CONTRIBUTIONS TO ELEMENTARY PARTICLE PHYSICS.

by

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P R E F A C E

This thesis has been carried out under the supervision of Professor A. Salam between October 1960 and August 1963, in the department of Theoretical Physics, Imperial College of Science and Technology, University of London. The material contained herein is original (except when stated in the text) and has not been previously presented for a degree in this or any other university. It consists of two distinct parts : (1) The Theory of final state interactions and (2) The calculation of the natural constants in pseudoscalar meson theory.

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PART I

FINAL STATE INTERACTIONS.

A B S T R A C T

The influence of final state interactions among pairs of particles on the cross section of production processes is analysed field theoretically. The solution which is obtained is exact when not more than one of the final state scattering amplitudes is large, and correctly reduces to Watson's theorem when all but one of these amplitudes are neglected.

INTRODUCTION

The last ten years have seen a big increase in the number of particle accelerators and the energy scales at which they can operate. Much data about production processes has accumulated as a result of these modern developments and it is quite evident now that the forces between pairs of particles in the final state can greatly influence the cross sections. The first paper dealing with this subject was written by Watson in 1952 who estimated the effect of a strong short range attractive force between one particular pair of particles near the threshold of production. His result is easily understood from the point of view that it is essentially a one-channel process so the phase of the amplitude is simply that of the interacting pair. Until recently all the applications were confined to problems where it was known (or assumed) that the interaction between a final pair predominated.

The purpose of this dissertation has been to generalise Watson's theorem to the interaction of more than one pair of final particles, irrespective of whether it is attractive or repulsive, strong or weak. Although we have set up the equations for such a system we have only been able to solve them in perturbative fashion. At low energies and up to first order in final state scattering the solution is of the type

$$M = M(\text{production}) \left[1 + \sum_{\substack{\text{pairs of} \\ \text{final particles}}} \{ c + iq \} \frac{e^{i\delta} \sin \delta}{q} \right]$$

q is the relative momentum between a final pair who scatter with the phase δ and c (≈ 1 for strong interactions) is a measure of how important are off-shell in relation to the on-shell processes. The expression will correctly describe the situation if not more than one of the final state interactions is large, but nevertheless constitutes a generalisation of the theorem since the other interactions are not omitted altogether. If the variation of the production part of the amplitude is neglected and one of the interactions is retained we have Watson's theorem $M \propto (c + q \cot \delta) \frac{e^{i\delta} \sin \delta}{q}$, while in the limit of small scattering $M \longrightarrow M$ (production) as we might expect.

The first section comprises the potential theoretic aspect of the problem and although much simpler than the latter sections because it is non-covariant, spin is neglected etc., embodies the main result. To begin with, total potentials between sets of particles are introduced and we obtain a series decomposition of the net scattering among all the particles in terms of separate total scatterings, according to a simple scheme which is closely related to the Fadeev equations. The specialisation to final state interactions follows readily. Section 2, which consists of the covariant field theory, is important in showing up the deficiencies of potential theory due to particle creation and annihilation, and in seeing what diagrams have been neglected and must be separately evaluated. The final state scattering contribution is recalculated covariantly with spin properly included and the most general expression is obtained in terms of helicity amplitudes. The form of the result is essentially the same as above including the presence of a factor c to describe the virtual processes. The reflexion of these final

state interactions on the Dalitz plot and the effect of competing resonances are then deduced from our solution. We include a short review of recent work on the subject based on unitarity and dispersion relations without which the thesis would be incomplete and especially the equivalence between formal scattering theory and axiomatic theory as shown by Jackson. The generalisation of the dispersion relations to all final pair channels which has been proposed by Peierls and Tarski in their model of final state interactions is compared with our perturbation solution.

With a view to deriving information about π-π scattering we apply the field theoretic solution to the problem of $K^{\pm}$ meson decay into three pions. The result $a_2 - a_0 \approx 1$, where a_T is the s-wave π-π scattering length in the isospin state T, compares very favourably with dispersion theory. However, the disagreement of this conclusion with other sources, about $\pi\pi$ forces such as the ABC experiment, shows that the situation in τ-decay is much more complicated than appears at first sight and is not a good means of deducing the pion-pion interaction.

CHAPTER I

Potential theory of final state interactions.

1. Scattering of free particles.

We begin by giving a brief description of the formalism and notation which we shall adopt in the subsequent work.

The eigenstates of the free Hamiltonian H_o will be written as $|\psi>$ and those of the total Hamiltonian H as $|\psi>_{\pm}$ corresponding to in and out-states. The Lippmann-Schwinger equation[2] then gives the construction of $|\psi>_{\pm}$ from $|\psi>$ when these have the same energy (this effectively implies an energy renormalisation):

$$|\psi>_{\pm} = |\psi> + \lim_{\varepsilon \to 0} \frac{1}{E-H_o \pm i\varepsilon} V|\psi>_{\pm} \qquad (1)$$

where $V = H - H_o$ is the interaction. If V is given at the outset then the problem is essentially determined. The description 'in' and 'out' is clear when we take the coordinate representative of the equation and take the limit $|\underline{r}| \to \infty$. Then $<\underline{r}|\psi>_{-}$ has incoming spherical waves and outgoing plane waves.

We can rewrite (1) more conveniently in the form :

$$|\psi>_{\pm} = (1 + \frac{1}{E-H\pm i\varepsilon} V) \quad |\psi> \qquad (2)$$

The scattering amplitude from a given initial state i to a final state specified by f is given by $_{-}<f|i>_{+} \equiv <f|S|i> \equiv S_{fi}$. Since

$$|i>_{+} = |i>_{-} + \left\{ \frac{1}{E_i - H + i\varepsilon} - \frac{1}{E_i - H - i\varepsilon} \right\} V |i>$$

$$S_{fi} = \underset{-}{<} f | i \underset{-}{>} + \left\{ \frac{1}{E_i - E_f + i\varepsilon} - \frac{1}{E_i - E_f - i\varepsilon} \right\} \underset{-}{<} f | V | i >$$

$$\therefore \quad S_{fi} = \delta_{fi} - 2\pi i \delta(E_i - E_f) \underset{-}{<} f | V | i > \qquad (3)$$

Putting $S = 1 + 2\pi i \delta(E_i - E_f) R$, (4)

$$<f| R |i> = -\underset{-}{<} f | V | i > = -<f | V | i>_{+} \qquad (5)$$

and $-R = V + V (E - H + i\varepsilon)^{-1} V$ where $E = E_i = E_f$.

We make use of the identity

$$(E - H + i\varepsilon)^{-1} = (E - H_o + i\varepsilon)^{-1} + (E - H + i\varepsilon)^{-1} V (E - H_o + i\varepsilon)^{-1}$$

to derive the relation $V (E - H + i\varepsilon)^{-1} = -R (E - H_o + i\varepsilon)^{-1}$. Defining the free causal propagator $\Delta = (E - H_o + i\varepsilon)^{-1}$ all the foregoing can be conveniently summarised as

$$R = R\Delta V - V \text{ . so } \quad V = R\Delta V - R \qquad (6)$$

This last formula will form the basis of all our work as it gives R in terms of V and vice versa, up to any given order, the order of any expression being defined as the number of times in which Δ appears. The series expansion of R in terms of V (which is a real operator, of course) demonstrates the unitarity of the S-matrix.

$$R - R^{+} = (V + V\Delta^{+}V + V\Delta^{+}V\Delta^{+}V + \ldots) - (V + V\Delta V + V\Delta V\Delta V + \ldots)$$

$$= V (\Delta^{+} - \Delta) V + (V\Delta^{+}V\Delta^{+}V - V\Delta V\Delta V) + \ldots$$

$$= (V + V\Delta V + V\Delta^{+} V\Delta^{+}V+..)\ (\Delta^{+}-\Delta)\ (V +V\Delta V + V\Delta V\Delta V + ..)$$

$$= R^{+}\ (\Delta^{+} -\Delta)R \ = 2\pi i\ R^{+}\delta\ (E-H_{o})R$$

i.e. the relation $R = R\Delta V - V$, alone, ensures a unitary theory, providing V is real and Δ is the Green's function, $(E-H_{o}+i\varepsilon)^{-1}$.

We note that because of overall three-momentum conservation the element R_{fi} will include the factor $(2\pi)^{3}\ \delta^{3}\ (\underline{K}_{i}- \underline{K}_{f})$

$$\therefore\ \ S_{fi} = \delta_{fi} + \ i(2\pi)^{4}\delta^{4}(k_{f}-k_{i})\ M_{fi} \qquad (7)$$

Hence the transition probability per unit volume and per unit time is

$$\omega_{fi} = (2\pi)^{4}\ \delta^{4}(k_{f}-k_{i})\ |M_{fi}|^{2} \qquad (8)$$

2. Scattering of bound state particles.

Equation (6) which is simply the potential theory analogue of the Bethe-Salpeter equation, is not immediately applicable to problems which involve bound state systems such as deuterons.[21] Therefore we shall derive equations which are very similar in structure to (6) and which will cover these cases as well. Firstly we introduce the binding potential v and denote states which involve the deuteron with the subscript b. Thus plane wave states which contain the deuteron are written $|\psi b\rangle$ and scattering states of it are denoted by $|\psi b\rangle_{\pm}$ ie.

$$(H_{o}+v)\ |\psi b\rangle = \ E\ |\psi\ b\rangle \text{ and}$$

$$(H_{o}+\ v\ +V)|\ \psi\ b\rangle_{\pm} \ = \ E\ \ |\psi b\rangle_{\pm}\ .$$

Let V now stand for the potential between any other particles and the deuteron. As before,

$$|\psi^{(\pm)}_b\rangle = \left(1 + \frac{1}{E-H_o-v-V\pm i\varepsilon}\, V\right) |\psi_b\rangle$$

$$|i_b\rangle_+ = |i_b\rangle + \left(\frac{1}{E_i-H_o-v-V+i\varepsilon} - \frac{1}{E-H_o-v-V-i\varepsilon}\right) V\, |i_b\rangle$$

The amplitude for a process in which the final state still contains the deuteron viz. the "elastic amplitude" is

$$\langle f_b | i_b\rangle_+ = \langle f_b\, i_b\rangle + \langle f_b | \left(\frac{1}{E_i-E_f+i\varepsilon} - \frac{1}{E_i-E_f-i\varepsilon}\right) | V | i_b\rangle$$

$$= \delta_{fi} - 2i\pi\delta(E_i-E_f)\, \langle f_b | V | i_b\rangle$$

$$= \delta_{fi} + 2i\pi\delta(E_i-E_f) \langle f_b | R_{ol} | i_b\rangle \qquad (9)$$

Thus $- R_{ol} = V+V(E-H_o-v-V+i\varepsilon)^{-1}V$ (10)

On the other hand the amplitude for a process where the final state consists of the disintegrated deuteron ie. an "inelastic amplitude" is

$$\langle f | i_b\rangle_+ = \langle f | i_b\rangle - 2\pi i\, \delta(E_i-E_f) \langle f | V | i_b\rangle$$

$$= 2\pi i\, \delta(E_i-E_f) \langle f | R_{in} | i_b\rangle \qquad (11)$$

Therefore - $R_{inel} = V+(V+v)(E-H_o-V-v+\iota\varepsilon)^{-1}V$ (12)

The integral equations (6),(10) and (12) cover all physical problems of interest.

3. Decomposition of the scattering element.

In potential theory it is usual to regard the potential V as the basic quantity from which all else is tobe determined. The same is true of field theory where the interaction Lagrangian is something which is given at the outset. One then determines R exactly through the integral equation, or if that is not feasible, then by means of a perturbation expansion. However, one might equally well choose R as the starting point, which, after all, is the quantity which is measured physically, and evaluate V from it. This is the point of view that we take and which is closely linked with modern S-matrix theory. This potential V is the total of all interactions and necessarily describes all the processes in nature; however certain parts of it are sufficient to describe the scattering between a chosen set of particles (at least in potential theory) and the remainder represents all other processes. In this section we will express the total R matrix in terms of the R matrix operating between this particular set of particles and the R matrix which arises from the remainder of V.

Suppose that we make the decomposition $V = V_o + V_1$. We have when $V_1 = 0$

$$- R = -R_o = V_o+V_o(E-H_o-V_o+\iota\varepsilon)^{-1} V_o \qquad (13a)$$

when $V_o = o$,

$$-R = -R_1 = V_1 + V_1(E-H_o-V_1+i\varepsilon)^{-1} V_1 \qquad (13b)$$

but in general, of course,

$$-R = (V_o+V_1) + (V_o+V_1)(E-H_o-V_o-V_1+i\varepsilon)^{-1}(V_o+V_1) \qquad (13c)$$

Our intention is to express the full scattering R in terms of the two independent scatterings R_o and R_1 and this we can do via (6) up to any chosen order. Firstly, we expand $-R = V+V\Delta V+..$, put V_o+V_1 and then substitute $-V= R_o+R_o\Delta R_o+..$, $-V_1=R_1+R_1\Delta R_1$, +.. Let $R^{(n)}$ represent the expansion up to n'th order.

e.g. $R^{(2)} = -V- V\Delta V - V\Delta V\Delta V$

$$=-(V_o+V_1) - (V_o+ V_1)\Delta(V_o+V_1) - (V_o+V_1)\Delta (V_o+V_1)\Delta(V_o+V_1)$$

$$= R_o+ R_o\Delta R_o + R_o\Delta R_o\Delta R_o + R_1 + R_1 \Delta R_1 + R_1\Delta R_1 \Delta R_1$$

$$- (R_o+ R_o\Delta R_o + R_1 + R_1 \Delta R_1) \Delta (R_o + R_o\Delta R_o +R_1 + R_1\Delta R_1)$$

$$+ (R_o+R_1) \Delta (R_o+R_1) \Delta(R_o+R_1)$$

$$= R_o+R_1 - R_o\Delta R_1 - R_1\Delta R_o + R_o\Delta R_1\Delta R_o + R_1\Delta R_o\Delta R_1$$

and it is clear that in general an R_1 must follow an R_o, and vice versa.

The very same procedure can be applied to the deuteron scattering problems to isolate the effect of the binding interaction :

e.g. $R_{el}^{(2)} = -V - V\Delta V - V\Delta (V+\omega)\Delta V$

$$= R + R\Delta R + R\Delta R\Delta R - (R+ R\Delta R)\Delta(R+R\Delta R)+R\Delta(R+r)\Delta R$$

$$= R + R\Delta r \Delta R$$

and $R^{(2)}_{inel} = -V - (V+v)\,\Delta V - (V+v)\,\Delta(V+v)\,\Delta V$

$= R + R\Delta R + R\Delta R\Delta R - (R+R\Delta R + r + r\Delta r)\,\Delta R + (R+r)\,\Delta(R+r)\,\Delta R$

$= R - r\,\Delta R + R\Delta r\,\Delta R$

It is evident what the general series is like upon examination. The diagrams below summarise the situation.

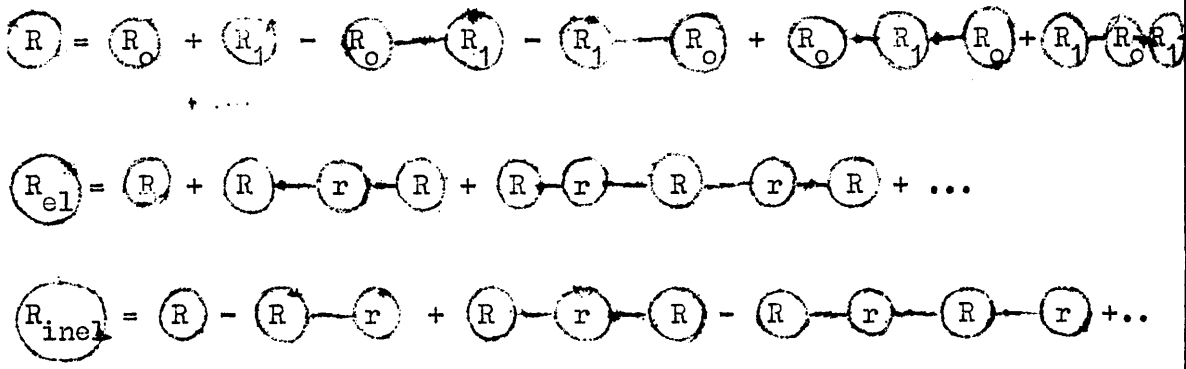


FIG. 1.

As yet we have not specified V_o or V_1 in any way, apart from stating that they are complementary to one another. In the particular case when the chosen set consists of 3 particles, '1','2' and '3' say, we can further decompose V_1 :

$$V_1 = V_{12} + V_{23} + V_{31} + V_{123} \qquad (14)$$

The statement is made precise by defining V_{ij} to be the total potential acting between partcles i and j, so that V_{123}, the remainder can be regarded as a 'three particle interaction'. (The recently discovered particle can be regarded as a resonance of a 3-pion interaction).

Properly speaking, the V_{ij} should be multiplied by the projection over ij particle states i.e. should be replaced by $\sum_{ij} V_{ij} \mid ij > < ij \mid$. Neglecting V_{123} for the moment and applying the O-1 method with two particle scatterings, we obtain the diagrams of Fig.2. Were we to include V_{123}, we should have to add the diagrams of Fig. 2b.

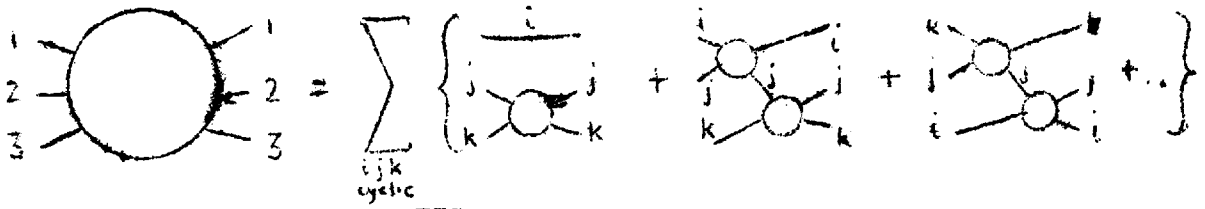


FIG. 2a.

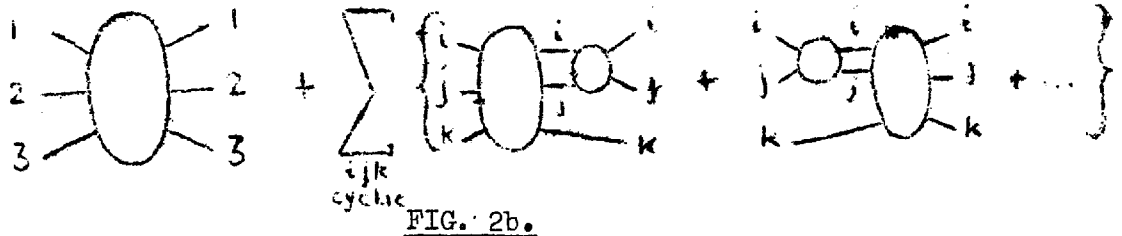


FIG. 2b.

Where there is no evidence for a 3-particle interaction, it is usual to neglect R_{123} and to attribute all the dynamical effects purely to the two particle collision operators R_{ij}.

For completeness sake we mention that the series contained in Figures 1 and 2 can bo obtained from the following equations by iteration:

$$R = \bar{R}_o + \bar{R}_1 \quad \text{where} \quad \begin{pmatrix} \bar{R}_o \\ \bar{R}_1 \end{pmatrix} = \begin{pmatrix} R_o \\ R_1 \end{pmatrix} + \begin{pmatrix} o & R_o \\ R_1 & 0 \end{pmatrix} \Delta \begin{pmatrix} \bar{R}_o \\ \bar{R}_1 \end{pmatrix}$$

$$R_{el} = \bar{R} \quad \text{where} \quad \begin{pmatrix} \bar{R} \\ \bar{r} \end{pmatrix} = \begin{pmatrix} R \\ r \end{pmatrix} + \begin{pmatrix} o & R \\ r & 0 \end{pmatrix} \Delta \begin{pmatrix} \bar{R} \\ \bar{r} \end{pmatrix}$$

$$R_{in} = R + R\Delta r \Delta R_{in}$$

In particular, the matrix equation for the three particle problem is nothing more than the Fadeev equation[15]

$$R = \bar{R}_{23}+\bar{R}_{31}+\bar{R}_{12} \quad \text{where} \quad \begin{pmatrix} \bar{R}_{23} \\ \bar{R}_{31} \\ \bar{R}_{12} \end{pmatrix} = \begin{pmatrix} R_{23} \\ R_{31} \\ R_{12} \end{pmatrix} + \begin{pmatrix} 0 & R_{23} & R_{23} \\ R_{31} & 0 & R_{31} \\ R_{12} & R_{12} & 0 \end{pmatrix} \Delta \begin{pmatrix} \bar{R}_{23} \\ \bar{R}_{31} \\ \bar{R}_{12} \end{pmatrix}$$

As an illustration of all these techniques we consider elastic and inelastic electron-deuteron scattering. v will be the p-n potential which binds them into the deuteron, and V will be the electron-nucleon potential. V itself is the sum of two parts: V_n the e-n potential and V_p the e-p potential.

Thus $R = R_n + R_p + R_n \Delta R_p + R_p \Delta R_n + \ldots$

Since v is a strong interaction while V is electromagnetic in origin, $V \approx \frac{1}{137} v$; therefore up to second order in e.m interactions R_N, it is a very simple matter to derive the diagrams of Fig. 3 by straight forward application of the iteration equations contained in Fig.1.

[Such diagrams have been obtained by Ferreira[16] for the problem of K-meson - deuteron scattering but his derivation is considerably more tedious

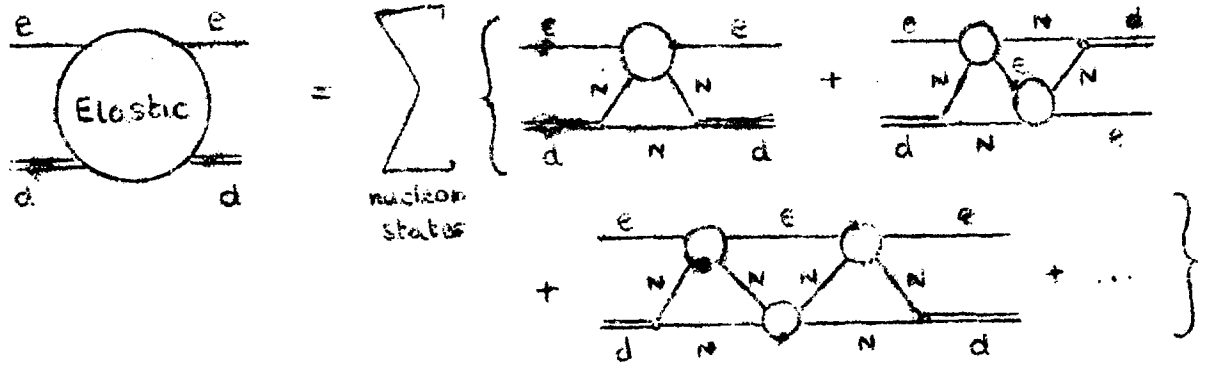

e
e
Elastic
d
d
=
nucleon
states
N
d
+
+ ...

FIG. 3a

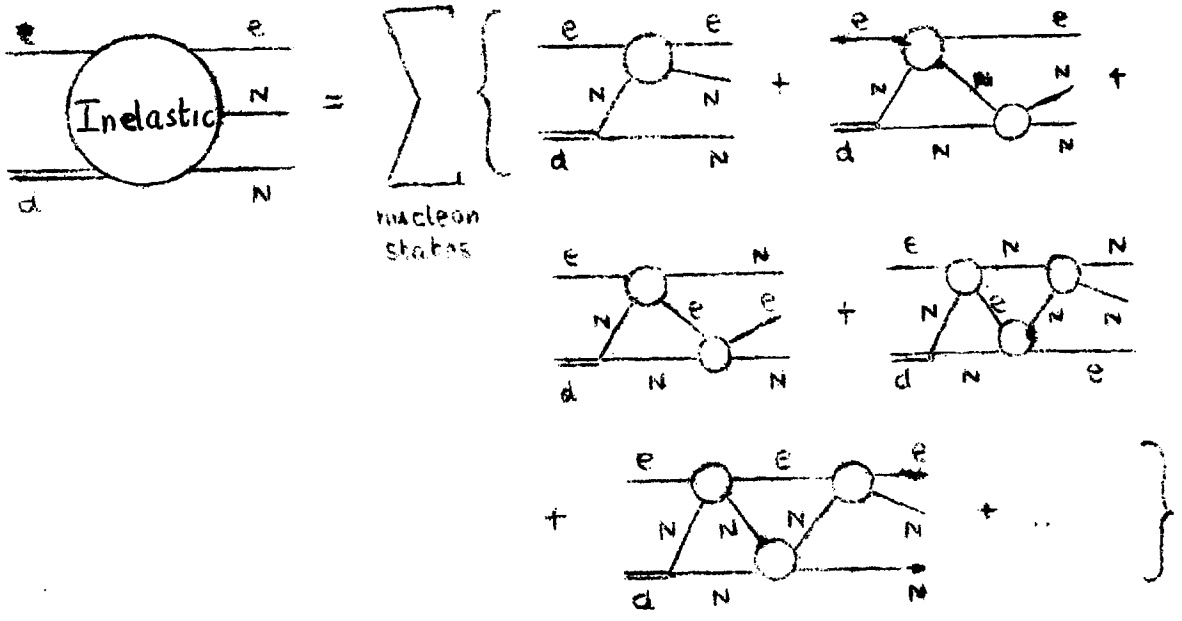

e
e
Inelastic
N
d
N
=
nucleon
states
N
d
+
+
+ ...

FIG. 3b

4. Final State Interactions.

We are now in a position to attack the main problem which is to isolate the effect on the total scattering amplitude of the interactions among the final particles that are emitted in a particular process. We will designate the initialparticles by A,B,.. and the final ones by 1,2,3 . The total potential V will act among all these particles, and we may imagine it to be a form factor multiplied by operators which annihilate the initial configuration and create the final particles. We define the decomposition into V_o and V_1 by defining V_1 to be the total potential which acts between the final particles only, so that the remainder V_o can be regarded as the production potential. To be definite let us take the initial set of particles to be different from the final set, so we need only consider the part of V_o which annihilates A,B,.. and creates 1,2,3,.. V_1 destroys and recreates 1,2,3,.. Because we are working in terms of potential theory these conditions apply to the R, so the only terms that contribute to the total R are $R_o+R_1 \Delta R_o$. Now

$$< f \mid (1+R_1 \Delta) R_o \mid i > = \leq f_1 \mid V_o \mid i_o >_+ \qquad (15)$$

where $\mid \psi_1 >_\pm$ is the in-or out-state in absence of V_o, while $\mid \psi o >_\pm$ is for $V_1 = o$. Equation (15) is just the term which Watson uses and it represents scattering from an in-state of no final state interaction to out-state of no primary interaction.

The first non-trivial situation in which it might be possible to detect the effect of V_1 is a final state consisting of three particles

when V_1 can be broken up into specific pairs of potentials; if these pairs interact differently in any way, these differences must show up kinematicall The effect of V_{123} will be apparent in the variation of the cross section as a function of the total energy of 1,2 and 3; we shall mostly neglect it when analysing the differential cross-section for a given incident energy, attributing all the fine structure to forces between pairs of particles. Before proceeding with an actual calculation of the final state scattering diagrams we will present the original argument given by Watson[20].

To begin with, it is assumed that the force between a final pair (say 12) is strong and attractive, so if we introduce a complete set of configuration states of that particular pair in equation (15),

$$M \sim \int d^3\underline{r}_{12} \; \langle f_1 | \underline{r}_{12} \rangle \; \langle \underline{r}_{12} | V_o | i_o \rangle_+$$

Further, if the production amplitude $\langle r_{12} | V_o | i_o \rangle_+$ is very short range it will be practically independent of the relative motion of 1 and 2 and is very well approximated by $M_o(\underline{o}) = \langle \underline{o} | V_o | i_o \rangle_+$; so $M \sim M_o(\underline{o}) M_1^*(\underline{o})$

$$\therefore \qquad M \sim M_o(\underline{o}) \; \frac{e^{i\delta} \sin\delta}{q} \; .$$

Indeed the result is quite obvious if considered in the light of unitarity: the process is governed essentially by one final state channel and hence must possess the phase of the final state scattering.

5. Evaluation of the lowest order scattering.

Unlike Watson[2c] who is only interested in the strong attraction limit of 1 and 2, we carry on with the summed up perturbation theory as given in the previous sections and proceed to estimate the effect of the final state scattering with particles 1,2 and 3 on an equal footing.

The decomposition represented by Figure 2a when combined with the fact that $R = R_o + R_1 \Delta R_o$ leads to the diagrams below up to first order in final state scatterings

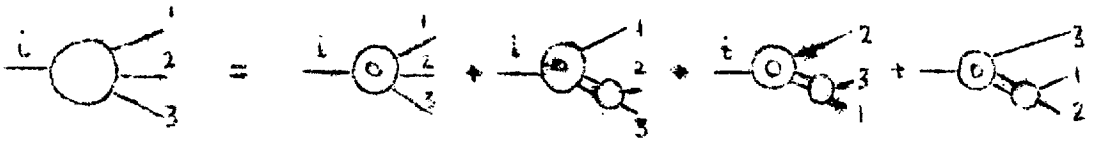


FIG.4.

Typically we shall have to evaluate $\langle 123 | R_{12} \Delta R_o | AB \rangle$. Introducing a complete set of intermediate states of particles 1 and 2, we find

$$M = \frac{1}{(2\pi)^3} \int \frac{d^3k'_1}{2E'_1} \frac{d^3k'_2}{2E'_2} \delta^3(\underline{k}_1+\underline{k}_2-\underline{k}'_1-\underline{k}'_2) \frac{\langle \underline{k}_1\underline{k}_2 | M_{12} | \underline{k}'_1\underline{k}'_2 \rangle \langle \underline{k}'_1\underline{k}'_2\underline{k}_3 | M_o | AB \rangle}{E-(E_3+E'_1+E'_2)+i\varepsilon} \qquad (16)$$

In this form of theory, each particle lies on its mass shell and three-momentum but not energy is conserved in intermediate processes.

As usual, we express our results in terms of the total and relative momenta

$$\underline{p}' = \underline{k}_1' + \underline{k}_2 \quad , \quad 2\underline{q}' = \underline{k}_1' - \underline{k}_2' \tag{17}$$

Evaluating in the barycentric frame of 1 and 2 where $\underline{p} = o$, we obtain

$$M = -\frac{1}{(2\pi)^3}\int \frac{M_{12}(\underline{q},\underline{q}')M_o(\underline{q}_1',\underline{k}_3;k_i)}{E_1+E_2-E_1'-E_2'+i\varepsilon} \frac{d^3\underline{q}'}{4E_1'E_2'}$$

$$= -\frac{1}{(4\pi)^3}\int \frac{M_{12}(\omega,\underline{\vartheta};\omega',\underline{\vartheta}')\ M_o(\omega',\underline{\vartheta}',\underline{k}_3;\underline{k}_i)}{\omega-\omega'+i\varepsilon} \cdot \frac{2q'^2 d\omega' d\Omega'}{\omega'} \tag{18}$$

where ω is the energy of 1 and 2 in this frame (ω' in the intermediate state) and $\underline{\vartheta}$ denotes the orientation (ϑ,ϕ) of $\underline{q}$ with respect to a fixed axis.

At this point we make a partial wave decomposition to bring out the angular dependence. For simplicity we shall assume 1 and 2 to be spinless so that orbital momentum is conserved - the characteristics of final state interaction should emerge irrespective of spin complications. The amplitude M_{12} will be independent of the orbital component m because of its invariance under rotations of (12); however M_o will depend on m because the final state of three particles need not be invariant under the separate rotations of (12) :

$$M_{12}(\underline{\vartheta},\underline{\vartheta}') = \sum_{l,m} (2l+1)\, D^{l\,*}_{mo}(\underline{\vartheta}) D^{l}_{mo}(\underline{\vartheta}') M_{12_l} \tag{19a}$$

$$M_{o}(\underline{\vartheta}') = \sum_{l,m} (2l+1)^{\frac{1}{2}} D^{l\,*}_{mo}(\underline{\vartheta}')\, M_{olm} \tag{19b}$$

The orthogonality relation of the rotation matrices,

$$\int d\Omega D^{*l}_{mo}(\underline{\vartheta})\, D^{*l'}_{m'o}(\underline{\vartheta}) = \frac{4\pi}{(2l+1)}\, \delta_{mm'}\, \delta_{ll'}$$

simplifies equation (18) to

$$M = \frac{1}{(4\pi)^2} \sum_{l,m} (2l+1)^{\frac{1}{2}}\, D^{l\,*}_{mo}(\underline{\vartheta})\, I_{lm} \tag{20a}$$

where

$$I_{lm} = \frac{2iq\pi}{\omega} M_{12l}(\omega,\omega) M_{olm}(\omega,k_3;k_i) - \mathcal{P}\int \frac{M_{12l}(\omega,\omega')M_{olm}(\omega',k_3;k_i)}{\omega - \omega'}\, \frac{2q'd\omega'}{\omega'} \tag{20b}$$

The first term is just the physical contribution (or the absorptive past, from the point of view of unitarity) and the second term, which presents us with the real problem represents the virtual contribution.

Because the off-shell elements are determined in some way by the on-shell elements, we write the virtual part for each partial wave as

$$\mathcal{P}\; M_{12}(\omega,\omega) M_{o}(\omega) \int \frac{M_{12}(\omega,\omega)M_{o}(\omega') / M_{12}(\omega,\omega)M_{o}(\omega)}{\omega' - \omega}\, \frac{2q'd\omega'}{\omega'}$$

and the numerator of the integrand is now dimensionless. If we suppose

that the off-shell elements become smaller the more unphysical becomes the intermediate scattering, then the numerator will act like a cut-off function. Because we do not know how rapidly or in what manner it cuts off, unless the potentials are given, we may regard this ratio as unity up to an ω_c after which it is zero. This sort of point interaction model has been adopted in the past; see ref[16] for example. Making a subtraction at the threshold of production,

$$I = M_{12}(\omega,\omega)M_o(\omega)\left[\frac{2iq\pi}{\omega} + \pi c + \mathcal{P}\int_{\cdot}^{\infty} \frac{\omega - m_1 - m_2}{(\omega'-m_1-m_2)(\omega'-\omega)} \cdot \frac{2q'd\omega'}{\omega'}\right]$$

$$\text{with} \quad c \equiv \frac{\mathcal{P}}{\pi}\int^{\omega_c} \frac{2q'd\omega'}{\omega'(\omega'-m_1-m_2)} \tag{21}$$

In general, q is a complicated function of ω. Thus

$$2q\omega = \sqrt{(\omega^2-m_1^2-m_2)^2-4m_1^2m_2^2} \tag{22}$$

Providing neither m_1 nor m_2 vanishes we can approximate the factor $\frac{2q}{\omega}$ as follows :

At low energies $\omega = m_1+m_2 + \frac{q^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$

and at high energies $\omega = 2q$. A function which has both these characteristics is

$$\frac{2q}{\omega} \longrightarrow \left\{\frac{\omega - m_1 - m_2}{\omega-m_1-m_2+\frac{(m_1+m_2)^3}{16m_1m_2}}\right\}^{\frac{1}{2}} \tag{23}$$

This expression does not deviate very much from the exact form providing

the ratio of the masses is not too large. In any case this is unimportant since the use of the model is itself an approximation to the truth.

The integrations are easily carried out now and we obtain

$$I_{lm} \approx M_{12_l}(\omega,\omega)\, Mo_{lm}(\omega)\left[\frac{2\imath q\pi}{\omega} + \pi c_{lm} + \frac{2q}{\omega}\log\left|\frac{\omega+2q}{\omega-2q}\right|\right] \tag{24a}$$

$$\text{with}\quad c_{lm} \approx \frac{1}{\pi}\cosh^{-1}\left[1 + \frac{16m_1m_2(\omega_{clm}-m_1-m_2)}{(m_1+m_2)^3}\right] \tag{24b}$$

In the low energy limit,

$$I_{lm} \approx M_{12_l}(\omega,\omega)\, Mo_{lm}(\omega)\left[\pi c_{lm} + \frac{2iq\,\pi}{\omega} + O\left(\frac{q^2}{\omega^2}\right)\right]$$

Therefore for the case where 1 and 2 interact, and near the threshold of their production, we have all in all.

$$M = \sum_{l,m}(2l+1)^{\frac{1}{2}}D^{l}_{mo}{}^{*}(\underline{\vartheta})M_{o_{lm}}(\omega)\left[1+\left(c_{lm}+\frac{2iq}{\omega}\right)\frac{M_{12l}(\omega,\omega)}{16\pi}\right]$$

$$= \sum_{l,m}(2l+1)^{\frac{1}{2}}D^{l*}_{mo}(\underline{\vartheta})M_{o_{lm}}(\omega)\left[1+\left(i+\frac{c_{lm}\,\omega}{2q}\right)e^{i\delta_l}\sin\delta_l\right] \tag{25a}$$

$$= \sum_{l,m}(2l+1)^{\frac{1}{2}}D^{l*}_{mo}(\underline{\vartheta})M_{olm}(\omega)\frac{e^{i\delta_l}\sin\delta_l}{2q/\omega}\left[\frac{2q}{\omega}\cot\delta_l + c_{lm}\right] \tag{25b}$$

$$= \sum_{l,m} (2l+1)^{\frac{1}{2}} D^{l\,*}_{mo}(\underline{\vartheta})\, M_{olm}(\omega) \cdot \frac{1+a_l c_{lm} \left(\frac{2q}{\omega}\right)^{2l}}{1 - ia_l \left(\frac{2q}{\omega}\right)^{2l+1}} \qquad (25c)$$

where we have defined the relativistic scattering length (dimensionless) as

$$\left(\frac{2q}{\omega}\right)^{2l+1} \cot \delta_l = \frac{1}{a_l} \qquad (26)$$

The physical significance of equation (25) is quite clear, there is interference between the unity of the production amplitude, and the final state scattering; the parameter c which describes the model can be regarded as the amount of final state interaction that takes place and occurs in the enhancement factor $(c + \frac{2q}{\omega} \cot \delta)$. Since we are presuming no knowledge about the potential, there is no very firm theoretical basis from which we can determine c. Nevertheless when the final state scattering is a strong interaction it is usual to invoke an energy cut-off of the order of 1 nucleon mass and then $c \approx 1$. At any rate, it should certainly be possible to determine c from accurate low energy experiments when the production mechanism is known.

We note a few characteristics of expression (25) when we retain the S-waves only. If we adopt a length parametrisation,

$$M_{fi} = M_o(\omega) \frac{(1 + ca)}{1 - ia \frac{2q}{\omega}} \qquad (27)$$

whereas if the final particles resonate at an energy ω_o like a spin zero

pseudo particle with lifetime Γ

$$\frac{2q}{\omega} \cot \delta = \frac{\omega^2 - \omega_o^2}{\omega_o \Gamma}$$

$$\therefore \quad M_{fi} = M_o(\omega) \frac{\frac{\omega^2 - \omega_o^2}{\omega_o \Gamma} + c}{\frac{\omega^2 - \omega_o^2}{\omega_o \Gamma} - \frac{2q}{\omega}} \tag{28}$$

Let us see the implications of (27) and (28) graphically, always assuming that M_o is independent of ω so that we are attributing all the structure of the amplitude to the interaction of a particular pair of final particles. Because we are only regarding that channel as the dominating part, we shall be repeating Watson's results to a very large extent. Notice that the enhancement in (27) is pronounced only when a is large in magnitude, and that when a = -c there is a destructive effect, so one cannot then neglect the influence of other channels.

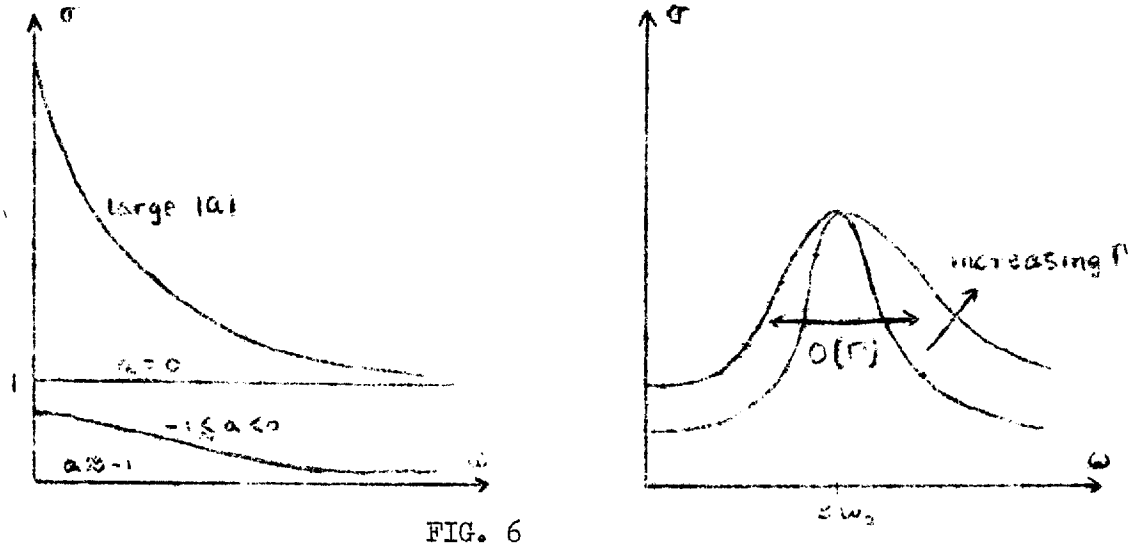


FIG. 6

6. Overlapping final state scatterings.

In separating out the total structure due to final state scatterings then, we must take an infinite string of perturbation diagrams into account; those represented in Figure 4, the double scattering diagrams such as shown in Figure 7 and so on. In practice we only retain the lowest order scatterings, and we can put forward the following heuristic argument to justify this approximation: suppose that all three final particles attract one another; then a pair of particles will scatter one another most for low relative momenta (and certain momenta where they resonate) causing the other particle to move with a high relative momentum in the opposite direction to the pair (if resonance effects of the pair which may cause the third particle to move slowlyare not appreciable); subsequent scattering of this odd particle with one of the particles of the pair will be small and unlikely. A similar sort of argument will apply if all the particles repel one another, but if some attract and others repel it breaks down. At any rate the retention of the lowest scattering diagrams will certainly be correct if we know that only the forces between one pair are significant, or if the magnitude of final state interactions

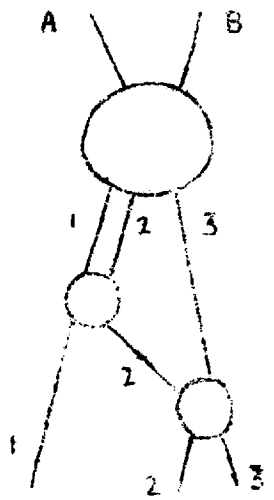


FIG.7

is small enough to justify a perturbative calculation. This would correspond to having small scattering lengths.

Quite apart from these considerations, there is the question of how correct is the use of the potential theory itself. The V are supposed to represent all the exchange effects between the particles in question although they do not allow for particle creation or annihilation. At high energies when many-particle states become increasingly important we can be sure that the model deviates more and more from the truth, and to the same extent as the peripheral model. (This is because the basic unit of the iteration $\langle E' | \Delta V | E \rangle = \frac{\langle E' | V | E \rangle}{E' - E}$ decreases in importance so that the first Born approximant - $R = V + V\Delta V$ is taken into account). Fortunately, equation (25) has been derived for near threshold energies and many particle structure will be quite insignificant.

A straightforward generalisation of (25) to all three lowest order scatterings with spinless particles is

$$M = M_o + \sum_{\substack{ijk \\ \text{perms}}} \sum_{l_{ij}, m_{ij}} (2l_{ij}+1) \tfrac{1}{2} D^{l_{ij}*}_{m_{ij}\, o}(\underline{\vartheta}) M_{o l_{ij} m_{ij}}(\omega_{ij}) \left[i + \frac{c_{ij}\omega_{ij}}{2q_{ij}} \right] e^{i\delta l_{ij}} \sin \delta l_{ij} \qquad (29)$$

If the energies are so low that just the S-waves need be retained,

$$M = M_o(\omega) \left\{ 1 + \sum_{ijk} \left[i + \frac{c_{ij}\omega_{ij}}{2q_{ij}} \right] e^{i\delta_{ij}} \sin \delta_{ij} \right\} \qquad (30)$$

and if the S-wave scattering is parametrised by a length,

$$M = M_o(\omega_{ij}) \left[1 + \sum \frac{a_{ij}\,(c_{ij} + 2iq_{ij}/\omega_{ij})}{1 - 2ia_{ij}q_{ij}/\omega_{ij}} \right] \tag{31}$$

As we have already stated, the result (31) will be particularly accurate if not more than one length is large in order that multiscattering effects in the final state be negligible.

