int_{E2}^{E1} \underline{E}_{es} \cdot d\underline{l}$$

where $\underline{E}_{es} = \underline{E} - \underline{v} \times \underline{B}$

The linearized set of equations can be written in the form

$$p_1 \frac{\partial n_e^*}{\partial t} + p_2 \frac{\partial n_e^*}{\partial x} + p_3 \frac{\partial n_e^*}{\partial y} + p_4 n_e^* + p_5 j_x' + p_6 j_y' = 0 \quad (4.1)$$

$$q_1 \frac{\partial n_e^*}{\partial x} + q_2 \frac{\partial n_e^*}{\partial y} + q_3 n_e^* + q_4 j_x' + q_5 E_x' + q_6 E_y' = 0 \quad (4.2)$$

$$r_1 \frac{\partial n_e^*}{\partial x} + r_2 \frac{\partial n_e^*}{\partial y} + r_3 n_e^* + r_4 j_y' + r_5 E_x' + r_6 E_y' = 0 \quad (4.3)$$

$$\frac{\partial j_x'}{\partial x} + \frac{\partial j_y'}{\partial y} = 0 \quad (4.4)$$

$$\frac{\partial E_x'}{\partial y} - \frac{\partial E_y'}{\partial x} = 0 \quad (4.5)$$

where (4.1) is the electron energy equation, (4.2) and (4.3) the two Ohm's Law components, and (4.4) and (4.5) the field equations.

Note that the T_e^* terms have been absorbed into the n_e^* terms.

The coefficients p , q and r are all functions of the steady state quantities, and are,

$$p_1 = \frac{3}{2} k T_{e0} + I_p + \frac{3}{2} n_{e0} k \left(\frac{\partial T_e}{\partial n_e} \right)_0 ,$$

$$p_2 = j_{ox} \left(\frac{p_{e0}}{e n_{e0}^2} - \frac{3}{2} \frac{k}{e} \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right) ,$$

$$p_3 = j_{0y} \left(\frac{p_e}{e n_{e0}^2} - \frac{3}{2} \frac{k}{e} \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right),$$

$$p_4 = \frac{j_0^2}{\sigma_0} \left[\frac{1}{\sigma_0} \left(\left(\frac{\partial \sigma}{\partial n_e} \right)_0 + \left(\frac{\partial \sigma}{\partial T_e} \right)_0 \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right) + \left(\frac{\partial \log \xi}{\partial n_e} \right)_0 - \left(\frac{\partial \log \xi}{\partial T_e} \right)_0 \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right]$$

$$p_5 = -\frac{2 j_{0x}}{\sigma_0 n_{e0}}, \quad p_6 = -\frac{2 j_{0y}}{\sigma_0 n_{e0}},$$

$$q_1 = \frac{\sigma_0}{n_{e0} e (1 + \beta_0^2)} \left(k T_{e0} + n_{e0} k \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right)$$

$$q_2 = -\beta_0 q_1$$

$$q_3 = \frac{j_{0x}}{\sigma_0} \left(\left(\frac{\partial \sigma}{\partial n_e} \right)_0 + \left(\frac{\partial \sigma}{\partial T_e} \right)_0 \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right) - \frac{(\beta_0 j_{0x} + j_{0y})}{1 + \beta_0^2} \times \\ \left(\left(\frac{\partial \beta}{\partial n_e} \right)_0 + \left(\frac{\partial \beta}{\partial T_e} \right)_0 \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right),$$

$$q_4 = -\frac{1}{n_{e0}}, \quad q_5 = \frac{\sigma_0}{n_{e0} (1 + \beta_0^2)}, \quad q_6 = -\beta_0 q_5$$

$$r_1 = \frac{\sigma_0 \beta_0}{n_{e0} e (1 + \beta_0^2)} \left(k T_{e0} + n_{e0} k \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right), \quad r_2 = r_1 / \beta_0$$

$$r_3 = \frac{j_{0y}}{\sigma_0} \left(\left(\frac{\partial \sigma}{\partial n_e} \right)_0 + \left(\frac{\partial \sigma}{\partial T_e} \right)_0 \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right) - \frac{(\beta_0 j_{0y} - j_{0x})}{1 + \beta_0^2} \times \\ \left(\left(\frac{\partial \beta}{\partial n_e} \right)_0 + \left(\frac{\partial \beta}{\partial T_e} \right)_0 \left(\frac{\partial T_e}{\partial n_e} \right)_0 \right)$$

$$r_4 = -\frac{1}{n_{e0}}, \quad r_5 = \frac{\sigma_0 \beta_0}{n_{e0} (1 + \beta_0^2)}, \quad r_6 = r_5 / \beta_0$$

Introducing the current stream function, $\mathbf{j} = \nabla \times \underline{\psi}$, where

$\underline{\psi} = (0, 0, \psi)$, and the electrostatic potential, $\underline{E} = -\nabla \phi$, we can

reduce the number of variables and equations to three, since (4.4)

and (4.5) are automatically satisfied by these forms for $\underline{j}$ and $\underline{E}$.

Assuming that the plasma is infinite in the y-direction, we can

consider one Fourier component in that direction and replace $\frac{\partial}{\partial y}$

by $-iK_y$. The walls bounding the plasma are assumed to be parallel to the y - z plane, and to lie at $x = 0$ and d (see Figure 4.1). The equations can now be written in the matrix form,

$$\begin{pmatrix} \frac{\partial}{\partial t} + \frac{p_4 - iK_y p_3}{p_1} & -\frac{iK_y p_5}{p_1} & 0 \\ \frac{q_3 - iK_y q_2}{q_1} & -\frac{iK_y q_4}{q_1} & \frac{iK_y q_6}{q_1} \\ \frac{r_3 - iK_y r_2}{r_1} & 0 & \frac{iK_y r_6}{r_1} \end{pmatrix} \begin{pmatrix} n_e^* \\ \psi' \\ \phi' \end{pmatrix} + \begin{pmatrix} \frac{p_2}{p_1} & -\frac{p_6}{p_1} & 0 \\ 1 & 0 & -\frac{q_5}{q_1} \\ 1 & -\frac{r_4}{r_1} & -\frac{r_5}{r_1} \end{pmatrix} \begin{pmatrix} \frac{\partial n_e^*}{\partial x} \\ \frac{\partial \psi'}{\partial x} \\ \frac{\partial \phi'}{\partial x} \end{pmatrix} = 0$$

In abbreviated notation this can be written,

$$A \left(\frac{\partial}{\partial x} \right) \xi + B \frac{\partial \xi}{\partial x} = 0 \quad - (4.6)$$

where ξ is the column vector $\begin{pmatrix} n_e^* \\ \psi' \\ \phi' \end{pmatrix}$ and A and B are the appropriate matrices.

The boundary conditions to be satisfied by the linearized quantities are

- a) $\psi' = \text{constant}$ on insulator walls
- b) $\phi' = \text{constant}$ on electrode walls
- c) $R_L \int_{\epsilon_1} \nabla \times \psi' \cdot d\mathbf{s} = -R_L \int_{\epsilon_2} \nabla \times \psi' \cdot d\mathbf{s} = [\phi']_{\epsilon_1}^{\epsilon_2}$

The problem then is to solve equation (4.6) for a given initial perturbation under the boundary conditions appropriate to the system considered.

The Laplace transform of equation (4.6) is,

$$A(Z) \bar{\xi} + B \frac{\partial \bar{\xi}}{\partial x} = \eta e^{-i\mathbf{k} \cdot \mathbf{r}} \quad (4.7)$$

where, $\bar{\xi}(x, y, Z) = \int_0^\infty \xi(x, y, t) e^{-Zt} dt,$

and $\eta = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$. We have assumed that $n_e^*(x, y, 0) = \alpha e^{-i\mathbf{k} \cdot \mathbf{r}}$, i.e. we are applying a plane wave perturbation in the electron density at $t = 0$ and calculating its development in time. (The initial perturbations in ψ' and ϕ' may be obtained by substituting $n_e^*(x, y, 0)$ into the two components of the Ohm's Law, and solving for ψ' and ϕ' under the appropriate boundary conditions). The boundary conditions for $\bar{\psi}'$ and $\bar{\phi}'$ are identical to those for ψ' and ϕ' .

The general solution of (4.7) is the sum of the general solution

of the reduced equation, i.e.

$$A(Z)\bar{\xi} + B \frac{\partial \bar{\xi}}{\partial x} = 0 \quad \text{--- (4.8)}$$

together with a particular solution of (4.7) itself. The general solution of (4.8) is a linear combination of the eigenfunctions $e^{\lambda x}$

where the eigenvalues (λ) are given by, $\det C(Z, \lambda) = 0$,

$$\text{where } C(Z, \lambda) = A(Z) + \lambda B.$$

In general this equation is a cubic in λ , with linear functions of z as coefficients. However, if $\det C(Z, \lambda)$ is a cubic this means that the inverse transform integrals, to transform from Z -space back into t -space, will involve branch cut integrals. These will be very difficult to evaluate and can be avoided by ignoring the gradient terms in the energy equation. This puts the element B_{11} equal to zero, and makes $\det C(Z, \lambda)$ a quadratic (the contribution to the λ^3 term from the cofactor of C_{12} is zero). With only two solutions to $\det C(Z, \lambda) = 0$ the multivaluedness, and hence the branch cut integrals, is removed. Since $B_{11} \propto \int_0^x$, it follows that for insulator walls $B_{11} = 0$ anyway and no approximation is necessary in this case.

Ignoring these gradient terms does not invalidate our stability analysis since they contribute only to the movement of the wave. Their neglect is justified a posteriori by the fact that the growth rates obtained from the calculations turn out to be greater than 10 times the real frequency from the infinite plasma theory. In other words the waves move very little in the time that it takes their amplitude to

e-fold, and hence it is valid to neglect the motion when analysing their stability.

We can assume then that the general solution of (4.8) is of the form

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} e^{\lambda_1 x} + \begin{pmatrix} \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix} e^{\lambda_2 x},$$

where the μ_i 's are constants.

which will be determined by the boundary conditions. A particular solution of (4.7) is $\rho e^{-iK \cdot x}$ where,

$$D \rho = \eta \quad \text{and} \quad D = C(Z, -iK_x),$$

$$\text{i.e. } \rho = D^{-1} \eta = \frac{\text{adj } D}{\det D} \eta$$

The general solution of (4.7) is therefore,

$$\bar{\xi}_i = \mu_i e^{\lambda_1 x - iK_y y} + \mu_{i+3} e^{\lambda_2 x - iK_y y} + \left(\frac{\text{adj } D}{\det D} \eta \right)_i e^{-iK \cdot x}$$

The μ_i 's are determined by the following six linear equations,

$$\begin{pmatrix} 0 & P_{12} & P_{13} & 0 & P_{15} & P_{16} \\ 0 & P_{22} & P_{23} & 0 & P_{25} & P_{26} \\ P_{31} & P_{32} & P_{33} & 0 & 0 & 0 \\ P_{41} & P_{42} & P_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{54} & P_{55} & P_{56} \\ 0 & 0 & 0 & P_{64} & P_{65} & P_{66} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix} = \frac{1}{\det D} \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The first two rows here are simply the formal representation of the boundary conditions, which are linear combinations of $\bar{\psi}'$ and $\bar{\phi}'$, one at $x = 0$, and one at $x = d$. Substituting each of the eigenfunctions into (4.8) we obtain two linearly independent homogeneous equations in $\mu_{1,2,3}$ and two in $\mu_{4,5,6}$, these constitute the last four rows. The form of the elements of P depends on the particular boundary type being considered, and a description of equation (4.9) for each boundary type considered here can be found in Appendix D, where more details of the calculations are given. In abbreviated form we have

$$P \mu = \frac{\zeta}{\det D} \quad \text{i.e.} \quad \mu = \frac{P^{-1} \zeta}{\det D}$$

Hence the solution for $\bar{\xi}(x, y, Z)$ is

$$\bar{\xi}_i = \left(\frac{(\text{adj } P \zeta)_i e^{\lambda_1 x} + (\text{adj } P \zeta)_{i+3} e^{\lambda_2 x}}{\det P \det D} \right) e^{-iK_y y} + \frac{(\text{adj } D \eta)_i}{\det D} e^{-iK_x x}$$

To transform back into t -space we extend the inverse transform integral,

$$\xi(x, y, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{\xi}(x, y, Z) e^{Zt} dZ,$$

to a path enclosing all space to the left of $\Re(Z) = \gamma$, using the fact that the integrand tends to zero on the semicircle of infinite radius centred at $Z = \gamma$, and lying to the left of $\Re(Z) = \gamma$. Using Laurent's theorem,

$$\xi_i(x, y, t) = \sum_{\text{poles}} (\text{residue of } \bar{\xi}_i \text{ at } Z_{\text{pole}}) e^{Z_{\text{pole}} t}$$

(γ is such that all singularities, which are simple poles for $\bar{\xi}$, lie to the left of $\mathcal{R}(Z) = \gamma$. For more details of the calculation for the various boundary types see Appendix D).

The poles of $\bar{\xi}$ are of course given by $\det D(Z) = 0$, and $\det P(Z) = 0$. However, the contribution from the pole at Z_0 , the solution of $\det D(Z) = 0$, can be shown to be identically zero for all boundary conditions (see Appendix E). $e^{Z_0 t}$ is in fact the time dependence of the perturbation $e^{-i \frac{K}{\omega} \cdot \underline{r}}$ obtained from an infinite plasma theory. Hence the boundary conditions remove the initial perturbation, and replace it by the modes which are the solution of the dispersion relation,

$$\det P(Z) = 0 \quad \text{--- (4.10) ,}$$

and the problem is reduced to solving this equation.

Unstable modes are associated with the solutions with positive real parts, and in the next three sections we will be concerned with the investigation of the spectrum of unstable modes for specific wall types.

4.3 Results for Insulator Walls.

This geometry corresponds to a discharge along the length of a long plasma contained by insulator walls, a situation similar to some instability experiments (1.24), (1.26). For insulator walls boundary condition (a) reduces to,

$$\psi'(x, y) = 0 \quad \text{at } x = 0, d$$

Because of the periodicity in the y -direction, the total perturbed current passing through the plasma is zero, hence $\psi'(x,0) = \psi'(x,d)$; and, since only differences in ψ' are significant, the absolute value of ψ' is arbitrary, and we put $\psi' = 0$ on the walls for convenience. Boundary condition (b) does not apply since the electrodes lie at $y = \pm \infty$ and (c) is automatically satisfied by the periodicity in y .

In the steady state $j_{ox} = 0$, and this, together with the boundary conditions, gives a dispersion relation of the form,

$$A_{11}(Z) \left(e^{\lambda_1(Z)d} - e^{\lambda_2(Z)d} \right) = 0.$$

Thus we get one pole at $Z = Z_1$, where $A_{11}(Z_1) = 0$, and an infinite number at $Z = Z_{2,3}^n$, $n = 1, 2, 3, \dots$, where $\lambda_1(Z_{2,3}^n) - \lambda_2(Z_{2,3}^n) = \frac{2\pi i n}{d}$.

We have two poles for each n , since the second equation is a quadratic.

The value $n = 0$ does not correspond to a pole since the numerator of is also zero at the corresponding value of Z ; and negative values of n give the same poles as positive values, hence only the latter need be considered. From the residues of ξ at these poles we can write

the solution for ξ in the following form,

$$\xi = \left[\Theta_1 e^{\lambda_2(Z_1)x + Z_1 t} + \sum_{n=1}^{\infty} \sum_{m=2}^3 \sum_{k=1}^2 \Gamma_{mk}^n e^{\lambda_k(Z_m^n)x + Z_m^n t} \right] \\ \chi \propto e^{-i K_y y} \quad \text{--- (4.11)}$$

where,

$$\Theta_1 = \begin{pmatrix} U_1 \\ V_1 \\ W_1 \end{pmatrix}, \quad \Gamma_{mk}^n = \begin{pmatrix} F_{mk}^n \\ G_{mk}^n \\ H_{mk}^n \end{pmatrix},$$

$$U_1 = \frac{i \lambda_2(Z_1)}{K_{2c}} \left(\frac{1 - e^{-i K_x d}}{1 - e^{\lambda_2(Z_1) d}} \right), \quad \lambda_1, \text{ say, } = 0 \text{ at } Z=Z_1,$$

$$F_{m k}^n = \frac{i \lambda_k(Z_m^n) B_{12} \pi n \det D_{22} (-1)^{n+m+k} ((-1)^{n+m} + e^{-(i K_x + \Delta \lambda(Z_m^n)) d}) (Q_1 Z_m^n + Q_2)^2}{\det D(Z_m^n) (Z_3^n - Z_2^n) Q_1 (Q_1 \frac{\pi^2 n^2}{d^2} - Q_4) A_{11}(Z_m^n) d^2},$$

$$V_1 = 0, \quad G_{m k}^n = - \frac{C_{11}(Z_m^n, \lambda_k(Z_m^n))}{C_{12}(Z_m^n, \lambda_k(Z_m^n))} F_{m k}^n,$$

$$W_1 = \frac{\det C_{13}(Z_1, \lambda_2(Z_1))}{\det C_{11}(Z_1, \lambda_2(Z_1))} U_1,$$

$$H_{m k}^n = \frac{\det C_{13}(Z_m^n, \lambda_k(Z_m^n))}{\det C_{11}(Z_m^n, \lambda_k(Z_m^n))} F_{m k}^n$$

$\Delta \lambda(Z) = \frac{1}{2} (\lambda_1(Z) - \lambda_2(Z))$ and the Q 's are defined by

$$\det C(Z, \lambda) = (Q_1 Z + Q_2) \lambda^2 + Q_3 \lambda + (Q_4 Z + Q_5).$$

Equation (4.11) represents the time development of the initial plane wave perturbation at every point in x and y . At $t = 0$ the sum of the modes is equal to the perturbation, $\propto e^{-i K_x x}$, however, the amplitude of each mode will vary in time, and in general the form of the initial perturbation will disappear.

The values of Z have been calculated for an Argon-Caesium plasma

with the following parameters: $n_n = 10^{25} \frac{1}{\text{m}^3}$, $\frac{n_s}{n_n} = 10^{-3}$,

$T = 1500^\circ\text{K}$, $T_{eo} = 2500^\circ\text{K}$, $d = 5 \text{ cm}$ (these parameters have been used in all the calculations in this Chapter). At this value of T_{eo} , $\left(\frac{\partial \log T_e}{\partial \log n_e}\right)_0 \sim 10^{-1}$, and terms with this factor have been neglected.

The real parts of Z_1 and Z_3^n , $\Re(Z_1)$ and $\Re(Z_3^n)$, have been found to be always negative. The Z_1 mode is stable since it has zero ψ' associated with it (see equation (4.11)), and hence zero $\underline{j}'$. The mode must therefore decay since we know that the source term of the electrothermal instability is the $\underline{j}'$ contribution to the perturbed Ohmic heating. Note also that $G_{m1}^n = -G_{m2}^n$ and therefore ψ' is a sum of $\sin\left(\frac{\pi n x}{d}\right)$ terms, i.e. the solution satisfies the boundary conditions.

$\Re(Z_2^n)$ is plotted in Figure (4.2) as a function of β_o for the least stable modes. (These curves are plotted for $\lambda_y = \frac{2\pi}{K_y} = 10 \text{ cm.}$, and the Hall parameter was varied by varying the magnitude of the magnetic field). We see that as β_o increases the modes are successively destabilized, starting with $n = 1$. For comparison, the broken curve shows $\Re(Z_{om})$. This is the growth rate from the infinite plasma theory for a perturbation $\propto e^{-i\mathbf{k}_m \cdot \mathbf{r}}$ where $\frac{\mathbf{j}_o \times \mathbf{K}_m \cdot \mathbf{B}}{\mathbf{j}_o \cdot \mathbf{K}_m \mathbf{B}} = 1$, i.e. a plane wave oriented in the direction for maximum growth (for large values of β_o). The dispersion relation (4.10) is in fact independent of K_x , and consequently so are Z_2^n and $\lambda_{1,2}(Z_2^n)$. However, the initial amplitudes of the modes are obviously K_x dependent.

Each mode is a sum of two plane waves, since the value of $\lambda_{1,2}(Z_2^n)$

are purely imaginary, and the form of the instability at $\beta_0 = 1.5$ and $\beta_0 = 2.2$ are shown in Figure (4.3). The second of these pictures is a combination of the modes $n = 1$ and 2, and is a projection of what the non-linear instability would look like with no mode interaction, and assuming that the non-linear amplitude of a mode is proportional to its linear growth rate. Mode-mode interaction as well as harmonic generation will of course be important in the non-linear phase of the instability, however these pictures give some idea of the qualitative features of the instability. In Figures (4.4) and (4.5) we plot, imposed upon n_e^* , contours of ψ' , i.e. j' lines, and ϕ' respectively.

The breakdown of the instability from an approximate plane wave, with one mode present, to apparently random turbulence, as more modes are destabilized, has been well established experimentally ^(1.26), and appears in the results of some of the non-linear, finite plasma computations. ^(1.19) That the turbulent structure is due to higher periodicities has been previously recognised; in particular a formula for the modes of ψ' which is equivalent to equation (4.11) without the time dependence, has been given by Shipuk and Pashkin ^(1.21). However, no justification of the successive destabilization of the modes has been given by Shipuk and Pashkin, while the theory presented here shows this phenomenon follows naturally from the application of boundary conditions.

A simple physical picture of why the modes have the behaviour described above can be given on the basis of the plane wave-infinite

plasma theory. We know from this theory that, if χ is the angle $\tan^{-1} \left(\frac{j_0 \times K_E \cdot B}{j_0 \cdot K_E \cdot B} \right)$ then only plane waves with χ in a range of angles ($\Delta\chi$) around $\frac{\pi}{4}$ or $\frac{5\pi}{4}$ will grow. Now usually the modes are approximately plane waves, and we therefore expect that a similar condition for the stability of a mode will apply if we can define an effective wave vector K_E .

For the Z_2^n modes, $\lambda_1 + \lambda_2 \approx \frac{2\pi i n}{d}$ and hence from the value of $\lambda_1 - \lambda_2$, in this section, $\lambda_1 \approx \frac{2\pi i n}{d}$ and $\lambda_2 \ll \lambda_1$. K_E can then be defined as $K_E = -\frac{2\pi n}{d} \hat{x} + K_y \hat{y}$, and, since j_0 is parallel to the y-axis,

$$\chi_E(n) = \tan^{-1} \left(\frac{2\pi n}{d K_y} \right) = \tan^{-1} \left(n \frac{1_y}{d} \right)$$

As n increases $\chi_E(n)$ tends to $\pi/2$ therefore the stability of the mode increases with n , since K_E moves further away from K_m . As β_0 increases $\Delta\chi$ increases (see Figure (2.12)), and therefore more and more modes get destabilized as they fall within the range of instability. Note that the fact that K_E is independent of K_x explains why the stability of the modes is independent of this parameter.

The Z_3^n modes are stable since $\lambda_1(Z_3^n) + \lambda_2(Z_3^n) \approx -\frac{2\pi i n}{d}$ and hence for them $\frac{j_0 \times K_E \cdot B}{j_0 \cdot K_E \cdot B} < 0$ i.e. K_E never falls within $\Delta\chi$.

If L_y is increased then, for constant n , $\chi_E(n)$ tends to $\pi/2$ for the Z_2^n modes, therefore we expect that perturbations of longer wavelengths will be more stable than shorter wavelength perturbations. In Figure (4.6) we plot the critical value of the Hall parameter,

β_{ocrit} , against $\frac{\lambda_y}{d}$ for various modes. We see that β_{ocrit} increases with $\frac{\lambda_y}{d}$, while for $\frac{\lambda_y}{d} \ll 1$ a large number of modes become unstable at, or nearly at, β_{∞} , which is β_{ocrit} for Z_0 . As λ_y tends to zero β_{ocrit} will go through a minimum for low values of n and increase towards infinity. This is because $\chi_E(n)$ tends to zero for low values of n ; but there will always be a high enough value of n such that $\tan \chi_E(n)$ is near enough to 1 for instability. However, the effects of energy transfer, i.e. thermal conduction and radiation, which have been neglected in this analysis, will dominate when λ_y is less than some characteristic length, typically of the order of 1 mm (see Figure (2.16)). And since these effects damp the wave they will cause β_{ocrit} to go to infinity as λ_y tends to zero for all values of n .

Finally $\frac{g(z)}{2\pi}$ is the same for all modes with the same wavelength in the j_0 direction, viz. λ_y , and this value for $\lambda_y = 10$ cm is $3.5 \times 10^2 \frac{1}{\text{sec}}$, giving a velocity in the y -direction of $35 \frac{\text{m}}{\text{sec}}$.

4.4 Results for Continuous Electrode Walls.

This case corresponds to a discharge between long electrodes, induced either by an external E.M.F. or by the usual $\underline{v} \times \underline{B}$ force generated by plasma flow. For continuous electrode walls boundary conditions (b) and (c) reduce to,

$$\phi'(x, y) = 0 \quad \text{at} \quad x = 0, d$$

This is independent of the external circuit connecting the two

electrodes, since the total perturbed current passing through the circuit is an average of $e^{-iK_y y}$ over y from $+\infty$ to $-\infty$ and is therefore zero. Boundary condition (a) does not apply since there are no insulators present.

In this case the steady state current will have a non-zero component perpendicular to the walls, and hence we have to neglect the linearized gradient terms in the electron energy equation to ensure that $\det C(Z, \lambda)$ is a quadratic. The dispersion relation then becomes,

$$\det C_{13}(Z, \lambda_1(Z)) (e^{\lambda_1(Z)d} - e^{\lambda_2(Z)d}) = 0, \text{ and we}$$

obtain the solution

$$\xi = \left[\sum_{m=4}^5 \mathcal{H}_m e^{\lambda_1(Z_m)x + Z_m t} + \sum_{n=1}^{\infty} \sum_{m=2}^3 \sum_{k=1}^2 \Gamma_{mk}^n e^{\lambda_k(Z_m^n)x + Z_m^n t} \right] e^{-ik_x x}$$

where $\mathcal{H}$ and Γ are defined as in section (4.3) with appropriate indices, and

$$U_m = \frac{\mathcal{C} D_{13}(Q_1 Z_m + Q_2)(\lambda_2(Z_m) - \lambda_1(Z_m))(-1)^m (e^{\lambda_2(Z_m)d} - e^{\lambda_1(Z_m)d})}{B_{32}(\lambda_1(Z_4) - \lambda_1(Z_5)) \det D(Z_m) (e^{\lambda_2(Z_m)d} - e^{\lambda_1(Z_m)d})}$$

$$F_{mk}^n = \frac{\mathcal{C} C_{11}(\lambda_k(Z_m^n))((-1)^{n+1} + e^{-(iK_x + \Delta\lambda(Z_m^n))d}) \mathcal{C} D_{13}(-1)^{n+m+k} i\pi n (Q_1 Z_m^n + Q_2)^2}{2 \det D(Z_m^n) \mathcal{C} C_{13}(\lambda_k(Z_m^n)) Q_1 (Q_1 \frac{\pi^2 n^2}{d^2} - Q_4)(Z_3^n - Z_2^n) d^2}$$

$$V_m = - \frac{C_{11}(Z_m, \lambda_1(Z_m))}{C_{12}(Z_m, \lambda_1(Z_m))} U_m, \quad G_{mk}^n = - \frac{C_{11}(Z_m^n, \lambda_k(Z_m^n))}{C_{12}(Z_m^n, \lambda_k(Z_m^n))} F_{mk}^n$$

$$W_m = 0, \text{ and } H_{mk}^n = \frac{cf C_{13}(Z_m^n, \lambda_k(Z_m^n))}{cf C_{11}(Z_m^n, \lambda_k(Z_m^n))} F_{mk}^n$$

$Z_{4,5}$ are of course the solutions of the quadratic $cf C_{13}(Z, \lambda, (Z)) = 0$. Although the solutions $Z_{2,3}^n$ of the dispersion relation look formally the same as for insulator walls, the different values of j_{ox} and j_{oy} for the two cases give rise to different results.

The two modes at $Z_{4,5}$ have no perturbed electric field associated with them, and, as for ψ' in the insulator wall case, ϕ' is a series of $\sin(\frac{\pi n x}{d})$ terms since $H_{m1}^n = H_{m2}^n$. The Z_4 mode is unstable, though it has a small growth rate. Fig. (4.7) shows $\mathcal{R}(Z_2^n)$ and $\mathcal{R}(Z_4)$ as a function of β_0 for $L_y = 10$ cm. These curves have been calculated using the fact that $E_{oy} = 0$ to obtain $j_{oy}/j_{ox} = \beta_0$. Thus as β_0 increases j_o tends towards the y-axis, as in the insulator wall case. Since T_{eo} is kept constant at 2500°K we are assuming that the magnitude of j_o is constant, since, having neglected the other energy losses, we have the usual unperturbed energy equation,

$$\frac{j_o^2}{\sigma_o} = 3 n_{eo} k (T_{eo} - T) \left(\nu_{co} \frac{m_e}{m_s} + \nu_{no} \frac{m_e}{m_n} \right)$$

To keep $|j_o|$ constant, while B is increased, either the flow velocity or the external circuit parameters would have to be varied. Figure (4.8) shows the variation of β_{ocrit} with $\frac{L_y}{d}$ for various modes.

The general dependence of the instability on β_0 and $\frac{L_y}{d}$ is the same as for insulator walls, except that Z_4 is unstable and the values

of β_{ocrit} are generally higher for the Z_2^n modes with continuous electrode walls.

The reason for the higher values of β_{ocrit} is that here

$$\tan \chi_{\text{E}(n)} = \frac{1 + \beta_0 n \lambda_y/d}{\beta_0 - n \lambda_y/d}$$
 and hence, for a given n , $\chi_{\text{E}(n)}$ is larger than for the insulator wall case, except when β_0 tends to infinity. In other words because of the different orientation of $\tilde{j}_0$ the direction of $\tilde{K}_m$ has moved further away from $\tilde{K}_E$ for any given n , and therefore the range of angles for instability, $\Delta\chi$, and the range of orientations of allowed modes do not overlap until a higher value of β_0 .

Figures (4.9), (4.10) and (4.11) show the form of the instability at $\beta_0 = 4.0$ and 7.0 . We have neglected the Z_4 mode since its low growth rate implies a low non-linear amplitude.

4.5 Results for Infinitely finely Segmented Electrode Walls.

This case corresponds to a long Hall or Faraday generator, and we are effectively assuming that the segmentation length is much less than λ_y . To present as simple results as possible we will assume that the electrodes are shorted externally, though they may be diagonally connected (see Figure 4.12). In this case boundary condition (c) reduces to,

$$\psi'(0,0) = \psi'(d,S)$$

$$\phi'(0,0) = \phi'(d,S)$$

Since the individual electrode and insulator lengths are assumed

to be negligibly small, boundary conditions (a) and (b) do not apply.