Anisovitch, Anselm and Gribov[29] have analysed the Feynman diagrams of f.s.i and their associated singularities to calculate the influence on the total amplitude. We are able to recover their results by assuming, like they do, that a and aq are small so that an expansion of (31) in powers of a is possible :

$$M \approx M_o(\omega_{ij}) \left[1 + \Sigma\, a_{ij}c_{ij} \right] \left[1 + 2i \Sigma \frac{a_{ij}q_{ij}}{\omega_{ij}} + O\left(\frac{q}{\omega}\right)^2 \right] \tag{32}$$

There is an exact one to one correspondence with their conclusions when we identify $f_o = M_o \left[1 + \Sigma\, ac \right]$ as the threshold amplitude.

The question arises as to whether one may extend equation (30) to apply even when all the f.s.i amplitudes are large. The first notion that springs to mind is the multiplicative form,

$$M = M_o(\omega_{ij}) \prod \left[\frac{2q_{ij}}{\omega_{ij}} \cot \delta_{ij} + c_{ij} \right] \frac{e^{i\delta_{ij}} \sin\delta_{ij}}{2q_{ij}/\omega_{ij}} \tag{33}$$

which certainly reduces to (30) for small a. It is easy to convince oneself however that (33) gives an incorrect description of f.s.i since it presupposes that each of the final channels is completely independent - of course this is not true as each of the channels are coupled through the integral equations of Fadeev[15]. In any case it is a simple matter to verify that the second order terms in an a_{ij} expansion of (33) do not agree with the second order contributions of Grigov et al[29].

CHAPTER II.

FIELD THEORY OF FINAL STATE INTERACTIONS.

1.Pole Terms in Production Processes.

Most of the considerations of the previous section preclude the possibility of particle creation or annihilation and assume that the production mechanism can be separated from the final state interaction. This last concept is very badly defined as soon as we begin to think in terms of field theory. For instance we may visualise pion production in nucleon-nucleon collisions as proceeding peripherally through the exchange of a pion. (See Figure 8).

In this Chew-Low mechanism[6], one of the vertices consists of $N\pi$ - $N\pi$ scattering, say through the N^*, so that the production amplitude itself involves forces between the final particles (1 and 2). Production can be followed by rescattering of the pion from the other nucleon (2 and 3), or else by the two nucleons (1 and 3). Can we allow for subsequent scattering of 1 and 2, although this accounts for the main reaction? It is not directly obvious how we have to proceed.

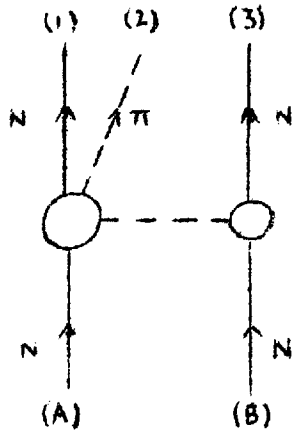


FIGURE 8.

Another point worth mentioning is that previously R arose from V by a succession of two-body interactions; in field theory one can imagine

some very trivial situations in which this may not be true e.g. πN scattering proceeding though a nucleon pole. Since this pole is not one of momentum transfer, the diagram cannot be associated with a potential unless the nucleon is itself regarded as the bound state of a pion and nucleon.

2. Connected and Irreducible Diagrams.

We wish to reformulate the theory in such a manner that it boils down to potential theory on condition that the pair intermediate states predominate. The way we shall carry out this reformulation will be to extract the one- and two-particle intermediate states in each of the final state channels in an obvious graphical manner along the lines of Symanzik[14], and then to approximate the two-particle irreducible production processes by a Chew-Low mechanism[6]. In order to achieve this end we must introduce a few basic concepts.

A diagram is 'disconnected' if it consists of at least two completely separated parts as shown in Figure 9. In the language of axiomatic theory this would mean that energy-momentum is conserved in at least two parts. i.e. that the amplitude will contain at least two δ function factors of four-momentum. Evidently a diagram will be named 'connected' if such a factorisation cannot be performed. Assuming that neither A nor B can decay into some part of the final configuration,

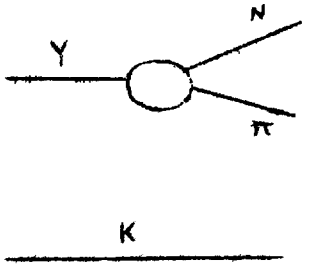


FIG. 9.

then it is clear that we were previously considering the connected diagram for the reaction A+B $\longrightarrow$ 1 + 2 + 3.

A diagram is said to be 'n-particle reducible' if it can be split into two disconnected parts by drawing a line (through its intermediate part) which cuts not more than n internal lines. If this cannot be done for any transversal, then the diagram is called 'n-particle irreducible'. The concept has significance in connexion with the singularities of the amplitude in a particular channel and because of this, is intimately related with the dispersion theory approach.

Reducible graphs can always be broken up to irreducible parts. With the convention that all lines which we draw are to be regarded as complete propagators (i.e. as containing all radiative corrections) and all bubbles as being amputated (i.e. of having the above singularities removed), then we have the fundamental reduction formula[14] for an n-particle irreducible diagram in one particular channel :

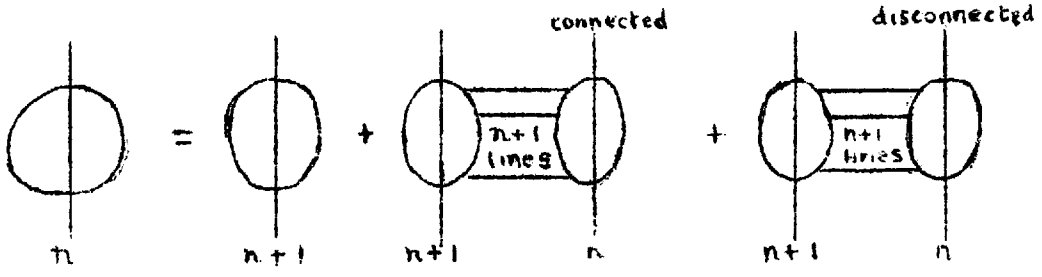


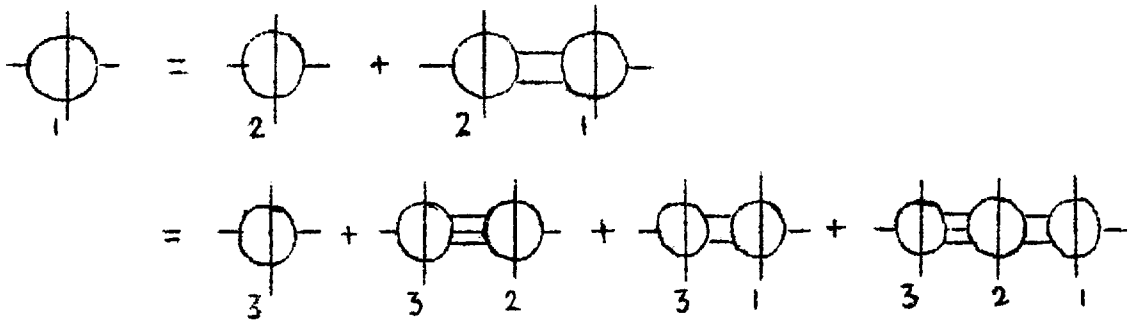
FIGURE 10.

This formula is trivially obvious looked at from a graphical point of view. For a diagram to be n-particle irreducible, we must cut at least (n+1) internal lines to disconnect it. On the r.h.s. of Fig. 10 we have separated out the diagrams that need at least (n+2) lines to be cut for disconnexion, and those that do not; the latter can themselves be written as the sum of two parts, each part of which is (n+1) particle irreducible, connected by (n+1) complete propagators to an n particle irreducible diagram (connected and disconnected).

Let us illustrate the reduction procedure by two simple examples in a self-explanatory diagrammatic way.

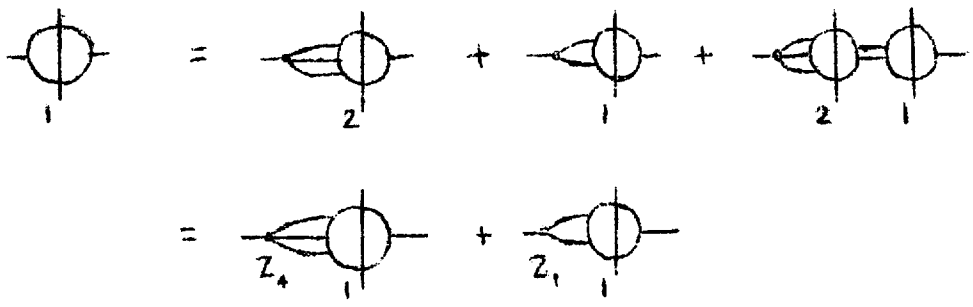
(i) Self Energy

We are investigating the radiative corrections to the propagator, thereby examining the 1 particle irreducible part of $-\langle 1 \mid 1 \rangle_+$.



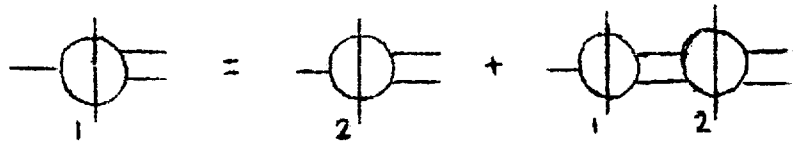
If these diagrams arise from the interaction Lagrangians $\bar{\psi}\psi\phi$, ϕ^4 then the only possible 3 particle irreducible diagrams are  and  so the equation becomes the Dyson equation[1] for renormalised pion-nucleon

interactions,

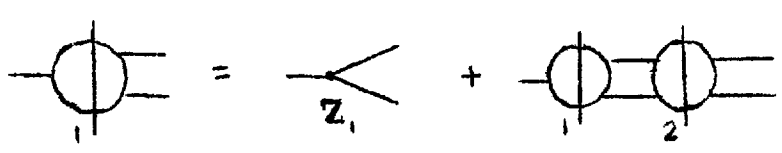


(ii) Vertex Part

Again in an obvious notation,



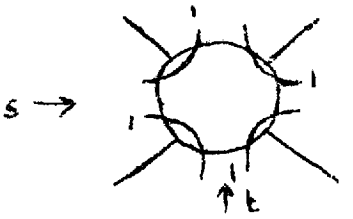
and in a ϕ^3 Lagrangian theory, we arrive at the Bethe-Salpeter equation:



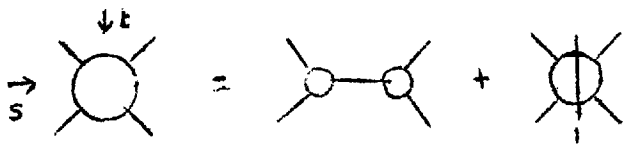
3, Two-Channel Irreducibility.

What we have discussed above is limited to only one particular channel, and needs to be generalised if we are to use it in connexion with final state interactions among at least three particles. It is not very hard to extend the analysis, as it happens, if we remember to impose the irreducibili

properties in each channel at every stage of the reduction. We show how this is done diagrammatically with reference to the four point function in two of the three possible channels and then generalise the foregoing to all three channels. Let us remark once again that all 'bubbles' drawn are amputated, and are of the form

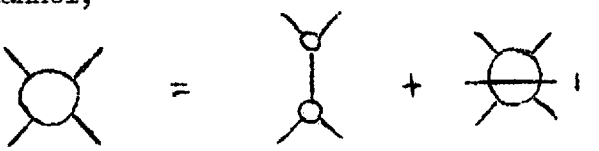


although not explicitly indicated.



in the s-channel

Neglecting the s-channel for the moment and pulling out the pole terms in the t-channel,



1 particle irreducibility in the s-channel will be satisfied by having

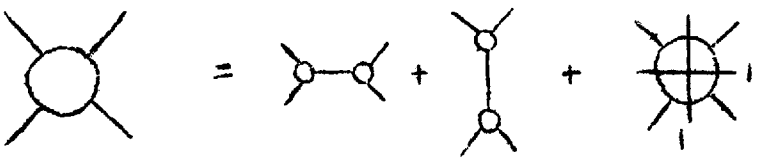
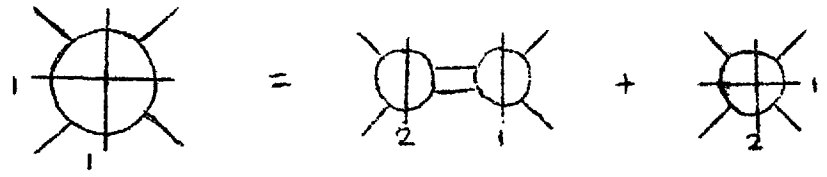
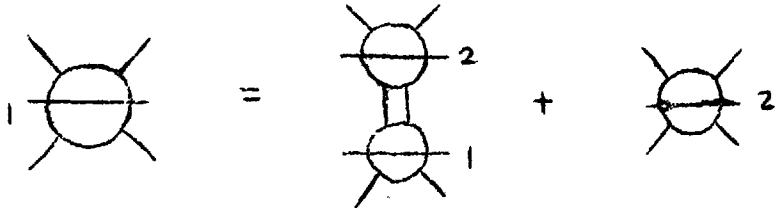


FIGURE 11.

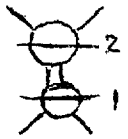
We now proceed to decompose further the last diagram on the r.h.s. of Figure 11 :



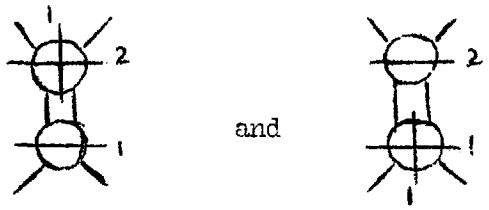
But,



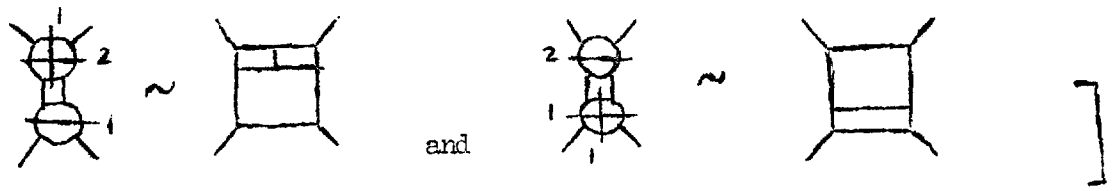
and if we have to maintain 2 particle irreducibility in the s-channel of



, there are two possible ways in which this can be done:



[As examples of Feynman diagrams which fall into these categories we may draw



]

Thus,

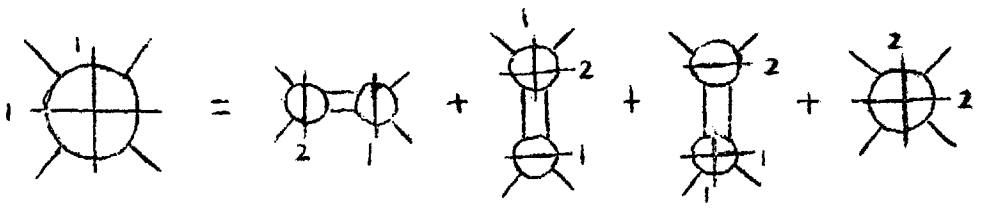
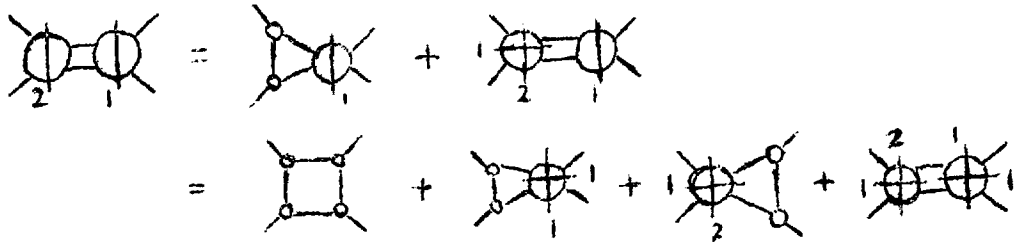
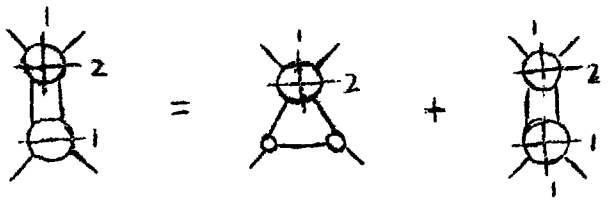


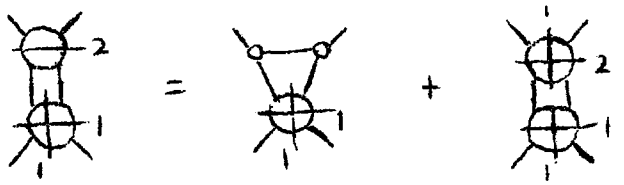
FIGURE 12.

A more symmetric form can be attained by carrying on with a few more reductions :





and



Collecting terms, the 2-channel decomposition has lead to the equation

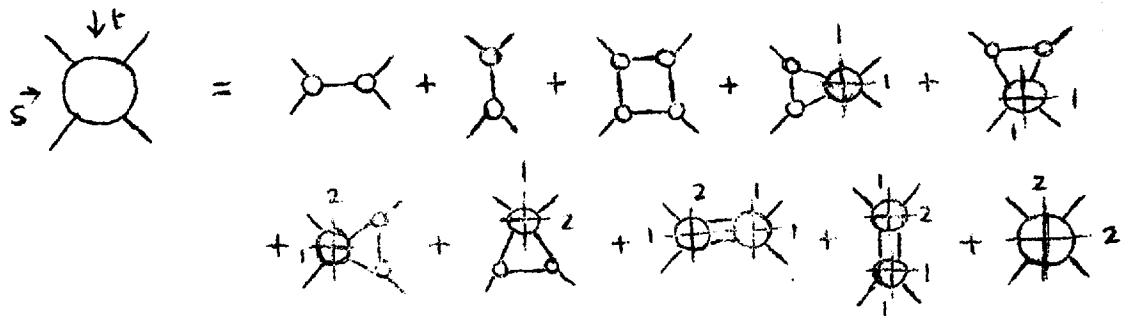


FIGURE 13.

which if given in terms of an integral equations, would correspond to the Mandelstam representation with dispersion relations in s and t only.

We shall state, without proof, the generalisation of Figure 16 to all 3 simultaneous channels, which is certainly plausible,

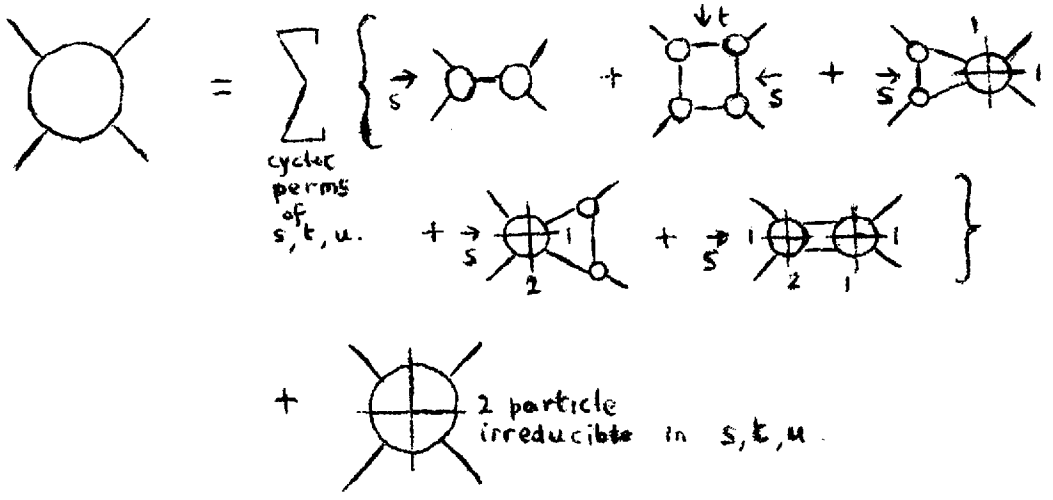


FIGURE 14.

4. Decomposition into final state interactions.

The expansion that we have carried out for the four-point function requires only a slight modification for the purpose of separating out the final state interactions, since we are interested in just the 23, 31 and 12 channel contributions :

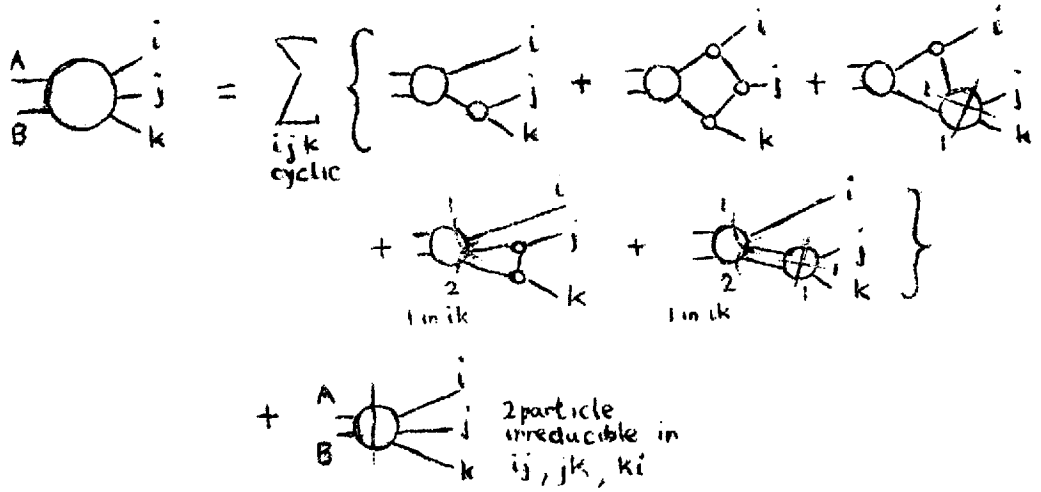


FIGURE 15.

To establish the correspondance with potential theory one simply discards all the three point functions that cannot give rise to potentials in any of the final state channels, so

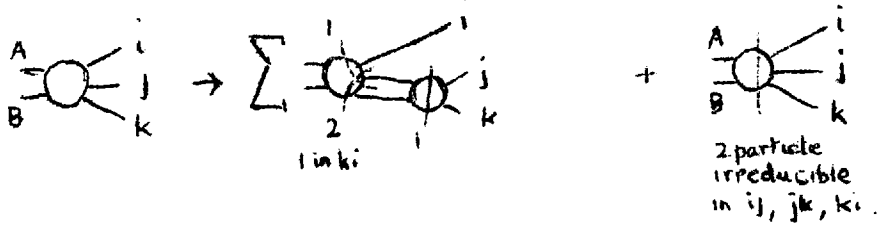


FIGURE 16.

The last term can be regarded as M_o while the first term might be further decomposed into a string of final state scatterings associated with $M_1 \Delta M_o$. We repeat once more that, of course, there are countless examples where one is uncertain how to apply potential theory. This uncertainty now disappears because the technique contained in Figure 15 properly takes into account pole terms and other types of singularities. Naturally, as soon as we wish to carry out a detailed calculation, we are obliged to make the approximation of neglecting certain singularities as compared with others. Confining ourselves to the threshold of production region it is reasonable to neglect all 3 particle intermediate states and therefore to throw away all diagrams which are two particle irreducible or more; but before doing so blindly, it is absolutely essential to retain certain classes of diagrams containing particle exchanges which would have been lost in the process. Thus the diagram below is two particle irreducible in every final state channel and may be of fundamental importance. Our approximation will then be completely defined if we keep only those terms of the two particle irreducible diagrams which are given by a Chew-Low mechanism as in Figure 17.

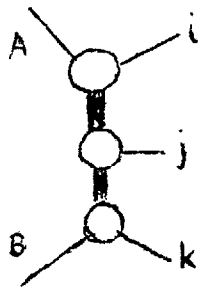


FIGURE 17.

On the basis of this approximation, the diagrams of Figure 15 reduce to

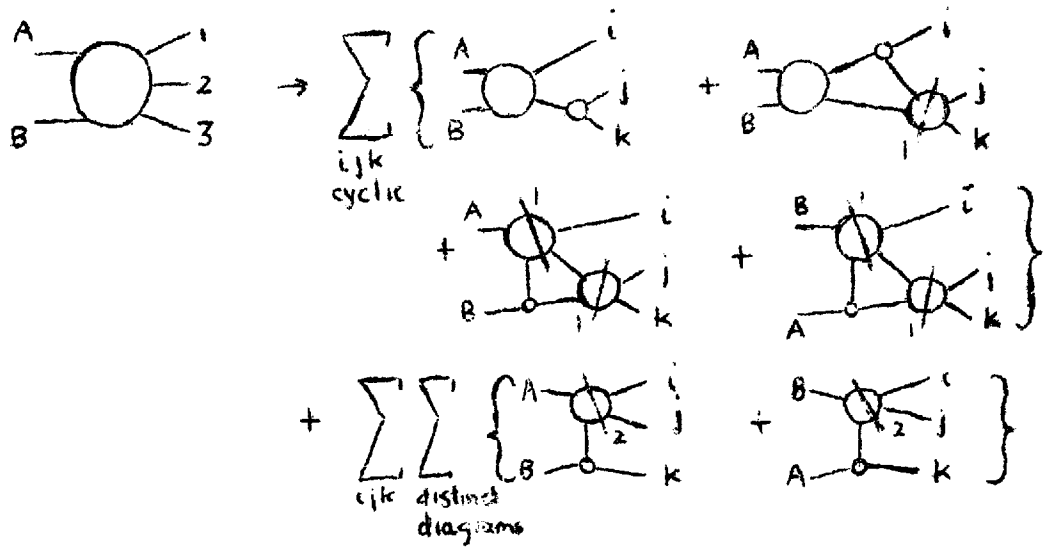


FIGURE 18.

Clearly, what intermediate states are possible depends upon the problem under consideration. In the light of this let us reexamine inelastic electron-deuteron scattering up to first order in e.m interaction; remember that now the deuteron is to be regarded as a particle with a field of its own. A straightforward application of the decomposition of Figure 18 and proper care in the counting, (which we have not exercised throughout), together with the fact that e-m processes must involve the exchange of photons lends to

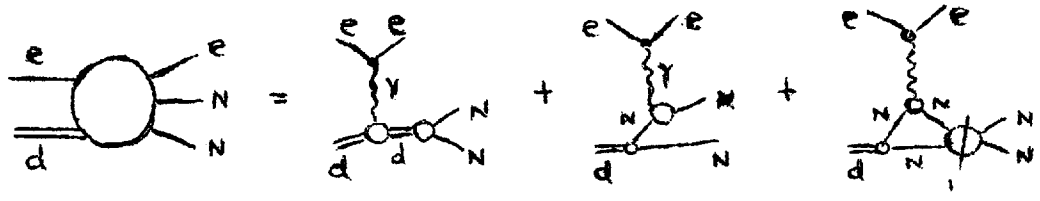
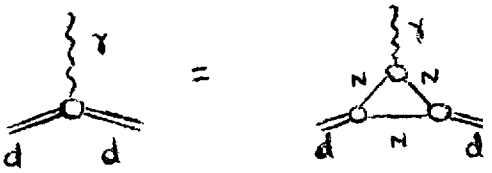


FIGURE (3c)

If  according to the impulse approximation, then the potential theory diagrams are exactly reproduced.

5. Covariant Calculation of Final State Interaction.

The contributions to the five point function as given in Figure 18, which we have obtained systematically by considering the final state channels, are of several types: firstly there are the Chew-Low graphs of Figures 19, the Born terms of Figure 20, the vertex graphs of Figure 21 and finally what we call the "final state interactions" graphs of Figure 22.

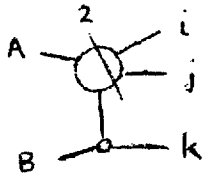


FIGURE 19.

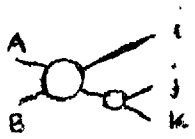


FIGURE 20.

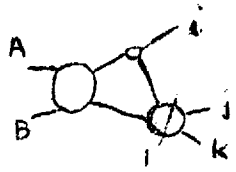


FIGURE 21.

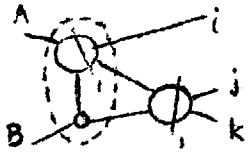


FIGURE 22.

The amplitudes of the first two are trivially easy to write, but Figure 21 produces much more complicated singularities in the s-variables and, indeed, these may well be complex [11]. (Although these singularities can be located in principle[17] and the contributions evaluated even when anomalous thresholds arise, a complete discussion of these questions is quite beyond

the scope of the present work). Lastly there are the diagrams of Figure 22 where our interest lies and we shall attempt to calculate these in a manner that is very reminescent of the non-covariant treatment, but which is complicated by the spin details.

Unlike potential theory, we shall have 2-particle intermediate states which need not be the same as the final ones, and for that reason we attach the distinguishing labels n and m to them. Let f_i represent the free field solution of particle i. The amputated matrix element will contain complete propagators, but in so far as we have been neglecting 3-particle intermediate states, it is in order to approximate the propagators by the free parts.

$$\therefore \quad M = -i \frac{d^4p_n d^4p_m}{(2\pi)^4} \int \frac{\mathcal{M}(p_n,p_m)}{(p_n^2-m_n^2+i\epsilon)(p_m^2-m_m^2+i\epsilon)} \delta^4(p_n+p_m-p_1-p_2) \qquad (36a)$$

$$\text{where } \mathcal{M}(p_n,p_m) = f_1^+ f_2^+ \; M(12,p_n p_m) \; f_3^+ \; M(3p_n p_m, AB) \; f_A f_B \qquad (36b)$$

The M are the separate interaction elements and their unphysicality is expressed by the labels p_n and p_m, denoting that the particles do not lie on their mass shells. These M may have vector indices and may indeed be matrices with factors $(\not{p}_n+m)$, .. absorbed.

To treat the problem more symmetrically we introduce the total and relative momentum vectors : $p = p_n+p_m$ and $2q = p_n-p_m$.

$$\therefore \quad M = -i \int \frac{d^4q}{(2\pi)^4} \quad \frac{\mathcal{M}(q)\ p_1=p_f}{\left[(\tfrac{1}{2}p+q)^2-m_n^2+i\varepsilon\right]\left[(\tfrac{1}{2}p-q)^2-m_m^2+i\varepsilon\right]} \tag{37a}$$

which diverges logarithmically if $\mathcal{M}$ is just constant, but is probably not so in actual fact. Keeping the δ -function of the propagators, we should have a physical process (or the absorptive part of the amplitude, in the language of dispersion theory) with

$$2i\,(M)_{abs} = -\,i \int \frac{d^4q}{(2\pi)^4}\, \mathcal{M}(q) \cdot -2\pi i\delta_+\left[(\tfrac{1}{2}p+q)^2-m_n^2\right] \cdot -2\pi i\delta_+\left[(\tfrac{1}{2}p-q)^2-m_m^2\right] \tag{38}$$

Note that only the proper root of the δ functions has been retained because of the causal method of integration. The integral (38) is evaluated in Appendix 2, and Appendix 1 explains the notation employed throughout this dissertation. Quoting the answer,

$$(M)_{abs} = \tfrac{1}{2}\int \frac{d^4q}{(2\pi)^2}\,\mathcal{M}(q)\ \delta_+\left[(\tfrac{1}{2}p+q)^2-m_n^2\right]\delta_+\left[(\tfrac{1}{2}p-q)^2-m_m^2\right]$$

$$= \frac{\left\{(p^2-m_n^2-m_m^2)^2 - 4m_n^2m_m^2\right\}^{\frac{1}{2}}}{16\pi p^2}\,\bar{\mathcal{M}}\ \vartheta\left\{p^2-(m_n+m_m)^2\right\} \tag{39a}$$

where $\bar{\mathcal{M}} \equiv \int \frac{d\Omega}{4\pi}\,\mathcal{M}(m_n,m_m)$ (39b)

is the average value of physical scattering in the intermediate state.

We now come to the problem of evaluating the complete integral (37a) which apart from an averaging over angles also comprises an integration over off-mass shell elements. We expect the result to be determined in some way by the mass-shell elements and for that reason we write (37a) as

$$M_{fi} = -i\bar{\mu}\int \frac{d^4q}{(2\pi)^4} \quad \frac{\mathcal{M}(p_n,p_m)\,/\,\bar{\mu}}{\left[(\tfrac{1}{2}p+q)^2 - m_n^2+i\varepsilon\right]\left[(\tfrac{1}{2}p-q)^2-m_m^2+i\varepsilon\right]} \qquad (37b)$$

and now we are integrating over a dimensionless quantity. $\mathcal{M}(p_n,p_m)/\bar{\mu}$ is a measure of the extent to which the off-shell elements compare with the on-shell ones, and is of course unknown. We might expect the ratio to become smaller the more unphysical becomes the intermediate scattering. We shall choose this ratio to be unity up to a cut-off $q^2 = -\Lambda^2$, in which range the M's can be regarded as two renormalised coupling constants for the primary and secondary interactions. This procedure parallels the one which we carried out in the non-covariant theory. Thus we will be computing

$$(M_{fi})_{\text{real}} = \bar{\mu} I\,(p^2) \text{ with}$$

$$I(p^2) = -i\int \frac{d^4q}{(2\pi)^4} \quad \frac{1}{(q^2-m_n^2+i\varepsilon)\left[\overline{p-q}^2-m_m^2+i\varepsilon\right]}$$

$$= -i\int \frac{d^4q}{(2\pi)^4}\int_0^1 \frac{dx}{\left[(q^2-m_n^2)(1-x)+\left\{(p-q)^2-m_m^2\right\}x\right]^2} \qquad (40)$$

The denominator in (40) is $(q-px)^2 - a^2(x)$ where

$$a^2(x) = p^2x^2 - (p^2+m_n^2-m_m^2)x + m_n^2 \ .$$

The logarithmic divergence permits a shift in origin through px :

$$\therefore \quad I(p^2) = -i\int_0^1 dx \ \frac{d^4q}{(2\pi)^4} \cdot \frac{1}{(q^2 - a^2(x))^2}$$

$$= -i\int^{q^2=-\Lambda^2} \frac{d^4q}{(2\pi)^4} \cdot \frac{1}{(q^2-m_n^2)} + \frac{\pi^2}{(2\pi)^4}\int \frac{2x \cdot \frac{da^2}{dx}}{2a^2}\, dx$$

$$= \frac{1}{16\,\pi^2}\left[2\log\frac{\Lambda}{m_n} - 1 + \int_0^1 x\,\frac{da^2}{dx}\,\frac{dx}{a}\right] .$$

Notice that, as before, we integrate the divergent part up to a cut-off Λ , but the convergent part up to infinity. The error which is introduced thereb is negligible for large Λ . Now

$$\int_0^1 x\,\frac{da^2}{dx}\cdot\frac{dx}{a^2} = \int_0^1 dx\left[2 + \frac{(p^2+m_n^2-m_m^2)x - 2m_n^2}{p^2x^2-(p^2+m_n^2-m_m^2)x+m_n^2}\right] dx$$

$$= 2 + \frac{p^2+m_n^2-m_m^2}{p^2}\log\left(\frac{m_m}{m_n}\right) + \frac{(p^2-m_n^2-m_m^2)^2-4m_n^2m_m^2}{4Bp^4}\log\left|\frac{A-B-\frac{m_n^2}{p^2}}{A+B-\frac{m_n^2}{p^2}}\right|$$

where $A = \frac{1}{2}\left(1+\frac{(m_n^2-m_m^2)}{p^2}\right)$ and $B^2 = \frac{1}{4p^4}\left[(p^2-m_n^2-m_m^2)^2-4m_n^2 m_m^2\right]$.

The answer is easily symmetrised in m_n and m_m by rewriting this last results with m_n and m_m interchanged. Halving the sum of these expressions, we arrive at

$$I(p^2)=\frac{1}{16\pi^2}\left[\log\frac{\Lambda^2}{m_n m_m}+1+\frac{m_m^2-m_n^2}{p^2}\log\left(\frac{m_n}{m_m}\right)+\frac{C}{p^2}\log\left|\frac{p^2-m_n^2-m_m^2+C}{p^2-m_n^2-m_m^2-C}\right|\right] \tag{41a}$$

with $C^2(p^2,m_n^2,m_m^2) \equiv p^4+m_n^4+m_m^4-2m_n^2p^2-2m_n^2m_m^2-2m_m^2p^2$

$$= (p^2-m_n^2-m_m^2)^2 - 4m_n^2m_m^2 \tag{41b}$$

At the threshold of production of n and m, $p^2=(m_n+m_m)^2$ and C=0, so

$$I\left((\overline{m_n+m_m}^2)\right) = \frac{1}{16\pi^2}\left[\log\frac{\Lambda^2}{m_n m_m}+1+\frac{m_m-m_n}{m_m+m_n}\log\left(\frac{m_n}{m_m}\right)\right] \tag{42}$$

and in terms of it the final result reads

$$M = \frac{\bar{\mathcal{M}}}{16\pi^2}\left[16\pi^2(\overline{M_n+m_m}^2)+\frac{(m_n-m_m)}{p^2(m_n+m_m)}\left\{p^2-(m_n+m_m)^2\right\}\log\left(\frac{m_n}{m_m}\right)+\frac{C}{p^2}\log\left|\frac{p^2-m_n^2-m_m^2+C}{p^2-m_n^2-m_m^2-C}\right| + \frac{i\pi C}{p^2}\right] \tag{43}$$

The imaginary part which we had disregarded in calculating I has been explicitly included and derives from the branch cut associated with the

logarithmic term, by attaching a small imaginary part to each mass according to Feynman's prescription. Notice the great similarity between (24) and (43), which becomes even more striking when we specialise to final state scattering : n=1 and m=2; Since $p^2 = \omega^2$ and $c = 2q\,\omega$,

$$M = \frac{\bar{\mathcal{M}}}{16\pi^2}\left[\pi c\,(\overline{m_1+m_2}^2) + \frac{m_1-m_2}{\omega^2(m_1+m_2)}\left\{\omega^2-(m_1+m_2)^2\right\} + \frac{2q}{\omega}\log\left|\frac{\omega^2-m_1^2-m_2^2+2q\,\omega}{\omega^2-m_1^2-m_2^2-2q\,\omega}\right| + \frac{2i\,q}{\omega}\right] \tag{44a}$$

where $$\pi\, c\,(\overline{m_1+m_2}^2) = \log\frac{\Lambda^2}{m_1 m_2} + 1 + \frac{m_2-m_1}{m_2+m_1}\log\left(\frac{m_1}{m_2}\right) \tag{44b}$$

In order to carry out a partial wave analysis of $\bar{\mathcal{M}}$ we must make a brief diversion into the question of constructing helicity states, and the properties which they possess.

6. Helicity decomposition.

The basic one particle-states are normalised in the usual way so that $\langle \underline{p}'s'\lambda' | \underline{p}\, s\lambda\rangle = (2\pi)^3\delta^3(\underline{p}-\underline{p}').2E\,\delta_{\lambda\lambda'}\delta_{ss'}$. Equivalently, the 1 particle projection operator is $\Sigma_{s,\lambda}\int |\underline{p}\, s\,\lambda\rangle \frac{d^3p}{(2\pi)^3 2E}\langle \underline{p}s\lambda|$.

In the frame of zero total momentum of two particles, the two-particle helicity states [8] are defined through

$$|\,\omega\underline{o}\;jm;\,s_1s_2,\lambda_1\lambda_2\rangle = \left\{\frac{(2j+1)q}{4\pi\omega}\right\}^{\frac{1}{2}}\int D^{j*}_{m\lambda}(\underline{\vartheta})\,|\underline{q}s_1\lambda_1, -\underline{q}s_2\lambda_2\rangle\,\frac{d\Omega}{4\pi}$$

$$\text{or } |\underline{q}\, s_1\lambda_1\, ;\, -\underline{q}\, s_2\lambda_2\rangle = \sum_{j,m}\left(\frac{4\pi\omega}{q}\right)^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}D^{j}_{m\lambda}(\underline{\vartheta})\,|\omega\,\underline{o}jm; s_1s_2\lambda_1\lambda_2\rangle \quad (46a)$$

where $\underline{q}$ is a vector of magnitude q, oriented at an angle $\underline{\vartheta}$,

In (45) and (46), $\lambda = \lambda_1 - \lambda_2$ and the zero vector within the helicity state shows that the definition is true in the barycentric frame. This state can be regarded as a one particle state of mass ω, spin j and spin component m. By applying a Lorentz transformation [18] on this pseudo-particle we can construct a general two particle state of total momentum $\underline{p}$ and whose 4-vector square is ω^2. This transformation brings in the rotation angles of the particle momenta as referred to the untransformed momenta, and thse angles are shown in Figure 23. O represents the original frame and O' is the new frame.

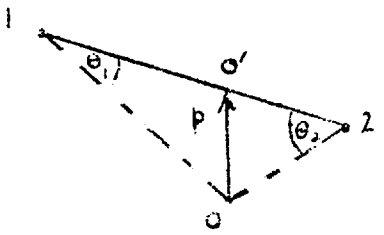


FIGURE 23.

$$\therefore |\underline{p}\; j\; m;\; s_1\lambda_1 s_2\lambda_2\rangle = \sum_{\mu_1,\mu_2} D^{s_1^*}_{\lambda_1\mu_1}(\vartheta_1)D^{s_2^*}_{\lambda_2\mu_2}(\vartheta_2)\,|\omega\,\underline{o}jm;\; s_1\lambda_1 s_2\lambda_2\rangle$$

$$\text{or } |p,jm;\; s_1\lambda_1 s_2\lambda_2\rangle = \left\{\frac{(2j+1)q}{4\pi\omega}\right\}^{\frac{1}{2}} \sum_{\mu_1\mu_2}\int\frac{d\Omega}{4\pi} D^{s_1^*}_{\lambda_1\mu_1}(\vartheta_1)D^{s_2^*}_{\lambda_2\mu_2}(\vartheta_2)\, D^{j}_{m\lambda}(\underline{\vartheta})$$

$$|\underline{q}\, s_1\lambda_1\, ;\, -\underline{q}\, s_2\lambda_2\rangle \qquad (47)$$

Showing the construction from the original states.

The normalisation properties can now be conveniently written, because

$$\int \frac{d\Omega}{4\pi} \cdot \frac{q}{4\pi\omega} = \int \frac{d^3p_1}{2E_1(2\pi)^3} \cdot \frac{d^3p^2}{2E_2(2\pi)^3} \cdot \delta^4(p_1+p_2-p)$$

$$< p'jm';\ s_1'\lambda_1' s_2'\lambda_2' \mid p\ j\ m;\ s_1\lambda_1\ s_2\lambda_2 >$$

$$= (2\pi)^4\delta^4(p-p')\delta_{jj'}\ \delta_{mm'}\delta_{s_1s_1'}\ \delta_{s_2s_2'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}\ 2\pi\delta(p^2-p'^2) \tag{48}$$

It is now straightforward to construct a three particle helicity state by coupling the $|\omega jm>$ to $|\ \underline{p}_3 s_3\lambda_3>$ as given by (45) and (46). In the overall barycentric frame for a total energy W,

$$|\ \omega\ \underline{p}\ jm;\ s_1\lambda_1\ s_2\lambda_2\ ;\ -\underline{p}\ s_3\lambda_3\ >$$

$$= \sum_{JM} \left(\frac{4\pi W}{Q}\right)^{\frac{1}{2}} (2J+1)^{\frac{1}{2}}\ D^J_{M\Lambda}(\theta)\ |W\underline{Q}JM;\ jm;s_1\lambda_1 s_2\lambda_2 s_3\lambda_3 > \tag{49a}$$

where $\Lambda = m - \lambda_3$ and θ denotes the orientation of $\underline{p}$ relative to some particular axis. The above is easily generalised to any number of particles, but the notation becomes too cumbersome if we have to label the states completely. For the same of simplicity we write $|ij>$ for a plane wave state and $|\ (ij)\ >$ for a helicity state coupled as in (45). At the same time it is best to label the angular momenta and energies by ij suffices as the order in which the particles are

coupled is then self evident. In this notation, (46a) and (49a) become

$$|\ 1\ 2\ \rangle = \sum_{j_{12} m_{12}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}}_{m_{12}\lambda_{12}}(\underline{\vartheta}_{12})\ |(12)\rangle \qquad (46b)$$

$$|(12)3\rangle = \sum_{j_{123} m_{123}} \frac{4\pi\omega_{123}}{q_{12,3}}\ (2j_{123}+1)^{\frac{1}{2}} D^{j_{123}}_{m_{123}\lambda_{12,3}}\ |((12)3)\rangle \qquad (49b)$$

so

$$|123\rangle = \sum_{\substack{j_{123}, m_{123} \\ j_{12}, m_{12} \\ \mu_1 , \mu_2}} \left(\frac{4\pi\omega_{123}}{q_{12,3}}\right)^{\frac{1}{2}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{123}+1)^{\frac{1}{2}}(2j_{12}+1)^{\frac{1}{2}}$$

$$D^{j_{123}}_{m_{123}\mu_{12,3}}(\underline{\vartheta}_{12,3})\ D^{j_{12}}_{m_{12}\mu_{12}}(\underline{\vartheta}_{12})\ e^{i\pi s_2}\ D^{s_1 *}_{\lambda_1\mu_1}(\vartheta_1) D^{s_2 *}_{\lambda_2\mu_2}(\vartheta_2)\,|((12)3)\rangle \qquad (50)$$

The angles involved in (50) are indicated in Figure 24, and the factor $e^{i\pi s_2}$ arises from the helicity convention of Wick. The reccupling coefficients $\langle((12)3)\,|\,((13)2)\rangle$ have been given by Wick,[18] but will not be used in the forthcoming work.