The dispersion relation is

$$(C_{12}(\lambda_2(Z)) C_{13}(\lambda_1(Z)) - C_{12}(\lambda_1(Z)) C_{13}(\lambda_2(Z))) \times \\ (1 - e^{\lambda_1(Z)d - iK_y S}) = 0.$$

The expression in the first set of brackets is a cubic in Z with

solutions $Z_{6,7,8}$, and from the other factor of the L.H.S. we get poles at Z_q^n which are the solutions of $\lambda_1(Z) = i \left(\frac{2\pi n + K_y S}{d} \right)$

The poles give the following solutions for ξ ,

$$\xi = \left[\sum_{m=6}^8 \sum_{k=1}^2 \textcircled{H} e^{\lambda_k(Z_m)x + Z_m t} + \sum_{n=-\infty}^{\infty} \Gamma^n e^{\lambda_1(Z_q^n)x + Z_q^n t} \right] \alpha e^{-iK_y x}$$

where $\textcircled{H}$ and Γ are defined as before with appropriate indices, and,

$$U_{mk} = \frac{(-1)^{k+1} C_{12}(\lambda_k(Z_m)) C_{23}(\lambda_k(Z_m)) [C_{33}(\lambda_1(Z_m)) C_{12}(\lambda_1(Z_m)) + C_{11}(\lambda_1(Z_m)) C_{23}(\lambda_1(Z_m)) C_{13}]}{\det D(Z_m) (\lambda_1(Z_m) - \lambda_2(Z_m)) (Z_m - Z_h) (Z_m - Z_j)}$$

$\lambda = \begin{matrix} 2, & k=1 \\ 1, & k=2 \end{matrix}$ and h, j = the combination of 6, 7, 8 not including m .

$$F^n = \frac{C_{12}(\lambda_1(Z_q^n)) C_{13}(\lambda_1(Z_q^n)) [C_{33}(\lambda_2(Z_q^n)) C_{12}(\lambda_2(Z_q^n)) + C_{23}(\lambda_2(Z_q^n)) C_{11}(\lambda_2(Z_q^n)) C_{13}]}{\det D(Z_q^n) B_{23} A_{22} (Z_q^n - Z_6) (Z_q^n - Z_7) (Z_q^n - Z_8) d} \\ \times \frac{(Q_1(Z_q^n) + Q_2)^2 (1 - e^{-iK_y S - iK_x d})}{Q_1 \left(\left(\frac{2\pi n + K_y S}{d} \right)^2 Q_1 - Q_4 \right)}$$

$$V_{mk} = - \frac{C_{11}(Z_m, \lambda_k(Z_m))}{C_{12}(Z_m, \lambda_k(Z_m))} U_{mk}, \quad G^n = - \frac{C_{11}(Z_q^n, \lambda_1(Z_q^n))}{C_{12}(Z_q^n, \lambda_1(Z_q^n))} F^n$$

$$W_{mk} = \frac{C_{13}(Z_m, \lambda_k(Z_m))}{C_{11}(Z_m, \lambda_k(Z_m))} U_{mk}$$

$$H^n = \frac{\mathcal{A} C_{13}(Z_q^n, \lambda_1(Z_q^n))}{\mathcal{A} C_{11}(Z_q^n, \lambda_1(Z_q^n))} F^n$$

To calculate the values of Z here we once again have to establish the ratio j_{oy}/j_{ox} ; the magnitude of j_o being fixed by T_{eo} . If the geometry represents a generator of length ℓ ($\gg d$) and width w , and the output current passes through a single load (R_L) connected between the ends of the duct, then

$$E_{oy} = \frac{-(j_{oy} - \frac{S}{d} j_{ox})}{\sigma_L} \quad \text{where } \sigma_L = \frac{1}{R_L w d}$$

On account of the external shorting circuits, we have, $E_{ox} d + E_{oy} S = 0$ i.e. $E_{ox} = -\frac{S}{d} E_{oy}$. Substituting these into the steady state Ohm's Law, i.e. equations (2.2) and (2.3), we obtain,

$$\frac{j_{oy}}{j_{ox}} = \frac{\beta_o + \frac{\sigma_o}{\sigma_L} \frac{S}{d}}{1 + \frac{\sigma_o}{\sigma_L}}$$

Putting $S = 0$ and maximising the power output, viz.

$\Pi_{out} = w \ell d (j_{oy} - j_{ox} S/d)^2 / \sigma_L$
with respect to σ_L , we obtain (for $\sigma_o = \text{constant}$),

$$\left(\frac{\sigma_o}{\sigma_L}\right)_{\Pi_{out} \max} = 1 + \beta_o^2, \quad \text{and} \quad \left(\frac{j_{oy}}{j_{ox}}\right)_{\Pi_{out} \max} = \frac{\beta_o}{2 + \beta_o^2}$$

Using this value for j_{oy}/j_{ox} we find that $\mathcal{R}(Z_{6,7,8})$ are all negative, while the values of $\mathcal{R}(Z_n^9)$ which are positive are drawn in Figure (4.13) as a function β_o ($L_y = 10$ cm). In Figure (4.14) β_{ocrit} is drawn as a function of L_y/d for the same values

of n , while Figures (4.15), (4.16) and (4.17) show the projected form of the instability at $\beta_0 = 3$ and 7.

The Z_n^9 modes are simple plane waves and the forms of the graphs are compatible with the infinite plasma stability condition on $\underline{j}_0$, $\underline{k}$ and $\underline{B}$. It can be seen that the successive destabilisation of modes, and the increase in stability with increasing $\mathcal{A}_y$ are features of this case also.

The results for all three types of wall presented here have established that the boundary conditions give an infinite set of modes, which are either plane waves or nearly plane waves. The effective wave vectors of the modes are all contained within a finite range of angles, $\Delta \theta$ say. We have seen that if $\Delta \theta$ and $\Delta \chi$, the range of angles for instability from an infinite plasma theory, overlap, then the plasma is unstable, with the number of growing modes depending on the amount of overlap.

Can we arrange that $\Delta \theta$ and $\Delta \chi$ do not overlap for arbitrary β_0 ? In the segmented electrode case we have a circuit parameter, viz. S , with which to attempt this. Here the modes are pure plane waves and the wave vector is given by,

$$\begin{aligned} K(n) &= -j(\lambda_1(Z_n^9)) \hat{x} + k_y \hat{y} \\ &= -\left(\frac{2\pi n + k_y S}{a}\right) \hat{x} + k_y \hat{y}, \quad n = -\infty, \dots, \infty \end{aligned}$$

if $\frac{K_y S}{2\pi} = N \pm \delta$, where N is the nearest integer to $\frac{K_y S}{2\pi}$, then, defining θ to be the angle $\tilde{K}(n)$ makes with the x-axis, $\Delta\theta$ is the range from $\tan^{-1}\left(\frac{\pm dK_y}{2\pi\delta}\right)$ through $\theta = 0$ to $\tan^{-1}\left(\frac{\pm dK_y}{2\pi(\delta-1)}\right)$

Thus $\Delta\theta$ is grouped around the x-axis, and to keep $\Delta\chi$ and $\Delta\theta$ as far apart as possible, for large values of β_o , we require K_m to lie along the y-axis. This requires $\frac{j_{oy}}{j_{ox}} = 1$, i.e. $\frac{\sigma_e}{\sigma_L} = \frac{\beta_o - 1}{1 - S/d}$; in order to keep σ_L positive we consider only negative values of S here. For K_m along the y-axis, $\Delta\chi$ and $\Delta\theta$ have least overlap, at large values of β_o , when δ takes its maximum value, i.e. $\delta = \frac{1}{2}$. Then, for $N = 0$ and negative S , we have $\frac{S}{\Lambda_y} = -\frac{1}{2}$.

Setting $\frac{j_{oy}}{j_{ox}} = 1$ we have plotted in Figure (4.18) the value of β_{ocrit} for the first mode to destabilize at each point as a function of S/Λ_y , using various values of Λ_y/d . Since the maximum value of $\Delta\chi$ is $\pi/2$, $\Delta\chi$ and $\Delta\theta$ never overlap when $\Lambda_y/d = 2$ and $\delta = S/\Lambda_y = -\frac{1}{2}$, for in this case $\Delta\theta$ is symmetrical about x and equal to $\pi/2$. Hence β_{ocrit} goes to infinity.

Experimentally Λ_y/d is usually observed to lie between 1 and 2, and for $\Lambda_y/d < 2$ β_{ocrit} is always finite, although a pronounced peak in β_{ocrit} is still present. The position of this peak moves further away from $\frac{S}{\Lambda_y} = -\frac{1}{2}$ as the peak value of β_{ocrit} decreases. This is because χ_m is really given by

$$\chi_m = \frac{\pi}{4} + (1 - r(1-u))/2\beta_o$$

or
$$\frac{5\pi}{4} + (1 - r(1-u))/2\beta_o$$

and hence, for low values of β_0 , the direction for maximum growth is not quite along the y-axis when $j_{0y}/j_{0x} = 1$. Consequently there is minimum overlap between $\Delta \theta$ and $\Delta \chi$ when $\Delta \theta$ is not quite symmetrical about $\theta = 0$, i.e. $S/\mathcal{L}_y$ is not quite $-\frac{1}{2}$, since minimum overlap occurs when $\Delta \theta$ is symmetrical about a line perpendicular to the direction for maximum growth.

We see therefore that, by a suitable choice of the parameters R_L and S , a significant stabilizing effect can be achieved for the case of shorted segmented electrodes. S must be taken near to $-\mathcal{L}_y/2$, and R_L must then be chosen such that K_m is near to the y-axis. This means, however, that the value of $\tilde{\sigma}_e/\sigma_L$ is not equal to $(\sigma_e/\sigma_L)_{max}$. If the object of stabilizing the waves is to increase the output power of the generator, a compromise between stabilization and load optimization may be necessary. It also must be noted that this technique will apply to only one value of $\mathcal{L}_y$; however, in practice only one dominant value of $\mathcal{L}_y$ seems to appear (1.21), (1.25), (1.26).

4.6 Summary.

The linear analysis of the electrothermal instability presented in this chapter shows that the boundary walls of a finite plasma can have a considerable effect on the instability. In general a plane wave perturbation in the electron density is split into an infinite number of modes. These modes are plane waves in the case of externally shorted, finely segmented electrodes, and approximate plane waves in the cases of

continuous insulator and continuous electrode walls. Usually only a finite number of the modes are unstable and this number increases with increasing Hall parameter. The modes have different orientations with respect to the walls, and the successive mode destabilisation with increasing Hall parameter provides a plausible explanation for the experimentally observed transition from near plane wave to turbulent structure.

Usually the effect of boundaries is to stabilize perturbations of long wavelength, so that, if the effects of energy transfer, i.e. radiation and conduction, are taken into account, the critical Hall parameter has a minimum as a function of wavelength. This is not the case however, for the segmented electrode geometry with $S = 0$ and external shortings, where β_{ocrit} for $n = 0$ does not increase with increasing $\mathcal{L}_y$.

Once again, as in Chapter 2, this instability of a uniform, i.e. infinite $\mathcal{L}_y$, perturbation is due to the persistence of the consequences of the periodicity. Another effect that is significant for uniform perturbations is the change in the electric field due to the perturbed current in the external circuitry, i.e. boundary condition (c). This does not appear in this chapter since for the cases of continuous insulators and continuous electrodes the total perturbed current passing through the external circuitry is zero, and in the case of segmented electrodes the external impedance between the segmented electrodes^{is} considered to be zero. However, the stability of

a uniform perturbation and the change in load voltage are important in the non-linear theory of electrothermal waves and will therefore be discussed in the next chapter.

To return to the linear boundary analysis, the stability of the individual modes can usually be explained in terms of an effective wave vector, $\underline{K_E}$, which is the wave vector of the plane wave to which the mode approximates. For the mode to be unstable, $\underline{K_E}$ must lie within the range of angles for instability, $\Delta\chi$, derived from the infinite plasma theory.

In general the $\underline{K_E}$'s of all the modes occupy a range of angles $\Delta\theta$, and the possibility arises of arranging the boundary conditions and external circuitry such that $\Delta\theta$ and $\Delta\chi$ do not overlap till as high a value of Hall parameter as possible. By varying the diagonal connections and the external load of the segmented electrode system considered in section (4.5), it was shown that enhanced stability could be achieved in this case.

Whether or not this could be achieved in a real system depends on whether or not the boundary conditions dominate over the effects neglected in this analysis. In particular if the steady state is non-uniform (for example there may exist a layer of hot electrons along the walls ^(4.1)) then new or modified constraints may be imposed on the instability. However the stabilization effect is quite marked for wavelengths equal to or greater than 1.25 times the wall separation, and it seems likely that some stability enhancement may be possible if the plasma steady state is not extremely non-uniform.

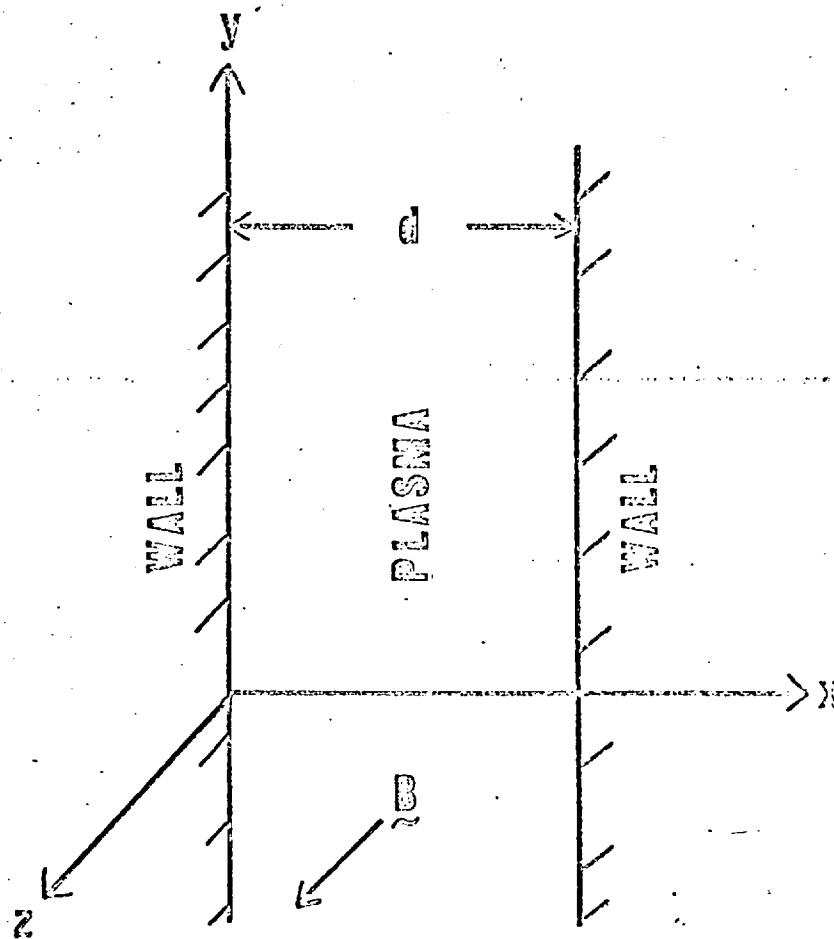


Fig.(4.1) Coordinate system and geometry.

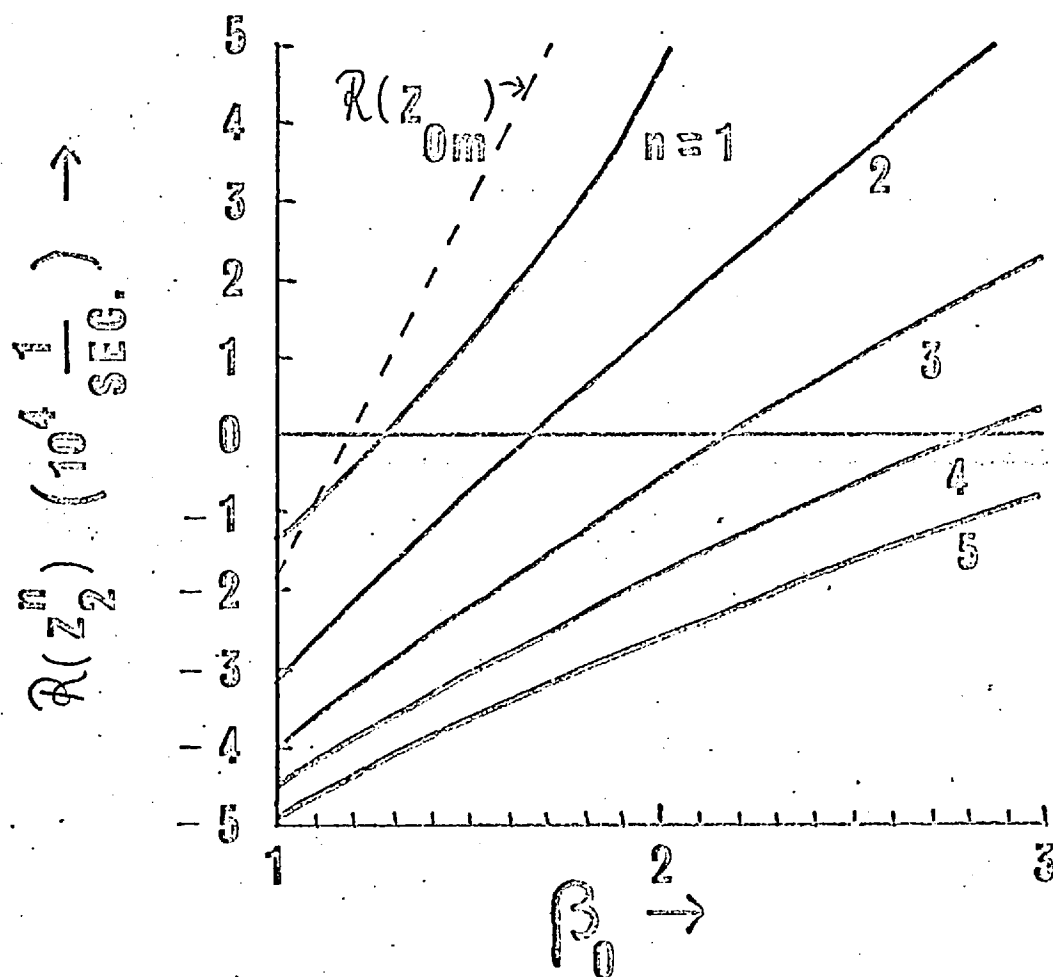


Fig.(4.2) $R(z_2^n)$ versus Hall parameter (continuous insulator walls).

a)

INSULATOR

← y

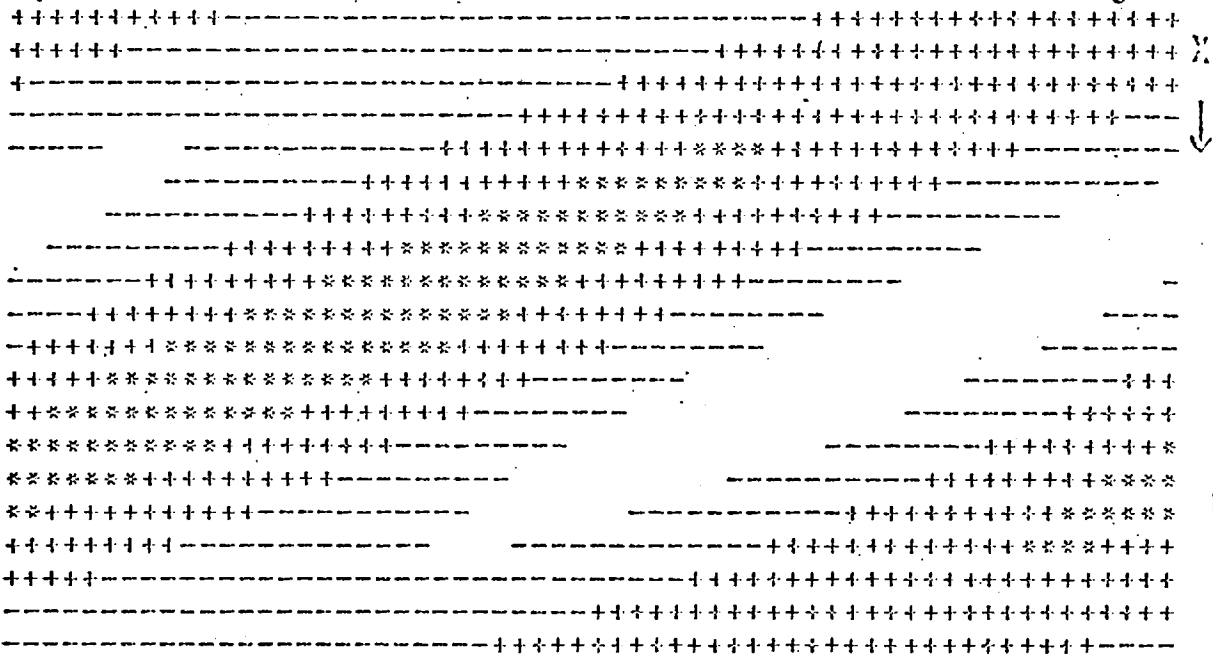


Fig (4.3)

INSULATOR

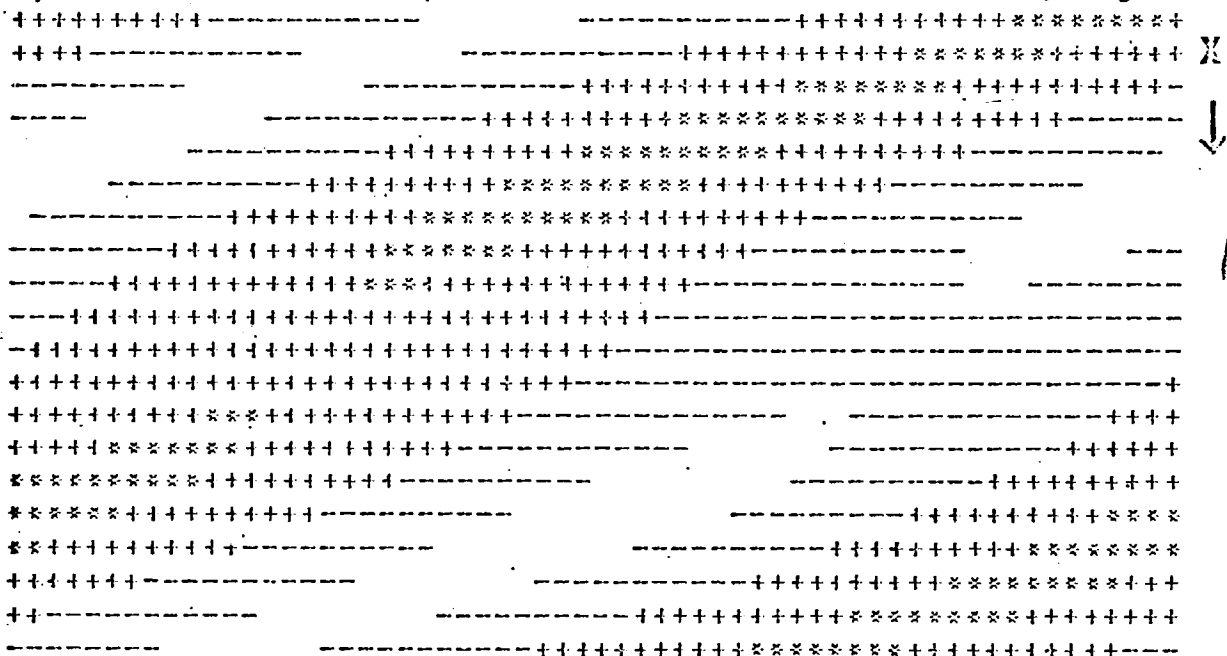
⊗ B

← j_0

Contour maps of n_e^* in the continuous insulator wall case. a) $\beta_0 = 1.5$,b) $\beta_0 = 2.2$. Asterisk:- $n_e^*/n_{e\max}^* > 0.9$, plus:- $0 < n_e^*/n_{e\max}^* < 0.9$,minus:- $-0.9 < n_e^*/n_{e\max}^* < 0$, blank:- $n_e^*/n_{e\max}^* < -0.9$ (for both (a) and(b), $\lambda_y = 10\text{cm.}$, $d = 5\text{cm.}$

b)

← y



← j_0

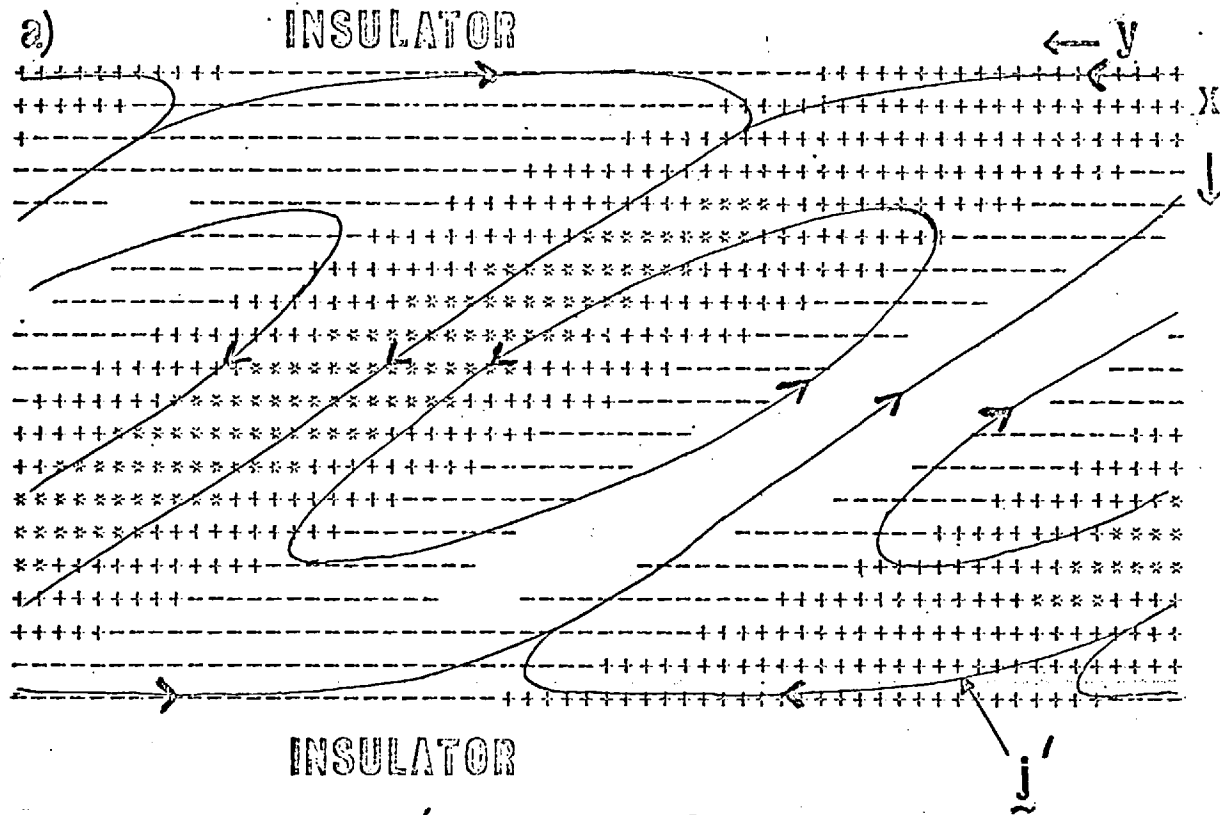
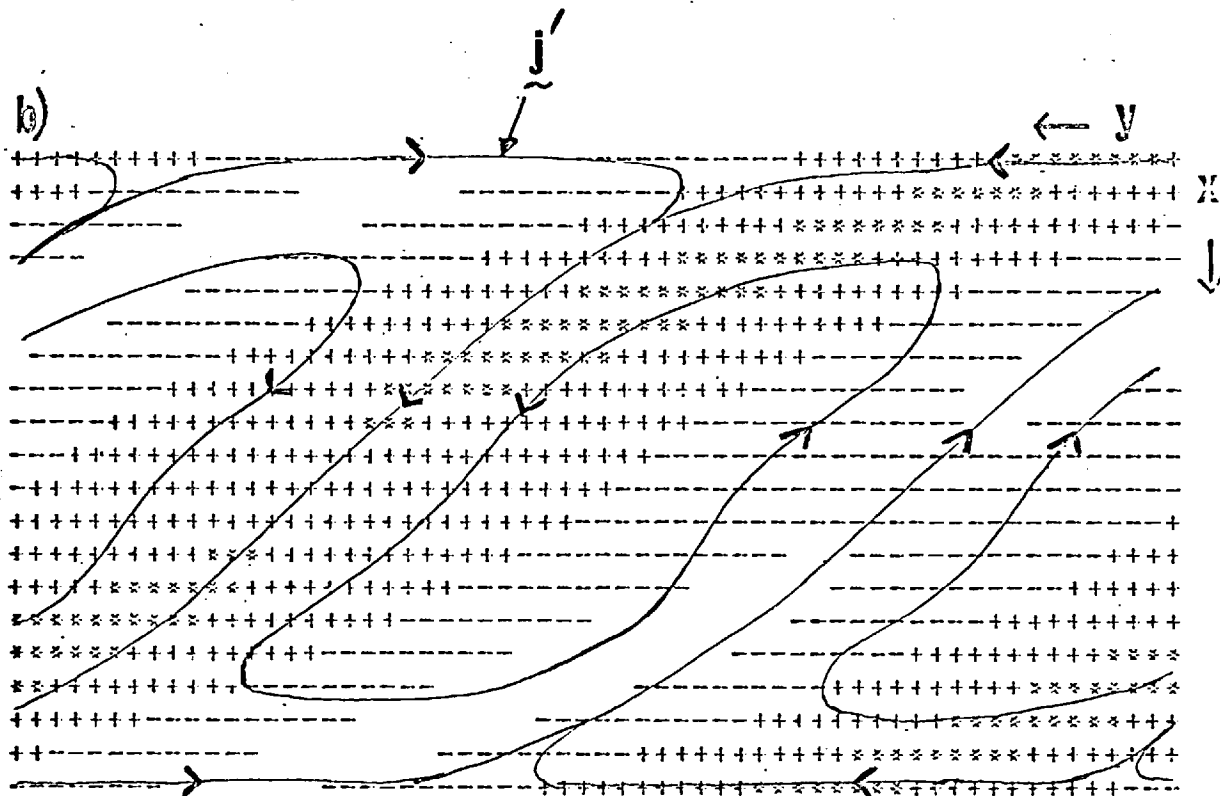


Fig.(4.4) Contours of ψ' (i.e. streamlines of $\vec{j}'$) in the continuous insulator case. a) $\beta_0 = 1.5$, b) $\beta_0 = 2.2$.



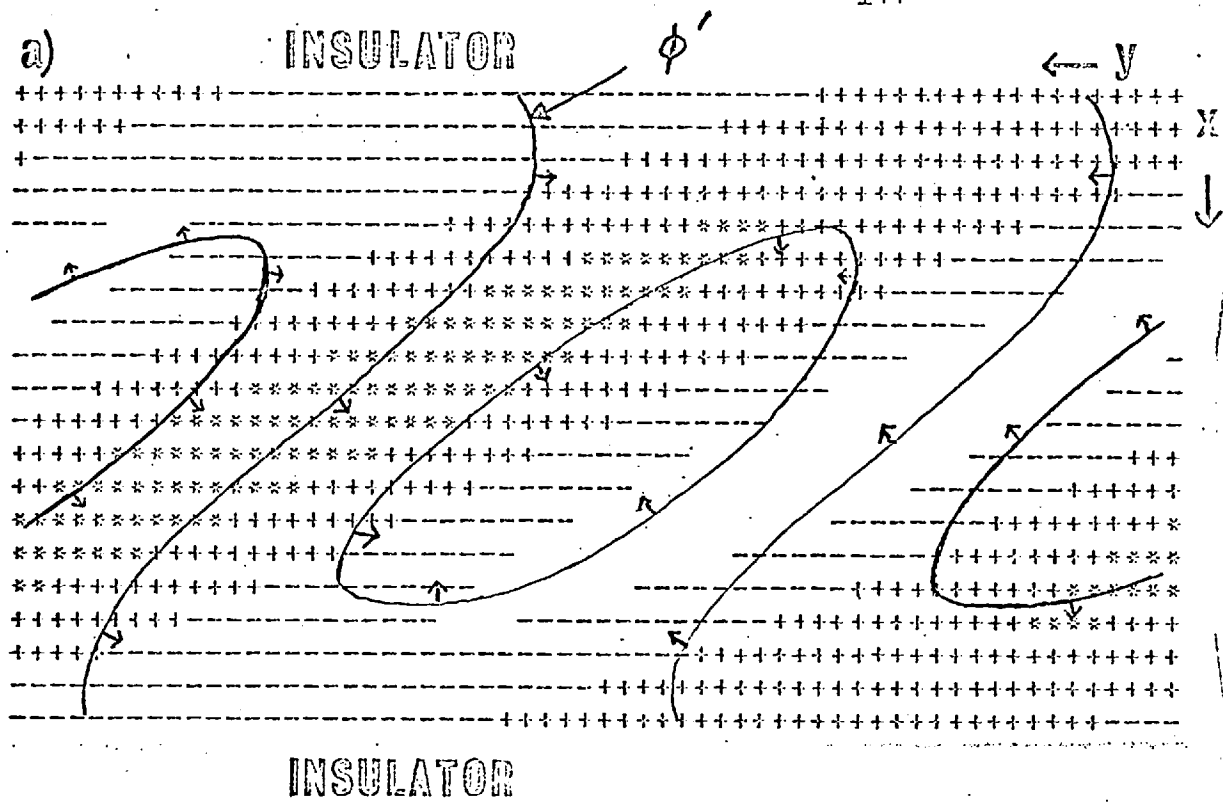
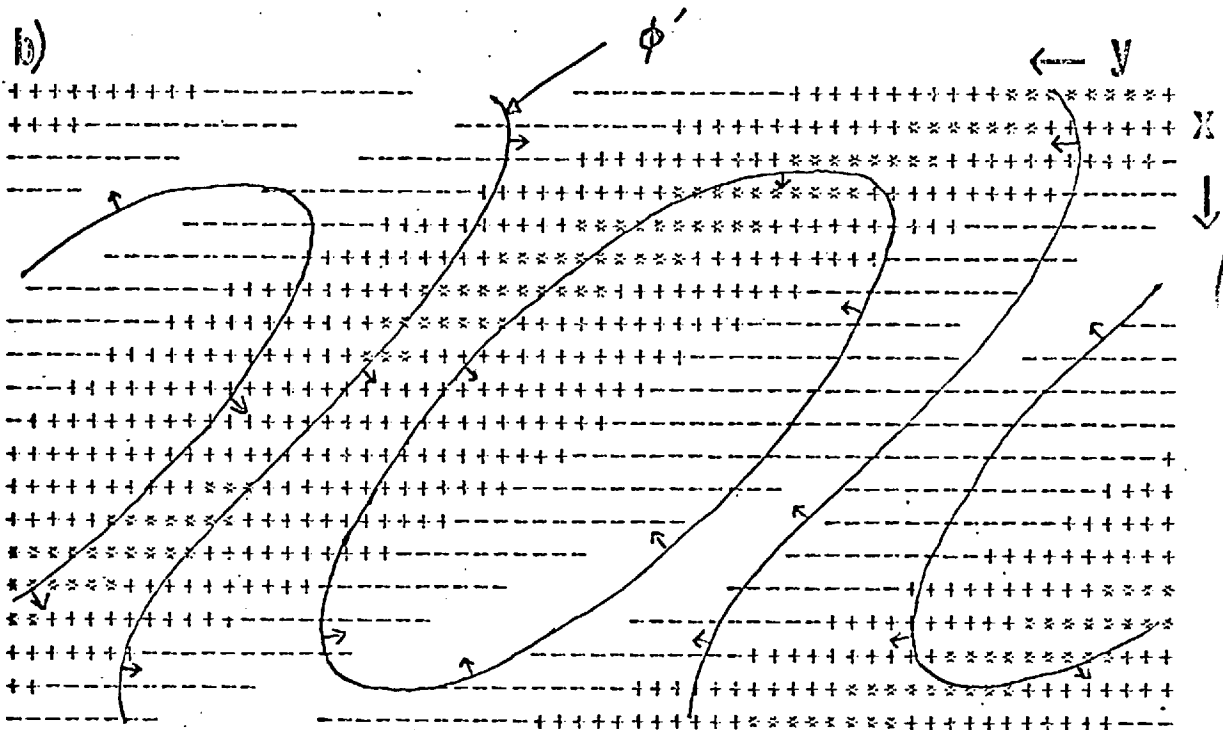


Fig.(4.5) Contours of ϕ' (arrows show direction of $\underline{E}'$) in the continuous insulator case. a) $\beta_0 = 1.5$, b) $\beta_0 = 2.2$.



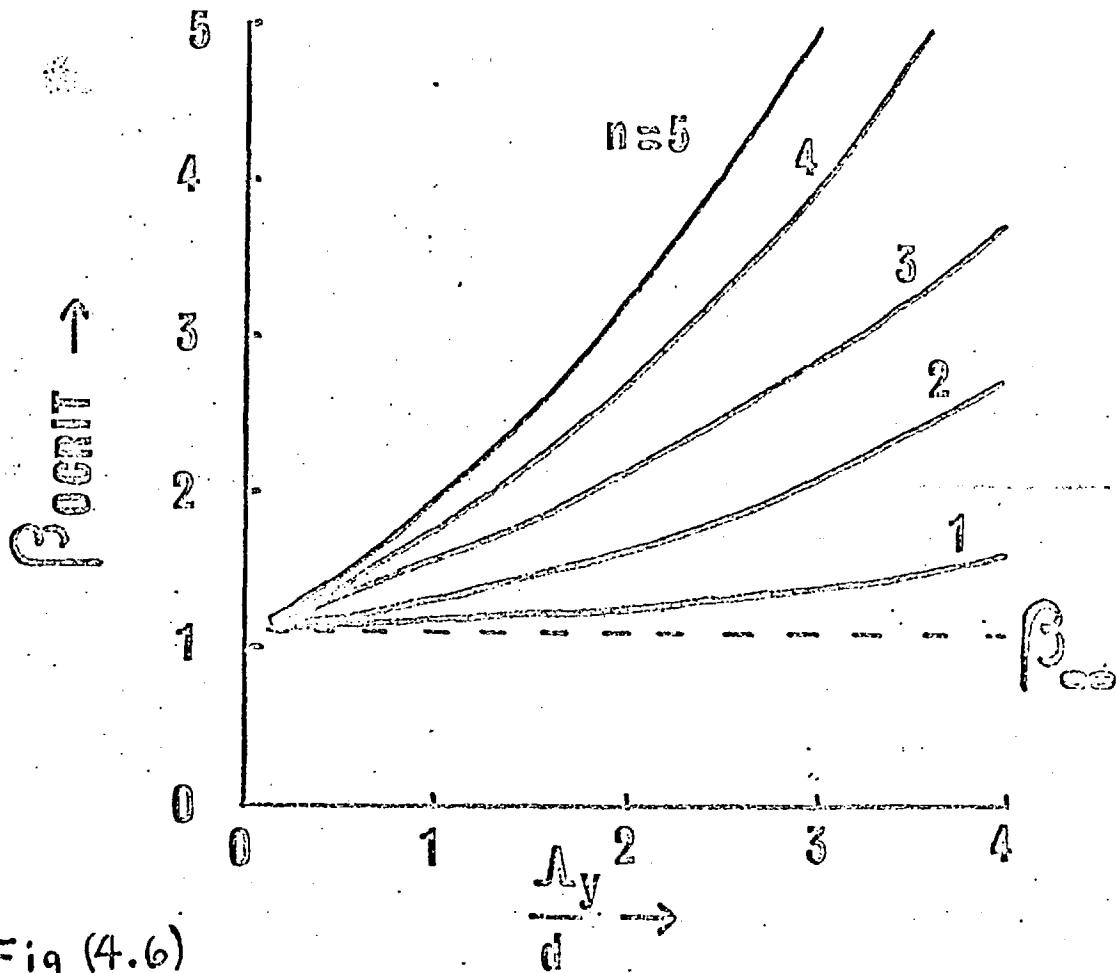


Fig (4.6)

Critical Hall parameter for various modes versus λ_y/d (continuous insulator walls). β_{∞} is the critical Hall parameter from the infinite plasma theory.

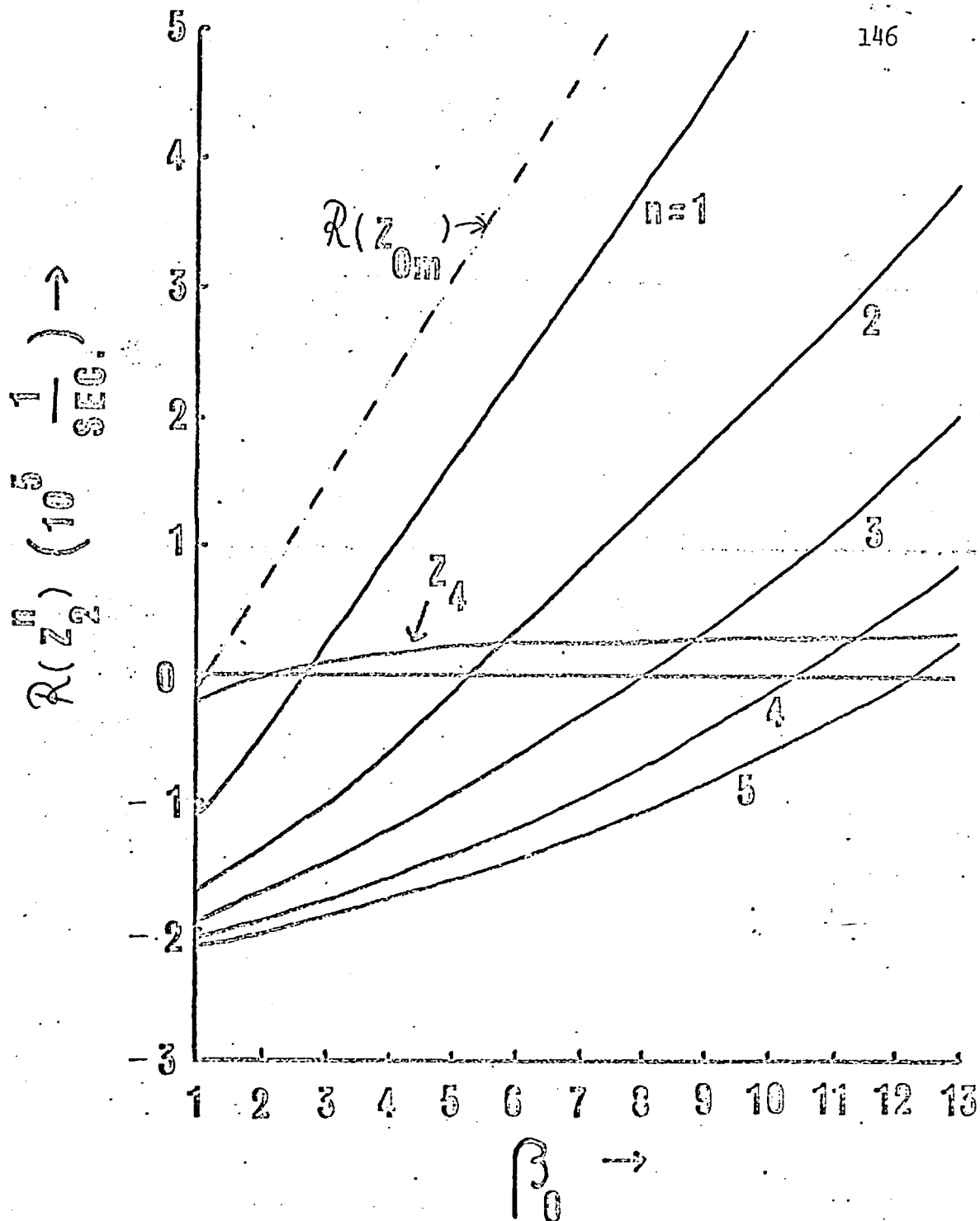


Fig.(4.7) $R(z_2^n)$ and $R(z_4)$ versus Hall parameter (continuous electrode walls).

↑
 β_{OCRIT}

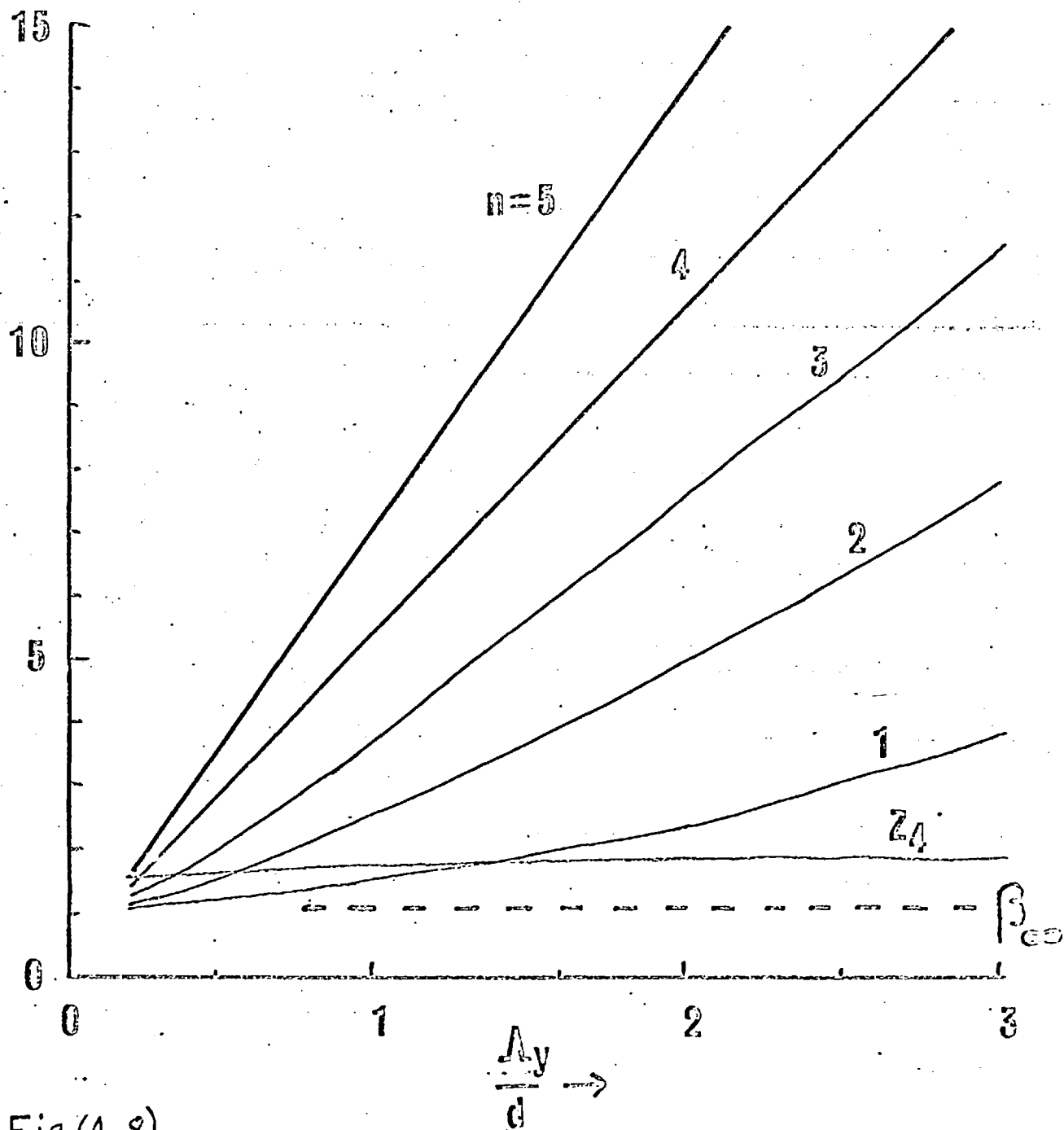


Fig (4.8)

Critical Hall parameter for various modes versus $\frac{\lambda_y}{d}$ (continuous electrode walls)

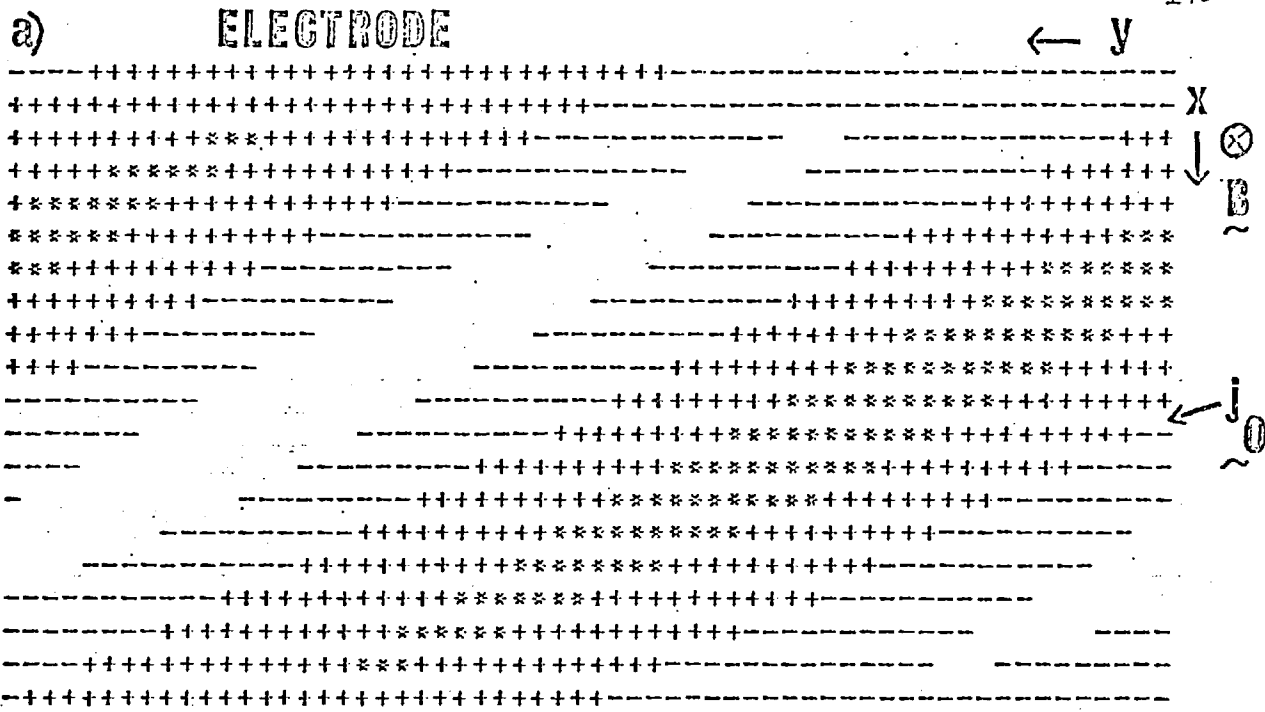
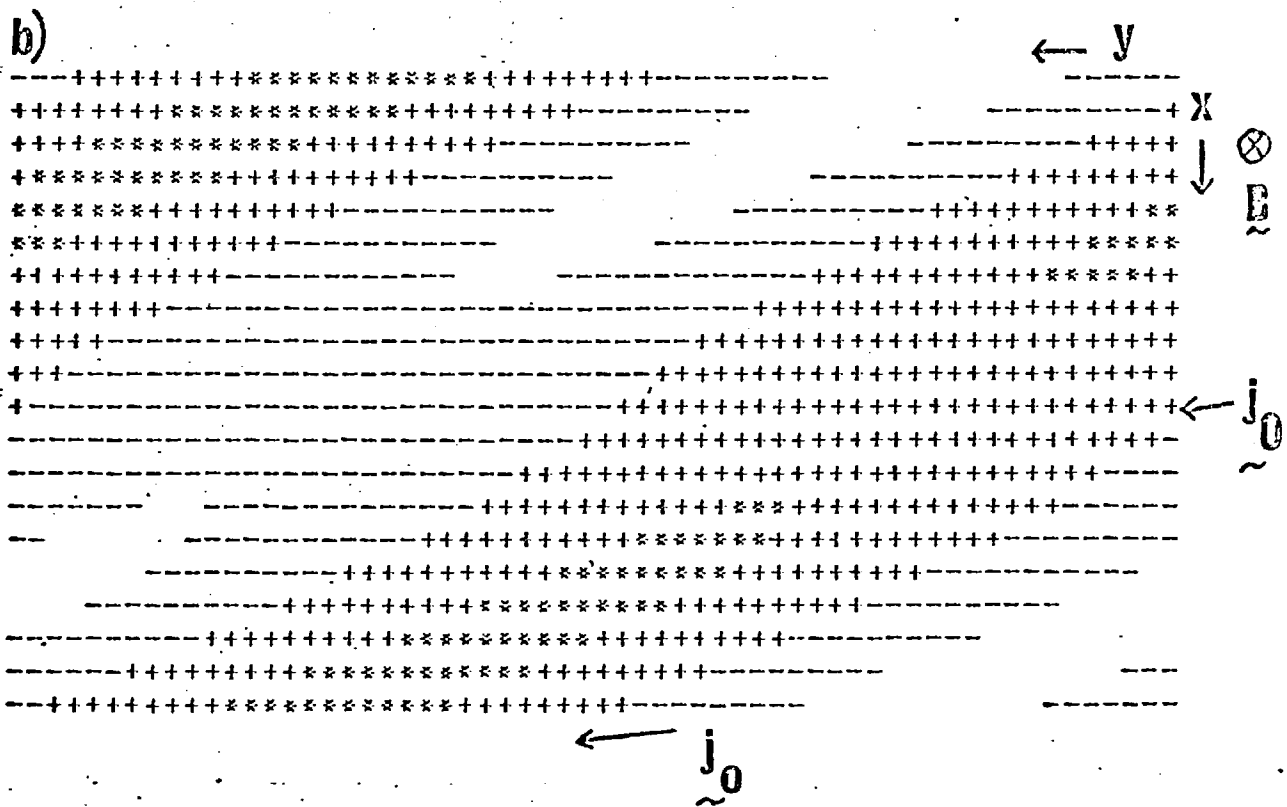
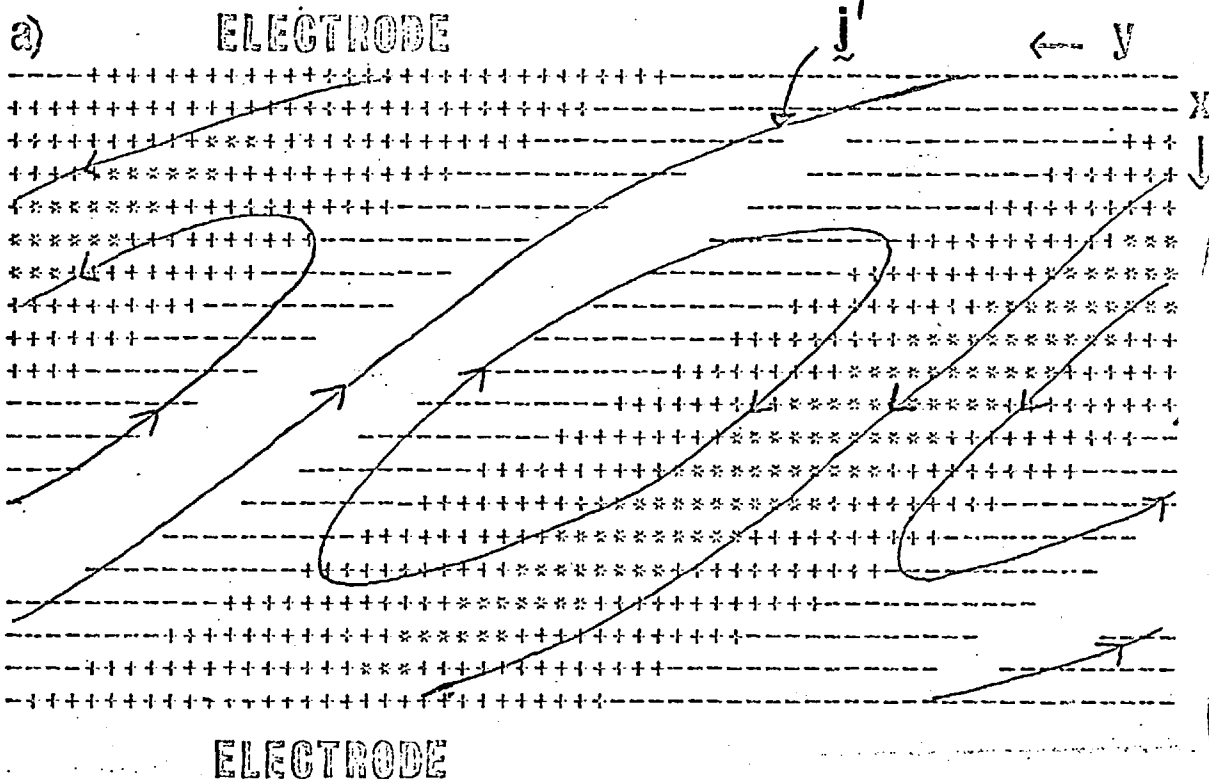


Fig.(4.9) ELECTRODE $\leftarrow j_0 \quad \otimes B$

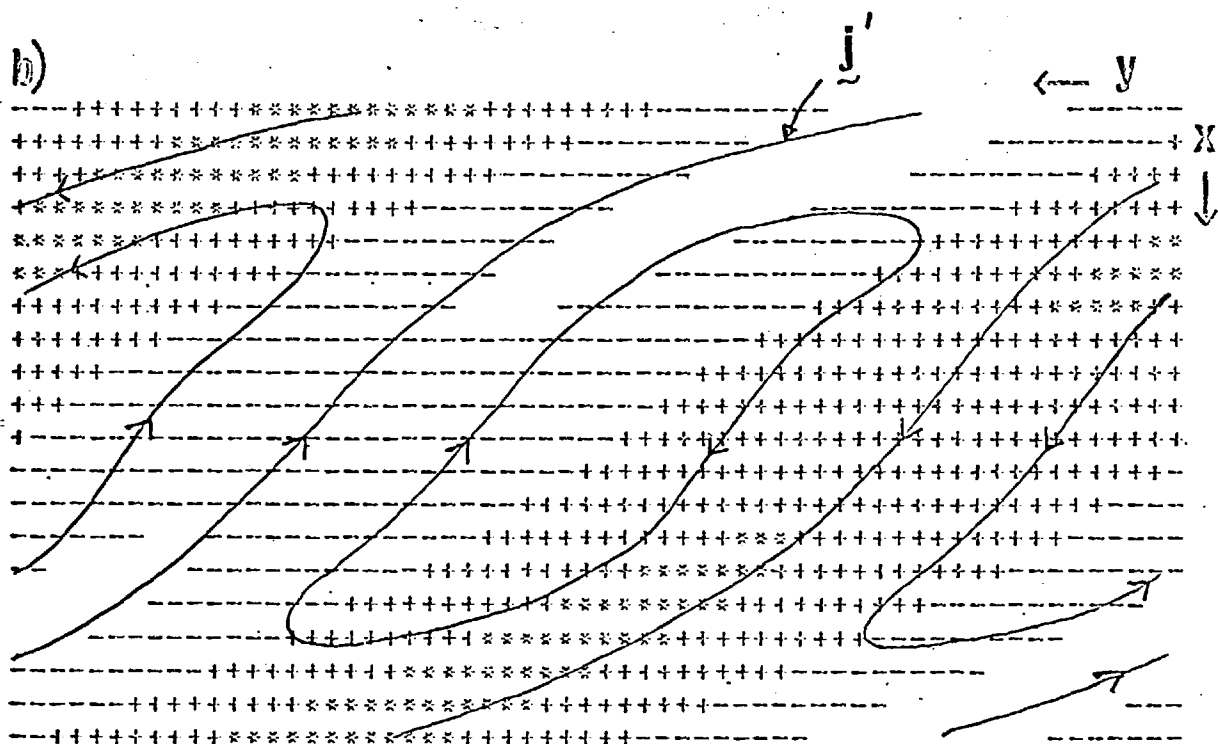
Contour maps of n_e^* in the continuous electrode wall case. a) $\beta_0 = 4$, b) $\beta_0 = 7$.

Asterisk: - $n_e^*/n_{e\max}^* > 0.9$, plus: - $0 < n_e^*/n_{e\max}^* < 0.9$, minus: - $-0.9 < n_e^*/n_{e\max}^* < 0$, blank: - $n_e^*/n_{e\max}^* < -0.9$ (for both (a) and (b), $L_y = 10$ cm., $d = 5$ cm).





Fig(4,10) Countours of ψ' (i.e. streamlines of $\underline{j}'$) in the continuous electrode case. a) $\beta_0 = 4$, b) $\beta_0 = 7$.



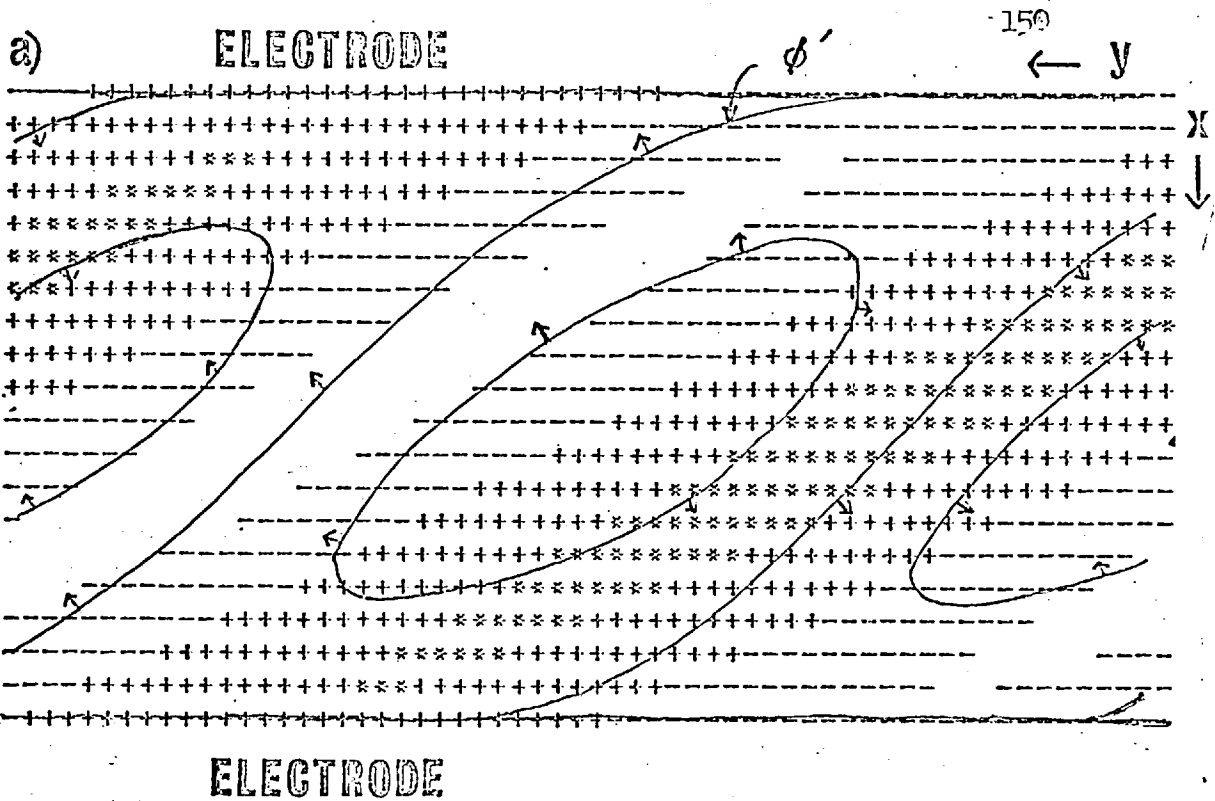
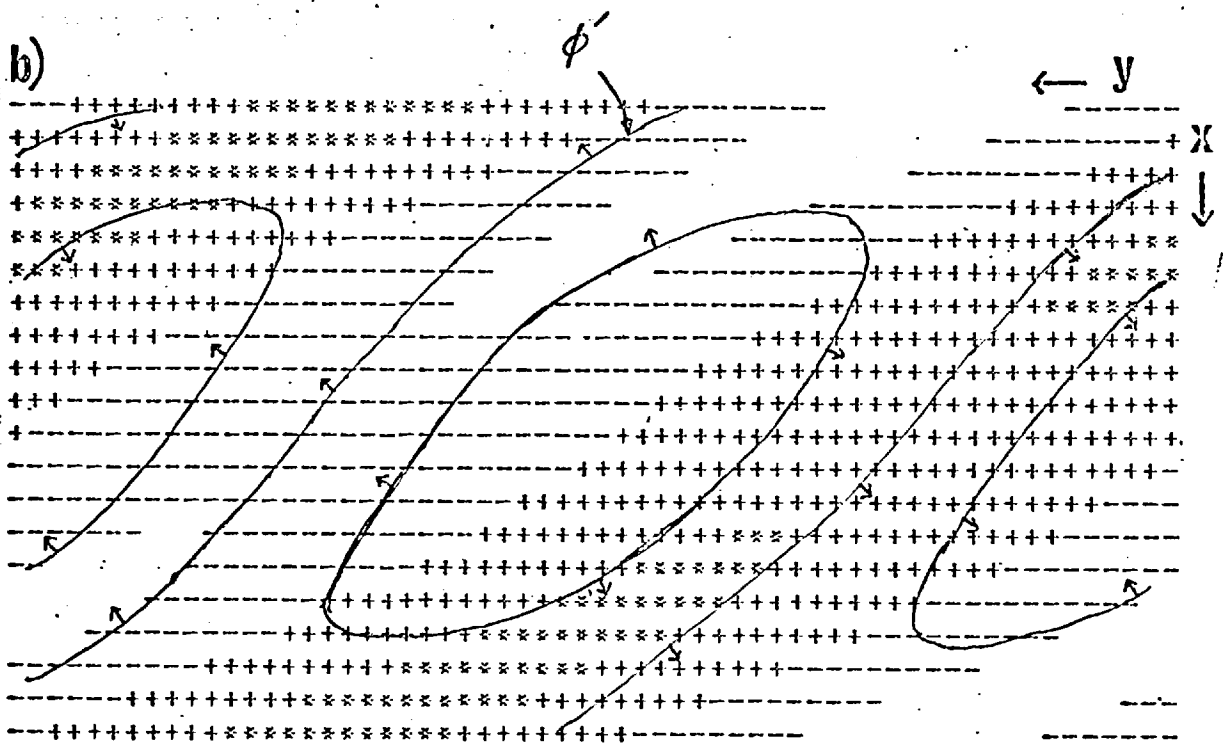


Fig.(4.11) Contours of ϕ' (arrows show direction of $\underline{E}'$) in the continuous electrode case. a) $\beta_0 = 4$, b) $\beta_0 = 7$.