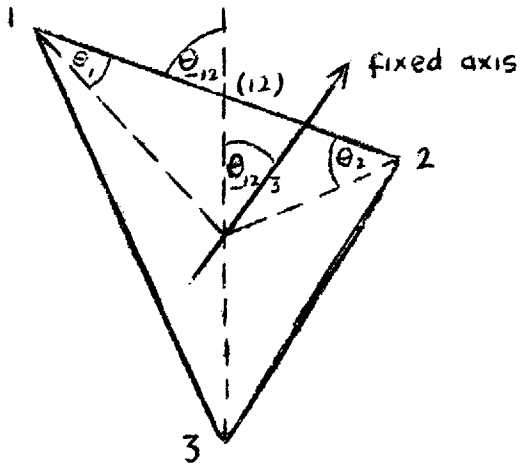


FIGURE 24.

7. Relativistic partial wave analysis.

We now return to equations (44) and (39) to carry out an angular momentum analysis and obtain complete correspondence with (25). Remember that $\bar{\mathcal{M}}$ is a physical average over angles and must contain a sum over intermediate spins. Now in the shorthand notation,

$$\langle 12 \mid M_{12} \mid 1'2' \rangle = \Sigma_{j_{12}'m_{12}'} \langle 12 \mid M_{12} \mid (1'2') \rangle \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}}_{m_{12}\lambda_{12}}(\underline{\vartheta}'_{12})$$

$$\langle 123 \mid M_o \mid AB \rangle = \Sigma_{j_{12}m_{12}} \langle (12)3 \mid M_o \mid AB \rangle \frac{4\pi\omega_{12}}{q_{12}}^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\lambda_{12}}(\underline{\vartheta}_{12})$$

both statements being true in the barycentric frame of 1 and 2

$$\therefore \bar{\mathcal{M}} = \Sigma_{\lambda_1'\lambda_2'} \int \frac{d\Omega'_{12}}{4\pi} \langle 12 \mid M_{12} \mid 1'2' \rangle \langle 1'2'3 \mid M_o \mid AB \rangle$$

$$= \Sigma_{\substack{\lambda_1'\lambda_2' \\ j_{12}\, m_{12} \\ j'_{12}\, m'_{12}}} \int \frac{d\Omega'_{12}}{4\pi} \langle 12 \mid M_{12} \mid (1'2') \rangle \langle (1'2')3 \mid M_o \mid AB \rangle \frac{4\pi\omega_{12}}{q_{12}} (2j_{12}+1)^{\frac{1}{2}} (2j'_{12}+1) D^{j_{12}}_{m_{12}\lambda'_{12}}(\underline{\vartheta}'_{12}) D^{j_{12}*}_{m_{12}\lambda_{12}}(\underline{\vartheta}'_{12})$$

$$= \Sigma_{\substack{\lambda_1'\lambda_2' \\ j_{12}\, m_{12}}} \langle 12 \mid M_{12} \mid (1'2') \rangle \langle (1'2')3 \mid M_o \mid AB \rangle \frac{4\pi\omega_{12}}{q_{12}}$$

$$= \sum_{\substack{\lambda_1'\lambda_2' \\ j_{12}m_{12}}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{3/2} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\lambda_{12}}(\underline{\vartheta}_{12}) \langle (12)|M_{12}|(1'2')\rangle \langle (1'2')3|M_o|AB\rangle \tag{51}$$

At low energies up to O (q) one can approximate (44) by

$$M \approx \frac{\bar{M}}{16\pi} \left[c + \frac{2iq}{\omega}\right] .$$ Together with the production amplitude,

$$M = \sum_{j_{12}m_{12}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\lambda_{12}}(\underline{\vartheta}_{12}) \langle (12)3|M_o|AB\rangle + \frac{\omega_{12}}{4q_{12}} \sum_{\lambda_1'\lambda_2'} \langle (12)|M_{12}|(1'2')\rangle \langle (1'2')3|M_o|AB\rangle \left\{c + \frac{2iq_{12}}{\omega_{12}}\right\}$$

or in greater detail,

$$M = \sum_{\substack{j_{12}m_{12} \\ \lambda_1'\lambda_2'}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\lambda_{12}}(\underline{\vartheta}_{12}) \langle \lambda_1'\lambda_2', 3|M_o{}^{j_{12}m_{12}}(\omega_{12})|AB\rangle \cdot \left\{\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} + \frac{c_{j_{12}m_{12}} + \frac{2iq_{12}}{\omega_{12}}}{4q_{12}/\omega_{12}} \langle \lambda_1\lambda_2|M_{12}{}^{j_{12}}(\omega_{12})|\lambda_1'\lambda_2'\rangle\right\} \tag{52}$$

We have chosen to label the parameter c with angular momentum as there is no reason to advocate the same cut-off for every partial wave.

Final state scattering among spinless particles is then only a special case of (52) with $\lambda_1 = \lambda_2 = 0$. Also j_{12} is just the orbital momentum ℓ_{12},

$$\therefore \quad M = \sum_{\ell_{12} m_{12}} \frac{4\pi\omega_{12}}{q_{12}} (2j_{12}+1)^{\frac{1}{2}} D^{12\,*}_{m_{12}\,o}(\underline{\vartheta}_{12}) \langle \omega_{12} \ell_{12} m_{12}, 3 | M_o | AB \rangle \cdot \left\{ 1 + \frac{(c_{\ell_{12} m_{12}} + \frac{2iq_{12}}{\omega_{12}}) e^{i\delta_{12}} \sin\delta_{12}}{2q_{12}/\omega_{12}} \right\} \tag{53}$$

where we have used unitarity to put $M = 2e^{i\delta}\sin\delta$. Actually the general equation (52) can always be written in the form (53) if M is diagonalised by clever manipulation of the symmetry properties of helicity states[8].

We can go one stage further with (53) and bring out the angular dependence $\underline{\vartheta}_{12,3}$ to give the completely reduced element $\langle ((12)3) | M_o | AB \rangle$ which may then be written linearly in terms of $\langle ((13)2) | M_o | AB \rangle$. (The latter is the natural way of coupling if we refer to Figure 22). In order to couple 3 to (12) we must first go to the overall barycentric frame by applying a suitable Lorentz transformation $\mathfrak{L}$ along the direction of motion of 3 with respect to the rest frame of (12). In equation (51) notice that the intermediate projection $\sum_{\substack{j_{12} m_{12} \\ \lambda_1 \lambda_2}} |j_{12} m_{12} \lambda_1' \lambda_2' \rangle\langle j_{12} m_{12} \lambda_1' \lambda_2' |$ will not bring in rotation angles of the intermediate particles. Thus, invariance

of the S(or M) matrix under Lorentz transformations means that

$$< (12)3 \mid M_o \mid AB > = <(12)\ 3|\mathcal{L}^{-1} M_o \mathcal{L}|AB >$$

$$= <(12)3 \mid M_o| AB> \ D\ (\vartheta_1)\ D(\vartheta_2) \text{ in the total rest frame}$$

$$<12 \mid M_{12} \mid (1'2')><(1'2')3|M_o|AB> = <12|\mathcal{L}^{-1}M_{12}|(1'2')><(1'2')3|M_o\mathcal{L}|AB>$$

$$= < 12 \mid M_o \mid (1'2') ><(1'2')3|M_o|AB> \text{in the total frame.}$$

The final result, after coupling the remaining particles, is

$$M = \sum_{\substack{j_{12}m_{12} \\ j_{123}m_{123} \\ \nu_1\nu_2 \\ \mu_1\mu_2}} e^{i\pi s_2} \left(\frac{4\pi\omega}{q_{12}}\right)^{\frac{1}{2}} \frac{4\pi\omega_{AB}}{q_{12,3}^{\frac{1}{2}}\, q_{AB}^{\frac{1}{2}}} (2j_{AB}+1)(2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\mu_{12}}(\underline{\vartheta}_{12,3}) D^{j_{AB}}_{m_{AB}\lambda_{12,3}}(\underline{\vartheta}_{12,3})$$

$$D^{j_{AB}}_{m_{AB}\lambda_{AB}}(\underline{\vartheta}_{AB}) \cdot D^{s_1}_{\lambda_1\mu_1}(\vartheta_1) D^{s_2}_{\lambda_2\mu_2}(\vartheta_2) <\nu_1\nu_2\lambda_3|M_o^{j_{12}m_{12}j_{AB}}(\omega_{12},\omega_{AB})|\lambda_A\lambda_B>$$

$$\cdot\left\{\delta_{1\mu_1\nu_2\mu_2} + \frac{c_{j_{12}m_{12}} + \dfrac{2iq_{12}}{\omega_{12}}}{4q_{12}/\omega_{12}} <\mu_1\mu_2|M_{12}^{j_{12}m_{12}}(\omega_{12})|\nu_1\nu_2>\right\}$$

(54)

We have used the total rotational invariance of M_o to put $j_{AB} = j_{123}$, $m_{AB} = m_{123}$ and translational invariance, or energy-momentum conservation, to put $\omega_{123} = \omega_{AB}$. $\lambda_{12,3} = m_{12} - \lambda_3$ and the angles are all measured with

respect to a given axis. If the axis is chosen to be along the direction of relative motion of A and B, equation (54) simplifies because $D^{j_{AB}}_{m_{AB}\lambda_{AB}}(\underline{0}) = \delta m_{AB}\lambda_{AB}$. Moreover choosing the direction of motion of 3 to define the ?ox axis, $D^{j_{AB}*}_{m_{AB}\lambda_{12,3}}(\underline{\vartheta}_{12,3}) = d^{j_{AB}}_{m_{AB}\lambda_{12,3}}(\vartheta_{12,3})$. The relation between the Wick variables ϑ_{12}, $\vartheta_{12,3}$, ω_{12}, ω_{AB} and the relativistic invariant s variables is given in the appendix as well as a short discussion about three-particle phase space and of Dalitz plots.

8. Simultaneous Interactions.

In writing the amplitude in equation (52) and (54), the pole terms and vertex graphs have been omitted altogether and the final state interactions involving 23 and 32 have been assumed negligible. The first two omissions can be remedied by separately calculating the contributions. On the other hand, the forces between 23 and 31 do complicate the siutation because of the different production amplitudes associated with their graphs. However, the results of Chapter I will necessarily be recovered if the two body scatterings are the only significant amplitudes, and we need only look back to equation (29) modified by spin details as given in (54) to deduce the solution up to lowest order in final state scattering. In fact the main results of this chapter has been to confirm covariantly the results of Chapter I and to indicate what extra contributions to the amplitude by particle creation and annihilation, must be explicitly included.

One final generalisation must be mentioned. This has to do with problems in which one pair channel predominates in the sense that the only

significant amplitudes are

$$A + B \longrightarrow (\text{Pair of particles}) + 3$$

In particular if we consider the two processes

$$A + B \longrightarrow 1 + 2 + 3 \quad \text{and} \quad M + n + 3 ,$$

we can still apply (52) by combining these amplitudes in the form of a two component vector and with the understanding that the final state interaction, as specified by $\cot\delta$, takes the form of a 2 x 2 symmetric matrix ie Watson's theorem acquires a matrix character. Nauenberg[28] has proceeded in just such a manner for the production of $\Sigma\Pi$ near the Y^* threshold; besides $\Sigma\Pi$ he includes $\bar{K}N$ intermediate states so $\cot\delta =$

$$\begin{pmatrix} \cot\delta(\Sigma\Pi \longrightarrow \Sigma\Pi) , & \cot\delta(\Sigma\Pi \rightarrow \bar{K}N) \\ \cot\delta(\bar{K}N \rightarrow \Sigma\Pi) , & \cot\delta(\bar{K}N \longrightarrow \bar{K}N) \end{pmatrix}$$

which he next parametrises by the resonance point and width of the Y^*.

CHAPTER III

DISPERSION THEORY OF FINAL STATE INTERACTIONS

1. Generalised Unitarity.

The work contained in the previous chapter being field theoretical, it naturally led to the appearance of off-shell matrix elements although these were later approximately related to on-shell elements via the point interaction model. By contrast, if we approach the subject from the dispersion theoretic angle, all we have to do is to locate the singularities of the relevant diagrams, evaluate the discontinuities across them and then to obtain the actual amplitude by a Hilbert transformation. The whole point of the dispersion relations is that these discontinuities are always on-shell amplitude and no models need be invoked for the purpose of computation. Actually the problem is vastly more complicated [17] and is associated with the question as to where all the dominating singularities lie, what discontinuities they possess. In the following we entirely simplify the situation by picking out of all this complexity only the cut originating from the final state interactions of pairs of particles.

Looking back at (37a), then if we assume that $M(q)$ possesses 'distant' singularities we have an oridinary Feynman integral for the self-energy of a particle carrying momentum p. A Landau-Cutkosky analysis [9,10] reveals that, regarded as a function of the complex invariant p^2, the diagram has a branch point at $p^2 = 4m^2$ and a branch cut along the real axis (by choice) running to infinity. The discontinuity across this cut is $2i(M_{fi})_{abs}$, given by expressions (38) and (39). Accordingly one can postulate the dispersion relation,

$$M(p^2) = \int_{(m_n+m_m)^2}^{\infty} \frac{\frac{1}{\pi}\,\mathrm{Im}\,M(p'^2)}{p'^2-p^2-i\varepsilon}\,dp'^2 = \frac{1}{\pi}\int_{(m_n+m_m)^2}^{\infty} \frac{M_{abs}(p'^2)dp'^2}{p'^2-p^2-i\varepsilon} \qquad (55)$$

which may now be explicitly calculated. We can expect equation (55) to be correct near $p^2 \lesssim (m_n+m_m)^2$ and probably up to the threshold of production of another particle, and providing other singularities, such as the complex type discowered by Landshoff and Tremain[11], lie well away from the physical region. Note that the amplitude M_o is not the physical amplitude, but just the 'production part' - this circumstance should be compared with axiomatic theory when the functions which occur are the full physical amplitudes.

Wishing to establish the connexion between (55) and our previous results on final state scattering we carry out a similar sort of approximation. With $p^2 = \omega^2$,

$$(M_{fi})_{abs} = \frac{q}{8\pi\omega}\,\vartheta\,(\omega - m_1 - m_2)\,\bar{\mathcal{M}}(\omega) \qquad (56)$$

$$\therefore \quad M(\omega) = \frac{1}{16\pi^2}\int_{(m_1+m_2)^2}^{\infty} \frac{2q'}{\omega'}\,\frac{d\omega'^2}{\omega'^2 - \omega^2 - i\varepsilon}\,\bar{\mathcal{M}}(\omega')$$

$$= \frac{1}{16\pi^2}\int_{(m_1+m_2)}^{\infty} \frac{4q'}{\omega'+\omega}\,\frac{d\omega'}{\omega - \omega - i\varepsilon}\,\bar{\mathcal{M}}(\omega') \qquad (57)$$

which should be compared with equation (18). The slight difference in the integrands would disappear if we consider the antiparticle diagram in addition to the particle diagram contribution (18). In (21) the point interaction model implied $\mathcal{M}(\omega,\omega') = \mathcal{M}(\omega,\omega)$ up to $\omega' = \omega_c$, where ω was the esternal and ω' the intermediate energy; since there is no reason for not putting $\mathcal{M}(\omega',\omega) = \mathcal{M}(\omega',\omega')$, apart from ω' being an integration variable, this would imply $\mathcal{M}(\omega,\omega) = \mathcal{M}(\omega',\omega')$ i.e. that the physical elements are constant up to a cut-off.

Writing $M(\omega) = \dfrac{\mathcal{M}(\omega)}{16\pi^2}\displaystyle\int_{m_1+m_2}^{\infty} \frac{4q}{\omega'+\omega}\cdot\frac{\bar{\mathcal{M}}(\omega')\,/\bar{\mathcal{M}}(\omega)}{\omega' - \omega + i\varepsilon}\,d\omega'$

this model gives $M(\omega) \approx \dfrac{\mathcal{M}(\omega)}{16\pi^2}\displaystyle\int_{m_1+m_2}^{\omega_c} \frac{4q'}{\omega+\omega'}\cdot\frac{d\omega'}{\omega'-\omega+i\varepsilon}$ (58)

or precisely the same thing as before.

In spite of the obvious comparison with the previous work, (57) provides us in principle with a more powerful method of calculation, as the integral can be explicitly evaluated if $\bar{\mathcal{M}}(\omega)$ is known. For instance, let us suppose that the production amplitude is structureless and that final state scattering is pure S-wave, so that

$$\bar{\mathcal{M}}(\omega) = \frac{g\, e^{i\delta} \sin\delta}{2q/\omega} = \frac{ga}{1 - \frac{2iaq}{\omega}} .$$

Strictly speaking we should replace $\bar{\mathcal{M}}(\omega)$ by $\mathrm{Re}\,\bar{\mathcal{M}}(\omega)$, because it has been assumed to have no complex singularities. The f.s.i. amplitude becomes

$$\therefore \quad M(\omega) = \frac{1}{16\pi^2}\left[\frac{2i\pi q}{\omega} + \mathcal{P}\int_{m_1+m_2}^{\infty} \frac{4q'}{\omega'+\omega}\,\frac{d\omega'}{\omega'-\omega}\,\frac{ga}{\sqrt{1+a^2\left(\frac{2q'}{\omega'}\right)^2}}\right] \qquad (59)$$

which tells us that $\left(\frac{2q_c}{\omega c}\right) = O\left(\frac{1}{a}\right)$ ie that the cut-off energy becomes larger as the final state scattering becomes smaller. This means that the principal part of the integral and therefore the parameter c should be somewhat insensitive to the amount of final scattering.

2. Axiomatic Theory.

In this sub-section we shall transcribe Jackson's work[25] into our own notation and will compare it with what has gone before. At the outset we will only let one pair of particles, 1 and 2 say, interact strongly in the final state. By the Lehmann, Symanzik and Zimmerman reduction procedure, one may cast the amplitude in the form

$$\leq 123 \,|\, AB >_{+} \;=\; f_3^{+} \leq 12 \,|\, J^{+}(o) \,|\, o > f_A f_B \,.$$

$$< 123 \,|\, M \,|\, AB > \;=\; i f_3^{+} \int d^4x \; f_2^{+}(x) < 1 \,|\, \vartheta(x) \left[j_2(x),\; J^{+}(o) \right] |\, o > f_A f_B \tag{60}$$

Here $J(o)$ is an effective current consisting of retarded commutators, which may itself be derived by reduction, and which depends on external invariants, while j_1 is the current of particle 1, viz $(\Box + m_2^2)\phi_2(x) = j_2(x)$ if particle 1 has integral spin. The neglect of an equal times commutator will be discussed later.

$$\mathrm{Im}\, <123|M|AB> \;=\; \tfrac{1}{2} f_3^{+} \int d^4x f_2^{+}(x) < 1 | \left[j_2(x),\; J^{+}(o) \right] | o > f_A f_B$$

$$= \tfrac{1}{2} f_3^{+} f_2^{+} \sum_n \int d^4x \; e^{ip_2 \cdot x} \left\{ \begin{array}{l} <1 \,|\, j_2(x) \,|\, n><n \,|\, J^{+}(o) \,|\, o> \\ -<1 \,|\, J^{+}(o) \,|\, n><n \,|\, j_2(x) \,|\, o> \end{array} \right\} f_A f_B$$

Translational invariance of Heisenberg fields represented by

$$\varphi(x) = e^{+iP \cdot x} \; \varphi(o) \; e^{-iP \cdot x} \tag{61}$$

allows the space-time integration to be carried out. Hence

$$\mathrm{Im}\, <123|M|AB> \;=\; \tfrac{1}{2} (2\pi)^4 f_3^{+} f_2^{+} \sum_n \left\{ \begin{array}{l} <1 \,|\, j_2(o) \,|\, n><n \,|\, J^{+}(o) \,|\, o> \delta^4(p_1 + p_2 - p_n) \\ -<1 \,|\, J^{+}(o) \,|\, n><n \,|\, j_2(o) \,|\, o> \delta^4(p_n + p_2) \end{array} \right\} \tag{62}$$

The second sum in (62) vanishes because the matrix elements can only be $\langle 1 \mid J(o)\mid \bar{2}\rangle\langle\bar{2}\mid J_2(o)\mid o\rangle$ and $\langle \bar{2}\mid j_2(o)\mid o\rangle = 0$. In the remaining sum, if pole terms do not exist,

$$\text{Im } M\,(123,AB) = \tfrac{1}{2}\sum_{n,m}\int \frac{d^4p_n d^4p_m}{(2\pi)^2}\,\delta^4(p_1+p_2-p_n-p_m)\langle 1\mid j_2(o)\mid n\rangle\langle n\mid J\,(o)\mid o\rangle$$
$$\delta_+(p_n^2-m_n^2)\,\delta_+\,(p_m^2-m_n^2) \qquad (63)$$

When n = 1, m= 2,

$$\text{Im } M\,(123,AB) = \frac{1}{8\pi^2}\int d^4q\,\langle 1\mid j_2(o)\mid q_{12}\rangle\langle q_{12}\mid J^+(o)\mid o\rangle\delta(p+q^2-m_1^2)\,\delta(q^2-m_2^2)$$

which is the same as (28) apart from $\langle q_{12}\mid J^+(o)\mid o\rangle$ being the physical amplitude M and not the production amplitude M_o.

$$\text{Hence}\quad \text{Im } M\,(123,AB) = \frac{q}{8\pi\omega}\int \frac{d\Omega'_{12}}{4\pi}\; M^+\,(1'2',\,12)\,M\,(1'2'3,AB) \qquad (64)$$

This is nothing more than a familiar restatement of the unitarity condition on the transition amplitude in the particular form used by Jacob et al. Expanding in partial waves, and proceeding as in (51)

$$\text{Im } M\,(123,AB) = \sum_{\substack{j,m\\ \lambda'_1,\lambda'_2}} \tfrac{1}{2}\left(\frac{4\pi\omega}{q}\right)^{\frac{1}{2}}(2j+1)\,{}^{\frac{1}{2}}D^{j\,*}_{m\lambda}(\underline{\vartheta}_{12})\;\langle \lambda'_1\lambda'_2\mid M^j(\omega)\mid \lambda_1\lambda_2\rangle^*$$
$$\langle jm\,\lambda'_1\lambda'_2\,3\mid M\mid AB\rangle$$

Because there is a dispersion relation in just the energy^2 invariant and no other, by assumption,

$$\text{Im}\langle jm\;\lambda_1\lambda_2\;3\mid M\mid AB\rangle = \tfrac{1}{2}\sum_{\lambda'_1\lambda'_2}\left\{\langle\lambda'_1\lambda'_2\mid M^j(\omega)\mid\lambda_1\lambda_2\rangle^*\langle jm\;\lambda'_1\lambda'_2\;3\mid M\mid AB\rangle\right\} \qquad (65)$$

Watson's theorem now follows trivially if a proper diagonalisation of the final state amplitude is carried out so $M^j(\omega) = 2e^{i\delta_j(\omega)}\sin\delta_j(\omega)$; for Im M^j having to be real means that the partial amplitude M^j has the phase factor $e^{i\delta_j}$; ie the phase of the amplitude is simply the phase of the final state scattering.

3. Omnes Solution, the N/D method and the equivalence with Formal Scattering Theory.

In writing a dispersion relation for each partial wave M^{jm}, we will lump all the effects from other intermediate states into a real function C^{jm}, which moreover will contain the equal time commutator term that was previously dropped in equation (60). Thus,

$$M^{jm}_{\mu}(s) = C^{jm}_{\mu}(s) + \frac{1}{\pi}\int \frac{\text{Im } M^{jm}_{\mu}(s')\, ds'}{s' - s - i\varepsilon} \tag{66a}$$

$$\text{where } \text{Im } M^{jm}_{\mu}(s) = \tfrac{1}{2}\sum_{\mu}\left\{M^{j\,*}_{\mu\mu'_1}(s)\, M^{jm}_{\mu'}(s)\right\} \tag{66b}$$

The label μ is to denote the base set of two particle helicity states which diagonalises the S matrix elements of the pair. In order not to cause any confusion with the 5 particle amplitude we put a suffix 1 to the final state scattering amplitude.

The essential features of the solution will remain even when we specialise to the S-wave interaction of spinless particles.

$$M(s) = C(s) + \frac{1}{\pi}\int \frac{\text{Im } M(s')\, ds'}{s'-s-i\varepsilon}\,; \quad \text{Im } M(s) = \tfrac{1}{2} M_1^{\,*}(s) M(s) \tag{67}$$

Jacob, Mahoux and Omnes[24] have investigated this problem, with the subsequent intention of applying the solution to $\pi\pi$ interactions in the ABC experiment[50]. Since Im M (s) = $e^{-i\delta(s)}$ sin (s).M(s), the Omnès-Mushkelishvili[7] solution is

$$M(s) = C(s) + \frac{g(s)}{\pi} \int \frac{C(s')\ e^{i\delta(s')} \sin\delta(s')\ ds'}{g(s')\ (s'-s-i\varepsilon)} \tag{68a}$$

$$\text{where } g(s) = \exp\left[\frac{1}{\pi}\int \frac{\delta(s')\ ds'}{s' - s - i\varepsilon}\right] \tag{68b}$$

If the many particle intermediate states cause C(s) to be a slowly varying function, the solution is well represented by

$$M(s) = C(s)\ g(s) \tag{69}$$

which is the solution adopted by Jackson[25].

A few words about the relation to the N/D method:
Chew and Mandelstam[53] write the solution M(s) on the ratio of two functions, the numerator N(s) having only the lefthand cut (none in the case of (67)) and the denominator D(s) having only the righthand cut. Clearly D(s) is just $\left[C(s)\ g(s)\right]^{-1}$.

The equivalence between the dispersive expression (69) and the field theoretical or Watson expression (25b) is obvious once we realise that the phases of each are thesame along the self same cut so the analytic function that describes them must be unique.

We reproduce Jackson and Kane's paper[23] which demonstrates this equivalence explicitly for pion-pion final state interactions. Putting $q^2 = \nu$ in pion mass units we have,

$$\text{dispersively,}\quad M_D(\nu) = M_D(o)\exp\left\{\frac{\nu}{\pi}\int_o^\infty \frac{\delta(\nu')\,d\nu'}{\nu'(\nu'-\nu-i\varepsilon)}\right\} \tag{70a}$$

$$\text{and field-theoretically},\quad M_W(\nu) = \sqrt{\frac{\nu+1}{\nu}}\; e^{i\delta(\nu)}\sin\delta(\nu) \tag{70b}$$

$$\text{Now Im}\left[M_W(\nu)\right]^{-1} = -\frac{\text{Im } M_W(\nu)}{|M_W(\nu)|^2} = -\sqrt{\frac{\nu}{\nu+1}}$$

Since the function $\sqrt{\frac{\nu}{\nu+1}}$ has two cuts, a.L.H. one running from $-\infty$ to -1 with a discontinuity $2\sqrt{\frac{|\nu|}{|\nu|-1}}$, and a R.H. one from o to ∞ with the discontinuity $2\sqrt{\frac{\nu}{\nu+1}}$., therefore the function

$F(\nu) = \frac{1}{M_W(\nu)} + i\sqrt{\frac{\nu}{\nu+1}} = \sqrt{\frac{\nu}{\nu+1}}\cot\delta$, is analytic except for a branch cut from $-\infty$ to -1.

Application of Cauchy's theorem yields the dispersion relation,

$$F(\nu) \quad F(o) = \frac{\nu}{\pi}\int \frac{d\nu'}{\nu'(\nu'+\nu)}\sqrt{\frac{\nu'}{\nu'-1}}$$

$$\therefore \sqrt{\frac{\nu}{\nu+1}}\cot\delta = F(o) + \frac{2}{\pi}\sqrt{\frac{\nu}{\nu+1}}\log\left(\sqrt{\nu}+\sqrt{\nu+1}\right) \tag{71}$$

which is simply the Chew-Mandelstam effective range formula obtained as an approximation to thier N/D method solution by replacing N by a constant i.e. from (70a). This establishes the connexion between (70a) and (70b). (Jackson has also shown the equivalence for the Schrödinger and Dirac

equations).

In the reaction $pd \to He^3 + \pi + \pi$, Abashian Booth and Crowe[50] observed an enhancement of the cross section when the pions were in a total isospin state 0 at low relative momenta. A.B.C.[50] and Truong[51] assumed that this effect was caused by strong S-wave $\pi\pi$ scattering in the final state and by application of the non-relativistic Watson enhancement formula obtained a best fit with a scattering length of about 2 pion wavelengths. The data has since been reanalysed by Jacob, Mahoux and Omnes[24] by a sophisticated calculation (based on the dispersion relation (70a)), who reduce this value to about 1.5, corresponding to a pion-pion coupling constant of $\lambda = -0.2$. What this means essentially, is that an $a \approx 1.5$ in the relativistic formula (70b) can fit the experiment quite well. This value moreover agrees with the work of Hamilton et al[54,55] on $N\pi$ scattering.

4. Dispersion Relations for overlapping interactions.

We have seen that final state interaction of just one pair of particles is defined by the dispersion relation (66). Difficulties immediately arise if we try to extend (66) as it stands to other interactions, because j and m are the angular momenta of a special pair of final particles. It is necessary to retrace our steps and rewrite the dispersion relation as

$$M(s_{12}) = C(s_{12}) + \frac{1}{\pi}\int \frac{\phi_{12}(s')\, ds'}{s' - s_{12} - i\varepsilon} \qquad (72a)$$

$$\phi_{12}(s) = \frac{q}{8\pi\omega}\int \frac{d\Omega'}{4\pi} M^{*}_{12}(s;\underline{\vartheta}', \text{ other } s) \tag{72b}$$

$$\longrightarrow \quad e^{-i\delta_{12}} \sin\delta_{12}\int \frac{d\Omega_{12}}{4\pi} M(s_{ij}, s_{1A}, s_{3A}) \tag{72c}$$

if we retain the s-wave part of the interaction for an analysis near the production threshold.

We follow the model of Peierls and Tarski[27] to postulate the dispersion representation for simultaneous final pair scatterings and later show its formal equivalence with the perturbation series obtained in the previous two chapters. Because it is assumed that the majority of the structure comes f.s.i. , we will neglect the dependence of the total amplitude on the momentum transfers, which means that we are picking out the j_{AB}= 0 partial wave in effect. Thus we assume

$$M(s_{ij}) = C(s_{ij}) + \sum_{\substack{123 \text{ cyclic} \\ \text{perms}}} \frac{1}{\pi}\int \frac{\phi_{12}(s')\,ds'}{s' - s_{12} - i\varepsilon} \tag{73a}$$

$$\text{with } \phi_{12}(s) = e^{-i\delta_{12}} \sin\delta_{12}\int \frac{d\Omega_{12}}{4\pi} M\left[s_{12}, s_{23}(\cos\vartheta_{12}), s_{31}(\cos\vartheta_{12})\right] \tag{73b}$$

The virtue of this Peierls-Tarski model is that it allows ϕ_{ij} to be explicitly written out; referring to the appendix,

$$2s_{13}s_{12} = 2s_{12}(m_1^2+m_3^2)+(s_{12}+m_1^2-m_2^2)(s_{AB}-s_{12}-m_3^2)+C(s_{AB},s_{12},m_3^2)C(s_{12},m_1^2,m_2^2)\cos\vartheta_{12}$$

$$2s_{23}s_{12} = 2s_{12}(m_2^2+m_3^2)+(s_{12}+m_1^2-m_2^2)(s_{AB}-s_{12}-m_3^2)-C(s_{AB},s_{12},m_3^2)C(s_{12},m_1^2,m_2^2)\cos\vartheta_{12}$$

where $C^2(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$.

$$\phi_{12}(s_{12}) = \frac{e^{-i\delta_{12}} \sin\delta_{12}.\ s_{12}}{2C(s_{AB},s_{12},m_3^2)C(s_{12},m_1^2,m_2^2)} \int_{L+}^{L-} d(s_{13}-s_{23})\ M\left[s_{12},s_{23},s_{31}\right] \tag{74a}$$

with limits $L_{\pm}(s_{AB},s_{12}) \equiv (m_1^2-m_2^2) \pm \frac{1}{s_{12}} C(s_{AB},s_{12},m_3^2)\ C(s_{12},m_1^2,m_2^2)$ (74b)

If one does not confine oneself to S-wave final state interaction, then the generalisation of equation (74) is

$$\phi_{12}(s_{12}) = \int_{L_+}^{L_-} \frac{d\ (s'_{13}-s'_{23})}{32\pi\, C(s_{AB},s_{12},m_3^2)}\ M^*_{12}(s_{12},s_{13}-s_{23},s'_{13}-s'_{23})M(s_{12},s_{23},s_{31}) \tag{75}$$

This construction has the advantage of giving a very clear picture when referred to the Dalitz plot (see Appendix)

For if we construct a set of three Cartesian systems centred at the symmetrical point

$s_{12}=s_{23}=s_{31} \equiv s_o = \frac{1}{3}(s_{AB}+m_1^2+m_2^2+m_3^2)$

by defining

$y_k = s_{ij}-s_o$, $2X_k = s_{ki}-s_{ij}$,

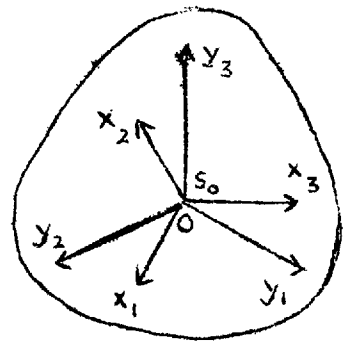


FIGURE 25.

the model can be compactly written as

$$M(x,y) = C(x,y) + \sum_{i=1}^{3} \frac{dx'_i\ dy'_i}{y'_i-y_i-i\varepsilon}\ \frac{M^*_{jk}(y'_i;x_i,x'_i)\ M(y',x',s_{AB})}{16\pi^2 C(s_{AB},s_o+y'_i,m_i^2)}$$

$$\text{or } C(x,y)+ \sum_{i=1}^{3} \frac{1}{\pi} \int \frac{dx_i' dy_i'}{y_i'-y_i-i\varepsilon} \cdot \frac{e^{-i\delta(y_i')} \sin\delta(y_i')}{C\,(s_{AB}, s_o+y_i', m_i^2)}\, M(y_i', x_i', s_{AB}) \quad (76)$$

if s-wave forces predominate. In subtracted form,

$$M(x,y) = M(o,o) + \sum_{i=1}^{3} \frac{y_i}{\pi} \int \frac{e^{-i\delta(y_i')} \sin\delta(y_i')}{C\,(s_{AB}, s_o+y_i', m_i^2)} \cdot \frac{M(y_i', x_i', s_{AB})}{y_i'(y_i'-y_i-i\varepsilon)}\, dy_i' dx_i'$$

summarises the dispersion relation model for s-wave f.s.i.

Peierls and Tarski carried on from equation (73) by assuming that particles 2 and 3 were identical and did not interact together. They were then able to solve the integral equations in the static limit $m_1^2 \to \infty$ and obtained

$$M(s_{ij}) = \tfrac{1}{2}\, g(s_{12}) g(s_{AB}-s_{12}) + \tfrac{1}{2}\, g\,(s_{13})\; g(s_{AB}-s_{13}) \approx g(s_{12}) g(s_{13}) \qquad (77)$$

with g(s) defined by(68b). The above can be regarded as a lowest order perturbation solution in $\frac{1}{m}$, but nevertheless does show that M = (enhancement in one channel)x(reflection of enhancement in the other channel).

5. The Interaction Solution

The dispersion relation (73) which defines the model was first obtained by Khuri and Treiman[33] in connexion with $K^{\pm}$ meson decay into three pions, and we later study that particular problem. There they could more or less justify the use of (73) by examining the axiomatics

of the retarded products which involved the three pionic fields and the weak Lagrangian causing the decay. The detailed axiomatics of the five point function are much too complicated, and one must have recourse to the sort of argument proposed by Jackson.

However, one way in which we can render the model plausible is to show that the iteration solution is formally similar to the perturbation series given in Section 2. In order to do this we consider the unsubtracted representation (73) and keep on substituting the dispersion relation for the total amplitude on the r.h.s. , so

$$M(s_{ij}) \sim C(s_{ij}) \left[1 + \Sigma \int \frac{M_{ij}\, ds'}{s'-s_{ij}-i\varepsilon} + \Sigma\Sigma \int \frac{M_{ij} M_{ij}^{*}\, ds'ds''}{(s'-s_{ij}-i\varepsilon)(s''-s_{ik}-i\varepsilon)} + \ldots \right]$$

The connexion becomes transparent if we collect all terms such as (12) + (12)(12) + (12)(12)(12)+.. and write the sum of these integrals as $M_{12}'(s) = M_{12}(s) + \int \frac{M_{12}(s')\, M_{12}'^{*}(s')\, ds'}{s' - s}$

which is essentially the dispersion relation of the Bethe-Salpeter equation. M_{12}' is the full and M_{12} the 1 particle irreducible amplitude. Then

$$M(s_{ij}) \sim C(s_{ij}) \left[1 + \Sigma \int \frac{M_{ij}'\, ds'}{s'-s_{ij}} + \sum_{\substack{\text{different} \\ \text{scatterings}}} \sum \int \frac{M_{ij}' M_{ik}'^{*}\, ds'\, ds''}{(s'-s_{ij})(s'-s_{ik})} + \ldots \right]$$

or $M \sim M$ (production) $\left[1 + \text{single scattering} + \text{double scattering} + \ldots \right]$ on making the correct identification.

Having established the relation with the field theoretical series, it follows that if we obtain an approximate solution of the dispersion equation by keeping up to the first iteration, we will be doing nothing more nor less than calculating up to first order in final state scattering in the field theoretical context, since the validity of both solutions depends on the assumption that not more than one pair amplitude is large.

6. Final State Interactions and the Dalitz Plot.

We recall the fact that the population of events within the Dalitz plot is proportional to the square of the invariant matrix element. The axes is such a description are a pair of energies or what amounts to the same thing, a pair of final state s-variables, such as illustrated in Figure 25. We must be very cautious before drawing any conclusions as to the nature of f.s.i. when we examine the density of points within the boundary. For let us suppose that in the first instance that $|M|^2$ appears to be constant; now it is easy to imagine a Chew-Low production mechanism, which on its own will give rise to some structure, followed by a strong final state interaction (see Figure 22) which happens to remove that particular energy bias.

Certainly if one has good reasons to suppose that the production does not give rise to asymmetries and that it is effectively a point interaction, then one can make more definite predictions. A pole term of the form $g(s_{12}-\mu^2)^{-1}$ will be apparent when μ is close to the

threshold $(m_1+m_2)^2$ and shows itself in an enhancement towards smaller values of s_{12} - but this is not easy to separate from a rescattering graph (with a scattering length approximation) which gives the same type of enhancement, unless g and μ are accurately known. There are numerous cases however, in which we know that pole terms are absent eg. $\pi\pi$, $\pi\gamma$,... scattering. The rest of the discussion in this section is confined to such problems and (29) can be adequately used at low energies.

Typically we need to consider the term $\left(c + \frac{2iq}{\omega}\right) \frac{e^{i\delta}\sin\delta}{2q/\omega}$

and it is clear from this expression that an analysis of the Dalitz plot must be carried out in terms of the q_{ij} and <u>not</u> s_{ij} variables. (This important point has been put forward by Sawyer and Wali[34] with reference to $\pi\pi$ interactions in τ-decay). Depending on whether the interaction is or is not resonant the term can assume the two forms

$$F = \frac{C + \frac{2iq}{\omega}}{\frac{1}{a} - \frac{2iq}{\omega}} \qquad \text{and} \qquad G = \frac{C + \frac{2iq}{\omega}}{\frac{\omega^2 - \omega_0^2}{\omega_0\Gamma} - \frac{2iq}{\omega}} \tag{78}$$

Now the relative momentum q_{ij} of a given pair ij can be related to the kinetic energy T_k of the remaining particle k, from the kinematics (see the appendix). For a total available energy $= m_1+m_2+m_3+Q$,

$$s_{ij} = (\omega - m_k)^2 - 2\omega T_k$$

$$\simeq (m_1+m_j)^2+2(m_i+m_j)\,Q - 2\,(m_1+m_j+m_k)T_k \quad \text{at low energies}$$

$$\therefore \quad \frac{2q_{ij}}{\omega_{ij}} = \frac{1}{s_{ij}}\left\{(s_{ij}-m_1^2-m_j^2)^2-4m_i^2m_j^2\right\}^{\frac{1}{2}} \frac{(8m_im_j)^{\frac{1}{2}}}{(m_i+m_j)^2}\left[(m_i+m_j)Q-(m_i+m_j+m_k)T\right.$$

$$\therefore \quad \frac{2q_{ij}}{\omega_{ij}} \equiv \rho_k\,(1-t_k)^{\frac{1}{2}} \quad \text{where } \rho_k^2 = \frac{8m_im_j\,Q}{(m_i+m_j)^3} \tag{79a}$$

and t_k is the K.E. of particle k in units of the maximum value of $\frac{(m_i+m_j)Q}{(m_i+m_j+m_k)}$. Similarly, because ω_{ij}^2 is linear in t_k, $\frac{\omega_{ij}^2-\omega_o^2}{\omega_o\Gamma_{ij}} \equiv \frac{t_k - t_k^{(o)}}{\gamma_{k/2}}$ with $t_k^{(o)}$ corresponding to the energy of k when ij resonate, and γ_k being the effective width.

It is interesting to discover the influence on the Dalitz plot of simultaneous s-wave resonances between particles 1 and 2, and 1 and 3. For that purpose, we shall need the variations of Re G and Im G as functions of t_k, γ_k and $t_k^{(o)}$. These are shown below for a resonance which occurs in the physical region. The gross features of the simultaneous interactions will remain even under the approximations $c_{ij} \simeq 1$ and, of course, when the structure of the production mechanism is neglected.

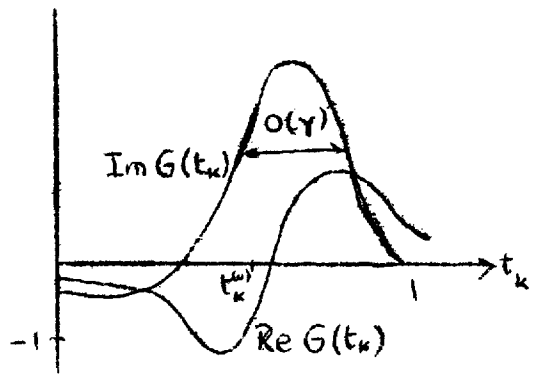


FIGURE 26.

Then (30) reduces to $M \alpha\beta + G(t_2) + G(t_3)$ and it is clear that in a graph of t_2 vs t_3 the points will mainly cluster along the lines $t_2 = t_2^{(o)}$ and $t_3 = t_3^{(o)}$, which fact is extremely well known. However, some finer effects can be perceived if along some part of these cluster lines, Im $G(t_3)$ has opposite sign to Im $G(t_2)$, as may well happen. One then notices a certain variation in density along these lines ie the resonance in one channel interferes with that in the other channel. This is to be compared with the Peierls-Tarski conclusion (77) which also gives a modulating effect of one resonance on the other.

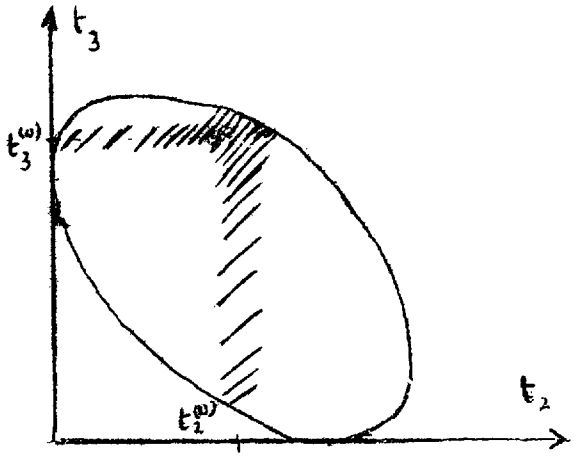


FIGURE 27.

7. Anomalous Thresholds in Production Processes.

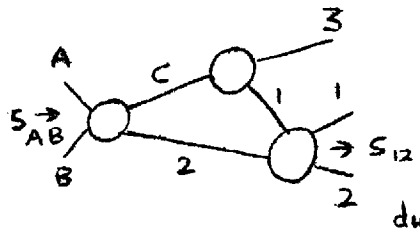
Throughout most of the preceding work it was assumed that the two particle cuts of each final state channel ran from the normal threshold to infinity. We will conclude this chapter with a brief discussion of some diagrams that do not meet this requirement; thus, for certain masses of intermediate states singularities below the two particle cuts can arise in the same way as the anomalous thresholds of the third order vertex function.[12] This fact was pointed out by Sawyer[22] with reference

to π production in π-N collisions as a possible complicating feature in so far as the anomalous threshold contribution leads to a logarithmic singularity at threshold. Recently, it has been shown by Anisovitch et al[29] that this singularity is exactly cancelled by the contribution arising from the normal two particle cut, so it is still perfectly feasible to expand in powers of the final state relative momenta.

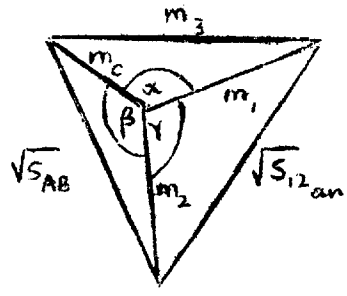
To elaborate on this consider Figure 28. The anomalous threshold cut in s_{12} is obtained in the normal way[12] by placing each of the internal particles on the mass shell. Referring to the dual diagram (which is a plane figure, if the sum of the Feynman auxiliary variables is to vanish at each vertex), the anomalous branch point is at $\cos\gamma = \cos(\alpha+\beta)$ if $\sin(\alpha+\beta) < 0$ (physical sheet)

ie at $\dfrac{m_1^2+m_2^2-s_{12}}{2m_1m_2} =$

$$\frac{m_1^2+m_c^2-m_3^2}{2m_1m_c} \cdot \frac{m_2^2+m_c^2-s_{AB}}{2m_2m_c} - \frac{C(s_{AB},m_2^2,m_c^2)\, C(s_{12},m_1^2,m_2^2)}{4m_1m_2m_c^2}$$



diagram



dual diagram

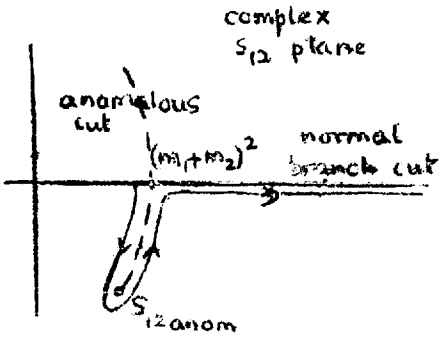
FIGURE 28

So when $s_{AB} > (m_1+m_2)^2+m_3^2 - \frac{(m_1+m_2)}{m_1}(m_1^2+m_3^2-m_c^2)$,

$$s_{12_{anom}} = m_1^2+m_2^2+ \frac{(m_1^2+m_c^2-m_3^2)(s_{AB}-m_2^2-m_c^2)}{2m_c^2} - \frac{i}{2m_c^2}\left[(s_{AB}-m_2^2-m_c^2)^2-4m_2^2m_c^2\right]^{\frac{1}{2}}$$

$$\cdot \left[4m_1^2m_c^2-(m_3^2-m_1^2-m_c^2)\right]^{\frac{1}{2}}$$

which is a complex singularity in the physical region $s_{AB} > (m_2+m_c)^2$.



In a Cauchy integral the contour of integration has to pass round this complex singularity which marks the "anomalous cut" before following the normal branch cut, and the representation will be

$$M = \int_{s_{12}anom}^{(m_1+m_2)^2} \frac{\text{disc } M_{anom}(s')ds'}{s'-s} + \int_{(m_1+m_2)^2}^{\infty} \frac{\text{disc } M_{normal}(s')ds'}{s'-s}$$

Disc $M_{anom}(s)$ is itself the discontinuity of the absorptive part and according to the Landau-Cutkosky rules[9,10] is evaluated by placing every internal line on the mass shell. (Equivalently disc M_{normal} becomes complex when analytically continued past $(m_1+m_2)^2$). It gives rise to a logarithmic singularity which precisely cancels the one arising rrom the normal branch integral[12] owing to disc $M_{normal} \rightarrow$ disc M_{anom} as $s \rightarrow (m_1+m_2)^2$. Because Sawyer assumed incorrectly that disc $M_{normal} \rightarrow 0$ his conclusions as regards the diagram with 1=2=A=π, B=c=3=N, are invalid.

CHAPTER IV

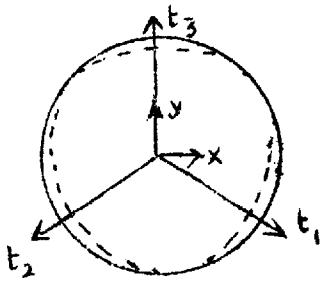
PION-PION INTERACTIONS IN $K^{\pm}$ DECAY

1. Experimental Situation

Until it was realised that parity need not be conserved in weak processes, the decay of the K meson into two pions (θ mode) and into three pions (τ mode) remained a mystery. It is customary now to invoke CP or T invariance as the constraint which governs weak interactions, and this appears to be quite adequate so far in its explanation of experimental facts. A great deal of literature is to be found on the subject of the non-leptonic decay of the K meson; a good summary of the situation up to 1957 is contained in the review article of Gell-Mann and Rosenfeld[31], where it is shown that all the evidence in $\tau^{\pm}$ decay points to a totally symmetric T=1 isotopic state of the 3π[30], which we shall take as the starting point of our analysis.

There are in reality two $\tau^{\pm}$ modes, τ : $K^{\pm} \rightarrow \pi^{\pm} + \pi^{\pm} + \pi^{\mp}$

τ' : $K^{\pm} \rightarrow \pi^{0} + \pi^{0} + \pi^{\pm}$



which we differentiate according to custom by a prime index. Owing to the fact that there are overwhelmingly more τ than τ' events [45] we will evidently be dealing more with the former process in attempting to deduce the $\pi - \pi$ interaction. In the appendix is shown the Fabri-Dalitz

plot of the three pion kinetic energies (t in units of their maximum) and a discussion is given there. Again we follow common practice in attaching the suffix 3 to the unlike pion. The latest experimental evidence[45] shows that $|M_\tau|^2$ is an increasing function of t_3 with a slope of about 50%, and all other structure is simply not present from the statistical point of view.

2. Isotopics and Kinematics.

Barton and Kacser[44] stress that the assumption of the symmetric T=1 final state is much weaker than making use of the $\Delta T = \frac{1}{2}$ rule, which indeed may not always hold; only if we wish to correlate the $K^{\pm}$ to the K_2^o experiments does the $\Delta T = \frac{1}{2}$ rule come into its own. The notation of Khuri and Treiman[33] will be adopted because it is so neat and compact. We affix the isovector components α, β, γ to pions 1,2 and 3 respectively; because we may imagine the decay to be parity conserving with the K-meson as an isovector with $T_3 = \pm 1$, we also give it a label ρ. Then we can expand the matrix element in the form

$$M_{\alpha\beta\gamma}(s_{23}, s_{31}, s_{12}) = A(s_{23}, s_{31}, s_{12})\delta_{\rho\alpha}\delta_{\beta\gamma} + B(s_{23}, s_{31}, s_{12})\delta_{\rho\beta}\delta_{\gamma\alpha} + C(s_{23}, s_{31}, s_{12})\delta_{\rho\gamma}\delta_{\alpha\beta} \qquad (80)$$

Bose-Einstein statistics means that under the typical interchange, $2 \leftrightarrow 3$, $\beta \leftrightarrow \gamma$, M should remain invariant. Therefore we have the symmetries

$$A(s_{23}, s_{31}, s_{12}) = B(s_{31}, s_{23}, s_{12}) = C(s_{12}, s_{31}, s_{23}) \text{ etc...} \qquad (81)$$

We may project out the τ and τ' amplitudes by giving the appropriate values to α, β, γ . For instance,

$$M_{\tau'} = \langle \pi^o \pi^o \pi^+ | M | K^+ \rangle = \tfrac{1}{2} \langle \pi_3, \pi_3, \pi_1 + i\pi_2 | M | K_1 + iK_2 \rangle$$

$$= \tfrac{1}{2} \left[M_{1331} - iM_{1332} + M_{2331} + M_{2332} \right] = C$$

Hence we have

$$M_{\tau}(s_{23}, s_{31}, s_{12}) = A(s_{23}, s_{31}, s_{12}) + B(s_{23}, s_{31}, s_{12}) \tag{82a}$$

$$M_{\tau'}(s_{23}, s_{31}, s_{12}) = C(s_{23}, s_{31}, s_{12}) \tag{82b}$$

which means that a knowledge of $M_{\tau'}$ is all that is needed.

$$M_{\tau}(s_{23}, s_{31}, s_{12}) = M_{\tau'}(s_{12}, s_{31}, s_{23}) + M_{\tau'}(s_{23}, s_{12}, s_{31}) \tag{82c}$$

In the above we have used the three relativistic invariants to describe the kinematics, and these are linearly related to the kinetic energies according to $s_{ij} = (m-\mu)^2 - 2mT_k$. m is the mass of the $K^{\pm}$ and we give all pions the same μ mass since we are attempting to explain just the gross features. The relative momentum of a particular pair of pions is related to the kinetic energy of the other pion through

$$4q_{ij}^2 = s_{ij} - 4\mu^2 = (m-\mu)^2 - 4\mu^2 - 2mT_k \doteq 2m(\bar{T} - T_k)$$

$$\doteq 2m\bar{T}(1-t_\mu) \tag{83}$$

$$\bar{T} = \frac{(m-\mu)^2 - 4\mu^2}{2m} \approx 50 \text{ MeV} \quad \text{is the maximum energy.}$$

As it happens, the Q value of the reaction ≈ 75 MeV, so in units of the maximum, $t_1+t_2+t_3 \approx \frac{3}{2}$.

The factor $2q_{ij}/\omega_{ij}$ recurs in the work and because q_{ij} is never large, we may put $\omega_{ij} \approx 2\mu$ so

$$4q_{ij}^2/\omega_{ij}^2 \approx m\bar{T}\,(1-t_k)/2\mu^2 \equiv \rho^2(1-t_k) \qquad (84)$$

with $\rho^2 \approx 0.64$.

3. Calculation of the Matrix Elements.

The simplest way that we can set about the problem is to attribute all structure in the spectrum to interactions among the pions in the final state. That is to say, the production gives no preference in energy to any pion ie we are visualising the effective interaction Lagrangian

$$\mathcal{L} = g\,(\underline{\phi}_k \cdot \underline{\phi}_\pi)\,(\underline{\phi}_\pi \cdot \underline{\phi}_\pi) \quad \text{where } g \text{ is a weak coupling constant.}$$

$$M_{\alpha\rho\alpha\beta\gamma} = g\,(\delta_{\rho\alpha}\delta_{\beta\gamma} + \delta_{\rho\beta}\delta_{\gamma\alpha} + \delta_{\rho\gamma}\delta_{\alpha\beta}) \qquad (85a)$$

Of course, the pion-pion scattering is given by (53)

$$M_{1\alpha'\beta',\alpha\beta} = \tfrac{1}{3}(M^{(0)}-M^{(2)})\,\delta_{\alpha'\beta'}\delta_{\alpha\beta} + \tfrac{1}{2}(M^{(1)}+M^{(2)})\,\delta_{\alpha'\alpha}\delta_{\beta'\beta} + \tfrac{1}{2}(M^{(2)}-M^{(1)})\,\delta_{\alpha'\beta}\delta_{\beta'\alpha} \qquad (85b)$$

where $M^{(0)}$ and $M^{(2)}$ are even in $\cos\vartheta$ while $M^{(1)}$ is odd in $\cos\vartheta$ - this follows from crossing symmetry. $M^{(T)}$ is for an isospin state T.

The evaluation of final state scattering was accomplished in Section 2 and 3. The details of isospin are now important and typically we must consider the final state scattering between 1 and 2 with absorptive part,

$$\overline{\mathcal{M}}_{\rho\alpha\beta\gamma}(12) = \sum_{\alpha'\beta'} \int \frac{d\Omega'_{12}}{4\pi} M_{\rho\alpha'\beta'\gamma}(\omega_{12}\vartheta'_{12})\, M_{1\alpha'\beta'\alpha\beta}(\omega_{12},\vartheta'_{12})$$

$$= g\int \frac{d\Omega'_{12}}{4\pi}\Bigg[\tfrac{1}{3}(M^{(o)}-M^{(2)})\delta_{\alpha\beta} + \tfrac{1}{3}(M^{(o)}-M^{(2)})\delta_{\rho\gamma}\delta_{\alpha\beta} +$$

$$(M^{(o)}-M^{(2)})\,\delta_{\alpha\beta}\,\delta_{\rho\gamma}$$

$$+ \tfrac{1}{2}(M^{(2)}+M^{(1)})(\delta_{\rho\alpha}\delta_{\beta} + \delta_{\rho\beta}\delta_{\gamma\alpha} + \delta_{\rho\gamma}\delta_{\alpha\beta})$$

$$+ \tfrac{1}{2}(M^{(2)}-M^{(1)})(\delta_{\rho\beta}\delta_{\gamma\alpha} + \delta_{\rho\alpha}\delta_{\beta\gamma} + \delta_{\rho\gamma}\delta_{\alpha\beta})\Bigg]$$

$$= g\int \frac{d\Omega'_{12}}{4\pi}\Big[\Big\{\tfrac{5}{3}M^{(o)}(\omega_{12},\vartheta'_{12}) - \tfrac{2}{3}M^{(2)}(\omega_{12},\vartheta'_{12})\Big\}\delta_{\alpha\beta}\delta_{\rho\gamma} + M^{(2)}(\omega_{12}\vartheta'_{12})\{\delta_{\rho\alpha}\delta_{\beta\gamma}+\delta_{\beta\rho}\delta_{\gamma\alpha}$$

which means that only the s-wave scattering contributes. This is not in the least bit surprising and might have been deduced immediately because the production amplitude is completely symmetric between the pions . On making the usual partial wave decomposition,

$$\overline{\mathcal{M}}_{\rho\alpha\beta\gamma}(12) = g.\frac{8\pi\omega_{12}}{q_{12}}\Big[\Big\{\tfrac{5}{3}e^{i\delta^{(o)}_{12}}\sin\delta^{(o)}_{12} - \tfrac{2}{3}e^{i\delta^{(o)}_{12(o)}}\sin\delta^{(o)}_{12}\Big\}\delta_{\alpha\beta}\delta_{\rho\gamma} + e^{i\delta^{(2)}_{12}}\sin\delta^{(2)}_{12}\cdot$$

$$\{\delta_{\alpha\rho}\delta_{\beta\gamma} + \delta_{\rho\beta}\delta_{\gamma\alpha}\}\Big] \qquad (86)$$

$$\therefore\ M_{\rho\alpha\beta\gamma}(12) = g\left[\left\{\frac{5}{3}F_{12}^{(o)} - \frac{2}{3}F_{12}^{(2)}\right\}\delta_{\alpha\beta}\delta_{\rho\gamma} + F_{12}^{(2)}\left\{\delta_{\rho\alpha}\delta_{\beta\gamma} + \delta_{\rho\beta}\delta_{\gamma\alpha}\right\}\right] \qquad (87a)$$

where
$$F_{ij}^{(Y)} \quad \frac{e^{i\delta^{(T)}(\omega_{ij})}\sin\delta^{(T)}(\omega_{ij})}{2q_{ij}/\omega_{ij}} \left\{\frac{2iq_{ij}}{\omega_{ij}} + c\right\} \qquad (87b)$$

$$= \frac{a_T\left(c + \frac{2iq_{ij}}{\omega_{ij}}\right)}{1 - ia_T\frac{2q_{ij}}{\omega_{ij}}} \qquad (87c)$$

if a length parametrisation is a good way of describing $\pi\pi$ scattering.

Adding all the diagrams up to first order in final state scatterings,

$$A = g\left[1 + \frac{5}{3}F_{23}^{(o)} - \frac{2}{3}F_{23}^{(2)} + F_{31}^{(2)} + F_{12}^{(2)}\right]$$

$$B = g\left[1 + F_{23}^{(2)} + \frac{5}{3}F_{31}^{(o)} - \frac{2}{3}F_{31}^{(2)} + F_{12}^{(2)}\right]$$

$$C = g\left[1 + F_{23}^{(2)} + F_{31}^{(2)} + \frac{5}{3}F_{12}^{(o)} - \frac{2}{3}F_{12}^{(2)}\right]$$

From equations (82),

$$M = 2g\left[1 + F_{12}^{(2)} + \frac{1}{6}(5F_{31}^{(o)} + F_{31}^{(2)}) + \frac{1}{6}(5F_{23}^{(o)} + F_{23}^{(2)})\right] \qquad (88a)$$

$$M = g\left[1 + F_{23}^{(2)} + F_{31}^{(2)} + \frac{5}{3}F_{12}^{(o)} - \frac{2}{3}F_{12}^{(2)}\right] \qquad (88b)$$

We should add that c is in reality a function of $2q/\omega$, and taking

the view that it is constant at low energies goes hand in hand with the length parametrisation or any other that is adopted (eg a resonance formula as used by Mitra[36]).

If we use (83) then,

$$F^{(T)}(\omega_{ij}) \simeq \frac{a_T(c + i\rho(1-t_k)^{\frac{1}{2}})}{1 - ia_T\rho(1-t_k)^{\frac{1}{2}}} = \frac{1 + a_T c}{1-ia_T\rho(1-t_k)^{\frac{1}{2}}} - 1 \qquad (87d)$$

The deviation from a constant being small, we shall expand about the symmetrical point $t_1=t_2=t_3 \approx \frac{1}{2}$.

$$|M(t)|^2 = |1+F(t)|^2 = |1+F(\tfrac{1}{2})|^2 \cdot \left| 1+ \frac{F(t)-F(\tfrac{1}{2})}{1 + F(\tfrac{1}{2})} \right|^2 \qquad (89)$$

$$\text{But Re}\left\{F(t) - F(\tfrac{1}{2})\right\} = \frac{1+ ac}{1+a^2\rho^2(1-t)} - \frac{1+ ac}{1+\frac{1}{2}a^2\rho^2} \quad \frac{a^2\rho^2(t-\frac{1}{2})(1+ac)}{(1+\frac{1}{2}\rho^2a^2)^2}$$

$$\text{and} \quad \text{Im}\left\{F(t) - F(\tfrac{1}{2})\right\} = \frac{a\rho(1-t)^{\frac{1}{2}}(1+ac)}{1+a^2\rho^2(1-t)} - \frac{\frac{1}{\sqrt{2}}a\rho(1+ac)}{1+\frac{1}{2}\rho^2a^2} \approx \frac{(1+ac)\frac{1}{\sqrt{2}}a\rho(\frac{1}{2}-t)}{(1+\frac{1}{2}\rho^2a^2)}$$

because near $t=\frac{1}{2}$ (and only near there), $(1-t)^{\frac{1}{2}} \simeq \frac{1}{\sqrt{2}}(\frac{3}{2} - t)$.

In the denominator of (89) we can replace $F(\frac{1}{2})$ by its real part without undue error because it constitutes the major part. Hence

$$\left|\frac{M(t)}{M(\tfrac{1}{2})}\right|^2 = \left\{ 1+ \frac{a^2\rho^2(t-\frac{1}{2})}{(1+\frac{1}{2}\rho^2a^2)^2} \quad \frac{1+ac}{1+ac} \right\}^2 + \frac{\frac{1}{2}a^2\rho^2(\frac{1}{2}-t)^2}{1+\frac{1}{2}\rho^2a^2}$$

$$\equiv 1 + \alpha(t-\tfrac{1}{2}) \qquad (90a)$$

where $\alpha = \frac{2a^2\rho^2}{(1+\frac{1}{2}\rho^2 a^2)^2} \cdot \frac{1+ac}{1+\Sigma ac}$ (90b)

Therefore, in detail, and normalising to 1 at the symmetric point for τ' decay, the Taylor expansion gives

$$|M_\tau|^2 = 4\left[1 + a_2(t_3-\tfrac{1}{2}) + (\tfrac{5}{6}a_o + \tfrac{1}{6}a_2)(t_2-\tfrac{1}{2}) + (\tfrac{5}{6}a_o - \tfrac{1}{6}a_2)(t_1-\tfrac{1}{2})\right]$$

$$= 4\left[1 + \frac{5}{12}(a_2-a_o)(2t_3-1)\right] \quad (91a)$$

$$|M_{\tau'}|^2 = 1 + a_2(t_1-\tfrac{1}{2}) + a_2(t_2-\tfrac{1}{2}) + (\tfrac{5}{3}a_o - \tfrac{2}{3}a_2)(t_3-\tfrac{1}{2})$$

$$= 1 + \frac{5}{6}(a_o-a_2)(2t_3-1) \quad (91b)$$

with $(1+\frac{4}{3}ca_2 + \frac{5}{3}ca_o)\,a_T = \frac{2(1+a_T c)\,a_T^2\rho^2}{(1+\frac{1}{2}a_T^2\rho^2)^2}$ (91c)

The branching ratio $\tau \to 3\pi / \tau' \to 3\pi$ is 4 and the ratio of the slopes is 1 : -2. Both these results are consistent with the $T = \frac{1}{2}$ rule,[35] but do not imply it. The α factors which appear in (91) look something like the Khuri-Treiman factors at first glance, but on closer inspection they differ appreciably and most especially in the power of the scattering length

4. Comparison with Experiment.

The latest experiments tell us that the difference a_2-a_o is approximately 1.2, and this statement can be converted into one which

involves the scattering length as soon as c is given. Now a nucleon mass cut-off gives $c \approx 1.25$ so allowing it a little latitude, we shall let it vary between 1 and 1.5. The solutions to

$$\frac{(1+a_2 c)a_2^2}{(1+\frac{1}{2}a_2^2 \rho^2)^2} - \frac{(1+a_o c)a_o^2}{(1+\frac{1}{2}a_o^2 \rho^2)^2} = \frac{3}{5\rho^2 c}\left[1 + \frac{4}{3}ca_2 + \frac{5}{3}ca_o\right]$$

will be plotted in the range $|a_T| \lesssim 1$ Referring to Figure 29 where they are graphed, we notice that when $a_2 \lesssim 0$ there is a strong destructive effect in the first order scattering and, in that region, the second order effects should make themselves felt. Therefore the solutions in the region $a_2 \lesssim 0$ are not consistent with the approximations. With $a_2 > 0$ we can crudely write $a_2 - a_o = 0(1)$ which compares very favourably with the first order iteration solution of the dispersion equations: $a_2 - a_o \approx 0.9$, obtained by Khuri and Treiman; on the whole, it inspires us with confidence in our treatment of final state interactions. Another pleasing aspect is the way in which Thomas and Holladay's solution[32] drops out. It is marked T.H. in Figure 29; at thatpoint $a_2 \approx 1$ and $a_o \approx -0.2$, so $\pi^+\pi^+$ scatter with a = 1 while

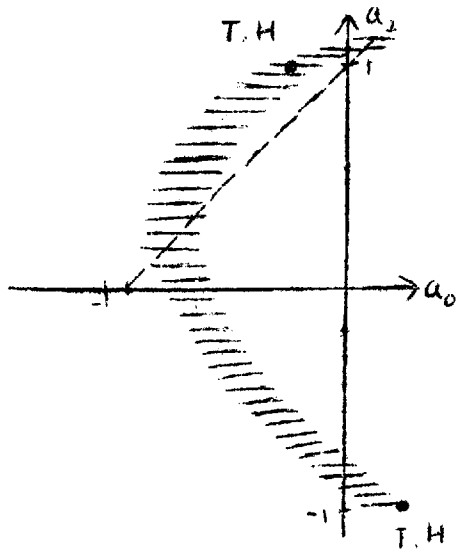


FIGURE 29.

$\pi^+\pi^-$ scatter with $a \approx 0$, which is precisely the assumption they make.

$a_2 - a_0 \approx 1$ is really rather obvious, as the enhancement takes place when the $\pi^+\pi^+$ 'stick together' (32 - 38); this can happen if π^+ attracts π^+ much more so than it does π^- or else if π^+ does not interact with π^+ but repels π^-. The trouble with this conclusion is that it is in disagreement with practically every source about $\pi\pi$ forces and especially the ABC experiment where a value $a_0 \approx 1.5$ is suggested — therefore the T=2 interaction would have to be even more attractive and this is extremely unlikely, to say the least! This contradiction has been well summarised in a recent paper by Smith, Prowse and Stork.(43)

5. Models for Meson Decay

We are obliged to conclude that the original model on which the whole calculation was based, is fundamentally wrong. Barton and Kacser(44) who incidentally give an excellent account of the dispersion theory, suggest that the production Lagrangian (which comprises nucleon-antinucleon loops,) contains derivative coupling :

$$\mathcal{L} = g_1(\underline{\phi}_K \cdot \underline{\phi}_\pi)\,(\underline{\phi}_\pi \cdot \underline{\phi}_\pi) + g_2(\underline{\phi}_K \cdot \overleftrightarrow{\partial_\mu} \underline{\phi}_\pi)\,(\underline{\phi}_\pi \cdot \overleftrightarrow{\partial_\mu}\, \underline{\phi}_\pi)$$

They go on to show that, g_2/g_1 being an unknown parameter, the first bit of information about a_T comes from the quadratic dependence of the

matrix element on t. Even with 6000 events it is impossible to say anything in this respect!



FIGURE 30.

On the other hand, Riazzuddin & Fayyazuddin[39] and Beg and DeCelles[41] consider specific P-wave models for the decay (see Figure 30). Beg and De Celles prove that the ρ contribution predominates over the K^*, and then break up the model further in the manner of Figure 31 to calculate the magnitude of the linear coefficient with remarkable agreement. This type of model is very much in fashion now and has been applied by several people[46-52] to the problem of K_o and η decays.

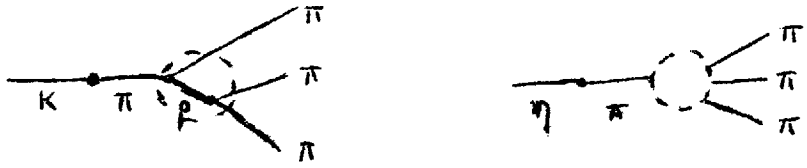


FIGURE 31.

PART II

CALCULATION OF THE NATURAL CONSTANTS IN PSEUDO-SCALAR MESON THEORY.

ABSTRACT.

By expressing Feynman diagrams as dispersion integrals with a finite cut-off we calculate to lowest order the renormalisation constants Z_1, Z_2, Z_3 and Z_4 of pseudoscalar meson theory, that is, with the interaction Lagrangian,

$$: g \bar{\psi} \gamma_5 \psi \phi \; : \; + \; : h \phi^4 :$$

By setting each of these Z equal to zero we obtain the approximate solutions $g^2/4\pi = 0\ (50)$, $h/16\pi = 0(0.5)$, $\mu/m \approx 0.5$ for uncharged mesons, and $g^2/4\pi = 0\ (20)$, $h/16\pi = 0(0.05)$, $\mu/m \lesssim 0.5$ for charged mesons, (when $\mu/m \approx 0.15$, $g^2/4\pi \approx 13$, $h/16\pi \approx 0.04$) with a cut-off of about $2\frac{1}{2}$ nucleon masses in each case.

INTRODUCTION

One of the outstanding problems of modern times has been the question of what distinguishes an elementary from a composite particle. At last it appears to be resolving itself, and the answer seems to be that since every particle is as elementary as every other in practice, it is pointless to contemplate the question. This notion was put forward by Feynman as an "impotence principle", and finds its expression in two contexts : thus in S-matrix theory the forces which act between sets of particles can give rise to poles in the amplitudes whose positions are identified as the masses of the ensuing bound states [13,19]; in field theory it has been suggested by Salam[10], Weinberg[13] and others[56,57,59,61] that the same indistinguishability criterion is realised by the vanishing of the wave function and vertex function renormalisation constants, making the Lagrangian of the theory "null" in some sense. More recently Rockmore [64] has demonstrated the equivalence of the two approaches in the lowest order of a perturbation expansion.

Therefore, the setting of each Z equal to zero will provide equations for the coupling constants and masses of particles and presumably if all particles and their associated quantum numbers are introduced, we should only obtain the actual physical solutions. As an interesting exercise we may suppose that all the particles in nature are fermions and pseudoscalar bosons which interact via the

conventional Yukawa coupling $g\bar{\psi}\gamma_5\psi\phi$: and we obtain the set of three equations $Z_{1,2,3}\ (g,\frac{\mu}{m}) = 0$. It is well-known, of course, that all these renormalisation constants diverge in perturbation theory, but if we include a finite cut-off Λ, we may well be able to solve[62]

$$Z_{1,2,3,}\ (g\ ,\frac{\mu}{m}\ ,\ \frac{\Lambda}{m}\ ) = 0$$

(In order not to make the calculation too arbitrary, we choose the same Λ for each constant).

Naturally, every Z is an infinite series in powers of g^2 and the simplest thing we can do is to terminate it at g^2, although we have no justification for doing this other than the curious fact that this procedure almost always leads to sensible answers - this is sometimes explained on the basis of the nearest singularities being the most important. At any rate the fact that reasonable answers do ensue justifies the method a posteriori. Next there is the problem of how one must introduce the cut-off in a more or less unique fashion. Thus we cannot proceed in the standard manner by separating out the divergent from the finite part because such techniques rely entirely on Λ being so large that various theorems on shifting the origin of integration[5] can be used. Rather, we express the Feynman diagrams through a dispersion representation by the Landau-Cutkosky prescription[9,10] and in this way, Λ appears only in the final integration over masses - the dispersion integrals will be nothing more nor less than the Lehmann spectral

representation[4] evaluated to lowest order in g^2. Even with the question of Λ settled, there remains the difficulty of whether to calculate Z_1 through the representation in the m.m. of the fermion or boson. We have selected the latter with both fermions on the mass shell because it is the more symmetrical, and estimate the correction to Z_1 if the asymmetric representation were adopted.

Next we consider the effect of the meson-meson scattering terms $h\varphi^4$ which are requisite for a proper renormalisation. Regarding these contributions as small perturbations (which we know them to be, in practice because h is small) we estimate them up to first order in h. In addition to $Z_{1,2,3}$, we have another renormalisation constant, Z_4, at our disposal. If we make it vanish too (as this again makes the Lagrangian null) we will be provided with an extra equation, and all in all,

$$Z_{1,2,3,4}\ \left(g,\ h, \frac{\mu}{m}, \frac{\Lambda}{m}\right) = 0.$$

In attempting to obtain the solutions of the above equations, we observe that Z_2 and Z_3 are independent of h (at least in so far as first order effects are concerned). Hence we have computed g^2 and Λ/m numerically in terms of μ/m for the range $0.001 \lesssim \mu/m \lesssim 1$. These results are inserted into $Z_4=0$ to obtain h as a function of μ/m. From this we calculate Z_1 (actually the rough range of values it may cover) in terms of μ/m, and where it is zero we have a solution. We find that μ/m must exceed about 0.3 for this to happen and $g^2/4\pi$ increases very rapidly from 40 to 100 ($\mu/m \simeq 0.6$) to infinity. Therefore we estimate the solution to

be

$$g^2/4\pi \quad = 0(50), \quad \mu/m \simeq 0.5, \quad h/16 \quad = 0(0.5), \Lambda/m \simeq 2.3$$

although the situation is not very clear cut.

Finally we introduce the ordinary charge properties of pions and nucleons viz. we specifically assume that the boson is an isovector and the fermion an isospinor which are coupled through

$$= \frac{g}{\sqrt{2}} \; : \bar{\Psi} \, \underline{\tau} . \underline{\Phi} \, \Psi \quad + \quad h : \left(\underline{\Phi} \cdot \underline{\Phi} \right)^2 :$$

The only modifications that arise are various numerical factors multiplying g and h. Proceeding in the same way as in the neutral case, we have here a much more clear cut situation and the solutions become

$$g^2/4\pi = 0(20), \quad h/16\pi = 0(0.05), \quad \mu/m \lesssim 0.5 \text{ and } \Lambda/m = 0(3).$$

In fact when $\mu/m \simeq 0.15$, $g^2/4\pi \simeq 13$!

Thus if we believe that a theory in which every particle is equally elementary is characterised by the vanishing of its renormalisation constants, it follows that a charged version of pseudoscalar meson theory is favoured (since a solution is decidedly easier to come by) and the comparison with the currently accepted values $g^2/4\pi \simeq 15$ and λ (Chew-Mandelstam constant[52]) $= -h/16\pi \simeq -0.2$ is embarassing! In fact one must criticize the results as being too good in view of the nature of the assumptions and approximations that have been applied to obtain them, especially as a similar calculation for quantum electrodynamics leads to $\frac{e^2}{4\pi} \simeq 0.8$.

CHAPTER I.

DISPERSION REPRESENTATION OF THE RENORMALISATION CONSTANTS.

1. Definition of the Lagrangian.

The field theory of neutral pseudoscalar mesons interacting with fermions is specified by the Lagrangian

$$\mathcal{L}(p) = :\psi_o(\not{p}-m_o)\psi_o: + \tfrac{1}{2}:\phi_o(p^2-\mu_o^2)\phi_o: + :g_o\bar{\psi}_o\gamma_5\psi_o\phi_o: + :h_o\phi_o^4: \qquad (1a)$$

The o suffices correspond to unrenormalised quantities; the renormalised quantities (which have no suffices) are then defined by the substitutions,

$$\psi_o = Z_2^{\frac{1}{2}}\psi\,, \quad \phi_o = Z_3^{\frac{1}{2}}\phi\,, \quad m_o = m-\delta m\,, \quad \mu_o^2=\mu^2-\delta\mu^2$$

$$g_o = Z_1Z_2^{-1}Z_3^{-1/2}g, \qquad h_o = Z_4Z_3^{-2}h.$$

$$\therefore \mathcal{L}(p) = :\bar{\psi}(\not{p}-m)\psi: + \tfrac{1}{2}\phi(p^2-\mu^2)\phi :$$

$$+ (Z_2-1)\,\bar{\psi}(\not{p}-m)\psi - Z_2\delta m + \tfrac{1}{2}(Z_3-1)\phi(p^2-\mu^2)\phi - \tfrac{1}{2}Z_3\delta\mu^2\phi\phi$$

$$+ Z_1g\bar{\psi}\gamma_5\psi\phi + Z_4h\phi^4. \qquad (1b)$$

In the method of renormalisation as given by Matthews and Salam[3] one works with the renormalised quantities from the very beginning. Thus from (1b) we have the vertex factors Z_1g, Z_4h and the propagators

$$\Delta(k) = \frac{1}{Z_3(k^2-\mu^2+\mu^2)}\,, \quad S(p) = \frac{1}{Z_2(\not{p}-m+\delta m)}$$

The constants Z_2, δm and $\delta\mu^2$ are chosen in conjunction with the four primitively divergent diagrams (meson and fermion propagators, vertex fn.

meson-meson scattering) in such a way that all infinities are exactly removed. Thus with all radiative corrections included, the complete functions are

$$\Delta'(k) = \frac{1}{Z_3(k^2-\mu^2+\mu^2) + \Pi^*(k^2)}$$

$$S'(p) = \frac{1}{Z_2(\not{p}-m+m)+\Sigma^*(p)}$$

$$g\,\Gamma(p,p') = Z_1 g + g\,\Lambda(p,p')$$

$$hM(k_1k_2k_3k_4) = Z_4 h + hN(k_1k_2k_3k_4) \qquad (2)$$

where, Π^*, Σ^*, Λ and N signify the respective corrections to the meson 2-pt function, (quadratically divergent), fermion 2-pt function (linearly div.) 3-pt function(logarithmically div.), meson 4-pt function (log. div). Making two substractions in Π^* and Σ^* (at the particle mass), one subtraction in Λ (all particles on mass shell) and one subtraction in N (at the symmetrical point with all mesons on the mass shell), and choosing the subtraction constants as follows, we have the Dyson equations(

$$\Sigma^*(p) \equiv -\delta m Z_2 + (1-Z_2)(\not{p}-m)+(\not{p}-m)^2\Pi_f^*(p)$$

$$\Pi^*(k) \equiv -\delta\mu^2 Z_3 + (1-Z_3)(k^2-\mu^2) + (k^2-\mu^2)^2\Pi_f^*(k)$$

$$\Lambda(p,p';k) \equiv (1-Z_1)+(k^2-\mu^2)\Lambda_{fk}(p,p';k)+(p^2-m^2)\Lambda_{fp}(p,p';k)+(p'^2-m^2)\Lambda_{fp'}(p,p';k) \qquad (3a)$$

$$N(k_r) \equiv (1-Z_4) + \sum_{i=1}^{4}(k_i^2-\mu^2)N_{fk}(k_r) + \sum_{12,33,31}(s_{ij}-\tfrac{4}{3}\mu^2)\,N_{fs_{ij}}(k_r)$$

$$\therefore \quad Z_2\delta m = -\Sigma^*(m), \qquad Z_2 = 1 - \frac{\partial\Sigma^*}{\partial \not{p}}\Big|_{\not{p}=m}$$

$$Z_3\,\delta\mu^2 = -\Pi^*(\mu^2), \quad Z_3 = 1 - \frac{\partial\Pi^*}{\partial k^2}\Big|_{k^2=\mu^2}$$

$$Z_1 = 1 - \Lambda(m,m,\mu), \quad Z_4 = 1 - N\left(\tfrac{4}{3}\mu^2, \tfrac{4}{3}\mu^2, \tfrac{4}{3}\mu^2\right) \qquad (35)$$

so, by choice,

$$S'(p) = \left[(\not{p}-m)\left\{1 + (\not{p}-m)\,\Pi_f^*(p)\right\}\right]^{-1}$$

$$\Delta'(k) = \left[(k^2-\mu^2)\left\{1 + (k^2-\mu^2)\Pi_f^*(k)\right\}\right]^{-1}$$

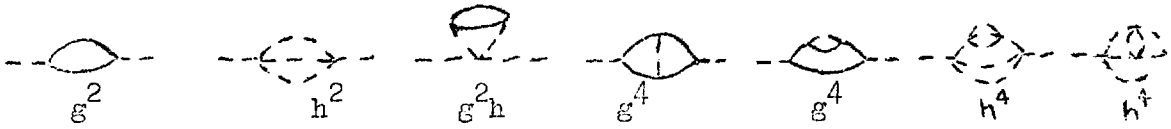
$$\Gamma(p,p',k) = 1 + (k^2-\mu^2)\Lambda_{fk}(p,p',k) + (p^2-m^2)\Lambda_{fp}(p,p'k) + (p'^2-m^2)\Lambda_{fp'}(p,p'$$

$$N(k_r) = 1 + \sum_i (k_i^2-\mu^2)\,N_{fk_i}(k_r) + \sum_{12,23,31}(s_{ij}-\tfrac{4}{3}\mu^2)N_{fs_{ij}}(k_r) \qquad (3c)$$

If there are no corrections, then of course Σ^*, Π^*, Λ and N all vanish so $Z_i = 1$, $m^2 = \mu^2 = o$ and the perturbation calculation starts from here by expansion in powers of g^2 and h.

2. Boson Self Energy

Up to order f^4 where f stands for g or h we would need to consider the graphs in the figure below



Up to first order in g^2 and subsequently to first order in h, the first and

third diagrams remain of which the third is just a constant which cannot affect Z_3. Thus we need only compute the single term in Figure 1 which is

$$\Pi^*(k^2) = \frac{-ig^2}{(2\pi)^4}\int d^4p \ \frac{T_r\left[\gamma_5(\not{p}-m)\gamma_5(\not{p}-\not{k}-m)\right]}{\left[p^2-m^2\right]\left[(p-k)^2-m^2\right]}$$

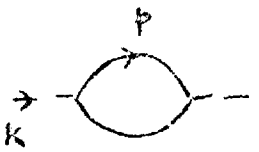


FIGURE 1.

$$= \frac{-ig^2}{(2\pi)^4}\int d^4p \ \frac{4\,(p^2-p.k-m^2)}{\left[p^2-m^2\right]\left[(p-k)^2-m^2\right]} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The singularity in Π^* arises from the denominator and is obtained by a pinch of the contour of integration[12]. ie the branch point occurs at $k^2=4m^2$ when $p^2=m^2$ and $(p-k)^2=m^2$. Across the normal branch cut the discontinuity is obtained by the replacements $(p^2-m^2)^{-1} \rightarrow -2\pi i\,\delta_+(p^2-m^2)$, $(k-p)^2-m^2 \rightarrow -2\pi i\,\delta_+(\overline{k-p}^2-m^2)$ according to the Cutkosky rule[10].

$$\therefore \frac{1}{\pi}\,\mathrm{Im}\Pi^*(k^2) = \frac{g^2}{(2\pi)^3}\int 4p.k\,\delta(p^2-m^2)\,\vartheta(p)\,\delta(\overline{k-p}^2-m^2)\,\vartheta\,(k-p)\,d^4p$$

$$= \frac{g^2}{8\pi^2}\,\vartheta\,(k^2-4m^2)\left\{k^4-4m^2k^2\right\}^{\frac{1}{2}} \qquad (5)$$

The evaluation of such 4-dimensional integrals is shown in Appendix 2. Writing down the usual dispersion relation with a cut-off Λ^2,

$$\Pi^*(k^2) = \frac{1}{\pi}\int_{4m^2}^{\Lambda^2} \frac{\mathrm{Im}\ \Pi^*(K^2)\,dK^2}{K^2-k^2-i\varepsilon} \qquad \text{and using (3b) and (5),}$$

$$Z_3\,\delta\mu^2 = -\frac{g^2}{8\pi^2}\int_{4m^2}^{\Lambda^2}\frac{K(K^2-4m^2)^{\frac{1}{2}}}{K^2-\mu^2}\,dK^2 \qquad (6)$$

$$Z_3 = 1 - \frac{g^2}{8\pi^2}\int_{4m^2}^{\Lambda^2}\frac{K(K^2-4m^2)^{\frac{1}{2}}}{(K^2-\mu^2)^2}\,dK^2 \qquad (7)$$

Equation (7) will stay unchanged even when the g^2h diagram is added, although (6) is altered, of course.