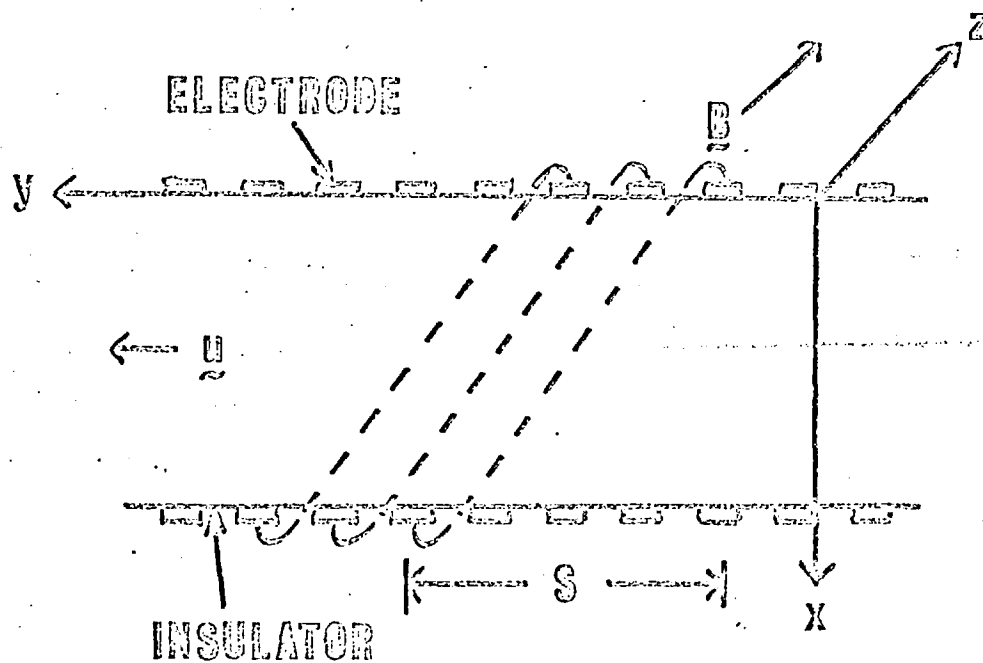


Fig.(4.12) External circuitry for segmented electrode case.

$$R(z_9^n) \left(10^5 \frac{1}{\text{SEC.}} \right)$$

152

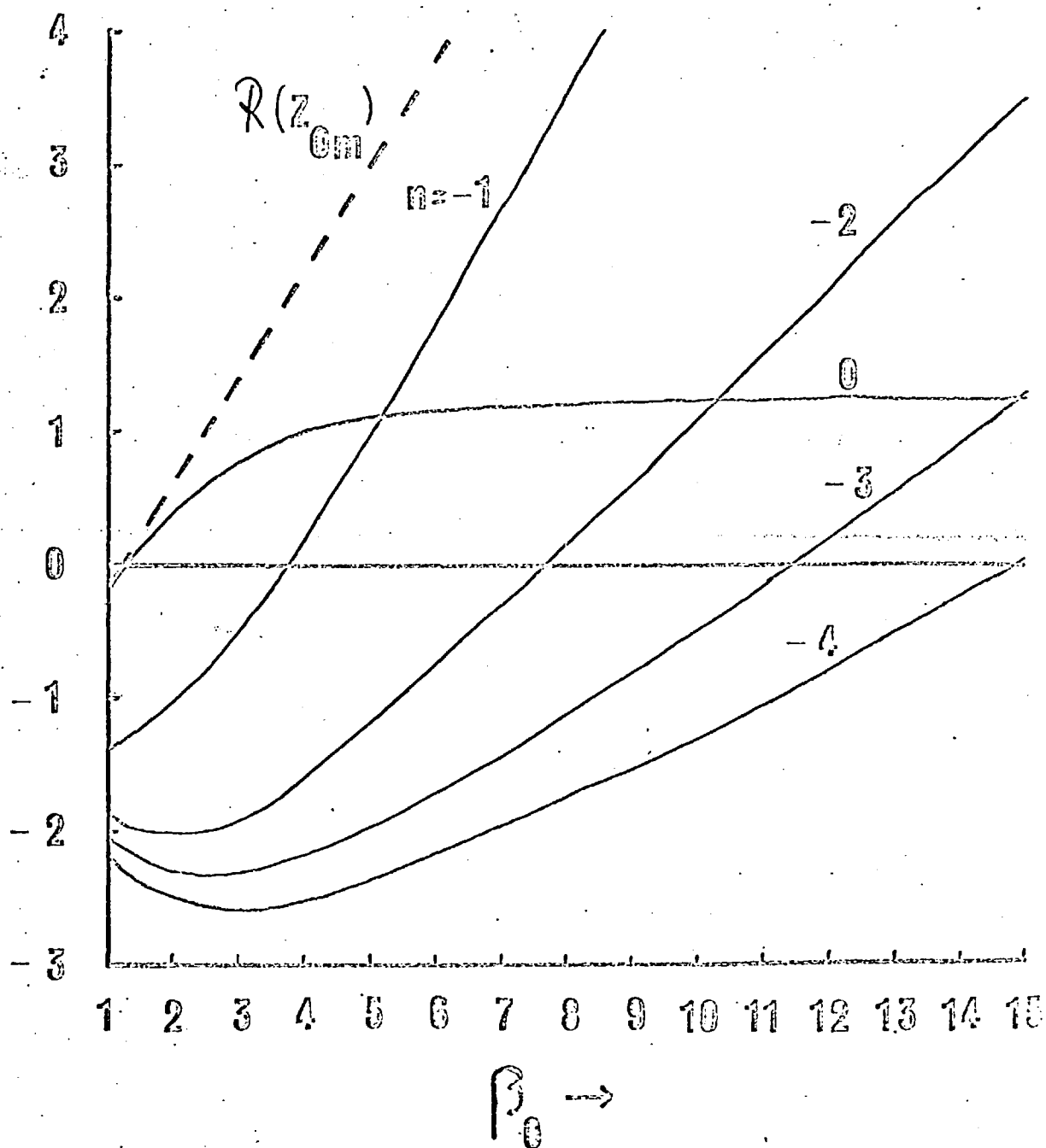


Fig.(4.13) $R(z_9^n)$ versus Hall parameter (segmented electrode walls).

β_{OCRIT}

153

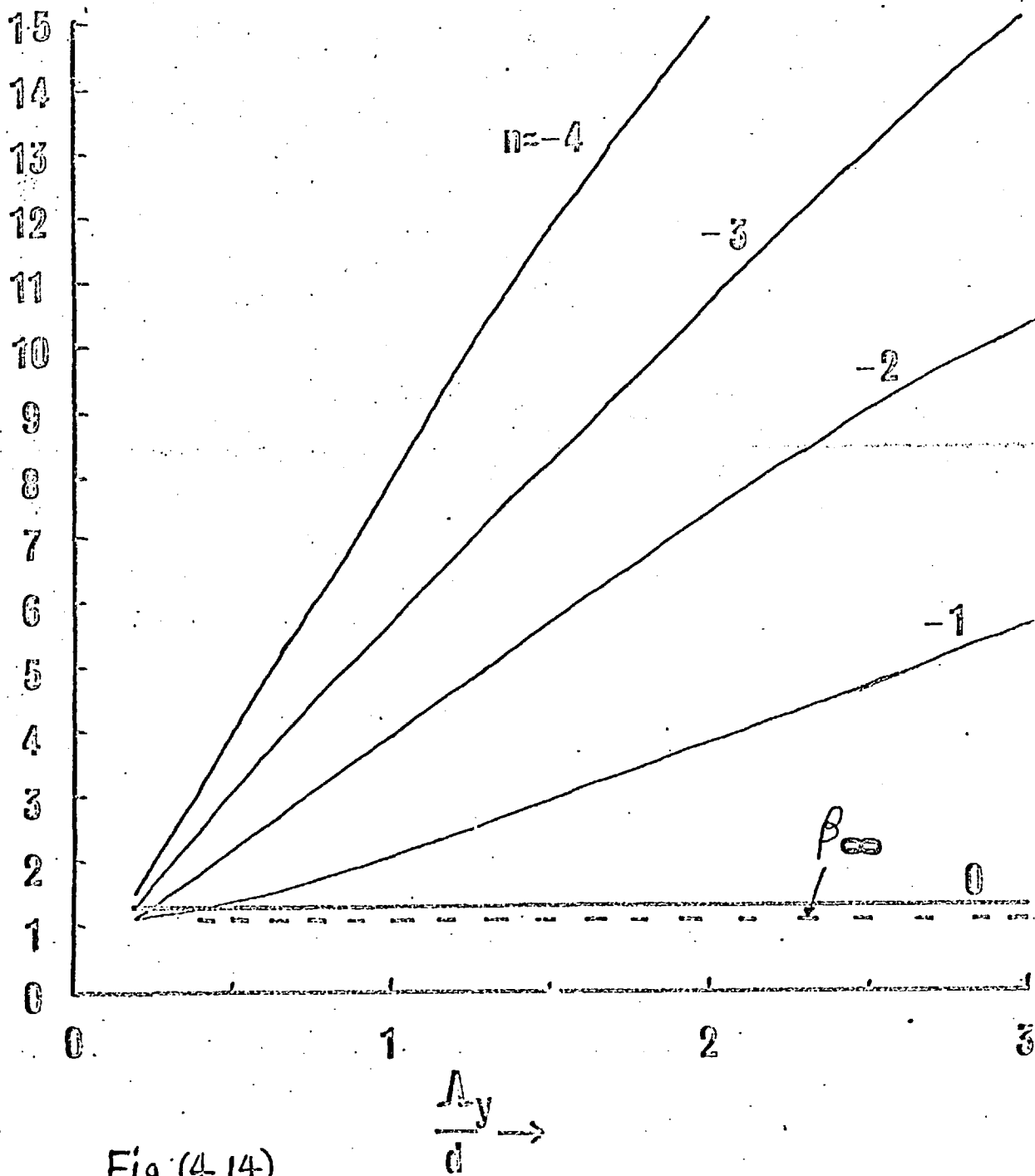
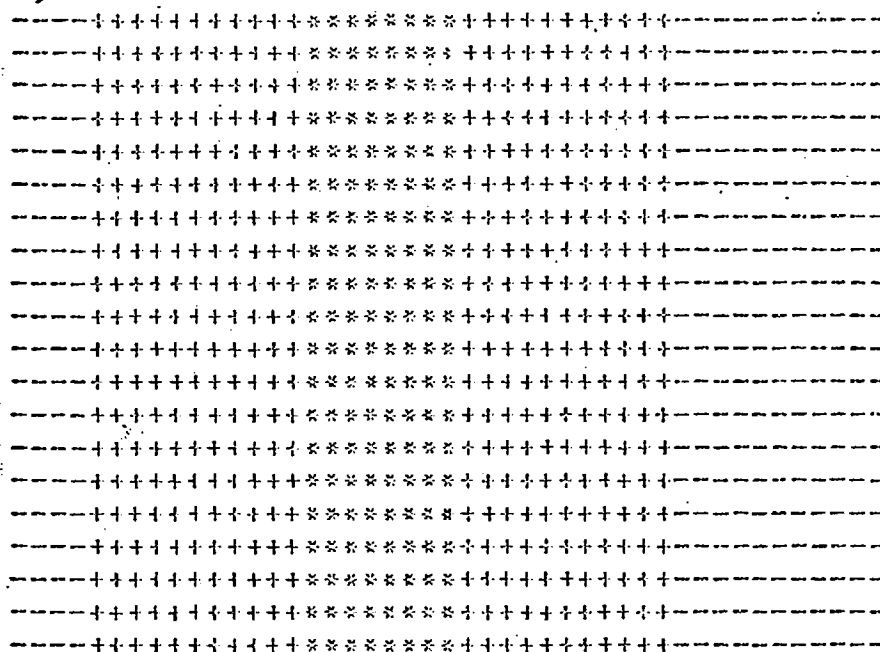


Fig.(4.14)

Critical Hall parameter for various modes versus $\frac{\lambda_y}{d}$ (segmented electrode walls).

a) SEGMENTED ELECTRODE



← y
X
↓

SEGMENTED ELECTRODE

⊗ B

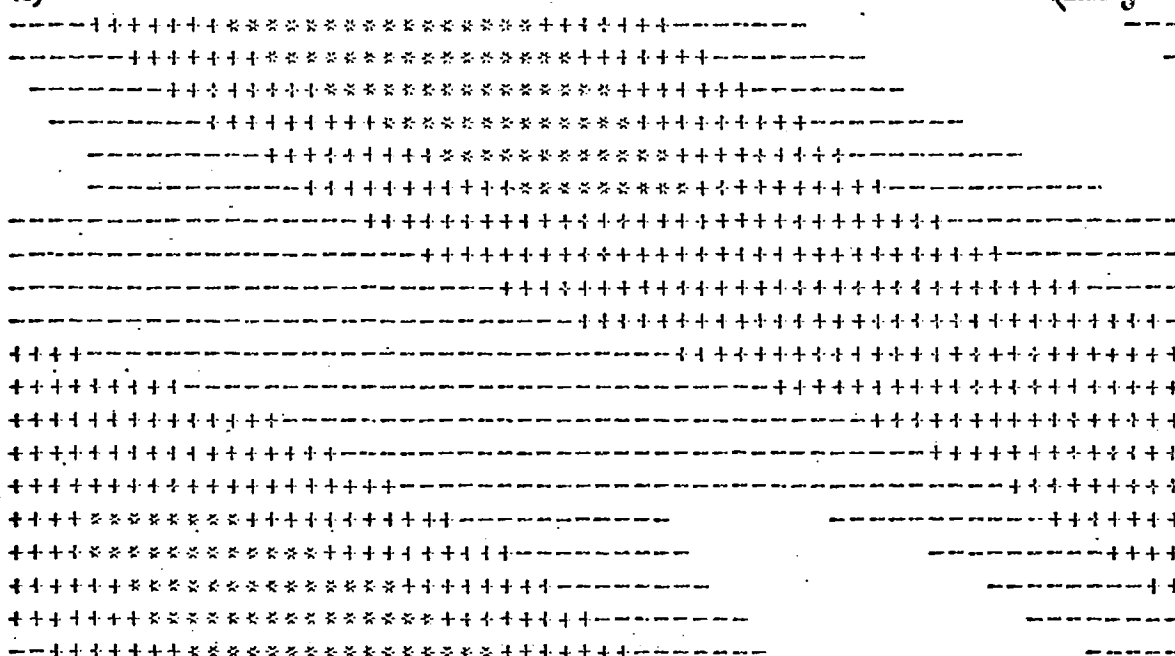
↓ j₀

Fig. (4.15)

Contour maps of n_e^* in the segmented electrode case. a) $\beta_0 = 3$, b) $\beta_0 = 7$.

Asterisk: - $n_e^*/n_{e\max}^* > 0.9$, plus: - $0 < n_e^*/n_{e\max}^* < 0.9$, minus: - $0 < n_e^*/n_{e\max}^* < -0.9$, blank: - $n_e^*/n_{e\max}^* < -0.9$. (for both (a) and (b), $L_y = 10$ cm., $d = 5$ cm).

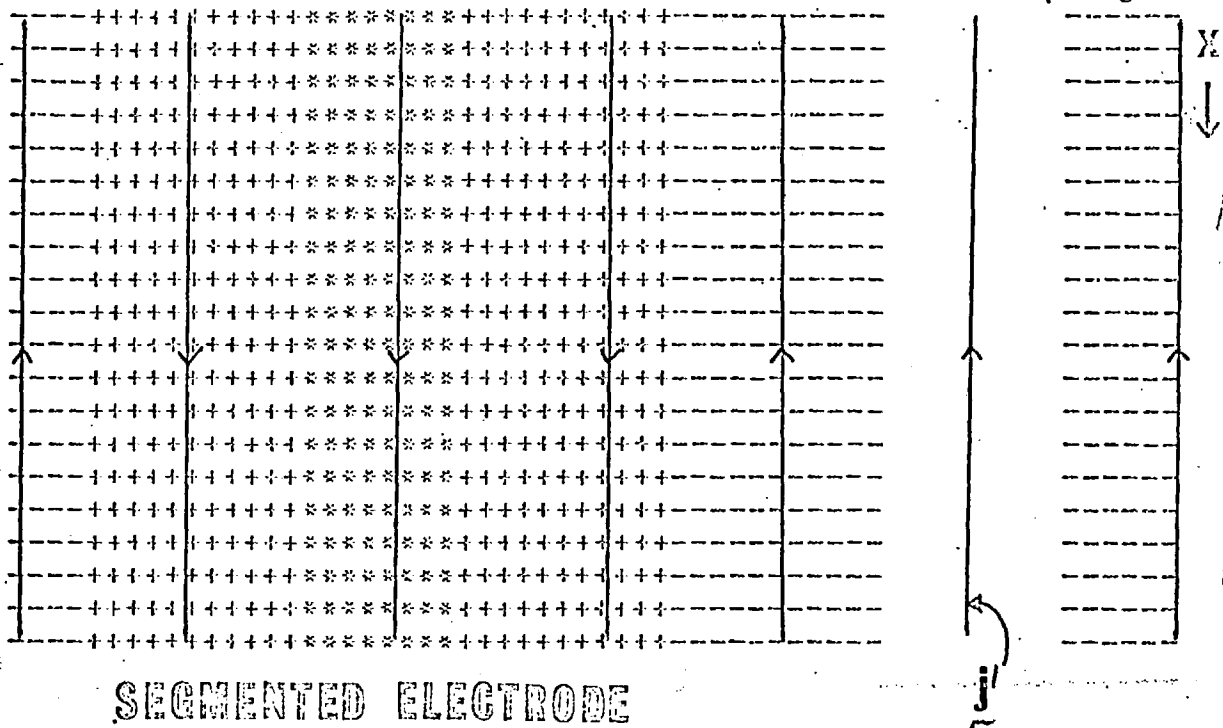
b)



← y
X
↓

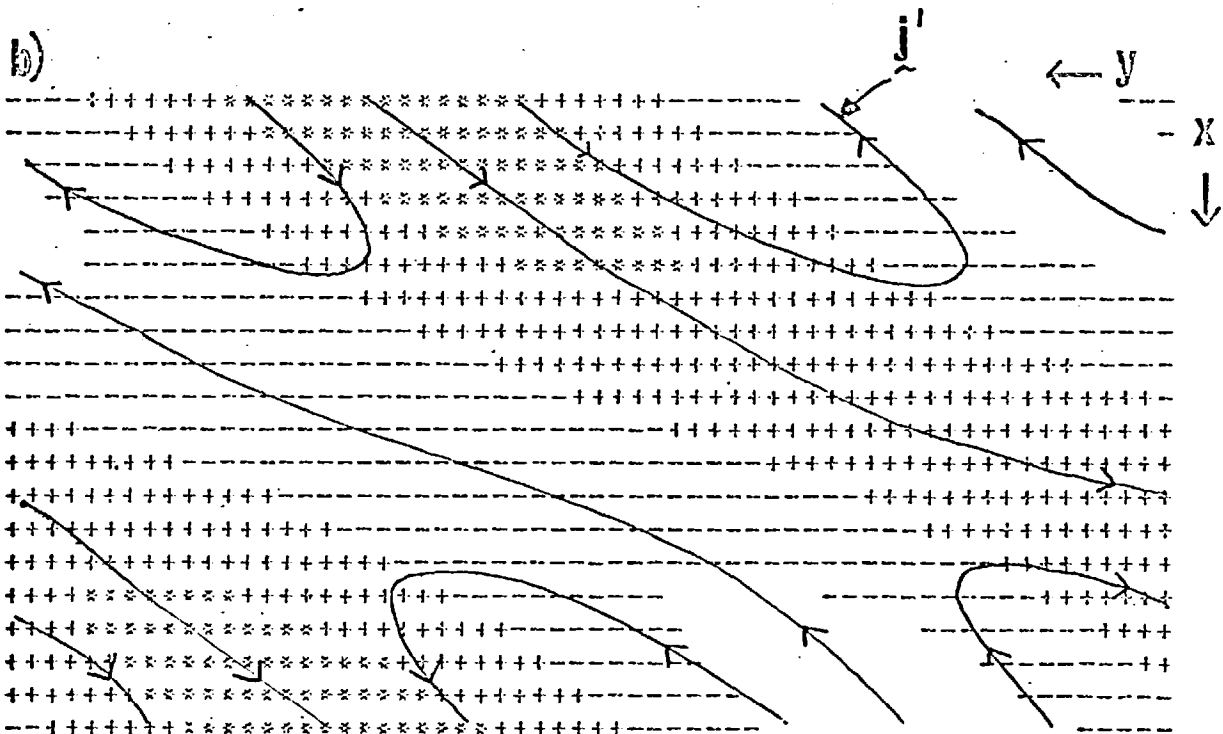
↓ j₀

a) SEGMENTED ELECTRODE



SEGMENTED ELECTRODE

Fig. (4.16) Contours of ψ' (i.e. streamlines of j') in the segmented electrode case. a) $\beta_0 = 3$, b) $\beta_0 = 7$.

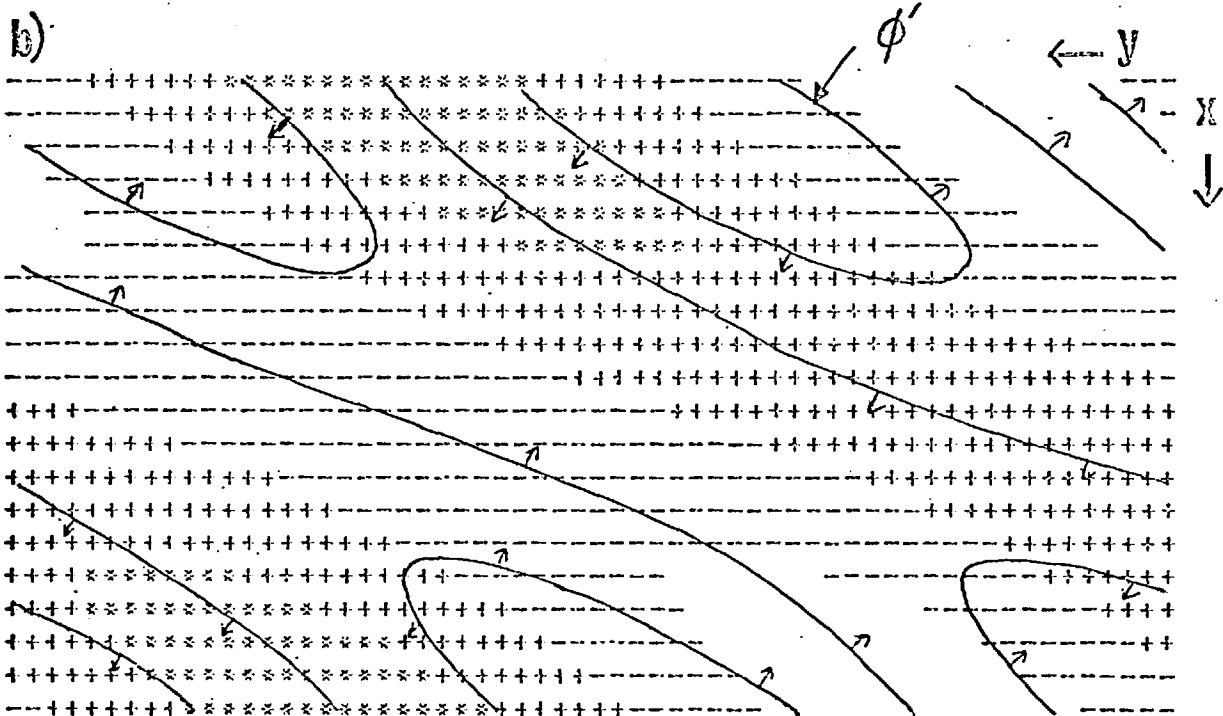


a) SEGMENTED ELECTRODE



SEGMENTED ELECTRODE

Fig.(4.17) Contours of ϕ' (arrows show direction of $\underline{E}'$) in the segmented electrode case. a) $\beta_o = 3$, b) $\beta_o = 7$.



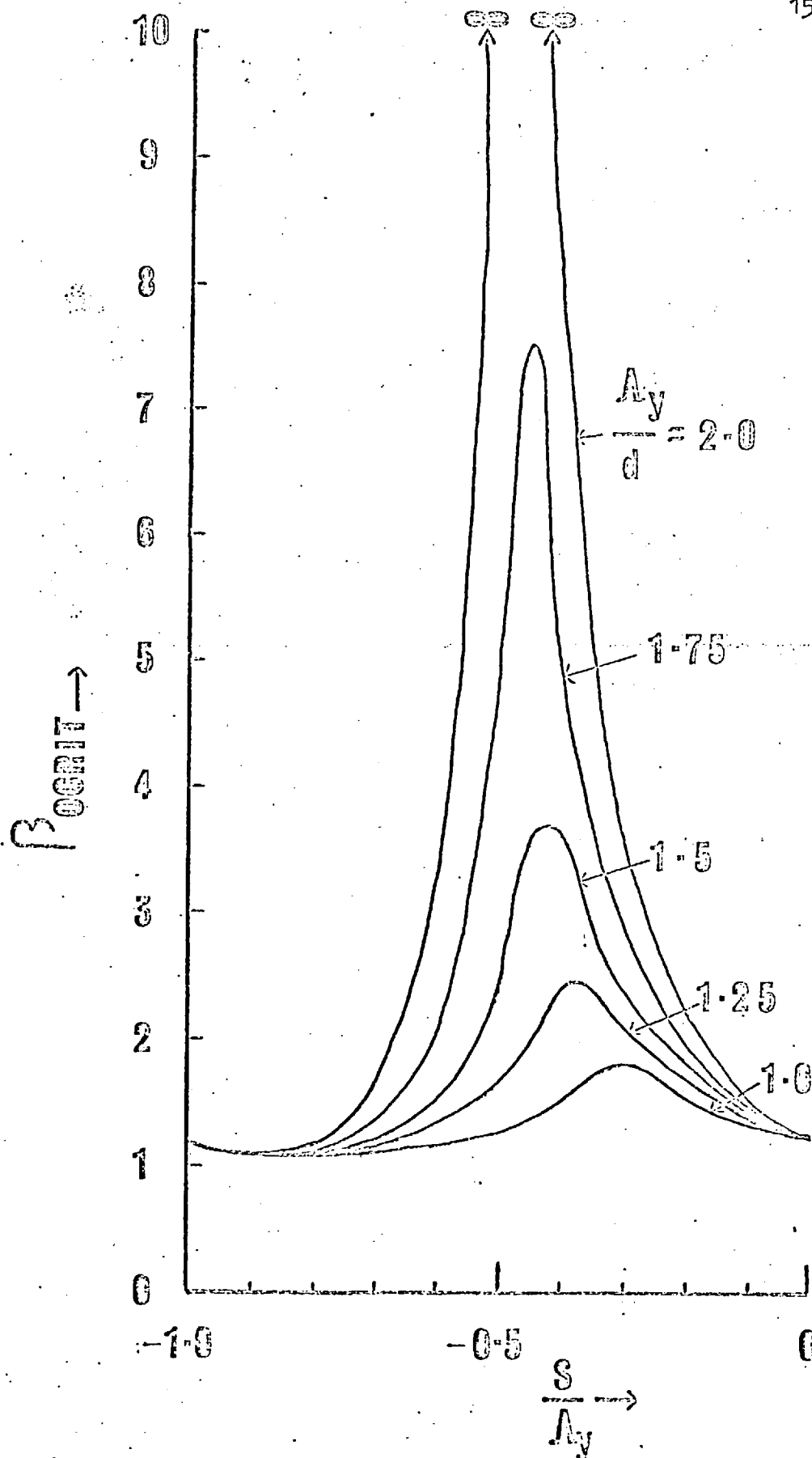


Fig.(4.18) Smallest critical Hall parameter versus S/Λ_y for various values of Λ_y

CHAPTER 5.

A NON - LINEAR THEORY OF ELECTROTHERMAL WAVES.

5.1 Introduction

Experimentally electrothermal waves are observed as fluctuations of finite, steady amplitude. Among the most useful experimental papers for details of the fluctuation amplitudes and the effective conductivity and Hall parameter due to the fluctuations are those of Kerrebrock and Dethlefsen ^(1.24), Brederlow and Hodgson ^(1.25), and Riedmuller ^(1.26). In these experiments the fluctuations are observed to give an effective conductivity and an effective Hall parameter below the average value of these parameters (the average is over space and is denoted by $\langle \rangle$).

The effective conductivity can be defined in two ways. We can define it to be the ratio of the square of the average current to the average Ohmic heating, i.e. $\frac{\langle \underline{j} \rangle^2}{\langle \underline{j}^2 / \sigma \rangle}$ or as the ratio of the average current to the average electric field parallel to $\langle \underline{j} \rangle$, i.e. $\frac{|\langle \underline{j} \rangle|}{\langle E_{\parallel} \rangle}$. It turns out that the two definitions are almost exactly the same. Usually the effective Hall parameter is defined as the ratio of the electric field perpendicular to $\langle \underline{j} \rangle$ to that parallel to $\langle \underline{j} \rangle$, i.e. $\frac{\langle E_{\perp} \rangle}{\langle E_{\parallel} \rangle}$.

Since a Faraday generator, for instance, requires a high Hall field i.e. $E_{\perp}$, along the duct to ensure that $\underline{j}$ is wholly transverse, the reduction of β_{eff} due to the fluctuations is damaging to the generator, and of course so also is the reduction in σ_{eff} . A quantitative theory to estimate the steady amplitude of the fluctuations and the values of σ_{eff} and β_{eff} is therefore desirable.

The experimental observations show that for a wide range of conditions the amplitude of the steady electric field fluctuations is less than 0.5 (i.e. $|\underline{E}^*| < 0.5$). Therefore it seems possible that an extension of the electrothermal perturbation equations of Chapter 2 to include second order terms, but neglecting higher order terms, will be sufficient to derive and describe the steady level of the fluctuations.

There have been several such analyses reported in the literature, most notable are those of Zampaglione (5.1) and Solbes (5.2). In both these analyses equations are derived and solved for the development of the average quantities and for the fluctuations around the average values.

Solbes for instance obtains solutions for the fluctuation amplitude and the effective parameters by assuming a square wave shape for the fluctuations. This assumption does not seem justified in view of the experimental profiles observed by Brederlow and Hodgson, and by Riedmuller.

A more logical approach is to represent the fluctuations by a cosine series i.e. we denote the perturbation in electron density say by $\sum_{n=0}^{\infty} a_n(\tau) \cos(n\omega\tau - nKy)$ where $K = \frac{2\pi}{\lambda}$, and λ = wavelength of the initial perturbation. The amplitudes a_n are considered to be functions of time. We are obviously assuming no dispersion, which is in fact true in the linear theory. However, non-linear effects introduce dispersion and this form of $\hat{n}_e$ is not in fact a proper

solution for the perturbation in the electron density. The movement terms from the time and space derivatives separate off from the growth terms since the operations make them sine series. The movement equation is then found not to be satisfied by a single value of ω_r / K but requires as many values as there are harmonics.