3. Fermion self energy.

Up to order f^4 there are the graphs



and we can discard all but the first diagram for our purpose.

$$\Sigma^*(p) = \frac{-ig^2}{(2\pi)^4}\int d^4k\,\frac{\gamma_5(\not{k}+m)\gamma_5}{\left[k^2-m^2\right]\left[(p-k)^2-\mu^2\right]}$$

FIGURE 2.



$$\equiv \quad mA(p^2) + \not{p}\,B(p^2) \quad \ldots\ldots\ldots (8)$$

From equation (3b)

$$\delta_m Z_2 = -m\left[A(m^2)+B(m^2)\right] \quad \ldots\ldots\ldots(9a)$$

$$Z_2 = 1 - 2m^2\left[A'(m^2)+B'(m^2)\right] - B(m^2)\ldots(9b)$$

and we can therefore determine δm and Z_2 from the integral representatio

of A and B :

$$A(p^2) = \frac{\mathrm{Tr}\,\Sigma^*(p)}{4m} = \frac{ig^2}{(2\pi)^4}\int \frac{d^4k}{[k^2-m^2][(k-p)^2-\mu^2]}$$

$$B(p^2) = \frac{\mathrm{Tr}[\not{p}\,\Sigma^*(p)]}{4p^2} = \frac{-ig^2}{p^2(2\pi)^4}\int \frac{p.k\,d^4k}{[k^2-m^2][(k-p)^2-\mu^2]}$$

By the Cutkosky prescription,

$$-\frac{1}{\pi}\,\mathrm{Im}\,A(p^2) = \frac{-g^2}{(2\pi)^3}\int \delta(k^2-m^2)\vartheta(k)\delta(p-k^2-\mu^2)\vartheta(p-k)d^4k$$

$$= \frac{-g^2}{16\pi^2p^2}\left\{(p^2-m^2-\mu^2)^2 - 4m^2\mu^2\right\}^{\frac{1}{2}}\vartheta\left[p^2-(m+\mu)^2\right] \qquad (10)$$

$$\frac{1}{\pi}\,\mathrm{Im}\,B(p^2) = \frac{g^2}{(2\pi)^3p^2}\int k.p\,\delta(k^2-m^2)\vartheta(k)\,\delta(p-k^2-\mu^2)\vartheta(p-k)d^4k$$

$$= \frac{g^2}{32\pi^2p^4}(p^2+m^2-\mu^2)\left\{(p^2-m^2-\mu^2)^2-4m^2\mu^2\right\}^{\frac{1}{2}}\vartheta\left[p^2-(m+\mu)^2\right] \qquad (11)$$

Hence from (9),

$$\delta m\, Z_2 = -\frac{m}{\pi}\int_{(m+\mu)^2}^{\Lambda^2} \frac{\mathrm{Im}A(K^2) + \mathrm{im}\,B\,(K^2)}{K^2-m^2}\,dK^2$$

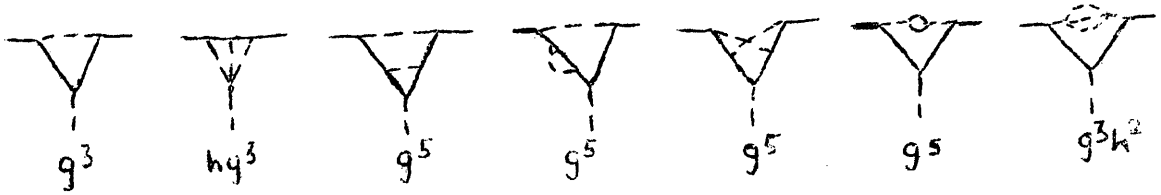
$$= -\frac{g^2m}{32\pi^2}\int_{(m+\mu)^2}^{\Lambda^2} \frac{\left\{(K^2-m^2-\mu^2)^2-4m^2\mu^2\right\}^{\frac{1}{2}}}{K^4}\left[1+\frac{\mu^2}{K^2-m^2}\right]dK^2 \qquad (12)$$

$$Z_2 = 1 - \frac{g^2}{32\pi^2}\int_{(m+\mu)^2}^{\Lambda^2} \frac{\left\{(K^2-m^2-\mu^2)^2-4m^2\mu^2\right\}^{\frac{1}{2}}}{K^4}\left[1-\frac{\mu^2(K^2+m^2)}{(K^2-m^2)^2}\right]dK^2 \qquad (13)$$

Of course these results are precisely those obtained by Lehmann from a perturbation expansion of the spectral representation for the propagator. Equations (12) and (13) do not need any correction up to first order in h.

4. Vertex Function

Here the relevant diagrams up to f^5 are



and just the first two will be the ones to calculate. Now the most symmetrical way of obtaining a dispersion relation for the vertex function is to investigate the analytic behaviour in the momentum square of the boson with both fermions put on their mass shells. Alternatively, we could make one boson and one fermion real, but besides being unsymmetrical, we expect the same result for Z_1 at large Λ as in the symmetrical treatment (Z_1 is essentially unique at ∞ cut-offs). We must mention that our Λ may not turn out large, so the result for Z_1 is certainly ambiguious ; — this ambiguity would be true in any treatment of the vertex part.

The loop contribution of Figure 3 is

$$g\,\gamma_5\Gamma(q^2) = \frac{ig^3}{(2\pi)^4}\int d^4k\ \frac{\gamma_5(\not p-\not k+m)\gamma_5(\not p'-\not k+m)\gamma_5}{\left[(p'-k)^2-m^2\right]\left[k^2-\mu^2\right]\left[(p-k)^2-m^2\right]}$$

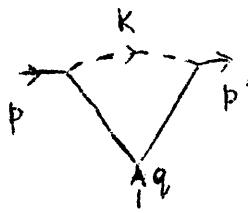


FIGURE 3.

$$\therefore\quad \bar{u}(p')\,\gamma_5 u(p)\,\Gamma(q^2) = \frac{ig^2 \bar{u}(P')\gamma_5 u(p)}{(2\pi)^4}\int d^4k\,\frac{k^2}{\left[(p'-k)^2-m^2\right]\left[k^2-\mu^2\right]\left[(p-k)^2\right.}$$

$$\text{or}\quad \Gamma(q^2) = \frac{ig^2}{(2\pi)^4}\int\frac{k^2 d^4k}{\left[(p'-k)^2-m^2\right]\left[k^2-\mu^2\right]\left[(p-k)^2-m^2\right]} \tag{14}$$

The integral (14) has a normal branch cut starting at $q^2 = 4m^2$ and the discontinuity across it is

$$\frac{1}{\pi}\,\mathrm{Im}\,\Gamma(q^2) = -\frac{g^2}{(2\pi)^3}\int d^4k\,\delta(k^2-2p.k)\,\vartheta(p'-k)\,\vartheta(k-p)\cdot\frac{k^2}{k^2-\mu^2}\;\delta(k^2 2p.k)$$

Referring to the appendix,

$$\frac{1}{\pi}\,\mathrm{Im}\,\Gamma(q^2) = \frac{-g^2}{(2\pi)^3}\int_{4m^2-q^2}^{o}\frac{\pi\,.\,dt}{2\sqrt{q^4-4m^2q^4}}\cdot\frac{t}{t-\mu^2}\,\vartheta(q^2-4m^2)$$

$$= -\frac{g^2}{16\pi^2}\cdot\frac{\vartheta\,(q^2-4m^2)}{\left\{q^4-4m^2q^2\right\}^{\frac{1}{2}}}\left[q^2-4m^2-\mu^2\log\frac{q^2-4m^2+\mu^2}{\mu^2}\right] \tag{15}$$

$$\therefore\quad Z_1 = 1-\Gamma(\mu^2) = 1+\frac{g^2}{16\pi^2}\int_{4m^2}^{\Lambda^2}\frac{K^2-4m^2-\mu^2\log\left(\frac{K^2-4m^2+\mu^2}{\mu^2}\right)}{K\,(K^2-\mu^2)\,(K^2-4m^2)^{\frac{1}{2}}}\,dK^2 \tag{16a}$$

As stated already, we obtain a different Z_1 for finite Λ when we write the representation in p^2 with $q^2 = \mu^2$ and $p'^2 = m^2$. In that case we obtain an integral with a threshold at $p^2 = (m+\mu)^2$ which will give

$Z_1 \neq 1$ even when $\Lambda^2 = 4m^2$. We have calculated the absorptive part γ of the form factor and it is not particularly enlightening to work out Z_1 in terms of it because it is extremely complicated. Instead we will estimate it roughly as follows :

We know that as $\Lambda \to \infty$, $Z_1 \to \frac{g^2}{16\pi^2} \log \frac{\Lambda^2}{m^2}$ and that as $\Lambda \to (m+\mu)^2$, $Z_1 \to 1$. Therefore since

$$Z_1 = 1 + \frac{g^2}{16\pi^2} \int_{(m+\mu)^2}^{4m^2} \frac{\gamma(K^2)\, dK^2}{K^2-(m+\mu)^2} + \int_{4m^2}^{2} \frac{\gamma(K^2)\, dK^2}{K^2-(m+\mu)^2}$$

we will assume (16) for the integral above $4m^2$. We are left with the correction

$$\frac{g^2}{16\pi^2} \int_{(m+\mu)^2}^{4m^2} \frac{\gamma(K^2)dK^2}{K^2 - (m+\mu)^2} \sim \frac{g^2}{16\pi^2} \log \left(\frac{2m}{m+\mu}\right)^2 , \text{ because}$$

$\log \left(\frac{\Lambda}{m+\mu}\right)^2$ has the right features at ∞ and correctly vanishes at threshold. Hence we crudely obtain as the asymmetric renormalisation constant,

$$Z_1' \simeq Z_1 + \frac{g^2}{16\pi^2} \log \left(\frac{2m}{m+\mu}\right)^2 \qquad (16b)$$

Expression (16) needs to be corrected even up to first order in h, by the graph of Figure 4. However as this correction is supposed to be small we will just attempt to estimate it in a rough and ready manner.

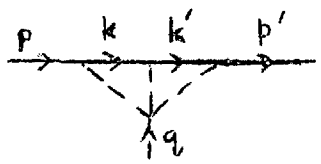


FIGURE 4.

$$\bar{u}(p')\ g\gamma_5\Gamma(q^2)u(p) = -\frac{g^3h}{(2\pi)^8}\int \frac{d^4kd^4k' \cdot \bar{u}(p')(\not{k}'-m)\gamma_5(\not{k}-m)u(p)}{(k^2-m^2)(k'^2-m^2)[(k-p)^2-\mu^2][(k-k')-\mu^2][(p'-k')^2-\mu^2]}$$

In order to evaluate Γ, we multiply the expressions on each side by $\bar{u}(p)\gamma_5\, u(p')$ and sum over spins.

$$Tr[\gamma_5\Gamma(q^2)(\not{p}+m)\gamma_5(\not{p}'+m)] = \frac{-g^2h}{(2\pi)^8}\int \frac{d^4kd^4k' Tr[(\not{k}'-m)\gamma_5(\not{k}-m)(\not{p}+m)\gamma_5(\not{p}'+m)]}{(k^2-m^2)(k'^2-m^2)[(k-p)^2-\mu^2][(k-k)^2-\mu^2][(p'-k')^2-\mu^2]}$$

$$\text{or } -\tfrac{1}{2}\ q^2\Gamma(q^2) = \frac{-g^2h}{(2\pi)^8}\int \frac{d^4kd^4k' \left[-\frac{1}{2}q^2(k.k'-m^2)+m^2(k-k').q+k'.qk-p-k'.pk\right.}{(k^2-m^2)(k'^2-m^2)[(k-p)^2-\mu^2][(k-k')^2-\mu^2][(p'-k')^2-\mu^2]}$$

In accordance with our object, we retain the leading term in the numerator and use the standard Feynman precedure of introducing the cut-off by rotating the contours of integration to lie along the imaginary case.

$$\therefore \Gamma(q^2) \approx \frac{g^2h}{(2\pi)^8}\int \frac{k.k'\ d^4kd^4k'}{k^4k'^4(k-k')^2} \quad \text{for large} \quad k \text{ and } k'$$

$$\approx \frac{g^2h}{(2\pi)^8}\cdot(i\pi^2)^2\int \frac{-K^2d(K^2)K'^2d(K'^2)KK'}{K^4K'^4(K^2+K'^2)} \quad \text{with } k^2 = -K^2$$

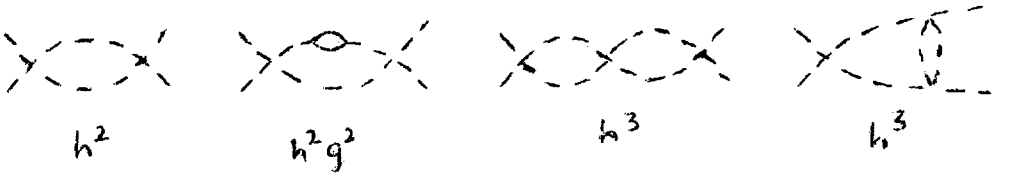
$$\approx \frac{g^2h}{8\pi^2.16\pi^2}\log\frac{\Lambda^2}{9\mu^2} \qquad (17)$$

where we have put the lower limit at $9\mu^2$ immediately, by analogy with the lower limit in the dispersion representation and by the same reasoning as (16b). We must remember to add (17) to (16) in the event where we do n

neglect meson meson scattering altogether.

5. Meson-Meson Scattering.

Up to f^3 we have the set of diagrams:



and only the first two come into the reckoning according to our approximation. In the computation of Z_4 we will put all the mesons on the mass shell.

The first diagram (Fig.5) is of the self energy type and has the matrix element

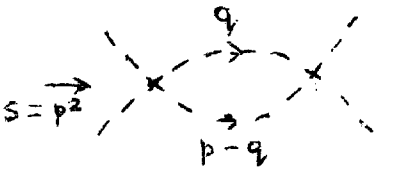


FIGURE 5

$$N(s) = \frac{-ih^2}{(2\pi)^4}\int d^4q \cdot \frac{1}{[q^2-\mu^2][(p-q)^2-\mu^2]}$$

$$\therefore \quad \frac{1}{\pi}\,\mathrm{Im}N(s) = \frac{h^2}{(2\pi)^3}\int d^4q\,\delta(q^2-\mu^2)\vartheta(q)\,\delta[(p-q)^2-\mu^2]\vartheta(p-q) = \frac{h^2}{(4\pi)^2}\sqrt{1-\frac{4\mu^2}{s}}\,\vartheta(s-4\mu^2) \tag{18}$$

$$\therefore \quad N(s) = \frac{h^2}{(4\pi)^2}\int_{4\mu^2}^{\Lambda^2}\left\{1-\frac{4\mu^2}{s'}\right\}^{1/2}\frac{ds'}{s'-s-i\varepsilon}\,, \quad \text{so}$$

$$N\left(\tfrac{4}{3}\mu^2\right) = \frac{h^2}{(4\pi)^2}\int_{4\mu^2}^{\Lambda^2}\left\{1-\frac{4\mu^2}{s'}\right\}^{1/2}\frac{ds'}{s'-\frac{4}{3}\mu^2} \sim \frac{h^2}{(4\pi)^2}\log\frac{\Lambda^2}{4\mu^2} \tag{19}$$

and of course this must be multiplied by 3 because of the 't' and 'u' channels.

Next we must consider Fig.6.

We must remember that the internal meson propagator is renormalised ie doubly subtracted.



FIGURE 6

$$\therefore \Delta(q)=\left[q^2-\mu^2+(q^2-\mu^2)^2\,\Pi_f^*(q)\right]^{-1}$$

As the correction is $-\Pi_f^*(q)$. Thus,

$$N(s) = \frac{-ih^2}{(2\pi)^4}\int d^4q \quad \frac{-\Pi_f^*(q)}{(p-q)^2-\mu^2} \tag{20}$$

But from (5), $\Pi_f^*(q) = \dfrac{g^2}{8\pi^2}\int_{4m^2}^{\Lambda^2} \dfrac{K\,(K^2-4m^2)^{\frac{1}{2}}dK^2}{(K^2-\mu^2)^2(K^2-q^2-i\varepsilon)}$

$$\therefore \frac{1}{\pi}\ \mathrm{Im}\ N(s)=\frac{h^2g^2}{(2\pi)^3 8\pi^2}\int_{4m^2}^{\Lambda^2}\frac{\pi}{2s}\left\{(s-\mu^2-K^2)^2-4\mu^2K^2\right\}^{\frac{1}{2}}\cdot\frac{K\sqrt{K^2-4m^2}}{(K^2-\mu^2)^2}\,dK^2\,\vartheta\left(s-(\mu+K)^2\right)$$

$$\therefore N(s) = \frac{h^2}{(4\pi)^2}\,\frac{g^2}{8\pi^2}\int_{(2m+\mu)^2}^{\Lambda^2} ds' \int_{4m^2}^{\Lambda^2} dK^2\,\frac{K(K^2-4m^2)^{\frac{1}{2}}}{(K^2-\mu^2)^2}\cdot\frac{\left\{(s-\mu^2-K^2)^2-4\mu^2K^2\right\}^{\frac{1}{2}}}{s'\,(s'-s-i\varepsilon)}\,ds' \tag{21}$$

$$\therefore N(\tfrac{4}{3}\mu^2) \approx \frac{h^2}{(4\pi)^2}\cdot\frac{g^2}{8\pi^2}\ \log\ \frac{\Lambda^2}{4m^2}\log\ \frac{\Lambda^2}{(2m+\mu)^2} \tag{22}$$

and there are altogether 5 more such terms as (22).

For the sake of reference we add that the Chew-Mandelstam constant[53] is related to our h by $-h/16\pi = \lambda_y$ since it is the negative of the scattering amplitude at the symmetrical point.

6. Summary of Results and Isospin Modifications.

It is convenient to introduce the abbreviation $\alpha=g^2/8\pi^2$, $\beta=h/16\pi^2$ $y = \mu^2/m^2$, $z = \Lambda^2/m^2$, in terms of which the equations for the renormalisation constants read :

$$Z_3 = 1 - \alpha \int_4^z \frac{(x^2-4x)^{\frac{1}{2}}}{(x-y)^2}\, dx \tag{23}$$

$$Z_2 = 1 - \frac{\alpha}{4}\int_{(1+\sqrt{y})^2}^{z} \frac{\{(x-1-y)^2-4y\}^{\frac{1}{2}}}{x^2}\left[1 - \frac{y\,(x+1)}{(x-1)^2}\right] dx \tag{24}$$

$$Z_1 \simeq 1 + \frac{\alpha}{2}\int_4^z \frac{x - 4 - y\log\left(\frac{x-4+y}{y}\right)}{(x^2-4x)^{\frac{1}{2}}(x-y)}\, dx - \alpha\beta\log\left(\frac{z}{q_y}\right) \tag{25a}$$

$$Z_1' \simeq Z_1 + \frac{\alpha}{2}\log\frac{4}{(1+\sqrt{y})^2} \tag{25b}$$

$$Z_4 \simeq 1 - 3\beta\log\left(\frac{z}{4y}\right) - \alpha\beta\log\left(\frac{z}{4}\right)\log\left(\frac{z}{(2+\sqrt{y})^2}\right) \tag{26}$$

Notice that we have inserted the same cut-off in every integral to make all Z_i dependent on the 4 variables α, β, y, z. There is no compelling reason for this and we have only done so on grounds of simplicity, besides which it would otherwise be impossible to make any further progress.

In order that the model be as realistic as possible we insert the normal isospin properties of fermions and bosons i.e. we specifically assume that the isovector boson and isospinor fermion are coupled through the Lagrangian $\frac{f}{\sqrt{2}} : \bar{\Psi}\, \underline{\tau} \cdot \underline{\Phi}\, \Psi :+ h.c.\ (\underline{\Phi}^{+} \cdot \underline{\Phi})^2 :$

The modifications that result thereby are simply various factors and they are easily obtained as follows :

(i) Pion self energy

In addition to (4) we have the extra factor

$$\mathrm{Tr}\left[\frac{\tau_r}{\sqrt{2}} \cdot \frac{\tau_s}{\sqrt{2}}\right] = \delta_{rs} \quad \therefore\ a \to a\ .$$



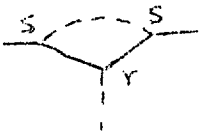
(ii) Nucleon self energy

$$a \to a \quad \frac{\tau_r}{\sqrt{2}} \frac{\tau_r}{\sqrt{2}} = \frac{3}{2}\, a$$



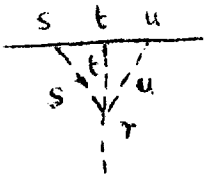
(iii) N N vertex

This acquires an extra label - the pion in ex r.



Now $\frac{\tau_s}{\sqrt{2}} \frac{\tau_r}{\sqrt{2}} \frac{\tau_s}{\sqrt{2}} = -\frac{\tau_r}{2\sqrt{2}} \quad \therefore\ a \to -\frac{1}{2}a$

The pion-pion scattering correction requires the computation of $M_{rstu} \frac{\tau_s}{\sqrt{2}} \frac{\tau_t}{\sqrt{2}} \frac{\tau_u}{\sqrt{2}}$



$$= h\,(\delta_{rs}\delta_{tu} + \delta_{rt}\delta_{su} + \delta_{ru}\delta_{st})\tau_s\tau_t\tau_u\,/2\sqrt{2} = \frac{5}{2\sqrt{2}}\, h\tau_r$$

$\therefore\ \beta \to -5\beta$ as well.

(iv) $\pi\pi$ SCATTERING



The matrix element has 4 pion suffices now

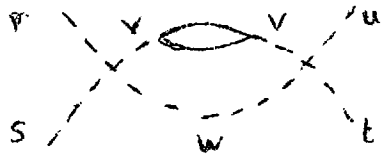
$$M_{rsvw} M_{vwut} = h^2 (\delta_{rs}\delta_{vw} + \delta_{rv}\delta_{sw} + \delta_{rw}\delta_{sv})(\delta_{vw}\delta_{ut} + \delta_{vu}\delta_{wt} + \delta_{vt}\delta_{wu})$$

$$= h^2 (7\delta_{rs}\delta_{tu} + 2\delta_{rt}\delta_{su} + 2\delta_{ru}\delta_{st})$$

Together with the 't' and 'u' channels, $M.M = 11h^2 (\delta\delta + \delta\delta + \delta\delta)$

$\therefore$ $3\beta \longrightarrow 11\beta$.

If the intermediate pion lines are corrected to lowest order in g^2, then because $\alpha \longrightarrow \alpha$ from (1) and a δ factor turns up in the pion propagator, $3\beta \longrightarrow 11\beta$ still.



Therefore isobaric spin causes equations (23), (24), (25) and (26) to change to

$$Z_3 = 1 - \alpha \int_4^z \frac{(x^2 - 4x)^{\frac{1}{2}}}{(x-y)^2}\, dx \qquad (23)$$

$$Z_2 = 1 - \frac{3\alpha}{8} \int_{(1+\sqrt{y})^2}^{z} \frac{\{(x-1-y)^2 - 4y\}^{\frac{1}{2}}}{x^2} \left[1 - \frac{y(x+1)}{(x-1)^2}\right] dx \qquad (24)$$

$$Z_1 = 1 - \frac{\alpha}{4} \int_4^z \frac{x - 4 - y \log\left(\frac{x - 4 + y}{y}\right)}{(x^2-4x)^{\frac{1}{2}}(x-y)} dx - \frac{5}{2} \alpha\beta \log\left(\frac{z}{9y}\right) \qquad (25a)$$

$$Z_1' = Z_1 - \frac{\alpha}{4} \log \frac{4}{(1+\sqrt{y})^2} \qquad (25b)$$

$$Z_4 = 1 - 11\beta \log\left(\frac{z}{4y}\right) - 22\alpha\beta \log\left(\frac{z}{4}\right) \log \frac{z}{(2+\sqrt{y})^2} \qquad (26)$$

The computation of the natural constants α, β and y is given in the next section.

CHAPTER II.

CALCULATION OF THE NATURAL CONSTANTS.

1 Method of Solution.

Let us recapitulate the way in which the renormalisation constants have been calculated: firstly, it has been assumed that a calculation to lowest order in g^2 and h is a good approximation to the truth; moreover we have adopted the same cut-off in each of the divergent integrals, a procedure which is open to criticism. However, that the solutions do come out with the correct order of magnitude justifies these approximations in a manner of speaking.

Bearing these assumptions in mind, we attempt to solve the set of four equations $Z_i=o$ from (23) - (26), first for neutral and then for charged meson field theories. The integrands are somewhat complicated and we have therefore evaluated all integrals numerically for the range $0.001 \leq y \leq 1$, which is the one of practical interest. In the first instance we notice that Z_2 and Z_3 are independent of β, so we determine α and z in terms of y through $Z_2=Z_3=o$. Inserting these values into $Z_4=o$ will enable us to evaluate the rough value of β and we may thereby estimate the magnitude Z_1 to Z_1' of the vertex function renormalisation constant. By seeing where the latter vanishes we will have arrived at the required solutions.

2. Numerical Computation of the Integrals.

While it is certainly possible to derive closed expressions for the integrals in (23) and (24), they will be much too long and impractical; when all is said and done, it is much simpler to evaluate them, as well as (25), numerically, and this we carry out for the range $0.001 \leq y \leq 1$. We have to consider the three integrals

$$I_3(y,z) = \int_4^z \frac{(x^2-4x)^{\frac{1}{2}}dx}{(x-y)^2} \qquad (27)$$

$$I_2(y,z) = \int_{(1+\sqrt{y})^2}^z \frac{\left\{(x-1-y)^2-4y\right\}^{\frac{1}{2}}}{x^2}\left[1-\frac{y(x+1)}{(x-1)^2}\right] \qquad (28)$$

$$I_1(y,z) = \int_4^z \frac{x-4-y\log\left(\frac{x-4+y}{y}\right)}{(x^2-4x)^{\frac{1}{2}}(x-y)}\,dx \qquad (29)$$

and compute them numerically up to $z = 25$. These are given in the tables below and have also been graphed for convenience.

TABLE 1.

z \ y	0.01	0.05	0.1	0.3	1
5	0.05	0.44	0.04	0.03	0.02
6	0.136	0.130	0.118	0.094	0.063
8	0.314	0.307	0.284	0.241	0.179
10	0.472	0.465	0.436	0.382	0.295
15	0.797	0.790	0.761	0.677	0.560
20	1.057	1.049	1.021	0.930	0.805
25	1.262	1.255	1.226	1.135	1.010

TABLE 2.

z \ y	0.005	0.01	0.05	0.1	0.3	1
4	0.597	0.582	0.487	0.350	0.159	0
5	0.771	0.756	0.595	0.510	0.287	0.035
6	0.029	0.905	0.741	0.650	0.408	0.099
8	1.170	1.153	0.983	0.888	0.625	0.245
10	1.374	1.357	1.178	1.080	0.807	0.385
15	1.744	1.727	1.543	1.443	1.160	0.728
20	2.007	1.990	1.806	1.703	1.418	0.958
25	2.222	2.205	2.021	1.918	1.623	1.148

TABLE 3.

z \ y	0.05	0.1	0.3	1
5	0.065	0.067	0.07	0.10
6	0.161	0.165	0.22	0.243
8	0.351	0.357	0.425	0.497
10	0.519	0.526	0.60	0.709
15	0.849	0.861	0.945	1.104
20	1.104	1.116	1.205	1.389
25	1.304	1.316	1.410	1.611

3. Solutions of the Equations.

(i) Neutral mesons.

The dependence of α and z on y from $Z_2=Z_3=0$ or $4I_3=I_2$, is tabulated as well as $\beta(y)$ (obtained from $Z_4=0$) and $Z_1(y)$. These quantities have also been plotted in graphs 1,2,3, and 4.

y=	0.005	0.01	0.05	0.1	0.3	1
$\alpha^{-1}=$	0.268	0.257	0.198	0.160	0.065	0
$g^2/4\pi=$	23.4	24.4	31.7	39.3	97	∞
z =	7.1	7	6.4	5.9	4.8	4
$\beta=$	0.040	0.046	0.065	0.108	0.240	∞
$h/16\pi=$	0.126	0.145	0.204	0.340	0.765	∞
Z_1 =	0.67	0.65	0.53	0.06	-0.95	?
Z_1' =	3.00	3.03	3.06	2,68	3.10	?

Therefore Z_1 vanishes when $y \gtrsim 0.1$. At the same time $g^2/4\pi$ shows a very sharp and physically unreasonable increase over this range, and we deduce that if a solution exists, it must be $g^2/4\pi = O(50)$, $h/16\pi = O(0.5)$, $\mu/m = O(0.3)$ with $(\Lambda/m)^2 \approx 2.3$.

(ii) Charged Mesons.

$Z_2=Z_3=o$ implies $\frac{8}{3} I_3 = I_2$ now. We proceed just like in (1) and plot $g^2/4\pi$, $h/16\pi$, z, $Z_1(y)$ versus y .

y =	0.005	0.01	0.05	0.1	0.3	1
a^{-1}=	0.515	0.508	0.408	0.310	0.120	0
$g^2/4\pi$=	12.2	12.4	15.4	20.2	52.4	∞
Z=	10.0	9.9	8.6	7.5	5.3	4
β =	0.010	0.011	0.015	0.021	0.062	∞
$h/16\pi$	0.031	0.035	0.051	0.066	0.195	∞
Z_1 =	0.51	0.52	0.50	0.47	0	?
Z_1' =	-0.09	-0.08	-0.10	-0.2	-1.1	?

By inspection of Z_1, $g^2/4\pi = O(20)$, $h/16\pi = O(0.05)$, $\mu/m \lesssim 0.5$ with $\Lambda/m = O(3)$, and this solution is decidedly more clear-cut than in the previous case because $g^2/4\pi$ is finite and has a comparatively small variation over this range of μ.

4. Generalisations and Conclusions.

In contrast to the neutral theory, the solution of the charged theory is straightforward to obtain, and therefore it appears that Z=o favours the normal π-N isospin properties. In fact the magnitude of

the calculated Yukawa coupling constant at the physical mass $\mu/m \simeq 0.15$, is $g^2/4\pi \simeq 13$, which is altogether too satisfactory! (Actually $\lambda = -h/16\pi = -0\ (0.05)$ does not compare too favourably with the current value of -0.2). We have not tried to answer the question as to whether the solutions to second order in g^2 and h - the equations $Z_i = C$ become biquadratic in these variables - are maintained, since the calculation is several degrees more difficult than the one given in this dissertation. Also, we have avoided all reference to the deuteron, ρ and ω meson resonances for the same reason.

With the apparent success of lowest order perturbation theory as applied to the πN system, we might attempt to do the same for another self contained theory, quantum electrodynamics. The photon mass is of course assumed to be zero and gauge invariance leaves us with only two renormalisation constants Z_2 and Z_3. To lowest order in e^2 and for very large cut-offs Λ, these are the well-known expressions[5],

$$Z_3 = 1 - \frac{2\alpha}{3}\left[\log z + \frac{5}{6}\right] \tag{30}$$

$$Z_2 = 1 - \frac{\alpha}{2}\left[\log z + \frac{11}{2} - 4\int_0^1 \frac{dx}{x}\right] \tag{31}$$

where $\alpha \equiv e^2/8\pi^2$, $z = \Lambda^2/m^2$, and m is the electron mass. Discarding the infrared divergence and setting $Z_2 = Z_3 = 0$ we arrive at $z = 0(10^6)$ and $e^2/4\pi \simeq 0.8$ which is out by a long way! In this light we must view the meson theory results as rather accidental.

APPENDIX I

NOTATION AND KINEMATICS.

UNITS AND NORMALISATION

We follow common practice and adopt the natural system of units whereby $\hbar = c = 1$. The loss of two dimensional units which allows all quantities to be expressed in terms of a length L or inverse mass, is offset by the simplicity of the ensuing formulae. Because action is dimensionless, the Lagrangian space time density has the dimensions $[\mathcal{L}] = L^{-4}$. From the construction,

$$\mathcal{L} = (\partial_\mu \phi \partial^\mu \phi + m^2\phi^2) + \bar{\psi}(i\not{\partial} - m)\psi + \text{etc.}$$

it follws that the dimensionalities of boson and fermion fields are

$$[\phi] = L^{-1} \ , \ [\psi] = L^{-3/2}$$

whence we may deduce the dimensionalities of coupling constants for various interactions :

(i) $\mathcal{L} = f\phi^3$ $\quad [f] = L^{-1}$

(ii) $\mathcal{L} = g\bar{\psi}\psi\phi$, $\quad [g] = L^0$

(iii) $\mathcal{L} = e\phi\partial_\mu\phi.\phi^\mu$, $\quad [e] = L^0$

(iv) $\mathcal{L} = G\bar{\psi}\psi\bar{\psi}\psi$ $\quad [G] = L^2$

(v) $\mathcal{L} = h\phi^4$ $\quad [h] = L^0$

As to the question of notation we maintain 'up' and 'down' indices with the metric $g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$; $\mu = 0,1,2,3$.

Thus $a.b = a_\mu b^\mu = a_\mu b_\nu g^{\mu\nu} = a_o b_o - a_i a_i = a_o b_o - \underline{a}.\underline{b}$

Our γ_μ matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$

and the Dirac equation reads $(i\gamma_\mu \partial^\mu - m)\psi(x) = 0$, where

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_o}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) .$$

The other eleven linearly independent matrices (apart from unity) are

$$\gamma_5 \equiv \gamma_o \gamma_1 \gamma_2 \gamma_3, \sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu], \sigma_{\mu 5} \equiv i\gamma_\mu \gamma_5 .$$

We shall not have occasion to use any particular representation, and by Schur's lemma we have the similarity transformations,

$$\left.\begin{aligned} \gamma_\mu^{+} &= A\gamma_\mu A^{-1}, & A^{+} &= A \\ \tilde{\gamma}_\mu &= -B\gamma_\mu B^{-1}, & B &= -B \\ \gamma_\mu^{*} &= -C\gamma_\mu C^{-1}, & C^{*} &= C^{-1} \end{aligned}\right\} \qquad A = C^{+}B.$$

Then the adjoint and charge conjugate spinors are defined as

$$\bar{\psi} = \psi^{+}A , \quad \psi^{c} = C^{*}\tilde{\psi}^{+} = -B^{-1}\tilde{\bar{\psi}} .$$

Next we consider matrix invariants and traces. We define $\not{p} = p^\mu \gamma_\mu$, so the Fourier transform of the free Dirac equation is $(\not{p}-m)u(\underline{p}) = 0$, $(\not{p}+m)\, v(\underline{p}) = o$.

u and v are the +ve and -ve energy solutions. Because $p^2-m^2=o$, $u_{\pm}$ will be taken to be the 1st and 2nd Columns of $(p+m)$ and $V_{\pm}$ will be taken to be the 3rd and 4th columns of $(\not{p}-m)$.

Thus $\bar{u}_\lambda u_\lambda = -v_\lambda v_\lambda = 2m$, and

$$\sum_\lambda u_\lambda \bar{u}_\lambda = (\not{p}+m), \qquad \sum v_\lambda v_\lambda = (\not{p}-m).$$

By straightforward application of the defining relation, we have

$$\mathrm{Tr}\left[\gamma_\mu \gamma_\nu\right] = 4g_{\mu\nu}$$

$$\mathrm{Tr}\left[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma\right] = 4\left(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}\right)$$

with traces of an odd no of γ-matrices vanishing.

$\mathrm{Tr}\left[\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma\right] = \varepsilon_{\mu\nu\rho\sigma}$, the completely antisymmetric 'unit tensor.

$\mathrm{Tr}\left[\gamma_5 \gamma_\mu \gamma_\nu\right] = 0$, by the group property of the γ_r matrices.

Finally we shall carry out Fourier transformations in such a way that all the 2π factors are associated with momenta. The result is that we are always concerned with factors such as $\dfrac{d^4p}{(2\pi)^4}$ and $2\pi\,\delta(p^2-m^2)$.

The normalisation of states in defined through the completeness relation of free particle (in-or-out) states,

$$1 = \sum_{s\lambda} \int |ps\lambda\rangle \frac{d^4p}{(2\pi)^4}\; 2\pi\theta(p)\delta(p^2-m^2)\, \langle ps\lambda|$$

$$+ \sum_{s\lambda, s'\lambda'} \int |ps\lambda, p's'\lambda\rangle \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\; 2\pi\theta(p)\delta(p^2-m^2)2\pi\theta(p)\delta(p^2-m^2)\langle ps\lambda, p's'\lambda'|$$

$$\text{ie } \langle \underline{p}s\lambda |\, \underline{p}'s'\lambda' \rangle = 2E_p\,(2\pi)^3\,\delta^3(\underline{p}-\underline{p}')\,\delta_{ss'}\delta_{\lambda\lambda'}.$$

This may be achieved by the LSZ reduction formula quite automatically

$$c^{+}_{\pm}(\underline{p}) = \lim_{x_o \to \mp\infty} i\int d^3\underline{x}\,\phi^{+}(x)\overleftrightarrow{\partial_o}\, e^{-ip.x} \mid o > \text{for boson fields,}$$

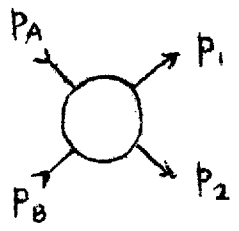
$$a^{+}_{\lambda\pm}(\underline{p}) = \lim_{x_o \to \mp\infty} \int d^3\underline{x}\,\psi(x)\,\gamma_o\, u_\lambda(\underline{p})\, e^{-ip.x} \mid o > \text{ for fermions,}$$

$$b^{+}_{\lambda\pm}(\underline{p}) = \lim_{x_o \to \mp\infty} \int d^3\underline{x}\, v_\lambda(\underline{p})\, e^{-ip.x}\, \gamma_o\psi(x) \mid o > \text{ for antifermions,}$$

$$\text{if} \quad <o \mid \phi(x) \mid \underline{p}> \;=\; e^{ip.x} \quad , \quad <o \mid \psi(x) \mid \underline{p}\,\lambda> = u_{\lambda}(p)\, e^{-ip.x}$$

THE PROCESS : A+B $\Rightarrow$ 1+2

Although the kinematics and phase space for the four-point function are exceedingly well-known, we shall discuss it briefly for the purpose of elucidating the notation that will be used later in the five-point function. We shall not direct all our arrows inwards, for the simple reason that when we afterwards investigate final state effects we will be working asymmetrically anyhow.



The three Mandelstam invariants are

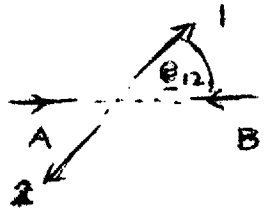
$$s_{AB} = s_{12} = (p_A + p_B)^2$$
$$s_{A1} = s_{2B} = (p_A - p_1)^2 \qquad \text{and} \quad s_{AB}+s_{A1}+s_{A2} = m_A^2 + m_B^2 + m_1^2 + m_2^2 .$$
$$s_{A2} = s_{1B} = (p_A - p_2)^2$$

In the channel A+B → 1+2, and in the c. of. m frame,

$$p_A = \left(\frac{s_{AB}+m_A^2-m_B^2}{2\sqrt{s_{AB}}} , \underline{q}_{AB} \right), p_B = \left(\frac{s_{AB}+m_B^2-m_A^2}{2\sqrt{s_{AB}}} , -\underline{q}_{AB} \right); 2q_{AB}\sqrt{s_{AB}} = C(s_{AB}, m_A^2, m_B^2)$$

$$p_1 = \left(\frac{s_{AB}+m_1^2-m_2^2}{2\sqrt{s_{AB}}} , \underline{q}_{12} \right), p_2 = \left(\frac{s_{AB}+m_2^2-m_1^2}{2\sqrt{s_{AB}}} , -\underline{q}_{12} \right); 2q_{12}\sqrt{s_{AB}} = C(s_{AB}, m_1^2, m_2^2)$$

where C is defined in (41b).



The scattering angle can be given in terms of s_{A1} or s_{A2}:

$$s_{1A} = -q_{AB}^2 - q_{12}^2 + 2q_{12}q_{AB}\cos\vartheta_{12} + \frac{1}{4s_{AB}}\left[(m_A^2-m_B^2)-(m_1^2-m_2^2)\right]^2$$

where we let $\underline{q}_{AB}$ point along the z-axis and make $\underline{q}_{12}$ lie in the xoz plane.

Now, in a helicity decompsotion after the manner of Jacob and Wick, we encounter the kinematic variables ω_{AB} and ϑ_{12} themselves. We shall

call ω_{AB} and θ_{12} the Wick variables. The relation with the invariants is extremely simple,

$$\omega_{AB} = \sqrt{s_{AB}} \quad , \quad \cos\theta_{12} = \frac{s_{1A}+q_{AB}^2+q_{12}^2-\frac{1}{4s_{AB}}(m_A^2+m_2^2-m_B^2-m_1^2)^2}{2q_{12}q_{AB}}$$

Finally because the relativistic flux is $2C(s_{AB},m_A^2,m_B^2)$, the differential cross section will be

$$d\sigma = \frac{1}{2C(s_{AB},m_A^2,m_B^2)} \int \frac{d^3\underline{p}_1 d^3\underline{p}_2}{(2\pi)^4 4E_1E_2} \; |M|^2 \; (2\pi)^4 \delta^4 (p_1-p_2-p_A-p_B)$$

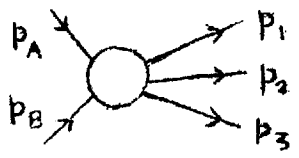
$$= \frac{1}{2C(s_{AB},m_A^2,m_B^2)} \int \frac{q_{12}\, d\Omega_{12}}{16\pi^2 \omega_{12}} |M|^2 = \frac{|M|^2}{32\pi s_{AB}} \cdot \frac{C(s_{AB},m_1^2,m_2^2)}{C(s_{AB},m_A^2,m_B^2)} \, d(\cos\theta_{12})$$

$$\text{or} \quad \frac{d\sigma}{ds_{1A}} = \frac{|M|^2}{16\pi C^2(s_{AB},m_A^2,m_B^2)} \; .$$

KINEMATIC VARIABLES FOR THE PROCESS : A+B → 1 + 2 + 3

Taking the scalar product of each four-momentum with

$$p_1+p_2+p_3 = p_A + p_B$$



$$p_1\cdot p_A+p_2\cdot p_A+p_3\cdot p_A=m_A^2+p_A\cdot p_B$$

$$p_1\cdot p_B+p_2\cdot p_B+p_3\cdot p_B=p_A\cdot p_B+m_B^2$$

$$m_1^2+p_1\cdot p_2+p_3\cdot p_1=p_1\cdot p_A+p_1\cdot p_B$$

$$p_1\cdot p_2+m_2^2+p_2\cdot p_3=p_2\cdot p_A+p_2\cdot p_B$$

$$p_3\cdot p_1+p_2\cdot p_3+m_3^2=p_3\cdot p_A+p_3 p_B \; .$$

Addition of all five equations would be equivalent to squaring the original conservation law,

$$s_{AB}=m_A^2+m_B^2+2p_A\cdot p_B= m_1^2+m_2^2+m_3^2+2p_1\cdot p_2+2p_2\cdot p_3+2p_3\cdot p_1$$

$$= s_{12}+s_{23}+s_{31}-m_1^2-m_2^2-m_3^2 ,$$

which is equivalent to the subsidiary equation for the Mandelstam invariants. As the s_{ij} are not sufficient to describe the amplitude (5 independent invariants are needed), we add four equations and subtract the remaining one. Thus, $m_B^2-m_A^2-m_1^2-m_2^2-m_3^2=2(p_1\cdot p_2+p_2\cdot p_3+p_3\cdot p_1-p_1\cdot p_A-p_2\cdot p_A-p_3\cdot p_A)$ which gives us directly one subsidiary equation for 6 invariants.

It is essential in any dispersion representation to know what invariants to select. Obviously $p_A\cdot p_B$ ($\sim$ incident energy) is a significant variable in final state interactions, and keeping it fixed means fixing $(p_1\cdot p_2+p_2\ p_3+p_3\cdot p_1)$ and $(p_1\cdot p_A+p_2\cdot p_A+p_3\cdot p_A)$. Therefore we may choose as our five invariants for final state interaction of 1 and 2 or 1 and 3,

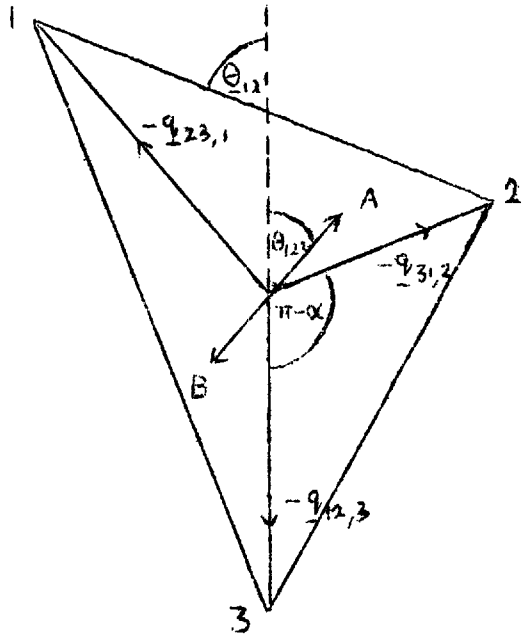
$$s_{AB}=(p_A+p_B)^2, s_{12}=(p_1+p_2)^2, s_{13}=(p_1+p_3)^2, s_{1A}=(p_1-p_A)^2, s_{3A}=(p_3-p_A)^2.$$

The first three correspond to energies, while the last two represent momentum transfers.

In the helicity specification which is most natural to describe the final state interaction between 1 and 2, the five Wick variables that enter into the description are ω_{AB}, the total energy in the overall b.c frame, ω_{12} the total energy of 1 and 2, in their own c of m frame, the

angle $\underline{\vartheta}_{12}$ between the relative motion of 1 and 2, and the direction of motion of 3, and $\vartheta_{12,3}$ the angle between the motion of (12) and of A. Here we have chosen $\underline{q}_{AB}$ to lie along the z-axis and $\underline{q}_{12,3}$ to lie in the xoz plane.

We now determine the connexion between the Wick variables ω_{12}, ω_{AB} $\vartheta_{12}, \phi_{12}, \vartheta_{12,3}$ and the relativistic invariants $s_{12}, s_{AB}, s_{13}, s_{3A}, s_{1A}$.



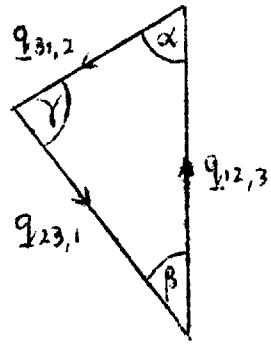
Firstly we notice that

$2q_{ij}\sqrt{s_{ij}} = C(s_{ij}, m_i^2, m_j^2)$ and

$2q_{ij,k}\sqrt{s_{AB}} = C(s_{AB}, s_{ij}, m_k^2)$,

by the method of construction of states as stated in section 2 ;

also that three angles which appear naturally in the total c of m system are those between the directions of motion of pairs as showin in the diagram.



eg. $2q_{12,3}q_{23,1}\cos\beta = q_{23,1}^2 + q_{12,3}^2 - q_{31,2}^2$

gives $\cos\beta$ uniquely in terms of s_{23}, s_{31} and s_{12}.

Clearly $s_{12} = \omega_{12}^2$ and $s_{AB} = \omega_{AB}^2$; however, to determine s_{13}, we must

go to the c of m.frame of 1 and 2 when $p_3 = (\xi_3, q_3)$ say.

$$\omega_{AB}^2=(p_1+p_2+p_3)^2=\omega_{12}^2+m_3^2+2\xi_3\omega_{12}$$

$$\therefore \xi_3 = \frac{\omega_{AB}^2-\omega_{12}^2-m_3^2}{2\omega_{12}} \quad \text{and} \quad |\underset{\sim}{q}_3| = \frac{\omega_{AB}}{\omega_{12}} q_{12,3}$$

$$\therefore s_{13}=(p_1+p_3)^2=m_1^2+m_3^2+\frac{1}{2\omega_{12}^2}(\omega_{12}^2+m_1^2-m_2^2)(\omega_{AB}^2-\omega_{12}^2-m_3^2)+\frac{2\omega_{AB}}{\omega_{12}} q_{12}q_{12,3}\cos\vartheta_{12}$$

$$= \tfrac{1}{2}(\omega_{AB}^2-\omega_{12}^2+m_1^2+m_2^2+m_3^2)+\frac{1}{2\omega_{12}^2}(m_1^2-m_2^2)(\omega_{AB}^2-m_3^2)+\frac{2\omega_{AB}}{\omega_{12}} q_{12}q_{12,3}\cos\vartheta_{12}$$

For later use we deduce

$$s_{23}=\tfrac{1}{2}(\omega_{AB}^2-\omega_{12}^2+m_1^2+m_2^2+m_3^2)+\frac{1}{2\omega_{12}^2}(m_2^2-m_1^2)(\omega_{AB}^2-m_3^2)-\frac{2\omega_{AB}}{\omega_{12}} q_{12}q_{12,3}\cos\vartheta_{12}.$$

In order to bring in $\vartheta_{12,3}$ we consider in the total b.c frame,

$$s_{3A}=(p_3-p_A)^2=m_3^2+m_A^2-2\frac{(\omega_{AB}^2+m_3^2-\omega_{12}^2)}{2\omega_{AB}}\cdot\frac{(\omega_{AB}^2+m_A^2-m_B^2)}{2\omega_{AB}} - 2q_{AB}q_{12,3}\cos\vartheta_{12,3}$$

$$= \tfrac{1}{2}(\omega_{12}^2-\omega_{AB}^2+m_3^2+m_A^2+m_B^2)+\frac{1}{2\omega_{AB}^2}(m_A^2-m_B^2)(\omega_{12}^2-m_3^2)-2q_{AB}q_{12,3}\cos\vartheta_{12,3}$$

Finally we come to the most complicated relation containing Φ_{12} (s_{23} being known),

$$s_{1A}=(p_1-p_A)^2=m_1^2+m_A^2-2\frac{(\omega_{AB}^2+m_1^2-s_{23})}{2\omega_{AB}}\cdot\frac{(\omega_{AB}^2+m_A^2-m_B^2)}{2\omega_{AB}} + 2\underset{\sim}{q}_{23',1}\cdot\underset{\sim}{q}_{AB}$$

But with reference to the chosen coordinate system, the component of $\underline{q}_{23}$ along the z-axis is such that

$$-\underline{q}_{23,1}\cdot\underline{q}_{AB} = q_{23,1}q_{AB}(\cos\beta\cos\vartheta_{12,3}+\sin\beta\sin\vartheta_{12,3}\cos\phi_{12})$$

$$\therefore\ s_{1A} = \tfrac{1}{2}(s_{23}-\omega_{AB}^2+m_1^2+m_A^2+m_B^2)+\frac{1}{2\omega_{AB}^2}(s_{23}-m_1^2)(m_A^2-m_B^2)$$

$$-2q_{23,1}q_{AB}(\cos\beta\cos\vartheta_{12,3}+\sin\beta\sin\vartheta_{12,3}\cos\phi_{12})$$

and the r.h.s. will itself involve ω_{12} and $\cos\vartheta_{12}$, by substitution of s_{23} and $\cos\beta$.

THREE-PARTICLE PHASE SPACE.

We are to evaluate $\dfrac{d^3\underline{p}_1}{2E_1}\ \dfrac{d^3\underline{p}_2}{2E_2}\ \dfrac{d^3\underline{p}_3}{2E_3}\ \dfrac{\delta^4(p_1+p_2+p_3-p_A-p_B)}{(2\pi)^5} \equiv d^5\chi$ in order to obtain the differential cross section. This can easily be done in terms of the Wick variables by integrating over the phase space of 1 and 2 in the (12) frame :

$$d^5\chi = \frac{1}{(2\pi)^5}\ \frac{q_{12}d\Omega_{12}}{4\omega_{12}}\ \frac{d^3\underline{p}_3}{2E_3}$$

$$= \frac{1}{(2\pi)^5}\ \frac{q_{12}q_{12,3}}{8\,\omega_{12}}\ dE_3\ d\Omega_{12}d\Omega_{12,3}\ .$$

The result may be expressed in terms of the relativistic invariants as was done for the four point function :

$$\frac{d\,(s_{12},s_{13},s_{3A},s_{1A})}{d\,(E_3,\cos\vartheta_{12},\cos\vartheta_{12,3},\phi_{12})} = 2\omega_{AB}\cdot\frac{2\omega_{AB}q_{12}q_{12,3}}{\omega_{12}}\,2q_{AB}q_{12,3}\cdot 2q_{23,1}q_{AB}\sin\beta\sin\vartheta_{12,3}\sin\phi_1$$

$$= \frac{16q_{AB}^2\omega_{AB}^2q_{12,3}^2q_{23,1}q_{12}}{\omega_{12}}\sin\beta\sin\vartheta_{12,3}\sin\phi_{12}$$

$$\therefore\quad d\sigma = \frac{|M|^2}{4q_{AB}\omega_{AB}}\,d^5\chi = \frac{M^2}{(2\pi)^5\cdot 2^9}\cdot\frac{ds_{12}ds_{13}ds_{3A}ds_{1A}d\phi_{12,3}}{q_{AB}^3\omega_{AB}^3\,q_{12,3}q_{23,1}\sin\beta\sin\vartheta_{12,3}\sin\phi_{12}}$$

Integrating over $\phi_{12,3}$ and rewriting,

$$d\sigma = \frac{|M|^2}{(4\pi)^4}\cdot\frac{s_{AB}ds_{12}ds_{13}ds_{3A}ds_{1A}}{C^3(s_{AB},m_A^2,m_B^2)C(s_{AB},m_3^2,s_{12})C(s_{AB},m_1^2,s_{23})\sin\beta\sin\vartheta_{12,3}\sin\phi_{12}}$$

is the final result, with $\beta,\vartheta_{12,3}$ and Φ_{12} obtainable in complicated fashion in terms of the same invariants.

In the above way, the phase space differentials consist of five angles and one energy; an alternative method is to have two energies and 3 angles, which is extremely relevant to the following discussion on Dalitz plots. We carry out an integration over $\underline{p}_3$ say in total b.c. frame which leaves us with

$$d^5\chi = \frac{d^3\underline{p}_1}{2E_1}\,\frac{d^3\underline{p}_2}{2E_2}\cdot\frac{\delta(E_1+E_2-E_3-\omega_{AB})}{2E_3\,(2\pi)^5} = \frac{d^3\underline{p}_1\,|\underline{p}_2|^2d|\underline{p}_2|\,d\Omega}{8\,E_1E_2E_3(2\pi)^5}\delta(E_1+E_2+E_3-\omega_{AB})$$

where $E_3 = \left\{(p_1+p_2)^2+m_3^2\right\}^{1/2} = \left\{|\underline{p}_1|^2+|\underline{p}_2|^2+2|\underline{p}_1\underline{p}_2|\cos\vartheta + m_3^2\right\}^{1/2}$

We may now carry out the integration over the last δ function since

$d(\cos\vartheta) = \dfrac{E_3 dE_3}{|\underline{p}_1||\underline{p}_2|}$, and the final phase space element is

$d^5\chi = \dfrac{1}{8\,(2\pi)^5}\, dE_1 dE_2 d\Omega_1 d\phi$, which means that if energies of particles 1 and 2 are observed experimentally with complete disregard of angles (ie integration over Ω_1 and ϕ) and events are graphed with coordinates E_1 and E_2 , the population of events is proportional to the modulus square of the matrix element. This latter fact explains the inherent advantage of such Dalitz plots over all other methods of mapping out the phase space.

DALITZ PLOTS.

Consider the c. of m. frame of the three particles 1,2,3 having a total available energy ω . $\therefore \omega = E_1+E_2+E_3$ and $\underline{q}_1+\underline{q}_2+\underline{q}_3 = 0$. Let us eliminate the momentum of particle 3,

$$E_3^2-m_3^2 = E_2^2-m_2^2+E_1^2-m_1^2+ 2\left[(E_2^2-m_2^2)(E_1^2-m_1^2)\right]^{1/2} C \quad \text{where } c = \cos\vartheta.$$

Putting $E_i = m_i+T_i$ and $\omega = m_1+m_2+m_3+ Q$, we can specialise to two cases :

(i) Non-relativistic limit ($T_i \ll m_i$)

$$Qm_3-(m_1T_2+m_2T_1) - (m_1+m_2+m_3)(T_1+T_2) = 2c\sqrt{m_1m_2T_1T_2}$$

Putting $(m_1+m_2+2m_3)y = 2(m_2+m_3)T_2+2(m_1+m_3)T_1$

$$(m_1+m_2+2m_3)x = 2(m_2+m_3)T_2-2(m_1+m_3)T_1$$

we obtain the equation $A - By = c\sqrt{y^2-x^2}$ where

$$A = \frac{2Qm_3}{(m_1+m_2+2m_3)}\left[\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}\right]^{1/2} > 0 \quad \text{and} \quad B = \left[\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}\right]^{1/2} > 1$$

ie $(B^2-c^2)y^2+c^2x^2-2ABy+A^2=0$ so the locus is a function of c^2 only.

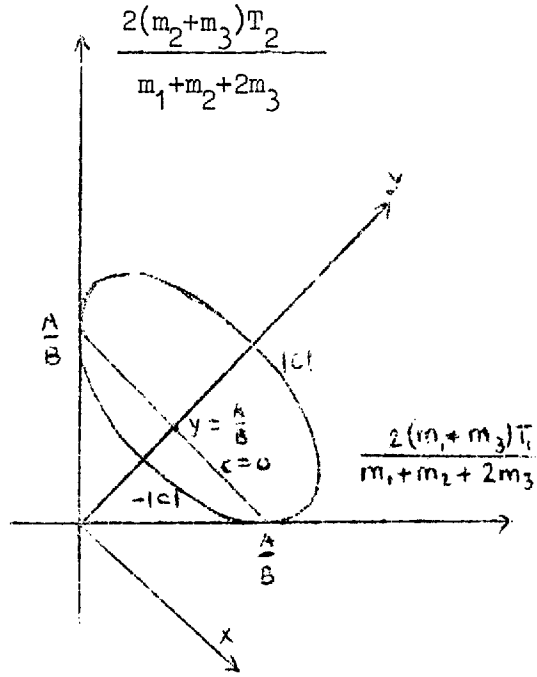
In particular for c=o,

$$y = \frac{A}{B} = \frac{2Qm_3}{m_1+m_2+2m_3}$$

while for $c\neq o$, we have the ellipse

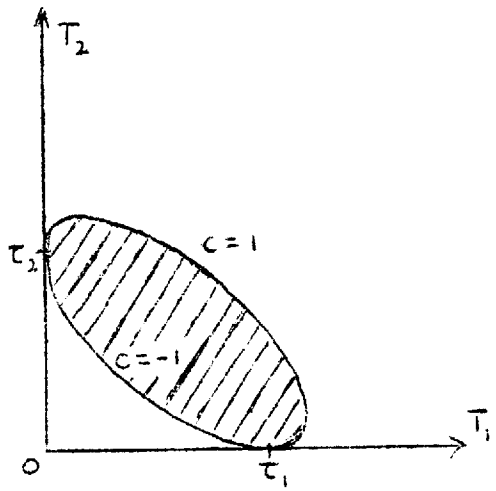
$$\frac{x^2}{\frac{A^2}{B^2-c^2}} + \frac{\left(y-\frac{AB}{B^2-c^2}\right)^2}{\frac{AC}{B^2-c^2}} = 1$$

The properties of the ellipse are summarised in the diagram.



Transforming to T_1 and T_2 the figure will remain elliptical but contracted and expanded along each axis, and retains its features. Now,

$$\tau_1 = \frac{Qm_3}{m_1+m_3} \text{ and } \tau_2 = \frac{Qm_3}{m_2+m_3} .$$



(ii) Relativistic limit ($T_i \gg m_i$)

Here $Q^2-2Q(T_1+T_2) = 2(c-1)T_1T_2$

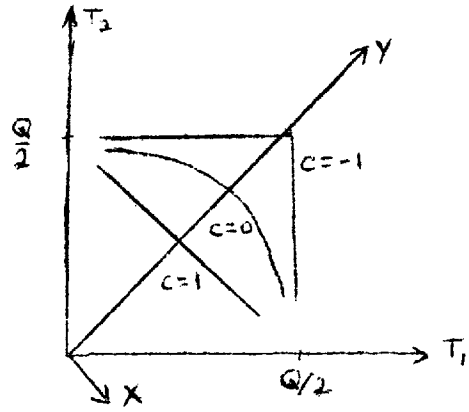
With $Y = T_1+T_2$ and $X = T_2-T_1$,

$Q^2-2YQ=\frac{1}{2}(1-c)(X^2-Y^2)$

ie the locus is a function of c now. When c=1, $x = \frac{Q}{2}$

$c=o, (Y-2Q)^2-X^2=2Q^2$

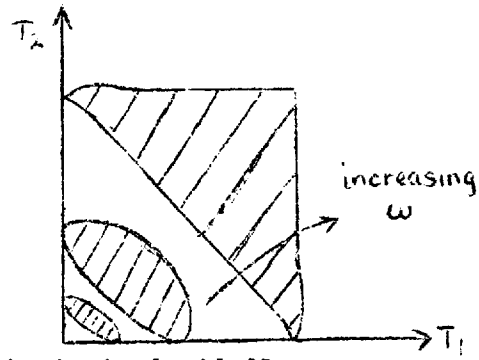
$c=-1, (X-Y+Q)(X+Y-Q)=o$



These high energy curves are exact only when T_1 and T_2 are both large. The behaviour when T_1 is small, say, is given by

$$2c\sqrt{2mT_1}\,.\,T_2 = q(Q-2T_2).$$

The figure shows the manner of distortion of the Dalitz plot as the available energy increases.



The form in which we have pursued this topic is decidedly non-covariant, but now we shall reexpress the results in terms of the relativistic invariants.

$$s_{ij} = (p_A+p_B-p_k)^2 = \omega^2-2E_k\omega+m_k^2 = (\omega-m_k)^2-2T_k\omega\,.$$

or $m_i^2+m_j^2+2E_iE_j-2\underline{q}_i\cdot\underline{q}_j$.

As we have already seen, the boundary of the Dalitz plot is obtained when all three final particles are moving along the same line, ie by putting various $\cos\theta = \pm 1$. Hence the boundary is given by

$$\pm 2q_1q_2 = 2E_1E_2 + m_1^2+m_2^2-s_{12}$$

or $4(E_1^2-m_1^2)(E_2^2-m_2^2) = (2E_1E_2+m_1^2+m_2^2-s_{12})^2$

By substituting for E_1 and E_2, s_{23} and s_{31} respectively, and carrying out a great deal of tedious algebraic simplication, we get

$$s_{23}s_{31}s_{12} \geq \omega^2 \left[m_1^2 s_{23} + m_2^2 s_{31} + m_3^2 s_{12} - 2m_2^2 m_3^2 - 2m_3^2 m_1^2 - 2m_1^2 m_2^2\right]$$

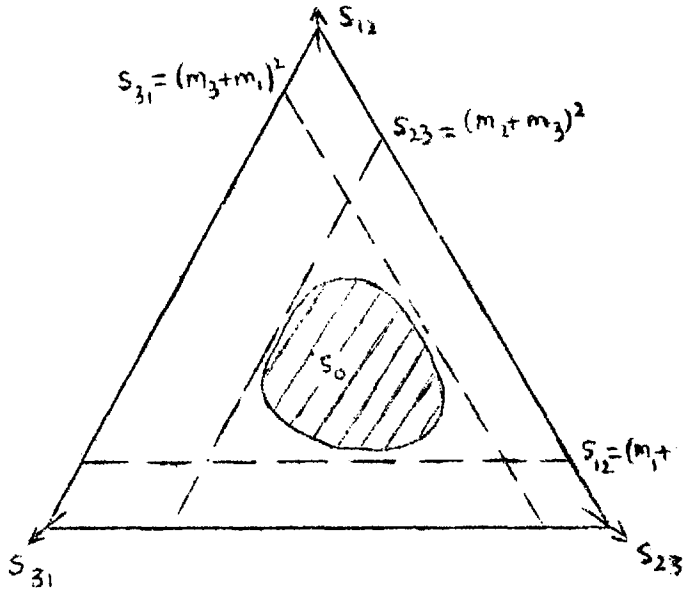
$$+ m_2^2 m_3^2 s_{23} + m_3^2 m_1^2 s_{31} + m_1^2 m_2^2 s_{12} - 2m_1^2 m_2^2 m_3^2 \,.$$

as our final symmetrical result for the Dalitz plot.

The subsidiary condition,

$$s_{12}+s_{23}+s_{31} = s_{AB} + m_1^2 + m_2^2 + m_3^2$$

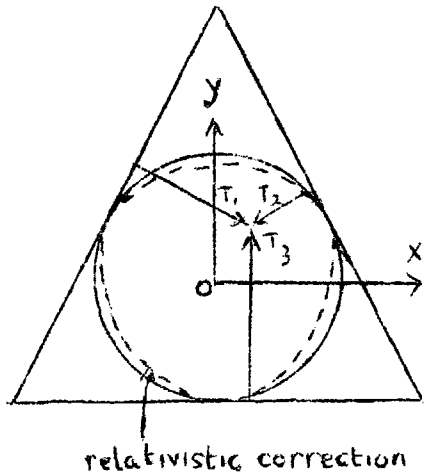
means that a triangular coordinate system (with a triangle of increasing height as ω increases) may be used. The Dalitz plot will lie within the boundary $s_{ij} \geq (m_i+m_j)^2$. and must shrink to nothing as $Q \to 0$.



The Fabri-Dalitz plot is derived when all three final masses are equal, m say. In the non-relativistic limit, A=Q, B=2 and $x = T_2 - T_1$, $y = T_2 + T_1$. The boundary equation,

$$\frac{3x^2}{Q^2} + \frac{\left(y - \frac{2Q}{3}\right)^2}{Q^2} = 1$$

becomes unit circle under the transformation $x \to \frac{\sqrt{3}}{Q}x, y \to \frac{Q}{3}\left(y + \frac{2Q}{3}\right)$



It is contained within the equilateral triangle of ht.Q corresponding to the coordim tes (T_1,T_2,T_3). The statistical distribution of events within the plot measures the transition probability, and this circumstance has been used in the investigation of the decays $K \to 3\pi$, $\eta \to 3\pi$, etc..

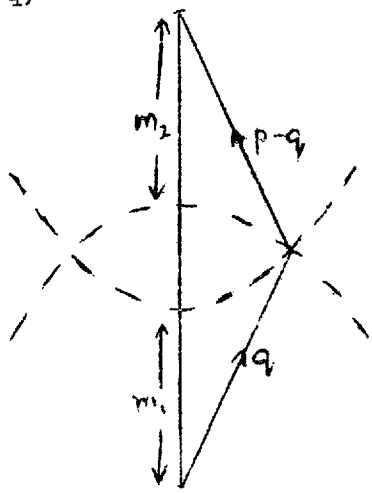
The relativistic analogue is $s_{12}s_{23}s_{31} = (\omega^2-m^2)^2m^2$

APPENDIX 2.

ON THE EVALUATION OF CERTAIN 4-DIMENSIONAL INTEGRALS.

SELF-ENERGY DIAGRAM.

$$J(p^2)= \int d^4q\, \delta(q^2-m_1^2)\vartheta(q)\delta\left[(p-q)^2-m_2^2\right]\vartheta\,(p-q)$$



We evaluate this integral in the frame where p is time-like. With the convention that in general

$k = (k_o,k_1,k_2,k_3)$ and $\sqrt{k^2} \equiv k$.

$p= (p;o,o,o)$ and

$q = (q\cosh\chi;\ q\sinh\chi\sin\vartheta\cos\Phi,$

$q\sinh\chi\sin\vartheta\sin\phi,\ q\sinh\chi\cos\vartheta)$

so

$$J(p^2)=\int\tfrac{1}{2}q^2d(q^2)\sinh\chi d(\cosh\chi)d(\cos\vartheta)d\phi.\,\delta(q^2-m_1^2)\delta(p^2+q^2-m_2^2-2pq\cosh\chi)\vartheta(p-q\cos$$

$$= 2\pi m_1^2\int \sinh\chi d\,(\cosh\chi)\,\delta(p^2+m_1^2-m_2^2-2m_1p\cosh\chi)\vartheta(p-m_1\cosh\chi)$$

$$= \frac{\pi\ m_1}{p}\left[\left(\frac{p^2+m_1^2-m_2^2}{2pm_1}\right)^2 - 1\right]^{1/2}\vartheta\left\{p^2-(m_1+m_2)^2\right\}$$

$$= \frac{\pi}{2p^2}\ \vartheta\left[p^2-(m_1+m_2)^2\right]\left\{(p^2-m_1^2-m_2^2)^2-4m_1^2m_2^2\right\}^{\frac{1}{2}}$$

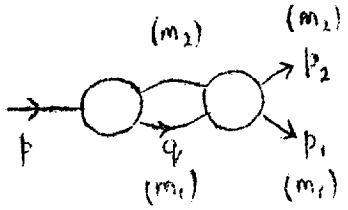
eg  has $J(p^2) = \frac{\pi}{2}\left\{1-\frac{4m^2}{p^2}\right\}^{1/2}\vartheta\ (p^2-4m^2)$

 has $J(p^2)= \frac{\pi}{2p^2}\vartheta\left[p^2-(m+\mu)^2\right]\left\{(p^2-m^2-\mu^2)-4m^2\mu^2\right\}^{\frac{1}{2}}$

[N.B. Choosing p timelike corresponds to the b.c. frame of 1 and 2 and in terms of the total energy ω_{12} and relative mtm. q_{12} we may identify $\mathfrak{J}(p^2) = \pi q_{12}/\omega_{12}$.]

VERTEX INTEGRALS.

$$J(q^2) = \int d^4q\ (q^2-m_1^2)\vartheta(q)\delta\left[(p-q)^2-m_2^2\right]\vartheta(p-q)$$
$$f\ (p^2,\ \overline{p_1-q}^2)$$



with $p^2=m_2^2$ and $p_1^2 = m_1^2$.

We evaluate in the frame where $\underline{p}$ = o and where p_1 and p_2 lies in the O-3 plane. Defining p^2 = s and $(p_1-q)^2$ = t, we have $2p_1\cdot p_2 = s-m_1^2-m_2^2$, $2p\cdot p_1 = s+m_1^2-m_2^2$, $2p\cdot p_2 = s+m_2^2-m_1^2$.

Since p = $(\sqrt{s}\ ;\ o,\ o,\ o)$

$$p_1 = \left(\frac{s+ m_1^2-m_2^2}{2\sqrt{s}}\ ;\ o,\ o,\ \frac{\left\{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2\right\}^{1/2}}{2\sqrt{s}}\right)$$

$$p_2 = \left(\frac{s + m_2^2-m_1^2}{2\sqrt{s}}\ ;\ o,\ o,\ -\frac{\left\{(s-m_1^2-m_2^2)-4m_1^2m_2^2\right\}^{1/2}}{2\sqrt{s}}\right)$$

$$\therefore\ J(s) = \pi\int q^2dq^2\ \sinh\chi d\ (\cosh\chi)\ d\ (\cos\vartheta)\ f(s,t)$$
$$\cdot\delta(q^2-m_2^2)\delta(s-m_1^2+m_2^2-2m_2\sqrt{s}\cosh\chi)\vartheta\ (s-m_1^2+m_2^2-2m_2\sqrt{s}\cosh\chi)$$

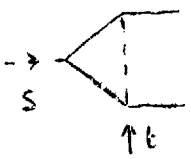
$$= \frac{\pi\, m_2^2}{2m_2\sqrt{s}} \cdot \frac{\left\{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2\right\}^{1/2}}{2m_2\sqrt{s}}\, \vartheta\left[s-(m_1+m_2)^2\right] \int_{-1}^{+1} d\,(\cos\vartheta)\, f(s,t)$$

But $t = m_1^2 + m_2^2 - \frac{(s+m_1^2-m_2^2)}{\sqrt{s}}\, m_2\cosh\chi + \frac{m_2}{\sqrt{s}}\left\{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2\right\}^{1/2}\sinh\chi\cos\theta$

$$= \frac{(m_1^2-m_2^2)^2}{s} + \frac{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2}{2s}\cdot(\cos\vartheta-1)$$

$$\therefore\quad J(s) = \frac{\pi\,\vartheta\left[s-(m_1+m_2)^2\right]}{2\left\{(s-m_1^2-m_2^2)^2-4m_1^2m_2^2\right\}^{1/2}} \int_{-s+2(m_1^2+m_2^2)}^{(m_1^2-m_2^2)^2/s} dt\, f(s,t)$$

eg. $$J(s) = \frac{\pi\,\vartheta\,(s-4m^2)}{2\left\{s^2-4m^2s\right\}^{1/2}} \int_{4m^2-s}^{0} dt\, \frac{t}{t-\mu^2}$$



$$= \frac{\pi\,\vartheta(s-4m^2)}{2\sqrt{s^2-4m^2s}}\left[\, s - 4m^2 - \mu^2\log\left(\frac{s-4m^2+\mu^2}{\mu^2}\right)\right]$$

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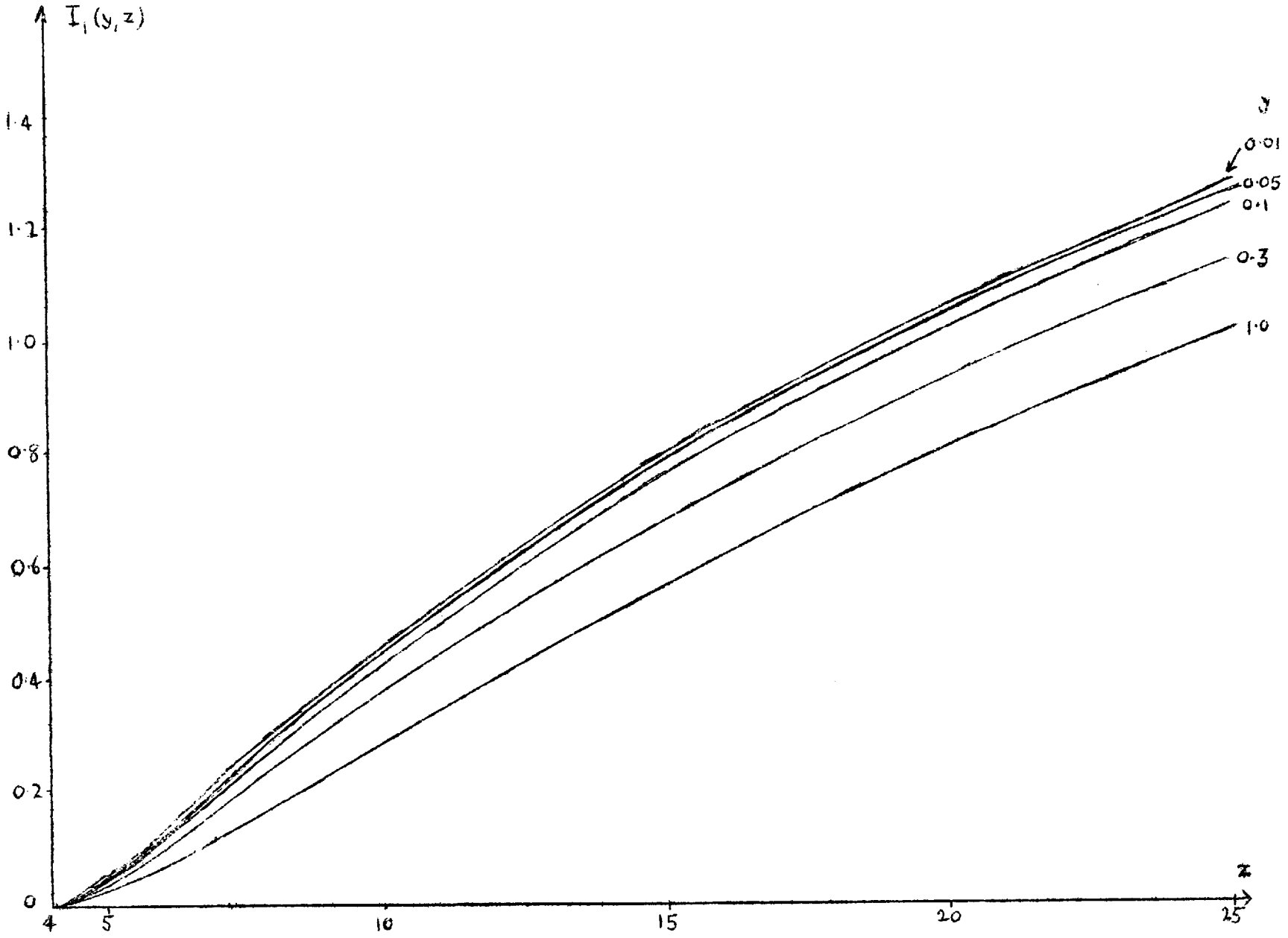
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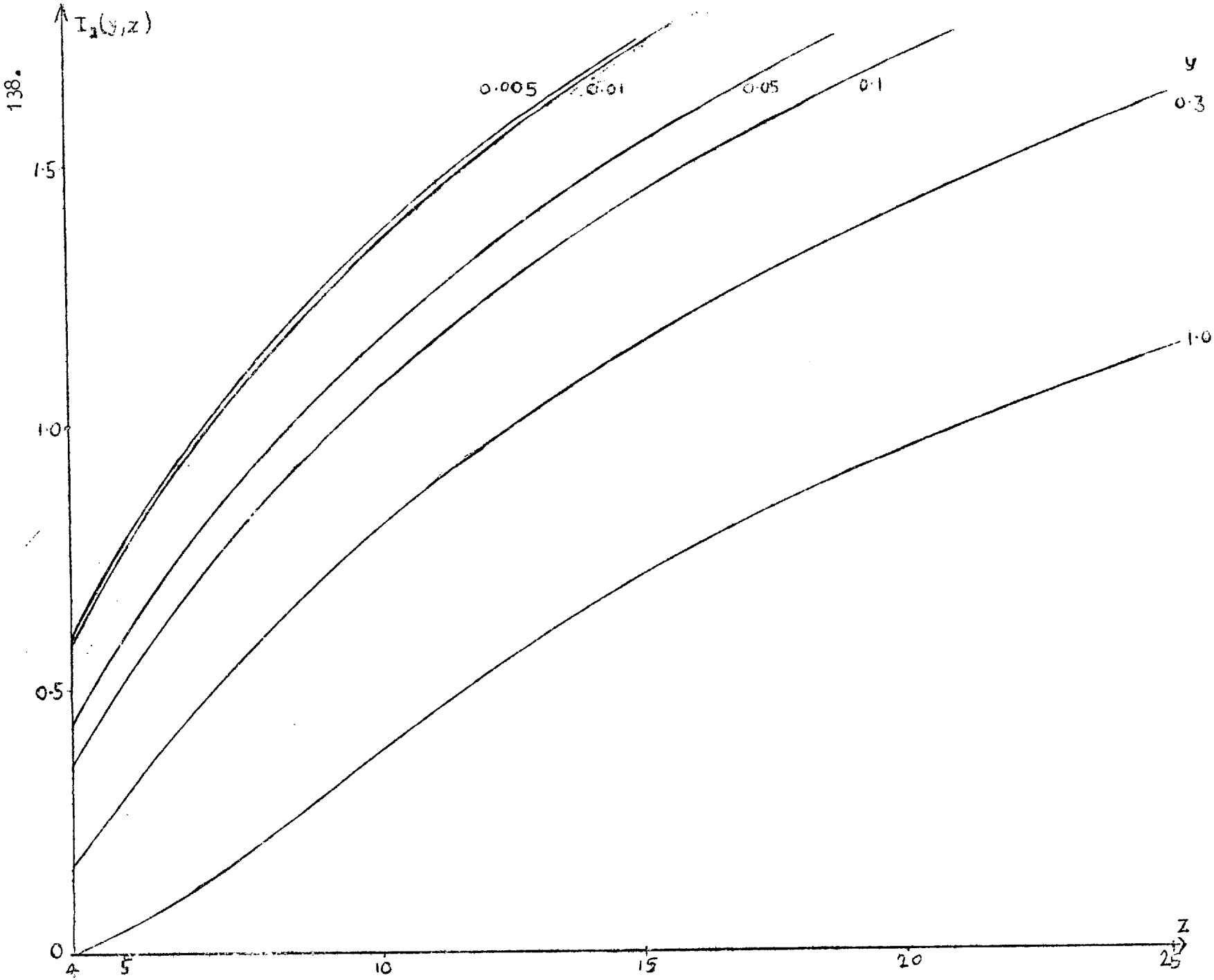
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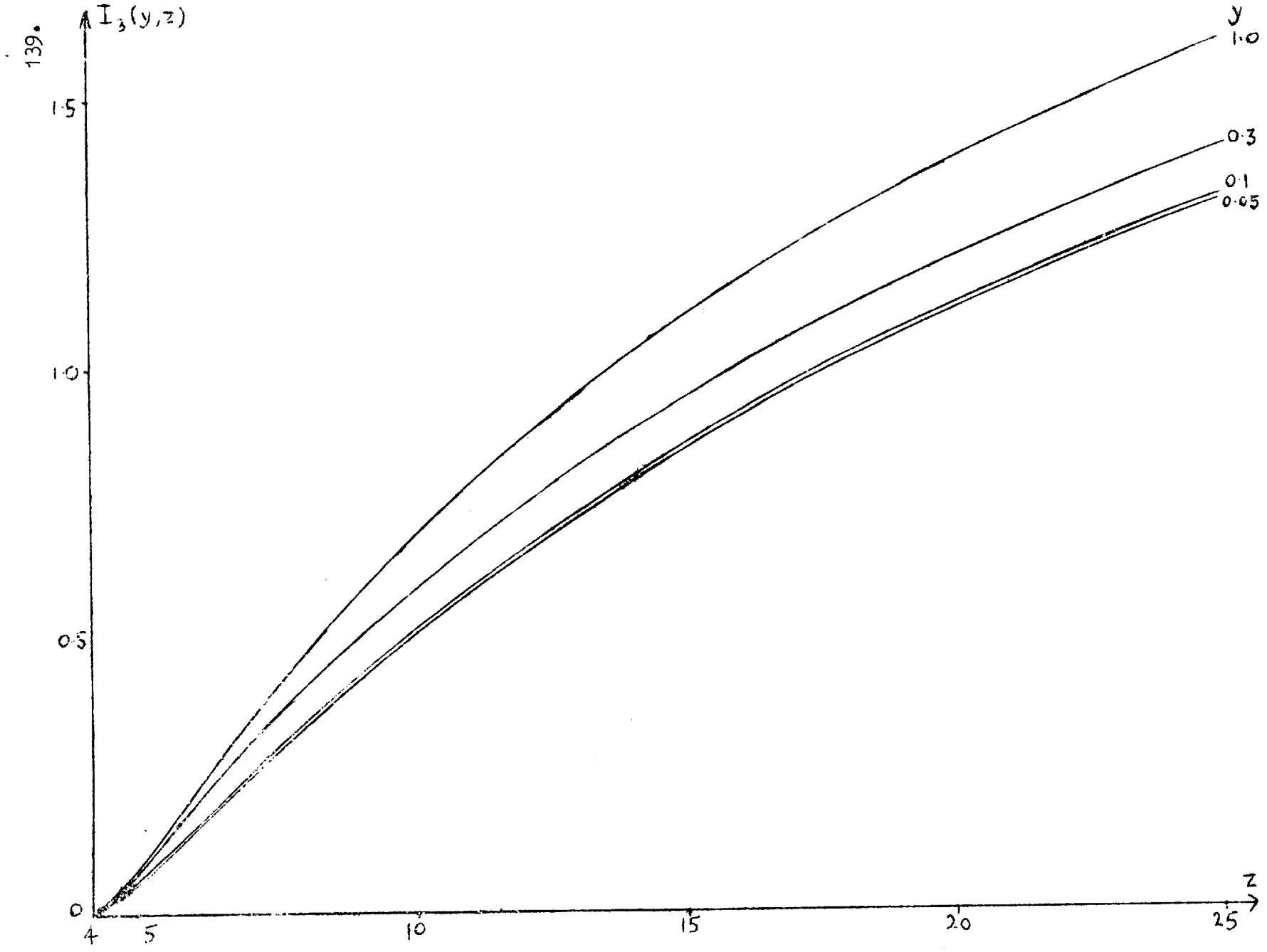
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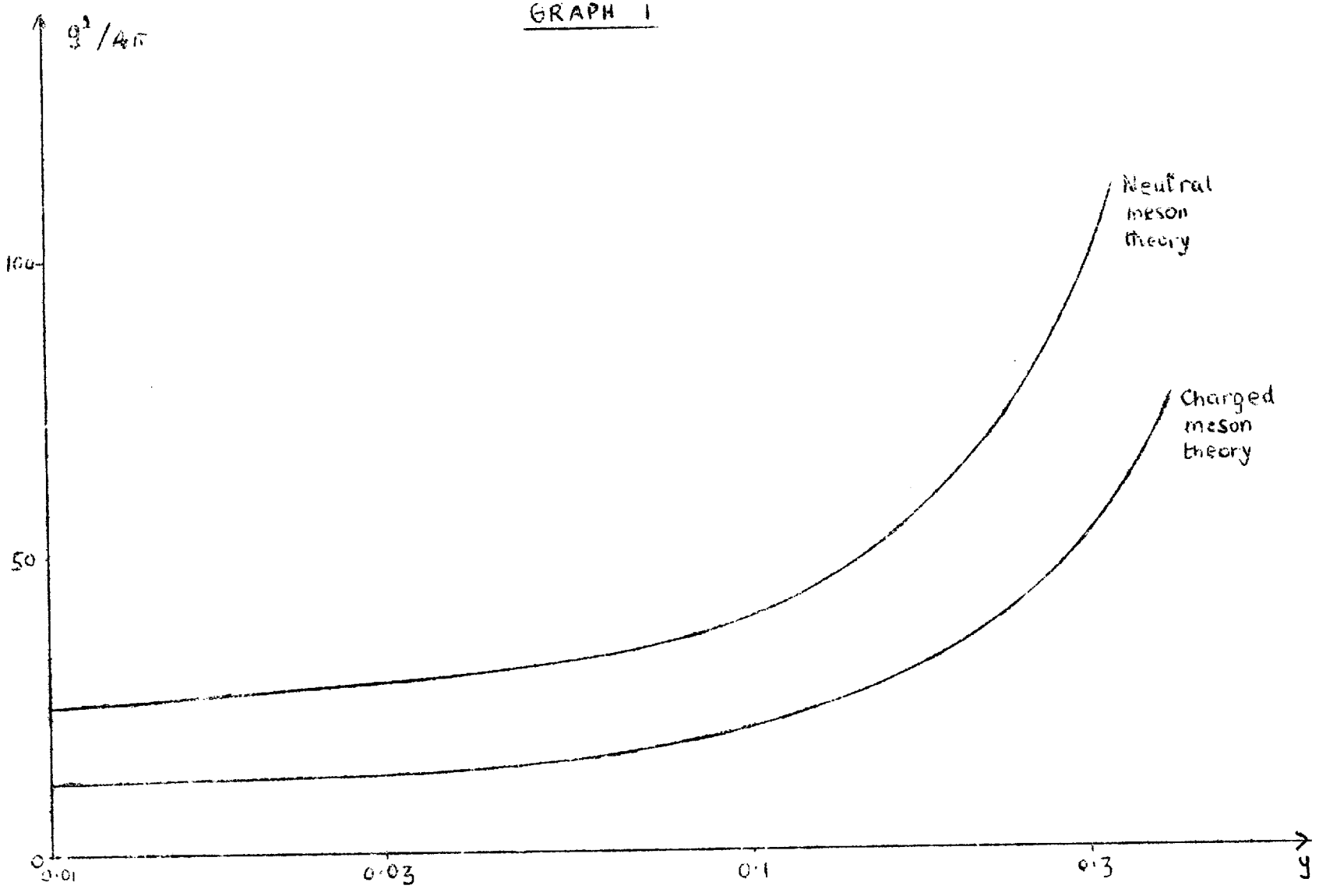
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$I_1(y,z)$
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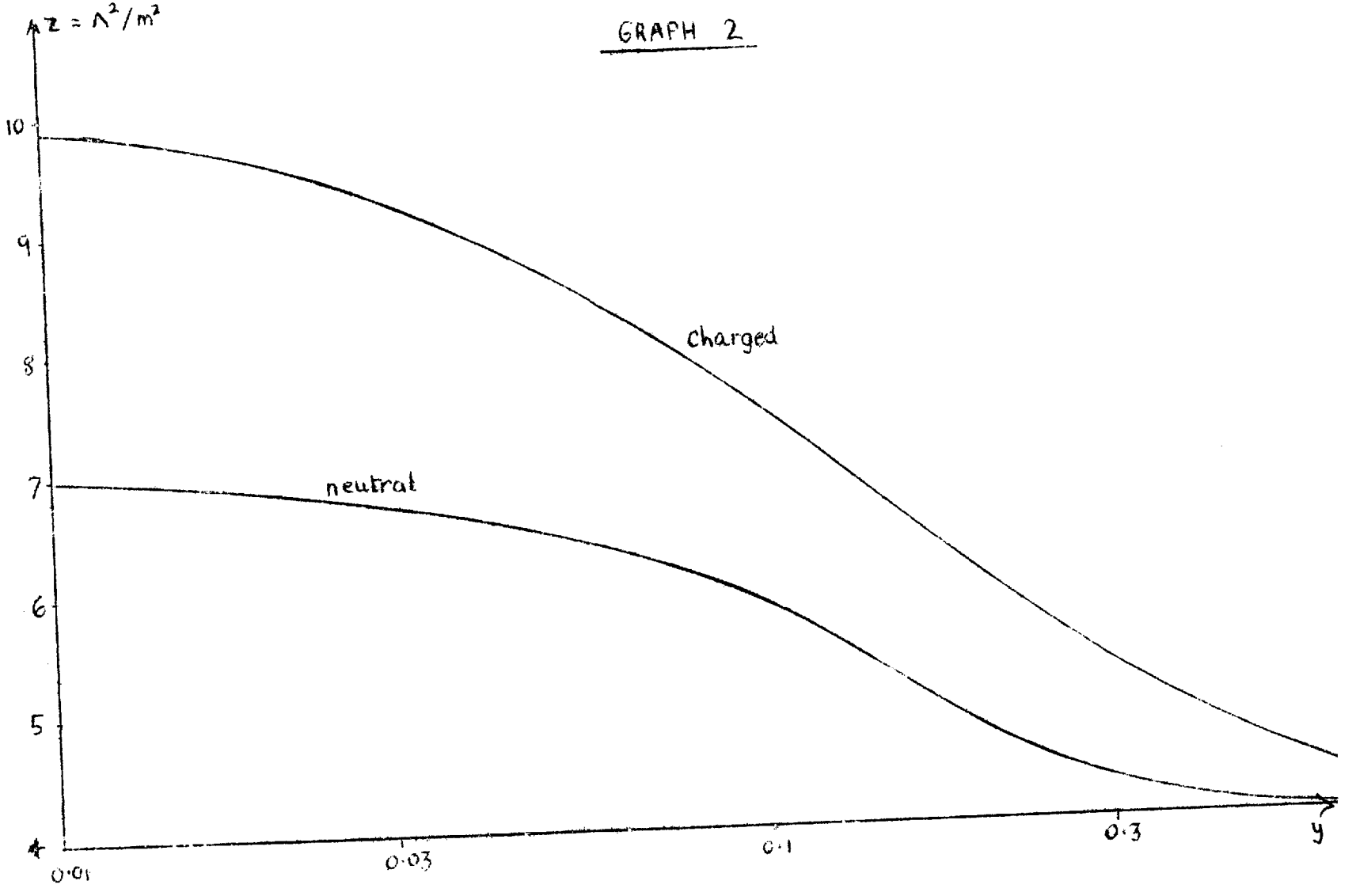