However we know that the movement of electrothermal waves is much slower than the growth, and the distance moved by the waves in the gas frame during the time it takes the steady amplitude to be reached is small compared with the wavelength. Hence it is a reasonable approximation to neglect the movement terms in the energy equation in establishing the steady amplitudes.

The perturbed electrothermal equations then give us a set of equations for $\frac{\partial a_n}{\partial t}$, $n=0,1,2,\dots$. On the assumption that a_n decreases as n increases we can curtail the cosine series at some finite n , equal to N say. We then equate the time derivatives of the harmonic amplitudes to zero to obtain $N + 1$ equations for the $N + 1$ unknowns $a_0, a_1, a_2, \dots, a_N$. No assumptions about the wave shape are necessary in this procedure.

In addition previous non-linear theories have not properly taken into account the response of the external circuitry to the perturbed currents. Since the fluctuations give rise to a change in the average current there will be a change in the voltage drop across any external load, due to the change in current passing through it. Thus the change in the average current leads to a change in the average electric field,

which alters the Ohm's Law and consequently the fluctuation behaviour. The dependence of the fluctuations and the effective quantities on the external impedance will therefore be investigated in this chapter.

It turns out that the external circuit has a significant influence on the final amplitude of the waves.

The change in the load voltage due to the changing current is the only way that the electrodes and insulators will be considered to influence the wave. No boundary conditions as in Chapter 4 will be applied. We will simply assume that we have a plane wave perturbation in the direction for maximum growth, plus higher harmonics and a uniform perturbation. We can then derive the steady values of the amplitudes from the method described above. The steady values of the uniform current and electric field perturbations, together with their unperturbed values, then enable us to evaluate σ_{eff} and β_{eff} .

5.2 Equations and Analysis.

a) Definitions and External Circuit.

We denote the displacement or perturbation of a quantity by the symbol $\hat{}$, emphasising that the displacement is assumed to be non-linear unlike the linear perturbations denoted by a dash in Chapter 2. We

write $\hat{n}_e$ as,
$$\hat{n}_e = \sum_{n=0}^{\infty} a_n(t) \cos(n\omega_r t - nKy)$$

where $K = \frac{2\pi}{\lambda}$, and we assume that at $t = 0$ all the a_n 's are zero except for a_1 . That is we assume a plane wave disturbance of initially a single wavelength with a peak at $y = 0$, $t = 0$.

The terms that generate the higher harmonics and the uniform perturbation are the second order terms in the electron energy equation. These are multiples of two cosines, e.g. $\cos m \cos n$ (Note: for brevity we will write $\cos (n\omega_\tau t - nKy) = \cos n$); and this product is $\frac{1}{2} (\cos (m+n) + \cos (m-n))$. Therefore the harmonics generated are cosine terms and hence the description of $\hat{n}_e$ at all times as a series of cosine terms is self consistent.

If we had taken the initial disturbance to be a sine term then both cosine and sine terms would arise, and, although the equations would be equivalent to the ones used here, they would be more complicated.

We can therefore write the displacements of the fields also as cosine series, i.e.

$$\hat{j}_x = \sum_{n=0}^{\infty} b_n \cos n, \quad \hat{j}_y = c_0$$

$$\hat{E}_x = d_0, \quad \hat{E}_y = \sum_{n=0}^{\infty} f_n \cos n$$

and the displacements of σ and β we can write as

$$\hat{\sigma} = \dot{\sigma} \hat{n}_e + \frac{\ddot{\sigma}}{2} \hat{n}_e^2, \quad \hat{\beta} = \dot{\beta} \hat{n}_e + \frac{\ddot{\beta}}{2} \hat{n}_e^2$$

where the dot represents total differentiation with respect to n_e . We assume Saha equilibrium so that $\hat{T}_e$ is eliminated from the equations using $\hat{T}_e = T_e \hat{n}_e + \frac{\ddot{T}_e}{2} \hat{n}_e^2$. The spatially varying terms of $\hat{j}_y$ and $\hat{E}_x$ are zero of course due to $\nabla \cdot \underline{j} = 0$ and $\nabla \times \underline{E} = 0$.

We will assume that in the unperturbed state $\underline{j}_0$ is re-circulated

through the plasma via a load R_L (see Fig. 5.1). We will also assume that no average current perpendicular to $\underline{j}_0$ is allowed to flow, i.e. no current flows through the plasma that does not pass through R_L , although currents circulating in the plasma will exist.

This geometry is very simple and corresponds to some of the instability experiments. In any case we are not going to do a full boundary condition analysis, but simply introducing load voltage response into the equations. The above prescription is the simplest way of doing this, and it is hoped that qualitatively the results obtained apply to several geometries.

We will consider a wave travelling at 45° to $\underline{j}_0$. This is not exactly the direction for maximum growth but it is a reasonable approximation, and taking $\chi = 45^\circ$ makes the equations somewhat simpler.

Since the average current perpendicular to $\underline{j}_0$ is zero (i.e. the uniform perturbation in $\underline{j}$ is parallel to $\underline{j}_0$) we must have $b_0 = c_0$. (The average of a quantity over space is assumed to be over an integral number of wavelengths, so that $\langle \hat{j}_x \rangle = b_0$ and of course $\langle \hat{j}_y \rangle = c_0$. Note that the zero subscript on the small alphabetic letters a, b, c etc. do not denote the unperturbed state, but rather the perturbation component of zero wave number). The perturbation of the current through R_L is therefore $\sqrt{2} b_0$, and if A is the cross section of the plasma and $\mathcal{L}$ is the distance between the electrodes we can define an equivalent conductivity of the external load by, $\sigma_L = \frac{\mathcal{L}}{A R_L}$

From this definition the change in the uniform electric field parallel to $\tilde{j}_0$ due to the load voltage can be written as $-\frac{\sqrt{2} b_0}{\sigma_L}$. We also have the possibility of a uniform perturbation of the Hall field, i.e. the field perpendicular to $\tilde{j}_0$, therefore we can write

$$d_0 = \langle \hat{E}_x \rangle = e_0 - \frac{b_0}{\sigma_L}$$

and $f_0 = \langle \hat{E}_y \rangle = -e_0 - \frac{b_0}{\sigma_L}$

where $-\sqrt{2} e_0 =$ perturbation of the Hall field.

b) Perturbation of the Ohm's Law.

To perturb the Ohm's Law, i.e. equations (2.2) and (2.3), it is convenient to write the x-component in the form,

$$j_x = \sigma E_x - \beta j_y \quad - (5.1)$$

and the y-component in the form,

$$\gamma j_y = \beta E_x + E_y \quad - (5.2) \quad , \text{ where } \gamma = \frac{1+\beta^2}{\sigma}.$$

The perturbed x and y components are then respectively

$$\hat{j}_x = \hat{\sigma} E_{0x} + \sigma_0 \hat{E}_x + \hat{\sigma} \hat{E}_x - \hat{\beta} j_{0y} - \beta_0 \hat{j}_y - \hat{\beta} \hat{j}_y \quad - (5.3)$$

and

$$\gamma \hat{j}_{0y} + \gamma_0 \hat{j}_y + \hat{\gamma} \hat{j}_y = \hat{\beta} E_{0x} + \beta_0 \hat{E}_x + \hat{\beta} \hat{E}_x + \hat{E}_y \quad - (5.4)$$

Substituting for the various displacements these can be written as,

$$\begin{aligned}
\sum b_n \cos(n) = & \left(\dot{\sigma} \sum a_n \cos n + \frac{\ddot{\sigma}}{2} (\sum a_n \cos n)^2 \right) E_{ox} \\
& + \sigma_o d_o + \left(\dot{\sigma} \sum a_n \cos n \right) d_o - \left(\dot{\beta} \sum a_n \cos n \right. \\
& \left. + \frac{\ddot{\beta}}{2} (\sum a_n \cos n)^2 \right) j_{oy} - \beta_o c_o - \left(\dot{\beta} \sum a_n \cos n \right) c_o
\end{aligned}
\quad \text{--- (5.5)}$$

for the x-component, and,

$$\begin{aligned}
& \left(\dot{\sigma} \sum a_n \cos n + \frac{\ddot{\sigma}}{2} (\sum a_n \cos n)^2 \right) j_{oy} + \sigma_o c_o \\
& + \left(\dot{\sigma} \sum a_n \cos n \right) c_o = \left(\dot{\beta} \sum a_n \cos n + \frac{\ddot{\beta}}{2} (\sum a_n \cos n)^2 \right) E_{ox} \\
& + \beta_o d_o + \left(\sum \dot{\beta} a_n \cos n \right) d_o + \sum f_n \cos n
\end{aligned}
\quad \text{--- (5.6)}$$

for the y-component. Note that

$$\begin{aligned}
\dot{\sigma} &= \frac{2 \beta_o \dot{\beta}}{\sigma_o} - \frac{1 + \beta_o^2}{\sigma_o^2} \dot{\sigma} \\
\text{and,} \quad \ddot{\sigma} &= \frac{2 \beta_o \ddot{\beta}}{\sigma_o} - \frac{2 \dot{\beta}^2}{\sigma_o} - \frac{4 \beta_o \dot{\beta} \dot{\sigma}}{\sigma_o^2} \\
&\quad - \frac{(1 + \beta_o^2) \ddot{\sigma}}{\sigma_o^2} + \frac{2(1 + \beta_o^2)}{\sigma_o^3} \dot{\sigma}^2
\end{aligned}$$

The quadratic terms in these equations can be written as simple cosine terms, and the coefficients of each particular cosine term can then be equated to zero to give a set of $N + 1$ equations in $a_o, a_1, \dots, a_N$, where a_n is taken to be zero if $n > N$. For instance, substituting b_o for c_o and $e_o - \frac{b_o}{\sigma_L}$ for d_o , equation (5.5) gives us; from $n = 0$:-

$$\begin{aligned}
b_o = & \left(\dot{\sigma} a_o + \frac{\ddot{\sigma}}{2} \left(a_o^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right) E_{ox} + \sigma_o \left(e_o - \frac{b_o}{\sigma_L} \right) \\
& + \dot{\sigma} a_o \left(e_o - \frac{b_o}{\sigma_L} \right) - \left(\dot{\beta} a_o + \frac{\ddot{\beta}}{2} \left(a_o^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right) j_{oy} \\
& - \beta_o b_o - \dot{\beta} a_o b_o
\end{aligned}
\quad \text{--- (5.7)}$$

from $n = 1$:-

$$b_1 = (\dot{\sigma} a_1 + \frac{\ddot{\sigma}}{2} (2a_0 a_1 + a_1 a_2)) E_{0x} + \dot{\sigma} a_1 (e_0 - \frac{b_0}{\sigma_L^2}) \\ - (\dot{\beta} a_1 + \frac{\ddot{\beta}}{2} (2a_0 a_1 + a_1 a_2)) j_{0y} - \dot{\beta} a_1 b_0 \quad \text{--- (5.8)}$$

from $n = 2$:-

$$b_2 = (\dot{\sigma} a_2 + \frac{\ddot{\sigma}}{2} (2a_0 a_2 + \frac{a_1^2}{2})) E_{0x} + \dot{\sigma} a_2 (e_0 - \frac{b_0}{\sigma_L^2}) \\ - (\dot{\beta} a_2 + \frac{\ddot{\beta}}{2} (2a_0 a_2 + \frac{a_1^2}{2})) j_{0y} - \dot{\beta} a_2 b_0 \quad \text{--- (5.9)}$$

Here we have neglected all terms in the cosine series with $n > 2$, i.e. we consider only an initial perturbation together with its first harmonic and a uniform displacement. Essentially then we have two ordering schemes. Firstly the fluctuation amplitudes are assumed small enough for third order terms to be negligible, and secondly we assume that the amplitudes of the harmonics decrease rapidly with increasing n .

What we want to get from these equations is the perturbed current, i.e. the b 's, as a function of the perturbed electron density, i.e. the a 's. We can then substitute for $\hat{j}$ in the perturbed electron energy equation and obtain equations for the time development of the a 's. We therefore have to determine e_0 as a function of a_0 , a_1 , a_2 , and b_0 . We can do this using the $n = 0$ equation of (5.6), which is, substituting

$$-e_0 - \frac{b_0}{\sigma_L} \text{ for } f_0,$$

$$\begin{aligned} & \left(\dot{\gamma} a_0 + \frac{\ddot{\gamma}}{2} \left(a_0^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right) j_{oy} + \gamma_0 b_0 \\ & + \dot{\gamma} a_0 b_0 = \left(\dot{\beta} a_0 + \frac{\ddot{\beta}}{2} \left(a_0^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right) E_{ox} \\ & + \beta_0 \left(e_0 - \frac{b_0}{\sigma_L} \right) + \dot{\beta} a_0 \left(e_0 - \frac{b_0}{\sigma_L} \right) - \left(e_0 + \frac{b_0}{\sigma_L} \right) \end{aligned}$$

— (5.10)

Equation (5.10) can be rearranged to give,

$$\begin{aligned} e_0 = & \frac{(1 + \beta_0 + \sigma_L \gamma_0)}{\beta_0 - 1} \left(\frac{b_0}{\sigma_L} - \frac{\dot{\beta} b_0 a_0}{\sigma_L (\beta_0 - 1)} \right) + \frac{a_0 b_0 (\ddot{\gamma} + \frac{\dot{\beta}}{\sigma_L})}{\beta_0 - 1} \\ & + \left(\dot{\gamma} a_0 - \frac{\dot{\gamma} \dot{\beta} a_0^2}{\beta_0 - 1} + \frac{\ddot{\gamma}}{2} \left(a_0^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right) \frac{j_{oy}}{\beta_0 - 1} - \left(\dot{\beta} a_0 - \frac{\dot{\beta}^2 a_0^2}{\beta_0 - 1} + \frac{\ddot{\beta}}{2} \left(a_0^2 + \frac{a_1^2}{2} + \frac{a_2^2}{2} \right) \right) \frac{E_{ox}}{\beta_0 - 1}, \end{aligned}$$

where terms higher than second order have been neglected.

Substituting this into (5.7), and rearranging, we obtain b_0 in the form,

$$b_0 = A_1 a_0 + A_2 a_0^2 + A_3 a_1^2 + A_4 a_2^2$$

where,

$$\begin{aligned} A_1 = & \left[E_{ox} \left(\dot{\gamma} - \frac{\sigma_0}{\beta_0 - 1} \dot{\beta} \right) + j_{oy} \left(\frac{\sigma_0}{\beta_0 - 1} \dot{\gamma} - \dot{\beta} \right) \right] \left(1 + \beta_0 - \frac{(2 + \gamma_0 \sigma_L) \sigma_0}{(\beta_0 - 1) \sigma_L} \right)^{-1} \\ A_2 = & \left\{ E_{ox} \left[\frac{\ddot{\gamma}}{2} - \frac{\sigma_0}{\beta_0 - 1} \frac{\ddot{\beta}}{2} + \frac{\sigma_0}{(\beta_0 - 1)^2} \dot{\beta}^2 - \frac{\dot{\beta} \dot{\gamma}}{\beta_0 - 1} \right. \right. \\ & \left. \left. - \left(\dot{\gamma} - \frac{\sigma_0}{\beta_0 - 1} \dot{\beta} \right) \left(\dot{\beta} \left(1 + \frac{(2 + \gamma_0 \sigma_L) \sigma_0}{(\beta_0 - 1)^2 \sigma_L} \right) - \frac{\sigma_0}{\beta_0 - 1} \dot{\gamma} - \right. \right. \right. \end{aligned}$$

continued over

$$\begin{aligned}
& - \frac{\dot{\sigma}}{\sigma_L} \frac{(2 + \gamma \sigma_L)}{(\beta_0 - 1)} \left(1 + \beta_0 - \frac{(2 + \gamma \sigma_L) \sigma_0}{(\beta_0 - 1) \sigma_L} \right)^{-1} \Big] + j_{oy} \left[\frac{\sigma_0}{\beta_0 - 1} \frac{\ddot{\sigma}}{2} \right. \\
& - \frac{\ddot{\beta}}{2} - \frac{\sigma_0 \dot{\sigma} \dot{\beta}}{(\beta_0 - 1)^2} + \frac{\dot{\sigma} \ddot{\sigma}}{\beta_0 - 1} - \left(\frac{\sigma_0 \ddot{\sigma}}{\beta_0 - 1} - \dot{\beta} \right) \left(\dot{\beta} \left(1 + \frac{(2 + \gamma \sigma_L) \sigma_0}{(\beta_0 - 1)^2 \sigma_L} \right) \right. \\
& \left. \left. - \frac{\sigma_0}{\beta_0 - 1} \dot{\sigma} - \frac{\dot{\sigma}}{\sigma_L} \frac{(2 + \gamma \sigma_L)}{(\beta_0 - 1)} \right) \left(1 + \beta_0 - \frac{(2 + \gamma \sigma_L) \sigma_0}{(\beta_0 - 1) \sigma_L} \right)^{-1} \right] \left(1 + \beta_0 - \frac{(2 + \gamma \sigma_L) \sigma_0}{(\beta_0 - 1) \sigma_L} \right)^{-1} \\
A_3 = & \left[E_{ox} \left(\frac{\ddot{\sigma}}{4} - \frac{\sigma_0}{\beta_0 - 1} \frac{\ddot{\beta}}{4} \right) + j_{oy} \left(\frac{\sigma_0}{\beta_0 - 1} \frac{\ddot{\sigma}}{4} - \frac{\ddot{\beta}}{4} \right) \right] \left(1 + \beta_0 - \frac{(2 + \gamma \sigma_L) \sigma_0}{(\beta_0 - 1) \sigma_L} \right)^{-1}
\end{aligned}$$

and $A_4 = A_3$.

Using equations (5.8) and (5.9) we can then write b_1 and b_2 as,

$$b_1 = B_1 a_1 + B_2 a_0 a_1 + B_3 a_1 a_2$$

and,

$$b_2 = C_1 a_2 + C_2 a_0 a_2 + C_3 a_1^2 + C_4 a_1 a_3$$

where,

$$B_1 = \dot{\sigma} E_{ox} - \dot{\beta} j_{oy}$$

$$\beta_2 = (\ddot{\sigma} E_{ox} - \ddot{\beta} j_{oy}) + \left(\frac{2 + \gamma \sigma_L}{\beta_0 - 1} \frac{A_1}{\sigma_L} + \frac{j_{oy} \dot{\sigma} - \dot{\beta} E_{ox}}{\beta_0 - 1} \right) \dot{\sigma} - A_1 \dot{\beta}$$

$$B_3 = \frac{\ddot{\sigma}}{2} E_{ox} - \frac{\ddot{\beta}}{2} j_{oy}$$

and $C_1 = B_1$, $C_2 = B_2$, $C_3 = \frac{B_3}{2}$

Note that the $n = 1, 2$ equations of (5.6) simply give us the electric field fluctuation amplitudes f_1 and f_2 .

c) Perturbation of the Energy Equation.

We will perturb the electron energy equation in the form (2.4), neglecting the spatial gradient terms and the time derivatives of the

cosines. Since the Ionization energy is ten times the thermal energy we can write the rate of change of electron density terms simply as $I_p \sum \frac{\partial a_n}{\partial t} \cos n$, neglecting the first and second order terms from the thermal energy. All time derivatives will be equated to zero and it is therefore not necessary to evaluate the terms involving the rate of change of the amplitudes exactly.

The Ohmic heating, elastic losses, thermal conduction and radiation transfer will however be evaluated accurately to second order. The

perturbed Ohmic heating can be written as

$$\left(\frac{\hat{j}^2}{\sigma} \right) = \frac{2 \dot{j}_{ox} \hat{j}_x + 2 \dot{j}_{oy} \hat{j}_y}{\sigma_0} + \frac{\hat{j}_x^2 + \hat{j}_y^2}{\sigma_0} - \frac{\dot{j}_0^2}{\sigma_0^2} \hat{\sigma} + \frac{\dot{j}_0^2}{\sigma_0^3} \hat{\sigma}^2 - \frac{(2 \dot{j}_{ox} \hat{j}_x + 2 \dot{j}_{oy} \hat{j}_y) \hat{\sigma}}{\sigma_0^2} = \sum h_n \cos n$$

substituting for $\hat{j}_x$, $\hat{j}_y$ and $\hat{\sigma}$ we obtain after some manipulation,

$$h_0 = G_1 a_0 + G_2 a_0^2 + G_3 a_1^2 + G_4 a_2^2$$

where,

$$G_1 = \frac{2}{\sigma_0} (\dot{j}_{ox} + \dot{j}_{oy}) A_1 - \frac{\dot{j}_0^2}{\sigma_0^2} \dot{\sigma}$$

$$G_2 = \frac{2}{\sigma_0} (\dot{j}_{ox} + \dot{j}_{oy}) (A_2 - \frac{\dot{\sigma}}{\sigma_0} A_1) - \frac{\dot{j}_0^2}{\sigma_0^2} \frac{\ddot{\sigma}}{2} + \frac{2 A_1^2}{\sigma_0} + \frac{\dot{j}_0^2}{\sigma_0^3} \dot{\sigma}^2$$

$$G_3 = \frac{2}{\sigma_0} (\dot{j}_{ox} + \dot{j}_{oy}) A_3 - \frac{\dot{j}_0^2}{\sigma_0^2} \frac{\ddot{\sigma}}{4} + \frac{1}{\sigma_0} \frac{B_1^2}{2} - \frac{\dot{\sigma}}{\sigma_0^2} \dot{j}_{ox} B_1 + \frac{\dot{j}_0^2}{\sigma_0^3} \frac{\dot{\sigma}^2}{2}, \quad G_4 = G_3,$$

$$h_1 = H_1 a_1 + H_2 a_0 a_1 + H_3 a_1 a_2$$

$$\text{where, } H_1 = \frac{2}{\sigma_0} \dot{j}_{ox} B_1 - \frac{\dot{j}_0^2}{\sigma_0^2} \dot{\sigma}$$

$$H_2 = \frac{2}{\sigma_0} \dot{j}_{ox} B_2 - \frac{\dot{j}_0^2}{\sigma_0^2} \ddot{\sigma} + \frac{2}{\sigma_0} A_1 B_1 - \frac{2\dot{\sigma}}{\sigma_0^2} (\dot{j}_{ox}(A_1+B_1) + \dot{j}_{oy} A_1) + \frac{2\dot{j}_0^2}{\sigma_0^3} \dot{\sigma}^2$$

$$H_3 = \frac{2}{\sigma_0} \dot{j}_{ox} B_3 - \frac{\dot{j}_0^2}{\sigma_0^2} \frac{\ddot{\sigma}}{2} + \frac{1}{\sigma_0} B_1 C_1 - \frac{\dot{\sigma}}{\sigma_0^2} \dot{j}_{ox}(B_1+C_1) + \frac{\dot{j}_0^2}{\sigma_0^3} \dot{\sigma}^2,$$

$$h_2 = I_1 a_2 + I_2 a_0 a_2 + I_3 a_1^2$$

$$I_1 = \frac{2}{\sigma_0} \dot{j}_{ox} C_1 - \frac{\dot{j}_0^2}{\sigma_0^2} \dot{\sigma}$$

$$I_2 = \frac{2}{\sigma_0} \dot{j}_{ox} C_2 - \frac{\dot{j}_0^2}{\sigma_0^2} \ddot{\sigma} + \frac{2A_1 B_1}{\sigma_0} + \frac{2\dot{j}_0^2}{\sigma_0^3} \dot{\sigma}^2 - \frac{2\dot{\sigma}}{\sigma_0^2} (\dot{j}_{ox} + \dot{j}_{oy}) A_1 - \frac{2\dot{\sigma}}{\sigma_0^2} C_1 \dot{j}_{ox}$$

$$I_3 = \frac{2}{\sigma_0} \dot{j}_{ox} C_3 - \frac{\dot{j}_0^2}{\sigma_0^2} \frac{\ddot{\sigma}}{4} + \frac{1}{\sigma_0} \frac{B_1^2}{2} - \frac{\dot{\sigma}}{\sigma_0^2} B_1 \dot{j}_{ox} + \frac{\dot{j}_0^2}{\sigma_0^3} \frac{\dot{\sigma}^2}{2}$$

The perturbation of the equipartition elastic losses can be written simply as,

$$\hat{\xi} = \dot{\xi} \sum a_n \cos n + \frac{\ddot{\xi}}{2} (\sum a_n \cos n)^2 = \sum p_n \cos n$$

therefore $p_0 = K_1 a_0 + K_2 a_0^2 + K_3 a_1^2 + K_4 a_2^2$

where $K_1 = \dot{\xi}$, $K_2 = \frac{\ddot{\xi}}{2}$, $K_3 = \frac{\ddot{\xi}}{4}$, $K_4 = \frac{\ddot{\xi}}{4}$,

$$p_1 = L_1 a_1 + L_2 a_0 a_1 + L_3 a_1 a_2$$

where $L_1 = \dot{\xi}$, $L_2 = \dot{\xi}$, $L_3 = \frac{\ddot{\xi}}{2}$

and $p_2 = M_1 a_2 + M_2 a_0 a_2 + M_3 a_1^2$

$$M_1 = \ddot{\xi}, \quad M_2 = \ddot{\xi}, \quad M_3 = \frac{\ddot{\xi}}{4}$$

The perturbation of the thermal conduction can be written as,

$$\begin{aligned} \left(\frac{\partial}{\partial y} \left(\hat{K}_e \frac{\partial \hat{T}_e}{\partial y} \right) \right) &= \frac{\partial \hat{K}_e}{\partial y} \frac{\partial \hat{T}_e}{\partial y} + (\kappa_{oe} + \hat{K}_e) \frac{\partial^2 \hat{T}_e}{\partial y^2} \\ &= (\dot{\kappa}_e \dot{T}_e + \kappa_{oe} \ddot{T}_e) (\sum a_n n K \sin n)^2 - [\kappa_{oe} \dot{T}_e \\ &\quad + (\kappa_{oe} \ddot{T}_e + \dot{\kappa}_e \dot{T}_e) (\sum a_n \cos n)] (\sum a_n n^2 K^2 \cos n) \\ &= \sum q_n \cos n \end{aligned}$$

and we have,

$$q_0 = 0,$$

$$q_1 = Q_1 a_1 + Q_2 a_0 a_1 + Q_3 a_1 a_3$$

where,

$$Q_1 = -\kappa_{oe} \dot{T}_e K^2$$

$$Q_2 = -(\kappa_{oe} \ddot{T}_e + \dot{\kappa}_e \dot{T}_e) K^2, \quad Q_3 = Q_2/2$$

and

$$q_2 = R_1 a_2 + R_2 a_0 a_1 + R_3 a_1^2$$

where,

$$R_1 = -4 \kappa_{oe} \dot{T}_e K^2$$

$$R_2 = -4 (\kappa_{oe} \ddot{T}_e + \dot{\kappa}_e \dot{T}_e) K^2$$

$$R_3 = R_2/4$$

The perturbation of the radiation transfer is due to the perturbation of the Planck function. The Planck function for the i^{th} line is approximately given by,

$$B_i = \frac{2 h \nu_i^3}{c^2} \exp\left(-\frac{h \nu_i}{k T_e}\right)$$

and the perturbation of this is,

$$\hat{B}_i = \frac{2h\nu_i^3}{c^2} \exp\left(-\frac{h\nu_i}{kT_e}\right) \left(\frac{h\nu_i}{kT_{e0}}\right) \left[\dot{T}_e \hat{n}_e + \left(\frac{1}{2} \ddot{T}_e + \frac{1}{T_e} \left(\frac{1}{2} \frac{h\nu_i}{kT_{e0}} - 1\right) \dot{T}_e^2\right) \hat{n}_e^2 \right]$$

Substituting this into the radiation transfer integral we get

(see Appendix 2) $\hat{R} = \sum r_n \cos n$

with $r_0 = 0$, $r_1 = T_1 a_1 + T_2 a_1 a_0 + T_3 a_1 a_2$

where $T_1 = \sum_i R_{ci} \sqrt{K} \dot{T}_e$
 $T_2 = 2 \sum_i R_{ci} \left[\frac{1}{2} \ddot{T}_e + \frac{1}{T_e} \left(\frac{1}{2} \frac{h\nu_i}{kT_{e0}} - 1 \right) \dot{T}_e^2 \right] \sqrt{K}$
 $T_3 = T_2/2$

$$R_{ci} = \frac{3k^2 T_{e0}}{2hc} \sqrt{\frac{n_{ag} \Delta\nu_i g_i}{\tau_i g_g}} \left(\frac{h\nu_i}{kT_{e0}} \right)^3 e^{-\frac{h\nu_i}{kT_{e0}}}$$

and,

$$r_2 = U_1 a_2 + U_2 a_0 a_2 + U_3 a_1^2$$

where,

$$U_1 = T_1 \sqrt{2}, \quad U_2 = T_2 \sqrt{2}, \quad U_3 = \frac{U_2}{4}$$

Putting all the terms of the energy equation together, and equating the coefficients of $\cos(n)$, $n = 0, 1, 2$, to zero we get the following set of equations for a_0 , a_1 , and a_2 ,

$$I_P \frac{\partial a_0}{\partial t} = P_1 a_0 + P_2 a_0^2 + P_3 a_1^2 + P_4 a_2^2 \quad \text{--- (5.11)}$$

$$I_P \frac{\partial a_1}{\partial t} = P_5 a_1 + P_6 a_0 a_1 + P_7 a_1 a_2 \quad \text{--- (5.12)}$$

$$I_P \frac{\partial a_2}{\partial t} = P_8 a_2 + P_9 a_0 a_2 + P_{10} a_1^2 \quad \text{--- (5.13)}$$

where the P's are defined by,

$$P_1 = G_1 - K_1, \quad P_2 = G_2 - K_2$$

$$P_3 = G_3 - K_3, \quad P_4 = G_4 - K_4$$

$$P_5 = H_1 - L_1 + Q_1 - T_1$$

$$P_6 = H_2 - L_2 + Q_2 - T_2$$

$$P_7 = H_3 - L_3 + Q_3 - T_3$$

$$P_8 = I_1 - M_1 + R_1 - U_1$$

$$P_9 = I_2 - M_2 + R_2 - U_2$$

$$P_{10} = I_3 - M_3 + R_3 - U_3$$

d) Solution of the equations.

The signs of the P's in equations (5.11) - (5.13) are of crucial importance. Table (5.1) shows a set of typical values for the P's.

P_5 and P_8 are of course positive since the initial perturbation and its first harmonic are unstable. P_3 is positive when $\frac{\alpha_o}{\alpha_L} > 1$, hence

P_1	-1.85	PLASMA PARAMETERS $T_{e0} = 2500^\circ K$ $T = 1500^\circ K$ $\frac{n_s}{n_n} = 10^{-3}$ (Argon Caesium Plasma), $B = 3 \text{ tesla}$, $\frac{\alpha_o}{\alpha_L} = 10$ $\mathcal{L} = 1 \text{ cm.}$ $(P_s^2 \text{ divided by } 10^5 I_p)$
P_2	.081	
P_3	1.77	
P_4	1.77	
P_5	1.08	
P_6	-5.8	
P_7	2.8	
P_8	.97	
P_9	-5.8	
P_{10}	1.44	

a positive a_0 is generated

by the initial perturbation, i.e. the electron density increases.

P_1 is negative, hence the uniform perturbation is stable. This is in contrast to the results obtained in Chapter 2. The reason is that, as stated in Chapter 2, the direction of the perturbed

TABLE (5.1)

current there was determined by the wave vector $\underline{K}$, even when $|\underline{K}| \rightarrow 0$.

Therefore if $\underline{K}$ is in the direction for maximum growth, i.e. $\underline{j}'$ at 45° to $\underline{j}_0$, then a perturbation with $|\underline{K}| = 0$ will be unstable. Whereas in

this chapter we recognise that the direction of a uniform perturbation of the current will be determined by the external circuitry. In the present case $\underline{j}'$ must be parallel to $\underline{j}_0$, and this gives stability.