$I_2(y,z)$
y
0.005
0.01
0.05
0.1
0.3
1.0
1.5
1.0
0.5
0
4
5
10
15
20
25
z

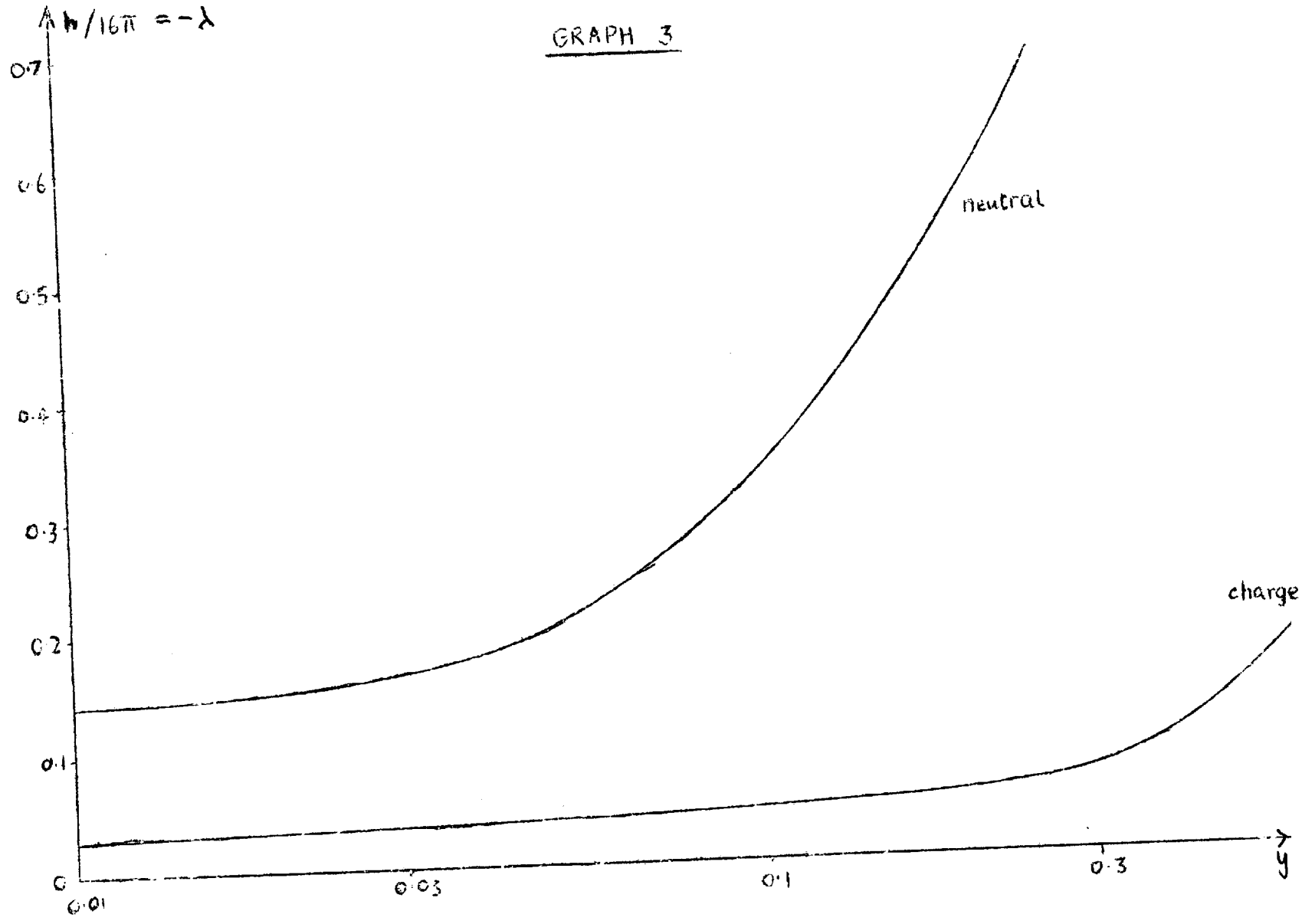

139.
$I_3(y,z)$
y
1·0
0·3
0·1
0·05
1·5
1·0
0·5
0
z
4
5
10
15
20
25

GRAPH 1

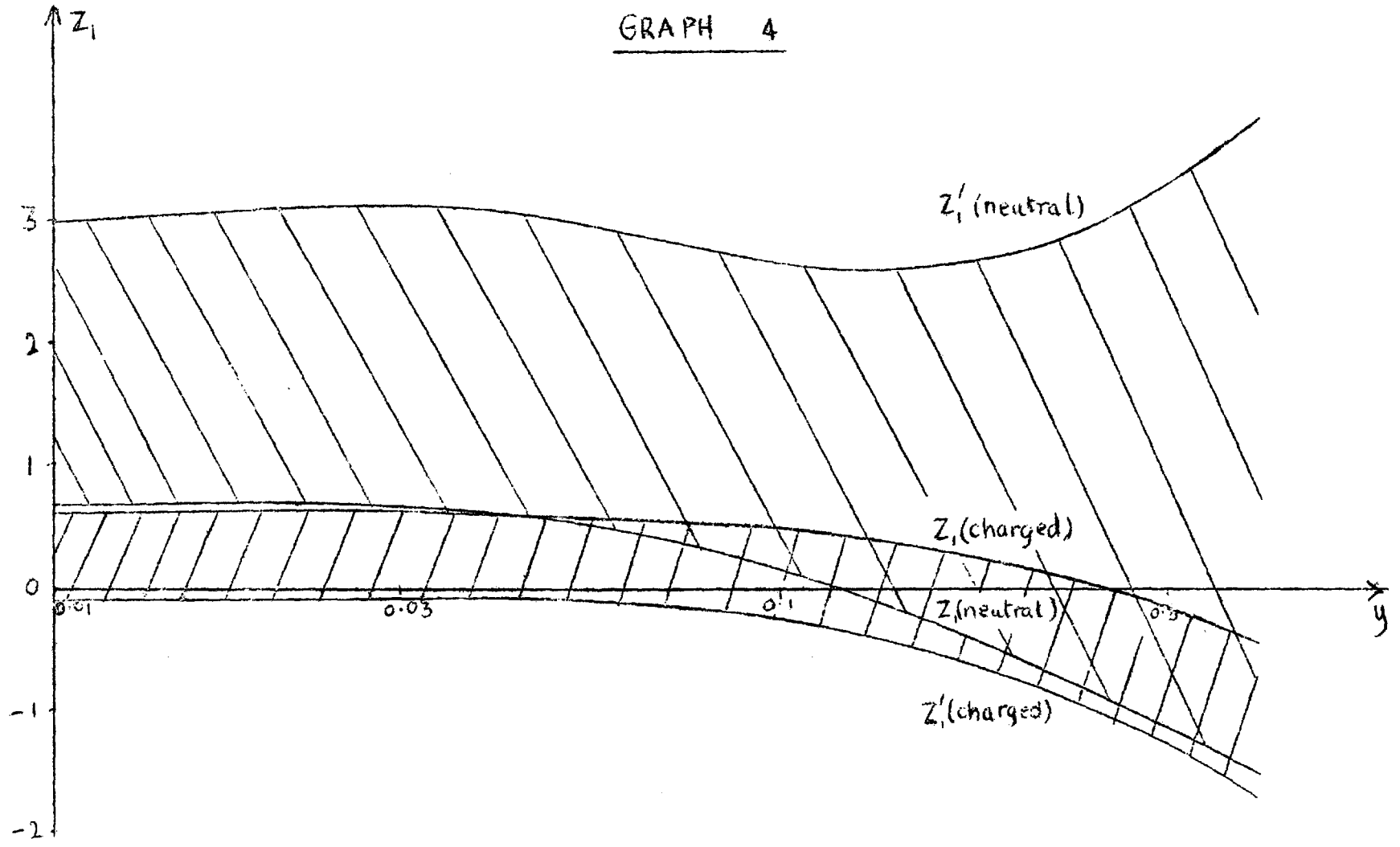

$g^2/4\pi$
100
50
0
0·01
0·03
0·1
0·3
y
Neutral meson theory
Charged meson theory

GRAPH 2




GRAPH 3
h/16π = −λ
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
0·01
0·03
0·1
0·3
y
neutral
charge

GRAPH 4


Z_1
3
2
1
0
-1
-2
0·01
0·03
0·1
0·3
y
Z_1' (neutral)
Z_1 (charged)
Z_1 (neutral)
Z_1' (charged)

R. DeLbourgo. Ph.D. 1963.

8.B

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FINAL STATE SCATTERING

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Abstract: Increasing evidence in recent experiments suggests the importance of interactions among the final particles in various production processes. The final state interactions may or may not be strong, and will in general take place among all the pairs. Watson's theorem has therefore been generalized by perturbation theory based on 'total potentials' to deal with such situations. A parameter $c \approx 1$ is found in this way, which characterises the 'amount' of final state scattering and which theoretically is a measure of the importance of the virtual processes in relation to the physical process. The result is applied to $K^{\pm}$ decay and two classes of solutions are found on the assumption of S-wave production of the pions: (i) $a_2 \approx 1.5$, $a_0 \approx -0.5$ and (ii) $a_2 \approx 0$, $a_0 \approx -0.5$, where the a are scattering lengths in the isospin states $T = 0$ and $T = 2$.

1. Introduction

Watson [1]) has given an expression for the effect of final state interactions on reaction cross-sections, basing his argument on the assumption that the final state force is strong in relation to the production process and takes place attractively between only one particular pair of particles. Then, since it is effectively a one-channel process, unitarity gives the matrix element to be

$$M_{fi} \propto e^{i\delta} \sin \delta/q \tag{1}$$

at low energies, where δ is the S-wave phase shift for the scattering between the pair, and q is their relative momentum. However, eq. (1) is inadequate when the attraction is not so pronounced, or even if the interaction is repulsive. Moreover, it is not immediately obvious how to apply this formula to more than one interacting pair.

In this paper we start with the non-covariant form of perturbation theory, and by introducing 'total potentials' among sets of particles, we obtain a series decomposition of the net scattering among all the particles in terms of separate total scatterings. (The method is also suitable for the description of reactions involving an initial bound state particle.) In this way, it becomes an easy matter to decompose diagrams in any manner one pleases, and it is hoped that the technique for doing so will prove generally useful.

We then specialise to the case of final state scattering, neglecting the antiparticle diagrams, and derive an expression for the matrix element of a production process which includes the interaction among two spinless particles in the final state. This

expression exhibits the general features we would expect at low energies:

$$M_{fi} = M(\text{production})\left[1+\frac{ce^{i\delta}\sin\delta}{q}\right], \tag{2}$$

where $c \approx 1$ and is a measure of how important are the virtual processes in relation to the real process. For instance, if the final state scattering is large, and the variation of M (production) is neglected, eq. (2) just reduces to eq. (1). On the other hand, in the limit of small scattering $M_{fi} = M$ (production) as we expect.

In the next section we calculate covariantly the effect of final state interaction – this would correspond to the inclusion of the antiparticle diagrams in the non-covariant theory – and derive the most general expression in terms of helicity amplitudes. The form is essentially the same as eq. (2) including the presence of a factor c.

Finally, we consider the simple example of $K^{\pm}$ meson decay and assume the observed spectrum to be due solely to the S-wave final state scattering among the pions, evaluated to lowest order. Characterising this scattering by a length a_T in the isospin state T, we find that it is possible to obtain a good agreement with experiment with three sets of solutions:

$$\text{(i)} \qquad a_2 \approx 1.7\pm0.5, \qquad a_0 \approx -0.7\mp0.5,$$

$$\text{(ii)} \qquad a_2 \approx 0\pm0.1, \qquad a_0 \approx -0.3\mp0.1 \text{ or } -0.7\pm0.1.$$

Set (i) means a strong $\pi^+-\pi^+$ attraction with almost no $\pi^+-\pi^-$ interaction [9-11]), while set (ii) means very small $\pi^+-\pi^+$ interaction and moderate $\pi^+-\pi^-$ repulsion [7, 8]).

2. Non-Covariant Theory

It is well-known that the Lippmann-Schwinger formalism [2]) for the description of scattering states yields for the S-matrix element between the free states f and i

$$S_{fi} = \delta_{fi}-2\pi i\delta(E_i-E_f)R_{fi} = \delta_{fi}-i(2\pi)^4\delta^4(k_f-k_i)M_{fi}, \tag{3}$$

$$\langle f|R|i\rangle = {}_-\langle f|V|i\rangle = \langle f|V|i\rangle_+, \tag{4}$$

where $|\psi\rangle_\pm$ denotes scattering eigenstates of the total Hamiltonian H(+ denotes an in-state while – denotes an out-state) and $V \equiv H-H_0$ is the total interaction. It follows from the Lippmann-Schwinger equation [2]) that

$$R = V+R\Delta V \qquad \text{or } V = R-R\Delta V, \tag{5}$$

where $\Delta = (E-H_0+i\varepsilon)^{-1}$ is the free causal propagator. In fact (5) will form the basis of all our work as it gives R in terms of V and vice versa, up to any given order, the order of an expression being defined as the number of times in which Δ appears.

Suppose that it is possible to make the decomposition $V = V_0+V_1$. We have when $V_1 = 0$,

$$R = R_0 = V_0+V_0(E-H_0-V_0+i\varepsilon)^{-1}V_0, \tag{6a}$$

when $V_0 = 0$,

$$R = R_1 = V_1 + V_1(E - H_0 - V_1 + i\varepsilon)^{-1} V_1, \tag{6b}$$

but, in general, of course,

$$R = (V_0 + V_1) + (V_0 + V_1)(E - H_0 - V_0 - V_1 + i\varepsilon)^{-1}(V_0 + V_1). \tag{6c}$$

We wish to express the full scattering R in terms of the two independent free scatterings R_0 and R_1, and this we can do via (5) up to any chosen order. Firstly, we write $R = V + V\Delta V + \ldots$, put $V = V_0 + V_1$ and then substitute $V_0 = R_0 - R_0\Delta R_0 + \ldots$, $V_1 = R_1 - R_1\Delta R_1 + \ldots$. For example up to second order,

$$R = R_0 + R_1 + R_1\Delta R_0 + R_0\Delta R_1 + R_0\Delta R_1\Delta R_0 + R_1\Delta R_0\Delta R_1 \tag{7}$$

and it is clear that in general and R_1 must follow an R_0, and vice versa. This is shown diagrammatically in fig. 1(a).

In order to allow our scheme to cover as many examples as possible, we must include the possibility of our initial and final states involving a bound state particle such as a deuteron [3,4]). Let v be the binding potential, and take $|\psi_b\rangle$ to represent plane wave states involving the deuteron, and $|\psi_b\rangle_\pm$ to be in- and out-states involving it. We then get for an 'elastic' process in which the initial and final states both contain the deuteron,

$$S_{fi} = {}_-\langle f_b | i_b \rangle_+ = \delta_{fi} - 2i\pi\delta(E_i - E_f)\langle f_b | R_{el} | i_b \rangle, \tag{8a}$$

$$R_{el} = V + V(E - H_0 - v - V + i\varepsilon)^{-1} V. \tag{8b}$$

The 'inelastic' amplitude for a process in which the final state contains the disintegrated deuteron is given by

$$S_{fi} = {}_-\langle f | i_b \rangle_+ = \delta_{fi} - 2i\pi\delta(E_i - E_f)\langle f | R_{inel} | i_b \rangle, \tag{9a}$$

$$R_{inel} = V + (V + v)(E - H_0 - V - v + i\varepsilon)^{-1} V. \tag{9b}$$

We express R_{el} and R_{inel} in terms of R and r (corresponding to separate interactions of V and v respectively) by the same method as before, and obtain the series of diagrams in figs. 1(b), (c).

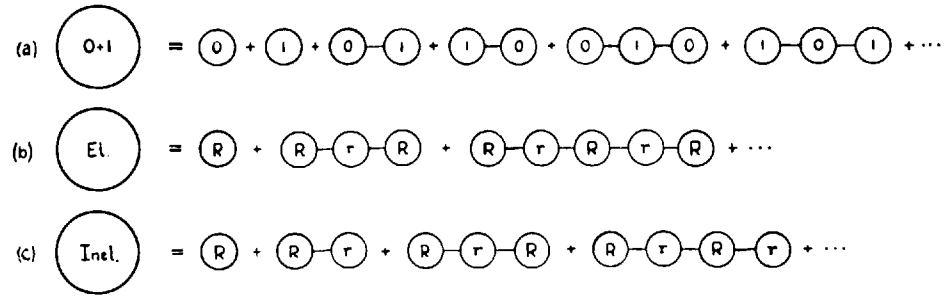


Fig. 1. Decomposition of total scattering in terms of separate net scatterings.

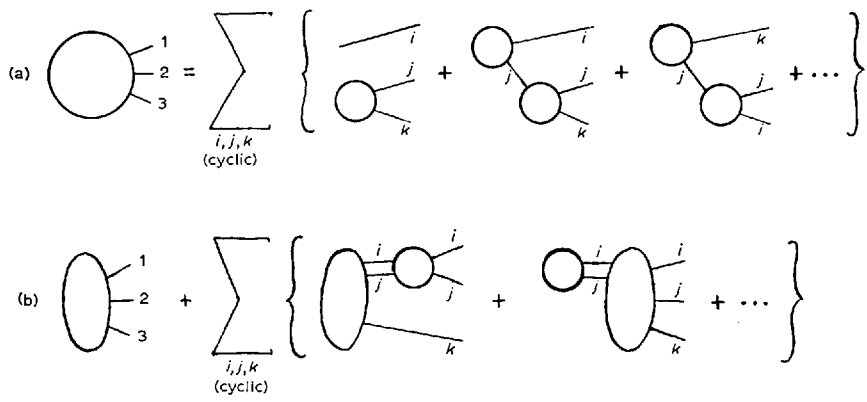


Fig. 2. Decomposition of 3-particle scattering into separate total scatterings among the 3 particles.

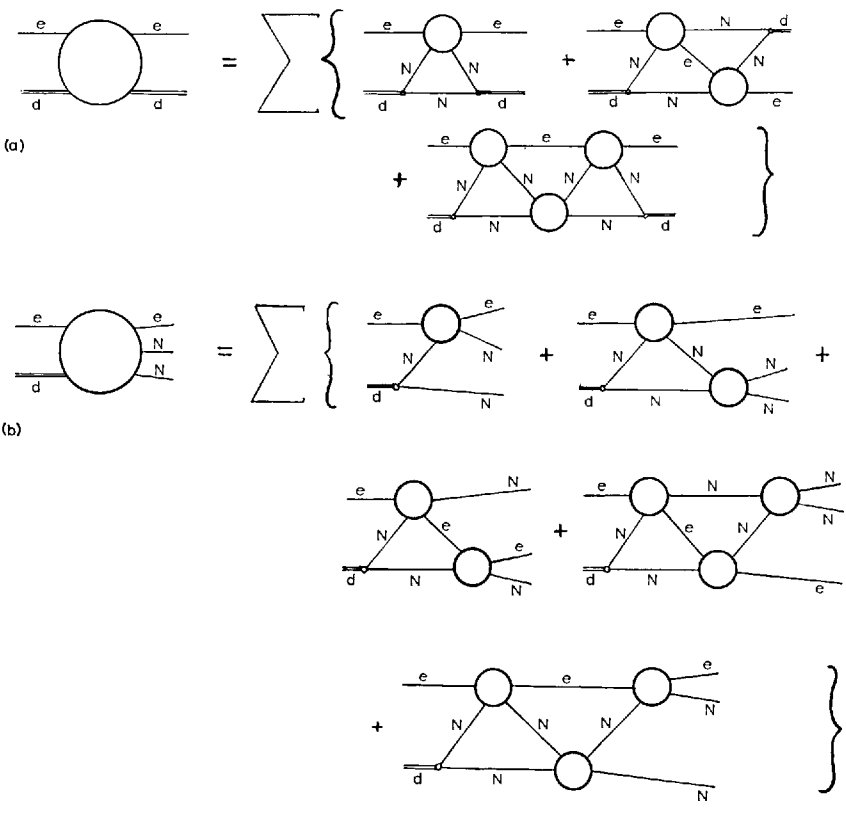


Fig. 3. Decomposition of electron-deuteron scattering into a series of electron-nucleon and nucleon-nucleon scatterings.

Finally, we complete the discussion of the decomposition technique by considering simultaneous interactions among pairs of particles. For simplicity we take the case of 3 particles and suppose that

$$V = V_{12} + V_{23} + V_{31} + V_{123}. \tag{10}$$

The statement is made precise by saying that V_{ij} is the net interaction acting between particles i and j only, so that V_{123}, the remainder, can be regarded as a '3-particle interaction'. Neglecting V_{123} for the moment, and applying the 0-1 method with two-particle scatterings, we obtain the diagrams of fig. 2(a). Were we to include V_{123}, we should have to add the diagrams of fig. 2(b) to those of 2(a).

We illustrate all these methods at once by considering elastic and inelastic electron-deuteron scattering: v will be the p-n binding potential and V_n and V_p are the electron-neutron and electron-proton potentials respectively. Since v is strong and V_N is electromagnetic, $V_N \approx \frac{1}{137} v$, so that up to second order in electromagnetic interactions, we derive the diagrams of figs. 3(a) and 3(b). In those figures the symbol $\sum$ denotes a sum over all the different permutations of the two possible nucleon states n and p. In particular we note that the inelastic process is essentially just a final scattering of the emergent particles. (These diagrams have been obtained by Ferreira [4]), but this method is rather more tedious).

3. Final State Interactions

We designate the inital particles by A, B, ... and the final ones by 1, 2, 3, ... The total interaction V must be capable of annihilating the initial configuration and creating the final configuration. We make the decomposition of V into V_0 and V_1 precise by requiring that V_1 be the total interaction that acts only between the final particles, so that the remainder V_0 can be regarded as the production potential. To be definite, let us take the initial set of particles to be different from the final set, so that we must allow only for the part of V_0 which annihilates A, B, ... and creates 1, 2, 3, ..., while V_1 annihilates 1, 2, 3, ... and recreates 1, 2, 3, ... These conditions apply by and large to the R, so that the only terms that contribute are $R_0 + R_1 \Delta R_0$.

Now

$$\langle f|\{1 + R_1 \Delta\} R_0|i\rangle = -\langle f_1|V_0|i_0\rangle_+, \tag{11}$$

where $|\psi_1\rangle_\pm$ is the in- or out- state in absence of V_0, while $|\psi_0\rangle_\pm$ is for $V_1 = 0$. Eq. (11) is just the term which Watson uses and it represents scattering from an in-state of no final state interaction to an out-state of no primary interaction.

The first non-trivial situation in which it might be possible to detect the effect of V_1, is a final state consisting of three particles, when V_1 can be broken up into specific pairs of potentials; if these pairs interact differently in any way, these differences must show up kinematically. The potential V_{123} will be neglected because we are

only interested in fine structure at a given incident energy, whereas we can expect its effects to become evident only in measurements of the cross-section as a function of the total energy of the system. Therefore, to first order in final state scattering the decomposition (10) leads to the fig. 4 diagrams.

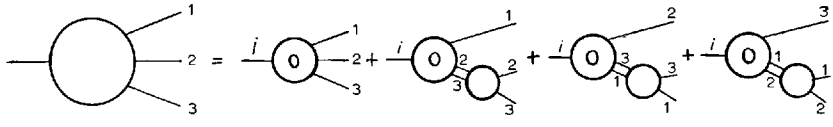


Fig. 4. Final state scattering among 3 particles evaluated to lowest order.

Unlike Watson [1]), who proceeds essentially to use unitarity and considers only the last diagram because he is only interested in the strong attraction limit of 1 and 2, we carry on with this summed-up perturbation theory and proceed to estimate the effect of the final state scatterings.

Typically, we shall have to evaluate $\langle 123|R_{12}\Delta R_0|\mathrm{AB}\rangle$. Introducing a complete set of intermediate states of particles 1 and 2, we find

$$M_{\mathrm{fi}} = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k_1'}{2E_1'}\frac{\mathrm{d}^3k_2'}{2E_2'}\,\delta^3(k_1'+k_2'-k_1-k_2)\,\frac{\langle k_1k_2|M_{12}|k_1'k_2'\rangle\langle k_1'k_2'k_3|M_0|\mathrm{AB}\rangle}{E-(E_3+E_1'+E_2')+i\varepsilon}. \tag{12}$$

In this form of theory each particle lies on the mass shell and three-momentum but not energy is conserved in intermediate processes. As usual, we express our results in terms of the total and relative momenta

$$\boldsymbol{p}' = \boldsymbol{k}_1'+\boldsymbol{k}_2', \qquad 2\boldsymbol{q}' = \boldsymbol{k}_1'-\boldsymbol{k}_2'. \tag{13}$$

Evaluating in the barycentric frame of 1 and 2, where $\boldsymbol{p} = 0$, we obtain

$$\begin{aligned} M_{\mathrm{fi}} &= \frac{1}{(2\pi)^3}\int \frac{M_{12}(\boldsymbol{q};\boldsymbol{q}')M_0(\boldsymbol{q}_1', k_3; k_{\mathrm{i}})}{E_1+E_2-E_1'-E_2'+i\varepsilon}\,\frac{\mathrm{d}^3\boldsymbol{q}'}{4E_1'E_2'} \\ &= \frac{1}{(4\pi)^3}\int \frac{M_{12}(\omega,\theta;\omega',\theta')M_0(\omega',\theta',k_3;k_{\mathrm{i}})}{\omega-\omega'+i\varepsilon}\,\frac{2q'\mathrm{d}\omega'\mathrm{d}\Omega'}{\omega'}, \end{aligned} \tag{14}$$

where ω is the energy of 1 and 2 in this frame (ω' in the intermediate state) and θ denotes the orientation (θ, ϕ) of $\boldsymbol{q}$ with respect to a fixed axis.

At this point we make a partial wave decomposition to bring out the angular dependences. For simplicity we take 1 and 2 to be spinless so that orbital momentum is conserved. The amplitude M_{12} will be independent of the orbital component m because of its invariance under rotations of (12); however, M_0 will depend on m because the final state of three particles need not be invariant under the separate rotations of (12):

$$M_{12}(\theta,\theta') = \sum_{l,m}(2l+1)D^{l*}_{m0}(\theta)D^{l}_{m0}(\theta')M_{12l}, \tag{15a}$$

$$M_0(\theta') = \sum_{l,m} (2l+1)^{\frac{1}{2}} D^{l*}_{m0}(\theta') M_{0lm}. \tag{15b}$$

Hence (14) simplifies to

$$M_{fi} = \frac{1}{(4\pi)^2} \sum_{l,m} (2l+1)^{\frac{1}{2}} D^{l*}_{m0}(\theta) I_{lm}, \tag{16a}$$

where

$$I_{lm} = \int \frac{M_{12l}(\omega, \omega') M_{0lm}(\omega', k_3; k_i)}{\omega - \omega' + i\varepsilon} \frac{2q' \mathrm{d}\omega'}{\omega'}. \tag{16b}$$

The real problem arises as to how to evaluate I. Because the off-shell elements are determined in some way by the on-shell elements, we write for each partial wave

$$I = M_{12}(\omega, \omega) M_0(\omega) \int \frac{M_{12}(\omega, \omega') M_0(\omega') / M_{12}(\omega, \omega) M_0(\omega)}{\omega - \omega' + i\varepsilon} \frac{2q' \mathrm{d}\omega'}{\omega'} \tag{16c}$$

and the numerator of the integrand is now dimensionless. If we suppose that the off-shell elements become smaller the more unphysical becomes the intermediate scattering, then the numerator will act like a cut-off function. Because we do not know how rapidly or in what manner it cuts off, we may regard this ratio as unity up to ω_c after which it is zero. This sort of point interaction model has already been adopted, more or less; see ref. [4]) for example.

The δ-function just gives us the physical contribution. Making a subtraction at threshold, $\omega = m_1 + m_2$, for the principal integral, we get

$$I = -M_{12}(\omega, \omega) M_0(\omega) \left[\frac{2iq}{\omega} + \pi c + \mathrm{P} \int \frac{\omega - m_1 - m_2}{(\omega' - m_1 - m_2)(\omega' - \omega)} \frac{2q' \mathrm{d}\omega'}{\omega'} \right], \tag{17a}$$

with

$$c \approx \frac{\mathrm{P}}{\pi} \int \frac{2q' \mathrm{d}\omega'}{\omega'(\omega' - m_1 - m_2)}. \tag{17b}$$

The integrals can be evaluated to a good approximation and we obtain

$$I \approx -M_{12}(\omega, \omega) M_0(\omega) \left[\frac{2iq}{\omega} + \pi c + \frac{2q}{\omega} \log \left(\frac{\omega + 2q}{\omega - 2q} \right) \right], \tag{18a}$$

with

$$c \approx \frac{1}{\pi} \cosh^{-1} \left[1 + \frac{16 m_1 m_2 (\omega_c - m_1 - m_2)}{(m_1 + m_2)^3} \right]. \tag{18b}$$

In the limit of low energies $I \to -\pi c M_{12}(\omega, \omega) M_0(\omega)$ and we believe this answer to be very good as it should be independent of the shape of cut-off then. Therefore, all in all, in the low energy limit for the case when 1 and 2 interact

$$M_{fi} = \sum_{l,m} (2l+1)^{\frac{1}{2}} D^{1*}_{m0}(\theta) \left[1 + c_{lm} \frac{e^{i\delta_l} \sin \delta_l}{2q/\omega} \right]. \tag{19}$$

The physical significance of (19) is quite clear: there is interference between the unity of the production amplitude and the final state scattering. The quantity c can be regarded as the amount of final state scattering that takes place.

The method of derivation of (19) is in fact independent of the strength of the scattering and of whether it is attractive or repulsive. Moreover it will give the correct answer in both the weak and strong interaction limits as remarked in the introduction. The only parameter that occurs is c, and this is easily estimated to be about 1 by invoking a cut off ω_c of the order of the nucleon mass for the case of strong processes, when one of the particles is a pion.

Retaining only the S-waves and characterising those by a scattering length a, we get

$$M(\omega) = M_0(\omega)\left[1+\frac{ca}{1-ia(2q/\omega)}\right]. \tag{20}$$

If we suppose that the spectrum shape is governed mainly by the final state scattering at these low energies so that M_0 can be taken to be structureless,

$$|M(\omega)|^2 \propto \frac{(1+ca)^2+a^2(2q/\omega)^2}{1+a^2(2q/\omega)^2}. \tag{21}$$

Figs. 5(a) and 5(b) show the spectrum shapes for $a > 0$ and $a < 0$ respectively, and in principle it should be possible to find a and c from accurate low energy experiments.

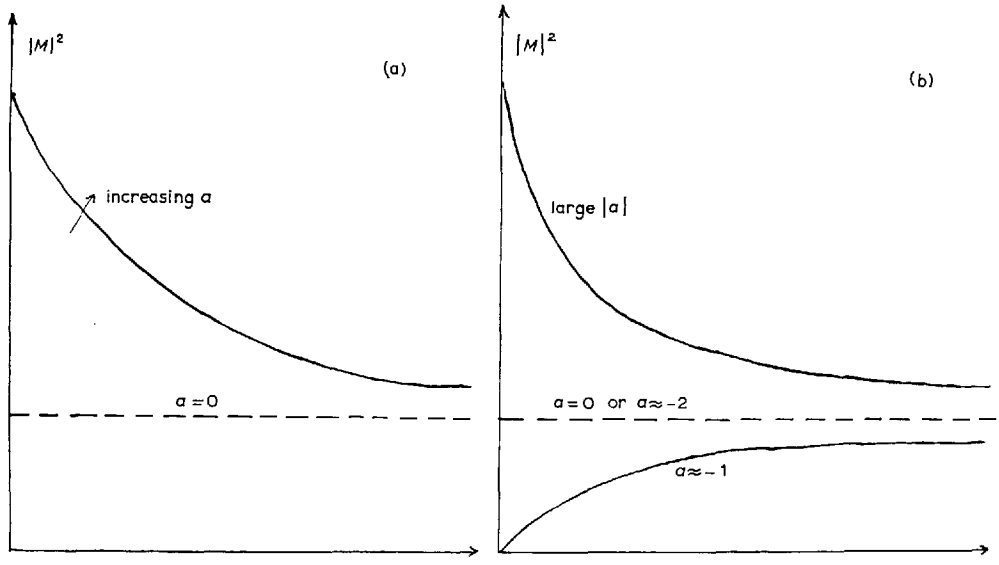


Fig. 5. The square of the amplitude $|M|^2$ as a function of the energy of the interacting particles in their barycentric frame, (a) for an attractive final state interaction, and (b) for a repulsive final state interaction.

4. Covariant Calculation

We now proceed to calculate covariantly the effect of final state scattering and we shall arivc at essentially the same result as (19). Carrying out this covariant calculation will be the same as including the antiparticle diagrams in the non-covariant theory. Let n and m in fig. 6 designate the two intermediate particles when we separate out

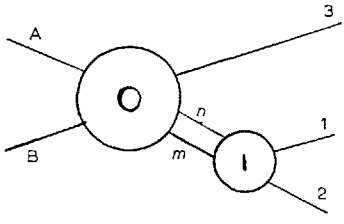


Fig. 6. Final state interaction between 12 and nm.

the potential between 12 nm and the remainder. As yet we do not choose them to be 1 and 2 in order that our result be applicable to $\Sigma\pi$ intermediate states in $\Lambda\pi$ final state interactions, say. Replacing the operator Δ of the non-covariant theory by the two propagators $(p^2-m^2+i\varepsilon)^{-1}$ we obtain covariantly

$$M_{\mathrm{fi}} = -i\int \frac{\mathrm{d}^4p_{\mathrm{n}}}{(2\pi)^4}\frac{\mathrm{d}^4p_{\mathrm{m}}}{(2\pi)^4}\frac{i\mathscr{M}(p_{\mathrm{n}}^2,\, p_{\mathrm{m}}^2)}{(p_{\mathrm{n}}^2-m_{\mathrm{n}}^2+i\varepsilon)(p_{\mathrm{m}}^2-m_{\mathrm{m}}^2+i\varepsilon)}(2\pi)^4\delta^4(p_{\mathrm{n}}+p_{\mathrm{m}}-p_1-p_2), \quad (22\mathrm{a})$$

where

$$\mathscr{M}(p_{\mathrm{n}}^2,\, p_{\mathrm{m}}^2) = f_1^{\dagger}f_2^{\dagger}M_1(12,\, p_{\mathrm{n}}^2p_{\mathrm{m}}^2)f_3^{\dagger}M_0(3p_{\mathrm{n}}^2p_{\mathrm{m}}^2,\, \mathrm{AB})f_{\mathrm{A}}f_{\mathrm{B}}. \quad (22\mathrm{b})$$

The quantity f_i is the free field solution of particle i (having given quantum indices) and the M will be matrices with attached vector indices, in general. Their unphysicality in m and n is expressed by the labels p_{n}^2 and p_{m}^2. Such factors as $(\gamma\cdot p+m)$, ... have been absorbed into the M.

To treat the problem symmetrically, use

$$p = p_{\mathrm{n}}+p_{\mathrm{m}}, \qquad 2q = p_{\mathrm{n}}-p_{\mathrm{m}}. \quad (23)$$

We find

$$M_{\mathrm{fi}} = i\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathscr{M}(p_{\mathrm{n}}^2,\, p_{\mathrm{m}}^2)_{k_1=k_{\mathrm{f}}}}{[(\frac{1}{2}p+q)^2-m_{\mathrm{n}}^2+i\varepsilon][(\frac{1}{2}p-q)^2-m_{\mathrm{m}}^2+i\varepsilon]}. \quad (24\mathrm{a})$$

Retaining the δ functions from the denominators in (24) we should have a physical process with

$$(M_{\mathrm{fi}})_{\mathrm{physical}} = -i\int \frac{\mathrm{d}^4q}{(4\pi)^2}\mathscr{M}(m_{\mathrm{n}}^2,\, m_{\mathrm{m}}^2)\delta[(\tfrac{1}{2}p+q)^2-m_{\mathrm{n}}^2]\delta[(\tfrac{1}{2}p-q)^2-m_{\mathrm{m}}^2].$$

The integral is easily evaluated through the use of 4-dimensional polar co-ordinates:

$$(M_{\mathrm{fi}})_{\mathrm{physical}} = \frac{i[\frac{1}{4}(p^2-m_{\mathrm{n}}^2-m_{\mathrm{m}}^2)^2-m_{\mathrm{n}}^2m_{\mathrm{m}}^2]^{\frac{1}{2}}}{16\pi p^2}\int \frac{\mathrm{d}\Omega_{\mathrm{mn}}}{4\pi}\mathscr{M}(m_{\mathrm{n}}^2,\, m_{\mathrm{m}}^2), \quad (25)$$

which, of course, just corresponds to an integration over intermediate scattering angles. To denote this average value we put

$$\bar{\mathscr{M}} \equiv \int \frac{\mathrm{d}\Omega_{mn}}{4\pi} \mathscr{M}(m_n^2, m_m^2). \tag{26}$$

The problem of evaluating the actual element (24) is tackled in much the same spirit as in the non-covariant theory. Apart from an integration over intermediate angles, M_{fi} involves an integration over off-mass-shell elements. We expect the result to be determined in some way by the mass-shell elements and for that reason, we write

$$M_{fi} = i\bar{\mathscr{M}} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\mathscr{M}(p_n^2, p_m^2)/\bar{\mathscr{M}}}{[(\frac{1}{2}p+q)^2 - m_n^2 + i\varepsilon][(\frac{1}{2}p-q)^2 - m_m^2 + i\varepsilon]} \equiv i\bar{\mathscr{M}} I(p^2). \tag{24b}$$

Again, we are integrating over a dimensionless quantity. The ratio $\mathscr{M}(p_n^2, p_m^2)/\bar{\mathscr{M}}$ is a measure of the extent to which the unphysical compare with the physical elements and is unknown in general. We may expect it to decrease the more unreal becomes the intermediate scattering and we adopt the point interaction model whereby this ratio is unity up to a cut-off $q = \Lambda$. In this range, the M can be regarded as renormalised coupling constants for the primary and secondary interactions.

We calculate $I(p^2)$ in (24b) in the normal way by introducing Feynman variables and express the answer in terms of the threshold value $I((m_n+m_m)^2)$:

$$I(p^2) = I((m_n+m_m)^2) + \frac{i}{16\pi^2}\left[\frac{m_n-m_m}{m_n+m_m}\frac{p^2-(m_n+m_m)^2}{p^2}\log\left(\frac{m_n}{m_m}\right) + \frac{C}{2p^2}\log\left(\frac{p^2-m_n^2-m_m^2-C}{p^2-m_n^2-m_m^2+C}\right)\right], \tag{27a}$$

where

$$I((m_n+m_m)^2) = \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m_m m_n} + 1 + \frac{m_m-m_n}{m_m+m_n}\log\left(\frac{m_n}{m_m}\right)\right], \tag{27b}$$

$$C^2 = (p^2-m_n^2-m_m^2) - 4m_n^2 m_m^2. \tag{27c}$$

Naturally $I(p^2)$ is symmetrical in m and n.

For the special case of final state scattering n = 1 and m = 2. In the barycentric frame of 1 and 2 let ω_{12} be their total energy, and q_{12} their relative momentum. Since $C = 2[(p_1\cdot p_2)^2 - m_1^2 m_2^2]^{\frac{1}{2}} = 2q_{12}\omega_{12}$, we have

$$I(\omega_{12}^2) = I((m_1+m_2)^2) + \frac{i}{16\pi^2}\left[\frac{m_1-m_2}{m_1+m_2}\frac{\omega_{12}^2-(m_1+m_2)^2}{\omega_{12}^2}\log\left(\frac{m_1}{m_2}\right) + \frac{q_{12}}{\omega_{12}}\log\left(\frac{\omega_{12}^2-m_1^2-m_2^2-2q_{12}\omega_{12}}{\omega_{12}^2-m_1^2-m_2^2+2q_{12}\omega_{12}}\right)\right], \tag{28a}$$

where

$$I((m_1+m_2)^2) = \frac{i}{16\pi^2}\left[\log\frac{\Lambda^2}{m_1 m_2} + 1 + \frac{m_2-m_1}{m_2+m_1}\log\left(\frac{m_1}{m_2}\right)\right]. \tag{28b}$$

Therefore near threshold, where we expect our result to be generally valid, eqs. (24b) and (28a) give

$$M_{\mathrm{fi}} \approx iI((m_1+m_2)^2)\int \frac{\mathrm{d}\Omega'_{12}}{4\pi}\,\mathscr{M}(m_1^2, m_2^2). \tag{29}$$

In order to make the comparison with the non-covariant theory more obvious it is necessary to give a helicity description to the element. Referring to Jacob and Wick [5]) and to Wick [6]), we can construct from the free states of two particles with given spins and helicities a superposition of states of total angular momentum according to

$$|p_1 s_1 \lambda_1, p_2 s_2 \lambda_2\rangle = \sum_{j_{12} m_{12}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}}_{m_{12}\lambda_{12}}(\theta_{12})|\theta_{12}\omega_{12} j_{12} m_{12}, s_1\lambda_1 s_2\lambda_2\rangle, \tag{30a}$$

where $\lambda_{12} = \lambda_1 - \lambda_2$, while j_{12} and m_{12} label the total angular momentum and its third component, and θ_{12} denotes the orientation of $\boldsymbol{q}_{12}$ with respect to the z-axis. More shortly we write

$$|12\rangle = \sum_{j_{12} m_{12}} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}}_{m_{12}\lambda_{12}}(\theta_{12})|(12)\rangle. \tag{30b}$$

Note that eqs. (30) are only true in the barycentric frame of 1 and 2.

The helicity states for 3 particles may be constructed similarly by coupling 3 with (12); see ref. [6]). We have

$$|123\rangle = \sum_{\substack{j_{123} m_{123} \\ j_{12} m_{12} \\ \mu_1 \mu_2}} \mathrm{e}^{-i\pi s_2} \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} \left(\frac{4\pi\omega_{123}}{q_{12,3}}\right)^{\frac{1}{2}} (2j_{123}+1)^{\frac{1}{2}}(2j_{12}+1)^{\frac{1}{2}} \times D^{j_{123}}_{m_{12,3}\lambda_{12,3}}(\theta_{12,3}) D^{j_{12}}_{m_{12}\mu_{12}}(\theta_{12}) D^{s_1 *}_{\lambda_1\mu_1}(\theta_1) D^{s_2}_{\lambda_2\mu_2}(\theta_2)|((12)3)\rangle, \tag{31}$$

where $\mu_{12} = \mu_1 - \mu_2$ and $\lambda_{12,3} = m_{12} - \lambda_3$.

The rotation angles are shown in the accompanying fig. 7. The quantities $q_{12,3}$ and ω_{123} represent the relative momentum and total energy of 3 and (12) in their

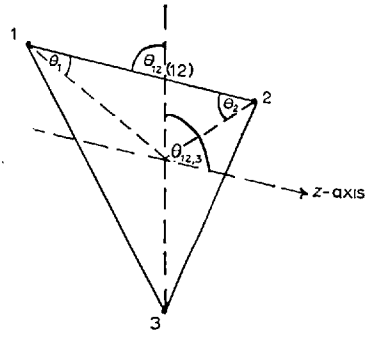


Fig. 7. Rotation angles for 3-particle states.

barycentric frame, (12) being regarded as a particle having mass ω_{12}. The recoupling coefficients $\langle((12))3|((13)2)\rangle$ have been given by Wick[6]). Using the above construction, we obtain in an obvious notation

$$\mathscr{M}(m_1^2, m_2^2) = \sum_{\lambda'_1\lambda'_2} \langle 12|M_{12}|1'2'\rangle\langle 1'2'3|M_0|\mathrm{AB}\rangle$$

$$= \sum_{\substack{j_{12}m_{12} \\ j'_{12}m'_{12} \\ \lambda'_1\lambda'_2}} \langle(12)|M_{12}|(1'2')\rangle\langle(1'2')3|M_0|\mathrm{AB}\rangle \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)(2j'_{12}+1)^{\frac{1}{2}}$$

$$\times D^{j_{12}*}_{m_{12}\lambda_{12}}(\theta_{12}) D^{j_{12}}_{m_{12}\lambda'_{12}}(\theta'_{12}) D^{j'_{12}*}_{m'_{12}\lambda'_{12}}(\theta'_{12}), \quad (32)$$

having used the rotational invariance of the M_{12} element. We are now in a position to integrate over θ'_{12}. The properties of the rotation matrices lead to

$$\bar{\mathscr{M}} = \sum_{\substack{j_{12}m_{12} \\ \lambda'_1\lambda'_2}} \langle(12)|M_{12}|(1'2')\rangle\langle(1'2')3|M_0|\mathrm{AB}\rangle \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}}_{m_{12}\lambda_{12}}(\theta_{12}). \quad (33)$$

On the other hand, for the simple production amplitude, we have

$$M_{0\mathrm{fi}} = \sum_{j_{12}m_{12}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\lambda_{12}}(\theta_{12}) \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} \langle(12)3|M_0|\mathrm{AB}\rangle. \quad (34)$$

Hence we obtain in the low energy limit of final state scattering between only 1 and 2

$$M_{\mathrm{fi}} = \sum_{\substack{j_{12}m_{12} \\ \lambda'_1\lambda'_2}} \left[1 - \frac{c_{j_{12}m_{12}}}{4q_{12}/\omega_{12}} \langle(12)|M_{12}|(1'2')\rangle\right] \langle(1'2')3|M_0|\mathrm{AB}\rangle$$

$$\times \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} (2j_{12}+1)^{\frac{1}{2}} D^{j_{12}*}_{m_{12}\lambda_{12}}(\theta_{12}), \quad (35a)$$

where

$$c_{j_{12}m_{12}} = \frac{1}{\pi}\left[\log\frac{\Delta^2_{j_{12}m_{12}}}{m_1 m_2} + 1 + \frac{m_2 - m_1}{m_2 + m_1}\log\left(\frac{m_1}{m_2}\right)\right]. \quad (36)$$

Completing the helicity description of (35a) and writing the details in full,

$$M_{\mathrm{fi}} = \sum_{\substack{j_{12}m_{12} \\ \nu_1\nu_2 \\ \mu_1\mu_2 \\ j_{\mathrm{AB}}m_{\mathrm{AB}}}} \left[1 - \frac{c_{j_{12}m_{12}}}{4q_{12}/\omega_{12}} \langle\lambda_1\lambda_2|M^{j_{12}}(\omega_{12})|\nu_1\nu_2\rangle\right] \left(\frac{4\pi\omega_{12}}{q_{12}}(2j_{12}+1)\right)^{\frac{1}{2}} \frac{4\pi\omega_{\mathrm{AB}}}{q^{\frac{1}{2}}_{\mathrm{AB}} q^{\frac{1}{2}}_{1,23}}$$

$$\times (2j_{\mathrm{AB}}+1)\mathrm{e}^{-i\pi s_2} D^{j_{12}*}_{m_{12}\lambda_{12}}(\theta_{12}) D^{j_{\mathrm{AB}}*}_{m_{\mathrm{AB}}\lambda_{12,3}}(\theta_{12,3}) D^{s_1}_{\nu_1\mu_1}(\theta_1) D^{s_2}_{\nu_2\mu_2}(\theta_2) D^{j_{\mathrm{AB}}}_{m_{\mathrm{AB}}\lambda_{\mathrm{AB}}}(\theta_{\mathrm{AB}})$$

$$\langle\mu_1\mu_2\lambda_3|M^{j_{\mathrm{AB}}j_{12}m_{12}}(\omega_{\mathrm{AB}})|\lambda_{\mathrm{A}}\lambda_{\mathrm{B}}\rangle, \quad (35b)$$

which is the final expression. We have used the translational and rotational invariance of the whole M_0 element to put $\omega_{123} = \omega_{\mathrm{AB}}$ and $j_{\mathrm{AB}} = j_{123}$ and $m_{\mathrm{AB}} = m_{123}$. By appropriate use of the symmetry properties of helicity states (see Jacob and Wick [5])), it is possible to diagonalise the elements and therefore to express them in the form

$-2e^{i\delta}\sin\delta$ through unitarity, the δ belonging to various symmetry assignments. Note that if Λ is of the order of the nucleon mass and one of the particles is a pion, $c \approx 1$ as before.

As the most trivial example, we write (35a) for the case when 1 and 2 are spinless. Because $\lambda_1 = \lambda_2 = \lambda_1' = \lambda_2' = 0$ and j_{12} is just the relative orbital momentum l_{12}, we have in that case

$$M_{\mathrm{fi}} = \sum_{l_{12}m_{12}} (2l_{12}+1)^{\frac{1}{2}} D^{l_{12}*}_{m_{12}0}(\theta_{12}) \left[1 + c_{l_{12}m_{12}} \frac{e^{i\delta_{l_{12}}} \sin\delta_{l_{12}}}{2q_{12}/\omega_{12}}\right] \left(\frac{4\pi\omega_{12}}{q_{12}}\right)^{\frac{1}{2}} \langle (12)3|M_0^{l_{12}m_{12}}|\mathrm{AB}\rangle, \tag{37}$$

which compares exactly with (19).

We conclude this section with various remarks about the application of (19) or (35) to experiments. Once the production amplitude and the final state scattering amplitudes are known it becomes a matter of estimating c_{jm} in some way and proceeding with the calculation, the constant c ≈ 1 if we accept a nucleon mass cut-off, but on the other hand there are certain processes[8]) where c_{jm} can be expected to be small because the final state scattering is almost real. In practice we shall only retain the lowest order in each final state scattering channel and we may justify this for attractive interactions as follows: a pair of particles will scatter one another most for low relative momenta, causing the other particle to move with a high relative momentum in the opposite direction to the pair; subsequent scattering of this odd particle with one of the particles of the pair will be small and unlikely. A similar sort of argument will apply if all the particles repel one another, but, if some attract and some repel the argument does not hold. A more rigorous justification for retaining the lowest order scatterings can probably only be given by dispersion relations. [This might apply for instance (see fig. 8) to the final state interaction between the pion from the

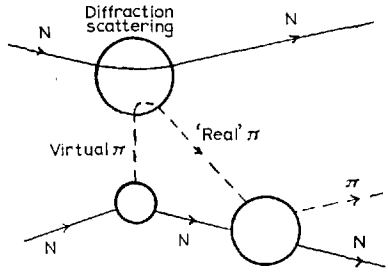


Fig. 8. An example of quasi-real final state scattering.

cloud of a nucleon (produced by a diffraction scattered nucleon) and the nucleon itself, as shown in the diagram. This was pointed out to me by Dr. M. Islam.]

5. $K^{\pm}$ Meson Decay

As an illustration of the previous work we try one of the simplest possible problems, $K^{\pm}$ meson decay, in the hope of obtaining information about π-π scattering. This

example has been approached heretofore in a variety of ways [7-11]), and roughly speaking, these have led to two different sets of conclusions: (i) that $T = 2$ is strongly attractive and $T = 0$ slightly repulsive, or (ii) that $T = 2$ is slightly repulsive and $T = 0$ much more so. We have obtained both sets of solutions by our method. Of particular interest to us is the analysis of Thomas and Holladay [9]) who assume that a pair of π^+ mesons attract in S-waves with a scattering length of about 1 pion Compton wavelength, and that π^+ and π^- interact very weakly. Watson's theorem then gives $|M_\tau|^2 \propto a_2^2/(1+a_2^2q^2)$. However, no account has been taken of the production amplitude, and as absence of π^+-π^- force implies some repulsion in $T = 0$, it is not directly obvious how to apply Watson's analysis to τ' decay, say.

Following the notation of Khuri and Treiman [7]), let 1 and 2 stand for the like pions, and 3 for the odd pion. The isovector labels are taken as α, β and γ respectively, and ρ for the $K^\pm$ meson because the latter acts effectively as a particle of isospin 1 under the $|\Delta T| = \frac{1}{2}$ rule. We split the total potential into a π-π part and the remainder, so that to first order, we shall just have the diagrams of fig. 4, where the initial particle is the K meson.

A typical term which we shall have to calculate is illustrated in fig. 9. By eqs. (29)

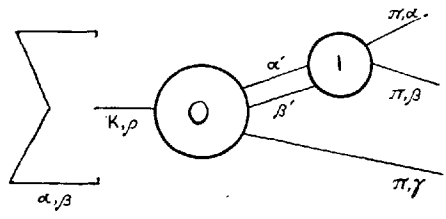


Fig. 9. A typical first order scattering in K±-meson decay.

and (28b) it will have the matrix element

$$M_{\rho\alpha\beta\gamma} = \sum_{\alpha'\beta'} - \frac{1}{16\pi^2}\left(1+2\log\frac{\Lambda}{\mu}\right)\int M_{0\rho\alpha'\beta'\gamma}(\omega_{12}, \theta'_{12})M_{1\alpha'\beta'\alpha\beta}(\omega_{12}, \theta'_{12})\frac{d\Omega'_{12}}{4\pi}$$

$$\equiv \sum_{\alpha',\beta'} - \frac{c}{16\pi}\int M_{0\rho\alpha'\beta'\gamma}(\omega_{12}, \theta'_{12})M_{1\alpha'\beta'\alpha\beta}(\omega_{12}, \theta'_{12})\frac{d\Omega'_{12}}{4\pi}. \quad (38)$$

Here $c = (1+2\log(\Lambda/\mu))/\pi$, where μ is the pion mass. Estimating it on the basis of a cut-off Λ when a nucleon is capable of being produced in the barycentric frame of the two pions, $\Lambda \approx 4.3\ \mu$, therefore $c \approx 1.25$ (of course this is not physically possible by angular momentum conservation, but it will do for our rough purpose).

The simplest thing we can do is take the M_0 production amplitudes to be structureless, i.e.

$$M_{0\rho\alpha\beta\gamma} = g(\delta_{\rho\alpha}\delta_{\beta\gamma}+\delta_{\rho\beta}\delta_{\gamma\alpha}+\delta_{\rho\gamma}\delta_{\alpha\beta}), \quad (39)$$

where g is some weak coupling constant. Actually, it is a simple matter to accomo-

date some structure by replacing $g\delta_{\rho\alpha}\delta_{\beta\gamma}$ by $g+g'(4q_{23}^2/\omega_{23}^2)Y_{lm}(\theta_{23})$ and similarly for $\delta_{\rho\beta}\delta_{\gamma\alpha}$ and $\delta_{\rho\gamma}\delta_{\alpha\beta}$. As this will only complicate matters, we shall adhere to (39). As for π-π scattering, it is well known that [12])

$$M_{1\alpha'\beta'\alpha\beta} = \tfrac{1}{3}(M^{(0)}-M^{(2)})\delta_{\alpha'\beta'}\delta_{\alpha\beta}+\tfrac{1}{2}(M^{(2)}+M^{(1)})\delta_{\alpha'\alpha}\delta_{\beta'\beta}+\tfrac{1}{2}(M^{(2)}-M^{(1)})\delta_{\alpha'\beta}\delta_{\beta'\alpha}, \quad (40)$$

where $M^{(T)}$ is the scattering element for isospin T. Combining (39) with (40) and summing over isovector labels, (38) reduces to

$$M_{\rho\alpha\beta\gamma} = -\frac{gc}{16\pi}\int\frac{d\Omega'_{12}}{4\pi}\,[\delta_{\rho\gamma}\delta_{\alpha\beta}\{\tfrac{5}{3}M^{(0)}(\omega_{12},\theta'_{12})-\tfrac{2}{3}M^{(2)}(\omega_{12},\theta'_{12})\} + \{\delta_{\rho\alpha}\delta_{\beta\gamma}+\delta_{\rho\beta}\delta_{\alpha\gamma}\}M^{(2)}(\omega_{12},\theta'_{12})]. \quad (41)$$

A partial wave decomposition of each isospin amplitude leads to

$$M_{\rho\alpha\beta\gamma} = \frac{gc\omega_{12}}{2q_{12}}\,[\delta_{\rho\gamma}\delta_{\alpha\beta}\{\tfrac{5}{3}e^{i\delta^{(0)}(\omega_{12})}\sin\delta^{(0)}(\omega_{12})-\tfrac{2}{3}e^{i\delta^{(2)}(\omega_{12})}\sin\delta^{(2)}(\omega_{12})\} + \{\delta_{\rho\alpha}\delta_{\beta\gamma}+\delta_{\rho\beta}\delta_{\alpha\gamma}\}e^{i\delta^{(2)}(\omega_{12})}\sin\delta^{(2)}(\omega_{12})], \quad (42)$$

where $\delta^{(T)}$ is the S-wave phase shift for isospin T. No other wave can remain because of the complete symmetry of the production amplitude. If we characterise the scattering by a length approximation over the small range of momenta of the pions, we have

$$M_{\rho\alpha\beta\gamma} = gc[\delta_{\rho\gamma}\delta_{\alpha\beta}\{\tfrac{5}{3}f^{(0)}(\omega_{12})-\tfrac{2}{3}f^{(2)}(\omega_{12})\}+\{\delta_{\rho\alpha}\delta_{\beta\gamma}+\delta_{\rho\beta}\delta_{\gamma\alpha}\}f^{(2)}(\omega_{12})], \quad (43a)$$

with

$$f^{(T)}(\omega_{ij}) = \frac{a_T}{1-ia_T(2q_{ij}/\omega_{ij})}, \quad (43b)$$

a_T being the scattering length.

Adding the four diagrams we can write the full amplitude in the form

$$M_{\rho\alpha\beta\gamma} = A\delta_{\rho\alpha}\delta_{\beta\gamma}+B\delta_{\rho\beta}\delta_{\gamma\alpha}+C\delta_{\rho\gamma}\delta_{\alpha\beta}, \quad (44)$$

$$A = g[1+c\{\tfrac{5}{3}f^{(0)}(\omega_{23})-\tfrac{2}{3}f^{(2)}(\omega_{23})+f^{(2)}(\omega_{31})+f^{(2)}(\omega_{12})\}], \quad (45a)$$

$$B = g[1+c\{f^{(2)}(\omega_{23})+\tfrac{5}{3}f^{(0)}(\omega_{31})-\tfrac{2}{3}f^{(2)}(\omega_{31})+f^{(2)}(\omega_{12})\}], \quad (45b)$$

$$C = g[1+c\{f^{(2)}(\omega_{23})+f^{(2)}(\omega_{31})+\tfrac{5}{3}f^{(0)}(\omega_{12})-\tfrac{2}{3}f^{(2)}(\omega_{12})\}]. \quad (45c)$$

In terms of A, B and C we have

$$M_\tau = A+B = 2g\left[1+c\left\{\frac{5f^{(0)}(\omega_{23})+f^{(2)}(\omega_{23})}{6}+\frac{5f^{(0)}(\omega_{31})+f^{(2(}(\omega_{31})}{6}+f^{(2(}(\omega_{12})\right\}\right], \quad (46a)$$

$$M_{\tau'} = C = g[1+c\{f^{(2)}(\omega_{23})+f^{(2)}(\omega_{31})+\tfrac{5}{3}f^{(0)}(\omega_{12})-\tfrac{2}{3}f^{(2)}(\omega_{12})\}], \quad (46b)$$

showing to what extent the elements are influenced by the two isospin scatterings.

Let m be the mass of the $K^{\pm}$ meson and T_k be the kinetic energy of pion k. The kinematics then give

$$4q_{ij}^2 = 2m(\overline{T}-T_k) \equiv 2m\overline{T}(1-t_k), \tag{47}$$

where $\overline{T}$ is the maximum pion energy. In units of $\overline{T}$, $t_1+t_2+t_3 \approx \frac{3}{2}$. Because q_{ij} is never large, we shall put $\omega_{ij} \approx 2\mu$ in $f(\omega_{ij})$. We find

$$\frac{4q_{ij}^2}{\omega_{ij}^2} \approx \frac{m\overline{T}}{2\mu^2}(1-t_k) \equiv \rho^2(1-t_k), \tag{48}$$

where $\rho^2 \approx 0.64$, and

$$f^{(T)}(\omega_{ij}) \approx \frac{a_T}{1-ia_T q_{ij}/\mu} = \frac{a_T}{1-ia_T\rho(1-t_k)^{\frac{1}{2}}}. \tag{49}$$

The spectrum is most easily shown in fig. 10, the Fabri-Dalitz plot, and the latest

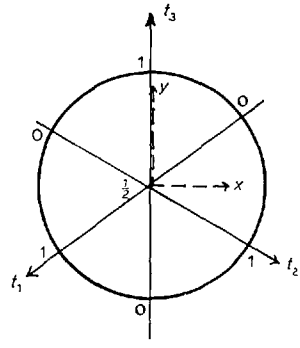


Fig. 10. The Fabri-Dalitz plot for τ and τ' decays. The subscript 3 stands for the unlike pion.

experimental evidence [13]) indicates that $|M_\tau|^2$ is a nearly linear function of t_3, rising with a slope of about 50%, and is apparently independent of x. Before proceeding with a detailed calculation, we can crudely ascertain the expected solutions with the help of figs. 5(a) and 5(b), if we neglect some interference terms upon squaring (46a). These are: (i) $a_2 \approx 2$ to give a rise of $|M_\tau|^2$ with t_3 (a small negative a_0 will then ensure that most of the variation along t_1 and t_2 is removed), (ii) $a_2 \approx 0$, so that the rise along t_3 occurs through the repulsive effect along t_1 and t_2 with $a_0 \approx -0.5$.

For the purposes of calculation we chose $c = 1.25$ and then varied a_2 and a_0, in such a manner that $|M_\tau|^2$ showed a slope of about 50% in t_3 and negligible variation ($\lesssim 5\%$) along the x axis. Not unreasonably, we restricted our scattering lengths to have magnitudes $\leqq 2$ pion wavelengths. It was found that

(i) $a_2 \approx 1.7 \pm 0.5$, $\quad a_0 \approx 0.7 \mp 0.5$,

(ii) $a_2 \approx 0 \pm 0.1$, $\quad a_0 \approx -0.3 \mp 0.1$ $\quad$ or $\quad -0.7 \pm 0.1$.

The two sets of solution in (ii) come about because there are two repulsive scattering lengths that give the same $|M|^2$, as we may see in fig. 5(b).

The answers are probably correct to within 0.2, allowing c a small latitude, and are plotted in fig. 11 for convenience. Graphs are drawn in fig. 12 in order to show the

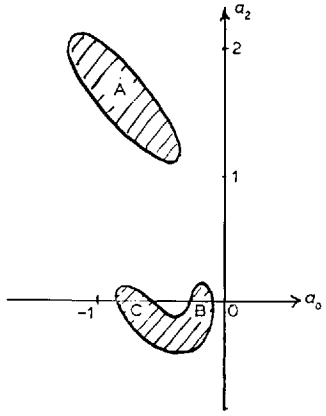


Fig. 11. The relationship between allowed values of the S-wave scattering lengths.

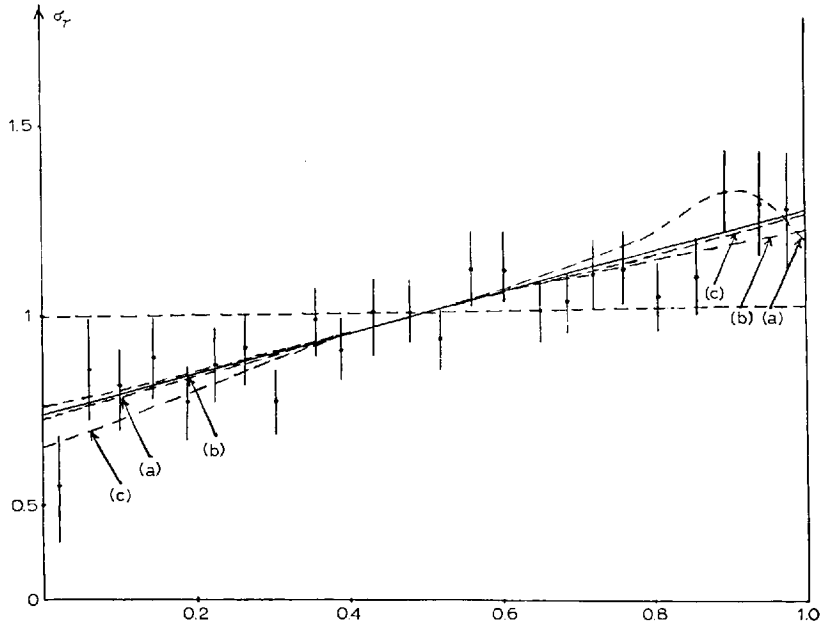


Fig. 12. The predicted spectrum for τ decay. The curves (a), (b) and (c) represent the three sets of solutions given by the values (50). The solid line shows the best straight line fit. The abscissa gives the kinetic energy of the π^- meson in units of the maximum.

fits with the data on $|M_\tau|^2$ of ref. [13]), while fig. 13 shows the predicted spectrum for τ' decay. The parameter c was fixed at 1.25 and the curves are for the following pairs

of values:

$$\begin{aligned} &\text{(a)} \quad a_2 = 1.7, \quad a_0 = -0.7, \\ &\text{(b)} \quad a_2 = 0, \quad a_0 = -0.38, \\ &\text{(c)} \quad a_2 = 0, \quad a_0 = -0.66. \end{aligned} \tag{50}$$

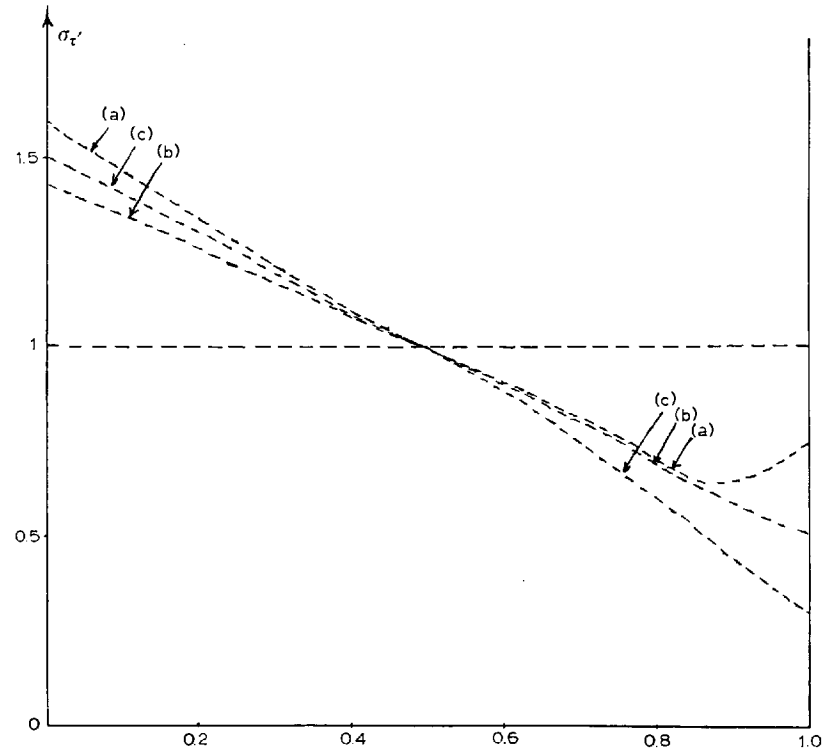


Fig. 13. The predicted spectrum for τ' decay. The curves (a), (b) and (c) represent the three sets of solutions given by the values (50). The abscissa gives the kinetic energy of the π^+ meson in units of the maximum.

There are thus two distinct regions in which it is possible to obtain good agreement with experiment. The region A has more or less been proposed by the group of people [9-11]) working in terms of potentials, while the region C corresponds to the dispersion-theory group [7,8]). As it is, both regions are compatible with the experiment and it will need some other evidence to allow us to decide which region is the correct one. The attractive a_2 that we propose is somewhat larger than the Thomas and Holladay [9]) prediction because the production part of the amplitude has not been overlooked. As a final remark, we note that the region C will compare with the Khuri and Treiman solution [7]) and will correspond to a pion-pion coupling constant of $\lambda \approx 0.1$. (ref. [12])).

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Calculation of the Yukawa Coupling Constant.

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Summary — By expressing Feynman digrams as dispersion integrals with a finite cut-off, we calculate to lowest order the renormalization constants Z_1, Z_2, Z_3, for a Yukawa coupling. By setting each Z equal to zero we obtain the approximate solutions $g^2/4\pi = 0$ (30) and $\mu/m \leqslant 0.2$ with a cut-off of about 3 nucleon masses.

1. – Introduction.

On the basis that each particle in nature is to be regarded as a bound state of all other existing particles, SALAM (1) has proposed that the three constants δm, $\delta\mu^2$ and Z_1 all vanish. (Here m is the fermion mass and μ is the boson mass.) For instance if all the particles in nature were either fermions or pseudoscalar bosons interacting via the Yukawa coupling $g\bar{\psi}\gamma_5\psi\varphi$ then

$$Z_1\,(g, m, \mu) = 0 \tag{1}$$

$$\delta m\,(g, m, \mu) = 0 \tag{2}$$

$$\delta\mu^2(g, m, \mu) = 0 \tag{3}$$

SALAM (1) has further pointed out that in order to satisfy eqs. (2) and (3) it is sufficient to put $Z_2 = 0$ and $Z_3 = 0$.

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(1) A. SALAM: *Nuovo Cimento*, **25**, 224 (1962).

It is well known, of course, that all these constants diverge, but if we carry out a perturbation calculation with the inclusion of a finite cut-off Λ, we may well be able to solve $Z_1(g, m, \mu, \Lambda) = 0$, etc.

The question remains as to how one must introduce this cut-off in a more or less unique fashion. Thus, we cannot proceed in the standard manner [2] by separating out the divergent from the finite part, because such methods rely entirely on the fact that Λ is very large, so that various theorems on shifting the origin of integration, can be established. Instead we shall express the Feynman diagrams through a dispersion representation by the CUTKOSKY prescription [3], and in this way, the cut-off appears only in the final integration over masses. In fact the dispersion integrals will just be equivalent to the LEHMANN spectral representation [4] in lowest order. The advantage of this procedure is that a calculation of even the fourth order diagrams is not too difficult, and in particular the self energy diagrams which contain overlapping divergences.

Having calculated the Z to lowest order in perturbation theory, we find that $(1 - Z_3) \geqslant 2(1 - Z_1)$ so that it is impossible for all three Z's to vanish simultaneously. However, we do solve the pairs of equations: (i) $Z_1 = Z_2 = 0$ and (ii) $Z_3 = Z_2 = 0$, for $g^2/4\pi$ and Λ^2/m^2 in terms of μ^2/m^2. By comparing magnitudes (see Section **5** for discussion of this point) we arrive at the estimates $g^2/4\pi = 0$ (30), $\mu/m \leqslant 0.2$, $\Lambda/m = 0$ (3).

Finally we take our fermion to be an isospinor, and our boson to be an isovector; in this case we have $Z_1 > 1$ and it is only possible to solve (ii). Moreover the arguments used in obtaining the previous solutions are no longer applicable, so one can no longer reach any conclusion. However, disregarding the difficulty due to the nonvanishing of Z_1, and choosing $\mu/m \leqslant 0.1$, we still have $g^2/4\pi \approx 25$ and $\Lambda/m \approx 3$.

2. – Boson self mass.

Figure 1*a* gives the loop contribution as

$$\Pi(k^2) = -\frac{ig^2}{(2\pi)^4}\int \mathrm{d}^4 p \, \frac{4(p^2 - p\cdot k - m^2)}{[p^2 - m^2][(p-k)^2 - m^2]}. \tag{4}$$

We relate this to the constants in question by the expansion

$$\Pi(k^2) = -\delta\mu^2 + (Z_3^{-1} - 1)(k^2 - \mu^2) + \Pi_f(k^2)(k^2 - \mu^2)^2, \tag{5}$$

(2) J. M. JAUCH and R. ROHRLICH: *The Theory of Photons and Electrons* (Cambridge, Mass., 1955).

(3) R. E. CUTKOSKY: *Journ. Math. Phys.*, **1**, 429 (1960).

(4) H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

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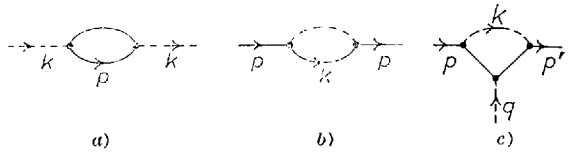
$$\delta\mu^2 = -\Pi(\mu^2) \tag{6}$$

and

$$Z_3^{-1} = 1 + \frac{\partial\Pi(k^2)}{\partial k^2}\Big|_{k^2=\mu^2}, \tag{7}$$

or, what we use instead,

$$Z_3 = 1 - \Pi'(\mu^2)\,. \tag{7'}$$



a) b) c)

Fig. 1. – *a*) Self mass of boson; *b*) self mass of fermion; *c*) vertex part.

The integral (4) has a normal threshold singularity (5) at $k^2 = 4m^2$ and across the normal branch cut, the Cutkosky prescription (3) gives

$$\frac{1}{\pi}\operatorname{Im}\Pi(k^2) = \frac{g^2}{(2\pi)^3}\int 4p\cdot k\,\delta(p^2-m^2)\theta(p)\,\delta(k^2-2p\cdot k)\theta(k-p)\,\mathrm{d}^4p = \tag{8}$$

$$= \frac{g^2}{8\pi^2}\,\theta(k^2-4m^2)\{k^4-4m^2k^2\}\,.$$

Writing down the usual dispersion relation with a cut-off Λ^2,

$$\Pi(k^2) = \frac{1}{\pi}\int_{4m^2}^{\Lambda^2}\frac{\operatorname{Im}\Pi(K^2)}{K^2-k^2-i\varepsilon}\,\mathrm{d}K^2\,, \tag{9}$$

and using (6) and (7)

$$\delta\mu^2 = -\frac{g^2}{8\pi^2}\int_{4m^2}^{\Lambda^2}\frac{K(K^2-4m^2)^{\frac{1}{2}}}{K^2-\mu^2}\,\mathrm{d}K^2\,, \tag{10}$$

$$Z_3 = 1 - \frac{g^2}{8\pi^2}\int_{4m^2}^{\Lambda^2}\frac{K(K^2-4m^2)^{\frac{1}{2}}}{(K^2-\mu^2)^2}\,\mathrm{d}K^2\,. \tag{11}$$

(5) See for instance, R. J. EDEN: *Lectures on the use of Perturbation Methods in Dispersion Theory*, technical report no. 211 (University of Maryland, 1961).

3. – Fermion self mass.

In Fig. 1*b*, the loop contribution is

$$\sum(p) = -\frac{ig^2}{(2\pi)^4}\int \mathrm{d}^4k \frac{\gamma\cdot k - m}{[k^2-m^2][(k-p)^2-\mu^2]} = mA(p^2) + \gamma\cdot pB(p^2)\,. \tag{12}$$

But, according to the usual identification,

$$\sum(p) = \delta m + (Z_2^{-1}-1)(\gamma\cdot p - m) + \sum_f (p)(\gamma\cdot p - m)^2, \tag{13}$$

so

$$\delta m = \sum(p)\,|_{\gamma\cdot p = m} = m[A(m^2) + B(m^2)] \tag{14}$$

and

$$Z_2^{-1} = 1 + \frac{p_\mu}{m}\frac{\partial\sum(p)}{\partial p_\mu}\Big|_{\gamma\cdot p = m}, \tag{15}$$

or equivalently

$$Z_2 = 1 - \frac{p_\mu}{m}\frac{\partial\sum(p)}{\partial p_\mu}_{\gamma\cdot p = m} = 1 - 2m^2[A'(m^2) + B'(m^2)] - B(m^2)\,. \tag{15'}$$

We can therefore determine δm and Z_2 from the integral representations of A and B:

$$A(p^2) = \frac{\mathrm{Tr}\,[\sum(p)]}{4m} = \frac{ig^2}{(2\pi)^4}\int \frac{1}{[k^2-m^2][(k-p)^2-\mu^2]}\,\mathrm{d}^4k\,,$$

$$B(p^2) = \frac{\mathrm{Tr}\,[\gamma\cdot p\sum(p)]}{4p^2} = -\frac{ig^2}{p^2(2\pi)^4}\int \frac{p\cdot k}{[k^2-m^2][(k-p)^2-\mu^2]}\,\mathrm{d}^4k\,,$$

$$\frac{1}{\pi}\,\mathrm{Im}\,A(p^2) = -\frac{g^2}{(2\pi)^3}\int \delta(k^2-m^2)\theta(k)\,\delta(p^2-2p\cdot k+m^2-\mu^2)\theta(p-k)\,\mathrm{d}^4k = \tag{16}$$

$$= -g^2\{(p^2-m^2-\mu^2)^2-4m^2\mu^2\}^{\frac{1}{2}}\theta[p^2-(m+\mu)^2]/16\pi^2p^2\,,$$

$$\frac{1}{\pi}\,\mathrm{Im}\,B(p^2) = \tag{17}$$

$$= \frac{g^2}{(2\pi)^3p^2}\int k\cdot p\,\delta(k^2-m^2)\theta(k)\,\delta(p^2-2p\cdot k+m^2-\mu^2)\theta(p-k)\,\mathrm{d}^4k =$$

$$= g^2(p^2+m^2-\mu^2)\{(p^2-m^2-\mu^2)^2-4m^2\mu^2\}^{\frac{1}{2}}\theta[p^2-(m+\mu)^2]/32\pi^2p^4\,.$$

Hence,

$$\delta m = \frac{m}{\pi}\int_{(m+\mu)^2}^{\Lambda^2} \frac{\operatorname{Im} A(K^2) + \operatorname{Im} B(K^2)}{K^2 - m^2}\,\mathrm{d}K^2\,, = \tag{18}$$

$$= -\frac{g^2 m}{32\pi^2}\int_{(m+\mu)^2}^{\Lambda^2} \frac{\{(K^2 - m^2 - \mu^2) - 4m^2\mu^2\}^{\frac{1}{2}}}{K^4}\left[1 + \frac{\mu^2}{K^2 - m^2}\right]\mathrm{d}K^2\,,$$

$$Z_2 = 1 - \frac{1}{\pi}\int_{(m+\mu)^2}^{\Lambda^2} \frac{2m^2 \operatorname{Im} A(K^2) + (K^2 + m^2)\operatorname{Im} B(K^2)}{(K^2 - m^2)}\,\mathrm{d}K^2 = \tag{19}$$

$$= 1 - \frac{g^2}{32\pi^2}\int_{(m+\mu)^2}^{\Lambda^2} \frac{\{(K^2 - m^2 - \mu^2)^2 - 4m^2\mu^2\}^{\frac{1}{2}}}{K^4}\left[1 - \frac{\mu^2(K^2 + m^2)}{(K^2 - m^2)}\right]\mathrm{d}K^2\,.$$

4. – Vertex part.

The most symmetrical way of obtaining a dispersion relation for the vertex function is to investigate the analytic behaviour in the momentum square of the boson, with both nucleons put on their mass shells. Alternatively we could make one boson and one fermion real, but besides being unsymmetrical we expect the same result for Z_1 at large cut-offs, as in the symmetrical treatment. We must mention that our Λ may not turn out large, so the result for Z_1 is certainly ambiguous. This ambiguity is true in any treatment of the vertex part.

The loop contribution of Fig. 1*c* is

$$g\Gamma_5(q^2) = -\frac{ig^3}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{[\gamma\cdot(p' - k) - m]\gamma_5[\gamma\cdot(p - k) - m]}{(p' - k)^2 - m^2][k^2 - \mu^2][(p - k)^2 - m^2]}\,,$$

so

$$\bar{u}(p')\Gamma_5^1(q^2)u(p) = \frac{ig^2\bar{u}(p')\gamma_5 u(p)}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{k^2}{[(p' - k)^2 - m^2][k^2 - \mu^2][(p - k)^2 - m^2]}\,. \tag{20}$$

The standard definition is $(Z_1^{-1} - 1)\gamma_5 = \Gamma_5(\mu^2)$, or instead

$$Z_1 = 1 - \Gamma_5(\mu^2)\gamma_5^{-1}\,. \tag{21}$$

The integral (20) has a normal branch cut starting at $q^2 = 4m^2$, and the discon-

tinuity across it is given by

$$(22)\qquad \frac{1}{\pi}\operatorname{Im}(\bar{u}\Gamma_5' u) = \frac{g^2\bar{u}\gamma_5 u}{(2\pi)^3}\int \delta(k^2-2p\cdot k)\,\delta(k^2-2p'\cdot k)\,\theta(p'-k)\,\theta(k-p)\,\frac{k^2}{k^2-\mu^2}\,\mathrm{d}^4k = \frac{g^2\bar{u}\gamma_5 u\,\theta(q^2-4m^2)}{16\pi^2\{q^4-4m^2q^2\}^{\frac{1}{2}}}\left[q^2-4m^2-\mu^2\log\left(\frac{q^2-4m^2+\mu^2}{\mu^2}\right)\right],$$

$$(23)\qquad \therefore\ Z_1 = 1-\frac{g^2}{16\pi^2}\int_{4m^2}^{\Lambda^2}\frac{K^2-4m^2-\mu^2\log[(K^2-4m^2+\mu^2)/\mu^2]}{K(K^2-\mu^2)\{K^2-4m^2\}^{\frac{1}{2}}}\,\mathrm{d}K^2\,.$$

5. – Calculation of the costants.

We summarize the results contained in eqs. (10), (11), (18), (19) and (23) by introducing the abbreviations,

$$x = K^2/m^2,\qquad y=\mu^2/m^2,\qquad Z=\Lambda^2/m^2,\qquad \text{and}\qquad \alpha = g^2/8\pi^2\,.$$

$$(10')\qquad \delta\mu^2 = -\alpha m^2\int_4^z\frac{(x^2-4x)^{\frac{1}{2}}}{x-y}\,\mathrm{d}x\,,$$

$$(11')\qquad Z_3 = 1-\alpha\int_4^z\frac{(x^2-4x)^{\frac{1}{2}}}{(x-y)^2}\,\mathrm{d}x\,,$$

$$(18')\qquad \delta m = -\frac{\alpha m}{4}\int_{(1+\sqrt{y})^2}^z\frac{\{(x-1-y)^2-4y\}^{\frac{1}{2}}}{x^2}\left[1+\frac{y}{x-1}\right]\mathrm{d}x\,.$$

$$(19')\qquad Z_2 = 1-\frac{\alpha}{4}\int_{(1+\sqrt{y})^2}^z\frac{\{(x-1-y)^2-4y\}^{\frac{1}{2}}}{x^2}\left[1-\frac{y(x+1)}{(x-1)^2}\right]\mathrm{d}x\,,$$

$$(23')\qquad Z_1 = 1-\frac{\alpha}{2}\int_4^z\frac{x-4-y\log[(x-4+y)/y]}{(x-y)(x^2-4x)^{\frac{1}{2}}}\,\mathrm{d}x\,.$$

As expected, the integrands in (10') and (18') are always positive within the ranges of integration and we cannot have $\delta m = \delta\mu^2 = 0$. However, it

is entirely feasible for the Z to vanish. Examination of the integrands in (11′) and (23′) shows that

$$(1 - Z_3) \geqslant 2(1 - Z_1)\,, \tag{24}$$

the equality sign holding in the limit $y \to 0$. Equation (24) means that it is impossible for Z_1 and Z_3 to vanish simultaneously, but we can still hope to solve the pairs of equations (i) $Z_1 = Z_2 = 0$ and (ii) $Z_3 = Z_2 = 0$, numerically, for α and z terms of y. These solutions are plotted in Fig. 2 and 3 for the range $0.001 \leqslant y \leqslant 1$. When $y \leqslant 0.01$, $Z_3 \approx -1$ in (i) and $Z_1 \approx \frac{1}{2}$ in (ii).

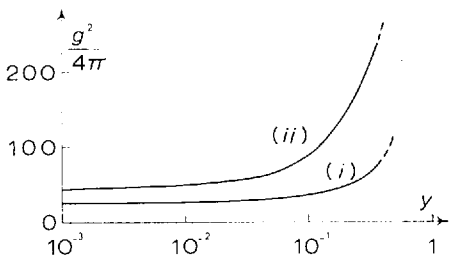


Fig. 2. – $g^2/4\pi$ as a function of $y = \mu^2/m^2$: (i) for $Z_1 = Z_2 = 0$; (ii) for $Z_3 = Z_2 = 0$.

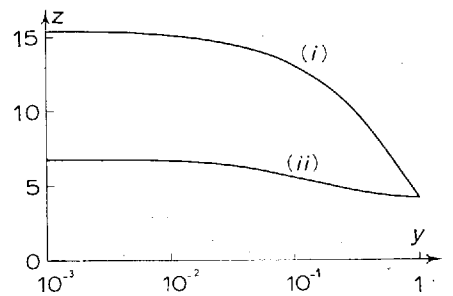


Fig. 3. – $z = \Lambda^2/m^2$ as a function of $y = \mu^2/m^2$: (i) for $Z_1 = Z_2 = 0$; (ii) for $Z_3 = Z_2 = 0$.

The fact that it is impossible to obtain exact solutions to $Z_1 = Z_2 = Z_3 = 0$ is perhaps not surprising, in the sense that a fixed cut-off for all three constants is probably too stringent. The existence of a cut-off corresponds to some nonloeality, and its value is always quoted to an order of magnitude. This suggests that we only compare the solutions of (i) with those of (ii), and in doing so, it is most important that the g^2 solutions (rather than z, which as we have stated, is a loose quantity) be close to one another. As another justification of this approach, we must add that the calculation of the Z, being done to lowest order only, must be subject to higher order corrections; therefore the solutions from the lowest order diagrams are only expected to be correct up to an order of magnitude.

As an artificial criterion, which is open to criticism, we have required the ratio of the g^2 solutions to be within $\frac{1}{2}$ and 2. Figure 2 then shows that $y \leqslant 0.05$, and in this range $g^2/4\pi = 0$ (30) and $z = 0$ (10), *i.e.* $\mu/m \leqslant 0.2$ and $\Lambda/m = 0$ (3). Surprisingly these values are very close to the meson-nucleon physical constants, although our model is unrealistic so far, as no account has been taken of isospin.

If we include the normal isospin properties into a new coupling $g\overline{\Psi}\gamma_5\tau\cdot\Phi\Psi$, the net effect is to leave α unchanged in Z_3 and to replace it by $\frac{3}{2}\alpha$ in Z_2 and $-\frac{1}{2}\alpha$ in Z_1. Immediately we fall into difficulties because $Z_1 \geqslant 1$. The new solutions to $Z_2 = Z_3 = 0$ are graphed in Fig. 4 and 5 and are much the same

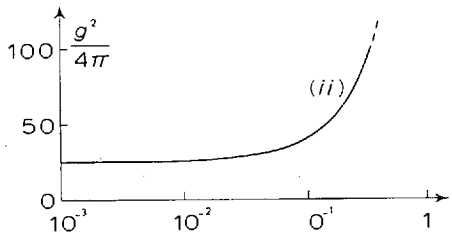


Fig. 4. – $g^2/4\pi$ as a function of $y = \mu^2/m^2$ for $Z_3 = Z_2 = 0$, isospin beig included.

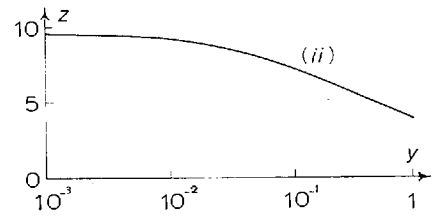


Fig. 5. – $z = A^2/m^2$ as a function of $y = \mu^2/m^2$ for $Z_3 = Z_2 = 0$, isospin being included

as before. However, the fact that $Z_1 > 1$ might lead us to conclude that at least another bound-state particle exists in nature, if we are to believe the approximate validity of lowest order perturbation theory and the truth of the theory [1].

* * *

I am indebted to Professor A. SALAM for proposing this problem, and numerous suggestions in connection with it; I also wish to thank Mr. J. STRATHDEE for many helpful comments.

RIASSUNTO (*)

Esprimendo i diagrammi di Feynman come integrali di dispersione con un taglio finito, si calcolano, nell'ordine più basso, le costanti di rinormalizzazione Z_1, Z_2, Z_3, per un accoppiamento di Yukawa. Ponendo ciascuna Z uguale a zero, si ottengono le soluzioni approssimate $g^2/4\pi = 0$ (30) ed $\mu/m \leqslant 0.2$ con un taglio di circa 3 masse nucleoniche.

(*) *Traduzione a cura della Redazione.*