P_6 and P_9 are negative so that the positive perturbation a_0 opposes the destabilizing $P_5 a_1$ and $P_8 a_2$ terms. Essentially P_6 is the rate of change of the linear growth rate with respect to a uniform perturbation of electron density. The fluctuations are therefore stabilized at a steady amplitude by the reduction of the growth rate to zero due to the increase in the average electron density.

The uniform increase in the electron density reaches a steady value when the stabilizing $P_1 a_0$ term cancels the second order driving terms in equation (5.11).

If $\frac{\sigma_0}{\sigma_L} < 1$ then P_3 is negative and the average electron density will decrease. It follows then that the fluctuations will not reach a steady level of fluctuations according to the equations used here, since $P_6 a_0 a_1$ is in this case destabilizing. This means that either higher order terms become important in stabilizing the fluctuations or the fluctuations simply grow until the seed is fully ionized in the peaks of n_e and unionized in the troughs. We expect then that the steady amplitudes of the fluctuations will increase with decreasing $\frac{\sigma_0}{\sigma_L}$.

Note that the stability of the uniform perturbations is necessary for the existence of steady finite amplitude fluctuations (i.e. of only second order significance). If P_1 were positive then the uniform displacement of the density would not stop growing with the first

and second order terms alone. Hence no steady state would be reached, at least until higher order terms became important.

To derive the steady amplitudes we equate the time derivatives in (5.11) - (5.13) to zero and obtain the set,

$$P_1 a_0 + P_2 a_0^2 + P_3 a_1^2 + P_4 a_2^2 = 0 \quad \text{--- (5.14)}$$

$$P_5 a_1 + P_6 a_0 a_1 + P_7 a_1 a_2 = 0 \quad \text{--- (5.15)}$$

$$P_8 a_2 + P_9 a_0 a_2 + P_{10} a_1^2 = 0 \quad \text{--- (5.16)}$$

which we can solve for a_0 , a_1 and a_2 .

a_1 cancels from (5.15) to give,

$$a_2 = -\frac{1}{P_7} (P_5 + P_6 a_0) \quad \text{--- (5.17)}$$

substituting for a_1^2 in (5.14) from (5.16), we obtain an equation in a_0 and a_2 and substituting for a_2 from (5.17) we obtain a quadratic in a_0 . This is

$$a_0^2 Z_1 + a_0 Z_2 + Z_3 = 0$$

$$\text{where, } Z_1 = \frac{P_3 P_6 P_9}{P_1 P_7 P_{10}} + \frac{P_2}{P_1} + \frac{P_4}{P_1} \left(\frac{P_6}{P_7} \right)^2$$

$$Z_2 = 1 + \frac{P_3 P_6 P_8}{P_1 P_7 P_{10}} + \frac{P_3 P_5 P_9}{P_1 P_7 P_{10}} + 2 \frac{P_4 P_5 P_6}{P_1 P_7^2}$$

and, $Z_3 = \frac{P_3 P_8 P_5}{P_1 P_7 P_{10}} + \frac{P_4}{P_1} \left(\frac{P_5}{P_7} \right)^2$

The solution of this quadratic gives two positive solutions for a_0 .

The larger of the two is the required solution since this is the one that gives a positive value for a_2 when a_0 is substituted into (5.17),

the other solution giving a negative value. (Note that P_{10} is positive and therefore a_2 should be positive). a_1 is then found from (5.16).

From the values of a_0 , a_1 and a_2 the field displacements, $\hat{j}$ and $\hat{E}$ can be evaluated, since we know these as a function of the a 's. The values of σ_{eff} and β_{eff} can therefore be obtained from the definitions in section (5.1). The results are discussed and compared with experiments in the next section.

5.3 Results.

a) Dependence of the Fluctuations on Magnetic Field.

Figure (5.2) shows the variation of the fluctuations with the average Hall parameter (parameters used are: $T = 1500^\circ\text{K}$, $n_n = 10^{25}$, $\frac{n_s}{n_n} = 10^{-3}$, $T_{eo} = 2500^\circ\text{K}$, $\lambda = 1 \text{ cm.}$, $\frac{\sigma_o}{\sigma_L} = 10$). The average Hall parameter is not exactly equal to β_o , the unperturbed Hall parameter, due to the increase in the average electron density. $\langle\beta\rangle$ is plotted against β_o in Fig. (5.3). We have plotted the fluctuation quantities as a function of $\langle\beta\rangle$ since usually the experimentalists deduce β from the measured average electron density and this gives them $\langle\beta\rangle$.

We see that the fluctuations in the electron density reach a maximum value at $\langle\beta\rangle$ just above β_{ocrit} and then decrease. However the uniform perturbation, a_o , increases monotonically with $\langle\beta\rangle$.

The quantity measured by Kerrebrock and Dethlefsen ^(1.24) was the fluctuating electric field. The calculations show that the amplitude of the first and second harmonic of the field parallel to $\hat{j}_o$ increase monotonically with $\langle\beta\rangle$, and this is in agreement with their observations.

We note therefore that one must be cautious in deducing the behaviour of the electron density fluctuations from the behaviour of the electric field fluctuations.

However from Riedmuller's (1.26) results the electric field fluctuations should level out at higher values of $\langle \beta \rangle$. Possibly this does not occur in the calculations here due to ^{the} large amplitude of a_0 at higher values of $\langle \beta \rangle$, indicating that higher order terms may be significant.

Both the effective conductivity and the effective Hall parameter decrease with increasing magnetic field (see Fig. (5.4)). The definition of σ_{eff} used was $\sigma_{\text{eff}} = \frac{|\langle \underline{j} \rangle|}{\langle E_{\parallel} \rangle}$, and the alternative Ohmic heating definition mentioned in section (5.1) was found to differ by less than ten per cent from this. These curves are in good qualitative agreement with the results reported in references (1.24), (1.25) and (1.26).

b) Dependence of the Fluctuations on Temperature.

The unperturbed temperature of the electrons was varied by varying the unperturbed electric current. The variation of the fluctuations is shown in Fig. (5.5) (same parameters as in (5.3a) except $B = 3$ tesla, T_{e0} varies). We see that at high electron temperatures the amplitudes decrease towards zero. This is of course consistent with the fall off in the growth rate due to the decreasing value of β_0 described in Chapter 2. It is also in agreement with some results presented by Kerrebrock and Dethlefsen which showed that the electric field fluctuations decreased with increasing current density.

Note that, as the electric field fluctuations decrease f_1 decreases more rapidly than f_2 , and over a small range T_{eo} the second harmonic has a large amplitude than the first harmonic. Hence the electric field fluctuations will have a dominant wavelength of $\frac{\Lambda}{2}$ in this region. As T_{eo} increases further the electric field fluctuations go through zero and become π out of phase with the electron density fluctuations. The first harmonic also becomes dominant again.

Not surprisingly both the effective conductivity and the effective Hall parameter increase with increasing temperature (see Fig (5.6)). $\langle \beta \rangle$, β_{eff} and β_o converge at about 1.8 where the fluctuations go to zero. This is β_{ocrit} .

c) Dependence of the Fluctuations on the external Load.

For all the previous results in this section the value of $\frac{\sigma_o}{\sigma_L}$ was 10, i.e. the load impedance is ten times the plasma impedance. When the load impedance is decreased the amplitude of the fluctuations increases. This can be seen from Fig. (5.7), where the various harmonic amplitudes are plotted as a function of $\frac{\sigma_o}{\sigma_L}$ (same parameters as in (5.3a) except $B = 3$ tesla and $\frac{\sigma_o}{\sigma_L}$ varies).

As $\frac{\sigma_o}{\sigma_L}$ tends to infinity, i.e. the load impedance goes to infinity, the amplitudes approach an asymptotic limit. With decreasing $\frac{\sigma_o}{\sigma_L}$ the amplitudes increase until, for $\frac{\sigma_o}{\sigma_L}$ near to 1, they move outside the limit where third and higher terms are negligible. The theory presented here is therefore unable to establish the fluctuation amplitudes for $\frac{\sigma_o}{\sigma_L}$ of the order of or less than 1.

Direct comparison of these results with the published experiment results is not possible, since the load impedance is not mentioned in any of the papers. However in private discussion Kerrebrock has disclosed that in the experiment he performed with Dethlefsen the external impedance was always much greater than the plasma impedance. On the basis of the theory presented here, this is consistent with the fluctuation amplitudes that they observe.

The calculations of Solbes were carried out for what he calls a current source with $\sigma_L = 0$, i.e. the amplitudes he calculates correspond to the asymptotic state when $\frac{\sigma_c}{\sigma_L} \rightarrow \infty$.

d) Conclusion.

The results presented here show that neglect of third and higher terms is a good approximation for a wide range of plasma and circuit parameters. However the neglect of the third and higher harmonics of the electron density fluctuations is not a good approximation. The ratio of a_1 to a_2 indicates that probably a_3 and a_4 will be significant, and a larger set of equations should be used in determining the value of the steady amplitudes.

On the other hand the qualitative agreement of the results with experimental observations is very good. In addition a smaller set of equations, involving only a_0 and a_1 was tried at first, and though different values for a_0 and a_1 appeared (about 100% difference for a_0 , and 20% for a_1), the qualitative behaviour of the waves as a function of B , T_{eo} , and σ_L was essentially the same. Hence extension of the

theory to include higher harmonics will probably not alter the form of the graphs presented in this chapter, but change, to a limited extent, only the numerical values.

In addition the results presented here in section (5.a) are similar to the results obtained by Solbes assuming a square wave profile for the waves. Even quantitatively the results of the two theories are near to the experimental results. This seems to indicate that the finite amplitude of the fluctuations and the reduction in σ_{eff} and β_{eff} are not very sensitive to the wave profile.

The results show that relatively small fluctuations in the electron density give rise to a considerable reduction in the effective conductivity. From the variation of the amplitudes with $\frac{\sigma_c}{\sigma_L}$ we see that this reduction in the effective conductivity is likely to be particularly marked when the plasma and load impedances are about equal. Even taking into account the strong dependence of the plasma conductivity on the current density, the optimization of the power output of a generator still requires the plasma and load impedances to be about the same order of magnitude. For instance if we use Kerrebrock's ^(1.16) model of $\sigma \propto j^\alpha$ where α lies between 0 and 1, and optimise $\frac{\sigma_c^2 E_G^2}{(1 + \sigma_c/\sigma_L)^2 \sigma_L}$ (see section (1.2a)) w.r.t. $\frac{\sigma_c}{\sigma_L}$, we obtain

$$\frac{\sigma_c}{\sigma_L} = 1 - \alpha$$

It follows that for an optimized generator we require $\frac{\sigma_c}{\sigma_L} < 1$, i.e. we require to operate in the region of large electrothermal fluctuations.

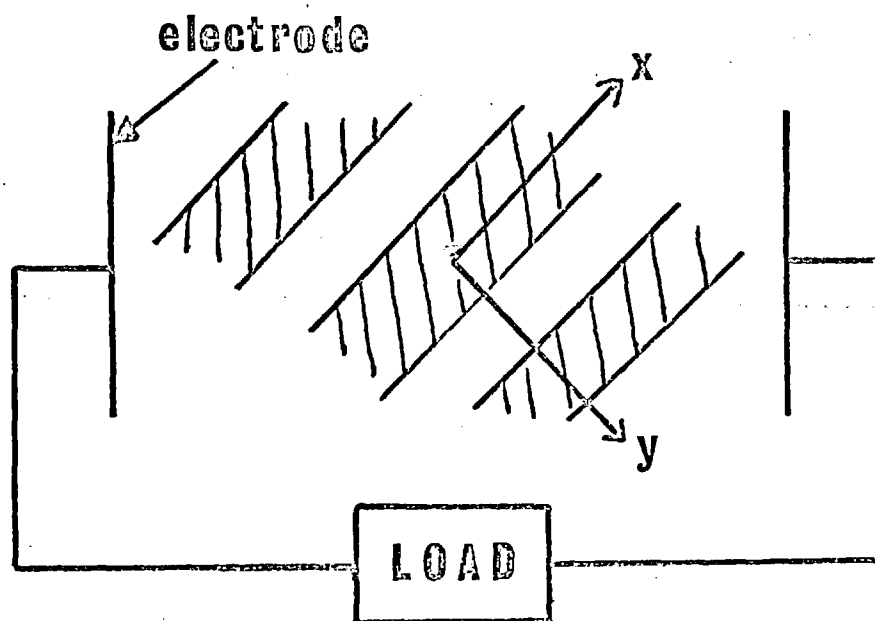


Fig.(5.1) External circuitry and co-ordinate system (shaded areas represent regions of higher electron density).

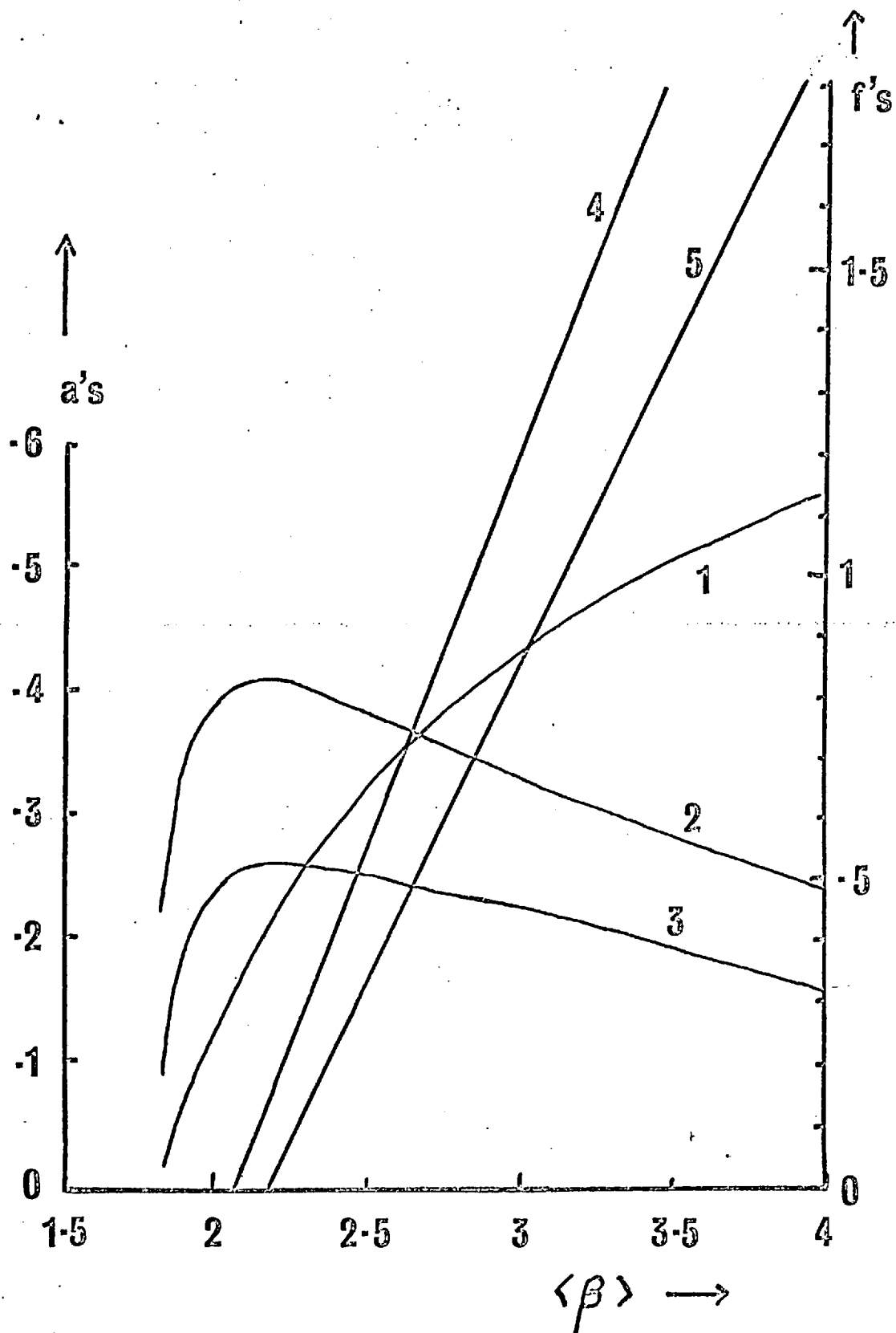


Fig.(5.2) Fluctuations of electron density (left hand scale) and electric field(right hand scale) versus $\langle \beta \rangle$.

1) $\frac{a_0}{\pi_{co}}$, 2) $\frac{a_1}{\langle n_0 \rangle}$, 3) $\frac{a_2}{\langle n_0 \rangle}$, 4) $\frac{f_1}{\langle E_{||} \rangle}$, 5) $\frac{f_2}{\langle E_{||} \rangle}$. ($E_{||}$ is the component of $\underline{E}$ parallel to $\underline{j}_0$)

Fig.(5.4) Effective Hall parameter and conductivity versus $\langle \beta \rangle$

- 1) β_{EFF} (broken line represents $\beta_{EFF} = \langle \beta \rangle$),
- 2) $\frac{\sigma_{EFF}}{\sigma_0}$

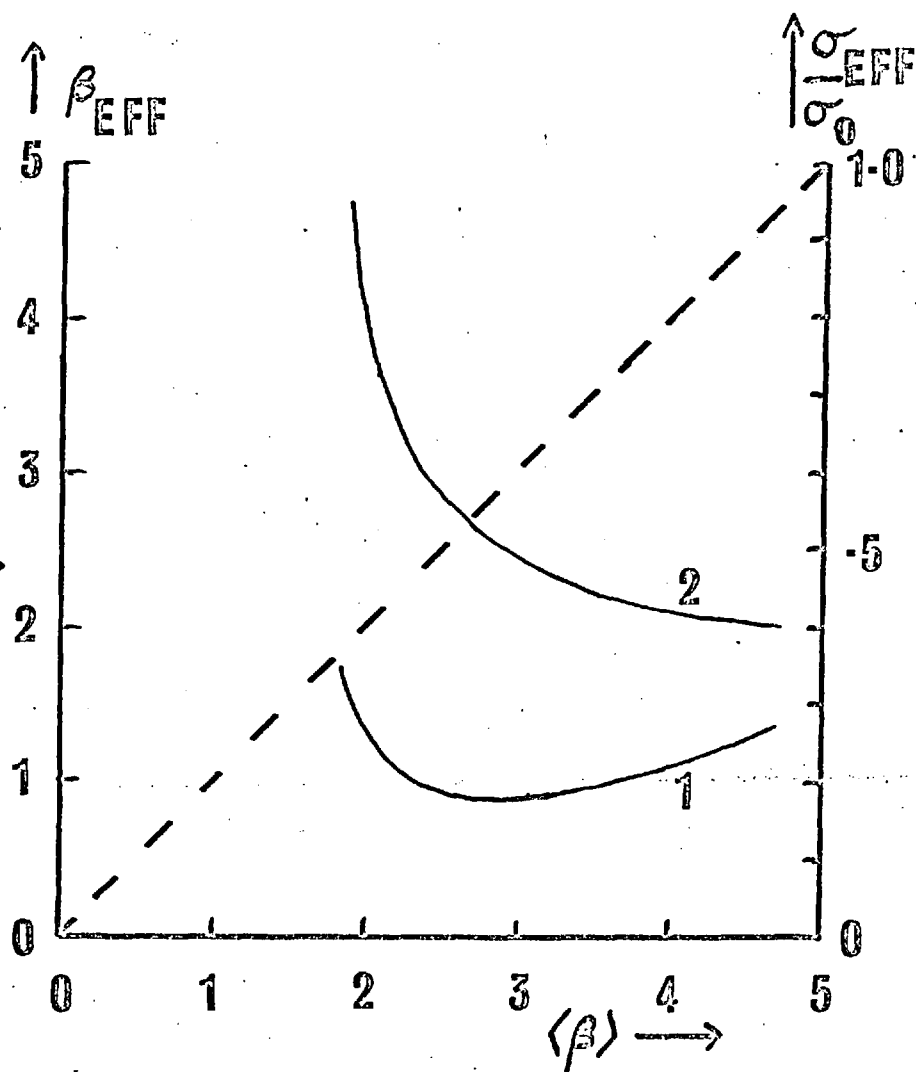
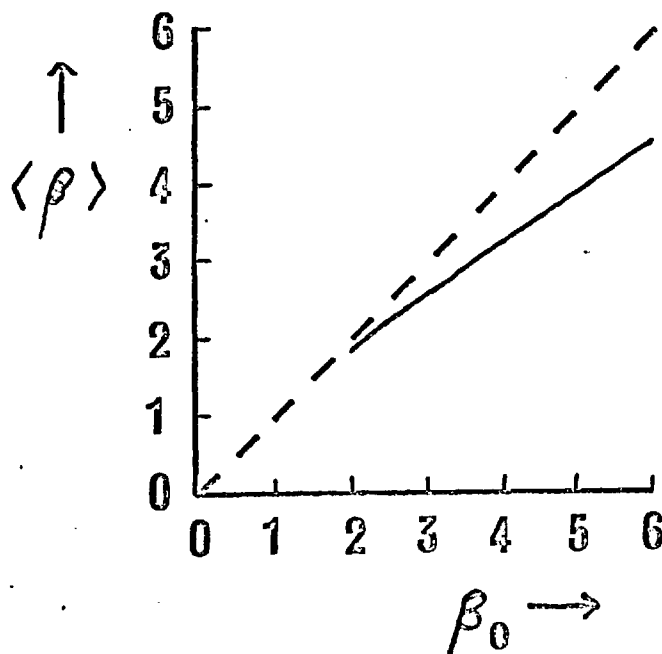


Fig.(5.3) $\langle \beta \rangle$ versus β_0
(broken line represents $\langle \beta \rangle = \beta_0$)



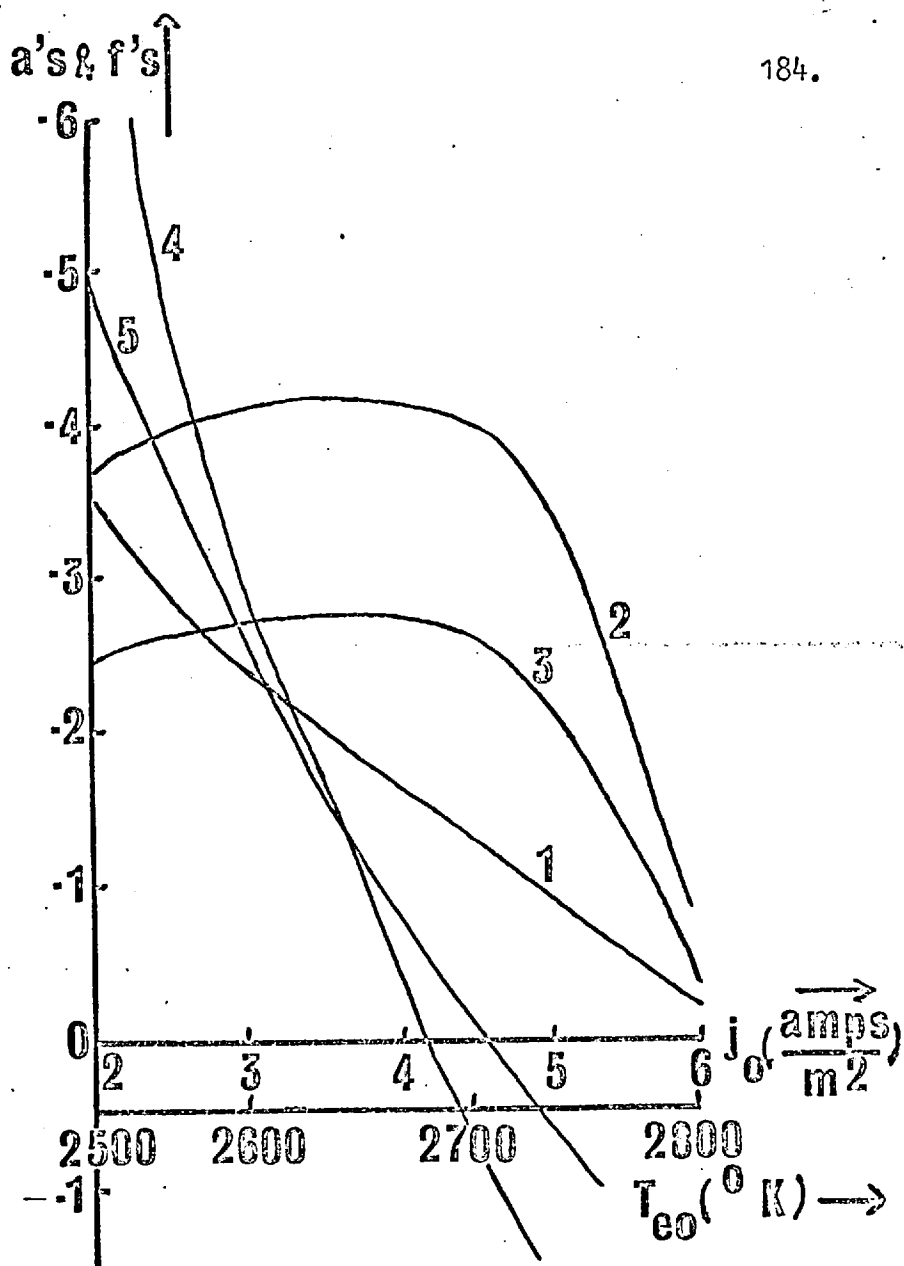


Fig.(5.5) Fluctuations of electron density and electric field versus T_{eo} (i.e. versus j_o for fixed gas temperature). 1) $\frac{a_o}{n_{eo}}$, 2) $\frac{a_1}{\langle n_e \rangle}$, 3) $\frac{a_2}{\langle n_e \rangle}$, 4) $\frac{f_1}{\langle E_{||} \rangle}$, 5) $\frac{f_2}{\langle E_{||} \rangle}$.

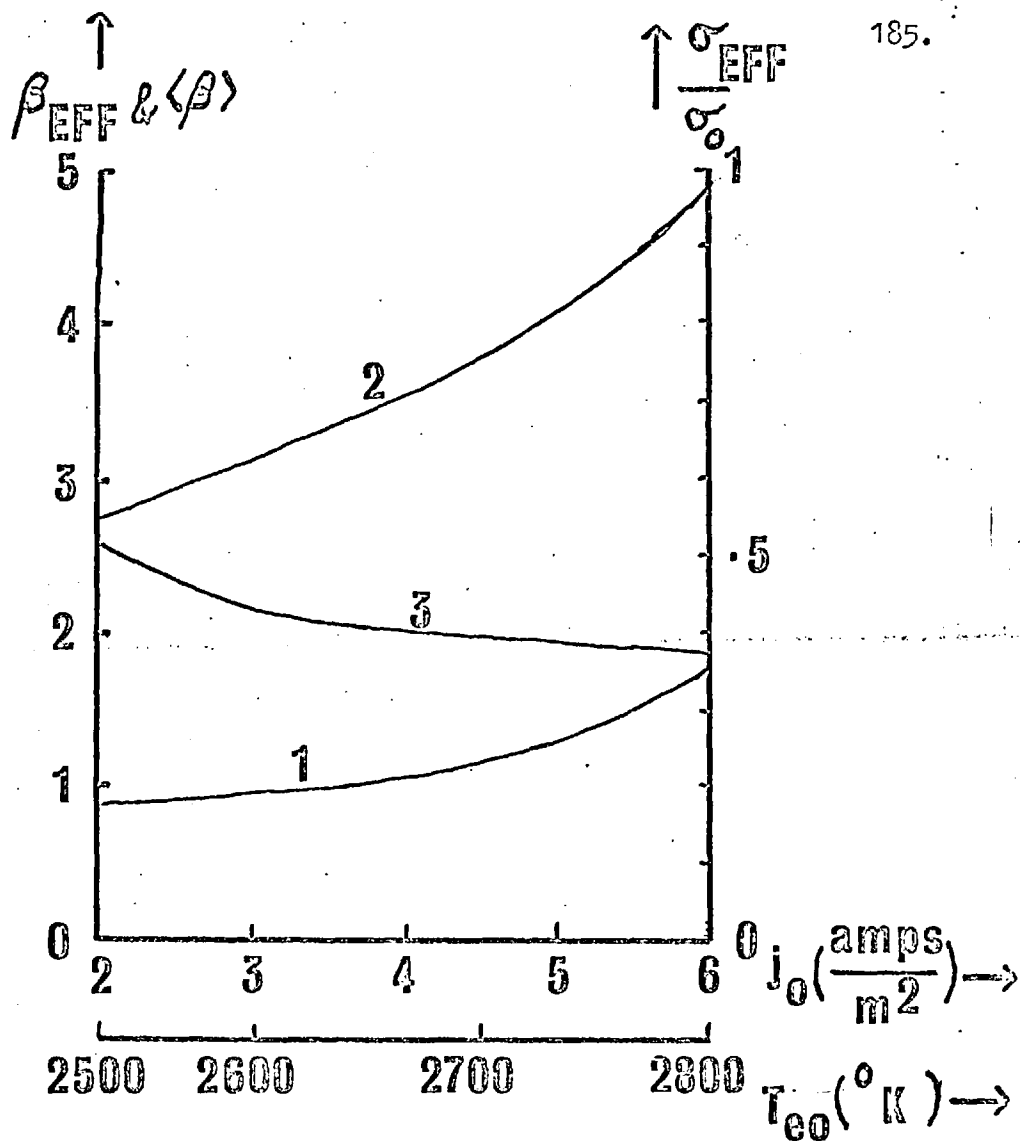


Fig.(5,6) Effective Hall parameter, conductivity and average Hall parameter versus T_{eo} . 1) β_{EFF} , 2) $\frac{\sigma_{EFF}}{\sigma_0}$, 3) $\langle \beta \rangle$.

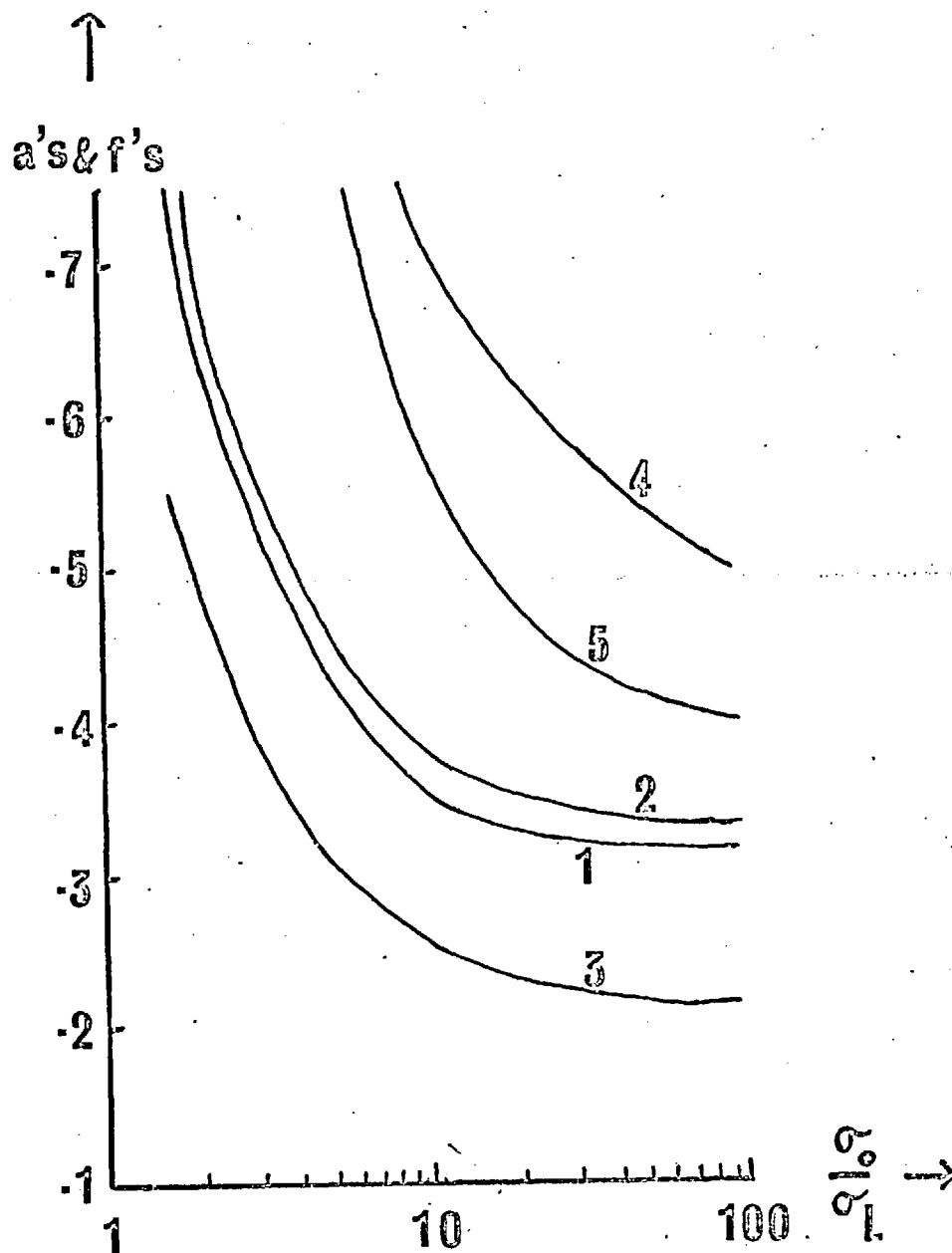


Fig.(5.7) Fluctuations in the electron density and electric field versus $\frac{\sigma_0}{\sigma_L}$. 1) $\frac{a_0}{n_{e0}}$, 2) $\frac{a_1}{\langle n_e \rangle}$, 3) $\frac{a_2}{\langle n_e \rangle}$, 4) $\frac{f_1}{\langle E_{||} \rangle}$, 5) $\frac{f_2}{\langle E_{||} \rangle}$.

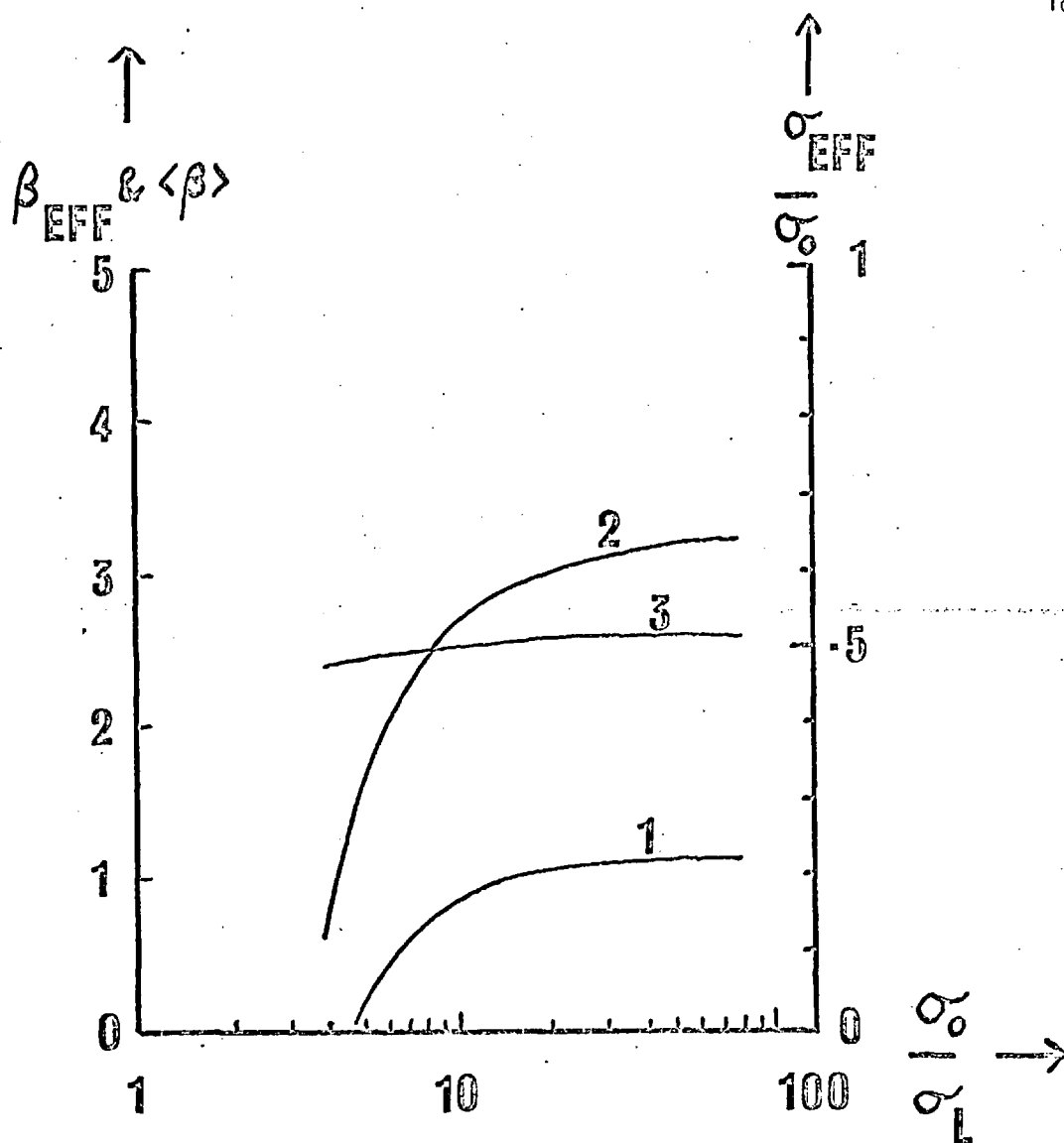


Fig.(5.8) Effective Hall parameter and conductivity and average Hall parameter versus $\frac{\sigma_0}{\sigma_L}$

1) β_{EFF} , 2) $\frac{\sigma_{EFF}}{\sigma_0}$, 3) $\langle \beta \rangle$.

APPENDIX A.The Radiative- Collisional Theory of Ionization.

In the Caesium plasma considered in this thesis, where the electron thermal energy is an order of magnitude below the ionization energy, the processes of ionization and recombination are very complicated. The dominant ionization process is stepwise transitions from the atomic ground state to the various excited states, and then ionization from an excited state.

The population densities of the excited states, and all the various mechanisms affecting them, are therefore of great importance. The problem of determining these densities and the ionization and recombination rates can be formulated in the following way. For each excitation level i (i , equal to $1 \rightarrow \infty$, is the principal quantum number with $i = 1$ representing the ground state i.e. $n_{a0} = n_{a1}$) we can write an equation for $\frac{\partial n_{ai}}{\partial t}$, where n_{ai} is the population density of the i th shell, in terms of the various transition processes into and out of that shell (Note that in Chapter 2 the subscript i refers to particular states not shells). This is,

$$\frac{\partial n_{ai}}{\partial t} = n_e \left[\sum_{j>i} n_{aj} S_{ji}^d - \sum_{j<i} n_{ai} S_{ij}^d + \sum_{j<i} n_{aj} S_{ji}^e - \sum_{j>i} n_{ai} S_{ij}^e + n_e^2 R_i^c - n_{ai} I_i^c \right] +$$

continued over

$$\begin{aligned}
& + \sum_{j>i} n_{aj} A_{ji} - \sum_{j<i} n_{ai} A_{ij} + n_e^2 R_i^r \\
& - n_i \int_0^\infty \epsilon_\nu I_i^r(\nu) d\nu + \sum_{j<i} n_{aj} \int_0^\infty \epsilon_\nu M_{ji}(\nu) d\nu \\
& - \sum_{j>i} n_{ai} \int_0^\infty \epsilon_\nu M_{ij}(\nu) d\nu - \sum_{j<i} n_{ai} \int_0^\infty \epsilon_\nu N_{ij}(\nu) d\nu \\
& + \sum_{j>i} n_{aj} \int_0^\infty \epsilon_\nu N_{ji}(\nu) d\nu
\end{aligned}$$

— (A.1)

The first six terms of this equation are collisional transitions, i.e. collisions with free electrons, and the last eight are radiative transitions. Individually the physical processes are:-

1) Collisional de-excitation (terms 1 and 2)

S_{ij}^d is the average rate coefficient for collisional de-excitation from level i to level j . By average we mean that if $s_{ij}^d(E_e)$ (E_e) is the de-excitation rate coefficient for electrons of energy E_e then,

$S_{ij}^d = \frac{1}{n_e} \int_0^\infty f_e(E_e) s_{ij}^d(E_e) dE_e$, where $f_e(E_e)$ is the electron distribution function.

2) Collisional excitation (terms 3 and 4)

S_{ij}^e is the average rate coefficient for collisional excitation from level i to level j .

3) Collisional Recombination (term 5)

R_i^c is the average rate coefficient for three body recombination

(i.e. two electrons and an ion) into the i^{th} level.

4) Collisional Ionization (term 6)

I_i^C is the average rate coefficient for collisional ionization from the i^{th} level.

5) Spontaneous Emission (terms 7 and 8)

A_{ij} is the spontaneous transition probability from the i^{th} to the j^{th} levels.

6) Radiative Recombination (term 9)

R_i^R is the average (over electron distribution function) radiative recombination rate coefficient. In this process the recombination energy is removed by an emitted photon, instead of a second electron as in three body recombination.

7) Photoionization (term 10)

$I_i^R(\nu)$ is the rate coefficient for photoionization from the i^{th} level in a radiation field of unit photon density of frequency ν . ϵ_ν is the density of photons of frequency ν .

8) Photon Absorption (terms 11 and 12)

$M_{ij}(\nu)$ is the rate coefficient for the induced transition from the i^{th} to the j^{th} level in a radiation field of unit photon density, of frequency ν , where $j > i$.

9) Induced Emission (terms 13 and 14)

$N_{ij}(\nu)$ is the rate coefficient for the induced transition from the i^{th} to the j^{th} level in a radiation field of unit photon density of

frequency ν , where $j < i$.

In addition to (A.1) we have the Boltzmann equation for the determination of $f(E_e)$ at all points in the plasma, and the equation of radiative transfer for the determination of ϵ_ν at all points in the plasma. Solving this set of three equations will then give us the time evolution of the population densities and the ionization in the plasma. This however is a very complicated, and so far unattempted, problem. It can fortunately be simplified.

The necessity to solve the equation of radiative transfer can be removed in two ways, either by assuming an optically thin or an optically thick plasma. In the first case the radiation field is weak and the last five terms in equation (A.1) can be neglected. In the second case the excitation levels are in equilibrium with the radiation field and the last eight terms cancel and can be dropped from the equations.

In fact it is necessary to use a combination of both assumptions in the density regime of Caesium that we consider for MHD generators. This is because the plasma is optically thick to resonance radiation and optically thin for all other radiation.

We then have a system of equations,

$$\frac{\partial n_{ai}}{\partial t} = F_i(n_{aj}, j=1 \rightarrow \infty, n_e, f_e)$$

for $i = 1 \rightarrow \infty$,

coupled with the Boltzmann equation. These equations have been solved numerically assuming a finite number of levels (1.4), (1.5). For instance Shaw et. al. assume a three level model, the ground state,

the first excitation level and a "lumped" level to simulate the effects of all the higher levels. At the time that these calculations were performed it was felt that the uncertainties in the various cross-sections and transition probabilities made a more complicated model unjustified.

Interesting though these calculations are, they do not provide us with a simple equation for the rate of ionization and radiation which we require for the electrothermal wave theory. The theory by Bates, Kingston, and McWhirter, however, points the way. (2.2), (2.3)

In their theory they assume that $f_e(E_e)$ is a Maxwellian so that the f_e dependence of F_i becomes a T_e dependence. From the available estimates of the various cross-sections and transition rates they concluded that the population densities of all levels n_{ai} where $i = 2, \dots, \infty$ would reach equilibrium from any perturbed state much faster than n_{ai} or n_e . The idea of their theory is then that all the excited levels are in a quasi-equilibrium in any changing situation and we could put $\frac{\partial n_{ai}}{\partial t} = 0$ for $i \geq 2$, giving us the set of equations

$$\frac{\partial n_{ai}}{\partial t} = F_i(n_{aj}, j=1 \rightarrow \infty, n_e, T_e)$$

and $0 = F_i(n_{aj}, j=1 \rightarrow \infty, n_e, T_e), i = 2, 3, \rightarrow \infty.$

If we once again consider only a finite level scheme of N levels ($i = 1, 2, \dots, N$), we can then solve the $N - 1$ equations.

$$F_i(n_{aj}, j=1 \rightarrow N, n_e, T_e) = 0, \quad i = 2 \rightarrow N$$

for n_{ai} , $i = 2, \dots, N$, as a function of n_{a1} , n_e and T_e . Substituting these solutions into the equation for $\frac{\partial n_{a1}}{\partial t}$ we obtain,

$$\frac{\partial n_{ai}}{\partial t} = F(n_{ai}, n_e, T_e)$$

(Note that,
$$\frac{\partial n_e}{\partial t} = - \sum_{i=1}^N \frac{\partial n_{ai}}{\partial t} ,$$

and on the basis of the quasi-equilibrium approximation,

$$\frac{\partial n_e}{\partial t} = - \frac{\partial n_{ai}}{\partial t})$$

The form of this equation turns out to be

$$\frac{\partial n_{ai}}{\partial t} = - \frac{\partial n_e}{\partial t} = A_2 n_e^3 - A_1 n_e n_{ai}$$

where A_1 and A_2 are functions of temperature. Bates et. al. use parameters α and S , where

$$\alpha = n_e A_2$$

$$\text{and } S = A_1.$$

They define α to be the collisional radiative recombination coefficient and S to be the collisional radiative ionization coefficient. The values of these coefficients for various densities and temperatures in hydrogen are tabulated numerically by Bates et. al., which is inconvenient for our purposes since we would really like an analytic formula for A_1 and A_2 as a function of T_e .

Using a fairly crude classical argument Hinnov and Hirschberg (2.1) have deduced a formula for the recombination coefficient A_2 . One important feature of their argument is that it demonstrates that A_2 is to a large extent independent of the atomic species involved.

Their argument is complicated but can be summarized as follows.

We define $i^{\dagger}$ to be the level such that the number of electrons transferred above it from the continuum which eventually recombine (i.e. to the grand state) is equal to the number transferred below it, which subsequently will be reionized. From statistical arguments Hinnoy and Hirschberg expect that $i^{\dagger}$ will be the level where the probability of a one level jump by collisional excitation is equal to the probability of a one level jump by collisional de-excitation. Hence they can determine $i^{\dagger}$.

Now using the classical Thomson formula for the energy transfer in electron - electron collisions a formula for the cross-section for ionization from the i^{th} level by an electron of energy E_e can be obtained. By averaging over a Maxwellian, including the appropriate electron velocity, an ionization rate from the i^{th} level for a given value of n_{ai} can be obtained. By assuming that the inverse recombination rate must be the same when n_{ai} is determined by the Saha function, we obtain a recombination rate into the i^{th} level.

Summing this over $i = 1$ to $i = i^{\dagger}$ a total recombination rate is obtained, giving

$$A_2 = 1.1 \times 10^{-20} T_e^{-9/2} \frac{m^3}{\text{sec.}}$$

A_1 can then be determined assuming again Saha equilibrium between n_{a1} and n_e . A_1 is of course very much dependent on the atomic species via I_p , the ionization potential.

Similar results have been obtained by Mackin and Keck ^(A.1) and D'Angelo ^(A,2) using similar heuristic arguments. The results in fact agree fairly well with the more exact evaluation of Bates et. al. We must remember however, that we have assumed a Maxwellian for the electron distribution function and Saha equilibrium for the state to which the plasma tends.

The detailed calculations of Shaw et. al. and Lo Surdo show that these are not good approximations below an electron density of $10^{19} \frac{1}{m^3}$ in a current carrying plasma. The ionization process is largely carried out by the electrons in the high energy tail. This therefore becomes depleted and, if the electron collisions are not sufficiently numerous to fill up the tail again, then the distribution function is markedly non Maxwellian in this energy region, and the quasi-equilibrium electron density in a current carrying plasma is well below Saha.

Bearing in mind this limitation of the collisional radiative theory we have used the Hinnoy and Hirschberg formulation in this thesis. Most of the results presented here are for $n_e > 10^{19} \frac{1}{m^3}$ and the results presented for $n_e < 10^{19} \frac{1}{m^3}$ (all are in Chapter 2), represent the worst possible behaviour of the electrothermal wave in terms of instability, since assuming a Maxwellian and Saha assumes an ionization rate much greater than the one that actually exists in the plasma.

In conclusion let us consider the excitation energy required to populate the energy levels of the Caesium atom. If we assume that the energy levels are at L.T.E. with respect to the electron temperature

then the total excitation energy of the plasma can be written as,

$$E_E = n_{el} \sum_{i=1}^{\infty} h\nu_i i^2 \exp\left(-\frac{h\nu_i}{kT_e}\right) \quad - (A.2)$$

where $h\nu_i$ = energy difference between i^{th} level and the ground state, i^2 is the ratio of the degeneracy of the i^{th} level to that of the ground state.

This energy varies rapidly with T_e and represents a component of the electron thermal capacity in the same way that the ionization energy does.

Bityurin and Ivanov ^(A.3) have argued that the excitation energy is significant compared to the electron thermal energy and must therefore be taken account of in the electron energy equation. In fact the sum (A.2) diverges to infinity, and the statistical theory is saved from catastrophe by the reduction of the ionization potential in a plasma. From Griem ^(2.7) (pp. 139-140) this reduction is given by, $\Delta I_p = \frac{e^2}{4\pi\epsilon_0\rho_D}$ where ρ_D is the Debye length, i.e. $\rho_D = \sqrt{\frac{\epsilon_0 k T_e}{n_e e^2}}$.

Putting in typical values for our plasma ($T_{eo} = 2500^\circ\text{K}$, $n_e = 2 \times 10^{20} \frac{1}{\text{m}^3}$) we obtain $\Delta I_p \approx .069 \text{ ev.}$, i.e. the reduction is about .00177 times the ionization potential (3.9 volts). Hence, since $h\nu_i \approx I_p \left(1 - \frac{0.7}{i^2}\right)$ we have only about 21 levels before the new ionization limit, therefore the sum in (A.2) is taken over only the first 21 levels. Defining i_{max} to be the principal quantum number before the reduced ionization limit we can therefore write,

$$E_E = n_{a1} \sum_{i=1}^{i_{\max}} h\nu_i i^2 \exp\left(-\frac{h\nu_i}{kT_e}\right)$$

where $i_{\max}$ is a function of n_{e0} and T_{e0} ($i_{\max} \propto \left(\frac{n_{e0}}{T_{e0}}\right)^{1/4}$),

Therefore we must add the term $I_E \frac{\partial T_e}{\partial t}$, where,

$$I_E = n_{a1} \sum_{i=1}^{i_{\max}} \left(\frac{h\nu_i}{kT_e}\right)^2 k i^2 \exp\left(-\frac{h\nu_i}{kT_e}\right)$$

to the energy equation.

This term is about 10% of the ionization term, and is therefore usually not very significant for the Ionization Mode of electrothermal waves. It is however, significant in the Fast Thermal Mode since it is the same order of magnitude as the thermal energy. The calculations of Chapter 2 do not include this term, but the calculations of Chapter 3 do.

Note that $\sum_{i=2}^{i_{\max}} n_{ai} \ll n_{a1}$ due to the Boltzmann exponential factors and it is a good approximation to write

$$n_{a1} = n_a = n_s - n_e \text{ as in Chapter 2.}$$

APPENDIX B.Linearization of the Radiation Transfer and Elastic Losses.

The Radiation transfer term used in this thesis is that derived by Lutz (2.4) for a nonequilibrium seeded plasma. The term is derived by solving the equation of radiative transfer, viz.

$$\frac{dI_{\nu}}{ds} = -m_{\nu} I_{\nu} + J_{\nu} \quad \text{--- (B.1)}$$

where I_{ν} = specific intensity, i.e. the radiative energy flux for photons of frequency ν per unit area, per second in some specified direction

s = path length in this direction

m_{ν} = absorption coefficient at frequency ν ,

and J_{ν} = rate of energy emission per unit volume in the specified direction, and at frequency ν ,

If the plasma properties vary in the y-direction only we can replace ds by $dy/\cos \theta$, where θ is the angle between the photon path and the y-axis. The theory assumes also that resonance radiation from the lowest doublet state dominates the energy transfer in the plasma. This follows from the fact that these states are by far the most densely populated excitation levels (for an L.T.E. plasma $n_{ai} \propto \exp(-\frac{h\nu_i}{kT_e})$), in this Appendix, as in Chapter 2, i refers to a specific state rather than a shell as in Appendix A).

The L.T.E. population of the lowest resonance states (i.e. $6^2P_{1/2}$ and

$6^2P_{3/2}$) is a consequence of the strong self absorption of the radiation from these levels.

The cross section for excitation and de-excitation from these states by electron collisions is of the order of 10^{-18} m^2 . Hence with $n_e \sim 10^{20} \frac{1}{\text{m}^3}$ and the electron thermal velocity of the order of $10^5 \frac{\text{m}}{\text{sec}}$, we obtain,

$$\tau_{\text{coll}} \sim \frac{1}{10^{20} \times 10^{-18} \times 10^5}$$

$$\sim 10^{-7} \text{ secs.}$$

for the characteristic time for collisional transitions to and from these states.

Now the characteristic time for radiative transitions is of the order of 10^{-8} secs, hence it would seem that the radiative transitions ought to be dominant. However the absorption coefficient for the lowest resonance radiation is of the order of $10^5 \frac{1}{\text{m}}$. Hence the probability of a photon being recaptured in a plasma of dimension larger than 1 cm, say, is fairly high. This increases the effective lifetime of the excited states, and collisions are the dominant transition mechanism.

Hence the resonance states are at L.T.E. with respect to the electron temperature, and the radiation field is in equilibrium with this excitation population. Therefore we can apply Kirchhoff's Law to obtain J_ν , i.e. $J_\nu = m_\nu B_\nu$, where B_ν is the Planck function at the electron temperature,

$$B_\nu = \left(\frac{2 h \nu^3}{c^2} \right) \frac{1}{e^{\frac{h\nu}{kT_e}} - 1}$$

Solving equation (B.1) and assuming that m_ν is constant in space, Lutz derives the following formula for the radiative energy loss per unit volume in an infinite plasma,

$$\begin{aligned}
 R &= \sum_i \left\{ 2\pi B_i(y) \Delta\nu_i m_{pi}^{1/2} \int_0^{m_{pi}} m_\nu^{-1/2} \left(1 + \frac{m_\nu}{2m_{pi}}\right) dm_\nu \right. \\
 &\quad - \pi \Delta\nu_i m_{pi}^{1/2} \int_0^{m_{pi}} m_\nu^{1/2} \left(1 + \frac{m_\nu}{2m_{pi}}\right) dm_\nu \int_1^\infty \frac{dz}{z} \times \\
 &\quad \left. \left[\int_y^\infty B_i(t) e^{-m_\nu z(t-y)} dt + \int_{-\infty}^y B_i(t) e^{-m_\nu z(y-t)} dt \right] \right\} \\
 &= \sum_i R_i(y)
 \end{aligned}$$

R_i is the sum of an emission term (the 1st integral) plus an absorption term (the 2nd and 3rd integrals), i.e. $R_i = R_{Ei} + R_{Ai}$

The perturbation of R comes from the perturbation of B_i ,

$$\begin{aligned}
 B_i(y) &= \frac{2h\nu_i^3}{c^2} \frac{1}{\exp\left(\frac{h\nu_i}{kT_e(y)}\right) - 1} \\
 &\approx \frac{2h\nu_i^3}{c^2} \exp\left(-\frac{h\nu_i}{kT_e(y)}\right)
 \end{aligned}$$

since $\frac{h\nu_i}{kT_e} \gg 1$ for the resonance doublet.

$$\therefore B'_i(y) = 2 \left(\frac{h\nu_i}{kT_{e0}} \right)^2 \frac{k}{\lambda_i^2} e^{-\frac{h\nu_i}{kT_{e0}}} T'_e(y)$$

now $T'_e(y)$ is proportional to $\exp(-iKy)$, hence to evaluate R'_{Ai} we have to evaluate the following double integral,

$$I(m_\nu) = \int_1^\infty \frac{dz}{z} \left[\int_y^\infty e^{-iKt - m_\nu z(t-y)} dt + \int_{-\infty}^y e^{-iKt - m_\nu z(y-t)} dt \right]$$

letting $u = t - y$, we have

$$\begin{aligned} I(m_\nu) &= e^{-iKy} \int_1^\infty \frac{dz}{z} \left[\int_0^\infty e^{-(iK + m_\nu z)u} du + \int_{-\infty}^0 e^{(-iK + m_\nu z)u} du \right] \\ &= e^{-iKy} \int_1^\infty \frac{dz}{z} \left[\frac{1}{iK + m_\nu z} + \frac{1}{m_\nu z - iK} \right] \\ &= e^{-iKy} \int_1^\infty \frac{2m_\nu}{K^2 + m_\nu^2 z^2} dz \\ &= e^{-iKy} \frac{2}{K} \tan^{-1} \frac{K}{m_\nu} \end{aligned}$$

Hence,

$$R'_i = R'_{Ei} + R'_{Ai} =$$

continued over

$$= 2\pi \Delta\nu_i m_{pi}^{1/2} B_i' \left[\int_0^{m_{pi}} m_\nu^{-1/2} \left(1 + \frac{m_\nu}{2m_{pi}}\right) dm_\nu \right. \\ \left. - \int_0^{m_{pi}} \frac{m_\nu^{1/2}}{K} \left(1 + \frac{m_\nu}{2m_{pi}}\right) \tan^{-1}\left(\frac{K}{m_\nu}\right) dm_\nu \right]$$

letting $\frac{m_\nu}{K} = v$, $\frac{K}{m_{pi}} = w$ the expression in the square brackets is given by

$$K^{1/2} \left\{ \left[2v^{1/2} + \frac{w}{3}v^{3/2} \right]_0^{1/w} - \left[\frac{2}{3}v^{3/2} \tan^{-1}\frac{1}{v} \right. \right. \\ \left. \left. + \frac{w}{5}v^{5/2} \tan^{-1}\frac{1}{v} \right]_0^{1/w} - \int_0^{1/w} \left(\frac{2}{3} \frac{v^{3/2}}{1+v^2} \right. \right. \\ \left. \left. + \frac{wv^{5/2}}{5(1+v^2)} \right) dv \right\}$$

Letting $r = v^2$ this becomes

$$K^{1/2} \left\{ \frac{13}{15} \frac{1}{w^{1/2}} - \frac{13}{15w^{3/2}} \tan^{-1}w - \int_0^{1/w} \frac{\frac{4}{3}u^4 + \frac{2w}{5}u^6}{1+u^4} du \right\} \\ = K^{1/2} \left\{ \frac{7}{3} \frac{1}{w^{1/2}} - \frac{13}{15w^{3/2}} \tan^{-1}w + \right. \\ \left. \frac{(4/3 - \frac{2w}{5})}{4\sqrt{2}} \log \left(\frac{w + \sqrt{2w} + 1}{w - \sqrt{2w} + 1} \right) + \frac{(4/3 + \frac{2w}{5})}{2\sqrt{2}} \left(\pi - \tan^{-1} \frac{\sqrt{2w}}{1-w} \right) \right\}$$

since $w \ll 1$, due to the large self absorption we can expand about

$$w = 0, \text{ to get, } R_i' = 2\pi \Delta\nu_i m_{pi}^{1/2} B_i' \frac{2}{3\sqrt{2}} \pi K^{1/2},$$

to lowest order in w .

$$\text{i.e. } R_i' \approx 3\pi \Delta \nu_i m_{pi}^{1/2} \left(\frac{h\nu_i}{kT_{e0}} \right)^2 \frac{k}{\lambda_i^2} K^{1/2} e^{-\frac{h\nu_i}{kT_{e0}}} T_e^*$$

When evaluating the τ_2 term of $\hat{R}_i$ in section 5 we obtain similar results as for r_1 only a factor $(2K)^{\frac{1}{2}}$ appears instead of $(K)^{\frac{1}{2}}$.

consist of losses

The equipartition elastic losses by Coulomb collisions to the Caesium ions and losses by neutral collisions to the noble gas. We can therefore write $\mathcal{E} = \mathcal{E}_c + \mathcal{E}_n$

Now

$$\mathcal{E}_c \propto n_e^2 T_e^{-3/2} \log \mathcal{L}_c (T_e - T)$$

where

$$\mathcal{L}_c \propto \frac{T_e^{3/2}}{n_e^{1/2}}$$

$$\begin{aligned} \therefore \mathcal{E}_c' &= \frac{\mathcal{E}_{c0}}{\log \mathcal{L}_{c0}} \left[\frac{3}{2} T_e^* - \frac{n_e^*}{2} \right] + \mathcal{E}_{c0} \left[2n_e^* - \frac{3}{2} T_e^* + \frac{T_{e0}}{T_{e0} - T} T_e^* \right] \\ &= \mathcal{E}_{c0} \left[n_e^* (2 - u) + T_e^* \left(3u + \frac{3T - T_{e0}}{2(T_{e0} - T)} \right) \right] \end{aligned}$$

where

$$u = \frac{1}{2 \log \mathcal{L}_{c0}}$$

and

$$\mathcal{E}_n \propto n_e T_e^{1/2} (T_e - T) q_{en}$$

$$\therefore \mathcal{E}_n' = \mathcal{E}_{n0} \left[n_e^* + T_e^* \left(\frac{5T_{e0} - 3T}{2(T_{e0} - T)} \right) \right],$$

where we assume

$$\frac{T_{e0}}{q_{en0}} \frac{\partial q_{en}}{\partial T_e} = 1.$$

hence,

$$\mathcal{E}' = \frac{j_o^2}{\sigma_o} \left[n_e^* (1 + (1-u)s) + T_e^* \left(\frac{5T_{eo} - 3T}{2(T_{eo} - T)} \right) \right. \\ \left. 3s(u-1) \right], \quad \text{where} \quad s = \left(1 + \frac{2n_o m_s}{\chi_{co} m_n} \right)^{-1}$$

since $\frac{j_o^2}{\sigma_o} = \mathcal{E}_o$

Note: In some papers on nonequilibrium plasmas an empirical factor δ appears in the expression for $\mathcal{E}$. This is in order to account for the electron energy loss due to inelastic collisions. Since this energy is removed from the plasma by radiation, we take account of these losses in the radiation transfer term in this thesis. Hence $\mathcal{E}$ represents the energy loss due solely to elastic collisions.

APPENDIX C.Elements of the Electrosonic Dispersion Relation Matrix.

The linearized equations for electrosonic waves is

$$M \xi = 0, \text{ where}$$

$$\xi^+ = (n_e^*, T_e^*, v_{ex}', v_{ey}', n^*, T^*, v_x', v_y', E_y')$$

The elements of M are:-

Row 1: electron density equation.

$$\begin{aligned} M_{11} &= n_{e0} (i\omega - iK v_{ey0}) + n_{e0} \left(\frac{\partial v_{ex0}}{\partial x} + \frac{\partial v_{ey0}}{\partial y} \right) \\ &+ A_{10} (2n_{e0} - n_{s0}) n_{e0} + 3 A_{20} n_{e0}^3, \\ M_{12} &= - \frac{\partial A_{10}}{\partial T_{e0}} T_{e0} n_{e0} (n_{s0} - n_{e0}) - \frac{\partial A_{20}}{\partial T_{e0}} n_{e0}^3 T_{e0}, \\ M_{13} &= \frac{\partial n_{e0}}{\partial x}, \quad M_{14} = \frac{\partial n_{e0}}{\partial y} - iK n_{e0}, \\ M_{15} &= - A_{10} n_{e0} n_{s0}, \\ M_{16} &= M_{17} = M_{18} = M_{19} = 0 \end{aligned}$$

Row 2: heavy particle density equation.

$$\begin{aligned} M_{21} &= M_{22} = M_{23} = M_{24} = M_{26} = M_{29} = 0, \\ M_{25} &= (i\omega + \frac{\partial v_{oy}}{\partial y} + \frac{\partial v_{ox}}{\partial x} - iK v_{oy}) n_0, \\ M_{27} &= \frac{\partial n_0}{\partial x}, \quad M_{28} = \frac{\partial n_0}{\partial y} - iK n_0 \end{aligned}$$

Row 3: Ohm's Law x-component.

$$\begin{aligned} M_{31} &= \frac{\partial \sigma}{\partial n_{e0}} n_{e0} \frac{j_{ox}}{\sigma_0} - \frac{\partial \beta_0}{\partial n_{e0}} \frac{n_{e0}}{(1 + \beta_0^2)} (j_{oy} + \beta_0 j_{ox}) \\ &- j_{ox} - \frac{\sigma}{n_{e0} e (1 + \beta_0^2)} \left[\frac{\partial p_{e0}}{\partial x} - \beta_0 \frac{\partial p_{e0}}{\partial y} \right] + \frac{\beta_0 \sigma_0}{1 + \beta_0^2} \frac{iK k T_{e0}}{e}, \end{aligned}$$

$$M_{32} = \frac{\partial \sigma_0}{\partial T_{e0}} \frac{j_{ox}}{\sigma_0} T_{e0} - \frac{\partial \beta_0}{\partial T_{e0} (1 + \beta_0^2)} T_{e0} (j_{oy} + \beta_0 j_{ox}) \\ + \frac{\beta_0 \sigma_0}{1 + \beta_0^2} i K \frac{k T_{e0}}{e},$$

$$M_{33} = n_{e0} e, \quad M_{34} = M_{36} = M_{38} = 0,$$

$$M_{35} = \frac{\partial \sigma_0}{\partial n_0} \frac{j_{ox}}{\sigma_0} n_0 - \frac{\partial \beta_0}{\partial n_0} \frac{n_0}{1 + \beta_0^2} (j_{oy} + \beta_0 j_{ox}),$$

$$M_{37} = -n_{e0} e, \quad M_{39} = -\frac{\beta_0 \sigma_0}{1 + \beta_0^2}$$

Row 4: Ohm's Law y-component.

$$M_{41} = \frac{\partial \sigma_0}{\partial n_{e0}} n_{e0} \frac{j_{oy}}{\sigma_0} - \frac{\partial \beta_0}{\partial n_{e0}} \frac{n_{e0}}{1 + \beta_0^2} [\beta_0 j_{oy} - j_{ox}] \\ - \frac{\sigma_0}{n_{e0} (1 + \beta_0^2)} \left(\frac{\partial p_{e0}}{\partial x} \beta_0 + \frac{\partial p_{e0}}{\partial y} \right) - \frac{\sigma}{1 + \beta_0^2} i K \frac{k T_{e0}}{e} - j_{oy},$$

$$M_{42} = \frac{\partial \sigma}{\partial T_{e0}} T_{e0} \frac{j_{oy}}{\sigma_0} - \frac{\partial \beta_0}{\partial T_{e0}} T_{e0} [\beta_0 j_{oy} - j_{ox}] - \frac{\sigma_0}{1 + \beta_0^2} i K \frac{k T_{e0}}{e},$$

$$M_{43} = M_{46} = M_{47} = 0,$$

$$M_{44} = n_{e0} e,$$

$$M_{45} = \frac{\partial \sigma}{\partial n_0} n_0 \frac{j_{oy}}{\sigma_0} - \frac{\partial \beta}{\partial n_0} \frac{n_0}{1 + \beta_0^2} [\beta_0 j_{oy} - j_{ox}],$$

$$M_{48} = -n_{e0} e, \quad M_{49} = \frac{\sigma_0}{1 + \beta_0^2}$$

Row 5: x-component of gas momentum equation.

$$M_{51} = k n_{e0} \frac{\partial T_{e0}}{\partial x} - j_{0y} B, \quad M_{52} = k T_{e0} \frac{\partial n_{e0}}{\partial x},$$

$$M_{53} = M_{59} = 0, \quad M_{54} = n_{e0} e B,$$

$$M_{55} = \rho_0 (\underline{v}_0 \cdot \nabla) v_{0x} + k n_0 \frac{\partial T_0}{\partial x}, \quad M_{56} = k \frac{\partial n_0}{\partial x} T_0,$$

$$M_{57} = \rho_0 i \omega + \rho_0 \frac{\partial v_{0x}}{\partial x} - \rho_0 v_{0y} i K + K^2 \mu_0,$$

$$M_{58} = \rho_0 \frac{\partial v_{0x}}{\partial y} - n_{e0} e B$$

Row 6: y-component of gas momentum equation.

$$M_{61} = n_{e0} k \left(\frac{\partial T_{e0}}{\partial y} - i K T_{e0} \right) + j_{0x} B$$

$$M_{62} = k T_{e0} \left(\frac{\partial n_{e0}}{\partial y} - i K n_{e0} \right), \quad M_{63} = -n_{e0} e B,$$

$$M_{64} = M_{69} = 0,$$

$$M_{65} = \rho_0 (\underline{v}_0 \cdot \nabla) v_{0y} + n_0 k \left(\frac{\partial T_0}{\partial y} - i K T_0 \right),$$

$$M_{66} = k T_0 \left(\frac{\partial n_0}{\partial y} - i K n_0 \right),$$

$$M_{67} = \rho_0 \frac{\partial v_{0y}}{\partial x} + n_{e0} e B,$$

$$M_{68} = i \omega \rho_0 + \rho_0 \left(\frac{\partial v_{0y}}{\partial y} - i K v_{0y} \right) + \frac{4}{3} \mu_0 K^2$$

Row 7: electron energy equation.

$$M_{71} = \frac{3}{2} n_{e0} k T_{e0} i\omega + \frac{5}{2} n_{e0} k T_{e0} \nabla \cdot \underline{v}_{e0} +$$

$$\frac{3}{2} n_{e0} k (\underline{v}_{e0} \cdot \nabla T_{e0}) - \frac{3}{2} n_{e0} k T_{e0} iK v_{e0y}$$

$$+ iK \frac{\partial T_{e0}}{\partial y} n_{e0} \frac{\partial \kappa_{e0}}{\partial n_{e0}} + n_{e0} \frac{\partial \xi}{\partial n_{e0}} - \frac{2j_o^2}{\sigma_o} + \frac{j_o^2}{\sigma_o^2} n_{e0} \frac{\partial \sigma_o}{\partial n_{e0}}$$

$$+ (i\omega + \nabla \cdot \underline{v}_{e0} - iK v_{e0y}) I_P n_{e0},$$

$$M_{72} = \left(\frac{3}{2} n_{e0} k T_{e0} + I_{E0} \right) i\omega + \frac{5}{2} n_{e0} k T_{e0} \nabla \cdot \underline{v}_{e0}$$

$$+ \frac{3}{2} k T_{e0} \underline{v}_{e0} \cdot \nabla n_{e0} - iK \frac{3}{2} n_{e0} k T_{e0} v_{e0y} + \kappa_{e0}^2 K T_{e0}$$

$$+ iK T_{e0} \frac{\partial T_{e0}}{\partial y} \frac{\partial \kappa_{e0}}{\partial T_{e0}} + T_{e0} \frac{\partial \xi}{\partial T_{e0}} + \frac{j_o^2}{\sigma_o^2} \frac{\partial \sigma_o}{\partial T_{e0}} T_{e0} + T_{e0} \frac{\partial R_o}{\partial T_{e0}},$$

$$M_{73} = \frac{3}{2} n_{e0} k \frac{\partial T_{e0}}{\partial x} + \frac{3}{2} k T_{e0} \frac{\partial n_{e0}}{\partial x} + \frac{2j_{ox}}{\sigma_o} n_{e0} e$$

$$+ \frac{\partial n_{e0}}{\partial x} I_P,$$

$$M_{74} = -iK \frac{5}{2} p_{e0} + \frac{3}{2} \frac{\partial p_{e0}}{\partial y} + \frac{2j_{oy}}{\sigma_o} n_{e0} e$$

$$+ \left(\frac{\partial n_{e0}}{\partial y} - iK n_{e0} \right) I_P,$$

$$M_{75} = iK \frac{\partial T_{e0}}{\partial y} \frac{\partial \kappa_{e0}}{\partial n_o} n_o + n_o \frac{\partial \xi}{\partial n_o}$$

$$+ \frac{j_o^2}{\sigma_o} n_o \frac{\partial \sigma_o}{\partial n_o} + n_o \frac{\partial R_o}{\partial n_o},$$

$$M_{76} = T_o \frac{\partial R_o}{\partial T_o} + T_o \frac{\partial \xi}{\partial T_o}, \quad M_{77} = -\frac{2j_{ox}}{\sigma_o} n_{e0} e,$$

$$M_{78} = -\frac{2j_{oy}}{\sigma_o} n_{e0} e, \quad M_{79} = 0.$$

Row 8: heavy particle energy equation.

$$M_{81} = -\frac{2}{3} n_{e0} \frac{\partial \mathcal{E}_0}{\partial n_{e0}}, \quad M_{83} = M_{84} = M_{89} = 0,$$

$$M_{82} = -\frac{2}{3} T_{e0} \frac{\partial \mathcal{E}_0}{\partial T_{e0}},$$

$$M_{85} = n_0 i \omega \left(k T_0 - a_0^2 \frac{\rho_0}{n_0} \right) + v_{ox} k \frac{\partial T_0}{\partial x} n_0 \\ + v_{oy} \left(k \frac{\partial T_0}{\partial y} - i K k T_0 \right) n_0 + a_0^2 v_{oy} \rho_0 i K \\ - \frac{2}{3} n_0 \frac{\partial \mathcal{E}_0}{\partial n_0} + \frac{2}{3} i K \frac{\partial T_0}{\partial y} \frac{\partial \kappa_{ho}}{\partial n_0}$$

$$M_{86} = i \omega n_0 k T_0 + v_{ox} k \frac{\partial n_0}{\partial x} T_0 + v_{oy} k \left(\frac{\partial n_0}{\partial y} - i K n_0 \right) T_0 \\ - a_0^2 \underline{v}_0 \cdot \nabla \rho_0 - \frac{2}{3} T_0 \frac{\partial \mathcal{E}}{\partial T_0} + \frac{2}{3} i K \frac{\partial T_0}{\partial y} \frac{\partial \kappa_{ho}}{\partial T_0} + \frac{2}{3} \kappa_{oh} K^2$$

$$M_{87} = -a_0^2 \frac{\partial \rho_0}{\partial x} + \frac{\partial p_{ho}}{\partial x} - \frac{2}{3} \mu_0 v_{ox} K^2$$

$$M_{89} = -a_0^2 \frac{\partial \rho_0}{\partial y} + \frac{\partial p_{ho}}{\partial y} - \frac{8}{9} \mu_0 v_{oy} K^2$$

Row 9: Conservation of Charge.

$$M_{91} = -i K \frac{j_{ey}}{e} + n_{e0} \nabla \cdot (\underline{v}_0 - \underline{v}_{e0}),$$

$$M_{92} = M_{95} = M_{96} = M_{99} = 0,$$

$$M_{93} = -\frac{\partial n_{e0}}{\partial x},$$

$$M_{94} = -\frac{\partial n_{e0}}{\partial y} + i K n_{e0},$$

$$M_{97} = \frac{\partial n_{e0}}{\partial x}, \quad M_{98} = \frac{\partial n_{e0}}{\partial y} - i K n_{e0}$$

APPENDIX D.Calculation of the Dispersion Relations for Boundary Effects.

The evaluation and solution of the dispersion relation, and the transformation of $\bar{\xi}_i$ back into real time involves a great deal of complicated algebra; a brief discription only will be given here.

Formulating the general boundary condition problem in terms of the matrix P (as in section (4.2)) gives a clear and succinct account of the analysis. In addition the proof that the contribution to the solution from the poles at $\det D = 0$ vanishes for all boundary conditions is most easily carried out in terms of this general formulation (see Appendix E).

However, $\bar{\xi}_i$ will not be derived here by explicitly deriving the inverse of P. Rather we shall use the last four rows of equation (4.9) to derive relations between the elements of μ . Then the first two rows, i.e. the boundary conditions, will be used to determine $\mu_{2,5}$ or $\mu_{3,6}$ in terms of the boundaries and the steady state. From this all the elements of μ can be derived from the above relations, and hence $\bar{\xi}_i$ can be written in terms of known quantities.

If we use the last two rows of (4.8) to derive the last four rows of (4.9) then we have,

$$P_{3i} = C_{2i}(\lambda_i) \quad i = 1, 2, 3$$

$$P_{4i} = C_{3i}(\lambda_1) \quad i = 1, 2, 3$$

$$P_{5i+3} = C_{2i}(\lambda_2) \quad i = 1, 2, 3$$

$$P_{6i+3} = C_{3i}(\lambda_2) \quad i = 1, 2, 3$$

Solving rows 3 and 4 of (4.9) for simple relations between μ_1 , μ_2 and μ_3 we obtain,

$$\mu_i = \frac{cf C_{1i}(\lambda_1)}{cf C_{1j}(\lambda_1)} \mu_j \quad i, j = 1, 2, 3 \quad \text{--- (D.1)}$$

Similarly, from rows 5 and 6,

$$\mu_{i+3} = \frac{cf C_{1i}(\lambda_2)}{cf C_{1j}(\lambda_2)} \mu_{j+3} \quad i, j = 1, 2, 3 \quad \text{--- (D.2)}$$

Having obtained one of the pairs $\mu_{2,5}$ or $\mu_{3,6}$ from the boundary conditions we can substitute for them in these relations to obtain all the elements of μ .

We can then carry out the inverse transform of $\bar{\xi}_1$, say, by finding all the poles of $\bar{\xi}_1$ and evaluating the residues at each pole. We then have

$$\bar{\xi}_1 = \sum_{\text{poles}} (\text{residue at pole}) e^{Z_{\text{pole}} t}$$

where the residue has two terms, one proportional to $e^{\lambda_1(Z_{\text{pole}})x}$ and the other to $e^{\lambda_2(Z_{\text{pole}})x}$, hence,

$$\bar{\xi}_1 = \sum_{\text{poles}} \sum_k \boxed{H}_1 e^{\lambda_k(Z_{\text{pole}})x + Z_{\text{pole}} t}$$

(for brevity we drop the subscripts "pole" and k of $\boxed{H}_1$)

If we express ξ_2 and ξ_3 in the same way, i.e.,

$$\xi_{2,3} = \sum_{\text{poles}} \sum_k \bar{H}_{2,3} e^{\lambda_k(Z_{\text{pole}})x + Z_{\text{pole}}t}$$

then the values of $\bar{H}_{2,3}$ can be determined from equation (4.6).

The first row of equation (4.6) gives,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - Z_1 \right) \xi_1 + (A_{12} + B_{12} \frac{\partial}{\partial x}) \xi_2 = 0 \\ \text{i.e. } \sum_{\text{poles}} \sum_k \{ C_{11}(\lambda_k(Z_{\text{pole}})) \bar{H}_1 + C_{12}(\lambda_k(Z_{\text{pole}})) \bar{H}_2 \} e^{\lambda_k(Z_{\text{pole}})x + Z_{\text{pole}}t} &= 0 \end{aligned}$$

equating the coefficient of each exponential term to zero we obtain,

$$\bar{H}_2 = - \frac{C_{11}(\lambda_k(Z_{\text{pole}}))}{C_{12}(\lambda_k(Z_{\text{pole}}))} \bar{H}_1 \quad \text{--- (D.3)}$$

The second row of (4.6) is

$$(A_{21} + B_{21} \frac{\partial}{\partial x}) \xi_1 + A_{22} \xi_2 + (A_{23} + B_{23} \frac{\partial}{\partial x}) \xi_3 = 0$$

i.e. we have,

$$\sum_{\text{poles}} \sum_k \left\{ \left(C_{21} - \frac{C_{11}C_{22}}{C_{12}} \right) \bar{H}_1 + C_{23} \bar{H}_3 \right\} e^{\lambda_k(Z_{\text{pole}})x + Z_{\text{pole}}t} = 0$$

where we have substituted from (D.3) for $\bar{H}_2$, and the $\lambda_k, Z_{\text{pole}}$ dependence of the elements of C is understood.

$$\begin{aligned} \text{Hence } \bar{H}_3 &= \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{12}C_{23}} \bar{H}_1 \\ &= \frac{C_{11}C_{33}}{C_{12}C_{31}} \bar{H}_1 \end{aligned}$$

(note that $C_{13} = 0$).

Therefore if we can derive ξ_1 by the inverse transform we can immediately obtain ξ_2 and ξ_3

We will now describe the dispersion relation for specific boundary conditions.

Insulator Walls.

The boundary conditions are,

$$\xi_2(x, y) = 0 \quad \text{at } x = 0, d$$

i.e. we have,

$$\mu_2 + \mu_5 = - \frac{c f D_{12}}{\det D} \alpha$$

and $\mu_2 e^{\lambda_1 d} + \mu_5 e^{\lambda_2 d} = - \frac{c f D_{12}}{\det D} \alpha e^{-i K x d}$

$$\therefore \mu_2 = \frac{c f D_{12}}{\det D} \alpha \left[\frac{e^{-i K x d} - e^{\lambda_2 d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right]$$

$$\text{and } \mu_5 = \frac{c f D_{12}}{\det D} \alpha \left[\frac{e^{\lambda_1 d} - e^{-i K x d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right]$$

using (D.1) and (D.2) we can then write,

$$\begin{aligned} \bar{\xi}_1 &= \frac{cf C_{11}(\lambda_1)}{cf C_{12}(\lambda_1)} \frac{cf D_{12}}{\det D} \alpha \left[\frac{e^{-ik_x d} - e^{\lambda_2 d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] e^{\lambda_1 x - i k_y y} \\ &+ \frac{cf C_{11}(\lambda_2)}{cf C_{12}(\lambda_2)} \frac{cf D_{12}}{\det D} \alpha \left[\frac{e^{\lambda_1 d} - e^{-ik_x d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] e^{\lambda_2 x - i k_y y} \\ &+ \frac{\alpha cf D_{11}}{\det D} e^{-i \underline{k} \cdot \underline{r}} \end{aligned}$$

$$\text{now } \det C(\lambda_1, Z) = \det C(\lambda_2, Z) = 0$$

$$\therefore C_{11} cf C_{11} + C_{12} cf C_{12} = 0 \quad \text{if } \lambda = \lambda_1 \text{ or } \lambda_2$$

$$\text{hence} \quad \frac{cf C_{11}}{cf C_{12}} = - \frac{C_{12}}{C_{11}} \quad \text{if } \lambda = \lambda_1 \text{ or } \lambda_2$$

$$\begin{aligned} \text{i.e. } \bar{\xi}_1 &= \frac{cf D_{12}}{\det D} \alpha \left\{ \frac{C_{12}(\lambda_1)}{C_{11}(\lambda_1)} \left[\frac{e^{\lambda_2 d} - e^{-ik_x d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] e^{\lambda_1 x - i k_y y} \right. \\ &+ \left. \frac{C_{12}(\lambda_2)}{C_{11}(\lambda_2)} \left[\frac{e^{-ik_x d} - e^{\lambda_2 d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] e^{\lambda_2 x - i k_y y} \right\} + \frac{\alpha cf D_{11}}{\det D} e^{-i \underline{k} \cdot \underline{r}} \end{aligned}$$

We know that the contribution from $\det D \neq 0$ is zero, hence the poles of $\bar{\xi}_1$ are given by

$$C_{11}(Z) (e^{\lambda_2 d} - e^{\lambda_1 d}) = 0$$

$$\text{i.e. } A_{11}(Z) (e^{\lambda_2 d} - e^{\lambda_1 d}) = 0$$

The zero of the first factor is at $z = z_1$, and the second factor is zero when

$$\lambda_1(z) - \lambda_2(z) = \frac{2\pi i n}{d} \quad - (D.4)$$

$$n = 1, 2, 3, \dots$$

If $\lambda_1 = \lambda_2$, i.e. $n = 0$, the numerator of $\bar{\xi}_1$ is zero also and hence $\bar{\xi}_1$ does not have a pole at $n = 0$. As stated in section (4.3) equation (D.4) is a quadratic in z and therefore there are two poles for each n . The residues at the poles give the solution for $\bar{\xi}_1$ (i.e. n_e^*) shown in section (4.3).

Continuous Electrode Walls.

The boundary conditions here are $\bar{\xi}_3(x, y) = 0$ at $x = 0, d$, i.e. we have,

$$\begin{aligned}\mu_3 + \mu_6 &= - \frac{cf D_{13}}{\det D} \alpha \\ \mu_3 e^{\lambda_1 d} + \mu_6 e^{\lambda_2 d} &= - \frac{cf D_{13}}{\det D} \alpha e^{-iK_x d} \\ \text{i.e. } \mu_3 &= \frac{cf D_{13}}{\det D} \alpha \left[\frac{e^{-iK_x d} - e^{\lambda_2 d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] \\ \mu_6 &= \frac{cf D_{13}}{\det D} \alpha \left[\frac{e^{\lambda_1 d} - e^{-iK_x d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right]\end{aligned}$$

i.e.

$$\begin{aligned}\bar{\xi}_i &= \frac{cf D_{13}}{\det D} \alpha \left\{ \frac{cf C_{11}(\lambda_1)}{cf C_{13}(\lambda_1)} \left[\frac{e^{-iK_x d} - e^{\lambda_2 d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] e^{\lambda_1 x - iK_y y} \right. \\ &+ \frac{cf C_{11}(\lambda_2)}{cf C_{13}(\lambda_2)} \left[\frac{e^{-iK_x d} - e^{\lambda_2 d}}{e^{\lambda_2 d} - e^{\lambda_1 d}} \right] e^{\lambda_2 x - iK_y y} \Big\} \\ &+ \frac{cf D_{11}}{\det D} \alpha e^{-i\vec{k} \cdot \vec{r}}\end{aligned}$$

In this case therefore the poles are given by $\text{cf } C_{13}(Z, \lambda_1(Z)) = 0$ and once again, the quadratic $\lambda_1(Z) - \lambda_2(Z) = \frac{2\pi i n}{d}$, $n = 1, 2, 3, \dots$

Since λ_1 and λ_2 are solutions of a quadratic they represent the two branches of a double valued function (due to the square root in the solution of a quadratic)

and if one branch is a root of $\text{cf } C_{13}(\lambda) = 0$, then the other is not.

Hence if we arbitrarily choose λ_1 to be the root, then we solve

$$\text{cf } C_{13}(\lambda_1) = \lambda_1^2 B_{32} + \lambda_1(A_{21}B_{22} - A_{22}B_{31}) - A_{22}A_{31} = 0$$

for λ_1 . We then substitute the two solutions into $\det C(Z, \lambda_1) = 0$ to obtain the positions of two poles, viz. Z_4 and Z_5 .

The residues at the various poles give the solution shown in section (4.4).

Infinitely Finely Segmented Electrodes.

The boundary conditions here are, for the case of shorted electrodes as shown in Fig. (4.12),

$$\bar{\xi}_2(0,0) = \bar{\xi}_2(d,s)$$

$$\bar{\xi}_3(0,0) = \bar{\xi}_3(d,s)$$

i.e. we have

$$\begin{aligned} & \mu_2(1 - e^{\lambda_1 d - iK_y s}) + \mu_5(1 - e^{\lambda_2 d - iK_y s}) \\ & = -\alpha \frac{\text{cf } D_{12}}{\det D} (1 - e^{-iK_x d - iK_y s}) \end{aligned}$$

and
$$\mu_3 (1 - e^{\lambda_1 d - i K_y S}) + \mu_6 (1 - e^{\lambda_2 d - i K_y S})$$

$$= -\alpha \frac{\mathcal{F} D_{13}}{\det D} (1 - e^{-i K_x d - i K_y S})$$

In this case we do not obtain $\mu_{2,5}$ or $\mu_{3,6}$ explicitly but instead substitute for these quantities in the B.C. using (D.1) & (D.2), i.e.

$$\begin{aligned} & \frac{\mathcal{F} C_{12}(\lambda_1)}{\mathcal{F} C_{11}(\lambda_1)} (1 - e^{\lambda_1 d - i K_y S}) \mu_1 + \\ & \frac{\mathcal{F} C_{12}(\lambda_2)}{\mathcal{F} C_{11}(\lambda_2)} (1 - e^{\lambda_2 d - i K_y S}) \mu_4 = - \frac{\alpha \mathcal{F} D_{12}}{\det D} (1 - e^{-i K_x d - i K_y S}) \\ & \& \frac{\mathcal{F} C_{13}(\lambda_1)}{\mathcal{F} C_{11}(\lambda_1)} (1 - e^{\lambda_1 d - i K_y S}) \mu_1 + \\ & \frac{\mathcal{F} C_{13}(\lambda_2)}{\mathcal{F} C_{11}(\lambda_2)} (1 - e^{\lambda_2 d - i K_y S}) \mu_4 = - \frac{\alpha \mathcal{F} D_{13}}{\det D} (1 - e^{-i K_x d - i K_y S}) \end{aligned}$$

$$\therefore \mu_1 = \frac{\mathcal{F} C_{11}(\lambda_1) (\mathcal{F} D_{12} \mathcal{F} C_{13}(\lambda_2) - \mathcal{F} D_{13} \mathcal{F} C_{12}(\lambda_2))}{(1 - e^{\lambda_1 d - i K_y S}) (\mathcal{F} C_{12}(\lambda_1) \mathcal{F} C_{13}(\lambda_2) - \mathcal{F} C_{13}(\lambda_1) \mathcal{F} C_{12}(\lambda_2))} \times$$

$$\alpha \frac{(e^{-i K_x d - i K_y S} - 1)}{\det D}$$

and,

$$\mu_4 = \frac{\operatorname{cof} C_{11}(\lambda_2) (\operatorname{cof} D_{12} \operatorname{cof} C_{13}(\lambda_1) - \operatorname{cof} D_{13} \operatorname{cof} C_{12}(\lambda_1))}{(1 - e^{\lambda_2 d - i k_y s}) (\operatorname{cof} C_{12}(\lambda_2) \operatorname{cof} C_{13}(\lambda_1) - \operatorname{cof} C_{13}(\lambda_1) \operatorname{cof} C_{12}(\lambda_1))} \times$$

$$\frac{\alpha \left(e^{-i k_x d - i k_y s} - 1 \right)}{\det D}$$

Hence the poles of $\bar{\xi}_1$ are given by $(1 - e^{\lambda_1(z)d - i k_y s}) = 0$, i.e.

$$\lambda_1(z) = i \left(\frac{2\pi n + k_y s}{d} \right)$$

$$n = -\infty \dots \infty$$

and by,

$$\operatorname{cof} C_{12}(\lambda_2(z)) \operatorname{cof} C_{13}(\lambda_1(z)) - \operatorname{cof} C_{12}(\lambda_1(z)) \operatorname{cof} C_{13}(\lambda_2(z)) = 0.$$

Once again only one branch of λ gives a pole for a given n , and we emphasise this by writing λ_1 . The first equation above is linear and gives one pole for each n , while the second equation is a cubic in z and gives three poles. The residues at the poles give the solutions shown in section (4.5).

The solutions for $\bar{\xi}(x, y, t)$ in the three cases of sections (4.3), (4.4), (4.5), have been checked to ensure that they satisfy the appropriate boundary conditions and the equations. It is however very difficult to check if they satisfy the initial conditions. This check has been made for simplified equations where the elements of A and B are chosen so that the resulting series for $\bar{\xi}$ are simple Fourier series. The forms

presented in Chapter 4 have been shown to be correct for such simplified cases.

APPENDIX E.

Proof that the contribution to $\bar{\xi}$ from $\det D = 0$ vanishes.

We want to show that in the inverse Laplace transformation of $\bar{\xi}_i$ the contribution from the pole at $\det D = 0$ vanishes if P is of the form derived in section (4.2).

We have,

$$\bar{\xi}_i = \mu_i e^{\lambda_1 x - i k_y y} + \mu_{i+3} e^{\lambda_2 x - i k_y y} + \frac{(\text{adj } D \eta)_i}{\det D} e^{-i k_x x}$$

where $\mu_i = \frac{(\text{adj } P \zeta)_i}{\det P \det D}$ & $\mu_{i+3} = \frac{(\text{adj } P \zeta)_{i+3}}{\det P \det D}$

Now $(\text{adj } D \eta)_i = \alpha (\text{adj } D)_i = \alpha \text{cf } D_{ii}$, and if P_{kj} is written as a function of λ , $P_{kj} = S_{kj}(\lambda)$, then we can write

$$\zeta_k = -\alpha \sum_{j=1}^3 S_{kj}(-i k_x) \text{cf } D_{ij}, \quad k=1,2$$

The absence of n_e^* in the boundary conditions is not necessary for this proof. For the sake of symmetry will write the first two rows of P as though all the elements are non zero.

Now when $Z = Z_0$ one of the roots of $\det C(z, \lambda) = 0$ is $-i k_x$. Let λ_1 be this root; it follows then that,

$$\zeta_k = -\alpha \sum_{j=1}^3 P_{kj} \text{cf } D_{ij}$$

We can write,

$$\mu_i = \frac{\sum_{k=1}^2 (\text{adj } P)_{ik} \zeta_k}{\det P \det D} = \frac{\sum_{k=1}^2 \text{cf } P_{ki} \zeta_k}{\det P \det D}$$

$$\therefore \text{ at } Z = Z_0, \text{ num } (\mu_i) = -\alpha \sum_{k=1}^2 \sum_{j=1}^3 P_{kj} \text{cf } P_{ki} \text{cf } D_{ij} \quad -(E.1)$$

now
$$\sum_{k=1}^4 P_{kj} \text{ of } P_{kj} = \det P, \quad j = 1, 2, 3$$

and
$$\sum_{k=1}^4 P_{kj} \text{ of } P_{ki} = 0, \quad j = 1, 2, 3, i \neq j$$

therefore

$$\text{num}(\mu_i) = -\alpha \det P \text{ of } D_{ii} + \alpha \sum_{k=3}^4 \sum_{j=1}^3 P_{kj} \text{ of } P_{ki} \text{ of } D_{ij}$$

Assuming that the last four rows of P are derived from the 2nd and 3rd rows of (4.8), we have

$$P_{kj} = C_{k-1, j}(Z, \lambda_1), \quad \text{for } k=3, 4, j=1, 2, 3$$

hence, when $Z = Z_0$,

$$P_{kj} = D_{k-1, j}$$

We therefore have

$$\begin{aligned} \text{num}(\mu_i) &= -\alpha \det P \text{ of } D_{ii} + \alpha \sum_{k=3}^4 \sum_{j=1}^3 \text{ of } P_{ki} D_{k-1, j} \text{ of } D_{ij} \\ &\& \sum_{j=1}^3 D_{ij} \text{ of } D_{ij} = 0 \quad \text{for } i=2, 3, \end{aligned} \quad \text{hence,}$$

$$\text{num}(\mu_i) = -\alpha \det P \text{ of } D_{ii}$$

Similarly to (E.1) we can write, at $Z = Z_0$,

$$\text{num}(\mu_{i+3}) = -\alpha \sum_{k=1}^2 \sum_{j=1}^3 P_{kj} \text{ of } P_{ki+3} \text{ of } D_{ij}, \quad i=1, 2, 3$$

Therefore,

$$\begin{aligned} \text{num}(\mu_{i+3}) &= \alpha \sum_{k=3}^4 \sum_{j=1}^3 P_{kj} \text{ of } P_{ki+3} \text{ of } D_{ij} \\ &= \alpha \sum_{k=3}^4 \sum_{j=1}^3 \text{ of } P_{ki+3} D_{k-1, j} \text{ of } D_{ij} = 0 \end{aligned}$$

Hence the μ_{i+3} term has no pole at $Z = Z_0$.

The residue of $\frac{1}{\det D}$ at $Z = Z_0$ is $\frac{1}{\text{cf } D_{11}}$. Therefore the residue of $\bar{\xi}_i$ at $Z = Z_0$ is given by,

$$\begin{aligned} \text{residue } (\bar{\xi}_i)_{Z_0} &= \text{residue } (\mu_i \text{ term})_{Z_0} + \text{residue } (\text{adj } D \eta)_i \text{ term}_{Z_0} \\ &= - \frac{\alpha \det P \text{ cf } D_{1i}}{\det P \text{ cf } D_{11}} e^{-iK_1 \cdot \mathbf{x} + Z_0 t} + \frac{\alpha \text{ cf } D_{1i}}{\text{cf } D_{11}} e^{-iK_1 \cdot \mathbf{x} + Z_0 t} \end{aligned}$$

the det P's cancel and,

$$\text{residue } (\bar{\xi}_i)_{Z_0} = 0$$

Q.E.D.

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